Abstract. Given a complex reductive group $G$, Borel subgroup $B \subset G$, and topological surface $S$ with boundary $\partial S$, we study the “Betti spectral category” $\text{DCoh}_N(\text{Loc}_G(S, \partial S))$ of coherent sheaves with nilpotent singular support on the character stack of $G$-local systems on $S$ with $B$-reductions along $\partial S$. Modifications along the components of $\partial S$ endow $\text{DCoh}_N(\text{Loc}_G(S, \partial S))$ with commuting actions of the affine Hecke category $\mathcal{H}_G$ in its realization as coherent sheaves on the Steinberg stack. We prove a “spectral Verlinde formula” identifying the result of gluing two boundary components with the Hochschild homology of the corresponding $\mathcal{H}_G$-bimodule structure. The equivalence is compatible with Wilson line operators (the action of $\text{Perf}(\text{Loc}_G(S))$ realized by Hecke modifications at points) as well as Verlinde loop operators (the action of the center of $\mathcal{H}_G$ realized by Hecke modifications along closed loops). The result reduces the calculation of such “Betti spectral categories” to the case of disks, cylinders, pairs of pants, and the Möbius band.

1. Introduction

In this paper, we describe structures predicted by four-dimensional topological field theory on the spectral side of the Geometric Langlands correspondence. We first state the main result, and then provide some context.

Let $S$ be a (not necessarily oriented) topological surface, $G$ a complex reductive group, and $\text{Loc}_G(S)$ the character derived stack of $G$-local systems on $S$.

For connected $S$ and any point $s \in S$, one has the “global complete intersection” presentation by group-valued Hamiltonian reduction

$$\text{Loc}_G(S) = (\text{Rep}_G(S \setminus \{s\}) \times_G \{e\})/G$$

starting from the smooth affine variety of representations

$$\text{Rep}_G(S \setminus \{s\}) = \text{Hom}(\pi_1(S \setminus \{s\}), G)$$

In other words, one starts with $G$-local systems on the punctured surface $S \setminus \{s\}$ trivialized at a base point, imposes that the monodromy around $s$ is the identity $e \in G$, and then quotients by the adjoint action of $G$ to forget the trivialization.

This presentation provides a description of the $-1$st cohomology of the cotangent complex

$$T^{*-1}\text{Loc}_G(S)|_E \simeq RT(S, ad(E)) \quad E \in \text{Loc}_G(S)$$

inside of which we single out the nilpotent cone

$$\mathcal{N} \subset T^{*-1}\text{Loc}_G(S)$$

consisting of nilpotent endomorphisms.

Our focus in this paper is the dg category $\text{DCoh}_N(\text{Loc}_G(S))$ of coherent sheaves with nilpotent singular support which sits between perfect complexes and all coherent sheaves

$$\text{Perf}(\text{Loc}_G(S)) \subset \text{DCoh}_N(\text{Loc}_G(S)) \subset \text{DCoh}(\text{Loc}_G(S))$$

It is a Betti version of the de Rham version proposed by Arinkin and Gaitsgory [AG] as the correct spectral side of the geometric Langlands correspondence.

Now let $S$ be a (not necessarily oriented) topological surface with boundary $\partial S$, $B \subset G$ a Borel subgroup, and $\text{Loc}_G(S, \partial S)$ the parabolic character derived stack of $G$-local systems on $S$ with $B$-reductions along $\partial S$. 

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For example, in the case of a cylinder $Cyl = S^1 \times [0,1]$ with boundary $\partial Cyl = S^1 \times \{0,1\}$, we obtain the Grothendieck-Steinberg stack

$$St_G = B/B \times G/G \simeq \text{Loc}_G(Cyl, \partial Cyl)$$

and the affine Hecke category in its spectral realization

$$\mathcal{H}_G = \text{D Coh}(St_G)$$

Here we suppress the nilpotent singular support from the notation since all codirections turn out to be nilpotent.

The concatenation of cylinders equips $\mathcal{H}_G$ with a natural monoidal structure, and by [BNP2, Theorem 1.4.6(1)] we have a monoidal equivalence

$$\mathcal{H}_G \simeq \text{End}_{\text{Perf}(G/G)}(\text{Perf}(B/B))$$

Once we identify a boundary component of $\partial S$ with the circle, modifications of the parabolic structure along that component provides a natural $\mathcal{H}_G$-action on $\text{D Coh}(\text{Loc}_G(S, \partial S))$.

Once we identify two distinct boundary components of $\partial S$ with the circle, on the one hand, we obtain a natural $\mathcal{H}_G$-bimodule structure on $\text{D Coh}(\text{Loc}_G(S, \partial S))$, and on the other hand, a new surface $\tilde{S} = S/\sim$ where we glue the two boundary components together.

Our main result allows us to recover the dg category $\text{D Coh}(\text{Loc}_G(\tilde{S}, \partial \tilde{S}))$ as the Hochschild homology category of the $\mathcal{H}_G$-bimodule structure on $\text{D Coh}(\text{Loc}_G(S, \partial S))$. It is compatible with natural symmetries, realized by Hecke modifications at points and along closed loops, which we do not state explicitly for now (see Section 5 below).

**Theorem 1.1** (Corollary 4.2 below). There is a canonical equivalence

$$\text{D Coh}_N(\text{Loc}_G(\tilde{S}, \partial \tilde{S})) \simeq \mathcal{H}_G \otimes_{\mathcal{H}_G \otimes \mathcal{H}_G^\op} \text{D Coh}_N(\text{Loc}_G(S, \partial S))$$

respecting commuting Hecke symmetries at points and along closed loops.

**Remark 1.2.** There is a straightforward generalization where $G$ is not necessarily a constant group-scheme over $S$. This arises naturally when $S$ is not orientable and the descent of the constant group-scheme $G$ from the two-fold orientation cover is given by an involution on $G$.

The theorem is a corollary of the following general assertion. Let $p : X \to Y$ and $q : Z \to Y \times Y$ be quasi-smooth morphisms of smooth derived stacks, and set $Z_X = Z \times_{Y \times Y} X \times X$. Assume $p$ is proper.

Introduce the fundamental correspondence

$$Z_X = Z \times_{Y \times Y} X \times X \xrightarrow{\delta} Z \times_{Y \times Y} X \xrightarrow{p} Z \times_{Y \times Y} Y$$

and the support condition

$$\Lambda_{-1} = p_* \delta^! T_{Z_X}^{s-1} \subset T_{Z \times_{Y \times Y} Y}^{s-1}$$

Consider the monoidal category $\mathcal{H}_{X,Y} = \text{D Coh}(X \times Y, X)$ and the $\mathcal{H}_{X,Y}$-bimodule $\text{D Coh}(Z_{X \times X})$.

**Theorem 1.3** (Theorem 3.1 below). There is a canonical equivalence of $\text{Perf}(Y)$-modules

$$\text{D Coh}_{\Lambda_{-1}}(Z \times_{Y \times Y} Y) \simeq \text{D Coh}(Z_X) \otimes_{\mathcal{H}_{X,Y} \otimes \mathcal{H}_{X,Y}^\op} \mathcal{H}_{X,Y}$$

The proof of Theorem 1.3 is an application of descent with singular support conditions, which was developed in our work [BNP2] with Toly Preygel. The assertion of Theorem 1.1 generalizes the calculation of the Hochschild homology category of $\mathcal{H}$ itself, arising when $S$ is the cylinder, which was the main application of [BNP2].
1.1. **Topological field theory interpretation.** We include here an informal discussion placing our results within topological field theory (TFT), and specifically the Geometric Langlands program. TFTs organize invariants of manifolds that satisfy strong locality properties, reducing their calculation to atomic building blocks. We will explain how Theorem 1.1 fits into this paradigm.

