A model in one-dimensional thermoelasticity

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Abstract

We study a one-dimensional nonlinear hyperbolic-parabolic initial boundary value problem occurring in the theory of thermoelasticity. We prove existence and uniqueness of the local-in-time strong solution. Also, some global-in-time weak measure valued solutions are proven to exist. To this end we introduce an auxiliary problem with artificial viscosity and prove its global-in-time well-posedness. Next, we show that solutions of the auxiliary problem converge, at some short time interval to the strong solution, and to our measure valued solution for an arbitrary time.

1 Introduction

In the thermoelasticity theory, it is assumed that a considered material is elastic and its response to an external load depends on the temperature. We consider the following system which is the simplest model of the one-dimensional thermoelasticity which takes into account the nonlinear coupling between the temperature and displacement of the elastic material (see [4, 2, 13, 16] for related 1D problems):

\begin{align*}
  u_{tt} - u_{xx} &= \mu \theta_x \quad \text{in} \quad (0, \infty) \times (a, b), \\
  \theta_t - \theta_{xx} - \mu \theta(u_t)_x &= 0, \quad \theta \geq 0 \quad \text{in} \quad (0, \infty) \times (a, b), \\
  u(t,a) = u(t,b) &= 0, \quad \theta_x(t,a) = \theta_x(t,b) = 0, \\
  u(0,.) &= u_0, \quad u_t(0,.) = u_1, \\
  \theta(0,.) &= \theta_0 \geq 0.
\end{align*}

In this system, $u$ denotes the displacement of the elastic material, $\theta$ is the material’s temperature and $\mu$ is a material’s constant. We are interested in positive solutions $\theta \geq 0$. The coupling term $\mu \theta(u_t)_x$ in the temperature equation enables us to use the comparison principle at a formal level and arrive at non-negativity of the temperature, unlike in the linear approximation where the coupling term is replaced by $\mu(u_t)_x$.

Let us briefly describe the derivation of the system (1.1) from the first principles. The balance of momentum for an elastic material reads

\begin{equation}
  \rho u_{tt} = \text{div}\sigma + f
\end{equation}
where \( u \) is the displacement of the material, \( \rho \) is the mass density, \( \sigma \) is the Piola-Kirchhoff stress tensor and \( f \) describes the density of external forces acting on the continuum. The linear elastic constitutive stress-strain relation is a generalization of the classical Hook’s law

\[
\sigma = D \text{sym}(\nabla u) - \mu(\theta - \theta^*) I,
\]

(1.3)

where \( D \) is the elasticity tensor (which is symmetric and positive definite), \( \text{sym}(\nabla u) = \frac{1}{2}(\nabla u + \nabla^T u) \) is the symmetrized gradient of \( u \) and \( \mu \) is a positive constant which depends on the material and is determined experimentally. The material’s temperature is denoted by \( \theta \), while \( \theta^* \) denotes a given, reference temperature. Assuming that there is no external load acting to the material, and by taking the elasticity tensor \( D \) to be equal to the identity and the mass density \( \rho \) to be equal to 1, the system (1.2)-(1.3) in 1D reduces exactly to (1.1).1.

In the thermoelasticity theory, the system of elastic equations (1.2)-(1.3) is coupled with the heat equation which describes the evolution of the material’s temperature, and is a consequence of the first law of thermodynamics: the time derivative of the total energy is equal to the sum of the power of external forces and the rate of heat received by the continuum

\[
\frac{dE}{dt}(t) = P_{\text{ext}}(t) + \frac{dQ}{dt}(t).
\]

This principle implies the differential equation

\[
\frac{de}{dt}(t) + \text{div} q = \sigma \cdot \nabla v,
\]

where \( e \) is the density of the internal energy, \( q \) is the heat flux and \( v \) is the velocity of the material’s displacement. Assuming that \( e = c\theta + D^{-1} T \cdot T \), where \( c \) is a constant depending on the material and \( T = D \text{sym}(\nabla u) \), and, furthermore, assuming that the heat transfer satisfies the Fourier law \( q = -\kappa \nabla \theta \), with \( \kappa \) being the material’s conductivity, we obtain the heat equation in the form

\[
cc\theta_t - \kappa \Delta \theta + \mu(\theta - \theta^*) \text{div} u_t = 0.
\]

(1.4)

Assuming all the physical constants to be equal to 1, i.e. \( c = \kappa = 1 \), the reference temperature \( \theta^* \) to be equal to zero, the equation (1.4) in 1D reduces to (1.1).1.

We prove the existence and uniqueness, locally in time, of a strong solution, with the temperature being non-negative, to this hyperbolic-parabolic coupling problem by using the artificial viscosity approach. We would like to emphasize that, physically, temperature is expected to be non-negative but the models studied so far do not seem to fulfill this expectation.

To be more precise, we first introduce an auxiliary, regularized problem, and passing to the limit as regularization parameter tends to zero, we obtain a solution to the original problem (1.1). Furthermore, we also prove the existence, globally in time, of a measure valued solution to problem (1.1), as defined below. These results are summarized in the following two theorems:

**Theorem 1.** Let \( u_0 \in H^2(a,b) \cap H_0^1(a,b) \), \( u_1 \in H_0^1(a,b) \), \( \theta_0 \in H^2(a,b) \), \( \theta_0 \geq 0 \). Then there exists a time \( T_0 > 0 \) (depending on data) and a unique solution \((u,\theta)\) to problem (1.1) on \((0,T_0)\) with the following regularity:

\[
\|u\|_{W^{2,\infty}(0,T_0;L^2(a,b))} + \|u\|_{W^{1,\infty}(0,T_0;H^1(a,b))} + \|	heta\|_{W^{1,\infty}(0,T_0;L^2(a,b))} + \|	heta\|_{H^1(0,T_0;H^1(a,b))} \leq C.
\]

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Theorem 2. Let \( u_0 \in H_0^1(a,b) \), \( u_1 \in L^2(a,b) \), \( \theta_0 \in L^2(a,b) \), \( \theta_0 \geq 0 \). Then for every \( T > 0 \) there exists a measure \( \gamma \in L^2(0,T;\mathcal{M}(a,b)) \) and functions \( (u,\theta) \) defined on \((0,T)\) satisfying the following equalities:

\[
\int_0^T \int_a^b u_{tt}v + \int_0^T \int_a^b u_xv_x - \mu \int_0^T \int_a^b \theta_xv = 0
\]

and

\[
\int_0^T \int_a^b \theta_tv + \int_0^T \int_a^b \theta_x\psi_x + \mu \int_0^T \int_a^b \theta_u\psi_x = -\mu \int_0^T \int_a^b \psi d\gamma,
\]

for all test functions \((v,\psi) \in H_0^1(a,b) \times H^1(a,b)\), with \((u,\theta)\) and \( \gamma \) being related in the following way: there exists a sequence \((u^n,\theta^n)\) such that

\[
(u^n_t,\theta^n_x) \rightrightarrows (u_t,\theta_x) \text{ weakly in } L^2(0,T;L^2(a,b)),
\]

\[
u^n_x \theta^n_x \rightrightarrows \gamma \text{ weakly in } L^2(0,T;\mathcal{M}(a,b)).
\]

Moreover, there exist a constant \( C > 0 \), depending on the initial data, such that the solution \((u,\theta)\) satisfies the following first order estimates:

\[
\|u\|_{W^{1,\infty}(0,T;L^2(a,b))} + \|u\|_{L^{\infty}(0,T;H^1(a,b))} + \|\theta\|_{L^{\infty}(0,T;H^1(a,b))} \leq C \left( \|u_0\|_{H^1(a,b)}, \|u_1\|_{L^2(a,b)}, \|\theta_0\|_{L^2(a,b)} \right)
\]

and

\[
\|\theta\|_{L^{\infty}(0,T;L^2(a,b))} + \|\theta\|_{L^2(0,T;H^1(a,b))} \leq C \left( \|u_0\|_{H^1(a,b)}, \|u_1\|_{L^2(a,b)}, \|\theta_0\|_{L^2(a,b)} \right).
\]

2 Literature review

It is well known that the equations of one-dimensional nonlinear thermoelasticity in general admit smooth, classical solution; locally for any data and globally for small data. This was investigated by various authors and one of the pioneer works was Slemrod’s paper [16] from 1981. He considered the following nonlinear thermoelasticity problem

\[
\begin{align*}
    u_{tt} - a(u_x,\theta)u_{xx} + b(u_x,\theta)\theta_x &= 0, \\
    c(u_x,\theta)\theta_t - d(u_x,\theta)\theta_{xx} + b(u_x,\theta)u_{tx} &= 0,
\end{align*}
\]

(2.1)

with mixed boundary conditions (Neumann for \( u \), Dirichlet for \( \theta \) and vice versa), where \( a, b, c, d \) are differentiable functions such that \( a, c, d > 0, b \geq \beta > 0 \). The author proved local existence and uniqueness of a solution. Furthermore, assuming the smallness of the initial data, he also proved global existence and uniqueness. We emphasize here that a strong assumption that \( b \) is bounded away from 0 prevents an expected nonnegativity result for \( \theta \). The methodology of the proof is based on application of the contraction mapping theorem to solutions of a related linear problem.

Later on, Racke [12] proved the local existence of classical smooth solutions to the equations of one-dimensional nonlinear thermoelasticity [11] for the physically reasonable Dirichlet boundary conditions (the boundary of the configuration is assumed to be rigidly clamped and held at constant temperature) for both bounded and unbounded domains assuming smooth data.

Racke and Shibata dealt with the same problem in [13] where they proved a global existence of smooth solutions by using the spectral analysis to estimate the decay rates of solutions to the linearized
problem, which are then used in a standard way to obtain energy estimates. Racke and Shibata \cite{13} as well as Racke \cite{12} again assume $b \geq \beta > 0$. Consequently, temperature $\theta$ constructed by them does not share any comparison principle and the nonnegativity result for temperature is missing. Our model is the simplest one addressing the potential nonnegativity of temperature. Indeed, our solutions are proven to be nonnegative.

By following the similar approach as in \cite{13}, the global existence of smooth solutions for small and smooth data in the case of Neumann boundary conditions was proved in \cite{15}. Some more general models were also considered. For instance, global existence of solutions for small initial data and decay of classical solutions for the equations of one-dimensional nonlinear thermoeelasticity was also considered by Hrusa and Tarabek \cite{7}, Jiang \cite{9} and Zheng and Shen \cite{17}. In their paper \cite{8}, Hu and Wang investigated the global solvability of smooth small solutions to the one-dimensional thermoelasticity problem with second sound in the half line. Their work was motivated by Jiang’s paper \cite{10} where the author obtained the global existence of smooth solutions to the system of classical thermoeelasticity under Dirichlet boundary conditions in the half line by directly using energy estimates.

