Discrete Torsion and Gerbes II

Eric R. Sharpe
Department of Physics
Box 90305
Duke University
Durham, NC 27708
ersharpe@cgtp.duke.edu

In a previous paper we outlined how discrete torsion can be understood geometrically, as an analogue of orbifold $U(1)$ Wilson lines. In this paper we shall prove the remaining details. More precisely, in this paper we describe gerbes in terms of objects known as stacks (essentially, sheaves of categories), and develop much of the basic theory of gerbes in such language. Then, once the relevant technology has been described, we give a first-principles geometric derivation of discrete torsion. In other words, we define equivariant gerbes, and classify equivariant structures on gerbes and on gerbes with connection. We prove that in general, the set of equivariant structures on a gerbe with connection is a torsor under a group which includes $H^2(\Gamma, U(1))$, where $\Gamma$ is the orbifold group. In special cases, such as trivial gerbes, the set of equivariant structures can furthermore be canonically identified with the group.

September 1999
# Contents

## 1 Introduction

## 2 Stacks

- 2.1 Presheaf of categories .................................................. 6
- 2.2 Sheaf of categories ....................................................... 8
  - 2.2.1 Gluing law for objects ........................................... 8
  - 2.2.2 Gluing law for morphisms ....................................... 10
  - 2.2.3 Examples ............................................................ 11
- 2.3 Cartesian functors ....................................................... 12
- 2.4 2-arrows ....................................................................... 13
  - 2.4.1 2-arrows ................................................................ 13
  - 2.4.2 Sheaves of natural transformations ......................... 14

## 3 Gerbes and stacks

- 3.1 Definitions and examples ............................................... 16
- 3.2 Equivalences of gerbes ................................................... 18
- 3.3 Sheaf cohomology and gerbes ........................................ 18
- 3.4 Gauge transformations of gerbes ..................................... 21
- 3.5 Connections on gerbes ................................................... 22
- 3.6 Transition functions for gerbes ....................................... 27
- 3.7 Nonabelian gerbes ........................................................ 28

## 4 Technical notes on stacks

- 4.1 Sheafification ............................................................... 29
  - 4.1.1 Sheafification of presheaves of sets ......................... 29
1 Introduction

Historically discrete torsion has been a rather mysterious aspect of string theory. Discrete torsion was originally discovered [1] as an ambiguity in the choice of phases to assign to different sectors of string orbifold partition functions. Although other work has been done on the subject, no work done to date has succeeded in giving any sort of genuinely deep understanding of discrete torsion. In fact, discrete torsion has sometimes been referred to as an inherently stringy degree of freedom, without any geometric analogue.

In this paper we shall give a purely geometric understanding of discrete torsion, as a precise analogue of orbifold $U(1)$ Wilson lines, but for (two-form) $B$-fields rather than vector fields. In [2] we outlined this description; in this paper we shall prove the technical details omitted in [2]. In an upcoming paper [3] we shall rederive these results in a more elementary fashion, and also describe how this picture can be used to derive other physical manifestations of discrete torsion.

More precisely, in this paper we shall argue that discrete torsion should be understood as a (discrete) ambiguity in lifting the action of an orbifold group $\Gamma$ on a space $X$ to a 1-gerbe with connection on $X$, just as orbifold Wilson lines can be understood as an ambiguity in lifting the action of $\Gamma$ to a bundle with connection. This description makes no assumptions on the nature of $\Gamma$ – it may or may not be freely-acting, it may or may not be abelian – this description holds true regardless.

Our description of discrete torsion hinges on a deeper understanding of type II $B$-fields than is common in the literature. More specifically, just as vector fields are understood as connections on bundles, we describe $B$-fields as connections on (1-)gerbes. Although gerbes seem to be well-known in some circles, their usefulness does not seem to be widely appreciated. As accessible accounts of gerbes which provide the level of detail we need do not seem to exist, we describe gerbes in considerable detail.

We begin this paper by describing stacks (essentially, sheaves of categories), in section 2. Next, we give a basic description of gerbes in terms of stacks, in section 3. In order to define and study equivariant structures on gerbes, we need some rather technical results, which we collect in sections 4 and 5 on stacks and gerbes. (Readers studying this paper for the first time are advised to skip sections 4 and 5.) Finally, we define equivariant structures on gerbes, and classify equivariant structures on gerbes and gerbes with connection, deriving $H^2(\Gamma, U(1))$ in the process.

For additional information on gerbes, the reader might consult [4, 5, 6, 7, 8, 9, 10, 11].

In passing, we should mention that in addition to the description of gerbes in terms of stacks, there exist alternative descriptions. For example, (equivalence classes of) gerbes can be described in terms of objects on loop spaces, as described in (for example) [6]. (We
briefly review this description, using it to check our results, in section 8.) One noteworthy description not in [6] is known as “bundle gerbes” [12, 13]. Because of certain technical difficulties with the description of gerbes in terms of bundle gerbes, notably the difficulty in determining whether two bundle gerbes are isomorphic, we shall not refer to them any further in this paper. In general, which description of gerbes is most useful clearly depends upon both the application in mind and personal preference.

In an earlier version of this paper, we argued that the difference between any two equivariant structures on a 1-gerbe with connection is an element of $H^2(\Gamma, U(1))$. We have since corrected a minor error in that calculation, and weakened the result to only claim that the difference between equivariant structures is an element of a group which includes $H^2(\Gamma, U(1))$. In other words, there are additional degrees of freedom beyond just those encoded in $H^2(\Gamma, U(1))$, which we missed previously.

2 Stacks

The reader might well ask what object should be associated to a gerbe – after all, in [2] we really only referred to gerbes in terms of sheaf cohomology groups. The answer is that gerbes can be understood in terms of sheaves of categories, also known as stacks, which we shall review in this section.

Our presentation of stacks closely follows [6, section 5], [7, section 3], and [10, chapter 1].

2.1 Presheaf of categories

Before defining a sheaf of categories, we shall first define a presheaf of categories, which are sometimes also called prestacks. We shall closely follow the definitions of [10]. In passing we shall note that stacks and related ideas are often defined with respect to Grothendieck topologies and sites, i.e., in the language of descent, rather than the topologies that most physicists are acquainted with. We feel that such definitions add an essentially irrelevant (for our purposes) layer of technical abstraction, and so have circumvented them. (For definitions of stacks in terms of sites, see for example [6], [9, exposé VI], or [14]; for a basic introduction to the ideas of Grothendieck topologies, see for example [15, 16].)

Before we define a presheaf of categories, let us take a moment to review the notion of a presheaf of sets, following [6, section 1.1]. A presheaf of sets $\mathcal{S}$ on a space $X$ is an assignment of a set $\mathcal{S}(U)$ to every open set $U \subseteq X$, together with a map $\rho^*_{UV} : \mathcal{S}(U) \to \mathcal{S}(V)$ associated to each inclusion $\rho_{VU} : V \hookrightarrow U$, such that if $W \subseteq V \subseteq U$ are open sets, then

$$\rho^*_{UV} \circ \rho^*_{WV} = \rho^*_{WU}$$
and such that $\rho_{UV}^*$, the map associated to the trivial inclusion $U \hookrightarrow U$, is the identity. To define a presheaf of categories we shall follow a similar pattern: we shall associate a category to each open set, and a functor to each inclusion of open sets, with certain constraints on the functors.

A presheaf of categories $\mathcal{C}$ on a topological space $X$ associates, to any open set $U \subseteq X$, a category $\mathcal{C}(U)$, and for every inclusion of open sets $\rho : U_2 \hookrightarrow U_1$, there is a restriction functor $\rho^* : \mathcal{C}(U_1) \to \mathcal{C}(U_2)$, which may be taken to be the identity whenever $U_1 = U_2 = U$ and $\rho = 1_U$. (A word on notation: we shall sometimes use the notation $\mathcal{C}(\rho)$ or $|_U$ instead of $\rho^*$ in this paper.)

In passing, we should point out that not every category $\mathcal{C}(U)$ need be nonempty – for example, we shall see later that a nontrivial 1-gerbe over a space $X$ necessarily has $\mathcal{C}(X)$ empty.

The restriction functors are required to satisfy two conditions. The first is that for every pair of composable inclusions of open sets $\rho_1 : U_2 \hookrightarrow U_1$ and $\rho_2 : U_3 \hookrightarrow U_2$, one is given an invertible natural transformation $\varphi_{\rho_1, \rho_2} : (\rho_1 \rho_2)^* \Longrightarrow \rho_2^* \circ \rho_1^*$

The second condition on restriction functors is that if one has three composable inclusions $\rho_1 : U_2 \hookrightarrow U_1$, $\rho_2 : U_3 \hookrightarrow U_2$, and $\rho_3 : U_4 \hookrightarrow U_3$, then the following diagram of natural transformations is required to commute:

$$
\begin{array}{ccc}
(\rho_1 \rho_2 \rho_3)^* & \Longrightarrow & \rho_3^* \circ (\rho_1 \rho_2)^* \\
\downarrow & & \downarrow \\
(\rho_2 \rho_3)^* \circ \rho_1^* & \Longrightarrow & \rho_3^* \circ \rho_2^* \circ \rho_1^*
\end{array}
$$

(1)

An assignment of categories to open sets, together with inverse image functors satisfying the two conditions above, defines a presheaf of categories.

In order to get a better grasp of this material, the reader is encouraged to compare the presheaf of categories defined above with the general definition of presheaf.

At this point an example might help the reader. It is possible to describe a presheaf of sets as a special kind of presheaf of categories. To do this, consider a set as a discrete category – a category whose objects are the elements of the set, and whose only morphisms are the identity morphisms mapping any object back to itself. It can then be shown fairly easily that if we think of a set as a discrete category, then a presheaf of sets is a special kind of presheaf of categories.

Readers closely watching related references will note that in [chapter 5], these natural transformations are defined in the opposite direction.

7
Before going on, we should also describe how any presheaf of categories contains within it, a number of presheaves of sets. More specifically, given a presheaf of categories \( \mathcal{C} \) on a space \( X \), we shall define, for any open \( U \subseteq X \) and any two objects \( P_a, P_b \in \text{Ob} \mathcal{C}(U) \), a presheaf (of sets) \( \text{Hom}_U(P_a, P_b) \) of local morphisms from \( P_a \) to \( P_b \). This presheaf of sets is defined as follows. For any open \( V \subseteq U \), the set of sections of the presheaf is given by \( \text{Hom}_C(V) (P_a|_V, P_b|_V) \). For any inclusion \( \rho : U_2 \hookrightarrow U_1 \) of open sets \( U_1, U_2 \subseteq U \), define the restriction map \( \rho^* \) by,

\[
\rho^*: \beta \mapsto \varphi_{1,2}^{-1} \circ \beta|_{U_2} \circ \varphi_{1,2} \in \text{Hom}(P_a|_{U_2}, P_b|_{U_2})
\]

for any \( \beta \in \text{Hom}(P_a|_{U_1}, P_b|_{U_1}) \). It is straightforward to check that the restriction map satisfies the axioms for a presheaf of sets, and so \( \text{Hom}_U(P_a, P_b) \) is a presheaf of sets. In order to distinguish this presheaf of morphisms from a set of morphisms, we shall always denote the presheaf by \( \text{Hom} \) and the set by \( \text{Hom} \).

In passing, we should mention that we shall sometimes use the notation \( \text{Aut}_{\mathcal{C}(U)}(P) \) to denote the set \( \text{Hom}_{\mathcal{C}(U)}(P, P) \), and \( \text{Aut}_U(P) \) to denote the presheaf \( \text{Hom}_U(P, P) \).

## 2.2 Sheaf of categories

We shall use the terms “sheaf of categories” and “stack” synonymously in this paper, though not all authors quite agree [10]. In the conventions of [10], where the two concepts are distinguished, what we shall define in this section will technically be a “stack,” rather than a “sheaf of categories” (which is required to satisfy a stronger gluing condition on objects in [10]).

Before giving the technical definitions, let us review the gluing conditions for a sheaf of sets, on which the gluing conditions below shall be modelled. Following [3, section 1.1], given a presheaf \( \mathcal{S} \) of sets over a space \( X \), we say the presheaf is a sheaf of sets if for any open set \( U \subseteq X \) and every open cover \( \{U_\alpha\} \) of \( U \), if \( \{s_\alpha \in \mathcal{S}(U_\alpha)\} \) is a family of elements such that

\[
\rho_{U_\alpha U_\beta}^* s_\alpha = \rho_{U_\alpha U_\beta}^* s_\beta
\]

then there exists a unique \( s \in \mathcal{S}(U) \) such that \( \rho_{U_\alpha U}^* s = s_\alpha \) for all \( \alpha \). The constraints for a presheaf of categories to be a sheaf of categories are closely related; one must give a rather similar gluing condition for both the objects and the morphisms of the categories.

### 2.2.1 Gluing law for objects

First, we shall define the gluing condition for objects. Let \( \{U_\alpha\} \) be an open cover of some open set \( U \subseteq X \), and suppose we are given a family of objects \( x_\alpha \in \text{Ob} \mathcal{C}(U_\alpha) \), and a family
of isomorphisms $\phi_{\alpha\beta}$:

$$\phi_{\alpha\beta} : x_\beta|_{U_{\alpha\beta}} \sim x_\alpha|_{U_{\alpha\beta}}$$

satisfying the compatibility condition $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ in $C(U_{\alpha\beta\gamma})$, and such that $\phi_{\alpha\alpha} = 1_\alpha$ in $C(U_\alpha)$.

Gluing holds for objects (more technically, the descent condition is effective) if there exists an object $x \in \text{Ob} C(U)$, together with a family of isomorphisms $\psi_\alpha : x|_{U_\alpha} \sim x_\alpha$, such that the following diagram commutes:

\[
\begin{array}{ccc}
  x|_{U_\beta}|_{U_{\alpha\beta}} & \xrightarrow{\phi_{\alpha\beta}} & x|_{U_\alpha}|_{U_{\alpha\beta}} \\
  \downarrow{\psi_{\beta}|_{U_{\alpha\beta}}} & & \downarrow{\psi_{\alpha}|_{U_{\alpha\beta}}} \\
  x_\beta|_{U_{\alpha\beta}} & \xrightarrow{\phi_{\beta\gamma}} & x_\alpha|_{U_{\alpha\beta}}
\end{array}
\]

where we have used the notation $|_U$ to indicate the restriction functor, and where $\varphi_{\alpha,\alpha\beta}$ indicates the natural transformation $|_{U_{\alpha\beta}} \Rightarrow |_{U_\alpha}$ appearing in the definition of presheaf of categories.

Before we proceed to the gluing law for morphisms, we shall take a moment to reflect on the gluing law for objects given above. First, note that one can not assume that the objects $x|_{U_\alpha}|_{U_{\alpha\beta}}$, $x|_{U_\beta}|_{U_{\alpha\beta}}$, and $x|_{U_{\alpha\beta}}$ in $\text{Ob} C(U_{\alpha\beta})$ are the same object. Rather, they are related by invertible natural transformations between the restriction functors, nothing more.

We should also mention that we were sloppy in part of the gluing law given above. Strictly speaking, the relation $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ does not make sense, even after naively using the restriction functors, as the morphisms in question act on distinct objects. In order to properly make sense out of this relation, one must make use of the natural transformations between restriction functors appearing in the definition of presheaf. More concretely, the relation $\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}$ should be replaced with the constraint that the following diagram commutes:

\[
\begin{array}{ccc}
  x_\gamma|_{U_{\beta\gamma}} |_{U_{\alpha\beta\gamma}} & \xrightarrow{\phi_{\beta\gamma}|_{U_{\alpha\beta\gamma}}} & x_\beta|_{U_{\beta\gamma}} |_{U_{\alpha\beta\gamma}} \\
  \downarrow{\varphi_{\beta\gamma,\alpha\beta\gamma}} & & \downarrow{\varphi_{\beta\gamma,\alpha\beta\gamma}} \\
  x_\gamma|_{U_{\alpha\beta\gamma}} & \xrightarrow{\phi_{\alpha\beta\gamma}} & x_\alpha|_{U_{\alpha\beta\gamma}}
\end{array}
\]

Finally, the reader might ask to what extent the object $x$ constructed in the gluing law above is unique. Certainly in the definition of sheaves of sets, the gluing law specified unique objects, but uniqueness was not mentioned in the definition above. In fact, it can be shown using the gluing law for morphisms defined in the next section that the object $x$ is unique up to unique isomorphism commuting with the $\psi_\alpha$. We shall not work through the
details, but the basic idea is as follows. If \( x' \in \text{Ob} \mathcal{C}(U) \) is another object with isomorphisms \( \psi'_\alpha : x'|_{U_\alpha} \sim x_\alpha \) that make diagram (2) commute, then define a set of morphisms \( f_\alpha : x|_{U_\alpha} \to x'|_{U_\alpha} \) by \( f_\alpha = \psi'^{-1}_\alpha \circ \psi_\alpha \). Using the gluing law for morphisms these can be glued together to form a unique morphism \( f : x \to x' \) whose restriction to each \( U_\alpha \) commutes with \( \psi_\alpha \) and \( \psi'_\alpha \), i.e.,

\[
\begin{array}{ccc}
x|_{U_\alpha} & \xrightarrow{f|_{U_\alpha}} & x'|_{U_\alpha} \\
\psi_\alpha & \downarrow & \psi'_\alpha \\
x_\alpha & = & x_\alpha
\end{array}
\]

commutes, and with further work it can be shown that \( f \) is an isomorphism. Note that if we dropped the constraint that the restriction of \( f \) commute with the \( \psi_\alpha \), then we would lose uniqueness – given any \( f \), we could compose with any automorphism of either \( x \) or \( x' \) to obtain another morphism \( x \to x' \).

2.2.2 Gluing law for morphisms

The gluing condition on morphisms can be stated as follows. Let \( U \subseteq X \) be an open set, \( x, y \in \text{Ob} \mathcal{C}(U) \), and \( \{U_\alpha\} \) be an open cover of \( U \). If \( \{f_\alpha : x|_{U_\alpha} \to y|_{U_\alpha}\} \) is a set of maps such that \( f_\alpha|_{U_\alpha \beta} = f_\beta|_{U_\alpha \beta} \), then there exists a unique \( f : x \to y \) such that \( f_\alpha = f|_{U_\alpha} \).

Unfortunately this phrasing is slightly sloppy. Strictly speaking, the relation \( f_\alpha|_{U_\alpha \beta} = f_\beta|_{U_\alpha \beta} \) does not make sense: the objects \( x|_{U_\alpha}|_{U_\alpha \beta} \) and \( x|_{U_\beta}|_{U_\alpha \beta} \) are (in general) distinct objects of \( \text{Ob} \mathcal{C}(U_\alpha \beta) \), so the morphisms \( f_\alpha|_{U_\alpha \beta} \) and \( f_\beta|_{U_\alpha \beta} \) are morphisms between (in general) distinct objects of \( \mathcal{C}(U_\alpha \beta) \), and so cannot be immediately compared. Thus, we must replace the condition that \( f_\alpha|_{U_\alpha \beta} = f_\beta|_{U_\alpha \beta} \) with the condition that the following diagram commutes:

\[
\begin{array}{ccc}
x|_{U_\beta}|_{U_\alpha \beta} & \xleftarrow{\varphi_{\alpha \beta}^{-1}} & x|_{U_\alpha}|_{U_\alpha \beta} \\
{f_\beta|_{U_\alpha \beta}} & \downarrow & {f_\alpha|_{U_\alpha \beta}} \\
y|_{U_\beta}|_{U_\alpha \beta} & \xrightarrow{\varphi_{\alpha \beta}} & y|_{U_\alpha}|_{U_\alpha \beta} \\
\end{array}
\]

Finally, note that we can rephrase the gluing condition for morphisms somewhat more elegantly by saying that morphisms satisfy the gluing law if for any pair of objects \( x, y \in \text{Ob} \mathcal{C}(U) \) and any open cover \( \{U_\alpha\} \) of \( U \), the ordinary sheaf (of sets) axiom for gluing in the presheaf of morphisms \( \text{Hom}_{\mathcal{C}}(x, y) \) is satisfied. Put another way, satisfying the gluing condition for morphisms is equivalent to the presheaf of sets \( \text{Hom} \) being a sheaf of sets. Phrased yet another way, in a sheaf of categories, each presheaf of sets of morphisms \( \text{Hom} \) is a sheaf of sets, not just a presheaf.
2.2.3 Examples

One easy example of a stack is the sheaf of discrete categories associated to a sheaf of sets. Recall we pointed out earlier that a presheaf of sets can be understood as a presheaf of discrete categories. (Identify the elements of each set with objects in each category. By definition, the only morphisms in a discrete category are the identity morphisms, so we have completely characterized the categories.) It is easy to check that if the presheaf of sets is actually a sheaf of sets, then the corresponding presheaf of (discrete) categories is actually a stack.

A trivial example of a stack is the stack of all principal $G$-bundles on $X$, for some Lie group $G$. More precisely, define a stack $C$ by associating to each open set $U \subseteq X$, the category $C(U)$ whose objects are all principal $G$ bundles over $U$, with morphisms all bundle isomorphisms:

It is straightforward to check that this defines a presheaf of categories (with the restriction functors defined naturally, and the natural transformations trivial), and furthermore this presheaf of categories is a stack. We shall denote this example of a stack by $\text{Tors}(G)$.

Now, let us describe an example of a presheaf of categories that is not a stack. Fix some principal $G$ bundle $P$ on $X$, and define a presheaf of categories as follows. To each open set $U$, associate a category with one object, equal to $P|_U$, and let the morphisms in this category be the automorphisms of $P|_U$. It is easy to check that this defines a presheaf of categories, with the restriction functors defined naturally, and the natural transformations trivial. Denote this presheaf of categories by $P$.

We shall now argue that the presheaf of categories $P$ is not a stack in general, by observing that it does not always satisfy the gluing law for objects. Let $U \subseteq X$ be an open subset of $X$ such that there exists a principal $G$ bundle $Q$ over $U$ such that $P|_U \otimes Q$ is not topologically equivalent to $P|_U$. Let $\{U_\alpha\}$ be a good cover of $U$. Let $g_{\alpha \beta} : U_{\alpha \beta} \rightarrow G$ denote the transition functions for $Q$, defined with respect to the cover $\{U_\alpha\}$. Define a family of isomorphisms $\phi_{\alpha \beta} : P|_{U_\beta}|_{U_{\alpha \beta}} \rightarrow P|_{U_\alpha}|_{U_{\alpha \beta}}$ by

$$\phi_{\alpha \beta} = g_{\alpha \beta} \circ \varphi_{\alpha, \alpha \beta} \circ \varphi^{-1}_{\beta, \alpha \beta}$$

where the $\varphi$ are the (trivial) natural transformations appearing in the definition of presheaf of categories $P$. (We have made them explicit for completeness.) It is easy to check these isomorphisms satisfy the gluing condition, and so by the gluing condition for objects we should find a corresponding object in $P(U)$. This new object was essentially created by tensoring local sections of $P|_U$ with local sections of $Q$, and so in general should be local sections of $P|_U \otimes Q$. (This argument is somewhat weak; rigorous versions can be found in

---

2In fact, any morphism of principal $G$-bundles for fixed $G$ over a fixed space $X$ is necessarily an isomorphism [17, section 4.3].

3In special cases, such as $X$ contractible, $P$ may be a stack. However, we shall consider more general $X$, for which $P$ will not be a stack.

4In writing this slightly loose statement, we are assuming that $G$ is abelian.
are equivalent if there exists a Cartesian functor $D$. It can be shown that $\chi$ of categories, that is, a morphism between stacks is precisely a Cartesian functor. A morphism between stacks is precisely a morphism between the underlying presheaves of categories, but rather a map between presheaves of categories.

A Cartesian functor is not precisely a functor, in the sense that it is not a map between categories. Let $\rho : \mathcal{C} \rightarrow \mathcal{D}$ denote a Cartesian functor. That is, $\rho$ is a family of functors $\rho_\mathcal{C} : \mathcal{C}(U) \rightarrow \mathcal{D}(U)$, together with, for every inclusion $\rho : U_2 \hookrightarrow U_1$, an invertible natural transformation $\chi_\rho : \rho_\mathcal{D} \circ F(U_1) \Rightarrow F(U_2) \circ \rho_\mathcal{C}$. These invertible natural transformations are required to have the property that for any pair of composable inclusions $\rho : U_3 \hookrightarrow U_2$, $\rho_1 : U_2 \hookrightarrow U_1$, the following diagram commutes:

$$\begin{array}{c}
\rho^*_\mathcal{D} \circ \rho^*_\mathcal{C} \circ F(U_1) \\
\varphi_{\rho_1,\rho_2} \Uparrow \\
(\rho_1 \rho_2)^*_\mathcal{C} \circ F(U_1) \\
\chi_{\rho_1,\rho_2} \\

\chi_{\rho,\rho_1,\rho_2} \\
\varphi_{\rho_1,\rho_2} \Uparrow \\
F(U_3) \circ (\rho_1 \rho_2)^*_\mathcal{C}
\end{array}$$

where the $\varphi$ are the invertible natural transformations appearing in the definition of presheaf of categories.

Such maps between presheaves of categories are called Cartesian functors. Note that a Cartesian functor is not precisely a functor, in the sense that it is not a map between categories, but rather a map between presheaves of categories.

A morphism between stacks is precisely a morphism between the underlying presheaves of categories, that is, a morphism between stacks is precisely a Cartesian functor.

Cartesian functors can be composed. That is, if $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are three presheaves of categories on $X$, and $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$ are two Cartesian functors, then one can define a Cartesian functor $G \circ F : \mathcal{C} \rightarrow \mathcal{E}$. We shall outline the definition. For any open set $U$, the functors $F(U) : \mathcal{C}(U) \rightarrow \mathcal{D}(U)$ and $G(U) : \mathcal{D}(U) \rightarrow \mathcal{E}(U)$ can certainly be composed. Given an inclusion $\rho : U_2 \hookrightarrow U_1$, we can define the invertible natural transformation $\chi_{\rho}^{FG}$ as the composition of the natural transformations associated to $F$ and $G$. In other words, $\chi_{\rho}^{FG} : \rho^*_\mathcal{E} \circ (GF)(U_1) \Rightarrow (GF)(U_2) \circ \rho^*_\mathcal{C}$ is defined by

$$\chi_{\rho}^{FG} : \rho^*_\mathcal{E} \circ G(U_1) \circ F(U_1) \xrightarrow{\chi^{FG}_G} G(U_2) \circ \rho^*_\mathcal{D} \circ F(U_1) \xrightarrow{\chi^{FG}_F} G(U_2) \circ F(U_2) \circ \rho^*_\mathcal{C}$$

It can be shown that $\chi_{\rho}^{FG}$ satisfies the pentagonal identity (5).

Now, what does it mean for two stacks to be equivalent? We say that two stacks $\mathcal{C}$, $\mathcal{D}$ are equivalent if there exists a Cartesian functor $F : \mathcal{C} \rightarrow \mathcal{D}$ such that the functor $F(U) : \mathcal{C}(U) \rightarrow \mathcal{D}(U)$ associated to any open set $U$ is an equivalence of categories.\footnote{Recall that a functor $F : \mathcal{E} \rightarrow \mathcal{F}$ is said to be an equivalence of the categories $\mathcal{E}$, $\mathcal{F}$ if there is a functor $G : \mathcal{F} \rightarrow \mathcal{E}$ such that $GF$ and $FG$ are naturally isomorphic to the identity functors on $\mathcal{E}$ and $\mathcal{F}$, respectively.}

section 5.3 in the discussion of gauge transformations for gerbes.) By assumption, however, no such object exists in $\mathcal{P}(U)$. Thus, the gluing law is not satisfied, and so the presheaf of categories $\mathcal{P}$ cannot be a stack.
We shall now show that a Cartesian functor $F : \mathcal{C} \to \mathcal{D}$ between presheaves of categories $\mathcal{C}, \mathcal{D}$ induces a morphism of presheaves
\[ \text{Hom}_{\mathcal{C}(U)}(P_a, P_b) \to \text{Hom}_{\mathcal{D}(U)}(F(U)(P_a), F(U)(P_b)) \]
for any open set $U$ and any two objects $P_a, P_b \in \text{Ob} \mathcal{C}(U)$. For any open $V \subseteq U$, define a map of sets
\[ \lambda(V) : \text{Hom}_{\mathcal{C}(V)}(P_a|_V, P_b|_V) \to \text{Hom}_{\mathcal{D}(V)}(F(U)(P_a)|_V, F(U)(P_b)|_V) \]
by,
\[ \lambda(V)(\beta) \equiv (\chi^F_{U,V}(P_b))^{-1} \circ F(V)(\beta) \circ \chi^F_{U,V}(P_a) \]
for all $\beta : P_a|_V \to P_b|_V$, where $\chi^F$ denotes the invertible natural transformation defining $F$ as a Cartesian functor. It is straightforward to check that $\lambda$ defines a morphism of presheaves of sets, in other words, that for every inclusion $\rho : U_2 \hookrightarrow U_1$ of open $U_1, U_2 \subseteq U$, the following diagram commutes:
\[ \begin{array}{ccc}
\text{Hom}_{\mathcal{C}}(P_a, P_b)(U_1) & \xrightarrow{\lambda(U_1)} & \text{Hom}_{\mathcal{D}}(F(U)(P_a), F(U)(P_b))(U_1) \\
\downarrow \rho^* & & \downarrow \rho^* \\
\text{Hom}_{\mathcal{C}}(P_a, P_b)(U_2) & \xrightarrow{\lambda(U_2)} & \text{Hom}_{\mathcal{D}}(F(U)(P_a), F(U)(P_b))(U_2)
\end{array} \]

2.4 2-arrows

Now that we have defined analogues of functors for presheaves of categories (namely, Cartesian functors), we shall define analogues of natural transformations between Cartesian functors. These analogues of natural transformations are known as 2-arrows [10, section 1.1]. We shall also define sheaves of sets describing local natural transformations between Cartesian functors.

2.4.1 2-arrows

Let $F, G : \mathcal{C} \to \mathcal{D}$ be Cartesian functors between a pair of presheaves of categories $\mathcal{C}, \mathcal{D}$ over a space $X$. A 2-arrow $\Psi : F \Rightarrow G$ is a family of natural transformations $\Psi(U) : F(U) \Rightarrow G(U)$ (one for each open $U \subseteq X$), such that, for any inclusion $\rho : U_2 \hookrightarrow U_1$, the following diagram commutes:
\[ \begin{array}{ccc}
\rho^*_D \circ F(U_1) & \xrightarrow{\Psi(U_1)} & \rho^*_D \circ G(U_1) \\
\chi^F \downarrow & & \downarrow \chi^G \\
F(U_2) \circ \rho^*_C & \xrightarrow{\Psi(U_2)} & G(U_2) \circ \rho^*_C
\end{array} \]

$G : \mathcal{F} \to \mathcal{E}$ and there are invertible natural transformations $\text{Id}_{\mathcal{E}} \Rightarrow GF$ and $\text{Id}_{\mathcal{F}} \Rightarrow FG$, i.e., $\text{Id}_{\mathcal{E}} \cong GF$ and $\text{Id}_{\mathcal{F}} \cong FG$. 

