COMPLEXITY OF SOME GRAPHS GENERATED BY SQUARE

MOHAMED R. ZEEN EL DEEN∗, WALAA A. ABOAMER

Department of Mathematics and Computer Science, Faculty of Science, Suez University, Suez 42524, Egypt

Abstract. Complexity plays a vital and significant role when designing communication networks (graphs). The more quality and perfect the network, the greater the number of trees spanning this network, which leads to greater possibilities of connection between two vertices, and this ensures good rigidity and resistance. In this work, we present nine network designs created by a square of different average degree 4, 6 and 8, then we deduce a simpler and evident formula expressing the number of spanning trees of these networks using some basic properties of orthogonal polynomials, block matrix analysis technique, and recurrence relations. In addition, we compute the entropy of each network and determine the best by comparing these designs using network entropy. Finally, we compare the entropy of spanning trees on our networks with other triangle and Apollonian networks and observe the entropy of our networks, which is the highest among the triangle and Apollonian networks studied.

Keywords: complexity; recurrence relation; Chebyshev polynomials; duplication of graphs; entropy.

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1. INTRODUCTION

In crowded places and occasions telecommunications companies facing significant problems, when someone makes contact with another person and the lines are busy. So another empty communication channel must be provided in order for the communication to take place.

∗Corresponding author

E-mail address: mohamed.zeeneldeen@suezuniv.edu.eg

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The good company try to increase the number of communication channels, these demands to calculate the number of different tracks (the total number of spanning trees) available within each network and choosing the best design to grantee this increasing.

A graph is a formal mathematical illustration of a network since any network can be modeled by a graph $G$ where nodes are represented by vertices $V(G)$ and links are represented by edges $E(G)$. Let $|E(G)|$ be the cardinality of $E(G)$ and $|V(G)|$ be the cardinality of $V(G)$. We deal with finite and undirected with multiple edges and loops permitted graphs. The degree of a vertex $x \in V(G)$ is the number of edges incident with the vertex, while the average degree of a graph is applied to measure the number of edges compared to the number of vertices which calculates by dividing the summation of all vertex degrees by the total number of nodes.

A spanning tree of any graph is a communication subgraph that guarantees connectivity between all vertices of the original graph with a minimum number of edges. In other words, a spanning tree ensures the existence and uniqueness of a connection between any pair of vertices. The number of spanning trees $\tau(G)$ is equal to the total number of various spanning subgraphs of $G$ that are trees, this quantity is also known as complexity $\tau(G)$ of $G$.

Many new graphs can be generated from a given pair of graphs using graph operations [1]. For every vertex $x \in V(G)$, the open neighborhood set $N(x)$ is the set of all adjacent vertices to $x$ in $G$. Duplication of an edge $e = xy$ in a graph $G$ by a new vertex $z$ creates a new graph $G'$ such that $N(z) = \{x, y\}$. Duplication of a vertex $x$ in a graph $G$ by a new edge $e = uv$ creates a new graph $G'$ such that $N(u) \cap N(v) = \{x\}$.

There are several methods of finding this number. The celebrated matrix tree theorem of Kirchhoff [2] tells us that: the complexity $\tau(G)$ of a graph $G$ is equal any cofactor of Laplace matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of $G$ and $A(G)$ is the adjacency matrix of $G$. $\tau(G)$ can also be calculated from the eigenvalues of the Kirchhoff matrix $H$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n(= 0)$ denote the eigenvalues of $H$ matrix. Kelmans and Chenlnokov [3], have shown that

\[
\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i .
\]

After that Temperley [4], has shown that:
\[ \tau(G) = \frac{1}{n^2} \det( H + J ) , \]

where \( J \) is the \( n \times n \) matrix all of whose elements are unity.

From Temperley’s Equation (2), it is easy to prove the following lemma.

**Lemma 1.1.** Let \( G \) be a graph with \( n \) vertices and \( D, A \) are the degree and adjacency matrices, respectively, of \( \overline{G} \), the complement of \( G \). Then,

\[ \tau(G) = \frac{1}{n^2} \det( nI - D + A ) . \]

The senior advantage of the formula (3) is that it directly expresses \( \tau(G) \) as a determinant rather than in terms of co-factors or eigenvalues.

For some special classes of graphs, there are simple closed formulas that make it much easier to calculate and determine the number of corresponding spanning trees, especially when these numbers are very large. Cayley showed that a complete graph \( K_n \) has \( n^{n-2}, n \geq 2 \) spanning trees [2]. Another result is due to Sedlacek [5], he derived a formula for the wheel with \( n + 1 \) vertices, \( W_{n+1} \), has a number of spanning trees \((\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2, n \geq 2\). Recently, several closed formulas have been published for counting and maximizing the number of spanning trees for some families of graphs (see, [6–11]).

2. **Basic Proof Tools**

There exists a powerful relation concerning orthogonal polynomials, especially the Chebyshev polynomials of the first and second kinds and determinants that we use in our computations. For positive integer \( n \), the Chebyshev polynomials of the first kind are defined by [12]:

\[ T_n(x) = \cos(n \arccos x) . \]

For positive integer \( m \), the Chebyshev polynomials of the first kind are defined by:

\[ T_n(x) = \cos(n \arccos x) . \]

The Chebyshev polynomials of the second kind are defined by

\[ U_{n-1}(x) = \frac{1}{n} \frac{d}{dx} T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)} . \]
The Chebyshev polynomials of the second kind satisfy the recursion relation

\[ U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0. \] \hspace{1cm} (6)

Let \( A_n(x) \) be \( n \times n \) matrix such that:

\[
A_n(x) = \begin{pmatrix}
2x & -1 & 0 & 0 & \cdots & 0 \\
-1 & 2x & -1 & 0 & \cdots & 0 \\
0 & -1 & 2x & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 2x \\
0 & 0 & 0 & \cdots & 0 & -1 & 2x
\end{pmatrix}.
\]

From this recursion relation and by expanding \( \det A_n(x) \), one obtains:

\[ U_n(x) = \det(A_n(x)), \quad n \geq 1. \] \hspace{1cm} (7)

Using standard methods for solving the recursion (6), one obtains the explicit formula

\[ U_n(x) = \frac{1}{2\sqrt{x^2-1}}[(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}], \quad n \geq 1, \quad x = \pm 1. \] \hspace{1cm} (8)

**Lemma 2.1.** [13] If

\[
B_n(x) = \begin{pmatrix}
x & -1 & 0 & 0 & \cdots & 0 \\
-1 & (x+1) & -1 & 0 & \cdots & 0 \\
0 & -1 & (x+1) & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1 & (x+1) \\
0 & 0 & 0 & \cdots & 0 & -1 & x
\end{pmatrix}, \text{ then } \det(B_n(x)) = (x-1)U_{n-1} \left( \frac{x+1}{2} \right).
\]

**Lemma 2.2.** [8] If

\[
E_n(x) = \begin{pmatrix}
x & 1 & 1 & 1 & \cdots & 1 \\
1 & x & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & x & 1 \\
1 & 1 & 1 & \cdots & 1 & x
\end{pmatrix}, \text{ then } \det(E_n(x)) = (x+n-1)(x-1)^{n-1}.
\]
Lemma 2.3. Let $P$, $Q$ and $R$ be matrices of dimension $n \times n$, then

$$
\det \begin{pmatrix}
P & Q & R & Q \\
Q & P & Q & R \\
R & Q & P & Q \\
Q & R & Q & P
\end{pmatrix} = [\det(P - R)]^2 \det(P + R + 2Q) \det(P + R - 2Q).
$$

Proof. Using the properties of determinants and matrix row and column operations [17] yields:

$$
\begin{pmatrix}
P & Q & R & Q \\
Q & P & Q & R \\
R & Q & P & Q \\
Q & R & Q & P
\end{pmatrix} = \begin{pmatrix}
P - R & O & R & Q \\
O & P - R & Q & R \\
R - P & O & P & Q \\
O & R - P & Q & P
\end{pmatrix} = \begin{pmatrix}
P - R & O & R & Q \\
O & P - R & Q & R \\
O & O & P + R & 2Q \\
O & O & 2Q & P + R
\end{pmatrix},
$$

then

$$
\det \begin{pmatrix}
P & Q & R & Q \\
Q & P & Q & R \\
R & Q & P & Q \\
Q & R & Q & P
\end{pmatrix} = \det \begin{pmatrix}
P - R & O \\
O & P - R
\end{pmatrix} \times \det \begin{pmatrix}
P + R & 2Q \\
2Q & P + R
\end{pmatrix}
$$

$$
= [\det(P - R)]^2 \det(P + R + 2Q) \det(P + R - 2Q)
$$

\[\square\]

3. Complexity of Some Families of Graphs Generated by a Square

Suppose we have four transmission sources in a broadcasting network, and we want to reach all the nodes in that network without closing that circuit linking those nodes. Therefore, we are going to study this problem in the presence of various models linking these nodes to determine which of these models are the best to connect all nodes.

