An Erdős-Ko-Rado theorem for the derangement graph of PGL(2, q) acting on the projective line

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Abstract

Let $G = \text{PGL}(2, q)$ be the projective general linear group acting on the projective line $\mathbb{P}_q$. A subset $S$ of $G$ is intersecting if for any pair of permutations $\pi, \sigma$ in $S$, there is a projective point $p \in \mathbb{P}_q$ such that $p^\pi = p^\sigma$. We prove that if $S$ is intersecting, then $|S| \leq q(q-1)$. Also, we prove that the only sets $S$ that meet this bound are the cosets of the stabilizer of a point of $\mathbb{P}_q$.

Keywords: derangement graph, independent sets, Erdős-Ko-Rado theorem

1. Introduction

The Erdős-Ko-Rado theorem\textsuperscript{[3]} is a very important result in extremal combinatorics. There are many different proofs and extensions of this theorem and we refer the reader to \textsuperscript{[4]} for a full account. There are also many applications of the Erdős-Ko-Rado theorem, for example to qualitatively independent sets\textsuperscript{[12, 13]}, problems in geometry\textsuperscript{[6]} and in statistics\textsuperscript{[16]}.

In this paper, we are concerned with an extension of the Erdős-Ko-Rado theorem to permutation groups. Let $G$ be a permutation group on $\Omega$. We let $\text{fix}(g)$ denote the number of fixed points of the permutation $g$ of $G$. A subset $S$ of $G$ is said to be intersecting if $\text{fix}(g^{-1}h) \neq 0$, for every $g, h \in S$. As with the Erdős-Ko-Rado theorem, we are interested in finding the size of the largest intersecting set of $G$ and classifying the sets that attain this bound. This problem can be formulated in a graph-theoretic terminology. We denote by $\Gamma_G$ the derangement graph of $G$, the vertices of this graph are the elements of $G$ and the edges are the pairs $\{g, h\}$ such that $\text{fix}(g^{-1}h) = 0$. An intersecting set of $G$ is simply an independent set of $\Gamma_G$. In \textsuperscript{[3]} and \textsuperscript{[15]} the natural extension of the Erdős-Ko-Rado theorem for the symmetric group $\text{Sym}(n)$ was independently

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proven. Indeed, it was shown that every independent set of permutations in \( \Gamma_{\text{Sym}(n)} \) has size at most \((n-1)!\). Also, the only sets that meet this bound are the cosets of the stabilizer of a point. The same result was also proved in [8] using the character theory of \( \text{Sym}(n) \).

In this paper, inspired by the approach used in [8], we prove a similar result for the permutation group \( G = \text{PGL}(2, q) \) acting on the projective line \( \mathbb{P}_q \). We show that an independent set \( S \) in \( \Gamma_G \) has size at most \( q(q-1) \). Also, we prove that the sets \( S \) that meet this bound are the cosets of the stabilizer of a point of \( \mathbb{P}_q \). In particular, these results are a natural \( q \)-analogue of the result for the symmetric group \( \text{Sym}(n) \). Here we report the main theorem.

**Theorem 1.** Every independent set \( S \) of the derangement graph of \( \text{PGL}(2, q) \) acting on the projective line \( \mathbb{P}_q \) has size at most \( q(q-1) \). Equality is met if and only if \( S \) is the coset of the stabilizer of a point.

Theorem 1 indicates that the argument described in [8] will likely be particularly fruitful for studying the derangement graph of a permutation group \( G \) whose character theory is well-understood.

In Section 2 we recall some general results on the eigenvalues and the independent sets of the graph \( \Gamma_G \) that will be used throughout the paper. In Section 3 the character table of \( \text{PGL}(2, q) \) is described. Section 4 includes technical lemmas that are used in Section 5 to prove Theorem 1. Finally, in Section 6 we conclude with some general remarks on the derangement graphs of 2-transitive and 3-transitive groups.

### 2. General results

Let \( G \) be a permutation group on \( \Omega \) and \( \Gamma_G \) its derangement graph. Since the right regular representation of \( G \) is a subgroup of \( \text{Aut}(\Gamma_G) \), we see that \( \Gamma_G \) is a Cayley graph. Namely, if \( D \) is the set of derangements of \( G \) (i.e. the fixed-point-free permutations of \( G \)), then \( \Gamma_G \) is the Cayley graph on \( G \) with connection set \( D \), i.e. \( \Gamma_G = \text{Cay}(G, D) \). Clearly, \( D \) is a union of \( G \)-conjugacy classes, so \( \Gamma_G \) is a normal Cayley graph.

As usual, we simply say that the complex number \( \xi \) is an eigenvalue of the graph \( \Gamma \) if \( \xi \) is an eigenvalue of the adjacency matrix of \( \Gamma \).

In this paper, we use \( \text{Irr}(G) \) to denote the irreducible complex characters of a group \( G \) and given \( \chi \in \text{Irr}(G) \) and a subset \( S \) of \( G \) we write \( \chi(S) \) for \( \sum_{s \in S} \chi(s) \). In the following lemma we recall that the eigenvalues of \( \Gamma_G \) are determined by the irreducible complex characters of the group \( G \).

**Lemma 2.** Let \( G \) be a permutation group on \( \Omega \) and \( D \) the set of derangements of \( G \). The spectrum of the graph \( \Gamma_G \) is \( \{ \chi(D)/\chi(1) \mid \chi \in \text{Irr}(G) \} \). Also, if \( \tau \) is an eigenvalue of \( \Gamma_G \) and \( \chi_1, \ldots, \chi_s \) are the irreducible characters of \( G \) such that \( \tau = \chi_i(D)/\chi_i(1) \), then the dimension of the \( \tau \) eigenspace of \( \Gamma_G \) is \( \sum_{i=1}^s \chi_i(1)^2 \).

**Proof.** Since \( \Gamma_G \) is a normal Cayley graph, the result follows from [1].
Since $\Gamma_G$ is a Cayley graph, the $\mathbb{C}$-vector space with basis elements labelled by the vertices of $\Gamma_G$ is the $\mathbb{C}$-vector space underlying the group algebra $\mathbb{C}G$. If $S$ is a subset of $G$, we simply write $v_S$ for the vector $\sum_{g \in S} g$ of $\mathbb{C}G$ (so $v_S$ is the characteristic vector for the set $S$). In particular, $v_G$ is the all-1 vector of $\mathbb{C}G$. Finally, we recall Hoffman’s bound for the size of an independent set of $\Gamma_G$ and a consequence for when equality holds in this bound.

**Lemma 3.** Let $G$ be a permutation group on $\Omega$. Let $S$ be an independent set of $\Gamma_G$ and $\tau$ be the minimum eigenvalue of $\Gamma_G$. Assume that the valency of $\Gamma_G$ is $d$. Then $|S| \leq |G|/(1 - \frac{d}{\tau})$. If the equality is met, then $v_S - \frac{|S|}{|G|}v_G$ is an eigenvector of $\Gamma_G$ with eigenvalue $\tau$.