Let us first focus on two-dimensional TFTs. Cutting surfaces along closed curves reduces the calculation of their TFT invariants to those assigned to the disk, cylinder, and pair of pants (along with the Möbius band in the unoriented case). This information is encoded in a commutative Frobenius algebra structure on the vector space assigned to the circle. For example, from class functions $\mathbb{C}[\Gamma/\Gamma]$ on a finite group $\Gamma$, two-dimensional Yang-Mills theory recovers the orbifold count $\#|\text{Loc}_\Gamma(S)|$ of $\Gamma$-local systems on any surface $S$.

Next let us turn to three-dimensional TFTs, but focus on their two-dimensional invariants. Here cutting surfaces along closed curves reduces the calculation of their TFT invariants to the balanced braided tensor structure on the category assigned to the circle. For example, from the category $\text{Vec}[\Gamma/\Gamma]$ of adjoint-equivariant vector bundles on a finite group $\Gamma$, Dijkgraaf-Witten theory recovers the vector space $\mathbb{C}[\text{Loc}_\Gamma(S)]$ of functions on $\Gamma$-local systems on any surface $S$.

To describe the gluing in more detail, let $\mathcal{Z}$ be a three-dimensional TFT, and suppose the balanced braided tensor category $\mathcal{Z}(S^1)$ is presented as a category of modules for an algebra $A$. Let $S$ be a surface with two boundary components each identified with $S^1$. Let $\tilde{S}$ be the closed surface obtained by gluing together the two boundary components of $S$ as identified with $S^1$. Let $\gamma \subset \tilde{S}$ be the distinguished closed curve given by the glued boundary components.

If $S = S_1 \coprod S_2$ is disconnected, then $\gamma \subset \tilde{S}$ is separating and $\tilde{S} \simeq S_1 \coprod \gamma S_2$. Here the invariants $\mathcal{Z}(S_1)$ and $\mathcal{Z}(S_2)$ define right and left $A$-modules, and the gluing is given by the tensor product

$$\mathcal{Z}(\tilde{S}) \simeq \mathcal{Z}(S_1) \otimes_A \mathcal{Z}(S_2)$$

In general, the invariant $\mathcal{Z}(S)$ is an $A$-bimodule, and the gluing is given by the Hochschild homology

$$\mathcal{Z}(\tilde{S}) \simeq \mathcal{Z}(S) \otimes_{A \otimes A^{op}} A$$

Iterating this, one arrives at a complete description of the vector space $\mathcal{Z}(S)$ assigned to a surface $S$ in terms of the balanced braided tensor category $\mathcal{Z}(S^1)$. In particular, compactifying to three-manifolds, the Verlinde formula expresses the dimension $\dim \mathcal{Z}(S) = \mathcal{Z}(S \times S^1)$ in terms of the structure constants of the Verlinde algebra $\mathcal{Z}(S^1 \times S^1)$ viewed as the center of the algebra $A$.

Returning to the surface $S$ itself, with the choice of a simple closed curve $\gamma \subset S$, one finds a compatible action of the Verlinde algebra $\mathcal{Z}(S^1 \times S^1)$ on the vector space $\mathcal{Z}(S)$ as loop operators along $\gamma \subset S$. For example, in the Dijkgraaf-Witten theory of a finite group $\Gamma$, the action on the vector space $\mathbb{C}[\text{Loc}_\Gamma(S)]$ of functions on $\Gamma$-local systems on any surface $S$ results from modifications of local systems along $\gamma$ as realized by the correspondence

$$\mathcal{L}_{\text{Loc}}(S) \times \mathcal{L}_{\text{Loc}}(S^1 \times S^1) \longrightarrow \mathcal{L}_{\text{Loc}}(S \coprod S_{\gamma \setminus S}) \longrightarrow \mathcal{L}_{\text{Loc}}(S)$$

where the torus $S^1 \times S^1$ appears in the unusual but homotopy equivalent form of the subspace of $S \coprod S \gamma S$ obtained by gluing a tubular neighborhood of $\gamma \subset S$ to itself along the complement of $\gamma$.

1.1.1. **Geometric Langlands and four-dimensional TFT.** Kapustin and Witten [KW] discovered that many structures of the Geometric Langlands program fit naturally into the framework of four-dimensional TFT, and more specifically, a topological twist of $\mathcal{N} = 4$ super Yang-Mills. In particular, in its spectral realization, the invariant assigned to a closed surface $S$ is a category of $B$-branes on the moduli $\text{Loc}_G(S)$ of $G$-local systems on $S$. To make the link with the Geometric Langlands program more precise, one needs to specify the category of $B$-branes.

In the traditional Geometric Langlands program of Beilinson and Drinfeld [BD], one assumes $S$ is a smooth projective complex curve algebraic curve, and works with the de Rham moduli $\text{Conn}_G(X)$ of flat $G$-connections on $X$. While $\text{Loc}_G(S)$ and $\text{Conn}_G(S)$ are analytically equivalence, they have different algebraic structures. On the one hand, categories of quasicoherent sheaves on $\text{Conn}_G(S)$ are
not locally constant in the algebraic curve $S$, and so are not the invariants of a TFT. On the other hand, categories of quasicoherent sheaves on $\mathcal{L}oc_G(S)$ manifestly depend only on the topological surface $S$. For example, it follows from the results of [BFN] that the category $QC(\mathcal{L}oc_G(S))$ of all quasicoherent sheaves, or more concretely, the small category $\text{Perf}(\mathcal{L}oc_G(S))$ of perfect complexes, fits into a fully extended $(3 + 1)$-dimensional oriented TFT.

Going further, as explained by Arinkin and Gaitsgory [AG], quasicoherent sheaves alone are too naive to be the spectral category in the Geometric Langlands correspondence. Most glaringly, they are not compatible with parabolic induction: the Eisenstein series and constant term constructions fail to give a continuous adjunction. Arinkin and Gaitsgory developed a beautiful solution to this problem by expanding from quasicoherent sheaves to ind-coherent sheaves with nilpotent singular support, showing it provided the minimal solution compatible with parabolic induction.

Following these developments, to find a spectral category that fits into a TFT, and is rich enough for a topological Geometric Langlands correspondence, we propose [BN] the category $QC_{cl}(\mathcal{L}oc_G(S))$ of ind-coherent sheaves with nilpotent singular support on the moduli of $G$-local systems on $S$, or more concretely, the small category $\text{DCoh}_{cl}(\mathcal{L}oc_G(S))$ formed by its compact objects. A substantial challenge is that coherent sheaves are much more complicated than perfect complexes: most notably, compatibilities between algebraic and geometric constructions for perfect complexes on derived stacks as appear in \cite{BFN} fail for coherent sheaves. To address this, in the papers [BNP1, BNP2], we developed new techniques to work with coherent sheaves, including descent with prescribed singular support. The main result of this paper, confirming the spectral category $\text{DCoh}_{cl}(\mathcal{L}oc_G(S))$ enjoys the gluing of a TFT, is an application of these techniques.

