ON A KIRCHHOFF WAVE MODEL WITH NONLOCAL NONLINEAR DAMPING

VANDO NARCISO
Nucleus of Exact and Technological Sciences
State University of Mato Grosso do Sul
79804-970 Dourados, MS, Brazil

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Abstract. This paper is concerned with the well-posedness as well as the asymptotic behavior of solutions for a quasi-linear Kirchhoff wave model with nonlocal nonlinear damping term \( \sigma \left( \int_{\Omega} |\nabla u|^2 \, dx \right) g(u_t) \), where \( \sigma \) and \( g \) are nonlinear functions under proper conditions. The analysis of such a damping term is presented for this kind of Kirchhoff models and consists the main novelty in the present work.

1. Introduction. In this paper we address well-posedness and long-time behavior to the following quasi-linear Kirchhoff wave model with nonlocal nonlinear damping

\[
 u_{tt} - \phi(\|\nabla u(t)\|_2^2) \Delta u + \sigma(\|\nabla u(t)\|_2^2) g(u_t) + f(u) = h \quad \text{in} \quad \Omega \times (0, \infty),
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \Gamma \), \( \phi \) and \( \sigma \) are scalar functions defined on \( \mathbb{R}^+ = [0, +\infty) \), \( f \) and \( g \) are nonlinear functions on \( \mathbb{R} \) corresponding to source and damping terms, and \( h \) is a external force. Here, \( \| \cdot \|_2 \) stands for \( L^2 \)-norm. The precise assumptions on \( \phi, \sigma, f \) and \( g \) shall be given later. The following initial-boundary conditions are considered:

\[
 u(\cdot, 0) = u_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot) \quad \text{in} \quad \Omega \quad \text{and} \quad u = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}^+.
\]

The main object of study of this paper is to analyze the existence and asymptotic behavior of solutions for (1)-(2) under the influence of the nonlocal nonlinear damping term

\[
 \sigma(\|\nabla u(t)\|_2^2) g(u_t).
\]

Since it consists in a product of two nonlinear terms, which is unusual of seeing for these kind of models, then it deserves some attention from mathematical viewpoint as we shall explain later. Before, let us consider some existing literature on the subject dealing with Kirchhoff (wave) models with nonlocal damping term.

In this direction, a first paper we found is due to Lazo [12] which considers the following abstract model

\[
 u_{tt} + \phi(\|A^\frac{1}{2} u(t)\|^2) Au + \sigma(\|A^\alpha u(t)\|^2) A^\theta u_t = 0, \quad 0 < \alpha, \theta \leq 1,
\]

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where $A$ is an operator defined in a real Hilbert space $(H, \| \cdot \|)$. Only existence of weak solutions for (4) is obtained by the author in [12] under proper conditions on $\phi$ and $\sigma$, and considering $\alpha = \theta$. The damping term in (4) is just nonlinear in one of its components. Besides, one of the main goals in [12] is to generalize the results of Medeiros and Milla Miranda [16] which studied equation (4) in the particular case $\sigma \equiv 1$. In such a case the dissipation becomes to the linear one and it is proved in [16] that uniqueness holds only when $\theta$ varies the range $\frac{1}{2} \leq \theta \leq 1$. Global existence and exponential decay of the energy are also addressed by the authors in [16].

Chueshov [3] studied the well-posedness and long-time dynamics of solutions for (4) by introducing a source term $f(u)$ and taking $\alpha = \frac{1}{2}$ and $\frac{1}{2} \leq \theta < 1$. Such a choice for $\theta$ allowed the author in [3] to consider existence and uniqueness of weak solutions as well as existence of finite-dimensional compact global and exponential attractors. Later, in Chueshov [4] is considered the strong case $\theta = 1$. More precisely, it is studied in [4] the following Kirchhoff wave model with nonlocal and nonlinear strong damping

$$u_{tt} - \phi(\|\nabla u(t)\|^2_2)\Delta u - \sigma(\|\nabla u(t)\|^2_2)\Delta u_t + f(u) = h.$$  

The strength of the nonlocal and nonlinear strong damping in (5) is capable to produce sufficient regularity in order to show the existence and uniqueness of weak solution as well as its regularizing property for a wide class of nonlinear source terms $f(u)$ with supercritical growth and a (possible) degenerate stiffness coefficient $\phi$. Moreover, global attractor and its properties are considered for strictly positive stiffness factors. See e.g. Chueshov [4]. See also the papers [8, 23, 24, 26, 27] where equation (5) is considered with linear strong damping $-\sigma \Delta u_t$, $\sigma \equiv \sigma_0 > 0$.

From the above mentioned works, e.g. [3, 4, 12, 16], one can see that it is unlikely to consider a nonlocal weak damping term

$$\sigma(\|\nabla u(t)\|^2_2) u_t$$

where

$$g(u) = au + |u|^b u_t$$

with $a, b > 0$.

What differentiates this work from others is that (3) consists in a product of the two nonlinearities $\sigma(\|\nabla u(t)\|^2_2)$ and $g(u_t)$. In this case, it is not possible to apply same arguments as used in [3, 4] in order to prove uniqueness of solutions for initial data in the natural weak energy space $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$. Indeed, a uniqueness result is only known for Kirchhoff wave models with nonlocal (strong) damping $g(u_t) := (-\Delta)\nu u_t$ with $\frac{1}{2} \leq \theta \leq 1$. But this is not our situation and then it seems to be a hard task to get the existence of global attractors for (1)-(2) on $\mathcal{H}$. This damping given by the product of two nonlinearities was first used by Jorge and Narciso in [7] for a class of extensible beams.

However, in spite of all difficulties produced by the quasi-linear model (1) under the presence of the nonlocal nonlinear damping (3), we prove in this work that (1)-(2) is globally well posed on the strong phase space $\mathcal{H}_1 = H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$.
for small initial data. In addition, we study the asymptotic behavior of its solution on $H_1$. To our best knowledge there are only a few works in this direction. For instance, one of them we found is due to Nakao [22] which considers the existence of a local attractor to the following Kirchhoff wave model with linear weak damping

$$u_{tt} - (1 + \|\nabla u(t)\|_2^2)\Delta u + u_t + f(u) = h(x). \quad (7)$$

It is worth noting that model (7) is a particular case of (1) with $\phi(s) = 1 + s$, $\sigma \equiv 1$, and $g(s) = s$. For this particular model (7) subject to initial-boundary conditions in (2) it is proved in [22] the existence of an attractor $A$ for the corresponding semigroup $S(t)$ in a neighborhood $V$ of $(0, 0)$ in $H_1$, namely, $A \subset V$ is compact, invariant and attracts any bounded set $B \subset V$. A related study on local attractor is also given by Nakao [21].

The main results of this paper are Theorem 2.1, Theorem 3.1 and Theorem 4.2. To their proofs new estimates and arguments are required in view of the damping term (3). When compared to existing results on the subject we note that (3) allows us to address the case $\theta = 0$ in (4) that complements the case not approached in [3, 4, 12, 16]. Moreover, in view of our conditions on the functions $\phi, \sigma$ and $g$ we also extend all results provided by Nakao [22] with respect to nonlocal damping and stiffness factor.

We finish this section by noting that (1) is a generalized equation of the classical model of vibrating strings given by equation

$$\partial_{tt} u - \left[ \frac{\tau_0}{m} + \frac{IE}{2mL} \int_0^L (\partial_x u)^2 \right] \partial_{xx} u = 0, \quad (8)$$

which was introduced by Kirchhoff [9] in 1883, where $u = u(x,t)$ is the lateral displacement, $m$ is the mass density, $\tau_0$ is the initial tension, $L$ is the length of the string in the rest position, $E$ is the Young’s modulus of the material of the string and $I$ the area of the cross section. After that, all kinds of generalizations of the Kirchhoff wave model (8) have been studied by many authors. Existing literature is truly long, see e.g. [1, 2, 5, 6, 10, 11, 15, 17, 18, 19, 24, 25, 28, 29] and references therein. A suitable survey on precise references dealing with Kirchhoff wave models like (1) with $\sigma \equiv 1$ (and also $\phi \equiv 1$) can be found in Chueshov [3, 4] and Nakao [20, 21, 22].

The remaining paper is organized as follows. In section 2 we introduce some initial assumptions and local existence of solution for (1)-(2). In section 3 we show the global well-posedness of solution under suitable additional assumptions. Finally, in Section 4 we establish our result on asymptotic behavior.

2. Local existence. We start this section by fixing some notations and assumptions that shall be used throughout this paper. Spaces $L^p(\Omega)$ stand for $p$-Lebesgue integrable functions with norm

$$\|u\|_p^p = \int_\Omega |u(x)|^p dx, \quad u \in L^p(\Omega),$$

and $W^{m,p}(\Omega)$ or $W^{m,p}_0(\Omega)$ denote well-known Sobolev spaces. In particular, if $p = 2$, then $L^2(\Omega)$ is a Hilbert space with inner-product and norm

$$(u, v) = \int_\Omega u(x)v(x) dx, \quad \|u\|_2^2 = \int_\Omega |u(x)|^2 dx, \quad u, v \in L^2(\Omega),$$
and $W^{m,2}(\Omega) := H^m(\Omega)$ as well as $W^{m,2}_0(\Omega) := H^m_0(\Omega)$. In the special case $H^1_0(\Omega)$ we have, in view of the Poincaré inequality, the following inner-product and norm

$$ (u, v)_{H^1_0(\Omega)} = (\nabla u, \nabla v), \quad \|u\|_{H^1_0(\Omega)} = \|\nabla u\|_2, \quad u, v \in H^1_0(\Omega). $$

We also set the following Hilbert phase spaces

$$ \mathcal{H} = H^1_0(\Omega) \times L^2(\Omega) \quad \text{and} \quad \mathcal{H}_1 = [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega), $$

equipped with their respective norms

$$ \|(u, v)\|_{\mathcal{H}}^2 = \|\nabla u\|_2^2 + \|v\|_2^2 \quad \text{and} \quad \|(u, v)\|_{\mathcal{H}_1}^2 = \|\Delta u\|_2^2 + \|\nabla v\|_2^2. $$

In order to consider the Hadamard well-posedness to the problem (1)-(2) in the strong topology of $\mathcal{H}_1$, we assume the following hypotheses on $\phi$, $\sigma$, $f$ and $g$.

**H1** The stiffness factor $\phi \in C^1(\mathbb{R}^+)$ satisfies

$$ \phi(s) \geq \phi_0 \quad \text{and} \quad \phi(s) s \geq \tilde{\phi}(s), \quad \forall \ s \in \mathbb{R}^+, \tag{9} $$

for some constant $\phi_0 > 0$, where we denote $\tilde{\phi}(s) := \int_0^s \phi(\tau) d\tau$.

