ANALYSIS OF HYBRID METHODS OF MIXED-SHEAR-PROJECTED TRIANGULAR AND QUADRILATERAL ELEMENTS FOR REISSNER-MINDLIN PLATES

GUOZHU YU\(^1\), XIAOPING XIE\(^2,\star\), AND YUANHUI GUO\(^3\)

\(^1\)School of Mathematics, Southwest Jiaotong University, Chengdu 610031, China
\(^2\)School of Mathematics, Sichuan University, Chengdu 610064, China
\(^3\)Experiment Center, China West Normal University, Nanchong, Sichuan 637009, China

Abstract. It is known that the 3-node hybrid triangular element MiSP3 and 4-node hybrid quadrilateral element MiSP4 presented by Ayad, Dhatt and Batoz (Int. J. Numer. Meth. Engng 1998, 42: 1149-1179) for Reissner-Mindlin plates behave robustly in numerical benchmark tests. These two elements are based on Hellinger-Reissner variational principle, where continuous piecewise linear/isoparametric bilinear interpolations, as well as the mixed shear interpolation/projection technique of MITC family, are used for the approximations of displacements, and piecewise-independent equilibrium modes are used for the approximation of bending moments/shear stresses. We show that the MiSP3 and MiSP4 elements are uniformly stable with respect to the plate thickness and thus free from shear-locking.

1. Introduction

Due to avoidance of \(C^1\)-continuity difficulty, the Reissner-Mindlin (R-M) plate model is today the dominating two-dimensional model used to calculate the bending of a thick/thin three-dimensional plate of thickness \(t\). It’s well-known that for values of \(t\) close to zero, the standard low-order finite element discretization of this model suffers from shear locking ([1, 23]).

To overcome the shear locking difficulty and derive ‘locking-free’ or robust plate bending elements that are valid for the analysis of thick and thin plates, significant efforts are devoted to the development of simple and efficient triangular and quadrilateral finite elements in the past few decades. The most common approach is to modify the variational formulation with some reduction operator so as to weaken the Kirchhoff constraint (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 17, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] and the references therein).

Among the existing elements, the family of finite elements named mixed interpolated tensorial components (MITC) by Bathe et. al [4, 5] is one of the most attractive representative. By virtue of an independent shear approximation and a discrete Mindlin technique along edges, MITC elements define the shear strains in terms of the edge tangential strains that are projected on the element degrees of freedom. As the lowest order quadrilateral MITC element, the 4-node plate element MITC4 is very likely the most used in practice. Unfortunately, there is no so called low order triangular ‘MITC3’ element. In other words, the 3-node

\*: Corresponding author.

Email addresses: yuguozhumail@gmail.com (G. Yu), xpxie@scu.edu.cn (X. Xie), gyh6209@sina.com (Y. Guo).
plate element MITC3 defined with the same technique of shear interpolation produces very unsatisfactory results, and, in general, it needs some kind of stabilization [12].

With the same technique of shear interpolation as in the element MITC family, Ayad, Dhatt and Batoz [3] presented an improved formulation for obtaining locking-free triangular and quadrilateral elements, which are called MiSP3 and MiSP4 elements respectively. It is based on Hellinger-Reissner variational principle, including variables of displacements, shear stresses and bending moments. For MiSP3 element continuous piecewise linear interpolation is used for the approximations of displacements, and a piecewise-independent equilibrium mode is used for the approximation of bending moments/shear stresses. While for MiSP4 element it adopts continuous isoparametric bilinear displacement interpolation. The numerical experiments in [3] showed that the MiSP3 and MiSP4 elements both avoid locking phenomenon. However, so far there is no uniform stability analysis for them with respect to plate thickness.

The main goal of this work is to establish uniform convergence for triangular MiSP3 element and quadrilateral MiSP4 element. The key to the analysis of MiSP3 is the discrete Helmholtz decomposition in Lemma 4.2, while for MiSP4 we use the property of the shear interpolation (Lemma 5.11) proved in [16].

We arrange the rest of this paper as follows. In Section 2 we give weak formulations of the model. Section 3 introduces the finite element spaces for MiSP3 and MiSP4 elements. We derive in Sections 4-5 uniform error estimates for MiSP3 and MiSP4 elements, respectively. Finally in Section 6 we provide some numerical results to verify the theoretical results.

For convenience, throughout the paper we use the notation $a \lesssim b$ to represent that there exists a generic positive constant $C$, independent of the mesh parameter $h$ and the plate thickness $t$, such that $a \leq Cb$. We also abbreviate $a \lesssim b \lesssim a \approx b$.

We will also use various standard differential operators:

$$\text{grad } r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}\right)^T, \quad \text{curl } p = \left(\frac{\partial p}{\partial y}, -\frac{\partial p}{\partial x}\right)^T, \quad \text{div } \psi = \frac{\partial \psi_1}{\partial x} + \frac{\partial \psi_2}{\partial y}, \quad \text{rot } \psi = \frac{\partial \psi_1}{\partial y} - \frac{\partial \psi_2}{\partial x}.$$  

2. WEAK PROBLEM

The Reissner-Mindlin model for the bending of a clamped isotropic elastic plate in equilibrium reads as: Find $(w, \beta) \in H^1_0(\Omega) \times H^1_0(\Omega)^2$ such that

\begin{align}
    -\text{div} D\varepsilon(\beta) - \lambda t^{-2}(\text{grad } w - \beta) &= 0 \quad \text{in } \Omega, \\
    -\lambda t^{-2}\text{div}(\text{grad } w - \beta) &= g \quad \text{in } \Omega.
\end{align}

Here $\Omega \subset \mathbb{R}^2$, assumed to be a convex polygon for simplicity, is the region occupied by the midsection of the plate with thickness $t$, $w$ and $\beta$ denote respectively the transverse displacement of the midplane and the rotation of the fibers normal to it, $\varepsilon(\beta)$ is the symmetric part of the gradient of $\beta$, $g$ is the transverse loading, $D$ is the elastic module tensor defined by

$$DQ = \frac{E}{12(1-\nu^2)}[(1-\nu)Q + \nu\text{tr}(Q)I]$$

with $Q$ a $2 \times 2$ symmetric matrix, $\lambda = \frac{E\kappa}{2(1+\nu)}$ with $E$ the Young’s modulus, $\nu$ the Poisson’s ratio, and $\kappa = \frac{5}{6}$ the shear correction factor.

Set

$$M := L^2(\Omega)^{2 \times 2}_{\text{sym}}, \quad \Gamma := L^2(\Omega)^2, \quad W := H^1_0(\Omega), \quad \Theta := H^1_0(\Omega)^2.$$
When introducing the shear stress vector \( \gamma = \lambda t^{-2}(\text{grad} w - \beta) \) and the bending moment tensor \( \mathbf{M} = -\mathcal{D}\mathbf{e}(\beta) \), the model problem (2.1)-(2.2) changes into the following system: Find \( (\mathbf{M}, \gamma, w, \beta) \in \mathbb{M} \times \Gamma \times W \times \Theta \) such that

\[
\begin{align*}
\text{div}\mathbf{M} - \gamma &= 0 \quad \text{in } \Omega, \\
\text{div}\gamma + g &= 0 \quad \text{in } \Omega, \\
\mathbf{M} + \mathcal{D}\mathbf{e}(\beta) &= 0 \quad \text{in } \Omega, \\
\gamma - \lambda t^{-2}(\text{grad} w - \beta) &= 0 \quad \text{in } \Omega.
\end{align*}
\]

The variational formulation of this system reads: Find \( (\mathbf{M}, \gamma, w, \beta) \in \mathbb{M} \times \Gamma \times W \times \Theta \) such that

\[
\begin{align*}
\mathbf{a}(\mathbf{M}, \gamma; Q, \tau) + \mathbf{b}(Q, \tau; w, \beta) &= 0 \quad \text{for all } (Q, \tau) \in \mathbb{M} \times \Gamma, \\
\mathbf{b}(\mathbf{M}, \gamma; v, \zeta) &= -\int_{\Omega} gvd\mathbf{x} \quad \text{for all } (v, \zeta) \in W \times \Theta,
\end{align*}
\]

where the bilinear forms

\[
\begin{align*}
\mathbf{a}(:, :, :, :) : (L^2(\Omega))^{2 \times 2}_{\sym} \times L^2(\Omega)^2 \times (L^2(\Omega))^{2 \times 2}_{\sym} \times L^2(\Omega)^2) &\rightarrow \mathbb{R}, \\
\mathbf{b}(\cdot, \cdot, \cdot, \cdot) : (L^2(\Omega))^{2 \times 2} \times L^2(\Omega)^2 \times (H^1_0(\Omega)) \times H^1_0(\Omega)^2) &\rightarrow \mathbb{R}
\end{align*}
\]

are defined by

\[
\begin{align*}
\mathbf{a}(\mathbf{M}, \gamma; Q, \tau) &= \int_{\Omega} \mathbf{M} : \mathcal{D}^{-1}Qd\mathbf{x} + \frac{t^2}{\lambda} \int_{\Omega} \gamma \cdot \tau d\mathbf{x}, \\
b(Q, \tau; v, \zeta) &= \int_{\Omega} Q : \mathbf{e}(\zeta)d\mathbf{x} - \int_{\Omega} \tau \cdot (\text{grad} v - \zeta)d\mathbf{x}.
\end{align*}
\]

In the latter analysis we will use the Helmholtz theorem: for any \( \tau \in L^2(\Omega)^2 \),

\[
\tau = \text{grad} s + \text{curl} q, \quad \text{with } (s, q) \in H^1_0(\Omega) \times \tilde{H}^1(\Omega),
\]

where \( \tilde{H}^1(\Omega) : = \{ q \in H^1(\Omega) : \int_{\Omega} qd\mathbf{x} = 0 \} \).

Then the shear strain vector \( \gamma \) can be decomposed as

\[
\gamma = \text{grad} r + \text{curl} p
\]

with \( (r, p) \in H^1_0(\Omega) \times \tilde{H}^1(\Omega) \). Moreover, since \( \gamma \cdot t = 0 \) on \( \partial \Omega \), the decomposition (2.12) indicates that \( p \) satisfies

\[
\text{grad} p \cdot \mathbf{n} = 0 \quad \partial \Omega,
\]

where \( \mathbf{t}, \mathbf{n} \) are respectively the unit tangent vector and unit outer normal vector along \( \partial \Omega \).

Then the model problem (2.1)-(2.2) is also equivalent to the following system:

Find \( (r, \beta, p, w) \in H^1_0(\Omega) \times H^1_0(\Omega)^2 \times \tilde{H}^1(\Omega) \times H^1_0(\Omega) \) such that

\[
\begin{align*}
\text{grad } r, \text{grad } v &= (g, v), \quad \forall v \in H^1_0(\Omega), \\
(\mathbf{e}(\beta), \mathbf{D}\mathbf{e}(\zeta)) - (\text{curl} p, \zeta) &= (\text{grad } r, \zeta), \quad \forall \zeta \in H^1_0(\Omega)^2, \\
-\beta, \text{curl } q - \frac{t^2}{\lambda} (\text{curl } p, \text{curl } q) &= 0, \quad \forall q \in \tilde{H}^1(\Omega), \\
(\text{grad } w, \text{grad } s) &= (\beta + \frac{t^2}{\lambda} \text{grad } r, \text{grad } s), \quad \forall s \in H^1_0(\Omega).
\end{align*}
\]

The following regularity results were proved by Arnold and Falk [2].
Theorem 2.1. Let $\Omega$ be a convex polygon or smoothly bounded domain in the plane. For any $t \in (0,1]$ and any $g \in L^2(\Omega)$, there exists a unique quadruple $(r, \beta, p, w) \in H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ solving problem (2.13)-(2.16). Moreover, there exists a constant $C$ independent of $t$ and $g$, such that

$$
\|w\|_2 + \|\beta\|_2 + \|r\|_2 + \|p\|_1 + t\|p\|_2 \leq C\|g\|_0.
$$

With the above theorem, we obtain some further results:

Theorem 2.2. Let $(r, \beta, p, w)$ be the solution of the problem (2.13)-(2.16). Then the following three conclusions (i)-(iii) hold.

