Decoherence of encoded quantum registers

Stefan Borghoff and Rochus Klesse
Universität zu Köln, Institut für Theoretische Physik, Züllicher Str. 77, D-50937 Köln, Germany
(Dated: August 28, 2007)

In order to eliminate disturbing effects of decoherence, encoding of quantum information in decoherence-free subspaces has been suggested. We analyze the benefits of this concept for a quantum register that is realized in a spin chain in contact with a common bosonic bath. Within a dissipation-less model, we provide explicit analytical results for the average fidelity of plain and encoded quantum registers. For the investigation of dissipative spin-boson couplings, we employ a master equation of Bloch-Redfield type.

PACS numbers: 02.50.Ga, 03.65.Yz, 03.67.-a, 03.67.Mn, 03.67.Pp, 75.10.Pq, 89.70.+c

I. INTRODUCTION

The main obstacle in utilizing the remarkable computational power of quantum systems is the omnipresent and fundamental phenomenon of decoherence. While this insight cast significant doubts about the idea of large-scale quantum computation, it also initiated extensive research on decoherence in quantum information systems, and, beyond that, it led to the development of quantum error-correcting and -avoiding methods. The latter ones, on which we focus in the present work, make use of possible symmetries in the interaction of, say, a quantum register and its surrounding environment. The idea is to encode quantum information in those register states that are protected by symmetry against the decohering interaction. Inasmuch as the symmetry is satisfied these states span a decoherence-free subspace in the register’s Hilbert space.

Almost necessarily physical realizations of this concept will have to rely on symmetries that hold only to some approximation. Encoding in subspaces that respect these symmetries can then provide only partial protection against decoherence, to an extent that will depend on the actual realization. The present work addresses this problem for the generic situation of a quantum register consisting of (effective) spin-1/2 particles in contact with a common bosonic bath.

In the main (Secs. II and III), we describe this system by the dissipation-less spin-boson model of Palma et al. which has been frequently used in similar contexts. For simplicity, we assume the spins to be arranged in a linear chain with inter-spin distance a. Furthermore, we will use a three-dimensional bosonic bath with an ohmic coupling density of states. In the limit of vanishing distance a the model exhibits a highly symmetric spin-boson interaction, allowing the construction of decoherence-free subspaces. Specifically, we consider subspaces that correspond to encoded quantum registers in which logical qubits are encoded in locally grouped physical qubits (spins) (cf. Sec. III). The main task is to analyze the decoherence which will appear in these encoded quantum registers when the distance a assumes finite values. In Sec. III we quantify the decoherence of encoded registers (with finite distance a) as well as of plain registers by means of the average register fidelity. Sec. IV is devoted to the effect of dissipative spin-boson couplings.

For a summary of our results we refer to the self-contained presentation in Sec. V.

II. DECOHERENCE OF QUANTUM REGISTERS

A. Physical quantum register

A physical n-qubit register may consist of n (effective) spin-1/2 particles located at sites r_0, r_2, ..., r_{n-1} of a one-dimensional lattice with lattice constant a and of finite length L = a(n−1). A homogeneous (effective) magnetic field in z-direction may lead to a Zeeman energy splitting ε. The corresponding register Hamiltonian is

$$H_R = \frac{\varepsilon}{2} \sum_{l=0}^{n-1} Z_l,$$

where $Z_l$ denotes the Pauli $\sigma_z$-operator for spin $l$. The register is supposed to be in contact with a thermal bath of three-dimensional bosons described by the Hamiltonian

$$H_B = \sum_k \hbar \omega_k b_k^\dagger b_k,$$

where $b_k^\dagger$ and $b_k$ are creation and annihilation operators of bosonic modes with linear dispersion $\omega_k = c|k|$. We assume a linear and local spin-boson interaction via $Z_l b_k^\dagger$ and $Z_l b_k$ operators. The corresponding coupling constant $g_{l,k}$ will acquire a phase $e^{ik r_l}$, reflecting the wave-like character of the bosonic modes. Apart from this phase the interactions may be isotropic and identical for each spin. Thus, $g_{l,k} = g_{|k|} e^{ik r_l}$, resulting in an interaction Hamiltonian

$$H_{RB} = \sum_l Z_l B(r_l),$$
where $B(r)$ is the hermitian bosonic field operator

$$B(r) = \sum_k g|k|e^{-iK_{r}r_k} + H.c.$$.

As customary, we describe the strength of the spin-boson coupling by a spectral density

$$J(\omega) = \sum_k \delta(\omega_k - \omega)|g|_k|^2 \equiv \alpha \omega^s e^{-\omega/\Omega}.$$.

It is characterized by a cut-off frequency $\Omega$, a constant $\alpha$ of appropriate dimension, and a non-negative spectral parameter $\alpha$. Since the spins interact with the bosons only via the energy conserving $Z_l$ operators, the model shows no dissipation but pure decoherence. This restriction makes the model analytically manageable.\textsuperscript{13,15,17,19} (Effects of additional dissipative couplings will be discussed in Sec.\textsuperscript{LV})

1. Decoherence of a physical quantum register

The register may be used to store quantum information in form of a state $\rho_0$ in which it is initially prepared. In general, $\rho_0$ is subjected to a non-unitary dynamics originating from the system’s own dynamics and its coupling to the bosonic bath. Assuming that the bath is initially in a thermal state $\rho_0$, the total initial state is $\rho = \rho_0 \otimes \rho_B$. During some time period $t$ this state will evolve unitarily according to $i\hbar \dot{\rho} = [H_R + H_B + H_{RB}, \rho]$ towards a final state $\rho_f$. Its partial trace with respect to the bosonic modes yields the reduced density operator $\rho_1 = Tr_B \rho_f$, which describes the final register state. This procedure defines a quantum operation $E$ on the register by

$$\rho_0 \rightarrow E(\rho_0) := \rho_1.$$.

The work of several authors\textsuperscript{5,11,14,15,17,19,21} established the operation $E$ to be of the form

$$E(\rho) = U \circ \mathcal{N}(\rho),$$

where $U$ is a purely unitary operation, and $\mathcal{N}$ is a non-unitary, completely positive map that can be written as

$$\mathcal{N}(\rho) = \sum_{\mu \nu \in \mathbb{Z}^2} e^{-D_{\mu\nu}}|\mu\rangle\langle\mu| \rho |\nu\rangle\langle\nu|.$$ (1)

Here, the double summation extends over all register eigenstates $|\nu\rangle$ which we label in the usual way by $n$-bit sequences $\nu \in \{0,1\}^n = \mathbb{Z}_2^n$ according to $Z_l|\nu\rangle = (-1)^{\nu_l}|\nu\rangle$. The decoherence coefficients $D_{\mu\nu}$ are

$$D_{\mu\nu} = \sum_{l=m=0}^{n-1} (\mu_l - \nu_l)(\mu_m - \nu_m)K(|r_l-r_m|, t),$$ (2)

with a distance and time dependent decoherence function

$$K(|r|, t) = 4Re \int_0^t dt' \int_0^{t''} dt'' \langle B(r, t'')B(r, 0) \rangle_T.$$ (3)

Here $\langle \ldots \rangle_T$ denotes the thermal average over the bosonic system at temperature $T$, and $B(r, t)$ is the bosonic field operator in interaction picture, i.e.

$$B(r, t) = \sum_k g|k|e^{-i(Kr_k - \omega_k)t} + H.c.$$ (4)

The unitary part $U$ of $E$ originates from the registers’s own dynamics but also includes the Lamb-shift caused by the bosonic bath. In principle, this part of the evolution $E$ can be reversed and therefore is not of major concern. In contrast to that, the operation $N$ gives rise to decoherence and so seriously affects the register in an irreversibly manner. Clearly the main attention has to be paid to $\mathcal{N}$. Therefore, it will be in the focus of our investigation.

