Data Rates for Network Linear Equations

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Abstract

In this paper, we study network linear equations subject to digital communications with a finite data rate, where each node is associated with one equation from a system of linear equations. Each node holds a dynamic state and interacts with its neighbors through an undirected connected graph, where along each link the pair of nodes share information. Due to the data-rate constraint, each node builds an encoder-decoder pair, with which it produces transmitted message with a zooming-in finite-level uniform quantizer and also generates estimates of its neighbors’ states from the received signals. We then propose a distributed quantized algorithm and show that when the network linear equations admit a unique solution, each node’s state is driven to that solution exponentially. We further establish the asymptotic rate of convergence, which shows that a larger number of quantization levels leads to a faster convergence rate but is still fundamentally bounded by the inherent network structure and the linear equations. When a unique least-squares solution exists, we show that the algorithm can compute such a solution with a suitably selected time-varying step size inherited from the encoder and zooming-in quantizer dynamics. In both cases, a minimal data rate is shown to be enough for guaranteeing the desired convergence when the step sizes are properly chosen. These results assure the applicability of various network linear equation solvers in the literature when peer-to-peer communication is digital.

1 Introduction

The pursuit of resilient and scalable solutions for the control and optimization of large-scale network systems has been one of the central themes in the field of systems and control in the past decade [1, 2]. For a group of interconnected agents (nodes), sensing and decision making can be carried out individually

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based on the information flow across the interconnections (links), under which collective goals such as consensus, formation and estimation can be achieved [3,4]. These distributed protocols provide resilience in the sense that nodes and links can join and leave the network without significantly affecting the performance of the network; they also provide scalability compared to centralized solutions because individual node sensing and decision are often quite simple. Simultaneously, control theory has embraced to a much greater degree than previously graph theory, communication theory, and complexity analysis, leading to many celebrated results for both theories and applications [5].

Particularly, systems of linear algebraic equations, as one of primary computation tasks, can be naturally defined over a network in the way that each node holds one or a few of the linear equations [6]. Network linear equations also arise from resource allocation problems when node cost functions are quadratic, see [7,8,9]. In the context of parallel computation, computer scientists aimed to develop algorithms that eventually compute part entries of the solutions [10,11,12,13,14]. On the other hand, in view of distributed gradient optimization [15,16,17], distributed algorithms that compute the entire solution vector at each node were also proposed for both discrete-time and continuous-time node dynamics [18,6,19,20,21,22,23,24,25]. In fact, when exact solutions exist for the linear equations, such first-order distributed solvers were generalized versions of the so-called alternation projection algorithms pioneered by von Neumann [26,15,27]. When no exact solution exists and one considers least-squares solutions, higher-order algorithms or algorithms using properly selected square-summable diminishing step-sizes are needed [21,22].

In this paper, we consider network linear equation solvers subject to digital node communications where only a finite data rate is available [28,29,30,31,32,33]. We use the convenient notion that each node holds one equation from a system of linear equations with \( m \) unknown variables. The nodes aim to reach consensus on the solution of the linear equations. The nodes interconnection is described by an undirected connected graph, where along each link the neighboring nodes exchange information constrained by a limited data rate measured in bits. Each node builds an encoder-decoder pair with the help of a zooming-in finite-level uniform quantization function, and is equipped with a dynamical internal encoder state co-evolving with the node states. At each step, each node’s encoder produces a quantized message with the node state and the current internal encoder state, which will be transmitted to its neighbors through the digital communication link. After receiving the quantized information from the neighbors, each node then decodes/estimates its neighbors’ states, based on which its own state is updated with the proposed algorithm. We have established the following results:

(i) When the network linear equation admits a unique exact solution, we show that the proposed encoder-decoder powered algorithm drives each node state to that solution asymptotically with an exponential convergence rate based on merely \( m \) bits information exchange between each pair of adjacent agents. Furthermore, we give an explicit form of the asymptotic rate of convergence, which is related to the
scale and the synchronizability of the network, the number of quantization levels, the dimension of the unknown variable, and the observation matrix. It is shown that a higher convergence rate is possible with higher data rates but is fundamentally bounded by the inherent network structure and the linear equations.

(ii) When the network linear equation admits a unique least-squares solution, we show that the same encoder-decoder pair enables the algorithm to compute such a solution with a time-varying step-size that comes from the dynamics of encoder internal states. Again, a data rate of $m$ bits per step can deliver such a convergence result, and an explicit form of the asymptotic convergence rate is established.

These results serve as assurance of the practical use of the various network linear equation solvers when digital point-to-point communications are subject to round-up errors. Generalizations to the scenarios where the solutions of the linear equations are not unique for both exact and least-squares cases are possibly along the same line of analysis, but are not included in the current paper for the ease of presentation. We also note that our results are closely related to the work on distributed optimization algorithms with quantized communication [34, 35]. However, new challenges for network linear equations arise compared to distributed optimization framework, although the problem appears to be a special case of quadratic program at first glance, lie in that gradients of the quadratic function associated with each node cannot be assumed to be globally bounded a priori, a key technical assumption for the convergence results of distributed (sub)gradient optimization [15, 35].

A preliminary version of the results will be presented at the IEEE CDC in 2018 [36]. Current manuscript compared to [36] makes the following improvements and extensions: (i) we future specify how the rate of convergence is influenced by the quantization levels, the scale and the synchronizability of the network, the variable dimension $m$ as well as the problem structure; (ii) we carry out more simulations to discuss how data rate influences algorithm parameter selection, and thereby, influences the converge rate; (iii) we also compare convergence rates for different types of communication graphs, and give the completed proofs of all results. The remainder of this paper is organized as follows. Section 2 defines the network linear equation, introduces the node encoders and decoders, and develops a distributed quantized algorithm. Section 3 presents the exact solver along with its convergence analysis and numerical examples. Section 4 further investigates the least-squares case. Finally, concluding remarks are given in Section 5.

Notation and Terminology. All vectors are column vectors and denoted by bold, lower case letters, i.e., $\mathbf{a}, \mathbf{b}, \mathbf{c}$, etc.; matrices are denoted with bold, upper case letters, i.e., $\mathbf{A}, \mathbf{B}, \mathbf{C}$, etc.; sets are denoted with $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc. Depending on the argument, $|\cdot|$ stands for the absolute value of a real number or the cardinality of a set. The Euclidean norm of a vector is denoted as $\| \cdot \|$. $\otimes$ denotes the Kronecker product. An undirected graph is an ordered pair of two sets denoted by $\mathcal{G} = \{ \mathcal{V}, \mathcal{E} \}$ where $\mathcal{V} = \{1, \ldots, N\}$ is a finite set of vertices (nodes), and each element in $\mathcal{E}$ is an unordered pair of two distinct nodes in $\mathcal{V}$,
called an edge. A path in $\mathcal{G}$ with length $p$ from $v_1$ to $v_{k+1}$ is a sequence of distinct nodes, $v_1v_2 \ldots v_{p+1}$, such that $(v_m, v_{m+1}) \in \mathcal{E}$, for all $m = 1, \ldots, p$. The graph $\mathcal{G}$ is termed connected if for any two distinct nodes $i, j \in V$, there is a path between them. The neighbor set of node $i$, denoted $\mathcal{N}_i$, is defined as $\mathcal{N}_i = \{j \in V : (i, j) \in \mathcal{E}\}$. Define the degree matrix $D_e = \text{diag}\{|\mathcal{N}_1|, \ldots, |\mathcal{N}_N|\}$ and the adjacency matrix $A$, where $[A]_{ij} = 1$ if $j \in \mathcal{N}_j$ and $[A]_{ij} = 0$ otherwise. Then $L = D_e - A$ is the Laplacian matrix of the graph $\mathcal{G}$.

2 Problem Statement and Algorithm Design

2.1 Linear Equations over Networks

Consider the following linear algebraic equation:

$$z = Hy$$  \hspace{1cm} (1)

with respect to unknown variable $y \in \mathbb{R}^m$, where $H \in \mathbb{R}^{N \times m}$ and $z \in \mathbb{R}^N$. The equation (1) has a unique exact solution if $\text{rank}(H) = m$ and $z \in \text{span}(H)$; an infinite set of solutions if $\text{rank}(H) < m$ and $z \in \text{span}(H)$; and no exact solutions if $z \notin \text{span}(H)$. When no exact solution exists, a least-squares solution of (1) can be defined via the following optimization problem:

$$\min_{y \in \mathbb{R}^m} \|z - Hy\|^2,$$  \hspace{1cm} (2)

which yields a unique solution $y^* = (H^T H)^{-1} H^T z$ if $\text{rank}(H) = m$.

We denote by

$$H = \begin{pmatrix} h_1^T \\ h_2^T \\ \vdots \\ h_N^T \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix},$$

where $h_i \in \mathbb{R}^m$ with $h_i^T$ being the $i$-th row vector of $H$.

Consider a network with $N$ nodes indexed as $\mathcal{V} = \{1, \ldots, N\}$, where node $i$ has access to the value of $h_i$ and $z_i$ without the knowledge of $h_j$ or $z_j$ from other nodes. The nodes interaction is described by a connected undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with the corresponding Laplacian matrix denoted by $L$. Time is slotted at $k = 0, 1, 2, \ldots$. Node $i$ at time $k$ holds an estimate $x_i(k) \in \mathbb{R}^m$ for the solution to equation (1) and exchanges information with its neighbors.

As the Euler approximation of the so-called “consensus + projection” flow proposed in [20], the following algorithm is an efficient distributed linear equations solver with a discrete recursion.

$$x_i(k+1) = x_i(k) + h \left[ \sum_{j \in \mathcal{N}_i} (x_j(k) - x_i(k)) - \gamma(k) \left( h_i h_i^T x_i(k) - z_i h_i \right) \right].$$  \hspace{1cm} (3)
It can be easily concluded from the analysis in [20, 22] that the following statements hold for the algorithm (3):

- When the linear equation (1) admits an exact solution $y^*$, it drives each $x_i(k)$ to $y^*$ exponentially with $\gamma(k) \equiv \gamma > 0$ provided that $h, \gamma$ are properly chosen.

- When the linear equation (1) has no exact solutions, it drives each $x_i(k)$ to a least-squares solution to (2) for small $h$ and $\gamma(k) = 1/k$.

It is clear that in the algorithm (3), nodes need to exchange their exact state values for the execution of the update. The aim of this paper is to develop algorithms that overcome such a constraint using quantized node communications, and to explore the corresponding convergence properties with minimal data rate statements.

### 2.2 Distributed Quantized Algorithm

Suppose that the communication channels corresponding to each edge in the network have a limited capacity or a finite bandwidth. As such, real-valued data should be quantized before transmitting. Thus, we propose a distributed quantized algorithm, in which each node is associated with an encoder while its neighbors possess a corresponding decoder. Let us begin by introducing a uniform quantization function $Q_K(\cdot)$.

**Definition 1 (Quantization Function)** A standard uniform quantizer is given by the function $Q_K(\cdot) : \mathbb{R} \rightarrow \{-K, \ldots, -1, 0, 1, \ldots, K\}$ where

$$Q_K(z) = \begin{cases} 0, & \text{if } -1/2 \leq z \leq 1/2, \\ i, & \text{if } 2i-1/2 < z \leq 2i+1/2, \ i = 1, \ldots, K, \\ K, & \text{if } z > \frac{2K+1}{2}, \\ -Q_K(-z), & \text{if } z > -1/2. \end{cases} \quad (4)$$

There is no need to send any information if the output of the quantizer is zero, so, for a $2K + 1$-level quantizer, the communication channel is required to be capable of transmitting $\lceil \log_2(2K) \rceil$ bits. With slight abuse of notation, we define $Q_K(a)$ for a vector $a = (a_1, \ldots, a_m)^T \in \mathbb{R}^m$ by

$$Q_K(a) = (Q_K(a_1), \ldots, Q_K(a_m))^T.$$

Next, we propose an encoder-decoder pair for each node to quantize its state, and to estimate the neighbors’ states. Suppose the nodes have a global scaling function $s(k)$. We still use $x_i(k)$ to denote the un-quantized state of node $i$ at time $k$, whose update will be specified at a later stage.
Encoder

Node $j \in V$ recursively generates $m$-vector quantized outputs $\{q_j(k)\}$ and $m$-vector internal states $\{b_j(k)\}$ from the exact $m$-vector state sequence $\{x_j(k)\}$ as follows for any $k \geq 1$:

$$q_j(k) \triangleq Q_K \left( \frac{1}{s(k-1)} (x_j(k) - b_j(k-1)) \right),$$

$$b_j(k) \triangleq s(k-1)q_j(k) + b_j(k-1),$$

(5)

where the initial value $b_j(0) = 0$.

