The Converse Part of The Theorem for Quantum Hoeffding Bound

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Abstract

We prove the converse part of the theorem for quantum Hoeffding bound on the asymptotics of quantum hypothesis testing, essentially based on an argument developed by Nussbaum and Szkola in proving the converse part of the quantum Chernoff bound. Our result complements Hayashi’s proof of the direct (achievability) part of the theorem, so that the quantum Hoeffding bound has now been established.

Keywords

quantum hypothesis testing, Hoeffding bound, error exponent

1 Introduction

Let $\rho$ and $\sigma$ be arbitrary density operators on a Hilbert space $\mathcal{H}$, and consider the hypothesis testing problem for $\rho^{\otimes n}$ and $\sigma^{\otimes n}$. Identifying a hermitian operator $0 \leq T_n \leq I$ on $\mathcal{H}^{\otimes n}$ with a POVM $(T_n, I - T_n)$ which represents a test of the hypotheses $\{\rho^{\otimes n}, \sigma^{\otimes n}\}$ on the true state, the error probabilities of the first and second kinds are defined by

$$\alpha_n[T_n] \overset{\text{def}}{=} 1 - \text{Tr}[\rho^{\otimes n}T_n] \quad \text{and} \quad \beta_n[T_n] \overset{\text{def}}{=} \text{Tr}[\sigma^{\otimes n}T_n].$$

Our concern is the following quantity:

$$B(r \mid \rho \parallel \sigma) \overset{\text{def}}{=} \sup_{\{T_n\}} \left\{ - \lim_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n] \mid \limsup_{n \to \infty} \frac{1}{n} \log \beta_n[T_n] \leq -r \right\},$$

(1)

where $r$ is an arbitrary positive number. Since the quantum Stein’s lemma established by [1] and [2] implies that

$$B(r \mid \rho \parallel \sigma) = 0 \quad \text{if} \quad r > D(\rho \parallel \sigma) \overset{\text{def}}{=} \text{Tr}[\rho(\log \rho - \log \sigma)],$$

(2)

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we can assume $0 < r \leq D(\rho \parallel \sigma)$. In the classical case where probability distributions $\{p, q\}$ on a common discrete set $\Omega$ are given instead of $\{\rho, \sigma\}$, we have (e.g., [3, 4]), for $0 < \forall r \leq D(p \parallel q) = \sum_\omega p(\omega) (\log p(\omega)/q(\omega))$,

$$B(r | p \parallel q) = \max_{0 \leq s < 1} \frac{-sr - \phi(s)}{1 - s},$$

where

$$\phi(s) = \phi(s | p \parallel q) = \log \sum_{\omega \in \Omega} p(\omega)^{1-s} q(\omega)^s.$$ 

This result is often referred to as the Hoeffding bound after [5]. Our aim is to show that the same expression holds for $B(r | \rho \parallel \sigma)$ as follows.

**Theorem 1** For any $0 < r \leq D(\rho \parallel \sigma)$ we have

$$B(r | \rho \parallel \sigma) = \max_{0 \leq s < 1} \frac{-sr - \phi(s)}{1 - s},$$

where

$$\phi(s) = \phi(s | \rho \parallel \sigma) = \log \text{Tr} \left[ \rho^{1-s} \sigma^s \right].$$

Finding such a compact expression as (4) for $B(r | \rho \parallel \sigma)$ has been a long-standing open problem; see [6] and section 3.4 of [7] for significant partial results on this problem. Recently, two remarkable results were reported on the error exponent for symmetric Bayesian discrimination of two quantum i.i.d. states, which had completed the theorem yielding the quantum Chernoff bound. That is, firstly Nussbaum and Szkola [8] proved the converse part of the theorem claiming that the exponent cannot exceed the bound, and then Audenaert et al. [9] proved the direct part for the achievability of the bound. It should be noted that the quantum Chernoff bound is represented by the use of the same function as (5). The approach made in [9] was immediately extended by Hayashi [10] to the asymmetric setting, whereby he proved that (LHS) $\geq$ (RHS) in (4). In this paper we show the converse inequality (LHS) $\leq$ (RHS) based on an argument developed in [8].

