Abstract

Recently, Behr [1] introduced a notion of the chromatic index of signed graphs and proved that for every signed graph \((G, \sigma)\) it holds that

\[ \Delta(G) \leq \chi'(G, \sigma) \leq \Delta(G) + 1, \]

where \(\Delta(G)\) is the maximum degree of \(G\) and \(\chi'\) denotes its chromatic index.

In general, the chromatic index of \((G, \sigma)\) depends on both the underlying graph \(G\) and the signature \(\sigma\). In the paper we study graphs \(G\) for which \(\chi'(G, \sigma) = \Delta(G)\) (respectively, \(\chi'(G, \sigma) = \Delta(G) + 1\)) for all possible signatures \(\sigma\).

We prove that all wheels, necklaces, complete bipartite graphs \(K_{r,t}\) with \(r \neq t\) and almost all cacti graphs are of class \(1^\pm\). Moreover, we give sufficient and necessary conditions for a graph to be of class \(2^\pm\), i.e. we show that these graphs must have odd maximum degree and give examples of such graphs with arbitrary odd maximum degree bigger than 1.
1. Introduction

In this paper we only consider simple, finite and undirected graphs. Graph $G$ has a set of vertices $V(G)$ and a set of edges $E(G)$ with $n(G)$, $m(G)$ denoting their cardinalities, respectively. By $\deg_G(v)$ we denote degree of a vertex $v$ in a graph $G$ and by $\Delta(G)$ the maximum degree of all vertices of $G$. Incidence is a pair $(v, e)$, where $v$ is a vertex, $e$ is an edge and $v$ is one of the endpoints of $e$. Incidence $(v, e)$ will be shortly denoted by $v: e$ and the set of all graph's incidences will be denoted by $I(G)$. All other definitions and symbols are consistent with those defined in Diestel [2].

Signed graphs were introduced in 1950s by Harary [3] as a generalization of simple graphs. Their main purpose was to better model social relations of disliking, indifference and liking. A signed graph is a pair $(G, \sigma)$ where $G$ is a graph and $\sigma: E(G) \to \{\pm 1\}$ is a function. $G$ and $\sigma$ are called the underlying graph of $(G, \sigma)$ and the signature of $(G, \sigma)$, respectively. Edge $e \in E(G)$ will be called positive (respectively, negative) if and only if $\sigma(e) = 1$ (respectively, $\sigma(e) = -1$). Cycle in $(G, \sigma)$ is called positive (respectively, negative) if product of signs of its edges is positive (respectively, negative). A signed graph with only positive cycles is balanced otherwise it is unbalanced.

Switching of a set $V' \subseteq V(G)$ of a signed graph $(G, \sigma)$ is an operation resulting in a signed graph $(G, \sigma')$ such that for every edge $uv \in E(G)$ we have

$$\sigma'(uv) = \begin{cases} -\sigma(uv), & \text{if exactly one of } u, v \text{ belongs to } V', \\ \sigma(uv), & \text{otherwise.} \end{cases}$$

For example, switching a vertex, i.e. switching a subset of vertices of cardinality one, negates the signs of its incident edges. If a signed graph $(G, \sigma')$ can be obtained by switching some of the vertices of $(G, \sigma)$, we say that $(G, \sigma')$ and $(G, \sigma)$ are switching equivalent. It is well-known that switching equivalence is an equivalence relation in the set of all signed graphs with a fixed underlying graph.

In 2020, Behr [1] introduced a concept of edge coloring of signed graphs as a generalization of ordinary graph edge coloring. Let $n$ be a positive integer and

$$M_n = \begin{cases} \{0, \pm 1, \ldots, \pm k\}, & \text{if } n = 2k + 1, \\ \{\pm 1, \ldots, \pm k\}, & \text{if } n = 2k. \end{cases}$$

An $n$-edge-coloring of a signed graph $(G, \sigma)$ is a function $f: I(G) \to M_n$ such that $f(u: uv) = -\sigma(uv)f(v: uv)$ for each edge $uv \in E(G)$ and $f(u: e_1) \neq f(u: e_2)$ for all edges $e_1 \neq e_2$ such that $u: e_1, u: e_2 \in I(G)$. By $\chi'(G, \sigma)$ we denote the chromatic index of a signed graph $(G, \sigma)$, i.e. the smallest $n$ for which $(G, \sigma)$ has an $n$-edge-coloring.
Behr [1] proved that every signed path can be colored using 2 colors and a signed cycle can be colored with 2 colors if and only if it’s balanced. The main result of Behr’s article [1] is the generalized Vizing’s theorem. We will call it the Behr’s theorem to emphasize the importance of Behr’s achievement.

Theorem 1 (Behr [1]). \( \Delta(G) \leq \chi'(G, \sigma) \leq \Delta(G) + 1 \) for all signed graphs \( (G, \sigma) \).

