One-loop Fierz transformations

Jason Aebischer and Marko Pesut

Physik-Institut, Universität Zürich,
CH-8057 Zürich, Switzerland

E-mail: jason.aebischer@physik.uzh.ch, marko.pesut@physik.uzh.ch

ABSTRACT: Fierz transformations for four-fermion operators are generalized to the one-loop level. A general renormalization scheme is used to compute QCD and QED corrections to the tree-level relations, which result from Fierz-evanescent operators. The results can be used to perform general one-loop basis transformations involving four-fermi and evanescent operators. We illustrate the usefulness of our results by discussing two examples from a matching calculation and a one-loop basis change.

KEYWORDS: Effective Field Theories, Renormalization Group

ArXiv ePrint: 2208.10513
Contents

1 Introduction 1

2 General procedure 2

3 Results 5
  3.1 Vector operators 5
  3.2 Scalar and vector LR operators 6
  3.3 Scalar and tensor LL operators 7

4 Application 8
  4.1 Matching 9
  4.2 Basis change 10

5 Conclusions 12

A Generalised greek identities 13

B Loop structures 14
  B.1 QCD 14
  B.2 QED 16

C General shifts 17

D Poles 21

E Renormalization constants 21
  E.1 Vector operators 22
    E.1.1 VLL: $q_1q_2g_3g_4$ 22
    E.1.2 VLL: $q_1q_2g_1g_2$ 23
    E.1.3 VLL: $q_1g_2\ell_1\ell_2$ 23
    E.1.4 VLL: $\ell_1\ell_2\ell_3\ell_4$ 24
  E.2 Scalar and vector LR operators 25
    E.2.1 SLR: $q_1q_2g_3g_4$ 25
    E.2.2 SLR: $q_1q_2g_1g_2$ 25
    E.2.3 SLR: $q_1g_2\ell_1\ell_2$ 25
    E.2.4 SLR: $\ell_1\ell_2\ell_3\ell_4$ 26
  E.3 Scalar and tensor LL operators 26
    E.3.1 SLL, TLL: $q_1q_2g_3g_4$ 26
    E.3.2 SLL, TLL: $q_1q_2g_1g_2$ 27
    E.3.3 SLL, TLL: semi-leptonic 28
    E.3.4 SLL, TLL: $\ell_1\ell_2\ell_3\ell_4$ 29
1 Introduction

Loop calculations including four-fermion operators often involve the use of Fierz transformations [1]. Applying these identities is however problematic when used in combination with dimensional regularisation, since Fierz identities only hold in \( D = 4 \) space-time dimensions, whereas the loop integral is continued to \( D \neq 4 \). Consequently, the loop integration and the Fierzing of an operator do not commute. This mismatch between the two operations is usually compensated by introducing evanescent operators [2–4]. These so-called Fierz-evanescent operators are defined by the difference of an operator and its Fierz-transformed version and therefore vanish in four space-time dimensions, but are non-zero in general \( D \) dimensions.

In this article we propose a simple way to deal with the issue of Fierz-evanescent contributions, by generalizing the Fierz transformations to the one-loop level. The one-loop corrections to the tree-level relations result from the insertion of Fierz-evanescent operators into one-loop diagrams. These contributions can be computed once and then added as shifts to the tree-level relations. We provide these simple shifts that generalize the standard Fierz relations to include one-loop QCD and QED corrections. Such one-loop Fierz identities can then be used in the context of one-loop matching computations, in which the resulting operators need to be fierzed in order to obtain the basis of choice. Another application is the basis change of two-loop anomalous dimension matrices. Such transformations play for instance an important role in the context of the Standard Model Effective Theory (SMEFT) [5] and its matching onto the Weak Effective Theory (WET) at the electroweak (EW) scale. The SMEFT-WET matching is known at tree-level [6, 7] and since recently also at one-loop [8], and the full one-loop Renormalization Group (RG) running is known above [9–11] and below [12, 13] the EW scale. However, in order to cancel the scheme-dependence present in the WET Wilson coefficients, which is introduced via the one-loop matching, the two-loop RG running below the EW scale has to be included. For this reason, in deriving the one-loop Fierz identities we will adopt a generalised BMU scheme, in which Greek identities\(^1\) [14] are used to reduce Dirac structures to the initial basis, but where the scheme-dependent parts (proportional to \( \epsilon \)) are kept general.\(^2\)\(^3\) The reason for this choice is that most of the known NLO anomalous dimension matrices (ADMs) in the WET are computed in the original BMU scheme: the two-loop QCD corrections for the \( \Delta F = 1 \) and \( \Delta F = 2 \) four-quark operators were computed in [16]. The \( \mathcal{O}(\alpha_s) \) corrections to the SM current-current operators have been derived in [17–20] and the two-loop ADMs for the SM QCD- and QED penguin operators are given in [17, 18, 21, 22]. In order to be able to perform a consistent NLO SMEFT analysis, these results have to be transformed into the JMS basis [7], in which the one-loop matching is computed. This step has already

\(^1\)The Greek identities were first used in the context of evanescent operators in [2].

\(^2\)The original BMU scheme is recovered from the generalized one by setting \( a_{1,2,3} = b_{1,2,3} = c_{1,2,3} = d_{1,2,3} = e_{1,2} = f_{1,2} = 1 \) in the Greek identities listed in appendix A.

\(^3\)We use the \( \overline{\text{MS}} \)-scheme together with NDR and the convention \( D = 4 - 2\epsilon \). Evanescent counterterms in the BMHV scheme have been discussed for instance in [15]. For our purpose the NDR scheme with an anticommuting \( \gamma_5 \) is a valid renormalization scheme, since in the computation we only encounter traces containing at most three gamma matrices, as can be seen from the “Feynman rules” in appendix B.
been performed for four-quark operators in the context of $\Delta F = 1$ processes in [23–25] and for $\Delta F = 2$ processes in [26]. The generalized Fierz identities derived in this article will however allow to transform the full set of operators at one-loop to the JMS basis. They therefore consist an important step in the pursuit of a full NLO SMEFT analysis.

Changing the BMU ADM into the JMS ADM is however a rather special case, since the JMS scheme can easily be transformed into the original BMU one. This makes the one-loop basis transformation rather simple, since only Fierz-evanescent operators have to be considered in the basis change. In full generality however the definition of evanescent operators involves general linear combinations of physical operators multiplied by $\epsilon$ [27, 28], as opposed to the generalised BMU scheme shown in appendix A. In order to be able to change also to bases containing such evanescent operators the renormalization constants $Z_{QQ}$, $Z_{QE}$ and $Z_{EE}$ are needed [28], which we report for Fierz-evanescent operators in appendix E.

The rest of the article is organized as follows: in section 2 we illustrate the procedure on how to obtain one-loop corrections of the Fierz identities. We report our results for all possible Dirac structures and fermion field combinations in section 3. The usefulness of our results is illustrated in section 4, were we show how they can be applied in two different contexts: first in a one-loop matching calculation involving Leptoquarks and secondly in a one-loop basis change of a two-loop ADM governing $\Delta F = 2$ processes. Finally we conclude in section 5. Several useful results are collected in the appendices: appendix A contains the generalized Greek identities used in this article. In appendix B we report all divergent structures encountered in the calculation. Appendix C lists the one-loop corrections to the tree-level Fierz relations in the generalized BMU scheme. In appendix D we collect one-loop corrections to the physical and Fierz-evanescent operators in our computation and in appendix E we provide the renormalization constants $Z_{QQ}$, $Z_{QE}$ and $Z_{EE}$ for bases containing Fierz-evanescent operators.

2 General procedure

In order to obtain one-loop corrections to Fierz identities let us consider an operator $O$ and its Fierz-transformed version $FO$, where $F$ denotes the application of tree-level Fierz identities. At tree-level the two structures are trivially related:

$$O = FO, \quad \text{(tree-level)}$$

(2.1)

whereas at one-loop the relation has to be generalized by introducing an evanescent operator $E$:

$$O = FO + E. \quad \text{(one-loop)}$$

(2.2)

The one-loop shift to the tree-level Fierz identity induced by the evanescent operator is then obtained by computing the one-loop corrections ($L$) to the operator and its Fierz-transformed version and taking the difference:

$$LO - LF O = LE. \quad \text{(2.3)}$$

The BMU scheme corresponds to $a_{ev} = b_{ev} = c_{ev} = d_{ev} = e_{ev} = f_{ev} = 1$ in [8].
As an example let us consider a four-quark operator
\[ Q = (\bar{q}_1 \Gamma_1 q_2)(\bar{q}_3 \Gamma_2 q_4), \quad (2.4) \]
with four different quarks \( q_{1,2,3,4} \) and two general Dirac structures \( \Gamma_{1,2} \). The Fierz-transformed operator has the form
\[ FQ = (\bar{q}_1^{\alpha} \tilde{\Gamma}_1 q_2^{\beta})(\bar{q}_3^{\beta} \tilde{\Gamma}_2 q_4^{\alpha}), \quad (2.5) \]
with different Dirac structures \( \tilde{\Gamma}_{1,2} \) and the colour indices \( \alpha, \beta \). In order to obtain the shift resulting from the Fierz-evanescent operator
\[ E_Q = Q - FQ, \quad (2.6) \]
one-loop corrections to the two operators have to be considered. They include genuine vertex corrections as well as contributions from Wavefunction renormalization and operator renormalization. As an example we consider the process \( q_2 q_4 \rightarrow q_1 q_3 \), for which the vertex corrections are depicted in figure 1 in the case of QCD and in figure 2 for QED. Since the two operators do not have the same flavour structure, one has to perform an additional (tree-level) Fierz transformation on one of the amplitudes after loop-integration in order to compare the two resulting amplitudes.

This Fierz transformation can be carried out at tree-level, since the amplitude is already at the one-loop order and therefore contributions from higher-order Fierz transformations can be neglected. Also Wavefunction renormalization contributions do not have to be considered, since they drop out in the difference after this additional Fierz transformation.

When computing the one-loop corrections to the operators new Dirac structures occur, which have to be reduced to the initial operator basis. To perform this projection we use...
the generalized Greek identities collected in App A. This choice of how to reduce the new Dirac structures by specifying the \( \epsilon \)-dependent pieces in the reduction defines the generalized BMU scheme, for which we report the obtained shifts in appendix C. The original BMU scheme has been introduced in [16] and its Greek identities can be found in [29]. Since the shifts from evanescent operators precisely result from these \( \epsilon \)-dependent parts of the projections, only the poles of the one-loop diagrams have to be computed. The insertion of an evanescent operator into these divergent structures will then yield the finite shift in the Fierz identities.

If two or more fermions of a four-fermi operator are the same, then also penguin diagrams have to be taken into account. Consider a four-quark operator of the form:

\[
P = (\bar{q}_1 \Gamma_1 q_2)(\bar{q}_3 \Gamma_2 q_3),
\]

and its Fierz-transformed version

\[
\mathcal{FP} = (\bar{q}_1 \tilde{\Gamma}_1 q_3)(\bar{q}_3 \tilde{\Gamma}_2 q_2),
\]

then \( P \) generates closed penguin diagrams and \( \mathcal{FP} \) open penguin diagrams, which for QCD are shown in figure 3. Similar diagrams are generated when QED corrections are considered.

In appendix B we report all divergent structures that can be obtained from vertex corrections and penguin diagrams. These structures, together with the Greek identities will then be used in the next section to obtain one-loop corrections to Fierz relations.
3 Results

In this section we report the results obtained, following the procedure outlined in the previous section. We compute the one-loop QCD- and QED-shifts to the tree-level Fierz relations for general four-fermi operators with vector-, scalar- and tensor Dirac structures and different combinations of fermion fields. All results obtained in this section are also valid for the Parity transformed operators which result from the interchange $P_L \leftrightarrow P_R$, since QCD and QED are invariant under Parity transformations. In order to simplify the presentation we will adopt the original BMU scheme in the following. The general results computed in the generalized BMU scheme are collected in appendix C.

