Turán-Type Reverse Markov Inequalities for Polynomials with Restricted Zeros

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Abstract
Let \( \mathcal{P}_n^c \) denote the set of all algebraic polynomials of degree at most \( n \) with complex coefficients. Let \( D^+ := \{ z \in \mathbb{C} : |z| \leq 1, \ \text{Im}(z) \geq 0 \} \).

For integers \( 0 \leq k \leq n \) let \( \mathcal{F}_{n,k}^c \) be the set of all polynomials \( P \in \mathcal{P}_n^c \) having at least \( n - k \) zeros in \( D^+ \). Let

\[
\|f\|_A := \sup_{z \in A} |f(z)|
\]

for complex-valued functions defined on \( A \subset \mathbb{C} \). We prove that there are absolute constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
c_1 \left( \frac{n}{k+1} \right)^{1/2} \leq \inf_P \frac{\|P'|_{[-1,1]}\|}{\|P\|_{[-1,1]}} \leq c_2 \left( \frac{n}{k+1} \right)^{1/2}
\]

for all integers \( 0 \leq k \leq n \), where the infimum is taken for all \( 0 \neq P \in \mathcal{F}_{n,k}^c \) having at least one zero in \([ -1, 1 ]\). This is an essentially sharp reverse Markov-type inequality for the classes \( \mathcal{F}_{n,k}^c \) extending earlier results of Turán and Komarov from the case \( k = 0 \) to the cases \( 0 \leq k \leq n \).

Keywords Turán type reverse Markov inequalities · polynomials with restricted zeros

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1 Introduction and Notation

Let \( P_n \) denote the set of all algebraic polynomials of degree at most \( n \) with real coefficients. Let \( P^c_n \) denote the set of all algebraic polynomials of degree at most \( n \) with complex coefficients. Let

\[
\| f \|_A := \sup_{z \in A} |f(z)|
\]

for complex-valued functions defined on \( A \subseteq \mathbb{C} \). Turán [32] proved that

\[
\| P' \|_{[-1,1]} \geq \frac{\sqrt{n}}{6} \| P \|_{[-1,1]} \quad (1.1)
\]

for all \( P \in P^c_n \) of degree \( n \) having all their zeros in the interval \([-1, 1]\). The examples \( P(x) = (x^2 - 1)^m \) and \( P(x) = (x^2 - 1)^m (x + 1) \) show that Turán’s reverse Markov-type inequality (1.1) is essentially sharp, even though the multiplicative constant \( 1/6 \) in (1.1) is not the best possible. Note that the best possible multiplicative constant \( c = c_n \) in (1.1) had been found by Erőd [10], see also [11]. Another simple observation of Turán [32] is the inequality

\[
\| P' \|_D \geq \frac{n}{2} \| P \|_D \quad (1.2)
\]

for all \( P \in P^c_n \) of degree \( n \) having all their zeros in the closed disk \( D \subseteq \mathbb{C} \). Malik [23] established an extension of (1.2) proving that

\[
\| P' \|_D \geq \frac{n}{1 + R} \| P \|_D
\]

for all \( P \in P^c_n \) of degree \( n \) having all their zeros in the closed disk \( D(0, R) \subseteq \mathbb{C} \) of radius \( R \leq 1 \) centered at 0, while Govil [16] showed that

\[
\| P' \|_D \geq \frac{n}{1 + R^n} \| P \|_D
\]

for all \( P \in P^c_n \) of degree \( n \) having all their zeros in the closed disk \( D(0, R) \subseteq \mathbb{C} \) of radius \( R \geq 1 \) centered at 0. See also [18, Sect. 4].

