Non-Hausdorff groupoids, proper actions and $K$-theory

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Abstract. Let $G$ be a (not necessarily Hausdorff) locally compact groupoid. We introduce a notion of properness for $G$, which is invariant under Morita-equivalence. We show that any generalized morphism between two locally compact groupoids which satisfies some properness conditions induces a $C^*$-correspondence from $C^*_r(G_2)$ to $C^*_r(G_1)$, and thus two Morita equivalent groupoids have Morita-equivalent $C^*$-algebras.

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Introduction

Very often, groupoids that appear in geometry, such as holonomy groupoids of foliations, groupoids of inverse semigroups [15, 6] and the indicial algebra of a manifold with corners [10] are not Hausdorff. It is thus necessary to extend various basic notions to this broader setting, such as proper action and Morita equivalence. We also show that a generalized morphism from $G_2$ to $G_1$ satisfying certain properness conditions induces an element of $KK(C^*_r(G_2), C^*_r(G_1))$.

In Section 2 we introduce the notion of proper groupoids and show that it is invariant under Morita-equivalence. Section 3 is a technical part of the paper in which from every locally compact topological space $X$ is canonically constructed a locally compact Hausdorff space $HX$ in which $X$ is (not continuously) embedded. When $G$ is a groupoid (locally compact, with Haar system, such that $G^{(0)}$ is Hausdorff), the closure $X'$ of $G^{(0)}$ in $HG$ is endowed with a continuous action of $G$ and plays an important technical rôle.

In Section 4 we review basic properties of locally compact groupoids with Haar system and technical tools that are used later. In Section 5 we construct, using tools of Section 3 a canonical $C^*_r(G)$-Hilbert module $E(G)$ for every (locally compact...) proper groupoid $G$. If $G^{(0)}/G$ is compact, then there exists a projection $p \in C^*_r(G)$ such that $E(G)$ is isomorphic to $pC^*_r(G)$. The projection $p$ is given by $p(g) = (c(s(g))c(r(g)))^{1/2}$, where $c: G^{(0)} \to \mathbb{R}_+$ is a “cutoff” function (Section 6). Contrary to the Hausdorff
case, the function $c$ is not continuous, but it is the restriction to $G^{(0)}$ of a continuous map $X' \to \mathbb{R}_+$ (see above for the definition of $X'$).

In Section 4 we examine the question of naturality $G \to C^*_r(G)$. Recall that if $f : X \to Y$ is a continuous map between two locally compact spaces, then $f$ induces a map from $C_0(Y)$ to $C_0(X)$ if and only if $f$ is proper. When $G_1$ and $G_2$ are groups, a morphism $f : G_1 \to G_2$ does not induce a map $C^*_r(G_2) \to C^*_r(G_1)$ (when $G_1 \subset G_2$ is an inclusion of discrete groups there is a map in the other direction). When $f : G_1 \to G_2$ is a groupoid morphism, we cannot expect to get more than a $C^*$-correspondence from $C^*_r(G_2)$ to $C^*_r(G_1)$ when $f$ satisfies certain properness assumptions: this was done in the Hausdorff situation by Macho-Stadler and O’Uchi [11, Theorem 2.1], see also [7, 13, 17], but the formulation of their theorem is somewhat complicated. In this paper, as a corollary of Theorem 7.8 we get that (in the Hausdorff situation), if the restriction of $f$ to $(G_1)^K$ is proper for each compact set $K \subset (G_1)^{(0)}$ then $f$ induces a correspondence $\mathcal{E}_f$ from $C^*_r(G_2)$ to $C^*_r(G_1)$. In fact we construct a $C^*$-correspondence out of any groupoid generalized morphism [5, 9] which satisfies some properness conditions. As a corollary, if $G_1$ and $G_2$ are Morita equivalent then $C^*_r(G_1)$ and $C^*_r(G_2)$ are Morita-equivalent $C^*$-algebras.

Finally, let us add that our original motivation was to extend Baum, Connes and Higson’s construction of the assembly map $\mu$ to non-Hausdorff groupoids; however, we couldn’t prove $\mu$ to be an isomorphism in any non-trivial case.

1. Preliminaries

1.1. Groupoids. Throughout, we will assume that the reader is familiar with basic definitions about groupoids (see [16, 15]). If $G$ is a groupoid, we denote by $G^{(0)}$ its set of units and by $r : G \to G^{(0)}$ and $s : G \to G^{(0)}$ its range and source maps respectively. We will use notations such as $G_x = s^{-1}(x)$, $G^y = r^{-1}(y)$, $G^x_y = G_x \cap G^y$. Recall that a topological groupoid is said to be étale if $r$ (and $s$) are local homeomorphisms.

For all sets $X$, $Y$, $T$ and all maps $f : X \to T$ and $g : Y \to T$, we denote by $X \times_{f,g} Y$, or by $X \times_T Y$ if there is no ambiguity, the set $\{(x, y) \in X \times Y | f(x) = g(y)\}$.

Recall that a (right) action of $G$ on a set $Z$ is given by

(a) a ("momentum") map $p : Z \to G^{(0)}$;
(b) a map $Z \times_{p,r} G \to Z$, denoted by $(z, g) \mapsto zg$

with the following properties:

(i) $p(zg) = s(g)$ for all $(z, g) \in Z \times_{p,r} G$;
(ii) $z(gh) = (zg)h$ whenever $p(z) = r(g)$ and $s(g) = r(h)$;
(iii) $zp(z) = z$ for all $z \in Z$.

Then the crossed-product $Z \rtimes G$ is the subgroupoid of $(Z \times Z) \rtimes G$ consisting of elements $(z, z', g)$ such that $z' = zg$. Since the map $Z \times G \to Z \times G$ given by $(z, z', g) \mapsto (z, g)$ is injective, the groupoid $Z \rtimes G$ can also be considered as a subspace of $Z \times G$, and this is what we will do most of the time.
1.2. Locally compact spaces. A topological space $X$ is said to be quasi-compact if every open cover of $X$ admits a finite sub-cover. A space is compact if it is quasi-compact and Hausdorff. Let us recall a few basic facts about locally compact spaces.

**Definition 1.1.** A topological space $X$ is said to be locally compact if every point $x \in X$ has a compact neighborhood.

In particular, $X$ is locally Hausdorff, thus every singleton subset of $X$ is closed. Moreover, the diagonal in $X \times X$ is locally closed.

**Proposition 1.2.** Let $X$ be a locally compact space. Then every locally closed subspace of $X$ is locally compact.

Recall that $A \subset X$ is locally closed if for every $a \in A$, there exists a neighborhood $V$ of $a$ in $X$ such that $V \cap A$ is closed in $V$. Then $A$ is locally closed if and only if it is of the form $U \cap F$, with $U$ open and $F$ closed.

**Proposition 1.3.** Let $X$ be a locally compact space. The following are equivalent:

$(i)$ there exists a sequence $(K_n)$ of compact subspaces such that $X = \bigcup_{n \in \mathbb{N}} K_n$;

$(ii)$ there exists a sequence $(K_n)$ of quasi-compact subspaces such that $X = \bigcup_{n \in \mathbb{N}} K_n$;

$(iii)$ there exists a sequence $(K_n)$ of quasi-compact subspaces such that $X = \bigcup_{n \in \mathbb{N}} K_n$ and $K_n \subset \mathring{K}_{n+1}$ for all $n \in \mathbb{N}$.

Such a space will be called $\sigma$-compact.

**Proof.** $(i) \implies (ii)$ is obvious. The implications $(ii) \implies (iii) \implies (i)$ follow easily from the fact that for every quasi-compact subspace $K$, there exists a finite family $(K_i)_{i \in I}$ of compact sets such that $K \subset \bigcup_{i \in I} K_i$. $\square$

1.3. Proper maps.

**Proposition 1.4.** [2, Théorème I.10.2.1] Let $X$ and $Y$ be two topological spaces, and $f: X \to Y$ a continuous map. The following are equivalent:

$(i)$ For every topological space $Z$, $f \times \text{Id}_Z: X \times Z \to Y \times Z$ is closed;

$(ii)$ $f$ is closed and for every $y \in Y$, $f^{-1}(y)$ is quasi-compact.

A map which satisfies the equivalent properties of Proposition 1.4 is said to be proper.

**Proposition 1.5.** [2, Proposition I.10.2.6] Let $X$ and $Y$ be two topological spaces and let $f: X \to Y$ be a proper map. Then for every quasi-compact subspace $K$ of $Y$, $f^{-1}(K)$ is quasi-compact.

**Proposition 1.6.** Let $X$ and $Y$ be two topological spaces and let $f: X \to Y$ be a continuous map. Suppose $Y$ is locally compact, then the following are equivalent:

$(i)$ $f$ is proper;
(ii) for every quasi-compact subspace $K$ of $Y$, $f^{-1}(K)$ is quasi-compact;
(iii) for every compact subspace $K$ of $Y$, $f^{-1}(K)$ is quasi-compact;
(iv) for every $y \in Y$, there exists a compact neighborhood $K_y$ of $y$ such that $f^{-1}(K_y)$ is quasi-compact.

Proof. (i) $\implies$ (ii) follows from Proposition 1.5. (ii) $\implies$ (iii) $\implies$ (iv) are obvious. Let us show (iv) $\implies$ (i).

Since $f^{-1}(y)$ is closed, it is clear that $f^{-1}(y)$ is quasi-compact for all $y \in Y$. It remains to prove that for every closed subspace $F \subset X$, $f(F)$ is closed. Let $y \in \overline{f(F)}$. Let $A = f^{-1}(K_y)$. Then $A \cap F$ is quasi-compact, so $f(A \cap F)$ is quasi-compact. As $f(A \cap F) \subset K_y$, it is closed in $K_y$, i.e. $K_y \cap f(A \cap F) = K_y \cap f(A \cap F)$. We thus have $y \in K_y \cap \overline{f(A \cap F)} = K_y \cap f(A \cap F) \subset f(F)$. It follows that $f(F)$ is closed. $\Box$

2. Proper groupoids and proper actions

2.1. Locally compact groupoids.

Definition 2.1. A topological groupoid $G$ is said to be locally compact (resp. $\sigma$-compact) if it is locally compact (resp. $\sigma$-compact) as a topological space.

Remark 2.2. The definition of a locally compact groupoid in [15] corresponds to our definition of a locally compact, $\sigma$-compact groupoid with Haar system whose unit space is Hausdorff, thanks to Propositions 1.5 and 1.6.

Example 2.3. Let $\Gamma$ be a discrete group, $H$ a closed normal subgroup and let $G$ be the bundle of groups over $[0,1]$ such that $G_0 = \Gamma$ and $G_t = \Gamma/H$ for all $t > 0$. We endow $G$ with the quotient topology of $([0,1] \times \Gamma)/([0,1] \times H)$. Then $G$ is a non-Hausdorff locally compact groupoid such that $(t,\gamma)$ converges to $(0,\gamma h)$ as $t \to 0$, for all $\gamma \in \Gamma$ and $h \in H$.

Example 2.4. Let $\Gamma$ be a discrete group acting on a locally compact Hausdorff space $X$, and let $G = (X \times \Gamma)/\sim$, where $(x,\gamma)$ and $(x,\gamma')$ are identified if their germs are equal, i.e. there exists a neighborhood $V$ of $x$ such that $y\gamma = y\gamma'$ for all $y \in V$. Then $G$ is locally compact, since the open sets $V_\gamma = \{(x,\gamma) | x \in X\}$ are homeomorphic to $X$ and cover $G$.

Suppose that $X$ is a manifold, $M$ is a manifold such that $\pi_1(M) = \Gamma$, $\tilde{M}$ is the universal cover of $M$ and $V = (X \times \tilde{M})/\Gamma$, then $V$ is foliated by $\{[x,\tilde{m}] | \tilde{m} \in \tilde{M}\}$ and $G$ is the restriction to a transversal of the holonomy groupoid of the above foliation.

Proposition 2.5. If $G$ is a locally compact groupoid, then $G^{(0)}$ is locally closed in $G$, hence locally compact. If furthermore $G$ is $\sigma$-compact, then $G^{(0)}$ is $\sigma$-compact.

Proof. Let $\Delta$ be the diagonal in $G \times G$. Since $G$ is locally Hausdorff, $\Delta$ is locally closed. Then $G^{(0)} = (\text{Id}, r)^{-1}(\Delta)$ is locally closed in $G$.

Suppose that $G = \sqcup_{n \in \mathbb{N}} K_n$ with $K_n$ quasi-compact, then $s(K_n)$ is quasi-compact and $G^{(0)} = \sqcup_{n \in \mathbb{N}} s(K_n)$.
Proposition 2.6. Let $Z$ a locally compact space and $G$ be a locally compact groupoid acting on $Z$. Then the crossed-product $Z \rtimes G$ is locally compact.

Proof. Let $p : Z \to G^{(0)}$ be the momentum map of the action of $G$. From Proposition 2.4, the diagonal $\Delta \subset G^{(0)} \times G^{(0)}$ is locally closed in $G^{(0)} \times G^{(0)}$, hence $Z \rtimes G = (p, r)^{-1}(\Delta)$ is locally closed in $Z \times G$. □

Let $T$ be a space. Recall that there is a groupoid $T \times T$ with unit space $T$, and product $(x, y)(y, z) = (x, z)$.

Let $G$ be a groupoid and $T$ be a space. Let $f : T \to G^{(0)}$, and let $G[T] = \{(t', t, g) \in (T \times T) \times G \mid g \in G_{f(t')}(0)\}$. Then $G[T]$ is a subgroupoid of $(T \times T) \times G$.

Proposition 2.7. Let $G$ be a topological groupoid with $G^{(0)}$ locally Hausdorff, $T$ a topological space and $f : T \to G^{(0)}$ a continuous map. Then $G[T]$ is a locally closed subgroupoid of $(T \times T) \times G$. In particular, if $T$ and $G$ are locally compact, then $G[T]$ is locally compact.

Proof. Let $F \subset T \times G^{(0)}$ be the graph of $f$. Then $F = (f \times \text{Id})^{-1}(\Delta)$, where $\Delta$ is the diagonal in $G^{(0)} \times G^{(0)}$, thus it is locally closed. Let $\rho : (t', t, g) \mapsto (t', r(g))$ and $\sigma : (t', t, g) \mapsto (t, s(g))$ be the range and source maps of $(T \times T) \times G$, then $G[T] = (\rho, \sigma)^{-1}(F \times F)$ is locally closed. □

Proposition 2.8. Let $G$ be a locally compact groupoid such that $G^{(0)}$ is Hausdorff. Then for every $x \in G^{(0)}$, $G_x$ is Hausdorff.

Proof. Let $Z = \{(g, h) \in G_x \times G_x \mid r(g) = r(h)\}$. Let $\varphi : Z \to G$ defined by $\varphi(g, h) = g^{-1}h$. Since $\{x\}$ is closed in $G$, $\varphi^{-1}(x)$ is closed in $Z$, and since $G^{(0)}$ is Hausdorff, $Z$ is closed in $G_x \times G_x$. It follows that $\varphi^{-1}(x)$, which is the diagonal of $G_x \times G_x$, is closed in $G_x \times G_x$. □

2.2. Proper groupoids.

Definition 2.9. A topological groupoid $G$ is said to be proper if $(r, s) : G \to G^{(0)} \times G^{(0)}$ is proper.

Proposition 2.10. Let $G$ be a topological groupoid such that $G^{(0)}$ is locally compact. Consider the following assertions:

(i) $G$ is proper;
(ii) $(r, s)$ is closed and for every $x \in G^{(0)}$, $G_x$ is quasi-compact;
(iii) for all quasi-compact subspaces $K$ and $L$ of $G^{(0)}$, $G^K_L$ is quasi-compact;
(iii)' for all compact subspaces $K$ and $L$ of $G^{(0)}$, $G^K_L$ is quasi-compact;
(iv) for every quasi-compact subspace $K$ of $G^{(0)}$, $G^K$ is quasi-compact;
(v) $\forall x, y \in G^{(0)}, \exists K_x, L_y$ compact neighborhoods of $x$ and $y$ such that $G^{L_y}_{K_x}$ is quasi-compact.