In this paper we introduce two new classes of graphs, namely \( 1^{\pm} \) and \( 2^{\pm} \), such that graph \( G \) is of class \( 1^{\pm} \) (respectively, \( 2^{\pm} \)) if and only if \( \chi'(G, \sigma) = \Delta(G) \) (respectively, \( \chi'(G, \sigma) = \Delta(G) + 1 \)) for all possible signatures \( \sigma \). In Section 2 we relate these classes to previously known notions. We also show sufficient and necessary conditions for a regular signed graph \( (G, \sigma) \) to be \( \Delta(G) \)-edge-colorable and use these results to obtain sufficient and necessary conditions for a regular signed graph to be of class \( 2^{\pm} \). Next, we show that graphs of class \( 2^{\pm} \) must be of odd maximum degree and there are such graphs for any greater than 1 odd value of the degree. In Section 2 we prove that all cacti except cycles are of class \( 1^{\pm} \). Section 4 deals with wheels—they are all of class \( 1^{\pm} \). In Section 5 we show that all necklaces except cycles are of class \( 1^{\pm} \). In Section 6 we deal with complete bipartite graphs by showing that graphs \( K_{r,t} \) are of class \( 1^{\pm} \) for any \( r \neq t \). The paper ends with some open problems and conjectures.

2. \( 1^{\pm} \) and \( 2^{\pm} \)

Behr [4] defined class ratio \( C(G) \) of graph \( G \) to be the number of possible signatures \( \sigma: E(G) \to \{\pm1\} \) such that the signed graph \( (G, \sigma) \) is \( \Delta(G) \)-colorable, divided by the number of all possible signatures defined on \( E(G) \), i.e. \( 2^{m(G)} \). The class ratio is a rational number satisfying \( 0 \leq C(G) \leq 1 \) and, by previous definitions, \( G \) is of class \( 1^{\pm} \) (respectively, \( 2^{\pm} \)) if and only if \( C(G) = 1 \) (respectively, \( C(G) = 0 \)). Clearly, there are graphs that are neither of class \( 1^{\pm} \) nor \( 2^{\pm} \).

The obvious example is a cycle—all signed cycles are 2-colorable if and balanced and require 3 colors otherwise. It is also easy to see that if \( G \) is of class \( 1^{\pm} \) (\( 2^{\pm} \)), then it is of class \( 1 \) (2), where class \( 1 \) (2) is defined as all graphs \( G \) whose ordinary chromatic index equals \( \Delta(G) \) (\( \Delta(G) + 1 \)). This relation provides justification for our notation: class \( X^{\pm} \) is both a signed counterpart of class \( X \) and a subclass of \( X \).

Recall that graph \( G \) is \( r \)-regular if and only if \( \deg_G(v) = r \) for all \( v \in V(G) \). Recall also that \( M \subseteq E(G) \) is a (perfect) matching in \( G \) if every vertex of \( G \) is incident to at most (exactly) one edge of \( M \).

Lemma 2. Let \( G \) be a \( k \)-regular graph and \( \sigma: E(G) \to \{\pm1\} \) be a signature. \( \chi'(G, \sigma) = \Delta(G) \) if and only if one of the following conditions holds:

1. \( k = 2r \) where \( r \) is a positive integer and \( (G, \sigma) \) admits a decomposition into exactly \( r \) spanning edge disjoint \( 2 \)-regular balanced subgraphs.
2. \( k = 2r + 1 \) where \( r \) is a positive integer and \( (G, \sigma) \) admits a decomposition into a perfect matching and exactly \( r \) spanning edge disjoint \( 2 \)-regular balanced subgraphs.
Proof. We prove the equivalence by proving both implications.

(⇐) In the \( k = 2r \) case, each of the \( r \) spanning edge disjoint \( 2 \)-regular balanced subgraphs can be colored using exactly 2 colors, so in total \( 2r \) colors are sufficient to color the whole graph. In the \( k = 2r + 1 \) case, the matching can be colored using color 0, each of the \( r \) spanning edge disjoint \( 2 \)-regular balanced subgraphs can be colored using 2 colors, so in total \( 2r + 1 \) colors are also sufficient. Since \( \Delta(G) = 2r \) in the first case and \( \Delta(G) = 2r + 1 \) in the second, we get \( \chi'(G, \sigma) = \Delta(G) \).

(⇒) We consider the two cases separately:

(1) \( G \) is a \( 2r \)-regular graph, so \( m(G) = r \cdot n(G) \). Let \( c \) be an arbitrary \( \Delta(G) \)-edge-coloring of \((G, \sigma)\). For every pair of opposite colors used by \( c \), edges colored with them form a subgraph with maximum degree not exceeding 2. Clearly, all such subgraphs are edge disjoint. Each of the subgraphs has at most \( n(G) \) edges. There are exactly \( r \) such subgraphs in \( G \) and \( G \) has \( r \cdot n(G) \) edges, so every subgraph contains exactly \( n(G) \) edges. If a subgraph with maximum degree not exceeding 2 contains \( n(G) \) edges, it clearly is spanning and \( 2 \)-regular. If it can be colored with two colors, it has to be balanced, too.