3.1. Complexity of some families of graphs generated by a square with average degree 4.

Theorem 3.1. For $n \geq 1$, the number of the spanning trees of the family of graphs $\Theta_n$ is given by:

$$
\tau(\Theta_n) = \frac{1}{144} \left[ (3 + \sqrt{3})(2 + \sqrt{3})^n + (3 - \sqrt{3})(2 - \sqrt{3})^n \right]^2 \left[ (2 + \sqrt{2})(3 + 2\sqrt{2})^n + (2 - \sqrt{2})(3 - 2\sqrt{2})^n \right].
$$

Proof. The Kirchhoff matrix associated with the graph $\Theta_n$ is,
Consider the family of graphs \( \Theta_n \), generated by a square, and constructed as shown in Fig. 1, according to the construction, the number of vertices in the graph \( \Theta_n \) are \( |V(\Theta_n)| = 4n + 1 \) and edges \( |E(\Theta_n)| = 8n \), \( n = 1, 2, \cdots \). When \( n \) is large, the average degree of \( \Theta_n \) is 4.

**Figure 1.** the graph \( \Theta_n \)
Thus, we get:

$$\tau(\Theta_n) = \det \begin{pmatrix} P & Q & O & Q \\ Q & P & O & \vdots \\ O & Q & P & \vdots \\ Q & O & P & \vdots \end{pmatrix} = \det \begin{pmatrix} \det(P) \\ \det(P + 2Q) \\ \det(P - 2Q) \end{pmatrix}.$$  

Applying Lemma (1.1), we get:

$$\tau(\Theta_n) = [\det(P)]^2 \times \det(P + 2Q) \times \det(P - 2Q).$$
det \( P \) = det \[
\begin{pmatrix}
4 & -1 & 0 & 0 & \ldots & \ldots & 0 \\
-1 & 4 & -1 & 0 & \ldots & \ldots & 0 \\
0 & -1 & 4 & -1 & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 4 & -1 & 0 \\
0 & 0 & 0 & \ldots & -1 & 3 & 0 \\
\end{pmatrix}
\].

Expanding the first seven determinants of \( P \) gives the values \( 3, 11, 41, 153, 571, 2131, \ldots \) for \( n = 1, 2, 3, 4, 5, 6, \ldots \), these values can be written in the form

\[
41 = 4 \times 11 - 3, \quad 153 = 4 \times 41 - 11, \\
571 = 4 \times 153 - 41, \quad 2131 = 4 \times 571 - 153, \ldots .
\]

Consequently, we have the following homogeneous recurrence relation

\[
(11) \quad a_{n+2} = 4a_{n+1} - a_n.
\]

Its characteristic equation is \( r^2 - 4r + 1 = 0 \), with two roots being \( 2 + \sqrt{3} \) and \( 2 - \sqrt{3} \).

The general solution of the Recurrence Relation (11), is \( a_n = \alpha (2 + \sqrt{3})^n + \beta (2 - \sqrt{3})^n \).

Using the initial conditions \( det(P) = 3, 11 \) at \( n = 1, 2 \), respectively, we have:

\[
3 = \alpha (2 + \sqrt{3}) + \beta (2 - \sqrt{3}), \\
11 = \alpha (2 + \sqrt{3})^2 + \beta (2 - \sqrt{3})^2.
\]

Solving these equations, we get: \( \alpha = \frac{3 + \sqrt{3}}{6} \) and \( \beta = \frac{3 - \sqrt{3}}{6} \). Therefore

\[
(12) \quad det(P) = \left[ \frac{3 + \sqrt{3}}{6} \right] (2 + \sqrt{3})^n + \left[ \frac{3 - \sqrt{3}}{6} \right] (2 - \sqrt{3})^n .
\]
Using induction and some properties of determinants, we obtain,

\[ \det(P + 2Q) = \det \begin{pmatrix}
2 & -1 & 0 & 0 & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 2 & -1 \\
0 & 0 & \ldots & \ldots & -1 & 1 \\
\end{pmatrix} = 1. \]

\[ \det(P - 2Q) = \det \begin{pmatrix}
6 & -1 & 0 & 0 & \ldots & 0 \\
-1 & 6 & -1 & 0 & \ldots & 0 \\
0 & -1 & 6 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 6 & -1 \\
0 & 0 & 0 & \ldots & -1 & 5 \\
\end{pmatrix} \]

Expanding the first seven determinants of \((P - 2Q)\), gives the values 5, 29, 169, 985, 5741, 33461, \ldots for \(n = 1, 2, 3, 4, 5, 6, \ldots\), these values can be written in the form,

\[ 169 = 6 \times 29 - 5, \quad 985 = 6 \times 169 - 29, \]
\[ 5741 = 6 \times 985 - 169, \quad 33461 = 6 \times 5741 - 985, \ldots \]

Consequently, we have the following homogeneous recurrence relation

\[ a_{n+2} = 6a_{n+1} - a_n. \]

Its characteristic equation is \(r^2 - 6r + 1 = 0\) with two roots being \(3 + 2\sqrt{2}\) and \(3 - 2\sqrt{2}\).

The general solution of the Recurrence Relation (14), is \(a_n = \alpha (3 + 2\sqrt{2})^n + \beta (3 - 2\sqrt{2})^n\).

Using the initial conditions, we get: \(\alpha = \frac{2 + \sqrt{2}}{4}\) and \(\beta = \frac{2 - \sqrt{2}}{4}\). Therefore:

\[ \det(P - 2Q) = \left[ \left( \frac{2 + \sqrt{2}}{4} \right)(3 + 2\sqrt{2})^n + \left( \frac{2 - \sqrt{2}}{4} \right)(3 - 2\sqrt{2})^n \right]. \]

From Eq.(12), Eq.(13) and Eq.(15) in Eq.(10), we get:
\[ \tau(\Theta_n) = \frac{1}{144} \left[ (3 + \sqrt{3})(2 + \sqrt{3})^n + (3 - \sqrt{3})(2 - \sqrt{3})^n \right]^2 \left[ (2 + \sqrt{2})(3 + 2\sqrt{2})^n + (2 - \sqrt{2})(3 - 2\sqrt{2})^n \right]. \]

\[ \Box \]

Consider the family of graphs \( \Delta_n \), generated by a square, and constructed as the cross product \( \Delta_n = C_4 \square P_n \) of a square \( C_4 \) and the path \( P_n \) [1], see Fig. 2. The graph \( \Delta_n \) has the number of vertices \( |V(\Delta_n)| = 4n \) and the edges \( |E(\Delta_n)| = 8n - 4, \ n = 1, 2, \cdots \). When \( n \) is large, the average degree of \( \Delta_n \) is 4.

**Theorem 3.2.** For \( n \geq 1 \), the number of the spanning trees of the graph \( \Delta_n = C_4 \times P_n \) is given by

\[ \tau(\Delta_n) = \frac{1}{12\sqrt{2}} \left[ (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right]^2 \left[ (3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n \right]. \]

**Proof.** Applying Lemma (1.1) and Lemma( 2.3), we have:

\[ \tau(\Delta_n) = \frac{1}{(4n)^2} \det \begin{bmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{bmatrix} = \frac{1}{(4n)^2} \left[ \det(P - J) \right]^2 \times \det(P + J - 2Q) \times \det(P + J + 2Q) \]
where \( P = \begin{pmatrix}
4 & 0 & 1 & \ldots & 1 & 1 \\
0 & 5 & 0 & 1 & \ldots & 1 \\
1 & 0 & 5 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & \ldots & 5 & 0 \\
1 & 1 & \ldots & \ldots & 0 & 4 \\
\end{pmatrix}_{n \times n} \), \( Q = \begin{pmatrix}
0 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 1 & 1 & \ldots & 1 \\
1 & 1 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & \ldots & 0 & 1 \\
1 & 1 & \ldots & \ldots & 1 & 0 \\
\end{pmatrix}_{n \times n} \),

\[
\tau(\Delta_n) = \frac{1}{(4n)^2} \left| \begin{pmatrix}
3 & -1 & 0 & \ldots & 0 \\
-1 & 4 & -1 & \ldots & 0 \\
0 & -1 & 4 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 4 & -1 \\
0 & 0 & \ldots & -1 & 3 \\
\end{pmatrix} - J \right| \times \left| \begin{pmatrix}
5 & -1 & 0 & \ldots & 0 \\
-1 & 6 & -1 & \ldots & 0 \\
0 & -1 & 6 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 6 & -1 \\
0 & 0 & \ldots & -1 & 5 \\
\end{pmatrix} \right| \times \left| \begin{pmatrix}
5 & 3 & 4 & \ldots & 4 \\
3 & 6 & 3 & \ldots & 4 \\
4 & 3 & 6 & \ldots & 4 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
4 & 4 & \ldots & 6 & 3 \\
4 & 4 & \ldots & 3 & 5 \\
\end{pmatrix} \right|.
\]

