A CHARACTERIZATION OF BRAIDED ENRICHED MONOIDAL CATEGORIES

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Abstract. We construct an equivalence between the 2-categories \( \mathcal{V} \text{MonCat}_G^A \) of rigid \( \mathcal{V} \)-monoidal categories for a braided monoidal category \( \mathcal{V} \) and \( \mathcal{V} \text{ModTens}_G^A \) of oplax braided functors from \( \mathcal{V} \) into the Drinfeld centers of ordinary rigid monoidal categories. The 1-cells in each are the respective lax monoidal functors, and the 2-cells are the respective monoidal natural transformations. Our proof also gives an equivalence in the case that we consider only strong monoidal 1-cells on both sides. The 2-categories \( \mathcal{V} \text{MonCat}_G^A \) and \( \mathcal{V} \text{ModTens}_G^A \) have \( G \)-graded analogues. We also get an equivalence of 2-categories between \( \mathcal{V} \)-module tensor category \( \mathcal{A} \), and \( \mathcal{G} \)-extensions of some fixed \( \mathcal{V} \)-monoidal category \( \mathcal{A} \), and \( \mathcal{G} \)-extensions of some fixed \( \mathcal{V} \)-module tensor category \( (\mathcal{A}, \mathcal{F}_A^Z) \).

1. Introduction

The theory of monoidal categories enriched in a symmetric monoidal category \( \mathcal{V} \) is well established [Str83, Kel05, Str05, GP18]. The articles ([MP19, MPP18, JMP19]) have developed the theory when \( \mathcal{V} \) need only be braided, without symmetry. Remark 5.2 of [JS93] outlines the idea of braided enriched categories and mentions that they form a 2-category. Braided enriched monoidal categories were classified in [MP19] in terms of strongly unital oplax braided monoidal functors from the enriching category \( \mathcal{V} \) into the Drinfeld center of an ordinary monoidal category:

**Theorem 1.1 ([MP19]).** Let \( \mathcal{V} \) be a braided monoidal category. There is a bijective correspondence

\[
\begin{array}{c}
\text{Rigid } \mathcal{V}\text{-monoidal categories } \mathcal{A}, \\
\text{satisfying that } x \to \mathcal{A}(1_{\mathcal{V}} \to x) \text{ admits a left adjoint}
\end{array}
\]

\[
\equiv 
\begin{array}{c}
\text{Pairs } (\mathcal{A}, \mathcal{F}_A^Z) \text{ with } \mathcal{A} \text{ a rigid monoidal category and } \mathcal{F}_A^Z : \\
\mathcal{V} \to Z(\mathcal{A}) \text{ braided oplax monoidal, such that } \mathcal{F}_A^Z \equiv R \text{ admits a right adjoint}
\end{array}
\]

where \( R : Z(\mathcal{A}) \to \mathcal{A} \) is the forgetful functor.

Note that we are writing composition from left to right, as in [MP19, MPP18], but reversed to the typical convention. From a \( \mathcal{V} \)-monoidal category \( \mathcal{A} \), we can take the underlying category \( \mathcal{A} \), which has the same objects as \( \mathcal{A} \) and hom sets \( \mathcal{A}(a \to b) \equiv \mathcal{V}(1_{\mathcal{V}} \to \mathcal{A}(a \to b)) \). The underlying category of \( \mathcal{A} \) is denoted by \( \mathcal{A}^\mathcal{V} \) in [MP19], but here we merely change the font; there are no ambiguous situations, and our focus is not on the distinction between \( \mathcal{A} \) and its underlying category \( \mathcal{A} \). As shown in [MP19], \( \mathcal{F}_A \) is taken to be the left adjoint of \( \mathcal{A}(1_{\mathcal{A}} \to -) \) and shown to lift to \( \mathcal{F}_A^Z : \mathcal{V} \to Z(\mathcal{A}) \). In the other direction, given \( (\mathcal{A}, \mathcal{F}_A^Z) \) we can construct a \( \mathcal{V} \)-monoidal category \( \mathcal{A} \) using the same objects as \( \mathcal{A} \), and hom objects defined via the Yoneda lemma by the natural isomorphisms

\[\mathcal{V}(v \to \mathcal{A}(a \to b)) \equiv \mathcal{A}(a \mathcal{F}_A(v) \to b),\]

which themselves follow from rigidity and the right adjoint of \( \mathcal{F} \). The article [MPP18] characterizes when the functor \( \mathcal{F}_A^Z \) is in fact strong monoidal, and also develops a notion of completeness for \( \mathcal{V} \)-monoidal categories. In the absence of monoidal structures, [MP19] states an equivalence of 2-categories; we go one categorical level higher to the case of \( \mathcal{V} \)-monoidal categories:

**Theorem 1.2.** Theorem 1.1 extends to an equivalence of 2-categories.

This theorem was announced in [JMP19]. A recent article, [KYZZ21], contains similar results with independent proofs. To start, we construct 2-categories with the sets from Theorem 1.1 as 0-cells as follows:

- The 1-cells on the left hand side are lax \( \mathcal{V} \)-monoidal functors.
- The 1-cells on the right are lax monoidal functors equipped with an extra action-coherence natural transformation, introduced in Section 3. The action-coherence natural transformation encodes the interaction between the functor \( \mathcal{F}_A^Z \), which is oplax, and the 1-cells, which are lax.
- On both sides, 2-cells are the corresponding monoidal natural transformations satisfying the appropriate coherences with the action coherence natural transformations.
We then construct a 2-functor between the 2-categories, proceeding as in [MP19] on 0-cells. The map on 2-cells turns out to be straightforward, so the bulk of our work goes into constructing the map on 1-cells between the two 2-categories, verifying that the maps together form a 2-functor, and showing that this 2-functor is a 2-equivalence. Starting with a lax \( \mathcal{V} \)-monoidal functor \( \mathcal{R} : \mathcal{A} \to \mathcal{B} \) between \( \mathcal{V} \)-monoidal categories, we get an ordinary lax monoidal functor by taking the underlying functor \( \mathcal{R} = R \), and we construct our action-coherence natural transformation \( r : F_B \to R \circ F_A \) via, for \( v \in \mathcal{V} \),

\[
\begin{array}{c}
\mathcal{B} (1_B \to R (F_A (v))) \\
\mathcal{R}_{1_B \to F_A (v)} \\
\eta_{F_A (v)}
\end{array}
\]

where the mate is taken under the adjunction

\[
\mathcal{B} (F_B (v) \to R (F_A (v))) \cong \mathcal{V} (v \to \mathcal{B} (1_B \to R (F_A (v))))
\]

Starting instead with a monoidal functor \( (R, \rho) \) between \( \mathcal{A} \) and \( \mathcal{B} \) in \( \mathcal{V} \text{-ModTens} \), with action-coherence \( r \), we construct a \( \mathcal{V} \)-monoidal functor \( R_{a \to b} \)

\[
\begin{array}{c}
\mathcal{B} (R (a) \to R (b)) \\
\mathcal{R}_{a \to b} \\
\mathcal{A} (a \to b)
\end{array}
\]

with mate taken under the adjunction

\[
\mathcal{V} (\mathcal{A} (a \to b) \to \mathcal{B} (R (a) \to R (b))) \cong \mathcal{B} (R (a) \mathcal{F}_B (\mathcal{A} (a \to b)) \to R (b))
\]

The laxitors for the 1-cells are the same on both sides, interpreted as elements of

\[
\mathcal{B} (R (a) R (b) \to R (a b)) = \mathcal{V} (1 \mathcal{V} \to \mathcal{B} (R (a) R (b) \to R (a b)))
\]

In [Lin76, MP19], Theorem 6.13 is addressed without any monoidal structure on the enriched categories by combining the \( (id_R) \circ \rho \) from the diagram above into a single coherence morphism to be included in the data of a 1-cell. Lastly, in [JMPP19] the authors introduce the notions of a \( G \)-grading and a \( G \)-extension of a \( \mathcal{V} \)-monoidal category for a finite group \( G \). In particular, they studied \( G \)-extensions of \( \mathcal{V} \)-fusion categories, which were also used by Kong and Zheng to present a unified theory of gapped and gapless edges for 2D topological orders [KZ18, KZ20, KZ21]. In Section 7 we introduce the necessary definitions and lift Theorem 1.2 to the \( G \)-graded setting. Then in Section 8 we go one step further to the case of \( G \)-extensions of \( \mathcal{V} \)-monoidal categories, and again lift Theorem 1.2.

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2. Background

Let \( \mathcal{V} \) be a braided monoidal category with braiding \( \beta \). We follow the notational conventions set in [MP19]; specifically, we suppress \( \otimes \) (in \( \mathcal{V} \)) and all associators and unitors in \( \mathcal{V} \), with compositions written explicitly as \( \circ \), and composition written from left to right.

2.1. \( \mathcal{V} \)-categories. A \( \mathcal{V} \)-category, or \( \mathcal{V} \)-enriched category, \( \mathcal{A} \) consists of a collection of objects along with:

- For each pair of objects \( a, b \in \mathcal{A} \), an associated hom object \( \mathcal{A} (a \to b) \in \mathcal{V} \),
- For each object \( a \in \mathcal{A} \), an identity element \( 1_a \in \mathcal{V} (1_{\mathcal{V}} \to \mathcal{A} (a \to a)) \), and
- For each triple of objects \( a, b, c \in \mathcal{A} \), a distinguished composition morphism

\[
\circ_{\mathcal{A}} \in \mathcal{V} (\mathcal{A} (a \to b) \mathcal{A} (b \to c) \to \mathcal{A} (a \to c))
\]

satisfying the axioms
• Unitality: For all \(a, b \in \mathcal{A}\),

\[
\mathcal{A}(a \to b) \to \mathcal{A}(a \to b)
\]

\[
\mathcal{A}(a \to b) \to \mathcal{A}(a \to b)
\]

• Associativity of \((- \circ -\)):

\[
\mathcal{A}(a \to d) \to \mathcal{A}(a \to d)
\]

\[
\mathcal{A}(a \to b) \to \mathcal{A}(b \to c) \to \mathcal{A}(c \to d)
\]

\[
\mathcal{A}(a \to b) \to \mathcal{A}(b \to c) \to \mathcal{A}(c \to d)
\]

For objects \(a, b\) in a \(\mathcal{V}\)-enriched category \(\mathcal{A}\) and \(v \in \mathcal{V}\), a \(v\)-graded morphism from \(a\) to \(b\) is a morphism \(f \in \mathcal{V}(v \to \mathcal{A}(a \to b))\). We will give particular attention to the case where \(v = 1\), the \(1\)-graded morphisms.

**Example 2.1 (Self-enrichment).** Given a closed monoidal category \(\mathcal{V}\), we can construct a \(\mathcal{V}\)-category \(\mathcal{V}\) by starting with the objects of \(\mathcal{V}\), then taking hom objects \(\mathcal{V}(u \to v) = [u, v]\), the internal hom in \(\mathcal{V}\) defined via \(\mathcal{V}(u \to v) \to \mathcal{V}(w \to \mathcal{V}(u \to v))\).

Under this identity adjunction, we also define \(j_v \) := mate \((\text{id}_u)\) and

\[
- \circ - : \mathcal{V}(u \to v) \to \mathcal{V}(w \to \mathcal{V}(u \to v))
\]

\[
\mathcal{V}(u \to v) \to \mathcal{V}(w \to \mathcal{V}(u \to v))
\]

**Definition 2.2.** Given a \(\mathcal{V}\)-category \(\mathcal{A}\), we can construct an ordinary category \(\mathcal{A}^\mathcal{V}\), the *underlying category* of \(\mathcal{A}\), by giving \(\mathcal{A}^\mathcal{V}\) the same objects as \(\mathcal{A}\), defining \(\mathcal{A}^\mathcal{V}(a \to b) := \mathcal{V}(1 \to \mathcal{A}(a \to b))\) for each \(a, b \in \mathcal{A}^\mathcal{V}\), and composition via \(f \circ g = (fg) \circ (- \circ -\)), for \(f \in \mathcal{A}^\mathcal{V}(a \to b)\) and \(g \in \mathcal{A}^\mathcal{V}(b \to c)\). In the following we will often denote the underlying category of a \(\mathcal{V}\)-category \(\mathcal{A}\) by a standard font \(\mathcal{A}\).