**Proof.** Set $v = |G|$, and $M = A - \tau I - (d - \tau)/vJ$, where $A$ is the adjacency matrix of $\Gamma_G$ and $J$ is the all-1 matrix. By definition of $\tau$ and by construction, $M$ is a positive semidefinite symmetric matrix. Hence, given a set $S \subseteq G$ of size $s$, it is true that

$$0 \leq v_S^T M v_S = v_S^T Av_S - \tau v_S^T v_S - \frac{d - \tau}{v} v_S^T J v_S = v_S^T Av_S - \tau s - \frac{d - \tau}{v} s^2.$$ 

If $S$ is an independent set of $\Gamma_G$, then $v_S^T Av_S = 0$ and the previous inequality yields the first part of the lemma.

Now, suppose that equality holds. Then $v_S^T Av_S = 0$. Since $M$ is positive semidefinite, we obtain $M v_S = 0$. Therefore

$$A \left( v_S - \frac{s}{v} v_G \right) = \tau \left( v_S - \frac{s}{v} v_G \right),$$

and the proof is completed.

### 3. PGL(2, $q$)

We recall that the character table of PGL(2, $q$) was computed by Jordan and Schur in 1907 and can be found in many textbooks, see [11]. Also, the character tables of PGL(3, $q$) and PGL(4, $q$) were found by R. Steinberg and the character table of PGL($n$, $q$) was finally determined by J. Green in 1955 in the celebrated paper [9].

We give the character table of PGL(2, $q$), first for $q$ even and second for $q$ odd. We note that by abuse of terminology we often refer to the elements of PGL(2, $q$) as matrices.

We briefly explain the notation used, but refer the reader to [11] for full details. We start by describing the conjugacy classes. The elements of PGL(2, $q$) can be collected in four sets: the set consisting only of the identity element, the set consisting of the non-scalar matrices with only one eigenvalue, the set consisting of the matrices with two distinct eigenvalues in $\mathbb{F}_q$ and the set of matrices with no eigenvalues in $\mathbb{F}_q$. The non-scalar matrices with only one eigenvalue form a conjugacy class of size $q^2 - 1$ and are represented by a unipotent matrix $u$. Every matrix with two distinct eigenvalues in $\mathbb{F}_q$ is conjugate to a
diagonal matrix $d_x$, with $x$ and $1$ along the main diagonal. Now, $d_x$ and $d_y$ are conjugate if and only if $y = x$ or $y = x^{-1}$. So, the label $x$ in the character table of $PGL(2, q)$ for $d_x$ represents an element of $\mathbb{F}_q \setminus \{0, 1\}$ up to inversion. Now, for $q$ odd, let $i \in \mathbb{F}_q^{*2}$ be an element of order 2 in $\mathbb{F}_q^{*2}/\mathbb{F}_q^{*}$. The matrices with no eigenvalues in $\mathbb{F}_q$ are conjugate in $PGL(2, q^2)$ to a diagonal matrix, with $r$ and $r^q$ along the main diagonal (for $r \in \mathbb{F}_q \setminus \mathbb{F}_q^*$). Each of these matrices is conjugate in $PGL(2, q)$ to the matrix $v_r$ depicted below. Also, $v_x$ and $v_y$ are conjugate if and only if $y \mathbb{F}_q^* = x \mathbb{F}_q^*$ or $y \mathbb{F}_q^* = x^{-1} \mathbb{F}_q^*$. So, the label $r$ in the character table of $PGL(2, q)$ for $v_r$ represents an element of $\mathbb{F}_q^{*2}/\mathbb{F}_q^{*}$ up to inversion. So,

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, d_x = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}, v_r = \begin{bmatrix} 0 & 1 \\ -r^1+q & r^1+q \end{bmatrix}.$$

We note here that in the permutation group $PGL(2, q)$ acting on the projective line $\mathbb{P}_q$, the permutations in the conjugacy class represented by $u$ fix exactly one point, the permutations in the conjugacy classes represented by $d_x$ fix exactly 2 points while the permutations in the conjugacy classes represented by $v_r$ fix no point.

Next we describe the characters of $PGL(2, q)$ by describing the maps $\varepsilon$, $\gamma$ and $\eta$. The map $\varepsilon$ is defined by $\varepsilon(x) = 1$ if $d_x \in PSL(2, q)$ and $\varepsilon(x) = -1$ otherwise. Similarly, $\varepsilon(r) = 1$ if $v_r \in PSL(2, q)$ and $\varepsilon(r) = -1$ otherwise. So,

| Name | $1$ | $u$ | $d_x$ | $v_r$ |
|------|-----|-----|-------|-------|
| Nr.  | $1$ | $1$ | $\frac{q^2-1}{2}$ | $\frac{q}{2}$ |
| Size | $1$ | $q^2-1$ | $q(q+1)$ | $q(q-1)$ |

Table 1: Character table of $PGL(2, q)$, for $q$ even.

| Name | $1$ | $u$ | $d_x$ | $d_{-1}$ | $v_r$ |
|------|-----|-----|-------|----------|-------|
| Nr.  | $1$ | $1$ | $\frac{q^2-1}{2}$ | $1$ | $\frac{q(q+1)}{2}$ | $q(q-1)$ |
| Size | $1$ | $q^2-1$ | $q(q+1)$ | $\frac{q(q+1)}{2}$ | $q(q-1)$ | $\frac{q(q-1)}{2}$ |

Table 2: Character table of $PGL(2, q)$, for $q$ odd.
\( \lambda_{-1} : \text{PGL}(2, q) \to \{\pm 1\} \) is the non-principal linear character. The letter \( \gamma \) represents a group homomorphism \( \gamma : \mathbb{F}_q^* \to \mathbb{C} \) of order greater than 2. Also, \( \nu_{\gamma_1} = \nu_{\gamma_2} \) if and only if \( \gamma_2 = \gamma_1 \) or \( \gamma_2 = \gamma_1^{-1} \). So, the label \( \gamma \) runs through the homomorphisms \( \gamma : \mathbb{F}_q^* \to \mathbb{C} \) of order greater than 2 up to inversion. The letter \( \beta \) stands for a group homomorphism \( \beta : \mathbb{F}_q^* / \mathbb{F}_q^* \to \mathbb{C} \) of order greater than 2. Also, \( \eta_{\beta_1} = \eta_{\beta_2} \) if and only if \( \beta_2 = \beta_1 \) or \( \beta_2 = \beta_1^{-1} \). So, the label \( \beta \) runs through the homomorphisms \( \beta : \mathbb{F}_q^* / \mathbb{F}_q^* \to \mathbb{C} \) of order greater than 2 up to inversion.

Note that given a non-principal linear character \( \psi : C \to \mathbb{C} \) of a cyclic group \( C \), we get \( \sum_{c \in C} \psi(c) = 0 \). In particular, given \( \gamma \) and \( \beta \) as in Table 1 and Table 2 we obtain \( \sum_{x \in \mathbb{F}_q^*} \gamma(x) = 0 \) and \( \sum_{r \in \mathbb{F}_q^* / \mathbb{F}_q^*} \beta(r) = 0 \).

In the following we simply denote by \( G_q \) the permutation group \( \text{PGL}(2, q) \) acting on the projective line \( \mathbb{P}_q \), and \( D \) the set of derangements of \( G_q \).

