Forced Response Approach on Parametric Vibration with Time Periodic Coefficient

Dishan Huang¹ᵃ*, Yingxue Li¹ᵇ

¹ School of Mechanical Engineering and Automation, Shanghai University, Shanghai 200444, China
ᵃ*hdishan@shu.edu.cn,ᵇliyingxuelingf@shu.edu.cn

Abstract—In this study, based on Galerkin discretization and frequency splitting, the forced vibration response of a large span cable with time periodic tension is approached as a closed-form solution, i.e. a special trigonometric series with mode functions. According to the principle of modulation feedback, the forced vibration response solution can be expressed as a series of linear combination of external excitation frequency and parametric excitation frequency with modes. Using orthogonality relation, the formula of forced vibration response approach is obtained. Investigations show that the presented approach computes the forced vibration response with higher accuracy. Therefore, this study is available to analyze the vibration of a large span cable with time periodic tension, and the characteristics of vibration response could be used as an important information source for the failure diagnosis and the health monitoring of cable structure.

1. Introduction

The vibration of large span cable in a stayed bridge is a problem of the second order partial differential system with time periodic coefficient. Large span cable is one of the most important stress parts of the cable-stayed bridge because of that most of the weight of bridge deck and load are transferred to the bridge tower through the cable. Due to the small mass, low damping, and low stiffness of cable-stayed bridge cables, they are prone to vibrate under time periodic tension and external excitation, such as wind, rain, earthquake and vehicle loads. The displacement excitation of the cable end is usually regarded as axial excitation. To guarantee the safety of stayed bridge, the forced vibration response of a stay cable is paid more attention by structure researchers.

There are many studies of the forced parametric vibration model of cable vibration. Galerkin method [¹] is widely used in the numerical calculation of string vibration to solve nonlinear partial differential equations. Takahashi and Konishi [²,³] proposed a method for calculating the nonlinear free vibration of cables based on the Galerkin method and Harmonic Wave Equilibrium method. Based on the nonlinear vibration theory for flexible cables, Xie et al. [⁴] proposed a numerical method to compute the nonlinear vibration of stayed cables due to support excitation and fluctuating wind loads, and studied the influencing factors of parametric vibration. Wu et al. [⁵] studied the dynamic responses of cable under simple harmonic axial excitation and the characteristics of parametric vibration. Macdonald [⁶,⁷] classified the support excitation into direct excitation and parametric excitation, which could lead to a traditional resonance vibration and a parametric vibration when the oscillation frequency of the support is twice of one of the natural frequencies of stayed cable. Chen and Sun [⁸] studied the nonlinear dynamic response of parabolic cable due to axial excitation considering the effects of inclination angle, damping and sag of cables. Ceshi Sun et al. [⁹] derived non-linear governing equations by Galerkin method and
solved numerically, also selected three longitudinal girder vibration amplitudes to illustrate their effect on cable non-linear response under primary and sub-harmonic resonance. Shouying Li et al. \cite{10} derived the equations of motion of a three-dimensional continuous stayed cable by considering the high-order nonlinear components of the dynamic cable tension, together with the equation of motion of the rivulet on the cable surface and solved the equations by using the finite difference method and the fourth-order Runge–Kutta algorithm, respectively.

In this paper, the parametric vibration is investigated without considering the cable inclination and gravity while considering the time periodic tension of stayed cables. The forced vibration solution form is given based on the system dynamic property and the vibration response with time periodic tension are expressed as a closed-form solution of a special trigonometric series with mode functions. The engineering example of forced vibration is listed to illustrate the availability of presented approach. The approach computation error of the vibration response is estimated.

2. Forced response of cable parametric vibration

According to the mechanical model of the horizontal cable, as shown in Fig. 1, the partial differential equation of forced vibration of the horizontal cable can be simplified to Eq.(1) \cite{11}, considering that the cable is simultaneously excited by axial displacement and distributed harmonic external force perpendicular to the axial direction.

2.1. Forced vibration response form

According to the mechanical model of the horizontal cable, as shown in Fig. 1, the partial differential equation of forced vibration of the horizontal cable can be simplified to Eq.(1) \cite{11}, considering that the cable is simultaneously excited by axial displacement and distributed harmonic external force perpendicular to the axial direction.

$$\frac{\partial^2 w}{\partial t^2} + \frac{c}{\rho} \frac{\partial w}{\partial t} - v^2 (1 + \beta \cos \omega_o t) \frac{\partial^2 w}{\partial x^2} = p \cos \omega_p t$$

(1)

where $c$ is the damp coefficient, $\rho$ density of the cable, $v$ the velocity of transverse vibration wave, $v^2 = \frac{H_0}{\rho}$, $H_0$ the static tension caused by preload and cable gravity action, $\beta$ the modulation coefficient $\beta = \frac{H_0}{H_0}$, the dynamic force $H_\omega = \frac{AE}{L} U_0$, $\omega_o$ the parametric excitation frequency, $\omega_p$ the external harmonic excitation frequency, $p$ distribution harmonic external force excitation amplitude.

