On zeros of polynomials and allied functions satisfying second order differential equations

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Abstract

We shall give bounds on the spacing of zeros of certain functions belonging to the Laguerre-Pólya class and satisfying a second order linear differential equation. As a corollary we establish new sharp inequalities on the extreme zeros of the Hermite, Laguerre and Jacobi polynomials, which are uniform in all the parameters.

1 Introduction

The aim of this paper is to establish new sharp inequalities on the extreme zeros of the classical orthogonal polynomials which are uniform in all the parameters. We shall use a modification of the method suggested in [7]. In fact, our result is more general, and deals with the solutions of the second order differential equation with variable coefficients

$$f'' - 2af' + bf = 0,$$

(1)

belonging to the Laguerre-Pólya class \( \mathcal{L} - \mathcal{P} \). The Laguerre-Pólya class consists of real polynomials having only real zeros and real entire functions having a representation of the form

$$cx^me^{-\alpha x^2 + \beta x} \prod_{i=1}^{\infty} (1 - \frac{x}{x_i})e^{x/x_i},$$

(2)

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where \( c, \beta, x_i \) are real, \( \alpha \geq 0 \), \( m \) is a nonnegative integer and \( \sum x_i^{-2} < \infty \). The well-known inequality of Laguerre (\[1\]), p.171) states that

\[
U(f) = f'^2 - ff'' = -\left(\frac{f'}{f}\right)' f^2 \geq 0, \tag{3}
\]

for any \( f \in \mathcal{L} - \mathcal{P} \).

We shall show that this simple inequality readily yields quite accurate bounds on the spacing of zeros of \( f \). Our main result, Theorems 1 and 3 below, will be proved in the next section. Then we use Theorem 3 to derive bounds on the extreme zeros of the Bessel function, generalized Hermite, Laguerre and Jacobi polynomials. As Theorem 3 does not provide any information concerning the precision of the inequalities, we will compare our results with the existing bounds, whenever the corresponding asymptotic or inequalities are known. It seems that the typical situation, at least for the classical orthogonal polynomials with parameters growing not faster than the degree, is as following. The extreme zero, say the largest one \( x_M \), of \( f(x) \), has the asymptotic expansion

\[
x_M \sim P - 3^{-1/3}t_{11}Qk^{-2/3} + ..., \tag{4}
\]

where \( k = \text{deg}(f) \) in the polynomial case. Here \( 3^{-1/3}t_{11} = 1.855757... \), and \( t_{11} \) denotes the least positive zero of the Airy function

\[
A(t) = \frac{\pi}{3} \sqrt{\frac{t}{3}} \left( J_{-1/3}(s) + J_{1/3}(s) \right),
\]

where \( s = 2(\frac{4}{3})^{3/2} \). On the other hand, in a few cases we were able to compare with, (7) gives \( x_M < P - \frac{3}{2}Qk^{-2/3} \), with the same values of \( P \) and \( Q \). Unfortunately, only the first term of the asymptotics is known when \( f \) contains parameters which can vary with \( k \). (see e.g. \[2, 3, 6, 9\] and the references therein). It is tempting to conjecture that even in this case our second term is still close to the correct value.

## 2 Main Theorem

Given an \( f \in \mathcal{L} - \mathcal{P} \) satisfying (4) we shall introduce two functions, the logarithmic derivative \( t(x) = \frac{f'(x)}{f(x)} \) and the discriminant \( \Delta(x) = b(x) - a^2(x) \). If \( f(x) \) is a polynomial of degree \( m \geq k \) with \( k \) distinct zeros \( x_1 < x_2 < ... < x_k \), counting without multiplicity, put formally \( x_0 = -\infty, x_{k+1} = \infty \). Then, by (3), \( t(x) \) consists of \( k + 1 \) decreasing branches \( B_0, B_1, ..., B_k \), where \( B_i \) is defined on \( (x_{i-1}, x_i) \). The same notation with an obvious modification will be used for the case of entire functions, when the sequences \( \{x_i\} \) and \( \{B_i\} \) are one or both side infinite.

**Theorem 1** Let \( f \in \mathcal{L} - \mathcal{P} \) satisfy (4) and suppose that \( a(x) \) intersects a branch \( B_i \) for \( x = c_i \). Let \( J \) be the region defined by \( \Delta(x) > 0 \). Then \( c_i \in J \), and moreover,

\[
x_i < c_i - \frac{1}{\sqrt{\Delta(c_i)}}, \tag{4}
\]

\[
x_{i+1} > c_i + \frac{1}{\sqrt{\Delta(c_i)}}. \tag{5}
\]

In particular, if \( a(x) \) intersects either \( B_0 \) or \( B_k \), then

\[
x_1 > \min_{x \in J} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\}, \tag{6}
\]
This implies
\[ x_k < \max_{x \in J} \left\{ x - \frac{1}{\sqrt{\Delta(x)}} \right\}, \quad (7) \]
respectively.