2.2. \(\mathcal{V}\)-functors and natural transformations. A \(\mathcal{V}\)-functor \(\mathcal{R} : \mathcal{A} \to \mathcal{B}\) between \(\mathcal{V}\)-categories consists of a function on objects and for each \(a, b \in \mathcal{A}\) a morphism \(\mathcal{R}_{a \to b} \in \mathcal{V}(\mathcal{R}(a) \to \mathcal{R}(b))\), satisfying the following axioms:

• Functoriality: For all \(a, b \in \mathcal{A}\),

\[
\mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(c)) \to \mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(b))
\]

\[
\mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(c)) \to \mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(b))
\]

• Unit preserving: For all \(a \in \mathcal{A}\),

\[
\mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(a)) \to \mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(a))
\]

\[
\mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(a)) \to \mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(a))
\]
We can compose two \( \mathcal{V} \)-functors \( \mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C} \) via \( (\mathcal{R} \circ \mathcal{J}) (a) = \mathcal{J} (\mathcal{R} (a)) \) and \( (\mathcal{R} \circ \mathcal{J})_{a \rightarrow b} = \mathcal{R}_{a \rightarrow b} \circ \mathcal{J}_{\mathcal{R}(a) \rightarrow \mathcal{R}(b)} \) for all objects \( a, b \in \mathcal{A} \). For \( \mathcal{V} \)-functors \( \mathcal{R}, \mathcal{J} : \mathcal{A} \rightarrow \mathcal{B} \), we define a \( 1_{\mathcal{V}} \)-graded natural transformation \( \theta : \mathcal{R} \Rightarrow \mathcal{J} \) by assigning to each object \( a \in \mathcal{A} \) a \( 1_{\mathcal{V}} \)-graded morphism \( \theta_{a} \in \mathcal{V} (1_{\mathcal{V}} \rightarrow \mathcal{B} (\mathcal{R} (a) \rightarrow \mathcal{J} (a))) \) such that for all \( a, b \in \mathcal{A} \),

\[
\mathcal{B} (\mathcal{R} (a) \rightarrow \mathcal{J} (b)) = \mathcal{B} (\mathcal{R} (a) \rightarrow \mathcal{J} (b))
\]

This condition is the usual naturality square, but written in terms of composition of \( 1_{\mathcal{V}} \)-graded morphisms.

**Definition 2.3.** Given a \( \mathcal{V} \)-functor \( \mathcal{R} : \mathcal{A} \rightarrow \mathcal{B} \) between \( \mathcal{V} \)-categories, we construct the underlying functor of \( \mathcal{R} \), an ordinary functor \( \mathcal{R}^{V} : \mathcal{A} \rightarrow \mathcal{B} \) between the underlying categories. We take \( \mathcal{R}^{V} (a) = \mathcal{R} (a) \) for each \( a \in \mathcal{A} \), and on morphisms \( f \in \mathcal{A} (a \rightarrow b) = \mathcal{V} (1_{\mathcal{V}} \rightarrow \mathcal{A} (a \rightarrow b)) \), we define \( \mathcal{R} (f) := f \circ \mathcal{R}_{a \rightarrow b} \in \mathcal{B} (\mathcal{R} (a) \rightarrow \mathcal{R} (b)) = \mathcal{V} (1_{\mathcal{V}} \rightarrow \mathcal{B} (\mathcal{R} (a) \rightarrow \mathcal{R} (b))) \). As with the underlying category, we will often denote \( \mathcal{R}^{V} \) by simply \( \mathcal{R} \).

**2.3. \( \mathcal{V} \)-Monoidal Categories.** Given a \( \mathcal{V} \)-category, we can add a monoidal structure to get a (strict) \( \mathcal{V} \)-monoidal category. We need the additional data of

- A unit object \( 1_{\mathcal{A}} \),
- For each pair \( a, b \in \mathcal{A} \), a tensor product object \( ab \in \mathcal{A} \), and
- For each \( a, b, c, d \in \mathcal{A} \), a tensor product morphism \( - \otimes_{\mathcal{A}} - \in \mathcal{V} (\mathcal{A} (a \rightarrow c) \mathcal{A} (b \rightarrow d) \rightarrow \mathcal{A} (ab \rightarrow cd)) \)

such that the following axioms hold:

- Tensor unit: For all \( a \in \mathcal{A} \), \( 1_{\mathcal{A}} a = a 1_{\mathcal{A}} = a \),
- Associative on objects: For all \( a, b, c \in \mathcal{A} \), \( (ab)c = a(bc) \),
- Unitality: For all \( a, b \in \mathcal{A} \),

\[
\mathcal{A} (a \rightarrow b) = \mathcal{A} (a \rightarrow b)
\]

- Associativity of \( (- \otimes -) \): For all \( a, b, c, d, e, f \in \mathcal{A} \),

\[
\mathcal{A} (abc \rightarrow def) \]

- Braided interchange: For all \( a, b, c, d, e, f \in \mathcal{A} \),

\[
\mathcal{A} (ad \rightarrow cf) = \mathcal{A} (ad \rightarrow cf)
\]
Definition 2.4. Given a $\mathcal{V}$-monoidal category $\mathcal{A}$, take the underlying category $A = \mathcal{A}^\mathcal{V}$. Then $A$ inherits the monoidal structure of $\mathcal{A}$ via the same tensor product on objects, and for $f \in A(a \rightarrow c)$ and $g \in A(b \rightarrow d)$, $fg := (fg) \circ (- \otimes -)$, where on the right hand side $f$ and $g$ are viewed as morphisms in $\mathcal{V}$. This is the underlying monoidal category of $\mathcal{A}$.

Example 2.5 (Monoidal self-enrichment). As before, when $\mathcal{V}$ is closed we can form the $\mathcal{V}$-category $\widehat{\mathcal{V}}$. We now additionally define, for $u, v, w, x \in \mathcal{V}$,

$$-\otimes_\mathcal{V} - := \text{mate}$$

$$\begin{pmatrix}
\begin{array}{c}
u \\ \psi \\
\ast \\
v \\
w \\
v \rightarrow v \\
w \rightarrow w \\
w \rightarrow x \\
\end{array}
\end{pmatrix}$$

to give $\widehat{\mathcal{V}}$ the structure of a $\mathcal{V}$-monoidal category.

2.4. $\mathcal{V}$-monoidal functors and natural transformations. We can similarly extend $\mathcal{V}$-functors to a monoidal setting: a strictly unital $\mathcal{V}$-monoidal functor $(\mathcal{R}, \rho^R) : \mathcal{A} \rightarrow \mathcal{B}$ consists of an underlying $\mathcal{V}$-functor $\mathcal{R} : \mathcal{A} \rightarrow \mathcal{B}$ with $\mathcal{R}(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and $j^A_1 \circ j^A_{1,a} = j^B_1$, and a family of 1-$\mathcal{V}$-graded isomorphisms $\rho_{a,b}^R \in \mathcal{V}(1_{\mathcal{V}} \rightarrow \mathcal{B}(\mathcal{R}(a) \mathcal{R}(b) \rightarrow \mathcal{R}(ab)))$ satisfying:

- Naturality:

$$\mathcal{B}(\mathcal{R}(a) \mathcal{R}(b) \rightarrow \mathcal{R}(cd))$$

$$\begin{pmatrix}
\rho^R_{ab} \\
\rho^R_{abc} \\
\mathcal{R}_{a,b} \\
\mathcal{R}_{a,b,c} \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\end{pmatrix}$$

- Associativity:

$$\mathcal{B}(\mathcal{R}(a) \mathcal{R}(b) \mathcal{R}(c) \rightarrow \mathcal{R}(abc))$$

$$\begin{pmatrix}
\rho^R_{ab} \\
\rho^R_{abc} \\
\mathcal{R}_{a,b} \\
\mathcal{R}_{a,b,c} \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\end{pmatrix}$$

We call $\mathcal{R}$ lax if the $\rho_{a,b}^R$ are not necessarily isomorphisms, and oplax if we have $\rho_{a,b}^R : 1_{\mathcal{V}} \rightarrow \mathcal{B}(\mathcal{R}(ab) \rightarrow \mathcal{R}(a) \mathcal{R}(b))$ instead, again not necessarily isomorphisms, along with the corresponding axioms in place of the above.

We can compose $(\mathcal{R}, \rho^R) : \mathcal{A} \rightarrow \mathcal{B}$ with $(\mathcal{S}, \sigma^S) : \mathcal{B} \rightarrow \mathcal{C}$ to get a $\mathcal{V}$-monoidal functor via the usual composition on $\mathcal{V}$-functors and laxitor

$$\mathcal{C}(\mathcal{T}(a) \mathcal{T}(b) \rightarrow \mathcal{T}(ab))$$

$$\begin{pmatrix}
\rho^R_{a,b} \\
\rho^S_{a,b,c} \\
\mathcal{R}_{a,b} \\
\mathcal{R}_{a,b,c} \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\end{pmatrix}$$

A 1-$\mathcal{V}$-graded monoidal natural transformation $\theta : \mathcal{R} \rightarrow \mathcal{S}$ is a 1-$\mathcal{V}$-graded natural transformation such that

$$\begin{pmatrix}
\rho^R_{a,b} \\
\rho^S_{a,b,c} \\
\mathcal{R}_{a,b} \\
\mathcal{R}_{a,b,c} \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\mathcal{A}(a \rightarrow c) \\
\mathcal{A}(b \rightarrow d) \\
\end{pmatrix}$$

(2.2)
Definition 2.6. Given a (lax) $V$-monoidal functor $(\mathcal{R}, \rho^R)$ between $V$-monoidal categories $\mathcal{A}$ and $\mathcal{B}$, there is a canonical ordinary (lax) monoidal functor $(R, \rho)$ between the underlying categories $A$ and $B$. We take $R$ to be the underlying functor, and we take $\rho \coloneqq \rho^R$; notice that for $a, b \in \mathcal{A}$,

$$\rho_{ab}^R \in V (1_V \otimes \mathcal{R} (a) \otimes \mathcal{R} (b) \otimes \mathcal{R} (ab)) = B (R (a) R (b) \to R (ab)).$$

We call $(R, \rho)$ the underlying monoidal functor of $(\mathcal{R}, \rho^R)$; verifying the axioms of a (lax) monoidal functor is a useful exercise.

2.5. The categorified trace. With a minor change in the notation of [MP19], given a $V$-monoidal category $\mathcal{A}$, we define a functor $\text{Tr} : \mathcal{A} \to \text{Tr}$, where $\mathcal{A}^V$ is the underlying category of $\mathcal{A}$. On objects, we set $\text{Tr} (a) = \mathcal{A} (1_{\mathcal{A}} \to a)$, and on morphisms $f \in \mathcal{A}^V (a \to b) = V (1_{\mathcal{A}} \to \mathcal{A} (a \to b))$ we define

$$\text{Tr} (f) = \begin{array}{c} \mathcal{A} (1_{\mathcal{A}} \to b) \\
\mathcal{A} (1_{\mathcal{A}} \to a) \
\end{array}.$$ 

In order for the name “trace” to make sense, we would expect an isomorphism $\text{Tr} (ab) \to \text{Tr} (ba)$. When $\mathcal{A}$ is rigid, we have

$$\text{Tr} (ab) = \mathcal{A} (1_{\mathcal{A}} \to ab)$$

$$\cong \mathcal{A} (b^* \to a)$$

$$\cong \mathcal{A} (1_{\mathcal{A}} \to b^{**}a),$$

so we get a natural isomorphism $\text{Tr} (ab) \cong \text{Tr} (b^{**}a)$. In the case that there is a $V$-graded natural isomorphism $1_{\mathcal{A}} \cong **$ ($\mathcal{A}$ is $V$-graded pivotal), this is then also isomorphic to $\mathcal{A} (1_{\mathcal{A}} \to ba) = \text{Tr} (ba)$. With this in mind, we continue the convention in [MP19] and [HPT16] to call the map a trace.

2.6. 2-categories, 2-functors, and 2-equivalences.

