Resummed corrections to the $\rho$ parameter due to a finite width of the top quark

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Abstract

We perform an all-order calculation of the $\rho$ parameter in a simplified framework, where the top propagator can be calculated exactly. Special emphasis is placed on the question of gauge invariance and the treatment of non-perturbative cut-off effects.

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1 Introduction

In the electroweak sector of the Standard Model (SM) every particle acquires its mass through an interaction with a scalar potential in a non-trivial vacuum. As a consequence, all the masses are proportional to a common scale, namely $G_F^{-1/2}$, which is fixed by low-energy measurements such as the $\mu$ decay rate. In this situation, the decoupling theorem \[1\] does not hold and thus there exist low-energy observables in which the quantum effects induced by virtual heavy particles do not vanish when the mass of these particles goes to infinity.

Most prominent among the non-decoupling effects is the $\rho$ parameter \[2\], which provides a measure of the relative strength of neutral and charged current interactions in four fermion processes at zero momentum transfer. At tree level $\rho = 1$ due to a global accidental $SU(2)$ symmetry, the so called custodial symmetry. $\rho$ can receive radiative corrections only by those sectors of the SM which break explicitly the custodial symmetry, namely the hypercharge and the Yukawa couplings which give a different mass to the components of fermion doublets. In this latter case, the contribution to the $\rho$ parameter is proportional to the mass splitting, therefore the leading contribution comes from the top-bottom doublet.

At one loop, the $\rho$ parameter has a quadratic dependence on the top quark mass, $\Delta \rho^{(1)} \approx G_F m_t^2$, and a logarithmic dependence on the Higgs mass, $\Delta \rho^{(1)} \approx g'^2 \log \left( \frac{m_H}{M_W} \right)$. Two loop corrections at the leading order, i.e. $\Delta \rho^{(2)} \approx G_F^2 m_t^4$, and at the next-to-leading order, i.e. $\Delta \rho^{(2)} \approx G_F^2 m_t^2 M_Z^2$, in the top quark mass were computed in the limits $m_H \to 0$ and $m_H \gg m_t$ in Refs.\[3, 4\] and for arbitrary Higgs mass in Ref.\[5\]. It turned out that due to accidental cancellations, the subleading corrections at two loop are larger than the leading ones \[6\]. At three loops, the computation of the leading top quark corrections, $\Delta \rho^{(3)} \approx G_F^3 m_t^6$, in the massless Higgs limit, was carried out in Ref.\[7\]. The complete dependence on the Higgs mass at three loops was obtained in Ref.\[8\]. Numerically it was found that this contribution to $\Delta \rho$ is quite large and provides a sizeable correction ($\approx 36\%$) to the leading electroweak correction at two loops. However, the size of the three loop correction is only about $2\%$ of the much larger two-loop subleading electroweak correction. Moreover, the perturbative series of the leading top quark contributions to the $\rho$ parameter has alternating signs up to three loops.

This raises the issue of the convergence of the perturbative expansion (it might be that this series is divergent, but Borel summable) and calls for a better understanding of higher order radiative corrections. It would be highly desirable to have a simplified framework in which the leading top quark contributions to the $\rho$ parameter can be computed to all orders in perturbation theory and eventually summed up. The actual calculation of the leading radiative corrections in the top quark mass is greatly simplified by the observation that to obtain them it is enough to consider the lagrangian of the SM in the limit of vanishing gauge coupling constants $g, g' \to 0$ \[4\]. This gaugeless limit provides an efficient way of reducing the number of Feynman diagrams to be computed and it has been used
in the two and three loop computations mentioned above.

The $SU(N_F) \times U(1)$ electroweak model in the large $N_F$-limit\[^9, 10\] provides an ideal framework in order to compute the leading top quark contributions to the $\rho$ parameter to all orders in perturbation theory. In fact, since only one-loop graphs contribute to the top quark self-energy at the leading order in the large $N_F$-limit, the exact top quark propagator can be obtained simply by resumming one-loop self-energy insertions. In this way, one takes into account the finite width effects due to the fact that the top quark is an unstable particle. The resulting Dyson propagator contains, in addition to the complex pole corresponding to the top quark, a tachyon pole in the euclidean region, $p^2 = -\Lambda_T^2$, which spoils causality and makes all the Wick-rotated Feynman integrals ill-defined.

Since the tachyon pole is a non-perturbative effect one can still compute the $\rho$ parameter by using the resummed top propagator instead of the Born one, provided an expansion in powers of the top Yukawa coupling is taken before performing the Wick rotation. All the coefficients of this perturbative expansion can be computed analytically. It turns out that they are all positive and grow factorially with the number of loops, thus their power series is divergent and not Borel summable.

In order to go beyond the perturbative approximation, one has to regularize the integrals containing the resummed top quark propagator. In this connection, the introduction of an UV cutoff at $\Lambda < \Lambda_T$ has been proposed in Ref.\[^11\]. However, this procedure breaks gauge invariance. We have adopted a different strategy devised in Ref.\[^12\]. Assuming that the occurrence of the tachyon pole is not due to the inconsistency of the theory under consideration, but of the intermediary expansion technique used, it is reasonable to circumvent the ill-defined part by an adequate subtraction of the tachyonic pole. In particular, we have chosen to subtract the tachyon minimally at its pole from the exact top propagator. One should be careful in doing this because the tachyon pole contributes to the Källén-Lehmann spectral function. The correct normalization of the latter, which is crucial in order to guarantee renormalizability, is recovered after the tachyonic subtraction by a suitable rescaling of the top propagator.

This procedure allows us to find a tachyon-free representation of the exact leading top contribution to the $\rho$ parameter which can be estimated numerically and compared with the expansion of $\Delta \rho$ at any fixed order in perturbation theory.

The paper is organized as follows. In Sect. 2 the $SU(N_F) \times U(1)$ model in the large $N_F$-limit is presented and its spectrum is briefly discussed. The top quark self-energy at the leading order in $N_F$ is computed in Sect. 3. The subtraction term for the removal of the tachyonic pole is discussed in Sect. 3.1. In Sect. 4 we check the validity of Ward identities connecting the self-energies of gauge bosons and of unphysical scalar particles computed with the resummed top propagator. The leading top quark contribution to the $\rho$ parameter is computed to all orders in perturbation theory in Sect. 5. In Sect. 6 the tachyon-free representation of the resummed top propagator is used in order to compute nonperturbatively the leading top contribution to the $\rho$ parameter. Finally the conclusions
are given in Sect. 7. In the Appendices we have collected technical details concerning the computation of the radiative corrections to the $\rho$ parameter.