3.1 Vector operators

In this subsection we discuss the shifts for four-fermi vector operators, for which we use the following notation:

$$V_{ij,kl}^{AB} \equiv \langle j_i \gamma_{\mu} P_A j_j \rangle \langle k_k \gamma_{\mu} P_B f_l \rangle , \quad \text{where } f = q, \ell . \quad (3.1)$$

For four-quark operators we define in addition the colour-crossed operators:

$$\tilde{V}_{ij,kl}^{AB} \equiv \langle \bar{q}_i^\alpha \gamma_{\mu} P_A q_j^\beta \rangle \langle \bar{q}_k^\gamma \gamma_{\mu} P_B q_l^\alpha \rangle , \quad (3.2)$$

where $\alpha, \beta$ denote colour indices.

The resulting shifts for vector operators are collected for four-quark operators in table 1, for semi-leptonic operators in table 2 and for four-lepton operators in table 3. Since the genuine vertex corrections of a four-fermi vector operator are the same as for its Fierz-transformed version, the first lines in the above mentioned tables vanish. Consequently, only operators with at least two equal fermions obtain a shift at one-loop, which results from penguin contributions. Finally, four-lepton operators with four equal leptons do not

---

5In addition we use the shorthand notation $V_{ij,kl}^{AB} \equiv \langle j_i \gamma_{\mu} f_j \rangle \langle k_k \gamma_{\mu} P_B f_l \rangle$, and analogous expressions.
These operators do not obtain any shift at the one-loop level, neither from QCD nor related through a Fierz relation. In addition to the vector operators defined in the previous section in [23] and for the QCD shift, our results for the QCD shift of the operators in Table 2. QCD and QED shifts in the original BMU scheme.

### Table 1. Fierz transformations for four-quark $\gamma_\mu P_L \otimes \gamma^\mu P_L$ operators together with their one-loop QCD and QED shifts in the original BMU scheme.

| Operator | Tree-level Fierz | QCD shift | QED shift |
|----------|------------------|-----------|-----------|
| $V^{LL}_{q_1q_2\ell_1\ell_2}$ | $V^{LL}_{q_1q_2\ell_1\ell_2}$ | 0 | 0 |
| $V^{LL}_{q_1q_1\ell_2}$ | $V^{LL}_{q_1q_1\ell_2}$ | 0 | $-\frac{2N_c}{3}Q^2_{q_1}V^{LL}_{q_1q_2\ell_1\ell_2}$ |
| $V^{LL}_{q_2q_1\ell_2}$ | $V^{LL}_{q_2q_1\ell_2}$ | 0 | $-\frac{2}{3}Q^2_{\ell_1}V^{LL}_{q_1q_1\ell_2}$ |
| $V^{LL}_{q_1q_1\ell_1\ell_2}$ | $V^{LL}_{q_1q_1\ell_1\ell_2}$ | 0 | $-\frac{2}{3}(N_c-1)Q^2_{q_1}V^{LL}_{q_1q_1\ell_2}$ |

### Table 2. Fierz transformations for semi-leptonic $\gamma_\mu P_L \otimes \gamma^\mu P_L$ operators together with their one-loop QCD and QED shifts in the original BMU scheme.

| Operator | Tree-level Fierz | QCD shift | QED shift |
|----------|------------------|-----------|-----------|
| $V^{LL}_{q_1q_2\ell_1\ell_2}$ | $V^{LL}_{q_1q_2\ell_1\ell_2}$ | 0 | 0 |
| $V^{LL}_{q_1q_2\ell_1\ell_2}$ | $V^{LL}_{q_1q_2\ell_1\ell_2}$ | 0 | $-\frac{2N_c}{3}Q^2_{q_1}V^{LL}_{q_1q_2\ell_1\ell_2}$ |
| $V^{LL}_{q_2q_2\ell_1}$ | $V^{LL}_{q_2q_2\ell_1}$ | 0 | $-\frac{2}{3}Q^2_{\ell_1}V^{LL}_{q_1q_2\ell_1\ell_2}$ |
| $V^{LL}_{q_1q_1\ell_1\ell_2}$ | $V^{LL}_{q_1q_1\ell_1\ell_2}$ | 0 | $-\frac{2}{3}(N_c-1)Q^2_{q_1}V^{LL}_{q_1q_1\ell_2}$ |

obtain any one-loop shift, since their Fierz-transformed version is equal to the initial operator. Our results for the QCD shift of $\Delta F = 1$ four-quark operators agree with the findings in [23] and for $\Delta F = 2$ with [30].

### 3.2 Scalar and vector LR operators

In this subsection we discuss scalar and vector four-fermi operators with the Dirac structures $P_L \otimes P_R$ and $\gamma_\mu P_R \otimes \gamma^\mu P_L$, respectively. At tree-level these types of operators are related through a Fierz relation. In addition to the vector operators defined in the previous subsection we define the analogous scalar operators as

$$S^{AB}_{\ell_1\ell_2\ell_1\ell_2} = \mathcal{J}_4 (P_A f_2) (P_B f_4) .$$

These operators do not obtain any shift at the one-loop level, neither from QCD nor from QED, which is indicated in table 4. The genuine vertex corrections drop out in
### Table 3.
Fierz transformations for four-lepton $\gamma_\mu P_L \otimes \gamma_\mu P_L$ operators together with their one-loop QCD and QED shifts in the original BMU scheme.

| Operator | Tree-level Fierz | QCD shift | QED shift |
|----------|------------------|-----------|-----------|
| $V_{LL}^{LL}$ | $\tilde{V}_{LL}^{LL}$ | 0 | 0 |
| $V_{LL}^{LL}$ | $V_{LL}^{LL}$ | 0 | $- \frac{2}{3} Q^2 \tilde{V}_{LL}^{LL}$ |
| $V_{LL}^{LL}$ | $V_{LL}^{LL}$ | 0 | $- \frac{2}{3} Q^2 V_{LL}^{LL}$ |
| $V_{LL}^{LL}$ | $V_{LL}^{LL}$ | 0 | 0 |

### Table 4.
Fierz transformations for scalar $P_L \otimes P_R$ operators together with their one-loop QCD and QED shifts in the original BMU scheme.

| Operator | Tree-level Fierz | QCD shift | QED shift |
|----------|------------------|-----------|-----------|
| $S^{LR}_{f_1f_2f_3f_4}$ | $-\frac{1}{2} \tilde{V}^{RL}_{f_1f_2f_3f_4}$ | 0 | 0 |
| $\tilde{S}^{LR}_{f_1f_2f_3f_4}$ | $-\frac{1}{2} \tilde{V}^{RL}_{f_1f_2f_3f_4}$ | 0 | 0 |

the difference, which was also the case for $VLL^6$ operators in the previous subsection. In addition, also the divergent parts of the penguin diagrams either vanish or cancel in the difference. Consequently, all shifts vanish for the LR operators, independently of the fermion combinations. This is a special feature of the original BMU scheme and does not hold anymore in the generalized BMU scheme, as can be seen in table 8 in the appendix.

### 3.3 Scalar and tensor LL operators

Finally we discuss the scalar and tensor operators with LL structure. Besides the notation for scalar operators defined in the previous subsection we define:

\[ T_{f_1f_2f_3f_4}^{LL} \equiv (\bar{f}_1 \sigma_{\mu\nu} P_L f_2)(\bar{f}_3 \sigma^{\mu\nu} P_L f_4), \quad \text{where } \sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu]. \quad (3.4) \]

and for colour-crossed four-quark operators we use:\[7\]

\[ \tilde{T}_{q_1q_2q_3q_4}^{LL} \equiv (\bar{q}_1^{\alpha} \sigma_{\mu\nu} P_L q_2^{\beta})(\bar{q}_3^{\beta} \sigma^{\mu\nu} P_L q_4^{\alpha}). \quad (3.5) \]

---

6We use the shorthand notation VLL to denote vector operators with $\gamma_\mu P_L \otimes \gamma_\mu P_L$ Dirac structure and analogous abbreviations for scalar and tensor operators.

7We refrain from including tensor operators with TLR structure in our discussion, since these structures vanish in $D = 4$ dimensions due to the relation $\frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \sigma^{\mu\nu} = \sigma_{\alpha\beta} \gamma_5$. Therefore they are usually not included in physical bases.
In this section we demonstrated the usefulness of the obtained shifts and also how to apply them by studying two examples from the existing literature. First we consider a one-loop matching calculation, in which a Fierz transformation is needed. Secondly we discuss a one-loop basis change of a two-loop anomalous dimension matrix for $\Delta F = 2$ transitions, in which the operators differ by a Fierz transformation.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Operator & Tree-level Fierz & QCD shift & QED shift \\
\hline
$S_{q_i q_j q_k q_L}^{LL}$ & $-\frac{1}{2} S_{q_i q_j q_k q_L}^{LL} + \frac{1}{8} T_{q_i q_j q_k q_L}^{LL}$ & $\frac{1}{2} (Q_1 + Q_2)(Q_3 + Q_4) S_{q_i q_j q_k q_L}^{LL}$ & $\frac{1}{8} (Q_{1234} + 2Q_{1423} + 3Q_{1324}) T_{q_i q_j q_k q_L}^{LL}$ \\
$S_{q_i q_j q_k q_L}^{LL}$ & $-\frac{1}{2} S_{q_i q_j q_k q_L}^{LL} + \frac{1}{8} T_{q_i q_j q_k q_L}^{LL}$ & $\frac{1}{2} (Q_1 + Q_2)(Q_3 + Q_4) S_{q_i q_j q_k q_L}^{LL}$ & $\frac{1}{8} (Q_{1234} + 2Q_{1423} + 3Q_{1324}) T_{q_i q_j q_k q_L}^{LL}$ \\
$S_{q_i q_j q_k q_L}^{LL}$ & $-\frac{1}{2} S_{q_i q_j q_k q_L}^{LL} + \frac{1}{8} T_{q_i q_j q_k q_L}^{LL}$ & $\frac{1}{2} (Q_1 + Q_2)(Q_3 + Q_4)$ & $\frac{1}{8} (Q_{1234} + 2Q_{1423} + 3Q_{1324}) T_{q_i q_j q_k q_L}^{LL}$ \\
$S_{q_i q_j q_k q_L}^{LL}$ & $-\frac{1}{2} S_{q_i q_j q_k q_L}^{LL} + \frac{1}{8} T_{q_i q_j q_k q_L}^{LL}$ & $\frac{1}{2} (Q_1 + Q_2)(Q_3 + Q_4)$ & $\frac{1}{8} (Q_{1234} + 2Q_{1423} + 3Q_{1324}) T_{q_i q_j q_k q_L}^{LL}$ \\
& & $\frac{1}{2} (Q_1 + Q_2)(Q_3 + Q_4)$ & $\frac{1}{8} (Q_{1234} + 2Q_{1423} + 3Q_{1324}) T_{q_i q_j q_k q_L}^{LL}$ \\
\hline
\end{tabular}
\caption{Fierz transformations for scalar $P_L \otimes P_L$ operators together with their one-loop QCD and QED shifts in the original BMU scheme.}
\end{table}

For these operators the shifts result exclusively from genuine vertex corrections, since the penguin diagrams again vanish, which can be deduced from the results in appendix B. For that reason the QED corrections have the same form for all fermion operators, since they only depend on the fermion charges. To shorten the notation we define:

$$ Q_{ijkl} = Q_i Q_j + Q_k Q_l. \quad (3.6) $$

Furthermore the first and fourth diagrams in figure 1 and figure 2 are finite for tensor operators and therefore do not have to be considered.

We report the one-loop shifts for scalar operators in table 5 and for tensor operators in table 6. Our QCD results for $\Delta F = 1$ and $\Delta F = 2$ operators again agree with the ones obtained in [23, 30]. Several entries in tables 5 and 6 are redundant, since the scalar and tensor operators mix under Fierz transformations. For convenience we report however all entries for scalar and tensor operators and remind the reader of this redundancy.

4 Application

In this section we demonstrated the usefulness of the obtained shifts and also how to apply them by studying two examples from the existing literature. First we consider a one-loop matching calculation, in which a Fierz transformation is needed. Secondly we discuss a one-loop basis change of a two-loop anomalous dimension matrix for $\Delta F = 2$ transitions, in which the operators differ by a Fierz transformation.
−QED shift
−Tree-level Fierz

\[ T_{LL}^{LL} \] \[ \sum_{Q_i} \sum_{Q_j} A_{Q_i} \cdot A_{Q_j} \] \[ \sum_{Q_i} \sum_{Q_j} B_{Q_i} \cdot B_{Q_j} \] \[ \sum_{Q_i} \sum_{Q_j} C_{Q_i} \cdot C_{Q_j} \]

\[ \sum_{Q_i} \sum_{Q_j} D_{Q_i} \cdot D_{Q_j} \] \[ \sum_{Q_i} \sum_{Q_j} E_{Q_i} \cdot E_{Q_j} \] \[ \sum_{Q_i} \sum_{Q_j} F_{Q_i} \cdot F_{Q_j} \] \[ \sum_{Q_i} \sum_{Q_j} G_{Q_i} \cdot G_{Q_j} \]

Table 6. Fierz transformations for tensor $\sigma_{\mu \nu} P_L \otimes \sigma^{\mu \nu} P_L$ operators together with their one-loop QCD and QED shifts in the original BMU scheme.

