INTEGRAL OPERATORS ARISING FROM THE RIEMANN ZETA FUNCTION

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Abstract. In this paper we have two issues coming from the same background. The first one is to describe a certain ratio of Fredholm determinants of integral operators arising from the Riemann zeta function by using the solution of a single integral equation. The second one is to introduce a new integral operator arising from the Riemann zeta function and to study its basic analytic properties.

Dedicated to Professor Kohji Matsumoto at the occasion of his 60th Birthday

1. Introduction

The Riemann xi-function \( \xi(s) = 2^{-s} \pi^{-s/2} \Gamma(s/2) \zeta(s) \) is an entire function satisfying the functional equation \( \xi(s) = \xi(1-s) \), where \( \zeta(s) \) is the Riemann zeta-function. In terms of \( \xi(s) \), the Riemann hypothesis (RH, for short) asserts that all zeros of \( \xi(1/2 - iz) \) are real. In [6], Lagarias pointed out a relationship between the RH and the theory of de Branges spaces which are reproducing kernel Hilbert spaces consisting of entire functions. The structure of subspaces of a given de Branges space is determined by a \( \text{Sym}_2(\mathbb{R}) \)-valued function \( H(t) \) on an interval which is called a Hamiltonian. Put \( E_\xi(z) = \xi(1/2 - iz) + \xi'(1/2 - iz) \), and suppose the RH and the simplicity of all zeros. Then \( E_\xi \) generates the de Branges space \( B(E_\xi) \). As suggested in [6], the problem to determine an explicit form of the Hamiltonian of \( B(E_\xi) \) is interesting and important for the study of the RH. However, it is as difficult as the case of general de Branges spaces. Therefore, we studied the family

\[
E^{\omega,\nu}_\xi(z) = \xi\left(\frac{1}{2} + \omega - iz\right)^\nu, \quad (\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0}
\]

instead of \( E_\xi \) in [8]. Under the RH, each \( E^{\omega,\nu}_\xi \) generates the de Branges space \( B(E^{\omega,\nu}_\xi) \), where the simplicity of zeros is unnecessary. An advantage of \( E^{\omega,\nu}_\xi \) is that the Hamiltonian of \( B(E^{\omega,\nu}_\xi) \) can be constructed explicitly as follows for each \( (\omega, \nu) \in \mathbb{R}_{>0} \times \mathbb{Z}_{>0} \) satisfying \( \omega \nu > 1 \).

For \( t \geq 0 \), we define the operator \( K[t](=K^{\omega,\nu}_\xi[t]) \) on \( L^2(-\infty, t) \) by

\[
(K[t]f)(x) = 1_{(-\infty, t]}(x) \int_{-\infty}^{t} K(x+y)f(y)\, dy,
\]

where \( 1_A(x) \) is the characteristic function of a subset \( A \subset \mathbb{R} \), and the integral kernel \( K(x):=K^{\omega,\nu}_\xi(x) \) is defined by its Fourier integral as follows

\[
\left(\frac{\xi(s-\omega)}{\xi(s+\omega)}\right)^\nu = \int_{-\infty}^{\infty} K^{\omega,\nu}_\xi(x)e^{ixz}\, dx, \quad s = \frac{1}{2} - iz
\]

for \( x \geq 0 \) and \( K(x) = 0 \) for \( x < 0 \). The RH implies that \( \det(1 \pm K[t]) \neq 0 \) for every \( t \geq 0 \) ([8] Prop. 4.4]). Thus,

\[
m(t) := \frac{\det(1 + K[t])}{\det(1 - K[t])}
\]

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defines a function of $t \in [0, \infty)$, where “\text{det}” stands for the Fredholm determinant. Then we proved that the diagonal matrix-valued function $H(t) := \text{diag}(m(t)^{-1/2}, m(t)^{1/2})$ is the Hamiltonian of the de Branges space $\mathcal{B}(E_\xi^{\omega, \nu})$ ([S] Thms 2.2 and 2.3)). The function $m(t)$ satisfies the formula

$$m(t) = \exp \left( \int_{-\infty}^{t} (\phi^+(\tau, \tau) + \phi^-(\tau, \tau)) \, d\tau \right),$$

where $\phi^\pm(t, x)$ consists of the unique solutions of the integral equations

$$\phi^\pm(t, x) \pm \int_{-\infty}^{t} K(x + y) \phi^\pm(t, y) \, dy = K(x + t)$$

on $L^2(-\infty, t)$ ([S] (3.32), (3.35)]. The solutions $\phi^\pm(t, x)$ are also important because their extensions to functions of $x$ on $\mathbb{R}$ describe the solution of the canonical system attached to $H(t)$.

In this paper, we consider the integral equations

$$\Phi(t, x) + \int_{-\infty}^{t} K(x + y) \Phi(t, y) \, dy = 1_{[-t, t]}(x)$$

and

$$\Psi(t, x) - \int_{-\infty}^{t} K(x + y) \Psi(t, y) \, dy = 1_{[-t, t]}(x)$$

instead of the pair of integral equations (1.4). Then, $m(t)$ and $\phi^\pm(t, x)$ are obtained from the solution of one of these single equation as follows.

