Spontaneous Symmetry Breaking and Chiral Symmetry

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Abstract

In this introductory lecture, some basic features of the spontaneous symmetry breaking are discussed. More specifically, \( \sigma \)-model, non-linear realization, and some examples of spontaneous symmetry breaking in the non-relativistic system are discussed in details. The approach here is more pedagogical than rigorous and the purpose is to get some simple explanation of some useful topics in this rather wide area.
I. Introduction

The symmetry principle is perhaps the most important ingredient in the development of high energy physics. Roughly speaking, the symmetries of the physical system lead to conservation laws, which give many important relations among physical processes. Many of the symmetries in Nature are however approximate symmetries rather than exact symmetries and are also very useful in the understanding of various phenomena in high energy physics. Among the broken approximate symmetries, the most interesting one is the Spontaneous Symmetry Breaking (SSB) which seems to have played a special role in high energy physics. Many important progress has come from the understanding of the SSB. The SSB is characterized by the fact that symmetry breaking shows up in the ground state rather than in the basic interaction. This makes it difficult to uncover this kind of approximate symmetries. Historically, SSB was first discovered around 1960 in the study of superconductivity in the solid state physics by Nambu and Goldstone. One of consequences of SSB is the presence of the massless excitation, called the Nambu-Goldstone boson, or just Goldstone boson for short. Later, Nambu applied the idea to the particle physics. In combination with $SU(3) \times SU(3)$ current algebra, SSB has been quite successful in the understanding of the chiral symmetry in the low energy phenomenology of strong interaction. More importantly, in 1964 it was discovered by Higgs and others that in the context of gauge theory, SSB has the remarkable property that it can convert the long range force in the gauge theory into a short range force. Thus it avoids both the massless Goldstone bosons and the massless gauge bosons. Weinberg and Salam then applied this ideas to construct a model of electromagnetic and weak interactions. The significance of this model was not realized until t’Hooft show in 1971 that it was renomalizable. Since then this model has enjoyed remarkable experimental success and now called the “Standard Model of Electroweak Interactions”. Undoubtedly, this will serve as benchmark for any new physics for years to come.

In this article I will give a simple introduction to the spontaneous symmetry breaking and its application to chiral symmetries in the hadronic interaction. The emphasis is on the qualitative understanding rather than completeness and mathematical rigor. Even though SSB has been quite successful in explaining many interesting phenomena, its implementation in the theoretical framework is more or less put in by hand and it is not at all clear what is the origin of SSB. Here I will also discuss some non-relativistic example where the physics is more tractable in the hope that they might give some hints about the true nature of SSB. Maybe good understanding of SSB might extend its applicability to some new frontier.

II. $SU(2) \times SU(2)$ $\sigma$-Model

The $\sigma$-model has a long and interesting history. It was originally constructed in 1960’s as a tool to study the chiral symmetry in the system with pions and
nucleons. Later the spontaneous symmetry breaking and PCAC (partially conserved axial current) were incorporated. Even though this model is not quite phenomenologically correct, it remains the simplest example which realizes many important aspects of broken symmetries. Even though the strong interaction is now described by QCD, the \( \sigma \)-model of pions and nucleons is still useful as an effective interaction in the low energies where it is difficult to calculate directly from QCD. In addition, the \( \sigma \)-model has also been used quite often as a framework to test many interesting ideas in field theory and string theory. Here we will discuss the most basic features of the \( \sigma \)-model.

The Lagrangian for \( SU(2) \times SU(2) \) \( \sigma \)-Model is given by

\[
\mathcal{L} = \frac{1}{2} \left[ (\partial_{\mu} \sigma)^2 + \left( \partial_{\mu} \vec{\pi} \right)^2 \right] + \frac{\mu^2}{2} \left( \sigma^2 + \vec{\pi}^2 \right) - \frac{\lambda}{4} \left( \sigma^2 + \vec{\pi}^2 \right)^2 + \overline{\eta} i \gamma^\mu \partial_{\mu} \eta + g \overline{N} \left( \sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi} \right) N
\]

where \( \vec{\pi} = (\pi_1, \pi_2, \pi_3) \) is the isotriplet pion fields, \( \sigma \) is the isosinglet field, and \( N \) is the isodoublet nucleon field. To discuss the symmetry property, it is more useful to write this Lagrangian as:

\[
\mathcal{L} = \frac{1}{2} \text{tr} \left( \partial_{\mu} \Sigma \partial^\mu \Sigma^\dagger \right) + \frac{\mu^2}{4} \text{tr} \left( \Sigma \Sigma^\dagger \right) - \frac{\lambda}{8} \left[ (\Sigma \Sigma^\dagger) \right]^2 + \overline{N}_L i \gamma^\mu \partial_{\mu} N_L + \overline{N}_R i \gamma^\mu \partial_{\mu} N_R + g (\overline{N}_L \Sigma N_R + \overline{N}_R \Sigma^\dagger N_L)
\]

where

\[
\Sigma = \sigma + i \vec{\tau} \cdot \vec{\pi}, \quad N_L = \frac{1}{2} (1 - \gamma_5) N, \quad N_R = \frac{1}{2} (1 + \gamma_5) N
\]

This Lagrangian is now clearly invariant under transformation,

\[
\Sigma \rightarrow \Sigma' = L \Sigma R^\dagger, \quad N_L \rightarrow N'_L = LN_L, \quad N_R \rightarrow N'_R = RN_R
\]

where

\[
L = \exp \left(-i \vec{\tau} \cdot \vec{\theta}_L \right), \quad R = \exp \left(-i \vec{\tau} \cdot \vec{\theta}_R \right)
\]

are two arbitrary \( 2 \times 2 \) unitary matrices. Thus the symmetry group is \( SU(2)_L \times SU(2)_R \) and representation contents under this group are

\[
\Sigma \sim \left( \begin{array}{c} \frac{1}{2} \\
\frac{1}{2} \end{array} \right), \quad N_L \sim \left( \begin{array}{c} \frac{1}{2} \\
0 \end{array} \right), \quad N_R \sim \left( \begin{array}{c} 0 \\
\frac{1}{2} \end{array} \right)
\]