(2) \( G \) is a \((2r + 1)\)-regular graph, so \( m(G) = r \cdot n(G) + \frac{1}{2} n(G) \). Let \( c \) be an arbitrary \( \Delta(G) \)-edge-coloring of \((G, \sigma)\). The set of edges colored in \( c \) with color 0 must form a matching in \( G \). Matching in \( G \) can have maximum size of \( \frac{1}{2} n(G) \), so there have to be at least \( r \cdot n(G) \) edges colored by \( c \) with colors different than 0. Such edges form \( r \) subgraphs with maximum degree not exceeding 2, each of them is colored with \( \{-k, k\} \) for some \( k \neq 0 \). Clearly, they are edge disjoint subgraphs with degree not exceeding 2, so they have in total at most \( r \cdot n(G) \) edges. Combining these observations, we conclude that these subgraphs have exactly \( r \cdot n(G) \) edges. This implies that they are \( 2 \)-regular, spanning and balanced – since otherwise they could not be colored using only colors \( \{-k, k\} \). This also implies that our matching has \( \frac{1}{2} n(G) \) edges, which means it is perfect.

\( \square \)

Corollary 3. If \( G \) is a graph with even \( \Delta(G) \), then \( G \) is not of class \( 2^\pm \).

Proof. Every graph has a regular supergraph with the same maximum degree. Let \( H \) be a \( \Delta(G) \)-regular supergraph of \( G \). It follows from Petersen’s 2-factor theorem \( [2] \) that \( H \) admits a decomposition into \( r \) edge disjoint \( 2 \)-regular, spanning subgraphs. Since all edges in a signed graph \((H, 1_{E(H)})\) are positive, these subgraphs are balanced. Lemma \( [2] \) implies that \( \chi'(H, 1_{E(H)}) = \Delta(H) \). Hence \( \chi'(G, 1_{E(G)}) \leq \chi'(H, 1_{E(H)}) = \Delta(H) = \Delta(G) \), which shows that \( G \) is not of class \( 2^\pm \). \( \square \)

Theorem 4. Let \( G \) be a graph. \( G \) is of class \( 2^\pm \) if and only if \( \Delta(G) \) is odd and there is no matching \( M \) in \( G \) such that \( \Delta(G \setminus M) < \Delta(G) \).
Proof. If follows from Corollary 8 that if graph is of class 2$^\pm$ then its maximum degree must be odd.

($\Rightarrow$) Let $G$ be a graph with odd maximum degree. Suppose that there is a matching $M \subseteq E(G)$ such that $\Delta(G \setminus M) < \Delta(G)$. We will show that $G$ is not of class 2$^\pm$. Clearly, $\Delta(G \setminus M)$ is even since $\Delta(G \setminus M) = \Delta(G) - 1$. The proof of Corollary 8 shows that a signed graph $(G \setminus M, 1_{E(G) \setminus M})$ is $\Delta(G \setminus M)$-colorable. Since $\Delta(G \setminus M)$ is even, $\Delta(G \setminus M)$-colorings of $G \setminus M$ do not use color 0. We can extend any of these colorings to a coloring of $G$ by using color 0 on $M$, so $(G, 1_{E(G)})$ is $\Delta(G)$-edge-colorable. This completes the proof.

($\Leftarrow$) Let us assume $\Delta(G)$ is odd, there is no matching $M \subseteq E(G)$ such that $\Delta(G \setminus M) < \Delta(G)$ and $G$ is not of class 2$^\pm$. We show that this leads to a contradiction. There must exist a signature $\sigma$ such that $(G, \sigma)$ is $\Delta(G)$-edge-colorable. Let $c$ be an arbitrary $\Delta(G)$-edge-coloring of $(G, \sigma)$. Edges colored in $c$ with color 0 must form a matching $M$. Clearly, $\Delta(G \setminus M) = \Delta(G)$, so graph $(G \setminus M, \sigma|_{E(G) \setminus M})$ requires at least $\Delta(G)$ colors different than 0. It follows that $(G, \sigma)$ cannot be colored using $\Delta(G)$ colors, so $G$ is of class 2$^\pm$.

It’s easy to observe that signed graphs with $\Delta = 1$ can be colored using color 0 regardless of their signature, so graphs with $\Delta = 1$ are of class 1$^\pm$.

Theorem 5. For every $k \geq 1$ there is a graph $G$ of class 2$^\pm$ such that $\Delta(G) = 2k + 1$.

Proof. We will describe the procedure of construction of a graph $G$ such that $\Delta(G) = 2k + 1$ and $G$ is of class 2$^\pm$. We start by constructing a complete bipartite graph $H$ with parts $\{u_1, \ldots, u_{2k}\}$ and $\{v_1, \ldots, v_{2k}\}$. Next we create a graph $H'$ by adding edges $u_{2i-1}u_{2i}$ for $1 \leq i \leq k$ to $H$, a new vertex $v$ and edges $vuv$ for $1 \leq i \leq 2k$. It is easy to see that

$$\deg_{H'}(w) = \begin{cases} 2k, & \text{if } v = w, \\ 2k + 1, & \text{otherwise.} \end{cases}$$

Now we construct the graph $G$ by creating two disjoint copies of $H'$ and connecting their lowest degree vertices with path $P_3$ in such a way that these vertices are the endpoints of the path (see Figure 1 for an example). By $v'_1, v_c, v'_2$ we denote vertices of that path and by $v'_1v_c, v_cv'_2$—its edges. Let’s observe that $\deg_G(v_c) = 2$ and other vertices of $G$ have degree $2k + 1$.

