OR-Gym: A Reinforcement Learning Library for Operations Research Problems

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Abstract

Reinforcement learning (RL) has been widely applied to game-playing and surpassed the best human-level performance in many domains, yet there are few use-cases in industrial or commercial settings. We introduce OR-Gym, an open-source library for developing reinforcement learning algorithms to address operations research problems. In this paper, we apply reinforcement learning to the knapsack, multi-dimensional bin packing, multi-echelon supply chain, and multi-period asset allocation model problems, as well as benchmark the RL solutions against MILP and heuristic models. These problems are used in logistics, finance, engineering, and are common in many business operation settings. We develop environments based on prototypical models in the literature and implement various optimization and heuristic models in order to benchmark the RL results. By re-framing a series of classic optimization problems as RL tasks, we seek to provide a new tool for the operations research community, while also opening those in the RL community to many of the problems and challenges in the OR field.

Keywords: Machine Learning, Reinforcement Learning, Optimization, Operations Research, Robust Optimization

1 Introduction

Reinforcement learning (RL) is a branch of machine learning that seeks to make a series of sequential decisions to maximize a reward [Sutton and Barto 2018]. The technique has

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received widespread attention in game-playing, whereby RL approaches have beaten some of the world’s best human players in Go and DOTA2 [Silver et al. 2017, Berner et al. 2019]. There is a growing body of literature that is applying RL techniques to existing OR problems. Kool et al. 2019 use the REINFORCE algorithm with attention layers to learn policies for the travelling salesman problem (TSP), vehicle routing problem (VRP), the orienteering problem (OP), and a prize collecting TSP variant. Oroojlooyjadid et al. 2017 use a Deep Q-Network (DQN) to manage the levels in the beer game and achieve near optimal results. Balaji et al. 2019 provided versions of online bin packing, news vendor, and vehicle routing problems as well as models for RL benchmarks. Hubbs et al. 2020 use RL to schedule a single-stage chemical reactor under uncertain demand which outperforms various optimization models. Martinez et al. 2011 approaches a flexible job shop scheduling problem with tabular Q-learning to outperform other algorithms. Li 2017 provides an overview of the progress and development of RL as well as a review of many different applications.

We seek to provide a standardized library for the research community who wish to explore RL applications by building on top of the preceding work and releasing OR-Gym, a single library that relies on the familiar OpenAI interface for RL [Brockman et al., 2016], but containing problems relevant to the operations research community. To this end, we have incorporated the benchmarks in Balaji et al. 2019, while extending the library to the KP, multi-period asset allocation, multi-echelon supply chain inventory management, and virtual machine assignment. RL problems are formulated as Markov Decision Processes (MDP), meaning they are sequential decision making problems, often times probabilistic in nature, and rely on the current state of the system to capture all relevant information for determining future states. This framework is not widely used in the optimization community, so we make explicit our thought process as we reformulate many optimization problems to fit into the MDP mold without loss of generality. Many current RL libraries, such as OpenAI Gym, have many interesting problems, but problems that are not directly relevant to industrial use. Moreover, many of these problems (e.g. the Atari suite) lack the same type of structure as classic optimization problems, and thus are primarily amenable to model-free RL techniques, that is RL algorithms that learn with little to no prior knowledge of the dynamics of the environment they are operating in. Bringing well-studied optimization problems to the RL community may encourage more integration of model-based and model-free methods to reduce sample complexity and provide better overall performance. It is our goal that this work encourages further development and integration of RL into optimization and the OR community while also opening the RL community to many of the problems and challenges that the OR community has been wrestling with for decades.

The library itself along with example code to enable reproducibility can be found at www.github.com/hubbs5/or-gym

2 Background

Included in the library are knapsack, bin packing, supply chain, travelling salesman, vehicle routing, news vendor, portfolio optimization, and scheduling problems. We provide RL benchmarks using the RLlib package for a selection of these problems, as well as heuristic and optimal solutions [Moritz et al. 2018]. Additionally, we discuss environmental design
considerations for each of the selected problems. Each of the environments we make available via the OR-Gym package is easily customizable via configuration dictionaries that can be passed to the environments upon initialization. This enables the library to be leveraged to wider research communities in both operations research and reinforcement learning.

All mathematical programming models are solved with Gurobi 8.2 and Pyomo 5.6.2 to optimality on a 2.9 GHz Intel i7-7820HQ CPU unless otherwise noted (Gurobi Optimization LLC [2018], Hart et al. [2017]).

2.1 Reinforcement Learning

Reinforcement learning (RL) is a machine learning approach that consists of an agent interacting with an environment over multiple time steps $t$, to maximize the cumulative sum or rewards, $R_t$, the agent receives (see Figure 1, Sutton and Barto [2018]). The agent plays multiple episodes - Monte Carlo simulations of the environment - and at each time step within an episode, observes the current state ($S_t$) of the environment and takes an action according to a policy ($\pi$) that maps states to actions ($a_t$). The goal of RL is to learn a policy that obtains high rewards. The problems are formulated as Markov Decision Processes (MDPs), thus RL can be viewed as a method for stochastic optimization [Bellman, 1957].

Deep reinforcement learning uses multi-layered neural networks to fit a policy function that will map states to actions. Here, we will use the Proximal Policy Optimization (PPO) algorithm for our RL comparisons [Schulman et al., 2016]. This is an actor-critic method, which consists of two networks, one to produce the actions at each time step (the actor) and one to produce a prediction of the rewards at each time step (critic). The actor learns a probabilistic policy that produces a probability distribution over available actions. This distribution is sampled from during training to encourage sufficient exploration of the state space. The critic learns the value of each state, and the difference between the predicted value from the critic and the actual rewards received from the environment is used in the loss function ($\mathcal{L}(\theta)$) to update the parameters of the networks. PPO limits the update of the network parameters by clipping the loss function (Equation 1) relative to the previous policy.

$$\mathcal{L}(\theta) = \mathbb{E}_t \left[ \min(r_t(\theta), \text{clip}(r_t(\theta), 1 - \epsilon, 1 + \epsilon)) \hat{A}_t \right]$$

where $r_t(\theta)$ is the probability ratio between the previous policy, $\pi_{k-1}(\theta)$, and the new policy $\pi_k(\theta)$ and $k$ denotes the updates to the policy parameters since initialization. The function clip enforces the constraint $1 - \epsilon \leq r_t(\theta) \leq 1 + \epsilon$ on the probability ratio. $\epsilon$ is a hyperparameter that limits the update of the policy, such that the probability of outputs does not change.
more than \( \pm \varepsilon \) at each update. \( \hat{A}_t \) denotes the advantage estimation of the state, which is the sum of the discounted prediction errors over \( T \) time steps given in Equation 2:

\[
\hat{A}_t = \sum_{t=1}^{T} \gamma^{T-t+1} \delta_t
\]

where \( \delta_t \) is the prediction error from the critic network at each time-step \( t \) and \( \gamma \) is the discount rate. This modification to the loss has shown more stable learning over other policy gradient methods across multiple environments [Schulman et al., 2016].