13
where the $\chi_\rho$ are natural transformations appearing in the definition of Cartesian functors.

In passing, note that 2-arrows can be composed. In other words, if $F, G, H : \mathcal{C} \to \mathcal{D}$ are Cartesian functors and $\Psi_1 : F \Rightarrow G$, $\Psi_2 : G \Rightarrow H$ are a pair of 2-arrows, then the composition $\Psi_2 \circ \Psi_1 : F \Rightarrow H$ is well-defined as a 2-arrow.

An invertible 2-arrow is a 2-arrow $\Psi$ such that $\Psi(U)$ is an invertible natural transformation for all open $U \subseteq X$.

2.4.2 Sheaves of natural transformations

Let $F, G : \mathcal{C} \to \mathcal{D}$ be Cartesian functors between a pair of presheaves of categories $\mathcal{C}$ and $\mathcal{D}$, over a space $X$. Fix some open set $U \subseteq X$. We shall define a presheaf of sets $2R_U(F, G)$ which shall describe local 2-arrows between $F$ and $G$, as well as a presheaf of sets $NT_U(F, G)$, a related sheaf which shall describe local natural transformations between $F$ and $G$.

We shall define a presheaf of 2-arrows from $F|_U$ to $G|_U$, which we shall denote $2R_U(F, G)$. To any open $V \subseteq U$, define the set $2R_U(F, G)(V)$ to be the set of all 2-arrows $F|_V \Rightarrow G|_V$. In other words, an element $\psi \in 2R_U(F, G)(V)$ is a collection of natural transformations $\psi(W) : F(W) \Rightarrow G(W)$ (one such for each open $W \subseteq V$), such that for any inclusion $\rho : W_2 \hookrightarrow W_1$ of open sets $W_1, W_2 \subseteq V$, the following diagram commutes:

$$\begin{array}{ccc}
\rho^* \circ F(W_1) & \xrightarrow{\psi(W_1)} & \rho^* \circ G(W_1) \\
\chi^F_\rho \downarrow & & \downarrow \chi^G_\rho \\
F(W_2) \circ \rho^* & \xrightarrow{\psi(W_2)} & G(W_2) \circ \rho^*
\end{array}$$

where the $\chi$ are the natural transformations defining $F, G$ as Cartesian functors.

Restriction maps in the presheaf $2R_U(F, G)$ are defined as follows. Let $\rho : V_2 \hookrightarrow V_1$ denote an inclusion of open sets $V_1, V_2 \subseteq U$. An element $\psi \in 2R_U(F, G)(V_1)$ is a collection of natural transformations $\psi(W) : F(W) \Rightarrow G(W)$, for $W \subseteq V_1$, as above. We define $\rho^* \psi$ to be the new collection of natural transformations obtained from the collection $\psi$ by removing all elements corresponding to open $W$ such that $V_2 \subset W \subseteq V_1$.

It should be clear that these definitions yield a presheaf of sets. For example, if $\rho_2 : V_3 \hookrightarrow V_2$ and $\rho_1 : V_2 \hookrightarrow V_1$ are a pair of composable inclusions of open sets, then for all $\psi \in 2R_U(F, G)(V_1)$, $\rho_2^* \rho_1^* \psi = (\rho_1 \rho_2)^* \psi$.

In the special case that $\mathcal{D}$ is a stack, not just a presheaf of categories, it can be shown that $2R_U(F, G)$ is a sheaf of sets, not just a presheaf. We shall outline the details here. Let
$V \subseteq U$ be an open set, and let \( \{U_\alpha\} \) be an open cover of \( V \). Let \( \{\psi_\alpha \in 2R_{U'}(F,G)(U_\alpha)\} \) be a set of elements such that \( \rho^*_\alpha,\alpha_\beta \psi_\alpha = \rho^*_\beta,\alpha_\beta \psi_\beta \) (where the \( \rho \) are the natural inclusions from \( U_\alpha \cap U_\beta \) into \( U_\alpha, U_\beta \)). We need to show that there exists a unique \( \psi \in 2R_{U'}(F,G)(V) \) such that \( \rho^*_\alpha \psi = \psi_\alpha \) (where \( \rho_\alpha : U_\alpha \hookrightarrow V \) is inclusion).

Finding a 2-arrow \( \psi \) means finding a set of natural transformations \( \psi(W) : F(W) \Rightarrow G(W) \), one for each open \( W \subseteq V \), obeying the usual compatibility condition. Let \( P \in \text{Ob} \mathcal{C}(W) \), then we can define \( \psi(W)(P) \) to be the unique morphism generated by gluing together morphisms

\[
(\chi^G_\alpha)^{-1} \circ \psi_\alpha(W \cap U_\alpha)(P|_{W \cap U_\alpha}) \circ \chi^F_\alpha : F(W)(P)|_{W \cap U_\alpha} \rightarrow G(W)(P)|_{W \cap U_\alpha}
\]

It is straightforward to check that this defines a natural transformation \( \psi(W) : F(W) \Rightarrow G(W) \) for all open \( W \subseteq V \), and also that these natural transformations satisfy diagram (8). Thus, we have defined a unique 2-arrow \( \psi \) such that \( \rho^*_\alpha \psi = \psi_\alpha \), and so we shown the gluing law for sheaves of sets is satisfied. Thus, when \( \mathcal{D} \) is a stack, the presheaf of sets \( 2R_{U'}(F,G) \) is a sheaf of sets.

In addition to the presheaf of sets \( 2R_{U'}(F,G) \), one can also define another presheaf of natural transformations, which we shall label \( NT_{U'}(F,G) \). Our use of this presheaf will be very limited; we mention it solely for completeness. For any open \( V \subseteq U \), define the set \( NT_{U'}(F,G)(V) \) to be the set of all natural transformations \( F(V) \circ |_V \Rightarrow G(V) \circ |_V \)

where \( |_V \) indicates the restriction functor from \( \mathcal{C}(U) \) to \( \mathcal{C}(V) \). For any inclusion \( \rho : U_2 \hookrightarrow U_1 \) between open sets \( U_1, U_2 \subseteq U \), define the restriction map

\[
\rho^* : NT_{U'}(F,G)(U_1) \rightarrow NT_{U'}(F,G)(U_2)
\]

by,

\[
\rho^* \eta \equiv \varphi^C_{1,2}^{-1} \circ \chi^G_\rho \circ \eta \circ (\chi^F_\rho)^{-1} \circ \varphi^C_{1,2}
\]

for \( \eta \in NT_{U'}(F,G)(U_1) \) (i.e., \( \eta : F(U_1) \circ |_{U_1} \Rightarrow G(U_1) \circ |_{U_1} \)), where the \( \varphi^C \) are the natural transformations defining \( \mathcal{C} \) as a presheaf of categories, and where the \( \chi \) are the natural transformations defining \( F, G \) as Cartesian functors.

It is straightforward to check that this defines a presheaf of sets. For example, for two composable inclusions \( \rho_1 : U_2 \hookrightarrow U_1, \rho_2 : U_3 \hookrightarrow U_2, U_1, U_2, U_3 \subseteq U \), the restriction maps obey \( \rho_2^* \rho_1^* \eta = (\rho_1 \rho_2)^* \eta \) for all \( \eta : F(U_1) \circ |_{U_1} \Rightarrow G(U_1) \circ |_{U_1} \).

In the case that \( \mathcal{D} \) is a stack, not just a presheaf of categories, it is straightforward to check that \( NT_{U'}(F,G) \) is a sheaf of sets, not just a presheaf, for all open \( U \subseteq X \).

In passing, note that any 2-arrow \( \psi : F|_U \Rightarrow G|_U \) defines an element of \( NT_{U'}(F,G)(V) \) for all open \( V \subseteq U \) for which \( \mathcal{D}(V) \neq \emptyset \), and moreover if \( \rho : U_2 \hookrightarrow U_1 \) is inclusion of open sets, then \( \rho^* \psi(U_1) = \varphi^C_{1,2}^{-1} \circ \psi(U_2) \circ \varphi^C_{1,2} \), i.e., the restriction functor relates these elements in a natural way.
3 Gerbes and stacks

For a brief but readable discussion of gerbes in terms of stacks, see for example [7, section 3]. More detailed information is available in [3, 5, 10]. This section reviews material that can be found in [3, chapter 5], [7, section 3], and [10, chapter 1].

3.1 Definitions and examples

A stack $C$ is called a (1-)gerbe if the following three conditions are satisfied:

1. For every open set $U \subseteq X$, every morphism in the category $C(U)$ is invertible. (In more technical language, this means $C(U)$ is a groupoid.)

2. Each point $x \in X$ has a neighborhood $U_x$ for which $C(U_x)$ is nonempty.

3. Any two objects $P_1, P_2$ of $C(U)$ are locally isomorphic. In other words, each $x \in U$ has a neighborhood $V_x$ such that the restrictions of $P_1$ and $P_2$ to $V_x$ are isomorphic.

One says that a gerbe $C$ is bound by a sheaf of abelian groups $A$ (or, that the gerbe has band $A$) if for any open set $U$ and object $P \in \text{Ob} C(U)$, there exists an isomorphism of sheaves of groups $\alpha_U(P) : A|_U \cong \text{Aut}_U(P)$. These isomorphisms are required to satisfy a constraint which we shall describe shortly. In the rest of this paper, when we speak of gerbes, we will implicitly refer to gerbes bound by some sheaf of abelian groups.

In passing, we should point out that since $\alpha(U)(P)$ is a morphism of sheaves of groups, it is compatible with restriction, i.e., the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{A}(U_1) & \xrightarrow{\alpha_{U_1}(P)} & \text{Hom}_{C(U_1)}(P, P) \\
\rho^* & \downarrow & \downarrow \rho^* \\
\mathcal{A}(U_2) & \xrightarrow{\alpha_{U_2}(P|_{U_2})} & \text{Hom}_{C(U_2)}(P|_{U_2}, P|_{U_2})
\end{array}
$$

where the $\rho^*$ are the restriction maps defining the sheaves of sets, $P \in \text{Ob} C(U_1)$, and $\rho : U_2 \hookrightarrow U_1$ is an inclusion between open sets $U_1, U_2 \subseteq U$.

We mentioned above that the isomorphisms $\alpha_U(P) : \mathcal{A}|_U \to \text{Aut}_U(P)$ are required to satisfy a condition, which we shall now explain. Let $U$ be an open set, let $P_1, P_2 \in \text{Ob} C(U)$, let $\beta \in \text{Hom}_{C(U)}(P_1, P_2)$, and let $g \in \text{Aut}(P_1)$. The morphism $\beta \circ g \circ \beta^{-1} \in \text{Aut}(P_2)$, and in fact inner automorphisms by $\beta$ of the form above clearly define a group homomorphism (in fact, an isomorphism) from $\text{Aut}(P_1)$ to $\text{Aut}(P_2)$. Put another way, for any $g \in \text{Aut}(P_1)$, there exists $g' \in \text{Aut}(P_2)$ such that $\beta \circ g = g' \circ \beta$, and $g$ and $g'$ are isomorphic to the same element of the group $\mathcal{A}(U)$. More generally, the morphism $\beta$ defines a morphism of sheaves

$$
\beta^* : \text{Aut}_U(P_1) \xrightarrow{\sim} \text{Aut}_U(P_2)
$$
We can now finally state the constraint on the isomorphisms $\alpha_U(P)$. The $\alpha_U(P)$ are required to be such that the following diagram commutes:

\[ \begin{array}{ccc}
\mathcal{A}|_U & \xrightarrow{\alpha_U(P_1)} & \text{Aut}_U(P_1) \\
\parallel & \downarrow & \beta^* \\
\mathcal{A}|_U & \xrightarrow{\alpha_U(P_2)} & \text{Aut}_U(P_2) \\
\end{array} \]

(10)

The 1-gerbes discussed in [2] all describe gerbes with band $C^\infty(U(1))$. Here we see that more general gerbes can be defined.

A trivial example of a 1-gerbe with band $C^\infty(U(1))$ is the stack $\text{Tors}(G)$, the stack of all principal $G$-bundles, introduced earlier. In fact, we shall see later that all gerbes with band $C^\infty(G)$ look locally like $\text{Tors}(G)$, in the same sense that all fiber bundles look locally like the trivial bundle.

A nontrivial example of a 1-gerbe with band $C^\infty(U(1))$ is the stack describing Spin$^c(n)$ lifts of a principal SO($n$) bundle, which we shall now describe. (Relevant information can be found in [18, appendix D].) Let $P$ be a principal SO($n$)-bundle on $X$. We shall describe a 1-gerbe, call it $\mathcal{C}$, that implicitly describes obstructions to lifting the structure group of $P$ from SO($n$) to Spin$^c(n)$. (We shall roughly follow [6, section 5.2] for this part.) To any open set $U \subset X$, define the objects of $\mathcal{C}(U)$ to be pairs $(Q, \phi)$, where $Q$ is a principal Spin$^c(n)$ bundle on $U$ which is a lift of $P|_U$, and where $\phi : Q \rightarrow P|_U$ is a morphism of principal bundles. (Note that if $P$ does not admit a global lift, then $\mathcal{C}(X) = \emptyset$, for example.) Morphisms between objects of $\mathcal{C}(U)$ are defined as follows. Let $(Q, \phi), (Q', \phi')$ be two objects of $\mathcal{C}(U)$. A morphism $u : (Q, \phi) \rightarrow (Q', \phi')$ is defined to be a morphism of principal Spin$^c(n)$ bundles, such that the following diagram commutes:

\[ \begin{array}{ccc}
Q & \xrightarrow{u} & Q' \\
\phi \downarrow & & \downarrow \phi' \\
P|_U & = & P|_U \\
\end{array} \]

(11)

It is straightforward to check that this structure defines a stack, and furthermore is a 1-gerbe. The element of $H^3(X, \mathbb{Z})$ associated to this 1-gerbe precisely classifies the obstruction to lifting the structure group of $P$ to Spin$^c(n)$. If the category $\mathcal{C}(X)$ is nonempty, then its objects are principal Spin$^c(n)$ bundles on $X$, globally defined lifts of the principal SO($n$) bundle $P$.

More generally, we shall see later that a nontrivial gerbe on a space $X$ can be distinguished from a trivial gerbe on $X$ by the category $\mathcal{C}(X)$. If this category is nonempty, then the gerbe is trivial – objects in $\mathcal{C}(X)$ define trivializations of the gerbe, just as global sections of a principal bundle trivialize the bundle. We shall return to this matter later.

\footnote{Recall [17, section 4.3] that if $u : Q_1 \rightarrow Q_2$ is a morphism of principal bundles over the same space with the same fiber, then it is necessarily an isomorphism. Thus, the morphisms we define in $\mathcal{C}(U)$ are all necessarily isomorphisms.}
3.2 Equivalences of gerbes

Let $\mathcal{C}$ and $\mathcal{D}$ be 1-gerbes, both with band $\mathcal{A}$ (a sheaf of abelian groups). Under what circumstances can we say that $\mathcal{C}$ is equivalent to $\mathcal{D}$?

A map between two gerbes $\mathcal{C}$ and $\mathcal{D}$ with specified band $\mathcal{A}$ is defined to be a Cartesian functor $F : \mathcal{C} \to \mathcal{D}$ such that for all open sets $U$ and for all $P \in \text{Ob} \mathcal{C}(U)$, the following diagram of sheaves of groups commutes (by definition of morphism of sheaves):

\[
\begin{array}{ccc}
\text{Aut}_{\mathcal{C}(U)}(P) & \xrightarrow{F(U)} & \text{Aut}_{\mathcal{D}(U)}(F(U)(P)) \\
\alpha_{\mathcal{C}(U)}(P) \downarrow \sim & & \sim \downarrow \alpha_{\mathcal{D}(U)}(P) \\
\mathcal{A}|_U & = & \mathcal{A}|_U
\end{array}
\] (12)

where the $\alpha$ are the isomorphisms between the band and the automorphisms of an object (given in the definition of band), and we have used $F(U)$ to denote the induced morphism of sheaves (here, sheaves of abelian groups) discussed in the section on Cartesian functors. More intuitively, this condition means that the action of the band on the gerbe commutes with the Cartesian functor.

An equivalence of two gerbes $\mathcal{C}$, $\mathcal{D}$ with band $\mathcal{A}$ is defined to be a Cartesian functor $F : \mathcal{C} \to \mathcal{D}$ obeying the constraint (12), such that the Cartesian functor defines an equivalence of stacks.

We shall show in section 5.3.8 that any map between two gerbes with the same band, over the same space, is necessarily an equivalence of gerbes. This is closely analogous to the result that any morphism of principal $G$-bundles, for fixed $G$, over the same space, is necessarily an isomorphism [17, section 4.3].

3.3 Sheaf cohomology and gerbes

In [3] we claimed that gerbes were classified by elements of sheaf cohomology groups. How can we derive an element of $H^2(X, \mathcal{A})$ from the description of gerbes given above? We shall work through the details in this subsection. More precisely, we shall show how to obtain a Čech representative of the relevant sheaf cohomology group, associated to some fixed open cover. For convenience, we shall assume the band $\mathcal{A} = C^\infty(U(1))$, though the reader should be able to easily extend to more general cases.

Before describing how to associate sheaf (and also, more usually, Čech) cohomology elements to gerbes, we shall take a moment to review how this procedure works for sheaves of local sections of bundles. Let $I$ be a sheaf of local sections of some principal $G$-bundle on a space $X$, and let $\{U_\alpha\}$ be a good open cover of $X$. Let $\{s_\alpha\}$ be any choice of local sections of $I$ with respect to the cover $\{U_\alpha\}$ (i.e., $s_\alpha \in I(U_\alpha)$ for all $\alpha$). Then on each overlap
$U_{\alpha\beta} = U_{\alpha} \cap U_{\beta}$, the sections $s_{\alpha}|_{U_{\alpha\beta}}$ and $s_{\beta}|_{U_{\alpha\beta}}$ will differ by some element of $C^\infty(G)$ over $U_{\alpha\beta}$. Denote each such element by $g_{\alpha\beta}$. It is straightforward to check that the $g_{\alpha\beta}$ define a cocycle representative of an element of $H^1(X, C^\infty(G))$ (or, rather, the corresponding Čech cohomology group associated to the cover $\{U_a\}$). Picking different local sections corresponds to changing the cocycle by a coboundary. Thus, we have derived an element of $H^1(X, C^\infty(G))$ classifying the sheaf $I$. Finally, note that $I$ admits a global section if and only if there exist sections $s_{\alpha}$ such that $s_{\alpha}|_{U_{\alpha\beta}}$ and $s_{\beta}|_{U_{\alpha\beta}}$ agree on overlaps, i.e., $g_{\alpha\beta}$ is the identity on each overlap, i.e., the corresponding element of sheaf and Čech cohomology is trivial.

Now that we have described how to associate Čech cohomology elements to any given sheaf of local sections of a bundle, we shall discuss how to associate cohomology elements to gerbes. We shall see that the details are closely analogous to the case above.

Let $\{U_a\}$ be a good cover of $X$, i.e., a cover such that every element and every intersection of elements is contractible. We shall assume this is sufficient for every object of any category $\mathcal{C}(U_a)$ to be isomorphic. (If not, pick a suitable refinement of $\{U_a\}$.) Then, let $P_a$ denote an object of $\mathcal{C}(U_a)$. (Since all objects in $\mathcal{C}(U_a)$ are isomorphic, the precise choice of $P_a$ is irrelevant.) Let $u_{\alpha\beta}$ denote the isomorphisms

$$u_{\alpha\beta} : P_{\alpha}|_{U_{\alpha\beta}} \xrightarrow{\sim} P_{\beta}|_{U_{\alpha\beta}}$$

where $U_{\alpha\beta} = U_{\alpha} \cap U_{\beta}$, and we implicitly assume $u_{\alpha\beta} = u_{\beta\alpha}^{-1}$. Suppose $U_{\alpha\beta\gamma} = U_{\alpha} \cap U_{\beta} \cap U_{\gamma} \neq \emptyset$. Define $h_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \to U(1)$ by

$$h_{\alpha\beta\gamma} = u_{\gamma\alpha} \circ u_{\beta\gamma} \circ u_{\alpha\beta} \in \text{Aut}(P_{\alpha}|_{U_{\alpha\beta\gamma}})$$

In fact, we have been slightly sloppy about the distinction between objects $x|_{U_{\alpha} \cap U_{\beta}}$ and $x|_{U_{\alpha\beta}}$, for example. To rigorously define $h_{\alpha\beta\gamma}$ we must introduce the invertible natural transformations $\varphi$ defining $\mathcal{C}$ as a presheaf of categories. Define maps $\varpi_{\alpha\beta,a\beta\gamma} : P_{\alpha}|_{U_{\alpha\beta\gamma}} \to P_{\beta}|_{U_{\alpha\beta}}$ by,

$$\overline{u}_{\alpha\beta,a\beta\gamma} = \varphi_{\alpha\beta,a\beta\gamma}^{-1} \circ u_{\alpha\beta}|_{U_{\alpha\beta\gamma}} \circ \varphi_{\alpha\beta,a\beta\gamma}$$

then the rigorous definition of $h_{\alpha\beta\gamma}$ is as

$$h_{\alpha\beta\gamma} = \varpi_{\gamma\alpha,a\beta\gamma} \circ \varpi_{\beta\gamma,a\beta\gamma} \circ \varpi_{\alpha\beta,a\beta\gamma} \in \text{Aut}(P_{\alpha}|_{U_{\alpha\beta\gamma}})$$

It is straightforward to check that $h_{\alpha\beta\gamma}$ defines a Čech 2-cocycle.

Naively, our description of $h_{\alpha\beta\gamma}$ above might appear to always be a coboundary, as $h_{\alpha\beta\gamma}$ naively appears to be the coboundary of a 1-cocycle defined by the $u_{\alpha\beta}$. However, there is an important distinction at work here. The $\varpi_{\alpha\beta,a\beta\gamma}$ are maps between (in general) distinct objects, not automorphisms of a single object. Since they are maps between distinct objects,
they will not (in general) be valued in the band of the gerbe. Thus, $h_{\alpha\beta\gamma}$ will not be a trivial Čech cocycle in general.

It can also be shown that if $\{P_\alpha, u_{\alpha\beta}\}$ and $\{P'_\alpha, u'_{\alpha\beta}\}$ are two choices of objects and isomorphisms, then the 2-cocycles $h_{\alpha\beta\gamma}$, $h'_{\alpha\beta\gamma}$ defined by either differ by a coboundary. We shall leave the details of this verification to the reader, but in passing we will mention three important points that come up. First, one needs the fact that elements of the band commute with morphisms, as demonstrated earlier. Second, if we let $\psi_\alpha : P'_\alpha \simto P_\alpha$ be a set of isomorphisms, then it is a useful fact that the $\psi_\alpha$ commute with the natural transformations used to define the presheaf of categories, by definition of natural transformation. Finally, note that $u_{\alpha\beta}$ and $\psi_\beta|_{U_{\alpha\beta}} \circ u'_{\alpha\beta} \circ \psi_\alpha^{-1}|_{U_{\alpha\beta}}$ may differ by an element of the band, in general.

Thus, any set of choices $\{P_\alpha, u_{\alpha\beta}\}$ will define the same element of cohomology. Furthermore, any two equivalent gerbes define the same element of cohomology. Let $F : \mathcal{C} \to \mathcal{D}$ be a map between two gerbes on $X$ with band $\mathcal{A}$, which also defines an equivalence of gerbes. Let $\{U_\alpha\}$ be a good open cover of $X$, let $\{P_\alpha \in \text{Ob} \mathcal{C}(U_\alpha)\}$ be a set of objects, and $\{u_{\alpha\beta} : P_\alpha|_{U_{\alpha\beta}} \simto P_\beta|_{U_{\alpha\beta}}\}$ be a set of isomorphisms. Then $\{F(U_\alpha)(P_\alpha) \in \text{Ob} \mathcal{D}(U_\alpha)\}$ is a set of objects in $\mathcal{D}$, and

$$\{F(U_\alpha)(u_{\alpha\beta}) : F(U_\alpha)(P_\alpha|_{U_{\alpha\beta}}) \simto F(U_\alpha)(P_\beta|_{U_{\alpha\beta}})\}$$

is a set of isomorphisms between objects in $\mathcal{D}$. These objects and isomorphisms in $\mathcal{D}$ define a cocycle $h^D_{\alpha\beta\gamma}$, but as noted above, the cohomology class of the cocycle is independent of the choice of objects and isomorphisms. Thus, any two equivalent gerbes define the same element of cohomology.

An astute reader may be slightly confused by the paragraph above. Nowhere in our discussion did we seem to use the fact that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of gerbes; we only used the fact that it is a map of gerbes. In particular, the reader might be concerned that if we could derive a contradiction: if $F$ were not an equivalence of gerbes, then we should not get the same cohomology element. However, we shall show in section 5.3.8 that any map of gerbes with the same band and over the same space is necessarily an equivalence of gerbes. Thus, it is not possible for $F$ to not be an equivalence of gerbes, and so any potential contradiction is averted.

So far we have described how a sheaf cohomology element can be associated to a given gerbe. The converse is also possible – given a sheaf cohomology element, we can construct an associated gerbe. This construction is carried out in, for example, [3, section 5.2]. We shall not repeat the construction here, but in passing we shall mention that it uses techniques closely akin to the descent categories we describe in section 4.1.2.

It should now be clear that equivalence classes of gerbes with band $\mathcal{A}$ on a space $X$ are in one-to-one correspondence with elements of $H^2(X, \mathcal{A})$.  

20
Let us take a moment to try to gain some intuition for the meaning of this sheaf cohomology description of gerbes. Suppose a gerbe $C$ on a space $X$ is described by a cohomologically-trivial cocycle $h_{\alpha\beta\gamma}$; what does this imply about the gerbe? If $h_{\alpha\beta\gamma}$ is a coboundary, then by slight redefinitions of the isomorphisms $u_{\alpha\beta} : P_\alpha|_{U_{\alpha\beta}} \sim \rightarrow P_\beta|_{U_{\alpha\beta}}$ we can arrange for $h_{\alpha\beta\gamma} = 1$. (Simply compose each isomorphism with an automorphism dictated by the cochain defining the coboundary; use the fact that elements of the band commute with morphisms.) Then, we can use the gluing law for objects to construct an object of $C(X)$. Thus, a gerbe described by a cohomologically-trivial cocycle has $C(X) \neq \emptyset$. Conversely, it is straightforward to check that if $C(X) \neq \emptyset$, then the corresponding element of cohomology is trivial. This is closely analogous to the fact that a global section of a principal $G$-bundle is a trivialization of the bundle; here, an object of $C(X)$ is a trivialization of the gerbe.

In particular, let us consider the example of the gerbe $\text{Tors}(G)$, for $G$ an abelian Lie group, on a space $X$. In this case, $\text{Tors}(G)(X) \neq \emptyset$, that is, there exists a globally defined object (in fact, several globally defined objects, in general), so it should be clear that on a good open cover $\{U_\alpha\}$ one can pick objects $\{P_\alpha\}$ and isomorphisms $\{u_{\alpha\beta}\}$ such that $h_{\alpha\beta\gamma} = 1$. In other words, $\text{Tors}(G)$ is an example of a gerbe with a globally defined object, which implies that the corresponding element of cohomology is trivial. $\text{Tors}(G)$ is an example of a trivial gerbe.

The Spin$^c$ gerbes, discussed earlier, are somewhat more interesting. From the general discussion above, as this class of gerbes has band $C^\infty(U(1))$, they should be topologically classified by elements of

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z})$$

Indeed, in discussing whether an $SO(n)$-bundle can be lifted to a Spin$^c$($n$)-bundle, there arises an element of $H^3(X, \mathbb{Z})$, usually labeled $W_3$, which is the image under a Bockstein homomorphism of the second Stiefel-Whitney class of the $SO(n)$-bundle in question. The characteristic class $W_3$ is precisely the integral characteristic class classifying the Spin$^c$ gerbe. An $SO(n)$-bundle admits a Spin$^c$ lift if and only if $W_3$ vanishes, in precise accord with the general framework above – a 1-gerbe admits a global trivialization if and only if the classifying integral characteristic class vanishes. A global trivialization of the Spin$^c$ gerbe associated to some $SO(n)$-bundle is precisely a Spin$^c$ lift of the bundle.

### 3.4 Gauge transformations of gerbes

Just as a gauge transformation of a principal $G$-bundle on a space $X$ is defined by a map $X \rightarrow G$, it turns out that a gauge transformation of a gerbe with band $\mathcal{A} = C^\infty(G)$ is a principal $G$-bundle. (In terminology introduced in [2], a gauge transformation of an $n$-gerbe


discussed earlier, are somewhat more interesting. From the general discussion above, as this class of gerbes has band $C^\infty(U(1))$, they should be topologically classified by elements of

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z})$$

Indeed, in discussing whether an $SO(n)$-bundle can be lifted to a Spin$^c$($n$)-bundle, there arises an element of $H^3(X, \mathbb{Z})$, usually labeled $W_3$, which is the image under a Bockstein homomorphism of the second Stiefel-Whitney class of the $SO(n)$-bundle in question. The characteristic class $W_3$ is precisely the integral characteristic class classifying the Spin$^c$ gerbe. An $SO(n)$-bundle admits a Spin$^c$ lift if and only if $W_3$ vanishes, in precise accord with the general framework above – a 1-gerbe admits a global trivialization if and only if the classifying integral characteristic class vanishes. A global trivialization of the Spin$^c$ gerbe associated to some $SO(n)$-bundle is precisely a Spin$^c$ lift of the bundle.

### 3.4 Gauge transformations of gerbes

Just as a gauge transformation of a principal $G$-bundle on a space $X$ is defined by a map $X \rightarrow G$, it turns out that a gauge transformation of a gerbe with band $\mathcal{A} = C^\infty(G)$ is a principal $G$-bundle. (In terminology introduced in [2], a gauge transformation of an $n$-gerbe


discussed earlier, are somewhat more interesting. From the general discussion above, as this class of gerbes has band $C^\infty(U(1))$, they should be topologically classified by elements of

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z})$$

Indeed, in discussing whether an $SO(n)$-bundle can be lifted to a Spin$^c$($n$)-bundle, there arises an element of $H^3(X, \mathbb{Z})$, usually labeled $W_3$, which is the image under a Bockstein homomorphism of the second Stiefel-Whitney class of the $SO(n)$-bundle in question. The characteristic class $W_3$ is precisely the integral characteristic class classifying the Spin$^c$ gerbe. An $SO(n)$-bundle admits a Spin$^c$ lift if and only if $W_3$ vanishes, in precise accord with the general framework above – a 1-gerbe admits a global trivialization if and only if the classifying integral characteristic class vanishes. A global trivialization of the Spin$^c$ gerbe associated to some $SO(n)$-bundle is precisely a Spin$^c$ lift of the bundle.

