ALGEBRAIC STRUCTURES AND DIFFERENTIAL GEOMETRY IN 2D STRING THEORY

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ABSTRACT

A careful treatment of closed string BRST cohomology shows that there are more discrete states and associated symmetries in $D = 2$ string theory than has been recognized hitherto. The full structure, at the $SU(2)$ radius, has a natural description in terms of abelian gauge theory on a certain three dimensional cone $Q$. We describe precisely how symmetry currents are constructed from the discrete states, explaining the role of the “descent equations.” In the uncompactified theory, we compute the action of the symmetries on the tachyon field, and isolate the features that lead to nonlinear terms in this action. The resulting symmetry structure is interpreted in terms of a homotopy Lie algebra.

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1. Introduction

Of the known soluble string theories in \( D \leq 2 \), the \( D = 2 \) model is in many ways particularly intriguing. It has a (two dimensional) space-time interpretation in terms of the interactions of a massless scalar field, somewhat misleadingly called the “tachyon,” and the space-time physics is realistic enough to include black holes.

In addition to the tachyon, the model also has “discrete states” which appeared in the matrix model calculations of Gross, Klebanov, and Newman [1]. Polyakov proposed [2] that these states, which are what survive in \( D = 2 \) from the infinite tower of string states for \( D > 2 \), should be described by a sort of stringy topological field theory. This suggestion was part of the motivation for subsequent efforts. The discrete states originally considered were spin one currents, but they have spin zero analogs found in the mathematical analysis [3–6]. The spin zero (and ghost number zero) states generate a “ground ring” [7] which is characteristic of \( D = 2 \) string theory. Simple considerations involving the ground ring explain many aspects of the free fermion description that comes from the matrix model; this was argued in [7] and enlarged upon (and extended to \( D < 2 \)) by Kutasov, Martinec, and Seiberg [8]. In fact, at the \( SU(2) \) radius, the ground ring is the ring of functions on a certain three dimensional cone \( Q \); the three dimensions correspond in the matrix model to the time, the matrix eigenvalue, and its canonical momentum.

Moreover, by combining the spin zero and spin one states, one can construct an enormous unbroken symmetry group of the \( D = 2 \) string theory. At the \( SU(2) \) radius, the group that arises is the group of volume preserving diffeomorphisms of \( Q \) (plus additional symmetries that we will find in this paper). In the uncompactified theory, one gets essentially the \( W_\infty \) symmetry of the matrix model free fermions, described by several groups [9–12]. For open strings, a \( W_\infty \) current algebra can be constructed, and the structure constants explicitly calculated, using the original spin one discrete states [13].

In the present paper, we will analyze some aspects of this story in somewhat more depth. In §2, we will look more closely at the chiral BRST cohomology at
the $SU(2)$ radius. The chiral ground ring is the ring of functions on a certain $x - y$ plane. We will show how certain peculiarities originally uncovered in mathematical analysis of this cohomology can be described in terms of the differential geometry of the $x - y$ plane. In §3, we combine left and right movers. We show that this process is more subtle than usually supposed; the proper analysis depends on the results of §2 and on certain details of closed string theory that are known in principle but do not usually arise in practice [14–17]. As a result we find that (if the BRST cohomology is taken in the usual space of conformal fields) there are more physical discrete states and symmetries at the $SU(2)$ radius than has been usually supposed. All basic formulas can be written in terms of the differential geometry of $Q$. For instance (overcoming a contradiction implicit in [7]), the cubic couplings of discrete states at the $SU(2)$ point can be generated by a Lagrangian

$$L = \int \sigma \cdot F \wedge F,$$  \hspace{1cm} (1.1)

where $\sigma$ is a scalar and $F = dA$ is the field strength of an abelian gauge field $A$. The $SU(2)$ point is $\sigma = F = 0$.

The enhanced symmetry of two dimensional string theory at the $SU(2)$ radius* is thus somewhat analogous to the enhanced symmetry of four dimensional general relativity, with Lagrangian

$$L = \frac{1}{16\pi G} \int \epsilon_{abcd} e^a \wedge e^b (d\omega + \omega \wedge \omega)^{cd}$$  \hspace{1cm} (1.2)

at $e = \omega = 0$. This accounts for the theoretical significance of the $SU(2)$ point and justifies our focussing on it in much of this paper.

Of course, it is hard to make sense of general relativity expanded around $e = \omega = 0$. (In three space-time dimensions one can do this [18], but this is related to the absence of local dynamics in three dimensional general relativity.) Two

* Or any rational multiple of that radius, where a similar structure arises.
dimensional string theory seems to be our best example, at the moment, of a model in which one can make some sense (at the $SU(2)$ radius) of the analog of $\epsilon = \omega = 0$, and one can also explicitly see (in the uncompactified theory) a phase with local dynamics in the form of tachyon scattering amplitudes.

It will be clear that our discussion of these matters is preliminary and barely scratches the surface. We will point out a few of the more obvious gaps at the end of §3.2 and elsewhere.

In §4, we explain how symmetry currents are constructed from the discrete states – using the “descent equations.” Of special significance are the symmetry currents of ghost number zero which are seen to arise from the BRST invariant states of ghost number one. We also work out a number of explicit examples.

In §5, we consider the uncompactified theory. In particular, we analyze how the symmetries associated with the discrete states act on the tachyon field in the linear approximation. In a suitable sense the tachyon field has spin one. We also show that this result agrees with the prediction of the matrix model description.

From the matrix model it is clear that the symmetries have the unusual property of acting nonlinearly on the tachyon field. In §6, we analyze in principle how this nonlinear action arises from the point of view of conformal field theory, and illustrate the point by a simple calculation. We interpret the nonlinear terms in the Ward identities in terms of homotopy Lie algebras [19–21]. In §7 we give some additional comments on our results.

The main points in the present work are probably that many of the structures we find can be naturally described in terms of the differential geometry of $Q$, and that many of them are similar to the structures arising in BRST closed string field theory.
2. The Chiral BRST Cohomology

At the $SU(2)$ radius, with world-sheet cosmological constant zero, the world-sheet Lagrangian of $D = 2$ string theory is

\[ L = \frac{1}{8\pi} \int d^2x\sqrt{h} (h^{ij} \partial_i X \partial_j X + h^{ij} \partial_i \phi \partial_j \phi) - \frac{1}{2\pi \sqrt{2}} \int d^2x \sqrt{h} \cdot \phi R^{(2)}. \]  

(\( h \) is the world-sheet metric, \( R^{(2)} \) is the Ricci scalar, and \( \phi \) is the Liouville field.) At the $SU(2)$ radius, the allowed values of the $X$ momentum are \( p = n\sqrt{2}, n \in \mathbb{Z}/2 \). Since the world-sheet cosmological constant $\mu$ is zero, $X$ and $\phi$ are free, the left and right movers of the theory are decoupled, and the BRST cohomology can be computed by first studying the chiral problem, that is the right movers (or left movers) only. This has been worked out in detail [3,4,5]; we will recall the relevant points and work out some relevant consequences. The reader may want to consult [7] for some background.

At spin zero and ghost number zero, BRST cohomology classes arise precisely at discrete momenta \( (p_X, p_\phi) = (n, iu) \cdot \sqrt{2} \), with \( u = 0, 1/2, 1, \ldots \) and \( n = u, u - 1, \ldots, -u \). They are denoted as \( O_{u,n} \) and the first few are

\[ O_{0,0} = 1 \]
\[ x = O_{1/2,1/2} = \left( cb + \frac{i}{\sqrt{2}} (\partial X - i \partial \phi) \right) \cdot e^{i(X+i\phi)/\sqrt{2}} \]
\[ y = O_{1/2,-1/2} = \left( cb - \frac{i}{\sqrt{2}} (\partial X + i \partial \phi) \right) \cdot e^{-i(X-i\phi)/\sqrt{2}}. \]  

(2.2)

These states generate under operator products the chiral ground ring, which is simply the ring of polynomials in $x$ and $y$.

At spin zero and ghost number one, one has discrete states of the form

\[ Y_{s,n}^\pm = cV_{s,n} \cdot e^{\sqrt{2} s \pm \phi \sqrt{2}}, \]  

(2.3)

with \( s = 0, 1/2, 1, \ldots \) and \( n = s, s - 1, \ldots, -s \). Here $V_{s,n}$ is a primary field constructed from $X$. For the time being we are mainly interested in the $Y_{s,n}^+$. Note
that for every $O_{u,n}$, there is a $Y_{u+1,n}^+$ with the same momentum; in addition there are the $Y_{s,±s}^+$ that do not have partners. These are the “discrete tachyons,” that is, the tachyon modes that survive at the $SU(2)$ radius.

At these values of the momenta, the $O$’s and $Y^+$’s actually make up only half of the BRST cohomology. This follows from the existence of the operator

$$a = [Q, \phi] = c\partial\phi + \sqrt{2}\partial c. \quad (2.4)$$

$\phi$ is not a conformal field in the usual sense, but $a$ is. Obviously, $a$ is BRST invariant, and in the usual space of conformal fields it cannot be written as $[Q,\ldots]$. $a$ is also a conformal primary field. In the present paper, we will consider only the BRST cohomology in the usual space of conformal fields, though this may be a restriction that should be relaxed.

This being so, we can form new families of BRST invariant vertex operators. At ghost number one we can extract a new operator which we will call $aO_{u,n}$ from the operator product of $a$ and $O_{u,n}$. To be precise,

$$aO_{u,n}(0) = \frac{1}{2\pi i} \oint \frac{dz}{z} a(z) \cdot O_{u,n}(0). \quad (2.5)$$

Obviously, $aO_{u,n}$ has the same momenta as $O_{u,n}$. Similarly by multiplying $a$ and $Y_{s,n}^+$ we make a new family of spin zero, ghost number two operators, which we will call $aY_{s,n}^+$. (The case $s = 0$ is exceptional here and in many later statements.) The operators made this way are all nonzero and independent of the old ones. This follows from the treatment of the BRST cohomology in [5], and we will in any case soon exhibit an inverse to multiplication by $a$. These states and their conjugates with opposite Liouville dressing are known to exhaust the BRST cohomology.

The BRST cohomology at typical discrete momenta can be arranged in a dia-
There are four states, of ghost numbers \((0, 1, 1, 2)\). At the special tachyon momenta \((\pm s, i(s - 1)) \cdot \sqrt{2}\), there are only two states, namely \(Y_{s,\pm s}^+\) and \(aY_{s,\pm s}^+\):

\[ Y_{s,\pm s}^+ \xrightarrow{a} aY_{s,\pm s}^+ . \]

The latter turns out to be (for \(s \neq 0\))

\[ aY_{s,\pm s}^+ \sim c \partial c V_{s,\pm s} \cdot e^{\sqrt{2}\phi(1-s)}. \] (2.6)

The “Minus” States

Now let us briefly discuss the “minus” states, that is the states whose Liouville dependence is \(e^{\sqrt{2}\phi(1+s)}\) with \(s \geq 0\). The two point function on the sphere gives a pairing

\[ \langle V_i^- W_j^+ \rangle \] (2.7)

between the minus and plus states. This pairing is nondegenerate and has ghost number three (from the three chiral ghost zero modes on the sphere) and Liouville momentum \((-2i\sqrt{2})\) (from the Liouville background charge). So the “dual” of a “plus” state with ghost number \(j\) and Liouville factor \(e^{\sqrt{2}\phi(1-s)}\) is a state with ghost number \(3 - j\) and Liouville factor \(e^{\sqrt{2}\phi(1+s)}\). As the ghost numbers of the “plus” BRST cohomology range from 0 to 2, those of the “minus” states range
from 1 to 3. The “minus” states form patterns dual to those of the “plus” states

\[ aY_{u+1,n} \]

\[ Y_{u+1,n} \]

\[ aP_{u,n} \]

\[ P_{u,n} \]

\[ a \]

\[ Y_{s,\pm s} \xrightarrow{a} aY_{s,\pm s} \]

The minus states of most immediate interest to us will be those of ghost number 1. These are simply the \( Y_{s,n}^- \) (which are the duals of the \( aY_{s,n}^+ \)). As explained in [7, §2.4], these states transform under area preserving diffeomorphisms as derivatives of a delta function supported at the origin of the \( x - y \) plane.

### 2.1. Interpretation of the Discrete States

The discrete states can be given the following interpretation, which may seem a nicety to begin with but will prove to be important. The states at ghost number 0 are just the polynomial functions on the \( x - y \) plane, as explained in [7]. (The operator \( O_{u,n} \) corresponds to the function \( x^{u+n}y^{u-n} \).) The \( x - y \) plane is endowed with the area form

\[ \omega = dx \wedge dy. \]  \hspace{1cm} (2.8)

The \( Y_{s,n}^+ \) with \( s > 0 \) correspond, in a sense explained in [7], to polynomial vector fields on the \( x - y \) plane that generate area-preserving diffeomorphisms. (If \( O_{u,n} \) corresponds to a function \( f \), then \( Y_{u,n}^+ \) corresponds to the area preserving vector field \( V^i = \omega^{ij}\partial_j f \).) Let \( V \) be any vector field on the \( x - y \) plane, and \( \mathcal{L}_V \) the corresponding Lie derivative. Then \( \mathcal{L}_V(\omega) = f \cdot \omega \) where \( f = \partial_i V^i \) is a function. \( V \) is uniquely determined by \( f \) modulo addition of an area-preserving vector field; and every \( f \) can arise. Thus, once the area preserving vector fields have been
identified with the $Y$’s, the additional vector fields on the $x − y$ plane have the quantum numbers of the functions. Since $a$ has momentum zero, the states $aO_{u,n}$ have the same momenta as the $O$’s and are thus in natural correspondence with the functions on the $x − y$ plane. As regards the momentum quantum numbers, they are the missing operators that we need to make up all the (polynomial) vector fields on the $x − y$ plane. So, for counting states, the ghost number one operators $Y_{s,n}^+$ and $aO_{u,n}$ make up the arbitrary vector fields on the plane.

At ghost number two, we have the $aY_{s,n}^+$, which have the same momenta as the $O_{s,n}$, except for a shift in the Liouville momentum by $-i\sqrt{2}$. Since $∂/∂x_i$ has the opposite quantum numbers of $x^i$, $(0,-i\sqrt{2})$ are precisely the $(p_X,p_φ)$ values of the bivector

$$\frac{∂}{∂x} \wedge \frac{∂}{∂y}$$

(2.9)

So, as the $O_{u,n}$’s have the quantum numbers of polynomial functions, the $aY_{s,n}^+$’s have the quantum numbers of the polynomial bivector fields on the plane.

These results, which we have obtained piecemeal, can be presented in the following unified way. Let $T$ be the tangent bundle of the $x − y$ plane. Let $\wedge^iT$, $0 ≤ i ≤ 2$ be its $i^{th}$ exterior power. Then the discrete states of ghost number $i$, for $i = 0,1,2$, transform as the polynomial sections of $\wedge^iT$. In other words, one has functions, vector fields, and bivectors for $i = 0,1,2$.

The BRST cohomology has a natural ring structure – induced from the operator products – even if we do not restrict to ghost number zero. The multiplication law going from ghost number $i$ times ghost number $j$ to ghost number $i + j$ is just the natural “wedge product” $\wedge^iT \times \wedge^jT \to \wedge^{i+j}T$, for $0 ≤ i,j ≤ 2$. This can be verified using arguments similar to the ones given for $i = j = 0$ in [7].

The Cosmological Constant

There is, in addition, one more operator that does not fit into this framework, namely the cosmological constant operator $ι = Y_{s,n}^+ = Y_{s,n}^-$. This is of ghost number one, and so should correspond to a symmetry charge (as we will show in
§4). However, as explained in [7], this operator is central. This is why we give it the name $\iota$, which is meant to be suggestive of the fact that the central operator in ordinary quantum mechanics is customarily called $i$.

One cannot make an operator of ghost number two by acting on $\iota$ with $a$, since, as is easily verified, $a \cdot \iota = 0$. Nevertheless, a corresponding operator of ghost number 2, namely $c\partial ce^{2\phi}$, does exist. This is the operator which, in the construction above, corresponds to the constant bivector

$$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$  

(2.10)

We will call this operator $\tilde{\iota}$; occasionally, to simplify various statements, we may somewhat inconsistently call it $a \cdot Y_{0,0}^i$.

**The Dual Version**

There is a dual version of this, making use of the area form $\omega$, which will prove to be essential. Contraction with $\omega$ is a natural operation mapping $i$-vectors, for $i = 0, 1, 2$, to differential forms of degree $2 - i$. Thus, one maps the 0-vector or function $f$ to the two form $f\omega$; the vector field $V^i\partial_i$ to the one form $V^i\omega_{ij}dx^j$; and the bivector

$$p^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$  

(2.11)

to the function $p^{ij}\omega_{ij}/2$. So in particular, the constant bivector (2.10) is mapped to the function 1.

Hence, we could think of the discrete states of ghost number $i$ as $2 - i$ forms. This point of view seems unnatural from what we have said so far, because it obscures the ring structure. But facts that will presently appear shed a different light.
2.2. Operators And Currents

Under some circumstances, from a BRST cohomology class of ghost number $g$, one can construct a conserved current of ghost number $g - 1$.

This is done as follows. Consider a BRST cohomology class represented by a highest weight BRST invariant state $|\psi\rangle$ of $L_0 = 0$. Set $|\alpha\rangle = b_{-1}|\psi\rangle$ (and assume $|\alpha\rangle \neq 0$). Then $L_0|\alpha\rangle = |\alpha\rangle$, so if $|\alpha\rangle$ is a highest weight state, the highest weight is one, and the operator corresponding to $|\alpha\rangle$ is a current. Moreover,

$$Q|\alpha\rangle = Qb_{-1}|\psi\rangle = L_{-1}|\psi\rangle.$$  \hspace{1cm} (2.12)

Hence, although the current corresponding to $|\alpha\rangle$ is not BRST invariant, its BRST commutator is a total derivative. To see whether $|\alpha\rangle$ is a highest weight state at least modulo BRST commutators, note that

$$L_n|\alpha\rangle = L_n b_{-1}|\psi\rangle = b_{n-1}|\psi\rangle.$$  \hspace{1cm} (2.13)

These states may not vanish, but we do at least have $Qb_{n-1}|\psi\rangle = L_{n-1}|\psi\rangle = 0$, $n > 0$, as $|\psi\rangle$ is BRST invariant and highest weight. Therefore, if the BRST invariant states $b_{n-1}|\psi\rangle$ are BRST commutators, then $|\alpha\rangle$ is of highest weight, at least modulo BRST commutators.

