Melnikov method for non-conservative perturbations of the restricted three-body problem

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Abstract
We consider the planar circular restricted three-body problem, as a model for the motion of a spacecraft relative to the Earth–Moon system. We focus on the collinear equilibrium points \( L_1 \) and \( L_2 \). There are families of Lyapunov periodic orbits around either \( L_1 \) or \( L_2 \), forming Lyapunov manifolds. There also exist homoclinic orbits to the Lyapunov manifolds around either \( L_1 \) or \( L_2 \), as well as heteroclinic orbits between the Lyapunov manifold around \( L_1 \) and the one around \( L_2 \). The motion along the homoclinic/heteroclinic orbits can be described via the scattering map, which gives the future asymptotic of a homoclinic orbit as a function of the past asymptotic. In contrast to the more customary Melnikov theory, we do not need to assume that the asymptotic orbits have a special nature (periodic, quasi-periodic, etc.). We add a non-conservative, time-dependent perturbation, as a model for a thrust applied to the spacecraft for some duration of time, or for some other effect, such as solar radiation pressure. We compute the first-order approximation of the perturbed scattering map, in terms of fast convergent integrals of the perturbation along homoclinic/heteroclinic orbits of the unperturbed system. As a possible application, this result can be used to determine the trajectory of the spacecraft upon using the thrust.

Keywords Melnikov method · Homoclinic and heteroclinic orbits · Three-body problem · Astrodynamics

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1 Introduction

1.1 Motivation

A motivation for this work is the following situation from astrodynamics. Consider a spacecraft traveling between the Earth and the Moon. Assume that the spacecraft is coasting along the stable/unstable hyperbolic invariant manifolds associated with the periodic orbits near one of the center-saddle equilibrium points, at some fixed energy level. Such an orbit is driven by the gravitational fields of the Earth and the Moon and does not require using the thrusters. The total energy is preserved along the orbit. One can describe the motion of the spacecraft in terms of some geometrically defined coordinates: an ‘action’ coordinate describing the size of the periodic orbit associated with the stable/unstable invariant manifold, an ‘angle’ coordinate describing the asymptotic phase of the motion, and a pair of ‘hyperbolic’ coordinates describing the position of the spacecraft relative to the corresponding stable/unstable manifold.

Suppose now that we want to make a maneuver in order to jump from the hyperbolic invariant manifold on the given energy level to another hyperbolic invariant manifold on a different energy level. Mathematically, turning on the thrusters amounts to adding a small, non-conservative, time-dependent perturbation to the original system. If such a perturbation is given, we would like to estimate its effect on the orbit of the spacecraft. More precisely, we would like to compute the change in the action and angle coordinates associated with the orbit as a result of applying the perturbation. We want to obtain such an estimate in terms of the original trajectory of the unperturbed system, and of the particular perturbation.

Other non-Hamiltonian perturbations, for instance, due to solar radiation pressure and solar wind, can be considered (see Milani 1987).

In this paper, we investigate the general problem of adding a non-Hamiltonian perturbation to a homoclinic/heteroclinic trajectory and computing the effect on the homoclinic/heteroclinic orbits.

Note that adding a non-Hamiltonian perturbation (e.g., a small dissipation) may destroy all the periodic orbits. Nevertheless, the normally hyperbolic manifold, formed by the collection of periodic orbits, persists (see details later). In contrast to the most customary versions of the Melnikov theory, which assume that the asymptotic orbits are periodic or quasiperiodic and that they preserve their nature under perturbation, we consider homoclinic excursions to a normally hyperbolic manifold. The asymptotic orbits could change their nature under the perturbations. For example, a family of Lyapunov periodic orbits subject to a small dissipation could get transformed into a family of orbits that converge to a critical point (see de la Llave and Kogelbauer 2019).

1.2 Brief description of the main results and methodology

While we consider a general setup and derive some general results on the effect of non-conservative, time-dependent perturbation on homoclinic orbits, the main motivation of this work resides with the PCRTBP.
In the PCRTB, the family of periodic orbits about either $L_1$ or $L_2$ forms a normally hyperbolic invariant manifold (NHIM). Each NHIM has hyperbolic stable and unstable manifolds. We will assume that the stable and unstable manifolds of the NHIM corresponding to either $L_1$ or $L_2$ intersect transversally and also that the unstable (stable) manifold of the NHIM corresponding to $L_1$ intersects transversally the stable (unstable) manifold of the NHIM corresponding to $L_2$. This assumption can be verified numerically for a wide range of energy levels and mass parameters in the PCRTBP (see, e.g., Koon et al. 2000). Thus, there exist homoclinic orbits to either one of the NHIMs, as well as heteroclinic orbits between the two NHIMs. There exist scattering maps associated with the transverse homoclinic intersections, as well as to the transverse heteroclinic intersections.

There exist some neighborhoods of $L_1$ and of $L_2$ where the Hamiltonian of the PCRTBP can be written in a normal form, via some suitable symplectic action–angle and hyperbolic coordinates $(I, \theta, y, x)$. In particular, each NHIM can be parametrized in terms of the action–angle coordinates $(I, \theta)$. Therefore, the scattering map can also be described in terms of these coordinates. In the unperturbed case, the scattering map is particularly simple: it is a shift in the angle coordinate (a phase shift).

The fact that we use normal form coordinates to express the Hamiltonian, and we subsequently estimate the scattering map in terms of the action–angle coordinates, is a matter of practical convenience. Normal forms are often used to compute numerically, with high precision, the periodic orbits and the NHIMs around the equilibrium points, as well as the corresponding stable and unstable manifolds, e.g., Jorba (1999) and Gómez et al. (2001a, b, c, d).

For applications, it is important to note that the scattering map for the PCRTBP can be computed numerically with high precision; see Canalias et al. (2006), Delshams et al. (2016) and Capinski et al. (2016).

We study the effect of a small, non-Hamiltonian, time-dependent perturbation that is added to the system. Provided that the perturbation is small enough, the NHIMs will persist (Fenichel 1971), although periodic orbits inside the NHIMs may disappear. Also, the transverse homoclinic/heteroclinic orbits, hence the scattering map, will survive in the perturbed system.

The main contribution is that we compute the effect of the non-conservative, time-dependent perturbation on the action and angle components of the scattering map. More precisely, we use Melnikov theory to provide explicit estimates—up to first order with respect to the size of the perturbation—for the difference between the perturbed scattering map and the unperturbed one, relative to the action and angle coordinates. The resulting expressions are given in terms of fast convergent improper integrals of the perturbation evaluated along segments of homoclinic/heteroclinic orbits of the unperturbed system. One important aspect in the computation is that, in the perturbed system, the action is a slow variable, while the angle is a fast variable.

We stress that, unlike the usual treatments of Melnikov theory, when one studies orbits homoclinic to hyperbolic fixed points, periodic or quasi-periodic orbits, here we study orbits homoclinic to NHIMs. The asymptotic dynamics inside the NHIMs may change under the perturbation.

The effect of the perturbation on the action–angle components of the scattering map can be interpreted, in the context of astrodynamics, as follows. The difference in the action coordinates between the perturbed and the unperturbed scattering map can be interpreted as the change in energy due to the maneuver, or equivalently, the change in the ‘size’ of the periodic orbit which the homoclinic/heteroclinic orbit is asymptotic to. The difference in the angle coordinates between the perturbed and the unperturbed scattering map can be interpreted as the change in asymptotic phase due to the maneuver.
We mention here that there are numerous works on using hyperbolic invariant manifolds to design low-energy space mission, see, for example, Koon et al. (2000), Gómez et al. (2001a, b, c, d), Belbruno (2004), Belbruno et al. (2010) and Parker and Anderson (2014), and the references listed there. We hope that our results can be used to optimize the thrust that needs to be applied in order to maneuver between hyperbolic invariant manifolds on different energy levels.

1.3 Related works

The study of Hamiltonian systems subject to non-conservative perturbations is of practical interest in physical models, such as in celestial mechanics, where dissipation leads to migration of satellites and spacecrafts (Milani 1987; Gkolias et al. 2017; de la Llave and Kogelbauer 2019; Calleja et al. 2020).

Computation of the scattering map, similar to the one in this paper, has been done in the case of the pendulum–rotator model subject to Hamiltonian perturbations, e.g., Delshams et al. (2008) and Gidea and de la Llave (2018). The rotator–pendulum model is a product system and is naturally endowed with action–angle and hyperbolic coordinates. It has two conserved quantities: the action of the rotator and the total energy of the pendulum. The effect of the perturbation on the action component of the scattering map is relatively easy to compute directly. On the other hand, the effect on the angle component of the scattering map is more complicated to compute, since this is a fast variable. To circumvent this difficulty, the papers Delshams et al. (2008) and Gidea and de la Llave (2018) use the symplecticity of the scattering map to estimate indirectly the effect of the perturbation on the angle component of the scattering map.

The perturbed scattering map has been computed in the case of the pendulum–rotator model subject to non-conservative perturbations in Gidea et al. (2021). Since the perturbations are not Hamiltonian, the symplectic argument used in Delshams et al. (2008) can no longer be applied. Therefore, to determine the effect of the perturbation on the angle component of the scattering map, a direct computation is performed in Gidea et al. (2021).

The PCRTBP model considered in this paper presents some significant differences from the pendulum–rotator model. First, it has only one conserved quantity, namely the total energy. Second, the PCRTBP is not a product system and does not carry a globally defined system of action–angle and hyperbolic coordinates. Third, in the unperturbed case the stable and unstable manifolds associated with a NHIM do not coincide. For these reasons, we construct locally defined systems of action–angle and hyperbolic coordinates along the unstable manifold as well as along the stable manifold, respectively. At the intersection of the unstable and stable manifolds, the two coordinate systems do not agree in general. So the computation of the perturbed scattering map has to take into account the ‘mismatch’ between these coordinate systems. The dynamics in these coordinate systems fails to be of product type, as there is a coupling between the action–angle and the hyperbolic coordinates, which also needs to be taken into account in the computation. All of these features make the computation of the perturbed scattering map for the planar circular restricted three-body problem more intricate than for the rotator–pendulum system. Some of the calculations are simplified taking advantage that some of the variables in the unperturbed system are slow variables, but we can deal with perturbation theory for fast variables by observing that, near the NHIMs, the difference between the variables and their asymptotic values is slow (a technique already used in Gidea et al. (2021).
1.4 Structure of the paper

In Sect. 2, we describe a general setup for two-degrees-of-freedom Hamiltonian systems subject to non-conservative, time-dependent perturbations. We also state the main result, Theorem 2.1. It provides an expansion of the perturbed scattering map in terms of the unperturbed scattering map, where the first-order term in the expansion is given explicitly in Sect. 5, in Proposition 5.2 and Proposition 5.7. Section 3 describes how to verify the hypotheses of Theorem 2.1 in the context of the PCRTBP. Section 4 defines some suitable coordinate systems, which we use to describe the geometric objects of interest. The proof of the main result is given in Sect. 4.

2 Setup and main result

Consider a (real analytic) \( C^\omega \), symplectic manifold \( (M, \Omega) \) of dimension 4. Each point \( z \in M \) is described via a system of local coordinates \( z = (p, q) \) with \( (p, q) \in \mathbb{R}^2 \times \mathbb{R}^2 \), such that \( \Omega \) relative to these coordinates is the standard symplectic form

\[
\Omega = dp \wedge dq = \sum_{i=1}^{2} dp_i \wedge dq_i. \tag{2.1}
\]

On \( M \), we consider a non-autonomous system of differential equations

\[
\frac{d}{dt} z = \chi^0(z) + \varepsilon \chi^1(z, t; \varepsilon), \tag{2.2}
\]

where \( \chi^0 : M \to TM \) is a \( C^\omega \)-smooth vector field on \( M \), \( \chi^1 : M \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to TM \) is a time-dependent, parameter-dependent \( C^r \)-smooth vector field on \( M \), with \( r \) sufficiently large, and \( \varepsilon \in \mathbb{R} \) is a ‘smallness’ parameter, taking values in the interval \( (-\varepsilon_0, \varepsilon_0) \) around 0. The dependence of \( \chi^1(z, t; \varepsilon) \) on the time \( t \) is assumed to be of a general type—not necessarily periodic or quasi-periodic.

The flow of (2.2) will be denoted by \( \Phi^t_\varepsilon \).

The assumptions that the manifold \( M \) is 4-dimensional and that \( \chi^0 \) is analytic are motivated by applications to celestial mechanics. (We will consider the special case when the unperturbed system is given by the PCRTBP.) We will use the assumption that \( \chi^0 \) is analytic only to be able to quote several normal form theorems. We believe it could be weakened to finitely differentiability at the price of providing some new normal form theorems.

The assumption that \( \chi^1 \) is only \( C^r \) and possibly non-Hamiltonian is also motivated by applications, as \( \chi^1 \) can be chosen to model the thrust applied to a spacecraft for some time. In particular, \( \chi^1 \) can have compact support in space and in time. Note that even if the perturbation were analytic, the NHIMs that play a role in our treatment can only be assumed to be finite differentiable. The regularity is limited by ratios between the tangential and normal contraction rates, as well as by the regularity of the perturbation. Since the motion on the unperturbed NHIM is integrable, we have that for \( \varepsilon \) small enough, the tangential rates are close to zero, so that the limitations of regularity due to the rates become irrelevant.

Below, we will require that the vector fields \( \chi^0, \chi^1 \) satisfy additional assumptions.
2.1 The unperturbed system

We assume that the vector field $\mathcal{X}^0$ represents an autonomous Hamiltonian vector field, that is, $\mathcal{X}^0 = J \nabla H_0$ for some $C^\infty$ Hamiltonian function $H_0 : M \to \mathbb{R}$, where $J$ is an almost complex structure compatible with the standard symplectic form given by (2.1), and the gradient $\nabla := \nabla_z$ is with respect to the associated Riemannian metric. The Hamilton equation for the unperturbed system is:

$$\frac{d}{dt} z = J \nabla H_0(z). \quad (2.3)$$

2.1.1 Unperturbed NHIM and action–angle coordinates

We assume that the Hamiltonian flow associated with $H_0$ satisfies the conditions (A-i) and (A-ii) below. In Sect. 3, we will see that these conditions can be verified in the PCTBP.

(A-i) There exists an equilibrium point $L_1$ of saddle–center type, that is, the linearized system $DJ \nabla H_0$ at $L_1$ has eigenvalues of the type $\pm \lambda, \pm i \omega$, with $\lambda, \omega \neq 0$.

Consequently, by the Lyapunov center theorem (Moser 1958), there exists a 1-parameter family of periodic orbits $\lambda_0(h)$, parametrized by the energy level $H = h$, for $h \in (H(L_1), h_1)$ with $h_1$ sufficiently small, such that $\lambda_0(h)$ shrinks to $L_1$ as $h \to H(L_1)$. This family of periodic orbits determines a 2-dimensional manifold $\Lambda_0 \simeq D \times \mathbb{T} \subseteq M$, which is a normally hyperbolic invariant manifold (NHIM) with boundary for the Hamiltonian flow $\Phi^t_0$ of $H_0$, where $D$ is closed interval with non-empty interior contained in $(H(L_1), h_1)$. The notion of a NHIM is recalled in Definition A.1, “Appendix A”.

The NHIM $\Lambda_0$ is symplectic when endowed with the form $\Omega|_{\Lambda_0}$, where $\Omega$ is given by (2.1). Moreover, $\Lambda_0$ is foliated by the periodic orbits $\lambda_0(h)$, i.e., $\Lambda_0 = \bigcup_{h \in D} \lambda_0(h)$. The flow $\Phi^t_0$ on each $\lambda_0(h)$ is a constant speed flow. In particular, the dynamics restricted to the NHIM is integrable. Therefore, $\Lambda_0$ can be parametrized in terms of symplectic action–angle variables $(I, \theta)$, so that each periodic orbit $\lambda_0(h)$ represents a level set $I_h$ of the action variable. The action $I_h$ is uniquely determined by the energy level $H_0 = h$. In fact, as we will see in Sect. 4.1, there exists a system of action–angle and hyperbolic variables $(I, \theta, y, x)$ in a neighborhood of $L_1$ such the Hamiltonian $H_0$ can be written in a normal form, and $(I, \theta)$ on $\Lambda_0$ coincide—up to a phase shift—with the action-angle coordinates described above.

2.1.2 Homoclinic connections

(A-ii) There exists a relatively compact open set $\mathcal{K}$ in $M$ such that the unstable manifold $W^u_\mathcal{K}(\Lambda_0)$ and the stable manifold $W^s_\mathcal{K}(\Lambda_0)$ inside $\mathcal{K}$ intersect transversally along a homoclinic channel $\Gamma_0 \subset \mathcal{K}$.

The definition of a homoclinic channel is given in “Appendix B”, Definition B.1.

The unstable and stable manifolds of each periodic orbit $\lambda_0(h)$ are contained in the same energy level as $\lambda_0(h)$, i.e., $W^u(\lambda_0(h)), W^s(\lambda_0(h)) \subseteq M_h$, where $M_h = \{z \in M \mid H = h\}$. By (A-ii), these manifolds intersect transversally within the energy level. Hence, each homoclinic orbit is asymptotic, in both forward and backwards times, to the periodic orbit $\lambda_0(h)$. The homoclinic channel $\Gamma_0$ consists of a 1-dimensional family of homoclinic orbit segments.

In Sect. 4.1, we will see that the normal form coordinates $(I, \theta, y, x)$ can be extended via the flow $\Phi^t_0$ along neighborhoods of $W^u(\Lambda_0)$ and $W^s(\Lambda_0)$, yielding two systems of action–angle and hyperbolic variables $(I^u, \theta^u, y^u, x^u), (I^s, \theta^s, y^s, x^s)$, respectively. Relative to these...
two systems of coordinates \( W^u(\Lambda_0) \) can be described locally by \( y^u = 0 \) and \( W^s(\Lambda_0) \) can be described locally by \( x^s = 0 \). The coordinate systems \((I^u, \theta^u, y^u, x^u)\) and \((I^s, \theta^s, y^s, x^s)\) do not agree with one another in a neighborhood of the homoclinic channel \( \Gamma_0 \).