### 3.4 Gauge transformations of gerbes

Just as a gauge transformation of a principal $G$-bundle on a space $X$ is defined by a map $X \rightarrow G$, it turns out that a gauge transformation of a gerbe with band $\mathcal{A} = C^\infty(G)$ is a principal $G$-bundle. (In terminology introduced in [2], a gauge transformation of an $n$-gerbe


discussed earlier, are somewhat more interesting. From the general discussion above, as this class of gerbes has band $C^\infty(U(1))$, they should be topologically classified by elements of

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z})$$

Indeed, in discussing whether an $SO(n)$-bundle can be lifted to a Spin$^c$($n$)-bundle, there arises an element of $H^3(X, \mathbb{Z})$, usually labeled $W_3$, which is the image under a Bockstein homomorphism of the second Stiefel-Whitney class of the $SO(n)$-bundle in question. The characteristic class $W_3$ is precisely the integral characteristic class classifying the Spin$^c$ gerbe. An $SO(n)$-bundle admits a Spin$^c$ lift if and only if $W_3$ vanishes, in precise accord with the general framework above – a 1-gerbe admits a global trivialization if and only if the classifying integral characteristic class vanishes. A global trivialization of the Spin$^c$ gerbe associated to some $SO(n)$-bundle is precisely a Spin$^c$ lift of the bundle.

### 3.4 Gauge transformations of gerbes

Just as a gauge transformation of a principal $G$-bundle on a space $X$ is defined by a map $X \rightarrow G$, it turns out that a gauge transformation of a gerbe with band $\mathcal{A} = C^\infty(G)$ is a principal $G$-bundle. (In terminology introduced in [2], a gauge transformation of an $n$-gerbe


discussed earlier, are somewhat more interesting. From the general discussion above, as this class of gerbes has band $C^\infty(U(1))$, they should be topologically classified by elements of

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z})$$

Indeed, in discussing whether an $SO(n)$-bundle can be lifted to a Spin$^c$($n$)-bundle, there arises an element of $H^3(X, \mathbb{Z})$, usually labeled $W_3$, which is the image under a Bockstein homomorphism of the second Stiefel-Whitney class of the $SO(n)$-bundle in question. The characteristic class $W_3$ is precisely the integral characteristic class classifying the Spin$^c$ gerbe. An $SO(n)$-bundle admits a Spin$^c$ lift if and only if $W_3$ vanishes, in precise accord with the general framework above – a 1-gerbe admits a global trivialization if and only if the classifying integral characteristic class vanishes. A global trivialization of the Spin$^c$ gerbe associated to some $SO(n)$-bundle is precisely a Spin$^c$ lift of the bundle.

### 3.4 Gauge transformations of gerbes

Just as a gauge transformation of a principal $G$-bundle on a space $X$ is defined by a map $X \rightarrow G$, it turns out that a gauge transformation of a gerbe with band $\mathcal{A} = C^\infty(G)$ is a principal $G$-bundle. (In terminology introduced in [2], a gauge transformation of an $n$-gerbe


discussed earlier, are somewhat more interesting. From the general discussion above, as this class of gerbes has band $C^\infty(U(1))$, they should be topologically classified by elements of

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z})$$

Indeed, in discussing whether an $SO(n)$-bundle can be lifted to a Spin$^c$($n$)-bundle, there arises an element of $H^3(X, \mathbb{Z})$, usually labeled $W_3$, which is the image under a Bockstein homomorphism of the second Stiefel-Whitney class of the $SO(n)$-bundle in question. The characteristic class $W_3$ is precisely the integral characteristic class classifying the Spin$^c$ gerbe. An $SO(n)$-bundle admits a Spin$^c$ lift if and only if $W_3$ vanishes, in precise accord with the general framework above – a 1-gerbe admits a global trivialization if and only if the classifying integral characteristic class vanishes. A global trivialization of the Spin$^c$ gerbe associated to some $SO(n)$-bundle is precisely a Spin$^c$ lift of the bundle.
is defined by an \((n - 1)\)-gerbe.) (Note that we are implicitly assuming \(G\) is an abelian Lie group.)

Strictly speaking, only equivalence classes of principal \(G\)-bundles will have distinct actions, but we shall defer discussion of this technicality until later.

How precisely does a principal \(G\)-bundle act on a gerbe with band \(C^\infty(G)\)? A complete discussion of the technical details is beyond the intended scope of this section – see instead section 5.3 for a complete discussion. However, we can give some general intuition. Suppose, for example, the objects of \(\mathcal{C}(U)\) are line bundles on \(U\). In this case, the action of \(I\) amounts to tensoring each object with \(I\). This yields, for any object \(P\), a map \(P \mapsto P \times I\). (We use the notation \(P \times I\) instead of something like \(P \otimes I\) because at the end of the day, we need not be manipulating bundles. Again, see section 5.3 for details.)

Not only does a principal \(G\)-bundle yield an action on the objects of \(\mathcal{C}(U)\), but it can also be used to define a self-equivalence of the category \(\mathcal{C}(U)\).

In section 5.3 we will give a number of results related gauge transformations of gerbes. One result, as noted above, is that a gauge transformation of a gerbe is defined by a bundle (a 0-gerbe), just as a gauge transformation of a bundle is given by a function (a (-1)-gerbe). We also argue that gauge transformations of gerbes commute with gerbe maps, just as gauge transformations of principal bundles commute with principal bundle maps, and that any map between two gerbes of the same band, over the same space, is necessarily an isomorphism, just as any morphism of principal \(G\)-bundles, for fixed \(G\) and over a fixed space, is necessarily an isomorphism. Finally, we shall argue that any gerbe with band \(C^\infty(G)\) looks locally like the trivial gerbe \(\text{Tors}(G)\), just as any principal bundle looks locally like the trivial principal bundle.

### 3.5 Connections on gerbes

In this subsection we shall restrict to gerbes with band \(C^\infty(U(1))\), for convenience.

Now, how does one define a connection on a 1-gerbe, defined in terms of stacks as above? It is tempting to proceed as follows. (This description will be wrong, but useful pedagogically.) Let \(\{U_\alpha\}\) be an open cover. Identify \(\mathcal{C}(U_\alpha)\) with \(\text{Tors}(U(1))(U_\alpha)\), i.e., identify objects of \(\mathcal{C}(U_\alpha)\) with principal \(U(1)\)-bundles on \(U_\alpha\). To each open set \(U_\alpha\) and principal \(U(1)\) bundle \(\mathcal{L}_\alpha\), associate a connection \(\nabla^\alpha\). Note that since \(\mathcal{L}_\alpha\) need not be trivial, and since \(U_\alpha\) need not be contractible, there is no reason why \(\nabla^\alpha\) should be expressible in terms of a single 1-form definable over all of \(U_\alpha\). Let \(F^\alpha\) denote the curvature of \(\nabla^\alpha\) on \(U_\alpha\). It is very tempting (and incorrect) to then identify the \(B\)-field associated to \(U_\alpha\) with \(F^\alpha\). Then, on overlaps, \(F^\alpha - F^\beta = 0\) in cohomology, so there exists 1-forms \(A^{\alpha\beta}\) such that \(F^\alpha - F^\beta = dA^{\alpha\beta}\). We could then clearly build up the \(\check{\text{C}}\text{ech-de Rham}\) complex describing a connection on a gerbe,
as described in \cite{2}.

Unfortunately, this natural-looking idea will not work in general. The essential problem is that this would define a 3-form $H$ that was always zero in cohomology, as $H|_{U_\alpha} = dF^\alpha = 0$. Put more simply, the $B$-field associated to any open set need not be closed, whereas the idea described in the paragraph above would always necessarily associate a closed 2-form to each $U_\alpha$.

The correct way to associate a connection to a 1-gerbe is somewhat more complicated to explain.

In this section we shall make frequent use of the idea of a torsor. We have strenuously avoided speaking of torsors in previous sections, but at this point their use becomes unavoidable. Torsors are defined in section 5.2; we shall assume henceforward that the reader is acquainted with the material in that section.

Connections on gerbes are defined in \cite{3} in terms of “connective structures” and “curvings” on the gerbe. In order to get some intuition for the meaning of these concepts, we shall take a moment to define analogues of “connective structure” and “curving” for a bundle. On a fixed principal $U(1)$-bundle on a space $X$, a “connective structure” is defined to be an $\Omega^1(X)$-torsor consisting of all the connections on the principal $U(1)$-bundle. This connective structure is required to obey the constraint that any gauge transformation $\phi : X \to U(1)$ defines an automorphism $\phi_*$ of the connective structure, such that any section $\nabla$ of the connective structure (a single connection on the bundle) transforms under $\phi_*$ as, $\phi_*(\nabla) = \nabla - d\ln\phi$. One then defines a “curving,” which is a map that assigns a closed 2-form $K(\nabla)$ to any section $\nabla$ of the connective structure, such that

1. if $\phi : X \to U(1)$ is a gauge transformation, then $K(\phi_*(\nabla)) = K(\nabla)$ for any $\nabla$
2. for any $\alpha \in \Omega^1(X)$, $K(\nabla + \alpha) = K(\nabla) + d\alpha$

Clearly the curving corresponds to the curvature of the connection $\nabla$.

These notions of connective structure and curving for a bundle seem quite clumsy, however they are more useful when discussing gerbes.

Now that we have given some basic intuitions, we shall give the rigorous definition of a connection on a 1-gerbe. This is formally described as assigning a connective structure and curving to a gerbe, call it $\mathcal{C}$. We shall closely follow the presentation of \cite[section 5.3]{3}.

Let $\Omega^1(U)$ denote the sheaf of 1-forms on an open set $U$. (Note that $\Omega^1(U)$ is a sheaf of abelian groups on $U$.)

A connective structure on a gerbe $\mathcal{C}$ is defined to be a Cartesian functor $\text{Co} : \mathcal{C} \to \text{Tors}(\Omega^1)$ between the underlying stacks, subject to the following constraint. Let $U$ be an open set and
$P \in \text{Ob } \mathcal{C}(U)$, let $\nabla$ be a section of the $\Omega^1(U)$-torsor $\text{Co}(U)(P)$, and let $g : P \to P$ be an automorphism of $P$, which we shall identify with an element of the band. Then we demand $\text{Co}(U)(g)(\nabla) = \nabla - d\ln g$.

At this point we shall introduce some notation. For any isomorphism $\phi : P_1 \cong P_2$ of objects of $\mathcal{C}(U)$, its image under the functor $\text{Co}(U)$ is denoted $\phi_*$. In other words, $\phi_* = \text{Co}(U)(\phi)$. For any inclusion $\rho : U_1 \hookrightarrow U$, let $\chi_\rho$ denote the invertible natural transformation $\rho^* \circ \text{Co}(U) \Rightarrow \text{Co}(U_1) \circ \rho^*$ appearing in the definition of $\text{Co}$ as a Cartesian functor. (Readers also studying [6, section 5.3] will note that in that reference, $\text{Co}(U)(P)$ is abbreviated to $\text{Co}(P)$, and the natural transformation we denote by $\chi_\rho$ is there denoted $\alpha_\rho$.)

An example of a gerbe with connective structure is in order at this point. Consider the (trivial) gerbe $\text{Tors}(G)$ of all principal $G$-bundles, where we assume $G$ is an abelian Lie group. An example of a connective structure on $\text{Tors}(G)$ is the one obtained by assigning, to each principal $G$-bundle $P$ over any open set $U$, the $\Omega^1(U)$-torsor of all connections on $P$. In other words, define $\text{Co}(U)(P)$ to be the $\Omega^1(U)$-torsor of all connections on $P$, and for any isomorphism $f : P_1 \to P_2$ define $\text{Co}(U)(f)$ to be the morphism such that for any $\nabla \in \Gamma(U, \text{Co}(U)(P_1))$, the morphism

$$f : (P_1, \nabla) \longrightarrow (P_2, \text{Co}(U)(f)(\nabla))$$

is an equivalence of bundles with connection. One can then define the rest of the structure of a Cartesian functor in the obvious way.

It can be shown that any gerbe with band $C^\infty(U(1))$ admits a connective structure. We shall not work through the details of this argument here; see instead, for example, [6, section 5.3].

Note that [6, section 5.3] if $\Phi : \mathcal{G} \to \mathcal{G}'$ defines an equivalence of two gerbes $\mathcal{G}$, $\mathcal{G}'$ on a space $X$, both with band $C^\infty(U(1))$, and the gerbe $\mathcal{G}'$ has a connective structure, call it $\text{Co}'$, then the connective structure $\text{Co}'$ on $\mathcal{G}'$ can be pulled back to form a connective structure $\text{Co}$ on $\mathcal{G}$. More specifically, $\text{Co} = \text{Co}' \circ \Phi$.

It should now be clear that there is a natural notion of equivalence of gerbes with connection structure. Let $(\mathcal{G}_1, \text{Co}_1)$ and $(\mathcal{G}_2, \text{Co}_2)$ be a pair of gerbes with connective structure. We say that a Cartesian functor $\Phi : \mathcal{G}_1 \to \mathcal{G}_2$ defines an equivalence of gerbes with connective structure if

1. $\Phi$ defines an equivalence of gerbes, and
2. there exists an invertible 2-arrow $\Psi : \text{Co}_1 \Rightarrow \text{Co}_2 \circ \Phi$ between the Cartesian functors $\text{Co}_1$ and $\text{Co}_2 \circ \Phi$.

For example, we shall see later that a principal $U(1)$ bundle $I$ defines a gerbe automorphism
$I_C : C \to C$, and in such a case, the 2-arrow $\Psi$ defined above is equivalent to a choice of connection on the bundle.

Just as the difference between any two connections on a principal $U(1)$ bundle on $X$ is a 1-form on $X$, i.e., an element of $\Omega^1(X)$, it can be shown [3, prop. 5.3.6] that the difference between any two connective structures on the same gerbe on $X$ is given by an $\Omega^1(X)$-torsor.

So far we have yet to describe precisely how to associate a $B$ field to a 1-gerbe. $B$ fields are described as a “curving” of the connective structure introduced above.

More precisely, given some gerbe $C$ on $X$ with connective structure $C_0$, we can define a curving of the connective structure as follows. A curving of the connective structure is a rule that assigns to any object $P \in \text{Ob} C(U)$ and to any section $\nabla$ of the $\Omega^1(U)$-torsor $C_0(U)(P)$, a ($\mathbb{R}$-valued) 2-form $K(\nabla)$ on $U$, called the curvature of $\nabla$, such that the following three properties are satisfied:

1. Given an inclusion $\rho : U_1 \hookrightarrow U$, the curvature $K(\chi_\rho(\rho^*\nabla))$ of the section $\chi_\rho(\rho^*\nabla)$ of $C_0(U_1)(\rho^*P)$ is equal to $\rho^*K(\nabla)$, where $\chi_\rho$ denotes the natural transformation defining $C_0$ as a Cartesian functor.

2. Let $\phi : P \cong P'$ be an isomorphism with another object $P' \in \text{Ob} C(U)$. Let $\phi_*(\nabla)$ be the corresponding section of $C_0(U)(P')$. Then $K(\nabla) = K(\phi_*(\nabla))$.

3. Let $\omega \in \Omega^1(U)$. Then $K(\nabla + \omega) = K(\nabla) + d\omega$.

From the last two conditions on the curving, we see that $K$ associates to any isomorphism class of objects in $C(U)$, a 2-form (an element of $\Omega^2(U)$, not necessarily closed) modulo exact 2-forms.

It can be shown [3, section 5.3] that, given a connective structure $C_0$ on a gerbe, there always exist curvings.

In passing, we should mention that if $\Phi_1, \Phi_2 : G_1 \to G_2$ are any pair of gerbe maps between the gerbes $G_1, G_2$, and $\psi : \Phi_1 \Rightarrow \Phi_2$ is any 2-arrow, then for any curving $K$ on $(G_2, C_0)$, we have as an immediate consequence of the definition of curving that

$$K(\nabla) = K(\psi_*(\nabla))$$

where $\nabla \in \Gamma(U, (C_0 \circ \Phi_1)(U)(P))$, for any open $U$ and any object $P \in \text{Ob} G_1(U)$. Intuitively, this means that if $(I, \nabla), (I', \nabla')$ are two bundles with connection which are isomorphic (as

---

8The reader should note that the 2-form which we here denote the “curvature” of a connection, is not the curvature in the usual sense. Rather, it is merely some 2-form – not necessarily closed – associated to the connection. The nomenclature is unfortunate, but seems to be standard.
bundles with connection), then the associated 2-forms $K(\nabla)$ and $K(\nabla')$ should be identical – the 2-forms $K$ should be the same on equivalence classes of bundles with connection.

Suppose that two gerbes with connective structure $(G_1, Co_1)$ and $(G_2, Co_2)$ come with specified curvings $K_1, K_2$, respectively. Then we say that $\Phi$ defines an equivalence of gerbes with connective structure and curving if

1. $(\Phi, \Psi)$ defines an equivalence of gerbes with connective structure, where $\Psi : Co_1 \Rightarrow Co_2 \circ \Phi$ is the associated 2-arrow between connective structures, and
2. for all open $U$, for all objects $P \in \text{Ob } G_1(U)$, and for all $\nabla \in \Gamma(U, Co_1(U)(P))$, we have that
   $$K_1(\nabla) = K_2(\Psi(\nabla))$$

Earlier, we mentioned that for a gerbe automorphism $I_C$ defined by a principal bundle $I$, specifying a 2-arrow $\Psi : Co \Rightarrow Co \circ I_C$ is equivalent to specifying a connection on the bundle. In order for such an automorphism of a gerbe with connective structure to be an automorphism of a gerbe with connective structure and curving, the constraint on the $K$'s implies that the connection on $I$ must be flat.

We define an “equivalence of gerbes with connection” to be an equivalence of gerbes with connective structure and curving. We shall usually use the former notation rather than the latter, as it is briefer.

How can we make contact with the description of connection given earlier in section 3? Let $\{U_\alpha\}$ be a good cover of $X$, such that the objects in any one category $\mathcal{C}(U_{\alpha_1,\ldots,\alpha_n})$ are all isomorphic. Then to each open set $U_\alpha$, the curving $K$ associates a 2-form (not necessarily closed), defined up to the addition of an exact 2-form. These 2-forms are precisely the 2-forms appearing in the earlier definition of connections on 1-gerbes. The rest of the earlier description – connections on principal $U(1)$ bundles on overlaps – can be understood directly in terms of the transition functions.

Earlier we argued that principal $G$-bundles define gauge transformations on gerbes with band $C^\infty(G)$; how does such a gauge transformation act on the connective structure and curving? We shall examine this in detail in section 5.3.3; we shall outline the results here. Let $I$ denote a principal $G$-bundle defining a gauge transformation of a 1-gerbe. Let $P \in \text{Ob } \mathcal{C}(U)$. We argued earlier that $I$ defines a map $P \mapsto P \times (I|_U)$. Now, a precise specification of how a particular section $\nabla \in \Gamma(U, Co(U)(P))$ is mapped by $I$ is equivalent to a specification of a connection on $I$ [2, section 5.3, equ’n (5-11)]. (Note in passing that this statement dovetails with the earlier observation that any two connective structures differ by an $\Omega^1$-torsor.)

More explicitly, let $\{A^\alpha\}$ be a connection on $I|_U$, defined with respect to an open cover $\{U_\alpha\}$ of $U$. In other words, each $A^\alpha$ is a 1-form on $U_\alpha$. Then if $\nabla_\alpha \in \Gamma(U_\alpha, Co(U_\alpha)(P|_\alpha))$, then under the action of $I$ on the gerbe, $\nabla_\alpha \mapsto \nabla_\alpha + A^\alpha$. 

26
So far we have described how the bundle $I$ defining a gauge transformation acts on the connective structure. How does $I$ act on the curving? It is clear from the definition of curving that $K(\nabla_\alpha + A^\alpha) = K(\nabla_\alpha) + dA^\alpha$.

Thus, we have recovered the description of gauge transformations on gerbe connections outlined in [2].

3.6 Transition functions for gerbes

In section 5.3.5, we show that if $\mathcal{C}$ is a gerbe with band $\mathcal{A} = C^\infty(G)$, then for any open $U$ such that $\mathcal{C}(U)$ is nonempty, the category $\mathcal{C}(U)$ is equivalent to the category $\text{Tors}(G)(U)$.

We can use this fact to define transition functions for gerbes. Now, such a term should be explained — we have described gerbes in terms of sheaves of categories, analogously to describing bundles in terms of sheaves of sections. Transition functions are not a necessary component of such a description. However, we can certainly recover transition functions, if we choose to do so. (For a discussion of sheaves of sets in terms of transition-function-like language, see for example [19, ch. I.A.iii, cor. I-11].)

The transition functions for a gerbe should be clear. Given some open cover $\{U_\alpha\}$ of $X$ such that $\mathcal{C}(U_\alpha) \neq \emptyset$ for all $\alpha$, and a set of equivalences of categories from $\mathcal{C}(U_\alpha)$ into $\text{Tors}(G)(U_\alpha)$, it should be clear that we could describe the gerbe in terms of transition functions between the categories $\text{Tors}(G)(U_\alpha)$. In other words, we can specify the gerbe by specifying principal $G$-bundles on overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$. Each such bundle determines a functor, and so determines how $\text{Tors}(G)(U_\alpha)$ is mapped into $\text{Tors}(G)(U_\beta)$. Elements of Čech cohomology classifying the gerbe are determined from tensor products of the bundles on triple overlaps; conversely, if one wishes to describe a gerbe with fixed Čech cohomology, one can demand that tensor products of bundles on triple overlaps have appropriate canonical trivializations. This description is precisely analogous to describing bundles in terms of gauge transformations on overlaps, describing how the local trivializations are mapped into one another. Note furthermore that this description is precisely the description of 1-gerbes given in [4, 5].

It should be clear that in order to describe a 1-gerbe with connection, one would specify a principal $G$-bundle with connection on each overlap $U_{\alpha\beta}$.

Morphisms of principal $G$-bundles (of same structure group, over same space) can be described in local trivializations as a set of gauge transformations [17, section 5.5]

$$\phi_\alpha : U_\alpha \to G$$

More generally, for any open $U$ such that $\mathcal{C}(U)$ is nonempty, the category $\mathcal{C}(U)$ is equivalent to the category $\text{Tors}(\mathcal{A})(U)$ of $\mathcal{A}$-torsors on $U$. 

27
one for each element $U_\alpha$ of an open cover $\{U_\alpha\}$. It should be clear that one can describe a map of gerbes in a similar fashion: for each $U_\alpha$, associate a principal $G$-bundle $T_\alpha$. This bundle determines an automorphism of the category $\text{Tors}(G)(U_\alpha)$, which describes how the gerbes are mapped into one another at the level of local trivializations.

It should also be clear that a map of gerbes with connection can be described in terms of a set of principal $G$-bundles $T_\alpha$ with connection.

### 3.7 Nonabelian gerbes

In the physics literature it is sometimes claimed\footnote{For example, see [20].} that certain physical theories have an understanding in terms of nonabelian gerbes. In the rest of this paper we have specialized to abelian gerbes, that is, gerbes with abelian band.

Just as an abelian gerbe can be described in terms of a stack that locally looks like a stack of principal $G$-bundles for abelian $G$, it is presumably the case that a nonabelian gerbe can be described in terms of a stack that locally looks like a stack of principal $G$-bundles for nonabelian $G$.

Unfortunately, more than this is difficult to say within the present framework. Much of our discussion of gerbes has hinged, either implicitly or explicitly, on the assumption that the band is a sheaf of abelian groups. We have not made a thorough study of how matters would be altered if the band became nonabelian.

Considerably more information on nonabelian gerbes can be found in [8]. Related material can be found in [21].

### 4 Technical notes on stacks

In this section we give some highly technical material on stacks. A reader perusing this paper for the first (or even the second or third) time is strongly encouraged to skip this section entirely.

In particular, we shall describe sheafification of presheaves of categories, pullbacks of stacks, and stalks of stacks. The basics of these topics are outlined in [6, section 5].
4.1 Sheafification

Given any presheaf of sets, it is possible to construct a sheaf of sets through a process sometimes called sheafification. Similarly, given any presheaf of categories, it is possible to construct a sheaf of categories. In this section we shall describe this procedure.

We should warn the reader that this section is extremely technical in nature. A reader visiting this material for the first time is urged to skip ahead to the next section.

First, we shall review sheafification for presheaves of sets, then we shall describe the process for presheaves of categories. In sheafification of presheaves of categories, one defines a family of “descent categories,” then takes a direct limit of descent categories. We shall devote subsections to both of these notions. Finally, we shall describe how to lift a Cartesian functor between presheaves of categories to a Cartesian functor between their sheafifications.

The bulk of our discussion will be based on material in [6, section 5.2].

4.1.1 Sheafification of presheaves of sets

First, let us take a moment to review how sheafification works for presheaves of sets, following [6, section 5.1]. (For a less technical review, see for example [22, section 0.3].) Let $F$ be a presheaf of sets on a space $X$, and pick some open set $U \subseteq X$. We want to build a sheaf of sets $\tilde{F}$, i.e., a presheaf of sets $\tilde{F}$ such that there is a bijective correspondence between elements of $\tilde{F}(U)$ and collections $\{s_{\alpha} \in F(U_{\alpha})\}$, $\{U_{\alpha}\}$ an open cover of $U$, such that $s_{\alpha}|_{U_{\alpha\beta}} = s_{\beta}|_{U_{\alpha\beta}}$.

The basic idea will be to build $\tilde{F}(U)$ by taking as elements, collections $\{s_{\alpha} \in F(U_{\alpha})\}$ for $\{U_{\alpha}\}$ an open cover, such that $s_{\alpha}|_{U_{\alpha\beta}} = s_{\beta}|_{U_{\alpha\beta}}$. In other words, we will build a set $\tilde{F}(U)$ in which the bijective correspondence between elements of the set and local elements satisfying compatibility conditions is built in from the definition.

More formally, each set $\tilde{F}(U)$ is constructed as follows. Take $\tilde{F}(U)$ to be the disjoint union over all open covers $\{U_{\alpha}\}$ of $U$ of collections $\{s_{\alpha} \in F(U_{\alpha})\}$ such that $s_{\alpha}|_{U_{\alpha\beta}} = s_{\beta}|_{U_{\alpha\beta}}$, modulo an equivalence relation $\sim$. In other words,

$$\tilde{F}(U) = \bigsqcup_{\{U_{\alpha}\}} \left\{ \{s_{\alpha} \in F(U_{\alpha})\} \mid s_{\alpha}|_{U_{\alpha\beta}} = s_{\beta}|_{U_{\alpha\beta}} \right\} / \sim$$  \hspace{1cm} (13)

The equivalence relation $\sim$ is defined as follows. Identify two collections $\{U_{\alpha}, s_{\alpha}\}$ and $\{U_{\alpha}', s_{\alpha}'\}$ if there exists another open cover $\{U_{\alpha}'\}$, a refinement of both $\{U_{\alpha}\}$ and $\{U_{\alpha}'\}$, such that whenever $U_{\alpha}' \subseteq U_{\beta} \cap U_{\gamma}'$, we have $s_{\beta}|_{U_{\alpha}'} = s_{\gamma}'|_{U_{\alpha'}}$. 


We can rewrite equation (13) in terms of a direct limit over open covers, as

\[
\tilde{F}(U) = \lim_{\{U_{\alpha}\}} \left\{ \{s_{\alpha} \in F(U_{\alpha})\} \mid s_{\alpha}|_{U_{\alpha\beta}} = s_{\beta}|_{U_{\alpha\beta}} \right\}
\]

where the partial ordering on open covers \{U_{\alpha}\} is provided by the notion of refinement.

Note that if \(F\) is a sheaf, not just a presheaf, then \(\tilde{F} = F\), i.e., for all open sets \(U\), \(\tilde{F}(U) = F(U)\).

### 4.1.2 Descent categories

Let \(\mathcal{C}\) denote a presheaf of categories on a space \(X\), and let \(U\) denote an open set in \(X\). In order to define the category \(\tilde{\mathcal{C}}(U)\) in the associated sheaf \(\tilde{\mathcal{C}}\), we shall take a direct limit over “descent categories” defined for the open set \(U\). In this subsection we shall describe descent categories. In subsequent subsections we shall describe the process of taking direct limits, and the resulting sheafification.

To each open cover \(\{U_{\alpha}\}\) of \(U\), we shall define a category, which we shall denote by

\[\text{Desc}_{U}(\mathcal{C}, \{U_{\alpha}\})\]

and which is called a descent category.

The objects of \(\text{Desc}(\mathcal{C}, \{U_{\alpha}\})\) are collections \(\{(x_{\alpha}), (\phi_{\alpha\beta})\}\), where each \(x_{\alpha} \in \text{Ob} \ \mathcal{C}(U_{\alpha})\), and where \(\phi_{\alpha\beta} : x_{\beta}|_{U_{\alpha\beta}} \xrightarrow{\sim} x_{\alpha}|_{U_{\alpha\beta}}\) are isomorphisms, such that (schematically\(^{11}\))

\[\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma}\]

in \(\mathcal{C}(U_{\alpha\beta\gamma})\), and \(\phi_{\alpha\alpha} = 1_{\alpha}\) in \(\mathcal{C}(U_{\alpha})\).

Morphisms \(\{(x_{\alpha}), (\phi_{\alpha\beta})\} \rightarrow \{(y_{\alpha}), (\phi'_{\alpha\beta})\}\) in \(\text{Desc}(\mathcal{C}, \{U_{\alpha}\})\) are collections of morphisms \(\{f_{\alpha} : x_{\alpha} \rightarrow y_{\alpha}\}\) in \(\mathcal{C}(U_{\alpha})\) such that the following diagram commutes:

\[
\begin{array}{ccc}
x_{\beta}|_{U_{\alpha\beta}} & \xrightarrow{\phi_{\alpha\beta}} & x_{\alpha}|_{U_{\alpha\beta}} \\
f_{\beta}|_{U_{\alpha\beta}} \downarrow & & \downarrow f_{\alpha}|_{U_{\alpha\beta}} \\
y_{\beta}|_{U_{\alpha\beta}} & \xrightarrow{\phi'_{\alpha\beta}} & y_{\alpha}|_{U_{\alpha\beta}}
\end{array}
\]

Note that if \(\mathcal{C}\) is a sheaf of categories, not just a presheaf, then for any open cover \(\{U_{\alpha}\}\) of \(U\), the category \(\text{Desc}(\mathcal{C}, \{U_{\alpha}\})\) is equivalent to \(\mathcal{C}(U)\).

\(^{11}\)The correct statement is commutativity of diagram \([\text{3}].\)
Suppose that \( \{U'_a\} \) is a refinement of \( \{U_a\} \) and \( \rho : \{U'_a\} \to \{U_a\} \) is the inclusion, i.e., \( \rho \) is a collection of maps \( \{\rho_i : U'_i \to U_{a(i)}\} \). Then \( \rho \) induces a functor

\[
\rho^* : \text{Desc}(C, \{U_a\}) \to \text{Desc}(C, \{U'_a\})
\]

The action of \( \rho^* \) on an object \( \{(x_{\alpha}), (\phi_{\alpha\beta})\} \in \text{Ob} \text{Desc}(C, \{U_a\}) \) is given by

\[
\rho^* : (x_{\alpha}) \mapsto (x_i \equiv x_{a(i)}|_{U'_i})
\]

\[
\rho^* : (\phi_{\alpha\beta}) \mapsto (\phi_{ij} \equiv \varphi_{ij} \circ \varphi^{-1}_{a(i)a(j);ij} \circ \phi_{a(i)a(j)}|_{U'_i} \circ \varphi_{a(i)a(j),ij} \circ \varphi^{-1}_{j,ij})
\]

where the \( \varphi \) are the invertible natural transformations appearing in the definition of the presheaf of categories \( C \). One can verify (after an extremely lengthy diagram chase) that this functor is well-defined on objects, i.e., that the image under \( \rho^* \) of the set of morphisms \( (\phi_{\alpha\beta}) \) satisfies the compatibility relation \( \mathbf{3} \).

