REPRESENTATION THEOREMS FOR NORMED MODULES

SIMONE DI MARINO, DANKA LUČIĆ, AND ENRICO PASQUALETTO

Abstract. In this paper we study the structure theory of normed modules, which have been introduced by Gigli. The aim is twofold: to extend von Neumann’s theory of liftings to the framework of normed modules, thus providing a notion of precise representative of their elements; to prove that each separable normed module can be represented as the space of sections of a measurable Banach bundle. By combining our representation result with Gigli’s differential structure, we eventually show that every metric measure space (whose Sobolev space is separable) is associated with a cotangent bundle in a canonical way.

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1. Introduction

In recent years, a great deal of attention has been devoted to the study of weakly differentiable objects on abstract metric measure spaces. In this regard, an important contribution is represented by N. Gigli’s book [10], where he proposed a first-order differential calculus tailored to this framework. A key role was played by the notion of normed $L^0$-module, which has been subsequently refined in [11]; such notion is a variant of a similar concept introduced by N. Weaver in [27], who was in turn inspired by J.-L. Sauvageot’s papers [22, 23]. A strictly related notion is that of ‘randomly normed space’, which was extensively studied in [18].

Let $(X, d, m)$ be a metric measure space. We denote by $L^0(m)$ the commutative ring of real-valued Borel functions on $X$, quotiented up to $m$-a.e. equality. Then a normed $L^0(m)$-module is an algebraic $L^0(m)$-module $\mathcal{M}$ equipped with a pointwise norm operator $|\cdot| : \mathcal{M} \to L^0(m)$, which induces a complete distance $d_\mathcal{M}$; we refer to Definition 2.11 for the precise definition.

Roughly speaking, a normed $L^0(m)$-module can be thought of as the space of measurable sections (up to $m$-a.e. equality) of some notion of measurable Banach bundle. Nevertheless, this interpretation might be not entirely correct, since it is currently unknown whether every normed $L^0(m)$-module actually admits this sort of representation. The main purpose of the present paper is, in fact, to show that all those normed $L^0(m)$-modules for which the distance $d_\mathcal{M}$ is separable can be written as spaces of sections of a separable Banach bundle. The result generalises a previous theorem, obtained by the second and third named authors in [20], for normed $L^0(m)$-modules that are ‘locally finitely-generated’, in a suitable sense.

We now pass to a more detailed description of the contents of this manuscript. We point out that, even though in the rest of this introduction we will just focus (for simplicity) on the case of a metric measure space, most of the results can be formulated and proven in the more general framework of $\sigma$-finite measure spaces, as we will see later on in the paper.

Liftings of normed modules. A prototypical example of normed $L^0(m)$-module is the space $L^0(m)$ itself, which is generated by the function constantly equal to 1. Its key feature is that it is possible to ‘take precise representatives’ of its elements, by just considering Borel versions. A similar property is – a priori, at least – not shared by all normed $L^0(m)$-modules, which are intrinsically defined in the $m$-a.e. sense. This non-trivial issue needs to be addressed in order to be able to provide a representation of (separable) normed $L^0(m)$-modules as spaces of sections of measurable Banach bundles, since the latter certainly have this sort of property. The whole §3 is dedicated to achieve such a goal, as we are now going to describe.
We denote by $\mathcal{L}^\infty(\Sigma)$ the space of bounded, $\Sigma$-measurable, real-valued functions on $X$, where $\Sigma$ stands for the completion of the Borel $\sigma$-algebra $\mathcal{B}(X)$. Then there exist linear continuous mappings $\mathcal{L}: L^\infty(m) \to \mathcal{L}^\infty(\Sigma)$, called *liftings*, which preserve products and select $m$-a.e. representatives; this is the statement of a highly non-trivial result by von Neumann–Maharam (cf. §2.1 below). A natural question arises:

Can we generalise von Neumann–Maharam’s theorem to normed modules? \( (1.1) \)

For technical reasons – namely, due to the fact that von Neumann–Maharam’s liftings cannot be defined on $L^0(m)$, according to [25] – we need to work with normed modules over $L^\infty(m)$ and $\mathcal{L}^\infty(\Sigma)$. The former have been introduced in [10], but essentially never used nor studied; we will investigate their properties in §2.5. The latter will be introduced and studied in §3.1. With these tools at our disposal, we will prove (in Theorem 3.5) that, given any normed $L^\infty(m)$-module $\mathcal{M}$, there exist a normed $\mathcal{L}^\infty(\Sigma)$-module $\tilde{\mathcal{M}}$ and a lifting map $\mathcal{L}: \mathcal{M} \to \tilde{\mathcal{M}}$. (Notice that the space $\tilde{\mathcal{M}}$ is not given a priori, but its existence is part of the statement.)

Given that the elements of $\tilde{\mathcal{M}}$ are ‘everywhere defined’, the above result shows that it is possible to select precise representatives of the elements of $\mathcal{M}$, thus providing a positive answer to the question (1.1); the fact of working with normed $L^\infty(m)$-modules rather than normed $L^0(m)$-modules is harmless, thanks to Lemma 2.24. Observe that liftings exist on *every* normed $L^\infty(m)$-module, not just on the ones induced by a separable $L^0(m)$-module. Moreover, given any $x \in X$, we might consider the fiber $\tilde{\mathcal{M}}_x := \chi_x \cdot \tilde{\mathcal{M}}$ of $\tilde{\mathcal{M}}$ at $x$, which turns out to be a Banach space. The properties of these fibers – that we will study in §4.3 – play an important role in §4, in the proof of the representation theorem for separable $L^0(m)$-modules.

**Representation theorems for normed modules.** First of all, we propose in §4.1 a notion of separable Banach bundle: if $\mathbb{B}$ is a given universal separable Banach space – i.e., wherein all separable Banach spaces can be embedded linearly and isometrically – then we define the *separable Banach $\mathbb{B}$-bundles* over $X$ as those weakly measurable (multi-valued) mappings that associate to any $x \in X$ a closed linear subspace of $\mathbb{B}$; see Definition 4.1.

Given a separable Banach $\mathbb{B}$-bundle $E$, there is a natural way to define the space $\Gamma(E)$ of its $L^0(m)$-sections; see Definition 4.2 and (4.7). Moreover, it is straightforward to check that the space $\Gamma(E)$ is a separable normed $L^0(m)$-module, cf. Remark 4.3 and Lemma 4.9.

On the other hand, the difficult task is actually to prove that every separable normed $L^0(m)$-module $\mathcal{M}$ is isomorphic to the space of measurable sections $\Gamma(E)$ of some separable Banach $\mathbb{B}$-bundle $E$ over $X$. Let us briefly outline the strategy that we will adopt:

- Consider the space $\text{R}(\mathcal{M})$ of all bounded elements of $\mathcal{M}$, which is a normed $L^\infty(m)$-module. Fix a sequence $(v_n)_n \subseteq \text{R}(\mathcal{M})$ generating $\mathcal{M}$ and a lifting $\mathcal{L}: \text{R}(\mathcal{M}) \to \tilde{\mathcal{M}}$.
- For any $x \in X$ and $n \in \mathbb{N}$, we can evaluate $\mathcal{L}(v_n)$ at $x$, thus obtaining an element $\mathcal{L}(v_n)_x$ of $\tilde{\mathcal{M}}_x$. The closure $E(x)$ of $\{ \mathcal{L}(v_n)_x : n \in \mathbb{N} \}$ is a Banach subspace of $\tilde{\mathcal{M}}_x$.
- The resulting family $\{ E(x) \}_{x \in X}$ is a measurable collection of separable Banach spaces; namely, even if the spaces $E(x)$ live in different fibers $\tilde{\mathcal{M}}_x$, they depend on $x$ in a measurable way, in a weak sense. This working definition is given in Definition 4.5.
The key step is to embed all the spaces $E(x)$ in the ambient space $B$ in a measurable fashion, thus obtaining a separable Banach $B$-bundle $E$. This technical result, achieved in Proposition 4.6 and Corollary 4.7, follows from a careful analysis of Banach–Mazur theorem (cf. §2.3), which states that $C([0,1])$ is a universal separable Banach space.

The only remaining fact to check is that $\Gamma(E)$ is isomorphic to $M$; cf. Theorem 4.13. Once we have proven that each separable normed $L^0(m)$-module is the space of sections of some separable Banach $B$-bundle, it is natural to investigate the relation between bundles and modules at the level of categories. It turns out that the functor associating to any bundle its module of sections is an equivalence of categories: in analogy with the terminology used in [20], we refer to this result as the Serre–Swan theorem; see Theorem 4.15. In §C, we will introduce the concept of pullback bundle and investigate its relation with the pullback of a normed module, a notion extensively used in the field of nonsmooth differential geometry.

The duality bundle-module described above resembles – and somehow extends – the theory of direct integrals of Hilbert spaces [26]. However, it is worth pointing out that our results provide a full correspondence between the ‘horizontal’ notion of normed module and the ‘(purely) vertical’ notion of Banach bundle, in the sense that we do not need to require the existence of a countable dense family of sections in order to obtain the Serre–Swan theorem.

At a late stage of development of the present paper, we discovered that a comprehensive theory of measurable Banach bundles has been thoroughly studied by A.E. Gutman (see, e.g., the paper [17]). It would be definitely interesting – but outside the scopes of this manuscript – to investigate the relations between our approach and Gutman’s one.

Finally, we will propose in §A an alternative (but less descriptive) method to represent separable normed $L^0(m)$-modules as spaces of sections of some bundle. More precisely, the idea is the following: since separable normed $L^0(m)$-modules can be approximated by finitely-generated ones, a representation theorem can be achieved by using the variant of Serre–Swan theorem for finitely-generated modules that has been proven in [20]. See Theorem A.6 below.

A notion of cotangent bundle on metric measure spaces. As already mentioned at the beginning of this introduction, the interest towards the theory of normed modules was mainly motivated by the development of a significant measure-theoretical tensor calculus on metric measure spaces. Although in this paper we just focus on ‘abstract’ normed modules, let us spend a few words about some possible applications of our results.

Several (essentially) equivalent concepts of Sobolev space on $(X,d,m)$ were introduced and studied in the last two decades, cf. [3,5,8,24]. A common point of all these approaches is that each Sobolev function $f$ is associated with a minimal object $|Df|$, which behaves more like the modulus of some weak differential of $f$ than the differential itself. Due to this reason, Gigli proposed the theory of normed modules with the aim of providing a linear structure underlying the Sobolev calculus; in a few words, the idea was to define the differential $df$ rather than the modulus of the differential. In this regard, the key object is the so-called
cotangent module $L^0(T^*X)$, which can be thought of as the space of measurable 1-forms and contains the differentials of all Sobolev functions. The module $L^0(T^*X)$ has a rich and flexible functional-analytic structure, whence it plays a central role in many works (e.g., [14, 16]).

In §B we propose an alternative viewpoint on the differential structure of a vast class of metric measure spaces (i.e., those having separable Sobolev space, which is a quite mild assumption). By combining our Serre–Swan Theorem 4.15 with the results proven in [10], we show in Theorem B.1 that $(X, d, m)$ is canonically associated with a cotangent bundle $T^*_X$, which does not need the language of normed modules to be formulated. This way, at $m$-almost every point $x \in X$ we obtain a weak notion of cotangent space $T^*_X$, which has a Banach space structure. However, in this manuscript we do not investigate further the possible geometric and analytic consequences of this more ‘concrete’ representation of the cotangent module.

2. Preliminaries

In this section we collect many preliminary definitions and results, which will be needed in the sequel. The contents of §2.1, §2.2, and §2.3 are classical. The material in §2.4 is mostly taken from [10, 11], apart from a few technical statements. The discussion in §2.5 is new, but strongly inspired by already known results.

Throughout the whole paper, we will always tacitly adopt the following convention: given any measurable space $(X, \Sigma)$, it holds that

$$ \{x\} \in \Sigma \quad \text{for every } x \in X. $$

In fact, this assumption plays a role only when considering fibers of normed $L^\infty(\Sigma)$-modules (starting from §3.3). Notice that in the setting of metric measure spaces (cf. §B), the condition in (2.1) is always in force, since all singletons are closed and thus Borel measurable.

2.1. Liftings of measurable functions. Let us recall the concept of ‘lifting of a measure space’, in the sense of [9, Definition 341A]:

**Definition 2.1** (Lifting). Let $(X, \Sigma, m)$ be a measure space. Then a map $\ell: \Sigma \to \Sigma$ is said to be a lifting of the measure $m$ provided the following properties are satisfied:

i) $\ell$ is a Boolean homomorphism, i.e., it holds that $\ell(\emptyset) = \emptyset$, $\ell(X) = X$, and

$$ \ell(A \Delta B) = \ell(A) \Delta \ell(B), $$

$$ \ell(A \cap B) = \ell(A) \cap \ell(B), $$

for every $A, B \in \Sigma$,

ii) $\ell(N) = \emptyset$ for every $N \in \Sigma$ such that $m(N) = 0$,

iii) $m(A \Delta \ell(A)) = 0$ for every $A \in \Sigma$.

Liftings can be proven to exist in high generality. For a proof of the following extremely deep result, we refer to [9, Theorem 341K].

**Theorem 2.2** (von Neumann–Maharam). Let $(X, \Sigma, m)$ be a complete, $\sigma$-finite measure space such that $m(X) > 0$. Then there exists a lifting $\ell: \Sigma \to \Sigma$ of the measure $m$. 
Remark 2.3. We point out that Theorem 2.2 strongly relies upon the Axiom of Choice. Consequently, all our results concerning liftings of normed modules will rely upon the Axiom of Choice as well. Nevertheless, in the study of separable normed modules, the usage of liftings (thus, of the Axiom of Choice) can be avoided; see Remark 4.14 for the details.

Given any measure space \((X, \Sigma, \mu)\), let us denote by \(L^\infty(\Sigma)\) the space of all bounded measurable functions \(\overline{f}: X \to \mathbb{R}\), which is a vector space and a commutative ring with respect to the natural pointwise operations. It turns out that \(L^\infty(\Sigma)\) is a Banach space and a topological ring when endowed with the norm \(\|\overline{f}\|_{L^\infty(\Sigma)} := \sup_{x} |\overline{f}|\). Consider the equivalence relation on \(L^\infty(\Sigma)\) given by \(\mu\)-a.e. equality: for any \(\overline{f}, \overline{g} \in L^\infty(\Sigma)\), we set

\[
\overline{f} \sim_{\mu} \overline{g} \iff \overline{f}(x) = \overline{g}(x) \text{ for } \mu\text{-a.e. } x \in X.
\]

Then we denote by \(L^\infty(\mu)\) the quotient space \(L^\infty(\Sigma)/ \sim_{\mu}\). It holds that \(L^\infty(\mu)\) is a Banach space and a topological ring when endowed with the quotient norm \(\|\overline{f}\|_{L^\infty(\mu)} := \text{ess sup}_{x} |\overline{f}|\).

We denote by \([\cdot]_{\mu}: L^\infty(\Sigma) \to L^\infty(\mu)\) the projection map, which turns out to be a linear and continuous operator. Let us define the family of simple functions \(S_f(\Sigma) \subseteq L^\infty(\Sigma)\) as

\[
S_f(\Sigma) := \left\{ \sum_{i=1}^{n} a_i \chi_{A_i} \middle| n \in \mathbb{N}, (a_i)_{i=1}^{n} \subseteq \mathbb{R}, (A_i)_{i=1}^{n} \subseteq \Sigma \text{ partition of } X \right\}.
\]

Here, \(\chi_A\) stands for the characteristic function of the set \(A \subseteq X\), namely, we set \(\chi_A(x) := 1\) for every \(x \in A\) and \(\chi_A(x) := 0\) for every \(x \in X \setminus A\). It can be readily proved that \(S_f(\Sigma)\) is a dense vector subspace and subring of \(L^\infty(\Sigma)\).

We define \(S_f(\mu) \subseteq L^\infty(\mu)\) as the image of \(S_f(\Sigma)\) under the map \([\cdot]_{\mu}\), namely

\[
S_f(\mu) := \left\{ \sum_{i=1}^{n} a_i [\chi_{A_i}]_{\mu} \middle| n \in \mathbb{N}, (a_i)_{i=1}^{n} \subseteq \mathbb{R}, (A_i)_{i=1}^{n} \subseteq \Sigma \text{ partition of } X \right\}.
\]

Therefore, \(S_f(\mu)\) is a dense vector subspace and subring of \(L^\infty(\mu)\).

As we are going to prove, any lifting of a measure space can be promoted to a ‘lifting of measurable functions’. This statement is taken from [9, Exercise 341X(e)]:

**Theorem 2.4 (Lifting of measurable functions).** Let \((X, \Sigma, \mu)\) be a measure space and let \(\ell: \Sigma \to \Sigma\) be a lifting of the measure \(\mu\). Then there exists a unique linear and continuous operator \(\mathcal{L}: L^\infty(\mu) \to L^\infty(\Sigma)\) such that

\[
\mathcal{L}([\chi_A]_{\mu}) = \chi_{\ell(A)} \quad \text{for every } A \in \Sigma. \tag{2.3}
\]

Moreover, the following properties hold:

i) \(\mathcal{L}\) is an isometry, i.e., \(\|\mathcal{L}(f)\|_{L^\infty(\Sigma)} = \|f\|_{L^\infty(\mu)}\) for every \(f \in L^\infty(\mu)\).

ii) \(\mathcal{L}([c]_{\mu}) = c\) for every constant \(c \in \mathbb{R}\).

iii) \(\mathcal{L}\) is a right inverse of \([\cdot]_{\mu}\), i.e., \([\mathcal{L}(f)]_{\mu} = f\) for every \(f \in L^\infty(\mu)\).

iv) \(\mathcal{L}(fg) = \mathcal{L}(f)\mathcal{L}(g)\) for every \(f, g \in L^\infty(\mu)\).

v) \([\mathcal{L}(f)]_{\mu} = \mathcal{L}([f]_{\mu})\) for every \(f \in L^\infty(\mu)\).

vi) If \(f, g \in L^\infty(\mu)\) satisfy \(f \geq g\) in the \(\mu\)-a.e. sense, then \(\mathcal{L}(f) \geq \mathcal{L}(g)\).
Proof. First of all, we are forced to define the operator \( \mathcal{L} : \text{Sf}(m) \to \overline{\text{Sf}(\Sigma)} \) as follows:

\[
\mathcal{L}\left(\sum_{i=1}^{n} a_i [X_{A_i}]_m\right) := \sum_{i=1}^{n} a_i \chi_{\ell(A_i)} \quad \text{for every } \sum_{i=1}^{n} a_i [X_{A_i}]_m \in \text{Sf}(m).
\] (2.4)

We need to prove the well-posedness of \( \mathcal{L} \). It thus remains to show that

\[
\sum_{i=1}^{n} a_i [X_{A_i}]_m = \sum_{j=1}^{m} b_j [X_{B_j}]_m \implies \sum_{i=1}^{n} a_i \chi_{\ell(A_i)} = \sum_{j=1}^{m} b_j \chi_{\ell(B_j)}.
\] (2.5)

The left-hand side of (2.5) is equivalent to saying that \( m(A_i \cap B_j) = 0 \) for all \( i, j \) with \( a_i \neq b_j \); this implies that \( \ell(A_i) \cap \ell(B_j) = \ell(A_i \cap B_j) = \emptyset \) for any such \( i, j \), which is equivalent to the right-hand side, whence (2.5) is proven. Moreover, it can be readily checked that \( \mathcal{L} \) is linear.

Given any simple function \( f = \sum_{i=1}^{n} a_i [X_{A_i}]_m \in \text{Sf}(m) \), we have that \( |f| = \sum_{i=1}^{n} |a_i| [X_{A_i}]_m \) \( m \)-a.e. and \( |\mathcal{L}(f)| = \sum_{i=1}^{n} |a_i| \chi_{\ell(A_i)} \), thus accordingly

\[
\|\mathcal{L}(f)\|_{L^\infty(\Sigma)} = \sup_X |\mathcal{L}(f)| = \max_{i=1, \ldots, n; \ell(A_i) \neq \emptyset} |a_i| = \max_{i=1, \ldots, n; m(A_i) > 0} |a_i| = \text{ess sup}_X |f| = \|f\|_{L^\infty(m)}.
\]

Then the map \( \mathcal{L} \) is an isometry from \( (\text{Sf}(m), \| \cdot \|_{L^\infty(m)}) \) to \( (\text{Sf}(\Sigma), \| \cdot \|_{L^\infty(\Sigma)}) \), so it can be uniquely extended to a linear isometry \( \mathcal{L} : L^\infty(m) \to L^\infty(\Sigma) \). This proves existence, uniqueness, and item i). Item ii) immediately follows from the 1-homogeneity of \( \mathcal{L} \) and (2.3).

To prove item iii), it suffices to observe that \([\mathcal{L}(f)]_m = f\) holds for every \( f \in \text{Sf}(m)\) by construction, thus also for any \( f \in L^\infty(m) \) by continuity of \([\cdot]_m\) and \( \mathcal{L} \). Item iv) can be proved for \( f, g \in \text{Sf}(m) \) by direct computation: if \( f = \sum_{i=1}^{n} a_i [X_{A_i}]_m \) and \( g = \sum_{j=1}^{m} b_j [X_{B_j}]_m \), then \( fg = \sum_{i,j} a_i b_j [X_{A_i \cap B_j}]_m \), so that accordingly

\[
\mathcal{L}(fg) = \sum_{i,j} a_i b_j \chi_{\ell(A_i \cap B_j)} = \left(\sum_{i=1}^{n} a_i \chi_{\ell(A_i)}\right) \left(\sum_{j=1}^{m} b_j \chi_{\ell(B_j)}\right) = \mathcal{L}(f) \mathcal{L}(g).
\]

Since \( L^\infty(m) \) and \( L^\infty(\Sigma) \) are topological rings, we deduce from the density of \( \text{Sf}(m) \) in \( L^\infty(m) \) that \( \mathcal{L}(fg) = \mathcal{L}(f) \mathcal{L}(g) \) holds for every \( f, g \in L^\infty(m) \), thus proving item iv) in full generality. Item v) can be readily checked for \( f \in \text{Sf}(m) \), whence the general case follows from an approximation argument. Finally, let us prove item vi). By linearity of \( \mathcal{L} \), it suffices to consider \( f \in L^\infty(m) \) such that \( f \geq 0 \) in the \( m \)-a.e. sense. Choose any sequence \((f_n)_n \subseteq \text{Sf}(m)\) such that \( f_n \geq 0 \) holds \( m \)-a.e. for each \( n \in \mathbb{N} \) and \( \lim_n \|f_n - f\|_{L^\infty(m)} = 0 \). Given that \( \mathcal{L} \) is continuous and \( \mathcal{L}(f_n) \geq 0 \) holds for all \( n \in \mathbb{N} \) by construction, we conclude that \( \mathcal{L}(f) \geq 0 \), as it is a uniform limit of non-negative functions. The proof of the statement is complete. \( \Box \)

2.2. Measurable correspondences. Aim of this subsection is to recall some basic definitions and results concerning measurable correspondences. The whole material we are going to discuss can be found, e.g., in [1].

Let \( (X, \Sigma) \) be a measurable space and let \( (Y, d_Y) \) be a separable metric space. Then any map \( \varphi : X \to 2^Y \) is said to be a correspondence from \( X \) to \( Y \) and is denoted by \( \varphi : X \to Y \).

**Definition 2.5 (Measurable correspondence).** A correspondence \( \varphi : X \to Y \) is said to be:
a) weakly measurable, provided \( \{ x \in X : \varphi(x) \cap U \neq \emptyset \} \in \Sigma \) for every \( U \subseteq Y \) open,
b) measurable, provided \( \{ x \in X : \varphi(x) \cap C \neq \emptyset \} \in \Sigma \) for every \( C \subseteq Y \) closed.

