Metric for two arbitrary Kerr sources

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Abstract

The full metric describing a stationary axisymmetric system of two arbitrary Kerr sources, black holes or hyperextreme objects, located on the symmetry axis and kept apart in equilibrium by a massless strut is presented in a concise explicit form involving five physical parameters. The binary system composed of a Schwarzschild black hole and a Kerr source is a special case not covered by the general formulas, and we elaborate the metric for this physically interesting configuration too.

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I. INTRODUCTION

In our recent paper [1] we have outlined an approach towards elaborating a physically attractive description of two arbitrary Kerr sources separated by a massless strut. We have shown that obtaining such a description is a feasible task because the axis condition – the only one needed to be solved – reduces to an algebraic equation of the order not higher than quartic. There we have also hinted that some clever trick might help eventually obviate the resolution of the cumbersome quartic equation and get concise expressions for the quantities $\sigma_1$ and $\sigma_2$ related to the horizons of black holes in the subextreme case. Writing that, we first of all had in mind our earlier work [2] where we obtained a simple representation of the well-known Kinnersley-Chitre metric [3, 26] and used it for solving the axis condition, thus finding all possible binary configurations of extreme Kerr black holes separated by a strut; the use of the NUT parameter $J_0$ in that analysis was a key ingredient for choosing a convenient parametrization by solving the asymptotic flatness condition $J_0 = 0$, and so it looked to us plausible to explore the expression of $J_0$ in the more general situation involving non-extreme Kerr constituents in order to obtain a second constraint which, together with the axis condition, would provide a system of two algebraic equations for finding the aforementioned quantities $\sigma_1$ and $\sigma_2$ uniquely.

It appears that the above idea has been successfully accomplished recently by Cabrera-Munguia [6] who, being well familiar with the paper [2], has succeeded in obtaining the desired expressions for $\sigma_1$ and $\sigma_2$ through a straightforward algebra involving the resolution of a quadratic equation, thus making an important step forward towards a definitive description of binary configurations of Kerr sources. At the same time, despite the author’s claim that the metric given by him covers both the black-hole and hyperextreme sectors of the double-Kerr solution, it can be shown that this is not true, and consequently, like in the author’s earlier paper on two identical Kerr sources [7], his formulas have conceptual problems that make them inappropriate for the description of hyperextreme Kerr objects. Such a situation is obviously undesirable. Indeed, first of all, since both the black holes and naked singularities are able to form configurations with struts, it is of course likely to have solutions able to describe both types of the Kerr sources. But more importantly, provided that some binary configurations with struts may contain configurations without struts as particular limiting cases corresponding to vanishing interaction force, the former configurations must
be reducible to the latter ones, the majority of which are actually configurations involving hyperextreme Kerr constituents [8, 9]. It might also be noted that in view of the discovery of the black hole-naked singularity dualism [10], the particular binary systems representing this phenomenon cannot be correctly treated by the solutions restricted exclusively to the black-hole sector.

The objective of the present paper is to combine together our previous results on the binary configurations of Kerr sources with the recent expressions for $\sigma_1$ and $\sigma_2$ found in [6] for obtaining a concise physical representation of the metric for two arbitrary Kerr sources separated by a massless strut which would describe in a unified manner both the black-hole and hyperextreme constituents. We shall also work out a particular 4-parameter metric describing a physically interesting ‘Schwarzschild-Kerr’ configuration which is not covered by the general formulas and for which only an equilibrium solution without a strut was obtained earlier in the literature [11]; it will illustrate well the transformation of a black-hole binary system into a configuration involving a hyperextreme Kerr source within the same extended solution.