4.1 Matching

In this subsection we apply our method to the matching results obtained in [31], where a general scalar Leptoquark (LQ) was integrated out at the one-loop level in QCD. The resulting semi-leptonic operators are obtained in the LQ basis, meaning that each current of the four-fermi operators contains a quark and a lepton. As an example we consider the scalar and tensor operators in the LQ basis, which in the notation of [31] read

\[ \tilde{O}_{S}^{AB} = (\bar{q} P_{A} \ell)(\bar{\ell} P_{B} q), \]

\[ \tilde{O}_{T}^{A} = (\bar{q} \sigma_{\mu \nu} P_{A} \ell)(\bar{\ell} \sigma^{\mu \nu} P_{A} q). \]  (4.1)  (4.2)

For most purposes it is however more favorable to work in the SM basis, which has the following form:

\[ O_{S}^{AB} = (\bar{q} P_{A} q)(\bar{\ell} P_{B} \ell), \]

\[ O_{T}^{A} = (\bar{q} \sigma_{\mu \nu} P_{A} q)(\bar{\ell} \sigma^{\mu \nu} P_{A} \ell). \]  (4.3)  (4.4)

At the operator level using the QCD relations for the semi-leptonic operators in tables 5 and 6 one finds the one-loop relation between the two bases:

\[ \left( \begin{array}{c} O_{S}^{AA} \\ O_{T}^{A} \end{array} \right) = R_{0} \left( \begin{array}{c} \tilde{O}_{S}^{AA} \\ \tilde{O}_{T}^{A} \end{array} \right) + \frac{\alpha_{s}}{4\pi} R_{1} \left( \begin{array}{c} O_{S}^{AA} \\ O_{T}^{A} \end{array} \right) \]  (4.5)

with the matrices

\[ R_{0} = \left( \begin{array}{cc} -\frac{1}{2} & -\frac{1}{2} \\ -6 & 1 \end{array} \right), \quad R_{1} = \left( \begin{array}{cc} 0 & \frac{N_{c}^{2} - 1}{12 N_{c}} \\ \frac{7 - 7 N_{c}^{2}}{N_{c}} & 0 \end{array} \right). \]  (4.6)
Solving the relation in eq. (4.5) for the SM basis and expanding in $\alpha_s$ one finds:

$$\begin{pmatrix} O_{AA}^A \\ O_{AT}^A \end{pmatrix} = \begin{bmatrix} R_0 + \frac{\alpha_s}{4\pi} R_1 R_0 \end{bmatrix} \begin{pmatrix} \tilde{O}_{AA}^A \\ \tilde{O}_{AT}^A \end{pmatrix}.$$  

(4.7)

To obtain the corresponding transformation for Wilson coefficients one simply has to take the inverse-transpose of the transformation in eq. (4.7). Expanding in $\alpha_s$ one finds:

$$\begin{pmatrix} C_{AA}^A \\ C_{AT}^A \end{pmatrix} = \begin{bmatrix} R_0 + \frac{\alpha_s}{4\pi} R_1 R_0 \end{bmatrix}^{-T} \begin{pmatrix} \tilde{C}_{AA}^A \\ \tilde{C}_{AT}^A \end{pmatrix} = \begin{bmatrix} \left( -\frac{1}{2} - 6 \right) + \frac{\alpha_s}{4\pi} \left( -\frac{7}{4} C_F 0 \right) \end{bmatrix} \begin{pmatrix} \tilde{C}_{AA}^A \\ \tilde{C}_{AT}^A \end{pmatrix},$$  

(4.8)

where we used $C_F = \frac{N_c^2 - 1}{2N_c}$ and set the second column of the second matrix to zero, since the Wilson coefficient of the tensor operator starts at $\mathcal{O}(\alpha_s)$, as can be seen in eq. (24) of [31]. The transformation in eq. (4.8) corresponds exactly to the results in eq. (26) of [31] and shows the advantage of our method. Having the one-loop Fierz relations at hand allows to perform the matching onto the LQ basis and fierzing them at one-loop, without the need to compute the contributions from Fierz-evanescent operators. The one-loop basis change is therefore reduced to a simple algebraic problem. Note however, that the BMU scheme was used in [31], which makes the transformation particularly simple. We will discuss a more general example in the next subsection.

### 4.2 Basis change

In appendix C.1 of [32] a one-loop basis change was performed for $\Delta F = 2$ operators and subsequently the two-loop QCD ADM was derived in the new basis. The initial basis in which the two-loop ADM was already known [16] is given by:

$$Q_{SLL}^1 = (\bar{b} R q_L)(\bar{b} R q_L),$$

(4.9)

$$Q_{SLL}^2 = -(\bar{b} R \sigma_{\mu \nu} q_L)(\bar{b} R \sigma_{\mu \nu} q_L),$$

(4.10)

$$E_{SLL}^1 = (\bar{b} R q^\mu_L)(\bar{b} R q^\nu_L) + \frac{1}{2} Q_{SLL}^1 - \frac{1}{8} Q_{SLL}^2,$$

(4.11)

$$E_{SLL}^2 = -(\bar{b} R \sigma_{\mu \nu} q^i_L)(\bar{b} R \sigma_{\mu \nu} q^j_L) - 6 Q_{SLL}^1 - \frac{1}{2} Q_{SLL}^2,$$

(4.12)

where $i, j$ denote colour-indices. This basis is then translated into the operator basis

$$Q_{SLL}^1 = (\bar{b} R q_L)(\bar{b} R q_L),$$

(4.13)

$$\tilde{Q}_{SLL}^1 = (\bar{b} R q^i_L)(\bar{b} R q^j_L),$$

(4.14)

$$E_{SLL}^1 = (\bar{b} R \gamma_{\mu} q_L)(\bar{b} R \gamma_{\nu} q_L) + 8(1 - \epsilon) Q_{SLL}^1,$$

(4.15)

$$E_{SLL}^2 = (\bar{b} R \gamma_{\mu} q^i_L)(\bar{b} R \gamma_{\nu} q^j_L) + 8(1 - \epsilon) Q_{SLL}^1.$$

(4.16)
The basis change between the two bases can be written in terms of two two-component vectors containing physical and evanescent operators:

\[
\vec{Q} = R(\vec{Q} + W\vec{E}), \quad \vec{E} = M(\epsilon U\vec{Q} + (1 + \epsilon W)\vec{E}), \tag{4.17}
\]

where the $2 \times 2$ matrices $R, W, M$ and $U$ are obtained from simple tree-level relations given in [32]. The NLO ADM of the barred basis is then translated into the second basis. In order to transform the NLO ADM from the first to the second basis a change of scheme has to be performed. It is parametrized by the quantity $Z_{QQ}^{(1,0)}$, which is a finite renormalization constant that has to be introduced to remove the finite matrix elements introduced by the physical and evanescent operators in the initial basis. This renormalization constant is given by:

\[
Z_{QQ}^{(1,0)} = R\left[ WZ_{EQ}^{(1,0)} - \left( Z_{QE}^{(1,1)} + WZ_{EE}^{(1,1)} - \frac{1}{2}\sigma^{(0)}W \right) U \right] R^{-1}, \tag{4.18}
\]

which depends on the renormalization constants and the anomalous dimension matrix in the barred basis [27, 28]. Adopting a similar notation as in the previous subsection we write:

\[
\vec{Q} = \left[ R_0 + \frac{\alpha_s}{4\pi} R_1 \right] \vec{Q}. \tag{4.19}
\]

In this notation the LO and NLO ADMs transform in the following way:

\[
\gamma^{(0)} = R_0 \gamma^{(0)} R_0^{-1}, \quad \gamma^{(1)} = R_0 \gamma^{(1)} R_0^{-1} + R_1 \gamma^{(0)} R_0^{-1} - R_0 \gamma^{(0)} R_1 R_0^{-1} + 2\beta_0 R_1 R_0^{-1}, \tag{4.20}
\]

where $\beta_0$ denotes the LO $\beta$-function of the strong coupling constant. Comparing these expressions with the results in [32] one finds:

\[
R_0 = R, \quad R_1 = -Z_{QQ}^{(1,0)} R_0. \tag{4.22}
\]

The matrix $R_1$ therefore encodes the change of scheme between the two bases. Similar to eq. (4.18) it can be split into different contributions, namely:

\[
R_1 = R_1^{\text{shift}} + R_1^{U}. \tag{4.23}
\]

The first matrix $R_1^{\text{shift}}$ results from the shifts computed in the previous section and $R_1^{U}$ depends on the renormalization constants of the barred basis, which can be derived from the results in appendix E.

Let us first derive the matrix $R_1^{\text{shift}}$. The contribution from the Fierz-evanescent operator $\bar{E}_1^{\text{LL}}$ to $R_1$ is given by the shift of $S_{bqbq}^{LL}$, which can be derived from the one of $S_{q1q2q3q4}^{LL}$ in table 5, leading to:

\[
R_1^{\text{shift}} = \begin{pmatrix}
0 & 0 \\
20+2N_c-7N_c^2 & 4-8N_c-N_c^2
\end{pmatrix}, \tag{4.24}
\]

which for $N_c = 3$ corresponds to the first term in eq. (4.18)

\[
R_1^{\text{shift}} = -RWZ_{EQ}^{(1,0)}. \tag{4.25}
\]
where we used the relations in eq. (4.22). Consequently $R_{1}^{U}$ is given by:

$$R_{1}^{U} = R \left( Z_{QE}^{(1,1)} + W Z_{EE}^{(1,1)} - \frac{1}{2} \gamma^{(0)} W \right) U,$$  \hspace{1cm} (4.26)

where the relevant renormalization constants are given in appendix E.\footnote{We use a different sign convention than \cite{32}.} We find

$$R_{1}^{U} = \begin{pmatrix} \frac{23}{6} & -\frac{1}{24} \\ \frac{43}{12} & -\frac{19}{38} \end{pmatrix}, \hspace{1cm} (4.27)$$

leading to

$$Z_{QQ}^{(1,0)} = -(R_{1}^{\text{shift}} + R_{1}^{U}) R_{0}^{-1} = \begin{pmatrix} -\frac{11}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}, \hspace{1cm} (4.28)$$

which agrees with the findings of \cite{32}. This simple exercise shows that if the matrix $U$ is non-zero, which corresponds to physical operators mixing into the evanescent operators in the basis change, the transformation is slightly more involved and besides the one-loop shifts one also needs to take into account the renormalization constants given in appendix E.

5 Conclusions

We present for the first time one-loop Fierz transformations that include QCD and QED corrections, which result from Fierz-evanescent operators. The corrections are computed in a general renormalisation scheme, that can easily be transformed into the original BMU scheme, in which most of the two-loop ADMs are known.

Our results simplify one-loop matching calculations in which Fierz transformations are needed, since only the matching of physical operators has to be computed, whereas the evanescent part is given by our formulae. They therefore facilitate the projection onto the physical basis and might for example be relevant for common matching tools such as \texttt{matchmaker} \cite{33}.

A second use case of our results are basis transformations involving two-loop ADMs. Changing a two-loop ADM from one basis into another is particularly simple in the absence of physical operators mixing into evanescent ones in the transformation. In that case the change of scheme can be read of directly from the one-loop Fierz relations. A general one-loop basis transformation can however be performed using the one-loop Fierz relations in combination with the renormalisation constants for the Fierz-evanescent operators provided in the appendix. One-loop basis changes of two-loop ADMs play for instance an important role in the NLO SMEFT analysis, since in that case the one-loop matching is known in the JMS basis whereas the two-loop running is given in the BMU basis. The results of this article allow to perform the full JMS-BMU translation at one-loop, which paves the way for a complete and scheme-independent NLO SMEFT analysis. For that purpose we plan to implement our results in the common SMEFT computing tools like \texttt{WCxf} \cite{34} and \texttt{wilson} \cite{35} and finally in the basis change package \texttt{abc\_eft} \cite{36}.