From Lemma (2.1), we obtain:

\[
(17) \quad \det(P - J) = \det \left| \begin{pmatrix}
3 & -1 & 0 & \ldots & 0 \\
-1 & 4 & -1 & \ldots & 0 \\
0 & -1 & 4 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 4 & -1 \\
0 & 0 & \ldots & -1 & 3 \\
\end{pmatrix} \right| = 2 \, U_{n-1}(2) = \frac{1}{\sqrt{3}} \left[(2 + \sqrt{3})^n - (2 - \sqrt{3})^n\right].
\]
Using the induction and the properties of determinants, we obtain the value

\[ \det(P + J + 2Q) = 4n^2. \]

From Eq. (17), Eq. (18) and Eq. (19) in Eq. (16), we get:

\[ \tau(\Lambda_n) = \frac{1}{12\sqrt{2}} \left[ (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right]^2 \left[ (3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n \right]. \]

Consider the family of graphs \( \Lambda_n \), generated by a square, and constructed starting with a square and finding its line graph [1], one gets an inner square. Repeating this process to the new interior square given the graph shown in Fig. 3, the graph \( \Lambda_n \) has the number of vertices \( |V(\Lambda_n)| = 4n \) and the edges \( |E(\Lambda_n)| = 8n - 4, \ n = 2,3,\ldots \). When \( n \) is large, the average degree of \( \Lambda_n \) is 4.
Theorem 3.3. For $n \geq 2$, the number of the spanning trees of the graph $\Lambda_n$ is given by

$$\tau(\Lambda_n) = 3 \cdot 2^{3n-7} \left[ (6 + 4\sqrt{2})^n + (6 - 4\sqrt{2})^n + 2^{n+1} \right].$$

Proof. Applying Lemma 1.1, we have: $\tau(\Lambda_n) = \frac{1}{(4n)^2} \det \left[ 4nI - \mathcal{D} + \mathcal{A} \right].$

\begin{equation}
\tau(\Lambda_n) = \frac{1}{(4n)^2} \det \begin{pmatrix}
P & Q & J & Q' \\
Q' & P & Q & J \\
J & Q' & P & Q \\
Q & J & Q' & P
\end{pmatrix} = \frac{1}{(4n)^2} \det \begin{pmatrix}
L & M \\
M & L
\end{pmatrix} = \frac{1}{(4n)^2} [\det(L+M) \times \det(L-M)]
\end{equation}

where $P = \begin{pmatrix}
3 & 0 & 1 & \ldots & 1 & 1 \\
0 & 5 & 0 & 1 & \ldots & 1 \\
1 & 0 & 5 & 0 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & \ldots & 5 & 0 \\
1 & 1 & \ldots & 0 & 5
\end{pmatrix}_{n,n}$, $Q = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
1 & 1 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & 1 & 1 \\
1 & 1 & \ldots & 0 & 0
\end{pmatrix}_{n,n}$.

$$\det(L+M) = \det \begin{pmatrix}
4 & 1 & 2 & \ldots & 2 & 2 & 1 & 2 & \ldots & 2 \\
1 & 6 & 1 & 2 & \ldots & 2 & 1 & 2 & 1 & 2 & \ldots & 2 \\
2 & 1 & 6 & 1 & \ldots & 2 & 2 & 1 & 2 & 1 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 & 2 & \ldots & 6 & 1 & 2 & 2 & \ldots & 2 & 1 \\
2 & 2 & \ldots & 1 & 6 & 2 & 2 & \ldots & 1 & 0 \\
2 & 1 & 2 & \ldots & 2 & 4 & 1 & 2 & \ldots & 2 \\
1 & 2 & 1 & 2 & \ldots & 2 & 1 & 6 & 1 & 2 & \ldots & 2 \\
2 & 1 & 2 & 1 & \ldots & 2 & 2 & 1 & 6 & 1 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
2 & 2 & \ldots & 2 & 1 & 2 & 2 & \ldots & 6 & 1 \\
2 & 2 & \ldots & 1 & 0 & 2 & 2 & \ldots & 1 & 6
\end{pmatrix} = \det \begin{pmatrix}
X & Y \\
Y & X
\end{pmatrix}.$$
Using the properties of the determinants, we arrive at:

\[
\begin{bmatrix}
3 & 1 & 2 & 2 & \ldots & 2 \\
1 & 4 & 1 & 2 & \ldots & 2 \\
2 & 1 & 4 & 1 & \ldots & 2 \\
2 & 2 & 1 & 4 & \ldots & 2 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
2 & 2 & 2 & \ldots & 4 & 1 \\
2 & 2 & 2 & \ldots & 1 & 3
\end{bmatrix}
\times 3^{2n-2} = 3n^2 2^{3n-1}.
\]
Expanding the first seven determinants of \((L - M)\), gives the values
\[36, 400, 4624, 53824, 627264, 7311616, 85229824 \cdots\]
for \(n = 2, 3, 4, 5, 6, 7, 8, \cdots\). These values can be written in the form:
\[53824 = 12 \times 4624 - 4 \times 400 - 64, \quad 627264 = 12 \times 53824 - 4 \times 4624 - 128, \]
\[7311616 = 12 \times 627264 - 4 \times 53824 - 256, \quad 85229824 = 12 \times 7311616 - 4 \times 627264 - 512.\]

Consequently, we have the following non-homogeneous recurrence relation

\[a_{n+2} = 12a_{n+1} - 4a_n - 82^n.\]  

Let \(a_n = b_n + d 2^n\) be the solution of the non-homogeneous Recurrence Relation Eq. (22), where \(b_n\) is the solution of the homogeneous recurrence relation \(b_{n+2} - 12b_{n+1} + 4b_n = 0\). Substituting in Eq. (22), we get, \(d = \frac{1}{2}\). The characteristic equation of homogeneous recurrence relation is \(r^2 - 12r + 4 = 0\) with two roots being \(6 + 4\sqrt{2}\) and \(6 - 4\sqrt{2}\), then
\[b_n = \alpha (6 + 4\sqrt{2})^n + \beta(6 - 2\sqrt{2})^n.\]
The general solution of the non-homogeneous Recurrence Relation Eq. (22), is:
\[a_n = \alpha (6 + 4\sqrt{2})^n + \beta(6 - 4\sqrt{2})^n + 2^{n-1}.\]

Using the initial conditions \(\text{det}(L - M) = 400, 4624\) to \(n = 3, 4,\) respectively, we get \(\alpha = \beta = \frac{1}{4}\). Therefore

\[\text{det}(L - M) = a_n = \frac{1}{4} \left[ (6 + 4\sqrt{2})^n + (6 - 4\sqrt{2})^n \right] + 2^{n-1}.\]  

From Eq. (21) and Eq. (23) in Eq. (37), we get:
\[\tau(\Lambda_n) = 3 \cdot 2^{3n-7} \left[ (6 + 4\sqrt{2})^n + (6 - 4\sqrt{2})^n \right] + 2^{n+1}.\]
Consider the family of graphs $\Psi_n$, generated by a square, and constructed by duplicating each vertex $u$ of the square by vertices $u_i$ such that $\mathcal{N}(u) = \mathcal{N}(u_i)$, see Fig. 4. Based on construction, the total number of vertices $|V(\Psi_n)|$ and edges $|E(\Psi_n)|$ are $|V(\Psi_n)| = 4n$ and $|E(\Psi_n)| = 8n - 4$, $n = 2, 3, 4, \cdots$.

It is notice that, when $n$ is large, the average degree of $\Psi_n$ is 4.

**Theorem 3.4.** For $n \geq 2$, the number of the spanning trees of the graph $\Psi_n$ is given by $\tau(\Psi_n) = n^2 2^{4n-2}$.

