Anomalies, Dualities, and Topology of $D = 6 \, \mathcal{N} = 1$ Superstring Vacua

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Abstract

We consider various aspects of compactifications of the Type I/heterotic $Spin(32)/\mathbb{Z}_2$ theory on K3. One family of such compactifications includes the standard embedding of the spin connection in the gauge group, and is on the same moduli space as the compactification of the heterotic $E_8 \times E_8$ theory on K3 with instanton numbers (8,16). Another class, which includes an orbifold of the Type I theory recently constructed by Gimon and Polchinski and whose field theory limit involves some topological novelties, is on the moduli space of the heterotic $E_8 \times E_8$ theory on K3 with instanton numbers (12,12). These connections between $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ models can be demonstrated by T duality, and permit a better understanding of non-perturbative gauge fields in the (12,12) model. In the transformation between $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ models, the strong/weak coupling duality of the (12,12) $E_8 \times E_8$ model is mapped to T duality in the Type I theory. The gauge and gravitational anomalies in the Type I theory are canceled by an extension of the Green-Schwarz mechanism.
1 Introduction

In the ongoing development of string duality, six-dimensional vacua with $N = 1$ supersymmetry have been a recent focus of interest. In such vacua, supersymmetry plus Lorentz invariance are not so strong as to prevent interesting dynamics, but are strong enough to allow much of the dynamics to be understood. This was exploited, for example, in the study of small instantons in the $SO(32)$ heterotic string [1].

There have been three broad categories of approaches developed for obtaining six-dimensional string vacua with $N=1$ supersymmetry. One begins with the $E_8 \times E_8$ theory compactified on K3. Using the recently developed 11-dimensional understanding of this string theory [2], this approach has a natural extension to M-theory compactified on $K3 \times S^1/Z_2$. The second approach begins with the $SO(32)$ theory compactified on K3. Here, one has the option of viewing the $SO(32)$ theory either as a heterotic string theory or a Type I superstring theory, since they are nonperturbatively equivalent [3]. The third and most recent approach, which goes under the name of F-theory, begins with the Type IIB superstring in ten dimensions [4]. By associating a two-torus to the complex scalar field of this theory, and allowing it to vary in non-trivial ways over a four-dimensional base space B, one forms a Calabi-Yau space with an elliptic fibration. Many different classes of vacua can be constructed by each of these approaches, and in many cases it is clear that constructions obtained by the various different approaches are dual to one another – i.e., nonperturbatively equivalent.

Six-dimensional theories with $N=1$ supersymmetry contain four distinct kinds of massless multiplets: the gravity multiplet, tensor multiplets, vector multiplets, and hypermultiplets. The vector multiplets are characterized by the choice of a gauge group and the hypermultiplets by the choice of a representation of the group. A tensor multiplet, which is always a singlet of the group, contains a two-form potential with an anti-self-dual field strength. Since the gravity multiplet contains a two-form potential with a self-dual field strength, it is only possible to give a manifestly covariant effective action when there is exactly one tensor multiplet. In this case the self-dual
and anti-self-dual tensors can be described together by a single two-form $B$ or a three-form field strength $H$. In traditional compactifications of string theory from ten dimensions there is exactly one tensor multiplet. Additional tensor multiplets are obtained in compactifications of M-theory by adding small instantons, which can also be interpreted as five-branes. In the $SO(32)$ approach, on the other hand, the addition of small instantons (or five-branes) leads to additional vector multiplets and enhanced gauge symmetry, but no additional tensor multiplets \[\text{[1]}\]. In the F-theory constructions there are also vacua without any tensor multiplet \[\text{[5]}\]. Here we will only consider vacua with one tensor multiplet.

Since all $N=1$ models in six dimensions are chiral, the cancellation of anomalies is an important requirement. In particular, in the case of one tensor multiplet, a necessary requirement is that the 8-form anomaly polynomial factorize in the product of two 4-forms: $I_8 \sim X_4 \wedge \tilde{X}_4$, where $X_4$ and $\tilde{X}_4$ have the structure

$$X_4 = \text{tr}R^2 - \sum_\alpha v_\alpha \text{tr}F^2_\alpha \tag{1.1}$$
$$\tilde{X}_4 = \text{tr}R^2 - \sum_\alpha \tilde{v}_\alpha \text{tr}F^2_\alpha \tag{1.2}$$

and $\alpha$ labels the various factors in the gauge group. Actually, as we will discuss in the next section, there can also be additional terms in $I_8$ of the form $X_2 \wedge X_6$ when the group contains $U(1)$ factors. As is well-known, the four-forms also appear in the Bianchi identity $dH = X_4$ and field equation $d^* H = \tilde{X}_4$. It has been realized recently that whenever any of the $\tilde{v}_\alpha$’s is negative there is a value of the dilaton for which the coupling constant of the corresponding gauge group diverges. This singularity is believed to signal a phase transition associated with the appearance of tensionless strings \[\text{[3, 4]}\]. This is a very interesting phenomenon, which is the subject of much current discussion. However, in most of this paper we will focus on models (or parameter ranges) for which this phenomenon does not occur.

In this paper we would like to elaborate on two recently-discussed classes of $D = 6, N = 1$ vacua and to establish the connection between them. The first class is the $SO(32)$ Type I string on $K3$ in the $T^4/\mathbb{Z}_2$ orbifold limit \[\text{[7]}\]. The second is the $E_8 \times E_8$ heterotic string on K3, with symmetric embedding.
of the instantons in the two $E_8$'s \cite{8}. For convenience we will refer to these as GP and DMW vacua respectively. Along the way we will also consider the $E_8 \times E_8$ string with other embeddings, and other vacua of the $Spin(32)/\mathbb{Z}_2$ heterotic string.

In ref. \cite{7}, consistency conditions for open superstring compactifications were studied, and the general solution was found for the case of a $T^4/\mathbb{Z}_2$ orbifold. Cancellation of tadpoles for massless Ramond-Ramond 10-form and 6-form potentials required 32 Chan-Paton nine-brane indices and 32 five-brane indices, the five-branes oriented so as to fill the 6-dimensional spacetime).\footnote{There is a semantic problem in this subject. In the GP models the five-brane Chan-Paton index takes 32 values, but half can be regarded as the images of the others under the orbifold $\mathbb{Z}_2$, and (as discussed in refs. \cite{9,9}) half again can be regarded as images under world-sheet parity $\Omega$. Thus there are only 8 dynamical five-branes, each having the minimum unit of 6-form charge. So we will refer to five-branes when, as will usually be the case, we mean the dynamical objects, and to ‘indices’ when we count the images separately.} The gauge group associated with nine-branes was found to be $U(16)$ (or a combination of unitary and symplectic subgroups, obtained by adding Wilson lines) and the same for the five-branes with the five-brane positions playing the role of the Wilson lines.

The world-sheet consistency conditions, from closure of the operator product expansion and cancellation of one-loop divergences, were conjectured to be a complete set. If so, spacetime gauge and gravitational anomaly cancellation, normally a tight constraint on the spectra of $D = 6$, $N = 1$ Yang-Mills theories, will hold automatically. In ref. \cite{7} the quartic terms in the spacetime anomalies were found to cancel for all GP models.

In section 2 we study the full anomaly; as expected, the GP models are (perturbatively) consistent, but the details are interesting. As is familiar from refs. \cite{9}, in order that the anomalies be canceled in the usual way (via tree-level exchange of the 2-form $B_{\mu\nu}$), the anomaly 8-form for the non-abelian gauge fields and gravity must factorize into a product of two 4-forms. In the consistent models of ref. \cite{7}, we find that part of the anomaly polynomial involving abelian gauge fields does not factorize in this fashion. We identify an extended version of the GS mechanism in which the anomalies involving $U(1)$ gauge fields are canceled by tree-level exchange of certain 0-form fields. These
fields are identified as R-R closed string twisted sector states. The requisite inhomogeneous transformation of these fields under gauge transformations leads to spontaneous symmetry breaking of some of the $U(1)$ factors (just as occurs for the Green-Schwarz mechanism in four dimensions \[^{10}\]), so that some of the states identified as massless in ref. \[^{7}\] receive masses of order the open string coupling.

Although the world-sheet considerations of ref. \[^{7}\] evidently guarantee a perturbatively consistent theory, we show that there is a potential nonperturbative inconsistency. For many of the GP models, spinors of the gauge group cannot be defined. Such spinors do not exist in perturbation theory but do arise nonperturbatively as D-branes, so the inconsistency appears first in this way.

In the GP construction there are two types of gauge groups: those carried by open strings that connect pairs of five-branes and those carried by open strings that connect pairs of nine-branes. As we will explain, there is a $T$ duality that interchanges the five-branes and nine-branes and thereby interchanges the two types of open strings and the gauge groups that they carry.

We will claim that these theories are on the same moduli space as the class of vacua considered by DMW. The latter exhibit an interesting phenomenon: heterotic/heterotic string duality. That is, they have an $S$-duality symmetry that interchanges perturbative heterotic strings with non-perturbative heterotic strings.

We will argue that the GP and DMW are equivalent under a sort of ‘$U$-duality’ that maps the $S$-duality of the DMW description to the $T$ duality of the GP description. This is an example of the widespread phenomenon of “duality of dualities.” Several pieces of evidence support the proposed correspondence between these models. One of these is a comparison of the structure of the factorized anomaly polynomials in each case. Another is the construction of solitonic strings in the GP setup with the correct properties to be identified with the two kinds of heterotic strings in the DMW description. Finally, we will exhibit at the end of this paper a direct proof of this relation using $T$ duality between certain orbifolds.
By integrating the Bianchi identity over K3 one learns that, of \( n_1 \) and \( n_2 \) are the number of instantons in the two factors, then \( E_8 \times E_8 \) compactifications must satisfy \( n_1 + n_2 + n_5 = 24 \). As we have already said, setting \( n_5 \), the number of \( E_8 \times E_8 \) fivebranes, to zero ensures that there is just one tensor multiplet. The DMW models correspond to the symmetric embedding \( n_1 = n_2 = 12 \). In the case of the \( SO(32) \) theory compactified on K3, there is a similar requirement, namely \( n_1 + n_5 = 24 \), where \( n_1 \) is the number of instantons embedded in the \( SO(32) \) group and \( n_5 \) is the number of \( SO(32) \) five-branes (magnetic duals of the heterotic string in ten dimensions). Perturbative string vacua have \( n_5 = 0 \), of course, but by now we have become accustomed to considering non-perturbative possibilities. The GP construction uses the K3 orbifold \( T^4/\mathbb{Z}_2 \), rather than a smooth K3. It turns out that the 16 orbifold singularities each contain a “hidden” instanton, a fact that we will demonstrate by studying the behavior upon blowing up the singularity. The models constructed by GP satisfy the counting rule by introducing eight five-branes in addition to the hidden instantons.

In section 3, we use \( D = 10 \) heterotic - Type I duality \([3]\) to relate the GP models to the \( SO(32) \) heterotic string on K3. From the anomaly polynomial one learns that the nine-brane gauge group maps to a perturbative gauge symmetry of the heterotic theory, and the five-brane gauge group to a non-perturbative gauge symmetry. It is therefore not surprising that Type I T-duality, which interchanges the nine- and five-branes (for a review see ref. \([11]\) ), maps to the heterotic weak/strong duality discussed in ref. \([12]\) and realized in the DMW construction. As discussed in ref. \([12]\), the dual heterotic string is a wrapped five-brane; in the Type I theory we construct both the heterotic string and its dual as D-branes \([13]\). Finally, we show that for one particular GP model, the heterotic dual can be constructed as a free orbifold.

In section 4 we consider the effect of blowing up the \( \mathbb{Z}_2 \) fixed points by turning on twisted sector fields. A count of instanton number implies that there must be one instanton hidden at each fixed point. The gauge bundle that results when the fixed point is blown up is a \( Spin(32)/\mathbb{Z}_2 \) bundle but not an \( SO(32) \) bundle. That is, fields in the vector representation of \( SO(32) \)
cannot be defined. This is not an inconsistency because the Type I string has only tensor representations (in perturbation theory) and one class of spinor representation (nonperturbatively). A calculation of the dimension of instanton moduli space agrees with the GP spectrum.

In section 5 we consider $T$-duality between the $SO(32)$ and $E_8 \times E_8$ heterotic strings on $K3$. We first classify $\mathbb{Z}_2$ subgroups of $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$. We then discuss $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ bundles (of instanton number 24) on $K3$ and the possible $T$-dualities between them. We resolve a puzzle remaining from ref. [8]. The nonperturbative gauge symmetries of the DMW model, which had been tentatively attributed to small $E_8 \times E_8$ instantons, in fact arise from small $Spin(32)/\mathbb{Z}_2$ instantons in the $T$-dual theory. Finally, we construct explicitly the $T$-duality between the $SO(32)$ and $E_8 \times E_8$ strings on $T^4/\mathbb{Z}_2$ orbifolds, with various embeddings of $\mathbb{Z}_2$ in the gauge groups, and in particular complete the connection between the GP and DMW models.

