Running of the Spectral Index and Inflationary Dynamics of $F(R)$ Gravity

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In this work we shall provide a model-independent general calculation of the running of the spectral index for vacuum $F(R)$ gravities. We shall exploit the functional form of the spectral index and of the tensor-to-scalar ratio in order to present a general $n_s - r$ relation for vacuum $F(R)$ gravity theories. As we show, viable $F(R)$ gravity theories can be classified to two classes of models regarding their prediction for the running spectral index. The $R^2$-attractor models predict a running of the spectral index in the range $-10^{-3} < a_s < -10^{-4}$, which classifies them in the same universality class that most inflationary scalar field models belong to. We provide three models of this sort, for which we verify our claims in detail. However there exist viable $F(R)$ gravity models with running of spectral index outside the range $-10^{-3} < a_s < -10^{-4}$ and in some cases it can be positive. We also present an $R^2$-corrected scalar field model, which also predicts a running of the spectral index in the range $-10^{-3} < a_s < -10^{-4}$. For all the cases we studied, we found no evidence for the most phenomenologically interesting scenario of having $r < 10^{-4}$ and a running $a_s < -10^{-3}$, which in principle could be realized.

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I. INTRODUCTION

The next two decades are expected to be the most sensational for theoretical cosmology, theoretical physics and theoretical astrophysics. Many experiments are expected to finally commence, which will probe the primordial Universe, and are expected to shed light to mysterious hypothetical eras, such as the inflationary era and the primordial dark ages that start from the reheating era and beyond. There are a variety of experiments with the first being the stage four Cosmic Microwave Background (CMB) experiments, such as the CMB-S4 [1] and the Simons observatory [2]. These experiments will probe directly the $B$-modes of inflation, or will put tighter constraints on the inflationary parameters. Apart from these two experiments, in the 2030’s decade several interferometers and other experiments [3-10] will seek for a stochastic primordial gravitational wave background.

It is apparent that inflation will be put into strong tests and will be scrutinized to a great extent during the next two decades. Inflation is an appealing hypothetical scenario for the primordial era, which solves all the problems of the standard Big Bang cosmology [11-14]. However it is possible that no primordial gravitational waves signal is discovered nor a direct $B$-mode is detected by the experiments. This will put inflation to a severely difficult position, and it is highly likely for this to occur, if inflation is controlled solely by a scalar field or some simple modified gravity, such as $f(R)$ gravity. This is because both the aforementioned theories yield an undetectable prediction for the energy spectrum of the primordial gravitational waves, unless the reheating era is abnormal, in which case a signal will be detected for the aforementioned theories too. Of course, there exist a large variety of modified gravity models that can accommodate the observational data [13-15], but in the case of complete absence of future observations, things will be quite difficult for cosmologists. In view of this pessimistic but nevertheless possible scenario, it is vital to have a backup plan looking for inflation in other parameters than the CMB constraints. One parameter with profound importance is the running of the spectral index $a_s$. Most of the inflationary single scalar field models predict that $a_s$ is in the range $-10^{-3} < a_s < -10^{-4}$. Thus if the running of the spectral index is ever discovered in this range, it is highly likely that the underlying inflationary theory might be a single scalar field theory. The single scalar field descriptions of inflation form a universality class in which many models belong. In fact, in some cases it is possible to detect a signal for the running of the spectral index in the next generation of CMB experiments, without even detecting the $B$-modes of inflation directly (thus without knowing the tensor-to-scalar ratio) [20]. This scenario also corresponds to another universality class of single scalar field models [20].