We rely on the implementation of the PPO algorithm found in the RLlib package [Moritz et al., 2018]. All RL solutions use the same algorithm and a 3-layer fully-connected network with 128 hidden nodes at each layer, for both the actor and critic networks. Although some hyperparameter tuning is inevitable, we sought to minimize our efforts in this regard in order to reduce over fitting our results.

3 Knapsack

The Knapsack Problem (KP) was first introduced by Mathews [1896], and a classic exposition of the problem can be found in Dantzig [1957] where a hiker who is packing his bag for a hike is used as the motivating example. Obvious applications of the KP include determining what cargo to load into a plane or truck to transport. Other applications come from finance, we may imagine an investor with limited funds who is seeking to build a portfolio, or apply the framework to warehouse storage for retailers [Ma et al., 2019].

There are a few versions of the problem in the literature, the unbounded KP, bounded, multiple choice, multi-dimensional, quadratic, and online versions [Kellerer et al., 2004]. It is a combinatorial optimization problem that seeks to maximize the value of items \( (v_i) \) with a given non-negative weight \( (w_i) \) contained in a knapsack subject to a weight limit \( W \). The problem has been well studied and is typically solved by dynamic programming approaches or via mathematical programming algorithms such as branch-and-bound.

We provide three versions of the knapsack problem, the Binary (or 0-1) Knapsack Problem (BinKP), Bounded Knapsack Problem (BKP), and the Online Knapsack Problem (OKP). The first two are deterministic problems where the complete set of items, weights, and values are known from the outset. The OKP is stochastic; each item appears one at a time with a given probability and must be either accepted or rejected by the algorithm. The online version is studied by Marchetti-Spaccamela and Vercellis [1995], who propose an approximation algorithm such that the expected difference between this algorithm and the optimal value is, on average, \( O(\log^{3/2} n) \). Lueker [1995] later improved this result with an algorithm that closes the gap to within \( O(\log n) \) on average, using an on-line greedy algorithm.
3.1 Binary (0-1) Knapsack

The binary version of the knapsack problem (BinKP) can be formulated as an optimization problem as follows:

$$\max_x z = \sum_{i=1}^{n} v_i x_i$$  \hspace{1cm} (3a)

$$\text{s.t.} \sum_{i=1}^{n} w_i x_i \leq W$$  \hspace{1cm} (3b)

$$x_i \in \{0, 1\}$$  \hspace{1cm} (3c)

where $x_i$ denotes a binary decision variable to include or exclude an item from the knapsack and the weights and values, $v_i$ and $w_i$ respectively, are positive, real numbers. This can be solved as an integer programming problem using algorithms such as branch-and-bound to maximize the objective function.

3.2 Bounded Knapsack

The bounded knapsack problem (BKP) differs from the binary case in that $x_i$ becomes an integer decision variable ($x_i \in \mathbb{Z}_0^+$) and we introduce a new constraint based on the number of each item $i$ we have available.

$$x_i \leq N_i$$  \hspace{1cm} (4)

where $N_i$ is the number of times an item can be selected.

3.3 Online Knapsack

A version of the Online Knapsack Problem (OKP) is found in [Kong et al. 2019]. This requires the algorithm to either accept or reject a given item that it is presented with. There are a limited number of items that the algorithm can choose from and each is drawn randomly with a probability $p_i$. After $M$ items have been drawn, the episode terminates leaving the knapsack with the items inside. The goal here is the same as for the traditional knapsack problems, namely to maximize the value of the items in the knapsack while staying within the weight limit.

3.4 Problem Formulation

The RL model’s state is defined as a concatenation of vectors: item value, item weight, number of items remaining, as well as the knapsack’s current load and maximum capacity. For the OKP cases, we provide the agent with the current item’s weight and value, and the knapsack’s load and maximum capacity. The reward function for our RL algorithm will simply be the value of each item that is placed within the knapsack for all cases. At each step, the RL algorithm must select one of the items to be placed into the knapsack, at which point the state is updated to reflect that selection and the value of the selection is returned as the reward to the agent. The episode continues until the knapsack is full or no items fit.
The BinKP and BKP are solved as integer programs using Gurobi 8.2 and Pyomo 5.6.2 to optimality ([Gurobi Optimization LLC 2018], [Hart et al. 2017]). Additionally, we will employ a heuristic solution that uses the value to weight ratio of each item to fill the knapsack first with the items that have the highest value to weight ratio before moving to the second item in the list and so forth until the weight limit is reached. For BinKP and BKP we use a simple, greedy heuristic given in [Dantzig 1957], which orders the items by value/weight ratio, and selects the next item that fits in this order. If an item does not fit - or the item has already been selected \(N\) times - the algorithm will continue through the list until all possibilities for packing while remaining within the weight constraint are exhausted, at which point the algorithm will terminate. The OKP algorithm used is a greedy, online algorithm based on the TwoBins algorithm given in [Han et al. 2015].

**Algorithm 1 TwoBins Online Knapsack Algorithm**

Require: Choose \(r \in \{0, 1\}\) with equal probability

1: for Each item \(i \in N\) in order do:
2:   if \(r = 1\) then:
3:     if \(w_i \leq W\) then:
4:       Select \(i\) for the knapsack
5:     else:
6:       Reject \(i\)
7:   end if
8: end if
9: if \(r = 0\) then:
10:   if \(\sum^n w_i \leq W\) then:
11:     Reject \(i\)
12:   else:
13:     Select \(i\) for the knapsack
14:   end if
15: end if
16: end for

While IP solutions seek to solve the problem simultaneously, RL requires an MDP formulation which relies on sequential decision making. In this case, the RL model must select successive items to place in the knapsack until the limit is reached. This approach seems more akin to how a human would pack a bag, placing one in at a time until the knapsack is full. The human knapsack-packer can always remove an item if she determines it does not fit or finds a better item, our RL system, however cannot. Once an item is selected for inclusion, it cannot be removed from the knapsack.