**Theorem 1.1.** Let $\tau > 0$ be a real number such that $\text{det}(1 \pm K[t]) \neq 0$ for every $0 \leq t < \tau$ for the operator $K$ (resp. $\Psi (m)$ Props. 4.2 and 4.4]. Then the unique solution $\Phi(t, x)$ (resp. $\Psi(t, x)$) of the integral equation (1.5) (resp. (1.6)) is a continuous function of $x$, $\Phi(t, x) := \lim_{x \to -t^+} \Phi(t, x) \neq 0$ (resp. $\Psi(t, x) := \lim_{x \to t^-} \Psi(t, x) \neq 0$) for every $0 < t < \tau$, and the following formulas hold for $0 < t < \tau$:

$$\phi^+(t, x) = -\frac{1}{\Phi(t, t)} \frac{\partial}{\partial t} \Phi(t, x) = \frac{1}{\Psi(t, t)} \frac{\partial}{\partial x} \Psi(t, x),$$

$$\phi^-(t, x) = -\frac{1}{\Phi(t, t)} \frac{\partial}{\partial t} \Phi(t, x) = \frac{1}{\Psi(t, t)} \frac{\partial}{\partial x} \Psi(t, x),$$

$$m(t) = \frac{\partial}{\Phi(t, t)} \Phi(t, t) = \frac{\partial}{\Psi(t, t)} \Psi(t, t).$$

As for assumptions of Theorem 1.1 it is proved unconditionally that there exists $\tau > 0$ such that $\text{det}(1 \pm K[t]) \neq 0$ for every $0 \leq t < \tau$, and $\tau = \infty$ under the RH ([S] Props. 4.2 and 4.4]).

Now, we change the issue to the second one. As proved in [S] Thm. 2.4, an equivalent condition for the RH is stated by using the family of operators $K^{\omega, \nu}_\xi[t]$. However, it would better, if it could be improved finding an equivalent condition using only one single operator avoiding parameters $\omega$ and $\nu$. As an attempt to achieve such a result, we consider the following matters. If $\nu = \theta/\omega$, the left-hand side of (1.2] has the limit

$$\lim_{\omega \to 0} \left( \frac{\xi(s - \omega)}{\xi(s + \omega)} \right)^{\theta/\omega} = \exp \left( -2\theta \frac{\xi'}{\xi}(s) \right).$$

Then we expect that the kernel $K_\theta(x)$ of the Fourier integral formula

$$\exp \left( -2\theta \frac{\xi'}{\xi}(s) \right) = \int_{-\infty}^{\infty} K_\theta(x) e^{izx} \, dx, \quad s = \frac{1}{2} - iz$$

plays a role similar to $K^{\omega, \nu}_\xi(x)$ of (1.2]. In fact, $K_\theta(x)$ satisfies the following properties corresponding to the latter four of five properties (K1)~(K5) required for $K^{\omega, \nu}_\xi(x)$ in [S].
Theorem 1.2. If \( \theta > 1 \), \( K_\theta(x) \) of (1.10) has the following properties:

(K-ii) \( K_\theta(x) \) is a real-valued continuous function on \( \mathbb{R} \) such that \( K_\theta(x) \ll \exp(x/2) \) as \( x \to +\infty \) and (1.10) holds for \( \Im(z) > 1/2 \),

(K-iii) \( K_\theta(x) = 0 \) for \( x < 0 \),

(K-iv) \( K_\theta(x) \) is continuously differentiable on \( \mathbb{R} \setminus \{ \log n \mid n \in \mathbb{N} \} \) and \( |K_\theta'(x)| \) is locally integrable on \( \mathbb{R} \),

(K-v) there exists \( 0 < \tau \leq \infty \) such that \( \det(1 \pm K_\theta[t]) \neq 0 \) for \( 0 \leq t < \tau \), where \( K_\theta[t] \) is the operator defined by (1.1) for \( K_\theta(x) \).

We make some more comments on the function \( K_\theta(x) \). We have

\[
\exp \left( -2\theta \frac{\zeta'(s)}{\zeta(s)} \right) = \exp \left( -2\theta \frac{\gamma'(s)}{\gamma(s)} \right) \left( -2\theta \frac{\zeta'(s)}{\zeta(s)} \right)
\]

if we put \( \gamma(s) = 2^{-1}s(s-1)\pi^{-s/2}\Gamma(s/2) \). The right factor on the right-hand side has the Dirichlet series expansion

\[
\exp \left( -2\theta \frac{\zeta'(s)}{\zeta(s)} \right) = \sum_{n=1}^{\infty} \frac{\lambda_\theta(n)}{n^s} \tag{1.11}
\]