### 2 Statement of the main result and some preliminary arguments

Our goal is to prove that for any sequence of tests $\{T_n\}$ and for any $0 < r \leq D(\rho \parallel \sigma)$ the following implication holds:

$$\limsup_{n \to \infty} \frac{1}{n} \log \beta_n[T_n] \leq -r \implies \liminf_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n] \geq -b(r),$$

where

$$b(r) = b(r | \rho \parallel \sigma) = \max_{0 \leq s < 1} \frac{-sr - \phi(s)}{1 - s}$$

(7)
with \( \phi(s) \) defined by (5). Let us define \( \Phi(a) \) and \( \Psi(a) \) for \(-D(\rho \parallel \sigma) \leq a \leq D(\sigma \parallel \rho)\) by

\[
\Phi(a) \overset{\text{def}}{=} \max_{s \in \mathbb{R}} (as - \phi(s)) = \max_{0 \leq s \leq 1} (as - \phi(s)),
\]

\[\Phi(a) \overset{\text{def}}{=} \max_{s \in \mathbb{R}} (as - \phi(s + 1)) = \Phi(a) - a.\] (9)

Then we can see\footnote{For the derivation of these properties of \( \Phi \) and \( \Psi \), refer to \cite{2, 6, 7, 10}.} that \( \Phi \) (\( \Psi \), resp.) is continuous and monotonically increasing (decreasing, resp.) on the domain \([-D(\rho \parallel \sigma), D(\sigma \parallel \rho)\]) and that

\[
\Phi(-D(\rho \parallel \sigma)) = 0, \quad \Phi(D(\sigma \parallel \rho)) = D(\sigma \parallel \rho),
\]

\[
\Psi(-D(\rho \parallel \sigma)) = D(\rho \parallel \sigma), \quad \Psi(D(\sigma \parallel \rho)) = 0.
\]

Hence, every \( r \in [0, D(\rho \parallel \sigma)] \) is uniquely represented as \( r = \Psi(a) \) by an \( a \in [-D(\rho \parallel \sigma), D(\sigma \parallel \rho)] \). Furthermore, it can be shown that

\[
r = \Psi(a) \iff b(r) = \Phi(a) = a + \Psi(a).
\] (10)

Therefore, the implication (6) for \( 0 < r \leq D(\rho \parallel \sigma) \) is equivalent to

\[
\limsup_{n \to \infty} \frac{1}{n} \log \beta_n[T_n] \leq -\Psi(a) \implies \liminf_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n] \geq -a - \Psi(a)
\] (11)

for \(-D(\rho \parallel \sigma) \leq a < D(\sigma \parallel \rho)\), which we shall prove in the next section.

### 3 Proof of (11)

Let the spectral (Shatten) decompositions of \( \rho, \sigma \) be denoted by

\[
\rho = \sum_i \lambda_i \lvert x_i \rangle \langle x_i \rvert, \quad \sigma = \sum_j \gamma_j \lvert y_j \rangle \langle y_j \rvert,
\] (12)

and define

\[
p(i, j) \overset{\text{def}}{=} \lambda_i \lvert \langle x_i | y_j \rangle \rvert^2, \quad q(i, j) \overset{\text{def}}{=} \gamma_j \lvert \langle x_i | y_j \rangle \rvert^2.
\] (13)

Then \( p \) and \( q \) form probability distributions on the range \( \Omega = \{(i, j)\} \) of the pair of indices \((i, j)\). In proving the converse part of the theorem of quantum Chernoff bound, Nussbaum and Szkola \cite{8} effectively used the following three facts on the relation between \{\( \rho, \sigma \)\} and \{\( p, q \)\}. Firstly, we have the identity

\[
\phi(s | \rho \parallel \sigma) = \phi(s | p \parallel q).
\] (14)