Let \( \varepsilon \in [0, 1] \) and let \( D_\varepsilon \) be the closed region bounded by the ellipse of the complex plane with large axis \([-1, 1]\) and small axis \([-i\varepsilon, i\varepsilon]\). Let \( P^c_n(D_\varepsilon) \) denote the collection of all \( P \in P^c_n \) of degree \( n \) having all their zeros in \( D_\varepsilon \). Extending Turán’s reverse Markov-type inequality (1.1), Erőd [10, Thm III] proved that there are absolute constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
c_1(n\varepsilon + \sqrt{n}) \leq \inf_P \frac{\| P' \|_{D_\varepsilon}}{\| P \|_{D_\varepsilon}} \leq c_2(n\varepsilon + \sqrt{n}),
\]
where the infimum is taken for all \( P \in \mathcal{P}_n^c(D_\varepsilon) \). Levenberg and Poletsky [21] proved that

\[
\frac{\sqrt{n}}{20 \ \text{diam} K} \leq \inf_P \frac{\|P'\|_K}{\|P\|_K}
\]

for all compact convex sets \( K \subset \mathbb{C} \), where the infimum is taken for all \( P \in \mathcal{P}_n^c \) of degree \( n \) having all their zeros in \( K \).

Let \( \varepsilon \in [0, 1] \) and let \( S_\varepsilon \) be the diamond of the complex plane with diagonals \([-1, 1]\) and \([-i\varepsilon, i\varepsilon]\). Let \( \mathcal{P}_n^c(S_\varepsilon) \) denote the collection of all \( P \in \mathcal{P}_n^c \) of degree \( n \) having all their zeros in \( S_\varepsilon \). It has been proved in [5] that there are absolute constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
c_1(n\varepsilon + \sqrt{n}) \leq \inf_P \frac{\|P'\|_{S_\varepsilon}}{\|P\|_{S_\varepsilon}} \leq c_2(n\varepsilon + \sqrt{n}),
\]

where the infimum is taken for all \( P \in \mathcal{P}_n^c(S_\varepsilon) \) with the property

\[
|P(z)| = |P(-z)|, \quad z \in \mathbb{C},
\]

or where the infimum is taken for all \( P \in \mathcal{P}_n^c(S_\varepsilon) \) with real coefficients. It is an interesting question whether or not the lower bound in the above inequality holds for all \( P \in \mathcal{P}_n^c(S_\varepsilon) \). Another result in [5] shows that this is the case at least when \( \varepsilon = 1 \), that is, there are absolute constants \( c_1 > 0 \) and \( c_2 > 0 \) such that

\[
c_1 n \leq \inf_P \frac{\|P'\|_{S_1}}{\|P\|_{S_1}} \leq c_2 n,
\]

where the infimum is taken for all (complex) \( P \in \mathcal{P}_n^c(S_1) \). Motivated by the above results Révész [28] established the right order Turán-type reverse Markov inequalities on convex domains of the complex plane. His main theorem contains the above mentioned results in [5] as special cases. It states that

\[
\frac{\|P'\|_K}{\|P\|_K} \geq c(K)n \quad \text{with} \quad c(K) = 0.0003 \frac{w(K)}{d(K)^2},
\]

for all \( P \in \mathcal{P}_n^c \) of degree \( n \) having all their zeros in a bounded convex set \( K \subset \mathbb{C} \), where \( d(K) \) is the diameter of \( K \) and

\[
w(K) := \min_{\gamma \in [-\pi, \pi]} \left( \max_{z \in K} \text{Re}(ze^{-i\gamma}) - \min_{z \in K} \text{Re}(ze^{-i\gamma}) \right)
\]

is the minimal width of \( K \). The proof given by Révész is elementary, but rather subtle. Results on Turán and Erőd type reverses of Markov- and Bernstein-type inequalities include [9,19,21,25,27,33,34]. The research on Turán and Erőd type reverses of Markov- and Bernstein-type inequalities got a new impulse suddenly in 2006 in large part by the work of Sz. Révész [28], see [5,6,8,14,15,29], for example.
G.G. Lorentz, M. von Golitschek, and Y. Makovoz devote Chapter 3 of their book [22] to incomplete polynomials. E.B. Saff and R.S. Varga were among the researchers having contributed significantly to this topic. See [1,30], and [31], for instance.