To complete the proof it suffices to show that $G$ is of class 2$^\pm$. Clearly, $\Delta(G)$ is odd, so Theorem 4 tells us that we need to show that there is no matching $M \subseteq E(G)$ such that $\Delta(G \setminus M) < \Delta(G)$. Let’s assume there is such a matching $M$ in $G$. We show that it leads to a contradiction. Clearly, $M$ must cover all the vertices of $G$ other than $v_c$—it must cover all the vertices of both copies of $H'$. Let’s observe that $H'$ has an odd number of vertices, so there is no perfect matching in it. That means that in order to cover vertices of both copies of $H'$, $M$ must contain edges $v'_1v_c$ and $v_cv'_2$. Then $M$ contains two edges incident with $v_c$, so $M$ is not a matching, a contradiction that completes the proof. \qed
3. Cacti

A connected graph $G$ is called a cactus if and only if every edge of $G$ belongs to at most one cycle. Clearly every tree is a cactus as it doesn’t have cycles at all. It is a well known fact that graph $G$ is a cactus if and only if there exists a sequence of graphs $G_1, \ldots, G_k$ called a decomposition of $G$, such that:

1. $G_i$ is a cycle or a path $P_2$ for $1 \leq i \leq k$;
2. $G_i$ has exactly one vertex in common with $G_1 \cup \ldots \cup G_{i-1}$ for $2 \leq i \leq k$;
3. $G = G_1 \cup \ldots \cup G_k$.

We prove that almost all signed cacti are of class $1^\pm$.

**Theorem 6.** Let $G$ be a cactus. If $G$ is not a cycle, then $G$ is of class $1^\pm$.

**Proof.** Let $\sigma : E(G) \to \{\pm 1\}$ be a function. If $\Delta(G) \leq 2$, then $G$ is a path, so our claim is obvious. Therefore we assume that $\Delta(G) \geq 3$ in the remainder of the proof.

Let $G_1, \ldots, G_k$ be an arbitrary decomposition of $G$ meeting the three above conditions. By $G'_l$ we denote $G_1 \cup \ldots \cup G_l$. Clearly, $G'_k = G$. We will use induction on $l$ to prove that $(G'_l, \sigma|_{E(G'_l)})$ is $\Delta(G)$-edge-colorable. Let us observe that $G'_1$ is a cycle or path, so $(G'_1, \sigma|_{E(G'_1)})$ can be colored using $\Delta(G) \geq 3$ colors. Let’s assume that $(G'_l, \sigma|_{E(G'_l)})$ is $\Delta(G)$-edge-colorable for any $l < x$. We will show that $(G'_{x-1}, \sigma|_{E(G'_{x-1})})$ can be colored using at most $\Delta(G)$ colors. Let us note that $G'_x = G'_{x-1} \cup G_x$ and $(G'_{x-1}, \sigma|_{E(G'_{x-1})})$ is $\Delta(G)$-edge-colorable. By $c$ we denote an arbitrary $\Delta(G)$-edge-coloring of $(G'_{x-1}, \sigma|_{E(G'_{x-1})})$. It follows from the definition of the decomposition of $G$ that $G_x$ is either a path $P_2$ or a cycle and $G_x$ has exactly one vertex in common with $G'_{x-1}$. Let us consider the two cases separately.

1. $G_x$ is a path $P_2$. By $u, v$ we denote vertices of $G_x$ and without loss of generality we assume $u \in V(G'_{x-1})$. Clearly, there are at most $\Delta(G) - 1$ edges incident to $u$ in $G'_{x-1}$, so there must be a color $\alpha \in M_{\Delta(G)}$ not used on any of these edges by $c$. Let’s color $u: uv$ with $\alpha$ and $v: uv$ with $-\sigma(uv)\alpha$. That way $(G'_x, \sigma|_{E(G'_x)})$ can be colored using $\Delta(G)$ colors.

2. $G_x$ is a cycle. By $u \in V(G_x)$ we denote a vertex such that $u \in V(G'_{x-1})$. By $uv_1, uv_2$ we denote two different edges of $G_x$. Clearly, there are at
most $\Delta(G) - 2$ edges incident to $u$ in $G'_{x-1}$, so there must be two different colors $\alpha, \beta \in M_{\Delta(G)}$ not used on any of these edges by $c$. Let us consider three possible subcases.

(a) $\alpha = -\beta$ and $\Delta(G) = 3$. Since $\Delta(G) = 3$, $0 \in M_{\Delta(G)}$ and $\alpha \neq 0$, $\beta \neq 0$. By $w$ we denote vertex in $G_x$ such that $w$ is adjacent to $v_1$ and $w \neq u$. Incidences $v_1: v_1w$ and $w: v_1w$ of $G'_x$ can be colored with color 0. Let us observe that other edges of $G_x$ form a path, so their incidences can be colored with colors $\pm \alpha$ in the coloring of $(G'_x, \sigma|_{E(G'_x)})$.

(b) $\alpha = -\beta$ and $\Delta(G) > 3$. Since $\Delta(G) > 3$, there must be colors $\gamma, -\gamma \in M_{\Delta(G)}$ such that $\gamma \neq 0$, $\gamma \neq \alpha$, $\gamma \neq \beta$. Let us observe that edges $v_1u, uv_2$ form a path in $G_x$, so their incidences can be colored with colors $\pm \alpha$ in the coloring of $(G'_x, \sigma|_{E(G'_x)})$. Other edges of $G_x$ also form a path—can be colored with colors $\pm \gamma$.