Regarding singularities, Dafermos and Hsiao in \cite{4} considered a special one-dimensional model taking into account the whole real line as a reference configuration. They proved that for large data a smooth solution blows up in a finite time. Similar problem was also considered by Hrusa and Messaoudi \cite{6}. They showed the existence of smooth initial data for which the solution will develop singularities in finite time.

Comparing our work to the existing literature, we see that we assume less regularity on the initial data which leads to a different functional framework. Moreover, we do not assume that $b(u_x, \theta)$ is bounded away from zero, which is the case in e.g. \cite{12,13,16}. Furthermore, none of the works mentioned above dealt with the non-negativity of the temperature, which is, besides the proof of existence and uniqueness of the solution, one of the main novelties of our paper.

3 Preliminaries

In the paper, we will repeatedly use the following interpolation inequality (often called Agmon’s inequality), stated and proven below (in one-dimensional framework) for completeness.

**Proposition 1.** For any $f \in H^1_0(a,b)$ the following inequality holds

$$
\|f\|_{L^\infty} \leq \sqrt{\|f\|_{L^2} \|f_x\|_{L^2}},
$$

while if only $f \in H^1(a,b)$, then

$$
\|f\|_{L^\infty} \leq \sqrt{2\|f\|_{L^2} \|f_x\|_{L^2}} + \sqrt{\frac{1}{b-a} \int_a^b f^2(s) ds}.
$$

**Proof.** First we notice that for any $f \in C^1[a,b]$ and $x_0 < x$ one has

$$
f^2(x) = \int_{x_0}^x (f^2)'(s) ds + f^2(x_0),
$$
hence for $x_0$ such that the value of $f^2$ at $x_0$ is smaller than the average
\[
f^2(x) \leq 2 \int_{x_0}^{x} f(s)f'(s)ds + f^2(x_0) \leq 2\|f\|_{L^2}\|f_x\|_{L^2} + \frac{1}{b-a} \int_{a}^{b} f^2(s)ds
\]
where we used the Cauchy-Schwarz inequality. Next, the usual density argument allows us to state (3.2) for any $f \in H^1(a,b)$. In order to arrive at (3.1) we pick up first $x_0$ being $a$ in (3.3) to arrive at
\[
f^2(x) = \int_{a}^{x} (f^2(s))' ds,
\]
next for $x_0$ being $b$ in (3.3) we have
\[
f^2(x) = \int_{b}^{x} (f^2(s))' ds.
\]
Hence, for smooth functions $f$ supported in $(a, b)$ one has
\[
2f^2(x) \leq 2 \left( \int_{a}^{x} |f(s)||f'(s)|ds + \int_{x}^{b} |f(s)||f'(s)|ds \right) = 2 \int_{a}^{b} |f(s)||f'(s)|ds.
\]
By applying the Cauchy-Schwarz inequality to the right-hand side we obtain
\[
f^2(x) \leq \|f\|_{L^2}\|f_x\|_{L^2},
\]
and using the density of compactly supported smooth functions in $H^1_0(a,b)$ we arrive at (3.1).

Moreover, in Section 4 we shall need a version of the Schaefer’s theorem for the subspace of nonnegative functions. Since we could not find a proper reference we attach a version of this theorem (together with the proof) which is applicable in our case.

**Proposition 2.** Let $\bar{X}$ be the Banach space of real-valued functions, by $X$ we denote its subset $\bar{X} \cap \{ f \geq 0 \}$. Let $A : X \to X$ be compact and continuous mapping, assume moreover that the set of all points for which there exists $\lambda \in [0,1]$ so that
\[
B := \{ x \in X : x = \lambda Ax \}
\]
is bounded. Assume next that the topology in $\bar{X}$ is order preserving (closure of the set of non-negative functions consists of non-negative functions). Then $A$ has a fixed point $x \in X$.

**Proof.** The proof follows the lines of the original proof of Schaefer (see [14] or compare to [5]). We only have to ensure that the fixed point is a non-negative function. After choosing a constant $M$ in such a way that
\[
\|u\| < M \text{ for all } u \in B \text{ (the definition of } B \text{ is given in (3.4))},
\]
one can define $A_M(u) := \frac{MA(u)}{\|A(u)\|}$ for $u$ such that $\|A(u)\| \geq M$, while $A_M = A$ for $u$ such that $\|A(u)\| \leq M$. We observe that $A_M : B(0,M) \cap X \to B(0,M) \cap X$, $A_M$ inherits continuity and compactness from $A$. Hence
\[
A_M : \text{conv}(B(0,M) \cap X) \to \text{conv}(B(0,M) \cap X).
\]
We are in a position to use classical Schauder’s fixed point theorem and say that \(A_M\) has a fixed point in \(\text{conv}(B(0,M) \cap X)\). On the one hand the latter set consists of non-negative functions only. Indeed, a convexification of the set of non-negative functions consists of non-negative functions only. Non-negativity is also preserved by the closure, as assumed in the statement of the proposition. Finally, we show that a fixed point of \(A_M\) is also a fixed point of \(A\) in a standard way.

\[\square\]

4 Auxiliary problem

In this section we introduce an auxiliary problem which we shall utilize in order to construct solutions of (1.1). More precisely, we follow the artificial viscosity approach (see e.g. [1]), i.e. we find a global solution to the regularized system:

\[
\begin{align*}
\tau_{tt} - \tau_{xx} - \nu \tau_{txx} &= \mu \theta_x \quad \text{in} \quad (0,T) \times (a,b), \\
\theta_t - \theta_{xx} - \mu \theta(u_t)_x &= 0, \quad \theta \geq 0 \quad \text{in} \quad (0,T) \times (a,b), \\
u(t,a) &= u(t,b) = 0, \quad \theta_x(t,a) = \theta_x(t,b) = 0, \\
u(0,.) &= u_0, \quad u_t(0,.) = u_1, \\
\theta(0,.) &= \theta_0 \geq 0,
\end{align*}
\]

(4.1)

and then construct a solution to the system (1.1) as a limit of solutions to (4.1) when \(\nu \to 0\), \(\nu > 0\) being a regularization parameter (artificial structural viscoelasticity). The initial conditions \(\theta_0, u_0\) and \(u_1\) belong to the following function spaces:

\[
u_0 \in H^1_0(a,b), \quad u_1 \in L^2(a,b), \quad \theta_0 \in L^2(a,b).
\]

(4.2)

Remark 1. Full thermoviscoelastic system has also an additional term in the temperature equation (see e.g. [3]). Equation (4.1) 2 with physical viscosity would be:

\[
\theta_t - \theta_{xx} - \mu \theta(u_t)_x = \nu(u_{tx})^2.
\]

Definition 1. We say that \((u, \theta)\) is a weak solution to problem (4.1) if the following conditions are satisfied:

1. 

\[
u \in W^{1,\infty}(0,T; L^2(a,b)) \cap H^1(0,T; H^1_0(a,b)), \quad u_{tt}, u_{xx} \in L^2(0,T; L^2(a,b))
\]

2. 

\[
\theta \in H^1(0,T; L^2(a,b)) \cap L^2(0,T; H^2(a,b)), \quad \theta \geq 0.
\]

3. \((u, \theta)\) satisfies the following variational equation:

\[
\int_a^b \tau_{tt} v + \int_a^b u_{xx} v_x + \nu \int_a^b u_{tx} v_x - \mu \int_a^b \theta_x v = 0,
\]

(4.3)

\[
\int_a^b \theta_t \psi + \int_a^b \theta_x \psi_x - \mu \int_a^b \theta u_{tx} \psi = 0,
\]

(4.4)

where \((v, \psi) \in H^1_0(a,b) \times H^1(a,b)).
Remark 2. Due to required regularity of the temperature $\theta \in L^2(0,T;H^1(a,b))$ as well as $\theta_t \in L^2(0,T;L^2(a,b))$, we have that
\[ \theta \in C([0,T];L^2(a,b)) \]
(see e.g. [5]). Similarly, from the regularity of the displacement $u$, we get that:
\[ u \in C([0,T];H^1(a,b)) \text{ and } u_t \in C([0,T];L^2(a,b)). \]

Therefore solutions belong to a regularity class in which initial conditions (4.1), (4.6) can be understood in the strong sense.

We will prove the existence of global-in-time solution $(u, \theta)$ of (4.1) by using the second order energy estimate and the Schaefer’s fixed point theorem as introduced in Section 3. Let us choose an arbitrary (but fixed) $T > 0$ and first define suitable function space:
\[ \mathcal{H}(0,T) = \{ \theta \in L^2(0,T;H^1_0(a,b)) : \theta \geq 0 \}. \quad (4.5) \]

The strong topology in $\mathcal{H}$ satisfies the order-preserving assumption in Proposition 2.

Next we define operator $F : \mathcal{H}(0,T) \to \mathcal{H}(0,T)$ in the following way. Take $\tilde{\theta} \in \mathcal{H}(0,T)$ and define $\tilde{u}$ as a solution of the following initial-boundary value problem:
\[
\begin{align*}
\tilde{u}_{tt} - \tilde{u}_{xx} - \nu \tilde{u}_{txx} &= \mu \tilde{\theta}_x \text{ in } (0,T) \times (a,b), \\
\tilde{u}(.,a) &= \tilde{u}(.,b) = 0 \text{ on } (0,T), \\
\tilde{u}(0,.) &= u_0, \quad \tilde{u}_t(0,.) = u_1 \text{ on } (a,b).
\end{align*}
\quad (4.6)
\]

Now, with given $\tilde{u}$, we define $\theta$ as a solution of the following initial-boundary value problem:
\[
\begin{align*}
\theta_t - \theta_{xx} &= \mu \theta \tilde{u}_{tx} \text{ in } (0,T) \times (a,b), \\
\theta_x(.,a) &= \theta_x(.,b) = 0 \text{ on } (0,T), \\
\theta(0,.) &= \theta_0 \geq 0 \text{ on } (a,b).
\end{align*}
\quad (4.7)
\]

We define $F(\tilde{\theta}) := \theta$.

Theorem 3. Let $u_0 \in H^1_0(a,b), u_1 \in L^2(a,b), \theta_0 \in H^1(a,b), \theta_0 \geq 0$. Then $F : \mathcal{H}(0,T) \to \mathcal{H}(0,T)$ is continuous. Moreover, we have the following estimates for $\theta = F(\tilde{\theta})$
\[
\|\theta\|_{H^1(0,T;L^2(a,b))} + \|\theta\|_{L^2(0,T;H^2(a,b))} \leq C\|\tilde{\theta}\|_{\mathcal{H}(0,T)}, \quad (4.8)
\]
where $C$ depends only on initial data and $T$.

Proof. We will prove the theorem in series of lemmas.