Let \( P_{n,k} \) be the set of all algebraic polynomials, with real coefficients, of degree at most \( n + k \) having at least \( n + 1 \) zeros at 0. That is, every \( P \in P_{n,k} \) is of the form
\[
P(x) = x^{n+1} R(x), \quad R \in \mathcal{P}_{k-1}.
\]

Let
\[
V_b^a(f) := \int_a^b |f'(x)| \, dx
\]
denote the total variation of a continuously differentiable function \( f \) on an interval \([a, b]\). A question [12] asked by A. Eskenazis and P. Ivanisvili related to their paper [13] [26] is answered in [7] by proving that there are absolute constants \( c_3 > 0 \) and \( c_4 > 0 \) such that
\[
c_3 \frac{n}{k} \leq \min_{0 \neq P \in P_{n,k}} \frac{\|P'\|_{[0,1]}}{V_0^1(P)} \leq \min_{0 \neq P \in P_{n,k}} \frac{\|P'\|_{[0,1]}}{|P(1)|} \leq c_4 \left( \frac{n}{k} + 1 \right)
\]
for all integers \( n \geq 1 \) and \( k \geq 1 \). Here \( c_3 = 1/12 \) is a suitable choice.

In [7] we also proved that there are absolute constants \( c_3 > 0 \) and \( c_4 > 0 \) such that
\[
c_3 \left( \frac{n}{k} \right)^{1/2} \leq \min_{0 \neq P \in P_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{V_0^1(P)} \leq \min_{0 \neq P \in P_{n,k}} \frac{\|P'(x)\sqrt{1-x^2}\|_{[0,1]}}{|P(1)|} \leq c_4 \left( \frac{n}{k} + 1 \right)^{1/2}
\]
for all integers \( n \geq 1 \) and \( k \geq 1 \). Here \( c_3 = 1/8 \) is a suitable choice.

Let
\[
D^+ := \{ z \in \mathbb{C} : |z| \leq 1, \ \text{Im}(z) \geq 0 \}.
\]

In [20] Komarov proved that
\[
\|P'\|_{[-1,1]} \geq A\sqrt{n} \|P\|_{[-1,1]} , \quad A = \frac{2}{3\sqrt{210e}} = 0.0279 \ldots,
\]
for all polynomials \( P \) of degree \( n \) having all their zeros in the closed upper half-disk \( D^+ \).

For integers \( 0 \leq k \leq n \) let \( \mathcal{F}^c_{n,k} \) be the set of all polynomials \( P \in \mathcal{P}^c_n \) having at least \( n - k \) zeros in \( D^+ \). In this paper we prove an essentially sharp reverse Markov-type inequality for the classes \( \mathcal{F}^c_{n,k} \), extending the above mentioned results of Turán and Komarov from the case \( k = 0 \) to the cases \( 0 \leq k \leq n \).
2 New Results

The lower bound of Theorem 2.1 below is quite a new result even in the case when the infimum is taken for polynomials $P \in \mathcal{P}_{c}^{n}$ having at least $n - k$ zeros only in $[-1, 1]$ rather than $D^+$. Theorem 2.1

There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \left( \frac{n}{k + 1} \right)^{1/2} \leq \inf_{P} \frac{\|P'\|[-1,1]}{\|P\|[-1,1]} \leq c_2 \left( \frac{n}{k + 1} \right)^{1/2}$$

for all integers $0 \leq k \leq n$, where the infimum is taken for all $0 \neq P \in \mathcal{F}_{n,k}^{c}$ having at least one zero in $[-1, 1]$. Here $c_1 = 1/636$ is a suitable choice. When $0 \leq k \leq n/100000$ the lower bound remains valid even if the infimum is taken for all $0 \neq P \in \mathcal{F}_{n,k}^{c}$.

Theorem 2.1 follows from the results below.

Theorem 2.2

Let $1 \leq k \leq n/100000$. We have

$$\|P'\|[-1,1] \geq \frac{1}{144e} \left( \frac{n - k}{2k} \right)^{1/2} \|P\|[-1,1]$$

for all $P \in \mathcal{F}_{n,k}^{c}$.