In view of the importance of the running of the spectral index for future experiments, in this article we shall calculate the most general form of the running of the spectral index for vacuum $F(R)$ gravity in the Jordan frame [21-46]. Using a model independent approach, which will enable us to quantify in simple parameters the viability of the $F(R)$ gravity, we shall calculate the running of the spectral index for a general vacuum $F(R)$ gravity. The viability constraint of the vacuum $F(R)$ gravity will constrain appropriately the possible values of the running of the spectral index. As we shall see, there are two classes of viable vacuum $F(R)$ gravity theories, one is the $R^2$ attractors and the other class deviates from this class significantly. As we shall demonstrate, the $R^2$ attractors yield a running of the spectral index in the range $-10^{-3} < a_s < -10^{-4}$, thus vacuum $F(R)$ gravity theories also belong to
the universality class that the most inflationary scalar field models belong too. We exemplify this important class of models by using three characteristic $F(R)$ gravity models, and we explicitly calculate the running of the spectral index for these models to further support our model-independent approach. Also the viable $F(R)$ gravities which deviate from the $R^2$-attractors do not belong to the aforementioned universality class, and for these theories the predictions for the running is larger than $|a_s| > 10^{-3}$ and in some cases it can be positive. Thus with this work we aim to point out the fact that some classes of $F(R)$ gravity models are indistinguishable from single scalar field models. However there exist viable $F(R)$ models that deviate from the single scalar field inflationary universality class. We also present in brief an $R^2$-corrected scalar field theory, and we calculate the running of the spectral index in this class of theories. $R^2$ corrections in scalar theories are frequently studied in the literature and are highly motivated because the effective inflationary Lagrangian described by a single scalar field might have $R^2$-corrections. As we shall demonstrate, it is possible for $R^2$-corrected scalar field models to belong to the universality class of the inflationary single scalar field models.

This paper is organized as follows: In section II we present the essential features of vacuum $F(R)$ gravity theory. We calculate the spectral index of the primordial curvature perturbations and the tensor-to-scalar ratio and we provide a model independent universal $n_s - r$ relation for $F(R)$ gravity. A thorough study of the running of the spectral index, along with explicit examples is presented too. In section III we perform the same analysis for $R^2$-corrected single scalar field theories and the conclusions are presented in the last section of the article.

For the purposes of this article, we shall assume that the geometric background is that of a flat Friedmann-Robertson-Walker metric, with line element,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2,$$

where $a(t)$ is the scale factor.

## II. THE VACUUM $F(R)$ GRAVITY INFLATIONARY DYNAMICS AND THE RUNNING OF THE SPECTRAL INDEX

As we mentioned in the introduction, one of our aims in this article is to investigate whether $F(R)$ gravity produces a running of the spectral index $a_s$ of the primordial scalar perturbations in the range $-10^{-3} < a_s < -10^{-4}$. Many viable scalar field theories produce a running of the spectral index belonging in the above range, so these basically constitute a characteristic universality class of models. As we will show in this section, using a model independent approach, some classes of $F(R)$ gravity models belong in the same universality class as the scalar models, however, it is possible to have viable $F(R)$ gravity models which deviate from the universality class.

To start off, we consider a vacuum $F(R)$ gravity theory with its gravitational action being,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R),$$

where $\kappa^2 = 8\pi G = \frac{1}{\sqrt{\rho}}$ and with $M_p$ denoting the reduced Planck mass. In the context of the metric formalism, upon varying the gravitational action with respect to the metric, we obtain the following field equations,

$$F_R(R) R_{\mu\nu} (g) - \frac{1}{2} F(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F_R(R) + g_{\mu\nu} \Box F_R(R) = 0,$$

with $F_R = \frac{dF}{dR}$. Eq. (3) can be rewritten as follows,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{\kappa^2}{F_R(R)} \left( T_{\mu\nu} + \frac{1}{\kappa^2} \left( \frac{F(R) - R F_R(R)}{2} g_{\mu\nu} + \nabla_\mu \nabla_\nu F_R(R) - g_{\mu\nu} \Box F_R(R) \right) \right).$$