Let us collect a few important properties of (weakly) measurable correspondences:

i) Every measurable correspondence from \( X \) to \( Y \) is weakly measurable. Conversely, every weakly measurable correspondence from \( X \) to \( Y \) with compact values is measurable.

ii) Let \( \varphi : X \to Y \) be a single-valued correspondence, i.e., for every \( x \in X \) there exists an element \( \varphi(x) \in Y \) such that \( \varphi(x) = \{ \varphi(x) \} \). Then \( \varphi \) is a measurable correspondence if and only if \( \varphi : X \to Y \) is a measurable map.

iii) Let \( \varphi, \varphi' : X \to Y \) be two measurable correspondences with compact values. Then the intersection correspondence \( \varphi \cap \varphi' : X \to Y \), defined as \( (\varphi \cap \varphi')(x) := \varphi(x) \cap \varphi'(x) \) for every \( x \in X \), is measurable.

iv) A correspondence \( \varphi : X \to Y \) with non-empty values is weakly measurable if and only if \( X \ni x \mapsto d_Y(y, \varphi(x)) \) is a measurable function for every \( y \in Y \).

v) Let \( f : X \times Y \to \mathbb{R} \) be a Carathéodory function, i.e.,
\[
\begin{align*}
  f(\cdot, y) : X \to \mathbb{R} & \quad \text{is measurable for every } y \in Y, \\
  f(x, \cdot) : Y \to \mathbb{R} & \quad \text{is continuous for every } x \in X.
\end{align*}
\]
Define the correspondence \( \varphi : X \to Y \) as \( \varphi(x) := \{ y \in Y : f(x, y) = 0 \} \) for every \( x \in X \). If the metric space \( (Y, d_Y) \) is compact, then \( \varphi \) is a measurable correspondence.

vi) Let \( \varphi : X \to Y \) be a weakly measurable correspondence with closed values. Then the graph of \( \varphi \) is measurable, i.e., it holds that
\[
\{(x, y) \in X \times Y \mid y \in \varphi(x)\} \in \Sigma \otimes \mathcal{B}(Y).
\]

vii) Kuratowski–Ryll-Nardzewski theorem. Suppose that the metric space \( (Y, d_Y) \) is complete. Let \( \varphi : X \to Y \) be a weakly measurable correspondence with non-empty closed values. Then \( \varphi \) admits a measurable selector \( s : X \to Y \), i.e., the map \( s \) is measurable and satisfies \( s(x) \in \varphi(x) \) for every \( x \in X \).

viii) Let \( \varphi : X \to Y \) be a weakly measurable correspondence. Then its closure correspondence \( \text{cl}_Y(\varphi) : X \to Y \), which is defined as \( \text{cl}_Y(\varphi)(x) := \text{cl}_Y(\varphi(x)) \) for every \( x \in X \), is weakly measurable.

Furthermore, we will also need the following standard results about preimages and compositions of measurable correspondences:

**Lemma 2.6.** Let \( (X, \Sigma) \) be a measurable space. Let \( (Y, d_Y), (Z, d_Z) \) be separable metric spaces, with \( (Y, d_Y) \) compact. Let \( \varphi : X \to Z \) be a measurable correspondence and let \( \psi : Y \to Z \) be a continuous map. Define the preimage correspondence \( \psi^{-1}(\varphi) : X \to Y \) as
\[
\psi^{-1}(\varphi)(x) := \psi^{-1}(\varphi(x)) \subseteq Y \quad \text{for every } x \in X.
\]
Then \( \psi^{-1}(\varphi) \) is a measurable correspondence.
Proof. Let \( C \subseteq Y \) be a closed set. In particular, \( C \) is compact, whence \( \psi(C) \subseteq Z \) is compact (and thus closed) by continuity of \( \psi \). Therefore, the measurability of \( \varphi \) grants that
\[
\{ x \in X \mid \psi^{-1}(\varphi)(x) \cap C \neq \emptyset \} = \{ x \in X \mid \varphi(x) \cap \psi(C) \neq \emptyset \} \in \Sigma.
\]
By arbitrariness of \( C \), we conclude that \( \psi^{-1}(\varphi) \) is a measurable correspondence. \( \square \)

**Lemma 2.7.** Let \( (X, \Sigma_X) \), \( (Y, \Sigma_Y) \) be measurable spaces and \( (Z, d_Z) \) a separable metric space. Let \( u : X \to Y \) be a measurable map and \( \varphi : Y \to Z \) a weakly measurable correspondence. Consider the correspondence \( \varphi \circ u : X \to Z \), given by
\[
(\varphi \circ u)(x) := \varphi(u(x)) \subseteq Z \quad \text{for every } x \in X.
\]
Then \( \varphi \circ u \) is a weakly measurable correspondence.

**Proof.** Let \( U \subseteq Z \) be an open set. Since \( \{ y \in Y : \varphi(y) \cap U \neq \emptyset \} \in \Sigma_Y \) and \( u \) is measurable, it holds that
\[
\{ x \in X \mid (\varphi \circ u)(x) \cap U \neq \emptyset \} = u^{-1}\{ y \in Y \mid \varphi(y) \cap U \neq \emptyset \} \in \Sigma_X.
\]
By arbitrariness of \( U \), we conclude that \( \varphi \circ u \) is a weakly measurable correspondence. \( \square \)

2.3. **Linear isometric embeddings of separable Banach spaces.** In this section we collect some results about linear isometric embeddings of Banach spaces.

Given any Banach space \( E \), we shall denote by \( B_E \) its closed unit ball \( \{ v \in E : \|v\|_E \leq 1 \} \).

We use the notation \( E' \) to denote the continuous dual space of \( E \), which is a Banach space. We begin by recalling some classical definitions and results (which can be found, e.g., in [1]):

i) **Cantor set.** The Cantor set is the product \( \Delta := \{0, 2\}^\mathbb{N} \), where each factor \( \{0, 2\} \) is endowed with the discrete topology. The topology of \( \Delta \) is induced by the distance
\[
d_\Delta(a, b) := \sum_{n=1}^{\infty} \frac{|a_n - b_n|}{3^n} \quad \text{for every } a = (a_n)_n, b = (b_n)_n \in \Delta.
\]

It holds that the Cantor set \( \Delta \) is compact. Moreover, \( \Delta \) is homeomorphic to the closed subset \( C := \{ \sum_{n=1}^{\infty} a_n/3^n : a \in \Delta \} \) of \([0, 1]\) via the map \( \Delta \ni a \mapsto \sum_{n=1}^{\infty} a_n/3^n \in C \).

ii) **Hilbert cube.** The Hilbert cube is the product topological space \( I^\infty := [-1, 1]^\mathbb{N} \). It is compact and its topology is induced by the distance
\[
d_{I^\infty}(\alpha, \beta) := \sum_{k=1}^{\infty} \frac{|\alpha_k - \beta_k|}{2^k} \quad \text{for every } \alpha = (\alpha_k)_k, \beta = (\beta_k)_k \in I^\infty.
\]

Moreover, there exists a continuous surjective map \( \psi : \Delta \to I^\infty \).
iv) Let $E$ be a separable Banach space. Let $(v_k)_{k \in \mathbb{N}}$ be a fixed dense subset of $B_E$. Then the map $\iota: B_{E'} \to I^\infty$, defined as $\iota(\omega) := (\omega[v_k])_k$ for every $\omega \in B_{E'}$, is a homeomorphism with its image (when the domain $B_{E'}$ is endowed with the restricted weak* topology).

v) Let $K$ be a compact Hausdorff topological space. Then the space $C(K)$ of real-valued continuous functions on $K$ is a separable Banach space if endowed with the norm

$$
\|g\|_{C(K)} := \sup_{t \in K} |g(t)| \quad \text{for every } g \in C(K).
$$

We now recall the classical Banach–Mazur theorem [7], which states that any separable Banach space can be embedded linearly and isometrically into $C([0,1])$. We also report the proof of this result, as the explicit construction of such an embedding will be needed in §4.

**Theorem 2.8** (Banach–Mazur). Let $E$ be a separable Banach space. Then there exists a linear isometric embedding $I: E \to C([0,1])$.

**Proof.** Let $\iota: B_{E'} \to I^\infty$ be a continuous embedding as in item iv). Let $\psi: \Delta \to I^\infty$ be a continuous surjection as in item iii). Note that $\psi^{-1}(\iota(B_{E'})) \subseteq \Delta$ is closed (thus compact) by continuity of $\psi$ and weak* compactness of $B_{E'}$ (the latter is granted by Banach–Alaoglu theorem). Then consider a retraction $r: \Delta \to \psi^{-1}(\iota(B_{E'}))$ as in item ii). We claim that the operator $I': E \to C(\Delta)$, which is defined as

$$
I'[v](a) := (\iota^{-1} \circ \psi \circ r)(a)[v] \quad \text{for every } v \in E \text{ and } a \in \Delta,
$$

is well-posed, linear, and isometric. The fact that $I'[v] \in C(\Delta)$ for every $v \in E$ can be easily checked: if a sequence $(a^i)_i \subseteq \Delta$ converges to some element $a \in \Delta$, then $(\iota^{-1} \circ \psi \circ r)(a^i)$ converges to $(\iota^{-1} \circ \psi \circ r)(a)$ in the weak* topology, and accordingly $I'[v](a) = \lim_i I'[v](a^i)$. Linearity of $I'$ immediately follows from the definition. Moreover, it holds that

$$
\|I'[v]\|_{C(\Delta)} = \sup_{a \in \Delta} |I'[v](a)| = \sup_{\omega \in B_{E'}} |\omega[v]| = \|v\|_E \quad \text{for every } v \in E,
$$

where the last equality follows from the fact that the canonical embedding of the space $E$ into its bidual $E''$ is an isometric operator. This shows that the map $I'$ is an isometry.

Finally, denote by $h: C \to \Delta$ the homeomorphism described in item i) and write $[0,1] \setminus C$ as a disjoint union $\bigsqcup_{i \in \mathbb{N}} (t_i, s_i)$. We define the map $e: C(\Delta) \to C([0,1])$ in the following way: given any $g \in C(\Delta)$, we set

$$
e(g)(t) := \begin{cases} 
(g \circ h)(t) & \text{if } t \in C, \\
(g \circ h)(t_i) + \frac{t - t_i}{s_i - t_i}(g \circ h)(s_i) & \text{if } t \in (t_i, s_i) \text{ for some } i \in \mathbb{N}.
\end{cases}
$$

It holds that $e$ is linear and isometric. Then the map $I: E \to C([0,1])$, given by $I := e \circ I'$, is a linear isometric embedding as well. Therefore, the statement is achieved. $\square$

The statement of Theorem 2.8 can be reformulated by using the following definition:
Definition 2.9 (Universal separable Banach space). A separable Banach space $\mathcal{B}$ is said to be a universal separable Banach space (up to linear isometry) provided for any separable Banach space $E$ there exists a linear isometric embedding $1: E \to \mathcal{B}$.

Therefore, Theorem 2.8 reads as follows: $C([0,1])$ is a universal separable Banach space.

Remark 2.10. As a byproduct of the proof of Theorem 2.8, we see that $C(\Delta)$ is a universal separable Banach space. □

2.4. Normed $L^0(\mathfrak{m})$-modules. Let $(X, \Sigma, \mathfrak{m})$ be a given $\sigma$-finite measure space. We shall denote by $L^0(\mathfrak{m})$ the space of all equivalence classes (up to $\mathfrak{m}$-a.e. equality) of measurable functions from $X$ to $\mathbb{R}$, which is a vector space and a commutative ring with respect to the natural pointwise operations.

Given a probability measure $\mathfrak{m}'$ on $(X, \Sigma)$ satisfying $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ – for instance, pick any sequence $(A_n)_n \subseteq \Sigma$ such that $0 < \mathfrak{m}(A_n) < +\infty$ for every $n \in \mathbb{N}$ and $X = \bigcup_{n \in \mathbb{N}} A_n$, and consider the measure $\mathfrak{m}' := \sum_{n \in \mathbb{N}} \frac{\mathfrak{m}(A_n)}{\mathfrak{m}'(A_n)}$ on $X$ – we define the complete distance $d_{L^0(\mathfrak{m})}$ as

$$d_{L^0(\mathfrak{m})}(f, g) := \int |f - g| \wedge 1 \, d\mathfrak{m}'$$

for every $f, g \in L^0(\mathfrak{m})$.

It turns out that $L^0(\mathfrak{m})$ is a topological vector space and a topological ring when endowed with the distance $d_{L^0(\mathfrak{m})}$. The distance $d_{L^0(\mathfrak{m})}$ depends on the chosen measure $\mathfrak{m}'$, but its induced topology does not. Observe also that the space $L^\infty(\mathfrak{m})$ is $d_{L^0(\mathfrak{m})}$-dense in $L^0(\mathfrak{m})$.

We recall the notion of normed $L^0(\mathfrak{m})$-module, which has been introduced by Gigli in [10]:

Definition 2.11 (Normed $L^0(\mathfrak{m})$-module). Let $(X, \Sigma, \mathfrak{m})$ be a $\sigma$-finite measure space. Then a normed $L^0(\mathfrak{m})$-module is a couple $(\mathcal{M}, | \cdot |)$ with the following properties:

i) $\mathcal{M}$ is an algebraic $L^0(\mathfrak{m})$-module.

ii) The map $| \cdot |: \mathcal{M} \to L^0(\mathfrak{m})$, which is called a pointwise norm on $\mathcal{M}$, satisfies

$$|v| \geq 0 \quad \text{for every } v \in \mathcal{M}, \text{ with equality if and only if } v = 0,$$

$$|v + w| \leq |v| + |w| \quad \text{for every } v, w \in \mathcal{M},$$

$$|f : v| = |f||v| \quad \text{for every } f \in L^0(\mathfrak{m}) \text{ and } v \in \mathcal{M},$$

where all (in)equalities are intended in the $\mathfrak{m}$-a.e. sense.

iii) The distance $d_{\mathcal{M}}$ on $\mathcal{M}$ associated with $| \cdot |$, which is defined as

$$d_{\mathcal{M}}(v, w) := d_{L^0(\mathfrak{m})}(|v - w|, 0)$$

for every $v, w \in \mathcal{M}$, is complete.

Remark 2.12 (Locality/glueing property). Let $(X, \Sigma, \mathfrak{m})$ be a $\sigma$-finite measure space. Then any normed $L^0(\mathfrak{m})$-module $\mathcal{M}$ has the following properties:

• LOCALITY. If $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ is a partition of $X$ and $v, w \in \mathcal{M}$ are two elements such that $[X_{A_n}]_m \cdot v = [X_{A_n}]_m \cdot w$ holds for every $n \in \mathbb{N}$, then $v = w$. 

• **Glueing.** If \((A_n)_{n\in\mathbb{N}} \subseteq \Sigma\) is a partition of \(X\) and \((v_n)_{n\in\mathbb{N}}\) is any sequence in \(\mathcal{M}\), then there exists \(v \in \mathcal{M}\) such that \([\chi_{A_n}]_m \cdot v = [\chi_{A_n}]_m \cdot v_n\) for every \(n \in \mathbb{N}\). The element \(v\) — which is uniquely determined by the locality property — will be denoted by \(\sum_{n\in\mathbb{N}} [\chi_{A_n}]_m \cdot v_n\). It holds \(d_\mathcal{M} (\sum_{n=1}^N [\chi_{A_n}]_m \cdot v_n, \sum_{n\in\mathbb{N}} [\chi_{A_n}]_m \cdot v_n) \to 0 \) as \(N \to \infty\).

Both the locality property and the glueing property have been proved in [10].

Let us fix some useful notation about normed \(L^0(\mathfrak{m})\)-modules:

a) **Morphism.** A map \(\Phi: \mathcal{M} \to \mathcal{N}\) between two normed \(L^0(\mathfrak{m})\)-modules \(\mathcal{M}, \mathcal{N}\) is said to be a morphism provided it is an \(L^0(\mathfrak{m})\)-linear contraction, i.e.,

\[
\Phi(f \cdot v + g \cdot w) = f \cdot \Phi(v) + g \cdot \Phi(w)
\]

for all \(f, g \in L^0(\mathfrak{m})\) and \(v, w \in \mathcal{M}\),

\[
|\Phi(v)| \leq |v| \quad \text{m-a.e. for all } v \in \mathcal{M}.
\]

This allows us to speak about the category \(\text{NMod}(X, \Sigma, \mathfrak{m})\) of normed \(L^0(\mathfrak{m})\)-modules.

b) **Dual.** The dual of \(\mathcal{M}\) is defined as the space \(\mathcal{M}^*\) of all \(L^0(\mathfrak{m})\)-linear continuous maps from \(\mathcal{M}\) to \(L^0(\mathfrak{m})\). It holds that \(\mathcal{M}^*\) is a normed \(L^0(\mathfrak{m})\)-module if endowed with the natural operations and the pointwise norm

\[
|\omega| := \text{ess sup} \left\{ \omega(v) \mid v \in \mathcal{M}, |v| \leq 1 \text{ m-a.e.} \right\} \in L^0(\mathfrak{m}) \quad \text{for all } \omega \in \mathcal{M}^*.
\]

c) **Generators.** We say that a set \(S \subseteq \mathcal{M}\) generates \(\mathcal{M}\) on some set \(A \in \Sigma\) provided the smallest (algebraic) \(L^0(\mathfrak{m})\)-module containing \([\chi_A]_m \cdot S\) is dense in \([\chi_A]_m \cdot \mathcal{M}\).

d) **Linear Independence.** Some elements \(v_1, \ldots, v_n \in \mathcal{M}\) are said to be independent on a set \(A \in \Sigma\) provided for any \(f_1, \ldots, f_n \in L^0(\mathfrak{m})\) one has \(\sum_{i=1}^n [\chi_A]_m f_i \cdot v_i = 0\) if and only if \(f_1, \ldots, f_n = 0\) holds m-a.e. on \(A\).

e) **Local Basis.** We say that \(v_1, \ldots, v_n \in \mathcal{M}\) constitute a local basis for \(\mathcal{M}\) on \(A\) provided they are independent on \(A\) and \(\{v_1, \ldots, v_n\}\) generates \(\mathcal{M}\) on \(A\).

f) **Local Dimension.** The module \(\mathcal{M}\) has local dimension equal to \(n \in \mathbb{N}\) on \(A\) if it admits a local basis on \(A\) made of exactly \(n\) elements.

g) **Dimensional Decomposition.** It holds that the normed \(L^0(\mathfrak{m})\)-module \(\mathcal{M}\) admits a unique dimensional decomposition \(\{D_n\}_{n \in \mathbb{N} \cup \{\infty\}} \subseteq \Sigma\), i.e., \(\mathcal{M}\) has local dimension \(n\) on \(D_n\) for every \(n \in \mathbb{N}\) and is not finitely-generated on any measurable subset of \(D_\infty\) having positive \(\mathfrak{m}\)-measure. Uniqueness here is intended up to \(\mathfrak{m}\)-negligible sets.

h) **Proper Module.** A normed \(L^0(\mathfrak{m})\)-module \(\mathcal{M}\), whose dimensional decomposition is denoted by \(\{D_n\}_{n \in \mathbb{N} \cup \{\infty\}}\), is said to be proper provided \(\mathfrak{m}(D_\infty) = 0\).

We refer to [10, 11] for a thorough discussion about normed \(L^0(\mathfrak{m})\)-modules.

**Definition 2.13** (Countably-generated module). Let \((X, \Sigma, \mathfrak{m})\) be any \(\sigma\)-finite measure space. Let \(\mathcal{M}\) be a normed \(L^0(\mathfrak{m})\)-module. Then we say that \(\mathcal{M}\) is countably-generated provided there exists a countable family \(\mathcal{C} \subseteq \mathcal{M}\) that generates \(\mathcal{M}\) (on \(X\)).

We call \(\text{NMod}_{\text{cg}}(X, \Sigma, \mathfrak{m})\) the category of countably-generated normed \(L^0(\mathfrak{m})\)-modules. Another class of modules we are interested in is that of separable normed \(L^0(\mathfrak{m})\)-modules, i.e., those normed \(L^0(\mathfrak{m})\)-modules \(\mathcal{M}\) for which \((\mathcal{M}, d_\mathcal{M})\) is a separable metric space. We denote
by \(\text{NMod}_{d}(X, \Sigma, m)\) the category of separable normed \(L^0(m)\)-modules. In the forthcoming discussion, we investigate the relation between countably-generated and separable modules.

Let \((X, \Sigma, m)\) be a \(\sigma\)-finite measure space. Given any \(A, B \in \Sigma\), we declare that \(A \sim_m B\) if and only if \(m(A \Delta B) = 0\). This way, we obtain an equivalence relation \(\sim_m\) on \(\Sigma\). Given any finite measure \(m'\) on \((X, \Sigma)\) such that \(m \ll m' \ll m\) (thus in particular \(\sim_m\) and \(\sim_m'\) coincide), we define the distance \(d_{m'}\) on the quotient set \(\Sigma/\sim_m\) as
\[
d_{m'}([A]_{\sim_m}, [B]_{\sim_m}) := m'(A \Delta B)
\]
for every \([A]_{\sim_m}, [B]_{\sim_m} \in \Sigma/\sim_m\).

Then we say that the measure space \((X, \Sigma, m)\) is separable provided \((\Sigma/\sim_m, d_{m'})\) is a separable metric space for some finite measure \(m'\) on \((X, \Sigma)\) such that \(m \ll m' \ll m\).

**Lemma 2.14.** Let \((X, \Sigma, m)\) be a given \(\sigma\)-finite measure space. Then \((X, \Sigma, m)\) is a separable measure space if and only if \((L^0(m), d_{L^0(m)})\) is separable.

**Proof.**

**Necessity.** Suppose \((X, \Sigma, m)\) is separable. Choose any probability measure \(m'\) on \((X, \Sigma)\) such that \(m \ll m' \ll m\) and \(d_{m'}\) is separable. Observe that \(L^0(m')\) and \(L^0(m)\) coincide. Fix a countable \(d_{m'}\)-dense subset \(\mathcal{C}\) of \(\Sigma/\sim_m\). Now consider \(f \in L^0(m)\) and \(\varepsilon > 0\). As already observed, we can find a simple function \(g \in \mathcal{S}(m)\) – see (2.2) – such that \(d_{L^0(m)}(f, g) < \varepsilon/2\).

Say that \(g = \sum_{i=1}^{n} a_i [\chi_{A_i}]_m\). Without loss of generality, we can assume that \(a_1, \ldots, a_n \in \mathbb{Q}\). For any \(i = 1, \ldots, n\), pick a set \(B_i \subset \Sigma\) such that \([B_i]_{\sim_m} \in \mathcal{C}\) and \(m'(B_i \Delta A_i) \leq \varepsilon/(2|a_i|n)\).

Therefore, it holds that
\[
d_{L^0(m)}\left(g, \sum_{i=1}^{n} a_i [\chi_{B_i}]_m\right) = \sum_{i=1}^{n} |a_i| \int |[\chi_{A_i}]_m - [\chi_{B_i}]_m| \, dm' = \sum_{i=1}^{n} |a_i| m'(A_i \Delta B_i) \leq \frac{\varepsilon}{2}.
\]

This means that the function \(h := \sum_{i=1}^{n} a_i [\chi_{B_i}]_m\) satisfies \(d_{L^0(m)}(f, h) < \varepsilon\). Given that
\[
\left\{ \sum_{i=1}^{n} a_i [\chi_{B_i}]_m \mid n \in \mathbb{N}, (a_i)_{i=1}^{n} \subseteq \mathbb{Q}, (B_i)_{i=1}^{n} \subseteq \mathcal{C} \right\}
\]
is a countable family, we conclude that \(L^0(m)\) is separable, as desired.

**Sufficiency.** Suppose \(L^0(m)\) is separable. Observe that the map
\[
(\Sigma/\sim_m, d_{m'}) \ni [A]_{\sim_m} \mapsto [\chi_A]_m \in (L^0(m), d_{L^0(m)})
\]
is an isometry. Consequently, we conclude that \((\Sigma/\sim_m, d_{m'})\) is separable, as required. \(\square\)

Observe that, trivially, any separable normed \(L^0(m)\)-module \(\mathcal{M}\) is countably-generated. The following result aims at determining in which cases the converse implication is satisfied.

**Proposition 2.15.** Let \((X, \Sigma, m)\) be any \(\sigma\)-finite measure space. Then a countably-generated normed \(L^0(m)\)-module \(\mathcal{M}\) is separable if and only if \((X, \Sigma, m|_{X \setminus D_0})\) is a separable measure space, where \(\{D_n\}_{n \in \mathbb{N}, |l| < \infty} \subset \Sigma\) stands for the dimensional decomposition of the module \(\mathcal{M}\).

In particular, if \((X, \Sigma, m)\) is a separable measure space, then any countably-generated normed \(L^0(m)\)-module is separable.
Proof. Suppose \((X, \Sigma, m|_{X\setminus D_0})\) is separable. Let \(C\) be a countable set generating \(\mathcal{M}\), i.e.,
\[
\left\{ \sum_{i=1}^{n} f_i \cdot v_i \left| n \in \mathbb{N}, (f_i)_{i=1}^{n} \subseteq L^0(m|_{X\setminus D_0}), (v_i)_{i=1}^{n} \subseteq C \right. \right\} \text{ is dense in } \mathcal{M}.
\]
(2.8)

Lemma 2.14 ensures that the space \(L^0(m|_{X\setminus D_0})\) is separable, thus we can pick a countable dense subset \(D\) of \(L^0(m|_{X\setminus D_0})\). Since the multiplication map \(L^0(m) \times \mathcal{M} \ni (f, v) \mapsto f \cdot v \in \mathcal{M}\) is continuous, we immediately conclude from (2.8) that the countable family
\[
\left\{ \sum_{i=1}^{n} f_i \cdot v_i \left| n \in \mathbb{N}, (f_i)_{i=1}^{n} \subseteq D, (v_i)_{i=1}^{n} \subseteq C \right. \right\} \text{ is dense in } \mathcal{M}.
\]

This proves that the metric space \((\mathcal{M}, d_{\mathcal{M}})\) is separable, as desired.