## 2 The model

In this Section we shall consider a $SU(N_F) \times U(1)$ gauge theory in the large $N_F$-limit. For the sake of simplicity only, one generation of fermions will be taken into account. The non-abelian gauge fields $W^a_\mu$ transform according to the adjoint representation of $SU(N_F)$, thus $a = 1, 2, \ldots, (N_F^2 - 1)$. The abelian gauge field is denoted by $B_\mu$. Left handed fermions transform according to the fundamental representation of $SU(N_F)$, therefore they are sorted in $N_F$-plets,

\[ L^L = \omega_- L = \left( \begin{array}{c} \nu^L(x) \\ \ell^L_j(x) \end{array} \right), \quad Q^L = \omega_- Q = \left( \begin{array}{c} t^L(x) \\ b^L_j(x) \end{array} \right), \quad j = 1, 2, \ldots N_F - 1, \]

while right handed fermions are singlets as in the SM

\[ \ell^R_j = \omega_+ \ell_j, \quad t^R = \omega_+ t, \quad b^R_j = \omega_+ b_j, \quad j = 1, 2, \ldots N_F - 1, \]

where $\omega_\pm = \frac{1 \pm \gamma_5}{2}$ are the chiral projectors. All fermions are taken to be massless except for the top quark. Finally, the scalar sector of the model is a gauged $O(2N_F)$ linear sigma model in the broken phase. The scalar fields are sorted in a complex $N_F$-plet

\[ \Phi = \left( \begin{array}{c} \frac{1}{\sqrt{2}} (v + H(x) + i\chi(x)) \end{array} \right), \quad j = 1, 2, \ldots N_F - 1, \]

which transforms according to the fundamental representation of $SU(N_F)$. The classical lagrangian of the model is given by

\[
\mathcal{L} = -\frac{1}{4} \left( \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu \right)^2 - \frac{1}{4} \left( \partial_\mu B_\nu - \partial_\nu B_\mu \right)^2 \\
+ (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{\lambda}{4} \left( \Phi^\dagger \Phi - \frac{v^2}{2} \right)^2 \\
+ \bar{L} i \gamma^\mu L + \bar{Q} i \gamma^\mu Q + \bar{l}^R_j i \gamma^\mu l^R_j + \bar{t}^R i \gamma^\mu t^R + \bar{b}^R_j i \gamma^\mu b^R_j \\
+ \frac{1}{\sqrt{2}} y_t (v + H) \bar{t} t - \frac{i}{\sqrt{2}} y_t \chi \bar{t} \gamma_5 t - y_t \bar{\phi}_j^L t^L_j - y_t \phi_j^L \bar{t}^L_j, \]

where $f^{abc}$ are the structure constants of $SU(N_F)$, $g$ and $g'$ are the $SU(N_F)$ and $U(1)$ gauge coupling constant respectively, $y_t$ is the Yukawa coupling constant of the top quark, $\lambda$ and $v$ are the quartic coupling constant and the v.e.v. of the Higgs $N_F$-plet respectively, while $D_\mu$ is the covariant derivative for matter fields.

The spectrum of the theory has been discussed in Ref. [9, 10]. For our purposes it is enough to mention here that there are $N_F - 1$ copies of the $W$ boson and of the charged unphysical scalar field, $\phi$. In the quark $N_F$-plet the upper component, which we can identify with the top quark, is massive, while the remaining $N_F - 1$ components are
massless. These latter are essentially copies of the bottom quark. The massless lepton $N_F$-plet does not play a role in the following.

In order to perform the large $N_F$ limit we make the following assumptions:

$$g^2N_F, g'^2N_F, y^2_t N_F, \lambda N_F, \frac{v^2}{N_F}$$

will be held fixed while taking $N_F$ to infinity. This amounts to say that the mass of every particle in the SM is of order $O(1)$ in the large $N_F$-limit. For the sake of simplicity in what follows we shall consider a $SU(1 + N_F) \times U(1)$ gauge theory. In this way the SM corresponds to the case $N_F = 1$.

### 3 Top quark self-energy

In this Section we shall discuss the one-loop top quark self-energy in the $SU(1 + N_F) \times U(1)$ model and its renormalization in the on shell scheme. Moreover, the subtraction of the tachyonic pole from the renormalized top propagator will be presented.

Due to the presence of $N_F$ copies of the $W$ boson, the unphysical charged scalar $\phi$ and the $b$ quark in the model under consideration, two graphs contributing to the top quark self-energy, $\Sigma_t(p)$, are enhanced by a factor $N_F$. These graphs are depicted in Fig.1. The remaining one-loop self-energy graphs are of order $O(1)$. Moreover it is straightforward to prove that higher order 1-PI graphs give subleading contribution to the top quark self-energy in the large $N_F$-limit. This means that the computation of the two graphs in Fig.1 is enough to get the exact top quark self-energy at the leading order in the large $N_F$-limit.

By direct inspection it turns out that graph (1) is proportional to the squared mass of the top quark, while graph (2) is proportional to the squared mass of the $W$ gauge boson. We are interested in the case in which the top quark is the heaviest (and only) mass scale, thus we neglect the contribution of graph (2). The dimensionally regularized contribution

![Figure 1: Enhanced one-loop self-energy graphs for the top quark.](image)
of graph (1) to $\Sigma_t(p)$ is given by

$$\Sigma_t(p) = \sqrt{2} \frac{G_F}{16\pi^2} m_{t,0}^2 N_F \left[ \frac{2}{D-4} + \log \left( -\frac{p^2}{\Lambda_B^2} - i\epsilon \right) \right] \frac{\hat{p} \omega_+}{\omega},$$

where $m_{t,0} = \frac{1}{\sqrt{2}} y_t v$ is the bare top quark mass, while $\Lambda_B$ is a regulator dependent quantity with the dimension of a mass whose explicit expression is not needed in the subsequent analysis.

The natural way to incorporate the finite width effects of an unstable particle is the resummation of the corresponding self-energy insertions. This leads us to consider the Dyson resummed top quark propagator instead of the Born one. We report here only the component of the resummed propagator with positive chirality because it is the only one which gives a non vanishing contribution to the $\rho$ parameter.

$$D_t(p) = \frac{i \hat{p} a_0(p^2) \omega_+}{a_0(p^2) p^2 - m_{t,0}^2 + i\epsilon},$$

where

$$a_0(p^2) = 1 - \sqrt{2} \frac{G_F}{16\pi^2} m_{t,0}^2 N_F \left[ \frac{2}{D-4} + \log \left( -\frac{p^2}{\Lambda_B^2} - i\epsilon \right) \right].$$

It should be noted that the above expression is valid to all orders in the interaction strength and provides the exact bare top quark propagator at the leading order in the large $N_F$-limit.

In order to renormalize the top quark self-energy, we require that the real part of the denominator of the Dyson propagator in Eq.(7) vanishes when computed at $p^2 = m_t^2$. This allows us to express the bare top quark mass in terms of the subtraction point $m_t$

$$m_{t,0}^2 = \text{Re}[a_0(m_t^2)] m_t^2 = m_t^2 - \alpha_t \left[ \frac{2}{D-4} + \log \left( \frac{m_t^2}{\Lambda_B^2} \right) \right] m_t^2,$$

where we have introduced a shorthand notation: $\alpha_t = \frac{\sqrt{2}}{16\pi^2} G_F m_t^2 N_F$.