Corollary 2.3

Let $1 \leq k \leq n$. We have

$$\|P'\|[-1,1] \geq \max \left\{ \frac{1}{2}, \frac{1}{448} \left( \frac{n - k}{2k} \right)^{1/2} \right\} \|P\|[-1,1]$$

for all $P \in \mathcal{F}_{n,k}^{c}$ with at least one zero in $[-1, 1]$.

Theorem 2.4

There is an absolute constant $c_2 > 0$ and there are polynomials $0 \neq P = P_{n,k} \in \mathcal{F}_{2n,2k}^{c}$ of the form

$$P(x) = (x^2 - 1)^{n-k}R(x), \quad R \in \mathcal{P}_{2k},$$

such that

$$\frac{\|P'\|[-1,1]}{\|P\|[-1,1]} \leq c_2 \left( \frac{n}{k} \right)^{1/2}$$

for every $1 \leq k \leq n$.

We remark that the upper bound of Theorem 2.1 remains valid if we replace the closed upper half-disk $D^+$ with the closed unit disk $D$ in the definition of $\mathcal{F}_{n,k}^{c}$, as then the infimum is taken for a larger class of polynomials. However, the lower bound of
Theorem 2.1 does not remain valid if we replace the closed upper half-disk $D^+$ with the closed unit disk $D$ in the definition of $\mathcal{F}_{n,k}^c$, not even in the case that $k = 0$. This can be seen by the example given in [20] (see also [21], where the case of star-shaped compact sets was considered). For completeness we present here a slight modification of the calculation made in [20] in a few lines. Given $\varepsilon > 0$, let $m$ be the even integer for which $1/\varepsilon < m \leq 1/\varepsilon + 2$. We claim that for every $\varepsilon > 0$ and for every integer $n \geq 1$ there is a $P_n \in \mathcal{P}_{mn}^c$ of degree $mn$ having all its zeros on the unit circle $\partial D$ such that

$$\|P_n\|_{[-1,1]} \leq (1/\varepsilon + 2)^{1-\varepsilon}(mn)^\varepsilon \|P_n\|_{[-1,1]}.$$ 

To see this let $P_n \in \mathcal{P}_{mn}^c$ be defined by $P_n(z) := (z^m - 1)^n$. Observe that $\|P_n\|_{[-1,1]} = 1$ (as $m$ is even), and the function

$$|P_n'(x)| = mn(1 - x^m)^{n-1}|x|^{m-1}$$

achieves its maximum on $[-1, 1]$ at the point $a \in (0, 1)$, where

$$a^m = \frac{m - 1}{mn - 1} \leq \frac{1}{n}.$$ 

Hence

$$|P_n'(a)| \leq mn a^{m-1} \leq mn^{1/m-1} \leq mn^{\varepsilon} \leq m^{1-\varepsilon}(mn)^\varepsilon \leq (1/\varepsilon + 2)^{1-\varepsilon}(mn)^\varepsilon.$$ 

### 3 Lemmas

Our proof of Theorem 2.2 is based on the following two non-trivial results. Lemma 3.1 below is proved in [17].

**Lemma 3.1** If $Q \in \mathcal{F}_{n,0}^c$ and

$$E_\delta := \left\{ x \in [-1, 1] : \left| \frac{Q'(x)}{Q(x)} \right| \leq n\delta \right\}, \quad \delta > 0,$$

then

$$m(E_\delta) < A\delta, \quad \delta > 0,$$

where $A := 70e$ is a suitable choice.

Lemma 3.2 below was first proved in [24]. Its proof may also be found in [4, Sect. 7.2] with the larger constant $B = 8\sqrt{2}$. 

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Lemma 3.2 If \( R \in \mathcal{P}_k^c \) and

\[
F_\alpha := \left\{ x \in \mathbb{R} : \left| \frac{R'(x)}{R(x)} \right| \geq \alpha \right\}, \quad \alpha > 0,
\]

then

\[
m(F_\alpha) \leq \frac{Bk}{\alpha}, \quad \alpha > 0,
\]

where \( B := 2e \) is a suitable choice.