II. THE SOLUTION IN PHYSICAL PARAMETERS

As it follows from the papers [1, 6], the Ernst complex potential $E$ [12] of the exact solution for two aligned Kerr sources is defined on the symmetry axis by the expression

$$ e(z) = \frac{z^2 - (M + ia)z + s - \mu + i(\tau + \delta)}{z^2 + (M - ia)z + s + \mu + i(\tau - \delta)}, $$

where $M$ is the total mass of the binary system, $a$ the rotational parameter, and the real quantities $s$, $\mu$, $\tau$, $\delta$ are related to the individual physical characteristics of the sources in the following way:

$$ s = -\frac{1}{4}[R^2 + 2(\sigma_1^2 + \sigma_2^2 - M^2 + a^2)], \quad \delta = Ma - m_1a_1 - m_2a_2, $$

$$ \tau = \frac{1}{2}R(a_2 - a_1) + \frac{(R + M)[m_2a_1(R + 2m_1) - m_1a_2(R + 2m_2)]}{(R + M)^2 + a^2}, $$

$$ \mu = -\frac{1}{2M}[R(\sigma_1^2 - \sigma_2^2) - 2a\tau]. $$

The set of five arbitrary real parameters is $\{m_1, m_2, a_1, a_2, R\}$, $m_1$ and $m_2$ being the individual Komar [13] masses of the Kerr constituents, $a_1 = j_1/m_1$ and $a_2 = j_2/m_2$ their individual
Komar angular momenta per unit mass, and $R$ the coordinate distance between the centers of the sources. The total mass $M$ and total angular momentum $J$ of the system therefore have the form

$$M = m_1 + m_2, \quad J = m_1 a_1 + m_2 a_2,$$

while $a$ satisfies the cubic equation

$$\frac{(a_1 + a_2 - a)(R^2 - M^2 + a^2)}{2(R + M)} - Ma + J = 0. \quad (4)$$

The remaining quantities $\sigma_1$ and $\sigma_2$, which represent the half-lengths of the horizons of black holes in the subextreme case, are given by the formulas

$$\sigma_1 = \sqrt{m_1^2 - a_1^2 + 4m_2 a_1 \frac{[m_1(a_1 - a_2 + a) + Ra][(R + M)^2 + a^2] + m_2 a_1 a^2}{[(R + M)^2 + a^2]^2}},$$

$$\sigma_2 = \sqrt{m_2^2 - a_2^2 + 4m_1 a_2 \frac{[m_2(a_2 - a_1 + a) + Ra][(R + M)^2 + a^2] + m_1 a_2 a^2}{[(R + M)^2 + a^2]^2}}, \quad (5)$$

and it is clear that $\sigma_1$ and $\sigma_2$ can take on real or pure imaginary values.

With the axis data thus defined, the equation

$$e(z) + \bar{e}(z) = 0, \quad (6)$$

(a bar over a symbol means complex conjugation) has four roots $\alpha_n$, $n = 1, 2, 3, 4$, of the form

$$\alpha_1 = \frac{1}{2} R + \sigma_1, \quad \alpha_2 = \frac{1}{2} R - \sigma_1, \quad \alpha_3 = -\frac{1}{2} R + \sigma_2, \quad \alpha_4 = -\frac{1}{2} R - \sigma_2, \quad (7)$$

and the above $\alpha_n$ define the location of the sources on the symmetry axis, a pair of real-valued $\alpha$’s determining a black hole, and a pair of complex conjugate $\alpha$’s determining a hyperextreme object. Fig. 1 illustrates that there are three generic types of binary configurations of non-extreme Kerr constituents: the “subextreme-subextreme”, “subextreme-hyperextreme” and “hyperextreme-hyperextreme”, among which only the first one is entirely composed of black holes.

The expression of the Ernst complex potential $\mathcal{E}$ in the whole $(\rho, z)$ space is obtainable from (1) by means of Sibgatullin’s integral method (for details the reader is referred to [1] and references therein), and it reads as follows

$$\mathcal{E} = (A - B)/(A + B),$$

4
where the functions \( R_\pm \) and \( r_\pm \) have been found to be defined by the formulas

\[
R_\pm = \mu_0 \left( \pm \sigma_1 - m_1 - ia_1 \right) \left[ (R + M)^2 + a^2 \right] + 2a_1 \left[ m_1 a + iM(R + M) \right] \bar{r}_\pm,
\]

\[
r_\pm = \mu_0 \left( \pm \sigma_2 + m_2 - ia_2 \right) \left[ (R + M)^2 + a^2 \right] - 2a_2 \left[ m_2 a - iM(R + M) \right] \bar{r}_\pm,
\]

\[
\bar{r}_\pm = \sqrt{\rho^2 + \left( z - \frac{1}{2}R \pm \sigma_1 \right)^2}, \quad \bar{R}_\pm = \sqrt{\rho^2 + \left( z + \frac{1}{2}R \pm \sigma_2 \right)^2}, \quad \mu_0 := \frac{R + M - ia}{R + M + ia}.
\]