Conversely, suppose \(\mathcal{M}\) is separable. It can be readily checked that there exists \(v \in \mathcal{M}\) such that \(|v| = 1\) holds \(m\)-a.e. on \(X \setminus D_0\). (Notice that \([\chi_{D_0}]_m \cdot v = 0\) by definition of \(D_0\).) Therefore, it holds that the module generated by \(\{v\}\) can be identified with \(L^0(m|_{X\setminus D_0})\) (considered as a normed \(L^0(m)\)-module), whence the space \(L^0(m|_{X\setminus D_0})\) is separable. We can conclude that \((X, \Sigma, m|_{X\setminus D_0})\) is separable by Lemma 2.14, thus completing the proof. \(\Box\)

In §A we shall need the concept of direct limit of normed \(L^0(m)\)-modules. The following result – which is taken from [21, Theorem 2.1] – states that direct limits always exist in the category of normed \(L^0(m)\)-modules. (Actually, the statement below was proven in the case in which \(X\) is a Polish space, \(\Sigma\) is the Borel \(\sigma\)-algebra of \(X\), and \(m\) is a Radon measure on \(X\); however, the very same proof works in the more general case of a \(\sigma\)-finite measure space.)

**Theorem 2.16** (Direct limits of normed \(L^0\)-modules). Let \((X, \Sigma, m)\) be any \(\sigma\)-finite measure space. Let \(\{(\mathcal{M}_i)_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\}\) be a direct system of normed \(L^0(m)\)-modules, i.e.,

i) \((I, \leq)\) is a directed set,

ii) \(\{\mathcal{M}_i : i \in I\}\) is a family of normed \(L^0(m)\)-modules,

iii) \(\{\varphi_{ij} : i, j \in I, i \leq j\}\) is a family of normed \(L^0(m)\)-module morphisms \(\varphi_{ij} : \mathcal{M}_i \to \mathcal{M}_j\) such that \(\varphi_{ii} = \text{id}_{\mathcal{M}_i}\) for all \(i \in I\) and \(\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}\) for all \(i, j, k \in I\) with \(i, j \leq k\).

Then there exists a unique couple \((\varprojlim \mathcal{M}_* \ni \varphi_i, \{\varphi_i\}_{i \in I})\), where \(\varprojlim \mathcal{M}_*\) is a normed \(L^0(m)\)-module and each map \(\varphi_i : \mathcal{M}_i \to \varprojlim \mathcal{M}_*\) is a normed \(L^0(m)\)-module morphism, such that:

a) \((\varprojlim \mathcal{M}_* \ni \varphi_i, \{\varphi_i\}_{i \in I})\) is a target for \(\{(\mathcal{M}_i)_{i \in I}, \{\varphi_{ij}\}_{i \leq j}\}\), i.e.,

\[
\begin{array}{ccc}
\mathcal{M}_i & \xrightarrow{\varphi_{ij}} & \mathcal{M}_j \\
\downarrow{\varphi_i} & & \downarrow{\varphi_j} \\
\varprojlim \mathcal{M}_* & & \varprojlim \mathcal{M}_*
\end{array}
\]

is a commutative diagram for every \(i, j \in I\) with \(i \leq j\).
Given any other target \((\mathcal{N}, \{\psi_i\}_{i \in I})\) for \((\{M_i\}_{i \in I}, \{\varphi_{ij}\}_{i \leq j})\), there exists a unique normed \(L^0(m)\)-module morphism \(\Phi: \lim\downarrow M_n \to \mathcal{N}\) such that

\[
\begin{array}{ccc}
M_i & \xrightarrow{\varphi_i} & \lim\downarrow M_n \\
\downarrow & & \downarrow \Phi \\
\psi_i & \searrow & \\
& \mathcal{N}
\end{array}
\]

is a commutative diagram for every \(i \in I\).

A simple example of direct limit of normed \(L^0(m)\)-modules is given by the ensuing result:

**Lemma 2.17.** Let \((X, \Sigma, m)\) be a \(\sigma\)-finite measure space. Let \(\mathcal{M}\) be a normed \(L^0(m)\)-module. Let \((\mathcal{M}_n)_n\) be an increasing sequence of normed \(L^0(m)\)-submodules of \(\mathcal{M}\) such that \(\bigcup_{n \in \mathbb{N}} \mathcal{M}_n\) is dense in \(\mathcal{M}\). Call \(\iota_n: \mathcal{M}_n \hookrightarrow \mathcal{M}\) the inclusion map for every \(n, m \in \mathbb{N}\) with \(n \leq m\). Then \((\{\mathcal{M}_n\}_{n \in \mathbb{N}}, \{\iota_{nm}\}_{n \leq m})\) is a direct system of normed \(L^0(m)\)-modules and

\[
\lim\downarrow \mathcal{M}_n \cong \mathcal{M}.
\]

**Proof.** Calling \(\iota_n: \mathcal{M}_n \hookrightarrow \mathcal{M}\) the inclusion map for any \(n \in \mathbb{N}\), we see that \((\mathcal{M}, \{\iota_n\}_{n \in \mathbb{N}})\) is a target for \((\{\mathcal{M}_n\}_{n \in \mathbb{N}}, \{\iota_{nm}\}_{n \leq m})\). Moreover, fix any other target \((\mathcal{N}, \{\psi_n\}_{n \in \mathbb{N}})\). Since the vector space \(\bigcup_{n \in \mathbb{N}} \iota_n(\mathcal{M}_n)\) is \(d_{\mathcal{M}}\)-dense in \(\mathcal{M}\) by assumption, there clearly exists a unique linear continuous map \(\Phi: \mathcal{M} \to \mathcal{N}\) such that \(\Phi(\psi_n(v)) = \psi(v)\) for every \(n \in \mathbb{N}\) and \(v \in \mathcal{M}_n\). Finally, this map can be readily proven to be a morphism of normed \(L^0(m)\)-modules.

We conclude the subsection by reminding the key notion of pullback of a normed \(L^0\)-module:

**Theorem 2.18 (Pullback of normed \(L^0\)-modules).** Let \((X, \Sigma_X, m_X)\) and \((Y, \Sigma_Y, m_Y)\) be \(\sigma\)-finite measure spaces. Let \(\varphi: X \to Y\) be a measurable map satisfying \(\varphi_* m_X \ll m_Y\). Let \(\mathcal{M}\) be a normed \(L^0(m_Y)\)-module. Then there exists a unique couple \((\varphi^* \mathcal{M}, \varphi^*)\) such that \(\varphi^* \mathcal{M}\) is a normed \(L^0(m_X)\)-module and \(\varphi^*: \mathcal{M} \to \varphi^* \mathcal{M}\) is a linear map with the following properties:

i) It holds that \(|\varphi^* v| = |v| \circ \varphi\) in the \(m_X\)-a.e. sense for every \(v \in \mathcal{M}\).

ii) The family \(\{\varphi^* v: v \in \mathcal{M}\}\) generates \(\varphi^* \mathcal{M}\).

Uniqueness has to be intended up to unique isomorphism: given any other couple \((\mathcal{N}, T)\) satisfying the same properties, there exists a unique isomorphism \(\Phi: \varphi^* \mathcal{M} \to \mathcal{N}\) of normed \(L^0(m_X)\)-modules such that

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\varphi^*} & \varphi^* \mathcal{M} \\
T \downarrow & & \downarrow \Phi \\
\mathcal{N}
\end{array}
\]

is a commutative diagram.

The pullback of a normed module has been introduced in [10], but the variant for normed \(L^0\)-modules presented above has been considered in [15] and [6].
2.5. Normed $L^\infty(m)$-modules. We recall the notion of normed $L^\infty(m)$-module, which has been introduced in [10]:

**Definition 2.19** (Normed $L^\infty(m)$-module). Let $(X, \Sigma, m)$ be any $\sigma$-finite measure space. Then a normed $L^\infty(m)$-module is a couple $(M, | \cdot |)$ having the following properties:

i) $M$ is an algebraic $L^\infty(m)$-module.

ii) The map $| \cdot | : M \to L^\infty(m)$, which is called a pointwise norm on $M$, satisfies

\begin{align}
|v| \geq 0 & \quad \text{for every } v \in M, \text{ with equality if and only if } v = 0, \\
|v + w| \leq |v| + |w| & \quad \text{for every } v, w \in M, \\
|f : v| = |f||v| & \quad \text{for every } f \in L^\infty(m) \text{ and } v \in M,
\end{align}

where all (in)equalities are intended in the $m$-a.e. sense.

iii) The space $M$ has the glueing property, i.e., given a partition $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $X$ and a sequence $(v_n)_{n \in \mathbb{N}} \subseteq M$ with $\sup_n \text{ess sup}_{A_n} |v_n| < +\infty$, there is (a unique) $v \in M$ such that $[X_{A_n}]_m \cdot v = [X_{A_n}]_m \cdot v_n$ for all $n \in \mathbb{N}$. The element $v$ is denoted by $\sum_{n \in \mathbb{N}} [X_{A_n}]_m \cdot v_n$.

iv) The norm $\| \cdot \|_M$ on $M$ associated with $| \cdot |$, which is defined as

$$\|v\|_M := \|v\|_{L^\infty(m)}$$

for every $v \in M$,

is complete.

We point out that in iii) we do not require $\lim_N \| \sum_{n=1}^N [X_{A_n}]_m \cdot v_n - \sum_{n \in \mathbb{N}} [X_{A_n}]_m \cdot v_n \|_M = 0$.

**Remark 2.20.** The above notion of normed $L^\infty(m)$-module is equivalent to the concept of $L^\infty(m)$-normed $L^\infty(m)$-module introduced in [10]. Here we do not specify – for the sake of brevity – that the pointwise norm operator takes values into the space $L^\infty(m)$, the reason being that in the present manuscript this is the only type of pointwise norm we are going to consider over $L^\infty(m)$-modules. However, in [10, 11] also $L^p(m)$-normed $L^\infty(m)$-modules, for any exponent $p \in [1, \infty)$, are studied.

**Remark 2.21** (Locality property). It can be readily checked that normed $L^\infty(m)$-modules have the locality property: if $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ is a partition of $X$ and $v, w \in M$ are two elements such that $[X_{A_n}]_m \cdot v = [X_{A_n}]_m \cdot w$ for every $n \in \mathbb{N}$, then $v = w$. This ensures that in item iii) of Definition 2.19 the element $\sum_{n \in \mathbb{N}} [X_{A_n}]_m \cdot v_n$ is uniquely determined.

**Remark 2.22** (Lack of glueing). We point out that item iii) of Definition 2.19 is not granted by i), ii), and iv). For instance, let us consider the Radon measure $m := \sum_{n \in \mathbb{N}} \delta_n$ on $\mathbb{N}$ and the space $c_0$ of all real-valued sequences that converge to 0, which is an algebraic module over the ring $\ell^\infty \cong L^\infty(m)$. We define the pointwise norm $| \cdot | : c_0 \to \ell^\infty$ as $|(a_n)| := (|a_n|)_n$ for every $(a_n) \in c_0$, which clearly satisfies items ii) and iv). Nevertheless, the glueing property fails: the elements $e_n := (\delta_{in})_{i \in \mathbb{N}}$ with $n \in \mathbb{N}$ belong to $c_0$, but by ‘glueing’ them we would obtain the sequence constantly equal to 1, which is not an element of $c_0$.

We now aim to investigate the relation between normed $L^\infty(m)$-modules and normed $L^0(m)$-modules. For the sake of clarity, we denote by $M^\infty$ the former, by $M^0$ the latter.
Definition 2.23 (Completion/restriction). Let \((X, \Sigma, m)\) be a \(\sigma\)-finite measure space.

i) **Completion.** Let \(\mathcal{M}^\infty\) be a normed \(L^\infty(m)\)-module. Then its completion \(C(\mathcal{M}^\infty)\) is defined as the metric completion of \((\mathcal{M}^\infty, d_{\mathcal{M}^\infty})\), where the distance \(d_{\mathcal{M}^\infty}(v, w) := d_{L^0(m)}(v, w)\) for every \(v, w \in \mathcal{M}^\infty\).

ii) **Restriction.** Let \(\mathcal{M}^0\) be a normed \(L^0(m)\)-module. Then its restriction \(R(\mathcal{M}^0)\) is defined as \(R(\mathcal{M}^0) := \{v \in \mathcal{M}^0 : |v| \in L^\infty(m)\}\).

It can be readily checked – by arguing as in [11, Theorem/Definition 2.7] – that \(C(\mathcal{M}^\infty)\) inherits a normed \(L^0(m)\)-module structure. Moreover, \(R(\mathcal{M}^0)\) is a normed \(L^\infty(m)\)-module.

The following result says ‘the completion map is the inverse of the restriction map’:

Lemma 2.24 (\(C = R^{-1}\)). Let \((X, \Sigma, m)\) be a \(\sigma\)-finite measure space. Then it holds that:

i) \(C(R(\mathcal{M}^0)) \cong \mathcal{M}^0\) for every normed \(L^0(m)\)-module \(\mathcal{M}^0\),

ii) \(R(C(\mathcal{M}^\infty)) \cong \mathcal{M}^\infty\) for every normed \(L^\infty(m)\)-module \(\mathcal{M}^\infty\).

Proof.

i) Let \(\mathcal{M}^0\) be a normed \(L^0(m)\)-module. Observe that \(C(R(\mathcal{M}^0))\) can be identified with the \(d_{\mathcal{M}^0}\)-closure of \(R(\mathcal{M}^0)\) in \(\mathcal{M}^0\), thus to conclude it suffices to show that \(R(\mathcal{M}^0)\) is \(d_{\mathcal{M}^0}\)-dense in \(\mathcal{M}^0\). To this aim, fix \(v \in \mathcal{M}^0\) and call \(A_n := \{|v| \leq n\}\) for every \(n \in \mathbb{N}\). Hence, it clearly holds that \(([\chi_{A_n}]_m \cdot v)_n \subseteq R(\mathcal{M}^0)\) and \(\lim_n d_{\mathcal{M}^0}([\chi_{A_n}]_m \cdot v, v) = 0\), as required.

ii) Let \(\mathcal{M}^\infty\) be a normed \(L^\infty(m)\)-module. We call \(\iota\) the \(L^\infty(m)\)-linear isometric embedding of \((\mathcal{M}^\infty, d_{\mathcal{M}^\infty})\) into \(C(\mathcal{M}^\infty)\). Notice that \(\iota(\mathcal{M}^\infty) \subseteq R(C(\mathcal{M}^\infty))\) by definition of \(R\). To conclude, it is enough to prove that actually \(\iota(\mathcal{M}^\infty) = R(C(\mathcal{M}^\infty))\). Fix any \(w \in R(C(\mathcal{M}^\infty))\) and \(\varepsilon > 0\). Pick a sequence \((v_n)_n \subseteq \mathcal{M}^\infty\) with \(\lim_n d_{\mathcal{M}^\infty}(\iota(v_n), w) = 0\). By using Egorov theorem, we can find a partition \((A_i)_{i \in \mathbb{N}} \subseteq \Sigma\) of \(X\) and a sequence \((n_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}\) such that

\[
\text{ess sup}_{A_i} |\iota(v_{n_i})| - w | \leq \varepsilon \quad \text{for every } i \in \mathbb{N}.
\]

Notice that \(\text{sup}_{A_i} |v_{n_i}| \leq \varepsilon + \text{sup}_{X}|w| < +\infty\), thus the glueing property of \(\mathcal{M}^\infty\) grants the existence of \(v := \sum_{i \in \mathbb{N}} [\chi_{A_i}]_m \cdot v_{n_i}\). It holds that \(\sum_{i=1}^k [\chi_{A_i}]_m \cdot v_{n_i} \to v\) as \(k \to \infty\) with respect to the distance \(d_{\mathcal{M}^\infty}\) (but not with respect to \(\| \cdot \|_{\mathcal{M}^\infty}\), in general). This ensures that \(\sum_{i=1}^k [\chi_{A_i}]_m \cdot \iota(v_{n_i}) = \iota\sum_{i=1}^k [\chi_{A_i}]_m \cdot v_{n_i} \to \iota(v)\) as \(k \to \infty\), thus accordingly the inequality \(|\iota(v) - w| \leq \varepsilon\) holds \(m\)-a.e.. We conclude that \(\iota(\mathcal{M}^\infty) = R(C(\mathcal{M}^\infty))\). \(\square\)

Let us mention that both the correspondences \(\mathcal{M}^\infty \mapsto C(\mathcal{M}^\infty)\) and \(\mathcal{M}^0 \mapsto R(\mathcal{M}^0)\) can be made into functors, which turn out to be equivalences of categories – one the inverse of the other. We omit the details, referring to [20, Appendix B] for a similar discussion.

3. Liftings of normed modules

Aim of this section is to generalise the theory of liftings to the setting of normed modules. In §3.1 we introduce and study a notion of normed module over \(L^\infty(\Sigma)\), whose elements are ‘everywhere defined’. In §3.2 we prove that every normed \(L^\infty(m)\)-module can be lifted to a
normed $\mathcal{L}^\infty(\Sigma)$-module. In §3.3 we focus our attention on the functional-analytic properties of the fibers of a normed $\mathcal{L}^\infty(\Sigma)$-module.

3.1. Definition of normed $\mathcal{L}^\infty(\Sigma)$-module. We propose a notion of normed module over the commutative ring $\mathcal{L}^\infty(\Sigma)$:

**Definition 3.1** (Normed $\mathcal{L}^\infty(\Sigma)$-module). Let $(X, \Sigma, \mathcal{m})$ be a σ-finite measure space. Then a normed $\mathcal{L}^\infty(\Sigma)$-module is a couple $(\mathcal{M}, |\cdot|)$ that satisfies the following properties:

i) $\mathcal{M}$ is an algebraic $\mathcal{L}^\infty(\Sigma)$-module.

ii) The map $|\cdot| : \mathcal{M} \to \mathcal{L}^\infty(\Sigma)$, which is called a pointwise norm on $\mathcal{M}$, satisfies

\[ |v| \geq 0 \quad \text{for every } v \in \mathcal{M}, \text{ with equality if and only if } v = 0, \]

\[ |v + w| \leq |v| + |w| \quad \text{for every } v, w \in \mathcal{M}, \]

\[ |f \cdot v| = |f| |v| \quad \text{for every } f \in \mathcal{L}^\infty(\Sigma) \text{ and } v \in \mathcal{M}. \]

iii) $\mathcal{M}$ satisfies the glueing property, i.e., given any partition $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $X$ and a sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ such that $\sup_n \sup_{A_n} |v_n| < +\infty$, there is (a unique) $v \in \mathcal{M}$ such that $\chi_{A_n} \cdot v = \chi_{A_n} \cdot v_n$ for all $n \in \mathbb{N}$. The element $v$ is denoted by $\sum_{n \in \mathbb{N}} \chi_{A_n} \cdot v_n$.

iv) The norm $\|\cdot\|_{\mathcal{M}}$ on $\mathcal{M}$ associated with $|\cdot|$, which is defined as

\[ \|v\|_{\mathcal{M}} := \|v\|_{\mathcal{L}^\infty(\Sigma)} \quad \text{for every } v \in \mathcal{M}, \]

is complete.

**Remark 3.2** (Locality property). It can be readily checked that normed $\mathcal{L}^\infty(\Sigma)$-modules have the locality property: if $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ is a partition of $X$ and $v, w \in \mathcal{M}$ are two elements such that $\chi_{A_n} \cdot v = \chi_{A_n} \cdot w$ for every $n \in \mathbb{N}$, then $v = w$. This ensures that in item iii) of Definition 3.1 the element $\sum_{n \in \mathbb{N}} \chi_{A_n} \cdot v_n$ is uniquely determined.

**Remark 3.3** (Lack of gluing). The glueing property is not granted by items i), ii), iv) of Definition 3.1, as shown by the same counterexample we provided in Remark 2.22.

Given any normed $\mathcal{L}^\infty(\Sigma)$-module $\mathcal{M}$, we can introduce the following equivalence relation: two elements $v, w \in \mathcal{M}$ are equivalent – shortly, $v \sim w$ – provided $|v - w| = 0$ holds $\mathcal{m}$-a.e..

Then we define the space $\Pi_m(\mathcal{M})$ as follows:

\[ \Pi_m(\mathcal{M}) := \mathcal{M} / \sim. \quad (3.2) \]

Given any element $v \in \mathcal{M}$, we will denote by $[v]_\sim \in \Pi_m(\mathcal{M})$ its equivalence class modulo $\sim$. The canonical projection map $v \mapsto [v]_\sim$ will be denoted by $\pi_m : \mathcal{M} \to \Pi_m(\mathcal{M})$.

**Lemma 3.4**. Let $(X, \Sigma, \mathcal{m})$ be a σ-finite measure space. Let $\mathcal{M}$ be a normed $\mathcal{L}^\infty(\Sigma)$-module. Then $\mathcal{M} := \Pi_m(\mathcal{M})$ is a normed $L^\infty(\mathcal{m})$-module and $\pi_m : \mathcal{M} \to \mathcal{M}$ is linear and continuous.

**Proof.** Given any $v, w \in \mathcal{M}$ and $f \in L^\infty(\mathcal{m})$ – say $v = [v]_\sim$, $w = [w]_\sim$, and $f = [f]_\sim$ – we set

\[ v + w := [v + w]_\sim \in \mathcal{M}, \]

\[ f \cdot v := [f \cdot v]_\sim \in \mathcal{M}, \]

\[ |v| := |[v]|_m \in L^\infty(\mathcal{m}). \]
It can be readily checked that the above operations are well-posed, meaning that they do not depend on the specific choice of the representatives \( \bar{v}, \bar{w} \), and \( \bar{f} \). Moreover, we have that the operator \(|\cdot|: \mathcal{M} \to L^\infty(\mathfrak{m})\) satisfies (2.6) as an immediate consequence of (3.1). To prove the glueing property of \( \mathcal{M} \), fix a partition \((A_n)_{n \in \mathbb{N}} \subseteq \Sigma\) of \( X \) and a sequence \((v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}\) such that \( \sup_n \operatorname{ess sup} A_n |v_n| < +\infty \). Choose a representative \( \bar{v}_n \) of \( v_n \) for each \( n \in \mathbb{N} \), so that there exists an \( \mathfrak{m} \)-negligible set \( N \in \Sigma \) such that \( \sup_n \sup A_n \setminus N |\bar{v}_n| < +\infty \). Since \( \mathcal{M} \) has the glueing property, there is \( \bar{v} \in \mathcal{M} \) such that \( \chi_{A_n \setminus N} \cdot \bar{v} = \chi_{A_n} \cdot \bar{v}_n \) holds for all \( n \in \mathbb{N} \). Therefore,

\[
\left[ \chi_{A_n} \right]_{\mathfrak{m}} \cdot \pi_{\mathfrak{m}}(\bar{v}) = \pi_{\mathfrak{m}}(\chi_{A_n \setminus N} \cdot \bar{v}) = \pi_{\mathfrak{m}}(\chi_{A_n} \cdot \bar{v}_n) = \left[ \chi_{A_n} \right]_{\mathfrak{m}} \cdot v_n \quad \text{holds for every } n \in \mathbb{N},
\]

which shows that \( \mathcal{M} \) has the glueing property. Now let us define \( \|v\|_{\mathcal{M}} := \operatorname{ess sup}_X |v| \) for every element \( v \in \mathcal{M} \). We aim to prove that \((\mathcal{M}, \|\cdot\|_{\mathcal{M}})\) is a Banach space, so fix a Cauchy sequence \((v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}\), say \( v_n = [\bar{v}_n] \) for all \( n \). Hence, there exists \( N \in \Sigma \) with \( \mathfrak{m}(N) = 0 \) such that \( \sup_{X \setminus N} |\bar{v}_n - \bar{v}_m| \to 0 \) as \( n, m \to \infty \). This means that the sequence \((\chi_{X \setminus N} \cdot \bar{v}_n)_{n \in \mathbb{N}}\) is Cauchy in \( \mathcal{M} \), thus \( \|\chi_{X \setminus N} \cdot \bar{v}_n - \bar{v}_m\|_{\mathcal{M}} \to 0 \) for some \( \bar{v} \in \mathcal{M} \). Then we have that

\[
\|v_n - \pi_{\mathfrak{m}}(\bar{v})\|_{\mathcal{M}} = \operatorname{ess sup}_X |v_n - \pi_{\mathfrak{m}}(\bar{v})| \leq \sup_X |\chi_{X \setminus N} \cdot \bar{v}_n - \bar{v}| \to 0 \quad \text{as } n \to \infty,
\]

as required. Finally, linearity and continuity of \( \pi_{\mathfrak{m}}: \mathcal{M} \to \mathcal{M} \) can be trivially verified. \( \square \)

3.2. **Liftings of normed** \( L^\infty(\mathfrak{m}) \)-**modules.** The following result shows that any lifting of measurable functions can be made into a ‘lifting of normed modules’, much like in Theorem 2.4 we ‘raised’ a lifting of a measure space to a lifting of measurable functions.