It is an interesting problem to construct a fully extended $(3 + 1)$-dimensional TFT that assigns $\text{DCoh}_{cl}(\mathcal{L}oc_G(S))$ to a surface $S$. Results of [BNP1, BNP2], as extended by the main result of this paper, highlight that such a TFT could assign the 2-category of small $H_G$-module categories to the circle $S^1$. Finding a suitable 3-category to assign to the point is the subject of ongoing work.

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2. Recollections

2.1. Singular support. We recall here some notions and results from [AG] (see also [BNP2] for a summary).

First, recall that a derived scheme $Z$ is quasi-smooth if and only if it is a derived local complete intersection in the sense that it is Zariski-locally the derived zero-locus of a finite collection of polynomials. Equivalently, a derived scheme $Z$ is quasi-smooth if and only if its cotangent complex $\mathbb{L}_Z$ is perfect of tor-amplitude $[-1, 0]$. More generally, we work with derived stacks that are quasi-smooth in the sense that they admit a smooth atlas of quasi-smooth derived schemes (for example the character stack is a quotient of a quasi smooth scheme by the action of an affine group).

Let $X$ be a quasi-smooth derived stack and $\mathbb{L}_X$ its cotangent complex. Let $X_{cl}$ denote the underlying classical stack of $X$. Introduce the shifted cotangent complex

$$T_X^{-1} = \text{Spec}_{X_{cl}} \text{Sym}_{X_{cl}} H^1(\mathbb{L}_X^\vee) \simeq (\text{Spec}_X \text{Sym}_X \mathbb{L}_X^\vee[1])_{cl}$$

There is a natural affine projection $T_X^{-1} \to X_{cl}$ with fiberwise $\mathbb{G}_m$-action and the fiber $T_X^{-1}|_x$ at a point $x \in X_{cl}$ is the degree $-1$ cohomology of $\mathbb{L}_Z|_x$. We denote by $\{0\}_X \subset T_X^{-1}$ the zero-section.

An important invariant of any $F \in QC^1(X)$ is its singular support

$$\text{supp} F \subset T_X^{-1}$$

It is a conic Zariski-closed subset when $F \in \text{DCoh}(X)$ and in general a union of conic Zariski-closed subsets. For $F \in \text{DCoh}(X)$, one has $\text{supp} F \subset \{0\}_X$ if and only if $F \in \text{Perf}_X$. 

Let $\text{Con} X$ denote the set of conic Zariski-closed subsets of $T^*_X - 1$. For any $\Lambda \in \text{Con} X$, one defines the full subcategory

$$i_\Lambda : \text{QC}^i_X(X) \hookrightarrow \text{QC}^i(X)$$

of ind-coherent complexes supported along $\Lambda$. The inclusion $i_\Lambda$ admits a right adjoint

$$R \Gamma_\Lambda : \text{QC}^i(X) \to \text{QC}^i_\Lambda(X)$$

We will often regard $\text{QC}^i_\Lambda(X)$ as a subcategory of $\text{QC}^i(X)$ via the embedding $i_\Lambda$, and likewise regard $R \Gamma_\Lambda$ as an endofunctor of $\text{QC}^i(X)$.

We can define functors between categories of sheaves with prescribed singular support by enforcing the support condition:

**Definition 2.1.** Suppose $f : X \to Y$ is a map of quasi-smooth stacks. Fix $\Lambda_X \in \text{Con} X$, $\Lambda_Y \in \text{Con} Y$, and define functors with support conditions

$$f_* : \text{QC}^i_{\Lambda_X}(X) \to \text{QC}^i_{\Lambda_Y}(Y) \quad f^! : \text{QC}^i_{\Lambda_Y}(Y) \to \text{QC}^i_{\Lambda_X}(X)$$

$$f_* = R \Gamma_{\Lambda_Y} \circ f_* \circ i_{\Lambda_X} \quad f^! = R \Gamma_{\Lambda_X} \circ f^! \circ i_{\Lambda_Y}$$

**Remark 2.2.** If the traditional functors preserve support conditions, then the above compositions agree with their traditional counterparts.

Associated to a map $f : X \to Y$ is a correspondence

$$T^*_X - 1 \xleftarrow{df^*} T^*_Y - 1 \times_Y X \xrightarrow{\tilde{f}} T^*_Y - 1$$

Given a subset $U \subset T^*_X - 1$, we may form the subset

$$f_* U = \tilde{f}((df^*)^{-1}(U)) \subset T^*_Y - 1$$

If $f : X \to Y$ is proper, then $\tilde{f}$ is proper, and this defines a map

$$f_* : \text{Con} X \to \text{Con} Y$$

Similarly, given a subset $V \subset T^*_Y - 1$, we may form the subset

$$f^! V = df^*(X \times_Y V) \subset T^*_X - 1$$

If $f : X \to Y$ is quasi-smooth, then $df^*$ is a closed immersion, and this defines a map

$$f^! : \text{Con} Y \to \text{Con} X$$

2.1.1. **Pushforwards.** For $F \in \text{QC}^i(X)$, and $f$ schematic and quasi-compact, [$\text{AG}$ Lemma 7.4.5] ensures

$$\text{supp} f_* F \subset f_* \text{supp} F$$

and therefore if $f_* \Lambda_X \subset \Lambda_Y$, then

$$f_*(\text{QC}^i_{\Lambda_X}(X)) \subset \text{QC}^i_{\Lambda_Y}(Y)$$

Following [BNP2], we codify this condition into a definition:

**Definition 2.3.** Let $X, Y$ be quasi-smooth stacks, and $\Lambda_X \in \text{Con} X, \Lambda_Y \in \text{Con} Y$.

Define a map of pairs $f : (X, \Lambda_X) \to (Y, \Lambda_Y)$ to be a map $f : X \to Y$ such that $f_* \Lambda_X \subset \Lambda_Y$.

In this case, we say “$f$ takes $\Lambda_X$ to $\Lambda_Y$.”
Remark 2.4. For a map of pairs \( f : (X, \Lambda_X) \to (Y, \Lambda_Y) \), we require
\[
(df^*)^{-1}(\Lambda_X) \subset X \times_Y \Lambda_Y
\]
If \( f : X \to Y \) is quasi-smooth, so that \( df^* \) is a closed immersion, then we can equivalently require
\[
df^*(X \times_Y T_Y^{-1}) \cap \Lambda_X \subset df^*(X \times_Y \Lambda_Y)
\]
With our previous notation, this can be rephrased in the form
\[
f^* T_Y^{-1} \cap \Lambda_X \subset f^* \Lambda_Y
\]
2.1.2. Pullbacks. Likewise, for \( F \in \mathcal{QC}^!_X(Y) \), [AG, Lemma 7.4.2] ensures
\[
\text{supp} f^! F \subset f^! \text{supp} F
\]
and therefore if \( f^! \Lambda_Y \subset \Lambda_X \), then
\[
f^!(\mathcal{QC}^!_X(Y)) \subset \mathcal{QC}^!_{f^! \Lambda_X}(X)
\]
This condition is implied by the following strong compatibility condition from [BNP2]:

Definition 2.5. Let \( X, Y \) be quasi-smooth stacks, and \( \Lambda_X \in \text{Con} X, \Lambda_Y \in \text{Con} Y \).
Define a strict map of pairs \( f : (X, \Lambda_X) \to (Y, \Lambda_Y) \) to be a map \( f : X \to Y \) such that
\[
(df^*)^{-1}(\Lambda_X) = X \times_Y \Lambda_Y
\]
In this case, we say “the \( f \)-preimage of \( \Lambda_Y \) is precisely \( \Lambda_X \).”

