Predictability Exponent of Stochastic Dynamical Systems

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Abstract—Predicting the trajectory of stochastic dynamical systems (SDSs) is an intriguing problem in numerous fields, where characterizing the predictability of SDSs is of fundamental importance. Prior works have tackled this issue by indirectly investigating the uncertainty of distribution in each prediction. How accurately the trajectory of SDSs can be directly predicted still remains open. This paper proposes a new metric, namely predictability exponent, to characterize the decaying rate of probability that the prediction error never exceeds arbitrary \( \epsilon \). To evaluate predictability exponent, we begin with providing a complete framework for model-known cases. Then, we bring to light the explicit relationship between predictability exponent and entropy by discrete approximation techniques. The definition and evaluation on predictability exponent are further extended to model-unknown cases by optimizing over model spaces, which build a bridge between the accuracy of trajectory predictions and popular entropy-based uncertainty measures. Examples of unpredictable trajectory design are presented to elaborate the applicability of the proposed predictability metric. Simulations are conducted to illustrate the efficiency of the obtained results.

Index Terms—Stochastic Dynamical Systems, Predictability Exponent, Unpredictable Trajectory, Probabilistic Perspective

I. INTRODUCTION

A Stochastic Dynamical System (SDS) is a dynamical system subjected to the effects of stochastic noise \( [2] \). Such effects of fluctuations have been of interest for over a century since the seminal work of Einstein \( [3] \). Predicting the trajectory of SDS is an intriguing problem in numerous fields, where characterizing the predictability of SDSs is of fundamental importance. Predictability of stochastic dynamical system aims to characterize to what extent the trajectory of an SDS can be accurately predicted. With rapidly increasing blooming of various modern prediction techniques, such as deep learning and reinforcement learning, it brings emerging theoretical challenges to the predictability analysis of SDSs. On the one hand, different prediction methods have different performance limits, black-box structures of deep neural networks make prediction performance analysis difficult. On the other hand, large-scale stochastic dynamical systems suffer from uncertainty and volatility, thus making huge costs for prediction and validation tasks in practice.

A. Related Work

A large amount of research (e.g., \( [4]–[11] \)) has been devoted to characterizing predictability by analyzing the growth of initial observation errors, and this characterization has found wide applications in the field of climatology \( [8] \), \( [10] \). Although the uncertainty caused by initial observation errors will significantly influence the prediction accuracy of a SDS, it can not directly reflect the predictability of a SDS itself. Therefore, we focus on the inherent uncertainty of a SDS in this work. It’s both challenging and interesting to understand how the inherent uncertainty of an SDS will affect its predictability.

Entropy, as an information-theoretic quantity, is intuitively used to measure the predictability of outcomes in a stochastic event. Many works (e.g., \( [12]–[17] \)) have been devoted to characterizing predictability by entropy. For instance, \( [6] \), \( [7] \), \( [9] \) consider predictability as system’s sensitivity to initial error and have found great applications in meteorology. \( [12]–[15] \) either directly use entropy or define entropy-based quantities to measure the predictability of various Markov decision processes (MDP), and then use these entropy-based indexes to give performance analysis on specific prediction tasks. These analyses view the prediction performance from the perspective of information theory rather than from traditional methods, such as root-mean-square-error, patter correlation (PC) and probability, etc. \( [16] \), \( [17] \) characterize the uncertainty of system trajectories by entropy, then use Fano’s inequality to derive a probabilistic upper bound to estimate predictability. Due to the probabilistic description, these works circumvent the indirect information-theoretic explanation caused by entropy.

Despite exciting results achieved by the works about predictability, most of them are based on stochastic systems with finite states, such as finite-state MDP, finite targets of motion, etc. This characterization cannot describe general stochastic dynamical systems whose states could be arbitrary. To evaluate the predictive performance for SDSs with infinite states, \( [18] \) specified a statistical scoring rule for time series based on entropy, and evaluate the optimal performance in an asymptotic sense. \( [19] \) use the Gibbs-Shannon entropy to quantify the inferability.

Although entropy provides an effective and intuitive measurement for the prediction uncertainty, it cannot well account for the prediction accuracy directly, which is also critical to characterize the predictability of SDSs. A more practical and rigorous characterization of predictability which directly takes the prediction accuracy into consideration is needed. Recently, \( [20] \), \( [21] \) have studied the one-step accurate prediction probability and designed unpredictable strategies accordingly, and their min-max strategies are robust when noises are not necessarily bounded. However, an appropriate definition of predictability considering prediction accuracy is still an open and challenging issue.

B. Motivations

In this article, we aim to fill the gap between the prediction accuracy and entropy measure of predictions, and extend

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the characterization on predictability of stochastic dynamical systems from the finite state case to the infinite state case. Specifically, our metric is defined directly from prediction performance, i.e., the asymptotic exponential decaying rate for the probability that prediction errors never exceed a fixed $\epsilon$.

This new metric is partially motivated by the following reasons. First, previous entropy-based metric mostly focus on Markov chain model, which cannot be directly used in the case where stochastic systems have infinite states. A simple example is that Shannon entropy of a uniform distribution on $N$ states will explode to infinity as $N$ approaches to infinity. In this case, a metric of infinite value is not practically useful. Second, even in the finite case, entropy is indirect in quantifying prediction performance because it characterize performance by indirect information-theoretic explanations rather than direct probabilistic explanations. Therefore, we need a predictability metric that can directly quantify the prediction performance, e.g., the probability of making accurate predictions. Third, the trajectory of states generated by an SDS, instead of completely irregular, actually possess some statistical features, e.g., central limit theorem. These statistical features usually do not rely on the realization of state trajectory, but rather, the stochastic system itself. Hence, it is reasonable to include the trajectory statistics into the predictability metric. Finally, the predictability metric should best the optimal prediction performance which only depend on the SDS to be predicted.

C. Contributions

The differences between this paper and its conference version [1] include i) definition, evaluation and estimation of predictability exponent have been extended to the case where model is unknown to the predictor, ii) applications of predictability exponent are provided, based on which we have designed unpredictable SDS, iii) extended simulations are provided, the main contributions are summarized as follows.

- We propose a new metric, predictability exponent, to characterize the predictability of SDSs with infinite states. The proposed metric describes the best prediction performance by the probability of accurate trajectory predictions with maximum error $\epsilon$, without the dependency on knowledge of any specific prediction method.
- An elegant approximated expression of the proposed metric for model-known cases is established, by which we build a bridge between the accuracy of trajectory prediction and differential entropy. We prove that the approximation error of the metric is bounded of scale $O(\epsilon)$, and demonstrate that the decaying rate of the accuracy probability on $K$-step predictions converging to the asymptotic one is $O(\epsilon^{-K})$.
- We further extend the predictability metric to model-unknown cases by optimizing over model spaces. We apply the predictability exponent to unpredictable trajectory design problems, and show that the proposed metric is more general and can be easily reduced to other existing metrics by finding the optimal solutions. Extensive simulation tests corroborate the theoretical findings.

D. Organization

The remainder of this paper is organized as follows. Section II gives basic preliminaries and describes the problem of interest. The discretization method and evaluation of predictability in the model-known case are presented in Sec. III, and discussions on the model-unknown case are presented in Sec. IV. Then, in Sec. V, we apply the predictability exponent to the design of unpredictable stochastic dynamical system. Simulations are shown in Sec. VI, followed by the concluding remarks and further research issues in Sec. VII. Throughout this paper, we use capital letters to denote stochastic variables and lower-case letters to denote deterministic variables.

II. PROBLEM FORMULATION AND PRELIMINARIES

A. Problem Formulation

Consider a discrete stochastic dynamical system, denoted by $\Phi$, as follows

$$\Phi : \begin{cases} X_{k+1} = f(X_k, u_k) + W_k, & W_k \sim q, \\ Y_k = g(X_k, u_k), \end{cases} \tag{1}$$

where $X_k \in \mathbb{R}^d$ is state vector, $W_k \in \mathbb{R}^d$ is noise, $u_k$ is input, $Y_k$ is output, $q$ is the $d$-dimensional probability density function of $W_k$, and $f$ and $g$ are two given functions but may be unknown to the predictor. We make the following assumption for the predictor.

**Assumption 1.** $\Phi$ is an observable system, which means using historical data $Y_{1:k}$ accurately decides the value of $x_k$.

For ease notation, we denote by $x_{[1:K]}$ the $K$-step state trajectory to be predicted after $t_0$, $\{x_{t_0+1}, x_{t_0+2}, \cdots, x_{t_0+K}\}$, regardless of the collected observations and the value of $t_0$. To make a rolling prediction on $x_{[1:K]}$ step-by-step, the predictor will learn the system and update iteratively during the rolling prediction process. The learned SDS model, $\tilde{\Phi}$, is given by

$$\tilde{\Phi} : \begin{cases} X_{k+1} = \tilde{f}(X_k, \tilde{u}_k) + \tilde{W}_k, & \tilde{W}_k \sim \tilde{q}, \\ Y_k = \tilde{g}(X_k, \tilde{u}_k). \end{cases} \tag{2}$$

Since the prediction object is stochastic, the prediction error is unavoidable. This unavoidable error is an inherent property of SDSs, whose upper bound can be used to characterize the rolling prediction performance regardless of prior knowledge, learning algorithm or training data, etc. The upper bound of prediction error is widely referred to as predictability, but badly in short of quantitative research. To explicitly characterize the upper bound of rolling prediction performance, we use probability as prediction performance.

**Definition 1 (Rolling prediction probability).** Let $\tilde{X}_{k|k-1}$ be the prediction of $k$-th state in the trajectory $x_{[1:K]}$. Then, the probability that the prediction error of each step does not exceed $\epsilon$ is:

$$P_\epsilon(x_{[1:K]}; \tilde{\Phi}) \triangleq \prod_{k=1}^{K} \mathbb{P}(\|\tilde{X}_{k|k-1} - x_k\|_\infty \leq \epsilon). \tag{3}$$

Although the rolling prediction probability $P_\epsilon(x_{[1:K]}; \tilde{\Phi})$ is intuitive to characterize how accurate $\Phi$ can be predicted, it
fails to characterize the general predictability because of its dependence on the given trajectory \(x_{[1:K]}\).

Notice that the prediction error of each step on \(x_{[1:K]}\) is independent and one-step prediction probability is bounded, thus \(P_e(x_{[1:K]}; \hat{\Phi})\) will converge to 0 when \(K\) approaches infinity regardless of \(x_{[1:K]}\). Therefore, a more meaningful and important question behind the intuitive convergence is whether the converging rate of \(P_e(x_{[1:K]}; \hat{\Phi})\) going to zero still depends on \(x_{[1:K]}\). This question is the first one that this paper aims to answer. Based on above observations, we characterize the system predictability by evaluating the asymptotic exponential decaying rate of \(P_e(x_{[1:K]}; \hat{\Phi})\), which is defined as follows.