As a possible extension of our results the computations can be performed for a general gauge group. Furthermore Yukawa one-loop corrections to the tree-level Fierz identities could be considered, which we will leave for the future.
Acknowledgments

We thank Gino Isidori and Andrzej Buras for comments on the manuscript. J. A. and M. P. acknowledge financial support from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement 833280 (FLAY), and from the Swiss National Science Foundation (SNF) under contract 200020-204428.

A Generalised greek identities

In this appendix we report the generalised Greek identities used in our calculations. The original Greek identities used in the BMU scheme can be found in the appendix of [29] and are obtained from the generalised version by setting all constants to one, $a_{1,2,3} = b_{1,2,3} = c_{1,2,3} = d_{1,2,3} = e_{1,2} = f_{1,2} = 1$. Another difference is the definition of $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}]$, which introduces a negative sign for the tensor contributions.

**VLL**

\[
\gamma_{\alpha}\gamma_{\beta}\gamma_{\mu} (1 \pm \gamma_5) \gamma^{\beta} \gamma^{\alpha} \otimes \gamma^{\mu} (1 \pm \gamma_5) = 4(1 - 2a_1 \epsilon) \gamma_{\mu} (1 \pm \gamma_5) \otimes \gamma^{\mu} (1 \pm \gamma_5),
\]
\[
\gamma_{\mu} (1 \pm \gamma_5) \gamma_{\alpha} \gamma_{\beta} \otimes \gamma^{\mu} (1 \pm \gamma_5) \gamma^{\alpha} \gamma^{\beta} = 4(4 - a_2 \epsilon) \gamma_{\mu} (1 \pm \gamma_5) \otimes \gamma^{\mu} (1 \pm \gamma_5),
\]
\[
\gamma_{\mu} (1 \pm \gamma_5) \gamma_{\alpha} \gamma_{\beta} \otimes \gamma^{\beta} \gamma^{\alpha} \gamma^{\mu} (1 \pm \gamma_5) = 4(1 - 2a_3 \epsilon) \gamma_{\mu} (1 \pm \gamma_5) \otimes \gamma^{\mu} (1 \pm \gamma_5).
\]

**VLR**

\[
\gamma_{\alpha} \gamma_{\beta} \gamma_{\mu} (1 \pm \gamma_5) \gamma^{\beta} \gamma^{\alpha} \otimes \gamma^{\mu} (1 \mp \gamma_5) = 4(1 - 2b_1 \epsilon) \gamma_{\mu} (1 \pm \gamma_5) \otimes \gamma^{\mu} (1 \mp \gamma_5),
\]
\[
\gamma_{\mu} (1 \pm \gamma_5) \gamma_{\alpha} \gamma_{\beta} \otimes \gamma^{\mu} (1 \mp \gamma_5) \gamma^{\alpha} \gamma^{\beta} = 4(1 + b_2 \epsilon) \gamma_{\mu} (1 \pm \gamma_5) \otimes \gamma^{\mu} (1 \mp \gamma_5),
\]
\[
\gamma_{\mu} (1 \pm \gamma_5) \gamma_{\alpha} \gamma_{\beta} \otimes \gamma^{\beta} \gamma^{\alpha} \gamma^{\mu} (1 \pm \gamma_5) = 16(1 - b_3 \epsilon) \gamma_{\mu} (1 \pm \gamma_5) \otimes \gamma_{\mu} (1 \mp \gamma_5).
\]

**SLR**

\[
\gamma_{\nu} \gamma_{\mu} (1 \mp \gamma_5) \gamma^{\mu} \gamma^{\nu} \otimes (1 \pm \gamma_5) = 16(1 - c_1 \epsilon) (1 \pm \gamma_5) \otimes (1 \pm \gamma_5),
\]
\[
(1 \mp \gamma_5) \gamma_{\mu} \gamma_{\nu} \otimes (1 \pm \gamma_5) \gamma^{\mu} \gamma^{\nu} = 4(1 + c_2 \epsilon) (1 \mp \gamma_5) \otimes (1 \pm \gamma_5),
\]
\[
(1 \mp \gamma_5) \gamma_{\nu} \gamma_{\mu} \otimes \gamma^{\mu} \gamma^{\nu} (1 \mp \gamma_5) = 4(1 - 2c_3 \epsilon) (1 \pm \gamma_5) \otimes (1 \pm \gamma_5).
\]

**SLL**

\[
\gamma_{\nu} \gamma_{\mu} (1 \pm \gamma_5) \gamma^{\mu} \gamma^{\nu} \otimes (1 \pm \gamma_5) = 16(1 - d_1 \epsilon) (1 \pm \gamma_5) \otimes (1 \pm \gamma_5),
\]
\[
(1 \pm \gamma_5) \gamma_{\mu} \gamma_{\nu} \otimes (1 \pm \gamma_5) \gamma^{\mu} \gamma^{\nu} = (4 - 2d_2 \epsilon) (1 \pm \gamma_5) \otimes (1 \pm \gamma_5) - \sigma_{\mu\nu} (1 \pm \gamma_5) \otimes \sigma^{\mu\nu} (1 \pm \gamma_5),
\]
\[
(1 \pm \gamma_5) \gamma_{\mu} \gamma_{\nu} \otimes \gamma^{\mu} \gamma^{\nu} (1 \pm \gamma_5) = (4 - 2d_3 \epsilon) (1 \pm \gamma_5) \otimes (1 \pm \gamma_5) + \sigma_{\mu\nu} (1 \pm \gamma_5) \otimes \sigma^{\mu\nu} (1 \pm \gamma_5).
\]
Note that we introduce the same evanescent structures for operators with colour-crossed SU(3)$_c$ structure as well as for Parity-flipped operators.

B Loop structures

In this appendix we report the loop structures that are encountered in our calculation. If another scheme is used, these structures can be used to compute the shifts in the new scheme. We list the “Feynman-rules” for the divergent parts of these structures in QCD and QED below.

B.1 QCD

We consider the vertex corrections of a four-quark operator with the following flavour content

\[ \mathcal{O} = (\bar{q}_1 \Gamma_1 q_2)(\bar{q}_3 \Gamma_2 q_4). \]  

To be general we suppress the colour contraction of the operator $\mathcal{O}$, which then has to be added to the “Feynman-rules” listed below. We only list three out of the six vertex corrections, since the poles of the other three amplitudes are identical to the ones shown.
For the penguin contributions we consider the operator

$$\mathcal{P} = (\bar{q}_1 \Gamma_1 q_2)(\bar{q}_3 \Gamma_2 q_3), \quad (B.2)$$

where we again suppress the colour contraction, which has to be added to the “Feynman-rules” below. One finds:

$$-\frac{\alpha_s}{4\pi 12\epsilon} (T^A)_{\delta\gamma} (T^A)_{\alpha\beta} (\bar{q}_1 \Gamma_1 q_2) (\bar{q}_3 \Gamma_2 q_3) \text{Tr} \left[ \gamma_\mu q_4 \right]$$

$$\frac{\alpha_s}{4\pi 12\epsilon} (T^A)_{\delta\gamma} (T^A)_{\alpha\beta} (\bar{q}_1 \Gamma_1 q_2) (\bar{q}_3 \Gamma_2 q_3) \text{Tr} \left[ \gamma_\mu q_4 \right]$$
B.2 QED

For the vertex corrections we consider the operator

\[ \mathcal{O}_{\text{QED}} = (\bar{f}_1 \Gamma_1 f_2)(\bar{f}_3 \Gamma_2 f_4), \]  

(B.3)

with four different fermions. We find:

\[
\frac{\alpha}{4\pi} Q_1 Q_2 \frac{1}{4\epsilon} (\bar{f}_1 \gamma_\mu \gamma_\nu \Gamma_1 \gamma^{\mu} \gamma^{\nu} f_2)(\bar{f}_3 \Gamma_2 f_4)
\]

\[
- \frac{\alpha}{4\pi} Q_2 Q_4 \frac{1}{4\epsilon} (\bar{f}_1 \Gamma_1 \gamma_\mu f_2)(\bar{f}_3 \Gamma_2 \gamma^{\mu} \gamma^{\nu} f_4)
\]

\[
\frac{\alpha}{4\pi} Q_2 Q_3 \frac{1}{4\epsilon} (\bar{f}_1 \Gamma_1 \gamma_\mu \gamma_\nu f_2)(\bar{f}_3 \Gamma_2 \gamma^{\nu} \gamma^{\mu} f_4)
\]

For the penguin contributions we consider insertions of the operator

\[ \mathcal{P}_{\text{QED}} = (\bar{f}_1 \Gamma_1 f_2)(\bar{f}_3 \Gamma_2 f_3), \]  

(B.4)

For closed and open penguins we find the following divergent structures:
In this appendix we report the one-loop shifts that result from the generalised BMU scheme. The results for VLL operators stemming from genuine vertex corrections are given in table 7. Those results do not depend on the flavour structure of the operators. If however two or more fermions have the same flavour, then the corresponding entries from table 1 have to be added to results in table 7 in order to obtain the full shift in the generalised BMU scheme.

The shifts for SLR and VRL operators are non-vanishing in the generalised BMU scheme, which can be seen in table 8. For the QED corrections to these operators we define the quantity

\[ A_{vwxy} = [2Q_{1423}(b_1 - c_3) + Q_{1324}(b_2 - c_2) + 4Q_{1234}(b_3 - c_1)]S^{LR}_{vwxy}. \]  

(C.1)
Tree-level Fierz

\[\begin{align*}
\text{Operator} & \quad \text{Tree-level Fierz} & \quad \text{QCD shift} & \quad \text{QED shift} \\
V_{1234}^{LL} & \quad V_{1234}^{LL} & \quad 2(a_1 - a_3) & \quad -2(a_1 - a_3) (Q_{1234} - Q_{1423}) V_{1234}^{LL} \\
\tilde{V}_{1234}^{LL} & \quad V_{1234}^{LL} & \quad 2(a_3 - a_1) & \quad -2(a_1 - a_3) (Q_{1234} - Q_{1423}) \tilde{V}_{1234}^{LL} \\
V_{1234}^{LL} & \quad V_{1234}^{LL} & \quad (a_1 - a_3) \frac{1-N_c^2}{N_c} V_{1234}^{LL} & \quad -2(a_1 - a_3) (Q_{1234} - Q_{1423}) V_{1234}^{LL} \\
V_{1234}^{LL} & \quad V_{1234}^{LL} & \quad (a_3 - a_1) \frac{1-N_c^2}{N_c} V_{1234}^{LL} & \quad -2(a_1 - a_3) (Q_{1234} - Q_{1423}) V_{1234}^{LL} \\
V_{1234}^{LL} & \quad V_{1234}^{LL} & \quad 0 & \quad -2(a_1 - a_3) (Q_{1234} - Q_{1423}) V_{1234}^{LL} \\
\end{align*}\]

Table 7. Fierz transformations for four-quark \(\gamma\mu P_L \otimes \gamma^\mu P_L\) operators together with their one-loop QCD and QED shifts resulting from genuine vertex corrections in the generalised BMU scheme.

\[\begin{align*}
\text{Operator} & \quad \text{Tree-level Fierz} & \quad \text{QCD shift} & \quad \text{QED shift} \\
S_{1234}^{LR} & \quad \frac{1}{2} V_{1234}^{RL} & \quad \frac{2b_3 - c_1}{N_c} S_{1234}^{LR} & \quad A_{1234} \\
S_{1234}^{LR} & \quad \frac{1}{2} V_{1234}^{RL} & \quad \frac{2b_3 - c_1}{N_c} S_{1234}^{LR} & \quad A_{1234} \\
S_{1234}^{LR} & \quad \frac{1}{2} V_{1234}^{RL} & \quad \frac{2b_3 - c_1}{N_c} S_{1234}^{LR} & \quad A_{1234} \\
S_{1234}^{LR} & \quad \frac{1}{2} V_{1234}^{RL} & \quad \frac{2b_3 - c_1}{N_c} S_{1234}^{LR} & \quad A_{1234} \\
\end{align*}\]

Table 8. Fierz transformations for scalar \(P_L \otimes P_R\) operators together with their one-loop QCD and QED shifts in the generalised BMU scheme.

