A note on upper bounds for the maximum span in interval edge colorings of graphs

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An edge coloring of a graph $G$ with colors $1, 2, \ldots, t$ is called an interval $t$-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of $G$ colored by $i$, the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. In 1994 Asratian and Kamalian proved that if a connected graph $G$ admits an interval $t$-coloring, then $t \leq (d + 1)(\Delta - 1) + 1$, and if $G$ is also bipartite, then this upper bound can be improved to $t \leq d(\Delta - 1) + 1$, where $\Delta$ is the maximum degree in $G$ and $d$ is the diameter of $G$. In this paper we show that these upper bounds can not be significantly improved.

Keywords: edge coloring, interval coloring, bipartite graph, diameter of a graph

1. Introduction

An edge coloring of a graph $G$ with colors $1, 2, \ldots, t$ is called an interval $t$-coloring if for each $i \in \{1, 2, \ldots, t\}$ there is at least one edge of $G$ colored by $i$, the colors of edges incident to any vertex of $G$ are distinct and form an interval of integers. The concept of interval edge colorings was introduced by Asratian and Kamalian [1]. In [1] they proved that if a triangle-free graph $G = (V, E)$ has an interval $t$-coloring, then $t \leq |V| - 1$. Furthermore, Kamalian [9] showed that if $G$ admits an interval $t$-coloring, then $t \leq 2|V| - 3$. Giaro, Kubale and Malafiejski [5] proved that this upper bound can be improved to $2|V| - 4$ if $|V| \geq 3$. For a planar graph $G$, Axenovich [4] showed that if $G$ has an interval $t$-coloring, then $t \leq \frac{11}{6}|V|$. In [8, 13] interval edge colorings of complete graphs, complete bipartite graphs, trees and $n$-dimensional cubes were investigated. The $NP$-completeness of the problem of existence of an interval edge coloring of an arbitrary bipartite graph was shown in [14]. In papers [2, 3, 6, 7, 9, 10, 11] the problem of existence and construction of interval edge colorings was considered and some bounds for the number of colors in such colorings of graphs were given. In particular, it was proved in [2] that if a connected graph $G$ admits an interval $t$-coloring, then $t \leq (d + 1)(\Delta - 1) + 1$, and if
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If \( G \) is a connected bipartite, then the upper bound can be improved to \( t \leq d(\Delta - 1) + 1 \), where \( \Delta \) is the maximum degree in \( G \) and \( d \) is the diameter of \( G \). In this paper we show that these upper bounds cannot be significantly improved.

2. Definitions and preliminary results

All graphs considered in this paper are finite, undirected and have no loops or multiple edges. Let \( V(G) \) and \( E(G) \) denote the sets of vertices and edges of \( G \), respectively. The maximum degree in \( G \) is denoted by \( \Delta(G) \), the chromatic index of \( G \) by \( \chi'(G) \) and the diameter of \( G \) by \( \text{diam}(G) \). A partial edge coloring of \( G \) is a coloring of some of the edges of \( G \) such that no two adjacent edges receive the same color. If \( \alpha \) is a partial edge coloring of \( G \) and \( v \in V(G) \), then \( S(v, \alpha) \) denotes the set of colors of colored edges incident to \( v \).

Let \( \lfloor a \rfloor \) denote the largest integer less than or equal to \( a \). Given two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), the Cartesian product \( G_1 \square G_2 \) is a graph \( G = (V, E) \) with the vertex set \( V = V_1 \times V_2 \) and the edge set \( E = \{((u_1, u_2), (v_1, v_2)) | u_1 = v_1 \text{ and } (u_2, v_2) \in E_2 \text{ or } u_2 = v_2 \text{ and } (u_1, v_1) \in E_1 \} \).

The set of all interval colorable graphs is denoted by \( \mathfrak{N} \) \cite{1, 9}. For a graph \( G \in \mathfrak{N} \), the greatest value of \( t \), for which \( G \) has an interval \( t \)-coloring, is denoted by \( W(G) \).

The terms and concepts that we do not define can be found in \cite{15}.

We will use the following results.

