Large volume quantum correction in loop quantum cosmology: Graviton illusion?

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The leading quantum correction to Einstein-Hilbert Hamiltonian coming from large volume vacuum isotropic loop quantum cosmology, is independent of quantization ambiguity parameters. It is shown here that this correction can be viewed as finite volume gravitational Casimir energy due to one-loop ‘graviton’ contributions. In vacuum case sub-leading quantum corrections and in non-vacuum case even leading quantum correction depend on ambiguity parameters. It may be recalled that these are in fact analogous features of perturbative quantum gravity where it is well-known that pure gravity (on-shell) is one-loop finite whereas higher-loops contributions are not even renormalizable. These features of the quantum corrections coming from non-perturbative quantization, sheds a new light on a major open issue; how to communicate between non-perturbative and perturbative approaches of quantum gravity.

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The Standard Model of particle physics, a description of matter fields based on perturbative quantum field theory, has been shown to be one of the most predictive physical theory ever constructed. The physical predictions of this theory have been verified experimentally with an outstanding accuracy. Unfortunately, the techniques of perturbative quantum field theory when applied to theory of gravity, fail quite miserably. Thus, in a quest for a quantum theory of gravity one is being compelled to take different courses. In the most popular applications of this theory have been verified experimentally physical theory ever constructed. The physical predictions of this theory have been shown to be one of the most predictive theories of quantum gravity known as loop quantum gravity (LQG). It is a quantization of cosmological (homogeneous) models using techniques of loop quantum gravity. Thanks to its simplicity, it allows explicit calculation to study its consequences. The loop quantum cosmology so far has lead to several impressive results. It has been shown that loop quantum cosmology cures the problem of classical singularity, along with quantum suppression of classical chaotic behaviour near singularities in Bianchi-IX models. Further, it has been shown that this model has a in-built generic phase of inflation. The corresponding power spectrum of density perturbation contains a distinguishing feature. However, the issues related with physical observables, external time evolution, physical Hilbert space are still in nascent stage. Nevertheless, it is possible to derive an effective Hamiltonian using WKB techniques.

We consider here spatially flat isotropic loop quantum cosmology, as we are interested in vacuum solution of it. The spatially closed model does not have vacuum solution. In loop quantum cosmology, a kinematical state is written as $|s\rangle = \sum s_\mu|\mu\rangle$, where $|\mu\rangle$’s are eigenstates of volume (densitized triad) operator. It is important to emphasize the meaning of volume in this context. In particular, the volume $V = \int d^3x\sqrt{-g}$ of the space is infinite, as it is non-compact. To avoid this trivial divergence in loop quantum cosmology, one considers the volume of a finite cell of universe (see Fig 1) and studies its evolution. This feature plays the central role in the arguments presented here. In loop quantum cosmology, the underlying (internal time) dynamics is described by a difference equation. This discrete evolution faithfully represents the underlying discrete geometry, a feature of full theory of loop quantum gravity. In the effective description of loop quantum cosmology, one tries to understand the dynamics from a perspective based on continuum geometry. In other words, one tries to approximate the fundamentally discrete dynamics by a continuum dynamics. In this process, the discrete dynamics effectively provides a new potential term in the continuum description. This feature can be seen rather easily by considering the gravitational term in the difference equation $A_{\mu+4\rho_0}s_{\mu+4\rho_0} - 2A_{\mu}s_{\rho_0} + A_{\mu-4\rho_0}s_{\mu-4\rho_0}$, where $A_{\mu} = |\mu + \rho_0|^{3/2} - |\mu - \rho_0|^{3/2}$. In deriving the effective Hamiltonian, in first step one approximates the solution of the difference equation $s_{\mu}$, by a smooth (differentiable) function say $\psi(p := \gamma|\mu\rangle^2/6)$. Using WKB approximation, one derives a Hamilton-Jacobi equation from it. The corresponding Hamiltonian then contains an effective potential term, referred as quantum geometry potential, which is $(l_p^2 p_\rho^3/288)(A_{\rho+4\rho_0} - 2A_{\rho} + A_{\rho-4\rho_0})$. Thus, the quantum geometry potential term arises when one tries to view a fundamentally discrete dynamics through a continuum description. In effective description, the quantum geometry potential leads to a generic
bounce \cite{13}. In large volume \(V (= p^{3/2}, \ p \ \text{is p density}}\) and small extrinsic curvature \(K\) (conjugate variable of \(p\)) regime, the gravitational part of the effective Hamiltonian \cite{11,12} can be expanded as