**Proof.** Applying Lemma 1.1, we have:

$$\tau(\Psi_n) = \frac{1}{(4n)^2} \det \left[ 4nI - \bar{D} + \bar{A} \right] = \frac{1}{(4n)^2} \det \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix}.$$  

Applying Lemma 2.3, with the matrix $R = J$ where $J$ is the unit matrix, we obtain:

$$\tau(\Psi_n) = \frac{1}{(4n)^2} [\det(P - J)]^2 \times \det(P + J + 2Q) \times \det(P + J - 2Q),$$

where $P = \begin{pmatrix} 2n+1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 3 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 3 & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 3 & 1 & 1 \\ 1 & 1 & \ldots & 1 & 3 \end{pmatrix}_{n,n}$, $Q = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 1 & \vdots \\ 0 & 1 & 1 & \ldots & 1 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 1 & \ldots & 1 & 1 & 1 \\ 0 & 1 & \ldots & 1 & 1 & \end{pmatrix}_{n,n}$.
\[ \tau(\Psi_n) = \frac{1}{(4n)^2} \left( \det \begin{pmatrix} 2n & 0 & 0 & \ldots & 0 \\ 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2 & 0 \\ 0 & 0 & \ldots & 0 & 2 \end{pmatrix} \right) \times \det \begin{pmatrix} 2n+2 & 2 & 2 & \ldots & 2 \\ 2 & 6 & 4 & \ldots & 4 \\ 2 & 4 & 4 & \ldots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 4 & \ldots & 6 & 4 \\ 2 & 4 & \ldots & 4 & 6 \end{pmatrix} \times \begin{pmatrix} 2n+2 & 2 & 2 & \ldots & 2 \\ 2 & 6 & 4 & \ldots & 4 \\ 2 & 4 & 4 & \ldots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 4 & \ldots & 6 & 4 \\ 2 & 4 & \ldots & 4 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 4 & 2 & \ldots & 2 \\ 1 & 2 & 4 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & 2 & 4 \\ 1 & 2 & \ldots & 2 & 4 \end{pmatrix}_{n,n} \times 2^{n+1}, \]

\[ = \frac{1}{(4n)^2} \left( n \ 2^n \right)^2 \times 4n \det \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 4 & 2 & \ldots & 2 \\ 1 & 2 & 4 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & \ldots & 2 & 4 \\ 1 & 2 & \ldots & 2 & 4 \end{pmatrix}_{n,n} \times 2^{n+1}, \]

\[ = \frac{1}{(4n)^2} \left( n \ 2^n \right)^2 \times 4n \ 2^{n-1} \det \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 2 & 1 \\ 1 & 1 & \ldots & 1 & 2 \end{pmatrix}_{n-1,n-1} \times 2^{n+1}. \]

Applying Lemma (2.2), with \( x = 2 \), we get:

\[ \tau(\Psi_n) = \frac{1}{(4n)^2} \left( n \ 2^n \right)^2 \times 2^{n+1} n^2 \times 2^{n+1} = n^2 \ 2^{4n-2}. \]
Consider the family of graphs $\Gamma_n$, generated by a square and constructed as shown in Fig. 5. It is noticed that according to the construction, the number of total vertices $|V(\Gamma_n)| = 4n$ and edges $|E(\Gamma_n)| = 8n - 4$, $n = 1, 2, 3, 4, \cdots$. When $n$ is large, the average degree of $\Gamma_n$ is 4.

**Theorem 3.5.** For $n \geq 1$, the number of the spanning trees of the graph $\Gamma_n$ is given by:

$$\tau(\Gamma_n) = 2^{6n - 4}.$$  

**Proof.** Applying Lemma (1.1), we have:

\[
\tau(\Gamma_n) = \frac{1}{(4n)^2} \det \left[ 4nI - \overline{D} + \overline{A} \right] = \frac{1}{(4n)^2} \det \left( \begin{pmatrix} P & Q & J & Q \\ Q & P & Q & J \\ J & Q & P & Q \\ Q & J & Q & P \end{pmatrix} \right).
\]

(24) \[
\tau(\Gamma_n) = \frac{1}{(4n)^2} \det \left( \begin{pmatrix} L & M \\ M & L \end{pmatrix} \right) = \frac{1}{(4n)^2} [\det(L + M) \times \det(L - M)].
\]

where $P = \begin{pmatrix} 5 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 5 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 5 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 5 & 1 \\ 1 & 1 & \cdots & 1 & 3 \end{pmatrix}_{n \times n}$, $Q = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 0 & 1 \end{pmatrix}_{n \times n}$. 

**Figure 5.** The graph $\Gamma_n$. 

\[
det(L + M) = \det \begin{pmatrix}
6 & 2 & \ldots & \ldots & 2 & 0 & 0 & 2 & \ldots & 2 \\
2 & 6 & 2 & \ldots & \ldots & 2 & 0 & 2 & \ldots & 2 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2 & 2 & \ldots & 6 & 2 & 2 & 2 & \ldots & 2 & 0 \\
2 & 2 & \ldots & 2 & 4 & 2 & 2 & \ldots & 0 & 2 \\
0 & 0 & 2 & \ldots & 2 & 6 & 2 & \ldots & 2 & 0 \\
0 & 2 & 0 & \ldots & 2 & 2 & 6 & 2 & \ldots & 2 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2 & 2 & \ldots & 2 & 0 & 2 & 2 & \ldots & 6 & 2 \\
2 & 2 & \ldots & 0 & 2 & 2 & 2 & \ldots & 2 & 4
\end{pmatrix} = \det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix},
\]

\[
= \det(X + Y) \det(X - Y) = \det \begin{pmatrix}
6 & 2 & 4 & \ldots & \ldots & 4 \\
2 & 8 & 2 & \ldots & \ldots & 4 \\
4 & 2 & 8 & 2 & \ldots & 4 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
4 & 4 & \ldots & \ldots & 8 & 2 \\
4 & 4 & \ldots & \ldots & 2 & 6
\end{pmatrix} \times \det \begin{pmatrix} 6 & 2 & 0 & \ldots & \ldots & 0 \\ 2 & 4 & 2 & 0 & \ldots & 0 \\
0 & 2 & 4 & 2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \ldots & 4 & 2 \\
0 & 0 & \ldots & \ldots & 2 & 2
\end{pmatrix},
\]

\[
= 2^{n+1} \det \begin{pmatrix}
2 & 0 & 1 & \ldots & \ldots & 1 \\
0 & 3 & 0 & \ldots & \ldots & 1 \\
1 & 0 & 3 & 0 & \ldots & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
1 & 1 & \ldots & \ldots & 3 & 0 \\
1 & 1 & \ldots & \ldots & 0 & 2
\end{pmatrix} \times 2^{n+1} = 2^{n+1} n^2 \times 2^{n+1} = n^2 2^{2n+2}.
\]
COMPLEXITY OF SOME GRAPHS GENERATED BY SQUARE

\begin{equation}
\det(L - M) = \det \begin{pmatrix}
4 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 4 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 4 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 4 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & 2 & 0 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 4 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 4 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 4 & 0 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 2 \\
\end{pmatrix}_{2n\times 2n} = 4^{2n-1}.
\end{equation}

From Eq. (25) and Eq. (26) in Eq. (24), we get:

\[
\tau(\Gamma_n) = \frac{1}{(4n)^2} 4^{2n-1} n^2 2^{2n+2} = 2^{6n-4}.
\]

\[\Box\]

3.2. Complexity of some families of graphs generated by a square with average degree 6.

Consider the family of graphs \( \Upsilon_n \) generated by a square, and constructed as shown in Fig. 6.