**Theorem 1** \cite{8}. \( W(K_{\Delta, \Delta}) = 2\Delta - 1 \) for any \( \Delta \in \mathbb{N} \).

**Theorem 2** \cite{12, 13}. \( W(K_2q) \geq 2^{q+1} - 2 - q \) for any \( q \in \mathbb{N} \).

**Theorem 3** \cite{12}. If \( G, H \in \mathfrak{N} \), then \( G \square H \in \mathfrak{N} \).

**Theorem 4** \cite{2}. (1) If \( G \) is a connected graph and \( G \in \mathfrak{N} \), then

\[
W(G) \leq (\text{diam}(G) + 1) (\Delta(G) - 1) + 1.
\]

(2) If \( G \) is a connected bipartite graph and \( G \in \mathfrak{N} \), then

\[
W(G) \leq \text{diam}(G) (\Delta(G) - 1) + 1.
\]

3. Main results

**Theorem 5** For any integers \( d, q \in \mathbb{N} \), there is a connected graph \( G \) with \( \text{diam}(G) = d \),

\[
\Delta(G) = \begin{cases} 
2^q - 1, & \text{if } d = 1, \\
2^q, & \text{if } d = 2, \\
2^q + 1, & \text{if } d \geq 3,
\end{cases}
\]
such that $G \in \mathfrak{N}$ and

$$W(G) \geq \begin{cases} (d + 1)(\Delta(G) - 1) - q + 2, & \text{if } d = 1, \\ (d + 1)(\Delta(G) - 1) - q + 1, & \text{if } d = 2, \\ (d + 1)(\Delta(G) - 1) - q - 2, & \text{if } d \geq 3. \end{cases}$$

**Proof.** For the proof we construct a graph $G_{d,q}$ which satisfies the condition of the theorem. We define a graph $G_{d,q}$ as follows: $G_{d,q} = P_d \Box K_{2q}$. Clearly, $G_{d,q}$ is a connected graph of diameter $d$ and

$$\Delta(G_{d,q}) = \begin{cases} 2^q - 1, & \text{if } d = 1, \\ 2^q, & \text{if } d = 2, \\ 2^q + 1, & \text{if } d \geq 3. \end{cases}$$

Since $P_d, K_{2q} \in \mathfrak{N}$, by Theorem 3 we have $G_{d,q} \in \mathfrak{N}$.

Let us show that

$$W(G_{d,q}) \geq \begin{cases} (d + 1)(\Delta(G_{d,q}) - 1) - q + 2, & \text{if } d = 1, \\ (d + 1)(\Delta(G_{d,q}) - 1) - q + 1, & \text{if } d = 2, \\ (d + 1)(\Delta(G_{d,q}) - 1) - q - 2, & \text{if } d \geq 3. \end{cases}$$

Let $V(K_{2q}) = \{v_1, v_2, \ldots, v_{2q}\}$ and

$$V(G_{d,q}) = \bigcup_{i=1}^{d} V^i(G_{d,q}),$$

$$E(G_{d,q}) = \bigcup_{i=1}^{d} E^i(G_{d,q}) \cup \bigcup_{j=1}^{2q} E_j(G_{d,q})$$

where

$$V^i(G_{d,q}) = \{v_{j}^{(i)} \mid 1 \leq j \leq 2^q\},$$

$$E^i(G_{d,q}) = \{(v_{j}^{(i)}, v_{k}^{(i)}) \mid 1 \leq j < k \leq 2^q\},$$

$$E_j(G_{d,q}) = \{(v_{j}^{(i)}, v_{j}^{(i+1)}) \mid 1 \leq i \leq d - 1\}.$$ 

For $i = 1, 2, \ldots, d$ define a subgraph $G_i$ of the graph $G_{d,q}$ in the following way:

$$G_i = (V^i(G_{d,q}), E^i(G_{d,q})).$$

Clearly, $G_i$ is isomorphic to $K_{2q}$ for $i = 1, 2, \ldots, d$. By Theorem 2 there exists an interval $(2^{q+1} - 2 - q)$-coloring $\alpha$ of the graph $K_{2q}$.

Define an edge coloring $\beta$ of the subgraphs $G_1, G_2, \ldots, G_d$.

For $i = 1, 2, \ldots, d$ and for every $(v_{j}^{(i)}, v_{k}^{(i)}) \in E(G_i)$ we set:

$$\beta((v_{j}^{(i)}, v_{k}^{(i)})) = \alpha((v_j, v_k)) + (i - 1)2^q,$$

where $1 \leq j < k \leq 2^q$.