Although we shall not explore the subject here, we should point out that this class of models has also been constructed in F-theory [14]. The key there is to choose the base space of the elliptic fibration to be $\mathbb{P}_1 \times \mathbb{P}_1$. In this description the inversion of heterotic string coupling is realized geometrically as the interchange of the two $\mathbb{P}_1$ factors.

## 2 Anomalies

### 2.1 Anomalies in the $U(16) \times U(16)$ Model

Reference [7] constructed a class of $N = 1$ six-dimensional Type I superstring vacua with maximal gauge group $U(16)_9 \times U(16)_5$. The first $U(16)_9$ factor comes from nine-branes while the second arises when all 8 five-branes are at a single fixed point of the orbifold. The massless spectrum includes the hypermultiplets

$$2 \times \begin{pmatrix} 120, 1 \end{pmatrix}_{(2/4,0)}$$

$$1 \times \begin{pmatrix} 16, 16 \end{pmatrix}_{(1/4,1/4)}$$

\(^2\text{In the heterotic dual the same representations appear, but all perturbatively.}\)
\[ 2 \times (1, 120)_{(0, 2/4)} \]
\[ 4 \times (1, 1)_{(0, 0)} \]
\[ 16 \times (1, 1)_{(0, 0)} \]

where the subscripts refer to the \(U(1)\) charges. \(^3\) In this section we will illustrate the cancellation of anomalies for this special case of maximal gauge group, and in the next we will consider the general case.

In order for the anomalies of the theory (2.1) to be canceled by the standard GS mechanism \([9]\), it is necessary that the anomaly 8-form factorize into a product of two 4-forms. The anomaly is then canceled, in the low energy theory, by the appearance of a gauge-variant counterterm coupling the field \(B_{\mu\nu}\) to one of the two 4-forms.

Recall that for \(N=1\) models in six dimensions cancellation of the coefficient of the \(\text{tr} \, R^4\) term in the anomaly polynomial requires that \(n_H - n_V = 273 - 29n_T\), where \(n_H, n_V, \) and \(n_T\) are the number of hyper multiplets, vector multiplets, and tensor multiplets, respectively. In this paper we will only be considering models with \(n_T = 1\). In addition, the hypermultiplet representations have to appear in such a way as to ensure that all \(\text{tr} \, F^4\) terms can be eliminated. In the theory with the field content given above, the \(\text{tr} \, R^4\) and \(\text{tr} \, F^4\) terms do cancel, and the full gravitational + gauge anomaly may be written, using standard formulas, \([15, 16]\), in the form

\[ I_8 = X^{(9)}_4 \wedge X^{(5)}_4 + X_2 \wedge X^{(9)}_6 + X'_2 \wedge X^{(5)}_6. \] (2.2)

The last two terms can appear only in the presence of \(U(1)\) factors, because they involve \(\text{tr} \, F\), which is nonvanishing for \(U(1)\) only. The various forms appearing here are

\[ X^{(a)}_4 = \text{tr} \, F^2 - \frac{1}{2} \text{tr} \, R^2 \]
\[ X^{(a)}_6 = \frac{8}{3} \left( -\frac{1}{4} \text{tr} \, R^2 \cdot \text{tr} \, F_a + \frac{3}{4} \text{tr} \, F^3_a - \frac{9}{4} \text{tr} \, F_a \cdot \text{tr} \, F^2_a + \frac{3}{2} (\text{tr} \, F_a)^3 \right) \]
\[ X_2 = 4 \text{tr} \, F_0 + \text{tr} \, F_5 \]
\[ X'_2 = 4 \text{tr} \, F_5 + \text{tr} \, F_9 \]

\(^3\)The \(U(1)\) charges are obtained by noting that the endpoints transform as a \(16 + \overline{16}\) of \(U(16)\); the factor of 1/4 is included in order to give a canonical normalization.
where \( F_a \) with \( a = 5, 9 \) refer to the two \( U(16) \) field strength two-forms in the fundamental (16-dimensional) representation. A term of the form \( \text{tr} F_a \) only involves the \( U(1) \) subalgebra. We note that this is not of the usual factorized form, and thus the full anomaly cannot be canceled by exchange of a two-form, which involves GS interactions of the familiar form

\[
\Gamma_{c.t.} = \int B_2 \wedge X_4^{(5)} + a_0 \int X_3^{(9)} \wedge X_3^{(5)} \tag{2.4}
\]

for some (scheme-dependent) constant \( a_0 \). The \( B_2 \) field is the 2-form with gauge invariant field strength \( H = dB_2 - X_3^{(9)} \) where \( X_4^{(a)} = dX_3^{(a)} \).

The counterterm (2.4) succeeds in canceling the first term in (2.2). The remaining two terms may be canceled by additional counterterms (up to gauge-dependent terms)

\[
\Gamma'_{c.t.} = \int B_0^{(9)} X_6^{(9)} + \int B_0^{(5)} X_6^{(5)} \tag{2.5}
\]

if we assign the anomalous transformation laws:

\[
\begin{align*}
B_0^{(9)} &\rightarrow B_0^{(9)} + 4\epsilon_9 + \epsilon_5 \\
B_0^{(5)} &\rightarrow B_0^{(5)} + 4\epsilon_5 + \epsilon_9
\end{align*} \tag{2.6}
\]

under the \( U(1) \) gauge transformations \( A_a \rightarrow A_a + d\epsilon_a \). It follows that the scalar fields \( B_0 \) must appear in the low energy Lagrangian in the combinations \( dB_0^{(a)} - q_{a,b} A_{(b)} \), where \( q_{a,b} \) are the coefficients of the shifts in (2.6). The inhomogeneity of the transformation laws (2.6) implies that the two \( U(1)'s \) are spontaneously broken, just as in four-dimensional theories with anomalous \( U(1)'s \) where a similar mechanism appears [10]. The unbroken symmetry is therefore only \( SU(16) \times SU(16) \). The \( B_0 \) fields are linear combinations of R-R twisted-sector scalars, which we will refer to as \( \phi_I, I = 1, \ldots, 16 \) labeling the orbifold fixed points. By examining string amplitudes, we may identify the fields \( B_0 \) more concretely and deduce their kinetic terms and anomalous couplings (2.5). These all appear at disk order, as does the standard GS term (2.4). In the next section, we will see that these couplings arise also in the (equivalent) boundary-state tadpole formalism. It is easy to infer from diagrams that the \( B_0 \) fields that cancel the \( U(1) \) anomalies are twisted sector R-R states. Consider the mixing of a scalar mode with a \( U(1) \) gauge boson.
The Chan-Paton factor of the $U(1)$ gauge boson is a matrix in the algebra of $SO(32)$ of the form:

$$\lambda \equiv M = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix}$$  \hspace{1cm} (2.7)

($I_{16}$ is a $16 \times 16$ unit matrix). The orbifold projection $R$ acts on the Chan-Paton indices as matrices $\gamma_{R,a}$, which are identical in form to $M$; again the index $a$ denotes either the nine- or five-brane sector. Consider first a coupling of an untwisted closed-string mode to a $U(1)$ gauge boson. Such an amplitude is proportional to $\text{tr} \lambda$, and hence it vanishes trivially. However, insertion of a twist operator in the interior of the disk creates a cut which must be taken to run from its insertion point to the boundary. The fields jump across the cut by the orbifold operation $R$, which includes the matrix $\gamma_{R,a}$ on the Chan-Paton degrees of freedom. The disk amplitude is then proportional to $\text{tr} \gamma_{R,a} \lambda \neq 0$. The R-R twisted state associated to a given fixed point then couples to the $U(1)$ at that fixed point, while each of the sixteen R-R twisted states couple to the $U(1)_9$ gauge boson. We can therefore identify

$$B_0^{(5)} = \phi_1$$

and

$$B_0^{(9)} = \frac{1}{4} \sum_{i=1}^{16} \phi_i.$$  \hspace{1cm} (2.8)

This ensures that eqs. (2.6) are consistent, if $\phi_1$ refers to the special fixed point and each of the other 15 twisted fields shift by $\phi_I \rightarrow \phi_I + \epsilon_9$. These rules generalize in a straightforward way to all other models in Ref. [7], as we will see in the next section. Incidentally, the 4-to-1 ratio of charges in (2.6) has a simple explanation: given the natural normalization of fields in (2.8), it is the only choice consistent with $5 \leftrightarrow 9$ $T$-duality.

At each fixed point the twisted sector includes three NS-NS scalars and one R-R scalar. They belong to a single $N = 1 D = 6$ hypermultiplet, transforming as an $SU(2)_R$ doublet $\Phi$. Because supersymmetry remains unbroken, the Higgs mechanism that gives mass to the vector boson by eating the R-R field must also give mass to the NS-NS superpartners; let us see how this arises from the $D$-terms. For any $U(1)$ gauge symmetry supersymmetry requires that the three one-forms

$$2I^A\epsilon = i\delta\Phi^*\sigma^A d\Phi - i d\Phi^*\sigma^A \delta\Phi$$  \hspace{1cm} (2.9)
be closed. The $SU(2)^R$-triplet $D$-terms are then defined by $dD^A = I^A$. For a linearly realized $U(1)$, $\delta \Phi = i\epsilon q \Phi$ and $D^A = q \Phi^* \sigma^A \Phi$. For the spontaneously broken $U(1)$,
\[
\delta \Phi = \epsilon v \equiv \epsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
Parameterizing the doublet by $\Phi = (\phi - i H^A \sigma^A) v$, this gives $\delta \phi = \epsilon$ and $D^A = H^A$, so that $D^2$ is a mass term for $H$.

We will consider details of blowing up the orbifold points in section 4. Let us note briefly here the following. Because of the above remarks, the $D$-term contains a term linear in $H^A$ as well as the usual quadratic terms for charged fields. In particular the $D$-term for the 9-brane $U(1)$ contains the sum of all the $H^A_I$. Let us go to a generic point in the K3 moduli space by giving vevs to all the geometric moduli of K3. Since the sum of $H^A_I$ is not (generically) zero, then D-flatness requires the gauge groups to be broken, at least to $Sp(8)$. We are now on a large smooth K3, and we can try to identify the remaining massless modes in the $\sigma$-model classical limit. $SO(32)$ has an $SU(2) \times Sp(8)$ maximal subgroup and it is easy to check that if we assume we have 2 instantons in $SU(2)$ we obtain the correct spectrum. This corresponds to the following topological fact. As we will explain in section 4, the point in GP moduli space at which the unbroken gauge group is $U(16)$ (or $SU(16)$ allowing for the mechanism above) corresponds to a certain $U(1)$ instanton of instanton number 16, embedded in a very particular way in $Spin(32)/\mathbb{Z}_2$, on the K3 orbifold. It can be shown that on a smooth K3 manifold, there is no abelian instanton with the given instanton number and embedding in $Spin(32)/\mathbb{Z}_2$ (this is equivalent to the fact that upon blowing up, the orbifold singularities are replaced by two-spheres whose cohomology classes are not anti-self-dual) so the $SU(16)$ must be “Higgsed” if one turns on twisted sector modes to blow up the singularities of the K3 orbifold. The “$SU(2)$ instanton of instanton number two” mentioned above is really an $SO(3)$ instanton; for $SU(2)$ instantons on a smooth K3 surface, the minimum instanton number is 4, but for an $SO(3)$ instanton (of non-zero second Stiefel-Whitney class) the minimum instanton number is 2. (These last assertions can be understood

\footnote{This is an assertion about K3; the smooth, ALE Eguchi-Hansen manifold discussed in section four does have a self-dual holomorphic two-sphere and abelian instanton.}
using an argument given – in the $SO(32)$ case – at the end of section 5.2.)

Repeating the analysis for the 5-branes, and using the $U(1)_5$ D-term, we obtain an $Sp(8)$ unbroken 5-brane group after turning on twisted sector modes, as expected from eight 5-branes on a smooth K3.

### 2.2 Anomalies in the General Case

We now consider the problem in more generality. Ref. [7] described models with nine-brane gauge group $U(16)$ and five-brane gauge group $\prod_I U(m_I) \times \prod_J Sp(n'_J)$. The five-brane unitary factors arise from $m_I/2$ five-branes at fixed point $I$, while the symplectic factors are from 5-branes away from fixed points. The maximum rank is achieved when all 8 five-branes sit at fixed points. We will focus on this case, since the more general case is obtained by spontaneous breaking (moving 5-branes off the fixed points). In particular, we will consider the $U(1)^{16}$ subgroup of the five-brane gauge group, coming from open strings with both ends on the same five-brane.

$T$-duality interchanges five-branes and nine-branes, but the groups above are not $T$-dual because we have omitted the Wilson lines that are dual to the five-brane positions [4]. Let us now rectify this. The Wilson lines in the directions $m = 6, 7, 8, 9$ must satisfy

$$[W_m, W_n] = 0, \quad MW_m M^{-1} = W_m^{-1},$$

*i.e.* flatness and the orbifold projection. Here $M$ is the matrix in (2.7), but it is now regarded as an element of the group. We can take a basis in which the $W_m$ are made up of blocks of two types

$$W_m = \begin{bmatrix}
\mathcal{R}_2(\theta) \\
\pm 1 \\
\mathcal{R}_2^{-1}(\theta) \\
\pm 1
\end{bmatrix},$$

where $\mathcal{R}_2(\theta)$ is a $2 \times 2$ rotation matrix. The structure of large and small blocks must be the same for all $m$, by flatness. The large $2 \times 2$ blocks are
T-dual to the motions of five-branes in the bulk of the orbifold, while the small blocks are T-dual to half-five-branes fixed at orbifold points.