Remark 1 Note that $b_j(k)$ is a one-step predictor, and the encoder is a difference encoder with a zooming-in scaling $s(k)$ that quantizes the prediction error $x_j(k) - b_j(k-1)$ rather than the state $x_j(k)$. Generally speaking, the amplitude of the prediction error is smaller than that of the state itself, so it can be represented by fewer bits. We use the scaling function $s(k)$ to zoom-in each node’s prediction error and require that $s(k)$ decay gradually to make the quantizer persistently excited, such that the nodes gradually increase the accuracy of state recovery of their neighbors. On the other hand, $s(k)$ should be large enough such that the quantizer will not be saturated, in which case the quantization error is bounded. We revisit subsequently the issue of avoidance of saturation.

Node $j \in V$ at time $k$ sends its quantized output $q_j(k)$ to its neighboring nodes $i \in N_j$, which then recovers node $j$’s state using the decoder defined as follows.

Decoder

When node $i \in N_j$ receives the quantized data $q_j(k)$ from node $j$, a decoder recursively generates an estimate $\hat{x}_{ij}(k)$ for $x_j(k)$ by the following for any $k \geq 1$:

$$\hat{x}_{ij}(k) \triangleq s(k-1)q_j(k) + \hat{x}_{ij}(k-1),$$

(6)

where the initial value $\hat{x}_{ij}(0) \triangleq 0$.

Based on the encoder-decoder pair defined in (5) and (6), motivated by (3), we now propose the following distributed linear equation solver with quantized node communication.

Algorithm 1 Distributed quantized algorithm

$$x_i(k+1) = x_i(k) + h \left[ \sum_{j \in N_i} (\hat{x}_{ij}(k) - b_i(k)) - \gamma(k) (h_i h_i^\top x_i(k) - z_i h_i) \right].$$

(7)

It is worth noting that the difference between (3) and (7) lies in the fact that the exact state $x_j(k)$ is used in (3) while $\hat{x}_{ij}(k)$ is used in (7). It is clear that Algorithm 1 merely relies on quantized node communication since $q_j(k)$ takes values in the alphabet $\{-K, \ldots, -1, 0, 1, \ldots, K\}$ only. From the second
equation of \( \{5\} \), using Equ. \( \{6\} \) and the assumed initial conditions of zero for \( \hat{x}_{ij}(0) \) and \( b_j(0) \), we have the following for any \( k \geq 0 \):

\[
\hat{x}_{ij}(k) = b_j(k), \quad \forall j \in \mathcal{V}, \forall i \in \mathcal{N}_j.
\] (8)

3 Exact Solutions

In this section, we consider Algorithm 1 and investigate the case that equation \( \{1\} \) has a unique solution. We establish the convergence results regarding the quantization levels along with the rate analysis, and demonstrate the results with numerical simulations.

3.1 Convergence Result

We impose the following assumptions.

\( \mathbf{A_1} \) There exists a unique solution \( y^* \), i.e., \( \text{rank}(H) = m \) and \( z \in \text{span}(H) \).

\( \mathbf{A_2} \) \( \max_i ||x_i(0)||_\infty \leq C_2 \) and \( \max_i ||x_i(0) - y^*||_\infty \leq C_w \) for some positive constants \( C_2 \) and \( C_w \).

\( \mathbf{A_3} \) \( \gamma(k) \equiv 1 \), and \( s(k) \equiv s(0)\alpha^k \forall k \geq 0 \) for some \( s(0) > 0 \) and \( \alpha \in (0, 1) \).

We now introduce a few useful notations as follows:

\[
H_d \triangleq \text{diag}\left\{ h_1 h_1^\top, \ldots, h_N h_N^\top \right\} \in \mathbb{R}^{mN \times mN},
\]

\[
F_d \triangleq L \otimes I_m + H_d, \quad \rho_h \triangleq 1 - h\lambda_{\text{min}}(F_d),
\] (9)

where \( \lambda_{\text{min}}(F_d) \) denotes the smallest eigenvalue of \( F_d \). Note that both the Laplacian matrix \( L \) and the matrix \( H_d \) are positive semidefinite. With the assumption \( \mathbf{A_1} \) and the condition that the undirected graph \( \mathcal{G} \) is connected, the matrix \( F_d \) turns out to be positive definite \([20]\) Lemma 9, and hence all eigenvalues of \( F_d \) are positive. The eigenvalues of \( L \) in an ascending order are denoted by \( 0 = \lambda_1(L) < \lambda_2(L) \leq \cdots \leq \lambda_N(L) \). Let \( h \in \left( 0, \frac{2}{\lambda_{\text{min}}(F_d) + \lambda_{\text{max}}(F_d)} \right) \) and \( \alpha \in (1 - h\lambda_{\text{min}}(F_d), 1) \), and set

\[
M(\alpha, h) \triangleq \frac{1 + 2hd^*}{2\alpha} + \frac{h^2 \sqrt{mN}\lambda_N(L)\lambda_{\text{max}}(F_d)}{2\alpha(\alpha - \rho_h)}, \quad \text{and} \quad K(\alpha, h) \triangleq \left[ M(\alpha, h) - \frac{1}{2} \right],
\] (10)

where \( d^* = \max_i |\mathcal{N}_i| \) denotes the degree of \( \mathcal{G} \), and \( \lambda_{\text{max}}(F_d) \) denotes the largest eigenvalue of \( F_d \).

We now begin to investigate the convergence properties of Algorithm 1 as an exact solver for the network linear equation \( \{1\} \).

Proposition 1 (Non-Saturation) Let \( \mathbf{A_1} \), \( \mathbf{A_2} \) and \( \mathbf{A_3} \) hold. Consider Algorithm 1 where

\[
h \in \left( 0, \frac{2}{\lambda_{\text{min}}(F_d) + \lambda_{\text{max}}(F_d)} \right) \quad \text{and} \quad \alpha \in (1 - h\lambda_{\text{min}}(F_d), 1).
\]

Then for any \( K \geq K(\alpha, h) \), the quantizer will never be saturated provided that \( s(0) \) satisfies

\[
s(0) > \max \left\{ \frac{C_x + h\|H_d\|_\infty C_w}{K + \frac{1}{2}}, \frac{2(\alpha - \rho_h)(\rho_h C_w + h C_x \lambda_N(L))}{\alpha h\lambda_N(L)} \right\}.
\] (11)
Proposition 1 with the proof deferred to Section 3.4.2 establishes the nonsaturation of the uniform quantizer, based on which the following theorem with the proof given in Section 3.4.3 shows the asymptotic convergence of the generated sequences to the unique exact solution.

**Theorem 1 (High Data Rate)** Suppose A1, A2 and A3 hold. With $F_d$ as defined in (9), let $h \in \left(0, \frac{2}{\lambda_{\text{min}}(F_d) + \lambda_{\text{max}}(F_d)}\right)$ and $\alpha \in (1 - h\lambda_{\text{min}}(F_d), 1)$. Then for any $K \geq K(\alpha, h)$, see (10), along Algorithm 1 there holds

$$\lim_{k \to \infty} x_i(k) = y^* \quad \forall i \in V$$

provided $s(0)$ satisfying (11). The convergence is in fact exponential with

$$\limsup_{k \to \infty} \frac{||x(k) - 1_N \otimes y^*||_2}{\alpha^k} \leq \frac{hs(0)\sqrt{mN\lambda N(L)}}{2\alpha(\alpha - \rho_h)},$$

where $x(k) = \text{col}(x_1(k), \ldots, x_N(k)) \triangleq (x_1(k)^T, \ldots, x_N(k)^T)^T$.

**Remark 2** Theorem 1 shows that by using a scaling function decaying exponentially and a uniform quantizer, Algorithm 1 can ensure asymptotic convergence to the unique solution. It is worth pointing out that for any given $\alpha, h$, the obtained quantization level $K(\alpha, h)$ is conservative, while (10) gives us some intuition on the relationship between the number of bits required and the control gains and the scaling factor. In addition, Theorem 1 gives an estimate of the rate of convergence: the smaller the scaling factor $\alpha$, the faster the convergence rate from (13) but more bits have to be communicated by (10), and, if $\alpha \to \rho_h$, the required number of bits goes to infinity. Thus, an appropriate selection of $\alpha$ amounts to a tradeoff between the rate of convergence and the communication overhead.

From (10) we know that for fixed $\alpha$, the quantization level $K(\alpha, h)$ will tend to infinity as $N \to \infty$. Since in practical applications, the communication channel usually has finite bandwidth. To satisfy this requirement, we can use a fixed number of quantization levels at the cost of slower convergence. We present the result in the following theorem, for which the proof is given in Section 3.4.4.

**Theorem 2 (Low Data Rate)** Suppose A1, A2, and A3 hold, with $F_d$ and $M(\alpha, h)$ as defined in (9) and (10). Then the following hold.

(i) For any $K \geq 1$, $\Xi_K$ is nonempty with

$$\Xi_K \triangleq \left\{ (\alpha, h) : h \in \left(0, \frac{2}{\lambda_{\text{min}}(F_d) + \lambda_{\text{max}}(F_d)}\right), \alpha \in (1 - h\lambda_{\text{min}}(F_d), 1), M(\alpha, h) < K + \frac{1}{2} \right\}. \quad (14)$$

(ii) For any $K \geq 1$, let $(\alpha, h) \in \Xi_K$ and $s(0)$ satisfy (11). Then along Algorithm 1 there holds $\lim_{k \to \infty} x_i(k) = y^*$ for $\forall i \in V$ at an exponential rate characterized by

$$\limsup_{k \to \infty} \frac{||x(k) - 1_N \otimes y^*||_2}{\alpha^k} \leq \frac{hs(0)\sqrt{mN\lambda N(L)}}{2\alpha(\alpha - \rho_h)}.$$
Remark 3 From Theorem 2 it is clear that we can always design a distributed network linear equation solver to ensure exponential convergence to exact solution with $3$ levels quantizer (namely, $K = 1$), under which each node sends merely $m$ bits of information (minimum number of bits) to its neighbors at each step.

From definition (14) it is seen that the set $\Xi_K$ is defined by three nonlinear inequalities, for which an explicit solution of these inequalities might be difficult to obtain. Then in the following proposition with the proof given in Section 3.4.5, we give an explicit method for choosing parameters $(\alpha, h)$ from $\Xi_K$ for any given $K \geq 1$ by introducing a free parameter $\epsilon \in (0, 1)$.

Proposition 2 For any given $K \geq 1$ and $\epsilon \in (0, 1)$, define $\Xi_{K, \epsilon} \equiv \{ (\alpha, h) : \alpha = 1 - (1 - \epsilon)h\lambda_{\text{min}}(F_d), h \in (0, \hat{h}_{K, \epsilon}^*) \}$, with

$$
\hat{h}_{K, \epsilon}^* \equiv 2K\epsilon\lambda_{\text{min}}(F_d)\left(\sqrt{mN\lambda_N(L)\lambda_{\text{max}}(F_d)} + 2\epsilon\lambda_{\text{min}}(F_d)^d + \epsilon(1 - \epsilon)(2K + 1)\lambda_{\text{min}}^2(F_d)\right)^{-1}.
$$

Then we have that $\Xi_K = \bigcup_{\epsilon \in (0, 1)} \Xi_{K, \epsilon}$.

We note from Theorem 1 that the proposed distributed protocol ensures exponential convergence with parameter $\alpha$, which is coupled with another algorithm parameter $h$ while without explicit dependence on the linear equations and the network. In the following, we investigate the asymptotic property of $\alpha$ as $N \to \infty$, and give a very compendious expression for the asymptotic value of $\alpha$. The proof can be found in Section 3.4.6.