endowed with multiplicative coefficients \( \lambda_\theta(n) \) for \( \Re(s) > 1 \). Thus, the Fourier integral formula

\[
\exp \left( -2\theta \frac{\gamma'(s)}{\gamma(s)} \right) = \int_{-\infty}^{\infty} g_\theta(x) e^{izx}dx, \quad s = \frac{1}{2} - iz, \tag{1.12}
\]

shown later, implies the series representation

\[
K_\theta(x) = \sum_{n=1}^{\infty} \frac{\lambda_\theta(n)}{\sqrt{n}} g_\theta(x - \log n) \tag{1.13}
\]

consisting of the “non-archimedean” or “arithmetic” part \( \lambda_\theta \) and the “archimedean” part \( g_\theta \). The Dirichlet series (1.11) and its coefficients \( \lambda_\theta(n) \) are studied in detail by Ihara [2] and Ihara–Matsumoto [3, 4] to investigate the value distribution of \( \zeta'/\zeta \). Therefore, to prove Theorem 1.2, we mainly study \( g_\theta(x) \) in §3.

As shown in §3, \( g_\theta(x) = 0 \) for \( x < 0 \), thus the sum on the right-hand side of (1.13) is finite for a bounded range \( 0 \leq x \leq x_0 \). Further, we show in §4 that \( g_\theta(x) \) is well approximated by series consisting of Bessel functions in such a range. Combining such approximations of \( g_\theta(x) \) with (1.13), we can calculate \( m(t) \), \( \Phi(t, x) \) and \( \phi^t(t, x) \) easily at least in a computational sense (cf. [1]).

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2. Proof of Theorem (1.1)

Proofs of (1.7) and (1.8) Formulas in (1.7) and (1.8) are proved by a way similar to Krein [5]. First we prove \( \Phi(t, t) \neq 0 \) for \( 0 < t < \tau \). The function \( \Phi(t, x) \) is a continuous function of \( x \) on \([-t, t]\) by the continuity of \( K(x) \) and \( 1_{[-t,t]}(x) \). Differentiating (1.5) with respect to \( x \), we have

\[
\frac{\partial}{\partial x} \Phi(t, x) + \int_{-\infty}^{t} K'(x + y)\Phi(t, y) dy = 0 \tag{2.1}
\]

for \(-t < x < t\). This shows that \( (\partial/\partial x)\Phi(t, x) \) is a continuous function of \( x \) on \([-t, t]\), since \( |K'(x)| \) is locally integrable [5, Prop. 4.1]. Thus, \( (\partial/\partial x)\Phi(t, x) \in L^2(-t, t) \). Applying integration by parts to (2.1),

\[
\frac{\partial}{\partial x} \Phi(t, x) + K(x + t)\Phi(t, t) - \int_{-\infty}^{t} K(x + y)\frac{\partial}{\partial y} \Phi(t, y) dy = 0. \tag{2.2}
\]
Therefore, if we suppose that $\Phi(t, t) = 0$,
\[
\frac{\partial}{\partial x} \Phi(t, x) - \int_0^t K(x + y) \frac{\partial}{\partial y} \Phi(t, y) \, dy = 0.
\]
This asserts that $(\partial/\partial x) \Phi(t, x)$ is a solution of the homogeneous equation $(1 - K[t])f = 0$ on $L^2(-t, t)$, and thus $(\partial/\partial x) \Phi(t, x) = 0$. If $\Phi(t, x) = c$ for $-t < x < t$, $c(1 + \int_0^{x-t} K(y) \, dy) = 1$. Hence, $K(x) = 0$ on $[0, 2t]$ which implies that $\Phi(t, x) = 1_{[-t, t]}(x)$ for $-t < x < t$ by (1.5). Therefore, $\Phi(t, t) = 1$ by the continuity of $\Phi(t, x)$ for $x$. This is a contradiction. Similar arguments also prove $\Psi(t, t) \neq 0$ for $0 < t < \tau$.

Equation (2.2) implies that $-(\partial/\partial x) \Phi(t, x)/\Phi(t, t)$ solves (1.4) for the minus sign. Hence the uniqueness of the solution concludes the first equality of (1.8). The second equality of (1.7) is also proved by the same way. On the other hand, by differentiating (1.5) with respect to $t$,
\[
\frac{\partial}{\partial t} \Phi(t, x) + K(t + x) \Phi(t, t) + \int_{-\infty}^t K(t + y) \frac{\partial}{\partial t} \Phi(t, y) \, dy = 0.
\]
This shows that $-(\partial/\partial t) \Phi(t, x)/\Phi(t, t)$ solves (1.4) for the plus sign. Hence the uniqueness of the solution concludes the first equality of (1.7). The second equality of (1.8) is also proved by the same way.

