Stationary Spacetime from Intersecting M-branes

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Abstract. We study a stationary “black” brane in M/superstring theory. Assuming BPS-type relations between the first-order derivatives of metric functions, we present general stationary black brane solutions with a traveling wave for the Einstein equations in D-dimensional. The solutions are given by a few independent harmonic equations (and plus the Poisson equation). General solutions are constructed by superposition of a complete set of these harmonic functions. Using the hyperspherical coordinate system, we explicitly give the solutions in 11-dimensional M theory for the case with M2 ⊥ M5 intersecting branes and a traveling wave. Compactifying these solutions into five dimensions, we show that these solutions include the BMPV black hole and the Brinkmann wave solution. We also find new solutions similar to the Brinkmann wave. We prove that the solutions preserve the 1/8 supersymmetry if the gravi-electromagnetic field $F_{ij}$, which is a rotational part of gravity, is self-dual. We also discuss non-spherical “black” objects (e.g., a ring topology and an elliptical shape) by use of other curvilinear coordinates.

1. Introduction

Black holes are now one of the most important subjects in string theory. The Beckenstein-Hawking black hole entropy of an extreme black hole is obtained in string theory by statistical counting of the corresponding microscopic states [1]. While, we have found several interesting black hole solutions in supergravity theories [2], which are obtained as an effective theory of a superstring model in a low energy limit. We also know black hole solutions in a higher-dimensional spacetime [3], which play a key role in a unified theory such as string theory. In higher dimensions, because there is no uniqueness theorem of black holes [4], we have a variety of “black” objects such as a black brane [5]. One of the most remarkable solutions is a black ring, which horizon has a topology of $S^1 \times S^2$ [6].

Among such “black” objects, supersymmetric ones are very important. The black hole solutions in a supergravity include the higher-order effects of a string coupling constant, although these are solutions in a low energy limit. On the other hand, the counting of states of corresponding branes is performed at the lowest order of a string coupling. The results of these two calculations need not coincide each other. However, if there is supersymmetry, these
should be the same because the numbers of dynamical freedom cannot be different in these BPS representations. Therefore, supersymmetric black hole (or black ring) solutions are often discussed in many literature [7].

The classification of supersymmetric solutions in minimal $\mathcal{N} = 2$ supergravity in $D = 4$ was first performed by a time-like or null Killing spinor [8]. Recently, solutions in minimal $\mathcal{N} = 1$ supergravity in $D = 5$ have been classified into two classes by use of G-structures analysis [9]. The six-dimensional minimal supergravity has also been discussed [10].

However, the fundamental theory is constructed in either ten or eleven dimensions. When we discuss the entropy of black holes, we have to show the relation between those supersymmetric black holes and more fundamental “black” branes in either $D = 10$ or 11, from which we obtain “black” holes (or rings) via compactification. The entropy is microscopically described by the charges of branes [11]. A supersymmetric rotating solution is obtained by compactification from M or type II supergravity [12, 13]. The supersymmetric rotating black ring solution is found [14]. Such solutions are obtained also in lower dimensions. These solutions are in fact new classes of rotating solutions in four- or five-dimensional supergravity. The existence of such solutions suggests that the uniqueness theorem of black holes is no longer valid even in supersymmetric spacetime if the dimension is five or higher [15]. Thus we may need to construct more generic “black” brane solutions in the fundamental theory and the black holes by some compactification. M-theory is the best candidate for such a unified theory. Since its low energy limit coincides with the eleven-dimensional supergravity, it provides a natural framework to study “black” brane or BPS brane solutions.

In this paper we study a class of intersecting brane solutions in $D$-dimensions with a $(d-1)$-dimensional transverse conformally flat space. We start with a generic form of the metric and solve the field equations of the supergravity (the Einstein equations and the equations for form fields). Assuming the intersection rule for the intersecting branes, which is the same as that derived in a spherically symmetric case [16], we derive the equations for each metric. We find that most metric components are described by harmonic functions, which are independent. One metric component $f$, which corresponds to a traveling wave, is usually given by the Poisson equation, which source term is given by the quadratic form of the “gravi-electromagnetic” field $F_{ij}$. In some configuration of branes, e.g., for two intersecting charged branes (M2$\perp$M5), the source term vanishes. As a result, we find only independent harmonic functions. Hence, we can easily construct arbitrary solution by superposing those harmonics. In order to preserve 1/8 supersymmetry, we have to impose that $F_{ij}$ is self-dual.