(c) $\alpha \neq -\beta$. Without loss of generality let us assume $\beta \neq 0$. Incidence $u: uv_1$ can be colored with $\alpha$ in the coloring of $(G'_x, \sigma|_{E(G'_x)})$ and $v_1: uv_1$—with $-\sigma(uv_1)\alpha$. Edges of $G_x$ other than $uv_1$ form a path and can be colored with $\pm \beta$ in such a way that $u: uv_2$ gets color $\beta$.

The decomposition of an $n$-vertex cactus can be done in $O(n)$ time. This implies that the coloring procedure being a part of the above proof is also linear.

4. Wheels

Wheel $W_n$ is a graph on $n$ vertices consisting of a cycle $C_{n-1}$, a vertex $v \notin V(C_{n-1})$ and edges between $v$ and all the vertices of a cycle $C_{n-1}$.

**Theorem 7.** All wheels are of class $1^\pm$.

**Proof.** We consider three cases separately:

1. $n = 4$. Wheel $W_4$ is a complete graph $K_4$. It’s easy to observe that $W_4$ has a decomposition into three perfect matchings and edges of any two of
them create a spanning cycle. Let us consider an arbitrary signed graph
\((W_4, \sigma)\) and it’s decomposition \(D\) into three perfect matchings. If there
is an odd number of negative edges in \((W_4, \sigma)\), there must be a matching
in \(D\) with an odd number of negative edges. Remaining edges create a
balanced spanning cycle. In the opposite case—when there is an even
number of negative edges in \((W_4, \sigma)\), there must be a matching in \(D\)
with an even number of negative edges. Remaining edges create a balanced
spanning cycle. In both cases, \((W_4, \sigma)\) has a decomposition into a perfect
matching and a balanced spanning cycle. Obviously, \(W_4\) is 3-regular, so
it follows from Lemma 2 that \((W_4, \sigma)\) is \(\Delta\)-edge-colorable.

2. \(n = 2k + 1, k \geq 2\). Let \(V(W_n) = \{u, v_0, \ldots, v_{n-2}\}\) and \(E(G) = \{uv_0, \ldots, uv_{n-2}\} \cup \{v_0v_1, v_1v_2, \ldots, v_{n-2}v_0\}\). For \(0 \leq i \leq k - 2\), let \(G_i\) be
a subgraph of \(W_n\), such that \(V(G_i) = \{v_i, v_{i+1}, u, v_{i+k}, v_{i+k+1}\}\) and
\(E(G_i) = \{v_{i+1}v_i, v_{i+k}v_{i+k+1}\}\). Clearly, \(G_i\) is a path and for
any different \(i_1, i_2\) paths \(G_{i_1}, G_{i_2}\) are edge disjoint.
Let \(P\) be a subgraph of \(W_n\), such that \(V(P) = \{v_{k-1}, v_k, u, v_{n-2}, v_0\}\)
and \(E(P) = \{v_kv_{k-1}, v_{k-1}u, uv_{n-2}, v_{n-2}v_0\}\). Clearly, \(P\) is a path.
We will show that paths \(P, G_i\) are edge disjoint for any \(0 \leq i \leq n - 2\). It’s
sufficient to show that \(\{v_0u, uv_{i+k}\} \cap \{v_{k-1}u, uv_{n-2}\} = \emptyset\) and \(\{v_{i+1}v_i, v_{i+k}v_{i+k+1}\} \cap \{v_kv_{k-1}, v_{n-2}v_0\} = \emptyset\). The first equality follows from the
fact that \(i < k - 1 < i + k < n - 2\). Let’s assume that the second equality
is false, so one of the following cases must hold:
(a) \(v_{i+1}v_i = v_kv_{k-1}\). Impossible because \(i < k - 1 < k\).
(b) \(v_{i+1}v_i = v_{n-2}v_0\). Impossible because \(i < i + 1 < n - 2\).
(c) \(v_{i+k}v_{i+k+1} = v_kv_{k-1}\). Impossible because \(k - 1 < i + k < i + k + 1\).
(d) \(v_{i+k}v_{i+k+1} = v_{n-2}v_0\). Impossible because \(i + k \neq 0\) and \(i + k + 1 \neq 0\).

Let’s observe that \(m(\bigcup_{i=0}^{k-2} G_i) + m(P) = 4(k - 1) + 4 = 4k = m(W_n)\), so
\(\bigcup_{i=0}^{k-2} G_i \cup P = W_n\) and \(W_n\) has a decomposition into exactly \(k = \Delta(W_n)/2\)
paths. It follows that any signed graph with \(W_n\) as an underlying graph
can be colored using \(\Delta(W_n)\) colors because edges of each path can be
colored with exactly two colors.

