BOUNDING THE GENUS OF SUBVARIETIES OF GENERIC HYPERSURFACES FROM BELOW

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Abstract. A second-order invariant of C. Voisin gives a powerful method for bounding from below the geometric genus of a k-dimensional subvariety of a degree-d hypersurface in complex projective n-space. This work uses the Voisin method to establish a general bound, which lies behind recent results of G. Pacienza and Z. Ran.

1. Introduction

This paper does not seek to improve recently established lower bounds of the geometric genus of a subvariety of a generic hypersurface in complex projective space \( \mathbb{P}^n \), but rather to highlight the fundamental role in such bounds played by a new second-order invariant introduced by C. Voisin in [V]. It is Voisin’s invariant which has permitted these recent improvements and the invariant is as interesting in its own right as are its applications. It is our purpose to here to distinguish its role.

The invariant probes the scheme of lines of high contact with a hypersurfaces. It is perhaps a bit surprising that lines play any special role in determining all subvarieties of low genus but it turns out that they play a central role, a role enhanced by the geometric effect that second-order variation of the hypersurface has on the lines of high contact. Roughly the program is as follows. Let \( F \) be a generic homogeneous form of degree \( d \) on \( \mathbb{P}^n \) and let \( X_F \) denote the corresponding hypersurface. One wants to bound from below the geometric genus of any \( k \)-dimensional smooth variety \( Y_F \) can be mapped

\[ f : Y_F \to X_F \]

generically. Such a bound will of course depend on \( n, d, \) and \( k \). A rather straightforward argument by adjunction is available when \( d \) is fairly large with respect to \( n \) but the situation is more delicate for smaller \( d \). As \( d \) decreases, adjunction is not enough to produce global section of the canonical bundle of \( Y_F \). Using this very weakness to advantage, at a general point \( y \in Y_F \) Voisin uses the “possible degeneracy of adjunction” to canonically distinguish a line \( l(y) \) among those passing through a point

\[ x = x(y) \in X_F. \]

She then builds a second-order invariant out of the Lie bracket restricted to a distinguished distribution inside \( T_Y \), a kind of “second fundamental form” with values in \( d \)-forms on \( l(y) \). Using this tool, one shows in many situations, that if \( Y_F \) has low (e.g. zero) genus, \( l(y) \) must have contact order at least \( d \) with \( X_F \) at

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x(y). Adjunction then implies, in the range of values considered, that such a $Y_F$
must actually lie inside the locus of those lines which lie entirely inside $X_F$.

A rather mysterious point in the development of this program is how lines enter the picture in the first place. The idea is the following. A subbundle of $T_X$ over which one has some control is the so-called “vertical tangent space,” that is, the vectors at a point $(x, F) \in X$ corresponding to directions in which $F$ moves but $x$ stays fixed. This is just the space $M^d_x$ of homogeneous forms $G$ vanishing at $x$. Thus the critical issue is the positivity of the bundle $M^d_{X^f}$ whose fiber at $x$ is $M^d_x$.

Since one is studying adjunction relating the canonical bundle of $X^f$ to that of $Y^f$, one studies, for general $y \in Y_F$ and $x = x(y)$, the mapping

$\mu : M^d_x \to \frac{TX|_x}{f_*\left(TY|_y\right)}$

where $Y$ is the versal family of deformations of $Y_F$. A sub-bundle of $M^d_{P^1}$ whose positivity (and hence global generation) is easily controlled is

$P_1 \cdot M^1_{P^1}$

where $P_1$ is a (generically chosen) fixed homogeneous form of degree $d - 1$. However the map

$\frac{M^1_x \cdot P_1}{f_*\left(TX|_x\right)} \to \frac{TY|_y}{f_*\left(TY|_y\right) + \mu(P_1 \cdot M^1_x)}$

is not surjective necessarily. If a second map

$\frac{M^1_x}{f_*\left(TX|_x\right)} \to \frac{TY|_y}{f_*\left(TX|_y\right) + \mu(P_1 \cdot M^1_x)}$

is required to achieve surjectivity, the positivity or global-generation conclusion that can be drawn is weaker. In fact, the large the number of linearly independent $P$’s one needs to the filling up of the image of $[\mathbb{P}]$, the weaker the bounds one gets. On

the other hand, for each additional $P$, the ranks of the successive maps

$M_x^1 \to \frac{TX|_x}{f_*\left(TY|_y\right) + \mu(P_1 \cdot M^1_x + \ldots + P_{s-1} \cdot M^1_x)}$

are non-increasing. Lines enter for $n, d, k$ for which $s$ is large enough that the last two maps in this sequence have rank 1, which turns out to be exactly the range which positivity is insufficient to apply adjunction directly.. In this case one associates to $x = x(y)$ the line $l(y)$ whose ideal is given by the kernel of the (first) rank-1 mapping.

The appearance of the rational mapping

$l : Y \to G$

where $G$ is the Grassmann of projective lines in $\mathbb{P}^n$ allows us to lift the mapping $f$ to a (generically injective) mapping

$g = (l, x, F) : Y \to \Delta := L \times_{\mathbb{P}^n} X$
where
\[
\begin{array}{c}
L \\
\downarrow p \\
G
\end{array}
\xrightarrow{q} \mathbb{P}^n
\]
(2)
is the universal line. This line in turn determines a distinguished (vertical) distribution in \(T_Y\). To describe it, let \(G_x \subseteq G\) denote the variety of lines through \(x\). For \(l \in G\), let \(M^d_l\) denote homogeneous forms of degree \(d\) which vanish on \(l\). Then one can write
\[
T_\Delta|_{(l,x,F)} = T_{G_x}|_l \oplus T_{\mathbb{P}^n}|_x \oplus M^d_x
\]
and consider the subspace
\[
T'|_y := (0 \oplus 0 \oplus M^d) \cap g_* \left( T_Y|_y \right)
\]
of
\[
T_\Delta|_{((y),x(F))}.
\]

The key point which requires Voisin’s second-order construction is the proof that the distribution \(T'\) is integrable. Said another way, one shows that, when \(y\) moves (infinitesimally) so that the derivative of \(F(y)\) is a polynomial vanishing on \(l(y)\), then, in that direction, the line \(l(y)\) is (infinitesimally) stationary. This, the fact that \(Y\) can be taken to be \(GL(n+1)\)-invariant, and a little algebra yields the conclusion that \(l(y)\) must have contact at least \(d\) with \(X_{F(y)}\) at \(x(y)\).

The construction of the second-order invariant which shows the integrability of the distribution \(T'\) goes roughly as follows. Referring to (2), let \(E^d\) denote the bundle
\[
p_* q^* \mathcal{O}_{\mathbb{P}^n}(d)
\]
on \(G\) and let
\[
E^d_{\mathbb{P}^n} \subseteq p^* E^d
\]
denote the sub-bundle whose fiber at \((l,x)\) is given by the set of \(d\)-forms on \(l\) which vanish at \(x\). Now at a general point
\[
(l, x, F) = (l(y), x(y), F(y))
\]
in the image of \(Y\) in \(\Delta\), \(M^d_x\) sits as a summand of \(T_\Delta|_{(l,x,F)}\) and maps to the fiber \(E^d_{(l,x)}\) of \(E^d_{\mathbb{P}^n}\) by evaluation. Voisin constructs her second-order invariant out of the composition
\[
[T', T'] \to E^d_{\mathbb{P}^n}
\]
of Lie bracket in \(g_* T_Y\) with this evaluation map, giving it a beautiful geometric interpretation. This allows the conclusion that the co-rank of (3) is incompatible with previously established bounds unless \(\omega_{\mathbb{P}^n}(3)\) is zero, which in turn implies that the distribution \(T'\) is integrable.

G. Pacienza [P] and Z. Ran [R] have taken the Voisin technique even further and independently produce stronger results than those given in Main Theorem and Corollary below for subvarieties \(Y_F\) such that, for some \(a \geq -1\),
\[
h^0(\omega_{Y_F}(a)) = 0.
\]

Though earlier versions of this manuscript contributed to the stronger results of both authors, its primary justification at this point is its focus on Voisin’s new second-order invariant..
1.1. The formal setting. Before we can state the main theorem, we need to complete the notation we will need throughout the paper. For fixed $d$ and $n$ and variable $r$ let

$$
\begin{align*}
M^r &= H^0(\mathcal{O}_{\mathbb{P}^n}(r)) \\
S &= H^0(\mathcal{O}_{\mathbb{P}^n}(d)) - \{0\}.
\end{align*}
$$

We let

$$
N = \left( \begin{array}{c} n+d \\ d \end{array} \right) = \dim S.
$$

Again as above let

$$
X \subseteq \mathbb{P}^n \times S
$$
denote the universal hypersurface and

$$
\Delta = L \times_{\mathbb{P}^n} X \subseteq G \times \mathbb{P}^n \times S
$$

the incidence variety. We have the commutative diagram of natural maps:

$$
\begin{array}{cccccc}
\Delta & \nearrow \pi & \searrow \rho & \uparrow \pi & \downarrow \rho & \\ \\
L & \nearrow \pi & \searrow \rho & X & \nearrow \pi & \downarrow \rho & \\ \\
G & \nearrow \pi & \searrow \rho & \mathbb{P}^n & \nearrow \pi & \downarrow \rho & S
\end{array}
$$