It is noticed that according to the construction, the total number of vertices and edges are \(|V(\Upsilon_n)| = 4n \) and \(|E(\Upsilon_n)| = 12n - 8\), \( n = 1, 2, 3, 4, \ldots \). When \( n \) is large, the average degree of \( \Upsilon_n \) is 6.

\[\text{Figure 6. The graph } \Upsilon_n.\]
Theorem 3.6. For \( n \geq 1 \), the number of the spanning trees of the family of graphs \( \Upsilon_n \) is given by: 
\[
\tau(\Upsilon_n) = n^2 \left(2n + 4 \right) 3^{n-2} 2^{6n-5}.
\]

Proof. Applying Lemma 1.1, we have:
\[
\tau(\Upsilon_n) = \frac{1}{(4n)^2} \det \left[ 4nI - \overline{D} + \overline{A} \right] = \frac{1}{(4n)^2} \det \left( \begin{pmatrix} P & Q & J & J \\ Q & P & Q & J \\ J & Q & P & J \\ J & Q & P & J \end{pmatrix} \right).
\]

(27) 
\[
\tau(\Upsilon_n) = \frac{1}{(4n)^2} \left[ \det(P - J) \right]^2 \times \det(P + J + 2Q) \times \det(P + J - 2Q) .
\]

where 
\[
P = \begin{pmatrix}
2n+1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 5 & 1 & 1 & \ldots & 1 \\
1 & 1 & 5 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
1 & 1 & \ldots & 5 & 1 \\
1 & 1 & \ldots & 1 & 5
\end{pmatrix}_{n \times n},
\]

\[
Q = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 1 & \ldots & 1 \\
0 & 1 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 1 & \ldots & \ldots & 0 & 1 \\
0 & 1 & \ldots & 1 & 0
\end{pmatrix}_{n \times n}.
\]

\[
\tau(\Upsilon_n) = \frac{1}{(4n)^2} \det \left( \begin{pmatrix}
2n & 0 & \ldots & \ldots & 0 \\
0 & 4 & 0 & \ldots & 0 \\
0 & 0 & 4 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 4 & 0 \\
0 & 0 & \ldots & 0 & 4
\end{pmatrix} \right) \times \det \left( \begin{pmatrix}
2n+2 & 2 & \ldots & \ldots & 2 \\
2 & 6 & 4 & \ldots & 4 \\
2 & 4 & 6 & \ldots & 4 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
2 & 4 & \ldots & 6 & 4 \\
2 & 4 & \ldots & 4 & 6
\end{pmatrix} \right).
\]
\[
= \frac{1}{(4n)^2} \left( n \ 2^{2n-1} \right)^2 \times 4n \ \det \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 4 & 2 & \ldots & 2 \\
1 & 2 & 4 & \ldots & 2 \\
1 & 2 & 2 & \ldots & 2 \\
\vdots & \ldots & \ddots & \ddots & \vdots \\
1 & 2 & 2 & \ldots & 2 \\
1 & 2 & 2 & \ldots & 2 \\
\end{pmatrix}_{n \times n} \\
\times 2^n \ 3^{n-2} (2n+4),
\]

\[
= \frac{1}{(4n)^2} \left( n \ 2^{2n-1} \right)^2 \times 4n \ 2^{n-1} \det \begin{pmatrix}
2 & 1 & 1 & \ldots & 1 \\
1 & 2 & 1 & \ldots & 1 \\
1 & 1 & 2 & \ldots & 1 \\
1 & 1 & 1 & \ldots & 2 \\
\vdots & \ldots & \ddots & \ddots & \vdots \\
1 & 1 & \ldots & 2 & 1 \\
1 & 1 & \ldots & 1 & 2 \\
\end{pmatrix}_{n-1 \times n-1} \\
\times 2^n \ 3^{n-2} (2n+4).
\]

From Lemma 2.2, with \( x = 2 \), we obtain:
\[
\tau(\Upsilon_n) = \frac{1}{(4n)^2} \left( n \ 2^{2n-1} \right)^2 \times 2^{n+1} n^2 \times 2^n \ 3^{n-2} (2n+4) = n^2 (2n+4) 2^{6n-5} 3^{n-2}. \quad \square
\]

Consider the family of graphs \( \Pi_n \), generated by a square, and constructed as shown in Fig. 7. The graph \( \Pi_n \) has a number of vertices and edges are \( |V(\Pi_n)| = 4n \) and \( |E(\Pi_n)| = 12n - 8 \), \( n = 1, 2, \ldots \). When \( n \) is large, the average degree of \( \Pi_n \) is 6.

**Theorem 3.7.** For \( n \geq 2 \), the number of the spanning trees of the family of graphs \( \Pi_n \) is given by:
\[ \tau(\Pi_n) = \frac{9 \times 2^{4n-8}}{7} \left[ (16 - 5\sqrt{7}) (16 + 6\sqrt{7})^n + (16 + 5\sqrt{7}) (16 - 6\sqrt{7})^n - 9 \times 2^{n+1} \right]. \]

**Proof.** Applying Lemma 1.1, we have:

\[ \tau(\Pi_n) = \frac{1}{(4n)^2} \det \left[ 4nI - \overline{D} + \overline{A} \right] = \frac{1}{(4n)^2} \det \begin{pmatrix} P & Q & J & Q' \\ Q' & P & Q & J \\ J & Q' & P & Q \\ Q & J & Q' & P \end{pmatrix}. \]

(28) \[ \tau(\Pi_n) = \frac{1}{(4n)^2} \det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2} [\det(L + M) \times \det(L - M)] \]

where \( P = \begin{pmatrix} 5 & 0 & 1 & \ldots & 1 & 1 \\ 0 & 7 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 7 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 7 & 0 \\ 1 & 1 & \ldots & 0 & 5 \end{pmatrix}_{n \times n} \), \( Q = \begin{pmatrix} 0 & 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 1 \\ 0 & 1 & \ldots & 0 & 0 \end{pmatrix}_{n \times n} \),

\[ \det(L + M) = \det \begin{pmatrix} 6 & 1 & 2 & 2 & \ldots & 2 & 0 & 1 & 2 & 2 & \ldots & 2 \\ 1 & 8 & 1 & 2 & \ldots & 2 & 1 & 0 & 1 & 2 & \ldots & 2 \\ 2 & 1 & 8 & 1 & \ldots & 2 & 2 & 1 & 0 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \ldots & 8 & 1 & 2 & 2 & \ldots & 0 & 1 \\ 2 & 2 & \ldots & 1 & 6 & 2 & 2 & \ldots & 1 & 0 \\ 0 & 1 & 2 & 2 & \ldots & 2 & 6 & 1 & 2 & 2 & \ldots & 2 \\ 1 & 0 & 1 & 2 & \ldots & 2 & 1 & 8 & 1 & 2 & \ldots & 2 \\ 2 & 1 & 0 & 1 & \ldots & 2 & 2 & 1 & 8 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2 & 2 & \ldots & 0 & 1 & 2 & 2 & \ldots & 8 & 1 \\ 2 & 2 & \ldots & 1 & 0 & 2 & 2 & \ldots & 1 & 6 \end{pmatrix} = \det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix} \]
Using properties of determinants and straightforward induction, we get:

\( \det(L + M) = \det \begin{pmatrix} 6 & 2 & 4 & 4 & \ldots & 4 & 4 \\ 2 & 8 & 2 & 4 & \ldots & 4 & 4 \\ 4 & 2 & 8 & 2 & \ldots & 4 & 4 \\ 4 & 4 & 2 & \ldots & \ldots & \ldots & \ldots \\ 4 & 4 & 4 & 4 & \ldots & 2 & 8 \\ 4 & 4 & 4 & 4 & \ldots & 4 & 2 & 6 \end{pmatrix} \times \det \begin{pmatrix} 6 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 8 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 8 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 8 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 8 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 6 \end{pmatrix} \)
Consequently, we have the following non-homogeneous recurrence relation

\[(30) \quad a_{n+2} = 32a_{n+1} - 4a_n + 92^{n+3}.\]

Let \(a_n = b_n + d \cdot 2^{n+3}\) be the solution of the non-homogeneous recurrence relation, where \(b_n\) is the solution of homogeneous recurrence relation \(b_{n+2} - 32b_{n+1} + 4b_n = 0\). Substituting in Eq. (30), we get \(d = -\frac{9}{56}\). The characteristic equation of homogeneous recurrence relation is \(r^2 - 32r + 4 = 0\) with two roots being \((16 + 6\sqrt{7})\) and \((16 - 6\sqrt{7})\), then

\[b_n = \alpha (16 + 6\sqrt{7})^n + \beta (16 - 6\sqrt{7})^n.\]

The general solution of the non-homogeneous recurrence relation is

\[a_n = \alpha (16 + 6\sqrt{7})^n + \beta (16 - 6\sqrt{7})^n - \frac{9}{56} \cdot 2^{n+3}.\]

Using the initial conditions \(\det(L - M) = 196, 6400\) at \(n = 2, 3\), respectively, we have:

\[
\begin{align*}
196 &= \alpha (16 + 6\sqrt{7})^2 + \beta (16 - 6\sqrt{7})^2 - \frac{36}{7}, \\
6400 &= \alpha (16 + 6\sqrt{7})^3 + \beta (16 - 6\sqrt{7})^3 - \frac{72}{7}.
\end{align*}
\]

Solving these equations, we get \(\alpha = \frac{1}{14} [16 - 5\sqrt{7}]\), \(\beta = \frac{1}{14} [16 + 5\sqrt{7}]\). Therefore

\[(31) \quad \det(L - M) = a_n = \frac{1}{14} [(16 - 5\sqrt{7}) (16 + 6\sqrt{7})^n + (16 + 5\sqrt{7}) (16 - 6\sqrt{7})^n - 9 \cdot 2^{n+1}]\]

From Eq. (29) and Eq. (31) in Eq. (28), we get:

\[
\tau(\Pi_n) = \frac{9 \times 2^{4n-8}}{7} [(16 - 5\sqrt{7}) (16 + 6\sqrt{7})^n + (16 + 5\sqrt{7}) (16 - 6\sqrt{7})^n - 9 \times 2^{n+1}].
\]
Consider the family of graphs $\Phi_n$ generated by a square and constructed as shown in Fig. 8, the number of total vertices $|V(\Phi_n)|$ and edges $|E(\Phi_n)|$ are $|V(\Phi_n)| = 4n$ and $|E(\Phi_n)| = 12n - 8$, $n = 3, 4, \cdots$ according to the construction. Note that for $n = 2$ the graph $\Phi_n$ is isomorphic with the graph $\Upsilon_n$. When $n$ is large, the average degree of $\Phi_n$ is 6.