### 3.5 Results

The knapsack problem and its variations have been widely studied by computer scientists and operations researchers. Thus, good heuristics exist for these problems, making it challenging for model-free methods to compete. However, as shown in Table [II], the RL method comes close to the heuristics and optimal solutions in the deterministic, offline cases, and outperforms
Table 1: RL versus heuristic and optimal solutions. All results are averaged over 100 episodes.

| Knapsack Version | Metric                  | RL   | Heuristic | MILP |
|------------------|-------------------------|------|-----------|------|
|                  | Mean Rewards            | 1,072| 1,368     | 1,419|
| BinKP            | Standard Deviation      | 66.2 | 0         | 0    |
|                  | Performance Ratio vs MILP| 1.32 | 1.04      | 1    |
|                  | Mean Rewards            | 2,434| 2,627     | 2,696|
| BKP              | Standard Deviation      | 207  | 0         | 0    |
|                  | Performance Ratio vs MILP| 1.11 | 1.03      | 1    |
|                  | Mean Rewards            | 309  | 262       | 531  |
| OKP              | Standard Deviation      | 54.2 | 104       | 54.6 |
|                  | Performance Ratio vs MILP| 1.8  | 2.02      | 1    |

Figure 2: Training curves for three variations of the Knapsack problem. Shaded areas indicate variance.

the heuristic in the stochastic case. While RL does learn a good policy in the deterministic cases, the policy is probabilistic, thus it continues to exhibit some variance once trained.

In the case of the Online Knapsack, the RL model learns a policy that is close to the theoretical performance ratio for the TwoBins algorithm. The TwoBins algorithm under performs its theoretical level ($\approx 1.7$) because each episode has a step limit, which causes the TwoBins algorithm to pack too little before the limit is reached, bringing down its mean performance.

Overall, the RL algorithms do perform well and show they can solve the traditional Knapsack Problem and its variations. For simple, off-line cases it seems best to resort to heuristics or optimal solutions rather than RL. However, for stochastic, online cases, there may be benefits in pursuing a reinforcement learning solution (Figure 2).

4 Virtual Machine Packing

The Bin Packing Problem (BP) is a classic problem in operations research. In its most common form, there are a series of items in a list $i \in I$, each with a given size, $s_i$. The
algorithm must then place these items into one of a potentially infinite number of bins with a size $B$ [Coffman et al., 2013]. The objective minimizes the number of bins used, the unused space in the bins, or some other, related objective, while ensuring that the bins stay within their size constraints. For problems with identical bin sizes, we can formulate the problem as:

$$\min_{x,y} \sum_{j=1}^{n} y_j$$  \hspace{1cm} (5a)$$

s.t. \sum_{i=1}^{m} s_i x_{ij} \leq B y_j \hspace{.2cm} \forall j \in J$$  \hspace{1cm} (5b)$$

\sum_{j=1}^{n} x_{ij} = 1 \hspace{.2cm} \forall i \in I$$  \hspace{1cm} (5c)$$

x_{ij}, y_j \in \{0,1\}$$  \hspace{1cm} (5d)$$

where $x_{ij}$ are binary assignment variables for each item that must be assigned to a given bin, and where $y_j$ are bins to be opened by the packing algorithm.

Applications of BPs are common in numerous fields, from loading pallets [Ram, 1992], to robotics, box packing [Courcoubetis and Weber, 1990], stock cutting [Gilmore and Gomory, 1961], logistics, and data centers [Song et al., 2014].

BP’s are primarily divided into groups based on their dimensionality, although even 1-D problems are NP-hard (Christensen et al. [2017], Johnson [1974]). 1-D problems only consider a size or weight metric, whereas multi-dimensional problems consider an item’s area, volume, or combination of features.

4.1 Problem Formulation

We provide multiple versions of the bin packing problem in the OR-Gym library, including the environments implemented by Balaji et al. [2019], which rely on the examples and heuristic algorithms provided in Gupta and Radovanovic [2012]. We refer readers to their work for details and results. In addition to this environment, we implement a multi-dimensional version of the bin packing problem as applied to virtual machines with data from Cortez et al. [2017]. This data was collected over a 30-day period from Microsoft Azure data centers containing multiple physical machines (PM) to host virtual machine (VM) instances. Each VM has certain compute and memory requirements, and the algorithm must map a VM instance to a particular PM without exceeding either the compute or memory requirement. In this case, we model the normalized demand at 20-minute increments over a 24-hour period, whereby a new VM instance must be assigned every 20 minutes.

The environment requires mapping of each VM instance to one of 50 PMs at each time step. The objective is to minimize unused capacity on each PM with respect to both the compute and memory dimensions for each time step. The episode runs for a single, 24-hour period and ends if the model exceeds the limitations of a given PM. At each time step, the agent can select from one of the 50 PMs available. If a selection causes a PM to become overloaded, the agent incurs a large penalty and the episode ends. It is important to ensure
that the penalty is sufficiently large to prevent the agent from ending the episode prematurely, otherwise the agent will find a locally optimal strategy by simply ending the episode as quickly as possible.

For the RL implementation, we define the state is with two matrices. The first is a $50 \times 3$ matrix with one entry per PM. The first column is a binary value to denote whether the PM is in use or not. The remaining columns provide the capacity level for the CPU and memory of each PM. The second matrix contains the time and the incoming, normalized demand that needs to be packed with the CPU and memory requirements given as elements in the matrix. In practice, we concatenate these matrices for a single, $51 \times 3$ matrix before passing it to the RL agent.

### 4.2 OR Methods

Heuristics such as Best Fit (BF) [Johnson, 1974], and Sum of Squares (SS) [Csirik et al., 2006], have been proposed and are well studied, providing theoretical optimality bounds in the limit for one-dimensional BPs. For multi-dimensional problems, other heuristics such as Next Fit Decreasing Height (NFDH) and First Fit Decreasing Height (FFDH) are early and common heuristics approximation ratios of 3 and 2.7, respectively [Christensen et al., 2017]. Currently, Bansal et al. [2016] provides the best results in the 2-D case with an asymptotic approximation guarantee of $\approx 1.405$. We employ the First Fit algorithm for our VM packing case, which was shown to have an asymptotic performance ratio of 1.7 [Baker and Schwarz, 1983].