(i) The quadruple $(M = -D\epsilon(\beta), \gamma = \text{grad} r + \text{curl} p, w, \beta) \in M \times \Gamma \times W \times \Theta$ is the unique solution of the problem (2.7)-(2.8);

(ii) If $M \in H(\text{div}; \Omega) := \{Q \in L^2(\Omega)^{2\times 2} : \text{div} Q \in L^2(\Omega)^2\}$, then the equilibrium relation (2.3) holds;

(iii) Provided that $g \in L^2(\Omega)$, it holds

$$
\|w\|_2 + \|\beta\|_2 + \|M\|_1 + \|\gamma\|_0 + t\|\gamma\|_1 + \|r\|_2 + \|p\|_1 + t\|p\|_2 \lesssim \|g\|_0.
$$

3. Finite element formulations for MiSP method

This section is devoted to the finite element formulations of the MiSP element on triangular and quadrilateral meshes. Let $T_h$ be a regular family of finite element subdivisions of the polygonal domain $\Omega$. We denote by $h_K$ the diameter of a triangle or a quadrilateral $K \in T_h$, and denote $h := \max_{K \in T_h} h_K$.

Let $M_h \subset M$, $\Gamma_h \subset \Gamma$, $W_h \subset W$, $\Theta_h \subset \Theta$ be finite dimensional spaces for the bending moment, shear stress, transverse displacement, and rotation approximations. Then the corresponding finite element scheme for the problem (2.7)-(2.8) reads as: Find $(M_h, \gamma_h, w_h, \beta_h) \in M_h \times \Gamma_h \times W_h \times \Theta_h$ such that

$$
a(M_h, \gamma_h; Q_h, \tau_h) + \bar{b}(Q_h, \tau_h; w_h, \beta_h) = 0 \quad \text{for all } (Q_h, \tau_h) \in M_h \times \Gamma_h,
$$

$$
\bar{b}(M_h, \gamma_h; v_h, \zeta_h) = -\int_{\Omega} g v_h dx \quad \text{for all } (v_h, \zeta_h) \in W_h \times \Theta_h,
$$

and the reduction operator

$$
R_h : H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega) \to Z_h
$$

is defined by [16]

$$
\int_e R_h \psi \cdot t_e = \int_e \psi \cdot t_e, \forall \text{ edge } e \text{ of } T_h,
$$

where

$$
H_0(\text{rot}, \Omega) := \{\psi \in L^2(\Omega)^2 : \text{rot} \psi \in L^2(\Omega), \psi \cdot t|_{\partial \Omega} = 0\},
$$

$Z_h$ is to be defined in (3.13) for MiSP3 and in (3.21) for MiSP4, respectively, and $t_e$ denotes a unit vector tangent to $e$. 
For both MiSP3 and MiSP4 elements, we define
\begin{equation}
\Gamma_h = \text{div}_h \mathbb{M}_h, \quad \text{with } (Q_h, \tau_h) = (\text{div}_h Q_h, \text{div}_h T_h)
\end{equation}
for \( Q_h \in \mathbb{M}_h \). Here \( \text{div}_h \) denotes the divergence operator piecewise defined with respect to \( T_h \).

From the definition of the space \( \Gamma_h \), we have an equivalent form of the discrete scheme (3.1)-(3.2): Find \((M_h, w_h, Q_h) \in \mathbb{M}_h \times W_h \times \Theta_h \) such that
\begin{align}
& a(M_h, \text{div}_h M_h; Q_h, \text{div}_h Q_h) + \tilde{b}(Q_h, \text{div}_h Q_h; w_h, \beta_h) = 0 \quad \text{for all } Q_h \in \mathbb{M}_h, \quad (3.8) \\
& \tilde{b}(M_h, \text{div}_h M_h; v_h, \zeta_h) = -\int_{\Omega} g v_h \, dx \quad \text{for all } (v_h, \zeta_h) \in W_h \times \Theta_h. \quad (3.9)
\end{align}

### 3.1. Finite Dimensional Subspaces for MiSP3.

Let \( T_h \) be a conventional triangular mesh of \( \Omega \). For element MiSP3, the continuous piecewise linear interpolation is used for the transverse displacement and rotation approximation, i.e. the transverse displacement space \( W_h \) and rotation space \( \Theta_h \) are chosen as
\begin{align}
& W_h := \{v_h \in H^1_0(\Omega) \cap C(\bar{\Omega}): v_h|_K \in P_1(K) \text{ for all } K \in T_h\}, \quad (3.10) \\
& \Theta_h := \{\zeta_h \in (H^1_0(\Omega) \cap C(\bar{\Omega}))^2:\zeta_h|_K \in P_1(K)^2 \text{ for all } K \in T_h\}. \quad (3.11)
\end{align}

Here \( P_1(K) \) denotes the set of linear polynomials on \( K \).

For the approximation of bending moment tensor, we define
\begin{equation}
\mathbb{M}_h := \{Q_h \in L^2(\Omega)^{2 \times 2}_{\text{sym}} : (Q_h|_K)_{i,j} \in P_1(K) \text{ for all } K \in T_h, i, j = 1, 2\}. \quad (3.12)
\end{equation}

We take the space \( Z_h \) in (3.4) as
\begin{equation}
Z_h := \left\{\psi_h \in H_0(\text{rot}, \Omega): \psi_h|_K = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 0 \\ -x \end{pmatrix} \right\}, \text{ for all } K \in T_h \right\}. \quad (3.13)
\end{equation}

We also need the space
\begin{equation}
P_h := \{q_h \in L^2_0(\Omega): q_h|_K \in P_1(K) \text{ for all } K \in T_h, q_h \text{ is continuous at midpoints of element edges}\}. \quad (3.14)
\end{equation}

### 3.2. Finite Dimensional Subspaces for MiSP4.

Let \( T_h \) be a conventional quadrilateral mesh of \( \Omega \). Let \( Z_i(x_i, y_i), 1 \leq i \leq 4 \) be the four vertices of \( K \), and \( T_i \) be the sub-triangle of \( K \) with vertices \( Z_{i-1}, Z_i \) and \( Z_{i+1} \) (the index on \( Z_i \) is modulo 4). Define
\begin{equation}
\rho_K = \min_{1 \leq i \leq 4} \{\text{diameter of circle inscribed in } T_i\}. \quad (3.15)
\end{equation}

Throughout the paper, we assume that the partition \( T_h \) satisfies the following ‘shape-regularity’ hypothesis: There exists a constant \( \varrho > 2 \) independent of \( h \) such that for all \( K \in T_h \),
\begin{equation}
h_K \leq \varrho \rho_K. \quad (3.15)
\end{equation}

Let \( \hat{K} = [-1, 1] \times [-1, 1] \) be the reference square with vertices \( \hat{Z}_i, 1 \leq i \leq 4 \). For a quadrilateral \( K \in T_h \), there exists a unique invertible mapping \( F_K \) that maps \( \hat{K} \) onto \( K \) with \( F_K(\xi, \eta) \in Q^2_1(\xi, \eta) \) and \( F_K(Z_i) = Z_i, 1 \leq i \leq 4 \) (Figure 3.1). Here \( \xi, \eta \in [-1, 1] \) are the local isoparametric coordinates.
This isoparametric bilinear mapping \((x, y) = F_K(\xi, \eta)\) is given by
\[
(3.16) \quad x = \sum_{i=1}^{4} x_i N_i(\xi, \eta), \quad y = \sum_{i=1}^{4} y_i N_i(\xi, \eta),
\]
where
\[
N_1 = \frac{1}{4}(1 - \xi)(1 - \eta), \quad N_2 = \frac{1}{4}(1 + \xi)(1 - \eta), \quad N_3 = \frac{1}{4}(1 + \xi)(1 + \eta), \quad N_4 = \frac{1}{4}(1 - \xi)(1 + \eta).
\]

We can rewrite (3.16) as
\[
(3.17) \quad x = a_0 + a_1 \xi + a_2 \eta + a_{12} \xi \eta, \quad y = b_0 + b_1 \xi + b_2 \eta + b_{12} \xi \eta,
\]
with
\[
\begin{pmatrix}
  a_0 & b_0 \\
  a_1 & b_1 \\
  a_2 & b_2 \\
  a_{12} & b_{12}
\end{pmatrix} = \frac{1}{4}
\begin{pmatrix}
  1 & 1 & 1 & 1 \\
  -1 & 1 & 1 & -1 \\
  -1 & -1 & 1 & 1 \\
  1 & -1 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
  x_1 & y_1 \\
  x_2 & y_2 \\
  x_3 & y_3 \\
  x_4 & y_4
\end{pmatrix}.
\]

The Jacobi matrix and the Jacobian of the transformation \(F_K\) are respectively given by
\[
DF_K(\xi, \eta) = \begin{pmatrix}
  \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
  \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{pmatrix} = \begin{pmatrix}
  a_1 + a_{12} \eta & a_2 + a_{12} \xi \\
  b_1 + b_{12} \eta & b_2 + b_{12} \xi
\end{pmatrix},
\]
\[
J_K = \det(DF_K) = J_0 + J_1 \xi + J_2 \eta,
\]
where
\[
J_0 = a_1 b_2 - a_2 b_1, \quad J_1 = a_1 b_{12} - a_{12} b_1, \quad J_2 = a_{12} b_2 - a_2 b_{12}.
\]

**Remark 3.1.** Notice that when \(K\) is a parallelogram, we have \(a_{12} = b_{12} = 0\), and \(F_K\) is reduced to an affine mapping. Especially, when \(K\) is a rectangle, we further have \(a_2 = b_1 = 0\).

For element MiSP4, the continuous isoparametric bilinear interpolation is used for the transverse displacement and rotation approximation, i.e. the transverse displacement space \(W_h\) and rotation space \(\Theta_h\) are chosen as
\[
W_h := \{ v_h \in H^1_0(\Omega) \bigcap C(\overline{\Omega}) : v_h|_K \circ F_K \in Q_1(\hat{K}) \text{ for all } K \in \mathcal{T}_h \},
\]
\[
(3.18) \quad \Theta_h := \{ \zeta_h \in (H^1_0(\Omega) \bigcap C(\Omega))^2 : \zeta_h|_K \circ F_K \in Q_1(\hat{K})^2 \text{ for all } K \in \mathcal{T}_h \}.
\]
Here $Q_1(\hat{K})$ denotes the set of bilinear polynomials on $\hat{K}$. For the approximation of bending moment tensor, we define

\begin{equation}
(3.20) \quad M_h := \{Q_h \in L^2(\Omega)^{2 \times 2}_{sym} : (Q_h|_K \circ F_K)_{i,j} \in Q_1(\hat{K}) \text{ for all } K \in T_h, i, j = 1, 2\}.
\end{equation}

We take the space $Z_h$ in (3.4) as

\begin{equation}
(3.21) \quad Z_h := \{\psi_h \in H_0(\text{rot}, \Omega) : \psi_h|_K \circ F_K = \text{span}\{DF^{-1}_K \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi \end{pmatrix} \}, \text{ for all } K \in T_h\}.
\end{equation}

4. Error Analysis for MiSP3

In this section we will derive error estimates for the MiSP3 element. The corresponding subspaces in this section are defined as in subsection 3.1. We first give the following properties for the operator $R_h$.