By substituting the above equation into Eq.(7), we obtain the on shell renormalized top quark propagator at the leading order in the large-$N_F$ limit

$$\tilde{D}_t(p) = \frac{i \hat{p} a(p^2) \omega_+}{a(p^2) p^2 - m_t^2 + i\epsilon},$$

where $a(p^2) = 1 - \alpha_t \log \left( -\frac{p^2}{m_t^2} - i\epsilon \right)$.

Notice that in order to achieve the above result a suitable wave function renormalization for the left- and right-handed components of the top quark field must be imposed. On the top quark mass shell, $p^2 \simeq m_t^2$, we have

$$\tilde{D}_t(p) = \frac{i \hat{p} (1 + i\pi \alpha_t) \omega_+}{p^2 - m_t^2 + i m_t \Gamma_t},$$

where $\Gamma_t = \pi \alpha_t m_t$ is the total decay width of the top quark. We remark that this is the exact width of the top quark at the leading order in the large $N_F$-limit in the narrow width approximation, i.e. for $\Gamma_t \ll m_t$. 

6
3.1 Tachyonic regularization

Besides the complex pole corresponding to the unstable top quark, the propagator in Eq. (9) contains a tachyon pole. Its euclidean position, $p^2 = -\Lambda_T^2$, can be obtained by solving numerically the following equation

$$1 + \frac{1}{\Lambda_T^2} = \alpha_t \log (\lambda_T^2), \quad \text{where} \quad \lambda_T^2 = \frac{\Lambda_T^2}{m_t^2}.$$  

(11)

The tachyon pole induces causality violation effects in the theory and makes all the Wick-rotated Feynman integrals ill-defined. Thus in order to obtain sensible results, it must be removed from the integrals involving the top propagator.

One simple and consistent way to deal with the tachyon is to regard the model under consideration as a low-energy effective theory and to introduce explicitly a cutoff under the tachyon scale, $\Lambda < \Lambda_T$. However, this procedure breaks gauge invariance and thus we prefer to use a different approach. Assuming that the tachyon is a mere artifact of perturbation theory, we modify the top propagator in Eq. (9) by subtracting minimally from it the tachyonic pole. In order to determine the correct normalization of this tachyon-free representation of the top propagator, let us consider the spectral representation of the resummed top propagator

$$\hat{D}_t(p) = i \hat{p} \omega_+ \int_{-\infty}^{\infty} ds \frac{\rho(s)}{p^2 - s + i\epsilon}, \quad \text{where} \quad \rho(s) = \rho_-(s) + \rho_+(s) \theta(s).$$  

(12)

Notice that due to the tachyonic contribution to the spectral function,

$$\rho_-(s) = \kappa \delta(s + \Lambda_T^2), \quad \text{where} \quad \kappa = \frac{1}{1 + \alpha_t \lambda_T^2} \approx \frac{1}{\alpha_t} \exp \left( -\frac{1}{\alpha_t} \right),$$  

(13)

is the residuum at the tachyon pole, the exact top propagator does not satisfy the usual Källén-Lehmann representation. The other contribution to the spectral function, which comes from the positive part of the spectrum, is given by

$$\rho_+(s) = \frac{\alpha_t}{m_t^2} \left[ \frac{s}{m_t^2} - \alpha_t \frac{s}{m_t^2} \log \left( \frac{s}{m_t^2} \right) - 1 \right]^2 + \pi^2 \alpha_t^2 \frac{s^2}{m_t^4}.$$  

(14)

Clearly the removal of the tachyonic pole is necessary in order to find an expression for the resummed top quark propagator which respects causality and satisfies the Källén-Lehmann representation. On the other hand, the contribution of the tachyon pole is crucial in order to ensure the correct normalization of the spectral function, as can be easily checked numerically

$$\int_{-\infty}^{\infty} ds \rho(s) = \kappa + \int_0^{+\infty} ds \rho_+(s) = 1.$$  

(15)

In order to compensate the tachyonic contribution to the spectral function a suitable rescaling of the top quark propagator should be performed. This leads us to the following
tachyon-free representation of the exact top quark propagator
\[
\tilde{D}_t(p) = \frac{i \not{p} \omega_+}{1 - \kappa} \left[ \frac{a(p^2)}{1 - \kappa} \left( \frac{1 - \kappa}{p^2 - m_t^2} \right) - \frac{\kappa}{p^2 + \Lambda_T^2} \right].
\] (16)

Some comments are in order. i) Since the subtraction term vanishes to all orders in perturbation theory, this tachyonic regularization can be regarded as a prescription for summing up the perturbative expansion in which a tachyon does not show up in the theory’s spectrum. ii) Although not unique, our prescription preserves gauge invariance to all orders in perturbation theory. iii) The prefactor \(1/(1 - \kappa)\), which ensures the correct normalization of the spectral function after the subtraction of the tachyon pole, is crucial in order to prevent the appearance of spurious divergences in the radiative corrections to the \(\rho\) parameter.

4 Ward identities

In this Section we present the exact top quark contribution to the one-loop vector (i.e. \(Z\) and \(W\)) and scalar (i.e. \(\chi\) and \(\phi\)) self-energies at the leading order in the large \(N_F\)-limit. Since, in general, resummation of self-energy insertions can spoil gauge invariance, we will also check the validity of the Ward identities connecting these self-energies.

It is convenient to perform the computation of vector and scalar self-energies in a pure \(SU(1 + N_F)\) gauge theory without hypercharge. This simplifies the actual computation without modifying the leading top quark contribution to the \(\rho\) parameter.

Let us recall the expression of the sine and cosine of the Weinberg angle and of the electric charge in the zero hypercharge limit, \(g' = 0\).

\[
c_W = \frac{g}{\sqrt{g^2 + g'^2}} = 1, \quad s_W = \frac{g'}{\sqrt{g^2 + g'^2}} = 0, \quad e = \frac{g g'}{\sqrt{g^2 + g'^2}} = 0. \tag{17}
\]

As a consequence of the above relations, as expected, \(M_W = M_Z\).

The coupling of the \(Z\) boson to the top quark in the SM is given by

\[
e \gamma_\mu (g_t^- \omega_- + g_t^+ \omega_+) . \tag{18}
\]

If we switch off the hypercharge, we find

\[
e g_t^- = e \left( - \frac{2}{3} \frac{s_W}{c_W} + \frac{1}{2} \frac{1}{s_W c_W} \right) = g \frac{2}{3}, \quad e g_t^+ = - \frac{2}{3} \frac{e s_W}{c_W} = 0 . \tag{19}
\]

The couplings of the \(W\) boson to fermions do not change if we set \(g' = 0\).

For our purposes it will be enough to consider only those graphs contributing to the vector and scalar self-energies where at least one virtual top quark is exchanged. At the one-loop level this task amounts to the computation of two Feynman graphs which are shown in Fig. (2) for the case of the vector self-energies (the other graphs are the same with scalar external legs). The required amplitudes can be easily obtained by using the
Figure 2: Top contribution to the Z and W boson self-energies at one loop. The solid line with a bubble denotes the resummed top propagator.