In parallel with the five-brane discussion, we focus for the remainder on Wilson lines consisting entirely of small blocks—T-dual to all five-branes on fixed points. Thus,

$$W_m = \text{diag}(w_{m,i}), \quad w_{m,i} = \pm 1, \quad w_{m,i} = w_{m,i+17}. \quad (2.13)$$

For each fixed Chan-Paton factor $i$, the $(w_{1,i}, w_{2,i}, w_{3,i}, w_{4,i})$ form one of $2^4$ possible sequences, which we label $w_{m,i}, \tilde{I} = 1, \ldots, 16$. The sequences correspond to the fixed points of the T-dual theory, which are Fourier-dual (on $\mathbb{Z}_4^2$) to the original fixed points (after picking a fixed point around which to T-dualize); again, the latter are labeled $I = 1, \ldots, 16$. Note that if $w_{m,i}$ are equal for different $i$, this is T-dual to having several half-five-branes at the same fixed point. In any case, the choice (2.13) gives a maximal number of $U(1)$’s (more generally, Cartan elements of $U(n)$).

Now let us obtain the spectrum. The $U(1)^{16}_5 \times U(1)^{16}_9$ gauge transformations are respectively

$$\epsilon = \begin{bmatrix} 0 & \Delta \\ -\Delta & 0 \end{bmatrix}, \quad \tilde{\epsilon} = \begin{bmatrix} 0 & \tilde{\Delta} \\ -\tilde{\Delta} & 0 \end{bmatrix} \quad (2.14)$$

with $\Delta = \text{diag}(\epsilon_i), \quad \tilde{\Delta} = \text{diag}(\tilde{\epsilon}_i)$. These act on the Chan-Paton wavefunctions $\lambda$ as $\delta \lambda = \epsilon \lambda - \lambda \epsilon$ in the 55-sector, with appropriate tildes in the 99- and 59-sectors. As noted in [7], the 55 scalars have wavefunctions

$$\lambda = \begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix} \quad (2.15)$$

for antisymmetric matrices $A_{1,2}$. The gauge transformations act on these as

$$\delta(A_1 + iA_2) = -i\{\Delta, A_1 + iA_2\}, \quad (2.16)$$

implying charges $\epsilon_i + \epsilon_j$ for $i \neq j$. The content is that of two hypermultiplets so we have counted $i \leftrightarrow j$ as independent. Similarly in the 99-sector we have charges $\tilde{\epsilon}_i + \tilde{\epsilon}_j$ for $i \neq j$.

Now consider the 59-sector with the five-brane at fixed point $I$. This is fixed by the operation $T_{2t}R$, where $R$ is the orbifold $\mathbb{Z}_2$ and $T_{2t}$ is a
lattice translation by twice the coordinate of the fixed point. This acts on
the nine-brane Chan-Paton factor as $W_I \gamma^9_{9,R}$ where $W_I$ is the Wilson line
corresponding to the translation and $\gamma^9_{9,R} = M$.5

Thus the 59 wavefunctions satisfy

$$\lambda = M \lambda M^{-1} W_I^{-1}$$

with solution

$$\lambda = \begin{bmatrix} X_1 & X_2 \\ -X_2 W_I & X_1 W_I \end{bmatrix}.$$ (2.18)

The gauge transformation is

$$\delta (X_1 + iX_2) = i \Delta (X_1 + iX_2) W_I - i (X_1 + iX_2) \Delta.$$ (2.19)

The element $(X_1 + iX_2)_{ij}$ thus has charge $\bar{\epsilon}_j - \epsilon_i w_{I,j}$, where $w_{I,j}$ is the $j^{th}$
diagonal element of $W_I$.

Including the contribution of the hyper and vector multiplets, the $U(1)$
gauge anomaly is

$$- \sum_I \sum_{i,j \in I} (F_i - F_j)^4 - \sum_I \sum_{i,j \in I} (\bar{F}_i - \bar{F}_j)^4$$
$$+ \sum_I \sum_{i,j \in I} (F_i + F_j)^4 + \sum_I \sum_{i \neq j \in I} (\bar{F}_i + \bar{F}_j)^4$$
$$+ \sum_I \sum_{i,j \in I} \sum_{j \neq j' \in I} (F_i - w_{I,j} \bar{F}_{j'})^4$$
$$= 6 \sum_i F_i^2 \sum_j \bar{F}_j^2$$
$$+ \sum_I \left( 4 \sum_{i \in I} F_i - \sum_{j \in I} w_{I,j} \sum_{j' \in I} \bar{F}_{j'} \right) \left( 4 \sum_{i \in I} F_i^3 - \sum_{j \in I} w_{I,j} \sum_{j' \in I} \bar{F}_{j'}^3 \right).$$ (2.20)

We have used $\sum_I w_{I,j} w_{I,j'} = 16 \delta_{j,j'}$. There are also mixed gravitational-$U(1)$ anomalies:

$$-3 \text{tr} \ R^2 \left( \sum_i F_i^2 + \sum_j \bar{F}_j^2 \right).$$ (2.21)

5That is, $T_{2I}$ is a translation by one lattice unit in some subset of the directions $m$, and $W_I$ is a product of those $W_m$. Similarly, $w_{I,j}$, the $j^{th}$ diagonal element of $W_I$, is a product of the corresponding elements $w_{m,j}$, and $w_{I,j}$ is a product of the corresponding $w_{m,j}$ defined below equation (2.13).

6The dimensions of the $M$’s on the two sides are in general different, since $\lambda$ need not be square, but to avoid burdening the notation the same symbol is used.
\[-\frac{1}{16} \text{tr} R^2 \sum_i \left( 4 \sum_{i \in I} F_i - \sum_j w_{I,j} \sum_{j \in j} \tilde{F}_j \right) \left( 4 \sum_{i \in I} F_i - \sum_j w_{I,j} \sum_{j \in j} \tilde{F}_j \right) \]  

These reduce to the anomaly (2.22) when the Wilson lines are removed.

The first term on the right-hand side of eq. (2.20) factorizes in the usual form and can be canceled by the exchange of the $B_{\mu\nu}$ field. The second term can be canceled by exchange of R-R zero-form fields. The last line of eq. (3.30) in ref. [7] gives the tadpole of the R-R twisted sector six-form at fixed point $I$. Generalizing to include the nine-brane Wilson line, it becomes

$$B_{6,I} \left( 4 \text{Tr}_I (\gamma_{R,I}) - \text{Tr}(W_I \gamma_{R,9}) \right).$$  

As noted in ref. [7], the traces vanish automatically once all other constraints are satisfied. In fact, this is simply due to orientation symmetry, under which $B_{6,I}$ is odd and therefore projected out. From this calculation we can easily deduce the additional terms that are needed. As is clear from the boundary-state formalism in gauge backgrounds, one gets all 6-forms that can be formed out of the R-R forms and the gauge field strengths [17]. The nonvanishing terms are thus

$$B_{0,I} \left( 4 \text{Tr}_I (\gamma_{R,I} F^3) - \text{Tr}(W_I \gamma_{R,9} \tilde{F}^3) \right) + B_{4,I} \left( 4 \text{Tr}_I (\gamma_{R,I} F) - \text{Tr}(W_I \gamma_{R,9} \tilde{F}) \right) \propto B_{0,I} \left( 4 \sum_{i \in I} F_i^3 - \sum_j w_{I,j} \sum_{j \in j} \tilde{F}_j^3 \right) + B_{4,I} \left( 4 \sum_{i \in I} F_i - \sum_j w_{I,j} \sum_{j \in j} \tilde{F}_j \right).$$  

with similar terms involving $\text{tr} R^2$. These are of just the form required to cancel the $F F^3$ anomaly found in (2.21). Note that $B_{0,I}$ and $B_{4,I}$ are electric and magnetic potentials for the same field strength. We must thus carry out a duality transformation to put the action in local form, schematically

$$B_0 \wedge F^3 + B_4 \wedge F \rightarrow (dB_0 + A) \wedge *(dB_0 + A) + B_0 F^3.$$  

The modified gauge transformation $\delta B_0 = -\epsilon$ implied by the kinetic term gives an $F F^3$ anomaly. We see that the couplings of the twisted-sector fields are of the precise form needed to cancel the one-loop anomaly.

The $B_4 \wedge F$ coupling (or equivalently $A \wedge *dB_0$) makes the would-be anomalous $U(1)$ fields massive as discussed above. For example, let the five-branes be distributed half per fixed point to give the maximum $U(1)^{16}$. At the
massless level one can think of $B_{4,I}$ as a Lagrange multiplier determining the
gauge field $F_I$ in terms of the five-brane fields, so the gauge group is identical
to the nine-brane group, though the actual fields are linear combinations of
the five- and nine-brane fields. We are considering models with a rank-32
group, which contains at least two $U(1)$’s (the case considered in the previous
section) and at most 32 (when there is one five-brane at each $I$ and one nine-
brane with each $\tilde{I}$). One can see that if there are 16 or fewer $U(1)$’s, all are
broken, while if there are more than 16, exactly 16 are broken.

2.3 A Nonperturbative Anomaly

The models described in the previous section fall into several classes. The
number of half five-branes on each of the 16 fixed points can be odd or even,
and this is fixed because only whole five-branes can move off the fixed point.
Similarly, the number of Chan-Paton factors with each $\tilde{I}$ can be odd or even.
Although these models are all consistent in perturbation theory, there is a
nonperturbative inconsistency in most of the classes. Consider transporting
a charged field around the fixed point at $X^m = 0$, from $X^m$ to $-X^m$. On
this nontrivial path the field picks up the $SO(32)$ transformation $M$.
Now transport a field twice around the fixed point, so that it comes back to itself
times $M^2$. This path is topologically trivial so fields must come back to their
original values. Now $M^2 = -1$, but this is no problem because this is in the
$SO(32)$ vector representation, and there are no fields in this representation.
The matrix $M$ can be thought of as a rotation by $\frac{1}{2}\pi$ in each of the 16 planes
$(k, k + 16)$. It follows that $M^2$ is $+1$ in one spinor representation and $-1$ in
the other. This is precisely the spectrum given by the GSO projection, so
the holonomy is consistent.

Now consider the fixed point $I$. This is fixed by $T_{2I}R$, so the holonomy
is $W_I M$ and the consistency condition is $W_I MW_I M = 1$ in the relevant rep-
resentations. By construction, the Wilson line (2.12) satisfies $MW_I M^{-1} = W_I^{-1}$
and so $W_I MW_I M = M^2$ in the vector representation. In spinor repre-

\[ \text{The matrix } M \text{ has been redefined in eq. (2.7) by a factor of } i \text{ relative to ref. [7] so as to lie in the group } SO(32). \text{ This is irrelevant for the tensor representations that appear in perturbation theory, but necessary to extend the transformation to other representations.} \]
sentations it is not hard to see that the large blocks (being connected to the identity) have no net effect. Consider a small block with $-1$’s in rows $k$ and $k+16$. One can think of this as a rotation by $\pi$ in the plane $(k, k+16)$. This commutes with $M$ and squares to a $2\pi$ rotation. Thus it is trivial in the vector representation but $-1$ in both spinors. If there are an odd number of such blocks, then $W_I MW_I M = -M^2$ in the spinor representation and there is no spinor representation that can be consistently defined at both fixed points. Consistency of the model thus requires that for all $I$ the number of small $-1$ blocks is even. The $T$-dual statement for the five-branes is this: consider any 8 fixed points lying in a 3-plane (such as the 3-plane $x^6 = 0$). The total number of five-branes on these fixed points must be integer.

Since the spinor representations appear only as D-branes, this is a non-perturbative inconsistency. In section 5 we will see that it has a simple topological interpretation, and we will actually find a slightly stronger condition: consider any 4 fixed points lying in a 2-plane (such as the 2-plane $x^6 = x^7 = 0$). The total number of five-branes on these fixed points must be integer.

Taking into account the nonperturbative constraint, there appear to be six connected sets of GP orbifolds. There can be zero half-five-branes, 8 half-five-branes in a 3-plane, or 16 half-five-branes, times the corresponding three sectors of Wilson lines, giving nine sectors that are reduced to six by T-duality. We will see in section 5 that all of them are connected via smooth $K3$’s.