**Proof of (1.9)** Taking $x = t$ in equation (1.5) and then differentiating it with respect to $t$,
\[
0 = \frac{d}{dt} (\Phi(t, t)) + 2K(2t)\Phi(t, t)
- \int_{-\infty}^t K(t + y) \frac{\partial}{\partial y} \Phi(t, y) \, dy + \int_{-\infty}^t K(t + y) \frac{\partial}{\partial t} \Phi(t, y) \, dy.
\]
Using (1.7) on the right-hand side,
\[
\frac{d}{dt} (\Phi(t, t)) + 2K(2t)\Phi(t, t)
- \Phi(t, t) \int_{-\infty}^t K(t + x)(\phi^+(t, x) - \phi^-(t, x)) \, dx = 0.
\]
(2.3)

On the other hand, by the proof of [8, Thm. 6.1], we have
\[
\frac{1}{2} (\phi^+(t, x) + \phi^-(t, x))
= K(x + t) - \int_{-\infty}^t K(x + y) \frac{1}{2} (\phi^+(t, y) - \phi^-(t, y)) \, dy.
\]
Substituting this into (2.3) after taking $x = t$, we get
\[
\frac{d}{dt} (\Phi(t, t)) + \Phi(t, t)(\phi^+(t, t) + \phi^-(t, t)) = 0.
\]
(2.4)

Therefore, $\Phi(t, t) = C \exp \left( - \int_0^t (\phi^+(\tau, \tau) + \phi^-(\tau, \tau)) \, d\tau \right) = Cm(t)^{-1}$ by (1.3). To determine $C$, we take $x = t = 0$ in equation (1.5). Then $\Phi(0, 0) = 1$, since the integral on the left-hand side is zero because $K(x) = 0$ for $x < 0$, and thus $C = 1$ by $m(0) = 1$ (since $K[0]$ is the zero operator). Hence we obtain the first equality of (1.9). The second equality of (1.9) is proved by the same way. □
3. Proof of Theorem 1.2

Let \( \psi(s) = \Gamma'(s)/\Gamma(s) \) be the digamma function. To study \( g_\theta(x) \) from (1.12), we need the following result.

**Proposition 3.1.** For \( \theta > 0 \), we define

\[
\Psi_\theta(x) := \frac{1}{2\pi i} \lim_{U \to \infty} \int_{c-iU}^{c+iU} \left[ s^\theta \cdot \exp \left(-\theta \psi(s) - \frac{\theta}{s} \right) - 1 \right] e^{-izx} dz,
\]

where \( s = 1/2 - iz \). Then the right-hand side converges for \( c > -1/2 \) and the integral is independent of such \( c \). \( \Psi_\theta(x) \) is a real-valued continuous function on \((0, \infty)\), \( \Psi_\theta(x) = 0 \) on \((-\infty, 0)\), \( \Psi_\theta(x) \ll 1 \) as \( x \to 0^+ \), \( \Psi_\theta(x) \ll \exp(-x/2) \) as \( x \to +\infty \), and

\[
s^\theta \cdot \exp \left(-\theta \psi(s) - \frac{\theta}{s} \right) - 1 = \int_0^\infty \Psi_\theta(x)e^{izx} dx, \quad s = \frac{1}{2} - iz
\]

holds for \( \Im z > -1/2 \).

**Proof.** The integrand of (3.1) is holomorphic in \( \Im z \geq -1/2 \) except for \( z = -i/2 \), since \( \psi(s) + s^{-1} = \psi(s + 1) \) is holomorphic in \( \Re(s) > -1 \). The Stirling formula of \( \log \Gamma(s) \) for \( \Im s \leq \pi - \delta \) and \( \Im s \geq \delta \) (9, §12.33, §13.6) derives the asymptotic expansion

\[
\psi(s) = \log s - \frac{1}{2s} - \sum_{n=1}^{N-1} \frac{B_{2n}}{2n s^{2n}} + O(|s|^{-2N})
\]

in the same region. Using this with \( \psi(s) + s^{-1} = \psi(s + 1) \), we have

\[
s^\theta \cdot \exp \left(-\theta \psi(s) - \frac{\theta}{s} \right) - 1
\]

\[
= \exp \left(\theta \log \frac{s}{s+1} + \frac{\theta}{2(s+1)} + \sum_{n=1}^{N-1} \frac{\theta B_{2n}}{2n(s+1)^{2n}} + O(|s|^{-2N})\right) - 1
\]

\[
= \exp \left(-\theta \sum_{n=1}^{\infty} \frac{1}{n(s+1)^n} + \frac{\theta}{2(s+1)} + \sum_{n=1}^{N-1} \frac{\theta B_{2n}}{2n(s+1)^{2n}} + O(|s|^{-2N})\right) - 1
\]

\[
= - \frac{\theta}{2(s+1)} + \frac{\theta(3\theta - 10)}{24(s+1)^2} + \sum_{n=3}^{2N-1} \frac{C_{n-1}(\theta)}{(s+1)^n} + O(|s|^{-2N})
\]

