Robust state estimation for uncertain linear discrete systems with d-step state delay

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Abstract
This paper discusses the state estimation problems of an uncertain linear discrete time-varying state space model with d-step state delay. Based on the principle of minimising the expectation of estimation errors and the method of state augmentation, a robust state estimation algorithm is proposed. Specially, this estimator retains the form of Kalman-like filter and the characteristics of fast recursive calculation. Moreover, the conditions of bounded estimation error covariance and the proof of asymptotic unbiasedness of the filter are given. Finally, numerical examples are used to verify the effectiveness and the wide applicability of this estimator.

1 INTRODUCTION
In the past several decades, state estimation has been extensively utilised in system control, the existing achievements play important roles in signal processing, communication, target tracking and other fields [1–3]. With the deepening of the research, the state estimation methods emerge one after another, the most classic of them is the Kalman filter based on the minimum mean square error [4]. In [6], a two-step estimation scheme in the sense of minimum mean square error (MMSE) is given for systems with unknown dynamic deviations. However, the traditional Kalman filtering theory can only be applied to systems where model parameters and external interference signals are determined, and is limited to situations where there are no time delay. In many practical applications, not only do we lack an understanding of the statistical characteristics of external interference signals, but the system model itself also has a range of disturbances. It has stimulated widespread attention from scholars on robust state estimation and many results were obtained [44]. In [42], the author solves the filtering problem of a two-dimensional uncertain linear discrete time-varying system with random noise by solving a robust regularised least squares problem. In [45], a state estimator of Markov jump neural network is studied by the persistent dwell-time switching rule. In [17], both the interpolation theory approach and the Riccati equation approach are used to solve the estimation problem. Some robust deconvolution filters by the minimax robust estimation [29] and the polynomial method [30] are presented. In
[18–20], some optimal filtering methods for uncertain discrete-time systems based on regularised least squares (RLS) algorithm or the sensitivity penalisiation for estimation errors to parameter variations are presented respectively, but these methods still have problems in that the applicable situation is relatively conservative and the parameters are not convenient to be determined. In [46], by solving the convex optimisation problem, the author derives an extended dissipative state estimator for a class of Markov jump neural networks. A robust estimator based on expectation minimisation of estimation errors is released in [21], which better solves the problem of parameter uncertainty in the state space model. A variance-constrained state estimation algorithm was proposed in [43]. In [39], by appropriately determining the estimator gain to estimate the estimation error covariance, a state estimator is constructed. But they are limited to specific types of model uncertainties.

At the same time, the filters of time-delay system are paid much attention [7, 9, 10]. For the discrete time-invariant systems with time delay, the method of state augmentation [1] is directly, it is widely used in dealing with the time-delay problem of linear and nonlinear systems [36], and has achieved good results. In addition, linear matrix inequality (LMI) [5, 13] and the minimum mean square error (MMSE) estimation criteria [11, 12] are also used to estimate the current time state. Besides, the optimal filters based on the corresponding partial differential equation (PDE) are given in [31, 32]. In [41], the author explored the relevant stabilisation methods for discrete systems with distributed state delay and input delay. In the case of state time-delay systems with uncertain parameters, many researchers have investigated the robust control problem for time-delay systems with uncertainties using the $H_{\infty}$ control approach and obtained many significant results [14–16]. Meanwhile, $H_{\infty}$ robust state estimation for state time-delay systems has also been extensively studied [22], but there are still some limitations, such as some filters are only suitable for the cases where the system has a special type of model error [27], or the estimators should be verified at every instant when it is estimated online. In [37], scholars studied the robust state estimation problem of systems with time-correlated multiplicative noises (TCMNs) based on the minimum mean square error. By solving Riccati difference equation and Lyapunov equation, the robust estimation problem for uncertain systems with single delayed measurement can be solved [34]. Apart from this, the robust state estimator can be designed by solving a linear matrix inequality (LMI) [35], but this method needs to construct optimisation problems and is not easy to implement. In [33], a robust state estimation method that considers modelling noise and uncertainty by integrating quadratic constraints is proposed, but it is only applicable for a specific class of uncertain time-delay systems. A new approach through the reorganisation of measurements is proposed in [28], while only the case where time delays occur in the measurement equation is considered [8]. In [40], a delay compensation-based state estimator (DCBSE) was proposed, which uses a prediction-based estimation mechanism to replace the delayed estimation transmission method, effectively solving the communication delay and fading observation in the discrete time-varying system. So far, it should be pointed out that there are few available methods to solve the state estimation problem of time-delay systems affected by uncertain model uncertainties. Which is worthy of our further study.

In view of the above discussion, this paper mainly studies the robust state estimation problem of discrete linear systems with state delay and uncertain parameters. Compared with existing methods, the main difficulties we encountered are: (1) For systems with time delay, the system model becomes complicated and it is difficult to directly derive the state estimator. How to simplify the model for analysis? (2) How to ensure stable performance of the estimator when the model uncertainties affect the parameter matrix in arbitrary form? (3) When there are model uncertainties in the system model, the true error covariance matrix cannot be obtained. How to approximate or replace the error covariance matrix? (4) From the theoretical viewpoint, the application range of the traditional Kalman filter is extremely wide. How to define the application range of this new robust state estimator and analyse the progressive nature of the system based on this? In response to the above problems, we have made the following contributions: (1) The state augmentation method is used to simplify the system model, making the system easy to analyse. (2) Based on the idea of finding the minimum value of the cost function, some key matrices that can be calculated offline are constructed by calculating expectations to solve the problem of uncertain model uncertainties in the system. (3) We construct a “pseudo-error-covariance matrix” through the idea of regularised least squares (RLS). When the system without model uncertainties, the “pseudo-error-covariance matrix” is equal to the real error covariance matrix, we also proved that the constructed “pseudo-error-covariance matrix” is bounded. (4) We have assumed the scope of application of the estimator in the theoretical proof process, and according to the given conditions, the properties of the derived estimator are proved. Finally, several numerical simulations reflecting the estimation effect of the estimator under different conditions are given to prove its performance.

In the second section of this article, the state space model with state delay and uncertain parameters is given and converted, and then in the Section 3, the recursive process of the robust state estimator for uncertain linear discrete systems with state delay is introduced. In the second half of this text, some important properties and some numerical examples are used to prove the asymptotic stability and effectiveness of this algorithm. Finally, the full text is summarised in Section 6. In addition, the derivation process of the estimator and the proofs of some theorems are given in detail in Appendix.