According to the equivalent dynamic system of the parametric system, namely the modulated feedback model \cite{12-14}, when the parametric excitation frequency and the external excitation frequency are acting at the same time, there are many harmonics of the linear combination form of the parametric frequency $\omega_o$ and the external excitation frequency $\omega_p$ in the vibration response. According to Galerkin discretization, the solution of forced parameter vibration response can be expressed as:

$$w(x, t) = \varphi_l(x) \left[ \sum_{k=-\infty}^{\infty} A_k e^{j(\omega_p + k\omega_o)t} + \sum_{k=-\infty}^{\infty} B_k e^{-j(\omega_p + k\omega_o)t} \right]$$

(2)

where $\varphi_l(x)$ is the mode function. Because the energy of the vibration response is limited and it is distributed in a narrow band of the frequency $\omega_o$, as $k \to \infty$, the harmonic coefficients $A_k = 0, B_k = 0$. 

![Fig.1 Mechanical model of horizontal stayed cable](image-url)
2.2. Orthogonality relation
Substituted \( w(x, t) \) in the forced vibration response Eq.(2) into Eq.(1), the recurrence formula Eq.(3) and Eq. (4) can be obtained:

\[
- \frac{1}{2} v^2 \beta \varphi''_i(x) [A_{-1} + A_1] + [-\varphi_i(x) \omega_p^2 + j \frac{c}{\rho} \varphi_i(x) \omega_p - v^2 \varphi''_i(x)] A_0 = \frac{p}{2}
\]

(3)

\[
- \frac{1}{2} v^2 \beta \varphi''_i(x) [A_{k-1} + A_{k+1}] + [-\varphi_i(x) (\omega_p + k \omega_o)^2 + j \frac{c}{\rho} \varphi_i(x) (\omega_p + k \omega_o)] - v^2 \varphi''_i(x)] A_k = 0
\]

(4)

The stayed-cable is a continuous system. Consider the mode function can be selected as a set of functions conforming to boundary conditions and the boundary conditions are simply supported at both ends, the \( i \)-th order modal function is that:

\[ \varphi_i(x) = \sin \frac{\pi i x}{l} \]

(5)

The definite integral of mode function \( \varphi_i(x) \) from 0 to \( l \) can be calculated as follows:

\[
\int_0^l \varphi_i^2(x) dx = \frac{l}{2}
\]

(6)

\[
\int_0^l \varphi_i(x) \varphi''_i(x) dx = - \frac{\pi^2 i^2}{2l}
\]

(7)

\[
\int_0^l \varphi_i(x) \frac{p}{2} dx = \begin{cases} \frac{pl}{\pi i} & i = 1, 3, 5 \ldots \\ 0 & i = 2, 4, 6 \ldots \end{cases}
\]

(8)

Multiply \( \varphi_i(x) \) with the terms in both sides of Eq.(3)-(4) and doing the definite integral from 0 to \( l \), they are changed as:

\[
\frac{\pi^2 i^2 v^2 \beta}{4l} A_{-1} + \left( \frac{\pi^2 i^2 v^2}{2l} - \frac{l}{2} \omega_p^2 + j \frac{c}{2\rho} \omega_p \right) A_0 + \frac{\pi^2 i^2 v^2 \beta}{4l} A_1 = \frac{pl}{\pi i}
\]

(9)

\[
\frac{\pi^2 i^2 v^2 \beta}{4l} A_{k-1} + \left[ \frac{\pi^2 i^2 v^2}{2l} - \frac{l}{2} (\omega_p + k \omega_o)^2 + j \frac{c}{2\rho} (\omega_p + k \omega_o) \right] A_k + \frac{\pi^2 i^2 v^2 \beta}{4l} A_{k+1} = 0
\]

(10)

\( k = -n, -(n-1), \ldots, -3, -2, -1, 1, 2, 3 \ldots, n \); \( i = 1, 3, 5 \ldots \)

2.3. Harmonic coefficients
To compute the coefficient \( A_k \), put \( 2n+1 \) coefficient relations together, and let

\[
\Gamma = \frac{\pi^2 i^2 v^2 \beta}{4l}
\]