**Definition 2** (Predictability exponent). Given the prediction error bound \(\epsilon\), the asymptotic exponential decaying rate of the rolling prediction probability \(P_e(x_{[1:K]}; \hat{\Phi})\) is defined as

\[
\lambda_e(\Phi; \hat{\Phi}) \triangleq \lim_{K \to \infty} \inf \frac{1}{K} \ln \frac{P_e(x_{[1:K]}; \hat{\Phi})}{P_e(x_{[1:K]}; \Phi)}.
\]

The predictability exponent of \(\Phi\), \(\lambda_e(\Phi)\), is defined as the minimum exponential decaying rate of \(P_e(x_{[1:K]}; \hat{\Phi})\), given by

\[
\lambda_e(\Phi) \triangleq \min_{\hat{\Phi}} \lambda_e(\Phi; \hat{\Phi}).
\]

**Remark 1.** Since the predictability exponent is defined as the lower limit of the exponential decaying rate of \(P_e(x_{[1:K]}; \hat{\Phi})\), an upper bound of \(P_e(x_{[1:K]}; \hat{\Phi})\) can be established by \(e^{-K\lambda_e(\Phi)}\), which represents the best prediction performance. Therefore, \(\lambda_e(\Phi)\) is reasonable to characterize the predictability of an SDS.

Moreover, \(\lambda_e(\Phi)\) is actually the limit of the exponential decaying rate of rolling prediction probability. It not only has an elegant expression related with entropy but also has a fast converging speed of \(O(e^{-K})\). To fully delineate the interconnection between dynamical system and information theory, some preliminaries on entropy is necessary. To bridge the discrete world and continuous world and successfully evaluate \(\lambda_e(\Phi)\), some preliminaries on discrete approximation theory are indispensable.

**B. Preliminaries on Entropy**

In this part, we introduce Shannon entropy, differential entropy and KL-divergence in reference to literature [22].

**Definition 3** (Shannon entropy). The Shannon entropy of a probability vector \(\nu = [\nu_1, \cdots, \nu_n]^T\) is

\[
H_\nu(\nu) \triangleq -\sum_{i=1}^{n} \nu_i \ln \nu_i.
\]

**Definition 4** (KL-divergence). The KL-divergence of a probability vector \(\nu = [\nu_1, \cdots, \nu_n]^T\) with respect to another probability vector \(\mu = [\mu_1, \cdots, \mu_n]^T\) is given by

\[
D_{KL}(\nu||\mu) \triangleq \sum_{i=1}^{n} \nu_i \ln \frac{\nu_i}{\mu_i}.
\]

The KL-divergence of two continuous probability distribution \(\nu\) and \(\mu\) is given by

\[
D_{KL}(\nu||\mu) = \int \nu(x) \ln \frac{\nu(x)}{\mu(x)} dx.
\]

An important property of KL-divergence is that

\[
D_{KL}(\nu||\mu) \geq 0,
\]

where the equality holds if and only if (iff) \(\nu = \mu\).

**Definition 5** (Differential entropy). The differential entropy of a continuous probability distribution \(q\) is defined as

\[
H_d(q) = -\int_S q(x) \ln q(x) dx,
\]

where \(S\) is the support set of \(q\).

**C. Preliminaries on Discrete Approximation**

**Definition 6** (Partition). A partition of a space \(\Omega\), denoted by \(\Sigma\), is defined as

\[
\Sigma \triangleq \left\{ A_1, \cdots, A_{|\Sigma|} \mid \bigcup_{i=1}^{|\Sigma|} A_i = \Omega, A_i \cap A_j = \emptyset, \forall i \neq j \right\},
\]

where \(|\Sigma|\) denotes the number of elements in \(\Sigma\).

Based on the partition, the space \(\Omega\) can be treated as a set of \(|\Sigma|\) small regions, and any point in \(\Omega\) partitioned by \(\Sigma\) only belongs to one region. For ease notation, we define the following label function of the associated region of a point.

**Definition 7** (Label function induced from \(\Sigma\)). Given point \(x\) in \(\Omega\) partitioned by \(\Sigma\), the label function of \(x\)’s associated region is defined as

\[
\Theta(x) = \sum_{i=1}^{|\Sigma|} \mathbb{1}_{A_i}(x)i,
\]

where the indicator function \(\mathbb{1}_{A_i}(x) = 1\) if \(x \in A_i\) or 0 for otherwise.

Apparently, if \(x \in A_i\), then its region label is \(\Theta(x) = i\).

**Definition 8** (Discrete approximation). Suppose \(\Omega\) is the value space of the stochastic noise \(W\). Based on partition \(\Sigma\), the
discrete approximation of the $W$’s probability density function, $q(x)$, is given by

$$q_{\Sigma}(x) = \sum_{i=1}^{\left|\Sigma\right|} m_{A_i}(x) \int_{A_i} q(u)du.$$  

**Example 1.** Given a continuous distribution $q(x)$ on $[-5, 5]$, a partition $\Sigma_0 = \bigcup_{i=1}^{n} A_i$ is made to $[-5, 5]$. Then, a discrete approximation $q_{\Sigma}(x)$ is induced from $q(x)$ and $\Sigma_0$, as shown in Fig. 2. The areas under red step-wise curve is the discrete approximation $q_{\Sigma}(x)$ of $q(x)$ based on partition $\Sigma_0$.

The major theoretical results are summarized into Fig. 2, where we will evaluate the predictability exponent considering the SDS’s model is known and unknown, respectively, and then used the obtained results to design the unpredictable SDSs.

### III. Model-known Case

According to the definition of predictability exponent, the major challenges of evaluating the predictability exponent $\mathcal{L}(\Phi)$ are two-fold: the convergence rate as $K$ increases and the optimization on $\Phi$. In this section, we focus on the former one under the assumption that $\Phi$ is known to the predictor.

First, we apply discretization methods to the rolling prediction probability $\mathcal{P}_e(x_{[1:K]}; \Phi)$ to obtain its discrete version $\mathcal{P}_\Sigma(x_{[1:K]}; \Phi)$, which is evaluated and used to approximate the continuous one. Then, the asymptotic analysis is made to derive both an explicit expression of $\mathcal{L}_e(\Phi; \Phi)$ and the exponential converging rate of $\mathcal{P}_e(x_{[1:K]}; \Phi)$. Finally, we provide an estimation of $\mathcal{L}(\Phi; \Phi)$.

Since $\Phi$ is known, some simplifications on $\mathcal{P}_e(x_{[1:K]}; \Phi)$ is available. We have

$$\mathcal{P}_e(x_{[1:K]}; \Phi) = \prod_{k=1}^{K} \mathbb{P}(\|X_{k|k-1} - x_k\|_\infty \leq \epsilon)$$

$$= \prod_{k=1}^{K} \mathbb{P}(\|W_k + f(x_{k-1}, u_k) - f(x_{k-1}, u_k) - w_k\|_\infty \leq \epsilon)$$

$$= \prod_{k=1}^{K} \mathbb{P}(\|W_k - w_k\|_\infty \leq \epsilon).$$

Here, we provide a simple example to illustrate how to calculate $\mathcal{P}_e(x_{[1:K]}; \Phi)$.

**Example 2.** Suppose the SDS, $\Phi$, to be predicted has state dimension $d = 1$, and the prediction error tolerance $\epsilon = 0.6$. It follows that

$$\mathcal{P}_e(x_{[1:K]}; \Phi) = \prod_{k=1}^{K} \int_{\|s - w_k\|_\infty \leq \epsilon} q(s)ds.$$  

When $K = 5$, calculating the rolling prediction probability is multiplying the area of blue strips in Fig. 3.

This example intuitively admits the two-fold difficulties in computing rolling prediction probability: integration of continuous function and random locations of blue strips. Fortunately, these difficulties can be overcome by applying discretization method to rolling prediction probability.

#### A. Discretization

Consider the partition $\Sigma$ on the value space of $x$, and the corresponding discrete rolling prediction $\mathcal{P}_\Sigma(x_{[1:K]}; \Phi)$ is defined as follows.
Definition 9 (Model-known discrete rolling prediction). During time $1$ to time $K$, $\Phi$ has generated a series of states $x_{1:K}$ with noises $w_{1:K}$, where $x_i \sim x_i \forall 1 \leq i \leq K$. Then,

$$
P_{\Sigma}(x_{1:K}; \Phi) \triangleq \prod_{k=1}^{K} \mathbb{P}[\Theta(W_k) = \Theta(w_k)].$$

Remark 2. The only difference between the discrete rolling prediction probability $P_{\Sigma}(x_{1:K}; \Phi)$ and the continuous version $P_{\epsilon}(x_{1:K}; \Phi)$ is the accuracy requirement: $P_{\Sigma}(x_{1:K}; \Phi)$ requires predicted states and real states to be located in the same region defined by partition $\Sigma$, while $P_{\epsilon}(x_{1:K}; \Phi)$ requires the distance between predicted states and real states to be less than a given tolerance $\epsilon$. The discretization method makes the evaluation of $P_{\Sigma}(x_{1:K}; \Phi)$ tractable.

Next, we demonstrate how to build the bridge between the probability $P_{\Sigma}(x_{1:K}; \Phi)$ and $P_{\epsilon}(x_{1:K}; \Phi)$.