Theorem 3 (Network Scalability) Adopt the same hypothesis as Theorem 2. Let $K \geq 1$ and $(\alpha, h) \in \Xi_K$. Then

$$
\lim_{N \to \infty} \inf_{(\alpha, h) \in \Xi_K} \alpha \exp\left(-\frac{K\lambda_{\text{min}}^2(F_d)}{2\sqrt{mN\lambda_N(L)\lambda_{\text{max}}(F_d)}}\right) = 1.
$$

Remark 4 Theorem 3 together with Eq. (13) suggests that for large network, the highest possible rate of convergence tends to scale according to $O\left(\exp\left(-k\Theta_N K\right)\right)$, where

$$
\Theta_N = \frac{\lambda_{\text{min}}^2(F_d)}{2\sqrt{mN\lambda_N(L)\lambda_{\text{max}}(F_d)}}
$$

is some constant relying only on the number of nodes, the network structure and the equations.
3.2 Numerical Examples

Example 1. Let the linear equation (1) be given by

\[
H = \begin{pmatrix}
0.5 & -0.1 \\
-0.4 & 0.2 \\
0.3 & -0.7 \\
0.6 & 0.3 \\
-0.3 & 0.5
\end{pmatrix}, \quad z = \begin{pmatrix}
0.2 \\
0.2 \\
-1.8 \\
1.5 \\
1.2
\end{pmatrix}
\]

(17)

which yields a unique exact solution

\[
y^* = \begin{pmatrix}
1 \\
3
\end{pmatrix}.
\]

The network structure is shown in Figure 1.

Figure 1: Communication graph.

[Validation of Theorem 1] Let \( h = \frac{0.98}{\lambda_{\min}(\mathbf{F}_d)^++\lambda_{\max}(\mathbf{F}_d)} = 0.4215 \). Here one can compute \( \rho_h = 0.9554 \).

Set \( \alpha = 0.98 \) so that \( K(\alpha, h) = 225 \). Let \( K \) be 100, 300, 1000, respectively. We set \( s(0) = 1 \) and implement Algorithm 1. Figure 2 displays the trajectories of \( ||x(k) - 1_N \otimes y^*||_2 \) along with the theoretical upper bound \( B(k) = \frac{hs(0)\alpha^k\sqrt{mN\lambda_N(L)}}{2\alpha(\alpha-\rho_h)} \) given by (13). The trajectory with \( K = 300 \) verifies that Theorem 1 provides a sufficient condition on the data rate to ensure convergence, while the trajectories for \( K = 100 \) and \( K = 1000 \) coincide with that of \( K = 300 \). Therefore, it implies that (i) with the same algorithm parameters \( h, \alpha \), a higher data rate \( (K = 1000) \) cannot guarantee a faster convergence rate; (ii) there is some degree of conservativeness in the sufficient condition of Theorem 1.

[Validation of Theorem 2] Let \( K \) be \( K_1 = 3 \), \( K_2 = 6 \) and \( K_3 = 12 \), respectively. We choose \( (\alpha, h) \in \Xi_K \) with Proposition 2. Set \( \epsilon = 0.5 \), and we then choose \( (\alpha_1, h_1) = (0.9998, 0.0038) \in \Xi_{K_{1.0.5}} \), \( (\alpha_1, h_1) = (0.9996, 0.0077) \in \Xi_{K_{2.0.5}} \), and \( (\alpha_3, h_3) = (0.9992, 0.0154) \in \Xi_{K_{3.0.5}} \). We set \( s_1(0) = 1500 \), \( s_2(0) = 1200 \), \( s_3(0) = 1000 \) for \( K_1, K_2, K_3 \), respectively, to ensure (11). The trajectories of \( ||x(k) - 1_N \otimes y^*||_2 \) under the three sets of parameters are shown in Figure 3, which demonstrates the convergence of Algorithm 1 to the exact solution. A higher data rate allows us to choose a larger \( h \) and a smaller \( \alpha \), and therefore, leads to a faster convergence rate. Figure 3 is also consistent with the upper bound of convergence rate \( B(k) = \frac{hs(0)\alpha^k\sqrt{mN\lambda_N(L)}}{2\alpha(\alpha-\rho_h)} \) given by (13) in all three parameter settings.

Example 2. [Validation of Theorem 3]. We let \( N = 100 \) and \( m = 5 \). We randomly generate a matrix \( \mathbf{H} \) and \( \mathbf{z} \) such that \( \mathbf{z} = \mathbf{H}\mathbf{y} \) has a unique solution. We set \( \mathbf{L} \) as the Laplacian of a cycle graph. Then
data rate, and implies that a higher data allows the selection of a smaller \( \lambda \), i.e., \( \alpha_K^* = 100 \) and \( \Theta_N \) for Erdős-Rényi random graphs \( \mathcal{G}(N, p) \), where the possible connection between any two nodes is generated with a probability of \( p \), independently of every other edge. We let \( p \) increase from 0.1 to 0.9 in steps of 0.05. For each probability \( p \), we randomly generate \( 10^3 \) connected graphs with \( \mathcal{G}(N, p) \), and compute the mean \( \bar{\Theta}_N \). Figure 5 shows how \( \bar{\Theta}_N \) varies along with probability \( p \), which decreases as the connection probability \( p \) increases. This implies that \( \alpha_K^* \).
the fastest possible convergence rate under a fixed data rate $K$, might increase with the increase of the connectivity of the graphs.

### 3.3 Discussion: Improve Robustness with Damping

Convergence of Algorithm 1 relies on the equivalence between node $i$’s decoder output $\hat{x}_{ij}(k)$ of its neighbor $j$’s state and node $j$’s one-step prediction $b_j(k)$, which is characterized by (8). The theoretical and numerical results have shown the effectiveness of Algorithm 1 when (8) is satisfied. In fact, (8) holds when the encoder/decoder update (5)-(6) is exact and the following initialization condition is satisfied,

$$b_j(0) = 0, \hat{x}_{ij}(0) = 0, \forall j \in V, \forall i \in N_j,$$  \hspace{1cm} (18)

However, there could exist initialization errors in (18). And due to round-off noises in the storage and manipulation of real-valued vectors in digital computers, (5)-(6) may not be executed exactly. With initialization errors in (18) and the round-off noises, the update of $b_j(k), \hat{x}_{ij}(k)$ in encoder/decoder (5)-(6) is changed to

$$b_j(k) = I_j^e, \hat{x}_{ij}(0) = I_{ij}^e, \forall j \in V, \forall i \in N_j,$$

$$b_j(k) \triangleq s(k-1)q_j(k) + b_j(k-1) + e_j^b(k),$$

$$\hat{x}_{ij}(k) \triangleq s(k-1)q_{ij}(k) + \hat{x}_{ij}(k-1) + e_{ij}^b(k).$$ \hspace{1cm} (19)

The initialization errors $I_j^e, I_{ij}^e$, and round-off noises $e_j^b(k), e_{ij}^b(k)$ will persist during the algorithm.

**Performance of Algorithm 1 with initialization errors and round-off noises.** We continue to use the same $H$ and $z$ in (17). We set $h = 0.0213, \alpha = 0.998, K = 300$ and $s(0) = 10$. The initialization errors $I_j^e$ and $I_{ij}^e$ are independent and are randomly drawn from a uniform distribution on $[0,0.5]$, and the round-off noises $e_j^b(k), e_{ij}^b(k)$ are mutually independent random i.i.d. sequences with each value drawn from a uniform distribution on $[-1,1] \times 10^{-4}$. Figure 6 shows that Algorithm 1 with (19) cannot ensure convergence when there exists initialization errors or round-off noises. In fact, the error is very substantial in comparison to the average noise magnitude and the value of $\|1_N \otimes y^*\|_2$.

We propose to improve algorithm robustness by adding a damping term to encoder/decoder, where $b_j(k)$ and $\hat{x}_{ij}(k)$ are updated with

$$b_j(0) = I_j^e, \hat{x}_{ij}(0) = I_{ij}^e, \forall j \in V, \forall i \in N_j,$$

$$b_j(k) \triangleq s(k-1)q_j(k) + \phi b_j(k-1) + e_j^b(k),$$

$$\hat{x}_{ij}(k) \triangleq s(k-1)q_{ij}(k) + \phi \hat{x}_{ij}(k-1) + e_{ij}^b(k),$$ \hspace{1cm} (20)

where $\phi \in (0,1)$ is a damping factor, $I_j^e, I_{ij}^e$ are initialization errors, and $e_j^b(k), e_{ij}^b(k)$ are round-off noises.

Now, we adopt the same setting as Example 1. We run Algorithm 1 with (20) when there are initialization errors and round-off noises, and also run Algorithm 1 with (5)-(6) where there are no initialization errors and round-off noises, both with the same algorithm parameters. The damping factor
Figure 6: Trajectories of $||x - 1_N \otimes y^*||_2$ generated by Algorithm 1 for the following cases: (i) there exist initialization errors; (ii) there exist round-off noises; (iii) there exist both initialization errors and round-off noises.

is $\rho = 0.95$. Figure 7 displays the simulation results, which shows that (i) the damping can significantly reduce but not fully eliminate the effect of initialization errors in the final computed output (ii) the effect of round-off noises can be tolerated in the sense that $x_i(k)$ will converge to a neighborhood of the exact solution within a distance of similar magnitude to the round-off noises.

Figure 7: Trajectories of $||x - 1_N \otimes y^*||_2$ for (i) Algorithm 1 with a damped encoder/decoder (20) with initialization errors; (ii) Algorithm 1 with a damped encoder/decoder (20) with round-off noises; (iii) Algorithm 1 with a damped encoder/decoder (20) with both initialization errors and round-off noises; (iv) Algorithm 1 without errors or noises.

The formal convergence analysis of Algorithm 1 with a damped encoder/decoder update (20) is challenging because there will be a nonlinear coupling between the damping factor $\rho$ and all other parameters, and the errors and noises as well as $\rho$ will enter the update equation of $x(k)$ and $b(k)$ in (25)-(26). Thereby, we leave the theoretical analysis of (20) as a future research problem.

3.4 Proof of Statements

3.4.1 Preliminary Lemmas

We first give a reformulation of the recursion for Algorithm 1
Lemma 1 Let A1 and A3 hold. Define
\[
\mathbf{w}(k) = \mathbf{1}_N(k) - \mathbf{y}^*, \quad \mathbf{e}(k) = \mathbf{1}_N(k) - \mathbf{b}(k), \quad \mathbf{e}(k) = \{\mathbf{e}_1(k), \ldots, \mathbf{e}_N(k)\}, \quad \mathbf{w}(k) = \frac{\mathbf{w}(k)}{s(k)}, \quad \mathbf{e}(k) = \frac{\mathbf{e}(k)}{s(k)}.
\]
Then the following hold:
\[
\begin{align*}
\mathbf{w}(k + 1) &= \mathbf{P}_h \mathbf{w}(k) + \mathbf{L}, \\
\mathbf{e}(k + 1) &= \mathbf{P}_h \mathbf{e}(k) + \mathbf{L},
\end{align*}
\]
where \( \mathbf{P}_h \equiv \mathbf{I}_N - h \mathbf{F}_d \) with \( \mathbf{F}_d = \mathbf{L} \otimes \mathbf{I}_m + \mathbf{H}_d \), and \( \mathbf{e}(k) \) is defined as
\[
\mathbf{e}(k) = \mathbf{1}_N + \mathbf{h} \mathbf{L} \otimes \mathbf{I}_m \mathbf{e}(k) - h \mathbf{F}_d \mathbf{w}(k).
\]