We still need to define the action of the proposed functor \( \rho^* \) on morphisms. Given objects

\[
\{(x_{\alpha}), (\phi_{\alpha\beta})\}, \{(y_{\alpha}), (\phi'_{\alpha\beta})\} \in \text{Ob} \text{Desc}(C, \{U_a\})
\]

and a morphism

\[
\{f_{\alpha} : x_{\alpha} \to y_{\alpha}\} : \{(x_{\alpha}), (\phi_{\alpha\beta})\} \to \{(y_{\alpha}), (\phi'_{\alpha\beta})\}
\]

define

\[
\rho^* \{f_{\alpha}\} = \{f_i \equiv f_{a(i)}|_{U'_i} : x_{a(i)}|_{U'_i} \to y_{a(i)}|_{U'_i}\}
\]

It can be shown that this map is well-defined, i.e., the appropriate version of diagram \( \mathbf{4} \) commutes.

It is now straightforward to check that \( \rho^* \) does indeed define a functor

\[
\rho^* : \text{Desc}(C, \{U_a\}) \to \text{Desc}(C, \{U'_a\})
\]

Now, suppose \( \rho_1 : \{U'_a\} \to \{U_a\} \) and \( \rho_2 : \{U''_a\} \to \{U'_a\} \) are a pair of families of inclusions, associated to the refinement \( \{U'_a\} \) of \( \{U_a\} \), and to the refinement \( \{U''_a\} \) of \( \{U'_a\} \). Then there exists an invertible natural transformation \( \lambda : (\rho_1 \rho_2)^* \Rightarrow \rho_2^* \circ \rho_1^* \).

This invertible natural transformation is defined as follows. To each object

\[
\{(x_{\alpha}), (\phi_{\alpha\beta})\} \in \text{Ob} \text{Desc}(C, \{U_a\})
\]

\( \lambda \) associates a morphism

\[
\lambda(\{(x_{\alpha}), (\phi_{\alpha\beta})\}) : (\rho_1 \rho_2)^* \{(x_{\alpha}), (\phi_{\alpha\beta})\} \to (\rho_2^* \circ \rho_1^*) \{(x_{\alpha}), (\phi_{\alpha\beta})\}
\]

given by

\[
\lambda(\{(x_{\alpha}), (\phi_{\alpha\beta})\}) = \{\varphi_{i(a),a}(x_{a(a)}|_{U''_a}) : x_{a(a)}|_{U''_a} \to x_{a(a)}|_{U'_{i(a)}|_{U''_a}}\}
\]
where the $\varphi$ are the invertible natural transformations appearing in the definition of $\mathcal{C}$ as a presheaf of categories. It is straightforward to check that this indeed is a morphism, i.e., the appropriate version of diagram (14) commutes, and moreover $\lambda$ is an invertible natural transformation.

Finally, it is straightforward to check that if we are given three composable inclusions of open covers $\rho_1,$ $\rho_2,$ $\rho_3,$ then the following diagram (corresponding to diagram (1)) of natural transformations commutes:

$$
(\rho_1\rho_2\rho_3)^* \implies \rho_3^* \circ (\rho_1\rho_2)^*
$$

$$
(\rho_2\rho_3)^* \circ \rho_1^* \implies \rho_3^* \circ \rho_2^* \circ \rho_1^*
$$

So far we have discussed functors associated to sets of inclusion maps between two open covers of a fixed open set $U$. Other functors can also be constructed, as we shall now discuss. Suppose $\rho : V \hookrightarrow U$ is an inclusion of open sets. Then, for any open cover $\{U_\alpha\}$ of $U$, $\rho$ induces a functor $\rho_{V,\{U_\alpha\}}$ between descent categories associated to $U$ and $V$:

$$
\rho_{V,\{U_\alpha\}} : \text{Desc}_U(\mathcal{C}, \{U_\alpha\}) \to \text{Desc}_V(\mathcal{C}, \{U_\alpha | V\})
$$

where we have used the fact that $\{U_\alpha | V\}$ is an open cover of $V$. This functor can be defined in precise analogy with the previously-described functor between descent categories between open covers of a single open set $U$; we shall not repeat the details, as they are virtually identical. Similarly, given two inclusions $\rho_2 : U_3 \hookrightarrow U_2$ and $\rho_1 : U_2 \hookrightarrow U_1$, one can define (as above) an invertible natural transformation $\kappa_{12} : (\rho_1\rho_2)^*_{U_3,\{U_\alpha\}} \Rightarrow \rho_2^*_{U_3,\{U_\alpha \cup U_2\}} \circ \rho_1^*_{U_2,\{U_\alpha\}}$, such that analogues of diagram (1) commute.

Finally, we can also construct natural transformations between compositions of the two types of functors listed above. Let $\rho_{12} : \{U_2^{1}\alpha\} \to \{U_1^{1}\alpha\}$ be a set of inclusions between covers of an open set $U \subseteq X$, and let $\rho : V \hookrightarrow U$ be an inclusion map between open sets, inducing functors $\rho_{V,\{U_1^{1}\alpha\}}$ and $\rho_{V,\{U_2^{1}\alpha\}}$. Let $\rho_{12} : \{U_2^{1}\alpha | V\} \to \{U_1^{1}\alpha | V\}$ be the set of inclusion maps between covers of $V$ that makes the following diagram commute:

$$
\begin{array}{ccc}
\{U_1^{1}\alpha\} & \xleftarrow{\rho_{V,\{U_1^{1}\alpha\}}} & \{U_1^{1}\alpha | V\} \\
\rho_{12} \uparrow & & \uparrow \rho_{V,\{U_1^{1}\alpha\}} \\
\{U_2^{1}\alpha\} & \xleftarrow{\rho_{V,\{U_2^{1}\alpha\}}} & \{U_2^{1}\alpha | V\}
\end{array}
$$

Then we can define the following two invertible natural transformations:

$$
\delta_{V,12} : (\rho_{12} \circ \rho_{V,\{U_2^{1}\alpha\}})^* \Rightarrow \rho_{V,\{U_2^{1}\alpha\}}^* \circ \rho_{12}^*
$$

$$
\delta_{12,V} : (\rho_{V,\{U_1^{1}\alpha\}} \circ \rho_{V,\{U_2^{1}\alpha\}})^* \Rightarrow \rho_{V,\{U_2^{1}\alpha\}}^* \circ \rho_{V,\{U_1^{1}\alpha\}}^*
$$

These natural transformations also make analogues of diagram (1) commute.
4.1.3 Direct limits

Given our presheaf of categories $\mathcal{C}$ and for any open $U$, the descent categories $\text{Desc}(\mathcal{C}, \{U_\alpha\})$, we can now construct the associated sheaf of categories.

The sheaf of categories $\tilde{\mathcal{C}}$ associated to the presheaf of categories $\mathcal{C}$ is defined by

$$\tilde{\mathcal{C}}(U) = \lim_{\rightarrow} \text{Desc}(\mathcal{C}, \{U_\alpha\})$$

(16)

In words, $\tilde{\mathcal{C}}(U)$ is defined to be the direct limit over open covers of $U$ of descent categories.

We shall now give the definition of the direct limit used above.

The objects of the direct limit (16) are the disjoint union of the objects of all the descent categories associated to open covers of $U$. In other words,

$$\text{Ob } \lim_{\rightarrow} \text{Desc}(\mathcal{C}, \{U_\alpha\}) = \bigsqcup \text{Ob } \text{Desc}(\mathcal{C}, \{U_\alpha\})$$

(17)

Now, we shall describe morphisms of the direct limit category. Let $P_1, P_2$ be objects:

$$P_1 \in \text{Ob } \text{Desc}(\mathcal{C}, \{U_\alpha^1\})$$
$$P_2 \in \text{Ob } \text{Desc}(\mathcal{C}, \{U_\alpha^2\})$$

Let $\{U_{\alpha}^3\}$ be any (open cover) refinement of both $\{U_{\alpha}^1\}$ and $\{U_{\alpha}^2\}$, let $\rho_{13} : \{U_{\alpha}^3\} \to \{U_{\alpha}^1\}$ and $\rho_{23} : \{U_{\alpha}^3\} \to \{U_{\alpha}^2\}$ be the two sets of inclusions, and define $S_{\{U_{\alpha}^3\}}$ to be the set of all morphisms $\beta : \rho_{13}^* P_1 \to \rho_{23}^* P_2$ in $\text{Desc}(\mathcal{C}, \{U_{\alpha}^3\})$.

Define the set of all morphisms $\text{Hom}(P_1, P_2)$ to be the disjoint union of all the $S_{\{U_{\alpha}^3\}}$ (for $\{U_{\alpha}^3\}$ an (open cover) refinement of both $\{U_{\alpha}^1\}$ and $\{U_{\alpha}^2\}$), modulo an equivalence relation $\sim$ to be defined momentarily. In other words,

$$\text{Hom}(P_1, P_2) = \bigsqcup S_{\{U_{\alpha}^3\}} / \sim$$

The equivalence relation $\sim$ is defined as follows. If $\{U_{\alpha}\}$ and $\{U'_{\alpha}\}$ are two open covers which both refine both $\{U_{\alpha}^1\}$ and $\{U_{\alpha}^2\}$, and

$$\rho_i : \{U_{\alpha}\} \to \{U_{\alpha}^i\}$$
$$\rho'_i : \{U'_{\alpha}\} \to \{U_{\alpha}^i\}$$

are the sets of inclusion maps ($i \in \{1, 2\}$), then we say $\beta : \rho_1^* P_1 \to \rho_2^* P_2$ in $S_{\{U_{\alpha}\}}$ is equivalent to $\beta' : \rho_1'^* P_1 \to \rho_2'^* P_2$ in $S_{\{U'_{\alpha}\}}$ if and only if there exists an (open cover) refinement $\{U_{\alpha}''\}$ of both $\{U_{\alpha}\}$ and $\{U_{\alpha}'\}$, with sets of inclusion maps

$$\gamma : \{U_{\alpha}''\} \to \{U_{\alpha}\}$$
$$\gamma' : \{U_{\alpha}''\} \to \{U_{\alpha}'\}$$

such that
1. \( \rho_i \gamma = \rho_i' \gamma' \) for \( i \in \{1, 2\} \),

2. the following diagram commutes:

\[
\begin{array}{ccc}
(\rho_1 \gamma)^*(P_1) & \xrightarrow{\gamma^* \circ \rho_1^*(P_1)} & \gamma^* \circ \rho_2^*(P_2) \\
\downarrow{\lambda_1} & & \downarrow{\lambda_2} \\
\gamma'^* \circ \rho_1'^*(P_1) & \xrightarrow{\gamma'^*(\beta)} & \gamma'^* \circ \rho_2'^*(P_2)
\end{array}
\]

(18)

where the \( \lambda \) are the natural transformations defined in the previous subsection.

We have defined objects and morphisms in the categories \( \tilde{\mathcal{C}}(U) \). We shall now take a moment to discuss composition of morphisms in this category, as the correct definition might not be completely obvious to the reader. Let \( P_1, P_2, P_3 \in \text{Ob } \tilde{\mathcal{C}}(U) \), i.e., \( P_i \in \text{Ob } \text{Desc}_U(\mathcal{C}, \{U^i_\alpha\}) \) for some open covers \( \{U^i_\alpha\} \) of \( U \), \( i \in \{1, 2, 3\} \). Let \( \beta \in \text{Hom}_{\tilde{\mathcal{C}}(U)}(P_1, P_2) \), \( \alpha \in \text{Hom}_{\tilde{\mathcal{C}}(U)}(P_2, P_3) \). In other words, there exist refinements \( \{U^4_\alpha\} \) of \( \{U^1_\alpha\} \) and \( \{U^5_\alpha\} \), \( \{U^2_\alpha\} \), and \( \{U^3_\alpha\} \), such that

\[
\beta \in \text{Hom}_{\text{Desc}(\mathcal{C}, \{U^4_\alpha\})}(\rho_{14}^* P_1, \rho_{24}^* P_2), \quad \alpha \in \text{Hom}_{\text{Desc}(\mathcal{C}, \{U^5_\alpha\})}(\rho_{25}^* P_2, \rho_{35}^* P_3)
\]

We define the composition \( \alpha \circ \beta \) as follows. Let \( \{U^6_\alpha\} \) be a refinement of both \( \{U^4_\alpha\} \) and \( \{U^5_\alpha\} \), such that \( \rho_{24} \rho_{46} = \rho_{25} \rho_{56} \). Define \( \alpha \circ \beta \) to be

\[
\alpha \circ \beta \equiv \lambda_{356}^{-1} \circ \rho_{56}^* \alpha \circ \lambda_{256} \circ \lambda_{246}^{-1} \circ \rho_{46}^* \beta \circ \lambda_{146}
\]

\[
\in \text{Hom}_{\text{Desc}(\mathcal{C}, \{U^6_\alpha\})}(\rho_{16}^* P_1, \rho_{36}^* P_3)
\]

where the \( \lambda \) are the natural transformations defined in the previous subsection. It is straightforward to check that this composition is well-defined, i.e., \( \alpha \sim \alpha' \) and \( \beta \sim \beta' \) implies \( \alpha \circ \beta \sim \alpha' \circ \beta' \).

So far we have discussed how to construct the categories \( \tilde{\mathcal{C}}(U) \) appearing in the sheaf associated to the presheaf of categories \( \mathcal{C} \). To completely define the sheaf we must also specify restriction functors and natural transformations, which we shall now do.

Let \( \rho : V \hookrightarrow U \) be an inclusion of open sets. We shall now show that \( \rho \) induces a functor

\[
\rho^* : \tilde{\mathcal{C}}(U) = \lim_{\{U_\alpha\}} \text{Desc}_U(\mathcal{C}, \{U_\alpha\}) \longrightarrow \tilde{\mathcal{C}}(V) = \lim_{\{V_\alpha\}} \text{Desc}_V(\mathcal{C}, \{U_\alpha\})
\]

First we shall describe how \( \rho^* \) acts on objects. Recall that an object of \( \tilde{\mathcal{C}}(U) \) is an object, call it \( P \), of \( \text{Desc}_U(\mathcal{C}, \{U_\alpha\}) \) for some open cover \( \{U_\alpha\} \) of \( U \). The functor \( \rho^* \) acts on the object \( P \) as,

\[
P \mapsto \rho^*_{\psi_\alpha}(P) \in \text{Ob } \text{Desc}_V(\mathcal{C}, \{U_\alpha | V\})
\]
Now we shall describe how $\rho^*$ acts on morphisms. Let $P_1, P_2 \in \text{Ob } \tilde{C}(U)$. In other words,

\[
P_1 \in \text{Ob } \text{Desc}_U(\mathcal{C}, \{U_1^\alpha\})
\]
\[
P_2 \in \text{Ob } \text{Desc}_U(\mathcal{C}, \{U_2^\alpha\})
\]

for some open covers $\{U_1^\alpha\}$, $\{U_2^\alpha\}$ of $U$. Recall

\[
\text{Hom}(P_1, P_2) = \coprod_{\{U_3^\alpha\}} \text{Hom}(\rho_{13}^* P_1, \rho_{23}^* P_2)/\sim
\]

Let $\beta \in \text{Hom}(\rho_{13}^* P_1, \rho_{23}^* P_2)$ for some refinement $\{U_3^\alpha\}$ of $\{U_1^\alpha\}$ and $\{U_2^\alpha\}$. Define $\rho^*(\beta)$ as,

\[
\rho^*(\beta) = \delta_{23,V} \circ \delta_{V,23}^{-1} \circ \rho_{V,(U_3^\alpha)}^* (\beta) \circ \delta_{V,13} \circ \delta_{13,V}^{-1}
\]

so that

\[
\rho^*(\beta) \in \text{Hom}_{\text{Desc}_U(\mathcal{C}, \{U_3^\alpha\})} \left( \rho_{V,13}^* \circ \rho_{V,(U_3^\alpha)}^* (P_1), \rho_{V,23}^* \circ \rho_{V,(U_3^\alpha)}^* (P_2) \right)
\]

It is straightforward to check that this functor is well-defined – for example, $\beta \sim \beta'$ implies $\rho^*(\beta) \sim \rho^*(\beta')$.

Let $\rho_V : V \hookrightarrow U$ and $\rho_W : W \hookrightarrow V$ be inclusions between open sets. It is easy to check that the natural transformations $\kappa$ defined in the previous subsection give a set of invertible natural transformations $\tilde{\phi}_{VW} : (\rho_V \rho_W)^* \Rightarrow \rho_V^* \circ \rho_V^*$ between the restriction functors acting on the direct limit categories. Moreover, these natural transformations make analogues of diagram (1) commute. (Note that we have chosen to denote these natural transformations by $\tilde{\phi}$ instead of $\kappa$, in keeping with our general tendency to denote sheafified objects with a tilde.)

We have now given $\tilde{C}$ the structure of a presheaf of categories – we have associated categories (direct limits of descent categories) to each open set, and given restriction functors and appropriate natural transformations.

Furthermore, it can be shown that $\tilde{C}$ is a sheaf of categories, not just a presheaf.

Note that if $\mathcal{C}$ is a sheaf of categories, not just a presheaf, then $\tilde{C}$ is equivalent to $\mathcal{C}$ as a stack. In other words, if you sheafify a sheaf of categories, then you recover the original sheaf.

4.1.4 Lifts of Cartesian functors

Suppose $\Phi : \mathcal{C} \to \mathcal{D}$ is a Cartesian functor between presheaves of categories $\mathcal{C}, \mathcal{D}$ on a space $X$. In this section we shall show that $\Phi$ lifts to a Cartesian functor $\tilde{\Phi} : \tilde{\mathcal{C}} \to \mathcal{D}$ between the sheafifications of $\mathcal{C}$ and $\mathcal{D}$. 

35
Let $U$ be an open set, and $\{U_\alpha\}$ an open cover of $U$. We shall first define a functor

$$\Phi(U, \{U_\alpha\}) : \text{Desc}_U(C, \{U_\alpha\}) \longrightarrow \text{Desc}_U(D, \{U_\alpha\})$$

We shall define the functor $\Phi(U, \{U_\alpha\})$ on objects as follows. Let $\{(x_\alpha), (\phi_{\alpha\beta})\} \in \text{Ob} \text{Desc}_U(C, \{U_\alpha\})$. Define the action of $\Phi(U, \{U_\alpha\})$ by,

$$x_\alpha \mapsto \Phi(U_\alpha)(x_\alpha)$$

$$\phi_{\alpha\beta} \mapsto \chi_{\alpha,\alpha\beta}^{-1} \circ \Phi(U_{\alpha\beta})(\phi_{\alpha\beta}) \circ \chi_{\beta,\alpha\beta}$$

$$\in \text{Hom}_{D(U_{\alpha\beta})}(\Phi(U_\beta)(x_\beta)|_{U_{\alpha\beta}}, \Phi(U_\alpha)(x_\alpha)|_{U_{\alpha\beta}})$$

where the $\chi$ are the natural transformations appearing in the definition of $\Phi$ as a Cartesian functor. It is straightforward to check that this action is well-defined, i.e., that the images of the $\phi_{\alpha\beta}$ make the analogue of diagram (3) commute.

Now we shall define the functor $\Phi(U, \{U_\alpha\})$ on morphisms. Given objects

$$\{(x_\alpha), (\phi_{\alpha\beta})\}, \{(y_\alpha), (\phi'_{\alpha\beta})\} \in \text{Desc}_U(C, \{U_\alpha\})$$

and a morphism

$$\{f_\alpha : x_\alpha \rightarrow y_\alpha\} : \{(x_\alpha), (\phi_{\alpha\beta})\} \longrightarrow \{(y_\alpha), (\phi'_{\alpha\beta})\}$$

between these two objects, we define the action of $\Phi(U, \{U_\alpha\})$ on this morphism as,

$$f_\alpha \mapsto \Phi(U_\alpha)(f_\alpha)$$

It is straightforward to check that this map is well-defined, and moreover that

$$\Phi(U, \{U_\alpha\}) : \text{Desc}_U(C, \{U_\alpha\}) \longrightarrow \text{Desc}_U(D, \{U_\alpha\})$$

is a well-defined functor.

In passing, note that if $\Lambda : D \rightarrow E$ is another Cartesian functor between presheaves of categories $D, E$ on $X$, then we have the relation

$$(\Lambda \Phi)(U, \{U_\alpha\}) = \Lambda(U, \{U_\alpha\}) \circ \Phi(U, \{U_\alpha\})$$

Now, let $\{U^2_i\}$ be a refinement of $\{U^1_\alpha\}$, both open covers of the open set $U$, and let $\rho : \{U^2_i\} \rightarrow \{U^1_\alpha\}$ be the set of inclusions. We shall define an invertible natural transformation

$$\Xi_\rho : \rho_D^* \circ \Phi(U, \{U^1_\alpha\}) \Rightarrow \Phi(U, \{U^2_i\}) \circ \rho_C^*$$

We define this natural transformation as follows. To each object

$$\{(x_\alpha), (\phi_{\alpha\beta})\} \in \text{Ob} \text{Desc}_U(C, \{U^1_\alpha\})$$

36
the natural transformation $\Xi_\rho$ associates the morphism
\[
\left\{ \chi_{\alpha(i)},(x,_{\alpha(i)}) : \Phi(U^1_{\alpha(i)})(x_{\alpha(i)})|_{U^2_i} \to \Phi(U^2_i)(x_{\alpha(i)})|_{U^2_i} \right\}
\]
where the $\chi$ are the natural transformations defining $\Phi$ as a Cartesian functor. It is straightforward to check that this is a well-defined morphism, and moreover it defines a natural transformation, in that it commutes with morphisms $\{(x,_{\alpha}), (\phi_{\alpha\beta})\} \to \{(y,_{\alpha}), (\phi_{\alpha\beta})\}$ between objects in $\text{Desc}_{U}(C, \{U^1_a\})$.

If $\rho_1 : \{U^2_1\} \to \{U^1_a\}$ and $\rho_2 : \{U^2_2\} \to \{U^1_a\}$ are two sets of inclusions from refinements of open covers of $U$, then the following diagram (closely analogous to diagram (5)) commutes:
\[
\begin{array}{ccc}
\rho_2^D \circ \rho_1^* \circ \Phi(U, \{U^1_1\}) & \overset{\Xi_{12}}{\to} & \rho_2^D \circ \Phi(U, \{U^2_1\} \circ \rho_1^* \circ \Phi(U, \{U^3_1\}) \circ \rho_1^* \circ \rho_1^* \\
\chi_{12}^a \uparrow & & \uparrow \chi_{12}^a \\
(\rho_1 \rho_2)_D \circ \Phi(U, \{U^1_2\}) & \equiv_{13} & \Phi(U, \{U^3_2\}) \circ (\rho_1 \rho_2)^* 
\end{array}
\]
where the $\lambda$ are the natural transformations defined in the section on descent categories.

We are finally ready to start defining a Cartesian functor $\tilde{\Phi} : \tilde{C} \to \tilde{D}$. Let $U \subseteq X$ be an open set, and define a functor $\tilde{\Phi}(U) : \tilde{C}(U) \to \tilde{D}(U)$ as follows.

Let $P$ be an object of $\tilde{C}(U)$, that is, $P \in \text{Ob} \text{Desc}_{U}(C, \{U^1_a\})$ for some open cover $\{U^1_a\}$ of $U$. Define
\[
\tilde{\Phi}(U)(P) \equiv \Phi(U, \{U^1_1\})(P) \in \text{Ob} \text{Desc}_{U}(D, \{U^1_a\}) \subseteq \text{Ob} \tilde{D}(U)
\]

Let $P_1$ and $P_2$ be objects of $\tilde{C}(U)$, meaning
\[
P_i \in \text{Ob} \text{Desc}_{U}(C, \{U^1_a\})
\]
for open covers $\{U^1_a\}$ of $U$, and $i \in \{1, 2\}$. Let $\beta \in \text{Hom}_{\tilde{C}(U)}(P_1, P_2)$, which means that for some refinement $\{U^3_a\}$ of both $\{U^1_a\}$ and $\{U^2_a\}$,
\[
\beta \in \text{Hom}_{\text{Desc}_{U}(C, \{U^3_a\})}(\rho_{13}^*P_1, \rho_{23}^*P_2)
\]
Then define
\[
\tilde{\Phi}(U)(\beta) \equiv \Xi_{23}^{-1} \circ \Phi(U, \{U^3_1\})(\beta) \circ \Xi_{13}
\]
\[
\in \text{Hom}_{\text{Desc}_{U}(D, \{U^3_a\})}(\rho_{13}^*\Phi(U, \{U^1_a\})(P_1), \rho_{23}^*\Phi(U, \{U^2_a\})(P_2))
\]
With the definitions above, it can be shown that
\[
\tilde{\Phi}(U) : \tilde{C}(U) \to \tilde{D}(U)
\]
is a well-defined functor between the categories $\tilde{C}(U)$ and $\tilde{D}(U)$.

In order to define a Cartesian functor $\tilde{\Phi} : \tilde{C} \to \tilde{D}$, we need to specify more than the functors $\tilde{\Phi}(U)$. Specifically, for any inclusion $\rho : V \hookrightarrow U$, we need to specify an invertible natural transformation

$$\tilde{\chi}_\rho : \rho^*_C \circ \tilde{\Phi}(U) \Longrightarrow \tilde{\Phi}(V) \circ \rho^*_D$$

satisfying certain identities. This natural transformation is defined as follows. To each object

$$(x_\alpha, (\phi_{\alpha\beta})) \in \text{Ob} \text{Desc}_U(C, \{U_\alpha\}) \subseteq \tilde{C}(U)$$

(defined with respect to some open cover $\{U_\alpha\}$ of $U$) the natural transformation $\tilde{\chi}_\rho$ assigns the following morphism:

$$\tilde{\chi}_\rho((x_\alpha, (\phi_{\alpha\beta}))) \equiv \{ \chi_{U_\alpha, U_\alpha \cap V}(x_\alpha) : \Phi(U_\alpha)(x_\alpha)|_{U_\alpha \cap V} \to \Phi(U_\alpha \cap V)(x_\alpha|_{U_\alpha \cap V}) \}$$

where the $\chi$ are the invertible natural transformations defining $\Phi$ as a Cartesian functor.

It is straightforward to check that the definition given above for $\tilde{\chi}_\rho$ does in fact yield a well-defined natural transformation. Moreover, $\tilde{\chi}_\rho$ satisfies the usual pentagonal identity (5). Specifically, if $\rho_1 : V \hookrightarrow U$ and $\rho_2 : W \hookrightarrow V$ are inclusions of open sets, then the following diagram commutes:

$$\rho^*_2 \circ \rho^*_1 \circ \tilde{\Phi}(U) \xrightarrow{\tilde{\chi}_1} \rho^*_2 \circ \tilde{\Phi}(V) \circ \rho^*_1 \circ \tilde{\Phi}(U) \xrightarrow{\tilde{\chi}_2} \tilde{\Phi}(W) \circ \rho^*_2 \circ \rho^*_1 \circ \tilde{\Phi}(U)$$

$$\xrightarrow{\tilde{\phi}_{12}^\rho} \tilde{\Phi}(W) \circ (\rho_1 \rho_2)_C^*$$

where the $\tilde{\phi}$ are the natural transformations defined in the previous subsection.

Thus, we have now defined a Cartesian functor $\tilde{\Phi} : \tilde{C} \to \tilde{D}$. Put another way, given a Cartesian functor $\Phi : C \to D$ between two presheaves of categories $C, D$, we have now constructed a lift of $\Phi$ to a Cartesian functor between the sheafifications $\tilde{C}$ and $\tilde{D}$.

In passing, we shall mention that it is straightforward to check that if $\Lambda : D \to E$ is another Cartesian functor between presheaves of categories $D, E$, then the lift is compatible with composition – in other words,

$$\Lambda \circ \tilde{\Phi} = \tilde{\Lambda} \circ \tilde{\Phi}$$

4.1.5 Lifts of 2-arrows

In this section, we shall demonstrate that 2-arrows can also be lifted to sheafifications. Let $\Phi_1, \Phi_2 : C \to D$ be a pair of Cartesian functors between presheaves of categories $C, D$, and let $\psi : \Phi_1 \Rightarrow \Phi_2$ be a 2-arrow. We shall now define a lift $\tilde{\psi} : \tilde{\Phi}_1 \Rightarrow \tilde{\Phi}_2$ of the 2-arrow $\psi$ to a 2-arrow between the sheafifications of the Cartesian functors.
First, for any open set \( U \) and any open cover \( \{U_\alpha\} \) of \( U \), we shall define a natural transformation
\[
\psi(U, \{U_\alpha\}) : \Phi_1(U, \{U_\alpha\}) \longrightarrow \Phi_2(U, \{U_\alpha\})
\]
between functors from \( \text{Desc}_U(\mathcal{C}, \{U_\alpha\}) \) to \( \text{Desc}_U(\mathcal{D}, \{U_\alpha\}) \), as follows. Let \( \{(x_\alpha), (\phi_{\alpha\beta})\} \) be an object in \( \text{Desc}_U(\mathcal{C}, \{U_\alpha\}) \). Define a morphism
\[
\psi(U, \{U_\alpha\}) \left( \{(x_\alpha), (\phi_{\alpha\beta})\} \right) : \Phi_1(U, \{U_\alpha\}) \left( \{(x_\alpha), (\phi_{\alpha\beta})\} \right) \longrightarrow \Phi_2(U, \{U_\alpha\}) \left( \{(x_\alpha), (\phi_{\alpha\beta})\} \right)
\]
by,
\[
\psi(U, \{U_\alpha\}) \left( \{(x_\alpha), (\phi_{\alpha\beta})\} \right) \equiv \{ \psi(U_\alpha)(x_\alpha) : \Phi_1(U_\alpha)(x_\alpha) \longrightarrow \Phi_2(U_\alpha)(x_\alpha) \}
\]
It is easy to check that this is a well-defined morphism in the category \( \text{Desc}_U(\mathcal{D}, \{U_\alpha\}) \), and moreover that this defines a natural transformation \( \Phi \) by,
\[
\rho^* \circ \Phi_1(U, \{U_\alpha\}) \xrightarrow{\Xi_{\psi}} \Phi_1(U, \{U_\alpha\}) \circ \rho^* \quad \text{and} \quad \rho^* \circ \Phi_2(U, \{U_\alpha\}) \xrightarrow{\Xi_{\psi}} \Phi_2(U, \{U_\alpha\}) \circ \rho^*
\]
where \( \Xi \) is the natural transformation defined earlier, relating \( \Phi(U, \{U_\alpha\}) \) defined with respect to distinct open covers.

