DARK ENERGY AND CONDENSATE STARS:
CASIMIR ENERGY IN THE LARGE

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Vacuum fluctuations and the Casimir effect are considered in a cosmological setting. It is suggested that the dark energy, which recent observations suggest make up 73% of our universe, is vacuum energy due to a causal boundary effect at the cosmological horizon. After a discussion of the similarities and differences between material boundaries in flat spacetime and causal horizons in general relativity, a simple model with a purely vacuum energy de Sitter interior and Schwarzschild exterior, separated by a thin boundary layer is outlined. The boundary layer is a quantum transition region which replaces the event horizons of the classical de Sitter and Schwarzschild solutions, through which the vacuum energy changes.

1 Vacuum Fluctuations and the Cosmological Term

Vacuum fluctuations are an essential feature of quantum theory. The attractive force between uncharged metallic conductors in close proximity, discovered and discussed by Casimir more than half a century ago, is due to the vacuum fluctuations of the electromagnetic field in the region between the conductors. At first viewed perhaps as a theoretical curiosity, the Casimir effect is now being measured with increasing accuracy and sophistication in the laboratory. The Casimir force directly confirms the existence of quantum fluctuations and our methods for handling the ultraviolet divergences they generate, to obtain meaningful finite answers at macroscopic distance scales. Hence these Proceedings may be a good occasion to emphasize that the prediction and measurement of the Casimir effect is one of the striking successes of relativistic quantum field theory. When combined with the Equivalence Principle, this success gives us some confidence that we should also be able to treat the effects of vacuum fluctuations at macroscopic distances in a general as well as a special relativistic setting.

When the gravitational effects of vacuum energy are considered, we encounter one of the most interesting issues at the heart of any attempt to bring general relativity into concordance with quantum principles. Some sixty years
ago W. Pauli realized that the zero-point energy of quantum fluctuations in the vacuum should contribute to the stress-energy tensor of Einstein’s theory as would an effective cosmological constant \( \Lambda > 0 \), permeating all of space uniformly with a negative pressure, \( p_\Lambda = -\Lambda/(8\pi G) \). Such a cosmological term in Einstein’s equations leads to spacetime becoming curved on a scale of order \( \Lambda^{-\frac{1}{2}} \). Estimating the influence of the zero-point energy of the radiation field—the same vacuum fluctuations responsible for the Casimir force between conductors—with an short distance cutoff of the order of the classical electron radius, Pauli came to the conclusion that the radius of the universe ‘could not even reach to the moon.’ In other words, the expected curvature of space from any naive estimate of \( \Lambda \) based on vacuum fluctuations at microscopic scales is far greater than that actually observed.

This ‘cosmological constant problem’ has evolved in recent years from a theoretical question of fundamental importance to one at the center of observational cosmology as well. Observations of type Ia supernovae at moderately large redshifts (\( z \sim 0.5 \) to 1) have led to the conclusion that the expansion of the universe is accelerating. This is possible in Einstein’s theory only if the dominant energy in the universe has an effective equation of state with \( \rho + 3p < 0 \), i.e. assuming its energy density \( \rho > 0 \), it must have negative pressure. Taken at face value the observations imply that some 73% of the energy in the universe is of this hitherto undetected (dark) variety, which leads to a measured cosmological term in Einstein’s equations of

\[
\Lambda_{\text{meas}} \simeq (0.73) \frac{3H_0^2}{c^2} \simeq 1.4 \times 10^{-56}\text{cm}^{-2} \simeq 3.8 \times 10^{-122}\frac{c^3}{\hbar G}.
\]

Here \( H_0 \) is the present value of the Hubble parameter, approximately \( 75 \text{km/sec/Mpc} \simeq 2.4 \times 10^{-18}\text{sec}^{-1} \). The last number in (1) expresses the value of the cosmological dark energy inferred from the SN Ia data in terms of Planck units, \( L_{\text{pl}}^{-2} = \frac{c^2}{\hbar G} \). If instead of the Planck scale, we use the scale of spontaneous symmetry breaking in the experimentally well-tested standard model of electroweak interactions, namely \( 1\text{TeV}^4 \simeq 10\text{ cm}^{-2} \), then \( \Lambda_{\text{meas}} \) is still some 57 orders of magnitude smaller than this much more conservative ‘natural’ scale. Such an enormous mismatch of scales and gross error of estimates strongly suggest that some basic error is being made in estimating supposedly very weak quantum effects in gravity.

