The double contravariant powerset monad in the Goguen category of fuzzy sets

Sijia Lu, Dexue Zhang
School of Mathematics, Sichuan University, Chengdu, China
sijialu1027@qq.com, dxzhang@scu.edu.cn

Abstract

A monad is constructed in the Goguen category of fuzzy sets valued in a unital quantale, which is an analog of the double contravariant powerset monad in the category of sets. With help of this monad it is proved that the Goguen category of fuzzy sets is dually monadic over itself.

Keywords: Fuzzy set, Quantale, Monad

MSC(2020): 03E72, 18C15, 18C20

1 Introduction

In order to construct a foundation for fuzzy set theory, Goguen [10, 11] introduced a category Set(L), now called the Goguen category of L-fuzzy sets (or L-sets), where L is a complete lattice, often endowed with some extra structures. Goguen has obtained a characterization of such categories by a relatively simple system of axioms.

Let L be a complete lattice. Then an L-set is a pair (X, α), where X is a set and α: X → L is a function. The set X is the carrier of the L-set, the complete lattice L is the truth value set, and the value α(x) is the membership degree of the point x in the L-set. A morphism from (X, α) to (Y, β) is a function f: X → Y such that α ≤ β ◦ f, in which case we say that f: (X, α) → (Y, β) is a Goguen map or satisfies the Goguen condition. L-sets and their morphisms constitute a category Set(L) — the Goguen category of L-sets. The category Set(L) provides a nice framework for the study and application of fuzzy sets, it enjoys many pleasant categorical properties, particularly when L possesses rich structures, as demonstrated in Goguen [9, 11], Pultr [24, 25], Höhle and Stout [13], and Stout [28].

The double contravariant powerset monad in the category of sets (see e.g. [21] Example 2.11] and its submonads, the filter monad and the ultrafilter monad in particular, are among the fundamental constructions of sets and play important roles in category theory, algebra, topology, and other disciplines, see e.g. [14, 19, 21, 22]. Since the Goguen category Set(L) is not a topos unless L is a singleton set, its behavior is quite different from the category of sets when “powerobjects” are concerned [25]. It is natural to ask, though not a topos, whether there exist analogous constructions in the category Set(L). In this paper, in the circumstance that the truth value set is a unital quantale Q, a monad (P, µ, η) is constructed in the category Set(Q) of (Q-valued) fuzzy sets. This monad is an analog of the double contravariant powerset monad of sets, and it is a lifting of the double contravariant Q-powerset monad in [12] Remark 1.2.7. Some basic properties of the monad (P, µ, η) are examined. The main results include: (i) The category of the Eilenberg-Moore algebras of (P, µ, η) is equivalent to the opposite category of Set(Q), hence the Goguen category of fuzzy sets is dually monadic over itself, adding another one to the list
of pleasant properties of the Goguen category. (ii) For a commutative quantale, the (covariant) powerset monad in $\text{Set}(Q)$ constructed in [6] is a submonad of the double contravariant $Q$-powerset monad.

2 Preliminaries

For category theory we refer to Mac Lane [17] or Riehl [26]; for monads in the category of sets we refer to Manes [21]; for quantale theory we refer to Rosenthal [27]. In this preliminary section, we just recall some basic ideas about quantales and monads, the aim is to fix notations.

Quantales

A unital quantale [27] (also known as a complete residuated lattice [7, page 178])

$$Q = (Q, \& , k)$$

is a monoid with $k$ being the unit, such that the underlying set $Q$ is a complete lattice (with a top element $1$ and a bottom element $0$) and the multiplication $\&$ distributes over arbitrary suprema, i.e.,

$$p \& \left( \bigvee_{i \in I} q_i \right) = \bigvee_{i \in I} p \& q_i \quad \text{and} \quad \left( \bigvee_{i \in I} p_i \right) \& q = \bigvee_{i \in I} p_i \& q$$

for all $p, q, p_i, q_i \in Q$ ($i \in I$).

Quantales abound in mathematics; numerous examples are presented in [5, 27]. As advocated in Goguen [8, 9, 11] (where quantales are called complete lattice ordered semigroup), quantales are natural candidates for truth-value tables in the theory of fuzzy sets.

For each $q \in Q$, the map $- \& q: Q \to Q$ has a right adjoint

$$- / q: Q \to Q, \quad r / q = \bigvee \{ p \in Q \mid p \& q \leq r \},$$

called the left implication of $\&$.

For each $p \in Q$, the map $p \& - : Q \to Q$ has a right adjoint

$$p \& - : Q \to Q, \quad p \& r = \bigvee \{ q \in Q \mid p \& q \leq r \},$$

called the right implication of $\&$.

The left and the right implications satisfy that

$$p \leq r / q \iff p \& q \leq r \iff q \leq p \& r$$

for all $p, q, r \in Q$.

A quantale $Q$ is commutative if $p \& q = q \& p$ for all $p, q \in Q$, in which case we write

$$p \to q := q / p = p \& q$$

for all $p, q \in Q$.

Some basic properties of the left and the right implications are listed below.

Proposition 2.1. ([7, 27]) Let $(Q, \& , k)$ be a unital quantale.

(i) $k \leq y / x \iff x \leq y \iff k \leq x \& y$. 


(ii) \( x / k = x = k \setminus x \).
(iii) \( (y \setminus z) / x = y \setminus (z / x) \).
(iv) \( (y / x) \& x \leq y, x \& (x \setminus y) \leq y \).
(v) \( (z / y) / x = z / (x \& y), x \setminus (y \setminus z) = (y \& x) \setminus z \).
(vi) \( (z / y) \& (y / x) \leq z / x, (x \setminus y) \& (y \setminus z) \leq x \setminus z \).

**Standing Assumption.** Throughout this paper, \( Q = (Q, \&; k) \) always denotes a unital quantale with at least two elements, unless otherwise specified.

For each \( r \in Q \) and each element \( x \) of a set \( X \), we write \( r_x \) for the element of \( Q^X \) given by

\[
r_x(y) = \begin{cases} r & y = x, \\ 0 & y \neq x. \end{cases}
\]

Let \( X \) be a set. For all \( \lambda, \gamma \in Q^X \), we write \( \lambda /\gamma \) and \( \gamma \setminus \lambda \) for elements of \( Q \) given by

\[
\lambda /\gamma = \bigwedge_{x \in X} \lambda(x) / \gamma(x)
\]
and

\[
\gamma \setminus \lambda = \bigwedge_{x \in X} \gamma(x) \setminus \lambda(x).
\]

It is easily verified that

- \( (\bigwedge_i \lambda_i) /\gamma = \bigwedge_i (\lambda_i /\gamma) \); \( \lambda /\bigvee_i \gamma_i = \bigvee_i (\lambda /\gamma_i) \).
- \( \gamma \setminus (\bigwedge_i \lambda_i) = \bigwedge_i (\gamma \setminus \lambda_i) \); \( \bigvee_i \gamma_i \setminus \lambda = \bigvee_i (\gamma_i \setminus \lambda) \).

Let \( X, Y \) be sets and \( f : X \longrightarrow Y \) be a map. For each \( \gamma \in Q^X \), we write \( f(\gamma) \) for the image of \( \gamma \); that is,

\[
f(\gamma)(y) = \bigvee \{ \gamma(x) \mid f(x) = y \}
\]
for all \( y \in Y \). For each \( \lambda \in Q^Y \), we write \( f^{-1}(\lambda) \) for the inverse image of \( \lambda \); that is, \( f^{-1}(\lambda) = \lambda \circ f \).

**Lemma 2.2.** Let \( f : X \longrightarrow Y \) be a map. Then for all \( \alpha, \gamma \in Q^X \) and \( \beta \in Q^Y \),

(i) \( \alpha \setminus \gamma \leq f(\alpha) \setminus f(\gamma), \gamma /\alpha \leq f(\gamma) / f(\alpha) \);
(ii) \( f(\alpha) \setminus \beta = \alpha \setminus f^{-1}(\beta), \beta / f(\alpha) = f^{-1}(\beta) / \alpha \);
(iii) \( f(\alpha) \leq \beta \iff \alpha \leq \beta \circ f \).

For each unital quantale \( Q \), the Goguen category \( \text{Set}(Q) \) refers to the category for which

- an object is a fuzzy set \((X, \alpha)\);
- a morphism (called a Goguen map) \( f \) from \((X, \alpha)\) to \((Y, \beta)\) is a map \( f : X \longrightarrow Y \) such that \( \alpha \leq \beta \circ f \);
- composition is the usual composition of maps.
Monads and their algebras

A monad \( T = (T, \mu, \eta) \) in a category \( \mathcal{A} \) consists of a functor \( T: \mathcal{A} \to \mathcal{A} \) and two natural transformations \( \mu: T^2 \to T, \quad \eta: \text{id}_\mathcal{A} \to T \) which make the following diagrams commutative:

\[
\begin{array}{cccc}
T^3 & T^2 & & T^2 \\
\mu T & & & \mu \\
\downarrow & & & \downarrow \\
T^2 & & & T
\end{array}
\quad
\begin{array}{cccc}
T & T^2 & T^2 & T \\
\eta T & & & \eta \\
\downarrow & & & \downarrow \\
T & & & T
\end{array}
\]

The natural transformations \( \eta \) and \( \mu \) are called the unit and the multiplication of the monad, respectively.