Remark: The nucleon mass term \( \overline{N}_L N_R + \text{h.c.} \) transforms as \( \left( \frac{1}{2}, \frac{1}{2} \right) \) representation and is not invariant. One way to construct invariant nucleon mass term is to introduce another doublet of fermions with opposite parity,

\[
N'_L \sim \left( 0, \frac{1}{2} \right), \quad N'_R \sim \left( \frac{1}{2}, 0 \right)
\]
so that the term \((N_L'N_R + N_R'N_L + h.c.)\) is invariant. This will give same mass to both doublets and is usually called parity doubling. As we shall see later, another way to give mass to nucleon is by spontaneous symmetry breaking which does not require another doublet.

The general form of Noether current is of the form

\[
J_\mu \sim \sum_i \frac{\partial L}{\partial (\partial_\mu \phi_i)} \delta \phi_i
\]

where \(\delta \phi_i\) is the infinitesimal change of the fields under the symmetry transformations. We have for the left-handed transformation,

\[
\begin{align*}
\delta_L \sigma &= \overrightarrow{\theta}_L \cdot \overrightarrow{\pi}, \\
\delta_L \pi &= -\overrightarrow{\theta}_L \sigma + \overrightarrow{\theta}_L \times \overrightarrow{\pi}, \\
\delta_L N_L &= -i \overrightarrow{\theta}_L \cdot 2 N_L, \\
\delta_L N_R &= 0
\end{align*}
\]

and

\[
J^a_{L\mu} = \varepsilon^{abc} x^b \partial_\mu \pi^c + [\sigma \partial_\mu \pi^a - \pi^a \partial_\mu \sigma] + \overline{N}_L \gamma_\mu \frac{\pi^a}{2} N_L
\]

Similarly,

\[
\begin{align*}
\delta_R \sigma &= -\overrightarrow{\theta}_R \cdot \overrightarrow{\pi}, \\
\delta_R \pi &= \overrightarrow{\theta}_R \sigma + \overrightarrow{\theta}_R \times \overrightarrow{\pi}, \\
\delta_R N_L &= 0, \\
\delta_R N_R &= -i \overrightarrow{\theta}_R \cdot 2 N_R
\end{align*}
\]

and

\[
J^a_{R\mu} = \varepsilon^{abc} x^b \partial_\mu \pi^c - [\sigma \partial_\mu \pi^a - \pi^a \partial_\mu \sigma] + \overline{N}_R \gamma_\mu \frac{\pi^a}{2} N_R
\]

The corresponding charges are given by

\[
Q^a_L = \int d^3 x J^a_{L0}, \quad Q^a_R = \int d^3 x J^a_{R0}
\]

Using the canonical commutation relations, we can derive

\[
[Q^a_L, Q^b_L] = i \varepsilon_{ijk} Q^i_L, \quad [Q^a_R, Q^b_R] = i \varepsilon_{ijk} Q^i_R, \quad [Q^a_L, Q^b_L] = 0
\]

which is the \(SU_L(2) \times SU_R(2)\) algebra.

The vector and axial charges are given by

\[
Q^a = Q^a_R + Q^a_L, \quad Q^5 = Q^a_R - Q^a_L
\]

In particular, the axial charges are

\[
Q^5_i = \int d^3 x A^0_i(x) = \int d^3 x \left[ i (\sigma \partial_0 \pi_i - \pi_i \partial_0 \sigma) + N \gamma_j N \right]
\]
Remark: Another way to describe the symmetry of the $\sigma$-model is the $O(4)$ symmetry, which is isomorphic to $SU(2) \times SU(2)$ locally and is characterized by $4 \times 4$ orthogonal matrix,

$$RR^T = R^T R = 1$$

The infinitesimal transformation is

$$R_{ij} = \delta_{ij} + \varepsilon_{ij}, \quad \text{with} \quad \varepsilon_{ij} = -\varepsilon_{ji}$$

The scalar field $\phi_i = (\pi_1, \pi_2, \pi_3, \sigma)$ transform as 4-dimensional vector,

$$\phi_i \rightarrow \phi'_i = R_{ij} \phi_j \simeq \phi_i + \varepsilon_{ij} \phi_j$$

The combination $\phi_i \phi_i = \sigma^2 + \vec{\pi}^2$ is just the length of the vector $\phi_i$ and is clearly invariant under the rotations in 4-dimension. If we take

$$\varepsilon_{ij} = \varepsilon_{ijk} \alpha_k, \quad \varepsilon_{4i} = \beta_i, \quad i, j, k = 1, 2, 3$$

we get

$$\vec{\pi}' = \vec{\pi} + \vec{\alpha} \times \vec{\pi} + \vec{\beta} \cdot \vec{\pi} \quad \sigma' = \sigma + \vec{\beta} \cdot \vec{\pi}$$

Thus we see from Eqs(6,8) that the parameters $\vec{\alpha}$ correspond to vector transformations and $\vec{\beta}$ the axial transformation.