2. Ohmic decoherence function

In order to make $\mathcal{N}$ more explicit, we have to determine the decoherence function Eq. (3). For the sake of simplicity, we restrict ourself to an ohmic spectral function $J(\omega) = \omega \alpha e^{-\omega/\Omega}$. Then, determining the correlator $\langle B(r, t'')B(0, 0) \rangle_T$ by standard methods and passing the continuum limit for the bosonic modes, the decoherence function $E$ becomes

$$K(r, t) = \frac{\alpha}{\Omega} \int_0^{\infty} \frac{1 - \cos \omega t}{\omega} \coth \left( \frac{\omega}{2T} \right) \sin \omega r e^{-\omega/\Omega}.$$ (5)

(Henceforth we use units in which $c = 1$, $\hbar = 1$, and $k_B = 1$.) This integral can be better dealt with if we distinguish between the case of strictly vanishing distance $r$ and the case of a finite distance that is large compared to the cut-off wavelength $\sim 1/\Omega$, which we assume to be the smallest scale in the problem.

For vanishing distance we obtain

$$K(0, t) = 2\alpha \ln \left| \frac{\Gamma(T/\Omega)}{\Gamma(T/\Omega - itT)} \right| - \frac{\alpha}{2} \ln(1 + t^2\Omega^2).$$ (6)

At small times $t \ll 1/T$ this is in good approximation

$$K(0, t) \simeq \frac{\alpha}{2} \ln(1 + t^2\Omega^2),$$ (7)

whereas for large times $t \gg 1/T$ we have

$$K(0, t) \simeq \alpha \pi Tt + \alpha \ln \frac{\Omega}{2\pi T}. \quad \text{(8)}$$

Physically, $K(0, t)$ determines the decoherence of a single spin, as it is seen by Eqs. (2) and (1) for $n = 1$, according to which

$$\rho_0(t) = e^{-K(0, t)} \rho_0(0).$$
Asymptotically, the single-spin decoherence decays exponentially with a rate $\gamma = \alpha \pi T$, by Eq. (8).

For finite $r \gg 1/\Omega$ the oscillations of the spherical Bessel function $\sin(\omega r)/\omega r$ damp the integrand in Eq. (6) more effectively than the regular cut-off $\exp(-\omega/\Omega)$. This allows us to take the limit $\Omega \to \infty$. The resulting integral can be solved by contour integration, which finally leads us to

$$K(r, t) = \alpha \pi T \left( t - \frac{r}{2} + \frac{1}{2T^2 r} \right) + \frac{\alpha}{4\pi Tr} \left[ f_T(t + r) - f_T(t - r) - 2f_T(r) \right],$$  

for $r < t$, and to

$$K(r, t) = \alpha \pi T \frac{t^2}{2r} + \frac{\alpha}{4\pi Tr} \left[ f_T(t + r) + f_T(r - t) - 2f_T(r) \right],$$

for $r > t$. For convenience, we introduced a temperature dependent function $f_T(t) := \text{Li}_2(e^{-2\pi T t})$, where $\text{Li}_2(x) = \sum_{j=1}^{\infty} x^j / j^2$ is the dilogarithm of $x$.

In the large temperature regime characterized by $t, r, |t - r| \gg 1/T$, the $f_T(\cdot)$-terms in Eqs. (10) and (11) become exponentially suppressed, and so, additionally omitting an $\alpha \pi / 24Tr$ term,

$$K(r, t) \simeq \alpha \pi T \left( t - \frac{r}{2} \right), \quad \text{for } r < t$$

$$K(r, t) \simeq \alpha \pi T \frac{t^2}{2r}, \quad \text{for } r > t$$

Fig. 1 shows $K(0, t)$ and $K(r_0, t)$ as a function of dimensionless time $\tau = t/r_0$ together with their approximations Eqs. (9), (8), (11), and (12), for $\Omega = 10^3T$.

### B. Encoded quantum register

For vanishing lattice constant $a$ the locations of all spins of the quantum register introduced in Fig. 1 fall onto a single point $r_0$. This implies a highly symmetric spin-boson interaction

$$H_{RB}^{(0)} = \left( \sum_i Z_i \right) B(r_0)$$

that exactly annihilates all states with vanishing total spin-$z$ component. As a consequence, any linear subspace $C$ of the register’s state space $H_n$ that is spanned by such states is not affected by the bosonic bath at all. It represents a decoherence-free subspace.

At a finite lattice constant $a$ the former symmetry is absent and consequently $C$ ceases to be decoherence-free. However, by reasons of continuity the decoherence of states in $C$ will be still much lower than for arbitrary states as long as the lattice constant $a$ is not too large.

Of course, the decoherence reduction at a finite lattice constant $a$ will strongly vary for different choices of the subspace $C$. Here we will investigate subspaces that result from encoding (logical) qubits in local groups of physical qubits. This is supposed to be done in a regular manner such that the resulting structure forms a regular encoded quantum register.

To be specific, let us consider a one-dimensional physical $2n$-qubit register $R_{2n}$ whose $2n$ spins $S_0, \ldots, S_{2n-1}$ are grouped in $n$ pairs of neighboring spins as sketched in Fig. 2. The spin pair $S_{2i}S_{2i+1}$ has a four-dimensional Hilbert space of which the two orthonormal states

$$|0\rangle_1 := |0\rangle_{2i} |1\rangle_{2i+1}, \quad |1\rangle_1 := |1\rangle_{2i} |0\rangle_{2i+1}$$

are annihilated by the spin-boson interaction if $a = 0$. In this case the subspace $C_i$ spanned by states $|0\rangle_1$ and $|1\rangle_1$ is decoherence-free. We call $C_i$ the state space of the encoded (logical) qubit $Q_i$, and we further define an encoded $n$-qubit register $R_n^1$ to consist of the $n$ encoded qubits $Q_0, \ldots, Q_{n-1}$. Its Hilbert space $H_n^1 = C_0 \otimes \cdots \otimes C_{n-1}$ is by construction decoherence-free with respect to $H_{RB}^{(0)}$. We will denote a state $\rho$ as a state of the encoded register $R_n^1$ if the support of $\rho$ lies entirely in $H_n^1$.

![FIG. 2: Spin array representing qubits and encoded logical qubits of 1st and 2nd order.](image)

Clearly, pairing up remote spins instead of adjacent ones would lead to encoded qubits that will be more sensitive to an increasing lattice constant $a$. We will therefore
exclude this possibility from our considerations. Instead, one may speculate that the protection against decoherence improves if we iterate the pairing in order to built encoded qubits and registers of higher order (cf. Fig. 2).

More precisely, we recursively define encoded qubits of order \( \chi = 1, 2, \ldots \) by their logical states

\[
|0\rangle_\chi := |0\rangle_{2\chi-1}^1 |1\rangle_{2\chi-1}^1, \quad |1\rangle_\chi := |1\rangle_{2\chi-1}^1 |0\rangle_{2\chi-1}^1,
\]

where \( 0^\text{th} \) order states are identified with plain spin states \(|0\rangle_0 = |0\rangle_i \) and \(|1\rangle_0 = |1\rangle_i \) of spin \( S_i \). An encoded \( n \)-qubit register \( R^n_\chi \) of order \( \chi \) is then built from the encoded qubits of a \( 2n \)-qubit register \( R^{2n-1}_\chi \) of order \( \chi - 1 \). A state \( \rho_{\chi} \) of the encoded register \( R^n_\chi \) is by definition a state whose support lies in \( H^n_\chi \).