Instead of the dark energy being tied to some fixed short distance scale, as the above estimates assume, our experience with the Casimir effect suggests a quite different possibility, namely that the vacuum energy is determined instead by boundary conditions of the universe in the large. Just as the quadratically divergent vacuum energy is not what is measured in the Casimir effect, but only the difference of vacuum energies with and without the conducting plates present, so too \( \Lambda_{\text{meas}} \) may not be related at all to any short distance cutoff, but instead be determined dynamically by the deviation of the universe from globally flat Minkowski spacetime. In that case it would not be
surprising to find that $\Lambda_{\text{meas}}$ depends on the Hubble parameter $H_0$ as in (1) for purely dimensional reasons, and a non-zero value of $\rho_\Lambda$ comparable to the closure density would be expected. In such a framework the dimensionless number to be explained becomes 0.73, rather than $10^{-122}$ or $10^{-57}$. In these proceedings we would like to outline a simple model of this kind.

2 Event Horizons

One important clue that quantum effects in the large could alter the predictions of classical general relativity comes from the calculation of the stress-energy tensor of vacuum fluctuations in curved space backgrounds possessing event horizons. Two familiar examples of such spacetimes are the Schwarzschild metric of an uncharged non-rotating black hole, and the de Sitter metric, both of which can expressed in static, spherically symmetric coordinates in the form,

$$ds^2 = -f(r) c^2 dt^2 + \frac{dr^2}{h(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \, (2)$$

For both the Schwarzschild and de Sitter cases, the functions $f = h$ and

$$f_s(r) = h_s(r) = 1 - \frac{r_s}{r}, \quad r_s = \frac{2GM}{c^2};$$
$$f_{dS}(r) = h_{dS}(r) = 1 - \frac{r^2}{r_H^2}, \quad r_H = \left(\frac{3}{\Lambda}\right)^{\frac{1}{2}}, \, (3)$$

respectively. The first metric approaches flat space at large $r$ but becomes singular at the finite radius $r_s$. The second is regular in the interior of a spherical region but becomes singular at a finite radius $r_H$, depending on the cosmological vacuum energy $\rho_\Lambda = \Lambda/8\pi G$. This is the cosmological horizon of an observer at the origin in a universe containing 100% dark energy, with Hubble parameter $H = c \sqrt{\Lambda/3}$. In both cases the metric singularities may be regarded as pure coordinate artifacts, in the sense that they can be removed entirely by making a coordinate transformation to a different set of well-behaved coordinates in the vicinity of the horizons. However, it is important to recognize that this analytic extension of spacetime by a (singular) coordinate transformation involves a physical assumption, namely that there are no stress-energy sources or discontinuities of any kind at the event horizon. Even in the classical theory the hyperbolic nature of Einstein’s equations allows for sources and/or discontinuities transmitted at the speed of light on a null hypersurface, such as the Schwarzschild or de Sitter horizon.

When quantum fluctuations are considered there is an additional possibility of discontinuities at the horizons. This is because the behavior of the renormalized expectation value of the stress-energy tensor, $\langle \Psi | T_{ab} | \Psi \rangle$ as $r \to r_s$ or $r_H$ depends on the quantum state $|\Psi\rangle$ of the field theory, specified by choosing particular solutions of the wave equation which the quantum field satisfies. The Schwarzschild or de Sitter horizon is a characteristic surface and regular singular point of the wave equation $\Box \Phi = 0$ in static coordinates.
and its general solution is singular there; hence so is $\langle \Psi | T^{b}_{a} | \Psi \rangle$ in the corresponding quantum state. Since a photon with frequency $\omega$ and energy $\hbar \omega$ far from the horizon has a local energy $E_{\text{loc}} = \hbar \omega f^{-\frac{3}{2}}$, and the stress-energy is dimension four, its generic behavior near the event horizon is $E_{\text{loc}}^4 \sim f^{-2}$.

Detailed calculations bear this out, with the stress-energy in the vacuum state defined by absence of quanta with respect to the static time coordinate $t$ in behaving like the negative of a $p = \rho / 3 \sim T_{\text{loc}}^4$ radiation fluid at the local temperature, $T_{\text{loc}} = \hbar c | f' | f^{-\frac{3}{2}} / 4 \pi k_B$.