To prove Theorem 2.4 we need the following two lemmas. Lemma 3.3 below is stated and proved as Theorem 2.1 in [7] by using deep results from [2] and [3]. Recall that \( \mathcal{P}_{n-k,k}, 0 \leq k \leq n \), denotes the set of all algebraic polynomials with real coefficients, of degree at most \( n \) having at least \( n-k+1 \) zeros at 0.

Lemma 3.3 There are absolute constants \( c_3 > 0 \) and \( c_4 > 0 \) such that

\[
c_3 \frac{n-k}{k} \leq \min_{0 \neq P \in \mathcal{P}_{n-k,k}} \frac{\| P' \|_{[0,1]} V_0^1(P)}{\| P \|_{[0,1]} P(1)} \leq \min_{0 \neq P \in \mathcal{P}_{n-k,k}} \frac{\| P' \|_{[0,1]} |P(1)|}{\| P \|_{[0,1]} P(1)} \leq c_4 \frac{n}{k}
\]

for all integers \( 1 \leq k \leq n-1 \). Here \( c_3 = 1/12 \) is a suitable choice.

Lemma 3.4 below follows directly from Lemma 3.2 in [7].

Lemma 3.4 Let \( 1 \leq k \leq n/11 \) and let \( S(x) := x^{n-k} R(x) \) with \( R \in \mathcal{P}_k \). We have

\[
|S(x)| < \| S \|_{[0,1]}, \quad x \in \left[ 0, 1 - \frac{10k}{n-k} \right].
\]

Lemma 3.5 below follows simply from Lemma 3.4.

Lemma 3.5 Let \( 1 \leq k \leq (n-10)/20 \) and let \( W(x) := (1-x)^{n-k} V(x) \) with \( 0 \neq V \in \mathcal{P}_k \). We have

\[
|y^{1/2} W(y)| < \| u^{1/2} W(u) \|_{[0,1]}, \quad y \in \left[ \frac{10(2k+1)}{n}, 1 \right].
\]

Proof of Lemma 3.5 Replacing \( n \) by \( 2n+1 \) and \( k \) by \( 2k+1 \) in Lemma 3.4 we obtain

\[
|S(x)| < \| S \|_{[0,1]}, \quad x \in \left[ 0, 1 - \frac{10(2k+1)}{n} \right] \subset \left[ 0, 1 - \frac{10(2k+1)}{2n-2k} \right], \quad (3.1)
\]

whenever \( 1 \leq k \leq (n-10)/20 \leq n/2 \) and \( S(x) := x^{2n-2k} R(x) \) with \( R \in \mathcal{P}_{2k+1} \). Replacing the variable \( x \) by \( 1-x \) in (3.1) yields

\[
|S(x)| < \| S \|_{[0,1]}, \quad x \in \left[ \frac{10(2k+1)}{n}, 1 \right], \quad (3.2)
\]
whenever $1 \leq k \leq (n - 10)/20$ and $S(x) := (1 - x)^{2n-2k} R(x)$ with $R \in \mathcal{P}_{2k+1}$. Now let $1 \leq k \leq (n - 10)/20$ and let $W(x) := (1 - x)^{n-k} V(x)$ with $0 \neq V \in \mathcal{P}_k$. Applying (3.2) to $S$ defined by

$$S(x) = x W(x)^2 = (1 - x)^{2n-2k}(x V(x)^2), \quad V \in \mathcal{P}_k,$$

we get the conclusion of the lemma. \qed

4 Proof of the Theorems

Proof of Theorem 2.2 Let $0 \neq P \in \mathcal{F}^{c}_{n,k}$, that is, $P = QR$, where $Q \in \mathcal{F}^{c}_{n-k,0}$ and $R \in \mathcal{P}^{c}_k$. We have

$$\frac{P'}{P} = \frac{Q'}{Q} + \frac{R'}{R}. \quad (4.1)$$

By Lemma 3.1 we have

$$m(E_\delta) < A \delta, \quad \delta > 0, \quad A := 70e, \quad (4.2)$$

where

$$E_\delta := \left\{ x \in [-1, 1] : \left| \frac{Q'(x)}{Q(x)} \right| \leq (n - k)\delta \right\}, \quad \delta > 0. \quad (4.3)$$