Now, we shall define the 2-arrow \( \tilde{\psi} : \tilde{\Phi}_1 \Rightarrow \tilde{\Phi}_2 \), lifting \( \psi \) to the sheafifications. For each open set \( U \), let \( P \in \text{Ob} \tilde{\mathcal{C}}(U) \), that is, \( P \in \text{Ob} \text{Desc}_U(\mathcal{C}, \{U_\alpha\}) \) for some open cover \( \{U_\alpha\} \) of \( U \). Define a morphism
\[
\tilde{\psi}(U)(P) : \tilde{\Phi}_1(U)(P) \longrightarrow \tilde{\Phi}_2(U)(P)
\]
by,
\[
\tilde{\psi}(U)(P) \equiv \psi(U, \{U_\alpha\})(P)
\]
It is straightforward to check that \( \tilde{\psi}(U) \) is a natural transformation for each open \( U \), and moreover satisfies the usual compatibility relation, so \( \tilde{\psi} \) is a 2-arrow.

In passing, we shall mention that it is easy to check that if \( \Phi_3 : \mathcal{C} \rightarrow \mathcal{D} \) is another Cartesian functor between presheaves of categories \( \mathcal{C}, \mathcal{D} \), and if \( \psi_1 : \Phi_1 \Rightarrow \Phi_2 \) and \( \psi_2 : \Phi_2 \Rightarrow \Phi_3 \) are a pair of 2-arrows between the Cartesian functors, then the sheafification is compatible with composition:
\[
(\psi_2 \circ \psi_1) = \tilde{\psi}_2 \circ \tilde{\psi}_1
\]

### 4.2 Pullbacks of stacks

In this section we shall define pullbacks of stacks and some associated technology. We shall begin by defining pullbacks of stacks themselves, then go on to describe pullbacks of Cartesian
functors, analogues of natural transformations between composed pullbacks of stacks, and more.

### 4.2.1 Pullbacks of stacks

Let $f : X \to Y$ be a continuous map, and let $\mathcal{C}$ be a presheaf of categories on $Y$. In this subsection we shall describe how to construct a stack $f^*\mathcal{C}$ on $X$.

We should warn the reader that this section is extremely technical; the reader should probably skip this subsection on a first reading.

Before we describe the construction of pullbacks of stacks, we shall take a moment to review the definition of pullbacks of sheaves of sets, which are closely analogous (and much simpler technically). If $F$ is a sheaf of sets on a space $Y$ and $f : X \to Y$ is a continuous map, then we define the presheaf of sets $f^{-1}F$ to be the direct limit $f^{-1}F(U) = \lim_{\overset{V \supseteq f(U)}{\text{open}}} F(V)$ over open subsets $V \subseteq Y$ containing $f(U)$, for any open set $U$. Restriction maps are defined in a straightforward manner. To recover a sheaf, we sheafify $f^{-1}F$. We shall follow a very closely analogous procedure in defining pullbacks of stacks.

In order to construct the stack $f^*\mathcal{C}$, we shall first construct a presheaf of categories which we shall denote $f^{-1}\mathcal{C}$. Once we have constructed the presheaf $f^{-1}\mathcal{C}$ on $X$, we shall sheafify $f^{-1}\mathcal{C}$ to recover a sheaf of categories we shall denote $f^*\mathcal{C}$. The construction we will give for $f^{-1}\mathcal{C}$ will work for $\mathcal{C}$ a presheaf of categories, not necessarily a stack. Readers following [6, section 5] will note that our usage of the notation $f^*$ and $f^{-1}$ differs slightly from that reference.

In order to define $f^{-1}\mathcal{C}$ for a presheaf of categories $\mathcal{C}$ on $Y$, one first has to describe how to construct categories $f^*\mathcal{C}(U)$ associated to each open set $U \subseteq X$, then how to build restriction functors and invertible natural transformations.

For each open set $U \subseteq X$, the category $f^{-1}\mathcal{C}(U)$ is defined to be the direct limit $f^{-1}\mathcal{C}(U) = \lim_{\overset{V \supseteq f(U)}{\text{open}}} \mathcal{C}(V)$ where the direct limit is over open sets $V \subseteq Y$ such that $f(U) \subseteq V$.

We shall now describe precisely how one defines a direct limit of categories. We shall closely follow the prescription of [6, section 5.2].
Take as objects in \( f^{-1}\mathcal{C}(U) \), the disjoint union of all objects in all the categories \( \mathcal{C}(V) \). In other words,

\[
\text{Ob } \lim_{f(U) \subseteq V} \mathcal{C}(V) = \bigsqcup \text{Ob } \mathcal{C}(V)
\]

It remains to define the set of morphisms in the category \( f^{-1}\mathcal{C}(U) \). Let \( P_1 \in \text{Ob } \mathcal{C}(V_1) \), \( P_2 \in \text{Ob } \mathcal{C}(V_2) \). For any diagram

\[
V_1 \xleftarrow{\rho'_1} W \xrightarrow{\rho'_2} V_2
\]  

(22)

for \( W \) an open set in \( Y \) containing \( f(U) \), define \( S_W \) to be the set of morphisms \( \beta : \rho'_1 P_1 \to \rho'_2 P_2 \) in \( \mathcal{C}(W) \). Now, define an equivalence relation \( \sim \) on the disjoint union \( \coprod_W S_W \) as follows: for any diagram (22) and a similar diagram

\[
V_1 \xleftarrow{\rho'_1} W' \xrightarrow{\rho'_2} V_2
\]

\( (W' \text{ open}, f(U) \subseteq W') \) we say the morphism \( \beta : \rho'_1 P_1 \to \rho'_2 P_2 \) is equivalent to the morphism \( \beta' : \rho'^*_1 P_1 \to \rho'^*_2 P_2 \) if and only if there exists a diagram

\[
W \xleftarrow{\gamma} Z \xrightarrow{\gamma'} W'
\]

\( (Z \text{ open}, f(U) \subseteq Z) \) such that

1. \( \rho_i \gamma = \rho'_i \gamma' \) for \( i \in \{1, 2\} \),

2. there is a commutative diagram

\[
\begin{array}{ccc}
(p_1 \gamma)^* P_1 & \xrightarrow{\varphi_{\gamma,1}(P_1)} & \gamma^* p_1^* P_1 & \xrightarrow{\gamma^* (\beta)} & \gamma^* p_2^* P_2 & \xleftarrow{\varphi_{\gamma,2}(P_2)} & (p_2 \gamma)^* P_2 \\
\varphi_{\gamma,1}(P_1) & \downarrow & \gamma^* p_1^* P_1 & \xrightarrow{\gamma^* (\beta')} & \gamma^* p_2^* P_2 & \xleftarrow{\varphi_{\gamma,2}(P_2)} & (p_2 \gamma)^* P_2 \\
\gamma^* p_1^* P_1 & \xrightarrow{\gamma^* (\beta')} & \gamma^* p_2^* P_2 & \xleftarrow{\varphi_{\gamma,2}(P_2)} & (p_2 \gamma)^* P_2 \\
\end{array}
\]

(23)

where the \( \varphi \) are the natural transformations appearing in the definition of \( \mathcal{C} \) as a presheaf of categories.

The set of morphisms \( \text{Hom}(P_1, P_2) \) in the category \( f^{-1}\mathcal{C}(U) \) is defined to be the disjoint union \( \coprod_W S_W \), modulo the equivalence relation \( \sim \) described above.

So far we have defined objects and morphisms in the direct limit category \( f^{-1}\mathcal{C}(U) \). We shall now take a moment to describe composition of morphisms, as the correct definition might not be obvious to the reader. Let \( P_1, P_2, P_3 \in \text{Ob } (f^{-1}\mathcal{C})(U) \), i.e., \( P_i \in \text{Ob } \mathcal{C}(V_i) \) for some open sets \( V_i \supseteq f(U) \). Let \( \beta \in \text{Hom}_{f^{-1}\mathcal{C}(U)}(P_1, P_2) \), and \( \alpha \in \text{Hom}_{f^{-1}\mathcal{C}(U)}(P_2, P_3) \). In other words, there exists open sets \( V_{\alpha}, V_{\beta} \), such that \( f(U) \subseteq V_{\beta} \subseteq V_1 \cap V_2 \) and \( f(U) \subseteq V_{\alpha} \subseteq V_2 \cap V_3 \), with

\[
\begin{align*}
\beta & \in \text{Hom}_{\mathcal{C}(V_{\beta})}(P_1|_{V_{\beta}}, P_2|_{V_{\beta}}) \\
\alpha & \in \text{Hom}_{\mathcal{C}(V_{\alpha})}(P_2|_{V_{\alpha}}, P_3|_{V_{\alpha}})
\end{align*}
\]

41
We define $\alpha \circ \beta$ as follows. Let $V$ be an open set such that $f(U) \subseteq V \subseteq V_\alpha \cap V_\beta$. Then, define
\[
\alpha \circ \beta \equiv \varphi_{V_\alpha,V}^{-1} \circ \alpha|_V \circ \varphi_{V,V_\beta} \circ \varphi_{V_\beta,V}^{-1} \circ \beta|_V \circ \varphi_{V,V_\beta}
\in \text{Hom}_{\mathcal{C}(V)}(P_1|_V, P_3|_V)
\]

It can be shown that this definition is well-defined, i.e., if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha \circ \beta \sim \alpha' \circ \beta'$.

So far we have described the categories $f^{-1}\mathcal{C}(U)$ associated to any open set $U \subseteq X$. In order to define $f^{-1}\mathcal{C}$ as a presheaf of categories, we still need to define restriction functors and appropriate natural transformations.

Before we describe restriction functors, however, we shall take a moment to reflect on the meaning of the definition of the categories $f^{-1}\mathcal{C}(U)$ given above. Consider, for example, the special case of the identity map $\text{Id} : X \to X$. It can be shown that $\text{Id}^{-1}\mathcal{C}(U)$ is equivalent (as a category) to $\mathcal{C}(U)$. Naively, the reader might find this result quite surprising – the direct limit defining $\text{Id}^{-1}\mathcal{C}(U)$ contains more objects than $\mathcal{C}(U)$. However, although it contains more objects, there are also more isomorphisms, and in fact the number of isomorphism classes of objects in both categories is the same. For example, if $P_1, P_2 \in \text{Ob } \text{Id}^{-1}\mathcal{C}(U)$ are two objects, then it can be shown that $P_1$ is isomorphic to $P_2$ if and only if $P_1|_U$ is isomorphic to $P_2|_U$.

We shall not work through the details of proving that these are equivalent categories, though we shall take a moment to outline the general idea. Define a functor $F : \mathcal{C}(U) \to \text{Id}^{-1}\mathcal{C}(U)$ by, $F$ maps an object $P \mapsto P$, and $F$ maps a morphism $\beta \mapsto \beta$. Define a functor $G : \text{Id}^*\mathcal{C}(U) \to \mathcal{C}(U)$ as follows. $G$ is defined to map an object $P \mapsto P|_U$. Suppose $P_1 \in \text{Ob } \mathcal{C}(V_1)$ and $P_2 \in \text{Ob } \mathcal{C}(V_2)$ for open sets $V_1, V_2$ such that $U \subseteq V_1 \cap V_2$ – in other words, let $P_1$ and $P_2$ be objects of the category $\text{Id}^{-1}\mathcal{C}(U)$. Let $\beta \in \text{Hom}_{\mathcal{C}(V)}(P_1|_V, P_2|_V)$ for some open set $V, U \subseteq V \subseteq V_1 \cap V_2$, i.e., $\beta \in \text{Hom}_{\text{Id}^{-1}\mathcal{C}(U)}(P_1, P_2)$. Then define $G(\beta) = \varphi_{V,V}^{-1} \circ \beta|_U \circ \varphi_{V,V}$. It can be shown that $G$ is a well-defined functor (meaning, for example, that $\beta \sim \beta'$ implies $G(\beta) = G(\beta')$), and that $F$ and $G$ are inverses to one another, in the sense that there exist invertible natural transformations $F \circ G \Rightarrow \text{Id}_{\text{Id}^{-1}\mathcal{C}(U)}$ and $G \circ F \Rightarrow \text{Id}_{\mathcal{C}(U)}$.

In fact, more generally it can be shown that if $f : X \to Y$ is any open map (meaning, the image of any open set is open), then for all open $U \subseteq X$, the category $f^{-1}\mathcal{C}(U)$ is equivalent to the category $\mathcal{C}(f(U))$.

Now that we have given some intuition for the meaning of the direct limits used above, we shall describe the restriction functors and natural transformations needed to describe $f^{-1}\mathcal{C}$ as a presheaf of categories.

Given the categories $f^{-1}\mathcal{C}(U)$, we shall now define the pullback functors $\rho^* : f^{-1}\mathcal{C}(U_1) \to f^{-1}\mathcal{C}(U_2)$ for $\rho : U_2 \hookrightarrow U_1$. In fact, these restriction functors are straightforward to define.
Note that \( \rho \) induces a map
\[
\{ V \mid f(U_1) \subseteq V \} \longrightarrow \{ V \mid f(U_2) \subseteq V \}
\]
given by \( V \mapsto V \) (any open set containing \( f(U_1) \), also contains \( f(U_2) \)). We define \( \rho^* \) to act on objects \( P \in f^{-1}\mathcal{C}(U_1) \) as, \( P \mapsto P \), and on morphisms \( \beta \) as, \( \beta \mapsto \beta \). It should be clear that this map yields a well-defined functor \( f^{-1}\mathcal{C}(U_1) \rightarrow f^{-1}\mathcal{C}(U_2) \).

It should be clear that the invertible natural transformations \((\rho_1\rho_2)^* \Rightarrow \rho_2^* \circ \rho_1^*\) needed to define a presheaf of categories are trivial. In other words, if \( P \in \text{Ob} \ f^{-1}\mathcal{C}(U) \), meaning \( P \in \text{Ob} \mathcal{C}(V) \) for some open \( V \supseteq f(U) \), then both the functors \((\rho_1\rho_2)^* \) and \( \rho_2^* \circ \rho_1^* \), for any pair of composable inclusions \( \rho_1, \rho_2 \), map \( P \mapsto P \), and so the morphism that the requisite natural transformation should assign to \( P \) is given by the identity morphism.

So far we have defined a presheaf of categories \( f^{-1}\mathcal{C} \). Even if \( \mathcal{C} \) is a sheaf, the presheaf \( f^{-1}\mathcal{C} \) is, in general, not itself a sheaf. In order to get a sheaf from the presheaf \( f^{-1}\mathcal{C} \), we must sheafify the presheaf. We shall denote the result of the sheafification by \( f^*\mathcal{C} \).

As an illuminating example, consider \( \text{Id}^*\mathcal{C} \), where \( \text{Id} : X \rightarrow X \) is the identity map and \( \mathcal{C} \) is a stack. We pointed out earlier that each category \( \text{Id}^*\mathcal{C}(U) \) is equivalent to the category \( \mathcal{C}(U) \), for any open set \( U \). It should now be clear that \( \text{Id}^*\mathcal{C} \) is equivalent to \( \mathcal{C} \) as a stack.

If \( f \) is a homeomorphism, for example, and \( \mathcal{C} \) is a stack, then \( f^{-1}\mathcal{C} \) is already a stack, not just a presheaf of categories. Stronger statements can be made, but this is all we need for later use.

### 4.2.2 Pullbacks of Cartesian functors

Let \( f : X \rightarrow Y \) be a continuous map, and let \( \Phi : \mathcal{C} \rightarrow \mathcal{D} \) be a Cartesian functor between presheaves of categories \( \mathcal{C}, \mathcal{D} \) on \( Y \). We shall construct a Cartesian functor
\[
f^*\Phi : f^*\mathcal{C} \longrightarrow f^*\mathcal{D}
\]
by first constructing a Cartesian functor
\[
f^{-1}\Phi : f^{-1}\mathcal{C} \longrightarrow f^{-1}\mathcal{D}
\]
between the presheaf pullbacks \( f^{-1}\mathcal{C}, f^{-1}\mathcal{D} \) on \( X \), and using the fact that a Cartesian functor between presheaves of categories lifts to a Cartesian functor between the sheafifications.

We shall first define functors
\[
f^{-1}\Phi(U) : f^{-1}\mathcal{C}(U) \rightarrow f^{-1}\mathcal{D}(U)
\]
\footnote{For example, if \( \beta \sim \beta' \), then it should be clear that \( \rho^*(\beta) \sim \rho^*(\beta') \).}
for $U \subseteq X$ an open set.

We define $f^{-1}\Phi(U)$ on objects as follows. Recall

$$\text{Ob } f^{-1}\mathcal{C}(U) = \coprod_{f(U) \subseteq V} \text{Ob } \mathcal{C}(V)$$

so $P \in \text{Ob } f^{-1}\mathcal{C}(U)$ means, $P \in \text{Ob } \mathcal{C}(V)$ for some open $V \supseteq f(U)$. Define the action of $f^{-1}\Phi(U)$ on $P$ as,

$$P \mapsto \Phi(V)(P) \in \text{Ob } \mathcal{D}(V) \subseteq \text{Ob } f^{-1}\mathcal{D}(U)$$

Now we shall define $f^{-1}\Phi(U)$ on morphisms. Let $P_1, P_2 \in \text{Ob } f^{-1}\mathcal{C}(U)$, meaning, $P_i \in \text{Ob } \mathcal{C}(V_i)$ for open $V_i \supseteq f(U_i)$ and $i \in \{1, 2\}$. Let $\beta \in \text{Hom}_{f^{-1}\mathcal{C}(U)}(P_1, P_2)$. In other words, there exists open $V_\beta$, $f(U) \subseteq V_\beta \subseteq V_1 \cap V_2$, such that $\beta \in \text{Hom}_{\mathcal{C}(V_\beta)}(P_1|_{V_\beta}, P_2|_{V_\beta})$. Define $f^{-1}\Phi(U)$ on $\beta$ as,

$$\beta \mapsto \chi_{V_\beta}^{-1} \circ \Phi(V_\beta)(\beta) \circ \chi_{V_\beta} \circ (P_1)$$

where $\chi$ is the invertible natural transformation appearing in the definition of $\Phi : \mathcal{C} \rightarrow \mathcal{D}$ as a Cartesian functor.

It is straightforward to check that $f^{-1}\Phi(U)$, as given above, yields a well-defined functor $f^{-1}\mathcal{C}(U) \rightarrow f^{-1}\mathcal{D}(U)$.

In order to create a Cartesian functor $f^{-1}\Phi$, it remains to specify invertible natural transformations

$$f^{-1}\chi_\rho : \rho_{f^{-1}\mathcal{D}} \circ f^{-1}\Phi(U_1) \Longrightarrow f^{-1}\Phi(U_2) \circ \rho_{f^{-1}\mathcal{C}}$$

for inclusions $\rho : U_2 \hookrightarrow U_1$.

In fact, the requisite natural transformations are trivial. Let $P \in \text{Ob } f^{-1}\mathcal{C}(U_1)$, meaning $P \in \text{Ob } \mathcal{C}(V)$ for some open $V \supseteq f(U_1)$. First, note $P|_{U_2}$ is given by the same $P \in \text{Ob } \mathcal{C}(V)$, from the definition of restriction functor for pullbacks. Moreover,

$$f^{-1}\Phi(U_1)(P)|_{U_2} = f^{-1}\Phi(U_2)(P|_{U_2}) = \Phi(V)(P) \in \text{Ob } \mathcal{D}(V) \subseteq \text{Ob } f^{-1}\mathcal{D}(U_2)$$

Clearly, the morphism that the natural transformation $f^{-1}\chi_\rho$ assigns to the object $P$ is the identity morphism on $\Phi(V)(P)$. This yields a well-defined natural transformation, satisfying the usual pentagonal identity.

Thus, we have defined a Cartesian functor $f^{-1}\Phi : f^{-1}\mathcal{C} \rightarrow f^{-1}\mathcal{D}$.

Now that we have defined a Cartesian functor between the presheaves $f^{-1}\mathcal{C}$ and $f^{-1}\mathcal{D}$, we can use the fact that Cartesian functors between presheaves lift to Cartesian functors between sheafifications to immediately recover a Cartesian functor $f^*\Phi$:

$$f^*\Phi : f^*\mathcal{C} \rightarrow f^*\mathcal{D}$$
In passing, we shall mention that if we have another Cartesian functor \( \Lambda : D \to E \) between presheaves of categories on \( X \), then it is straightforward to check that pullbacks of Cartesian functors are compatible with composition of Cartesian functors. In other words,

\[
f^{-1}(\Lambda \circ \Phi) = f^{-1}\Lambda \circ f^{-1}\Phi
\]

and so, using an analogous result for sheafifications, we find

\[
f^*(\Lambda \circ \Phi) = f^*\Lambda \circ f^*\Phi
\]

### 4.2.3 Pullbacks of 2-arrows

Let \( f : X \to Y \) be a continuous map, and let \( \Phi_1, \Phi_2 : C \to D \) be Cartesian functors between presheaves of categories \( C, D \). Let \( \psi : \Phi_1 \Rightarrow \Phi_2 \) be a 2-arrow between the Cartesian functors. In this section we shall construct a 2-arrow

\[
f^*\psi : f^*\Phi_1 \Rightarrow f^*\Phi_2
\]

by first constructing a 2-arrow

\[
f^{-1}\psi : f^{-1}\Phi_1 \Rightarrow f^{-1}\Phi_2
\]

between the pullbacks \( f^{-1}\Phi_1, f^{-1}\Phi_2 \), and using the fact that a 2-arrow lifts to a 2-arrow between sheafifications.

We shall construct \( f^{-1}\psi \) as follows. Let \( U \subseteq X \) be an open set, and let \( P \in \text{Ob} f^{-1}C(U) \), that is, \( P \in \text{Ob} C(V) \) for some open \( V \supseteq f(U) \). Define a morphism

\[
(f^{-1}\psi)(U)(P) : (f^{-1}\Phi_1)(U)(P) \to (f^{-1}\Phi_2)(U)(P)
\]

by,

\[
(f^{-1}\psi)(U)(P) \equiv \psi(V)(P) : \Phi_1(V)(P) \to \Phi_2(V)(P)
\]

It is easy to check that this defines a natural transformation

\[
(f^{-1}\psi)(U) : (f^{-1}\Phi_1)(U) \Rightarrow (f^{-1}\Phi_2)(U)
\]

and moreover, for any inclusion \( \rho : U_2 \hookrightarrow U_1 \) of open subsets of \( X \), these natural transformations are compatible with the natural transformations \( f^{-1}\chi_\rho \) defining \( f^{-1}\Phi_1 \) and \( f^{-1}\Phi_2 \) as Cartesian functors.

Thus, we have defined a 2-arrow \( f^{-1}\psi : f^{-1}\Phi_1 \Rightarrow f^{-1}\Phi_2 \).

Now that we have defined a 2-arrow between the Cartesian functors \( f^{-1}\Phi_1 \) and \( f^{-1}\Phi_2 \), we can use the fact that 2-arrows lift to 2-arrows between sheafifications to immediately recover a 2-arrow \( f^*\psi \):

\[
f^*\psi : f^*\Phi_1 \Rightarrow f^*\Phi_2
\]
In passing, we shall mention that if we have another Cartesian functor \( \Phi : \mathcal{C} \to \mathcal{D} \) and a pair of 2-arrows \( \psi_1 : \Phi_1 \Rightarrow \Phi_2, \psi_2 : \Phi_2 \Rightarrow \Phi_3 \), then pullback is compatible with composition. In other words,

\[
f^{-1}(\psi_2 \circ \psi_1) = (f^{-1}\psi_2) \circ (f^{-1}\psi_1)
\]

By using the analogous result for sheafifications, we find

\[
f^*(\psi_2 \circ \psi_1) = (f^*\psi_2) \circ (f^*\psi_1)
\]

### 4.2.4 Analogues of natural transformations

In this section we shall define an analogue of natural transformation for compositions of pullbacks. More precisely, let \( f : X \to Y \) and \( g : Y \to Z \) be continuous maps between topological spaces. For any stack \( \mathcal{C} \) on \( Z \), we shall define a Cartesian functor

\[
\Psi_{gf}^\mathcal{C} : (gf)^*\mathcal{C} \to f^*g^*\mathcal{C}
\]

with the properties

1. For any Cartesian functor \( \Phi : \mathcal{C} \to \mathcal{D} \) between stacks on \( Z \), the following diagram commutes:

\[
\begin{array}{ccc}
(gf)^*\mathcal{C} & \overset{(gf)^*\Phi}{\longrightarrow} & (gf)^*\mathcal{D} \\
\Psi_{gf} & \downarrow & \Psi_g^D \\
f^*g^*\mathcal{C} & \overset{f^*g^*\Phi}{\longrightarrow} & f^*g^*\mathcal{D}
\end{array}
\]

(24)

2. If \( f : X \to Y, g : Y \to Z, h : Z \to W \) are three continuous maps, then for any stack \( \mathcal{C} \) on \( W \) we have the commuting diagram

\[
\begin{array}{ccc}
(hgf)^*\mathcal{C} & \overset{\Psi_{hgf}}{\longrightarrow} & (f^* \circ (hg)^*)\mathcal{C} \\
\Psi_{h,g,f} & \downarrow & \Psi_{h,g} \\
((gf)^* \circ h^*)\mathcal{C} & \overset{(f^* \circ g^* \circ h^*)\mathcal{C}}{\longrightarrow}
\end{array}
\]

(25)

which the reader should immediately recognize as being analogous to diagram (1).

Moreover, the Cartesian functor \( \Psi_{gf} : (gf)^*\mathcal{C} \to f^*g^*\mathcal{C} \) will be invertible, in the sense that there exists a Cartesian functor \( \Psi^{-1}_{gf} : f^*g^*\mathcal{C} \to (gf)^*\mathcal{C} \) and invertible 2-arrows \( \Psi \circ \Psi^{-1} \Rightarrow \text{Id} \) and \( \Psi^{-1} \circ \Psi \Rightarrow \text{Id} \).

In order to define \( \Psi \), we shall work at the level of presheaves of categories and define a Cartesian functor

\[
\Psi_{gf} : (gf)^{-1}\mathcal{C} \to f^{-1}g^{-1}\mathcal{C}
\]
Furthermore, we shall implicitly assume that the map \( f \) is open (i.e., images of open sets are open), and that \( g \) is such that \( g^{-1}C \) is a stack, not just a presheaf of categories. These restrictions could almost certainly be weakened; however, in this paper we shall only be interested in cases in which both \( f \) and \( g \) are homeomorphisms, so we shall not investigate these conditions further.

To define the Cartesian functor \( \Psi_{gf} : (gf)^{-1}C \to f^{-1}g^{-1}C \), we shall first define functors

\[
\Psi_{gf}(U) : (gf)^{-1}C(U) \to f^{-1}g^{-1}C(U)
\]

for open sets \( U \subseteq X \).

We define the functor \( \Psi_{gf}(U) \) on objects as follows. Let \( P \) be an object in \( (gf)^{-1}C(U) \), which is to say, \( P \in \text{Ob} \ C(V) \) for some open \( V \supseteq (gf)(U) \). The functor \( \Psi_{gf}(U) \) acts as,

\[
P \mapsto \Psi_{gf}(U)(P) \equiv P \in \bigoplus_{V \supseteq (gf)(U)} \text{Ob} \ C(V) \subseteq \bigoplus_{V' \supseteq f(U)} \bigoplus_{V'' \supseteq g(U')} \text{Ob} \ C(V) = \text{Ob} f^{-1}g^{-1}C(U)
\]

We define the functor \( \Psi_{gf}(U) \) on morphisms as follows. Let \( P_1, P_2 \) be objects in \( (gf)^{-1}C(U) \), which is to say, \( P_i \in \text{Ob} \ C(V_i) \) for open \( V_i \supseteq (gf)(U) \), \( i \in \{1, 2\} \). Let \( \beta \in \text{Hom}_{(gf)^{-1}C(U)}(P_1, P_2) \), which means that for some open \( V_{\beta} \), \( (gf)(U) \subseteq V_{\beta} \subseteq V_1 \cap V_2 \), \( \beta \in \text{Hom}_{C(V_{\beta})}(P_1|_{V_{\beta}}, P_2|_{V_{\beta}}) \). The functor \( \Psi_{gf}(U) \) acts on \( \beta \) as,

\[
\beta \mapsto \Psi_{gf}(U)(\beta) \equiv \beta
\]

In terms of equivalence classes, denoted by brackets [ ], the functor \( \Psi_{gf}(U) \) maps \([\beta]_{gf}\) to \([\beta]_{gf}\).

In order to define a Cartesian functor \( \Psi_{gf} : (gf)^{-1}C \to f^{-1}g^{-1}C \), we must specify an invertible natural transformation

\[
\chi_{gf} : \rho_{f^{-1}g^{-1}}^* \circ \Psi_{gf}(U) \Rightarrow \Psi_{gf}(V) \circ \rho_{(gf)^{-1}}^*
\]

for every inclusion \( \rho : V \hookrightarrow U \). We define this natural transformation to be the trivial one. In other words, given an object \( P \) in \( (gf)^{-1}C(U) \), which is to say, \( P \in \text{Ob} \ C(V) \) for some open \( V \supseteq (gf)(U) \), we define \( \chi_{gf}(P) = \text{Id}_{C(V)}(P) \). This assignment of morphisms to objects clearly defines a natural transformation, and moreover it is easy to see that \( \chi \) satisfies the pentagonal identity for natural transformations in Cartesian functors.

Thus, we have now defined a Cartesian functor \( \Psi_{gf} : (gf)^{-1}C \to f^{-1}g^{-1}C \). It is straightforward to check that this definition satisfies the two properties listed at the beginning of this section, and moreover that this Cartesian functor admits an inverse Cartesian functor.
Ψ_{gf}^{-1}$, such that the composite functors $\Psi \circ \Psi^{-1}$ and $\Psi^{-1} \circ \Psi$ can be identified with identity functors, up to invertible 2-arrows.

Finally, we can lift this Cartesian functor to a Cartesian functor between sheafifications

$$\Psi_{gf} : (gf)^*\mathcal{C} \rightarrow f^*g^*\mathcal{C}$$

satisfying the properties listed at the beginning of this section.

### 4.3 Stalks of stacks

For a presheaf of sets $\mathcal{F}$ on a space $X$, recall that one can define a stalk of the sheaf $\mathcal{F}$ at the point $x \in X$ to be the direct limit

$$\mathcal{F}_x \equiv \lim_{U \ni x} \mathcal{F}(U)$$

over open sets $U \subseteq X$ containing the point $x \in X$.

We can perform the analogous construction for presheaves of categories. Define the stalk of a presheaf of categories $\mathcal{C}$ at a point $x \in X$ to be the direct limit

$$\mathcal{C}_x \equiv \lim_{U \ni x} \mathcal{C}(U)$$

over open sets $U \subseteq X$ containing the point $x \in X$. This direct limit is defined in precise analogy with the direct limits defined in the previous two sections. Note that just as the stalk of a presheaf of sets is a set, the stalk of a presheaf of categories is a category.