Although this has been known for some time, the attitude usually adopted is that the states which lead to such divergences on the horizon are pathological, and only states which are regular on the horizon should be considered. In Schwarzschild spacetime the Hartle-Hawking (HH) state and in de Sitter spacetime the Bunch-Davies (BD) state has a regular stress tensor on the horizon. These are often considered preferred ‘vacuum’ states because they possess the maximal symmetry allowed by the geodesically complete background spacetime. However, the essence of an event horizon is that it divides spacetime into regions which are causally disconnected from each other. In both the Schwarzschild and de Sitter cases, the HH and BD states respectively specify that precise quantum correlations be set up and maintained in regions of the globally extended spacetimes which never have been in causal contact with each other. Despite their mathematical appeal, it is by no means clear physically why one should restrict attention to quantum states in which exact phase correlations between regions which have never been in any causal contact are rigorously enforced. As soon as one drops this acausal requirement, and on the contrary restricts attention to states with correlations that could have been arranged by some causal process in the past, then states with divergent $\langle \Psi | T^{b}_{a} | \Psi \rangle$ on the horizon become perfectly admissible.

Again our experience with the Casimir effect suggests a physical interpretation and resolution of these divergences. In the calculation of the local stress tensor $\langle \Psi | T^{b}_{a} | \Psi \rangle$ in flat spacetime with boundaries, one also finds divergences in the generic situation of non-conformally invariant fields and/or curved boundaries. The divergences may be traced to the hard boundary conditions imposed on modes of all frequencies, even the very highest. As the theory and applications of the Casimir effect have developed, it became clear that the material properties of the real conductors involved in the experiments must be taken into account. Casimir’s idealized boundary conditions on the electromagnetic field, appropriate for a perfect conductor with infinite conductivity, have given way to more detailed models incorporating the conductivity response function of real metals. The idealized boundary conditions which led to the divergences are not to be excluded from consideration; they are the correct ones at low frequencies. Finite stresses due to vacuum fluctuations are obtained by the modification of these boundary conditions at higher frequencies, required by taking into account the properties of the actual material
at the boundary in an average continuum description. At still smaller length and time scales approaching atomic dimensions, the approximation of a continuous or average conductivity response function will have to be modified, to take account the electron band structure and microscopic graininess of the conductors, which are composed after all of atomic constituents.

Actually, the divergences in $\langle \Psi | T_a^b | \Psi \rangle$ at the horizon does not require that special or artificial boundary conditions be imposed. This may be seen from the form of the wave equation in the static coordinates (2). Separating variables by writing $\Phi = e^{-i\omega t} Y_{lm} \psi_{\omega l}$, we find the second order ordinary differential equation for the radial function, $\psi_{\omega l}$,

$$\left[ -\frac{d^2}{dr^**2} + V_l \right] \psi_{\omega l} = \omega^2 \psi_{\omega l},$$

(4)

where we have made the usual change of radial variable from $r$ to $r** = \int r \, df(r)$, in order to put the second derivative term into standard form, and the scattering potential for the mode with angular quantum number $l$ is

$$V_l = f \left[ \frac{1}{r} \frac{df}{dr} + \frac{l(l+1)}{r^2} \right].$$

(5)

Since $f \to 0$ linearly as $r$ approaches the horizon, the variable $r** \to -\infty$ logarithmically, and the potential vanishes there as well (exponentially in $r**$). Hence the solutions of (4) define one dimensional scattering states on an infinite interval (in $r**$) with free wave boundary conditions at the horizon. The vacuum state defined by zero occupation number with respect to these scattering states is the Boulware vacuum $| \Psi_B \rangle$, which has a divergent $\langle \Psi_B | T_a^b | \Psi_B \rangle$, behaving like $-T_{loc}^b \text{diag}(-3,1,1,1) \sim f^{-2}$ on the horizon. In contrast to the Casimir effect in flat spacetime with curved boundaries, this divergence does not arise from hard Dirichlet or Neumann boundary conditions, but from an infinite redshift surface with free boundary conditions. Hence the properties of no ordinary material at the boundary can remove this divergence, and the effective cutoff of horizon divergences can arise only from new physics in the gravitational sector at ultrashort scales, i.e. in the structure of spacetime itself very near to the horizon.

3 The Model

The suggestion that a quantum phase transition may occur in the vicinity of the classical Schwarzschild horizon $r_s$ has been made recently. Earlier a superfluid analogy for gravitation had been discussed by Volovik. The present authors have suggested that this phase transition is analogous to a Bose-Einstein condensate (BEC) phase transition, observed in very cold atomic systems. At a phase transition in which the quantum ground state rearranges itself, the vacuum energy of the state also changes in general. Hence
the interior region may have a different effective value of \( \Lambda \) than the exterior region. As is now well known from discussions of vacuum energy in the accelerating universe, when \( p_V = -\rho_V < 0 \), then \( \rho_V + 3p_V < 0 \) and this behaves like an effective repulsive term in Einstein’s equations. Hence, a positive value of \( \Lambda \) in the interior serves to support the system against further collapse. This may be viewed as the gravitational analog of the model of an electron which was one of the motivations of some of the original investigations of the Casimir effect. It was found that the Casimir force on a sphere is repulsive, and therefore cannot cancel the repulsive Coulomb self-force on a charged sphere. However a repulsive Casimir force with interior vacuum energy \( p_V = -\rho_V < 0 \) is exactly what is needed to balance the attractive force of gravity to prevent collapse to a singularity. The Casimir proposal to model an elementary particle such as the electron as a conducting spherical shell does not work as originally proposed, but an analogous model can work for the non-singular final state of gravitational collapse.