### 2.1.3 Unperturbed scattering map associated with a homoclinic channel

Let \( \Omega^- : W^u(\Lambda_0) \to \Lambda_0 \) be the projection mapping defined by \( \Omega^-(z_0) = z_0^- \), where \( z_0^- \in \Lambda_0 \) is the footpoint of the unstable fiber through \( z_0 \in W^u(\Lambda_0) \). Similarly, let \( \Omega^+ : W^s(\Lambda_0) \to \Lambda_0 \) be the projection mapping defined by \( \Omega^+(z_0) = z_0^+ \), where \( z_0^+ \in \Lambda_0 \) is the footpoint of the stable fiber through \( z_0 \in W^s(\Lambda_0) \). The stable and unstable fibers are defined in “Appendix A”, Eq. (A.4).

Consider the homoclinic channel \( \Gamma_0 \) from condition (A-ii). By the definition of a homoclinic channel, \( \Omega^\pm \) restricted to \( \Gamma_0 \) is a diffeomorphism onto its image. To any homoclinic channel, we can associate a scattering map, which is defined in “Appendix B”, Definition B.2. Specifically, the scattering map \( \sigma_0 \) associated with \( \Gamma_0 \) is given by:

\[
\sigma_0 : \Omega^-(\Gamma_0) \subseteq \Lambda_0 \to \Omega^+(\Gamma_0) \subseteq \Lambda_0,
\]

provided that there exists a homoclinic point \( z_0 \in \Gamma_0 \) such that \( \Omega^-(z_0) = z_0^- \) and \( \Omega^+(z_0) = z_0^- \). For more details on the scattering map, see “Appendix B”.

The energy preservations along the stable and unstable manifolds of periodic orbits imply that \( \sigma_0 \) leaves each periodic orbit \( \lambda_0(h) \) invariant, that is, \( \sigma_0(\lambda_0(h)) \subseteq \lambda_0(h) \).

Fixing a point \( z_0 \in \Gamma \), we have that \( \sigma_0(z_0^-) = z_0^+ = z_0^- + \Delta \), for some \( \Delta \) depending on \( z_0 \). The invariance property of the scattering map (B.2), and the fact that \( \Phi^t_0 \) restricted to \( \lambda_0(h) \) is linear implies that

\[
\sigma_0(\Phi^t_0(z_0^-)) = \Phi^t_0(z_0^+) = \Phi^t_0(z_0^- + \Delta) = \Phi^t_0(z_0^-) + \Delta,
\]

for all \( t \) for which \( \Phi^t_0(z_0) \) remains in \( \Gamma_0 \). This implies that, in terms of the action-angle coordinates \((I, \theta)\), \( \sigma_0 \) is given by a shift in the angle

\[
\sigma_0(I, \theta) = (I, \theta + \Delta(I)),
\]

for some function \( \Delta \) that depends differentiable on \( I \). We stress that \( \Delta \) also depends on the choice of the homoclinic channel \( \Gamma_0 \).

### 2.2 The perturbation

The vector field \( \lambda^1 \) is a time-dependent, parameter-dependent vector field on \( M \).

(A-iii) We assume that \( \lambda^1 = \lambda^1(z, t; \varepsilon) \) is \( C^r \)-differentiable in all variables with uniformly bounded derivatives on \( \mathcal{H} \times \mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \), where the set \( \mathcal{H} \) is as in the condition (A-ii).

Above, we assume that \( r \) is suitably large. We will not assume that \( \lambda^1 \) is Hamiltonian. Thus, our setting can be used to model dissipation or forcing applied to a Hamiltonian system. Note that non-Hamiltonian perturbations are very singular, in the sense that periodic and homoclinic orbits may disappear. On the other hand, the NHIMs and their stable and unstable manifolds persist and can be used as the basis for perturbative calculations.
As a particular case, we will also write our results for the case when the perturbation $X^1$ in (2.2) is Hamiltonian, given by

$$X^1(z, t; \varepsilon) = J \nabla_z H_1(z, t; \varepsilon), \quad (2.5)$$

where $H_1$ is a time-dependent, parameter-dependent $C^r$-smooth Hamiltonian function on $M$.

When the perturbation is added to the system, as we will see in Sect. 5.1, the NHIM for the unperturbed system can be continued to a NHIM for the perturbed system, and the transverse homoclinic/heteroclinic orbits for the unperturbed system can be continued to transverse homoclinic/heteroclinic orbits for the perturbed system, provided that the perturbation is sufficiently small. Hence, there exists an associated scattering map for the perturbed system.

The goal is to quantify the effect of the perturbation on the corresponding scattering map.

### 2.3 Extended system

We consider system (2.2) under the conditions (A-i), and (A-ii). We associate to it the extended system

$$\frac{d}{d\tau} z = X^0(z) + \varepsilon X^1(z, t; \varepsilon),$$

$$\frac{d}{d\tau} t = 1, \quad (2.6)$$

which is defined on the extended phase space $\tilde{M} = M \times \mathbb{R}$. We denote $\tilde{z} = (z, t) \in \tilde{M}$. The independent variable will be denoted by $\tau$ from now on. We will denote by $\tilde{\Phi}^\tau_\varepsilon$ the extended flow of (2.6). We have

$$\tilde{\Phi}^\tau_\varepsilon(z, t) = (\Phi^\tau_\varepsilon(z), t + \tau).$$

In the extended phase space, we have the following:

$$\tilde{\Lambda}_0 = \Lambda_0 \times \mathbb{R}$$

is a NHIM with boundary for the extended unperturbed flow $\tilde{\Phi}^\tau_0$, and

$$\tilde{\Gamma}_0 = \Gamma_0 \times \mathbb{R}$$

is a homoclinic channel for $\tilde{\Phi}^\tau_0$.

The scattering map associated with $\tilde{\Gamma}_0$ is given by

$$\tilde{\sigma}_0(I, \theta, t) = (I, \theta + \Delta(I), t).$$

### 2.4 Main result

We describe a general setup for two-degrees-of-freedom Hamiltonian systems subject to non-conservative, time-dependent perturbations. We stress that some of the results below hold in a more general setting. Theorem 2.1 gives a first-order approximation of the perturbed scattering map $\tilde{\sigma}_\varepsilon$, where the 0-th order term is the unperturbed scattering map $\tilde{\sigma}_0$, and the 1-st order term is given by explicit Melnikov-type formulas. The unperturbed scattering map $\tilde{\sigma}_0$ is defined on the unperturbed NHIM $\tilde{\Lambda}_0$. The manifold $\tilde{\Lambda}_0$ is parametrized by $(I, \theta, t)$, where $(I, \theta)$ are the action–angle coordinates described in Sect. 2.1.1. The perturbed scattering map $\tilde{\sigma}_\varepsilon$ is defined on the perturbed NHIM $\tilde{\Lambda}_\varepsilon$ which persists from $\tilde{\Lambda}_0$, and whose existence is
given by Theorem 2.1-(i). Moreover, the perturbed NHIM $\tilde{\Lambda}_\varepsilon$ can be parametrized in terms of the coordinates $(I, \theta, t)$. See Sect. 5.1.1. Therefore, in Theorem 2.1-(iii)-(2.7), we express both the unperturbed and the perturbed scattering maps in terms of the $(I, \theta, t)$-coordinates.

**Theorem 2.1** Consider the system (2.2).
Assume that the unperturbed system $X^0$ satisfies the conditions (A-i) and (A-ii) and that the perturbation $X^1$ satisfies the condition (A-iii).

Then, there exists $\varepsilon_1$, with $0 < \varepsilon_1 < \varepsilon_0$, such that the following hold true:

(i) For all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, there is a $C^\ell$-family of NHIMs $\tilde{\Lambda}_\varepsilon$ for the extended flow $\tilde{\Phi}_\varepsilon^t$, for some $\ell \geq 1$;
(ii) For all $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, the unstable and stable manifolds of $\tilde{\Lambda}_\varepsilon$, $W^u(\tilde{\Lambda}_\varepsilon)$ and $W^s(\tilde{\Lambda}_\varepsilon)$, respectively, intersect transversally, in the extended phase space $\tilde{M}$, along a homoclinic channel $\tilde{\Gamma}_\varepsilon$;
(iii) The perturbed scattering map $\tilde{\sigma}_\varepsilon$ associated with $\tilde{\Gamma}_\varepsilon$ can be written as:

$$\tilde{\sigma}_\varepsilon(I, \theta, t) = \tilde{\sigma}_0(I, \theta, t) + \varepsilon \tilde{S}(I, \theta, t) + O_{C^\ell}(\varepsilon^2),$$

where $\tilde{S} = (\tilde{S}^1, \tilde{S}^\theta, \tilde{S}^t)$ is a mapping from some domain in $D \times \mathbb{T} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{T} \times \mathbb{R}$ as follows:

(iii-a) the components $\tilde{S}^1$ and $\tilde{S}^\theta$ are given by (5.17) and (5.40), respectively, and
(iii-b) the component $\tilde{S}^t$ is given by $\tilde{S}^t(I, \theta, t) = t$.

We recall the notation $O_{C^\ell}(\cdot)$ used above: $f = O_{C^\ell}(g)$ means that $\|f\|_{C^\ell} \leq M \|g\|_{C^\ell}$ for some $M > 0$, where $\ell \geq 0$, and $\|\cdot\|_{C^\ell}$ is the $C^\ell$-norm. In the sequel, to simplify the notation we will write $O(\cdot)$ without the subscript indicating the function space topology, whenever this can be inferred from the context.

### 2.5 Heteroclinic connections

Instead of the conditions (A-i) and (A-ii), we assume that the Hamiltonian flow associated with $H_0$ satisfies the conditions (A’-i) and (A’-ii) from below.

Condition (A’-i) has two parts (A’-i-a) and (A’-i-b).

(A’-i-a) There exist two equilibrium points $L_1, L_2$ of saddle-center type.

We do not assume that the two equilibrium points are on the same energy level, that is $H(L_1) \neq H(L_2)$ in general. Consequently, for each equilibrium point $L_1, L_2$ there exists a 1-parameter family of closed orbits $\lambda^1_0(h)$, for $h \in D^1$, and $\lambda^2_0(h)$ for $h \in D^2$, where $D^1, D^2$ are some closed intervals contained in some neighborhoods of $H(L_1), H(L_2)$, respectively.

(A’-i-b) There exists an interval of energies $D \subseteq D^1 \cap D^2$, with non-empty interior, such that there are periodic orbits $\lambda^1_0(h), \lambda^2_0(h)$ for all $h \in D$. Moreover, there exist normal form coordinates $(I^1, \theta^1, y^1, x^1)$ and $(I^2, \theta^2, y^2, x^2)$ defined around $\Lambda^1_0 := \bigcup_{h \in D} \lambda^1_0(h)$ and $\Lambda^2_0 := \bigcup_{h \in D} \lambda^2_0(h)$, respectively. These normal form coordinates are as in Sect. 4.1.1.

We remark that the normal form coordinates in (A’-i-b) already exist in small neighborhoods of $L_1$ and $L_2$, respectively. Condition (A’-i-b) requires these normal form coordinates to extend to the specified energy range.

Condition (A’-ii) has two parts (A’-ii-a) and (A’-ii-b).
(A'-ii-a) There exists a relatively compact open set $\mathcal{K}$ in $M$ such that the unstable manifold $W_u^\mathcal{K}(\Lambda_0^1)$ and the stable manifold $W_s^\mathcal{K}(\Lambda_0^2)$ inside $\mathcal{K}$ intersect transversally along a heteroclinic channel $\Gamma_0 \subset \mathcal{K}$.

The definition of a heteroclinic channel is given in Definition B.3, “Appendix B”. As a consequence, there exist transverse heteroclinic orbits from $\Lambda_1^1$ to $\Lambda_2^0$. Each such heteroclinic orbit is asymptotic in backwards time to a periodic orbit $\lambda_1^1(h)$ and is asymptotic in forward time to a periodic orbit $\lambda_2^0(h)$.

We require an additional non-degeneracy condition (A'-ii-b) formulated in terms of normal forms, which will be given in Sect. 4.1, Remark 4.4. As we will see, (A'-ii-b) consists of some explicit and verifiable conditions that the derivatives of certain functions are nonzero. In the case of homoclinic connections, we do not need a separate assumption analogous condition to (A'-ii-b), as this is automatically satisfied; see Sect. 4.1.

2.5.1 Unperturbed scattering map associated with a heteroclinic channel

As before, we define $\Omega^{-1} : W_u^0(\Lambda_0^1) \to \Lambda_0^1$ by $\Omega^{-1}(z_0) = z_0^-$, where $z_0^- \in \Lambda_0^1$ is the footpoint of the unstable fiber through $z_0 \in W_u^0(\Lambda_0^1)$, and $\Omega^{+2} : W_s^0(\Lambda_0^2) \to \Lambda_0^2$ by $\Omega^{+2}(z_0) = z_0^+$, where $z_0^+ \in \Lambda_0^2$ is the footpoint of the stable fiber through $z_0 \in W_s^0(\Lambda_0^2)$.

Associated with the heteroclinic channel $\Gamma_0$, we can define the scattering map as in Definition B.4 in “Appendix B”. Specifically,

$$\sigma_0 : \Omega^{-1}(\Gamma_0) \subseteq \Lambda_0^1 \to \Omega^{+2}(\Gamma_0) \subseteq \Lambda_0^2,$$

is given by

$$\sigma_0(z_0^-) = z_0^+, \quad \text{provided that there exists a } z_0 \in \Gamma_0 \text{ such that } \Omega^{-1}(z_0) = z_0^- \text{ and } \Omega^{+2}(z_0) = z_0^+.$$

Since $H_0$ is constant along heteroclinic orbits, we have that $I(z_0^-) = I(z_0^+)$. Then, the scattering map, expressed in terms of the action–angle coordinates, is given by

$$\sigma_0(I, \theta) = (I, \theta + \Delta(I)),$$

for some function $\Delta$ that depends smoothly on $I$.

In this case, we can obtain a result similar to Theorem 2.1. For brevity, we will not provide the precise formulas for the components of the corresponding expansion of the perturbed scattering map, which is analogous to (2.7). Such formulas are analogues of (5.17) and (5.40).

3 Geometric structures in the planar circular restricted three-body problem.

In this section, we survey the status of the verification of the conditions (A-i) and (A-ii), from Sect. 2.1, or the conditions (A'-i) and (A'-ii) from Sect. 2.5, in the concrete model of the PCRTBP. Some of the verifications in the literature are rigorous and some of them are numerical.

Of course, the verification of the hypothesis of Theorem 2.1 in a concrete model does not affect the validity of the rigorous arguments establishing Theorem 2.1, and the reader interested only in rigorous results may safely skip this section.
We note that, since our hypothesis is mainly transversality conditions, they can be verified with finite precision calculations, which seem to be safe calculations for today’s standard and could well be accessible to “computer assisted proofs”. We hope that this work could stimulate more extensive verifications.

The PCRTBP is a model describing the motion of an infinitesimal body under the Newtonian gravity exerted by two heavy bodies moving on circular orbits about their center of mass, under the assumption that these orbits are not affected by the gravity of the infinitesimal body.

We can think of the heavy bodies (referred to as primaries) representing the Earth and Moon, and the infinitesimal mass representing a spaceship.

It is convenient to study the motion of the infinitesimal body relative to a co-rotating frame which rotates with the primaries around the center of mass, and to use normalized units. Henceforth, the masses of the heavy bodies are $1 - \mu$ and $\mu$, where $\mu \in (0, 1/2]$. Relative to the co-rotating frame, the heavier mass $1 - \mu$ is located at $(\mu, 0)$, and the lighter mass $\mu$ is located at $(-1 + \mu, 0)$. The motion of the infinitesimal body relative to the co-rotating frame is described via the autonomous Hamiltonian

$$H_0(p_1, p_2, q_1, q_2) = \frac{(p_1 + q_2)^2 + (p_2 - q_1)^2}{2} - V(q_1, q_2),$$  \hspace{1cm} (3.1)

where $(p, q) = (p_1, p_2, q_1, q_2) \in \mathbb{R}^4$ represents the momenta and the coordinates of the infinitesimal body with respect to the co-rotating frame.

$$V(q_1, q_2) = \frac{q_1^2 + q_2^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},$$

$$r_1 = \left( (q_1 - \mu)^2 + q_2^2 \right)^{1/2},$$

$$r_2 = \left( (q_1 + 1 - \mu)^2 + q_2^2 \right)^{1/2}. \hspace{1cm} (3.2)$$

Above $V(q_1, q_2)$ represents the effective potential, and $r_1, r_2$ represent the distances from the infinitesimal body to the masses $1 - \mu$ and $\mu$, respectively. The phase space

$$M = \{(p, q) \in \mathbb{R}^4 \mid q \neq (\mu, 0), \text{ and } q \neq (-1 + \mu, 0)\}$$

is endowed with the symplectic form

$$\Omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2.$$ 

Note that the phase space $M$ is not compact.

The equations of motion of the infinitesimal body are given by the Hamilton equations corresponding to (3.1), that is

$$\frac{d}{dt} z = J \nabla H_0(z), \hspace{1cm} (3.3)$$

where $z = z(p_1, p_2, q_1, q_2)$ and $J$ represents the standard almost complex structure.

The Hamiltonian $H_0$ is an integral of motion, so the flow $\Phi^t_0$ of (3.3) leaves invariant each energy hyper-surface

$$M_h := H_0^{-1}(h) = \{(p_1, p_2, q_1, q_2) \in M \mid H_0(p_1, p_2, q_1, q_2) = h\}. \hspace{1cm} (3.4)$$

The system has three equilibrium points $L_1, L_2, L_3$ located on the $q_1$-axis, and two other equilibrium points $L_4, L_5$, each lying in the $(q_1, q_2)$-plane and forming an equilateral triangle with the primaries. The first three of the equilibrium points are attributed to Euler, and the...
last two are attributed to Lagrange. In our convention, $L_1$ is located between the primaries, $L_2$ is on the side of the lighter primary, and $L_3$ is on the side of the heavier primary. The linearized stability of $L_1$, $L_2$, $L_3$ is of center-saddle type. The linearized stability of $L_4$, $L_5$ is of center–center type, for $\mu$ less than some critical value $\mu_{\text{cr}}$.

We note that condition (A-i) is satisfied for each of the equilibrium points $L_1, L_2, L_3$.