For $n > 1$, this is automatically so, as

$$b_{n-1}|\psi\rangle = -\frac{1}{n-1}Qb_0 b_{n-1}|\psi\rangle.$$  \hspace{1cm} (2.14)

The crucial case is therefore $n = 1$. $|\alpha\rangle$ is of highest weight (modulo BRST commutators) if $b_0|\psi\rangle = Q|\ldots\rangle$.

Thus, the linear transformation $|\psi\rangle \rightarrow b_0|\psi\rangle$ of the BRST cohomology plays a crucial role. The cohomology classes that give rise to currents, in the chiral theory, are those that are in the kernel of $b_0$. 
Now let us apply this to the discrete states of the $D = 2$ model.

First of all, the states corresponding to the operators $O_{u,n}$ would be mapped by $b_0$ to ghost number $-1$, where the BRST cohomology vanishes. So they are annihilated by $b_0$, at least at the level of cohomology. In the appendix, we show that this is true exactly. These states therefore give rise to highest weight currents of ghost number $-1$.

The $Y^+_{s,n}$ are likewise annihilated by $b_0$. To see this, note that at the level of operators, the operation of multiplying by $b_0$ can be written

$$V(0) \to b_0 \cdot V(0) \equiv \frac{1}{2\pi i} \oint dz \, z \, b(z) \cdot V(0). \quad (2.15)$$

This removes from the states a factor of $\partial c$, wherever this is present, and otherwise gives zero. The explicit form of the wave functions of the $Y$’s shows that there are no terms in $\partial c$, and hence $b_0 \cdot Y^+_{s,n} = 0$. The $Y^+_{s,n}$ thus give rise to currents; these generate area preserving diffeomorphisms (of the $x - y$ plane), as shown in [7,13].

Multiplication by $b_0$ is not a trivial operation, however. One immediately sees that, with $a$ as defined in (2.4),

$$b_0 \cdot a = \sqrt{2}. \quad (2.16)$$

This strongly suggests that if a discrete state $X$ is annihilated by $b_0$, then $a \cdot X$ is not. This is actually true (except for the cosmological constant operator), as follows from the analysis of the absolute and relative BRST cohomology in [3,4]. The relevant calculation can be conveniently done on a cylinder with angular variable $\theta$. We have

$$b_0 = \frac{1}{2\pi} \oint d\theta \, b(\theta). \quad (2.17)$$

Multiplication by $a$ can be represented by the operator

$$\tilde{a} = \frac{1}{2\pi} \oint d\theta \, a(\theta) = \frac{1}{2\pi} \oint d\theta \left( c \partial \phi + \sqrt{2} \partial c \right). \quad (2.18)$$
Hence
\[ [b_0, \tilde{a}] = \frac{1}{2\pi} \oint d\theta \partial \phi. \]

(2.19)

This is the operator that measures the Liouville momentum. Hence if \( b_0 \cdot \mathcal{O} = 0 \), and \( \mathcal{O} \) corresponds to a Fock space state \( |\psi\rangle \) of Liouville momentum \( p_\phi(\psi) \), then \( b_0 \cdot (a\mathcal{O}) \) corresponds to the Fock space state \( p_\phi|\psi\rangle \). Bearing in mind the shift in Liouville momentum by \( -i\sqrt{2} \) in going from operators to states defined on the cylinder, \( p_\phi = 0 \) only for the cosmological constant operator \( \iota \); and so that is the only case for which \( b_0 \) annihilates \( a\mathcal{O} \). Actually, \( a\iota = 0 \), and that is why \( b_0 \) annihilates \( a\iota \); \( b_0 \) does not annihilate the operator \( \tilde{\iota} = c\partial c e^{\sqrt{2}\phi} \).

Notice that the current \( \partial \phi \) that appeared in the above computation is not a primary field (as \( a \) is not annihilated by \( b_0 \)). The above computation was formulated from the start with an angular variable \( \theta \) on the cylinder, and it was not guaranteed that the operators arising in the computation would be primary.

2.3. The \( b_0 \) Operator as an Exterior Derivative

What sort of operator is \( b_0 \)? For instance, does it commute with the action of the ground ring?

Let \( \mathcal{O} \) be a spin zero, ghost number zero operator of the ground ring. We can represent \( \mathcal{O} \) by the operator
\[ \hat{\mathcal{O}} = \oint_c \frac{d\omega}{2\pi i} \frac{\mathcal{O}}{w}. \]

(2.20)

The commutator of this with \( b_0 \) is \( [b_0, \hat{\mathcal{O}}] \). This object is actually BRST invariant: \( \{Q, [b_0, \hat{\mathcal{O}}]\} \) vanishes because \( \hat{\mathcal{O}} \) is BRST invariant and, being of dimension zero, it commutes with \( L_0 = \{Q, b_0\} \). Our object \( [b_0, \hat{\mathcal{O}}] \) is actually a charge, that is the integral of a spin one operator, since in computing this commutator a contour integral remains. There are non-trivial BRST invariant charges of ghost number \(-1\) (as we have seen in discussing the existence of currents), and there is no reason
to expect \([b_0, \hat{O}] = 0\). Indeed, this is not true, as one readily sees in simple examples (e.g., \(O = x\)).

The fact that the \(b_0 = 0\) condition does not commute with the \(O\)’s means that the currents – which correspond to the part of the cohomology annihilated by \(b_0\) – do not form a module for the ground ring. For essentially this reason, when we combine left and right movers, the discrete moduli of the theory will not form a ground ring module. One might think that the utility of the ground ring would thereby be lost. What saves the day is that \(b_0\) obeys the following condition. Let \(O\) and \(O'\) be any two (spin zero, ghost number zero) ground ring states. Then

\[
[[b_0, \hat{O}], \hat{O}'] = 0 \quad (2.21)
\]

modulo \{\(Q, \ldots\)\}. The point is that \([[b_0, \hat{O}], \hat{O}'(w)]\) would be a BRST invariant local operator of ghost number \(-1\) and \(L_0 = 0\); but the BRST cohomology with those quantum numbers is trivial.

What is the significance of (2.21)? We recall that the \(O\)’s have the interpretation of multiplication by functions on the \(x - y\) plane. In general, if an operator \(X\) has the property that \([[X, f], g] = 0\) for any two functions \(f\) and \(g\), then \(X\) is a first order differential operator. Hence, (2.21) means that \(b_0\) can be interpreted as a first order differential operator on the \(x - y\) plane.

\(b_0\) maps states of ghost number \(q\) to states of ghost number \(q - 1\), for \(q = 1, 2\), so (from our analysis of the quantum numbers) it maps \(\wedge^q T\) to \(\wedge^{q-1} T\). Dually, if we use the area form to identify \(\wedge^q T\) with the space \(\Omega^{2-q}\) of \(2 - q\) forms, then \(b_0\) maps \(\Omega^i\) to \(\Omega^{i+1}\). Furthermore, \(b_0^2 = 0\). In addition, \(b_0\) must commute with symmetries of the theory, which are a central extension of the polynomial area preserving diffeomorphisms of the plane. The central extension is described by the formula

\[
\left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] = \iota \quad (2.22)
\]

of [7, eqn. 2.23]. Ignoring the central extension (setting \(\iota\) to 0, so to speak),
there is only one first order operator on the $x - y$ plane that commutes with area-preserving diffeomorphisms; this is the exterior derivative $d$. The central extension necessitates the following slight modification: if $\alpha$ is a $j$ form, then

$$b_0 \alpha = d\alpha + \delta_{j=0} \alpha(0) \iota.$$  \hspace{1cm} (2.23)

(That is, the $\iota$ dependent correction vanishes unless $\alpha$ is a zero form, in which case $\alpha(0)$ is the function $\alpha$ evaluated at $x = y = 0$.)

Let us check that this identification gives the right answer for the kernel of $b_0$. Acting on zero forms, the exterior derivative annihilates only the constants, which however are not annihilated by the $\iota$ dependent part of (2.23). In our identification of discrete states with differential forms on the plane, zero forms correspond to ghost number two, and we have indeed seen that no states of ghost number two are annihilated by $b_0$. As for one forms, as the $x - y$ plane is contractible, a closed one form $\lambda$ on the $x - y$ plane is $\lambda = df$ for some $f$. In the relation between $\Omega^1$ and $\Lambda^1 T = T$, $\lambda$ corresponds to the vector field

$$V^i \frac{\partial}{\partial x^i} = -\partial_t f \cdot \omega^{ij} \cdot \frac{\partial}{\partial x^j}.$$  \hspace{1cm} (2.24)

This is an area preserving vector field derived from the Hamiltonian function $f$. Thus, the currents, which are associated with the kernel of $b_0$, correspond to those vector fields on the plane which are area preserving. Indeed, the currents derived from the $Y_{s,n}^+$ were seen in [7,13] to generate the group of area preserving transformations (of the $x - y$ plane).

We summarize in the table below the interpretation we have derived in this section for the states in the BRST chiral cohomology.
| Ghost # | Interpretation                  | Dual Interpretation |
|---------|--------------------------------|---------------------|
| $\mathcal{O}$ | 0 functions                 | two-forms           |
| $Y^+$  | 1 area preserving vector fields | closed one-forms    |
| $a\mathcal{O}$ | 1 area non-preserving vector fields | one-forms that are not closed |
| $aY^+$ | 2 bivectors                  | zero-forms          |

3. The Closed String Cohomology

In this section, we will analyze the BRST cohomology of the closed string in $D = 2$, beginning with the $SU(2)$ point. It is here that we will enjoy the payoff from our work in the last section.

If $|\psi_L\rangle$ and $|\psi_R\rangle$ are BRST invariant states in the left and right moving Fock spaces, with the same value of the Liouville momentum $p_{\phi}$, then the tensor product $|\Psi\rangle = |\psi_L\rangle \otimes |\psi_R\rangle$ is certainly a BRST invariant state of the closed string. Moreover, all closed string states arise this way.*

The closed string ground ring, at ghost number zero, is therefore easy to construct. Let us recall how this is done [7]. The left and right moving ghost number zero states are linear combinations of $x^n y^m$ and $x'^{n'} y'^{m'}$, with $x, y$ and $x', y'$ being the generators of the left and right moving chiral ground rings. The equality of Liouville momenta is the condition $n + m = n' + m'$. Products $x^n y^n x'^{m'} y'^{m'}$ with $n + m = n' + m'$ are monomials in $a_1 = xx'$, $a_2 = yy'$, $a_3 = xy'$, and $a_4 = yx'$. The $a_i$ obey one relation $a_1 a_2 - a_3 a_4 = 0$. This equation defines a quadric cone $Q$ in

* Without the $b_0 - \overline{b}_0$ condition that we will introduce presently, this is trivial. We will prove at the end of this section that it is true even when this condition is imposed.
the $a_i$ space. The closed string ground ring is the ring of polynomials in the $a_i$, or in other words the ring of polynomial functions on $Q$.

It is useful to introduce a four dimensional space $W$ with coordinates $x, y, x', y'$. On $W$ there is a one parameter group action $x, y, x', y' \to tx, ty, t^{-1}x', t^{-1}y'$. We will call this group $H$. $H$ is generated by the vector field

$$S = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - x' \frac{\partial}{\partial x'} - y' \frac{\partial}{\partial y'}$$

which measures the difference between the left and right moving Liouville momenta. $Q$ is just the quotient $W/H$. Many constructions below are most conveniently described in terms of $H$ invariant objects on the four dimensional space $W$, rather than explicitly working out the reduction to $Q$.

**The $b_0 - \overline{b}_0$ Condition**

The construction of the ground ring is so trivial that one may well wonder if there is anything non-trivial to be done in combining the left and right movers.

There is; this becomes clear when one considers the discrete moduli of the closed string. Moduli are usually constructed from primary fields of highest weights $(1, 1)$ and left and right moving ghost numbers $(0, 0)$. To construct these by multiplying left and right movers, one needs a left moving current $Y$ and a right moving current $\overline{Y}$, each of spin one and ghost number zero. As we have seen in §2.2, these will arise as $|Y\rangle = b_{-1}|Z\rangle$ and $|\overline{Y}\rangle = \overline{b}_{-1}|\overline{Z}\rangle$, where $|Z\rangle$ and $|\overline{Z}\rangle$ have spin zero and ghost number one and are annihilated by $b_0$ and $\overline{b}_0$, respectively. The moduli that one gets this way correspond thus to $|\mathcal{Y}\rangle = b_{-1}\overline{b}_{-1}|\mathcal{Z}\rangle$, where $|\mathcal{Z}\rangle = |Z\rangle \otimes |\overline{Z}\rangle$. Here $|\mathcal{Z}\rangle$ corresponds to an operator of spin $(0, 0)$ and ghost number $(1, 1)$. Moreover,

$$b_0|\mathcal{Z}\rangle = \overline{b}_0|\mathcal{Z}\rangle = 0.$$  

(3.2)

The discrete moduli that have been discussed hitherto are the ones constructed as just explained.
That this cannot be the whole story can be seen in a number of ways. First of all, in explicitly acting on moduli of the above type with symmetry generators, new operators appear. This computation will be presented in §5.3 below. Another reason that new operators are needed comes from consideration of the $\beta$ function; see §3.2. The reason for the problem is that although the construction of closed string moduli recalled in the last paragraph is the most familiar one, the general rules of closed string theory permit a more general construction. Closed string moduli are always derived from BRST invariant operators $Z$ of ghost number 2 and spin zero. However, the left and right moving ghost numbers need not both be 1 in general. Moreover, condition (3.2) is stronger than necessary; the essential condition is only

$$ (b_0 - \bar{b}_0) |Z\rangle = 0. $$

That these are the only really necessary conditions is known in the operator formalism [15–17] (where it arises due to the absence of a global section on $P_{g,n}$) and is actually obvious in closed string field theory [14] (where this restriction is necessary to write a kinetic term). In many situations, nothing is lost (and life is easier) if one restricts to $Z$’s that obey (3.2) and have ghost numbers $(1, 1)$. But in $D = 2$, the story is completely different.

3.1. Closed String Moduli

We have seen that the left moving discrete states of ghost number $i$ can be identified with the $2 - i$ forms on the $x - y$ plane. Similarly the right moving discrete states of ghost number $j$ are the $2 - j$ forms on the $x' - y'$ plane.

The product of the $x - y$ plane and the $x' - y'$ plane is the four manifold $W$ introduced above. By taking the tensor product of arbitrary left and right moving discrete states, one obtains the (polynomial) differential forms on $W$. The degree of the differential form is $(2 - i) + (2 - j) = 4 - i - j$, where $i$ and $j$ are the left and right moving ghost numbers. If we denote the space of $n$ forms on $W$ as $\Omega^n$,
and the space of forms of type $i, j$ (with $i$ indices tangent to the $x - y$ plane and $j$ to the $x' - y'$ plane) as $\Omega^{i,j}$, then

$$\Omega^n = \bigoplus_{i=0}^{n} \Omega^{i,n-i}. \quad (3.4)$$

This just says that the total ghost number is the sum of the independent left and right moving contributions.

In tensoring together left and right moving states, we should restrict to states of equal left and right moving Liouville momenta. This just means taking the $H$ invariant states. If, therefore, we do not worry about the $b_0 - \overline{b}_0$ condition, then the discrete states of the closed string are just the $H$ invariant differential forms on $W$.

What about $b_0 - \overline{b}_0$? We have identified $b_0$ as the exterior derivative on the $x - y$ plane – let us call it $d_L$. (For simplicity, we will ignore the central extension.) Likewise $\overline{b}_0$ is the exterior derivative on the $x' - y'$ plane, say $d_R$. Hence $b_0 - \overline{b}_0 = d_L - d_R$. By conjugating with $(-1)^{Fr}$ (the operator that multiplies $j$ forms on the $x' - y'$ plane by $(-1)^j$), we can transform this into $d = d_L + d_R$, the standard exterior derivative on $W$. Note that

$$0 = d_L^2 = d_R^2 = \{d_L, d_R\}. \quad (3.5)$$

As we have discussed, closed string moduli correspond to discrete states $Z$ of ghost number two that are annihilated by $b_0 - \overline{b}_0$. These are in other words, in view of what we have just said, the closed two forms on $W$ which are also $H$ invariant.

A closed two form $F$ is naturally regarded as the curvature of an abelian gauge field. The simple topology of $W$ or $Q$ does not support Dirac monopoles, so we can write simply $F = dA$, with $A$ being the vector potential. We have found that the closed string moduli (in the space of conformal fields) can be represented by an $H$ invariant abelian gauge field on $W$. Imposing the $H$ invariance means carrying
out a dimensional reduction from $W$ to the quotient space $Q = W/H$. Under this process, $A$ reduces to an abelian gauge field and a scalar on $Q$. Many formulas are simpler if written in terms of $H$ invariant objects on $W$, and we need not always carry out the dimensional reduction explicitly.

Let us summarize the closed string states that obey the $b_0 - \overline{b}_0$ condition. For convenience we split the states according to the type of Liouville dressing. The “plus” states are

\[
G = 0 : \quad \mathcal{O}_{u,n} \overline{\mathcal{O}}_{u,n'}
\]
\[
G = 1 : \quad Y_{s,n}^+ \overline{\mathcal{O}}_{s-1,n'}, \quad \mathcal{O}_{s-1,n'} Y_{s,n}^+, \quad (a + \overline{a}) \cdot (\mathcal{O}_{u,n} \overline{\mathcal{O}}_{u,n'}),
\]
\[
G = 2 : \quad Y_{s,n}^+ \overline{Y}_{s,n'}, \quad (a + \overline{a}) \cdot (Y_{s,n}^+ \overline{\mathcal{O}}_{s-1,n'}), \quad (a + \overline{a}) \cdot (\mathcal{O}_{s-1,n'} \overline{Y}_{s,n}^+),
\]
\[
G = 3 : \quad (a + \overline{a}) \cdot (Y_{s,n}^+ \overline{Y}_{s,n'}),
\]

and the “minus” states are

\[
G = 2 : \quad Y_{s,n}^- \overline{Y}_{s,n'},
\]
\[
G = 3 : \quad Y_{s,n}^- \overline{P}_{s-1,n'}, \quad P_{s-1,n'} \overline{Y}_{s,n}, \quad (a + \overline{a}) \cdot (Y_{s,n}^- \overline{Y}_{s,n'}),
\]
\[
G = 4 : \quad P_{s,n} \overline{Y}_{s,n'}, \quad (a + \overline{a}) \cdot (Y_{s,n}^- \overline{P}_{s-1,n'}), \quad (a + \overline{a}) \cdot (P_{s-1,n'} \overline{Y}_{s,n}),
\]
\[
G = 5 : \quad (a + \overline{a}) \cdot (P_{s,n} \overline{P}_{s,n'}).
\]

3.2. Comparison To The Usual Operators

In [7,§2.6], it is pointed out that the closed string moduli coming from conventional $(1,1)$ vertex operators of ghost number zero (which were the only ones considered there) correspond to functions on $Q$. Let us see how these fit in.