Lemma 1. There exists a unique solution to problem (4.6) with the following properties:
\[
\|\tilde{u}\|_{L^\infty(0,T;H^1(a,b))} + \|\tilde{u}\|_{W^{1,\infty}(0,T;L^2(a,b))} + \nu \|\tilde{u}\|_{H^1(0,T;H^1(a,b))} \leq C\|\tilde{\theta}\|_{\mathcal{H}(0,T)}.
\]
Proof. Since (4.6) has a fixed right-hand side, it is a parabolic equation for \( \tilde{u}_t \). We follow the standard existence proof by using the Duhamel’s formula (see e.g. [5]). Next we take \( \tilde{u}_t \) as a test function to obtain:

\[
\frac{1}{2} \frac{d}{dt} (\| \tilde{u}_t \|^2_{L^2} + \| \tilde{u}_x \|^2_{L^2}) + \nu \| \tilde{u}_{tx} \|^2_{L^2} \leq \frac{1}{2} (\| \tilde{u}_t \|^2_{L^2} + \| \tilde{\theta}_x \|^2_{L^2})
\]

and the uniqueness follows from Gronwall’s inequality. Additionally, we arrive at the required regularity for \( \tilde{u}_{tt} \) in Definition 1 as a consequence of parabolic regularity estimates for \( \tilde{u}_t \).

\[\square\]

Remark 3. Throughout the rest of the manuscript we will use \( \| \cdot \|_{L^2} \) to denote \( L^2 \)-norm in space (and analogously for any other norm). For the norms in space and time, the shortened notation \( \| \cdot \|_{L^2_t L^2_x} \) will be used (and analogously for any other norm).

Lemma 2. There exists a unique solution to problem (4.7) with the following properties:

\[
\| \theta \|_{H^1(0,T;L^2(a,b))} + \| \theta \|_{L^2(0,T;H^2(a,b))} \leq C \| \tilde{\theta} \|_{H(0,T)}.
\]

Proof. This lemma follows the classical results in parabolic equations (see [11], Chapter III). More precisely, since \( \tilde{u}_{tx} \in L^2_t(L^2_x) \), the existence and uniqueness of a weak solution is a consequence of Theorem 3.1 (p. 145) and Theorem 4.1 (p. 153). Moreover, Theorem 7.1 (p. 181) and Corollary 7.1 (p. 186) imply that \( \theta \in L^\infty_t(L^\infty_x) \) and therefore \( \theta_{ux} \in L^2_t(L^2_x) \). Now, the statement of the lemma follows from Theorem 6.1 (p. 178).

\[\square\]

Lemma 3 (Positivity). \( \theta \geq 0 \) on \((0,T) \times (a,b)\).

Proof. Let us consider the following problem

\[
\theta_t - \theta_{xx} = \mu \theta_+ \tilde{u}_{tx},
\]

where \( \theta_+ = \max\{\theta,0\} \) is a positive part of \( \theta \). By taking \( \theta_- = \max\{-\theta,0\} \) as a test function we obtain:

\[
-\frac{1}{2} \frac{d}{dt} \| \theta_- \|^2_{L^2} - \| (\theta_-)_x \|^2_{L^2} = 0.
\]

Since \( \theta_-(0,\cdot) = 0 \), we obtain that \( \theta_- = 0 \). Positivity of \( \theta \) now follows from Lemma 2 which guarantees the uniqueness of the solution of (4.7).

\[\square\]

Lemma 4 (Continuity). \( F : \mathcal{H}(0,T) \to \mathcal{H}(0,T) \) is continuous.

Proof. The continuity follows from Lemma 1 and Lemma 2. Let us take sequence \( \tilde{\theta}_n \to \tilde{\theta} \) in \( \mathcal{H}(0,T) \) and denote by \( \tilde{u}_n, \tilde{u} \) the solutions of (4.6) corresponding to \( \tilde{\theta}_n, \tilde{\theta} \). Set \( \theta_n = F(\tilde{\theta}_n), \theta = F(\tilde{\theta}) \).

Since problem (4.6) is linear, functions \( v_n = \tilde{u}_n - \tilde{u} \) satisfy the following equation with zero initial and boundary conditions:

\[
(v_n)_{tt} - (v_n)_{xx} - \nu (v_n)_{txx} = \mu (\tilde{\theta}_n - \tilde{\theta})_x \quad \text{in} \quad (0,T) \times (a,b).
\]
By taking \((v_n)_t\) as a test function, similarly as in Lemma 1 we conclude \((\tilde{u}_n)_t\to \tilde{u}_t\) in \(L^2(0,T;L^2(a,b))\). Now, let \(\psi = \theta - \theta_n\). Then functions \(\psi_n\) satisfy the following equation with zero initial and boundary data:

\[
(\psi_n)_t - (\psi_n)_{xx} - \mu \psi_n \tilde{u}_t = \mu \theta((\tilde{u}_n)_t - \tilde{u}_t).
\]

From Lemma 2 it follows that \(\theta\) is bounded and therefore the right-hand side of the last equation converges to zero in \(L^2(0,T;L^2(a,b))\). The statement of the lemma follows from the stability of a weak solution (see e.g. [11], Theorem 4.5 (p. 166)).

By summing up all the results obtained in the previous lemmas, we see that Theorem 3 is proved.

**Theorem 4.** Let \(u_0 \in H^1_0(a,b), u_1 \in L^2(a,b), \theta_0 \in H^1(a,b), \theta_0 \geq 0\). Then for any \(T > 0\) there exists a solution \((u,\theta)\) to problem (4.1) on \((0,T)\) satisfying the following estimate:

\[
\|u\|_{W^{1,\infty}(0,T;L^2(a,b))} + \|u\|_{H^1(0,T;H^1(a,b))} + \|\theta\|_{H^1(0,T;L^2(a,b))} + \|\theta\|_{L^2(0,T;H^2(a,b))} \leq C(\mu,\text{data}).
\]  

(4.9)

**Proof.** We use Schaefer’s fixed point theorem (Proposition 2). From Theorem 3 we conclude that \(F\) is a continuous and compact mapping. This follows from the fact that the set

\[
\{f \in \mathcal{H}(0,T) : \|f\|_{\mathcal{H}(0,T)} + \|f\|_{H^1(0,T;L^2(a,b))} + \|f\|_{L^2(0,T;H^2(a,b))} \leq C\}
\]

is compact in \(\mathcal{H}(0,T)\) as a consequence of Aubin-Lions lemma for the triplet

\[
H^2(a,b) \subset H^1(a,b) \subset L^2(a,b).
\]

It remains to prove the boundedness of the following set:

\[
\{f \in \mathcal{H}(0,T) : f = \lambda F(f), \lambda \in [0,1]\}.
\]

Let \(\lambda \in [0,1]\) and \(\theta = \lambda F(\theta)\). Then \(\theta\) is a solution to system (4.1). Because of Lemma 2 equation is satisfied in strong sense and therefore we can integrate equation (4.1)_2 over \((a,b)\). Furthermore, we take \(u_t\) as a test function in (4.1)_1 and sum the resulting equations to obtain the following estimate:

\[
\frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2_{L^2} + \frac{1}{2} \|u_x\|^2_{L^2} + \|\theta\|_{L^1} \right) + \nu \|u_{tx}\|^2_{L^2} = 0.
\]

Therefore, we conclude that \(\|u_{tx}\|_{L^2(0,T;L^2(a,b))} \leq C\), where \(C\) depends only on the initial data. Now, the bound for \(\theta\) can be obtained by using the standard results for parabolic equation [11] analogously as in Lemma 2.

**Proposition 3.** Weak solution of the regularized problem (4.1) obtained in Theorem 4 is unique.

**Proof.** We start a proof by recalling that due to (4.8), there exists a constant \(C > 0\), which depends on \(\nu\) and data, such that

\[
\|\theta_x\|_{L^2(0,T;H^1(a,b))} < C,
\]

(4.10)

