Roles of the transition amplitude phases in photoelectron asymmetry of single strong attosecond pulse

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Abstract

The angular distributions of the photoelectrons in ionization of hydrogen atom by both circularly and linearly polarized intense extreme ultraviolet (XUV) attosecond pulse are investigated by numerically solving the time-dependent Schrödinger equation. We clearly identify nonperturbative features in studying the asymmetrical photoelectron angular distributions in the polarization plane for the XUV photon energy (16.3 eV) close to the ionization threshold, while such nonperturbative features are absent for higher photon energy (36 eV) in the same pulse intensity region. In addition to the carrier-envelope phase (CEP) dependence, the ejection asymmetry of the photoelectron is also sensitive to the relative phases of transition amplitudes in absorbing one photon and two photons. As a consequence, the CEPs corresponding to the maximal (or zero) asymmetry obviously vary as the pulse intensity increases in a moderately large region from $1 \times 10^{15}$ W cm$^{-2}$ to $30 \times 10^{15}$ W cm$^{-2}$. We attribute the intensity dependence of the transition amplitude phases to a consequence of the depletion of population as well as the Stark energy shift of the initial state. We show that the relative phases of transition amplitudes can be precisely decoded from the pulse intensity dependence of the ejection asymmetry and those phases are insensitive to the ellipticity of the laser pulse.

1. Introduction

The production of few-cycle laser pulse [1] opens the door to steer the photoionization dynamics by varying the carrier-envelope phase (CEP) of the laser pulse. A prominent effect in applying few-cycle laser pulses is the asymmetrical angular distribution of the photoelectron in the polarization plane. Depending on the pulse parameters, different pictures can be utilized to explain the symmetry breaking in ejecting the photoelectrons. In the regime of tunneling ionization of infrared (IR) field, the ionization rate is decided by the instantaneous electric field strength $F(t)$ whose waveform is sensitive to the CEP of the few-cycle laser pulses. Therefore, the ejection asymmetry of photoelectron directly mirrors the asymmetry of the waveform of the pulse [2–10]. Utilizing the asymmetrical angular distribution of the photoelectron in above-threshold ionization, it is possible to develop a phasemeter [11] to record the CEP of the few-cycle linearly polarized (LP) laser pulses. For circularly polarized (CP) IR pulses, the lack of inversion symmetry of the field also results in a pronounced left–right asymmetry in the angle-resolved photoelectron spectrum [12], which has been used to determine the duration of few-cycle pulses and its absolute CEP [13]. Except for the photoemission asymmetry, the CEP effects are also identified in the spectra of stimulated radiation [14] and in the chemical bond breaking process [15–18].

The experimental generations of the few-cycle [19] and even single-cycle [20, 21] attosecond pulses in the extreme ultraviolet (XUV) regime with stable CEP have already been possible nowadays. Coherent and ultrashort pulse radiation is widely generated from x-ray free-electron lasers [22] and laser-driven XUV and x-ray sources [23]. Though these XUV pulses in the intensities of the order of $10^{15}$–$10^{16}$ W cm$^{-2}$ is not available experimentally, they are expected to become reality by the high-order frequency mixing (HFM)
[24, 25] or high-order parametric generation process [26]. For the tuning of CEP of XUV pulses, reference [20] pointed out that the attosecond CEP could be finely tuned by using aluminum foils with variable thickness. Moreover, recently, Birulia et al [27] proved that the CEP of XUV pulse via HFM can be easily controlled by the phases of the generating fields. Although ultrashort XUV pulses with tunable CEP is not experimentally available yet, a large number of precedent theoretical studies about CEP effects in the XUV regime for atoms [28–34] and H$_2^+$ molecule [35, 36] have been performed. Ionization in the XUV regime is usually dominated by multiphoton absorption picture, in which the ejection asymmetry can be treated as a result of the interference of the electrons with opposite parities ejected by absorbing even and odd number of photons, respectively [29, 31, 32, 37]. The ejection asymmetry in the multiphoton ionization regime is routinely well described by the standard perturbation theory (PT) [31, 32]. Since the relative phases of transition amplitudes predicted in PT do not change with the pulse intensity, the maximal or zero ejection asymmetry appears at a fixed CEP as the laser intensity increases [31, 38]. This character of intensity insensitivity plays an important role in determining the CEP of the pulse from the ejection asymmetry.

However, if the dynamic Stark shift and the depletion of the initial state become relevant, which is responsible for the appearance of the dynamic interference [39–42] for ultrastrong XUV pulses with duration of few femtoseconds, the ionization process would enters into the deep nonperturbative regime. It has been demonstrated that the CEPs corresponding to maximal or zero ejection asymmetry do show obvious intensity dependence in the deep nonperturbative regime of IR pulse [38, 43]. Similarly, for attosecond XUV pulses, the intensity dependence of the phases of transition amplitude in nonperturbative regime also affects the CEPs corresponding to maximal or zero ejection asymmetry as discussed later in this work. This feature provides a possible way to retrieve the intensity dependence of nonperturbative transition phases, which has not attracted much attention so far. Previous schemes on retrieve of transition phases mostly employ two time-delayed pulses (XUV + IR), such as attosecond streaking [20, 44, 45] and RABBIT (reconstruction of attosecond harmonic beating by interference of two-photon transitions) [46–48]. More recently, the 1–2 quantum beat method is suggested to retrieve phase variations of the photoelectron as a continuous function of energy [49]. However, in this work, we will show that the intensity dependence of phases of transition amplitude can be retrieved from asymmetry of angular distribution using only one intense pulse.