**H2** The damping coefficient $\sigma \in C^1(\mathbb{R}^+)$ is a positive function, namely,

$$ \sigma(s) > 0, \quad \forall \ s \in \mathbb{R}^+. \tag{10} $$

**H3** The source $f \in C^1(\mathbb{R})$ fulfills

$$ f(0) = 0, \quad |f'(s)| \leq C_f (1 + |s|^\rho), \quad \forall \ s \in \mathbb{R}, \tag{11} $$

where we consider $C_f > 0$ and the growth $\rho$ satisfying

$$ \rho \geq 0 \quad \text{if} \quad n = 1, 2 \quad \text{or} \quad 0 \leq \rho \leq \frac{2}{n-2} \quad \text{if} \quad n \geq 3. \tag{12} $$

In addition, for some $\eta \in (0, \lambda_1)$ and $C_f \geq 0$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$, we assume that

$$ f(s)s \geq \tilde{f}(s) - C_f \geq -\frac{\phi_0 \gamma}{2} s^2 - 2C_f, \quad \forall \ s \in \mathbb{R}, \tag{13} $$

where $\tilde{f}(s) := \int_0^s f(\tau)d\tau$.

**H4** The damping $g \in C^1(\mathbb{R})$ satisfies

$$ g(0) = 0, \quad \kappa_1 |s|\gamma \leq g'(s) \leq \kappa_2 (1 + |s|\gamma), \quad \forall \ s \in \mathbb{R}, \tag{14} $$

for some $\kappa_1, \kappa_2 > 0$, and $\gamma \geq 0$.

**Remark 1.** Condition (11) implies that $W^{m+1,2}(\Omega) \hookrightarrow W^{m,2(\rho+1)}(\Omega)$. So we denote by $C_{m+1,\rho} > 0$ the embedding constant for

$$ \|u\|_{W^{m+1,2}+1(\Omega)} \leq C_{m+1,\rho} \|u\|_{W^{m+1,2}}. $$

We also observe the choice for $\eta$ implies that

$$ \eta_0 := 1 - \frac{\eta}{\lambda_1} > 0. \tag{15} $$

**Theorem 2.1.** Let us assume that assumptions (H1)-(H4) hold and take $h \in H^1_0(\Omega)$. If $(u_0, u_1) \in \mathcal{H}_1$, then there exists a $T > 0$ such that the problem (1)-(2) has a unique solution $u = u(x, t)$ in the class

$$ u \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap W^{1,\infty}(0, T; H^1_0(\Omega)), $$

$$ u_t \in L^{\gamma+2}(0, T; L^{\gamma+2}(\Omega)) \quad \text{and} \quad u_{tt} \in L^{\frac{\gamma+2}{\gamma+1}}(0, T; L^{\frac{\gamma+2}{\gamma+1}}(\Omega)). $$
Proof. The proof on existence relies on Faedo-Galerkin method which was exten-
sively applied in the study of wave equations, see for instance Lions et al. [13, 14].
Let \((\omega_j)\in\mathbb{N}\) be eigenfunctions of the following problem
\[-\Delta \omega_j = \lambda_j \omega_j, \quad \omega_j \mid_{\Gamma} = 0, \quad j \in \mathbb{N}.
\]
From elliptic regularity we have \(\omega_j \in H^2(\Omega) \cap H_0^1(\Omega), \quad j \in \mathbb{N}\). Let \(V_m\) be a subspace
generated by the first \(m\) vectors \(\omega_1, \ldots, \omega_m\). For each \(m \in \mathbb{N}\), we can find a function
\[u^m(x, t) = \sum_{i=1}^{m} y_{im}(t) \omega_i(x), \quad x \in \Omega, \quad t \geq 0,
\]
which is a solution to the approximate problem
\[\begin{align*}
(u^m(t), \omega_j) + \phi \left( \|\nabla u^m(t)\|_2^2 \right) (-\Delta u^m(t), \omega_j) \\
+ \sigma \left( \|\nabla u^m(t)\|_2^3 \right) (g(u^m(t)), \omega_j) + (f(u^m(t)), \omega_j) = (h, \omega_j),
\end{align*}
\]
on \([0, t_m), \quad t_m > 0, \quad 1 \leq j \leq m, \quad \text{with initial condition}
\[u^m(0) = u_{0m} = \sum_{j=1}^{m} (u_0, \omega_j) \omega_j \to u_0 \in H^2(\Omega) \cap H_0^1(\Omega),
\]
\[u^m_t(0) = u_{1m} = \sum_{j=1}^{m} (u_1, \omega_j) \omega_j \to u_1 \in H_0^1(\Omega),
\]
by using standard methods in ODE. In what follows, the first estimate shall allow
us to extend the local solution to the interval \([0, T]\), for any given \(T > 0\). The
remaining estimates shall allow us to conclude the proof of Theorem 2.1.

A Priori Estimate I. Taking \(\omega_j = u^m_t(t)\) in the approximate problem (15) yields
\[\frac{d}{dt} E(u^m(t), u^m_t(t)) + \sigma \left( \|\nabla u^m(t)\|_2^2 \right) \int_{\Omega} g(u^m_t(t)) u^m_t(t) dx = 0,
\]
where
\[E(u^m(t), u^m_t(t)) = \frac{1}{2} \|u^m_t(t)\|_2^2 + \frac{1}{2} \|\nabla u^m(t)\|_2^2 + \int_{\Omega} f(u^m(t)) dx - \int_{\Omega} h u^m(t) dx.
\]
From assumption (9) we have
\[\int_0^1 \phi(\|\nabla u^m(t)\|_2^2) d\tau \geq \frac{\phi_0}{2} \|\nabla u^m(t)\|_2^2.
\]
Using assumption (12), Hölder inequality and \(\|u^m\|_2^2 \leq \frac{1}{\lambda_1} \|\nabla u^m\|_2^2\), we get
\[\int_{\Omega} f(u^m(t)) dx \geq -\frac{\phi_0 \eta_0}{2} \int_{\Omega} u^m(t) dx - C_f |\Omega| \geq -\frac{\phi_0 \eta_0}{2\lambda_1} \|\nabla u^m(t)\|_2^2 - C_f |\Omega| \]
and
\[-\int_{\Omega} h u^m(t) dx \geq -\frac{1}{\lambda_1 \phi_0 \eta_0} \|h\|_2^2 - \frac{\phi_0 \eta_0}{4} \|\nabla u^m(t)\|_2^2.
\]
Combining (17)-(19) and using assumption (14), we obtain
\[E(u^m(t), u^m_t(t)) \geq \frac{1}{2} \|u^m_t(t)\|_2^2 + \frac{\phi_0 \eta_0}{4} \|\nabla u^m(t)\|_2^2 - C_f |\Omega| - \frac{1}{\lambda_1 \phi_0 \eta_0} \|h\|_2^2.
\]
On the other hand, using assumption (13) we have
\[\int_{\Omega} g(u^m_t(t)) u^m_t(t) dx \geq \frac{\kappa_1}{\gamma + 1} \int_{\Omega} |u^m_t(t)|^{\gamma+2} dx.
\]
Thus, integrating (16) from 0 to \( t \leq t_m \) and using (21), we get

\[
E(u^m(t), u_t^m(t)) + \frac{\kappa_1}{\gamma + 1} \int_0^t \sigma \left( \|\nabla u^m(s)\|_2^2 \right) \|u_t^m(s)\|_{\gamma + 2}^2 \, ds \leq E(u_0, u_1),
\]

(22)

Therefore, from (20) and (22), we obtain

\[
\frac{1}{2} \|u_t^m(t)\|_2^2 + \frac{\phi_0 \eta_0}{4} \|\nabla u^m(t)\|_2^2 \leq E(u_0, u_1) + C_f|\Omega| + \frac{1}{\lambda_1 \phi_0 \eta_0} \|h\|_2^2 \equiv C_0,
\]

(23)

for all \( t \in [0, t_m] \). This estimate allows us to extend solution \( u^m \) to interval \([0, T]\). Moreover, since (23) implies that \( \|\nabla u^m(t)\|_2^2 \in [0, \frac{\lambda_1 \phi_0 \eta_0}{4}] \) and \( \sigma \) is a strictly positive continuous function on \( \mathbb{R}^+ \), then there exists a constant \( \sigma_0 = \sigma_0(\|(u_0, u_1)\|_H) > 0 \) such that

\[
\sigma \left( \|\nabla u^m(t)\|_2^2 \right) \geq \sigma_0 > 0, \quad \forall \ t \in [0, T].
\]

(24)

Then, going back to (22) we gain

\[
\frac{1}{2} \|u_t^m(t)\|_2^2 + \frac{\phi_0 \eta_0}{4} \|\nabla u^m(t)\|_2^2 + \frac{\sigma_0 \kappa_1}{\gamma + 1} \int_0^t \|u_t^m(s)\|_{\gamma + 2}^2 \, ds \leq C_0,
\]

(25)

for all \( t \in [0, T] \) and \( m \in \mathbb{N} \). Therefore, from (23) and (25) we have

\[
\begin{align*}
(u^m) & \text{ is bounded in } L^\infty(0, T; H^1_0(\Omega)), \\
(u_t^m) & \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \\
(u_{tt}^m) & \text{ is bounded in } L^{\gamma+2}(0, T; L^{\gamma+2}(\Omega)).
\end{align*}
\]

(26) (27) (28)

**A Priori Estimate II.** We first set

\[
F(t) = \frac{1}{2} \left[ \|\nabla u^m(t)\|_2^2 + \phi(\|\nabla u^m(t)\|_2^2) \|\Delta u^m(t)\|_2^2 \right] - \int_\Omega \nabla h \nabla u^m(t) \, dx
\]

and

\[
F(t) = F(t) + \frac{1}{\phi_0 \lambda_1} \|\nabla h\|_2^2.
\]

From condition (9) and Hölder inequality, we have

\[
\int_\Omega \nabla h \nabla u^m(t) \, dx \leq \frac{1}{\lambda_1^{1/2}} \|\nabla h\|_2 \|\Delta u^m(t)\| \leq \frac{1}{\phi_0 \lambda_1} \|\nabla h\|_2^2 + \frac{\phi_0}{4} \|\Delta u^m(t)\|_2^2.
\]

Hence

\[
F(t) \geq \frac{1}{2} \|\nabla u^m(t)\|_2^2 + \frac{\phi_0}{4} \|\Delta u^m(t)\|_2^2.
\]

(29)

Since \( h = h(x) \), we have \( \frac{d}{dt} F(t) = \frac{d}{dt} F(t) \). Thus, replacing \( \omega_j \) by \(-\Delta u_{ij}^m(t)\) in the approximate equation (15) yields

\[
\frac{d}{dt} F(t) + I_1 = I_2 + I_3,
\]