In the description below, the notation $R^n$ denotes the $n$-dimensional Euclidean space, $R^{n \times m}$ denotes the set of $n \times m$ real matrices, $E[\cdot]$ denotes the mathematical expectation of a random variable, $\mathbf{Cov}(a,b)$ denotes the covariance of $a$ and $b$, $\| \cdot \|$ denotes the Euclidean norm, $I_n$ denotes the $n \times n$ identity matrix, $0_{n \times m}$ denotes the $n \times m$ zero matrix, $\text{diag}(\cdot)$ denotes the block-diagonal matrix.
2 | PROBLEM DESCRIPTION

2.1 | State space model

Consider the following uncertain linear discrete-time system with state time delay:

$$
\begin{align*}
\dot{x}_{i+1} &= A_1(\xi_i)x_i + A_2(\xi_i)x_{i-d} + B_1(\xi_i)w_i, \\
y_i &= C_i(\xi_i)x_i + v_i,
\end{align*}
$$

where $d$ is discrete time, $x_i \in \mathbb{R}^n$ is the state vector, $w_i$ is the process noise, $y_i \in \mathbb{R}^m$ and $v_i \in \mathbb{R}^r$ are the measurement output and measurement error, respectively. $d$ indicates the number of delay steps, $A_1(\xi_i) \in \mathbb{R}^{n \times n}$, $A_2(\xi_i) \in \mathbb{R}^{n \times n}$, $B_1(\xi_i) \in \mathbb{R}^{n \times r}$, $C_i(\xi_i) \in \mathbb{R}^{m \times n}$ are matrices of corresponding dimensions related to the model parameter error $\xi_i$. The model parameter error $\xi_i$ changes with time $i$ and consists of $L$ independent real valued scalar bounded uncertainties $\varepsilon_{ij}$, $j = 1, \ldots, L$. In addition, to get closer to the real situation, we assume that the system parameter matrix error can affect the model parameters in any way and are not limited to certain conditions. So this model is closer to the actual industrial system than the models described in many existing literatures like [19], [20], [23], [24]. If we remove $A_2(\xi_i)x_{i-d}$, then System (1) becomes a standard state-space model. Initial conditions $x_0, w_i$ and $v_i$ are mutually independent random vectors, they meet the following relationships:

$$
\begin{align*}
E(w_i) &= 0, \quad \text{Cov}(w_i, w_j) = Q_1 \delta_{ij}, \\
E(v_i) &= 0, \quad \text{Cov}(v_i, v_j) = R_1 \delta_{ij}, \\
E(w_i, v_i) &= 0, \\
E[(x_0 - E(x_0))(x_0 - E(x_0))^T] &= \Pi_0
\end{align*}
$$

Where $\Pi_0, Q_1$ and $R_1$ are known positive definite matrices, $\delta_{ij}$ is the Kronecker delta function.

2.2 | Model transformation

This paper uses the method of state augmentation to transform this state delay model. Introduce augmented state $X_i = [x_i^T \ x_{i-1}^T \ \cdots \ x_{i-d}^T]^T$, then System (1) can be reconstructed into the following equivalent model:

$$
\begin{align*}
\dot{X}_{i+1} &= \mathcal{A}_i(\xi_i)X_i + \mathcal{B}_i(\xi_i)w_i, \\
y_i &= \mathcal{C}_i(\xi_i)x_i + v_i,
\end{align*}
$$

Among them:

$$
\mathcal{A}_i(\xi_i) = \begin{bmatrix}
A_1(\xi_i) & 0_{nxu} & \cdots & 0_{nxu} \\
I_u & 0_{nxu} & \cdots & 0_{nxu} \\
& I_u & \cdots & 0_{nxu} \\
& \cdots & \cdots & \cdots \\
& & & I_u
\end{bmatrix},
$$

It can be seen that the input dimension of the system increases from $n$ to $n(d + 1)$. The state estimation problem to be solved in this paper is: finding the optimal estimate of state $X_i$ by the observation data to time $i$.

3 | THE DESIGN OF ROBUST STATE ESTIMATOR FOR STATE DELAY SYSTEM

3.1 | The derivation of the robust state estimator

To obtain the robust state estimator of the system described above, we first define $\hat{X}_{i|j}$ and $P_{i|j}$ to represent the optimal estimate of $X_i$ based on $j_{i|j}$ and the covariance matrix of $X_i - \hat{X}_{i|j}$ respectively. Then according to [19], the Kalman filter can be fully explained by the solution of the regularised least squares (RLS) problem, as shown in the following Formulas (7) and (8).

$$
\hat{X}_{i+1|j|+1} = \hat{A}_i(0)\hat{X}_{i|j|+1} + \hat{B}_i(0)\hat{w}_{i|j|+1},
$$

$$
\begin{align*}
\begin{bmatrix}
\hat{X}_{i|j|+1} \\
\hat{w}_{i|j|+1}
\end{bmatrix} &= \arg \min_{\hat{X}_{i|j|}, \hat{w}_{i|j|}} \left\{ \| X_i - \hat{X}_{i|j|} \|^2_{P_{i|j}} + \| w_i - \hat{w}_{i|j} \|^2_{Q_{i|j}} + \| y_{i+1} - \hat{C}_{i+1|j|}(X_{i+1|j|}) \|^2_{R_{i+1}} \right\}.
\end{align*}
$$

As expressed in Equation (8), the cost function of the RLS problem is the square of the regular residual norm. These equations can be understood as follows: Given an initial estimation of $X_i$, the estimation is constantly revised by the measurement data at each moment. If there are model uncertainties in the system, the influence of them on the filtering performance should be taken seriously. To solve this problem, we can optimise the cost function by defining the following matrices:

$$
\begin{align*}
\Psi_i &= (P_{i+1}^{-1}, \Phi_i = \text{diag}\{P_{i|j|}^{-1}, Q_{i|j}^{-1}\}, \\
H_i(\xi_i, \xi_{i+1}) &= \mathcal{C}_{i+1}(\xi_{i+1})[\mathcal{A}_i(\xi_i)\mathcal{B}_i(\xi_i)], \\
\beta_i(\xi_i, \xi_{i+1}) &= \theta_{i+1} - \hat{C}_{i+1}(\xi_{i+1})\hat{A}_i(\xi_i)\hat{X}_{i|j}, \\
\alpha_j &= \text{col}\{X_i - \hat{X}_{i|j}, w_i\}.
\end{align*}
$$

So the cost function can be simplified as

$$
J(\alpha_j) = E \left\{ \| X_i - \hat{X}_{i|j} \|^2_{P_{i|j}^{-1}} + \| w_i \|^2_{Q_{i|j}^{-1}} + \| y_{i+1} - \hat{C}_{i+1}(\xi_{i+1})X_{i+1} \|^2_{R_{i+1}} \right\}.
\[
E \left\{ \| \alpha_i \|^2_{\Psi_i} + \left\| H_i(\epsilon_i, \epsilon_{i+1}) \alpha_i - \beta_i(\epsilon_i, \epsilon_{i+1}) \right\|^2_{\Psi_i} \right\}.
\]

The cost function above considers the impact of model uncertainties on the estimation performance. It can be found that if modelling errors do not exist, the simplified cost function will evolve into the standard Kalman filter. Continue to simplify, we can get,

\[
\begin{align*}
J(\alpha_i) &= \| \alpha_i \|^2_{\Phi_i} + E \left\{ \left\| H_i(\epsilon_i, \epsilon_{i+1}) \alpha_i - \beta_i(\epsilon_i, \epsilon_{i+1}) \right\|^2_{\Psi_i} \right\} \\
&= \alpha_i^T \Phi_i \alpha_i + E \left\{ \left( H_i(\epsilon_i, \epsilon_{i+1}) \alpha_i - \beta_i(\epsilon_i, \epsilon_{i+1}) \right)^T \Psi_i \right\} \\
&= \alpha_i^T \Phi_i \alpha_i + E \left\{ \left( H_i(\epsilon_i, \epsilon_{i+1}) \alpha_i - \beta_i(\epsilon_i, \epsilon_{i+1}) \right)^T \Psi_i \right\}.
\end{align*}
\]