A. Spontaneous Symmetry Breaking

The classical ground state is determined by minimum of the self interaction of scalars,

$$V(\sigma, \vec{\pi}) = -\mu^2 \left( \sigma^2 + \vec{\pi}^2 \right) + \frac{\lambda}{4} \left( \sigma^2 + \vec{\pi}^2 \right)^2$$

The minimum of the potential is located at

$$\sigma^2 + \vec{\pi}^2 = \frac{\mu^2}{\lambda} \equiv v^2$$

which is a 3-sphere, $S^3$ in the 4-dimensional space formed by the scalar fields. Each point on $S^3$ is invariant under $O(3)$ rotations. For example, the point $(0, 0, 0, v)$ is invariant under the rotations of the first 3 components of the vector. Then, after a point on $S^3$ is chosen to be the classical ground state, the symmetry is broken spontaneously from $O(4)$ to $O(3)$. Note that different points on $S^3$ are related to each other by the action of those rotations which are in $O(4)$ but not in $O(3)$. These rotations are usually denoted by $O(4)/O(3)$.(This is called the coset space.). Thus we can identify 3-sphere with $O(4)/O(3)$.

For the quantum theory, we need to expand the fields around the classical values

$$\sigma = v + \sigma', \quad \vec{\pi}' = \vec{\pi}, \quad \text{where} \quad <\sigma> = v$$
Here $v$ is usually called the vacuum expectation value (VEV). Then we see that

$$V \left( \sigma, \vec{\pi} \right) = \mu^2 \sigma^2 + \lambda \nu \sigma' \left( \sigma'^2 + \vec{\pi}'^2 \right) + \frac{\lambda}{4} \left( \sigma'^2 + \vec{\pi}'^2 \right)^2 \quad (17)$$

$$gN \left( \sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi} \right) N = g\nu N + gN \left( \sigma' + i\gamma_5 \vec{\tau} \cdot \vec{\pi}' \right) N$$

Thus $\pi'$s are massless and $N$'s are massive.

Remark: If we had made another choice for VEV, e.g.

$$< \pi_3 >= v, \quad < \pi_1 >= < \pi_2 >= < \sigma >= 0$$

The physics is still the same as we will now illustrate. In this case, we write $\pi_3 = \pi_3 + v$ to get

$$V \left( \sigma, \vec{\pi} \right) = \mu^2 \pi'^2 + \text{(cubic terms and higher)}$$

$$gN \left( \sigma + i\gamma_5 \vec{\tau} \cdot \vec{\pi} \right) N = g\nu N + gN \tau_3 \gamma_5 N + \cdots$$

Thus we still have 3 massless scalar fields. For the nucleon, if we define

$$N_L = \exp \left( -i\pi_3 \frac{\tau_3}{2} \right) N'_L, \quad N_R = N'_R$$

we get

$$g\nu N \tau_3 \gamma_5 N = g\nu N'$$

which the mass term for the new field. It is easy to see that $Q^4_1, Q^5_2, Q^3$ form an unbroken $SU(2)$ algebra.

There are several interesting features worth noting:

(1) The $\pi'$s are massless. This is a consequence of the Goldstone theorem which states that spontaneous symmetry breaking (SSB) of a continuous symmetry will give massless particle or zero energy excitation. This theorem will be discussed in more detail in next subsection.

(2) After SSB, the original multiplet $\left( \sigma, \vec{\pi} \right)$ splits into massless $\pi'$s and massive $\sigma$. Also the nucleons become massive. Thus even though the interaction is $SU(2)_L \times SU(2)_R$ symmetric the spectrum is only $SU(2)$ symmetric. This is the typical consequence of SSB. In some sense, the original symmetry is realized by combining the $SU(2)$ multiplet, e.g. $N$, with the massless Goldstone bosons to form the multiplets of $SU(2)_L \times SU(2)_R$. This, as we will discuss later, is the basis of the low energy theorem.

(3) The axial current in Eq(13) after the SSB will have a term linear in $\pi$ field,

$$A_{\mu}^i = iv\partial_{\mu} \pi_i + \cdots \quad (18)$$

which is responsible for the matrix element,

$$< 0 | A_\mu^i | \pi_j (p) > = ip^{\mu}v \quad (19)$$
Using this matrix element in $\pi$ decay, we can identify the VEV $v$ with the pion decay constant $f_\pi$. This coupling between axial current $A^\mu_i$ and $\pi_i$ will give rise to a massless pole.

(4) The appearance of the cubic term $\sigma' \left( \sigma'^2 + \pi'^2 \right)$ and the mass term $gv\bar{NN}$ is the result of the spontaneous symmetry breaking. Since these terms have dimension 3, they are usually called soft breaking, in contrast to the dimension 4 hard breaking terms.

(5) In the scalar self interaction, quartic, cubic, and quadratic terms have only 2 parameters, $\lambda, \mu$. This means that these 3 terms are not independent, and there is a relation among them. This is an example of low energy theorem for theory with spontaneous symmetry breaking.

B. Low energy theorem

The SSB leads to many relations which are quite different from the usual symmetry breaking. The most distinct ones are relations among amplitudes involving Goldstone bosons in low energies. As we have mentioned before, these relations are consequence of the fact that Goldstone bosons are massless and can be tagged on to other particles to form a larger multiplet. Since Goldstone bosons do carry energies, this is possible only in limit that Goldstone bosons have zero energies.

Consider the following processes involving the Goldstone bosons in the external states.

(i) $\pi^0 (p_1) + \sigma (p_2) \rightarrow \pi^0 (p_3) + \sigma (p_4)$

The tree-level contributions are coming from diagrams in Fig1.