For $i = 1, 2, 3$, for each energy level $h$, with $H(L_i) < h$ and $h$ sufficiently close to $H(L_i)$, there exists a unique periodic orbit $\lambda_0(h)$ near the equilibrium point $L_i$, which is referred to as a Lyapunov orbit. The existence of such periodic orbits follows from the Lyapunov center theorem (see, e.g., Moser 1958). Moreover, there exists a neighborhood of $L_i$ in the phase space where the Hamiltonian $H_0$ can be written in a normal form relative to some suitable coordinates $(I, \theta, y, x)$; see Sect. 4.1.

Each Lyapunov orbit is hyperbolic in the energy surface, so it has associated 2-dimensional unstable and stable manifolds denoted $W^u(\lambda_0(h))$ and $W^s(\lambda_0(h))$.

Numerical evidence, as well as computer-assisted proofs, shows that these periodic orbits can be continued for energy levels $h > H(L_i)$ that are not close to $H(L_i)$ (see, e.g., Broucke 1968; Capinski and Roldán 2012).

**Normally hyperbolic invariant manifold for the unperturbed system.** For an energy range $h \in D$ sufficiently close to the energy of $L_i$, the family of Lyapunov orbits

$$\Lambda_0 = \bigcup_{h \in D} \lambda_0(h), \quad (3.5)$$

defines a 2-dimensional NHIM with boundary for the Hamiltonian flow of (3.3). The NHIM carries the symplectic structure $\Omega_{|\Lambda_0}$, and it can be described in terms of the action–angle coordinates $(I, \theta)$. The action $I = I_h$ is uniquely defined by the energy $h$, and $\theta$ is symplectically conjugate with $I$ with respect to $\Omega_{|\Lambda_0}$. The variable $I$ is a first integral along the trajectories of the flow on $\Lambda_0$, and the action-level sets are in fact the Lyapunov orbits $\lambda_0(h)$. The motion restricted to each Lyapunov orbit is a rigid rotation in the variable $\theta$, with the frequency depending on the energy level.

The NHIM $\Lambda_0$ and its unstable and stable manifolds $W^u(\Lambda_0)$ and $W^s(\Lambda_0)$ have simple descriptions in terms of the normal form coordinates $(I, \theta, y, x)$ in a neighborhood of $L_1$: $\Lambda_0$ corresponds to $x = y = 0$, $W^u(\Lambda_0)$ corresponds to $y = 0$, and $W^s(\Lambda_0)$ corresponds to $x = 0$.

**Homoclinic connections.** We first focus on the dynamics around the equilibrium point $L_1$. There are analytic arguments (see Llibre et al. 1985) showing that, for a discrete set of values of $\mu$ that are sufficiently small, and for each $h$ sufficiently close to $H(L_1)$, the branches of $W^u(\lambda_0(h))$ and $W^s(\lambda_0(h))$ on the side of the heavier primary do not collide with the primary and intersect transversally along some homoclinic orbit $\gamma_0(h)$, not necessarily unique. Numerical evidence, as well as computer-assisted proofs, shows that this property holds in fact for a large range of values of masses $\mu$ and energies $h$; see, e.g. Koon et al. (2000), Gómez et al. (2001a, b, c, d) and Capinski (2012).

Each choice of a transverse homoclinic orbit $\gamma_0(h)$ can be continued in energy $h$ to a family of such homoclinic orbits, which forms a homoclinic manifold $\bigcup_{h \in D} \gamma_0(h)$. Moreover, we can ensure that $W^u(\lambda_0(h))$ and $W^s(\lambda_0(h))$ are contained inside some compact subset $\mathcal{K}$ of the phase space.

We define the transverse homoclinic manifold $\Gamma_0$ as an open disk contained in $\bigcup_{h \in D} \gamma_0(h)$. It consists of a 1-dimensional family of segments of homoclinic orbits. In order to define the scattering map, we need that the stable (resp. unstable) fibers of $\Lambda_0$ are transverse to $\Gamma_0$ relative to $W^s(\Lambda_0)$ (resp. $W^u(\Lambda_0)$), as per condition (B.1). This is automatically satisfied.
provided $h$ is sufficiently close to $H(L_1)$, since the flow is transverse to the corresponding fibers relative to $W^s(\Lambda_0)$ (resp. $W^u(\Lambda_0)$). This can be easily seen, for instance, from (4.6).

To define the scattering map, we also need that $\Omega^\pm$ are diffeomorphisms from $\Gamma_0$ onto their images; see Definition B.1. We can achieve this by further restricting to a suitable submanifold

$$\Gamma_0 \subseteq \bigcup_{h \in D} \gamma_0(h),$$

so that $\Gamma_0$ is a homoclinic channel.

In this way, we can ensure condition (A-ii).

In summary, for the PCRTBP, we can verify the existence of the geometric structures of interest and the corresponding conditions (A-i) and (A-ii) from Sect. 2.1.

**Heteroclinic connections.** We now focus on the dynamics around the equilibrium points $L_1$ and $L_2$. They satisfy condition (A'-i-a). For energy levels $h$ with $H(L_1) \lesssim h$, there exists a family $\lambda_0^1(h)$ of Lyapunov orbits near $L_1$, and for $H(L_2) \lesssim h$ there exists a family $\lambda_0^2(h)$ of Lyapunov orbits near $L_2$. Moreover, there exist normal form coordinates $(I^1, \theta^1, y^1, x^1)$ and $(I^2, \theta^2, y^2, x^2)$ defined around $L_1$ and $L_2$, respectively, for some suitable energy ranges. These normal form coordinates are as in Sect. 4.1.

Numerical evidence shows that families of periodic orbits near $L_1$ and $L_2$ can exist simultaneously, for some energy range. Therefore, we consider an interval of energies $D$ such that, for $h \in D$ we have the following: there exist families of periodic orbits $\lambda_0^1(h)$ near $L_1$, and $\lambda_0^2(h)$ near $L_2$, the following sets

$$\Lambda_0^1 = \bigcup_{h \in D} \lambda_0^1(h),$$

$$\Lambda_0^2 = \bigcup_{h \in D} \lambda_0^2(h),$$

are NHIMs for the Hamiltonian flow of (3.3), and the normal form coordinates $(I^1, \theta^1, y^1, x^1)$ and $(I^2, \theta^2, y^2, x^2)$ are defined in some neighborhoods of $\Lambda_0^1$ and $\Lambda_0^2$, respectively, granting (A'-i-b).

Numerical evidence, as well as computer-assisted proofs, shows that there exist transverse heteroclinic connections determined by $W^u(\lambda_0^1(h)) \cap W^s(\lambda_0^2(h))$ and $W^u(\lambda_0^2(h)) \cap W^s(\lambda_0^1(h))$ for certain ranges of energies. See, e.g., Koon et al. (2000), Gómez et al. (2001a,b,c,d), Wilczak and Zgliczynski (2003), Canalias and Masdemont (2006) and Belbruno et al. (2010). In either case, we denote the corresponding family of heteroclinic orbits by $\gamma_0(h)$. We consider a range of energies $h \in D$ for which this additional condition on the existence of transverse heteroclinic connections is satisfied.

The transverse intersection of the unstable manifold $W^u(\Lambda_0^1)$ with the stable manifold $W^s(\Lambda_0^2)$ defines a heteroclinic manifold $\bigcup_{h \in D} \gamma_0(h)$, which depends on the choice of the family of heteroclinic orbits $\gamma_0(h)$. One can always restrict to a submanifold

$$\Gamma_0 \subseteq \bigcup_{h \in D} \gamma_0(h),$$

that is a heteroclinic channel.

If that is the case, the condition (A'-ii-a) is verified.

The remaining condition (A'-ii-b), which will be given in Sect. 4.1, Remark 4.4, consists of some explicit non-degeneracy conditions on the normal forms. Such condition can also be verified numerically.
Thus, the conditions \((A'-i)\) and \((A'-ii)\) from Sect. 2.5 are checkable numerically.

It would be of interest to verify if Theorem 2.1 can be applied when the equilibrium point \(L_3\) is also considered; some numerical results concerning the dynamics around \(L_3\) can be found in Barrabés and Ollé (2006), Ceccaroni et al. (2016), Jorba and Nicolás (2020), Gómez et al. (2001a, b, c, d).

In summary, in this section we have outlined how the conditions of Theorem 2.1 can be verified in the PCRTBP. The theoretical results of Theorem 2.1 are independent on the application to the PCRTBP.

4 Coordinate systems and evolution equations

4.1 New coordinate systems for the unperturbed system

We consider the case of homoclinic connections described by conditions \((A-i)\) and \((A-ii)\). Under these assumptions, the manifolds \(W^u(\Lambda_0)\) and \(W^s(\Lambda_0)\) intersect transversally along the homoclinic channel \(\Gamma_0\).

The next proposition states that, in a neighborhood of each \(W^u(\Lambda_0)\) and \(W^s(\Lambda_0)\), there exists a system of symplectic coordinates such that \(\Lambda_0, W^u(\Lambda_0)\) and \(W^s(\Lambda_0)\) have very simple descriptions, and moreover, the unperturbed Hamiltonian \(H_0\) can be written in a normal form relative to the corresponding coordinates. As before, for \(z \in W^{s,u}(\Lambda_0)\), we denote \(z^\pm = \Omega^\pm(z)\).

Informally, we have three systems of coordinates near the homoclinic intersection. One is given by the normal form. Two other systems of coordinates are obtained propagating—by the unperturbed dynamics—the coordinate system along the stable and unstable manifolds.

As it is standard, we will think of the collections of coordinates as defining a geometric point. At the same time, we note that the coordinates are functions on a manifold. Therefore, we can compute the evolution of the coordinates using the usual calculus formulas in the ODE. As we will see, in perturbative calculations, it will be enough to use the unperturbed evolution.

Hence, near the NHIM we have three coordinate systems, but on the NHIM the three coordinate systems agree.

Near the homoclinic intersection, we have two systems of coordinates. The two systems of coordinates do not agree because the propagation happens in two different ways. Nevertheless, the coordinate change can be computed from the unperturbed dynamics.

Proposition 4.1 (Normal Forms) There exist three systems of analytic, symplectic coordinates\(^1\), and some analytic functions \(h_0 = h_0(I), g_1 = g_1(I)\), and \(g_2 = g_2(I, xy)\), defined for some range of energies \(h \in D\), as follows:

\((N)\) A coordinate system \((I, \theta, y, x)\) in a neighborhood \(\mathcal{N}\) of \(\Lambda_0\) such that for \(z \in \mathcal{N}\) we have:

\((N-i)\) \(z \in \Lambda_0\) if and only if \(x(z) = y(z) = 0\);
\((N-ii)\) \(z \in W^u(\Lambda_0)\) if and only if \(y = 0\), and \(z \in W^s(\Lambda_0)\) if and only if \(x = 0\);
\((N-iii)\) for \(z \in W^u(\Lambda_0)\) we have \(I(z) = I(z^-)\) and \(\theta(z) = \theta(z^-)\), and for \(z \in W^s(\Lambda_0)\) we have \(I(z) = I(z^+)\) and \(\theta(z) = \theta(z^+)\);

\(^1\) coordinates obtained from \((p, q)\) via a canonical transformation
\( \text{Proof} \) Part (N) follows from Giorgilli (2001), so we will not give a detailed proof.

We only summarize the procedure to obtain the normal form in a neighborhood of a center–saddle equilibrium point for a 2-degrees of freedom Hamiltonian \( H_0 \). First, the Hamiltonian \( H_0 \) is expanded in a Taylor series around that equilibrium point (shifted to the origin) as

\[
H_0(p, q) = H_2(p, q) + H_3(p, q) + H_4(p, q) + \cdots,
\]

where \( H_j(p, q) \) is an homogeneous polynomial of degree \( j \) in the variables \( (p_1, p_2, q_1, q_2) \). Then, by making a linear canonical change of coordinates \( (p, q) \mapsto (x, y) \), with the eigenvectors of the linearized system given by \( J \nabla H_2(0, 0) \) as the axes of the new system, the quadratic part \( H_2 \) of \( H_0 \) can be written in the new coordinates \( (x, y) \in \mathbb{R}^4 \) as

\[
H_2(x, y) = \lambda x_1 y_1 + \frac{\omega}{2} (x_2^2 + y_2^2),
\]

where \( \pm\lambda_1 = \pm \lambda \) and \( \pm\lambda_2 = \pm i \omega \) are the eigenvalues of \( J \nabla H_2(0, 0) \). Then, via another linear canonical change of coordinates

\[
x_1 = \xi_1, \quad y_1 = \eta_1, \quad x_2 = \frac{\xi_2 + i \eta_2}{\sqrt{2}}, \quad y_2 = \frac{i \xi_2 + \eta_2}{\sqrt{2}},
\]

we obtain \( H_2 \) written in complex variables as

\[
H_2^N(\xi, \eta) = \lambda \xi_1 \eta_1 + i \omega \xi_2 \eta_2 = \lambda_1 \xi_1 \eta_1 + \lambda_2 \xi_2 \eta_2.
\]
The next step is to apply a sequence of changes of coordinates to kill all monomials for which the exponent of $\xi_j$ is different from the exponent of $\eta_j$. Since the eigenvalue $\lambda_1 = \lambda$ is real and $\lambda_2 = i\omega$ is imaginary, there are no small divisors. The only possible source of divergence is due to the use of Cauchy’s estimates for the derivatives required by the normalization procedure. In Giorgilli (2001), the accumulation of derivatives is controlled via a KAM technique. The process can be continued to any order.

Thus, in the limit $H_0$ can be written as an expansion

$$H_0(\xi, \eta) = H_2^N(\xi_1\eta_1, \xi_2\eta_2) + H_3^N(\xi_1\eta_1, \xi_2\eta_2) + H_4^N(\xi_1\eta_1, \xi_2\eta_2) + \cdots,$$

where $H_j$ is an homogeneous polynomial in $\xi_1\eta_1, \xi_2\eta_2$ of degree $j$. The series expansion of $H_0$ is convergent in a neighborhood $\mathcal{N}$ of the origin, and the coordinate change $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ is canonical and given in terms of convergent series.

There exist periodic orbits $\lambda_0(h)$ around the equilibrium point for all energy levels $h$ sufficiently close to the energy level of the equilibrium point. This implies that the NHIM $\bar{\Lambda}_0 = \bigcup_{h \in \tilde{D}} \lambda_0(h)$ is contained in $\mathcal{N}$, for some suitable energy range $h \in \tilde{D}$.

To express $H_0$ in action–angle coordinates, one applies the canonical transformation

$$\bar{\xi}_2 = \sqrt{T} \exp(i\theta), \quad \eta_2 = -i \sqrt{T} \exp(-i\theta).$$

Finally, denote $x = \xi_1, y = \eta_1$. We obtain the normal form

$$H_0(I, xy) = \lambda xy + \omega I + H_3(I, xy) + H_4(I, xy) + \cdots.$$  \hspace{1cm} (4.5)

Moreover, in these coordinates the following hold:

(i) The normally hyperbolic invariant manifold $\bar{\Lambda}_0$ is given by $x = y = 0$, and each periodic orbit in $\Lambda_0$ corresponds to a level set of $I$;

(ii) The local unstable fibers $W^u_{\mathcal{N}}(z)$ for $z \in \bar{\Lambda}_0$ are given by $\theta = \text{const.}$ and $y = 0$;

(iii) The local stable fibers $W^s_{\mathcal{N}}(z)$ for $z \in \bar{\Lambda}_0$ are given by $\theta = \text{const.}$ and $x = 0$;

(iv) The local unstable invariant manifold $W^u_{\mathcal{N}}(\bar{\Lambda}_0)$ is given by $y = 0$;

(v) The local stable invariant manifold $W^s_{\mathcal{N}}(\bar{\Lambda}_0)$ is given by $x = 0$.

The equations of motion are:

$$\frac{d}{dt} I = 0,
\frac{d}{dt} \theta = \omega + \frac{\partial H_3^N}{\partial I} + \frac{\partial H_4^N}{\partial I} + \cdots,
\frac{d}{dt} y = -\lambda y - \frac{\partial H_3^N}{\partial x} - \frac{\partial H_4^N}{\partial x} + \cdots,
\frac{d}{dt} x = \lambda x + \frac{\partial H_3^N}{\partial y} + \frac{\partial H_4^N}{\partial y} + \cdots.$$

(4.6)

Note that Hamiltonian $H_0$ on $\mathcal{N}$ has two first integrals $I$ and $xy$, which are independent and in involution.

This implies that, if $z \in W^u_{\mathcal{N}}(\bar{\Lambda}_0)$ (resp. $z \in W^s_{\mathcal{N}}(\bar{\Lambda}_0)$), since $xy = 0$, we have $I(z) = I(z^-)$ (resp. $I(z) = I(z^+)$).

The Hamiltonian $H_0$ restricted to $\bar{\Lambda}_0$ is given by

$$h_0(I) := \omega I + H_3^N(I) + H_4^N(I) + \cdots,$$

where we denote $H_0(I, 0) = h_0(I)$, and $H_j^N(I, 0) = H_j^N(I)$ for all $j$. Each Lyapunov orbit $\lambda_h$ in $\bar{\Lambda}_0$ corresponds to a unique level set of $I$, so we can write $\lambda_0(h) = \lambda_0(I)$.
By regrouping the terms of the polynomials $H^N_j(I, xy)$ for $j \geq 3$, we can rewrite (4.5) in powers of $xy$ up to the second order as

$$H_0(I, \theta, y, x) = h_0(I) + (xy)g_1(I) + (xy)^2g_2(I, xy), \quad (4.7)$$

for some analytic functions $h_0 = h_0(I)$, $g_1 = g_1(I)$, and $g_2 = g_2(I, xy)$.

For points $z \in W^u_N(\tilde{\Lambda}_0)$ or $z \in W^s_N(\tilde{\Lambda}_0)$, since $xy = 0$, we have

$$\frac{d}{dt}\theta(z) = \frac{\partial h_0}{\partial I}(I(z)).$$

This implies that if $z \in W^u_N(\tilde{\Lambda}_0)$ (resp. $z \in W^s_N(\tilde{\Lambda}_0)$), we have $\theta(z) = \theta(z^-)$ (resp. $\theta(z) = \theta(z^+)$).

The coordinate system $(I, \theta, y, x)$ constructed above is the coordinate system from part (N).

Now we construct the coordinate system claimed in part (U).