and

\[
\|\theta_t\|_{L^2(0,T;L^2(a,b))} < C.
\]
From the last inequality we infer
\[
\|\theta_{tx}\|_{L^2(0,T;H^{-1}(a,b))} \leq C \|\theta_t\|_{L^2(0,T;L^2(a,b))} < C. \tag{4.11}
\]
In view of (4.10) and (4.11), we arrive at (see [5])
\[
\theta_x \in C([0,T];L^2(a,b)). \tag{4.12}
\]
Let \((u_1,\theta_1), (u_2,\theta_2)\) be two weak solutions of problem (4.1) and set \(u = u_1 - u_2, \theta = \theta_1 - \theta_2\). By subtracting (4.1) for \((u_1,\theta_1)\) and \((u_2,\theta_2)\) we get that \(u\) satisfies the following differential equation with zero initial and boundary conditions:
\[
u u_t - u_{xx} - \nu u_{txx} = \mu \theta_x.
\]
By multiplying the above equation by \(u_t\), integrating over space and time interval, and using Young’s and Gronwall’s inequalities, we get:
\[
\|u_t\|_{L^\infty_tL^2_x} + \|u_x\|_{L^\infty_tL^2_x} + \nu \|u_{tx}\|_{L^2_tL^2_x} \leq C\|\theta_x\|_{L^2_tL^2_x}.
\]
The equation for \(\theta = \theta_1 - \theta_2\) reads:
\[
\theta_t - \theta_{xx} = \mu(\theta_1u_{tx} + \theta(u_2)_{tx}).
\]
We multiply the above equation by \(\theta_t\) and integrate over \((a,b)\) to obtain:
\[
\|\theta_t\|_{L^2_x}^2 + \frac{1}{2} \frac{d}{dt} \|\theta_x\|_{L^2_x}^2 = \mu \int_a^b (\theta_1u_{tx}\theta_t + \theta(u_2)_{tx}\theta_t) dt \leq C (\|\theta_1\|_{L^\infty_tL^\infty_x}) \|u_{tx}\|_{L^2_tL^2_x} \|\theta_t\|_{L^2_x}
\]
for a.e. \(t \in (0,T)\). From the proof of Lemma 2 we know that \(\|\theta_1\|_{L^\infty_tL^\infty_x} \leq C\). Having that in mind, we use Young’s inequality to bound the right-hand side of the previous inequality:
\[
\|\theta_t\|_{L^2_x}^2 + \frac{1}{2} \frac{d}{dt} \|\theta_x\|_{L^2_x}^2 \leq C\varepsilon \|\theta_t\|_{L^2_x}^2 + \frac{C}{\varepsilon}\|u_{tx}\|_{L^2_x}^2 + \frac{C}{\varepsilon}\|\theta\|_{L^2_tL^2_x} \|u_{tx}\|_{L^2_tL^2_x} \|\theta\|_{L^2_tL^2_x} \|u_{tx}\|_{L^2_tL^2_x},
\]
where we also used that \(\|\theta\|_{L^\infty_x} \leq c (\|\theta\|_{L^2_x}^2 + \|\theta\|_{L^2_x}^2)\). By choosing \(\varepsilon\) such that \(C\varepsilon < \frac{1}{2}\), the first term on the right-hand side can be absorbed into the first term on the left-hand side, and after integrating from 0 to \(t\) we obtain:
\[
\frac{1}{2} \left( \int_0^t \|\theta_t(s)\|_{L^2_x}^2 ds + \|\theta_x(t)\|_{L^2_x}^2 \right) \leq \frac{C}{\varepsilon} \|\theta_t\|_{L^\infty_tL^\infty_x}^2 \int_0^t \|u_{tx}\|_{L^2_x}^2 dt + \frac{C}{\varepsilon} \|u_{tx}\|_{L^2_x}^2 \|\theta\|_{L^2_tL^2_x} \|u_{tx}\|_{L^2_tL^2_x} \|\theta\|_{L^2_tL^2_x} \|u_{tx}\|_{L^2_tL^2_x}.
\]
Multiplying the inequality with 2 and using the estimate \(\|u_{tx}\|_{L^2_tL^2_x} \leq C \|\theta_x\|_{L^2_tL^2_x}\), we obtain:
\[
\int_0^t \|\theta_t(s)\|_{L^2_x}^2 ds + \|\theta_x(t)\|_{L^2_x}^2 \leq \frac{C}{\varepsilon} \|\theta_x\|_{L^2_tL^2_x}^2 + \frac{C}{\varepsilon} \|\theta(t)\|_{L^2_tL^2_x}^2 \int_0^t \|(u_2)_{tx}\|_{L^2_x}^2 dt + \frac{C}{\varepsilon} \|\theta_x\|_{L^2_tL^2_x} \|u_{tx}\|_{L^2_tL^2_x} \|\theta\|_{L^2_tL^2_x} \|u_{tx}\|_{L^2_tL^2_x}. \tag{4.13}
\]
Next, using \( \theta(0) = 0 \), we have
\[
\|\theta(t)\|_{L^2} = \left\| \int_0^t \theta_t(s)ds \right\|_{L^2} \leq \sqrt{t} \int_0^t \|\theta_t\|^2_{L^2}.
\]
Hence, (4.13) turns into
\[
\int_0^t \|\theta_t(s)\|^2_{L^2} ds + \|\theta_x(t)\|^2_{L^2} \leq \frac{C}{\varepsilon} \|\theta_x\|^2_{L^2 L^2} + \frac{C}{\varepsilon} t \int_0^t \|\theta_t\|^2_{L^2} \int_0^t \|(u_2)_t\|_{L^2}^2 + \frac{C}{\varepsilon} \int_0^t \|\theta_x\|^2_{L^2} \|(u_2)_t\|_{L^2}^2. \tag{4.14}
\]
We are now in a position to finish the proof by a variant of an argument used in the proof of Lemma 2.1 in [11, p. 140]. First we partition interval \((0,T)\) into a finite number of subintervals \((t_{k-1}, t_k)\), \(k = 1, \ldots, N\), such that
\[
\frac{1}{4} \leq \frac{C}{\varepsilon} \int_{t_{k-1}}^{t_k} \|(u_2)_t\|_{L^2}^2 \leq \frac{1}{2}, \quad t_k - t_{k-1} < 1, \quad k = 1, \ldots, N. \tag{4.15}
\]
This can be done because \( \|u_{tx}\|_{L^2} \) is square integrable.
Now we proceed inductively, first we prove that \( \theta_x = 0 \) on \((t_0 = 0, t_1)\). Keeping in mind (4.15), from (4.14) we have
\[
\frac{1}{2} \int_0^t \|\theta_t(s)\|^2_{L^2} ds + \|\theta_x(t)\|^2_{L^2} \leq \frac{C}{\varepsilon} \int_0^t \|\theta_x\|^2_{L^2} + \frac{C}{\varepsilon} \|\theta_x\|^2_{L^2 L^2} \int_0^t \|(u_2)_t\|_{L^2}^2 \tag{4.16}
\]
for any \( t \leq t_1 \).
Let us fix \( 0 < t < t_1 \). Due to (4.12), we notice that \( f(t) := \sup_{0 \leq s \leq t} \|\theta_x(s,x)\|_{L^2(a,b)} \) is bounded, in particular integrable in \((0,t)\). In view of (4.16), for \( 0 < s < t \) we have
\[
\|\theta_x(s)\|^2_{L^2} \leq \frac{C}{\varepsilon} \int_0^s \|\theta_x\|^2_{L^2} + \frac{C}{\varepsilon} \sup_{0 \leq z \leq s} \|\theta_x(z,x)\|^2_{L^2} \int_0^s \|(u_2)_t\|_{L^2}^2 \leq \frac{C}{\varepsilon} \int_0^s \sup_{0 \leq z \leq s} \|\theta_x(z,x)\|^2_{L^2} ds + \frac{C}{\varepsilon} \sup_{0 \leq s \leq t} \|\theta_x(s,x)\|^2_{L^2} \int_0^t \|(u_2)_t\|_{L^2}^2.
\]
Next, we take supremum of the left-hand side over \( s \leq t \)
\[
\sup_{0 \leq s \leq t} \|\theta_x(s)\|^2_{L^2} \leq \frac{C}{\varepsilon} \int_0^t \sup_{0 \leq z \leq s} \|\theta_x(z,x)\|^2_{L^2} ds + \frac{C}{\varepsilon} \sup_{0 \leq s \leq t} \|\theta_x(s,x)\|^2_{L^2} \int_0^t \|(u_2)_t\|_{L^2}^2.
\]
Finally, utilizing (4.15), the last inequality turns into
\[
\frac{1}{2} \sup_{0 \leq s \leq t} \|\theta_x(s)\|^2_{L^2} \leq \frac{C}{\varepsilon} \int_0^t \sup_{0 \leq z \leq s} \|\theta_x(z,x)\|^2_{L^2} ds.
\]
Using Gronwall’s inequality we have
\[
\theta_x = 0 \quad \text{on} \quad [0,t_1]. \tag{4.17}
\]
Next, Poincaré’s inequality tells us that \( \theta(t, x) = c(t) \). On the other hand, (4.16) shows, in view of (4.17), that \( \theta = 0 \) on \([0, t_1]\). In particular, \( \theta(t_1) = 0 \) and therefore we can further iterate the argument to prove that \( \theta = 0 \) on \([t_{k-1}, t_k]\), \( k = 1, \ldots, N \), and thus finish the uniqueness proof.

We shall next prove that our unique solution is actually regular provided initial data is more regular. Indeed, if we impose more restrictive assumptions on initial data, Theorem 4 still gives a unique solution. We give formal estimates which can be consequently used to arrive at more regular solutions via a Schaefer’s theorem like in the proof of Theorem 4. This time, since we require more regularity from our solutions we search for a fixed point in the set

\[
\mathcal{H}_1(0, T) = \{ \theta \in H^1(0, T; H^1_0(a, b) : \theta \geq 0 \}.
\]

In such a case an obtained solution overlaps with the solution constructed in Theorem 4, see a proposition below.

**Proposition 4.** Let \( u_0 \in H^2(a, b) \cap H^1_0(a, b) \), \( u_1 \in H^2(a, b) \cap H^1_0(a, b) \), \( \theta_0 \in H^3(a, b) \), \( \theta_0 \geq 0 \). Then the unique solution \((u, \theta)\) to problem (4.1) given by Theorem 4 satisfies the following regularity properties:

\[
\|u\|_{W^{2,\infty}(0,T;L^2(a,b))} + \|u\|_{H^2(0,T;H^1(a,b))} + \|\theta\|_{H^3(0,T;L^2(a,b))} + \|\theta\|_{L^2(0,T;H^3(a,b))} \leq C(\nu, \text{data}).
\]

**Proof.** First of all, by Theorem 4 for any \( t > 0 \) we have

\[
\int_0^t \int_a^b u_{tx}^2 \, dx < C.
\]  

Next, multiplying (4.1) by \( \theta_t \) and then integrating in space and time, using (4.19), we arrive at

\[
\int_0^t \int_a^b \theta_t^2 \, dx \, dt < C \text{ for any } t > 0.
\]

Next, we apply \( \partial_t \) to (4.1), multiply the resulting equation by \( u_{tt} \) and integrate in both space and time to get:

\[
\|u_{tt}\|_{L^\infty_t L^2_x} + \|u_{tx}\|_{L^\infty_t L^2_x} + 2\nu\|u_{txx}\|_{L^2_t L^2_x} \leq C\|\theta_{tx}\|_{L^2_t L^2_x} + \|u_{tt}(0)\|_{L^2} + \|u_{tx}(0)\|_{L^2}.
\]

Since, \( u_{tt} = u_{xx} + \nu u_{txx} + \mu \theta_x \), we conclude:

\[
\|u_{tt}\|_{L^\infty_t L^2_x} + \|u_{tx}\|_{L^\infty_t L^2_x} + 2\nu\|u_{txx}\|_{L^2_t L^2_x} \leq C(\|\theta_{tx}\|_{L^2_t L^2_x} + \|u_0\|_{H^2} + \|u_1\|_{H^2} + \|\theta_0\|_{H^3}).
\]

Now, we apply \( \partial_t \) to both sides, multiply the resulting equation by \( \theta_{tt} \) and integrate to get:

\[
\|\theta_{tt}\|_{L^2_t L^2_x} + \frac{1}{2}\|\theta_{tx}(t)\|_{L^2}^2 = \mu \int_0^t \int_a^b \theta_t u_{tx} \theta_t + \mu \int_0^t \int_a^b \theta u_{tx} \theta_t + \frac{1}{2}\|\theta_{tx}(0)\|_{L^2}^2 = I + II + III.
\]
Let us estimate terms on the right-hand side separately. Using (3.2) and Young’s inequality we get:

\[ |I| \leq \int_0^t \|u_{tx}\|_{L^2} \|\theta_t\|_{L^\infty} \|\theta_{tt}\|_{L^2} \leq \int_0^t \|u_{tx}\|_{L^2} (\|\theta_t\|_{L^2}^{1/2} \|\theta_{tt}\|_{L^2}^{1/2} + \|\theta_t\|_{L^2}) \|\theta_{tx}\|_{L^2} \]
\[
\leq C\varepsilon \|\theta_t\|^2_{L^2_t L^2} + \frac{C}{\varepsilon} \int_0^t \|u_{tx}\|^2_{L^2} \|\theta_{tx}\|^2_{L^2} \]
\[
\leq C\varepsilon \|\theta_t\|^2_{L^2_t L^2} + \frac{C}{\varepsilon} \left( \sup_{0 \leq s \leq t} \|u_{tx}(s)\|^2_{L^2} \int_0^t \|\theta_t\|^2_{L^2} + \int_0^t \|u_{tx}\|^2_{L^2} \|\theta_{tx}\|^2_{L^2} \right) \]
\[
\leq C\varepsilon \|\theta_t\|^2_{L^2_t L^2} + \frac{C}{\varepsilon} \left( \int_0^t \|u_{tx}\|^2_{L^2} \|\theta_{tx}\|^2_{L^2} + \left( \int_0^t \|\theta_{tx}\|^2_{L^2} + \|u_0\|_{H^2} + \|u_1\|_{H^2} + \|\theta_0\|_{H^1} \right) \right) \int_0^t \|\theta_t\|^2_{L^2}. \]

where in the last inequality we used (4.21). Consequently, in view of (4.19) and (4.20)

\[ |I| \leq C\varepsilon \|\theta_t\|^2_{L^2_t L^2} + \frac{C}{\varepsilon} \int_0^t \|u_{tx}\|^2_{L^2} \|\theta_{tx}\|^2_{L^2} + C(\|u_0\|_{H^2}, \|u_1\|_{H^2}, \|\theta_0\|_{H^1}) \int_0^t \|\theta_{tx}\|^2_{L^2}. \]