1. Decoherence of encoded quantum register

How will the decohering operation \( \mathcal{N} \) affect the encoded registers which we have just introduced? First we observe that a state \( \rho_{\chi} \) of an encoded register \( R^n_\chi \) remains a state of \( R^n_\chi \) under \( \mathcal{N} \), simply because the spin-boson interaction \( H_{RB} \) does not flip spins. Moreover, in App. A we show the operation \( \mathcal{N} \) on an encoded register \( R^n_\chi \) to be formally given again by Eqs. (1) and (2). What changes is the decoherence function \( K(\rho_{\chi} - \rho_{\chi}|\tau\rangle \langle \tau|) \), which has to be replaced by an effective decoherence function \( K^{\chi}_{\rho_{\chi}}(\tau) \), and the summation, which now extends over logical register states \( |\mu\rangle_\chi, |\nu\rangle_\chi \) given by

\[
|\mu\rangle_\chi = |\mu_0\rangle_{\chi}^1 \cdots |\mu_{n-1}\rangle_{\chi}^{n-1}, \quad \mu \in \mathbb{Z}_2^n.
\]

Explicitly, for a state \( \rho_{\chi} \) of an encoded register \( R^n_\chi \) we have

\[
\mathcal{N}(\rho_{\chi}) = \sum_{\mu\nu \in \mathbb{Z}_2^n} e^{-D_{\mu\nu}^{\chi}} |\mu\rangle_\chi \langle \mu| \rho_{\chi} \langle \nu| \nu\rangle_\chi, \quad (15)
\]

with effective decoherence coefficients

\[
D_{\mu\nu}^{\chi} = \sum_{l,m=0}^{n-1} (\mu_l - \nu_l)(\mu_m - \nu_m)K^{\chi}_{\rho_{\chi}|\tau\rangle \langle \tau|}(t). \quad (16)
\]

The effective decoherence functions \( K^{\chi}_{\rho_{\chi}}(t) \) for \( \chi \geq 1 \) are recursively defined by

\[
K^{\chi}_{\rho_{\chi}}(t) = 2K^{\chi-1}_{2\chi-1}(t) - K^{\chi-1}_{|2\chi-1|}(t) - K^{\chi-1}_{|2\chi+1|}(t), \quad (17)
\]

with \( K^{0}_{\rho_{\chi}}(t) = K(\rho_{\chi}, t) \).

C. Discussion

The formal analogy of Eqs. (15), (16) and Eqs. (1) and (2) allows for a first comparison of the decoherence in encoded and plain quantum registers by simply comparing the corresponding decoherence functions given by Eq. (17) and Eq. (5).

---

**FIG. 3:** Decoherence functions \( K(0, t) \) (dashed) and \( K_0^1(t) \) (solid) as function of dimensionless time \( \tau = t/a \) at high (a) and low (b) temperatures with cut-off \( \Omega = 10^3/a \), and \( T = 10^2/a \) in (a) and \( T = 0.1/a \) in (b).

We begin with the \( 1^\text{st} \) order decoherence function \( K_0^1(t) \), which describes the effective decoherence of a single encoded qubit \( R_1^1 \). According to Eq. (17)

\[
K_0^1(t) = 2(K(0, t) - K(a, t)).
\]

In the high temperature regime \( t, a \gg 1/T \) we may use approximations (5), (11), and (12) to derive

\[
K_0^1(t) = 2\alpha \ln \frac{\Omega}{2\pi T} + \begin{cases} \alpha \pi Ta & : t > a \\ \alpha \pi T(2t - \frac{\pi^2}{6}) & : t \leq a \end{cases}
\]

The effective decoherence function \( K_0^1(t) \) increases twice as fast with time as \( K(0, t) \) for small times, but quickly saturates to a constant value at time \( t \approx a \) (cf. Fig. 3b). Qualitatively, this remains to be also true at lower temperatures. For \( a \ll 1/T < t \) we can extract from relation (8) and the exact expression (9) an asymptotic value

\[
K_0^1(\infty) \approx 2\alpha \ln \frac{\Omega a}{e}
\]

that is reached again at \( t \approx a \) (cf. Fig. 3b).

We conclude that for any finite distance \( a \) the coherence \( e^{-K_0^1(t)} \) of an encoded qubit approaches a finite asymptotic value at times \( t \gtrsim a \). In the long-time limit the coherence of the encoded qubit will therefore largely exceed the exponentially decaying coherence
\( e^{-K(0,t)} = (2\pi T/\Omega)^{\alpha} e^{-\alpha \pi T t} \) of a plain qubit. At short times, however, the encoded qubit performs worse than the plain qubit. The crossover time \( t_c \) can be easily determined to be \( t_c \approx a \) in the high temperature regime \( T \gg 1/a \), and \( t_c \approx \frac{1}{\pi} \ln(2\pi a^2 \Omega T/e^2) \gg a \) in the low temperature limit \( T \ll 1/a \). Thus, whether it is beneficial to encode or not also depends on the time period over which the qubit is supposed to store information. Here, it is important to observe that with lowering the temperature \( T \) one eventually reaches the low temperature regime where the crossover time \( t_c \) increases with \( 1/T \).

Do things further improve when one goes to higher-order encoded qubits? Interestingly, this is not the case, for the reason that the 1st order qubits of an encoded register \( R_1 \) are already essentially decoupled (see below), and pairing up these independent qubits to higher-order qubits would not further reduce their effective decoherence. The decoupling of the 1st order-qubits is seen from the effective decoherence functions \( K_l^1(t) \) for \( l \geq 1 \). By Eq. (17) we find

\[
K_l^1(t) = 2K(2la,t) - K(2la - a,t) - K(2la + a,t).
\]

In the high temperature regime this predicts by Eq. (11) actually a vanishing \( K_l^1(t) \) for times \( t > 2la + a \). More precisely, for \( l \geq 1 \)

\[
K_l^1(t) = O \left( \frac{\pi}{24T(2l)} \right) \ll K_0^0(t) \approx \alpha \pi T \left( t - \frac{la}{2} \right).
\]

Alternatively, we can also directly calculate the zero-distance decoherence functions \( K_0^l(t) \) according to relation (17). In the high temperature regime we obtain in the long-time limit for \( \chi \geq 1 \)

\[
K_0^\chi(\infty) = 2^{\chi-1} K_0^1(\infty),
\]

which obviously strongly increases with \( \chi \). Qualitatively similar behaviour is found also at low temperatures, where we used the exact result Eq. (13) to numerically determine \( K_0^\chi(\infty) \). The results are plotted in Fig. 4.

So far the discussion is restricted solely on a comparison of (effective) decoherence functions. While this suffices to characterize the decoherence of (encoded) qubits, it does not necessarily provide full insight in the performance of an entire (encoded) quantum register under the noise operation \( \mathcal{N} \). The next section addresses this point by a systematic investigation of average register fidelities.

Another important point which deserves further investigation is dissipation. The above analysis is based to a large extent on exact results that are available only for the dissipation-less spin-boson model. Therefore, it might be possible that essential features of encoded registers – particularly the saturation of the decoherence – will not survive when dissipative spin-boson couplings are taken into account. We will investigate this problem in Sec. IV.

![Graph](image)

**Fig. 4**: Asymptotic value of the decoherence functions \( K_0^\chi(\infty) \) as function of dimensionless temperature \( \Theta = Ta \) for \( \chi = 1, 2, 3 \) and cut-off \( \Omega = 5 \cdot 10^3/a \).