**Algorithm 2** FirstFit VM Packing Algorithm

| Require: Initialize environment with $J$ empty bins. |
|-----------------------------------------------------|
| 1: **for** Each item $i \in I$ in order **do:** |
| 2: **for** Each open bin $j \in J$ in order **do:** |
| 3: **if** $i$ fits in bin $j$ **then:** |
| 4: Place item $i$ in bin $j$ |
| 5: **end if** |
| 6: **end for** |
| 7: **if** $i$ does not fit into any open bin **then:** |
| 8: Open new bin. Place item into new bin. |
| 9: **end if** |
| 10: **end for** |

The optimization model solves the VM packing problem deterministically, off-line to provide an upper bound on performance. As in the standard case, the objective is to minimize the total number of bins (or physical machines) in use for all time steps. The full model is
Table 2: Comparison of RL versus heuristic model and shrinking horizon model solutions. All results are averaged over 100 episodes.

| VM Packing | RL (No Masking) | RL (Masking) | Heuristic | MILP |
|------------|-----------------|--------------|-----------|------|
| Mean Rewards | -1,040 | -511 | -556 | -439 |
| Standard Deviation | 17 | 110 | 111 | 111 |
| Performance Ratio vs MILP | 2.37 | 1.16 | 1.27 | 1 |

given below.

\[
\min_{x, z} \sum_{i=1}^{M} \sum_{j=1}^{T} \sum_{k=1}^{N} x_{ijk} \left( D_{j}^{\text{mem}} + D_{j}^{\text{cpu}} \right) - 2z_{ik} \\
\text{s.t.} \sum_{j}^{j} D_{j}^{\text{mem}} x_{ijk} \leq B_{\text{mem}} \quad \forall j \in N, k \in T \quad (6a) \\
\sum_{j}^{j} D_{j}^{\text{cpu}} x_{ijk} \leq B_{\text{cpu}} \quad \forall j \in N, k \in T \quad (6b) \\
\sum_{k}^{k} x_{ijk} \leq \tau_{j} y_{ij} \quad \forall i \in M, j \in N \quad (6c) \\
\sum_{j}^{j} y_{ij} = 1 \quad \forall i \in M \quad (6d) \\
\sum_{j}^{j} x_{ijk} \leq 1 \quad \forall k \geq t \in T \quad (6e) \\
z_{ik} \geq x_{ijk} \quad \forall i \in M, j \in N, k \in T \quad (6f) \\
x_{ijk}, y_{ij}, z_{ik} \in 0, 1 \quad (6g) \\
\]

where \( i, j, \) and \( k \) are indices that denote the physical machine in the set \( M \), the virtual machine requirements in \( N \), and the time step in \( T \) respectively. \( D_{j}^{\text{mem}} \) and \( D_{j}^{\text{cpu}} \) indicate the memory and CPU demand for each VM instance, which lasts for time \( \tau \). Each physical machine may host multiple processes simultaneously, but these are restricted by normalized memory and compute capacities, denoted by \( B_{\text{mem}} \) and \( B_{\text{cpu}} \) respectively.

The model also makes use of multiple binary, decision variables, \( x, y, \) and \( z \). \( x \) maps VMs to PMs for each time period. \( y \) relates VMs to PMs, while \( z \) enables the time periods of the PMs to be linked and used in the objective.

4.3 Results

The RL agent with masking quickly reaches peak performance with rewards hovering around -500 per episode. This surpasses the results for the heuristic benchmark by about 9%, but underperforms the more computationally intensive shrinking horizon MILP by 16%. All three methods show very similar variance.
As shown in Figure 3, the RL agent without masking greatly underperforms relative to all other methods. The agent was given the same hyperparameters and set up as the RL agent with masking, the only difference being that it was able to violate the packing constraints, thereby receiving a large negative reward and immediately ending to the episode. This example shows the use of a “soft constraint” versus a hard constraint in RL set up and design. The agent without masking quickly learned to take the -1000 reward as fast as possible, thus getting stuck in a local optimum policy because exploration yielded eventual termination by constraint violation and larger total rewards. This behavior could be mitigated with further hyperparameter tuning, environment design, or other techniques, but it seems far more effective to simply mask actions that would violate constraints and let the agent learn from there.

5 Supply Chain Inventory Management

Managing inventory levels is critical to supply chain sustainability. A clear relation exists between inventory levels and order fulfillment service levels (i.e. the more inventory you have, the better you can satisfy your customer’s requests). However, high inventory levels come at a cost, referred to as holding costs. The key is to strike a balance between the trade-off between service level and holding costs. To explore this, we provide two variations of a Multi-Echelon Inventory Management problem for optimization and training RL agents.

In these problems, a retailer faces uncertain consumer demand from day to day, and must hold inventory at a cost in order to meet that demand. If the retailer fails to meet that demand, it will either be marked as a backlog order, whereby it may be fulfilled at a later date and lower profit (InvManagement-v0), or simply chalked up as a lost sale with zero profit (InvManagement-v1). Each day, the retailer must decide how much inventory to purchase from its distributor, who will manufacture and ship the product to the retailer with a given lead time. In the multi-echelon case, the distributor will have a supplier, which may also have another supplier above it, and so forth until the supply chain terminates at the original party that consumes the raw materials required for the product, \( M \) steps away from...
the retailer.

In a decentralized supply chain, coordination is key to effectively managing the supply chain. Each stage faces uncertainty in the amount of material requested by the stage succeeding it. A lack of inter-stage coordination can result in bullwhip effects [Lee et al. 1997]. In the IMP, each stage operates according to its own unique costs, constraints, and lead times. The challenge is then to develop a re-order policy for each of the participants in the supply chain to minimize costs and maintain steady operations.

![Multi-echelon supply chain](image)

Figure 4: Multi-echelon supply chain.

The IMP environments presented are based on the work by Glasserman and Tayur [1995]. In this work, a multi-echelon system with both inventory holding areas and capacitated production areas for each stage is used. The inventory holding areas store intermediates that are transformed into other intermediates or final products in the respective production areas. The default configuration is to have both inventory and production areas at each stage in the supply chain, except for the retailer, which only holds inventory of the final product, and the supplier furthest upstream, which has virtually unlimited access to the raw material. However, production areas can be removed if desired by setting the production capacity to a large value at those stages.

The standard approach taken in industry to improve the performance in such systems is the use of IPA (infinitesimal perturbation analysis) to determine the optimal parameters for the desired inventory policy. Although it is acknowledged that the base stock policy may not be optimal for multi-stage capacitated systems, its simplicity makes it attractive for practical implementations. Determining the optimal policy for such a system requires dynamic programming and results in a state dependent policy, which is likely to face many challenges in being accepted by a company’s operations division in practice.

Relevant literature addressing other approaches to the IMP are those of Bertsimas and Thiele [2006], Chu et al. [2015], and Mortazavi et al. [2015]. In the work by Bertsimas and Thiele [2006], a general optimization methodology is proposed using robust optimization...
techniques for both capacitated and uncapacitated systems. Their work includes capacity limits on the orders and inventory, but not on production. Their model shows benefits in terms of tractability and is solved as either a linear program (LP) or a mixed-integer program (MIP), depending on whether fixed costs are included or not. Chu et al. [2015] use agent based modeling coupled with a cutting plane algorithm to optimize a multi-echelon supply chain with an \( (r, Q) \) reorder policy under a simulated environment. Monte Carlo simulations are used to determine expectations followed by hypothesis testing to deal with the effects of noise when accepting the improvements. Mortazavi et al. [2015] develop a four-echelon supply chain model with a retailer, distributor and manufacturer. They apply Q-learning to learn a dynamic policy to re-order stock over a 12-week cycle with non-stationary demand drawn from a Poisson distribution. Several reference papers in the area of inventory optimization are also available. Of note are those by Eruguz et al. [2016] and Simchi-Levi and Zhao [2012].