We have checked that the potential (8) satisfies identically the Ernst equation (12), namely,

\[
(\mathcal{E} + \bar{\mathcal{E}})\Delta \mathcal{E} = 2(\nabla \mathcal{E})^2,
\]

for any type of the Kerr constituents in the binary configuration.

The metric functions \( f, \gamma \) and \( \omega \) of the solution for two Kerr sources, entering the line element

\[
ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2,
\]

have the form (11)

\[
f = \frac{A\bar{A} - B\bar{B}}{(A + B)(A + B)}, \quad e^{2\gamma} = \frac{A\bar{A} - B\bar{B}}{16|\sigma_1|^2|\sigma_2|^2K_0^2\bar{R}_+\bar{R}_-\bar{r}_+\bar{r}_-}, \quad \omega = 2a - \frac{2\text{Im}[G(A + \bar{B})]}{AA - BB},
\]

\[
G = -zB + \sigma_1(R - \sigma_1^2 + \sigma_2^2)(R_+ - R_+)(r_+ + r_- + R)
\]

\[
+ \sigma_2(R^2 + \sigma_1^2 - \sigma_2^2)(r_- - r_+)(R_+ + R_- - R)
\]

\[
-2\sigma_1\sigma_2(2R[r_+r_- - R_+R_- - \sigma_1(r_- - r_+) + \sigma_2(R_+ - R_-)]
\]

\[
+(\sigma_1^2 - \sigma_2^2)(r_+ + r_- - R_+ - R_-)),
\]

where for \( K_0 \) we have got the following very simple formula

\[
K_0 = \frac{[(R + M)^2 + a^2][R^2 - (m_1 - m_2)^2 + a^2] - 4m_1^2m_2^2a^2}{m_1m_2[(R + M)^2 + a^2]}.
\]

Therefore, we have given, in terms of five physical parameters, a complete description of the Ernst potential and entire metric for two arbitrary Kerr sources separated by a massless strut (by construction, the axis condition is satisfied automatically). The metric
is asymptotically flat and is valid for all three types of binary configurations involving subextreme and/or hyperextreme Kerr constituents, and we have confirmed this with a computer check.

At this point, it is worth noting that the expressions for the Ernst potential and metric functions given in the papers [6, 7] are not valid in the hyperextreme case. One may come to this conclusion by observing that for instance the quantities $\bar{s}_\pm/s_\pm$ introduced in [7] as unitary complex objects for all types of binary systems, should not be unitary when these describe hyperextreme constituents (see, e.g., Ref. [1] for the definition and properties of the objects $X_i$).

Since the metric presented in [6] is plagued with conceptual errors and does not satisfy the field equations when one of the constituents is hyperextreme, it is necessary to verify whether some formulas derived in [6] for the subextreme case are also applicable to the configurations with hyperextreme sources. This first of all refers to the expression of the interaction force $F = (e^{-\gamma_0} - 1)/4$ [15, 16], $\gamma_0$ being the constant value of the metric function $\gamma$ on the strut, which in the black-hole case was shown to have the form

$$F = m_1 m_2 [(R + M)^2 - a^2] \bigg/ \left[ (R^2 - M^2 + a^2) [(R + M)^2 + a^2] \right].$$

but for which a formal use of the formulas of the paper [6] would yield a different expression in the hyperextreme case. Fortunately, the formulas of the present paper reveal that the above formula (14) still holds when the Kerr sources are hyperextreme. This conclusion is important for performing a correct transition to the case of equilibrium binary configurations without struts [17] when $a = \pm (R + M)$. Let us also mention that vanishing of $F$ when $m_1$ (or $m_2$) is equal to zero means the reduction of the two-body metric to a single Kerr solution [18] independently of the value of $a_1$ (or $a_2$).