### III. AVERAGE REGISTER FIDELITIES

#### A. Definition

Let a physical or encoded quantum register \( R_n \) with Hilbert space \( H_n \) be exposed to some noise operation \( \mathcal{N} \) (for instance, the one studied above), under which an initial pure register state \( \psi = |\psi\rangle \psi \) evolves to a final \( \mathcal{N}(\psi) \). The channel fidelity of \( \psi \) with respect to \( \mathcal{N} \),

\[
F(\psi,\mathcal{N}) := |\langle \psi|\mathcal{N}(\psi)\psi \rangle|,
\]

captures how well the state is preserved in this process. A quantum register will be suitable for information storage under the noise \( \mathcal{N} \) if, in average, the channel fidelity \( F(\psi,\mathcal{N}) \) for register states is large. A reasonable figure of merit is therefore given by the average fidelity of register \( R_n \) with respect to \( \mathcal{N} \),

\[
F := \frac{1}{N} \int_{H_n} d\psi \ F(\psi,\mathcal{N}) , \quad N = \int_{H_n} d\psi \ 1,
\]

where the integrals extend over all pure code states \( \psi \) with state vectors \( |\psi\rangle \in H_n \) with respect to a unitary invariant measure.

More precisely, we can express the average by an integration over the group \( U(H_n) \) of unitaries on \( H_n \) with the normalized Haar measure \( \mu \),

\[
F = \int_{U(H_n)} d\mu(U) \ F(U\psi_0U^\dagger,\mathcal{N}) , \quad (18)
\]

where \( \psi_0 \) is any fixed pure register state.

For large qubit number \( n \) the average register fidelity is known to agree with the entanglement fidelity of \( \mathcal{N} \) with respect to \( H_n \), which plays a prominent role in quantum information theory.
B. General expressions

1. Sum representation

From now on we consider the noise operation $\mathcal{N}$ of the dissipation-less spin-boson model which we introduced in Sec. III. Our aim is to derive an expression for the average fidelity for physical or encoded registers in terms of the decoherence coefficients (16). Using representation

$$\mathcal{N}(\rho) = \sum_{\mu\nu=0}^{2^n-1} e^{-D_{\mu\nu}} |\mu\nu\rangle \langle \mu\nu|,$$

(we omit the order-index $\chi$, and identify $\mathbb{Z}_2^n$ with integer numbers $0, 1, 2, \ldots, n - 1$) and relation (18) with $\psi_0 = |0\rangle |0\rangle$ we immediately obtain

$$F = \sum_{\mu\nu=0}^{2^n-1} e^{-D_{\mu\nu}} \int_{U(H_n)} d\mu(U) |U_{\mu\nu}|^2 |U_{\nu\mu}|^2.$$

The integral can be calculated by standard methods, leading to

$$\int_{U(H_n)} d\mu(U) |U_{\mu\nu}|^2 |U_{\nu\mu}|^2 = \frac{1 + \delta_{\mu\nu}}{4^n} + O(2^{-6n}).$$

This outcome is easily understood once one recognizes the integration as an average over all $2^n$-dimensional complex unit vectors $u_\mu \in H_n$ of which $U_{\mu\nu}$ and $U_{\nu\mu}$ are the $\mu$th and $\nu$th component, respectively. For dimension $2^n$ and $\mu \neq \nu$ the squared absolute values $|U_{\mu\nu}|^2$ and $|U_{\nu\mu}|^2$ are nearly independent gaussian variables $X_1$ and $X_2$ of mean $\langle X_1 \rangle = \langle X_2 \rangle = 2^{-n}$, by the normalization of $u_\mu$. For $\mu \neq \nu$ the integral over the product $|U_{\mu\nu}|^2 |U_{\nu\mu}|^2$ therefore amounts to the expectation value $\langle X_1 X_2 \rangle = (2^n)^2$. For $\mu = \nu$ we instead obtain the second moment $\langle X_1^2 \rangle = 2 \langle X_1 \rangle^2 = 2(n-2)^2$. The exact calculation reveals rather tiny corrections to these estimates of order $2^{-6n}$, which we will neglect in the following. Then, by the last two equations we find

$$F = \frac{1}{4^n} \sum_{\mu\nu=0}^{2^n-1} e^{-D_{\mu\nu}} (1 + \delta_{\mu\nu}).$$

Since the sum over the extra diagonal terms $e^{-D_{\mu\nu}} \delta_{\mu\nu}$ contributes at most $2^{-n}$ to the fidelity we can omit these terms as well and thus are left with

$$F = \frac{1}{4^n} \sum_{\mu\nu=0}^{2^n-1} e^{-D_{\mu\nu}}.$$

(19)

In general, the double summation over the exponentially large range $0, \ldots, 2^n - 1$ makes a direct numerical or analytical evaluation of this expression difficult. However, progress can always be made if we proceed similar as in Ref. 24 and employ a Hubbard-Stratonovich transformation. This will factorize the double sum into $n$ trivial sums, at the expense of an $n$-dimensional integration over auxiliary continuous degrees of freedoms.

2. Integral representation

We rewrite the decoherence coefficients Eq. (16) as

$$D_{\mu\nu} = v_{\mu\nu}^\dagger K v_{\mu\nu},$$

where we introduced real, $n$-dimensional vectors $v_{\mu\nu}$ with components

$$(v_{\mu\nu})_m = \mu_m - \nu_m,$$

and a real and positive $n \times n$ decoherence matrix $K$ whose entries are determined by the (effective) decoherence functions Eq. (17),

$$K_{lm} = K_{|l-m|}(t).$$

Then, with Gauss’s identity

$$e^{-v_{\mu\nu}^\dagger x K_{\mu\nu} v_{\mu\nu}} = \int \frac{d^n x}{N} e^{-x^\dagger K^{-1} x + 2i v_{\mu\nu}^\dagger x},$$

where $N = (\pi^n \det K)^{1/2}$, the average fidelity Eq. (19) becomes

$$F = \int \frac{d^n x}{N} e^{-x^\dagger K^{-1} x} \sum_{\mu\nu=0}^{2^n-1} e^{2iv_{\mu\nu}^\dagger x}.$$

The sum is readily determined to be

$$\sum_{\mu\nu=0}^{2^n-1} e^{2iv_{\mu\nu}^\dagger x} = \prod_{l=1}^{n} \sum_{\mu_1 \mu_2 \cdots \mu_l=0}^{1} e^{2i(\mu_1 - \nu_1) x_1} = 4^n \prod_{l=1}^{n} \cos^2 x_l,$$

such that the average fidelity becomes

$$F = \int \frac{d^n x}{N} e^{-x^\dagger K^{-1} x} \prod_{l=1}^{n} \cos^2 x_l.$$

(20)

This relatively well-behaved integral representation of the average fidelity can serve as starting point for numerical or analytical calculations (cf. Sec. III C). Furthermore, in contrast to the sum representation (19), the integral (20) indicates how to obtain approximative expressions.

3. Weak coupling approximation

For weak couplings $\alpha$ the inverse eigenvalues of $K$ become large and hence the integrand sharply peaks at the global maximum at $x = 0$. In this case it is appropriate to expand the integrand as

$$e^{-x^\dagger K^{-1} x} \prod_{l=1}^{n} \cos^2 x_l = e^{-x^\dagger (K^{-1} + 1) x} + O(|x|^4)$$

and to omit the $O(|x|^4)$ corrections in the exponent. Inserting this in Eq. (20) we arrive at a proper Gauss integral which yields a surprisingly simple weak coupling approximation

$$F_{\text{wc}} = \det(1 + K)^{-1/2}.$$
of the average fidelity. By consideration of a diagonal matrix $\mathbf{K}$ we estimate the relative error $|F - F_{\text{wc}}|/F$ of order $\text{tr}\mathbf{K}^2$, which one has to keep in mind when using this approximation.