**Lemma 4.1.** The operator $R_h : H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega) \rightarrow Z_h$ satisfies

\begin{equation}
(4.1) \quad R_h(\text{grad } v_h) = \text{grad } v_h, \forall v_h \in W_h,
\end{equation}

\begin{equation}
(4.2) \quad \|\eta - R_h \eta\|_0 \lesssim h \|\eta\|_1, \forall \eta \in H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega),
\end{equation}

\begin{equation}
(4.3) \quad \|\text{rot}(R_h \eta)\|_0 \lesssim \|\eta\|_1, \forall \eta \in H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega),
\end{equation}

\begin{equation}
(4.4) \quad \text{rot}(R_h \eta_h) = \text{rot}(\eta_h), \forall \eta_h \in \Theta_h.
\end{equation}

**Proof.** It is easy to verify $\text{grad } W_h \subset Z_h$ and $R_h \psi_h = \psi_h, \forall \psi_h \in Z_h$. Then (4.1) holds. The estimate (4.2) follows from a scaling argument and the definition of $R_h$.

For $\eta \in H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega)$, let $\Pi_h \eta$ be the Scott-Zhang interpolation [?] of $\eta$. Then we have

\begin{equation}
\|\text{rot}(R_h \eta)\|_0 \leq \|\text{rot}(R_h \eta) - \text{rot}(\Pi_h \eta)\|_0 + \|\text{rot}(\Pi_h \eta)\|_0 \\
\lesssim h^{-1} \|R_h \eta - \Pi_h \eta\|_0 + \|\eta\|_1 \\
\leq h^{-1} (\|R_h \eta - \eta\|_0 + \|\eta - \Pi_h \eta\|_0) + \|\eta\|_1 \\
\lesssim \|\eta\|_1.
\end{equation}

Here, the second inequality is based on an inverse inequality and the stability of Scott-Zhang interpolation. Hence (4.3) holds.

For any $K \in T_h$, it holds

\begin{equation}
\int_K \text{rot}(R_h \eta_h) dx = \int_{\partial K} R_h \eta_h \cdot t ds = \int_{\partial K} \eta_h \cdot t ds = \int_K \text{rot}(\eta_h) dx,
\end{equation}

since $\text{rot}(\eta_h)|_K$ and $\text{rot}(R_h \eta_h)|_K$ are constants, we have $\text{rot}(\eta_h)|_K = \text{rot}(R_h \eta_h)|_K$, which yields (4.4).

For the latter error analysis, we need the following discrete Helmholtz decomposition given in Theorem 4.1 of [14].

**Lemma 4.2.** For any $Q_h \in M_h$, there exist $s_h \in W_h$ and $q_h \in P_h$ such that

\begin{equation}
(4.5) \quad \text{div}_h Q_h = \text{grad } s_h + \text{curl}_h q_h.
\end{equation}
In the latter analysis, we will use the discrete Helmholtz decomposition (4.5) for \( Q \in \mathbb{M}_h \) and the Helmholtz decomposition (2.11) with \( \tau = \text{div} Q \) for \( Q \in (H^1(\Omega))^2 \). For convenience, we denote the decomposition as \( \text{div}_h Q = \text{grad} s + \text{curl}_h q \) in both cases.

We introduce two mesh-dependent norms as follows: for any \( Q \in (H^1(\Omega))^2 \cup \mathbb{M}_h \), \( v \in (H^2(\Omega) \cap H^1_0(\Omega)) \cup W_h \), \( \zeta \in H^1_0(\Omega)^2 \cup \Theta_h \),

\[
|||Q|||_{h,1} := ||Q||_0 + (h + t)||\text{curl}_h q||_0 + ||q||_0 + ||\text{grad} s||_0,
\]

\[
|||v, \zeta|||_{h,2} := ||\epsilon(\zeta)||_0 + ||R_h(\text{grad} v)||_0.
\]

We are now ready to give the error analysis. Basing on the standard error theory for mixed methods, we first show continuity results in Lemmas 4.3-4.4, then derive coercivity results in Lemmas 4.5-4.6, we finally give the desired estimates in Theorem 4.7.

**Lemma 4.3.** It holds

\[
a(M, \text{div}_h M; Q, \text{div}_h Q) \lesssim |||M|||_{h,1}|||Q|||_{h,1} \quad \text{for all } M, Q \in \mathbb{M} \cup \mathbb{M}_h.
\]

**Proof.** It is trivial. \( \square \)

**Lemma 4.4.** For any \( Q \in (H^1(\Omega))^2 \cup \mathbb{M}_h \), \( v \in (H^2(\Omega) \cap H^1_0(\Omega)) \cup W_h \), \( \zeta \in H^1_0(\Omega)^2 \cup \Theta_h \), it holds

\[
\bar{b}(Q, \text{div}_h Q; v, \zeta) \lesssim |||Q|||_{h,1}|||v, \zeta|||_{h,2}.
\]

**Proof.** Given \( Q \in (H^1(\Omega))^2 \cup \mathbb{M}_h \), by (2.11) and (4.5), we have

\[
\text{div}_h Q = \text{grad} s + \text{curl}_h q.
\]

We first show

\[
\text{curl}_h q, R_h(\text{grad} v) = 0
\]

holds. By integration by parts, we have

\[
\text{curl}_h q, R_h(\text{grad} v) = \sum_{K \in \mathcal{T}_h} \left( -(q, \text{rot}(R_h(\text{grad} v)))_K + \int_{\partial K} q R_h(\text{grad} v) \cdot t \, ds \right)
\]

\[
= - \sum_{K \in \mathcal{T}_h} (q, \text{rot}(R_h(\text{grad} v)))_K + \sum_{e \in \mathcal{E}_h} \int_e [q] R_h(\text{grad} v) \cdot t \, ds,
\]

here, \( \varepsilon_h \) denotes the set of interior edges for \( \mathcal{T}_h \), and \([q]_e\) means the jump across the edge \( e \). We only need to verify the two terms of (4.11) both vanish.

Since \( \text{rot}(R_h(\text{grad} v)) \) is a piecewise constant, and, for any \( K \in \mathcal{T}_h \),

\[
\int_K \text{rot}(R_h(\text{grad} v)) \, dx = \int_{\partial K} R_h(\text{grad} v) \cdot t \, ds = \int_{\partial K} (\text{grad} v) \cdot t \, ds = 0,
\]

we have \( \text{rot}(R_h(\text{grad} v))|_K = 0 \). So, the first term of (4.11) equals zero.

For the second term, if \( Q \in (H^1(\Omega))^2 \), it equals zero by continuity. Otherwise if \( Q \in \mathbb{M}_h \), since \( q \in P_h \), \([q]_e\) vanishes at the midpoint of \( e \) and \([q] R_h(\text{grad} v) \cdot t |_e\) is linear, then by one-point Gauss integration we know the second term equals zero.
Now with (4.10), we can deduce the desired result:

\[ \tilde{b}(Q_h, \text{div}_h Q_h; v, \zeta) = (Q_h, \epsilon(\zeta)) - (\text{div}_h Q_h, R_h(\text{grad} v - \zeta)) \]

\[ = (Q_h, \epsilon(\zeta)) - (\text{grad} s + \text{curl}_h q_h, R_h(\text{grad} v - \zeta)) \]

\[ = (Q_h, \epsilon(\zeta)) - (\text{grad} \cdot R_h(\text{grad} v - \zeta) - q_h, \text{rot}(R_h \zeta)) \]

\[ = \|Q_h\|_0 \epsilon(\zeta) + \|\text{grad} s\|_0 \|R_h(\text{grad} v - \zeta)\|_0 + \|q_h\|_0 \|\text{rot}(R_h \zeta)\|_0 \]

\[ \lesssim (\|Q_h\|_0 + \|\text{grad} s\|_0 + \|q_h\|_0) (\|\epsilon(\zeta)\|_0 + \|R_h(\text{grad} v - \zeta)\|_0 + \|\zeta\|_1) \]

\[ \lesssim \|Q_h\|_{h,1} \|\zeta\|_{h,2}. \]

□

**Lemma 4.5.** It holds

\[ a(Q_h, \text{div}_h Q_h; Q_h, \text{div}_h Q_h) \gtrsim \|Q_h\|_{h,1}^2, \text{ for all } Q_h \in \text{Ker} B, \]

here,

\[ \text{Ker} B = \left\{ Q_h \in M_h : \tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h) = 0, \text{ for all } (v_h, \zeta_h) \in W_h \times \Theta_h \right\}. \]

**Proof.** We want to check the property of \( Q_h \in \text{Ker} B \). Based on Lemma 4.2, there exist \( s_h \in W_h \) and \( q_h \in P_h \) such that

\[ \text{div}_h Q_h = \text{grad} s_h + \text{curl}_h q_h. \]

It is easy to have

\[ a(Q_h, \text{div}_h Q_h; Q_h, \text{div}_h Q_h) \gtrsim \|Q_h\|_0^2 + t^2 \|\text{div}_h Q_h\|_0^2. \]

By the inverse inequality \( \|Q_h\|_0^2 \gtrsim h^2 \|\text{div}_h Q_h\|_0^2 \) and the relation \( \|\text{div}_h Q_h\|_0^2 = \|\text{grad} s_h\|_0^2 + \|\text{curl}_h q_h\|_0^2 \), we have

\[ a(Q_h, \text{div}_h Q_h; Q_h, \text{div}_h Q_h) \gtrsim \|Q_h\|_0^2 + (t + h)^2 \|\text{curl}_h q_h\|_0^2. \]

We next need to bound \( \|\text{grad} s_h\|_0 \) and \( \|q_h\|_0 \). For any \( (v_h, \zeta_h) \in W_h \times \Theta_h \), it holds

\[ \tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h) = (Q_h, \epsilon(\zeta_h)) - (\text{grad} s_h, \text{grad} v_h - R_h \zeta_h) + (\text{curl}_h q_h, R_h \zeta_h) = 0. \]

On one hand, choose \( \zeta_h = 0 \) and \( v_h = s_h \), then \( \text{grad} s_h, \text{grad} s_h = 0 \). Since \( s_h \in H^1_0(\Omega) \), we have \( s_h = 0 \).