Using the FRW metric $[\text{II}]$, the field equations take the following form,

$$0 = - \frac{F(R)}{2} + 3 \left( H^2 + \dot{H} \right) F_R(R) - 18 \left( 4H^2 \dot{H} + H \ddot{H} \right) F_{RR}(R),$$

$$0 = \frac{F(R)}{2} - \left( \dot{H} + 3H^2 \right) F_R(R) + 6 \left( 8H^2 \dot{H} + 4H \ddot{H} + 6H \dot{H} + \dddot{H} \right) F_{RR}(R) + 36 \left( 4H \dot{H} + \dddot{H} \right)^2 F_{RRR},$$

where $F_{RR} = \frac{d^2 F}{dR^2}$, and $F_{RRR} = \frac{d^3 F}{dR^3}$. Moreover, $H$ denotes as usual the Hubble rate $H = \dot{a}/a$ and the Ricci scalar for the FRW metric $[\text{II}]$ is $\mathcal{R} = 12H^2 + 6\dot{H}$. 

In order to reveal the functional form of the running of the spectral index for vacuum $F(R)$ theories in a model-independent way, one needs to find the general form of the $n_s - r$ relation, where $n_s$ and $r$ are the spectral index of the scalar primordial perturbations and $r$ is the tensor-to-scalar ratio. This analysis was performed in Ref. [57], however for the sake of self-completeness, we present here in brief the general $n_s - r$ relation for vacuum $F(R)$ gravity.

We shall assume that the slow-roll approximations hold true,

$$\ddot{H} \ll H\dot{H}, \quad \frac{\dddot{H}}{H^2} \ll 1. \quad (7)$$

For vacuum $F(R)$ gravity, the slow-roll indices are defined as follows [15, 58, 59],

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_2 = 0, \quad \epsilon_3 = \frac{\dot{F}_R}{2HF_R}, \quad \epsilon_4 = \frac{\dot{F}_R}{HF_R}. \quad (8)$$

Accordingly, the spectral index and the tensor-to-scalar ratio for a general $F(R)$ theory read [15, 58, 59],

$$n_s = 1 - \frac{4\epsilon_1 - 2\epsilon_3 + 2\epsilon_4}{1 - \epsilon_1}, \quad r = 48 \frac{\epsilon_3^2}{(1 + \epsilon_3)^2}. \quad (9)$$

We can further simplify the tensor-to-scalar ratio for vacuum $F(R)$ gravity by using the Raychaudhuri equation,

$$\epsilon_1 = -\epsilon_3 (1 - \epsilon_4), \quad (10)$$

hence by using the slow-roll assumptions, we have $\epsilon_1 \simeq -\epsilon_3$, and in effect, the spectral index becomes,

$$n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4, \quad (11)$$

and accordingly the tensor-to-scalar ratio becomes $r \simeq 48\epsilon_3^2$, and in conjunction with the fact that $\epsilon_1 \simeq -\epsilon_3$, we finally have,

$$r \simeq 48\epsilon_1^2. \quad (12)$$

The slow-roll index $\epsilon_4$ is the most complicated and the most interesting one. We have,

$$\epsilon_4 = \frac{\dot{F}_R}{HF_R} = \frac{\dot{R}}{HF_{RRR}} \frac{F_{RRR}}{H} \frac{\dot{F}_R}{H_F_{RRR}} = \frac{F_{RRR}\dot{R}^2 + F_{RR}d(\dot{R})}{HF_{RR}\dot{R}}, \quad (13)$$

and due to the slow-roll assumptions, $\dot{R}$ is,

$$\dot{R} = 24\dot{H} \dot{H} + 6\ddot{H} \simeq 24\dot{H} \dot{H} = -2AH^3. \quad (14)$$