4. Small deviations

Finally, let us consider the practically relevant situation where the average fidelity deviates only by a small amount $\varepsilon$ from unity. By the weak coupling approximation we find

$$F = 1 - \varepsilon = \det(1 + \mathbf{K})^{-1/2}(1 + O(\text{tr}\mathbf{K}^2)),$$

and, taking the logarithms of both sides,

$$\ln(1 - \varepsilon) = -\frac{1}{2}\text{tr} \ln(1 + \mathbf{K}) + O(\text{tr}\mathbf{K}^2).$$

When we expand the logarithms we observe that in leading order $\varepsilon = \frac{1}{2}\text{tr} \mathbf{K} + O(\text{tr}\mathbf{K}^2)$, and hence, since the decoherence matrix $\mathbf{K}$ has constant diagonal elements $K_0(t)$,

$$F = 1 - \frac{1}{2}nK_0(t) + O(\text{tr}\mathbf{K}^2).$$

Small deviations of $F$ from unity are thus determined by the zero-distance decoherence function $K_0(t)$, describing the decoherence of a single (encoded) qubit, and they grow linearly with the number $n$ of qubits.

C. Examples

The following discussion of two illustrative examples will provide more insights in the average register fidelity. The results will be also useful in the subsequent comparison of plain and encoded quantum registers.

1. Independent qubits

The first example is a quantum register consisting of independent qubits, as it is reflected in vanishing decoherence functions $K_l(t)$ for $l > 0$. For instance, this is realized in a plain, physical register $R_n^0$ in the limit of a diverging lattice constant $a$, but also holds to good approximation for an encoded register $R_n^1$ in the long-time limit (cf. discussion in Sec. II C).

As a consequence of $K_l(t) = 0$ for $l > 0$ the decoherence matrix $\mathbf{K}$ of such a register is

$$\mathbf{K} = \kappa \mathbf{1}_n,$$

where $\kappa = K_0(t)$, and $\mathbf{1}_n$ is the $n \times n$ unit matrix. Because of this trivial matrix $\mathbf{K}$ the integral in nicely factorizes into $n$ one-dimensional Gaussian integrals,

$$F = \prod_{l=1}^{n} \int \frac{dx_l}{\sqrt{\pi \kappa}} e^{-x_l^2/\kappa \cos^2 x_l} = \left(1 + e^{-\kappa}/2\right)^n. \quad (23)$$

Not unexpected, the average fidelity of the $n$-qubit register is exactly the $n$th power of the average fidelity of a single qubit, $F_1 = (1 + e^{-\kappa})/2$. We note in passing that this result could have been derived also directly from the sum representation (cf. Fig. 5).

We can also employ the weak coupling expression (21), predicting

$$F_{\text{wc}} = (1 + \kappa)^{-n/2}.$$ 

While this is not quite the exact result (23), we indeed observe good agreement for small couplings $\kappa \ll \sqrt{1/n}$. Notice that the weak coupling approximation particularly holds in the regime $1/n \ll \kappa \ll \sqrt{1/n}$, where the average fidelity is already exponentially small. (cf. Fig. 5).

2. Symmetrically coupled qubits

As the extreme opposite to the first, our second example is a register whose qubits are symmetrically coupled to the bosonic bath by an interaction Hamiltonian $\mathbf{1}_n$. This is realized for a physical register in the limit of a vanishing lattice constant $a$, where all qubits are located at the same position. Consequently, here the decoherence matrix $\mathbf{K}$ becomes a uniform matrix with constant entries

$$K_{lm} = K_{|l-m|}(t) = K_0(t) \equiv \kappa. \quad (24)$$

Up to a factor $nk$ the matrix $\mathbf{K}$ describes the orthogonal projection on the diagonal $\mathbf{d} = (1, \ldots, 1)/\sqrt{n}$. $\mathbf{K}$ has therefore a non-degenerate eigenvalue $nk$ with an eigenvector $\mathbf{d}$, and an $(n - 1)$-fold degenerated eigenvalue 0 with eigenspace $\mathbf{d}^\perp$. It follows that the integrand in (20) has its entire weight on the diagonal $\mathbf{d}$, as an effect
of which the $n$-dimensional integral collapses to a one-dimensional one. In this way the average fidelity results in

$$F = \frac{1}{\sqrt{\pi n \kappa}} \int dx \ e^{-x^2/4n^2} \cos^{2n} x \ \frac{x}{\sqrt{n}}$$

$$= \frac{1}{4^n} \sum_{l=0}^{2n} \binom{2n}{l} e^{-\kappa(n-l)^2}.$$  

The result can be better interpreted in the limit of large $n \gg 1$ and small $\kappa \ll 1$ (independent of $n$). When $n$ is large, we can approximate the binomial factor $4^{-n} \binom{2n}{n}$ by a Gaussian, $\exp(-(n-l)^2/\pi n)$, and further, when $\kappa$ is small, we are allowed to replace the sum by an integral. This yields an average fidelity

$$F = \frac{1}{\sqrt{1 + n \kappa}}. \quad (25)$$

We notice that this expression also results from the weak coupling approximation Eq. (21), since here $\det(1 + K) = 1 + n \kappa$.

The algebraical decay with $n$ in Eq. (25) strongly contrasts with the exponential decay of the average register fidelity Eq. (23) observed for independent qubits. This marked difference must be attributed to the high degree of symmetry in the present case. In fact, the symmetric qubit-boson coupling (13) entails that states with a small total spin-$z$ component couple much less effectively to the bosonic bath as than they would do in the case of independent qubits. Apparently, for numbers of qubits and couplings with $n \gg 1/\kappa$ this results in a strongly enhanced averaged fidelity.

Remarkably, a register with $n \ll 1/\kappa$ does not benefit from these effects of symmetry. In this regime, the average fidelity for independent and symmetrically coupled qubits actually coincide (cf. Fig. 6). We are lacking a simple explanation for that, however, since in this regime also $1 - F \ll 1$ we can refer to the general result Eq. (22) for small deviations. According to this relation, here the fidelity is dominated solely by the zero-distance decoherence function $K_0(t)$, and hence all details concerning the spatial structure of the register do not matter.

**D. Comparison of plain and encoded quantum register**

In this subsection we will compare a plain, physical register $R_0^n$ with a $1^{st}$ order encoded register $R_1^n$ by means of their respective average register fidelities $F_0$ and $F_1$. Thereby, we will make good use of the results for the two preceding examples. We will restrict the comparison to the high temperature regime $a \gg 1/T$, where we can use the relatively simple expressions Eqs. (8), (11), and (12) for the decoherence function. Furthermore, the time $t$ will be assumed to be larger than $L_0 = (n - 1)a$ for the plain register $R_0^n$ and larger than $L_1 = (2n - 1)a$ for the encoded register $R_1^n$.