Definition 2.7. A 2-category $\mathscr{A}$ is a category enriched in $\text{Cat}$. Explicitly, $\mathscr{A}$ consists of the data

- A collection of objects. In the 2-categorical context, we will refer to objects as 0-cells to avoid ambiguity.
- For each $A, B \in \mathscr{A}$, a hom-category $\mathscr{A} (A \to B)$. We avoid the term “morphism” here to avoid confusion, and refer to objects of such a hom-category simply as 1-cells. The morphisms of a hom-category are called 2-cells. We write $R : A \to B$ for 1-cells, and $\theta : R \Rightarrow S$ for 2-cells. Vertical composition of 2-cells is suppressed in our notation.
- For each $A \in \mathscr{A}$, a unit 1-cell $\text{id}_A \in \mathscr{A} (A \to A)$.
- For each $A, B, C \in \mathscr{A}$, a composition bifunctor $\circ : \mathscr{A} (A \to B) \times \mathscr{A} (B \to C) \to \mathscr{A} (A \to C)$. On 1-cells $\circ$ is the expected composition, and it is horizontal composition on 2-cells.

satisfying the following axioms:

- Unitality: For all $A, B \in \mathscr{A}$ and each $f \in \mathscr{A} (A \to B)$, $\text{id}_A \circ f = f \circ \text{id}_B = f$.
- Associativity: For each $A, B, C, D \in \mathscr{A}$,

$$\mathscr{A} (A \to B) \times \mathscr{A} (B \to C) \times \mathscr{A} (C \to D) \xrightarrow{\text{id}_B \times \circ} \mathscr{A} (A \to B) \times \mathscr{A} (B \to D)$$

$$\mathscr{A} (A \to C) \times \mathscr{A} (C \to D) \xrightarrow{\circ} \mathscr{A} (A \to D)$$

commutes.

Definition 2.8. A 2-functor $P : \mathscr{A} \to \mathscr{B}$ between 2-categories $\mathscr{A}$ and $\mathscr{B}$ is a $\text{Cat}$-functor between $\text{Cat}$-categories. Explicitly, $P$ consists of:

- For each 0-cell $A \in \mathscr{A}$, a 0-cell $P(A)$ in $\mathscr{B}$.
- For each pair of 0-cells $A, B$ in $\mathscr{A}$, a functor

$$P_{A \to B} : \mathscr{A} (A \to B) \to \mathscr{B} (P(A) \to P(B))$$

such that

- For each 0-cell $A$ in $\mathscr{A}$, $P_{A \to A} (\text{id}_A) = \text{id}_{P(A)}$.
• the composition functor is preserved, i.e., given 0-cells $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in $\mathcal{A}$:
  - For 1-cells $\mathcal{R} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{S} : \mathcal{B} \to \mathcal{C}$,
    \[ P_{\mathcal{A} \to \mathcal{B}} (\mathcal{R}) \circ P_{\mathcal{B} \to \mathcal{C}} (\mathcal{S}) = P_{\mathcal{A} \to \mathcal{C}} (\mathcal{R} \circ \mathcal{S}) . \]
  - For 1-cells $\mathcal{R}_1, \mathcal{R}_2 : \mathcal{A} \to \mathcal{B}$ and $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{B} \to \mathcal{C}$, and 2-cells $\theta : \mathcal{R}_1 \Rightarrow \mathcal{R}_2$ and $\psi : \mathcal{S}_1 \Rightarrow \mathcal{S}_2$,
    \[ P_{\mathcal{A} \to \mathcal{C}} (\theta \psi) = P_{\mathcal{A} \to \mathcal{B}} (\theta) P_{\mathcal{B} \to \mathcal{C}} (\psi) . \]

**Definition 2.9 ([JY20])**. We say that a 2-functor $P : \mathcal{A} \to \mathcal{B}$ is a 2-equivalence if it is:
- Essentially surjective: For every 0-cell $B \in \mathcal{B}$, there is a 0-cell $A \in \mathcal{A}$ and an equivalence of categories $A \cong P(A)$.
- Fully faithful: For each pair of 0-cells $A, B \in \mathcal{A}$, $P_{A \to B}$ is an equivalence of categories.

3. The 2-categories $\mathcal{VMonCat}$ and $\mathcal{VModTens}$

In this section we will construct the 2-categories $\mathcal{VMonCat}$ and $\mathcal{VModTens}$, based on the 0-cells studied in [MP19]. First, however, we record some terminology from [MP18, JPP19].

**Definition 3.1 ([MP19, MPP18, JPP19])**. We call a $\mathcal{V}$-category $\mathcal{A}$
- weakly tensored if for each $a \in \mathcal{A}$, the representable functor $\mathcal{A} (a \to -) : A \to \mathcal{V}$ admits a left adjoint.
- tensored if for each $a \in \mathcal{A}$, the $\mathcal{V}$-representable functor $\mathcal{A} (a \to -) : \mathcal{A} \to \mathcal{V}$ admits a left $\mathcal{V}$-adjoint.

Note that we must be able to form the self-enrichment $\hat{\mathcal{V}}$ for the definition of a tensored $\mathcal{V}$-category to make sense; this requires $\mathcal{V}$ to be closed (see [MPP18]). In this article we assume $\mathcal{V}$ is rigid, and being rigid is a stronger property of a $\mathcal{V}$-category than being closed.

**Definition 3.2 ([JPP19] defn 2.6, HPT).** A $\mathcal{V}$-module tensor category is a pair $(A, \mathcal{F}^Z_A)$ where $A$ is a monoidal category and $\mathcal{F}^Z_A : \mathcal{V} \to Z(A)$ is an oplax, strongly unital braided monoidal functor. We call the $\mathcal{V}$-module tensor category oplax or strongly unital if $\mathcal{F}^Z_A$ is oplax or strongly unital, respectively.

We say that a $\mathcal{V}$-module tensor category $(A, \mathcal{F}^Z_A)$ is
- rigid if $A$ is rigid,
- weakly tensored if $\mathcal{F} := \mathcal{F}^Z \circ \text{Forget}_Z$ admits a right adjoint, with $\text{Forget}_Z : Z(A) \to A$ the forgetful functor,
- tensored if it is weakly tensored and $\mathcal{F}$ is strong monoidal.

**Definition 3.3.** The following data defines a 2-category, which we call $\mathcal{VMonCat}$:
- 0-Cells are rigid $\mathcal{V}$-monoidal categories ($\mathcal{A}, \mathcal{B}$, etc.) such that the functor $x \mapsto \mathcal{A} (1, x \to x)$ admits a left adjoint.
- For each pair of 0-cells $\mathcal{A}, \mathcal{B}$, the hom-category $\mathbb{VMonCat}(\mathcal{A} \to \mathcal{B})$ consists of
  - 1-cells: Strictly unital lax $\mathcal{V}$-monoidal functors $(\mathcal{R}, \rho^\mathcal{R}) : \mathcal{A} \to \mathcal{B}$, where $\rho^\mathcal{R}$ is the laxitor for $\mathcal{R}$.
  - 2-cells: 1-$\mathcal{V}$-graded monoidal natural transformations $\theta : (\mathcal{R}, \rho^\mathcal{R}) \to (S, \sigma^S)$.
  - Composition is the usual vertical composition of 1-$\mathcal{V}$-graded natural transformations:

\[
\begin{array}{ccc}
B (\mathcal{R}(a) \to S(a)) & \xrightarrow{(\theta \circ \varphi)_a} & B (\mathcal{R}(a) \to S(a)) \\
\downarrow \sigma^S_{\rho^\mathcal{R}(a)} & & \downarrow \sigma^S_{\rho^\mathcal{R}(a)} \\
S (\rho^\mathcal{R}(a)) & & S (\rho^\mathcal{R}(a))
\end{array}
\]

- Unit 2-cells are the identity 1-$\mathcal{V}$-graded natural transformations.
- Composition functors:
  - Two 1-cells $(\mathcal{R}, \rho^\mathcal{R}) : \mathcal{A} \to \mathcal{B}$ and $(S, \sigma^S) : \mathcal{B} \to \mathcal{C}$ have composition defined via $(\mathcal{R} \circ S)(a) := S (\mathcal{R}(a))$,
  - $(\mathcal{R} \circ S)_{a \to b} := \mathcal{R}_{a \to b} \circ S_{\mathcal{R}(a) \to \mathcal{R}(b)}$, and composite laxitor

\[
\begin{array}{ccc}
C (\mathcal{R}(a) \to S(a)) & \xrightarrow{C (\theta \circ \varphi)_a} & C (\mathcal{R}(a) \to S(a)) \\
\downarrow \sigma^S_{\rho^\mathcal{R}(a)} & & \downarrow \sigma^S_{\rho^\mathcal{R}(a)} \\
S (\rho^\mathcal{R}(a)) & & S (\rho^\mathcal{R}(a))
\end{array}
\]
For each pair of 0-cells $A \xrightarrow{1} B$, $R_1 \xrightarrow{R_2} S_1$, the horizontal composition of $\theta : R_1 \Rightarrow R_2$ and $\varphi : S_1 \Rightarrow S_2$ is given by

$$
\begin{align*}
\phi_{R_2} & = (\theta \varphi)_{R_1} \\
\phi_{S_2} & = (\theta \varphi)_{S_1}
\end{align*}
$$

- Lastly, unit 1-cells are the usual identity $V$-functor $\text{id}_A : A \to A$ along with the identity natural transformations as the tensorator.

**Remark 3.4.** The definition above of $\mathbf{VMonCat}$ does in fact define a 2-category.

- First, $\mathbf{VMonCat} (\mathcal{A} \to \mathcal{B})$ is a category:
  - Unit 2-cells are the usual identity natural transformations, and thus composition preserves units.
  - Vertical composition of 2-cells is associative by Lemma 2.2 of [MP19].
  - Composition functors are unital because unit 1-cells are identity $V$-monoidal functors, with their usual composition.
  - Composition functors are associative: Horizontal composition of 2-cells is associative by Lemma 2.3 of [MP19], and composition of 1-cells is just ordinary $V$-monoidal functor composition, which is associative.

**Definition 3.5.** We define a second 2-category, which we call $\mathbf{VModTens}$, from the following data:

- 0-Cells are weakly tensored $V$-module tensor categories $(A, \mathcal{F}_A^Z)$.

  - For each pair of 0-cells $(A, \mathcal{F}_A^Z), (B, \mathcal{F}_B^Z)$, we define the hom-category $\mathbf{VModTens} ((A, \mathcal{F}_A^Z) \to (B, \mathcal{F}_B^Z))$ as follows:
    - 1-cells are triples $(R, \rho, r) : (A, \mathcal{F}_A^Z) \to (B, \mathcal{F}_B^Z)$, where $(R, \rho)$ is a strictly unital lax monoidal functor, and $r : \mathcal{F}_B \Rightarrow \mathcal{F}_A \circ R$ is a strictly unital monoidal natural transformation, i.e., $R(1_A) = 1_B$ and $r_{uv} = \text{id}_{1_B}$; they must also satisfy the half-braiding coherence and action coherence conditions:

$$
\begin{align*}
\varphi_{(A, \mathcal{F}_A^Z)} & = R(\mathcal{F}_A(\varphi)) \\
\rho_{(A, \mathcal{F}_A^Z)} & = R(\mathcal{F}_A(\rho))
\end{align*}
$$

where $e_{(A, \mathcal{F}_A^Z)} : B \to (A, \mathcal{F}_A^Z)$ is the half-braiding on $\mathcal{F}_A(\varphi)$ (from the Drinfeld center), and similarly for $R(\mathcal{F}_A(\rho))$.

- 2-cells $(R, \rho, r) \Rightarrow (S, \sigma, s)$ are monoidal natural transformations $\Theta : (R, \rho) \to (S, \sigma)$ such that $r_{uv} \circ \Theta_{R(\varphi)} = s_{uv}$.

- Composition is the usual vertical composition of natural transformations: $(\Theta \varphi)_{uv} = \Theta_{uv} \circ \varphi_{uv}$.

- Unit 2-cells are the identity natural transformations.

- Composition functors:
  - Given $(R, \rho, r) : A \to B$ and $(S, \sigma, s) : B \to C$, we form their composite via functor $R \circ S$, laxitor $\sigma_{R(a), R(b)} \circ S (\rho_{ab})$ and coherence $r_{uv} \circ S (s_{uv})$.
  - If $A, B, C \in \mathbf{VModTens}, R_1, R_2 \in \mathbf{VModTens} (A \to B), S_1, S_2 \in \mathbf{VModTens} (B \to C)$, and $\Theta : R_1 \Rightarrow R_2$ and $\varphi : S_1 \Rightarrow S_2$ are 2-cells, the horizontal composite $\Theta \varphi$ is defined via

$$
\Theta \varphi = \varphi_{R_1(a)} \circ S_2 (\Theta_a).
$$

- Unit 1-cells: For each 0-cell $(A, \mathcal{F}_A^Z) \in \mathbf{VModTens}$ we define the unit 1-cell to be the triple consisting of the identity functor on $A$, the identity laxitor, and the identity natural transformation on $\mathcal{F}_A$.