(30)

where

\[
\begin{align*}
I_1 &= \sigma(\|\nabla u^m(t)\|_2^2) \int_\Omega g(u_t^m(t))(-\Delta u_t^m(t)) \, dx, \\
I_2 &= \phi(\|\nabla u^m(t)\|_2^2)(\nabla u^m(t), \nabla u_t^m(t)) \|\Delta u^m(t)\|_2^2, \\
I_3 &= \int_\Omega f(u^m(t))(\Delta u_t^m(t)) \, dx.
\end{align*}
\]
Integrating by parts, using assumption (13) and estimate (24), we have
\[
\mathcal{I}_1 = \sigma(\|\nabla u_m(t)\|_2^2)^{3/2} \sum_{i=1}^{n} \int_{\Omega} \frac{\partial}{\partial x_i} \left( g(u_i^m(t)) \right) \frac{\partial u_i^m(t)}{\partial x_i} \, dx
\]
\[
= \sigma(\|\nabla u_m(t)\|_2^2)^{3/2} \sum_{i=1}^{n} \int_{\Omega} g'(u_i^m(t)) \left( \frac{\partial u_i^m(t)}{\partial x_i} \right)^2 \, dx
\]
\[
\geq \sigma_0 \kappa_1 \sum_{i=1}^{n} \int_{\Omega} \left( u_i^m(t) \right)^{\frac{m}{2}} \frac{\partial u_i^m(t)}{\partial x_i} \left( \frac{\partial u_i^m(t)}{\partial x_i} \right) \, dx
\]
\[
= \frac{\sigma_0 \kappa_1}{(2 + 1)^2} \sum_{i=1}^{n} \int_{\Omega} \left( \frac{\partial}{\partial x_i} \right)^2 \left( u_i^m(t) \right)^{\frac{m}{2}} \, dx.
\]
Using again that \( \phi \in C^1 \) we have, observing from (25),
\[
\phi'(\|\nabla u_m(t)\|_2^2) \leq \max_{0 \leq \lambda \leq \frac{\Omega}{\phi_{\Omega}}} \phi'(\lambda) := \phi_1, \quad \forall \ t \in [0, T].
\]
Thus, from (23) and (29)
\[
\mathcal{I}_2 \leq \phi_1 \|\nabla u_m(t)\|_2 \|\nabla u_m(t)\|_2 \|\Delta u_m(t)\|_2^2
\]
\[
\leq \phi_1 \left[ \frac{2C_0^{1/2}}{(\phi_0 \eta_0)^{1/2}} \right] 2^{1/2} F^{1/2}(t) \left[ \frac{4}{\phi_0} \right] F(t)
\]
\[
\leq \frac{2^{7/2} \phi_1 C_0^{1/2} F(t)^{3/2}}{\phi_0^{3/2} \eta_0^{1/2}}.
\]
Integrating by parts, using assumption (10), Hölder inequality with \( \frac{\rho}{2(\rho+1)} + \frac{1}{2} = 1 \), embedding \( W^{m+1,2}(\Omega) \to W^{m,2(\rho+1)}(\Omega) \) with \( m = 0 \) and \( m = 1 \), and inequality (29), we deduce
\[
\mathcal{I}_3 = - \sum_{i=1}^{n} \int_{\Omega} f'(u(t)) \frac{\partial u(t)}{\partial x_i} \frac{\partial u(t)}{\partial x_i} \, dx
\]
\[
\leq C_f' \sum_{i=1}^{n} \int_{\Omega} (1 + |u(t)|^\rho) \left| \frac{\partial u(t)}{\partial x_i} \right| \left| \frac{\partial u(t)}{\partial x_i} \right| \, dx
\]
\[
\leq 2C_f'(\|\Omega\|_p^{\frac{\rho}{2(\rho+1)}} + \|u(t)\|_p^\rho) \|u(t)\|_W^{1,2(\rho+1)}(\Omega) \|\nabla u(t)\|_2
\]
\[
\leq 2C_f'(\|\Omega\|_p^{\frac{\rho}{2(\rho+1)}} + C_{1,\rho}^p \|\nabla u(t)\|_2^p) C_{2,\rho} \|u(t)\|_W^{2,2}(\Omega) \|\nabla u(t)\|_2
\]
\[
\leq 2C_f'C_{2,\rho} \left[ \|\Omega\|_p^{\frac{\rho}{2(\rho+1)}} + C_{1,\rho}^p \left( \frac{2C_0^{1/2}}{(\phi_0 \eta_0)^{1/2}} \right)^\rho \right] \mu \|\Delta u(t)\|_2 \|\nabla u(t)\|_2
\]
\[
\leq 2 \mu_1 \left[ \|\Omega\|_p^{\frac{\rho}{2(\rho+1)}} + C_{1,\rho}^p \left( \frac{2C_0^{1/2}}{(\phi_0 \eta_0)^{1/2}} \right)^\rho \right] \left[ 2^{1/2} F^{1/2}(t) \right] \left[ \frac{2}{\phi_0^{1/2}} \right] F(t),
\]
where \( \mu_1 > 0 \) is the embedding constant for \( \|u\|_W^{2,2}(\Omega) \leq \mu_1 \|\Delta u\|_2 \) once we have an equivalence of the norms \( \|\cdot\|_W^{2,2}(\Omega) \) and \( \|\Delta \cdot\|_2 \) in \( H^2(\Omega) \cap H_0^1(\Omega) \).
Replacing $I_1, I_2, I_3$ in (30) we infer
\[
\frac{d}{dt} F(t) + \sum_{i=1}^{n} \int_{\Omega} \left( \frac{\partial}{\partial x_i} (|u_i^m(t)|^2 u_i^m(t)) \right) dx \leq \varrho_0 F(t) + \varrho_1 F(t)^{3/2},
\]
where \( \varrho_0 = \frac{2^{5/2} C_\mu C_\rho \mu_1}{\phi_0^{1/2}} \left[ |\Omega| C^{5/2} + C_\rho^2 \left( \frac{2C_\mu^{3/2}}{(\phi_0^{1/2})^{1/2}} \right)^{\phi_0} \right] \) and \( \varrho_1 = \frac{2^{7/2} \phi_0^3}{\phi_0^{1/2} \phi_0^{1/2}} \).

Multiplying (31) by \( e^{-\varrho_0 t} \), using that \( \chi \geq 0 \) and \( e^{-\varrho_0 t} \leq 1 \), we obtain
\[
\frac{d}{dt} (F(t)e^{-\varrho_0 t}) \leq \varrho_1 F(t)^{3/2}.
\]
Integrating the above inequality \( 0 \) to \( t \leq T \) we arrive at
\[
F(t) \leq \varrho_2 + \varrho_3 \int_0^t F(s)^{3/2} ds,
\]
where
\[
\varrho_2 = F(0)e^{\varrho_0 T} \quad \text{and} \quad \varrho_3 = \varrho_1 e^{\varrho_0 T}.
\]

Now we define the functional \( \Psi(t) = \int_0^t F(s)^{3/2} ds \). Then, it follows that \( \Psi(0) = 0 \) and \( \Psi'(t) = F(t)^{3/2} \), from where we see
\[
\frac{\Psi'(t)}{(\varrho_2 + \varrho_3 \Psi(t))^{3/2}} \leq 1.
\]
Choosing \( T \) such that \( (\varrho_2 + \varrho_3 \Psi(T))^{3/2} \leq C_1 \), we obtain from (32) the following estimate
\[
F(t) \leq C_1, \quad \forall \ t \in [0, T].
\]

Therefore, from (29), we conclude
\[
\frac{1}{2} \| \nabla u_i^m(t) \|^2 + \frac{\phi_0}{4} \| \Delta u_i^m(t) \|^2 \leq C_1, \quad \forall \ t \in [0, T].
\]

Finally, estimates (31) and (33) imply
\[
(u^m) \text{ is bounded in } L^\infty(0, T; H^2(\Omega)),
\]
\[
(u^m) \text{ is bounded in } L^\infty(0, T; H^1_0(\Omega)),
\]
\[
\partial_{x_i} (|u_i^m| \mathcal{T}_i u_i^m) \text{ is bounded in } L^2(0, T; L^2(\Omega)).
\]

**Passage to the limit.** From (26)-(28) and (34)-(36) we can extract a subsequence of \((u^m)\), still denoted by \((u^m)\), such that
\[
u^m \rightharpoonup u \text{ weakly in } L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)),
\]
\[
u_i^m \rightharpoonup u_i \text{ weakly in } L^\infty(0, T; H^1_0(\Omega)),
\]
\[
u_i^m \rightharpoonup u_i \text{ weakly in } L^\frac{7+2}{5+2}(0, T; L^\frac{7+2}{5+2}(\Omega)),
\]
\[|u_i^m| \mathcal{T}_i u_i^m \rightharpoonup \xi \text{ weakly in } L^2(0, T; H^1_0(\Omega)).
\]

According to Lemma 1.3 in Lions [13] we have
\[\xi = |u_i| \mathcal{T}_i u_i.\]

The above convergence are sufficient to pass the limit in the approximate problem (15). Unless the nonlocal nonlinear damping term, which will be analyzed as follows, the remaining limits are standard.
Analysis of the nonlocal nonlinear term $\sigma(\|\nabla u^m(t)\|_2^2)g(u^m)$. Using compact embedding $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow H^1_0 \hookrightarrow L^2(\Omega)$ and since 

$$(u^m) \text{ is bounded in } L^\infty(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \hookrightarrow L^{\gamma+2}(0,T;H^2(\Omega) \cap H^1_0(\Omega)), $$

$$(u_t^m) \text{ is bounded in } L^\infty(0,T;H^1_0(\Omega)) \hookrightarrow L^{\gamma+2}(0,T;H^1_0(\Omega)), $$

then from Aubin-Lions theorem (cf. Lions [13]) there exists a subsequence of $(u^m)$, still denoted by $(u^m)$, such that

$$u^m \rightarrow u \text{ strongly } L^{\gamma+2}(0,T;H^1_0(\Omega)), \quad (37)$$

In addition, since $\sigma \in C^1(\mathbb{R}^+)$ we note

$$|\sigma(\|\nabla u^m(t)\|_2^2) - \sigma(\|\nabla u(t)\|_2^2)| \leq C\|\nabla u^m(t) - \nabla u(t)\|_2.$$ 

Thus, from convergence (37) we have

$$\int_0^T |\sigma(\|\nabla u^m(t)\|_2^2) - \sigma(\|\nabla u(t)\|_2^2)|^{\gamma+2}dt \leq C\int_0^T \|\nabla u^m(t) - \nabla u(t)\|_2^{\gamma+2}dt \rightarrow 0.$$ 

This implies in the next limit

$$\sigma(\|\nabla u^m(t)\|_2^2) \rightarrow \sigma(\|\nabla u(t)\|_2^2) \text{ em } L^{\gamma+2}(0,T). \quad (38)$$

On the other hand, using that $g(0) = 0$, the Mean Value Theorem and (13), we obtain

$$|g(u_t^m)| = |g'(\theta u_t^m)u_t^m| \leq \kappa_2[1 + |u^m_t|^\gamma]|u^m_t| \text{ for some } \theta \in (0,1).$$