**Theorem 3.5** (Lifting of normed \( L^\infty(\mathfrak{m}) \)-modules). Let \((X, \Sigma, \mathfrak{m})\) be a \( \sigma \)-finite measure space. Let \( \mathcal{M} \) be a normed \( L^\infty(\mathfrak{m}) \)-module. Let \( \ell \) be a lifting of \( \mathfrak{m} \) and call \( \mathcal{L}: L^\infty(\mathfrak{m}) \to L^\infty(\Sigma) \) its associated operator (as in Theorem 2.4). Then there exists a unique couple \((\mathcal{M}, \mathcal{L})\), called the \( \mathcal{L} \)-lifting of \( \mathcal{M} \), where \( \mathcal{M} \) is a normed \( L^\infty(\Sigma) \)-module and \( \mathcal{L}: \mathcal{M} \to \mathcal{M} \) is a linear map that satisfies the following properties:

i) It holds that \( |\mathcal{L}(v)| = \mathcal{L}(|v|) \) for every \( v \in \mathcal{M} \).

ii) The linear subspace \( \mathcal{V} \) of all elements \( \bar{v} \in \mathcal{M} \) of the form \( \bar{v} = \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n) \), where \((A_n)_{n \in \mathbb{N}} \subseteq \Sigma\) is a partition of \( X \) and \((v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}\), is dense in \( \mathcal{M} \).

Uniqueness is intended up to unique isomorphism: given any other couple \((\mathcal{N}, \mathcal{L}')\) with the same properties, there exists a unique \( L^\infty(\Sigma) \)-module isomorphism \( \Psi: \mathcal{M} \to \mathcal{N} \) preserving the pointwise norm such that

\[
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\mathcal{L}} & \mathcal{M} \\
\downarrow \Psi & & \downarrow \\
\mathcal{N} & & \mathcal{N}
\end{array}
\]

is a commutative diagram.
Proof.

**Existence.** First of all, let us define the pre-module $\mathcal{P}_m$ as

$$\mathcal{P}_m := \{ (A_n, v_n) \}_{n \in \mathbb{N}} \mid (A_n)_{n \in \mathbb{N}} \subseteq \Sigma \text{ is a partition of } X, \quad (v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}, \quad \sup_n \esssup_{A_n} |v_n| < \infty \}. $$

We declare that $\{ (A_n, v_n) \}_{n} \sim \{ (B_m, w_m) \}_{m}$ provided $\chi_{A_n \cap B_m} \cdot L(|v_n - w_m|) = 0$ holds for every $n, m \in \mathbb{N}$. We denote by $[A_n, v_n]_n$ the equivalence class of the sequence $\{ (A_n, v_n) \}_n$ with respect to the equivalence relation $\sim$. We endow $\mathcal{P}_m/\sim$ with the following operations:

\begin{align*}
[A_n, v_n]_n + [B_m, w_m]_m & := [A_n \cap B_m, v_n + w_m]_{n, m}, \\
\left( \sum_{i=1}^k c_i \chi_{C_i} \right) \cdot [A_n, v_n]_n & := [A_n \cap C_i, c_i v_n]_{n, i}, \\
\Vert [A_n, v_n]_n \Vert_{\mathcal{M}} & := \sup_{\llbracket \Sigma \rrbracket} \Vert [A_n, v_n]_n \Vert,
\end{align*}

for every $[A_n, v_n]_n, [B_m, w_m]_m \in \mathcal{P}_m/\sim$ and $\sum_{i=1}^k c_i \chi_{C_i} \in \overline{\mathcal{S}}(\Sigma)$. Then we define $\mathcal{M}$ as the completion of the normed space $(\mathcal{P}_m/\sim, \Vert \cdot \Vert_{\mathcal{M}})$. It can be readily checked – since the space $\overline{\mathcal{S}}(\Sigma)$ is dense in $L^\infty(\Sigma)$ – that the operations in (3.4) can be uniquely extended to $\mathcal{M}$, which has a natural structure of $L^\infty(\Sigma)$-module endowed with a pointwise norm $| \cdot | : \mathcal{M} \to L^\infty(\Sigma)$ satisfying items ii) and iv) of Definition 3.1. Define $\mathcal{L} : \mathcal{M} \to \mathcal{M}$ as $\mathcal{L}(v) := [X, v] \in \mathcal{P}_m/\sim$ for every $v \in \mathcal{M}$. It is clear that $\mathcal{L}$ is a linear operator that satisfies i). To show that $\mathcal{M}$ has the gluing property, fix a sequence $(\bar{w}_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ and a partition $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $X$. Given any $n, k \in \mathbb{N}$, there exist a sequence $(v_{n,i})_{i \in \mathcal{M}}$ and a partition $(A_{n,i})_{i \in \mathcal{M}} \subseteq \Sigma$ of $A_n$ such that $\Vert [A_{n,k}, v_{n,k}]_{n,i} - \chi_{A_n} \cdot \bar{w}_n \Vert_{\mathcal{M}} \leq 2^{-k}$. Call $z^k := [A_{n,i}^k, v_{n,i}^k]_{n,i} \in \mathcal{M}$ for every $k \in \mathbb{N}$. Given that $\sup_{A_n} |z^k - z^{k+1}| \leq 2^{-k+1}$ for all $n, k \in \mathbb{N}$, we have $\Vert z^{k+1} - z^k \Vert_{\mathcal{M}} \leq 2^{-k+1}$ for all $k \in \mathbb{N}$. Then $(z^k)_{k \in \mathbb{N}}$ is a Cauchy sequence, whence it converges to some element $\bar{w} \in \mathcal{M}$. Since for any $n \in \mathbb{N}$ it holds that $\chi_{A_n} \cdot \bar{w} = \lim_k \chi_{A_n} \cdot z^k = \chi_{A_n} \cdot \bar{w}_n$, the gluing property is proved. Finally, it only remains to show ii). Notice that it is enough to prove that $\mathcal{V} = \mathcal{P}_m/\sim$. To this aim, fix a partition $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma$ of $X$ and a sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$. We denote by $\bar{v}$ the element $\sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n) \in \mathcal{M}$. Then it holds that $\bar{v} = [A_n, v_n]_n$: for any $m \in \mathbb{N}$, we have $\chi_{A_m} \cdot \bar{v} = \chi_{A_m} \cdot \mathcal{L}(v_m) = \chi_{A_m} \cdot [X, v_m] = [A_m \cap A_n, v_n]_n = \chi_{A_m} \cdot [A_n, v_n]_n$. This proves that $\mathcal{V} = \mathcal{P}_m/\sim$ and accordingly that ii) is verified.

**Uniqueness.** First of all, observe that we are forced to set

$$\Psi \left( \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n) \right) := \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}'(v_n) \quad \text{for every } \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n) \in \mathcal{V}. $$

Its well-posedness stems from the fact that $\sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n)$ and $\sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}'(v_n)$ have the same pointwise norm by i). Hence, by ii) the map $\Psi$ can be uniquely extended to an $L^\infty(\Sigma)$-linear operator $\Psi : \mathcal{M} \to \mathcal{N}$ preserving the pointwise norm. Finally, since $\Psi(\mathcal{V})$ is dense in $\mathcal{N}$ again by ii), we conclude that $\Psi$ is a normed $L^\infty(\Sigma)$-module isomorphism. \qed
Remark 3.6. We highlight an important byproduct of the proof of Theorem 3.5:
\[
\mathcal{L}(f \cdot v) = \mathcal{L}(f) \cdot \mathcal{L}(v) \quad \text{for every } v \in \mathcal{M} \text{ and } f \in L^\infty(\mathcal{M}).
\] (3.5)

By density of $\text{Sf}(\mathcal{M})$ in $L^\infty(\mathcal{M})$, it is enough to prove it for $f = \sum_{i=1}^n a_i [X_{A_i}]_m \in \text{Sf}(\mathcal{M})$. Since $\chi_{\ell(A_i)} \cdot \mathcal{L}([a_i v - f \cdot v]) = \mathcal{L}([a_i v - f \cdot v]) = \mathcal{L}([a_i v - f \cdot v]) = \mathcal{L}([\chi_{\ell(A_i)} a_i v - f \cdot v]) = 0$
holds for every $i = 1, \ldots, n$, we see that $[\ell(A_i), a_i v] = [X, f \cdot v]$. Therefore, we deduce from the second line of (3.4) that
\[
\mathcal{L}(f) \cdot \mathcal{L}(v) = \left( \sum_{i=1}^n a_i \chi_{\ell(A_i)} \right) [X, v] = [\ell(A_i), a_i v] = [X, f \cdot v] = \mathcal{L}(f \cdot v),
\]
as required. 

A natural question arises: given a normed $L^\infty(\mathcal{M})$-module $\mathcal{M}$ and calling $\hat{\mathcal{M}}$ its $\mathcal{L}$-lifting, do $\Pi_{\mathcal{M}}(\hat{\mathcal{M}})$ and $\mathcal{M}$ coincide? The following result shows that the answer is positive.

Lemma 3.7. Let $(X, \Sigma, \mathcal{M})$ be a $\sigma$-finite measure space. Let $\mathcal{M}$ be a normed $L^\infty(\mathcal{M})$-module. Let $\ell$ be any lifting of $\mathcal{M}$, with associated operator $\mathcal{L}: L^\infty(\mathcal{M}) \to L^\infty(\Sigma)$. Denote by $(\hat{\mathcal{M}}, \hat{\mathcal{L}})$ the $\mathcal{L}$-lifting of $\mathcal{M}$. Then it holds that $\Pi_{\mathcal{M}}(\hat{\mathcal{M}}) \cong \mathcal{M}$.

Proof. To prove the statement, it suffices to show that the map $T := \pi_{\mathcal{M}} \circ \mathcal{L} : \mathcal{M} \to \Pi_{\mathcal{M}}(\hat{\mathcal{M}})$ is an isomorphism of normed $L^\infty(\mathcal{M})$-modules. We know that $T$ is $L^\infty(\mathcal{M})$-linear: it is linear as composition of linear operators, while for any $f \in L^\infty(\mathcal{M})$ and $v \in \mathcal{M}$ it holds that
\[
T(f \cdot v) = \pi_{\mathcal{M}}(\mathcal{L}(f \cdot v)) = \pi_{\mathcal{M}}(\mathcal{L}(f) \cdot \mathcal{L}(v)) = [\mathcal{L}(f) \cdot \mathcal{L}(v)] = f \cdot [\mathcal{L}(v)] = f \cdot T(v).
\]
Furthermore, for every $v \in \mathcal{M}$ we have that
\[
|T(v)| = ||\mathcal{L}(v)|| = ||\mathcal{L}(v)||_m = ||\mathcal{L}(v)||_m = |v|
\]
in other words, the map $T$ preserves the pointwise norm. In order to conclude, it suffices to prove that $T$ is surjective. Let $[\tilde{w}]_\sim \in \Pi_{\mathcal{M}}(\hat{\mathcal{M}})$ be fixed. Then for any $n \in \mathbb{N}$ we can pick an element $\tilde{w}_n = \sum_{i \in \mathbb{N}} X_{A_i} \cdot \mathcal{L}(v_i^n) \in \hat{\mathcal{M}}$ in such a way that $\lim_n ||\tilde{w}_n - \tilde{w}||_{\hat{\mathcal{M}}} = 0$. Now let us set $v_n := \sum_{i \in \mathbb{N}} [X_{A_i}]_m \cdot v_i^n \in \mathcal{M}$ for all $n \in \mathbb{N}$, which is well-defined as
\[
\sup \text{ess sup} \sup_{i \in \mathbb{N}} |v_i^n| \leq \sup \text{ess sup} \sup_{i \in \mathbb{N}} |\mathcal{L}(v_i^n)| = \|\tilde{w}_n\|_{\hat{\mathcal{M}}} < +\infty \quad \text{for every } n \in \mathbb{N}.
\]
Since for any $n, m \in \mathbb{N}$ it holds that
\[
|v_n - v_m| = \sum_{i,j \in \mathbb{N}} [X_{A_i} \cap A_j^n]_m |v_i^n - v_j^m| = \sum_{i,j \in \mathbb{N}} [X_{A_i} \cap A_j^n]_m \left[ |\mathcal{L}(v_i^n) - \mathcal{L}(v_j^m)| \right]_m = |\tilde{w}_n - \tilde{w}_m|_m,
\]
we see that the sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}$ is Cauchy, thus it converges to some element $v \in \mathcal{M}$. In general $\mathcal{L}(v)$ and $\tilde{w}$ may be different, but for sure one has that $||\mathcal{L}(v) - \tilde{w}||_m = 0$, indeed
\[
\left[ |\mathcal{L}(v) - \tilde{w}| \right]_m \leq \left[ |\mathcal{L}(v - v_n)| \right]_m + \left[ |\mathcal{L}(v_n) - \tilde{w}_n| \right]_m + \left[ |\tilde{w}_n - \tilde{w}| \right]_m = |v - v_n| + ||\tilde{w}_n - \tilde{w}||_m
\]
is satisfied $\mathcal{M}$-a.e. and $|v - v_n| + ||\tilde{w}_n - \tilde{w}||_m \to 0$ with respect to the $L^\infty(\mathcal{M})$-norm. This means that $T(v) = [\tilde{w}]_\sim$, whence the operator $T$ is surjective, as desired. \hfill \square
3.3. Fibers of a normed $L^\infty(\Sigma)$-module. Let $(X, \Sigma, m)$ be a $\sigma$-finite measure space and let $\mathcal{M}$ be a normed $L^\infty(\Sigma)$-module. Given any $x \in X$, we define the submodule $\mathcal{M}_x$ of $\mathcal{M}$ as

$$\mathcal{M}_x := \chi_{\{x\}} \cdot \mathcal{M}.$$  

Since the ideal $(\chi_{\{x\}}) \subseteq L^\infty(\Sigma)$ generated by $\chi_{\{x\}}$ can be identified with the real field $\mathbb{R}$, we deduce that $\mathcal{M}_x$ inherits a vector space structure. Moreover, let us define

$$\|\mathcal{v}\|_x := |\mathcal{v}|(x) \quad \text{for every } \mathcal{v} \in \mathcal{M}_x.$$  

Therefore, $(\mathcal{M}_x, \| \cdot \|_x)$ is a Banach space. We call it the fiber of $\mathcal{M}$ over the point $x$.

**Remark 3.8.** Given any element $\mathcal{v} \in \mathcal{M}$, we shall make use of the shorthand notation

$$\mathcal{v}_x := \chi_{\{x\}} \cdot \mathcal{v} \in \mathcal{M}_x \quad \text{for every } x \in X.$$  

Then $\mathcal{v}_x \in \mathcal{M}_x$ can be thought of as a map assigning to any $x \in X$ a vector $\mathcal{v}_x \in \mathcal{M}_x$. 

Fix a normed $L^0(m)$-module $\mathcal{M}$ and consider the duality pairing

$$\langle \cdot, \cdot \rangle : \mathcal{M}^* \times \mathcal{M} \to L^0(m), \quad \langle \omega, v \rangle := \omega(v) \quad \text{for every } \omega \in \mathcal{M}^* \text{ and } v \in \mathcal{M}.$$  

(3.6)

Choose any lifting $\ell$ of $m$. Denote by $\mathcal{L}$ the operator associated with $\ell$ as in Theorem 2.4. Call $(\mathcal{M}, \mathcal{L})$ and $(\mathcal{M}^*, \mathcal{L}^*)$ the $\mathcal{L}$-liftings of the normed $L^\infty(m)$-modules $R(\mathcal{M})$ and $R(\mathcal{M}^*)$, respectively. Notice that the duality pairing in (3.6) restricts to an $L^\infty(m)$-bilinear and continuous map $\langle \cdot, \cdot \rangle : R(\mathcal{M}^*) \times R(\mathcal{M}) \to L^\infty(m)$. We now want to lift it to a duality pairing between $\mathcal{M}$ and $\mathcal{N}$. Given any two sequences $(v_n)_{n \in \mathbb{N}} \subseteq R(\mathcal{M})$, $(\omega_m)_{m \in \mathbb{N}} \subseteq R(\mathcal{M}^*)$ and any two partitions $(A_n)_{n \in \mathbb{N}}, (B_m)_{m \in \mathbb{N}} \subseteq \Sigma$ of $X$, let us define

$$\left\langle \sum_{m \in \mathbb{N}} \chi_{B_m} \cdot \mathcal{L}^*(\omega_m), \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n) \right\rangle := \sum_{n,m \in \mathbb{N}} \chi_{A_n \cap B_m} \mathcal{L}\left(\langle \omega_m, v_n \rangle\right) \in L^\infty(\Sigma).$$  

(3.7)

The well-posedness of the previous definition stems from the following inequality:

$$\left| \sum_{n,m \in \mathbb{N}} \chi_{A_n \cap B_m} \mathcal{L}\left(\langle \omega_m, v_n \rangle\right) \right| = \sum_{n,m \in \mathbb{N}} \chi_{A_n \cap B_m} \mathcal{L}\left(\left| \langle \omega_m, v_n \rangle \right|\right) \leq \sum_{n,m \in \mathbb{N}} \chi_{A_n \cap B_m} \mathcal{L}\left(\langle \omega_m, v_n \rangle\right) \mathcal{L}\left(\left| \langle \omega_m, v_n \rangle \right|\right) = \sum_{n,m \in \mathbb{N}} \chi_{A_n \cap B_m} \mathcal{L}\left(\langle \omega_m, v_n \rangle\right) \mathcal{L}\left(\left| \langle \omega_m, v_n \rangle \right|\right) = \sum_{m \in \mathbb{N}} \chi_{B_m} \cdot \mathcal{L}^*(\omega_m) \mathcal{L}(v_n) \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \mathcal{L}(v_n).$$

The same inequality grants that the map in (3.7) can be uniquely extended to a pairing

$$\langle \cdot, \cdot \rangle : \mathcal{N} \times \mathcal{M} \to L^\infty(\Sigma),$$

for every $\omega \in \mathcal{N}^*$ and $v \in \mathcal{M}$.
Proof. It is clear that $R\colon \tilde{N}_x$ is an isometry when restricted to any dense subset of $\bar{\tilde{N}}_x$. Thus proving the statement. □

Remark 3.9. We underline that
\[
\langle \mathcal{L}^* (\omega), \mathcal{L} (v) \rangle = \mathcal{L} (\omega (v)) \quad \text{for every } v \in \mathcal{R} (\mathcal{M}) \text{ and } \omega \in \mathcal{R} (\mathcal{M}^*),
\]
by the very definition (3.7) of $\langle \cdot, \cdot \rangle : \tilde{N} \times \tilde{M} \to \mathcal{L}^* (\Sigma)$.

Now fix a point $x \in X$. Then the pairing $\langle \cdot, \cdot \rangle : \tilde{N} \times \tilde{M} \to \mathcal{L}^* (\Sigma)$ naturally induces a pairing $\langle \cdot, \cdot \rangle_x : \tilde{N}_x \times \tilde{M}_x \to \mathbb{R}$, which is the bilinear and continuous function given by
\[
\langle \bar{\omega}, \bar{v} \rangle_x := \langle \bar{\omega}, \bar{v} \rangle (x) \quad \text{for every } \bar{\omega} \in \tilde{N}_x \text{ and } \bar{v} \in \tilde{M}_x.
\]
Observe that $\tilde{M}_x \ni \bar{v} \mapsto \langle \bar{\omega}, \bar{v} \rangle_x \in \mathbb{R}$ is a linear and continuous function for every $\bar{\omega} \in \tilde{N}_x$.

Therefore, it makes sense to define the map $R_x : \tilde{N}_x \to (\tilde{M}_x)'$ as
\[
R_x (\bar{\omega}) := \langle \bar{\omega}, \cdot \rangle_x \quad \text{for every } \bar{\omega} \in \tilde{N}_x.
\]
It is clear that $R_x$ is a linear operator.

Proposition 3.10. Under the above assumptions, it holds that the map $R_x : \tilde{N}_x \to (\tilde{M}_x)'$ is an isometric embedding.

Proof. First of all, given any $\bar{\omega} \in \tilde{N}_x$ it holds that $|\langle \bar{\omega}, \bar{v} \rangle_x| \leq \|\bar{\omega}\|_x \|\bar{v}\|_x$ for every $\bar{v} \in \tilde{M}_x$ as a consequence of (3.8), whence accordingly
\[
\|R_x (\bar{\omega})\|_{(\tilde{M}_x)'} \leq \|\bar{\omega}\|_x \quad \text{for every } \bar{\omega} \in \tilde{N}_x.
\]
(3.11)

In particular, the operator $R_x$ is continuous. Then to prove the statement it suffices to show that $R_x$ is an isometry when restricted to any dense subset of $\tilde{N}_x$.

Observe that $D := \{ \mathcal{L}^* (\omega) : \omega \in \mathcal{R} (\mathcal{M}^*) \}$ is a dense subspace of $\tilde{N}_x$ by the very definition of $\mathcal{L}$-lifting. Fix any $\omega \in \mathcal{R} (\mathcal{M}^*)$. Then it holds that
\[
\left\| R_x (\mathcal{L}^* (\omega) x) \right\|_{(\tilde{M}_x)'} = \sup_{\bar{v} \in \tilde{M}_x : \|\bar{v}\|_x \leq 1} \langle \mathcal{L}^* (\omega) x, \bar{v} \rangle_x \geq \sup_{v \in \mathcal{R} (\mathcal{M}) : |v| \leq 1} \mathcal{L} (\omega (v)) (x)
\]
(3.9)
\[
= \sup_{v \in \mathcal{R} (\mathcal{M}) : |v| \leq 1} \mathcal{L} (\omega (v)) (x).
\]
(3.12)

Given any $\varepsilon > 0$, there exists $v \in \mathcal{R} (\mathcal{M})$ such that $|v| \leq 1$ and $\omega (v) \geq |\omega| - \varepsilon$ are satisfied m-a.e. by (2.7). Thus $\mathcal{L} (\omega (v)) \geq \mathcal{L} (|\omega| - \varepsilon)$ and accordingly $\mathcal{L} (\omega (v)) (x) \geq \|\mathcal{L}^* (\omega) x\|_x - \varepsilon$. This fact – if read in conjunction with (3.11) and (3.12) – ensures that the map $R_x$ is an isometry when restricted to $D$, thus proving the statement. □

Problem 3.11. Are the maps $R_x : \tilde{N}_x \to (\tilde{M}_x)'$ defined in (3.10) isomorphisms?
4. Representation of normed modules via embedding

In this section we study separable Banach $\mathbb{B}$-bundles and their strong connections with separable normed $L^0(m)$-modules. In §4.1 we introduce our notion of separable Banach bundle and we describe a constructive procedure to obtain one such bundle. In §4.2 we show that the space of sections of a given bundle is a normed $L^0(m)$-module; also, we extend the operation of ‘taking sections’ to the level of categories, thus obtaining a well-behaved section functor. In §4.3 we prove that the section functor is actually an equivalence of categories; this means, roughly speaking, that there is a full correspondence between separable Banach bundles and separable normed $L^0(m)$-modules.

4.1. Separable Banach bundles. Given a Banach space $\mathbb{B}$, we denote by $\text{Gr}(\mathbb{B})$ the family of its closed linear subspaces.

Definition 4.1 (Separable Banach bundle). Let $(X, \Sigma)$ be a measurable space and $\mathbb{B}$ a universal separable Banach space. Then a map $E : X \to \text{Gr}(\mathbb{B})$ is said to be a separable Banach $\mathbb{B}$-bundle over $X$ provided $E : X \to \mathbb{B}$ is a weakly measurable correspondence.

Given a separable Banach $\mathbb{B}$-bundle $E$ over $X$, let us define $T_E := \bigcup_{x \in X} \{x\} \times E(x) \subseteq X \times \mathbb{B}$.

(4.1)

Observe that $T_E \in \Sigma \otimes \mathcal{B}(\mathbb{B})$ by item vi) of §2.2.

Definition 4.2 (Sections of a separable Banach bundle). Let $(X, \Sigma)$ be a measurable space and $\mathbb{B}$ a universal separable Banach space. Let $E$ be a separable Banach $\mathbb{B}$-bundle over $X$. Then by section of $E$ we mean a measurable selector of $E$, namely, a measurable map $\bar{s} : X \to \mathbb{B}$ such that $\bar{s}(x) \in E(x)$ for every $x \in X$. The vector space of sections of $E$ is denoted by $\bar{\Gamma}(E)$.