Moreover, from the proof of Lemma 2 we have \(\|\theta\|_{L^\infty L^\infty_t} \leq C\). Therefore using (4.21) we have:

\[ |II| \leq C\varepsilon \|\theta_t\|^2_{L^2_t L^2} + \frac{C}{\varepsilon} \|u_{tx}\|^2_{L^2_t L^2} \]
\[
\leq C\varepsilon \|\theta_t\|^2_{L^2_t L^2} + \frac{C}{\varepsilon} \left( \|\theta_{tx}\|^2_{L^2_t L^2} + \|u_0\|_{H^2} + \|u_1\|_{H^2} + \|\theta_0\|_{H^1} \right). \]

Finally,

\[ |III| \leq C(\|\theta_{xx}(0)\|^2_{L^2} + \|\theta_x u_{tx}\|(0)\|^2_{L^2} + \|\theta u_{txx}\|(0)\|^2_{L^2}) \leq C(\nu, \text{data}). \]

Picking up \(\varepsilon\) small enough and summing up the estimates of I, II and III, we arrive at

\[ \|\theta_{tx}(t)\|^2_{L^2} \leq \frac{C}{\varepsilon} \int_0^t \|u_{tx}\|^2_{L^2} \|\theta_{tx}\|^2_{L^2} + C(\varepsilon, \text{data}) \int_0^t \|\theta_{tx}\|^2_{L^2} + C(\nu, \varepsilon, \text{data}). \]

We are now in a position to finish the proof by using slightly less standard version of Gronwall’s inequality. For reader’s convenience, we give the details. First, we simplify the notation and rewrite (4.22) as

\[ \|\theta_{tx}(t)\|^2_{L^2} \leq C(\varepsilon, \text{data}) \int_0^t (\|u_{tx}(s)\|^2_{L^2} + 1)\|\theta_{tx}(s)\|^2_{L^2} ds + C(\nu, \varepsilon, \text{data}). \]

Taking into account (4.19), multiplying both sides by \((\|u_{tx}(t)\|^2_{L^2} + 1)e^{-\int_0^t (\|u_{tx}(s)\|^2_{L^2} + 1)ds}\), the latter can be transformed into

\[
\frac{d}{dt} \left( \int_0^t \|\theta_{tx}(s)\|^2_{L^2} (\|u_{tx}(s)\|^2_{L^2} + 1) ds \right) e^{-\int_0^t (\|u_{tx}(s)\|^2_{L^2} + 1)ds} \leq -C(\nu, \varepsilon, \text{data}) \frac{d}{dt} e^{-\int_0^t (\|u_{tx}(s)\|^2_{L^2} + 1)ds},
\]
and consequently after integration

\[
\int_0^t \|\theta_{tx}(s)\|^2_{L^2} (\|u_{tx}(s)\|^2_{L^2} + 1) ds \leq C(\nu, \varepsilon, \text{data}) \left( e^{\int_0^t (\|u_{tx}(s)\|^2_{L^2} ds e^t} - 1 \right)
\]
and hence, in view of (4.22)
\[ \|\theta_{tx}(t)\|_{L^2}^2 \leq C(\varepsilon) \int_0^t \left( \|u_{tx}\|_{L^2}^2 + 1 \right) \|\theta_{tx}\|_{L^2}^2 + C(\nu, \varepsilon, \text{data}) \]
\[ \leq C(\nu, \varepsilon, \text{data}) \left( e^{\int_0^t \|u_{tx}(s)\|_{L^2}^2 ds} e^t - 1 \right) + C(\nu, \varepsilon, \text{data}). \]

5 Time-independent estimates

In this section we derive the time-independent estimates for the solution \((u, \theta)\) of (4.1). We will use them to construct global-in-time weak solutions to (1.1).

We first get the same estimate as in the proof of Theorem 4. We multiply (4.1)\(^1\) by \(u_{tx}\) and integrate from \(a\) to \(b\), next integrate (4.1)\(^2\), and sum the resulting expressions to obtain the basic energy equality:
\[ \frac{d}{dt} \left( \frac{1}{2} \|u_{tx}\|_{L^2}^2 + \frac{1}{2} \|u_x\|_{L^2}^2 + \|\theta\|_{L^1}^2 \right) + \nu \|u_{tx}\|_{L^2}^2 = 0 \] (5.1)
which yields the following estimate:
\[ \|u\|_{W^{1,\infty}(0,T;L^2(a,b))} + \|u\|_{L^\infty(0,T;H^1(a,b))} + \|\theta\|_{L^\infty(0,T;L^1(a,b))} \leq C \left( \|u_{tx}(0)\|_{L^2}, \|u_x(0)\|_{L^2}, \|\theta(0)\|_{L^2} \right). \] (5.2)

To derive the second estimate we multiply (4.1)\(^2\) by \(\theta\) and integrate:
\[ \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 = \mu \int_a^b \theta^2(u_t)_x = -2\mu \int_a^b \theta_x \theta u_t \leq 2\mu \|\theta\|_{L^\infty} \|\theta_x\|_{L^2} \|u_t\|_{L^2}. \]

Now, from (5.2) we conclude that \(\|u_{tx}(t)\|_{L^2}\) is bounded for a.e. \(t \in [0,T]\). Next, (5.2) and Young’s inequality yield
\[ \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \|\theta_x\|_{L^2}^2 \leq C_1 \|\theta\|_{L^2}^{1/2} \|\theta_x\|_{L^2}^{1/2} + C_2 \|\theta\|_{L^2} \|\theta_x\|_{L^2} \]
\[ \leq \left( \frac{C_1}{\varepsilon^2} \|\theta\|_{L^2}^2 + C_1 \varepsilon \|\theta_x\|_{L^2}^2 \right) + \left( \frac{C_2}{\varepsilon} \|\theta\|_{L^2}^2 + C_2 \varepsilon \|\theta_x\|_{L^2}^2 \right) \]
\[ \leq C \frac{1}{\varepsilon^2} \|\theta\|_{L^2}^2 + C \varepsilon \|\theta_x\|_{L^2}^2, \]
where \(C = 2\mu \max\{\sqrt{2}, \sqrt{\frac{1}{\nu}}\}\). We choose \(\varepsilon\) such that the last term can be absorbed into the right-hand side and use Gronwall’s inequality to obtain:
\[ \|\theta\|_{L^\infty(0,T;L^2(a,b))} + \|\theta\|_{L^2(0,T;H^1(a,b))} \leq \exp(Ct) \|\theta(0)\|_{L^2}. \] (5.3)
6 Time-dependent estimates

This section is devoted to higher order estimates of solutions to (4.1). Those estimates will hold only on properly short time intervals. We will need them to obtain local-in-time well-posedness of (1.1). The following theorem holds:

**Theorem 5.** Let \( u_0 \in H^2(a,b) \cap H^1_0(a,b) \), \( u_1 \in H^2(a,b) \cap H^1_0(a,b) \), \( \theta_0 \in H^2(a,b) \). Then there exists short enough time \( T_0 > 0 \) and a constant \( C > 0 \) such that \((u,\theta)\), solutions of (4.1) given by Proposition 4, satisfy

\[
\|u\|_{W^{2,\infty}(0,T_0;L^2(a,b))} + \|u\|_{W^{1,\infty}(0,T_0;H^1(a,b))} \leq C \left( \|u(0)\|_{L^2(a,b)} + \|u_t(0)\|_{L^2(a,b)} + \|\theta(0)\|_{L^2(a,b)} \right)
\]

and

\[
\|	heta\|_{W^{1,\infty}(0,T_0;L^2(a,b))} + \|	heta\|_{H^1(0,T_0;H^1(a,b))} \leq C \left( \|u(0)\|_{L^2(a,b)} + \|u_t(0)\|_{L^2(a,b)} + \|\theta(0)\|_{L^2(a,b)} \right).
\]

**Proof.** The proof of the theorem will consist of a few steps.

**Step 1.** We apply \( \partial_t \) to (4.1), next multiply the outcome by \( u_{tt} \) and integrate over \((a,b)\) to obtain:

\[
\frac{1}{2} \frac{d}{dt} \|u_{tt}\|_L^2 + \int_a^b u_{txx} u_{tt} - \nu \int_a^b u_{tx} u_{tt} \leq \frac{1}{2} \left( \|u_t\|_L^2 + \mu \|\theta_x\|_L^2 \right).
\]

Next, integrating by parts the second and third term on the left-hand side and utilizing the boundary conditions \( u_{tt}(a) = u_{tt}(b) = 0 \), being a consequence of the fact that the value of \( u \) at those points is equal to zero (by (4.3)), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \left( \|u_t\|_L^2 + \|u_{tx}\|_L^2 \right) + \nu \|u_{tx}\|_L^2 \leq \frac{1}{2} \left( \|u_t\|_L^2 + \mu \|\theta_x\|_L^2 \right).
\]

Dropping the \( \nu \)-term and using Gronwall’s inequality we arrive at

\[
\|u_{tt}\|_{L^\infty_t L^2_x} + \|u_{tx}\|_{L^\infty_t L^2_x} \leq \exp(t) \left( \|u_{tt}(0)\|_{L^2_x} + \|u_{tx}(0)\|_{L^2_x} + \mu \|\theta_x\|_{L^2_x} \right). \tag{6.3}
\]