The topological interpretation of the non-perturbative inconsistency that we have described really has to do with the second Stiefel-Whitney class $w_2$ of the $SO(32)$ bundle. In the above discussion, the key point was that a reflection $R$ of $\mathbb{R}^4$ and a translation $T_I$ in the $I$th direction obey geometrically

$$RT_I = T_I^{-1} R.$$ 

In dividing by the group $G$ generated by $R$ and $T_I$ to make a K3 orbifold from $T^4$, we represented $T$ and $T_I$ in the gauge group $SO(32)$ by matrices $M$ and $W_I$, in such a way that the desired relation

$$MW_I = W_I^{-1} M,$$
is satisfied in $SO(32)$ but not after lifting to $Spin(32)/\mathbb{Z}_2$. In such a situation, if $G$ acted freely on $\mathbb{R}^4$, then after dividing $\mathbb{R}^4$ by $G$, one would get a smooth manifold $\mathbb{R}^4/G$ with a flat $SO(32)$ bundle of non-zero $w_2$. Here, we are dealing with a somewhat more abstract string-theoretic version of $w_2$, as the $G$ action is not free.

### 3 Duality: Type I and Heterotic $SO(32)$

In ten dimensions the $SO(32)$ Type I and heterotic strings are dual with a specific transformation of the metric and other fields [3], and so they remain equivalent when compactified on corresponding smooth manifolds. Finding the dual of an orbifold compactification is less straightforward. In the limit of a large orbifold, one can use the adiabatic principle [18]: the dual theory looks geometrically like an orbifold away from the fixed point. Also, by transporting charged fields around a fixed point, it follows that the gauge holonomy (here $M$) is the same in both theories. But it need not be that a free orbifold CFT in one theory maps to a free orbifold theory in the other—there might, for example, be twisted-state backgrounds. This arose in ref. [7], where the heterotic orbifold with spin and gauge connections equal did not have a free-CFT Type I dual. In the present case, we can see immediately that most of the Type I models do not have a free-CFT dual. The point is that the antisymmetric tensor 6-form charge is canceled locally only for one set of models, the $U(1)^{16}$ models with a half-five-brane at each fixed point. For the other models there is a local charge, hence a field strength and a dilaton gradient. This is a higher order (disk) effect in the Type I CFT, but tree level in the heterotic dual.

In section 3.1 we first consider some generalities that do not depend on such details, specifically the relation between Type I $T$-duality and heterotic weak/strong duality [12, 8]. In section 3.2 we study those models which do have free-CFT heterotic duals.
3.1 Type I - Heterotic Duality

The quadratic-quadratic anomaly polynomial is

\[(\text{tr } R^2 - 2\text{tr } \tilde{F}^2)(\text{tr } R^2 - 2\text{tr } F^2), \quad (3.1)\]

where (as before) a tilde is used for the nine-brane fields, though now the trace is in the fundamental representation of \(U(m)\) rather than the Chan-Paton representation. In the heterotic string description, either \(F\) or \(\tilde{F}\) is identified as a perturbative gauge field at level one, whereas the other is identified as a nonperturbative gauge field \([19, 8]\). It follows that \(T\)-duality, which interchanges \(F\) and \(\tilde{F}\) in the Type I description, maps to strong/weak duality \([12, 8]\) in the heterotic description.

To develop this further, let us identify some of the scalars that appear in heterotic and Type I compactifications to six dimensions. The main point is to organize them in terms of supersymmetry multiplets. This gives a preferred (and most convenient) basis. Field redefinitions are constrained by gauge invariance and supersymmetry. Since the tensor multiplet includes a gauge field, the scalar \(\phi\) in the multiplet cannot be redefined by an infinitesimal transformation. The only freedom is to multiply the tensor multiplet by minus one, which acts on the bosons in the multiplet as

\[\phi \rightarrow -\phi, \quad B^- \rightarrow -B^- . \quad (3.2)\]

This transformation on \(B^-\) is equivalent to a duality transformation on the two-form \(B\). In general, this duality transformation is not a symmetry of the Lagrangian. It maps it to another Lagrangian.

We are now going to identify a particular scalar field \(\phi\) (and one hypermultiplet) in two cases:

1. The heterotic string on K3. The hypermultiplets include the moduli of K3, including the radius \(r_h\). Since supersymmetry transformations act simply on the background fields in the world-sheet sigma model, \(r_h\) is the radius in the heterotic string metric. The ten dimensional dilaton \(D_h\) affects the scalar \(\phi\) in the tensor multiplet—the latter is a function of \(D_h\) and \(r_h\). To identify \(\phi\), consider the ten dimensional
Lagrangian expressed in terms of the heterotic string metric. One of the terms there is \( e^{-2D_h} H \cdot H \). Therefore, the coefficient of \( H \cdot H \) in the six dimensional Lagrangian in the string metric is \( r^A_h e^{-2D_h} \). Since this term is Weyl invariant, this is also the answer in the Einstein metric, and hence

\[ r^A_h e^{-2D_h} = e^{-2\phi}. \]  

(3.3)

2. The Type I theory on K3. The scalar \( \phi \) is again a function of two scalars: the ten dimensional dilaton \( D_I \) and the radius of K3, \( r_I \), in the Type I string metric. To identify \( \phi \), consider the ten dimensional Lagrangian expressed in terms of the Type I string metric. Since \( B \) is a RR field, the \( H \cdot H \) term in that Lagrangian is not multiplied by an exponential of \( D_I \). Therefore, the coefficient of \( H \cdot H \) in the six dimensional Lagrangian in the string metric is \( r_I^4 \). Since this term is Weyl invariant, this is also the answer in the Einstein metric, and hence

\[ r_I^4 = e^{-2\phi}. \]  

(3.4)

Note that \( \phi \) is independent of the ten dimensional dilaton \( D_I \). The same conclusion can also be reached by studying the action of spacetime supersymmetry transformations on the world-sheet sigma model. It acts on the moduli without mixing with \( D_I \) which multiplies the two dimensional curvature term. Therefore, \( r_I \) should appear as a field in a separate multiplet. The ten dimensional dilaton \( D_I \) affects the hypermultiplets. A straightforward dimensional reduction shows that the appropriate combination is \( e^{-2D_I} r_I^4 \).

In fact, given the identification of the fields in the heterotic compactification we could have derived it in the Type I compactification using the change of variables in the ten dimensional Lagrangian used in heterotic/Type I duality (for this purpose we need only the change of variables, not the assumption of complete duality)

\[ r_I^2 = r_h^2 e^{-D_h}, \quad D_I = -D_h, \]  

(3.5)
which identifies the two fields as

\[ e^{-2\phi} = r_I^4 = r_h e^{-2D_h}, \quad r_I^4 e^{-2D_I} = r_h^4, \]  

(3.6)
i.e. as the Type I radius and the heterotic radius.

The duality transformation (3.2) in the Type I variables inverts the Type I radius \( r_I \) while holding \( r_h = r_I e^{-D_I/2} \) fixed. This is precisely the action of \( T \)-duality in this theory. So Type I \( T \)-duality is the image of heterotic weak/strong duality, which inverts the six-dimensional heterotic coupling \( e^\phi \). Note that the fact that \( r_h \) is held fixed and therefore \( D_I \) transforms is standard in \( T \) duality, which keeps the Newton constant \( G_N = r_h^4 = r_I^4 e^{-2D_I} \) fixed.

Heterotic duality gives two different perturbative heterotic limits of the GP theory. The latter should therefore have two different heterotic string solitons, one carrying the current algebra of the nine-brane gauge group and one that of the five-brane gauge group, and these should be interchanged by \( T \)-duality. The first is just the Dirichlet one-brane, extended in a non-compact direction. Its \( T \)-dual is the Dirichlet five-brane, wrapped in the compact directions and extended in one non-compact direction. The analysis in GP can be extended to include these objects. The Chan-Paton algebra gives

\[ \gamma_{\Omega,1} = \gamma_{\Omega R,5'} = I, \quad \gamma_{R,1} = \gamma_{\Omega R,1} = \gamma_{R,5'} = \gamma_{\Omega,1} = M. \]  

(3.7)

There are no constraints from divergences, because the R-R flux is free to spread in the non-compact directions. A prime is used to distinguish the heterotic five'-branes, which are wrapped on the \( K3 \) and localized in four non-compact directions, from the GP five-branes, which are extended in all the non-compact directions. Since \( M \) is even-dimensional, the minimal five'-brane thus has a two-valued Chan-Paton index as found in refs. [1, 7]. The minimal one-brane also has a two-valued Chan-Paton index; this simply labels the one-brane, at a given point on the compact space, and its image under \( R \).

Quantization of the one-brane is precisely as in ref. [2], as long as the one-brane is not coincident with a fixed point or five-brane. One finds right- and left-moving oscillations, in both the noncompact and compact directions,
together with right-moving Green-Schwarz superpartners. There are also
real left-moving fermions from the 19-open strings. These carry the 32-valued
nine-brane index. They generate an \( SO(32) \) current algebra, since
the Wilson lines and orbifold projections do not affect the local structure on
the macroscopic one-brane. There is no current algebra from the 15-strings
because these are massive, being stretched.

When the one-brane sits at a fixed point there are additional massless
degrees of freedom from strings stretched between the one-brane and its
image. These are found to be a \( U(1) \) gauge field on the one-brane; also,
the oscillations in the compact directions are enlarged to a complex field
carrying the \( U(1) \). Their right-moving superpartners also become complex,
and a real left-moving fermion appears. Similarly, when the one-brane is
coincident with a five-brane there are massless right- and left-moving fields
in the 15 sector. These carry the 32-valued five-brane index, and the NS
and R strings transform respectively as spinors 2 under the noncompact and
compact \( SO(4) \) tangent groups. Quantum fluctuations of a one-dimensional
object will take it away from these special points, but it may be possible to
fix it by turning on vev’s of the non-generic massless fields, and in this way
find new strings.

For the five'-brane everything is the same by \( T \)-duality, with the five-
branes and nine-branes interchanged. In particular, the transverse position
of the one-brane in the compact direction is related by \( T \)-duality to a \( U(1) \)
gauge field living on the five'-brane tangent to the non-compact directions.

It is straightforward to calculate the tension of the two strings. In the
Type I theory the electric heterotic string (the ten-dimensional one-brane)
has tension
\[
T_e = \frac{1}{\lambda_1} = e^{-D_1} = \frac{r_h^2}{r_1^2}.
\]
(3.8)
The magnetic heterotic string (the ten-dimensional five-brane) wraps around
K3 and therefore its tension is proportional to the volume of K3
\[
T_m = \frac{r_1^4}{\lambda_1} = r_1^4 e^{-D_1} = r_h^2 r_1^2.
\]
(3.9)
Clearly, they are exchanged under \( r_1 \rightarrow 1/r_1 \) holding \( r_h \) fixed, which is our
duality transformation.
3.2 An Orbifold Dual

The only GP theories for which the $H$ charge is canceled locally, and which therefore might map to free theories on the heterotic side, are those with the eight five-branes distributed half per fixed point. Indeed, as we will now describe, there is a consistent $T^4/Z_2$ orbifold of the heterotic string, constructed by embedding $Z_2$ in $Spin(32)/Z_2$ using the matrix $M$; its massless spectrum matches that of a GP model with half a five-brane at each fixed point.

Let us focus on the case with no Wilson lines. First we describe what one sees on the Type I side. The Type I gauge group, taking into account the results of section 2, is $SU(16) \times U(1)$. Turning to the hyper multiplets, the 99-sector contributes two antisymmetric 120’s, and the 59-sector contributes a 16 for each fixed point. The 16 at fixed point $I$ couples to the $U(1)$ linear combination $A_I + \tilde{A}$. It follows from section 2 that the massless gauge bosons have $A_I = -4\tilde{A}$, so the 16 couples to $-3\tilde{A}$; the $U(1)$ charge of each 16 is $-3$, where that of each 120 is $+2$.

Now we consider the heterotic string orbifold in which dividing by the reflection $R$ is accompanied by a transformation $M$ in the gauge group. In the untwisted sector, the projection to $R$-invariant states reduces the gauge group from $SO(32)$ to $U(16)$, while the surviving hypermultiplets (coming from the components of the gauge field tangent to $R^4$) are two 120’s. Note that the holonomy is order 4 in the twisted sector, $M^2 = -1$ being the current algebra GSO projection. The current algebra fermions are thus moded in integers $+\frac{1}{4}$ and the zero point energy on the left side is

$$- \frac{32}{192} - \frac{4}{24} + \frac{4}{48} = -\frac{1}{4}. \quad (3.10)$$

This is level-matched, the massless level having one $\lambda^{a}_{-1/4}$ excitation with $a$ an $SU(16)$ 16 index, showing the consistency of the theory. The zero point shift of the $U(1)$ charge is $-\frac{16}{4}$, giving the net $-3$ as above. Similarly the sector twisted by $M^3$ gives the $\overline{16}$ with charge $+3$. So the twisted sector massless states agree with what is found in the Type I description.
4 Topology Of The GP Model

The GP models are, of course, Type I orbifolds with target space $\mathbb{R}^6 \times T^4/\mathbb{Z}_2$. One expects intuitively that the twisted sector fields supported at the $\mathbb{Z}_2$ fixed points should include blowing up modes associated with a deformation from $T^4/\mathbb{Z}_2$ to a smooth K3 surface; indeed, the closed string spectrum found in \cite{7} contains appropriate twisted sector fields to do the job. There are, however, a variety of puzzles about the connection of the GP model with K3 compactifications that we wish to unravel here.