In this paper, we investigate the photoelectron ejection asymmetry in the ionization of hydrogen atom by an intense few-cycle XUV pulse based on the numerical calculations of the full-dimensional time-dependent Schrödinger equation (TDSE). The photon energies are chosen as 16.3 and 36 eV, for which the ionization process is in the nonperturbative and perturbative regimes, respectively. In our cases, the ejection asymmetry can be attributed to the interference of the ionization channels in absorbing one photon and two photons. We find the maximal or zero ejection asymmetry is not fixed at a particular CEP as the increase in laser intensity. Instead, the CEPs corresponding to the maximal or zero ejection asymmetry monotonously vary as the laser intensity increases from $1 \times 10^{15}$ W cm$^{-2}$ to $30 \times 10^{15}$ W cm$^{-2}$. Such an intensity sensitivity clearly demonstrates the breakdown of the standard PT. To address the origination of this intensity sensitivity, we extract the nonperturbative transition amplitudes in absorbing one photon and two photons from the final electron wave packet obtained in numerical TDSE calculations [50, 51], and compare with the results calculated by nonperturbative strong field model (SFM) which accounts for the depletion of population and Stark energy shift of the initial state [52]. The coherent addition of the one-photon and two-photon amplitudes obtained from SFM well reproduces the energy-resolved asymmetry in the full TDSE calculations. We further show that the variation of the CEPs of the maximal or zero ejection asymmetry directly mirrors the intensity dependence of the relative transition amplitude phases. Interestingly, those transition amplitude phases are nearly independent of the ellipticity of the laser pulse. The variations of the CEP of the maximal or zero ejection asymmetry are nearly identical for CP and LP laser pulses.

The rest of this paper is organized as follows. In section 2, we briefly describe the TDSE numerical method, and define the left–right asymmetry in the angular distribution of ionized electrons. In section 3, we propose the nonperturbative SFM for the calculation of the asymmetry parameter for CP and LP pulses. In section 4, we show our numerical results and analyze in detail the intensity dependence of transition amplitude phase which controls the ejection asymmetry in angular distribution. In the last section, our main conclusions of this paper are summarized. In addition, appendix gives the derivation of perturbative transition amplitude phases of the one-photon ionization amplitude. Atomic units are used throughout the paper unless explicitly stated.
Figure 1. (a) The temporal profile of the attosecond XUV pulses in z direction. (b) Spectrum of the XUV pulses (black line) shown in relation to the ionization potentials (13.6 eV) of hydrogen atom. Red line denotes two-photon absorption signal. The green overlap region indicates the photoelectron energy range of interference between one-photon and two-photon amplitudes. The photon energy $\omega = 16.3$ eV, and the FWHM $\tau = 253$ as. The spectral width of XUV pulse $\Delta \omega = 10.2$ eV is marked in (b).

2. TDSE method

The time evolution of the single-electron wave function $\Psi(r, t)$ in dipole approximation and velocity gauge is described by the following TDSE

$$i \frac{\partial}{\partial t} \Psi(r, t) = \left( -\frac{\nabla^2}{2} + V(r) - i A(t) \cdot \nabla \right) \Psi(r, t), \tag{1}$$

where $V(r) = -1/r$ and $A(t)$ denote the Coulomb potential of the H atom and the vector potential of the pulse, respectively. We use the split-Lanczos propagator [53–55] to propagate the wave function in time. In our calculations, the wave function is expressed uniquely as a sum of spherical harmonic functions, and the radial part of the wave function is discretized using the finite-element discrete variable representation (FE-DVR) method [56–58]. The maximal angular momentum of $l_{\text{max}} = 10$, the radial box size of $r_{\text{max}} = 100$ a.u., and the time step of $\Delta t = 0.01$ a.u. are sufficient to ensure the convergence for laser parameters chosen in this work.

In our simulation, the laser pulse propagates along the positive z axis and is polarized in the x–y plane. The vector potential $A(t)$ with CEP $\phi$ is written as [59]

$$A(t) = \frac{A_0}{\sqrt{2}} f(t) \left[ \cos(\omega t + \phi) e_x + \xi \sin(\omega t + \phi) e_y \right], \tag{2}$$

where $\omega$ is the photon energy, and $A_0 = \sqrt{I_0}/\omega$ is the amplitude of the vector potential with the peak intensity $I_0$. $\xi = \pm 1$ correspond to right- (+1) or left- (-1) handed CP pulses, and $\xi = 0$ corresponds to LP pulse. The Gaussian pulse, whose envelope is described by

$$f(t) = \exp \left( -2 \ln 2 \frac{t^2}{\tau^2} \right) \tag{3}$$

for the time interval $(-T/2, T/2)$ and $f(t) = 0$ elsewhere. $\tau$ is the full width at half maximum (FWHM) of the laser pulse, and $\tau \equiv n_p(2\pi/\omega) \approx 253$ as with $n_p = 1$ optical cycle when $\omega = 16.3$ eV (see figure 1(a)). This corresponds to a spectral width $\Delta \omega = 4\sqrt{2} \ln 2/\tau = 10.2$ eV (see figure 1(b)). The green overlap of one-photon and two-photon signals in figure 1(b) describes photoelectron energy range of interference is from 6.4 to 16.4 eV with strongest interference at $E = 10.9$ eV. The time span $T$ is set as six optical cycles in our calculation.

The differential ionization probability is obtained by projecting the final wave function at time $t_f$ (i.e., after interaction with the laser pulse) to the scattering state $\Psi_k(r)$ [29], i.e.,

$$P(k) = P(k, \theta, \phi) = |\langle \Psi_k(r) | \Psi(r, t_f) \rangle|^2, \tag{4}$$

with $k$ the asymptotic momentum of the photoelectron. The differential ionization probability in the polarization plane (x–y plane) is given by
\[ P(E, \varphi) = kP(k, \theta = \pi/2, \varphi), \] (5)

where \( E = \frac{k^2}{2} \) is electron’s kinetic energy.

To quantify the asymmetry in the ionized electron momentum distributions in the polarization plane, we label ejection direction along the negative x axis (\( k_x < 0 \)) as the ‘left’ and the positive \( k_x \) axis (\( k_x > 0 \)) the ‘right’, and define the energy-resolved normalized left–right asymmetry parameter as

\[ A(E, \phi) = \frac{P_{\text{right}}(E) - P_{\text{left}}(E)}{P_{\text{right}}(E) + P_{\text{left}}(E)}, \] (6)

with

\[ P_{\text{right}}(E) = \int_{-\pi/2}^{\pi/2} P(E, \varphi) d\varphi, \] (7)

and

\[ P_{\text{left}}(E) = \int_{-\pi/2}^{\pi/2} P(E, \varphi) d\varphi. \] (8)

The energy-integrated asymmetry parameter is given by

\[ A(\phi) = \frac{\int dEP_{\text{right}}(E) - \int dEP_{\text{left}}(E)}{\int dEP_{\text{right}}(E) + \int dEP_{\text{left}}(E)}. \] (9)

3. Nonperturbative SFM

In this section, we present a simple non-perturbative SFM with considering the depletion of population as well as the Stark energy shift of the initial state, which provides insights into the origination of the intensity dependence of the phases of transition amplitude when the electron absorbs one or two photons.