**Proof.** Let \( x_j \) be a zero of \( f \), consider \( g(x) = f(x)/(x-x_j) \). Using \( f'' = 2af' - bf \), we get
\[
0 \leq (x-x_j)^4 U(g) = (x-x_j)^4 (g'^2 - gg'') = (x-x_j)^2 f'^2 - 2(x-x_j)^2 af f' + ((x-x_j)^2 b - 1)f^2.
\]
Dividing by \( f^2 \) yields
\[
(x-x_j)^2 (t^2(x) - a(x)t(x) + b(x)) \geq 1.
\]
Therefore, for any \( c \) being a solution of \( t(x) = a(x) \), one has
\[
(c-x_j)^2 (b(c) - a^2(c)) \geq 1. \quad (8)
\]
This implies \( b(c) > a^2(c) \). Now, if \( c = c_i \), that is \( x_i < c_i < x_{i+1} \), choosing \( j = i \) or \( i+1 \) in (8), we conclude \( x_i < c_i - \frac{1}{\sqrt{\Delta(c_i)}} \), \( x_{i+1} > c_i + \frac{1}{\sqrt{\Delta(c_i)}} \), thus proving (3). Finally (3),(7) follow by
\[
x_1 > c_0 + \frac{1}{\sqrt{\Delta(c_0)}} \geq \min_{x \in J} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\},
\[
x_k < c_k - \frac{1}{\sqrt{\Delta(c_k)}} \leq \max_{x \in J} \left\{ x - \frac{1}{\sqrt{\Delta(x)}} \right\}.
\]
\[ \square \]
To understand what type of bounds maybe derived from (3) and (7), suppose that \( \Delta(x) \) has precisely two zeros \( y_1 < y_2 \). Let the minimum in (8) be attained for \( x = y_1 + \epsilon \). On omitting the higher order terms we have \( \Delta(y_1 + \epsilon) \approx \epsilon \Delta'(y_1) \), and so
\[
x_1 > \min \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\} \approx y_1 + \min_{\epsilon > 0} \left\{ \epsilon + \frac{1}{\sqrt{\Delta'(y_1)}} \right\} = y_1 + 3(4\Delta'(y_1))^{-1/3}.
\]
Similarly, we get \( x_k \lesssim y_2 - 3(4\Delta'(y_2))^{-1/3} \).

Therefore one could expect, say, for the least zero, that there are constants \( A, B, C \) such that \( A < \frac{x_1 - y_1}{\Delta(y_1)} < B \) and \( x_1 - y_1 = C(\Delta'(y_1))^{-1/3} \).

Theorem 3 also implies that
\[
x_{i+1} - x_i > \min_{x_i < x < x_{i+1}} \frac{2}{\sqrt{\Delta(x)}}. \quad (9)
\]
Slightly stronger result can be proved if we consider \( U(g) \) with \( g(x) = \frac{f(x)}{(x-x_i)(x-x_j)} \).

**Theorem 2** Let \( f \in \mathcal{L} - \mathcal{P} \) satisfy (3), have only simple zeros, and suppose that \( a(x) \) intersects \( t(x) \) between zeros \( x_i < x_j \) of \( f \). Then
\[
(x_i - x_j)^2 \geq \min_{x_i < x < x_j} \frac{8}{\Delta(x)}. \quad (10)
\]

**Proof.** Set \( g(x) = \frac{f(x)}{(x-x_i)(x-x_j)} \), then using (3) to eliminate higher derivatives of \( f \), we obtain
\[
U(g) = \frac{(x-x_i)^2(x-x_j)^2 (f'^2(x) - 2a(x)f'(x)f(x) + b(x)f^2(x)) - ((x-x_i)^2 + (x-x_j)^2) f^2(x)}{(x-x_i)^2(x-x_j)^2}.
\]
Consider this expression at a point \( c, a(c) = t(c), \) and \( x_i < c < x_j. \) Using \((c - x_i)^2 + (c - x_j)^2 \geq 2(c - x_i)(x_j - c),\) we obtain
\[
\Delta(c) \geq \frac{2}{(c - x_i)(x_j - c)} \geq \frac{8}{(x_j - x_i)^2},
\]
yielding (10).
\[QED\]

We expect that in many cases (10) is of the correct order besides the factor 8. For example, for \( f = \sin x \) ( \( f'' + f = 0 \)) it gives \( x_{i+1} - x_i \geq 2\sqrt{2}, \) instead of \( \pi. \) It is easy also to check that for the Chebyshev polynomials \( T_k \) the true answer can be at most \( \sqrt{\frac{x^2 - 1}{2}} \approx 2.1 \) times greater than that given by (10).