If we are given the function $\phi$ on $Q$, which we will think of as an $H$ invariant function on $W$, we can make the two form

\[
F = d_L d_R \phi.
\]
Explicitly

\[ F_{ij'} = \frac{\partial^2 \phi}{\partial x^i \partial x'^{j'}} \], \quad F_{ij} = F_{i'j'} = 0. \quad (3.9) \]

Thus, \( F \) is a differential form of type \((1, 1)\), and obviously (using (3.5)) \( d_L F = d_R F = 0 \). Thus, the operator corresponding to \( F \) has left and right moving ghost numbers \((1, 1)\), and is annihilated by both \( b_0 \) and \( \overline{b}_0 \). These are then the conventional \((1, 1)\) moduli, the ones considered in previous discussions of the discrete states. Thus we have explained the embedding of the functions on \( Q \) – considered in [7] – in the closed \( F \) invariant two forms on \( W \), which is the space of closed string moduli in the sense we are considering here.

To first order the states we have been discussing are all closed string moduli. In quadratic order, one meets a variety of effects including the quadratic part of the beta function. As was noted in [7, eqn. (2.47)], the quadratic part of the \( \beta \) function for the \( \phi \) field is

\[ \beta = \epsilon^{ij} \epsilon^{i'j'} \frac{\partial^2 \phi}{\partial x^i \partial x'^{j'}} \frac{\partial^2 \phi}{\partial x^i \partial x'^{j'}}. \quad (3.10) \]

There is a puzzle here. This formula is not invariant under volume preserving diffeomorphisms of \( Q \), which were claimed in [7] to be a symmetry of the theory. The puzzle is one reason that the extra states that we have been describing here are necessary. With the formula (3.9) for the map from the function \( \phi \) to the two form \( F \), we see that in terms of \( F \), the quadratic beta function can be written

\[ \beta = F \wedge F. \quad (3.11) \]

This formula is invariant, as it should be, under volume preserving diffeomorphisms; in fact it is invariant under arbitrary diffeomorphisms of \( W \) (that commute with \( H \) if one restricts to the \( H \) invariant subspace).
The Minus Operators

We now want to write down a Lagrangian from which (3.11) can be derived. To this aim, we must consider the “minus” operators, the ones with Liouville dressing $e^{\sqrt{2}\phi(1+s)}$, $s \geq 0$. Here no new moduli appear, compared to previous analyses, for the following reason. As we have noted at the beginning of §2, the left or right moving minus operators have ghost numbers in the range between 1 and 3. Moreover, the states of ghost number 1 in the chiral theory are annihilated by $b_0$ (or $\bar{b}_0$). In combining such left and right moving modes, the total ghost number can be 2 only if the left and right moving pieces are $(1,1)$. The modes made this way are annihilated by $b_0$ and $\bar{b}_0$ separately and are thus modes of the type considered in the past.

As the left and right moving zero modes can be interpreted as “functions” (or really distributions) on the $x - y$ or $x' - y'$ plane with delta function support at the origin, when one combines left and right movers the “minus” moduli make up an $H$ invariant “function” on $W$ with support at the origin (or a function on $Q$ with delta function support at the apex). Let us call this function $\sigma$.

The Lagrangian

The Lagrangian from which (3.11) is to be derived should involve both $\sigma$ and $F$, and is obviously

$$L = \int \sigma \cdot F \wedge F. \quad (3.12)$$

The $\sigma$ equation of motion is the $F \wedge F$ part of the beta function; the $A$ equation of motion gives terms in the beta function bilinear in $\sigma$ and $F$. (We have verified these terms in part.)

The Lagrangian (3.12) is oddly reminiscent of Lagrangians sometimes studied in works on topological field theory. It very much looks like the Lagrangian of a theory that does not have local dynamics. In many ways the most remarkable and mysterious aspect of the theory is that the same model that at the $SU(2)$
point gives rise to (3.12) also gives rise in the decompactified theory to the local
dynamics of the “tachyon.”

The absence of a quadratic term in (3.12) is unsettling but, given the framework
for the computation, inevitable: (3.12) is by definition a cubic coupling of modes
that are moduli in lowest order and hence appear in no quadratic coupling.

Several caveats should be stated here:

(1) If we take the analysis at the $SU(2)$ point literally, the moduli contained
in $F$ correspond to forms with a polynomial dependence on the coordinates $a_i$ of
$Q$, while $\sigma$ (which is constructed from states with the dual Liouville dressing) is
a “function” (or really a distribution) supported at $a_i = 0$. It is, however, very
tempting to believe that with better understanding, perhaps after making some
generic perturbation of the model, $F$ and $\sigma$ should be permitted to be general
fields on $Q$, rather than polynomials on the one hand and derivatives of a delta
function on the other.

(2) The Lagrangian (3.12) correctly describes the cubic couplings of discrete
states at the $SU(2)$ point. Whether it correctly describes the restrictions needed to
maintain conformal invariance under departure from the $SU(2)$ point is less clear.
Once one starts to depart from the $SU(2)$ point, it is not clear that the BRST
cohomology should be computed in the standard space of conformal fields that we
have been using.

(3) From the point of view of the Lagrangian (3.12), the symmetries of $Q$ (vol-
ume preserving diffeomorphisms, say) are a peculiar sort of big global symmetry
group; no gauge fields are in sight. However, in string theory, one believes that the
unbroken symmetries at, say, the $SU(2)$ point are the residue of a large underlying
gauge group. In particular, perturbations of the $SU(2)$ point that are equivalent
under volume preserving symmetries of $Q$ (and other closed string symmetries dis-
cussed presently) should be identified in the string theory; but this is not automatic
in (3.12).
**Dimensional Reduction**

The Lagrangian (3.12) has been written in terms of \( H \) invariant quantities on \( W \). If we wish, we can carry out explicitly the dimensional reduction to \( Q \). The abelian gauge field \( A \) on \( W \) reduces (after picking, non-invariantly, a section of the bundle \( W \to Q \)) to \( A = u \cdot dt + a \), where \( a \) is the pullback of an abelian gauge field from \( Q \), \( u \) is a function on \( Q \), and \( t \) is a parameter along the fibers of \( W \to Q \). \( t \) is determined only up to \( t \to t - f \), where \( f \) is a function on \( Q \). Under this transformation, we have

\[
\begin{align*}
    u &\to u \\
    a &\to a - u \cdot df,
\end{align*}
\]

(3.13)

The Lagrangian when written in terms of these variables becomes

\[
L = 2 \int_Q \sigma \cdot du \wedge da,
\]

(3.14)

which is invariant under (3.13). We presently will understand better the origin of this symmetry.

**3.3. Symmetries Of The Closed String**

To study the closed string symmetries, we must look at BRST invariant states of ghost number one (more on this in §4). In keeping with our general analysis, such a state \( |\Psi\rangle \) represents an \( H \) invariant vector field, \( V = V^I \partial_I \) (\( I = 1 \ldots 4 \) runs over \( x, y, x', y' \)) on \( W \). Dually, \( |\Psi\rangle \) determines the three form \( \lambda = \epsilon_{IJKL} V^I dx^J dx^K dx^L \) on \( W \).

Such a \( |\Psi\rangle \) determines a symmetry of the closed string theory if and only if \((b_0 - \overline{b_0})|\Psi\rangle = 0\). In keeping with our general analysis, this translates into \( d\lambda = 0 \)

\* \( H \) invariance of \( a \) implies \((dS + i_S d)a = 0\). \( a \) being a pullback means \( i_S a = 0\).
or, dually,

$$\partial_I V^I = 0. \quad (3.15)$$

The latter condition means that $V$ generates a volume-preserving diffeomorphism of $W$, that is a diffeomorphism that preserves the volume form $dx \wedge dy \wedge dx' \wedge dy'$.

The symmetries found in [7] are determined by $\Psi$’s that obey $b_0|\Psi\rangle = \overline{b}_0|\Psi\rangle = 0$. Such $\Psi$’s correspond to generators of diffeomorphisms that preserve both the left and right moving volume forms $\omega = dx \wedge dy$ and $\omega' = dx' \wedge dy'$. After imposing the $H$ invariance, they are equivalent to volume preserving diffeomorphisms of $Q = W/H$. The novelty now is that by requiring $\Psi$ to be annihilated only by $b_0 - \overline{b}_0$, we include diffeomorphisms that preserve $\omega \wedge \omega'$ but do not preserve $\omega$ or $\omega'$ separately.

The extra symmetries can be described as follows. The basic one is the $H$ generator

$$S = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - x' \frac{\partial}{\partial x'} - y' \frac{\partial}{\partial y'} \quad (3.16)$$

$S$ is clearly volume preserving, $\partial_I S^I = 0$. It preserves $\omega \wedge \omega'$ but not $\omega$ or $\omega'$. More generally, if $f$ is any $H$ invariant function, then $f \cdot S$ generates a diffeomorphism that is also volume preserving; indeed, $\partial_I (fS^I) = S^I \partial_I f = 0$ by $H$ invariance of $f$.

How do these new symmetries act on the states? $S$ annihilates all of them, this being the condition of $H$ invariance, but this is not true of $U = fS$. Its action can be deduced from the description of the states as $H$ invariant differential forms on $W$. The transformation of a differential form $\lambda$ under the symmetry generated by a vector field $U$ is in general

$$\delta \lambda = L_U \lambda = i_U (d\lambda) + d(i_U \lambda) \quad (3.17)$$

where $i_U$ is the operation of contraction of $U$ with one index of $\lambda$. The closed string states are closed differential forms on $W$, so we can delete the $i_U (d\lambda)$ term in (3.17).
Also, $H$ invariance of $\lambda$ means that (3.17) vanishes if $U = S$, so $d(i_S \lambda) = 0$. In general, if $U = f S$, then $i_U \lambda = f \cdot i_S \lambda$. Hence (3.17) reduces for symmetries of this type to

$$\delta \lambda = df \wedge i_S \lambda.$$  

(3.18)

The new symmetries act trivially on the ground ring. If indeed $\lambda$ is a ghost number zero ring state, then it corresponds to a four form, so $\delta \lambda$ is also a four form. From $i_S(df) = 0$ and $i_S^2 = 0$ it follows that $i_S \delta \lambda = 0$. A four form annihilated by $i_S$ must vanish, so $\delta \lambda = 0$.

In general, $\delta \lambda$ is given by a differential operator of degree zero acting on $\lambda$; symmetries of this type are usually called “internal symmetries.”

The new symmetry action on the closed string moduli can be made very explicit. In fact, after making explicitly the dimensional reduction to $Q$, the new symmetries correspond precisely to those described in (3.13). This is no accident. The new symmetries are generated by $H$ invariant vector fields on $W$ which (as they act trivially on the ground ring) project to zero on $Q$; thus they translate the section of $W \to Q$ that was used in the dimensional reduction.

3.4. Combining Left And Right Movers

In this discussion, we have been building closed string states by taking tensor products of left and right moving states and then selecting states annihilated by $b_0 - \bar{b}_0$. We will now discuss the justification for this more fully. This subsection can be omitted by readers willing to accept the procedure that we have followed above on faith.

First of all, in considering the BRST cohomology, we can assume that all states discussed below are annihilated by $L_0$ and $\overline{L}_0$. The reason for this is really that (i) $L_0$ and $\overline{L}_0$ generate a compact group over which one can average; (ii) as $L_0$ and $\overline{L}_0$ are BRST commutators, the cohomology class of a state is not changed in this averaging.
To begin with, we review some facts about BRST cohomology in the chiral case [22]. There are two relevant kinds of BRST cohomology. First, there is the “absolute” cohomology $H^n$; an element of $H^n$ is represented by a BRST invariant state $|\Psi\rangle$ of ghost number $n$, modulo $|\Psi\rangle \to |\Psi\rangle + Q|\Lambda\rangle$ where $|\Lambda\rangle$ has ghost number $n - 1$. Second, there is the relative cohomology $H^n_R$. An element of $H^n_R$ is represented by a BRST invariant state $|\Psi\rangle$ of ghost number $n$, annihilated by $b_0$, modulo $|\Psi\rangle \to |\Psi\rangle + Q|\Lambda\rangle$, where $|\Lambda\rangle$ has ghost number $n - 1$ and is annihilated by $b_0$.

There are standard maps between absolute and relative cohomology, that fit together into an exact sequence. First, there is a map

$$i : H^n_R \to H^n$$

which consists of forgetting the $b_0$ condition. Second, there is a map

$$b_0 : H^n \to H^{n-1}_R$$

of multiplication by $b_0$ (which maps arbitrary states to states annihilated by that operator, and preserves the $Q$ invariance since we are working in the null eigenspace of $L_0$). Finally there is a map

$$\{Q, c_0\} : H^n_R \to H^{n+2}_R,$$

that consists of multiplication by $\{Q, c_0\}$ (which is not a BRST commutator in the relative cohomology). It is easy to see that these maps are all well defined.

These maps fit together into the “exact sequence”

$$\ldots H^n_R \xrightarrow{i} H^n \xrightarrow{b_0} H^{n-1}_R \xrightarrow{\{Q, c_0\}} H^{n+1}_R \xrightarrow{i} H^{n+1} \ldots.$$ 

The statement that this is an exact sequence means (i) the composition of two successive maps is zero; (ii) a state in the kernel of one map is in the image of the
one before. The first statement is easy to see and left to the reader; the second will now be verified.

For instance, let us verify exactness of (3.22) at $H^n$. An element $w$ of $H^n$ is in the kernel of $b_0$ if any representative BRST invariant state $|\Psi\rangle$ obeys $b_0|\Psi\rangle = Q|\Lambda\rangle$ for some $|\Lambda\rangle$ which is annihilated by $b_0$. If so, pick for $w$ the representative $|\Psi'\rangle = |\Psi\rangle + Qc_0|\Lambda\rangle$; then $|\Psi'\rangle$ is annihilated by both $Q$ and $b_0$, so represents a state $w'$ in $H^n_R$; and $w = i(w')$.

To verify exactness at $H^{n-1}_R$, an element of the kernel of $\{Q,c_0\}$ can be represented by a state $|\Psi\rangle$, of ghost number $n - 1$, annihilated by both $Q$ and $b_0$, and such that $\{Q,c_0\}|\Psi\rangle = Q|\Theta\rangle$, where $|\Theta\rangle$ is annihilated by $b_0$. This being so, let $|\tilde{\Psi}\rangle = c_0|\Psi\rangle - |\Theta\rangle$; as $|\tilde{\Psi}\rangle$ is annihilated by $Q$ and $|\Psi\rangle = b_0|\tilde{\Psi}\rangle$, $|\Psi\rangle$ is in the image of $b_0$, as we wished to show.

Finally, to show exactness at $H^{n+1}_R$, suppose a state $|\Psi\rangle$ annihilated by $Q$ and $b_0$ is in the kernel of $i$. This means that $|\Psi\rangle = Q|\Lambda\rangle$ (but $b_0|\Lambda\rangle$ may be nonzero, so the element of $H^{n+1}_R$ represented by $|\Psi\rangle$ may be nonzero). Any state $|\Lambda\rangle$ can be written uniquely as $|\Lambda\rangle = c_0|\Lambda'\rangle + |\Lambda''\rangle$ where $|\Lambda'\rangle$ and $|\Lambda''\rangle$ are annihilated by $b_0$. Since $|\Psi\rangle$ and $|\Psi\rangle - Q|\Lambda''\rangle$ represent the same element of $H^{n+1}_R$, we can suppose that $|\Lambda''\rangle = 0$, so $|\Psi\rangle = Qc_0|\Lambda'\rangle$, with $|\Psi\rangle$, $|\Lambda'\rangle$ annihilated by $b_0$. From $b_0|\Psi\rangle = 0$, we learn that $Q|\Lambda'\rangle = 0$, a formula that implies $|\Psi\rangle = \{Q,c_0\}|\Lambda'\rangle$, which now finally tells us that $|\Psi\rangle$ is in the image of $\{Q,c_0\}$.

What we have said so far is true for arbitrary bosonic string vacua. For the $D = 2$ string vacuum we are studying, it is also true that the map $\{Q,c_0\}$ is zero since it adds two units of ghost number and relative cohomology exists only for ghost numbers 0, 1 (plus states) and 2, 3 (minus states). Since the map in question cannot mix plus and minus states it follows that it must equal zero.

**Closed Strings**

In discussing closed strings, we consider models in which the total Hilbert space is a tensor product of left and right moving Hilbert spaces and the BRST operator
Q is a corresponding sum $Q = Q_L + Q_R$, with $Q_L$ and $Q_R$ the BRST operators of left and right movers.

There are now several types of BRST cohomology to consider. The absolute closed string cohomology $H^n$ consists of $Q$ invariant states of ghost number $n$, modulo $Q\ket{\Lambda}$ for arbitrary $\ket{\Lambda}$. What we will call the relative closed string cohomology $H^n_R$ consists of $Q$ invariant states of ghost number $n$, annihilated by $b_0$ and $\overline{b}_0$, modulo $Q\ket{\Lambda}$ where $\ket{\Lambda}$ is annihilated by $b_0$ and $\overline{b}_0$. And what we will call the semirelative cohomology $H^n_S$ consists of $Q$ invariant states of ghost number $n$ annihilated by $b_0 - \overline{b}_0$, modulo $Q\ket{\Lambda}$ where $\ket{\Lambda}$ is annihilated by $b_0 - \overline{b}_0$. It is convenient to set $t_0^\pm = b_0 \pm \overline{b}_0$, and $c_0^\pm = c_0 \pm \overline{c}_0$.

Let $H^n_+$ and $H^n_-$ be the absolute BRST cohomology of right movers and left movers, respectively; and similarly let $H^n_{R,\pm}$ be the relative cohomology of right and left movers. If these are known, then $H^n$ and $H^n_R$ can be readily extracted, since they are determined by conditions that factorize between left and right movers. In fact

$$H^n = \bigoplus_m H^n_+ \otimes H^{n-m}_-,$$

$$H^n_R = \bigoplus_m H^m_{R,+} \otimes H^{n-m}_{R,-}.$$  \hfill (3.23)

But what we really need is $H^n_S$, for which we must work a little bit harder.