The SLL operators and their shifts in the generalised BMU scheme are collected in table 9. For the QED shifts to these operators we define the quantity:

\[
B_{vwxy} \equiv \frac{1}{32} \left[ (-128d_1 + 4d_3 + 40e_2 + 84f_2) Q_{1234} + (12d_2 + 40e_1 - 36f_1) Q_{1324} \\
+ (32d_1 - 16d_3) Q_{1423} \right] S_{vwxy}^{LL}
\]

\[
\frac{1}{32} \left[ (d_3 + 10e_2 - 7f_2) Q_{1234} + (-d_2 + 10e_1 + 3f_1) Q_{1324} + 8d_1 Q_{1423} \right] T_{vwxy}^{LL}.
\]
| Operator | Tree-level Fierz | QCD shift | QED shift |
|----------|-----------------|-----------|-----------|
| $S_{q_1q_23q_3q_4}^{LL}$ | $-\frac{1}{2} S_{q_1q_23q_3q_4}^{LL} - \frac{1}{8} T_{q_1q_23q_3q_4}^{LL}$ | $\frac{1}{8}[8d_1 + 3d_2 - 4d_3 + 10e_1 - 9f_1 - (32d_1 - d_3 - 10e_2 + 21f_2)(N_c^2 - 1)]S_{q_1q_23q_3q_4}^{LL}$ | $B_{q_1q_23q_3q_4}$ |
| $S_{q_1q_23q_3q_4}^{LL}$ | $-\frac{1}{2} S_{q_1q_23q_3q_4}^{LL} - \frac{1}{8} T_{q_1q_23q_3q_4}^{LL}$ | $\frac{1}{8}[8d_1 + 3d_2 - 4d_3 + 10e_1 - 9f_1 - (32d_1 - d_3 - 10e_2 + 21f_2)(N_c^2 - 1)]S_{q_1q_23q_3q_4}^{LL}$ | $\tilde{B}_{q_1q_23q_3q_4}$ |
| $S_{q_1q_2\ell_1\ell_2}^{LL}$ | $-\frac{1}{2} S_{q_1q_2\ell_1\ell_2}^{LL} - \frac{1}{8} T_{q_1q_2\ell_1\ell_2}^{LL}$ | $\frac{1}{16N_c}(32d_1 - d_3 - 10e_2 - 21f_2)(N_c^2 - 1)S_{q_1q_2\ell_1\ell_2}^{LL}$ | $B_{q_1q_2\ell_1\ell_2}$ |
| $S_{q_1q_2\ell_1\ell_2}^{LL}$ | $-\frac{1}{2} S_{q_1q_2\ell_1\ell_2}^{LL} - \frac{1}{8} T_{q_1q_2\ell_1\ell_2}^{LL}$ | $\frac{1}{16N_c}(32d_1 - d_3 - 10e_2 - 21f_2)(N_c^2 - 1)S_{q_1q_2\ell_1\ell_2}^{LL}$ | $B_{q_1q_2\ell_1\ell_2}$ |
| $S_{\ell_1\ell_2\ell_3\ell_4}^{LL}$ | $-\frac{1}{2} S_{\ell_1\ell_2\ell_3\ell_4}^{LL} - \frac{1}{8} T_{\ell_1\ell_2\ell_3\ell_4}^{LL}$ | $0$ | $B_{\ell_1\ell_2\ell_3\ell_4}$ |

Table 9. Fierz transformations for scalar $P_L \otimes P_L$ operators together with their one-loop QCD and QED shifts in the generalised BMU scheme.

Finally we report the shifts for the TLL operators in table 10. To abbreviate the QED shifts we use the following definition:

$$D_{vwxy} \equiv$$

$$\frac{1}{8}\left[ (12d_3 - 10e_2 - 7f_2)Q_{1234} + (12d_3 - 10e_2 + 9f_1)Q_{1234} + (32d_1 - 160e_2)Q_{1423} \right]S_{vwxy}^{LL} $$

$$\frac{1}{8}\left[ (3d_3 - 10e_2 + 7f_2)Q_{1234} + (3d_3 - 10e_2 + 9f_1)Q_{1234} + (24d_1 - 28f_2)Q_{1423} \right]T_{vwxy}^{LL}.$$  \hfill (C.3)
| Operator | Tree-level Fierz | QCD shift | QED shift |
|----------|-----------------|-----------|-----------|
| $T_{q,q;LL,LL}^{q,q}$ | $-6S_{q,q;LL,LL}^{q,q} + \frac{1}{2} T_{q,q;LL,LL}^{q,q}$ | $\frac{1}{2\mu_c} [-24d_1 + 3d_2 + 50e_1 + 40e_2 - 9f_1$ | $D_{q,q;LL,LL}$ |
|            |                  | $+ (3d_3 - 10e_2 - 21f_2)(N_c^2 - 1)] S_{q,q;LL,LL}^{q,q}$ |           |
|            |                  | $+ \frac{1}{2} [-24d_1 - 3d_2 - 50e_1 - 40e_2 + 9f_1 S_{q,q;LL,LL}^{q,q}$ |           |
|            |                  | $+ \frac{1}{8N_c} [-24d_1 + 3d_2 + 10e_1 - 9f_1 + 28f_2$ |           |
|            |                  | $+ (3d_3 - 10e_2 + 7f_2)(N_c^2 - 1)] T_{q,q;LL,LL}^{q,q}$ |           |
|            |                  | $+ \frac{1}{8} [-24d_1 - 3d_2 + 10e_1 - 9f_1 - 28f_2 T_{q,q;LL,LL}^{q,q}$ |           |
| $\tilde{T}_{q,q;LL,LL}^{q,q}$ | $-6S_{q,q;LL,LL}^{q,q} + \frac{1}{2} T_{q,q;LL,LL}^{q,q}$ | $\frac{1}{2} [-3d_2 + 3d_3 - 50e_1 - 10e_2 + 9f_1 + 21f_2 S_{q,q;LL,LL}^{q,q}$ | $\tilde{D}_{q,q;LL,LL}$ |
|            |                  | $+ \frac{1}{2N_c} [3d_2 - 3d_3 + 50e_1 + 50e_2 - 9f_1$ |           |
|            |                  | $+ 21f_2 - 40e_2 N_c^2 + 24d_1 (N_c^2 - 1)] \tilde{S}_{q,q;LL,LL}^{q,q} |           |
|            |                  | $+ \frac{1}{8N_c} [-3d_2 + 3d_3 - 10e_1 - 10e_2 + 9f_1 + 7f_2 T_{q,q;LL,LL}^{q,q}$ |           |
|            |                  | $+ \frac{1}{8} [3d_2 - 3d_3 + 10e_1 + 10e_2$ |           |
|            |                  | $- 9f_1 - 7f_2 + 4 (6d_1 - 7f_2)(N_c^2 - 1)] \tilde{T}_{q,q;LL,LL}^{q,q}$ |           |
| $T_{LL}^{LL}^{T_{LL}}$ | $-6S_{LL}^{LL} + \frac{1}{2} T_{LL}^{LL}$ | $\frac{1}{N_c} [3d_3 - 10e_2 - 21f_2](N_c^2 - 1) S_{LL}^{LL}$ | $D_{q,q;LL,LL}$ |
|            |                  | $+ \frac{1}{16N_c}[3d_3 - 10e_2 + 7f_2](N_c^2 - 1) T_{LL}^{LL}$ |           |
| $T_{LL}^{LL}^{T_{LL}}$ | $-6S_{LL}^{LL} + \frac{1}{2} T_{LL}^{LL}$ | $\frac{1}{N_c} [3d_1 - 5e_2](N_c^2 - 1) \tilde{S}_{LL}^{LL}$ | $D_{q,q;LL,LL}$ |
|            |                  | $+ \frac{1}{16N_c} [6d_1 - 7f_2](N_c^2 - 1) \tilde{T}_{LL}^{LL}$ |           |
| $T_{LL}^{LL}^{T_{LL}}$ | $-6S_{LL}^{LL} + \frac{1}{2} T_{LL}^{LL}$ | $0$ | $D_{q,q;LL,LL}$ |

Table 10. Fierz transformations for tensor $\sigma_{\mu\nu}P_L \otimes \sigma^{\mu\nu}P_L$ operators together with their one-loop QCD and QED shifts in the generalised BMU scheme.
Table 11. Loop structure for $\gamma\mu P_L \otimes \gamma^\mu P_L$ operators: the columns correspond to the divergent WFR contributions and vertex corrections as well as the finite contributions from evanescent operators, computed in the generalised BMU scheme. The first four rows result from QCD and are given in units of $\alpha_s/(4\pi)$ whereas the last one is a QED correction given in units of $\alpha/(4\pi)$.

### D Poles

In this appendix we report the pole structures as well as the finite terms that result from the operator insertions into the diagrams collected in appendix B. For simplicity we define the QED Wavefunction renormalization constant for an operator containing four different fermions by

$$P = -\frac{1}{2}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2). \quad (D.1)$$

### E Renormalization constants

In this appendix we consider all possible operator bases for the different Dirac structures VLL, SLR, SLL and TLL and fermion combinations discussed above and augment the bases with the corresponding Fierz-evanescent operators. We report for these sets of operators the renormalisation constants, which are used in general basis transformations. In this appendix we only consider contributions from genuine vertex corrections and neglect penguin contributions.
The renormalization constants for these operators read:

$$Z_{QQ}^{(1,1)} = Z_{EE}^{(1,1)} = \begin{pmatrix} -3N_c & 3 \\ -3N_c & 3 \end{pmatrix} , \quad Z_{QE}^{(1,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

and the finite renormalization is given by

$$Z_{EQ}^{(1,0)} = \begin{pmatrix} 2(a_1 - a_3)N_c & -2(a_1 - a_3) \\ 2(a_1 - a_3) & -2(a_1 - a_3)N_c \end{pmatrix} .$$

### E.1 Vector operators

#### E.1.1 VLL: $q_1 q_2 q_3 q_4$

Starting with the most general set of VLL operators we consider the case of four different quarks:

$$\{ V_{q_1 q_2 q_3 q_4}^{LL}, \tilde{V}_{q_1 q_2 q_3 q_4}^{LL}, E_{q_1 q_2 q_3 q_4}^{VLL}, \tilde{E}_{q_1 q_2 q_3 q_4}^{VLL} \} ,$$

with the evanescent operators

$$E_{q_1 q_2 q_3 q_4}^{VLL} \equiv V_{q_1 q_2 q_3 q_4}^{LL} - F_{q_1 q_2 q_3 q_4}^{VLL} , \quad \tilde{E}_{q_1 q_2 q_3 q_4}^{VLL} \equiv \tilde{V}_{q_1 q_2 q_3 q_4}^{LL} - \tilde{F}_{q_1 q_2 q_3 q_4}^{VLL} .$$

The renormalization constants for these operators read:

$$Z_{QQ}^{(1,1)} = Z_{EE}^{(1,1)} = \begin{pmatrix} -3N_c & 3 \\ -3N_c & 3 \end{pmatrix} , \quad Z_{QE}^{(1,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

and the finite renormalization is given by

$$Z_{EQ}^{(1,0)} = \begin{pmatrix} 2(a_1 - a_3)N_c & -2(a_1 - a_3) \\ 2(a_1 - a_3) & -2(a_1 - a_3)N_c \end{pmatrix} .$$

---

**Table 12.** Loop structure for $P_L \otimes P_R$ operators: the columns correspond to the divergent WFR contributions and vertex corrections as well as the finite contributions from evanescent operators, computed in the generalised BMU scheme. The first four rows result from QCD and are given in units of $\alpha_s/(4\pi)$ whereas the last one is a QED correction given in units of $\alpha/(4\pi)$. 