Then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iii)' $\iff$ (v) $\implies$ (iv). If $G^{(0)}$ is Hausdorff, then (i)–(v) are equivalent.
Proposition Let $G$ be a topological groupoid. Let $r: G \to G(0)$ be open then the canonical mapping $\pi: G(0) \to G(0)/G$ is open.

Proof. Let $V \subset G(0)$ be an open subspace. If $r$ is open, then $r(s^{-1}(V)) = \pi^{-1}(\pi(V))$ is open. Therefore, $\pi(V)$ is open. □

Proposition 2.12. Let $G$ be a topological groupoid such that $G(0)$ is locally compact and $r: G \to G(0)$ is open. Suppose that $(r,s)(G)$ is locally closed in $G(0) \times G(0)$, then $G(0)/G$ is locally compact. Furthermore,

(a) if $G(0)$ is $\sigma$-compact, then $G(0)/G$ is $\sigma$-compact;
(b) if $(r,s)(G)$ is closed (for instance if $G$ is proper), then $G(0)/G$ is Hausdorff.

Proof. Let $R = (r,s)(G)$. Let $\pi: G(0) \to G(0)/G$ be the canonical mapping. By Proposition 2.11, $\pi$ is open, therefore $G(0)/G$ is locally quasi-compact. Let us show that it is locally Hausdorff. Let $V$ be an open subspace of $G(0)$ such that $(V \times V) \cap R$ is closed in $V \times V$. Let $\Delta$ be the diagonal in $\pi(V) \times \pi(V)$. Then $(\pi \times \pi)^{-1}(\Delta) = (V \times V) \cap R$ is closed in $V \times V$. Since $\pi \times \pi: V \times V \to \pi(V) \times \pi(V)$ is continuous open surjective, it follows that $\Delta$ is closed in $\pi(V) \times \pi(V)$, hence $\pi(V)$ is Hausdorff. This completes the proof that $G(0)/G$ is locally compact and of assertion (b).

Assertion (a) follows from the fact that for every $x \in G(0)$ and every compact neighborhood $K$ of $x$, $\pi(K)$ is a quasi-compact neighborhood of $\pi(x)$. □

2.3. Proper actions.

Definition 2.13. Let $G$ be a topological groupoid. Let $Z$ be a topological space endowed with an action of $G$. Then the action is said to be proper if $Z \times G$ is a proper groupoid. (We will also say that $Z$ is a proper $G$-space.)

A subspace $A$ of a topological space $X$ is said to be relatively compact (resp. relatively quasi-compact) if it is included in a compact (resp. quasi-compact) subspace of $X$. This does not imply that $\overline{A}$ is compact (resp. quasi-compact).

Proposition 2.14. Let $G$ be a topological groupoid. Let $Z$ be a topological space endowed with an action of $G$. Consider the following assertions:

(i) $G$ acts properly on $Z$;
(ii) $(r,s): Z \times G \to Z \times Z$ is closed and $\forall z \in Z$, the stabilizer of $z$ is quasi-compact;
(iii) for all quasi-compact subspaces $K$ and $L$ of $Z$, $\{ g \in G \mid Lg \cap K \neq \emptyset \}$ is quasi-compact;
(iii)' for all compact subspaces $K$ and $L$ of $Z$, \( \{ g \in G \mid Lg \cap K \neq \emptyset \} \) is quasi-compact;
(iv) for every quasi-compact subspace $K$ of $Z$, \( \{ g \in G \mid Kg \cap K \neq \emptyset \} \) is quasi-compact;
(v) there exists a family \( (A_i)_{i \in I} \) of subspaces of $Z$ such that $Z = \bigcup_{i \in I} A_i$ and \( \{ g \in G \mid A_i g \cap A_j \neq \emptyset \} \) is relatively quasi-compact for all $i, j \in I$.

Then (i) \( \iff \) (ii) \( \iff \) (iii) \( \iff \) (iii)' and (iii) \( \iff \) (iv). If $Z$ is locally compact, then (iii)' \( \implies \) (v) and (iv) \( \implies \) (v). If $G^{(0)}$ is Hausdorff and $Z$ is locally compact Hausdorff, then (i)–(v) are equivalent.

**Proof.** (i) \( \iff \) (ii) follows from Proposition 2.10(i) \( \iff \) (iii). Implication (i) \( \implies \) (iii) follows from the fact that if \( (Z \times G)^K \) is quasi-compact, then its image by the second projection \( Z \times G \to G \) is quasi-compact. (iii) \( \implies \) (iii)' and (iii) \( \implies \) (iv) are obvious.

Suppose that $Z$ is locally compact. Take $A_i \subset Z$ compact such that $Z = \bigcup_{i \in I} A_i$. If (iii)' is true, then \( \{ g \in G \mid A_i g \cap A_j \neq \emptyset \} \) is quasi-compact, hence (v). If (iv) is true, then \( \{ g \in G \mid A_i g \cap A_j \neq \emptyset \} \) is a subset of the quasi-compact set \( \{ g \in G \mid Kg \cap K \neq \emptyset \} \), where $K = A_i \cup A_j$, hence (v).

Suppose that $Z$ is locally compact Hausdorff and that $G^{(0)}$ is Hausdorff. Let us show (v) \( \implies \) (ii). Let $C_{ij}$ be a quasi-compact set such that \( \{ g \in G \mid A_i g \cap A_j \neq \emptyset \} \subset C_{ij}$.

Let $z \in Z$. Choose $i \in I$ such that $z \in A_i$. Since $Z$ and $G^{(0)}$ are Hausdorff, stab($z$) is a closed subspace of $C_{ii}$, therefore it is quasi-compact.

It remains to prove that the map \( \Phi: Z \times_{G^{(0)}} G \to Z \times Z \) given by \( \Phi(z, g) = (z, zg) \) is closed. Let $F \subset Z \times_{G^{(0)}} G$ be a closed subspace, and \( (z, z') \in \Phi(F) \). Choose $i$ and $j$ such that $z \in A_i$ and $z' \in A_j$. Then \( (z, z') \in \Phi(F) \cap (A_i \times A_j) \subset \Phi(F \cap (A_i \times_{G^{(0)}} C_{ij}))) \subset \Phi(F \cap (Z \times_{G^{(0)}} C_{ij})). \)

There exists a net \( (z_\lambda, g_\lambda) \in F \cap (Z \times_{G^{(0)}} C_{ij}) \) such that \( (z, z') \) is a limit point of \( (z_\lambda, z_\lambda g_\lambda) \). Since $C_{ij}$ is quasi-compact, after passing to a universal subnet we may assume that $g_\lambda$ converges to an element $g \in C_{ij}$. Since $G^{(0)}$ is Hausdorff, $F \cap (Z \times_{G^{(0)}} C_{ij})$ is closed in $Z \times C_{ij}$, so \( (z, g) \) is an element of $F \cap (Z \times_{G^{(0)}} C_{ij})$. Using the fact that $Z$ is Hausdorff and $\Phi$ is continuous, we obtain \( (z, z') = \Phi(z, g) \in \Phi(F) \).

**Remark 2.15.** It is possible to define a notion of slice-proper action which implies properness in the above sense. The two notions are equivalent in many cases \[1\, \[3\].

**Proposition 2.16.** Let $G$ be a locally compact groupoid. Then $G$ acts properly on itself if and only if $G^{(0)}$ is Hausdorff. In particular, a locally compact space is proper if and only if it is Hausdorff.

**Proof.** It is clear from Proposition 2.10(ii) that $G$ acts properly on itself if and only if the product $\varphi: G^{(2)} \to G \times G$ is closed. Since $\varphi$ factors through the homeomorphism $G^{(2)} \to G \times_{r,t} G$, $(g, h) \mapsto (g, gh)$, $G$ acts properly on itself if and only if $G \times_{r,t} G$ is a closed subset of $G \times G$. 

If \( G^{(0)} \) is Hausdorff, then clearly \( G \times_{r,r} G \) is closed in \( G \times G \). Conversely, if \( G^{(0)} \) is not Hausdorff, then there exists \((x, y) \in G^{(0)} \times G^{(0)}\) such that \( x \neq y \) and \((x, y)\) is in the closure of the diagonal of \( G^{(0)} \times G^{(0)} \). It follows that \((x, y)\) is in the closure of \( G \times_{r,r} G \), but \((x, y) \not\in G \times_{r,r} G \), therefore \( G \times_{r,r} G \) is not closed.

2.4. Permanence properties.

**Proposition 2.17.** If \( G_1 \) and \( G_2 \) are proper topological groupoids, then \( G_1 \times G_2 \) is proper.

**Proof.** Follows from the fact that the product of two proper maps is proper [2, Corollaire I.10.2.3].

**Proposition 2.18.** Let \( G_1 \) and \( G_2 \) be two topological groupoids such that \( G_1^{(0)} \) is Hausdorff and \( G_2 \) is proper. Suppose that \( f : G_1 \to G_2 \) is a proper morphism. Then \( G_1 \) is proper.

**Proof.** Denote by \( r_1 \) and \( s_1 \) the range and source maps of \( G_1 \) \((i = 1, 2)\). Let \( \bar{f} \) be the map \( G_1^{(0)} \times G_1^{(0)} \to G_2^{(0)} \times G_2^{(0)} \) induced from \( f \). Since \( \bar{f} \circ (r_1, s_1) = (r_2, s_2) \circ f \) is proper and \( G_1^{(0)} \) is Hausdorff, it follows from [2, Proposition I.10.1.5] that \((r_1, s_1)\) is proper.

**Proposition 2.19.** Let \( G_1 \) and \( G_2 \) be two topological groupoids such that \( G_1 \) is proper. Suppose that \( f : G_1 \to G_2 \) is a surjective morphism such that the induced map \( f' : G_1^{(0)} \to G_2^{(0)} \) is proper. Then \( G_2 \) is proper.

**Proof.** Denote by \( r_i \) and \( s_i \) the range and source maps of \( G_i \) \((i = 1, 2)\). Let \( F_2 \subset G_2 \) be a closed subspace, and \( F_1 = f^{-1}(F_2) \). Since \( G_1 \) is proper, \((r_1, s_1)(F_1)\) is closed, and since \( f' \times f' \) is proper, \((f' \times f')(r_1, s_1)(F_1)\) is closed. By surjectivity of \( f \), we have \((r_2, s_2)(F_2) = (f' \times f')(r_1, s_1)(F_1)\). This proves that \((r_2, s_2)\) is closed. Since for every topological space \( T \), the assumptions of the proposition are also true for the morphism \( f \times 1 : G_1 \times T \to G_2 \times T \), the above shows that \((r_2, s_2) \times 1_T\) is closed. Therefore, \((r_2, s_2)\) is proper.

**Proposition 2.20.** Let \( G \) be a topological groupoid with \( G^{(0)} \) Hausdorff, acting on two spaces \( Y \) and \( Z \). Suppose that the action of \( G \) on \( Z \) is proper, and that \( Y \) is Hausdorff. Then \( G \) acts properly on \( Y \times_{G^{(0)}} Z \).

**Proof.** The groupoid \((Y \times_{G^{(0)}} Z) \times G\) is isomorphic to the subgroupoid \( \Gamma = \{(y, y', z, g) \in (Y \times Y) \times (Z \rtimes G) | p(y) = r(g), y' = yg\} \) of the proper groupoid \((Y \times Y) \times (Z \rtimes G)\). Since \( Y \) and \( G^{(0)} \) are Hausdorff, \( \Gamma \) is closed in \((Y \times Y) \times (Z \rtimes G)\), hence by Proposition 2.10(ii), \((Y \times_{G^{(0)}} Z) \times G\) is proper.

**Corollary 2.21.** Let \( G \) be a proper topological groupoid with \( G^{(0)} \) Hausdorff. Then any action of \( G \) on a Hausdorff space is proper.

**Proof.** Follows from Proposition 2.20 with \( Z = G^{(0)} \).
**Proposition 2.22.** Let $G$ be a topological groupoid and $f: T \to G^{(0)}$ be a continuous map.

(a) If $G$ is proper, then $G[T]$ is proper.

(ii) If $G[T]$ is proper and $f$ is open surjective, then $G$ is proper.

**Proof.** Let us prove (a). Suppose first that $T$ is a subspace of $G^{(0)}$ and that $f$ is the inclusion. Then $G[T] = G^{(0)}$. Since $(r_T, s_T)$ is the restriction to $(r, s)^{-1}(T \times T)$ of $(r, s)$, and $(r, s)$ is proper, it follows that $(r_T, s_T)$ is proper.

In the general case, let $\Gamma = (T \times T) \times G$ and let $T' \subset T \times G^{(0)}$ be the graph of $f$.

Then $\Gamma$ is a proper groupoid (since it is the product of two proper groupoids), and $G[T] = \Gamma[T']$.

Let us prove (b). The only difficulty is to show that $(r, s)$ is closed. Let $F \subset G$ be a closed subspace and $(y, x) \in (r, s)(F)$. Let $\tilde{F} = G[T] \cap (T \times T) \times F$. Choose $(t', t) \in T \times T$ such that $f(t') = y$ and $f(t) = x$. Denote by $\tilde{r}$ and $\tilde{s}$ the range and source maps of $G[T]$. Then $(t', t) \in (\tilde{r}, \tilde{s})(\tilde{F})$. Indeed, let $\Omega \ni (t', t)$ be an open set, and $\Omega' = (f \times f)(\Omega)$. Then $\Omega'$ is an open neighborhood of $(y, x)$, so $\Omega' \cap (r, s)(F) \neq \emptyset$. It follows that $\Omega \cap (\tilde{r}, \tilde{s})(\tilde{F}) \neq \emptyset$.

We have proved that $(t', t) \in (\tilde{r}, \tilde{s})(\tilde{F}) = (\tilde{r}, \tilde{s})(\tilde{F})$, so $(y, x) \in (r, s)(F)$. 

**Corollary 2.23.** Let $G$ be a groupoid acting properly on a topological space $Z$, and let $Z_1$ be a saturated subspace. Then $G$ acts properly on $Z_1$.

**Proof.** Use the fact that $Z_1 \times G = (Z \times G)[Z_1]$. 

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**2.5. Invariance by Morita-equivalence.** In this section, we will only consider groupoids whose range maps are open. We thus need a stability lemma:

**Lemma 2.24.** Let $G$ be a topological groupoid whose range map is open. Let $Z$ be a $G$ space and $f: T \to G^{(0)}$ be a continuous open map. Then the range maps for $Z \times G$ and $G[T]$ are open.

To prove Lemma 2.24, we need a preliminary result:

**Lemma 2.25.** Let $X, Y, T$ be topological spaces, $g: Y \to T$ an open map and $f: X \to T$ continuous. Let $Z = X \times_T Y$. Then the first projection $\pi_1: X \times_T Y \to X$ is open.

**Proof.** Let $\Omega \subset Z$ open. There exists an open subspace $\Omega'$ of $X \times Y$ such that $\Omega = \Omega' \cap Z$. Let $\Delta$ be the diagonal in $X \times X$. One easily checks that $(\pi_1, \pi_1)(\Omega) = (1 \times f)^{-1}(1 \times g)(\Omega') \cap \Delta$, therefore $(\pi_1, \pi_1)(\Omega)$ is open in $\Delta$. This implies that $\pi_1(\Omega)$ is open in $X$.

**Proof of Lemma 2.24.** This is clear for $Z \times G = Z \times G^{(0)} G$ using Lemma 2.25. For $G[T]$, first use Lemma 2.25 to prove that $T \times f,s G \xrightarrow{pr_2} G$ is open. Since the range map is open by assumption, the composition $T \times f,s G \xrightarrow{pr_2} G \xrightarrow{\pi_1} G^{(0)}$ is open. Using again Lemma 2.25, $G[T] \simeq T \times f,r \pi_2 (T \times f,s G) \xrightarrow{pr_1} T$ is open. 