The various $H$’s are related by exact sequences analogous to (3.22). First, one has

$$\ldots \to H^n_R \xrightarrow{i'} H^n_S \xrightarrow{b_0^+} H^{n-1}_R \xrightarrow{\{Q,c_0^+\}} H^{n+1}_R \xrightarrow{i'} H^{n+1}_S \to \ldots,$$  \hfill (3.24)

where $i'$ is the map that forgets the extra $b_0$ condition, and the next two maps are simply defined by multiplication with the indicated operators. Similarly, one has

$$\ldots \to H^n_S \xrightarrow{i''} H^n_R \xrightarrow{b_0^-} H^{n-1}_S \xrightarrow{\{Q,c_0^-\}} H^{n+1}_S \xrightarrow{i''} H^{n+1}_S \to \ldots,$$  \hfill (3.25)

where $i''$ is the map that forgets the $b_0^-$ condition and the next two maps are defined by multiplication with the indicated operators. The proof of exactness follows the discussion of (3.22) line by line.
The formulas just written hold for arbitrary bosonic string vacua. In the standard $D=2$ vacua, one has the further fact that in (3.24), $\{Q,c_0^+\}$ vanishes. The reason for this is that in the decomposition $H^n_R = \oplus H^m_{R,+} \oplus H^m_{R,-}$, $\{Q,c_0^+\}$ would increase $m$ or $n-m$ by 2, but $H^m_{R,\pm}$ is zero except for $m=0,1$ for plus states (and $m=1,2$ for minus states). Hence (3.24) reduces to exact sequences

\[ 0 \to H^n_R \xrightarrow{i'} H^n_S \xrightarrow{b_0^+} H^{n-1}_R \to 0. \] (3.26)

This means that as a vector space, $H^n_S$ has a (non-canonical) isomorphism

\[ H^n_S \cong i'(H^n_R) \oplus H^{n-1}_R. \] (3.27)

This has a simple interpretation. $i'(H^n_R)$ consists of closed string states with representatives that are annihilated by both $b_0$ and $\overline{b_0}$. These are the traditional closed string states, the ones discussed for instance in [7]. (3.26) shows that the other closed string states, which are annihilated by $b_0^-$ but not by $b_0^+$, are determined modulo $i'(H^n_R)$ by their image under multiplication by $b_0^+$, this image being an arbitrary element of $H^{n-1}_R$. The recipe that we followed intuitively at the beginning of this section to construct the closed string states can readily be seen to agree with this: the reader can see in equations (3.6) and (3.7) that at ghost number $n$ we have included the states in the relative cohomology, plus extra states that are simply obtained from the relative cohomology at ghost number $(n-1)$ upon multiplication by $(a+\overline{a})$, which acts as an inverse for $b_0^+$. We have therefore verified that this recipe is correct.

The recipe that we used is actually related in a more obvious way to (3.25). If one knows that $\{Q,c_0^-\} = 0$, then (3.25) implies that

\[ H^n_S = \ker b_0^- : H^n \to H^{n-1}_S. \] (3.28)

In other words, $H^n_S$ consists of closed string states annihilated by $b_0^-$. That $\{Q,c_0^-\} = 0$ depends on additional facts (coming from explicit calculations of
the BRST cohomology \([3,5]\)). \((3.24)\) is easier to work with, even though it gives the result in a less transparent form, because the unknown object \(\mathcal{H}_S^*\) appears only once, and not twice as in \((3.25)\).

4. The Closed String Symmetry Generators

The purpose of the present section is the construction of the generators of symmetries for the closed string theory. As we will see, there are subtleties concerning the proper combination of the left and right sectors of the theory. Most of our attention will concentrate on the symmetries that preserve total ghost number. One novel fact of these symmetries is that they do not in general preserve both the left and the right moving ghost numbers. There are also conserved charges that change the total ghost number, and we will show how to construct them.

For the purely holomorphic case the chiral symmetry algebra is generated by the currents \(J^\pm(z) = W_{s,n}^\pm(z)\), which are dimension one, primary operators that give rise to the charges \(A_{s,n}\) defined by

\[
A_{s,n}^\pm = \frac{1}{2\pi i} \oint_C W_{s,n}^\pm(z) dz
\]  
(4.1)

Since \(W\) is built completely from the matter sector, one has \(\{Q, W\} = \partial(cW)\), and as a consequence the above charges map BRST invariant states to BRST invariant states since

\[
\{Q, A\} = \oint_C \{Q, W\} dz = \oint C \partial(cW) dz = 0.
\]

Let us now turn to the case of the closed string. Conventionally charges are built by integrating around a closed curve a a holomorphic \((1,0)\) operator or an antiholomorphic \((0,1)\) operator.

In the present model, we can construct \((1,0)\) and \((0,1)\) operators, but because of the requirement of matching the Liouville momenta, they are not holomorphic
or antiholomorphic. The natural way to obtain a (1,0) operator is to multiply a purely holomorphic current $W_{s,n}(z)$ by an antiholomorphic operator of dimension zero, of zero ghost number and carrying the same Liouville momentum, in other words an operator in the antichiral ground ring. (0,1) operators are constructed similarly. So we have [7]

\begin{equation}
(1,0) : J_{s,n}(z,\bar{z}) = W_{s,n}(z) \overline{O}_{s-1,n}(\bar{z}),
\end{equation}

\begin{equation}
(0,1) : \overline{J}_{s,n}(z,\bar{z}) = O_{s-1,n}(z) \overline{W}_{s,n}(\bar{z}).
\end{equation}

These operators satisfy neither $\overline{\partial} J = 0$ nor $\overline{\partial} \overline{J} = 0$. In order to understand properly the nature of this complication let us recall the general framework for conserved currents.

4.1. Conserved Currents via Descent Equations

In general a strictly conserved current corresponds to a pair $(J_z, J_{\overline{z}})$ such that the one form

\begin{equation}
\Omega^{(1)} = J_z dz - J_{\overline{z}} d\overline{z},
\end{equation}

is closed

\begin{equation}
d\Omega^{(1)} = 0 \iff \partial_{\overline{z}} J_z + \partial_z J_{\overline{z}} = 0.
\end{equation}

The conserved charge $A$ is then given by

\begin{equation}
A(\mathcal{C}) = \oint_{\mathcal{C}} \Omega^{(1)} = \oint_{\mathcal{C}} dz J_z - \oint_{\mathcal{C}} d\overline{z} J_{\overline{z}},
\end{equation}

where the two integrals are taken on the same contour and with the same orientation. We take henceforth the orientation to be such that

\begin{equation}
\oint_{\mathcal{C}} \frac{dz}{2\pi i \bar{z}} = -\oint_{\mathcal{C}} \frac{d\bar{z}}{2\pi i z} = 1.
\end{equation}

Conservation is the statement that for two curves $\mathcal{C}$ and $\mathcal{C}'$ that are homologous,
namely $\partial M = C - C'$ for some surface $M$, one has

$$0 = \int_M d\Omega^{(1)} = \int_{\partial M} \Omega^{(1)} = \int_{\mathcal{C}} \Omega^{(1)} - \int_{\mathcal{C}'} \Omega^{(1)} = A(C) - A(C'). \quad (4.7)$$

In BRST quantization, to have a conserved charge that is well defined in the physical Hilbert space, we do not really need $A(C) = A(C')$. It is enough for this to hold modulo BRST commutators,

$$A(C') = A(C) + \{Q, B\}. \quad (4.8)$$

For then $A(C')$ and $A(C)$ have the same action on physical states. This possibility corresponds to the case when the form $\Omega^{(1)}$ is closed only up to a BRST commutator. Instead of (4.4) we have

$$d\Omega^{(1)} = \{Q, \Omega^{(2)}\}, \quad (4.9)$$

for some two-form $\Omega^{(2)}$. More explicitly, if $\Omega^{(2)} = \Omega^{(2)}_{z \bar{z}} dz \wedge d\bar{z}$, then

$$\partial_{\bar{z}} J_z + \partial_z J_{\bar{z}} = -\{Q, \Omega^{(2)}_{z \bar{z}}\}. \quad (4.10)$$

In this situation, (4.7) is replaced with

$$A(C) - A(C') = \{Q, \int_M \Omega^{(2)}\}, \quad (4.11)$$

as in (4.8). This is good enough to give a conserved charge at the physical level, but as we will see in §6, Ward identities can have unusual properties when $\Omega^{(2)} \neq 0$. 

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We must also formulate the condition for a conserved charge to commute with $Q$. This will happen if there is a zero-form $\Omega^{(0)}$ such that

$$d\Omega^{(0)} = \{Q, \Omega^{(1)}\}$$  \hspace{1cm} (4.12)$$

since in this case

$$\{Q, \oint_C \Omega^{(1)}\} = \oint_C \{Q, \Omega^{(1)}\} = \oint_C d\Omega^{(0)} = 0$$  \hspace{1cm} (4.13)$$
as desired. It is then necessarily also true that $\{Q, \Omega^{(0)}\} = 0$. Indeed, (4.13) implies that $d\{Q, \Omega^{(0)}\} = 0$, so $\{Q, \Omega^{(0)}\}$ is a $c$-number; this $c$-number must vanish, or the identity would be a BRST commutator, and the whole BRST machinery would break down.

Summarizing, the general framework for conserved charges is given by the descent equations

$$0 = \{Q, \Omega^{(0)}\}$$
$$d\Omega^{(0)} = \{Q, \Omega^{(1)}\}$$
$$d\Omega^{(1)} = \{Q, \Omega^{(2)}\}.$$ \hspace{1cm} (4.14)$$

These imply that $A = \oint_C \Omega^{(1)}$ is a BRST invariant charge conserved up to BRST trivial operators.

Let us close this section with a few comments. To derive the symmetry charges we have to choose BRST invariant zero forms $\Omega^{(0)}$. We will use below the BRST cohomology classes. Since this model has cohomology classes of several ghost numbers we can obtain symmetry charges of various ghost numbers and different statistics. While we will concentrate next on the case of the ghost number zero charges, the other charges play a role on the larger symmetry structure discussed in §6.
The solution of the descent equations is not unique. If we have one solution, we can generate another one via the replacements

\[
\begin{align*}
\Omega^{(0)} & \rightarrow \Omega^{(0)} + \{Q, \alpha^{(0)}\} \\
\Omega^{(1)} & \rightarrow \Omega^{(1)} + d\alpha^{(0)} + \{Q, \alpha^{(1)}\} \\
\Omega^{(2)} & \rightarrow \Omega^{(2)} + d\alpha^{(1)} + \{Q, \alpha^{(2)}\},
\end{align*}
\]

where \( \alpha^{(i)} \) is an arbitrary form of degree \( i \). It is clear from these equations that the charge \( \oint \Omega^{(1)} \) is independent of the choice of BRST representative for \( \Omega^{(0)} \).

The descent equations can be discussed effectively (and solved!) using states. In terms of states the zero form \( \Omega^{(0)}(z) \) correspond to the state \( |\Omega^{0}\rangle = \Omega^{(0)}|1\rangle \) (where \( |1\rangle \) is the SL(2,C) vacuum). Similarly, the one form \( \Omega^{(1)} = \Omega_{z}^{1} dz + \Omega_{z}^{1} d\bar{z} \) defines the states \( |\Omega^{1}_{z}\rangle = \Omega_{z}^{1}|1\rangle \), and \( |\Omega^{1}_{\bar{z}}\rangle = \Omega_{\bar{z}}^{1}|1\rangle \). Finally \( \Omega^{(2)} = \Omega^{2}dz \wedge d\bar{z} \) gives rise to the state \( |\Omega^{2}\rangle = \Omega^{2}|1\rangle \). If we have a suitable energy momentum tensor \( (T(z), \overline{T}(\bar{z})) \) such that \( |\partial \mathcal{O}\rangle = L_{-1} |\mathcal{O}\rangle \) and \( |\bar{\partial} \mathcal{O}\rangle = \overline{L}_{-1} |\mathcal{O}\rangle \) then we can write the descent equations as:

\[
\begin{align*}
0 & = Q |\Omega^{0}\rangle, \\
L_{-1} |\Omega^{0}\rangle & = Q |\Omega^{1}_{z}\rangle \\
\overline{L}_{-1} |\Omega^{0}\rangle & = Q |\Omega^{1}_{\bar{z}}\rangle \\
L_{-1} |\Omega^{1}_{z}\rangle - \overline{L}_{-1} |\Omega^{1}_{\bar{z}}\rangle & = Q |\Omega^{2}\rangle.
\end{align*}
\]

(4.16)

Since the Virasoro operators can be written as \( L_{-1} = \{Q, b_{-1}\} \) and \( \overline{L}_{-1} = \{Q, \bar{b}_{-1}\} \) it follows that the above equations are readily solved by

\[
|\Omega^{1}_{z}\rangle = b_{-1} |\Omega^{0}\rangle, \quad |\Omega^{1}_{\bar{z}}\rangle = \bar{b}_{-1} |\Omega^{0}\rangle, \quad |\Omega^{2}\rangle = b_{-1} \bar{b}_{-1} |\Omega^{0}\rangle.
\]

(4.17)

Again, the descent equations for the states do not give a unique solution. Other solutions can be obtained using the analog of (4.15) for states.
4.2. Currents of Ghost Number Zero

We will now derive the currents and associated charges of ghost number zero by using the descent equations. In order to motivate the choice we will make for the zero-forms, it is useful to understand how symmetry charges arise in string field theory [23]. In the BRST closed string field theory [24, 25] the gauge symmetries are written as

\[ \delta b^0_0 |\Psi\rangle = Qd b^0_0 |\Lambda\rangle + g |\Psi \star \Lambda\rangle + \cdots \] (4.18)

where the dots represent higher order contributions due to the nonpolynomiality of the theory. Here \( b^0_0 |\Psi\rangle \), the string field, is an element of the Hilbert space of ghost number (+2) (in the conventions that the ghost number of the SL(2,C) vacuum is zero) annihilated by \( b^0_0 \), and \( b^0_0 |\Lambda\rangle \), the gauge parameter, has ghost number (+1) and is also annihilated by \( b^0_0 \). Unbroken symmetries correspond to transformations for which \( Qb^0_0 |\Lambda\rangle = 0 \). In this case, to first order, the symmetry acts on the string field linearly, in a manner determined by the gauge parameter and the string product. A BRST trivial state \( b^0_0 |\Lambda\rangle \), however, will generate a symmetry which vanishes on-shell. Thus the nontrivial unbroken symmetries correspond to the closed string BRST semi-relative cohomology classes at ghost number (+1). This is a general statement in string field theory. The existence of these cohomology classes must explain the origin of the symmetry transformations we are looking for. The relation is actually very simple: the BRST classes simply give us the zero forms \( \Omega^{(0)} \) that determine the charges!

We can now understand the current whose (1, 0) piece is \( J_{s,n} = W_{s,n} \varOmega_{s-1,n'} \). (The analysis of the operator whose (0, 1) piece is \( \varOmega J \) is of course precisely analogous.) In this subsection, we will suppress the subscripts \( s, n, n' \). The current we are seeking will arise from the zero form

\[ \Omega^{(0)} = cW \varOmega. \] (4.19)

To find the higher components of this operator, we need the chiral descent equations
for $cW$ and $\mathcal{O}$. For $cW$, we have

$$0 = \{Q_L, cW\} = \{Q_R, cW\}$$

$$\{Q_L, W\} = \partial(cW) \quad (4.20)$$

$$0 = \{Q_R, W\} = \overline{\mathcal{O}}(cW).$$

Of course, $Q_L$ and $Q_R$ are the holomorphic and antiholomorphic parts of the BRST operator; thus $Q = Q_L + Q_R$. Similarly,

$$0 = \{Q_L, \overline{\mathcal{O}}\} = \{Q_R, \mathcal{O}\}$$

$$0 = \partial \overline{\mathcal{O}}$$

$$\{Q_R, X\} = \overline{\partial} \mathcal{O}, \quad (4.21)$$

for some spin $(0, 1)$ operator $X$. In terms of states it is simply given by $|X\rangle = \bar{b}_{-1} |\mathcal{O}\rangle$. This is actually a highest weight state, and the corresponding operator $X$ is primary. This is the case (see the discussion at the beginning of §2.2) since $|\mathcal{O}\rangle$ is annihilated by $\bar{b}_n$ for all $n \geq 0$ (Appendix A).

Using (4.20) and (4.21), one readily constructs the higher components of $\Omega^{(0)}$:

$$\Omega^{(1)} = W\overline{\mathcal{O}}dz - cW\overline{X}d\overline{z}$$

$$\Omega^{(2)} = -W\overline{X} dz \wedge d\overline{z}. \quad (4.22)$$

The current that is conserved up to $\{Q, \ldots\}$ thus has $(1, 0)$ component $W\overline{\mathcal{O}}$, as expected. In addition, it has a $(0, 1)$ piece $cW\overline{X}$, which may be less expected. Restoring the indices, the conserved charges are thus

$$A_{s,n,n'} = \frac{1}{2\pi i} \oint c dz W_{s,n} \mathcal{O}_{s-1,n'} - \frac{1}{2\pi i} \oint c d\overline{z} cW_{s,n} \overline{X}_{s-1,n'}. \quad (4.23)$$

In §5.3, we do an explicit computation showing that the second piece is needed to obtain sensible (BRST invariant) results when these charges act on discrete moduli.
As we will see in §6, the two-form $\Omega^{(2)}$ derived from the zero-form $\Omega^{(0)}$ will be responsible for the nonlinear action of the symmetry transformations on states. In terms of states the two form given in (4.22) is indeed given by

$$\ket{\Omega^2} = b_{-1} \overline{b}_{-1} \ket{\Omega^0}. \tag{4.24}$$

as predicted by our analysis leading to equation ((4.17)). Equation (4.24) gives a simple test for nonlinear symmetries; if the two form vanishes the symmetry will act linearly on states.