**Example 2.3.** ([21 Example 2.16]) The (covariant) powerset functor on \( \text{Set} \) is the functor

\[
\text{exp}: \text{Set} \to \text{Set}
\]

that assigns to each \( f: X \to Y \) the map

\[
\text{exp} f: 2^X \to 2^Y, \quad A \mapsto f(A).
\]

The functor \( \text{exp} \) gives rise to a monad

\[
(\text{exp}, m, e)
\]

in \( \text{Set} \), where for each set \( X \),

- \( e_X : X \to 2^X \) sends each \( x \in X \) to the singleton set \( \{x\} \);
- \( m_X : 2^X \to 2^X \) sends each \( \mathcal{F} \in 2^X \) to the union of \( \mathcal{F} \).

If \( F: \mathcal{A} \to \mathcal{B} \) is left adjoint to \( G: \mathcal{B} \to \mathcal{A} \), then the adjunction \( F \dashv G \) defines a monad

\[
T = (GF, G\varepsilon F, \eta)
\]

in \( \mathcal{A} \), where \( \eta \) and \( \varepsilon \) are the unit and counit of the adjunction, respectively.

**Example 2.4.** ([21 Example 2.11]) The contravariant powerset functor on \( \text{Set} \) is the functor

\[
\text{exp}^{-1}: \text{Set}^{\text{op}} \to \text{Set}
\]

that assigns to each \( f: X \to Y \) the map

\[
\text{exp}^{-1} f = f^{-1}: 2^Y \to 2^X, \quad B \mapsto f^{-1}(B).
\]

The functor \( \text{exp}^{-1} \) is right adjoint to its opposite

\[
(\text{exp}^{-1})^{\text{op}}: \text{Set} \to \text{Set}^{\text{op}}.
\]

The monad defined by the adjunction \( (\text{exp}^{-1})^{\text{op}} \dashv \text{exp}^{-1} \) is called the double contravariant powerset monad in \( \text{Set} \).
Let $\mathbb{T} = (T, \mu, \eta)$ be a monad in $\mathcal{A}$. A $\mathbb{T}$-algebra is a pair $(A, h)$ consisting of an object $A$ and a morphism $h: TA \to A$ in $\mathcal{A}$ such that the diagrams

\[
\begin{array}{ccc}
T^2 A & \xrightarrow{Th} & TA \\
\mu_A & \searrow & \downarrow h \\
TA & \xrightarrow{h} & A
\end{array} \quad \quad \begin{array}{ccc}
A & \xrightarrow{\eta h} & TA \\
\eta & \searrow & \downarrow h \\
A & \xrightarrow{h} & A
\end{array}
\]

are commutative. A morphism $f: (A, h) \to (A', f')$ of $\mathbb{T}$-algebras is a morphism $f: A \to A'$ of $\mathcal{A}$ that makes the square

\[
\begin{array}{ccc}
TA & \xrightarrow{h} & A \\
TF & \searrow & \downarrow f \\
TA' & \xrightarrow{h'} & A'
\end{array}
\]

commutative. The category of $\mathbb{T}$-algebras and their morphisms is called the Eilenberg-Moore category of the monad $\mathbb{T} = (T, \mu, \eta)$.

Assume that $F: \mathcal{A} \to \mathcal{B}$ is left adjoint to $G: \mathcal{B} \to \mathcal{A}$; assume that $\mathbb{T} = (GF, G\varepsilon F, \eta)$ is the monad in $\mathcal{A}$ defined by the adjunction $F \dashv G$. Then, for each object $B$ of $\mathcal{B}$, the pair $(GB, G\varepsilon_B)$ is a $\mathbb{T}$-algebra and the assignment

\[
B \xrightarrow{f} B' \mapsto (GB, G\varepsilon_B) \xrightarrow{Gf} (GB', G\varepsilon_{B'})
\]

defines a functor from $\mathcal{B}$ to the Eilenberg-Moore category of $\mathbb{T}$, called the comparison functor.

**Definition 2.5.** (26)

(i) An adjunction $F \dashv G$ is monadic if the comparison functor is an equivalence of categories.

(ii) A functor $G: \mathcal{B} \to \mathcal{A}$ is monadic if it admits a left adjoint that defines a monadic adjunction.

(iii) A category $\mathcal{B}$ is monadic over a category $\mathcal{A}$ if there exists a functor $G: \mathcal{B} \to \mathcal{A}$ that is monadic.

**Example 2.6.** (19, 21) The Eilenberg-Moore category of the powerset monad $(\exp, m, e)$ in $\mathcal{Set}$ is isomorphic to the category of complete lattices and join-preserving map. An algebra for the double contravariant powerset monad in $\mathcal{Set}$ is a complete and atomic Boolean algebra. The functor $\exp^{-1}: \mathcal{Set}^{\text{op}} \to \mathcal{Set}$ is monadic, hence the category of sets is dually monadic over itself.

Let $\mathbb{S} = (S, m, e)$ and $\mathbb{T} = (T, \mu, \eta)$ be monads in a category $\mathcal{A}$. A *monad map* from $\mathbb{S}$ to $\mathbb{T}$ is a natural transformation $\kappa: S \to T$ for which the diagrams

\[
\begin{array}{ccc}
id & \xrightarrow{e} & S \\
\eta & \searrow & \downarrow \kappa \\
T & \xrightarrow{} & S
\end{array} \quad \begin{array}{ccc}
S^2 & \xrightarrow{\kappa \kappa} & T^2 \\
m & \searrow & \downarrow \mu \\
S & \xrightarrow{} & T
\end{array}
\]
are commutative, where $\kappa \ast \kappa$ stands for the horizontal composite of $\kappa$ with itself. If $\kappa$, as a morphism between functors, is a monomorphism, then we say that $S = (S, m, e)$ is a submonad of $T = (T, \mu, \eta)$.

The covariant powerset monad in $\text{Set}$ can be made into a submonad of the double contravariant powerset monad in $\text{Set}$, see e.g. [21, page 79].

**Lifting of monads**

By a concrete category over sets [1] we mean a pair $(A, U)$, where $A$ is a category and $U : A \rightarrow \text{Set}$ is a faithful functor (called the forgetful functor). In the case that the forgetful functor is evident, we just say that $A$ is a concrete category. Together with the forgetful functor $U : \text{Set}(Q) \rightarrow \text{Set}$, the Goguen category $\text{Set}(Q)$ becomes a concrete category.

Let $(A, U)$ and $(B, V)$ be concrete categories over sets.

(i) A functor $\mathcal{F} : A \rightarrow B$ is a lifting of a functor $F : \text{Set} \rightarrow \text{Set}$ if $F \circ U = V \circ \mathcal{F}$; that is, the square

```
    A ----> B
    |     |     |
    |     V     V
    |     U     V
  Set ----> Set
```

commutes.

Similarly, a functor $\mathcal{K} : A^{\text{op}} \rightarrow B$ is a lifting of a functor $K : \text{Set}^{\text{op}} \rightarrow \text{Set}$ if $K \circ U^{\text{op}} = V \circ \mathcal{K}$.

(ii) A natural transformation

```
    A ----> B
    \mathcal{F} \quad \tau \quad \delta
```

is a lifting of a natural transformation

```
    \text{Set} ----> \text{Set}
    F \quad \kappa \quad \eta
```

if $\mathcal{F}$ is a lifting of $F$, $\delta$ is a lifting of $G$, and $V \tau = \kappa U$.

(iii) (C.f. [12] page 87) A monad $(\mathcal{F}, \mu, \eta)$ in $A$ is a lifting of a monad $(T, m, e)$ in $\text{Set}$ if the functor $\mathcal{F}$ is a lifting of the functor $T$, and the natural transformations $\mu$ and $\eta$ are lifting of $m$ and $e$, respectively.

**Proposition 2.7.** Let $(A, U)$ be a concrete category over sets. Suppose that

(i) $(T, m, e)$ is a monad in $\text{Set}$;

(ii) $\mathcal{F} : A \rightarrow A$ is a lifting of $T : \text{Set} \rightarrow \text{Set}$;

(iii) $\mu : \mathcal{F}^2 \rightarrow \mathcal{F}$ is a lifting of $m : T^2 \rightarrow T$ and $\eta : \text{id}_A \rightarrow \mathcal{F}$ is a lifting of $e : \text{id}_{\text{Set}} \rightarrow T$.