Then, using Hölder inequality, and since $(u^m_t)$ is bounded in $L^{\gamma+2}(0,T;L^{\gamma+2}(\Omega))$ from (28), and also $L^{\gamma+2}(\Omega) \hookrightarrow L^{\frac{2\gamma+2}{\gamma+1}}(\Omega)$, we get

$$\int_0^T \int_\Omega |g(u_t^m(t))|^{\frac{2\gamma+2}{\gamma+1}}dxdt \leq \kappa_2 \int_0^T \int_\Omega (|u_t^m(t)| + |u^m_t(t)|^{\gamma+1})^{\frac{2\gamma+2}{\gamma+1}}dxdt$$

$$\leq 2^{\frac{\gamma+2}{\gamma+1}}\kappa_2 \int_0^T \int_\Omega (|u_t^m(t)|^{\frac{2\gamma+2}{\gamma+1}} + |u^m_t(t)|^{\gamma+2})^{\frac{1}{\gamma+1}}dxdt$$

$$\leq 2^{\frac{\gamma+2}{\gamma+1}}\kappa_2 ||T||^{\frac{1}{\gamma+1}} \left( \int_0^T \int_\Omega |u_t^m(t)|^{\gamma+2}dxdt \right)^{\frac{1}{\gamma+1}}$$

$$+ 2^{\frac{\gamma+2}{\gamma+1}}\kappa_2 \int_0^T \int_\Omega |u_t^m(t)|^{\gamma+2}dxdt < \infty,$$

from where it follows that

$$g(u_t^m) \text{ is bounded in } L^{\frac{2\gamma+2}{\gamma+1}}(0,T;L^{\frac{2\gamma+2}{\gamma+1}}(\Omega)) = [L^{\gamma+2}(0,T;L^{\gamma+2}(\Omega))]'.$$

Then, using that $g$ is $C^1$ and uniqueness of the weak limit is easy to see that

$$g(u_t^m) \rightarrow g(u_t) \text{ weak in } L^{\frac{2\gamma+2}{\gamma+1}}(0,T;L^{\frac{2\gamma+2}{\gamma+1}}(\Omega)) = [L^{\gamma+2}(0,T;L^{\gamma+2}(\Omega))]'. \quad (39)$$

From (39) we have

$$\int_0^T \int_\Omega g(u_t^m(t))wdxdt \rightarrow \int_0^T \int_\Omega g(u_t(t))wdxdt, \quad \forall \ w \in L^{\gamma+2}(0,T;L^{\gamma+2}(\Omega)).$$

Taking $w = v\theta$, $\theta \in L^{\gamma+2}(0,T)$, $v \in L^2(\Omega)$, then

$$\int_0^T \left( \int_\Omega g(u_t^m(t))vdx \right) \theta dt \rightarrow \int_0^T \left( \int_\Omega g(u_t(t))vdx \right) \theta dt,$$
and so
\[
(g(u^n_0(t)), v) \to (g(u_t(t)), v) \quad \text{weak in} \quad L^{\frac{2+n}{n+1}}(0, T).
\] (40)
From convergence (38) and (40) we conclude
\[
\int_0^T \sigma(\|u^m(t)\|_2^2)(g(u^n_0(t)), v)dt \to \int_0^T \sigma(\|u(t)\|_2^2)(g(u_t(t)), v)dt.
\]
Therefore, using standard arguments, we can pass to the limit to obtain a function
\[
u_{tt} = -\phi(\|\nabla u(t)\|_2^2)\Delta u - \sigma(\|\nabla u(t)\|_2^2)g(u_t) - f(u) + h \quad \text{in} \quad \mathcal{D}'(\Omega \times (0, T)).
\]
Since
\[
\phi(\|\nabla u(t)\|_2^2)\Delta u \in L^\infty(0, T; L^2(\Omega)) \hookrightarrow L^{\frac{n+2}{n+1}}(0, T; L^{\frac{n+2}{n+1}}(\Omega)),
\]
\[
f(u) \in L^\infty(0, T; L^2(\Omega)) \hookrightarrow L^{\frac{n+2}{n+1}}(0, T; L^{\frac{n+2}{n+1}}(\Omega)),
\]
\[
\sigma(\|\nabla u(t)\|_2^2)g(u_t) \in L^{\frac{n+2}{n+2}}(0, T; L^{\frac{n+2}{n+2}}(\Omega)),
\]
we deduce that
\[
\nu_{tt} \in L^{\frac{n+2}{n+2}}(0, T; L^{\frac{n+2}{n+2}}(\Omega)).
\]
Hence,
\[
\nu_{tt} - \phi(\|\nabla u(t)\|_2^2)\Delta u + \sigma(\|\nabla u(t)\|_2^2)g(u_t) + f(u) = h \in L^{\frac{n+2}{n+2}}(0, T; L^{\frac{n+2}{n+2}}(\Omega)).
\]
Continuous dependence. We introduce the following notation
\[
\Psi_M := M(\|\nabla u(t)\|_2^2) - M(\|\nabla u(t)\|_2^2).
\]
Taking two solutions \(z_1 = (u, u_t), z_2 = (v, v_t)\) of (1)-(2) corresponding to initial data \(z_1(0) = (u_0, u_1), z_2(0) = (v_0, v_1) \in \mathcal{H}_1\), respectively, then function \(w = u - v\) verifies
\[
w_{tt} - \phi(\|\nabla u(t)\|_2^2)\Delta w + \sigma(\|\nabla u(t)\|_2^2)(g(u_t) - g(v_t))
\]
\[= -\Psi_\sigma(g(v_t)) + \Psi_\phi \Delta v - (f(u) - f(v)).\] (41)
Taking the multiplier \(w_1\) in (41) we infer
\[
\frac{1}{2} \frac{d}{dt} \mathcal{N}(t) + \sigma(\|\nabla u(t)\|_2^2) \int_\Omega (g(u_t(t)) - g(v_t(t)))w_1(t)dx
\]
\[= -\Psi_\sigma \int_\Omega g(v_t(t))w_1(t)dx + \phi'(\|\nabla u(t)\|_2^2)(\nabla u(t), \nabla u_t(t))\|\nabla w(t)\|_2^2
\]
\[+ \Psi_\phi \int_\Omega \Delta v(t)w_1(t)dx - \int_\Omega (f(u(t)) - f(v(t)))w_1(t)dx,
\] (42)
where
\[
\mathcal{N}(t) = \|w_2(t)\|_2^2 + \phi(\|\nabla u(t)\|_2^2)\|\nabla w(t)\|_2^2.
\]
From assumption (9) it easy to see that
\[
\mathcal{N}(t) \sim \|w_2(t)\|_2^2 + \|\nabla w(t)\|_2^2.
\] (43)
Now, from Mean Value Theorem there exists \(\theta \in (0, 1)\) such that
\[
g(u_t) - g(v_t) = \int_0^1 g'(\theta u_t + (1 - \theta)v_t)d\theta w_t.
\] (44)
From assumption (13) and a direct computation of the integral show
\[
\int_0^1 g'(\theta u_t + (1 - \theta)v_t)d\theta \geq \kappa_1 \int_0^1 |\theta u_t + (1 - \theta)v_t|^\gamma d\theta \geq C_{\kappa_1, \gamma} |u_t|^\gamma + |v_t|^\gamma.
\] (45)
Combining (44) and (45), and since (24) also holds with $u$ instead of $u^m$, we have
\[
\sigma(\|\nabla u(t)\|^2_2) \int_\Omega (g(u_t(t)) - g(v_t(t)))w_t dx \\
\geq\sigma_0 C_{k,\gamma} \int_\Omega \left( |u_t(t)|^\gamma + |v_t(t)|^\gamma \right) w_t^2(t) dx.
\] (46)

Now let us estimate the terms on the right side of (42). Initially, since $\sigma \in C^1$, then applying the Mean Value Theorem and (13) one obtains
\[
-\Psi_\sigma \int_\Omega g(v_t(t)) w_t(t) dx \leq C \|\nabla w(t)\|_2 \int_\Omega |v_t(t)||w_t(t)| dx \\
+ C \|\nabla w(t)\|_2 \int_\Omega |\gamma v_t(t)|^\gamma \|w_t(t)\| dx \\
\leq C \left( \|w_t(t)\|_2^2 + \|\nabla w(t)\|_2^2 \right) + \chi,
\] (47)

where
\[
|\chi| = C \|\nabla w(t)\|_2 \int_\Omega |v_t(t)|^{\gamma+1} \|w_t(t)\| dx \\
\leq C \|v_t(t)\|^{\gamma+2} \|\nabla w(t)\|_2^2 + \epsilon \int_\Omega \left( |u_t(t)|^\gamma + |v_t(t)|^\gamma \right) w_t^2(t) dx.
\] (48)

Further, using that $\phi \in C^1$ and applying again Mean Value Theorem one has
\[
\phi'(\|\nabla u(t)\|^2_2)(\nabla u(t), \nabla u_t(t)) \|\nabla w(t)\|_2^2 \leq K_0 \|\nabla w(t)\|_2^2
\] (49)

and
\[
\Psi_\phi \int_\Omega \Delta w_t dx \leq K_1 \left( \|w_t(t)\|_2^2 + \|\nabla w(t)\|_2^2 \right),
\] (50)

where $K_0 = K_0(\|(u_0, u_1)\|_{H_1})$ and $K_1 = K_1(\|(u_0, u_1)\|_{H_1})$ are constants depending on the strong initial data.

Last, using (10), the generalized Hölder inequality with $\frac{\rho}{2(\rho+1)} + \frac{1}{2} = 1$, Young inequality and embedding $H^1_0(\Omega) \hookrightarrow L^2(\rho+1)(\Omega)$, one sees
\[
- \int_\Omega (f(u) - f(v)) w_t dx \\
\leq 2 C_f \left( \|u\|_{2(\rho+1)}^\rho + \|v\|_{2(\rho+1)}^\rho \right) \|w\|_{2(\rho+1)} \|w_t\|_2 \\
\leq K_2 \left( \|w_t(t)\|_2^2 + \|\nabla w(t)\|_2^2 \right),
\] (51)

where $K_2$ depends only on the weak initial data.