It is easy to see that the cost function is a strictly convex function. To find the global unique minimum \( \alpha_{i,\text{opt}} \), calculate the partial derivative of Formula (11), we have

\[
\frac{\partial J(\alpha_i)}{\partial \alpha_i} = 2\Phi_i \alpha_i + 2E \left\{ H_i^T(\epsilon_i, \epsilon_{i+1}) \Psi_i H_i(\epsilon_i, \epsilon_{i+1}) \alpha_i \right\} \\
- E \left\{ H_i^T(\epsilon_i, \epsilon_{i+1}) \Psi_i \beta_i(\epsilon_i, \epsilon_{i+1}) \right\} \\
- E \left\{ \beta_i^T(\epsilon_i, \epsilon_{i+1}) \Psi_i H_i(\epsilon_i, \epsilon_{i+1}) \right\},
\]

then make it equal to zero and continue to simplify, we can get

\[
\begin{align*}
\Phi_i + E \left\{ H_i^T(\epsilon_i, \epsilon_{i+1}) \Psi_i H_i(\epsilon_i, \epsilon_{i+1}) \right\} \alpha_{i,\text{opt}} \\
= E \left\{ H_i^T(\epsilon_i, \epsilon_{i+1}) \Psi_i \beta_i(\epsilon_i, \epsilon_{i+1}) \right\}.
\end{align*}
\]

Bring the definition in Equation (9) into Equation (13), we can get that

\[
\begin{align*}
\begin{bmatrix}
P_{\dot{q}_{i}}^{-1} & 0_{n_i(d_i)+1 \times n_i} \\
0_{n_i(d_i)+1 \times n_i} & Q_{i}^{-1}
\end{bmatrix}
+ E \left\{ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right\} \hat{C}_i(\epsilon_i + 1) \\
\times \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \right\} \\
= E \left\{ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right\} \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right]
\end{align*}
\]

then define several key matrices as follows,

\[
\begin{align*}
\left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \right\} = H_{i1},
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \right\} = H_{i2},
\end{align*}
\]

\[
\begin{align*}
\left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right] \right\} = H_{i3},
\end{align*}
\]

\[
\begin{align*}
G_i = \begin{bmatrix} G_{i11} & G_{i12} \\
G_{i12}^T & G_{i22} \end{bmatrix} = H_{i1} - \left[ \begin{bmatrix} A_i(0) \\ B_i(0) \end{bmatrix} \right] \hat{C}_i(0) \left[ \begin{bmatrix} A_i(0) \\ B_i(0) \end{bmatrix} \right].
\end{align*}
\]

Next we continue to derive the state estimator. First of all, we define the initial state estimate \( \hat{X}_{0|0} \) and the pseudo-variance matrix estimation error \( P_{\dot{q}_{i}}^{-1} \). Different from the definition above, \( P_{\dot{q}_{i}}^{-1} \) is named the pseudo-variance matrix here because it is not equivalent to \( E\{\epsilon(X_i - \hat{X}_{i|0})^T(X_i - \hat{X}_{i|0})\} \).

When we calculate the initial state, the process noise does not need to be considered, so the cost function is defined as follows,

\[
J(\alpha_0) = E \left\{ \| X_0 \|^2_{\Delta_0} + \| \epsilon_0 - C_0(\epsilon_0)X_0 \|^2_{R_0^{-1}} \right\}.
\]

In which, \( \Delta_0 = E\{(X_0 - E(X_0))(X_0 - E(X_0))^T\} \). Then we can find the initial value of the estimator by simplifying Equation (16) as follows,

\[
\begin{align*}
\hat{X}_{0|0} = P_{0|0} E \left\{ \hat{C}_0(\epsilon_0) \right\} R_0^{-1} \epsilon_0, \\
P_{0|0} = (\Delta_0^{-1} + E\{\hat{C}_0(\epsilon_0)R_0^{-1}C_0(\epsilon_0)\})^{-1}.
\end{align*}
\]

Define \( \hat{\dot{Q}}_{i}^{-1} = Q_{i}^{-1} + G_{i12} - C_{i12} P_{i|0} G_{i12} \) and \( P_{\dot{q}_{i}}^{-1} = P_{\dot{q}_{i}}^{-1} + G_{i11} \), then there is an algebraic relationship as follows,

\[
\begin{align*}
\begin{bmatrix}
P_{\dot{q}_{i}}^{-1} & 0_{n_i(d_i)+1 \times n_i} \\
0_{n_i(d_i)+1 \times n_i} & Q_{i}^{-1}
\end{bmatrix}
+ E \left\{ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right\} \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right]
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
P_{\dot{q}_{i}}^{-1} & 0_{n_i(d_i)+1 \times n_i} \\
0_{n_i(d_i)+1 \times n_i} & Q_{i}^{-1}
\end{bmatrix}
+ E \left\{ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right\} \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right]
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
P_{\dot{q}_{i}}^{-1} & 0_{n_i(d_i)+1 \times n_i} \\
0_{n_i(d_i)+1 \times n_i} & Q_{i}^{-1}
\end{bmatrix}
+ E \left\{ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right\} \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right]
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix}
P_{\dot{q}_{i}}^{-1} & 0_{n_i(d_i)+1 \times n_i} \\
0_{n_i(d_i)+1 \times n_i} & Q_{i}^{-1}
\end{bmatrix}
+ E \left\{ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right\} \hat{C}_i(\epsilon_i + 1) \left[ \begin{bmatrix} A_i(\epsilon_i) \\ B_i(\epsilon_i) \end{bmatrix} \right]
\end{align*}
\]
According to the definition of $G_i$ above, it can be seen that Equation (14) can be written as follows:

\[
\begin{pmatrix}
P_{ij}^{-1} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\times R_{ij+1}^{-1} C_{ij+1}(0) [\hat{\xi}_i(0) \hat{B}_i(0)]
\times E \left\{ \frac{\partial^T(x_i)}{\partial^T(y_i)} \left( \hat{C}_{ij+1}(0) \hat{A}_i(y_i) \right) \right\} X_{ij}.
\]

Then define the optimal value $\alpha_{opt} = \text{col}[\hat{X}_{i|j+1} - \hat{X}_{i|j}, \hat{\nu}_{i|j+1}]$. Substituting Equation (18) into the equation above, we can get,

\[
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\begin{pmatrix}
P_{ij}^{-1} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\left( \frac{\partial^T(x_i)}{\partial^T(y_i)} \right) \left( \hat{C}_{ij+1}(0) \hat{A}_i(y_i) \right)
\times \left( \hat{X}_{i|j+1} - \hat{X}_{i|j} \right) = H_{i2} R_{i2}^{-1} y_{i+1} - H_{i3} \hat{X}_{i|j}.
\]  

(19)

Then we multiply

\[
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
^{-1}
\]

from the left sides of Equation (19) and simplify this equation. It is easy to get that

\[
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
^{-1}
\]

from the left sides of Equation (19) and simplify this equation. It is easy to get that

\[
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
^{-1}
\]