Proof. Since \( j \in \mathcal{N}_i \Leftrightarrow i \in \mathcal{N}_j \), by using (23) and \( \mathbf{e}_i(k) = \mathbf{x}_i(k) - \mathbf{b}_i(k) \), we have the following:
\[
\begin{align*}
\sum_{j \in \mathcal{N}_i} (\mathbf{x}_{ij}(k) - \mathbf{b}_i(k)) &= \sum_{j \in \mathcal{N}_i} (\mathbf{b}_j(k) - \mathbf{b}_i(k)) \\
&= \sum_{j \in \mathcal{N}_i} \left[ (\mathbf{x}_j(k) - \mathbf{x}_i(k)) - (\mathbf{x}_j(k) - \mathbf{b}_j(k)) + (\mathbf{x}_i(k) - \mathbf{b}_i(k)) \right] \\
&= \sum_{j \in \mathcal{N}_i} \left[ (\mathbf{x}_j(k) - \mathbf{x}_i(k)) - (\mathbf{e}_j(k) - \mathbf{e}_i(k)) \right].
\end{align*}
\]
Recall that \( \mathbf{y}^* \) is the unique solution to (11) such that \( \mathbf{h}_i^T \mathbf{y}^* = \mathbf{z}_i \forall i \in \mathcal{V} \). Then by \( \mathbf{w}_i(k) = \mathbf{x}_i(k) - \mathbf{y}^* \), there holds
\[
\mathbf{h}_i \mathbf{h}_i^T \mathbf{w}_i(k) = \mathbf{h}_i \mathbf{h}_i^T (\mathbf{x}_i(k) - \mathbf{y}^*) = \mathbf{h}_i \mathbf{h}_i^T \mathbf{x}_i(k) - \mathbf{h}_i \mathbf{z}_i.
\]
Also, using (23), (7), (24), \( \gamma(k) \equiv 1 \), and the definition of \( \mathbf{H}_d \) in (9), leads to
\[
\begin{align*}
\mathbf{x}(k + 1) &= \mathbf{x}(k) - \mathbf{h} \mathbf{L} \otimes \mathbf{I}_m \mathbf{x}(k) + \mathbf{h} \mathbf{L} \otimes \mathbf{I}_m \mathbf{e}(k) - \mathbf{h} \mathbf{F}_d \mathbf{w}(k), \\
\mathbf{b}(k + 1) &= s(k) \mathbf{Q}_k \left( \frac{\mathbf{x}(k + 1) - \mathbf{b}(k)}{s} \right) + \mathbf{b}(k).
\end{align*}
\]
Because \( \mathbf{L} \mathbf{1}_N = \mathbf{0}_N \), the following holds:
\[
\begin{align*}
\mathbf{L} \otimes \mathbf{I}_m \mathbf{x}(k) &= \mathbf{L} \otimes \mathbf{I}_m \mathbf{x}(k) - (\mathbf{L} \mathbf{1}_N \otimes \mathbf{I}_m) \mathbf{y}^* \\
&= \mathbf{L} \otimes \mathbf{I}_m (\mathbf{x}(k) - \mathbf{1}_N \otimes \mathbf{I}_m) \mathbf{y}^* = \mathbf{L} \otimes \mathbf{I}_m \mathbf{w}(k).
\end{align*}
\]
Now, by subtracting \( \mathbf{1}_N \otimes \mathbf{I}_m \mathbf{y}^* \) from both sides of (24) and by substituting (27), using \( \mathbf{w}(k) = \mathbf{x}(k) - \mathbf{1}_N \otimes \mathbf{I}_m \mathbf{y}^* \) and \( \mathbf{F}_d = \mathbf{L} \otimes \mathbf{I}_m + \mathbf{H}_d \), we obtain that
\[
\mathbf{w}(k + 1) = (\mathbf{I}_N - h \mathbf{F}_d) \mathbf{w}(k) + h \mathbf{L} \otimes \mathbf{I}_m \mathbf{e}(k).
\]
Dividing both sides of the above equation by \(s(k + 1)\), using \(s(k + 1) = \alpha s(k)\) and definitions of \(P_h, \omega(k)\) and \(\varepsilon(k)\), we obtain (21).

By subtracting \(b(k)\) from both sides of (25), using (27) and \(e(k) = x(k) - b(k)\), we obtain that

\[
x(k + 1) - b(k) = (I_{mN} + hL \otimes I_m) e(k) - h (L \otimes I_m + H_d) w(k).
\]

Then by using \(w(k) = s(k) \omega(k)\), \(e(k) = s(k) \varepsilon(k)\), \(F_d = L \otimes I_m + H_d\), and (23), we obtain that

\[
x(k + 1) - b(k) = s(k) \theta(k).
\]

Now recalling that \(e(k + 1) = x(k + 1) - b(k + 1)\) together with (26), the following holds:

\[
e(k + 1) = x(k + 1) - b(k) - s(k) Q_K \left( \frac{x(k + 1) - b(k)}{s(k)} \right) = s(k) (\theta(k) - Q_K(\theta(k))).
\]

Dividing both sides of the above equation by \(s(k + 1)\) and using \(s(k + 1) = \alpha s(k)\), we obtain (22).

\[\square\]

3.4.2 Proof of Proposition 1

The proof of non-saturation of the uniform quantizer is equivalent to showing that for any \(k \geq 0\), \(\theta(k)\) defined by (23) satisfies

\[
\|\theta(k)\|_\infty < K + \frac{1}{2}.
\]

The proof of Proposition 1 will use induction, and we begin by showing the quantizer is not saturated at \(k = 0\).

By using \(b_i(0) = 0 \forall i \in V\), we obtain that \(e(0) = x(0)\) and \(\varepsilon(0) = x(0)/s(0)\). Then by A2, we have

\[
\|\varepsilon(0)\|_\infty = \frac{\|x(0)\|_\infty}{s(0)} \leq C_x/s(0).
\]

(29)

By (27), and by recalling that \(\omega(0) = w(0)/s(0)\) and \(\varepsilon(0) = x(0)/s(0)\), we obtain that

\[
L \otimes I_m \omega(0) = L \otimes I_m x(0)/s(0) = L \otimes I_m \varepsilon(0).
\]

Then by definition (23) there holds

\[
\theta(0) = (I_{mN} + hL \otimes I_m) \varepsilon(0) - h (L \otimes I_m + H_d) \omega(0) = \varepsilon(0) - hH_d w(0)/s(0).
\]

As a result, by A2, (11) and (29) we have the following:

\[
\|\theta(0)\|_\infty \leq \|\varepsilon(0)\|_\infty + h \|H_d w(0)\|_\infty /s(0) \leq (C_x + h \|H_d\|_\infty C_w) /s(0) < K + \frac{1}{2}.
\]

Hence, when \(k = 0\), the quantizer is unsaturated. Now for the induction, we assume that when \(k = 0, \ldots, p\), the quantizer is not saturated. Then by (22) we have that

\[
\sup_{1 \leq k \leq p+1} \|\varepsilon(k)\|_\infty \leq \frac{1}{2\alpha}.
\]

(30)

We proceed to show that the quantizer is unsaturated for \(k = p + 1\).
From (21) it follows that

$$\omega(p + 1) = (\alpha^{-1}P_h)^{p+1}\omega(0) + \alpha^{-1}h\langle \alpha^{-1}P_h \rangle^p L \otimes I_m \varepsilon(0) + \alpha^{-1}h \sum_{i=0}^{p-1} \langle \alpha^{-1}P_h \rangle^i L \otimes I_m \varepsilon(p - i).$$  \hspace*{1cm} (31)

We now estimate the three terms on the right-hand side of the above equation separately. Note that any given \(h > 0\), the eigenvalues of \(P_h = I_{mN} - hF_d\) are sorted in an ascending order as \(1 - h\lambda_{\max}(F_d) \leq \cdots \leq 1 - h\lambda_{\min}(F_d)\), and there exists a unitary matrix \(U\) such that \(U^T P_h U = \text{diag}\{1 - h\lambda_{\max}(F_d), \ldots, 1 - h\lambda_{\min}(F_d)\} \equiv A\). Therefore,

\[
(P_h)^k = (UAU^T)^k = UA^kU^T.
\]  \hspace*{1cm} (32)

By using the definition of \(A\), we obtain that

$$\|A\|_2 = \max \{ |1 - h\lambda_{\min}(F_d)|, |1 - h\lambda_{\max}(F_d)| \}.$$  

Thus, by using \(h \in (0, \frac{2}{\lambda_{\min}(F_d) + \lambda_{\max}(F_d)})\) and Lemma 3.1, there holds \(\|A\|_2 = 1 - h\lambda_{\min}(F_d) = \rho_h\). For the first term, using (32), \(\|U\|_2 = 1\) and \(\|x\|_\infty \leq \|x\|_2 \leq \sqrt{m}\|x\|_\infty\) for any \(x \in \mathbb{R}^m\), we have

\[
\| (\alpha^{-1}P_h)^{p+1}\omega(0) \|_2 \leq \left\| U (A/\alpha)^{p+1} U^T \right\|_2 \|\omega(0)\|_2 \\
\leq \left( \frac{\rho_h}{\alpha} \right)^{p+1} \frac{\|w(0)\|_2}{s(0)} \leq \left( \frac{\rho_h}{\alpha} \right)^{p+1} \frac{\sqrt{mN}\|w(0)\|_\infty}{s(0)} < \frac{\sqrt{mN}C_w}{s(0)} \left( \frac{\rho_h}{\alpha} \right)^{p+1} \hspace{1cm} \text{(by A2)},
\]  \hspace*{1cm} (33)

For the second term of (31), using (32), (29), and \(\|L\|_2 = \lambda_N(L)\) we obtain the following:

\[
\|\alpha^{-1}h\langle \alpha^{-1}P_h \rangle^p L \otimes I_m \varepsilon(0)\|_2 \\
\leq \alpha^{-1}h \left\| U (A/\alpha)^p U^T \right\|_2 \|L\|_2 \|\varepsilon(0)\|_2 \leq \frac{hC_x\sqrt{mN}}{\alpha s(0)} \lambda_N(L) \left( \frac{\rho_h}{\alpha} \right)^p \hspace{1cm} \text{(34)}
\]

Similarly, for the last term of (31), by

\[
\left\| \sum_{i=0}^{p-1} \langle \alpha^{-1}P_h \rangle^i L \otimes I_m \varepsilon(p - i) \right\|_2 \\
\leq \sum_{i=0}^{p-1} \left\| \langle \alpha^{-1}P_h \rangle^i L \otimes I_m \varepsilon(p - i) \right\|_2 \\
\leq \sum_{i=0}^{p-1} \left( \frac{\rho_h}{\alpha} \right)^i = \frac{1 - (\rho_h/\alpha)^p}{1 - \rho_h/\alpha},
\]

and by (30) we have that

\[
\left\| \alpha^{-1}h \sum_{i=0}^{p-1} \langle \alpha^{-1}P_h \rangle^i L \otimes I_m \varepsilon(p - i) \right\|_2 \\
\leq \frac{\sqrt{mN}}{\alpha} \|L\|_2 \sup_{1 \leq k \leq p+1} \|\varepsilon(k)\|_\infty \| \sum_{i=0}^{p-1} \langle \alpha^{-1}P_h \rangle^i \|_2 \leq \frac{\sqrt{mN}\lambda_N(L)}{2(\alpha - \rho_h)} \left( \frac{\rho_h}{\alpha} \right)^p.
\]  \hspace*{1cm} (35)

Since \(\alpha \in (\rho_h, 1)\), by using (31) and (33)-(35), we have that

\[
\|\omega(p + 1)\|_\infty \leq \|\omega(p + 1)\|_2 \leq \frac{\sqrt{mN}}{\alpha} \max \left\{ \frac{\rho_h C_w + hC_x \lambda_N(L)}{s(0)}, \frac{h\lambda_N(L)}{2(\alpha - \rho_h)} \right\} \leq \frac{h\sqrt{mN}\lambda_N(L)}{2(\alpha - \rho_h)},
\]  \hspace*{1cm} (36)

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where the last inequality follows by (11). This together with \( \|x\|_\infty \leq \|x\|_2 \), (10), (23), and (30) leads to

\[
\|\theta(p + 1)\|_\infty \leq \| (I_{mN} + hL \otimes I_m) \varepsilon (p + 1)\|_\infty + \| hF_d \omega (p + 1)\|_\infty
\]

\[
\leq \|I_{mN} + hL \otimes I_m\|_\infty \|\varepsilon (p + 1)\|_\infty + h\|F_d\|_2 \|\omega (p + 1)\|_2
\]

\[
\leq \frac{1 + 2hd^*}{2\alpha} + \frac{h^2 \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)}{2\alpha(\alpha - \rho_h)} = M(\alpha, h) \leq K(\alpha, h) + \frac{1}{2} \leq K + \frac{1}{2}.
\]

As a result, when \( k = p + 1 \), the quantizer is also unsaturated. Therefore, by induction, we conclude that if a \((2K + 1)\)-levels uniform quantizer is applied, then the quantizer will never be saturated. ■

3.4.3 Proof of Theorem 1

Since the conditions required by Proposition 1 are the same as those used in Theorem 1, the quantizer will never be saturated by Proposition 1. Then by (22) we conclude that \( \sup_{k \geq 1} \|\varepsilon(k)\|_\infty \leq 1/2\alpha \), and hence (30) holds for any \( p \geq 0 \). Thus,

\[
\limsup_{k \to \infty} \|\omega(k)\|_2 \leq \frac{h\sqrt{mN} \lambda_N(L)}{2\alpha(\alpha - \rho_h)}.
\]

Then by using \( w(k) = s(0)\alpha^k \omega(k) \) and \( w(k) = x(k) - 1_N \otimes y^* \), we obtain (13) and (12). ■

3.4.4 Proof of Theorem 2

(i) By using \( \rho_h = 1 - h\lambda_{\min}(F_d) \) and \( M(\alpha, h) \) defined in (10), there holds:

\[
M(\alpha, h) = \frac{1 + 2hd^*}{2\alpha} + \frac{h^2 \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)}{2\alpha(\alpha - (1 - h\lambda_{\min}(F_d)))}.
\]

(37)

Noting that

\[
\lim_{h \to 0} \frac{1 + 2hd^*}{2} + \frac{h\sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)}{2\lambda_{\min}(F_d)} = \frac{1}{2},
\]

then for any given \( K \geq 1 \) there exists \( h^* \in \left(0, \frac{2}{\lambda_{\min}(F_d) + \lambda_{\max}(F_d)}\right) \) such that

\[
\frac{1 + 2h^*d^*}{2} + \frac{h^* \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)}{2\lambda_{\min}(F_d)} \leq K.
\]

By (37) it follows that

\[
\lim_{\alpha \to 1} M(\alpha, h^*) = \frac{1 + 2h^*d^*}{2} + \frac{h^* \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)}{2\lambda_{\min}(F_d)} \leq K,
\]

and hence there exists \( \alpha^* \in (1 - h^*\lambda_{\min}(F_d), 1) \) such that \( M(\alpha^*, h^*) < K + \frac{1}{2} \). Thus, \((\alpha^*, h^*) \in \Xi_K \), and hence \( \Xi_K \) is nonempty.