Theorem 3 has one shortcoming. The graph of \( t(x) \) consists of cotangent-shape branches in the middle and hyperbolic branches at the ends. Whenever the condition on intersection of \( t(x) \) with \( a(x) \) at a cotangent-shape branch is almost automatically fulfilled, the intersection with the uttermost hyperbolic branches at the ends. Whenever the condition on intersection of \( t(x) \) with \( a(x) \) at a cotangent-shape branch is almost automatically fulfilled, the intersection with the uttermost hyperbolic branches at the ends.

We can get rid of the intersection conditions if we restrict the class of functions \( f \) and assume that \( a(x) \) and \( b(x) \) are sufficiently smooth in a vicinity of zero \( x_i \) of \( f. \) Namely, we consider entire functions of order less than 2 with only distinct real zeros. By Hadamard’s factorization theorem (see e.g. [17]), such functions are either polynomials or have a canonical product representation
\[
f(x) = c^x \beta e^{\beta x} \prod_{i=1}^{\infty} \left(1 - \frac{x}{x_i}\right) e^{x/x_i}, \tag{11}
\]
where \( c, \beta, x_i \) are real, \( m = 0 \) or 1, and \( \sum x_i^{-2} < \infty, \) We denote this class by \( \mathcal{L} - \mathcal{P}(I). \) It is well known (the result usually attributed to Laguerre, see e.g. [17], p. 266, the polynomial case is given in [13], chapter 5, problems 62, 63), that \( f \in \mathcal{L} - \mathcal{P}(I) \) implies
\[
f - \lambda f' = -\lambda e^{x/\lambda} \left(e^{-x/\lambda} f(x) \right)' \in \mathcal{L} - \mathcal{P}(I).
\]
Iterating this yields

**Lemma 1** Let \( f \in \mathcal{L} - \mathcal{P}(I), \) then for any polynomial
\[
p(x) = \prod_{i=1}^{n} (x - \lambda_i) = \sum_{i=0}^{n} q_i x^i,
\]
\( \lambda_1, ..., \lambda_n, \) are real, the function \( g = \sum_{i=0}^{n} q_i f^{(i)} \in \mathcal{L} - \mathcal{P}(I), \) and thus \( U(g) \geq 0. \)

**Theorem 3** Let \( f \in \mathcal{L} - \mathcal{P}(I) \) satisfy (11).

(i) If \( a(x) \) and \( b(x) \) are differentiable in a vicinity of a zero \( x_i \) of \( f, \) then
\[
\Delta(x_i) - 2a'(x_i) \geq 0. \tag{12}
\]

(ii) If \( a(x) \) and \( b(x) \) are two times differentiable in a vicinity of a zero \( x_i \) of \( f, \) then for \( x = x_i, \)
\[
\min_{\lambda} \{(b^2 - 8a^2 a' - 4ba' + 4a^2 + 4ab' - 4aa'')\lambda^4 - 4(ab - 4aa' + b' - a'')\lambda^3 + 2(2a^2 + b - 2a')\lambda^2 - 4a\lambda + 1\} \geq 0. \tag{13}
\]
Proof. Consider $U(g)$, where $g = f - \lambda f'$. We have for $x = x_i$,

$$U(g) = \left((b(x_i) - 2a'(x_i))\lambda^2 - 2\lambda a(x_i) + 1\right)f''(x_i) \geq 0,$$

for any real $\lambda$. Obviously, $b(x_i) - 2a'(x_i)$ must be positive, hence we can choose $\lambda = \frac{a(x_i)}{b(x_i) - 2a'(x_i)}$. This yields (12).

To prove (13) we apply the previous lemma with $n = 2$ and $\lambda_1 = \lambda_2 = \lambda$. Then $g(x) = f(x) - 2\lambda f'(x) + \lambda^2 f''(x)$, and the result follows by calculating $U(g) \geq 0$ for $x = x_i$. □

It seems that (13) leads to the same type of bounds for the extreme zeros as (6) and (7). Moreover, numerical evidences suggest that one can reach the true value of the first two terms in the asymptotic expansion of the extreme zeros as a limiting case. For we choose $p(x) = (x - \lambda)$ in Lemma 1, e.g.

$$g = \sum_{i=0}^{n}(-\lambda)i^{(n)} f^{(i)},$$

and consider the inequality $U(g) > 0$ at a zero $x_i$. We will illustrate this for the case of Hermite polynomial.