We extend the coordinate system $(I, \theta, y, x)$ along the flow to a neighborhood $\mathcal{N}^u$ of $W^u(\tilde{\Lambda}_0)$, up to a neighborhood of the homoclinic manifold, as follows. Let $T_u > 0$ be a time such that $\Phi_{T_u}^0(\mathcal{N}) \supseteq \Gamma$. Let $\mathcal{N}^u := \Phi_{T_u}^0(\mathcal{N})$. Each point $z \in \mathcal{N}^u$ is of the form $z = \Phi_{T_u}^0(\xi)$ with $\xi \in \mathcal{N}$. We define the coordinates $(I^u, \theta^u, y^u, x^u)$ of $z$ to be equal to the coordinates $(I, \theta, y, x)$ of $z$, or equivalently

$$(I^u, \theta^u, y^u, x^u)(z) = (I, \theta, y, x)(\Phi_{T_u}^0(z)). \quad (4.8)$$

Since the coordinates $(I^u, \theta^u, y^u, x^u)$ of a point $z$ are the coordinates $(I, \theta, y, x)$ of $\Phi_{-T_u}^0(z)$, then the coordinate change in symplectic.

The restriction of the coordinates $(I^u, \theta^u, y^u, x^u)$ to $\mathcal{N} \cap \mathcal{N}^u$ is given by $(I, \theta, y, x) \circ \Phi_{-T_u}^0$. For every $z \in \tilde{\Lambda}_0 \cap \mathcal{N}^u$, we have $I^u(z) = I(z)$ and $\theta^u(z) = \theta \circ \Phi_{-T_u}^0(z) = \theta(z) - \omega(I(z))T_u$. We make a symplectic coordinate change in the action–angle variable

$$(I^u, \theta^u) \mapsto (I^u, \theta^u + \omega(I)T_u),$$

where we use the same notation for the old and for the new coordinates. Since $x = y = x^u = y^u = 0$ on $\tilde{\Lambda}_0$, we obtain that $(I^u, \theta^u, y^u, x^u) = (I, \theta, x, y)$ on $\tilde{\Lambda}_0 \cap \mathcal{N}^u$.

We obtain that the normal form expansion of $H_0$ in the coordinates $(I^u, \theta^u)$ is the same as in (4.7):

$$H(I^u, \theta^u, y^u, x^u) = h_0(I \circ \Phi_{-T_u}^0) + \left((x \circ \Phi_{-T_u}^0) \cdot (y \circ \Phi_{-T_u}^0)\right)g_1(I \circ \Phi_{-T_u}^0) + \left((x \circ \Phi_{-T_u}^0) \cdot (y \circ \Phi_{-T_u}^0)\right)^2 g_2(I \circ \Phi_{-T_u}^0, (x \circ \Phi_{-T_u}^0) \cdot (y \circ \Phi_{-T_u}^0)) = h_0(I^u) + (x^u y^u)g_1(I) + (x^u y^u)^2 g_2(I^u, x^u y^u).$$

Now we construct the coordinate system claimed in part (S). Let $T_s > 0$ be a time such that $\Phi_{-T_s}^0(\mathcal{N}) \supseteq \Gamma$. In general, $T_s \neq T_u$.

We extend the coordinate system $(I, \theta, y, x)$ along the flow to a neighborhood $\mathcal{N}^s$ of $W^s(\tilde{\Lambda}_0)$, up to the homoclinic manifold, as follows. Start with the coordinates $(I, \theta, y, x)$ defined on the neighborhood $\mathcal{N}$ of $\tilde{\Lambda}_0$. Let $\mathcal{N}^s := \Phi_{-T_s}^0(\mathcal{N})$. Each point $z \in \mathcal{N}^s$ is of the form $z = \Phi_{-T_s}^0(\xi)$ with $\xi \in \mathcal{N}$. We define the coordinates $(I^s, \theta^s, y^s, x^s)$ of $z$ to be equal to the coordinates $(I, \theta, y, x)$ of $z$, or equivalently

$$(I^s, \theta^s, y^s, x^s)(z) = (I, \theta, y, x)(\Phi_{T_s}^0(z)). \quad (4.9)$$
Since the coordinates \((I^s, \theta^s, y^s, x^s)\) of a point \(z\) are the coordinates \((I, \theta, y, x) \circ \Phi^T_{0}\) of \(z\), then the coordinate change is symplectic. Finally, we make the symplectic coordinate change

\[(I^s, \theta^s) \leftrightarrow (I^s, \theta^s - \omega(I^s) T_s),\]

where we use the same notation for the old and for the new coordinates.

We obtain that the normal form expansion of \(H_0\) in the coordinates \((I^u, \theta^u)\) is the same as in

\[H^N_{\lambda^0}(I^s, \theta^s, y^s, x^s) = h_0(I^s) + (x^s y^s) g_1(I^s) + (x^s y^s)^2 g_2(I^s, x^s y^s).\]

Since \(x = y = x^s = y^s = 0\) on \(\tilde{\Lambda}_0 \cap \mathcal{N}^s\), we have \((I^s, \theta^s, y^s, x^s) = (I, \theta, x, y)\) on \(\tilde{\Lambda}_0 \cap \mathcal{N}^s\).

Finally, we restrict the Hamiltonian to an energy range \(h \in D\) such that \(\Lambda_0 = \bigcup_{h \in D} \lambda_h \subseteq \mathcal{N} \cap \mathcal{N}^u \cap \mathcal{N}^s\). Moreover, we restrict \(D\) such that \(W^u(\Lambda_0), W^s(\Lambda_0)\) are contained in \(\mathcal{H}\), where the set \(\mathcal{H}\) is as in condition \((A-\text{i})\).

By the above constructions, the coordinates \((I, \theta, y, x), (I^u, \theta^u, y^u, x^u)\) and \((I^s, \theta^s, y^s, x^s)\) satisfy the properties listed in Proposition 4.1. \(\square\)

We note that \(H_0\) satisfies the following non-degeneracy condition, in terms of the above normal form coordinates:

\[
\frac{\partial h_0}{\partial I}(I_0) \neq 0, \quad g_1(I_0) \neq 0,
\]

for all \(I_0 = I_0(h)\) with \(h \in D\), provided \(D\) is contained in a sufficiently small neighborhood of the energy value of the equilibrium point \(L_1\).

Indeed, by condition \((A-\text{i})\) we have

\[
h_0(I) = \omega I + O(I^3),
\]

\[
g_1(I) = \lambda + O(I),
\]

in a sufficiently small neighborhood of the equilibrium point \(L_1\), which imply (4.10).

**Remark 4.2** It is important to note that the coordinates \((I^u, \theta^u, y^u, x^u)\) and \((I^s, \theta^s, x^s, y^s)\) do not generally agree at homoclinic points away from the Lyapunov orbit, where both coordinate systems are well defined. Nevertheless, for any homoclinic point \(z \in M_h \cap W^u(\Lambda_0) \cap W^s(\Lambda_0)\), we have that \(I^u(z) = I^s(z) = I_h\).

**Remark 4.3** The above result on the existence of a convergent normal form in a neighborhood of a center–saddle point, obtained via a convergent canonical coordinate transformation, is valid for 2-degrees-of-freedom Hamiltonian systems. For higher degree of freedom Hamiltonian systems, a similar result is true under some additional non-resonance conditions (see Giorgilli 2001). A related approach to the normal form that we use here can be found in Moser (1958). A numerical methodology for the effective computations of normal forms is developed in Jorba (1999).

**Remark 4.4** We now discuss the case when we have heteroclinic connections between two NHIMs \(\Lambda^1_0\) around \(L_1\) and \(\Lambda^2_0\) around \(L_2\), as in Sect. 2.5. The manifolds \(W^u(\Lambda^1_0)\) and \(W^s(\Lambda^2_0)\) are assumed to intersect transversally.

The construction of the normal form coordinates from Proposition 4.1 only works in a small neighborhood of the equilibrium point. In the case of heteroclinic connections, since
$L_1$ and $L_2$ are on different energy levels, the theory does not guarantee the simultaneous existence two normal form coordinate system around $L_1$ and $L_2$, respectively, for some common energy range.

In this case, we need to make a separate assumption that there exist two systems of normal form coordinates around $L_1$ and $L_2$, for some common energy range. Indeed, this assumption is already made in (A’-i-b).

Based on this assumption, we can construct, as in the proof of Proposition 4.1, two systems of coordinates

- $(I^{1,u}, \theta^{1,u}, y^{1,u}, x^{1,u})$ in a neighborhood $\mathcal{N}^{1,u} \subseteq \mathcal{K}$ of $W^u(\Lambda^1_0)$,
- $(I^{2,s}, \theta^{2,s}, y^{2,s}, x^{2,s})$ in a neighborhood $\mathcal{N}^{2,s} \subseteq \mathcal{K}$ of $W^s(\Lambda^2_0)$,

so that that $H_0$ can be written relative to these coordinate as

\[
H_0 = h^1_0(I^{1,u}) + (x^{1,u} y^{1,u})g^1_1(I^{1,u}) + (x^{1,u} y^{1,u})^2 g^1_2(I^{1,u}, x^{1,u} y^{1,u}),
\]
\[
H_0 = h^2_0(I^{2,s}) + (x^{2,s} y^{2,s})g^2_1(I^{2,s}) + (x^{2,s} y^{2,s})^2 g^2_2(I^{2,s}, x^{2,s} y^{2,s}).
\]

In the case of heteroclinic connections, we explicitly require the following non-degeneracy condition, analogous to (4.10):

(A’-ii-b) The Hamiltonian $H_0$, written in the corresponding normal form coordinates, satisfies:

\[
\frac{\partial h^1_0}{\partial I^{1,u}}(I_0) \neq 0,
\]
\[
g^1_1(I_0) \neq 0,
\]
\[
\frac{\partial h^2_0}{\partial I^{2,s}}(I_0) \neq 0,
\]
\[
g^2_1(I_0) \neq 0,
\]

for all $I_0 = I_0(h)$ with $h \in D$.

We remark that these conditions are automatically satisfied in small neighborhoods of $L_1$ and $L_2$, respectively. Condition (A’-ii-b) requires these conditions to hold on the specified energy range.

4.2 The scattering map for the unperturbed system

Consider the scattering map $\sigma_0$ associated with $\Gamma_0$. We will express the scattering map in terms of the coordinates $(I^u, \theta^u, y^u, x^u)$ and $(I^s, \theta^s, y^s, x^s)$.

Consider a homoclinic point $z_0 \in W^u(\lambda_h) \cap W^s(\lambda_h)$. Both coordinate systems $(I^u, \theta^u, y^u, x^u)$ and $(I^s, \theta^s, y^s, x^s)$ are defined around $z_0$.

By Proposition 4.1, the action coordinate of $z_0 \in \Gamma_0$ is the same as the action coordinate of the unstable and stable footpoints $z^-_0, z^+_0 \in \Lambda_0$, that is $I(z^-_0) = I^u(z^-_0) = I^u(z_0) = I^s(z_0) = I^s(z^+_0) = I(z^+_0)$. Therefore, the scattering map $\sigma_0$ preserves the I-coordinate. Hence, $\sigma_0$ is a phase shift on $I$-level-sets in $\Lambda_0$ wherever it is defined:

\[
\sigma_0(I, \theta) = (I, \theta + \Delta(I)).
\]  

(4.12)

In general, $\theta^u(z_0) \neq \theta^s(z_0)$. It is easy to see that the phase shift determined by the unperturbed scattering map is given by the ‘mismatch’ between the two angle coordinates.

Proposition 4.5 Let $h$ be a fixed energy level and let $z_0 \in \Gamma_0 \cap M_h$. 

(i) The angle mismatch $\theta^u(z_0) - \theta^u(z_0)$ is a constant that depends only on $h$, so we write it as $\theta^u(h) - \theta^u(h)$.

(ii) The scattering map $s_0$ is given by $(I, \theta) \mapsto s_0(I, \theta) = (I, \theta + \Delta(I))$, where $\Delta(I) = \theta^h(I) - \theta^u(h)$ for $I = I_h$.

**Proof** (i) If $z_0$ is a point in $\Gamma_0 \cap M_h$, then $\Gamma_0 \cap M_h$ consists of points of the form $\Phi_0^t(z_0)$, $t \in [t_1, t_2]$, for some interval $[t_1, t_2]$ containing 0.

We now turn the attention to the unperturbed homoclinic channel $\tilde{\Gamma}_0$ in the extended phase space. By the transversality conditions (B.3) on a homoclinic channel, we have that the fibers of $\tilde{\Lambda}_0$ intersect $\tilde{\Gamma}_0$ transversally. Therefore, $\tilde{\Gamma}_0$ can be parametrized in terms of the coordinates $(I^u, \theta^u, t)$, as well as in terms of the coordinates $(I^s, \theta^s, t)$:

$$\tilde{\Gamma}_0 = \{ (I^u, \theta^u, y^u, x^u, t) \mid y^u = 0, x^u = x_0^u(I^u, \theta^u, t) \},$$

$$\text{and} \quad \{ (I^s, \theta^s, x^s, y^s, t) \mid y^s = y_0^s(I^s, \theta^s, t), x^s = 0 \}. \quad (4.13)$$

Each homoclinic point $\tilde{z}_0 \in \tilde{\Gamma}_0$ is associated with unique $I^u = I^s = I_0$, $\theta^u$, and $\theta^s$, with $\theta^u(\tilde{z}_0) - \theta^u(\tilde{z}_0) = \Delta(I_0)$.

The corresponding scattering map is given by:

$$\tilde{s}_0(I, \theta, t) = (I, \theta + \Delta(I), t).$$

Thus, we have that $\tilde{\Gamma}_0$ is a graph over the $(I^u, \theta^u, t)$-variables, as well as a graph over the $(I^s, \theta^s, t)$-variables. On $\tilde{\Gamma}_0$, we have two coordinate systems $(I^u, \theta^u, t)$ and $(I^s, \theta^s, t)$, with the corresponding coordinate change given by

$$\begin{aligned}
(I^s, \theta^s, t) &= (I^u, \theta^u + \Delta(I^u), t) .
\end{aligned} \quad (4.14)$$

**4.3 Perturbed evolution equations**

In the sequel, we will identify the vector fields $\mathcal{X}^0$ and $\mathcal{X}^1$ with derivative operators acting on functions. In general, given a smooth vector field $\mathcal{X}$ and a smooth function $f$ on a manifold $M$, and $(z_j)_{j \in \{1, \ldots, \dim(M)\}}$ a system of local coordinates, then

$$(\mathcal{X}f)(z) = \sum_j (\mathcal{X})_j(z)(\partial_{z_j} f)(z). \quad (4.15)$$

Consider one of the coordinate systems defined in Sect. 4.1. To simplify notation, we will denote such a coordinate system by $(I, \theta, x, y)$. Below we provide evolution equations of these coordinates, expressing the time derivative of each coordinate along a solution of the
perturbed system. We include the expression for a general perturbation, as well as for the case when the perturbation is Hamiltonian:

\[
\frac{d}{dt} I = (\lambda^0 + \varepsilon \lambda^1)(I) = -\frac{\partial H_0}{\partial \theta} + \varepsilon \lambda^1(I) \\
= -\frac{\partial H_0}{\partial \theta} - \varepsilon \frac{\partial H_1}{\partial \theta}, \tag{4.16}
\]

\[
\frac{d}{dt} \theta = (\lambda^0 + \varepsilon \lambda^1)(\theta) = \frac{\partial H_0}{\partial I} + \varepsilon \lambda^1(\theta) \\
= \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I}. \tag{4.17}
\]

\[
\frac{d}{dt} y = \lambda^0(y) + \varepsilon \lambda^1(y) = -\frac{\partial H_0}{\partial x} + \varepsilon \lambda^1(y) \\
= -\frac{\partial H_0}{\partial x} - \varepsilon \frac{\partial H_1}{\partial x}. \tag{4.18}
\]

\[
\frac{d}{dt} x = \lambda^0(x) + \varepsilon \lambda^1(x) = \frac{\partial H_0}{\partial y} + \varepsilon \lambda^1(x) \\
= \frac{\partial H_0}{\partial y} + \varepsilon \frac{\partial H_1}{\partial y}. \tag{4.19}
\]

5 Proof of the main result

In this section, we prove Theorem 2.1.

5.1 Perturbed normally hyperbolic invariant manifolds

In this section, we prove the assertions (i) and (ii) of Theorem 2.1. Still, we note that those assertions hold under more general conditions than the ones assumed.

We only give the details in the case when \( H_0 \) satisfies the conditions \((A-i)\) and \((A-ii)\). The case when \( H_0 \) satisfies \((A'-i)\) and \((A'-ii)\) follows similarly.

5.1.1 Persistence of the normally hyperbolic invariant manifold under perturbation

We have that \( \Lambda_0 \) is a NHIM for the flow \( \Phi_0^t \) of \( \lambda^0 \). Then \( D\Phi_0^t(z) \) satisfies expansion/contraction rates as in “Appendix A”, for all \( z \in \Lambda_0 \), where we denote the constant and the expansion and contraction rates by \( C, \lambda_-, \lambda_+, \lambda_c, \mu_c, \mu_-, \mu_+ \), respectively.

It is immediate that \( \Lambda = \Lambda_0 \times \mathbb{R} \) is a NHIM for the flow \( \Phi_0^t \) of the extended system (2.6). We note that \( \Lambda_0 \) is a non-compact manifold with boundary.

The theory of normally hyperbolic invariant manifolds, Fenichel (1971), Hirsch et al. (1977) and Pesin (2004) (a handy summary of the results of the theory is Delshams et al. 2006), asserts the persistence of NHIMs under small perturbations. The persistence of non-compact manifolds when the perturbation has uniformly bounded derivatives in all variables is shown in Hirsch et al. (1977, Section 6). This generality is crucial for infinite-dimensional manifolds (Bates et al. 1999, 2008). When the unperturbed NHIM has a boundary, in the literature, one can find the results of persistence for inflowing or overflowing manifolds on the boundary. (These conditions are automatic when the boundary is invariant.) In such a
case, the persistent manifold is not necessarily invariant but only locally invariant and is not necessarily unique. See “Appendix A”.

Under the assumption (A-iii) of Theorem 2.1, $X^1 = X^1(z, t; \varepsilon)$ has uniformly bounded derivatives in all variables. Therefore, there exists $\varepsilon_1$ such that the manifold $\Lambda_0$ persists as a normally hyperbolic manifold $\tilde{\Lambda}_e$, for all $|\varepsilon| < \varepsilon_1$, which is locally invariant under the flow $\Phi^\tau_{\varepsilon}$.