To explain the issues, we may start by noting the following. A smooth K3 compactification needs a vacuum $SO(32)$ gauge bundle with instanton number 24, to make it possible to obey the familiar equation $dH = \text{tr} \ R \wedge R - \text{tr} \ F \wedge F$. The GP model does not have any explicit instantons in the vacuum. Rather than instantons, the model has five-branes; there are eight five-branes, at arbitrary positions on $T^4/\mathbb{Z}_2$. Since a Type I five-brane is equivalent to the small size limit of an instanton \cite{1}, eight instantons are implicit in the five-branes of the GP model. Since 24 are expected, $24 - 8 = 16$ seem to be missing.

As there are 16 $\mathbb{Z}_2$ orbifold singularities in $T^4/\mathbb{Z}_2$, one might intuitively think that one instanton is “hiding” at each singularity, and that blowing up one of these singularities will bring an instanton out into the open. Understanding this will be our first goal.

The construction in \cite{7}, as we have explained above, involves a twist operator that acts on the $SO(32)$ Chan-Paton label by multiplication by a $32 \times 32$ matrix which in $16 \times 16$ blocks looks like

$$M = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$ \hfill (4.1)

As noted earlier, this matrix obeys not $M^2 = 1$, as one might expect in constructing a $\mathbb{Z}_2$ orbifold, but $M^2 = -1$. Thus, this orbifold would not be possible if the gauge group of the Type I superstring were really $SO(32)$. Its viability depends on the fact that the gauge group is really $Spin(32)/\mathbb{Z}_2$. The

\footnote{As noted in footnote 1, these eight five-branes come from 32 Chan-Paton indices, counting the images under the orbifold $\mathbb{Z}_2$ and world-sheet parity $\Omega$.}
$\mathbb{Z}_2$ in question is generated by an element $w$ of the center of $\text{Spin}(32)$ that acts as $-1$ on the 32 dimensional vector representation, $-1$ on one spinor, of, say, negative chirality, and $+1$ on the other spinor. Only representations with $w=1$ are present in the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string, or equivalently, in the Type I superstring. The matrix $M$ obeys $M^2 = w$, and in the Type I or $\text{Spin}(32)/\mathbb{Z}_2$ heterotic theory, this is equivalent to $M^2 = 1$.

Thus, a $\mathbb{Z}_2$ orbifold such as this one is possible. But its existence depends on the fact that the gauge group is $\text{Spin}(32)/\mathbb{Z}_2$ rather than $\text{SO}(32)$, and we will have to use this fact in comparing the model to what can be seen geometrically.

We begin with some remarks about the difference between $\text{SO}(32)$ and $\text{Spin}(32)/\mathbb{Z}_2$ vector bundles on a manifold $X$. It is possible to have a $\text{Spin}(32)/\mathbb{Z}_2$ vector bundle that is not associated with any $\text{SO}(32)$ vector bundle. This can be achieved if on some two-cycle $S \subset X$, Dirac quantization is obeyed for the adjoint representation, and the positive chirality spinor, but not for the vector or negative chirality spinor. If for some $\text{Spin}(32)/\mathbb{Z}_2$ bundle, precisely the representations that are present for $\text{Spin}(32)/\mathbb{Z}_2$ obey Dirac quantization, then this bundle cannot be derived from an $\text{SO}(32)$ bundle. For an explicit example, suppose that the gauge field lives in an abelian subgroup of $\text{Spin}(32)/\mathbb{Z}_2$, generated by a matrix $Q$ which is the sum of 16 copies of

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\] (4.2)

Suppose moreover that the integrated magnetic flux is $\pi$ – that is precisely one-half of a Dirac quantum. Then Dirac quantization is violated for the vector, but is obeyed for the adjoint or the positive chirality spinor (for which the sum of the 16 $U(1)$ charges is even). This is the basic example of a $\text{Spin}(32)/\mathbb{Z}_2$ bundle that is not associated with an $\text{SO}(32)$ bundle. Note that the fact that $F/2\pi$ has a half-integral number of Dirac quanta for every element of the vector representation is essential in ensuring that Dirac quantization is obeyed for the adjoint and positive chirality spinor representations. This is the reason for using the embedding via $Q$.

Given a $\text{Spin}(32)/\mathbb{Z}_2$ bundle $E$, one can define a mod two cohomology class $\tilde{w}_2(E)$, which assigns the value $+1$ to a two-cycle on which Dirac quanti-
tization is obeyed for the vector representation and $-1$ to a two-cycle on which it is not obeyed. Thus $\tilde{w}_2(E) \in H^2(X, \mathbb{Z}_2)$ measures the obstruction to associating $E$ with an $SO(32)$ bundle. The notation $\tilde{w}_2$ is motivated by the fact that the obstruction to deriving a $Spin(n)$ bundle from an $SO(n)$ bundle $F$ is conventionally called $w_2(F)$ ($w_2$ is the second Stiefel-Whitney class). $\tilde{w}_2$ is quite analogous to $w_2$; in fact, for $n = 8$, $Spin(8)$ triality exchanges them. One might describe $\tilde{w}_2$ as the obstruction to a bundle having “vector structure,” just as $w_2$ is the obstruction to “spin structure.”

### 4.1 Blow-Up of a $\mathbb{Z}_2$ Orbifold Singularity

Now we want to consider the behavior of a $Spin(32)/\mathbb{Z}_2$ bundle near one of the $\mathbb{Z}_2$ orbifold singularities $P \in T^4/\mathbb{Z}_2$. If $P$ is blown up, one gets a two-sphere $S$, of self-intersection number $S \cdot S = -2$. The structure near $S$ looks like the Eguchi-Hansen space, which is an ALE hyper-Kahler manifold $X$ with fundamental group at infinity $\mathbb{Z}_2$. We will call the fundamental group at infinity $\tilde{\pi}_1(X)$. We want to guess what kind of gauge theory on $X$ one gets as a local description near $S$ after just slightly blowing up the singularities of the GP model.

First of all, the Eguchi-Hansen space admits a $U(1)$ instanton field $A$ whose field strength $F = dA$ vanishes at infinity. If we think of $S$ as the complex manifold $\mathbb{P}^1$, then $X$ can be regarded as a complex line bundle over $S$; in fact, $X$ is the total space of the line bundle $O(-2)$. This means that if $Y$ is a fiber of $X \to S$, then

$$\int_S \frac{F}{2\pi} = -2 \int_Y \frac{F}{2\pi}.$$  \hspace{1cm} (4.3)

We want to eventually embed the $U(1)$ instanton in $Spin(32)/\mathbb{Z}_2$, using the matrix $Q$, in such a way that Dirac quantization is obeyed for the adjoint or positive chirality spinor but not for the vector. With this in mind, we normalize $F$ so that

$$\int_S \frac{F}{2\pi} = \frac{1}{2},$$  \hspace{1cm} (4.4)

and hence

$$\int_Y \frac{F}{2\pi} = -\frac{1}{4}.$$  \hspace{1cm} (4.5)
Therefore
\[ \int_X \frac{F \wedge F}{16\pi^2} = -\frac{1}{32}. \]  
(4.6)
(In doing the integral, one can, using (4.5), think of one factor of \( F/2\pi \) as \(-1/4\) of a delta function supported on \( S \), after which the integral over \( S \) is done using (4.4).)

If now this gauge field is embedded in \( \text{Spin}(32)/\mathbb{Z}_2 \) via the embedding \( Q \) of the \( U(1) \) Lie algebra into \( SO(32) \) (that is, using the sum of sixteen copies of (4.2)), then the \( SO(32) \) instanton number becomes
\[ \int_X \text{tr} \frac{F \wedge F}{16\pi^2} = 1. \]  
(4.7)
Thus, this is an instanton of instanton number one that obeys Dirac quantization for the adjoint or the spinor, but not for the vector. It is an instanton without “vector structure.”

Because \( F \) is square-integrable, this instanton approaches a flat connection at infinity. We can determine which flat connection it is. The generator of the fundamental group at infinity \( \tilde{\pi}_1(X) \) is simply a large circle at infinity in \( Y \). The monodromy \( W \) of the connection around this circle is simply \( \exp \int_Y F \), and with the embedding (4.2) and the factor of \(-1/4\) in (4.5), this is equivalent to
\[ W = M, \]
with \( M \) the twisting matrix (4.4) used in [7].

The gauge field that is related – upon slight blowing up – to the GP model must have monodromy \( M \) around the generator of \( \tilde{\pi}_1(X) \), since in dividing by the \( \mathbb{Z}_2 \) that creates this cycle, the Chan-Paton factors were multiplied by the matrix \( M \). We also expect this gauge field to have instanton number one, since as we explained above, in the GP construction there seems to be one “missing” instanton buried in each fixed point. Moreover, it is very natural to suspect that this gauge field must commute with a \( U(16) \) subgroup of \( SO(32) \), because GP models get an unbroken \( U(16) \) gauge symmetry from the nine-branes. (The \( U(16) \) is broken to \( SU(16) \) by quantum corrections discussed in section 2.) The instanton we have constructed does indeed break \( SO(32) \) to \( U(16) \), because \( U(16) \) is the subgroup of \( SO(32) \) that commutes
with $Q$. Moreover, for a $U(16)$-invariant instanton that admits spinors but not vectors, the minimum instanton number is one, since in (4.4) we used the smallest half-integer. (From the analysis below of the dimension of instanton moduli space, it will be clear that an instanton $E$ on $X$ with $\tilde{w}_2(E) \neq 0$ has instanton number at least one even if one does not assume unbroken $U(16)$.) The field we have constructed is clearly the unique $U(16)$-invariant instanton with instanton number one and monodromy $M$ at infinity, and moreover is overdetermined by those properties. We regard these facts as compelling evidence that this is the gauge field related, after slight blow-up, to the structure of the GP model near the orbifold singularities.

4.2 Dimension Of The Moduli Space

To probe somewhat more deeply, we will need to understand some facts about instanton moduli spaces both on the non-compact hyper-Kahler manifold $X$ and on a compact K3 manifold. In general, with a simple gauge group $G$ on a compact four-manifold $Y$ without boundary, the index formula for the dimension of instanton moduli space for instanton number $k$ is

$$\dim \mathcal{M}_k = 4hk - \dim G \left( b_0 - b_1 + b_2^+ \right), \quad (4.8)$$

where $h$ is the dual Coxeter number of $G$, $b_0$ and $b_1$ are the dimensions of the spaces of harmonic zero-forms and one-forms on $Y$, and $b_2^+$ is the dimension of the space of self-dual harmonic two-forms on $Y$. On K3, $b_0 = 1$, $b_1 = 0$, and $b_2^+ = 3$, so the formula becomes

$$\dim \mathcal{M}_k = 4hk - 4\dim G. \quad (4.9)$$

Actually, the formulas $(4.8)$ and $(4.9)$ only coincide with the actual dimension of instanton moduli space if the generic instanton number $k$ field completely breaks the gauge symmetry; this will be so if $k$ is large enough.

Note that if $\pi_1(G) \neq 0$, giving the instanton number $k$ does not uniquely fix the topological class of the instanton; one will also meet two-dimensional characteristic classes such as $w_2$ and $\tilde{w}_2$. (If $G$ is not connected, one also meets one-dimensional characteristic classes.) These do not, however, appear
in the index formula (4.8) (except indirectly via the fact that \( k \) is sometimes shifted from integral values when classes such as \( w_2 \) are present).

Now if one wants to consider not a compact manifold \( Y \) but an ALE hyper-Kahler manifold \( X \), there are a few modifications in the formula. The moduli problem we want is one in which the instanton is required to be flat at infinity. Also, we do not want to divide by global gauge transformations at infinity; this has the happy consequence that moduli space always has the dimension suggested by the index formula, since there are no constant gauge transformations to worry about (and the relevant \( H^2 \) group can likewise be shown to vanish using the fact that the metric is hyper-Kahler).

Another change is that \( k \) might not be an integer (even when \( \pi_1(G) = 0 \)). In fact, in addition to specifying \( k \) (and classes such as \( w_2 \)), a component of instanton moduli space is labeled by the choice of a flat connection at infinity, or equivalently the choice of a representation \( \rho \) (in \( G \)) of the fundamental group at infinity \( \tilde{\pi}_1(X) \). The values of \( k \) are shifted from integers by an amount equal to the Chern-Simons invariant of the flat connection. This has the intuitively expected consequence that if the fundamental group at infinity is \( \mathbb{Z}_n \) (corresponding to the blow-up of a \( \mathbb{Z}_n \) orbifold singularity), then \( k \) is not necessarily an integer but takes values in \( \mathbb{Z}/n \). Note, though, that (for appropriate \( G \)) several choices of \( \rho \) may give the same shift in \( k \) (we give examples later), so specifying \( k \) does not determine the problem.