Then $(\mathcal{F}, \mu, \eta)$ is a monad in $A$ and it is a lifting of $(T, m, e)$. 6
Proof. We check for example the commutativity of the square:

\[
\begin{array}{c}
\mathcal{F}^3 \xrightarrow{\mathcal{F}\mu} \mathcal{F}^2 \\
\mu \mathcal{F} \downarrow \downarrow \downarrow \mu \\
\mathcal{F}^2 \xrightarrow{\mu} \mathcal{F}
\end{array}
\]

For each object \( A \) of \( \mathcal{A} \), since

\[
U((\mu \circ \mathcal{F})_A) = U(\mu_A \circ \mathcal{F}_A)
= U(\mu_A) \circ U(\mathcal{F}_A)
= m_{U(\mathcal{A})} \circ Tm_{U(A)}
= (m \circ Tm)_{U(A)}
= (m \circ m T)_{U(A)}
= U((\mu \circ \mu \mathcal{F})_A)
\]

then \( \mu \circ \mathcal{F} \mu = \mu \circ \mu \mathcal{F} \) because \( U \) is faithful.  

3 The powerset monad in \( \text{Set}(Q) \)

This section recalls the construction of the powerset monad \((\mathcal{U}, m, e)\) in \( \text{Set}(Q) \), which first appeared in [6, Section 4] under the name unbalanced powerobject monad. In next section, we shall see that for a commutative quantale, \((\mathcal{U}, m, e)\) is a submonad of the double contravariant powerset monad in \( \text{Set}(Q) \).

For each object \((X, \alpha)\) of \( \text{Set}(Q) \), let

\[
\mathcal{U}(X, \alpha) = (Q^X, \alpha^!) = (Q^X, \lambda \alpha \gamma = \bigwedge_{x \in X} \alpha(x) / \gamma(x)).
\]

For each Goguen map \( f : (X, \alpha) \rightarrow (Y, \beta) \), the map

\[
\mathcal{U} f : (Q^X, \alpha^!) \rightarrow (Q^Y, \beta^!), \quad \gamma \mapsto f(\gamma)
\]

satisfies the Goguen condition since

\[
\alpha^!(\gamma) = \alpha \mathcal{U} \gamma \leq f(\alpha) \mathcal{U} f(\gamma) \leq \beta \mathcal{U} f(\gamma) = \beta^!(f(\gamma)).
\]

Thus, the assignment \( f \mapsto \mathcal{U} f \) defines a functor

\[
\mathcal{U} : \text{Set}(Q) \rightarrow \text{Set}(Q),
\]

called the (covariant) powerset functor on \( \text{Set}(Q) \).
Lemma 3.1. For each object \((X, \alpha)\) of \(\text{Set}(Q)\), both
\[
e_{(X, \alpha)} : (X, \alpha) \to (Q^X, \alpha^\perp), \quad e_{(X, \alpha)}(x) = k_x
\]
and
\[
m_{(X, \alpha)} : (Q^{Q^X}, \alpha^{\perp\perp}) \to (Q^X, \alpha^\perp), \quad m_{(X, \alpha)}(\Lambda) = \bigvee_{\gamma \in Q^X} \Lambda(\gamma) & \gamma
\]
are Goguen maps.

Proof. The conclusion is contained in [6], the verification is included here for convenience of the reader.

For each \(x \in X\),
\[
\alpha^\perp(e_{(X, \alpha)}(x)) = \alpha \upharpoonright k_x = \alpha(x) / k = \alpha(x),
\]
hence \(e_{(X, \alpha)}\) is a Goguen map.

For each \(\Lambda \in Q^{Q^X}\),
\[
\alpha^{\perp\perp}(\Lambda) = \bigwedge_{\gamma \in Q^X} (\alpha \upharpoonright \gamma) / \Lambda(\gamma)
\]
\[
= \bigwedge_{x \in X} \bigwedge_{\gamma \in Q^X} (\alpha(x) / \gamma(x)) / \Lambda(\gamma)
\]
\[
= \bigwedge_{x \in X} \left(\alpha(x) / \bigvee_{\gamma \in Q^X} \Lambda(\gamma) & \gamma\right)
\]
\[
= \alpha^\perp(m_{(X, \alpha)}(\Lambda)),
\]
hence \(m_{(X, \alpha)}\) is a Goguen map. \(\square\)

The triple
\[(\mathcal{U}, m, e)\]
is a monad in \(\text{Set}(Q)\). Instead of verifying directly that \(e\) and \(m\) are natural transformations and satisfy the monad requirements, we show that it is a lifting of a monad in \(\text{Set}\), namely, the (covariant) \(Q\)-powerset monad described below.

Assigning to each \(f : X \to Y\) the map
\[
\exp_Q f : Q^X \to Q^Y, \quad \gamma \mapsto f(\gamma)
\]
defines a functor
\[
\exp_Q : \text{Set} \to \text{Set},
\]
called the (covariant) \(Q\)-powerset functor.

The functor \(\exp_Q\) gives rise to a monad
\[(\exp_Q, m, e)\]
in the category of sets [2, 18, 20], where for each set \(X\),

- \(e_X : X \to Q^X\) is the map such that for all \(x \in X\), \(e_X(x) = k_x\);
- \(m_X : Q^{Q^X} \to Q^X\) is the map such that for all \(\Lambda \in Q^{Q^X}\) and \(x \in X\),
\[
m_X(\Lambda)(x) = \bigvee_{\gamma \in Q^X} \Lambda(\gamma) & \gamma(x).
\]
When $Q$ is the Boolean algebra $2 = \{0, 1\}$, the monad $(\exp_Q, m, e)$ is then the powerset monad $(\exp, m, e)$ in Example 2.3. Thus, for a general unital quantale $Q$, we call $(\exp_Q, m, e)$ the $Q$-powerset monad in $\text{Set}$.

For each set $X$, define $\kappa_X : 2^X \to Q^X$ by

$$\kappa_X(A)(x) = \begin{cases} k & x \in A, \\ 0 & x \notin A. \end{cases}$$

Then $\kappa = \{\kappa_X\}_X$ is a monad map, exhibiting $(\exp, m, e)$ as a submonad of $(\exp_Q, m, e)$.

It is clear that

- the functor $\mathcal{W}$ is a lifting of the functor $\exp_Q$;
- for each object $(X, \alpha)$ of $\text{Set}(Q)$, $U(m_{(X, \alpha)}) = m_X$;
- for each object $(X, \alpha)$ of $\text{Set}(Q)$, $U(e_{(X, \alpha)}) = e_X$.

Then by Proposition 2.7, $(\mathcal{W}, m, e)$ is a monad in $\text{Set}(Q)$, a lifting of $(\exp_Q, m, e)$. Because of this fact, we call $(\mathcal{W}, m, e)$ the powerset monad in $\text{Set}(Q)$, instead of the unbalanced powerobject monad as in [6].

**Remark 3.2.** The monad $(\exp_Q, m, e)$ can be lifted to a monad in $\text{Set}(Q)$ in different ways, one lifting different from $\mathcal{W}$ is given below.

For each object $(X, \alpha)$ of $\text{Set}(Q)$, define $\alpha^o : Q^X \to Q$ by

$$\alpha^o(\gamma) = \bigvee_{x \in X} \gamma(x) \& \alpha(x).$$

The assignment $(X, \alpha) \mapsto (Q^X, \alpha^o)$ yields a functor

$$\mathcal{W} : \text{Set}(Q) \to \text{Set}(Q).$$

Both $e_X : (X, \alpha) \to (Q^X, \alpha^o)$ and $m_X : (Q^{Q^X}, \alpha^{oo}) \to (Q^X, \alpha^o)$ are Goguen maps (verifications are left to the reader), so the triple

$$(\mathcal{W}, m, e)$$

is a monad in $\text{Set}(Q)$ and it is also a lifting of $(\exp_Q, m, e)$.

The fact that $(\mathcal{W}, m, e)$ is a lifting of $(\exp_Q, m, e)$ is very useful. In the following we use this fact to determine the Eilenberg-Moore algebras of $(\mathcal{W}, m, e)$. Let $((X, \alpha), h)$ be an algebra of the monad $(\mathcal{W}, m, e)$. By definition $h : (Q^X, \alpha^\uparrow) \to (X, \alpha)$ is a Goguen map. Since $(\mathcal{W}, m, e)$ is a lifting of $(\exp_Q, m, e)$, it is readily verified that $(X, h)$ is an algebra of the monad $(\exp_Q, m, e)$. Conversely, if $(X, h)$ is an algebra of the monad $(\exp_Q, m, e)$ and $h : (Q^X, \alpha^\uparrow) \to (X, \alpha)$ satisfies the Goguen condition, then $((X, \alpha), h)$ is an algebra of the monad $(\mathcal{W}, m, e)$. Therefore, in order to determine algebras of the monad $(\mathcal{W}, m, e)$, we need to determine algebras of $(\exp_Q, m, e)$ first.