Fig 1 Tree graphs for $\pi\pi$ scattering
The amplitudes for these diagrams are given by,

\[
M_a = (-2i\lambda v)^2 \frac{i}{s}, \quad M_b = 3 (-2i\lambda v)^2 \frac{i}{t - m_\sigma^2}, \quad M_c = (-2i\lambda v)^2 \frac{i}{u}, \quad M_d = -2i\lambda
\]

\[
M = M_a + M_b + M_c + M_d = 4i\lambda^2 v^2 \left[ \frac{1}{s} + \frac{3}{t - m_\sigma^2} + \frac{1}{u} + \frac{1}{2\lambda v^2} \right]
\]  

(20)

Here \(s\), \(t\), and \(u\) are the usual Mandelstam variables, \(s = (p_1 + p_2)^2\), \(t = (p_1 - p_3)^2\), \(u = (p_1 - p_4)^2\). In the limit where pions have zero momenta, \(p_1 = p_3 = 0\), we get \(s = u = m_\sigma^2\), \(t = 0\) and

\[
M = 4i\lambda^2 v^2 \left[ \frac{1}{m_\sigma^2} + \frac{1}{m_\sigma^2} - \frac{3}{m_\sigma^2} + \frac{1}{m_\sigma^2} \right] = 0
\]  

(22)

where we have used \(m_\sigma^2 = 2\lambda v^2\). Thus the amplitude vanishes in the soft pion limit, i.e. \(p_\pi \to 0\).

(iii) \(\pi^0\pi^0 \to \pi^0\pi^0\)

Similar calculation gives

\[
M = M_a + M_b + M_c + M_d = -2i\lambda \left[ \frac{s}{s - m_\sigma^2} + \frac{t}{t - m_\sigma^2} + \frac{u}{u - m_\sigma^2} \right].
\]

In the soft pion limit, \(p_i \to 0\), we get

\[
M \simeq \frac{2i\lambda}{m_\sigma^2} (s + t + u) = \frac{i}{v^2} (s + t + u) \to 0.
\]  

(23)

This is the same as the limit, \(m_\sigma^2 \to \infty\), because soft pion means pion momentum much smaller than \(m_\sigma^2\). These are simple examples of the low energy theorem which says that physical amplitudes vanish in the limit where myonata of Goldstone bosons go to zero.

In examples above, the vanishing of these amplitudes results from some cancellation among different contributions. Since this is a general property of the Goldstone boson, there should be a better way of getting this. It turns that one can change the variables representing the scalar fields such that Goldstone bosons always enter with derivative coupling. Then the vanishing of the amplitudes involving Goldstone boson is manifest. This can be accomplished by the field redefinition which we will now describe briefly(7). Suppose we start from a Lagrangian with field \(\phi\) and make a transformation to a new field \(\eta\), with the relation,

\[
\phi = \eta F(\eta)
\]

where \(F(\eta)\) is some power series in \(\eta\). If we impose the condition that \(F(0) = 1\), the free Lagrangian for \(\eta\) will be the same as that for \(\phi\). Then according to a general theorem valid with rather weak restrictions on the Lagrangian and \(F(\eta)\), the on-shell matrix elements calculated with \(\eta\) fields and with \(\phi\) fields
are the same. We will use this field redefinition to write the Goldstone boson interaction in terms of derivative coupling. Consider a simplified Lagrangian given by

\[
L = \frac{1}{2} \left[ (\partial_{\mu} \sigma)^2 + (\partial_{\mu} \pi)^2 \right] + \frac{\mu^2}{2} (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 \tag{24}
\]

which is just the \(O(2)\) version of the \(\sigma\)-model without the nucleon. As before, the SSB will require the shift of the \(\sigma\)-field, as in Eq(16) and \(\pi\) field is massless (Goldstone boson). Equivalently, we can use a complex field defined by \(\phi = \frac{1}{\sqrt{2}} (\sigma + i\pi)\) so that the Lagrangian is of the form

\[
L = \partial_{\mu} \phi^{\dagger} \partial^{\mu} \phi + \mu^2 \phi^{\dagger} \phi - \lambda (\phi^{\dagger} \phi)^2 \tag{25}
\]

The symmetry transformation is then \(\phi \to \phi' = e^{i\alpha} \phi\), \(\alpha\) is some constant. Now we use the polar coordinates for the complex field

\[
\phi(x) = \frac{1}{\sqrt{2}} (\rho(x) + v) \exp \left( \frac{i\theta(x)}{v} \right) \tag{26}
\]

to write the Lagrangian in the form,

\[
L = \frac{1}{2} (\partial_{\mu} \rho)^2 + \frac{(\rho + v)^2}{2v^2} (\partial_{\mu} \theta)^2 + \frac{\mu^2}{2} (\rho + v)^2 - \frac{\lambda}{4} (\rho + v)^4 \tag{27}
\]

This clearly shows that \(\theta(x)\) is massless and has only derivative couplings. This follows from the fact that the \(U(1)\) (or \(SO(2)\)) symmetry \(\phi \to e^{i\alpha} \phi\), corresponds to \(\theta \to \theta + v\alpha\), which is inhomogeneous. So \(\theta(x)\) needs to have a derivative in order to be invariant under such inhomogeneous transformation. Note that this Lagrangian, due to the presence of terms like \((\partial_{\mu} \theta)^2 \rho^2\) is not renormalizable. But this Lagrangian will be used only as an effective theory to study the low energy phenomenology while the renomalizability deals with high energy behavior.

### C. Goldstone Theorem

From Eq(17) we see that the pions \(\pi\)' are massless. This is a consequence of the Goldstone theorem which states that spontaneous breaking of a continuous symmetry will give a massless particle or zero energy excitation. We will first illustrate this by showing that quadratic terms in \(\pi\) are absent in the tree level as a consequence of spontaneous symmetry breaking of the original chiral symmetry.. We will use the \(O(4)\) notation, \(\phi^i = (\pi_1, \pi_2, \pi_3, \sigma)\). The invariance under the chiral transformation implies that