As above we put

$$
\begin{align*}
M^r_{\mathbb{P}^n} &= \{(x, P) \in \mathbb{P}^n \times M^r : P(x) = 0\} \\
M^r_x &= M^r_{\mathbb{P}^n} |_x \\
M^r_G &= \{(l, P) \in G \times M^r : P|_l = 0\} \\
M^r_l &= M^r_G |_l.
\end{align*}
$$

Then for

$$
E^r = p_* \circ q^* \mathcal{O}_{\mathbb{P}^n}(r)
$$

we have the exact sequence

$$
0 \to M^r_G \to G \times M^r \overset{e^r}{\to} E^r \to 0.
$$

1.2. Versal subvarieties.

**Definition 1.1.** Call we call $Y/S'$ a versal sub-family of $k$-folds of $X/S$ if

$$
Y \subseteq \mathbb{P}^n \times S'
$$
is a smooth and projective family over $S'$ of fiber dimension $k$ admitting a $GL(n + 1)$-action such that, for some étale map

$$
S' \to S,
$$

there is a generically injective, $GL(n + 1)$-equivariant map

$$
f : Y/S' \to X/S.
$$
Here we use the word “versal” to suggest the fact that, by assumption, $Y_F$ deforms with every local deformation of $F$. Also the consideration of $S'/S$ etale is necessitated by Stein factorization. Several of the subvarieties we are studying may occur on the same (general) $X_F$, for example, lines on the general cubic surface. In order that the notation not become even more combersome we shall to refer to any of these components as $Y_F$.

For any integer $a \geq 0$ let
\[
\omega_X (a) = \omega_X \otimes t^* \mathcal{O}_{\mathbb{P}^n} (a)
\]
\[
\omega_Y (a) = \omega_Y \otimes (t \circ f)^* \mathcal{O}_{\mathbb{P}^n} (a).
\]

We will be interested in the case in which
\[
(6) \quad h^0 (\omega_{Y_F} (a)) = 0
\]
where
\[
Y_F
\]
is a fiber of $Y/S'$ lying over a generic $F \in S$.

1.3. The Main Theorem. The purpose of this paper is to prove:

**Theorem 1.1.** Suppose $Y/S'$ is a versal subfamily of $X/S$ of $k$-folds such that (6) holds for some $a \geq 0$. If the inequalities
\[
d + a \geq \max \left\{ \frac{7n - 3k - 3}{4}, \frac{3n - k + 1}{2} \right\}
\]
and
\[
\frac{d(d + 1)}{2} \geq 3n - k - 1
\]
are satisfied, then the image of $Y/S'$ lies inside the locus of lines on $X/S$.

Notice that, if the codimension of $Y$ in $X$ is at most 4 then the first inequality in the theorem becomes
\[
d + a \geq \frac{3n - k + 1}{2}
\]
while, if the codimension is at least 4, it becomes
\[
d + a \geq \frac{7n - 3k - 3}{4}.
\]

1.4. Corollary of the proof of the Main Theorem. Let
\[
\iota_F = (t \circ f|_{Y_F}) : Y_F \to \mathbb{P}^n.
\]

Since $S'/S$ is etale, the natural mapping
\[
(7) \quad T_S|_F \cong T_{S'}|_F \to H^0 (N_{i_F})
\]
composes with
\[
H^0 (N_{i_F}) \otimes \mathcal{O}_{Y_F} \to N_{i_F}
\]
to give a map
\[
(8) \quad \nu : M^d \otimes \mathcal{O}_{Y_F} \to N_{i_F}
\]
which is generically surjective by $GL(n+1)$-equivariance. Considering $M^d$ as a subspace of $T_{\mathbb{P}^n \times S}|_{(x,F)}$, we have for generic $x \in f (Y_F)$ that
\[
(9) \quad F' \in M^d \cap T_Y|_x \Leftrightarrow \nu (F')|_x = 0.
\]

An analysis of the proof of Theorem 1.1 yields the following.
Corollary 1.2. Suppose that \( d(d + 1) \geq 4n + 4 \) and \((\mathbb{F})\) holds for some \( a \geq 0 \). Let \( Y/S' \) be a versal subfamily of \( X/S \) of \((n - 3)\)-folds which does not lie inside the locus of lines on \( X/S \). Then, for generic \( P \in M^{d-1} \), the map
\[
\nu' = \nu|_{P \cdot M^1} : M^1 \otimes \mathcal{O}_{Y_F} \to N_{1,F}
\]
is generically surjective and so \( \nu' \) induces a rational map
\[
\mu : Y_F \to Gr(n - 2, n + 1).
\]
Furthermore
\[
h^0(\omega_{Y_F}(n + 2 - d)) \geq \left( \frac{n + 1}{n - 2} \right) - r
\]
where
\[
(10) \quad r = \dim \left\{ h \in H^0(\mathcal{O}_{Gr(n-2,n+1)}(1)) : h|_{\mu(Y_K)} = 0 \right\}.
\]
For example, if \( X_F \) is a sextic threefold and \( k = 1 \), then
\[
h^0(\omega_{Y_F}) = 10 - r
\]
and so
\[
h^0(\omega_{Y_F}) > 1
\]
unless \( \mu(Y_F) \) is a point.

1.5. Notation for relative tangent spaces. An important piece of notation throughout this paper is that for the “vertical tangent space” \( T_g \) to a smooth (surjective) map \( g : Y \to Z \).

\( T_g \) is defined by the exact sequence
\[
0 \to T_g \to T_Y \to g^*T_Y \to 0.
\]

For a composition
\[
W \xrightarrow{f} Y \xrightarrow{g} Z
\]
where \( g \) and \( g \circ f \) are both smooth (surjective), the Snake Lemma gives the exact sequence
\[
0 \to T_f \to T_{g\circ f} \to f^*T_g \to 0.
\]

Also, for a fibered product of smooth morphisms
\[
X \times_Z Y \xrightarrow{\alpha} Y \xrightarrow{\alpha'} Y
\]
we have isomorphisms
\[
T_{\alpha} \to T_{\alpha \alpha a} \to (a')^*T_{b'}
\]
\[
T_{\alpha'} \to T_{\alpha' \alpha a'} \to a^*T_{b}
\]
so that
\[
T_{\alpha \alpha a} = a^*T_b \oplus (a')^*T_{b'}.
\]

For example referring to \((\mathbb{F})\)
\[
(11) \quad T_{q_0} = \pi^*T_\theta \oplus (t \circ \rho)^* M^d_{\mathbb{A}^n}.
\]
2. Conditions for lifting a versal family \( Y/S' \)

2.1. Positivity results.

**Lemma 2.1.** i) The sheaf
\[
\Omega^h_{\mathbb{P}^n} (h + 1)
\]
is generated by global sections.

ii) \( M^1_{\mathbb{P}^n} = \Omega^1_{\mathbb{P}^n} (1) \)
so the sheaf
\[
\left( \wedge^h M^1_{\mathbb{P}^n} \right) (1)
\]
is generated by global sections for all \( h \geq 0 \).

iii) \( M^r_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n} (1) \)
is generated by global sections for all \( r \geq 0 \).

iv) The sheaf
\[
M^r_G (1)
\]
is generated by global sections for all \( r \geq 0 \).

**Proof.** i) This is a corollary of Mumford’s \( m \)-regularity theorem as follows. By the classical vanishing results of Bott (see [3], page 246),
\[
H^i \left( \Omega^h_{\mathbb{P}^n} (k') \right) = 0
\]
unless
\[
i = h, \ k' = 0 \\
i = 0, \ k' > h \\
i = n, \ k' < h - n
\]
so, in particular, for \( i > 0 \) and \( k' \geq h + 1 - i \). Therefore, by Mumford’s regularity theorem ([M], page 99), the maps
\[
H^0 \left( \Omega^h_{\mathbb{P}^n} (k') \right) \otimes H^0 (\mathcal{O}_{\mathbb{P}^n} (1)) \rightarrow H^0 \left( \Omega^{h}_{\mathbb{P}^n} (k' + 1) \right)
\]
are surjective for \( k' \geq h + 1 \).

ii) The isomorphism
\[
M^1_{\mathbb{P}^n} = \Omega^1_{\mathbb{P}^n} (1)
\]
is immediate from the (dual of the) Euler sequence for the tangent bundle of projective space. Then use i).

iii) Use the surjection
\[
M^{r-1} \otimes M^1_{\mathbb{P}^n} \rightarrow M^r_{\mathbb{P}^n}.
\]

iii) Again it suffices to check the case \( r = 1 \). Using the irreducibility of the action of \( GL(n + 1) \) it suffices to construct a single non-trivial meromorphic section of \( M^1_G \) with simple pole along the zero set of a Plücker coordinate. To do this, for all lines \( l \) not meeting
\[
X_1 = X_2 = 0
\]
the Plücker coordinate \( p_{12} (l) \neq 0 \) so there is, by Cramer’s rule, a unique \( A = X_0 + aX_1 + bX_2 \in M^1_G \) containing \( l \).