3.1. One-photon and two-photon transition amplitudes

For the present XUV fields, one-photon transition to the continuum still dominates over multiphoton ionization. The one-photon transition amplitude in the velocity gauge is given by [52]

\[ A_1(k) = -\frac{1}{\pi} \int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_k |A(t') \cdot \nabla|\Psi_i| e^{iE' t} s_0(t') dt', \] (10)

where \( |\Psi_k\rangle \) is the final scattering state with asymptotic momentum \( k \), \( E = \frac{k^2}{2} \), \( |\Psi_i\rangle \) is the initial ground state with energy \( E_i = -13.6 \text{ eV} \) (−0.5 a.u.), and \( s_0(t') \) is the amplitude of the initial state at \( t' \). The occupation amplitude \( s_0(t') \) includes the depletion of population as well as the Stark energy shift of the initial state,

\[ s_0(t') = g(t') \exp \left[ -i \left( E_{t'} + \int_{-T/2}^{t'} \delta E(t') dt' \right) \right] \] (11)

with \( g(t') \) the modulus of \( s_0(t') \) and \( \delta E(t') \) the instantaneous energy shift induced by the XUV pulse.

The interaction operator \( A(t') \cdot \nabla \) in the rotating wave approximation (RWA) can be written as

\[ A(t') \cdot \nabla = \frac{A_0}{2\sqrt{2}} f(t') e^{-i\omega t'} (\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}). \] (12)

Substituting equation (12) into equation (10), the amplitude \( A_1 \) can be rewritten as

\[ A_1 = -\frac{A_0 e^{-i\phi}}{2\sqrt{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(E' - \omega t') t} f(t') s_0(t') dt'. \] (13)

It can be easily seen from equation (13) that the one-photon transition amplitude phase is contributed by the radial dipole matrix element and the Fourier transform of the pulse envelope. Obviously, the transition phase from the radial dipole matrix element is independent of laser intensity.

To explicitly show the intensity dependence of transition amplitude phase, the Fourier transform of the pulse envelope can be rewritten as

\[ \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{i(E' - \omega t') t} f(t') s_0(t') dt' = \int_{-\frac{T}{2}}^{\frac{T}{2}} \{ \cos[h(t')] + i \sin[h(t')] \} f(t') g(t') dt', \] (14)
where

$$h(t') = (E - E_i - \omega)t' - \int_{-T/2}^{t'} \delta E_i(\tau')d\tau'.$$

(15)

If the depletion of population and Stark shift of initial state are ignored, i.e., $g(t') = 1$ and $\delta E_i(t') = 0$, then the conventional transition amplitude in first-order PT would be recovered, and the time integration over $(-T/2, T/2)$ of the imaginary part of equation (14) is zero since $\sin[(E - E_i - \omega)t']f(t')$ is an odd function. Thus, equation (14) can be rewritten as $\int_{-T/2}^{T/2} \cos[(E - E_i - \omega)t']f(t')dt'$, which is a positive real and intensity-independent number. Hence, in perturbative regime, the one-photon transition amplitude phase can be regarded as intensity-independent number. However, in deep nonperturbative regime, intensity dependence of $g(t')$ and $\delta E_i(t')$ results in the imaginary part of equation (14) not zero, leading to the intensity dependence of transition amplitude phases.

The two-photon transition amplitude is given by

$$A_2(k) = \sum_{q \neq 1} \int_{-T/2}^{T/2} dt' \langle \Psi_k | \mathbf{A}(t') \cdot \nabla | \Psi_q \rangle e^{i(E-E_q)t'} \int_{-T/2}^{t'} dt'' \langle \Psi_q | \mathbf{A}(t'') \cdot \nabla | \Psi_i \rangle e^{iE_i t''} s_0(t''),$$

(16)

where $| \Psi_q \rangle$ with energy $E_q$ is the intermediate state. Similar to one-photon transition amplitude, equation (16) also can be decomposed into the product of the radial dipole matrix element and the Fourier transform of the pulse envelope

$$A_2 = \frac{A_2^2 e^{-2\delta_0}}{8} \sum_{q \neq 1} \langle \Psi_k | \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} | \Psi_q \rangle \langle \Psi_q | \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} | \Psi_i \rangle \int_{-T/2}^{T/2} e^{i(E-E_q)t'} f(t') \chi(t')dt',$$

(17)

where

$$\chi(t') = \int_{-T/2}^{t'} e^{i(E-E_q-t')f(t'')}s_0(t'')dt''.$$

(18)

If the depletion of population and Stark shift of initial state are ignored, i.e., $s_0(t'') = e^{iE_0 t''}$, the conventional transition amplitude in 2nd-order PT would be recovered, then $\chi(t') = \int_{-T/2}^{t'} e^{i(E-E_i-t')f(t'')}f(t'')dt''$ is an intensity-independent number. Hence, in perturbative regime, the two-photon transition amplitude phase is also independent of laser intensity.

In our SFM and PT calculations, the wave functions of initial state and each intermediate eigenstate are obtained by the diagonalization of free-field Hamiltonian, which is discretized using the FE-DVR representation method. In the calculation of equation (17), the radial box size of $r_{\text{max}} = 50$ a.u., the first 100 intermediate eigenstates are chosen within the total 300 intermediate eigenstates, which already make sure the convergence of calculation. The numbers of finite element and basis function are 50 and 7, respectively. The final scattering states are obtained analytically from reference [29].