Remark 2.6. If \( f : X \to Y \) is quasi-smooth, so that \( df^* \) is a closed immersion, then \( f : (X, \Lambda_X) \to (Y, \Lambda_Y) \) is a strict map of pairs if and only if
\[
df^*(X \times_Y T_Y^{-1}) \cap \Lambda_X = df^*(X \times_Y \Lambda_Y)
\]
With our previous notation, this can be rephrased in the form
\[
f^* T_Y^{-1} \cap \Lambda_X = f^! \Lambda_Y
\]
2.2. Descent with singular supports. Next, we recall two results from [BNP2].

The first is the microlocal description of sheaves on fiber products:

Proposition 2.7. [BNP2, Proposition 2.1.9] Let \( X_1, X_2 \) be quasi-smooth stacks over a smooth separated base \( Y \). Then the functor of exterior product over \( Y \) induces an equivalence

\[
\begin{array}{ccc}
\text{D Coh}(X_1) \otimes \text{Perf}(Y) & \xrightarrow{\otimes_Y} & \text{D Coh}(X_2) \\
\downarrow \sim & & \downarrow \sim \\
\text{D Coh}_\Lambda (X_1 \times_Y X_2) & \xrightarrow{i^!} & \text{D Coh}(X_1 \times_Y X_2)
\end{array}
\]

where \( \Lambda = i^!(T_{X_1 \times X_2}^{-1}) \) for \( i : X_1 \times_Y X_2 \to X_1 \times X_2 \).

The most significant result of [BNP2] we will need is descent for sheaves with prescribed singular support.

Definition 2.8. A strict Cartesian diagram of pairs is a Cartesian diagram of quasi-smooth stacks which is also a commutative diagram of maps of pairs

\[
(Z = X \times_{S} X', \Lambda_Z) \xrightarrow{p_2} (X', \Lambda_{X'})
\]
\[
\xrightarrow{p_1} (X, \Lambda_X) \xrightarrow{p} (Y, \Lambda_Y)
\]
satisfying the strictness condition
\[
\Lambda_Z \supset p_1^! \Lambda_X \cap p_2^! \Lambda_{X'}.
\]
**Theorem 2.9.** [BNP2, Theorem 2.4.1, Corollary 2.4.2] Suppose \( f : (X_\bullet, \Lambda_\bullet) \to (X_{-1}, \Lambda_{-1}) \) is an augmented simplicial diagram of maps of pairs with all stacks quasi-smooth and maps proper. Suppose further that:

1. The face maps are quasi-smooth.
2. For any map \( g : [m] \to [n] \) in \( \Delta_+ \), the induced commutative square
   \[
   \begin{array}{ccc}
   (X_{n+1}, \Lambda_{n+1}) & \to & (X_n, \Lambda_n) \\
   g \downarrow & & \downarrow g \\
   (X_{m+1}, \Lambda_{m+1}) & \to & (X_m, \Lambda_m)
   \end{array}
   \]
   is a strict Cartesian diagram of pairs.
3. Pullback along the augmentation \( f !_\Lambda : \text{QC}^1_{\Lambda_{-1}}(X_{-1}) \to \text{QC}^1_{\Lambda_0}(X_0) \) is conservative.
4. Each \( \text{QC}^1_{\Lambda_k}(X_k) \) is compactly generated for each \( k \geq 0 \).

Then \( \text{QC}^1_{\Lambda_{-1}}(X_{-1}) \) is compactly generated as well, and pushforward along the augmentation provides an equivalence

\[
\text{DCoh}_{\Lambda_{-1}}(X_{-1}) \xrightarrow{\sim} |\text{DCoh}_{\Lambda_0}(X_\bullet), f_\bullet|.
\]

### 2.3. Bar and Čech constructions.

Let us now recall the relative bar construction in algebra and geometry (see [BFN] for a review in the \( \infty \)-categorical setting).

Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category. Given an algebra \( A \in \mathcal{C} \), the trace of an \( A \)-bimodule \( M \in \mathcal{C} \) is defined to be the tensor product of bimodules

\[
\text{Tr}(A, M) = M \otimes_{A \otimes A^{op}} A
\]

Suppose \( B \to A \) is a morphism of algebra objects. Viewing \( A \) as an algebra in \( B \)-bimodules, we can identify \( A \) with the geometric realization of the relative bar resolution

\[
A \simeq |A^\otimes B(\bullet+2)|.
\]

Note the two extreme cases: when \( B = A \), then we recover the constant resolution; when \( B \) is the monoidal unit, we recover the absolute bar resolution

\[
A \simeq |A^\otimes (\bullet+2)|.
\]

The relative bar resolution can be used to calculate the trace

\[
\text{Tr}(A, M) = A \otimes_{A \otimes A^{op}} M \simeq |A^\otimes B(\bullet+2)| \otimes_{A \otimes A^{op}} M \simeq |A^\otimes B(\bullet+2) \otimes_{A \otimes A^{op}} M|
\]

Given a correspondence \( Z \to Y \times Y \) of derived stacks, its geometric trace is defined to be the fiber product

\[
\text{Tr}^{\text{geom}}(Y, Z) = Z \times_{Y \times Y} Y.
\]

Given a map \( p : X \to Y \) of derived stacks, we can form its Čech construction

\[
X_\bullet = X \times_Y (\bullet+1) \to Y.
\]

viewed as an augmented simplicial object. In general, this is not a colimit diagram, but we will only encounter situations where it is.

Note that we can identify the Čech construction of the base change

\[
Z \times_{Y \times Y} X \to Z \times_{Y \times Y} Y
\]
with the substitution of the Čech construction of \( p : X \to Y \) into the definition of the trace

\[
Z \times_{Y \times Y} X^{\times_Y (\bullet + 1)} \to \text{Tr}^{\text{geom}}(Y, Z)
\]

Again, in general, this is not a colimit diagram, but we will only encounter situations where it is.

To guide later discussion, let us informally relate the bar and Čech constructions. We will work in the category of spans with objects derived stacks and morphisms correspondences of derived stacks.

Any derived stack \( Y \) is naturally an algebra object with multiplication

\[
Y \times Y \xrightarrow{\delta} Y \xrightarrow{\text{id}_Y} Y
\]

More generally, any map \( q : Z \to Y \) of derived stacks provides a \( Y \)-module with action

\[
Y \times Z \xrightarrow{q \times \text{id}_Z} \xrightarrow{\text{id}_Z} Z
\]

Given a map \( p : X \to Y \), the fiber product \( X \times_Y X \) is also an algebra object with multiplication

\[
X \times_Y X \xrightarrow{\delta} X \times_Y X \xrightarrow{p} X \times_Y X
\]

The relative diagonal \( X \to X \times_Y X \) is a map of algebra objects, and \( X \times_Y X \) descends to an algebra object in \( X \)-bimodules with multiplication

\[
X \times_Y X \xrightarrow{\sim} X \times_Y X \xrightarrow{p} X \times_Y X
\]

Note that here the multiplication can be viewed as an honest map.