4.3. CHARGES PRESERVE THE $b_0 - \overline{b}_0 = 0$ CONDITION

As we have discussed in §3, physical states $\ket{\psi}$, in addition to being annihilated by the BRST operator, must satisfy the condition

$$(b_0 - \overline{b}_0) \ket{\psi} = 0, \tag{4.25}$$

where

$$b_0 - \overline{b}_0 = b_0^- = \oint_C \frac{dz}{2\pi i} z b(z) + \oint_C \frac{d\overline{z}}{2\pi i} \overline{z} \overline{b}(z), \tag{4.26}$$

and the contour surrounds the origin. We now want to verify that if $\ket{\psi}$ is acted on by a charge $A_{s,n,n'}$, it still satisfies (4.25). (One should also verify this for the new symmetries that we have constructed, but we have not undertaken this.) We must then check that $[b_0 - \overline{b}_0, A_{s,n,n'}] = 0$. Using (4.23) with the integration label $z$ changed into $w$ and with the contour chosen to be a circle around the origin ($w = 0$) we find

$$[b_0 - \overline{b}_0, A_{s,n,n'}] = \oint_C \frac{dw}{2\pi i} W_{s,n}(w) \oint_C \frac{d\overline{z}}{2\pi i} \overline{z} \overline{b}(\overline{z}) \overline{O}_{s-1,n'}(\overline{w})$$

$$- \oint_C \frac{dw}{2\pi i} w W_{s,n}(w) \overline{X}_{s-1,n'}(\overline{w})$$

$$+ \oint_C \frac{dw}{2\pi i} c(w) W_{s,n}(w) \oint_C \frac{d\overline{z}}{2\pi i} \overline{z} \overline{b}(\overline{z}) \overline{X}_{s-1,n'}(\overline{w}), \tag{4.27}$$

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where $C$ refers to the circle around the origin in the $w$ plane, and $\oint$ is an integral around the $w$ points. (In writing the first term in (4.27), we have used the fact that the $W\bar{\Omega}$ term in the charge can only be acted upon by $\bar{b}_0$ since $W$ is constructed without ghosts.) Vanishing of (4.27) can be established using the explicit expression for $X_{k,k}(z)$ given in Appendix A, which shows that it is built with matter and antighost fields only; it contains no ghost fields. Using the SU(2) lowering operators, the same must hold for $X_{s,n}$ and of course also for $\bar{X}_{s,n}$. Thus the last term in the above equation vanishes as there is no short distance singularity in the operator product expansion of $\bar{b}$ and $\bar{X}$. It also follows from Eqn. (A.17) that

$$b(z) O_{s,n}(w) = \frac{1}{z-w} X_{s,n}(w) + \cdots$$

Using this result for the first term in the right hand side of (4.27) we then find

$$[b_0 - \bar{b}_0, A_{s,n,n'}] = -\oint_C \left( \frac{w}{2\pi i} dw + w \frac{d\bar{w}}{2\pi i} \right) W_{s,n} \bar{X}_{s-1,n'} = 0$$

where in the last step we used the fact that the contour was a circle, on which $|w|^2 =$ constant so $\bar{w}dw + wd\bar{w} = 0$. This establishes the desired result.

There is one more point to be discussed. The descent equations show that upon contour deformation a conserved charge acquires an additional term of the form $\{Q, \Omega^{(2)}\}$ integrated over the part of the surface bounded by the contours. Thus acting on a physical state $|\psi\rangle$ the additional term we get is

$$Q \int \Omega^{(2)} |\psi\rangle = -Q \int dz \wedge d\bar{z} \ W \bar{X} \ |\psi\rangle,$$

where we made use of (4.22). While this term is clearly trivial in the absolute cohomology we must show it is trivial in the semi-relative cohomology. It is so. The state $|\psi\rangle$ can be written as $b^-_0 |\chi\rangle$, and $b^-_0$ commutes with $W \bar{X}$ (recall both $W$ and $\bar{X}$ do not contain the ghost field $c$); therefore the above state is of the form $Q b^-_0 \cdots |\chi\rangle$, which is trivial in the semi-relative cohomology. Thus the charges are
well-defined on the physical states. The attentive reader will have noticed that we made implicit use of this result in choosing a convenient contour for the charge $A$ in this section.

4.4. Discussion Of New Moduli and Symmetry Transformations

In §3, we found new closed string states, annihilated by $Q$ and by $b_0 - \bar{b}_0$, at ghost number 1 and 2. Explicitly, they are

$$
G = 1 : (a + \bar{a})\mathcal{O}_{u,n} \mathcal{O}_{u,n'}
$$

$$
G = 2 : (a + \bar{a})Y^+_{s,n} \mathcal{O}_{s-1,n'}, \quad (a + \bar{a})\mathcal{O}_{s-1,n} \mathcal{Y}^+_{s,n'},
$$

(4.31)

where $G$ is the ghost number. (We recall that $Y$ is the same as $cW$.) We will first discuss briefly the problem of explicitly describing the operators corresponding to the new moduli, and then we will give the symmetry charges associated with the states at $G = 1$.

We recall that the moduli come from spin zero operators of ghost number two. For the case of the ghost number two states annihilated by $b_0$ and $\bar{b}_0$, the transition from the states to the moduli (dimension $(1,1)$ operators that can be added to the action of the conformal field theory) was simple. The states were of the type $Y^+_{s,n} \mathcal{Y}^+_{s,n'}$ which are just $cW^+_{s,n} \mathcal{W}^+_{s,n'}$; the corresponding moduli are $W^+_{s,n} \mathcal{W}^+_{s,n'}$ which are simply obtained by deleting the ghost fields.

As for the new states, the first nontrivial examples are given by

$$
(a + \bar{a})(c\partial X \cdot 1) = c\partial X (\mathcal{O} + \frac{1}{\sqrt{2}}c\mathcal{O} + c\partial c \partial X,
$$

(4.32)

and similarly with left and right movers reversed. It is clear that the simple rule to obtain the corresponding moduli does not apply. Nevertheless, the $c\mathcal{O} \partial X \overline{\partial \phi}$ term in (4.32) must correspond to a piece in the modulus of the form

$$
\partial X \overline{\partial \phi} + \cdots.
$$

(4.33)

This is not satisfactory as it stands, since the field $\overline{\partial \phi}$ is not primary, because of the background charge of the Liouville field, nor it is Weyl invariant. The $c\overline{\partial \mathcal{O}} \partial X$
term in (4.32) correspond roughly to an additional contribution $\partial X \bar{\omega}$, where $\bar{\omega}$ is the $(0, 1)$ part of the spin connection. With the right coefficient, this term restores Weyl invariance at the cost of local Lorentz invariance. To save the situation, one must take due account of the $c \partial c \partial X$ term. This term enters in verifying the $b_0 - \bar{b}_0$ condition and so is bound to enter in constructing the appropriate marginal operator. (New contributions to the local Lorentz and Weyl transformation laws of $X$ may also be part of the story.) We do not understand how to incorporate it in constructing the operator, and so we leave this matter for later consideration.

Let us now turn to the new symmetries. In order to find the expressions for the currents corresponding to the new symmetries, we apply the descent equations. The beginning point is the zero form corresponding to the new BRST class of operators at $G = 1$ (eqn. (4.31))

$$\Omega^{(0)} = (a + \pi) \mathcal{O}_{u,n} \bar{\mathcal{O}}_{u,n'} = a \mathcal{O}_{u,n} \bar{\mathcal{O}}_{u,n'} + \mathcal{O}_{u,n} a \bar{\mathcal{O}}_{u,n'}.$$  

(4.34)

We will use the notation

$$\partial \mathcal{O} = \{Q, X_\mathcal{O}\}, \quad \partial(a \mathcal{O}) = \{Q, X_{a\mathcal{O}}\},$$

(4.35)

where $X_\mathcal{O}$ was simply denoted as $X$ before, and $X_{a\mathcal{O}}$ is an operator that actually fails to be primary, since $a\mathcal{O}$ is not annihilated by $b_0$ (recall that as for $X$, we have that $|X_{a\mathcal{O}}\rangle = b_{-1} |a\mathcal{O}\rangle$). The formulas we are constructing here for the new symmetry charges are therefore only adequate on the cylinder (flat metric). Maintaining current conservation in general will involve adding to the currents additional terms involving coupling to the world-sheet curvature; we do not know an efficient way to generate these terms.

A short calculation with the descent equations gives us

$$\Omega^{(1)} = (X_{a\mathcal{O}} \mathcal{O} + X_{\mathcal{O}} \bar{\mathcal{O}}) dz + (\mathcal{O} \overline{X_{a\mathcal{O}}\mathcal{O}} - a\mathcal{O} \overline{X_{\mathcal{O}}}dz) \overline{\mathcal{O}}$$

(4.36)
\[
\Omega^{(2)} = (X_\mathcal{O} \overline{X}_{\overline{a} \mathcal{O}} - X_{\overline{a}} \overline{X}_\mathcal{O}) \, dz \wedge d\overline{z},
\]

(4.37)

The simplest charge is that corresponding to \(\Omega^{(0)} = (a + \overline{a}) \mathcal{O}_{0,0} \overline{\mathcal{O}}_{0,0} = (a + \overline{a}) \, 1 \cdot 1\). We then have \(X_1 = 0\), and \(X_{a \cdot 1} = \partial \phi\). It therefore follows that

\[
\Omega^{(1)} = \partial \phi \, dz + \overline{\partial} \phi \, d\overline{z},
\]

(4.38)

and recalling our sign convention (4.6) we conclude that the corresponding charge simply measures the difference between left and right components of Liouville momentum. This is the symmetry generator we denoted as \(S\) in (3.16), and which actually annihilates all states. The two-form corresponding to this symmetry vanishes. The other “new” charges, as discussed in §3.3, correspond to vector fields \(fS\), and have a nonvanishing action.

5. Symmetry Transformations of Tachyons and Discrete States

In this section we show explicitly the linearized action of the symmetry transformations on the states of the theory. In the case of the uncompactified theory we will pay particular attention to the transformation of the tachyon. The symmetry generators which map the tachyon states to themselves generate the subalgebra of the Virasoro algebra corresponding to the \(L_n\’s\) with \(n \geq 0\). The transformation law of the tachyon under these generators will turn out to show that it is an object of dimension one. In showing this, we will need to make a suitable rescaling of the tachyon field \(\mathcal{T}_p\) with a momentum dependent factor. It is very interesting that this factor coincides with the external leg factor appearing in \(S\)-matrix calculations of the \(c = 1\) theory. Thus, if the tachyon is defined to have standard Virasoro transformations under the discrete symmetries, the external leg factors go away from \(S\)-matrix elements. We also give an alternative derivation of the transformation law of the tachyon using the picture of tachyon states as perturbations of the Fermi surface in the matrix model. Finally, we give examples where we compute discrete symmetry transformations of the discrete states. These examples illustrate clearly...
what would have gone wrong with the naive charges, and the necessity of including new states in the semirelative cohomology.

5.1. Transformation of the Tachyon

The calculation of the action of the discrete symmetries on the holomorphic tachyon is useful in preparation for the closed string case, so we consider this case first.

The holomorphic case. Consider the holomorphic part of a tachyon on the right branch of the spectrum ($p \geq 0$)

$$T_p(z) = e^{ipX(z)} e^{(\sqrt{2}-p)\phi(z)}$$  \hspace{1cm} (5.1)

It is clear from the momentum dependence of this field that the momentum dependence of a symmetry generator that maps a tachyon into a tachyon must be of the form $\exp(iaX^+)$ where we have defined $X^\pm = (X \pm i\phi)/\sqrt{2}$. Since the $W^+_{s,n}$ operators are of the form

$$W^+_{s,n} \sim (\text{Polynomial in } \partial X) e^{i(n+s-1)X^+} e^{i(n-s+1)X^-},$$  \hspace{1cm} (5.2)

the relevant operators are the ones generated by the currents $W_{s,s-1}$. We find an explicit expression for these generators by acting with the SU(2) lowering operator $\oint dz \ e^{-iX\sqrt{2}}$ on $W_{s,s}$. Thus

$$W_{s,s-1} = \frac{1}{2\pi i} \oint dz e^{-iX\sqrt{2}} e^{i\sqrt{2}sX(w)} e^{-\sqrt{2}(s-1)\phi(w)},$$

from which it follows that

$$W_{s,s-1}(w) = S_{2s-1}(-i\sqrt{2}X^{(j)}/j!) e^{i\sqrt{2}(s-1)X^+(w)}$$  \hspace{1cm} (5.3)

where $S_k$ denotes an elementary Schur polynomial (for definition and notation see the Appendix) and $X^{(j)} \equiv \partial^{(j)} X$. We are now ready to compute the action of this
generator on the tachyon

\[ \{A_{s,s-1}, T_p(w)\} = \frac{1}{2\pi i} \oint W_{s,s-1}(z)T_p(w) \]

The contraction of the exponentials in \(W\) and \(T_p\) give a zero \((z - w)^{2s-2}\) and this implies that the complete Schur polynomial in \(W\) must be contracted with the exponential in \(T_p\) in order to give a contribution. A small calculation gives

\[ \{A_{s,s-1}, T_p(w)\} = S_{2s-1}(\sqrt{2} \sqrt{p}) T_{p+\sqrt{2(s-1)}}(w), \] (5.4)

where \(c_k = (-1)^k/k\). It is possible, for our particular coefficients \(c_k\), to simplify the expressions of the Schur polynomials down to factorials:

\[ S_k(\sqrt{a}) = \frac{(-1)^k \Gamma(a + k)}{k! \Gamma(a)}, \] (5.5)

as derived in the Appendix. We can now write our result (5.4) as

\[ \{A_{s,s-1}, T_p(w)\} = \frac{(-1)^{2s-1} \Gamma(\sqrt{2} p + 2s - 1)}{(2s - 1)! \Gamma(\sqrt{2} p)} T_{p+\sqrt{2(s-1)}}(w). \] (5.6)

The normalization of the operators \(A_{s,s-1}\) has not been fixed yet. We fix it by introducing the operators \(Q_{2s}\) via the relation

\[ A_{s+1,s} = \frac{(-1)^{2s}}{(2s + 1)!} Q_{2s}. \] (5.7)

Using (5.6) and (5.7) the transformation for the tachyon now takes the form

\[ \{Q_{2s}, T_p(w)\} = -\frac{\Gamma(\sqrt{2} p + 2s + 1)}{\Gamma(\sqrt{2} p)} T_{p+\sqrt{\frac{p}{2}}} . \] (5.8)

The operators \(Q_{2s}\) for \(s = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\), generate half of the Virasoro algebra. Indeed one readily verifies that acting on the tachyon the \(Q_{2s}\) operators satisfy the
commutation relations

$$\{Q_{2s}, Q_{2s'}\} = (2s - 2s')Q_{2s+2s'}.$$  \hfill (5.9)

The transformation indicated in (5.8) is not yet of the standard form. To this end we must redefine the tachyon by a momentum dependent factor. Letting

$$T_p = \frac{\hat{T}_p}{\Gamma(\sqrt{2p} + 1)},$$  \hfill (5.10)

the transformation law for the tachyon becomes

$$\{Q_{2s}, \hat{T}_p(w)\} = \sqrt{2p} \frac{\hat{T}_p + 2s}{\sqrt{2}} (w).$$  \hfill (5.11)

This is the simplest form for the transformation law. Consider now the standard transformation law for a field of dimension \(d\) under Virasoro:

$$\{Q_{2s}, \phi_n\} = [2s(d − 1) − n]\phi_{n+2s}.$$  \hfill (5.12)

The absence of an \(s\) dependent term in the right hand side of (5.11) indicates that the tachyon transforms as a field of dimension one under the symmetry transformations. The transformation mixes tachyons whose momenta differ by an integer times \(1/\sqrt{2}\).

**The Closed String Case.** The field we want to study now is the closed string tachyon. In fact, we must consider the BRST invariant tachyon field. While in the holomorphic case there was no necessity to deal with the BRST invariant field because the symmetry transformations do not involve the ghosts, the closed string charges involve the ghost fields in a nontrivial way. We must therefore consider
the transformation of
\[ T_p(z, \bar{z}) = c(z) T_p(z) \bar{c}(\bar{z}) \bar{T}_p(\bar{z}), \]
\[ = c(z) e^{ipX(z)} e^{(\sqrt{2}-p)\phi(z)} \bar{c}(\bar{z}) e^{ip\bar{X}(\bar{z})} e^{(\sqrt{2}-p)\bar{\phi}(\bar{z})}. \] (5.13)

The operators that map tachyons into tachyons in this case are
\[ A_{s,s-1} = \frac{1}{2\pi i} \oint dz W_{s,s-1} \mathcal{O}_{s-1,s-1} - \frac{1}{2\pi i} \oint d\bar{z} cW_{s,s-1} \bar{X}_{s-1,s-1} \] (5.14)

We now wish to calculate \( \{A_{s,s-1}, T_p\} \). Let us begin with the first term in the right hand side of (5.14). Recall from the holomorphic calculation that the operator product expansion of \( W(z) \) with the holomorphic part of the tachyon \( T_p(w) \) gave terms of the form \((z-w)^{-1}, 1, (z-w), \cdots \). This implies that only the regular terms in the antiholomorphic product of operators can contribute. We therefore have to compute the regular terms in
\[ \mathcal{O}_{s-1,s-1}(\bar{z}) \bar{c} T_p(w). \] (5.15)

It is actually sufficient to determine the regular terms up to BRST commutators of the form \( \{Q, \star\} \). This is the case because the holomorphic part of the calculation gave a \( Q \)-invariant tachyon field, and therefore the antiholomorphic BRST commutators can be recast as closed string BRST commutators. We will use the properties of the ring to perform this calculation. Note that since the momentum factors are the same as in the holomorphic calculation we get a zero of the form \((\bar{\tau} - \bar{\pi})^{2s-2} \) that must be cancelled in order to get a contribution. Since \( \mathcal{O}_{s-1,s-1} \sim \bar{\tau}^{2s-2} \), where
\[ \bar{\tau} = \mathcal{O}_{s-1,s-1} = [\bar{\tau}b + i\bar{\tau}X^-]e^{i\bar{X}^+} \]
we first compute
\[ \bar{\tau}(\bar{z}) \bar{c} T_p(w) \sim (-\sqrt{2}p) \bar{\tau} T_{p+\frac{1}{\bar{\tau}}} (w) + \mathcal{O}(\bar{\tau} - \bar{\pi}). \] (5.16)
It is now straightforward to complete the calculation

\[ x^n(z) \bar{T}_p(w) \sim (x \cdot \bar{x} \cdot \cdots \cdot \bar{x})(\bar{T}_p) \]

\[ = x \cdot (x \cdot (x \cdot (\bar{T}_p) \cdots) \cdots) \]

\[ = (-)^n(\sqrt{2p})(\sqrt{2p} + 1) \cdots (\sqrt{2p} + n - 1) \bar{T}_{p+n/\sqrt{2}(w)} \]

\[ = (-)^n \frac{\Gamma(\sqrt{2p} + n)}{\Gamma(\sqrt{2p})} \bar{T}_{p+n/\sqrt{2}(w)} + \mathcal{O}(\bar{z} - \bar{w}). \]  

(5.17)

where in the second step we used associativity of the operator product expansion, and then we made repeated use of (5.16). We have therefore obtained the following result

\[ \mathcal{O}_{s-1,s-1}(\bar{z}) \bar{T}_p(w) \sim (-1)^{2s-2} \frac{\Gamma(\sqrt{2p} + 2s - 2)}{\Gamma(\sqrt{2p})} \bar{T}_{p+(2s-2)/\sqrt{2}} + \mathcal{O}(\bar{z} - \bar{w}). \]

(5.18)

The reader may enjoy verifying that the same result is obtained by direct calculation using the representative for \( \mathcal{O}_{k,k} \) given in (A.9) and equations (A.8) and (A.5).