| Operator | WFR $[\frac{1}{\tau}]$ | pole $[\frac{1}{\tau}]$ | evanescent cont. |
|----------|-----------------------|------------------|-----------------|
| $S_{q_1 q_2 q_3 q_4}^{LR}$ | $-2CFS_{q_1 q_2 q_3 q_4}^{LR}$ | $8CF S_{q_1 q_2 q_3 q_4}^{LR}$ | $(\frac{c_2 + c_3}{N_c} - 8CFc_1)S_{q_1 q_2 q_3 q_4}^{LR} + (c_2 - 2c_3)S_{q_1 q_2 q_3 q_4}^{LR}$ |
| $S_{q_1 q_2 q_3 q_4}^{LR}$ | $-2CFS_{q_1 q_2 q_3 q_4}^{LR}$ | $3S_{q_1 q_2 q_3 q_4}^{LR}$ | $(-4c_1 - c_2)S_{q_1 q_2 q_3 q_4}^{LR} + (\frac{2c_1 + c_2}{N_c} - 4c_1 CF)S_{q_1 q_2 q_3 q_4}^{LR}$ |
| $S_{q_1 q_2 q_3 q_4}^{LR}$ | $-CFS_{q_1 q_2 q_3 q_4}^{LR}$ | $4CF S_{q_1 q_2 q_3 q_4}^{LR}$ | $-4CFC_1S_{q_1 q_2 q_3 q_4}^{LR}$ |
| $S_{q_1 q_2 q_3 q_4}^{LR}$ | $-CFS_{q_1 q_2 q_3 q_4}^{LR}$ | $CFS_{q_1 q_2 q_3 q_4}^{LR}$ | $-2CFc_3S_{q_1 q_2 q_3 q_4}^{LR}$ |
| $S_{q_1 q_2 q_3 q_4}^{LR}$ | $P_{q_1 q_2 q_3 q_4}^{LR}$ | $(4Q_{1234} - Q_{1324} + Q_{1423})S_{q_1 q_2 q_3 q_4}^{LR}$ | $(-4c_1 Q_{1234} - c_2 Q_{1324} - 2c_3 Q_{1423})S_{q_1 q_2 q_3 q_4}^{LR}$ |
\[\begin{array}{|c|c|c|c|}
\hline
\text{Operator} & \text{WFR } [\frac{1}{e}] & \text{pole } [\frac{1}{e}] & \text{evanescent cont.} \\
\hline
S_{\ell_1,\ell_2}^{LL} & -2C_F S_{\ell_1,\ell_2}^{LL} & \frac{8C_F S_{\ell_1,\ell_2}^{LL}q_0}{\pi^2} + \frac{1}{2} T_{\ell_1,\ell_2}^{LL} & \left(\frac{d_2 - d_4}{2} - 8C_F d_1\right) S_{\ell_1,\ell_2}^{LL} \\
\hline
S_{\ell_1,\ell_2}^{LL} & -2C_F S_{\ell_1,\ell_2}^{LL} & \frac{3S_{\ell_1,\ell_2}^{LL}q_0}{\pi^2} + \frac{1}{4} T_{\ell_1,\ell_2}^{LL} & \left(\frac{d_2 - d_4}{2} - 4d_1\right) S_{\ell_1,\ell_2}^{LL} \\
\hline
S_{\ell_1,\ell_2}^{LL} & 4C_F S_{\ell_1,\ell_2}^{LL} & \frac{4C_F S_{\ell_1,\ell_2}^{LL}q_0}{\pi^2} + \frac{1}{4} T_{\ell_1,\ell_2}^{LL} & -4C_F d_3 S_{\ell_1,\ell_2}^{LL} \\
\hline
S_{\ell_1,\ell_2}^{LL} & -C_F S_{\ell_1,\ell_2}^{LL} & \frac{C_F S_{\ell_1,\ell_2}^{LL}q_0}{\pi^2} + \frac{1}{4} T_{\ell_1,\ell_2}^{LL} & -\frac{1}{2} C_F d_3 S_{\ell_1,\ell_2}^{LL} \\
\hline
S_{\ell_1,\ell_2}^{LL} & P_{\ell_1,\ell_2}^{LL} & \frac{8C_F S_{\ell_1,\ell_2}^{LL}q_0}{\pi^2} + \frac{1}{4} T_{\ell_1,\ell_2}^{LL} & \left(-4d_1 Q_{1234} + \frac{d_2 q_0}{\pi^2} - d_4 Q_{1423}\right) S_{\ell_1,\ell_2}^{LL} \\
\hline
\end{array}\]

Table 13. Loop structure for $P_L \otimes P_L$ operators: the columns correspond to the divergent WFR contributions and vertex corrections as well as the finite contributions from evanescent operators, computed in the generalised BMU scheme. The first four rows result from QCD and are given in units of $\alpha_s/(4\pi)$ whereas the last one is a QED correction given in units of $\alpha/(4\pi)$.

E.1.2 VLL: $q_1 q_2 g_1 g_2$

For operators generating $\Delta F = 2$ transitions we define the operator basis

\[\{V_{q_1 q_2 g_1 g_2}^{LL}, E_{q_1 q_2 g_1 g_2}^{VLL}\},\]

where $E_{q_1 q_2 g_1 g_2}^{VLL}$ is defined analogously as in eq. (E.2). One finds for the renormalization constants:

\[Z_{QQ}^{(1,1)} = 3 - \frac{3}{N_c}, \quad Z_{QE}^{(1,1)} = -3, \quad Z_{EE}^{(1,0)} = 2(a_1 - a_3)(N_c - 1), \quad Z_{EE}^{(1,1)} = -3 - \frac{3}{N_c}.\]

The same relations hold if the basis $\{\tilde{V}_{q_1 q_2 g_1 g_2}^{LL}, E_{q_1 q_2 g_1 g_2}^{VLL}\}$ is used instead.

E.1.3 VLL: $q_1 q_2 \ell_1 \ell_2$

For semi-leptonic operators we define the operator basis

\[\{V_{q_1 q_2 \ell_1 \ell_2}^{LL}, E_{q_1 q_2 \ell_1 \ell_2}^{VLL}\},\]

where $E_{q_1 q_2 \ell_1 \ell_2}^{VLL}$ is defined analogously as in eq. (E.2). One finds for the renormalization constants:

\[Z_{QQ}^{(1,1)} = 0, \quad Z_{QE}^{(1,1)} = 0, \quad Z_{EE}^{(1,0)} = 2(a_1 - a_3)C_F, \quad Z_{EE}^{(1,1)} = 0.\]
For four-lepton operators we define the operator basis

\[ \{ V_{\ell_1 \ell_2 \ell_3 \ell_4}^{LL}, E_{\ell_1 \ell_2 \ell_3 \ell_4}^{VLL} \}, \]

(E.10)

where \( E_{\ell_1 \ell_2 \ell_3 \ell_4}^{VLL} \) is defined analogously as in eq. (E.2). One finds for the QED renormalization constants:

\[ Z_{QQ}^{(1,1)} = -P - Q_{1234} + 4Q_{1324} - Q_{1423} , \]  

(E.11)

\[ Z_{Q}^{(1,1)} = 0 , \]  

(E.12)

\[ Z_{EQ}^{(1,0)} = 2(a_1 - a_3)(Q_1 - Q_3)(Q_2 - Q_4) , \]  

(E.13)

\[ Z_{EE}^{(1,1)} = Z_{QQ}^{(1,1)} . \]  

(E.14)

### Table 14

| Operator | WFR $[\frac{1}{t}]$ | pole $[\frac{1}{t}]$ | evanescent cont. |
|----------|-----------------|-----------------|-----------------|
| \( T_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) | \(-2C_F T_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) | \(-24 S_{q_1 q_2 q_3 q_4}^{LL} + 24 S_{q_1 q_2 q_3 q_4}^{LL} \) | \( \frac{24}{N_c} (e_1 + e_2) S_{q_1 q_2 q_3 q_4}^{LL} - 20(1 + e_2) S_{q_1 q_2 q_3 q_4}^{LL} + \frac{7}{N_c} T_{LL}^{\tilde{q}_1 q_2 q_3 q_4} + \frac{11}{N_c} T_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) |
| \( \tilde{T}_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) | \(-2C_F \tilde{T}_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) | \(12 S_{q_1 q_2 q_3 q_4}^{LL} + \left(24C_F - \frac{12}{N_c}\right) S_{q_1 q_2 q_3 q_4}^{LL} - 3 T_{LL}^{\tilde{q}_1 q_2 q_3 q_4} + \left(6C_F + \frac{3}{N_c}\right) \tilde{T}_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) | \(-20e_1 S_{q_1 q_2 q_3 q_4}^{LL} + \left(-40C_F e_2 + \frac{20}{N_c}\right) S_{q_1 q_2 q_3 q_4}^{LL} + \frac{3}{N_c} T_{LL}^{\tilde{q}_1 q_2 q_3 q_4} - \left(\frac{3}{N_c} + 7f_2 C_F\right) \tilde{T}_{LL}^{\tilde{q}_1 q_2 q_3 q_4} \) |
| \( T_{LL}^{\tilde{q}_1 \ell_1 \ell_2 \ell_2} \) | \(-C_F T_{LL}^{\tilde{q}_1 \ell_1 \ell_2 \ell_2} \) | 0 | 0 |
| \( T_{LL}^{\tilde{q}_1 \ell_1 \ell_2 \ell_3} \) | \(-C_F T_{LL}^{\tilde{q}_1 \ell_1 \ell_2 \ell_3} \) | \(12 C_F S_{\ell_1 \ell_2 \ell_3}^{LL} + 3 C_F T_{LL}^{\tilde{q}_1 \ell_1 \ell_2 \ell_3} \) | \(-20C_F e_2 S_{\ell_1 \ell_2 \ell_3}^{LL} - \frac{7}{2} C_F f_2 T_{LL}^{\tilde{q}_1 \ell_1 \ell_2 \ell_3} \) |
| \( T_{LL}^{\ell_1 \ell_2 \ell_3 \ell_4} \) | \( P T_{LL}^{\ell_1 \ell_2 \ell_3 \ell_4} \) | \(12Q_{1324} + 12Q_{1423} \) | \(-20e_1 Q_{1324} - 20e_2 Q_{1423} \) |

The same relations hold if the basis \( \{ V_{\ell_1 \ell_2 \ell_3 q_2}^{LL}, E_{\ell_1 \ell_2 \ell_3 q_2}^{VLL} \} \) is used instead with the only difference that \( Z_{EQ}^{(1,0)} \) changes sign.

#### E.1.4 VLL: $\ell_1 \ell_2 \ell_3 \ell_4$

For four-lepton operators we define the operator basis

\[ \{ V_{\ell_1 \ell_2 \ell_3 \ell_4}^{LL}, E_{\ell_1 \ell_2 \ell_3 \ell_4}^{VLL} \}, \]
E.2 Scalar and vector LR operators

E.2.1 SLR: $q_1 q_2 q_3 q_4$

Starting with the most general set of SLR operators we consider the case of four different quarks:

$$\{S_{q_1 q_2 q_3 q_4}^{LR}, S_{q_1 q_2 q_3 q_4}^{LR}, E_{q_1 q_2 q_3 q_4}^{SLR}, E_{q_1 q_2 q_3 q_4}^{SLR}\},$$

(E.15)

with the evanescent operators

$$E_{q_1 q_2 q_3 q_4}^{SLR} = S_{q_1 q_2 q_3 q_4}^{LR} - F_{q_1 q_2 q_3 q_4}^{SR},$$

(E.16)

$$E_{q_1 q_2 q_3 q_4}^{SR} = S_{q_1 q_2 q_3 q_4}^{LR} - F_{q_1 q_2 q_3 q_4}^{SR},$$

(E.17)

The renormalization constants for these operators read:

$$Z_Q^{(1,1)} = Z_E^{(1,1)} = \begin{pmatrix} -6C_F & 0 \\ -3 & \frac{3}{N_c} \end{pmatrix},$$

$$Z_Q^{(1,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(E.18)

and the finite renormalization is given by

$$Z_{EQ}^{(1,0)} = \begin{pmatrix} \frac{b_2 + 2b_1 - 2c_1 - c_2}{N_c} - 8b_3 C_F + 8c_1 C_F & c_2 + 2c_3 - 2b_1 - b_2 \\ -b_2 - 4b_1 + 4c_1 + c_2 & \frac{b_2 + 4b_1 - 4c_1 - c_2}{N_c} - 4b_1 C_F + 4c_3 C_F \end{pmatrix}. $$

(E.19)