**Remark 3.6.** With the above definition, $\mathbf{VModTens}$ is a 2-category:

- For each pair $(A, \mathcal{F}_A^Z), (B, \mathcal{F}_B^Z)$ of 0-cells in $\mathbf{VModTens}$, $\mathbf{VModTens} ((A, \mathcal{F}_A^Z) \to (B, \mathcal{F}_B^Z))$ is a category:
  - Unit 2-cells are units under composition, since they are identity natural transformations with the usual composition of natural transformations.
  - Composition of natural transformations is associative.
• Composition functors are unital because the unit 1-cells are identity functors with identity laxitors and the identity natural transformation for action-coherence.
• Composition functors are associative because horizontal composition of natural transformations, composition of monoidal functors, and vertical composition of natural transformations are all associative.

4. Calculations using mates

Given an adjunction \( \mathcal{A} (\mathcal{L} (x) \to b) \cong \mathcal{B} (x \to \mathcal{R} (b)) \) and a morphism \( f \in \mathcal{A} (\mathcal{L} (x) \to b) \), the mate of \( f \) is the corresponding morphism \( \text{mate}(f) \in \mathcal{B} (x \to \mathcal{R} (b)) \).

Remark 4.1. A large number of our discussions focus on the following adjunction (Adjunction 4.2 in [MP19]): Let \( \mathcal{A} \) be a 0-cell in \( \mathcal{V}\text{MonCat} \). Then for each \( a, b \in \mathcal{A} \) and each \( v \in \mathcal{V} \), we have the adjunction

\[
A (a \mathcal{F}_A (v) \to b) \cong V (v \to \mathcal{A} (a \to b))
\]

where \( A = \mathcal{A}^V \) is the underlying category of \( \mathcal{A} \). In particular, note that when \( v = 1_V \) the adjunction becomes \( \mathcal{A}^V (a \to b) \cong V (1_V \to \mathcal{A} (a \to b)) \). In this case, we actually have equality by definition of \( \mathcal{A}^V \), and, although slightly redundant, we often use a superscript to keep track of which side of this equality adjunction a morphism lives on. There is no ambiguity even without a notational difference, as a quick check of source and target will clear up any confusion.

Definition 4.2. We define the unit of this adjunction for \( v \in \mathcal{V} \) as

\[ \eta_v^{\mathcal{F}A} := \text{mate}(\text{id}_{\mathcal{F}_A (v)}) \]

and the counit of this adjunction for \( a, b \in A \) as

\[ \epsilon_{a \to b}^{\mathcal{F}A} := \text{mate}(\text{id}_{\mathcal{F}_A (a \to b)}) , \]

both under the corresponding version of Adjunction 4.1.

Notation 4.3. We extend a useful notation from [MP19]:

\[ [a \to b; c \to d; \cdots]^{\mathcal{F}A} := \mathcal{F} (\mathcal{A} (a \to b) \mathcal{A} (c \to d) \cdots) , \]

where \( \mathcal{A} \) is a \( \mathcal{V} \)-monoidal category, and \( \mathcal{F} \) is any functor from \( \mathcal{V} \). Note that \( \mathcal{F} \) need not be specifically \( \mathcal{F}_A \), e.g., we use

\[ [a \to b; c \to d; \cdots]^{\mathcal{F}A} := \mathcal{F}_B (\mathcal{A} (a \to b) \mathcal{A} (c \to d) \cdots) . \]

Remark 4.4. Under Adjunction 4.1, the mate of \( - \circ \mathcal{A} \in \mathcal{V} (\mathcal{A} (a \to b) \mathcal{A} (b \to c) \to \mathcal{A} (a \to c)) \) is

\[
\begin{tikzpicture}
  \node (a) at (0,0) [shape=rectangle,draw] {a \to b; c \to d};
  \node (b) at (1,1) [shape=rectangle,draw] {c \to d};
  \node (c) at (2,2) [shape=rectangle,draw] {d};
  \node (d) at (1,3) [shape=rectangle,draw] {e};
  \node (e) at (2,4) [shape=rectangle,draw] {f};
  \draw [->] (a) -- (b);
  \draw [->] (b) -- (c);
  \draw [->] (a) -- (d);
  \draw [->] (b) -- (e);
  \draw [->] (c) -- (e);
  \draw [->] (d) -- (e);
  \draw [->] (a) -- (e);
  \draw [->] (b) -- (e);
  \draw [->] (c) -- (e);
\end{tikzpicture}
\]

and the mate of \( - \otimes \mathcal{A} \circ - : \mathcal{A} (a \to b) \mathcal{A} (c \to d) \to \mathcal{A} (ac \to bd) \) is

\[
\begin{tikzpicture}
  \node (a) at (0,0) [shape=rectangle,draw] {a \to b; c \to d};
  \node (b) at (1,1) [shape=rectangle,draw] {b \to d};
  \node (c) at (2,2) [shape=rectangle,draw] {c \to d};
  \node (d) at (1,3) [shape=rectangle,draw] {e};
  \node (e) at (2,4) [shape=rectangle,draw] {f};
  \draw [->] (a) -- (b);
  \draw [->] (b) -- (c);
  \draw [->] (a) -- (d);
  \draw [->] (b) -- (e);
  \draw [->] (c) -- (e);
  \draw [->] (d) -- (e);
  \draw [->] (a) -- (e);
  \draw [->] (b) -- (e);
  \draw [->] (c) -- (e);
\end{tikzpicture}
\]

Note that when going from \( \mathcal{V}\text{ModTens} \) to \( \mathcal{V}\text{MonCat} \) this is a definition, and in the other direction it is a proposition requiring proof (Proposition 4.9 in [MP19] and Proposition 5.3 in [MP19]).

We now record several useful lemmas in computing mates; we start with a consequence of naturality.
Lemma 4.5. Let $A(x \to b) \cong B(x \to R(b))$ be an adjunction. If $f_1 \in A(x \to b)$ and $f_2 \in A(b \to c)$, then $\text{mate}(f_1 \circ f_2) = \text{mate}(f_1) \circ R(f_2)$. Similarly, if $g_1 \in B(x \to y)$ and $g_2 \in B(y \to A(b))$, then $\text{mate}(g_1 \circ g_2) = \mathcal{L}(g_1) \circ \text{mate}(g_2)$. In diagrams,

\[
\begin{align*}
\text{mate} & \left( \begin{array}{c}
\mathcal{L}(x) \\
\downarrow f_1 \\
\downarrow f_2 \\
\end{array} \right) = \mathcal{A}(c) \\
\text{mate}(f_1) & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
Proof. From the composition lemma, the mate of $- \otimes_A -$, and naturality of $\mu$, we have

Next, $F_A$ lifts to $Z (A)$ and so $F_A (f)$ can go under the crossing, and using lemma 4.6, we have

To end the section we notice one more straightforward lemma that will be useful:

Lemma 4.9. Given $f \in V (u \to v)$, we can consider $F_A (f)$ as a morphism in $V (1_V \to A (F_A (u) \to F_A (v))) = A (F_A (u) \to F_A (v))$. Then using Lemmas 4.7 and 4.5, respectively, we note $F_A (f)$ is the mate of each of the following expressions, so they are equal:

$\mathcal{A} (1_A \to F_A (v)) = \mathcal{A} (1_A \to F_A (v)) \triangleright \mathcal{A} (1_A \to F_A (v)) = \mathcal{A} (1_A \to F_A (v))$

5. The 2-functor

In this section, we construct a map $P : \textbf{VMonCat} \to \textbf{VModTens}$ and prove that $P$ is a 2-functor. In the following section we show that $P$ is in fact a 2-equivalence.

5.1. Map on 0-cells. Our map on 0-cells is the bijective correspondence from [MP19]: we set $P (\mathcal{A}) = \left( A, F_A^Z \right)$, with $A := \mathcal{A}^V$ the underlying category of $\mathcal{A}$ and $F_A^Z$ the left adjoint of $x \mapsto \mathcal{A} (1_A \to x)$ lifted to the center via the half-braidingings

$\eta_{\mathcal{A} (1_A \to u)}$

$\varepsilon_{F_A (1_A \to v)}$

(5.1)

Remark 5.1. To lift $F_A$ to $F_A^Z$, it is also necessary to show that the images of morphisms in $V$ are morphisms in $Z (A)$. This was omitted from [MP19], so we include the short proof here.
**Proof.** To prove the remark, we need to show that we can pull morphisms in the image of $\mathcal{F}_A$ under the half-braiding, i.e.,

$$
\mathcal{F}_A(v) \xrightarrow{\rho} \mathcal{F}_A(f) = \mathcal{F}_A(u)
$$

for all $f \in \mathcal{V}(u \to v)$. We take the mate of the left hand side using Lemma 4.5 and Lemma 4.6 from [MP19], then apply a braided interchange, Lemma (4.9), and Corollary 4.7 from [MP19] in turn:

From here we use the definition of the half-braiding, apply Lemma (4.9) again, use another braided interchange after adding an identity, and recognize the result as the mate of the right hand side, so that $\mathcal{F}_A(f)$ is in fact a morphism in $\mathcal{Z}(A)$

$$
\mathcal{F}_A(v) \xrightarrow{\rho} \mathcal{F}_A(f) = \mathcal{F}_A(u)
$$

\[\Box\]

### 5.2. Map on 1-cells: Definition.

Given 0-cells $\mathcal{A}, \mathcal{B} \in \mathcal{VMonCat}$ and a 1-cell $(\mathcal{R}, \rho^R) \in \mathcal{VMonCat} (\mathcal{A} \to \mathcal{B})$, we construct a 1-cell $(\mathcal{R}, \rho, r)$ in $\mathcal{VModTens} (P(\mathcal{A}) \to P(\mathcal{B}))$. Define $R : A \to B$ to be the underlying functor of $\mathcal{R}$, and define the laxitor $\rho$ via, for $a, b \in A$, $\rho_{a,b} \coloneqq \text{mate}(\rho^R_{a,b}) = \rho^R_{a,b}$ under the identity adjunction

$$
\mathcal{B}(R(a) \cdot R(b)) = \mathcal{V}(1_B \to B(R(a) \cdot R(b)))
$$

Lastly, we define the natural transformation $r$ via

\[r \in \mathcal{Z}(A)
\]

(5.2)

**Remark 5.2.** Following immediately from the corresponding properties in $\mathcal{VMonCat}$, we have:

- The pair $(R, \rho)$ is a monoidal functor.
- The functor $R$ is strictly unital: $R(1_A) = 1_B$.
- The laxitor $\rho$ is strictly unital: $\rho_{1_A,a} = \id_{R(a)} = \rho_{a,1_B}$.
- The natural transformation $r$ is strictly unital: $r_{1_B} = \id_{1_B}$. 

Proposition 5.3. The natural transformation \( r \) is in fact natural, i.e., for each \( f \in \mathcal{V} (u \to v) \), we have

\[
\rho_f : R (F_A (f)) = R (F_B (f)) \]

Proof. The mate of the left hand side under Adjunction 4.1, using Lemmas 4.5 and 4.9 along with functoriality of \( R \), is

\[
\mathcal{B} (1_B \to R (F_A (v))) = \mathcal{B} (1_B \to R (F_A (u))) \]

which is exactly the mate of the right hand side. \( \square \)

Proposition 5.4. The natural transformation \( r \) plays nicely with the half braiding \( e_{a,F_A (v)} \) defined in Equation 5.1, i.e.,

\[
\rho : R (F_A (v)) = R (F_A (u)) \]

Proof. Taking the mate of the left hand side, then using braided interchange, simplifying, and recognizing the mate of \( r \) via lemma 4.6 gives

\[
\mathcal{B} (1_B \to R (F_A (v))) = \mathcal{B} (1_B \to R (F_A (u))) \]

Next, use unitality and monoidality of \( R \), then simplify to see that this is equal to
Taking the mate of the right hand side, then applying braided interchange followed by lemma 4.6 for the mate of \( r_n \), we get

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

Lastly, using naturality of \( \mathcal{R} \) and unitality of \( \rho^R \), then functoriality of \( \mathcal{R} \) and Corollary 4.7 from [MP19], we see that this is equal to the mate of the left hand side from above:

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

\[
\mathcal{B}(R(a) \to R(\mathcal{F}_A(v) a))
\]

Proposition 5.5. The action-coherence condition is satisfied:

\[
R(\mathcal{F}_A(u) \mathcal{F}_A(v))
\]

\[
= \mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]

Proof. Using Lemma 4.7, functoriality of \( \mathcal{R} \), and Lemma 4.6, the mate of the left hand side is

\[
\mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]

\[
= \mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]

\[
= \mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]

\[
= \mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]

\[
= \mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]

\[
= \mathcal{B}(1_B \to R(\mathcal{F}_A(u) \mathcal{F}_A(v)))
\]
After expanding using the mate of $\mu$ and applying braided interchange, the mate of the right hand side becomes:

$$
\begin{array}{c}
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v)))
\end{array}
$$

Then we use lemma 4.6 twice, the definition of $r_\alpha$, and naturality and unitality of $\rho$ to see that this is equal to

$$
\begin{array}{c}
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to R(\mathcal{F}_\mathcal{A}(u) \mathcal{F}_\mathcal{A}(v)))
\end{array}
$$

where the last equality is the definition of $\mu$, matching the mate of the left hand side. \hfill \Box

All together this proves:

**Proposition 5.6.** The triple $(R, \rho, r)$ is a 1-cell in $\mathbf{VModTens}$.

**5.3. Map on 2-cells.** Suppose we have a $\mathbf{V}$-monoidal natural transformation $\theta : R \Rightarrow S$, i.e., a 2-cell in $\mathbf{VMonCat}$. We define a 2-cell $P_{\mathcal{A} \to \mathcal{B}}(\theta) : R \Rightarrow S$ between the corresponding 1-cells in $\mathbf{VModTens}$ via, for each $a \in \mathcal{A}$, $P_{\mathcal{A} \to \mathcal{B}}(\theta)_a := \text{mate}(\theta_a)$ under the identity adjunction

$$
\mathcal{V}(1_\mathcal{V} \to \mathcal{B}(\mathcal{R}(a) \to \mathcal{S}(a))) = \mathcal{B}(\mathcal{R}(a) \to \mathcal{S}(a)) .
$$

We write $\Theta_a := P_{\mathcal{A} \to \mathcal{B}}(\theta)_a$ for brevity.