3 Applications

In this section we shall use (6) and (7) to give new bounds on the extreme zeros of classical orthogonal polynomials. We refer to [2, 16], and the references therein for the known asymptotic results, and to [1, 16] for all formulae concerning special functions which are used in the sequel.

To gain some impression about the sharpness of the inequalities of Theorem 1 we start with the Bessel function. In this case extremely precise bounds, far better than can be obtained by our method, are known [5, 12, 13].

**Bessel functions** $J_\nu(x)$ can be defined by the following product representation

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu + 1)} \prod_{i=1}^{\infty} \left(1 - \frac{x^2}{j_{\nu,i}^2}\right),$$

where $j_{\nu,1} < j_{\nu,2} < \ldots$, are the positive zeros of $J_\nu(x)$. Thus, $u = u(x) = x^{-\nu} J_\nu(x)$ is an entire function and moreover $u \in L - P$. It can be shown directly, using

$$x^2 J_\nu''(x) + x J_\nu'(x) + (x^2 - \nu^2) J_\nu(x) = 0,$$

that

$$xu'' + (2\nu + 1)u' + xu = 0.$$

The corresponding calculations are very simple.

**Theorem 4**

$$j_{\nu,1} > \frac{((2\nu + 1)^{2/3} + 2^{2/3})^{3/2}}{2}$$

provided $\nu > -\frac{1}{2}$.

Proof. Assuming $\nu > -\frac{1}{2}$ and $x > 0$, one readily sees that $a(x) = -\frac{2\nu+1}{x}$ intersects all the branches $B_i$ of $t(x)$ for $i \geq 1$. Using the power series representation

$$u(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{2^{\nu+2i}i!\Gamma(i + \nu + 1)},$$

we find
we find \( t(0) = 0 \). Since \( t(x) \) is a decreasing function tending to \( -\infty \) for \( x \to j_{1,\nu}^{(-)} \), it follows that \( a(x) \) intersects the branch \( B_0 \) as well. Now, the condition \( \Delta(x) > 0 \) yields \( j_{\nu,1} > \nu + \frac{1}{2} \), and moreover

\[
\begin{align*}
\nu_1 > \min_{0 < x < \nu + 1/2} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\} = \min_{x > 0} \left\{ x + \frac{2x}{\sqrt{4x^2 - (2\nu + 1)^2}} \right\} = \frac{(2\nu + 1)^{2/3} + 2^{2/3}3^{1/3}}{2},
\end{align*}
\]

where the minimum is attained for

\[
x = \frac{(2\nu + 1)^{2/3} \sqrt{2\nu + 1}^3 + 2^{2/3}3^{1/3}}{2}.
\]

The bound given by (14) is \( \nu + \frac{3}{2}\nu^{1/3} + O(1) \) for \( k \to \infty \). On the other hand, it is known [5] that the first two (in fact three, see [12, 13]) terms of the asymptotic expansion of \( j_{\nu,1} \) provide a lower bound for it. Namely, for \( \nu > 0 \), \( j_{\nu,1} > \nu + 3^{-1/3} i_{11}^{1/3} \). Thus, (14) gives the correct answer up to the value of the constant at the second term, 1.5 instead of 1.855757....

**Generalized Hermite polynomials** \( H_k^\mu(x) \) are polynomials orthogonal on \( (-\infty, \infty) \) for \( \mu > -\frac{1}{2} \), with respect to the weight function \( |x|^{2\mu}e^{-x^2} \). The corresponding ODE is

\[
u'' - 2(x - \mu x^{-1})u' + (2k - \theta_k x^{-2})u = 0, \quad u = H_k^\mu(x),
\]

where \( \theta_2 = 0, \theta_{2i+1} = 2\mu \).

The following result is an improvement on asymptotics given in [3, 8].

**Theorem 5** Let \( x_m \) and \( x_M \) be the least and the largest positive zero of \( H_k^\mu(x) \) respectively, \( \mu > -\frac{1}{2} \). Then

\[
x_m > \sqrt{k + \mu - r} + \frac{3}{2} \left( k + \mu - r \right)^{1/6}, \quad (15)
\]

\[
x_M < \sqrt{k + \mu + r} - \frac{3}{2} \left( k + \mu + r \right)^{1/6}, \quad (16)
\]

where \( r = \sqrt{k^2 + 2k\mu - \theta_k} \).