Given any $\bar{s} \in \bar{\Gamma}(E)$, we define the function $|\bar{s}| : X \to [0, +\infty)$ as $|\bar{s}|(x) := \|\bar{s}(x)\|_{\mathbb{B}}$ for every $x \in X$.

(4.2)

Moreover, we define the linear subspace $\bar{\Gamma}_b(E) \subseteq \bar{\Gamma}(E)$ as $\bar{\Gamma}_b(E) := \{\bar{s} \in \bar{\Gamma}(E) : |\bar{s}| \in L^\infty(\Sigma)\}$.

Remark 4.3. It is straightforward to check that $\bar{\Gamma}_b(E)$ is a normed $\mathcal{L}^\infty(\Sigma)$-module when endowed with the natural pointwise operations and the pointwise norm $|\cdot| : \bar{\Gamma}_b(E) \to \mathcal{L}^\infty(\Sigma)$ that has been defined in (4.2).

Proposition 4.4. Let $(X, \Sigma)$ be a measurable space and $\mathbb{B}$ a universal separable Banach space.

i) Let $(\bar{s}_n)_n$ be any sequence of measurable maps $\bar{s}_n : X \to \mathbb{B}$. Define $E : X \to \mathbb{B}$ as $E(x) := \text{cl}_\mathbb{B} \left( \text{span} \{\bar{s}_n(x) : n \in \mathbb{N}\} \right)$ for every $x \in X$. Then $E$ is a separable Banach $\mathbb{B}$-bundle over $X$.

ii) Let $E$ be a separable Banach $\mathbb{B}$-bundle over $X$. Then there exists a countable $\mathbb{Q}$-linear subspace $\mathcal{C}$ of $\bar{\Gamma}(E)$ such that $E(x) = \text{cl}_\mathbb{B} \{\bar{s}(x) : \bar{s} \in \mathcal{C}\}$ for every $x \in X$. 

■
Proof.
i) Given any open set $U \subseteq \mathbb{B}$, it holds that
\[
\{ x \in X \mid E(x) \cap U \neq \emptyset \} = \bigcup_{n \in \mathbb{N}} \bigcup_{(q_1, \ldots, q_n) \in \mathbb{Q}} \left\{ x \in X \mid \sum_{i=1}^{n} q_i s_i(x) \in U \right\} \subseteq \Sigma.
\]
By arbitrariness of $U$, we conclude that $E : X \to \text{Gr}(\mathbb{B})$ is a separable Banach $\mathbb{B}$-bundle.

ii) Let $(v_n)_n \subseteq \mathbb{B}$ be a fixed dense sequence. Given any $n, k \in \mathbb{N}$, let us define the correspondence $\varphi_{nk} : X \to \mathbb{B}$ as
\[
\varphi_{nk}(x) := \left\{ \begin{array}{ll}
\text{cl}_B(E(x) \cap B_{1/k}(v_n)) & \text{if } E(x) \cap B_{1/k}(v_n) \neq \emptyset, \\
0_B & \text{otherwise}.
\end{array} \right.
\]
Notice that $\varphi_{nk}$ is weakly measurable as a consequence of item viii) of §2.2. Then by applying Kuratowski–Ryll-Nardzewski theorem (item vii) of §2.2) we obtain a measurable selector $\bar{s}_{nk}$ of the correspondence $\varphi_{nk}$. Given any $x \in X, v \in E(x)$, and $\varepsilon > 0$, we can find $n, k \in \mathbb{N}$ such that $1/k < \varepsilon/2$ and $\|v - v_n\|_B < 1/k$. Therefore, it holds that $\|v - \bar{s}_{nk}(x)\|_B < \varepsilon$. This shows that $\{\bar{s}_{nk}(x) : n, k \in \mathbb{N}\}$ is a dense subset of $E(x)$ for every $x \in X$. The claim follows by taking as $\bar{E}$ the $\mathbb{Q}$-linear subspace of $\overline{\Gamma(E)}$ generated by $\{\bar{s}_{nk} : n, k \in \mathbb{N}\}$. \qed

In the sequel, we will need the following working definition of a measurable collection of Banach spaces. Roughly speaking, it is an intermediate construction that will be used to cook up the separable Banach bundle underlying a given separable normed $L^0(\mathfrak{m})$-module.

Definition 4.5 (Measurable collection of separable Banach spaces). Let $(X, \Sigma)$ be a given measurable space. Then a family $\{E(x)\}_{x \in X}$ of separable Banach spaces is said to be a measurable collection of separable Banach spaces provided there exist elements $v_n(x) \in E(x)$ and $\omega_n(x) \in E(x)'$, with $n \in \mathbb{N}$ and $x \in X$, such that the following properties hold:

a) $(v_n(x))_n$ is a dense subset of $E(x)$ for every $x \in X$,

b) $\|\omega_n(x)\|_{E(x)} = 1$ and $\omega_n(x)[v_n(x)] = \|v_n(x)\|_{E(x)}$ whenever $n \in \mathbb{N}$ and $x \in X$ are such that $v_n(x) \neq 0_{E(x)}$,

c) $X \ni x \mapsto \omega_n(x)[v_k(x)] \in \mathbb{R}$ is a measurable function for every $n, k \in \mathbb{N}$.

In particular, the function $X \ni x \mapsto \|v_n(x)\|_{E(x)} \in \mathbb{R}$ is measurable for every $n \in \mathbb{N}$.

In the next two technical results we explain how to get a separable Banach bundle out of a measurable collection of separable Banach spaces. Here, the explicit construction in the proof of Banach–Mazur Theorem 2.8 plays a role.

Theorem 4.6 (Measurable family of embeddings). Let $(X, \Sigma)$ be a measurable space. Let $\mathbb{B}$ be a universal separable Banach space and $\{E(x)\}_{x \in X}$ a measurable collection of separable Banach spaces. Choose elements $(v_n(x))_{n \in \mathbb{N}} \subseteq E(x)$ for $x \in X$ as in Definition 4.5. Then there exists a family $\{I_x\}_{x \in X}$ of linear isometric embeddings $I_x : E(x) \to \mathbb{B}$ such that
\[
X \ni x \mapsto I_x[v_n(x)] \in \mathbb{B}
\]
is a measurable map for every $n \in \mathbb{N}$. (4.3)

We say that $\{I_x\}_{x \in X}$ is a measurable family of (linear isometric) embeddings.
First of all, let us define the objects we will need throughout the proof:

i) Given any \( k \in \mathbb{N} \), we denote by \( \pi_k : I^\infty \to [-1,1] \) the projection on the \( k \)th component of the Hilbert cube \( I^\infty \), namely, we define \( \pi_k(\alpha) := \alpha_k \) for every \( \alpha = (\alpha_k)_k' \in I^\infty \). Each \( \pi_k \) is continuous, as \( I^\infty \) is endowed with the restricted weak* topology.

ii) Given any \( x \in X \), we define the map \( \iota_x : B_{E(x)'} \to I^\infty \) as

\[
\iota_x(\omega) := \left( \frac{\omega \left[ v_k(x) \right]}{\|v_k(x)\|_{E(x)} \vee 1} \right)_k \in I^\infty \quad \text{for every } \omega \in B_{E(x)'}.
\] (4.4)

Then it holds that \( \iota_x \) is a homeomorphism with its image (when the domain \( B_{E(x)'} \) is endowed with the restricted weak* topology). Recall item iv) of \( \S 2.3 \).

iii) Let us define the correspondence \( K' : X \to I^\infty \) as \( K'(x) := \iota_x(B_{E(x)'} \setminus \{0\}) \) for every \( x \in X \).

Observe that \( K' \) has compact values by virtue of Banach–Alaoglu theorem.

iv) Fix a continuous surjective map \( \psi : \Delta \to I^\infty \) (recall item iii) of \( \S 2.3 \).

v) Denote by \( K : X \to \Delta \) the preimage correspondence \( \psi^{-1}(K') \), defined as in Lemma 2.6. Since \( \psi \) is continuous and \( \Delta \) is compact, it holds that \( K \) has compact values.

vi) Given any \( x \in X \), we define the retraction \( r_x : \Delta \to K(x) \) as in item ii) of \( \S 2.3 \). Namely, for any point \( a \in \Delta \) we have that \( r_x(a) \) is the unique element of \( K(x) \) satisfying the identity \( d_{\Delta}(a, r_x(a)) = d_{\Delta}(a, K(x)) \).

vii) Given any \( x \in X \), we define the operator \( I_x' : E(x) \to C(\Delta) \) as

\[
I_x'[v](a) := (\iota_x^{-1} \circ \psi \circ r_x)(a)[v] \quad \text{for every } v \in E(x) \text{ and } a \in \Delta.
\] (4.5)

Then each map \( I_x' \) is a linear isometric embedding (recall the proof of Theorem 2.8).

viii) Fix a linear isometric map \( I : C(\Delta) \to \mathbb{B} \). Given any \( x \in X \), we define the linear isometric embedding \( I_x : E(x) \to \mathbb{B} \) as \( I_x[v] := (I \circ I_x')[v] \) for every \( v \in E(x) \).

It remains to prove that \( X \ni x \mapsto I_x[v_k(x)] \in C([0,1]) \) is a measurable map for any \( k \in \mathbb{N} \).

Fix \( (\omega_n(x))_{n \in \mathbb{N}} \subseteq B_{E(x)'} \) for \( x \in X \) as in Definition 4.5. We set the family of indexes \( Q \) as

\[
Q := \left\{ q = (q_n)_n \in \bigoplus_{n \in \mathbb{N}} Q \mid q_n \geq 0 \text{ for every } n \in \mathbb{N} \text{ and } \sum_{n \in \mathbb{N}} q_n = 1 \right\},
\]

where \( \bigoplus_{n \in \mathbb{N}} Q \) stands for the set of all sequences \( q = (q_n)_n \in Q^\mathbb{N} \) such that \( q_n = 0 \) for all but finitely many \( n \in \mathbb{N} \). Let us define

\[
\omega^q(x) := \sum_{n \in \mathbb{N}} q_n \omega_n(x) \in B_{E(x)'} \quad \text{for every } q \in Q \text{ and } x \in X.
\]

Observe that \( \{\omega^q(x)\}_{q \in Q} \) is weak* dense in \( B_{E(x)'} \) for every \( x \in X \), whence it follows that \( \{\iota_x(\omega^q(x))\}_{q \in Q} \) is \( d_{I^\infty} \)-dense in \( K'(x) \). Moreover, given any \( \alpha \in I^\infty \) and \( \lambda > 0 \), we have that

\[
\left\{ x \in X \mid d_{I^\infty}(\alpha, K'(x)) < \lambda \right\} = \bigcup_{q \in Q} \left\{ x \in X \mid d_{I^\infty}(\alpha, \iota_x(\omega^q(x))) < \lambda \right\}
\]

\[= \bigcup_{q \in Q} \left\{ x \in X \mid \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \alpha_k - \sum_{n \in \mathbb{N}} q_n \omega_n(x) \left[ v_k(x) \right] \right| \frac{1}{\|v_k(x)\|_{E(x)} \vee 1} < \lambda \right\} \in \Sigma,
\]
as a consequence of the measurability of each function $X \ni x \mapsto \omega_n(x)[v_k(x)]$. Therefore, the function $X \ni x \mapsto d_I \omega_n(\alpha, K'(x))$ is measurable for every $\alpha \in I^\infty$, thus accordingly $K'$ is a weakly measurable correspondence by item iv) of §2.2. Thanks to item i) of §2.2, we deduce that $K'$ is a measurable correspondence, whence $K$ is a measurable correspondence as well by Lemma 2.6. For any $a \in \Delta$, let us consider the correspondence $Z_a : X \to \Delta$ given by

$$Z_a(x) := \left\{ b \in \Delta \mid d_\Delta(a, b) = d_\Delta(a, K(x)) \right\} \quad \text{for every} \quad x \in X.$$  

It holds that $X \times \Delta \ni (x, b) \mapsto d_\Delta(a, b) - d_\Delta(a, K(x)) \in \mathbb{R}$ is a Carathéodory function, since $\Delta \ni b \mapsto d_\Delta(a, b)$ is continuous and $X \ni x \mapsto d_\Delta(a, K(x))$ is measurable (the latter follows from the measurability of $K$, by taking items i) and iv) of §2.2 into account). Therefore, we have that $Z_a : X \to \Delta$ is a measurable correspondence by item v) of §2.2, whence the intersection correspondence $Z_a \cap K : X \to \Delta$ is measurable as well by item iii) of §2.2. Since it holds that $Z_a(x) \cap K(x) = \{r_x(a)\}$ for every $x \in X$, we deduce from item ii) of §2.2 that $X \ni x \mapsto r_x(a) \in \Delta$ is a measurable map for every $a \in \Delta$. (4.6)

Let us fix $k \in \mathbb{N}$, a dense subset $(a^i)_{i \in \mathbb{N}}$ of $\Delta$, and an element $g \in C(\Delta)$. Observe that

$$I'_x[v_k(x)](a^i) = (t_x^{-1} \circ \psi \circ r_x)(a^i)[v_k(x)] = (\pi_k \circ \psi)(r_x(a^i)) \quad \text{for all} \quad x \in X \text{ and } i \in \mathbb{N}.$$  

Being $\pi_k \circ \psi : \Delta \to [-1, 1]$ continuous, we deduce from (4.6) that $X \ni x \mapsto I'_x[v_k(x)](a^i) \in \mathbb{R}$ is measurable for every $i \in \mathbb{N}$. Since we have that

$$\left\| g - I'_x[v_k(x)] \right\|_{C(\Delta)} = \sup_{i \in \mathbb{N}} \left| g(a^i) - I'_x[v_k(x)](a^i) \right| \quad \text{for every} \quad x \in X,$$

we deduce that $X \ni x \mapsto \left\| g - I'_x[v_k(x)] \right\|_{C(\Delta)} \in \mathbb{R}$ is measurable for every element $g \in C(\Delta)$. Therefore, it holds that $X \ni x \mapsto I'_x[v_k(x)] \in C(\Delta)$ is a measurable map for all $k \in \mathbb{N}$. Recalling that $I_x = I \circ I_x$ for every $x \in X$ and that $I$ is continuous, we can finally conclude that $X \ni x \mapsto I_x[v_k(x)] \in \mathcal{B}$ is a measurable map for all $k \in \mathbb{N}$, as required.

**Corollary 4.7.** Let $(X, \Sigma)$ be a measurable space and $\mathcal{B}$ a universal separable Banach space. Let $\{E(x)\}_{x \in X}$ be a measurable collection of separable Banach spaces. Consider the associated measurable family $\{I_x\}_{x \in X}$ of linear isometric embeddings $I_x : E(x) \to \mathcal{B}$ as in Theorem 4.6. Then the map $X \ni x \mapsto E(x) := I_x(E(x)) \in \text{Gr}(\mathcal{B})$ is a separable Banach $\mathcal{B}$-bundle over $X$.

**Proof.** Choose elements $(v_n(x))_{n \in \mathbb{N}} \subseteq E(x)$ for $x \in X$ as in Definition 4.5. Let us define

$$\tilde{s}_n(x) := I_x[v_n(x)] \quad \text{for every} \quad n \in \mathbb{N} \text{ and } x \in X.$$  

Theorem 4.6 guarantees that each map $\tilde{s}_n : X \to \mathcal{B}$ is measurable, thus accordingly the correspondence $E : X \to \mathcal{B}$, which is given by

$$E(x) = I_x(E(x)) = \text{cl}_\mathcal{B}\left(\text{span}\{\tilde{s}_n(x) \mid n \in \mathbb{N}\}\right) \quad \text{for every} \quad x \in X,$$

is a separable Banach $\mathcal{B}$-bundle over $X$ by item i) of Proposition 4.4. 

**\square**
4.2. The section functor. Let \((X, \Sigma, \mathfrak{m})\) be a \(\sigma\)-finite measure space and \(\mathcal{B}\) a universal separable Banach space. Let \(E\) be a separable Banach \(\mathcal{B}\)-bundle over \(X\). Then we define

\[
\Gamma_b(E) := \Pi_m(\bar{\Gamma}_b(E)), \quad \Gamma(E) := \mathcal{C}(\Gamma_b(E)),
\]

where the operations \(\Pi_m\) and \(\mathcal{C}\) are defined as in (3.2) and Definition 2.23, respectively.

Remark 4.8. Observe that \(\Gamma(E)\) can be identified with the quotient space \(\bar{\Gamma}(E)/\sim\), where the equivalence relation \(\sim\) on \(\bar{\Gamma}(E)\) is defined in the following way: given any \(\bar{s}, \bar{t} \in \bar{\Gamma}(E)\), we declare that \(\bar{s} \sim \bar{t}\) provided \(m(\{x \in X : \bar{s}(x) \neq \bar{t}(x)\}) = 0\).

Lemma 4.9. Let \((X, \Sigma, \mathfrak{m})\) be a \(\sigma\)-finite measure space and \(\mathcal{B}\) a universal separable Banach space. Let \(E\) be a separable Banach \(\mathcal{B}\)-bundle over \(X\). Then the normed \(L^0(\mathfrak{m})\)-module \(\Gamma(E)\) is countably-generated. In particular, if \((X, \Sigma, \mathfrak{m})\) is separable, then \(\Gamma(E)\) is separable.

Proof. Thanks to item ii) of Proposition 4.4, we can find a sequence \((\bar{s}_n)_n \subseteq \bar{\Gamma}(E)\) such that the identity \(E(x) = \text{cl}_\mathcal{B}\{\bar{s}_n(x) : n \in \mathbb{N}\}\) is satisfied for every \(x \in X\). Given any \(n, k \in \mathbb{N}\), we define \(B_{nk} := \{x \in X : |\bar{s}_n(x)| \leq k\}\) and \(s_{nk} := \pi_m(\chi_{B_{nk}} \cdot \bar{s}_n)\), where \(\pi_m: \bar{\Gamma}_b(E) \to \Gamma_b(E)\) stands for the canonical projection map. We claim that \((s_{nk})_{n,k}\) generates \(\Gamma(E)\). To prove it, fix \(\bar{t} \in \bar{\Gamma}(E)\) and \(\varepsilon > 0\). Then there exist \(k \in \mathbb{N}\) and \(A \subseteq \Sigma\) with \(|\bar{t}| < k - \varepsilon/2\) m-a.e. on \(A\) and \(d_{\bar{\Gamma}(E)}(t_0, \bar{t}) < \varepsilon/2\), where we set \(t_0 := [\chi_A]_m \cdot t \in \Gamma_b(E)\). Choose any element \(\bar{t}_0 \in \bar{\Gamma}_b(E)\) such that \(t_0 = \pi_m(\bar{t}_0)\). Pick a partition \((A_n)_n \subseteq \Sigma\) of \(A\) such that \(|\bar{s}_n - \bar{t}_0| \leq \varepsilon/2\) on \(A_n\) for every \(n \in \mathbb{N}\). Therefore, we have that \(|\sum_{n \in \mathbb{N}} [\chi_{A_n}]_m \cdot s_{nk} - t_0| \leq \varepsilon/2\) holds m-a.e. on \(X\). This implies that \(d_{\bar{\Gamma}(E)}(\sum_{n \in \mathbb{N}} [\chi_{A_n}]_m \cdot s_{nk}, \bar{t}) < \varepsilon\), thus proving the first claim. The second one is then an immediate consequence of Proposition 2.15. \(\square\)

Definition 4.10 (Morphism of separable Banach bundles). Let \((X, \Sigma, \mathfrak{m})\) be a \(\sigma\)-finite measure space and \(\mathcal{B}\) a universal separable Banach space. Let \(E, F\) be two separable Banach \(\mathcal{B}\)-bundles over \(X\). Then a pre-morphism \(\varphi\) from \(E\) to \(F\) is a measurable map \(\varphi: TE \to TF\) such that \(\varphi(\{x\} \times E(x)) \subseteq \{x\} \times F(x) \cong F(x)\) for every \(x \in X\) and

\[
\varphi(x, \cdot): E(x) \to F(x)
\]

is a linear contraction for every \(x \in X\).

We declare two pre-morphisms \(\varphi_1, \varphi_2\) from \(E\) to \(F\) to be equivalent if there exists a set \(N \subseteq \Sigma\) with \(m(N) = 0\) such that \(\varphi_1(x, \cdot) = \varphi_2(x, \cdot)\) for every \(x \in X \setminus N\). This defines an equivalence relation, whose equivalence classes are called morphisms and usually denoted by \(\varphi: E \to F\).

We denote by \(\mathbf{SBB}_\mathcal{B}(X, \Sigma, \mathfrak{m})\) the category having the separable Banach \(\mathcal{B}\)-bundles over the space \(X\) as objects and the morphisms of separable Banach \(\mathcal{B}\)-bundles as arrows.

Let us consider two separable Banach \(\mathcal{B}\)-bundles \(E, F\) over \(X\) and a morphism \(\varphi: E \to F\). Fix any pre-morphism \(\bar{\varphi}: TE \to TF\) that is a representative of \(\varphi\). Then we define the morphism of normed \(L^0(\mathfrak{m})\)-modules \(\Gamma(\varphi): \Gamma(E) \to \Gamma(F)\) as follows: given any \(s \in \Gamma(E)\), we define \(\Gamma(\varphi)(s)\) as the equivalence class (under the relation \(\sim\) introduced in Remark 4.8) of

\[
X \ni x \mapsto \bar{\varphi}(x, \bar{s}(x)) \in F(x),
\]
where $\bar{s} \in \bar{\Gamma}(E)$ is any representative of $s$. It can be readily checked that this way we obtain a covariant functor $\Gamma: SBB_B(X, \Sigma, m) \to N\text{Mod}_{\mathcal{B}}(X, \Sigma, m)$, which we call the \textit{section functor}. (For brevity, in our notation the dependence of $\Gamma$ on the space $\mathcal{B}$ is omitted.)

\textbf{Lemma 4.11} ($\Gamma$ is full). \textit{Let $(X, \Sigma, m)$ be a $\sigma$-finite measure space and $\mathcal{B}$ a universal separable Banach space. Let $E, F$ be two separable Banach $\mathcal{B}$-bundles and $\Phi: \Gamma(E) \to \Gamma(F)$ a morphism of normed $L^0(m)$-modules. Then there exists a morphism $\varphi: E \to F$ of separable Banach $\mathcal{B}$-bundles such that $\Gamma(\varphi) = \Phi$.}

\textbf{Proof.} Thanks to item ii) of Proposition 4.4, there is a countable $\mathbb{Q}$-linear subspace $\mathcal{C}$ of $\bar{\Gamma}(E)$ such that the $\mathbb{Q}$-linear space $\mathcal{C}(x) := \{ \bar{s}(x) : \bar{s} \in \mathcal{C} \}$ is dense in $E(x)$ for every $x \in X$. Given any $\bar{s} \in \mathcal{C}$, choose a representative $\Phi(\bar{s}) \in \bar{\Gamma}(F)$ of $\Phi([\bar{s}]_\sim) \in \Gamma(F)$. Then there exists $N \in \Sigma$ with $m(N) = 0$ such that the following properties hold:

$$
\Phi(\bar{s} + \bar{t})(x) = \Phi(\bar{s})(x) + \Phi(\bar{t})(x),
$$

$$
\Phi(q \bar{s})(x) = q \Phi(\bar{s})(x),
$$

for every $x \in X \setminus N$, $\bar{s}, \bar{t} \in \mathcal{C}$, and $q \in \mathbb{Q}$. (4.8)

Given any $x \in X$, we define the map $\varphi_x: \mathcal{C}(x) \to F(x)$ as

$$
\varphi_x(\bar{s}(x)) := \begin{cases} 
\Phi(\bar{s})(x) & \text{if } x \in X \setminus N, \\
0_F(x) & \text{if } x \in N.
\end{cases}
$$

The properties in (4.8) grant that each map $\varphi_x$ is a $\mathbb{Q}$-linear contraction, thus it can be uniquely extended to an $\mathbb{R}$-linear contraction $\varphi_x: E(x) \to F(x)$. Then we define $\varphi: TE \to TF$ as $\varphi(x, v) := (x, \varphi_x(v))$ for every $x \in X$ and $v \in E(x)$. In order to prove that $\varphi$ is a pre-morphism, it is sufficient to check its measurability. To this aim, we just have to show that $\varphi^{-1}(A \times \bar{B}_r(w)) \in \Sigma \otimes \mathcal{B}(\mathcal{B})$ for any $A \in \Sigma$, $w \in \mathcal{B}$, and $r > 0$. This follows from the identity

$$
\varphi^{-1}(A \times \bar{B}_r(w)) = S \cup \left\{ (x, v) \in ((A \setminus N) \times \mathcal{B}) \cap TE \mid \varphi_x(v) \in \bar{B}_r(w) \right\} = S \cup \bigcup_{n \in \mathbb{N}} \bigcap_{s \in \mathcal{C}} A_{n, s},
$$

where we set $S := ((A \setminus N) \times \mathcal{B}) \cap TE \in \Sigma$ if $\|w\|_{\mathcal{B}} \leq r$, while $S := \emptyset$ if $\|w\|_{\mathcal{B}} > r$, and

$$
A_{n, s} := \left\{ (x, v) \in ((A \setminus N) \times \mathcal{B}) \cap TE \mid \|v - \bar{s}(x)\|_{\mathcal{B}} < \frac{1}{k}, \|w - \Phi(\bar{s})(x)\|_{\mathcal{B}} < r + \frac{1}{k} \right\} \in \Sigma.
$$

Therefore, $\varphi$ is a pre-morphism from $E$ to $F$. We denote by $\varphi: E \to F$ its equivalence class. Observe that $\Gamma(\varphi)([\bar{s}]_\sim) = \Phi([\bar{s}]_\sim)$ for every $\bar{s} \in \mathcal{C}$ by construction. Finally, by arguing exactly as in the proof of Lemma 4.9, we deduce that $\{[\bar{s}]_\sim : \bar{s} \in \mathcal{C}\}$ generates $\Gamma(E)$, whence we can conclude that $\Gamma(\varphi) = \Phi$. Consequently, the statement is achieved. \hfill $\Box$

\textbf{Lemma 4.12} ($\Gamma$ is faithful). \textit{Let $(X, \Sigma, m)$ be a $\sigma$-finite measure space and $\mathcal{B}$ a universal separable Banach space. Let $E, F$ be separable Banach $\mathcal{B}$-bundles. Let $\varphi, \psi: E \to F$ be two morphisms of separable Banach $\mathcal{B}$-bundles such that $\varphi \neq \psi$. Then $\Gamma(\varphi) \neq \Gamma(\psi)$.}
Proof. Choose representatives \( \varphi, \psi \) of \( \varphi, \psi \), respectively. Pick a set \( P' \in \Sigma \) such that \( m(P') > 0 \) and \( \varphi(x, \cdot) \neq \psi(x, \cdot) \) for every \( x \in P' \). By item ii) of Proposition 4.4, there exists \( (s_n)_n \subseteq \Gamma(E) \) such that \( (s_n(x))_n \) is dense in \( E(x) \) for all \( x \in X \). Therefore, there exist \( n \in \mathbb{N} \) and \( P \subseteq P' \), \( m(P) > 0 \), and \( \varphi(x, s_n(x)) \neq \psi(x, s_n(x)) \) for every \( x \in P \). This ensures that \( \Gamma(\varphi)([s_n]_\sim) \neq \Gamma(\psi)([s_n]_\sim) \) and thus accordingly \( \Gamma(\varphi) \neq \Gamma(\psi) \), as required. \( \square \)

4.3. Representation theorem. We are finally in a position – by combining the whole machinery developed so far – to prove that every separable normed \( L^0(m) \)-module is the space of sections of a separable Banach bundle.