\[
H_{\text{eff}}^{\text{grav}} = H_{\text{EH}} + \frac{l_p^2}{p} \left[ -\frac{1}{24} p^{5/2} + \frac{2\mu_0^2\gamma^2}{9\sqrt{p}} H_{\text{EH}} \right] - \left( \frac{49 \mu_0^2\gamma^2}{864 \ p^2} H_{\text{EH}} \right) - \left( \frac{5 \mu_0^2\gamma^2}{708 \ p^{7/2}} + \ldots \right) + \ldots, \tag{1}
\]

where \(H_{\text{EH}} = -(3/2\kappa)\sqrt{p} K^2\), is the Einstein-Hilbert Hamiltonian for the homogeneous and isotropic spacetime. In natural units \((c = \hbar = 1)\), \(\kappa = 16\pi G = l_p^2\), is the gravitational coupling constant. \(\gamma\) is the Barbero-Immirzi parameter. \(\mu_0\) here is viewed as a quantization ambiguity parameter. \(\mu_0\) appears as the length of the edges, while expressing curvature tensor in terms of holonomies around a square. It essentially plays the role of a regulator \cite{13}. Both of these parameters are generally assumed to be order of unity numbers but there is no unique way to fix their values within loop quantum cosmology itself. The accuracy of WKB approximation here increases with increasing volume. So in large volume (scale set by the step-size of the difference equation), the effective Hamiltonian \cite{11} is quite trustworthy.

Let’s now consider the pure gravity case i.e without any matter field. The Einstein-Hilbert Hamiltonian \(H_{\text{EH}}\) vanishes (on-shell) for pure gravity. It is then clear from the expression \cite{11} that leading quantum correction is independent of the parameters \(\mu_0\) and \(\gamma\). This quantum correction comes solely from the quantum geometry potential. In the volume goes to infinity limit this quantum correction vanishes. We will refer this term as gravitational Casimir energy. Later we will show that this term can indeed be viewed as gravitational Casimir energy due to the finite volume of the system. We now re-write the term as

\[
E_{\text{Cas}}^{\text{grav}} = -\frac{1}{24} \frac{l_p^2}{d^2}, \tag{2}
\]

where the volume \(d^3 = p^{3/2}\).

Traditionally, one computes Casimir energy by computing the shift of vacuum polarisation energy due to imposition of an external boundary condition. In particular, the quantum electrodynamic Casimir energy between two conducting plates of surface area \(A\) and separated by a distance \(d\), is \(- (\pi^2/240)(A/d^3)\). Surprisingly, this expression does not have explicit dependence on fine structure constant. Although, in principle possible but in reality there is no conductor that can enforce such boundary conditions for modes of all wavelength. On the other hand, the experimental result seems to agree with the traditional expression extremely well \cite{13}. Thus, reconciling these two facts may appear as a conceptually difficult task. However, in recent approach of computing Casimir energy \cite{13,14}, instead of imposing boundary condition, one considers the plates as a classical static background field. Then one introduces an interaction of the type \(-C_{\text{int}} = \lambda \sigma(x)\phi^2(x)\) where \(\sigma(x)\) is classical (non-dynamical) background field and \(\phi(x)\) is the dynamical field whose vacuum fluctuations contribute to Casimir energy. The background field \(\sigma(x)\) is represented by delta functions peaked around the positions of the conducting plates. Using the techniques of perturbative quantum field theory, one computes order by order contributions of this interaction. It is possible to sum up all order contributions to give a ‘close’ form expression for the Casimir energy. In strong coupling \((\lambda \rightarrow \infty)\) limit, the explicit dependence of coupling constant drops out from this expression and it reduces exactly to the traditional expression of Casimir energy. The real experimental system that one considers for measuring Casimir force, the fine structure constant effectively appears as strong coupling \(\lambda\). Thus, the recent approach of computing Casimir energy addresses the mentioned conceptual difficulty quite well. On the other hand in weak coupling \((\lambda \rightarrow 0)\) limit, the Casimir energy computed using this method, scales as \(~ (\lambda^2/d)\). It is essentially the contribution from one-loop diagram with two-point insertion.