\[
\delta V = \frac{\partial V}{\partial \phi_i} \delta \phi_i = \frac{\partial V}{\partial \phi_i} \varepsilon_{ij} \phi_j = 0 \tag{28}
\]
Differentiating Eq(28) with respect to $\phi_k$ and then evaluating this at the minimum, we see that
\[
\frac{\partial^2 V}{\partial \phi_i \partial \phi_k} \mid_{\text{min}} \langle \phi_j \rangle = 0,
\] (29)
For the case $\langle \phi_j \rangle = \delta_{ij} v$, we see that in the expansion of $V$ around the minimum, $\sigma = v, \pi_i = 0$, there are no terms of the form, $\pi_i \pi_j, \sigma' \pi_i$, with $\sigma' = \sigma - v$. Therefore $\pi_i'$s are massless in the tree level. We can extend this to more general case where the effective potential is written as $V(\phi_i)$. This potential is invariant under some symmetry group $G$, which transforms $\phi_i$ as
\[
\phi_i \rightarrow \phi'_i = \phi_i + \alpha^a t^a_{ij} \phi_j \quad \text{or} \quad \delta \phi_i = \alpha^a t^a_{ij} \phi_j
\] (30)
where $|\alpha_i| \ll 1$ are the parameters for the infinitesimal transformations and $t^a$ matrix for the representation where $\phi_i$ belongs. The invariance under these transformation implies that
\[
\frac{\partial V}{\partial \phi_i} \alpha^a t^a_{ij} \phi_j = 0
\] (31)
The minimum of the potential is located at $\phi_i = v_i$, which satisfies the equation,
\[
\left( \frac{\partial V}{\partial \phi_i} \right) \mid_{\phi=v} = 0
\] (32)
Differentiating Eq (31) with respect to $\phi_k$ and evaluating this at the minimum, $\phi_i = v_i$, we get
\[
\left( \frac{\partial^2 V}{\partial \phi_k \partial \phi_i} \right) \mid_{\phi=v}(t^a_{ij} v_j) = 0
\] (33)
This means that the vector $u^a_i = t^a_{ij} v_j$, if non-zero, is an eigenvector of the mass matrix
\[
m^2_{ij} = \left( \frac{\partial^2 V}{\partial \phi_k \partial \phi_i} \right) \mid_{\phi=v}
\] (34)
with zero eigenvalue(massless). Thus the number of massless Goldstone bosons is just the number of independent vectors of the form, $u^a_i = t^a_{ij} v_j$. In other words, if $u^a_i \neq 0$, the combination
\[
\chi^a = \sum_{ij} \phi_i t^a_{ij} v_j
\] (35)
is the Goldstone boson, up to a normalization constant.

These arguments only show that $\pi$’s are massless in the tree level. It turns out that this property is true independent of perturbation theory and can be illustrated in case of $\sigma$-model as follows. The axial charge is of the form,
\[
Q^\rho_i = i \int d^3x \left[ \pi_i \partial^\rho \sigma - \sigma \partial^\rho \pi_i + \cdots \right]
\] (36)
Since $\partial_0 \pi_i$ and $\partial_0 \sigma$ are just the momenta conjugate to $\pi_i$ and $\sigma$, we can derive,

\[ [Q_i^5, \pi_j(0)] = \delta_{ij} \sigma(0). \]  

(37)

Between vacuum states, this yields

\[ <0| [Q_i^5, \pi_j(0)] |0> = \delta_{ij}. \]  

(38)

For the case of SSB, we have

\[ <0| \sigma(0) |0> \neq 0 \]  

(39)

Note that this condition implies that the axial charges $Q_i^5$’s do not annihilate the vacuum, $Q_i^5 |0> \neq 0$. Using $Q_i^5 = \int A_i^0(x) d^3x$ we can write the LHS of Eq (38) as

\[ \langle 0| [Q_i^5, \pi_j(0)]|0> = i \int d^3x \langle 0| [A_i^0(x), \pi_j]|0> = i \sum_n \delta^3(\vec{p}_n) \langle 0|A_i^0(0)|n> \langle n|\pi_j|0> e^{-iE_n t} - \langle n|\pi_j|0> \langle 0|A_i^0(x)|n> e^{iE_n t} \]  

(40)

This has explicit dependence on $t$, while the right hand side, $<0|\sigma|0>$, is independent of time. The only way these two features can be consistent is to have a state with the property that

\[ E_n \rightarrow 0 \quad \text{as} \quad \vec{p}_n \rightarrow 0. \]  

(41)

This is the content of the Goldstone theorem. For the relativistic system, the energy and momentum is related by $E_n = \sqrt{\vec{p}_n^2 + m_n^2}$. Then Eq (11) implies the existence of massless particle in the system. More specifically, there are physical states $|\pi_i>$ with the property that

\[ \langle 0|A_i^0(0)|\pi_i> \langle \pi_i|\pi_j|0> \neq 0 \]

and are massless from Goldstone theorem. It is convenient to choose the normalization such that $<\pi_i|\pi_j|0> = \delta_{ij}$ and write

\[ \langle 0|A_i^\mu(0)|\pi_i(p)> = i f_\pi p^\mu \delta_{ij} \quad \text{with} \quad f_\pi \text{ a consntnat} \]  

(42)

It is easy to see that

\[ f_\pi = \langle 0|\sigma|0> \]  

(43)

For the non-relativistic system Eq(11) simply says that the dispersion relation $E(p)$ has zero energy excitation.

D. Non-linear $\sigma$-model

In the $\sigma$-model without the nucleons, we have 3 massless $\pi$’s and a massive $\sigma$ field. For the energies much smaller than $m_\sigma$, the massless Goldstone bosons are
the important physical degrees of freedoms and it is desirable to write down an effective theory with \( \pi \)'s only. As we have seen, the theory with SSB has many physical consequences, e.g. low energy theorem for the Goldston e bosons. The removal of \( \sigma \)-field should preserve symmetry so that these results are maintained.