\[\square\]
2.2. **Global generation in the vertical tangent space.** Next we wish to study how close
\[ \bigwedge^h M_P^n \]
is to being generated by global sections. Fix \( x \in P^n \). Let \( T \subseteq M^r_x \)
be a subspace of corank \( h \). For each \( s \leq h \), consider the map
\[ \mu_{T,s} : \sum_{m=1}^s t^* M_P^n \cdot P_m \to \frac{M^r_x}{T}. \]
for generically chosen \( P_1, \ldots, P_s \in M^{r-1} \). This map is surjective for \( s = h \) and its
rank \( \gamma(s) \)
must be a strictly increasing function of \( s \) until surjectivity is reached. Furthermore
\[ \gamma(s) - \gamma(s - 1) \]
is a non-increasing function. Let \( s_T \leq h \) denote the smallest \( s \) such that \( \mu_{T,s} \) is
surjective and \( s'_T \leq s_T \) be the smallest value such that
\[ \gamma(s + 1) - \gamma(s) \leq 1. \]
Either
i) \( s'_T = s_T \) and
\[ s_T \leq \frac{h}{2} \]
or
ii). \( s'_T < s_T \) and for generic \( P \in M^{r-1} \)
\[ \nu_{T,P} : M_P^n \to \frac{t^* M_P^n}{T + \text{image} (\mu_{T,s'_T})} \]
has rank 1. Also
\[ h - (s_T - s'_T) = \dim (\text{image} (\mu_{T,s'_T})). \]
**Remark 2.1.** Notice that, by this last equality,
\[ (s_T - s'_T) \geq \text{rank} (\mu_{T,s'_T}) \]
if and only if
\[ \frac{h}{2} \leq (s_T - s'_T). \]
Of course we always have
\[ s_T + s'_T = (s_T - s'_T) + 2s'_T \leq h. \]
**Lemma 2.2.** i) If
\[ s'_T + 1 < s_T, \]
there exists \( l_T \in G \) such that
\[ M^r_{l_T} \subseteq T + \text{image} (\mu_{T,s'_T}). \]
ii) The line \( l_T \) in i) is independent of the choice of \( P_1, \ldots, P_{s'_T} \).
iii) More generally (even when \( s'_T \geq s_T - 1 \), given any \( l \) such that
\[ M^r_{l_T} \subseteq T + \text{image} (\mu_{T,s'_T}), \]
we have
\[ \dim \frac{T}{\cap M_{l_T}} = \dim \frac{T + M_{l_T}^r}{M_{l_T}^r} = r - s_T, \]
and
\[ \dim \frac{M_{l_T}^r}{\cap M_{l_T}^r} = \dim \frac{T + M_{l_T}^r}{T} = \text{rank} (\mu_{T,s_T}) - s_T. \]

Proof. i) The rank of
\[ \nu_P : M^1_{\mathbb{P}^n} \to \frac{T}{\text{image} (\mu_{T,s_T})} \]
is locally constant as we vary generic \( P_s \in M^{r-1} \). Thus
\[ \frac{\partial \nu_P}{\partial P_s} \in \text{Hom} (\text{kernel} (\nu_P), \text{image} (\nu_P)) \]
so that
\[ Q \cdot K_s \subseteq \text{image} (\nu_P) \]
for each \( Q \in M^{r-1} \). So
\[ \frac{K_s \cdot M^{r-1} + T + \text{image} (\mu_{T,s_T})}{T + \text{image} (\mu_{T,s_T})} \subseteq \text{image} (\nu_P). \]

Leaving \( x \) and \( T \) fixed but varying \( P_s \in M^{r-1} \), suppose that \( K_s \) varies. Then differentiating (16) with respect to \( P_s \), constant rank again implies that
\[ \frac{\frac{\partial K_s}{\partial P_s} \cdot M^{r-1} + T + \text{image} (\mu_{T,s_T})}{T + \text{image} (\mu_{T,s_T})} \subseteq \text{image} (\nu_P). \]

Now use that
\[ \text{rank} (\mu_{T,s_T+1}) = 1 \]
so that
\[ K_{P_{s_T+1}} \big|_x = M^1_{l_T} \]
for some line \( l_T \) passing through \( x \). To see that \( l \) does not depend on the (generic) choice of \( P_{s_T+1} \), notice that (17) and \( s_T + 1 < s_T \) imply that
\[ M^1_{l_T} \subseteq K_{P_{s_T+1}} + \frac{\partial K_{P_{s_T+1}}}{\partial P_s} \big|_x \neq M^1_x \]
so that
\[ \frac{\partial K_{P_{s_T+1}}}{\partial P_s} \subseteq M^1_{l_T}. \]

So
\[ \frac{M^{1}_{l_T} \cdot M^{r-1} + T}{T} \subseteq \bigcap_{P_{s_T+1}} \text{image} (\mu_{T,s_T+1}) = \text{image} (\mu_{T,s_T}) \]
since \( \text{image} (\mu_{T,s_T+1}) \) cannot be independent of \( P_{s_T+1} \) and
\[ \dim \frac{\text{image} (\mu_{T,s_T+1})}{\text{image} (\mu_{T,s_T})} = 1. \]
Thus
\[ \frac{M^{1}_{l_T} + T}{T} \subseteq \text{image} (\mu_{T,s_T}). \]
For ii), notice that $l_T$ is the unique line $l$ such that

$$M_r^l \cdot M_r^{-1} \subseteq T + M^1 \cdot P_1 + \ldots + M^1 \cdot P_{s_T'} + M^1 \cdot P_{s_T'+1}$$

and so, for $s \leq s_T'$, we can reverse the roles of $P_s$ and $P_{s_T'}$ in (14) and we must get the same line $l$ defining the kernel. By repeating this argument for each $s \leq s_T'$ we see that the line $l_T$ is also independent of the choice of $P_1, \ldots, P_{s_T'}$.

For iii), let $l$ be any line such that

$$M_r^l + T \subseteq \text{image } (\mu_{T,s_T'}).$$

Let

$$E_r^l = \frac{M_r^l}{M_r^l}$$

and denote the image of $T$ in $E_r^l$ as $\hat{T}$. Now notice that, for $P_s$ generic, the image $M_r^l \cdot P_s$ in

$$(18)$$

has rank 1 for each $s$ until we reach the situation in which

$$M_r^l + \text{image } (\mu_s)$$

generates (18). Since

$$M_r^l \subseteq \text{image } (\mu_{s_T'})$$

and $s_T' \leq s_T$, this cannot happen until $s = s_T$. So

$$s_T = \dim \frac{M_r^l}{T + M_r^l}.$$  

The second equality follows since

$$\frac{\text{image } (\mu_s)}{\text{image } (\mu_{s-1})}$$

must have a codimension 1 subspace generated by

$$\frac{M_r^l + T}{T} \cap \text{image } (\mu_s)$$

for each $s \leq s_T'$.

2.3. Geometry of versal families. We are now ready to apply the results of §2.1 in the case of a versal family $Y/S'$ with

$$r = d$$

$$h = (n - 1) - k$$

$$T = f_*T_{\text{to } f} \subseteq t^*M_{\mathbb{P}^n}|_{(x,F)}$$

for generic $x \in f(Y_F)$. From the exact sequence

$$0 \to T_t \to f^*T_X \to t^*T_{\mathbb{P}^n} \to 0$$

and the surjectivity of

$$T_Y \to (t \circ f)^* T_{\mathbb{P}^n}$$

we have

$$f_*T_{\text{to } f} \subseteq T_t$$

is a subspace of codimension $h = n - 1 - k$. 
First suppose that 
\[ s_T = 1. \]
Since \( Y \) is \( GL(n+1) \)-equivariant, the normal vectors at generic \( x \in f(Y) \) can be chosen in \( M^d_x \) and, since \( s_T = 1 \), we have that, for general \( A_1, \ldots, A_h \in M^1_x \) and for general \( P \in M^{d-1} \),
\[ P \cdot A_1, \ldots, P \cdot A_h \in M^d_x \]
generate
\[ N_f|_x = \frac{M^d}{T}. \]
Then for any \( A \in M^1 \) not vanishing at \( x \), \( P \cdot A \) generates \( M^d/M^d_x \) so, for general \( B_1, \ldots, B_{h+1} \in M^1 \) and general \( P \in M^{d-1} \),
\[ (19) \quad P \cdot B_1, \ldots, P \cdot B_{h+1} \in M^d \]
generate
\[ \frac{M^d}{T}. \]
Referring to (8) let
\[ \nu' : M^1 \otimes \mathcal{O}_{Y_F} \to N_{i_F} \]
be the generically surjective map defined by
\[ \nu' (B) = \nu (P \cdot B). \]
The pointwise kernels of \( \nu' \) define a rational map
\[ \mu : Y_F \to Gr(k+1, M^1) . \]
Under the perfect pairing
\[ \bigwedge^{h+1} M^1 \otimes \bigwedge^{k+1} M^1 \to \bigwedge^{n+1} M^1 = \mathbb{C} \]
induced by wedge product, we have a natural isomorphism
\[ \bigwedge^{h+1} M^1 = \left( \bigwedge^{k+1} M^1 \right)^\vee, \]
By (9), the kernel of the linear map
\[ (20) \quad \bigwedge^{h+1} M^1 \to \text{Hom} \left( f^* \omega_{P^n \times S}|_{Y_F}, \omega_{Y_F} \right) \]
\[ B_1 \wedge \ldots \wedge B_{h+1} \mapsto (\eta \mapsto \langle P \cdot B_1 \wedge \ldots \wedge P \cdot B_{h+1} | \eta \rangle) \]
consists in those elements of \( \left( \bigwedge^{k+1} M^1 \right)^\vee \) which vanish identically on \( \mu (Y_F) \). But the image of (20) actually lies in
\[ \text{Hom} \left( f^* \omega_{P^n \times S}|_{Y_F}, \omega_{Y_F} \left( -(d-1) \right) \right) = \omega_{Y_F} (n+2-d) \]
since all the vectors \( P \cdot B_1, \ldots, P \cdot B_{h+1} \) are tangent to \( X \) at points at which \( P \) is zero. Thus
\[ (21) \quad h^0 (\omega_{Y_F} (n+2-d)) \geq \left( \begin{array}{c} n+1 \\ k+1 \end{array} \right) - r \]
where
\[ (22) \quad r = \dim \left\{ h \in H^0 (\mathcal{O}_{Gr(k+1,n+1)} (1)) : h|_{\mu (Y_F)} = 0 \right\}. \]
We conclude the following.
Lemma 2.3. Suppose that, for generic \( x \in f(Y_F) \),
\[ s_{T \circ f} = 1. \]