The algebras of $(\exp_Q, m, e)$ have been determined in [18, 23]. These algebras can be described either as cocomplete $Q$-lattices or as $Q$-modules. A sketch of the ideas is included here for convenience of the reader.

A $Q$-order on a set $X$ is a map $o : X \times X \to Q$ such that for all $x, y, z \in X$,

$$k \leq o(x, x) \quad \text{and} \quad o(y, z) \& o(x, y) \leq o(x, z).$$
The pair \((X, o)\) is called a Q-ordered set or a Q-category \([15][29]\).

A map \(f: (X, o_X) \longrightarrow (Y, o_Y)\) between Q-ordered sets is said to preserve Q-order, if for all \(x_1, x_2 \in X\),

\[ o_X(x_1, x_2) \leq o_Y(f(x_1), f(x_2)). \]

A Q-order-preserving map \(f: (X, o_X) \longrightarrow (Y, o_Y)\) is a left adjoint, if there is a Q-order-preserving map \(g: (Y, o_Y) \longrightarrow (X, o_X)\) such that for all \(x \in X\) and \(y \in Y\),

\[ o_Y(f(x), y) = o_X(x, g(y)). \]

Let \((X, o)\) be a Q-ordered set, \(a \in X\), and \(\gamma \in Q^X\). We say that

\begin{itemize}
\item \((X, o)\) is separated if \(x = y\) whenever \(k \leq o(x, y) \land o(y, x)\).
\item \(a\) is a supremum of \(\gamma\) if for all \(y \in X\), \(o(a, y) = \bigwedge_{x \in X} o(x, y) / \gamma(x)\).
\item \((X, o)\) is cocomplete if every \(\gamma \in Q^X\) has a supremum.
\item \((X, o)\) is a cocomplete Q-lattice if it is both separated and cocomplete.
\end{itemize}

**Definition 3.3.** \((15)\) A Q-module (precisely, a left Q-module) is a pair \((X, \otimes)\), where \(X\) is a complete lattice and \(\otimes: Q \times X \longrightarrow X\) is a map, called a (left) Q-action on \(X\), subject to the following conditions: for all \(x \in X\) and \(r, s \in Q\),

\begin{enumerate}
\item \(k \otimes x = x\), where \(k\) is the unit of \(Q\);
\item \(s \otimes (r \otimes x) = (s \& r) \otimes x\);
\item \(r \otimes - : X \longrightarrow X\) preserve joins;
\item \(- \otimes x : Q \longrightarrow X\) preserve joins.
\end{enumerate}

A homomorphism \(f: (X, \otimes) \longrightarrow (Y, \otimes)\) between Q-modules is a join-preserving map \(f: X \longrightarrow Y\) that preserves the action, i.e., \(r \otimes f(x) = f(r \otimes x)\) for all \(r \in Q\) and \(x \in X\).

**Example 3.4.** \((15)\) For each set \(X\), define \(\otimes: Q \times Q^X \longrightarrow Q^X\) by \((r \otimes \gamma)(x) = r \& \gamma(x)\), then \((Q^X, \otimes)\) is a Q-module.

Given a Q-module \((X, \otimes)\), define \(o: X \times X \longrightarrow Q\) by

\[ o(x, y) = \bigvee \{ r \in Q \mid r \otimes x \leq y \}. \]

Then \((X, o)\) is a cocomplete Q-lattice with supremum of \(\gamma \in Q^X\) given by

\[ \sup \gamma = \bigvee_{x \in X} \gamma(x) \otimes x. \]

Conversely, given a cocomplete Q-lattice \((X, o)\), define a binary relation \(\leq\) on \(X\) by letting \(x \leq y\) if \(k \leq o(x, y)\). Then \((X, \leq)\) is a complete lattice. Furthermore, the assignment \((r, x) \mapsto \sup x\) defines a Q-action on the complete lattice \((X, \leq)\). These processes are inverse to each other, hence the category of cocomplete Q-lattices and left adjoints is isomorphic to that of Q-modules and Q-module homomorphisms. Details of these claims can be found in \([30, \text{Section 4}]\) or \([5, \text{Section 3.3}]\).

Let \((X, o)\) be cocomplete Q-lattice. Then \(X\) together with the map \(\sup: Q^X \longrightarrow X\) is an algebra of the monad \((\exp_Q, m, e)\). Conversely, let \((X, h)\) be an algebra of \((\exp_Q, m, e)\). Since
the powerset monad \((\exp, m, e)\) is a submonad of \((\exp_Q, m, e)\), \(X\) together with the restriction of \(h\) on \(2^X\) is an algebra of \((\exp, m, e)\), hence \(X\) is a complete lattice and \(h\) maps each subset of \(X\) to its join \([17, \text{page } 142]\). Define \(\otimes : Q \times X \rightarrow X\) by \(r \otimes x = h(r_x)\). Then \((X, \otimes)\) is a \(Q\)-module. Therefore, an algebra of the monad \((\exp_Q, m, e)\) is essentially a cocomplete \(Q\)-lattice, or equivalently, a \(Q\)-module.

Now we are able to describe algebras of the monad \((U, m, e)\).

**Proposition 3.5.** An algebra of the monad \((U, m, e)\) is a fuzzy set \(\alpha : X \rightarrow Q\) of a cocomplete \(Q\)-lattice \((X, o)\) such that the map

\[
\sup : (Q^X, \alpha^\uparrow) \rightarrow (X, \alpha)
\]

satisfies the Goguen condition; that is, for all \(\gamma \in Q^X\),

\[
\alpha \preceq \gamma \leq \alpha(\sup \gamma).
\]

A homomorphism is a map \(f : X \rightarrow Y\) that is simultaneously a Goguen map \(f : (X, \alpha) \rightarrow (Y, \beta)\) and a left adjoint \(f : (X, o_X) \rightarrow (Y, o_Y)\).

The value \(\alpha \preceq \gamma\) can be viewed as the degree that the fuzzy set \(\gamma\) is contained in the fuzzy set \(\alpha\) (c.f. \([9, \text{page } 369]\)), the inequality

\[
\alpha \preceq \gamma \leq \alpha(\sup \gamma)
\]

says that \(\alpha\) is closed under formation of suprema in the cocomplete \(Q\)-lattice \((X, o)\). So, an algebra of \((U, m, e)\) is a fuzzy set of a cocomplete \(Q\)-lattice that is closed under formation of suprema.

In terms of \(Q\)-modules, we have:

**Proposition 3.6.** An algebra of \((U, m, e)\) is a fuzzy set \(\alpha : X \rightarrow Q\) of a \(Q\)-module \((X, \otimes)\) such that

(i) for each subset \(A\) of \(X\), \(\bigwedge_{x \in A} \alpha(x) \leq \alpha(\bigvee A)\);

(ii) for each \(r \in Q\) and \(x \in X\), \(\alpha(x) / r \leq \alpha(r \otimes x)\).

A homomorphism is a map \(f : X \rightarrow Y\) that is simultaneously a Goguen map \(f : (X, \alpha) \rightarrow (Y, \beta)\) and a \(Q\)-module homomorphism \(f : (X, \otimes_X) \rightarrow (Y, \otimes_Y)\).

### 4 The double contravariant powerset monad in \(\text{Set}(Q)\)

For each object \((X, \alpha)\) of \(\text{Set}(Q)\), let

\[
\mathcal{P}(X, \alpha) = (Q^X, \alpha^\uparrow) \quad \text{and} \quad \mathcal{P}^\dagger(X, \alpha) = (Q^X, \alpha^\downarrow),
\]

where for all \(\gamma \in Q^X\),

\[
\alpha^\uparrow(\gamma) = \gamma \preceq \alpha \quad \text{and} \quad \alpha^\downarrow(\gamma) = \alpha \succeq \gamma.
\]

**Lemma 4.1.** For each Goguen map \(f : (X, \alpha) \rightarrow (Y, \beta)\), both

\[
\mathcal{P}f : (Q^Y, \beta^\uparrow) \rightarrow (Q^X, \alpha^\uparrow), \quad \lambda \mapsto \lambda \circ f
\]

and

\[
\mathcal{P}^\dagger f : (Q^Y, \beta^\downarrow) \rightarrow (Q^X, \alpha^\downarrow), \quad \lambda \mapsto \lambda \circ f
\]

satisfy the Goguen condition.
Proof. We verify the case of $\mathcal{P}f$ for example. Since $f : (X, \alpha) \rightarrow (Y, \beta)$ is a Goguen map, then $f(\alpha) \leq \beta$, hence by Lemma 2.2,
\[ \lambda \lor \beta \leq \lambda \lor f(\alpha) = (\lambda \circ f) \lor \alpha \]
for all $\lambda \in \mathbb{Q}^Y$, which shows that $\mathcal{P}f : (\mathbb{Q}^Y, \beta^\dagger) \rightarrow (\mathbb{Q}^X, \alpha^\dagger)$ satisfies the Goguen condition. □

Therefore, we obtain two contravariant functors:
\[ \mathcal{P} : \text{Set}(\mathbb{Q})^{\text{op}} \rightarrow \text{Set}(\mathbb{Q}) \]
and
\[ \mathcal{P}^\dagger : \text{Set}(\mathbb{Q}) \rightarrow \text{Set}(\mathbb{Q})^{\text{op}}. \]

**Proposition 4.2.** $\mathcal{P} : \text{Set}(\mathbb{Q})^{\text{op}} \rightarrow \text{Set}(\mathbb{Q})$ is right adjoint to $\mathcal{P}^\dagger : \text{Set}(\mathbb{Q}) \rightarrow \text{Set}(\mathbb{Q})^{\text{op}}$.