**Proposition 5.7.** As defined above, $\Theta$ is a 2-cell in $\mathbf{VModTens}$

**Proof:** Naturality and monoidality of $\Theta$ follow easily from the corresponding properties in the enriched setting: taking the appropriate mate of each side of the required conditions gives exactly the corresponding condition in $\mathbf{VMonCat}$. It remains to check that $s_v \circ \Theta_{\mathcal{F}_\mathcal{A}(v)} = \rho_v$ for all $v \in \mathcal{V}$.

From the hexagon axiom in $\mathbf{VMonCat}$, with $a = 1_{\mathcal{A}}$ and $b = \mathcal{F}_\mathcal{A}(v)$, we have

$$
\begin{array}{c}
\mathcal{B}(1_\mathcal{B} \to S(\mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to S(\mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to S(\mathcal{F}_\mathcal{A}(v))) \\
\mathcal{B}(1_\mathcal{B} \to S(\mathcal{F}_\mathcal{A}(v)))
\end{array}
$$
Next, precompose with $\eta^F_\mu$ on both sides, and use strict unitality of $\Theta$:

$$
\begin{align*}
\mathcal{B} (1_g \to S (F_A (v))) & \\
\eta^F_\mu \circ \eta^S_\mu & = \\
\sigma_{S, F_A (v)} & \\
R_{1_g \to F_A (v)} & \\
\mathcal{B} (1_g \to S (F_A (v))) & \\
\eta^S_\mu
\end{align*}
$$

Now the right hand side is exactly the mate of $r_\mu$, and by Lemma 4.7, the mate of the left hand side is

$$
\begin{align*}
S (F_A (v)) & \\
\text{mate} (\theta_{F_A (v)}) & \\
\text{mate} \left( \eta^F_\mu \circ R_{1_g \to F_A (v)} \right) & \\
F_B (v) & \\
\text{mate} (\Theta_{F_A (v)}) & \\
\mathcal{F}_B (v)
\end{align*}
$$

which then proves that $s_\mu \circ \Theta_{F_A (v)} = r_\mu$, as desired. \hfill \Box

**Proposition 5.8.** With the above definitions, $P : \mathcal{VMonCat} \to \mathcal{VModTens}$ is a 2-functor.

We will take the remainder of this section to prove the proposition. We need to show that

- Each $P_{\mathcal{A} \to \mathcal{B}}$ preserves unit 1-cells,
- Each $P_{\mathcal{A} \to \mathcal{B}}$ is a functor, i.e., vertical composition of 2-cells is preserved
- Horizontal composition of 2-cells is preserved, and
- Composition of 1-cells is preserved.

First, note that both id$_{P (\mathcal{A})}$ and $P_{\mathcal{A} \to \mathcal{A}} (\text{id}_\mathcal{A})$ are the identity 1-cell on $P (\mathcal{A})$, so unit 1-cells are preserved. Next, since $P_{\mathcal{A} \to \mathcal{B}}$ on 2-cells is simply taking mates under the identity adjunction, it follows easily from the definitions that horizontal and vertical composition of 2-cells are both preserved. The bulk of the work is involved in showing that composition of 1-cells is preserved, i.e.,

$$
P_{\mathcal{A} \to \mathcal{B}} \left( (\mathcal{R}, \rho^\mathcal{R}) \right) \circ P_{\mathcal{B} \to \mathcal{C}} \left( (\mathcal{S}, \sigma^\mathcal{S}) \right) = P_{\mathcal{A} \to \mathcal{C}} \left( (\mathcal{R}, \rho^\mathcal{R}) \circ (\mathcal{S}, \sigma^\mathcal{S}) \right).
$$

The left hand side is given by

$$
P_{\mathcal{A} \to \mathcal{B}} \left( (\mathcal{R}, \rho^\mathcal{R}) \right) \circ P_{\mathcal{B} \to \mathcal{C}} \left( (\mathcal{S}, \sigma^\mathcal{S}) \right) = (\mathcal{R}^\mathcal{V}, \rho, r) \circ (\mathcal{S}^\mathcal{V}, \sigma, s),
$$

where $\rho$, $\sigma$, $r$, and $s$ are as defined in 5.2, and the right hand side is

$$
P_{\mathcal{A} \to \mathcal{C}} \left( (\mathcal{R}, \rho^\mathcal{R}) \circ (\mathcal{S}, \sigma^\mathcal{S}) \right) = P_{\mathcal{A} \to \mathcal{C}} \left( \mathcal{R} \circ \mathcal{S}, \sigma^\mathcal{S} \circ S (\rho^\mathcal{R}) \right) = \left( \mathcal{R} \circ \mathcal{S}, \text{mate} (\sigma^\mathcal{S} \circ S (\rho^\mathcal{R})) \right),
$$

where $\sigma^\mathcal{S} \circ S (\rho^\mathcal{R})$ is as defined in 3.5 and $t := \text{mate} \left( \eta^F_\mu \circ (R_{1_g \to F_A (v)} \circ S_{1_g \to R (F_A (f))}) \right)$. Now, $(\mathcal{R} \circ \mathcal{S})^\mathcal{V} = \mathcal{R}^\mathcal{V} \circ \mathcal{S}^\mathcal{V}$, so these are equal as functors. Further, since $\text{mate} (\sigma_{c,d}) = \sigma_{c,d}^\mathcal{S}$, we directly have

$$
\text{mate} \left( S (R (ab)) \right) = \left( \mathcal{R} \circ \mathcal{S}, \text{mate} (\sigma^\mathcal{S} \circ S (\rho^\mathcal{R})) \right),
$$

so they are also equal as lax monoidal functors. It remains to check that for each $v \in \mathcal{V}$,

$$
s_\mu \circ S (r_\mu) = \text{mate} \left( \eta^F_\mu \circ (R_{1_g \to F_A (v)} \circ S_{1_g \to R (F_A (f))}) \right).
$$
To unpack this line, we expand definitions and mates, starting with the right hand side using Lemma 4.5:

Next, expanding the left hand side, we get

where the large boxes act as brackets for multiple morphisms in our diagrams. Then expanding these mates and using unitality of $\rho$ and $\sigma$, we have

To show these are equal, we break down the calculation into a few lemmas.
Lemma 5.9.

\[ \mathcal{F}_C(\mathcal{B}(1_A \rightarrow R(\mathcal{F}_A(o)))) \]

\[ \mathcal{F}_C(\mathcal{R}_{1_A \rightarrow \mathcal{F}_A(o)}) \]

\[ \mathcal{F}_C(\eta^{F_C}) \]

\[ \mathcal{F}_C(o) \]

\[ = \]

\[ \mathcal{F}_C(\mathcal{B}(1_A \rightarrow R(\mathcal{F}_A(o)))) \]

\[ \mathcal{F}_C(\mathcal{R}_{\mathcal{B}(r_o)}) \]

\[ \mathcal{F}_C(\eta^{F_C}) \]

\[ \mathcal{F}_C(o) \]

Proof. Since \( \mathcal{F}_C \) is a functor, it suffices to show that

\[ \mathcal{B}(1_A \rightarrow R(\mathcal{F}_A(o))) \]

\[ \mathcal{R}_{1_A \rightarrow \mathcal{F}_A(o)} \]

\[ \eta^{F_C} \]

\[ \mathcal{B}(1_A \rightarrow R(\mathcal{F}_A(o))) \]

\[ \mathcal{R}_{\mathcal{B}(r_o)} \]

\[ \eta^{F_C} \]

The left hand side is exactly the mate of \( r_o \), so it suffices to show that the mate of the right hand side is \( r_o \) as well. Taking the mate of the right hand side above using Lemma 4.5, we get

\[ R(\mathcal{F}_A(o)) \]

\[ R(\mathcal{F}_A(o)) \]

\[ \text{mate}(R(\mathcal{F}_A(o))) \]

\[ \text{mate}(\text{r}_o) \]

\[ \eta^{F_C} \]

\[ \mathcal{F}_B(o) \]

\[ \text{mate}(\text{r}_o) \]

\[ \mathcal{F}_B(o) \]

\[ = \]

\[ R(\mathcal{F}_A(o)) \]

\[ R(\mathcal{F}_A(o)) \]

\[ \text{mate}(\text{r}_o) \]

\[ \mathcal{F}_B(o) \]

\[ \text{mate}(\text{r}_o) \]

\[ \mathcal{F}_B(o) \]

Then by Lemma 4.6, this is equal to

\[ R(\mathcal{F}_A(o)) \]

\[ R(\mathcal{F}_A(o)) \]

\[ \text{mate}(\eta^{F_B}) \]

\[ \mathcal{F}_B(o) \]

\[ \text{mate}(\text{r}_o) \]

\[ \mathcal{F}_B(o) \]

as desired.

Lemma 5.10.
Proof. On the left hand side we expand definitions, apply lemma 4.7 and lemma 4.6, and use the definition of the counit: 

\[
\begin{array}{c}
R(F_A(\alpha)) \\
F_B(Tr_B(\tau_B)) \\
F_B(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
\mathcal{B}(1_B \rightarrow R(F_A(\alpha))) \\
\text{mate} \\
\mathcal{B}(1_B \rightarrow F_B(\alpha))
\end{array}
\]

\[
\begin{array}{c}
R(F_A(\alpha)) \\
F_B(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
F_B(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

Lemma 5.11.

Proof. Expanding definitions and applying Lemma 4.5, \(r_\alpha\) is equal to

\[
\begin{array}{c}
\mathcal{B}(1_B \rightarrow R(F_A(\alpha))) \\
\text{mate} \\
R(F_A(\alpha))
\end{array}
\]

\[
\begin{array}{c}
R(F_A(\alpha)) \\
F_B(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
R(F_A(\alpha)) \\
F_B(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
F_B(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

Thus, putting together the above lemmas, along with naturality of \(s\), we get the equality:

\[
\begin{array}{c}
S(R(F_A(\alpha))) \\
S(F_A(\alpha)) \\
F_C(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
S(R(F_A(\alpha))) \\
S(F_A(\alpha)) \\
F_C(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
S(R(F_A(\alpha))) \\
S(F_A(\alpha)) \\
F_C(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

\[
\begin{array}{c}
S(R(F_A(\alpha))) \\
S(F_A(\alpha)) \\
F_C(\mathcal{B}(1_B \rightarrow F_B(\alpha)))
\end{array}
\]

Therefore composition of 1-cells is preserved, and \(P\) is a 2-functor.
6. Equivalence

Proposition 6.1. The 2-functor $P : \mathcal{V}\text{MonCat} \to \mathcal{V}\text{ModTens}$ as defined above is essentially surjective on 0-cells.

Proof. To say that $P$ is essentially surjective is just to say that for every 0-cell $(A, \mathcal{F}^Z_A) \in \mathcal{V}\text{ModTens}$, there exists a 0-cell $\mathcal{A} \in \mathcal{V}\text{MonCat}$ such that $P(\mathcal{A}) \cong (A, \mathcal{F}^Z_A)$. This is exactly the main theorem in [MP19].