Combining Eqs. (14) and (13) we obtain,

$$\epsilon_4 \simeq -\frac{24F_{RRR}H^2}{F_{RR}} \epsilon_1 - 3\epsilon_1 + \frac{\dot{\epsilon}_1}{H\epsilon_1}, \quad (15)$$

however $\dot{\epsilon}_1$ is,

$$\dot{\epsilon}_1 = \frac{\ddot{H}H^2 - 2\dot{H}^2 H}{H^4} = -\frac{\ddot{H}}{H^2} + \frac{2\dot{H}^2}{H^3} \simeq 2H\epsilon_1^2, \quad (16)$$

hence $\epsilon_4$ reads,

$$\epsilon_4 \simeq -\frac{24F_{RRR}H^2}{F_{RR}} \epsilon_1 - \epsilon_1. \quad (17)$$

Upon introducing the dimensionless parameter $x$, defined in the following way,

$$x = \frac{48F_{RRR}H^2}{F_{RR}}, \quad (18)$$
the parameter $\epsilon_4$ takes the final form,

$$\epsilon_4 \simeq -\frac{x}{2}\epsilon_1 - \epsilon_1.$$  (19)

Hence, upon substituting $\epsilon_4$ from Eq. (19) in Eq. (11), we have,

$$n_s - 1 = -4\epsilon_1 + x\epsilon_1,$$  (20)

and we can solve the above equation with respect to $\epsilon_1$ to obtain,

$$\epsilon_1 = \frac{1 - n_s}{4 - x}.$$  (21)

Upon combining Eqs. (21) and (12), we obtain,

$$r \simeq \frac{48(1 - n_s)^2}{(4 - x)^2}.$$  (22)

The above relation is a universal relation holding true for all the vacuum $F(R)$ gravity models. It is crucial to note that the viability of an $F(R)$ gravity model depends on the values of the parameter $x$ at the first horizon crossing of a mode during the inflationary era, since $x$ affects both the values of $r$ via Eq. (22) and of the spectral index via Eq. (20).

At this point, let us derive the running of the spectral index for general vacuum $F(R)$ gravity. The running of the spectral index is defined as,

$$a_s = \frac{d n_s}{d \ln k},$$  (23)

where $k$ is the comoving wavenumber of a primordial mode. We can rewrite $a_s$ as follows,

$$a_s = \frac{d n_s}{d \ln k} = \frac{d n_s}{d N} \frac{d N}{d \ln k},$$  (24)

where $N$ is the e-foldings number, and by using $\frac{d N}{d \ln k} = \frac{1}{1 - \epsilon_1}$, the final expression for the running of the spectral index $a_s$ is,

$$a_s = \frac{1}{1 - \epsilon_1} \frac{d n_s}{d N}.$$  (25)

The above relation is general and holds for any $F(R, \phi)$ model, but let us specify our analysis focusing on vacuum $F(R)$ gravity, in which case we have,

$$\frac{d n_s}{d N} = \frac{d \epsilon_1}{d N} (-4 + x) + \frac{d x}{d N} \epsilon_1.$$  (26)

The term $\frac{d x}{d N}$ for vacuum $F(R)$ gravity is,

$$\frac{d \epsilon_1}{d N} = \frac{\dot{\epsilon}_1}{H},$$  (27)

so by using (16) we finally get,

$$\frac{d \epsilon_1}{d N} = 2\epsilon_1^2.$$  (28)

Now let us calculate $\frac{d x}{d N}$ and after some algebra we obtain,

$$\frac{d x}{d N} = -1152\frac{F_{RRRR}}{F_{RR}} H^4 - 2x\epsilon_1 + \frac{x^2}{2}\epsilon_1,$$  (29)

where $F_{RRRR} = \frac{\partial^4 F}{\partial R^4}$. Thus the term $\frac{d n_s}{d N}$ related to the running of the spectral index for a general vacuum $F(R)$ gravity is equal to,