3.2. Asymmetry parameter for a right-handed CP pulse

According to the selection rules for the dipole transitions in RWA, the orbital angular momentum number $l$ can increase or decrease by one in absorbing one right-handed CP photon, while the magnetic quantum number $m$ must increase by 1. Therefore, equation (10) can be factorized as

$$A_i(k) = M_i(k) Y_i^m(\theta, \phi)e^{-i\alpha_i},$$

(19)

corresponding to the state $| l = 1, m = 1 \rangle$, where $Y_i^m(\theta, \phi)$ is the spherical harmonics function. The $k$-dependent part $M_i(k) \equiv M_i(k)e^{i\alpha_1}$, where $M_i$ and $\alpha_1$ are the modulus and phase of $M_i$, respectively. In appendix, we give the phase $\alpha_1$ calculated by 1st-order PT. Its value is $3\pi/2 + \sigma_1$ for CP pulse, where $\sigma_1$ is the Coulomb phase shift. Clearly, $\alpha_1$ is independent of the intensity. Similar to equation (19), the second-order transition amplitude for CP pulse takes the following factorized form

$$A_2(k) = M_2(k) Y_2^m(\theta, \phi)e^{-i\alpha_2},$$

(20)

corresponding to the final state of $| l = 2, m = 2 \rangle$, where $M_2(k) \equiv M_2(k)e^{i\alpha_2}$.

Note that the asymmetry parameter equals to zero if only the $A_1(k)$ or $A_2(k)$ contributes. The asymmetrical distribution can originate from the interference of the amplitudes $A_1(k)$ and $A_2(k)$,

$$P_{12}(E, \phi) = k |A_1 + A_2|^2 = k [A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi + a_{21} - \phi)].$$

(21)

where $a_{21} = a_2 - a_1$, and
$X_1 = |M_1(k)Y_1^0|$, $X_2 = |M_2(k)Y_2^0|$, (22)

for $\theta = \pi/2$. Based on equations (6) and (21), the energy-resolved asymmetry parameter from the interference between one-photon and two-photon ionization is given by

$$A_{12} = \frac{4A_1A_2}{\pi(X_1^2 + X_2^2)} \cos(a_{21} - \phi)$$

(23) for CP pulse.

3.3. Asymmetry parameter for a LP pulse

For a LP pulse, the one-photon amplitude in the polarization plane can be expressed as

$$A_1(k) = M_1(k)(-Y_1^1 + Y_1^{-1})e^{-i\phi},$$

(24) corresponding to final scattering states with $|l = 1, m = 1\rangle$ and $|l = 1, m = -1\rangle$, respectively, where $M_1(k) \equiv M_1(k)e^{i\phi}$. The two-photon amplitude of final scattering states with $|l = 0\rangle$ and $|l = 2\rangle$ is

$$A_2(k) = M_0(k)Y_0^0e^{-2i\phi} + M'_2(k)\left(Y_2^0 - \frac{1}{3}Y_2^0 + Y_2^{-2}\right)e^{-2i\phi},$$

(25) where $M'_0(k) \equiv M'_0(k)e^{i\phi}$ is the transition amplitude of final scattering states $|l = 0, m = 0\rangle$, and $M'_2(k) \equiv M'_2(k)e^{2\phi}$. Hence, the energy-resolved asymmetry parameter from the interference between one-photon and two-photon amplitudes is given by

$$A_{12} = \frac{24X'_0X'_1\cos(a_{10} + \phi) + 32X'_1X'_2\cos(a_{21} - \phi) + \pi(3X'_0^2 + 6X'_1^2 + 4X'_0X'_2 + 4X'_0X'_2\cos a_{20})}{|X'_1|^2 + |X'_2|^2}$$

(26) for LP pulse, where $a_{10} = a_1 - a_0, a_{20} = a_2 - a_0$, and

$$X'_0 = |M'_0(k)Y_0^0|, \quad X'_1 = |M'_1(k)Y_1^0|, \quad X'_2 = |M'_2(k)Y_2^0|$$

(27) for $\theta = \pi/2$.

If the contribution of state $|l = 0, m = 0\rangle$ is ignored, i.e., $|X'_0| = 0$, then the energy-resolved asymmetry parameter from the interference between $|l = 1\rangle$ and $|l = 2\rangle$ amplitudes is given by

$$\tilde{A}_{12} = \frac{48X'_1X'_2}{\pi(9X'_1^2 + 11X'_2^2)} \cos(a_{21} - \phi)$$

(28) for LP pulse.

From equations (23) and (28), we see that the maximal asymmetry $|A_{12}|_{\max} = 2/\pi$, and $|\tilde{A}_{12}|_{\max} = 2.41/\pi$ are reached when $X_2/X_1 = 1$ for CP pulse and $X'_2/X'_1 = \sqrt{9/11} \approx 0.9$ for LP pulse, respectively. It is clear that the asymmetry parameter is controlled by both the CEP $\phi$ and the relative transition amplitude phase $a_{21}$ regardless of the polarization of the pulses.

Note that equations (23) and (26) can be also calculated by TDSE, where for CP pulse the amplitudes $M_1(k)$ and $M_2(k)$ can be extracted from TDSE by only including, in final scattering state, the partial waves with $|l = 1, m = 1\rangle$, and $|l = 2, m = 2\rangle$, respectively. Similarly, for LP pulse, the amplitudes $M'_0(k)$, $M'_1(k)$, and $M'_2(k)$ are also obtained by projecting calculation, by only including the final partial waves with $|l = 0, m = 0\rangle$, $|l = 1, m = 1\rangle$, and $|l = 2, m = 2\rangle$, respectively. After the real and imaginary part of amplitudes are obtained, the phases $a_1$ and $a_2$ are known by subtracting $\varphi$ and $2\varphi$, respectively.

4. Results and discussion

Figure 2 shows the momentum distribution of differential ionization probability of electrons for the H atom in the polarization plane with two kinds of polarization at laser intensities $I_0 = 20 \times 10^{15}$ W cm$^{-2}$. It is clearly seen that the momentum distribution has obvious left–right asymmetry for two polarizations, where a large proportion of electrons are emitted along the negative $k_x$ direction (‘left’). The projection of the momentum distribution onto the $k_x$ axis exhibits a peak structure at $k_x \approx -0.6$ a.u., which is different from the predication from energy conservation where one photon is absorbed (see the inner circle), indicating the one-photon ionization does not dominate here.

To investigate the CEP and intensity dependence of the left–right asymmetry parameter, we show the energy-integrated asymmetry originating from the angular distribution of ionized electron in figure 3. Here we introduce two photon energies 16.3 eV and 36 eV, for which the ionizations are in the nonperturbative
Figure 2. Momentum distribution of differential ionization probability of electrons in the polarization plane calculated by TDSE for (a) CP pulse and (b) LP pulse. The laser intensity is $20 \times 10^{15} \text{ W cm}^{-2}$, the central carrier frequency $\omega = 13.6 \text{ eV}$ and the corresponding CEP $\phi = \pi$. Two magenta circles denote the peaks of one-photon and two-photon ionization amplitudes, respectively.