On the other hand, choose \( v_h = 0 \), then

\[ (Q_h, \epsilon(\zeta_h)) + (\text{curl}_h q_h, R_h \zeta_h) = (Q_h, \epsilon(\zeta_h)) - (q_h, \text{rot}(R_h \zeta_h)) \]

\[ = (Q_h, \epsilon(\zeta_h)) - (q_h, \text{rot}(\zeta_h)) = 0, \text{ for any } \zeta_h \in \Theta_h. \]

For the above \( q_h \), there exists \( \zeta \in H^1_0(\Omega) \), such that

\[ \text{rot} \zeta = q_h, \text{ and } \|\zeta\|_1 \lesssim \|q_h\|_0. \]

So we get

\[ \|q_h\|_0 \lesssim \frac{(q_h, \text{rot} \zeta)}{\|\zeta\|_1} = \frac{(q_h, \text{rot}(\Pi_0 \zeta))}{\|\zeta\|_1} + \frac{(q_h, \text{rot}(R_h \zeta - \Pi_0 \zeta))}{\|\zeta\|_1} + \frac{(q_h, \text{rot}(\zeta - R_h \zeta))}{\|\zeta\|_1}. \]

For the first term in the right-hand side of this relation, it holds

\[ \frac{(q_h, \text{rot}(\Pi_0 \zeta))}{\|\zeta\|_1} = \frac{(Q_h, \epsilon(\Pi_0 \zeta))}{\|\zeta\|_1} \lesssim \frac{(Q_h, \epsilon(\Pi_0 \zeta))}{\|\Pi_0 \zeta\|_1} \leq \|Q_h\|_0. \]
For the second term, it holds
\[
\frac{(q_h, \text{rot}(R_h \zeta - \Pi_h \zeta))}{\|\zeta\|_1} = -\frac{\langle \text{curl}_h q_h, R_h (\zeta - \Pi_h \zeta) \rangle}{\|\zeta\|_1}
\]
\[
\lesssim h \|\text{curl}_h q_h\|_0.
\]
For the third term, it holds
\[
\frac{(q_h, \text{rot}(\zeta - R_h \zeta))}{\|\zeta\|_1} = \frac{\Sigma_{K \in T_h}(q_h, \text{rot}(\zeta - R_h \zeta))_K}{\|\zeta\|_1}
\]
\[
\lesssim h \|\text{curl}_h q_h\|_0.
\]
So, for $Q_h \in \text{Ker} B$ with the decomposition (4.14), we have $s_h = 0$ and $\|q_h\|_0 \lesssim \|Q_h\|_0 + h \|\text{curl}_h q_h\|_0$, which, together with (4.15), imply the coercivity (4.12).

Lemma 4.6. The inf-sup condition
\[
(4.16) \quad \sup_{Q_h \in \mathcal{M}_h} \frac{\tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h)}{\|Q_h\|_{h,1}} \gtrsim \|\| (v_h, \zeta_h) \| \|_{h,2}, \text{ for all } (v_h, \zeta_h) \in W_h \times \Theta_h
\]
holds.

Proof. Given $\zeta_h \in \Theta_h$, let $Q^1_h = C_1 \varepsilon(\zeta_h)$ (the constant $C_1$ to be determined), then $\text{div}_h Q^1_h = 0$. Given $v_h \in W_h$, there exists $Q^2_h \in \mathcal{M}_h$, such that $\text{div}_h Q^2_h = -\text{grad} v_h$ and $\|Q^2_h\|_0 \leq C_2 h \|\text{div}_h Q^2_h\|_0$. Suppose $\|R_h \zeta_h\|_0 \leq C_3 \|\varepsilon(\zeta_h)\|_0$.

Take $Q_h = Q^1_h + Q^2_h$, then
\[
\tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h)
\]
\[
= (Q_h, \varepsilon(\zeta_h)) - (\text{div}_h Q_h, \text{grad} v_h - R_h \zeta_h)
\]
\[
= (Q^1_h, \varepsilon(\zeta_h)) + (Q^2_h, \varepsilon(\zeta_h)) + (\text{grad} v_h, \text{grad} v_h) - (\text{grad} v_h, R_h \zeta_h)
\]
\[
\geq C_1 \|\varepsilon(\zeta_h)\|_0^2 - C_2 h \|\text{grad} v_h\|_0 \|\varepsilon(\zeta_h)\|_0 + \|\text{grad} v_h\|_0^2 - C_3 \|\text{grad} v_h\|_0 \|\varepsilon(\zeta_h)\|_0
\]
\[
\geq (C_1 - C_2^2 h^2 - C_3^2) \|\varepsilon(\zeta_h)\|_0^2 + \frac{1}{2} \|\text{grad} v_h\|_0^2.
\]
Let $C_1 \geq 2(C_2^2 h^2 + C_3^2)$, then
\[
\tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h) \geq \frac{C_1}{2} \|\varepsilon(\zeta_h)\|_0^2 + \frac{1}{2} \|\text{grad} v_h\|_0^2 \gtrsim \|\| (v_h, \zeta_h) \| \|_{h,2}^2.
\]
On the other hand,
\[
\|Q_h\|_{h,1} = \|Q^1_h + Q^2_h\|_{h,1} = \|Q^1_h + Q^2_h\|_0 + \|\text{grad} v_h\|_0
\]
\[
\leq \|Q^1_h\|_0 + \|Q^2_h\|_0 + \|\text{grad} v_h\|_0
\]
\[
\leq C_1 \|\varepsilon(\zeta_h)\|_0 + (C_2 h + 1) \|\text{grad} v_h\|_0 \lesssim \|\| (v_h, \zeta_h) \| \|_{h,2}.
\]
Then the result (4.16) holds.
Theorem 4.7. Let \((M, \gamma = \text{div}M, w, \beta) \in M \times \Gamma \times W \times \Theta\) be the solution of the problem (2.7)-(2.8). Then the discretization problem (3.8)-(3.9) admits a unique solution \((M_h, w_h, \beta_h) \in M_h \times W_h \times \Theta_h\) such that

\[
\|M - M_h\|_{h, 1} + \|(w - w_h, \beta - \beta_h)\|_{h, 2} \leq \inf_{Q_h \in M_h} \|M - Q_h\|_{h, 1} + \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \|(w - v_h, \beta - \zeta_h)\|_{h, 2} + ht\|\gamma\|_1 + h\|\gamma\|_0.
\]

Proof. Since

\[
a(M, \gamma; Q_h, \text{div}Q_h) + \tilde{b}(Q_h, \text{div}Q_h; w, \beta) - (\text{div}Q_h, \text{grad}w - \beta - R_h(\text{grad}w - \beta)) = 0, \forall Q_h \in M_h,
\]

then for all \(Q_h \in M_h\), it holds

\[
\begin{align*}
a(M - M_h, \gamma - \text{div}M_h; Q_h, \text{div}Q_h) + \tilde{b}(Q_h, \text{div}Q_h; w - w_h, \beta - \beta_h) & - (\text{div}Q_h, \text{grad}w - \beta - R_h(\text{grad}w - \beta)) = 0. \\
\end{align*}
\]

Denote

\[
Z_h(g) = \{Q_h \in M_h : \tilde{b}(Q_h, \text{div}Q_h; v_h, \zeta_h) = -(g, v_h), \forall (v_h, \zeta_h) \in W_h \times \Theta_h\}.
\]

Let \(\tilde{Q}_h\) be any element of \(Z_h(g)\). Since \(\tilde{Q}_h - M_h \in Z_h(0) = \text{Ker}B\), then

\[
\|\tilde{Q}_h - M_h\|_{h, 1} \leq a(\tilde{Q}_h - M_h, \text{div}(\tilde{Q}_h - M_h); \tilde{Q}_h - M_h, \text{div}(\tilde{Q}_h - M_h)) = a(M - M_h, \gamma - \text{div}M_h; \tilde{Q}_h - M_h, \text{div}(\tilde{Q}_h - M_h)) + a(M - M_h, \gamma - \text{div}M_h; \tilde{Q}_h - M_h, \text{div}(\tilde{Q}_h - M_h)) - \tilde{b}(\tilde{Q}_h - M_h, \text{div}(\tilde{Q}_h - M_h); w - w_h, \beta - \beta_h) + (\text{div}(\tilde{Q}_h - M_h), \text{grad}w - \beta - R_h(\text{grad}w - \beta)) \leq \|\tilde{Q}_h - M_h\|_{h, 1} + \|(w - v_h, \beta - \zeta_h)\|_{h, 2} + ht\|\gamma\|_1.
\]

So we have

\[
\|\tilde{Q}_h - M_h\|_{h, 1} \leq \|\tilde{Q}_h - M\|_{h, 1} + \|(w - v_h, \beta - \zeta_h)\|_{h, 2} + ht\|\gamma\|_1.
\]

Then, by using the triangle inequality, we get

\[(4.17) \|M - M_h\|_{h, 1} \leq \inf_{Q_h \in Z_h(g)} \|\tilde{Q}_h - M\|_{h, 1} + \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \|(w - v_h, \beta - \zeta_h)\|_{h, 2} + ht\|\gamma\|_1.
\]

For any \(Q_h \in M_h\), there exists \(\tilde{Q}_h \in M_h\), such that, for all \((v_h, \zeta_h) \in W_h \times \Theta_h\),

\[
\tilde{b}(Q_h, \text{div}Q_h; v_h, \zeta_h) = \tilde{b}(M - Q_h, \text{div}(M - Q_h); v_h, \zeta_h) - (\gamma, \text{grad}v_h - \zeta_h - R_h(\text{grad}v_h - \zeta_h))
\]
This estimate and (4.17) imply
\[
\sup_{(v_h, \zeta_h) \in W_h \times \Theta_h} \frac{\tilde{b}(M - Q_h, \text{div}_h(M - Q_h); v_h, \zeta_h)}{||| (v_h, \zeta_h) |||_{h, 2}}
\]
\[
\sup_{(v_h, \zeta_h) \in W_h \times \Theta_h} \frac{\tilde{b}(M - Q_h, \text{div}_h(M - Q_h); v_h, \zeta_h) - (\gamma, \text{grad} v_h - \zeta_h - R_h, \text{grad} v_h - \zeta_h)}{||| (v_h, \zeta_h) |||_{h, 2}}
\]
\[
= \sup_{(v_h, \zeta_h) \in W_h \times \Theta_h} \frac{\tilde{b}(M - Q_h, \text{div}_h(M - Q_h); v_h, \zeta_h) + (\gamma, \zeta_h - R_h, \zeta_h)}{||| (v_h, \zeta_h) |||_{h, 2}}
\]
\[
\lesssim ||| M - Q_h |||_{h, 1} + h ||\gamma||_0.
\]

Choose \( \bar{Q}_h = Q_h + M \), then \( \bar{Q}_h \in Z_h(g) \). Thus we get
\[
||| M - \bar{Q}_h |||_{h, 1} = ||| M - Q_h - \bar{Q}_h |||_{h, 1} \leq ||| M - Q_h |||_{h, 1} + ||| \bar{Q}_h |||_{h, 1} \lesssim ||| M - Q_h |||_{h, 1} + h ||\gamma||_0.
\]

This estimate and (4.17) imply
\[
||| M - M_h |||_{h, 1} \lesssim \inf_{Q_h \in \mathcal{M}_h} ||| M - Q_h |||_{h, 1} + \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} ||| (w - v_h, \beta - \zeta_h) |||_{h, 2} + ht ||\gamma||_1 + h ||\gamma||_0.
\]

On the other hand, from the coercivity and continuity properties we get
\[
\sup_{Q_h \in \mathcal{M}_h} \frac{\tilde{b}(Q_h, \text{div}_h Q_h; v_h - w_h, \zeta_h - \beta_h)}{||| Q_h |||_{h, 1}}
\]
\[
\leq \left\{ \begin{aligned}
&-a(M - M_h, \text{div}_h(M - M_h); Q_h, \text{div}_h Q_h) - \tilde{b}(Q_h, \text{div}_h Q_h; w - v_h, \beta - \zeta_h) \\
&+ (\text{div}_h Q_h, \text{grad} w - \beta - R_h, \text{grad} w - \beta))
\end{aligned} \right\}
\]
\[
\lesssim ||| M - M_h |||_{h, 1} + ||| (w - v_h, \beta - \zeta_h) |||_{h, 2} + ht ||\gamma||_1.
\]

This inequality and (4.18) imply
\[
||| (w - w_h, \beta - \beta_h) |||_{h, 2} \lesssim \inf_{Q_h \in \mathcal{M}_h} ||| M - Q_h |||_{h, 1} + \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} ||| (w - v_h, \beta - \zeta_h) |||_{h, 2} + ht ||\gamma||_1 + h ||\gamma||_0.
\]

A combination of (4.18) and (4.19) completes the proof. □

To obtain the convergence order, we first need to consider error estimates for the approximations of finite element spaces in Lemma 4.8-4.9.