Finally, and crucially in what follows, the index formula on an ALE space is not obtained simply by shifting \( k \) as needed. There is a crucial contribution involving the eta invariant (for the operator \( d + d^* \) restricted to self-dual forms) of the flat connection \( \rho \) at infinity. This contribution depends only on \( \rho \) (and not on the instanton number or other characteristic classes); in fact, it only depends on how the adjoint representation of \( G \) transforms under \( \rho \). Further, the quantities \( b_0, b_1, \) and \( b_2^+ \) cannot simply be replaced by their \( L^2 \) counterparts (which in fact vanish). One must go back to the index theorem, and use the \( R^2 \) curvature integral which on a compact manifold would equal \( b_0 - b_1 + b_2^+ \), or equivalently the eta invariant of the trivial flat connection at infinity.

In this paper, the only ALE space that we will consider in detail is the
Eguchi-Hansen manifold $X$, with fundamental group at infinity $\mathbb{Z}_2$. The representation $\rho$ just corresponds to the choice of an element $x \in G$ with $x^2 = 1$. The eta invariant is a linear combination of the numbers $n_+$ and $n_-$ of generators of $G$ that are even or odd under $x$; of course, $n_+ + n_- = \dim G$. The $R^2$ curvature integral gives a contribution proportional to $\dim G$. So the terms mentioned in the last paragraph are linear combinations of $n_+$ and $n_-$. The coefficient of $n_+$ is actually zero, since for abelian $G$ the moduli space has dimension zero. The coefficient of $n_-$ is actually such that the dimension of moduli space is

$$\dim \mathcal{M}_k = 4hk - \frac{1}{2}n_-.$$  \hfill (4.10)

As an example, take $G = \text{Spin}(32)/\mathbb{Z}_2$, and set $x$ equal to the matrix $M$ that appeared earlier. As $M$ breaks $\text{Spin}(32)$ to $U(16)$, and $\dim \text{Spin}(32) = 496$, $\dim U(16) = 256$, we have $n_- = 496 - 256 = 240$. Also, for $\text{Spin}(32)$, $h = 30$. The formula thus becomes in this case

$$\dim \mathcal{M}_k = 120(k - 1).$$  \hfill (4.11)

So the $k = 1$ instanton that we constructed earlier with monodromy $M$ at infinity has no moduli. Indeed, none were manifest in the construction of this instanton (and it is not hard to prove directly that there are none). Moreover, the spectrum of the GP model, when all five-branes are safely away from the orbifold singularities, contains no massless $U(16)$ non-singlets that would be naturally interpreted as moduli of the instanton gauge bundle at the singularity. So the fact that in this case $\dim \mathcal{M}_1 = 0$ is further evidence that the instanton we constructed is related to the gauge bundle of the GP model.

5 Duality: Heterotic $SO(32)$ and $E_8 \times E_8$

5.1 Instantons On The ALE Space

For further illustration of these ideas, we will need to understand the various types of $E_8$ or $\text{Spin}(32)/\mathbb{Z}_2$ instantons on the ALE space with fundamental group $\mathbb{Z}_2$ at infinity. For a recent discussion on instantons on ALE spaces from the point of view of string theory see [21].
We will need to classify $\mathbb{Z}_2$ subgroups of $\text{Spin}(32)/\mathbb{Z}_2$ and $E_8$, since the monodromy of the instanton at infinity generates such a subgroup. First, begin with $\text{Spin}(32)/\mathbb{Z}_2$. There are two types of elements of order two in $\text{Spin}(32)/\mathbb{Z}_2$: those that would square to one in $\text{SO}(32)$, and those that would square to $-1$ in $\text{SO}(32)$. A basic topological fact is that the monodromy at infinity on the ALE space is of the second kind – it squares to $-1$ in $\text{SO}(32)$ – if and only if the bundle does not have vector structure, that is $\tilde{w}_2(E) \neq 0$. One might suspect this intuitively, and a proof can go as follows. The region $T$ at infinity in the Eguchi-Hansen space is homotopic to a circle bundle over the two-sphere $S$; this circle bundle has Euler class $-2$ (because $S \cdot S = -2$). Since the Euler class of the bundle reduces to 0 mod 2, the spectral sequence (for the fibration $T \to S$) that computes the mod 2 cohomology of $T$ is trivial, and the pullback $H^2(S, \mathbb{Z}_2) \to H^2(T, \mathbb{Z}_2)$ is an isomorphism. So the bundle lacks vector structure when restricted to $T$ if and only if it lacks vector structure when restricted to $S$. A flat bundle at infinity with monodromy that squares to one in $\text{SO}(32)$ obviously corresponds to a bundle with vector structure at infinity, and from the special case we examined of an instanton without vector structure on $S$ with monodromy $M$ at infinity, it is clear that monodromy that squares to $-1$ corresponds to lack of vector structure at infinity. So in short, the monodromy at infinity squares to $-1$ in $\text{SO}(32)$ if and only if $\tilde{w}_2(E) \neq 0$.

Now to classify the $\mathbb{Z}_2$ subgroups, consider first the case of a bundle with vector structure where we are dealing with $\mathbb{Z}_2$ subgroups of $\text{SO}(32)$. Such a group is generated by a matrix that we can take to be

$$x = \text{diag}(-1, -1, \ldots, -1, 1, 1, \ldots, 1)$$

with $p$ eigenvalues $-1$ and $32 - p$ eigenvalues 1. For this to be in $\text{SO}(32)$ rather than $\text{O}(32)$, $p$ must be even. Requiring $x$ to be of order 2 (and not order 4) when lifted to $\text{Spin}(32)$, $p$ must be divisible by four; after dividing by $\mathbb{Z}_2$ to get $\text{Spin}(32)/\mathbb{Z}_2$, we can identify $x$ with $-x$ and so take $p \leq 16$. The non-trivial cases are thus $p = 4, 8, 12, \text{ and } 16$. This gives four choices of $\mathbb{Z}_2$ subgroup. Of course $n_\pm = p(32 - p)$.

We now want to show that with such monodromy at infinity, the instanton number is $k = p/8 \mod \mathbb{Z}$ (so that integer $k$ corresponds to the three cases
\( p = 0, 8, 16 \) and half-integer \( k \) corresponds to the two cases \( p = 4, 12 \). One method to do this is to simply exhibit a special case of an instanton with that instanton number modulo \( \mathbb{Z} \). Take the gauge group to be \( SO(4) \), and ask for the monodromy at infinity to be \(-1\). With \( SO(4) \) regarded as \((SU(2) \times SU(2))/\mathbb{Z}_2\), take a standard \( SU(2) \) one-instanton solution on \( \mathbb{R}^4 \), centered at the origin, and embedded in one of the \( SU(2) \) factors of \( SO(4) \). Such a field is invariant under the \( \mathbb{Z}_2 \) symmetry \( x^i \rightarrow -x^i \) of \( \mathbb{R}^4 \), and descends to an instanton number \( 1/2 \) field on \( \mathbb{R}^4/\mathbb{Z}_2 \) with monodromy at infinity \(-1 \in SO(4)\)

Thus if the monodromy at infinity has 4 eigenvalues \(-1\), the instanton number is \( 1/2 \) modulo \( \mathbb{Z} \). With \( 4n \) eigenvalues \(-1\) at infinity, one can take \( n \) copies of the half-instanton just described in commuting \( SU(2) \) subgroups of \( SO(4n) \), giving instanton number \( n/2 \) modulo \( \mathbb{Z} \), in agreement with the claim above. The same method, applied to an \( SO(16) \) subgroup of \( E_8 \), can be used to justify the claims made presently for \( E_8 \).

In this analysis of \( \mathbb{Z}_2 \) subgroups of \( SO(32) \), there is one subtlety that we do not have to face. The \( SO(32) \) element \( x \) can be lifted to \( Spin(32)/\mathbb{Z}_2 \) in two different ways (depending on the sign of the action on spinors), but because the gauge bundle with monodromy \( x \) can be considered as an \( SO(32) \) bundle, the instanton number and eta invariant can be computed in \( SO(32) \), and one does not need to worry about the choice of lifting. For bundles without vector structure, the choice of lifting does matter.

Now we consider elements of order two in \( Spin(32)/\mathbb{Z}_2 \) that square to \(-1\) in \( SO(32) \). An \( SO(32) \) matrix that squares to \(-1\) is equivalent to the matrix \( M \). \( M \) can be lifted to \( Spin(32)/\mathbb{Z}_2 \) in two ways, giving two group elements that we will call \( M \) and \( M' \). Note that the formula (4.11) applies equally well to \( M \) or \( M' \) (as they act the same way in the adjoint representation), so in either case the eta invariant shifts the effective instanton number by \(-1\). However, \( M \) and \( M' \) have different values of the allowed instanton number.

Recall that we constructed above an explicit instanton with monodromy \( M \)

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The following facts, which make this clear, may be familiar. The region at infinity in \( \mathbb{R}^4 \) is homotopic to \( S^3 \), which is isomorphic to \( SU(2) \). The \( SU(2) \) one-instanton is asymptotic at infinity to a pure gauge \( A = dg \cdot g^{-1} \), where in a suitable gauge \( g \) is the “identity map” from the \( S^3 \). Therefore, under the \( \mathbb{Z}_2 \) transformation of \( \mathbb{R}^4 \) or \( S^3 \), which acts by “multiplication by \(-1\),” one has \( g \rightarrow -g \), making clear that the monodromy at infinity, after dividing by this \( \mathbb{Z}_2 \), is the element \(-1 \) of \( SU(2) \) or equivalently of \( SO(4) \).
and instanton number 1. This was done by taking in $SO(32)$ a total of 16 commuting copies of an $SO(2)$ instanton with magnetic flux

$$
\int_Y \frac{F}{2\pi} = \frac{1}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

(5.2)

Each $SO(2)$ factor contributed $1/16$ to the instanton number. Now to get monodromy $M'$ instead of $M$, we want to make at infinity an extra $2\pi$ rotation in one of the $SO(2)$ subgroups. This will occur if we add one Dirac quantum to the magnetic flux (integrated over $Y$) in that subgroup, so that one will have

$$
\int_Y \frac{F}{2\pi} = \frac{5}{4} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
$$

(5.3)

This subgroup will now contribute $5^2/16$ to the instanton number, so in going from $M$ to $M'$ the instanton number has changed by $(5^2 - 1^2)/16 = 3/2$, showing that with monodromy $M'$ at infinity, the instanton number is half-integral.

Now we move on to $E_8$. $E_8$ actually has only two subgroups of order two. One of them is obtained by considering a subgroup $(SU(2) \times E_7)/\mathbb{Z}_2$ of $E_8$ and taking the $\mathbb{Z}_2$ generated by the element $-1$ of $SU(2)$. This breaks $E_8$ to $(SU(2) \times E_7)/\mathbb{Z}_2$ and gives half-integer $k$ and $n_- = 112$. The other case is obtained by taking the $\mathbb{Z}_2$ to be the center of a $Spin(16)/\mathbb{Z}_2$ subgroup of $E_8$, breaking $E_8$ to $Spin(16)/\mathbb{Z}_2$. This gives integer $k$ and $n_- = 128$.

To prove that these are the only $\mathbb{Z}_2$ subgroups of $E_8$, note that one can assume that the generator $x$ of $\mathbb{Z}_2$ is in $Spin(16)/\mathbb{Z}_2$, which contains a maximal torus. If $x^2 = 1$ in $SO(16)$, even without dividing by the $\mathbb{Z}_2$, then as above one can realize $x$ as a diagonal $SO(16)$ matrix with $p$ eigenvalues $-1$ and $16 - p$ eigenvalues 1. By arguments as above, the only cases that one needs to consider are $p = 4$ and $p = 8$. These can be seen to correspond respectively to the unbroken groups $(SU(2) \times E_7)/\mathbb{Z}_2$ and $Spin(16)/\mathbb{Z}_2$. One can also consider the case in which in $SO(16)$, $x^2 = -1$, so that $x^2 = 1$ only in $Spin(32)/\mathbb{Z}_2$. This corresponds to the case that $x$ is a matrix $N$ which is the sum of eight blocks of the form

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

(5.4)
(analogous to the \(SO(32)\) matrix \(M\) that entered earlier). The subgroup of \(SO(16)\) left unbroken by \(N\) is \(U(8)\), but additional unbroken symmetries come from generators of \(E_8\) in the spinor of \(SO(16)\). In defining the action of \(N\) on the spinor representation of \(SO(16)\), there is an arbitrary minus sign, and what unbroken group one gets depends on how this sign is chosen. With one choice of sign one gets the unbroken \((SU(2) \times E_7)/\mathbb{Z}_2\) seen earlier, while the other choice gives another way to construct the \(\mathbb{Z}_2\) symmetry with unbroken \(Spin(16)/\mathbb{Z}_2\). We will label these two lifts of \(N\) to \(E_8\) as \(N\) and \(N'\), respectively. This uniform construction of the two inequivalent \(\mathbb{Z}_2\) subgroups of \(E_8\), differing only by how the action of \(N\) is lifted to spinors, will be convenient when we analyze \(T\)-dualities.