Using Table 1 (for \( q \) even), Table 2 (for \( q \) odd) and the fact that the elements in the conjugacy classes labelled by \( \nu_r \) are the derangements of \( G_q \), it is possible to compute \( \chi(D) / \chi(1) \), for all \( \chi \in \text{Irr}(G_q) \). For example, if \( \chi = \lambda_1 \) then we easily see that

\[
\frac{\lambda_1(D)}{\lambda_1(1)} = \frac{q(q - 1)(q - 1)}{2} + \frac{q(q - 1)}{2} = \frac{q^2(q - 1)}{2},
\]

which is equal to \(|D|\), is an eigenvalue of \( \Gamma_{G_q} \). The only cases where this is not trivial is when \( \chi = \eta_\beta \) and \( \chi = \lambda_{-1} \). If \( \chi = \eta_\beta \), then we have

\[
\frac{\chi(D)}{\chi(1)} = \frac{q(q - 1)}{q - 1} \left( \sum_{r \in \mathbb{F}_q^* / \mathbb{F}_q^*} \beta(r) \right) = -q \left( \sum_{r \in \mathbb{F}_q^* / \mathbb{F}_q^*} \beta(r) \right) = q
\]

since \( \mathbb{F}_q^* / \mathbb{F}_q^* \) is a cyclic group. By direct calculation or using the description of the conjugacy classes of \( \text{PSL}(2, q) \) given in [11], we see that \( \text{PSL}(2, q) \) contains \( q(q - 1)^2 / 4 \) derangements. As \(|D| = q^2(q - 1)/2\), we get that \( \text{PGL}(2, q) \setminus \text{PSL}(2, q) \) contains \( q(q^2 - 1)/4 \) derangements. So, if \( \chi = \lambda_{-1} \), we have

\[
\frac{\lambda_{-1}(D)}{\lambda_{-1}(1)} = \frac{q(q - 1)^2}{4} - \frac{q(q^2 - 1)}{4} = \frac{q(q - 1)}{2}.
\]

In particular, we obtain the following table:

| Character | \( \lambda_1 \) | \( \lambda_{-1} \) | \( \psi_1 \) | \( \psi_{-1} \) | \( \eta_\beta \) | \( \nu_r \) |
|-----------|-----------------|-----------------|-------------|-------------|-------------|-------------|
| Eigenvalue | \( q^2(q - 1) \) | \( -q(q - 1) \) | \( -q^{-1}(q - 1) \) | \( q^{-1} \) | \( q \) | \( 0 \) |
| Dimension | 1 | 1 | \( q^2 \) | \( q^2 \) | \( (q - 1)(q + 1)^2(q - 3) \) | \( (q + 1)^2(q - 3) \) |

Table 3: Eigenvalues of \( \Gamma_{G_q} \)

In summary, the valency of the graph \( \Gamma_{G_q} \) is \( q^2(q - 1)/2 \) and the minimum eigenvalue \( \tau \) is \( -q(q - 1)/2 \). If \( q \) is even, then \( \psi_1 \) is the only character \( \chi \) such that \( \tau = \chi(D)/\chi(1) \). If \( q \) is odd, then \( \lambda_{-1} \) and \( \psi_1 \) are the only characters \( \chi \) such that \( \tau = \chi(D)/\chi(1) \).
Lemma 4. An independent set of maximal size in $\Gamma_{G_q}$ has size $q(q - 1)$.

Proof. The coset of the stabilizer of a point of $G_q$ is an independent set in $\Gamma_{G_q}$ of size $|G_q|/(q + 1) = q(q - 1)$. From the eigenvalues of $\Gamma_{G_q}$ and Lemma 3 we see that such an independent set is an independent set of maximal size.

Similar to the case of the symmetric group (and also the standard Erdős-Ko-Rado theorem for sets), finding the bound in Theorem 1 is not difficult, but it is in the characterization of the sets that meet this bound that the difficulty lies. Indeed, it is not difficult to also establish a similar bound on the size of the independent sets in the derangement graph for the group $\text{PSL}(2, q)$.

Lemma 5. An independent set of maximal size in $\Gamma_{\text{PSL}(2, q)}$ has size $q(q - 1)/2$.

Proof. This can be proved using Lemma 3 with the information on the character table of $\text{PSL}(2, q)$ in [11] and the fact that a point-stabilizer in $\text{PSL}(2, q)$ has size $q(q - 1)/2$.

The next lemma will be used in Lemma 7 to limit the search of independent sets of maximal size in $\Gamma_{G_q}$.

Lemma 6. Assume $q$ odd. If $S$ is an independent set of maximal size of $\Gamma_{G_q}$, then $\lambda_{-1}(S) = 0$.

Proof. By Lemma 4 we have $|S| = q(q - 1)$. Consider the two sets $S_+ = S \cap \text{PSL}(2, q)$ and $S_- = S \setminus S_+$. Clearly $S_+$ is an independent set in $\Gamma_{\text{PSL}(2, q)}$ and, for $g \in G_q \setminus \text{PSL}(2, q)$, the set $gS_-$ is also an independent set in $\Gamma_{\text{PSL}(2, q)}$. Thus we obtain that $|S_+| = |S_-| = q(q - 1)/2$ and in particular $\lambda_{-1}(S) = |S_+| - |S_-| = 0$.

4. Auxiliary lemmas

Consider the $\{0, 1\}$-matrix $A$, where the rows are indexed by the elements of $G_q$, the columns are indexed by the ordered pairs of points of $\mathbb{P}_q$ and $A_{(p_1, p_2)} = 1$ if and only if $p_1^2 = p_2$. In particular, $A$ has $|G_q| = q(q^2 - 1)$ rows and $|\mathbb{P}_q|^2 = (q + 1)^2$ columns.

The entry $(A^T A)_{(p_1, q_1), (p_2, q_2)}$ equals the number of permutations of $G_q$ mapping $p_1$ into $q_1$ and $p_2$ into $q_2$. Since $G_q$ is 2-transitive, we get by a simple counting argument that

$$(A^T A)_{(p_1, q_1), (p_2, q_2)} = \begin{cases} 
q(q - 1) & \text{if } p_1 = p_2 \text{ and } q_1 = q_2, \\
q - 1 & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2, \\
0 & \text{otherwise.}
\end{cases}$$

This shows that with the proper ordering of the columns of $A$,

$$A^T A = q(q - 1)I_{(q + 1)^2} + (q - 1)(J_{q+1} - I_{q+1}) \otimes (J_{q+1} - I_{q+1}),$$

(in here $I_{q+1}, J_{q+1}$ denote the identity matrix and the all-1 matrix of size $q + 1$, respectively). The matrix $J_{q+1}$ has eigenvalue 0 (with multiplicity $q$) and $q + 1$
(with multiplicity 1). Hence \((J_{q+1} - I_{q+1}) \otimes (J_{q+1} - I_{q+1})\) has eigenvalue 1 (with multiplicity \(q^2\)), \(-q\) (with multiplicity \(2q\)), and \(q^2\) (with multiplicity 1). So, \(A^T A\) is diagonalizable with eigenvalues \(q(q - 1) + (q - 1)q^2 = |G_q|\) (with multiplicity 1), \(q(q - 1) + (q - 1) = q^2 - 1\) (with multiplicity \(q^2\)) and \(q(q - 1) - (q - 1)q = 0\) (with multiplicity \(2q\)). This shows that the kernel of \(A^T A\) has dimension \(2q\).