E.2.2 SLR: $q_1 q_2 q_1 q_2$

For operators generating $\Delta F = 2$ transitions we choose to eliminate the Parity-flipped operators $S_{q_1 q_2 q_1 q_2}^{RL}, S_{q_1 q_2 q_1 q_2}^{RL}$ and therefore the initial basis together with the renormalization constants stays the same as in the previous subsection. The basis takes the form

$$\{S_{q_1 q_2 q_1 q_2}^{LR}, S_{q_1 q_2 q_1 q_2}^{LR}, E_{q_1 q_2 q_1 q_2}^{SLR}, E_{q_1 q_2 q_1 q_2}^{SLR}\}. $$

(E.20)

E.2.3 SLR: $q_1 q_2 \ell_1 \ell_2$

For semi-leptonic operators we define the operator basis

$$\{S_{q_1 q_2 \ell_1 \ell_2}^{LR}, E_{q_1 q_2 \ell_1 \ell_2}^{SLR}\},$$

(E.21)

where $E_{q_1 q_2 \ell_1 \ell_2}^{SLR}$ is defined analogously as in eq. (E.16). One finds for the renormalization constants:

$$Z_Q^{(1,1)} = -3C_F, \quad Z_Q^{(1,1)} = 0, \quad Z_E^{(1,0)} = 4(c_1 - b_3) C_F, \quad Z_E^{(1,1)} = -3C_F. $$

(E.22)

For the basis $\{S_{q_1 \ell_1 \ell_2}^{LL}, E_{q_1 \ell_1 \ell_2}^{SLL}\}$ one finds the following renormalization constants:

$$Z_Q^{(1,1)} = 0, \quad Z_Q^{(1,1)} = 0, \quad Z_E^{(1,0)} = 2(c_3 - b_1)C_F, \quad Z_E^{(1,1)} = 0. $$

(E.23)
E.2.4 SLR: $\ell_1 \ell_2 \ell_3 \ell_4$

For four-lepton operators we define the operator basis

$$\{S^{SLR}_{\ell_1 \ell_2 \ell_3 \ell_4}, E^{SLR}_{\ell_1 \ell_2 \ell_3 \ell_4}\},$$  \hspace{1cm} (E.24)

where $E^{\alpha LL}_{\ell_1 \ell_2 \ell_3 \ell_4}$ is defined analogously as in eq. (E.16). One finds for the QED renormalization constants:

$$Z_{QQ}^{(1,1)} = -P - 4Q_{1234} + Q_{1324} - Q_{1423},$$  \hspace{1cm} (E.25)

$$Z_{QE}^{(1,1)} = 0,$$  \hspace{1cm} (E.26)

$$Z_{EQ}^{(1,0)} = 2Q_{1423}(c_3 - b_1) + Q_{1324}(c_2 - b_2) + 4Q_{1234}(c_1 - b_3),$$  \hspace{1cm} (E.27)

$$Z_{EE}^{(1,1)} = Z_{QQ}^{(1,1)}.$$  \hspace{1cm} (E.28)

E.3 Scalar and tensor LL operators

E.3.1 SLL, TLL: $q_1 q_2 q_3 q_4$

Starting with the most general set of SLL and TLL operators we consider the case of four different quarks:

$$\{S^{\alpha LL}_{q_1 q_2 q_3 q_4}, T^{\alpha LL}_{q_1 q_2 q_3 q_4}, E^{\alpha LL}_{q_1 q_2 q_3 q_4}, E^{\tilde{\alpha} LL}_{q_1 q_2 q_3 q_4}, E^{TLL}_{q_1 q_2 q_3 q_4}, E^{TLL}_{q_1 q_2 q_3 q_4}\},$$  \hspace{1cm} (E.29)

with the evanescent operators

$$E^{SLL}_{q_1 q_2 q_3 q_4} \equiv S^{LL}_{q_1 q_2 q_3 q_4} - F^{SLL}_{q_1 q_2 q_3 q_4},$$  \hspace{1cm} (E.30)

$$E^{TLL}_{q_1 q_2 q_3 q_4} \equiv T^{LL}_{q_1 q_2 q_3 q_4} - F^{TLL}_{q_1 q_2 q_3 q_4},$$  \hspace{1cm} (E.31)

$$E^{TLL}_{q_1 q_2 q_3 q_4} \equiv T^{LL}_{q_1 q_2 q_3 q_4} - F^{TLL}_{q_1 q_2 q_3 q_4},$$  \hspace{1cm} (E.32)

$$E^{\tilde{T}LL}_{q_1 q_2 q_3 q_4} \equiv \tilde{T}^{LL}_{q_1 q_2 q_3 q_4} - F^{\tilde{T}LL}_{q_1 q_2 q_3 q_4}.$$  \hspace{1cm} (E.33)

The renormalization constants for these operators read:

$$Z_{QQ}^{(1,1)} = Z_{EE}^{(1,1)} = \begin{pmatrix}
-6C_F & 0 & \frac{1}{2N_c} & -\frac{1}{2} \frac{N_c^2 - 2}{4N_c} \\
3 & \frac{1}{N_c} & \frac{1}{2} & \frac{N_c^2 - 2}{4N_c} \\
24 & -24 & 2C_F & 0 \\
-12 & \frac{12(N_c^2 - 2)}{N_c} & 3 & 2C_F - 3N_c
\end{pmatrix},$$  \hspace{1cm} (E.34)

and the finite renormalization we find:

$$\left(Z_{EQ}^{(1,0)}\right)_{11} = -\frac{32d_1 N_c^2 + 24d_1 - 3d_2 + d_3 N_c^2 + 3d_3 - 10e_1 + 10e_2 N_c^2 - 10e_2}{8N_c},$$  \hspace{1cm} (E.35)

$$\left(Z_{EQ}^{(1,0)}\right)_{12} = \frac{1}{8}\left(-8d_1 - 3d_2 + 4d_3 - 10e_1 + 9f_1\right),$$  \hspace{1cm} (E.36)
\[
(Z_{\text{E}Q}^{(1,0)})_{13} = \frac{8d_1 - d_2 - d_3 N_c^2 + d_3 + 10e_1 - 10e_2 N_c^2 + 10e_2 + 3f_1}{32N_c} \\
+ \frac{7f_2 N_c^2 - 7f_2}{32N_c},
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{14} = \frac{1}{32}(-8d_1 + d_2 - 10e_1 - 3f_1),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{21} = \frac{1}{8}(32d_1 - 3d_2 - d_3 - 10e_1 - 10e_2 + 9f_1 - 21f_2),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{22} = \frac{-8d_1 (N_c^2 + 3) + 3d_2 + 4d_3 N_c^2 - 3d_3 + 10e_1 + 10e_2}{8N_c} \\
+ \frac{-9f_1 + 21f_2}{8N_c},
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{23} = \frac{1}{32}(d_2 - d_3 - 10e_1 - 10e_2 - 3f_1 + 7f_2),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{24} = \frac{-8d_1 N_c^2 + 8d_1 - d_2 + d_3 + 10e_1 + 10e_2 + 3f_1 - 7f_2}{32N_c},
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{31} = \frac{24d_1 - 3d_2 - 3d_3 N_c^2 + 3d_3 - 50e_1 + 10e_2 N_c^2 - 50e_2}{2N_c} \\
+ \frac{9f_1 + 21f_2 N_c^2 - 21f_2}{2N_c},
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{32} = \frac{1}{2}(-24d_1 + 3d_2 + 50e_1 + 40e_2 - 9f_1),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{33} = \frac{-24d_1 + 3d_2 + 3d_3 N_c^2 - 3d_3 + 10e_1 - 10e_2 N_c^2 + 10e_2 - 9f_1}{8N_c} \\
- \frac{7f_2 N_c^2 + 21f_2}{8N_c},
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{34} = \frac{1}{8}(-24d_1 + 3d_2 + 10e_1 - 9f_1 + 28f_2),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{41} = \frac{1}{2}(3d_2 - 3d_3 + 50e_1 + 10e_2 - 9f_1 + 21f_2),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{42} = \frac{-24d_1 (N_c^2 - 1) + 3d_2 - 3d_3 + 50e_1 - 40e_2 N_c^2 + 50e_2}{2N_c} \\
- \frac{-9f_1 + 21f_2}{2N_c},
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{43} = \frac{1}{8}(3d_2 - 3d_3 + 10e_1 + 10e_2 - 9f_1 - 7f_2),
\]
\[
(Z_{\text{E}Q}^{(1,0)})_{44} = \frac{-24d_1 (N_c^2 - 1) + 3d_2 - 3d_3 + 10e_1 + 10e_2 - 9f_1}{8N_c} \\
- \frac{-28f_2 N_c^2 + 21f_2}{8N_c}.
\]

**E.3.2 SLL, TLL: \(q_1 q_2 q_1 q_2\)**

For operators generating \(\Delta F = 2\) transitions we choose the basis which is closely related to the one chosen in \([32]\), namely

\[
\{ S_{q_1 q_2 q_1 q_2}^{LL}, T_{q_1 q_2 q_1 q_2}^{LL}, \tilde{F}_{q_1 q_2 q_1 q_2}^{SLL}, \tilde{F}_{q_1 q_2 q_1 q_2}^{TLL} \},
\]
where the two evanescent operators are defined according to eq. (E.30). We find for the renormalization constants:

$$Z_{QQ}^{(1,1)} = \begin{pmatrix} 3 - 6C_F & \frac{2N_c}{12(N_c+2)} \\ \frac{2N_c}{2C_F + 3} & 0 \end{pmatrix}, \quad Z_{QE}^{(1,1)} = \begin{pmatrix} 0 & -\frac{1}{3} \\ -\frac{24}{3} & 0 \end{pmatrix},$$

(E.52)

and the finite renormalization reads:

$$Z_{EQ}^{(1,0)}_{11} = \frac{N_c(16d_1(N_c+2) - 3d_2 - 2d_3N_c - d_3 + 9f_1 - 21f_2) - 10e_1(N_c+2)}{8N_c},$$

(E.53)

$$Z_{EQ}^{(1,0)}_{12} = \frac{16d_1 + d_2(N_c-2) - N_c(2d_3N_c + d_3 + 10e_1 + 10e_2 + 3f_1 - 7f_2)}{32N_c} + \frac{2(d_1 + 3f_1 - 7f_2)}{32N_c},$$

(E.54)

$$Z_{EQ}^{(1,0)}_{21} = \frac{N_c(-3d_1 + 50e_1 - 20e_2N_c + 10e_2 - 9f_1 - 42f_2N_c + 21f_2)}{2N_c} + \frac{48d_1 (N_c^2 - 1) + 3d_2 (N_c + 2) - 6d_3 + 40e_1 + 40e_2 - 18f_1 + 42f_2}{2N_c},$$

(E.55)

$$Z_{EQ}^{(1,0)}_{22} = \frac{N_c(3d_2 - 3d_3 - 9f_1 + 14f_2N_c - 7f_2) + 10e_1(N_c+2)}{8N_c} + \frac{10e_2 (-2N_c^2 + N_c + 2)}{8N_c},$$

(E.56)

$$Z_{EE}^{(1,1)} = \begin{pmatrix} -3 + \frac{3}{N_c} & \frac{1}{2} \\ -12N_c + \frac{24}{N_c} + 12 & 2C_F - 3(N_c+1) \end{pmatrix}.$$

(E.57)

### E.3.3 SLL, TLL: semi-leptonic

**SLL and TLL** $q_1q_2\ell_1\ell_2$: for semi-leptonic operators we define the operator basis

$$\{S_{q_1q_2\ell_1\ell_2}^{SLL}, T_{q_1q_2\ell_1\ell_2}^{SLL}, E_{q_1q_2\ell_1\ell_2}^{SLL}, E_{q_1q_2\ell_1\ell_2}^{TLL}\},$$

(E.58)

where the two evanescent operators are defined according to eq. (E.30). One finds for the renormalization constants:

$$Z_{QQ}^{(1,1)} = Z_{EE}^{(1,1)} = \begin{pmatrix} -3C_F & 0 \\ 0 & C_F \end{pmatrix}, \quad Z_{QE}^{(1,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

(E.59)

$$Z_{EQ}^{(1,0)} = \begin{pmatrix} \frac{C_F}{\xi} (32d_1 - d_3 - 10e_2 - 21f_2) & \frac{C_F}{\xi} (-d_3 - 10e_2 + 7f_2) \\ \frac{C_F}{\xi} (-3d_3 + 10e_2 + 21f_2) & \frac{C_F}{\xi} (32d_1 - d_3 - 10e_2 - 21f_2) \end{pmatrix}.$$