**Lemma 4.8.** It holds
\[
\inf_{Q_h \in \mathcal{M}_h} ||| M - Q_h |||_{h, 1} \lesssim h(||| M |||_1 + ||r||_2 + ||p||_1 + t||p||_2).
\]

**Proof.** For the exact solution \( M \), first let \( Q_h^1 \) be its piecewise constant \( L^2 \) projection, then
\[
||| M - Q_h^1 |||_0 \lesssim h ||| M |||_1.
\]
Basing on Theorem 2.2, we have $\gamma = \mathrm{div} \mathbf{M} = \mathrm{grad} \, r + \mathrm{curl} \, p$, with $(p, r) \in H^1_0(\Omega) \times \hat{H}^1(\Omega)$. Choose $Q^2_h$ satisfying $\mathrm{div}_h Q^2_h = \mathrm{grad} \left( I_h r \right) + \mathrm{curl}_h (\Pi_h p)$ (we recall that $I_h$ and $\Pi_h$ are respectively the nodal interpolation and the Scott-Zhang interpolation operators), and
\[
\|Q^2_h\|_0 \approx h \|\mathrm{div}_h Q^2_h\|_0 \lesssim h(\|r\|_2 + \|p\|_1).
\]
Take $Q_h = Q^1_h + Q^2_h$, then we can obtain the desired result
\[
\left\| \mathbf{M} - Q_h \right\|_{h,1} \leq \left( \|\mathbf{M} - Q^1_h\|_0 + \|Q^2_h\|_0 + (h + t)\|\mathrm{curl} \, p - \mathrm{curl}_h (\Pi_h p)\|_0 + \|\mathrm{grad} \, r - \mathrm{grad} \left( I_h r \right)\|_0 + \|p - \Pi_h p\|_0 \right) \lesssim h\|\mathbf{M}\|_1 + h(\|r\|_2 + \|p\|_1) + h\|p\|_1 + h\|r\|_2 + h\|p\|_1 + h\|\mathrm{grad} \, r\|_2 + h\|\mathrm{curl} \, p\|_2,
\]
where we have used the approximation properties
\[
\|p - \Pi_h p\|_0 \lesssim h\|p\|_1, \quad \|\mathrm{curl}_h (\Pi_h p)\|_0 \lesssim \|p\|_1, \quad \text{and} \quad \|\mathrm{curl} \, p - \mathrm{curl}_h (\Pi_h p)\|_0 \lesssim h\|p\|_2.
\]

Lemma 4.9. It holds
\[
\inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \left\| (w - v_h, \beta - \zeta_h) \right\|_{h,2} \lesssim h(\|\beta\|_2 + \|w\|_2).
\]

Proof. By the definition of mesh-dependent norm, we immediately get
\[
\inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \left\| (w - v_h, \beta - \zeta_h) \right\|_{h,2} = \inf_{\zeta_h \in \Theta_h} \left\| \epsilon(\beta) - \epsilon(\zeta_h) \right\|_0 + \inf_{v_h \in W_h} \left\| R_h(\mathrm{grad} \, w) - R_h(\mathrm{grad} \, v_h) \right\|_0 \lesssim \inf_{\zeta_h \in \Theta_h} \left\| \epsilon(\beta) - \epsilon(\zeta_h) \right\|_0 + \left\| R_h(\mathrm{grad} \, w) - \mathrm{grad} \, w \right\|_0 + \inf_{v_h \in W_h} \left\| \mathrm{grad} \, w - \mathrm{grad} \, v_h \right\|_0 \lesssim h(\|\beta\|_2 + \|w\|_2).
\]

Theorem 4.10. The discretization problem (3.8)-(3.9) admits a unique solution $(\mathbf{M}_h, w_h, \beta_h) \in \mathbf{M}_h \times W_h \times \Theta_h$ such that
\[
\left\| \mathbf{M} - \mathbf{M}_h \right\|_{h,1} + \left\| (w - w_h, \beta - \beta_h) \right\|_{h,2} \lesssim h \left( \|\mathbf{M}\|_1 + \|\beta\|_2 + \|w\|_2 + \|r\|_2 + \|p\|_1 + t\|p\|_2 \right) \lesssim h\|g\|_0.
\]
Furthermore, it holds
\[
\left\| \mathbf{M} - \mathbf{M}_h \right\|_0 + (t + h)\|\gamma - \gamma_h\|_0 + \|w - w_h\|_1 + \|\beta - \beta_h\|_1 \lesssim h \left( \|\mathbf{M}\|_1 + \|\beta\|_2 + \|w\|_2 + \|r\|_2 + \|p\|_1 + t\|p\|_2 \right) \lesssim h\|g\|_0.
\]

Proof. (4.21) follows from Theorem 4.7, Lemma 4.8 and Lemma 4.9 directly. For (4.22), basing on the definition of mesh-dependent norms, we only need to estimate $(t + h)\|\gamma - \gamma_h\|_0$ and $\|w - w_h\|_1$.

In fact, from the decomposition $\gamma = \mathrm{grad} \, r + \mathrm{curl}_h p$ and $\gamma_h = \mathrm{grad} \, r_h + \mathrm{curl}_h p_h$, we have
\[
(t + h)\|\gamma - \gamma_h\|_0 = (t + h)\|\mathrm{grad} \, (r - r_h) + \mathrm{curl}_h (p - p_h)\|_0 \lesssim \|\mathrm{grad} \, (r - r_h)\|_0 + (t + h)\|\mathrm{curl}_h (p - p_h)\|_0 \lesssim \|\mathbf{M} - \mathbf{M}_h\|_{h,1}.
\]
And the error estimate for \( \| w - w_h \|_1 \) can be obtained from the triangle inequality:

\[
\| \text{grad} \ w - \text{grad} \ w_h \|_0 = \| \text{grad} \ w - R_h \text{grad} \ w + R_h (\text{grad} \ w - \text{grad} \ w_h) \|_0 \\
\leq \| \text{grad} \ w - R_h \text{grad} \ w \|_0 + \| (w - w_h, \beta - \beta_h) \|_{h,2} \\
\lesssim h \| w \|_2 + \| (w - w_h, \beta - \beta_h) \|_{h,2}.
\]

Then an application of (4.21) implies (4.22). \( \square \)

5. Error Analysis for MiSP4

This section is denoted to the error estimates for the MiSP4 element. The corresponding subspaces in this section are defined as in subsection 3.2. The error analysis for MiSP4 is similar as for MiSP3. And first we also give the following properties for the operator \( R_h \).

**Lemma 5.1.** [16, Lemma 2.1] \( R_h (\text{grad} \ v_h) = \text{grad} \ v_h, \forall v_h \in W_h. \)

**Lemma 5.2.** [18, Theorem III 3.4] \( \| \eta - R_h \eta \|_0 \lesssim h \| \eta \|_1, \forall \eta \in H^1(\Omega)^2 \cap H_0(\text{rot}, \Omega). \)

We introduce two mesh-dependent norms for the finite dimensional spaces:

For any \( Q \in (H^1(\Omega))^{2 \times 2} \cup M_h, v \in (H^2(\Omega) \cap H_0^1(\Omega)) \cup W_h, \zeta \in H_0^1(\Omega)^2 \cup \Theta_h, \) define

(5.1) \[
\| Q \|_{h,1} := \| Q \|_0 + (t + h) \| \text{div} \, Q \|_0,
\]

(5.2) \[
\| (v, \zeta) \|_{h,2} := \| e(\zeta) \|_0 + (t + h)^{-1} \| R_h (\text{grad} \ v - \zeta) \|_0.
\]

With the definition of mesh-dependent norms, it is easy to check the continuity results in Lemma 5.3. While the corresponding coercivity results are deduced in Lemma 5.4-5.6. Lemma 5.5 is a preparation for Lemma 5.6.

**Lemma 5.3.** For any \( M, Q \in (H^1(\Omega))^{2 \times 2} \cup M_h, v \in (H^2(\Omega) \cap H_0^1(\Omega)) \cup W_h, \zeta \in H_0^1(\Omega)^2 \cup \Theta_h, \) it holds uniformly the continuity conditions

(5.3) \[
a(M, \text{div} \, M; Q, \text{div} \, Q) \lesssim \| M \|_{h,1} \| Q \|_{h,1},
\]

(5.4) \[
\tilde{b}(Q, \text{div} \, Q; v, \zeta) \lesssim \| Q \|_{h,1} \| (v, \zeta) \|_{h,2}.
\]

**Lemma 5.4.** It holds uniformly the discrete coercivity condition

(5.5) \[
a(Q_h, \text{div} \, Q_h; Q_h, \text{div} \, Q_h) \gtrsim \| Q_h \|_{h,1} \quad \text{for all } Q_h \in M_h.
\]

**Proof.** The proof immediately follows from the inverse inequality \( \| \text{div} \, Q_h \|_0 \leq |Q_h|_1 \lesssim h^{-1} \| Q_h \|_0. \)

**Lemma 5.5.** The following two conclusions hold:

1. For any given \( \zeta_h \in \Theta_h, \) there exists \( Q_h^1 \in M_h, \) such that

(5.6) \[
\| Q_h^1 \|_0^2 \approx \| e(\zeta_h) \|_0^2, \quad \text{and} \quad \text{div} \, Q_h^1 = 0;
\]

2. For any given \( v_h \in W_h, \zeta_h \in \Theta_h, \) there exists \( Q_h^2 \in M_h, \) such that

(5.7) \[
\| \text{div} \, Q_h^2, R_h (\text{grad} \, v_h - \zeta_h) \|_0 \approx -\frac{1}{t^2 + h^2} \| R_h (\text{grad} \, v_h - \zeta_h) \|_0^2.
\]

and

(5.8) \[
\| \text{div} \, Q_h^2 \|_0 \approx h^{-1} \| Q_h^2 \|_0.
\]
Proof. The proof is similar to that in [13].

(1) Given \( \zeta_h \in \Theta_h \), choose \( Q_{\mathbf{h}} \) as the 5-parameter PS element in [13]. The proof for (5.6) can be found in [13, Lemma 4.4].