Given a correspondence \( Z \to Y \times Y \), note that its algebraic and geometric traces agree

\[
\text{Tr}(Y, Z) \simeq Z \times_{Y \times Y} Y = \text{Tr}^{\text{geom}}(Y, Z)
\]

Now consider the \( X \times_Y X \)-bimodule given by the base change

\[
Z_X = Z \times_{Y \times Y} X \times X
\]

Let us calculate its trace \( \text{Tr}(X \times_Y X, Z_X) \) using the relative bar resolution

\[
X \times_Y X \simeq \left| (X \times_Y X)^{\times_Y (\bullet + 2)} \right| \simeq \left| X^{\times_Y (\bullet + 3)} \right|
\]

of the map of algebras \( X \to X \times_Y X \): we find

\[
Z_X \times_{(X \times_Y X)^{\times_Y (\bullet + 2)}} (X \times_Y X)^{\times_Y (\bullet + 2)} \simeq Z_X \times_{X \times X} X^{\times_Y (\bullet + 1)} \simeq Z \times_{Y \times Y} X^{\times_Y (\bullet + 1)}.
\]

We identify the result with the Čech construction of the map

\[
Z \times_{Y \times Y} X \xrightarrow{\text{id}_Y} Z \times_{Y \times Y} Y
\]

but with the alternative augmentation

\[
Z \times_{Y \times Y} X^{\times_Y (\bullet + 1)} \xrightarrow{\text{Tr}(X \times_Y X, Z_X)}
\]

In situations where the Čech construction calculates \( \text{Tr}(Y, Z) \simeq Z \times_{Y \times Y} Y \), we then have Morita-invariance of the trace

\[
\text{Tr}(X \times_Y X, Z_X) \simeq \text{Tr}(Y, Z)
\]

We will only encounter situations where this holds, but will pass to categories of sheaves where an interesting failure of Morita-invariance occurs in the form of singular support conditions.
3. Gluing geometric bimodules

We now prove our main theorem, a gluing result for geometric bimodules. We will use the notation of Section 2.3.
Let \( p : X \to Y \) and \( q : Z \to Y \times Y \) be quasi-smooth morphisms of smooth derived stacks, and set \( Z_X = Z \times_{Y \times Y} X \times X \). Assume \( p \) is proper.

Let \( Z^{-1} \) denote the geometric trace \( Z \times Y \times Y \) of the \( Y \)-bimodule \( Z \).

Recall the fundamental correspondence
\[
Z_X = Z \times Y \times Y \times X \xrightarrow{\delta} Z \times Y \times Y \xrightarrow{p} Z \times Y \times Y = Z^{-1}
\]
and introduce on \( Z^{-1} \) the support condition
\[
\Lambda^{-1} = p_* \delta^! T^{-1}_{Z_X}
\]

Introduce the monoidal category \( H = H_{X,Y} = \text{DCoh}(X \times Y \times X) \) and the \( H_{X,Y} \)-bimodule \( \text{DCoh}(Z_X \times X) \).

**Theorem 3.1.** There is a canonical equivalence of \( \text{Perf}(Y) \)-modules
\[
\text{Tr}(H_{X,Y}, \text{DCoh}(Z_X)) \simeq \text{DCoh}(\Lambda^{-1} Z^{-1})
\]

**Proof.** We would like to compare sheaves on the diagram
\[
Z_* = Z \times Y \times Y \times Y \times (X \times (X \times (X \times X)))
\]
with, on the one hand, the category \( \text{DCoh}(\Lambda^{-1} Z^{-1}) \) and, on the other hand, the trace of the \( A = H_{X,Y} \)-bimodule \( \text{DCoh}(Z_X) \) as calculated via the bar construction relative to \( B = \text{Perf}(X) \). The face maps in the simplicial diagram \( Z_* \) are all proper and quasi-smooth maps, being base changes of the proper and quasi-smooth map \( \pi \). The degeneracy maps (given by relative diagonals) are likewise proper since \( \pi \) is representable and separated. Let
\[
g_* : Z_* \simeq Z_X \times_{X \times X} (X \times Y \times X)^{\times \bullet} \to W_* = Z_X \times (X \times Y)^{\times \bullet}
\]
be the map to the absolute two-sided bar construction, and define
\[
\Lambda_* = g_*^! T^{\times -1} W_*
\]
to be the resulting support condition on \( Z_* \), so that we have a simplicial diagram of pairs \((Z_*, \Lambda_*)\).

We now pass to categories using (\( \text{DCoh}(\Lambda_* f_* \text{Perf}(X)) \)), obtaining an augmented simplicial category
\[
\mathcal{C}_* = \text{DCoh}(\Lambda_* Z_*) \to \text{DCoh}(\Lambda^{-1} Z^{-1}).
\]
By repeated application of Proposition 2.7 we have the identification
\[
\text{DCoh}(Z_n) \simeq \text{DCoh}(Z_X \otimes \text{Perf}(X) \otimes \text{Perf}(X)^{\bullet})
\]
on simplices compatibly with structure maps, and thus an identification of simplicial objects
\[
\mathcal{C}_* = \text{DCoh}(Z_X) \otimes \text{Perf}(X) \otimes \text{Perf}(X)^{\bullet} \simeq \text{DCoh}(Z_X) \otimes \text{Perf}(X)^{\otimes 2}
\]
with the relative bar construction. Thus we have identified
\[
|\mathcal{C}_*| \simeq \text{Tr}(H, \text{DCoh}(Z_X)).
\]

We will now verify the hypotheses of Theorem 2.9 are satisfied for the augmented simplicial diagram
\[
(Z_*, \Lambda_*) \to (Z^{-1}, \Lambda^{-1})
\]
As already noted, the face maps are quasi-smooth and proper, the degeneracy maps are proper, and the requisite squares are Cartesian. Next, note that \( p \) is a representable proper map, so that applying [AG Prop. 7.4.19], we see the augmentation is conservative, since by definition the support condition on the target \( Z^{-1} \) is the image of the support condition on the source \( Z_0 \). Next, we need
to see that the categories $\text{QC}^i_{\Lambda_n}(Z_n)$ are compactly generated for each $n \geq 0$. First, we can identify $\text{QC}^i_{\Lambda_n}(Z_n)$ as the essential image of

$$\text{QC}^i(Z_X) \otimes_{\text{QC}(X \times X)} \text{QC}^i(X \times Y \times X)^{\otimes_{\text{QC}(X)}} \to \text{QC}^i(Z_n).$$

This follows directly from \[AC\] Proposition 7.4.12] (as in the proof of Proposition \[2.7\] given in \[BNP2\] Proposition 2.1.9].) Since $\text{QC}^i(Z_X)$, $\text{QC}^i(X)$, and $\text{QC}^i(X \times Y)$ are compactly generated and all structure maps preserve compact objects by our hypotheses, it follows that the left hand side is compactly generated, hence so is $\text{QC}^i_{\Lambda_n}(Z_n)$.

It remains to establish that the diagram $(Z_\bullet, \Lambda_\bullet)$ is a strict diagram of pairs, which we now prove separately as Proposition \[3.2\].

**Proposition 3.2.** The diagram $(Z_\bullet, \Lambda_\bullet)$ is a strict Cartesian diagram of pairs.

**Proof.** The proof closely mimics the proof of \[BNP2\] Proposition 3.3.8], which is the case $Z = Y$. We indicate the idea and modifications necessary for the general case.