(E.60)

**SLL and TLL** $q_1\ell_2\ell_1q_2$: similarly, we can choose to work in the fierzed basis with respect to eq. (E.58):

$$\{S_{q_1\ell_2\ell_1q_2}^{SLL}, T_{q_1\ell_2\ell_1q_2}^{SLL}, E_{q_1\ell_2\ell_1q_2}^{SLL}, E_{q_1\ell_2\ell_1q_2}^{TLL}\},$$

(E.61)
where the two evanescent operators are defined according to eq. (E.30). One finds for the renormalization constants:

\[
Z_{QQ}^{(1,1)} = Z_{EE}^{(1,1)} = \begin{pmatrix}
0 & -\frac{C_F}{2} \\
-12C_F & -2C_F
\end{pmatrix}, \quad Z_{QE}^{(1,1)} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}, \quad (E.62)
\]

\[
Z_{EQ}^{(1,0)} = \begin{pmatrix}
\frac{C_F}{2} (d_3 - 2d_1) & -\frac{C_F}{2} d_1 \\
C_F (20e_2 - 12d_1) & \frac{C_F}{2} (7f_2 - 6d_1)
\end{pmatrix}. \quad (E.63)
\]

### E.3.4 SLL, TLL: $\ell_1\ell_2\ell_3\ell_4$

For four-lepton operators we define the operator basis

\[
\{S_{\ell_1\ell_2\ell_3\ell_4}^{LL}, T_{\ell_1\ell_2\ell_3\ell_4}^{LL}, E_{\ell_1\ell_2\ell_3\ell_4}^{SLL}, E_{\ell_1\ell_2\ell_3\ell_4}^{TLL}\}, \quad (E.64)
\]

where the two evanescent operators are defined according to eq. (E.30). One finds for the QED renormalization constants:

\[
Z_{QQ}^{(1,1)} = Z_{EE}^{(1,1)} = \begin{pmatrix}
-P + Q_{1324} - 4Q_{1234} - Q_{1423} & -\frac{1}{4} (Q_{1324} + Q_{1423}) \\
-12Q_{1324} - 12Q_{1423} & -P + 3Q_{1324} - 3Q_{1423}
\end{pmatrix}, \quad (E.65)
\]

\[
Z_{QE}^{(1,1)} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}. \quad (E.66)
\]

\[
\left(Z_{EQ}^{(1,0)}\right)_{11} = \frac{1}{8} \left[ 32d_1 - d_3 - 10e_2 - 21f_2 \right] Q_{1234} + \left( -3d_2 - 10e_1 + 9f_1 \right) Q_{1324} + \left( 4d_3 - 8d_1 \right) Q_{1423}, \quad (E.67)
\]

\[
\left(Z_{EQ}^{(1,0)}\right)_{12} = \frac{1}{32} \left[ (-d_3 - 10e_2 + 7f_2) Q_{1234} + (d_2 - 10e_1 - 3f_1) Q_{1324} - (8d_1) Q_{1423} \right], \quad (E.68)
\]

\[
\left(Z_{EQ}^{(1,0)}\right)_{21} = \frac{1}{2} \left[ (-3d_3 + 10e_2 + 21f_2) Q_{1234} + (3d_2 + 50e_1 - 9f_1) Q_{1324} + (-24d_1 + 40e_2) Q_{1423} \right], \quad (E.69)
\]

\[
\left(Z_{EQ}^{(1,0)}\right)_{22} = \frac{1}{8} \left[ (-3d_3 + 10e_2 - 7f_2) Q_{1234} + (3d_2 + 10e_1 - 9f_1) Q_{1324} + (28f_2 - 24d_1) Q_{1423} \right]. \quad (E.70)
\]

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

[1] M. Fierz, *Force-free particles with any spin*, Helv. Phys. Acta **12** (1939) 3 [nSPIRE].

[2] A.J. Buras and P.H. Weisz, *QCD Nonleading Corrections to Weak Decays in Dimensional Regularization and 't Hooft-Veltman Schemes*, Nucl. Phys. B **333** (1990) 66 [nSPIRE].

[3] M.J. Dugan and B. Grinstein, *On the vanishing of evanescent operators*, Phys. Lett. B **256** (1991) 239 [nSPIRE].
[4] S. Herrlich and U. Nierste, *Evanescent operators, scheme dependences and double insertions*, Nucl. Phys. B **455** (1995) 39 [hep-ph/9412375] [inSPIRE].

[5] B. Grzadkowski, M. Iskrzynski, M. Misiak and J. Rosiek, *Dimension-Six Terms in the Standard Model Lagrangian*, JHEP **10** (2010) 085 [arXiv:1008.4884] [inSPIRE].

[6] J. Aebischer, A. Crivellin, M. Fael and C. Greub, *Matching of gauge invariant dimension-six operators for $b \to s$ and $b \to c$ transitions*, JHEP **05** (2016) 037 [arXiv:1512.02830] [inSPIRE].

[7] E.E. Jenkins, A.V. Manohar and P. Stoffer, *Low-Energy Effective Field Theory below the Electroweak Scale: Operators and Matching*, JHEP **03** (2018) 016 [arXiv:1709.04486] [inSPIRE].

[8] W. Dekens and P. Stoffer, *Low-energy effective field theory below the electroweak scale: matching at one loop*, JHEP **10** (2019) 197 [arXiv:1908.05295] [inSPIRE].

[9] R. Alonso, E.E. Jenkins, A.V. Manohar and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators III: Gauge Coupling Dependence and Phenomenology*, JHEP **04** (2014) 159 [arXiv:1312.2014] [inSPIRE].

[10] E.E. Jenkins, A.V. Manohar and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators I: Formalism and lambda Dependence*, JHEP **10** (2013) 087 [arXiv:1308.2627] [inSPIRE].

[11] E.E. Jenkins, A.V. Manohar and M. Trott, *Renormalization Group Evolution of the Standard Model Dimension Six Operators II: Yukawa Dependence*, JHEP **01** (2014) 035 [arXiv:1310.4838] [inSPIRE].

[12] E.E. Jenkins, A.V. Manohar and P. Stoffer, *Low-Energy Effective Field Theory below the Electroweak Scale: Anomalous Dimensions*, JHEP **01** (2018) 084 [arXiv:1711.05270] [inSPIRE].

[13] J. Aebischer, M. Fael, C. Greub and J. Virto, *B physics Beyond the Standard Model at One Loop: Complete Renormalization Group Evolution below the Electroweak Scale*, JHEP **09** (2017) 158 [arXiv:1704.06639] [inSPIRE].

[14] N. Tracas and N. Vlachos, *Two Loop Calculations in QCD and the $\Delta I = 1/2$ Rule in Nonleptonic Weak Decays*, Phys. Lett. B **115** (1982) 419 [inSPIRE].

[15] H. Bélusca-Maïto, A. Iškrovac, M. Mador-Božinović and D. Stöckinger, *Dimensional regularization and Breitenlohner-Maison/'t Hooft-Veltman scheme for $\gamma_5$ applied to chiral YM theories: full one-loop counterterm and RGE structure*, JHEP **08** (2020) 024 [arXiv:2004.14398] [inSPIRE].

[16] A.J. Buras, M. Misiak and J. Urban, *Two loop QCD anomalous dimensions of flavor changing four quark operators within and beyond the standard model*, Nucl. Phys. B **586** (2000) 397 [hep-ph/0005183] [inSPIRE].

[17] A.J. Buras, M. Jamin and M.E. Lautenbacher, *Two loop anomalous dimension matrix for Delta S = 1 weak nonleptonic decays. 2. O(alpha-alpha_s)*, Nucl. Phys. B **400** (1993) 75 [hep-ph/9211321] [inSPIRE].

[18] M. Ciuchini, E. Franco, G. Martinelli and L. Reina, *The Delta S = 1 effective Hamiltonian including next-to-leading order QCD and QED corrections*, Nucl. Phys. B **415** (1994) 403 [hep-ph/9304287] [inSPIRE].
[19] F.J. Gilman and M.B. Wise, Effective Hamiltonian for Delta s = 1 Weak Nonleptonic Decays in the Six Quark Model, Phys. Rev. D 20 (1979) 2392 [inSPIRE].

[20] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nonleptonic Decays of K Mesons and Hyperons, Sov. Phys. JETP 45 (1977) 670 [inSPIRE].

[21] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Effective Hamiltonians for \( \Delta s = 1 \) and \( \Delta B = 1 \) nonleptonic decays beyond the leading logarithmic approximation, Nucl. Phys. B 370 (1992) 69 [Addendum ibid. 375 (1992) 501] [inSPIRE].

[22] A.J. Buras, M. Jamin, M.E. Lautenbacher and P.H. Weisz, Two loop anomalous dimension matrix for \( \Delta S = 1 \) weak nonleptonic decays I: \( \mathcal{O}(\alpha_s) \), Nucl. Phys. B 400 (1993) 37 [hep-ph/9211304] [inSPIRE].

[23] J. Aebischer, A.J. Buras and J. Kumar, Simple Rules for Evanescent Operators in One-Loop Basis Transformations, arXiv:2202.01225 [inSPIRE].

[24] J. Aebischer, C. Bobeth, A.J. Buras, J. Kumar and M. Misiak, General non-leptonic \( \Delta F = 1 \) WET at the NLO in QCD, JHEP 11 (2021) 227 [arXiv:2107.10262] [inSPIRE].

[25] J. Aebischer, C. Bobeth, A.J. Buras and J. Kumar, BSM master formula for \( \epsilon'/\epsilon \) in the WET basis at NLO in QCD, JHEP 12 (2021) 043 [arXiv:2107.12391] [inSPIRE].

[26] J. Aebischer, A.J. Buras and J. Kumar, SMEFT ATLAS of \( \Delta F = 2 \) transitions in the SMEFT, Phys. Rev. D 106 (2022) 035003 [arXiv:2203.11224] [inSPIRE].

[27] K.G. Chetyrkin, M. Misiak and M. Münnz, \( |\Delta F| = 1 \) nonleptonic effective Hamiltonian in a simpler scheme, Nucl. Phys. B 520 (1998) 279 [hep-ph/9711280] [inSPIRE].

[28] M. Gorbahn and U. Haisch, Effective Hamiltonian for non-leptonic \( |\Delta F| = 1 \) decays at NNLO in QCD, Nucl. Phys. B 713 (2005) 291 [hep-ph/0411071] [inSPIRE].

[29] J. Aebischer, C. Bobeth, A.J. Buras and J. Kumar, Complete NLO QCD Corrections for Tree Level \( \Delta F = 2 \) FCNC Processes, JHEP 03 (2012) 052 [arXiv:1201.1302] [inSPIRE].

[30] J. Aebischer, C. Bobeth, A.J. Buras and J. Kumar, SMEFT ATLAS of \( \Delta F = 2 \) transitions, JHEP 12 (2020) 187 [arXiv:2009.07276] [inSPIRE].

[31] J. Aebischer, A. Crivellin and C. Greub, QCD improved matching for semileptonic B decays with leptoquarks, Phys. Rev. D 99 (2019) 055002 [arXiv:1811.08907] [inSPIRE].

[32] M. Gorbahn, S. Jager, U. Nierste and S. Trine, The supersymmetric Higgs sector and \( B -\bar{B} \) mixing for large tan \( \beta \), Phys. Rev. D 84 (2011) 034030 [arXiv:0901.2065] [inSPIRE].

[33] A. Carmona, A. Lazopoulos, P. Olgoso and J. Santiago, Matchmaker\textit{e}: automated tree-level and one-loop matching, SciPost Phys. 12 (2022) 198 [arXiv:2112.10787] [inSPIRE].

[34] J. Aebischer et al., WCxf: an exchange format for Wilson coefficients beyond the Standard Model, Comput. Phys. Commun. 232 (2018) 71 [arXiv:1712.05298] [inSPIRE].

[35] J. Aebischer, J. Kumar and D.M. Straub, Wilson: a Python package for the running and matching of Wilson coefficients above and below the electroweak scale, Eur. Phys. J. C 78 (2018) 1026 [arXiv:1804.05033] [inSPIRE].

[36] J. Aebischer, M. Fael, A. Lenz, M. Spannowsky and J. Virto eds., Computing Tools for the SMEFT, (2019) [inSPIRE].