$\square$
In order to define the notion of Morita-equivalence for topological groupoids, we introduce some terminology:

**Definition 2.26.** Let $G$ be a topological groupoid. Let $T$ be a topological space and $\rho: G^{(0)} \to T$ be a $G$-invariant map. Then $G$ is said to be $\rho$-proper if the map $(r, s): G \to G^{(0)} \times_T G^{(0)}$ is proper. If $G$ acts on a space $Z$ and $\rho: Z \to T$ is $G$-invariant, then the action is said to be $\rho$-proper if $Z \times G$ is $\rho$-proper.

It is clear that properness implies $\rho$-properness. There is a partial converse:

**Proposition 2.27.** Let $G$ be a topological groupoid, $T$ a topological space, $\rho: G^{(0)} \to T$ a $G$-invariant map. If $G$ is $\rho$-proper and $T$ is Hausdorff, then $G$ is proper.

**Proof.** Since $T$ is Hausdorff, $G^{(0)} \times_T G^{(0)}$ is a closed subspace of $G^{(0)} \times G^{(0)}$, therefore $(r, s)$, being the composition of the two proper maps $G \to G^{(0)} \times G^{(0)}$, is proper. $\Box$

**Remark 2.28.** When $T$ is locally Hausdorff, one easily shows that $G$ is $\rho$-proper iff for every Hausdorff open subspace $V$ of $T$, $G^{\rho^{-1}(V)}$ is proper.

**Proposition 2.29.** [14] Let $G_1$ and $G_2$ be two topological (resp. locally compact) groupoids. Let $r_i, s_i (i = 1, 2)$ be the range and source maps of $G_i$, and suppose that $r_i$ are open. The following are equivalent:

(i) there exist a topological (resp. locally compact) space $T$ and $f_i: T \to G_i^{(0)}$ open surjective such that $G_1[T]$ and $G_2[T]$ are isomorphic;

(ii) there exists a topological (resp. locally compact) space $Z$, two continuous maps $\rho: Z \to G_1^{(0)}$ and $\sigma: Z \to G_2^{(0)}$, a left action of $G_1$ on $Z$ with momentum map $\rho$ and a right action of $G_2$ on $Z$ with momentum map $\sigma$ such that

(a) the actions commute and are free, the action of $G_2$ is $\rho$-proper and the action of $G_1$ is $\sigma$-proper;

(b) the natural maps $Z/G_2 \to G_1^{(0)}$ and $G_1 \backslash Z \to G_2^{(0)}$ induced from $\rho$ and $\sigma$ are homeomorphisms.

Moreover, one may replace (b) by

(b)' $\rho$ and $\sigma$ are open and induce bijections $Z/G_2 \to G_1^{(0)}$ and $G_1 \backslash Z \to G_2^{(0)}$.

In (i), if $T$ is locally compact then it may be assumed Hausdorff.

If $G_1$ and $G_2$ satisfy the equivalent conditions in Proposition 2.29, then they are said to be Morita-equivalent. Note that if $G_i^{(0)}$ are Hausdorff, then by Proposition 2.27 one may replace “$\rho$-proper” and “$\sigma$-proper” by “proper”.

To prove Proposition 2.29, we need preliminary lemmas:

**Lemma 2.30.** Let $G$ be a topological groupoid. The following are equivalent:

(i) $r: G \to G^{(0)}$ is open;

(ii) for every $G$-space $Z$, the canonical mapping $\pi: Z \to Z/G$ is open.
Proof. To show (ii) $\implies$ (i), take $Z = G$: the canonical mapping $\pi: G \to G/G$ is open. Therefore, for every open subspace $U$ of $G$, $r(U) = G^{(0)} \cap \pi^{-1}(\pi(U))$ is open.

Let us show (i) $\implies$ (ii). By Lemma 2.21, the range map $r: Z \times G \to Z$ is open. The conclusion follows from Proposition 2.11. \qed

**Lemma 2.31.** Let $G$ be a topological groupoid such that the range map $r: G \to G^{(0)}$ is open. Let $X$ be a topological space endowed with an action of $G$ and $T$ a topological space. Then the canonical map

$$f: (X \times T)/G \to (X/G) \times T$$

is an isomorphism.

**Proof.** Let $\pi: X \to X/G$ and $\pi': X \times T \to (X \times T)/G$ be the canonical mappings. Since $\pi$ is open (Lemma 2.30), $f \circ \pi' = \pi \times 1$ is open. Since $\pi'$ is continuous surjective, it follows that $f$ is open. \qed

**Lemma 2.32.** Let $G$ be a topological groupoid whose range map is open and $f: Y \to Z$ a proper, $G$-equivariant map between two $G$-spaces. Then the induced map $\bar{f}: Y/G \to Z/G$ is proper.

**Proof.** We first show that $\bar{f}$ is closed. Let $\pi: Y \to Y/G$ and $\pi': Z \to Z/G$ be the canonical mappings. Let $A \subset Y/G$ be a closed subspace. Since $f$ is closed and $\pi$ is continuous, $(\pi')^{-1}(f(A)) = f(\pi^{-1}(A))$ is closed. Therefore, $\bar{f}(A)$ is closed.

Applying this to $f \times 1$, we see that for every topological space $T$, $(Y \times T)/G \to (Z \times T)/G$ is closed. By Lemma 2.31, $\bar{f} \times 1_T$ is closed. \qed

**Lemma 2.33.** Let $G_2$ and $G_3$ be topological groupoids whose range maps are open. Let $Z_1, Z_2$ and $X$ be topological spaces. Suppose there are maps

$$X \xleftarrow{\rho_1} Z_1 \xrightarrow{\sigma_1} G_2^{(0)} \xleftarrow{\rho_2} Z_2 \xrightarrow{\sigma_2} G_3^{(0)},$$

a right action of $G_2$ on $Z_1$ with momentum map $\sigma_1$, such that $\rho_1$ is $G_2$-invariant and the action of $G_2$ is $\rho_1$-proper, a left action of $G_2$ on $Z_2$ with momentum map $\rho_2$ and a right $\rho_2$-proper action of $G_3$ on $Z_2$ with momentum map $\sigma_2$ which commutes with the $G_2$-action.

Then the action of $G_3$ on $Z = Z_1 \times_{G_2} Z_2$ is $\rho_1$-proper.

**Proof.** Let $\varphi: Z_2 \times G_3 \to Z_2 \times_{G_2^{(0)}} Z_2$ be the map $(z_2, \gamma) \mapsto (z_2, z_2 \gamma)$. By assumption, $\varphi$ is proper, therefore $1_{Z_1} \times \varphi$ is proper. Let $F = \{(z_1, z_2, z_2') \in Z_1 \times Z_2 \times Z_2 | \sigma_1(z_1) = \rho_2(z_2) = \rho_2(z_2')\}$. Then $1_{Z_1} \times \varphi: (1 \times \varphi)^{-1}(F) \to F$ is proper, i.e. $Z_1 \times_{G_2^{(0)}} (Z_2 \times G_3) \to Z_1 \times_{G_2^{(0)}} (Z_2 \times G_2^{(0)})$ is proper. By Lemma 2.32 taking the quotient by $G_2$, we get that the map

$$\alpha: Z \times G_3 \to Z_1 \times_{G_2} (Z_2 \times_{G_2^{(0)}} Z_2)$$

defined by $(z_1, z_2, \gamma) \mapsto (z_1, z_2, z_2 \gamma)$ is proper.
By assumption, the map $Z_1 \times G_2 \to Z_1 \times Z_1$ given by $(z_1, g) \mapsto (z_1, z_1g)$ is proper. Endow $Z_1 \times G_2$ with the following right action of $G_2 \times G_2$: $(z_1, g) \cdot (g', g'') = (z_1g', (g'')^{-1}gg'')$. Using again Lemma 2.32, the map

$$\beta: Z_1 \times G_2 \times G_2 \to (Z_1 \times G_2) \times G_2 \times G_2$$

$$\beta = (Z_1 \times G_2) \times G_2 = (Z_1 \times Z_1) \times G_2 \times G_2$$

is proper. By composition, $\beta \circ \alpha: Z \times G_3 \to Z \times Z$ is proper. □

**Proof of Proposition 2.34**

Let us treat the case of topological groupoids. Assertion (b') follows from the fact that the canonical mappings $Z \to Z/G_2$ and $Z \to G_1 \backslash Z$ are open (Lemma 2.33).

Let us first show that (ii) is an equivalence relation. Reflexivity is clear (taking $Z = G$, $\rho = r$, $\sigma = s$), and symmetry is obvious. Suppose that $(Z_1, \rho_1, \sigma_1)$ and $(Z_2, \rho_2, \sigma_2)$ are equivalences between $G_1$ and $G_2$, and $G_2$ and $G_3$ respectively. Let $Z = Z_1 \times G_2$ be the quotient of $Z_1 \times G_2$ by the action $(z_1, z_2, \gamma = (z_1r, \gamma^{-1}z_2)$ of $G_2$. Denote by $\rho: Z \to G_1(0)$ and $\sigma: Z \to G_3(0)$ the maps induced from $\rho_1 \times 1$ and $1 \times \sigma_2$. By Lemma 2.32 the first projection $p_1: Z_1 \times G_2 \to Z_1$ is open, therefore $\rho = \rho_1 \circ p_1$ is open. Similarly, $\sigma$ is open. It remains to show that the actions of $G_3$ and $G_1$ are $\rho$-proper and $\sigma$-proper respectively. For $G_3$, this follows from Lemma 2.33 and the proof for $G_1$ is similar.

This proves that (ii) is an equivalence relation. Now, let us prove that (i) and (ii) are equivalent.

Suppose (ii). Let $\Gamma = G_1 \times Z \times G_2$ and $T = Z$. The maps $\rho: T \to G_1(0)$ and $\sigma: T \to G_3(0)$ are open surjective by assumption. Since $G_1 \times Z \simeq Z \times G_2$ and $Z \times G_2 \simeq Z \times G_1(0)$, we have $G_2[T] = (T \times T) \times G_1(0) \times G_3(0)$, which is $\simeq G_1 \times (Z \times G_2) \times \sigma \circ p_2 \circ \rho$. This implies that $\Gamma$ is an equivalence relation. Let $Z = T \times G$. Let us check that the action of $G$ is $p_1$-proper. Write $Z \times G = \{(t, g, h) \in T \times G, f(t) = r(g) \text{ and } s(g) = r(h)\}$. One needs to check that the map $Z \times G \to (T \times T, G)^2$ defined by $(t, g, h) \mapsto (t, g, t, h)$ is a homeomorphism onto its image. This follows easily from the facts that the diagonal map $T \to T \times T$ and the map $G^{(2)} \to G \times G, (g, h) \mapsto (g, gh)$ are homeomorphisms onto their images.

Let us check that the action of $G[T]$ is $s \circ p_2$-proper. One easily checks that the groupoid $G' = G[T] \times (T \times T, G)$ is isomorphic to a subgroupoid of the trivial groupoid $(T \times T) \times (G \times G)$. It follows that if $r'$ and $s'$ denote the range and source maps of $G'$, the map $(r', s')$ is a homeomorphism of $G'$ onto its image.

Let us now treat the case of locally compact groupoids. In the proof that (ii) is a transitive relation, it just remains to show that $Z$ is locally compact.
Let $U_3$ be a Hausdorff open subspace of $G_3^{(0)}$. We show that $\sigma^{-1}(U_3)$ is locally compact. Replacing $G_2$ by $(G_2)^{(0)}_{U_3}$, we may assume that $G_2$ acts freely and properly on $Z_2$. Let $\Gamma$ be the groupoid $(Z_1 \times_{G_2^{(0)}} Z_2) \rtimes G_2$, and $R = (r,s)(\Gamma) \subset (Z_1 \times_{G_2^{(0)}} Z_2)^2$. Since the action of $G_2$ on $Z_2$ is free and proper, there exists a continuous map $\varphi: Z_2 \times_{G_2^{(0)}} Z_2 \to G_2$ such that $z_2 = \varphi(z_2, z_2')$. Then $R = \{(z_1, z_2, z_1', z_2') \in (Z_1 \times_{G_2^{(0)}} Z_2)^2; z_1' = z_1 \varphi(z_2, z_2')\}$ is locally closed. By Proposition 2.12, $Z = (Z_1 \times_{G_2^{(0)}} Z_2)/G$ is locally compact.

Finally, if (i) holds with $T = \cup_i V_i$ with $V_i$ open Hausdorff, let $T' = \Pi V_i$. It is clear that $G_1[T'] \simeq G_2[T']$. □

Let us examine standard examples of Morita-equivalences:

**Example 2.34.** Let $G$ be a topological groupoid whose range map is open. Let $(U_i)_{i \in I}$ be an open cover of $G^{(0)}$ and $U = \Pi_{i \in I} U_i$. Then $G[U]$ is Morita-equivalent to $G$.

**Example 2.35.** Let $G$ be a topological groupoid, and let $H_1$, $H_2$ be subgroupoids such that the range maps $r_i: H_i \to H_i^{(0)}$ are open. Then $(H_1 \setminus G_{s(H_2)}^{s(H_1)}) \rtimes H_2$ and $H_1 \rtimes (G_{s(H_2)}^{s(H_1)}/H_2)$ are Morita-equivalent.

**Proof.** Take $Z = G_{s(H_2)}^{s(H_1)}$ and let $\rho: Z \to Z/H_2$ and $\sigma: H_1 \setminus Z$ be the canonical mappings. The fact that these maps are open follows from Lemma 2.30. □

The following proposition is an immediate consequence of Proposition 2.22.

**Proposition 2.36.** Let $G$ and $G'$ be two topological groupoids such that the range maps of $G$ and $G'$ are open. Suppose that $G$ and $G'$ are Morita-equivalent. Then $G$ is proper if and only if $G'$ is proper.

**Corollary 2.37.** With the notations of Example 2.34, $G$ is proper if and only if $G[U]$ is proper.

3. A Topological Construction

Let $X$ be a locally compact space. Since $X$ is not necessarily Hausdorff, a filter $^1 \mathcal{F}$ on $X$ may have more than one limit. Let $S$ be the set of limits of a convergent filter $\mathcal{F}$. The goal of this section is to construct a Hausdorff space $\mathcal{H}X$ in which $X$ is (not continuously) embedded, and such that $\mathcal{F}$ converges to $S$ in $\mathcal{H}X$.

3.1. The space $\mathcal{H}X$.

**Lemma 3.1.** Let $X$ be a topological space, and $S \subset X$. The following are equivalent:

(i) for every family $(V_s)_{s \in S}$ of open sets such that $s \in V_s$, and $V_s = X$ except perhaps for finitely many $s$’s, one has $\cap_{s \in S} V_s \neq \emptyset$;

1or a net; we will use indifferently the two equivalent approaches
versely, suppose there exists a singleton subspace of \( V \).

We introduce the notations \( \Omega_V \)'s and \( \Omega^Q \)'s which satisfy the equivalent conditions of Lemma 3.1 and \( \mathcal{H}X = \mathcal{H}X \cup \{ \emptyset \} \).

**Lemma 3.2.** Let \( X \) be a locally Hausdorff space. Then every \( S \in \mathcal{H}X \) is locally finite. More precisely, if \( V \) is a Hausdorff open subspace of \( X \), then \( V \cap S \) has at most one element.