In terms of the normal form coordinates $(I, \theta, y, x)$ from Proposition 4.1–(N), the unperturbed NHIM is given by:

$$\tilde{\Lambda}_0 = \{(I, \theta, y, x, t)| x = y = 0\}.$$  

Then, the perturbed NHIM can be written as a graph over the $(I, \theta, t)$-coordinates over a suitable domain, i.e.,

$$\tilde{\Lambda}_e = \{(I, \theta, y, x, t)| x = x_e(I, \theta, t), y = y_e(I, \theta, t)\}.$$  

See “Appendix A”. Consequently, every point $\tilde{z}_e \in \tilde{\Lambda}_e$ is determined by its coordinates $(I, \theta, t)$.

The NHIM $\tilde{\Lambda}_e$ is $O(\varepsilon)$ close in the $C^\ell$-topology to $\tilde{\Lambda}_0$, where $\ell$ is as in (A.3). The locally invariant manifolds are in fact invariant manifolds for an extended system, and they depend on the extension. Hence, they do not need to be unique. Nevertheless, given a smooth family of systems, it is possible to choose the invariant manifolds in such a way that the invariant manifolds depend smoothly on parameters, as well as the stable and unstable bundles and the stable and unstable manifolds.

For the perturbed NHIM $\tilde{\Lambda}_e$, $|\varepsilon| < \varepsilon_1$, there exists an invariant splitting of the tangent bundle $T\tilde{\Lambda}_e$, similar to that in (A.1), so that $D\tilde{\Phi}_\varepsilon^\tau(\tilde{z})$ satisfies expansion/contraction relations similar to those in (A.2) for all $\tilde{z} \in \tilde{\Lambda}_e$, for some constants $\tilde{C}, \tilde{\lambda}_-, \tilde{\lambda}_+, \tilde{\lambda}_e, \tilde{\mu}_-, \tilde{\mu}_+, \tilde{\mu}_e$. These constants are independent of $\varepsilon$ and can be chosen as close as desired to the unperturbed ones, that is, to $C, \lambda_-, \lambda_+, \lambda_e, \mu_-, \mu_+, \mu_e$, respectively, by choosing $\varepsilon_1$ suitably small.

There exist unstable and stable manifolds $W^u(\tilde{\Lambda}_e)$, $W^s(\tilde{\Lambda}_e)$ associated with $\tilde{\Lambda}_e$, and there exist corresponding projection maps $\Omega^- : W^u(\tilde{\Lambda}_e) \rightarrow \tilde{\Lambda}_e$ and $\Omega^+ : W^s(\tilde{\Lambda}_e) \rightarrow \tilde{\Lambda}_e$.

For $\tilde{z}^+ = \Omega^+(\tilde{z})$, with $\tilde{z} \in W^s(\tilde{\Lambda}_e)$, we have

$$(5.1) \quad d(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}), \tilde{\Phi}_\varepsilon^\tau(\tilde{z}^+)) \leq C_{\tilde{z}} e^{t\tilde{\lambda}_+}, \quad \text{for all } \tau \geq 0,$$

and for $\tilde{z}^- = \Omega^-(\tilde{z})$, with $\tilde{z} \in W^u(\tilde{\Lambda}_e)$, we have

$$(5.2) \quad d(\tilde{\Phi}_\varepsilon^\tau(\tilde{z}), \tilde{\Phi}_\varepsilon^\tau(\tilde{z}^-)) \leq C_{\tilde{z}} e^{t\tilde{\mu}_-}, \quad \text{for all } \tau \leq 0,$$

for some $C_{\tilde{z}} > 0$. The constant $\tilde{C}_{\tilde{z}}$ can be chosen uniformly bounded, provided that we restrict to $z$ to the compact neighborhood $\mathcal{K}$ given by (A-ii), and we use the fact that $X^1 = X^1(z, t; \varepsilon)$ is uniformly differentiable in all variables.

To simplify notation, from now on we will drop the symbol from $\tilde{C}, \tilde{C}_{\tilde{z}} \tilde{\lambda}_-, \tilde{\lambda}_+, \tilde{\mu}_-, \tilde{\mu}_+, \tilde{\lambda}_e, \tilde{\mu}_e$.

In the sequel, we will fix a choice $\tilde{\Lambda}_e$, and all computations will be performed relative to that choice. Nevertheless, the estimate for the perturbed scattering map $\tilde{\sigma}_e$ is independent of the choice of the locally invariant manifold $\tilde{\Lambda}_e$.
5.1.2 Persistence of the transverse intersection of the hyperbolic invariant manifolds under perturbation

For the unperturbed system, the unstable and stable manifolds $W^u(\tilde{\Lambda}_0)$, $W^s(\tilde{\Lambda}_0)$ intersect transversally along the 3-dimensional homoclinic channel $\tilde{\Gamma}_0$. By the persistence of transversality under small perturbations, it follows that $W^u(\tilde{\Lambda}_e)$, $W^s(\tilde{\Lambda}_e)$ intersect transversally along $\tilde{\Gamma}_e$, for all $|\varepsilon| < \varepsilon_1$, provided $\varepsilon_1$ is chosen small enough. Condition (B.1) in the definition of a homoclinic/heteroclinic channel is also a transversality-type condition, so it is also persistent under small perturbations. We conclude that (i) and (ii) from Theorem 2.1 hold true for all $|\varepsilon| < \varepsilon_1$, provided $\varepsilon_1$ is chosen small enough.

Recall from Sect. 4.2 that the unperturbed homoclinic channel $\tilde{\Gamma}_0$ can be described as a graph over the $(I^u, \theta^u, t)$-variables, as well as a graph over the $(I^s, \theta^s, t)$-variables. It follows that for the perturbed system, for $|\varepsilon| < \varepsilon_1$ with $\varepsilon_1$ sufficiently small, the perturbed homoclinic channel $\tilde{\Gamma}_e$ can also be described as a graph over the $(I^s, \theta^s, t)$-variables, as well as a graph over the $(I^u, \theta^u, t)$-variables. More precisely, we have

\begin{equation}
\tilde{\Gamma}_e = \{(I^u, \theta^u, y^u, x^u, t) \mid y^u = y^u_e(I^u, \theta^u, t), x^u = x^u_e(I^u, \theta^u, t), (5.3)\}
\end{equation}

with $y^u_e(I^u, \theta^u, t) = O(\varepsilon)$, $x^u_e(I^u, \theta^u, t) = x^u_0(I^u, \theta^u, t) + O(\varepsilon)$, $y^s_e(I^s, \theta^s, t) = y^s_0(I^s, \theta^s, t) + O(\varepsilon)$, and $x^s_e(I^s, \theta^s, t) = O(\varepsilon)$.

Therefore, each homoclinic point $\tilde{z}_e \in \tilde{\Gamma}_e$ is associated with unique coordinate triples $(I^u, \theta^u, t)$ and $(I^s, \theta^s, t)$, which satisfy (4.14).

We now describe how to match a perturbed homoclinic point $\tilde{z}_e \in \tilde{\Gamma}_e$ to an unperturbed homoclinic point $\tilde{z}_0 \in \tilde{\Gamma}_0$. As we represent both $\tilde{\Gamma}_e$ and $\tilde{\Gamma}_0$ as graphs over the same coordinates, we have a canonical way to match the homoclinic points.

To each point $\tilde{z}_0 \in \tilde{\Gamma}_0$, of local coordinates $(I^s, \theta^s, t)$ and $(I^u, \theta^u, t)$, with $(I^s, \theta^s, t) = (I^u, \theta^u + \Delta(I^u), t)$, we assign the point $\tilde{z}_e \in \tilde{\Gamma}_e$ that has the same local coordinates $(I^s, \theta^s, t)$ and $(I^u, \theta^u, t)$, via the graph representations in (5.3).

More precisely, if $\tilde{z}_0 \in \tilde{\Gamma}_0$ is given by

\begin{align*}
(I^u_0, \theta^u_0, 0, x^u_0(I^u, \theta^u, t_0), t_0), & \quad \text{and} \\
(I^s_0, \theta^s_0, y^s_0(I^s, \theta^s, t_0), 0, t_0),
\end{align*}

with $I^s_0 = I^s_0$ and $\theta^s_0 = \theta^s_0 + \Delta(I^u_0)$, then we associate to it $\tilde{z}_e \in \tilde{\Gamma}_e$ given by

\begin{align*}
(I^u_0, \theta^u_0, y^u_e(I^u_0, \theta^u_0, t_0), x^u_e(I^u_0, \theta^u_0, t_0), t_0) & \quad \text{and} \\
(I^s_0, \theta^s_0, y^s_e(I^s_0, \theta^s_0, t_0), x^s_e(I^s_0, \theta^s_0, t_0), t_0),
\end{align*}

respectively. Note also that the representation of the NHIM in these coordinates will also change.

Therefore, to $\tilde{z}_0 \in \tilde{\Gamma}_0$ we associate $\tilde{z}_e \in \tilde{\Gamma}_e$ with

\begin{equation}
I^s(\tilde{z}_e) - I^u(\tilde{z}_e) = 0, \quad \theta^s(\tilde{z}_e) - \theta^u(\tilde{z}_e) = \theta^s(\tilde{z}_0) - \theta^u(\tilde{z}_0), \quad (5.4)
\end{equation}

and with the same time-coordinate $t_0$.

This choice can be understood geometrically. The intersection $\Gamma_e$ is a smooth manifold. By the theory of normally hyperbolic manifolds, it depends smoothly on parameters. This manifold is parameterized by the $(I, \theta, t)$ coordinates. To specify a family of points, we impose a condition on these coordinates.
By letting all the choices vary—over all the \((I, \theta, t)\) that satisfy (5.4)—we can compute the whole manifold.

Choice (5.4) is one out of many possible choices. We have made it to simplify several calculations.

In the sequel, we will compare the scattering map associated with \(\tilde{z}_0\) with the scattering map associated with \(\tilde{z}_e\) satisfying (5.4). The method of calculation we will use is to compute the leading terms of the change of the coordinates of the asymptotic points \(z_{e\pm}\). We note that these points exist because of the theory of smooth dependence of the NHIM on parameters. Once we know that these points exist and that they are asymptotic, we compute the change of coordinates using the unperturbed dynamics. Since the unperturbed system conserves energy, the changes of energy are caused by the perturbation, so they are a slow variable. For the fast variables, we will need to introduce extra cancellations between the dynamics in the asymptotic trajectory and the homoclinic one.

5.2 Perturbed scattering map

In this section, we prove the assertion (iii) of Theorem 2.1.

We recall that the existence of the scattering map and its smooth dependence on parameters follow from the standard theory of normally hyperbolic invariant manifolds. The only thing we have to do is to compute the formulas for the derivatives, knowing that they exist. This was done by estimating the change of the coordinate functions along the connecting orbits.

We start with the unperturbed system (2.3). We recall that for a given homoclinic channel \(\Gamma_0\), the corresponding scattering map \(\sigma_0\) is a phase shift of the form:

\[
\sigma_0(I, \theta) = (I, \theta + \Delta(I)).
\]

We choose and fix an energy level \(h\) of \(H_0\), and a point \(z_0 \in \Gamma_0 \cap M_h\). In the \((I^u, \theta^u, y^u, x^u)-coordinates\), \(z_0\) is given by

\[
z_0 = (I_0^s, \theta_0^s, y_0^s, 0) = (I_0^u, \theta_0^u, 0, x_0^u),
\]

where \(I_0^s = I_0^u = I_0\). The effect of the flow \(\Phi_0^+\) on \(z_0\) in these coordinates is given by

\[
\Phi_0^+(z_0) = (I_0^s, \theta_0^s + \omega(I_0)\tau, y_0^s(\tau), 0) = (I_0^u, \theta_0^u + \omega(I_0)\tau, 0, x_0^u(\tau)),
\]

where \(y_0^s(\tau)\) and \(x_0^u(\tau)\) are the \(y^s\)-component and the \(x^u\)-component, respectively, of \(\Phi_0^+(z_0)\) evaluated at \(\tau\), and \(\omega(I_0) = \partial \theta_0^u / \partial \tau\).

There exist uniquely defined points \(z_0^-, z_0^+\) in \(\lambda_0(h)\) such that \(W^u(z_0^-) \cap (\Gamma_0 \cap M_h) = W^s(z_0^+) \cap (\Gamma_0 \cap M_h) = \{z_0\}\). In the \((I, \theta, y, x)-coordinates\), the footpoints \(z_0^\pm\) are given by

\[
z_0^- = (I_0, \theta_0^-, 0, 0), \quad z_0^+ = (I_0, \theta_0^+, 0, 0),
\]

where \(\theta_0^- = \theta_0^u\) and \(\theta_0^+ = \theta_0^s\). The effect of the flow \(\Phi_0^\pm\) on \(z_0^\pm\) in these coordinates is given by:

\[
\Phi_0^-(z_0^-) = (I_0, \theta_0^- + \omega(I_0)\tau, 0, 0), \quad \Phi_0^+(z_0^+) = (I_0, \theta_0^+ + \omega(I_0)\tau, 0, 0).
\]

The scattering map \(\sigma_0\) takes \(z_0^- \in \lambda_0(h)\) into \(z_0^+ \in \lambda_0(h)\).

In the extended system (2.6), the corresponding homoclinic point is \(\tilde{z}_0 = (z_0, t_0)\) for some \(t_0 \in \mathbb{R}\). The scattering map \(\tilde{\sigma}_0\) takes \(\tilde{z}_0^- = (z_0^-, t_0)\) into \(\tilde{z}_0^+ = (z_0^+, t_0)\).
We will compute the effect of the perturbation on the scattering map \( \tilde{\sigma}_0 \).

When we add the perturbation, there exists a homoclinic point \( \tilde{z}_e \in \tilde{\Gamma}_e \) corresponding to \( \tilde{z}_0 = (z_0, t_0) \) from the unperturbed case, such that \( \tilde{z}_e \) satisfies the condition (5.4). Associated to \( \tilde{z}_e \in \tilde{\Gamma}_e \), we have the points \( \tilde{z}^-_e, \tilde{z}^+_e \) in \( \tilde{\Lambda}_e \) such that \( W^u(\tilde{z}^-_e) \cap \tilde{\Gamma}_e = \{ \tilde{z}_e \} \), and \( W^s(\tilde{z}^+_e) \cap \tilde{\Gamma}_e = \{ \tilde{z}_e \} \). The scattering map \( \tilde{\sigma}_e \) takes \( \tilde{z}^-_e \in \tilde{\Lambda}_e \) into \( \tilde{z}^+_e \in \tilde{\Lambda}_e \).

In the sequel, we will make a quantitative comparison between

\[
\tilde{z}_0 \mapsto \tilde{\sigma}_0(\tilde{z}_0^-) := \tilde{z}_0^+,
\]

and

\[
\tilde{z}_e \mapsto \tilde{\sigma}_e(\tilde{z}_e^-) := \tilde{z}_e^+.
\]

### 5.2.1 Estimates

Below we will refer to the notation in (5.5)–(5.8). To simplify notation, we denote \( I_e^s = I^s(\tilde{z}_e) \), \( I_e^u = I^u(\tilde{z}_e) \), \( I_e^{s+} = I^s(\tilde{z}^+_e) \), \( I_e^{u-} = I^u(\tilde{z}^-_e) \), \( \xi_e^s = (x^s y^s)(\tilde{z}_e) \), \( \xi_e^{s+} = (x^s y^s)(\tilde{z}^+_e) \), \( \xi_e^u = (x^u y^u)(\tilde{z}_e) \), \( \xi_e^{u-} = (x^u y^u)(\tilde{z}^-_e) \).

Note that in the following, the coordinates of the scattering map can be considered as functions of the point. Hence, the symbols \( O(\varepsilon) \) can be interpreted as relative to the \( C^r \) norm.

**Lemma 5.1**

(i) Estimates on \( I \):

\[
I_e^{s+} - I_e^s = O(\varepsilon), \quad I_e^{u-} - I_e^u = O(\varepsilon), \quad I_e^{s+} - I_e^{u-} = O(\varepsilon).
\]

(ii) Estimates on \( h \):

\[
h_0(I_e^{s+}) - h_0(I_e^s) = (I_e^{s+} - I_e^s) \left( \frac{\partial h_0}{\partial I}(I_0) \right) + O(\varepsilon^2),
\]

\[
h_0(I_e^{u-}) - h_0(I_e^u) = (I_e^{u-} - I_e^u) \left( \frac{\partial h_0}{\partial I}(I_0) \right) + O(\varepsilon^2).
\]

(iii) Estimates on \( \xi \):

\[
\xi_e^s = O(\varepsilon), \quad \xi_e^u = O(\varepsilon), \quad \xi_e^{s+} = O(\varepsilon^2), \quad \xi_e^{u-} = O(\varepsilon^2).
\]

(iv) Estimates on \( g_1 \):

\[
g_1(I_e^{s+}) = g_1(I_0) + \frac{\partial g_1}{\partial I}(I_0)(I_e^{s+} - I_0) + O(\varepsilon^2),
\]

\[
g_1(I_e^{u-}) = g_1(I_0) + \frac{\partial g_1}{\partial I}(I_0)(I_e^{u-} - I_0) + O(\varepsilon^2).
\]

(v) Estimates on \( \frac{\partial g_1}{\partial I} \):

\[
\frac{\partial g_1}{\partial I}(I_e^{s+}) = \frac{\partial g_1}{\partial I}(I_0) + \frac{\partial^2 g_1}{\partial I^2}(I_0)(I_e^{s+} - I_0) + O(\varepsilon^2),
\]

\[
\frac{\partial g_1}{\partial I}(I_e^{u-}) = \frac{\partial g_1}{\partial I}(I_0) + \frac{\partial^2 g_1}{\partial I^2}(I_0)(I_e^{u-} - I_0) + O(\varepsilon^2).
\]

**Proof** (i) Due to (5.3), we have \( \tilde{z}_e = z_0 + O(\varepsilon) \) and \( \tilde{z}^\pm_e = \begin{pmatrix} z^\pm_0 + O(\varepsilon) \end{pmatrix} \), we have \( I_e^{s,u}(\tilde{z}_e) = I^{s,u}(z_0) + O(\varepsilon) \), and \( I_e^{s,u}(\tilde{z}_e) = I^{s,u}(z_0) + O(\varepsilon) \). We note the graph property of \( \tilde{\Gamma}_e \) (5.3), guaranteeing that \( \tilde{z}_e \) and \( z_0 \) have the same time-coordinate \( t_0 \in \mathbb{R} \).