By Lemma 3.2 we have

$$m(F_\delta) \leq B \delta, \quad \delta > 0, \quad B := 2e, \quad (4.4)$$

where

$$F_\delta := \left\{ x \in [-1, 1] : \left| \frac{R'(x)}{R(x)} \right| \geq \frac{k}{\delta} \right\}, \quad \delta > 0. \quad (4.5)$$

Now we choose $\delta > 0$ such that

$$\frac{k}{\delta} = \frac{1}{2} (n - k)\delta; \quad (4.6)$$

that is,

$$\delta := \left( \frac{2k}{n-k} \right)^{1/2}. \quad (4.7)$$
Then, combining (4.1)–(4.7), we can deduce that
\[
\left\| \frac{P'(x)}{P(x)} \right\| \geq \left\| \frac{Q'(x)}{Q(x)} \right\| - \left\| \frac{R'(x)}{R(x)} \right\| \\
\geq (n - k)\delta - \frac{k}{\delta} = \left(\frac{(n - k)k}{2}\right)^{1/2}, \; x \in [-1, 1] \setminus H_\delta,
\]
where \( H_\delta := E_\delta \cup F_\delta \) with
\[
m(H_\delta) < (A + B)\delta = 72e\delta.
\]
(4.8)

Note that
\[
1 \leq k \leq \frac{n}{100000}
\]
implies that
\[
72e\delta = 72e \left(\frac{2k}{n - k}\right)^{1/2} \leq 72e \left(\frac{2}{99999}\right)^{1/2} < 1.
\]
(4.10)

Choose an \( x_0 \in [-1, 1] \) such that \( |P(x_0)| := \|P\|_{[-1,1]} \). It follows from (4.10) that the length of the interval \([x_0 - 72e\delta, x_0 + 72e\delta] \cap [-1, 1]\) is at least \( 72e\delta \), and hence (4.9) implies that there is a
\[
y \in [x_0 - 72e\delta, x_0 + 72e\delta] \cap [-1, 1]
\]
(4.11)
such that
\[
y \in [-1, 1] \setminus H_\delta.
\]
(4.12)

If
\[
|P(y)| \geq \frac{1}{2} \|P\|_{[-1,1]},
\]
(4.13)
then combining (4.12), (4.8) and (4.13), we obtain
\[
\|P'\|_{[-1,1]} \geq |P'(y)| \geq \left(\frac{1}{2} (n - k)k\right)^{1/2} |P(y)| \\
\geq \left(\frac{1}{2} (n - k)k\right)^{1/2} \frac{1}{2} \|P\|_{[-1,1]} \geq \frac{1}{144e} \left(\frac{n - k}{2k}\right)^{1/2} \|P\|_{[-1,1]},
\]
and the theorem follows. If (4.13) does not hold, that is, \( |P(y)| < \frac{1}{2} \|P\|_{[-1,1]} \), then it follows from the mean value theorem and (4.11) that there is a value \( \xi \) in the open
interval between $y$ and $x_0$ such that
\[ \|P'\|_{[-1,1]} \geq |P'(\xi)| \geq \left| \frac{P(y) - P(x_0)}{y - x_0} \right| \geq \frac{1}{2} \|P\|_{[-1,1]} |y - x_0|^{-1} \]
\[ \geq (144e\delta)^{-1} \|P\|_{[-1,1]} = \frac{1}{144e} \left( \frac{n - k}{2k} \right)^{1/2} \|P\|_{[-1,1]}, \]
and the theorem follows. \qed