(ii) For any \((\alpha, h) \in \Xi_K \), from (14) it follows that \( h \in \left(0, \frac{2}{\lambda_{\min}(F_d) + \lambda_{\max}(F_d)}\right) \), \( \alpha \in (1 - h\lambda_{\min}(F_d), 1) \), and \( M(\alpha, h) < K + \frac{1}{2} \). Then \( K(\alpha, h) = \left[ M(\alpha, h - \frac{1}{2}) \right] \leq K \) together with Theorem 1 leads to the result (ii). ■
3.4.5 Proof of Proposition 2

We first prove \( \bigcup_{\epsilon \in (0,1)} \Xi_{K,\epsilon} \subset \Xi_K \). For any given \( K \geq 1 \) and \( \epsilon \in (0,1) \), let \( (\alpha, h) \in \Xi_{K,\epsilon} \). Then \( \alpha - \rho_h = c\epsilon h \lambda_{\min}(F_d) > 0 \) by \( \rho_h = 1 - h \lambda_{\min}(F_d) \), and \( \alpha \in (1 - h \lambda_{\min}(F_d), 1) \). Also, by the definition \( M(\alpha, h) \) in (10), we obtain the following:

\[
S(\epsilon, h) \triangleq M(\alpha, h) = \frac{1 + 2hd^*}{2(1 - (1 - \epsilon)h \lambda_{\min}(F_d))} + \frac{h \sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)}{2\epsilon \lambda_{\min}(F_d)(1 - (1 - \epsilon)h \lambda_{\min}(F_d))}
\]

(38)

Then by using the definition of \( \hat{h}_{K,\epsilon} \) in (15) and \( h < \hat{h}_{K,\epsilon} \), there holds \( M(\alpha, h) < K + \frac{1}{2} \). It is clear that for any \( (\alpha, h) \in \Xi_{K,\epsilon}, h \in \left(0,\frac{\lambda_{\min}(F_d)^2}{2\lambda_{\max}(F_d)}\right) \). In summary, we have shown that for any given \( K \geq 1 \) and \( \epsilon \in (0,1) \), \( \Xi_{K,\epsilon} \subset \Xi_K \). Thus, \( \bigcup_{\epsilon \in (0,1)} \Xi_{K,\epsilon} \subset \Xi_K \).

We now validate \( \Xi_K \subset \bigcup_{\epsilon \in (0,1)} \Xi_{K,\epsilon} \). For any \( (\alpha_0, h_0) \in \Xi_K, \) by (14) we have \( h_0 \in \left(0,\frac{\lambda_{\min}(F_d)^2}{2\lambda_{\max}(F_d)}\right) \) and \( \rho_{h_0} = 1 - h_0 \lambda_{\min}(F_d) \). Note that \( \alpha_0 = 1 - (1 - \epsilon_0)h_0 \lambda_{\min}(F_d) \) with \( \epsilon_0 = 1 - \frac{1 - \alpha_0}{h_0 \lambda_{\min}(F_d)} \). Then \( \epsilon_0 \in (0,1) \) and \( M(\alpha_0, h_0) = S(\epsilon_0, h_0) \), where \( S(\epsilon_0, h_0) \) is given by (38) with \( (\epsilon, h) \) replaced by \( (\epsilon_0, h_0) \).

This together with \( M(\alpha_0, h_0) < K + \frac{1}{2} \) leads to \( S(\epsilon_0, h_0) < K + \frac{1}{2} \). This is equivalent to

\[ h_0 < \hat{h}_{K,\epsilon_0} = 2K\epsilon_0 \lambda_{\min}(F_d) \left(\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)\right) \]

\[ + 2\epsilon_0 \lambda_{\min}(F_d)d^* + \epsilon_0(1 - \epsilon_0)(2K + 1)\lambda_{\min}(F_d)^2 \]

Then by \( h_0 \in \left(0,\frac{\lambda_{\min}(F_d)^2}{2\lambda_{\max}(F_d)}\right) \), we conclude that \( h_0 \in (0, h_{K,\epsilon_0}) \). This together with \( \epsilon_0 \in (0,1) \) implies that \( (\alpha_0, h_0) \in \Xi_{K,\epsilon_0} \), and hence \( \Xi_K \subset \bigcup_{\epsilon \in (0,1)} \Xi_{K,\epsilon} \). This completes the proof of Lemma 2.

3.4.6 Proof of Theorem 3

For any given \( K \geq 1 \), define

\[
\Gamma_K \triangleq \{ \alpha : \alpha = 1 - (1 - \epsilon)h \lambda_{\min}(F_d), \epsilon \in (0,1), h \in (0, h_{K,\epsilon}^*) \}. \]

(39)

By \( h_{K,\epsilon}^* \leq \hat{h}_{K,\epsilon} \) and (15), we know for any \( h \in (0, h_{K,\epsilon}^*), h \leq 2K\epsilon \lambda_{\min}(F_d) \left(\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)\right)^{-1} \).

Then for any \( \alpha \in \Gamma_K \) with \( \epsilon \in (0,1), h \in (0, h_{K,\epsilon}^*), k \), from \( (1 - \epsilon)\epsilon \leq \frac{1}{4} \) \( \forall \epsilon \in (0,1) \) it follows that

\[
\alpha = 1 - (1 - \epsilon)h \lambda_{\min}(F_d) > 1 - \frac{2K(1 - \epsilon)\lambda_{\min}^2(F_d)}{\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)} \geq 1 - \frac{K\lambda_{\max}^2(F_d)}{2\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)},
\]

Thus, the following holds for fixed \( K \):

\[
\inf_{\alpha \in \Gamma_K} \alpha \geq \frac{1 - \frac{K\lambda_{\min}^2(F_d)}{2\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)}}{\exp\left(-\frac{K\lambda_{\min}^2(F_d)}{2\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)}\right)}
\]

\[
\geq \frac{1 - \frac{K\lambda_{\max}^2(F_d)}{2\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)}}{\exp\left(-\frac{K\lambda_{\max}^2(F_d)}{2\sqrt{mN} \lambda_N(L)\lambda_{\max}(F_d)}\right)}
\]

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which together with \( \lim_{x \to 0} \frac{1-x}{\exp(-x)} = 0 \) produces
\[
\lim_{N \to \infty} \inf_{\alpha \in \Gamma_K} \frac{\exp \left( - \frac{K \lambda_{\min}^2(F_d)}{2 \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)} \right)}{\exp \left( - \frac{K \lambda_{\min}^2(F_d)}{2 \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)} \right)} \geq 1. \tag{40}
\]

From \( \text{(13)} \) it follows that
\[
\hat{h}_{K,\epsilon} \geq 2K \epsilon \lambda_{\min}(F_d) \left( \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d) + 2\lambda_{\min}(F_d) d^* + (2K + 1) \lambda_{\min}^2(F_d) \right)^{-1}.
\]

This together with
\[
\inf_{h \in (0, \hat{h}_{K,\epsilon})} \alpha \leq 1 - (1 - \epsilon) \hat{h}_{K,\epsilon} \lambda_{\min}(F_d)
\]
implies
\[
\inf_{\alpha \in \Gamma_K} \alpha \leq 1 - \max_{\epsilon \in (0, 1)} \frac{2\epsilon(1 - \epsilon)K \lambda_{\min}^2(F_d)}{2\lambda_{\min}(F_d) d^* + \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d) + (2K + 1) \lambda_{\min}^2(F_d)} = 1 - \frac{K \lambda_{\min}^2(F_d)}{2\lambda_{\min}(F_d) d^* + \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d) + (2K + 1) \lambda_{\min}^2(F_d)}.
\]

Then we have that for any given \( K \geq 1, \)
\[
\inf_{\alpha \in \Gamma_K} \alpha \leq 1 - \frac{K \lambda_{\min}^2(F_d)}{2\lambda_{\min}(F_d) d^* + \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d) + (2K + 1) \lambda_{\min}^2(F_d)} \leq 1 - \frac{K \lambda_{\min}^2(F_d)}{2\lambda_{\min}(F_d) d^* + \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d) + (2K + 1) \lambda_{\min}^2(F_d)}.
\]

which together with \( \lim_{N \to \infty} \frac{1-c_1/(\sqrt{N}+c_2)}{1-c_1/\sqrt{N}} = 1 \) and \( \lim_{x \to 0} \frac{1-x}{\exp(-x)} = 0 \) gives
\[
\limsup_{N \to \infty} \frac{\inf_{\alpha \in \Gamma_K} \alpha}{\exp \left( - \frac{K \lambda_{\min}^2(F_d)}{2 \sqrt{mN} \lambda_N(L) \lambda_{\max}(F_d)} \right)} \leq 1. \tag{41}
\]

Using Lemma 2 and (39), we have the following:
\[
\inf_{(\alpha, h) \in \Xi_K} = \inf_{(\alpha, h) \in \cup_{\epsilon \in (0,1)} \Xi_{K,\epsilon}} = \inf_{\epsilon \in (0,1)} \inf_{(\alpha, h) \in \Xi_{K,\epsilon}} = \inf_{\alpha \in \Gamma_K}.
\]

By this, using (40) and (41), we obtain (16).

\section{Least-Squares Solver}

In this section, we investigate the case \( \text{rank}(H) = m \) and \( z \not\in \text{span}(H) \). Then equation (11) does not have exact solutions, while a least-squares solution is defined as the solution to the optimization problem (2). We consider Algorithm 1 then show the convergence results regarding the quantization level along with the data rate analysis, and demonstrate the results with numerical simulations.
4.1 Convergence Results

Assumptions A4, A5 and A6 are no longer in force, instead, we impose the following conditions on the initial states and step-size.

A4 rank(H) = m and z /∈ span(H).

A5 \( \max_k \| x_k \|_\infty \leq C_x \) for constant \( C_x > 0 \).

A6 (i) \( \gamma(0) = 1, \gamma(k) ↓ 0, \sum_{k=1}^{\infty} \gamma(k) = \infty \), (ii) \( s(k) = s_r \gamma(k) \) for some \( s_r > 0 \), and (iii) \( 1 < \beta(k+1) < \beta(k) \) for any \( k \geq 0 \), where \( \beta(k) \triangleq \frac{\gamma(k)}{\gamma(k+1)} \).

Remark 5 We now specify how to choose \( \gamma(k) \) to make A6 hold. Set \( \gamma(k) = \frac{k^d}{(k+k_0)^{d}} \) for some \( d \in (\frac{1}{2}, 1] \), where \( k_0 = \frac{1}{\beta(0)^{\frac{1}{2}} - 1} \). Then it is seen that \( \gamma(0) = 1, \gamma(k) ↓ 0 \) and \( \sum_{k=1}^{\infty} \gamma(k) = \infty \). By definition we obtain that

\[ \beta(k) = \frac{\gamma(k)}{\gamma(k+1)} = \frac{(k+k_0+1)^d}{(k+k_0)^d} = \left( 1 + \frac{1}{k+k_0} \right)^d > 1. \]

Then \( \{ \beta(k) \} \) is a monotonely decreasing sequence, and \( (1 + 1/k_0)^d = \beta(0) \). Thus, A6 (i) and (iii) hold.

