Elastic wave propagation in a cylinder with external active friction

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Abstract. Wave propagation in an elastic cylindrical body interacting with an external moving absolutely non-deformable medium (a cage) under a shock load on the front end is considered. The friction force arising on the contact surface directly depends on the parameters of the propagating waves and, due to the external medium motion, - it is active. The stated one-dimensional problem is solved analytically by the method of characteristics. Exact solutions are obtained for the stress wave propagation in a cylinder. An increase in the wave parameters with time and distance was determined; the influence of the parameters of external load and the physical and mechanical characteristics of the cylinder on the parameters of the increase in propagating waves were determined. The place and time of equalization of the velocities of the cylinder particles and the external medium motion, and the stress state of the cylinder particles, were discussed.

1. Introduction
In mechanical engineering, there are structures that interact with an external body or medium. As an example, we can refer to a two-layer (in radius) rod system, shaft-sleeve structure and others. In construction, underground pipelines or piles embedded into the ground interact with the external medium - soil. In these structures, under loading in the direction of the cylindrical body or pipeline axes, movements of the external medium (a bush, an outer cage, or soil around an underground pipeline) are possible. The propagation of longitudinal waves in rods or pipelines in the case of interaction with the external medium is affected by the friction force on the contact surface. In the case of a stationary or moving external medium (a bush or soil) against the direction of the wave, this friction force is passive and reduces the amplitude of the propagating elastic wave [1-9]. If the external medium moves along the direction of wave propagation, then the friction force on the contact surface turns into an active force [3-4,10-12]. In [11,12], the problems of longitudinal wave propagation in a cylindrical body in the presence of an incompressible external medium in a non-one-dimensional axisymmetric statement were solved numerically. According to the results of calculations [12], an increase in the parameters of waves (stress and particle velocity) with time and distance were revealed, as well as the reasons for the mechanism of increasing stresses in the cylinder. The influence of the parameters of external load and the physical-mechanical characteristics of the cylinder on the parameters of propagating waves was determined. According to the results obtained in [12], the equalization of the velocities of the external medium and cylinder particles depends on all parameters...
of the problem (geometrical and mechanical characteristics of the cylinder, the parameters of acting load).

From the given numerical solutions [8], it is possible to determine approximately the point of time and the place where the equality of the velocities of the cylinder particles and the external medium motion occurs. With numerical solutions obtained, it is possible to show with sufficient accuracy the direction of lateral forces induced by the amplitude of propagating waves. In addition, from [12] we come to the important conclusion that the hypothesis of flat sections for a structure with active external friction forces is fully satisfied. Figure 1 confirms this statement (followed from the solution to the problem in an axisymmetric two-dimensional statement [12]); the profiles of the meridional cross-section of the cylinder remain straight even for different diameters of the cylinder. The complete fulfillment of the hypothesis of flat sections can also be explained by the coincidence of the directions of longitudinal stress and active external friction.

![Figure 1. Cross-sectional profiles x = 0.4 m at different values of the cylinder radius at t=5R_0/c (R_0 – is the radius of the cylinder, c is the propagation velocity of the longitudinal wave in the cylinder).](image)

These conclusions and a comparison of the results of a two-dimensional problem with a similar one-dimensional problem [1-2] with braking external friction, where the hypotheses of flat sections can be violated up to 10-15%, should be reduced to a one-dimensional similar problem; then the places and time of equality of the velocities of the medium and cylinder particles are accurately determined.

This study is a continuation of a study given in [12] in a one-dimensional statement. The aim of the study is the analytical (precise) determination of the influence of the external medium motion on the propagating waves in an elastic cylindrical body. Thus, let us investigate the propagation of waves in a semi-infinite cylindrical body surrounded by an incompressible medium moving with velocity \( v_{\text{ext}} \).

Let us assume that the velocity of the external medium motion is greater than the velocity of the cylinder particles induced by the load on the front end. The surface friction force arising on the contact surface is due to the amplitude of propagating waves [2] and it is active.