**1. Average fidelity of a plain quantum register**

We consider a physical $n$-qubit register $(R_0^n)$ as defined in Sec. 1A. In the high temperature regime and for $t > L_1$, its time dependent decoherence matrix $K(t)$ follows by Eqs. (8) and (11) to be given by

$$K_{lm}(t) = \gamma t \left( 1 - \frac{a|m - m|}{2t} \right), \quad \gamma = \alpha \pi T, \quad (26)$$

where we suppressed a logarithmic term $\ln(\Omega/2\pi T) \ll \gamma t$ in the diagonal matrix elements. Since the non-trivial structure of $K(t)$ does not allow for a simple evaluation of the exact formula Eq. (20), we immediately switch to a numerical evaluation of the weak coupling approximation Eq. (21).

$$F_0(t) = \det(1 + K(t))^{-1/2}.$$  

The dashed curve in Fig. 6 shows the average fidelity $F_0(t)$ of a linear qubit register with $n = 125$ qubits at a decoherence rate $\gamma = 10^{-4}/a$. The time domain $130a < t < 1000a$ is chosen such that Eq. (26) and the weak coupling approximation is applicable. For comparison, Fig. 7 also shows the average register fidelities of independent and symmetrically coupled qubits with a time dependent parameter $\kappa = \gamma t$.

$$F_1(t) = \left( 1 + e^{-\gamma t} \right)^n,$$

$$F_s(t) = \frac{1}{\sqrt{1 + n \gamma t}} \quad (27)$$

(cf. Eqs. (23) and (25)).

As expected, the average register fidelity $F_0(t)$ lies between $F_1(t)$ and $F_s(t)$. It might be more surprising that for large times $t \gg L_0$ the fidelity $F_0(t)$ is much closer to the fidelity $F_s(t)$ of a register of symmetrically coupled
shows the fidelity of encoded and independent qubits which we analyzed in Sec. III C 2. Hence, by Eq. (11), the rate 
\[ \gamma = \alpha \pi T, \]

where \( K_0(t) \) and \( K_1(t) \) are decoherence functions for plain and encoded qubits, respectively. We therefore expect encoding to be advantageous when \( K_0(t) > K_1(t) \).

To resolve the time dependency of the average fidelities at shorter times we may use approximation Eq. (22), valid for small deviations \( 1 - F \ll 1 \), according to which

\[ F_0(t) = 1 - \frac{n}{2} K_0(t), \quad F_1(t) = 1 - \frac{n}{2} K_1(t), \]

where \( K_0(t) \) and \( K_1(t) \) are decoherence functions for plain and encoded qubits, respectively.

The saturation of the effective decoherence functions \( K_0(t) \) at times \( t > a \) entails the saturation of the averaged register fidelity \( F_1(t) \) to the asymptotic value

\[ F_1(\infty). \]

In contrast to that, the fidelity \( F_0(t) \) of the plain quantum register keeps decaying with increasing time. As a trivial consequence, the average fidelity of the encoded register will always exceed the fidelity of a plain register if time \( t \) becomes sufficiently large.

To begin with, we consider the average fidelity \( F_1(t) \) of an encoded register \( R_n^1 \) at large temperatures \( T \gg 1/a \). For times \( t > L_1 \) we find from relation Eq. (17) with Eqs. (8), and (11) a decoherence matrix

\[ K_{ll}(t) = K_1^0(t) = \gamma a, \quad \gamma = \alpha \pi T, \]

\[ K_{l \neq m}(t) = 0, \]

where again we omitted a logarithmic term \( \alpha \ln(\Omega/2\pi T) \ll \gamma a \) in the diagonal elements. Note that the time dependence has dropped out. This is precisely the decoherence matrix of a register with independent qubits which we analyzed in Sec. III C 1. Hence, by Eq. (20), for times \( t > L_1 \) the average fidelity becomes a constant

\[ F_1(\infty) = F_1 = \left(1 + e^{-\gamma a}/2\right)^n. \tag{28} \]

For instance, for \( \gamma = 10^{-4}/a \) and \( n = 125 \), which are the parameters of the fidelity \( F_0(t) \) plotted in Fig. (7), we obtain an \( F_0(t) \) largely exceeding asymptotic value

\[ F_1(\infty) \simeq 1 - 0.0062. \]
IV. DISSIPATIVE COUPLINGS

The preceding sections have shown that the decoherence of an encoded qubit and also the fidelity of an encoded register (of 1st order) is essentially determined by the effective decoherence function

\[ K^1_0(t) = 2(K(0, t) - K(a, t)). \]

In accordance with previous work\textsuperscript{11,12} we observed that for the dissipation-less model the decoherence coefficient \( K^1_0(t) \) saturates to a finite constant value \( K^1_0(\infty) \) in the long time limit. It is important to find out whether this saturation holds for general physical interactions, or merely is a feature of the dissipation-less interaction.

Here we cannot address this question in full generality. However, as a first step we will analyze a two-spin system that is dissipatively coupled to a bosonic environment. Lacking an exact analytical solution, we will treat this system by a quantum master-equation of Bloch-Redfield type (Sec. IV A). Its viability in the present context is demonstrated for the dissipationless case, where the exact results can be reproduced (Sec. IV B). In Sec. IV C we will then consider the decay of a state that would remain invariant under the spin-boson interaction in the limit \( a \to 0 \). Contrary to what is observed in the dissipationless case, here we find that for any finite \( a \) the state continues to decay with a constant rate for large times. The asymptotic rate appears to be the given by the decay rate of a single spin multiplied with a universal factor \( 2 - 2 \sin(\varepsilon a)/(\varepsilon a) \).

A. Bloch-Redfield master equation

We consider the general situation of a system \( S \) that is coupled to a bath \( B \) via an interaction \( H_I \). The Liouville-von-Neumann equation for the density operator \( \rho_{SB} \) of the total system in the interaction picture is

\[ \dot{\rho}_{SB}(t) = -i[H_I(t), \rho_{SB}(t)]. \]

It follows that the reduced density operator \( \rho \) (interaction picture) of the system obeys an equation of motion\textsuperscript{12}

\[ \dot{\rho}(t) = -\int_0^t \text{d}s \text{tr}_B[H_I(t), [H_I(s), \rho_{SB}(s)]] \]

\[ -i\text{tr}_B[H_I(t), \rho_{SB}(0)]. \]  

We assume that initially \( \rho_{SB}(0) \) is a product of an initial \( \rho(0) \) and a thermal bath state \( \rho_B \), and further take for granted that

\[ \text{tr}_B[H_I(t), \rho(0) \otimes \rho_B] = 0, \]  

which in many cases is satisfied, in particular in those to be analyzed below. The remaining term on the r.h.s. depends on the total state \( \rho_{SB}(s) \) at times \( 0 \leq s \leq t \). In general, this does no allow to exactly determine \( \rho(t) \) from Eq. (29). One therefore frequently invokes the Born-Markov approximation by substituting

\[ \rho_{SB}(s) \rightarrow \rho(s) \otimes \rho_B \rightarrow \rho(t) \otimes \rho_B, \]

Obviously, this approximation is good in the limit of weak couplings. This results in the Bloch-Redfield master equation

\[ \dot{\rho}(t) = \mathcal{R}_t(\rho(t)), \]

where \( \mathcal{R}_t \) denotes the Redfield super-operator defined by

\[ \rho \rightarrow \mathcal{R}_t(\rho) = -\int_0^t \text{d}s \text{tr}_B[H_I(t), [H_I(s), \rho \otimes \rho_B]]. \]  

Note that the Redfield operator is explicitly time-dependent and therefore the resulting dynamics does not exhibit a semigroup structure. In this sense, the Bloch-Redfield equation Eq. (33) is non-Markovian, notwithstanding the fact that the Born-Markov approximation has been used to derive it. In many cases it is justified to eliminate this “deficiency” by simply extending the domain of integration in Eq. (33) from \([0, t]\) to \([−∞, t]\) (cf. Ref. \textsuperscript{12}). However, as it has been stressed by Doll et al.\textsuperscript{17} when dealing with a spatially extended quantum
object this procedure would lead to noncausal behavior and thus to spurious results.