Let \( h \in (0, \frac{2}{\lambda_2(L)+\lambda_N(L)}) \) and \( \beta(0) \in (1, \frac{1}{1-h\lambda_2(L)}) \). We introduce some useful notations:

\[
\hat{\rho}_h \triangleq 1 - h\lambda_2(L), \quad z_H \triangleq (z_h^T, \ldots, z_Nh_N^T)^T, \\
M'(h, \beta(0)) \triangleq (1 + 2hd^*)\beta(0) + 2hM_2(h, \beta(0)), \quad K'(h, \beta(0)) \triangleq \left[ M'(h, \beta(0)) - \frac{1}{2} \right] 
\]

(42)

with

\[
M_1(h, \beta(0)) \triangleq \left( \sqrt{mNC_x(1 + h\lambda_N(L))} + \frac{2\|z_H\|_2}{\lambda_{\min}(F_d)} \right) \\
\times \left( \|H_d\|_\infty + \frac{h\lambda_N(L)\|H_d\|_2}{1/\beta(0) - \rho_h} + \|z_H\|_\infty + \lambda_N(L)\left( \sqrt{mNC_x(1 + h\lambda_N(L))} + \frac{h\|z_H\|_2}{1/\beta(0) - \rho_h} \right) \right), 
\]

M_2(h, \beta(0)) \triangleq \beta(0)\sqrt{mNC_x(L)}\left( \frac{h\lambda_N(L)}{2(1/\beta(0) - \rho_h)} + \frac{1}{\lambda_{\min}(F_d)} \left( \|H_d\|_\infty + \frac{h\lambda_N(L)\|H_d\|_2}{1/\beta(0) - \rho_h} \right) \right),

(43)

where \( H_d \) and \( F_d \) are defined in (9). We now ready to state the main result of the algorithm (7).

Proposition 3 Suppose A4, A5, and A6 hold. Let Algorithm 1 be applied to the least-squares problem (2). Suppose \( h \in (0, \min \left\{ \frac{2}{\lambda_2(L)+\lambda_N(L)}, \frac{1}{\lambda_{\min}(F_d)} \right\} \) and \( \beta(0) \in \left( 1, \frac{1}{1-h\lambda_2(L)} \right) \). Then for any given \( K \geq K'(h, \beta(0)) \), the quantizer will never be saturated provided that

\[ s_r > \max \left\{ \frac{C_x + h(C_x\|H_d\|_\infty + \|z_H\|_\infty)}{K + \frac{1}{2}}, \frac{M_1(h, \beta(0))/M_2(h, \beta(0))}{K + \frac{1}{2}} \right\}. \]

(44)

Proposition 3 establishes the nonsaturation of the uniform quantizer, for which the proof is given in Section 4.3.2. Although the least-squares problem (2) seems like a special case of distributed optimization, the main challenge lies in that gradients of the quadratic function associated with each node cannot be
assumed to be globally bounded a priori, a key technical assumption for the convergence analysis of distributed (sub)gradient optimization \[15, 35\]. This is because the gradient function takes a linear form of the generated sequence \(x(k)\), which might be unbounded with inappropriate algorithmic parameters. Thus, the main effort of the proof lies in suitably choosing the parameters and proving the boundedness of the generated sequence.

**Theorem 4 (High Data Rate)** Suppose A4, A5, and A6 hold. Let \(h \in \left(0, \min \left\{ \frac{2}{\lambda_2(L) + \lambda_N(L)}, \frac{1}{\lambda_{\min}(\mathbf{F}_d)} \right\} \right)\) and \(\beta(0) \in (1, 1/(1 - h\lambda_2(L)))\). Then for any given \(K \geq K'(h, \beta(0))\), along Algorithm \(\mathcal{A}\) there hold:

\[
\lim_{k \to \infty} x_i(k) = y_{LS}^* \triangleq (H^T H)^{-1} H^T z \quad \forall i \in \mathcal{V}, \tag{45}
\]

\[
\limsup_{k \to \infty} \frac{\|x_i(k) - y_{LS}^*\|_\infty}{\gamma(k)} < \infty \tag{46}
\]

provided that \(s_r\) satisfies (44).

Theorem 4 shows that the Algorithm \(\mathcal{A}\) can ensure asymptotic convergence to the unique least-squares solution \(y_{LS}^*\). Its proof is deferred to Section 4.3.3.

**Remark 6** Note by Theorem 4 that slow rate of convergence is obtained by Algorithm \(\mathcal{A}\) with decreasing step-sizes for the least-squares solver, as opposed to the exponential convergence of the exact solver shown in Theorem 2 for Algorithm \(\mathcal{A}\) with constant step-size. This is mainly because for the distributed least-squares problem even with un-quantized communication channel, the primal domain algorithm cannot guarantee exact convergence with constant step-size \[27\]. While it is noticed by \[17\] and \[38\] that the exact convergence or even the linear rate of convergence can be obtained by primal-dual domain algorithms. As such, we might be able to find the least-squares solution with limited communication data rate at an exponential rate by the primal-dual domain methods. We leave the problem of designing least-squares solver with non-decreasing step-size for future research.

Similar to Theorem 2 for the exact solver case, in the following theorem we show that we can also design a distributed protocol for the least-squares solver to converge to a least-squares solution with 3-level quantizers, which uses the minimum number of quantization levels.

**Theorem 5 (Low Data Rate)** Suppose A4, A5 and A6 hold. Then the following hold:

(i) For any \(K \geq 1\), \(\Xi'_K\) is nonempty with

\[
\Xi'_K \triangleq \left\{ (h, \beta(0)) : \beta(0) \in (1, 1/(1 - h\lambda_2(L))) \right\}, \quad h \in \left(0, \min \left\{ \frac{2}{\lambda_2(L) + \lambda_N(L)}, \frac{1}{\lambda_{\min}(\mathbf{F}_d)} \right\} \right), \quad M'(h, \beta(0)) \leq K + \frac{1}{2}, \tag{47}
\]

(ii) For any \(K \geq 1\), let \((h, \beta(0)) \in \Xi'_K\) and \(s_r\) satisfy (44). Then along Algorithm \(\mathcal{A}\) there hold for all \(i \in \mathcal{V}\) that \(\lim_{k \to \infty} x_i(k) = y_{LS}^*\) with the rate of convergence characterized by

\[
\limsup_{k \to \infty} \frac{\|x_i(k) - y_{LS}^*\|_\infty}{\gamma(k)} < \infty.
\]
The proof of Theorem 5 is given Section 4.3.3. Similarly to Proposition reflem-rate, the following result with the proof given in Section 4.3.3 gives an explicit method for choosing algorithm parameters \((h, \beta(0)) \in \Xi'_K\) for any given \(K \geq 1\) by introducing a free parameter \(\epsilon \in (0,1)\).

**Proposition 4**  For any given \(K \geq 1\) and \(\epsilon \in (0,1)\), define \(\hat{h}_{K,\epsilon} \triangleq \min \left\{ \frac{2}{\lambda_2(L) + \lambda_N(L)} \lambda_{\min}(F_d), \hat{h}_{K,\epsilon} \right\}\) and 
\(\Xi'_{K,\epsilon} \triangleq \{ (\beta(0), h) : \beta(0)^{-1} = 1 - (1-\epsilon)h\lambda_2(L), h \in (0, \hat{h}_{K,\epsilon}) \}\), where \(\hat{h}_{K,\epsilon}\) is defined in the following

\[
\begin{align*}
\hat{h}_{K,\epsilon} &\triangleq 2K \epsilon \lambda_{\min}(F_d) \left(2d^* \epsilon \lambda_{\min}(F_d) + (2K+1)(1-\epsilon)\lambda_{\min}(F_d)\lambda_2(L)\right) \\
&\quad + 2\sqrt{mA_N} \lambda_N(L) \left(2\epsilon \|H_d\|_\infty + \kappa_N (2\|H_d\|_2 + \lambda_{\min}(F_d))\right)^{-1}
\end{align*}
\]

with \(\kappa_N \triangleq \frac{\lambda_N(L)}{\lambda_2(L)}\). Then \(\Xi'_{K} = \bigcup_{\epsilon \in (0,1)} \Xi'_{K,\epsilon}\).

### 4.2 Numerical Examples

**Example 4** Let \(H, z\) be given as follows:

\[
H = \begin{pmatrix}
1.7889 & -1.0764 \\
-1.0764 & 0.1903 \\
0.4707 & 0.1008 \\
0.8356 & -0.1716 \\
0.5978 & -1.6668
\end{pmatrix}, \quad z = \begin{pmatrix}
-0.2854 \\
1.2038 \\
1.1032 \\
0.7088 \\
-0.9495
\end{pmatrix}
\]

then the unique least square solution of \(y^* = \arg \min \|z - Hy\|^2\) is \(y^* = \begin{pmatrix} 0.1415 \\ 0.6391 \end{pmatrix}\). The nodes again communicate according to the graph shown in Fig 1.

[Validation of Theorem 4] Set \(h = 0.0853\) and \(\gamma(k) = \left(\frac{26}{K + 20}\right)^{0.85}\) such that \(\beta(0) \in (1, \tilde{\rho}^{-1})\). Hence, \(K(h, \beta(0)) = 870\). We set \(K_1 = 900\), \(K_2 = 300\) and \(K_3 = 1800\), respectively. We set \(s_r = 0.82\) to meet (44) fin all three cases. We then run Algorithm 1 with the quantization levels \(K_1 K_2\) and \(K_3\), respectively, while with the same parameters \(h, \gamma(k)\). The simulation results are displayed in Figure 8 which shows that the trajectories of \(\|x - 1_N \otimes y\|^2\) coincide in all three cases. It then implies that i) Once the sufficient condition of Theorem 4 is satisfied, increasing data rate solely cannot speed up convergence; ii) The condition in Theorem 4 is sufficient for convergence but not necessary. Figure 8 also shows the trajectory of \(\frac{\|x(k) - y^*\|_\infty}{\gamma(k)}\), which verifies the convergence rate described by (46).

[Validation of Theorem 5] We set the quantization level \(K\) to be \(K_1 = 10\), \(K_2 = 30\) and \(K_3 = 90\), respectively. Then we utilize Proposition 4 to select algorithm parameters \(h\) and \(s(k) = s_r \gamma(k) = \frac{s_r h^{k^*}}{(k+\lambda_0)}\) such that \((h, \beta(0)) \in \Xi'_{K}\) and \(s_r\) satisfies (44) for the three cases. By setting \(\epsilon = 0.5\), the derived parameters for the three cases are given in Table 1. Figure 9 shows the trajectories of \(\|x - 1_N \otimes y^*\|^2\) for the three cases. It demonstrates the convergence of the algorithm with the chosen parameters, verifying Theorem
5. It also shows that with a higher data rate, the convergence could be faster if algorithm parameters are properly chosen.

| $k_0$ | $\delta$ | $h$  | $s_r$ |
|------|-------|------|------|
| K=10 | 120   | 0.85 | 0.0055 | 0.9583 |
| K=30 | 36    | 0.75 | 0.0164 | 0.6934 |
| K=90 | 9     | 0.55 | 0.0492 | 0.6968 |

Table 1: Parameter settings

4.3 Proofs of Statements

4.3.1 Preliminary Lemmas

The following lemma gives a new but equivalent recursion of Algorithm 1.