The fact that \( I_e^{s,u}(\tilde{z}_0) = I_e^{s,u}(z_0^\pm) = I_0 \) yields the estimates in (i).
(ii) We estimate the term \( h_0(I_{e}^{s+}) - h_0(I_{e}^{s}) \). Applying the integral form of the mean value theorem, we have

\[
h_0(I_{e}^{s+}) - h_0(I_{e}^{s}) = (I_{e}^{s+} - I_{e}^{s}) \int_0^1 \frac{dh_0}{dt} (tI_{e}^{s+} + (1-t)I_{e}^{s}) \, dt.
\] (5.14)

We write the integrand of (5.14) as a Taylor expansion

\[
\frac{\partial h_0}{\partial I}(tI_{e}^{s+} + (1-t)I_{e}^{s}) = \frac{\partial h_0}{\partial I}(I_0) + \frac{\partial^2 h_0}{\partial I^2}(tI_{e}^{s+} + (1-t)I_{e}^{s}) + O(\varepsilon^2)
\] (5.15)

where we used

\[
\frac{\partial h_0}{\partial I}(tI_{e}^{s+} + (1-t)I_{e}^{s}) = \frac{\partial h_0}{\partial I}(I_0) \quad \text{and} \quad \frac{\partial^2 h_0}{\partial I^2}(tI_{e}^{s+} + (1-t)I_{e}^{s}) = \frac{\partial^2 h_0}{\partial I^2}(I_0).
\]

Since \( I_{e}^{s+} - I_{e}^{s+} = O(\varepsilon) \) and \( I_{e}^{s} - I_{e}^{s} = O(\varepsilon) \), we have

\[
\frac{\partial h_0}{\partial I}(tI_{e}^{s+} + (1-t)I_{e}^{s}) = \frac{\partial h_0}{\partial I}(I_0) + O(\varepsilon).
\] (5.16)

Since \( I_{e}^{s+} - I_{e}^{s} = O(\varepsilon) \) from (5.14), we obtain the first estimate in (5.10). The other estimate follows similarly.

(iii) The fact that \( \xi^{s,u} = O(\varepsilon) \) follows from \( \tilde{z}_e = \tilde{z}_0 + O(\varepsilon) \), and \( \xi^{s,u}(\tilde{z}_0) = 0 \), hence \( \xi^{s,u}(\tilde{z}_e) = \xi^{s,u}(\tilde{z}_0) + O(\varepsilon) = O(\varepsilon) \).

In the same way, we obtain \( \xi^{s+} = \xi^{u-} = O(\varepsilon) \). To prove that in fact \( \xi^{s+} = \xi^{u-} = O(\varepsilon^2) \), we proceed as follows. For \( \xi^{s+}_e \), we use the Taylor expansion:

\[
\xi^{s+}_e = \xi^{s+}_0 + D\xi^{s+}_0 \cdot (\xi^{s+}_e - \xi^{s+}_0) + O(\varepsilon^2),
\]

where \( \cdot \) denotes the dot product, and we used that \( \xi^{s+}_e - \xi^{s+}_0 = O(\varepsilon) \). We have

\[
\xi^{s+}_0 = 0
\]

and

\[
D\xi^{s+}_0 \cdot (\xi^{s+}_e - \xi^{s+}_0) = x^s(\tilde{z}_0^s) (y^{s+}_e - y^{s+}_0) + y^s(\tilde{z}_0^s) (x^{s+}_e - x^{s+}_0) = 0,
\]

since \( y^s(\tilde{z}_0^s) = x^s(\tilde{z}_0^s) = 0 \) on \( \Lambda_0 \). Therefore (5.2.1) implies

\[
\xi^{s+}_e = O(\varepsilon^2).
\]

Similarly, we obtain

\[
\xi^{u-}_e = O(\varepsilon^2).
\]

(iv) We write \( g_1(I_{e}^{s+}) \) as a Taylor expansion, using that \( I_{e}^{s+} - I_{e}^{s+} = O(\varepsilon) \), obtaining

\[
g_1(I_{e}^{s+}) = g_1(I_{e}^{s+}) + \frac{\partial g_1}{\partial I}(I_{e}^{s+})(I_{e}^{s+} - I_{e}^{s+}) + O(\varepsilon^2).
\]

Since \( I_{e}^{s+} = I_0 \), the first equation in (5.12) follows.

Similarly

\[
g_1(I_{e}^{u-}) = g_1(I_{e}^{u-}) + \frac{\partial g_1}{\partial I}(I_{e}^{u-})(I_{e}^{u-} - I_{e}^{u-}) + O(\varepsilon^2),
\]
and \( I_0^{u-} = I_0 \) yield the second equation in (5.12).

(v) The proof is similar to that of (iv).

\[ \square \]

5.2.2 Change in action by the scattering map

We now give the expression of the action–coordinate \( \tilde{S}^I \) of the mapping \( \tilde{S} \) in (2.7), where \( I \) is the action–coordinate described in Proposition 4.1-(N).

Proposition 5.2 The change in \( I \) by the scattering map \( \tilde{\sigma}_\varepsilon \) is given by the following formula:

\[
I(\tilde{z}_\varepsilon^+) - I(\tilde{z}_\varepsilon^-) = -\varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \left( (\mathcal{X}^1 H_0)(\Phi_0^+(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\Phi_0^-(\tilde{z}_0^-)) \right) \, d\tau \\
- \varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_{-\infty}^0 \left( (\mathcal{X}^1 H_0)(\Phi_0^+(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\Phi_0^-(\tilde{z}_0^-)) \right) \, d\tau + O(\varepsilon^2). \tag{5.17}
\]

When the perturbation is Hamiltonian \( \mathcal{X}^1 = J \nabla H_1 \), in (5.17), we have \( \mathcal{X}^1 H_0 = \{H_0, H_1\} \), where \( \{\cdot, \cdot\} \) denotes the Poisson bracket.

Proposition 5.2 implies that

\[ \tilde{\sigma}_\varepsilon^I(I, \theta, t) = \tilde{\sigma}_0^I(I, \theta, t) + \varepsilon \tilde{S}^I(I, \theta, t) + O(\varepsilon^2), \]

where \( \varepsilon \tilde{S}^I \) is given by the first two terms on the right-hand side of (5.17). The expression of \( \varepsilon \tilde{S}^I \) is particularly simple since \( I \) is a slow variable.

Below we will refer to the notation in (5.5)–(5.8).

We note that we can express the right-hand side of (5.17) in terms of the \( (I^{u,s}, \theta^{u,s}, y^{u,s}, x^{u,s}) \) coordinates, by making the following substitutions:

\[ \Phi_0^+(\tilde{z}_0^+) = (I_0^s, \theta_0^s + \omega(I_0) \tau, 0, 0, t_0 + \tau), \]
\[ \Phi_0^-(\tilde{z}_0^-) = (I_0^s, \theta_0^u + \omega(I_0) \tau, 0, 0, t_0 + \tau), \]
\[ \Phi_0^-(\tilde{z}_0^-) = (I_0^s, \theta_0^u + \omega(I_0) \tau, y^s(\tau), 0, t_0 + \tau) \]
\[ = (I_0^u, \theta_0^u + \omega(I_0) \tau, 0, x^u(\tau), t_0 + \tau). \]

To prove Proposition 5.2, we will use the following:

Lemma 5.3 The change in \( H_0 \) by the scattering map \( \tilde{\sigma}_\varepsilon \) is given by the following equation:

\[
H_0(\tilde{z}_\varepsilon^+) - H_0(\tilde{z}_\varepsilon^-) = -\varepsilon \int_0^{+\infty} \left( (\mathcal{X}^1 H_0)(\Phi_0^+(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\Phi_0^-(\tilde{z}_0^-)) \right) \, d\tau \\
- \varepsilon \int_{-\infty}^0 \left( (\mathcal{X}^1 H_0)(\Phi_0^+(\tilde{z}_0^+)) - (\mathcal{X}^1 H_0)(\Phi_0^-(\tilde{z}_0^-)) \right) \, d\tau + O(\varepsilon^2). \tag{5.18}
\]

Proof of Lemma 5.3 First, note that

\[ (\mathcal{X}^0 + \varepsilon \mathcal{X}^1)H_0 = \mathcal{X}^0 H_0 + \varepsilon \mathcal{X}^1 H_0 = \{H_0, H_0\} + \varepsilon \mathcal{X}^1 H_0 = \varepsilon \mathcal{X}^1 H_0. \]
Second, applying Lemma D.1 and Lemma D.2 from “Appendix D”, for $F = H_0$, we have

$$H_0(z^+_e) - H_0(z^-_e) = -\varepsilon \int_{0}^{+\infty} \left( (\partial^1 H_0)(\Phi^T_0(z^+_e)) - (\partial^1 H_0)(\Phi^T_0(z^-_e)) \right) d\tau + O(\varepsilon^{1+\rho})$$

(5.19)

$$H_0(z^-_e) - H_0(z^-_e) = \varepsilon \int_{-\infty}^{0} \left( (\partial^1 H_0)(\Phi^T_0(z^-_e)) - (\partial^1 H_0)(\Phi^T_0(z^-_e)) \right) d\tau + O(\varepsilon^{1+\rho}).$$

(5.20)

Subtracting the two equations from above, after cancelling out the common term $H_0(\tilde{z}_e)$ representing the value of $H_0$ at the homoclinic point $\tilde{z}_0$, we obtain (5.18) with an error term of order $O(\varepsilon^{1+\rho})$. Since the function $H_0(z^+_e) - H_0(z^-_e)$ can be expanded as a Taylor series in $\varepsilon$, by matching the corresponding terms of this Taylor expansion with the terms in (5.19) minus (5.20), it follows that the error term $O(\varepsilon^{1+\rho})$ must equal $O(\varepsilon^2)$.

\[\square\]

**Proof of Proposition 5.2** By Proposition 4.1, we have

$$H_0(z^+_e) = h_0(I^s_e) + (\xi_s^+ g_1(I^s_e) + (\xi_s^+)^2 g_2(I^s_e, \xi_s^+).$$

(5.21)

$$H_0(z^-_e) = h_0(I^-_e) + (\xi_u^- g_1(I^-_e) + (\xi_u^-)^2 g_2(I^-_e, \xi_u^-).$$

(5.22)

Subtracting we obtain

$$H_0(z^+_e) - H_0(z^-_e) = (h_0(I^s_e) - h_0(I^-_e))$$

$$+ ((\xi_s^+ g_1(I^s_e) - (\xi_u^- g_1(I^-_e))$$

$$+ O(\varepsilon^2).$$

(5.23)

where the error term $O(\varepsilon^2)$ in the above is due to Lemma 5.1 Eq. (5.11).

The term $h_0(I^s_e) - h_0(I^-_e)$ in (5.23) is given, by Lemma 5.1 Eq. (5.10), as

$$h_0(I^s_e) - h_0(I^-_e) = (I^s_e - I^-_e) \left( \frac{\partial h_0}{\partial I} (I) \right) + O(\varepsilon^2).$$

(5.24)

Since, by Lemma 5.1 Eq. (5.11), we have

$$\xi_s^+ = O(\varepsilon^2), \quad \xi_u^- = O(\varepsilon^2),$$

(5.25)

we obtain

$$(\xi_s^+ g_1(I^s_e) - (\xi_u^- g_1(I^-_e) = O(\varepsilon^2).$$

(5.26)

Thus, from (5.23) and using (5.24) we have

$$H_0(z^+_e) - H_0(z^-_e) = (I^s_e - I^-_e) \left( \frac{\partial h_0}{\partial I} (I) \right) + O(\varepsilon^2).$$

(5.27)
As the left-hand side of (5.27) is given by (5.18), since $\frac{\partial h_0}{\partial I} \neq 0$ by (4.10), solving for $I^{s+}_e - I^{u-}_e$ yields

\[
I^{s+}_e - I^{u-}_e = \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \left( H_0(\tilde{z}^+_e) - H_0(\tilde{z}^-_e) \right)
\]

\[
= -\varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \left( \chi^1 H_0 \left( \tilde{\Phi}_0^T(\tilde{z}^+_0) \right) - \chi^1 H_0 \left( \tilde{\Phi}_0^T(\tilde{z}^-_0) \right) \right) \, d\tau
\]

\[
- \varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_{-\infty}^{0} \left( \chi^1 H_0 \left( \tilde{\Phi}_0^T(\tilde{z}^-_0) \right) - \chi^1 H_0 \left( \tilde{\Phi}_0^T(\tilde{z}^-_0) \right) \right) \, d\tau
\]

\[
+ O(\varepsilon^{1+\varepsilon}).
\]

(5.28)

By the same argument as in the proof of Lemma 5.3, the error term $O(\varepsilon^{1+\varepsilon})$ in (5.28) must equal $O(\varepsilon^2)$. This shows (5.17).

\[\square\]

**Proposition 5.4**

\[
I^{s+}_e - I^s_e = \varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \left( \chi^1 H_0(\tilde{\Phi}_0^T(\tilde{z}^+_0)) - \chi^1 H_0(\tilde{\Phi}_0^T(\tilde{z}^-_0)) \right) \, d\tau
\]

\[
- \varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} g_1(I_0) \int_0^{+\infty} \left( \chi^1 \tilde{\xi}^s(\tilde{\Phi}_0^T(\tilde{z}^+_0)) - \chi^1 \tilde{\xi}^s(\tilde{\Phi}_0^T(\tilde{z}^-_0)) \right) \, d\tau
\]

\[
+ O(\varepsilon^2).
\]

(5.29)

\[
I^{u-}_e - I^u_e = \varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_{-\infty}^{0} \left( \chi^1 H_0(\tilde{\Phi}_0^T(\tilde{z}^-_0)) - \chi^1 H_0(\tilde{\Phi}_0^T(\tilde{z}^-_0)) \right) \, d\tau
\]

\[
- \varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} g_1(I_0) \int_{-\infty}^{0} \left( \chi^1 \tilde{\xi}^u(\tilde{\Phi}_0^T(\tilde{z}^-_0)) - \chi^1 \tilde{\xi}^u(\tilde{\Phi}_0^T(\tilde{z}^-_0)) \right) \, d\tau
\]

\[
+ O(\varepsilon^2).
\]

(5.30)

**Proof** The formula for $H_0(\tilde{z}^+_e) - H_0(\tilde{z}^-_e)$ is given by (5.19). The normal form expansion of $H_0$ at a homoclinic point $\tilde{z}_e$ can be written with respect to the two sets of coordinates as

\[
H_0(\tilde{z}_e) = h_0(I^s_e) + (\tilde{\xi}^s_z) g_1(I^s_e) + (\tilde{\xi}^s_z)^2 g_2(I^s_e, \tilde{\xi}^s_z)
\]

(5.31)

\[
H_0(\tilde{z}_e) = h_0(I^u_e) + (\tilde{\xi}^u_z) g_1(I^u_e) + (\tilde{\xi}^u_z)^2 g_2(I^u_e, \tilde{\xi}^u_z)
\]

(5.32)

Subtracting (5.31) from (5.21), we obtain

\[
H_0(\tilde{z}^+_e) - H_0(\tilde{z}^-_e) = h_0(I^{s+}_e) - h_0(I^s_e) + (\tilde{\xi}^{s+}_z) g_1(I^{s+}_e) - (\tilde{\xi}^s_z) g_1(I^s_e) + O(\varepsilon^2)
\]

\[
= \left( \frac{\partial h_0}{\partial I}(I_0) \right) (I^{s+}_e - I^s_e)
\]

\[
+ (\tilde{\xi}^{s+}_z) g_1(I^{s+}_e) - g_1(I^s_e)
\]

\[
+ (\tilde{\xi}^{s+}_z - \tilde{\xi}^s_z) g_1(I^s_e) + O(\varepsilon^2)
\]

(5.33)

\[
= \left( \frac{\partial h_0}{\partial I}(I_0) \right) (I^{s+}_e - I^s_e)
\]

\[
+ (\tilde{\xi}^{s+}_z - \tilde{\xi}^s_z) g_1(I_0) + O(\varepsilon^2).
\]

In the above, we have used Lemma 5.1, Eqs. (5.10)–(5.12).
Applying Lemma D.1 and Lemma D.2 from “Appendix D”, for $F = \xi^s$, we have

$$
\xi^{s+} - \xi^s = -\varepsilon \int_0^{+\infty} \left( (\lambda^1 \xi^s)(\Phi^+_0(z_0^+)) - (\lambda^1 \xi^s)(\Phi^+_0(z_0)) \right) \, d\tau + O(\varepsilon^{1+\varepsilon}).
$$

(5.34)

Thus, using (5.33), (5.19), and (5.34), we obtain (5.29).

Equation (5.30) follows similarly.

The argument that the error term $O(\varepsilon^{1+\varepsilon})$ can be replaced by $O(\varepsilon^2)$ follows in the same way as in the proof of Lemma 5.3.