(11)

\[
\Omega_k = \frac{\pi^2 i^2 v^2}{2l} - \frac{l}{2} (\omega_p + k \omega_o)^2 + j \frac{c}{2\rho} (\omega_p + k \omega_o)
\]

(12)

\[ p = \frac{pl}{\pi i} \]

(13)

Substituting Eqs.(11-13) into Eqs.(9)&(10) yields the following the \( (2n+1) \) order linear algebra equations related to the coefficient \( A_k \).
It can be simplified as:

$$\mathbf{H} \mathbf{A} = \mathbf{P}_1$$  \hspace{1cm} (15)

where $\mathbf{H}$ is a $(2n+1) \times (2n+1)$ matrix, $\mathbf{A}$ is a $(2n+1)$ order coefficient vector and $\mathbf{P}_1$ is a $(2n+1)$ order force vector.

Let

$$\Omega_k^* = \frac{\pi^2 l^2 v^2}{2l} - \frac{l}{2} (\omega_p + k \omega_0)^2 - j \frac{c l}{2\rho} (\omega_p + k \omega_0)$$

Similarly, we construct $(2n+1)$ order linear algebra equations related to the coefficient $\mathbf{B}_k$.

$$\Omega_{n}^* \Gamma \mathbf{A} = \mathbf{P}_2$$  \hspace{1cm} (18)

where $\mathbf{H}^*$ is a conjugate matrix of matrix $\mathbf{H}$, $\mathbf{B}$ is a $(2n+1)$ order coefficient vector and $\mathbf{P}_2$ is a force vector.

2.4. Forced vibration response solution

When $n \to \infty$ in Eqs.(14)&(17), the vectors $-\Gamma \mathbf{A}_{n-1}$, $-\Gamma \mathbf{A}_{n+1}$, $-\Gamma \mathbf{B}_{n-1}$ and $-\Gamma \mathbf{B}_{n+1}$ all tend to be zero. Thus $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{R}$, where $\mathbf{R}$ is a real vector expressed as:

$$\mathbf{R} \approx [0 \cdots -1 0 0 \cdots 0]^T$$  \hspace{1cm} (19)

When $n \to \infty$, Eqs.(15) and (18) can be written as:

$$\mathbf{H} \mathbf{A} = \mathbf{R}$$  \hspace{1cm} (20)

$$\mathbf{H}^* \mathbf{B} = \mathbf{R}$$  \hspace{1cm} (21)

Therefore, the vector $\mathbf{B}$ is a conjugate of vector $\mathbf{A}$ when $n \to \infty$.

$$\mathbf{B} = \mathbf{A}^*$$  \hspace{1cm} (22)

Therefore, the vector $\mathbf{A}$ is a conjugate of vector $\mathbf{B}$ when $n \to \infty$. When the number of approximation terms is large enough and mode $i=1, 3, 5, \cdots$, the forced vibration response solution $w(x, t)$ can be simplified to a mode linear superposition of vibration response.
\[ w(x, t) \approx \sum_{i=1}^{m} \sin\left(\frac{\pi i x}{l}\right) \left[ \sum_{k=-n}^{n} A_{ik} e^{j(\omega_p + k\omega_p)t} + \sum_{k=-n}^{n} A^*_{ik} e^{-j(\omega_p + k\omega_p)t} \right] \]  

(23A)

\[ w(x, t) \approx 2 \sum_{i=1}^{m} \sin\left(\frac{\pi i x}{l}\right) \left[ \sum_{k=-n}^{n} Re(A_{ik}) \cos(\omega_p + k\omega_p)t + Im(A_{ik}) \sin(\omega_p + k\omega_p)t \right] \]  

(23B)

The forced parametric vibration response is computed by Eq. (23), where the range of mode is from 1 to \( m \). The coefficients \( A_{ik} \) and \( B_{ik} \) are computed from Eqs. (20) & (21), and \( k \) determines the series approach terms. When \( k \) is large enough, \( A_{ik} \to 0 \). When \( i \) is large enough, the modal force \( P = \frac{\pi l}{t} \to 0 \). According to the requirements of engineering, the number of modes \( i = 1, 3, \ldots, m \) is chosen.