2. Statement of the problem and solution method

Consider an elastic cylinder compressed by an external body without pressure. A load in the form of stress is applied along the front end of the cylinder (\( x = 0 \)); the value of this shock load is determined by the following relations

\[
\sigma = \sigma_{\text{max}} H(T - t) \quad \text{at } t > 0, \tag{1}
\]

here \( \sigma_{\text{max}} \) is the set stress, \( H \) is the Heaviside function, \( T \) is the duration of the load acting on the front end. Before applying the load, the cylinder in question is considered to be at rest and stress-free. From the moment the load (1) is applied, a longitudinal elastic wave propagates through the cylinder.
Behind the wave front, normal stresses appear in the cylinder, acting on the cross-section. A friction force appears on the lateral surface, acting against the direction of the relative velocity, due to the non-deformability of the external body.

The equation of motion, taking into account the forces of friction on the lateral surface of the cylinder, has the following form:

\[
\rho S \frac{\partial v}{\partial t} = S \frac{\partial \sigma}{\partial x} + \tau P, \tag{2}
\]

where \( \rho \) is the density; \( v \) is the velocity; \( \sigma \) is the stress; \( P, S \) – are the perimeter and cross-sectional area; \( \tau \) is the shear stress directed along the \( x \) axis and acting on the lateral surface of the cylindrical body. We assume that the friction force, due to the external medium motion, is directed along the velocity of a cylindrical body motion and changes according to the Coulomb dry friction law [1, 2]:

\[
\tau = f|\sigma_N|, \tag{3}
\]

where \( \sigma_N \) is the normal stress, and \( u \) is the displacement of the rod particles.

Under the assumption of the absence of strain (the external medium is assumed to be non-deformable) in the transverse direction, the following values are obtained for an elastic cylindrical body:

\[
\sigma = \rho c^2 \frac{\partial u}{\partial x},
\]

\[
c^2 = \frac{1 - \mu}{(1 + \mu)(1 - 2\mu)} \frac{E}{\rho}, \tag{4}
\]

\[
\sigma_N = -\frac{\mu}{1 - \mu} \sigma,
\]

where \( E \) is the modulus of elasticity, \( \mu \) is the Poisson's ratio.

Eliminating the displacement from (2) - (4), we obtain a system of two hyperbolic equations for stress and velocity in the form

\[
\rho \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = \frac{\mu}{1 - \mu} \frac{Pf}{S} |\sigma|, \tag{5}
\]

with the displacement this system is reduced to one second-order equation:

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\mu}{1 - \mu} \frac{c^2}{S} |\sigma_N|, \tag{6}
\]

Solving the system of equations (5) or (6) with zero initial and boundary conditions (1) an exact one-dimensional analytical solution to the problem can be derived.

Before solving these equations, let us analyze the obtained equations. First, we obtain the equation for the change in wave energy. We multiply the first equation in (5) by \( v \) and after transforms using the second equation in (5), we obtain

\[
\frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + \frac{\sigma^2}{2\rho c^2} \right) = \frac{\partial (\sigma v)}{\partial x} + \frac{\mu}{1 - \mu} \frac{Pf}{S} |\sigma v| \tag{7}
\]
If we integrate (7) over the entire volume of the body in which the wave propagates, then the first term on the right-hand side of equation (7) disappears. Hence, we get:

\[
\frac{\partial}{\partial t} \left( \frac{\rho v^2}{2} + \frac{\sigma^2}{2\rho c^2} \right) = \frac{\mu}{1 - \mu} \frac{P_f}{S} \sigma_v.
\] (8)

Here, angular brackets denote volume averaging. The left-hand side of equation (8) contains a change in the wave energy per unit volume, and the right-hand side contains a positive value - the work of the forces of active friction per unit time, which shows an increase in the wave energy. The increase in the wave energy depends on the mechanical and geometrical characteristics of the cylinder, on the friction coefficient, and on the wave parameters; this fully confirms the solutions obtained in [12].

Now let us make some remarks about the application of equation (4). Here, it is assumed that the hypothesis of flat sections is fulfilled. This conclusion was obtained by solving the problem in a two-dimensional statement [8,12].