SM Feynman rules for vertices in the zero hypercharge limit (see Eq. (19)), the resummed top propagator in Eq. (19) and the Born propagator of a massless b quark. In order to avoid cumbersome expressions and since they do not play any role in the proof of the validity of Ward identities, we do not show in this Section the contributions coming from the tachyon subtraction term. Their discussion is deferred to Sect. 6.

The Z self-energy in the zero hypercharge limit reads

\[
\Sigma_Z^{\mu
u}(p^2) = \frac{i}{2} g^2 N_c \int \frac{d^D q}{(2\pi)^D} \frac{a(q^2) a((q - p)^2) \left[g^{\mu\nu}(q^2 - q \cdot p) - 2g^{\mu q}q^\nu + q^\mu p^\nu + q^\nu p^\mu\right]}{[a(q^2) q^2 - m_t^2] [a((q - p)^2)(q - p)^2 - m_t^2]}, \tag{20}
\]

where \(N_c\) is the number of colours.

The \(\chi\) self-energy, whose expression does not depend on the hypercharge \(g'\), is given by

\[
\Sigma_\chi(p^2) = ig^2 N_c \frac{m_t^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{a((q - p)^2) q \cdot (q - p) - m_t^2}{[a(q^2) q^2 - m_t^2][a((q - p)^2)(q - p)^2 - m_t^2]}. \tag{21}
\]

By dotting \(p_\mu, p_\nu\) in Eq. (20) we get

\[
p_\mu p_\nu \Sigma_Z^{\mu\nu}(p^2) = \frac{i}{2} g^2 N_c \frac{m_t^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{a(q^2) a((q - p)^2) \left[q^2 p^2 + ((q - p)^2 - q^2) q \cdot p\right]}{[a(q^2) q^2 - m_t^2][a((q - p)^2)(q - p)^2 - m_t^2]}
= ig^2 N_c m_t^2 \int \frac{d^D q}{(2\pi)^D} \frac{a((q - p)^2) (p^2 - q \cdot p)}{[a(q^2) q^2 - m_t^2][a((q - p)^2)(q - p)^2 - m_t^2]} \tag{22}
\]

Notice that in Eq. (21) there is a term proportional to \(m_t^4\) which is absent in the second line of Eq. (22). This term, however, cancels if one takes into account the contribution of Higgs tadpoles to the scalar self-energy \(\Sigma_\chi\).

\[
\Sigma_\chi^{\text{tad}} = -ig^2 N_c \frac{m_t^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{a(q^2) q^2 - m_t^2}. \tag{23}
\]

If we add \(\Sigma_\chi^{\text{tad}}\) to the self-energy in Eq. (21) we get

\[
\Sigma_\chi(p^2) = ig^2 N_c \frac{m_t^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{a((q - p)^2) (q - p) \cdot p}{[a(q^2) q^2 - m_t^2][a((q - p)^2)(q - p)^2 - m_t^2]} \tag{24}
\]
By comparing Eqs. (22) and (24) we find
\[ \frac{p_\mu p_\nu}{M_Z^2} \Sigma_\mu^\nu (p^2) + \Sigma (p^2) = 0. \] (25)

We consider now the one-loop $W$ self-energy.
\[ \Sigma_{W}^{\mu\nu} (p^2) = i g^2 N_c \int \frac{d^D q}{(2\pi)^D} \frac{a(q^2)[g^{\mu\nu}(q^2 - q \cdot p) - 2q^\mu q^\nu + q^\mu p^\nu + q^\nu p^\mu]}{[a(q^2)q^2 - m_i^2](q - p)^2}. \] (26)

The one-loop $\phi$ self-energy is given by
\[ \Sigma_{\phi} (p^2) = ig^2 N_c \frac{m_i^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{q \cdot (q - p)}{[a(q^2)q^2 - m_i^2](q - p)^2}. \] (27)

By dotting $p_\mu, p_\nu$ in Eq. (26) we get
\[ p_\mu p_\nu \Sigma_{W}^{\mu\nu} (p^2) = ig^2 N_c \frac{m_i^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{q^\mu q^\nu p^2 + a(q^2)[(q - p)^2 - q^2]q \cdot p}{[a(q^2)q^2 - m_i^2](q - p)^2}. \] (28)

By comparing Eqs. (28) and (27) one finds that a relation analogous to the one in Eq. (25) does not hold. However if one adds the contribution of Higgs tadpoles
\[ \Sigma_{\phi}^{tad} = -ig^2 N_c \frac{m_i^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{a(q^2)q^2 - m_i^2}, \] (29)

to the self-energy in Eq. (27) one gets
\[ \Sigma_{\phi} (p^2) = ig^2 N_c \frac{m_i^2}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{(q - p) \cdot p}{[a(q^2)q^2 - m_i^2](q - p)^2}. \] (30)

Now it is straightforward to prove the following Ward identity
\[ \frac{p_\mu p_\nu}{M_W^2} \Sigma_{W}^{\mu\nu} (p^2) + \Sigma_{\phi} (p^2) = 0. \] (31)

5 Perturbative top contributions to the $\rho$ parameter

In this Section we shall use the resummed top propagator in Eq. (9) in order to compute the leading contributions in the top quark mass to the $\rho$ parameter. It turns out that due to the presence of a tachyonic pole, the resulting expression for $\Delta \rho$ is ill-defined. However, while the tachyon is a nonperturbative effect, all the coefficients of the perturbative expansion of $\Delta \rho$ in powers of the interaction strength $\alpha_t$ are well-defined and can be computed analytically.

The $\rho$ parameter is usually defined as the ratio between the neutral and charged current coupling constants at zero momentum transfer
\[ \rho = \frac{J_{NC}(0)}{J_{CC}(0)} = \frac{1}{1 - \Delta \rho} . \] (32)
$J_{CC}(0)$ is given by the Fermi coupling constant, $G_F$, determined from the $\mu$-decay rate, while $J_{NC}(0)$ can be measured in neutrino scattering on electrons or hadrons. Notice that this definition of the $\rho$ parameter is process dependent since, in general, the radiative corrections depend on the hypercharge of the particles involved in the scattering process. However, the leading contributions in the top quark mass to $\Delta \rho$ are universal.