We now determine the kernel of \(A\) with multiplicity 2, labelled by the identity element of \(G_q\) in \(A\), we see that the all-1 vector is also in the column space of \(A\). Also, by the discussion.

Proof. The claims on the rank and on the kernel follow from the previous discussion.

Note that the columns of \(A\) are the characteristic vectors of cosets of stabilizers of points of \(G_q\). In particular, these columns are characteristic vectors of independent sets of size \(q(q - 1)\) in \(\Gamma_{G_q}\). By taking the sum of all the columns in \(A\), we see that the all-1 vector is also in the column space of \(A\).

Let \(S\) be any independent set of \(\Gamma_{G_q}\) of size \(q(q - 1)\). By Lemma 3 and 4 the characteristic vector of \(S\) is in the direct sum of the \(q^2(q - 1)/2\)-eigenspace and the \(-q(q - 1)/2\)-eigenspace.

If \(q\) is even, from the multiplicities of the eigenvalues in Table 3 it is clear that the vector space spanned by the characteristic vectors of independent sets of size \(q(q - 1)\) has dimension at most \(q^2 + 1\). So the lemma follows.

If \(q\) is odd then Lemma 6 yields that \(v_S\) is orthogonal to the eigenspace arising from the character \(\lambda_{-1}\). So \(v_S\) must lie in the direct sum of the eigenspaces arising from \(\lambda_1\) and \(\psi_1\). Again, the multiplicities of the eigenvalues in Table 3 show that the vector space spanned by the characteristic vectors of independent sets of size \(q(q - 1)\) has dimension at most \(q^2 + 1\) and the lemma follows.

Now we fix a particular ordering of the rows of \(A\) so that the first row is labelled by the identity element of \(G_q\), then we label the next \(q^2(q - 1)/2\) rows by the derangements of \(G_q\) and the final \((q^2 - 2)(q + 1)/2\) rows are labelled by the remaining permutations. Similarly, we fix a particular ordering of the

\[
\begin{align*}
\text{Proof.} & \quad \text{The claims on the rank and on the kernel follow from the previous discussion.} \\
& \quad \text{Note that the columns of } A \text{ are the characteristic vectors of cosets of stabilizers of points of } G_q. \text{ In particular, these columns are characteristic vectors of independent sets of size } q(q - 1) \text{ in } \Gamma_{G_q}. \text{ By taking the sum of all the columns in } A, \text{ we see that the all-1 vector is also in the column space of } A. \\
& \quad \text{Let } S \text{ be any independent set of } \Gamma_{G_q} \text{ of size } q(q - 1). \text{ By Lemma 3 and 4 the characteristic vector of } S \text{ is in the direct sum of the } q^2(q - 1)/2\text{-eigenspace and the } -q(q - 1)/2\text{-eigenspace.} \\
& \quad \text{If } q \text{ is even, from the multiplicities of the eigenvalues in Table 3 it is clear that the vector space spanned by the characteristic vectors of independent sets of size } q(q - 1) \text{ has dimension at most } q^2 + 1. \text{ So the lemma follows.} \\
& \quad \text{If } q \text{ is odd then Lemma 6 yields that } v_S \text{ is orthogonal to the eigenspace arising from the character } \lambda_{-1}. \text{ So } v_S \text{ must lie in the direct sum of the eigenspaces arising from } \lambda_1 \text{ and } \psi_1. \text{ Again, the multiplicities of the eigenvalues in Table 3 show that the vector space spanned by the characteristic vectors of independent sets of size } q(q - 1) \text{ has dimension at most } q^2 + 1 \text{ and the lemma follows.} \\
\end{align*}
\]
columns of $A$ so that the first $q + 1$ columns are labelled by the ordered pairs of the form $(p, p)$ and then the last $q(q + 1)$ columns are labelled by the ordered pairs of the form $(p_1, p_2)$, with $p_1 \neq p_2$. With this ordering, we get that the matrix $A$ is a block matrix. Namely,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & M \\ B & C \end{pmatrix}.$$ 

In particular, the rows of the submatrix $M$ are labelled by the derangements of $G_q$ and the columns of $M$ are labelled by the ordered pairs of distinct elements of $\mathbb{F}_q$.

Now, set $N = M^T M$. In particular, $N$ is a square $(q + 1)q$ matrix whose rows and columns are indexed by the ordered pairs of distinct points.

From now on, we identify the elements of $\mathbb{F}_q$ with the elements of $\mathbb{F}_q \cup \{\infty\}$. Namely, the point $[1, a]$ corresponds to the element $a$ of $\mathbb{F}_q$ and the point $[0, 1]$ corresponds to the element $\infty$. Now, given four distinct points $\alpha, \beta, \gamma, \delta$, we recall that the cross-ratio, denoted by $\text{crr}(\alpha, \beta, \gamma, \delta)$, is defined by

$$\frac{\alpha - \gamma}{\alpha - \beta} \frac{\delta - \beta}{\delta - \gamma}.$$ 

We recall that the cross-ratio is $G_q$-invariant, i.e. if $g \in G_q$ and $\alpha, \beta, \gamma, \delta \in \mathbb{F}_q$, then $\text{crr}(\alpha^g, \beta^g, \gamma^g, \delta^g) = \text{crr}(\alpha, \beta, \gamma, \delta)$.

In the following proposition we prove that the entries of the matrix $N$ are determined by the cross-ratio.

**Proposition 8.** If $q$ is even, then

$$N_{(\alpha, \beta), (\gamma, \delta)} = \begin{cases} q(q - 1)/2 & \text{if } \alpha = \gamma \text{ and } \beta = \delta, \\ 0 & \text{if } \alpha = \gamma \text{ and } \beta \neq \delta, \\ 0 & \text{if } \alpha \neq \gamma \text{ and } \beta = \delta, \\ 0 & \text{if } \alpha = \delta \text{ and } \beta = \gamma, \\ q/2 & \text{otherwise.} \end{cases}$$

If $q$ is odd, then

$$N_{(\alpha, \beta), (\gamma, \delta)} = \begin{cases} q(q - 1)/2 & \text{if } \alpha = \gamma \text{ and } \beta = \delta, \\ 0 & \text{if } \alpha = \gamma \text{ and } \beta \neq \delta, \\ 0 & \text{if } \alpha \neq \gamma \text{ and } \beta = \delta, \\ (q - 1)/2 & \text{if } \text{crr}(\alpha, \delta, \gamma, \beta) \text{ is a square in } \mathbb{F}_q, \\ (q + 1)/2 & \text{if } \text{crr}(\alpha, \delta, \gamma, \beta) \text{ is not a square in } \mathbb{F}_q. \end{cases}$$

**Proof.** Note that $N_{(\alpha, \beta), (\gamma, \delta)}$ is the number of derangements mapping $\alpha$ to $\beta$ and $\gamma$ to $\delta$. Write $n$ for $N_{(\alpha, \beta), (\gamma, \delta)}$. If $\alpha = \gamma$ and $\beta = \delta$, then $n$ is the number of derangements mapping $\alpha$ to $\beta$. Since $G_q$ is transitive of degree $q + 1$ and since $G_q$ contains $q^2(q - 1)/2$ derangements, we have $n = q^2(q - 1)/2q = q(q - 1)/2$.