Replacing (46)-(51) in (42), choosing $\epsilon = \frac{\sigma_0 C_{k,\gamma}}{2} > 0$ and using (43), we arrive at
\[
\frac{1}{2} \frac{d}{dt} \mathcal{N}(t) + \frac{\sigma_0 C_{k,\gamma}}{2} \int_\Omega \left( |u_t|^\gamma + |v_t|^{\gamma} \right) w_t^2 dx \leq K_3 (1 + \|v_t(t)\|^{\gamma+2}_{\gamma+2}) \mathcal{N}(t),
\] (52)

for any $t \in [0, T]$, where $K_3 = K_3(\|(u_0, u_1)\|_{H_1}) > 0$. From (25) the function $\zeta(t) = K_3(1 + \|v_t(t)\|^{\gamma+2}_{\gamma+2})$ has the property $\zeta \in L^1(0, T)$. Then integrating (52) on $[0, t]$, using Grownall inequality and (43) we conclude
\[
\|z_1(t) - z_2(t)\|_H \leq K_4 e^{K_3 t} \|z_1(0) - z_2(0)\|_H, \quad t \in [0, T],
\] (53)

for some constants $K_3 > 0$ depending on strong initial data and $K_4 > 0$ depending on weak initial data.
Therefore, as a consequence of (53), we conclude that problem (1)-(2) has a unique “local” strong solution \((u, u_t)\). The proof of Theorem 2.1 is now complete.

**Remark 2.** The same conclusion of Theorem 2.1 can be obtained if we replace condition \((H3)\) by the following:

\((H5)\) \(f \in C^2(\mathbb{R})\) is a monotonous nondecreasing function such that \(f(0) = 0\),

\[
f'(s) \geq -\phi_0 \eta \quad \text{and} \quad |f''(s)| \leq C_f \rho (1 + |s|^{\rho-1}), \quad \forall \ s \in \mathbb{R},
\]

where \(C_f \rho > 0\) and \(\rho\) satisfies

\[
\rho \geq 1 \quad \text{if} \quad n = 1, 2 \quad \text{or} \quad 1 \leq \rho \leq \frac{4}{n-2} \quad \text{if} \quad n \geq 3.
\] (55)

It is worth noting that assumption \((H5)\) is stronger than \((H3)\). Here assumption (55) implies that \(H^1_0(\Omega) \hookrightarrow L^{p+2}(\Omega)\). It allows us to work with proper estimates on the second derivative of \(f\) and Hölder inequality, and recover assumption (12) with \(C_f = 0\). The proof relies on similar arguments which shall be given later in Theorem 3.1.

**Remark 3.** In addition to assumption \((H4)\) if also consider either

\[
\gamma \geq 0 \quad \text{if} \quad n = 1, 2 \quad \text{or} \quad 0 \leq \gamma \leq \frac{2}{n-2} \quad \text{if} \quad n \geq 3,
\] (56)

then the solution \(u\) obtained in Theorem 2.1 has the following regularity

\[u \in L^\infty(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap W^{1,\infty}(0, T; H^1_0(\Omega)) \cap W^{2,\infty}(0, T; L^2(\Omega)).\] (57)

This is possible since condition (56) allows us to show that \(g(u_t) \in L^\infty(0, T; L^2(\Omega))\).

3. **Invariant set and global existence.** The energy functional \(E(t)\) associated with problem (1)-(2) is given by

\[
E(t) = \frac{1}{2} \|u_t(t)\|^2_2 + \frac{1}{2} \phi (\|\nabla u(t)\|^2_2) + \int_\Omega \hat{f}(u(t))dx - \int_\Omega hu(t)dx.
\] (58)

We also set the modified energy

\[
\tilde{E}(t) = E(t) + C_f |\Omega| + \frac{1}{\lambda_1 \phi_0 \eta_0} \|h\|^2_2.
\] (59)

**Proposition 1.** Under assumptions of Theorem 2.1, there exists an open bounded set \(B_R \subset \mathcal{H}\) such that if \((u_0, u_1) \in B_R\), we have \((u(t), u_t(t)) \in B_R, 0 \leq t < T,\) where \(u(t)\) is the local in time solution in Theorem 2.1.

**Proof.** First of all, we note that \(\tilde{E}(t)\) is non-negative. Indeed, from assumptions (9), (12) and Hölder inequality, it is easy to see that

\[
\tilde{E}(t) \geq \frac{1}{2} \|u_t(t)\|^2_2 + \frac{\phi_0 \eta_0}{4} \|\nabla u(t)\|^2_2 \geq \vartheta \|(u(t), u_t(t))\|^2_\mathcal{H},
\] (60)

where \(\vartheta = \min \{\frac{1}{2}, \frac{\phi_0 \eta_0}{4}\}\). Taking the multiplier \(u_t\) with (1) and using (13) we obtain

\[
\frac{d}{dt} E(t) + \kappa_1 \sigma_0 \frac{\|u_t\|^2_\mathcal{H}}{\gamma + 1} \int_\Omega |u_t|^{\gamma+2} dx \leq 0.
\] (61)

Let \(R > 0\), we define \(B_R\) by

\[
B_R = \{(u, v) \in \mathcal{H}; \tilde{E}(u, v) < \vartheta R\} \subset \mathcal{H}.
\] (62)
If \((u_0, u_1) \in B_R \cap H_1\), using that \(\frac{d}{dt} E(t) = \frac{d}{dt} \bar{E}(t)\), we have from (60) and (61) that
\[
\|(u(t), u_t(t))\|_H \leq \frac{1}{\varrho} \bar{E}(u(t), u_t(t)) \leq \frac{1}{\varrho} \bar{E}(u_0, u_1) < R, \quad 0 \leq t < T.
\]
Therefore, \((u(t), u_t(t)) \in B_R\) for all \(t \in [0, T)\). This completes the proof of Proposition 1.

**Proposition 2.** Under assumptions of Theorem 2.1, we additionally assume that
\[
\gamma \geq 0 \quad \text{if} \quad n = 1, 2 \quad \text{or} \quad 0 \leq \gamma \leq \frac{4}{n - 2} \quad \text{if} \quad n \geq 3.
\] (63)

Then,
\[
\bar{E}(t) \leq \left[ \frac{\gamma}{2K} (t - 1)^+ + \left( \bar{E}(0) \right)^{-\frac{2}{\gamma}} \right]^{-\frac{\gamma}{2}} + L \quad \text{for} \quad \gamma > 0;
\] (64)
and
\[
\bar{E}(t) \leq \bar{E}(0) \left( \frac{1 + K}{K} \right) e^{-\beta t} + L \quad \text{for} \quad \gamma = 0,
\] (65)
where \(\beta = \ln \left( \frac{1 + K}{K} \right), a^+ = (a + |a|)/2, K = K(\|(u_0, u_1)\|_H) > 0\) and
\[
L = 4 \left[ 2C_f|\Omega| + \frac{1}{\lambda_1 \phi_0 \eta_0} \|h\|_2^2 \right].
\]

**Proof.** Integrating (61) from \(t\) to \(t + 1\), we obtain
\[
\int_t^{t+1} \int_{\Omega} |u_t|^2 \, dx \, ds \leq E(t) - E(t + 1) := W(t)^2.
\] (66)
Then, using Hölder inequality with \(\frac{\gamma}{\gamma + 2} + \frac{2}{\gamma + 2} = 1\) and (66), we have
\[
\int_t^{t+1} \int_{\Omega} |u_t|^2 \, dx \, ds \leq |\Omega|^\frac{\gamma}{\gamma + 2} \left( \int_t^{t+1} \int_{\Omega} |u_t|^2 \, dx \, ds \right)^\frac{2}{\gamma + 2}
\leq \left( \frac{\gamma + 1}{\lambda_1 \phi_0} \right)^\frac{\gamma}{\gamma + 2} W(t)^\frac{2}{\gamma + 2}.
\] (67)
Using (67), from the Mean Value Theorem for integrals, there exist \(t_1 \in [t, t + \frac{1}{4}]\) and \(t_2 \in [t + \frac{1}{4}, t + 1]\) such that
\[
\|u_i(t_i)\|_2^2 \leq \frac{4(\gamma + 1)^\frac{\gamma}{\gamma + 2} |\Omega|^\frac{\gamma}{\gamma + 2}}{(\lambda_1 \phi_0)^\frac{\gamma}{\gamma + 2}} W(t)^\frac{2}{\gamma + 2}, \quad i = 1, 2.
\] (68)
On the other hand, taking the multiplier \(u\) with (1) we get
\[
\phi(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 + \int_{\Omega} f(u) u \, dx + \int_{\Omega} h u \, dx
\] (69)
\[
= \|u_t(t)\|_2^2 - \frac{d}{dt} \int_{\Omega} u_t(t) u(t) \, dx - \sigma(\|\nabla u(t)\|_2^2) \int_{\Omega} g(u(t)) u(t) \, dx.
\]
From assumptions (9) and (12),
\[
\phi(\|\nabla u(t)\|_2^2)\|\nabla u(t)\|_2^2 + \int_{\Omega} f(u(t)) u(t) \, dx
\geq \frac{1}{2} \hat{\phi}(\|\nabla u(t)\|_2^2) + \int_{\Omega} \hat{f}(u(t)) \, dx - C_f |\Omega|.
\] (70)
Combining (59) with (69)-(70) we deduce
\[ \bar{E}(t) \leq \frac{3}{2} \|u(t)\|_2^2 - \frac{d}{dt} \int_\Omega u(t)u(t)dx - \sigma(\|\nabla u(t)\|_2^2) \int_\Omega g(u(t))u(t)dx + \frac{L}{4}. \] (71)

Integrating (71) from \( t_1 \) to \( t_2 \), we obtain
\[ \int_{t_1}^{t_2} \bar{E}(s)ds \leq \frac{3}{2} \int_{t_1}^{t_2} \|u_0(t)\|_2^2ds + \mathcal{L}_1 + \mathcal{L}_2 + \frac{L}{4}, \] (72)
where
\[ \mathcal{L}_1 = -\int_\Omega u_2(t_2)u(t_2)dx + \int_\Omega u_1(t_1)u(t_1)dx, \]
\[ \mathcal{L}_2 = -\int_{t_1}^{t_2} \sigma(\|\nabla u(s)\|_2^2) \int_\Omega g(u(s))u(s)dxds. \]

Let us estimate the right hand side of (72) in terms of \( W(t) \). First, from (68) and (60) we have
\[ |\mathcal{L}_1| \leq \frac{4(\gamma + 1)^{\frac{1}{\gamma+2}}}{(\kappa_1 \sigma_0)^{\frac{1}{\gamma+2}} \lambda_1^{1/2}} W(t) \sup_{t \leq s \leq t+1} \|\nabla u(s)\|_2 \]
\[ \leq C_1 W(t)^{\frac{2}{\gamma+2}} + \epsilon \sup_{t \leq s \leq t+1} \bar{E}(s). \]