In Ref. [4], this problem has been circumvented by using a *causal* master equation in which causality is explicitly taken care of by step functions in the time domain that truncate acausal contributions. The resulting dynamics has been shown to approximate quite well the known exact solution. Here, we will simply stay with the non-Markovian Bloch-Redfield equation as given by Eqs. (35) and (36).

### B. Dissipation-less two-spin system

First, in order to demonstrate its viability, we use the Bloch-Redfield master equation to reanalyze the dissipation-less model of Section II A for $n = 2$. The spin-boson Hamiltonian in the interaction picture is

$$ H_I(t) = \sum_{l=0,1} Z_l \otimes B(r_l, t), \quad (35) $$

where $Z_l$ is the (time-independent) Pauli-$z$-operator on the $l$-th spin, and $B(r_l, t)$ as in Eq. (11). Condition (31) is satisfied and the Redfield operator determines to be

$$ \mathcal{R}_I(\rho) = \sum_{m,l=0,1} C(|r_l - r_m|, t) (Z_m \rho Z_l - Z_l Z_m \rho) + \text{h.c.}, $$

where

$$ C(|r|, t) = \int_0^t dt' \langle B(r, t') B(0, 0) \rangle_T. $$

Presenting $\rho(t)$ in the computational basis $|\mu\rangle$,

$$ \rho(t) = \sum_{\mu, \nu} \rho_{\mu\nu}(t) |\mu\rangle \langle \nu|, $$

and again omitting imaginary parts, which would contribute only to the unitary pure $U$ of the time evolution, the Bloch-Redfield master equation (33) predicts the coefficients $\rho_{\mu\nu}(t)$ to obey independent differential equations

$$ \dot{\rho}_{\mu\nu}(t) = \sum_{m,l=0}^1 (\mu_l - \nu_l)(\mu_m - \nu_m) 4 \text{Re} C(|\mu\rangle \langle \mu|, t) \rho_{\mu\nu}(t). $$

After integration we observe that the master master-equation reproduces the exact result Eqs. (2) and (3) obtained in Sec. III (for $n = 2$). This is more than one could have expected and we believe that the exactness must be ascribed to the fact that the present model is lacking dissipative couplings. We do not expect that the Bloch-Redfield theory exactly describes the dynamics in the dissipative model. Nevertheless, the positive outcomes for the present dissipation-less model still encourages us to use the Bloch-Redfield master equation also for the dissipative model that we are investigating next.

### C. Dissipative two-spin system

Let now the two spins interact with the bosonic bath via the dissipative Hamiltonian (interaction picture)

$$ H_I(t) = \sum_{l=0,1} X_l(t) \otimes B(r_l, t), \quad (36) $$

where $B(r, t)$ is as in Eq. (11). $X_l(t)$ can be conveniently written with operators

$$ u_l \equiv \sigma_{\pm}^{(l)} = X_l + i Y_l, $$

$$ d_l \equiv \sigma_{\pm}^{(l)} = X_l - i Y_l $$

as

$$ X_l(t) = e^{-i\epsilon t} d_l + e^{+i\epsilon t} u_l. $$

#### 1. Master equation

A straightforward calculation shows that in rotating wave approximation — which is valid as $\epsilon t \gg 1$ — the corresponding Redfield operator Eq. (34) is given by

$$ \mathcal{R}_I(\rho) = \sum_{m,l} [C_-(|r_l - r_m|, t) \{ u_m \rho d_l - d_l u_m \rho \} $$

$$ + C_+(|r_l - r_m|, t) \{ d_m \rho u_l - u_l d_m \rho \} $$

$$ + \text{h.c.}], \quad (37) $$

with energy dependent correlation functions

$$ C_{\pm}(|r|, t) = \int_0^t ds e^{\pm i\epsilon s} \langle B(r, s) B(0, 0) \rangle. $$

The real part $C_r'(r, t)$ of $C_r(r, t)$ has a surprisingly simple structure in the limit of large $T t \gg 1$. Here we find the zero-distance correlations to be given by the familiar expressions

$$ C_r'(0, t) = \alpha \pi n(\epsilon) $$

$$ C_{\pm}'(0, t) = \alpha \pi (n(\epsilon) + 1), \quad (38) $$

where $n(\epsilon) = (e^{\epsilon/T} - 1)^{-1}$. The finite-distance correlations follow to be ($|r| = a$)

$$ C_{\pm}'(a, t) = \frac{\sin(\epsilon a)}{\epsilon a} \theta(t - a) C_{\pm}'(0, t). \quad (39) $$

We notice that the finite-distance correlation deviates from the zero-distance correlation only by a temperature-independent factor that, of course, approaches unity for $a \to 0$, but also for $\epsilon \to 0$ (when $a < t$), what is somewhat unexpected.

Anticipating the discussion given below, we remark that the deviation of the first factor $\frac{\sin(\epsilon a)}{\epsilon a}$ from unity for any finite $\epsilon t$ reflects the decay of coherence in a symmetric subspace in the long time limit. Note that for
$\varepsilon \to 0$, which precisely corresponds to the transition to a dissipation-less model, the first factor remains unity also for finite $a$. Clearly, this corresponds to the saturation of the subspace fidelity observed in the previous section. The second factor, $\theta(t - a)$, which is responsible for causality, here as well as in the previous case leads to an initial drop of the subspace fidelity until the time $t = a$ is reached.

2. Asymptotic decay of subspace fidelity

The anti-symmetric state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

is annihilated by $Z_0 + Z_1$ and $X_0 + X_1$, and therefore remains invariant under both, the dissipationless interaction Eq. (35) and the dissipative interaction Eq. (36), provided that $a = 0$. We are interested in the decay of the state $|\psi_0\rangle$ under the dissipative spin-boson interaction Eq. (36) for finite distance $a$. To this end we consider the fidelity

$$F(t) := \langle \psi_0 | \rho(t) | \psi_0 \rangle,$$

where $\rho(t)$ is the reduced spin state at time $t$ (in interaction picture) that evolved via the interaction Eq. (36) with the bath from the initial state $\rho(0) = |\psi_0\rangle \langle \psi_0|$. Here we will approximately determine $\rho(t)$ by the Bloch-Redfield master equation with the dissipative Redfield operator Eq. (37).

To obtain a first impression of the dynamics we integrated the master equation numerically. Characteristic outcomes for the fidelity $F(t)$ are shown in Fig. 9. We chose energy $\varepsilon = 5\Gamma$ and a small overall coupling constant $a = 0.01$. Similar to the previously observed behavior, also here we see a relatively strong decay of the fidelity at times $t < a$. However, in contrast to the dissipation-less model, here we clearly see a decay of $F_0(t)$ for large times with a rate that increases with distances ranging from $aT = 0.1$ to 0.3.

At times $t \gg a$ and for sufficiently small couplings the decay rate $\gamma_1 = dF/dt$ of the fidelity $F(t)$ is in good approximation determined by the expression

$$\gamma_1 = -\langle \psi_0 | R_x (|\psi_0\rangle \langle \psi_0|) | \psi_0 \rangle.$$

Using expressions Eqs. (38) and (39) we obtain after some algebra

$$\gamma_1 = 2 \left(1 - \frac{\sin (\varepsilon a)}{\varepsilon a}\right) \gamma_0,$$

where $\gamma_0 = 2\pi a (n(\varepsilon) + 1/2)$ is the decay rate of a single spin. This is a simple and quite general result that -- in the light of the discussions of the preceding sections -- quantifies the benefits of using a symmetric subspaces in a dissipative system. It identifies

$$p = \varepsilon a \equiv \frac{\varepsilon a}{\hbar c}$$

as the relevant parameter that captures the achievable reduction of the decay rates of encoded qubits in comparison to plain physical qubits. Fig. 10 shows $\gamma_1/\gamma_0$ as a function of $p$.