$$\frac{d n_s}{d N} = 2\epsilon_1^2 (x - 4) - 1152\frac{F_{RRRR}}{F_{RR}} H^4 - 2x\epsilon_1 + \frac{x^2}{2}\epsilon_1.$$  (30)
The above expression for the running of the spectral index has to be evaluated at the first horizon crossing, and crucially depends on the values of $x$ and on the values of $\frac{F_{\text{Planck}}}{H^4}$. Clearly the values of $x$ at horizon crossing determine the viability of the $F(R)$ gravity model and the $F(R)$ gravity models can be distinguished in attractors depending the value of $x$. One important class of $F(R)$ gravity models are the $R^2$ attractors, which can be obtained in the case that $x = 0$ or $x \ll 1$. The case $x = 0$ corresponds to the $R^2$ gravity and for both the $x = 0$ and $x \ll 1$ cases, the universal relation (22) becomes $n \simeq 3(1 - n_s)^2$. Also the term $\frac{F_{\text{Planck}}}{H^4}$ for the $R^2$ model, but also when $x \ll 1$, the term $\frac{F_{\text{Planck}}}{H^4}$ has also to satisfy $\frac{F_{\text{Planck}}}{H^4} < 1$ or to be zero in order to have consistency with the condition $x \ll 1$. In effect, all the $R^2$ attractor models ($x = 0$ or $x \ll 1$) yield a running of the spectral index which has the following approximate form at leading order,

$$a_s \simeq -\frac{8\epsilon_1^2}{1 - \epsilon_1}.$$  (31)

Since the $R^2$ attractor $F(R)$ gravity models are viable, this means that their first slow-roll index $\epsilon_1$ satisfies the Planck 2018 constraints [60] which constrains $\epsilon_1 < 0.0097$. Hence for the $R^2$ attractor models, this means that the running of the spectral index is of the order $a_s \sim -7.44 \times 10^{-4}$. The current Planck constraints on the scale dependence of the spectral index are $a_s = -0.006 \pm 0.0013$ thus all the $R^2$ attractor models are viable as expected. More importantly, the $R^2$ attractors belong to the same universality class that many scalar models belong to, which predict $-10^{-3} < a_s < -10^{-4}$ [21, 61]. Let us consider a brief three characteristic examples of this category, with the first being the pure $R^2$ model [62, 63], and the second being a deformation of the pure $R^2$ model, namely, models (32) and (33) yield both $a_s \sim -0.007035$ with $x \sim O(10^{-29})$ and $\frac{F_{\text{Planck}}}{H^4} \sim 10^{-28}$ at horizon crossing. Finally, the model (32) yields $a_s \sim -0.0007035$ with $x \sim O(10^{-29})$ and $\frac{F_{\text{Planck}}}{H^4} \sim 10^{-28}$ at horizon crossing. Thus all the models indeed fall in the category of the $R^2$ attractors and predict the same running for the spectral index as the universality classed for the scalar models do, in the range $-10^{-3} < a_s < -10^{-4}$.