We give an explicit description of the shifted cotangents to $Z_n$, on the level of $k$-points of the derived stack. Such points can be represented by tuples

$$(y, \{x_0, \ldots, x_n\}, z, \gamma)$$

with $y \in Y$, $x_i \in p^{-1}y \subset X$, $z \in Z$ with $\mu_i(z) = y$ and $\gamma : \mu_i(z) \sim \mu_r(z)$, and $\mu_i(z) = \mu_r(z) = y$. Here we denote by $\mu_i \times \mu_r : Z \to Y \times Y$ the defining projection. We represent points of $W_n$ by tuples

$$(y_0, x_0, x'_0, \ldots, y_{n-1}, x_{n-1}, x'_{n-1}; y, x_n, x'_n) : p(x_i) = p(x'_i) = y_i, \mu_i(z) = p(x_n), \mu_r(z) = p(x'_n).$$

The map $\eta_n : Z_n \to W_n$ is thus represented by

$$\eta_n(y, \{x_0, \ldots, x_n\}, z) = (y, x_0, x_1; y, x_1, x_2, \ldots, y, x_{n-1}, x_n; z, x_n, \gamma \circ x_0)$$

where we use the path $\gamma$ to identify $\mu_r(z) \sim p(x_0)$.

Under these identifications, we write at a geometric point $\eta = (y, \{x_0, \ldots, x_n\}, z, \gamma)$ of $Z_n$

$$T_{Z_n}^{-1}|_{\eta} \sim \{v_0, \ldots, v_{n+1} \in \Omega_Y : dp^{1*}_{x_i}v_0 = dp^{1*}_{z_i}v_1, \ldots\}$$

while at a geometric point $\eta' = (y_0, x_0, x'_0, \ldots, y_{n-1}, x_{n-1}, x'_{n-1}; z, x_n, x'_n)$ of $W_n$

$$T_{W_n}^{-1}|_{\eta'} \sim \{v_0, \ldots, v_{n+1} \in \Omega_Y : dp^{1*}_{x_0}v_0 = 0 = dp^{1*}_{x_0}v_0, \ldots\}$$

Combining these descriptions, we find at a geometric point $\eta = (y, \{x_0, \ldots, x_n\}, z, \gamma)$ of $Z_n$

$$\Lambda_n|_{\eta} \sim \{v_0, \ldots, v_{n+1} \in \Omega_Y : dp^{1*}_{x_i}v_0 = 0 = dp^{1*}_{x_i}v_1, \ldots\}$$

$$\Lambda_n|_{\eta} \sim \{v_0, \ldots, v_{n+1} \in \Omega_Y : dp^{1*}_{x_i}v_0 = 0 = dp^{1*}_{x_i}v_1, \ldots\}$$
We now need to check for any \( \psi : [m] \to [n] \in \Delta \) that the induced diagram

\[
\begin{array}{ccc}
(Z_{n+1}, \Lambda_{n+1}) & \xrightarrow{d_0} & (Z_n, \Lambda_n) \\
\uparrow \phi & & \downarrow \psi \\
(Z_{m+1}, \Lambda_{m+1}) & \xrightarrow{d_0} & (Z_m, \Lambda_m)
\end{array}
\]

is a strict Cartesian diagram of pairs, in other words that for any geometric point \( \eta \) we have

\( ((d_0)^! \Lambda_n)|_\eta \cap (\tilde{\phi}^! \Lambda_{m+1})|_\eta \subset \Lambda_{n+1}|_\eta \)

We first consider the case of face maps, i.e., of \( \psi \) an inclusion. The simplicial map \( \tilde{\psi} : [m + 1] \to [n + 1] \) inducing \( \tilde{\phi} \) is given by \( \tilde{\psi}(0) = 0, \tilde{\psi}(i) = 1 + \psi(i - 1) \) for \( i \geq 1 \). In this case the support condition \( (\tilde{\phi}^! \Lambda_{m+1})|_\eta \) consists of the one equation \( d(\mu)_1^* v_{m+1} = d(\mu)_2^* v_{m+2} \) coming from \( Z \) and the subset of the equations \( d p_*^* v_{i-1} = 0 = d p_*^* v_i \) corresponding to indices \( i \) in the image of \( \tilde{\psi} \), together with additional degeneracy identities among the complementary \( v_j \). Likewise the support condition \( ((d_0)^! \Lambda_n)|_\eta \) consists of the \( Z \)-equation and the equations \( d p_*^* v_{i-1} = 0 = d p_*^* v_i \) for \( i \geq 1 \), plus a degeneracy condition relating \( v_0 \) and \( v_{n+1} \). Since \( \tilde{\psi} \) has 0 in its image, the intersection of these two conditions imposes all the equations defining \( \Lambda_{n+1} \), as desired.

The general case follows the argument of [BNP2, Proposition 3.3.8] verbatim. We factor \( \psi : [m] \to [n] \) (in a unique fashion)

\[
\psi : [m] \xrightarrow{\pi} [k] \xrightarrow{\iota} [n]
\]

as a surjection followed by an injection. This gives rise to an extended diagram

\[
\begin{array}{ccc}
(Z_{n+1}, \Lambda_{n+1}) & \xrightarrow{d_0} & (Z_n, \Lambda_n) \\
\downarrow \phi & & \downarrow \psi \\
(Z_{k+1}, \Lambda_{k+1}) & \xrightarrow{d_0} & (Z_k, \Lambda_k) \\
\downarrow \phi & & \downarrow \psi \\
(Z_{m+1}, \Lambda_{m+1}) & \xrightarrow{d_0} & (Z_m, \Lambda_m)
\end{array}
\]

where \( p \) correspond to the injection \( \iota \), and \( q \) corresponds to the surjection \( \pi \).

We need to show that the large square satisfies the required strictness. By the case of a surjection, we know that the top square satisfies the required strictness. Thus it suffices to show that \( (\tilde{\psi})^! \Lambda_{m+1} \) already equals \( \Lambda_{k+1} \) since then

\[
(\tilde{\psi} \circ \tilde{\phi})^! \Lambda_{m+1} = (\tilde{\phi})^! (\tilde{\psi})^! \Lambda_{m+1} = (\tilde{\phi})^! \Lambda_{k+1}
\]

Define \( \pi' : [k] \to [m] \) to be the section of \( \pi \) given by its break points

\[
\pi'(i) = \sup \pi^{-1}(i)
\]

Thus the pullback map admits the description

\[
(v_0, \ldots, v_{m+1}) \mapsto (v_0, v_1 + \pi'(0), \ldots, v_1 + \pi'(k))
\]

and thus itself admits a section by repeating terms.

It is now elementary to see that \( (\tilde{\psi})^! \Lambda_{m+1} = \Lambda_{k+1} \): the inclusion \( (\tilde{\psi})^! \Lambda_{m+1} \subset \Lambda_{k+1} \) is evident, while the inclusion \( (\tilde{\psi})^! \Lambda_{m+1} \supset \Lambda_{k+1} \) follows from the fact that the noted section takes \( \Lambda_{k+1} \) into \( \Lambda_{m+1} \). This completes the proof.

\( \square \)
4. Gluing parabolic local systems

Let us introduce the notation $\mathbf{G} = G/G \simeq LBG \simeq \text{Loc}_G(S^1)$ and $\mathbf{B} = B/B \simeq LBG \simeq \text{Loc}_G(S^1)$ for the adjoint quotients, and $p : \mathbf{B} \to \mathbf{G}$ for the Grothendieck-Springer resolution.