A significant reduction requires $p \ll 1$, corresponding to distances

$$a \ll \frac{\hbar}{\varepsilon}.$$

For instance, for atomic qubits this implies that the distance $a$ between atoms should be small compared to the
wavelength of the light that is emitted in a bit-flip transition.

V. SUMMARY OF RESULTS

In Secs. and [we studied a quantum register physically realized as a linear spin-chain that interacts with a three-dimensional bosonic bath via a dissipation-less spin-boson coupling with an ohmic coupling spectral density. Within this framework we analyzed the benefits of using encoded qubits and encoded quantum registers in order to reduce effects of decoherence.

In agreement with previous work, we found that the coherence of a 1st order encoded qubit converges to a finite asymptotic value at times \( a/c \), where \( a \) is the inter-spin distance and \( c \) is the velocity of sound or light. As a consequence, the coherence of the encoded qubit exceeds the one of a plain qubit for times larger than a crossover time \( t_c \). At high temperatures \( k_BT \gg hc/a \), the crossover time is \( t_c \approx a/c \), whereas at low temperatures \( k_BT \ll hc/a \) it increases roughly as \( t_c \approx h/(k_BT) \) with decreasing temperature \( T \). Moreover, we observe the 1st order encoded qubits to be effectively decoupled from each other, meaning that higher-order encoding becomes counterproductive. On the other hand, this decoupling of encoded qubits should be advantageous for quantum error correction, which is known to be significantly hampered when the qubits are coupled via the bosonic bath. This aspect of using encoded qubits may deserve further investigation.

In Sec. [we derived a convenient integral representation for the averaged register fidelity with respect to the noise of the dissipation-less spin-boson model. This allowed us to investigate the decoherence of an entire plain or encoded quantum register. Small deviations of the averaged register fidelity from unity are proportional to the number of (encoded) qubits and to the (effective) decoherence function \( K_0(t) \) of a single (encoded) qubit. The improved performance of 1st order encoded qubits therefore carries over to an entire 1st order encoded quantum register. This is confirmed by more detailed analytical and numerical results presented in Sec. [.

Finally, in Sec. [we addressed the role of dissipation within a two-spin model. Its dynamics was determined by employing a master equation of Bloch-Redfield type. In the presence of dissipative spin-boson couplings the coherence of an encoded qubit seems no longer to converge to a finite value. Instead, here we expect an exponential decay with an, however, reduced asymptotic effective decay rate \( \gamma_1 = 2(1 - \sin(p)/p)\gamma_0 \), where the dimensionless parameter \( p \) is determined by the energy splitting \( \epsilon \) of the spins and the inter-spin distance \( a, p = \epsilon a/(hc) \).

This work is supported by DFG grant No. KL2159.

APPENDIX A: DECOHERENCE OF AN ENCODED QUANTUM REGISTER

To begin with, let us consider a 1st order encoded register \( R_1 \). Its Hilbert space \( H_1^n \) is spanned by \( 2^n \) state vectors
\[
|\mu\rangle^1 = |\mu_0\rangle_0 \cdots |\mu_{n-1}\rangle_{n-1} \in H_{2n}^1, \quad \mu \in Z_2^n.
\]
Each \( \mu \in Z_2^n \) corresponds one-to-one to a \( \mu' \in Z_2^n \) by demanding \( |\mu\rangle^1 = |\mu'\rangle \), which by definition (A1) means
\[
\mu_{2i} = \mu_i, \quad \mu_{2i+1} = 1 - \mu_i.
\]
Let \( \rho_1 \) be a state of the encoded register, i.e. \( \text{supp} \rho_1 \subset H_1^n \). By Eq. (1) we find
\[
N(\rho_1) = \sum_{\mu \nu \in Z_2^n} e^{D_{\mu'\nu'}}|\mu\rangle\rho_1|\nu\rangle\langle\mu|,
\]
where \( \nu' \) relates to \( \nu \) as \( \nu' \) to \( \mu \), and, by Eq. (2),
\[
D_{\mu'\nu'} = \sum_{l,m=0}^{2n-1} (\mu'_l - \nu'_l)(\mu_m - \nu_m)K(|l-m|a,t).
\]
Making use of relation and rearranging the sum we can rewrite \( D_{\mu'\nu'} \) in terms of \( \mu \) and \( \nu \) as an effective decoherence coefficient
\[
D_{\mu\nu}^1 = \sum_{l,m=0}^{n-1} (\mu_l - \nu_l)(\mu_m - \nu_m)K_1^l(t),
\]
when effective decoherence functions \( K_1^l(t) \) are defined as
\[
K_1^l(t) = 2K_2^0(t) - K_2^{2l-1}(t) - K_2^{2l+1}(t),
\]
with
\[
K_2^0(t) = K(|a,t|).
\]
It is straightforward to generalize this analysis to higher orders, which eventually leads us to relations and .

1 R. P. Feynman, International Journal of Theoretical Physics 21,467 (1982).
2 D. Deutsch, Proc. Roy. Soc., London A400, 97 (1985).
3 P. W. Shor, SIAM J. Sci. Statist. Comput. 26, 1484 (1997), arXiv:quant-ph/9508027v2.
4 D. Giulini et al., Decoherence and the Appearance of a Classical World in Quantum Theory (Springer-Verlag, Berlin, 1996).
5 W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003), arXiv:quant-ph/0105127v3.
6 W. G. Unruh, Phys. Rev. A 51, 992 (1994), arXiv:hep-th/9406058v1.
I. Chuang, R. Laflamme, P. Shor, and W. Zurek, Science 270, 1633 (1995), arXiv:quant-ph/9503007v1
8 P. W. Shor, Phys. Rev. A 52, R2493 (1995).
9 A. R. Calderbank and P. W. Shor, Phys. Rev. A 54, 1098 (1996), arXiv:quant-ph/9512032v2
10 A. M. Steane, Phys. Rev. Lett. 77, 793 (1996).
11 G. Palma, K.-A. Suominen, and A. Ekert, Proc. Roy. Soc., London A 452, 567 (1997), arXiv:quant-ph/9702001v1
12 P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997), arXiv:quant-ph/9705044v2
13 D. Lidar, I. Chuang, and K. Whaley, Phys. Rev. Lett. 81, 2594 (1998), arXiv:quant-ph/9807004v2
14 L.-M. Duan and G.-C. Guo, Phys. Rev. A 57, 737 (1998), arXiv:quant-ph/9811058v1
15 J. H. Reina, L. Quiroga, and N. F. Johnson, Phys. Rev. A 65, 032326 (2002), arXiv:quant-ph/0105029v2
16 R. Klesse and S. Frank, Phys. Rev. Lett. 95, 230503 (2005), arXiv:quant-ph/0505153v3
17 R. Doll, M. Wubs, P. Hänggi, and S. Kohler, Phys. Rev. B 76, 045317 (2007), arXiv:cond-mat/0703075v3
18 U. Weiss, Quantum Dissipative Systems (World Scientific, Singapore, 1999).
19 H.-P. Breuer and F. Petruccione, The theory of open quantum systems (University Press, Oxford, 2002).
20 M. Nielsen and I. Chuang, Quantum Computation and Quantum Information (University Press, Cambridge, 2000).
21 D. Bacon, D. A. Lidar, and K. B. Whaley, Phys. Rev. A 60, 1944 (1999), arXiv:quant-ph/9902041v2
22 P. Pereyra and P. Mello, Journal of Physics A 16, 237 (1982).