Theorem 4.13 (Representation theorem). Let \((X, \Sigma, m)\) be a complete, \( \sigma \)-finite measure space. Let \( \mathcal{M} \) be a countably-generated normed \( L^0(m) \)-module. Let \( \mathbb{B} \) be a universal separable Banach space. Then there exists a separable Banach \( \mathbb{B} \)-bundle \( E \) over \( X \) such that \( \Gamma(E) \cong \mathcal{M} \).

Proof.

STEP 1. First, fix a countable \( \mathbb{Q} \)-linear subspace \((v_n)_n\) of \( R(\mathcal{M}) \) that generates \( \mathcal{M} \). Choose a sequence \((\omega_n)_n \subseteq R(\mathcal{M}^*)\) such that the identities \(|\omega_n| = 1\) and \( \omega_n(v_n) = |v_n| \) hold \( m \)-a.e. for every \( n \in \mathbb{N} \). Let \( \ell \) be any lifting of \( m \), whose existence is granted by Theorem 2.2. Consider the operator \( \mathcal{L}: L^\infty(m) \to L^\infty(\Sigma) \) associated with \( \ell \) as in Theorem 2.4. Denote by \((\mathcal{M}, \mathcal{L})\) and \((\mathcal{N}, \mathcal{L}^*)\) the \( \mathcal{L} \)-liftings of \( R(\mathcal{M}) \) and \( R(\mathcal{M}^*) \), respectively; recall Theorem 3.5. Therefore,

\[
|\mathcal{L}^*(\omega_n)(x)| = 1, \quad \langle \mathcal{L}^*(\omega_n), \mathcal{L}(v_n) \rangle(x) = |\mathcal{L}(v_n)(x)|, \quad \text{for every } n \in \mathbb{N} \text{ and } x \in X. \tag{4.9}
\]

(Recall the discussion about the duality pairing \( \langle \cdot, \cdot \rangle \) between \( \mathcal{M} \) and \( \mathcal{N} \) in \S 3.3.)

Given any point \( x \in X \), we define the separable Banach subspace \( E(x) \) of \( \mathcal{M}_x \) as

\[
E(x) := \overline{\mathcal{L}(v_n)_x \mid n \in \mathbb{N}}. \quad \tag{4.10}
\]

Consider the isometric embedding \( R_x: \mathcal{N}_x \to (\mathcal{M}_x)' \), which has been introduced in (3.10) and studied in Proposition 3.10. Let us define

\[
\bar{v}_n(x) := \mathcal{L}(v_n)_x \in E(x),
\]

\[
\bar{\omega}_n(x) := R_x(\mathcal{L}^*(\omega_n)_x)|_{E(x)} \in E(x)', \quad \text{for every } n \in \mathbb{N} \text{ and } x \in X.
\]

It follows from the second line in (4.9) that

\[
\bar{\omega}_n(x)[\bar{v}_n(x)] = \|\bar{v}_n(x)\|_{E(x)} \quad \text{for every } n \in \mathbb{N} \text{ and } x \in X. \tag{4.11}
\]

Moreover, observe that for any \( n \in \mathbb{N} \) and \( x \in X \) one has that

\[
\|\bar{\omega}_n(x)\|_{E(x)'} \leq \|R_x(\mathcal{L}^*(\omega_n)_x)\|_{(\mathcal{M}_x)'} = \|\mathcal{L}^*(\omega_n)_x\|_{\mathcal{M}_x} \tag{4.9} = 1.
\]

Hence, if \( n \in \mathbb{N} \) and \( x \in X \) satisfy \( \bar{v}_n(x) \neq 0_{E(x)} \), then (4.11) forces \( \|\bar{\omega}_n(x)\|_{E(x)'} = 1 \). Finally, given any \( n, k \in \mathbb{N} \), we have that the function \( X \ni x \mapsto \bar{\omega}_n(x)[\bar{v}_k(x)] = \langle \mathcal{L}^*(\omega_n), \mathcal{L}(v_k) \rangle(x) \) is measurable. All in all, we have proven that \( \{E(x)\}_{x \in X} \) is a measurable collection of separable Banach spaces (in the sense of Definition 4.5) when equipped with \((\bar{v}_n)_n, (\bar{\omega}_n)_n\). Therefore, let us consider a measurable family \( \{I_x\}_{x \in X} \) of linear isometric embeddings \( I_x: E(x) \to \mathbb{B} \),
whose existence is granted by Theorem 4.6. We thus denote by $E: X \to \text{Gr}(\mathbb{B})$ the map $X \ni x \mapsto \text{I}_x(E(x))$, which is a separable Banach $\mathbb{B}$-bundle over $X$ thanks to Corollary 4.7.

**STEP 2.** Let $v \in R(\mathcal{M})$ be fixed. We claim that

\[ \mathcal{L}(v)_x \in E(x) \quad \text{for m-a.e. } x \in X. \]  

(4.12)

Indeed, we can find a sequence $(u_k)_k$ - where $u_k = \sum_{i=1}^{m_k} f_i^k \cdot u_i^k$ for some $(f_i^k)_{i=1}^{m_k} \subseteq L^\infty(\mathbb{m})$ and $(u_i^k)_{i=1}^{m_k} \subseteq \{v_i\}_n$ - such that $\lim_k \text{d}_{\mathcal{M}}(u_k, v) = 0$. Then (up to taking a not relabelled subsequence) we have that $|\mathcal{L}(u_k) - \mathcal{L}(v)|_x \to 0$ for m-a.e. point $x \in X$, or equivalently that $\lim_k \|\mathcal{L}(u_k)_x - \mathcal{L}(v)_x\|_x = 0$ for m-a.e. $x \in X$. Since

\[ \mathcal{L}(u_k)_x = \sum_{i=1}^{m_k} \mathcal{L}(f_i^k)(x) \mathcal{L}(u_i^k)_x \in E(x) \quad \text{for every } k \in \mathbb{N} \text{ and } x \in X, \]

we obtain (4.12). Now let us define the map $\overline{\mathcal{L}}(v): X \to \mathbb{B}$ as

\[ \overline{\mathcal{L}}(v)(x) := \begin{cases} \text{I}_x[\mathcal{L}(u_k)_x] & \text{if } \mathcal{L}(v)_x \in E(x), \\ 0_\mathbb{B} & \text{otherwise}. \end{cases} \]  

(4.13)

Choose any set $N \in \Sigma$ such that $m(N) = 0$ and $\mathcal{L}(v)_x = \lim_k \mathcal{L}(u_k)_x$ for every $x \in X \setminus N$. Hence, we have that

\[ \overline{\mathcal{L}}(v)(x) = \lim_{k \to \infty} \text{I}_x[\mathcal{L}(u_k)_x] = \lim_{k \to \infty} \sum_{i=1}^{m_k} \mathcal{L}(f_i^k)(x) \text{I}_x[\mathcal{L}(u_i^k)_x] \quad \text{for every } x \in X \setminus N. \]

By recalling Theorem 4.6 and the fact that the measure space $(X, \Sigma, m)$ is complete, we deduce that $\overline{\mathcal{L}}(v)$ is a measurable map from $X$ to $\mathbb{B}$. In other words, it holds that $\overline{\mathcal{L}}(v) \in \Gamma(\mathcal{E})$. Then let us denote by $\overline{\mathcal{J}}: R(\mathcal{M}) \to \Gamma(\mathcal{E})$ the map given by $\overline{\mathcal{J}}(v) := [\overline{\mathcal{L}}(v)]_\mathbb{B}$ for every $v \in R(\mathcal{M})$.

**STEP 3.** We aim to prove that $\overline{\mathcal{J}}$ maps $R(\mathcal{M})$ to $\Gamma_b(\mathcal{E})$ and that $\overline{\mathcal{J}}: R(\mathcal{M}) \to \Gamma_b(\mathcal{E})$ is an isomorphism of normed $L^\infty(\mathbb{m})$-modules. This is sufficient to conclude that the spaces $\mathcal{M}$ and $\Gamma(\mathcal{E})$ are isomorphic as normed $L^0(\mathbb{m})$-modules by item i) of Lemma 2.24. We first check the $L^\infty(\mathbb{m})$-linearity of $\overline{\mathcal{J}}$: if $v, w \in R(\mathcal{M})$ and $f, g \in L^\infty(\mathbb{m})$, then for m-a.e. $x \in X$ we have

\[ \overline{\mathcal{J}}(f \cdot v + g \cdot w)(x) = \text{I}_x[\mathcal{L}(f \cdot v + g \cdot w)_x] = \text{I}_x[\mathcal{L}(f)(x) \cdot \mathcal{L}(v)_x + \mathcal{L}(g)(x) \cdot \mathcal{L}(w)_x] \]

\[ = \mathcal{L}(f)(x) \cdot \overline{\mathcal{J}}(v)(x) + \mathcal{L}(g)(x) \cdot \overline{\mathcal{J}}(w)(x), \]

thus accordingly $\overline{\mathcal{J}}(f \cdot v + g \cdot w) = f \cdot \overline{\mathcal{J}}(v) + g \cdot \overline{\mathcal{J}}(w)$. Moreover, given any $v \in R(\mathcal{M})$ one has

\[ \|\overline{\mathcal{J}}(v)(x)\|_\mathbb{B} = \|\mathcal{L}(v)_x\|_x = \|\mathcal{L}(v)(x)\|_x = \mathcal{L}(|v|(x)) \quad \text{for m-a.e. } x \in X, \]

whence $|\overline{\mathcal{J}}(v)| = |v|$ holds in the m-a.e. sense. This ensures that $\overline{\mathcal{J}}$ maps $R(\mathcal{M})$ to $\Gamma_b(\mathcal{E})$ and that $\overline{\mathcal{J}}: R(\mathcal{M}) \to \Gamma_b(\mathcal{E})$ is a morphism of normed $L^\infty(\mathbb{m})$-modules preserving the pointwise norm. Finally, to prove that the map $\overline{\mathcal{J}}$ is surjective, it is enough to show that its image is dense in $\Gamma_b(\mathcal{E})$. Fix $s \in \Gamma_b(\mathcal{E})$ and $\varepsilon > 0$. Choose any representative $\tilde{s} \in \Gamma_{b, \mathbb{B}}(\mathcal{E})$ of $s$. Given that the sequence $(\overline{\mathcal{J}}(v_n)(x))_n$ is dense in $\mathcal{E}(x)$ for every $x \in X$ by (4.10) and (4.13), we can find a partition $(A_n)_n \subseteq \Sigma$ of $X$ such that $\|\tilde{s}(x) - \overline{\mathcal{J}}(v_n)(x)\| \leq \varepsilon$ for every $n \in \mathbb{N}$ and $x \in A_n$. This implies that the inequality $|\tilde{s} - \sum_{n \in \mathbb{N}} \chi_{A_n} \cdot \overline{\mathcal{J}}(v_n)| \leq \varepsilon$ holds everywhere on $X$. 
Now let us define \( v := \sum_{n \in \mathbb{N}} |X_{A_n}|_m \cdot v_n \in R(\mathcal{M}) \). Clearly \(|s - \mathcal{I}(v)| \leq \varepsilon\) holds \( m \)-a.e. by construction, so that \( \|s - \mathcal{I}(v)\|_{\Gamma(E)} \leq \varepsilon\). This yields surjectivity of \( \mathcal{I} \), which is consequently an isomorphism of normed \( L^\infty(m) \)-modules. Hence, the statement is finally achieved. \( \square \)

**Remark 4.14** (Proof of the representation theorem without the Axiom of Choice). In the above proof of Theorem 4.13, we made use of the theory of liftings of normed modules that we developed in §3. Nevertheless, it is possible to provide an alternative proof which does not rely upon the Axiom of Choice (and, thus, without appealing to von Neumann’s theory of lifting). Some weaker form of the Axiom of Choice is needed anyway, e.g., in the proof of Banach–Mazur Theorem 2.8, where Banach–Alaoglu theorem is used. We now sketch the argument of the alternative proof of Theorem 4.13, leaving its verification to the reader.

Fix a countable \( \mathbb{Q} \)-linear subspace \( (v_n)_n \) of \( \mathcal{M} \) that generates \( \mathcal{M} \). Choose \( (\omega_n)_n \subseteq \mathcal{M}^* \) such that \( |\omega_n| = 1 \) and \( \omega_n(v_n) = |v_n| \) hold \( m \)-a.e. for every \( n \in \mathbb{N} \). Given any \( n, k, \in \mathbb{N} \), pick a measurable representative \( \widetilde{\omega}_n(v_k) \) of \( \omega_n(v_k) \). Then we can find a \( m \)-null set \( N \subseteq X \) such that

\[
\widetilde{\omega}_n(v_k + v_{k'}) (x) = \widetilde{\omega}_n(v_k)(x) + \widetilde{\omega}_n(v_{k'})(x),
\]

\[
\widetilde{\omega}_n(qv_k)(x) = q \widetilde{\omega}_n(v_k)(x),
\]

\[
\widetilde{\omega}_n(v_k)(x) \leq \omega_k(v_k)(x)
\]

for every \( n, k, k' \in \mathbb{N}, q \in \mathbb{Q}, \) and \( x \in X \setminus N \). Observe that for any \( x \in X \setminus N \) the family

\[
V_x := \left\{ (\widetilde{\omega}_n(v_k)(x))_{n \in \mathbb{N}} \mid k \in \mathbb{N} \right\} \subseteq \ell^\infty
\]

is a \( \mathbb{Q} \)-linear subspace of \( \ell^\infty \). Hence, calling \( E(x) := \{0_{\ell^\infty}\} \) for all \( x \in N \) and \( E(x) := \cl_{\ell^\infty}(V_x) \) for all \( x \in X \setminus N \), we have that \( \{E(x)\}_{x \in X} \) is a family of separable Banach subspaces of \( \ell^\infty \).

Moreover, for any \( x \in X \) and \( n, k \in \mathbb{N} \), we define \( \tilde{v}_k(x) \in E(x) \) and \( \tilde{\omega}_n(x) \in E(x)' \) as follows: trivially, \( \tilde{v}_k(x) := 0_{E(x)} \) and \( \tilde{\omega}_n(x) := 0_{E(x)'} \) if \( x \in N \); if \( x \notin N \), then we set

\[
\tilde{v}_k(x) := (\widetilde{\omega}_n(v_k)(x))_{n \in \mathbb{N}} \in E(x),
\]

while we denote by \( \tilde{\omega}_n(x) : E(x) \to \mathbb{R} \) the unique linear and continuous operator satisfying \( \tilde{\omega}_n(x)[\tilde{v}_k(x)] = \omega_n(v_k)(x) \) for every \( k' \in \mathbb{N} \). Then it holds that \( \{E(x)\}_{x \in X} \) is a measurable collection of separable Banach spaces – together with \( \tilde{v}_k(x) \) and \( \tilde{\omega}_n(x) \). Finally, one can also prove that the associated separable Banach \( \mathcal{B} \)-bundle \( E \) satisfies \( \Gamma(E) \cong \mathcal{M} \), as desired. \( \blacksquare \)

In analogy with [20], we have a Serre–Swan theorem for separable normed \( L^0(m) \)-modules:

**Theorem 4.15** (Serre–Swan theorem). Let \( (X, \Sigma, m) \) be a complete, \( \sigma \)-finite measure space. Let \( \mathcal{B} \) be a universal separable Banach space. Then the section functor

\[
\Gamma : \text{SBB}_\mathcal{B}(X, \Sigma, m) \longrightarrow \text{NMod}_{\mathcal{B}}(X, \Sigma, m)
\]

is an equivalence of categories. In particular, if \( (X, \Sigma, m) \) is a separable measure space, then

\[
\Gamma : \text{SBB}_\mathcal{B}(X, \Sigma, m) \longrightarrow \text{NMod}_{\mathcal{B}}(X, \Sigma, m)
\]

is an equivalence of categories.
Proof. The first claim follows from Lemma 4.11, Lemma 4.12, and Theorem 4.13. The second claim follows from the first one by taking Proposition 2.15 into account. □

Problem 4.16. Does there exist some notion of measurable Banach bundle that is sufficient to describe also the duals of separable normed $L^0(m)$-modules? In this regard, it is well-known that all duals of separable Banach spaces can be embedded linearly and isometrically into $\ell^\infty$, thus the space $\ell^\infty$ would be a good candidate for the ‘ambient space’ of the fibers of the bundle.

Finally, we now extend Banach–Mazur Theorem 2.8 to the setting of separable normed $L^0(m)$-modules; i.e., as we are going to see, we prove the existence of universal such modules.

Definition 4.17 (Universal separable normed $L^0$-module). Let $(X, \Sigma, m)$ be a $\sigma$-finite measure space. Let $M$ be a separable normed $L^0(m)$-module. Then we say that $M$ is a universal separable normed $L^0(m)$-module if for any separable normed $L^0(m)$-module $N$ there exists a normed $L^0(m)$-module morphism $I: N \rightarrow M$ that preserves the pointwise norm.

Given a measurable space $(X, \Sigma)$ and a universal separable Banach space $B$, we shall denote by $\Gamma(B)$ the space of sections of the separable Banach $B$-bundle $X \ni x \mapsto B \in \text{Gr}(B)$.

Theorem 4.18 (Existence of universal modules). Let $(X, \Sigma, m)$ be a complete, $\sigma$-finite, separable measure space. Let $B$ be a universal separable Banach space. Then $\Gamma(B)$ is a universal separable normed $L^0(m)$-module.

Proof. Let $M$ be any given separable normed $L^0(m)$-module. Theorem 4.15 grants the existence of a separable Banach $B$-bundle $E$ over $X$ such that $\Gamma(E) \cong M$. With a slight abuse of notation, we use the symbol $B$ to denote the separable Banach $B$-bundle $X \ni x \mapsto B \in \text{Gr}(B)$. Consider the pre-morphism $\varphi$ from $E$ to $B$ defined as follows: given any $x \in X$, we declare that $\varphi(x, \cdot): E(x) \rightarrow B$ is the inclusion map. Call $\varphi: E \rightarrow B$ the equivalence class of $\varphi$. Therefore, it holds that $\Gamma(\varphi): \Gamma(E) \rightarrow \Gamma(B)$ is a morphism of normed $L^0(m)$-modules that preserves the pointwise norm. The statement is achieved. □

Appendix A. Representation of normed modules via direct limits

In this section we provide an alternative description of separable normed $L^0(m)$-modules, which builds upon the representation results for proper modules that have been proven in [20].

Roughly speaking, the strategy we will adopt is the following: any separable normed $L^0(m)$-module $M$ can be obtained as direct limit of finite-dimensional normed $L^0(m)$-modules $(M_k)_k$; each module $M_k$ is the space of sections of some finite-dimensional Banach bundle $E_k$, thus by ‘patching together’ the bundles $E_k$ we obtain some notion of separable Banach bundle $E$, whose space of sections can be eventually identified with $M$. Even though this approach is much more ‘implicit’ than the one proposed in §4, it has the advantage of clarifying how to approximate separable normed $L^0(m)$-modules by proper ones.

A word on notation: for simplicity, we will use some terminology that has been already used in §4 (such as ‘separable Banach bundle’ and so on), but with a different meaning. Since this section is independent of §4, we believe that this will not cause any ambiguity.
Let \((X, \Sigma, m)\) be a fixed \(\sigma\)-finite measure space. For the sake of simplicity, we shall write \(m_A := m|_A\) for every \(A \in \Sigma\) such that \(m(A) > 0\).

As observed in [14, Section 2], any normed \(L^0(m_A)\)-module \(\mathcal{N}\) can be canonically viewed as a normed \(L^0(m)\)-module; we shall denote it by \(\text{Ext}_A(\mathcal{N})\) and call it the \textit{extension} of \(\mathcal{N}\).

On the other hand, a normed \(L^0(m)\)-module \(\mathcal{M}\) can be ‘localised’ on \(A\) as follows: we define \(\mathcal{M}|_A\) as the pullback of \(\mathcal{M}\) under the identity map from \((X, \Sigma, m_A)\) and \((X, \Sigma, m)\) (cf. [10, Section 1.6]), so that \(\mathcal{M}|_A\) is a normed \(L^0(m_A)\)-module. It is clear that \(\text{Ext}_A(\mathcal{M}|_A)\) is isomorphic to the normed \(L^0(m)\)-submodule \([X_A]|_m \cdot \mathcal{M} = \{[X_A]|_m \cdot v : v \in \mathcal{M}\}\) of \(\mathcal{M}\).

A.1. \textbf{Separable Banach bundles}. We propose an alternative notion of separable Banach bundle over \(X\), which extends the one that has been introduced in [20]. The language we adopt here is slightly different from that of [20], but it can be readily checked that the two resulting theories are fully consistent.

We fix the notation \(\mathbb{N} := \mathbb{N} \cup \{\infty\}\). Given any \(n \in \mathbb{N}\), we define the vector space \(V_n\) as
\[
V_n := \begin{cases} \mathbb{R}^n & \text{if } n < \infty, \\ c_{00} & \text{if } n = \infty, \end{cases}
\]
where \(c_{00}\) stands for the space of all sequences in \(\mathbb{R}\) having only finitely many non-zero terms. Calling \((e_n)_{n \in \mathbb{N}}\) the canonical basis of \(V_\infty\) (i.e., \(e_n := (\delta_{im})_{i \in \mathbb{N}}\) for all \(n \in \mathbb{N}\)), we shall always implicitly identify \(V_n\) with the subspace of \(V_\infty\) spanned by \(e_1, \ldots, e_n\).

The topology we shall consider on the space \(V_\infty\) is the one induced by the \(\ell^\infty\)-norm. It is straightforward to check that a set \(S \subseteq c_{00}\) belongs to the Borel \(\sigma\)-algebra associated to such topology if and only if \(S \cap V_n\) is a Borel subset of \((\mathbb{R}^n, d_{\text{Eucl}})\) for every \(n \in \mathbb{N}\).