Figure 3. CEP- and intensity-dependent asymmetry map by integrating over the whole electron energy (equation (9)) obtained by TDSE for CP pulse (a) and (b) and LP pulse (c) and (d) with two different photon energies: $\omega = 13.6 \text{ eV}$ (left column) and $36 \text{ eV}$ (right column). $\phi_{\text{max}}$ denotes the CEP when the asymmetry parameter takes a maximum value for each intensity (dashed lines with filled circles).

and perturbative regimes, respectively. For the two photon energies, the asymmetry parameter oscillates with a period of $2\pi$ as the CEP varies, and $A(\phi + \pi) = -A(\phi)$. Besides, since at low laser intensity one-photon ionization dominates, the energy-integrated asymmetry parameter is close to zero. At the largest laser intensity, the asymmetry parameter increases to about 0.43 for CP and 0.57 for LP pulses due to the contribution of the high-order ionization transition when $\omega = 13.6 \text{ eV}$. However, for higher photon energy $\omega = 36 \text{ eV}$ the maximum asymmetry parameter just reaches 0.08 and 0.13 for CP and LP pulses, respectively. The asymmetry in angular distribution of $\omega = 13.6 \text{ eV}$ is in general larger than that of $\omega = 36 \text{ eV}$, which can be attributed to that the overlap between one-photon and two-photon transition amplitudes decreases with increasing $\omega$ (as shown in figures 5 and 6). Another remarkable difference between the two frequencies is that for $\omega = 13.6 \text{ eV}$ the CEP which corresponds to the maximum asymmetry (indicated by $\phi_{\text{max}}$ here) increases as the laser intensity increases (see the dashed lines in figures 3(a) and (c)), but for $\omega = 36 \text{ eV}$ $\phi_{\text{max}}$ is fixed at about $\pi$ as intensity varies (see the dashed lines in figures 3(b) and (d)). As reference [31] stated that the applicability of the PT description of ionization is governed not only by the intensity but also by the smallness of the parameter $\zeta = I_0/(4\omega^3)$. If the parameter $\zeta$ is much smaller than 1, the ionization can be regarded as in the perturbative regime. On the other hand, if $\zeta$ is close to or even larger than 1, the ionization enters into the nonperturbative regime. For pulses with frequency 36 eV and peak intensity $I_0 = 20 \times 10^{15} \text{ W cm}^{-2}$, the parameter $\zeta$ is about 0.25, thus indicating
that we are in the perturbative regime. However, if $\omega = 13.6 \text{ eV}$, the parameter $\zeta = 2.64$ at the same intensity. Therefore, the ionization process cannot be well described by the standard PT any more.

For these two frequencies, the population of the ground state differs remarkably as time varies, as shown in figures 4(a) and (b). For $\omega = 13.6 \text{ eV}$ the population of the ground state starts to decrease rapidly at about $t = -11 \text{ a.u.}$, followed by a stable value of $9.26\%$ at the end of the electric field for CP pulse and $9.57\%$ for LP pulse. However, for $\omega = 36 \text{ eV}$, the final populations of the ground state are $95.2\%$ for both polarizations. Hence, at the same intensity, the depletion of the ground state should be taken into account for $\omega = 13.6 \text{ eV}$, while for $\omega = 36 \text{ eV}$ the depletion of the ground state can be ignored. Except the depletion of the ground state, its Stark energy shift also plays an important role in transition amplitude phases. In figures 4(c) and (d), for $\omega = 13.6 \text{ eV}$ the energy of the ground state $E_i$ of H atom as a function of time shows a obvious oscillation deviated from $-0.5 \text{ a.u.}$ ($-13.6 \text{ eV}$) for CP pulse, and for LP pulse this oscillation becomes more drastic. However, similar to the depletion of the ground state, for $\omega = 36 \text{ eV}$, the energy of the ground state just displays a slight shift as the time changes for CP and LP pulses.

From figure 5, we can clearly see that the energy dependence of transition amplitudes $X_i$ and phases $a_i$ ($i = 1, 2$) extracted from TDSE, which can be done by only choosing the partial waves $|l = 1, m = 1\rangle$ and $|l = 2, m = 2\rangle$ due to dipole selection rule for CP pulse, respectively, can be well reproduced by PT when $\omega = 36 \text{ eV}$. Figures 5(a) and (b) show that the maximum value of the transition amplitudes $X_i$ is higher than that of $X_1$ by an order of magnitude, hence, at energy range $0 < E < 40 \text{ eV}$ the one-photon ionization dominates. One-photon ionization phase $a_1$ increases rapidly as electron’s energy increases when the energy is small, but increases smoothly at higher energy. From PT, we know that $a_1$ is equal to $3 \pi/2 + \sigma_1$, where $\sigma_1 = \arg \Gamma(2 - i/k)$ is the Coulomb phase shift, is independent of the laser intensity. Two-photon ionization phase $a_2$ also does not depend on the laser intensity, resulting in the CEP of maximum asymmetry is fixed as intensity increases for CP pulse when $\omega = 36 \text{ eV}$ (see equation (23)).

As for situation where $\omega = 13.6 \text{ eV}$, the transition amplitudes $k^{1/2}X_i$ and phases $a_i$ ($i = 1, 2$) can not be well reproduced by PT, as shown by figure 6. Even though PT results show that the transition amplitudes $k^{1/2}X_i$ peaked around $E = i\omega + E_i$ ($i = 1, 2$) just as the results obtained from TDSE, the peaks calculated by PT are almost twice the magnitude of TDSE results (figures 6(a) and (b)). One-photon transition phase $a_1$ extracted from TDSE undergoes a transient increase, and then drops smoothly until $E \approx 10.8 \text{ eV}$, while for PT results, $a_1$ always increases. Hence, the difference between TDSE and PT results enlarges with increasing electron’s energy (figure 6(c)). Similarly, two-photon transition phase $a_2$ extracted from TDSE also increases at first and then decreases gradually, but PT results does not follow this trend (figure 6(d)). However, the SFM which accounts for the depletion of population as well as the Stark energy shift of the
Figure 5. Energy-dependent one-photon transition amplitudes $k^{1/2}X_1$ and phases $a_1$ (left column) extracted from TDSE and calculated by 1st-order PT. Two-photon transition amplitudes $k^{1/2}X_2$ and phases $a_2$ (right column) extracted from TDSE and calculated by 2nd-order PT. The laser pulse is CP and the intensity is $5 \times 10^{15}$ W cm$^{-2}$, the photon energy is $\omega = 36$ eV, and the CEP $\phi = 0$.