At tree-level the $\rho$ parameter is given by

$$\rho = \frac{M_Z^2}{M_W^2},$$

where

$$\Sigma_{V}^{\mu \nu}(p^2 = 0) = \Pi_{V} g_{\mu \nu}, V = W, Z.$$  (33)

$\Pi_Z$ and $\Pi_W$ can be obtained by setting $p = 0$ in Eqs. (20), (26) respectively.

$$\Pi_Z = \frac{i}{2} g^2 N_c \left(1 - \frac{2}{D}\right) \int \frac{d^D q}{(2\pi)^D} \frac{a^2(q^2) q^2}{[a(q^2) q^2 - m_t^2]^2}. $$  (34)

$$\Pi_W = ig^2 N_c \left(1 - \frac{2}{D}\right) \int \frac{d^D q}{(2\pi)^D} \frac{m_t^2}{[a(q^2) q^2 - m_t^2]^2 q^2}. $$  (35)

By substituting Eqs. (34) and (35) into Eq. (33), it is now straightforward to derive the leading top quark contribution to the $\rho$ parameter. Since it turns out that the result is both IR- and UV-convergent we can compute it directly in $D = 4 - \epsilon$

$$\Delta \rho = \frac{i}{4} g^2 N_c \frac{m_t^4}{M_W^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{[a(q^2) q^2 - m_t^2]^2 q^2}. $$  (36)

In the free field case ($\alpha_t = 0$), the above equation gives the well known result

$$\Delta \rho = \frac{g^2 N_c m_t^2}{64\pi^2 M_W^2} = \frac{N_c \sqrt{2}}{16\pi^2} G_F m_t^2 = \frac{N_c}{N_F} \alpha_t. $$  (37)

After expanding the denominator in Eq. (36) about $\alpha_t = 0$, one can perform a Wick rotation and integrate over the solid angle

$$\Delta \rho = \frac{N_c}{N_F} \alpha_t \sum_{j=0}^{\infty} (j + 1) \alpha_t^j \int_0^\infty dx \frac{x^j (\log x)^j}{(1 + x)^{(j+2)}}, $$  (38)

where we have introduced the dimensionless variable $x = -\frac{g^2}{m_t^2}$.

Let us consider a slightly more general class of integrals

$$I_l(j) = (j + 1) \int_0^\infty dx \frac{x^{(j-l)} (\log x)^l}{(1 + x)^{(j-l+2)}}, $$  (39)

with $j, l \in \mathbb{N}$, $j \geq l + 1$.

---

3 Notice, however, that the integral in Eq. (39) is ill-defined due to the tachyonic pole present in the resummed top propagator [9]. As a consequence, the Wick rotation cannot be performed because the resulting integral would be divergent.
Clearly, we are interested in $I_0(j)$, but if we integrate by parts the latter, we immediately find a linear combination of $I_0(j - 1)$ and $I_1(j)$ (see the identity in Eq. (10)). The case $j = l$ must be dealt with separately because boundary terms contribute. Integration by parts can be applied to $I_l(j)$ giving rise to the following recurrence relation

$$I_l(j) = \frac{(j + 1)}{j - l + 1} I_l(j - 1) + \frac{(j + 1)(j - l)}{j - l + 1} I_{l+1}(j).$$  \hspace{1cm} (40)

Eq. (40) allows us to express $I_0(j)$ in closed form by means of a linear combination of simpler integrals

$$I_0(j) = \sum_{l=0}^{j} c_l(j) I_l^{(2)}(l), \text{ where } I_l^{(2)}(l) = \int_0^\infty dx \frac{(\log x)^l}{(1 + x)^2}.$$  \hspace{1cm} (41)

The coefficients $c_l(j)$ are related to the combinatorial problem of grouping together $l$ objects out of a total of $j$ without repetitions. Their explicit expression and some useful properties are reported in Appendix A.

In order to compute $I_l^{(2)}(l)$, let us consider the following change of variable: $\log x = t$

$$I_l^{(2)}(l) = \int_{-\infty}^{+\infty} dt \frac{t^l}{e^t + e^{-t} + 2}. \hspace{1cm} (42)$$

Since the integrand in Eq. (42) is an odd function if $l$ is odd and an even function otherwise we immediately obtain the following result

$$I_{2l}^{(2)}(2l) = \int_{-\infty}^{+\infty} dt \frac{t^{(2l)}}{e^t + e^{-t} + 2}, \hspace{0.5cm} I_{2l+1}^{(2)}(2l + 1) = 0.$$  \hspace{1cm} (43)

The integrals in Eq. (43) can be computed analytically by exploiting the properties of polylogarithms, as shown in Appendix B. We report here only the final result

$$I_{2l}^{(2)}(2l) = 2(2l)! \left(1 - 2^{(1-2l)}\right) \zeta(2l). \hspace{1cm} (44)$$

Putting all the pieces together we obtain

$$\Delta \rho = \frac{N_c}{N_F} \sum_{j=0}^\infty r_j \alpha_t^{(j+1)}, \text{ where } r_j = \sum_{l=0}^{[j/2]} 2c_{2l}(j) (2l)! \left(1 - 2^{(1-2l)}\right) \zeta(2l),$$  \hspace{1cm} (45)

where with $[\cdot]$ we denote the integer part of a real number. As an example, we report here the perturbative expansion of the leading top contribution to the $\rho$ parameter up to terms of order five in $\alpha_t$

$$\Delta \rho = \frac{N_c}{N_F} \alpha_t \left[1 + \alpha_t + \left(1 + \frac{1}{3} \pi^2\right) \alpha_t^2 + \left(1 + \frac{11}{6} \pi^2\right) \alpha_t^3 + \left(1 + \frac{35}{6} \pi^2 + \frac{7}{15} \pi^4\right) \alpha_t^4 + O(\alpha_t^5)\right].$$  \hspace{1cm} (46)

By making use of the asymptotic estimates of the combinatorial coefficients in Eqs. (70) and (73) we can easily find the leading order behaviour of the coefficients of the perturbative expansion of the $\rho$ parameter. In particular, it turns out that $r_j \approx (j + 1)!$ for $j \gg 1$. Thus the perturbative expansion of the $\rho$ parameter is factorially divergent and not Borel summable, being a fixed sign power series.
5.1 Wave function renormalization of unphysical scalars

In order to perform a cross check of our result in Eq. (45), we compute the radiative corrections to the $\rho$ parameter by means of the wave function renormalization of the unphysical scalar fields. This latter approach is related to the one pursued in Sect. 5 by gauge invariance.

Let us consider the kinetic terms of the scalar part of the SM lagrangian. The UV divergences that show up in radiative corrections can be reabsorbed by introducing suitable wave function renormalization constants in the following way

$$\mathcal{L}_{KS} = Z_\phi \left[ \partial_\mu \phi^- + i \frac{g_v}{2} W^-_\mu \right]^2 + \frac{Z_\chi}{2} \left( \partial_\mu \chi + \frac{g_v}{2c_W} Z_\mu \right)^2 + \text{other terms} \quad .$$  

The self-energies of the unphysical scalars have been computed in Sect. 4 (see Eqs. (24), (25)). The renormalized masses of the gauge bosons are given by $M_W = \sqrt{Z_\phi \frac{g_v}{2}}$ and $M_Z = \sqrt{Z_\chi \frac{g_v}{2c_W}}$, thus

$$\rho = \frac{Z_\phi}{Z_\chi} \Rightarrow \frac{d}{dp^2} \rho \left( \Sigma_\phi(p^2) - \Sigma_\chi(p^2) \right) \bigg|_{p^2=0} ,$$

since $Z_{S} = 1 + \frac{d}{dp^2} \Sigma_{S}(p^2) \bigg|_{p^2=0} \Rightarrow S = \phi, \chi$ .