Using that \( \sigma \) is continuous we also have
\[ \sigma(\|\nabla u(t)\|_2^2) \leq \max_{0 \leq \lambda \leq \frac{4C_0}{\omega_0}} \sigma(\lambda) = \sigma_1, \]
where \( \sigma_1 = \sigma(\|u_0, u_1\| H) > 0 \). Besides, condition (63) also implies that \( H^1_0(\Omega) \hookrightarrow L^{\gamma+2}(\Omega) \). Thus, from Hölder inequality, (66) and (67) we infer
\[ |\mathcal{L}_2| \leq \sigma_1 \kappa_2 \int_{t_1}^{t_2} \int_\Omega \|u_1\|^{\gamma+1} dxds + \sigma_1 \kappa_2 \int_{t_1}^{t_2} \int_\Omega \|u_1\| dxds \]
\[ \leq \sigma_1 \kappa_2 \left( \int_{t_1}^{t_2} \int_\Omega \|u_1\|^{\gamma+2} dxds \right)^{\frac{\gamma+1}{\gamma+2}} \left( \int_{t_1}^{t_2} \int_\Omega \|u_1\|^2 dxds \right)^{\frac{1}{\gamma+2}} \]
\[ + \sigma_1 \kappa_2 \left( \int_{t_1}^{t_2} \int_\Omega \|u_1\|^2 dxds \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} \int_\Omega \|u_1\|^2 dxds \right)^{\frac{1}{2}} \]
\[ \leq C_3 W(t)^{\frac{2(\gamma+1)}{\gamma+2}} \sup_{t \leq s \leq t+1} \|u(s)\|_2^{\gamma+2} + C_4 W(t)^{\frac{2}{\gamma+2}} \sup_{t \leq s \leq t+1} \|\nabla u(s)\|_2 \]
\[ \leq C_5(\epsilon) \left( W(t)^{\frac{2(\gamma+1)}{\gamma+2}} + W(t)^{\frac{2(\gamma+1)}{\gamma+2}} \right) + \epsilon \sup_{t \leq s \leq t+1} \bar{E}(s). \]

Thereby, inserting in (72) these estimates for \( |\mathcal{L}_1| \) and \( |\mathcal{L}_2| \) we obtain
\[ \int_{t_1}^{t_2} \bar{E}(s)ds \leq C_6 \left( W(t)^{\frac{2}{\gamma+2}} + W(t)^{\frac{2(\gamma+1)}{\gamma+2}} \right) + 2 \epsilon \sup_{t \leq s \leq t+1} \bar{E}(s) + \frac{L}{4}. \] (73)

Using again the Mean Value Theorem there exists \( \tau \in [t_1, t_2] \) such that
\[ \int_{t_1}^{t_2} \bar{E}(s)ds = \bar{E}(\tau)(t_2 - t_1) \geq \frac{1}{2} \bar{E}(\tau + 1), \]
once we have that $\tilde{E}(t)$ is decreasing as well as $E(t)$. Then, from equality $\tilde{E}(t+1) = \tilde{E}(t) - W(t)^2$, we conclude

$$\tilde{E}(t) \leq W(t)^2 + 2\int_{t_1}^{t_2} \tilde{E}(s)ds. \quad (74)$$

Combining (73) and (74) we get

$$\tilde{E}(t) \leq W(t)^2 + 2C_6 \left( \frac{4}{\gamma+2} + W(t)\frac{4(\gamma+1)}{\gamma+2} \right) + 2\epsilon \sup_{t \leq s \leq t+1} \tilde{E}(s) + \frac{L}{2}. $$

Using \( \frac{4}{\gamma+2} < 2 < \frac{4(\gamma+1)}{\gamma+2} \) and taking $\epsilon = \frac{1}{8}$, we obtain

$$\tilde{E}(t) \leq W(t)\frac{4}{\gamma+2} \left( 4C_6 + W(t)\frac{2\gamma}{\gamma+2} + 4C_6W(t)\frac{4\gamma}{\gamma+2} \right) + L. \quad (75)$$

In addition, observing definition of $W(t)$, we infer

$$\left( 4C_6 + W(t)\frac{2\gamma}{\gamma+2} + 4C_6W(t)\frac{4\gamma}{\gamma+2} \right) \leq C_7,$$

where $C_7 = C_7(\|u_0, u_1\|_H) > 0$. Thus, (75) can be rewritten as

$$\tilde{E}(t) \leq C_7W(t)\frac{4}{\gamma} + L,$$

from where we arrive at

$$\tilde{E}(t)^{1+\frac{2}{\gamma}} \leq K(E(t) - E(t+1)) + [L]^{1+\frac{2}{\gamma}}, \quad (76)$$

where $K = C_7^{\frac{\gamma+2}{\gamma}} (\|u_0, u_1\|_H) > 0$. Therefore, applying Nakao’s Lemma (cf. Nakao [20, Lemma 2.1]) we finally conclude

$$\tilde{E}(t) \leq \left[ \frac{\gamma}{2K} (t-1)^{+} + \left( \tilde{E}(0) \right)^{-\frac{2}{\gamma}} \right]^{1+\frac{2}{\gamma}} + L \quad \text{if} \quad \gamma > 0,$$

and

$$\tilde{E}(t) \leq \tilde{E}(0) \left( \frac{1+K}{K} \right) e^{-\beta t} + L \quad \text{if} \quad \gamma = 0,$$

which implies in the estimates (64)-(65). This concludes the proof of Proposition 2. \qed

**Corollary 1.** Under assumptions of Proposition 2 with $h \equiv 0$ and $C_f = 0$ in (12), then the energy $E(t)$ defined in (58) satisfies the following estimates:

(i) if $\gamma > 0$, then there exists a constant $C = C(\|u_0, u_1\|_H) > 0$ such that

$$E(t) \leq C(1+t)^{-\frac{2}{\gamma}} \quad 0 \leq t < T; \quad (77)$$

(ii) if $\gamma = 0$, then there exist constants $C, \beta > 0$ depending on $\|u_0, u_1\|_H$ such that

$$E(t) \leq Ce^{-\beta t}, \quad 0 \leq t < T. \quad (78)$$

**Proof.** The proof is an immediate consequence of Proposition 2 by noting that (76) holds with $L = 0$ and $\tilde{E}(t) = E(t)$ when one takes $h \equiv 0$ and $C_f = 0$. Therefore, conclusions (77)-(78) follows from Nakao’s Lemma (cf. Nakao [18, 19]). \qed
Remark 4. In what follows, under suitable additional assumptions, we are going to show that there exists a “small” invariant set \( I \) such that the local solution given by Theorem 2.1 can be extended to a global one for initial data belonging to \( I \). More precisely, one of the additional assumptions to be considered is of the form \( g'(0) > 0 \). With this hypothesis on \( g \), one sees that it is not superlinear at the origin, which rules out interesting cases such as those considered in \([10, 11]\) for wave models with nonlinear boundary dissipation. However, due to the difficulties caused by the nonlinear nonlocal damping, it seems to be a hard task to apply the same ideas as introduced in \([10, 11]\) on damping \( g \) at the origin in the present problem. Summarizing, the next condition assumed on \( g \) is much weaker than that in \([10, 11]\).

More precisely, we have:

**Theorem 3.1.** Let us assume that \( \phi, \sigma, f \) and \( g \) satisfy the following hypotheses: \( \phi \in C^2(\mathbb{R}^+) \) is a positive monotonous nondecreasing function, \( \sigma \) satisfies \((H2)\), \( f \) fulfills condition \((H5)\), \( g \) satisfies \((H4)\) with \( g'(0) > 0 \) and \( \gamma \) fulfilling \((50)\). Then, there exists an open set \( B \subset H_1 \) such that if \((u_0, u_1) \in B \) the solution \((u(t), u_t(t))\) of the problem \((1)-(2)\) exists on \([0, \infty)\) and \((u(t), u_t(t)) \in B \) for all \( t \geq 0 \).

**Proof.** We first note that assumptions of Theorem 3.1 are enough to conclude all statements made in Theorem 2.1 and Proposition 2. In addition, we perform formally the computations below since they hold for Galerkin approximate solutions and so for their limits.

Let us consider \((u_0, u_1) \in B_R \cap H_1 \) with \( B_R \) given in \((62)\). From Proposition 1 we have \((u(t), u_t(t)) \in B_R, 0 \leq t < T\). Moreover, from \((60)\) we also get

\[
\|\(u(t), u_t(t)\| \leq \frac{1}{g} E(u(t), u_t(t)) < R, \quad \forall \ (u_0, u_1) \in B_R.
\]

Deriving equation \((1)\) with respect to variable \( t \) we have

\[
\begin{align*}
&\frac{d}{dt} \|\(u_0(t), u_t(t)\|_2^2 + \phi(\|\nabla u(t)\|_2^2)\|\nabla u_t(t)\|_2^2 + \int f'(u)u_t^2 dx \\
&+ \sigma(\|\nabla u(t)\|_2^2) \int g'(u)u_t^2 dx - \phi(\|\nabla u(t)\|_2^2)(\nabla u, \nabla u_t)2(\nabla u, \nabla u_t) \delta(t, u_t(t)) = 0.
\end{align*}
\]

Taking the multiplier \( u_{tt} \) with \((79)\) we obtain

\[
\begin{align*}
&\frac{d}{dt} \left[ \|u_{tt}(t)\|_2^2 + \phi(\|\nabla u(t)\|_2^2)\|\nabla u_t(t)\|_2^2 + \int f'(u)u_t^2 dx \right] \\
&+ \sigma(\|\nabla u(t)\|_2^2) \int g'(u)u_t^2 dx - \phi(\|\nabla u(t)\|_2^2)(\nabla u, \nabla u_t)2(\nabla u, \nabla u_t) \delta(t, u_t(t)) \\
&= \phi'(\|\nabla u(t)\|_2^2)(\nabla u, \nabla u_t)\|\nabla u_t(t)\|_2^2 - \sigma'(\|\nabla u(t)\|_2^2)2(\nabla u, \nabla u_t) \int g(u_t)u_{tt} dx \\
&+ \frac{1}{2} \int f''(u)u_t^3 dx.
\end{align*}
\]

Since \( g' \) is continuous and we are additionally assuming that \( g'(0) > 0 \), then there exists \( \delta > 0 \) such that \( g'(s) > 0 \) for all \(|s| < \delta\), which implies in the existence of a constant \( \kappa_3 > 0 \) such that

\[
g'(s) \geq \kappa_3 > 0, \quad \forall \ s \in I_\delta := [-\delta, \delta].
\]

Moreover, using \((13)\) in particular on \( \mathbb{R} \setminus I_\delta \), we also have

\[
g'(s) \geq \kappa_1 |s|^\gamma \geq \kappa_1 |\delta|^\gamma > 0, \quad \forall \ s \in \mathbb{R} \setminus I_\delta.
\]
Thus, taking \( \kappa = \min \{ \kappa_3, \kappa_1 | \delta > \} \), we conclude from the above that
\[ g'(s) \geq \kappa, \quad \forall s \in \mathbb{R}. \] (81)