Now let us proceed to the cases that $x$ is not small. In this case there are three distinct possible scenarios, firstly $0 < x \leq 1$, secondly $x \sim O(4)$ and thirdly $x > 1$ and $x \neq 4$ or larger, with $x$ being evaluated at the first horizon crossing for $N \sim 60$. The last two scenarios do not correspond to viable $F(R)$ gravities because the former leads to a blow-up in the tensor-to-scalar ratio, while the latter although yielding a small value for the tensor-to-scalar ratio, it yields a large scalar index $n_s$. Hence the only case that is worth studying is $x \sim O(1)$. In this case, the model can be viable and yield $n_s$ and $r$ compatible with the Planck constraints. Indeed, the presence of the term $x^2$ in the spectral index [21], can affect the spectral index even if $0 < x \leq 1$. On the contrary, the running of the spectral index contains the term $\sim x^2\epsilon_1$, hence if $\epsilon_1 = 0.0097$ which is the largest value allowed from the Planck 2018 data [60], with $0 < x \leq 1$, the term $x^2\epsilon_1$ will be of the order $x^2\epsilon_1 \approx (0, 0.006208)$ (values larger than 0.007 in total for the running of the spectral index are excluded by the Planck 2018 data). Thus the running of the spectral index is dramatically affected in this case, since the running can be positive and large, and certainly this class of $F(R)$ gravity models do not fall into the same universality class that the most of the viable inflationary scalar models and the $R^2$ attractors belong to. However, although this class of models certainly exists theoretically, we have no model that belongs to this class to present, due to the lack of analyticity. Only a few $F(R)$ gravity models can be worked out analytically, unless the Hubble rate is given. The same conclusions can be obtained if $-1 \geq x < 0$, thus this case too is covered by the conclusions obtained for the case $0 < x \leq 1$. Also notice that the case $-1 \geq x < 0$ yields the same running for the spectral index as the $0 < x \leq 1$, with different spectral index though.
Thus in this section we basically demonstrated that there exist two classes of viable $F(R)$ gravity models that can be classified according to their predictions for the running of the spectral index. The first class contains all $R^2$ attractor models with $x = 0$ or $x \ll 1$, while the second class contains models with $0 < x \leq 1$. The attractor models belong to the universality class that most of the known inflationary scalar models belong to, while the other viable $F(R)$ models constitute a class of their own, with possible significant effects on the running of the spectral index. None however belongs to the class presented in Ref. [20], where for a small $r$ with $r \leq 10^{-4}$, then $a_s < -10^{-3}$. If the small $r$ and $a_s < -10^{-3}$ combination is verified experimentally, and specifically if the $a_s < -10^{-3}$ case is verified experimentally, this can be in favor of inflation and of specific inflationary theories which have too small tensor-to-scalar ratio to be detected. In the next section we shall also present an $f(R, \phi)$ gravity which has a large tensor-to-scalar ratio and also satisfies $-10^{-3} < a_s < -10^{-4}$. As we show, that model too belongs to the universality class of the scalar field inflationary models, and also deviates from the class of models with $r \leq 10^{-4}$, and $a_s < -10^{-3}$. This further supports the claim of [20] that if a measurement occurs with $a_s < -10^{-3}$ and no hint for tensor perturbations is found, then this will indicate that a small $r$ scalar theory might be controlling inflation.

III. THE RUNNING OF THE SPECTRAL INDEX IN $f(R, \phi)$ GRAVITY: $R^2$ QUANTUM-CORRECTED SCALAR FIELD THEORY

In this section we shall consider an $f(R, \phi)$ gravity instead of an vacuum $F(R)$ gravity which we considered in the previous section. Specifically, this $f(R, \phi)$ gravity will be an $R^2$ quantum corrected scalar field theory. We shall be interested in investigating whether this class of theories also belongs to the universality classes that most of the scalar inflationary theories belong, regarding their predictions on the running of the spectral index. The full analysis of the $R^2$ quantum corrected scalar field theory is developed elsewhere [65], here we are mainly interested on the running of the spectral index.

Scalar field theories are of fundamental importance since scalar fields are natural predictions of string theory, which is to date the most appealing UV completion of classical gravitation and particle physics. Although string theory is rather impossible to be verified experimentally on terrestrial accelerators, it might be possible to find its imprints at the effective inflationary Lagrangian. Consider the most general scalar Lagrangian,

$$S_\varphi = \int d^4x \sqrt{-g} \left( \frac{1}{2} Z(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \mathcal{V}(\varphi) + h(\varphi) R \right).$$