3. \(n = 2k, k \geq 3\). Let us consider wheel \(W_{n-1}\). By \(u\) we denote the center
vertex of \(W_{n-1}\)—the one with degree \(\Delta(W_{n-1})\) and by \(v_1, v_2\) we denote

Figure 3: The decomposition of \(W_7\) into three paths. Distinct paths are marked with solid,
dashed and dotted lines.
arbitrary vertices adjacent to $u$ such that $v_1v_2 \in E(W_{n-1})$. It follows from the previous case that $W_{n-1}$ admits a decomposition $D$ into $k - 1$ paths. We construct graph $W'_{n-1}$ from $W_{n-1}$ by adding new vertex $v$ and replacing edge $v_1v_2$ by path $v_1, v, v_2$. It’s easy to observe that $W'_{n-1}$ also admits some decomposition $D'$ into $k - 1$ paths. It can be constructed directly from $D$—path containing edge $v_1v_2$ in $D$ contains edges $v_1v, vv_2$ in $D'$. Clearly, graph $W_n$ can be obtained from $W'_{n-1}$ by adding edge $vu$. It follows that $W_n$ admits a decomposition into $k - 1$ paths and a single edge, so every signed graph with $W_n$ as an underlying graph can be colored using $2k - 1 = \Delta(W_n)$ colors.

\[ \square \]

5. Necklaces

Necklace is a connected graph having a decomposition into $k \geq 2$ paths, in which all vertices except selected two—$u, v$—are different. Vertices $u, v$ are starting and ending vertices of all paths. If $k = 2$, a necklace is a cycle, so it’s possible that a signed graph on such necklace is negative and requires $\Delta + 1$ colors to be properly colored. We prove necklaces are $\Delta$-colorable in all other cases.

**Theorem 8.** Let $G$ be a necklace. If $G$ is not a cycle, then $G$ is of class $1^\pm$.

**Proof.** $G$ is not a cycle, so $\Delta(G) > 3$. By $D = G_1, \ldots, G_{\Delta(G)}$ we denote the decomposition of $G$ into paths. By $u, v$ we denote the starting and ending vertices of all the paths from $D$. Without loss of generality, we assume that $m(G_i) \leq m(G_{i+1})$ for $1 \leq i < \Delta(G)$. It follows that if there is a path of length 1 in $D$, it must be $G_1$. By $u^j_i$ we denote a vertex belonging to path $G_i$ such that the distance between $u^j_i$ and $u$ in $G_i$ is $j$. The definition of $v^j_i$ is analogous. By $S = (G, \sigma)$ we denote an arbitrary signed graph with $G$ being its underlying graph. We consider two cases separately:

1. $\Delta(G) = 2k + 1$, $k \geq 1$. We will construct necklace $S$ starting with necklace $S_0$ that contains three paths from $D—G_1, G_2, G_3$ and extending it with
consecutive pairs of paths. By $S_p$ we denote the necklace constructed from $S_{p-1}$ by adding paths $G_{2p+2}$, $G_{2p+3}$. Clearly, $\Delta(S_0) = 3$. We color edges $uu_2^1$, $vv_3^1$ with color 0. Let us observe that the remaining edges of $S_0$ span a path that contains following vertices: $u_2^1$, $v_2^1$, $v_1$, $u_1$, $u_1^1$, ..., $v_2^1$. It can be colored with colors $\pm 1$.

Necklace $S_p$ is constructed from $S_{p-1}$ by adding paths $G_{2p+2}$, $G_{2p+3}$. Clearly, $\Delta(S_p) = \Delta(S_{p-1}) + 2$, so there are two new colors $\pm \alpha$ available for coloring of $S_p$. Let us assume there is an edge $uu_1^1$ in $S_{p-1}$ colored with color 0. We can color edge $uu_2^1$ with color 0 and path $u_1^1$, $u_1^2$, $v_2^1$, $v_2^2$, ..., $v_2^2$, previously colored with 1, was recolored. Let us observe that this gives us a $\Delta(S_p)$-edge-coloring of $S_p$. Let us also observe that there is still an edge incident to $u$ such that it’s colored with color 0 in the coloring of $S_p$, so such an edge must exist in the colorings of all $S_0$, ..., $S_p$.

2. $\Delta(G) = 2k$, $k \geq 2$. We will construct necklace $S$ starting with necklace $S_0$ that contains four paths from $D$—$G_1$, ..., $G_4$ and extending it with consecutive pairs of paths. By $S_p$ we denote the necklace constructed from $S_{p-1}$ by adding paths $G_{2p+3}$, $G_{2p+4}$. Clearly, $\Delta(S_0) = 4$. Let us observe that vertices $u_3^1$, $v_2^1$, $u_1^1$, ..., $v_2^1$, $v_1$, $v_2^2$ span a path in $S_0$ and its edges can be colored with colors $\pm 1$. We can assume that incidence $u$: $uu_3^1$ gets color 1. Let us observe that the remaining edges also span a path, end its edges can be colored with colors $\pm 2$.