Also, phenomenologically there are no good evidence for the existence of the \( \sigma \) meson which is the partner of \( \pi \)'s in the chiral symmetry. We now discuss the explicit steps for carrying out this process. Write the scalar fields as a vector in 4-dimensional space,

\[
\phi_i = (\phi_1, \phi_2, \phi_3, \phi_4) = \left( \frac{\pi}{\sqrt{s}}, \sigma \right)
\]

We want to parametrize the \( \phi \) fields in such a way that the non-Goldstone field to be eliminated later is \( O(4) \) invariant. One simple parametrization for this purpose is

\[
\phi_i = R_{i4}(x) s(x), \quad i = 1, \cdots, 4
\]

where \( R_{ab} \) a 4 \( \times \) 4 orthogonal matrix, \( RRT = RT R = 1 \), which gives \( R_{i4}R_{4i} = 1 \) and

\[
\phi_i \phi_i = s^2.
\]

So \( s(x) \) is the magnitude of the vector \( \phi_i \) and is clearly \( O(4) \) invariant. Thus it can be eliminated without effecting the symmetry. One simple choice for \( R_{i4} \) is,

\[
R_{a4}(x) = \frac{2\eta_a(x)}{1 + \eta^2}, \quad a = 1, 2, 3 \quad R_{44}(x) = \frac{1 - \eta^2}{1 + \eta^2}
\]

Note that we can invert these relations to get

\[
\eta = \frac{\pi}{\sigma + s}
\]

The Lagrangian is of the form

\[
L = \frac{1}{2} \left[ (\partial_\mu s)^2 + 4s^2 \left( \frac{\partial_\mu \eta}{1 + \eta^2} \right)^2 \right] + \frac{1}{2} \mu^2 s^2 - \frac{\lambda}{4} s^4
\]

So \( \eta \)'s are the massless Goldstone bosons. To study the physics of Goldstone bosons at low energies, \( E \ll m_\sigma \), we can replace the \( s \) field by a constant, \( s(x) = v \) to get

\[
L = 2v^2 \left( \frac{\partial_\mu \eta}{1 + \eta^2} \right)^2
\]

In order to get the correct normalization we rescale \( \eta \) so that

\[
L = \frac{1}{2} \left( \frac{\partial_\mu \eta'}{1 + \eta'^2} \right)^2 = \frac{1}{2} \left( \partial_\mu \eta' \right)^2 - \frac{1}{4v^2} \left( \eta' \right)^2 \left( \partial_\mu \eta' \right)^2 + \cdots
\]
Here the interaction terms will always contain derivatives and amplitudes involving $\pi'$s will vanish in the limit of zero momenta (low energy theorem). According to the theorem of field redefinition, this describes the same physics as the usual $\sigma$-model Lagrangian in the Goldstone sector. For example, we can check that in the simple case of $\pi^0\pi^0$ scattering in the tree level the amplitude from this Lagrangian in Eq (51) is

$$M = \frac{2i}{v^2} [p_1 \cdot p_2 - p_1 \cdot p_3 - p_1 \cdot p_4] = \frac{i}{v^2} (s + t + u).$$ (52)

This is the same result as in Eq (23), obtained in the $\sigma$-model with $m_\sigma \to \infty$.

The Lagrangian in Eq (51) which contains on the Goldstone boson fields, is one example of non-linear realization of chiral symmetry, which will be discussed in detail in the next section. Here we want to mention a useful geometric interpretation of Lagrangian in Eq (51). When we eliminate the $O(4)$ invariant field by setting $s(x) = v$, the fields $\phi_i$ satisfy the relation,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = v^2$$

which is just the sphere with radius $v$ in 4-dimensional Euclidean space, $S^3$. The variables $\eta_1, \eta_2$, and $\eta_3$ are just one particular choice of the coordinates of the space $S^3$. The transformation of Lagrangian from $\phi$ fields to $\eta$ fields can be understood in terms of metric tensor in $S^3$. For simplicity, consider just the kinetic terms in $\mathcal{L}$,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^i (\partial^\mu \phi^j) g_{ij}$$

where $g_{ij} = \delta_{ij}$ is the trivial metric in the 4-dimensional Euclidean space. Then the transformation $\phi^i \to \phi^i(\eta)$ gives

$$\partial_\mu \phi^i = \frac{\partial \phi^i}{\partial \eta^a} \partial_\mu \eta^a$$ (53)

and

$$\mathcal{L} = \frac{1}{2} \delta_{ij} \frac{\partial \phi^i}{\partial \eta^a} \frac{\partial \phi^j}{\partial \eta^b} (\partial^\mu \eta^a) (\partial^\mu \eta^b) = \frac{1}{2} g_{ab}(\eta) (\partial^\mu \eta^a) (\partial^\mu \eta^b)$$ (54)

where

$$g_{ab}(\eta) = g_{ij} \frac{\partial \phi^i}{\partial \eta^a} \frac{\partial \phi^j}{\partial \eta^b} = \frac{4v^2 \delta_{ab}}{(1 + \eta^2)^2}$$ (55)

is the induced metric on $S^3$. Thus in the Lagrangian the coefficient of $(\partial_\mu \eta^a) (\partial^\mu \eta^b)$ is the metric of the space $S^3$, which is just the coset space $O(4)/O(3)$. In the general case where the symmetry breaking is of the form, $G \to H$, the non-linear Lagrangian can be written down with the metric on the manifold $G/H$ as in Eq (54).