The vacuum configuration of scalar fields compels the scalar fields to be either minimally or conformally coupled [66]. The quantum corrections of a minimally coupled or of a conformally coupled scalar field Lagrangian can be of the form [66],

$$S_{\text{eff}} = \int d^4x \sqrt{-g} \left( \Lambda_1 + \Lambda_2 R + \Lambda_3 R^2 + \Lambda_4 R_{\mu\nu} R^{\mu\nu} + \Lambda_5 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \Lambda_6 \Box R 
+ \Lambda_7 R \Box R + \Lambda_8 R_{\mu\nu} \Box R^{\mu\nu} + \Lambda_9 R^3 + \mathcal{O}(\partial^8) + ... \right),$$

which contains up to fourth order derivative and is diffeomorphically invariant. We shall consider the $R^2$ corrections in this paper, so the effective Lagrangian is of the form,

$$S = \int d^4x \sqrt{-g} \left( \frac{f(R)}{2\kappa^2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \mathcal{V}(\varphi) \right),$$

where,

$$f(R) = R + \frac{R^2}{36M^2},$$

where $M$ is a mass scale which in the context of $R^2$-corrected scalar field theory is a free parameter, in contrast to the vacuum $R^2$ model [62, 63]. For the FRW metric, the field equations read,

$$3 f_R \dot{\mathcal{H}}^2 = \frac{R f_R - f}{2} - 3 \mathcal{H} f_R + \kappa^2 \left( \frac{1}{2} \dot{\varphi}^2 + \mathcal{V}(\varphi) \right),$$

$$- 2 f_R \dot{\mathcal{H}} = \kappa^2 \dot{\varphi}^2 + \dot{f}_R - \mathcal{H} f_R,$$
\[ \dot{\phi} + 3H \dot{\phi} + V' = 0, \quad (40) \]

with the “dot” and the “prime” denoting differentiation with respect to the cosmic time and the scalar field respectively, while \( f_R = \frac{\partial f}{\partial R} \). By assuming the slow-roll conditions,

\[ \dot{H} \ll H^2, \quad \ddot{H} \ll H\dot{H}, \quad (41) \]

and also the following approximations,

\[ \frac{\dot{H}^2}{M^2} \ll H^2, \quad \frac{\dot{H}^2}{M^2} \ll V(\phi), \quad (42) \]

the field equations at leading order read (the details are given in [65]):

\[ H^2 \simeq \frac{\kappa^2 V(\phi)}{3} + \mathcal{O}(\frac{\kappa^2 \dot{\phi}^2}{2} H^2), \quad (43) \]

\[ \dot{H} \simeq -\frac{\kappa^2 \dot{\phi}^2}{2} - \frac{\kappa^4 \dot{\phi}^4}{4M^2}. \quad (44) \]

Furthermore the approximate forms of the slow-roll indices for the \( R^2 \)-corrected scalar theory are [65],

\[ \epsilon_1 = \frac{1}{2\kappa^2} \left( \left( \frac{V'}{V} \right)^2 + \frac{1}{6M^2} \left( \frac{V'}{V} \right)^2 \frac{V''}{V} \right). \quad (45) \]

\[ \epsilon_2 = -\frac{V''}{\kappa^2 V} + \epsilon_1, \quad (46) \]

\[ \epsilon_3 = \frac{\epsilon_1}{1 - \frac{3M^2}{2\kappa^2} + \epsilon_1}, \quad (47) \]

while \( \epsilon_4 \) is \( \epsilon_4 = \frac{\dot{E}}{2H\dot{E}} \), with,

\[ E = 1 + \frac{2R}{36M^2} + \frac{8}{3\kappa^2 M^4} \frac{H^2 \dot{H}^2}{\dot{\phi}^2}, \quad (48) \]

\[ \dot{E} = \frac{4H\dot{H}}{3M^2} + \frac{16}{3\kappa^2 M^4 \dot{\phi}^2} (H H^3 \dot{\phi}^2 - H^2 \dot{H}^2 \dot{\phi} \ddot{\phi}), \quad (49) \]

and we omit the final form of \( \epsilon_4 \) for brevity. Having the slow-roll indices, one can easily confront the theory with the Planck observations. The spectral index for the \( f(R, \phi) \) theory is the same as in the vacuum \( F(R) \) gravity of the previous section, however, the tensor-to-scalar ratio is different at leading order in the slow-roll indices [15, 58],

\[ r = 16(\epsilon_1 + \epsilon_3). \quad (50) \]