Figure 6. Energy-dependent one-photon transition amplitude $k^{1/2}X_i$ and phase $a_i$ (left column) extracted from TDSE, in comparison with results calculated by 1st-order PT and SFM. Energy-dependent two-photon transition amplitude $k^{1/2}X_i$ and phase $a_2$ (right column) extracted from TDSE, in comparison with results calculated by 2nd-order PT and SFM. The laser pulse is CP and the intensity is $5 \times 10^{15}$ W cm$^{-2}$, the photon energy is $\omega = 13.6$ eV, and the CEP $\phi = 0$.

initial state can well reproduce TDSE results, including the transition amplitudes $k^{1/2}X_i$ and phases $a_i$ ($i = 1, 2$) as shown by figure 6. Hence, in the non-perturbative regime, it can be concluded that intensity dependence of transition amplitude phases originates from the depletion of population as well as the Stark energy shift of the initial state. The deviation of $a_1$ between SFM and TDSE beyond $E > 12$ eV is due to that this energy range is far away from one-photon transition peak.
The energy-resolved asymmetry mainly originates from the interference between one-photon and two-photon ionization amplitudes, as shown in figure 7. The asymmetry parameters defined by equation (6) calculated by TDSE, PT and SFM for CP and LP pulses are plotted in figures 7(a) and (b). The ‘extracted’ asymmetry parameters are equivalent to the results by choosing the partial waves \(|l| = 1\) and \(|l| = 2\) from TDSE. The agreement between ‘extracted’ and TDSE indicates that the asymmetry is caused by the interference between one-photon and two-photon amplitudes. The energy-resolved asymmetry parameters and transition amplitude phases in numerical TDSE results are well reproduced by the SFM, instead of PT. Energy dependence of the left-right asymmetry is determined by the relative magnitude \(|\mathcal{M}_1|/|\mathcal{M}_2|\) (or \(|\mathcal{M}'_1|/|\mathcal{M}'_2|\)) of the two interfering transition amplitudes. At intensity \(I_0 = 5 \times 10^{15} \text{ W cm}^{-2}\) for CP pulse, from figure 7(a), we see that the left-right asymmetry appears at energy range \(3 < E < 20 \text{ eV}\) due to the interference of the first-order and the second-order transition amplitudes. Such asymmetry reaches its maximum at \(E = 9.5 \text{ eV}\), which corresponds to \(X_1 = X_1\) at this intensity. For LP pulse, the asymmetry reaches its maximum at \(X'_2 \approx 0.9X'_1\), and its absolute value is larger than that for CP pulse. Since the LP laser field can be treated as the superposition of a right-handed CP laser field and a left-handed CP laser field, it is not surprising that the transition amplitude phases \(a_{21}\) for LP pulse have similar energy dependence as that for CP pulse (figures 7(c) and (d)).

The intensity-dependence of \(\phi_{\text{max}}\) of the maximal asymmetry directly mimics the intensity-dependence of the transition amplitude phase (compare the black line and the dashed line in figure 8). Here we assume that the interference between one-photon and two-photon ionization amplitudes dominates in determining the energy-integrated asymmetry parameter. From equation (23) it is easy to find that for CP pulse the asymmetry parameter is proportional to \(\cos(a_{21} - \phi)\) and, thus, the maximal asymmetry is reached when \(a_{21} - \phi = \pm n\pi\) with \(n\) an arbitrary integer. So we get \(\phi_{\text{max}} = a_{21}^{\text{max}} \pm n\pi\) (figure 8(a)), where \(a_{21}^{\text{max}}\) represents the phase difference between amplitudes \(\mathcal{M}_1(k)\) and \(\mathcal{M}_2(k)\) when \(X_1 = X_2\). For LP pulse, the asymmetry parameter is also proportional to \(\cos(a_{21} - \phi)\), it is clearly seen from figure 8(b) that \(\phi_{\text{max}} = a_{21}^{\text{max}} \pm n\pi\) for each intensity. The difference between \(\phi_{\text{max}}\) and \(a_{21}^{\text{max}} + \pi\) for LP pulse at higher intensity demonstrates that the contribution from scattering state \(|l| = 0, m = 0\) for LP pulse is much larger than that for CP pulse.

At last, we show the ellipticity dependence of energy-integrated asymmetry as CEP varies in figure 9. For a fixed intensity \(I_0 = 20 \times 10^{15} \text{ W cm}^{-2}\), the asymmetry is maximum for a fixed, particular value of \(\phi = \phi_0 \approx 1.05\pi\) as the ellipticity increases from 0 to 1 (figure 9(a)). It indicates that energy-resolved transition amplitude phases is independent of the ellipticity of laser pulse. Figure 9(b) shows that as the ellipticity increases the absolute value of maximum asymmetry decreases monotonously. This regularity
Figure 8. Comparison of the CEP which corresponds to maximum asymmetry $\phi_{\text{max}}$ and intensity-dependent transition amplitude phases $a_{\text{21}}$ as intensity varies for (a) CP pulse and (b) LP pulse. $a_{\text{21}}^{\text{max}}$ denotes the transition amplitude phases where the one-photon amplitude is equal to two-photon amplitude.