Definition 6.2. Let $(A, \mathcal{F}^Z_A)$, $(B, \mathcal{F}^Z_B) \in \mathcal{V}\text{ModTens}$ be 0-cells, and let $(R, \rho, r) : (A, \mathcal{F}^Z_A) \to (B, \mathcal{F}^Z_B)$ be a 1-cell between them. We construct a 1-cell $(R, \rho_R)$ in $\mathcal{V}\text{MonCat}$ such that $P(\mathcal{A}) \to (A, \mathcal{F}^Z_A) = (R, \rho_R)$. Since $\mathcal{A}$ and $\mathcal{B}$ share the same objects as $A$ and $B$, respectively, we can define $\mathcal{R}(a) = R(a)$ for all $a \in \mathcal{A}$. Next, for $a, b \in \mathcal{A}$, we make $\mathcal{R}$ into a $\mathcal{V}$-functor via

\[
\mathcal{R}_{a \to b} := \text{mate} \left( \begin{array}{c}
R(a) \\
\rho(a \to b) \\
\sigma(a \to b)
\end{array} \right)
\]

under Adjunction 4.1, and tensorator $\rho_R$ given by

\[
\rho^R_{a,b} := \text{mate} (\sigma_{a,b}) = \rho_{a,b}
\]

under the equality adjunction

\[
B(\mathcal{R}(a) \mathcal{R}(b) \to \mathcal{R}(ab)) = \mathcal{V}(1_{\mathcal{V}} \to \mathcal{B}(\mathcal{R}(a) \mathcal{R}(b) \to \mathcal{R}(ab))).
\]

Lemma 6.3. The map $\mathcal{R}$ defined above is functorial: For all $a, b, c \in \mathcal{A}$,

\[
\mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(c)) = \mathcal{B}(\mathcal{R}(a) \to \mathcal{R}(c))
\]

Proof. Using lemma 4.7, expanding definitions, and using naturality of $\rho$, the left hand side has mate
Next, use associativity of $\rho$, the action-coherence condition, and naturality of $\rho$ again to see that this is

Lastly, we recognize the mate of $- \circ \mathcal{A} - (R$ is a functor), use lemma 4.6, and finish with naturality of $\rho$ and $r$:

This is exactly the mate of the right hand side via the composition lemma.

Lemma 6.4. We also have that $R$ preserves identity objects, and thus is a $\mathcal{V}$-functor: For all $a \in A$,

Proof. Using Lemma 4.5 followed by naturality and unitality of $r$ and $\rho$, the mate of the left hand side is

Lemma 6.5. The $\mathcal{V}$-functor $R$ is strictly unital, and the laxitor $\rho^R$ is strongly unital, i.e., $R(1_\mathcal{A}) = 1_\mathcal{B}$ and $\rho^R_{1_\mathcal{A},a} = \text{id}_{R(a)} = \rho^R_{a,1_\mathcal{A}}$.

Proof. First, $R$ is strictly unital, so $R(1_\mathcal{A}) = R(1_\mathcal{A}) = 1_\mathcal{B}$.

Then $\rho$ is strongly unital, so
\[ \rho^R_{1,3,a} = \text{mate} (\rho^R_{1,3,a}) = \text{mate} (\text{id}_{R(a)}) = j^R_a \]

and similarly for the other side.

**Lemma 6.6.** The laxitor \( \rho^R \) is natural: For all \( a, b, c, d \in A \),

\[ \mathcal{B} (R(a)R(b) \rightarrow R(cd)) = \mathcal{B} (R(a)R(b) \rightarrow R(cd)) \]

\[ \mathcal{A} (a \rightarrow c) \quad \mathcal{A} (b \rightarrow d) \quad \mathcal{A} (a \rightarrow c) \quad \mathcal{A} (b \rightarrow d) \]

**Proof.** Let \( a, b, c, d \in A \). Using Lemma 4.7 and Lemma 4.5, the left hand side has mate

\[ R(a)R(b) \quad \text{mate} (R(ab \rightarrow cd)) \]

\[ [a \rightarrow c; b \rightarrow d]^4_{\gamma_a} \]

Next, we use naturality of \( r \) and \( \rho \) to get

\[ R(a)R(b) \quad \text{mate} (R(ab \rightarrow cd)) \]

\[ [a \rightarrow c; b \rightarrow d]^4_{\gamma_a} \]

and lastly we use lemma 4.6 and the definition of \( - \otimes A \) to see that this is
On the other hand, taking the mate of the right hand side using Lemma 4.5, Lemma 4.8, and expanding the mates of $\mathcal{R}$, we have

Lastly, using naturality and associativity of $\rho$ along with the half-braiding coherence to get

Lastly, using naturality and associativity of $\rho$ again, we have

by the action-coherence condition on $\tau$ and naturality of $\rho$. 

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Lemma 6.7. The laxitor $\rho^R$ is associative: For all $a, b, c \in A$,

\[
\mathcal{B} (R (a) R (b) R (c) \rightarrow R (abc)) = \mathcal{B} (R (a) R (b) R (c) \rightarrow R (abc))
\]

Proof. Taking the mate of the left hand side and applying associativity of $\rho$ and lemma 4.8 twice, we get

\[
\text{mate (} \rho^R_{abc} \text{)} = \text{mate (} \rho^R_{abc} \text{)}
\]

which is exactly the mate of the right hand side. □

We have proven:

Proposition 6.8. The pair $(\mathcal{R}, \rho^R)$ defined above is a 1-cell in $\mathbf{VMonCat}$.

We’ve constructed a map from 1-cells of $\mathbf{VModTens}$ to 1-cells of $\mathbf{VMonCat}$. To show that $P$ is essentially surjective on 1-cells, it now suffices to prove the following proposition.

Proposition 6.9. For each pair $\mathcal{A}, \mathcal{B}$ of 0-cells in $\mathbf{VMonCat}$, the above map is an essential inverse to $P_{\mathcal{A} \rightarrow \mathcal{B}}$ on 1-cells.

Proof. Starting with a 1-cell $(R, \rho, r)$ in $\mathbf{VModTens} (\mathcal{A}, \mathcal{F}^Z_B) \rightarrow (\mathcal{B}, \mathcal{F}^Z_{\mathcal{B}})$, we have

\[ P_{\mathcal{A} \rightarrow \mathcal{B}} (\mathcal{R}, \rho^R) = (\mathcal{R}, \rho^R), \]

where for $a, b \in A$, $\mathcal{R} (a) = R (a)$, $\rho^R_{ab} = \text{mate (} \rho_{ab} \text{)} = \rho_{ab}$ and $\mathcal{R}_{a \rightarrow b}$ is defined in 6.1. Then $(\mathcal{R}, \rho^R)$ maps back to $(R, \rho, r)$, another 1-cell in $\mathbf{VModTens}$. Now we show that $(\mathcal{R}, \rho^R)$ and $(R, \rho, r)$ are equal as 1-cells.

- As functors. For $a \in A$, $\mathcal{R} (a) = R (a) = R (a)$. For $a, b \in A$ and a morphism $f \in A (a \rightarrow b) = \mathcal{V} (1_V \rightarrow \mathcal{A} (a \rightarrow b))$, we take the same mate of $f \circ \mathcal{R}_{a \rightarrow b}$ in two different ways. Under adjunction 4.1, using the definition of $R$ as the underlying functor of $\mathcal{R}$, we have

\[
\text{mate (} f \circ \mathcal{R}_{a \rightarrow b} \text{)} = R (f).
\]

Next, we take the same mate via Lemma 4.5, making use of naturality and unitality of $r$ and $\rho$:
By Lemma 4.6 this is equal on the nose to $R(f)$.

- As lax monoidal functors: for $a, b \in A$, $\rho_{a,b} = R_{\rho_{a,b}} = \rho_{a,b}$.
- As 1-cells in $\mathcal{V}\text{ModTens}$: for $v \in \mathcal{V}$, we compute

\[
\begin{align*}
\mathfrak{r}_v &= \text{mate} \\
&= \left( \text{mate} \right) \\
&= \left( \text{mate} \right)
\end{align*}
\]

by unitality of $\rho$ and lemma 4.5. We now show that this is equal to $r_v$ using Lemma 4.6 with $f = \eta^\mathcal{V}_v$ and naturality of $r$:

\[
\begin{align*}
\mathfrak{r}_v &= \text{mate} \\
&= \left( \text{mate} \right) \\
&= \left( \text{mate} \right)
\end{align*}
\]

Therefore composing $P_{A \to B}$ with this inverse map is the same as taking mates twice.

Definition 6.10. Given a monoidal natural transformation $\Theta : (R, \rho, r) \Rightarrow (S, \sigma, s)$, where $(R, \rho, r)$ and $(S, \sigma, s)$ are 1-cells in $\mathcal{V}\text{ModTens}$, we define a $\mathcal{V}$-monoidal natural transformation $\theta : (R, \rho^R) \Rightarrow (S, \sigma^S)$ via $\theta_a := \text{mate}(\Theta_a) = \Theta_a$ under the identity adjunction

\[
\mathcal{V} (1_\mathcal{V} \to B (R (a) \to S (a))) = B (R (a) \to S (a)).
\]

Proposition 6.11. As defined, $\theta$ is a $\mathcal{V}$-monoidal natural transformation.

Proof. First, naturality: We need to show that

\[
\begin{align*}
\mathcal{B} (R (a) \to S (b)) &= \mathcal{B} (R (a) \to S (b)) \\
\mathcal{A} (a \to b) &= \mathcal{A} (a \to b)
\end{align*}
\]
By lemma 4.7 and expanding, the mate of the left hand side is

\[
S(b) \Rightarrow \Sigma_\alpha \Rightarrow R(a) \Rightarrow F_B(A(a \to b))
\]

where the last equality uses naturality and monoidality of \( \Theta \). Now appealing to the coherence on \( \rho \) and \( \sigma \) and recognizing the mate of \( S_{a \to b} \), this becomes

\[
S(b) \Rightarrow \Sigma_\alpha \Rightarrow R(a) \Rightarrow F_B(A(a \to b))
\]

which is exactly the mate of the right hand side. To show monoidality, we need to check that

\[
B(R(a) R(b) \to S(ab))
\]

The mate of the left hand side, after using lemma 4.7, monoidality of \( \Theta \), and lemma 4.8, becomes

\[
S(ab) \Rightarrow \Theta_{ab} \Rightarrow R(a) R(b) \Rightarrow \rho_{a,b} \Rightarrow \sigma_{a,b}
\]

which is exactly the mate of the right hand side under lemma 4.7.

Thus we have proven

**Proposition 6.12.** The 2-functor \( P \) is fully faithful on 2-cells.

In turn, this finishes the proof of our main theorem:

**Theorem 6.13.** The 2-functor \( P : \mathbf{VMonCat} \Rightarrow \mathbf{VModTens} \) is a 2-equivalence.

**Remark 6.14.** If we instead demand that all of our lax functors are in fact strong, i.e., we restrict to the subcategories where 1-cells are only the strong monoidal and strong \( \mathbf{V} \)-monoidal functors, then we get a 2-equivalence by the same proof. Showing that a lax monoidal functor is in fact strong is simply a check that the laxitors are invertible, and this translates directly through \( P \), since \( P \) is the identity on each of the laxitors.
7. $G$-gradings

For the remainder of the article, fix a finite group $G$ and assume that $\mathcal{V}$ is linear.

**Definition 7.1.** We say that a $\mathcal{V}$-category $\mathcal{C}$ is additive if its underlying category is additive. We say a $\mathcal{V}$-functor between additive $\mathcal{V}$-categories is additive if its underlying functor is additive.

**Definition 7.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. We define the external direct sum $\mathcal{A} \oplus \mathcal{B}$ to be the $\mathcal{V}$-category with objects $a \oplus b$ for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, and hom objects given by $(\mathcal{A} \oplus \mathcal{B})(a \oplus b_1 \to a \oplus b_2) := \mathcal{A}(a_1 \to a_2) \oplus \mathcal{B}(b_1 \to b_2)$, using the direct sum on objects of $\mathcal{V}$.