for some polynomials \( C_n(\theta) \in \mathbb{Q}[\theta] \). Hence

\[
\Psi_\theta(x) = \left( -\frac{\theta}{2} u(x) + \sum_{n=2}^{2N-1} \frac{C_{n-1}(\theta)}{(n-1)!} x^{n-1} \right) e^{-3x/2} 1_{(0,\infty)}(x) +
\]

\[
+ \frac{1}{2\pi} \int_{-\infty+ic}^{\infty+ic} R_N(s)e^{-izx} dz
\]

by [7] p. 167, where \( u(x) = 1 \) for \( x > 0 \), \( u(0) = 1/2 \), and

\[
R_N(s) = \left( s^\theta \exp \left(-\theta \psi(s) - \frac{\theta}{s} \right) - 1 \right) - \left( -\frac{\theta}{2(s+1)} + \sum_{n=2}^{2N-1} \frac{C_{n-1}(\theta)}{(s+1)^n} \right)
\]

which is holomorphic in \( \Re(s) \geq 0 \) except for \( s = 0 \) and satisfies \( R_N(s) = O(|s|^{-2N}) \). This estimate enables us to move the path of integration as \( \Im z = c \to +\infty \) if \( N \geq 1 \). Therefore, \( \int_{-\infty+ic}^{\infty+ic} R_N(s)e^{-izx} dz = 0 \) for \( x < 0 \) and the integral is bounded as \( x \to 0^+ \). On the other hand, by moving the path of integration as \( \Im z = c \to -1/2, \int_{-\infty+ic}^{\infty+ic} R_N(s)e^{-izx} dz \ll \exp(-x/2) \) as \( x \to +\infty \). Hence, (3.2) holds for \( \Im z > -1/2 \) by the Fourier inversion formula. Moreover, we find that \( \Psi_\theta(x) \) is real-valued by considering (3.2) for pure-imaginary \( z \). \( \square \)
Proposition 3.2. Let $\theta > 0$. There exists a real-valued continuous function $\Psi_{\theta}$, continuously differentiable on $(0, \infty)$, such that
\[
\exp(-\theta \psi(s)) = \int_0^\infty \Psi_{\theta}(x) e^{ixx} \, dx, \quad s = \frac{1}{2} - iz \tag{3.5}
\]
holds for $\Im z > -1/2$, $\Psi_{\theta}(x) = 0$ for $x < 0$, $\Psi_{\theta}(x) = \Gamma(\theta)^{-1} x^{\theta-1} + O(x^\theta)$ as $x \to 0^+$, and $\Psi_{\theta}(x) \ll \exp(-\kappa x)$ as $x \to \infty$ for any $\kappa < 1/2$.

Proof. Define $\Psi_{0,\alpha}(x) = 0$ for $x < 0$ and
\[
\Psi_{0,\alpha}(x) = e^{-x/2 (x/\alpha)^{(\theta-1)/2}} I_{\theta-1}(2\sqrt{\alpha x}) \tag{3.6}
\]
for $x \geq 0$, where $I_\nu(z)$ is the modified Bessel function of the first kind. Then $\Psi_{0,\alpha}(x)$ is continuously differentiable on $(0, \infty)$ and
\[
\frac{1}{s^\theta} \exp \left( \frac{\alpha}{s} \right) = \int_0^\infty \Psi_{0,\alpha}(x) e^{ixx} \, dx, \quad s = \frac{1}{2} - iz, \tag{3.7}
\]
holds if $\Im z > -1/2$ and $\theta > 0$ by [4, p. 173] (and the changing of variable $\log(1 + \alpha) - \theta$ as $\alpha \to 0$).