The Nonperturbative Inconsistency

We can now obtain a better understanding of the nonperturbative inconsistency found in section 2.3. Thus far we have considered the Dirac quantization condition on the small two-sphere \(S\) located at each orbifold point. There are other closed 2-cycles on \(K3\). Consider four fixed points lying in a plane (for example \(x^6 = x^7 = 0\)); label them by \(\alpha\). There is sphere \(S'\) which intersects each of the \(S_\alpha\) once. It follows that

\[
\int_{S'} \frac{F}{2\pi} = \sum_\alpha \int_{Y_\alpha} \frac{F}{2\pi}.
\]  

For holonomy \(M\), this is just the matrix (5.4) in each \(U(1)\) and so is integer-valued in the positive chirality spinor. However, if \(m\) of the fixed points have holonomy \(M'\), then in one \(O(2)\) one obtains \((m + 1)\) times (5.4), shifting the value in the spinor by \(\pm m/2\). It follows that for \(m\) odd the positive chirality spinor does not satisfy Dirac quantization on \(S'\); there is an obstruction \(\tilde{w}_2\). From the discussion above, the number of instantons in the plane is \(m/2\) plus an integer, so the condition for the positive chirality spinor to exist is that the number of instantons in each plane be an integer. This is slightly stronger than the \((T\text{-dual of the})\) condition found in section 2.3.

To summarize, the nonperturbative inconsistency has a simple origin. A vector bundle which admits tensors but not spinors of \(SO(32)\) can be a
consistent background for the Type I string in perturbation theory, but not nonperturbatively.

5.2 $\text{Spin}(32)/\mathbb{Z}_2$ Instantons on K3

We would now like to study in more detail $\text{Spin}(32)/\mathbb{Z}_2$ instantons on K3. In addition to the instanton number $k$, a $\text{Spin}(32)/\mathbb{Z}_2$ bundle $E$ on K3 is classified by the characteristic class $\tilde{w}_2(E)$, which is the obstruction to $E$ admitting “vector structure.” $\tilde{w}_2(E)$ takes values in $H^2(\text{K3}, \mathbb{Z}_2)$, which has $2^{22}$ elements, so for a fixed K3 there are $2^{22}$ topological classes of $E$ to consider, for given $k$. However, if one classifies $E$'s only up to diffeomorphism (which may be appropriate if one plans to let the gravitational moduli of the K3 vary arbitrarily), there are only a few cases. In fact, being a mod two cohomology class, $\tilde{w}_2$ can be lifted to an integral cohomology class that is well-defined modulo two. Its square (which is even because K3 is a spin manifold) is therefore well-defined modulo four. K3 has a very large diffeomorphism group (see for instance chapter six of [22]), and it can be shown that if $\tilde{w}_2$ is non-zero, its only invariant is the value of $\tilde{w}_2^2$ modulo four. So there are three cases: $\tilde{w}_2 = 0$, which corresponds to the $SO(32)$ bundles that have been assumed in the past; $\tilde{w}_2$ non-zero and $\tilde{w}_2^2$ congruent to 2 modulo four; and $\tilde{w}_2$ non-zero but $\tilde{w}_2^2$ congruent to 0 modulo four. Of the three cases, we will only study two in this paper: the conventional case with $\tilde{w}_2 = 0$, and the bundle relevant to the GP model, which (with $\tilde{w}_2$ supported on two-spheres obtained by blowing up 16 $\mathbb{Z}_2$ fixed points) has $\tilde{w}_2 \neq 0$, but $\tilde{w}_2^2$ congruent to 0 modulo four.

The index formula for the dimension of instanton moduli space says that (regardless of the value of $\tilde{w}_2$)

\[
\dim \mathcal{M}_k = 120k - 992, \quad (5.6)
\]

for $k$ large enough that the generic instanton completely breaks the gauge symmetry. Is this true for the value $k = 24$ that is relevant to K3 compactification of the $\text{Spin}(32)/\mathbb{Z}_2$ heterotic string? A necessary but not sufficient condition is that the right hand side of (5.3) should be positive. For the physical value, $k = 24$, the necessary condition is obeyed, but nonetheless
complete breaking of the symmetry is not possible for the standard bundle with \( \tilde{w}_2 = 0 \). This has been argued as follows. Take an \( SU(2) \) instanton of instanton number 24, embedded in \( SO(32) \) so as to break \( SO(32) \) to \( SU(2) \times SO(28) \). This gives an \( SU(2) \times SO(28) \) theory with 10 copies of the \((2,28)\) of \( SU(2) \times SO(28) \), a spectrum that makes possible Higgsing to an \( SO(8) \) without charged fields.

This argument shows that there is a branch of the moduli space on which the generic unbroken gauge group is \( SO(8) \), but does not show that there is not another branch with, for instance, complete symmetry breaking. The following simple argument shows this and exhibits directly that “vector structure” is the key issue.

Given an \( SO(32) \) bundle \( E \) over \( K3 \) with instanton number \( k \), let \( D \) be the Dirac operator on spinors \emph{with values in the vector representation of} \( SO(32) \). One of the two spin bundles of \( K3 \) is trivial; let us call this the bundle of positive chirality spinors. The index of \( D \), that is the number of zero modes of positive chirality minus the number with negative chirality, is from the index theorem

\[
I = 2(32 - k). \tag{5.7}
\]

For \( k < 32 \) there are thus at least \( 2(32 - k) \) positive chirality zero modes. Let \( \psi \) be such a mode. Using the fact that the curvature of \( K3 \) and of the instanton bundle are both anti-self-dual and that \( \psi \) has positive chirality, one gets

\[
0 = \int_{K3} (D\psi,D\psi) = \int_{K3} (\psi,D^2\psi) = -\int_{K3} (\psi,D_{\alpha}\psi) = \int_{K3} (D_{\alpha}\psi,D^\alpha\psi), \tag{5.8}
\]

with the gamma matrix terms canceling out (since for instance \( \bar{\psi}\Gamma^{\alpha\beta}F_{\alpha\beta}\psi = 0 \) because of considerations of self-duality and chirality). So such a zero mode is covariantly constant, \( D_{\alpha}\psi = 0 \). Since the positive chirality spin bundle of \( K3 \) is of rank two, to get \( 2(32 - k) \) covariantly constant positive chirality spinors, the structure group of the gauge connection on \( E \) must leave fixed a \( 32 - k \) dimensional subspace of the vector representation of \( SO(32) \). So for any \( k \leq 30 \), there is always an unbroken \( SO(32 - k) \).

This argument clearly uses heavily the existence of vector structure, and one may ask what happens for \( \tilde{w}_2 \neq 0 \). For the \( k = 24 \) bundle relevant to
the GP model, complete Higgsing is possible; this is clear from the explicit spectrum found in ref. [7]. This is also true for a $k = 24$ bundle with $\tilde{w}_2^2$ congruent to two modulo four. In fact, one can construct such a bundle with 23 $SO(32)$ instantons, breaking to $SO(9)$; the last instanton can be a $Spin(32)/\mathbb{Z}_2$ instanton, supported on a two-sphere $S$ of $S \cdot S = -2$, as constructed above, and breaking $Spin(32)/\mathbb{Z}_2$ to $U(16)$. If the $SO(9)$ and $U(16)$ are aligned generically in $Spin(32)/\mathbb{Z}_2$, their intersection is trivial, showing that complete Higgsing is possible.

5.3 Comparison to $E_8$ Instantons

Upon toroidal compactification, the $SO(32)$ heterotic string is equivalent to the $E_8 \times E_8$ heterotic string. One may ask to what extent that is also true upon K3 compactification.

In $E_8 \times E_8$ compactification on K3, one may place $12 + n$ instantons in one $E_8$ and $12 - n$ in the other, with $0 \leq n \leq 12$. The generic unbroken gauge symmetry depends on $n$. One generically has complete Higgsing for $n = 0, 1, 2$, while for $n > 2$ there is a generic unbroken gauge group. The $n = 0$ and $n = 2$ models are actually equivalent. This is known from F-theory [14], though it is conceivable that there might exist a more down-to-earth explanation via some sort of T-duality.

We would like to know whether the various $Spin(32)/\mathbb{Z}_2$ models are equivalent to some of the $E_8 \times E_8$ models. The conventional model based on the $SO(32)$ bundle with vector structure has – as we have explained above – a generic unbroken $SO(8)$ symmetry, with no massless charged hypermultiplets. The $E_8 \times E_8$ model with those characteristics is the $n = 4$ model, so one is led to conjecture that (as independently suggested in [14]) the $Spin(32)/\mathbb{Z}_2$ model with vector structure is equivalent to the $E_8 \times E_8$ model with instanton numbers $(16, 8)$. Later we will demonstrate, via T-duality, that this is so.

We would also like to identify GP models with $E_8 \times E_8$ models. In particular, a special case of the GP models, as explained in section 4.2, has a description as a $Spin(32)/\mathbb{Z}_2$ heterotic string orbifold. This model has the gauge group completely broken generically, so the $E_8 \times E_8$ models to which it might be equivalent are $n = 0$ and 1. In fact, we will argue later using
T-duality that this particular $Spin(32)/\mathbb{Z}_2$ model is equivalent to the $n = 0$ model, that is to the $E_8 \times E_8$ model with equal instanton numbers in the two $E_8$'s.

Comparison to $E_8 \times E_8$ Perturbation Theory

This last-mentioned equivalence actually makes it possible to resolve some puzzles that were left hanging in [8]. We will pause to explain this here, before going on in the next subsection to construct the T-dualities by which the $Spin(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ models can be related. In [8], it was shown that the $E_8 \times E_8$ model with $n = 0$, that is with instanton numbers $(12, 12)$, has a strong-weak coupling duality that exchanges perturbative massless gauge fields, which arise via conventional symmetry restoration when Higgs expectation values are turned off, with massless gauge fields that arise non-perturbatively at singularities.

An attempt was made in section 4 of [8] to match particular unbroken gauge groups with particular singularities, associated with small instantons. The paradox is that extensive numerical evidence was offered for such matching, which however was based on assumptions that did not appear sound.

For example, it was proposed that un-Higgsing of a perturbative $Sp(n)$ gauge group for $n = 1, 2, 3$ was dual to the collapse of $n$ small instantons at a point. This proposal neatly fits the facts if the small instantons in question generate just the non-perturbative gauge groups that actually arise from small $Spin(32)/\mathbb{Z}_2$ instantons. Since the model in question was an $E_8 \times E_8$ model, it was assumed in [8] that the small instantons would have to be small $E_8$ instantons. But it is now clear [23, 3] that small $E_8$ instantons behave in a very different (and more exotic) way than small $Spin(32)/\mathbb{Z}_2$ instantons. The fact that the $(12, 12)$ $E_8 \times E_8$ model turns out to be equivalent to a $Spin(32)/\mathbb{Z}_2$ model resolves the contradiction. Because of this relation, in addition to singularities due to small $E_8$ instantons, the model also has singularities due to small $Spin(32)/\mathbb{Z}_2$ instantons. The facts presented in [8] all fit neatly if we reinterpret the claim to be that the restoration of a perturbative $Sp(n)$ symmetry is dual to the collapse of $n$ coincident $Spin(32)/\mathbb{Z}_2$ instantons.

Another contradiction in [8] concerned the special case of $n = 3$, that is
un-Higgsing of an $Sp(3)$ subgroup. This should be dual to collapse of three instantons at a point. If they are $E_8$ instantons, then in the $(12,12)$ model, this leaves only 9 instantons in one of the two $E_8$’s, which would lead to the un-Higgsing of a perturbative $SU(3)$ group that is not predicted by the duality. Resolutions of this puzzle suggested in [8] were not very convincing, but the situation is now clear: the instantons in question are $Spin(32)/\mathbb{Z}_2$ instantons, and collapse of three of them need not lead to restoration of any perturbative gauge symmetry.

We can likewise now resolve a number of contradictions in the discussion in [8] of the restoration of an $SU(n)$ gauge symmetry, for $n = 3, \ldots, 6$. This was interpreted in terms of the collapse of $n/2$ instantons at a $\mathbb{Z}_2$ orbifold singularity. The difficulty here is that in the numerical evidence in [8], it was necessary to claim that the moduli space of $E_8$ instantons of instanton number $n/2$ on the Eguchi-Hansen space has dimension $60n$, for even or odd $n$. While this is true for even $n$ (provided we take the monodromy at infinity to be trivial – recall that there is another choice that gives integral instanton number), it is, because of the correction involving the eta invariant, false for odd $n$. The correction is $-n_-/2 = -56$, given that $n_- = 112$ for the monodromy at infinity that gives half-integral instanton number.

If, however, one reinterprets the discussion in terms of $Spin(32)/\mathbb{Z}_2$ instantons, then all becomes clear. We consider in the neighborhood of a $\mathbb{Z}_2$ orbifold singularity a bundle without vector structure (since that is the sort of bundle that arises in the GP model near orbifold singularities, as we have discussed). The monodromy at infinity can therefore be $M$ or $M'$, corresponding as we have seen to integer or half-integer instanton number. The eta invariant is not zero, but has the effect, as we calculated in (4.11), of shifting the instanton number by one. Thus, we modify the proposal in [8] to assert that the un-Higgsing of an $SU(n)$ perturbative gauge symmetry, for $n = 3, \ldots, 6$, is dual to collapse of $1 + n/2$ $Spin(32)/\mathbb{Z}_2$ instantons at a $\mathbb{Z}_2$ orbifold singularity without vector structure. Not only is this revised proposal free of the previous contradictions, but (given the $T$-duality that we will discuss in the next subsection) it is indeed in agreement with the spectrum found in ref. [9]. There it was found (without restriction to $n \leq 6$)
that with $n/2$ five-branes (equivalent to $2n$ Chan-Paton indices) at an $A_1$ orbifold singularity without vector structure, one gets an $SU(n)$ gauge symmetry. We claim that there is an instanton hidden in this kind of orbifold singularity even before five-branes come near, so the total instanton number carried by the orbifold singularity to give $SU(n)$ symmetry is indeed $1 + n/2$.