Now we define an edge coloring $\gamma$ of the graph $G_{d,q}$ in the following way: for all $e \in E(G_{d,q})$
\[ \gamma(e) = \begin{cases} 
\beta(e), & \text{if } e \in E(G_i), 1 \leq i \leq d, \\
\max S(v_j^{(i)}, \beta) + 1, & \text{if } e = (v_j^{(i)}, v_j^{(i+1)}) \in E_j(G_{d,q}), 1 \leq i \leq d - 1, 1 \leq j \leq 2^q. 
\end{cases} \]

It can be verified that if \( d = 1 \), then \( \gamma \) is an interval \((2^q+1-2-q)\)-coloring of the graph \( G_{1,q} \); if \( d = 2 \), then \( \gamma \) is an interval \((3 \cdot 2^q - 2 - q)\)-coloring of the graph \( G_{2,q} \) and \( \gamma \) is an interval \(((d+1)2^q - 2 - q)\)-coloring of the graph \( G_{d,q} \) for \( d \geq 3 \). This implies the necessary lower bounds for \( W(G_{d,q}) \). \( \square \)

**Theorem 6** For any integers \( d, \Delta \geq 2 \), there is a connected bipartite graph \( G \) with \( diam(G) = d \), \( \Delta(G) = \Delta \), such that \( G \in \mathcal{H} \) and \( W(G) = d(\Delta - 1) + 1 \).

**Proof.** For the proof we are going to construct a graph \( G_{d,\Delta} \) which satisfies the condition of the theorem. We consider some cases.

Case 1: \( \Delta = 2 \) or \( d = 2 \).

If \( \Delta = 2 \) and \( d \geq 2 \), then we take \( G_{d,2} = C_{2d} \). Clearly, \( C_{2d} \in \mathcal{H} \), \( diam(C_{2d}) = d \) and \( \Delta(C_{2d}) = 2 \). From Theorem 4 we have \( W(C_{2d}) \leq d + 1 \) for \( d \geq 2 \).

Now let us show that \( W(C_{2d}) = d + 1 \) for \( d \geq 2 \).

Let \( V(C_{2d}) = \{v_1, v_2, \ldots, v_{2d}\} \) and \( E(C_{2d}) = \{(v_i, v_{i+1})| 1 \leq i \leq 2d - 1\} \cup \{(v_1, v_{2d})\} \).

Define an edge coloring \( \alpha \) of the graph \( C_{2d} \) as follows:

1. \( \alpha((v_1, v_{2d})) = 1 \),
2. \( \alpha((v_i, v_{i+1})) = \alpha((v_{2d-i+1}, v_{2d-i})) = i + 1 \) for \( i = 1, 2, \ldots, d \).

It is easy to see that \( \alpha \) is an interval \((d + 1)\)-coloring of the graph \( C_{2d} \), thus \( W(C_{2d}) = d + 1 \).

If \( d = 2 \) and \( \Delta \geq 2 \), then we take \( G_{2,\Delta} = K_{\Delta,\Delta} \). Clearly, \( K_{\Delta,\Delta} \in \mathcal{H} \), \( diam(K_{\Delta,\Delta}) = 2 \) and \( \Delta(K_{\Delta,\Delta}) = \Delta \). From Theorem 4 we have \( W(K_{\Delta,\Delta}) = 2\Delta - 1 \).

Case 2: \( \Delta, d \geq 3 \).

Subcase 2.1: \( d \) is even.

Define the graph \( G_{d,\Delta} \) as follows:

\[
V(G_{d,\Delta}) = \{u_j^{(i)}, v_j^{(i)}| 1 \leq i \leq \frac{d}{2}, 1 \leq j \leq \Delta\},
\]

\[
E(G_{d,\Delta}) = E_1 \cup E_2 \cup E_3 \cup E_4,
\]

where

\[
E_1 = \{(u_j^{(1)}, v_j^{(1)}| 1 \leq i \leq \Delta, 1 \leq j \leq \Delta\} \setminus \{(u_1^{(1)}, v_1^{(1)})\},
\]

\[
E_2 = \bigcup_{i=2}^{\frac{d}{2}-1} \{(u_j^{(i)}, v_j^{(i)}| 1 \leq j \leq \Delta, 1 \leq k \leq \Delta\} \setminus \{(u_1^{(i)}, v_1^{(i)}), (u_1^{(i)}, v_1^{(i)})\},
\]

\[
E_3 = \{(u_j^{(i-1)}, v_j^{(i)}, (v_j^{(i-1)}, u_1^{(i)})| 2 \leq i \leq d\},
\]

\[
E_4 = \{(u_j^{(4)}, v_j^{(4)}| 1 \leq i \leq \Delta, 1 \leq j \leq \Delta\} \setminus \{(u_1^{(4)}, v_1^{(4)})\}.
\]