The expression of effective Hamiltonian \cite{11} is valid in large volume \((d^2 >> l_p^2)\) regime. Naturally in the regime of interest, the gravitational coupling constant \(l_p\) is very weak. We have already mentioned that in isotropic loop quantum cosmology, one essentially studies the evolution of a finite cell of the universe. So one can expect to get finite volume Casimir energy due to quantum fluctuations of geometry. The homogeneous and isotropic vacuum solution of Einstein equation is Minkowski spacetime. Since we are considering vacuum isotropic loop quantum cosmology, the system is essentially a finite patch of the Minkowski spacetime. This feature makes it conceptually easier to use the techniques of perturbative quantum field theory.

To compute the gravitational Casimir energy for the system i.e. a finite cell of the universe, here we use the recent method. Essential difference in this case is that one should consider the vacuum fluctuations of spin 2 field. Also, instead of one pair of boundary, here one has three pairs of boundary, one pair in each spatial direction. For simplicity, however we will compute Casimir energy due to a massless spin 0 field (massless Klein-Gordon field). The computational scheme can be extended for the spin 2 field as well. The result will differ by a numerical factor of 2 because of its two helicities. So to study the qualitative behaviour (as it is not expected to have quantitative match; in loop quantum cosmology one consider only the temporal fluctuations of geometry. Imposition of high symmetry essentially freezes the spatial fluctuations.), use of spin 0 field suffices.

We consider the background field \(\sigma(x)\) to be represented by three dimensional delta functions, as there are
boundaries along all three spatial directions. We ‘normalize’ non-dynamical background field \( \sigma(x) \) as

\[
\int d^3 \bar{x} \sigma(x) := \int d^3 \bar{x} \frac{1}{2} [\delta^3(\bar{x} - \frac{\bar{d}}{2}) + \delta^3(\bar{x} + \frac{\bar{d}}{2})] = 1 , \tag{3}
\]

where \( \bar{d} = \{d,d,d\} \). It is worth pointing out that a different ‘normalization’ essentially alters the coupling constant \( \lambda \). In this approach, the Casimir energy is read off from the one-loop effective action computed using the background field method \( [17] \). Naturally, the Casimir energy is defined as

\[
\int dt \ E_{\text{Cas}}(\sigma) := -\frac{i}{2} \ log \det \left[ -\delta^3 \mathcal{L} \right] , \tag{4}
\]

where \( \mathcal{L} \) is the full interacting Lagrangian. The functional determinant can be expressed in terms of Feynman diagrams. Being independent of \( d \), the one-loop diagram with one-point insertion does not contribute to Casimir force which is a physically measurable quantity. The contribution from one-loop diagram with two-point insertion (see Fig.1) can be computed in a straightforward manner, leading to

\[
E_{\text{Cas}}[\sigma] = -\frac{1}{2} \frac{\alpha^2 \lambda^2}{d^3} , \tag{5}
\]

where \( \alpha^2 = (\sqrt{3}/2\pi)^{-1} \). In the calculation, the \( d \) independent contributions (formally divergent) have been dropped, as they do not contribute to Casimir force.

Before we compare the expression \( \mathcal{L} \) of Casimir energy computed using perturbative quantum field theory, with the expression \( \mathcal{L} \) extracted from isotropic loop quantum cosmology, a caution is appropriate. In strong coupling limit the method used here gives unambiguous expression for Casimir energy as coupling constant dependence drops out. However in weak coupling limit, it depends on the choice of ‘normalization’ \( \mathcal{L} \) which needs to be provided from outside. Also, the contribution from isotropic loop quantum cosmology itself may not account for the full gravitational Casimir energy. So here we restrict to the qualitative comparison of these two expressions. Comparing the expressions \( \mathcal{L} \) and \( \mathcal{L} \), it is clear that the expression \( \mathcal{L} \) can indeed be viewed as contributions from virtual quanta whose coupling strength is Planck constant \( \hbar \). So we refer these quanta as ‘gravitons’. However, due to the ‘normalization’ uncertainty involved in the perturbative method used here, it is not yet possible to conclude definitively about the spin degrees of freedom of these quanta.