Now, using again (13), estimate (24) (with \( u \) instead of \( u^m \)) and (81), we obtain
\[ \sigma(\| \nabla u(t) \|^2) \int_{\Omega} g'(u_t)u_t^2 dx \geq \frac{\sigma \alpha_0}{2} \| u_{tt}(t) \|^2 + \frac{\sigma \alpha_1}{2} \int_{\Omega} |u_t|^m (u_{tt})^2 dx \]
\[ = \frac{\sigma \alpha_0}{2} \| u_{tt}(t) \|^2 + \frac{\sigma \alpha_1}{2} \int_{\Omega} \left[ \partial_t (|u_t|^m u_t) \right]^2 dx. \] (82)

In addition, the term \( J_1 := -\phi'(\| \nabla u(t) \|^2) (\nabla u, \nabla u_t) 2(\Delta u, u_{tt}) \) can be rewritten as
\[ J_1 = \phi'(\| \nabla u(t) \|^2) (\nabla u, \nabla u_t) 2(\nabla u, u_{tt}) \]
\[ = \phi'(\| \nabla u(t) \|^2) \frac{d}{dt} (\nabla u, \nabla u_t^2) - 2\phi'(\| \nabla u(t) \|^2) (\nabla u, u_t) (\nabla u_t) \]
\[ = \frac{d}{dt} [\phi'(\| \nabla u(t) \|^2) (\nabla u, u_t^2)] - 2\phi''(\| \nabla u(t) \|^2) (\nabla u, u_t) \]
\[ - 2\phi''(\| \nabla u(t) \|^2) (\nabla u, u_t) \| u_{tt}(t) \|^2. \]

Substituting (82)-(83) in (80) we infer
\[ \frac{1}{2} \frac{d}{dt} E(t) + \frac{\sigma \alpha_0}{2} \| u_{tt}(t) \|^2 + \frac{\sigma \alpha_1}{2} \| u_t \|^2 \int_{\Omega} \left[ \partial_t (|u_t|^m u_t^m) \right]^2 dx \]
\[ \leq 2\phi''(\| \nabla u(t) \|^2) (\nabla u, u_t^2) + 3\phi'(\| \nabla u(t) \|^2) (\nabla u, u_t) \| u_{tt}(t) \|^2 \]
\[ - \sigma'(\| \nabla u(t) \|^2) 2(\nabla u, u_t) \int_{\Omega} g(u_t) u_{tt} dx + \frac{1}{2} \int_{\Omega} f''(u) u_t^2 dx. \]

where
\[ E(t) = \| u_{tt}(t) \|^2 + \phi(\| \nabla u(t) \|^2) \| \nabla u_t(t) \|^2 \]
\[ + 2\phi'(\| \nabla u(t) \|^2) (\nabla u, \nabla u_t)^2 + \int_{\Omega} f'(u) u_t^2 dx. \] (85)

Since \( \phi \) is monotonous nondecreasing we have \( \phi'(\| \nabla u(t) \|^2) (\nabla u, \nabla u_t)^2 \geq 0 \). In addition, from assumptions (9), (54) and (14), and \( \| u \| \leq \frac{1}{\lambda_1^2} \| \nabla u \| \), we have
\[ E(t) \geq \| u_{tt}(t) \|^2 + \phi_0 \| u_{tt}(t) \|^2. \] (86)

In what follows, we estimate the terms on the right side of (84). Firstly, since \( \phi \in C^{2}(\mathbb{R}^+), \sigma \in C^{1}(\mathbb{R}^+) \) and \( (u_0, u_1) \in E_R \), there exists a constant \( C_R > 0 \) such that
\[ \max_{0 \leq \lambda \leq R} \{ |\phi'(\lambda)|, |\phi''(\lambda)|, \sigma(\lambda), |\sigma'(\lambda)| \} \leq C_R. \]

Thus, the term \( J_2 := 2\phi''(\| \nabla u(t) \|^2) (\nabla u, \nabla u_t^2) + 3\phi'(\| \nabla u(t) \|^2) (\nabla u, \nabla u_t) \| u_{tt}(t) \|^2 \)

is estimated as
\[ J_2 \leq 2C_R R^{3/2} \| \nabla u_t(t) \|^2 + 3C_R R^{1/2} \| \nabla u_t \|^2 \]
\[ \leq 2C_R R^{3/2} + 3C_R R^{1/2} \| \nabla u_t(t) \|_2 \frac{1}{\phi_0 \eta_0} E(t) \]
\[ \leq C_{1,R} \| \nabla u_t(t) \|_2 E(t), \]

where \( C_{1,R} = \frac{1}{\phi_0 \eta_0} [2C_R R^{3/2} + 3C_R R^{1/2}] \).
From condition (57), the Mean Value Theorem, assumption (13) and embeddings \(H^1_0(\Omega) \hookrightarrow L^2(\Omega)\) and \(H^1_0(\Omega) \hookrightarrow L^{2(\gamma+1)}(\Omega)\) with \(\|u\|_{2(\gamma +1)} \leq C_\gamma \|\nabla u\|_2\), we can estimate the term \(J_3 := -\sigma'(\|\nabla u(t)\|_2^2) 2(\nabla u, \nabla u_t) \int_\Omega g(u_t) u_{tt} dx\) as follows

\[
J_3 \leq 2C_R \|\nabla u(t)\|_2 \|\nabla u_t(t)\|_2 \int_\Omega |g(u_t)| \|u_{tt}\| dx
\]

\[
\leq 2\kappa_2 C_R R^{1/2} \|\nabla u_t(t)\|_2 \int_\Omega (|u_t| + |u_t|^\gamma) |u_{tt}| dx
\]

\[
\leq 4\kappa_2 C_R R^{1/2} \|\nabla u_t(t)\|_2 \|u_t(t)\|_2 \|\nabla u(t)\|_2 \|u_{tt}\|_2
\]

\[
\leq C_{2,R} \left[ \|\nabla u_t(t)\|_2 + \|\nabla u(t)\|_2 \right] E(t).
\]

where \(C_{2,R} = 4\kappa_2 C_R R^{1/2} \left[ \frac{1}{\lambda_{\gamma/2}} + C_{\gamma+1}^2 \right].\) Finally, from assumption (54), Hölder inequality with \(\frac{p-1}{p+2} + \frac{3}{p+2} = 1\) and embedding \(H^1_0(\Omega) \hookrightarrow L^{p+2}(\Omega)\) with \(\|u\|_{p+2} \leq C_p \|\nabla u\|_2\), we have

\[
\frac{1}{2} (f''(u), u_t^2) \leq \frac{C_{j''}}{2} \int_\Omega (1 + |u|^{p-1}) |u_t|^3 dx
\]

\[
\leq \frac{C_{j''}}{2} \left[ \|\nabla u(t)\|_p + \|u(t)\|_p \right] \|u_t(t)\|_p^3
\]

\[
\leq C_{j''} C_p^3 \left[ \|\nabla u(t)\|_p + \|u(t)\|_p \right] \|\nabla u(t)\|_2^3
\]

\[
\leq C_{j''} C_p^3 \left[ \|\nabla u(t)\|_p + \|u(t)\|_p \right] \|\nabla u(t)\|_2^3
\]

\[
\leq C_{3,R} \|\nabla u_t(t)\|_2 E(t),
\]

where \(C_{3,R} = \frac{C_{j''} C_p^3}{\lambda_{\gamma/2}} \left[ \|\nabla u(t)\|_p + \|u(t)\|_p \right] \). Replacing (87)-(89) in (84) we obtain

\[
\frac{1}{2} \frac{d}{dt} E(t) + \frac{\sigma_0 \mu_0}{2} \|u_{tt}(t)\|_2^2 + \frac{\sigma_0 \mu_1}{2} \left( \frac{\|u(t)\|_2^2}{\|u(t)\|_2^2} \right) \int_\Omega \left[ \partial_t (|u| \nabla u \nabla u_t) \right]^2 dx
\]

\[
\leq C_{4,R} \left[ \|\nabla u_t(t)\|_2 + \|\nabla u(t)\|_2 \right] E(t).
\]

where \(C_{4,R} = \sum_{i=1}^3 C_{i,R}.\) Now, we choose \(\mu > 0\) so that

\[
0 < \mu \leq \mu_0, \quad \text{where} \quad \mu_0 = \min \left\{ \frac{\sigma_0 \mu_1}{2} \left[ \frac{(\lambda_1 \phi_0 \eta_0)_{\gamma/2}^{1/2}}{2 \lambda_1 \phi_0 \eta_0} \right] \right\} > 0.
\]

Then, multiplying the equation (79) by \(\mu u_t\), we have

\[
\frac{d}{dt} [\mu(u_{tt}(t)) + \mu E(t)] = 2\mu \|u_{tt}(t)\|_2^2 - \mu \sigma(\|\nabla u(t)\|_2^2) \int_\Omega g'(u_t) u_{tt} u_t dx
\]

\[
- \mu \sigma'(\|\nabla u(t)\|_2^2) 2(\nabla u, \nabla u_t) \int_\Omega g(u_t) u_t dx.
\]

Then, from assumption (13), embeddings \(H^1_0(\Omega) \hookrightarrow L^2(\Omega), H^1_0(\Omega) \hookrightarrow L^{2(\gamma+1)}(\Omega)\) and Young inequality, we can estimate the term

\[
J_4 := -\mu \sigma(\|\nabla u(t)\|_2^2) \int_\Omega g'(u_t) u_{tt} u_t dx
\]
as follows.

\[ J_4 \leq \mu C_{R\kappa_2} \int_\Omega (1 + |u(t)|^\gamma) |u(t)| u_{tt} \, dx \]
\[ \leq \mu C_{R\kappa_2} \|u(t)\|_2 \|u_{tt}(t)\|_2 + \mu C_{R\kappa_2} \|u(t)\|_{2(\gamma + 1)} \|u_{tt}(t)\|_2 \]
\[ \leq \frac{\mu C_{R\kappa_2}}{\lambda_1^{1/2}} \|\nabla u(t)\|_2 \|u_{tt}(t)\|_2 + \mu C_{R\kappa_2} \gamma \frac{\|u(t)\|_{2\gamma + 1}}{\|u_{tt}(t)\|_2} \|u_{tt}(t)\|_2 \]
\[ \leq \frac{\mu C_{R\kappa_2}}{\lambda_1^{\gamma/2}} E(t)^{1/2} \|u_{tt}(t)\|_2 + \mu \frac{C_{R\kappa_2}}{\lambda_1^{\gamma/2}} \|\nabla u(t)\|_2^2 E(t) \]
\[ \leq \frac{\mu}{2} E(t) + \frac{\mu}{2} \left( \frac{C_{R\kappa_2}}{\lambda_1^{\gamma/2}} \|u_{tt}(t)\|_2 \right)^2 + \mu C_{5,R} \|\nabla u(t)\|_2^2 E(t), \]

where \( C_{5,R} = \frac{C_{R\kappa_2} \gamma + 1}{\lambda_1^{\gamma/2}} > 0 \). In addition, we can estimate

\[ J_5 := -\mu \sigma'(\|\nabla u(t)\|_2^2) 2 \langle \nabla u, \nabla u \rangle \int_\Omega g(u(t)) u_{tt} \, dx \]
as follows

\[ J_5 \leq 2 \mu \kappa_2 C_R \|\nabla u(t)\|_2 \|\nabla u_{tt}(t)\|_2 \int_\Omega \left( 1 + \frac{|u(t)|^\gamma}{\gamma + 1} \right) |u(t)|^2 \, dx \]
\[ \leq 2 \mu \kappa_2 C_R \frac{R^{1/2}}{\lambda_1} \|\nabla u(t)\|_2^3 + \frac{2 \mu \kappa_2 C_R \gamma + 2}{\gamma + 1} \|\nabla u_{tt}(t)\|_2 \frac{R^{1/2}}{\lambda_1} \|\nabla u(t)\|_2^2 \]
\[ \leq \mu C_{6,R} \|\nabla u(t)\|_2^2 E(t) + \mu C_{7,R} \|\nabla u(t)\|_2^2 + \mu C_{8,R} \left( \|\nabla u(t)\|_2 + \sum_{i=0}^\gamma \|\nabla u(t)\|_2^{\gamma + i} \right) E(t), \]

where \( C_{6,R} = \frac{2 \mu \kappa_2 C_R R^{1/2}}{\lambda_1^{\gamma/2}} > 0 \) and \( C_{7,R} = \frac{2 \mu \kappa_2 C_R \gamma + 2}{\gamma + 1} \lambda_1^{\gamma/2} > 0 \).