Secondly, the quantum i.i.d. extensions \( \{\rho^\otimes n, \sigma^\otimes n\} \) correspond to the classical i.i.d. extensions \( \{p^n, q^n\} \) by (12) and (13). Thirdly, it holds for any projection \( T \) that

\[
\alpha[T] + \beta[T] \geq \frac{1}{2} \sum_{\omega \in \Omega} \min\{p(\omega), q(\omega)\},
\] (15)
where $\alpha[T] \overset{\text{def}}{=} \text{Tr}[\rho(I - T)]$ and $\beta[T] \overset{\text{def}}{=} \text{Tr}[\sigma T]$. The last one is the most ingenious finding in [8], which is derived by combining the general inequality

$$\lambda|u - v|^2 + \gamma|v|^2 \geq \frac{1}{2}|u|^2 \min\{\lambda, \gamma\} \quad (\forall \lambda, \gamma \geq 0, \ \forall u, v \in \mathbb{C}) \quad (16)$$

with

$$\alpha[T] = \sum_{i,j} \lambda_i |\langle x_i |(I - T)y_j \rangle|^2 \quad \text{and} \quad \beta[T] = \sum_{i,j} \gamma_j |\langle x_i |Ty_j \rangle|^2. \quad (17)$$

In the following lemma we present a slight extension of (19) with a seemingly different form, which is more convenient for the present purpose.

**Lemma 1** For any test $0 \leq T \leq I$ and any positive number $\delta$ we have

$$\alpha[T] + \delta \beta[T] \geq \frac{1}{2} \left[ p \{ p \leq \delta q \} + \delta q \{ p > \delta q \} \right], \quad (18)$$

where

$$p \{ p \leq \delta q \} \overset{\text{def}}{=} \sum_{\omega: p(\omega) \leq \delta q(\omega)} p(\omega),$$

$$q \{ p > \delta q \} \overset{\text{def}}{=} \sum_{\omega: p(\omega) > \delta q(\omega)} q(\omega).$$

**Proof:** It is easy to see that (16) and (17) yield

$$\alpha[T] + \delta \beta[T] \geq \frac{1}{2} \sum_{\omega \in \Omega} \min\{p(\omega), \delta q(\omega)\} \quad (19)$$

for any projection $T$ and any $\delta > 0$. In addition, this inequality holds for any test $0 \leq T \leq I$, because $\min_T (\alpha[T] + \delta \beta[T])$ is attained by the projection, which is denoted by $\{ \rho - \delta \sigma > 0 \}$ following [11], onto the linear subspace spanned by the eigenvectors of $\rho - \delta \sigma$ corresponding to positive eigenvalues [12, 13]. It is obvious that (19) is equivalent to (18).

Considering the nth i.i.d. case in (18) and letting $\delta = e^{-nb}$ for an arbitrary $b \in \mathbb{R}$, we have

$$\alpha_n[T_n] + e^{-nb} \beta_n[T_n] \geq \frac{1}{2} \left[ f_n(b) + e^{-nb} g_n(b) \right], \quad (20)$$

where

$$f_n(b) \overset{\text{def}}{=} p^n \{ p^n \leq e^{-nb} q^n \} = p^n \left\{ \frac{1}{n} \log \frac{q^n}{p^n} \geq b \right\}, \quad (21)$$