Let $\tau \in \mathbb{R}$ be some value of the time variable. Applying formula (5.29) to $\tilde{\Phi}^+_e(z_+^e)$ and $\tilde{\Phi}^+_e(z_e^e)$ instead, respectively, and formula (5.30) to $\tilde{\Phi}^-_e(z^-_e)$ and $\tilde{\Phi}^-_e(z_e)$ instead, respectively, we obtain:

**Corollary 5.5** For any time $\tau \in \mathbb{R}$, we have

$$
I_e^s(\tilde{\Phi}^+(z_+^e)) - I_e^s(\tilde{\Phi}^+(z_e^e)) = -\varepsilon \left( \frac{\partial h_0}{\partial I} (I_0) \right)^{-1} \int_0^{+\infty} \left( (\lambda^1 H_0)(\tilde{\Phi}^+_0(z_0^+)) - (\lambda^1 H_0)(\tilde{\Phi}^+_0(z_0)) \right) \, d\xi + O(\varepsilon^2). \tag{5.35}
$$

$$
I_e^u(\tilde{\Phi}^+(z_+^e)) - I_e^u(\tilde{\Phi}^+(z_e^e)) = \varepsilon \left( \frac{\partial h_0}{\partial I} (I_0) \right)^{-1} \int_{-\infty}^{0} \left( (\lambda^1 H_0)(\tilde{\Phi}^+_0(z_0^+)) - (\lambda^1 H_0)(\tilde{\Phi}^+_0(z_0)) \right) \, d\xi \tag{5.36}
$$

and

$$
I_e^u(\tilde{\Phi}^-(z_-^e)) - I_e^u(\tilde{\Phi}^-(z_e)) = \varepsilon \left( \frac{\partial h_0}{\partial I} (I_0) \right)^{-1} \int_{-\infty}^{0} \left( (\lambda^1 H_0)(\tilde{\Phi}^+_0(z_0^-)) - (\lambda^1 H_0)(\tilde{\Phi}^+_0(z_0)) \right) \, d\xi + O(\varepsilon^2). \tag{5.37}
$$

**Corollary 6.**

$$
\int_0^{+\infty} \left( (\lambda^1 \xi^s)(\Phi^+_0(z_0^+)) - (\lambda^1 \xi^s)(\Phi^+_0(z_0)) \right) \, d\tau - \int_{-\infty}^{0} \left( (\lambda^1 \xi^u)(\Phi^+_0(z_0^-)) - (\lambda^1 \xi^u)(\Phi^+_0(z_0)) \right) \, d\tau = 0. \tag{5.38}
$$

**Proof** Let us denote

$$
J^+ = \int_0^{+\infty} \left( (\lambda^1 \xi^s)(\Phi^+_0(z_0^+)) - (\lambda^1 \xi^s)(\Phi^+_0(z_0)) \right) \, d\tau, \tag{5.39}
$$

$$
J^- = -\int_{-\infty}^{0} \left( (\lambda^1 \xi^u)(\Phi^+_0(z_0^-)) - (\lambda^1 \xi^u)(\Phi^+_0(z_0)) \right) \, d\tau. \tag{5.40}
$$

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Recall that, by condition (5.4), we have $I^u_0 = I^s_0$. Subtracting (5.36) from (5.35), and comparing with (5.17), we should have
\[
\left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} g_1(I_0)(J^+ + J^-) = 0.
\]
By (4.10), we have that $\frac{\partial h_0}{\partial I}(I_0) \neq 0$ and $g_1(I_0) \neq 0$, therefore
\[ J^+ + J^- = 0. \tag{5.39} \]

### 5.2.3 Change in angle by the scattering map

We now give the expression of the angle-component $\tilde{S}^\theta$ of the mapping $\tilde{S}$ in (2.7), where $\theta$ is the angle coordinate described in Proposition 4.1-(N).

**Proposition 5.7** The change in $\theta$ by the scattering map $\tilde{\sigma}_\varepsilon$ is given by the following equation:
\[
\theta^s(\tilde{z}^+_{\varepsilon}) - \theta^u(\tilde{z}^-_{\varepsilon}) = \Delta(I_0) - \varepsilon \int_0^\infty \left( \Lambda^1 \theta^s(\Phi^+_{\varepsilon}(\tilde{z}^+_{\varepsilon})) - \Lambda^1 \theta^s(\Phi^+_{\varepsilon}(\tilde{z}^-_{\varepsilon})) \right) d\tau
\]
\[- \varepsilon \int_{-\infty}^{0} \left( \Lambda^1 \theta^u(\Phi^+_{\varepsilon}(\tilde{z}^+_{\varepsilon})) - \Lambda^1 \theta^u(\Phi^+_{\varepsilon}(\tilde{z}^-_{\varepsilon})) \right) d\tau
\]+ $\varepsilon \left( \frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \left( \Lambda^1 H_0 \Phi^+_{\varepsilon}(\tilde{z}^+_{\varepsilon}) \right) d\tau$
\[- \left( \Lambda^1 H_0 \Phi^+_{\varepsilon}(\tilde{z}^-_{\varepsilon}) \right) d\tau
\]+
$\varepsilon \left( \frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_{-\infty}^{0} \left( \Lambda^1 H_0 \Phi^+_{\varepsilon}(\tilde{z}^-_{\varepsilon}) \right) d\tau$
\[- \left( \Lambda^1 H_0 \Phi^+_{\varepsilon}(\tilde{z}^-_{\varepsilon}) \right) d\tau + O(\varepsilon^2),$ \tag{5.40}

where $\Delta(I_0)$ is the phase shift on the action level set $I_0$ that defines the unperturbed scattering map $\tilde{\sigma}_0$ (see Proposition 4.5).

Proposition 5.7 implies that
\[
\tilde{\sigma}^\theta_e(I, \theta, t) = \tilde{\sigma}^\theta_0(I, \theta, t) + \Delta(I_0) + \varepsilon \tilde{S}^\theta(I, \theta, t) + O(\varepsilon^2),
\]
where $\varepsilon \tilde{S}^\theta$ is given by the first four terms on the right-hand side of (5.40). The expression of $\varepsilon \tilde{S}^\theta$ is more complicated than the one for $\varepsilon \tilde{S}^I$ since $\theta$ is a fast variable.

**Proof** We will begin by computing the difference of the $\theta^s$ evaluated at a homoclinic point and at the footpoint of the stable fiber through the homoclinic point. By Lemma D.2 from “Appendix D”, we have
\[
\theta^s(\tilde{z}^+_{\varepsilon}) - \theta^s(\tilde{z}^-_{\varepsilon}) = - \int_0^{+\infty} \frac{d}{d\tau} \left[ \theta^s(\Phi^+_{\varepsilon}(\tilde{z}^+_{\varepsilon})) - \theta^s(\Phi^+_{\varepsilon}(\tilde{z}^-_{\varepsilon})) \right] d\tau.
\]
Now by Eq. (4.17), we have:

$$\frac{d\theta_s}{d\tau} = \frac{\partial H_0}{\partial I^s} + \varepsilon \chi^1 \theta_s.$$  

We can break the integral up into two parts:

$$A = -\int_0^{+\infty} \left( \frac{\partial H_0}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+)) - \frac{\partial H_0}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) \right) d\tau$$  

and

$$B = -\varepsilon \int_0^{+\infty} \left( \chi^1 \theta^s(\Phi^r_0(\tilde{z}_0^+)) - \chi^1 \theta^s(\Phi^r_0(\tilde{z}_0)) \right) d\tau.$$  

As for the integral $B$, $\varepsilon \chi^1 \theta^s$ is $O(\varepsilon)$. So, by Lemma D.4 from “Appendix D”, we can express the integral in terms of the unperturbed system plus an error term:

$$B = -\varepsilon \int_0^{+\infty} \left( \chi^1 \theta^s(\Phi^r_0(\tilde{z}_0^+)) - \chi^1 \theta^s(\Phi^r_0(\tilde{z}_0)) \right) d\tau + O(\varepsilon^{1+\varepsilon}).$$

Returning to the integral (5.41), we now use the normal form (4.3) of $H_0$ given by Proposition 4.1, yielding

$$\frac{\partial H_0}{\partial I^s} = \frac{\partial h_0}{\partial I^s}(I^s) + (\xi^s) \frac{\partial g_1}{\partial I^s}(I^s) + (\xi^s)^2 \frac{\partial g_2}{\partial I^s}(I^s, \xi^s),$$

where $\xi^s = x^s y^s$.

Thus, the integral $A$ given by (5.41) breaks into three parts

$$A_1 = -\int_0^{+\infty} \left( \frac{\partial h_0}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+)) - \frac{\partial h_0}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) \right) d\tau,$$

$$A_2 = -\int_0^{+\infty} \left( (\xi^s) \frac{\partial g_1}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+)) - (\xi^s) \frac{\partial g_1}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) \right) d\tau,$$

$$A_3 = -\int_0^{+\infty} \left( (\xi^s)^2 \frac{\partial g_2}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+)) - (\xi^s)^2 \frac{\partial g_2}{\partial I^s}(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) \right) d\tau.$$  

From Lemma 5.1 Eq. (5.11), we have $\xi^s(\Phi(\tilde{z}_\varepsilon^+)) = O(\varepsilon^2)$ and $\xi^s(\Phi(\tilde{z}_\varepsilon)) = O(\varepsilon)$ in (5.46). Thus, we can immediately obtain that $A_3$ is $O(\varepsilon^2)$.

We use the integral form of the mean value theorem to rewrite the integral $A_1$. Recall that

$$\frac{\partial F}{\partial x}(b) - \frac{\partial F}{\partial x}(a) = (b - a) \int_0^1 \frac{\partial^2 F}{\partial x^2}(a + t(b - a)) d\tau$$

Applying this result to $A_1$ for $b = I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+))$, $a = I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon))$, and $F = \frac{\partial h_0}{\partial I^s}$, yields

$$A_1 = -\int_0^{+\infty} \left( I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+)) - I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) \right) C_\varepsilon(\tau) d\tau,$$  

where $C_\varepsilon$ stands for the integral

$$C_\varepsilon(\tau) = \int_0^1 \frac{\partial^2 h_0}{\partial (I^s)^2} \left[ I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) + t \left( I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon^+)) - I^s(\Phi^r_\varepsilon(\tilde{z}_\varepsilon)) \right) \right] d\tau.$$
We evaluate the expression $I^s(\Phi^+_{\varepsilon}(\bar{z}_{\varepsilon}^+)) - I^s(\Phi^+_{\varepsilon}(\bar{z}_{\varepsilon}))$ in (5.47) by invoking Corollary 5.5, obtaining

$$-\varepsilon \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \left( (\lambda^1 H_0)(\Phi^+_{\varepsilon}(z_{\varepsilon}^+)) - (\lambda^1 H_0)(\Phi^+_{\varepsilon}(z_{\varepsilon})) \right) d\varsigma + \varepsilon \left( \frac{\partial h_0}{\partial I^2}(I_0) \right)^{-1} g_1(I_0) \int_0^{+\infty} \left( (\lambda^1 \xi^s)(\Phi^+_{\varepsilon}(z_{\varepsilon}^+)) - (\lambda^1 \xi^s)(\Phi^+_{\varepsilon}(z_{\varepsilon})) \right) d\varsigma + O(\varepsilon^2).$$

(5.49)

The next part of the integrand is

$$C_\varepsilon(\tau) = \int_0^{1} \frac{\partial^2 h_0}{\partial I^2}(I_0) \left[ I^s(\Phi^+_{\varepsilon}(\bar{z}_x)) + t \left( I^s(\Phi^+_{\varepsilon}(\bar{z}_x)) - I^s(\Phi^+_{\varepsilon}(\bar{z}_x)) \right) \right] d\tau.$$

Using Gronwall’s inequality—Lemma C.1—we can write

$$C_\varepsilon(\tau) = C_0(\tau) + O(\varepsilon^q),$$

where $0 < q < 1$. However, when $\varepsilon = 0$, $I^s$ along the flow of the footpoint $\bar{z}_{0x}^+$ is equal to $I^s$ along the flow of the homoclinic point $\bar{z}_0$. Since $I^s(\bar{z}_0) = I_0$, we obtain

$$C_0(\tau) = \frac{\partial^2 h_0}{\partial I^2}(I_0),$$

which is a constant.

Putting these expressions together, we can write $A_1$ as

$$\varepsilon \left( \frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} \int_0^{+\infty} \int_0^{+\infty} \left( (\lambda^1 H_0)(\Phi^+_{\varepsilon}(z_{\varepsilon}^+)) - (\lambda^1 H_0)(\Phi^+_{\varepsilon}(z_{\varepsilon})) \right) d\varsigma d\tau - \varepsilon \left( \frac{\partial^2 h_0}{\partial I^2}(I_0) \right) \left( \frac{\partial h_0}{\partial I}(I_0) \right)^{-1} g_1(I_0) \int_0^{+\infty} \int_0^{+\infty} \left( (\lambda^1 \xi^s)(\Phi^+_{\varepsilon}(z_{\varepsilon}^+)) - (\lambda^1 \xi^s)(\Phi^+_{\varepsilon}(z_{\varepsilon})) \right) d\varsigma d\tau + O(\varepsilon^{1+q}).$$

(5.50)

We will now write the double integrals in (5.50) in a simpler form. We show only the details of the computation from the first double integral that appears in (5.50), since the second double integral can be treated in a similar fashion. Denote by $s^s$ the following improper integral

$$s^s(\tau) = -\int_0^{+\infty} \left( (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x}^+) - (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x}) \right) d\nu.$$

(5.51)

Since $(\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x}^+) - (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x})$ approaches 0 exponentially as $\nu \rightarrow +\infty$, the above integral is convergent and moreover

$$\frac{d}{d\tau} (s^s(\tau)) = (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x}^+) - (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x}).$$

That is, $s^s(\tau)$ is the antiderivative of $\tau \mapsto (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x}^+) - (\lambda^1 H_0)(\Phi_0^+)(\bar{z}_{0x})$ satisfying the condition that it equals 0 at $+\infty$.
Making the change of variable \( \nu = \sigma + \tau \) with \( d\nu = d\sigma \) the double integral in (5.50) becomes

\[
\int_0^{+\infty} \int_{\tau}^{+\infty} \left( (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0)) \right) d\nu d\tau = -\int_0^{+\infty} \mathcal{J}_s(\tau) d\tau.
\] (5.52)

Using Integration by Parts, we obtain

\[
-\int_0^{+\infty} \mathcal{J}_s(\tau) d\tau = -\tau \mathcal{J}_s(\tau) \bigg|_0^{+\infty} + \int_0^{+\infty} \left( (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0)) \right) d\tau
\] (5.53)

\[
= \int_0^{+\infty} \left( (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0)) \right) d\tau.
\]

In the above, the quantity \( \tau \mathcal{J}_s(\tau) \) obviously equals 0 at \( \tau = 0 \) and equals 0 when \( \tau \to +\infty \) since, by l’Hospital Rule

\[
\lim_{\tau \to +\infty} \frac{\mathcal{J}_s(\tau)}{\tau^{-1}} = \lim_{\tau \to +\infty} -\frac{(\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0))}{\tau^{-2}} = 0,
\]

since \((\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0))\) approaches 0 at exponential rate as \( \tau \to +\infty \).

A similar computation can be done to write the second double integral that appears in (5.50) as a single integral.

Thus, we obtain the following expression of \( A_1 \):

\[
\varepsilon \left( \frac{\partial^2 h_0}{\partial \tau^2}(I_0) \right) \left( \frac{\partial h_0}{\partial \tau}(I_0) \right)^{-1} \int_0^{+\infty} \left( (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0)) \right) \tau d\tau
\]

\[
- \left( (\chi^1 H_0)(\hat{\Phi}_0^\nu(\tilde{z}_0)) \right) \tau d\tau
\]

\[
- \varepsilon \left( \frac{\partial^2 h_0}{\partial \tau^2}(I_0) \right) \left( \frac{\partial h_0}{\partial \tau}(I_0) \right)^{-1} g_1(I_0) \int_0^{+\infty} \left( (\chi^1 \xi^s)(\hat{\Phi}_0^\nu(\tilde{z}_0^+)) - (\chi^1 \xi^s)(\hat{\Phi}_0^\nu(\tilde{z}_0)) \right) \tau d\tau
\]

\[
+ O(\varepsilon^{1+\varepsilon}).
\] (5.54)

Finally, we turn to the integral \( A_2 \) given by (5.45). Using Lemma 5.1 Eqs. (5.11) and (5.13), (5.34), as well as integration by parts similarly to above, we express \( A_2 \) as

\[
-\int_0^{+\infty} \left( (\xi^s \frac{\partial g_1}{\partial \tau})(\hat{\Phi}_e^\nu(\tilde{z}_e^+)) - (\xi^s \frac{\partial g_1}{\partial \tau})(\hat{\Phi}_e^\nu(\tilde{z}_e)) \right) d\tau
\]

\[
= -\left( \frac{\partial g_1}{\partial \tau}(I_0) \right) \int_0^{+\infty} \left( (\xi^s((\hat{\Phi}_e^\nu(\tilde{z}_e^+))) - (\xi^s(\hat{\Phi}_e^\nu(\tilde{z}_e)))) \right) d\tau
\]

\[
+ O(\varepsilon^{1+\varepsilon}).
\]
In the above, we have also replaced the error term argument as in the proof of Proposition 5.2.

Subtracting (5.56) from (5.57) yields

\[
\theta^s(\bar{z}_e^+) - \theta^u(\bar{z}_e^-) = \theta^s(\bar{z}_e) - \theta^u(\bar{z}_e),
\]

plus an epsilon order term consisting of the sum of six integrals, plus an error term of order \(O(\epsilon^2)\). Using
notation (5.38), four of these integrals are
\[-\varepsilon \left( \frac{\partial^2 h_0}{\partial I^2} (I_0) \right) \left( \frac{\partial h_0}{\partial I} (I_0) \right)^{-1} g_1(I_0) J^+,\]
\[-\varepsilon \left( \frac{\partial^2 h_0}{\partial I^2} (I_0) \right) \left( \frac{\partial h_0}{\partial I} (I_0) \right)^{-1} g_1(I_0) J^-,\]
\[\varepsilon \left( \frac{\partial g_1}{\partial I} (I_0) \right) J^+,\]
\[\varepsilon \left( \frac{\partial g_1}{\partial I} (I_0) \right) J^-.
\]

By Corollary 5.6, since $J^+ + J^- = 0$, the sum of the first two expressions in (5.58) equals 0, and the sum of the last two expressions in (5.58) equals 0.

Also, we recall from Sect. 5.1.2 that for a given unperturbed homoclinic point $\tilde{z}_0$ we selected a perturbed homoclinic point $\tilde{z}_\varepsilon$ satisfying condition (5.4), that is $\theta^s(\tilde{z}_\varepsilon) - \theta^u(\tilde{z}_\varepsilon) = \theta^s(\tilde{z}_0) - \theta^u(\tilde{z}_0) = \Delta(I(\tilde{z}_0))$.

Combining these results, we obtain (5.40).

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Appendix A: Normally hyperbolic invariant manifolds

We briefly recall the notion of a normally hyperbolic invariant manifold.