Variations of the IMP include single period and multi-period systems, as well as single product and multi-product systems. The IMP environments currently available (InvManagement-v0 and InvManagement-v1) support single product systems with with stationary demand and either single or multiple time periods. It is assumed that the product is non-perishable and sold in discrete quantities. A depiction of the multi-echelon system is given in Figure 4. Stage 0 is the retail site, which is an inventory location that sells the final product to external customers. As mentioned previously, stages 1 through \( M - 1 \) have both an inventory area and a manufacturing area, and stage \( M \) has only a manufacturing area. For each unit of inventory transformed, one unit of intermediate or final product is obtained. Material produced at a stage is shipped to the inventory area of the stage succeeding it. Lead times may exist in the production/transfer of material between stages. Each manufacturing site has a limited production capacity. Each inventory holding area also has a limited holding capacity. It is assumed that the last stage has immediate access to an unlimited supply of raw materials and thus a bounded inventory area is not designated for this stage.

5.1 Problem Formulation

At each time period in the IMP, the following sequence of events occurs:

1. Stages 0 through \( M - 1 \) place replenishment orders to their respective suppliers. Replenishment orders are filled according to available production capacity and available inventory at the respective suppliers. Lead times between stages include both production times and transportation times.

2. Stages 0 through \( M - 1 \) receive incoming inventory replenishment shipments that have made it down the product pipeline after the associated lead times have passed.

3. Customer demand occurs at stage 0 (the retailer) and is filled according to the available inventory at that stage.

4. One of the following occurs at each stage,

   (a) Unfulfilled sales and replenishment orders are backlogged at a penalty. Note: Backlogged sales take priority in the following period.
6. Any inventory remaining at the end of the last period is lost.

\begin{align*}
I_{t+1}^{m} &= I_t^{m} + R_{t-L_m}^{m} - S_t^{m} & \forall m \in \mathcal{M}, t \in \mathcal{T} \\
T_{t+1}^{m} &= T_t^{m} - R_{t-L_m}^{m} + R_t^{m} & \forall m \in \mathcal{M}, t \in \mathcal{T} \\
R_t^{m} &= \min \left( c^{m+1}, I_t^{m+1}, \hat{R}_t^{m} \right) & \forall m \in \mathcal{M} \setminus \{|\mathcal{M}|\}, t \in \mathcal{T} \\
S_t^{m} &= \begin{cases} R_t^{m-1} & \text{if } m > 0 \\ \min \left( I_t^{0} + R_t^{0}, D_t + B_{t-1}^{0} \right) & \text{if } m = 0 \end{cases} & \forall m \in \mathcal{M}, t \in \mathcal{T} \\
U_t^{m} &= \hat{R}_t^{m-1} - S_t^{m} & \forall m \in \mathcal{M}, t \in \mathcal{T} \\
P_t^{m} &= (\alpha)^n \cdot \left( p^m S_t^{m} - r^m R_t^{m} - k^m U_t^{m} - h^m I_t^{m+1} \right) & \forall m \in \mathcal{M}, t \in \mathcal{T}
\end{align*}

Six equations govern the behavior of the IMP. The first two (Equations 7a - 7b) are material balances for the on hand \((I)\) and pipeline \((T)\) inventory at the beginning of each period \(n\), respectively. The on-hand inventory at each stage \(m\) in the beginning of the next period is equal to the initial inventory in the current period, plus the reorder quantity arrived \((R\), placed \(t - L_m\) periods ago\), minus the sales in the current period \((S\)). The pipeline inventory at each stage in the beginning of the next period is equal to the pipeline inventory in the current period, minus the delivered reorder quantity \((\text{placed } t - L_m \text{ periods ago})\), plus the current reorder quantity placed. Equation 7c relates the accepted reorder quantity \((R)\) to the requested reorder quantity \((\hat{R})\). In the absence of production capacity \((c)\) and inventory constraints in the stage above the stage \(m\), the accepted reorder quantity would be equal to the requested order quantity. However, when these constraints exist, they place an upper bound on the reorder quantity that is accepted. At stage \(M - 1\), the term \(I_t^{m+1}\) can be ignored or set to \(\infty\) since it is assumed that the inventory of raw materials at stage \(M\) is infinite.

Equation 7d gives the sales \(S\) at each period, which equal the accepted reorder quantities from the succeeding stages for stages 1 through \(M\) and equal the fulfilled customer demands for the retailer (stage 0). The fulfilled customer demand is equal to the demand plus the previous period’s backlog, unless there is insufficient on-hand inventory at the retailer, in which case, all of the inventory on hand is sold. The unfulfilled demand \(U\) or unfulfilled reorder requests are given by Equation 7e. It should be noted that \(\hat{R}_t^{-1} \equiv D_t + B_{t-1}^{0}\). The profit \(P\) is given by Equation 7f which discounts the profit (sales revenue minus procurement costs, unfulfilled demand costs, and excess inventory holding costs) with a discount factor \(\alpha\). \(p\), \(r\), \(k\), and \(h\) are the unit sales price, unit procurement cost, unit penalty for unfulfilled demand, and unit inventory holding cost at each stage \(m\), respectively. For stage \(M\), \(R_t^{m} \equiv S_t^{m}\) and \(r^m\) represents a raw material procurement cost. If backlogging is not allowed, any unfulfilled demand or procurement orders are lost sales and all \(B\) terms are set to 0.

The supply chain inventory management problems were modeled as MDPs, whereby the agent must decide how much stock to re-order from the higher levels at each time step. At each time step \(t\), an action \(a_t^{m} \in A\) is taken at each stage \(m\). The action corresponds to
Figure 5: Illustration of state for inventory management environments where \( \max(L) \) denotes the system’s maximum lead time.

Each reorder quantity at each stage in the supply chain. The actions are integer values and maintain the supply capacity and inventory constraints of the form \( A \leq C \).

The states are denoted by the inventory on hand for each level, as well as the previous actions for each of the \( \max(L) \) time steps in order to capture the inventory in the pipeline (see Figure 5).

The RL agent seeks to maximize the profit of the supply chain as defined in the reward function given in Equation 7f. The simulation lasts for 30 periods (days), and transitions from one state to the next as material transfers are performed through the supply chain and orders are fulfilled at the retailer.