$$g_n(b) \overset{\text{def}}{=} q^n \{ p^n > e^{-nb} q^n \} = q^n \left\{ \frac{1}{n} \log \frac{q^n}{p^n} < b \right\}. \quad (22)$$
Noting that
\[ \frac{1}{n} \log \frac{q^n(\omega^n)}{p^n(\omega^n)} = \frac{1}{n} \sum_{i=1}^{n} \log \frac{q(\omega_i)}{p(\omega_i)} \quad \text{for} \quad \omega^n = (\omega_1, \ldots, \omega_n) \]
and that \( \Phi \) is the Legendre transformation of \( \phi(s) = E_p[e^{s \log q/p}] \), we see that Cramér’s theorem in large deviation theory (e.g., see [4]) yields
\[ \lim_{n \to \infty} \frac{1}{n} \log f_n(b) = -\Phi(b) = -b - \Psi(b) \quad (23) \]
if
\[ b > E_p \left[ \log \frac{q}{p} \right] = -D(p \| q) = -D(\rho \| \sigma). \]
Similarly, since \( \Psi \) is the Legendre transformation of \( \phi(s + 1) = E_q[e^{s \log q/p}] \), we have
\[ \lim_{n \to \infty} \frac{1}{n} \log g_n(b) = -\Psi(b) \quad (24) \]
if
\[ b < E_q \left[ \log \frac{q}{p} \right] = D(q \| p) = D(\sigma \| \rho). \]
Thus we obtain
\[ \lim_{n \to \infty} \frac{1}{n} \log \left[ f_n(b) + e^{-nb} g_n(b) \right] = -b - \Psi(b) \]
for \(-D(\rho \| \sigma) < \forall b < D(\sigma \| \rho)\). Hence, (20) implies
\[ -b - \Psi(b) \leq \liminf_{n \to \infty} \frac{1}{n} \log \left( \alpha_n[T_n] + e^{-nb} \beta_n[T_n] \right) \]
\[ \leq \max \left\{ \liminf_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n], -b + \limsup_{n \to \infty} \frac{1}{n} \log \beta_n[T_n] \right\}. \quad (25) \]
Now we assume that
\[ \limsup_{n \to \infty} \frac{1}{n} \log \beta_n[T_n] \leq -\Psi(a) \]
for an arbitrarily fixed \( a \in [-D(\rho \| \sigma), D(\sigma \| \rho)) \). Then, substituting \( b = a + \epsilon \) into (25) with \( 0 < \epsilon < D(\sigma \| \rho) - a \), we have
\[ -a - \epsilon - \Psi(a + \epsilon) \leq \max \left\{ \liminf_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n], -a - \epsilon - \Psi(a) \right\}. \]
Moreover, since \( \Psi \) is monotonically decreasing, the RHS cannot be \(-a - \epsilon - \Psi(a)\). Therefore
\[ -a - \epsilon - \Psi(a + \epsilon) \leq \liminf_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n]. \]
Letting \( \epsilon \downarrow 0 \), we have \(-a - \Psi(a) \leq \liminf_{n \to \infty} \frac{1}{n} \log \alpha_n[T_n] \), which completes the proof of (11).
4 Concluding remarks

We have proved (11) by extending an argument of Nussbaum and Szkola [8], yielding the converse part of Theorem 1 for the quantum Hoeffding bound. Combined with the direct part which was proved by Hayashi [10], the theorem has been established. Several remarks on the theorem are now in order.

Remark 1 Besides (3), \( B(r | p \parallel q) \) in the classical case has another expression:

\[
B(r | p \parallel q) = \min_{\hat{p} : D(\hat{p} \parallel q) \leq r} D(\hat{p} \parallel p), \tag{26}
\]

which is also called the Hoeffding bound as well as (3). In the quantum case, as Hayashi showed in [7] (sections 3.4 and 3.7), the inequality

\[
B(r | \rho \parallel \sigma) \leq \min_{\hat{\rho} : D(\hat{\rho} \parallel \sigma) \leq r} D(\hat{\rho} \parallel \rho) \quad (=: \tilde{b}(r)) \tag{27}
\]

follows from the (strong) converse part of the quantum Stein’s lemma [2]. The RHS can be represented as

\[
\tilde{b}(r) = \max_{0 \leq s < 1} \frac{-sr - \tilde{\phi}(s)}{1 - s}, \tag{28}
\]

where

\[
\tilde{\phi}(s) \overset{\text{def}}{=} \log \text{Tr} \left[ e^{(1-s) \log \rho + s \log \sigma} \right].
\]

It then follows from the Golden-Thompson inequality \( \text{Tr} \left[ e^{A+B} \right] \leq \text{Tr} \left[ e^A e^B \right] \) that \( \tilde{\phi}(s) \leq \phi(s) \), which gives another proof of (27) by (1), with showing that the equality in (27) does not hold in general; see [2] for a similar remark on a slightly different context.