Figure 9. (a) CEP-dependent energy-integrated asymmetry in the ionized electron distribution for the H atom as a function of the ellipticity $\xi$. The laser intensity is $I_0 = 20 \times 10^{15}$ W cm$^{-2}$, and the photon energy is $\omega = 13.6$ eV. (b) Ellipticity dependence of maximum asymmetry $|A_{\text{max}}|$ extracted from (a) for a fixed CEP $\phi = 1.05\pi$ and calculated by equation (29) with the known maximum asymmetries of $\xi = 0$ and $\xi = 1$.

would allow one to measure the ellipticity of laser pulse. For the elliptically polarized laser field, we can treat it as the superposition of a LP laser field and a CP laser field, i.e., $A(t) = (1 - \xi)A_0(t)\sqrt{1 + \xi^2} - \frac{\xi A_0(t)}{\sqrt{1 + \xi^2}}(\cos \omega t e_x + \sin \omega t e_y)$. Hence, the maximum energy-integrated asymmetry of arbitrary ellipticity can be written as

$$A_{\text{max}}(\xi) = \frac{(1 - \xi)}{\sqrt{1 + \xi^2}}A_{\text{max}}(\xi = 0) + \frac{\sqrt{2\xi}}{\sqrt{1 + \xi^2}}A_{\text{max}}(\xi = 1),$$

with neglecting the interference of electrons ionized from LP pulse and CP pulse, which is in good agreement with the extracted results from TDSE, as shown in figure 9(b) (dashed red line). So, if we obtain the maximum energy-integrated asymmetries of two ellipticity, the maximum energy-integrated asymmetries of arbitrary ellipticity can be obtained directly.

5. Conclusions

In summary, we have comprehensively studied the photoelectron ejection asymmetry in photoionization of hydrogen atom by moderately strong few-cycle XUV pulses with two photon energies. The features beyond the prediction of standard PT are successfully identified. The proposed SFM, which accounts for the depletion of population as well as the Stark energy shift of the initial state, well reproduces the transition amplitude phases extracted from TDSE in absorbing one photon and two photons. Due to the intensity dependence of transition amplitude phases, the CEPs corresponding to the maximal or zero asymmetry...
vary as the pulse intensity increase. This interesting phenomenon appears when either the CP or LP laser pulse is used. In addition, we also show that the intensity dependence of relative phases of the transition amplitude can be precisely extracted from the electron’s ejection asymmetry. Experimental study of these phenomena, requiring achieving broadband ultrashort XUV pulse with tunable CEP, is currently challenging. But given the rapid progress in experimental attosecond science, we believe the present predictions are able to be verified in the near future.

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Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix. Derivation of one-photon ionization transition amplitudes in perturbative regime

The one-photon ionization transition amplitude of 1st-order PT in the velocity gauge for a right-handed CP pulse followed by equation (13) is given by

\[ A_1 = -\frac{A_0 e^{-i\phi}}{2\sqrt{2}} \langle \psi_i | \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} | \Psi_f \rangle \int_{-T/2}^{T/2} e^{i(E-E_i)\rho} f(t')dt', \tag{A1} \]

where \( |\psi_i\rangle \) is the initial state with energy \( E_i = -0.5 \) a.u., and \( |\Psi_f\rangle \) is the final scattering state with momentum \( k \) and energy \( E = k^2/2 \). By using

\[ \langle \psi_i | \frac{\partial}{\partial \rho} | \psi_i \rangle = (E - E_i) \cdot \langle \psi_i | \rho | \psi_i \rangle, \quad \rho = x, y, z \tag{A2} \]

and

\[ x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \tag{A3} \]

the amplitude \( A_1 \) for CP pulse can be rewritten as

\[ A_1 = \frac{A_0}{4\sqrt{2\pi}} (E - E_i) \sin \theta e^{i\varphi} e^{-i\phi} \langle \psi_i | r | \psi_i \rangle \int_{-T/2}^{T/2} e^{i(E-E_i)\rho} f(t')dt', \tag{A4} \]

where \( |\psi_i\rangle \) and \( |\psi_k\rangle \) are radial wave function of initial and final states, respectively. \( |\psi_i\rangle \) is explicitly given by \( |\psi_i\rangle = 2e^{-r} \). \( |\psi_k\rangle \) is explicitly given by \([29]\]

\[ |\psi_k\rangle = i e^{-ik} \frac{1}{\sqrt{2\pi}} \frac{R_{kl}(r)}{k}, \tag{A5} \]

where \( \sigma_1 = \arg \Gamma(2 - i/k) \) is the Coulomb phase shift, and the radial wave function \( R_{kl} \) is given by

\[ R_{kl} = \frac{2e^{\pi/2k}}{3} |\Gamma(2 - i/k)| e^{-ikr} \frac{e^{-ikr}}{r} F_1(2 + i/k, 4, 2ikr), \tag{A6} \]

with \( F_1 \) the Kummer confluent hypergeometric function. The \( R_0 \) defined by equation (A6) is normalized by \( \int_0^\infty r^2 R_{kl}(r)R_{kl}(r) dr = 2\pi \delta(k - k') \). Then the radial coupling can be expressed as

\[ \langle \psi_k | r | \psi_i \rangle = (-i e^{i\varphi}) \frac{2}{3} \sqrt{2} |\Gamma(2 - i/k)| ke^{i\varphi} \int_0^\infty e^{i\varphi} F_1(2 + i/k, 4, 2ikr) r^3 e^{-r} dr. \tag{A7} \]

The Fourier transform of the pulse envelope \( \int_{-T/2}^{T/2} e^{i(E-E_i)\rho} f(t') dt' \) is a positive real and intensity-independent number, hence, from equation (A7), we find that in perturbative limit the transition amplitude phase
which is independent of laser intensity.

Similar to right-handed CP pulse, for LP pulse, the one-photon ionization transition amplitude of 1st-order PT can be expressed as

\[ A_1 = \frac{A_0}{2\sqrt{2}} e^{-i\phi} |\psi_i\rangle \langle \psi_i' | e^{i(E - E_i)'} f(t') dt' \]

\[ = \frac{A_0}{8\sqrt{2}\pi} (E - E_i) \sin \theta (e^{i\phi} + e^{-i\phi}) e^{-i\phi} |\psi_i\rangle |\psi_i' \rangle \int_{-T/2}^{T/2} e^{i(E - E_i)'} f(t') dt'. \]  

(A9)

Comparing equation (A9) with equation (A4), we find that the one-photon transition amplitude phase \( a_1 \) for LP pulse is the same as that for CP pulse.