Then, for \( r \) as in (22),
\[ h^0(\omega_{Y_F} (n + 2 - d)) \geq \left( \frac{n + 1}{k + 1} \right) - r. \]

2.4. Versal sub-families, first reduction. We next return to the case of arbitrary \( s_T \). From the global generation of (??) we conclude that the map
\[ H^0 \left( (t \circ \rho)^* \bigwedge^h M^r_{P^m} \otimes O_{P^n} (a') \right) \rightarrow \left( \bigwedge^h \frac{t^* M^r_{P^m}}{F} \right) \otimes O_{P^n} (a') \]
is (generically) surjective whenever \( a' \geq s_T \).

Lemma 2.4. Let \( Y/S' \) be a versal sub-family of \( k \)-folds. Suppose that, for generic \( F \in S \),
\[ \left( \bigwedge^{n-1-k} t^* M^d_{P^m} \otimes \omega_X (a) \right)_{X_F} \]
is generated by global sections. Then
\[ H^0 (\omega_{Y_F} (a)) \neq 0. \]

Proof. At generic \( (x, F) \in f(Y) \), there is a vector
\[ \nu \in H^0 \left( \bigwedge^{n-1-k} t^* M^d_{P^m} \otimes \omega_X (a) \right)_{X_F} \]
such that
\[ \left\langle \nu \wedge w \wedge z \left| \omega_X (a) \right|_{(x,F)} \right\rangle \neq 0 \]
where
\[ w \in \bigwedge^N T_S|_F \]
\[ z \in \bigwedge^k f_* T_{Y_F}|_{(x,F)}. \]

But this means that
\[ \left\langle \nu \wedge w \left| \omega_X (a) \right|_{Y_F} \right\rangle \]
gives a non-zero element of
\[ H^0 (\omega_{Y_F} (a)). \]
\[ \square \]

We conclude:

Corollary 2.5. Let
\[ f = (\tilde{x}, \tilde{F}) : Y \rightarrow X \]
be a versal sub-family of \( k \)-folds for which (??) holds.

i) \[ d + a - (n + 1) < s_{T \circ f} \leq h = (n - 1) - k. \]

ii) If \( s^f_{T \circ f} + 1 < s_{T \circ f} \), then the map \( f \) lifts to a \( GL(n+1) \)-equivariant map
\[ g = (\tilde{i}, \tilde{x}, \tilde{F}) : Y \rightarrow \Delta \]
such that the corank of
\[ \varepsilon = e^d \circ g_* : T_{top} \to T_{top} \to \pi^* E_{n}^d \]
is
\[ s_{T_{top}} \]
and, for \( T = T_{top} \) at a general point of \( Y \) we have
\[ \dim \frac{M_{d_T}^d}{T \cap M_{d_T}^d} = \dim \frac{T + M_{d_T}^d}{T} = \text{rank} (\mu_{T,s'_T}) - s'_T. \]

Proof. i) See (23).
ii) Define the lifting \( g \) by
\[ \tilde{l} : Y \to G \]
\[ y \mapsto l_{T_{top}}. \]
and use Lemma 2.2ii). \( \square \)

Remark 2.2. For \( T = T_{top} \) we recall the inequality
\[ d + a - (n + 1) < s_T \leq n - 1 - k \]
coming from Corollary 2.5. Referring to remark 2.1
\[ (s_T - s'_T) \geq \text{rank} (\mu_{T,s'_T}) \Leftrightarrow (s_T - s'_T) \geq \frac{n - 1 - k}{2} \]
and
\[ s_T + s'_T \leq (n - 1 - k). \]
So
\[ s_T - s'_T = 2s_T - (s_T + s'_T) \]
\[ \geq 2s_T - (n - 1 - k) \]
\[ \geq 2(d + a - n) - (n - 1 - k). \]
Combining these inequalities we have that
\[ s_T - s'_T \geq \text{rank} (\mu_{T,s'_T}) \]
whenever
\[ d + a \geq \frac{7n - 3 - 3k}{4}. \]
Also
\[ (s_T - s'_T) \geq 2 \]
whenever
\[ s_T \geq \frac{n - 1 - k}{2} + 1 \]
which is insured by
\[ d + a - n \geq \frac{n - 1 - k}{2} + 1 \]
that is, by
\[ d + a \geq \frac{3n + 1 - k}{2}. \]
Thus the rest of this paper is devoted to proving the assertion of the Main Theorem in the case in which:

**Condition 1:** The map \( f \) lifts to a \( GL(n+1) \)-equivariant map

\[
g = \left( \tilde{I}, \tilde{x}, \tilde{F} \right) : Y \to \Delta.
\]

such that

\[
M^d_I \subseteq T_{tof} + \text{image} \left( \mu_{T_{tof}, s_{T_{tof}}} \right).
\]

**Condition 2:**

\[
(s_{T_{tof}} - s'_{T_{tof}}) \geq \text{rank} \left( \mu_{T_{tof}, s'_{T_{tof}}} \right).
\]

(Notice that (27) implies Condition 2 and (28) implies Condition 1.) The critical point in what follows will be that, under these assumptions, the map

\[
g = \left( \tilde{I}, \tilde{x}, \tilde{F} \right) : Y \to \Delta
\]

generically has the property

\[
(p \circ \pi) M^d_G \cap f_* \left( T_Y \right) \subseteq f_* T_{\pi og}
\]

where as above \( T_{\pi og} \) is the tangent space to the fibers of

\[
\pi \circ g = \left( \tilde{I}, \tilde{x} \right).
\]

Establishing this last property is perhaps the deepest part of the proof. It is here that the Lie bracket computation introduced by Voisin is the central ingredient.

3. **Voisin’s bracket map**

3.1. **The contact isomorphism.** We define

\[
M^d_{kG} \subseteq G \times M'
\]

as the sub-bundle whose fiber at \( l \) is given by those forms which vanish to order \( k \) along \( l \). Let

\[
F (l')
\]

be any local section of the bundle \( M^d_G \) near some fixed \( l \in G \). Differentiating the equation

\[
F (l') = 0,
\]

the restriction of

\[
- \frac{\partial F (l')}{\partial l'} \bigg|_{l'=l},
\]

to the line \( l \) itself depends only on the value of \( F (l) \). So it gives a well defined bilinear map

\[
(29) \quad M^d_G \times T_G \to E^d.
\]

Alternatively we can understand this pairing by viewing

\[
l' = l'_u : l \to \mathbb{P}^n
\]

as a family of maps from the fixed projective line \( l \) with parameter \( u \) (which is given by the identity map at \( u = 0 \)) and differentiating the relation

\[
F (l'_u) \circ l'_u = 0.
\]
We then see that
\[
\lim_{u \to 0} \frac{F(l) \circ l'(u)}{u} = - \frac{\partial F(l'_u)}{\partial u}_{u=0} \circ l.
\]

The map \((29)\) factors through
\[
\frac{\mathcal{M}_G^d}{\mathcal{M}_G^d_2} \otimes T_G \to E^d
\]
and the induced map
\[
\frac{\mathcal{M}_G^d}{\mathcal{M}_G^d_2} \to \text{Hom}(T_G, E^d)
\]
is injective. If we restrict the lifted map to
\[
\frac{\mathcal{M}_q^d}{\mathcal{M}_G^d_2} \to \text{Hom}(T_q, p^* E^d)
\]
it remains injective and the image lies the sub-bundle
\[
E_{P^n}^d = \{ (\alpha, (l, x)) \in p^* E^d : \alpha(x) = 0 \}.
\]