If $\alpha = \gamma$ and $\beta \neq \delta$ or if $\alpha \neq \gamma$ and $\beta = \delta$, then clearly $n = 0$. 
From now on we can assume that $\alpha \neq \gamma$ and $\beta \neq \delta$. Assume $\alpha = \delta$ and $\beta = \gamma$. Since $G_q$ is 2-transitive, without loss of generality, we can assume that $\alpha = 0$ and $\beta = \infty$. In this case, the elements $g$ such that $0^g = \infty$ and $\infty^g = 0$ are the matrices of the form

$$g = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix}$$

with $\lambda \in \mathbb{F}_q^*$. Further, $q$ is a derangement if and only if $g$ has no eigenvalue in $\mathbb{F}_q$, i.e. the characteristic polynomial $p_\lambda(t) = t^2 - \lambda$ of $g$ is irreducible over $\mathbb{F}_q$. If $q$ is even, then $p_\lambda(t)$ is reducible for every value of $\lambda$, and so $n = 0$. If $q$ is odd, then $\mathbb{F}_q^*$ has $(q-1)/2$ non-square elements. Hence there exist $(q-1)/2$ values of $\lambda$ such that $p_\lambda(t)$ is irreducible over $\mathbb{F}_q$, and so $n = (q-1)/2$. Note that $\text{crr}(\alpha, \delta, \gamma, \beta) = \text{crr}(0, 0, \infty, \infty) = 1$ is a square in $\mathbb{F}_q$.

From now on we can assume that $|\{\alpha, \beta, \gamma, \delta\}| \geq 3$.

As $N$ is symmetric, up to interchanging the pairs $(\alpha, \beta)$, $(\gamma, \delta)$, we may assume that $\beta \neq \gamma$. Since $G_q$ is 3-transitive, without loss of generality, we may assume that $\alpha = 0$, $\beta = 1$, $\gamma = \infty$ and $\delta = d$, for some $d \in \mathbb{F}_q \setminus \{1\}$. The elements $g$ such that $0^g = 1$ and $\infty^g = d$ are the matrices of the form

$$g = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda d \end{bmatrix}$$

with $\lambda \in \mathbb{F}_q^*$. The permutation $g$ is a derangement if and only if $g$ has no eigenvalue in $\mathbb{F}_q$, i.e. the characteristic polynomial $p_\lambda(t) = t^2 - (1 + \lambda d)t + \lambda d - \lambda$ of $g$ is irreducible over $\mathbb{F}_q$. Now, we determine the number of values of $\lambda$ such that $p_\lambda(t)$ is irreducible. Assume that $p_\lambda(t)$ is reducible over $\mathbb{F}_q$ with roots $a, b \in \mathbb{F}_q$. So, $p_\lambda(t) = (t - a)(t - b).$ This yields $a + b = 1 + \lambda d$ and $ab = \lambda d - \lambda$. We get $b = (1 + \lambda - a)/(1 - a)$ (note that $a$ cannot be 1, because otherwise $\lambda d - \lambda = b = \lambda d$, which yields $\lambda = 0$). From this we obtain $\lambda = (a - a^2)/(d - 1 - ad)$ (note that $d - 1 - ad \neq 0$, because otherwise $a \in (0,1)$, which yields $\lambda = 0$). Consider the function $\varphi : \mathbb{F}_q \setminus X \to \mathbb{F}_q$, where $\varphi(a) = (a - a^2)/(d - 1 - ad)$ and $X = \{a \mid d - 1 - ad = 0\}$. Note that $X = \{1 - d^{-1}\}$ if $d \neq 0$, and $X = 0$ if $d = 0$.

We have proved so far that $p_\lambda(t)$ is reducible if and only if $\lambda$ lies in the image of $\varphi$. We now compute the size of $\text{Im} \varphi$. It is easy to check that $\varphi(a_1) = \varphi(a_2)$ if and only if $a_2 = a_1$ or $a_2 = (1 - a_1)(d - 1)/(d - 1 - a_1 d)$. This shows that the fiber of $\varphi(a_1)$ contains two points if $a_1 \neq (1 - a_1)(d - 1)/(d - 1 - a_1 d)$ and only one point if $a_1 = (1 - a_1)(d - 1)/(d - 1 - a_1 d)$. Note that $a_1 = (1 - a_1)(d - 1)/(d - 1 - a_1 d)$ if and only if $a_1^2 d - 2(d - 1) a_1 + (d - 1) = 0$.

If $q$ is even and $d \neq 0$, then this happens if $a_1^2 = d^{-1} - 1$ (i.e. for a unique value of $a_1$). Thence the image of $\varphi$ contains $(q - 1 - 1)/2 + 1 = q/2$ elements, and $n = |\mathbb{F}_q \setminus \text{Im} \varphi| = q/2$. Similarly, if $q$ is even and $d = 0$, the image of $\varphi$ contains $q/2$ elements, and $n = |\mathbb{F}_q \setminus \text{Im} \varphi| = q/2$.

If $q$ is odd, then $a_1^2 d - 2(d - 1) a_1 + (d - 1) = 0$ if and only if $a_1 = (d - 1 - d)/2$ is a square (in this case there are two distinct solutions for $a_1$). So, if $1 - d$ is a square, then the image of $\varphi$ contains $(q - 1 - 1)/2 + 2 = (q + 1)/2$ elements and so $n = |\mathbb{F}_q \setminus \text{Im} \varphi| = (q - 1)/2$. But, if $1 - d$ is not a square, then
the image of $\varphi$ contains $(q - 1)/2$ elements and so $n = |\mathbb{F}_q \setminus \text{Im } \varphi| = (q + 1)/2$. Finally, we note that $\text{crr}(0, d, \infty, 1) = 1 - d$.

In the next proposition we use the character table of $G_q$ to find the rank of $M$.

**Proposition 9.** The matrix $M$ has rank $q(q - 1)$.

**Proof.** Let $\Omega$ be the set of ordered pairs of distinct elements of $\mathbb{F}_q$ and $V$ be the vector space with basis $\{e_\omega\}_{\omega \in \Omega}$. Clearly, $V$ is a $G_q$-module. Namely, $V$ is the permutation module of the action of $G_q$ on $\Omega$. Let $\pi$ be the character afforded by $V$, so

$$\pi(g) = |\{\omega \in \Omega \mid \omega^g = \omega\}|.$$

We have $\pi(1_{G_q}) = q(q + 1)$, $\pi(1_{G_q}) = 2$ for every element $g$ conjugate to $d_x$ (for some $x$) and $\pi(1_{G_q}) = 0$ otherwise. As $\pi = \sum_{\chi \in \text{Irr}(G_q)} \langle \chi, \pi \rangle \chi$, by direct calculation of $\langle \chi, \pi \rangle$ with the Tables 1 and 2, we get that

$$\pi = \lambda_1 + 2\psi_1 + \sum_\beta \eta_\beta + \sum_\gamma \nu_\gamma, \quad \text{for } q \text{ even},$$

$$\pi = \lambda_1 + 2\psi_1 + \psi_{-1} + \sum_\beta \eta_\beta + \sum_\gamma \nu_\gamma, \quad \text{for } q \text{ odd}.$$

Let $C \subseteq \text{Irr}(G_q)$ be the set of constituents of $\pi$. Then $V = \bigoplus_{\chi \in C} V_\chi$, where $V_\chi$ is an irreducible $G_q$-submodule of $V$, unless $\chi = \psi_1$, and $V_{\psi_1}$ is the sum of two isomorphic irreducible $G_q$-submodules of $V$ of dimension $q$. Clearly, $V_{\psi_1} \cong V_1 \oplus V_2$ (see Lemma 7).