5.4 T-Dualities between K3 Compactifications

Here we will, finally, justify the claims made earlier about $T$-dualities between certain K3 compactifications of the $\text{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ heterotic strings.

It is helpful first to recall how the $\text{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ theories are related after compactification on $\mathbb{R}^3 \times S^1$. One can interpolate in three steps from the vacuum with unbroken $\text{Spin}(32)/\mathbb{Z}_2$ to the vacuum with unbroken $E_8 \times E_8$:

(a) Starting with unbroken $\text{Spin}(32)/\mathbb{Z}_2$ with a very large $S^1$, one continuously turns on a Wilson line that breaks this group to a subgroup that is locally $SO(16) \times SO(16)$. The requisite Wilson line is a diagonal matrix $W = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ with 16 eigenvalues 1 and 16 eigenvalues $-1$.

(b) Then one makes an $r \to 1/r$ transformation on the $S^1$. In other words, one continuously reduces the radius of the $S^1$ until it is very small; at that point, the theory is better described via a $T$-duality transformation which makes $r$ large again. The dual theory is an $E_8 \times E_8$ theory with a Wilson line $W'$ that breaks $E_8 \times E_8$ to a subgroup that is locally $SO(16) \times SO(16)$. The Wilson line in question is a product, in each $E_8$, of the group element $(-1)^F$ corresponding to a $2\pi$ rotation in an $SO(16)$ subgroup; this group element acts as $+1$ in the part of the adjoint representation of $E_8$ that transforms as the adjoint of $SO(16)$, and $-1$ on the rest.

(c) Finally, one can continuously turn off the Wilson line $W'$ and restore the $E_8 \times E_8$ gauge symmetry.

Note that, though all three steps are needed to interpolate from unbroken $\text{Spin}(32)/\mathbb{Z}_2$ to unbroken $E_8 \times E_8$, step (b) is the only step that is really necessary to show that the $\text{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ theories are equivalent. Step (b) shows by itself that the $\text{Spin}(32)/\mathbb{Z}_2$ theory with the gauge group
weakly broken by a certain Wilson line (Wilson line symmetry breaking is
weak if \( r \) is large) is continuously connected to the \( E_8 \times E_8 \) theory with the
gauge group similarly weakly broken. We mention this because, when we get
to K3 orbifolds, there may be an obstruction to steps (a) or (c); it is only
step (b) that we need to implement.

**Model with Vector Structure**

In actually studying \( T \)-dualities between \( Spin(32)/\mathbb{Z}_2 \) and \( E_8 \times E_8 \) het-

erotic strings, we will consider K3 manifolds constructed as \( \mathbb{Z}_2 \) orbifolds of a

four-torus \( T^4 \). On the four-torus we will sometimes have Wilson lines.

When one divides the four-torus by \( \mathbb{Z}_2 \), one must also make a \( \mathbb{Z}_2 \) twist in
the gauge group in order to preserve level matching. We first consider the
case of a \( Spin(32)/\mathbb{Z}_2 \) model with vector structure, so that the generator \( x \)
of the \( \mathbb{Z}_2 \) twist can be regarded as an element of \( SO(32) \) (but \( x \) is equivalent
to \( -x \) because the gauge group is really \( Spin(32)/\mathbb{Z}_2 \)). In the fermionic
construction of the \( Spin(32)/\mathbb{Z}_2 \) heterotic string, the \( SO(32) \) is carried by
32 left-moving Majorana-Weyl fermions, and the twist simply multiplies \( p \)
of them by \( -1 \); because \( x \) is equivalent to \( -x \) we can take \( p \leq 16 \). For
level matching, \( p \) must be congruent to 4 modulo 8, so the possibilities are
\( p = 4 \) and \( p = 12 \). \( p = 4 \) corresponds to the “standard embedding of the
spin connection in the gauge group” (and breaks \( SO(32) \) to \( SO(28) \times SO(4) \),
which becomes \( SO(28) \times SU(2) \) if one replaces the orbifold by a smooth K3),
but we will here consider the other case \( p = 12 \).

The \( p = 12 \) model has unbroken \( SO(20) \times SO(12) \) (where the two fac-
tors act respectively on left-moving fermions that are even or odd under \( x \)).
The spectrum of massless hypermultiplets consists of a \( (20, 12) \) from the un-
twisted sector and sixteen \( (1, 32) \) half-hypermultiplets from twisted sectors
(the \( 32 \) is a chiral spinor of \( SO(12) \)). The unbroken gauge group of this model
after generic Higgsing (at least on one obvious branch) is easily determined.
After using the \( (1, 32) \)’s to completely break the \( SO(12) \), the \( (20, 12) \) suf-
fices to break \( SO(20) \) down to \( SO(8) \). So the model, at least on this branch,
has a generic unbroken \( SO(8) \), as expected for a K3 compactification of the
\( Spin(32)/\mathbb{Z}_2 \) heterotic string with vector structure.

Now we want to turn on a Wilson line \( W \) that breaks \( SO(32) \) to \( SO(16) \times
$SO(16)$ (and together with $x$ breaks $SO(32)$ to a smaller group). To continuously turn on $W$, we write

$$W = \exp(\pi b),$$

(5.9)

where $b$ is an $SO(32)$ generator that is conjugate to eight copies of

$$
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$

(5.10)

plus a $16 \times 16$ identity matrix. Since $x$ multiplies all four coordinates of the four-torus by $-1$, we can continuously turn on $W$ by setting $W_t = \exp(\pi tb)$, $0 \leq t \leq 1$, provided

$$xb = -bx.$$  

(5.11)

This can be achieved as follows. Break up the 32 fermions of the $Spin(32)/\mathbb{Z}_2$ heterotic string into two groups of 16, where $x$ has four eigenvalues $-1$ in the first group and eight in the second group. Take $b$ to be zero in the first group, and in the second group (in a basis in which $x$ is $-1$ on the first eight basis elements and $+1$ on the others) take $b$ to be the matrix

$$b = 
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
$$

(5.12)

in eight by eight blocks. This $b$ has the desired properties.

So starting with this $\mathbb{Z}_2$ orbifold of the $Spin(32)/\mathbb{Z}_2$ heterotic string, we can continuously turn on the Wilson line $W$, implementing step (a) in the above scenario. Then we can implement the crucial step (b), making an $r \to 1/r$ transformation on the circle that has the Wilson line, thereby mapping the $Spin(32)/\mathbb{Z}_2$ model to an $E_8 \times E_8$ model.

The $E_8 \times E_8$ heterotic string has a convenient fermionic construction with two groups of 16 left-moving free fermions and a separate GSO-like projection on each. When the $E_8 \times E_8$ model is obtained as just described, the matrix $x$ acts with four eigenvalues $-1$ on the first group and with eight on the second. Such elements of $E_8$ were discussed in our classification of $\mathbb{Z}_2$ subgroups of $E_8$, and break $E_8$ to $SU(2) \times E_7$ and $SO(16)$, respectively. After arriving at an $E_8 \times E_8$ model in this fashion, we also have a Wilson line that would break $E_8 \times E_8$ to $SO(16) \times SO(16)$ and breaks $SU(2) \times E_7 \times SO(16)$.
to $SU(2) \times SU(2) \times SO(12) \times SO(8) \times SO(8)$. (Note that the unbroken subgroup of the second $E_8$ is $SO(8) \times SO(8)$, since this is the intersection of the two $SO(16)$’s that commute respectively with $x$ and with $W$.)

One may ask whether the Wilson line symmetry breaking can be turned off to get a pure $\mathbb{Z}_2$ orbifold of the $E_8 \times E_8$ heterotic string. To do this, one wants to write in each $E_8$

$$(-1)^F = e^{\pi b}, \quad (5.13)$$

where $b$ is an $E_8$ generator with $xb = -bx$; this enables one to interpolate from the 1 to $(-1)^F$ via $W_t = \exp(\pi t b)$, as before. In a basis in which the $-1$ eigenvalues of $x$ are the first four of the first group of sixteen fermions and the first eight in the second group, one can take $b$ to generate a rotation of the 4-5 plane of the first group and the 8-9 plane of the second.

**Model without Vector Structure**

Now we want to relate $E_8 \times E_8$ compactification with instanton numbers $(12,12)$ to a $Spin(32)/\mathbb{Z}_2$ model without vector structure.

It will be convenient to start on the $E_8 \times E_8$ side. The way we will ensure that an orbifold corresponds to a $(12,12)$ compactification is by showing that it is symmetric between the two $E_8$’s.

There is actually no $\mathbb{Z}_2$ orbifold of the four-torus that is symmetric in the two $E_8$’s. To see this, use the fermionic construction of $E_8 \times E_8$, with two groups of sixteen fermions. The two possible $\mathbb{Z}_2$ generators of $E_8$, say $N$ and $N'$, can be realized by matrices that act as $-1$ on four or eight fermions, respectively, breaking $E_8$ to $SU(2) \times E_7$ or $SO(16)$. Level matching requires that the number of twisted fermions is 4 modulo 8, so the possible $\mathbb{Z}_2$ twists are $N \times 1$ (the standard embedding of the spin connection in the gauge group, with instanton numbers $(24,0)$) or $N \times N'$, with unbroken gauge group $SU(2) \times E_7 \times SO(16)$. No level-matched choice is symmetric between the two $E_8$’s.

There is, however, a $\mathbb{Z}_2$ orbifold with an additional $\mathbb{Z}_2$ Wilson line (one might describe it as a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold) that is symmetric in the two $E_8$’s. (This was pointed out independently in [24].) To see this, use another construction of $N$ and $N'$ that was described in our classification of $\mathbb{Z}_2$ subgroups.
In this construction, \( N \) is an \( SO(16) \) matrix that breaks \( SO(16) \) to \( U(8) \), and \( N' = (-1)^F N \).

Consider a \( \mathbb{Z}_2 \) orbifold of \( T^4 \) with the \( \mathbb{Z}_2 \) acting on gauge fermions by \( N \times N' \). This is equivalent to the model described two paragraphs ago with unbroken gauge group \( SU(2) \times E_7 \times SO(16) \), though the unbroken subgroup of \( SO(16) \times SO(16) \) is only \( U(8) \times U(8) \). It is not symmetric in the two \( E_8 \)'s. Consider, however, a related model with in addition a \( \mathbb{Z}_2 \) Wilson line \( (-1)^F \times (-1)^F \) in, say, the \( x^{10} \) direction, which we take to have period one.

Thus, the transformation \( x^{10} \rightarrow -x^{10} \) (with also inversion of \( x^7, \ldots, x^9 \)) acts in \( E_8 \times E_8 \) as \( N \times N' \). The transformation \( x^{10} \rightarrow x^{10} + 1 \) acts as \( (-1)^F \times (-1)^F \). The combined transformation \( x^{10} \rightarrow -x^{10} + 1 \) therefore acts as \( N(-1)^F \times N'(-1)^F \). But this equals \( N' \times N \), which differs from \( N \times N' \) by exchange of the two \( E_8 \)'s. Moreover, \( x^{10} \rightarrow -x^{10} + 1 \) is conjugated to \( x^{10} \rightarrow -x^{10} \) by the symmetry \( x^{10} \rightarrow x^{10} + 1/2 \). The conclusion, then, is that this particular \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) orbifold (or \( \mathbb{Z}_2 \) orbifold with Wilson line) is invariant under exchange of the two \( E_8 \)'s accompanied by \( x^{10} \rightarrow x^{10} + 1/2 \). Therefore, it corresponds to a K3 compactification with equal instanton numbers \((12, 12)\).

(One can verify that the massless hypermultiplet spectrum makes complete Higgsing possible.)

Moreover, because the Wilson line we used is just the one that by itself would break \( E_8 \times E_8 \) to \( SO(16) \times SO(16) \), we are in the right situation to interpolate to \( Spin(32)/\mathbb{Z}_2 \): all we need to do is to make the usual \( r \rightarrow 1/r \) transformation in the direction with the Wilson line. In the resulting \( Spin(32)/\mathbb{Z}_2 \) model, the twist \( N \times N' \) (which is represented on the 32 fermions by the direct sum of 16 copies of the \( 2 \times 2 \) matrix in (5.10)) is our friend \( M \), the \( \mathbb{Z}_2 \) twist of \( Spin(32)/\mathbb{Z}_2 \) that forbids vector structure. So we have succeeded in showing that the \( E_8 \times E_8 \) compactification with equal instanton numbers is equivalent to a \( Spin(32)/\mathbb{Z}_2 \) compactification without vector structure.

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