Proof. It suffices to check that $f : (X, \alpha) \rightarrow (\mathbb{Q}^Y, \beta^\dagger)$ is a Goguen map if and only if so too is its transpose
\[ \bar{f} : (Y, \beta) \rightarrow (\mathbb{Q}^X, \alpha^\dagger), \quad \bar{f}(y)(x) = f(x)(y). \]
This is easy since
\[ f : (X, \alpha) \rightarrow (\mathbb{Q}^Y, \beta^\dagger) \text{ is a Goguen map} \]
\[ \iff \forall x \in X, \forall y \in Y, \alpha(x) \leq f(x)(y) / \beta(y) \]
\[ \iff \forall x \in X, \forall y \in Y, \alpha(x) \& \beta(y) \leq f(x)(y) \]
\[ \iff \forall y \in Y, \forall x \in X, \beta(y) \leq \alpha(x) \setminus \bar{f}(y)(x) \]
\[ \iff \bar{f} : (Y, \beta) \rightarrow (\mathbb{Q}^X, \alpha^\dagger) \text{ is a Goguen map}. \]

In the adjunction $\mathcal{P}^\dagger \dashv \mathcal{P}$:

- the unit $\eta$ assigns to each object $(X, \alpha)$ of $\text{Set}(\mathbb{Q})$ the map
  \[ \eta_{(X, \alpha)} : (X, \alpha) \rightarrow \mathcal{P}\mathcal{P}^\dagger(X, \alpha), \quad \eta_{(X, \alpha)}(x)(\gamma) = \gamma(x); \]
- the counit $\epsilon$ assigns to each object $(Y, \beta)$ of $\text{Set}(\mathbb{Q})^{\text{op}}$ the opposite of
  \[ \epsilon_{(Y, \beta)} : (Y, \beta) \rightarrow \mathcal{P}^\dagger\mathcal{P}(Y, \beta), \quad \epsilon_{(Y, \beta)}(y)(\lambda) = \lambda(y). \]

We call the monad defined by the adjunction
\[ \mathcal{P}^\dagger \dashv \mathcal{P} \]
the **double contravariant powerset monad** in $\text{Set}(\mathbb{Q})$ and denote it by
\[ \mathfrak{P} = (\mathcal{P}\mathcal{P}^\dagger, \mu, \eta). \]

As usual, we write $\mathfrak{P}$ for both the monad $(\mathcal{P}\mathcal{P}^\dagger, \mu, \eta)$ and the functor $\mathcal{P}\mathcal{P}^\dagger$. We spell out the details of the monad $\mathfrak{P}$ for later use. For each object $(X, \alpha)$ of $\text{Set}(\mathbb{Q})$,
• \( \Psi(X, \alpha) = (Q^{Q^X}, (\alpha^\dagger_\Lambda)^\dagger) \), where for all \( \Lambda: Q^X \to Q \),

\[
(\alpha^\dagger_\Lambda)^\dagger = \bigwedge_{\gamma \in Q^X} (\Lambda(\gamma)/(\alpha \setminus \gamma));
\]

• the unit \( \eta \) assigns to \( (X, \alpha) \) the Goguen map

\[
\eta_{(X, \alpha)}: (X, \alpha) \to (Q^{Q^X}, (\alpha^\dagger_\Lambda)^\dagger)
\]
given by

\[
\eta_{(X, \alpha)}(x)(\gamma) = \gamma(x)
\]
for all \( x \in X \) and \( \gamma \in Q^X \);

• the multiplication \( \mu \) assigns to \( (X, \alpha) \) the Goguen map

\[
\mu_{(X, \alpha)}: \Psi^2(X, \alpha) \to \Psi(X, \alpha)
\]
given by

\[
\mu_{(X, \alpha)}(\mathbb{H})(\gamma) = \hat{\mathbb{H}}(\hat{\gamma}), \quad \hat{\gamma}(\Lambda) = \Lambda(\gamma)
\]
for all \( \mathbb{H}: Q^{Q^X} \to Q \), \( \gamma \in Q^X \) and \( \Lambda \in Q^{Q^X} \).

The monad \( \Psi \) is a lifting of a monad in the category of sets, namely, a lifting of the double contravariant \( Q \)-powerset monad that we describe now.

By the **contravariant \( Q \)-powerset functor** on \( \text{Set} \) we mean the functor

\[
\exp^{-1}_Q: \text{Set}^{\text{op}} \to \text{Set}
\]
that sends a map \( f: X \to Y \) to

\[
f^{-1}: Q^Y \to Q^X, \quad \lambda \mapsto \lambda \circ f.
\]

The contravariant \( Q \)-powerset functor \( \exp^{-1}_Q \) is right adjoint to its opposite \( (\exp^{-1}_Q)^{\text{op}}: \text{Set} \to \text{Set}^{\text{op}} \).

In the adjunction \( (\exp^{-1}_Q)^{\text{op}} \dashv \exp^{-1}_Q \),

• the unit \( \eta \) assigns to each set \( X \) the map

\[
\eta_X: X \to Q^{Q^X}, \quad \eta_X(x)(\gamma) = \gamma(x);
\]

• the counit \( \epsilon \) assigns to each set \( Y \) the map

\[
\epsilon_Y: Y \to Q^{Q^Y}, \quad \epsilon_Y(y)(\lambda) = \lambda(y).
\]

The monad \( (\exp^{-2}_Q, \mu, \eta) \)
defined by the adjunction \( (\exp^{-1}_Q)^{\text{op}} \dashv \exp^{-1}_Q \) is called the **double contravariant \( Q \)-powerset monad** (c.f. [12, Remark 1.2.7]) in \( \text{Set} \).1 When \( Q \) is the Boolean algebra \{0,1\}, this monad is just the double contravariant powerset monad in \( \text{Set} \).

We spell out details of the monad \( (\exp^{-2}_Q, \mu, \eta) \) for later use:

---

1The double contravariant \( Q \)-powerset functor \( \exp^{-2}_Q \) already appeared in [H page 112].
• the functor \( \exp_{Q^2} \) assigns to each \( f : X \rightarrow Y \) the map
  \[ \exp_{Q^2} f : Q^{Q^X} \rightarrow Q^{Q^Y} \]
given by
  \[ \exp_{Q^2} f(\Lambda)(\gamma) = \Lambda(\gamma \circ f) \]  \hspace{1cm} (4.i)
for all \( \Lambda \in Q^{Q^X} \) and \( \gamma \in Q^Y \);
• the unit \( \eta \) assigns to each set \( X \) the map \( \eta_X : X \rightarrow Q^{Q^X} \) given by \( \eta_X(x)(\gamma) = \gamma(x) \);
• the multiplication \( \mu \) assigns to each set \( X \) the map
  \[ \mu_X : \exp_{Q^4}^4(X) \rightarrow \exp_{Q^2}^2(X) \]
given by
  \[ \mu_X(\overline{\Lambda})(\overline{\gamma}) = \overline{\Lambda(\overline{\gamma})}, \quad \overline{\gamma}(\overline{\Lambda}) = \Lambda(\gamma) \]  \hspace{1cm} (4.ii)
for all \( \overline{\Lambda} : Q^{Q^X} \rightarrow Q, \gamma \in Q^X \) and \( \Lambda \in Q^{Q^X} \).

It is clear that
• the functor \( \mathcal{P} : \text{Set}(Q)^{\text{op}} \rightarrow \text{Set}(Q) \) is a lifting of \( \exp_{Q^1}^{-1} : \text{Set}^{\text{op}} \rightarrow \text{Set} \);
• the functor \( \mathcal{P}^\dagger : \text{Set}(Q) \rightarrow \text{Set}(Q)^{\text{op}} \) is a lifting of \( (\exp_{Q^1}^{-1})^{\text{op}} : \text{Set} \rightarrow \text{Set}^{\text{op}} \);
• the multiplication and the unit of the monad \( (\mathcal{P}, \mu, \eta) \) are lifting of that of the monad \( (\exp_{Q^2}^{-2}, \mu, \eta) \), respectively.

Therefore, the monad \( (\mathcal{P}, \mu, \eta) \) is a lifting of \( (\exp_{Q^2}^{-2}, \mu, \eta) \).