We must now consider the second term in the charge \( A_{s,s-1} \), namely,

\[ -\frac{1}{2\pi i} \oint d\bar{z} cW_{s,s-1} X_{s-1,s-1}. \]

The holomorphic calculation involved here is essentially a copy of the previous one since we just have an extra \( c(z) \) factor. This factor implies that this time we can only get terms regular or vanishing in \( (z - w) \). Therefore, in order to get a contribution, the antiholomorphic contraction

\[ X_{s-1,s-1}(\bar{z}) \bar{T}_p(w) \]

must give at least a first order pole in \( (\bar{z} - \bar{w}) \). The contraction of the exponentials again gives us the zero \( (\bar{z} - \bar{w})^{2s-2} \). From the expression for \( X_{k,k} \) in the Appendix
we see that the generic term in $X_{s-1,s-1}$ is of the form $S_{2s-2,q} (\bar{\partial} X^{-}) \partial^{q-1} b$ (times an exponential). The contractions with the ghost factor and the exponential in the tachyon will therefore give the pole $(\bar{z} - \bar{w})^{2s-2}$. This is not singular enough to yield a residue, and we therefore conclude that the second term in the charge does not contribute in the present case.

Thus summarizing, we have from (5.6) and (5.18) that the symmetry transformation on the tachyon reads:

$$\{A_{s,s-1}, T_p\} = \frac{(-1)^{2s}}{(2s-1)!} \frac{\Gamma(\sqrt{2p} + 2s - 1)}{\Gamma(\sqrt{2p})} \frac{\Gamma(\sqrt{2p} + 2s - 2)}{\Gamma(\sqrt{2p})} T_{p+(2s-2)/\sqrt{2}}. \quad (5.19)$$

As in the holomorphic case, we introduce charges $Q_{2s}$ which are suitably normalized, via the relation

$$A_{s+1,s} = \frac{(-1)^{2s}}{(2s+1)!} Q_{2s}. \quad (5.20)$$

(The sign factor has been chosen to agree with standard conventions.) We then obtain the transformation law

$$\{Q_{2s}, T_p\} = -(-1)^{2s} \frac{\Gamma(\sqrt{2p} + 2s + 1)}{\Gamma(\sqrt{2p})} \frac{\Gamma(\sqrt{2p} + 2s)}{\Gamma(\sqrt{2p})} T_{p+2s/\sqrt{2}}. \quad (5.21)$$

It is straightforward to verify that, acting on the closed string tachyon, the above operators $Q_{2s}$ with $s = 0, \frac{1}{2}, 1, \cdots$, form a subalgebra of Virasoro.

It is possible to redefine the tachyon field $T_p$ with suitable momentum dependent functions, so as to bring the above representation of Virasoro into a standard form. We introduce a new tachyon field $\hat{T}_p$ via:

$$T_p = \frac{\Gamma(1 - \sqrt{2p})}{\Gamma(\sqrt{2p})} \frac{\hat{T}_p}{\sqrt{2p}}, \quad (5.22)$$

where the first factor in the right hand side is precisely the external leg factor of $S$ matrix elements in the $c = 1$ model and contains all of the expected poles at
discrete momenta (cf. the second paper in [1]). We then obtain

\[ \{Q_{2s}, \hat{T}_p\} = -\sqrt{2}p \hat{T}_{p+2s/\sqrt{2}}. \]  

(5.23)

This is the final form for the transformation and proves that the field whose \( S \) matrix elements do not show discrete leg poles transforms as a dimension one field under Virasoro. The same would be true for any field differing from this one by a momentum dependent factor \( h(p) \) satisfying the periodicity condition \( h(p + 1/\sqrt{2}) = f_0 h(p) \), with \( f_0 \) a constant. This follows from (5.23) and the fact that the Virasoro commutation relations are invariant under the replacement \( Q_{2s} \rightarrow (f_0)^{2s} Q_{2s} \).

5.2. Comparison To The Matrix Model

Now we want to compare this result to the prediction of the \( c = 1 \) matrix model.

In the matrix model, the tachyon is described as a curve in the phase space of the matrix eigenvalue, representing the fermi surface. We will call this phase space the \( a_1 - a_2 \) plane. At zero cosmological constant, the fermi surface is the curve \( a_1 a_2 = 0 \). This has two branches, and we will look at the branch \( a_1 = 0 \). A small deformation perturbs this equation to an equation

\[ a_1 - f(a_2) = 0 \]  

(5.24)

where \( f(a_2) \) is the tachyon field. Symmetries of the matrix model are generated by appropriate time dependent canonical transformations. For our purposes, we can ignore the time dependence and consider charges acting at \( t = 0 \). The symmetries are then transformations of the \( a_1 - a_2 \) plane that preserve the area form

\[ \omega = da_1 \wedge da_2. \]  

(5.25)

Such transformations are generated by Hamiltonian functions \( h(a_1, a_2) \) in the standard fashion. As explained in [7, §3.1], the symmetries of the matrix model are
generated not by arbitrary \( h \)'s, but just by those \( h \)'s that vanish on the fermi surface \( a_1 a_2 = 0 \) defining the vacuum state. Since we are working near \( a_1 = 0 \), we simply restrict \( h \) to be divisible by \( a_1 \).

First, we will work to leading order near \( a_1 = 0 \). This means that we consider \( h \) to have a simple zero there:

\[
h(a_1, a_2) = a_1 u(a_2). \tag{5.26}
\]

The Hamiltonian vector field generated by this \( h \) is

\[
V = u(a_2) \frac{\partial}{\partial a_2} - \frac{\partial u}{\partial a_2} a_1 \frac{\partial}{\partial a_1}. \tag{5.27}
\]

Notice that the vector fields of this form generate reparametrizations of the \( a_2 \) line and thus obey (part of) a Virasoro algebra. This precisely corresponds to the Virasoro algebra that we have been looking at earlier in the conformal field theory.

To determine how such symmetries act on the tachyon field, we compute

\[
V(a_1 - f(a_2)) = -\frac{\partial u}{\partial a_2} a_1 - u \frac{\partial f}{\partial a_2}. \tag{5.28}
\]

Hence acting by, say, \( 1 + \epsilon V \) transforms the equation \( a_1 - f(a_2) = 0 \) into the equation

\[
a_1 - \left( f + \epsilon u \frac{\partial f}{\partial a_2} + \epsilon \frac{\partial u}{\partial a_2} f \right) = 0 \tag{5.29}
\]

(up to order \( \epsilon^2 \)). The transformation law of \( f \) is thus

\[
\delta f = \epsilon \left( u \frac{\partial f}{\partial a_2} + \frac{\partial u}{\partial a_2} f \right). \tag{5.30}
\]

This is the transformation law of a field of spin 1, in agreement with what we found in the conformal field theory.
This conclusion could have been obtained without any computation, as follows. The fact that we are only considering area preserving diffeomorphisms means that it is natural to think of the $a_1 - a_2$ plane as the cotangent bundle of the $a_2$ line. The equation $a_1 - f(a_2) = 0$ thus defines a section of the cotangent bundle of the $a_2$ line or in other words a differential form on the $a_2$ line. Such a differential form transforms, of course, as a field of spin one.

Had we considered instead Hamiltonians with a higher order dependence on $a_1$, say $\tilde{h} = a_1^2 v(a_2)$, the same calculation would show that the symmetry generated by $\tilde{h}$ acts nonlinearly on the tachyon field $f$. In the quantum theory, this means that the symmetries change the number of particles; for instance, a symmetry transformation might map a two particle state to a one particle state or vice-versa.

This is a very unusual situation. In fact, kinematically it is impossible for massive particles or for massless particles in $D > 2$. The dispersive propagation of waves would make it impossible for a symmetry to map an $n$ particle state to an $m$ particle state with $m \neq n$ except for massless particles in $D = 2$.

We are, happily, dealing with just that case, so the behavior just indicated is not impossible, but it is nonetheless very strange from the point of view of conformal field theory. Conventionally, in conformal field theory, a symmetry comes from a world-sheet conserved current $J$. In quantizing on a cylinder, the conserved charge is defined as

$$ A = \oint J, $$

with the integral taken over any contour running once around the cylinder. Current conservation means that $A$ does not depend on the contour. $A$ is then a well-defined operator in the one string Hilbert space. It maps one string states to one string states. It appears that there is no room for $A$ to map a one string state to a two string state. In §6, we will discuss how this can come about.
5.3. Discrete Charges on Discrete States

The purpose of this subsection is to show that the second term in the conserved charges $A_{s,n}$, which did not give a contribution when acting on the tachyon of the uncompactified theory, is essential to obtain sensible results for the action on discrete states. At the same time, we will show explicitly that the new moduli that we have introduced (corresponding to operators that are not annihilated separately by $b_0$ and $\overline{b}_0$) are necessary by showing that they are created when symmetries act on the “old” moduli.

We will consider the simplest example, involving the discrete charge

$$A_{\frac{1}{2}, \frac{1}{2}} = \oint \frac{dz}{2\pi i} W_{\frac{3}{2}, \frac{1}{2}} \sigma_{\frac{1}{2}, \frac{1}{2}} - \oint \frac{dz}{2\pi i} c W_{\frac{3}{2}, \frac{1}{2}} \bar{X}_{\frac{1}{2}, \frac{1}{2}}, \quad (5.32)$$

where $\bar{X}_{\frac{1}{2}, \frac{1}{2}} = \overline{b} \exp(iX^+)$.  

Let us act on the discrete tachyon:

$$D_1(w, \overline{w}) = c(w)W_{\frac{1}{2}, -\frac{1}{2}} (w) \overline{c}(\overline{w}) \overline{W}_{\frac{1}{2}, -\frac{1}{2}} (\overline{w}). \quad (5.33)$$

There are no complications in calculating the commutator; after some work one finds

$$\{A_{\frac{1}{2}, \frac{1}{2}}, D_1(w, \overline{w})\} = i\sqrt{2} c \partial X \overline{c} \partial X + 2c \partial X (\overline{c} \partial c + \frac{1}{\sqrt{2}} \overline{c} \partial \phi) + 2c \partial c \partial X. \quad (5.34)$$

Let us analyze the result. The first term on the right hand side was expected; it corresponds to a closed string cohomology class at ghost number (1,1), with the corresponding state annihilated both by $b_0$ and $\overline{b}_0$. This is therefore a conventional modulus. The second term on the right hand side also has ghost number (1,1), but while the holomorphic part corresponds to a conventional modulus, the antiholomorphic part corresponds to the operator $\overline{\tau}$ discussed in §2. This term is BRST
invariant, but not annihilated by $b_0 - \bar{b}_0$. What saves the day is the last term on the right hand side of (5.34). This term comes from the second term in the charge, showing that the second term is necessary. This last term in (5.34) is of ghost number (2,0), and can be recognized as the BRST invariant operator $(a \cdot c \partial X)$. The last two terms in (5.34) add up to

$$2c\partial X (\overline{\partial c} + \frac{1}{\sqrt{2}} \overline{c\partial \phi}) + 2c\partial c \partial X = (a + \overline{a}) (c\partial X \cdot 1),$$

which corresponds to the first “new” modulus.

6. Non-Linear Symmetries In Conformal Field Theory

This final section will be denoted to answering the question raised at the end of §5.2 – how can symmetries that act nonlinearly on the states arise in conformal field theory?

To state the answer in the simplest possible form, this can occur when one has a current $B$ that is not strictly conserved, but conserved only up to a BRST commutator. If we think of $B$ as an operator valued one-form, then

$$\text{dB}^{(1)} = \{Q, B^{(2)}\},$$

with $B^{(2)}$ a two-form.

(6.1) leads to a possibility of non-linear symmetry action, as follows. Recall the conventional derivation of linear Ward identities. We start with the insertion of $0 = \text{dB}^{(1)}(z)$ in a correlation function:

$$0 = \int_z \langle \text{dB}^{(1)}(z) \prod_{i=1}^n T_i(w_i) \rangle.$$

Here $T_i, \ i = 1 \ldots n$ are some additional $Q$ invariant operators, inserted at points $w_i$. The symbol $\int_z$ means to integrate in $z$, without integrating over the $w_i$ (or
other moduli of the surface $\Sigma$ on which all this is happening). Instead of using the fact that $dB^{(1)} = 0$, we think of $dB^{(1)}$ as an exact differential, and by picking up in the standard way singularities in the operator products $B^{(1)}(z)T_i(w_i)$, we get

$$0 = \sum_{i=1}^{n} \langle [A, T_i(w_i)] \cdot \prod_{j \neq i} T_j(w_j) \rangle$$

(6.3)

where $A$ is the conserved charge $A = \oint B^{(1)}$. This is a conventional linear Ward identity.

In the derivation of (6.3), $z$ was the only integration variable, so the only singularities encountered were singularities in the operator product of $B$ with a single $T_i$. This is the reason that the Ward identity is linear; that is, it involves only terms coming from $B \cdot T_i \sim T_i'$ with one initial and one final $T$.

We are now interested in a situation in which, instead of $dB^{(1)} = 0$, we have (6.1), with some $B^{(2)}$. The derivation of the Ward identity then begins with

$$\int_{\mathcal{M}_{g,n+1}} \langle dB^{(1)}(z) \prod_{i=1}^{n} T_i(w_i) \rangle = \int_{\mathcal{M}_{g,n+1}} \langle \{Q, B^{(2)}\} \prod_{i=1}^{n} T_i(w_i) \rangle.$$  

(6.4)

Here instead of just integrating over $z$, we are integrating over $\mathcal{M}_{g,n+1}$, the moduli space of Riemann surfaces of genus $g$ with $n + 1$ punctures. Equation (6.4) is an equality between two total derivatives. The left hand side has been written in (6.4) explicitly as a total derivative $dB$; the right hand side is implicitly a total derivative since in string theory an insertion of $\{Q, B^{(2)}\}$ gives a total derivative on moduli space. (More exactly, it gives a form that becomes a total derivative after integrating over the position of insertion of $B^{(2)}$.)

The Ward identity – the generalization of (6.3) – arises now as follows. Integrating by parts in (6.4), one picks up surface terms at the various components of infinity. The surface terms from the left hand side of (6.4) are the ones that we have already encountered in the ordinary, linear Ward identity (6.3). However, the
right hand side of (6.4) may contribute additional surface terms. For instance, as
in figure (1(ii)), one component of infinity in \( \mathcal{M}_{g,n+1} \) parametrizes surfaces which
have a genus zero component containing \( B \) and two of the \( T \)'s. In the Ward identity,
this would be interpreted as a matrix element \( \langle T'' | A | T, T' \rangle \) or \( \langle T, T' | A | T'' \rangle \)
depending on whether the \( T \)'s are positive or negative energy (incoming or outgo-
ing) states. “Non-linear” terms in the Ward identity can thus arise from the right
hand side of (6.4).

6.1. Invariant Formulation

We will now restate the foregoing in a more invariant language. To begin with,
obviously a key role was played by the equation (6.1). This equation was, in fact,
part of the “descent” equations

\[
0 = \{ Q, \mathcal{O}^{(0)} \} \\
d\mathcal{O}^{(0)} = \{ Q, \mathcal{O}^{(1)} \} \\
d\mathcal{O}^{(1)} = \{ Q, \mathcal{O}^{(2)} \}.
\]  

(6.5)

We can thus give a succinct explanation of where currents come from and
when they act non-linearly. Currents arise as one-form components derived from a
BRST invariant zero-form observable \( \mathcal{O}^{(0)} \) (in the semirelative cohomology); they
act non-linearly when the associated two-form component is non-vanishing. For in
that case, the right hand side of (6.4) must be included.

Now, the discussion of (6.4) was rather asymmetrical in several respects. We
singled out one operator, the “current,” for separate treatment from the other
operators \( T_i \). This is unnatural since, just like the \( T_i \), the current is derived from
a basic zero-form via the descent equations. Also, in our discussion of (6.4), the
exact differentials on the left and right hand sides appeared to have quite different
origins.

\* These will not involve a coincidence of \( B \) with just one other operator, since the position
of insertion of \( B^{(2)} \) has already been integrated to reduce the right hand side of (6.4) to a
total derivative.
We will now give a more symmetric account. We start with some BRST invariant operator-valued zero-forms $O_a$, $a = 1 \ldots s$, of ghost number $w_a$ and annihilated by $b_0 - \bar{b}_0$. It may be that one of the $O$’s has ghost number one, and if so the associated one-form might be called a ghost number zero “current,” but whether this is so is immaterial.

Naively speaking, we wish to consider the “correlation function” $\Theta = \langle \prod_{a=1}^{s} O_a \rangle$ on a fixed Riemann surface $\Sigma$ with $s$ marked points. Because of antighost zero modes, $\Theta$ cannot be interpreted as a function on the moduli space $\mathcal{M}_{g,s}$; rather it is a differential form. In fact, the operator formalism, for instance, constructs $\Theta$ as a closed differential form of degree

$$ \text{deg } \Theta = 6g - 6 + \sum_{a=1}^{s} w_a. \quad (6.6) $$

(This is the number of antighost insertions needed to get a non-zero result given that the total ghost number inside the correlator must be $6 - 6g$.)

Here is a brief sketch of how one proves that $\Theta$ is closed. For $\eta$ a Beltrami differential representing a tangent vector to $\mathcal{M}_{g,s}$ let $\int_{\Sigma} \eta b$ be the corresponding mode of the ghost field $b$. If the ghost quantum numbers are such that $\Theta$ is an $n$-form, consider $n+1$ Beltrami differential $\eta(i)$, $i = 1 \ldots n + 1$. The $n$ form $\Theta$ is defined by

$$ \Theta(\eta(1), \ldots, \eta(n)) = \langle \prod_{a=1}^{s} O_a \prod_{j=1}^{n} \int_{\Sigma} \eta(j)b \rangle. \quad (6.7) $$

Consider the identity

$$ 0 = \langle \prod_{a=1}^{s} O_a \cdot \{Q, \prod_{j=1}^{n+1} \int_{\Sigma} \eta(i)b \} \rangle. \quad (6.8) $$

By use of $\{Q, b\} = T$ ($T$ being here the stress tensor), this takes the form

$$ 0 = \langle \prod_{a=1}^{s} O_a \sum_{j=1}^{n+1} (-1)^{j-1} \int_{\Sigma} \eta(j) T \prod_{1 \leq k \leq n+1, k \neq j} \int_{\Sigma} \eta(k)b \rangle. \quad (6.9) $$

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Since the insertion of $T$ is a derivative on $\mathcal{M}_{g,s}$, the right hand side of (6.9) is the antisymmetrized sum of the first derivatives of $\Theta$, and thus is $d\Theta(\eta(1), \ldots, \eta(n+1))$.