**Proof.** Suppose \( a \neq b \) and \( \{ a, b \} \subset V \cap S \). Then there exist \( V_a, V_b \) open disjoint neighborhoods of \( a \) and \( b \) respectively; this contradicts Lemma 3.1(ii). Suppose that \( X \) is locally compact. We endow \( \hat{H}X \) with a topology. Let us introduce the notations \( \Omega_V = \{ S \in \mathcal{H}X | V \cap S \neq \emptyset \} \) and \( \Omega^Q = \{ S \in \mathcal{H}X | Q \cap S = \emptyset \} \). The topology on \( \hat{H}X \) is generated by the \( \Omega_V \)'s and \( \Omega^Q \)'s (\( V \) open and \( Q \) quasi-compact). More explicitly, a set is open if and only if it is a union of sets of the form \( \Omega^Q_{(V_i)_{i \in I}} = \Omega^Q \cap (\cap_{i \in I} \Omega_{V_i}) \) where \( (V_i)_{i \in I} \) is a finite family of open Hausdorff sets and \( Q \) is quasi-compact.

**Proposition 3.3.** For every locally compact space \( X \), the space \( \hat{H}X \) is Hausdorff.

**Proof.** Suppose \( S \not\subset S' \) and \( S, S' \in \mathcal{H}X \). Let \( s \in S - S' \). Since \( S' \) is locally finite and since every singleton subspace of \( X \) is closed, there exist \( V \) open and \( K \) compact such that \( s \in V \subset K \) and \( K \cap S' = \emptyset \). Then \( \Omega_V \) and \( \Omega_K \) are disjoint neighborhoods of \( S \) and \( S' \) respectively.

For every filter \( \mathcal{F} \) on \( \hat{H}X \), let

\[
L(\mathcal{F}) = \{ a \in X | \forall V \ni a \text{ open, } \Omega_V \in \mathcal{F} \}.
\]

**Lemma 3.4.** Let \( X \) be a locally compact space. Let \( \mathcal{F} \) be a filter on \( \hat{H}X \). Then \( \mathcal{F} \) converges to \( S \in \mathcal{H}X \) if and only if properties (a) and (b) below hold:

(a) \( \forall V \text{ open, } V \cap S \neq \emptyset \Rightarrow \Omega_V \in \mathcal{F} \);
(b) \( \forall Q \text{ quasi-compact, } Q \cap S = \emptyset \Rightarrow \Omega^Q \in \mathcal{F} \).

If \( \mathcal{F} \) is convergent, then \( L(\mathcal{F}) \) is its limit.

**Proof.** The first statement is obvious, since every open set in \( \hat{H}X \) is a union of finite intersections of \( \Omega_V \)'s and \( \Omega^Q \)'s.

Let us prove the second statement. It is clear from (a) that \( S \subset L(\mathcal{F}) \). Conversely, suppose there exists \( a \in L(\mathcal{F}) - S \). Since \( S \) is locally finite and every singleton subspace of \( X \) is closed, there exists a compact neighborhood \( K \) of \( a \) such that \( K \cap S = \emptyset \). Then \( a \in L(\mathcal{F}) \) implies \( \Omega_K \in \mathcal{F} \), and condition (b)
implies $\Omega^K \in \mathcal{F}$, thus $\emptyset = \Omega^K \cap \Omega_K \in \mathcal{F}$, which is impossible: we have proved the reverse inclusion $L(\mathcal{F}) \subset S$. \hfill \Box

**Remark 3.5.** This means that if $S_\lambda \to S$, then $a \in S$ if and only if $\forall \lambda$ there exists $s_\lambda \in S_\lambda$ such that $s_\lambda \to a$.

**Example 3.6.** Consider Example 3.5 with $\Gamma = \mathbb{Z}_2$ and $H = \{0\}$. Then $\mathcal{H}G = G \cup \{S\}$ where $S = \{(0,0), (0,1)\}$. The sequence $(1/n,0) \in G$ converges to $S$ in $\mathcal{H}G$, and $(0,0)$ and $(0,1)$ are two isolated points in $\mathcal{H}G$.

**Proposition 3.7.** Let $X$ be a locally compact space and $K \subset X$ quasi-compact. Then $L = \{S \in \mathcal{H}X \mid S \cap K \neq \emptyset\}$ is compact. The space $\mathcal{H}X$ is locally compact, and it is $\sigma$-compact if $X$ is $\sigma$-compact.

**Proof.** We show that $L$ is compact, and the two remaining assertions follow easily. Let $\mathcal{F}$ be a ultrafilter on $L$. Let $S_0 = L(\mathcal{F})$. Let us show that $S_0 \cap K \neq \emptyset$: for every $S \in L$, choose a point $\varphi(S) \in K \cap S$. By quasi-compactness, $\varphi(\mathcal{F})$ converges to a point $a \in K$, and it is not hard to see that $a \in S_0$.

Let us show $S_0 \in \mathcal{H}X$: let $(V_s) (s \in S_0)$ be a family of open subspaces of $X$ such that $s \in V_s$ for all $s \in S_0$, and $V_s = X$ for every $s \notin S_1$ ($S_1 \subset S_0$ finite).

By definition of $S_0$, $\Omega_{(V_s) \in S_1} = \cap_{s \in S_1} \Omega_{V_s}$ belongs to $\mathcal{F}$, hence it is non-empty. Choose $S \in \Omega_{(V_s) \in S_1}$, then $S \cap V_s \neq \emptyset$ for all $s \in S_1$. By Lemma 3.1(ii), $\cap_{s \in S_1} V_s \neq \emptyset$. This shows that $S_0 \in \mathcal{H}X$.

Now, let us show that $\mathcal{F}$ converges to $S_0$.

- **If** $V$ is open Hausdorff such that $S_0 \in \Omega_V$, then by definition $\Omega_V \in \mathcal{F}$.
- **If** $Q$ is quasi-compact and $S_0 \in \Omega_Q$, then $\Omega_Q \in \mathcal{F}$, otherwise one would have $\{S \in \mathcal{H}X \mid S \cap Q \neq \emptyset\} \in \mathcal{F}$, which would imply as above that $S_0 \cap Q \neq \emptyset$, a contradiction.

From Lemma 3.1 $\mathcal{F}$ converges to $S_0$. \hfill \Box

**Proposition 3.8.** Let $X$ be a locally compact space. Then $\hat{\mathcal{H}}X$ is the one-point compactification of $\mathcal{H}X$.

**Proof.** It suffices to prove that $\hat{\mathcal{H}}X$ is compact. The proof is almost the same as in Proposition 3.7. \hfill \Box

**Remark 3.9.** If $f: X \to Y$ is a continuous map from a locally compact space $X$ to any Hausdorff space $Y$, then $f$ induces a continuous map $\mathcal{H}f: \mathcal{H}X \to \mathcal{H}Y$. Indeed, for every open subspace $V$ of $Y$, $(\mathcal{H}f)^{-1}(V) = \Omega_{f^{-1}(V)}$ is open.

**Proposition 3.10.** Let $G$ be a topological groupoid such that $G^{(0)}$ is Hausdorff, and $r: G \to G^{(0)}$ is open. Let $Z$ be a locally compact space endowed with a continuous action of $G$. Then $\mathcal{H}Z$ is endowed with a continuous action of $G$ which extends the one on $Z$.

**Proof.** Let $p: Z \to G^{(0)}$ such that $G$ acts on $Z$ with momentum map $p$. Since $p$ has a continuous extension $\mathcal{H}p: \mathcal{H}Z \to G^{(0)}$, for all $S \in \mathcal{H}Z$, there exists $x \in G^{(0)}$ such that $S \subset p^{-1}(x)$. For all $g \in G^*$, write $Sg = \{sg \mid s \in S\}$.
Let us show that $Sg \in \mathcal{H}Z$. Let $V_s$ ($s \in S$) be open sets such that $sg \in V_s$. By continuity, there exist open sets $W_s \ni s$ and $W_g \ni g$ such that for all $(z,h) \in W_s \times G(0) W_g$, $zh \in V_s$. Let $V'_s = W_s \cap p^{-1}(r(W_g))$. Then $V'_s$ is an open neighborhood of $s$, so there exists $z \in \cap_{s \in S} V'_s$. Since $p(z) = r(h)$, there exists $h \in W_g$ such that $p(z) = r(h)$. It follows that $zh \in \cap_{s \in S} V_s$. This shows that $Sg \in \mathcal{H}Z$.

Let us show that the action defined above is continuous. Let $\Phi: \mathcal{H}Z \times_{G(0)} G \to \mathcal{H}Z$ be the action of $G$ on $\mathcal{H}Z$. Suppose that $(S_\lambda, g_\lambda) \to (S, g)$ and let $S' = L((S_\lambda, g_\lambda))$. Then for all $a \in S$ there exists $s_\lambda \in S_\lambda$ such that $s_\lambda \to a$. This implies $s_\lambda g_\lambda \to ag$, thus $ag \in S'$. The converse may be proved in a similar fashion, hence $Sg = S'$.

Applying this to any universal net $(S_\lambda, g_\lambda)$ converging to $(S, g)$ and knowing from Proposition 3.11 that $\Phi(S_\lambda, g_\lambda)$ is convergent in $\mathcal{H}Z$, we find that $\Phi(S_\lambda, g_\lambda)$ converges to $\Phi(S, g)$. This shows that $\Phi$ is continuous in $(S, g)$.

3.2. THE SPACE $\mathcal{H}'X$. Let $X$ be a locally compact space. Let $\Omega'_V = \{S \in \mathcal{H}X \mid S \subset V\}$. Let $\mathcal{H}'X$ be $\mathcal{H}X$ as a set, with the coarsest topology such that the identity map $\mathcal{H}'X \to \mathcal{H}X$ is continuous, and $\Omega'_V$ is open for every relatively quasi-compact open set $V$. The space $\mathcal{H}'X$ is Hausdorff since $\mathcal{H}X$ is Hausdorff, but it is usually not locally compact.

**Lemma 3.11.** Let $X$ be a locally compact space. Then the map 
$$
\mathcal{H}'X \to \mathbb{N}^* \cup \{\infty\}, \quad S \mapsto \#S
$$
is upper semi-continuous.

**Proof.** Let $S \in \mathcal{H}'X$ such that $\#S < \infty$. Let $V_s$ ($s \in S$) be open relatively compact Hausdorff sets such that $s \in V_s$, and let $W = \cup_{s \in S} V_s$. Then $S' \in \mathcal{H}'X$ implies $\#(S' \cap V_s) \leq 1$, therefore $S' \in \Omega'_W$ implies $\#S' \leq \#S$. □

**Proposition 3.12.** Let $X$ be a locally compact space such that the closure of every quasi-compact subspace is quasi-compact. Then

(a) the natural map $\mathcal{H}'X \to \mathcal{H}X$ is a homeomorphism,

(b) for every compact subspace $K \subset X$, there exists $C_K > 0$ such that
$$
\forall S \in \mathcal{H}X, \ S \cap K \neq \emptyset \implies \#S \leq C_K.
$$

(c) If $G$ is a locally compact proper groupoid with $G(0)$ Hausdorff then $G$ satisfies the above properties.

**Proof.** To prove (b), let $K_1$ be a quasi-compact neighborhood of $K$ and let $K' = \overline{K_1}$. Let $a \in K \cap S$ and suppose there exists $b \in S - K'$. Then $K_1$ and $X - K'$ are disjoint neighborhoods of $a$ and $b$ respectively, which is impossible. We deduce that $S \subset K'$.

Now, let $(V_i)_{i \in I}$ be a finite cover of $K'$ by open Hausdorff sets. For all $b \in S$, let $I_b = \{i \in I \mid b \in V_i\}$. By Lemma 3.2, the $I_b$’s ($b \in S$) are disjoint, whence one may take $C_K = \#I$. 


To prove (a), denote by $\Delta \subset X \times X$ the diagonal. Let us first show that $pr_1: \Delta \to X \times X$ is proper.

Let $K \subset X$ compact. Let $L \subset X$ quasi-compact such that $K \subset \hat{L}$. If $(a, b) \in \Delta \cap (K \times X)$, then $b \in \overline{L}$; otherwise, $L \times L^c$ would be a neighborhood of $(a, b)$ whose intersection with $\Delta$ is empty. Therefore, $pr_1^{-1}(K) = \Delta \cap (K \times \overline{L})$ is quasi-compact, which shows that $pr_1$ is proper.

It remains to prove that $\Omega'_V$ is open in $\mathcal{H}X$ for every relatively quasi-compact open set $V \subset X$. Let $S \in \Omega'_V$, $a \in S$ and $K$ a compact neighborhood of $a$. Let $L = pr_2(\Delta \cap (K \times X))$. Then $Q = L - V$ is quasi-compact, and $S \in \Omega'_K \subset \Omega'_V$, therefore $\Omega'_V$ is a neighborhood of each of its points.

To prove (c), let $K \subset G$ be a quasi-compact subspace. Then $L = r(K) \cup s(K)$ is quasi-compact, thus $G^L_K$ is also quasi-compact. But $\overline{K}$ is closed and $\overline{K} \subset G^L_K$, therefore $\overline{K}$ is quasi-compact.  

4. HAAR SYSTEMS

4.1. THE SPACE $C_c(X)$. For every locally compact space $X$, $C_c(X)_0$ will denote the set of functions $f \in C_c(V)$ ($V$ open Hausdorff), extended by 0 outside $V$. Let $C_c(X)$ be the linear span of $C_c(X)_0$. Note that functions in $C_c(X)$ are not necessarily continuous.

Proposition 4.1. Let $X$ be a locally compact space, and let $f: X \to \mathbb{C}$. The following are equivalent:

(i) $f \in C_c(X)$;

(ii) $f^{-1}(\mathbb{C}^*)$ is relatively quasi-compact, and for every filter $\mathcal{F}$ on $X$, let $\tilde{\mathcal{F}} = i(\mathcal{F})$, where $i: X \to \mathcal{H}X$ is the canonical inclusion; if $\tilde{\mathcal{F}}$ converges to $S \in \mathcal{H}X$, then $\lim_{\mathcal{F}} f = \sum_{s \in S} f(s)$.

Proof. Let us show (i) $\implies$ (ii). By linearity, it is enough to consider the case $f \in C_c(V)$, where $V \subset X$ is open Hausdorff. Let $K$ be the compact set $\overline{f^{-1}(\mathbb{C}^*)} \cap V$. Then $f^{-1}(\mathbb{C}^*) \subset K$. Let $\mathcal{F}$ and $S$ as in (ii). If $S \cap V = 0$, then $S \in \Omega^K$, hence $\Omega^K \in \mathcal{F}$, i.e. $X - K \in \mathcal{F}$. Therefore, $\lim_{\mathcal{F}} f = 0 = \sum_{s \in S} f(s)$. If $S \cap V \neq \emptyset$, then $a$ is a limit point of $\mathcal{F}$, therefore $\lim_{\mathcal{F}} f = f(a) = \sum_{s \in S} f(s)$.

Let us show (ii) $\implies$ (i) by induction on $n \in \mathbb{N}^*$ such that there exist $V_1, \ldots, V_n$ open Hausdorff and $K$ quasi-compact satisfying $f^{-1}(\mathbb{C}^*) \subset K \subset V_1 \cup \cdots \cup V_n$. For $n = 1$, for every $x \in V_1$, let $\mathcal{F}$ be a ultrafilter convergent to $x$. By Proposition 3.8, $\tilde{\mathcal{F}}$ is convergent; let $S$ be its limit, then $\lim_{\mathcal{F}} f = \sum_{s \in S} f(s) = f(x)$, thus $f_{|x}$ is continuous.