**Theorem 3.8.** For $n \geq 3$, the number of the spanning trees of the graph $\Phi_n$ is given by:

$$
\tau(\Phi_n) = 3^{2n-5}2^{4n+1} \left[ (45 + 26\sqrt{3})(2 + \sqrt{3})^{n-3} + (45 - 26\sqrt{3})(2 - \sqrt{3})^{n-3} \right].
$$

**Proof.** Let us applying Lemma 1.1, we have: $\tau(\Phi_n) = \frac{1}{(4n)^2} \det \left[ 4nI - D + A \right].$

(32)
where $P = \begin{pmatrix} 5 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 1 & 1 \\ 1 & 1 & \ldots & 1 & 1 & 5 \end{pmatrix}_{n \times n}$, $Q = \begin{pmatrix} 0 & 0 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 1 & 0 & \ldots & 1 & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 0 & 0 \\ 1 & 1 & \ldots & 1 & 0 & 0 \end{pmatrix}_{n \times n}$.

\[
\text{det}(P - J) = \text{det} \begin{pmatrix} 4 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 6 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 6 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 6 & 0 \\ 0 & 0 & \ldots & 0 & 4 \end{pmatrix}_{n \times n} = 2^{n+2} 3^{n-2}.
\]

\[
\text{det}(P + J + 2Q) = \text{det} \begin{pmatrix} 6 & 2 & 4 & \ldots & 4 & \ldots & 4 \\ 2 & 8 & 2 & 4 & \ldots & 4 & \ldots & 4 \\ 4 & 2 & 8 & 2 & \ldots & 4 & \ldots & 4 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 4 & 4 & \ldots & 8 & 2 \\ 4 & 4 & \ldots & 2 & 6 \end{pmatrix}_{n \times n} = 2^{n+1} \text{det} \begin{pmatrix} 2 & 0 & 1 & \ldots & 1 \\ 0 & 3 & 0 & 1 & \ldots & 1 \\ 2 & 0 & 3 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 1 & 1 & \ldots & 3 & 0 \\ 1 & 1 & \ldots & 0 & 2 \end{pmatrix}_{n \times n} = 2^{n+1} n^2.
\]

\[
\text{det}(P + J - 2Q) = \text{det} \begin{pmatrix} 6 & 2 & 0 & 0 & \ldots & 0 & \ldots & 0 \\ 2 & 8 & 2 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & 2 & 8 & 2 & \ldots & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 8 & 2 \\ 0 & 0 & \ldots & 2 & 6 \end{pmatrix}_{n \times n} = 2^n \text{det} \begin{pmatrix} 3 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 4 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 4 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 4 & 1 \\ 0 & 0 & \ldots & 1 & 3 \end{pmatrix}_{n \times n} = 2^n \text{det}(K).
\]

By expanding the first seven determinants of $K$, gives the values 30, 112, 418, 1560, 5822, \ldots for $n = 3, 4, 5, 6, 7, \ldots$, these values can be written in the form

\[418 = 4 \times 112 - 30,\]
1560 = 4 \times 418 - 112, \\
5822 = 4 \times 1560 - 418. \\

Consequently, we have the following homogeneous recurrence relation

\[ a_{n+2} = 4a_{n+1} - a_n. \]

The general solution of the recurrence relation (35), is

\[ a_n = \alpha (2 + \sqrt{3})^n - 3 + \beta (2 - \sqrt{3})^n. \]

Using the initial conditions \( \text{det}(k) = 30, 112 \) at \( n = 3, 4 \), respectively, we get \( \alpha = \frac{45 + 26\sqrt{3}}{3} \) and \( \beta = \frac{45 - 26\sqrt{3}}{3} \). Therefore,

\[ \text{det}(P+J-2Q) = 2^n \text{det}(K) = 2^n \left[ \left( \frac{45 + 26\sqrt{3}}{3} \right)(2 + \sqrt{3})^{n-3} + \left( \frac{45 - 26\sqrt{3}}{3} \right)(2 - \sqrt{3})^{n-3} \right]. \]

From Eq. (33), Eq. (34) and Eq. (36) in Eq. (32), we get:

\[ \tau(\Phi_n) = 3^{2n-5} 2^{4n+1} \left[ (45 + 26\sqrt{3})(2 + \sqrt{3})^{n-3} + (45 - 26\sqrt{3})(2 - \sqrt{3})^{n-3} \right] \]

- For \( n = 2 \), the graph \( \Phi_2 \) is isomorphic to the graph \( \Upsilon_2 \) and \( \tau(\Phi_2) = \tau(\Upsilon_2) = 4096 \).

### 3.3. Complexity of a graph generated by a square with average degree 8.

Consider the family of graphs \( \Omega_n \), generated by a square, and constructed as a composition \( C_4[P_n] \) of the graphs \( C_4 \) and \( P_n \) as shown in Fig. 9, the graph \( \Omega_n \) has number of vertices \( |V(\Omega_n)| = 4n \) and edges \( |E(\Omega_n)| = 4(4n - 3), n = 2, 3, \cdots \). When \( n \) is large, the average degree of \( \Omega_n \) is 8.

![Figure 9](image-url)
Theorem 3.9. For $n \geq 2$, the number of the spanning trees of the graph $\Omega_n$ is given by:

$$
\tau[\Omega_n] = \left(\frac{3^{n-3}}{500}\right)[(4 + \sqrt{15})^n - (4 - \sqrt{15})^n][(-60 + 17\sqrt{15})(4 + \sqrt{15})^n + (-60 - 17\sqrt{15})(4 - \sqrt{15})^n]^2.
$$

Proof. Let us applying Lemma (1.1), we have:

$$
\tau(\Omega_n) = \frac{1}{(4n)^2} \det [4nI - \overline{D} + A].
$$

(37)

$$
\tau[\Omega_n] = \frac{1}{(4n)^2} \det \begin{pmatrix} P & Q & Q & J \\ Q & P & J & Q \\ Q & J & P & Q \\ J & Q & Q & P \end{pmatrix} = \frac{1}{(4n)^2} \det \begin{pmatrix} L & M \\ M & L \end{pmatrix} = \frac{1}{(4n)^2} \left( \det(L + M) \times \det(L - M) \right).
$$

where $P = \begin{pmatrix} 6 & 0 & 1 & \ldots & 1 & 1 \\ 0 & 9 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 9 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 9 & 0 \\ 1 & 1 & \ldots & 0 & 6 \end{pmatrix}_{n \times n}$, $Q = \begin{pmatrix} 0 & 0 & 1 & \ldots & 1 & 1 \\ 0 & 0 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 0 & 0 \\ 1 & 1 & \ldots & 0 & 0 \end{pmatrix}_{n \times n}$,

$$
\det(L + M) = \det \begin{pmatrix} 6 & 0 & 2 & 2 & \ldots & 2 & 1 & 1 & 2 & 2 & \ldots & 2 \\ 0 & 9 & 0 & 2 & \ldots & 2 & 1 & 1 & 1 & 2 & \ldots & 2 \\ 2 & 0 & 9 & 0 & \ldots & 2 & 2 & 1 & 1 & 1 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \ldots & 9 & 0 & 2 & 2 & \ldots & 2 & 1 \\ 2 & 2 & \ldots & 0 & 6 & 2 & 2 & \ldots & 2 & 1 \\ 1 & 1 & 2 & 2 & \ldots & 2 & 6 & 0 & 2 & 2 & \ldots & 2 \\ 1 & 1 & 1 & 2 & \ldots & 2 & 0 & 9 & 0 & 2 & \ldots & 2 \\ 2 & 1 & 1 & 1 & \ldots & 2 & 2 & 0 & 9 & 0 & \ldots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \ldots & 1 & 1 & 2 & 2 & \ldots & 2 & 9 & 0 \\ 2 & 2 & \ldots & 1 & 1 & 2 & 2 & \ldots & 0 & 6 \end{pmatrix} = \det \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}.
$$
= \det \begin{pmatrix} 7 & 1 & 4 & 4 & \ldots & 4 \\ 1 & 10 & 1 & 4 & \ldots & 4 \\ 4 & 1 & 10 & 1 & \ldots & 4 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 4 & 4 & 4 & \ldots & 10 & 1 \\ 4 & 4 & 4 & \ldots & 1 & 7 \end{pmatrix} \times \det \begin{pmatrix} 5 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 8 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 8 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 8 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 5 \end{pmatrix},

= 4 \cdot 3^{n-1} \det \begin{pmatrix} 2 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 3 & 0 & 1 & \ldots & 1 \\ 1 & 0 & 3 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 3 & 0 \\ 1 & 1 & 1 & 2 & \ldots & 0 & 2 \end{pmatrix} \times \det \begin{pmatrix} 5 & -1 & 0 & 0 & \ldots & 0 \\ -1 & 8 & -1 & 0 & \ldots & 0 \\ 0 & -1 & 8 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 8 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 5 \end{pmatrix},

= 4 \cdot 3^{n-1} n^2 \times \det(X - Y).