(2) Given \( v_h \in W_h \), \( \zeta_h \in \Theta_h \), for any \( K \in \mathcal{T}_h \), \( R_h(\text{grad } v_h - \zeta_h)|_K \) can be expressed as

\[
R_h(\text{grad } v_h - \zeta_h)|_K = \frac{1}{J_K} \begin{pmatrix}
 b_2 + b_1 \xi & -(b_1 + b_2 \eta) \\
 -(a_2 + a_1 \xi) & a_1 + a_2 \eta
\end{pmatrix} \begin{pmatrix} 1 & \eta & 0 \\ 0 & 0 & 1 \xi \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}, \text{here } \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} \text{ depends on } v_h, \zeta_h.
\]

Some calculations show

\[
\|R_h(\text{grad } v_h - \zeta_h)|_K\|^2_{0,K} = \frac{4}{J_K(\xi_1, \eta_1)} \left[(b_2c_1 - b_1c_3)^2 + \frac{1}{3}(b_2c_2 - b_1c_3)^2 + \frac{1}{3}(b_1c_1 - b_1c_4)^2 + \frac{1}{9}(b_1c_2 - b_1c_4)^2\right. \\
\left. + (a_2c_1 - a_1c_3)^2 + \frac{1}{3}(a_2c_2 - a_1c_3)^2 + \frac{1}{3}(a_1c_1 - a_1c_4)^2 + \frac{1}{9}(a_1c_2 - a_1c_4)^2\right]
\]

\[
= \frac{C_1}{J_K(\xi_1, \eta_1)} \left[(b_2c_1 - b_1c_3)^2 + (b_2c_2 - b_1c_3)^2 + (a_2c_1 - a_1c_3)^2 + (a_1c_1 - a_1c_4)^2\right].
\]

Take \( Q_h|_K = \begin{pmatrix} c_1 \xi + c_3 \eta + c_2 \xi \eta \\ c_1 \xi + c_3 \eta + c_2 \xi \eta \\ 0 \end{pmatrix} \), then we have

\[
\text{div}_h Q_h|_K = \frac{1}{J_K} \begin{pmatrix} (b_2c_1 - b_1c_3) + (b_1c_1 - b_1c_4)\xi + (b_2c_2 - b_1c_3)\eta \\
-(a_2c_1 - a_1c_3) + (a_1c_1 - a_1c_4)\xi + (a_2c_2 - a_1c_3)\eta \end{pmatrix}
\]

and

\[
\|\text{div}_h Q_h|_K\|^2_{0,K} = \frac{4}{J_K(\xi_2, \eta_2)} \left[(b_2c_1 - b_1c_3)^2 + \frac{1}{3}(b_2c_2 - b_1c_3)^2 + \frac{1}{3}(b_1c_1 - b_1c_4)^2\right. \\
\left. + (a_2c_1 - a_1c_3)^2 + \frac{1}{3}(a_2c_2 - a_1c_3)^2 + \frac{1}{3}(a_1c_1 - a_1c_4)^2\right]
\]

\[
= \frac{C_2}{J_K(\xi_2, \eta_2)} \left[(b_2c_1 - b_1c_3)^2 + (b_2c_2 - b_1c_3)^2 + (a_2c_1 - a_1c_3)^2 + (a_1c_1 - a_1c_4)^2\right].
\]

On the other hand, it holds

\[
\int_K \text{div}_h Q_h \cdot R_h(\text{grad } v_h - \zeta_h) dxdy = \frac{4}{J_K(\xi_3, \eta_3)} \left[(b_2c_1 - b_1c_3)^2 + \frac{1}{3}(b_2c_2 - b_1c_3)^2 + \frac{1}{3}(b_1c_1 - b_1c_4)^2\right. \\
\left. + (a_2c_1 - a_1c_3)^2 + \frac{1}{3}(a_2c_2 - a_1c_3)^2 + \frac{1}{3}(a_1c_1 - a_1c_4)^2\right]
\]

\[
= \frac{C_3}{J_K(\xi_3, \eta_3)} \left[(b_2c_1 - b_1c_3)^2 + (b_2c_2 - b_1c_3)^2 + (a_2c_1 - a_1c_3)^2 + (a_1c_1 - a_1c_4)^2\right].
\]
Let $C_0 = -\frac{C_2 J_K(\xi,\eta)}{C_2 J_K(\xi,\eta) + \frac{1}{t+h^2}}$, and choose $Q_h^2|_K = C_0 Q_h|_K$, i.e. $\text{div}_h Q_h^2|_K = C_0 \text{div}_h Q_h|_K$, then a summation over all elements in $T_h$ completes the proof for (5.7). The result (5.8) follows from the construction of $Q_h^2$.

**Lemma 5.6.** It holds the inf-sup condition

\begin{equation}
\sup_{Q_h \in M_h} \frac{\tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h)}{||Q_h||_{h,1}} \gtrsim ||(v_h, \zeta_h)||_{h,2}, \text{ for all } (v_h, \zeta_h) \in W_h \times \Theta_h.
\end{equation}

**Proof.** For $\zeta_h \in \Theta_h$, from (5.6) there exists a positive constant $C_1$ and $Q_h^1 \in M_h$, such that

\begin{equation}
(Q_h^1, \epsilon(\zeta_h)) = ||Q_h^1||_0^2 = C_1 ||\epsilon(\zeta_h)||_0^2, \text{ and } \text{div}_h Q_h^1 = 0.
\end{equation}

For $v_h \in W_h$, $\zeta_h \in \Theta_h$, from (5.7) for any positive constant $C_2$ there exists $Q_h^2 \in M_h$, such that

\begin{equation}
(\text{div}_h Q_h^2, R_h(\text{grad} \ v_h - \zeta_h)) = -C_2 (t^2 + h^2) ||\text{div}_h Q_h^2||_0^2 = -C_2^{-1} (t^2 + h^2)^{-1} ||R_h(\text{grad} \ v_h - \zeta_h)||_0^2,
\end{equation}

and there exists a positive constant $C_3$ independent of $h$ and $t$, such that

\begin{equation}
||\text{div}_h Q_h^2||_0^2 = C_3 h^{-2} ||Q_h^2||_0^2.
\end{equation}

Let $Q_h = Q_h^1 + Q_h^2$, then we have

\begin{align*}
\tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h) &= (Q_h^1, \epsilon(\zeta_h)) - (\text{div}_h Q_h^1, \text{div}_h Q_h^1; R_h(\text{grad} \ v_h - \zeta_h)) \\
&= (Q_h^1, \epsilon(\zeta_h)) + (Q_h^2, \epsilon(\zeta_h)) - (\text{div}_h Q_h^2, R_h(\text{grad} \ v_h - \zeta_h)) \\
&\geq ||Q_h^1||_0^2 - ||Q_h^2||_0^2 ||\epsilon(\zeta_h)||_0 - \frac{1}{2C_1} ||Q_h^1||_0^2 + C_2 (t^2 + h^2) ||\text{div}_h Q_h^2||_0^2 \\
&\geq ||Q_h^1||_0^2 - \frac{C_1}{2} ||\epsilon(\zeta_h)||_0^2 - \frac{1}{2C_1 C_3} ||\text{div}_h Q_h^2||_0^2 + C_2 (t^2 + h^2) ||\text{div}_h Q_h^2||_0^2 \\
&\geq \frac{C_1}{2} ||Q_h^1||_0^2 + \frac{C_2}{2} (t^2 + h^2) ||\text{div}_h Q_h^2||_0^2 \quad \text{(by taking } C_2 = \frac{1}{C_1 C_3}) \\
&\approx ||\epsilon(\zeta_h)||_0^2 + (t^2 + h^2)^{-1} ||R_h(\text{grad} \ v_h - \zeta_h)||_0^2 \\
&\approx ||Q_h^1||_0^2 + (t^2 + h^2) ||\text{div}_h Q_h^1||_0^2 + ||Q_h^2||_0^2 = ||Q_h||_0^2 + (t^2 + h^2) ||\text{div}_h Q_h||_0^2.
\end{align*}

This immediately indicates

\[\sup_{Q_h \in M_h} \frac{\tilde{b}(Q_h, \text{div}_h Q_h; v_h, \zeta_h)}{||Q_h||_{h,1}} \gtrsim ||(v_h, \zeta_h)||_{h,2}.\]

With the above continuity and coercivity results, we can obtain the following error estimates for MiSP4 element by following the same way as in Theorem 4.7.

**Theorem 5.7.** Let $(M, \gamma = \text{div}_h M_h, w, \beta) \in M \times \Gamma \times W \times \Theta$ be the solution of the problem (2.7)-(2.8). Then the discretization problem (3.8)-(3.9) admits a unique solution


\((M_h, w_h, \beta_h) \in M_h \times W_h \times \Theta_h\) such that

\[
|||M - M_h|||_{h,1} + |||(w - w_h, \beta - \beta_h)|||_{h,2} \\
\leq \inf_{Q_h \in M_h} |||M - Q_h|||_{h,1} + \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} |||(w - v_h, \beta - \zeta_h)|||_{h,2} + h t \|\gamma\|_1 + h \|\gamma\|_0.
\]

Next we consider the approximation properties of finite element spaces. Lemma 5.8 gives the error estimates for space \(M_h\), and Lemma 5.12 is for space \(W_h \times \Theta_h\). We need to notice here the key for Lemma 5.12 is the property of the operator \(R_h\) described in Lemma 5.11. Finally the convergence theorem, i.e. Theorem 5.13, follows from these lemmas.

**Lemma 5.8.** It holds

\[
\inf_{Q_h \in M_h} |||M - Q_h|||_{h,1} \lesssim h (||M||_1 + ||\gamma||_0 + t ||\gamma||_1).
\]

**Proof.** For the exact solution \(M\), first let \(Q^1_h\) be its piecewise constant \(L^2\) projection, then

\[
||M - Q^1_h||_0 \lesssim h ||M||_1.
\]

For the exact solution \(\gamma\), secondly choose \(Q^2_h\) satisfying:

1. \(\text{div}_h Q^2_h\) is the piecewise constant \(L^2\) projection of \(\gamma\), then

\[
||\gamma - \text{div}_h Q^2_h||_0 \approx h ||\gamma||_1, \quad ||\text{div}_h Q^2_h||_0 \lesssim ||\gamma||_0;
\]

2. \(||Q^2_h||_0 \approx h ||\text{div}_h Q^2_h||_0\), then \(||Q^2_h||_0 \lesssim h ||\gamma||_0\).

Take \(Q_h = Q^1_h + Q^2_h\), then we get the desired result

\[
||M - Q_h||_{h,1} \leq ||M - Q^1_h||_0 + ||Q^2_h||_0 + (h + t) ||\gamma - \text{div}_h Q^2_h||_0 \\
\lesssim h ||M||_1 + h ||\gamma||_0 + h ||\gamma - \text{div}_h Q^2_h||_0 + t ||\gamma||_1 \\
\lesssim h (||M||_1 + ||\gamma||_0 + t ||\gamma||_1).
\]

\(\square\)

**Remark 5.9.** We note that with the same technique as in Lemma 5.8, the condition \(t \lesssim h\) in [13, Lemma 3.2] and in [13, Theorem 4.3] can be removed.

**Assumption 5.10.** [16] The mesh \(T_h\) is a refinement of a coarser partition \(T_{2h}\), obtained by joining the midpoints of each opposite edge in each \(K_{2h} \in T_{2h}\) (called macroelement). In addition, \(T_{2h}\) is a similar refinement of a still coarser regular partition \(T_{4h}\).