Defining $\hat{B}_i(0) - \hat{A}_i(0) \hat{B}_i G_{i12}$, $\hat{C}_{i+1}(0) [\hat{A}_i(0) \hat{B}_i(0)]$ and $\hat{X}_{i|j+1} + \hat{P}_i G_{i12} \hat{\nu}_{i|j+1}$ are $\hat{B}_i(0), H_i$ and $\hat{X}_{i|j+1}$, we can get that

\[
\begin{pmatrix}
P_{ij}^{-1} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
\begin{pmatrix}
I_{a(d+1)} & 0_{a(d+1) \times a} \\
0_{a(d+1) \times a} & Q_{ij}^{-1}
\end{pmatrix}
^{-1}
\]

Defining the variable $\hat{X}_{i|j+1} = \hat{A}_i(0) \hat{X}_{i|j+1} + \hat{B}_i(0) \hat{\nu}_{i|j+1}$, then we can continue to simplify and gain the following expression,

\[
\hat{X}_{i|j+1} = \hat{X}_{i|j} - \hat{P}_{ij} \hat{A}_i(0) \hat{C}_{i+1}(0) R_{i+1}^{-1} \hat{C}_{ij+1}(0)
\times \left( \hat{X}_{i|j+1} - \hat{A}_i(0) \hat{X}_{i|j} \right)
\times \left( \hat{X}_{i|j+1} - \hat{A}_i(0) \hat{X}_{i|j} \right)
\times \left( \hat{X}_{i|j+1} - \hat{A}_i(0) \hat{X}_{i|j} \right).
\]  

(24)
Similarly, Equation (22) can be simplified as follows,

$$
\begin{aligned}
& (-G_{ij2}^{T} P_{ij} \ L_i) (H_{ij3} R_{ji}^{-1} \psi_{j+1} - H_{ij3} \tilde{X}_{ij}) \\
& = \tilde{B}_i(0) \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) \tilde{A}_i(0) (X_{ij+1} - \tilde{X}_{ij}) \\
& + \tilde{C}_{ij+1}^{T}(0) \tilde{X}_{ij+1} + \tilde{B}_i(0) \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) \tilde{B}_i(0) \tilde{\psi}_{j+1} \\
& = \tilde{B}_i(0) \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) \\
& \times (X_{ij+1} - \tilde{A}_i(0) \tilde{X}_{ij}) + \tilde{C}_{ij+1}^{T}(0) \tilde{X}_{ij+1},
\end{aligned}
$$

Note that the expression of $\tilde{C}_{ij+1}^{T}(0)$ can be obtained by shifting the term.

$$
\begin{aligned}
\tilde{C}_{ij+1}^{T}(0) &= (-G_{ij2}^{T} P_{ij} \ L_i) (H_{ij3} R_{ji}^{-1} \psi_{j+1} - H_{ij3} \tilde{X}_{ij}) \\
& - \tilde{B}_i(0) \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) \\
& \times (X_{ij+1} - \tilde{A}_i(0) \tilde{X}_{ij}).
\end{aligned}
$$

Then, we can get the expression of $\tilde{\psi}_{j+1}$ as follows:

$$
\begin{aligned}
\tilde{\psi}_{j+1} &= \tilde{X}_{ij} \times (-G_{ij2}^{T} P_{ij} \ L_i) (H_{ij3} R_{ji}^{-1} \psi_{j+1} - H_{ij3} \tilde{X}_{ij}) \\
& - \tilde{B}_i(0) \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) \\
& \times (X_{ij+1} - \tilde{A}_i(0) \tilde{X}_{ij}).
\end{aligned}
$$

Taking Equations (24) and (26) back to the definition of $X_{ij+1}$ in Equation (27), we gain:

$$
\begin{aligned}
X_{ij+1} &= \tilde{A}_i(0) X_{ij+1} + \tilde{B}_i(0) \tilde{\psi}_{j+1} \\
& = \tilde{A}_i(0) P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \Psi_{j+1} \\
& - \tilde{A}_i(0) P_{ij} \tilde{A}_i^{T} (0) \tilde{C}_{ij+1}^{T}(0) \tilde{C}_{ij+1}(0) X_{ij+1} \\
& + \tilde{A}_i(0) I_{n(d+1)} - P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \\
& + P_{ij} [A_{ij2}^{T}(0) \tilde{C}_{ij+1}(0) \tilde{C}_{ij+1}(0) \tilde{A}_i(0)] \\
& + \tilde{B}_i(0) \tilde{\psi}_{j+1} \\
& - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \\
& + \tilde{B}_i(0) \left( \tilde{\psi}_{j+1} \right) \tilde{C}_{ij+1}(0) \tilde{C}_{ij+1}(0) \tilde{A}_i(0) \\
& = \left( \tilde{A}_i(0) P_{ij} I_{n(d+1)} - P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \Psi_{j+1} \\
& + \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \\
& - \tilde{A}_i(0) P_{ij} \tilde{A}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \\
& + \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \right)
\end{aligned}
$$

According to Equation (15), the following relationship is obtained:

$$
\begin{aligned}
H_{ij2} &= H_{ij2} \left[ I_{n(d+1)} I_{n(d+1) \times e} \right], \\
G_{ij2} &= \left[ G_{ij21} G_{ij22} \right], \\
& = H_{ij2} \left[ \tilde{A}_i^{T}(0) B_i^{T} (0) \tilde{C}_{ij+1}(0) R_{ji}^{-1} \right] \\
& \times \tilde{C}_{ij+1}(0) \tilde{A}_i(0) B_i(0).
\end{aligned}
$$

Then, the following equation can be obtained by shifting the term:

$$
\begin{aligned}
\tilde{A}_i(0) P_{ij} I_{n(d+1)} - P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \Psi_{j+1} \\
+ \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \\
& - \tilde{A}_i(0) \tilde{A}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \\
& + \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \\
& = \left( \tilde{A}_i(0) P_{ij} I_{n(d+1)} - P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \Psi_{j+1} \\
& + \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \right)
\end{aligned}
$$

According to the matrix inversion lemma in [25], we have

$$
\begin{aligned}
I_{n(d+1)} + P_{ij} \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) = \left( \tilde{A}_i(0) P_{ij} I_{n(d+1)} - P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \Psi_{j+1} \\
+ \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \right)
\end{aligned}
$$

Therefore, defining $P_{ij} \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) y_{ij+1} = C_{ij+1}(0),$ we can get that:

$$
\begin{aligned}
I_{n(d+1)} + P_{ij} \tilde{C}_{ij+1}^{T}(0) R_{ji}^{-1} \tilde{C}_{ij+1}(0) = \left( \tilde{A}_i(0) P_{ij} I_{n(d+1)} - P_{ij} \left[ I_{n(d+1) \times e} \right] H_{ij2} \Psi_{j+1} \\
+ \tilde{B}_i(0) \tilde{\psi}_{j+1} - \tilde{B}_i(0) \tilde{B}_i^{T} (0) \tilde{C}_{ij+1}(0) \tilde{X}_{ij+1} \right)
\end{aligned}
$$
In which $y_{i+1} = \tilde{C}_{i+1}(0)P_{i+1|0} \tilde{C}_{i+1}^T(0) + R_{i+1}$. Then it is easy to get that

$$X_{i+1|0} = \tilde{A}_i(0) \hat{X}_{i|0} + P_{i+1|0}$$

$$\times \left( p_{i+1|0}^{-1} \begin{bmatrix} \tilde{A}_i(0) \tilde{P}_i \left[ I_{(d+1)} 0_{(d+1) \times s} \right] \\ + \hat{B}_i(0) \tilde{Q}_i \left[ -G_{i+1|0}^{-1} P_{i|0} \right] I_s \end{bmatrix} \right)$$

It can be found that the forms of Equations (24) and (26) are similar to Equations (46) and (47) in [19] and the Equation (A2) and Equation (A3) in [20], so we designate $\hat{X}_{i+1|0}$ as $\hat{X}_{i+1|0}$. Then the derivation of the robust state estimator for linear discrete systems with state delay is completed.