This notion of stalk may give the reader some degree of intuition for stacks. Furthermore, one ought to be able to work out many stack-variants of other concepts from ordinary sheaves. As we shall not use stalks of presheaves of categories in this paper, or other stack-theoretic versions of other sheaf theory concepts, we shall not speak about such matters any further.

### 5 Technical notes on gerbes

In this section we shall make some highly technical remarks on gerbes. In particular, after discussing pullbacks of gerbes and defining torsors, we shall speak in detail about gauge transformations of gerbes, and use such ideas to derive some basic facts about gerbes which were mentioned earlier in this paper. Many of the basic ideas are taken from [section 5].

Readers studying this paper for the first time are urged to skip this section.
5.1 Pullbacks of gerbes

Let $f : X \to Y$ be a continuous map, and $\mathcal{C}$ a gerbe on $Y$ with band $\mathcal{A}$. It is easy to check that $f^*\mathcal{C}$ is a gerbe on $X$, with band $f^*\mathcal{A}$ \cite[prop. 5.2.6]{ref}. Moreover, the element of the sheaf cohomology group $H^2(X, f^*\mathcal{A})$ characterizing $f^*\mathcal{C}$ is precisely the pullback of the element of the sheaf cohomology group $H^2(Y, \mathcal{A})$ characterizing $\mathcal{C}$ \cite[section 5.2]{ref}, as the reader might have guessed.

If $\mathcal{C}$ has a connective structure, then (at least for $f$ a diffeomorphism) one naturally obtains a connective structure on $f^*\mathcal{C}$. Let $\text{Co}$ denote the connective structure on $\mathcal{C}$, meaning that $\text{Co}$ is a Cartesian functor

\[ \text{Co} : \mathcal{C} \to \text{Tors}_Y(\Omega^1) \]

which is compatible with the bands of either gerbe. This Cartesian functor can be lifted to a Cartesian functor

\[ f^*\text{Co} : f^*\mathcal{C} \to f^*\text{Tors}_Y(\Omega^1) \]

and at least in the special case that $f$ is a diffeomorphism (the only case we shall need), $f^*\text{Tors}_Y(\Omega^1) \cong \text{Tors}_X(\Omega^1)$. Thus, at least for $f$ a diffeomorphism we find that a connective structure on $\mathcal{C}$ naturally defines a connective structure on $f^*\mathcal{C}$.

Let $K$ denote a curving on $(\mathcal{C}, \text{Co})$. At least in the case that $f$ is a diffeomorphism, we can pullback $K$ to a curving $f^*K$ on $(f^*\mathcal{C}, f^*\text{Co})$. We shall outline the details here. Since $f$ is assumed to be a diffeomorphism, we know that $f^*\mathcal{C} \cong f^{-1}\mathcal{C}$, and so it suffices to define $f^*K$ at the level of $f^{-1}\mathcal{C}$ and $f^{-1}\text{Co}$. Let $U \subseteq X$ be open, and let $P \in \text{Ob} (f^{-1}\mathcal{C})(U)$, i.e., $P \in \text{Ob} \mathcal{C}(V)$ for some open $V \supseteq f(U)$. Recall that by definition,

\[ (f^{-1}\text{Co})(U)(P) = \text{Co}(V)(P) \]

Let $\nabla$ be a section of $(f^{-1}\text{Co})(U)(P)$, which is to say, a section of $\text{Co}(V)(P)$. We are now finally ready to define $f^*K$. Define

\[ (f^*K)(\nabla) \equiv (f|_U)^* [K(\nabla)] \]

It is straightforward to check that this definition satisfies the defining axioms for a curving.

To summarize, there exist natural notions of pullback for both gerbes and connections on gerbes.

5.2 Torsors

Most mathematical descriptions of gerbes rely heavily on torsors. For the most part, we have strenuously avoided speaking of torsors in this text, but at times their use is unavoidable.
In this section we shall define torsors. Our discussion will largely follow [6, section 5.1] and [23].

A torsor with respect to a group $G$ is a set with an action of $G$ that is free and transitive. An example of a $G$-torsor for a topological group $G$ is the group $G$ itself (though when described as a torsor, one implicitly drops the group structure). Let $C^\infty(G)$ denote the group of smooth maps from a manifold $X$ into a Lie group $G$, then an example of a $C^\infty(G)$-torsor is any principal $G$-bundle. (Some authors abuse notation and refer to a principal $G$-bundle as a $G$-torsor, rather than a $C^\infty(G)$-torsor; we shall specifically avoid such mangled usage.) Let $\Omega^1(X)$ denote the group of smooth maps from a manifold $X$ into a Lie group $G$, then an example of a $C^\infty(G)$-torsor is any principal $G$-bundle. (Any two such connections differ by a 1-form, which can be seen as follows: Let $A_\mu^\alpha$ and $A'_\mu^\alpha$ denote two locally-defined connections on the bundle with respect to a good open cover $\{U_\alpha\}$. Then $A_\alpha^\alpha - A_\beta^\beta = A'_\alpha^\alpha - A'_\beta^\beta = d\ln g_{\alpha\beta}$ on overlaps, so in particular $A_\alpha^\alpha - A'_\alpha^\alpha$ is a globally-defined 1-form.)

The torsors that we use in this paper are torsors with respect to a sheaf of abelian groups, not just a group. A torsor with respect to a sheaf of groups (say, $A$) is a sheaf of sets, say, $F$, such that the set $F(U)$ associated to any open set $U$ is a torsor with respect to the group $A(U)$, and such that the action of $A$ commutes with restriction maps.

A morphism of $A$-torsors $\phi : F \to G$ is a morphism of sheaves of sets, such that the action of $A$ commutes with $\phi$.

An example of a torsor with respect to the sheaf $A = C^\infty(G)$ of $C^\infty$ maps into $G$ is the sheaf of local sections of a smooth principal $G$-bundle.

The set of isomorphism classes of $A$-torsors has an (abelian) group structure. Let $F$, $G$ be a pair of $A$-torsors. The product of sheaves $F \times G$ is an $(A \times A)$-torsor. We define the $A$-torsor $F \cdot G$ (the result of the group operation) to be the sheaf associated to the presheaf

$$
\frac{(F \times G) \times A}{A \times A}
$$

where $A \times A$ acts on $A$ via the product map $A \times A \to A$.

Given any $A$-torsor $F$, there is a natural definition of $F^{-1}$. Specifically, the $A$-torsor $F^{-1}$ is defined by the property that for any open $U \subseteq X$, the set $F^{-1}(U)$ is the set of torsor isomorphisms $F(U) \xrightarrow{\sim} A(U)$.

It can be shown that the group of isomorphism classes of $A$-torsors, over a space $X$, is in natural bijection with the group $H^1(X, A)$.

\[\text{Note our notation is slightly ambiguous: we use } C^\infty(G) \text{ to denote both the group of smooth maps into } G, \text{ and the sheaf of smooth local maps into } G. \text{ The correct interpretation should be clear from context.}\]
It should be clear that any morphism of $\mathcal{A}$-torsors, over the same space, is necessarily an isomorphism. This is a generalization of a similar result for principal $G$-bundles \cite[section 4.3]{17}, namely that any morphism of principal $G$-bundles for fixed $G$ over the same base space is necessarily an isomorphism.

We should also mention a technical lemma which we shall use in what follows. Let $I, K$ be a pair of $\mathcal{A}$-torsors over a space $X$, for some sheaf of abelian groups $\mathcal{A}$. We shall show that in order to define an isomorphism $I \to K$ of $\mathcal{A}$-torsors $I, K$, it suffices to show how a set of local sections $\{s_\alpha\}$ of $I$, defined with respect to an open cover $\{U_\alpha\}$ of $X$, are mapped. To define the isomorphism for other elements of the sets $I(U_\alpha), K(U_\alpha)$, use the action of the group $\mathcal{A}(U_\alpha)$ — in other words, any other element of $I(U_\alpha)$ will differ from $s_\alpha$ by an element of the group $\mathcal{A}(U_\alpha)$, so we can define its image to be the image of $s_\alpha$ modulo the same group element, thus explicitly recovering an isomorphism of sets which, by construction, commutes with the action of $\mathcal{A}(U_\alpha)$. We can construct maps $I(W) \to K(W)$ for $W \subseteq U_\alpha$ for some $U_\alpha$ by using restriction in the obvious way, and we can construct maps for $W \supseteq U_\alpha$ by performing all possible gluings of elements of $\{I(U_\alpha)\}$.

More information on torsors can be found in, for example, \cite[section VIII.2]{23}, or \cite[section 5.1]{24}.

5.3 Gauge transformations of gerbes

Just as maps $X \to G$ define gauge transformations of principal $G$-bundles on a space $X$, we will see explicitly in this section that (equivalence classes of) bundles define gauge transformations of gerbes of band $C^\infty(G)$.

To fully explain these ideas will take some time. Much of the material we present in subsections 5.3.1 and 5.3.3 is taken from \cite[section 5.2]{3}.

5.3.1 Gauge transformations of objects

Let $\mathcal{C}$ denote a gerbe on a manifold $X$, with band $\mathcal{A}$. We shall assume $\mathcal{A} = C^\infty(G)$ for some abelian Lie group $G$, for simplicity. Let $U \subseteq X$ be an open subset of a space $X$, and let $I$ be a principal $G$-bundle on $U$. Strictly speaking, we should take $I$ to be a $\mathcal{A}|_U$-torsor, not a principal $G$-bundle; however, where possible, we are trying to avoid the language of torsors.

Given any object $P \in \text{Ob } \mathcal{C}(U)$, we shall show how to use $I$ to construct another object we shall denote $P \times I \in \text{Ob } \mathcal{C}(U)$. Let $\{U_\alpha\}$ be a good open cover of $U$, so that $I(U_\alpha) \neq \emptyset$ for all $\alpha$. Let $\{s_\alpha \in I(U_\alpha)\}$ be a set of local sections of $I$, over the open subsets $U_\alpha$. Define $g_{\alpha\beta}$ to be the (unique) element of $\mathcal{A}(U_{\alpha\beta})$ that transports $s_\beta|_{U_{\alpha\beta}}$ to $s_\alpha|_{U_{\alpha\beta}}$. (Recall that strictly speaking, we should interpret $I$ as a torsor, so that there is no natural group law on the $\{s_\alpha\}$
per se.)

We can now define the object \( P \times I \), using the gluing law for objects, as follows. Define a set of isomorphisms

\[
\phi_{\alpha\beta} : P|_{U_\beta}|_{U_{\alpha\beta}} \longrightarrow P|_{U_\alpha}|_{U_{\alpha\beta}}
\]

by,

\[
\phi_{\alpha\beta} \equiv \varphi_{\alpha\beta} \circ g_{\alpha\beta} \circ \varphi_{\beta\alpha\beta}^{-1}
\]

It is straightforward to check that these satisfy the axioms for the gluing law for objects, and so from said gluing law we recover a new object in \( \mathcal{C}(U) \) which we shall denote \( P \times I \).

In passing, we should mention a minor technical problem with the description above, namely that the object \( P \times I \) is almost, but not quite, uniquely specified by a set of local sections of \( I \). Recall that the gluing law for objects yields objects that are uniquely only up to unique isomorphism commuting with the gluing maps \( \psi_\alpha \). Thus, the object \( P \times I \) is not uniquely defined – however, there exists a unique isomorphism (commuting with the gluing maps \( \psi_\alpha \)) between any two objects that one might label “\( P \times I \).” Thus, in order to uniquely specify an action of a torsor \( I \) together with a set of local sections of \( I \) on objects, we must choose specific examples of \( P \times I \) for each object \( P \). In the next subsection we shall derive a functor describing the action of the torsor \( I \), and it is straightforward to check that for any two such functors differing only in the choices made of objects “\( P \times I \),” the isomorphisms between the choices define a 2-arrow between functors. We shall not speak further about this issue, except when absolutely necessary.

In defining the action of a torsor \( I \) on objects \( P \), we referred to a specific choice of a set of local sections of \( I \). What happens if we choose a distinct set of local sections of \( I \)? The answer is that any two sets of local sections of \( I \) will define isomorphic objects \( P \times I \). We shall outline how this is proven in the special case that the two sets of local sections in question are defined with respect to the same open cover \( \{U_\alpha\} \). (It is straightforward to check that the same result also holds for local sections defined with respect to distinct open covers, but the details are more cumbersome.) Let \( U \subseteq X \) be an open set, \( I \) an \( \mathcal{A}|_{U_\alpha} \)-torsor, and \( \{s_\alpha\}, \{s'_\alpha\} \) two sets of local sections of \( I \), both defined over the same open cover \( \{U_\alpha\} \) of \( U \). Let \( P \times I \) and \( (P \times I)' \) denote the two objects obtained from gluing via the local sections \( \{s_\alpha\}, \{s'_\alpha\} \), and let

\[
\psi_\alpha(P) : (P \times I)|_{U_\alpha} \longrightarrow P|_{U_\alpha}
\]

\[
\psi'_\alpha(P) : (P \times I)'|_{U_\alpha} \longrightarrow P|_{U_\alpha}
\]

denote the corresponding isomorphisms. Define \( f_\alpha : (P \times I)|_{U_\alpha} \rightarrow (P \times I)'|_{U_\alpha} \) by,

\[
f_\alpha \equiv (\psi'_\alpha(P))^{-1} \circ (s'_\alpha - s_\alpha) \circ \psi_\alpha(P)
\]

where we have used \( (s'_\alpha - s_\alpha) \) to denote the element of the band mapping \( s_\alpha \mapsto s'_\alpha \). It is straightforward to check that the \( \{f_\alpha\} \) satisfy the gluing axiom for morphisms, and so there exists a (unique) morphism \( P \times I \rightarrow (P \times I)' \). Thus, distinct local sections of \( I \) define isomorphic objects \( P \times I \).
5.3.2 Induced equivalences of categories

A $\mathcal{A}|_U$-torsor $I$ not only induces a map between objects of the category $\mathcal{C}(U)$, but it also induces a self-equivalence of the category $\mathcal{C}(U)$, which we shall now explain.

We shall denote the proposed functor by $I(U)$. We explained the action of $I(U)$ on objects, namely $P \mapsto P \times I$, in the previous subsection. We define the action of $I(U)$ on morphisms as follows.

Let $P_1, P_2$ be objects of $\mathcal{C}(U)$, and let $\beta \in \text{Hom}_{\mathcal{C}(U)}(P_1, P_2)$. Let $\psi_\alpha(P_i)$ denote the isomorphisms $\psi_\alpha(P_i) : (P_i \times I)|_{U_\alpha} \sim P_i|_{U_\alpha}$ (where $i \in \{1, 2\}$) constructed at the same time as the $P_i \times I$, in the gluing law for objects, for $\{U_\alpha\}$ an open cover of $U$. Define morphisms

$$(\beta \times I)|_{U_\alpha} : (P_1 \times I)|_{U_\alpha} \longrightarrow (P_2 \times I)|_{U_\alpha}$$

by

$$(\beta \times I)|_{U_\alpha} \equiv (\psi_\alpha(P_2))^{-1} \circ \beta|_{U_\alpha} \circ \psi_\alpha(P_1)$$

One can then use the gluing law for morphisms to glue together the $(\beta \times I)|_{U_\alpha}$ to form a (unique) morphism $\beta \times I : P_1 \times I \rightarrow P_2 \times I$.

It is straightforward to check that the map we have just defined, namely $I(U) : \beta \mapsto \beta \times I$, completes the definition of a functor $I(U) : \mathcal{C}(U) \rightarrow \mathcal{C}(U)$.

In passing, note that the maps $\psi_\alpha$ defined above, define a natural transformation $\rho^* \circ I_{\mathcal{C}(U)} \Rightarrow \rho^* \circ \text{Id}_{\mathcal{C}(U)}$, for $\rho : U_\alpha \hookrightarrow U$ inclusion.

It is easy to check that there exist invertible natural transformations

$$I^{-1}(U) \circ I(U) \Longrightarrow \text{Id}_{\mathcal{C}(U)}$$

$$I(U) \circ I^{-1}(U) \Longrightarrow \text{Id}_{\mathcal{C}(U)}$$

where the $I^{-1}(U)$ are the functors associated to the dual torsors, so each functor $I(U)$ defines a self-equivalence of the category $\mathcal{C}(U)$.

So far we have shown how an $\mathcal{A}|_U$-torsor $I$ (together with a specific choice of local sections of $I$) defines an equivalence of categories $I(U) : \mathcal{C}(U) \rightarrow \mathcal{C}(U)$. We shall now use $I$ to define a Cartesian functor.

Let $I$ be an $\mathcal{A}$-torsor, let $\{U_\alpha\}$ be a good open cover of $X$, and let $\{s_\alpha\}$ be a set of local sections of $I$, defined with respect to $\{U_\alpha\}$. We have already demonstrated how to define a family of functors $I(U) : \mathcal{C}(U) \rightarrow \mathcal{C}(U)$, each defined by the $\mathcal{A}|_U$-torsor $I|_U$ and local sections $\{s_\alpha|_U\}$ of $I|_U$, defined with respect to the open cover $\{U_\alpha \cap U\}$ of $U$. In order to
define a Cartesian functor $I : \mathcal{C} \to \mathcal{C}$, it remains to define invertible natural transformations $\chi_\rho : \rho^* \circ I(U) \Rightarrow I(V) \circ \rho^*$ for each inclusion $\rho : V \hookrightarrow U$, obeying the usual constraint.

Let $\rho : U_2 \hookrightarrow U_1$ be an inclusion of open sets, and let $P \in \text{Ob} \mathcal{C}(U_1)$. Let

$$\psi^1_\alpha(P) : (P \times I_{U_1})_{U_1 \cap U_\alpha} (= I(U_1)(P)|_{U_1 \cap U_\alpha}) \rightarrow P|_{U_1 \cap U_\alpha}$$
$$\psi^2_\alpha(P) : (P|_{U_2} \times I_{U_2})_{U_2 \cap U_\alpha} (= I(U_2)(P)|_{U_2 \cap U_\alpha}) \rightarrow P|_{U_2 \cap U_\alpha}$$

be the isomorphisms appearing in the gluing law for objects. Define $f_\alpha : I(U_1)(P)|_{U_2 \cap U_\alpha} \rightarrow I(U_2)(P)|_{U_2 \cap U_\alpha}$ by,

$$f_\alpha \equiv (\psi^2_\alpha(P))^{-1} \circ \varphi_{2,\alpha} \circ \varphi_{1,\alpha}^{-1} \circ \psi^1_\alpha(P)|_{U_2 \cap U_\alpha} \circ \varphi_{1,\alpha} \circ \varphi_{2,\alpha}^{-1}$$

where $\varphi$ are the natural transformations defining $\mathcal{C}$ as a presheaf of categories, and we have implicitly used the notation $U_{\alpha i} = U_i \cap U_\alpha (i \in \{1, 2\})$. It is straightforward to check that the $f_\alpha$ satisfy the gluing axiom for morphisms, and so there exists a unique morphism

$$\chi_\rho(P) : I(U_1)(P)|_{U_2} \rightarrow I(U_2)(P|_{U_2})$$

such that $\chi_\rho(P)|_{U_2 \cap U_\alpha} = f_\alpha$. Moreover, it is straightforward to check that the $\chi_\rho$ define a natural transformation $\rho^* \circ I(U_1) \Rightarrow I(U_2) \circ \rho^*$, and moreover that this set of natural transformations makes diagram (3) commute.

Thus, we have just defined the natural transformations needed to describe $I : \mathcal{C} \rightarrow \mathcal{C}$ as a Cartesian functor. Moreover, it should be clear that $I$ defines a map of gerbes, i.e., commutes with the action of the band.

The Cartesian functor $I : \mathcal{C} \rightarrow \mathcal{C}$ we defined above depends explicitly upon a choice of local sections $\{s_\alpha\}$ of $I$, with respect to some open cover $\{U_\alpha\}$ of $X$. How do two Cartesian functors defined by distinct choices of local sections of the same $\mathcal{A}$-torsor $I$ differ? It is straightforward to check that any two such Cartesian functors $I : \mathcal{C} \rightarrow \mathcal{C}$ differ by a 2-arrow. In other words, if $I$ and $I'$ are two Cartesian functors $\mathcal{C} \rightarrow \mathcal{C}$, both associated to the same $\mathcal{A}$-torsor $I$ but differing in the choice of local sections, then there exists an invertible 2-arrow $\eta : I \Rightarrow I'$. The natural transformations over each open set $U$ are defined by the morphisms $P \times I \rightarrow (P \times I)'$ we defined earlier in section 5.3.1 in studying this same issue in the context of gauge transformations of individual objects.

We should also mention an interesting special case of the matter above. Let $I^1_\mathcal{C}$ denote an automorphism of the gerbe $\mathcal{C}$, associated to the $\mathcal{A}$-torsor $I$ and defined by local sections $\{s^1_\alpha\}$ defined over an open cover $\{U^1_\alpha\}$ of $X$. Let $\{U^2_\alpha\}$ be a refinement of $\{U^1_\alpha\}$, and consider the automorphism $I^2_\mathcal{C}$ defined by the local sections $\{s^2_\alpha \equiv s^1_{\alpha(i)}|_{U^2_\alpha}\}$ of $I$, defined with respect to the cover $\{U^2_\alpha\}$. It is straightforward to check that, in this special case, the distinction between $I^1_\mathcal{C}$ and $I^2_\mathcal{C}$ is identical to the ambiguity in defining $I_\mathcal{C}$ on objects, for fixed choices of local sections. Thus, for judicious choices in the definition of $I^1_\mathcal{C}$, $I^1_\mathcal{C}$ is the same functor as $I^2_\mathcal{C}$.
Suppose \( I^1 \) and \( I^2 \) are a pair of \( \mathcal{A} \)-torsors. Let \( I^1_\mathcal{C} \) and \( I^2_\mathcal{C} \) be corresponding automorphisms of the gerbe \( \mathcal{C} \). We shall now show that (under the correct circumstances) the gerbe automorphism \( (I^1 \cdot I^2)_\mathcal{C} \) is identical to the automorphism \( I^1_\mathcal{C} \circ I^2_\mathcal{C} \). Assume (without loss of generality) that the local sections defining \( I^1_\mathcal{C} \) and \( I^2_\mathcal{C} \) are defined with respect to the same open cover \( \{U_\alpha\} \) of \( X \) (if not, restrict to a mutual refinement); let \( \{s^1_\alpha\} \) and \( \{s^2_\alpha\} \) denote the local sections of \( I^1, I^2 \), respectively, defining \( I^1_\mathcal{C} \) and \( I^2_\mathcal{C} \). Then it is straightforward to check that (with the usual judicious choices) the gerbe automorphism \( (I^1 \cdot I^2)_\mathcal{C} \) defined with respect to the local sections \( \{s^1_\alpha \cdot \alpha_{\mathcal{A}(U_\alpha)} s^2_\alpha\} \) of the torsor \( I^1 \cdot I^2 \) is identical to the composition \( I^1_\mathcal{C} \circ I^2_\mathcal{C} \), i.e.,

\[
(I^1 \cdot I^2)_\mathcal{C} = I^1_\mathcal{C} \circ I^2_\mathcal{C}
\]

Suppose \( I^1 \) and \( I^2 \) are a pair of \( \mathcal{A} \)-torsors. We shall show here that a torsor isomorphism \( \omega : I^1 \to I^2 \) is equivalent to a 2-arrow \( \overline{\omega} : I^1_\mathcal{C} \to I^2_\mathcal{C} \). First, we shall define some notation. Assume \( \{s^1_\alpha\} \) and \( \{s^2_\alpha\} \) are local sections of \( I^1, I^2 \) defining \( I^1_\mathcal{C}, I^2_\mathcal{C} \), assumed to both \( \mathcal{C} \)-be defined over an open cover \( \{U_\alpha\} \) of \( X \). Assume first that we are given \( \omega : I^1 \to I^2 \); we shall describe how to construct \( \overline{\omega} : I^1_\mathcal{C} \to I^2_\mathcal{C} \). Let \( U \) be an open set, and denote the natural transformations associated to \( I^1_\mathcal{C}, I^2_\mathcal{C} \) by \( \psi_\alpha \), i.e.,

\[
\psi^i_\alpha(P) : I^i_\mathcal{C}(U)(P)|_{U \cap U_\alpha} \to P|_{U \cap U_\alpha}
\]

for \( P \in \text{Ob} \mathcal{C}(U) \), and \( i \in \{1, 2\} \). Then we define

\[
\overline{\psi}_\alpha(U)(P) : I^1_\mathcal{C}(U)(P)|_{U \cap U_\alpha} \to I^2_\mathcal{C}(U)(P)|_{U \cap U_\alpha}
\]

by,

\[
\overline{\psi}_\alpha(U)(P) \equiv \left(\psi^2_\alpha(P)\right)^{-1} \circ \left( s^2_\alpha - \omega(s^1_\alpha) \right) \circ \psi^1_\alpha(P)
\]

It is easy to check that the \( \overline{\psi}_\alpha(U)(P) \) satisfy the gluing axiom for morphisms, and so can be glued together to form a unique morphism

\[
\overline{\psi}(U)(P) : I^1_\mathcal{C}(U)(P) \to I^2_\mathcal{C}(U)(P)
\]

whose restriction to \( U \cap U_\alpha \) is given by \( \overline{\psi}_\alpha(U)(P) \). Moreover, it is straightforward to check that \( \overline{\psi}(U) : I^1_\mathcal{C}(U) \Rightarrow I^2_\mathcal{C}(U) \) is a natural transformation, and finally that \( \overline{\psi} : I^1_\mathcal{C} \Rightarrow I^2_\mathcal{C} \) is a 2-arrow. Thus, given a torsor isomorphism \( \omega : I^1 \to I^2 \), we have constructed a 2-arrow \( \overline{\psi} : I^1_\mathcal{C} \Rightarrow I^2_\mathcal{C} \).

Conversely, given a 2-arrow \( \overline{\psi} : I^1_\mathcal{C} \Rightarrow I^2_\mathcal{C} \), we shall now construct a torsor isomorphism \( \omega : I^1 \to I^2 \). We shall define \( \omega \) by describing the action of \( \omega \) on the local sections \( \{s^1_\alpha\} \) of \( I^1_\mathcal{C} \). Using the same conventions as in the paragraph above, for any open \( U \) and \( P \in \text{Ob} \mathcal{C}(U) \),

\[\text{Our notation is slightly sloppy, in that } \mathcal{A}(U_\alpha) \text{ is an abelian group, not a ring, and } P(U_\alpha) \text{ is a set, not a module.}\]

\[\text{If they are not both defined over the same open cover, then restrict to a mutual refinement.}\]
define \( g(U \cap U_\alpha)(P) \) to be the element of the abelian group \( A(U \cap U_\alpha) \) associated to the automorphism

\[
\psi^2_\alpha(P) \circ \varpi(U)(P) \circ \left( \psi^1_\alpha(P) \right)^{-1} : P|_{U_\alpha} \rightarrow P|_{U_\alpha}
\]

Assume without loss of generality\(^{16}\) that \( \{U_\alpha\} \) is a good cover, and so each category \( C(U_\alpha) \) contains a single isomorphism class of objects\(^{17}\). Then, for \( U = U_\alpha \), the band element \( g(U_\alpha) \) defined above is independent of the choice of \( P \). Define

\[
\omega(s^1_\alpha) \equiv g(U_\alpha)^{-1} \cdot s^2_\alpha
\]

To define a torsor isomorphism, it suffices to describe the action on a set of local sections associated to a cover, which we have just done. Thus, we have just associated a torsor isomorphism \( \omega : I^1 \rightarrow I^2 \) to the 2-arrow \( \varpi : I^1_C \Rightarrow I^2_C \). To summarize the results of this paragraph and the last, a torsor isomorphism \( \omega : I^1 \rightarrow I^2 \) is equivalent to a 2-arrow \( \varpi : I^1_C \Rightarrow I^2_C \).

As a consequence of the results in the last paragraph, we see that isomorphic torsors define isomorphic gerbe automorphisms. Thus, distinct gerbe automorphisms are defined by equivalence classes of torsors, not individual torsors \( \text{per se} \). This fact will be quite important when discussing how the group cohomology group \( H^2(\Gamma, U(1)) \) appears when describing equivariant structures on \( B \) fields – modding out by group coboundaries ultimately comes from the fact that only equivalence classes of torsors define distinct gerbe automorphisms.

In passing, we should also speak briefly on pullbacks, and how the pullback of an automorphism associated to a torsor \( I \) is related to the automorphism associated to the pullback of the torsor. Let \( f : X \rightarrow Y \) be a continuous map, and let \( C \) be a gerbe on \( Y \), with band \( \mathcal{A} \). Let \( I \) be an \( \mathcal{A} \)-torsor, and let \( I_C \) denote an associated automorphism of \( C \) (defined with respect to local sections \( \{s_\alpha\} \), over an open cover \( \{U_\alpha\} \) of \( Y \)). Let \( (f^*I)_C \) denote the automorphism of the gerbe \( f^*C \) defined by the \( f^*\mathcal{A} \)-torsor \( f^*I \), with respect to the same open sections \( \{s_\alpha\} \) of \( f^*I \), over the open cover \( \{f^{-1}(U_\alpha)\} \) of \( X \). It is straightforward to check that, at least in the case that \( f \) is a homeomorphism (so that \( f^{-1}C \cong f^*C \) and \( f^{-1}I \cong f^*I \)), the gerbe automorphisms \( f^*(I_C) \) and \( (f^*I)_C \) coincide\(^{18}\).

\(^{16}\)If not, restrict to a suitable refinement.

\(^{17}\)We shall prove later that isomorphism classes of objects in each category \( C(U) \) are in one-to-one correspondence with elements of \( H^1(U, A|_U) \).

\(^{18}\)In fact, we are being slightly sloppy. In defining the action of \( I \), recall there was a minor ambiguity in the definition of the action on the objects of \( C \). Distinct choices differ by unique morphisms, so we assumed such choices were made, and thereafter ignored the point. Essentially the same problem arises here. The correct statement is that \( f^*(I_C) \) and \( (f^*I)_C \) differ by at most a unique 2-arrow; but by making judicious choices, they can be assumed to coincide.
5.3.3 Action on connections

In this section we shall describe how gauge transformations of gerbes act on connective structures and curvings. We briefly touched on these matters in section 3.5; here we re-examine them in more detail.

To fix notation, let $C$ be a gerbe with connective structure $C_0$. Let $U \subseteq X$ be open, and $P \in \mathrm{Ob} \; C(U)$. Let $I$ be an $A|_U$-torsor, let \{${U}_\alpha$\} be a good open cover of $U$, and let \{${s}_\alpha$\} be a set of local sections of $I$, defined with respect to \{${U}_\alpha$\}. Let $I_C : C|_U \to C|_U$ denote the gerbe automorphism corresponding to $I$ with sections \{${s}_\alpha$\}.