**Step 2.** Let us now differentiate (4.1) with respect to time, multiply the resulting equation by \( \theta_t \) and integrate over \((a,b)\) to obtain:

\[
\frac{1}{2} \frac{d}{dt} \|\theta_t\|_L^2 + \|	heta_{tx}\|_L^2 = \mu \int_a^b \left( \theta_t^2(u_t) + \theta_t(u_t) \right). \tag{6.2}
\]

The right-hand side can be rewritten by using integration by parts and Hölder’s inequality:

\[
\frac{1}{2} \frac{d}{dt} \|\theta_t\|_L^2 + \|	heta_{tx}\|_L^2 = \mu \int_a^b \left( \theta_t^2(u_t) - u_{tt}(\theta_x \theta_t + \theta \theta_{tx}) \right)
\]

\[
\leq \mu \|u_{tx}\|_{L^2} \|\theta_t\|_{L^\infty} + \mu \|u_{tt}\|_{L^2} \|\theta_x\|_{L^\infty} \|\theta_t\|_{L^2} + \mu \|u_{tt}\|_{L^2} \|\theta\|_{L^\infty} \|\theta_{tx}\|_{L^2}.
\]

Notice that the boundary terms coming from integration by parts vanish due to \( u_{tt}(a) = u_{tt}(b) = 0 \). We integrate the resulting inequality from 0 to \( t \), use (6.3) to estimate \( \|u_{tx}\|_{L^2} \) and \( \|u_{tt}\|_{L^2} \) and use
Agmon’s inequality \((3.1), (3.2)\) to obtain:

\[
\frac{1}{2} \| \theta_t(t) \|^2_{L^2} + \| \theta_{tx} \|^2_{L^2L^2_x} \\
\leq \frac{1}{2} \| \theta_t(0) \|^2_{L^2} + \mu \exp(t) \left( \| u_{tx}(0) \|_{L^2} + \| u_{tx}(0) \|_{L^2} + \mu \| \theta_{tx} \|_{L^2L^2_x} \right) \\
\int_0^t \left( \| \theta_t \|_{L^\infty} \| \theta_{tx} \|_{L^2} + \| \theta_x \|_{L^\infty} \| \theta_{tx} \|_{L^2} + \| \theta \|_{L^\infty} \| \theta_{tx} \|_{L^2} \right) \\
\leq \frac{1}{2} \| \theta_t(0) \|^2_{L^2} + c \mu \exp(t) \left( \| u_{tx}(0) \|_{L^2} + \| u_{tx}(0) \|_{L^2} + \mu \| \theta_{tx} \|_{L^2L^2_x} \right) \\
\int_0^t \left( \| \theta_t \|^{3/2}_{L^2} \| \theta_{tx} \|^{1/2}_{L^2} + \| \theta_x \|^{1/2}_{L^2} \| \theta_{tx} \|^{1/2}_{L^2} + \| \theta \|^{1/2}_{L^2} \| \theta_{tx} \|^{1/2}_{L^2} + \| \theta \|_{L^2} \| \theta_{tx} \|_{L^2} \right). 
\tag{6.4}
\]

We set

\[
A(t) := c \mu^2 \exp(t), 
\tag{6.5}
\]

where \(c = \max \left\{ \sqrt{2}, \sqrt{\frac{1}{b^2 a}} \right\} \), and firstly estimate the integral in \((6.4)\) multiplied by \(A(t) \| \theta_{tx} \|_{L^2L^2_x} \).

Using Hölder’s and Young’s inequality (with \(p = 4/3\) and \(q = 4\)), we estimate the first term:

\[
I = A(t) \| \theta_{tx} \|_{L^2L^2_x} \int_0^t \| \theta_t(s) \|^{3/2}_{L^2} \| \theta_{tx}(s) \|^{1/2}_{L^2} ds \\
= A(t) \left( \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds \right)^{1/2} \left( \int_0^t \| \theta_t(s) \|^{3/2}_{L^2} \| \theta_{tx}(s) \|^{1/2}_{L^2} ds \right) \\
\leq A(t) \left( \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds \right)^{1/2} \left[ \left( \int_0^t \| \theta_t(s) \|^2_{L^2} ds \right)^{3/4} \left( \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds \right)^{1/4} \right] \\
= A(t) \left( \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds \right)^{3/4} \left( \int_0^t \| \theta_t(s) \|^2_{L^2} ds \right)^{3/4} \\
\leq A(t) \varepsilon \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds + \frac{A(t)}{\varepsilon^3} \left( \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds \right)^3 \\
\leq A(t) \varepsilon \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds + \frac{A(t) \cdot t^2}{\varepsilon^3} \int_0^t \| \theta_t(s) \|^6_{L^2} ds
\tag{6.6}
\]

In the last inequality, Hölder with \(p = 3/2\) and \(q = 3\) was used.

The second term is estimated in a similar way

\[
II = A(t) \| \theta_{tx} \|_{L^2L^2_x} \int_0^t \| \theta_t(s) \|^2_{L^2} ds \\
\leq A(t) \varepsilon \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds + \frac{A(t)}{\varepsilon} \left( \int_0^t \| \theta_{tx}(s) \|_{L^2} ds \right)^2 \\
\leq A(t) \varepsilon \int_0^t \| \theta_{tx}(s) \|^2_{L^2} ds + \frac{A(t) \cdot t^2}{\varepsilon} \int_0^t \| \theta_t(s) \|^2_{L^2} ds. 
\tag{6.7}
\]
To estimate the third term, we must first estimate $\|\theta_{xx}\|_{L^2}$, which can be rewritten using (4.1) and further estimated by (3.2), as follows:

$$
\|\theta_{xx}\|_{L^2} \leq \|\theta_t\|_{L^2} + \|\theta u_{tx}\|_{L^2} \leq \|\theta_t\|_{L^2} + \|\theta\|_{L^2}^{1/2} \|\theta_x\|_{L^2}^{1/2} \|u_{tx}\|_{L^2} + \|\theta\|_{L^2} \|u_{tx}\|_{L^2}.
$$

(6.8)

Using (6.8) we can rewrite the third term:

$$
\text{III} = A(t)\|\theta_t\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2} \|\theta_x(s)\|_{L^2}^{1/2} \|\theta_{xx}(s)\|_{L^2}^{1/2} ds \\
\leq A(t)\|\theta_t\|_{L^2} \int_0^t \left(\|\theta_t(s)\|_{L^2}^{3/4} \|\theta_x(s)\|_{L^2}^{1/4} + \|\theta(s)\|_{L^2}^{1/4} \|\theta_t(s)\|_{L^2} \|\theta_x(s)\|_{L^2} \|u_{tx}(s)\|_{L^2}^{1/2} \right) ds.
$$

We estimate all the terms from the right-hand side separately by using Young’s inequality:

$$
\text{III.1} = A(t)\|\theta_t\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2}^{3/4} \|\theta_x(s)\|_{L^2}^{1/2} ds \leq A(t)\|\theta_t\|_{L^2}^{2} \|\theta_x\|_{L^2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2}^{3} ds,
$$

$$
\text{III.2} = A(t)\|\theta_t\|_{L^2} \int_0^t \|\theta(s)\|_{L^2}^{1/4} \|\theta_t(s)\|_{L^2} \|\theta_x(s)\|_{L^2} \|u_{tx}(s)\|_{L^2}^{1/2} ds \leq A(t) \sqrt{\exp(t)} \|u_{tt}(0)\|_{L^2}^{1/2} \|\theta_t\|_{L^2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2} \|\theta_x(s)\|_{L^2}^{3/4} ds \\
+ A(t) \sqrt{\exp(t)} \|u_{tx}(0)\|_{L^2}^{1/2} \|\theta_t\|_{L^2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2} \|\theta_x(s)\|_{L^2}^{3/4} ds \\
+ A^{3/2}(t) \|\theta_t\|_{L^2} \|\theta\|_{L^2}^{3/2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2} \|\theta_x(s)\|_{L^2}^{3/4} ds \leq A(t) \sqrt{\exp(t)} \|u_{tt}(0)\|_{L^2}^{1/2} \|\theta_t\|_{L^2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2}^{4} ds \\
+ A(t) \sqrt{\exp(t)} \|u_{tx}(0)\|_{L^2}^{1/2} \|\theta_t\|_{L^2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2}^{4} ds \\
+ A^{3/2}(t) \|\theta_t\|_{L^2} \|\theta\|_{L^2}^{3/2} \|\theta\|_{L^2} \int_0^t \|\theta_t(s)\|_{L^2}^{4} ds,
$$

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III.3 = A(t)∥θtx∥_{L^2_t L^2_s} \int_0^t \|θ(s)\|\frac{1}{2}∥θt(s)∥\frac{1}{2}∥θx(s)∥\frac{1}{2}∥utx(s)∥\frac{1}{2}d s

≤ A(t)\sqrt{\exp(t)}∥ut(0)∥\frac{1}{2}∥θtx∥_{L^2_t L^2_s}∥θ∥\frac{1}{2} L^\infty_t L^2_s \int_0^t∥θt(s)∥L^2∥θx(s)∥\frac{1}{2}d s

+ A(t)\sqrt{\exp(t)}∥utx(0)∥\frac{1}{2}∥θtx∥_{L^2_t L^2_s}∥θ∥\frac{1}{2} L^\infty_t L^2_s \int_0^t∥θt(s)∥L^2∥θx(s)∥\frac{1}{2}d s

+ A^{3/2}(t)∥θtx∥\frac{3}{2}∥θ∥\frac{1}{2} L^\infty_t L^2_s \int_0^t∥θt(s)∥L^2∥θx(s)∥\frac{1}{2}d s

≤ A(t)\sqrt{\exp(t)}∥ut(0)∥\frac{1}{2}∥θtx∥_4^{\frac{1}{2}}∥θ∥_4^{\frac{3}{2}}∥θ∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}} + A(t)\sqrt{\exp(t)}∥utx(0)∥\frac{1}{2} L^\infty_s \int_0^t∥θt(s)∥L^2∥θx(s)∥\frac{1}{2}d s

+ A(t)\sqrt{\exp(t)}∥utx(0)∥\frac{1}{2}∥θtx∥_4^{\frac{1}{2}}∥θ∥_4^{\frac{3}{2}}∥θ∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}} + A(t)\sqrt{\exp(t)}∥utx(0)∥\frac{1}{2} L^\infty_s \int_0^t∥θt(s)∥L^2∥θx(s)∥\frac{1}{2}d s

+ A^{3/2}(t)∥θtx∥_4^{\frac{1}{2}}∥θ∥_4^{\frac{3}{2}}∥θ∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}} + A^{3/2}(t)∥θt∥L^\infty_s \int_0^t∥θt(s)∥L^2∥θx(s)∥\frac{1}{2}d s.