The trivial equality
\[
\exp(-\theta \psi(s)) = \frac{1}{s^\theta} \exp \left( \frac{\theta}{s} \right) + \frac{1}{s^\theta} \exp \left( \frac{\theta}{s} \right) \cdot \left[ s^\theta \exp \left( -\theta \psi(s) - \frac{\theta}{s} \right) - 1 \right]
\]
implies that (3.5) holds for
\[
\Psi_{\theta}(x) = \Psi_{0,\theta}(x) + \int_0^x \Psi_{0,\theta}(y) \Psi_{\theta}(x - y) \, dy. \tag{3.8}
\]
In fact, the integral on the right-hand side exists by Proposition 3.1 and
\[
\Psi_{0,\theta}(x) = x^{\theta-1} (\Gamma(\theta)^{-1} + O(x)) \tag{3.9}
\]
which is derived from the series expansion
\[
I_\nu(z) = \sum_{m=0}^\infty \frac{(z/2)^{2m+\nu}}{m! \Gamma(\nu + m + 1)}. \tag{3.10}
\]
Clearly, $\Psi_{\theta}(x)$ is continuous on $(0, \infty)$, vanishes on $(-\infty, 0)$ and is continuously differentiable on $(0, \infty)$. Therefore it remains to show the upper bound for large $x > 0$. By (3.9),
\[
\Psi_{\theta}(x) = \Psi_{0,\theta}(x) + O(x) = x^{\theta-1} (\Gamma(\theta)^{-1} + O(x)) + O(x)
\]
as $x \to 0^+$. The asymptotic formula
\[
I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} (1 + O(|z|^{-1})), \quad |\arg z| < \pi/2, \quad |z| \to \infty
\]
derives
\[
\Psi_{0,\theta}(x) \ll x^{(2\theta-3)/4} \exp \left( -\frac{x}{2} + 2\sqrt{\theta x} \right) \ll \exp(-\kappa x) \tag{3.11}
\]
as $x \to +\infty$ for any $\kappa < 1/2$. By Proposition 3.1, $e^{x/2} \Psi_{0,\theta}(x)$ is uniformly bounded on $[0, \infty)$. Therefore,
\[
\int_0^x \Psi_{0,\theta}(y) \Psi_{\theta}(x - y) \, dy = \int_0^x \Psi_{0,\theta}(y) e^{-(x-y)/2} \cdot e^{(x-y)/2} \Psi_{\theta}(x - y) \, dy
\]
\[
\ll e^{-x/2} \int_0^x e^{y/2} \Psi_{0,\theta}(y) \, dy = e^{-x/2} \int_0^x e^{y/2} \Psi_{0,\theta}(y) \, dy,
\]
since $|I_{\nu}(z)| = I_{\nu}(z)$ for real $z,$ $\theta$ by (3.10). Using (3.10) again,
\[
\int_{0}^{x} e^{y/2} \Psi_{\theta, \alpha}^{0}(y) \, dy = \int_{0}^{x} y^{\theta-1} \sum_{m=0}^{\infty} \frac{1}{m! \Gamma(\theta + m)} (\theta y)^{m} \, dy
\]
\[
= \sum_{m=0}^{\infty} \frac{\theta^{m}}{m! \Gamma(\theta + m + 1)} (\theta y)^{m+\theta} = (x/\theta)^{\theta/2} I_{\theta}(2\sqrt{\theta} x) = e^{x/2} \Psi_{\theta + 1, \theta}^{0}(x).
\]
Hence $\int_{0}^{s} \Psi_{\theta, \alpha}^{0}(y) \Psi_{\theta}^{0}(x - y) \, dy \ll \Psi_{\theta + 1, \theta}^{0}(x)$ as $x \rightarrow \infty$ which implies the estimate $\Psi_{\theta}(x) \ll \exp(-\kappa x)$ by (3.11).

**Proof of Theorem 1.2** (K-ii), (K-iii) We have
\[
\exp\left(-\frac{\alpha}{s}\right) = \int_{0}^{\infty} J_{0}(2\sqrt{\alpha x}) \, s e^{-sx} \, dx,
\]
for $\Re s > 0$ by [1] p. 173 (and the changing of variable log(1/x) → x). Therefore, by integration by parts,
\[
\exp\left(-\frac{\alpha}{s}\right) - 1 = \int_{0}^{\infty} \frac{d}{dx} J_{0}(2\sqrt{\alpha x}) \, e^{-sx} \, dx
\]
\[
= -\alpha \int_{0}^{\infty} J_{1}(2\sqrt{\alpha x}) \sqrt{\alpha x} e^{-sx} \, dx.
\]
Combining this with (3.3) and the equality
\[
\exp\left(-2\theta \frac{\alpha}{s}(s)\right) = \pi^{\theta} \exp\left(-\theta \psi\left(\frac{s}{2} + 1\right)\right)
\]
\[
+ \pi^{\theta} \exp\left(-\theta \psi\left(\frac{s}{2} + 1\right)\right) \left[ \exp\left(-\frac{2\theta}{s} - 1\right) - 1 \right],
\]
we obtain
\[
g_{\theta}(x) = 2\pi^{\theta} e^{-\frac{\pi^{2}}{4} x} \Psi_{\theta}(2x)
\]
\[
- 4\pi^{\theta} e^{-\frac{\pi^{2}}{4} x} \int_{0}^{x} \Psi_{\theta}(2(x - y)) \frac{e^{2y} J_{1}(2\sqrt{2y})}{\sqrt{2y}} \, dy.
\]
In particular, $g_{\theta}(x)$ is a real-valued continuous function on $\mathbb{R}$ vanishing on $(-\infty, 0)$ by Proposition 3.2 and the assumption $\theta > 1.$ Thus formula (1.13) implies (K-iii). The Dirichlet series (1.11) converges absolutely for $\Re(s) > 1$ ([2] Prop. 3.9.5 or [3] Thm. 2). Therefore, (1.10) holds for $3z > 1/2$ by Proposition 3.2 and the Fubini theorem. Moreover, we have $K_{0}(x) \ll \exp(x/2)$ by moving the path of integration in the inversion formula of (1.10) noting the growth of $\psi(s)$ and the non-vanishing of $\zeta(s)$ for $\Re(s) \geq 1.$