**Definition 7.3.** A $G$-graded $\mathcal{V}$-monoidal category is a $\mathcal{V}$-monoidal category $\mathcal{C}$ along with a decomposition $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, such that for each $c_g \in \mathcal{C}_g$ and $c_h \in \mathcal{C}_h$, we have $c_g c_h \in \mathcal{C}_{gh}$. We say that the $G$-grading is faithful if for each $g \in G$ there exist $a_g, b_g \in \mathcal{C}_g$ such that $\mathcal{C}_g(a_g \to b_g) \neq 0_V$. If $\mathcal{C}$ and $\mathcal{D}$ are $G$-graded $\mathcal{V}$-monoidal categories, a $\mathcal{V}$-monoidal functor $(F, v) : \mathcal{C} \to \mathcal{D}$ is called $G$-graded if the underlying functor $F$ is $G$-graded, i.e., $F$ is additive and for every $g \in G$ and every $c_g \in \mathcal{C}_g$ we have $F(c_g) \in \mathcal{D}_g$. We say that a $G$-graded $\mathcal{V}$-monoidal functor $(F, v)$ is an equivalence if it is a $\mathcal{V}$-monoidal equivalence of $\mathcal{V}$-monoidal categories. Note that in this case, $F|_{\mathcal{C}_g} : \mathcal{C}_g \to \mathcal{D}_g$ is a $V$-equivalence for all $g \in G$, and is a $\mathcal{V}$-monoidal equivalence when $g = e$.

**Definition 7.4.** ([JMPP19]) A $G$-graded $\mathcal{V}$-module tensor category consists of a linear monoidal category $\mathcal{C}$, a decomposition $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ into linear categories $\mathcal{C}_g$, and a braided strong monoidal functor $\mathcal{F}_C : \mathcal{V} \to Z(\mathcal{C})$ such that $\mathcal{F}_C^2(\mathcal{V}) \subseteq \text{Rep}(G)' \subseteq Z(\mathcal{C})$ (equivalently, $\mathcal{F}_C(\mathcal{V}) \subseteq C_G$). Given $(\mathcal{C}, \mathcal{F}_C)$ and $(\mathcal{D}, \mathcal{F}_D)$ $G$-graded tensored $\mathcal{V}$-module tensor categories, a morphism $(R, \rho, r) : \mathcal{C} \to \mathcal{D}$ is called $G$-graded if $R$ is additive and for $g \in G$ and $c_g \in \mathcal{C}_g$, $R(c_g) \in \mathcal{D}_g$. We say that $(R, \rho, r)$ is an equivalence if $(R, \rho)$ is a strong monoidal equivalence and $r$ is a natural isomorphism.

7.1. $G$-graded correspondence. On the way to a $G$-graded version of Theorem 1.2, we define $G$-graded analogues to $\mathcal{V}\text{MonCat}$ and $\mathcal{V}\text{ModTens}$ and construct a correspondence on $0$-cells.

**Definition 7.5.** Define the $2$-category $\mathcal{V}\text{MonCat}_G$ as follows:

- $0$-cells are $G$-graded (weakly) tensored $\mathcal{V}$-monoidal categories
- $1$-cells are $G$-graded $\mathcal{V}$-monoidal functors
- $2$-cells are $\mathcal{V}$-monoidal natural transformations.

The same proofs as in Section 3 show that $\mathcal{V}\text{MonCat}_G$ is again a $2$-category under the same compositions. Similarly, we define the $2$-category $\mathcal{V}\text{ModTens}_G$ via

- $0$-cells are $G$-graded (weakly) tensored $\mathcal{V}$-module tensor categories
- $1$-cells are $G$-graded monoidal functors with action-coherence natural transformations
- $2$-cells are monoidal natural transformations satisfying the coherence in 3

Working towards extending Theorem 1.2, we first extend Theorem 1.1 for the $0$-cells:

**Theorem 7.6.** There is a bijective correspondence between equivalence classes:

\[
\begin{align*}
\{ \text{Faithfully } G\text{-graded (weakly) tensored rigid } \mathcal{V}\text{-monoidal categories } \mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \} & \cong \\
\{ \text{Faithfully } G\text{-graded (weakly) tensored rigid } \mathcal{V}\text{-module tensor categories } (\mathcal{C} = \bigoplus_{g \in G} C_g, \mathcal{F}_C^{\mathcal{V}}) \}\ .
\end{align*}
\]

**Proof.** In the following, we revert to notation of [MP19] for clarity, with the underlying category of $\mathcal{C}$ denoted by $\mathcal{C}^V$, and the $\mathcal{V}$-monoidal category associated to a $\mathcal{V}$-module tensor category $(\mathcal{C}, \mathcal{F}_C)$ being denoted by $\mathcal{C}_F^\mathcal{V}$. We use this to check that our abuse of notation is valid up to equivalence - here a $G$-graded monoidal equivalence. We will continue to use ordinary font in subscripts and superscripts to specify relation to the underlying category, e.g., $C$ refers to $\mathcal{C}^V$. We first show that the correspondence in Theorem 1.1 can be extended to preserve $G$-gradings.

Suppose we start with a (weakly) tensored $G$-graded $\mathcal{V}$-monoidal category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$. We show first that $\mathcal{F}_C(\mathcal{V}) \subseteq \mathcal{C}_G$. Fix $b \in \mathcal{C}_g$ for some nonidentity $g \in G$; we know $\mathcal{C}(1_\mathcal{C} \to b) \cong 0_V$ since $\mathcal{C}$ is $G$-graded (and $1_\mathcal{C} \in \mathcal{C}_e$), so for all $v \in V$,

\[0_{\text{vec}} \cong \mathcal{V}(v \to \mathcal{C}(1_\mathcal{C} \to b)) \cong \mathcal{C}^V(\mathcal{F}_C(v) \to b) .\]
Thus $\mathcal{C}^V (\mathcal{F}_C (v) \to b) \cong 0_{\text{vec}}$ for all $b$ not in $\mathcal{C}^V$, so $\mathcal{F}_C (V) \subseteq \mathcal{C}_e$. We also have, with all direct sums over $g \in G$,

$$
\mathcal{C}^V \left( \bigoplus a_g \to \bigoplus b_g \right) = \mathcal{V} \left( 1 \to \mathcal{C} \left( \bigoplus a_g \to \bigoplus b_g \right) \right)
$$

$$
= \mathcal{V} \left( 1 \to \bigoplus \mathcal{C}_g \left( a_g \to b_g \right) \right)
$$

$$
\cong \bigoplus \mathcal{V} \left( 1 \to \mathcal{C}_g \left( a_g \to b_g \right) \right)
$$

so $\mathcal{C}^V$ is equivalent to $\bigoplus \mathcal{C}_g^V$, and thus $(\mathcal{C}^V, \mathcal{F}_C)$ is a (weakly) tensored $G$-graded $V$-module tensor category.

Now suppose we have a $G$-graded $V$-module tensor category $(C = \bigoplus C_g, \mathcal{F}_C)$. Then we get a $V$-monoidal category $C_{\mathcal{F} \mathcal{F}}$ by 5.1. Further, $C \cong \bigoplus C_g$ as $V$-module categories with the actions $(v, c_g) \mapsto c_g \mathcal{F}_C (v)$; this is valid because $\mathcal{F}_C (v) \subseteq \mathcal{C}_e$, so $c_g \mathcal{F}_C (v)$ is again in $\mathcal{C}_e$. Since every element of $\mathcal{C}$ can be written as a direct sum of objects from the $\mathcal{C}_g$, we have the following natural isomorphisms:

$$
\mathcal{V} \left( v \to \mathcal{C} \left( \bigoplus a_g \to \bigoplus b_g \right) \right) \cong \mathcal{C}^V \left( \left( \bigoplus a_g \right) \mathcal{F}_C (v) \to \bigoplus b_g \right)
$$

$$
\cong \bigoplus \mathcal{C}_g^V \left( (a_g \mathcal{F}_C (v)) \to b_g \right)
$$

$$
\cong \bigoplus \mathcal{V} \left( v \to \mathcal{C}_g \left( a_g \to b_g \right) \right)
$$

$$
\cong \mathcal{V} \left( v \to \bigoplus \mathcal{C}_g \left( a_g \to b_g \right) \right)
$$

$$
= \mathcal{V} \left( v \to \left( \bigoplus \mathcal{C}_g \right) \left( \bigoplus a_g \to \bigoplus b_g \right) \right).
$$

Now by the Yoneda Lemma we see that $\mathcal{C}$ is equivalent to $\bigoplus \mathcal{C}_g$ as $V$-monoidal categories, so $\mathcal{C}$ is a $G$-graded $V$-monoidal category.

To finish the proof, we see that in [MP19, sections 6 and 7] the authors construct a strong monoidal equivalence of categories $\mathcal{C} \cong C_{\mathcal{F} \mathcal{F}}$, and extend it to an equivalence of $V$-module tensor categories. They analogously construct an equivalence of the $V$-monoidal categories $\mathcal{C} \cong \mathcal{C}^V$. Both equivalences are the identity on objects, so both are $G$-graded.

\[\Box\]

**Theorem 7.7.** Theorem 1.2 extends to an equivalence of 2-categories between $\mathcal{V}\text{MonCat}_G$ and $\mathcal{V}\text{ModTens}_G$.

**Proof.** We construct a 2-functor $P_G : \mathcal{V}\text{MonCat}_G \to \mathcal{V}\text{ModTens}_G$ in the same way as in Section 5. The 0-cells in the image of $P_G$ are $G$-graded by the above, and the 1-cells are $G$-graded since $P_G$ doesn’t change how 1-cells act on objects; 2-cells are unchanged from the non-$G$-graded case. Since $\mathcal{V}\text{MonCat}_G$ and $\mathcal{V}\text{ModTens}_G$ have the same composition as $\mathcal{V}\text{MonCat}$ and $\mathcal{V}\text{ModTens}$, respectively, it follows that $P_G$ is a 2-functor.

To check that $P_G$ is a 2-equivalence, we again just need to check that it is essentially surjective on 0-cells, essentially surjective on 1-cells, and fully faithful on 2-cells. Essential surjectivity on 0-cells is exactly Theorem 7.6.

Next, let $\mathcal{C}$ and $\mathcal{D}$ be 0-cells in $\mathcal{V}\text{MonCat}_G$ and consider a 1-cell $(R, \rho^R) : \mathcal{C} \to \mathcal{D}$ in $\mathcal{V}\text{MonCat}$. Since $R (a) = P (R) (a)$ for all objects $a \in \mathcal{C}$, it follows that $R$ is $G$-graded if and only if $P (R)$ is $G$-graded. Thus the same construction as for 1-cells in $\mathcal{V}\text{MonCat}$ and $\mathcal{V}\text{ModTens}$ gives us a map from 1-cells in $\mathcal{V}\text{MonCat}_G$ to 1-cells in $\mathcal{V}\text{ModTens}_G$. Since $P$ is essentially surjective on 1-cells, it now follows that $P_G$ is as well.

Given two 1-cells $R, S$ between $G$-graded $V$-monoidal categories, note that $R$ and $S$ are still 1-cells in $\mathcal{V}\text{MonCat}$ as well. Since 2-cells don’t have any inherent $G$-grading, we have $\mathcal{V}\text{MonCat}_G (R \to S) = \mathcal{V}\text{MonCat} (R \to S)$. This also holds for $\mathcal{V}\text{ModTens}$ and $\mathcal{V}\text{ModTens}_G$, so $P_G$ is fully faithful on 2-cells as before.

\[\Box\]

**8. G-extensions**

Now fix a finite group $G$, a tensored $V$-monoidal category $\mathcal{A}$ and its corresponding tensored $V$-module tensor category $(A, F_A^\mathcal{C})$. For this section assume that $V$ is linear and $F_A^\mathcal{C}$ is strong monoidal.

In this section we extend Theorem 7.7 to the setting of $G$-extensions of $\mathcal{A}$ and $(A, F_A)$. We construct $\mathcal{V}\text{MonCat}_G^\mathcal{A}$ and $\mathcal{V}\text{ModTens}_G^\mathcal{A}$, prove a correspondence between zero cells, and extend the correspondence to an equivalence of 2-categories.

**Definition 8.1.** A $G$-graded extension of $\mathcal{A}$ is a faithfully $G$-graded $V$-monoidal category $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ equipped with a $V$-monoidal equivalence $(I^C, \iota^C) : \mathcal{A} \to \mathcal{C}$. For example, any faithfully $G$-graded $V$-monoidal category $\mathcal{C}$, equipped
with the identity on \( \mathcal{C} \) is a \( G \)-extension of \( \mathcal{C} \). If \((\mathcal{C}, I^C, I^C)\) and \((D, I^D, I^D)\) are \( G \)-extensions of \( \mathcal{A} \). We define a morphism from \( \mathcal{C} \) to \( D \) to be a triple \((\mathcal{R}, \rho, p)\), where \((\mathcal{R}, \rho)\) is a \( G \)-graded strong \( V \)-monoidal functor, and \( p \) is a \( V \)-monoidal natural transformation \( p : I^D \Rightarrow I^C \circ R_e \).