Clearly, \( G_{d,\Delta} \) is a connected \( \Delta \)-regular bipartite graph of diameter \( d \). It is easy to see that \( G_{d,\Delta} \in \mathcal{H} \). By Theorem 4 we have \( W(G_{d,\Delta}) \leq d(\Delta - 1) + 1 \).

Now we show that \( W(G_{d,\Delta}) = d(\Delta - 1) + 1 \).

Define an edge coloring \( \beta \) of the graph \( G_{d,\Delta} \) in the following way:

1. for every \( (u_j^{(i)}, v_k^{(j)}) \in E(G_{d,\Delta}) \),
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\[ \beta((u^{(i)}_j, v^{(i)}_k)) = (i - 1)(2\Delta - 1) + j + k - i, \]

where \( 1 \leq i \leq \frac{d}{2}, 1 \leq j \leq \Delta, 1 \leq k \leq \Delta; \)

2. for \( i = 2, \ldots, \frac{d}{2} \)

\[ \beta((u^{(i-1)}_\Delta, v^{(i)}_1)) = \beta((v^{(i-1)}_\Delta, u^{(i)}_1)) = (i - 1)(2\Delta - 1) - i + 2. \]

It is not difficult to check that \( \beta \) is an interval \((d(\Delta - 1) + 1)-\)coloring of the graph \( G_{d,\Delta} \).

Subcase 2.2: \( d \) is odd.

If \( d = 3 \), then define the graph \( G_{3,\Delta} \) as follows:

\[ V(G_{3,\Delta}) = \{u_i, u'_i| 1 \leq i \leq \Delta - 1\} \cup \{v_j, v'_j| 1 \leq j \leq \Delta\}, \]

\[ E(G_{3,\Delta}) = \{(u_i, v_j), (u'_i, v'_j)| 1 \leq i \leq \Delta - 1, 1 \leq j \leq \Delta\} \cup \{(u_i, v'_i)| 1 \leq i \leq \Delta\}. \]

Clearly, \( G_{3,\Delta} \) is a connected \( \Delta \)-regular bipartite graph of diameter 3. It is easy to see that \( G_{3,\Delta} \in \mathcal{G} \). By Theorem \[ \square \] we have \( W(G_{3,\Delta}) \leq 3\Delta - 2. \)

Now we show that \( W(G_{3,\Delta}) = 3\Delta - 2. \)

Define an edge coloring \( \gamma \) of the graph \( G_{3,\Delta} \) in the following way:

1. for \( i = 1, 2, \ldots, \Delta - 1, j = 1, 2, \ldots, \Delta \)

\[ \gamma((u_i, v_j)) = i + j - 1, \]

2. for \( i = 1, 2, \ldots, \Delta - 1, j = 1, 2, \ldots, \Delta \)

\[ \gamma((u'_i, v'_j)) = \Delta + i + j - 1, \]

3. for \( i = 1, 2, \ldots, \Delta \)

\[ \gamma((u_i, v'_i)) = \Delta + i - 1. \]

It is not difficult to check that \( \gamma \) is an interval \((3\Delta - 2)-\)coloring of the graph \( G_{3,\Delta} \).

Assume that \( d \geq 5. \)

Define the graph \( G_{d,\Delta} \) as follows:

\[ V(G_{d,\Delta}) = \{a, b_1, b_2, \ldots, b_{\Delta-3}, c, d_1, d_2, \ldots, d_{\Delta-3}\} \cup \{u^{(i)}_j, v^{(i)}_j| 1 \leq i \leq \left\lfloor \frac{d}{2} \right\rfloor, 1 \leq j \leq \Delta\}, \]

\[ E(G_{d,\Delta}) = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5, \]

where

\[ E_1 = \{(u^{(i)}_1, v^{(i)}_1)| 1 \leq i \leq \Delta, 1 \leq j \leq \Delta\} \setminus \{(u^{(1)}_\Delta, v^{(1)}_\Delta)\}, \]