**Proof.** The zeros of \( H_k^\mu(x) \) are symmetric with respect to the origin, hence we may assume \( x > 0 \). Since \( a(x) = x - \mu x^{-1} \), is a continuous increasing function for \( \mu \geq 0 \), and positive tending to \( \infty \) for \( \mu < 0 \), in this region, it intersects all the branches \( B_i \) corresponding to the positive zeros of \( H_k^\mu(x) \). The discriminant

\[
\Delta(x) = (2kx^2 - \theta_k - (x^2 - \mu)^2) x^{-2},
\]

has two positive roots \( y_{1,2} = \sqrt{k + \mu \pm r} \). Solving \( \Delta(x) > 0 \), by Theorem 4, we obtain, \( y_1 < x_m < x_M < y_2 \). Moreover,

\[
x_m > \min_{y_1 < x < y_2} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\}, \quad (17)
\]

\[
x_M < \max_{y_1 < x < y_2} \left\{ x - \frac{1}{\sqrt{\Delta(x)}} \right\}, \quad (18)
\]

We shall prove here (15), the proof of (16) is similar. Suppose that the minimum in (17) is attained for \( x = y_1 + \epsilon \). Since \( \Delta(y_1) = 0 \), and \( \Delta''(y_1) = -\frac{2(3x^2 + 3\mu^2 + 3\delta)}{x^4} < 0 \), we have \( \Delta(y_1 + \epsilon) < \epsilon \Delta'(y_1) \), and so

\[
x_m > \min_{y_1 < x < y_2} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\} > y_1 + \epsilon + \frac{1}{\sqrt{\epsilon \Delta'(y_1)}} \geq y_1 + \min_{\epsilon > 0} \left\{ \epsilon + \frac{1}{\sqrt{\epsilon \Delta'(y_1)}} \right\} =
\]

\[
6
\]
Calculated yield $\Delta'(y_1) = 4r(k + \mu - r)^{-1/2}$, and the result follows.

Now we suppose that $k$ is large and consider the asymptotics corresponding to (17) and (16). If $\mu$ is fixed then

$$x_m > 2\sqrt{\mu^2 + \theta + 3(\mu^2 + \theta)^{1/6}} \left(1 - O(k^{-1})\right).$$

$$x_M < \sqrt{2k} - \frac{3}{2} (2k)^{-1/6} \left(1 - O(k^{-1/3})\right).$$

If $\frac{k}{k} = \delta$ is fixed, then

$$x_m > (\sqrt{2\delta + 1} - 1) \sqrt{\frac{k}{2}} + \frac{3}{2\sqrt{2}} \left(\sqrt{2\delta + 1} - 1\right)^{1/3} \left((2\delta + 1)^{-1/6} k^{-1/6}\right) \left(1 - O(k^{-1/3})\right).$$

$$x_M < (\sqrt{2\delta + 1} + 1) \sqrt{\frac{k}{2}} - \frac{3}{2\sqrt{2}} \left(\sqrt{2\delta + 1} + 1\right)^{1/3} \left((2\delta + 1)^{-1/6} k^{-1/6}\right) \left(1 - O(k^{-1/3})\right).$$

For $\mu = 0$, a sharper result is known (see e.g. [16], sec.6.32), which indicates the same loss of the precision in (16) as for the Bessel function,

$$x_M < \sqrt{2k + 1} - 6^{-1/3} (2k + 1)^{-1/6} \min_{s} \sqrt{\frac{k}{2}} - \frac{3}{2\sqrt{2}} \left((2\delta + 1)^{-1/6} k^{-1/6}\right) s. \quad (19)$$

**Laguerre Polynomials** $L_k^{(\alpha)}(x)$ are polynomials orthogonal on $[0, \infty)$ for $\alpha > -1$, with respect to the weight function $x^\alpha e^{-x}$. The corresponding ODE is

$$u'' - (1 - (\alpha + 1)x^{-1})u' + kx^{-1}u = 0, \quad u = L_k^{(\alpha)}(x).$$

**Theorem 6** Let $x_m$ and $x_M$ be the least and the largest zero of $L_k^{(\alpha)}(x)$ respectively, $\alpha > -1$. Then

$$x_m > r^2 + 3r^{4/3} (s^2 - r^2)^{-1/3}, \quad (20)$$

$$x_M < s^2 - 3s^{4/3} (s^2 - r^2)^{-1/3} + 2, \quad (21)$$

where $r = \sqrt{k + \alpha + 1} - \sqrt{k}$, $s = \sqrt{k + \alpha + 1} + \sqrt{k}$.