**Remark 4.3.**

(i) Though \( \mathcal{P} \) is a lifting of \( \exp_{Q^1}^{-1} \) and \( \mathcal{P}^\dagger \) is a lifting of \( (\exp_{Q^1}^{-1})^{\text{op}} \), the functor \( \mathcal{P}^\dagger \) is not the opposite of \( \mathcal{P} \) unless the quantale \( Q \) is commutative.

(ii) The construction of the adjunction \( \mathcal{P}^\dagger \dashv \mathcal{P} \), hence that of the monad \( \mathcal{P} \), makes use of the quantale structure of \( Q \). But, the construction of the adjunction \( (\exp_{Q^1}^{-1})^{\text{op}} \dashv \exp_{Q^1}^{-1} \) does not depend on the quantale structure of \( Q \); that means, \( Q \) can be replaced by any nonempty set in this construction.

(iii) The functors \( \exp_{Q} : \text{Set} \rightarrow \text{Set} \) and \( \exp_{Q}^{-1} : \text{Set}^{\text{op}} \rightarrow \text{Set} \) are closely related to each other. Of particular interest is the following fact for which the verification is left to the reader: for any pullback square in the category of sets, as displayed on the left,

\[
\begin{array}{ccc}
A & \xrightarrow{f} & C \\
\downarrow h & & \downarrow j \\
B & \xrightarrow{g} & D
\end{array}
\quad \quad \begin{array}{ccc}
Q^C & \xrightarrow{\exp_{Q}^{-1}f} & QA \\
\downarrow \exp_{Q}j & & \downarrow \exp_{Q}h \\
Q^D & \xrightarrow{\exp_{Q}^{-1}g} & QB
\end{array}
\]

the right square is commutative. In the case that \( Q \) is the Boolean algebra \( 2 = \{ 0, 1 \} \), this fact is just the Beck-Chevalley condition of the category of sets (see e.g. [26, page 179]). It should be warned that though \( \mathcal{U} \) is a lifting of \( \exp_{Q} \) and \( \mathcal{P} \) is a lifting of \( \exp_{Q}^{-1} \), the nice connection between \( \exp_{Q} \) and \( \exp_{Q}^{-1} \) does not carry over. For instance, it does not make sense to formulate a square for \( \mathcal{U} \) and \( \mathcal{P} \) as displayed on the right for \( \exp_{Q} \) and \( \exp_{Q}^{-1} \), since \( \mathcal{U} \) sends an object \( (X, \alpha) \) of \( \text{Set}(Q) \) to \( (Q^X, \alpha^\dagger) \), while \( \mathcal{P} \) sends it to \( (Q^X, \alpha^\dagger) \).
The following theorem implies that the category of Eilenberg-Moore algebras of the monad \((\mathcal{P}, \mu, \eta)\) is equivalent to the opposite category of \(\text{Set}(Q)\), hence \(\text{Set}(Q)\) is dually monadic over itself.

**Theorem 4.4.** The functor \(\mathcal{P} : \text{Set}(Q)^{\text{op}} \to \text{Set}(Q)\) is monadic.

**Proof.** We apply the “reflexive tripleability theorem” (see e.g. [26, Proposition 5.5.8]) to prove the conclusion. Since \(\text{Set}(Q)\) is a complete category, we only need to show that the functor

\[
\mathcal{P} : \text{Set}(Q)^{\text{op}} \to \text{Set}(Q)
\]

reflects isomorphisms and preserves coequalizers of reflexive pairs.\(^2\)

Suppose that \(f : (X, \alpha) \to (Y, \beta)\) is a Goguen map such that

\[
\mathcal{P}f : (Q^Y, \beta^+) \to (Q^X, \alpha^+), \quad \lambda \mapsto \lambda \circ f
\]

is an isomorphism in \(\text{Set}(Q)\). Then \(f\) is a bijection and

\[
\lambda \leftrightarrow \beta = (\lambda \circ f) \leftrightarrow \alpha
\]

for all \(\lambda \in Q^Y\). Putting \(\lambda = \alpha \circ f^{-1}\) gives that \(\beta \leq \alpha \circ f^{-1}\), hence \(\beta \circ f \leq \alpha\) and consequently \(\beta \circ f = \alpha\). Therefore, \(f\) is an isomorphism in \(\text{Set}(Q)\), hence an isomorphism in \(\text{Set}(Q)^{\text{op}}\). This proves that \(\mathcal{P}\) reflects isomorphisms.

Now we show that \(\mathcal{P}\) preserves coequalizers of reflexive pairs. Consider a coequalizer in \(\text{Set}(Q)^{\text{op}}\) of a reflexive pair; this means we have an equalizer

\[
(Z, \gamma) \xrightarrow{e} (X, \alpha) \xrightarrow{f} (Y, \beta)
\]

in \(\text{Set}(Q)\) together with a Goguen map \(h : (Y, \beta) \to (X, \alpha)\) such that both \(h \circ f\) and \(h \circ g\) are the identity map on \((X, \alpha)\). We wish to prove that

\[
\mathcal{P}(Y, \beta) \xrightarrow{\mathcal{P}f} \mathcal{P}(X, \alpha) \xrightarrow{\mathcal{P}e} \mathcal{P}(Z, \gamma)
\]

is a coequalizer in \(\text{Set}(Q)\).

Since \(h\) is a common left inverse for \(f\) and \(g\), then

- both \(f\) and \(g\) are injective;
- \(\beta \circ f = \alpha = \beta \circ g\);
- for all \(x_1, x_2 \in X\), \(f(x_1) = g(x_2) \implies x_1 = x_2\).

Since \(e : (Z, \gamma) \to (X, \alpha)\) is an equalizer of \(f\) and \(g\), we may identify \(Z\) with the subset

\[
\{x \in X \mid f(x) = g(x)\}
\]

of \(X\) and identify \(\gamma\) with the restriction of \(\alpha\) on \(Z\); that is, \(\gamma = \alpha|Z\).

\(^2\)A parallel pair of morphisms \(r, s : A \to B\) in a category is reflexive if they have a common right inverse; that is, there is a morphism \(i : B \to A\) such that \(r \circ i = 1_B = s \circ i\).
Suppose that \( d: \mathcal{P}(X, \alpha) \longrightarrow (W, \lambda) \) is a Goguen map such that \( d \circ \mathcal{P} f = d \circ \mathcal{P} g \). We need to show that there is a unique Goguen map

\[ \overline{d}: \mathcal{P}(Z, \gamma) \longrightarrow (W, \lambda) \]

satisfying \( d = \overline{d} \circ \mathcal{P} e \). Uniqueness is obvious since \( \mathcal{P} e \) is an epimorphism.

Before proving the existence of \( \overline{d} \), we show that for all \( \xi_1, \xi_2 \in Q^X \),

\[ \xi_1|Z = \xi_2|Z \implies d(\xi_1) = d(\xi_2). \]

To see this, define \( \xi \in Q^Y \) by

\[ \xi(y) = \begin{cases} 
\xi_1(x) & y = f(x) \text{ for some } x \in X, \\
\xi_2(x) & y = g(x) \text{ for some } x \in X, \\
1 & \text{otherwise.}
\end{cases} \]

That \( \xi \) is well-defined follows from that \( \xi_1|Z = \xi_2|Z \) and the aforementioned facts about \( f \) and \( g \). Since \( \mathcal{P} f(\xi) = \xi_1 \) and \( \mathcal{P} g(\xi) = \xi_2 \), it follows that \( d(\xi_1) = d(\xi_2) \), as desired.

For each \( \zeta \in Q^Z \), define \( E(\zeta) \in Q^X \) by

\[ E(\zeta)(x) = \begin{cases} 
\zeta(x) & x \in Z, \\
1 & x \notin Z.
\end{cases} \]

Then \( E: \mathcal{P}(Z, \gamma) \longrightarrow \mathcal{P}(X, \alpha) \) is a Goguen map, because for each \( \zeta \in Q^Z \),

\[ \gamma^\uparrow(\zeta) = \bigsqcap_{x \in Z} \zeta(x) / \gamma(x) = \bigsqcap_{x \in X} E(\zeta)(x) / \alpha(x) = \alpha^\uparrow(E(\zeta)). \]

Let \( \overline{d} = d \circ E \). We claim that \( \overline{d} \) satisfies the requirement. For each \( \xi \in Q^X \), since the restrictions of \( E \circ \mathcal{P} e(\xi) \) and \( \xi \) on \( Z \) are equal, i.e., \( (E \circ \mathcal{P} e(\xi))|Z = \xi|Z \), it follows that

\[ \overline{d} \circ \mathcal{P} e(\xi) = d(E \circ \mathcal{P} e(\xi)) = d(\xi), \]

which completes the proof.

Next, we show that for a commutative quantale, the (covariant) powerset monad \( (\mathcal{U}, m, e) \) is a submonad of the double contravariant powerset monad \( (\mathcal{P}, \mu, \eta) \).