Moreover, when analyzing the thermodynamical properties of the black-hole constituent in a “subextreme-hyperextreme” binary configuration, one has to be sure that the formulas for the horizon area $A_i$, surface gravity $\kappa_i$ and horizon’s angular velocity $\Omega_i^H$ of $i$th black-hole constituent, $i = 1, 2$, derived in [6] with the aid of Tomimatsu’s formulas [19], namely,

$$A_1 = \frac{\sigma_1}{4\pi \kappa_1} = \frac{(m_1 + \sigma_1)(R + M)^2 + a^2 - 2m_1 a_1 a^2 + a_1^2 (R^2 - M^2 + a^2)^2}{[(R + M)^2 + a^2][(R + \sigma_1)^2 - \sigma_1^2]},$$

$$A_2 = \frac{\sigma_2}{4\pi \kappa_2} = \frac{(m_2 + \sigma_2)(R + M)^2 + a^2 - 2m_2 a_2 a^2 + a_2^2 (R^2 - M^2 + a^2)^2}{[(R + M)^2 + a^2][(R + \sigma_2)^2 - \sigma_2^2]},$$

$$\Omega_1^H = \frac{m_1 - \sigma_1}{2m_1 a_1}, \quad \Omega_2^H = \frac{m_2 - \sigma_2}{2m_2 a_2},$$

(15)
and satisfying the Smarr mass formula [20]

\[ m_i = \frac{1}{4\pi} \kappa_i A_i + 2\Omega_i^H j_i = \sigma_i + 2\Omega_i^H j_i, \quad i = 1, 2, \]

are also valid for a black hole in the presence of a hyperextreme object. A direct check shows after some endeavor that this is indeed the case.

We now turn to consideration of a nontrivial special case not covered by the general formulas.

III. THE ‘SCHWARZSCHILD-KERR’ BINARY CONFIGURATION

The formulas presented in the previous section provide one with a powerful tool for studying the behavior and physical properties of two interacting arbitrary Kerr sources with almost the same ease as the known single black hole spacetimes. For the analysis of a particular binary configuration one only needs to assign concrete values to the physical parameters \( m_1, m_2, a_1, a_2 \) and \( R \), then find the corresponding \( a \) from the cubic equation (4), the quantities \( \sigma_1 \) and \( \sigma_2 \) from (5), and determine the functions \( r_\pm \) and \( R_\pm \) with the aid of formulas (9); these being substituted into (8) give the form of the Ernst potential, while the substitutions into (12) and (13) yield the metric of that particular binary configuration. However, one cannot assign zero value to \( a_1 \) or to \( a_2 \) in the general formulas (9) because a subtle degeneration then occurs and these formulas fail to describe the binary configuration correctly. This explains in particular why the case of a system comprised of a rotating and a non-rotating sources evaded the researchers for so many years like a real ghost, and for example was not extracted from the general expressions in the paper [8]. Note that by setting, say, \( a_1 = 0 \) in (9) we get \( r_\pm/\tilde{r}_\pm = \mu_0^{-1} \) and the solution becomes problematic. Therefore, this special case needs a separate analysis.

As a preliminary, let us first note that for some applications with yet unknown precise values of the masses and angular momenta of the constituents the resolution of the cubic equation (4) can be circumvented after introducing the constant \( a \) as arbitrary parameter instead of \( a_1 \) or \( a_2 \). Indeed, equation (4) is linear in \( a_i \) and hence can be trivially solved, say, for \( a_2 \), thus changing the set of arbitrary parameters \( \{m_1, m_2, a_1, a_2, R\} \) into a new set \( \{m_1, m_2, a_1, a, R\} \), the physical characteristic \( a_2 \) being related to the parameters of the latter set in a simple way by (4). Such a redefinition of the parameters, as will be seen below,
allows to describe correctly and in a concise form the desired long-searched-for binary system composed of a Schwarzschild black hole and a Kerr source separated by a massless strut.