The model we arrive at is one with the de Sitter interior matched to a Schwarzschild exterior, sandwiching a thin shell which straddles the region near to \( r_H \approx r_S \), cutting off the divergences in \( \langle \Psi|T_{ab}|\Psi \rangle \) as \( r_H \) is approached from inside and \( r_S \) is approached from outside. This thin shell is the boundary layer where the new physics of a quantum phase transition takes place. A true quantum boundary layer requires a full quantum treatment. However as a first approximation we may treat the boundary layer in a mean field approximation in which Einstein’s eqs. continue to hold, but with an effective equation of state of the ‘material’ making up the layer. The fact that this boundary layer cuts off and replaces an infinite redshift surface at the causal boundaries of the interior and exterior suggests that the most extreme equation of state consistent with causality should play a role here, namely the Zel’dovich equation of state \( p = \rho \), where the speed of sound becomes equal to the speed of light. This is the critical equation of state at the limit of stability for a phase transition to a new phase with a different value of the vacuum energy. It also arises naturally as one component of the stress-energy tensor, \( \langle \Psi|T_{ab}|\Psi \rangle = \text{diag}(-\rho, p, p, p) \) which satisfies the conservation equation,

\[
\nabla_a T^a_r = \frac{dp}{dr} + \frac{\rho + p}{2f} \frac{df}{dr} + 2 \frac{p - p \rho}{r} = 0,
\]

and has three independent components in the most general static, spherically symmetric case. It is clear from (6) that the three independent components can be taken to be that with \( p = \rho/3 \), behaving like \( f^{-2} \), \( p = \rho \), behaving like \( f^{-1} \), and \( p = -\rho \), behaving like \( f^0 \), reflecting the allowed dominant and subdominant classical scaling behaviors of the stress tensor near the horizon.

In the simplest model possible we set the tangential pressure \( p_\perp = p \) and consider only two independent components of the stress tensor in non-
overlapping regions of space. In that case we have three regions, namely,

I. Interior (deSitter) : \(0 \leq r < r_1\), \(\rho = -p\).
II. Thin Shell : \(r_1 < r < r_2\), \(\rho = +p\).
III. Exterior (Schwarzschild) : \(r_2 < r\), \(\rho = p = 0\).

Because of (6), \(p = -\rho\) is a constant in the interior, which becomes a patch of de Sitter space in the static coordinates (14), for \(0 \leq r \leq r_1 < r_H\). The exterior region is a patch of Schwarzschild spacetime for \(r_S < r_2 \leq r < \infty\).

The location of the interfaces at \(r_1\) and \(r_2\) can be estimated by the behavior of the stress tensor near the Schwarzschild and de Sitter horizons. If \(1 - r_S/r_1\) is a small parameter \(\epsilon\), then the location of the outer interface occurs at an \(r_1\) where the most divergent term in the local stress-energy \(\propto M^{-4} \epsilon^{-2}\), becomes large enough to affect the classical curvature \(\sim M^{-2}\), i.e. for

\[
\epsilon \approx \frac{M_{pl}}{M} \approx 10^{-38} \left(\frac{M_\odot}{M}\right),
\]

where \(M_{pl}\) is the Planck mass \(\sqrt{\hbar c/G} \approx 2 \times 10^{-5}\) gm. Thus \(\epsilon \ll 1\) for an object of the order of a solar mass, \(M = M_\odot\), with a Schwarzschild radius of order of a few kilometers. If instead of a collapsed star one considers the interior de Sitter region to be a model of cosmological dark energy, then the radius \(r_H\) is set by the Hubble scale \(cH_0^{-1} \approx 10^{28}\) cm., and \(M \approx 3 \times 10^{22} M_\odot\) is the order of the total mass-energy in the visible universe.