The self-energies of the unphysical scalars have been computed in Sect. 4 (see Eqs. (24), (25)). The $\rho$ parameter reads

$$\Delta \rho = i g^2 N_c \frac{m_t^4}{M_W^2} \frac{d}{dp^2} \int \frac{d^D q}{(2\pi)^D} \frac{q \cdot p}{[a(q^2) q^2 - m_t^2][a((q-p)^2) (q-p)^2 - m_t^2] q^2} \quad .$$  

We now develop the denominator about $p = 0$ . For the computation of the derivative w.r.t. $p^2$ it is enough to keep terms of the order of $q \cdot p$, because a term proportional to $p_\mu$ is already present in the numerator

$$\Delta \rho = 2i g^2 N_c \frac{m_t^4}{M_W^2} \frac{d}{dp^2} \int \frac{d^D q}{(2\pi)^D} \frac{(q \cdot p)^2 (a(q^2) - \alpha_t)}{[a(q^2) q^2 - m_t^2]^3 q^2} \quad .$$  

The tensor reduction can be performed immediately. In fact, since the denominator does not depend on $p$, $q^\mu q^\nu$ must be proportional to $g^{\mu\nu}$. Moreover we can work in four dimensions because the above integral is both IR- and UV-convergent.

$$\Delta \rho = \frac{i}{2} g^2 N_c \frac{m_t^4}{M_W^2} \int \frac{d^4 q}{(2\pi)^4} \frac{q^2 (a(q^2) - \alpha_t)}{[a(q^2) q^2 - m_t^2]^3 q^2}$$

$$\quad = \frac{i}{2} g^2 N_c \frac{m_t^4}{M_W^2} \left[ \int \frac{d^4 q}{(2\pi)^4} \frac{m_t^2 - \alpha_t q^2}{[a(q^2) q^2 - m_t^2]^3 q^2} + \int \frac{d^4 q}{(2\pi)^4} \frac{1}{[a(q^2) q^2 - m_t^2]^3 q^2} \right] \quad .$$  

Notice that the second term in the last line of the above equation is simply $2\Delta \rho$, as can be seen from Eq. (50), thus finally we are left with

$$\Delta \rho = \frac{i}{2} g^2 N_c \frac{m_t^4}{M_W^2} \int \frac{d^4 q}{(2\pi)^4} \frac{\alpha_t q^2 - m_t^2}{[a(q^2) q^2 - m_t^2]^3 q^2} \quad .$$  

\footnote{Notice that this works because the derivative w.r.t. $p^2$ is both IR- and UV-convergent, otherwise one looses finite parts by computing the derivative in this way.}
After expanding about $\alpha_t = 0$, one can Wick rotate the above expression and integrate over the solid angle, finding

$$\Delta \rho = \frac{N_c}{N_F} \alpha_t \sum_{j=0}^{\infty} (j + 2)(j + 1) \alpha_j^j \int_0^\infty dx \frac{(1 + \alpha_t x) x^j (\log x)^j}{(1 + x)^{(j+3)}}.$$  \hspace{1cm} (53)

At the leading order in the interaction strength $\alpha_t$, we immediately find the result in Eq.(37)

$$\Delta \rho = \frac{N_c}{N_F} \alpha_t \int_0^\infty dx \frac{2}{(1 + x)^3} = \frac{N_c}{N_F} \alpha_t.$$  \hspace{1cm} (54)

Let us consider the coefficient of $\alpha_j^j$, with $j \geq 1$ (for the sake of brevity in the following equations the common prefactor $\frac{N_c}{N_F} \alpha_t$ is omitted)

$$(j + 2)(j + 1) \int_0^\infty dx \frac{x^j (\log x)^j}{(1 + x)^{(j+3)}} + (j + 1) j \int_0^\infty dx \frac{x^{(j-1)} (\log x)^{(j-1)}}{(1 + x)^{(j+1)}}$$

$$- (j + 1) j \int_0^\infty dx \frac{x^{(j-1)} (\log x)^{(j-1)}}{(1 + x)^{(j+2)}}.$$  \hspace{1cm} (55)

Notice that the second term in the first line of the above equation is given by $(j+1) I_0(j-1)$. After integrating by parts the first term in the first line of Eq.(55), we find

$$(j + 1) j \int_0^\infty dx \frac{x^{(j-1)} (\log x)^j}{(1 + x)^{(j+2)}} + (j + 1) I_0(j - 1).$$  \hspace{1cm} (56)

By applying iteratively integration by parts on the first term of the above equation and by using Eq.(41), we get

$$\sum_{l=1}^{j} c_l (j - 1) \left[ \frac{j}{l} I^{(3)}_l(l) + (j + 1) I^{(2)}_{l-1}(l - 1) \right],$$  \hspace{1cm} (57)

where the integrals $I^{(3)}_l(l)$ are given by

$$I^{(3)}_l(l) = 2 \int_0^\infty dx \frac{(\log x)^l}{(1 + x)^3} = 2 \int_{-\infty}^{+\infty} dt \frac{e^t t^l}{(1 + t)^3}.$$  \hspace{1cm} (58)

In the above equation a change of variable $t = \log x$ has been performed. We integrate by parts the integrals in Eq.(58) and we postpone the evaluation of the boundary contributions (for the notations see Appendix[13])

$$2 \int dt \frac{e^t t^l}{(1 + e^t)^3} = - \frac{t^l}{(1 + e^t)^2} + t^l - l \int dt \frac{e^t t^{l-1}}{(1 + e^t)^2} - l \int dt \frac{e^t t^{l-1}}{1 + e^t} =$$

$$- t^l [Li_{-1}(- e^t) + Li_0(- e^t)] + l \left[ \int dt t^{l-1} Li_{-1}(- e^t) + \int dt t^{l-1} Li_0(- e^t) \right].$$  \hspace{1cm} (59)

Some comments are in order. i) The term $- t^l Li_{-1}(- e^t)$ vanishes when evaluated at the boundaries, i.e. $\pm \infty$, therefore it can be neglected. ii) The first integral in the second line
of the above equation vanishes when \( l \) is even, while for \( l \) odd its result is given in Eq.(79).

iii) The last integral in the second line of the above equation is divergent if evaluated at (positive) infinity, however this divergence is exactly compensated by \(-tl_l\text{Li}_0(-e^t)\). We compute this latter integral by exploiting the properties of the polylogarithms.

\[
\int dt\ t^{(l-1)}\text{Li}_0(-e^t) = l\sum_{r=1}^{l}(-)^{r+1}\left[\prod_{s=0}^{r-1}(l-s)\right] t^{(l-r)}\text{Li}_{r}(-e^t) = -\int dt\ t^l\ \text{Li}_{-1}(-e^t) + tl_l\text{Li}_0(-e^t) .
\] (60)

The second line of the above equation follows immediately by a comparison between the sum in the first line of the same equation and Eq.(79). Putting all the pieces together, we find

\[
I^{(3)}_l(l) = -l I^{(2)}_{(l-1)}(l-1) + I^{(2)}_1(l) \quad \text{for} \quad l \geq 1 .
\] (61)

If we plug the above result into Eq.(57), we get

\[
\sum_{l=1}^{j} c_l(l-1) \left[ j I^{(2)}_l(l) + I^{(2)}_{l-1}(l-1) \right] = \sum_{l=0}^{j} c_l(j) I^{(2)}_l(l) .
\] (62)

In the above equation we have used the identity in Eq.(68).