\[ E_2 = \{(u^{(1)}_\Delta, a), (a, u^{(2)}_1), (v^{(1)}_\Delta, c), (c, v^{(2)}_1), (a, c)| 1 \leq i \leq \Delta - 3\}, \]

\[ E_3 = \bigcup_{i=2}^{\left\lfloor \frac{d}{2} \right\rfloor - 1}\{(u^{(i)}_j, v^{(i)}_j)| 1 \leq j \leq \Delta, 1 \leq k \leq \Delta\} \setminus \{(u^{(i)}_1, v^{(i)}_1), (u^{(i)}_\Delta, v^{(i)}_\Delta)\}, \]

\[ E_4 = \{(u^{(i)}_j, v^{(i)}_j)| 1 \leq i \leq \Delta - 3, 1 \leq j \leq \Delta, 1 \leq k \leq \Delta\}, \]

\[ E_5 = \{(u^{(i)}_j, v^{(i)}_j)| 1 \leq i \leq \Delta, 1 \leq j \leq \Delta, 1 \leq k \leq \Delta\} \setminus \{(u^{(i)}_1, v^{(i)}_1), (u^{(i)}_\Delta, v^{(i)}_\Delta)\}. \]
\begin{equation*}
E_4 = \{(u_\Delta^{(i-1)}, v_1^{(i)}), (v_\Delta^{(i-1)}, u_1^{(i)}) | 3 \leq i \leq \lfloor \frac{d}{2} \rfloor \},
\end{equation*}

\begin{equation*}
E_5 = \{(u_1^{(i)}, v_1^{(j)}) | 1 \leq i \leq \Delta, 1 \leq j \leq \Delta \} \setminus \{(u_1^{(\lfloor \frac{d}{2} \rfloor)}, v_1^{(\lfloor \frac{d}{2} \rfloor)})\}.
\end{equation*}

Clearly, \(G_{d,\Delta}\) is a connected bipartite graph with a maximum degree \(\Delta\) and a diameter \(d\).

Now we show that \(G_{d,\Delta} \in \mathcal{N}\) and \(W(G_{d,\Delta}) = d(\Delta - 1) + 1\).

Define an edge coloring \(\lambda\) of the graph \(G_{d,\Delta}\) in the following way:

1. for every \((u_j^{(i)}, v_k^{(j)}) \in E(G_{d,\Delta})\)
\[
\lambda((u_j^{(i)}, v_k^{(i)})) = j + k - 1,
\]

where \(1 \leq j \leq \Delta, 1 \leq k \leq \Delta\);

2. 
\[
\lambda((u_\Delta^{(i)}, a)) = \lambda((v_\Delta^{(i)}, c)) = 2\Delta - 1, \lambda((a, c)) = 2\Delta;
\]

3. 
\[
\lambda((a, b_i)) = \lambda((c, d_i)) = 2\Delta + i, i = 1, 2, \ldots, \Delta - 3;
\]

4. 
\[
\lambda((a, u_1^{(2)})) = \lambda((c, v_1^{(2)})) = 3\Delta - 2;
\]

5. for every \((u_j^{(i)}, v_k^{(i)}) \in E(G_{d,\Delta})\)
\[
\lambda((u_j^{(i)}, v_k^{(i)})) = (i - 1)(2\Delta - 1) + j + k - i + \Delta - 1,
\]

where \(2 \leq i \leq \lfloor \frac{d}{2} \rfloor, 1 \leq j \leq \Delta, 1 \leq k \leq \Delta\);

6. for \(i = 3, \ldots, \lfloor \frac{d}{2} \rfloor\)
\[
\lambda((u_\Delta^{(i-1)}, v_1^{(i)})) = \lambda((v_\Delta^{(i-1)}, u_1^{(i)})) = (i - 1)(2\Delta - 1) - i + \Delta + 1.
\]

It can be verified that \(\lambda\) is an interval \((d(\Delta - 1) + 1)-\)coloring of the graph \(G_{d,\Delta}\), thus \(G_{d,\Delta} \in \mathcal{N}\) and \(W(G_{d,\Delta}) \geq d(\Delta - 1) + 1\). On the other hand, from Theorem 4 we have \(W(G_{d,\Delta}) \leq d(\Delta - 1) + 1\), hence \(W(G_{d,\Delta}) = d(\Delta - 1) + 1\). □

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