**Lemma 4.5.** For each set \( X \), the map

\[ j_X: Q^X \longrightarrow Q^{Q^X}, \quad j_X(\lambda)(\gamma) = \gamma \not= \lambda \]

is injective. The assignment \( X \mapsto j_X \) defines a natural transformation from \( \text{exp}_Q \) to \( \text{exp}_{Q^2} \).

**Proof.** That \( j_X \) is injective is clear. It remains to check that \( \{j_X\}_X \) is a natural transformation; that is, for each map \( f: X \longrightarrow Y \), the square

\[
\begin{array}{ccc}
\text{exp}_Q X & \xrightarrow{j_X} & \text{exp}_{Q^2} X \\
\downarrow \text{exp}_Q f & & \downarrow \text{exp}_{Q^2} f \\
\text{exp}_Q Y & \xrightarrow{j_Y} & \text{exp}_{Q^2} Y
\end{array}
\]

is commutative. This is easy since for all \( \lambda \in Q^X \) and \( \gamma \in Q^Y \), by Lemma 2.2 and equation (4.1) we have

\[ j_Y \circ \text{exp}_Q f(\lambda)(\gamma) = \gamma \not= f(\lambda) = \gamma \circ f \not= \lambda = \text{exp}_{Q^2} f(j_X(\lambda))(\gamma). \]
If \( Q \) is commutative, then for all set \( X \) and all \( \lambda, \gamma \in Q^X \),
\[
\lambda \downarrow \gamma = \bigwedge_{x \in X} (\lambda(x) \rightarrow \gamma(x)) = \gamma \uparrow \lambda.
\]
In this case we write
\[
\text{sub}_X(\lambda, \gamma) := \lambda \downarrow \gamma = \gamma \uparrow \lambda = j_X(\lambda)(\gamma).
\]

**Theorem 4.6.** Let \( Q \) be a commutative quantale. Then the powerset monad \((\mathcal{U}, m, e)\) is a submonad of the double contravariant powerset monad \((\mathcal{P}, \mu, \eta)\).

**Proof.** We prove the conclusion in two steps.

**Step 1.** For each object \((X, \alpha)\) of \(\text{Set}(Q)\) and each \(\lambda \in Q^X\), \(\alpha \uparrow (\lambda) = \text{sub}_X(\lambda, \alpha) = j_X(\lambda)(\alpha)\)

and
\[
\text{sub}_X(\gamma_1, \gamma_2) \leq j_X(\lambda)(\gamma_1) \rightarrow j_X(\lambda)(\gamma_2)
\]
for all \(\gamma_1, \gamma_2 \in Q^X\), it suffices to show that
\[
(\alpha_1 \uparrow (\Lambda)) = \Lambda(\alpha)
\]
whenever \(\Lambda: Q^X \rightarrow Q\) satisfies
\[
\text{sub}_X(\gamma_1, \gamma_2) \leq \Lambda(\gamma_1) \rightarrow \Lambda(\gamma_2).
\]

Since \(Q\) is commutative, by definition we have
\[
(\alpha_1 \uparrow (\Lambda)) = \bigwedge_{\gamma \in Q^X} \text{sub}_X(\alpha, \gamma) \rightarrow \Lambda(\gamma).
\]

Since \(\text{sub}_X(\alpha, \alpha) \geq k\), then
\[
(\alpha_1 \uparrow (\Lambda)) \leq k \rightarrow \Lambda(\alpha) = \Lambda(\alpha).
\]

Conversely, since
\[
\text{sub}_X(\alpha, \gamma) \rightarrow \Lambda(\gamma) \geq (\Lambda(\alpha) \rightarrow \Lambda(\gamma)) \rightarrow \Lambda(\gamma) \geq \Lambda(\alpha)
\]
for all \(\gamma \in Q^X\), then
\[
(\alpha_1 \uparrow (\Lambda)) = \bigwedge_{\gamma \in Q^X} \text{sub}_X(\alpha, \gamma) \rightarrow \Lambda(\gamma) \geq \Lambda(\alpha).
\]

**Step 2.** \((\mathcal{U}, m, e)\) is a submonad of \((\mathcal{P}, \mu, \eta)\).

By **Step 1** one sees that for each \((X, \alpha)\) of \(\text{Set}(Q)\), the map
\[
\kappa_{(X, \alpha)}: \mathcal{U}(X, \alpha) \rightarrow \mathcal{P}(X, \alpha), \quad \lambda \mapsto \text{sub}_X(\lambda, -)
\]
satisfies the Goguen condition, hence \(\kappa = \{\kappa_{(X, \alpha)}\}\) is a natural transformation from \(\mathcal{U}\) to \(\mathcal{P}\), and it is a lifting of the natural transformation \(j = \{j_X\}\) in the above lemma.
It is clear that, as a morphism between functors, \( \kappa \) is a monomorphism and \( \eta = \kappa \circ e \). So, to see that \((\mathcal{U}, m, e)\) is a submonad of \((\mathcal{P}, \mu, \eta)\), we only need to show that the square

\[
\begin{array}{ccc}
\mathcal{U}^2 & \xrightarrow{\kappa \ast \kappa} & \mathcal{P}^2 \\
m & & \mu \\
\downarrow \kappa & & \downarrow \mu \\
\mathcal{U} & \xrightarrow{\kappa} & \mathcal{P}
\end{array}
\]

is commutative. Since \( \kappa \) is a lifting of \( j \), it suffices to show that for each set \( X \), the following square is commutative:

\[
\begin{array}{ccc}
\exp_Q^2 X & \xrightarrow{(j \ast j)_X} & \exp_Q^{-4} X \\
m_X & & \mu_X \\
\downarrow & & \downarrow \\
\exp_Q X & \xrightarrow{j_X} & \exp_Q^{-2} X
\end{array}
\]

For this we calculate: for all \( \Lambda \in Q^{Q^X} \) and \( \lambda \in Q^X \),

\[
\mu_X \circ (j \ast j)_X (\Lambda)(\lambda) = \mu_X \circ j_{Q^X} \circ \exp_Q j_X (\Lambda)(\lambda) = j_{Q^X} (j_X (\Lambda))(\tilde{\lambda}) = \bigwedge_{\Xi \in Q^{Q^X}} j_X (\Lambda)(\Xi) \to \Xi(\lambda) = \bigwedge_{\gamma \in Q^X} \Lambda(\gamma) \to \text{sub}_X (\gamma, \lambda) = \text{sub}_X \left( \bigvee_{\gamma \in Q^X} \Lambda(\gamma) \& \gamma, \lambda \right) = j_X \circ m_X (\Lambda)(\lambda).
\]

The fact that the monad \((\mathcal{P}, \mu, \eta)\) is a lifting of \((\exp_Q^{-2}, \mu, \eta)\) is useful. As an application, we describe here a submonad of \((\mathcal{P}, \mu, \eta)\) by lifting the \(Q\)-filter monad \(Q\text{-Fil}\) in the category of sets, the latter is a submonad of \((\exp_Q^{-2}, \mu, \eta)\). The same idea can be used to construct some other monads in \(\text{Set}(Q)\).

The following definition is a slight modification of that of \(Q\text{-filter}\) in \([4, 12, 16]\).

**Definition 4.7.** A \(Q\)-filter on a set \( X \) is a map \( F: Q^X \to Q \) subject to the following conditions: for all \( \lambda, \gamma \in Q^X \),

(F1) \( F(k_X) \geq k \), where \( k_X \) is the constant map \( X \to Q \) with value \( k \);
(F2) \( F(\lambda) \land F(\gamma) \leq F(\lambda \land \gamma) \);
(F3) \( \gamma \preceq \lambda \leq F(\gamma) / F(\lambda) \).
(F4) \( F(r_X) \leq r \) for all \( r \in Q \), where \( r_X \) is the constant map \( X \to Q \) with value \( r \).

In presence of \([F3]\) the inequalities in \([F2]\) and \([F4]\) are actually equalities.

For each set \( X \), write

\[
Q\text{-Fil}(X)
\]
for the set of \( Q \)-filters on \( X \). For each \( f: X \to Y \) and each \( F \in Q\text{-Fil}(X) \), define
\[
f(F): Q^Y \to Q
\]
by
\[
f(F)(\gamma) = F(\gamma \circ f).
\]
Then \( f(F) \) is a \( Q \)-filter on \( Y \). In this way we obtain a functor
\[
Q\text{-Fil}: \text{Set} \to \text{Set}.
\]

The \( Q \)-filter functor \( Q\text{-Fil} \) is a subfunctor of \( \exp^{-2}_Q: \text{Set} \to \text{Set} \), indeed, it can be made into a submonad of \( (\exp^{-2}_Q, \mu, \eta) \), as we see below.