Again we use the matrix $N = MTM$ and in order to prove that $M$ has rank $q(q - 1)$ it suffices to prove that $N$ has rank $q(q - 1)$. By Lemma 7 we get that $V_{\psi_1}$ is contained in the kernel of $N$. Also, as $N_{\omega_1, \omega_2} = N_{\omega_2, \omega_1}$ for every $g \in G_q$, we obtain that every eigenspace of $N$ is a $G_q$-submodule of $V$. Therefore, since for $\chi \neq \psi_1$ the module $V_\chi$ is irreducible, we get that $V_\chi$ is an eigenspace of $N$. Thus, to conclude the proof it suffices to show that the eigenvalue $s_\chi$ of the eigenspace $V_\chi$ is not zero, for all $\chi \neq \psi_1$.

By Wedderburn Theorem 10, we get that $\mathbb{C}G_q = \bigoplus_{\chi \in \text{Irr}(G_q)} I_\chi$, where $I_\chi$ are minimal two-sided ideals of the semisimple algebra $\mathbb{C}G_q$. Also, each $I_\chi$ is generated (as an ideal) by the idempotent $E_\chi = \frac{1}{|G_q|} \sum_{g \in G_q} \chi(g^{-1})g$. Set $v_\chi = \sum_{g \in G_q} \chi(g^{-1})e_{(0, \infty)}$. As $V_\chi = VI_\chi$ and $v_\chi = \frac{1}{|G_q|} e_{(0, \infty)} E_\chi \in V_\chi$, we obtain that $v_\chi$ is an eigenvector of $N$ with eigenvalue $s_\chi$. Note that, given $\chi \in C$, we have

\begin{align*}
(\dagger) \quad (NV_\chi)_{(0, \infty)} &= \sum_{(a, b) \in \Omega} N_{(0, \infty), (a, b)}(v_\chi)_{(a, b)} = \sum_{(a, b) \in \Omega} \sum_{g \in G_q} \chi(g^{-1})N_{(0, \infty), (a, b)} \\
(\ddagger) \quad (v_\chi)_{(0, \infty)} &= \sum_{g \in G_q} \chi(g^{-1}) = (q - 1)(\text{Res}_T(\chi), 1) = (q - 1)(\chi, \pi) = q - 1,
\end{align*}
where $T$ is the stabilizer in $G_q$ of $0, \infty$ and $\text{Res}_T(\chi)$ is the restriction of $\chi$ to $T$ (note that in the fourth equality in (‡) we are using Frobenius Reciprocity).

In the rest of the proof, we do not determine (for $q$ odd) the eigenvalue $s_\chi$ of $v_\chi$, but we simply prove that $s_\chi > 0$, for $\chi \neq \psi_1$. If $\chi = \lambda_1$, then by (‡) the vector $v_\chi$ is $(q - 1)$ times the all-1 vector. By Proposition \[ N \] $N$ is a stochastic matrix with row sum $q(q^2 - 1)/2$, so $s_{\lambda_1} = q(q^2 - 1)/2$. Now, for the remaining characters in $C$ we distinguish two cases depending on whether $q$ is even or $q$ is odd.

Assume $q$ even. Let $\chi \in C$, with $\chi \neq \lambda_1, \psi_1$. Now, as \( \sum_{g \in G_q} \chi(g - 1) = 0 \), subtracting $q/2 \sum_{g \in G_q} \chi(g - 1)$ from (†) and using Proposition \[8\] we get

\[
(Nv_\chi)(0, \infty) = \frac{q^2}{2} \sum_{g \in T} \chi(g^{-1}) - \frac{q}{2} \left( \sum_{g \text{ s.t. } 0^g = 0} \chi(g^{-1}) + \sum_{g \text{ s.t. } \infty^g = \infty} \chi(g^{-1}) + \sum_{g \text{ s.t. } \infty^g = 0} \chi(g^{-1}) \right)
\]

On the right-hand side of Equation (†) the first, the second and the third summands are

\[
\frac{q^2}{2}(q - 1)\langle \text{Res}_T(\chi), 1 \rangle = \frac{q^2(q - 1)}{2} \langle \chi, \pi \rangle = \frac{q^2(q - 1)}{2},
\]
\[
\frac{q}{2}q(q - 1)\langle \text{Res}(G_q)_0(\chi), 1 \rangle = \frac{q^2(q - 1)}{2} \langle \chi, \lambda_1 + \psi_1 \rangle = 0,
\]
\[
\frac{q}{2}q(q - 1)\langle \text{Res}(G_q)_{\infty}(\chi), 1 \rangle = \frac{q^2(q - 1)}{2} \langle \chi, \lambda_1 + \psi_1 \rangle = 0.
\]

Also, if $g$ is a permutation such that $0^g = \infty$ and $\infty^g = 0$, then $g$ has order 2. Therefore $g$ is conjugate to $u$ and so the fourth summand in Equation (†) is $\frac{q(q - 1)}{2} \chi(u)$. Summing up, we obtain

\[
(Nv_\chi)(0, \infty) = \begin{cases} 
\frac{(q^2-1)q}{2} & \text{if } \chi = \eta_\beta, \\
\frac{(q-1)^2q}{2} & \text{if } \chi = \nu_\gamma.
\end{cases}
\]

Hence, (‡) yields $s_{\nu_\gamma} = q(q - 1)/2 > 0$ (for every $\gamma$) and $s_{\eta_\beta} = q(q + 1)/2 > 0$ (for every $\beta$).

We point out that the matrix $N$ has only 4 eigenvalues and actually $N$ is the matrix of an association scheme of rank 4, see [17] for more details.

Assume $q$ odd. Let $\chi \in C$, with $\chi \neq \lambda_1, \psi_1$. Now, as $\sum_{g \in G_q} \chi(g^{-1}) = 0$,
subracting $(q - 1)/2 \sum_{g \in G_q} \chi(g^{-1})$ from (1) and using Proposition 3, we get

$$(Nv_{\chi})(0, \infty) = \frac{q^2 - 1}{2} \sum_{g \in T} \chi(g^{-1}) - \frac{q - 1}{2} \left( \sum_{g \text{ s.t. } \|g\| = 0} \chi(g^{-1}) + \sum_{g \text{ s.t. } \|g\| = \infty} \chi(g^{-1}) \right)$$

\[+ \sum_{(a, b) \in \Omega} \chi(g^{-1}) \text{if } (a, b) \in \Omega \text{ not square} \text{ and } 0^g = a, \infty^g = b \]

Arguing as in the case of $q$ even, we get that the first three summands in Equation 2 are $(q^2 - 1)(q - 1)/2, 0$ and 0. Now, consider the subset $\Delta = \{(a, b) \in \Omega \mid \text{crr}(0, b, a, \infty) \text{ not a square}\}$ of $G_q$. Since there are $(q - 1)^2/2$ elements in $\Delta$ we have that $|\Delta| = (q - 1)^2/2$.