Remark 2 Rewriting (4) into

\[
B(r | \rho \parallel \sigma) = \max_{t \geq 0} (-tr - \xi(t))
\]

by

\[
\xi(t) \overset{\text{def}}{=} (t + 1) \phi \left( \frac{t}{t + 1} \right)
\]

and invoking that \( \xi : [0, \infty) \rightarrow \mathbb{R} \) is a convex function with the derivative \( \xi'(t) \) ranging over \([-D(\rho \parallel \sigma), 0]\), we can show that, for any \( t \geq 0 \),

\[
\xi(t) = \max_{0 < r \leq D(\rho \parallel \sigma)} (-tr - B(r | \rho \parallel \sigma)) = \max_{r > 0} (-tr - B(r | \rho \parallel \sigma)),
\]

where the second equality follows from (2). This leads to the following conversion formula for (1):

\[
\phi(s | \rho \parallel \sigma) = \max_{r > 0} \left( -sr - (1 - s) B(r | \rho \parallel \sigma) \right), \quad 0 \leq \forall s \leq 1. \tag{29}
\]
From the definition (1) of $B(r \mid \rho \| \sigma)$ it is obvious that, for any $r > 0$ and any quantum channel (trace-preserving completely positive map) $\mathcal{E}$,

$$B(r \mid \rho \| \sigma) \geq B(r \mid \mathcal{E}(\rho) \| \mathcal{E}(\sigma)).$$

(30)

Thus (29) yields

$$\phi(s \mid \rho \| \sigma) \leq \phi(s \mid \mathcal{E}(\rho) \| \mathcal{E}(\sigma)), \quad 0 \leq s \leq 1,$$

or equivalently

$$\operatorname{Tr}\left[\rho^{1-s}\sigma^s\right] \leq \operatorname{Tr}\left[\mathcal{E}(\rho)^{1-s}\mathcal{E}(\sigma)^s\right], \quad 0 \leq s \leq 1.$$  (32)

Two renowned proofs of this inequality are that of Uhlmann [14] based on an interpolation theory and that of Petz [15] which, in this case, relies upon the operator concavity of the function $f(u) = u^s$. Our proof seems to be new. Note that the monotonicity of quantum relative entropy

$$D(\rho \| \sigma) \geq D(\mathcal{E}(\rho) \| \mathcal{E}(\sigma))$$

(33)

is obtained by differentiating (32) at $s = 0$.

**Remark 3** For an arbitrary $a \in \mathbb{R}$, let

$$F_n(a) \overset{\text{def}}{=} \operatorname{Tr}\left[\rho^{\otimes n} \{ \rho^{\otimes n} - e^{-na}\sigma^{\otimes n} \leq 0 \} \right],$$

(34)

$$G_n(a) \overset{\text{def}}{=} \operatorname{Tr}\left[\sigma^{\otimes n} \{ \rho^{\otimes n} - e^{-na}\sigma^{\otimes n} > 0 \} \right],$$

(35)

where $\{ A \leq 0 \}$ ($\{ A > 0 \}$, resp.) for a hermitian operator $A$ is defined as the projection onto the subspace spanned by the eigenvectors of $A$ corresponding to nonpositive (positive, resp.) eigenvalues. Assume that the following limits exist:

$$\mathcal{F}(a) \overset{\text{def}}{=} - \lim_{n \to \infty} \frac{1}{n} \log F_n(a),$$

(36)

$$\mathcal{G}(a) \overset{\text{def}}{=} - \lim_{n \to \infty} \frac{1}{n} \log G_n(a),$$

(37)

Then, as is shown in [11] we have

$$B(r \mid \rho \| \sigma) = \sup_{a : \mathcal{G}(a) \geq r} \mathcal{F}(a) = \inf_{a : \mathcal{G}(a) < r} (a + \mathcal{G}(a)).$$

On the other hand, since $\Phi$ ($\Psi$, resp.) is continuous and monotonically increasing (decreasing, resp.), it follows from (11) and (10) that

$$B(r \mid \rho \| \sigma) = \sup_{a : \Phi(a) \geq r} \Phi(a) = \inf_{a : \Phi(a) < r} (a + \Phi(a)).$$

Comparing these expressions, we are led to the following conjecture:

$$\mathcal{F}(a) = \Phi(a) \quad \text{and} \quad \mathcal{G}(a) = \Psi(a).$$

(38)

If $\rho$ and $\sigma$ commute, these relations are equivalent to (23) and (24), and hence are true. In the general case, however, they have no mathematical proof at present.

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2Note that our $B(r \mid \rho \| \sigma)$ corresponds to $B_c(r \mid \bar{\sigma} \| \bar{\rho})$ in [11].
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