Let’s now go back to the expression \( \mathcal{L} \) of the effective Hamiltonian. In vacuum case, the leading quantum correction is unambiguous. However, the sub-leading corrections depends on ambiguity parameters. With inclusion of matter, the Einstein-Hilbert Hamiltonian \( H_{\text{EH}} \) does not vanish. Clearly, the leading quantum correction then also becomes ambiguous. There will also be contributions from the direct coupling of matters with gravity. It is important to observe that these features of large volume quantum corrections coming from non-perturbative quantization, in fact closely resemble the features of perturbative quantum gravity. It is well-known through the work of ‘t Hooft and Veltman \( [18] \) that pure gravity (on-shell) is one-loop (order \( \hbar^2 \) finite. In other words, one-loop contributions from perturbative quantum gravity without matter, is unambiguous. However, higher-loops contributions from pure gravity are not even renormalizable i.e. it is not possible to obtain unambiguous results from such computations. With inclusion of matter, perturbative quantum gravity is not even one-loop renormalizable. Now, as we have already mentioned that a severe criticism that often haunts the advocates of non-perturbative quantum gravity, is its relation to the ‘low energy’ world. Although for symmetric models as shown here, the quantum cosmology based on loop quantum gravity not only reproduces the Einstein-Hilbert Hamiltonian as the leading term but also its quantum corrections resemble the qualitative features of perturbative quantum gravity in the regime where the later should be a reasonable effective description.

In computing Casimir energy using perturbative quantum field theory, the interaction term was introduced rather by hand. We now argue that the form of the interaction used in the calculation arises quite naturally. The gravitational Lagrangian involves term of the form \( g(x)g(x)\partial g(x)\partial g(x) \). In the background field method of quantum field theory, one expands the field \( g(x) \) around a given classical background say \( \eta(x) \); \( g(x) = \eta(x) + h(x) \), where \( h(x) \) is the fluctuating field. Inserting the decomposition into the gravitational Lagrangian, it is easy to see that it contains a term of the form \( \sigma(x)h(x)h(x) \). The \( \sigma(x) \sim \partial \eta(x)\partial \eta(x) \), can indeed be treated as a classical background field. For small extrinsic curvature regime, one can simply consider background \( \eta(x) \) as static while computing perturbative corrections. For a finite cell of the universe, the use of delta function potential peaked
around the boundaries is also well-motivated. For example, one crude way to make the volume of flat space finite, is by multiplying the metric component with a Heaviside step function say \( \theta(\mu - |\mu|) \) where \( \mu = \{ \infty, \tilde{d} \} \). It is easy to see that the background field \( \sigma(x) \) then involves delta functions peaked around boundaries. However, the mentioned term need not be the only boundary interaction term that can contribute to the Casimir energy. So, it is necessary to perform a ‘first principle’ computation of gravitational Casimir energy for the system. It may also help to eventually settle the issue of spin degrees of freedom through quantitative comparison or at least to specify what to expect from a computation using the full theory of loop quantum gravity.

To summarize, the leading quantum correction to Einstein-Hilbert Hamiltonian coming from vacuum isotropic loop quantum cosmology is unambiguous and can be viewed as gravitational Casimir energy due to one-loop ‘graviton’ contributions. However, based on arguments presented here, it is not yet possible to conclude definitively about the spin degrees of freedom of these quanta. The sub-leading quantum corrections depend on quantization ambiguity parameters. In non-vacuum case even leading quantum correction depends on ambiguity parameters. Importantly, these are analogous features of perturbative quantum gravity. In other words, the quantum corrections coming from loop quantum cosmology whose quantization relies on non-perturbative techniques, closely resemble the qualitative features of perturbative quantum gravity in the regime where the later should be a reasonable effective description.

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[1] M.B. Green, J.H. Schwarz and E. Witten, “Superstring theory”, vol 1 and 2, Cambridge Univ. Press,(1987); J. Polchinski, “String theory”, vol 1 and 2, Cambridge Univ. Press (1998).

[2] C. Rovelli, Living Rev. Rel. 1, 1 (1998) [arXiv:gr-qc/9710008]. T. Thiemann, [arXiv:gr-qc/0110034]. T. Thiemann, Lect. Notes Phys. 631, 41 (2003) [arXiv:gr-qc/0210094]. A. Ashtekar and J. Lewandowski, Class. Quant. Grav. 21, R53 (2004) [arXiv:gr-qc/0404018].

[3] M. Varadarajan, Phys. Rev. D 66, 024017 (2002) [arXiv:gr-qc/0204067]. M. Varadarajan, Class. Quant. Grav. 22, 1207 (2005) [arXiv:gr-qc/0401120].