**Lemma 4.8.** For each \( Q \)-filter \( F \) on \( Q\text{-Fil}(X) \), the map
\[
\sigma(F): Q^X \to Q, \quad \sigma(F)(\lambda) = F(\lambda)
\]
is a \( Q \)-filter on \( X \), where \( \hat{\lambda}: Q\text{-Fil}(X) \to Q \) is given by \( \hat{\lambda}(F) = F(\lambda) \).

**Proof.** That \( \sigma(F) \) satisfies \([F1]\)\([F2]\) and \([F3]\) is clear, it remains to check that it satisfies \([F3]\).
We calculate: for all \( \lambda, \gamma \in Q^X \),
\[
\gamma \not\prec \lambda \leq \bigwedge_{F \in Q\text{-Fil}(X)} F(\gamma) / F(\lambda)
= \hat{\gamma} \not\prec \hat{\lambda}
\leq \hat{F}(\hat{\gamma}) / \hat{F}(\hat{\lambda})
= \sigma(F)(\gamma) / \sigma(F)(\lambda),
\]
which completes the proof. \( \square \)

The \( Q \)-filter \( \sigma(F) \) is called the *diagonal \( Q \)-filter*, or the *Kowalsky sum*, of \( F \). The diagonal \( Q \)-filter is closely related to the multiplication of the monad \( (\exp^{-2}_Q, \mu, \eta) \). Let \( i \) be the inclusion transformation of the functor \( Q\text{-Fil} \) in \( \exp^{-2}_Q \). Then for each \( F \in Q\text{-Fil}^2(X) \),
\[
\sigma(F) = \mu_X \circ (i \ast i)_X(F),
\]
where \( i \ast i \) stands for the horizontal composite of \( i \) with itself. This shows that the functor \( Q\text{-Fil} \) is closed under the multiplication \( \mu \), hence \( \mu \) induces a natural transformation from \( Q\text{-Fil}^2 \) to \( Q\text{-Fil} \), which is also denoted by \( \mu \).

For each \( x \) of \( X \),
\[
\eta_X(x): Q^X \to Q, \quad \eta_X(x)(\lambda) = \lambda(x)
\]
is a \( Q \)-filter, hence the unit of the monad \( (\exp^{-2}_Q, \mu, \eta) \) factors through \( Q\text{-Fil} \). This means that \( \eta \) can be viewed as a natural transformation from the identity functor to \( Q\text{-Fil} \).

Since \( \eta \) factors through \( Q\text{-Fil} \) and \( Q\text{-Fil} \) is closed under the multiplication \( \mu \), the triple
\[
(Q\text{-Fil}, \mu, \eta)
\]
is a monad in the category of sets, a submonad of \( (\exp^{-2}_Q, \mu, \eta) \).

Now we lift the monad \( (Q\text{-Fil}, \mu, \eta) \) to a monad in \( \text{Set}(Q) \). For each object \((X, \alpha) \) of \( \text{Set}(Q) \), let
\[
\mathcal{G}(X, \alpha) = (Q\text{-Fil}(X), (\alpha_i^\uparrow)^\uparrow),
\]
where for each $Q$-filter $F$ on $X$,

$$(\alpha^\uparrow)(F) = \bigwedge_{\gamma \in Q^X} \left( F(\gamma) / \bigwedge_{x \in X} \alpha(x) \setminus \gamma(x) \right).$$

Then we obtain a functor

$$\mathcal{F}: \text{Set}(Q) \to \text{Set}(Q),$$

which is a subfunctor of the functor $\mathcal{P}$.

**Proposition 4.9.** The triple $(\mathcal{F}, \mu, \eta)$ is a submonad of the monad $(\mathcal{P}, \mu, \eta)$ in $\text{Set}(Q)$, and it is a lifting of the $Q$-filter monad $(Q-\text{Fil}, \mu, \eta)$.

Besides the covariant $Q$-powerset monad, the $Q$-filter monad, and the double contravariant $Q$-powerset monad, some other monads in $\text{Set}$ can also be lifted to $\text{Set}(Q)$. For instance, the powerset monad in Example 2.3 and the list monad (see e.g. [26, page 156]). For each object $(X, \alpha)$ of $\text{Set}(Q)$ and each subset $A \subseteq X$, let

$$\alpha_P(A) = \bigwedge_{a \in A} \alpha(a).$$

Then the assignment $(X, \alpha) \mapsto (2^X, \alpha_P)$ gives rise to a functor on $\text{Set}(Q)$, which leads to a lifting of the powerset monad to $\text{Set}(Q)$. The list monad in $\text{Set}$ can be lifted to $\text{Set}(Q)$ in a similar way.

It should be noted that there exist monads in $\text{Set}(Q)$ that are not lifting of any monad in the category of sets, the monad $\mathcal{E}^2$ constructed in Demirci [3, Section 4] provides such an example.

**References**

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, Wiley, 1990.

[2] M.A. Arbib, E.G. Manes, Fuzzy machines in a category, Bulletin of Australian Mathematical Society 13 (1975) 169-210.

[3] M. Demirci, Many valued topologies on L-sets, Fuzzy Sets and Systems 437 (2022) 97-113.

[4] P. Eklund, W. Gähler, Fuzzy filter functions and convergence, in: *Applications of Category Theory to Fuzzy Subsets*, Kluwer, Dordrecht, 1992, pp.109-136.

[5] P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen, *Semigroups in Complete Lattices. Quantales, Modules and Related Topics*, Springer, 2018.

[6] P. Eklund, J. Kortelainen, L.N. Stout, Adding fuzziness to terms and power objects using a monadic approach, Fuzzy Sets and Systems 192 (2012) 104-122.

[7] N. Galatos, P. Jipsen, T. Kowalski, H. Ono, *Residuated Lattices, An Algebraic Glimpse at Substructural Logics*, Elsevier, 2007.

[8] J.A. Goguen, L-fuzzy sets, Journal of Mathematical Analysis and Applications 18 (1967) 145-174.

[9] J.A. Goguen, The logic of inexact concepts, Synthese 19 (1969) 325-373.

[10] J.A. Goguen, Categories of $V$-sets, Bulletin of the American Mathematical Society 75 (1969) 622-624.

[11] J.A. Goguen, Concept representation in natural and artificial languages: Axioms, extensions and applications for fuzzy sets, International Journal of Man-Machine Studies 6 (1974) 513-561.
[12] U. Höhle, *Many Valued Topology and Its Applications*, Kluwer Academic Publishers, 2001.
[13] U. Höhle, L.N. Stout, Foundations of fuzzy sets, Fuzzy Sets and Systems 40 (1991) 257-296.
[14] D. Hofmann, G. J. Seal, W. Tholen (eds.), *Monoidal Topology: A Categorical Approach to Order, Metric, and Topology*, Cambridge University Press, 2014.
[15] A. Joyal, M. Tierney, *An Extension of the Galois Theory of Grothendieck*, Memoirs of the American Mathematical Society, No. 309, 1984.
[16] H. Lai, D. Zhang, G. Zhang, The saturated prefilter monad, Topology and its Applications 301 (2021) 107525.
[17] S. Mac Lane, *Categories for the Working Mathematician*, Second Edition, Graduate Texts in Mathematics, Volume 5, Springer, 1998.
[18] J. Machner, T-algebras of the monad $L$-Fuzz, Czechoslovak Mathematical Journal 35 (1985) 515-528.
[19] E.G. Manes, *Algebraic Theories*, Graduate Texts in Mathematics, Volume 26, Springer, 1976.
[20] E.G. Manes, A class of fuzzy theories, Journal of Mathematical Analysis and Applications 85 (1982) 409-451.
[21] E.G. Manes, Monads of sets, in: *Handbook of Algebra, Volume 3*, Elsevier, 2003, pp. 67-153.
[22] E.G. Manes, Monads in topology, Topology and its Applications 157 (2010) 961-989.
[23] M.C. Pedicchio, W. Tholen, Multiplicative structures over sup-lattices, Archivum Mathematicum 25 (1989) 107-114.
[24] A. Pultr, Fuzzy mappings and fuzzy sets, Commentationes Mathematicae Universitatis Carolinae 17 (1976) 441-459.
[25] A. Pultr, On categories over the closed categories of fuzzy sets, in: Abstracta. 4th Winter School on Abstract Analysis, Praha, Czechoslovak Academy of Sciences, 1976, pp. 47-63.
[26] E. Riehl, *Category Theory in Context*, Dover Publications, 2016.
[27] K.I. Rosenthal, *Quantales and Their Applications*, Longman, 1990.
[28] L.N. Stout, The logic of unbalanced subobjects in a category with two closed structures, in: *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publishers, 1992, pp.73-106.
[29] I. Stubbe, Categorical structures enriched in a quantaloid: categories, distributors and functors, Theory and Applications of Categories 14 (2005) 1-45.
[30] I. Stubbe, Categorical structures enriched in a quantaloid: tensored and cotensored categories, Theory and Applications of Categories 16 (2006) 283-306.