Notice that in III.2 and III.3 we made use of (6.3). Set

\[ B(t) = \max \left\{ A(t)\sqrt{\exp(t)}∥ut(0)∥\frac{1}{2}, A(t)\sqrt{\exp(t)}∥utx(0)∥\frac{1}{2}, A^{3/2}(t) \right\} \]

and apply (5.3) to see that for any \( t > 0 \) we can estimate the term III by

\[ \text{III} \leq B(t)\varepsilon \cdot C^{4/3}(t)∥θtx∥_4^{\frac{1}{2}} + \frac{B(t)}{\varepsilon^3} \max \left\{ \int_0^t∥θt(s)∥^3L^2ds, \int_0^t∥θt(s)∥^4L^2ds \right\}, \]

where

\[ C(t) = ∥θ(0)∥L^2 \exp(2\mu c t). \]

The fourth and fifth term are estimated by using Cauchy-Schwarz inequality and (5.3):

\[ \text{IV} = A(t)∥θtx∥L^\infty_t L^2_s \int_0^t∥θ(s)∥\frac{1}{2}∥θx(s)∥_L^\infty_t L^2_s \]

\[ \leq A(t)∥θtx∥L^\infty_t L^2_s \left( \int_0^t∥θ(s)∥L^2∥θx(s)∥L^2ds \right)^{\frac{1}{2}} \left( \int_0^t∥θtx(s)∥^2L^2ds \right)^{\frac{1}{2}} \]

\[ \leq A(t)∥θtx∥_4^{\frac{1}{2}}∥θ∥_4^{\frac{3}{2}}∥θ∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}}∥θx∥_4^{\frac{3}{2}} \]

\[ \leq A(t) \cdot \sqrt[7]{A^4}∥θtx∥L^\infty_t L^2_s ∥θ∥L^\infty_t L^2_s ∥θ∥L^\infty_t L^2_s ∥θ∥L^\infty_t L^2_s \]

\[ \leq A(t) \cdot \sqrt[7]{A^4} \cdot C(t)∥θtx∥L^\infty_t L^2_s. \]
In this section we show that the sequence of solutions to system (4.1), whose existence is guaranteed

Local-in-time well-posedness

Assume that the initial data have the following regularity

where

We claim that there exists a sequence

such that:

\[ E(t) := \max \left\{ \frac{A(t) \cdot t^2}{\varepsilon^3}, \frac{A(t) \cdot t}{\varepsilon}, \frac{B(t)}{\varepsilon^3} \right\}, \]

where \( A(t), B(t) \) and \( C(t) \) are given by (6.9), (6.10) and (6.11) respectively.

Step 3. In order to prove Theorem 5 we notice that choosing \( \varepsilon \) and short enough time so that

\[ D(t) < 1/2 \]

enables us to absorb the second term from the right-hand side in (6.14) into the left-hand side. Next, applying Gronwall’s inequality yields existence of \( T_0 \) such that the claim of Theorem 5 holds for all \( t \in (t_0, T_0) \).

7 Local-in-time well-posedness

In this section we show that the sequence of solutions to system (4.1), whose existence is guaranteed

by Theorem 4, converges to a solution of (1.1) as the artificial viscoelasticity parameter tends to zero. Assume that the initial data have the following regularity \( u_0 \in H^2(a, b) \cap H^1_0(a, b), u_1 \in H^3_0(a, b) \) and \( \theta_0 \in H^2(a, b), \theta_0 \geq 0 \). Set \( \nu = \frac{1}{4} \) and let \( (u^n, \theta^n) \) be the sequence of solutions to (4.1) corresponding to initial data \( (u_0, u_1^n, \theta_0) \) introduced by a regularization procedure described below, where \( u_1^n \) and \( \theta_0^n \) are the regularized initial elastic velocity and temperature.

We claim that there exists a sequence \( u_1^n \) such that:

(i) \( u_1^n \in H^2(a, b) \cap H^1_0(a, b), \)

(ii) \( u_1^n \to u_1 \) in \( H^1(a, b), \)

(iii) \( \nu \| u_1^n \|_{H^2(a, b)} \to 0. \)
Without loss of generality we firstly assume that the interval \((a, b)\) is symmetric around zero (we can always achieve that with an appropriate composition) and extend \(u\) to \(\mathbb{R}\) by 0. Set \(c := \frac{b-a}{2}\) and define
\[
\sigma_n = \frac{c \sqrt{n} + 1}{c \sqrt{n} - 1}.
\]
Then the sequence \(\tilde{u}^n_1\) defined in the following way
\[
\tilde{u}^n_1(x) = u_1(\sigma_n x).
\]
satisfies \(\tilde{u}^n_1 = 0\) on \(\mathbb{R} \setminus (\frac{a}{\sigma_n}, \frac{b}{\sigma_n})\). Finally, set
\[
u^n = \tilde{u}^n_1 \ast \eta_n,
\]
where \(\eta_n\) is the sequence of standard mollifiers (with \(\varepsilon = \frac{1}{\sqrt{n}}\))
\[
\eta_n(x) = \sqrt{n} \eta(\sqrt{n} x).
\]
Since \(\frac{1}{\sqrt{n}} < c - \frac{c}{\sigma_n}\) we see that supp \(u^n_1 = (\frac{a}{\sigma_n}, \frac{b}{\sigma_n}) + (-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}})\) is a subset of \((a, b)\). Thus \(u^n_1 \in H^1_0(a, b)\). Statement (ii) follows from the definition of mollifiers and the fact that \(\lim_{n \to \infty} \sigma_n = 1\), while statement (iii) is a direct consequence of the fact that \(\|u^n_1\|_{H^2(a, b)}\) behaves like \(\sqrt{n}\).

We next regularize the initial temperature \(\theta_0\) by using the extension operator to extend \(\theta_0\) to \(H^2\)-function on the real line which we then compose with the mollifiers defined above. Finally, we employ Theorem 5 to see that the solutions of (4.1) possess the following regularity for any \(t\) smaller than \(T_0 > 0\):
\[
\begin{align*}
\|u^n_t\|_{W^{1,\infty}(0, T_0; L^2(a, b))} + \|\nu^n\|_{W^{1,\infty}(0, T_0; L^2(a, b))} + \|\theta^n\|_{W^{1,\infty}(0, T_0; L^2(a, b))} + \|\theta^n\|_{H^1(0, T_0; L^2(a, b))} &
\leq C \left(\|u^n_t(0)\| + \|u^n_{tx}(0)\| + \|\theta^n(0)\|_{L^2}\right). 
\end{align*}
\]
(7.1)

The right-hand side is estimated as follows:
\[
\begin{align*}
\|u^n_t(0)\|_{L^2} &\leq \|u^n_{tx}(0)\|_{L^2} + \nu\|u^n_{xx}(0)\|_{L^2} + \mu\|\theta^n(0)\|_{L^2} \leq C \left(\|u^n(0)\_{H^2} + \nu\|u^n_{tx}(0)\|_{H^2} + \mu\|\theta^n(0)\|_{L^1}\right) \leq C,
\end{align*}
\]
Thus, letting \(n \to \infty\), (7.1) implies the following convergences:
\[
\begin{align*}
\|u^n_{tx}\|_{L^2} &\to \|u_{tx}\| \quad \text{weakly in} \quad L^2(0, T_0; L^2(a, b)), \\
\|\theta^n\|_{L^2} &\to \|\theta\| \quad \text{in} \quad C([0, T_0]; C[a, b]).
\end{align*}
\]
(7.2)

We are now in a position to pass to the limit in all the terms of the weak formulation. Notice that the convergences obtained in (7.2) enable us to pass to the limit in nonlinear term \(\int_a^b \theta^n u^n_{tx} \psi\). Furthermore, the boundedness of the sequence \(u^n_{tx}\) implies vanishing of the regularization term \(\nu\int_a^b u^n_{tx} v_x\). Therefore we have proven the existence part of Theorem 1 which we state again for completeness:

**Theorem 1.** Let \(u_0 \in H^2(a, b) \cap H^1_0(a, b)\), \(u_1 \in H^1_0(a, b)\), \(\theta_0 \in H^2(a, b)\), \(\theta_0 \geq 0\). Then there exists \(T_0 > 0\) (depending on data) and a solution \((u, \theta)\) to problem (1.1) on \((0, T_0)\) with the following regularity:
\[
\begin{align*}
\|u\|_{W^{1,\infty}(0, T_0; L^2(a, b))} + \|u\|_{W^{1,\infty}(0, T_0; H^1(a, b))} + \|\theta\|_{W^{1,\infty}(0, T_0; L^2(a, b))} + \|\theta\|_{H^1(0, T_0; H^1(a, b))} &\leq C.
\end{align*}
\]
To complete the proof of Theorem 4 we still need to show uniqueness. The below proposition is devoted to this issue.