Expanding the first seven determinants of \( (X - Y) \) yields the values 24, 190, 1496, 11778, 92728, 730046, 5747640 \ldots for \( n = 2, 3, 4, 5, 6, 7, 8, \ldots \), these values have the following recurrence relation

\( a_{n+2} = 8a_{n+1} - a_n \). Its characteristic equation is

\( r^2 - 8r + 1 = 0 \), with the roots being \( 4 + \sqrt{15} \) and \( 4 - \sqrt{15} \).

The general solution of the recurrence relation is

\( a_n = \alpha (4 + \sqrt{15})^n + \beta (4 - \sqrt{15})^n \). Using the initial conditions \( a_2 = 24 \), \( a_3 = 190 \) at \( n = 2, 3 \), respectively, we get: \( \alpha = \frac{-60 + 17\sqrt{15}}{15} \) and \( \beta = \frac{-60 - 17\sqrt{15}}{15} \).

Therefore, \( \det(X - Y) = a_n = \left(\frac{-60 + 17\sqrt{15}}{15}\right)(4 + \sqrt{15})^n + \left(\frac{-60 - 17\sqrt{15}}{15}\right)(4 - \sqrt{15})^n \),

(38) \( \det(L + M) = 4n^2 3^{n-1} \left[ \left(\frac{-60 + 17\sqrt{15}}{15}\right)(4 + \sqrt{15})^n + \left(\frac{-60 - 17\sqrt{15}}{15}\right)(4 - \sqrt{15})^n \right] \).
\[\det(L - M) = \det \begin{pmatrix} 6 & 0 & \ldots & \ldots & 0 & -1 & -1 & 0 & \ldots & 0 \\ 0 & 9 & 0 & \ldots & \ldots & 0 & -1 & -1 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 9 & 0 & \ldots & 0 & 0 & -1 & -1 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 9 & 0 & 0 & 0 & 0 & \ldots & -1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 6 & 0 & \ldots & \ldots & -1 & -1 & \ldots \\ -1 & -1 & 0 & \ldots & 0 & 6 & 0 & \ldots & \ldots & 0 & \ldots \\ -1 & -1 & -1 & 0 & \ldots & 0 & 0 & 9 & 0 & \ldots & 0 & \ldots \\ 0 & -1 & -1 & -1 & \ldots & 0 & 0 & 0 & 9 & 0 & \ldots & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -1 & -1 & 0 & 0 & \ldots & 9 & 0 & \ldots \\ 0 & 0 & \ldots & -1 & -1 & 0 & 0 & \ldots & \ldots & 0 & 6 \end{pmatrix}\]

\[= \det \begin{pmatrix} K \\ H \\ \hline H & K \end{pmatrix} = \det(K + H) \times \det(K - H) ,\]

\[= \det \begin{pmatrix} 5 & -1 & 0 & 0 & \ldots & \ldots & 0 \\ -1 & 8 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 8 & -1 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 8 & -1 \\ 0 & 0 & 0 & \ldots & -1 & -1 & 5 \end{pmatrix} \times \det \begin{pmatrix} 7 & 1 & 0 & 0 & \ldots & \ldots & 0 \\ 1 & 8 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 8 & 1 & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 8 & 1 \\ 0 & 0 & 0 & \ldots & 1 & 7 \end{pmatrix} .\]

Expanding the first seven determinants of \((K - H)\) yields the values 48, 378, 2976, 23430, 184464, 1452282, 11433792 \cdots for \(n = 2, 3, 4, 5, 6, 7, 8, \cdots\), these values have the recurrence relation

\[a_{n+2} = 8a_{n+1} - a_n.\] Its characteristic equation is \(r^2 - 2r + 1 = 0\), with the roots being \(4 + \sqrt{15}\) and \(4 - \sqrt{15}\).

The general solution of the recurrence relation is \(a_n = \alpha (4 + \sqrt{15})^n + \beta (4 - \sqrt{15})^n\).

Using the initial conditions \(a_n = 48, 378\) at \(n = 2, 3\), respectively, we get: \(\alpha = \frac{\sqrt{15}}{5}\) and \(\beta = -\frac{\sqrt{15}}{5}\). Therefore, \(\det(K - H) = a_n = (\frac{\sqrt{15}}{5}) \left[ (4 + \sqrt{15})^n - (4 - \sqrt{15})^n \right] .\)
(39)
\[
det(L - M) = \left(\frac{\sqrt{15}}{5}\right) \left[(4 + \sqrt{15})^n - (4 - \sqrt{15})^n\right] \left[(\frac{-60 + 17\sqrt{15}}{15})(4 + \sqrt{15})^n + (\frac{-60 - 17\sqrt{15}}{15})(4 - \sqrt{15})^n\right].
\]

From Eq. (38) and Eq. (39) in Eq. (37), we get:
\[
\tau[\Omega_n] = \left(\frac{3^{n-3}}{500}\right)\left[(4 + \sqrt{15})^n - (4 - \sqrt{15})^n\right]\left[(-60 + 17\sqrt{15})(4 + \sqrt{15})^n + (-60 - 17\sqrt{15})(4 - \sqrt{15})^n\right]^2.
\]

\[\square\]

4. NUMERICAL RESULTS

In the following table 1 we enumerate some of the values of \(\tau(G)\), the number of spanning trees, of our graphs \(\Psi_n, \Gamma_n, \Lambda_n, \Delta_n, \Theta_n, \Upsilon_n, \Pi_n, \Phi_n\) and \(\Omega_n\).

| \(n\) | \(\tau(\Psi_n)\) | \(\tau(\Gamma_n)\) | \(\tau(\Lambda_n)\) | \(\tau(\Delta_n)\) | \(\tau(\Theta_n)\) | \(\tau(\Upsilon_n)\) | \(\tau(\Pi_n)\) | \(\tau(\Phi_n)\) | \(\tau(\Omega_n)\) |
|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 4 | 4 | 4 | 45 | 4 | 4 | 4 | 4 |
| 2 | 256 | 256 | 216 | 384 | 3509 | 4096 | 3528 | 4096 | 20736 |
| 3 | 9216 | 16384 | 19200 | 31500 | 284089 | 2211840 | 1843200 | 2211840 | 30703050 |
| 4 | 262144 | 1048576 | 1775616 | 2558796 | 23057865 | 909569664 | 941432832 | 1189085184 | 44957265408 |
| 5 | 6553600 | 67108864 | 165347328 | 207746836 | 1871801381 | 317089382400 | 219 \times 3^2 \times 101761 | 222 \times 3^6 \times 209 | 3^7 \times 3009731890 |

**TABLE 1.** Some values of spanning trees of our graphs.

According to the results in table 1, it is easily seen that:

(i) Of all the graphs of average degree 4, the graph \(\Theta_n\) has a number of spanning trees greater than the other four graphs \(\Psi_n, \Gamma_n, \Lambda_n, \Delta_n\).

(ii) Of all the graphs of average degree 6, the graph \(\Phi_n\) has a number of spanning trees greater than the other two graphs \(\Upsilon_n, \pi_n\).