### 3.2 The iterative process of this robust state estimator

**Initial conditions.**

$$P_{0|0} = (\Delta_0^{-1} + E \{ \tilde{C}_0^T(\xi_0) \tilde{R}_0^{-1} \tilde{C}_0(\xi_0) \})^{-1}$$

$$\hat{X}_{0|0} = P_{0|0} E \{ \tilde{C}_0(\xi_0) \} \tilde{R}_0^{-1} \xi_0$$

**Parameter modification.**

$$p_{i+1|0}^{-1} = P_{i|0}^{-1} + G_{i+1}$$

$$\hat{Q}_i^{-1} = Q_i^{-1} + G_{i+1}^{-T} \tilde{P}_i G_{i+1}$$

$$\hat{A}_i(0) = (\hat{A}_i(0) - \hat{B}_i(0) \hat{Q}_i G_{i+1}) (I_{(d+1)} - \hat{P}_i) G_{i+1}$$

$$\hat{B}_i(0) = \hat{B}_i(0) - \hat{A}_i(0) \hat{P}_i G_{i+1}$$

**State estimate and covariance matrix updating.**

$$P_{i+1|0} = \hat{A}_i(0) \hat{X}_{i|0} + P_{i+1|0}$$

$$\gamma_{i+1} = \tilde{C}_{i+1}(0) P_{i+1|0} \tilde{C}_{i+1}^T(0) + R_{i+1}$$

$$P_{i+1|0} = \hat{P}_{i+1|0} - \tilde{C}_{i+1}(0) \gamma_{i+1} \tilde{C}_{i+1}^T(0) \hat{P}_{i+1|0}$$

$$\hat{X}_{i+1|0} = \hat{A}_i(0) \hat{X}_{i|0} + P_{i+1|0}$$

$$\times \left( p_{i+1|0}^{-1} \begin{bmatrix} \hat{A}_i(0) \hat{P}_i \left[ I_{(d+1)} 0_{(d+1) \times s} \right] \\ + \hat{B}_i(0) \hat{Q}_i \left[ -G_{i+1|0}^{-1} \hat{P}_i \right] I_s \end{bmatrix} \right)$$

**4 SOME PROPERTIES OF THE ESTIMATOR**

In this section, we will put forward the asymptotic properties of the robust state estimator of uncertain linear discrete systems with state delay. Assume modelling errors $\xi_{i,j}$ are normalised in magnitude and form set $\mathfrak{w}$, that is, $\mathfrak{w} = \{ \xi | |\xi_{i,j}| \leq 1, j = 1, 2, \ldots, L \}$. Moreover, to make the certification process more convenient and tidy, retain the index of the time-invariant coefficient, then abbreviate $A_{1i}(0), A_{2i}(0), B_i(0)$ and $C_i(0)$ as $A_{1i}, A_{2i}, B_i$ and $C_i$. To make the process simple, we define $M_i, N_i$ and $O_i$ as $[ A_{1i}, 0_{\text{loc}(d+1)} A_{2i} ]$ and $[ C_i, 0_{\text{loc}(d+1)} ]$. Meanwhile, define some matrices as follows:

$$U_i = (Q_i^{-1} + G_{i+1}^{-1} G_{i+1}^{-T} G_{i+1}^{-1})^{-1}, D_i = N_i U_i^{-1},$$

$$J_i = \begin{bmatrix} 0_{\text{loc}(d+1)} & U_i^{-1/2} G_{i+1}^{-1/2} \end{bmatrix},$$

$$F_i = \begin{bmatrix} G_{i+1}^{1/2} \\ 0_{\text{loc}(d+1)} \end{bmatrix},$$

$$T_{i1} = M_i - N_i U_i^{-1} (I_{(d+1)} + G_{i+1}^{-1} G_{i+1} U_i G_{i+1}^{-1})^{-1},$$

$$T_{i2} = D_i (I_{s} + U_i^{-1/2} G_{i+1}^{-1/2} G_{i+1} U_i^{-1/2})^{-1/2},$$

$$W_i = \begin{bmatrix} \tilde{I}^{-1} \\ 0_{\text{loc}(d+1) \times s} \end{bmatrix}.$$
The proof of Theorem 1 is deferred to Appendix A.

**Theorem 2.** If the conditions in Theorem 1 are satisfied, then the robust state estimator is asymptotically unbiased, and the estimation error covariance matrix is bounded at each sampled instant $i$.

The proof of Theorem 2 is deferred to Appendix B.

**5. SIMULATION RESULTS AND RELATED ANALYSIS**

In order to test the performance of the robust state estimator above, some numerical simulation examples are used to compare the effects of this estimator and the Kalman filter. To obtain more realistic comparison results, Kalman filter is applied to the system after model conversion during the simulation process, including comparing its effect based on actual parameters and nominal parameters respectively. Consider the following uncertain linear discrete system with two-step state delay:

\[
\begin{align*}
    x_{i+1} &= A_1(\varepsilon_i)x_i + A_2(\varepsilon_i)x_{i-2} + B_i(\varepsilon_i)w_i, \\
    y_i &= C_i(\varepsilon_i)x_i + r_i,
\end{align*}
\]

in which,

\[
A_1(\varepsilon_i) = \begin{bmatrix} 0.9802 & 0.0100 + 0.4802\varepsilon_i \\ 0.0000 & 0.9802 \end{bmatrix},
\]

\[
A_2(\varepsilon_i) = \begin{bmatrix} -0.2802 & 0.0060 + 0.4802\varepsilon_i \\ 0.0000 & -0.2802 \end{bmatrix},
\]

\[
C_i(\varepsilon_i) = \begin{bmatrix} 1.0000 & -1.0000 \end{bmatrix},
\]

\[
R_i = 1.0000,
\]

\[
Q = \begin{bmatrix} 1.9608 & 0.0195 \\ 0.0195 & 1.9605 \end{bmatrix},
\]

\[
\Pi_0 = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}.
\]

The initial value conditions are as follows,

\[
x_0 = [0, 0]^T, x_1 = [0, 0]^T, x_2 = [0, 0]^T, x_0 = 0.
\]

Meanwhile, the value of $w_i$ follows a normal distribution, so does $r_i$. In the simulations, we designed and generated $1 \times 10^3$ pairs of input and output data to compare the filtering effect. At the same time, in order to ensure the accuracy of the simulation, we perform five hundred tests on each set of numerical experiment to take the average estimation errors variance value. In this paper, the average value of the squared Euclidean distance from the estimation value to the actual value is used as the final data to compare the filtering performance, which is:

\[
E\|X_i - \hat{X}_i\|^2 \approx \frac{1}{n} \sum_{j=1}^{n} \|X_i - \hat{X}_i\|^2.
\]

Obviously, $n$ represents the number of simulations, and the value here is $5 \times 10^2$. For the setting of model uncertainties, the generation of them follows a uniform distribution of $(-1, 1)$. To investigate the impact of changes in model uncertainties on estimator performance, we performed simulations in two different cases. The first case is that the model uncertainties are kept unchanged for each experiment, that is, they do not change with instants. In the following we will call this situation fixed value case. The other case is that the model uncertainties are generated with
every sampled instant in each experiment, we call this situation time-varying case. Then the comparison results are reflected in Figures 1–3.