For a closed (not necessarily oriented) surface with boundary $S$, consider the restriction of local systems to the boundary

$$\text{Loc}_G(S) \rightarrow \text{Loc}_G(\partial S) \simeq (\mathbf{G})^{\pi_0(\partial S)}$$

Write $\partial S = \coprod_{\alpha \in \pi_0(\partial S)} \partial_\alpha S$ for the decomposition of $\partial S$ into connected components. For $A \subset \pi_0(\partial S)$, denote by $\partial_A S = \coprod_{\alpha \in A} \partial_\alpha S$ the union of those connected components.

Define the stack of parabolic local systems to be the base change

$$\text{Loc}_G(S, \partial_A S) = \text{Loc}_G(S) \times_{\text{Loc}_G(\partial_A S)} \text{Loc}_B(\partial_A S) \simeq \text{Loc}_G(S) \times_{(\mathbf{G})^A} (\mathbf{B})^A$$

so in other words, the stack of local systems with a Borel reduction along $\partial_A S$.

Define the parabolic spectral category to be

$$\text{DCoh}_\mathcal{X}(\text{Loc}_G(S, \partial_A S))$$

**Example 4.1.** The Steinberg stack

$$\text{St}_G = \mathbf{B} \times G \mathbf{B} \simeq \mathcal{L}(B \setminus G/B) \simeq \text{Loc}_G(Cyl, \partial Cyl)$$

is the special case of the cylinder ($Cyl = S^1 \times [0, 1], \partial Cyl = S^1 \times \{0, 1\}$).

It carries an $(S^1 \times S^1)$-action separately rotating the boundary components, with the diagonal rotation identified with the rotation of the cylinder.

The affine Hecke category is the corresponding parabolic spectral category

$$\mathcal{H}_G = \text{DCoh}(\text{St}_G) \simeq \text{DCoh}_\mathcal{X}(\text{Loc}_G(Cyl, \partial Cyl))$$

since all odd codirections of $\text{St}_G$ are nilpotent.

For $A \subset \pi_0(\partial S)$, define a marking of $\partial_A S$ to be the data of a marked point $x_\alpha \in \partial_\alpha S$ and orientation of $\partial_\alpha S$, for $\alpha \in A$. Note that an orientation of $S$ can be used to induce an orientation of $\partial S$ all at once.

A marking of $\partial_A S$ provides identifications $\partial_\alpha S \simeq S^1$, for $\alpha \in A$, up to contractible choices. Given two distinct $\alpha \neq \beta \in A$, set $\bar{A} = A \setminus \{\alpha, \beta\}$, and introduce the glued surface

$$\bar{S} = S \coprod_{\partial_\alpha S} \coprod_{\partial_\beta S} S^1$$

where we identify the two corresponding boundary components. Note that the image of the glued circles provides a canonical circle $\gamma : S^1 \hookrightarrow S$ in the interior.

Passing to local systems, we obtain the presentation

$$\text{Loc}_G(\bar{S}, \partial_{\bar{A}} \bar{S}) \simeq \text{Loc}_G(S, \partial_{\bar{A}} S) \times_{G \times G} G$$

Observe that the spectral category $\text{DCoh}_\mathcal{X}(\text{Loc}_G(\bar{S}, \partial_{\bar{A}} \bar{S}))$ is naturally a module over

$$\text{Perf}(\text{Loc}_G(\bar{S}, \partial_{\bar{A}} \bar{S})) \simeq \text{Perf}(\text{Loc}_G(S, \partial_{\bar{A}} S)) \otimes_{\text{Perf}(G \times G)} \text{Perf}(G)$$

Now recall that the standard convolution diagrams equip the affine Hecke category

$$\mathcal{H}_G = \text{DCoh}(\text{St}_G) \simeq \text{DCoh}_\mathcal{X}(\text{Loc}_G(Cyl, \partial Cyl))$$

with a monoidal structure compatible with rotations of the cylinder. By [BNP2, Theorem 1.4.6(1)], we have a monoidal equivalence

$$\mathcal{H}_G \simeq \text{End}_{\text{Perf}(G)}(\text{Perf}(\mathbf{B}))$$

compatible with rotations of the cylinder. Geometrically, the monoidal structure is realized by gluing cylinders along consecutive boundary components. We will use the orientation-reversing
diffeomorphism of the cylinder given by reversing the interval to fix an equivalence of the affine Hecke category with its opposite algebra.

For $A \subset \pi_0(\partial S)$, a marking of $\partial A S$ equips $\text{DCoh}_{\mathcal{N}'}(\text{Loc}\mathcal{G}(S, \partial_A S))$ with the structure of $\mathcal{H}_{\mathcal{G}} \otimes_A$ -module. In particular, an ordered pair of distinct $\alpha \neq \beta \in A$ equips $\text{DCoh}_{\mathcal{N}'}(\text{Loc}\mathcal{G}(S, \partial_A S))$ with the structure of $\mathcal{H}_{\mathcal{G}}$-bimodule.

Observe that the resulting trace

$$\text{Tr}(\mathcal{H}_{\mathcal{G}}, \text{DCoh}_{\mathcal{N}'}(\text{Loc}\mathcal{G}(S, \partial_A S))) = \text{DCoh}_{\mathcal{N}'}(\text{Loc}\mathcal{G}(S, \partial_A S)) \otimes_{\mathcal{H}_{\mathcal{G}}} \mathcal{H}_{\mathcal{G}}$$

is naturally a module over

$$\text{Perf}(\text{Loc}\mathcal{G}(S, \partial_A S)) \simeq \text{Perf}(\text{Loc}\mathcal{G}(S, \partial_A S)) \otimes_{\text{Perf}(G \times G)} \text{Perf}(G)$$

**Corollary 4.2.** There is a canonical equivalence of $\text{Perf}(\text{Loc}\mathcal{G}(S, \partial_A S))$-modules

$$\text{Tr}(\mathcal{H}_{\mathcal{G}}, \text{DCoh}_{\mathcal{N}'}(\text{Loc}\mathcal{G}(S, \partial_A S))) \simeq \text{DCoh}_{\mathcal{N}'}(\text{Loc}\mathcal{G}(S, \partial_A S))$$

between the trace of the parabolic spectral category and the spectral category of the glued surface.

**Proof.** This follows from Theorem 3.1 once we identify the support condition $\Lambda_{-1}$ with the nilpotent cone $\mathcal{N}$. For this, consider the fundamental correspondence specialized to the current situation

$$\text{Loc}\mathcal{G}(S, \partial_A S) \overset{\delta}{\leftarrow} \text{Loc}\mathcal{G}(\tilde{S}, \partial_{\tilde{A}} S) \times_G B \overset{p}{\rightarrow} \text{Loc}\mathcal{G}(\tilde{S}, \partial_{\tilde{A}} S)$$

Given a geometric point $\rho \in \text{Loc}\mathcal{G}(\tilde{S}, \partial_{\tilde{A}} S)$ with monodromy $\rho(\gamma) \in G$ around the glued circles, one calculates

$$T^{*-1}_{\text{Loc}\mathcal{G}(\tilde{S}, \partial_{\tilde{A}} S)}|_\rho \simeq \{ v \in g^* : Ad(\rho)v = v \}$$

$$\Lambda_{-1}|_\rho = \{ v \in g^* : \exists g \in \rho|_x, g \cdot \rho(\gamma) \in B, g \cdot v \in n \}$$

i.e., there is a frame for the $G$-torsor given by the fiber of $\rho$ at $x \in S$ taking the monodromy around $\gamma$ into $B$ and the convector $v$ into $n$.