Now assume the implication is true for $n - 1$ ($n \geq 2$) and let us prove it for $n$. Since $K$ is quasi-compact, there exist $V'_1, \ldots, V'_n$ open sets, $K_1, \ldots, K_n$ compact such that $K \subset V'_1 \cup \cdots \cup V'_n$ and $V'_i \subset K_i \subset V_i$. Let $F = (V'_1 \cup \cdots \cup V'_n) - (V'_1 \cup \cdots \cup V'_{n-1})$. Then $F$ is closed in $V'_n$ and $f_{|F}$ is continuous. Moreover, $f_{|F} = 0$ outside $K' = K - (V'_1 \cup \cdots \cup V'_{n-1})$ which is closed in $K$, hence quasi-compact, and Hausdorff, since $K' \subset V'_n$. Therefore, $f_{|F} \in C_c(F)$. It follows that there exists an extension $h \in C_c(V'_n)$ of $f_{|F}$. By considering $f - h$, we
may assume that \( f = 0 \) on \( F \), so \( f = 0 \) outside \( K' = K_1 \cup \cdots \cup K_{n-1} \). But \( K' \subset V_1 \cup \cdots \cup V_{n-1} \), hence by induction hypothesis, \( f \in C_c(X) \). \( \square \)

**Corollary 4.2.** Let \( X \) be a locally compact space, \( f : X \to \mathbb{C} \), \( f_n \in C_c(X) \). Suppose that there exists fixed quasi-compact set \( Q \subset X \) such that \( f_n^{-1}(C^*) \subset Q \) for all \( n \), and \( f_n \) converges uniformly to \( f \). Then \( f \in C_c(X) \).

**Lemma 4.3.** Let \( X \) be a locally compact space. Let \( (U_i)_{i \in I} \) be an open cover of \( X \) by Hausdorff subspaces. Then every \( f \in C_c(X) \) is a finite sum \( f = \sum f_i \), where \( f_i \in C_c(U_i) \).

**Proof.** See [6, Lemma 1.3]. \( \square \)

**Lemma 4.4.** Let \( X \) and \( Y \) be locally compact spaces. Let \( f \in C_c(X \times Y) \). Let \( V \) and \( W \) be open subspaces of \( X \) and \( Y \) such that \( f^{-1}(C^*) \subset Q \subset V \times W \) for some quasi-compact set \( Q \). Then there exists a sequence \( f_n \in C_c(V) \otimes C_c(W) \) such that \( \lim_{n \to \infty} \| f - f_n \|_\infty = 0 \).

**Proof.** We may assume that \( X = V \) and \( Y = W \). Let \((U_i)\) (resp. \((V_j)\)) be an open cover of \( X \) (resp. \( Y \)) by Hausdorff subspaces. Then every element of \( C_c(X \times Y) \) is a linear combination of elements of \( C_c(U_i \times V_j) \) (Lemma 4.3). The conclusion follows from the fact that the image of \( C_c(U_i) \otimes C_c(V_j) \to C_c(U_i \times V_j) \) is dense. \( \square \)

**Lemma 4.5.** Let \( X \) be a locally compact space and \( Y \subset X \) a closed subspace. Then the restriction map \( C_c(X) \to C_c(Y) \) is well-defined and surjective.

**Proof.** Let \((U_i)_{i \in I} \) be a cover of \( X \) by Hausdorff open subspaces. The map \( C_c(U_i) \to C_c(U_i \cap Y) \) is surjective (since \( Y \) is closed), and \( \oplus_{i \in I} C_c(U_i \cap Y) \to C_c(Y) \) is surjective (Lemma 4.3). Therefore, the map \( \oplus_{i \in I} C_c(U_i) \to C_c(Y) \) is surjective. Since it is also the composition of the surjective map \( \oplus_{i \in I} C_c(U_i) \to C_c(X) \) and of the restriction map \( C_c(X) \to C_c(Y) \), the conclusion follows. \( \square \)

### 4.2. Haar Systems

Let \( G \) be a locally compact proper groupoid with Haar system (see definition below) such that \( G(0) \) is Hausdorff. If \( G \) is Hausdorff, then \( C_c(G(0)) \) is endowed with the \( C^*_r(G) \)-valued scalar product \( \langle \xi, \eta \rangle(g) = \xi(r(g)) \eta(s(g)) \). Its completion is a \( C^*_r(G) \)-Hilbert module. However, if \( G \) is not Hausdorff, the function \( g \mapsto \xi(r(g)) \eta(s(g)) \) does not necessarily belong to \( C_c(G) \), therefore we need a different construction in order to obtain a \( C^*_r(G) \)-module.

**Definition 4.6.** [16, pp. 16-17] Let \( G \) be a locally compact groupoid such that \( G^* \) is Hausdorff for every \( x \in G(0) \). A Haar system is a family of positive measures \( \lambda = \{ \lambda^x \mid x \in G(0) \} \) such that \( \forall x, y \in G(0), \forall \varphi \in C_c(G) \):

(i) \( \text{supp}(\lambda^x) = G^* \);
(ii) \( \lambda(\varphi) : x \mapsto \int_{g \in G^*} \varphi(g) \lambda^x(g) \, dg \in C_c(G(0)) \);
(iii) \( \int_{h \in G^*} \varphi(gh) \lambda^x(h) \, dh = \int_{h \in G^*} \varphi(h) \lambda^y(h) \, dh \).

Note that \( G^* \) is automatically Hausdorff if \( G(0) \) is Hausdorff (Prop. 2.8). Recall also [15, p. 36] that the range map for \( G \) is open.
**Lemma 4.7.** Let $G$ be a locally compact groupoid with Haar system. Then for every quasi-compact subspace $K$ of $G$, \( \sup_{x \in G^{(0)}} \lambda^x(K \cap G^x) < \infty \).

*Proof.* It is easy to show that there exists $f \in C_c(G)$ such that $1_K \leq f$. Since\( \sup_{x \in G^{(0)}} \lambda(f)(x) < \infty \), the conclusion follows. \(\square\)

**Lemma 4.8.** Let $G$ be a locally compact groupoid with Haar system such that $G^{(0)}$ is Hausdorff. Suppose that $Z$ is a locally compact space and that $p : Z \to G^{(0)}$ is continuous. Then for every $f \in C_c(Z \times_{p,r} G)$, $\lambda(f) : z \mapsto \int_{g \in G^{p(z)}} f(z, g) \lambda^g(z)(dg)$ belongs to $C_c(Z)$.

*Proof.* By Lemma 4.5, $f$ is the restriction of an element of $C_c(Z \times G)$.
If $f(z, g) = f_1(z)f_2(g)$, then $\psi(x) = \int_{g \in G^x} f_2(g) \lambda^g(dx)$ belongs to $C_c(G^{(0)})$, therefore $\psi \circ p \in C_c(Z)$. It follows that $\lambda(f) = f_1(\psi \circ p)$ belongs to $C_c(Z)$.
By linearity, if $f \in C_c(Z) \otimes C_c(G)$, then $\lambda(f) \in C_c(Z)$.
Now, for every $f \in C_c(Z \times G)$, there exist relatively quasi-compact open subspaces $V$ and $W$ of $Z$ and $G$, and a sequence $f_n \in C_c(V \otimes C_c(W))$ such that $f_n$ converges uniformly to $f$. From Lemma 4.7, $\lambda(f_n)$ converges uniformly to $\lambda(f)$, and $\lambda(f_n) \in C_c(Z)$. From Corollary 1.2, $\lambda(f) \in C_c(Z)$. \(\square\)

**Proposition 4.9.** Let $G$ be a locally compact groupoid with Haar system such that $G^{(0)}$ is Hausdorff. If $G$ acts on a locally compact space $Z$ with momentum map $p : Z \to G^{(0)}$, then $(\lambda^g)_{g \in G}$ is a Haar system on $Z \times G$.

*Proof.* Results immediately from Lemma 4.8. \(\square\)

5. The Hilbert Module of a Proper Groupoid

5.1. The space $X'$. Before we construct a Hilbert module associated to a proper groupoid, we need some preliminaries. Let $G$ be a locally compact groupoid such that $G^{(0)}$ is Hausdorff. Denote by $X'$ the closure of $G^{(0)}$ in $\mathcal{H}G$.

**Lemma 5.1.** Let $G$ be a locally compact groupoid such that $G^{(0)}$ is Hausdorff. Then for all $S \in X'$, $S$ is a subgroup of $G$.

*Proof.* Since $r$ and $s : G \to G^{(0)}$ extend continuously to maps $\mathcal{H}r = \mathcal{H}s$ on $X'$, i.e. $\exists x_0 \in G^{(0)}$, $S \subset G_{x_0}^{Z_0}$. Let $F$ be a filter on $G^{(0)}$ whose limit is $S$. Then $a \in S$ if and only if $a$ is a limit point of $F$. Since for every $x \in G^{(0)}$ we have $x^{-1}x = x$, it follows that for every $a, b \in S$ one has $a^{-1}b \in S$, whence $S$ is a subgroup of $G_{x_0}^{Z_0}$. \(\square\)

Denote by $q : X' \to G^{(0)}$ the map such that $S \subset G_{q(S)}$. The map $q$ is continuous since it is the restriction to $X'$ of $\mathcal{H}r$.

**Lemma 5.2.** Let $G$ be a locally compact proper groupoid such that $G^{(0)}$ is Hausdorff. Let $F$ be a filter on $X'$, convergent to $S$. Suppose that $q(F)$ converges to $S_0 \in X'$. Then $S_0$ is a normal subgroup of $S$, and there exists $\Omega \in F$ such that $\forall S' \in \Omega, S'$ is group-isomorphic to $S/S_0$. In particular, $\{S' \in X' \mid \#S = \#S_0 \#S'\} \in F$. 

**Lemma 5.3.** Let $G$ be a locally compact proper groupoid such that $G^{(0)}$ is Hausdorff. Let $F$ be a filter on $X'$, convergent to $S$. Suppose that $q(F)$ converges to $S_0 \in X'$. Then $S_0$ is a normal subgroup of $S$, and there exists $\Omega \in F$ such that $\forall S' \in \Omega, S'$ is group-isomorphic to $S/S_0$. In particular, $\{S' \in X' \mid \#S = \#S_0 \#S'\} \in F$. 

**Lemma 5.4.** Let $G$ be a locally compact proper groupoid such that $G^{(0)}$ is Hausdorff. Let $F$ be a filter on $X'$, convergent to $S$. Suppose that $q(F)$ converges to $S_0 \in X'$. Then $S_0$ is a normal subgroup of $S$, and there exists $\Omega \in F$ such that $\forall S' \in \Omega, S'$ is group-isomorphic to $S/S_0$. In particular, $\{S' \in X' \mid \#S = \#S_0 \#S'\} \in F$. 

**Lemma 5.5.** Let $G$ be a locally compact proper groupoid such that $G^{(0)}$ is Hausdorff. Let $F$ be a filter on $X'$, convergent to $S$. Suppose that $q(F)$ converges to $S_0 \in X'$. Then $S_0$ is a normal subgroup of $S$, and there exists $\Omega \in F$ such that $\forall S' \in \Omega, S'$ is group-isomorphic to $S/S_0$. In particular, $\{S' \in X' \mid \#S = \#S_0 \#S'\} \in F$.
Proof. Using Proposition 5.12 we see that $S$ is finite. We shall use the notation $\Omega_{(V_s)_{s \in S}} = \Omega_{(V_s)_{s \in S}} \cap \Omega_{1_{s=2}V}$. Let $V_s' \subset V_s$ ($s \in S$) be Hausdorff, open neighborhoods of $s$, chosen small enough so that for some $\Omega \in F$,

(a) $\Omega \subset \Omega_{(V_s')_{s \in S}}$;
(b) $V_s', V_s'_{s_2} \subset V_s_{s_1s_2}, \forall s_1, s_2 \in S$;
(c) $\forall s \in S - S_0, \forall S' \in \Omega, q(S') \notin V$;
(d) $q(\Omega) \subset \Omega_{(V_s')_{s \in S_0}}$.

Let $S' \in \Omega$. Let $\varphi: S \to S'$ such that $\{\varphi(s)\} = S' \cap V_s'$. Then $\varphi$ is well-defined since $S' \cap V_s' \neq \emptyset$ (see (a)) and $V_s'$ is Hausdorff.

If $s_1, s_2 \in S$ then $\varphi(s_i) \in S' \cap V_s'$. By (b), $\varphi(s_1)\varphi(s_2) \in S' \cap V_s_{s_1s_2}$. Since $V_s_{s_1s_2}$ is Hausdorff and also contains $\varphi(s_1s_2) \in S'$, we have $\varphi(s_1s_2) = \varphi(s_1)\varphi(s_2)$.

This shows that $\varphi$ is a group morphism.

The map $\varphi$ is surjective, since $S' \subset \cup_{s \in S} V_s'$ (see (a)). By (c), $\ker(\varphi) \subset S_0$ and by (d), $S_0 \subset \ker(\varphi)$. \hfill \qed

Suppose now that the range map $r: G \to G^{(0)}$ is open. Then $X'$ is endowed with an action of $G$ (Prop. 5.10) defined by $S \cdot g = g^{-1}Sg = \{g^{-1}sg | s \in S\}$.

5.2. Construction of the Hilbert module. Now, let $G$ be a locally compact, proper groupoid. Assume that $G$ is endowed with a Haar system, and that $G^{(0)}$ is Hausdorff. Let

$$E^0 = \{ f \in C_c(X') | f(S) = \sqrt{\#S}f(q(S)) \forall S \in X'\}.$$ 

$(q(S) \in G^{(0)}$ is identified to $\{q(S)\} \in X'\.)$

Define, for all $\xi, \eta \in E^0$ and $f \in C_c(G)$:

$$(\xi f)(S) = \int_{g \in G^{(0)}} \xi(g^{-1}Sg)f(g^{-1})\lambda^x(dg).$$

Proposition 5.3. With the above assumptions, the completion $E(G)$ of $E^0$ with respect to the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$ is a $C^*_r(G)$-Hilbert module.

We won’t give the direct proof here since this is a particular case of Theorem 6.8 (see Example 7.7(c)).

6. Cutoff functions

If $G$ is a locally compact Hausdorff proper groupoid with Haar system. Assume for simplicity that $G^{(0)}/G$ is compact. Then there exists a so-called “cutoff” function $c \in C_c(G^{(0)})_+$ such that for every $x \in G^{(0)}$, $\int_{g \in G^{(0)}} c(s(g))\lambda^x(dg) = 1$, and the function $g \mapsto \sqrt{c(r(g))c(s(g))}$ defines projection in $C^*_r(G)$. However, if $G$ is not Hausdorff, then the above function does not belong to $C_c(G)$ in general, thus we need another definition of a cutoff function.

Let $X_{\geq k}' = \{ S \in X' | \#S \geq k\}$. By Lemma 5.14, $X_{\geq k}'$ is closed.

Lemma 6.1. Let $G$ be a locally compact, proper groupoid with $G^{(0)}$ Hausdorff. Let $X_{\geq k} = q(X_{\geq k}')$. Then $X_{\geq k}$ is closed in $G^{(0)}$.
Proof. It suffices to show that for every compact subspace $K$ of $G^{(0)}$, $X_{\geq k} \cap K$ is closed. Let $K' = G^K$. Then $K'$ is quasi-compact, and from Proposition 3.7 $K'' = \{ S \in \mathcal{H}G | S \cap K' \neq \emptyset \}$ is compact. The set $q^{-1}(K) \cap X'_{\geq k} = K'' \cap X'_{\geq k}$ is closed in $K''$, hence compact; its image by $q$ is $X_{\geq k} \cap K$. \hfill \Box

Lemma 6.2. Let $G$ be a locally compact, proper groupoid, with $G^{(0)}$ Hausdorff. Let $(\alpha) \in \mathbb{R}$. For every compact set $K \subset G^{(0)}$, there exists $f: X'_K \to \mathbb{R}^*_+$ continuous, where $X'_K = q^{-1}(K) \subset X'$, such that

$$\forall S \in X'_K, \quad f(S) = f(q(S))(\#S)^\alpha.$$ 

Proof. Let $K' = G^K$. It is closed and quasi-compact. From Proposition 3.7 $X'_K$ is quasi-compact. For every $S \in X'_K$, we have $S \subset K'$. By Proposition 5.12 there exists $n \in \mathbb{N}^*$ such that $X'_{\geq n+1} \cap X'_K = \emptyset$. We can thus proceed by reverse induction: suppose constructed $f_{k+1}: X'_K \cap q^{-1}(X'_{\geq k+1}) \to \mathbb{R}^*_+$ continuous such that $f_{k+1}(S) = f_{k+1}(q(S))(\#S)^\alpha$ for all $S \in X'_K \cap q^{-1}(X'_{\geq k+1})$.