As can be seen from Figure 1, in the first set of simulations, the estimation performance of the filter described in this paper is better than the Kalman filter based on nominal parameters.

In the second set of simulations, the uncertainty of the system model is increased: the parameter of $A_1(\varepsilon)$ and $A_2(\varepsilon)$ related to the model uncertainties are modified to $0.01 + 0.9802\varepsilon$ and $0.006 + 0.9802\varepsilon$. Figure 2 shows that when the model uncertainties are fixed values in each experiment, the performance of the Kalman filter based on nominal parameters drops sharply, and the robust state estimator in this paper having a performance improvement of approximately 4.0000 dB compared to it.

In the last set of simulations, the uncertainty of the system model is reduced: the parameter of $A_1(\varepsilon)$ and $A_2(\varepsilon)$ related to the model errors are modified to $0.09 + 0.4802\varepsilon$ and $0.06 + 0.4802\varepsilon$. Figure 3 illustrates the broad applicability of this state estimator: when the uncertainty of model uncertainties becomes small, it still maintains good performance.

By comparing the effects of system state estimation in three different situations above, the following two points can be obtained: on the one hand, the performance of this estimator for the state-delayed system is significantly better than the
traditional Kalman filter, especially when the system contains large unknown model uncertainties; On the other hand, to some extent, the robust state estimator in this paper has relatively stable performance, and can maintain good effect even if the system has time-varying model uncertainties.

6 Conclusion

For the uncertain systems with state delay and random parametric uncertainties, this paper develops a robust state estimation algorithm based on the minimum expected error in estimation. Under some assumptions, this state estimator converges to a time-invariant stable system, and the estimation error covariance matrix is bounded. The effectiveness of this filter is verified by some numerical simulations, so this estimator has great application potential.

It is worth pointing out that, as the degree of time delay increases, the computational burden of this system will indeed increase accordingly, but generally speaking, in actual production, the degree of time delay will not be too large, so the impact of this problem is not very serious. But we will also actively solve such problems in future work. In addition, it is meaningful to consider the time-varying delay in the state equation.

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APPENDIX A

Before giving the proof of Theorem 1, we first present two lemmas as follows:

Lemma 1. If \( A_{ij} \) and \( A_{kj} \) are full rank matrices, \( \langle A_{ij}, C_i \rangle \) is detectable, then \( \langle M_{ij}, C_i \rangle \) is detectable.

Suppose that \( M_{ij} = v_i \lambda_i \), in which \( \lambda_i \) is the eigenvalue of \( M_{ij} \), \( v_i \) is the feature vector of \( M_{ij} \), then

\[
\text{rank} \begin{bmatrix} M_{ij} - \lambda_i I \\ F_i \end{bmatrix} = \text{rank} \begin{bmatrix} A_{ij} - \lambda_i I_n & 0_{n \times 2} & \cdots & 0_{n \times 2} & A_{kj} \\ I_n & -\lambda_i I_n & 0_{n \times 2} \\ I_n & -\lambda_i I_n & 0_{n \times 2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ I_n & -\lambda_i I_n & 0_{n \times 2} \\ R_i^{-1/2} O_i \\ G_{ij/11} \end{bmatrix}
\]

Because the rank of the matrix is equal to the rank of the row and the rank of the column, so \( \text{rank} \begin{bmatrix} M_{ij} - \lambda_i I \\ F_i \end{bmatrix} \leq n(d + 1) \). By \( R_i \neq 0 \), it is easy to know that,

\[
\text{rank} \begin{bmatrix} M_{ij} - \lambda_i I \\ F_i \end{bmatrix} \geq \text{rank} \begin{bmatrix} A_{ij} - \lambda_i I_n & 0_{n \times 2} & \cdots & 0_{n \times 2} & A_{kj} \\ I_n & -\lambda_i I_n & 0_{n \times 2} \\ I_n & -\lambda_i I_n & 0_{n \times 2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ I_n & -\lambda_i I_n & 0_{n \times 2} \\ R_i^{-1/2} O_i \\ G_{ij/11} \end{bmatrix}
\]
Suppose $A_{11}, A_{22}$ are full rank matrices, according to the definition of $M_j, M_i$ is a full rank matrix, so $A_i \neq 0$. Then the right-hand side of Equation (37) satisfies that,

\[
\begin{bmatrix}
A_{11} - \lambda_i I_n & 0_{d \times n} & \cdots & 0_{d \times n} & A_{22} \\
I_n & -\lambda_i I_n & & & \\
I_n & -\lambda_i I_n & 0_{d \times n} & & \\
\vdots & & & & \cdots \\
I_n & -\lambda_i I_n & & & 0_{d \times n}
\end{bmatrix}
\begin{bmatrix}
R_i^{-1/2} O_i
\end{bmatrix}
= \text{rank}(A_{11} - \lambda_i I_n) + nd + m.
\tag{38}
\]

If $(A_{11}, C_i)$ is detectable, according to the equivalence relation in [38], \[\text{rank}(A_{11} - \lambda_i I_n) = n,\] because that the rank of $C_i$ is less than or equal to $m$, then \[\text{rank}(A_{11} - \lambda_i I_n) \geq n - m.\] According to Equation (37), we can get

\[
\begin{bmatrix}
A_{11} - \lambda_i I_n & 0_{d \times n} & \cdots & 0_{d \times n} & A_{22} \\
I_n & -\lambda_i I_n & & & \\
I_n & -\lambda_i I_n & 0_{d \times n} & & \\
\vdots & & & & \cdots \\
I_n & -\lambda_i I_n & & & 0_{d \times n}
\end{bmatrix}
\begin{bmatrix}
M_i - \lambda_i I \\
F_i
\end{bmatrix}
\geq n(d + 1),
\tag{39}
\]

In summary, \[\text{rank}(M_i - \lambda_i I) = n(d + 1).\] According to the equivalence relation in [38], $(M_i, F_i)$ is detectable. Therefore, the proof of Lemma 1 is accomplished.