Theorem 1 (Evaluation of $P_{\epsilon}(x_{1:K}; \Phi)$). Considering the SDS $\Phi$ with partition $\Sigma$ on $x$, the model-known discrete rolling prediction probability for trajectory $x_{1:K}$ is

$$
P_{\Sigma}(x_{1:K}; \Phi) = \exp\{-K[H_s(q_{\Sigma}; K) + D_{KL}(q_{\Sigma}; q_{\Sigma})]\},$$

where $q_{\Sigma}$ is the discrete approximation of $q$ based on $\Sigma$, $q_{\Sigma}$ is the empirical distribution of $w_{1:K}$, satisfying

$$q_{\Sigma, K}(u) = \frac{1}{K} \sum_{j=1}^{N} \sum_{k=1}^{K} I_{A_j}(w_k).$$

Proof. See Appendix $\Box$

The introduction of empirical distribution $q_{\Sigma, K}$ gives $P_{\epsilon}(x_{1:K}; \Phi)$ an elegant expression related with information theory. This theorem is quite instructive in two perspectives: i) discrete rolling prediction is decided by the uncertainty of $q_{\Sigma, K}$ and the distance between $q_{\Sigma, K}$ and $q_{\Sigma}$, as $K$ goes to infinity, $q_{\Sigma, K}$ is fixed and $q_{\Sigma, K}$ is converging to $q_{\Sigma}$; ii) discrete rolling prediction probability has an exponential decaying trend against time, and the decaying rate at each time step is explicitly expressed. The from of discrete rolling prediction probability drives us to find the relationship between it and continuous version. An inequality relationship between $P_{\Sigma}(x_{1:K}; \Phi)$ and $P_{\epsilon}(x_{1:K}; \Phi)$ is provided:

Lemma 1 (Sandwich lemma),

$$\max_{\{\Sigma\} \text{diam}(\Sigma) \leq \epsilon} P_{\Sigma}(x_{1:K}; \Phi) \leq P_{\epsilon}(x_{1:K}; \Phi) \leq \max_{\Sigma} P_{\Sigma}(x_{1:K}; \Phi),$$

where $\text{diam}(\Sigma) \triangleq \max_{A \in \Sigma} \max_{x \in A} \|x - y\|_\infty$.

Proof. See Appendix $\Box$

This sandwich lemma provides a coarse way to bound rolling prediction probability by discrete rolling prediction probability from both upper and lower directions. Just knowing $P_{\epsilon}(x_{1:K}; \Phi)$ can be bounded from both upper and lower directions by $P_{\Sigma}(x_{1:K}; \Phi)$ is not convenient for calculation, a more satisfying result should be that there exist a partition such that calculating the discrete case is sufficient to know the value of continuous case. Motivated by inequality of Lemma $\Box$ we confirm the existence of a special partition to evaluate $P_{\Sigma}(x_{1:K}; \Phi)$, and obtain a theorem as follows.

Theorem 2 (Existence of $\Sigma^*(K)$). Consider an SDS $\Phi$, there exists a partition, $\Sigma^*(K)$, such that

$$P_{\epsilon}(x_{1:K}; \Phi) = P_{\Sigma^*(K)}(x_{1:K}; \Phi).$$

Proof. See Appendix $\Box$

The existence of $\Sigma^*(K)$ ensures that it is reasonable to transform the calculation of the continuous probability into calculation of a discrete one. However, this transformation is only a formal method, and the existence theorem itself does not provide a tractable algorithm to design $\Sigma^*(K)$.

The discretization method proposed has released us from integration because we can now concentrate on

$$\lim_{K \to \infty} \frac{1}{K} \ln P_{\Sigma^*(K)}(x_{1:K}; \Phi),$$

which is more convenient to be calculated.

B. Evaluation

When $K$ is approaching infinity, the law of large number ensures that empirical distribution $q_{\Sigma^*(K), K}$ is converging toward $q_{\Sigma^*(K)}$ almost surely, thus the asymptotic decaying rate of rolling prediction probability only rely on $q_{\Sigma^*(K)}$. Define a functional $L$ by

$$L(q_{\Sigma^*(K)}) = \max_{x} - \frac{\ln(q_{\Sigma^*(K)}(x))}{q_{\Sigma^*(K)}(x)}.$$

Then, we have a theorem as follow.

Theorem 3 (Evaluation of $\mathcal{I}_e(\Phi; \Phi)$). The asymptotic decaying rate of rolling prediction $\mathcal{I}_e(\Phi; \Phi)$ equals to the converging value of sequence $\{H_s(q_{\Sigma^*(K)}(x))\}_{K=1}^{\infty}$, i.e.,

$$\mathcal{I}_e(\Phi; \Phi) = \lim_{K \to \infty} H_s(q_{\Sigma^*(K)}).$$
Meanwhile, the converging speed toward $\mathcal{I}_c(\Phi; \Phi)$ is $O(e^{-K})$ in the sense of probability:

$$\mathbb{P}(\left| -\frac{1}{K} \ln \mathcal{P}_{I^0(\Sigma)}(x_{[1:K]}; \Phi) - \mathcal{I}_c(\Phi; \Phi) \right| \geq t) \leq 2 \exp \left\{ -\frac{2Kt^2}{L(\mathcal{G}^0(\Sigma))^2} \right\}.$$  

Proof. See Appendix [D].

Remark 3. This theorem not only shows that $\mathcal{I}_c(\Phi; \Phi)$ is the limit of an entropy sequence, but also introduces an efficient way to estimate it because of the exponential convergence of $-\frac{1}{K} \ln \mathcal{P}_{I^0(\Sigma)}(x_{[1:K]}; \Phi)$ toward $\mathcal{I}_c(\Phi; \Phi)$. Conversely, $\mathcal{P}_{I^0(\Sigma)}(x_{[1:K]}; \Phi)$ can also be approximated by $e^{-K\mathcal{I}_c(\Phi; \Phi)}$.

If $\Sigma^*(K)$ is given, then an explicit expression of $\mathcal{I}_c(\Phi; \Phi)$ is derived. However, usually we only know the existence of $\Sigma^*(K)$ and do not have the specific form, so an approximation of $\mathcal{I}_c(\Phi; \Phi)$ is needed. Define an error functional $\delta_\epsilon(q)$ by

$$\delta_\epsilon(q) \triangleq \max_{x_1, x_2 \in \mathbb{R}^d} |\ln(q(x_1)) - \ln(q(x_2))|,$$

where $x_1, x_2 \in \mathbb{R}^d$ are two arbitrary states of the SDS $\Phi$.

Given any distribution $q$, define $\rho(q)$ as the maximum value of the solution set to an inequality,

$$\rho(q) = \max_{\epsilon \geq 0} \left\{ z : d(\ln(\frac{z}{\epsilon})) \leq \delta_\epsilon(q) \right\}.$$

On the other hand, when $z = 2$ the above inequality holds, thus the solution set is not empty. On the other hand, note the fact that $\delta_\epsilon(q)$ is bounded and monotonically increase with $z$, the solution set must be upper bounded. Therefore, $\rho(q)$ is well-defined. Then, we have a theorem as follows.

Theorem 4 (Approximation of $\mathcal{I}_c(\Phi; \Phi)$). For a $d$ dimensional stochastic dynamical system $\Phi$, $\mathcal{I}_c(\Phi; \Phi)$ can be approximated by $H_\Delta(q) - d\log(2e)$ such that

$$|\mathcal{I}_c(\Phi; \Phi) - H_\Delta(q) + d\log(2e)| \leq 2\delta_\epsilon(q).$$

Proof. See Appendix [E].

Remark 4. This theorem gives an accurate approximation to $\mathcal{I}_c(\Phi; \Phi)$. The error is controlled by $2\delta_\epsilon(q)$.

In the last section, we have derived the explicit expression of $\mathcal{I}_c(\Phi; \Phi)$. Now, we shall deal with the optimization over $\Phi$ on $\mathcal{I}_c(\Phi; \Phi)$. In other words, when both determined and stochastic parts of $\Phi$ are both unknown, the optimal SDS $\Phi^*$ need to be found out. Naturally, we may guess the optimal model for prediction is exactly the original model $\Phi$. If it is true, then the evaluation of predictability exponent in model-known case can be utilized and a lot of work can be saved.

Therefore, our main work in this section is to prove the hypothesis that $\Phi^* = \Phi$. Before the proof, it’s still necessary to apply our discretization method to the model-unknown case similarly to the model-known case. After discretization and a series of preparations, we proceed proof in two steps: the first one is to prove that the determined part of $\Phi^*$ must be the same as $\Phi$, and the second one is to prove that the stochastic part of $\Phi^*$ should share the same distribution as $\Phi$.

A. Discretization

Some simplifications on rolling prediction probability is available. Let $\epsilon r_k$ be the model mismatch error of the determined part, satisfying

$$\epsilon r_k = f(x_{k-1}, u_{k-1}) - \tilde{f}(x_{k-1}, \tilde{u}_{k-1}).$$

It may be caused by learning algorithm, training data, knowledge of physical system, etc. Note that

$$\mathcal{P}_c(x_{[1:K]}; \tilde{\Phi}) = \prod_{k=1}^{K} \mathbb{P}(\|\tilde{X}_{k|k-1} - x_{k}\|_\infty \leq \epsilon) = \prod_{k=1}^{K} \mathbb{P}(\|\tilde{f}(x_{k-1}, \tilde{u}_{k-1}) + \tilde{W}_k - f(x_{k-1}, u_{k-1}) - w_k\|_\infty \leq \epsilon) = \prod_{k=1}^{K} \mathbb{P}(\|\tilde{W}_k - (\epsilon r_k + w_k)\|_\infty \leq \epsilon).$$

It is reasonable to describe $\epsilon r_k$ by a random variable obeying some unknown distribution $q_\epsilon$. Therefore, the distribution of $\epsilon r_k + w_k$, denoted as $\hat{q}$, is the convolution of $q(\cdot)$ and $q_\epsilon$, i.e.,

$$\hat{q}(x) = \int_{\mathbb{R}^d} q(y) q_\epsilon(x-y) dy.$$

This transformation provides a more compact expression for rolling prediction probability in model-unknown case, and is more suitable for discretization.

Given a partition $\Sigma$ of $\mathbb{R}^d$ and a label function $\Theta(\cdot)$ induced from it, the discrete version of rolling prediction in model-unknown case is introduced.

Definition 10 (Model-unknown discrete rolling prediction). During time $1$ to time $K$, $\Phi$ has generated a series of states $x_{[1:K]}$ with noises $u_{[1:K]}$ and model error $\epsilon r_{[1:K]}$, where $x_i \sim X_i \forall 1 \leq i \leq K$. Then,

$$\mathcal{P}_{\Sigma}(x_{[1:K]}; \tilde{\Phi}) \triangleq \prod_{k=1}^{K} \mathbb{P}(\Theta(\tilde{W}_k) = \Theta(w_k + \epsilon r_k)).$$

Remark 5. Different from the definition of model-known discrete rolling prediction probability, model-unknown case needs a tricky handling of model mismatch error which is compressed as $\epsilon r_k$. As a result, partition is available to the support of $\tilde{W}_k$. Directly making discretization on the space of $x_k$ is technically acceptable but harder to approximate the continuous rolling prediction probability by discrete ones, and that’s the necessity behind above transformations (6).

Moreover, this discretization method helps to derive similar expressions for $\mathcal{P}_c(x_{[1:K]}; \tilde{\Phi})$ just as the expressions for $\mathcal{P}_c(u_{[1:K]})$, and we need this similarity to prove the optimality.