2. Basic Equations for a Stationary Spacetime with Branes

We first present the basic equations for a stationary spacetime with intersecting branes and describe how to construct generic solutions. We consider the following bosonic sector of a low energy effective action of superstring theory or M-theory in $D$ dimensions ($D \leq 11$):

$$S = \frac{1}{16\pi G_D} \int d^D X \sqrt{-g} \left[ \mathcal{R} - \frac{1}{2} (\nabla \phi)^2 - \sum_A \frac{1}{2} n_A e^{a_A \phi} F_{nA}^2 \right],$$

where $\mathcal{R}$ is the Ricci scalar of a spacetime metric $g_{\mu\nu}$, $F_{nA}$ is the field strength of an arbitrary form with a degree $n_A (\leq D/2)$, and $a_A$ is its coupling constant with a dilaton field $\phi$. Each index $A$ describes a different type of brane. Although we leave the spacetime dimension $D$ free, the present action is most suitable for describing the bosonic part of $D = 10$ or $D = 11$ supergravity.

As for a metric form for a spacetime with intersecting branes, we assume the following metric
form [13]:

\[ ds^2 = 2\hat{\theta}^i\theta^i + \sum_{i=1}^{d-1} (\theta^i)^2 + \sum_{\alpha=2}^{p} (\theta^\alpha)^2, \]

where \( D = d + p \) and the dual basis \( \theta^A \) are given by

\[ \theta^u = e^\xi du, \quad \theta^i = e^\xi (dv + f du + \frac{A}{\sqrt{2}}), \quad \theta^j = e^\eta dx^j, \quad \theta^\alpha = e^{\zeta_\alpha} dy^\alpha. \]

Here we have used light-cone coordinates; \( u = -(t - y_1)/\sqrt{2} \) and \( v = (t + y_1)/\sqrt{2} \). This metric form includes rotation of spacetime and a traveling wave. Since we are interested in a stationary solution, we assume that the metric components \( f, A = A_i dx^i, \xi, \eta \) and \( \zeta_\alpha \) depend only on the spatial coordinates \( x^i \) in \( d \)-dimensions, which coordinates are given by \( \{t, x^i(i = 1, 2, \ldots, d - 1)\} \). In this setting, we set each brane \( A \) in a submanifold of \( p \)-spatial dimensions, which coordinates are given by \( \{y_\alpha(\alpha = 1, 2, \ldots, p)\} \). Note that the solution in this metric form is invariant under the gauge transformation, \( A \to A + d\Lambda \), \( v \to v - \Lambda/\sqrt{2} \).

As for the \( n_A \)-form field with a \( q_A \)-brane, we assume that the source brane exists in the coordinates \( \{y_1, y_{a_2}, \ldots, y_{a_q}\} \). The form field generated by an “electric” charge is given by the following form:

\[ F_{n_A} = \partial_j E_A dx^j \wedge du \wedge dv \wedge dy_2 \wedge \cdots \wedge dy_{q_A} + \frac{1}{\sqrt{2}} \partial_i B_j^A dx^i \wedge dx^j \wedge du \wedge dy_2 \wedge \cdots \wedge dy_{q_A} \]

where \( n_A = q_A + 2 \) and \( E_A \) and \( B_j^A \) are scalar and vector potentials. This setting automatically guarantees the Bianchi identity.

We can also discuss the form field generated by a “magnetic” charge by use of a dual \(*q_A\)-brane, which is obtained by a dual transformation of the \( n_A \)-field with a \( q_A \)-brane (\( *n_A = D - n_A \), \( *q_A = *q_A - 2 \)). In other words, the field components of \( F_{n_A} \) generated by a “magnetic” charge are described by the same form of (4) of the dual field \( *F_{n_A} = F_{n_A}^* \). We then treat \( F_{n_A} \), which is generated by a “magnetic” charge, as another independent form field with a different brane from \( F_{n_A} \), which is generated by an “electric” charge, when we sum up by the types of branes \( A \).

The solution obtained in this section is summarized as follows:

\[ ds^2 = \prod_A H_A^{2\Delta_A + 1} \left[ 2 \prod_B H_B^{-2 \frac{D-2}{2\Delta_B}} du \left( dv + f du + \frac{A}{\sqrt{2}} \right) + \sum_{\alpha=2}^p \prod_B H_B^{-2 \frac{\gamma_{\alpha B}}{2\Delta_B}} dy_\alpha^2 + \sum_{i=1}^{d-1} dx_i^2 \right], \]

\[ \varphi = (D - 2) \sum_A e_i A_A \ln H_A, \]

where

\[ \Delta_A = (q_A + 1)(D - q_A - 3) + \frac{D - 2}{2} a_A^2, \]

\[ \gamma_{\alpha A} = \delta_{\alpha A} + q_A + 1 = \begin{cases} D - 2 & \alpha = \alpha_2, \ldots, \alpha_{q_A} \\ 0 & \text{otherwise} \end{cases}. \]