So, by dimension, the induced map
\[
(31)
\]
is an isomorphism which we call the contact isomorphism. We rewrite \((31)\) as a multiplication
\[
\bullet : \frac{\mathcal{M}_G^d}{\mathcal{M}_G^d_2} \otimes T_q \to E_{P^n}^d.
\]

Since
\[
E_{P^n}^r = E^{r-1} \otimes \left( \frac{U}{C \cdot q} \right)^\vee
\]
and
\[
T_q = \text{Hom}\left( \frac{U}{C \cdot q}, \frac{(M^1)^\vee}{U} \right),
\]
we can rewrite \((31)\) as
\[
(33)
\]
and \((32)\) as
\[
\bullet : \frac{\mathcal{M}_G^d}{\mathcal{M}_G^d_2} \otimes \left( U^\perp \right)^\vee \to E^{d-1}.
\]
3.2. The vertical contact distribution. Referring to the exact sequences
\[ 0 \to M_1^d = T_1 \to T_X \to T^{*}_{\mathbb{P}^n} \to 0 \]
we next wish to examine the lift of the distribution
\[ (p \circ \pi)^* M_G^d \subset \pi^* M_{\mathbb{P}^n}^d \subset T^{*}_{\mathbb{P}^n} \]
under maps
\[ g : Y \to \Delta \]
such that \( f = \rho \circ g \) is a versal sub-family.
First we can consider an element
\[ F \in M^d \]
as
\[ dC_{n+1} F = \sum_{i=0}^{n} \frac{\partial F}{\partial X_i} \otimes dX_i \]
\[ \in H^{d_0}(\mathcal{O}_{\mathbb{P}^n}(d-1)) \otimes H^{d_0}(\mathcal{O}_{\mathbb{P}^n}(1)). \]
If we are evaluating \( dC_{n+1} F \) at points of a line \( l \subseteq X_F \) then
\[ dC_{n+1} F|_l \in E^{d-1} \otimes U_{\perp}|_l \]
so we get a map
\[ \delta_{C_{n+1}} : (p \circ \pi)^* M_G^d \to E^{d-1} \otimes U_{\perp}. \]
The associated map
\[ \pi^* (U_{\perp})^{\vee} \otimes (p \circ \pi)^* M_G^d \to E^{d-1} \]
is just the multiplication map given in (34), that is,
\[ \upsilon \otimes F \mapsto \upsilon \cdot F. \]
Suppose now we are given a \( GL(n+1) \)-equivariant map
\[ g : Y \to \Delta \]
\[ y \mapsto (l, \hat{x}, \hat{F}) \]
which is an immersion at \( y \mapsto (l, x, F) \). Recalling (11) inside
\[ T_q \circ \pi \circ g \subseteq (p \circ \pi \circ g)^* M_{\mathbb{P}^n}^d. \]
define the distribution
\[ (36) \quad T' = T_q \circ \pi \circ g \cap (p \circ \pi \circ g)^* M_{\mathbb{P}^n}^d. \]
Recalling (32) and the isomorphism
\[ T_q = \text{Hom}\left( \frac{U}{U \cdot q}, \frac{(M^1)^{\vee}}{U} \right), \]
we have the composition map
\[ T_q \circ \pi \circ g \xrightarrow{\partial} T_x \oplus \pi^* T_q \to T_x \xrightarrow{\pi^*} \pi^* E_{\mathbb{P}^n}^d \xrightarrow{(p \circ \pi)^*} E^{d-1} \otimes \frac{U}{U \cdot q} \]
\[ \tau \mapsto \tilde{F}_*(\tau) \oplus \tilde{l}_*(\tau) \mapsto \tilde{F}_*(\tau) \mapsto \tilde{F}_*(\tau)|_l \]
which we denote as
\[ \varepsilon : T_q \circ \pi \circ g \to \pi^* E_{\mathbb{P}^n}^d. \]
We are finally ready to present the second-order tool and its geometric interpretation which are the central ingredients of everything which follows.

**Lemma 3.1.** For vector fields $\tau, \tau'$ in $T'$, 

$$
\varepsilon ([\tau, \tau']) = \hat{l}_* (\tau') \cdot \hat{F}_* (\tau) - \hat{l}_* (\tau) \cdot \hat{F}_* (\tau').
$$

**Proof.** Once the machinery is set up, the verification of the Lemma is straightforward and most easily checked by an elementary computation in normalized local coordinates. Let $\{y_j\}$ be local coordinates for $Y$ such that

$$
\hat{l}_* (0) = l \quad \quad \hat{F}_* (0) = F.
$$

Suppose that 

$$
x = (1, 0, \ldots, 0) 
$$

$$
l = \{X_2 = \ldots = X_n = 0\}.
$$

Then 

$$
X_i = b_i X_1, \quad i = 2, \ldots, n,
$$

become the local coordinates for a small neighborhood of $l$ in $G_x$. Letting $J$ denote a multi-index for the variables $\{y_j\}$ we write 

$$
\hat{l}_* (\{y_j\}) \mapsto \{b_i (\{y_j\})\} = \{\sum_j b_{i,j} \cdot y^j\}.
$$

Also if $I$ denotes multi-indices for the variables $\{X_2, \ldots, X_n\}$ we write 

$$
\hat{F}_* (\{y_j\}) = \sum_{I,J} y^J \hat{F}_{I,J} X^I
$$

where 

$$
\hat{F}_{I,J} = \hat{F}_{I,J} (X_0, X_1)
$$

is a homogeneous form of degree $d - |I|$. The condition that 

$$
\sum_k a'_k \frac{\partial}{\partial y_k} = \tau' \in T'
$$

is the equation, for all $\{y_j\}$ that 

$$
(37) \quad \sum_{k,I} a'_k \frac{\partial}{\partial y_k} \hat{F}_{0:k} + \sum_k a'_k (1 + \delta_{jk}) \hat{F}_{0:jk} + \sum_{i=2}^n \sum_k a'_k \frac{\partial b_i}{\partial y_j} (0) \cdot \hat{F}_{i:k} = 0.
$$

Applying $\frac{\partial}{\partial y_j}$ to $\hat{F}_*$ and then setting $\{y_j\} = 0$ we obtain 

$$
\frac{\partial a_k'}{\partial y_j} \hat{F}_{0:k} + \sum_k a'_k (1 + \delta_{jk}) \hat{F}_{0:jk} + \sum_{i=2}^n \sum_k a'_k \frac{\partial b_i}{\partial y_j} (0) \cdot \hat{F}_{i:k} = 0.
$$

So for 

$$
\sum_j a_j \frac{\partial}{\partial y_j} = \tau \in T',
$$

we can compute 

$$
[\tau, \tau'] \hat{F} = \sum_j a_j \frac{\partial a'_k}{\partial y_j} \hat{F}_{0:k} - \sum_j a'_j \frac{\partial a_k}{\partial y_j} \hat{F}_{0:k}
$$

at $\{y_j\} = 0$ as 

$$
(38) \quad - \sum_{i=2}^n \sum_{j,k} a_j \frac{\partial b_i}{\partial y_j} (0) \cdot a'_k \hat{F}_{i:k} + \sum_{i=2}^n \sum_{j,k} a'_j \frac{\partial b_i}{\partial y_j} (0) \cdot a_k \hat{F}_{i:k}.
$$
Rewriting (38) as

\[
\sum_{j,k} \sum_{i=2}^{n} \left( a'_k \frac{\partial b_i}{\partial y_k} (0) \cdot a_j \frac{\partial^2 \tilde{F}}{\partial X_i \partial y_j} (0) - a_j \frac{\partial b_i}{\partial y_j} (0) \cdot a'_k \frac{\partial^2 \tilde{F}}{\partial X_i \partial y_k} (0) \right)
\]

and restricting to \( \tilde{l} (0) \) we see that we obtain exactly

\[
\tilde{l} (\tau') \bullet \tilde{F}_* (\tau) - \tilde{l} (\tau) \bullet \tilde{F}_* (\tau').
\]

\[\square\]

**Corollary 3.2.** Suppose that Conditions 1 and 2 above hold. Let \( T' \) be as in the lemma. Then at generic \( y = (l, x, F) \in Y \),

\[
\tilde{l}_* \left( T'|_y \cap M^d_{2l} \right) = 0.
\]

**Proof.** Recall that

\[
\text{codim} (Y, X) = \dim \frac{M^d_T}{T_{q \circ \sigma \circ g}|_y} = n - 1 - k.
\]

Also by Corollary 2.5ii) we have for \( T = T_{q \circ \sigma \circ g}|_y \) that

\[
\dim \left( \varepsilon \left( T_{q \circ \sigma \circ g}|_y \right) \right) = d - s_T.
\]

Suppose \( \tau' \in T_{q \circ \sigma \circ g}|_y \cap \tilde{l}^* M^d_{2l} \). Then for every \( \tau \in T'_y \) the element

\[
\tilde{l}_* (\tau') \bullet \tilde{F}_* (\tau) - \tilde{l}_* (\tau) \bullet \tilde{F}_* (\tau') = \tilde{l}_* (\tau') \bullet \tilde{F}_* (\tau)
\]

lies in the image of \( \varepsilon \). If \( \tilde{l}_* (\tau') \neq 0 \), then the map

\[
\tilde{\psi} : \frac{M^d_T}{M^d_{2l}} = E^d_{p^n} \otimes T'_y \Big|_{(l, x)} \tilde{l}_* (\tau') \rightarrow E^d_{p^n} \Big|_{(l, x)}
\]

is surjective and so

\[
\dim \left( \ker \tilde{\psi} \right) = (n - 1) d - d.
\]