The discrete version of rolling prediction is much more tractable to be evaluated.
Theorem 5 (Evaluation of \( P_{\Sigma}(x_{[1:K]}; \hat{\Phi}) \)). Considering the SDS \( \Phi \) with partition \( \Sigma \) on \( x \), the model-unknown discrete rolling prediction probability using \( \hat{\Phi} \) for trajectory \( x_{[1:K]} \) is
\[
P_{\Sigma}(x_{[1:K]}; \hat{\Phi}) = \exp\left\{-K[H_s(q_{\Sigma,K}) + D_{KL}(\hat{q}_{\Sigma,K}||\hat{q}_{\Sigma,K})]\right\},
\]
where \( \hat{q}_{\Sigma,K} \) is the discrete approximation of \( \hat{q} \) based on \( \Sigma \). \( q_{\Sigma,K} \) is the empirical distribution of \( \{w_k + \text{err}_k\} \), satisfying
\[
\hat{q}_{\Sigma,K}(u) = \frac{1}{K} \sum_{j=1}^{[\Sigma]} I_{A_j}(u) \sum_{k=1}^{K} I_{A_j}(w_k + \text{err}_k).
\]
Proof. See Appendix G \( \Box \)

Remark 6. It should be noted that \( \hat{q}_{\Sigma} \) only depend on prediction model \( \hat{\Phi} \) and partition \( \Sigma \), while in contrast \( \hat{q}_{\Sigma,K} \) depends on real model \( \Phi \) and the mismatch error \( \text{err}_k \).

Theorem 6 (Existence of \( \hat{\Sigma}^*(K) \)). Consider a learned system \( \Phi \), there exists a partition, \( \hat{\Sigma}^*(K) \), such that
\[
P_{\Sigma}(x_{[1:K]}; \hat{\Phi}) = P_{\hat{\Sigma}^*(K)}(x_{[1:K]}; \hat{\Phi}).
\]
Proof. See Appendix G \( \Box \)

B. Evaluation

Prepared with discrete rolling prediction \( P_{\Sigma}(x_{[1:K]}; \hat{\Phi}) \), we can perform optimization over \( \Phi \) and derive the evaluation and approximation results on \( I_{e}(\Phi) \).

Theorem 7 (Optimization). Consider an SDS \( \Phi \), for any learned model \( \Phi \), we have
\[
I_{e}(\Phi; \hat{\Phi}) - I_{e}(\Phi; \hat{\Phi}) = \lim_{K \to \infty} H_s(q_{\hat{\Sigma}^*(K)}) - H_s(q_{\Sigma^*(K)}) + D_{KL}(\hat{q}_{\Sigma^*(K)}||\hat{q}_{\Sigma^*(K)}).
\]
It can be approximated by \( H_d(\hat{q}) - H_d(q) + D_{KL}(\hat{q}||\hat{q}) \) and bounded by \( 2[\delta_\rho(q) + \delta_\rho(\hat{q})] \), i.e.,
\[
\left| I_{e}(\Phi; \hat{\Phi}) - I_{e}(\Phi; \hat{\Phi}) \right| 
\leq 2 \left[ \delta_\rho(q) + \delta_\rho(\hat{q}) \right].
\]
Moreover, the optimal model \( \hat{\Phi}^* \) for rolling prediction that attains the slowest asymptotic decaying rate is the real model, which means \( \hat{\Phi}^* = \Phi \), i.e., \( \Phi = \arg \min_{\hat{\Phi}} I_{e}(\Phi; \hat{\Phi}) \).

Proof. See Appendix H \( \Box \)

Remark 7. Although \( \hat{\Phi} \) is the optimal model for rolling prediction that attains the slowest asymptotic decaying rate, it’s not necessarily the optimal model that attains the largest rolling prediction probability for a specific time step \( K \) or depends on the empirical distribution of \( x_{[1:K]} \) and thus no explicit expression is available. Nevertheless, if we consider the asymptotic decaying rate, the optimality is clear.

Finally, combining the optimality conclusion and results in model-known case the predictability exponent can be well evaluated and approximated in the following two theorems. Using the fact that \( I_{e}(\Phi; \Phi) = I_{e}(\Phi) \), the following two theorems directly follow from Theorem 3 and Theorem 4, respectively.

Theorem 8 (Evaluation of \( I_{e}(\Phi) \)). The minimum asymptotic decaying rate of rolling prediction \( I_{e}(\Phi) \) equals to the converging point of sequence \( \{H_s(q_{\Sigma^*(K)})\} \), i.e.,
\[
I_{e}(\Phi) = \lim_{K \to \infty} H_s(q_{\Sigma^*(K)}).
\]
Meanwhile, the converging speed toward \( I_{e}(\Phi) \) is \( O(e^{-K}) \) in the sense of probability and
\[
P\left\{ I_{e}(\Phi; \hat{\Phi}) - I_{e}(\Phi) \right\} \leq 2 \exp\left\{ -2\left( \frac{Kt^2}{L} \right) \right\}.
\]

Theorem 9 (Approximation of \( I_{e}(\Phi) \)). For a \( d \) dimensional stochastic dynamical system \( \Phi \), its predictability exponent can be approximated by \( H_d(q) - d \log(2e) \), such that
\[
\left| I_{e}(\Phi) - H_d(q) + d \log(2e) \right| \leq 2\delta_\rho(q,e)(q).
\]

In summary, we have proposed a metric \( I_{e}(\Phi) \) to quantitatively characterize the predictability of \( \Phi \). Although defined as the asymptotic decaying rate of \( P_{\Sigma}(x_{[1:K]}; \hat{\Phi}) \), it is the limit of \( H_s(q_{\Sigma^*(K)}) \) in essence. Moreover, the decaying rate converge to predictability exponent in speed of scale \( O(e^{-K}) \). Finally, \( I_{e}(\Phi) \) can be approximated by \( H_d(q) - d \log(2e) \) within error of scale \( O(e) \).

In the next section, we will apply predictability exponent to the design of unpredictable stochastic dynamical system, and compare the result to previous work [20].

V. Application: Unpredictable Stochastic Dynamical System

In this section, we discuss the design of unpredictable stochastic dynamical system. If a clear definition on predictability is available, designing unpredictable SDS is equivalent to optimizing on predictability metric. Once the optimization problem is solved, the solution will serve as an unpredictable SDS in the sense of this predictability metric. In this section, we will find the predictability exponent \( I_{e}(\Phi) \) to design an unpredictable stochastic dynamical system, and then compare these design to another unpredictable design in
work [20]. Finally, an equivalence relation between these two designs is proved in Theorem [11].

A. Design of Unpredictable Stochastic Dynamical System

Just as defined before, we consider the prediction on Φ using ˆΦ, and we use the asymptotic decaying rate \( \mathcal{I}_e(\Phi; ˆ\Phi) \) to characterize the prediction performance. Obviously, the smaller the asymptotic decaying rate is, the better the prediction performs.

To design an unpredictable stochastic dynamical system Φ, we need to make \( \mathcal{I}_e(\Phi; ˆ\Phi) \) as large as possible regardless of the choice of ˆΦ. Therefore, a max-min optimization problem is formulated as follows.

\[
\max_{\Phi} \min_{\Phi} \mathcal{I}_e(\Phi; ˆ\Phi). \tag{7}
\]

Since the inner min optimization is embedded in the definition of predictability exponent, therefore what we need to do now is solving the outer max optimization on \( \mathcal{I}_e(\Phi) \). Substitute the predictability exponent by \( H_a(q) = d \log \epsilon \), we have the optimization object as

\[
\max_q H_a(q). \tag{8}
\]

Furthermore, if we add some common constraints to q, we have the following functional optimization problem

\[
\max_q H_a(q) \quad \text{s.t. } E(q) = 0, \text{Var}(q) = \sigma^2, M(\text{supp}(q)) < \infty, \tag{9}
\]

where \( \text{supp}(q) \) denotes the support of distribution q and \( M(\text{supp}(q)) \) is the Lebesgue measure of set \( \text{supp}(q) \).

Remark 8. We give some notes as follows.

1) If \( E(W) = E(q) \neq 0 \), we can let \( ˆf(x, u) = f(x, u) + E(q) \), then \( X_{k+1} = ˆf(X_k, u_k) + W - E(q) \). Let \( ˆq \) be the distribution of \( W - E(q) \), there is \( E(ˆW) = E(ˆq) = 0 \). Hence, we assume \( E(q) = 0 \) without loss of generality.

2) We focus on the class of distributions with the same variance.

3) Practically the noises embedded in a SDS are always bounded, we therefore make optimization with respect to the distributions with finite support.

Theorem 10. The optimal noise distribution \( q^* \) with fixed variance \( \sigma^2 \), zero expectation and finite support that can make a SDS unpredictable is the solution to problem (9), i.e.

\[
q^* \sim \frac{1}{2\sqrt{3\sigma}} \mathcal{L}[-\sqrt{3\sigma}, \sqrt{3\sigma}].
\]

Proof. See Appendix I

This conclusion is consistent with our intuition: SDS with uniformly distributed noise should be the most difficult to be predicted. However, this conclusion can be different if we choose different constraints in the optimization problem.

B. Equivalence with Another Unpredictable SDS

In [20], the design of unpredictable SDS is based on point prediction, which is modeled as the following min-max optimization problem.

\[
\min_q \max_u \int_{B_u(r)} q(x)dx \quad \text{s.t. } E(q) = 0, \text{Var}(q) = \sigma^2, M(\text{supp}(f)) < \infty, \forall r > 0. \tag{10}
\]

More specifically, \( u \) in the inner maximization represents the predicted point, the objective function, \( \int_{B_u(r)} q(x)dx \), is the probability that the distance between \( u \) and the real sample is less than \( r \); inner maximization attains the best performance of this prediction method based on point, while the outer minimization on \( q \) helps to find the best \( q \) to make the SDS unpredictable. In fact, this design is equivalent to our design based on predictability exponent, which is ensured by the following theorem.

Theorem 11. The unpredictable SDS design based on \( \mathcal{I}_e(\Phi) \) and the design based on one-step probability is equivalent, i.e., optimization (10) can be transformed to be optimization (9), and they share the same optimal solution \( q^* \).

Proof. See Appendix K

VI. SIMULATION

In this section, we simulate rolling prediction on a randomly generated linear stochastic dynamical system driven by Gaussian noises. On the one hand, under the assumption that \( \Phi \) is known, we verify that the optimal asymptotic prediction probability decaying rate can be approximated by \( H_a(q) = -d \log(2\epsilon) \) with error of scale \( O(\epsilon) \). Moreover, we verify the fast converging speed of decaying rate. On the other hand, under the assumption that \( \Phi \) is unknown, we choose Gaussian distributions to describe the model mismatch error. Then, we simulate the effect of different model mismatch errors on \( \mathcal{I}_e(\Phi; ˆ\Phi) \).