Since the functions \(h\) and \(f\) are of order \(\epsilon \ll 1\) in the transition region II, the proper thickness of the shell is

\[
\ell = \int_{r_1}^{r_2} dr h^{-\frac{1}{2}} \sim \epsilon^{\frac{1}{2}} r_S \sim \sqrt{L_{pl} r_S} \ll r_S.
\]

Although very small, the thickness of the shell is very much larger than the Planck scale \(L_{pl} \approx 2 \times 10^{-33}\) cm. The energy density and pressure in the shell are of order \(M^{-2}\) and far below Planckian for \(M \gg M_{pl}\), so that the geometry can be described reliably by Einstein’s equations. The details of the solution in region II, the matching at the interfaces, \(r_1\) and \(r_2\), and analysis of the thermodynamic stability of the configuration have also been studied.[11]

4 Conclusions

The simplified model illustrates the general features of bulk vacuum energy arising from a gravitational Casimir-like boundary effect with \(\rho_v \approx M_{pl}/L_{pl} r_H^2\). The basic assumption required for a solution of this kind to exist is that gravity, i.e. spacetime itself, must undergo a quantum vacuum rearrangement phase transition in the vicinity of the horizon, \(r \approx r_H\).
where the value of the vacuum energy $\rho_V$ can change. This cannot occur in the strictly classical Einstein theory of general relativity with $\Lambda$ constant. However, quantum fluctuations alter the low energy effective theory of gravity and through the effects of the trace anomaly the conformal part of the metric becomes dynamical, unlike in the Einstein theory where it is constrained.\textsuperscript{14} The conformal part of the metric may provide the order parameter(s) that would make the analogy with atomic Bose-Einstein condensation the apt one, and the non-singular solution we have discussed could be called a gravitational vacuum condensate ‘star.’ At the edge of the ‘star,’ the vacuum condensate disorders, due to the quantum fluctuations of the conformal degrees of freedom in the metric. In a mean field treatment these can be accounted for approximately by the contributions of the anomaly to the quantum stress-energy tensor, $\langle \Psi | T^a_b | \Psi \rangle$ near the horizons of the classical Schwarzschild and de Sitter geometries. A fully consistent mean field model would require that this interior vacuum energy density be derived from the consideration of the long wavelength collective modes within the cavity. At a finer level of resolution this continuum mean field description must give way to a more fundamental treatment in terms of the analogs of the atomistic degrees of freedom that make up the ‘material’ in the boundary layer, and which condense into the vacuum condensate of the interior.\textsuperscript{15} A step towards such a description would be to include fluctuations about the mean field treatment, which will give rise to time dependent dissipative effects as well.\textsuperscript{16} The details of this relaxation to flat space will determine if a realistic model of cosmological dark energy as a finite size Casimir effect of the universe in the large is possible.

References

1. See e.g. M. Bordag, U. Mohideen, and V. M. Mostepanenko, Phys. Rept. 353, 1 (2001) for a recent review.
2. W. Pauli (unpublished); N. Straumann, e-print ArXiv: gr-qc/0208027
3. See e.g. A. V. Filippenko, e-print ArXiv: astro-ph/0309739
4. D. G. Boulware, Phys. Rev. D 11, 1404 (1975); 13, 2169 (1976).
5. S. M. Christensen and S. A. Fulling, Phys. Rev. D 15, 2088 (1977).
6. J. B. Hartle and S. W. Hawking, Phys. Rev. D 13, 2188 (1976).
7. T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London A 360, 117 (1978).
8. D. Deutsch and P. Candelas, Phys. Rev. D 20, 3063 (1979).
9. G. Chapline, E. Hohlfeld, R. B. Laughlin, and D. I. Santiago, Phil. Mag. B 81, 235 (2001); Int. J. Mod. Phys. A 18, 3587 (2003); R. B. Laughlin, Int. J. Mod. Phys. A 18, 831 (2003).
10. G. E. Volovik, Phys. Rept. 351, 195 (2001).
11. P. O. Mazur and E. Mottola, e-print arXiv: gr-qc/0100035
12. T. H. Boyer, Phys. Rev., 174, 1764 (1968).
13. P. A. M. Dirac, Proc. Roy. Soc. Lond. 270, 354 (1962).
14. P. O. Mazur and E. Mottola, *Phys. Rev.* D 64, 104022 (2001).
15. P. O. Mazur, *Acta Phys. Polon.* B 27, 1849 (1996); e-print arXiv: [hep-th/9712208](http://arxiv.org/abs/hep-th/9712208).
16. E. Mottola in *Physical Origins of Time-Asymmetry*, eds. J. J. Halliwell *et al.* (Cambridge Univ. Press, Cambridge, 1993).