\( H_A \) for each \( q_A \)-brane and \( A = A_i dx^i \) are arbitrary harmonic functions, while the vector potential \( B_j^A \) can be chosen either as \( B_j^A \propto A_i/H_A \) (when \( H_A \neq 1 \)), or an arbitrary harmonic function (when \( H_A = 1 \)). The “wave” metric \( f \) usually satisfies the Poisson equation

\[ \partial^2 f = \frac{1}{2} \left[ 1 - \sum_A \frac{D - 2}{2\Delta_A} \right] \prod_B H_B^{-\frac{2(D-2)}{2\Delta_A}} (\partial_{ij} A_i)^2, \]
with some source term originated by the “rotation”-induced metric $A_i$, although it can be also an arbitrary harmonic function for some specific configuration of branes.

It is worth noticing that we have independent Laplace equations for $H_A$ and $A_i$. This makes the construction of solutions very easy. The superposition of any solutions also provides us an exact solution. Hence we can construct an infinite number of solutions. We can also show that a part of supersymmetry is preserved if $F_{ij}$ is self-dual.

3. “black” brane solutions with M2-M5 (D1-D5) branes: the case of $d = 5$

We consider solutions in five-dimensions. There are two branes (M2 and M5). In the ten-dimensional type IIB case, we find the exactly the same as what we show below, when we replace M2 and M5 with D1 and D5 (the indices $A = 2, 5$ with the indices $A = 1, 5$).

The metric in five-dimensions is written by

$$ds_5^2 = -\Xi^2 \left( dt + \frac{A}{2} \right)^2 + \Xi^{-1} ds_{E4}^2,$$

where $\Xi = [H_2H_5(1 + f)]^{-1/3}$. The unknown functions $H_A(A = 2, 5)$, $A_i$ and $f$ satisfy the following equations:

$$\partial^2 H_A = 0 \quad (A = 2, 5), \quad \partial_j F^{ij} = 0, \quad \partial^2 f = 0,$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$.

Giving an explicit form of a solution, we obtain the properties of a “black” object. For example, assuming the asymptotic behaviors $(r \equiv \left( \sum_{i=1}^{4} x_i^2 \right)^{1/2} \to \infty)$ for $H_A$ and $f$ as

$$H_A \to 1 + \frac{Q_H^{(A)}}{r^2}, \quad f \to \frac{Q_0}{r^2},$$

we find

$$M_{ADM} = \frac{\pi}{4G_5} \left( Q_0 + Q_H^{(2)} + Q_H^{(5)} \right).$$

The entropy of a black hole, if it exists, is defined by

$$S = \frac{A_h}{4G_5},$$

where $A_h$ is the area of horizon.

In what follows, adopting the hyperspherical coordinates as a curvilinear coordinate system, we show explicitly how to construct the exact solutions.

3.1. hyperspherical coordinates

We adopt the hyperspherical coordinates:

$$x_1 + ix_2 = r \cos \theta e^{i\phi}, \quad x_3 + ix_4 = -r \sin \theta e^{i\psi},$$

where $0 \leq \phi, \psi < 2\pi$ and $0 \leq \theta \leq \pi/2$. The line element of 4D flat space is

$$ds_{E4}^2 = dr^2 + r^2 \left( d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2 \right).$$

The symmetric axis is described by $\theta = 0$ and $\pi/2$, and the infinity corresponds to $r = \infty$. 


From the asymptotically flatness condition and the regularity conditions on the symmetric axis ($\theta = 0, \pi/2$), the solution is given by

$$H_A = 1 + \sum_{\ell=0}^{\infty} h_\ell^{(A)} r^{-2(\ell+1)} P_\ell(\cos 2\theta).$$

(15)

The spherically symmetric solution ($\ell = 0$) is given by

$$H_A = 1 + \frac{Q_H^{(A)}}{r^2},$$

(16)

where $Q_H^{(A)}$ is a constant, which corresponds to a conserved charge.

Assuming asymptotically flatness and regularity condition, the solution for $A_\ell$ is given by

$$A_\phi = \sum_{m=1}^{\infty} b_m^{(\ell)} r^{2m} F(-m, m, 1, \sin^2 \theta), \quad A_\psi = \sum_{n=1}^{\infty} b_n^{(\ell)} r^{2n} F(-n, n, 1, \cos^2 \theta).$$

(17)

If we take the first two terms in the general solution, we obtain a simple solution as

$$A_\phi = \frac{\cos^2 \theta}{r^2} \left[ J_1^{(\psi)} + J_2^{(\psi)} (1 - 3 \sin^2 \theta) \right], \quad A_\psi = \frac{\sin^2 \theta}{r^2} \left[ J_1^{(\psi)} + J_2^{(\psi)} (1 - 3 \cos^2 \theta) \right],$$

(18)

where $J_1^{(\psi)}, J_2^{(\psi)}$ are constants. The first two constants describe angular momenta of a “black” object. As we show in Appendix A, if $F^{ij}_0$ is self-dual, the spacetime is supersymmetric. This condition implies $J_1^{(\psi)} = -J_2^{(\psi)}$ and $J_2^{(\psi)} = J_1^{(\psi)}$.