A. Simulation Setup

We consider a linear SDS as follows.

\[
\Phi : \begin{cases}
X_{k+1} = AX_k + W_k, & W_k \sim \mathcal{N}(\mu, \Sigma), \\
Y_k = X_k,
\end{cases}
\]

where \( X_k \in \mathbb{R}^2 \). We randomly generate \( A, \mu, \Sigma \) and normalize \( \Sigma \) to have spectral radius 1 without loss of generality. Setting the accuracy tolerance \( \epsilon \) to be 0.1 and the time step to be 400, we generate several state paths of \( \Phi \) starting from a random initial state. Then we calculate the decaying rates of \( P_e(x_{1:K}; \Phi) \) for each state path. Next, we consider the effect of model mismatch error \( \epsilon_{\text{M}} \sim \mathcal{N}(\tau [1, 1]^T, (1 + \eta)I) \). \( \tau \) ranges across 0, 0.5, 1, 1.5 and \( \eta \) ranges across 0, 0.5, 1, 1.5.
In Theorem 3 in the sense of probability. This fact has verified the optimality of $\Phi$, which is of the same scale as $P_{\epsilon}(\Phi)$. Therefore, Theorem 4 indeed gives an both elegant and accurate approximation to the predictability exponent $I_{\epsilon}(\Phi)$, whose value is approximated by the red dotted line.

B. Results and Analysis

1) Optimal asymptotic decaying rate: Driven by the stochastic noises, the same initial state can evolve into totally different trajectories even they have the same SDS model. Fig. 4(a) has simulated three possible state trajectories of $\Phi$, and Fig. 4(b) presents the relationship between their exponential decaying rate and time steps.

First, the approximation performance of $H_{d}(q) - d \log(2\epsilon)$ is verified to be correct. As Fig. 4(b) shows, all the trajectories of decaying rate converge to the same red dotted line (which is the theoretical asymptotic decaying rate). Moreover, if we zoom in the graph, it is clear that the approximation error is bounded by 0.05, which is of the same scale as $\epsilon$. Therefore, Theorem 4 indeed gives an both elegant and accurate approximation to the predictability exponent $I_{\epsilon}(\Phi)$.

Second, the converging rate is quite fast. Fig. 4(b) shows that all trajectories approaches red dotted line in less than 50 time steps, this quick convergence is ensured by the concentration inequality in Theorem 3 in the sense of probability.

Third, the fact that different trajectories generated from the same SDS hold the same asymptotic decaying rate has reconfirmed the advantage of predictability exponent. This metric is defined directly from probabilistic prediction performance and does not depend on specific state trajectory generated from a SDS, therefore it views different trajectories in Fig. 4(a) as having the same predictability.

2) Effect of model error: On the one hand, both Fig. 5(a) and Fig. 5(b) have shown us the effect of model error in expectation and variance respectively, obviously the larger the model error is the larger the asymptotic decaying rate will be. This fact has verified the optimality of $\Phi$ in rolling prediction.

On the other hand, expectation error (Fig. 5(b)) and variance error (Fig. 5(a)) do have different effects, and expectation error seems to have greater influence on decaying rate than variance error. The deviation from optimal value grows faster as $\tau$ becomes larger, while the deviation from optimal value grows equally as $\eta$ becomes larger. This provides an insight for the model selection: we should put more attention on the
expectation error of our model, which means an unbiased model is better for rolling prediction.

VII. CONCLUSION

In this paper, we studied predictability of SDSs by introducing a new probabilistic performance metric, i.e., the exponential decaying rate of the probability that the prediction error never exceed a given $\epsilon$. The proposed metric is defined directly from the prediction performance rather than intuitive entropy-based characterization, therefore able to directly quantify to what extent an SDS can be accurately predicted. Moreover, an elegant approximation of this metric is available, which is deeply related with the differential entropy of SDSs. This impressive fact provides a better understanding of the relationship between predictability and information theory.

APPENDIX A
PROOF OF THEOREM

Note that $\mathcal{P}_p(x_{[1:K];\Phi})$ and $\mathcal{P}_\Sigma(x_{[1:K];\Phi})$ only depends on $w_{[1:K]}$ when $\Phi$ is known. We thus use $\mathcal{P}_p(w_{[1:K]})$ and $\mathcal{P}_\Sigma(w_{[1:K]})$ to represent them, respectively, in following proofs for ease notation, which will not cause any confusion.

Suppose the discrete distribution on $\{A_i\}_{i=1}^{[\Sigma]}$ induced by the step-wise function $q_\Sigma$ is

$$p(i) = q_\Sigma(x_0),$$

where $x_0$ is any point belonging to $A_i$. Similarly, we have the discrete distribution on $\{A_i\}_{i=1}^{[\Sigma]}$ induced by the step-wise function $q_{\Sigma,K}$ is

$$p_K(i) = q_{\Sigma,K}(x_0),$$

where $x_0$ is any point belonging to $A_i$.

On the one hand, according the definition of discrete rolling prediction probability, there is

$$\mathcal{P}_\Sigma(w_{[1:K]}) = \prod_{k=1}^{K} \mathbb{P}[\Theta(W_k) = \Theta(w_k)] = \prod_{k=1}^{K} q_\Sigma(w_k) = \prod_{i=1}^{[\Sigma]} p(n)^{N_n},$$

where $N_n = \sum_{k=1}^{K} I_{A_n}(w_k)$. The point to be emphasized is that, the last equality is performing classification on sequence $\{q_\Sigma(w_k)\}_{k=1}^{K}$ based on their values, i.e., gathering those terms possessing the same values together. For example, after enumeration of this sequence we found there are $N_n$ values being exactly $p(n)$, so their product can be gathered simply as $p(n)^{N_n}$.

On the other hand, given any $n$ s.t. $1 \leq n \leq [\Sigma]$ and any point $u \in A_n$, there is

$$p_K(n) = \frac{1}{K} \sum_{j=1}^{[\Sigma]} I_{A_j}(u) \sum_{k=1}^{K} I_{A_j}(w_k) = \frac{1}{K} \sum_{k=1}^{K} I_{A_n}(w_k) = \frac{N_n}{K}.$$ 

Therefore, we continue to find that

$$\mathcal{P}_\Sigma(w_{[1:K]}) = \prod_{n=1}^{[\Sigma]} p(n)^{N_n} = \exp \left\{ \sum_{n=1}^{[\Sigma]} N_n \ln(p(n)) \right\} = \exp \left\{ K \sum_{n=1}^{[\Sigma]} p_K(n) \ln(p(n)) \right\} = \exp \left\{ -K \left[ -\sum_{n=1}^{[\Sigma]} p_K(n) \ln(p_K(n)) + \sum_{n=1}^{[\Sigma]} p_K(n) \ln\left(\frac{p_K(n)}{p(n)}\right) \right] \right\} = \exp \left\{ -K \left[ H_n(q_\Sigma;K) + D_{KL}(q_{\Sigma,K}||q_\Sigma) \right] \right\}.$$ 

The proof is completed.

APPENDIX B
PROOF OF LEMMA

For simplicity, we call a discrete rolling prediction a $\Sigma$-rolling prediction if for each prediction there is certain region $A \in \Sigma$ where the predicted point and the real point locate. Similarly, we call a continuous rolling prediction an $\epsilon$-rolling prediction if for each prediction there is a region $[u,u+\epsilon)$ where both the predicted point and the real point locate. As a result, we have $\mathcal{P}_\Sigma(w_{[1:K]}) \leq \mathcal{P}_\epsilon(w_{[1:K]})$ under the assumption that $\max_{\Sigma\mid \text{diam}(\Sigma) \leq \epsilon} \mathcal{P}_\Sigma(w_{[1:K]}) \leq \mathcal{P}_\epsilon(w_{[1:K]})$.

On the other hand, when it comes to the extreme case where $\Sigma = A_1$ and $A_1 = \mathbb{R}^d$, there is $\mathcal{P}_\Sigma(w_{[1:K]}) = 1$. As a result, $\mathcal{P}_\epsilon(w_{[1:K]}) \leq \max_{\Sigma} \mathcal{P}_\Sigma(w_{[1:K]})$.

APPENDIX C
PROOF OF THEOREM

First, we prepare some necessary settings. Let $\mathcal{S}$ be the space composed of all partitions of $\mathbb{R}^d$. To make $\mathcal{S}$ a metric
space, we implement $S$ with a partition distance metric $D(\cdot, \cdot)$ which is defined in [23], [24]:

$$D(P, Q) = \min \left\{ M(A^c) : \emptyset \subset A \subset \mathbb{R}^d, P^A = Q^A \right\},$$

where $P^A$ is a partition of set $A$ induced by $P$, i.e., if $P = \bigcup_{i=1}^m B_i$ then $P^A = \bigcup_{i=1}^m (B_i \cap A)$.

Intuitively, partition distance $D(P, Q)$ is the minimum measure of set that must be deleted from $\mathbb{R}^d$, so that the two induced partitions ($P$ and $Q$ restricted to the remaining elements) are identical to each other. It’s trivial to verify that $D(\cdot, \cdot)$ satisfies all three requirements of a distance metric.

Consider this functional operator:

$$\mathcal{F} : Y \to \mathbb{R}$$

$$\Sigma \mapsto \mathcal{P}_\Sigma(x_{[1:K]}).$$

According to Theorem 1 there is

$$\mathcal{F}(\Sigma) = \exp (-N[H(q_{\Sigma,K}) + D_{KL}(q_{\Sigma,K} \| |q_{\Sigma}])] ,$$

where $q_{\Sigma,K}$ is the empirical distribution of $w_{[1:K]}$:

$$q_{\Sigma,K}(u) = \frac{1}{K} \sum_{j=1}^{[\Sigma]} \mathbb{I}_{A_j}(u) \sum_{i=1}^{K} \mathbb{I}_{A_j}(w_i),$$

and $q_{\Sigma}$ is the discretized step-wise function of $q$:

$$q_{\Sigma}(x) = \sum_{i=1}^{n} \mathbb{I}_{A_i}(x) \int_{A_i} q(u) du .$$

Second, we define the continuity in metric space $S$ with distance metric $D(\cdot, \cdot)$. Continuity means for any $\Sigma = \bigcup_{i=1}^{[\Sigma]} A_i \subset Y$ and any converging partition sequence

$$\left\{ \Sigma^n = \bigcup_{i=1}^{[\Sigma^n]} A^n_i \right\}_{n=1}^{\infty},$$

where $\lim_{n \to \infty} D(\Sigma^n, \Sigma) = 0$, there is $\lim_{n \to \infty} \mathcal{F}(\Sigma^n) = \mathcal{F}(\Sigma)$.