**Lemma 2.** Based on the knowledge about linear algebra and matrix inversion lemma [25], for any matrices $A, B, C$ and $D$, suppose the inverse of the matrix exists, it is easy to know that there is the following relationship,

\[
(A + B C)^{-1} = (I + A B)^{-1} A,
\]

Then by Equation (41) and the parameter modification (33), we can get

\[
P_{ij} = \left(p_{ij}^{-1} + O_{j}^{T} R_{j}^{-1} O_{j}\right)^{-1}
= p_{ij}^{-1} - p_{ij}^{-1} O_{j}^{T} \left(R_{j} + p_{ij}^{-1} O_{j}^{T}\right)^{-1} O_{j} P_{ij}^{-1}.
\]

Using Equation (41), first we define that $\eta$ is

\[
\tilde{Q}_{i} = \left(Q_{i+1}^{-1} - C_{i+1}^{-1} G_{i+1}^{-1} C_{i+1}^{T} G_{i+1}^{-1} C_{i+1}^{-1} \right)^{-1}
\times \left(I_{n(d+i)} + G_{i+1}^{-1} P_{i+1}^{1/2} G_{i+1}^{1/2} P_{i+1}^{-1} G_{i+1}^{-1/2} \right)
\]

Next, we carry out the transformation of $P_{ij}$ in Equation (42):

\[
P_{ij} = p_{ij}^{-1} - p_{ij}^{-1} C_{i11}^{1/2}
\times \left(I_{n(d+i)} + G_{i11}^{1/2} P_{i11}^{1/2} G_{i11}^{1/2} P_{i11}^{-1} G_{i11}^{-1/2} \right)
\times \left(I_{n(d+i)} + G_{i11}^{1/2} P_{i11}^{1/2} G_{i11}^{1/2} P_{i11}^{-1} G_{i11}^{-1/2} \right)^{-1} G_{i11}^{-1/2} G_{i11}^{-1/2} U_{ij}.\]
Then following equation can be obtained:

\[ N_i \dot{\hat{Q}}_i \hat{G}_{i12}^T \hat{p}_{jT} M_{i}^T = N_i \dot{U}_i \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} G_{11}^{1/2} P_{ih} M_{i}^T \]

Using the formula in Equation (41), we can find that

\[ M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} G_{11}^{1/2} P_{ih} M_{i}^T \]

\[ = M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} G_{11}^{1/2} P_{ih} M_{i}^T \]

Combine with Equation (42), we can get

\[ P_{+1|T} = M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} G_{11}^{1/2} P_{ih} M_{i}^T \]

Substituting Lemma 2 into the Equation (45), we have

\[ P_{+1|T} = M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} G_{11}^{1/2} P_{ih} M_{i}^T \]

According to the formula in Equations (33), (34) and (42), we can get

\[ P_{+1|T} = \bar{X}_i(0) \hat{B}_i A_i^T (0) + \bar{X}_i(0) \hat{B}_i \hat{B}_i^T (0) \]

\[ = M_i \left( P_{ih} - P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} G_{11}^{1/2} P_{ih} \right) \]

\[ \times G_{i12} \hat{Q}_i N_{i}^T - M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} \]

\[ \times G_{i11}^{1/2} P_{ih} M_{i}^T + M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} \]

\[ \times G_{i11}^{1/2} P_{ih} M_{i}^T \]

\[ \times G_{i12} \hat{Q}_i N_{i}^T - M_i P_{ih} G_{i11}^{-1/2} \left( I_{i(d+1)} + G_{11}^{1/2} P_{ih} G_{11}^{1/2} \right)^{-1} \]

\[ \times G_{i11}^{1/2} P_{ih} M_{i}^T \mathbf{1} \]
simplify Equation (47), Equation (48) can be gain

\[ P_{\|i-1|j} = M_j P_{\|i-1} M_j^T - M_j P_{\|i-1} O_{j1} R_{j1}^{-1/2} \]

\[ \times \left( I_{\|i-1} O_{j1} R_{j1}^{-1/2} \right)^{-1} \]

\[ \times R_{j1}^{-1/2} O_{j1} P_{\|i-1} O_{j1} R_{j1}^{-1/2} \]

\[ \times \left( I_{\|i-1} + G_{j1}^{-1/2} G_{j12} U_{j1}^T G_{j12}^{-1/2} + G_{j11}^{-1/2} P_{\|i-1} G_{j11}^{-1/2} \right)^{-1} \]

\[ \times \left( G_{j11}^{-1/2} P_{\|i-1} M_j^T + G_{j11}^{-1/2} G_{j12} U_{j1} N_{j1}^T \right) + N_{j1} U_{j1} N_{j1}^T \]

\[ = M_j P_{\|i-1} M_j^T + N_{j1} U_{j1} N_{j1}^T \]

Substituting the variables defined in Equation (35), then \( P_{\|i-1} \) can be further transformed into the following form:

\[ P_{\|i-1|j} = M_j P_{\|i-1} M_j^T + N_{j1} U_{j1} N_{j1}^T - (M_j P_{\|i-1} F_j + D_j J_j) \]

\[ = \left[ \begin{array}{c}
I_{\|i-1} + G_{j1}^{-1/2} G_{j12} U_{j1}^T G_{j12}^{-1/2} + G_{j11}^{-1/2} P_{\|i-1} G_{j11}^{-1/2}
\end{array} \right]^{-1} \]

\[ \times \left[ \begin{array}{c}
I_{\|i-1} O_{j1} R_{j1}^{-1/2}
\end{array} \right] \]

\[ \times \left[ \begin{array}{c}
R_{j1}^{-1/2} O_{j1} P_{\|i-1} O_{j1} R_{j1}^{-1/2}
\end{array} \right] \]

\[ \times \left[ \begin{array}{c}
I_{\|i-1} + G_{j1}^{-1/2} G_{j12} U_{j1}^T G_{j12}^{-1/2} + G_{j11}^{-1/2} P_{\|i-1} G_{j11}^{-1/2}
\end{array} \right]^{-1} \]

\[ \times \left[ \begin{array}{c}
G_{j11}^{-1/2} P_{\|i-1} M_j^T + G_{j11}^{-1/2} G_{j12} U_{j1} N_{j1}^T
\end{array} \right] \]

Since the last term of Equation (49) is in the form of the Riccati regression equation, according to the same arguments as those asymptotic properties analysis of the Kalman filter in [1], if the conclusion of Lemma 1 is satisfied, and \((T_{i1}, T_{i2})\) is stabilisable, then we can get the convergence of \( P_{\|i-1|j} \).

From Formula (34), it is obvious that the convergence of \( P_{\|i-1|j} \) and \( P_{\|i} \) are equivalent, so that \( P_{\|i} \) also converges to a unique positive semidefinite matrix.

Define the following matrix, \( L_j = M_j - (M_j P_{\|i-1} F_j + D_j J_j) \)

\[ \left( W_j + F_j P_{\|i-1} F_j^T \right)^{-1} F_j \]

Using the formula in Equation (41), the following equations are established,

\[ L_j = M_j - (M_j P_{\|i-1} F_j + D_j J_j) \left( W_j + F_j P_{\|i-1} F_j^T \right)^{-1} F_j \]

\[ = M_j \left( P_{\|i}^{-1} + G_{j11} \right)^{-1} G_{j11} - M_j \left( P_{\|i}^{-1} + G_{j11} \right)^{-1} \]

\[ \times G_{j12} \left( I_{\|i-1} + P_{\|i} G_{j11} \right)^{-1} \]

\[ \times \left[ \begin{array}{c}
I_{\|i} G_{j12}^{-1/2} G_{j12} U_{j1}^T G_{j12}^{-1/2} + G_{j11}^{-1/2} P_{\|i} G_{j11}^{-1/2}
\end{array} \right]^{-1} \]

\[ \times \left[ \begin{array}{c}
I_{\|i} + G_{j1}^{-1/2} G_{j12} U_{j1}^T G_{j12}^{-1/2} + G_{j11}^{-1/2} P_{\|i} G_{j11}^{-1/2}
\end{array} \right]^{-1} \]

\[ \times \left[ \begin{array}{c}
G_{j11}^{-1/2} P_{\|i} M_j^T + G_{j11}^{-1/2} G_{j12} U_{j1} N_{j1}^T
\end{array} \right] \]