Finally we find the Laplace equation for $f$, which gives us a simple solution:

$$f = \sum_{\ell=0}^{\infty} Q_\ell r^{-2(\ell+1)} P_\ell(\cos 2\theta),$$

(19)

where $Q_\ell$’s are constants.

In this case, the solution with the lowest multipole moment is given by

$$H_A = 1 + \frac{Q_H^{(A)}}{r^2}, \quad (A = 2, 5), \quad f = \frac{Q_0}{r^2},$$

$$A_\phi = J_\phi \cos^2 \theta r^2, \quad A_\psi = J_\psi \sin^2 \theta r^2.$$  

(20)

The mass and the entropy of this spacetime are

$$M_{\text{ADM}} = \frac{\pi}{4G_5} (Q_0 + Q_H^{(2)} + Q_H^{(5)}), \quad S = \frac{A_\phi}{4G_5} = \frac{\pi^2}{3G_5} \frac{\Lambda_+^2 + \Lambda_+ \Lambda_- + \Lambda_-^2}{\Lambda_+^{3/2} + \Lambda_-^{3/2}},$$

(21)

where

$$\Lambda_+ = Q_0 Q_H^{(2)} Q_H^{(5)} - \frac{J^2}{8} + \frac{\Delta J^2}{16}, \quad \Lambda_- = Q_0 Q_H^{(2)} Q_H^{(5)} - \frac{J^2}{8} - \frac{\Delta J^2}{16}.$$  

(22)

$J^2$ and $\Delta J^2$ are defined by $J^2 \equiv (J_\phi^2 + J_\psi^2)/2$ and $\Delta J^2 \equiv J_\phi^2 - J_\psi^2$, respectively.

Fixing $J^2$, if we maximize entropy $S$, we find the maximum entropy with

$$S = S_{\text{max}} = \frac{\pi^2}{2G_5} \sqrt{\frac{Q_0 Q_H^{(2)} Q_H^{(5)}}{8} - \frac{J^2}{8}},$$

(23)

if $\Delta J^2 = 0$, i.e., $J_\phi^2 = J_\psi^2 = J^2$. Note that supersymmetry implies $J_\phi = -J_\psi = J$, which corresponds to the BMPV solution [12, 13]. If $J_\phi \neq -J_\psi$, the above solution describes a regular rotating non-BPS black hole spacetime in five dimensions.
4. Concluding Remarks

In this paper, we have studied a stationary “black” brane solution in M/superstring theory. Assuming a BPS type relation between the first-order derivatives of metric functions, we have shown how to construct a stationary “black” brane solution with a traveling wave. We consider two types of intersecting brane: (1) charged branes and (2) neutral branes with a current. The solutions are given by harmonic functions $H_A$ and $A_i$ plus a wave metric $f$ which satisfies the Poisson equation or the Laplace equation. Since those differential equations are linear and independent except for the Poisson equation for $f$, we can easily construct general solutions by superposition of harmonic functions.

Using the hyperspherical coordinate system, we present exact solutions in eleven-dimensional M theory for the case with M2$\perp$M5 intersecting branes and a traveling wave. Compactifying these solutions into five dimensions, we show that these solutions include the BMPV black hole and the Brinkmann wave solution.

The charges of branes of the BMPV black hole correspond to the numbers of D-brane tension. While SO(4) rotational symmetries, which describe angular momenta of the black hole, corresponds to endmorphisms in the graded algebra that rotate the fermionic generators $G^i_m$ [13]. By this correspondence (AdS/CFT correspondence), we can discuss the properties of our solutions in the SCFT side.

Although we assume the BPS type relations for the metric, we have to solve the elliptic type differential equations if we want to find most general solutions, especially non-BPS spacetimes. For this purpose, we need a completely different approach such as a soliton technique to generate new solutions [19, 20, 21].

We have found that the BPS and non-BPS rotating asymptotically flat stringy black holes, from which we may learn more about connections between microscopic and macroscopic states of gravitating objects. In our framework, we consider a toroidally compactified string theory, but one may embed the BMPV type geometry in M-theory compactified on generic Calabi-Yau spaces, which would be more interesting.

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