**References**

[1] Nisoli M, De Silvestri S, Svelto O, Szpöcs R, Ferencek V, Spielmann C, Sartania S and Krausz F 1997 Opt. Lett. 22 522
[2] de Bohan A, Antoine P, Milschi D B and Piraux B 1998 Phys. Rev. Lett. 81 1837
[3] Cormier E and Lambropoulos P 1998 Eur. Phys. J. D 2 15
[4] Brabec T and Krausz F 2000 Rev. Mod. Phys. 72 545
[5] Paulus G G, Grasbon F, Walther H, Villoresi P, Nisoli M, Stagira S, Priori E and De Silvestri S 2001 Nature 414 182
[6] Milosović D B, Paulus G G, Bauer D and Becker W 2006 J. Phys. B: At. Mol. Opt. Phys. 39 R203
[7] Tong X M, Hino K and Toshima N 2006 Phys. Rev. A 74 013405(R)
[8] Li Y, Li M, Zhou Y, Xie H, Lan P and Lu P 2017 Opt. Express 25 11233
[9] Zille D, Adolph D, Moller M, Sayler A M and Paulus G G 2018 New J. Phys. 20 063018
[10] Kühnel M et al 2021 Phys. Rev. Lett. 126 113001
[11] Paulus G G et al 2004 Phys. Rev. Lett. 91 253004
[12] Milosović D B, Paulus G G and Becker W 2002 Phys. Rev. Lett. 89 153001
[13] Fukahori S et al 2017 Phys. Rev. A 95 053410
[14] Nakajima T and Watanabe S 2006 Phys. Rev. Lett. 96 213001
[15] Kling M F et al 2006 Science 312 246
[16] Xie X et al 2012 Phys. Rev. Lett. 109 243001
[17] Alnasr A S et al 2014 Nat. Commun. 5 3800
[18] Kühnel M et al 2016 Phys. Rev. Lett. 116 193001
[19] Baltuska A et al 2003 Nature 421 611
[20] Sansone G et al 2006 Science 314 443
[21] Goulielmakis E et al 2008 Science 320 1614
[22] Shimotake T et al 2008 Nat. Photon. 2 555
[23] Krausz F and Ivanov M 2009 Rev. Mod. Phys. 81 183
[24] Heyl C M, Rudawski P, Brizuela F, Bengtsson S N, Mauritssson J and L’Huillier A 2014 Phys. Rev. Lett. 112 143902
[25] Khokhlova M A and Strelkov V V 2020 New J. Phys. 22 093030
[26] Hört O, Dubrouil A, Khokhlova M A, Descamps D, Petit S, Burgay F, Mével F, Constant E and Strelkov V V 2021 Opt. Express 29 5982
[27] Birulia V A, Khokhlova M A and Strelkov V V 2022 Phys. Rev. A 106 023514
[28] Peng L-Y and Starace A F 2007 Phys. Rev. A 76 043401
[29] Peng L-Y, Pronin E A and Starace A F 2008 New J. Phys. 10 025030
[30] Peng L-Y, Tan F, Gong Q H, Pronin E A and Starace A F 2009 Phys. Rev. A 80 013407
[31] Pronin E A, Starace A F and Peng L-Y 2009 Phys. Rev. A 80 063403
[32] Pronin E A, Starace A F and Meng L-Y 2011 Phys. Rev. A 84 013417
[33] Dijokap J M N, Hu S X, Jiang W-C, Peng L-Y and Starace A F 2012 New J. Phys. 14 095010
[34] Ngoko Djiokap J M, Manakov N L, Meremianin A V, Hu S X, Madsen L B and Starace A F 2014 Phys. Rev. Lett. 113 223002
[35] Yuan K-J and Bandraud A D 2012 Phys. Rev. A 85 031413
[36] Yuan K-J and Bandraud A D 2016 J. Phys. B: At. Mol. Opt. Phys. 49 065601
[37] Roudnev V and Erny B D 2007 Phys. Rev. Lett. 99 220406
[38] Chelkowski S, Bandraud A D and Apolonski A 2004 Phys. Rev. A 70 013813
[39] Demekhin P V and Cederbaum I S 2012 Phys. Rev. Lett. 108 253001
[40] Bagheri M, Saalmann U and Rost J M 2017 Phys. Rev. Lett. 118 143202
[41] Jiang W-C and Burgdörfer J 2018 Opt. Express 26 19921
[42] Jiang W-C, Chen S-G, Peng L-Y and Burgdörfer J 2020 Phys. Rev. Lett. 124 043203
[43] Shvetsov-Shilovski N I, Räisänen E, Paulus G G and Madsen L B 2014 Phys. Rev. A 89 043431
[44] Drescher M, Hentschel M, Kienberger R, Tempea G, Spielmann C, Reider G A, Corkum P B and Krausz F 2001 Science 291 1923
[45] Itatani J, Quéré F, Yudin G L, Ivanov M Y, Krausz F and Corkum P B 2002 Phys. Rev. Lett. 88 173903
[46] Muller H G 2002 Appl. Phys. B 74 s17
[47] Caillat J et al 2011 Phys. Rev. Lett. 106 093002
[48] Klünder K et al 2011 Phys. Rev. Lett. 106 143002
[49] Fuchs J et al 2021 Phys. Rev. Res. 3 013195
[50] Ngoko Djiokap J M, Hu S X, Madsen L B, Manakov N L, Meremianin A V and Starace A F 2015 Phys. Rev. Lett. 115 113004
[51] Ngoko Djiokap J M et al 2016 Phys. Rev. A 94 013408
[52] Jiang W-C, Wang M-X, Peng L-Y and Burgdörfer J 2022 Phys. Rev. A 105 023104
[53] Jiang W-C and Tian X-Q 2017 Opt. Express 25 26832
[54] Wang S, Jiang W-C, Tian X-Q and Sun H-B 2020 Phys. Rev. A 101 033417
[55] Jiang W-C, Liang H, Wang S, Peng L-Y and Burgdörfer J 2021 Phys. Rev. Res. 3 L032052
[56] Rescigno T N and McCurdy C W 2000 Phys. Rev. A 62 032706
[57] Schneider B I and Collins L A 2005 J. Non-Cryst. Solids 351 1351
[58] Rayson M J 2007 Phys. Rev. E 76 026704
[59] Liang J, Jiang W-C, Wang S, Li M, Zhou Y and Lu P 2020 J. Phys. B: At. Mol. Opt. Phys. 53 095601