**Proof of Theorem 1.2** (K-iv) By (1.13) and Proposition 3.2 it remains to show that $|(d/dx)g_{\theta}(x)|$ is integrable around $x = 0.$ We have
\[
\frac{d}{dx} g_{\theta}(x) = -3 e^{-\frac{\pi^{2}}{4} x} \left( \Psi_{\theta}(2x) - 2\theta \int_{0}^{x} \Psi_{\theta}(2(x - y)) \frac{e^{2y} J_{1}(2\sqrt{2y})}{\sqrt{2y}} \, dy \right)
\]
\[
+ 2 e^{-\frac{\pi^{2}}{4} x} \left( \frac{d}{dx} \Psi_{\theta}(2x) - 2\theta \int_{0}^{x} \frac{d}{dx} \Psi_{\theta}(2(x - y)) \frac{e^{2y} J_{1}(2\sqrt{2y})}{\sqrt{2y}} \, dy \right)
\]
and
\[
\frac{d}{dx} \Psi_{\theta}(x) = \frac{d}{dx} \Psi_{\theta, \alpha}(x) + \int_{0}^{x} \frac{d}{dy} \Psi_{\theta, \alpha}(y) \Psi_{\theta}(x - y) \, dy.
\]
Applying the series expansion (3.10) to definition (3.9) of $\Psi_{\theta, \alpha}$, we easily find that $|(d/dx)\Psi_{\theta, \alpha}(x)|$ is integrable around $x = 0$ by the assumption $\theta > 1.$ Therefore, the above two equalities implies that $|(d/dx)g_{\theta}(x)|$ is integrable around $x = 0.$

\[\square\]
Proof of Theorem 1.2 (K-v) Put $s = 1/2 - i(u + iv)$. For $\delta > 0$, the estimate
\[
\exp\left(-2\theta \frac{\delta^s}{\zeta}(s)\right) \ll \exp\left(-\theta \Re \psi(s/2)\right) \ll (1 + v)^{-\theta}
\] (3.14)
holds uniformly for $u \in \mathbb{R}$ and $v \geq 1/2 + \delta$ with the implied constant depending only on $\delta > 0$. On the other hand, it will be shown that
\[
(FK_\theta f)(z) = \exp\left(-2\theta \frac{\delta^s}{\zeta}(s)\right) (Ff)(z), \quad s = \frac{1}{2} - iz
\] (3.15)
holds for $f \in L^2(-\infty, t)$ and $\Im z > 1/2$, where $(Ff)(z) = \int_0^\infty f(x)e^{izx}dx$. Then, (K-v) is proved by a way similar to the proof of [8, Prop. 4.2] if we use (3.14) (resp. (3.15)) instead of (4.3) (resp. (3.3)) of [8]. Hence it remains to show that (3.15) holds for $f \in L^2(-\infty, t)$ and $\Im z > 1/2$.

Let $f \in L^2(-\infty, t)$, Then, $(Ff)(u + iv) = \int_t^{t+1} f(x)e^{izx}dx$ is defined if $v \geq 0$. On the other hand, $K_\theta f$ is defined and has a support in $[-t, \infty)$ (but not necessarily $L^2(-t, \infty)$), since we see $(F_\theta f)(x) = \int_{-x}^x K_\theta(x+y)f(y)dy$. Moreover, $K_\theta(x) \ll \exp(x/2)$ implies $(K_\theta f)(x) \ll \exp(x/2)$, and hence $(FK_\theta f)(u + iv) = \int_{-x}^x (K_\theta f)(x)e^{-izx}dx$ is defined if $v > 1/2$. As a consequence, the calculation
\[
(FK_\theta f)(z) = \int_{-\infty}^\infty \int_{-\infty}^\infty K_\theta(x+y)e^{izx}dx f(y)dy
\]
\[
= \int_{-\infty}^\infty K_\theta(x)e^{izx}dx \int_{-\infty}^\infty f(y)e^{-izy}dy
\]
\[
= \exp\left(-2\theta \frac{\delta^s}{\zeta}(s)\right) (Ff)(-z)
\]
is justified if $\Im z > 1/2$. Hence we complete the proof. $\square$

4. Approximate formulas

In this section, we study approximate formulas of $\Psi_\theta(x)$ and $g_\theta(x)$ in a bounded range $0 \leq x \leq x_0$. This is because formulas (3.8) and (3.13) are not so explicit from a computational point of view, since $\Psi_\theta(x)$ is given by the inversion formula (3.1).