Hence (6.9) is the desired result $d\Theta = 0$.

Now, the case that is most often considered is the case in which

$$\sum_{a=1}^{s} w_a = 2s.$$  \hspace{1cm} (6.10)

In this case, $\Theta$ is a form of degree $6g - g + 2s$, i.e., a top form or measure on moduli space. In this case, one can integrate $\Theta$ over $\mathcal{M}_{g,s}$ to get a number, the string theory correlation function. (The integral will converge if there are no infrared divergences.)

The second most important case is

$$\sum_{a=1}^{s} w_a = 2s - 1.$$  \hspace{1cm} (6.11)

In this case, $\Theta$ is a form of codimension 1. Since $d\Theta = 0$, certainly

$$0 = \int_{\mathcal{M}_{g,s}} d\Theta.$$  \hspace{1cm} (6.12)

On the other hand, by Stokes’s theorem, if $W_\alpha$ are the components at infinity in $\mathcal{M}_{g,s}$, then

$$\int_{\mathcal{M}_{g,s}} d\Theta = \sum_\alpha \int_{W_\alpha} \Theta.$$  \hspace{1cm} (6.13)

Combining these, we get the Ward identity

$$0 = \sum_\alpha \int_{W_\alpha} \Theta.$$  \hspace{1cm} (6.14)

The most typical way to obey (6.11) is to take one of the $\mathcal{O}$’s, say $\mathcal{O}_1$, to have ghost number one, and the others to have ghost number two. Then $\mathcal{O}_1$ corresponds
to a “current,” and the others are “fields.” But it is not necessary to have this arrangement. In general, one might be able to obey (6.11) with no operator having ghost number one (or two). One would then obtain the Ward identity (6.14), but one would have no temptation to single out one of the operators as a “current” and the others as “fields.” There is complete symmetry, in the basic formalism, between the currents and the fields.

6.2. An Example

We now want to give a simple example of the extraction of nonlinear contributions in a Ward identity. We will do this for the uncompactified $D = 2$ model, in genus $g = 0$, with the cosmological constant $\mu = 0$. We therefore must recall a few facts about that model.

The components of the tachyon vertex operator are

\begin{align*}
G = 2 & : \quad T^{(0)} = c \bar{c} e^{ipX} e^{(\sqrt{2} - |p|)\phi} \\
G = 1 & : \quad T^{(1)} = (dz - d\bar{z} c) e^{ipX} e^{(\sqrt{2} - |p|)\phi} \\
G = 0 & : \quad T^{(2)} = dz \wedge d\bar{z} e^{ipX} e^{(\sqrt{2} - |p|)\phi}.
\end{align*}

(6.15)

Here $p$ is the momentum. Roughly speaking $p > 0$ and $p < 0$ correspond to incoming or outgoing states. It is known [2,27,28] that for $g = \mu = 0$, the amplitudes $\langle T_{p_1} \ldots T_{p_n} \rangle$ vanish unless precisely one $p_i$ is positive or precisely one is negative. Moreover, the one positive (or one negative) momentum must be an “exceptional momentum,” an integral multiple of $1/\sqrt{2}$.

To get a simple, non-trivial example, we will take one tachyon of positive $X$ momentum $p = 1/\sqrt{2}$, and three tachyons of negative momenta $p_i = -q_i$, with $q_i > 0$, $\sum_{i=1}^{3} q_i = \sqrt{2}$, and the $q_i$ otherwise generic. To keep things clear,

\* After we had worked out the conceptual framework explained above for this calculation, we received a new paper by Klebanov [26] in which essentially the same Ward identity is obtained with a slightly different justification. Klebanov also works out some consequences of this Ward identity. Some of the considerations of Kutasov, Martinec, and Seiberg [8] are also closely related.
we will call the vertex operator of the positive momentum tachyon \( \hat{T} \), and we will call the others \( T_i \), \( i = 1 \ldots 3 \). Then we have \( T_i^{(0)} = \sqrt{2} e^{-i q_i X} e^{i \phi} \) and \( \hat{T}^{(0)} = \sqrt{2} e^{i \phi} \).

We also will include one other operator, which will be one of the discrete currents of the \( D = 2 \) model. In fact, we will pick a current with \( (p_X, p_\phi) = (1, i) / \sqrt{2} \). This operator, call it \( B \), has components

\[
\begin{align*}
G = 1 & : & B^{(0)} &= 0_{1/2,1/2} \bar{Y}_{3/2,1/2} \\
G = 0 & : & B^{(1)} &= d_1 X_{1/2,1/2} \bar{Y}_{3/2,1/2} + d_2 \bar{O}_{1/2,1/2} \bar{W}_{3/2,1/2} \\
G = -1 & : & B^{(2)} &= d_3 \wedge d_4 X_{1/2,1/2} \bar{W}_{3/2,1/2}.
\end{align*}
\]

(6.16)

Introducing the explicit expressions for the operators we find

\[
\begin{align*}
B^{(0)} &= \left( c b + \frac{i}{\sqrt{2}} \partial (X - i \phi) \right) \cdot \left( \bar{c} \left( \overline{(\partial X)^2} + \frac{i}{\sqrt{2}} \overline{\partial^2 X} \right) \right) \cdot e^{i(X+i\phi)/\sqrt{2}} \\
B^{(1)} &= \left( d_1 b \bar{c} + d_2 (c b + \frac{i}{\sqrt{2}} \partial (X - i \phi)) \right) \cdot \left( \overline{(\partial X)^2} + \frac{i}{\sqrt{2}} \overline{\partial^2 X} \right) \cdot e^{i(X+i\phi)/\sqrt{2}} \\
B^{(2)} &= d_3 \wedge d_4 \cdot b \cdot \left( \overline{(\partial X)^2} + \frac{i}{\sqrt{2}} \overline{\partial^2 X} \right) \cdot e^{i(X+i\phi)/\sqrt{2}}.
\end{align*}
\]

(6.17)

In the notation of [7, eqn. (2.32)], \( B^{(1)} \) corresponds to \( \bar{Y}_{3/2,1/2} \). We will sometimes use the notation \( \Omega^{(1)} = \Omega_+^{(1)} + \Omega_-^{(1)} \) where we decompose any one-form into its \( dz \) and \( d\bar{z} \) components. The sum of the momenta of the five operators we are considering are \( \sum p_X = 0 \) and \( \sum p_\phi = -i 2 \sqrt{2} \), as they should be. Figure 2 shows the positions of the five relevant operators in the usual diagram for the states of the theory.

We now want to study the amplitude \( \Theta = \langle T_1^{(0)} T_2^{(0)} T_3^{(0)} B^{(0)} \hat{T}^{(0)} \rangle \). Here all the operator valued zero forms are of ghost number two except for \( B^{(0)} \) which is of ghost number one. Thus the ghost numbers add up to nine and we need three antighost insertions to bring down the ghost number to the standard value of six. Therefore, as expected \( \Theta \) must be a three form, and since the moduli space of the five-punctured sphere is four-dimensional, \( \Theta \) is indeed a form of codimension one.
The Ward identity is to be obtained by constructing this three form and computing its boundary contributions.

We do the calculation on the complex \( z \) plane. We insert \( T_1^{(0)}, T_2^{(0)}, \) and \( T_3^{(0)} \) at 0, 1, and \( \infty \), the operator \( B^{(0)} \) at \( x \) and \( \hat{T}^{(0)} \) at \( y \). The form \( \Theta \) will include three antighost insertions. These can be represented as \( b(v) = \oint b(z)v(z)dz \) where \( v(z) \) is the vector field on the surface that generates, via the Schiffer variation, the desired change in modulus of the surface (see [16]). For example, a change of position of a puncture will be given by a vector field \( v(z) = \epsilon \) (a constant), for which \( b(v) = \epsilon b_{-1} \). But \( b_{-1} \) and \( \bar{b}_{-1} \) are precisely the operators that acting on zero form states give the one form states, and acting on the one form states gives us the two form states of the descent equations (eq. (4.17)). For our present case, since all the moduli of the surface can be associated to motions of the punctures, the antighost insertions simply turn the zero forms appearing in \( \Theta \) into their descendents. For the fixed tachyons we use their zero forms \( T_i^{(0)} \). The \( B \) and \( \hat{T} \) operators can appear as the three forms \( B^{(1)}(x)\hat{T}^{(2)}(y) \) or \( B^{(2)}(x)\hat{T}^{(1)}(y) \).

The term \( B^{(1)}(x)\hat{T}^{(2)}(y) \) would have pieces of the form \( dx \wedge dy \wedge d\bar{y} \) and \( d\bar{x} \wedge dy \wedge d\bar{y} \). The first vanishes because the correlator is zero due to wrong left and right movers ghost number. The second piece appears at first sight to be nonzero, but it turns out to vanish because the various possible contractions of \( \left( cb + \frac{i}{\sqrt{2}} \partial (X - i\phi) \right) \) with vertex operators and with ghost zero modes add up to zero. The term \( B^{(2)}(x)\hat{T}^{(1)}(y) \) would have pieces \( dx \wedge d\bar{x} \wedge dy \) and \( dx \wedge d\bar{x} \wedge d\bar{y} \). The first one vanishes again due to ghost number and the second one will give us a nonvanishing result. It follows, then, that the closed three form we are trying to construct is of the form

\[
\Theta = q \ dx \wedge d\bar{x} \wedge d\bar{y} \tag{6.18}
\]

and can be computed by evaluating

\[
\langle T_1^{(0)}(0)T_2^{(0)}(1)T_3^{(0)}(\infty)B^{(2)}(x)\hat{T}^{(1)}(y) \rangle. \tag{6.19}
\]

(Notice that not only is \( B^{(2)} \neq 0 \), making nonlinear contributions to the Ward
identity possible, but with this way of doing the calculation, all contributions come from $B^{(2)}$. The general equation $d\Theta = 0$ reduces to

$$\frac{\partial q}{\partial y} = 0. \tag{6.20}$$

In calculating $q$ various factors must be evaluated. The Wick contractions of the exponential factors in the vertex operators give a factor of

$$|x|^{2-2q_1 \sqrt{2}} |x - 1|^{2-2q_2 \sqrt{2}} \cdot \frac{|x - y|^2}{|y|^2 |y - 1|^2}. \tag{6.21}$$

This certainly does not obey (6.20). An additional important factor comes from evaluating the ghost matrix elements. The left moving ghost fields are $c(0)c(1)c(\infty)b(x)c(y)$, and the ghost zero mode wave functions are $1, z, z^2$. Using $c(\infty)$ to absorb the $z^2$ zero mode, and taking all contractions of the other fields $c(0)c(1)c(y)b(x)$ with each other and with the remaining ghost zero modes, we find that the amplitude of the left moving ghosts is precisely

$$\frac{y(y - 1)}{x(1 - x)(x - y)}. \tag{6.22}$$

Notice that this precisely cancels the unwanted $y$-dependent factors in (6.21) that violate (6.20). The right-moving ghost amplitude is just 1. The remaining factor comes from contractions of the right-moving oscillator factor $(\overline{\partial} X)^2 + \frac{i}{\sqrt{2}} \overline{\partial}^2 X$ in $\hat{T}^{(1)}_{\overline{y}}$. The sum over contractions gives

$$\left( \frac{i q_1}{\overline{x}} + \frac{i q_2}{\overline{x} - 1} - \frac{i}{\sqrt{2} \overline{x} - \overline{y}} \right)^2 + \frac{1}{\sqrt{2} \overline{x}^2} + \frac{1}{\sqrt{2} (\overline{x} - 1)^2} - \frac{1}{2(\overline{x} - \overline{y})^2}. \tag{6.23}$$

Multiplying these factors, the final result is then

$$\Theta = dx \wedge d\overline{x} \wedge d\overline{y} \cdot \frac{|x|^{2-2q_1 \sqrt{2}} |x - 1|^{2-2q_2 \sqrt{2}}}{x(1 - x) (\overline{x} - \overline{y}) \overline{y} \cdot (\overline{y} - 1)} \cdot \left( \left( \frac{i q_1}{\overline{x}} + \frac{i q_2}{\overline{x} - 1} - \frac{i}{\sqrt{2} \overline{x} - \overline{y}} \right)^2 + \frac{1}{\sqrt{2} \overline{x}^2} + \frac{1}{\sqrt{2} (\overline{x} - 1)^2} - \frac{1}{2(\overline{x} - \overline{y})^2} \right). \tag{6.24}$$

Now, we have to extract the boundary terms from the various three dimen-
sional submanifolds corresponding to degenerations of the moduli space of the five-punctured sphere. If $C_\epsilon$ is the ball $|x| = \epsilon$ (or $|x - 1| = \epsilon$, or $|1/x| = \epsilon$) with $y$ unconstrained, then

$$\int_{C_\epsilon} \Theta = 0,$$

just because when restricted to $C_\epsilon$, $dx$ and $d\overline{x}$ are proportional (and $dy$ is missing). This is not surprising since such terms would correspond to linear terms in the Ward identity where $B$ meets one of the tachyons $T_i$ of negative $X$ momentum. Such terms should vanish since there is no cohomology at the corresponding total momentum. Nonvanishing terms arise for $y$ near $a = 0, 1, \infty, \text{or } x$. Notice that in each case, for $y$ near $a$, $\Theta$ is proportional to $d\overline{y}/(\overline{y} - \overline{x})$ (or $d\overline{y}/\overline{y}$ for $a = \infty$). The surface term near $y = a$ is just the residue of $\Theta$, that is the coefficient of $d\overline{y}/(\overline{y} - \overline{x})$.

The Ward identity is

$$0 = \int_x \sum_{a=0,1,\infty,x} \text{Res}_{y=a} \Theta.$$  \hspace{1cm} (6.26)

The symbol $\int_x$ refers to an integration over the complex $x$ plane; the residue of $\Theta$ is a two form proportional to $dx \wedge d\overline{x}$ which can be so integrated.

The term in (6.26) with $a = x$ gives an ordinary linear term in the Ward identity, of the form $A|\hat{T}\rangle \sim |\hat{T}'\rangle$, where $A = \oint B^{(1)}$ is the conserved charge and $\hat{T}'$ is a tachyon state of exceptional $X$ momentum $+\sqrt{2}$.

More interesting for our present purposes are the terms with $a = 0, 1, \infty$. These arise in the above computation from short distance singularities of $\hat{T}$ with one of the $T_i$. Such singularities arise in the region of moduli space near a degeneration of the type in figure (1(ii)) in which $\hat{T}$ and one of the $T_i$ are in one branch, say $\Sigma_1$, while $B$ and the other two $T_i$ are contained in the other branch $\Sigma_2$. In the spirit of the conventional linear Ward identity (6.3), the effect of the branch $\Sigma_2$ as seen by an observer on $\Sigma_1$ is to insert on $\Sigma_1$ a new operator at the point $P$. In the problem at hand, this operator turns out to be a tachyon operator, so the two tachyon operators on $\Sigma_2$ are effectively replaced by one. The interpretation of this
in terms of matrix elements of the conserved charge must therefore involve terms in which the number of tachyons is not conserved. For instance, the term with \( a = \infty \) corresponds to a process

\[
A \left| T_1^{(0)}(p_1) T_2^{(0)}(p_2) \right\rangle = \lambda \left| T^{(0)}(p_1 + p_2 + 1/\sqrt{2}) \right\rangle,
\]

(6.27)

in which \( A \) maps two tachyons of \( X \) momentum \( p_1 \) and \( p_2 \) to a tachyon of momentum \( p_1 + p_2 + 1/\sqrt{2} \) with an amplitude \( \lambda \). (The Liouville momentum automatically works out correctly in this process.) Thus, this operator \( A \) has matrix elements that reduce by one the number of tachyons of negative \( p_X \). Indeed, by considering \( p_X \) and \( p_\phi \), one can easily show that the only non-zero matrix elements of \( A \), when acting on tachyons of negative \( p_X \), are those that reduce the number of tachyons by one. This is the justification for writing (6.27) as we have, and not just as a much weaker statement \( \langle T(p_1 + p_2 + 1/\sqrt{2}) | A | T(p_1) T(p_2) \rangle = \lambda \).

We will now give a theoretical explanation for how this arises and how to determine the value of \( \lambda \).

Let \( \epsilon \) be a measure of the distance from the degeneration at \( a = \infty \). At \( \epsilon = 0 \), the Riemann sphere breaks up into the two components \( \Sigma_1 \) and \( \Sigma_2 \), joined at a double point \( P \). In addition to \( P \), \( \Sigma_1 \) contains \( \hat{T}^{(0)} \) and \( T_3^{(0)} \), and \( \Sigma_2 \) contains the other three operators. \( \Sigma_1 \) has the moduli space \( \mathcal{M}_{0,3} \) consisting of one point, and \( \Sigma_2 \) has a two dimensional moduli space \( \mathcal{M}_{0,4} \). Superficially, factorization of physical states at \( P \) appears to involve the sum over states \( \langle \mathcal{O}^a \rangle \langle \mathcal{O}_a \rangle \), where \( \mathcal{O}^a \) is the dual state to \( \mathcal{O}_a \) (so on the sphere \( \langle \mathcal{O}^a \mathcal{O}_b \rangle = \delta^a_b \)). However, the computation determining \( \lambda \) involved integrating \( \Theta \) not over \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \) but over the cycle \( \mathcal{C}_\epsilon \) in \( \mathcal{M}_{0,5} \). The third dimension in \( \mathcal{C}_\epsilon \) (relative to \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \) over which it is fibered) is the twist angle joining \( \Sigma_1 \) and \( \Sigma_2 \). In the above computation, extraction of the residue at \( y = \infty \) was a way of integrating over the twist angle and reducing the integral over \( \mathcal{C}_\epsilon \) to an integral over \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \). As for the three Beltrami differentials on \( \mathcal{C}_\epsilon \), two of them are the ones we want for eventually integrating on \( \mathcal{M}_{0,3} \times \mathcal{M}_{0,4} \). The third one, associated with the twist angle, is an insertion
of \( b_0 - \bar{b}_0 \) on the long neck which near \( \epsilon = 0 \) joins \( \Sigma_1 \) and \( \Sigma_2 \). Because of this insertion, the sum over physical states arising at the double point \( P \) is not the naive \( \sum_a |O^a\rangle \langle O^a| \) but \( \sum_a |\tilde{O}^a\rangle \langle O^a| \) where \( \tilde{O}^a = (b_0 - \bar{b}_0)|O^a\rangle \). In later work, we will restrict the symbol \( O^a \) to run over plus states only (and not all states as above), and then the sum over states at a degeneration is

\[
\sum_a |O^a\rangle \langle \tilde{O}^a| + \sum_a |\tilde{O}^a\rangle \langle O^a|.
\] (6.28)

We therefore have that

\[
\int \text{Res}_{y=\infty} \Theta = \sum_a \langle \langle \hat{T}^{(0)} T_{3}^{(0)} \tilde{O}^a \rangle \rangle \langle \langle O^a T_{1}^{(0)} T_{2}^{(0)} B^{(2)} \rangle \rangle.
\] (6.29)

where the symbol \( \langle \langle \cdots \rangle \rangle \) indicates that the relevant integral over moduli space should be done (the object inside such correlator must have the right ghost number to be a top form). The states appearing at \( P \) are BRST invariant, since they appear in the operator product of the BRST invariant operators on \( \Sigma_1 \) or \( \Sigma_2 \), and can be considered to represent BRST cohomology classes, since BRST trivial states would decouple. (If we were considering loops, BRST non-invariant states would circulate around the loops and cancel in pairs.) It is natural to define the nonlinear action of the charge \( A = \oint B \) by saying that for \( T_i^{(0)}(p_i) \) states on the incoming branch

\[
A | T_1^{(0)}(p_1) T_2^{(0)}(p_2) \rangle \rangle = \sum_a \lambda_a |\tilde{O}^a\rangle,
\] (6.30)

where \( \lambda_a \equiv \langle \langle O^a T_1^{(0)} T_2^{(0)} B^{(2)} \rangle \rangle. \) A similar formula, with kets replaced by bras, holds for \( T_i^{(0)}(p_i) \) on the outgoing branch. (It is no accident that the nonlinear term arises only when \( p_1 \) and \( p_2 \) are both on the same branch.)