If $\chi = \psi_{a, t}$, then $|\chi(g^{-1})| \in \{0, 1\}$, for every $g \in S$. From Equation 2 and the previous paragraph, we have $(Nv_{\chi})(0, \infty) \geq (q^2 - 1)(q - 1)/2 - |\Delta| > 0$, so $s_{\psi_{a, t}} > 0$.

If $(a, b)$ is in $\Delta$ and $g_{ab}$ is in $G_q$ such that $(0, \infty)^{g_{ab}} = (a, b)$, then the set of elements of $G_q$ mapping $(0, \infty)$ to $(a, b)$ is the coset $T_{g_{ab}}$. Since, crr$(0, b, a, \infty) = \text{crr}(0, b, a, \infty)^{-1}$ and $(a, b) \in \Delta$, we see that crr$(0, b, a, \infty)$ is not a square. So, by Proposition 3, we get that $T_{g_{ab}}$ contains exactly $(q + 1)/2$ elements conjugate to $v_r$ for some $r$. Therefore $T_{g_{ab}}$ contains at most $(q - 3)/2$ elements conjugate to $d_x$, for some $x$.

Assume $\chi = \nu_r$. Since $\chi(g) = 0$ if $g$ is conjugate to $v_r$ (for some $r$), $|\chi(g)| \leq 2$ if $g$ is conjugate to $d_x$ (for some $x$) and $|\Delta| = (q - 1)^2/2$, we obtain that the last summand on the right-hand side of Equation 2 is greater than or equal to

$$\frac{(q - 1)^2}{2} \cdot \frac{q - 3}{2} \cdot (-2) = -\frac{(q - 1)^2(q - 3)}{2} > -\frac{(q^2 - 1)(q - 1)}{2}.$$  

Hence $s_{\nu_r} > 0$.

Assume $\chi = \eta_{\beta}$. Now $|\chi(g)| \leq 2$ if $g$ is conjugate to $v_r$ (for some $r$), $\chi(g) = 0$ if $g$ is conjugate to $d_x$ (for some $x$) and $|\Delta| = (q - 1)^2/2$. Checking Table 2, we see that $-2 = \chi(v_r) = -\beta(r) - \beta(r^{-1})$ if and only if $rF_q^\times \in \text{Ker } \beta$. Since $\beta$ has order greater than 2, we get $|\text{Ker } \beta| < (q + 1)/2$. Hence the last summand on the right-hand side of Equation 2 is greater than

$$\frac{(q - 1)^2}{2} \cdot \frac{q + 1}{2} \cdot (-2) = -\frac{(q^2 - 1)(q - 1)}{2}.$$  

So, $s_{\eta_{\beta}} > 0$.

Now, we construct a submatrix $\overline{A}$ of $A$. In the submatrix $\overline{A}$, we keep all the rows of $A$ and we delete the columns indexed by the ordered pairs $(\infty, t), (t, \infty)$, for every $t \in F_q$. In particular, we get

$$\overline{A} = \begin{pmatrix} 1 & 0 \\ 0 & M \\ B & C \end{pmatrix},$$
where the matrix $\overline{M}$ and $\overline{C}$ are obtained by deleting the appropriate columns of $M$ and $C$. The matrix $\overline{A}$ has $q^2 + 1$ columns and $\overline{M}$ has $q(q - 1)$ columns.

**Proposition 10.** We have $\text{rank}(A) = \text{rank}(\overline{A})$ and $\overline{M}$ has full column rank.

**Proof.** We start by proving that the columns indexed by $(\infty, t), (t, \infty)$ (for $t \in \mathbb{F}_q$) of $A$ are a linear combination of the columns of $\overline{A}$. We denote by $a_{xy}$ the column of $A$ indexed by the ordered pair $(x, y)$, for $x, y \in \mathbb{P}_q$. Since $G_q$ is 2-transitive, it suffices to prove that $a_{0\infty}$ is a linear combination of the columns of $\overline{A}$. Set $v = \sum_{x \neq 0, \infty} \sum_{y \neq \infty} a_{xy}$ and $w = (q - 2) \sum_{x \neq \infty} a_{0x} + a_{\infty\infty}$. By construction, $v$ and $w$ are a linear combination of the columns of $\overline{A}$. Also, it is easy to check that

$$v_g = \begin{cases} 
q - 1 & \text{if } 0^g = \infty \text{ or } \infty^g = \infty, \\
q - 2 & \text{otherwise,}
\end{cases} \quad w_g = \begin{cases} 
0 & \text{if } 0^g = \infty, \\
q - 1 & \text{if } \infty^g = \infty, \\
q - 2 & \text{otherwise.}
\end{cases}$$

Hence $(q - 1)a_{0\infty} = v - w$ and we get that $a_{0\infty}$ is a linear combination of the columns of $\overline{A}$. Thence $\text{rank}(A) = \text{rank}(\overline{A})$.

Since $\overline{M}$ has $q(q - 1)$ columns, Proposition 9 shows that $\overline{M}$ has full column rank.

5. **Proof of Theorem 1**

At this point, we have all the tools to conclude the proof of Theorem 1. By Lemma 4, it remains to prove that if $S$ is an independent set of maximal size of $\Gamma_{G_q}$, then $S$ is the coset of the stabilizer of a point. Up to multiplication of $S$ by a suitable element of $G_q$, we may assume that the identity element of $G_q$ is in $S$. In particular, we have to prove that $S$ is the stabilizer of a point. By Lemma 7 and Proposition 10, we have that the characteristic vector $v_S$ of $S$ is a linear combination of the columns of $\overline{A}$. Hence

$$\begin{pmatrix} 1 & 0 \\
0 & \overline{M} \\
B & \overline{C} \end{pmatrix} \begin{pmatrix} v \\
w \end{pmatrix} = v_S,$$

for some vectors $v, w$. As the identity element of $G_q$ is in $S$ and by the ordering of the rows of $\overline{A}$, we get

$$v_S = \begin{pmatrix} 1 \\
0 \\
t \end{pmatrix}.$$  

So, $1^Tv = 1$, $\overline{M}w = 0$ and $Bv + \overline{C}w = t$. By Proposition 10 the matrix $\overline{M}$ has full column rank. Thence $w = 0$ and $Bv = t$.

Now, for every point $x$ of $\mathbb{P}_q$, there exists a permutation $g_x$ of $G_q$ fixing $x$ and acting fixed-point-freely on $\mathbb{P}_q \setminus \{x\}$ (indeed, $g_x$ can be chosen any non-identity unipotent matrix of $G_q$ fixing $x$). Order the rows of $B$ so that the first $q + 1$
rows are labelled by the permutations \( \{ g_x \}_x \). In particular, up to permuting the rows of \( B \), we get

\[
B = \begin{pmatrix}
I_{q+1} \\
B'
\end{pmatrix}
\quad \text{and} \quad
Bv = \begin{pmatrix}
v \\
B'v
\end{pmatrix}.
\]

Since \( Bv \) is equal to the \( \{0,1\} \)-vector \( t \), we obtain that \( v \) is a \( \{0,1\} \)-vector. But \( 1^Tv = 1 \) and so \( v \) must be the characteristic vector of a point \( p \) of \( \mathbb{P}_q \). This shows that \( v_\mathcal{S} \) is the stabilizer of the point \( p \) and the proof is complete.