### 5.2 OR Methods

The base stock policy has been shown to be optimal for capacitated production-inventory systems under certain conditions [Kapuscinski and Tayur, 1999]. For multi-stage systems, these conditions are that backlogging is allowed (no lost sales), lead times are fixed, and the capacity at a stage does not exceed the capacity at the stage below it. Although the base-stock policy is not necessarily optimal under other conditions, it is one of the valid OR approaches used in practice due to its simplicity. Under this policy, the requested reorder quantity is given by Equation 8, where \( z^m \) is the base-stock level at stage \( m \) and the term in the summation is the current inventory position at the beginning of period \( t \).

\[
\hat{R}_t^m = \max \left( 0, z^m - \sum_{m' = 1}^{m} \left( I_t^{m'} + T_t^{m'} - D_{t-1}^{m'} \right) \right) \quad \forall m \in \mathcal{M}, t \in \mathcal{T} \tag{8}
\]

Glasserman and Tayur [1995] propose a numerical method to determine the optimal base-stock level called infinitesimal perturbation analysis (IPA). IPA is a gradient descent approach that minimizes the expected cost over a sample path. IPA relies on perturbing a simulated sample path iteratively until the desired improvement in an objective function is obtained. Derivatives are calculated or estimated explicitly to provide the updates necessary for the gradient descent. To guarantee convergence, the optimization is performed offline. In the IPA implementation for the base stock policy, a simulated sample path of \( T \) periods is run with fixed base stock levels. The gradients for the state variables (inventory positions, reorder
quantities, etc) are determined from the recursive relations in Equations 7a - 7f. These are used to update the base stock levels. The updated levels are applied to the same simulated sample path over \( T \) periods (iteration number 2). The process is repeated until the base stock levels stabilize or the change in the objective function is below a certain tolerance.

\[
f = \frac{1}{|T|} \mathbb{E}_D \left[ \sum_{m \in \mathcal{M}} P^m_t \right]
\]

(9)

In the present work, we follow this idea of optimizing over a sample path, but apply more robust approaches for the optimization. Since the objective function for the IMP (normalized expected profit over the sample path, Equation 9) is non-smooth and non-differentiable for discrete demand distributions, line search algorithms based on the Wolfe conditions [Nocedal and Wright 2006] often fail to converge when determining the step size for the classical IPA approach. Instead of settling on taking full step sizes or using a subpar approach when determining the step sizes, derivative-free optimization (DFO) based on Powell’s method [Powell 1964] is made available. Another second approach available for IMP is that of mixed-integer programming (MIP), which readily expresses the discontinuities in the model equations using binary variables and can guarantee finding the optimal base stock levels over a sample path.

The benefit of using Powell’s method [Powell 1964] is that it does not rely on gradients and can be applied to non-differentiable systems. Since the demand distributions are discrete, Powell’s method is followed by either rounding the base stock levels to the nearest integer or performing a local search to find the nearest optimal integer solution. In the latter case, enumeration is used to compare all neighboring integer solutions to the optimum found by Powell’s method. Using Powell’s method provides the advantage of being much faster to solve than the MIP alternative, which is NP-hard.

In the MIP approach, the IMP system is modelled as a mixed-integer linear program (MILP) and a MIP solver is used to find the optimal base-stock levels. The disjunctions arising from the minimum operators in Equations 7c - 7d and the maximum operator in Equation 8 are reformulated into algebraic inequalities using Big-M reformulations (Equations 10a - 10f). Furthermore, to ensure the standard multi-echelon condition \( z_m \leq z_{m+1} \forall m \in \mathcal{M} \setminus \{|\mathcal{M}|\} \), the base stock levels are written in terms of the total inventory level, \( x^m \geq 0 \), at each stage (\( z^m = \sum_{m'=1}^m x^{m'} \)).

\[
x = \max(A_1, ..., A_{|I|})
\]
\[
A_i \leq x \leq A_i + M_i(1 - y_i) \quad \forall i \in I
\]
\[
x = \min(A_1, ..., A_{|I|})
\]
\[
A_i + M_i(1 - y_i) \leq x \leq A_i \quad \forall i \in I
\]
\[
\sum_{i \in I} y_i = 1
\]
\[
y \in \{0, 1\}^{|I|}
\]

(10a)
(10b)
(10c)
(10d)
(10e)
(10f)

A third approach to the IMP is to use a dynamic reorder policy that is determined by solving a linear program (LP) at each time period with a shrinking horizon (SHLP). This
approach requires prior knowledge of the demand probability distribution and assumes the expected value of the demand for all time periods. After solving the optimization for the current time period in the simulation, the optimal reorder action found for the current period is implemented. All future reorder actions are discarded and the process is repeated for the next time period. The shrinking horizon model has no binary variables as a result the removal of Eq 8. In the absence of targeted base stock levels, the requested reorder quantity $R^m_t$ becomes the same as the accepted reorder quantity $R^m_t$. As a result, Eq 7c can be replaced with $R^m_t \leq c^{m+1}$ and $R^m_t \leq I^m_t$ and the second part of Eq 7d can be replaced with $S^0_t \leq I^0_t + R^0_{t-L_m}$ and $S^0_t \leq D_t + B^0_{t-1}$.

5.3 Results

Two examples are run to compare the OR (DFO, MIP, and SHLP) and RL approaches in the inventory management problem. It should be noted that the DFO and MIP approaches use static base stock policies, whereas the SHLP and RL approaches use dynamic reorder policies. Both approaches are compared to an oracle model, which is a LP model that determines the optimal dynamic reorder policy for that run by using the actual demand values for each time period. InvManagement-v0 is a 30 period, 4 stage supply chain with backlog. A Poisson distribution is used for the demand with a mean of 20. A discount factor for the time value of money of 97% percent is used (3% discount). The second problem, InvManagement-v1, is identical to InvManagement-v0, except it does not have a backlog. The parameters used in both environments are: $2 unit sales price for the final product; $1.50, $1.00, $0.75, and $0.50 unit replenishment costs for stages 0 through 3, respectively; $0.100, $0.075, $0.050, and $0.025 unit penalty costs for unfulfilled orders in stages 0 through 3, respectively; $0.15, $0.10, and $0.05 unit holding costs for stages 1 through 2, respectively; 100, 90, and 80 unit manufacturing capacities at stages 1 through 3, respectively; and 3, 5, and 10 period lead times for reorders placed at stages 0 through 2, respectively.

