Relativistic corrections to the Pionium Lifetime

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Abstract: Next to leading order contributions to the pionium lifetime are considered within non-relativistic effective field theory. A more precise determination of the coupling constants is then needed in order to be consistent with the relativistic $\pi^-\pi^+$ scattering amplitude which can be obtained from chiral perturbation theory. The relativistic correction is found to be 4.1% and corresponds simply to a more accurate value for the non-relativistic decay momentum.

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In the DIRAC experiment which is underway at CERN, one plans to measure the pionium lifetime with an accuracy of 10% or better[1][2]. The dominant decay proceeds through the strong annihilation $\pi^+ + \pi^- \rightarrow \pi^0 + \pi^0$. Since the momenta of the final state particles is given by the mass difference $\Delta m = m_+ - m_0$ between the charged and neutral pions, the process is strongly non-relativistic. In lowest order the decay rate follows directly from the corresponding non-relativistic scattering amplitude which can be written as

$$T_{NR} = \frac{8\pi}{3E_+E_0}(a + bp^2/m_+^2)$$

in the center-of-mass frame where the energies of the pions are $E_0 = E_+ = m_+ + p^2/2m_+$ when $p$ is the momentum of the charged pions. The S-wave scattering length $a$ and the slope parameter $b$ include both higher order chiral corrections and isospin-violating effects from the quark mass difference $m_u - m_d$ and short-range electromagnetic effects. At threshold the momentum $p = 0$ and the full scattering amplitude with long-range Coulomb interactions removed is then given by just this scattering length. It can be written in terms of the more conventional isospin-symmetric scattering lengths $a_0$ and $a_2$ in the isospin $I = 0$ and isospin $I = 2$ channels as $a = a_0 - a_2 + \Delta a$ where $\Delta a$ includes these symmetry-breaking effects.

Since the charged pions in pionium are supposed to be bound in a $1S$ Coulomb state $\Psi(r)$ with relative momentum $\gamma = \alpha m_+/2$, the annihilation takes place essentially at threshold. In lowest order the transition rate is then given by just the scattering length $a$,

$$\Gamma = \frac{16\pi}{9m_+^2} |\Psi(0)|^2 m_0 \sqrt{2\Delta mm_0} a^2$$

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where \(|\Psi(0)|^2 = \gamma^2/\pi\) gives the probability to find the two particles at the same point. A measurement of the pionium lifetime will thus give an experimental determination of the scattering length. For this to be meaningful, corrections to this lowest order formula must be calculated so that the theoretical lifetime has an uncertainty substantially smaller than the one in the experiment.

In a recent paper we have described this system in the new framework of non-relativistic effective field theory\[2\]. This approach offers a compact and more systematic approach to obtaining the many different higher order corrections to the pionium lifetime compared with previous methods which to a large extent were based on covariant methods\[4\]. As shown by Holstein\[5\] to lowest order in the interactions this approach is equivalent to just using effective couplings in non-relativistic quantum mechanics.

The effective field theory for non-relativistic pions is based upon the free Schrödinger Lagrangian \(L_0 = \pi^\dagger (i \partial_t + \nabla^2/2m_\pi) \pi\) which corresponds to the propagator

\[
G(E, k) = \frac{1}{E - k^2/2m_\pi + i\epsilon}
\]

for a particle with energy \(E\) and momentum \(k\). The interactions we consider are contained in the Lagrangian

\[
L_{\text{int}} = -\frac{1}{2}C_0 (\pi^*_+ \pi^*_- \pi_0 \pi_0) + \frac{1}{4}C_2 (\pi^*_+ \pi^*_- \pi_0 \nabla^2 \pi_0 + \pi^*_+ \nabla^2 \pi^*_- \pi_0 \pi_0) + \text{h.c.}
\]

where the gradient is defined as \(\nabla = 1/2(\nabla_+ - \nabla_-)\). It makes it possible to calculate the scattering amplitude for \(\pi^+ + \pi^- \to \pi^0 + \pi^0\) to order \(p^2\) which then must agree with the definition (3) to this order. The result of this matching for the first coupling constant is then

\[
C_0 = \frac{8\pi}{3m^2_+} \left[ a - (b - a) \frac{\Delta m}{m_+} \right]
\]

when we neglect smaller rescattering corrections\[2\]. Similarly, one finds for the derivative coupling \(C_2 = (8\pi/3m^4_+)(b - a)\). These values are more accurate than the ones used previously\[3\].

With the value of the coupling \(C_0\) now determined, one can calculate the second order correction \(\Delta E\) to the ground state energy of pionium from the bound state diagram in Fig.1 as first pointed out by Labelle and Buckley\[6\]. It is found to be imaginary with a resulting decay rate of \(\Gamma = -2 \text{Im} \Delta E\). To lowest order in the mass difference \(\Delta m\) and ignoring the small binding energy, this gives the zero-order result (2).