Let us choose a simple potential which can be shown [65] that it yields a viable phenomenology,

\[ V(\phi) = \frac{\mathcal{V}_0}{\kappa^4} (\kappa \phi)^2, \quad (51) \]

with \( \mathcal{V}_0 \) being a dimensionless parameter. We set \( M = \beta/\kappa \) where \( \beta \) is dimensionless. This theory is a viable one, and the viability is guaranteed when \( \mathcal{V}_0 \sim \mathcal{O}(10^{-13}) \) and \( \beta \sim \mathcal{O}(10^{-6}) \). Full details on the inflationary phenomenology of this model is given in [65] and one set of parameters that yield viability for this model is, \( \mathcal{V}_0 = 9.37 \times 10^{-13}, \beta = 6.8 \times 10^{-6} \) and \( N = 60 \), for which we get,

\[ n_S = 0.96611, \quad r = 0.063968. \quad (52) \]

For the same set of values for the free parameters we obtain \( a_s = -0.00056 \), which belongs in the range \( -10^{-3} < a_s < -10^{-3} \). This model, as all the viable \( F(R) \) gravity models, belong to the same universality class of inflationary scalar potentials. Hence, this result further supports the argument of Ref. [21], which states that if a measurement of the running of the spectral index is found with \( a_s < -10^{-3} \) and no sign of tensor perturbations is found, this will indicate that probably a scalar inflationary theory controls the dynamics of inflation, with characteristics different than the scalar models which constitute the universality class. Such a future observation will shed further light on the \( a_s - r \) relation.
IV. CONCLUSIONS

In this paper we provided a model independent theoretical framework for vacuum $F(R)$ gravity in order to predict the most general form of the running of the spectral index for these theories. Exploiting the functional form of the spectral index and of the tensor-to-scalar ratio for vacuum $F(R)$ gravity, we presented a quite general $n_s - r$ relation holding true for all vacuum $F(R)$ gravities. The viable $F(R)$ gravities can be classified in two distinct classes of models, which are characterized by small or $O(1)$ values for the parameter $x \sim \frac{\dot{H}}{H^2}$ when it is evaluated at the first horizon crossing during inflation. The small $x$ values belong to the $R^2$-attractor models, which have a running spectral index which takes values $-10^{-3} < x < -10^{-4}$. This feature classifies the $R^2$-attractor models in the same universality class that most inflationary single scalar field models belong to. We presented three models, which belong to this class, all of which were deformations of the $R^2$ model. We calculated in detail the running of the spectral index and we verified that the result was compatible with the model independent approach we used for this class of models. In contrast, the viable $F(R)$ gravity models with $x \sim O(1)$ predict a running of the spectral index which does not belong to the range $-10^{-3} < a_s < -10^{-4}$ and it is highly likely that it is also positive. However, we did not provide any example for this class of $F(R)$ gravity models, which must exist theoretically, however the lack of analyticity prevented us from finding an example of this sort. Finally, we also provided another theory which belongs to the universality class of the inflationary scalar models, namely an $R^2$-corrected single scalar field theory. In this case we used an explicit example for the calculation of the running of the spectral index, due to the perplexity of the theoretical framework. As we demonstrated, this theory too predicts $-10^{-3} < a_s < -10^{-4}$. Notably, in all the cases we studied, we found no evidence for models which predict $r < 10^{-4}$ and $a_s < -10^{-3}$, a class of theories pointed out in Ref. 20. This would be a particularly interesting scenario, but it seems that it is not easy to find analytically a vacuum $F(R)$ gravity which can realize such a scenario. It is thus worth investigating whether this scenario can be realized in non-minimally coupled theories or in mimetic scalar field theories or even in generalized $F(R, \phi)$ theories. We hope to address this issue in the near future.

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