**Lemma 5.11.** [16, Lemma 3.2, 3.4] Under Assumption 5.10, let \(W_h, \Theta_h, Z_h\) and the operator \(R_h\) be defined as before. Then for the given \((w, \beta)\), there exist \(\hat{w} \in W_h\) and \(\hat{\beta} \in \Theta_h\) and operator \(\Pi : H_0(\text{rot}, \Omega) \cap H^1(\Omega)^2 \to Z_h\) satisfying

\[
\beta - \hat{\beta} \lesssim h \|\beta\|_2,
\]

(5.13)

\[
R_h(\text{grad} \hat{w} - \hat{\beta}) = \Pi(\text{grad} w - \beta),
\]

(5.14)

\[
|||\eta - \Pi \eta|||_0 \lesssim h ||\eta||_1, \forall \eta \in H_0(\text{rot}, \Omega) \cap H^1(\Omega)^2.
\]

(5.15)
Lemma 5.12. Under Assumption 5.10, it holds
\[
(5.16) \quad \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \|\|(w - v_h, \beta - \zeta_h)\|\|_{h, 2} \lesssim h\|\beta\|_2 + \frac{ht^2}{t + h}\|\gamma\|_1.
\]

Proof. Choose \((v_h, \zeta_h) = (\hat{w}, \hat{\beta})\), with \((\hat{w}, \hat{\beta}) \in W_h \times \Theta_h\) as in Lemma 5.11, then we can get
\[
\inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \|\|(w - v_h, \beta - \zeta_h)\|\|_{h, 2} \\
= \inf_{(v_h, \zeta_h) \in W_h \times \Theta_h} \|\|\epsilon(\beta) - \epsilon(\zeta_h)\|\|_0 + \frac{1}{t + h}\|R_h(\text{grad } w - \beta) - R_h(\text{grad } v_h - \zeta_h)\|_0 \\
\leq \|\epsilon(\beta) - \epsilon(\hat{\beta})\|_0 + \frac{1}{t + h}\|R_h(\text{grad } w - \beta) - R_h(\text{grad } \hat{w} - \beta)\|_0 \\
= \|\epsilon(\beta) - \epsilon(\hat{\beta})\|_0 + \frac{1}{t + h}\|R_h(\text{grad } w - \beta) - \Pi(\text{grad } w - \beta)\|_0 \\
\leq h\|\beta\|_2 + \frac{ht^2}{t + h}\|\gamma\|_1.
\]
\[\square\]

Theorem 5.13. Under Assumption 5.10, the discretization problem (3.8)-(3.9) admits a unique solution \((M_h, w_h, \beta_h) \in M_h \times \Gamma_h \times W_h \times \Theta_h\) such that
\[
(5.17) \quad \|\|M - M_h\|\|_{h, 1} + \|\|(w - w_h, \beta - \beta_h)\|\|_{h, 2} \lesssim h(\|M\|_1 + \|\beta\|_2 + \|\gamma\|_0 + t\|\gamma\|_1) \lesssim h\|g\|_0.
\]
Furthermore, it holds
\[
(5.18) \quad \|\|M - M_h\|\|_0 + (t + h)\|\gamma - \gamma_h\|_0 + \|w - w_h\|_1 + \|\beta - \beta_h\|_1 \\
\lesssim h(\|M\|_1 + \|\gamma\|_1 + \|\beta\|_2 + t\|\gamma\|_1 + \|\gamma\|_0) \lesssim h\|g\|_0.
\]

Proof. The estimate (5.17) follows from the Theorem 5.7, Lemma 5.8 and Lemma 5.12.

For the second estimate, we only need to estimate \(\|w - w_h\|_1\). In fact,
\[
\|\text{grad } w - \text{grad } w_h\|_0 \\
= \|\text{grad } w - R_h\text{grad } w + R_h(\text{grad } w - \text{grad } w_h - \beta + \beta_h) + R_h(\beta - \beta_h)\|_0 \\
\leq \|\text{grad } w - R_h\text{grad } w\|_0 + \|R_h(\text{grad } w - \text{grad } w_h - \beta + \beta_h)\|_0 + \|R_h(\beta - \beta_h)\|_0 \\
\lesssim h(\|w\|_2 + \|M\|_1 + \|\beta\|_2 + h\|\gamma\|_1 + \|\gamma\|_0).
\]
\[\square\]

6. Numerical Results

We compute a square plate with analytical solution to show the convergence. This example is taken from [21]. The domain is the unit square \((0, 1)^2\), the material parameters are taken as \(E = 1.0, \nu = 0.3\) and \(\kappa = \frac{3}{6}\). The exact solution is: the first component of the rotation \(\beta_1 = 100y^3(1 - y)^3x^2(x - 1)(2x - 1)\), the second component of the rotation \(\beta_2 = 100x^3(1 - x)^3y^2(1 - y)(2y - 1)\), and the displacement \(w = 100(\frac{1}{3}x^3(1 - x)^3y^2(1 - y)(2y - 1) + \frac{2t^2}{\alpha(1 - \nu)}[y^3(1 - y)^3x(x - 1)(5x^2 - 5x + 1) + x^3(1 - x)^3y(y - 1)(5y^2 - 5y + 1)])\). Therefore, the transverse load \(g = \frac{200t^2}{\alpha(1 - \nu)}[x^3(1 - x)^3y^2(1 - y)(2y - 1) + y^3(1 - y)^3(5x^2 - 5x + 1) + x(x - 1)y(y - 1)(5x^2 - 5x + 1)(5y^2 - 5y + 1)]\). For the plate thickness \(t\), we consider four cases: \(t = 1.0, 0.1, 0.001, 1.0 - 8\).

The results for MiSP3 method under the uniform meshes (Figure 6.1) are reported in Table 6.1, while the results for MiSP4 method under the uniform meshes (Figure 6.2) are
reported in Table 6.2. These results are conformable to the error estimates in Theorem 4.10 and Theorem 5.13.

Table 6.1. Results of error on uniform mesh with MiSP3

| $t$  | $4 \times 4$ | $8 \times 8$ | $16 \times 16$ | $32 \times 32$ | $64 \times 64$ | rate  |
|------|--------------|--------------|----------------|----------------|----------------|-------|
| 1    | $|w - w_h|_1$ | 0.2834       | 0.1679         | 0.0877         | 0.0443         | 0.0222 | 0.9182 |
|      | $|\beta - \beta_h|_1$ | 0.0820       | 0.0461         | 0.0238         | 0.0120         | 0.0060 | 0.9427 |
|      | $\|M - M_h\|_0$ | 0.0070       | 0.0033         | 0.0015         | 0.0008         | 0.0004 | 1.0543 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0882       | 0.0525         | 0.0275         | 0.0139         | 0.0070 | 0.9156 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.1194       | 0.0618         | 0.0299         | 0.0145         | 0.0071 | 1.0169 |
| 0.1  | $|w - w_h|_1$ | 0.0132       | 0.0066         | 0.0032         | 0.0016         | 0.0008 | 1.0153 |
|      | $|\beta - \beta_h|_1$ | 0.0824       | 0.0460         | 0.0238         | 0.0120         | 0.0060 | 0.9445 |
|      | $\|M - M_h\|_0$ | 0.0069       | 0.0032         | 0.0015         | 0.0008         | 0.0004 | 1.0520 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0851       | 0.0501         | 0.0270         | 0.0138         | 0.0070 | 0.9031 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.0386       | 0.0139         | 0.0051         | 0.0020         | 0.0008 | 1.3764 |
| 0.001| $|w - w_h|_1$ | 0.0112       | 0.0053         | 0.0025         | 0.0012         | 0.0006 | 1.0520 |
|      | $|\beta - \beta_h|_1$ | 0.0838       | 0.0463         | 0.0238         | 0.0120         | 0.0060 | 0.9506 |
|      | $\|M - M_h\|_0$ | 0.0070       | 0.0033         | 0.0016         | 0.0008         | 0.0004 | 1.0569 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0840       | 0.0496         | 0.0294         | 0.0166         | 0.0094 | 0.7902 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.0298       | 0.0088         | 0.0026         | 0.0007         | 0.0002 | 1.7753 |
| 1e-8 | $|w - w_h|_1$ | 0.0112       | 0.0053         | 0.0025         | 0.0012         | 0.0006 | 1.0520 |
|      | $|\beta - \beta_h|_1$ | 0.0838       | 0.0463         | 0.0238         | 0.0120         | 0.0060 | 0.9506 |
|      | $\|M - M_h\|_0$ | 0.0070       | 0.0033         | 0.0016         | 0.0008         | 0.0004 | 1.0569 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0840       | 0.0497         | 0.0294         | 0.0167         | 0.0097 | 0.7781 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.0297       | 0.0088         | 0.0026         | 0.0007         | 0.0002 | 1.7781 |
Table 6.2. Results of error on uniform mesh with MiSP4

| $t$  | $4 \times 4$ | $8 \times 8$ | $16 \times 16$ | $32 \times 32$ | $64 \times 64$ | rate |
|------|-------------|-------------|---------------|--------------|--------------|------|
| 1    | $|w - w_h|_1$ | 0.2806      | 0.1460        | 0.0736       | 0.0369       | 0.0184| 0.9819 |
|      | $|\beta - \beta_h|_1$ | 0.0771      | 0.0383        | 0.0191       | 0.0095       | 0.0048| 1.0039 |
|      | $\|M - M_h\|_0$ | 0.0062      | 0.0020        | 0.0008       | 0.0003       | 0.0002| 1.2977 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0877      | 0.0458        | 0.0231       | 0.0116       | 0.0058| 0.9799 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.1187      | 0.0539        | 0.0252       | 0.0121       | 0.0059| 1.0812 |
| 0.1  | $|w - w_h|_1$ | 0.0117      | 0.0052        | 0.0025       | 0.0012       | 0.0006| 1.0610 |
|      | $|\beta - \beta_h|_1$ | 0.0775      | 0.0384        | 0.0191       | 0.0095       | 0.0048| 1.0057 |
|      | $\|M - M_h\|_0$ | 0.0061      | 0.0020        | 0.0008       | 0.0003       | 0.0002| 1.2957 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0870      | 0.0458        | 0.0231       | 0.0116       | 0.0058| 0.9771 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.0395      | 0.0127        | 0.0044       | 0.0017       | 0.0007| 1.4504 |
| 0.001| $|w - w_h|_1$ | 0.0095      | 0.0041        | 0.0019       | 0.0009       | 0.0005| 1.0896 |
|      | $|\beta - \beta_h|_1$ | 0.0777      | 0.0384        | 0.0191       | 0.0095       | 0.0048| 1.0065 |
|      | $\|M - M_h\|_0$ | 0.0061      | 0.0020        | 0.0008       | 0.0003       | 0.0002| 1.2944 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0866      | 0.0460        | 0.0234       | 0.0117       | 0.0059| 0.9704 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.0307      | 0.0082        | 0.0021       | 0.0005       | 0.0001| 1.9555 |
| 1e-8 | $|w - w_h|_1$ | 0.0095      | 0.0041        | 0.0019       | 0.0009       | 0.0005| 1.0896 |
|      | $|\beta - \beta_h|_1$ | 0.0777      | 0.0384        | 0.0191       | 0.0095       | 0.0048| 1.0065 |
|      | $\|M - M_h\|_0$ | 0.0061      | 0.0020        | 0.0008       | 0.0003       | 0.0002| 1.2944 |
|      | $\|\gamma - \gamma_h\|_0$ | 0.0866      | 0.0460        | 0.0234       | 0.0117       | 0.0059| 0.9703 |
|      | $(t + h)\|\gamma - \gamma_h\|_0$ | 0.0306      | 0.0081        | 0.0021       | 0.0005       | 0.0001| 1.9703 |

We note that the error analysis for MiSP4 element requires the partitions of domain to satisfy Assumption 5.10. However, numerical results in Table 6.3 show that this assumption seems not to be absolutely necessary for the uniform convergence, as is similar to the MITC4 element [16]. Here the used partitions (Figure 6.3) do not satisfy Assumption 5.10.