According to the converging definition of partition sequence, given any $\gamma > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$ we have $D(\Sigma^n, \Sigma) < \gamma$. Pick any $x \in \mathbb{R}^d$, and suppose $x \in A_i$. When $n > N$ we have $A_i \subset \Sigma^n$ such that $M(A_i \Delta A^n_i) \subseteq \gamma$, where $M(\cdot)$ denotes the Lebesgue measure, and $\Delta$ denotes the symmetric difference between two sets.

Third, we prove the continuity of $q_{\Sigma}$. Note that

$$|q_{\Sigma}(x) - q_{\Sigma^n}(x)|$$

$$= \sum_{i=1}^{n} \mathbb{I}_{A_i}(x) \int_{A_i} q(u) du - \sum_{i=1}^{n} \mathbb{I}_{A^n_i}(x) \int_{A^n_i} q(u) du$$

$$= \left| \int_{A_i} q(u) du - \int_{A^n_i} q(u) du \right|$$

$$\leq \int_{A_i \Delta A^n_i} q(u) du .$$

Since $\lim_{n \to \infty} M(A_i \Delta A^n_i) = 0$, we have

$$\lim_{n \to \infty} |q_{\Sigma}(x) - q_{\Sigma^n}(x)| = 0 .$$

Fourth, we prove the continuity of $q_{\Sigma,K}$:

$$|q_{\Sigma,K}(x) - q_{\Sigma^n,K}(x)|$$

$$= \frac{1}{K} \sum_{j=1}^{[\Sigma]} \mathbb{I}_{A_j}(x) \sum_{i=1}^{K} \mathbb{I}_{A_j}(w_i) - \frac{1}{K} \sum_{j=1}^{[\Sigma^n]} \mathbb{I}_{A_j}(x) \sum_{i=1}^{K} \mathbb{I}_{A_j}(w_i)$$

$$= \frac{1}{K} \sum_{i=1}^{K} \left( \mathbb{I}_{A_i}(w_i) - \mathbb{I}_{A^n_i}(w_i) \right)$$

Notice that $\lim_{n \to \infty} M(A_i \Delta A^n_i) = 0$, we have

$$\lim_{n \to \infty} \sum_{i=1}^{K} \left( \mathbb{I}_{A_i}(w_i) - \mathbb{I}_{A^n_i}(w_i) \right) = 0 .$$

Finally, the above two results lead to the continuity of $\mathcal{F}(\Sigma)$ because $H(\cdot)$ and $D_{KL}(\cdot)$ are continuous functions.

The bounded inequality

$$\max_{\{\Sigma : \text{diam}(\Sigma) \leq \epsilon\}} \mathcal{P}_\Sigma(w_{[1:K]}) \leq \mathcal{P}_\Sigma(w_{[1:K]}),$$

suggests the existence of $\Sigma_a, \Sigma_b \in Y$ such that

$$\mathcal{P}_{\Sigma_a}(w_{[1:K]}) \leq \mathcal{P}_{\Sigma_b}(w_{[1:K]}).$$

Then, the intermediate value theorem [25] admits the existence of $\Sigma^*(K) \in Y$ such that

$$\mathcal{P}_{\Sigma^*(K)}(w_{[1:K]}) = \mathcal{P}_\Sigma(w_{[1:K]}) .$$

The proof is completed.

### Appendix D

#### Proof of Theorem 3

For the first part of this theorem, suppose the discrete distribution on $\{A_i\}_{i=1}^{[\Sigma^*(K)]}$ induced by the step-wise function $q_{\Sigma}$ is:

$$p(i) = q_{\Sigma^*(K)}(x_0),$$

where $x_0$ is any point belonging to $A_1$. Similarly, the discrete distribution on $\{A_i\}_{i=1}^{[\Sigma^*(K)]}$ induced by the step-wise function $q_{\Sigma^*(K)}$ is:

$$p_K(i) = q_{\Sigma^*(K),K}(x_0),$$

where $x_0$ is any point belonging to $A_i$.

It follows that

$$\mathcal{L}(\Phi; \Phi) = \lim_{K \to \infty} -\frac{1}{K} \ln \mathcal{P}_\Sigma(w_{[1:K]}; \Phi)$$

$$= \lim_{K \to \infty} -\frac{1}{K} \ln \mathcal{P}_{\Sigma^*(K)}(w_{[1:K]}; \Phi)$$

$$= \lim_{K \to \infty} -\frac{1}{K} \sum_{n=1}^{[\Sigma^*(K)]} p_K(n) \ln(p(n)).$$

According to strong law of large number, we have

$$\lim_{K \to \infty} p_K(n) \overset{a.s.}{=} p(n).$$
Therefore, it follows that
\[ I_e(\Phi; \Phi) = \lim_{K \to \infty} \sup_n - \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p_K(n) \ln(p(n)) \]
\[ = \lim_{K \to \infty} - \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p_K(n) \ln(p(n)) \]
\[ = - \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p(n) \ln(p(n)) \]
\[ = H_s(q_{\Sigma^*(K)}). \]

For the second part of this theorem,
\[ \mathbb{P}\left( \left| - \frac{1}{K} \ln p_{\Sigma^*(K)}(w_{[1,K]}) - I_e(\Phi; \Phi) \right| \geq t \right) \]
\[ \leq \mathbb{P}\left( \left| \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p(n)(p(n) - p_K(n)) \frac{\ln(p(n))}{p(n)} \right| \geq t \right) \]
\[ \leq \mathbb{P}\left( \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p(n)|p(n) - p_K(n)| \geq \frac{t}{L(q_{\Sigma^*(K)})} \right). \]

According to Hoeffding inequality, for \( \forall 1 \leq n \leq |\Sigma^*(K)| \) there is
\[ \mathbb{P}\left( |p(n) - p_K(n)| \geq \frac{t}{L(q_{\Sigma^*(K)})} \right) \leq 2 \exp\left\{ - \frac{2Kt^2}{L(q_{\Sigma^*(K)})^2} \right\}. \]

Together with the fact that
\[ \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p(n)|p(n) - p_K(n)| \leq \max_n \{|p(n) - p_K(n)|\}, \]
one infers that
\[ \mathbb{P}\left( \sum_{n=1}^{\lfloor \Sigma^*(K) \rfloor} p(n)|p(n) - p_K(n)| \geq \frac{t}{L(q_{\Sigma^*(K)})} \right) \]
\[ \leq \mathbb{P}\left( \max_n \{|p(n) - p_K(n)|\} \geq \frac{t}{L(q_{\Sigma^*(K)})} \right) \]
\[ \leq 2 \exp\left\{ - \frac{2Kt^2}{L(q_{\Sigma^*(K)})^2} \right\}. \]

The proof is completed.

APPENDIX E

PROOF OF THEOREM 3

Suppose \( \Sigma^*(K) = \bigcup_{i=1}^{\infty} A_i(K), \) and there exists \( a_i(K) \in A_i(K) \) such that \( q(a_i(K))|A_i(K)| = q_{\Sigma^*(K)}(a_i(K)), \) where \( |A_i(K)| \) denotes the Lebesgue volume of \( A_i(K). \) For the convenience of analysis, we let \( A_i(K) \) be a cube with diameter equalling \( r(K)\epsilon. \) Then it follows that
\[ I_e(\Phi; \Phi) = \lim_{K \to \infty} H_s(q_{\Sigma^*(K)}) \]
\[ = \lim_{K \to \infty} - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{q(a_i(K))|A_i(K)|\} \]
\[ = \lim_{K \to \infty} - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{q(a_i(K))\} \]
\[ = \lim_{K \to \infty} - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{|A_i(K)|\} \]
\[ - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{|A_i(K)|\}. \]

Denote \( r^* = \lim_{K \to \infty} r(K), \) then the second term above can be organized as follows
\[ \lim_{K \to \infty} - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{|A_i(K)|\} \]
\[ = \lim_{K \to \infty} - \ln(r(\epsilon)) \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \]
\[ = \lim_{K \to \infty} - \ln(r(\epsilon)) = - \ln(r(\epsilon)). \]

The first term is actually a Riemann integration of the differential entropy of \( q, \)
\[ \lim_{K \to \infty} \left| - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{q(a_i(K))\} - H_d(q) \right| \]
\[ = \lim_{K \to \infty} \left| \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{q(a_i(K))\} \right| \]
\[ - \int_{A_i(K)} q(x) \ln(q(x)) dx \]
\[ = \lim_{K \to \infty} \left| \sum_{i=1}^{\infty} \int_{A_i(K)} q(x) \ln\left( \frac{q(x)}{q(a_i(K))} \right) dx \right| \]
\[ \leq \lim_{K \to \infty} \sum_{i=1}^{\infty} \int_{A_i(K)} q(x) dx \right| \delta_{r(K)}(q) \]
\[ = \lim_{K \to \infty} \delta_{r(K)}(q) \]
\[ = \delta_{r^*}(q). \]

It follows that
\[ |I_e(\Phi; \Phi) - H_d(q) + d \ln(r^*)| \]
\[ = \lim_{K \to \infty} \left| - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{q(a_i(K))\} - H_d(q) \right| \]
\[ - d \ln(r^*) + d \ln(r^*) \]
\[ = \lim_{K \to \infty} \left| - \sum_{i=1}^{\infty} q(a_i(K))|A_i(K)| \ln\{q(a_i(K))\} - H_d(q) \right| \]
\[ \leq \delta_{r^*}(q). \]
Denote the \( \epsilon \) neighborhood of \( x \) as set \( \mathcal{N}_\epsilon(x) \triangleq \{ y \mid \| y - x \|_\infty \leq \epsilon \} \). On the one hand, we have

\[
\mathcal{P}_\epsilon(w_{1:K]) = \prod_{k=1}^{K} \mathbb{P}(W_k \in \mathcal{N}_\epsilon(w_k)) = \prod_{k=1}^{K} (2\epsilon)^d q(\tilde{w}_k),
\]

where \( \tilde{w}_k \in \mathcal{N}_\epsilon(w_k) \) satisfying

\[
q(\tilde{w}_k) = \frac{1}{(2\epsilon)^d} \int_{\mathcal{N}_\epsilon(w_k)} q(u) du.
\]