Let

\[
\psi = \tilde{\psi} \big|_{T'_y / T'_y \cap M^d_{2l}}.
\]

But

\[
d - s_T \geq \dim \psi \left( \frac{T'_y}{T'_y \cap M^d_{2l}} \right).
\]

So

\[
d - s_T \geq \dim \left( \frac{T'_y}{T'_y \cap M^d_{2l}} \right) - (n - 1) d - d \]

\[
(n - 1) d - s_T \geq \dim \left( \frac{T'_y}{T'_y \cap M^d_{2l}} \right).
\]

On the other hand from Corollary 2.5ii) we have for \( T = T_{q \circ \sigma \circ g}|_y \) that

\[
\dim \frac{M^d_T}{T'_y} = \text{rank} \left( \mu_{T, s'_T} \right) - s'_T.
\]

(40)
So
\[ \dim \frac{T'_y}{T'_y \cap M^d_{2l}} = \dim \frac{T'_y + M^d_{2l}}{M^d_{2l}} = (n - 1) d - \dim \frac{M^d_l}{T'_y + M^d_{2l}} \]
and therefore by (10) we have
\[ \dim \frac{T'_y}{T'_y \cap M^d_{2l}} \geq (n - 1) d - \left( \text{rank} \left( \mu_{T', s'} \right) - s'_T \right) \cdot \]
Thus
\[ (n - 1) d - s_T \geq (n - 1) d - \text{rank} \left( \mu_{T', s'} \right) + s'_T \]
that is
\[ \text{rank} \left( \mu_{T', s'} \right) \geq s_T + s'_T. \]
But by (13)
\[ (n - 1 - k) - (s_T - s'_T) = \dim \left( \text{image} \left( \mu_{T', s'} \right) \right) \]
and so we have
(41)
\[ n - 1 - k \geq 2s_T \]
which contradicts (26) unless
\[ s'_T = 0 \]
But then
\[ s_T = n - 1 - k \]
which again contradicts (11). \qed

3.3. A critical lemma from linear algebra. We will need the following linear algebra computation. By Corollary 3.2 the map
(42) \[ \tilde{l}_*: T' \rightarrow T_q|_{(l, x)} \]
induces
\[ \frac{T'}{T' \cap M^d_{2G}} \bigg|_y \rightarrow T_q|_{(l, x)} \]
where, for notational simplicity, we denote \((p \circ \pi)^* M^d_{2G}\) simply as \(M^d_{2G}\). Since
\[ \frac{T'}{T' \cap M^d_{2G}} \bigg|_y \cong \frac{T' + M^d_{2G}}{M^d_{2G}} \bigg|_y \subseteq \frac{M^d_{G}}{M^d_{2G}} \bigg|_y \]
and, by (13),
\[ \frac{M^d_{G}}{M^d_{2G}} \bigg|_y \cong \pi^* \left( \left. E^d_{\varphi_{\alpha}} \otimes T^\vee_q \right|_{(l, x, F)} \right), \]
we can extend \(\tilde{l}_*\) to a map of the same rank
\[ \varphi : \left. E^d_{\varphi_{\alpha}} \otimes T^\vee_q \right|_{(l, x)} \rightarrow T_q|_{(l, x)} \cdot \]
Now let
\[ K^\vee = T_q|_x \]
\[ W = E^d_{\varphi_{\alpha}}|_{(l, x)} \cdot \]
On the other hand, considering
\[ T' \subseteq (p \circ \pi)^* M^d_{G} \bigg|_y \]
we have
\[ \vartheta : T'_y \to \frac{M_G}{M_{2G}}|_y = W \otimes K \]
whose image we denote by \( H \). The standard map
\begin{equation}
\psi : \bigwedge^2 (W \otimes K) \to W
\end{equation}
\[ A \wedge B \mapsto \langle \vartheta (A) | \varphi (B) \rangle - \langle \vartheta (B) | \varphi (A) \rangle \]
restricts to a map
\[ \bigwedge^2 H \to W \]
under which
\[ \vartheta (\tau) \wedge \vartheta (\tau') \mapsto \varepsilon ([\tau, \tau']) . \]

The needed linear algebra result is then:

**Lemma 3.3.** i) Let
\[ H \subseteq W \otimes K \]
be a subspace of codimension \( c \) and
\[ \varphi : H \to K^\vee \]
be any linear map and let
\[ \psi : H \wedge H \to W \]
be defined as in (43). Let \( J = \psi (H \wedge H) \). Then
\[ \dim \frac{W}{J} \leq c + 1 \]
and, if equality holds, \( \varphi \) factors as a composition
\[ H \to \frac{W \otimes K}{J' \otimes K} \to K^\vee \]
where \( J' \supseteq J \) is a hyperplane in \( W \), and the image of the associated morphism
\[ \frac{W}{J'} \to K^\vee \otimes K^\vee \]
lies in
\[ \text{Sym}^2 (K^\vee) . \]
ii) If \( c > 0 \) in i) then
\[ \dim \frac{W}{J} \leq c . \]

**Proof.** i) Pick a complementary subspace \( J^\perp \) to \( J \) in \( W \) and a basis \( \{w_j\} \) for \( W \) which is compatible with the decomposition
\[ W = J \oplus J^\perp . \]
Let \( \{k_i\} \) be a basis of \( K \) and use the induced isomorphism
\[ K \to K^\vee \]
\[ k_i \to (k_i \mapsto \delta_{i\nu}) \]
to identify \( K \) and \( K^\vee \). Pick a complementary space \( H^\perp \) to \( H \) in \( W \otimes K \) which has a basis consisting of monomials
\begin{equation}
(44) \quad w_{j(h)} \otimes k_{i(h)}
\end{equation}
for \( h = 1, \ldots, c \), and extend \( \varphi \) to \( W \) by setting 
\[
\varphi (w_{j(h)} \otimes k_{i(h)}) = 0.
\]
Then for  
\[
\psi : \bigwedge^2 (W \otimes K) \to W
\]
as in \( (43) \) we have 
\[
\dim (\psi (H^\perp \otimes (W \otimes K))) \leq c,
\]
so we are reduced to proving the assertion of the lemma in the case \( c = 0 \).

In that case \( \varphi \) is given by a matrix 
\[
\left( a_{j,i}^{j',i'} \right)
\]
and 
\[
\text{(45)} \quad \psi ((w_j \otimes k_i) \wedge (w_{j'} \otimes k_{i'})) = a_{j,j'}^{j',i} w_j - a_{j,i}^{j',i} w_{j'}.
\]
If \( J^\perp \neq 0 \), pick \( w_j \in J^\perp \) and conclude that \( a_{j,j'}^{j',i} = 0 \) unless \( j = j' \) whenever \( w_{j'} \in J^\perp \) and 
\[
a_{j,j'}^{j',i} = a_{j,i}^{j',i}.
\]

ii) Assume 
\[
2 \leq \dim \frac{W}{J} = c + 1.
\]
Then the hyperplane \( J' \) will vary non-trivially as the choice of \( (44) \) varies over all possible bases. \( \square \)

4. Line contact for lifted \( Y/S' \)

4.1. \( Y/S' \) lies in locus of osculating lines.

**Theorem 4.1.** Let \( Y/S' \) be a versal family of \( k \)-folds in \( X/S \) such that Condition 1 and Condition 2 above hold. Then, at generic \( y \in Y \):

i) Either 
\[
s_{\text{Tref}} \leq 1
\]
(in which case Lemma 2.3 applies) or, referring to \( (42) \), we have 
\[
\tilde{l}_* = 0.
\]

ii) If \( \tilde{l}_* = 0 \) in \( (42) \), then the distribution 
\[
T_{q \pi \omega} \cap M^d_G
\]
is integrable and 
\[
\tilde{l} (y)
\]
has contact with \( X_{\tilde{F}(y)} \) of order at least \( d \) at \( \tilde{x} (y) \).

**Proof.** i) Fix generic \( y \in Y \) and let \( (l, x, F) = g (y) \). Let \( \tilde{J} \) denote the image of the map \( \varepsilon \) given by 
\[
T_{q \pi \omega} \xrightarrow{g_*} T_\pi \oplus \pi^* T_q \to T_\pi \xrightarrow{ev_1} \pi^* E_{P^n}
\]
at \( y \) and let 
\[
W = \pi^* E_{P^n} \mid_{(l,x,F)}.
\]
Then by Lemma 2.3 ii) 
\[
\dim \frac{W}{J} = s_{\text{Tref}}.
\]
On the other hand, let

\[ H \]

denote the image of \( T' \) under the map

\[
T' \to \frac{T' + \tilde{l}^* M_{2G}}{l^* M_{2G}} \Big|_y \subseteq \frac{\tilde{l}^* M_{2G}}{l^* M_{2G}} \bigg|_y = \pi^* \left( E^d_{2n} \otimes T_q \right) \big|_{(l,x,F)}.
\]

Then Lemma 3.1 and Lemma 3.3 then imply that either \( \tilde{l}_* = 0 \) or

\[
\dim \frac{W}{J} \leq 1.
\]

But

\[
\dim \frac{W}{J} \geq \dim \frac{W}{J} = s_{T_{\text{tof}}}.
\]

So either \( s_{T_{\text{tof}}} \leq 1 \) or \( \varphi \) must be the zero map.

i) If \( \varphi = 0 \) then \( \psi = 0 \) on \( T' \) as well so that the distribution \( T' \) is integrable.