\textbf{Definition A.1} (Banach bundle of dimension \(n\)). Let \(n \in \mathbb{N}\) be given. Then we say that a couple \(E = (A, n)\) is a Banach bundle of dimension \(n\) over a given set \(A \in \Sigma\) provided the function \(n: A \times V_n \to [0, +\infty)\) is measurable and satisfies the following property:
\[
n(x, \cdot) \text{ is a norm on } V_n \text{ for every } x \in A.
\]

Notice that \((A, n)\) is a Banach bundle of dimension \(n \in \mathbb{N}\) if and only if \((A, n)|_{A \times V_k}\) is a Banach bundle of dimension \(k\) for every \(k \in \mathbb{N}\) satisfying \(k \leq n\).

\textbf{Remark A.2.} We observe that if \(n = \infty\), then the norms \(n(x, \cdot)\) on \(V_\infty\) cannot be complete, as the vector space \(c_{00}\) does not support any complete norm; cf. for instance [1].

Let us consider a Banach bundle \(E = (A, n)\) of dimension \(n \in \mathbb{N}\). Then the space \(\Gamma_A(E)\) of \textit{sections} of \(E\) is defined as the family of all measurable maps \(s: A \to \mathbb{R}^n\), considered up to \(m_A\)-a.e. equality. As shown in [20], it turns out that \(\Gamma_A(E)\) is a normed \(L^0(m_A)\)-module when endowed with the natural pointwise operations and the following pointwise norm:
\[
|s|(x) := n(x, s(x)) \text{ for } m_A\text{-a.e. } x \in A
\]
for every \(s \in \Gamma_A(E)\). More precisely, \(\Gamma_A(E)\) is a free \(L^0(m_A)\)-module of rank \(n\).
We now define the space of sections of a Banach bundle $E = (A, n)$ of any dimension $n \in \bar{\mathbb{N}}$, possibly $n = \infty$. Call $I$ the set of all $k \in \mathbb{N}$ with $k \leq n$. We set $E_k := (A, n|_{A \times V_k})$ for all $k \in I$. Given any $j, k \in I$ with $j \leq k$, we have a canonical inclusion map $\iota_{jk} : \Gamma_A(E_j) \hookrightarrow \Gamma_A(E_k)$. Namely, $\iota_{jk}$ is the map sending (the equivalence class of) any section $s = (s_1, \ldots, s_j)$ of $E_j$ to (the equivalence class of) the section $(s_1, \ldots, s_j, 0 \ldots, 0)$ of $E_k$. It can be readily checked that $\{(\Gamma_A(E_k))_{k \in I}, \{\iota_{jk}\}_{j \leq k}\}$ is a direct system in the category of normed $L^0(\mathfrak{m}_A)$-modules. Then we define

$$\Gamma_A(E) := \lim_{\longrightarrow} \Gamma_A(E_*)$$

whose existence is granted by Theorem 2.16. Notice that such definition of $\Gamma_A(E)$ is consistent with the previous one when $E$ is a Banach bundle of finite dimension; cf. [21, Lemma 2.10].

**Definition A.3** (Banach bundle). We say that $E = \{(A_n, E_n)\}_{n \in \mathbb{N}}$ is a separable Banach bundle over $X$ provided $\{A_n\}_{n \in \mathbb{N}} \subseteq \Sigma$ is a partition of $X$ and each $E_n = (A_n, n_n)$ is a Banach bundle of dimension $n$. Moreover, we say that $E$ is proper provided $\mathfrak{m}(A_\infty) = 0$.

Let $E = \{(A_n, E_n)\}_{n \in \mathbb{N}}$ be a separable Banach bundle over $X$. Then we define the space of its sections as

$$\Gamma(E) := \prod_{n \in \mathbb{N}} \operatorname{Ext}_{A_n}(\Gamma_{A_n}(E_n)).$$

The direct product $\Gamma(E)$ inherits an (algebraic) $L^0(\mathfrak{m})$-module structure. Moreover, the fact that the sets $A_n$ are pairwise disjoint grants that the following definition is meaningful:

$$|s| := \sum_{n \in \mathbb{N}} [X_{A_n}]_{\mathfrak{m}} |s_n| \quad \text{for every } s = \{s_n\}_{n \in \mathbb{N}} \in \Gamma(E). \quad (A.1)$$

It is straightforward to verify that (A.1) actually defines a pointwise norm on $\Gamma(E)$, whose associated distance $d_{\Gamma(E)}$ is complete. Therefore, $\Gamma(E)$ is a normed $L^0(\mathfrak{m})$-module.

**A.2. Representation theorem.** The purpose of this subsection is to show that any separable normed $L^0(\mathfrak{m})$-module $\mathcal{M}$ is isomorphic to the space of sections $\Gamma(E)$ of some separable Banach bundle $E$ over $X$.

In the sequel, we will need the following consequence of [20, Theorem 3], which we rephrase in the current language. Actually, the result was obtained for modules on metric measure spaces, but the same proof can be repeated verbatim in the case of $\sigma$-finite measure spaces.

**Theorem A.4** (Representation theorem for proper modules). Let $(X, \Sigma, \mathfrak{m})$ be a $\sigma$-finite measure space. Let $\mathcal{M}$ be a proper normed $L^0(\mathfrak{m})$-module. Then there exists a proper Banach bundle $E$ over $X$ such that $\mathcal{M}$ is isomorphic to $\Gamma(E)$ as a normed $L^0(\mathfrak{m})$-module.

**Remark A.5.** Let $\mathcal{M}$ be a separable normed $L^0(\mathfrak{m})$-module that is not finitely-generated on any measurable subset of $X$ having positive $\mathfrak{m}$-measure. We claim that there exists an increasing sequence $(\mathcal{N}_n)_{n \in \mathbb{N}}$ of normed $L^0(\mathfrak{m})$-submodules of $\mathcal{M}$ with the following properties:

1. Each $\mathcal{N}_n$ has local dimension $n$ on $X$.
2. The set $\bigcup_{n \in \mathbb{N}} \mathcal{N}_n$ is dense in $\mathcal{M}$.
We construct the desired modules $\mathcal{N}_n$ in a recursive way. Fix any dense subset $(v_n)_{n \in \mathbb{N}}$ of $\mathcal{M}$. We aim to build a sequence $(\mathcal{N}_n)_{n \in \mathbb{N}}$ of normed $L^0(\mathfrak{m})$-modules satisfying i) and such that each $\mathcal{N}_n$ contains the elements $v_1, \ldots, v_n$. This would clearly imply ii). First, we define $\mathcal{N}_1$ as the normed $L^0(\mathfrak{m})$-module generated by the element

$$\sum_{k \in \mathbb{N}} \left[ \chi_{\{|v_k|>0\}} \chi_{\{v_j<|v_k|>0\}} \right] \mathfrak{m} \cdot v_k \in \mathcal{M}.$$

Then $\mathcal{N}_1$ has dimension 1 (as $\mathcal{M}$ is not finitely-generated on any measurable set) and $v_1 \in \mathcal{N}_1$.

Now suppose to have already defined $\mathcal{N}_n$ for some $n \in \mathbb{N}$. We want to define $\mathcal{N}_{n+1}$. Fix a local basis $w_1, \ldots, w_n$ of $\mathcal{N}_n$. For any $k \geq n + 1$ we call $B'_k$ the set where $w_1, \ldots, w_n, v_k$ are independent and $B_k := B'_k \setminus \bigcup_{j=n+1}^{k-1} B'_j$; then we define $\mathcal{N}_{n+1}$ as the normed $L^0(\mathfrak{m})$-module generated by $\mathcal{N}_n \cup \{w_{n+1}\}$, where we put $w_{n+1} := \sum_{k=n+1}^{\infty} \chi_{B_k} \mathfrak{m} \cdot v_k$. Hence, $\mathcal{N}_{n+1}$ has local dimension equal to $n + 1$ on $X$ (as $\mathcal{M}$ is not finitely-generated on any measurable set) and contains the elements $v_1, \ldots, v_{n+1}$ by construction.

By building on top of Theorem A.4, we can eventually prove the following result:

**Theorem A.6 (Representation theorem).** Let $(X, \Sigma, \mathfrak{m})$ be any $\sigma$-finite measure space. Let $\mathcal{M}$ be a separable normed $L^0(\mathfrak{m})$-module. Then there exists a separable Banach bundle $E$ over the space $X$ such that $\Gamma(E) \cong \mathcal{M}$.

**Proof.** Let us call $\{A_n\}_{n \in \mathbb{N}}$ the dimensional decomposition of the module $\mathcal{M}$. Consider the normed $L^0(\mathfrak{m}_{A_n})$-module $\mathcal{N} := \mathcal{M}|_{A_n}$. As shown in Remark A.5, one can build an increasing sequence $(\mathcal{N}_k)_{k \in \mathbb{N}}$ of normed $L^0(\mathfrak{m}_{A_n})$-submodules of $\mathcal{N}$ such that each $\mathcal{N}_k$ has local dimension equal to $k$ on the set $A_n$ and $\bigcup_{k \in \mathbb{N}} \mathcal{N}_k$ is dense in $\mathcal{N}$. Let us pick any sequence $(v_k)_{k \in \mathbb{N}} \subseteq \mathcal{N}$ such that

$$v_1, \ldots, v_k$$

is a local basis for $\mathcal{N}_k$ on $A_n$ for every $k \in \mathbb{N}$.

(A.2)

Given $k \in \mathbb{N}$, we can find (by Theorem A.4) a Banach bundle $F'_k = (A_{\infty}, \mathfrak{n}'_k)$ of dimension $k$ such that $\Gamma_{A_{\infty}}(F'_k) \cong \mathcal{N}_k$. Under such isomorphism, the elements $v_1, \ldots, v_k \in \mathcal{N}_k$ correspond to some sections $s^1_k, \ldots, s^k_k \in \Gamma(F'_k)$, respectively. Pick representatives $\bar{s}^1_k, \ldots, \bar{s}^k_k$ of them. Then we know from (A.2) that there is a measurable set $N \subseteq A_{\infty}$ with $\mathfrak{m}(N) = 0$ such that

$$\bar{s}^1_k(x), \ldots, \bar{s}^k_k(x)$$

is a basis of $\mathbb{R}^k$ for every $k \in \mathbb{N}$ and $x \in A_{\infty} \setminus N$.

Therefore, for any $k \in \mathbb{N}$ we define a new Banach bundle $F_k = (A_{\infty}, \mathfrak{n}_k)$ of dimension $k$ as

$$\mathfrak{n}_k\left((\lambda_1, \ldots, \lambda_k)\right) := \begin{cases} \mathfrak{n}'_k(x, \lambda_1 \bar{s}^1_k(x) + \ldots + \lambda_k \bar{s}^k_k(x)) & \text{if } x \in A_{\infty} \setminus N, \\ (\lambda_1^2 + \ldots + \lambda_k^2)^{1/2} & \text{if } x \in N. \end{cases}$$

It follows that $\Gamma_{A_{\infty}}(F_k) \cong \mathcal{N}_k$ and $\mathfrak{n}_{k+1}|_{A_{\infty} \times V_k} = \mathfrak{n}_k$ for every $k \in \mathbb{N}$. Hence, we can consider the Banach bundle $E_{\infty} = (A_{\infty}, \mathfrak{n}_{\infty})$ of dimension $\infty$, where $\mathfrak{n}_{\infty} : A_{\infty} \times V_{\infty} \to [0, \infty)$ is defined as the unique function satisfying $\mathfrak{n}_{\infty}|_{A_{\infty} \times V_k} = \mathfrak{n}_k$ for all $k \in \mathbb{N}$. As granted by Lemma 2.17, the direct limit of $\Gamma_{A_{\infty}}(F_k) \cong \mathcal{N}_k$ is isomorphic to $\mathcal{N}$, thus $\Gamma_{A_{\infty}}(E_{\infty}) \cong \mathcal{M}|_{A_{\infty}}$. To conclude, notice that $\mathcal{M}|_{X \setminus A_{\infty}}$ is a proper normed $L^0(\mathfrak{m}_{X \setminus A_{\infty}})$-module, whence accordingly

$$\mathcal{M}|_{X \setminus A_{\infty}} \cong \Gamma_{X \setminus A_{\infty}}(E')$$

for some proper Banach bundle $E' = \{(A_n, E_n)\}_{n \in \mathbb{N}}$. 


as a consequence of Theorem A.4. Hence, the separable Banach bundle $E := \{(A_n, E_n)\}_{n \in \mathbb{N}}$ satisfies $\Gamma(E) \cong \mathcal{M}$, as required.

**Remark A.7** (‘Serre–Swan theorem’). Given a $\sigma$-finite measure space $(X, \Sigma, m)$ and two separable Banach bundles $E = \{(A_n, E_n)\}_{n \in \mathbb{N}}$ and $F = \{(B_m, F_m)\}_{m \in \mathbb{N}}$ over $X$, we can define a pre-morphism between $E$ and $F$ as a family $\varphi = \{\varphi_{nm}\}_{n,m \in \mathbb{N}}$ of measurable maps

$$\varphi_{nm}: (A_n \cap B_m) \times V_n \rightarrow (A_n \cap B_m) \times V_m$$

such that $\varphi_{nm}(x, V_n) \subseteq \{x\} \times V_m$ and

$$\varphi_{nm}(x, \cdot): (V_n, n^E_n(x, \cdot)) \rightarrow (V_m, n^F_m(x, \cdot))$$

is a linear contraction for every $n, m \in \mathbb{N}$ and $x \in A_n \cap B_m$, where we call $E_n = (A_n, n^E_n)$ and $F_m = (B_m, n^F_m)$.

We declare two pre-morphisms $\{\varphi_{nm}\}_{n,m \in \mathbb{N}}$ and $\{\psi_{nm}\}_{n,m \in \mathbb{N}}$ between $E$ and $F$ to be equivalent provided there exists a set $N \in \Sigma$ with $m(N) = 0$ such that

$$\varphi_{nm}(x, \cdot) = \psi_{nm}(x, \cdot)$$

for every $x \in X \setminus N$ and $n, m \in \mathbb{N}$.

Therefore, it makes sense to consider the category of separable Banach bundles over $X$ having the equivalence classes of pre-morphisms as arrows. We point out that – in analogy with what done in §4.2 – it is possible to promote the correspondence $E \mapsto \Gamma(E)$ to a functor (called the section functor) from the category of separable Banach bundles over $X$ to that of separable normed $L^0(m)$-modules. Furthermore, the section functor can be shown to be an equivalence of categories (the so-called ‘Serre–Swan theorem’). We omit the details. ■

**Appendix B. Cotangent bundle on metric measure spaces**

As mentioned in the introduction, in the study of the differential structure of metric measure spaces a key role is played by the so-called cotangent module $L^0(T^*X)$, which has been introduced by Gigli in [10]. We now propose an alternative axiomatisation, at least in the case in which $W^{1,2}(X)$ is separable. In the approach we are going to present, we directly introduce a notion of cotangent bundle $T^*X$ over $(X, d, m)$, which does not require the theory of normed modules to be formulated (cf. Remark B.2). Even though we just consider $p = 2$ for simplicity, a similar construction could be performed for any $p \in (1, \infty)$.

By **metric measure space** $(X, d, m)$ we mean a complete, separable metric space $(X, d)$, endowed with a non-negative Radon measure $m$. Calling $\Sigma$ the completion of the Borel $\sigma$-algebra $\mathcal{B}(X)$ and $\mathcal{m}$ the completion of the measure $m$, it holds that $(X, \Sigma, \mathcal{m})$ is a complete, $\sigma$-finite measure space, thus in particular the results of §4.3 can be applied. Observe that normed $L^0(m)$-modules and normed $L^0(\mathcal{m})$-modules can be identified in a canonical way.

We denote by $\text{LIP}(X)$ the space of all real-valued Lipschitz functions on $X$. Given any function $f \in \text{LIP}(X)$, its **slope** $\text{lip}(f): X \rightarrow [0, +\infty)$ is defined as

$$\text{lip}(f)(x) := \lim_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}$$

if $x \in X$ is an accumulation point,
and \( \text{lip}(f)(x) := 0 \) otherwise. We define the Cheeger energy \( \text{Ch}: L^2(m) \to [0, +\infty] \) on \( X \) as
\[
\text{Ch}(f) := \inf \left\{ \lim_{n \to \infty} \int \text{lip}^2(f_n) \, dm \mid (f_n)_n \subseteq \text{LIP}(X), [f_n]_m \in L^2(m), [f_n]_m \to f \text{ in } L^2(m) \right\}
\]
for every \( f \in L^2(m) \). Then (following [8]) we define the Sobolev space on \( (X, d, m) \) as
\[
W^{1,2}(X) := \left\{ f \in L^2(m) \mid \text{Ch}(f) < +\infty \right\}.
\]
Given any \( f \in W^{1,2}(X) \), there exists a distinguished function \( |Df| \in L^2(m) \) – called the minimal relaxed slope of \( f \) – which provides the integral representation \( \text{Ch}(f) = \int |Df|^2 \, dm \) of the Cheeger energy. It turns out that \( W^{1,2}(X) \) is a Banach space if endowed with the norm
\[
\|f\|_{W^{1,2}(X)} := \left( \int |f|^2 \, dm + \text{Ch}(f) \right)^{1/2}
\]
for every \( f \in W^{1,2}(X) \).

In the case in which the measure \( m \) is boundedly finite, some equivalent notions of Sobolev spaces have been introduced in [3, 5, 24]. It is worth pointing out that in most cases the Sobolev space \( W^{1,2}(X) \) is separable. Indeed, as proven in [2], the separability is granted by the reflexivity of the Sobolev space, which is in turn known to hold on a vast class of metric measure spaces; for instance, whenever the underlying metric space \( (X, d) \) is doubling (cf. [2]) or carries a ‘nice’ geometric structure (such as Euclidean spaces, Carnot groups, Finsler manifolds, subRiemannian manifolds, and locally CAT(\( \kappa \)) spaces; cf. the introduction of [19]). To the best of our knowledge, the only known examples of a non-separable Sobolev space are described in [2].

With the terminology introduced above at our disposal, we can now state and prove our existence and uniqueness result about the cotangent bundle of a metric measure space:

**Theorem B.1** (Cotangent bundle). Let \( (X, d, m) \) be a metric measure space such that the Sobolev space \( W^{1,2}(X) \) is separable. Let \( \mathcal{B} \) be a universal separable Banach space. Then there exists a unique couple \( (\mathbb{T}^*X, d) \), where \( \mathbb{T}^*X \) is a separable Banach \( \mathcal{B} \)-bundle over \( X \) (in the sense of Definition 4.1) called the cotangent bundle of \( (X, d, m) \) and \( d: W^{1,2}(X) \to \Gamma(\mathbb{T}^*X) \) is a linear operator called the differential, such that the following properties are satisfied:

i) It holds that \( |df| = |Df| \) in the \( m \)-a.e. sense for every \( f \in W^{1,2}(X) \).

ii) Given any dense sequence \( (f_n)_n \) in \( W^{1,2}(X) \), it holds that
\[
\{ df_n(x) \mid n \in \mathbb{N} \}
\]
is dense in \( \mathbb{T}^*_xX := \mathbb{T}^*X(x) \) for \( m \)-a.e. \( x \in X \).

Uniqueness is intended up to unique isomorphism: given any other couple \( (E, d') \) with the same properties, there exists a unique isomorphism \( \varphi: \mathbb{T}^*X \to E \) such that
\[
W^{1,2}(X) \xrightarrow{d} \Gamma(\mathbb{T}^*X) \xrightarrow{d'} \Gamma(\varphi) \xrightarrow{\Gamma(\varphi)} \Gamma(E)
\]
is a commutative diagram.
Proof. First of all, recall the notion of cotangent module introduced in [10, Definition 2.2.1] (cf. [13, Proposition 4.1.8] for the formulation we will present, via normed \(L^0(m)\)-modules): there exists a unique couple \((L^0(T^*X), d)\), where \(L^0(T^*X)\) is a separable normed \(L^0(m)\)-module and \(d: W^{1,2}(X) \to L^0(T^*X)\) is a linear operator, such that:

i') \(|df| = |Df|\) holds \(m\)-a.e. for every \(f \in W^{1,2}(X)\).

ii') Given any \((f_n)_n\) dense in \(W^{1,2}(X)\), the family \(\{df_n : n \in \mathbb{N}\}\) generates \(L^0(T^*X)\).

(The separability of \(L^0(T^*X)\) is granted, e.g., by [13, Lemma 3.1.17].) Uniqueness is intended in the following sense: given another couple \((M, d')\) having the same properties, there exists a unique isomorphism \(\Phi: L^0(T^*X) \to M\) of normed \(L^0(m)\)-modules such that \(\Phi \circ d = d'\).

By applying Lemma 4.11, we find a separable Banach \(\mathbb{B}\)-bundle \(T^*X\) over the space \(X\) such that \(\Gamma(T^*X) \cong L^0(T^*X)\). It is clear that the map \(d: W^{1,2}(X) \to \Gamma(T^*X)\) satisfies the item i) as a consequence of i'). Moreover, the validity of the item ii) can be deduced from ii') by suitably adapting the arguments in the proof of Proposition 4.4. Finally, the uniqueness of the couple \((T^*X, d)\) up to unique isomorphism can be obtained by combining the analogous property of \((L^0(T^*X), d)\) with the fact (Theorem 4.15) that the section functor is an equivalence of categories. Therefore, the statement is achieved. \(\square\)

Remark B.2. The theory of normed modules is not really used in Theorem B.1, in the sense that the spaces \(\Gamma(T^*X)\) and \(\Gamma(E)\) can be just considered as vector spaces, without looking at their module structure. The same observation applies to the morphism \(\Gamma(\varphi)\) as well. \(\blacksquare\)

**APPENDIX C. THE PULLBACK BUNDLE**

In the field of nonsmooth differential geometry, a prominent role is played by the notion of pullback of a normed module, which we recalled in Theorem 2.18. The primary purpose of this section is to show that separable Banach bundles come with a natural notion of pullback (see Definition C.1) which is consistent with that of normed modules (see Theorem C.3).

For technical reasons (namely, because we will need the Disintegration Theorem C.2), we will mostly work with metric measure spaces (instead of general \(\sigma\)-finite measure spaces). Furthermore, we study the projection operator \(\text{Pr}_\varphi\) associated with a normed \(L^0(m)\)-module \(\mathcal{M}\) (see Theorem C.9), which is a distinguished left inverse of the pullback map \(\varphi^*: \mathcal{M} \to \varphi^*\mathcal{M}\). In the setting of separable normed modules, we also provide a fiberwise description of the operator \(\text{Pr}_\varphi\); see Proposition C.11 for the details.

C.1. **Pullback and section functors commute.** Let us begin with the definition of pullback of a separable Banach bundle.

**Definition C.1 (Pullback bundle).** Let \((X, \Sigma_X), (Y, \Sigma_Y)\) be measurable spaces and \(\mathbb{B}\) a universal separable Banach space. Let \(\varphi: X \to Y\) be a measurable map. Let \(E\) be a separable Banach \(\mathbb{B}\)-bundle over \(Y\). Then we define the map \(\varphi^*E: X \to \text{Gr}(\mathbb{B})\) as

\[
\varphi^*E(x) := E(\varphi(x)) \quad \text{for every } x \in X.
\]
It follows from Lemma 2.7 that $\varphi^*E$ is a separable Banach $B$-bundle over $X$. We call it the pullback bundle of $E$ under the map $\varphi$.

Let $(X, d_X), (Y, d_Y)$ be separable metric spaces. By $\mathcal{P}(X)$ we mean the family of all Borel probability measures $\mu$ on $X$, i.e., with $\mu(X) = 1$. Then a family $\{\mu_y\}_{y \in Y} \subseteq \mathcal{P}(X)$ is said to be a Borel family of measures provided for any Borel function $f : X \to [0, +\infty]$ it holds that

$$Y \ni y \mapsto \int f \, d\mu_y \in [0, +\infty] \quad \text{is Borel measurable.}$$

We need the following classical result, whose proof can be found, e.g., in [4, Theorem 5.3.1].

**Theorem C.2** (Disintegration theorem). Let $(X, d_X, m_X), (Y, d_Y, m_Y)$ be metric measure spaces, with $m_X, m_Y$ finite. Let $\varphi : X \to Y$ be a Borel map satisfying $\varphi_* m_X = m_Y$. Then there exists a $m_Y$-a.e. uniquely determined Borel family of measures $\{m^y_Y\}_{y \in Y} \subseteq \mathcal{P}(X)$ such that

$$m^y_Y(X \setminus \varphi^{-1}(y)) = 0 \quad \text{for } m_Y\text{-a.e. } y \in Y,$$

$$\int f \, dm_X = \int \left( \int f \, dm^y_X \right) \, dm_Y(y) \quad \text{for every } f : X \to [0, +\infty] \ \text{Borel.}$$

**We abbreviate the conditions** (C.1a) and (C.1b) to the single expression $m_X = \int m^y_Y \, dm_Y(y)$.

We are now ready to prove that the sections of the pullback bundle can be identified with the elements of the pullback of the space of sections of the bundle itself.