## 6 Nonperturbative top contribution to the \( \rho \) parameter

In this Section we shall use the tachyon-free representation of the resummed top propagator \(16\) in order to compute nonperturbatively the exact leading top quark contribution to the \( \rho \) parameter at the leading order in the large \( N_F \)-limit.

The contribution of the tachyonic subtraction term in Eq.(16) to the one-loop self-energies of the \( W \) and \( Z \) vector bosons at zero external momentum can be easily computed. We show here the results.

\[
\Pi_Z = i^{\frac{2}{g^2 N_c}} \left( 1 - \frac{2}{D} \right) \int \frac{d^D q}{(2\pi)^D} \left\{ \frac{a^2(q^2)q^2}{[a(q^2)q^2 - m_W^2]q^2} + \frac{\kappa^2 - 2\kappa}{q^2 + \Lambda_T^2} - \frac{\kappa^2 \Lambda_T^2}{(q^2 + \Lambda_T^2)^2} - \frac{2\kappa m_W^2}{[a(q^2)q^2 - m_W^2](q^2 + \Lambda_T^2)q^2} \right\} .
\] (63)

\[
\Pi_W = i g^2 N_c \left( 1 - \frac{2}{D} \right) \int \frac{d^D q}{1 - \kappa} \left\{ \frac{m_W^2}{[a(q^2)q^2 - m_W^2]q^2} + \frac{\kappa \Lambda_T^2}{(q^2 + \Lambda_T^2)q^2} \right\} .
\] (64)

By using Eqs.(63) and (64) into Eq.(33) and by setting \( D = 4 \), one can write down a tachyon-free representation of the leading top contribution to the \( \rho \) parameter

\[
\Delta \rho = i^{\frac{4}{g^2 N_c}} M_W^2 \left( 1 - \frac{2}{D} \right) \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \left[ \frac{1}{a(q^2)q^2 - m_t^2} + \frac{\kappa \Lambda_T^2}{q^2 + \Lambda_T^2} \right]^2 .
\] (65)

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Notice that the perturbative expansion of the above result coincide order by order in $\alpha_t$ with the factorially divergent and not Borel summable perturbative series in Eq.(38) since the additional terms, proportional to $\kappa$, vanish to all orders in perturbation theory. However, the integral in Eq.(35) is now well-defined and thus the Wick-rotation and the integration over the solid angle can be performed directly on it. The result of these operations can be expressed in terms of the dimensionless variable $x = -\frac{q^2}{m_t^2}$ as follows

$$\Delta \rho = \frac{N_c}{N_F} \frac{\alpha_t}{(1-\kappa)^2} \int_0^\infty dx \left[ \frac{1}{a(-x)} \frac{1}{x+1} + \frac{\kappa \lambda_T^2}{x-\lambda_T^2} \right]^2.$$  \hspace{1cm} (66)

The above integral can be computed numerically (for instance with the help of Mathematica) for an arbitrary value of the interaction strength $\alpha_t$, allowing us to make a comparison between the exact nonperturbative result and its perturbative approximation at any fixed order in $\alpha_t$.

In Table 1 we show the position of the tachyonic pole (divided by $m_t$), the residuum at the tachyon pole, $\kappa$, and the leading top contribution to the $\rho$ parameter (omitting the prefactor $N_c/N_F$) for some values of $\alpha_t$.

| $\alpha_t$ | $\lambda_T$ | $\kappa$ | $\Delta \rho$ |
|------------|-------------|---------|--------------|
| 0.02       | 7.2 \cdot 10^{10} | 9.6 \cdot 10^{-21} | 0.021 |
| 0.04       | 2.7 \cdot 10^{5} | 3.5 \cdot 10^{-10} | 0.042 |
| 0.06       | 4.2 \cdot 10^{3} | 9.6 \cdot 10^{-7} | 0.065 |
| 0.08       | 518.0 | 4.7 \cdot 10^{-5} | 0.090 |
| 0.10       | 148.4 | 4.5 \cdot 10^{-4} | 0.118 |
| 0.20       | 12.38 | 0.032 | 0.249 |
| 0.40       | 3.805 | 0.147 | 0.329 |
| 0.60       | 2.602 | 0.198 | 0.341 |
| 0.80       | 2.141 | 0.214 | 0.344 |
| 1.00       | 1.895 | 0.218 | 0.344 |

Table 1: Numerical values for the tachyonic pole, its residuum and the leading top contribution to the $\rho$ parameter.

The exact numerical result, $\Delta \rho(\alpha_t)$, shows a typical saturation behaviour (see Fig. 3) for $\alpha_t > 0.2$ which cannot be reproduced by the perturbative expansion of the $\rho$ parameter (45) at any fixed order since all the expansion coefficients are positive (see Fig. 4). However for small enough values of the interaction strength, say $\alpha_t < 0.2$, the agreement between the nonperturbative exact result and its perturbative approximation (starting with terms of order $O(\alpha_t^4)$) is very good. Finally, it should be noted that since the perturbative expansion of the $\rho$ parameter is a divergent asymptotic series, the perturbative approximation of the exact result can be improved by adding further terms to the series only up to a certain order, beyond which the approximation gets worse and worse.
Figure 3: Exact leading top contribution to the $\rho$ parameter

Figure 4: Comparison between the exact result for $\Delta \rho$ and its perturbative expansion at 1, 2, 3 and 10 loops
7 Conclusions

In this paper the $SU(N_F) \times U(1)$ model at the leading order in the large $N_F$-limit has been used in order to compute the exact leading top quark contribution to the $\rho$ parameter and its perturbative expansion to all orders in the interaction strength $\alpha_t$.

Since only one-loop graphs contribute to the top quark self-energy at the leading order in the large $N_F$-limit, the exact top quark propagator can be obtained simply by resumming one-loop self-energy insertions. In this way, one takes into account the finite width effects due to the fact that the top quark is an unstable particle. On the other hand, this Dyson resummed propagator contains a tachyon pole in the euclidean region which spoils causality and makes all the Wick-rotated integrals ill-defined. We have regularized the resummed propagator by subtracting the tachyon minimally at its pole. Although not unique, this procedure allows to define a tachyon-free representation of the exact top propagator which respects gauge invariance.

The validity of the Ward identities connecting the self-energies of vector bosons and of unphysical scalar particles, computed by using the resummed top propagator instead of the Born one, have been checked. These vector and scalar self-energies then have been used in order to compute the leading top contribution to the $\rho$ parameter in two different ways as a further check of gauge invariance. It turns out that the perturbative expansion in powers of the interaction strength $\alpha_t$ of the $\rho$ parameter is factorially divergent and not Borel summable.