Consider the action of $I_C$ on $P$. By definition of $P \times I = I_C(U)(P)$, we have morphisms $\psi_\alpha(P) : (P \times I)|_{U_\alpha}$ such that the following diagram commutes:

$$
\begin{array}{cccc}
(P \times I)|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & (P \times I)|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & (P \times I)|_{U_\alpha} \\
\psi_\beta(P)|_{U_\alpha} & \downarrow & \psi_\beta(P)|_{U_\alpha} & \downarrow & \psi_\beta(P)|_{U_\alpha} \\
(\varphi_{\alpha,\beta})|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & (\varphi_{\alpha,\beta})|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & (\varphi_{\alpha,\beta})|_{U_\alpha}
\end{array}
$$

where the $\varphi$ are the natural transformations defining $C$ as a presheaf of categories, and $s_\beta \to s_\alpha$ denotes the band element mapping $s_\beta \to s_\alpha$.

By applying the Cartesian functor $C_0$, we can (with a bit of work) recover the following commutative diagram:

$$
\begin{array}{cccc}
C_0(P \times I)|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & C_0(P \times I)|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & C_0(P \times I)|_{U_\alpha} \\
C_0(P)|_{U_\alpha} & \downarrow & C_0(P)|_{U_\alpha} & \downarrow & C_0(P)|_{U_\alpha} \\
C_0(P)|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & C_0(P)|_{U_\alpha} & \xrightarrow{\varphi_{\alpha,\beta}} & C_0(P)|_{U_\alpha}
\end{array}
$$

where we have abbreviated $C_0(U)(P)$ by $C_0(P)$, for example, and where the map

$$
C_0(U)(P \times I)|_{U_\alpha} \to C_0(U)(P)|_{U_\alpha}
$$

is given by the composition

$$(\chi_\alpha)^{-1} \circ (\chi_\alpha)^{-1} \circ C_0(U_\alpha)(\psi_\alpha(P)|_{U_\alpha}) \circ \chi_\alpha \circ \chi_\alpha$$

The $\chi$ are the natural transformations defining $C_0$ as a Cartesian functor.

From diagram (26), it should be clear that $C_0(U)(P \times I)$ and $C_0(U)(P)$ differ by the $\Omega^1|_U$-torsor of connections on $I$. Moreover, a precise specification of how a particular section $\nabla \in \Gamma(U, C_0(U)(P))$ is mapped is determined by the local sections \{${s}_\alpha$\} determining $I_C$, and is equivalent to a choice of connection on $I$. (Compare, for example, [6, section 5.3, equ’n (5-11)].)
In other words, let \( \{ A^\alpha \} \) be a family of 1-forms on \( \{ U_\alpha \} \) defining a connection on \( I \). Assume the action of \( I_C \) on sections of \( \text{Co}(U)(P) \) is determined by \( \{ A^\alpha \} \). Then for any \( \nabla_\alpha \in \Gamma(U_\alpha, \text{Co}(U_\alpha)(P|_{U_\alpha})), \nabla_\alpha \mapsto \nabla_\alpha + A^\alpha \) under the action of \( I_C \).

Phrased more simply still, a gauge transformation of a gerbe with connective structure is defined by an equivalence class of bundles with connection. (After all, a 2-arrow between gerbe automorphisms lifts to a “connection-preserving” map between the connective structures, as noted earlier.)

How does \( I_C \) act on a curving \( K \) on \((C, \text{Co})\)? The answer should be immediately clear from the definition of curving:

\[
K(\nabla_\alpha + A^\alpha) = K(\nabla_\alpha) + dA^\alpha
\]

Much earlier in this paper we remarked that an automorphism of a gerbe with connection is defined by an equivalence class of bundles with flat connection. Here, we can see that more explicitly. If the connection is flat, then the curving \( K \) is invariant under the gauge transformation. Put another way, just as constant gauge transformations define bundle automorphisms that preserve the connection, a gauge transformation of a gerbe that preserves the connection is an equivalence class of bundles with flat connection.

### 5.3.4 Gauge transformations commute with gerbe maps

Let \( F : C \to D \) be a map between gerbes \( C, D \), that is, a Cartesian functor commuting with the action of the band. Assume \( C \) and \( D \) both have band \( \mathcal{A} \), and are both defined over a space \( X \).

In this section, we shall show that if \( I \) is any \( \mathcal{A} \)-torsor, then there exists an invertible 2-arrow

\[
\kappa : F \circ I_C \Rightarrow I_D \circ F
\]

where \( I_C : C \to C \) and \( I_D : D \to D \) are gerbe maps induced by the \( \mathcal{A} \)-torsor \( I \).

In fact, this is quite straightforward. Without loss of generality\(^\text{19}\), assume that the local sections of \( I \) defining \( I_C \) and \( I_D \) are identical. Let \( \{ U_\alpha \} \) be a good open cover of \( X \), and let \( \{ s_\alpha \} \) be the local sections of \( I \) over \( \{ U_\alpha \} \) which define \( I_C \) and \( I_D \). Let

\[
\psi^C_\alpha(P) : I_C(U)(P)|_{U \cap U_\alpha} \to P|_{U \cap U_\alpha} \quad \text{for } P \in \text{Ob } C(U)
\]

\[
\psi^D_\alpha(P') : I_D(U)(P')|_{U \cap U_\alpha} \to P'|_{U \cap U_\alpha} \quad \text{for } P' \in \text{Ob } D(U)
\]

(for open \( U \subseteq X \)) denote the isomorphisms obtained in the definition of gauge transformation. Define

\[
\kappa_\alpha(U)(P) : (F \circ I_C)(U)(P)|_{U \cap U_\alpha} \to (I_D \circ F)(U)(P)|_{U \cap U_\alpha}
\]

\(^{19}\)As the gerbe automorphisms defined by distinct choices of local sections of \( I \) differ by an invertible 2-arrow, clearly we are free to choose any convenient sets of local sections without changing the result.
for open $U \subseteq X$ and $P \in \text{Ob} \, C(U)$ by,

$$
\kappa_\alpha(U)(P) \equiv \psi_\alpha^D(F(U)(P))^{-1} \circ \left(\chi^F_\alpha\right)^{-1} \circ F(U \cap U_\alpha) \left(\psi^C_\alpha(P)\right) \circ \chi^F_\alpha
$$

where the $\chi^F$ are the natural transformations defining $F$ as a Cartesian functor. It is straightforward to check that the $\kappa_\alpha(U)(P)$ satisfy the gluing axiom for morphisms, so they can be glued together to form a unique morphism

$$
\kappa(U)(P) : (F \circ I_C)(U)(P) \rightarrow (I_D \circ F)(U)(P)
$$

such that $\kappa(U)(P)|_{U \cap U_\alpha} = \kappa_\alpha(U)(P)$.

It is straightforward to check that $\kappa(U)$ defines a natural transformation $(F \circ I_C)(U) \Rightarrow (I_D \circ F)(U)$, and moreover that $\kappa$ defines an invertible 2-arrow $F \circ I_C \Rightarrow I_D \circ F$.

Thus, loosely speaking, gauge transformations commute with gerbe maps.

### 5.3.5 Sheaves of morphisms as torsors

If $P_1$ and $P_2$ are any two objects of $C(U)$, then the sheaf of (iso)morphisms $\text{Hom}_U(P_1, P_2)$ is an $A|_U$-torsor\(^{20}\). In particular, $A|_U \cong \text{Aut}(P_1)$ acts on the left, and it should be clear that this action is free and transitive – for any open $V \subseteq U$, the set $\text{Hom}(P_1|_V, P_2|_V)$ is either empty (and so trivially an $A(V)$-torsor), or is nonempty and is manifestly a torsor under the group $A(V)$. Thus, as a sheaf of abelian groups, $A|_U$ acts freely and transitively on the sheaf $\text{Hom}_U(P_1, P_2)$, and moreover its action commutes with restriction maps, so $\text{Hom}_U(P_1, P_2)$ is an $A|_U$-torsor.

One can easily check that if $F : C \rightarrow D$ is a map of gerbes of band $A$ over the same space $X$, then for any open $U$ and for any two objects $P_a, P_b \in \text{Ob} \, C(U)$, the induced map

$$
\text{Hom}_U(P_a, P_b) \rightarrow \text{Hom}_U(F(U)(P_a), F(U)(P_b))
$$

is a morphism of torsors. As any morphism of $A$-torsors is an isomorphism, this means that the $A$-torsors $\text{Hom}_U(P_a, P_b)$ and $\text{Hom}_U(F(U)(P_a), F(U)(P_b))$ are isomorphic.

It is straightforward to check that the torsor $\text{Hom}_U(P_1, P_2)$ induces the gauge transformation on $C(U)$ that maps the object $P_1$ to the object $P_2$, as the reader has probably guessed.

Moreover, it is straightforward to see that $\text{Hom}_U(P, P \times I) \cong I$ as $A|_U$-torsors, for any $A|_U$-torsor $I$. Let $\{U_\alpha\}$ be a good open cover of $U$, and let $\{s_\alpha \in I(U_\alpha)\}$ be the set of local

\(^{20}\) Those readers also studying [1, section 5] will note that in that reference, $\text{Hom}(P_1, P_2)$ is instead denoted $\text{Isom}(P_1, P_2)$. 

59
sections of $I$ defining the map $P \mapsto P \times I$. Recall that we constructed $P \times I$ using the gluing law for objects, and the same gluing law also yields a set of isomorphisms

$$\psi_\alpha(P) : (P \times I)|_{U_\alpha} \rightarrow P|_{U_\alpha}$$

We can define an isomorphism of $\mathcal{A}|_U$-torsors by specifying that the sections $\{s_\alpha\}$ should map to the isomorphisms $\{\psi_\alpha^{-1}\}$. As mentioned in the section on torsors, to define an isomorphism between two torsors, it suffices to describe how a set of local sections (defined with respect to an open cover) are mapped, thus we have now defined an isomorphism $I \xrightarrow{\sim} \text{Hom}_U(P, P \times I)$ of $\mathcal{A}|_U$-torsors.

One implication of the fact that $\text{Hom}_U(P, P \times I) \cong I$ is that the action of $I$ is free on equivalence classes of objects in $\mathcal{C}(U)$: $P \times I$ is isomorphic to $P$ if and only if $I$ is trivializable, i.e., only if $I$ has a global section does there exist a morphism $P \rightarrow P \times I$, not just between their restrictions to open subsets.

Moreover, the action of torsors on objects is not only free, but transitive: any two objects $P_1, P_2 \in \text{Ob} \mathcal{C}(U)$ can be related by a set of local sections of the $\mathcal{A}|_U$-torsor $\text{Hom}_U(P_1, P_2)$.

Thus, the set of equivalence classes of objects of $\mathcal{C}(U)$ is a torsor under the action of the abelian group $H^1(U, \mathcal{A}|_U)$.

This means that there exists a (noncanonical) one-to-one correspondence between equivalence classes of objects of $\mathcal{C}(U)$ and equivalence classes of $\mathcal{A}|_U$-torsors. In the case that $\mathcal{A} = C^\infty(G)$ for some abelian $G$, we can rephrase this by saying that equivalence classes of objects of $\mathcal{C}(U)$, for any gerbe $\mathcal{C}$ with band $\mathcal{A}$, are in one-to-one correspondence with equivalence classes of principal $G$-bundles on $U$. This implies that all gerbes with band $\mathcal{A} = C^\infty(G)$ look locally like the trivial gerbe $\text{Tors}(G)$, just as all bundles look locally like a trivial bundle.

This last result gives a great deal of insight into gerbes, and so is worth repeating. Just as all bundles can be locally trivialized, all gerbes can be locally trivialized. Gerbes with band $\mathcal{A} = C^\infty(G)$ for abelian $G$ are locally isomorphic to the stack of principal $G$-bundles, a trivial gerbe.

We shall conclude this section by explicitly demonstrating such local trivializations. Local trivializations are not needed to define a sheaf – however, as they can be used to give insight into local structure, we shall work them out explicitly. More precisely, for any open set $U$ such that $\mathcal{C}(U) \neq \emptyset$, we shall construct functors

$$F(U) : \mathcal{C} \rightarrow \text{Tors}(\mathcal{A})(U)$$

$$F^*(U) : \text{Tors}(\mathcal{A})(U) \rightarrow \mathcal{C}(U)$$

such that there exist invertible natural transformations $F^*(U) \circ F(U) \Rightarrow \text{Id}$ and $F(U) \circ F^*(U) \Rightarrow \text{Id}$.
Fix an object $P_0 \in \text{Ob } \mathcal{C}(U)$, and a torsor $I_0 \in \text{Ob } \text{Tors}(\mathcal{A})(U)$. We define the functor $F(U)$ as follows. Let $P \in \text{Ob } \mathcal{C}(U)$ be any object, and define the $\mathcal{A}_{|U}$-torsor

$$ I \equiv \text{Hom}_U(P_0, P) $$

Define

$$ F(U)(P) \equiv I \cdot I_0 $$

Next, let $P_a, P_b \in \text{Ob } \mathcal{C}(U)$ be objects, and $\beta : P_a \to P_b$ be a morphism. Define the $\mathcal{A}_{|U}$-torsors

$$ I^a \equiv \text{Hom}_U(P_0, P_a) $$

$$ I^b \equiv \text{Hom}_U(P_0, P_b) $$

The morphism $\beta$ induces a morphism of torsors

$$ \beta^\# : I^a \longrightarrow I^b $$

and so we define

$$ F(U)(\beta) \equiv \beta^\# : I^a \cdot I_0 \longrightarrow I^b \cdot I_0 $$

With these definitions, $F(U)$ is a well-defined functor from $\mathcal{C}(U)$ to $\text{Tors}(\mathcal{A})(U)$.

Next, we shall define the functor $F^*(U)$. Let $I$ be a $\mathcal{A}_{|U}$-torsor in $\text{Ob } \text{Tors}(\mathcal{A})(U)$, and define

$$ F^*(U)(I) \equiv \left( I \cdot I_0^{-1} \right)_\mathcal{C}(U)(P_0) $$

Next, let $I^a, I^b$ be $\mathcal{A}_{|U}$-torsors in $\text{Ob } \text{Tors}(\mathcal{A})(U)$, and let $\beta : I^a \to I^b$ be a morphism of $\mathcal{A}_{|U}$-torsors. The morphism $\beta$ defines a 2-arrow

$$ \beta^* : \left( I^a \cdot I_0^{-1} \right)_\mathcal{C} \Longrightarrow \left( I^b \cdot I_0^{-1} \right)_\mathcal{C} $$

Define

$$ F^*(U)(\beta) \equiv \beta^*(U)(P_0) $$

With these definitions, $F^*(U)$ is a well-defined functor from $\text{Tors}(\mathcal{A})(U)$ to $\mathcal{C}(U)$.

Finally, it is straightforward to check that there exist invertible natural transformations

$$ F^*(U) \circ F(U) \Longrightarrow \text{Id}_{\mathcal{C}(U)} $$

$$ F(U) \circ F^*(U) \Longrightarrow \text{Id}_{\text{Tors}(\mathcal{A})(U)} $$

Thus, we have now finished demonstrating explicitly that for any $U$ such that $\mathcal{C}(U) \neq \emptyset$, the category $\mathcal{C}(U)$ is equivalent to the category $\text{Tors}(\mathcal{A})(U)$. 
5.3.6 Sheaves of natural transformations as torsors

We shall now show that sheaves of natural transformations are $\mathcal{A}$-torsors. More specifically, let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be a pair of Cartesian functors defining maps of gerbes, between the gerbes $\mathcal{C}$ and $\mathcal{D}$, assumed to both have band $\mathcal{A}$ and both be defined over the same space $X$. We shall show that the sheaf of local 2-arrows $2\mathcal{R}_U(F, G)$ is an $\mathcal{A}|_U$-torsor, for any open $U \subseteq X$. We shall also derive some useful properties of these torsors.

We shall now argue that the sheaf of sets $2\mathcal{R}_U(F, G)$ is an $\mathcal{A}|_U$-torsor. We define the action of $\mathcal{A}|_U$ as follows. Let $V \subseteq U$ be open, and let $g \in \mathcal{A}(V)$. Let $\psi$ be an element of the set $2\mathcal{R}_U(F, G)(V)$, which is to say, $\psi$ is a collection of natural transformations $F(W) \Rightarrow G(W)$, one for each open $W \subseteq V$. The element $g \in \mathcal{A}(V)$ acts on each natural transformation $\psi(W)$ as,

$$\psi(W) \mapsto \psi(W) \circ (\rho^*_W g)$$

where $\rho^*$ is the restriction map in the sheaf $\mathcal{A}$, and for each $P \in \text{Ob } \mathcal{C}(W)$, $\rho^* g$ is interpreted as an element of $\text{Aut}(F(W)(P))$.

It is easy to see that this action of $\mathcal{A}$ is free, and commutes with the restriction map. To check that it is transitive requires more work, which we shall outline here. Let $\psi_a, \psi_b$ be two elements of the set $2\mathcal{R}_U(F, G)(V)$. For any $P \in \text{Ob } \mathcal{C}(V)$, $\psi_b(V)(P)^{-1} \circ \psi_a(V)(P)$ is an automorphism of $F(V)(P)$, which we can identify with an element $g(P) \in \mathcal{A}(V)$. Any two objects $P_1, P_2$ which are related by a morphism will clearly define isomorphic group elements: $g(P_1) = g(P_2)$. More generally, one can show that $g(P_1) = g(P_2)$ even if $P_1, P_2$ are not related by a morphism, by restricting to an open subset $W \subseteq V$ such that there exists a morphism $\beta : P_1|_W \rightarrow P_2|_W$. Define a morphism

$$\lambda : F(V)(P_1)|_W \rightarrow F(V)(P_2)|_W$$

by,

$$\lambda \equiv (\chi_W^F)^{-1} \circ F(W)(\beta) \circ \chi_W^F$$

then it is straightforward to check that

$$\left(\psi_b(V)(P_2)^{-1} \circ \psi_a(V)(P_2)\right)|_W = \lambda \circ \left(\psi_b(V)(P_1)^{-1} \circ \psi_a(V)(P_1)\right)|_W \circ \lambda^{-1}$$

which implies that $g(P_1) = g(P_2)$. Thus, for any pair of 2-arrows $\psi_a, \psi_b \in 2\mathcal{R}_U(F, G)(V)$, there exists $g \in \mathcal{A}(V)$ such that $\psi_a = \psi_b \circ g$. This means that the action of $\mathcal{A}|_U$ on the sheaf of sets $2\mathcal{R}_U(F, G)$ is transitive, and so we have finished demonstrating that $2\mathcal{R}_U(F, G)$ is an $\mathcal{A}|_U$-torsor.

We shall now argue that if $V$ is an open subset of $U$ such that the categories $\mathcal{C}(V)$ and $\mathcal{D}(V)$ each only contain a single isomorphism class of objects, then the set of local 2-arrows $2\mathcal{R}_U(F, G)(V)$ is nonempty. First, we shall construct natural transformations $F(V) \Rightarrow G(V)$. Fix some $P_0 \in \text{Ob } \mathcal{C}(V)$, and a morphism $\psi(V)(P_0) : F(V)(P_0) \rightarrow G(V)(P_0)$. For any
Thus, given any natural transformation $\psi : F \Rightarrow G$, let $\beta : P \to P_0$ be a morphism, and define $\psi(V)(P) : F(V)(P) \to G(V)(P)$ by, $\psi(V)(P) \equiv G(V)(\beta)^{-1} \circ \psi(V)(P_0) \circ F(V)(\beta)$. It is straightforward to check that $\psi(V)(P)$ is independent of the choice$^{21}$ of $\beta$, and moreover that this defines a natural transformation $\psi(V) : F \Rightarrow G$. In fact, we can generate an entire $\mathcal{A}(V)$-torsor of natural transformations in this fashion, by choosing different $P_0$ and different $\psi(V)(P_0)$. In passing, note that this also implies that a natural transformation $F \Rightarrow G$ (assuming that, for $|V$ denoting a restriction functor associated to the inclusion $V \hookrightarrow U$, $\mathcal{C}(U)$ and $\mathcal{D}(U)$ are both nonempty).

So far we have constructed a natural transformation $\psi(V) : F \Rightarrow G$; we shall now show that any such natural transformation defines a local 2-arrow $F|_V \Rightarrow G|_V$, i.e., an element of $\mathbf{2}\mathbf{R}_I(F,G)(V)$. In order to construct an element of $\mathbf{2}\mathbf{R}_I(F,G)(V)$, we need to specify (suitably compatible) natural transformations $F(W) \Rightarrow G(W)$ for all open $W \subseteq V$. For any $\psi(V) : F \Rightarrow G$ and any open $W \subseteq V$, define $\psi(W)$ to be the natural transformation generated by the composition

$$F(W) \circ \rho^* \xleftarrow{\chi^F_{\rho}} \rho^* \circ F(V) \xrightarrow{\psi(V)} \rho^* \circ G(V) \xrightarrow{\chi^G_{\rho}} G(W) \circ \rho^*$$

where $\rho : W \hookrightarrow V$ is inclusion. The composition above only defines a natural transformation $F(W) \circ \rho^* \Rightarrow G(W) \circ \rho^*$, not $F(W) \Rightarrow G(W)$; however using the ideas in the previous paragraph, it should be clear that the composition can be used to determine a natural transformation $F(W) \Rightarrow G(W)$ (uniquely, in fact, if every object of $\mathcal{D}(W)$ is isomorphic to the restriction of an object of $\mathcal{D}(V)$). Moreover, it is straightforward to check that this collection of natural transformations has the property that for any inclusion of open sets $\rho : W_2 \hookrightarrow W_1$, the following diagram commutes:

$$
\begin{array}{ccc}
\rho^* \circ F(W_1) & \xrightarrow{\psi(W_1)} & \rho^* \circ G(W_1) \\
\chi^F_{\rho} \downarrow & & \downarrow \chi^G_{\rho} \\
F(W_2) \circ \rho^* & \xrightarrow{\psi(W_2)} & G(W_2) \circ \rho^*
\end{array}
\quad (27)
$$

Thus, given any natural transformation $\psi(V) : F \Rightarrow G$, we can construct a local 2-arrow $\psi : F|_V \Rightarrow G|_V$. In other words, we have constructed an element of $\mathbf{2}\mathbf{R}_I(F,G)(V)$, and so if $V \subseteq U$ is an open set such that $\mathcal{C}(V)$ and $\mathcal{D}(V)$ each only contain a single isomorphism class of objects, then the set of local 2-arrows $\mathbf{2}\mathbf{R}_I(F,G)(V)$ is nonempty.

We shall now argue that for any $\mathcal{A}|_U$-torsor $I$, there is an isomorphism

$$I \xrightarrow{\sim} \mathbf{2}\mathbf{R}_I(F,I_D \circ F)$$

of $\mathcal{A}|_U$-torsors. We shall define this isomorphism by describing how a distinguished set of local sections $\{s_\alpha\}$ of $I$, defined with respect to an open cover $\{U_\alpha\}$ of $U$, are mapped into local

\footnote{In other words, if $\psi_\alpha(V)(P)$ and $\psi_\beta(V)(P)$ denote the two isomorphisms defined by any two morphisms $\alpha, \beta : P \to P_0$, then $\psi_\alpha(V)(P) = \psi_\beta(V)(P)$.}
sections of $2\mathcal{R}_I(F, I_D \circ F)$. Denote the local sections of $I$ defining the gerbe map $I_D : \mathcal{D} \to \mathcal{D}$ by $\{s_\alpha\}$, and assume (without loss of generality) that they are defined with respect to an open cover $\{U_\alpha\}$ of $U$ such that $\mathcal{C}(U_\alpha)$ and $\mathcal{D}(U_\alpha)$ each only contain a single isomorphism class of objects, for all $\alpha$. By definition of $I_D$, there exist natural transformations

$$\psi_\alpha(U_\alpha) : |u_\alpha \circ I_D(U) \implies |u_\alpha \circ \text{Id}_D(U)$$

where the $|u_\alpha$ denote restriction functors associated to the inclusion $U_\alpha \hookrightarrow U$. Define new natural transformations

$$\psi_\alpha(U_\alpha) : F(U_\alpha) \implies (I_D \circ F)(U_\alpha)$$

as the natural transformations associated to the composition

$$F(U_\alpha) \circ |u_\alpha \xrightarrow{\psi_\alpha(U_\alpha)} |u_\alpha \circ F(U) \xrightarrow{\psi_\alpha(U_\alpha)} |u_\alpha \circ (I_D \circ F)(U) \xrightarrow{\psi_\alpha(U_\alpha)} (I_D \circ F)(U_\alpha) \circ |u_\alpha$$

as in the previous paragraphs. Furthermore, proceeding again as in the previous few paragraphs, extend $\psi_\alpha(U_\alpha)$ to an element $\psi_\alpha \in 2\mathcal{R}_I(F, G)(U_\alpha)$. Finally, define the isomorphism $I \to 2\mathcal{R}_I(F, I_D \circ F)$ of $\mathcal{A}_{|U}$-torsors by mapping $s_\alpha \mapsto \psi_\alpha$. As described in the section on torsors, to define an isomorphism it suffices to describe how a set of local sections, defined with respect to an open cover, is mapped. Thus, $I \cong 2\mathcal{R}_I(F, I_D \circ F)$.

We have just shown that $I \cong 2\mathcal{R}_I(F, I \circ F)$ as $\mathcal{A}_{|U}$-torsors. One important consequence which is immediately derived from this fact is that if there exists a 2-arrow $F \Rightarrow I \circ F$ for some $\mathcal{A}$-torsor $I$, then $I$ must have a global section, and so is trivial.

### 5.3.7 Gerbe maps differ by gauge transformations

In this subsection we shall show that any two gerbe maps differ by a gauge transformation of the gerbe (i.e., a bundle, modulo equivalence). There is a precisely analogous notion for bundles: any two morphisms $F, G : P_1 \to P_2$ between principal $G$-bundles differ by a gauge transformation (namely, $F \circ G^{-1}$, which is a bundle automorphism, and hence a gauge transformation).

Let $F, G : \mathcal{C} \to \mathcal{D}$ be any two gerbe maps, and open $U \subseteq X$ open. We shall first show that there exists an $\mathcal{A}_{|U}$-torsor $I$ and a local 2-arrow $\psi : G|_U \Rightarrow (I_D \circ F)|_U$ (i.e., a global section of the sheaf $2\mathcal{R}_I(G, I_D \circ F)$). Given $F$ and $G$, define $I$ to be the $\mathcal{A}_{|U}$-torsor $2\mathcal{R}_I(F, G)$. Let $\{U_\alpha\}$ be a good open cover of $U$, and let $\{s_\alpha\}$ be a set of local sections of $I$ defining the gerbe automorphism $I_D$. Let

$$\psi_\alpha(U_\alpha) : |u_\alpha \circ I_D(U) \implies |u_\alpha \circ \text{Id}_D(U)$$

(where $|u_\alpha$ denotes the restriction functors associated to the inclusions $U_\alpha \hookrightarrow U$) denote the natural transformations associated to the $\{s_\alpha\}$, appearing in the definition of $I_D$. Let\footnote{If this is not true, then simply restrict the $\{s_\alpha\}$ to elements of a suitable refinement.}
ψ^s : F|_{U_α} ⇒ G|_{U_α} denote the s_α, interpreted explicitly as local 2-arrows. Now, we shall define a natural transformation G(U) ⇒ (I_D ◦ F)(U), by using the gluing law. For any object P ∈ Ob C(U), define a morphism

\[ f_α(P) : G(U)(P)|_{U_α} → (I_D ◦ F)(U)(P)|_{U_α} \]

by,

\[ f_α(P) ≡ ψ^I_α(F(U)(P))^{-1} \circ (χ^F_α)^{-1} \circ ψ^s_α(U_α)(P|_{U_α})^{-1} \circ χ^G_α \]

where the χ denote natural transformations defining F, G as Cartesian functors. It is straightforward to check that the f_α(P) satisfy the gluing axiom for morphisms, and so define a morphism ψ(U)(P) : G(U)(P) → (I_D ◦ F)(U)(P). Moreover, it is straightforward to check that the ψ(U)(P) define a natural transformation ψ(U) : G(U) ⇒ (I_D ◦ F)(U). Finally, a few paragraphs earlier we argued that a natural transformation ψ(U) : G(U) ⇒ (I_D ◦ F)(U) defines a local 2-arrow G|_U ⇒ (I_D ◦ F)|_U. Thus, we have explicitly constructed a global section of the sheaf 2R^U(G, I_D ◦ F).

In other words, we have just shown that for any two gerbe maps F, G, and for any open U ⊆ X, there exists a A|_U-torsor I and a local 2-arrow ψ : G|_U ⇒ (I_D ◦ F)|_U. (Of course, I is only defined up to isomorphism, as always.)

Note that the result above implies that if Φ : C → C is any automorphism of the gerbe C, then there exists an A-torsor I such that Φ is equivalent to I_C : C → C, i.e., there exists an invertible 2-arrow Φ ⇒ I_C.

Next, suppose that F and G define equivalences of gerbes with connective structure. This means that we also have invertible 2-arrows

\[ \Psi^F : Co_C → Co_D ◦ F \]
\[ \Psi^G : Co_C → Co_D ◦ G \]

It is straightforward to check that the difference between these 2-arrows is defined by a choice of connection on I (up to equivalence, as always).

Finally, suppose that F and G define equivalences of gerbes with connective structure and curving. This means that in addition to also having invertible 2-arrows Ψ^F, Ψ^G as above, the curving K is invariant under the action of the 2-arrows. The difference between such data is again defined by a bundle I with connection, and the constraint that the curving is invariant becomes the constraint that the connection on I is flat. So, the difference between two equivalences of gerbes with connection (connective structure and curving) is defined by a bundle I with flat connection.

In passing, we should mention that there is a closely analogous notion for bundles. Given two morphisms F, G : P_1 → P_2 between, say, principal U(1)-bundles P_1, P_2 with connection,
the difference $F \circ G^{-1}$ is a gauge transformation which preserves the connection – in other words, a constant²³ gauge transformation.

5.3.8 Maps of gerbes are equivalences of gerbes

In this subsection we shall argue that any map between gerbes with the same band, over the same space, is necessarily an isomorphism. This is a direct analogue of the statement that any morphism of principal $G$-bundles, for fixed $G$ and over a fixed space, is necessarily an isomorphism [17, section 4.3].

Let $F : \mathcal{C} \to \mathcal{D}$ be a map between the gerbes $\mathcal{C}, \mathcal{D}$, both assumed to have band $\mathcal{A}$ and both be defined over a fixed space $X$. We shall argue that $F$ is an equivalence of gerbes.

More precisely, we shall show that each functor $F^*(U) : \mathcal{C}(U) \to \mathcal{D}(U)$ is an equivalence of categories, in that there exists a functor $F^\ast(U) : \mathcal{D}(U) \to \mathcal{C}(U)$ and invertible natural transformations

\begin{align*}
\psi_1 : & \quad F(U) \circ F^*(U) \Longrightarrow \text{Id}_{\mathcal{D}(U)} \\
\psi_2 : & \quad F^*(U) \circ F(U) \Longrightarrow \text{Id}_{\mathcal{C}(U)}
\end{align*}

We define the functor $F^*(U) : \mathcal{D}(U) \to \mathcal{C}(U)$ on objects as follows. Fix some arbitrary object $P_0 \in \text{Ob } \mathcal{C}(U)$. For any object $P \in \text{Ob } \mathcal{D}(U)$, define an $\mathcal{A}|_U$-torsor

\[ I = \text{Hom}_U(F(U)(P_0), P) \]

Define $F^*(U)(P) \equiv I_C(U)(P_0)$.