Necklace $S_p$ is constructed from $S_{p-1}$ by adding paths $G_{2p+3}$, $G_{2p+4}$. Clearly, $\Delta(S_p) = \Delta(S_{p-1}) + 2$, so there are two new colors $\pm \alpha$ available for coloring of $S_p$. Let us assume there is an incidence $u$: $uu_1^1$ in $S_{p-1}$ colored with color 1. We can color incidence $u$: $uu_2^1$ with color 1 and $u_2^1$: $uu_3^1$ with color $-\sigma(uu_3^1)$. Path $u_1^1$, $u_1^2$, $v_2^1$, $v_2^2$, ..., $v_2^2$, $v_1$, $v_2^2$, $v_2^2$, $u_1^3$ can be colored with colors $\pm \alpha$. That way incidence $u$: $uu_1^1$, previously colored with 1, was recolored. Let us observe that this gives us a $\Delta(S_p)$-edge-coloring of $S_p$. Let us also observe that there is
still an incidence incident to \( u \) such that it’s colored with color 1 in the coloring of \( S_p \), so such an incidence must exist in the colorings of all \( S_0, \ldots, S_p \).

\[ \square \]

6. Complete bipartite graphs

A graph is bipartite if and only if its vertices can be divided into two sets \( V_1, V_2 \) such that each edge has endpoints in both sets. These sets are usually called parts. Graph is a complete bipartite graph if and only if it’s bipartite and for every \( u \in V_1, v \in V_2 \) there is an edge \( uv \).

**Theorem 9.** Let \( r, t \) be positive integers and \( G \) be a complete bipartite graph \( K_{r,t} \). If \( r \neq t \), then \( G \) is of class \( 1^\pm \).

**Proof.** Without loss of generality, we assume that \( r < t \). Let \( t = 2s + k, s \in \mathbb{N} \) and \( k \in \{0, 1\} \). Let \( V(G) = \{u_0, \ldots, u_{r-1}\} \cup \{v_0, \ldots, v_{t-1}\} \) and \( E(G) = \{u_iv_j: 0 \leq i \leq r-1, 0 \leq j \leq t-1\} \).

Let \( 0 \leq j \leq s - 1 \). Let \( G_j \) be a subgraph of \( G \) such that \( V(G_j) = \{u_0, \ldots, u_{r-1}\} \cup \{v_{2j}, \ldots, v_{(2j+r) \mod t}\} \) and \( E(G_j) = \{u_iv_{(i+2j) \mod t} : 0 \leq i \leq r-1\} \cup \{u_iv_{(i+2j+1) \mod t} : 0 \leq i \leq r-1\} \). Let us observe that:

- \( \deg_{G_j}(u_i) = 2 \) for \( i = 0, \ldots, r-1 \);
- \( \deg_{G_j}(v_{2j}) = \deg_{G_j}(v_{(2j+r) \mod t}) = 1 \);
- \( \deg_{G_j}(v_{(2j+i) \mod t}) = 2 \) for \( i = 1, \ldots, r-1 \).
It follows from the definition of $G_j$ and above observations that $G_j$ is a path. We will show that any two distinct paths $G_{j_1}$, $G_{j_2}$ are edge disjoint. Let us assume it’s not true. Then one of the following cases must hold for some $i_1$, $i_2 \in \{0, \ldots, r-1\}$.

1. $u_{i_1}v_{(i_1+2j_1)} \equiv u_{i_2}v_{(i_2+2j_2)} \pmod{t}$. Clearly, $u_{i_1} = u_{i_2}$, so $i_1 = i_2$. Moreover, $v_{(i_1+2j_1)} \equiv v_{(i_2+2j_2)} \pmod{t}$, so $2j_1 \equiv 2j_2 \pmod{t}$. It follows that $0 \equiv 2(j_1 - j_2) \pmod{t}$. Let us consider two cases separately:
   
   (a) $t = 2s + 1$. Then $t \equiv 2s + 1 \pmod{t}$, so $0 \equiv 2s + 1 \pmod{t}$ and $2s \equiv -1 \pmod{t}$. We observe that if $0 \equiv 2(j_1 - j_2) \pmod{t}$, then $0 \equiv 2s(j_1 - j_2) \pmod{t}$, so $0 \equiv j_2 - j_1 \pmod{t}$ and $j_1 \equiv j_2 \pmod{t}$. It follows that $j_1 = j_2$, a contradiction.
   
   (b) $t = 2s$. If $0 \equiv 2(j_1 - j_2) \pmod{t}$, then $0 \equiv 2(j_1 - j_2) \pmod{2s}$ and $0 \equiv j_1 - j_2 \pmod{s}$, so $j_1 \equiv j_2 \pmod{s}$. It follows that $j_1 = j_2$, a contradiction.

2. $u_{i_1}v_{(i_1+2j_1+1)} \equiv u_{i_2}v_{(i_2+2j_2+1)} \pmod{t}$. We observe that $i_1 = i_2$, so $2j_1 + 1 \equiv 2j_2 + 1 \pmod{t}$ and then $2(j_1 - j_2) \equiv 0 \pmod{t}$. The remaining part of that case is identical to the previous case and leads to a contradiction.

3. Either $u_{i_1}v_{(i_1+2j_1)} \equiv u_{i_2}v_{(i_2+2j_2+1)} \pmod{t}$ or $u_{i_1}v_{(i_1+2j_1+1)} \equiv u_{i_2}v_{(i_2+2j_2)} \pmod{t}$. Without loss of generality, we assume that the first of two holds. Clearly, $i_1 = i_2$, so $2j_1 \equiv 2j_2 + 1 \pmod{t}$ and then $2(j_1 - j_2) \equiv 1 \pmod{t}$. We consider two cases separately:
   
   (a) $t = 2s+1$. Then $2s \equiv -1 \pmod{t}$. Let us observe that $2s(j_1 - j_2) \equiv s \pmod{t}$, so $j_2 - j_1 \equiv s \pmod{t}$. It’s a contradiction because $j_1$, $j_2 \in \{0, \ldots, s-1\}$.
   