Lemma 2 Let A4 and A6 (ii) hold. Define

$$\begin{align*}
P(k) &= I_{mN} - h (L \otimes I_m + \gamma(k)H_d), \\
\varepsilon(k) &\triangleq \frac{e(k)}{s(k)}, \quad \eta(k) \triangleq (D \otimes I_m) x(k)/\gamma(k),
\end{align*}$$

(49)

where $D \triangleq I_N - \frac{1_n 1_n^T}{N}$ and $e(k)$ is defined by (9). Then

$$\begin{align*}
x(k+1) &= P(k)x(k) + h\gamma(k)(s_r L \otimes I_m \varepsilon(k) + z_H), \\
\eta(k+1) &= \beta(k)(I_{mN} - hL \otimes I_m) \eta(k) + h s_r L \otimes I_m \varepsilon(k) + hD \otimes I_m (z_H - H_d x(k)), \\
\varepsilon(k+1) &= \beta(k) (\theta(k) - Q_K (\theta(k)) ),
\end{align*}$$

(50) (51) (52)

where $\theta(k)$ is defined as follows:

$$\theta(k) \triangleq (I_{mN} + h L \otimes I_m) \varepsilon(k) - h s_r^{-1}(L \otimes I_m \eta(k) + H_d x(k) - z_H).$$

(53)
Proof. By using (24) and \( e(k) = s(k)\gamma(k)\varepsilon(k) \), we obtain the following variant of (7):

\[
x(k+1) = x(k) - hL \otimes I_m x(k) + hL \otimes I_m e(k) - h\gamma(k)(H_d x(k) - z_H)
= (I_{mN} - hL \otimes I_m) x(k) + h s_r \gamma(k) L \otimes I_m e(k) - h\gamma(k) H_d x(k) + h\gamma(k) z_H.
\]

Hence (50) holds by using the definition of \( P(k) \) in (19). By multiplying both sides of (51) on the left with \( D \otimes I_m \), using \( DL = L, D(I_N - hL) = (I_N - hL)D \), and the definition of \( \eta(k) \) and \( \beta(k) = \frac{\gamma(k)}{(k+1)} \), we obtain (51). By subtracting \( b(k) \) from both sides of the first equality of (51), using \( e(k) = x(k) - b(k) \), \( \gamma(k) = s_r^{-1} s(k) \) and \( L \otimes I_m x(k) = L \otimes I_m (D \otimes I_m) x(k) = \gamma(k) L \otimes I_m \eta(k) \), we obtain that

\[
x(k+1) - b(k) = (I_{mN} + hL \otimes I_m) e(k) - hL \otimes I_m x(k) - h\gamma(k)(H_d x(k) - h z_H)
= s(k) (I_{mN} + hL \otimes I_m) \varepsilon(k) - h s_r^{-1} s(k) (L \otimes I_m \eta(k) + H_d x(k) - z_H).
\]

Recalling that \( e(k+1) = x(k+1) - b(k+1) \) together with (20), we then have the following:

\[
e(k+1) = x(k+1) - b(k) - s(k) Q_K \left( \frac{x(k+1) - b(k)}{s(k)} \right)
= s(k) (\theta(k) - Q_K (\theta(k)))
\]

and hence dividing both sides of the above equation by \( s(k+1) \) we obtain (52).

\[
\square
\]

4.3.2 Proof of Proposition 3

The proof of non-saturation of the uniform quantizer is equivalent to show that for any \( k \geq 0 \), \( \theta(k) \) defined by (53) satisfies \( \|\theta(k)\|_\infty < K + \frac{1}{2} \). Again, we use an induction proof.

Recalling that \( \gamma(0) = 1, b_i(0) = 0 \forall i \in V \), we obtain \( \varepsilon(0) = x(0) \) and \( \varepsilon(0) = x(0)/s_r \). Then by using \( L \otimes I_m \eta(0) = L \otimes I_m x(0) \) and (53), we obtain that

\[
\theta(0) = \varepsilon(0) + hL \otimes I_m \varepsilon(0) - h s_r^{-1} L \otimes I_m \eta(0) - h s_r^{-1} (H_d x(0) - z_H) = x(0)/s_r - h s_r^{-1} (H_d x(0) - z_H).
\]

Then from A5, and (44) it follows that

\[
\|\theta(0)\|_\infty \leq C_x/s_r + h(C_z H_d \|H_d\|_\infty + \|z_H\|_\infty)/s_r = C_x + h(C_z H_d \|H_d\|_\infty + \|z_H\|_\infty)/s_r = K + \frac{1}{2}.
\]

Hence, when \( k = 0 \), the quantizer is unsaturated. Next, for the induction, we assume that when \( k = 0, \ldots, p \), the quantizer is not saturated. Then by (52) and A6, there holds

\[
\|\varepsilon(k)\|_\infty \leq \frac{\beta(k)}{2} \leq \frac{\beta(0)}{2} \forall k : 1 \leq k \leq p + 1.
\]

We aim to show that the quantizer is unsaturated for \( k = p + 1 \). Define \( \Gamma(k, k + 1) \triangleq I_{mN} \) and

\[
\Gamma(k_1, k_2) \triangleq P(k_1) P(k_1 - 1) \ldots P(k_2) \forall k_1 \geq k_2 \geq 0.
\]
Then from (50) and $\gamma(0) = 1$ it follows that
\[
x(p + 1) = \Gamma(p, 0)x(0) + hs_r\Gamma(p, 1)\mathbf{L} \otimes \mathbf{I}_m\varepsilon(0) + hs_r \sum_{i=1}^{p} \gamma(i)\Gamma(p, i + 1)\mathbf{L} \otimes \mathbf{I}_m\varepsilon(i) + h \sum_{i=0}^{p} \gamma(i)\Gamma(p, i + 1)\mathbf{z}_H.
\]

We now estimate the bound of $x(p + 1)$. Using $\mathbf{F}_d = \mathbf{L} \otimes \mathbf{I}_m + \mathbf{H}_d$, the following holds:
\[
\min\{1, \gamma(k)\} x^T \mathbf{F}_d x \leq x^T (\mathbf{L} \otimes \mathbf{I}_m + \gamma(k)\mathbf{H}_d) x \leq \max\{1, \gamma(k)\} x^T \mathbf{F}_d x \quad \forall x \in \mathbb{R}^{mN}.
\]
By recalling that $0 < \gamma(k) \leq 1$, $\min\{1, \gamma(k)\} = \gamma(k)$ and $\max\{1, \gamma(k)\} = 1$. Thus, for the matrix $\mathbf{L} \otimes \mathbf{I}_m + \gamma(k)\mathbf{H}_d$, the smallest eigenvalue of is greater than or equal to $\gamma(k)\lambda_{\text{min}}(\mathbf{F}_d)$ while the largest eigenvalue is smaller than or equal to $\lambda_{\text{max}}(\mathbf{F}_d)$. Then by $\mathbf{P}(k)$ defined in (49), the eigenvalues of $\mathbf{P}(k)$ sorted in an ascending order satisfy
\[
1 - h\lambda_{\text{max}}(\mathbf{F}_d) \leq \lambda_1(\mathbf{P}(k)) \leq \cdots \leq \lambda_{mN}(\mathbf{P}(k)) \leq 1 - h\gamma(k)\lambda_{\text{min}}(\mathbf{F}_d).
\]
Thus, for any $k \geq 0$:
\[
\|\mathbf{P}(k)\|_2 \leq \max\{1 - h\gamma(k)\lambda_{\text{min}}(\mathbf{F}_d), 1 - h\lambda_{\text{max}}(\mathbf{F}_d)\}.
\]
Then by recalling that $0 < h < \frac{2}{\lambda_{\text{min}}(\mathbf{F}_d) + \lambda_{\text{max}}(\mathbf{F}_d)}$ and $\gamma(k) \leq 1$, the following holds:
\[
\|\mathbf{P}(k)\|_2 \leq 1 - h\gamma(k)\lambda_{\text{min}}(\mathbf{F}_d) \leq \exp(-h\gamma(k)\lambda_{\text{min}}(\mathbf{F}_d)),
\]
where the last inequality holds by $1 - x \leq \exp(-x)$ $\forall x \geq 0$. Then from (56) it follows that for any $k_1 \geq k_2 \geq 0$:
\[
\|\Gamma(k_1, k_2)\|_2 \leq \exp\left(-h\lambda_{\text{min}}(\mathbf{F}_d) \sum_{k=k_2}^{k_1} \gamma(k)\right).
\]
Also, using (55), (57), A5, $\|x\|_{\infty} \leq \|x\|_2 \leq \sqrt{m}\|x\|_{\infty}$ for any $x \in \mathbb{R}^m$, and $x(0) = s_r\varepsilon(0)$, we obtain that
\[
\|x(p + 1)\|_2 \leq hs_r\|\mathbf{L}\|_2 \sum_{i=1}^{p} \gamma(i)\|\Gamma(p, i + 1)\|_2\|\varepsilon(i)\|_2 + \|\Gamma(p, 0)\|_2\|x(0)\|_2
\]
\[
+ h\|\Gamma(p, 1)\|_2\|\mathbf{L}\|_2\|x(0)\|_2 + h\|\mathbf{z}_H\|_2 \sum_{i=1}^{p} \gamma(i)\|\Gamma(p, i + 1)\|_2
\]
\[
\leq \sqrt{mN}(1 + h\lambda_{N}(\mathbf{L}))C_x + \frac{hs_r\beta(0)\sqrt{mN}\lambda_{N}(\mathbf{L})}{2} \times \sum_{i=1}^{p} \gamma(i) \exp\left(-h\lambda_{\text{min}}(\mathbf{F}_d) \sum_{k=k_1+1}^{k} \gamma(k)\right)
\]
\[
+ h\|\mathbf{z}_H\|_2 \sum_{i=0}^{p} \gamma(i) \exp\left(-h\lambda_{\text{min}}(\mathbf{F}_d) \sum_{k=k_1+1}^{k} \gamma(k)\right).
\]
Since $h\gamma(k) \leq 1/\lambda_{\text{min}}(\mathbf{F}_d)$ for any $k \geq 0$, there holds:
\[
h\gamma(k) \leq 2 \left(h\gamma(k) - \frac{\lambda_{\text{min}}(\mathbf{F}_d)}{2}(h\gamma(k))^2\right) \quad \forall k \geq 0.
\]
Thus, by $x - x^2/2 < 1 - \exp(-x)$ $\forall x \in (0, 1)$, we have the following sequence of inequalities:

$$
\sum_{i=k_1}^{p} h\lambda_{\min}(F_d)\gamma(i) \exp \left( -h\lambda_{\min}(F_d) \sum_{k=i+1}^{p} \gamma(k) \right) \\
\leq 2 \sum_{i=k_1}^{p} \left( h\lambda_{\min}(F_d)\gamma(i) - \left( h\lambda_{\min}(F_d)\gamma(i) \right)^2/2 \right) \times \exp \left( -h\lambda_{\min}(F_d) \sum_{k=i+1}^{p} \gamma(k) \right) \\
\leq 2 \sum_{i=k_1}^{p} (1 - \exp\left( -\lambda_{\min}(F_d)\gamma(i) \right)) \times \exp \left( -h\lambda_{\min}(F_d) \sum_{k=i+1}^{p} \gamma(k) \right) \\
= 2 \sum_{i=k_1}^{p} \left[ \exp \left( -h\lambda_{\min}(F_d) \sum_{k=i+1}^{p} \gamma(k) \right) - \exp \left( -h\lambda_{\min}(F_d) \sum_{k=i}^{p} \gamma(k) \right) \right] \leq 2.
$$

Using this in (68) yields

$$
\|x(p+1)\|_2 \leq \sqrt{mNC_x(1 + h\lambda_N(L))} + \frac{s_r\beta(0)\sqrt{mN\lambda_N(L)}}{\lambda_{\min}(F_d)} + \frac{2\|z_H\|_2}{\lambda_{\min}(F_d)} \leq M_x.
$$

(59)

Since $L$ is symmetric, we can define an orthogonal matrix $T = \left( \begin{smallmatrix} \frac{1}{\sqrt{N}}, \phi_2, \ldots, \phi_N \end{smallmatrix} \right)$, where $L\phi_i = \lambda_i(L)\phi_i$ for every $i = 2, \ldots, N$. Let $\tilde{\eta}(k) = (T^{-1} \otimes I_m)\eta(k) = (T^T \otimes I_m)\eta(k)$ and decompose it as $\tilde{\eta}(k) = (\tilde{\eta}_1(k)^T, \tilde{\eta}_2(k)^T)^T$ with $\tilde{\eta}_1(k) = \frac{1}{\sqrt{N}}(1_N^T \otimes I_m)\eta(k)$ and $\tilde{\eta}_2(k) = (T_2^T \otimes I_m)\eta(k)$, where $T_2 = (\phi_2, \ldots, \phi_N)$. Then $\tilde{\eta}_1(k) = 0_m$ by $\eta(k) = (D \otimes I_m)x(k)_{\{1\}}$ and $1_N^T D = 0_m$. Then by multiplying both sides of (61) with $T_2^T \otimes I_m$ from the left, and noting $T_2^T L = \text{diag}\{\lambda_2(L), \ldots, \lambda_N(L)\}$ $T_2^T$ we have the following:

$$
\tilde{\eta}_2(k+1) = \beta(k)h\sigma_r \left( \sum_{i=0}^{\infty} (T_2^T \otimes I_m) (z_H - H_d x(k)) + \varepsilon(k) \right) \\
+ \beta(k) \left( \text{diag}\{1 - h\lambda_2(L), \ldots, 1 - h\lambda_N(L)\} \otimes I_m \right) \tilde{\eta}_2(k).
$$

(60)

Thus, there holds

$$
\tilde{\eta}_2(p+1) = \left( D_{h}^{p+1} \otimes I_m \right) \tilde{\eta}_2(0) \prod_{k=0}^{p} \beta(k) + h\sigma_r \sum_{k=0}^{p} \left( D_{h}^{k} T_2^T L \otimes I_m \right) \prod_{i=p-k}^{p} \beta(i) \varepsilon(p-k) \\
+ h \sum_{k=0}^{p} \left( D_{h}^{k} T_2^T D \otimes I_m \right) \prod_{i=p-k}^{p} \beta(i) \left( z_H - H_d x(p-k) \right).
$$