Training curves for both environments v0 and v1, as well the mean values for the oracle, SHLP, and MILP models are given in Figure 6. For the InvManagement-v0 environment, the RL model the static policy models, but was outperformed by the shrinking horizon model, achieving a performance ratio of 1.2 (Table 3). The daily rewards for each of the comparison models are given in Figure 7, where it can be seen that the RL model’s rewards track closely with the oracle and SHLP models before leveling off towards the end of the episode. The model does this because it has been trained on a 30-day episode, and, unlike the static base stock policy models, it does not need to continue to order additional inventory towards the end of the run because the holding costs become large and the extra stock will not be sold within the episode horizon to be worthwhile. This behavior is shown in the inventory plots given in Figure 8, where the RL inventory levels decrease as the episode progresses, as is also observed with the oracle and SHLP models, which are also aware of the limited horizon.

The static policy models do not exhibit this behavior and continue targeting the optimal base stock levels, which increases the holding costs and causes the profit to suffer. However, in a real application, the supply chain is likely to operate for more than 30 days, and dropping inventories towards the end of the run would be detrimental to the supply chain’s performance beyond the 30 days. The RL model could be trained on a continuous environment that cycles
Table 3: Total reward comparison for InvManagement problems and the various models used to solve them.

|                      | InvManagement-v0 | RL  | SHLP | DFO  | MIP  | Oracle |
|----------------------|------------------|-----|------|------|------|--------|
| Mean Rewards         | 438.8            | 508.0| 360.9| 388.0| 546.8|        |
| Standard Deviation   | 30.6             | 28.1 | 39.9 | 30.8 | 30.3 |        |
| Performance Ratio vs Oracle | 1.2      | 1.1  | 1.5  | 1.4  | 1.0  |        |

|                      | InvManagement-v1 | RL  | SHLP | DFO  | MIP  | Oracle |
|----------------------|------------------|-----|------|------|------|--------|
| Mean Rewards         | 409.8            | 485.4| 364.3| 378.5| 542.7|        |
| Standard Deviation   | 17.9             | 29.1 | 33.8 | 26.1 | 29.9 |        |
| Performance Ratio vs Oracle | 1.3      | 1.1  | 1.5  | 1.4  | 1.0  |        |

month after month, drawing demand from the same probability distribution to avoid this behavior in a real-world application.

The RL model in InvManagement-v1 (no backlog orders), performs slightly worse relative to the oracle with a performance ratio of 1.3 (Table 3). The same end-of-episode dynamics observed in InvManagement-v0 are also visible in Figure 9.

For both environment variations, the RL model outperforms the static policy models, scoring performance ratios between 1.2-1.3. This is shows the potential for RL to learn competitive, dynamic policies for multi-stage inventory management problems.

6 Asset Allocation

Asset allocation refers to the task of creating a collection (portfolio) of financial assets as to maximize an investors return, subject to some investment criteria and constraints. The most famous approach to portfolio optimization stems from Markowitz’s mean-variance framework from the 1950s that frames portfolio optimization as a trade-off between the return of an asset (modeled using its expected value) and the risk associated with the asset (modeled by its variance) [Markowitz, 1952]. Since then, the asset allocation problem has been extensively explored in the literature with a multitude of models that take into account many different practical considerations (see Black and Litterman 1992, Perold 1984, Fabozzi et al. 2007, Konno and Yamazaki 1991, Krokhmal et al. 2003, and DeMiguel et al. 2009).
Figure 7: Cumulative profit for both inventory management models. Results averaged over 10 episodes. Shaded areas indicate standard deviation of rewards.

Figure 8: Average inventory on hand at each echelon for InvManagement-v0.

Figure 9: Average inventory on hand at each echelon for InvManagement-v1.
6.1 Problem Formulation

Here, we focus on the multi-period asset allocation problem (MPAA) of Dantzig and Infanger [1993].

An investor starts with a portfolio $x^0 = [x_0^0, \ldots, x_n^0]$ consisting of $n$ assets, with an additional quantity of cash we call a “zero” asset, $x_0^1$. The investor wants to determine the optimal distribution of assets over the next $L$ periods to maximize her total wealth at the end of the final investment horizon of total length $L$.

The optimization variables are $b_l^i$ (the amount of asset $i$ bought at the beginning of period $l$) and $s_l^i$ (the amount sold) where $i = 1, \ldots, n$ and $l = 1, \ldots, L$.

The price of an asset $i$ in period $l$ is denoted by $P_l^i$. We assume that the cash account is interest-free meaning that $P_0^l = 1$, $\forall l$. The sales and purchases of an asset $i$ in period $l$ are given by proportional transaction costs $\alpha_l^i$ and $\beta_l^i$, respectively. These prices and costs are subject to change asset-to-asset and period-to-period.

The deterministic version of this problem is given by Formulation 11:

$$\max_{x,s,b} \sum_{i=0}^{n} P_l^i x_l^i$$  \hspace{1cm} (11a)

$$x_l^i \leq x_{l-1}^i + \sum_{i=1}^{n} (1 - \alpha_l^i) P_l^i s_l^i - \sum_{i=1}^{n} (1 + \beta_l^i) P_l^i b_l^i \hspace{1cm} \forall l \in L$$  \hspace{1cm} (11b)

$$x_l^i = x_{l-1}^i - s_l^i + b_l^i \hspace{1cm} \forall i \in n \hspace{1cm} \forall l \in L$$  \hspace{1cm} (11c)

$$s_l^i, b_l^i, x_l^i \geq 0 \hspace{1cm} \forall i \in n \hspace{1cm} \forall l \in L$$  \hspace{1cm} (11d)

This is a linear programming problem that can be easily solved by common optimization solvers.

In the RL environment, we consider the optimization of a portfolio derived closely from Formulation 11. This portfolio initially contains $100 in cash and three other assets and our goal is to optimize portfolio value over a time horizon of 10 periods. Of course, the prices of each asset for each period are not known in advance. Consequently, these prices are modeled as Gaussian random variables with a fixed mean and variance for each asset in each period. In each instantiation of the environment, these prices take on a new value. Transaction costs are fixed in all instances.

Each step in the RL environment corresponds to a single investment period. The state of the environment, $s$, at a particular period is described by the following vector of the current period, $l$, as well as cash and asset quantities and prices, corresponding to the notation of Formulation 11:

$$s = [l, x_0^l, x_1^l, x_2^l, x_3^l, P_1^l, P_2^l, P_3^l]$$

At each step, the RL agent must decide how much, if any, of each particular asset to buy or sell. The action, $a$, is described by the following vector of amounts of the three assets to buy or sell, corresponding to the notation of Formulation 11:

$$a = [\delta_1^l, \delta_2^l, \delta_3^l]$$
where,

\[
\delta^i_l = \begin{cases} 
sl^i_l & (b^i_l = 0), \\
bl^i_l & (b^i_l = 0), \\
s^i_l & (b^i_l = 0) 
\end{cases}
\]

for \(\delta^i_l < 0\)

\[
\delta^i_l = \begin{cases} 
sl^i_l & (b^i_l = 0) \\
bl^i_l & (b^i_l = 0) \\
s^i_l & (b^i_l = 0) 
\end{cases}
\]

for \(\delta^i_l > 0\)

\(\delta^i_l \in [-2000, 2000]\)

The reward \(r\) is given only at the very end of the episode and is equal to the value of the objective function of Equation (11a) (i.e. the portfolio value). An alternative design choice that we considered was to give a reward at each step (for example, the current portfolio value). This implementation did prove to facilitate learning but we chose the sparse-reward formulation to demonstrate the power of RL even in relatively challenging situations.