**Proof of Corollary 2.3** Let $1 \leq k \leq n$. Suppose $0 \not\equiv P \in \mathcal{F}_{n,k}^c$ has at least one zero in $[-1, 1]$. Choose $a, b \in [-1, 1]$ such that $P(a) = 0$, and $|P(b)| = \|P\|_{[-1,1]}$. By the mean value theorem there is a $\xi \in (-1, 1)$ between $a$ and $b$ such that
\[ \|P'\|_{[-1,1]} \geq |P'(\xi)| \geq \left| \frac{P(b) - P(a)}{b - a} \right| \geq \frac{1}{2} \|P\|_{[-1,1]} \cdot (4.14) \]
If $1 \leq k \leq \frac{n}{100000}$, the result follows from Theorem 2.2 and (4.14) as $1/448 \leq (144e)^{-1}$. If $\frac{n}{100000} < k \leq n$, then
\[ \frac{1}{448} \left( \frac{n - k}{2k} \right)^{1/2} \leq \frac{1}{448} \left( \frac{99999}{2} \right)^{1/2} < \frac{1}{2}, \]
and the result follows simply from (4.14). \qed

**Proof of Theorem 2.4** For $k = n$ the polynomials $P = P_{n,n} \in \mathcal{F}_{2n,2n}^c$ defined by $P(x) := x$ show the theorem with $c_2 = 1$. Let $1 \leq k \leq n - 1$. By the upper bound of Lemma 3.3 there is an absolute constant $c_4 > 0$ and there are polynomials
\[ 0 \not\equiv Q = Q_{n,k} \in \mathcal{P}_{n-k,k} \]
such that
\[ \frac{\|Q'\|_{[0,1]}}{\|Q\|_{[0,1]}} \leq c_4 \frac{n}{k} \cdot (4.15) \]
Let
\[ 0 \not\equiv R(x) = R_{n,k}(x) = Q(1 - x). \]
Obviously $R$ is of the form
\[ R(x) = (1 - x)^{n-k+1} U(x), \quad U \in \mathcal{P}_{k-1}, \]
and $R'$ is of the form
\[ R'(x) = (1 - x)^{n-k} V(x), \quad V \in \mathcal{P}_{k-1}, \]
\[ \square \]
Let \( 0 \neq P = P_{n,k} \) be defined by \( P(x) := R(x^2) \). Observe that \( P \) is of the form
\[
P(x) = (1 - x^2)^n \frac{U(x)}{x^n}, \quad U \in \mathcal{P}_{2k-2},
\]
hence \( P \in \mathcal{P}_{2n,2k} \). Observe that \( P(x) := R(x^2) \) and (4.16) imply that
\[
\|P\|_{[-1,1]} = \|R\|_{[0,1]} = \|Q\|_{[0,1]}
\tag{4.18}
\]
and
\[
P'(x) = 2xR'(x^2).
\tag{4.19}
\]
First assume that \( 1 \leq k \leq (n - 10)/20 \). Let \( y := x^2 \). Using (4.19), (4.17), \( R' \neq 0 \), and Lemma 3.5, we obtain
\[
|P'(x)| = |2xR'(x^2)| = |2y^{1/2}R'(y)| < \|2aR'(u)\|_{[0,1]} = \|P'\|_{[-1,1]}
\]
for every \( y = x^2 \in [10(2k + 1)/n, 1] \), and hence there is an
\[
a \in \left[0, \left(\frac{10(2k + 1)}{n}\right)^{1/2}\right] \subset [0, 1]
\tag{4.20}
\]
such that
\[
|P'(a)| = \|P'\|_{[0,1]}.
\tag{4.21}
\]
Note that \( 1 \leq k \leq (n - 10)/20 \) implies that \( a \in [0, 1] \). Using (4.19), (4.21), (4.19) again, (4.20), (4.15), and (4.18), we obtain
\[
\|P'\|_{[-1,1]} = \|P'\|_{[0,1]} = \|P'(a)| = |2aR'(a^2)|
\leq 2 \left(\frac{10(2k + 1)}{n}\right)^{1/2} \|R'\|_{[0,1]} = 2 \left(\frac{10(2k + 1)}{n}\right)^{1/2} \|Q'\|_{[0,1]}
\leq 2 \left(\frac{10(2k + 1)}{n}\right)^{1/2} c_4 \frac{n}{k} \|Q\|_{[0,1]}
\leq c_2 \left(\frac{n}{k}\right)^{1/2} \|Q\|_{[0,1]} = c_2 \left(\frac{n}{k}\right)^{1/2} \|P\|_{[-1,1]}
\]
with the absolute constant \( c_2 = 12c_4 > 0 \).