The ‘Schwarzschild-Kerr’ configuration is a 4-parameter specialization of the general case when one of the angular momenta is set equal to zero. By choosing $a_1 = 0$, we convert the upper constituent into a Schwarzschild black hole, while the lower constituent is a rotating Kerr source that could be a black hole (real $\sigma_2$) or a naked singularity (pure imaginary $\sigma_2$). It follows trivially from (5) that vanishing of $a_1$ implies $\sigma_1 = m_1$, which means that the upper constituent can never be a naked singularity, no matter how large is the angular momentum per unit mass $a_2$ of the lower constituent. Note that although the condition $a_1 = 0$ causes drastic simplification of the expression for $\sigma_1$, the use of this condition in the expression for $\sigma_2$ does not lead at first glance to a considerable change in the aspect of that quantity and actually does not simplify much the resolution of the cubic equation (1) for $a$. However, as we have already mentioned, the latter equation can be easily solved for $a_2$, yielding

$$a_1 = 0, \quad a_2 = \frac{a[(R + M)^2 + a^2]}{(R + m_2)^2 - m_1^2 + a^2},$$

which permits us to introduce $a$ as arbitrary parameter instead of $a_2$ in all the formulas and simplify considerably the form of $\sigma_2$, getting

$$\sigma_2 = \sqrt{m_2^2 - a^2 + 2a^2\rho_0}, \quad \rho_0 := \frac{2m_1m_2}{(R + m_2)^2 - m_1^2 + a^2},$$

as well as the form of $s$, $\delta$, $\tau$ and $\mu$ in the axis data (1):

$$s = \frac{\rho_0}{2}[(R + m_2)^2 - m_1^2 - a^2] - \frac{R^2}{4}, \quad \delta = \frac{\rho_0a(R^2 - M^2 + a^2)}{2m_2},$$

$$\tau = \frac{1}{2}Ra - \rho_0a(R + M), \quad \mu = \frac{1}{2}R(m_1 - m_2) + \rho_0a^2.$$  

Then the substitution of (19) into the formula (72) of [1] leads to the correct expressions for $r_\pm$ and $R_\pm$,

$$r_\pm = \pm \frac{R \mp m_1 + m_2 + ia}{R \mp m_1 + m_2 - ia} \tilde{r}_\pm, \quad R_\pm = \frac{\mp \sigma_2 + ia(1 - \rho_0)}{m_2 - ia\rho_0} \tilde{R}_\pm,$$

$$\tilde{r}_\pm = \sqrt{\rho^2 + \left(z - \frac{1}{2}R \pm m_1\right)^2}, \quad \tilde{R}_\pm = \sqrt{\rho^2 + \left(z + \frac{1}{2}R \pm \sigma_2\right)^2},$$

thus resolving the problem of the limit $a_1 = 0$ in the general expressions (9) of the previous section.
The Ernst potential and all metric functions of the ‘Schwarzschild-Kerr’ binary configuration, accounting for (8) and (12), assume the following final form

\[
\mathcal{E} = \frac{A - B}{A + B}, \quad f = \frac{A\tilde{A} - B\tilde{B}}{(A + B)(A + B)}, \quad e^{2\gamma} = \frac{A\tilde{A} - B\tilde{B}}{16|\sigma_2|^2 K_0^2 \tilde{R}_+ \tilde{R}_- \tilde{r}_+ \tilde{r}_-},
\]

\[
\omega = 2a - \frac{2\text{Im}[G(\tilde{A} + \tilde{B})]}{AA - BB},
\]

\[
A = [R^2 - (m_1 + \sigma_2)^2](R_+ - R_-)(r_+ - r_-) - 4m_1\sigma_2(R_+ - r_-)(R_- - r_+),
\]

\[
B = 2m_1(R^2 - m_1^2 + \sigma_2^2)(R_- - R_+)(r_+ + r_- + R)
\]