Since $X'_K \cap q^{-1}(X'_{\geq k+1})$ is closed in the compact set $X'_K \cap q^{-1}(X'_{\geq k})$, there exists a continuous extension $h: X'_K \cap q^{-1}(X'_{\geq k}) \to \mathbb{R}$ of $f_{k+1}$. Replacing $h(x)$ by $\sup(h(x), \inf f_{k+1})$, we may assume that $h(X'_K \cap q^{-1}(X'_{\geq k})) \subset \mathbb{R}^*_+$. Put $f_k(S) = h(q(S))(\#S)^\alpha$. Let us show that $f_k$ is continuous.

Let $\mathcal{F}$ be a ultrafilter on $X'_K \cap q^{-1}(X'_{\geq k})$, and let $S$ be its limit. Since $q(\mathcal{F})$ is a ultrafilter on $K$, it has a limit $S_0 \in X'_K$. For every $S_1 \in q^{-1}(X'_{\geq k})$, choose $\psi(S_1) \in X'_K$ such that $q(S_1) = q(\psi(S_1))$. Let $S' \in X'_K \cap X'_{\geq k}$ be the limit of $\psi(\mathcal{F})$.

From Lemma 5.2 $\Omega_1 = \{ S_1 \in X'_K \cap q^{-1}(X'_{\geq k}) | \#S = \#S_0 \#S_1 \}$ is an element of $\mathcal{F}$, and $\Omega_2 = \{ S_2 \in X'_{\geq k} | \#S' = \#S_0 \#S_2 \}$ is an element of $\psi(\mathcal{F})$.

- If $\#S_0 > 1$, then $S' \in X'_{\geq k+1}$, so $S$ and $S_0$ belong to $q^{-1}(X'_{\geq k+1})$. Therefore, $f_k(S_1) = (\#S_1)^\alpha h(q(S_1))$ converges with respect to $\mathcal{F}$ to

$$\frac{(\#S)^\alpha}{(\#S_0)^\alpha} h(S_0) = \frac{(\#S)^\alpha}{(\#S_0)^\alpha} f_{k+1}(S_0) = f_{k+1}(S) = f_{k+1}(q(S))(\#S)^\alpha = h(q(S))(\#S)^\alpha = f_k(S).$$

- If $S_0 = \{ q(S) \}$, then $f_k(S_1) = (\#S_1)^\alpha h(q(S_1))$ converges with respect to $\mathcal{F}$ to $(\#S)^\alpha h(q(S)) = f_k(S)$.

Therefore, $f_k$ is a continuous extension of $f_{k+1}$. \hfill \Box

Theorem 6.3. Let $G$ be a locally compact, proper groupoid such that $G^{(0)}$ is Hausdorff and $G^{(0)}/G$ is $\sigma$-compact. Let $\pi: G^{(0)} \to G^{(0)}/G$ be the canonical mapping. Then there exists $c: X' \to \mathbb{R}^*_+$ continuous such that

(a) $c(S) = c(q(S)) \#S$ for all $S \in X'$;
(b) $\forall \alpha \in G^{(0)}/G$, $\exists x \in \pi^{-1}(\alpha)$, $c(x) \neq 0$;
(c) $\forall K \subset G^{(0)}$ compact, $\text{supp}(c) \cap q^{-1}(F)$ is compact, where $F = s(G^K)$.

If moreover $G$ admits a Haar system, then there exists $c: X' \to \mathbb{R}^*_+$ continuous satisfying (a), (b), (c) and

(d) $\forall x \in G^{(0)}$, $\int_{g \in G^x} c(s(g)) \lambda^x(g) = 1$. 

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Proof. There exists a locally finite cover \((V_i)\) of \(G(0)/G\) by relatively compact open subspaces. Since \(\pi\) is open and \(G(0)\) is locally compact, there exists \(K_i \subset G(0)\) compact such that \(\pi(K_i) \supset V_i\). Let \((\varphi_i)\) be a partition of unity associated to the cover \((V_i)\). For every \(i\), from Lemma \ref{lem:partition} there exists \(c_i: X'_{K_i} \to \mathbb{R}^*_+\) continuous such that \(c_i(S) = c_i(q(S))\#S\) for all \(S \in X'_{K_i}\). Let
\[
c(S) = \sum_i c_i(S) \varphi_i(\pi(q(S))).
\]

It is clear that \(c\) is continuous from \(X'\) to \(\mathbb{R}^*_+\), and that \(c(S) = c(q(S))\#S\).

Let us prove (b): let \(x_0 \in G(0)\). There exists \(i\) such that \(\varphi_i(\pi(x_0)) \neq 0\). Choose \(x \in K_i\) such that \(\pi(x) = \pi(x_0)\), then \(c(x) \geq c_i(x) \varphi_i(\pi(x_0)) > 0\).

Let us show (c). Note that \(F = \pi^{-1}(\pi(K))\) is closed, so \(q^{-1}(F)\) is closed.

Let \(K_1\) be a compact neighborhood of \(K\) and \(F_1 = \pi^{-1}(\pi(K_1))\). Let \(J = \{i| V_i \cap \pi(K_1) \neq \emptyset\}\). Then for all \(i \notin J\), \(c_i(\varphi_i \circ \pi \circ q) = 0\) on \(q^{-1}(F_1)\), therefore \(c = \sum_{j \in J} c_j(\varphi_j \circ \pi \circ q)\) in a neighborhood of \(q^{-1}(F)\). Since for all \(i\), \(\text{supp}(c_i(\varphi_i \circ \pi \circ q))\) is compact and since \(J\) is finite, \(\text{supp}(c) \cap q^{-1}(F) \subset \bigcup_{i \in J} \text{supp}(c_i(\varphi_i \circ \pi \circ q))\) is compact.

Let us show the last assertion. Let \(\varphi(g) = c(s(g))\). Let \(\mathcal{F}\) be a filter on \(G\) convergent in \(\mathcal{H}G\) to \(A \subset G\). Choose \(a \in A\) and let \(S = a^{-1}A\). Then \(s(\mathcal{F})\) converges to \(S\) in \(\mathcal{H}G\), hence
\[
\lim_{\mathcal{F}} \varphi = \#S c(s(a)) = \sum_{g \in S} c(s(g)) = \sum_{g \in S} \varphi(g).
\]

For every compact set \(K \subset G(0)\),
\[
\begin{align*}
\{g \in G | r(g) \in K \text{ and } \varphi(g) \neq 0\} & \subset \{g \in G | r(g) \in K \text{ and } s(g) \in \text{supp}(c)\} \\
& \subset G^K_{q(s(g))},
\end{align*}
\]
so \(G^K \cap \{g \in G | \varphi(g) \neq 0\}\) is included in a quasi-compact set. Therefore, for every \(l \in C_c(G(0))\), \(g \mapsto l(r(g)) \varphi(g)\) belongs to \(C_c(G)\). It follows that \(h(x) = \int_{g \in G} \varphi(g) \lambda^x(dg)\) is a continuous function. Moreover, for every \(x \in G(0)\) there exists \(g \in G^x\) such that \(\varphi(g) \neq 0\), so \(h(x) > 0 \forall x \in G(0)\). It thus suffices to replace \(c(x)\) by \(c(x)/h(x)\). 

\[\qed\]

Example 6.4. In Example \ref{ex:locally_compact} with \(\Gamma = \mathbb{Z}_n\) and \(H = \{0\}\), the cutoff function is the unique continuous extension to \(X'\) of the function \(c(x) = 1\) for \(x \in (0,1]\), and \(c(0) = 1/n\).

Proposition 6.5. Let \(G\) be a locally compact, proper, groupoid with Haar system such that \(G(0)\) is Hausdorff and \(G(0)/G\) is compact. Let \(c\) be a cutoff function. Then the function \(p(g) = \sqrt{c(r(g))c(s(g))}\) defines a selfadjoint projection \(p \in C_r^*(G)\), and \(E(G)\) is isomorphic to \(pC_r^*(G)\).
Proof. Let $\xi_0(x) = \sqrt{c(x)}$. Then one easily checks that $\xi_0 \in \mathcal{E}^0$, $(\xi_0, \xi_0) = p$ and $\xi_0(\xi_0, \xi_0) = \xi_0$, therefore $p$ is a selfadjoint projection in $C^*_r(G)$. The maps

$$E(G) \rightarrow pC^*_r(G), \quad \xi \mapsto (\xi_0, \xi) = p(\xi_0, \xi)$$

$$pC^*_r(G) \rightarrow E(G), \quad a \mapsto \xi_0 a = \xi_0 pa$$

are inverses from each other. □

7. Generalized morphisms and $C^*$-algebra correspondences

Until the end of the paper, all groupoids are assumed locally compact, with open range map. In this section, we introduce a notion of generalized morphism for locally compact groupoids which are not necessarily Hausdorff, and a notion of locally proper generalized morphism.

Then, we show that a locally proper generalized morphism from $G_1$ to $G_2$ which satisfies an additional condition induces a $C^*_r(G_1)$-module $E$ and a $*$-morphism $C^*_r(G_2) \rightarrow K(E)$, hence an element of $KK(C^*_r(G_2), C^*_r(G_1))$.

7.1. Generalized morphisms.

Definition 7.1. Let $G_1$ and $G_2$ be two groupoids. A generalized morphism from $G_1$ to $G_2$ is a triple $(Z, \rho, \sigma)$ where

$$G_1^{(0)} \xleftarrow{\rho} Z \xrightarrow{\sigma} G_2^{(0)},$$

$Z$ is endowed with a left action of $G_1$ with momentum map $\rho$ and a right action of $G_2$ with momentum map $\sigma$ which commute, such that

(a) the action of $G_2$ is free and $\rho$-proper,

(b) $\rho$ induces a homeomorphism $Z/G_2 \simeq G_1^{(0)}$.

In Definition 7.1, one may replace (b) by (b)' or (b)" below:

(b)' $\rho$ is open and induces a bijection $Z/G_2 \rightarrow G_1^{(0)}$.

(b)" the map $Z \times G_2 \rightarrow Z \times G_1^{(0)}$ defined by $(z, \gamma) \mapsto (z, z\gamma)$ is a homeomorphism.

Example 7.2. Let $G_1$ and $G_2$ be two groupoids. If $f : G_1 \rightarrow G_2$ is a groupoid morphism, let $Z = G_1^{(0)} \times_{f,r} G_2$, $\rho(x, \gamma) = x$ and $\sigma(x, \gamma) = s(\gamma)$. Define the actions of $G_1$ and $G_2$ by $g : (x, \gamma) \cdot \gamma' = (r(g), f(g)\gamma' \gamma)$. Then $(Z, \rho, \sigma)$ is a generalized morphism from $G_1$ to $G_2$.

That $\rho$ is open follows from the fact that the range map $G_2 \rightarrow G_2^{(0)}$ is open and from Lemma 2.26. The other properties in Definition 7.1 are easy to check.

7.2. Locally proper generalized morphisms.

Definition 7.3. Let $G_1$ and $G_2$ be two groupoids. A generalized morphism from $G_1$ to $G_2$ is said to be locally proper if the action of $G_1$ on $Z$ is $\sigma$-proper.

Our terminology is justified by the following proposition:
Proposition 7.4. Let $G_1$ and $G_2$ be two groupoids such that $G_2^{(0)}$ is Hausdorff. Let $f : G_1 \to G_2$ be a groupoid morphism. Then the associated generalized groupoid morphism is locally proper if and only if the map $(f,r,s) : G_1 \to G_2 \times G_1^{(0)} \times G_1^{(0)}$ is proper.

Proof. Let $\varphi : G_1 \times_{f \circ r,s} G_2 \to (G_2 \times_{s,s} G_2) \times_{r \times r,f \times f} (G_1^{(0)} \times G_1^{(0)})$ defined by $\varphi(g_1,g_2) = (f(g_1)g_2,g_2,r(g_1),s(g_1))$. By definition, the action of $G_1$ on $Z$ is proper if and only if $\varphi$ is a proper map. Consider $\theta : G_2 \times_{s,s} G_2 \to G_2^{(2)}$ given by $(\gamma,\gamma') = (\gamma \gamma')^{-1}$. Let $\psi = (\theta \times 1) \circ \varphi$. Since $\theta$ is a homeomorphism, the action of $G_1$ on $Z$ is proper if and only if $\psi$ is proper.

Suppose that $(f,r,s)$ is proper. Let $f' = (f,r,s) \times 1 : G_1 \times G_2 \to G_2 \times G_1^{(0)} \times G_1^{(0)} \times G_2$. Then $f'$ is proper. Let $F = \{(\gamma,x,x',\gamma') \in G_2 \times G_1^{(0)} \times G_1^{(0)} \times G_2 \mid s(\gamma) = r(\gamma') = f(x'), r(\gamma) = f(x)\}$. Then $f' : (f')^{-1}(F) \to F$ is proper, i.e. $\psi$ is proper.

Conversely, suppose that $\psi$ is proper. Let $F' = \{(\gamma,y,x,x') \in G_2 \times G_1^{(0)} \times G_1^{(0)} \times G_1^{(0)} \mid s(\gamma) = y\}$. Then $\psi^{-1}(F') \to F'$ is proper, therefore $(f,r,s)$ is proper. \qed

Our objective is now to show the

Proposition 7.5. Let $G_1, G_2, G_3$ be groupoids. Let $(Z_1,\rho_1,\sigma_1)$ and $(Z_2,\rho_2,\sigma_2)$ be two generalized groupoid morphisms from $G_1$ to $G_2$ and from $G_2$ to $G_3$ respectively. Then $(Z,\rho,\sigma) = (Z_1 \times_{G_2} Z_2,\rho_1 \times 1, 1 \times \sigma_2)$ is a generalized groupoid morphism. If $(Z_1,\rho_1,\sigma_1)$ and $(Z_2,\rho_2,\sigma_2)$ are locally proper, then $(Z,\rho,\sigma)$ is locally proper.

Proposition 7.4 shows that groupoids form a category whose arrows are generalized morphisms, and that two groupoids are isomorphic in that category if and only if they are Morita-equivalent. Moreover, the same conclusions hold for the category whose arrows are locally proper generalized morphisms. In particular, local properness of generalized morphisms is invariant under Morita-equivalence.

All the assertions of Proposition 7.5 follow from Lemma 2.33.

7.3. Proper generalized morphisms.

Definition 7.6. Let $G_1$ and $G_2$ be groupoids. A generalized morphism $(Z,\rho,\sigma)$ from $G_1$ to $G_2$ is said to be proper if it is locally proper, and if for every quasi-compact subspace $K$ of $G_2^{(0)}$, $\sigma^{-1}(K)$ is $G_1$-compact.

Examples 7.7. (a) Let $X$ and $Y$ be locally compact spaces and $f : X \to Y$ a continuous map. Then the generalized morphism $(X,\text{Id},f)$ is proper if and only if $f$ is proper.

(b) Let $f : G_1 \to G_2$ be a continuous morphism between two locally compact groups. Let $p : G_2 \to \{e\}$. Then $(G_2,p,p)$ is proper if and only if $f$ is proper and $f(G_1)$ is co-compact in $G_2$. 

(c) Let $G$ be a locally compact proper groupoid with Haar system such that $G^{(0)}$ is Hausdorff, and let $\pi : G^{(0)} \to G^{(0)}/G$ be the canonical mapping. Then $(G^{(0)}, \text{Id}, \pi)$ is a proper generalized morphism from $G$ to $G^{(0)}/G$.