Definition A.1 Let $M$ be a $C^r$-smooth manifold, $\Phi^t$ a $C^r$-flow on $M$. A submanifold (with or without boundary) $\Lambda$ of $M$ is a normally hyperbolic invariant manifold (NHIM) for $\Phi^t$ if it is invariant under $\Phi^t$, and there exists a splitting of the tangent bundle of $TM$ into sub-bundles $Tz M = E^u_z \oplus E^s_z \oplus T_z \Lambda$, $\forall z \in \Lambda$ (A.1)

that are invariant under $D\Phi^t$ for all $t \in \mathbb{R}$, and there exist rates

$\lambda_- \leq \lambda_+ < \lambda_c < 0 < \mu_- < \mu_+ \leq \mu_c$

and a constant $C > 0$, such that for all $x \in \Lambda$ we have

$Ce^{t\lambda_-} \| v \| \leq \| D\Phi^t(z)(v) \| \leq Ce^{t\lambda_+} \| v \|$ for all $t \geq 0$, if and only if $v \in E^s_z$,

$Ce^{t\mu_+} \| v \| \leq \| D\Phi^t(z)(v) \| \leq Ce^{t\mu_-} \| v \|$ for all $t \leq 0$, if and only if $v \in E^u_z$, (A.2)

$Ce^{t|\lambda_c|} \| v \| \leq \| D\Phi^t(z)(v) \| \leq Ce^{t|\mu_c|} \| v \|$ for all $t \in \mathbb{R}$, if and only if $v \in T_z \Lambda$.

In the case when $\Phi^t$ is a Hamiltonian flow, the rates can be chosen so that

$\lambda_- = -\mu_+,
\lambda_+ = -\mu_-,
\lambda_c = -\mu_c.$

The regularity of the manifold $\Lambda$ depends on the rates $\lambda_-, \lambda_+, \mu-, \mu_+, \lambda_c,$ and $\mu_c$. More precisely, $\Lambda$ is $C^\ell$-differentiable, with $\ell \leq r - 1$, provided that

$\ell \mu_c + \lambda_+ < 0,$
$\ell \lambda_c + \mu_- > 0.$ (A.3)
The manifold \( \Lambda \) has associated unstable and stable manifolds of \( \Lambda \), denoted \( W^u(\Lambda) \) and \( W^s(\Lambda) \), respectively, which are \( C^\ell -1 \)-differentiable. They are foliated by 1-dimensional unstable and stable manifolds (fibers) of points, \( W^u(z) \), \( W^s(z) \), \( z \in \Lambda \), respectively, which are as smooth as the flow.

These manifolds are defined by:

\[
W^s(\Lambda) = \{ y | d(\Phi^t(y), \Lambda) \rightarrow 0 \text{ as } t \rightarrow +\infty \} \\
= \{ y | d(\Phi^t(y), \Lambda) \leq C_\ell e^{t\lambda^+}, t \geq 0 \}, \\
W^u(\Lambda) = \{ y | d(\Phi^t(y), \Lambda) \rightarrow 0 \text{ as } t \rightarrow -\infty \} \\
= \{ y | d(\Phi^t(y), \Lambda) \leq C_\ell e^{t\mu^-}, t \geq 0 \}, \\
W^s(x) = \{ y | d(\Phi^t(y), \Phi^t(x)) < C_\ell e^{t\lambda^+}, t \geq 0 \}, \\
W^u(x) = \{ y | d(\Phi^t(y), \Phi^t(x)) < C_\ell e^{t\mu^-}, t \leq 0 \}.
\]

The fibers \( W^u(x) \), \( W^s(x) \) are not invariant by the flow, but \textit{equivariant} in the sense that

\[
\Phi^t(W^u(z)) = W^u(\Phi^t(z)), \\
\Phi^t(W^s(z)) = W^s(\Phi^t(z)).
\]

Since \( W^{s,u}(\Lambda) = \bigcup_{z \in \Lambda} W^{s,u}(z) \), we can define the projections along the fibers

\[
\Omega^+: W^s(\Lambda) \rightarrow \Lambda, \quad \Omega^+(z) = z^+ \text{ iff } z \in W^s(z^+), \\
\Omega^-: W^u(\Lambda) \rightarrow \Lambda, \quad \Omega^-(z) = z^- \text{ iff } z \in W^u(z^-).
\]

The point \( z^+ \in \Lambda \) is characterized by:

\[
d(\Phi^t(z), \Phi^t(z^+)) \leq C_\ell e^{t\lambda^+}, \quad \text{for all } t \geq 0.
\]

and the point \( z^- \in \Lambda \) by

\[
d(\Phi^t(z), \Phi^t(z^-)) \leq C_\ell e^{t\mu^-}, \quad \text{for all } t \leq 0,
\]

for some \( C_\ell > 0 \).

For our applications, the most important result about NHIMs is that they persist when we perturb the flow. This is the fundamental result of Fenichel (1971), Hirsch et al. (1977) and Pesin (2004).

The standard assumption for persistence is that the unperturbed NHIM is a compact manifold without a boundary. The persistence of the NHIM also holds when the compactness assumption is replaced with the assumption that the perturbation has uniformly bounded derivatives in all variables (Hirsch et al. 1977, Section 6). There are also proofs of persistence in the infinite-dimensional case which do not require compactness, such as in Bates et al. (1999) and Bates et al. (2008).

We remark that the particular case when the perturbation is periodic or quasi-periodic in time can be reduced to the compact case. More precisely, we can rewrite system (2.2) as

\[
\frac{d}{dt}z = \lambda^0(z) + \varepsilon \lambda^1(z, \theta), \\
\frac{d}{dt}\theta = \omega,
\]

where \( \theta \) ranges over a torus \( \mathbb{T}^d \), and \( \omega \in \mathbb{R}^d \) is a rationally independent vector when \( d > 1 \). If the flow of \( \frac{d}{dt}z = \lambda^0(z) \) admits a compact NHIM \( \Lambda_0 \), the extended system for \( \varepsilon = 0 \) admits
a compact NHIM $\Lambda_0 \times \mathbb{T}^d$ which persists for small enough $\varepsilon$, using the standard theory. The torus $\mathbb{T}^d$ is sometimes called 'the clock manifold'.

In the case when the manifold has a boundary, the persistence result requires a step of extending the flow. This makes that the persistent manifold is not invariant but only locally invariant and not unique (it depends on the extension).

When we are given a family of flows, it is possible to choose the extensions depending smoothly on parameters and obtain that the manifolds depend smoothly on parameters.

The precise meaning of the smooth dependence is that the we can find parametrizations $k_\varepsilon : \Lambda_0 \to \Lambda_\varepsilon$. The parametrizations $k_\varepsilon$ can be chosen so that

$$\frac{d}{d\varepsilon} k_\varepsilon(z) \in E^u_z \oplus E^s_z,$$

where the splitting $E^u_z \oplus E^s_z$ corresponds to the invariant manifold of the perturbed system. In this case, we obtain $\Lambda_\varepsilon$ as a graph over the central variables on $\Lambda_0$; see Delshams et al. (2008).

The maps $k_\varepsilon(x)$ are jointly $C^r$ as functions of $x, \varepsilon$. The proof of this well-known result is not very difficult. It suffices to consider an extended flow $\hat{\Phi}_t(x, \varepsilon) = (\Phi^t_0(x), \varepsilon)$, which is a small perturbation of $\Phi^t_0(x, \varepsilon) = (\Phi^t_0(x), \varepsilon)$. The regularity of the NHIM of $\hat{\Phi}_t$ gives the claimed regularity of the NHIM of $\Phi_t$ with respect to parameters.

From the same proof (using the invariant objects of the extended flow), it easily follows the regularity with respect to parameters of the stable and unstable bundles and the stable and unstable manifolds.

**Appendix B: Scattering map**

Assume that $W^u(\Lambda), W^s(\Lambda)$ have a transverse intersection along a manifold $\Gamma$ satisfying:

$$T_z \Gamma = T_z W^s(\Lambda) \cap T_z W^u(\Lambda), \text{ for all } z \in \Gamma,$$

$$T_z M = T_z \Gamma \oplus T_z W^u(z^-) \oplus T_z W^s(z^+), \text{ for all } z \in \Gamma. \tag{B.1}$$

Under these conditions, the projection mappings $\Omega^\pm$ restricted to $\Gamma$ are local diffeomorphisms. We can restrict $\Gamma$ if necessary so that $\Omega^\pm$ are diffeomorphisms from $\Gamma$ onto open subsets $U^\pm$ in $\Lambda$.

**Definition B.1** A homoclinic channel is a homoclinic manifold $\Gamma$ satisfying the strong transversality condition B.1, and such that

$$\Omega^\pm_{|\Gamma} : \Gamma \to U^\pm := \Omega^\pm(\Gamma)$$

are $C^{\ell-1}$-diffeomorphisms.

**Definition B.2** Given a homoclinic channel $\Gamma$, the scattering map associated with $\Gamma$ is defined as

$$\sigma := \sigma^\Gamma : U^- \subseteq \Lambda \to U^+ \subseteq \Lambda,$$

$$\sigma = \Omega^+ \circ (\Omega^-)^{-1}.$$

Equivalently, $\sigma(z^-) = z^+$, provided that $W^u(z^-)$ intersects $W^s(z^+)$ at a unique point $z \in \Gamma$. 

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The meaning of the scattering map is that, given a homoclinic excursion, it has two orbits in the manifold is asymptotic to. It is asymptotic to an orbit in the past and to another orbit in the future. The scattering map considers the future asymptotic orbit as a function of the asymptotic in the past. When we consider all the homoclinic orbits in a homoclinic channel, we obtain a scattering map from an open domain. The intuition of the scattering map is that if we observe the orbit for long times, we just measure the effect of the homoclinic excursion on the asymptotic behavior. The scattering map is a very economical way of studying these excursions since it is a map only on the NHIM. Furthermore, as we will see now, it satisfies remarkable geometric properties.

Due to (A.5), the scattering map satisfies the following property

$$\Phi^T \sigma^T = \sigma \Phi^T$$

(B.2)

for any $T \in \mathbb{R}$.

If $M$ is a symplectic manifold, $\Phi^T$ is a Hamiltonian flow on $M$, and $\Lambda \subseteq M$ is symplectic, then the scattering map is symplectic. If the flow is exact Hamiltonian, the scattering map is exact symplectic. For details, see Delshams et al. (2008).

In a similar fashion, we can define heteroclinic channels and associated scattering maps. Given two NHIM’s $\Lambda^1$ and $\Lambda^2$, we can define the projection mappings $\Omega^{\pm,i} : W^{s,u}(\Lambda^i) \rightarrow \Lambda^i$ for $i = 1, 2$. Assume that $W^u(\Lambda^1)$ intersects transversally $W^s(\Lambda^2)$ along a heteroclinic manifold $\Gamma$ so that:

$$T_z \Gamma = T_z W^u(\Lambda^1) \cap T_z W^s(\Lambda^2),$$

for all $z \in \Gamma$,

$$T_z M = T_z \Gamma \oplus T_z W^u(z^-) \oplus T_z W^s(z^+),$$

for all $z \in \Gamma$,

(B.3)

where $z^- = \Omega^{-,1}(z) \in \Lambda^1$ and $z^+ = \Omega^{+,2}(z) \in \Lambda^2$.

We can restrict $\Gamma$ so that $\Omega^{-,1} : \Gamma \rightarrow \Lambda^1$ and $\Omega^{+,2} : \Gamma \rightarrow \Lambda^2$ are diffeomorphisms onto their corresponding images.

**Definition B.3** A heteroclinic channel is a heteroclinic manifold $\Gamma$ satisfying the strong transversality condition B.3, and such that

$$\Omega^{-,1}_\Gamma : \Gamma \rightarrow U^- := \Omega^{-,1}(\Gamma) \subseteq \Lambda^1,$$

$$\Omega^{+,2}_\Gamma : \Gamma \rightarrow U^+ := \Omega^{+,2}(\Gamma) \subseteq \Lambda^2,$$

are $C^{l-1}$-diffeomorphisms.

**Definition B.4** Given a heteroclinic channel $\Gamma$, the scattering map associated with $\Gamma$ is defined as

$$\sigma := \sigma^\Gamma : U^- \subseteq \Lambda^1 \rightarrow U^+ \subseteq \Lambda^2,$$

$$\sigma = \Omega^{+,2} \circ (\Omega^{-,1})^{-1}.$$
Appendix C: Gronwall’s inequality

In this section, we apply Gronwall’s Inequality to estimate the error between the solution of an unperturbed system and the solution of the perturbed system, over a time of logarithmic order with respect to the size of the perturbation.

Lemma C.1 Consider the following differential equations:
\[
\begin{align*}
\frac{d}{dt}z(t) &= \mathcal{X}^0(z, t) \quad \text{(C.1)} \\
\frac{d}{dt}z(t) &= \mathcal{X}^0(z, t) + \varepsilon \mathcal{X}^1(z, t, \varepsilon) \quad \text{(C.2)}
\end{align*}
\]

Assume that $\mathcal{X}^0$, $\mathcal{X}^1$ are uniformly Lipschitz continuous in the variable $z$, $C_0$ is the Lipschitz constant of $\mathcal{X}^0$, and $\mathcal{X}^1$ is bounded with $\|\mathcal{X}^1\| \leq C_1$, for some $C_0, C_1 > 0$. Let $z_0$ be a solution of Eq. (C.1) and $z_\varepsilon$ be a solution of Eq. (C.2) such that
\[
\|z_0(t_0) - z_\varepsilon(t_0)\| < c\varepsilon. \quad \text{(C.3)}
\]

Then, for $0 < \varrho_0 < 1$, $k \leq 1 - \varrho_0 C_0$, and $K = c + \frac{C_1}{C_0}$, we have
\[
\|z_0(t) - z_\varepsilon(t)\| < K\varepsilon^\varrho_0, \text{ for } 0 \leq t - t_0 \leq k \ln(1/\varepsilon). \quad \text{(C.4)}
\]

For a proof, see Gidea et al. (2021).

Appendix D: Master lemmas

In this section, we recall some abstract Melnikov-type integral operators and some of their properties from Gidea et al. (2021).

Consider a system as in (2.2) and the extended systems as in (2.6).

Assume that, for some $\varepsilon_1 > 0$, and for each $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$, there exists a normally hyperbolic invariant manifold $\tilde{\Lambda}_\varepsilon$ for $\tilde{\Phi}_\varepsilon$, as well as a homoclinic channel $\tilde{\Gamma}_\varepsilon$, which depend $C^\ell$-smoothly on $\varepsilon$. Associated with $\tilde{\Gamma}_\varepsilon$, we have projections $\Omega^\pm : \tilde{\Gamma}_\varepsilon \rightarrow \Omega^\pm(\tilde{\Gamma}_0) \subseteq \tilde{\Lambda}_\varepsilon$, which are local diffeomorphisms. We are thinking of $\tilde{\Phi}_\varepsilon$, $\tilde{\Lambda}_\varepsilon$, $\tilde{\Gamma}_\varepsilon$ as perturbations of $\tilde{\Phi}_0$, $\tilde{\Lambda}_0$, $\tilde{\Gamma}_0$.

For $\tilde{z}_0 \in \tilde{\Gamma}_0$ let $\tilde{z}_\varepsilon \in \tilde{\Gamma}_\varepsilon$ be the corresponding homoclinic point satisfying (5.4). Because of the smooth dependence of the normally hyperbolic manifold and of its stable and unstable manifolds on the perturbation, $\tilde{z}_\varepsilon$ is $O(\varepsilon)$-close to $\tilde{z}_0$ in the $C^\ell$-topology, that is
\[
\tilde{z}_\varepsilon = \tilde{z}_0 + O(\varepsilon). \quad \text{(D.1)}
\]

Let $(\tilde{z}_\varepsilon, \varepsilon) \in \tilde{M} \mapsto F(\tilde{z}_\varepsilon, \varepsilon) \in \mathbb{R}^k$ be a uniformly $C^1$-smooth mapping on $\tilde{M} \times \mathbb{R}$.

We define the integral operators
\[
\begin{align*}
\mathcal{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon, \tilde{z}_\varepsilon) &= \int_0^{+\infty} \left( F(\tilde{\Phi}_\varepsilon^+(\tilde{z}_\varepsilon^+)) - F(\tilde{\Phi}_\varepsilon^+(\tilde{z}_\varepsilon)) \right) \, d\tau, \\
\mathcal{J}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon, \tilde{z}_\varepsilon) &= \int_{-\infty}^{0} \left( F(\tilde{\Phi}_\varepsilon^-(\tilde{z}_\varepsilon^-)) - F(\tilde{\Phi}_\varepsilon^-(\tilde{z}_\varepsilon)) \right) \, d\tau. \quad \text{(D.2)}
\end{align*}
\]

Lemma D.1 (Master Lemma 1) The improper integrals (D.2) are convergent. The operators $\mathcal{J}^+(\mathbf{F}, \tilde{\Phi}_\varepsilon, \tilde{z}_\varepsilon)$ and $\mathcal{J}^-(\mathbf{F}, \tilde{\Phi}_\varepsilon, \tilde{z}_\varepsilon)$ are linear in $\mathbf{F}$.
Lemma D.2 (Master Lemma 2)
\[
F(z_e^+) - F(z_e^-) = -\mathcal{J}^+((\lambda^0 + \epsilon \lambda^1)F, \Phi_0^+, z_e^-), \\
F(z_e^-) - F(z_e^+) = \mathcal{J}^-((\lambda^0 + \epsilon \lambda^1)F, \Phi_0^-, z_e^-).
\]  
(D.3)

Lemma D.3 (Master Lemma 3)
\[
\mathcal{J}^+(F, \Phi_e^+, z_e^-) = \mathcal{J}^+(F, \Phi_0^+, z_0) + O(\epsilon^0), \\
\mathcal{J}^-(F, \Phi_e^-, z_e^-) = \mathcal{J}^-(F, \Phi_0^-, z_0) + O(\epsilon^0),
\]  
(D.4)

for 0 < \varrho < 1. The integrals on the right-hand side are evaluated with \( \lambda^1 = \lambda^1(\cdot; 0) \).

Lemma D.4 (Master Lemma 4) If \( F = O_{C^1}(\epsilon) \), then
\[
\mathcal{J}^+(F, \Phi_e^+, z_e^-) = \mathcal{J}^+(F, \Phi_0^+, z_0) + O(\epsilon^{1+\varrho}), \\
\mathcal{J}^-(F, \Phi_e^-, z_e^-) = \mathcal{J}^-(F, \Phi_0^-, z_0) + O(\epsilon^{1+\varrho}),
\]  
(D.5)

for 0 < \varrho < 1. The integrals on the right-hand side are evaluated with \( \lambda^1 = \lambda^1(\cdot; 0) \).

The proofs of the above lemmas can be found in Gidea et al. (2021), and similar arguments can be found in Gidea and de la Llave (2018).

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