6. Comments

Theorem 1 proves that in the derangement graph \( \Gamma_G \), where \( G \) is the group \( \text{PGL}(2,q) \), the independent sets of maximal size are the cosets of the stabilizer of a point. The same result holds if \( G \) is the symmetric group \( [3,15] \) or if \( G \) is the alternating group \( [14] \). It is interesting to ask for which other permutation groups does this result hold, and is there a way to characterize the permutation groups that have this property?

In Lemma 5, we saw that the cosets of the stabilizer of a point are independent sets of maximal size in the derangement graph of \( \text{PSL}(2,q) \). We further conjecture that, similar to the case for \( \text{PGL}(2,q) \), \( \text{Sym}(n) \) and \( \text{Alt}(n) \), all independent sets of maximal size in \( \Gamma_{\text{PSL}(2,q)} \) are cosets of the stabilizer of a point.

**Conjecture 1.** Every independent set \( S \) of the derangement graph of \( \text{PSL}(2,q) \) acting on the projective line \( \mathbb{P}_q \) has size at most \( q(q-1)/2 \). Equality is met if and only if \( S \) is the coset of the stabilizer of a point.

It seems likely that the methods used in this paper could be applied to \( \text{PSL}(2,q) \), since the character table of this group is well understood.

In our proof of Theorem 1, we used the fact that the group \( \text{PGL}(2,q) \) is 2-transitive (for example, in the proof of Proposition 8 and Proposition 10). It is a reasonable question to ask if this (being 2-transitive) could be a characterization of the groups that have the property that the independent sets of maximal size in the derangement graph are the cosets of the stabilizer of a point. It is not hard to show that this is not a characterization, and it is further interesting to see how this property can fail to hold for some 2-transitive groups.

If \( G \) is a 2-transitive group, then the permutation character of \( G \) is the sum of the trivial character and an irreducible character that we will call \( \psi \). The eigenvalue of the derangement graph \( \Gamma_G \) arising from \( \psi \) is \(-d_\psi(1)\), where \( d \) is the valency of \( \Gamma_G \). If this eigenvalue is indeed the least eigenvalue of \( \Gamma_G \), then by Lemma 3, we have that the size of an independent set is no bigger than the size of the stabilizer of a point in \( G \). Thus, if it were true that the eigenvalue arising from the character \( \psi \) is the least eigenvalue, then we would have the bound like in Theorem 1. But the characterization of the sets that meet this bound is another question entirely. In fact, there are examples of 2-transitive groups in which there are independent sets of maximal size that are not cosets of the stabilizer of a point.
For example, let $G_{n,q}$ be the 2-transitive group PGL$(n + 1, q)$ in its action on the projective space $\mathbb{P}_q^n$, with $n \geq 1$. Since $G_{n,q}$ contains a Singer cycle of length $(q^{n+1} - 1)/(q-1)$, the graph $\Gamma_{G_{n,q}}$ has a clique of size $|\mathbb{P}_q^n| = (q^{n+1} - 1)/(q-1)$. Thus, any independent set of $\Gamma_{G_{n,q}}$ has size at most $|G_{n,q}|/|\mathbb{P}_q^n|$. Naturally, the stabilizer of a point is an independent set for $\Gamma_{G_{n,q}}$ of size $|G_{n,q}|/|\mathbb{P}_q^n|$ and so it is an independent set of maximal size. But for $G_{n,q}$ with $n \geq 2$, the cosets of the stabilizer of a point are not the only independent sets of maximal size. Indeed, it is not hard to see that the stabilizer of a hyperplane of $\mathbb{P}_q^n$ in $G_{n,q}$ is also an independent set of maximal size for $\Gamma_{G_{n,q}}$. Moreover, if $n \geq 2$, then the stabilizer of a point and of a hyperplane are not conjugate subgroups of $G_{n,q}$. Therefore, for $n \geq 2$, the graph $\Gamma_{G_{n,q}}$ contains at least $2(|G_{n,q}|/|\mathbb{P}_q^n|)^2$ independent sets of maximal size. We make the following conjecture.

**Conjecture 2.** Any independent set of maximal size in the derangement graph of PGL$(n + 1, q)$ acting on the projective space $\mathbb{P}_q^n$ is either the coset of the stabilizer of a point or the coset of the stabilizer of a hyperplane.

It is not clear that the method in this paper could be used to prove this conjecture. In particular, even if the character table of PGL$(n + 1, q)$ is known [9], it would still be challenging to obtain the minimum eigenvalue of $\Gamma_{G_{n,q}}$ as in Lemma 4.

Further, there exist 2-transitive groups of degree $n$ where the number of independent sets of maximal size is $n^{n-1}$. Clearly, this means that there are many independent sets of maximal size which are not cosets of the stabilizer of a point. For example, let $n$ be a power of a prime and $\mathbb{F}_n$ the field with $n$ elements. The affine general linear group $G$ on $\mathbb{F}_n$ (i.e. the group generated by the permutations of $\mathbb{F}_n$ of the form $f_{a,b} : \xi \mapsto a\xi + b$, for $a,b \in \mathbb{F}_n$ and $a \neq 0$) is a 2-transitive group. Since $G$ is a Frobenius group with kernel of size $n$ and complement of size $n-1$, we obtain that $\Gamma_G$ is the disjoint union of $n-1$ complete graphs $K_n$. In particular, we get that $\Gamma_G$ has $n^{n-1}$ independent sets of maximal size.

From the above comments it is clear that a result similar to Theorem 1 does not hold for all 2-transitive groups, but perhaps we do better to consider 3-transitive groups. In particular, some information on the 3-transitive groups is listed in Table 4 and Table 5, see [2, page 195, 197]. Table 4 lists all possible socles of an almost simple 3-transitive group and Table 5 gives all possible point stabilizers of an affine 3-transitive group.

| Group | $M_{11}$ | $M_{12}$ | $M_{22}$ | $M_{23}$ | $M_{24}$ | PSL$(2,q)$ | $\text{Alt}(n)$ |
|-------|---------|---------|---------|---------|---------|-----------|-------------|
| Degree | 11      | 12      | 12      | 22      | 23      | 24        | $q + 1$     |

Table 4: Socale of an almost simple 3-transitive group.

| Group | $\text{SL}(n, 2) \leq H \leq \text{GL}(n, 2)$ | $\text{GL}(1, 3)$ | $\text{GL}(1, 4)$ | $\text{Alt}(7)$ |
|-------|---------------------------------------------|-------------------|-------------------|------------------|
| Degree | $2^n$                                      | 3                 | 4                 | 16               |

Table 5: Point stabilizer of an affine 3-transitive group.
Using GAP, it is straightforward to build the derangement graph for each of the 3-transitive groups of degree 11, 12, 22, 23, 24 and 16 and to then find all the independent sets of maximal size. Indeed, for every one of these groups, every independent set of maximal size of $\Gamma_G$ is the coset of the stabilizer of a point. Thus we conclude with the following conjecture.

**Conjecture 3.** Let $G$ be a 3-transitive group of degree $n$. Every independent set $S$ of the derangement graph of $G$ acting on $\Omega$ has size at most $|G|/n$. Equality is met if and only if $S$ is the coset of the stabilizer of a point.
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