7.4. Construction of a $C^*$-correspondence. Until the end of the section, our goal is to prove:

**Theorem 7.8.** Let $G_1$ and $G_2$ be locally compact groupoids with Haar system such that $G_1^{(0)}$ and $G_2^{(0)}$ are Hausdorff, and $(\mathbb{Z}, \rho, \sigma)$ a locally proper generalized morphism from $G_1$ to $G_2$. Then one can construct a $C^*_\rho(G_1)$-Hilbert module $\mathcal{E}_Z$ and a map $\pi : C^*_\rho(G_2) \to \mathcal{L}(\mathcal{E}_Z)$. Moreover, if $(\mathbb{Z}, \rho, \sigma)$ is proper, then $\pi$ maps to $\mathcal{K}(\mathcal{E}_Z)$. Therefore, it gives an element of $KK(C^*_\rho(G_2), C^*_\rho(G_1))$.

**Corollary 7.9.** (see [14]) Let $G_1$ and $G_2$ be locally compact groupoids with Haar system such that $G_1^{(0)}$ and $G_2^{(0)}$ are Hausdorff. If $G_1$ and $G_2$ are Morita-equivalent, then $C^*_\rho(G_1)$ and $C^*_\rho(G_2)$ are Morita-equivalent.

**Corollary 7.10.** Let $f : G_1 \to G_2$ be morphism between two locally compact groupoids with Haar system such that $G_1^{(0)}$ and $G_2^{(0)}$ are Hausdorff. If the restriction of $f$ to $(G_1)^K$ is proper for each compact set $K \subset (G_1)^{(0)}$ then $f$ induces a correspondence $\mathcal{E}_f$ from $C^*_\rho(G_2)$ to $C^*_\rho(G_1)$. If in addition for every compact set $K \subset (G_2)^{(0)}$ the quotient of $(G_1)^{(0)} \times_{f,r}(G_2)^K$ by the diagonal action of $G_1$ is compact, then $C^*_\rho(G_2)$ maps to $\mathcal{K}(\mathcal{E}_f)$ and thus $f$ defines a $KK$-element $[f] \in KK(C^*_\rho(G_2), C^*_\rho(G_1))$.

**Proof.** See Proposition 7.4 and Definition 7.6 applied to the generalized morphism $Z_f = G_1^{(0)} \times_{f,r} G_2$ as in Example 7.2.

The rest of the section is devoted to proving Theorem 7.8. Let us first recall the construction of the correspondence when the groupoids are Hausdorff [11]. It is the closure of $C_c(Z)$ with the $C^*_\rho(G_1)$-valued scalar product

\begin{equation}
\langle \xi, \eta \rangle(g) = \int_{\gamma \in (G_2)^{(0)}} \overline{\xi(g^{-1}z\gamma)} \eta(g^{-1}z\gamma) \lambda^\rho(z)(d\gamma),
\end{equation}

where $z$ is an arbitrary element of $Z$ such that $\rho(z) = r(g)$. The right $C^*_\rho(G_1)$-module structure is defined $\forall \xi \in C_c(Z), \forall a \in C_c(G_1)$ by

\begin{equation}
(\xi a)(z) = \int_{g \in (G_1)^{(0)}} \xi(g^{-1}z)a(g^{-1}) \lambda^\rho(z)(dg),
\end{equation}

and the left action of $C^*_\rho(G_2)$ is

\begin{equation}
(b\xi)(z) = \int_{\gamma \in (G_2)^{(0)}} b(\gamma)\xi(\gamma z) \lambda^\rho(z)(d\gamma)
\end{equation}

for all $b \in C_c(G_2)$.

We now come back to non-Hausdorff groupoids. For every open Hausdorff set $V \subset Z$, denote by $V'$ its closure in $\mathcal{H}((G_1 \times Z)^V)$, where $z \in V$ is identified
to \((ρ(z), z) ∈ \mathcal{H}((G_1 × Z)^0)\). Let \(E^0_V\) be the set of \(ξ ∈ C_c(V')\) such that 
\[
ξ(z) = \frac{ξ(S × \{z\})}{\sqrt{\# S}}
\]
for all \(S × \{z\} ∈ V'\).

**Lemma 7.11.** The space \(E^0_Z = \sum_{i∈I} E^0_{V_i}\) is independent of the choice of the cover \((V_i)\) of \(Z\) by Hausdorff open subspaces.

**Proof.** It suffices to show that for every open Hausdorff subspace \(V\) of \(Z\), one has \(E^0_V \subset \sum_{i∈I} E^0_{V_i}\). Let \(ξ ∈ E^0_V\). Denote by \(q_V : V' → V\) the canonical map defined by \(q_V(S × \{z\}) = z\). Let \(K ⊂ V\) compact such that \(supp(ξ) ⊂ q_V^{-1}(K)\). There exists \(J ⊂ I\) finite such that \(K ⊂ \bigcup_{i∈J} V_i\). Let \((ϕ_j)_{j∈J}\) be a partition of unity associated to that cover, and \(ξ_j = ξ(ϕ_j ∘ q_V)\). One easily checks that \(ξ_j ∈ E^0_{V_i}\) and that \(ξ = \sum_{j∈J} ξ_j\). □

We now define a \(C^*(G_1)\)-valued scalar product on \(E^0_Z\) by Eqn. 2 where \(z\) is an arbitrary element of \(Z\) such that \(ρ(z) = r(g)\). Our definition is independent of the choice of \(z\), since if \(z'\) is another element, there exists \(γ' ∈ G_2\) such that \(z' = zγ'\), and the Haar system on \(G_2\) is left-invariant.

Moreover, the integral is convergent for all \(g ∈ G_1\) because the action of \(G_2\) on \(Z\) is proper.

Let us show that \(⟨ξ, η⟩ \in C_c(G_1)\) for all \(ξ, η ∈ E^0_Z\). We need a preliminary lemma:

**Lemma 7.12.** Let \(X\) and \(Y\) be two topological spaces such that \(X\) is locally compact and \(f : X → Y\) proper. Let \(F\) be a ultrafilter such that \(f\) converges to \(y ∈ Y\) with respect to \(F\). Then there exists \(x ∈ X\) such that \(f(x) = y\) and \(F\) converges to \(x\).

**Proof.** Let \(Q = f^{-1}(y)\). Since \(f\) is proper, \(Q\) is quasi-compact. Suppose that for all \(x ∈ Q\), \(F\) does not converge to \(x\). Then there exists an open neighborhood \(V_x\) of \(x\) such that \(V_x^c ∈ F\). Extracting a finite cover \((V_1, ..., V_n)\) of \(Q\), there exists an open neighborhood \(V\) of \(Q\) such that \(V^c ∈ F\). Since \(f\) is closed, \(f(V^c)^c\) is a neighborhood of \(y\). By assumption, \(f(V^c)^c ∈ F\), i.e. \(∃ A ∈ F\), \(f(A) ⊂ f(V^c)^c\). This implies that \(A ⊂ V\), therefore \(V^c ∈ F\): this contradicts \(V^c ∈ F\).

Consequently, there exists \(x ∈ Q\) such that \(F\) converges to \(x\). □

To show that \(⟨ξ, η⟩ \in C_c(G_1)\), we can suppose that \(ξ ∈ E^0_U\) and \(η ∈ E^0_V\), where \(U\) and \(V\) are open Hausdorff. Let \(F(g, z) = \overline{ξ(z)}η(g^{-1}z)\), defined on \(Γ = G_1 ×_{r, ρ} Z\). Since the action of \(G_1\) on \(Z\) is proper, \(F\) is quasi-compactly supported. Let us show that \(F ∈ C_c(Γ)\).

Let \(F\) be a ultrafilter on \(Γ\), convergent in \(HΓ\). Since \(G_1^{(0)}\) is Hausdorff, its limit has the form \(S = S'g_0 × S''\) where \(S' ⊂ (G_1)^{r(g_0)}, S'' ⊂ r^{-1}(r(g_0))\). Moreover, \(S'\) is a subgroup of \((G_1)^{r(g_0)}\) by the proof of Lemma 5.1.

Suppose that there exist \(z_0, z_1 \in S''\) and \(g_1 ∈ S'g_0\) such that \(z_0 ∈ U\) and \(g_1^{-1}z_1 ∈ V\). By Lemma 7.12 applied to the proper map \(G_1 × Z → Z × Z\), there exists \(s_0 ∈ S'\) such that \(z_0 = s_0z_1\). We may assume that \(g_0 = s_0g_1\). Then
\[
\sum_{s \in S} F(s) = \sum_{s' \in S'} \xi(z_{0}) \eta(g_{0}^{-1}(s')^{-1}z_{0}). \quad \text{If } s' \notin \text{stab}(z_{0}), \text{then } g_{0}^{-1}(s')^{-1}z_{0} \notin V \text{ since } g_{0}^{-1}z_{0} \text{ and } g_{0}^{-1}(s')^{-1}z_{0} \text{ are distinct limits of } (g, z) \mapsto g^{-1}z \text{ with respect to } \mathcal{F} \text{ and } V \text{ is Hausdorff. Therefore,}
\]
\[
\sum_{s \in S} F(s) = \sqrt{\#(\text{stab}(z_{0}) \cap S') \xi(z_{0})} \frac{\#(\text{stab}(g_{0}^{-1}z_{0}) \cap (g_{0}^{-1}S')g_{0})) \eta(z_{0})}{\#(\text{stab}(z_{0}) \cap S') \xi(z_{0})} = \lim_{x \to z} F(x).
\]

If for all \(z_{0}, z_{1} \in S^{\prime \prime}\) and all \(g_{1} \in S'g_{0}, (z_{0}, g_{1}^{-1}z_{1}) \notin U \times V, \) then \(\sum_{s \in S} F(g, z) = 0 = \lim_{x \to z} F(x).\)

By Proposition 4.1, \( F \in C_{c}(\Gamma)\).

Since \((\xi, \eta)(g) = \int_{\gamma \in (G_{2})^{\sigma}(g)} F(g, z_{\gamma}) \lambda^{\sigma}(\gamma)(d\gamma), \) to prove that \((\xi, \eta) \in C_{c}(G_{1})\) it suffices to show:

**Lemma 7.13.** Let \(G_{1}\) and \(G_{2}\) be two locally compact groupoids with Haar system such that \(G_{1}^{(0)}\) are Hausdorff. Let \((Z, \rho, \sigma)\) be a generalized morphism from \(G_{1}\) to \(G_{2}\). Let \(\Gamma = \Gamma_{1} \times_{\rho, \sigma} Z. \) Then for every \(F \in C_{c}(\Gamma), \) the function

\[
g \mapsto \int_{\gamma \in (G_{2})^{\rho}(g)} F(g, z_{\gamma}) \lambda^{\rho}(\gamma)(d\gamma),
\]

where \(z \in Z\) is an arbitrary element such that \(\rho(z) = r(g)\), belongs to \(C_{c}(G_{1})\).

**Proof.** Suppose first that \(F(g, z) = f(g)h(z)\), where \(f \in C_{c}(G_{1})\) and \(h \in C_{c}(Z)\). Let \(H(z) = \int_{\gamma \in (G_{2})^{\rho}(g)} h(z_{\gamma}) \lambda^{\rho}(\gamma)(d\gamma). \) By Lemma 7.14 below (applied to the groupoid \(Z \times G_{2}\), \(H\) is continuous. It is obviously \(G_{2}\)-invariant, therefore \(H \in C_{c}(Z/G_{2})\). Let \(\tilde{H} \in C_{c}(G_{1}^{(0)}) \simeq C_{c}(Z/G_{2})\) correspond to \(H\). The map

\[
g \mapsto \int_{\gamma \in (G_{2})^{\rho}(g)} F(g, z_{\gamma}) \lambda^{\rho}(\gamma)(d\gamma) = f(g)\tilde{H}(s(g))
\]

thus belongs to \(C_{c}(G_{1})\).

By linearity, the lemma is true for \(F \in C_{c}(G_{1}) \otimes C_{c}(Z)\). By Lemma 4.1 and Lemma 4.3, \(F\) is the uniform limit of functions \(F_{n} \in C_{c}(G_{1}) \otimes C_{c}(Z)\) which are supported in a fixed quasi-compact set \(Q = Q_{1} \times Q_{2} \subset G_{1} \times Z\). Let \(Q' \subset Z\) quasi-compact such that \(\rho(Q') \supset r(Q_{1})\). Since the action of \(G_{2}\) on \(Z\) is proper, \(K = \{\gamma \in G_{2} | Q' \gamma \cap Q_{2} \neq \emptyset\}\) is quasi-compact. Using the fact that \(G_{1}^{(0)} \simeq Z/G_{2}\), it is easy to see that

\[
\sup_{(g, z) \in \Gamma} \int_{\gamma \in (G_{2})^{\rho}(g)} 1_{Q}(g, z_{\gamma}) \lambda^{\rho}(\gamma)(d\gamma) \leq \sup_{z \in Q'_{1}} \int_{\gamma \in G_{2}^{(0)}} 1_{Q_{2}}(z_{\gamma}) \lambda^{\rho}(\gamma)(d\gamma)
\]

\[
\leq \sup_{x \in G_{2}^{(0)}} \int_{\gamma \in G_{2}^{(0)}} 1_{K}(\gamma) \lambda^{\rho}(d\gamma) < \infty
\]
by Lemma 7.12. Therefore,
\[ \lim_{n \to \infty} \sup_{g \in G_1} \left| \int_{\gamma \in G^*_\varphi(s)} F(g, z\gamma) - F_n(g, z\gamma) \lambda^\varphi(z)(d\gamma) \right| = 0. \]
The conclusion follows from Corollary 4.2.

In the proof of Lemma 7.13 we used the

**Lemma 7.14.** Let \( G \) be a locally compact, proper groupoid with Haar system, such that \( G^x \) is Hausdorff for all \( x \in G^{(0)} \), and \( G^x = \{ x \} \) for all \( x \in G^{(0)} \). We do not assume \( G^{(0)} \) to be Hausdorff. Then \( \forall f \in C_c(G^{(0)}) \),
\[ \varphi: G^{(0)} \to \mathbb{C}, \quad x \mapsto \int_{g \in G^x} f(s(g)) \lambda^x(dg) \]
is continuous.

**Proof.** Let \( V \) be an open, Hausdorff subspace of \( G^{(0)} \). Let \( h \in C_c(V) \). Since \( (r, s): G \to G^{(0)} \times G^{(0)} \) is a homeomorphism from \( G \) onto a closed subspace of \( G^{(0)} \times G^{(0)} \), and \( (x, y) \mapsto h(x)f(y) \) belongs to \( C_c(G^{(0)} \times G^{(0)}) \), the map \( g \mapsto h(r(g))f(s(g)) \) belongs to \( C_c(G) \), therefore by definition of a Haar system, \( x \mapsto \int_{g \in G^x} h(r(g))f(s(g)) \lambda^x(dg) = h(x)\varphi(x) \) belongs to \( C_c(G^{(0)}) \).
Since \( h \in C_c(V) \) is arbitrary, this shows that \( \varphi|_V \) is continuous, hence \( \varphi \) is continuous on \( G^{(0)} \).