For our present case the only contribution will come from the (zero form) tachyon state \( \tilde{O}^a = T^{(0)}(-p_3) \), where \( p_3 = -p_1 - p_2 - 1/\sqrt{2} \). Then \( O^a = (\partial c + \)
\( \overline{\partial c} T^{(0)}(p_3) \), and therefore equation (6.30) just reduces to (6.27). The value of \( \lambda \) is therefore defined as

\[
\lambda = \langle \langle (\partial c + \overline{\partial c}) T^{(0)}(p_3) T_1^{(0)} T_2^{(0)} B^{(2)} \rangle \rangle.
\] (6.31)

One can use directly equation (6.29) to get

\[
\lambda \cdot \langle \langle \tilde{T}^{(0)} T_3^{(0)} T^{(0)}(-p_3) \rangle \rangle = \int \text{Res}_{y=\infty} \Theta
\]

\[
= \int dx \wedge d\bar{x} \frac{|x|^{2-2q_1 \sqrt{2}} |1-x|^{2-2q_2 \sqrt{2}}}{x(1-x)}
\]

\[
\cdot \left( \left( \frac{i q_1}{x} + \frac{i q_2}{x-1} \right)^2 + \frac{1}{\sqrt{2} \pi^2} + \frac{1}{\sqrt{2} (x-1)^2} \right).
\] (6.32)

This concludes our explanation on the origin of the nonlinear action of the symmetry charges.

6.3. Homotopy Lie Algebras

One may ask what happens to the Jacobi identity when there are nonlinear terms in the Ward identities of currents. Recently, Kontsevich [21] has used in Chern-Simons gauge theory the notion of homotopy Lie algebras, whose origins are in old work of Stasheff [19].\(^*\) We will now argue that the nonlinear terms in the Ward identities generate a homotopy Lie algebra.\(^\dagger\)

Given a graded Lie algebra with generators \( T_a \) (which may be even or odd), introduce objects \( \eta^a \) with dual quantum numbers and opposite statistics. (In our

\(^*\) Kontsevich showed that it is possible to do Chern-Simons gauge theory using homotopy Lie algebras instead of ordinary ones. In that application he considers only algebras preserving a metric, which we will not insist on. Homotopy Lie algebras are closely related to homotopy associative algebras, which have been suggested by some authors [29] to be relevant to string field theory and have been used by Kontsevich in roughly that connection.

\(^\dagger\) While we were checking the axioms of such an algebra, we learned of parallel ideas by E. Verlinde that in some respects go farther.
application, the reversal of statistics comes about because the symmetry generators have the statistics of the currents or one form components of operators, while the two form components have opposite statistics.) Introduce a vector field

\[ V = f^b_{a_1a_2}\eta^{a_1}\eta^{a_2}\frac{\partial}{\partial \eta^b} + f^b_{a_1a_2a_3}\eta^{a_1}\eta^{a_2}\eta^{a_3}\frac{\partial}{\partial \eta^b} + f^b_{a_1a_2a_3a_4}\eta^{a_1}\eta^{a_2}\eta^{a_3}\eta^{a_4}\frac{\partial}{\partial \eta^b} + \ldots \] (6.33)

The f’s are c-number “structure constants.” V is required to be odd, so if the \( T_a \) are all bosonic, and the \( \eta^a \) hence all fermionic, then \( f^b_{a_1a_2...a_n} \) vanishes for odd \( n \).

On \( V \) impose the equation

\[ 0 = \{V, V\}. \] (6.34)

The term in (6.34) cubic in \( \eta \) is the conventional Jacobi identity for the ordinary structure constant \( f^b_{a_1a_2} \). (This would no longer be so if one permits \( V \) to contain terms of less than quadratic order in \( \eta \).) (6.34) defines a “homotopy Lie algebra.”

We want to show how to formally extract such a homotopy Lie algebra from \( D = 2 \) string theory. In doing so, we will consider as generators just the plus states, though the minus states could be included.

Using the notation introduced in the previous subsection, for every BRST cohomology class \( \mathcal{O}_a \) of plus states, there is a dual minus state \( \mathcal{O}^a \). If \( \mathcal{O}_a \) is annihilated by \( b_0 - \overline{b}_0 \), then \( \mathcal{O}^a \) is not. Again, \( \tilde{\mathcal{O}}^a = (b_0 - \overline{b}_0)\mathcal{O}^a \). When one derives Ward identities, one picks up contributions when a surface \( \Sigma \) degenerates. We want to consider the contributions when several currents collide; then \( \Sigma \) splits (as in figure (1)) to two components \( \Sigma_L \) and \( \Sigma_R \), sharing a point \( P \). Also, \( \Sigma_L \) has genus zero. The operators appearing at \( P \) on the two sides are BRST invariant (since they are produced by “integrating out” the BRST invariant objects on \( \Sigma_L \) or \( \Sigma_R \)) and annihilated by \( b_0 - \overline{b}_0 \) (since insertions of other operators do not make sense) and can be considered as BRST cohomology classes (since a zero form operator \( \{Q, \ldots\} \) would decouple). The sum over cohomology classes appearing at \( P \) is

\[ \sum_a |\mathcal{O}_a\rangle \langle \tilde{\mathcal{O}}^a| + \sum_a |\tilde{\mathcal{O}}^a\rangle \langle \mathcal{O}_a|, \] (6.35)

as was explained in (6.28).
Define now

\[ f^b_{a_1 a_2 \ldots a_n} = \langle \langle O_{a_1} O_{a_2} \ldots O_{a_n} \tilde{O}^b \rangle \rangle, \quad (6.36) \]

where \( n \geq 2 \) and all the \( O \)'s are BRST classes (zero forms). Here \( \langle \ldots \rangle \) is a genus zero correlation function (integrated over the moduli of the \( n + 1 \) points in the standard way). We claim that if \( V \) is defined with these \( f \)'s, then the axiom \( \{ V, V \} = 0 \) of a homotopy Lie algebra is obeyed.

To prove this, note that \( f^b_{a_1 a_2 \ldots a_n} \) as defined in (6.36) will vanish unless the sum of the ghost numbers of the indicated states is \( 2n + 2 \) (= \( \dim (\mathcal{M}_{0,n+1}) + 6 \)). To derive a Ward identity, one considers the case in which the sum of the ghost numbers is \( 2n + 1 \), so that the “correlation function” \( \Theta = \langle O_{a_1} O_{a_2} \ldots O_{a_n} \tilde{O}^b \rangle \) is a closed differential form of codimension one. The Ward identity is then as usual

\[ \sum_\alpha \int_{\mathcal{M}_\alpha} \Theta = 0 \quad (6.37) \]

where \( \alpha \) labels the possible ways that the surface \( \Sigma \) can degenerate; these are in one to one correspondence with divisions of the set \( \{ a_1, \ldots, a_n \} \) into two disjoint subsets \( \{ u_1, \ldots, u_{n_\alpha} \} \) and \( \{ v_1, \ldots, v_{n-n_\alpha} \} \). The Ward identity is concretely

\[ 0 = \sum_\alpha \langle \langle O_{u_1} O_{u_2} \ldots O_{u_{n_\alpha}} \tilde{O}^{a_1} \rangle \rangle \cdot \langle \langle O_{v_1} O_{v_2} \ldots O_{v_{n-n_\alpha}} \tilde{O}^b \rangle \rangle, \quad (6.38) \]

where we have used (6.35) and the fact that by Liouville momentum counting, there must be at least one “minus” operator on each side to get a nonzero result.*

Expressed in terms of the \( f \)'s, (6.38) is the component of the relation \( \{ V, V \} = 0 \) proportional to \( \eta^{a_1} \ldots \eta^{a_n} \), and thus we have justified that relation.

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* Were it not for this fact, we would have had to include all of the operators as symmetry generators, which might be natural anyway.
6.4. Application In $D = 4$?

Finally, one might wonder any of these issues could be relevant in a realistic string theory with a macroscopic four dimensional target space.

The conventional Poincaré symmetries in string theory are derived from operators such as $c\partial X^\mu$ whose two-form components vanish. That is why they act linearly. However, one can at least imagine that there might exist a four dimensional Poincaré invariant vacuum state, with suitable Poincaré invariant couplings between “matter fields” and “ghosts” (once these are coupled the distinction between them is fuzzier) such that the Poincaré currents would come from operators with non-zero two-form components.

Even if this occurs, the Poincaré symmetries will act linearly on the particles of the theory, this being inevitable kinematically in $D > 2$. However, it is at least conceivable that the Poincaré currents could act nonlinearly on themselves. If the novel terms are entirely of higher than the usual order, such a theory would be much like an ordinary Poincaré invariant theory, but with some unusual higher order “anomalous” Ward identities presumably involving the spacetime stress tensor. One might speculate that such identities could shed light on the vanishing of the cosmological constant.

7. Some Additional Comments

One of the important questions within two dimensional string theory is that of formulating a string field theory capable of describing the various possible backgrounds. On the one hand we have the collective string field theory [30] which affords a very successful description of the tachyon dynamics, but by not incorporating explicitly the degrees of freedom corresponding to the discrete states makes it difficult to study the changes of backgrounds. On the other extreme, we have the BRST closed string field theory, which, by construction describes correctly the perturbative dynamics of the theory to all orders. A background is needed to define it, but it is at least formally background independent: classical solutions shift
backgrounds [31]. In this string field theory there is not only a tachyon field but also a field for every discrete state, and infinitely more fields one for each Fock space state in the Hilbert space.

For a solvable theory such as the two-dimensional string one could perhaps expect that a suitable truncation of the field content of the BRST string field theory may afford a complete and elegant formulation, which could even be completely background independent, in the sense that no specific background would be necessary to define the theory. It is clear that the discrete states we have considered must be included along with the tachyon to get a complete description. However, at discrete radii other than the $SU(2)$ point, other discrete states would appear. Perhaps they should be included also; an overall framework for doing this is not evident.

Another important issue is that of the spacetime interpretation of the model. It was proposed in [7] that the three-dimensional cone $Q$ plays the role of spacetime in the compactified model. We have indeed succeeded in describing many phenomena in terms of the differential geometry of $Q$. Hopefully, some generalization of this differential geometric setup will survive when the appropriate additional degrees of freedom are incorporated.

Throughout this paper we have had to deal with subtleties associated with the fact that many relations hold only up to BRST trivial terms. A very similar phenomenon is familiar in BRST closed string field theory [32]. The nonpolynomial structure of the theory arises because the failure of the string products to associate (more precisely, to give Jacobi identities) is repaired by the BRST operator acting on the higher order string products. This type of structure is called a homotopy Lie algebra [33,19], the higher string products defining the homotopies. The nonlinear symmetries, studied here seem to define a similar structure. It is of great interest to understand the relations clearly, as this may give us the tools to investigate concretely the enormous symmetry structure of critical closed string theory.
Acknowledgements: We are indebted to E. Verlinde for stimulating discussions.

APPENDIX Schur Polynomials and Explicit Representatives

Schur Polynomials. The elementary Schur polynomials $S_k$ with $k = 0, 1, 2, \ldots$, are defined via the following generating function:

$$
\sum_{k=0}^{\infty} S_k(x_j) z^k = \exp \left( \sum_{k=1}^{\infty} x_k z^k \right),
$$

where the $x_k$’s, with $k = 1, 2, \ldots$, are the arguments of the polynomials. The first few examples are:

$$
S_0 = 1, \quad S_1 = x_1, \quad S_2 = x_2 + \frac{1}{2} x_1^2, \quad S_3 = x_3 + x_2 x_1 + \frac{1}{3!} x_1^3.
$$

The Schur polynomials can be reduced to factorials under some circumstances. If

$$
x_k = \frac{(-1)^k}{k} a \equiv c_k a,
$$

use of the generating function above gives

$$
\sum_{k=1}^{\infty} S_k(c a) z^k = \exp \left( -a \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k \right) = \exp \left( -a \ln(1 + z) \right) = (1 + z)^{-a},
$$

where from we deduce

$$
S_k(c a) = \frac{1}{k!} \left( \frac{d}{dz} \right)^k (1 + z)^{-a} \bigg|_{z=0} = \frac{(-1)^k}{k!} \frac{\Gamma(a + k)}{\Gamma(a)}.
$$

Another simple consequence of the generating function is the identity

$$
S_k(x + y) = \sum_{j=0}^{k} S_j(x) S_{k-j}(y).
$$
Using this and (A.5) one finds that
\[ S_k[\vec{c}(\sqrt{2} - 1)] = S_k(\vec{c}\sqrt{2}p) + S_{k-1}(\vec{c}\sqrt{2}p), \] (A.7)

and this implies that
\[ \sum_{q=0}^{n} (-)^q S_{n-q}(\vec{c}[\sqrt{2}p - 1]) = S_n(\vec{c}\sqrt{2}p). \] (A.8)

**Ground Ring Representatives.** The state corresponding to the ground ring operator \( \mathcal{O}_{k,k} \) \((k > 0)\) was given by Bouwknegt et. al. and it reads*  
\[
|\mathcal{O}_{k,k}\rangle = -\sum_{q=1}^{2k+1} S_{2k+1-q} \left( \frac{\alpha_-}{j} \right) b_{-q} |\sqrt{2}k, i\sqrt{2}k; 0\rangle
\] (A.9)

where \( \alpha_- \equiv \frac{1}{\sqrt{2}}(\alpha_- + i\phi_-) \) and the vacuum \( |p_X, p_\phi; 0\rangle \) is taken to be equal to \( c_1 |p_X, p_\phi; 1\rangle \). The first two examples are
\[
|\mathcal{O}_{0,0}\rangle = b_{-1} |0\rangle = |1\rangle
\]
\[
|\mathcal{O}_{\frac{1}{2}, \frac{1}{2}}\rangle = \left( c_1 b_{-2} + \frac{1}{\sqrt{2}}(\alpha_{-1} - i\phi_{-1}) \right) |1/\sqrt{2}, i/\sqrt{2}; 1\rangle
\]

It is clear that the states \( |\mathcal{O}_{k,k}\rangle \) satisfy the conditions
\[
\{b_0, b_1, b_2, \cdots \} |\mathcal{O}_{k,k}\rangle = 0, \tag{A.10}
\]

and therefore the above representatives are automatically highest weight states (and define the primary operators \( \mathcal{O}_{k,k} \)). The above conditions, as shown in the

* We have changed the overall sign of the representative, for convenience.
Lemma (sect.4), imply that

\[ |\partial \mathcal{O}_{k,k}\rangle = Q |X_{k,k}\rangle, \quad \text{with} \quad |X_{k,k}\rangle = b_{-1} |\mathcal{O}_{k,k}\rangle. \quad (A.11) \]

The states \( |X_{k,k}\rangle \) are therefore given by

\[ |X_{k,k}\rangle = \sum_{q=1}^{2k} S_{2k-q} \left( \frac{-\alpha}{j} \right) b_{-(q+1)} b_{-1} |\sqrt{2k}, i\sqrt{2k}; 0\rangle \quad (A.12) \]

The first two examples for these states are

\[ |X_{0,0}\rangle = 0, \quad |X_{\frac{1}{2}, \frac{1}{2}}\rangle = b_{-2} b_{-1} |1/\sqrt{2}, i/\sqrt{2}; 0\rangle = b_{-2} |1/\sqrt{2}, i/\sqrt{2}; 1\rangle. \quad (A.13) \]

The operators \( \mathcal{O}_{k,k} \) and \( X_{k,k} \) corresponding to the states quoted above are simply given by

\[ \mathcal{O}_{k,k} = \left( -S_{2k} \left( \frac{-i}{j!} \partial^j X^- \right) + \sum_{q=1}^{2k} S_{2k-q} \left( \frac{-i}{j!} \partial^j X^- \right) c \frac{\partial^{q-1} b}{(q-1)!} \right) e^{i2k X^+} \quad (A.14) \]

\[ X_{k,k} = \sum_{q=1}^{2k} S_{2k-q} \left( \frac{-i}{j!} \partial^j X^- \right) \frac{\partial^{q-1} b}{(q-1)!} e^{i2k X^+} \quad (A.15) \]

The first examples for these operators are

\[ \mathcal{O}_{0,0} = 1, \quad \mathcal{O}_{\frac{1}{2}, \frac{1}{2}} = (cb + i\partial X^-) \exp(iX^+), \]

\[ X_{0,0} = 0, \quad X_{\frac{1}{2}, \frac{1}{2}} = b \exp(iX^+). \quad (A.16) \]

One final identity concerns the operator product expansion of an antighost with the \( \mathcal{O} \)’s:

\[ b(z) \mathcal{O}_{k,k}(w) = \frac{1}{z-w} X_{k,k}(w) + \text{regular.} \quad (A.17) \]
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FIGURE CAPTIONS

1) Linear contributions to Ward identities come from degenerations of Riemann surfaces of the type indicated in (i), where a genus zero branch containing a current and precisely one additional field splits off. Nonlinear terms come from more slightly more general degenerations (ii) with a current and more than one additional field splitting off. The example given here involves a coupling of a current to four tachyons.

2) Here we show the three tachyons $T_i$ with negative $X$ momentum, the exceptional tachyon $\hat{T}$ and the discrete state $B$. 

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