**Proof.** Using the variables $r$ and $s$ we get $a(x) = \frac{1 - rs}{2x}$, $b(x) = \frac{(s - r)^2}{4x}$, $\Delta(x) = \frac{(x - r^2)(s^2 - x)}{4r^2}$. Obviously, $a(x)$ is a continuous increasing function for $\alpha > -1$, and $x > 0$. Thus, it intersects all the branches of $t(x)$. By Theorem 3 we have $r^2 < x_m < x_M < s^2$, and also

$$x_m > \min_{r^2 < x < s^2} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\}, \quad (22)$$

$$x_M < \max_{r^2 < x < s^2} \left\{ x - \frac{1}{\sqrt{\Delta(x)}} \right\}. \quad (23)$$

To prove (20), we assume that the minimum is attained at $x = r^2 + \epsilon$. We obtain

$$\Delta(r^2 + \epsilon) = \frac{\epsilon(s^2 - r^2 - \epsilon)}{4(r^2 + \epsilon)^2} < \frac{\epsilon(s^2 - r^2)}{4r^4}.$$
Therefore
\[ x_m > \min_{r^2 < x < s^2} \left\{ x + \frac{1}{\sqrt{\Delta(x)}} \right\} > r^2 + \epsilon + \frac{2r^2}{\sqrt{\epsilon(s^2 - r^2)}} \geq r^2 + \min_{\epsilon > 0} \left\{ \epsilon + \frac{2r^2}{\sqrt{\epsilon(s^2 - r^2)}} \right\} = r^2 + 3s^{4/3}(s^2 - r^2)^{-1/3}. \]

To prove (21) we set \( s^2 - \epsilon \) for the extremal value of \( x \). Then
\[ \Delta(s^2 - \epsilon) = \frac{\epsilon(s^2 - r^2 - \epsilon)}{4(s^2 - \epsilon)^2} < \frac{\epsilon(s^2 - r^2)}{4(s^2 - \epsilon)^2}, \]
and by (23),
\[ x_M < s^2 - \epsilon - \frac{2(s^2 - \epsilon)}{\sqrt{\epsilon(s^2 - r^2)}} \leq s^2 - \min_{0 < s < s^2 - r^2} \left\{ \epsilon + \frac{2s^2}{\sqrt{\epsilon(s^2 - r^2)}} \right\} + 2 \max_{0 < s < s^2 - r^2} \sqrt{\frac{\epsilon}{s^2 - r^2}} = s^2 - 3s^{4/3}(s^2 - r^2)^{-1/3} + 2. \]

This completes the proof. \( \square \)

If \( \alpha \) is fixed (20) and (21) give
\[ x_m > \frac{(1 + \alpha)^2 + 3(1 + \alpha)^{4/3}}{4k} (1 - O(k^{-1})) , \]
\[ x_M < 4k - 3 \cdot 2^{2/3} k \cdot 1^{1/3} \left( 1 - O(k^{-1/3}) \right) . \]

If \( \frac{\alpha}{k} = \delta \), then
\[ x_m > (\sqrt{1 + \delta} - 1)^2 k + \frac{3}{22/3} (\sqrt{1 + \delta} - 1)^{4/3} (1 + \delta)^{-1/6} k^{1/3} \left( 1 - O(k^{-1/3}) \right) , \]
\[ x_M < (\sqrt{1 + \delta} + 1)^2 k - \frac{3}{22/3} (\sqrt{1 + \delta} + 1)^{4/3} (1 + \delta)^{-1/6} k^{1/3} \left( 1 - O(k^{-1/3}) \right) . \]

The classical inequality (see [16], sec.6.32) is
\[ x_M < \left( \sqrt{4k + 2\alpha + 2 - 6^{-1/3}} (4k + 2\alpha + 2)^{-1/6} \right)^2 , \]
provided \( |\alpha| \geq 1/4 \), \( \alpha > -1 \). This bound is sharp only if \( \alpha \) is fixed. Inequalities uniform in \( \alpha \) and \( k \), giving in fact the first terms of (20) and (21), has been established in [1]. Better bounds, practically coinciding with the main term of our inequalities, were given by [4]. Inequalities with the second term only slightly weaker than in (20), (21) were recently obtained by the author using a similar but more complicated approach [4].

Jacobi Polynomials \( P_k^{(\alpha, \beta)}(x) \) are polynomials orthogonal on \([-1, 1]\) for \( \alpha, \beta > -1 \), with respect to the weight function \((1 - x)^\alpha (1 + x)^\beta\). The corresponding ODE is
\[ u'' - \frac{(\alpha + \beta + 2)x + \alpha - \beta}{1 - x^2} u' + \frac{k(k + \alpha + \beta + 1)}{1 - x^2} u = 0, \quad u = P_k^{(\alpha, \beta)}(x). \]

For the known asymptotic results giving the main terms of the following Theorem [7], one should consult [2, 3, 14]. The inequalities of the same order of precision as asymptotics seems were known only for the ultraspherical case [4, 10].

To simplify the calculations we need the following claim showing that \( P_k^{(\alpha, \beta)}(x) \) has a negative zero.
Lemma 2 Let $x_m$ and $x_M$ be the least and the largest zero of $P_k^{(\alpha,\beta)}(x)$ respectively. Then $x_m + x_M \leq 0$, provided $\alpha \geq \beta$.