On the other hand,

\[
\mathcal{P}_{\Sigma^*}(w_{1:K]) = \prod_{k=1}^{K} \mathbb{P}(\Theta(W_k) = \Theta(w_k)) = \prod_{k=1}^{K} (r(K)\epsilon)^d q(s_{\Theta(w_k)}),
\]

where \( s_{\Theta(w_k)} \in A_{\Theta(w_k)} \) satisfying

\[
q(s_{\Theta(w_k)}) = \frac{1}{(r(K)\epsilon)^d} \int_{A_{\Theta(w_k)}} q(u) du.
\]

It follows that

\[
0 = \ln[\mathcal{P}_\epsilon(w_{1:K}]) \] \(- \ln[\mathcal{P}_{\Sigma^*}(w_{1:K})]\]

\[
dK \ln\left(\frac{2}{r(K)}\right) + \sum_{k=1}^{K} \ln[q(\tilde{w}_k)] - q(s_{\Theta(w_k)})].
\]

Furthermore,

\[
|d \ln\left(\frac{2}{r^*}\right)| \leq \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} |\ln[q(\tilde{w}_k)] - q(s_{\Theta(w_k)})] \leq \delta_{r^*}(q).
\]

Because \( \rho(q) \) is the maximum solution to equation \( |d \ln\left(\frac{2}{r^*}\right)| \leq \delta_{r^*}(q) \), we have \( \delta_{r^*}(q) \leq \delta_{\rho(q)}(q) \). Immediately,

\[
|\mathcal{I}_e(\Phi; \Phi) - H_d(q) + d \ln(2\epsilon)| \leq |\mathcal{I}_e(\Phi; \Phi) - H_d(q) + d \ln(r^*)| \leq \delta_{r^*}(q) + |d \ln(2\epsilon)| \leq 2\delta_{r^*}(q).
\]

The proof is completed.

**APPENDIX F**

**Proof of Theorem**

Suppose the discrete distribution on \( \{ A_i \}_{i=1}^{\Sigma} \) induced by the step-wise function \( \tilde{q}_{\Sigma} \):

\[
\tilde{p}(i) = \tilde{q}_{\Sigma}(x_0),
\]

where \( x_0 \) is any point belonging to \( A_i \). The discrete distribution on \( \{ A_i \}_{i=1}^{\Sigma} \) induced by the step-wise function \( \tilde{q}_{\Sigma,K} \) is:

\[
\tilde{p}_K(i) = \tilde{q}_{\Sigma,K}(x_0),
\]

where \( x_0 \) is any point belonging to \( A_i \).

According to the definition of discrete rolling prediction probability, there is

\[
\mathcal{P}_\Sigma(x_{1:K}; \tilde{\Phi}) = \prod_{k=1}^{K} \mathbb{P}(\Theta(\tilde{W}_k) = \Theta(w_k + \epsilon r_k)) = \prod_{n=1}^{N_n} \tilde{p}(n)^{N_n},
\]

where \( N_n = \sum_{k=1}^{K} \mathbb{I}_{A_\epsilon}(w_k + \epsilon r_k) \). It should be emphasized that, the last equality is doing a classification on sequence \( \{ \tilde{q}_{\Sigma}(w_k + \epsilon r_k) \}_{k=1}^{K} \) based on their values, which gathers the same values to one term. For example, after enumeration of this sequence we found that there are \( N_n \) values exactly \( \tilde{p}(n) \), so their product can be gathered simply as \( \tilde{p}(n)^{N_n} \).

Another observation is, the distribution induced by stepwise function \( \tilde{q}_{\Sigma,K} \) has the property that

\[
\tilde{p}_K(n) = \frac{1}{K} \sum_{j=1}^{K} \mathbb{I}_{A_\epsilon} \left( u \right) \mathbb{I}_{A_j} (w_i + \epsilon r_k) (u \in A_n)
\]

\[
= \frac{1}{K} \sum_{k=1}^{K} \mathbb{I}_{A_n} (w_k + \epsilon r_k) = \frac{N_n}{K}.
\]

Therefore, we continue to find that

\[
\mathcal{P}_\Sigma(x_{1:K}; \tilde{\Phi}) = \prod_{n=1}^{\Sigma} \tilde{p}(n)^{N_n} = \exp\left(\sum_{n=1}^{\Sigma} N_n \ln(\tilde{p}(n))\right) = \exp\left(\sum_{n=1}^{\Sigma} \tilde{p}_K(n) \ln(\tilde{p}(n))\right)
\]

\[
= \exp\left(-K \sum_{n=1}^{\Sigma} \ln(\tilde{p}_K(n)) + \sum_{n=1}^{\Sigma} \tilde{p}_K(n) \ln(\frac{\tilde{p}_K(n)}{\tilde{p}(n)})\right)
\]

\[
= \exp\left(-K[H_{\Sigma}(\tilde{q}_{\Sigma,K}) + D_{K,L}(\tilde{q}_{\Sigma,K} || \tilde{q}_{\Sigma})]\right).
\]

The proof is completed.

**APPENDIX G**

**Proof of Lemma**

For simplicity, we call a discrete rolling prediction a \( \Sigma \)-rolling prediction if each prediction locates in the same region as the true state; and we call a continuous rolling prediction an \( \epsilon \)-rolling prediction if each prediction is contained in the \( \epsilon \) ball centered at true state in the sense of infinity norm.

On the one hand, given a partition \( \Sigma \) with \( \text{diam}(\Sigma) \leq \epsilon \) it follows that

\[
\Theta(\tilde{W}_k) = \Theta(w_k + \epsilon r_k) \Leftrightarrow \exists A \in \Sigma \text{ s.t. } \tilde{W}_k, w_k + \epsilon r_k \in A
\]

\[
\Rightarrow \| \tilde{W}_k - w_k - \epsilon r_k \|_\infty \leq \epsilon.
\]
However, the inverse direction isn’t necessarily right. Therefore, a \( \Sigma \)-prediction with \( \text{diam}(\Sigma) \leq \epsilon \) must be an \( \epsilon \)-prediction.

As a result, we have \( \mathcal{P}_{\Sigma}(x_{[1:K]}; \tilde{\Phi}) \leq \mathcal{P}_{\epsilon}(x_{[1:K]}; \tilde{\Phi}) \). Moreover, since \( \Sigma \) can be any partition as long as \( \text{diam}(\Sigma) \leq \epsilon \), there is

\[
\max_{\{\Sigma: \text{diam}(\Sigma) \leq \epsilon\}} \mathcal{P}_{\Sigma}(x_{[1:K]}; \tilde{\Phi}) \leq \mathcal{P}_{\epsilon}(x_{[1:K]}; \tilde{\Phi}).
\]

On the other hand, when it comes to the extreme case, where \( \Sigma = A_1 \) and \( A_1 = \mathcal{R}^d \), there is \( \mathcal{P}_{\Sigma}(x_{[1:K]}; \tilde{\Phi}) = 1 \). As a result,

\[
\mathcal{P}_{\epsilon}(x_{[1:K]}; \tilde{\Phi}) \leq \max_{\Sigma} \mathcal{P}_{\Sigma}(x_{[1:K]}; \tilde{\Phi}).
\]

\textbf{APPENDIX H}

\textbf{PROOF OF THEOREM 5}

Just like the case in Theorem 2, the continuity of \( F(\Sigma) \triangleq \exp\{-K[H_s(q_{\Sigma}; K) + D_{KL}(q_{\Sigma}; K || q_{\Sigma})]\} \) can be easily proved. Then we use the intermediate value theorem to attain the existence conclusion.

\textbf{APPENDIX I}

\textbf{PROOF OF THEOREM 7}

First, we prove \( \Phi \) is the optimal solution. In fact, we need to prove \( \mathcal{I}_e(\Phi; \tilde{\Phi}) \geq \mathcal{I}_e(\Phi; \Phi) \), i.e.,

\[
\lim_{k \to \infty} -\frac{1}{K} \ln \mathcal{P}_{\epsilon}(x_{[1:K]}; \tilde{\Phi}) \geq \lim_{k \to \infty} -\frac{1}{K} \ln \mathcal{P}_{\epsilon}(x_{[1:K]}; \Phi).
\]

Suppose the discrete distribution on \( \{A_i\}_{i=1}^{\left|\Sigma\right|} \) induced by the step-wise function \( q_{\Sigma} \) is:

\[
\tilde{p}(i) = \tilde{q}_{\Sigma}(x_0) \quad \text{where} \quad x_0 \in A_i.
\]

And the discrete distribution on \( \{A_i\}_{i=1}^{\left|\Sigma\right|} \) induced by the step-wise function \( q_{\Sigma,K} \) is:

\[
\tilde{p}_K(i) = \tilde{q}_{\Sigma,K}(x_0) \quad \text{where} \quad x_0 \in A_i.
\]

It follows that

\[
\lim_{k \to \infty} -\frac{1}{K} \ln \mathcal{P}_{\epsilon}(x_{[1:K]}; \tilde{\Phi}) = \varliminf_{k \to \infty} -\frac{1}{K} \ln \prod_{k=1}^{K} \mathcal{P}(\| \tilde{W}_k - \text{err}_k + w_k \|_\infty \leq \epsilon) = \lim_{k \to \infty} H_s(\tilde{q}_{\Sigma';(K)}) + D_{KL}(\tilde{q}_{\Sigma';(K)} || \tilde{q}_{\Sigma';(K)}).
\]

Moreover, \( \mathcal{I}_e(\Phi; \tilde{\Phi}) - \mathcal{I}_e(\Phi; \Phi) \) equals

\[
\lim_{k \to \infty} H_s(q_{\Sigma';(K)}) - H_s(q_{\Sigma';(K)}) + D_{KL}(q_{\Sigma';(K)} || q_{\Sigma';(K)}).
\]

Because \( \hat{q} \) is actually the convolution of \( q \) and \( q_{\epsilon} \), there is

\[
H_s(q_{\hat{\Sigma};(K)}) = H_s\left((q \ast q_{\epsilon})(x_{[1:K]})\right) \geq H_s(q_{\Sigma';(K)}),
\]

and the equality holds if and only if \( \hat{q} = q \). Furthermore,

\[
D_{KL}(q_{\hat{\Sigma};(K)} || q_{\hat{\Sigma};(K)}) \geq 0,
\]

and the equality holds if and only if \( \hat{q} = q \). Then we have

\[
\lim_{k \to \infty} H_s(q_{\hat{\Sigma};(K)}) + D_{KL}(q_{\hat{\Sigma};(K)} || q_{\hat{\Sigma};(K)}) \geq \lim_{k \to \infty} H_s(q_{\Sigma';(K)}).
\]