However, after having subtracted consistently the tachyonic pole the expression for the leading top contribution to the $\rho$ parameter can be evaluated numerically and compared with its perturbative approximation. The agreement between the exact result and its perturbative expansion (starting with terms of order $O(\alpha_t^4)$) is very good for $\alpha_t < 0.2$ which in the SM, i.e. for $N_F = 1$, corresponds to a top quark mass of 1.4 TeV. Moreover, the exact numerical result shows a typical saturation behaviour which cannot be reproduced by the perturbative expansion of the $\rho$ parameter at any fixed order, since all the expansion coefficients are positive.

Though the subtraction of the tachyon pole is determined by the demand of causality, the procedure is not quite unique, since the correction factor that is needed to insure a properly normalized spectral density could have been different from a constant. However, one can consider this correction factor, which is given by the residuum of the tachyon pole, as an estimate for the uncertainty in the calculation due to non-perturbative effects or effects of new physics at high energy. The uncertainty is at most of the order of 20%.

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A Combinatorial coefficients

In this Appendix we give the explicit expression of the combinatorial coefficients $c_l(j)$ introduced in Eq.(41) and we show some of their properties.

The recurrence relation in Eq.(40) can be applied to the integrals $I_l(j)$ (see Eq.(39)) as long as $l \leq j$. Thus, starting from $I_0(j)$ and applying repeatedly the recurrence relation, one ends up with a linear combination of $I_l(l)$, with $0 \leq l \leq j$. The coefficients of this linear combination are

$$c_0(j) = 1,$$

$$c_l(j) = \frac{1}{l!} \prod_{i=1}^{l} \sum_{r_i = r_{(i-1)+1}}^{j-l+i} r_i,$$

with $r_1 = 1, 2, \ldots j - l + 1$. \hspace{1cm} (67)

By using the definition of $c_l(j)$ in Eq.(67), it is straightforward to show that $c_j(j) = 1$. Moreover, another useful relation which can be easily proven is the following

$$c_l(j) - c_l(j-1) = j l c_{l-1}(j-1).$$

The knowledge of the asymptotic behaviour of $c_l(j)$ for $j \gg 1$ will allow us to determine the large order behaviour of the perturbative expansion of the $\rho$ parameter in Eq.(45). For this purpose, by making use of the following identity

$$\sum_{k=r+1}^{j-l+i} k^s = \sum_{k=1}^{j-l+i} k^s - \sum_{k=1}^{r} k^s \approx \frac{j^{(s+1)}}{s+1} - \frac{r^{(s+1)}}{s+1},$$

we eventually find

$$c_l(j) = \frac{1}{2^l} \left( \frac{j^l}{l!} \right)^2 + O(j^{(l-1)}), \text{ for } j \gg 1, l \ll j.$$

In order to obtain an asymptotic estimate of $c_l(j)$ which holds for $l \simeq j$, it is convenient to write down an expression for the ‘last’ coefficients $c_{(j-l)}(j)$

$$c_{(j-l)}(j) = \left( \prod_{k=1}^{l} (j-k+1) \right) \prod_{i=1}^{l} \sum_{r_i = r_{(i-1)+1}}^{j-l+i} \frac{1}{r_i}. \hspace{1cm} (71)$$

The sums of the reciprocals of natural numbers in the above equation can be rewritten in terms of products of the harmonic numbers

$$l! \prod_{i=1}^{l} \sum_{r_i = r_{(i-1)+1}}^{j-l+i} \frac{1}{r_i} \approx (H(j))^l, \text{ where } H(j) = \sum_{r=1}^{j} \frac{1}{r}. \hspace{1cm} (72)$$

Since $H(j) \simeq \log j$ for $j \gg 1$, the leading order asymptotic behaviour of $c_{(j-l)}(j)$ is given by

$$c_{j-l}(j) = \frac{j!}{l! (j-l)!} (\log j)^l + O\left((\log j)^{(l-1)}\right), \text{ for } j \gg 1, l \ll j.$$

\hspace{1cm} (73)
In this Appendix we compute the integrals in Eq. (43) by making use of the properties of polylogarithmic functions.

The polylogarithm \( \text{Li}_s(z) \) is, in general, a special function defined by the following series

\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad \forall z, s \in \mathbb{C}, \text{ with } |z| < 1.
\]

(74)

By analytic continuation it is possible to extend the domain of the polylogarithm over a larger range of \( z \). Notice that for some values of the parameter \( s \), it is possible to express the polylogarithm by using elementary functions. For instance

\[
\text{Li}_0(z) = \sum_{k=1}^{\infty} \frac{z^k}{1} = \frac{z}{1 - z}, \quad \text{Li}_1(z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = -\log(1 - z).
\]

(75)

By using the definition in Eq. (74) and by integrating the series term by term it is straightforward to prove that

\[
\text{Li}_{s+1}(z) = \int_0^z \frac{\text{Li}_s(t)}{t} \, dt, \quad \text{thus} \quad \frac{d}{dz} \text{Li}_{s+1}(z) = \frac{\text{Li}_s(z)}{z}.
\]

(76)

We list here some properties of the polylogarithms which are needed for the computation of the above mentioned integrals.

\[
\lim_{|z| \to 0} \text{Li}_s(z) = 0. \quad \text{Li}_s(-1) = -\left(1 - 2^{1-s}\right) \text{Li}_s(1) = -\left(1 - 2^{1-s}\right) \zeta(s),
\]

(77)

where \( \zeta(s) \) is the Riemann zeta function.

In order to compute the integrals in Eq. (43), it is convenient to perform an indefinite integration by parts and evaluate the boundary contributions only at the very end of the computation.

\[
\int dt \frac{t^{(2l)}}{e^t + e^{-t} + 2} = -\frac{t^{(2l)}}{1 + e^t} + 2l \int dt \frac{e^t t^{(2l-1)}}{(1 + e^t) e^t} = -\frac{t^{(2l)}}{1 + e^t} + t^{(2l)}
\]

\[
-2l \int dt \frac{e^t t^{(2l-1)}}{1 + e^t} = -t^{(2l)} \text{Li}_0(-e^t) + 2l \int dt t^{(2l-1)} \text{Li}_0(-e^t).
\]

(78)

The procedure can be iterated thanks to the properties of the derivative of the polylogarithms. After 2\( l \) iterations we are left with

\[
\int dt \frac{t^{(2l)}}{e^t + e^{-t} + 2} = -t^{(2l)} \text{Li}_0(-e^t) + \sum_{k=1}^{2l} (-)^{k+1} \left[ \prod_{r=0}^{k-1} (2l - r) \right] t^{(2l-k)} \text{Li}_k(-e^t).
\]

(79)

It is now easy to compute the boundary contributions. Since the integrand is an even function, it is enough to evaluate the integral in Eq. (78) at \( t = 0 \) and \( t \to -\infty \) and then
doubling the result. By using the relations in Eq. (77), one sees that at $t = 0$ only the last term of the sum contributes, while all of the terms in Eq. (79) vanish in the limit $t \to -\infty$. Thus finally we find

$$\int_{-\infty}^{0} dt \frac{t^{(2l)}}{e^t + e^{-t} + 2} = -(2l)! \text{Li}_{2l}(-1) = (2l)! \left(1 - 2^{(1-2l)}\right) \zeta(2l).$$  

(80)

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