Since \( \tilde{l}_* = 0 \),

\[
\tilde{l} : Y \to G
\]

is constant along \( T' \), so that

\[
T' \subseteq T_{\pi \circ g}.
\]

Let \( y \) be a general point in \( Y \) with \( g(y) = (l,x,F) \). Now \( \tilde{l}, \tilde{x} \) and

\[
\tilde{F}_{\tilde{l}}
\]

are all constant on the leaf \( Y'_{(l,x,F)} \) through \( y \) which integrates \( T' \). On the other hand

\[
Y_{(l,x)} = (\pi \circ g)^{-1} (l,x)
\]

has tangent space \( T_{\pi \circ g}|_{Y_{(l,x)}} \). Thus

\[
\dim Y_{(l,x)} - \dim Y'_{(l,x,F)} = \operatorname{rank} \frac{T_{\text{tof}}}{T'} - (n - 1).
\]

By Lemma 2.2

\[
\operatorname{rank} \frac{T_{\text{tof}}}{T'} = d - s_{T_{\text{tof}}}
\]

so that

\[
\dim Y_{(l,x)} - \dim Y'_{(l,x)} = d - s_{T_{\text{tof}}} - (n - 1).
\]

On the other hand, by Corollary 2.5

\[
d - s_{T_{\text{tof}}} \leq n - a
\]

for \( a \geq 0 \). Thus \( a \leq 1 \) and

\[
\dim Y_{(l,x)} - \dim Y'_{(l,x)} \leq 1 - a.
\]

As above we assume that \( l \) is given by

\[
X_j = 0, \ j \geq 2
\]

and

\[
x = [1, 0, \ldots, 0].
\]
We consider the map
\[ Y_{(l,x)} \to E_{\mathbb{P}^n}^{d} \big|_{(l,x)}. \]
\[ y' \to \tilde{F}(y') \big|_{l}. \]

The fiber of this map containing the (generically chosen) basepoint \( y \) is exactly \( Y_{(l,x)}' \). So the image is of dimension at most \( 1 - a \).

On the other hand, the image is invariant under the action of the stabilizer of \((l, x)\) in \( GL(n + 1) \). Letting \( \mathcal{P}_{X_0, X_1}^{d-1} \) denote the space of homogeneous forms of degree \( d - 1 \) in \( X_0 \) and \( X_1 \), it is clear that the only such subsets of \( X_1 \cdot \mathcal{P}_{X_0, X_1}^{d-1} \) of dimension \( \leq 1 \) invariant under the group
\[
\left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}
\]
are \( \{0\} \) and \( \{C \cdot X_1^d\} \). Thus either
\[ \tilde{F}(y) \big|_{l} = cX_1^d \]
or
\[ \tilde{F}(y) \big|_{l} = 0. \]

\[ \square \]

4.2. **Line osculation hierarchy.** To complete the proof of Theorem [1.1], we lastly study the geometry of the hierarchy of varieties
\[ \Delta_r := \{(l, x, F) \in L \times \mathbb{P}^n : l \cdot X_F \geq r \cdot x\}. \]
(We only need the cases \( r = d \) and \( r = d + 1 \) but the fundamental calculations are the same for all \( r \) so we make them in general.) We have the following (commutative) diagram of maps:
\[
\begin{array}{ccc}
\Delta_{d+1} & \cap & \ldots \\
& \cap & \\
\Delta_1 & = & \Delta \\
L & \xleftarrow{\pi} & X \\
G & \xleftarrow{p} & \mathbb{P}^n & \xleftarrow{t} & S \\
\end{array}
\]

Write
\[ \pi_r = \pi|_{\Delta_r} : \Delta_r \to L \]
\[ \rho_r = \rho|_{\Delta_r} : \Delta_r \to X. \]

Since the fibers of
\[ \pi_r : \Delta_r \to L \]
are punctured vector spaces of dimension

\[ N - r, \]

\( \Delta_r \) is smooth and irreducible of dimension

\[ N - r + 2(n - 1) + 1 \]

for each \( r \).

**Lemma 4.2.** i) The map

\[ \rho_r : \Delta_r \to X \]

is surjective as long as

\[ r \leq n \]

and generically injective when

\[ r > n. \]

ii) The map

\[ s \circ \rho_r : \Delta_r \to S \]

is surjective if

\[ r \leq 2(n - 1) \]

so that

\[ \Delta_{r,F} := (s \circ \rho_r)^{-1}(F) \]

is smooth for generic \( F \in S^d \) in that case.

**Proof.** i) Assume

\[ x = [1,0,\ldots,0] \in Proj(\mathbb{C}[X_0,\ldots,X_n]). \]

The contact conditions for a line through \( x \) with respect to

\[ F = \sum_{j=1}^{n} X_0^{d-j} F_j(X_1,\ldots,X_n) \]

become

\[ \{F_1 = \ldots = F_{r-1} = 0\} \subseteq Proj(\mathbb{C}[X_1,\ldots,X_n]) \]

ii) A constant count shows that all hypersurfaces \( X_F \) in \( \mathbb{P}^n \) admit lines with a point of contact of order \( 2n - 2 \).

Let

\[ M^d_{\mathbb{P}^n} = \ker(M^r \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\text{eval}} \mathcal{O}_{\mathbb{P}^n}(r)) \]

and

\[ M^d_G = \ker(M^r \otimes \mathcal{O}_G \to E^r). \]

We have as spaces that

\[ \Delta_{d+1} = p^* M^d_{\mathbb{P}^n} - \{0\} \]

\[ \Delta = q^* M^d_G - \{0\}. \]
4.3. Canonical bundles of spaces of line-osculators. Now

\[ U^\vee = p_* q^* \mathcal{O}_{\mathbb{P}^n}(1) \]

so that

\[ c_1(U^\vee) = \mathcal{O}_G(1). \]

Also

\[ T_G = Hom \left( U, G \times \mathbb{C}^{n+1} \right) \]

so that

\[ c_1(\omega_G) = (n+1) \cdot c_1(U). \]

For \( E^d \) as defined above, we have

\[ E^d = p_* q^* \mathcal{O}_{\mathbb{P}^n}(d) = Sym^d U^\vee \]

so that

\[ c_1(E^d) = \mathcal{O}_G \left( \frac{d(d+1)}{2} \right). \]

\( p^* E^r \) has a distinguished line sub-bundle

\[ \mathcal{L}^r_L \]

whose fiber at \((x, l) \in L\) is

\[ H^0(\mathcal{O}_L(r)(-r \cdot x)). \]

We have

\[ c_1(\mathcal{L}^r_L) = r \cdot c_1(\mathcal{L}^1_L) = p^* \mathcal{O}_G(r) - q^* \mathcal{O}_{\mathbb{P}^n}(r). \]

For \( r \leq d \), define

\[ \mathfrak{M}^d_r = \mathcal{L}^r_L \otimes p^* E^{d-r} \subseteq p^* E^d. \]

so that

\[ c_1(\mathfrak{M}^d_r) = p^* c_1(E^{d-r}) + (d-r+1) \cdot c_1(\mathcal{L}^r_L) \]

\[ = p^* \mathcal{O}_G \left( \frac{(d-r)(d-r+1)}{2} \right) \]

\[ + (d-r+1) \cdot r \cdot (p^* \mathcal{O}_G(1) - q^* \mathcal{O}_{\mathbb{P}^n}(1)) \]

\[ = p^* \mathcal{O}_G \left( \frac{(d+r)(d-r+1)}{2} \right) - q^* \mathcal{O}_{\mathbb{P}^n}(r \cdot (d-r+1)). \]

Define

\[ \mathfrak{F}^d_r = \frac{p^* E^d}{\mathfrak{M}^d_r}. \]

and

\[ M^d_r = \ker \left( M^d \otimes \mathcal{O}_L \to \mathfrak{F}^d_r \right). \]

Notice that

\[ M^d_1 = M^d_{\mathbb{P}^n}, \]

\[ M^d_{d+1} = p^* M^d_G. \]