(iii) The number of the spanning trees of the graph \(\Omega_n\), of average degree 8 is the largest among all the graphs at different values of \(n \geq 2\).
5. Entropy of Our Networks

Because the complexity of a network $\tau(G)$ increases exponentially with the number of vertices, there exists a constant $\rho(G)$, called the entropy of spanning trees [15], described by this relation:

$$\rho(G) = \lim_{n \to \infty} \frac{\ln \tau(G)}{|V(G)|} .$$

By calculating the entropy of graphs (networks), we hope to determine the best one. The entropy of spanning trees of a network is a quantitative measure of the number of spanning trees to evaluate the goodness and the resistance of a network and to describe its structure. The most goodness and resistance network is the network that has the highest spanning-tree entropy. According to the definition of the entropy of spanning trees of a network, the bigger the entropy value, the more the number of spanning trees, so there are more possibilities of connections between two vertices.

$$\rho(\Psi_n) = \lim_{n \to \infty} \frac{\ln \tau(\Psi_n)}{4n} = \lim_{n \to \infty} \frac{\ln n^2 2^{4n-2}}{4n} = \ln 2 = 0.693147 .$$

$$\rho(\Gamma_n) = \lim_{n \to \infty} \frac{\ln \tau(\Gamma_n)}{4n} = \lim_{n \to \infty} \frac{\ln 2^{6n-4}}{4n} = \frac{3 \ln 2}{2} = 1.03972 .$$

$$\rho(\Lambda_n) = \lim_{n \to \infty} \frac{\ln \{ 3 2^{2n-7} [(6 + 4\sqrt{2})^n + (6 - 4\sqrt{2})^n + 2^{n+1}] \} }{4n} = \frac{7 \ln 2}{4} = 1.213007 .$$

$$\rho(\Upsilon_n) = \lim_{n \to \infty} \frac{\ln \left[ n^2 (2n+4) 2^{6n-5} 3^{n-2} \right] }{4n} = \frac{\ln 3 + 6 \ln 2}{4} = 1.314373 .$$

$$\rho(\Pi_n) = \lim_{n \to \infty} \frac{\ln \tau(\Pi_n)}{4n} = \frac{4 \ln 2 + \ln( 16 + 6 \sqrt{7})}{4} = 1.558598 .$$

$$\rho(\Phi_n) = \lim_{n \to \infty} \frac{\ln \tau(\Phi_n)}{4n} = \frac{2 \ln 3 + 4 \ln 2 + \ln(2 + \sqrt{3})}{4} = 1.571692 .$$

$$\rho(\Omega_n) = \lim_{n \to \infty} \frac{\ln \tau(\Omega_n)}{4n} = \frac{\ln 3 \times ( 3 \ln (4 + \sqrt{15})/\ln 3 + 1) }{4} = 1.82223 .$$

$$\rho(\Delta_n) = \lim_{n \to \infty} \frac{\ln \left[ \frac{1}{12\sqrt{2}} \left( (2 + \sqrt{3})^n - (2 - \sqrt{3})^n \right)^2 \left[ (3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n \right] \right] }{4n}$$

$$= \frac{\ln \left( 2 + \sqrt{3} \right) }{2} = 1.866025 .$$
\[ \rho(\Theta_n) = \lim_{n \to \infty} \frac{\ln \left( \tau(\Theta_n) \right)}{4n+1} = \frac{\ln \left( \frac{2 + \sqrt{3}}{2} \right)}{2} = 1.866025. \]

The following table 2 illustrates the values of entropy and average degree of studying networks.

| Graphs | $\Psi_n$ | $\Gamma_n$ | $\Lambda_n$ | $\Delta_n$ | $\Theta_n$ | $\Upsilon_n$ | $\Pi_n$ | $\Phi_n$ | $\Omega_n$ |
|--------|----------|-----------|-----------|-----------|-----------|-----------|------|------|--------|
| $|V(G)|$ | 4n       | 4n       | 4n       | 4n       | 4n+1      | 4n       | 4n   | 4n   | 4n     |
| $|E(G)|$ | 8n-4     | 8n-4     | 8n-4     | 8n-4     | 8n        | 12n-8    | 12n-8| 12n-8| 12n-8  |
| Average degree | 4        | 4        | 4        | 4        | 4         | 6        | 6    | 6    | 8      |
| $\rho(G)$ | 0.693147 | 1.03972  | 1.213007 | 1.866025 | 1.866025  | 1.314373 | 1.558598 | 1.571692 | 1.82223 |

**TABLE 2.** The entropy of the studied graphs.

Comparing the value of entropy in our design networks (graphs), we have found that:

(i) Of all our different design networks (graphs) of average degree 4, the entropy of the graphs $\Delta_n$ and $\Theta_n$ is the largest.

(ii) Among all our different design networks (graphs) of average degree 6, the entropy of the graph $\Phi_n$ is the largest.

(iii) The entropy of the graphs $\Delta_n$ and $\Theta_n$ of average degree 4 is larger than the entropy of all graphs of average degrees 6 and 8.

Now we compare the value of entropy in our design networks (graphs) with other networks:

In 2013, Zhang et all [16] proved that the Apollonian graph of average degree 6 has entropy 1.354. In 2019, Daoud [17] introduced some networks generated by a triangle with average degree 4 and 6, and proved that the graph $\Upsilon_n$ of average degree 6 has entropy 1.514280. It is clear that the entropy of our studied networks $\Pi_n$ and $\Phi_n$ with the same average degree 6 is larger than the entropy of the Apollonian graph, and the graphs generated by a triangle with the same average degree 6. Finally, the entropy of spanning trees of our studied (graphs) networks $\Theta_n$ and $\Delta_n$ of average degree 4 are the highest among these networks.
6. CONCLUSIONS

The complexity which is the number of spanning trees in the networks is a significant invariant. The enumerating of this number is not only helpful from a combinatorial standpoint, but it is also an important measure of the network reliability and electrical circuit design. In this paper, we introduce nine network designs that are created by a square of average degree 4, 6 and 8, then we obtain a simple and evident expression for the number of spanning trees of these networks using some basic properties of orthogonal polynomials, block matrix analysis technique, and recurrence relations. Finally, we compute the entropy of these networks and compare the entropy of spanning trees on our networks with the other triangle and Apollonian networks. We deduce that the entropy of our networks is the highest among the studied triangle and Apollonian networks and the networks $\Theta_n$ and $\Delta_n$ of average degree 4 are the highest among these networks.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] D.B. West, Introduction to Graph Theory, 2 Eds., Prentice-Hall of India, New Delhi, 2003.
[2] N.L. Biggs, Algebraic Graph Theory, 2 Eds., Cambridge University Press, Cambridge, 1993.
[3] A.K. Kelmans, V.M. Chelnokov, A certain polynomials of a graph and graphs with an external number of trees, J. Comb. Theory (B), 16 (1974), 197-214.
[4] H.N.V. Temperley, Graph theory and applications, Halsted Press, New York, 1981.
[5] J. Sedlacek, Lucas number in graph theory Mathematics (Geometry and Graph Theory) (Chech), 111–115, Univ. Karlova, Prague, 1970.
[6] L. Clark, On the enumeration of multipartite spanning trees of the complete graph, Bull. ICA, 38 (2003), 50-60.
[7] S.N. Daoud, Generating formulas of the number of spanning trees of some special graphs, Eur. Phys. J. Plus. 129 (2014), 146.
[8] S. N. Daoud, K. Mohamed, The complexity of some families of cycle-related graphs, J. Taibah Univ. Sci. 11 (2017), 205-228.

[9] J.-B. Liu, S.N. Daoud, Number of Spanning Trees in the Sequence of Some Graphs, Complexity. 2019 (2019), 4271783.

[10] S.N. Daoud, W. Saleh, Complexity trees of the sequence of some nonahedral graphs generated by triangle, Heliyon. 6 (2020), e04786.

[11] M.R.Z. El Deen, W.A. Aboamer, Complexity of Some Duplicating Graphs, IEEE Access. 9 (2021), 56736-56756.

[12] J.C. Mason, D.C. Handscomb, Chebyshev polynomials, Chapman & Hall/CRC, Boca Raton, 2003.

[13] Y. Zhang, X. Yong, M.J. Golin, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, Discrete Math. 298 (2005), 334–364.

[14] R.B. Bapat, Graphs and Matrices, 1st ed. Springer, New York, 2010.

[15] R. Mokhlissi, D. Lotfi, J. Debnath, M. El Marraki, N. EL Khattabi, The evaluation of the number and the entropy of spanning trees on generalized small-world networks, J. Appl. Math. 2018 (2018), 1017308.

[16] J. Zhang, W. Sun, G. Xu, Enumeration of spanning trees on Apollonian networks, J. Stat. Mech. 2013 (2013), P09015.

[17] S.N. Daoud, Number of spanning trees of some families of graphs generated by a triangle, J. Taibah Univ. Sci. 13 (2019), 731-739.