\[ = M_j P_{\|i-1} M_j^T + N_{j1} U_{j1} N_{j1}^T - (M_j P_{\|i-1} F_j + D_j J_j) \]

\[ = M_j P_{\|i-1} M_j^T + D_j J_j - (M_j P_{\|i-1} F_j + D_j J_j) \]

\[ = M_j P_{\|i-1} M_j^T + D_j J_j - (M_j P_{\|i-1} F_j + D_j J_j) \]

\[ = M_j P_{\|i-1} M_j^T + D_j J_j - (M_j P_{\|i-1} F_j + D_j J_j) \]

\[ = M_j P_{\|i-1} M_j^T + D_j J_j - (M_j P_{\|i-1} F_j + D_j J_j) \]
From the definition in Equations (51), (52) and (53), we can notice that

\[
\frac{d}{dt} x_i \rightarrow \frac{d}{dt} x_i + \left( I_{(d+1)} + G_{r11}^{1/2} P_{\parallel, j} G_{r11}^{1/2} \right)^{-1} \left( M_i - \left( M_i P_{\parallel, j} G_{r11}^{1/2} + N_i Q \right) G_{r12}^{1/2} \right) G_{r11}^{1/2} \left( I_{(d+1)} + P_{\parallel, j} G_{r11}^{1/2} \right)^{-1}
\]

In which, we can notice that

\[
M_i - \left( M_i P_{\parallel, j} G_{r11}^{1/2} + N_i Q \right) G_{r12}^{1/2} G_{r11}^{-1/2} \left( I_{(d+1)} + P_{\parallel, j} G_{r11}^{1/2} \right)^{-1}
\]

\[
\times \left( I_{(d+1)} + G_{r11}^{1/2} U_i G_{r12}^{1/2} G_{r11}^{-1/2} \right) G_{r11}^{1/2} \left( I_{(d+1)} + P_{\parallel, j} G_{r11}^{1/2} \right)^{-1}
\]

According to the equation in Equation (33) and the definition of \( M_i \) and \( N_i \), it can be noticed that the formula in Equation (55) is equivalent to \( \Delta_{(0)} \), then it is easy to gain that \( L_i = \Delta_{(0)}(I + P_{\parallel, j} G_{r11}^{1/2} O_{1/2} G_{r11}^{1/2})^{-1} \).

Define \( S_i \) and \( V_j \) as \([I_{(d+1)} - P_{\parallel, j} G_{r11}^{1/2} O_{1/2} G_{r11}^{1/2}] \Delta_{(0)}(0)\) and \([M_i P_{\parallel, j} G_{r11}^{1/2} O_{1/2} G_{r11}^{1/2}] \Delta_{(0)}(0)\), then the state update equation in Equation (34) can be written as

\[
\Delta_{(i+1)} = S_i \Delta_{(i)} + P_{\parallel, j} G_{r11}^{1/2} V_j \Delta_{(j)}.
\]
From Theorem 1, we know that \( P_{i|i-1} \) converges to \( P_i \) so that
\[
\lim_{i \to \infty} \left( L_{i(d+1)} + P_{i+1|i} R_i^{-1} R_{i+1}^{-1} O_{i+1} \right) = \lim_{i \to \infty} \left( L_{i(d+1)} + P_{i|i-1} O_i^{-1} R_i^{-1} O_{i+1} \right).
\]
(58)

According to Equation (57), with the increment of \( i \), the eigenvalue set of \( S_i \) converges to that of \( L_x \), from the above we know that \( L_x \) converges to the matrix \( L \). Then Theorem 1 is proved.

**APPENDIX B**

At the beginning, we define some variables as follows:
\[
\Gamma_i = P_{i|i-1} \hat{C}_i^T(0) R_i^{-1} \hat{C}_i(0), \quad \chi_i = [L_i + \Gamma_i(0)] X_i, \quad \bar{\chi}_i = [L_i + \Gamma_i(0)] \hat{X}_i, \quad \tilde{\chi}_i = \chi_i - \bar{\chi}_i.
\]
(59)

Define that
\[
\tilde{A}_i(\epsilon) = (I_{n(d+1)} + \Gamma_{i+1}(0)) \times \begin{bmatrix}
\tilde{A}_{11}(\epsilon) & 0_{nx(d-1)} & A_{21}(\epsilon) \\
0_{nx(d-1)} & 0_{nxn} & 0_{nxn} \\
(I_{n(d+1)} + \Gamma_{i+1}(0))^{-1}
\end{bmatrix}
\]
(60)

then it can be obtained through the proof process of Theorem 1 and Equation (56) that
\[
\begin{bmatrix}
\tilde{X}_{i+1|i+1} \\
\tilde{\chi}_{i+1|i+1}
\end{bmatrix} = \tilde{A}_i(\epsilon, \epsilon_{i+1}) \begin{bmatrix}
\tilde{X}_i \\
\tilde{\chi}_i
\end{bmatrix} + \tilde{B}_i(\epsilon, \epsilon_{i+1}) \begin{bmatrix}
\epsilon_i \\
\epsilon_{i+1}
\end{bmatrix} + \begin{bmatrix}
\epsilon_i \\
\epsilon_{i+1}
\end{bmatrix}.
\]
(61)

In which
\[
\begin{align*}
\tilde{A}_i(\epsilon, \epsilon_{i+1}) &= \begin{bmatrix}
\tilde{A}_{11}(\epsilon, \epsilon_{i+1}) & \tilde{A}_{12}(\epsilon, \epsilon_{i+1}) - I_r \\
\tilde{A}_{21}(\epsilon, \epsilon_{i+1}) & \tilde{A}_{22}(\epsilon, \epsilon_{i+1}) + I_r
\end{bmatrix}, \\
\tilde{B}_i(\epsilon, \epsilon_{i+1}) &= \begin{bmatrix}
(I_{n(d+1)} + \Gamma_{i+1}(0)) \\
-V^T \tilde{C}_{i+1}(\epsilon_{i+1}) \tilde{B}_i(\epsilon_i) - V_i \\
V_i \tilde{C}_{i+1}(\epsilon_{i+1}) \tilde{B}_i(\epsilon_i) & V_i
\end{bmatrix},
\end{align*}
\]
(62)

where
\[
\begin{align*}
\tilde{A}_{11}(\epsilon, \epsilon_{i+1}) &= (I_{n(d+1)} + \Gamma_{i+1}(0)) - V^T \tilde{C}_{i+1}(\epsilon_{i+1}) \\
\times (I_{n(d+1)} + \Gamma_{i+1}(0))^{-1} \tilde{A}_i(\epsilon), \\
\tilde{A}_{12}(\epsilon, \epsilon_{i+1}) &= V^T \tilde{C}_{i+1}(\epsilon_{i+1}) \left( I_{n(d+1)} + \Gamma_{i+1}(0) \right)^{-1} \tilde{A}_i(\epsilon).
\end{align*}
\]

Besides, Equation (61) and Equation (16) in [26] have similar forms, so Equation (61) can be proved in the same way and the proof process is omitted here. According to the stability of matrix \( A_{11}(\epsilon) \) and \( A_{22}(\epsilon) \), we can get the condition for the boundedness of estimation errors of the robust filter. So the proof of Theorem 2 is completed.