**Theorem C.3** (Pullback and section functors commute). Let $(X, d_X, m_X), (Y, d_Y, m_Y)$ be metric measure spaces, with $m_X, m_Y$ finite. Let $B$ be a universal separable Banach space and let $\varphi : X \to Y$ be a Borel map satisfying $\varphi_* m_X = m_Y$. Let $E$ be a separable Banach $B$-bundle over $Y$. Then it holds that

$$\varphi^* \Gamma(E) \cong \Gamma(\varphi^*E),$$

the pullback map $\varphi^* : \Gamma(E) \to \Gamma(\varphi^*E)$ being given by $(\varphi^* s)(x) := s(\varphi(x))$ for $m_X$-a.e. $x \in X$.

**Proof.** First of all, observe that $|\varphi^* s|(x) = \|(s \circ \varphi)(x)\|_B = \|(s \circ \varphi)(x)\|_B$ holds for every $s \in \Gamma(E)$ and $m_X$-a.e. $x \in X$. Consequently, in order to prove the statement, it suffices to show that the family $\{\varphi^* s : s \in \Gamma(E)\}$ generates $\Gamma(\varphi^*E)$ on $X$. To this aim, let $t \in \Gamma(\varphi^*E)$ and $\varepsilon > 0$ be fixed. Thanks to the separability of $B$, we can find a Borel partition $(A_n)_{n \in \mathbb{N}}$ of $X$ such that for every $n \in \mathbb{N}$ it holds that $\|t(x) - t(x')\|_B \leq \varepsilon$ for $m_X$-a.e. $x, x' \in A_n$. By using Theorem C.2, we can disintegrate the measure $m_X$ along $\varphi$ as $\int m^y_Y \, dm_Y(y)$. Then let us define

$$\tilde{t} := \sum_{n \in \mathbb{N}} [\chi_{A_n}] \cdot m_X \cdot \varphi^* s_n \in \Gamma(\varphi^*E),$$

where for every $n \in \mathbb{N}$ the section $s_n \in \Gamma(E)$ is given by

$$s_n(y) := \int_{A_n} t(x) \, dm^y_X(x) \in E(y) \quad \text{for } m_Y\text{-a.e. } y \in Y,$$

with the convention that $\int_{A_n} t \, dm^y_X := 0_{E(y)}$ if $m^y_X(A_n) = 0$. Some verifications are in order: in view of (C.1a), we know that for $m_Y$-a.e. $y \in Y$ it holds $t(x) \in E(y)$ for $m^y_Y$-a.e. $x \in X$. Therefore,
Being the map $t$ Borel (thus strongly Borel, as $\mathcal{B}$ is separable) and bounded on the set $A_n$, we have that the Bochner integral $\int_{A_n} t \, dm^Y_X \in E(y)$ exists for $m_Y$-a.e. $y \in Y$, whence the well-posedness of (C.2) follows. Moreover, given any element $\omega \in \mathcal{B}'$, it holds that
\[
\omega [s_n(y)] = \chi_{\{y' \in Y : m^Y_X(A_n) > 0\}}(y) \frac{\int \omega \{t(x)\} \, dm^y_X(x)}{\int \chi_{A_n} \, dm^y_X} \quad \text{for } m_Y\text{-a.e. } y \in Y.
\]
Given that $\{m^y_X\}_{y \in Y}$ is a Borel family of measures, we deduce that $Y \ni y \mapsto \omega [s_n(y)] \in \mathbb{R}$ is a Borel function, thus accordingly the map $s_n : Y \to \mathcal{B}$ is weakly Borel. Being $\mathcal{B}$ separable, we conclude that $s_n$ is Borel. Therefore, we have that $(s_n)_{n \in \mathbb{N}} \subseteq \Gamma(E)$, so that the definition in (C.2) is meaningful. Given any $n \in \mathbb{N}$ and $m_X$-a.e. $x \in A_n$, we may estimate
\[
\| \tilde{t}(x) - t(x) \|_B = \| t(x) - (\varphi^* s_n)(x) \|_B = \left\| t(x) - \int_{A_n} t(x') \, dm^{\varphi(x')}_X(x') \right\|_B
\]
\[
= \left\| \int_{A_n} (t(x) - t(x')) \, dm^{\varphi(x')}_X(x') \right\|_B \leq \int_{A_n} \| t(x) - t(x') \|_B \, dm^{\varphi(x')}_X(x') \leq \varepsilon,
\]
whence $d_{\Gamma(\varphi^* E)}(\tilde{t}, t) \leq \varepsilon$. This shows that $\{\varphi^* s : s \in \Gamma(E)\}$ generates $\Gamma(\varphi^* E)$ on $X$. \qed

Remark C.4. Some of the assumptions of Theorem C.3 might be dropped:

i) The result holds whenever $m_X$ and $m_Y$ are $\sigma$-finite. Indeed, by arguing as we did at the beginning of §2.4, one can find $m'_X \in \mathcal{P}(X)$ such that $m_X \ll m'_X \ll m_X$. Observe that $m_Y \ll \varphi_* m'_X \ll m_Y$. Given that $L^0(m'_X) = L^0(m_X)$ and $L^0(\varphi_* m'_X) = L^0(m_Y)$, the claim immediately follows from Theorem C.3.

ii) In addition, the assumption $\varphi_* m_X = m_Y$ can be relaxed to $\varphi_* m_X \ll m_Y$ (requiring that the measure $\varphi_* m_X$ is $\sigma$-finite when $m_X(X) = +\infty$). To show it, consider the following equivalence relation $\sim$ on a given normed $L^0(m_Y)$-module $\mathcal{M}$: for any $v, w \in \mathcal{M}$, we declare that $v \sim w$ if and only if $|v - w| > 0$ holds $\varphi_* m_X$-a.e. on $Y$. Then the quotient $\mathcal{M}/\sim$ inherits a natural structure of normed $L^0(\varphi_* m_X)$-module. Moreover, it is easy to prove that $\varphi^* \mathcal{M}_{\varphi_* m_X} \cong \varphi^* \mathcal{M}$. Hence, given a separable Banach $\mathcal{B}$-bundle $E$ over $Y$, we can deduce from Theorem C.3 that $\varphi^* \Gamma(E) \cong \varphi^* \Gamma(E)_{\varphi_* m_X} \cong \Gamma(\varphi^* E)$.

We omit the details and will not insist further on these observations. \hfill \qed

C.2. The projection operator $\text{Pr}_\varphi$. Let us begin by recalling a few basic notions and results about the projection operator $\text{Pr}_\varphi$ for bounded functions. Cf. [12] for a related discussion.

Definition C.5 (The operator $\text{Pr}_\varphi$ for functions). Let $(X, \Sigma_X, m_X)$, $(Y, \Sigma_Y, m_Y)$ be $\sigma$-finite measure spaces. Let $\varphi : X \to Y$ be a measurable map satisfying $\varphi_* m_X = m_Y$. Then we define the projection operator $\text{Pr}_\varphi : L^\infty(m_X) \to L^\infty(m_Y)$ as the linear, 1-Lipschitz mapping
\[
\text{Pr}_\varphi(f) := \frac{d\varphi_+(f^+ m_X)}{dm_Y} - \frac{d\varphi_-(f^- m_X)}{dm_Y} \in L^\infty(m_Y) \quad \text{for every } f \in L^\infty(m_X),
\]
where $\frac{d\varphi_+(f^+ m_X)}{dm_Y}$ stands for the Radon–Nikodým derivative of $\varphi_+(f^+ m_X)$ with respect to $m_Y$, while $f^+ := f \vee 0$ and $f^- := -f \wedge 0$ are the positive part and the negative part of $f$, respectively.
Let us briefly comment on the well-posedness of $\text{Pr}_\varphi$: given that

$$\varphi_*(f^\pm m_X) \leq \varphi_*(\|f\|_{L^\infty(m_X)} m_X) = \|f\|_{L^\infty(m_X)} \varphi_* m_X = \|f\|_{L^\infty(m_Y)} m_Y,$$

we know that $\varphi_*(f^\pm m_X)$ is $\sigma$-finite and absolutely continuous with respect to $m_Y$, so that the Radon–Nikodým derivative $\frac{d\varphi_*(f^\pm m_X)}{dm_Y}$ exists. Moreover, the same estimate shows that

$$|\text{Pr}_\varphi(f)| = \left| \frac{d\varphi_*(f^+ m_X)}{dm_Y} - \frac{d\varphi_*(f^- m_X)}{dm_Y} \right| \leq \frac{d\varphi_*(f^+ m_X)}{dm_Y} + \frac{d\varphi_*(f^- m_X)}{dm_Y}$$

holds $m_Y$-a.e. on $Y$ for any given function $f \in L^\infty(m_X)$, thus $\|\text{Pr}_\varphi(f)\|_{L^\infty(m_Y)} \leq \|f\|_{L^\infty(m_X)}$. Finally, the linearity of $\text{Pr}_\varphi$ can be directly checked from its definition.

**Remark C.6.** It is straightforward to check the validity of the following properties:

$$\begin{align*}
\text{Pr}_\varphi(c) &= c & \text{for every } c \in \mathbb{R}, \\
\text{Pr}_\varphi(f) &\leq \text{Pr}_\varphi(g) & \text{for every } f, g \in L^\infty(m_X) \text{ with } f \leq g, \\
\text{Pr}_\varphi(g \circ \varphi f) &= g \text{Pr}_\varphi(f) & \text{for every } f \in L^\infty(m_X) \text{ and } g \in L^\infty(m_Y),
\end{align*}$$

where all equalities and inequalities are intended in the almost everywhere sense.  

**Example C.7.** In general, the projection operator $\text{Pr}_\varphi: L^\infty(m_X) \to L^\infty(m_Y)$ cannot be extended to a continuous map from $L^0(m_X)$ to $L^0(m_Y)$, as shown by the following example.

Let us consider $X := \mathbb{N}$ and $Y := \{0\}$, endowed with the Borel measures $m_X := \sum_{i \in \mathbb{N}} 2^{-i} \delta_i$ and $m_Y := \delta_0$, respectively. The unique map $\varphi: X \to Y$ (sending all elements to 0) is Borel and satisfies $\varphi_* m_X = m_Y$. We argue by contradiction: suppose there exists a continuous extension $T: L^0(m_X) \to L^0(m_Y)$ of $\text{Pr}_\varphi$. Define $f_n := \sum_{i=1}^n 2^i \chi_i \in L^\infty(m_X)$ for every $n \in \mathbb{N}$ and $f := \sum_{i \in \mathbb{N}} 2^i \chi_i \in L^0(m_X)$. Notice that $f_n \to f$ in $L^0(m_X)$ as $n \to \infty$ and $L^0(m_Y) \cong \mathbb{R}$. Given that $\text{Pr}_\varphi(f_n) = n$ for every $n \in \mathbb{N}$, we deduce that

$$T(f) = \lim_{n \to \infty} T(f_n) = \lim_{n \to \infty} \text{Pr}_\varphi(f_n) = +\infty,$$

which leads to a contradiction. Therefore, $\text{Pr}_\varphi$ cannot be extended to such a map $T$.  

In the metric measure space setting, the operator $\text{Pr}_\varphi: L^\infty(m_X) \to L^\infty(m_Y)$ can be alternatively described (thanks to the disintegration theorem) in the following way:

**Proposition C.8 (Characterisation of $\text{Pr}_\varphi$ for functions).** Let $(X, d_X, m_X)$, $(Y, d_Y, m_Y)$ be metric measure spaces, with $m_X$, $m_Y$ finite. Let $\varphi: X \to Y$ be a Borel map with $\varphi_* m_X = m_Y$. Then for every function $f \in L^\infty(m_X)$ it holds that

$$\text{Pr}_\varphi(f)(y) = \int f \, dm_Y^y \quad \text{for } m_Y \text{-a.e. } y \in Y,$$

where $m_X = \int m_Y^y \, dm_Y(y)$. 

Proof. Given that $m_X$ is finite, it holds that $Pr_\varphi(f) = \frac{d\varphi_*(f|m_X)}{dm_Y}$. Therefore, in order to prove the statement it is sufficient to show that

$$\varphi_*(f|m_X)(A) = \int_A \left( \int f \, dm_X^y \right) \, dm_Y(y) \quad \text{for every } A \subseteq Y \text{ Borel}.$$  \hfill (C.6)

Since for $m_Y$-a.e. $y \in Y$ it holds $\varphi(x) = y$ for $m_X^y$-a.e. $x \in X$ by (C.1a), we have that

$$\varphi_*(f|m_X)(A) = \int \chi_A \circ \varphi \, f \, dm_X \overset{\text{(C.1b)}}{=} \int \left( \int \chi_A(\varphi(x)) \, f(x) \, dm_X^y(x) \right) \, dm_Y(y)$$

$$= \int_A \left( \int f \, dm_X^y \right) \, dm_Y(y),$$

thus proving (C.6). Consequently, the proof is complete. \hfill $\square$

We are now in a position to generalise the object $Pr_\varphi$ to the framework of normed modules. This construction has been first obtained in [10] and later studied in [12]. Here we work with normed $L^\infty$-modules equipped with a pointwise norm taking values in $L^\infty$, thus we need to provide a slightly different proof, but the ideas are essentially borrowed from [10, 12].

**Theorem C.9** (The operator $Pr_\varphi$ for modules). Let $(X, \Sigma_X, m_X)$, $(Y, \Sigma_Y, m_Y)$ be $\sigma$-finite measure spaces. Let $\varphi: X \rightarrow Y$ be a measurable map satisfying $\varphi_*, m_X = m_Y$. Let $\mathcal{M}^\infty$ be a normed $L^\infty(m_Y)$-module. Denote $\mathcal{M}^0 := C(\mathcal{M}^\infty)$. Then there exists a unique linear and continuous operator $Pr_\varphi: R(\varphi^*, \mathcal{M}^0) \rightarrow \mathcal{M}^\infty$ – called the projection operator – such that

$$Pr_\varphi(f \cdot \varphi^* v) = Pr_\varphi(f) \cdot v \quad \text{for every } f \in L^\infty(m_X) \text{ and } v \in \mathcal{M}^\infty.$$  \hfill (C.7)

Moreover, it holds that

$$\left|Pr_\varphi(w)\right| \leq Pr_\varphi(|w|) \quad m_Y\text{-a.e.} \quad \text{for every } w \in R(\varphi^*, \mathcal{M}^0).$$  \hfill (C.8)

**Proof.** Denote by $\mathcal{V}$ the family of all those elements $w \in R(\varphi^*, \mathcal{M}^0)$ that can be written as $w = \sum_{n \in \mathbb{N}} [\chi_{A_n}]_{m_X} \cdot \varphi^* v_n$, for a partition $(A_n)_{n \in \mathbb{N}} \subseteq \Sigma_X$ of $X$ and a sequence $(v_n)_{n \in \mathbb{N}} \subseteq \mathcal{M}^\infty$. Note that $\mathcal{V}$ is a dense linear subspace of $R(\varphi^*, \mathcal{M}^0)$. We are forced to set $Pr_\varphi: \mathcal{V} \rightarrow \mathcal{M}^\infty$ as

$$Pr_\varphi(w) := \sum_{n \in \mathbb{N}} Pr_\varphi([\chi_{A_n}]_{m_X} \cdot v_n) \in \mathcal{M}^\infty \quad \text{for every } w = \sum_{n \in \mathbb{N}} [\chi_{A_n}]_{m_X} \cdot \varphi^* v_n \in \mathcal{V},$$

where the sum is intended with respect to the distance $d_{\mathcal{M}^0}$. Let us check that such sum is actually well-defined: given any $k \in \mathbb{N}$, we have the $m_Y$-a.e. inequality

$$\sum_{n=1}^k \left|Pr_\varphi([\chi_{A_n}]_{m_X} \cdot v_n)\right| = \sum_{n=1}^k Pr_\varphi([\chi_{A_n}]_{m_X} \cdot v_n) \overset{\text{(C.4c)}}{=} \sum_{n=1}^k Pr_\varphi([\chi_{A_n}]_{m_X} \cdot v_n \circ \varphi)$$

$$= Pr_\varphi\left(\sum_{n=1}^k [\chi_{A_n}]_{m_X} \cdot \varphi^* v_n\right) \overset{\text{(C.4b)}}{\leq} Pr_\varphi(|w|),$$

whence it follows that $\sum_{n \in \mathbb{N}} |Pr_\varphi([\chi_{A_n}]_{m_X} \cdot v_n)| \leq Pr_\varphi(|w|)$ holds $m_Y$-a.e. on $Y$. This grants that the sum $\sum_{n \in \mathbb{N}} Pr_\varphi([\chi_{A_n}]_{m_X} \cdot v_n)$ exists in $\mathcal{M}^0$ and defines an element $Pr_\varphi(w) \in \mathcal{M}^\infty$. Moreover, the same estimates show that (C.8) holds for every $w \in \mathcal{V}$, thus $Pr_\varphi$ can be uniquely extended to a linear and continuous map $Pr_\varphi: R(\varphi^*, \mathcal{M}^0) \rightarrow \mathcal{M}^\infty$, which also satisfies (C.8).
By construction, the resulting map $\Pr_\varphi$ is the unique linear and continuous operator satisfying (C.7) for functions $f \in L^\infty(m_X)$ of the form $f = [\chi_A]_{m_X}$, where $A \in \Sigma_X$. Finally, since simple functions are dense in $L^\infty(m_X)$, one can easily deduce that (C.7) is verified. \qed

**Remark C.10.** Observe that $L^\infty(m_X)$ and $L^0(m_X)$ have a natural structure of normed $L^\infty(m_X)$-module and normed $L^0(m_X)$-module, respectively, and $C(L^\infty(m_X)) = L^0(m_X)$; the same holds for $L^\infty(m_Y)$ and $L^0(m_Y)$. Moreover, the pullback map $\varphi^* : L^0(m_Y) \to L^0(m_X)$ is given by $\varphi^* g = g \circ \varphi$ for every $g \in L^0(m_Y)$. Therefore, by using (C.4c) we deduce that the operator $\Pr_\varphi : L^\infty(m_X) \to L^\infty(m_Y)$ actually coincides with the projection operator between normed modules, thus accordingly no ambiguity may arise. \hfill \blacksquare

We conclude by showing that the projection operator $\Pr_\varphi$ (associated with a separable normed module) can be characterised in a fiberwise way, thus generalising Proposition C.8.

**Proposition C.11** (Characterisation of $\Pr_\varphi$ for modules). Let $(X, d_X, m_X)$, $(Y, d_Y, m_Y)$ be metric measure spaces, with $m_X$, $m_Y$ finite. Let $\varphi : X \to Y$ be a Borel map with $\varphi_* m_X = m_Y$. Let $B$ be a universal separable Banach space. Let $E$ be a separable Banach $B$-bundle over $Y$. Then the projection operator $\Pr_\varphi : \Gamma_b(\varphi^* E) \to \Gamma_b(E)$ is given by

$$\Pr_\varphi (t)(y) = \int t(x) \, dm^y_X(x) \in E(y) \quad \text{for every } t \in \Gamma_b(\varphi^* E) \text{ and } m_Y \text{-a.e. } y \in Y,$$

where $m_X = \int m^y_Y \, dm_Y(y)$.

**Proof.** First of all, let us observe that Theorem C.3 gives $\Gamma_b(\varphi^* E) = R(\Gamma(\varphi^* E)) \cong R(\varphi^* \Gamma(E))$, thus the projection operator $\Pr_\varphi$ associated with $\Gamma(\varphi^* E)$ can be seen as a mapping from $\Gamma_b(\varphi^* E)$ to $\Gamma_b(E)$. Given any $t \in \Gamma_b(\varphi^* E)$, we define $\Phi(t)(y) := \int t \, dm^y_X \in E(y)$ for $m_Y$-a.e. $y \in Y$. By arguing as in the proof of Theorem C.3, one can see that $\Phi(t) \in \Gamma_b(E)$. The resulting map $\Phi : \Gamma_b(\varphi^* E) \to \Gamma_b(E)$ is linear and 1-Lipschitz. For any $f \in L^\infty(m_X)$ and $s \in \Gamma_b(E)$ we have

$$\Phi(f \cdot \varphi^* s)(y) = \int f(x) \, s(\varphi(x)) \, dm^y_X(x) = \left( \int f \, dm^y_X \right) s(y) = (\Pr_\varphi(f) \cdot s)(y)$$

for $m_Y$-a.e. $y \in Y$, thus accordingly $\Phi = \Pr_\varphi$ by Theorem C.9, which yields the statement. \qed

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References

[1] C. Aliprantis and K. Border, *Infinite Dimensional Analysis: A Hitchhiker’s Guide*, Studies in Economic Theory, Springer, 1999.
[2] L. Ambrosio, M. Colombo, and S. Di Marino, *Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope*, (2015), pp. 1–58. Variational methods for evolving objects.
[3] L. Ambrosio and S. Di Marino, *Equivalent definitions of BV space and of total variation on metric measure spaces*, J. Funct. Anal., 266 (2014), pp. 4150–4188.
[4] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.
[5] *Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below*, Invent. Math., 195 (2014), pp. 289–391.
[6] L. Benatti, *A review of differential calculus on metric measure spaces via $L^0$-normed modules*, 2018. Master’s thesis, University of Trieste.
[7] C. Bessaga and A. Pelczyński, *Selected topics in infinite-dimensional topology*, Polska Akademia Nauk. Instytut Matematyczny. Monografie matematyczne, 1975.
[8] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces*, Geom. Funct. Anal., 9 (1999), pp. 428–517.
[9] D. Fremlin, *Measure Theory: Measure algebras. Volume 3*, Measure Theory, Torres Fremlin, 2011.
[10] N. Gigli, *Non-smooth differential geometry - an approach tailored for spaces with Ricci curvature bounded from below*, Mem. Amer. Math. Soc., 251 (2018), pp. 161.
[11] *Lecture notes on differential calculus on RCD spaces*, (2018). Publ. RIMS Kyoto Univ. 54.
[12] N. Gigli and E. Pasqualetto, *Behaviour of the reference measure on RCD spaces under charts*. To appear on Communications in Analysis and Geometry, arXiv:1607.05188.
[13] *Lectures on Non-smooth Differential Geometry*, SISSA Springer Series 2, 2020.
[14] N. Gigli, E. Pasqualetto, and E. Soultanis, *Differential of metric valued Sobolev maps*, Journal of Functional Analysis, 278 (2020), p. 108403.
[15] N. Gigli and C. Rigoni, *Recognizing the flat torus among RCD$^*(0, N)$ spaces via the study of the first cohomology group*, Calculus of Variations and Partial Differential Equations, 57(5) (2017).
[16] N. Gigli and A. Tyulenev, *Korevaar-Schoen’s energy on strongly rectifiable spaces*, (2020). Preprint, arXiv:2002.07440.
[17] A. Gutman, *Banach bundles in the theory of lattice-normed spaces. II. Measurable Banach bundles*, Siberian Adv. Math., 3 (1993), pp. 8–40.
[18] R. Haydon, M. Levy, and Y. Raynaud, *Randomly normed spaces*, vol. 41 of Travaux en Cours [Works in Progress], Hermann, Paris, 1991.
[19] E. Le Donne, D. Lučić, and E. Pasqualetto, *Universal infinitesimal Hilbertainity of sub-Riemannian manifolds*, (2019). Preprint, arXiv:1910.05962.
[20] D. Lučić and E. Pasqualetto, *The Serre-Swan theorem for normed modules*, Rendiconti del Circolo Matematico di Palermo Series 2, 68 (2019), pp. 385–404.
[21] E. Pasqualetto, *Direct and inverse limits of normed modules*, (2019). Preprint, arXiv:1902.04126.
[22] J.-L. Sauvageot, *Tangent bimodule and locality for dissipative operators on $C^*$-algebras*, in Quantum probability and applications, IV (Rome, 1987), vol. 1396 of Lecture Notes in Math., Springer, Berlin, 1989, pp. 322–338.
[23] J.-L. Sauvageot, *Quantum Dirichlet forms, differential calculus and semigroups*, in Quantum probability and applications, V (Heidelberg, 1988), vol. 1442 of Lecture Notes in Math., Springer, Berlin, 1990, pp. 334–346.
[24] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana, 16 (2000), pp. 243–279.
[25] W. Strauss, N. Macheras, and K. Musial, *Liftings*, Handbook on Measure Theory (E. Pap, ed.), Elsevier, Amsterdam, 2002.

[26] M. Takesaki, *Theory of operator algebras. I*, Springer-Verlag, New York-Heidelberg, 1979.

[27] N. Weaver, *Lipschitz algebras and derivations. II. Exterior differentiation*, J. Funct. Anal., 178 (2000), pp. 64–112.

Università di Genova (DIMA), MALGA, Via Dodecaneso 35, 16146 Genova, Italy
Email address: simone.dimarino@unige.it

Università di Pisa, Dipartimento di Matematica, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
Email address: danka.lucic@dm.unipi.it

Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy
Email address: enrico.pasqualetto@sns.it