Going back to (92) and replacing the above estimates for \( J_4 \) and \( J_5 \) we obtain

\[
\frac{d}{dt} \left[ \mu(u_{tt}, u_t) \right] + \frac{\mu}{2} E(t) \leq \mu \left[ 2 + \frac{(C_{R\kappa_2})^2}{2 \lambda_1^{\gamma/2}} \right] \|u_{tt}(t)\|_2^2 + \mu C_{8,R} \left( \|\nabla u(t)\|_2 + \sum_{i=0}^\gamma \|\nabla u(t)\|_2^{\gamma + i} \right) E(t) \leq 0, \]

where \( C_{8,R} = \sum_{i=5}^\gamma C_{i,R} \). Combining (90) and (93) we arrive at

\[
\frac{1}{2} \frac{d}{dt} E^\mu(t) + \frac{\mu}{4} E(t) + \left[ \frac{\sigma_0 \kappa_2}{2} - \mu \left( 2 + \frac{(C_{R\kappa_2})^2}{2 \lambda_1^{\gamma/2}} \right) \right] \|u_{tt}(t)\|_2^2 \]
\[ + \left[ \frac{\mu}{4} - K_R \left( \|\nabla u(t)\|_2 + \sum_{i=0}^\gamma \|\nabla u(t)\|_2^{\gamma + i} \right) \right] E(t) \leq 0, \]

where \( K_R = C_{5,R} + \mu C_{8,R} \) and

\[ E^\mu(t) = E(t) + 2 \mu \int_\Omega u_{tt} u_t \, dx \]

Now we set

\[ D_\mu := \left\{ (u, v) \in \mathcal{H}_1; \ K_R \left( \|\nabla v\|_2 + \sum_{i=0}^\gamma \|\nabla v\|_2^{\gamma + i} \right) < \frac{\mu}{8} \right\}. \]
We assume that \((u_0, u_1) \in D_\mu\). Then, as long as \((u(t), u_t(t)) \in D_\mu\), we have from (94) and (91) that
\[
\frac{d}{dt} E^\mu(t) + \frac{\mu}{2} E(t) \leq 0. \tag{96}
\]
Besides, from (86) and (95) we get
\[
|E^\mu(t) - E(t)| \leq \frac{2\mu}{\lambda_1^{1/2}} \|u_t(t)\|_2 \|\nabla u_t(t)\|_2 \leq \frac{2^{3/2} \mu}{(\lambda_1 \delta_0 \eta_0)^{1/2}} E(t),
\]
and using (91) we have
\[
\frac{1}{2} E(t) \leq E^\mu(t) \leq \frac{3}{2} E(t). \tag{97}
\]
Returning to (96) we obtain
\[
\frac{d}{dt} E^\mu(t) + \frac{\mu}{3} E^\mu(t) \leq 0,
\]
from where it follows that
\[
E^\mu(t) \leq E^\mu(0) e^{-\frac{\mu}{3} t}.
\]
Applying again (97) one has
\[
E(t) \leq 3E(0) e^{-\frac{\mu}{3} t}.
\]
From embedding \(H_0^1(\Omega) \hookrightarrow L^{\mu+2}(\Omega)\) and \(H_0^1(\Omega) \hookrightarrow L^{2(\gamma+1)}(\Omega)\), it is easy to see
\[
E(0) = E(u_0, u_1) \leq C.
\]
where \(C = C(\|u_0, u_1\|_{H_1})\). Denoting by \(E(t) = E(u(t), v(t))\) we define for \(K > 0\)
\[
B_K = \{(u, v) \in H_1; E(u, v) < K\}. \tag{98}
\]
Note that \(B_K \neq \emptyset\). Indeed, from (85) ones sees that \(E(0,0) = 0 < K\), and so \((0,0) \in B_K\).

The above argument assures that if \((u_0, u_1) \in B_K \cap D_\mu\) and \((u(t), u_t(t)) \in D_\mu\), \(0 \leq t \leq T\), for some \(T > 0\), then \((u(t), u_t(t)) \in B_K\), \(0 \leq t \leq T\).

We shall show that \(B_K \subset D_\mu\) for \(K > 0\) small enough. Indeed, let \((u(t), u_t(t)) \in B_K\), from (86) and (98) we have
\[
\|\nabla u_t(t)\|_2^2 \leq \frac{1}{\phi_0 \eta_0} E(u(t), u_t(t)) \leq \frac{K}{\phi_0 \eta_0}. \tag{99}
\]
Choosing \(K\) small enough, then from definition of \(D_\mu\) and (99) it follows that \((u(t), u_t(t)) \in D_\mu\), from where we deduce that \(B_K \subset D_\mu\). Hence, we obtain \((u(t), u_t(t)) \in B_K\) for all \((u_0, u_1) \in B_K\) as long as the solution \((u(t), u_t(t))\) exists.

Now, from (86)
\[
\|u_{tt}(t)\|_2^2 \leq K. \tag{100}
\]
In addition, since \(\phi(\|\nabla u(t)\|_2^2) \Delta u = u_{tt} + \sigma(\|\nabla u(t)\|_2^2) g(u_t) + f(u) - h\), from (12), assumption on \(g\) and inequality (100), we obtain
\[
\|\Delta u(t)\|_2^2 \leq C_K < \infty. \tag{101}
\]

Therefore, we conclude from (99) and (101) that if \((u_0, u_1) \in B_R \cap B_K\), then \((u(t), u_t(t))\) exists on \([0, \infty)\) and \((u(t), u_t(t)) \in B_R \cap B_K\), \(0 \leq t < \infty\). We finally set \(B = B_R \cap B_K\). Then \(B\) is an open bounded set in \(H_1\) and, if \((u_0, u_1) \in B\) then
\[
(u(t), u_t(t)) \in B, \quad 0 \leq t < \infty.
\]
This completes the proof of Theorem 3.1. \(\square\)
4. A "Local" attractor. In view of Theorems 2.1 and 3.1 we can define a one-parameter operator

\[ S(t) : (u_0, u_1) \mapsto (u(t), u_t(t)), \quad t \geq 0, \]

that is continuous with respect to \( H \)-topology on any bounded set \( B \subset \mathcal{H}_1 \) and enjoys usual semigroup properties like \( S(0) = I \) and \( S(t + \tau) = S(t) \circ S(\tau) \).

In the present section we construct a "local" attractor for \( S(t) \) in a neighborhood of \((0,0)\) in \( \mathcal{H}_1 \). To do so we consider some concepts, originally given in Nakao \cite{21, 22}, as follows.

Let \( H \) and \( \mathcal{H}_1 \) be two Banach spaces and \( S(t) \) be a semigroup such that \( S(t)B \subset B \) for an open set \( B \) in \( H \cap \mathcal{H}_1 \). We assume that \( S(t) \) is continuous on any bounded set \( D \subset B \) with respect to \( H_1 \) topology.

**Definition 4.1.** (Nakao \cite{22}) A bounded set \( \mathfrak{A} \subset B \) is a \((\mathcal{H}_1, H)\) "local" attractor associated with the semigroup \( S(t) \) iff the following conditions are fulfilled:

1. \( \text{dist}_H(S(t)D, \mathfrak{A}) \to 0 \) as \( t \to \infty \) for any bounded set \( D \subset B \), where \( \text{dist}_H(D, \mathfrak{A}) = \sup_{d \in D} \inf_{a \in \mathfrak{A}} \|d - a\|_H \),

2. \( S(t)\mathfrak{A} = \mathfrak{A} \) for any \( t \geq 0 \),

3. \( \mathfrak{A} \) is compact in \( H \).

In the next we set \( H = \mathcal{H}, \mathcal{H}_1 = \mathcal{H}_1 \) and \( S(t) \) the semigroup associated with problem (1)-(2). Now we present our final main result.

**Theorem 4.2.** Let us assume that assumptions of Theorem 3.1 hold. Then, the semigroup \( S(t) \) corresponding to problem (1)-(2) has a \((\mathcal{H}_1, \mathcal{H})\) "local" attractor \( \mathfrak{A} \subset V \), where \( V \) is a neighborhood of \((0,0)\) in \( \mathcal{H}_1 \).

**Proof.** Let us take \((u_0, u_1) \in \mathcal{B} \). Then Theorem 3.1 guarantees that \((u(t), u_t(t)) = S(t)(u_0, u_1)\) exists on \([0, \infty)\), \( S(t)\mathcal{B} \subset \mathcal{B} \) and \( \mathcal{B} \) is a bounded set in \( \mathcal{H}_1 \). Moreover, we observe that \( S(t) \) is continuous on \( \mathcal{B} \) with respect to \( \mathcal{H} \)-topology for any \( t \geq 0 \). Setting the \( \omega \)-limit set

\[ \mathfrak{A} := \omega(\mathcal{B}), \]

then \( \mathfrak{A} \) is not empty since \( \mathcal{H}_1 \) is compactly embedded in \( \mathcal{H} \) and \( \mathcal{B} \) is a bounded set in \( \mathcal{H}_1 \). Finally, using standard arguments as in Nakao and Zhijian \cite{23} (see also Nakao \cite{22}) we conclude that \( \mathfrak{A} \) is a \((\mathcal{H}_1, \mathcal{H})\) "local" attractor for \( S(t) \) which is contained in \( \mathcal{B} \). The proof of Theorem 4.2 is complete by taking \( V = \mathcal{B} \).

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E-mail address: vnarciso@uems.br