Proof. According to the Markoff theorem (see e.g. [16], sec. 6.21), $\frac{\partial x}{\partial \alpha} < 0$, $\frac{\partial x}{\partial \beta} > 0$, for any zero $x_i$ of $P_k^{(\alpha,\beta)}(x)$. As for the ultraspherical case $\alpha = \beta$, we obviously have $x_m + x_M = 0$, the result follows. □

Theorem 7 Let $x_m$ and $x_M$ be the least and the largest zero of $P_k^{(\alpha,\beta)}(x)$ respectively, $\alpha \geq \beta > -1$. Then

\begin{align*}
x_m > y_1 + 3(1 - y_1^2)^{2/3}(2R)^{-1/3}, \\
x_M < y_2 - 3(1 - y_2^2)^{2/3}(2R)^{-1/3} + \frac{4q(s + 1)}{(r^2 + 2s + 1)^{3/2}},
\end{align*}

(24)

where

\[ s = \alpha + \beta + 1, \quad q = \alpha - \beta, \quad r = 2k + \alpha + \beta + 1, \quad R = \sqrt{(r^2 - 2s + 1)(r^2 - s^2)}, \]

and

\[ y_1 = -\frac{R + q(s + 1)}{r^2 + 2s + 1}, \quad y_2 = \frac{R - q(s + 1)}{r^2 + 2s + 1}. \]

Proof. We may assume $|x| < 1$. In this interval $a(x)$ is a continuous function, and

\[ \lim_{x \to -1^+} u(x) = -\infty, \quad \lim_{x \to 1^-} u(x) = \infty. \]

Thus, $a(x)$ intersect all the branches of $\partial x$. The corresponding discriminant is

\[ \Delta(x) = -\frac{(r^2 + 2s + 1)x^2 + 2q(s + 1)x + q^2 + s^2 - r^2}{4(1 - x^2)^2}, \]

with the zeros $y_1, y_2$. Thus, we obtain $y_1 < x_m < x_M < y_2$. Observe that for $y_1 < x < y_2$,

\[ \Delta(x) = -\frac{(r^2 + 2s + 1)(x - y_1)(y_2 - x)}{4(1 - x^2)^2} < -\frac{(r^2 + 2s + 1)(x - y_1)(y_2 - y_1)}{4(1 - x^2)^2} = \frac{R(x - y_1)}{2(1 - x^2)^2}. \]

Set $x = y_1 + \epsilon$ for the extreme value in [B]. By the previous lemma $y_1 + \epsilon < x_m \leq 0$, and thus $2y_1 + \epsilon < 0$. This yields

\[ x_m > y_1 + \epsilon + \frac{\sqrt{2}(1 - (y_1 + \epsilon)^2)}{\sqrt{eR}} \geq y_1 + \min_{\epsilon > 0} \left\{ \epsilon + (1 - y_1^2) \sqrt{\frac{2}{eR}} \right\} - \frac{\sqrt{2e}(2y_1 + \epsilon)}{\sqrt{R}} > y_1 + 3(1 - y_1^2)^{2/3}(2R)^{-1/3}. \]

Similarly, using $x = y_2 - \epsilon > y_1$, as the extreme value in [B] and $\Delta(x) < \frac{R(y_2 - x)}{2(1-x^2)^2}$, we obtain

\[ x_M < y_2 - \epsilon - \frac{\sqrt{2}(1 - (y_2 - \epsilon)^2)}{\sqrt{eR}} \leq y_2 - \min_{\epsilon > 0} \left\{ \epsilon + (1 - y_2^2) \sqrt{\frac{2}{eR}} \right\} - \frac{\sqrt{2e}(2y_2 - \epsilon)}{\sqrt{R}} \leq y_2 - 3(1 - y_2^2)^{2/3}(2R)^{-1/3} - \sqrt{\frac{2}{R}} \min_{0<\epsilon\leq y_2-y_1} \left\{ \sqrt{e}(2y_2 - \epsilon) \right\}. \]
The last minimum is attained for $\epsilon = y_2 - y_1$, and equals to $-\frac{4\epsilon^{s+1}}{(x^2 + 2x + 1)^{3/2}}$. This completes the proof. □

If $\alpha, \beta$ are fixed and $k \to \infty$,

$$x_m > -1 + \frac{(1 + \beta)^2 + 3(1 + \beta)^{4/3}}{2k^2} (1 - O(k^{-1})),$$

$$x_M < 1 - \frac{(1 + \alpha)^2 + 3(1 + \alpha)^{4/3}}{2k^2} (1 - O(k^{-1})).$$

In particular, for the Chebyshev polynomial $T_k(x)$ ($\alpha = -1/2$) we get $x_M < 1 - \frac{1 + 3^{2/3}}{8k^2} + O(k^{-3}) \approx 1 - 0.72k^{-2}$, instead of the correct value $x_M = \cos \frac{k \pi}{2k} = 1 - \frac{2^2}{8k^2} + ... \approx 1 - 1.23k^{-2}$.