Second, we prove the approximation formula for the error \( \mathcal{I}_e(\Phi; \tilde{\Phi}) - \mathcal{I}_e(\Phi; \Phi) \). Suppose \( \hat{\Sigma}(K) = \bigcup_{i=1}^{\infty} A_i(K) \), where \( A_i(K) \) is a cube with diameter equaling \( \tilde{r}(K) \). It follows that

\[
\mathcal{I}_e(\Phi; \tilde{\Phi}) = \lim_{K \to \infty} -\sum_{i=1}^{\infty} \tilde{q}(A_i(K)) \ln \{\tilde{q}(A_i(K))\}
\]

\[
\leq \lim_{K \to \infty} -\sum_{i=1}^{\infty} \tilde{q}(A_i(K)) |A_i(K)| \ln \{\tilde{q}(A_i(K))|A_i(K)|\}
\]

\[
= \lim_{K \to \infty} -\sum_{i=1}^{\infty} \tilde{q}(A_i(K)) |A_i(K)| \ln \{\tilde{q}(A_i(K))\}
\]

\[
- \sum_{i=1}^{\infty} \tilde{q}(A_i(K)) |A_i(K)| \ln \{|A_i(K)|\},
\]

where \( \hat{a}_i(K), \tilde{a}_i(K) \in A_i(K) \) such that

\[
q(\hat{a}_i(K)) |A_i(K)| = \tilde{q}(\hat{a}_i(K)) |A_i(K)| = \tilde{q}(\tilde{a}_i(K)) (\hat{a}_i(K)).
\]

Similarly as the proof in Theorem 4 we have

\[
|\mathcal{I}_e(\Phi; \tilde{\Phi}) - H_{KL}(\tilde{q}) + D_{KL}(\tilde{q} || \tilde{q})| \leq 2 \delta_{\rho(\tilde{q})} (\tilde{q}).
\]

Immediately,

\[
|\mathcal{I}_e(\Phi; \tilde{\Phi}) - \mathcal{I}_e(\Phi; \Phi) - H_{KL}(\tilde{q}) + D_{KL}(\tilde{q} || \tilde{q})| \leq \delta_{\rho(\tilde{q})} (\tilde{q}) + d \ln(2\epsilon) \quad \text{and} \quad H_{KL}(\tilde{q} || \tilde{q}) + d \ln(2\epsilon)
\]

\[
\leq \delta_{\rho(\tilde{q})} (\tilde{q}) + d \ln(2\epsilon).
\]

The proof is completed.

\textbf{APPENDIX J}

\textbf{PROOF OF THEOREM 10}

Let \( \mathcal{L}(q, \lambda_0, \cdots, \lambda_2) \)

\[
= H_a(q) + \lambda_0 \left( \int_{-\infty}^{\infty} q(x) dx - 1 \right) + \lambda_1 \left( \int_{-\infty}^{\infty} qx(x) dx - 0 \right)
\]

\[
+ \lambda_2 \left( \int_{-\infty}^{\infty} x^2 q(x) dx - \sigma^2 \right),
\]

by KKT conditions we get

\[
\begin{cases}
\log(q) + 1 = \lambda_0 + \lambda_1 x + \lambda_2 x^2 \\
\int_{-N}^{N} q(x) dx = 1 \\
\int_{-N}^{N} qx(x) dx = 0 \\
\int_{-N}^{N} x^2 q(x) dx = \sigma^2.
\end{cases}
\]
The first KKT condition shows that \( q(x) = te^{\lambda x^2}I_{[-N,N]}(x) \). Take this into other KKT conditions we have

\[
\begin{align*}
\int_{-N}^N te^{\lambda x^2} \, dx &= 1 \\
\int_{-N}^N x^2 e^{\lambda x^2} \, dx &= \sigma^2.
\end{align*}
\]

Our goal is to solve these equations to get \( \lambda, t \), the second equation can be transformed as follows

\[
\int_{-N}^N x^2 e^{\lambda x^2} \, dx = 2 \left[ \frac{tx}{2\lambda} e^{\lambda x^2} \bigg|_{x=0}^{x=N} - \int_0^N \frac{t}{2\lambda} e^{\lambda x^2} \, dx \right] = tN e^{\lambda N^2} - \frac{1}{2\lambda} \sigma^2 = \frac{1}{2\lambda} + \frac{2\sigma^2}{N e^{\lambda N^2}} \Rightarrow t = \frac{1 + 2\sigma^2 \lambda}{2N e^{\lambda N^2}}.
\]

Substitute this into the first equation, we can just focus on the solution of this integral equation:

\[
\int_{-N}^N \frac{1 + 2\sigma^2 \lambda}{2N e^{\lambda N^2}} e^{\lambda x^2} \, dx = 1.
\]

1) \( \lambda = 0 \): In this case, it’s easy to know that only uniform distribution is possible, and \( N = \sqrt{3\sigma} \) is the solution. Therefore the distribution is

\[
\frac{1}{2\sqrt{3\sigma}}I_{[-\sqrt{3\sigma},\sqrt{3\sigma}]}.\]

2) \( \lambda \neq 0 \): If \( N \geq \sqrt{3\sigma} \), on the one hand we have

\[
\frac{1 + 2\sigma^2 \lambda}{2N e^{\lambda N^2}} \leq \frac{1 + 2\sigma^2}{2\sqrt{3\sigma} e^{\lambda^2 \sigma^2}} \leq \frac{1 + 2\sigma^2}{2\sqrt{3\sigma}(1 + 3\sigma^2 \lambda)} \leq \frac{1}{2\sqrt{3\sigma}}.
\]

This indicates that

\[
q(x) \leq \frac{1}{2\sqrt{3\sigma}}, \quad \forall x \in [-N,N].
\]

On the other hand,

\[
\int_{-N}^N x^2 q(x) \, dx = \sigma^2
\]

\[
\Rightarrow \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} x^2 \left( \frac{1}{2\sqrt{3\sigma}} - q(x) \right) \, dx = 2 \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} x^2 q(x) \, dx
\]

\[
\Rightarrow \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} x^2 q(x) \, dx = \frac{1}{2\sqrt{3\sigma}} \frac{3\sigma}{2} - q(x) \, dx = \frac{3\sigma}{2} q(x) \, dx.
\]

However, the fact that l.h.s \( \leq 3\sigma^2 \) and r.h.s \( > 3\sigma^2 \) leads to contradiction.

If \( N < \sqrt{3\sigma} \), on the one hand it follows that

\[
\int_{-N}^N q(x) \, dx = \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} q(x) \, dx,
\]

then there is

\[
q(0) > \frac{1}{2\sqrt{3\sigma}}.
\]

Suppose \( q(m) = \frac{1}{2\sqrt{3\sigma}} \), then

\[
q(x) \geq \frac{1}{2\sqrt{3\sigma}} \quad \forall x \in [-m,m],
\]

and

\[
q(x) \leq \frac{1}{2\sqrt{3\sigma}} \quad \forall x \notin [-m,m].
\]

On the other hand,

\[
\int_{-N}^N x^2 q(x) \, dx = \sigma^2
\]

\[
\Rightarrow \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} x^2 (q(x) - \frac{1}{2\sqrt{3\sigma}}) \, dx = 2 \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} \frac{1}{2\sqrt{3\sigma}} (q(x) - q) \, dx
\]

\[
\Rightarrow \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} x^2 (q(x) - \frac{1}{2\sqrt{3\sigma}}) \, dx = \int_{-\sqrt{3\sigma}}^{\sqrt{3\sigma}} x^2 (q(x) - q) \, dx
\]

Again, the fact that l.h.s < \( m^2 \) and r.h.s \( \geq m^2 \) leads to contradiction.

In summary, the solution to KKT conditions must be uniform distribution, which is \( \frac{1}{2\sqrt{3\sigma}}I_{[-\sqrt{3\sigma},\sqrt{3\sigma}]} \).

APPENDIX K

**Proof of Theorem**

Optimization problem \( \text{[10]} \) can be reformulated as

\[
\max_{q} \min_{u} g_r(q,u)
\]

s.t. \( g_r(q,u) = -\log \int_{B_r(r)} q(x) \, dx \),

\( E(q) = 0, \text{Var}(q) = \sigma^2, \mathcal{M}(\text{supp}(q)) < \infty \).

Furthermore,

\[
\min_{u} g_r(q,u)
\]

\[
= \min_{h} \int_{-\infty}^{\infty} g_r(q,u) h(u) \, du
\]

\[
= \min_{h} \int_{-\infty}^{\infty} -\log \left[ \int_{B_r(r)} q(x) \, dx \right] h(u) \, du
\]

\[
= \min_{h} \int_{-\infty}^{\infty} -\log \left[ \frac{\int_{B_r(r)} q(x) \, dx}{q(u) \cdot 2r} \right] h(u) \, du
\]

\[
= \min_{h} \int_{-\infty}^{\infty} \left[ H_d(h) - \log(2r) + K(q,h,r) \right] h(u) \, du
\]

Now we can consider this functional optimization problem

\[
\max_{q} \min_{h} D_{K\mathcal{C}}(h||q) + H_d(h) + \log(2r) + K(q,h,r)
\]

s.t. \( K(q,h,r) = \int (h(u) \log \frac{B_r(r)}{q(u) \cdot 2r} \, dx) \),

\( E(q) = 0, \text{Var}(q) = \sigma^2, \mathcal{M}(\text{supp}(q)) < \infty \).
Construct a decreasing convergent sequence \( \{ r_n \}_{n=1}^{\infty} \) such that \( \lim_{n \to \infty} r_n = 0 \) and it is easy to show that \( \lim_{n \to \infty} K(f, h, r_n) = 0. \) Therefore problem (10) is equivalent to the following problem

\[
\max_q \min_h D_{KL}(h||q) + H_d(h) \quad E(q) = 0, \quad \text{Var}(q) = \sigma^2, \quad \mathcal{M}(\text{supp}(q)) < \infty.
\]

There is

\[
\min_h D_{KL}(h||q) + H_d(h) \leq \max_q D_{KL}(q||q) + H_d(h) = H_d(q),
\]

and the equality holds when \( q \) is a uniform distribution. Moreover, Theorem 10 shows that the solution to problem (9) is uniform distribution, we have

\[
\max_q \min_h D_{KL}(h||q) + H_d(h) \leq \max_q H_d(q) = H_d(q^*),
\]

where \( q^* = \frac{1}{2\sqrt{\pi} r} \mathbf{1}_{[-\sqrt{3}\pi, \sqrt{3}\pi]} \). Therefore, these two optimization problems are actually equivalent to each other.

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