6.2 OR Methods

Robust optimization (RO) is an established field in operations-research that rigorously addresses decision-making under uncertainty. In the robust optimization paradigm, optimization problem parameters are treated as unknown quantities bounded by an uncertainty set. It is the objective of RO to find good solutions that are feasible no matter what values within the uncertainty set the problem parameters take Ben-Tal et al. [2009]. In essence, robust optimization allows the practitioner to optimize the worst-case scenario.

To benchmark the performance of the RL agent, we formulate and solve the corresponding robust-optimization problem to (11), originally from Ben-Tal et al. [2000]. Constraints on cash flow are modeled using a \(3 - \sigma\) approach such that they the chosen investment policy will be feasible in 99.7% of possible scenarios. The goal of the optimization is to find an investment policy such that the portfolio value will be greater than or equal to the objective function value of the optimization problem. We refer the reader to Cornuejols and Tutuncu [2006] for a clear description and explanation of the robust formulation.

We then take the investment policy (i.e. the amount of each asset to buy and sell in each period) and simulate this policy for several different realizations of the parameters within the uncertainty set. We use the average rewards achieved as a benchmark against the policy found by the RL agent.

A alternative to the robust optimization approach would have been to use multi-stage stochastic programming (MSP). To paraphrase the arguments of Ben-Tal et al. [2000], RO was highlighted instead because: it is much more tractable than the MSP, any simplifying implementation of MSP (e.g. by solving on a rolling-horizon) would be \textit{ad hoc} and would not necessarily yield a superior solution than the RO approach, and, although the RO approach seems ‘conservative’, the mathematical form of the RO is such that the RO solution is not as conservative as it may seem. As important, a central goal of asset allocation is not only to maximize expected rewards, but also to minimize risk. Robust optimization, which minimizes the downside-risk, is a natural approach to achieve portfolios with high returns under a variety of market conditions. As the results of Ben-Tal et al. [2000] show for this problem, not only does RO match the performance of MSP in terms of mean portfolio value, the standard-deviation (and hence the risk) of the return is substantially lower.

We solve this using BARON 19.7.13 on a 2.9 GHz Intel i7-7820HQ CPU Sahinidis [2019].
6.3 Results

The solution to the deterministic optimization problem, where all prices (equal to their mean values in the RO and RL formulations) are known with certainty, is $17,956.20. However, this value is not practically important because market prices are never known in advance.

The robust optimization solution is a more practically important metric for comparison. While the uncertainty set chosen is conservative, the RO approach multiplies the initial $100 in cash to a portfolio value of at least $610.17 for 99.7% of the parameter space. This value corresponds to the objective function value of the RO problem, and is the worst-case return.

The solution to the RO problem corresponds to an investment strategy. When we take the RO strategy and actually apply it in $10^3$ randomly generated instances of this problem, we see that the RO policy is able to multiply the initial cash to an average portfolio value of about $865 with a standard deviation of roughly $80. Results for the RL and RO approaches are given in Figure 10.

![Figure 10: Training rewards for RL algorithm smoothed over 100 episodes and average rewards given by implementing robust optimization solution. Shaded areas indicate standard deviation of rewards. Histogram compares results from 1,000 simulated episodes for RL and RO. Notice the wider distribution and skew of the RL solution which offers less downside protection.](image)

We see that, despite the sparse-reward formulation, the RL agent is able to successfully learn a policy that, on average, yields higher portfolio values than the strategy given by RO. Purely in the sense of maximizing expected portfolio value, RL is a strong alternative to robust optimization. On the downside however, the variance of the RL portfolio values is higher than in the case of RO. Consequently, RO seems to be the better alternative when our primary concern is risk, achieving worst-case returns that are much higher than those from RL. As a final note, the number of episodes required to train the RL agent was very high, resulting in a training time on the order of hours. On the other hand, the robust optimization policy was found in just a few minutes.

7 Conclusion

We have developed a library of reinforcement learning environments consisting of classic operations research and optimization problems. RL traditionally relies on environments being formulated as Markov Decision Processes (MDPs). These are sequential decision processes where a decision or action in one state influences the transition to the subsequent state. In
many optimization problems, sequential decisions are required making RL straightforward to apply to the problem. Hard constraints can be imposed on RL agents via action masking, whereby the probability associated with selecting an action that would lead to a constrain violation are set to 0 ensuring these actions are not selected. The use of action masking reduces the search space and can both improve the learned policy and reduce the amount of training time required by the model.

Additionally, we have provided benchmarks for four problem classes showing results for RL, heuristics, and optimization models. In all cases, RL is capable of learning a competent policy. RL outperforms the benchmark in many of the more complex and difficult environments where uncertainty plays a significant role, and may be able to provide value in similar, industrial applications. RL did not outperform the heuristic models in the off-line knapsack problems. This does not come as a complete surprise given that the knapsack heuristics were able to find the optimal solution, and that the knapsack problem has been well-studied for many decades, and numerous heuristics have been developed to solve the problem and increase the state-of-the-art solutions. In cases such as this, the additional complexity and computational costs of RL may not be worthwhile, however note that in formulation of the off-line knapsack problem, uncertainty also played no role.

RL provides a general framework to solve these various problems. Each system was solved with the same algorithm using three layers and 128 nodes each and minor hyperparameter tuning for the learning rate and entropy loss within a standard library. With new tools such as Ray, RL becomes increasingly accessible for solving classic optimization problems under uncertainty as well as industrial equivalents optimization models do not exist, or are too expensive to develop and compute online.

It may be possible to extract rules from the RL policies to develop better heuristic solutions for each of the problems. Additionally, given the structure of these problems and the fact that optimization models exist for them, it is rife for exploration of hybrid RL and mathematical programming approaches which can combine the speed and flexibility of RL with the solutions available from mathematical programming to improve results and time to solution. Another interesting avenue to pursue is the use of graph neural networks and RL, which may be able to benefit from the structure of these models. Finally, the OR-Gym library contains additional environments from the literature for experimentation.¹

¹The full list and download instructions can be found here: https://github.com/hubbs5/or-gym
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