**Definition 8.2.** A G-graded extension of \((A, F_A^Z)\) is a faithfully G-graded rigid tensored \( V \)-module tensor category \((C = \bigoplus_{g \in G} C_g, F^Z_C) : A \rightarrow \mathcal{C} \) of \( V \)-module tensor categories. If \((C, F^Z_C, I^C, I^C)\) and \((D, F^Z_D, I^D, I^D)\) are \( G \)-extensions of \((A, F_A^Z)\), we define a morphism from \( C \) to \( D \) to be a tuple \((R, \rho, r, p)\), where \((R, \rho, r) : C \rightarrow D\) is a 1-cell in \( \text{VModTens}_G \) \((C \rightarrow D)\), and \( p : I^D \Rightarrow I^C \circ R_e \) is a monoidal natural transformation compatible with \( r \): We require

\[
\begin{array}{c}
\begin{array}{c}
R_e (I^C (F_A (\psi))) \\
F_D (\varphi)
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
R_e (I^C (\varphi)) \\
F_D (\psi)
\end{array}
\end{array}
\]

\[(8.1)\]

**Definition 8.3.** Define the 2-category \( \text{VMonCat}_G^A \) via:

- 0-cells are the \( G \)-extensions of \( \mathcal{A} \).
- 1-cells are the morphisms defined above.
- 2-cells are the \( I_V \)-graded monoidal natural transformations \( \Theta : (R, \rho, p) \Rightarrow (S, \sigma, q) : \mathcal{C} \rightarrow \mathcal{D} \) such that \((p_a \Theta) \circ (- \circ \rho) = q_a\) for each \( a \in \mathcal{C} \).

We define the 2-category \( \text{VModTens}_G^A \) via:

- 0-cells are the \( G \)-extensions of \((A, F_A^Z)\).
- 1-cells are the morphisms defined above.
- 2-cells are the \( I_V \)-graded monoidal natural transformations \( \Theta : (R, \rho, r, p) \Rightarrow (S, \sigma, q, r) : C \rightarrow D \) such that for all \( v \in \mathcal{V} \), \( r_v \circ \Theta_{I_v, (\psi, (\varphi))} = s_v \), and for all \( a \in C \), \( p_a \circ \Theta_{I_a, (\varphi)} = q_a \).

We again use the same composition as for \( \text{VMonCat} \) and \( \text{VModTens} \), with the slight change that for 1-cells we now need to compose the new coherences \( p \) as well; we take the usual vertical composition of \( (I_V \text{-graded}) \)-monoidal natural transformations. All checks that \( \text{VMonCat}_G^A \) and \( \text{VModTens}_G^A \) are 2-categories are analogous to the checks for \( \text{VMonCat} \) and \( \text{VModTens} \).

**Theorem 8.4.** There is a bijective correspondence between equivalence classes

\[
\left\{ \begin{array}{c}
\text{\( V \)-monoidal \( G \)-extensions}
\end{array} \right\}
\cong
\left\{ \begin{array}{c}
\text{\( G \)-extensions}
\end{array} \right\}.
\]

**Proof.** From Theorem 7.6, we have a correspondence of 0-cells \( P_G : \text{VMonCat}_G^A \rightarrow \text{VModTens}_G^A \). Given a 0-cell \((\mathcal{C}, I, i)\) in \( \text{VMonCat}_G^A \), we get a \( G \)-graded \( V \)-module tensor category \((C, F_C) : A \rightarrow \mathcal{C} \) via the map on 0-cells induced by \( P_G \). By the map on 1-cells induced by \( P_G \), we also get a \( V \)-monoidal equivalence \((I, i, i) : (I, i)\) from \((I, i)\). Thus we have a 0-cell \((C, F_C, I, i, i)\) in \( \text{VModTens}_G^A \). Further, both of the induced maps from \( P_G \) are bijections on equivalence classes of 0-cells and 1-cells, respectively, it follows that we get a bijection from equivalence classes of 0-cells in \( \text{VMonCat}_G^A \) to 0-cells of \( \text{VModTens}_G^A \).

8.1. \( G \)-extension 2-functor. We lift our 2-functor \( P_G : \text{VMonCat}_G^A \rightarrow \text{VModTens}_G^A \) to a 2-functor \( P_G^A : \text{VMonCat}_G^A \rightarrow \text{VModTens}_G^A \). On 0-cells, we begin the definition of \( P_G^A \) with the correspondence in Theorem 8.4: From a 0-cell \((\mathcal{C}, I^C, I^C)\) in \( \text{VMonCat}_G^A \), we use Theorem 8.4 to get a 0-cell in \( \text{VModTens}_G^A \).

Given 0-cells \((\mathcal{C}, I^C, I^C), (\mathcal{D}, I^D, I^D)\) in \( \text{VMonCat}_G^A \) and a 1-cell \((\mathcal{R}, \rho, p) \in \text{VMonCat}_G^A (\mathcal{C} \rightarrow \mathcal{D})\), we construct the first three components of a 1-cell, \((R, \rho, r)\), as in Theorem 7.7. Lastly, we need a coherence natural isomorphism \( p \). We notice that

\[
p_a \in \mathcal{V} (1_V \Rightarrow \mathcal{D} (I_D (a) \Rightarrow S (\mathcal{R} (I_B (a)))) = D (I_D (a) \Rightarrow S (\mathcal{R} (I_B (a))))
\]

so we use the same one, i.e., we take the mate of \( p \) under this identity adjunction.
Lemma 8.5. Considered together, \((R, \rho, r, p)\) make up a 1-cell in \(\text{VModTens}_G^A\). That is, coherence 8.1 is satisfied:

\[
\begin{align*}
R_c(\mathcal{F}_A(a)) &\quad \quad R_c(\mathcal{F}_A(a)) \\
P_{\mathcal{F}_A(a)} &\quad \quad P_{\mathcal{F}_A(a)} \\
\mathcal{F}_D(a) &\quad \quad \mathcal{F}_D(a)
\end{align*}
\]

Proof. Taking the mate of the left hand side and applying naturality and unitality of \(p\), we have

\[
\begin{align*}
\mathcal{D}(1_{\mathcal{D}} &\rightarrow R_c(\mathcal{F}_A(a))) \\
\mathcal{D}(1_{\mathcal{D}} &\rightarrow R_c(\mathcal{F}_A(a)))
\end{align*}
\]

On the right hand side, taking mates, using functoriality of \(R\), and applying Lemma 4.6 gives

\[
\begin{align*}
\mathcal{D}(1_{\mathcal{D}} &\rightarrow R_c(\mathcal{F}_A(a))) \\
\mathcal{D}(1_{\mathcal{D}} &\rightarrow R_c(\mathcal{F}_A(a)))
\end{align*}
\]

which is the same as the mate of the left hand side above. \(\square\)

Lastly, we need to work with 2-cells. Given 1-cells \((\mathcal{R}, \rho, p), (\mathcal{S}, \sigma, q) : \mathcal{C} \rightarrow \mathcal{D}\) and a 2-cell \(\theta : \mathcal{R} \Rightarrow \mathcal{S}\), we construct \(\Theta : (R, \rho, r, p) \Rightarrow (S, \sigma, s, q)\) via \(\Theta_a = \theta_a\) using the identity adjunction

\[
\mathcal{V}(1_{\mathcal{V}} \rightarrow \mathcal{D}(\mathcal{R}(a) \rightarrow \mathcal{S}(a))) = D(R(a) \rightarrow S(a)).
\]

Lemma 8.6. Under these definitions, we have \(p_a \circ \Theta_{\mathcal{F}_A(a)} = q_a\).

Proof. Taking mates on both sides gives exactly the coherence condition on \(\theta\). \(\square\)

Proposition 8.7. The above construction of \(P_A^A : \text{VMonCat}_G^A \rightarrow \text{VModTens}_G^A\) defines a 2-functor.

Proof. We already have a map on 0-cells, and for each pair of 0-cells \(\mathcal{C}\) and \(\mathcal{D}\) in \(\text{VMonCat}\), we can use the above maps on 1-cells and 2-cells to construct the functor

\[
P_A^A : \text{VMonCat}_G^A \left( (\mathcal{C}, I^C, t^C) \rightarrow (\mathcal{D}, I^D, t^D) \right) \rightarrow \text{VModTens}_G^A \left( P_A^A \left( (\mathcal{C}, I^C, t^C) \right) \rightarrow P_A^A \left( (\mathcal{D}, I^D, t^D) \right) \right)
\]

Now we just need to check that \(P_A^A\) as defined here is a 2-functor:

- For each 0-cell \((\mathcal{C}, I^C, t^C)\), both \(id_{P_A^A(\mathcal{C})}\) and \(P_A^A(\mathcal{C})\) are the identity 1-cell on \(\mathcal{C}\).
- Composition of 1-cells is preserved by \(P_A^A\). To see this, notice that the only change from Section 7.1 is that we now have an extra coherence natural transformation \(p\). However, \(p\) is mapped directly to its mate under adjunction 4.1, and mates preserve composition (Lemma 4.7).
- Composition of 2-cells is unchanged from previous sections, so both horizontal and vertical composition of 2-cells are preserved.

\(\square\)
8.2. $G$-extension equivalence. Now to finish the equivalence, we need to show that $P^R_G$ is fully faithful and essentially surjective.

**Lemma 8.8.** The 2-functor $P^R_G$ is essentially surjective on 0-cells.

**Proof.** This is exactly Theorem 8.4. □

**Lemma 8.9.** The 2-functor $P^R_G$ is also essentially surjective on 1-cells.

**Proof.** Given $(R, p, r, p) : (C, I^C, i^C, i^C) \to (D, I^D, i^D, i^D)$ a 1-cell of $G$-graded extensions of $(A, F_A)$, we construct $(R, p)$ as before, and again take mates under the identity adjunction for $p$. The only check needed to prove that $(R, p, r, p)$ is a 1-cell in $\mathcal{V}MonCat^R_G$ is to show that $p$ is a $\mathcal{V}$-monoidal natural transformation. Monoidality follows directly from taking mates and applying monoidality of $p$, but naturality is more involved. We need to show that

\[ p_R \cdot i^D(a) \to R(I^C(b)) \]

starting with the right hand side, taking mates and using naturality of $p$ twice (note the composite laxitor), we get

\[ p_R \cdot i^D(a) \to R(I^C(b)) \]

Next, we use the coherence on $p$ and $r$ and recognize the mate of the composite functor $I^C \circ R$:

\[
\begin{align*}
R(I^C(b)) & \quad R(I^C(b)) & \quad R(I^C(b)) \\
p_R & \quad p_R & \quad p_R \\
D_{F_A(A(a \to b))} & \quad D_{F_A(A(a \to b))} & \quad D_{F_A(A(a \to b))} \\
D_{F_A(A(a \to b))} & \quad D_{F_A(A(a \to b))} & \quad D_{F_A(A(a \to b))} \\
D_{F_A(A(a \to b))} & \quad D_{F_A(A(a \to b))} & \quad D_{F_A(A(a \to b))} \\
j^D(a) & \quad j^D(a) & \quad j^D(a) \\
f_D(A(a \to b)) & \quad f_D(A(a \to b)) & \quad f_D(A(a \to b)) \\
\end{align*}
\]

and this is exactly the mate of the left hand side. Thus we get a 1-cell in $\mathcal{V}MonCat^R_G$. With the exception of the new coherence natural transformation $p$, this map was already shown to be an essential inverse to $P^R_G$ in Theorem 7.7. But we've also inverted $P^R_G$ for $p$, since $P^R_G$ takes $p$ to itself, so $P^R_G$ is essentially surjective on 1-cells. □

**Lemma 8.10.** Lastly, $P^R_G$ is fully faithful on 2-cells.
Proof. Given \((R, \rho, r, p), (S, \sigma, s, q) : (C, F^C, i^C, t^C) \to (D, F^D, i^D, t^D)\) 1-cells of \(G\)-extensions of \((A, F_A)\) and \(\Theta : R \Rightarrow S\) a 2-cell between them, we construct \(\theta : R \Rightarrow S\) via \(\theta_a = \Theta_a\), as before. We already know that \(\theta\) is a \(V\)-monoidal natural transformation by the work before we introduced \(G\)-extensions. The last coherence on \(\theta\) follows immediately from taking mates as when we constructed \(\Theta\) going the other way in the previous section. Lastly, since we’re defining \(\theta = \Theta\) at the component level, this map is a bijection, and thus \(P^A_G\) is fully faithful on 2-cells.

\(\square\)

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