[4] M. Bojowald, Class. Quant. Grav. 17, 1489 (2000) [arXiv:gr-qc/9910103]. M. Bojowald, Class. Quant. Grav. 17, 1509 (2000) [arXiv:gr-qc/9910104]. M. Bojowald, Class. Quant. Grav. 18, 1055 (2001) [arXiv:gr-qc/0008052]. M. Bojowald, Class. Quant. Grav. 18, 1071 (2001) [arXiv:gr-qc/0008053]. M. Bojowald, Class. Quant. Grav. 19, 2717 (2002) [arXiv:gr-qc/0202077]. M. Bojowald, Class. Quant. Grav. 18, L109 (2001) [arXiv:gr-qc/0105113]. M. Bojowald and H.A. Morales-Tecotl, Lect. Notes Phys. 646, 421 (2004) [arXiv:gr-qc/0306008]. M. Bojowald, [arXiv:gr-qc/0503020].

[5] A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. Phys. 7, 233 (2003) [arXiv:gr-qc/0304074].

[6] M. Bojowald, Phys. Rev. Lett. 86, 5227 (2001) [arXiv:gr-qc/0102060].

[7] M. Bojowald and G. Date, Phys. Rev. Lett. 92, 071302 (2004) [arXiv:gr-qc/0311003]. M. Bojowald, G. Date and M. G. Hossain, Class. Quant. Grav. 21, 3541 (2004) [arXiv:gr-qc/0404039].

[8] M. Bojowald, Phys. Rev. Lett. 89, 261301 (2002) [arXiv:gr-qc/0206054]. G. Date and M. G. Hossain, Phys. Rev. Lett. 94, 011301 (2005) [arXiv:gr-qc/0407069].

[9] M. G. Hossain, [arXiv:gr-qc/0411012]. M. G. Hossain, [arXiv:gr-qc/0503065].

[10] M. G. Hossain, Class. Quant. Grav. 21, 179 (2004) [arXiv:gr-qc/0308014]. M. Bojowald, P. Singh and A. Skirzewski, Phys. Rev. D 70, 124022 (2004) [arXiv:gr-qc/0408094]. K. Noui, A. Perez and K. Van-dersloot, [arXiv:gr-qc/0411039].

[11] G. Date and M. G. Hossain, “Effective Hamiltonian for isotropic loop quantum cosmology,” Class. Quant. Grav. 21, 4941 (2004) [arXiv:gr-qc/0407073].

[12] K. Banerjee and G. Date, “Discreteness corrections to the effective Hamiltonian of isotropic loop quantum cosmology,” Class. Quant. Grav. (to appear), [arXiv:gr-qc/0501102].

[13] G. Date and M. G. Hossain, Phys. Rev. Lett. 94, 011302 (2005) [arXiv:gr-qc/0407074].

[14] S. K. Lamoreaux, Phys. Rev. Lett. 78, 5 (1997).

[15] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, “Calculating vacuum energies in renormalizable quantum field theories: A new approach to the Casimir problem,” Nucl. Phys. B 645, 49 (2002) [arXiv:hep-th/0207120]. N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, O. Schroeder and H. Weigel, “The Dirichlet Casimir problem,” Nucl. Phys. B 677, 379 (2004) [arXiv:hep-th/0309130]. K. A. Milton, “The Casimir effect: Recent controversies and progress,” J. Phys. A 37, R209 (2004) [arXiv:hep-th/0406024].

[16] R. L. Jaffe, “The Casimir effect and the quantum vacuum,” [arXiv:hep-th/0503155].

[17] M.E. Peskin and D.V. Schroeder, “An introduction to Quantum Field Theory”, Addison-Wesley Publishing Company, (1995).

[18] G. ’t Hooft and M. J. G. Veltman, “One Loop Divergencies In The Theory Of Gravitation,” Annales Poincare Phys. Theor. A 20 (1974) 69; G. ’t Hooft, “Perturbative Quantum Gravity,” in Proceedings of the International School of Subnuclear Physics, Erice 2002, “From Quarks and Gluons to Quantum Gravity”, Subnuclear Series Vol. 40, ed. A. Zichichi, World Scientific, pp 249 - 269. Also available via [http://www.phys.uu.nl/~thooft/lectures/erice02.pdf](http://www.phys.uu.nl/~thooft/lectures/erice02.pdf)