If $R(s) \ll |s|^{-N}$ as $|s| \to \infty$ in $\Re(s) > 1$,
\[
\frac{1}{2\pi} \int_{-\infty + ic}^{\infty + ic} R\left(\frac{1}{2} - iz\right) e^{-izx}dz \ll e^{c}x_0\left(\frac{1}{2} + e\right)^{-N}
\]
for $c > 1/2$ and $0 \leq x \leq x_0$, where the implied constant depends only on $R(s)$. Therefore, the asymptotic expansion
\[
s^\theta \exp\left(-\theta \psi(s) - \frac{\theta}{s}\right) - 1 = \sum_{n=1}^{N-1} \frac{\tilde{C}_n(\theta)}{s^n} + O(|s|^{-N})
\] (4.1)
for $|\arg s| \leq \pi - \delta$, $|s| \geq \delta$ derived from (3.3) implies that $\Psi_\theta^1(x)$ is well approximated by
\[
\Psi_\theta^1(x) := 1_{[0, x_0]}(x)e^{-x/2} \sum_{n=1}^{N-1} \frac{\tilde{C}_n(\theta)}{s^n} x^{n-1}
\] (4.2)
in a bounded range $0 \leq x \leq x_0$ if $N$ is sufficiently large, where $\tilde{C}_1(\theta) = -\theta/2$, $\tilde{C}_2(\theta) = \theta(\theta + 2)/24$, etc. More precisely, $\Psi_\theta^1(x) - \Psi_\theta^1(x) \ll e^{(1/2 + \epsilon)x_0}(1 + \epsilon)^{-N}$ for $0 \leq x \leq x_0$, where $\epsilon > 0$ is a given constant. Thus $\Psi_\theta(x)$ is well approximated by
\[
\Psi_\theta^N(x) := \Psi_\theta^0(x) + \int_0^x \Psi_\theta^0(y)\Psi_\theta^1(x-y)dy
\] (4.3)
in a bounded range \(0 \leq x \leq x_0\) if \(N\) is sufficiently large, that is,

\[
\Psi_\theta(x) - \Psi_\theta^N(x) = \int_0^x \Psi^0_\theta(y) \left( \Psi^1_\theta(x-y) - \Psi^1_\theta(x-y) \right) dy \lesssim e^{(1/2+\epsilon)x} \int_0^{x_0} |\Psi^0_\theta(y)| dy \cdot (1 + \epsilon)^{1-N}.
\]

For the integral on the right-hand side of (4.3), the trivial equality

\[
\frac{1}{s^\theta} \exp \left( -\frac{\theta}{s} \right) \cdot \sum_{n=1}^{N-1} \frac{\hat{C}_{n-1}(\theta)}{s^n} = \sum_{n=1}^{N-1} \hat{C}_n(\theta) \frac{1}{s^{\theta+n}} \exp \left( -\frac{\theta}{s} \right)
\]

combined with (3.7) and (4.1) gives

\[
\int_0^x \Psi^0_\theta(y) \Psi^1_\theta(x-y) dy = \sum_{n=1}^{N-1} \hat{C}_n(\theta) \Psi^0_{\theta+n}(x).
\]

Hence, by taking \(\hat{C}_0(\theta) = 1\), we obtain

\[
\Psi^N_\theta(x) = \sum_{n=0}^{N-1} \hat{C}_n(\theta) \Psi^0_{\theta+n}(x).
\]

These sums are useful for calculating \(\Psi_\theta(x)\) in computational ways.

On the other hand, we note the decomposition

\[
\exp \left( -\theta \psi \left( \frac{s}{2} + 1 \right) + \frac{2\theta}{s-1} \right) = \frac{1}{w^\theta} \exp \left( \frac{\theta}{w} \right) \cdot w^\theta \exp \left( -\theta \psi(w) - \frac{\theta}{w} \right) \cdot \exp \left( -\frac{2\theta}{2w-3} \right),
\]

where \(w = (s+2)/2\), and the asymptotic expansion

\[
w^\theta \exp \left( -\theta \psi(w) - \frac{\theta}{w} \right) \exp \left( -\frac{2\theta}{2w-3} \right) = 1 + \sum_{n=1}^{N-1} A_n(\theta) w^n + O \left( |w|^{-N} \right)
\]

for large \(w\) derived from (3.3), where \(A_1(\theta) = -3\theta/2, A_2(\theta) = \theta(27\theta - 34)/24\), etc.

Then, we find that \(g_\theta(x)\) is well approximated by

\[
g^N_\theta(x) = \Psi^2_{\theta,\theta}(x) + \sum_{n=1}^{N-1} A_n(\theta) \Psi^2_{\theta+n,\theta}(x),
\]

with \(\Psi^2_{\theta,\alpha}(x) := 2e^{-3\alpha/2} \Psi^0_{\theta,\alpha}(2x)\) in a bounded range \(0 \leq x \leq x_0\), since \(\int_0^\infty \Psi^2_{\theta,\alpha}(x)e^{izx} dx = w^{-\theta} \exp(\alpha/w)\) with \(w = (s+2)/2\) and \(s = 1/2 - iz\) holds by (3.7). As before, “well approximated” means that \(g_\theta(x) - g^N_\theta(x) \ll_{x_0, \epsilon} (1 + \epsilon)^{-N}\) holds for \(0 \leq x \leq x_0\).

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