Figure 6.3. Quadrilateral mesh

Acknowledgements. The work of the first author was partly supported by National Natural Science Foundation of China (11401492 and 11226333). The work of the second author was partly supported by National Natural Science Foundation of China (11171239) and Major Research Plan of National Natural Science Foundation of China (91430105).
Table 6.3. Results of error on quadrilateral mesh with MiSP4

| \( t \) | \( 4 \times 4 \) | \( 8 \times 8 \) | \( 16 \times 16 \) | \( 32 \times 32 \) | \( 64 \times 64 \) | rate |
|---------|------------|------------|------------|------------|------------|-----|
| 1       | \( |w - w_h|_1 \) | 0.2873     | 0.1693     | 0.0881     | 0.0445     | 0.0223 | 0.9217 |
|         | \( |\beta - \beta_h|_1 \) | 0.0924     | 0.0528     | 0.0255     | 0.0122     | 0.0060 | 0.9872 |
|         | \( \|M - M_h\|_0 \) | 0.0066     | 0.0032     | 0.0012     | 0.0005     | 0.0002 | 1.1968 |
|         | \( \|\gamma - \gamma_h\|_0 \) | 0.0899     | 0.0531     | 0.0277     | 0.0140     | 0.0070 | 0.9203 |
|         | \( (t + h)\|\gamma - \gamma_h\|_0 \) | 0.1285     | 0.0645     | 0.0306     | 0.0147     | 0.0072 | 1.0398 |
| 0.1     | \( |w - w_h|_1 \) | 0.0118     | 0.0064     | 0.0031     | 0.0015     | 0.0008 | 0.9898 |
|         | \( |\beta - \beta_h|_1 \) | 0.0834     | 0.0496     | 0.0253     | 0.0122     | 0.0060 | 0.9506 |
|         | \( \|M - M_h\|_0 \) | 0.0065     | 0.0031     | 0.0012     | 0.0005     | 0.0002 | 1.1925 |
|         | \( \|\gamma - \gamma_h\|_0 \) | 0.0930     | 0.0574     | 0.0285     | 0.0141     | 0.0070 | 0.9318 |
|         | \( (t + h)\|\gamma - \gamma_h\|_0 \) | 0.0493     | 0.0181     | 0.0059     | 0.0022     | 0.0009 | 1.4475 |
| 0.001   | \( |w - w_h|_1 \) | 0.0096     | 0.0051     | 0.0024     | 0.0012     | 0.0006 | 1.0151 |
|         | \( |\beta - \beta_h|_1 \) | 0.0835     | 0.0475     | 0.0245     | 0.0120     | 0.0060 | 0.9525 |
|         | \( \|M - M_h\|_0 \) | 0.0066     | 0.0032     | 0.0013     | 0.0006     | 0.0003 | 1.1477 |
|         | \( \|\gamma - \gamma_h\|_0 \) | 0.0947     | 0.0702     | 0.0470     | 0.0355     | 0.0310 | 0.4031 |
|         | \( (t + h)\|\gamma - \gamma_h\|_0 \) | 0.0408     | 0.0152     | 0.0051     | 0.0019     | 0.0009 | 1.3908 |
| 1e-8    | \( |w - w_h|_1 \) | 0.0096     | 0.0051     | 0.0024     | 0.0012     | 0.0006 | 1.0151 |
|         | \( |\beta - \beta_h|_1 \) | 0.0835     | 0.0475     | 0.0245     | 0.0120     | 0.0060 | 0.9525 |
|         | \( \|M - M_h\|_0 \) | 0.0066     | 0.0032     | 0.0013     | 0.0006     | 0.0003 | 1.1466 |
|         | \( \|\gamma - \gamma_h\|_0 \) | 0.0947     | 0.0703     | 0.0470     | 0.0356     | 0.0315 | 0.3976 |
|         | \( (t + h)\|\gamma - \gamma_h\|_0 \) | 0.0407     | 0.0151     | 0.0051     | 0.0019     | 0.0008 | 1.3976 |

References

[1] D.N. Arnold. Discretization by finite elements of a model parameter dependent problem. *Numerische Mathematik*, 37(3):405–421, 1981.

[2] D.N. Arnold and R.S. Falk. A uniformly accurate finite element method for the Reissner-Mindlin plate. *SIAM Journal on Numerical Analysis*, 26(6):1276–1290, 1989.

[3] R. Ayad, G. Dhatt, and J.L. Batoz. A new hybrid-mixed variational approach for Reissner–Mindlin plates. The MiSP model. *International journal for numerical methods in engineering*, 42(7):1149–1179, 1998.

[4] K.J. Bathe, F. Brezzi, and S.W. Cho. The mitc7 and mitc9 plate bending elements. *Computers & Structures*, 32(3):797–814, 1989.

[5] K.J. Bathe and E.N. Dvorkin. A four-node plate bending element based on Mindlin/Reissner plate theory and a mixed interpolation. *International Journal for Numerical Methods in Engineering*, 21(2):367–383, 1985.

[6] J.L. Batoz, K.J. Bathe, and L.W. Ho. A study of three-node triangular plate bending elements. *International Journal for Numerical Methods in Engineering*, 15(12):1771–1812, 1980.

[7] J.L. Batoz and M.B. Tahar. Evaluation of a new quadrilateral thin plate bending element. *International Journal for Numerical Methods in Engineering*, 18(11):1655–1677, 1982.

[8] Daniele Boffi and Lucia Gastaldi. Mixed finite elements, compatibility conditions, and applications: lectures given at the CIME Summer School held in Cetraro, Italy, June 26–July 1, 2006, volume 1939. Springer, 2008.

[9] D. Braess. *Finite elements: Theory, fast solvers, and applications in solid mechanics*. Cambridge Univ Pr, 2001.

[10] F. Brezzi, K.J. Bathe, and M. Fortin. Mixed-interpolated elements for Reissner–Mindlin plates. *International Journal for Numerical Methods in Engineering*, 28(8):1787–1801, 1989.

[11] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer-Verlag, 1991.
[12] F. Brezzi, M. Fortin, and R. Stenberg. Error analysis of mixed-interpolated elements for Reissner-Mindlin plates. *Math. Models Methods Appl. Sci.*, 1(2):125–151, 1991.

[13] C. Carstensen, X. Xie, G. Yu, and T. Zhou. A priori and a posteriori analysis for a locking-free low order quadrilateral hybrid finite element for Reissner-Mindlin plates. *Computer Methods in Applied Mechanics and Engineering*, 200(9-12):1161–1175, 2011.

[14] X.L. Cheng. A simple finite element method for the reissner-mindlin plate. *J. Comput. Math.*, 12(1):46–54, 1994.

[15] R. Durán and E. Liberman. On mixed finite element methods for the Reissner-Mindlin plate model. *Mathematics of computation*, 58(198):561–573, 1992.

[16] R.G. Durán, E. Hernández, L. Hervella-Nieto, E. Liberman, and R. Rodríguez. Error estimates for low-order isoparametric quadrilateral finite elements for plates. *SIAM journal on numerical analysis*, 41:1751–1772, 2003.

[17] R.S. Falk and T. Tu. Locking-free finite elements for the Reissner-Mindlin plate. *Mathematics of computation*, 69(231):911–928, 2000.

[18] V. Girault and P.A. Raviart. Finite element methods for Navier-Stokes equations, Theory and algorithms, volume 5 of Springer Series in Computational Mathematics, 1986.

[19] J. Hu, P. Ming, and Z. Shi. Nonconforming quadrilateral rotated $q^1$ element for Reissner-Mindlin plate. *Journal of Computational Mathematics*, 21(1):25–32, 2003.

[20] J. Hu and Z.C. Shi. Two lower order nonconforming rectangular elements for the Reissner-Mindlin plate. *Mathematics of computation*, 76(260):1771–1786, 2007.

[21] J. Hu and Z.C. Shi. Error analysis of quadrilateral wilson element for Reissner–Mindlin plate. *Computer Methods in Applied Mechanics and Engineering*, 197(6):464–475, 2008.

[22] J. Hu and Z.C. Shi. Analysis for quadrilateral MITC elements for the Reissner-Mindlin plate problem. *Mathematics of computation*, 78(266):673–711, 2009.

[23] T.J.R. Hughes. *The finite element method: linear static and dynamic finite element analysis*. Prentice-hall, 1987.

[24] T.J.R. Hughes, M. Cohen, and M. Haroun. Reduced and selective integration techniques in the finite element analysis of plates. *Nuclear Engineering and Design*, 46(1):203–222, 1978.

[25] T.J.R. Hughes and R.L. Taylor. The linear triangular bending element. *The Mathematics of Finite Elements and Applications*, 4:127–142, 1981.

[26] T.J.R. Hughes, R.L. Taylor, and W. Kanoknukulchai. A simple and efficient finite element for plate bending. *International Journal for Numerical Methods in Engineering*, 11(10):1529–1543, 1977.

[27] T.J.R. Hughes and T.E. Tezduyar. Finite elements based upon Mindlin plate theory with particular reference to the four-node bilinear isoparametric element. *Journal of Applied Mechanics*, 48:587, 1981.

[28] C. Lovadina. A low-order nonconforming finite element for Reissner-Mindlin plates. *SIAM journal on numerical analysis*, 42(6):2688–2705, 2005.

[29] R.H. Macneal. Derivation of element stiffness matrices by assumed strain distributions. *Nuclear Engineering and Design*, 70(1):3–12, 1982.

[30] D.S. Malkus and T.J.R. Hughes. Mixed finite element methods–reduced and selective integration techniques: A unification of concepts. *Computer Methods in Applied Mechanics and Engineering*, 15(1):63–81, 1978.

[31] P.B. Ming and Z.C. Shi. Nonconforming rotated $q^1$ element for Reissner-Mindlin plate. *Mathematical Models and Methods in Applied Sciences*, 11(8):1311–1342, 2001.

[32] P.B. Ming and Z.C. Shi. Two nonconforming quadrilateral elements for the Reissner-Mindlin plate. *Mathematical Models and Methods in Applied Sciences*, 15(10):1503–1518, 2005.

[33] P.B. Ming and Z.C. Shi. Analysis of some low order quadrilateral reissner-mindlin plate elements. *Mathematics of computation*, 75(255):1043–1065, 2006.

[34] P. Papadopoulos and R.L. Taylor. A triangular element based on Reissner-Mindlin plate theory. *International journal for numerical methods in engineering*, 30(5):1029–1049, 1990.

[35] J. Pitkäranta and M. Suri. Design principles and error analysis for reduced-shear plate-bending finite elements. *Numerische Mathematik*, 75(2):223–266, 1996.

[36] Z. Zhang and S. Zhang. Wilson’s element for the Reissner-Mindlin plate. *Computer methods in applied mechanics and engineering*, 113(1-2):55–65, 1994.
[37] O.C. Zienkiewicz, R.L. Taylor, P. Papadopoulos, and E. Onate. Plate bending elements with discrete constraints: new triangular elements. *Computers & Structures*, 35(4):505–522, 1990.

[38] O.C. Zienkiewicz, R.L. Taylor, and J.M. Too. Reduced integration technique in general analysis of plates and shells. *International Journal for Numerical Methods in Engineering*, 3(2):275–290, 1971.