If $\frac{1}{2} = A, \frac{1}{2} = B$ are fixed and $k \to \infty$,

$$x_m > -S + \frac{3(1 - S^2)^{2/3}}{2((1 + A)(1 + B)(1 + A + B))^{1/6}} k^{-2/3} \left(1 - O(k^{-1/3})\right),$$

$$x_M < T - \frac{3(1 - T^2)^{2/3}}{2((1 + A)(1 + B)(1 + A + B))^{1/6}} k^{-2/3} \left(1 - O(k^{-1/3})\right),$$

where

$$S = \frac{A^2 - B^2 + 4\sqrt{(1 + A)(1 + B)(1 + A + B)}}{(A + B + 2)^2}, \quad T = \frac{B^2 - A^2 + 4\sqrt{(1 + A)(1 + B)(1 + A + B)}}{(A + B + 2)^2}.$$ 

In the ultraspherical case $\alpha = \beta$ this yields

$$x_M < \frac{\sqrt{1 + 2A}}{1 + A} - \frac{3A^{4/3}}{2(1 + A)^{5/3}(1 + 2A)^{1/6}} k^{-2/3} \left(1 - O(k^{-1/3})\right).$$

To demonstrate that Theorem 3 can give inequalities of the same order of precision as Theorem 1, consider the case of Hermite polynomials $H_k(x) = H^0_k(x)$. From (12) one obtains $2k - x_i^2 - 2 \geq 0$, that is $|x_i| \leq \sqrt{2k - 2}$. Furthermore, by (13) we have for any zero $x_i$ of $H_k(x)$, and any $\lambda$,

$$\phi = 4\lambda^2((1 - 2\lambda^2)x_i^2 - 4\lambda(2k\lambda^2 - 4\lambda^2 + 1)x_i + (2k\lambda^2 - 2\lambda^2 + 1)^2 \geq 0.$$ 

It is easy to see that the discriminant of this expression in $\lambda$, that is the resultant $\text{Result}_\lambda(\phi, \frac{\partial \phi}{\partial \lambda})$, must vanish for the extremal value of $x$. We have for the resultant

$$2^{14} (2x^2 - (k - 1)^2) (4x^6 - 24(k - 1)x^4 + (48k^2 - 96k + 75)x^2 - 32(k - 1)^2).$$

To obtain the answer in a closed form one can use the substitution $k = \frac{m^2 - 4m^6 - 1}{4m^6}$, that is $m = (2k + 2 + \sqrt{4k^2 + 8k + 5})^{1/6}$. This yields (using $x < \sqrt{2k}$),

$$x \leq \frac{(m^4 - 1)^{3/2}}{\sqrt{2} m^3} = \sqrt{2k} - 3 \cdot 2^{-11/6} k^{-1/6} + O(k^{-1/2}),$$

for

$$\lambda = \frac{\sqrt{2} m^3}{(m^4 - m^2 - 1)\sqrt{m^4 - 1}}.$$ 

This is only slightly weaker than the bound given by (13).

If one chooses $g = \sum_{i=0}^n (-\lambda)^i \binom{n}{i} f^{(i)}$, for $n > 2$, the expression for $U(g)$ contains polynomials of high
degree and becomes rather complicated. But for the Hermite polynomials and a few small values of $n$ the asymptotic for the bounds given by $U(g) > 0$, can be obtained rather easily. We have applied the following procedure (we used *Mathematika* for calculations). Given an $n$, set $k = m^6/2$, $x = m^3 - c(n)/m$, $\lambda = m^{-3} + d(n)m^{-5}$, and consider the coefficient of $U(g)$ at the greatest power of $m$. This is a polynomial, $\phi$ say, in variables $c$ and $d$. Moreover, the discriminant of $\phi$ in $d$ vanishes at the optimal value of $d$, thus enabling one to exclude $d$. It is left to find possible values of $c$ as the roots of the obtained algebraic equation and to select an appropriate one corresponding to the real value of $d$. The results of these calculations indicate that $c$ strictly increases with $n$ and tends to the ’true’ asymptotic value given by (19), i.e. $1.85575\ldots$. For instance, we get $c(5) \approx 1.73$, $c(6) \approx 1.79$, $c(7) \approx 1.82$, and $c(8) \approx 1.836$.

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