(61)

Note that $(T_2 \otimes I_m) \tilde{\eta}_2(k) = (T_2 T_2^T \otimes I_m) \eta(k) = \left( I_N - \frac{1_N^T}{N} \otimes I_m \right) \eta(k) = \eta(k)$. Then by multiplying both sides of (61) on the left with $(T_2 \otimes I_m)$, there holds

$$
\eta(p+1) = \left( T_2 D_{h}^{p+1} T_2^T \otimes I_m \right) \eta(0) \prod_{k=0}^{p} \beta(k) + h\sigma_r \sum_{k=0}^{p} \left( T_2 D_{h}^{k} T_2^T L \otimes I_m \right) \varepsilon(p-k) \prod_{i=p-k}^{p} \beta(i) \\
+ h \sum_{k=0}^{p} \left( T_2 D_{h}^{k} T_2^T D \otimes I_m \right) \left( z_H - H_d x(p-k) \right) \prod_{i=p-k}^{p} \beta(i).
$$

(62)
By the definition of $D_h$ in (60), $\|D_h\|_2 = \max \{1-h\lambda_2(L), |1-h\lambda_N(L)|\}$. Thus, by using $h \in (0, \frac{2}{\lambda_2(L)+\lambda_N(L)})$ and Lemma 3.1, we obtain that $\|D_h\|_2 = 1-h\lambda_2(L) = \rho_h$. Taking two-norms of (62), by recalling that $\beta(k) \leq \beta(0) \forall k \geq 0$, $\|D\|_2 = \|T_2\|_2 = 1$, we have the following:

$$
\|\eta(p+1)\|_2 \leq (\beta(0)\rho_h)^{p+1}\|\eta(0)\|_2 + hs_r\beta(0)\|L\|_2(\beta(0)\rho_h)^p\|\varepsilon(0)\|_2
$$

$$
+ hs_r\beta(0)\|L\|_2 \sum_{k=0}^{p-1} (\beta(0)\rho_h)^k\|\varepsilon(p-k)\|_2 + h\beta(0)\sum_{k=0}^{p} (\beta(0)\rho_h)^k\|z_H - H_dx(p-k)\|_2.
$$

Note by $\gamma(0) = 1, s(0) = s_r, \|D\|_2 = 1$, and $A5$ that

$$
\|\eta(0)\|_2 = \|D \otimes I_m x(0)\|_2 \leq \|D\|_2\|x(0)\|_2 \leq \sqrt{mNC_x},
$$

$$
\|\varepsilon(0)\|_2 \leq \|x(0)\|_2/s(0) \leq \sqrt{mNC_x}/s_r.
$$

Similar to (59) we can easily show that $\|x(k)\|_2 \leq m \forall k = 0, \ldots, p$. Then by using (59), $\beta(0)\rho_h < 1$ and $\sum_{k=0}^{p} (\beta(0)\rho_h)^k \leq \frac{1}{1-\beta(0)\rho_h}$, we obtain the following:

$$
\|\eta(p+1)\|_2 \leq \sqrt{mNC_x} (1 + h\beta(0)\lambda_N(L)) + \frac{\sqrt{mNHs_r\beta(0)^2\lambda_N(L)}}{2(1-\beta(0)\rho_h)}
$$

$$
+ (\|z_H\|_2 + \|H_d\|_2M_x) \frac{h\beta(0)}{1-\beta(0)\rho_h}. \quad (63)
$$

This together with (53), (55) and (59) leads to

$$
\|\theta(p+1)\|_\infty \leq \|(I_{mN} + hL \otimes I_m)x(p+1)\|_\infty + hs_r^{-1}(\lambda_N(L)\|\eta(p+1)\|_2 + \|z_H\|_\infty + \|H_d\|_\infty\|x(p+1)\|_\infty)
$$

$$
\leq \beta(0)(1/2 + hd^*) + h\sqrt{mN}\lambda_N(L) (\beta(0)\lambda_{\text{min}}(F_d) \times \left(\|H_d\|_\infty + \frac{h\beta(0)\lambda_N(L)}{1-\beta(0)\rho_h}\right))
$$

$$
+ hs_r^{-1}(\sqrt{mNC_x} (1 + h\beta(0)\lambda_N(L)) + \frac{2\|z_H\|_2}{\lambda_{\text{min}}(F_d)}) \times \left(\|H_d\|_\infty + \frac{h\beta(0)\lambda_N(L)}{1-\beta(0)\rho_h}\right) + hs_r^{-1}\|z_H\|_\infty
$$

$$
= \frac{\beta(0)(1 + 2hd^*)}{2} + h (s_r^{-1}M_1(h,\beta(0)) + M_2(h,\beta(0))) \quad \text{(by 43)}
$$

$$
\leq \beta(0)(1/2 + hd^*) + 2hM_2(h,\beta(0)) = M'(h,\beta(0)) \quad \text{(by 42) and 44}
$$

$$
\leq \left[M'(h,\beta(0)) - \frac{1}{2} + \frac{1}{2} = K'(h,\beta(0)) + \frac{1}{2} \leq K + \frac{1}{2}.
$$

As a result, when $k = p+1$, the quantizer is also unsaturated. Therefore, by induction, we conclude that if a $(2K+1)$-levels uniform quantizer is applied, then the quantizer will never be saturated. 

**4.3.3 Proof of Theorem 4**

From Proposition 3 it follows that (63) holds for any $p \geq 0$. This implies that $\sup_{k \geq 1} \|\eta(k)\|_\infty < \infty$. Then using the definition of $\eta(k)$, we obtain that $\|x_i(k) - y(k)\|_\infty = O(\gamma(k))$.  

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Define \( y_k = \sum_{i=1}^{N} x_{i,k} / N \). Then by multiplying both sides of (54) from the left by \( \frac{1}{N} (I_N \otimes I_m) \), there holds

\[
y_{k+1} = y_k - h\gamma(k)(H^T H y_k - H^T z) / N - \frac{h\gamma(k)}{N} \sum_{i=1}^{N} h_i h_i^\top (x_{i,k} - y_k).
\]

Then by recalling that \( y^* = (H^T H)^{-1} H^T z \), we obtain that

\[
y_{k+1} - y_{LS}^* = y_k - y_{LS}^* - \frac{h\gamma(k)}{N} H^T H (y_k - y_{LS}^*) - \frac{h\gamma(k)}{N} \sum_{i=1}^{N} h_i h_i^\top (x_{i,k} - y_k).
\]

Since \( (x_{i,k} - y_k) \to 0 \) and \( H^T H \) is positive definite by rank(\( H \)) = \( m \), from [39] Lemma 3.1.1 it follows that \( \lim_{k \to \infty} y_k = y_{LS}^* \), this together with \( x_i(k) - y(k) \to 0 \) implies (45). Then by \( \|x_i(k) - y(k)\|_\infty = O(\gamma(k)) \) and [39] Theorem 3.1.1 we obtain that \( \|y(k) - y_{LS}^*\|_\infty = O(\gamma(k)) \), and hence (46) holds.

**4.3.4 Proof of Theorem 5**

(i) Using \( \hat{p}_h = 1 - h\lambda_2(L) \), \( \kappa_N = \frac{\lambda_1(L)}{\lambda_2(L)} \), (42) and (43), we obtain the following:

\[
M'(h, 1) = \frac{1 + 2hd^*}{2} + 2h\sqrt{mN}\lambda_N(L)\left(\frac{1}{\lambda_{\min}(F_d)}\left(\|H_d\|_\infty + \|H_d\|_2\kappa_N\right) + \frac{\kappa_N}{2}\right)
\]

Thus, \( \lim_{h \to 0} M'(h, 1) = \frac{1}{2} \). Then for any \( K \geq 1 \), there exists \( h^* \in \left(0, \min\left\{\frac{2}{\lambda_2(L) + \lambda_N(L)}, \frac{1}{\lambda_{\min}(F_d)}\right\}\right) \) such that \( M'(h^*, 1) \leq K \). By recalling the definition of \( M'(h, \beta(0)) \) in (42), we have that

\[
\lim_{\beta(0) \to 1} M'(h^*, \beta(0)) = M'(h^*, 1) \leq K.
\]

Then there exists \( \beta^*(0) \in \left(1, \frac{1}{1 - h\lambda_2(L)}\right) \) such that \( M'(h^*, \beta^*(0)) \leq K + \frac{1}{2} \). Therefore, \( (h^*, \beta^*(0)) \in \Xi'_K \). Hence \( \Xi'_K \) is nonempty.

(ii) For any \( (h, \beta(0)) \in \Xi'_K \), from (47) it follows that \( h \in \left(0, \min\left\{\frac{2}{\lambda_2(L) + \lambda_N(L)}, \frac{1}{\lambda_{\min}(F_d)}\right\}\right) \), \( \beta(0) \in \left(1, \frac{1}{1 - h\lambda_2(L)}\right) \), and \( M'(h, \beta(0)) \leq K + \frac{1}{2} \). Then by definition (42),

\[
K'(h, \beta(0)) = \left[M'(h, \beta(0)) - \frac{1}{2}\right] \leq K,
\]

which together with Theorem 4 leads to assertion (ii).

**4.3.5 Proof of Proposition 4**

We first validate \( \bigcup_{\epsilon \in (0, 1)} \Xi_{K, \epsilon} \subset \Xi_K \). For any given \( K \geq 1 \) and \( \epsilon \in (0, 1) \), let \( (\beta(0), h) \in \Xi_{K, \epsilon} \). Then \( 1/\beta(0) - \hat{p}_h = eh\lambda_2(L) > 0 \) by \( \hat{p}_h = 1 - h\lambda_2(L) \), and hence \( \beta(0) < 1/\hat{p}_h \). Then by the definition of \( M'(h, \beta(0)) \) in (42), using \( \kappa_N = \frac{\lambda_1(L)}{\lambda_2(L)} \) and \( \beta(0)^{-1} = 1 - (1 - \epsilon)h\lambda_2(L) \), we obtain that

\[
M'(h, \beta(0)) = (1 + 2hd^*)\beta(0) + 2h\sqrt{mN}\lambda_N(L)\beta(0) \times \left(\frac{1}{\lambda_{\min}(F_d)}\left(\|H_d\|_\infty + \frac{\kappa_N\|H_d\|_2}{\epsilon}\right) + \frac{\kappa_N}{2\epsilon}\right)
\]

\[
= (2\epsilon\|H_d\|_\infty + 2\kappa_N\|H_d\|_2 + \kappa_N\lambda_{\min}(F_d)) \times \frac{(1 + 2hd^*)\epsilon\lambda_{\min}(F_d) + 2h\sqrt{mN}\lambda_N(L)}{2\epsilon(1 - (1 - \epsilon)h\lambda_2(L))\lambda_{\min}(F_d)}
\]

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Then by using the definition of $\hat{h}_{K,\epsilon}$ in (48) and $h < \hat{h}_{K,\epsilon}$, there holds $M'(h, \beta(0)) < K + \frac{1}{2}$. Obviously, for any $(\alpha, h) \in \Xi_{K,\epsilon}$, there holds $h \in \left(0, \min \left\{ \frac{2}{\lambda_2(L) + \lambda_N(L)}, \frac{1}{\lambda_{\min}(P_d)} \right\} \right)$. In summary, we have verified $\Xi'_{K,\epsilon} \subset \Xi'_K$ for any given $K \geq 1$ and any $\epsilon \in (0, 1)$. Thus, $\bigcup_{\epsilon \in (0, 1)} \Xi'_{K,\epsilon} \subset \Xi'_K$. Similar to that of Lemma 2 we can also prove that $\Xi'_K \subset \bigcup_{\epsilon \in (0, 1)} \Xi'_{K,\epsilon}$. Thus, we complete the proof.

\section{Conclusions}

We have studied solving linear equations over a network subject to digital node communications with a limited data rate. We propose a node encoder-decoder pair, based on which a distributed quantized algorithm is designed. For the unique exact solution case, the proposed encoder-decoder powered algorithm drove each node state to the solution asymptotically at an exponential rate. For the unique least-squares solution case, the same encoder-decoder pair enabled the algorithm to compute such a solution with a properly selected time-varying step size. A minimal data rate was shown to be enough for the desired convergence for both cases. These results suggest the practical applicability of various network linear equation solvers in the literature.

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