Thus $\mathcal{N}$ evidently contains $\Lambda_{-1}|_\rho$; conversely, for any conjugacy class $[\alpha] \in G$ and $v \in \mathcal{N}$ there exists a frame $g$ sending $\alpha$ to $B$ and $v$ to $n$.\[\square]\n
### 5. Verlinde Loop Operators

We record here the compatibility of the gluing of Corollary 4.2 with further natural symmetries available in the Betti setting.

Let $Z(\mathcal{H}_{\mathcal{G}}) = \text{End}_{\mathcal{H}_{\mathcal{G}} \otimes \mathcal{H}_{\mathcal{G}}}(\mathcal{H}_{\mathcal{G}})$ be the center of the affine Hecke category. Recall that $Z(\mathcal{H}_{\mathcal{G}})$ is naturally an $E_2$-monoidal category with a universal central map $Z(\mathcal{H}_{\mathcal{G}}) \rightarrow \mathcal{H}_{\mathcal{G}}$.

We will recall the geometric description of $Z(\mathcal{H}_{\mathcal{G}})$ obtained in [BNP2, Theorem 4.3.1].

Let $\text{D Coh}_{prop/\mathcal{G}}(\mathcal{L}(\mathcal{G}))$ denote the dg category of coherent sheaves on the loop space $\mathcal{L}(\mathcal{G}) \simeq \text{Loc}\mathcal{G}(S^1 \times S^1)$ with proper support over $G \simeq \text{Loc}\mathcal{G}(S^1)$.

Recall that convolution equips $\text{D Coh}_{prop/\mathcal{G}}(\mathcal{L}(\mathcal{G}))$ with a natural $E_2$-monoidal structure. Recall the fundamental correspondence

$$\mathcal{L}(\mathcal{G}) \overset{p}{\leftarrow} \mathcal{L}(\mathcal{G}) \times_G B \overset{\delta}{\rightarrow} B \times_G B$$

and the induced functor

$$\delta_* p^* : \text{D Coh}_{prop/\mathcal{G}}(\mathcal{L}(\mathcal{G})) \longrightarrow \text{D Coh}(B \times_G B) = \mathcal{H}_{\mathcal{G}}$$

**Theorem 5.1.** [BNP2, Theorem 4.3.1] The functor $\delta_* p^*$ is the universal central map underlying an $E_2$-monoidal equivalence

$$\text{D Coh}_{prop/\mathcal{G}}(\mathcal{L}(\mathcal{G})) \overset{\sim}{\longrightarrow} Z(\mathcal{H}_{\mathcal{G}})$$
Remark 5.2. It is useful to reformulate the universal central map of the theorem as a central action.

Let $Cyl = S^1 \times [0, 1]$ denote the cylinder, and $\gamma = S^1 \times \{1/2\} \subset Cyl$ the meridian. Modifications of local systems along $\gamma$ provides a correspondence

$$\Loc_G(S^1 \times S^1) \times \Loc_G(Cyl, \partial Cyl) \xrightarrow{p_1} \Loc_G(Cyl, \partial Cyl) \xrightarrow{p_2} \Loc_G(Cyl, \partial Cyl)$$

where the torus $S^1 \times S^1$ appears in the unusual but homotopy equivalent form of the gluing of a tubular neighborhood of $\gamma$ to itself along the complement of $\gamma$.

The universal central map of the theorem extends to a central $Z(H_G)$-action on $\mathcal{H}_G$ given by

$$\mathcal{A} \ast \mathcal{M} = p_2 \ast p_1^*(\mathcal{A} \boxtimes \mathcal{M})$$

Now let us focus on the equivalence of Corollary 4.2.

On the one hand, observe that $Z(H_G)$ naturally acts on the algebraic side

$$\text{Tr}(H_G, \DCoh_X(\Loc_G(S, \partial_A S))) = \DCoh_X(\Loc_G(S, \partial_A S)) \otimes_{H_G} \mathcal{H}_G$$

via its central action on the factor $\mathcal{H}_G$ in the tensor product.

On the other hand, as we will now explain, $Z(H_G)$ naturally acts on the geometric side

$$\DCoh_X(\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S}))$$

by what are called Verlinde loop operators. Recall the canonical curve $\gamma : S^1 \to \tilde{S}$ of glued boundary components. Modifications of local systems along $\gamma$ provides a correspondence

$$\Loc_G(S^1 \times S^1) \times \Loc_G(\tilde{S}) \xrightarrow{p_1} \Loc_G(\tilde{S} \coprod \gamma \tilde{S}) \xrightarrow{p_2} \Loc_G(\tilde{S})$$

where the torus $S^1 \times S^1$ appears in the unusual but homotopy equivalent form of the gluing of a tubular neighborhood of $\gamma$ to itself along the complement of $\gamma$.

This provides a $Z(H_G)$-action on $\DCoh_X(\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S}))$ with multiplication

$$\mathcal{A} \ast \mathcal{M} = p_2 \ast p_1^*(\mathcal{A} \boxtimes \mathcal{M})$$

Proposition 5.3. The equivalence of Corollary 4.2 respects the natural $Z(H_G)$-actions.

Proof. This is a straightforward comparison of the correspondence of Remark 5.2 with the correspondence defining Verlinde loop operators.

Returning to the setting of Corollary 4.2, it is convenient to express the glued surface in the form

$$\tilde{S} = S \coprod S^1 \times S^1 \times Cyl$$

using the provided identifications $\partial_\alpha S \coprod \partial_\beta S \simeq S^1 \times S^1 \simeq \partial Cyl$.

Now observe that the constructed equivalence

$$\DCoh_X(\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S})) \otimes_{H_G} \mathcal{H}_G \xrightarrow{\sim} \DCoh_X(\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S}))$$

is induced by the functor

$$q_2 \ast q_1^* : \DCoh_X(\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S})) \otimes \DCoh(\Loc_G(Cyl, \partial Cyl)) \longrightarrow \DCoh_X(\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S}))$$

defined by the correspondence

$$(1) \quad \Loc_G(S, \partial_A S) \times \Loc_G(Cyl, \partial Cyl) \xrightarrow{q_1} \Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S}) \times G \times \mathcal{B} \times \mathcal{B} \xrightarrow{q_2} \Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S})$$

where the projection $\Loc_G(\tilde{S}, \partial_{\tilde{A}} \tilde{S}) \to G \times \mathcal{B}$ is given by evaluation at the glued loops.

Now we can extend diagram (1) to also encode the modification of bundles along the distinguished curve $\gamma = S^1 \times \{1/2\} \subset Cyl \subset \tilde{S}$. Namely, let us take the fiber product over $\Loc_G(Cyl)$ of each term of diagram (1) with the following correspondence

$$(2) \quad \Loc_G(S^1 \times S^1) \times \Loc_G(Cyl) \xrightarrow{p_1} \Loc_G(Cyl, \partial Cyl) \xrightarrow{p_2} \Loc_G(Cyl)$$
Note that diagram (2) results from the correspondence of Remark 5.2 but without the $B$-reductions already found here in diagram (1).

Finally, by base change, the natural $Z(H_G)$-actions given by $p_2^*p_1^*$ are compatible with the gluing given by $q_2^*q_1^*$.

□

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