3. Examples

Taking the stayed-cable (number: C13) of Sifang Cable-stayed Bridge on Songhua River, Harbin as the research object \(^{[11]}\), regarded it as a horizontal cable and ignored the influence of gravity, the forced response of cable parametric vibration is calculated by considering the periodic excitation of cable axial displacement and vertical axial external force. The geometric parameter and material property of the cable are listed in Table 1:

| Content                      | Value       |
|------------------------------|-------------|
| Mass per unit length \( \rho \) | 120.4 kg/m  |
| Length of cable \( l \)       | 178 m       |
| Young modulus \( E \)         | 190 GPa     |
| Cross-sectional area \( A \)  | 0.014124 m\(^2\) |
| Parametric excitation frequency \( \omega_p \) | 3 rad/s    |
| Support excitation amplitude \( U_o \) | 0.1 m      |
| Damp coefficient \( c \)       | 0.5 Nm/s    |
| Static tension \( H_o \)       | 9352500 N   |

Table 1 Geometric parameter and material property of stay cable

where the modulation coefficient \( \beta = \frac{T}{H_o} = \frac{AEU_o}{T^2H_o} = 0.161199663 \), the velocity of transverse vibration wave \( v = \left(\frac{H_o}{p}\right)^\frac{1}{2} = 278.7085837 \) (m/s).

The trigonometric series approach is used to solve the forced response on the number of series expansion terms \( k = -14, -13, \ldots, -2, -1, 0, 1, 2, \ldots, 13, 14 \). In most cases, the excitation frequency \( \omega_p \) of external force is not equal to the parametric vibration frequency \( \omega_p \). The computation of forced vibration response under the first 9 order vibration modes \( i = 1, 3, 5, 7, 9 \) is divided into two cases.

3.1. Excitation of different frequency

When the external excitation frequency \( \omega_p = 2 \) (rad/s) and the amplitude of distributed external force \( p = 0.1 \) (N/m), substitute them into Eq. (20). In the numerical computation, the forced vibration response time varies from 0 to 100s, with a computation step of 0.001s. According to Eq. (23), the forced vibration response \( w(x, t) \) of the parametric system is given under the vibration mode \( i = 1, 3, 5, 7, 9 \).

When \( x = l/2 \), the time history of forced vibration \( w(l/2, t) \), its spectrum and its phase trace are shown in Fig. 2.
3.2. Excitation of same frequency

Let the parameter excitation frequency and the external excitation frequency equal, $\omega_p = \omega_o = 3\text{rad/s}$, other parameters remain unchanged, the coefficient $A_{ik}$ and $A_{ik}^*$ ($B_{ik}$) in the first 9 order modes can
also be obtained. The forced vibration response under the same frequency excitation is that. When \( x = l/2 \), the time history of forced vibration \( w(l/2, t) \), its spectrum and its phase trace are computed, as shown in Fig.3.

(a) Time history of forced parametric vibration \( w(l/2, t) \)

(b) Spectrum of forced parametric vibration \( w(l/2, \omega) \)

(c) Phase diagram of forced parametric vibration

Fig. 3 Forced vibration of stay cable in same frequencies \( (\omega_o = \omega_p = 3) \)
4. Computation Error

In order to estimate the calculation accuracy of forced parametric vibration response, the approach error is defined as follows:

$$e = \frac{\partial^2 w}{\partial t^2} + \frac{c}{\rho} \frac{\partial w}{\partial t} - v^2 (1 + \beta \cos \omega_0 t) \frac{\partial^2 w}{\partial x^2} - p \cos \omega_p t$$

(24)

The unit of approach error $e$ is acceleration unit, $\text{m/s}^2$.

Introducing relative error definition:

$$\Delta = e_{\text{max}} / \left( \frac{\partial^2 w}{\partial t^2} \right)_{\text{max}}$$

(25)

In given two examples, the relative approach errors $\Delta$ of the forced vibration response are 0.0153 and 0.0490 respectively at the cable length $l/2$. Also, in the parametric forced vibration response approach, if the number of modes involved in the approach increases, the relative approach error will decrease.

Because the cable can be seen as an infinite degrees of freedom system, only more modes superposition involved in the numerical computation of forced vibration can the approaching error become smaller, as shown in Fig.15 (the case of same frequency excitation).

Fig. 4 Approaching error reducing with number of modes involved

5. Conclusion

Based on the Galerkin discretization method and frequency splitting, a new approach of a special trigonometric series and mode function is presented and a closed-form solution of forced parametric vibration response under the external harmonic excitation can be fully determined without considering cable inclination and gravity.

The vibration response coefficient vectors can be quickly obtained by determining the modal force matrix and splitting matrix with the material characteristics, parametric excitation frequency and external harmonic excitation frequency and amplitude. And the computational approach and calculation error estimation of the forced response solution of the parametric vibration can be obtained. Combined with an example of C13 of the Sifang Cable-stayed Bridge, the calculation results of the forced response are given to verify the validity of the approach form. This study provides a new idea for the theoretical study and approach calculation of the forced response of continuum parametric vibration.

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