It is interesting to see how the transformations of $SU(2) \times SU(2)$ are realized on this manifold $S^3$. For the infinitesimal isospin rotation, we have

$$\delta \pi = \hat{\alpha} \times \pi, \quad \delta \sigma = 0 \quad \alpha: \text{group parameters}$$ (56)
which implies from Eq(58) that
\[ \delta \vec{\eta} = \vec{\alpha} \times \vec{\eta} \quad (57) \]
This is just a rotation on the vector \( \vec{\eta} \) and the Lagrangian in Eq(58) with metric given in Eq(55) is clearly invariant under such transformation. The axial transformation on \( \vec{\pi} \) and \( \vec{\sigma} \) is of the form

\[ \delta \vec{\pi} = \vec{\beta} \vec{\sigma}, \quad \delta \vec{\sigma} = - \vec{\beta} \cdot \vec{\pi}, \quad \vec{\beta}: \text{group parameters} \quad (58) \]
which gives

\[ \delta \vec{\eta} = \frac{\vec{\beta}}{2} (1 - \eta^2) + \vec{\eta} \left( \vec{\beta} \cdot \vec{\eta} \right). \quad (59) \]
This transformation is non-linear and inhomogeneous. But we can get simple transformation for the combination,

\[ \delta \left( \frac{\partial_{\mu} \vec{\eta}}{1 + \eta^2} \right) = \left( \vec{\eta} \times \vec{\beta} \right) \times \left( \frac{\partial_{\mu} \vec{\eta}}{1 + \eta^2} \right) \quad (60) \]
This looks very much like an isospin rotation except that the parameters for the rotation now depend on the fields \( \eta \) and it is easy now to see that the Lagrangian in Eq(54) is invariant under the axial transformations.

**Remark:** We can transform the metric in Eq(55) into the more familiar Robertson-Walker metric used in cosmology as follows. First we use the spherical coordinates for \( \vec{\eta} \) to write the metric in the line element as

\[ (dl)^2 = g_{ab} d\eta^a d\eta^b = \frac{4 \nu^2}{(1 + \eta^2)^2} \left[ (d\eta)^2 + \eta^2 (d\theta)^2 + \eta^2 \sin^2 \theta (d\phi)^2 \right]. \quad (61) \]
Define the new variable \( r \) by

\[ r = \frac{2 \eta}{1 + \eta^2}. \quad (62) \]
In terms of new variable the line element is of form,

\[ (dl)^2 = \left( \frac{dr}{1 - r^2} \right)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2 \quad (63) \]
which is just the usual Robertson-Walker metric for the case of positive curvature.

**III. Non-linear Realization**

As we have seen in the last section, it is useful to write down a Lagrangian with only Goldstone bosons as an effective theory to describe physics at low energies. In this section we will discuss the general description of this procedure, which is usually called the non-linear realization. The usual discussion of
this subject is generally rather formal and abstract. The discussion here will emphasize the intuitive understanding rather than the mathematical rigor.

In order to make the discussion here somewhat self contained, we will first discuss some simple results from group theory which are useful for the understanding of non-linear realization. Then we discuss the general features of non-linear realization.

A. Useful results from group theory

Here we will recall the rearrangement theorem which is central to most of the group theoretical result and then discuss the concept of coset space which forms the basis of the non-linear realization.

(i) Rearrangement Theorem
Let \( G = \{g_1, \cdots, g_n\} \) be a finite group. If we multiply the whole group by an arbitrary group element \( g_i \), i.e. \( \{g_1g_i, \cdots, g_ng_i\} \), the resulting set is just the group \( G \) itself.

(ii) Coset space
Coset space decomposes a group into non-overlapping sets with respect to a subgroup. Let \( H = \{h_1, \cdots, h_l\} \) be a non-trivial subgroup of \( G \). For any element \( g_i \) in \( G \) but not in \( H \), the left coset \( g_iH \), or coset for short, is just \( \{g_ih_1, \cdots, g_ih_l\} \). The coset \( g_iH \) will not have any element in common with the subgroup \( H \), and any two such cosets are either identical or have no elements in common. This can be seen as follows. Consider cosets \( g_1H \), and \( g_2H \). Suppose that there is one element in common, \( g_1h_i = g_2h_j \) for some \( i, j \)

Then we can write
\[
g_1^{-1}g_2 = h_ih_j^{-1}
\]
which means \( g_1^{-1}g_2 \) is one of the element of subgroup \( H \). Then by the rearrangement theorem applied to the subgroup \( H \), we get

\[
g_1^{-1}g_2H = H, \quad \Rightarrow \quad g_1H = g_2H
\]
i.e. these two cosets have the same group elements. The group \( G \) is now decomposed into these non-overlapping cosets and the collection of all the distinct cosets \( g_1H, \cdots, g_kH \), together with \( H \), will contain all the group elements of \( G \). This is denoted by \( G/H \). This can be generalized to Lie group where rearrangement theorem is valid.

Example of coset space: Consider points on 2-dimensional plane which form a group under the addition,

\[
(x_1, y_1) \cdot (x_2, y_2) = (x_1 + x_2, y_1 + y_2)
\]
Clearly, points on the \( y \)-axis, \((0, y)\), form a subgroup, denoted by \( H \). Then the vertical line of the form, \((a, y)\) with a fixed is a coset with respect to the
subgroup $H$. It is clear that the whole 2-dimensional plane can be decomposed into collection of such vertical lines. We can label these cosets, vertical lines, by choosing one element from each coset. Clearly there are many ways to choose such representatives. One convenient parametrization is to choose those points on the $x$-axis, so that the cosets are of the form $x_iH$. Each group element $(x, y)$ can be written as the product,

$$(x, y) = (x, 0) \cdot (0, y)$$

where $(0, y) \in H$. Under the action of an arbitrary group element $g = (a, b)$ this will give

$$g(x, y) = (a, b) \cdot (x, y) = (a, b) \cdot (x, 0) \cdot (0, y) = (a + x, 0) \cdot (0, b) \cdot (0, y)$$

$$= (a + x, 0) \cdot (0, b + y)$$

(the computation here is organized in such a way that it parallel to the more complicate case in the non-linear realization.) Thus the group element $g = (a, b)$ will move the points in the coset $xH$ to points in the coset $(a + x)H$. In terms of coset parameters, we have

$$g : x \rightarrow x + a.$$
i.e. $\vec{\xi}$ also labels the different vacua, which are degenerate. Recall that the different vacua form the manifold $G/H$. Thus the coset parameters are also the parameters for the manifold $G/H$. Under the action of an arbitrary group element $g_1 \in G$, we have the combination

$$g_1 g = g_1 e^{i \vec{\xi} \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}}. \quad (66)$$