Also

\[ \Delta_r = M^d_r - \{0\} \]

so that in the exact tangent bundle sequence

\[ 0 \to T_\pi \to T_{\Delta_r} \to \pi^* T_L \to 0. \]
we have
\[ T_\pi = \pi^* M^d_r. \]
Furthermore
\[
c_1 \left( M^d_r \right) = c_1 \left( \mathcal{G}_r^d \right) - c_1 \left( p^* E^d \right)
\]
\[
= -p^* \mathcal{O}_G \left( \frac{r(r-1)}{2} \right) - q^* \mathcal{O}_{\mathbb{P}^n} \left( r \cdot (d-r+1) \right).
\]
Notice that this formula also holds for \( r = d + 1 \).
Also we have
\[
\det \mathcal{G}_r^d = p^* \mathcal{O}_G \left( \frac{r(r-1)}{2} \right) + q^* \mathcal{O}_{\mathbb{P}^n} \left( r \cdot (d-r+1) \right).
\]
The image of the tautological section
\[
\Delta \mapsto (p \circ \pi)^* E^d
\]
\[ F \mapsto F|_l \]
in the quotient bundle
\[ \mathcal{G}_r^d \]
has zero-scheme \( \Delta_r \), and so, by adjunction,
\[
\omega_{\Delta_r} = \left( \pi^* \omega_{L/G} \right) \otimes \left( \pi^* \omega_G \right) \otimes \det \mathcal{G}_r^d.
\]
On the other hand, from the exact sequence
\[ 0 \to \mathcal{O}_L \to p^* U \otimes q^* \mathcal{O}_{\mathbb{P}^n} \to T_{L/G} \to 0 \]
we have
\[
\omega_{L/G} = p^* \det U^\vee \otimes q^* \mathcal{O}_{\mathbb{P}^n} \left( -2 \right) = p^* \mathcal{O}_G \left( 1 \right) \otimes q^* \mathcal{O}_{\mathbb{P}^n} \left( -2 \right).
\]
So
\[
\omega_{\Delta_r} = \pi^* \left( p^* \left( \mathcal{O}_G \left( \frac{r(r-1)}{2} - n \right) \right) \otimes q^* \mathcal{O}_{\mathbb{P}^n} \left( r \cdot (d-r+1) - 2 \right) \right).
\]
Thus, for example, whenever
\[
\frac{d(d-1)}{2} \geq n, \quad a + d \geq 2
\]
we conclude that
\[
\omega_{\Delta_d} (a) = \omega_{\Delta_r} \otimes (q \circ \pi)^* \mathcal{O}_{\mathbb{P}^n} (a)
\]
is the pull-back of a globally generated bundle on \( L \) and therefore is globally generated. Thus by adjunction, for generic \( F \in S^d \)
\[ \omega_{\Delta_r,F} (a) \]
is globally generated.
Notice that the analogous computation for \( \Delta_{d+1} \) gives
\[
\omega_{\Delta_{d+1}} = \left( \frac{d(d+1)}{2} - n \right) \pi^* p^* c_1 (U^\vee) + \pi^* c_1 (\mathcal{O}_L (-2)).
\]
so that (49) continues to hold for this case.

4.4. Maps to the $d$-th osculation space. Let $Y/S$ be a versal family of $k$-folds in $X/S$ such that Condition 1 and Condition 2 above hold. For example this is the case if

$$d + a \geq \max \left\{ \frac{7n - 3k - 3}{4}, \frac{3n - k + 1}{2} \right\}.$$ 

and $Y/S'$ is a versal sub-family of $k$-folds such that (3) holds. We finish the proof of Theorem 1.1 by showing:

**Lemma 4.3.** Suppose $Y/S$ is a versal family of $k$-folds in $X/S$ such that Condition 1 and Condition 2 above hold and

$$\frac{d(d - 1)}{2} - n \geq (2n - 1 - d) - k = \text{codim} (g(Y), \Delta_d).$$

Then

$$f(Y) \subseteq \rho_{d+1}(\Delta_{d+1}).$$

That is $f(Y_F)$ lies in the sub-variety cut out by the union of all lines on $X_F$.

**Proof.** At generic $(x, l, F) \in g(Y)$, suppose that

$$T_{g(Y)} \cap T_{\pi_d}|_{(x, l, F)} \not\subseteq \left((p \circ \pi)^* M^d_G\right)|_{(x, l, F)}.$$ 

Then letting $c = \text{codim} (g(Y), \Delta_d)$, there is a vector $v \in \left(\bigwedge^c (p \circ \pi)^* M^d_G\right)|_{(x, l, F)}$ such that

$$\langle v \wedge w \wedge z | \omega_{\Delta_d}|_{(x, l, F)} \rangle \neq 0$$

where

$$w \in \bigwedge^N T_{S|_F}$$

$$z \in \bigwedge^k T_{\Delta_d, F,(x, l, F)}.$$ 

From Lemma 2.4, 4.3, and (51), we have that $(\bigwedge^c \pi^* p^* M^d_G) \otimes \omega_{\Delta_d}$ is generated by global sections. So there is an element

$$v \in H^0 \left(\omega_{\Delta_d} \otimes \bigwedge^c (p \circ \pi_d)^* M^d_G\right)$$

such that $v \mapsto v$ under the map

$$H^0 \left(\omega_{\Delta_d} \otimes \bigwedge^c (p \circ \pi_d)^* M^d_G\right) \rightarrow \omega_{\Delta_d} \otimes \bigwedge^c (p \circ \pi_d)^* M^d_G|_{(x, l, F)}.$$ 

Let $\alpha$ denote the image of $v$ under the map

$$H^0 \left(\omega_{\Delta_d} \otimes \bigwedge^c (p \circ \pi_d)^* M^d_G\right) \rightarrow H^0 \left(\bigwedge^c T_{\Delta_d} \otimes \omega_{\Delta_d}\right) \rightarrow H^0 \left(\Omega^k_{\Delta_d, F}\right) \rightarrow H^0 \left(\omega^k_{Y_F}\right).$$

The hypothesis of the lemma implies that, if $(x, l, F)$ is generic in $g(Y)$

$$\langle z | \alpha_{(x, l, F)} \rangle = 0$$

contradicting (52). So

$$T_{g(Y)}|_{(x, l, F)} \cap T_{\pi_d}|_{(x, l, F)} \subseteq \left((p \circ \pi_d)^* M^d_G\right)|_{(x, l, F)}.$$
We claim that
\[ F \in (s \circ \rho_d)_* \left( T_{g(Y)}|_{(x,l,F)} \cap T_{\pi d}|_{(x,l,F)} \right) \subseteq M^d_{G|l} \]
which immediately gives
\[ (l, x, F) \in \Delta_{d+1}. \]
To see this notice that \( GL(n+1) \) acts on the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{L} & X \\
\downarrow g & & \downarrow (t, s) \\
\Delta_d & \xrightarrow{\rho} & L \times_{P^n} S \to \mathbb{P}^n \times S
\end{array}
\]
so that the stabilizer of \( C \cdot x \) acts on the map
\[
(s \circ f): (t \circ f)^{-1}(x) \to S.
\]
This in turn implies the following containment at \((l, x, F)\):
\[
\left( F + \sum_i \left( \frac{\partial F}{\partial X_i} M^1_x \right) \right) \subseteq (s \circ f)_* \left( T_{(t \circ f)^{-1}(x)} \right)
\]
\[
= (s \circ \rho_d \circ g)_* \left( T_{(t \circ f)^{-1}(x)} \right)
\]
\[
= (s \circ \rho_d)_* \left( T_{g(q \circ \pi d \circ \varphi)^{-1}(x)} \right)
\]
\[
= (s \circ \rho_d)_* \left( T_{g(Y) \cap T_{q \circ \pi d}} \right)
\]
where \( M^1_x \) denotes the linear forms vanishing on \( x \). On the other hand, the containment
\[
(\rho_d)_* \left( T_{g(Y) \cap T_{\pi d}} \right) \subseteq (\rho_d)_* \left( T_{g(Y) \cap T_{q \circ \pi d}} \right)
\]
is actually an equality at generic \((l, x, F) \in g(Y)\) since the composition
\[
Y \to \Delta_d \subseteq L \times_{P^n} X \subseteq L \times X \to X
\]
is an immersion there. So
\[
F \in (s \circ \rho_d)_* \left( T_{g(Y) \cap T_{\pi d}} \right) = M^d_{G|l}.
\]
But
\[
F \in M^d_{G|l}
\]
implies that
\[
F|_l = 0.
\]
\[\square\]

Corollary \( \text{2} \) is then obtained as follows.

**Corollary 4.4.** Suppose \((n - 1) - k = 2,\)
\[
\frac{d(d - 1)}{2} - n \geq n + 2 - d = \text{codim} \left( g(Y), \Delta_d \right),
\]
and
\[
f(Y) \not\subseteq \rho_{d+1} \left( \Delta_{d+1} \right).
\]
Then, for \( r \) as in \( \text{(10)} \),
\[
h^0 \left( \omega_{Y_F} (n + 2 - d) \right) \geq \left( \frac{n + 1}{k + 1} \right) - r.
\]
Proof. Since
\[ s_{T \circ f} \leq (n - 1) - k = 2 \]
the only possibilities are
\[ s_{T \circ f} = s'_{T \circ f} = 1 \]
and
\[ s_{T \circ f} = 2, \quad s'_{T \circ f} = 0. \]
In the latter case, Theorem 4.1 and Lemma 4.3 imply that
\[ f(Y) \subseteq \rho_{d+1} (\Delta_{d+1}). \]
So \( s_{T \circ f} = 1 \). Now apply Lemma 2.3. \[ \square \]

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