\[
+ 4RM_1\sigma_2(R_+ + R_- - r_+ - r_-),
\]

\[
G = -zB + m_1(R^2 - m_1^2 + \sigma_2^2)(R_- - R_+)(r_+ + r_- + R)
\]

\[
+ \sigma_2(R^2 + m_1^2 - \sigma_2^2)(r_- - r_+)(R_+ + R_- - R)
\]

\[
- 2m_1\sigma_2\{2R[r_+ r_- - R_+ R_- - m_1(r_- - r_+) + \sigma_2(R_- - R_+)]
\]

\[
+ (m_1^2 - \sigma_2^2)(r_+ + r_- - R_+ - R_-),
\]

with

\[
K_0 = \frac{[(R + m_1)^2 - \sigma_2^2][(R + m_2)^2 - m_1^2 + a^2]}{m_2[(R + M)^2 + a^2]}.
\]

It is worthwhile noting that the static limit \(a = 0\) in the above metric does not represent any difficulty and leads to the Bach-Weyl solution for two Schwarzschild black holes [21].

A peculiar feature of the ‘Schwarzschild-Kerr’ configuration is that, whereas the upper constituent in it is always a black hole, the lower constituent can be either a black hole or a naked singularity, depending on the values of the rotational parameter \(a\). When both constituents are subextreme, the areas of their horizons assume the form

\[
A_1 = \frac{16\pi m_2^2[(R + M)^2 + a^2]}{(R + m_1)^2 - \sigma_2^2},
\]

\[
A_2 = \frac{8\pi m_2[(R + M)^2 + a^2]((m_2 + \sigma_2)[(R + m_1)^2 - m_1^2 + a^2] - 2a^2\rho_0(R + M))}{[(R + m_2)^2 - m_1^2 + a^2][(R + m_1)^2 - \sigma_2^2]},
\]

and one can see that these are substantially affected by the black hole interaction. At the same time, in the case of balance \((a = \pm (R + M))\), when gravitational attraction is equal to spin-spin repulsion, the formula for \(A_2\) is no longer valid because \(\sigma_2\) then becomes a pure imaginary quantity [11]

\[
\sigma_2 = i\sqrt{R^2 - m_1^2 + 2R[M + m_1^2(R + m_2)^{-1}]},
\]
which means that in the equilibrium state the lower constituent is a naked singularity. Nevertheless, the expression for $A_1$ still holds and coincides with the one obtained in \[11\]: $A_1 = 16\pi m_1^2(1 + m_2/R)$. The solution (21) describes correctly both sectors of the ‘Schwarzschild-Kerr’ configuration, thus giving unified treatment of the subextreme and hyperextreme sources as inseparable ingredients of the same global spacetime.

\section*{IV. CONCLUSION}

The construction of the solution for two arbitrary Kerr sources separated by a massless strut in a completely physical parametrization may be considered as an important new contribution into the list of physically meaningful spacetimes of Einstein’s general relativity. To a certain extent, the 5-parameter metric presented in this paper draws a line beneath a long period of extensive studies of the famous double-Kerr solution that had started immediately after its discovery by Kramer and Neugebauer \[22\] almost four decades ago. Although many physical properties of the black-hole and hyperextreme sectors of the double-Kerr spacetime were clarified in the 80s of the last century \[8, 23–27\], it was eventually the extended version of the double-Kerr solution \[9, 28\] obtained within the framework of Sibgatullin’s integral method \[14\] that made possible the unified treatment of the sub- and hyperextreme Kerr constituents in binary configurations and the introduction of the physical parametrizations on the basis of the standard parameters of the extended solution. After obtaining a physical representation of the double-Kerr equilibrium problem \[17\], we were convinced that a more general binary configuration of arbitrary Kerr sources separated by a massless strut can be also rewritten in terms of physical parameters, and we are glad that our expectations have finally come true.

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FIG. 1: Three types of binary configurations composed of nonextreme Kerr sources: (a) ‘subextreme-subextreme’, (b)-(c) ‘subextreme-hyperextreme’, and (d) ‘hyperextreme-hyperextreme’.