In the problem under consideration, friction enters only the right-hand side of the obtained equations (5)-(6) and does not change their hyperbolicity. The slope of the characteristics remains constant, independent of the sought for values. Let us write the system of equations (5) along the characteristics \( C_+ \) and \( C_- \) as:

\[
\frac{d\sigma}{dx} - \rho \frac{dv}{dt} = -\frac{\mu}{1 - \mu} \frac{P_f}{S} \sigma \quad \text{at} \quad \frac{dx}{dt} = c \; ;
\] (9)

\[
\frac{d\sigma}{dx} + \rho \frac{dv}{dt} = -\frac{\mu}{1 - \mu} \frac{P_f}{S} \sigma \quad \text{at} \quad \frac{dx}{dt} = -c.
\] (10)

Here, it is meant that the friction force occurs only under compression. Therefore, equations (9)-(10) are valid. There can be no other case according to the results obtained in [12].

Using condition (9) at the wave front, we integrate equations (9) along the characteristics \( C_+ \):

\[
\sigma = \sigma_0(\alpha) \exp \left( \frac{\mu}{1 - \mu} \frac{P_f}{2S} x \right) \quad \text{at} \quad x = ct + \alpha, \quad (11)
\]

where \( \alpha \) and \( \sigma_0(\alpha) \) are the quantities that are constant along the characteristics \( C_+ \), but have different values for a different characteristic \( C_- \). They are determined from the initial and boundary conditions.

Thus, for the wave problems of a cylindrical body with active external friction, an analytical solution is obtained in the form (11).

3. Results and analysis of results

If condition (3) is taken into account, then solution (11) is valid only in the region where the velocity of the cylinder particles increases with the velocity of the external medium. Further, there is a joint motion of the particles of the cylinder and the external medium, therefore, there is no friction. Hence, taking into account the initial and boundary conditions of the problem, we obtain the solution up to the point of time \( \tau \) or in the section \([0; x]\) behind the wave front in the following form

\[
\sigma = \sigma_{\text{max}} \exp \left( \frac{\mu}{1 - \mu} \frac{P_f}{2S} x \right) \quad \text{at} \quad x = ct, \quad (12)
\]

where \( \tau \) is the moment of equality of the velocity of the cylinder particles with the velocity of the external medium. The rest of the stress components are easily determined with formulas (3) and (4), related to longitudinal stresses.
At $t > t_*$, behind the wave front:

$$\sigma = -\sigma_{\text{max}} \exp \left( \frac{\mu Pf}{1 - \mu 2S} x_0 \right)$$  \hspace{1cm} (13)$$

and, therefore, $v = v_{\text{ext}}$. Here, the constant value on the right-hand side of relation (13) is the maximum value of the longitudinal stresses set in (1). So, from (13) it is possible to determine the point of time and place where the equality of velocities occurs:

$$t_* = \frac{1 - \mu 2S}{c \mu Pf} \ln \left( \frac{c \rho v_{\text{ext}}}{\sigma_{\text{max}}} \right),$$

$$x_* = \frac{1 - \mu 2S}{\mu Pf} \ln \left( \frac{c \rho v_{\text{ext}}}{\sigma_{\text{max}}} \right).$$ \hspace{1cm} (14)$$

From this, it is seen that the beginning time of the joint motion of the cylinder particles with the external medium really depends on the geometrical and mechanical characteristics of the cylindrical body, on the parameters of acting load, on the friction coefficient, and on the velocity of motion of the external medium. And, finally, it completely confirms the numerical results of the problem considered in [12] in the non-one-dimensional statement. Figures 2 - 3 show analytical solutions in the form of graphs. Figure 2 corresponds to the change in longitudinal stresses with time in fixed sections of the cylinder. Here curves 1 - 4 correspond to points $x = 0.4$ m; 1.2 m; 2 m and 3 m, respectively. As seen from the changes in longitudinal stresses, the numerical solution derived in [12] completely coincides with the exact solution behind the wave front, and at the wave front, as is known, the smearing in the numerical solutions occurs.