We define $F^*(U)$ on morphisms as follows. Let $\beta : P_a \to P_b$ be a morphism between objects $P_a, P_b \in \text{Ob } \mathcal{D}(U)$. Define $\mathcal{A}|_U$-torsors

\[ I^a \equiv \text{Hom}_U(F(U)(P_0), P_a) \]
\[ I^b \equiv \text{Hom}_U(F(U)(P_0), P_b) \]

The morphism $\beta$ defines a morphism of torsors $\beta^* : I^a \to I^b$, and thus a 2-arrow $\beta^* : I^a_C \Rightarrow I^b_C$. Define

\[ F^*(U)(\beta) \equiv \beta^*(U)(P_0) \]

It is straightforward to check that with these definitions, $F^*(U)$ is a well-defined functor $\mathcal{D}(U) \to \mathcal{C}(U)$.

Before proving that $F^*(U)$ defines an inverse to $F(U)$, we shall briefly attempt to provide some intuition for this result. First, note that for any object $P \in \text{Ob } \mathcal{D}(U)$, there exists a 2-arrow such that

\[ (FF^*)(U)(P) = (FIC)(U)(P_0) \Rightarrow (IF)(U)(P_0) = P \]

Assuming the base space is connected. Locally constant, more generally.

²³Assuming the base space is connected.
This is not quite sufficient to prove that $F$ is an equivalence of categories, because the 2-arrow above depends upon $P$, whereas we need to find a single natural transformation. Similarly, using the fact that

$$\text{Hom}_U(F(U)(P_0), F(U)(P)) \cong \text{Hom}_C(P_0, P)$$

for any object $P \in \text{Ob} \ C(U)$, we see that

$$(F^*F)(U)(P) = I_C(U)(P_0) \cong P$$

Again, the remarks above are not intended to be proofs, but are intended merely to give the reader some intuition as to why our definition of $F^*(U)$ is a correct one.

We shall now construct invertible natural transformations

$$\psi_1 : F(U) \circ F^*(U) \Rightarrow \text{Id}_D(U)$$
$$\psi_2 : F^*(U) \circ F(U) \Rightarrow \text{Id}_C(U)$$

Existence of these natural transformations, together with $F^*(U)$, will suffice to prove that $F(U)$ is an equivalence of categories, and that $F : C \rightarrow D$ is an equivalence of gerbes.

We shall define $\psi_1 : (FF^*)(U) \Rightarrow \text{Id}$ as follows. Let $P_i$ be a family of objects of $D(U)$, one for each equivalence class of objects in $D(U)$. Define a family of $\mathcal{A}|_U$-torsors

$$I^i \equiv \text{Hom}_D(F(U)(P_0), P_i)$$

Fix a family of 2-arrows $\Psi_i$:

$$\Psi_i : (FI^i) \Rightarrow (I^i_D)$$

Now, for any object $P \in \text{Ob} \ D(U)$, let $f : P \rightarrow P_i$ be a morphism from $P$ to some $P_i$, and define $\psi_1(P) : (FF^*)(U)(P) \rightarrow P$ by,

$$\psi_1(P) \equiv f^{-1} \circ \Psi_i(U)(P_0) \circ (FF^*)(U)(f)$$

It is straightforward to check that $\psi_1(P)$ is independent of the choice of $f$, and moreover that $\psi_1(P)$ defines a natural transformation

$$\psi_1 : F(U) \circ F^*(U) \Rightarrow \text{Id}_{D(U)}$$

We shall define $\psi_2 : (F^*F)(U) \Rightarrow \text{Id}$ as follows. Let $P'_i$ be a family of objects of $C(U)$, one for each equivalence class of objects in $C(U)$. Define families of $\mathcal{A}|_U$-torsors

$$I^i \equiv \text{Hom}_C(P_0, P'_i)$$
$$I'^i \equiv \text{Hom}_C(F(U)(P_0), F(U)(P'_i))$$
and fix a family of 2-arrows

\[ \Psi'_i : I'_C \Rightarrow I_C \]

Now, for any object \( P \in \text{Ob} \mathcal{C}(U) \), let \( f : P \to P'_i \) for a morphism from \( P \) to some \( P'_i \), and define \( \psi_2(P) : (F^*F)(U)(P) \to P \) by,

\[ \psi_2(P) \equiv f^{-1} \circ \Psi'_i(U)(P_0) \circ (F^*F)(U)(f) \]

It is straightforward to check that \( \psi_2(P) \) is independent of the choice of \( f \), and moreover that \( \psi_2(P) \) defines a natural transformation

\[ \psi_2 : F^*(U) \circ F(U) \Rightarrow \text{Id}_{\mathcal{C}(U)} \]

Thus, any gerbe map \( F : \mathcal{C} \to \mathcal{D} \) between gerbes of the same band, over the same space, is necessarily an equivalence of gerbes.

### 6 Equivariant gerbes

In this section we shall define the notion of equivariance under a group \( \Gamma \) acting on a space \( X \) for gerbes defined on the space \( X \). We shall (loosely) follow \[6\], section 7.3]. We shall assume that \( X \) is connected. We should also mention that we shall often refer to pullbacks of stacks and gerbes, concepts which were defined in section 4.2. More generally, in both this section and the next, we shall often make use of results proven in sections 4 and 5, without specific attribution.

First, however, we shall define the notion of an equivariant stack. Let \( \Gamma \) be a group acting on a topological space \( X \) by homeomorphisms, and let \( \mathcal{C} \) be a stack on \( X \). Loosely, for \( \mathcal{C} \) to be equivariant under \( \Gamma \) means that \( g^*\mathcal{C} \) should be isomorphic (in the appropriate sense) to \( \mathcal{C} \) for all \( g \in \Gamma \). Technically, an equivariant structure on a stack \( \mathcal{C} \) consists of the following data:

1. A Cartesian functor \( \Phi_g : g^*\mathcal{C} \to \mathcal{C} \) defining an equivalence of stacks, for each \( g \in \Gamma \).
2. For each pair \( (g_1, g_2) \in \Gamma \times \Gamma \), an invertible 2-arrow

\[ \psi_{g_1,g_2} : \Phi_{g_1} \circ g_2^* \Phi_{g_1} \circ \Psi_{g_1,g_2}^C \]

between Cartesian functors \( (g_1g_2)^*\mathcal{C} \to \mathcal{C} \), where \( \Psi_{g_1,g_2}^C : (g_1g_2)^*\mathcal{C} \to g_2^*g_1^*\mathcal{C} \) is the analogue of a natural transformation defined in section 4.2.4.
Moreover, the 2-arrows $\psi_{g_1, g_2}$ are required to make the following diagram commute:

$$
\begin{array}{ccc}
\Phi_{g_1g_2g_3} & \xrightarrow{\psi_{g_1, g_2, g_3}} & \Phi_{g_3} \circ g_3^*\Phi_{g_1, g_2} \circ \Psi_{g_1g_2g_3} \\
\downarrow & & \downarrow \\
\Phi_{g_2g_3} \circ (g_2g_3)^*\Phi_{g_1} \circ \Psi_{g_1g_2g_3} & \xrightarrow{\psi_{g_2, g_3}} & \Phi_{g_3} \circ g_3^*(\Phi_{g_2} \circ g_2^*\Phi_{g_1} \circ \Psi_{g_1, g_2}) \circ \Psi_{g_1g_2g_3}
\end{array}
$$

(28)

Note that in order to make sense out of the diagram above, we are using the fact that the $\Psi$ obey

$$
\Psi_{g_1, g_2} \circ \Psi_{g_1g_2g_3} = \Psi_{g_2g_3} \circ \Psi_{g_1, g_2g_3}
$$

and also that

$$
\Psi_{g_2g_3} \circ (g_2g_3)^*\Phi_{g_1} = g_3^*g_2^*\Phi_{g_1} \circ \Psi_{g_2, g_3}
$$

as is discussed in section 4.2.4.

An equivariant structure on a gerbe is defined to be an equivariant structure on the underlying stack such that each $\Phi_g : g^*\mathcal{C} \to \mathcal{C}$ defines an equivalence of gerbes.

We shall now argue that any two distinct equivariant structures on a gerbe differ by a choice of principal $G$-bundles $T_g$ (for $A = \mathcal{C}^\infty(G)$), one for each $g \in \Gamma$, and a set of isomorphisms of principal $G$-bundles

$$
\omega_{g_1, g_2} : T_{g_1g_2} \longrightarrow T_{g_2} \cdot g_2^*T_{g_1}
$$

such that the following diagram commutes:

$$
\begin{array}{ccc}
T_{g_1g_2g_3} & \xrightarrow{\omega_{g_1, g_2, g_3}} & T_{g_3} \cdot g_3^*T_{g_1g_2} \\
\downarrow & & \downarrow \\
T_{g_2g_3} \cdot (g_2g_3)^*T_{g_1} & \xrightarrow{\omega_{g_2, g_3}} & T_{g_3} \cdot g_3^*(T_{g_2} \cdot g_2^*T_{g_1})
\end{array}
$$

(29)

modulo equivalences of bundles.

In passing, we should point out the close formal resemblance between diagram (29) and structures appearing in [11, section 1].

Suppose we have two distinct equivariant structures on a gerbe, that is, two sets of gerbe maps $\Phi_g, \Phi'_g : g^*\mathcal{C} \to \mathcal{C}$ and corresponding 2-arrows $\psi_{g_1, g_2}, \psi'_{g_1, g_2}$. First, from section 5.3.7, any two gerbe maps $\Phi_g, \Phi'_g$ differ by a gauge transformation, that is, (for band $A = \mathcal{C}^\infty(G)$) there exists a principal $G$-bundle we shall denote $T_g$ and a 2-arrow $\lambda_g : \Phi_g \Rightarrow T_g \circ \Phi'_g$. Note we have used $T_g$ to denote both a principal $G$-bundle and an associated gerbe automorphism.

Note that we have been slightly sloppy – to completely specify a gerbe automorphism associated to the bundle $T_g$, we would also need to specify a choice of local sections. Any two choices of local sections define automorphisms that differ by a 2-arrow, so changing the choice of local sections merely corresponds to changing $\lambda_g$. Thus, we shall not belabor the choice of local sections any further.
Next, let $\omega_{g_1,g_2} : T_{g_1g_2} \Rightarrow T_{g_2} \circ g_2^* T_{g_1}$ be a 2-arrow such that the following diagram commutes:

$$
\begin{array}{ccc}
\Phi_{g_1g_2} & \xrightarrow{\psi_{g_1g_2}} & \Phi_{g_2} \circ g_2^* \Phi_{g_1} \\
\lambda_{g_1g_2} & \Downarrow & \lambda_{g_2} \circ g_2^* \lambda_{g_1} \\
T_{g_1g_2} \circ \Phi'_{g_1g_2} & \xrightarrow{T_{g_1g_2}} & T_{g_1g_2} \circ (\Phi'_{g_2} \circ g_2^* \Phi'_{g_1}) \\
\end{array}
$$

(30)

where $\Psi$ are implicit (omitted for aesthetic reasons), and where we have used $\Upsilon_{1,2}$ to denote a 2-arrow

$$
\Upsilon_{1,2} : (T_{g_2} \circ g_2^* T_{g_1}) \circ (\Phi'_{g_2} \circ g_2^* \Phi'_{g_1} \circ \Psi_{g_1,g_2}) \Rightarrow (T_{g_2} \circ \Phi'_{g_2}) \circ g_2^*(T_{g_1} \circ \Phi'_{g_1}) \circ \Psi_{g_1,g_2}
$$

describing how we commute the gauge transformations past gerbe maps. The $\Upsilon$ should not be assumed to be completely arbitrary; we shall assume that the following diagram of $\Upsilon$ commutes:

$$
\begin{array}{ccc}
(T_{g_3} \cdot g_3^*(T_{g_2} \cdot g_2^* T_{g_1})) \circ \Phi'_{g_3} \circ g_3^* (\Phi'_{g_2} \circ g_2^* \Phi'_{g_1}) & \Rightarrow & T_{g_3} \circ \Phi'_{g_3} \circ (T_{g_2} \cdot g_2^* T_{g_1}) \circ (\Phi'_{g_2} \circ g_2^* \Phi_{g_1}) \\
\Downarrow & & \Downarrow \\
(T_{g_3} \cdot g_3^* T_{g_2}) \circ \Phi'_{g_3} \circ g_3^* (\Phi'_{g_2} \circ g_2^* T_{g_1} \circ g_2^* \Phi'_{g_1}) & \Rightarrow & T_{g_3} \circ \Phi'_{g_3} \circ g_3^* (T_{g_2} \circ \Phi'_{g_2} \circ g_2^* T_{g_1} \circ g_2^* \Phi'_{g_1})
\end{array}
$$

where we have omitted $\Psi_{g_1,g_2}$ and $\Psi_{g_1g_2}$. The specification of the 2-arrow $\omega_{g_1,g_2} : T_{g_1g_2} \Rightarrow T_{g_2} \circ g_2^* T_{g_1}$ is equivalent to a specification of an isomorphism of principal $G$-bundles

$$
\omega_{g_1,g_2} : T_{g_1g_2} \Rightarrow T_{g_2} \cdot g_2^* T_{g_1}
$$

and it is straightforward to check that the requirement that diagram (28) commute for both the $\psi_{g_1,g_2}$ and the $\psi'_{g_1,g_2}$ implies that the following diagram of 2-arrows commutes:

$$
\begin{array}{ccc}
T_{g_1g_2} & \xrightarrow{\omega_{g_1g_2} \cdot g_3} & T_{g_3} \circ g_3^* T_{g_1g_2} \\
\omega_{g_1g_2} \cdot g_3 & \Downarrow & \omega_{g_1g_2} \cdot g_3 \\
T_{g_2g_3} \circ (g_2g_3)^* T_{g_1} & \xrightarrow{T_{g_2g_3}} & T_{g_3} \circ g_3^* (T_{g_2} \circ g_2^* T_{g_1})
\end{array}
$$

(31)

which implies that diagram (29) commutes, as claimed.

To summarize our progress so far, we have discovered that the difference between two equivariant structures on a gerbe is described by the data $(T_g, \omega_{g_1g_2})$ such that diagram (29) commutes. However, we have been a little sloppy. We could replace any of the bundles principal $G$-bundles $T_g$ by isomorphic bundles $T'_g$, as only equivalence classes of principal $G$-bundles are relevant. Iff $\kappa_g : T_g \rightarrow T'_g$ are isomorphisms, then the difference between two equivariant structures can also be described by the data $(T'_g, \kappa_{g_1g_2} \circ \omega_{g_1g_2} \circ (\kappa_{g_2} \circ g_2^* \kappa_{g_1})^{-1})$.

At the end of the day, we will recover a classification of equivariant structures preserving the gerbe connection, in which we shall find $H^2(\Gamma, G)$. Before we begin working out the
details, we should take a moment to explain the general idea. First, since automorphisms of gerbes with connection are defined by equivalence classes of bundles with connection, we will also specify connections on the bundles $T_g$, and the isomorphisms $\omega_{g_1, g_2}$ will be forced to preserve those connections. Then, we will demand that the connection on the gerbe be invariant under all gerbe equivalences, in precise analogy with our strategy for studying equivariant bundles in \cite{2}. This will imply that any two equivariant structures differ by a set of bundles $\{T_g\}$ with flat connection. We find the group $H^2(\Gamma, U(1))$ by taking the bundles $T_g$ to be topologically trivial, not just flat, with gauge-trivial connections. Cocycle representatives of elements of $H^2(\Gamma, U(1))$ will be defined by the isomorphisms $\omega_{g_1, g_2}$; the group cocycle condition will come from commutivity of diagram \cite{24}.

7 Equivariant gerbes with connection

In the previous section we defined the notion of equivariant structure for gerbes. In this section we shall extend this notion to define equivariant structures for gerbes with connection (connective structure and curving). As we have only defined connections for gerbes with band $\mathcal{A} = C^\infty(U(1))$, we shall assume throughout this section that all gerbes have band $C^\infty(U(1))$.

Let $\mathcal{C}$ be a gerbe which is equivariant with respect to the action of a group $\Gamma$ acting by diffeomorphisms. Under what circumstances will the equivariant structure respect the connection on $\mathcal{C}$? As the reader has probably guessed, the gerbe maps $\Phi_g : g^*\mathcal{C} \rightarrow \mathcal{C}$ are required to be equivalences of gerbes with connection, so we must also specify 2-arrows

$$\kappa_g : g^*\mathcal{C} \Rightarrow \mathcal{C} \circ \Phi_g$$

(Recall that the pullback of a gerbe with connection is another gerbe with connection – a connection on $\mathcal{C}$ naturally induces a connection on $g^*\mathcal{C}$. In order for the equivariant structure to respect this connection, we must demand that the equivalence of gerbes $\Phi_g$ respect the connection.)

So far we have defined equivariant structures on gerbes with connection. How are these equivariant structures classified?

Suppose we have two equivariant structures on a gerbe with connection, that is, two sets of gerbe maps $(\Phi_g, \kappa_g), (\Phi'_g, \kappa'_g)$ defining equivalences of gerbes with connection, and corresponding 2-arrows $\psi_{g_1, g_2}, \psi'_{g_1, g_2}$ satisfying the conditions above. How are these two equivariant structures related?

In the previous section, we mentioned that for any pair $\Phi_g, \Phi'_g$, there exists\footnote{U p to isomorphism.} a principal

71
$U(1)$-bundle $T_g$ and a 2-arrow $\lambda_g : \Phi_g \Rightarrow T_g \circ \Phi'_g$, using results in section 5.3.7. Here, because the gerbe maps define equivalences of gerbes with connection, to describe the difference between the gerbe maps we must also specify a connection on each bundle $T_g$, and the connections are constrained to be flat. Moreover, from commutivity of diagram (30), as applied to the connective structures, we find that the morphisms $\omega_{g_1,g_2}$ must preserve the connection on each bundle.

In other words, so far we have found that the difference between two equivariant structures on a gerbe with connection is defined by a set of principal $U(1)$-bundles $T_g$, each with a flat connection, together with connection-preserving isomorphisms $\omega_{g_1,g_2} : T_{g_1} \to T_{g_2} \cdot g_2^* T_{g_1}$ such that diagram (29) commutes.

As before, the bundles with connection are only defined up to isomorphism. If $\kappa_g : T_g \to T'_g$ defines a set of isomorphisms of bundles with connection, then we can replace the data $(T_g, \omega_{g_1,g_2})$ with the data $(T'_g, \kappa_{g_1,g_2} \circ \omega_{g_1,g_2} \circ (\kappa_{g_2} \otimes g_2^* \kappa_{g_1})^{-1})$.

At the end of the day, we wish to find how the group $H^2(\Gamma, U(1))$ appears in describing the difference between two equivariant structures. This group appears as follows. Take all the bundles $T_g$ to be topologically trivial, with gauge-trivial connections. Then, we can use the fact that the bundles $T_g$ are only defined up to isomorphism to replace each $T_g$ with the canonical trivial bundle with identically zero connection. As the isomorphisms $\omega_{g_1,g_2}$ are constrained to preserve the connection, this means they must be constant (assuming the underlying space is connected). From diagram (28) we see that the $\omega_{g_1,g_2}$ define a group 2-cocycle. Now, even after making this choice of $T_g$’s, there is still a residual gauge invariant – we can gauge-transform each $T_g$ by a constant gauge transformation, which preserves the (identically zero) connection on each $T_g$. It is clear that these constant gauge transformations of each $T_g$ change the $\omega_{g_1,g_2}$ by a group coboundary. Thus, we have found elements of the group $H^2(\Gamma, U(1))$ lurking in the differences between any two equivariant structures on a gerbe with connection.

In general, however, there will be additional possible orbifold group actions, beyond those classified by elements of $H^2(\Gamma, U(1))$. In retrospect, we should not be surprised – for example the Cartan-Leray spectral sequence for $H^2(X/\Gamma, \mathbb{Z})$ (for $\Gamma$ freely-acting) contains contributions from more than just $H^2(\Gamma, U(1))$. We shall discuss this matter further in [3].

---

25 We get principal $U(1)$ bundles because we have assumed the band is $C^\infty(U(1))$ in this section. In the previous section we did not have such a constraint on the band.

26 Lest we give the wrong impression, classifying equivariant structures is not the same thing as calculating cohomology, but in very special cases, cohomology calculations can shed light.
8 Check: loop spaces

In this section we shall shed some light on the methods and results of the previous two sections by thinking about gerbes in terms of loop spaces.

First, note that a principal $U(1)$ bundle $P$ with connection on a manifold $M$ determines a $U(1)$-valued function on $LM$, the loop space of $M$. More precisely, for any loop in $M$, we can assign an element of $U(1)$ given by the value of the Wilson loop. Thus, Wilson loops assign elements of $U(1)$ to each loop in $M$, and so define a $U(1)$-valued function on $LM$.

The assignment above, of $U(1)$-valued functions on $LM$ to principal $U(1)$ bundles with connection on $M$, is a basic example of a more general principle. Namely, to any $n$-gerbe with connection on a manifold $M$, one can assign an $(n-1)$-gerbe with connection on the loop space $LM$. (This can be seen in terms of Deligne cohomology; see, for example, [6, section 6.5].) More relevantly to this paper, to any $(1)$-gerbe with connection (and band $C^\infty(U(1))$) on a manifold $M$, one can assign a principal $U(1)$ bundle with connection on the loop space $LM$. (For more details, see, for example, [6, section 6.2]; a derivation of this fact at the level of Deligne cohomology is also given in [25].)

Thus, the reader might naively be led to suspect that an equivariant gerbe on a space $M$ is equivalent to an equivariant bundle on $LM$. Unfortunately, this is not quite correct. The essential difficulty is that, in general, the map from $n$-gerbes on $M$ to $(n-1)$-gerbes on $LM$ is a many-to-one map. For example, consider the map from principal $U(1)$ bundles on $M$ to $U(1)$-valued functions on $LM$ defined by Wilson loops, as described at the beginning of this section. Specifying Wilson loops about every loop on a manifold $M$ (i.e., specifying a $U(1)$-valued function on $LM$) does not uniquely determine a principal $U(1)$ bundle with connection. Instead, such a set of Wilson loops only determines an equivalence class of principal $U(1)$ bundles with connection [26, prop. 1.12.3].

In the present situation, because a $(1)$-gerbe with connection on $M$ can not be uniquely determined by a principal $U(1)$ bundle with connection on $LM$, we can not completely describe equivariant gerbes with connection in terms of equivariant bundles with connection on $LM$.

However, it is true that an equivariant structure on a gerbe with connection on $M$ does determine an equivariant structure on the corresponding bundle with connection on $LM$. We shall merely outline the results. Recall that an equivariant structure on a gerbe $\mathcal{C}$ with

---

27Strictly speaking, to a $(1)$-gerbe with connective structure on $M$, we can assign a bundle on $LM$. A curving on that connective structure can be used to define a connection on the bundle on $LM$. See [6, section 6.2] for more details. In this framework, gerbe maps become morphisms of principal bundles (more generally, torsors) on $LM$, and gerbe maps related by invertible 2-arrows map to the same morphism of principal bundles on $LM$. 

---
A connection is defined by a collection of gerbe maps
\[ \Phi_g : g^*C \rightarrow C \]
and invertible 2-arrows
\[ \psi_{g_1, g_2} : \Phi_{g_1} \circ g_2^* \circ \Phi_{g_1} \circ \psi_{g_1, g_2} \]
subject to various constraints. We shall let \( P \) denote the bundle with connection on \( LM \) corresponding to the gerbe \( C \) on \( M \). The gerbe maps \( \Phi_g \) become bundle isomorphisms \( \phi_g : g^* P \rightarrow P \), and the existence of invertible 2-arrows \( \psi_{g_1, g_2} \) implies that \( \phi_{g_1, g_2} = \phi_{g_1} \circ g_2^* \phi_{g_1} \), which, recall from [2], defines an equivariant structure on the bundle \( P \).

Recall in [2] we pointed out that any equivariant structure on a bundle with connection could be obtained from any other equivariant structure via a set of constant gauge transformations. In the present context, however, there is a slight subtlety. Gauge transformations of \( P \) on \( LM \) are determined by principal \( U(1) \) bundles with connection on \( M \), that is, \( U(1) \)-valued functions on \( LM \). However, not all \( U(1) \)-valued functions on \( LM \) can be understood in terms of Wilson loops on bundles with connection on \( M \). In fact, the only constant \( U(1) \)-valued function on \( LM \) that can be understood in terms of Wilson loops on bundles with connection on \( M \), is the trivial constant function that is the identity for all points on \( LM \). To see this fact, suppose that \( L \) is a principal \( U(1) \) bundle with connection on \( M \) with the property that all Wilson loops are equal to some (single) element of \( U(1) \), call it \( x \). Let \( W(\gamma) \) denote the value of the Wilson loop about any loop \( \gamma \), then it should be clear that
\[ W(2\gamma) = W(\gamma)^2 \]
or, in other words, \( x = x^2 \). But as \( x \in U(1) \), the only way that this can be satisfied is if \( x \) is the identity. Thus, the only constant gauge transformations on \( LM \) that can be understood as coming from bundles with connection on \( M \) are those that are identically the identity.

If \( LM \) is connected, then there is only one constant gauge transformation allowed between equivariant structures on \( P \rightarrow LM \). From this the reader might incorrectly conclude that this must mean that there can only be one equivariant structure on a gerbe with connection. The problem with this argument is that the map from bundles (with connection) on \( M \) to functions on \( LM \) is a many-to-one map. At the level of the loop space \( LM \), there is only one equivariant structure; however, there are actually multiple equivariant structures on the gerbe on \( M \), all of which map to the same equivariant structure on the bundle \( P \) on \( LM \).

To classify equivariant structures on a gerbe with connection, we must return to the analysis of the previous section. In passing, however, there is one slight additional bit of insight that can be gained from thinking in terms of loop spaces. We argued that on \( LM \), for \( LM \) connected, there was only one equivariant structure on the bundle \( P \); any other differed from it by a constant gauge transformation on \( LM \). However, specifying a constant gauge transformation on \( LM \) only determines an equivalence class of bundles with connection on
In the previous section, we found that equivariant structures on a gerbe with connection on \( M \) were determined by a set of isomorphisms of (trivial) bundles with (trivial) connection; here we see that these isomorphisms must preserve the trivial connection, and so must be constant, in order to remain within the equivalence class determined by the trivial constant gauge transformation on \( LM \). This sheds some light on the constraints on these isomorphisms determined in the previous section.

In general, \( LM \) has one component for each element of \( \pi_1(M) \). Differences between equivariant structures on a bundle with connection on a non-simply-connected space are not described by constant gauge transformations, but rather by locally constant gauge transformations. These degrees of freedom correspond to taking the bundles \( T_a \) with connection to be flat, but nontrivial.

9 Conclusions

In this paper we have accomplished two things. First, we have given a thorough review of gerbes in terms of stacks, at a relatively basic level (i.e., without using the language of sites). Second, we have discussed the classification of equivariant structures on (1-)gerbes with connection, and proven that in general the set of such equivariant structures is a torsor under a group that includes \( H^2(\Gamma, U(1)) \), as claimed in [2], providing a simple geometric understanding of discrete torsion.

10 Acknowledgements

We would like to thank P. Aspinwall, D. Freed, A. Knutson, D. Morrison, and R. Plesser for useful conversations.

References

[1] C. Vafa, “Modular invariance and discrete torsion on orbifolds,” Nucl. Phys. B273 (1986) 592-606.

[2] E. Sharpe, “Discrete torsion and gerbes I,” hep-th/9909108.

[3] E. Sharpe, “Discrete torsion,” to appear.
[4] N. Hitchin, “Lectures on special lagrangian submanifolds,” math.DG/9907034, and lectures given at the Harvard Winter School on Vector Bundles, Lagrangian Submanifolds, and Mirror Symmetry, January 1999.

[5] D. Chatterjee, *On the Construction of Abelian Gerbs*, Ph.D. thesis, Cambridge University, 1998.

[6] J. L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Birkhäuser, Boston, 1993.

[7] J.-L. Brylinski and D. A. McLaughlin, “The geometry of degree-four characteristic classes and of line bundles on loop spaces I,” Duke Math. J. **75** (1994) 603-638.

[8] J. Giraud, *Cohomologie Non Abélienne*, Springer-Verlag, Berlin, 1971.

[9] A. Grothendieck, *Revêtements Etale et Groupe Fondamental*, SGA 1, Lecture Notes in Math. **224**, Springer-Verlag, Berlin, 1971.

[10] L. Breen, *On the Classification of 2-Gerbes and 2-Stacks*, Astérisque 225, Société Mathématique de France, 1994.

[11] D. Freed, “Higher algebraic structures and quantization,” Comm. Math. Phys. **159** (1994) 343-398, hep-th/9212113.

[12] M. Murray, “Bundle gerbes,” J. London Math. Soc. (2) **54** (1996) 403-416, dg-ga/9407015.

[13] A. Carey, M. Murray, and B. Wang, “Higher bundle gerbes and cohomology classes in gauge theories,” J. Geom. Phys. **21** (1997) 183-197, hep-th/9511169.

[14] A. Grothendieck, “Technique de descente et théorèmes d’existence en géométrie algébrique, I. Généralités. Descente par morphismes fidèlement plats,” Séminaire Bourbaki **190**.

[15] M. Artin, *Grothendieck Topologies*, Harvard University, 1962.

[16] D. Mumford, “Picard groups of moduli problems,” in *Arithmetical Algebraic Geometry*, ed. by O. F. G. Schilling, Harper & Row, New York, 1965.

[17] D. Husemoller, *Fibre Bundles*, third edition, Springer-Verlag, New York, 1966, 1994.

[18] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.

[19] D. Eisenbud and J. Harris, *Schemes: The Language of Algebraic Geometry*, Wadsworth & Brooks, Pacific Grove, California, 1992.

[20] R. Dijkgraaf, “The mathematics of fivebranes,” hep-th/9810157.
[21] P. Deligne, J. Milne, A. Ogus, and K. Shih, *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Math. 900, Springer-Verlag, Berlin, 1982.

[22] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York, 1978.

[23] B. Iversen, *Lectures on Torsors and Divisors*, Aarhus Universitet Lecture Notes Series 56, 1986.

[24] S. Mac Lane and I. Moerdijk, *Sheaves in Geometry and Logic*, Springer-Verlag, New York, 1992.

[25] K. Gawedzki, “Topological actions in two-dimensional quantum field theories,” pp. 101-141 in *Nonperturbative Quantum Field Theory*, ed. by G. ’t Hooft, A. Jaffe, G. Mack, P. K. Mitter, and R. Stora, NATO ASI Series vol. 185, Plenum Press, New York, 1988.

[26] B. Kostant, “Quantization and unitary representations,” pp. 87-208 of *Lectures in Modern Analysis and Applications III*, R. M. Dudley et al, Lecture Notes in Math. 170, Springer-Verlag, Berlin, 1970.