**Proposition 5.** The solution \((u, \theta)\) to problem (1.1) given by Theorem 4 is unique.

**Proof.** Let \((u_1, \theta_1), (u_2, \theta_2)\) be two weak solutions of problem (1.1) and set
\[
u = u_1 - u_2, \quad \theta = \theta_1 - \theta_2.
\]
By subtracting (1.1) for \((u_1, \theta_1)\) and \((u_2, \theta_2)\) we get that \(u\) satisfies the following differential equation with zero initial and boundary conditions:
\[
u_{tt} - \nu_{xx} = \mu \theta_x \text{ in } (0, T) \times (a, b).
\]
By multiplying the above equation by \(\nu_t\), integrating over space and time interval, and using Young’s and Gronwall’s inequality, we get:
\[
\|\nu_t\|_{L^\infty L^2_x} + \|\nu_x\|_{L^\infty L^2_x} \leq C\|\theta_x\|_{L^2_t L^2_x}. \tag{7.3}
\]
The equation for \(\theta = \theta_1 - \theta_2\) reads:
\[
\theta_t - \theta_x = \mu (\theta_1 u_{tx} + \theta (u_2)_{tx}).
\]
We multiply the above equation by \(\theta\) and integrate over \((a, b)\) to obtain:
\[
\frac{1}{2} \frac{d}{dt} \|\theta\|^2_{L^2_x} + \|\theta_x\|^2_{L^2_x} = \mu \int_a^b (\theta_1 u_{tx} \theta + \theta (u_2)_{tx} \theta) = \mu \int_a^b \theta_1 u_{tx} \theta + \mu \int_a^b \theta (u_2)_{tx} \theta. \tag{7.4}
\]
We integrate the previous equation with respect to time and estimate integrals on the right-hand side separately. The first integral is separated into two terms by using integration by parts. Using (3.2) we estimate the first term:
\[
\left| \int_0^t \int_a^b (\theta_1)_{x} u_{t} \theta \right| \leq \int_0^t \|(\theta_1)_{x}\|_{L^2} \|u_t\|_{L^2} \|\theta\|_{L^\infty} \leq \int_0^t \|(\theta_1)_{x}\|_{L^2} \|u_t\|_{L^2} \left( \|\theta\|^{1/2}_{L^2} \|\theta_x\|^{1/2}_{L^2} + C \|\theta\|_{L^2} \right)
\leq \|(\theta_1)_{x}\|_{L^\infty L^2_x} \|u_t\|_{L^\infty L^2_x} \|\theta\|^{1/2}_{L^2} \|\theta_x\|^{1/2}_{L^2} + C \|(\theta_1)_{x}\|_{L^\infty L^2_x} \|u_t\|_{L^\infty L^2_x} \|\theta\|_{L^2}.
\]
Since \(\theta_1\) is a solution, Theorem 4 implies that \(\|(\theta_1)_{x}\|_{L^\infty L^2_x} \leq C\). Furthermore, from estimate (7.3) we know that \(\|u_t\|_{L^\infty L^2_x} \leq C\|\theta_x\|_{L^2_t L^2_x}\), so we can use Young’s inequality (with \(p = 4\) and \(q = 4/3\)) to see:
\[
\left| \int_0^t \int_a^b (\theta_1)_{x} u_{t} \theta \right| \leq C\|\theta_x\|^2_{L^2_t L^2_x} + \frac{C}{\varepsilon^2} \|\theta\|^2_{L^2_t L^2_x}.
\]
The second term is estimated in a similar way:
\[
\left| \int_0^t \int_a^b \theta_1 u_{t} \theta \right| \leq \int_0^t \|\theta_1\|_{L^\infty} \|u_t\|_{L^2} \|\theta_x\|_{L^2} \leq \int_0^t C\|\theta_x\|^2_{L^2_x} + \frac{C}{\varepsilon} \|u_t\|^2_{L^2_x} \|\theta_1\|_{L^\infty} \leq C \|\theta_x\|^2_{L^2_t L^2_x} + \frac{C}{\varepsilon} \|\theta_x\|^2_{L^2_t L^2_x} \int_0^t \|\theta_1\|_{L^\infty}.
\]

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What is left is to estimate the second integral in (7.4):
\[
\left| \int_0^t \int_a^b \theta(u_2)_{tx} \, dx \, dt \right| \leq \int_0^t \int_a^b \theta_x u_2 \, dx \, dt + \int_0^t \int_a^b \theta(u_2)_t \theta_x \, dx \, dt \leq \int_0^t \|\theta\|_{L^2} \|u_2\|_{L^\infty} \|\theta_x\|_{L^2}.
\]
Since \( u_2 \) is a solution, we have that \( \|u_2\|_{L^\infty} \leq C \) so the second integral is estimated by using Young’s inequality:
\[
\left| \int_0^t \int_a^b \theta(u_2)_{tx} \, dx \, dt \right| \leq C \varepsilon \|\theta_x\|_{L^2}^2 + \frac{C}{\varepsilon} \|\theta\|_{L^2}^2.
\]
At last, we employ the obtained estimates into (7.4) to see that
\[
\|\theta(t)\|_{L^2}^2 + 2\|\theta_x\|_{L^2}^2 \leq C \varepsilon \|\theta_x\|_{L^2}^2 + \frac{C}{\varepsilon} \|\theta\|_{L^2}^2 + \frac{C}{\varepsilon} \|\theta_x\|_{L^2}^2 \int_0^t \|\theta_1\|_{L^\infty}^2.
\]
We choose \( \varepsilon \) small enough that the \( \theta_x \) term is absorbed in the left-hand side. Next, similarly as in Proposition 3, we will use a trick from [11]. We partition time interval \((0, T_0)\) into finitely many intervals \((t_{k-1}, t_k)\) in such a way that \( C \varepsilon \int_{t_{k-1}}^{t_k} \|\theta_1\|_{L^\infty}^2 < 1/2 \). Since \( \theta_1 \) is a solution,
\[
\|\theta_1\|_{L^2(0, T_0; L^\infty(a, b))} < \infty.
\]
We are thus in a position to proceed inductively as in the proof of Proposition 3 and absorb the term
\[
\frac{C}{\varepsilon} \|\theta_x\|_{L^2}^2 \int_0^t \|\theta_1\|_{L^\infty}^2
\]
to the left-hand side at each interval \((t_{k-1}, t_k)\). This allows us to use Gronwall’s inequality at each time interval and obtain \( \theta = 0 \) for \( t \in (t_{k-1}, t_k) \) for any \( k \). The proof is finished.

8 Global-in-time measure valued solution

In order to prove the existence of a global-in-time weak solution, we would like to pass to the limit as \( \nu \to 0 \), i.e. \( n \to \infty \), using only the first order estimates. The only difficulty lies in the nonlinear term:
\[
-\mu \int_a^b u^n_{tx} \theta^n \psi = \mu \int_a^b u^n (\theta^n \psi + \theta^n \psi x)
\]
Notice that the above expression is well-defined a.e. in \((0, T)\) because of the following estimate:
\[
\left| \mu \int_a^b u^n (\theta^n \psi + \theta^n \psi x) \right| \leq \|u^n\|_{L^2} \|\theta^n\|_{H^1} \|\psi\|_{H^1}.
\]
Because of the uniform convergence of \( \theta^n \) we have \( \int_a^b u^n \theta^n \psi x \to \int_a^b u \theta \psi x \). However, from the uniform estimates we can only conclude that \( u^n \theta^n \) is bounded in \( L^2 L^2 \) and therefore there exists a measure \( \gamma \) such that
\[
\int_0^T \int_a^b u^n \theta^n \psi \to \int_0^T \int_a^b \psi d\gamma.
\]
Therefore we have proven the second main theorem of the manuscript:
Theorem 2. Let $u_0 \in H^1_0(a,b)$, $u_1 \in L^2(a,b)$, $\theta_0 \in L^2(a,b)$, $\theta_0 \geq 0$. Then for every $T > 0$ there exists a measure $\gamma \in L^2(0,T; M(a,b))$ and functions $(u, \theta)$ defined on $(0,T)$ satisfying the following equalities:

$$\int_a^b u_{tt}v + \int_a^b u_x v_x - \mu \int_a^b \theta_x v = 0$$

and

$$\int_a^b \theta_t \psi + \int_a^b \theta_x \psi_x + \mu \int_a^b \theta u_t \psi_x = -\mu \int_a^b \psi d\gamma,$$

for all test functions $(v, \psi) \in H^1_0(a,b) \times H^1(a,b)$, with $(u, \theta)$ and $\gamma$ being related in the following way: there exists a sequence $(u^n, \theta^n)$ such that

$$(u^n_t, \theta^n_x) \rightharpoonup (u_t, \theta_x) \text{ weakly in } L^2(0,T; L^2(a,b)),$$

and

$$u^n_t \theta^n_x \rightharpoonup \gamma \text{ weakly in } L^2(0,T; M(a,b)).$$

Moreover, there exist a constant $C > 0$, depending on the initial data, such that the solution $(u, \theta)$ satisfies the following first order estimates:

$$\|u\|_{W^{1,\infty}(0,T;L^2(a,b))} + \|u\|_{L^\infty(0,T;H^1(a,b))} + \|\theta\|_{L^\infty(0,T;L^1(a,b))} \leq C \left( \|u_0\|_{H^1(a,b)}, \|u_1\|_{L^2(a,b)}, \|\theta_0\|_{L^2(a,b)} \right)$$

and

$$\|\theta\|_{L^\infty(0,T;L^2(a,b))} + \|\theta\|_{L^2(0,T;H^1(a,b))} \leq C \left( \|u_0\|_{H^1(a,b)}, \|u_1\|_{L^2(a,b)}, \|\theta_0\|_{L^2(a,b)} \right).$$

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References

[1] Gui Q. Chen and Constantine M. Dafermos. The vanishing viscosity method in one-dimensional thermoelasticity. *Trans. Amer. Math. Soc.*, 347(2):531–541, 1995.

[2] Constantine M. Dafermos. On the existence and the asymptotic stability of solutions to the equations of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 29:241–271, 1968.

[3] Constantine M. Dafermos. Global smooth solutions to the initial-boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity. *SIAM J. Math. Anal.*, 13(3):397–408, 1982.

[4] Constantine M. Dafermos and L. Hsiao. Development of singularities in solutions of the equations of nonlinear thermoelasticity. *Quart. Appl. Math.*, 44(3):463–474, 1986.
[5] Lawrence C. Evans. Partial differential equations, volume 19 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, second edition, 2010.

[6] William J. Hrusa and Salim A. Messaoudi. On formation of singularities in one-dimensional nonlinear thermoelasticity. Arch. Rational Mech. Anal., 111(2):135–151, 1990.

[7] William J. Hrusa and Michael A. Tarabek. On smooth solutions of the Cauchy problem in one-dimensional nonlinear thermoelasticity. Quart. Appl. Math., 47(4):631–644, 1989.

[8] Yuxi Hu and Na Wang. On global solutions in one-dimensional thermoelasticity with second sound in the half line. Commun. Pure Appl. Anal., 14(5):1671–1683, 2015.

[9] Song Jiang. Global existence of smooth solutions in one-dimensional nonlinear thermoelasticity. Proc. Roy. Soc. Edinburgh Sect. A, 115(3-4):257–274, 1990.

[10] Song Jiang. On global smooth solutions to the one-dimensional equations of nonlinear inhomogeneous thermoelasticity. Nonlinear Anal., 20(10):1245–1256, 1993.

[11] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uraltseva. Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[12] Reinhard Racke. Initial boundary value problems in one-dimensional nonlinear thermoelasticity. Math. Methods Appl. Sci., 10(5):517–529, 1988.

[13] Reinhard Racke and Yoshihiro Shibata. Global smooth solutions and asymptotic stability in one-dimensional nonlinear thermoelasticity. Arch. Rational Mech. Anal., 116(1):1–34, 1991.

[14] Helmut Schaefer. Über die methode der a priori-schranken. Math. Ann., 129:415–416, 1955.

[15] Yoshihiro Shibata. Neumann problem for one-dimensional nonlinear thermoelasticity. In Partial differential equations, Part 1, 2 (Warsaw, 1990), volume 2 of Banach Center Publ., 27, Part 1, pages 457–480. Polish Acad. Sci. Inst. Math., Warsaw, 1992.

[16] Marshall Slemrod. Global existence, uniqueness, and asymptotic stability of classical smooth solutions in one-dimensional nonlinear thermoelasticity. Arch. Rational Mech. Anal., 76(2):97–133, 1981.

[17] Song M. Zheng and Wei X. Shen. Global solutions to the Cauchy problem of quasilinear hyperbolic parabolic coupled systems. Sci. Sinica Ser. A, 30(11):1133–1149, 1987.