Figure 3 shows the change in the velocity of the cylinder particles with time in the sections $x = 0.4$ m (curve 1) and $x = 1.2$ m (curves 2 - 4) for various velocity options of the external medium (curves 2 - 4, respectively). It should be emphasized here that the velocities of particles in the section $x = 0.4$ m (curve 1) after reaching the motion velocity of the external medium coincide with the course of curves 2 - 4. The equality of the velocities of the cylinder particles and the medium for curves 2 - 4 were obtained at the following points: $t_* = 0.384$ ms (at $v_{\text{ext}} = 5$ ms$^{-1}$); $t_* = 0.565$ ms (at $v_{\text{ext}} = 7$ ms$^{-1}$); $t_* = 0.752$ ms (at $v_{\text{ext}} = 10$ ms$^{-1}$). We note that if $t_* > T$, then the velocities of the particles and the cylinder do not equalize and joint motion does not occur. We also note that the numerical solutions presented in [12] coincide with satisfactory accuracy with the exact solutions (Figure 3); this once again confirms the reliability of the numerical results obtained in [12].

**Figure 2.** Changes in longitudinal stresses of the cylinder with time at fixed points.
Figure 3. Changes in particle velocity with time at fixed points of the cylinder.

4. Discussion

Now let us consider the questions: when and where does the equality of the velocities of the cylinder particles and the external medium motion occur, and what is the stress state of the cylinder particles, i.e. defined in formulas (4). From the dynamic theory of elasticity it is known that under a constant load $\sigma_{\text{max}}$ acting on the end of the cylinder along the cylinder axis, leading to a plane longitudinal wave propagation, the particle velocity takes on the following values:

$$v_0 = -\frac{\sigma_0}{c\rho},$$

(15)

where $\sigma_0 = -\sigma_{\text{max}}$. Then, a plane longitudinal elastic wave propagates along the cylinder with velocity $c$, and the wave leading front is a wave of strong discontinuity. Naturally, jumps in longitudinal stresses $\langle \sigma \rangle$ and velocity $\langle v \rangle$ must satisfy the following conditions

$$\langle \sigma \rangle + c\rho\langle v \rangle = 0.$$

(16)

It follows from the research results that the maximum stress values are reached at the leading front of wave propagation at the value of particle velocity $v = v_{\text{ext}}$. Hence, for the maximum values of the longitudinal stresses of the cylinder particles from (16), we obtain

$$\sigma_{\text{max}} = -c\rho v_{\text{ext}}$$

(17)

Thus, the increasing maximum stress values are in proportional dependence to the motion velocity of the external medium and to the mechanical characteristics of the cylinder. The ratio of the maximum longitudinal stresses to the actual load is expressed with formulas (15) and (17):

$$\frac{\sigma_{\text{max}}}{\sigma_0} = \frac{v_{\text{ext}}}{v_0}$$

(18)

In general, the increase in the stresses of the cylinder in comparison with the falling load depends only on the ratio of the velocities of motion of the external medium and the velocity of the cylinder particles. The resulting formula (18) is most important for structures corresponding to the considered problem, which makes it possible to determine the reasons for the structure destruction. It also explains the multiple increases in stresses in the underground pipeline [3-4]. Studies in [10] led to the same conclusion, that the velocity of particles of an underground pipeline is equalized with the velocity of soil motion. If we take into account the difference in the physical and mechanical
characteristics of the pipeline and the surrounding soil, then the particle stresses in the underground pipeline increase up to 50 or more times, and, ultimately, leads to the destruction of the underground pipeline. The correctness of the solutions obtained in [3-4] is beyond doubt. Even calculations performed in a two-dimensional statement [10, 12] confirm this. Such an approach to a study of the pipeline-soil interactions draws attention to the relative simplicity of the problem statement.

5. Conclusion
An exact solution on the elastic wave propagation in a cylindrical body in the presence of moving absolutely non-deformable body was obtained by an analytical method. The law of Coulomb friction induced by the parameters of propagating waves is considered valid on the contact surface. The solutions obtained show an increase in the wave parameters until the velocities of the cylinder particles and the velocity of the external medium motion are equal. The places and times of equalization of the velocities of the cylinder particles and the external body, and the maximum stresses in the cylinder, depending on the applied load, are determined by an analytical expression.

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