   (b) $t = 2s$. We observe that $2(j_1 - j_2) - 1 \equiv 0 \pmod{2s}$, so the odd number $2(j_1 - j_2) - 1$ is divisible by an even number $2s$, a contradiction.
All of the cases lead to a contradiction, so paths \( G_{j_1}, G_{j_2} \) are edge disjoint. Clearly, \( \bigcup_{j=0}^{s-1} G_j \subseteq G \). We will continue the proof separately for two cases:

1. \( t = 2s \). Since \( G \) is a complete bipartite graph, \( m(G) = rt = 2rs \). We observe that \( m(G_j) = 2r \) for \( 0 \leq j \leq s - 1 \). It follows that \( m(\bigcup_{j=0}^{s-1} G_j) = \sum_{j=0}^{s-1} m(G_j) = 2rs \), so \( m(G) = m(\bigcup_{j=0}^{s-1} G_j) \). Since paths \( G_j \) are edge disjoint, \( G = \bigcup_{j=0}^{s-1} G_j \), so \( G \) admits a decomposition into exactly \( s \) paths. Let us observe that an arbitrary signed graph with \( G \) being its underlying graph can be colored with \( 2s = \Delta(G) \) colors, since each path can be colored with just two colors. It completes a proof for this case.

2. \( t = 2s + 1 \). It’s clear that \( m(G) = rt = r(2s + 1) \) and \( m(\bigcup_{j=0}^{s-1} G_j) = 2rs \). Let \( M \) be a subgraph of \( G \) such that \( V(M) = \{u_0, \ldots, u_{r-1}\} \cup \{v_{(i-1) \mod t}: 0 \leq i \leq r - 1\} \) and \( E(M) = \{u_{i}v_{(i-1) \mod t}: 0 \leq i \leq r - 1\} \).

It’s easy to observe that all the vertices of \( M \) have degree equal to 1, so \( M \) is a matching.

We will show that \( E(M) \cap E(\bigcup_{j=0}^{s-1} G_j) = \emptyset \). We assume that it’s not true.

Then for some \( j \) and \( i_1, i_2 \in \{0, \ldots, r - 1\} \) one of the following cases must hold:

(a) \( u_{i_1}v_{(i_1-1) \mod t} = u_{i_2}v_{(i_2+2j) \mod t} \). Clearly, \( i_1 = i_2 \). We observe that \( i_1 - 1 \equiv i_2 + 2j \pmod{t} \), so \( 2j + 1 \equiv 0 \pmod{t} \) and then \( 2js + s \equiv 0 \pmod{t} \). Since \( t = 2s + 1 \), \( 2s \equiv -1 \pmod{t} \), so \( j \equiv s \pmod{t} \). It’s a contradiction because \( j \in \{0, \ldots, s - 1\} \).

(b) \( u_{i_1}v_{(i_1-1) \mod t} = u_{i_2}v_{(i_2+2j+1) \mod t} \). It holds that \( i_1 - 1 \equiv i_2 + 2j + 1 \pmod{t} \), so \( 2j \equiv -2 \pmod{t} \) and then \( 2js \equiv -2s \pmod{t} \). Since \( t = 2s + 1 \), \( j + 1 \equiv 0 \pmod{t} \). It’s a contradiction because \( t = 2s + 1 \) and \( j \in \{0, \ldots, s - 1\} \).

All of the cases lead to a contradiction, so \( E(M) \cap E(\bigcup_{j=0}^{s-1} G_j) = \emptyset \).

Since \( m(M) = r, G = E(M) \cup E(\bigcup_{j=0}^{s-1} G_j) \), \( G \) admits a decomposition into exactly \( s \) paths and one matching. Let us observe that an arbitrary signed graph with \( G \) being its underlying graph can be colored with \( 2s+1 = \Delta(G) \) colors, since each path can be colored with two non-zero colors and a matching can be colored with color 0.

We briefly consider complete bipartite graphs with equal parts. Complete bipartite graph \( K_{r,r} \) is an \( r \)-regular graph, so it follows from Lemma 2 that if \( r \) is
even and $K_{r,r}$ has an odd number of negative edges, it cannot be colored using $\Delta$ colors. We hypothesize that for all other cases, complete bipartite graphs with equal parts are $\Delta$-colorable. If the conjecture is true, graphs with odd $r$ must have a perfect matching $M$ such that $K_{r,r} \setminus M$ have an even number of negative edges and have a decomposition into $(r-1)/2$ spanning 2-regular subgraphs with positive cycles being their components.

**Conjecture 10.** Let $r \in \mathbb{N}$ and $S$ be a signed complete bipartite graph $K_{r,r}$. $\chi'(S) = \Delta(S)$ if one of the following conditions holds:

1. $r$ is odd;
2. $r$ is even and there is an even number of negative edges in $S$.

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