Since the pion scattering lengths are set by the natural size \(1/m_\pi\), the counting rules needed to estimate the magnitudes of contributions appearing in different orders of perturbations theory, are simple. The energy \(E\) in the propagator will be of the order \(Q^2\) when

\[\text{We define here } C_0 \text{ with opposite sign to what we used in } [3].\]
the characteristic momentum in the process is $Q$. As a result, the propagator scales as $1/Q^2$. For the same reason the four-dimensional volume integration $\int d^4k$ will scale as $Q^5$. The loop diagram in Fig. 1 thus scales as $Q$ since it involves two one-particle propagators. This is the leading order contribution to the decay rate.

To next order in the effective theory we must include the contribution from the derivative coupling $C_2$ in (4). It will follow from the diagram in Fig.2 and is seen to scale as $Q^3$ again using dimensional regularization. The contributions to the decay rate from these two diagrams are thus found to be

$$\Gamma = \frac{m_0}{4\pi} |\Psi(0)|^2 \sqrt{2\Delta m m_0 (C_0^2 + 2C_0 C_2 \Delta m m_0)} \quad (6)$$

With the above value for the two coupling constants we then simply get the lowest order rate (2). The correction due to the slope parameter thus disappears with the more precise values for the matched coupling constants used here. It results from a cancellation between a next-to-leading order $p^2$ contribution and a $\Delta m$ contribution which in this particular process are of the same order.

To this order in perturbation theory relativistic effects must also be included. These have previously also been considered in more covariant approaches. Here they will arise from the lowest order relativistic correction to the free Lagrangian $L_0$ which now should be taken to be

$$L = \pi^1 \left( i \partial_t + \frac{\nabla^2}{2m_\pi} + \frac{\nabla^4}{8m_\pi^2} \right) \pi \quad (7)$$

This new interaction will modify the propagators in the bubble of Fig.1 as shown in Fig.3. Since the diagram now involves three propagators, it scales as $Q^5 Q^4 Q^{-6} = Q^3$. This is
Relativistic correction to the decay rate. The cross denotes the relativistic coupling.

\[ \Delta E^{(\text{rel})} = \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \Psi^*(p) C_0 \frac{-k^4/8m_0^3}{(2\Delta m - k^2/m_0 + i\epsilon)^2} C_0 \Psi(q) \]

Here \( \Psi(p) \) is the Fourier transform of the bound state wavefunction. The integrations over momenta \( p \) and \( q \) now give just \( |\Psi(0)|^2 \). Using dimensional regularization for the divergent integral over the loop momentum \( k \), we obtain the finite result

\[ \Delta E^{(\text{rel})} = -\frac{5iC_0^2}{32\pi} \Delta m \sqrt{\Delta mm_0} |\Psi(0)|^2 \]

In terms of the decay rate, it corresponds to

\[ \frac{\Delta \Gamma^{(\text{rel})}}{\Gamma} = \frac{5\Delta m}{4m_+} \]

and amounts to 4.1%. By its very nature it can also be derived from using more covariant methods. Finally, the same relativistic interaction will also act on the external legs of the diagram in Fig.1. But the resulting correction is then of the order \( \alpha^2 \) and can thus be neglected here.

The relativistic correction (10) follows also directly from the available phase space for the \( \pi^0, \pi^0 \) final state since there is no energy dependence in the annihilation amplitude to lowest order. Each \( \pi^0 \) has the energy \( E_0 = m_0 + k^2/2m_0 - k^4/8m_0^3 \) when we include the next-to-leading order term in the momentum expansion. The decay rate will then involve the integral

\[ \Gamma \propto \int \frac{d^3k}{(2\pi)^3} \delta(2m_+ - 2E_0) \]

when we ignore the small binding energy. The argument of the delta-function will now have two zeros of which one represents an unphysical, high-momentum state. Keeping only the contribution from the physical state, we recover exactly the additional term (10).

In a recent paper by Gall, Gasser, Lyubovitskij and Rusetsky the pionium lifetime is also calculated from the non-relativistic Lagrangian used above. But instead of using bound state perturbation theory based upon the standard Coulomb wavefunction as done here, they determine the properties of the bound state by calculating the complex pole
on the second Riemann sheet of the corresponding T-matrix. In this way they derive higher order corrections which are not considered here. Our results are consistent with their general form of the decay rate where the relativistic correction (10) is seen to be the first term in the expansion of their decay momentum $p^*$. The higher order corrections they have derived should follow in the present approach from including the Coulomb-interactions between the charged pions and rescattering effects corresponding to extra bubbles in the diagram Fig.1.

Almost at the same time another calculation of the lifetime was completed by Eiras and Soto[8]. This is again based upon the same effective Lagrangian which is now further reduced by integrating out degrees of freedom with momenta of the order $\sqrt{2\Delta m m_0}$. It then becomes an effective theory for only charged pions with contact and Coulomb interactions. Their result for the lifetime is also in agreement with what we have obtained here.

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