Now assume that in addition to \( 1 \leq k \leq n - 1 \) we have \( (n - 10)/20 \leq k \leq n - 1 \). Hence \( k \geq n/30 \) also holds. Choose an \( a \in [0, 1] \) such that (4.21) holds. Using (4.19), (4.21), (4.19) again, (4.15), \( k \geq n/30 \), (4.18), and \( 1 \leq k \leq n \), we obtain
\[
\|P'\|_{[-1,1]} = \|P'\|_{[0,1]} = \|P'(a)| = |2aR'(a^2)| \leq 2\|R'\|_{[0,1]} = 2\|Q'\|_{[0,1]}
\leq 2c_4 \frac{n}{k} \|Q\|_{[0,1]} = 60c_4 \|Q\|_{[0,1]} = 60c_4 \|P\|_{[-1,1]} \leq c_2 \left(\frac{n}{k}\right)^{1/2} \|P\|_{[-1,1]}
\]
\[
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\]
with the absolute constant \( c_2 = 60c_4 > 0 \).

**Proof of Theorem 2.1** The case \( k = 0 \) is the result of Komarov [20] mentioned in the Introduction, so we may assume that \( 1 \leq k \leq n \), in which case the lower bound of the theorem follows immediately from Corollary 2.3. To see that \( c_1 := 1/636 \) can be chosen in the lower bound of the theorem we distinguish three cases. If \( k = 0 \), then Komarov’s result mentioned in the Introduction gives the lower bound of the theorem with \( c_1 := 1/636 \) as

\[
\frac{1}{636} > \frac{2}{3\sqrt{210e}}.
\]

If \( 1 \leq k \leq n/318 \), then Corollary 2.3 gives the lower bound of the theorem with \( c_1 := 1/636 \) as

\[
\frac{1}{636} \left( \frac{n}{k+1} \right)^{1/2} \leq \frac{1}{636} \left( \frac{n}{k} \right)^{1/2} = \frac{1}{636} \left( \frac{2n}{n-k} \right)^{1/2} \left( \frac{n-k}{2k} \right)^{1/2}
\]

\[
= \frac{1}{636} \left( \frac{k}{n-k} \right)^{1/2} \left( \frac{n-k}{2k} \right)^{1/2}
\]

\[
\leq \frac{1}{449} \left( 1 + \frac{1}{317} \right)^{1/2} \left( \frac{n-k}{2k} \right)^{1/2}
\]

\[
\leq \frac{1}{448} \left( \frac{n-k}{2k} \right)^{1/2}.
\]

If \( n/318 \leq k \leq n \), then \( n/k \leq 318 \), and hence Corollary 2.3 gives the lower bound of the theorem with \( c_1 := 1/636 \) again as

\[
\frac{1}{636} \left( \frac{n}{k+1} \right)^{1/2} \leq \frac{1}{636} \left( \frac{n}{k} \right)^{1/2} \leq \frac{1}{636} \sqrt{318} \leq \frac{1}{2}.
\]

To see the upper bound of the theorem let \( f(n, k) \) be defined by

\[
f(n, k) := \min_{0 \neq P \in X_n^{c}} \frac{\|P'||[-1,1]|}{\|P\|[-1,1]}.
\]

When \( k = 0 \) and \( n = 2\nu \) is even the polynomial \( P \) defined by \( P(x) = (x^2 - 1)^\nu \) shows the upper bound of the theorem. Observe that for a fixed positive integer \( n \) the function \( f(n, k) \) is decreasing on the set of integers \( 0 \leq k \leq n \), and for a fixed integer \( 1 \leq k \leq n \) we have \( f(n, k) \leq f(n-1, k-1) \). So it is sufficient to show the upper bound of the theorem only for even numbers \( n = 2\nu \) and \( k = 2\kappa \) satisfying \( 1 \leq \kappa \leq \nu \) in which case the upper bound of the theorem follows from Theorem 2.4. \( \square \)
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