Now, let us show the positivity of the scalar product. Recall that for all \( x \in G^{(0)} \) there is a representation \( \pi_{G_1, x}: C^*(G_1) \to \mathcal{L}(L^2(G_1^x)) \) such that for all \( a \in C_c(G_1) \) and all \( \eta \in C_c(G_1^x) \),
\[ \langle \pi_{G_1, x}(a)\eta \rangle(g) = \int_{h \in G_1^{(0)}} a(h)\eta(gh) \lambda^\varphi(g)(dh). \]
By definition, \( \|a\|_{C^*_c(G_1)} = \sup_{x \in G^{(0)}} \|\pi_{G_1, x}(a)\| \).
\[ \langle \eta, \pi_{G_1, x}(a)\eta \rangle = \int_{g \in G_1^{(0)}} \overline{\eta(g)}a(h)\eta(gh) \lambda^\varphi(g)(dh) \lambda^x(dg) = \int_{g \in G_1^{(0)}} \overline{\eta(g)}a(g^{-1}h)\eta(h) \lambda^x(dg) \lambda^x(dh). \]
Fix \( z \in Z \) such that \( \rho(z) = x \). Replacing \( a(g^{-1}h) \) by
\[ \langle \xi, \xi \rangle(g^{-1}h) = \int_{\gamma \in G^*_\varphi(z)} \xi(g^{-1}z\gamma)\xi(h^{-1}z\gamma) \lambda^\varphi(z)(d\gamma), \]
we get
\[ (5) \quad \langle \eta, \pi_{G_1, x}(\langle \xi, \xi \rangle)\eta \rangle = \int_{\gamma \in G^*_\varphi(z)} \lambda^\varphi(z)(d\gamma) \left| \int_{g \in G^x} \eta(g)\xi(g^{-1}z\gamma) \lambda^x(dg) \right|^2. \]
It follows that \( \pi_{G_1, x}(\langle \xi, \xi \rangle) \geq 0 \) for all \( x \in G^{(0)}_1 \), so \( \langle \xi, \xi \rangle \geq 0 \) in \( C^*_c(G_1) \).
Now, let us define a $C^*_r(G_1)$-module structure on $E^0_Z$ by Eqn. 8 for all $\xi \in E^0_Z$ and $a \in C_c(G_1)$.

Let us show that $\xi a \in E^0_Z$. We need a preliminary lemma:

**LEMMA 7.15.** Let $X$ and $Y$ be quasi-compact spaces, $(\Omega_k)$ an open cover of $X \times Y$. Then there exist finite open covers $(X_i)$ and $(Y_j)$ of $X$ and $Y$ such that $\forall i, j \exists k, X_i \times Y_j \subset \Omega_k$.

**Proof.** For all $(x, y) \in X \times Y$ choose open neighborhoods $U_{x,y}$ and $V_{x,y}$ of $x$ and $y$ such that $U_{x,y} \times V_{x,y} \subset \Omega_k$ for some $k$. For $y$ fixed, there exist $x_1, \ldots, x_n$ such that $(U_{x_i,y})_{1 \leq i \leq n}$ covers $X$. Let $V_y = \bigcap_{i=1}^n U_{x_i,y}$. Then for all $(x, y) \in X \times Y$, there exists an open neighborhood $U'_{x,y}$ of $x$ and $k$ such that $U'_{x,y} \times V_y \subset \Omega_k$.

Let $(V_1, \ldots, V_m) = (V_{y_1}, \ldots, V_{y_m})$ such that $\cup_{1 \leq j \leq m} V_j = Y$. For all $x \in X$, let $U'_x = \bigcap_{j=1}^m U'_{x,y_j}$. Let $(U_1, \ldots, U_p)$ be a finite sub-cover of $(U'_x)_{x \in X}$. Then for all $i$ and for all $j$, there exists $k$ such that $U_i \times V_j \subset \Omega_k$. \[\square\]

Let $Q_1$ and $Q_2$ be quasi-compact subspaces of $G_1$ of $Z$ respectively such that $a^{-1}(C^*) \subset Q_1$ and $\xi^{-1}(C^*) \subset Q_2$. Let $Q$ be a quasi-compact subspace of $Z$ such that $\forall g \in Q_1, \forall z \in Q_2, g^{-1}z \in Q$. Let $(\mathcal{U})$ be a finite cover of $Q$ by Hausdorff open subspaces of $Z$. Let $Q' = Q_1 \times_{r,\rho} Q_2$. Then $Q'$ is a closed subspace of $Q_1 \times Q_2$. Let $\Omega'_k = \{(g, z) \in Q'| g^{-1}z \in U_k\}$. Then $(\Omega'_k)$ is a finite open cover of $Q'$. Let $\Omega_k$ be an open subspace of $Q_1 \times Q_2$ such that $\Omega'_k = \Omega_k \cap Q'$. Then $(Q_1 \times Q_2 - Q') \cup \{\Omega_k\}$ is an open cover of $Q_1 \times Q_2$. Using Lemma 7.15, there exist finite families of Hausdorff open sets $(W_i)$ and $(V_j)$ which cover $Q_1$ and $Q_2$, such that for all $i$ and for all $j$, there exists $k$ such that $U_i \times V_j \subset \Omega_k$.

Thus, we can assume by linearity and by Lemmas 4.8 and 7.11 that $\xi \in E^0_Z$, $a \in C_c(W)$, $U = W^{-1}V$, and $U$, $V$ and $W$ are open and Hausdorff.

Let $\Omega = \{(g, S) \in W^{-1} \times U| g^{-1}g_U(S) \in V\}$. Then the map $(g, S) \mapsto (g^{-1}, g^{-1}S)$ is a homeomorphism from $\Omega$ onto $W \times_{r,\rho,g_U} V'$. Therefore, the map $(g, z) \mapsto (g^{-1}z)a(g^{-1})$ belongs to $C_c(\Omega) \subset C_c(G_1 \times_{r,\rho,g_U} U')$. By Lemma 7.8

$$S \mapsto (\xi a)(S) = \int_{g \in C_c(W)} \xi(g^{-1}S)a(g^{-1})\lambda^{g_U}(S)(dg)$$

belongs to $C_c(U')$. It is immediate that $(\xi a)(S) = \sqrt{\xi^*}(\xi a)(g(S))$ for all $S \in U'$, therefore $\xi a \in E^0_Z$. This completes the proof that $\xi a \in E^0_Z$.

Finally, it is not hard to check that $(\xi, \eta)a = (\xi, \eta)^*a$. Therefore, the completion $E_Z$ of $E^0_Z$ with respect to the norm $||\xi|| = ||\xi, \xi||^{1/2}$ is a $C^*_r(G_1)$-Hilbert module.

Let us now construct a morphism $\pi: C^*_r(G_2) \rightarrow \mathcal{L}(E_Z)$. For every $\xi \in E^0_Z$ and every $b \in C_c(G_2)$, define $b\xi$ by Eqn. 8. Let us check that $b\xi \in E^0_Z$. As above, by linearity we may assume that $\xi \in E^0_Y$, $b \in C_c(W)$ and $VW^{-1} \subset U$, where $V \subset Z$ and $W \subset G_2$ are open and Hausdorff.

Let $\Phi(S, \gamma) = (S\gamma, \gamma)$. Then $\Phi$ is a homeomorphism from $\Omega = \{(S, \gamma) \in U' \times_{\sigma,\rho} V| g_U(S) \gamma \in V\}$ onto $V' \times_{\rho,g_U} W$. Let $F(z, \gamma) = b(\gamma)\xi(z\gamma)$. Since $F = (\xi \otimes b) \circ \Phi$, $F$ is an element of $C_c(\Omega) \subset C_c(U' \times_{\sigma,\rho} W)$. By Lemma 7.8 $b\xi \in C_c(U')$. 29
It is immediate that \((b\xi)(S) = \sqrt{F(S)(b\xi)(q(S))}\). Therefore, \(b\xi \in \mathcal{E}_0^0 \subset \mathcal{E}_Z^0\).

Let us prove that \(\|b\xi\| \leq \|b\| \|\xi\|\). Let

\[
\zeta(\gamma) = \int_{g \in G_1^x} \eta(g)\xi(g^{-1}z\gamma) \lambda^x(dg),
\]

where \(z \in Z\) such that \(\rho(z) = r(g)\) is arbitrary. From \(\mathbf{5}\),

\[
\langle \eta, \pi_{G_1, x}((\xi, \xi)) \eta \rangle = \|\zeta\|_2^2 |L^2(G_2^x)|.
\]

A similar calculation shows that

\[
\langle \eta, \pi_{G_1, x}((b\xi, b\xi)) \eta \rangle = \int_{\gamma \in G_2^x} \lambda^x(\gamma)(d\gamma) \int_{g \in G_1^x} \eta(g)\xi(g^{-1}z\gamma\gamma') b(\gamma') \lambda^x(\gamma')(d\gamma')^2 = \langle b\xi, b\xi \rangle \leq \|b\|^2 \|\zeta\|^2.
\]

By density of \(C_0(G_2^x)\) in \(L^2(G_2^x)\), \(\|\pi_{G_1, x}((b\xi, b\xi))\| \leq \|b\| \|\pi_{G_1, x}((\xi, \xi))\|\). Taking the supremum over \(x \in G_1^x\), we get \(\|b\xi\| \leq \|b\| \|\xi\|\). It follows that \(b \rightarrow (\xi \rightarrow b\xi)\) extends to a *-morphism \(\pi: C^*_e(G_2) \rightarrow \mathcal{L}(\mathcal{E}_Z)\).

Finally, suppose now that \((Z, \rho, \sigma)\) is proper, and let us show that \(C^*_e(G_2)\) maps to \(K(\mathcal{E}_Z)\).

For every \(\eta, \zeta \in \mathcal{E}_Z^0\), denote by \(T_{\eta, \zeta}\) the operator \(T_{\eta, \zeta}(\xi) = \eta(\zeta, \xi)\). Compact operators are elements of the closed linear span of \(T_{\eta, \zeta}\)'s. Let us write an explicit formula for \(T_{\eta, \zeta}\):

\[
T_{\eta, \zeta}(\xi)(z) = \int_{g \in G_1^x} \eta(g^{-1}z)\langle \zeta, \xi(g^{-1}) \rangle \lambda^x(dg) = \int_{g \in G_1^x} \eta(g^{-1}z) \int_{\gamma \in G_2^x} \zeta(g^{-1}z\gamma) \lambda^x(d\gamma) \lambda^x(\gamma)(d\gamma).
\]

Let \(b \in C_0(G_2)\), let us show that \(\pi(b) \in K(\mathcal{E}_Z)\). Let \(K\) be a quasi-compact subspace of \(G_2\) such that \(b^{-1}(C^*) \subset K\). Since \((Z, \rho, \sigma)\) is a proper generalized morphism, there exists a quasi-compact subspace \(Q\) of \(Z\) such that \(\sigma^{-1}(r(K)) \subset G_1Q\). Before we proceed, we need a lemma:

**Lemma 7.16.** Let \(G_2\) be a locally compact groupoid acting freely and properly on a locally compact space \(Z\) with momentum map \(\sigma: Z \rightarrow G_2(0)\). Then for every \((z_0, \gamma_0) \in Z \times G_2\), there exists a Hausdorff open neighborhood \(\Omega_{z_0, \gamma_0}\) of \((z_0, \gamma_0)\) such that

- \(U = \{z_1\gamma_1 \mid (z_1, \gamma_1) \in \Omega_{z_0, \gamma_0}\}\) is Hausdorff;
- there exists a Hausdorff open neighborhood \(W\) of \(\gamma_0\) such that \(\forall \gamma \in G_2, \forall z \in pr_1(\Omega_{z_0, \gamma_0}), \forall z' \in U, z' = z\gamma \implies \gamma \in W\).

**Proof.** Let \(R = \{(z, z') \in Z \times Z \mid \exists \gamma \in G_2, z' = z\gamma\}\). Since the \(G_2\)-action is free and proper, there exists a continuous function \(\phi: R \rightarrow G_2\) such that \(\phi(z, \gamma) = \gamma\). Let \(W\) be an open Hausdorff neighborhood of \(\gamma_0\). By continuity of \(\phi\), there exist open Hausdorff neighborhoods \(V\) and \(U_0\) of \(z_0\) and \(z_0\gamma_0\) such that for all \((z, z') \in R \cap (V \times U_0), \phi(z, z') \in W\). By continuity of the action,
there exists an open neighborhood $\Omega_{z_0,\gamma_0}$ of $(z_0, \gamma_0)$ such that $\forall (z_1, \gamma_1) \in \Omega_{z_0,\gamma_0}$, $z_1 \gamma_1 \in U_0$ and $z_1 \in V$.

By Lemma 4.16 there exist finite covers $(V_i)$ of $Q$ and $(W_j)$ of $K$ such that for every $i$, $j$, $(Z \times_{G} G_2) \cap (V_i \times W_j) \subset \Omega_{z_0,\gamma_0}$ for some $(z_0, \gamma_0)$.

By Lemma 6.2 applied to the groupoids $(G_1 \times Z)_{V_i}$, for all $i$ there exists $c_i' \in C_c(V_i')_+$ such that $c_i'(S) = (\# S)c_i'(q_{V_i}(S))$ for all $S \in V_i'$, and such that $\sum_i c_i' \geq 1$ on $Q$. Let

$$f_i(z) = \int_{g \in G_{i}^{\prime \prime}(z)} c_i'(g^{-1}z) \lambda^{\rho(z)}(dg)$$

and let $f = \sum_i f_i$. As in the proof of Theorem 6.3 one can show that for every Hausdorff open subspace $V$ of $Z$ and every $h \in C_c(V)$, $(g, z) \mapsto h(z)c_i'(g^{-1}z)$ belongs to $C_c(G \times Z)$, therefore $h f_i$ is continuous on $V$. Since $h$ is arbitrary, it follows that $f_i$ is continuous, thus $f$ is continuous. Moreover, $f$ is $G_1$-equivariant, nonnegative, and $\inf_Q f > 0$. Therefore, there exists $f_1 \in C_c(G_1 \setminus Z)$ such that $f_1(z) = 1/f(z)$ for all $z \in Q$. Let $c_i(z) = f_1(z)c_i'(z)$. Let

$$T_i(\xi)(z) = \int_{g \in G_{i}^{\prime \prime}(z)} \int_{g \in G_{i}^{\prime}(z)} c_i(g^{-1}z)b(\gamma)\xi(z\gamma)\lambda^{\rho(z)}(dg)\lambda^{\rho(z)}(d\gamma).$$

Then $\pi(b) = \sum_i T_i$, therefore it suffices to show that $T_i$ is a compact operator for all $i$.

By linearity and by Lemma 4.3 one may assume that $b \in C_c(W_j)$ for some $j$. Then, by construction of $V_i$ (see Lemma 4.16), there exist open Hausdorff sets $U \subset Z$ and $W \subset G_2$ such that $\{\gamma \in G_2 \mid \exists (z, z') \in V_i \times U, z' = z\gamma\} \subset W$, and $\{z\gamma \mid (z, \gamma) \in V_i \times W\} \subset U$.

The map $(z, z\gamma) \mapsto c(z)b(\gamma)$ defines an element of $C_c(V_i' \times U)$. Let $L_1 \times L_2 \subset V_i \times U$ compact such that $(z, z\gamma) \mapsto c(z)b(\gamma)$ is supported on $q_{V_i}^{-1}(L_1) \times L_2$.

By Lemma 6.2 applied to the groupoids $(G_1 \times Z)_{V_i}$ and $(G_1 \times Z)_{U}$, there exist $d_1 \in C_c(V_i')_+$ and $d_2 \in C_c(U')_+$ such that $d_1 > 0$ on $L_1$ and $d_2 > 0$ on $L_2$, $d_1(S) = \sqrt{\#S}d_1(q_{V_i}(S))$ for all $S \in V_i'$, and $d_2(S) = \sqrt{\#S}d_2(q_{U}(S))$ for all $S \in U'$. Let

$$f(z, z\gamma) = \frac{c(z)b(\gamma)}{d_1(z)d_2(z\gamma)}.$$ 

Then $f \in C_c(V_i \times_{G_1} U)$. Therefore, $f$ is the uniform limit of a sequence $f_n = \sum \alpha_{n,k} \otimes \beta_{n,k}$ in $C_c(V_i) \otimes C_c(U)$ such that all the $f_n$ are supported in a fixed compact set. Then $T_i$ is the norm-limit of $\sum_k T_{d_1\alpha_{n,k},d_2\beta_{n,k}}$, therefore it is compact.

**Remark 7.17.** The construction in Theorem 7.3 is functorial with respect to the composition of generalized morphisms and of correspondences. We don’t include a proof of this fact, as it is tedious but elementary. It is an easy exercise when $G_1$ and $G_2$ are Hausdorff.
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