Since $g_1 e^{i \vec{\xi} \cdot \vec{A}}$ is also a group element in $G$, we can write a coset decomposition,

$$g_1 e^{i \vec{\xi} \cdot \vec{A}} = e^{i \vec{\xi'} \cdot \vec{A}} e^{i \vec{\alpha'} \cdot \vec{V}} \quad \text{(67)}$$

and then

$$g_1 \left( e^{i \vec{\xi} \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}} \right) = e^{i \vec{\xi'} \cdot \vec{A}} e^{i \vec{\alpha'} \cdot \vec{V}} = e^{i \vec{\xi'} \cdot \vec{A}} e^{i \vec{\alpha'} \cdot \vec{V}} \quad \text{(68)}$$

where we have used the group property of $H$ to write

$$e^{i \vec{\alpha'} \cdot \vec{V}} e^{i \vec{\alpha} \cdot \vec{V}} = e^{i \vec{\alpha'} \cdot \vec{V}} \quad \text{(69)}$$

Note that the new coset parameters $\vec{\xi'}$ and parameters $\vec{\alpha'}$ for the subgroup $H$, all depend on the original coset parameters $\vec{\xi}$,

$$\vec{\xi'} = \vec{\xi} \left( \vec{\xi}, g_1 \right), \quad \vec{\alpha'} = \vec{\alpha} \left( \vec{\xi}, g_1 \right) \quad (70)$$

In this way, the group element $g_1$ transform the coset parameters from $\vec{\xi} \rightarrow \vec{\xi'}$ in the coset space $G/H$. As we will see later, these coset parameters will be identified with the Goldstone bosons. These transformations on $\vec{\xi'}$, and $\vec{\alpha'}$ induced by the group elements will have the same group properties as the group elements and are called the non-linear realization of the group. This is in contrast to the usual representation of the group where group elements are represented by matrices. In the transformation in Eq\textcolor{red}{(70)} $\vec{\xi'}$ is generally not a linear function of $\vec{\xi}$. But for the special case where $g = h$ is a group element from the unbroken subgroup $H$, we get, from Eq\textcolor{red}{(64)},

$$h g = h e^{i \vec{\xi} \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}} = \left( h e^{i \vec{\xi} \cdot \vec{A}} h^{-1} \right) \left( h e^{i \vec{\alpha} \cdot \vec{V}} \right) = \left( h e^{i \vec{\xi} \cdot \vec{A}} h^{-1} \right) e^{i \vec{\alpha} \cdot \vec{V}} \quad (71)$$

where

$$h e^{i \vec{\alpha} \cdot \vec{V}} = e^{i \vec{\alpha'} \cdot \vec{V}} \quad (72)$$

In general, the broken generators $\vec{A}$ transform as some representation $D$ with respect to the subgroup $H$, $hA_i h^{-1} = A_j D_{ji} (h)$ \textcolor{red}{(73)}

For example, in the case of $G = SU(2) \times SU(2)$ model, the broken generators, $A_1, A_2, A_3$ transform as triplet under the unbroken subgroup $H = SU(2)$. We can then write

$$h e^{i \vec{\xi} \cdot \vec{A}} h^{-1} = \exp \left( i \vec{\xi} h A_i h^{-1} \right) = \exp \left( i \vec{\xi} A_j D_{ji} (h) \right) = \exp \left( i \vec{\xi'} A_j \right) \quad \text{(74)}$$
where
\[ \xi'_j = D_{ji}(h) \xi_i \]  
(75)

This means that \( \xi'_i \)s transform linearly under the subgroup \( H \). Also it is easy to see that the parameters \( \vec{\alpha}' \) are independent of the coset parameters \( \xi_i \). But if the group element is of the form \( e^{i \vec{\zeta} \cdot \vec{A}} \) the transformation law for the coset parameters \( \xi \) is non-linear and quite complicated.

2. Chiral symmetry

For the case of chiral symmetry, there is significant simplification due to parity operation. We will illustrate this in the simple case of \( SU(2)_L \times SU(2)_R \) symmetry. The parity operation is of the form,
\[ P : \vec{V} \rightarrow \vec{V}, \quad \vec{A} \rightarrow -\vec{A} \]

Consider the case where group element \( g \) consists of left-handed transformation,
\[ g = \exp\left(i \theta \cdot (\vec{V} - \vec{A})\right) \equiv L \]  
(76)

and write the transformation of coset parameters as
\[ ge^{i \vec{\zeta} \cdot \vec{A}} = Le^{i \vec{\zeta} \cdot \vec{A}} = e^{i \vec{\zeta} \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}} \]  
(77)

Under the parity transformation, the left-handed transformation is changed into right-handed one,
\[ P(L) = P(e^{i \theta \cdot (\vec{V} - \vec{A})}) = e^{i \theta \cdot (\vec{V} + \vec{A})} \equiv R \]  
(78)

Then applying the parity transformation to Eq(77), we get
\[ Re^{-i \vec{\zeta} \cdot \vec{A}} = e^{-i \vec{\zeta} \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}} \]  
(79)

Using the notation
\[ \Sigma \equiv e^{i \vec{\zeta} \cdot \vec{A}}, \quad h = e^{i \vec{\alpha} \cdot \vec{V}} \]  
(80)

we can combine the Eqs(77,79) into
\[ \Sigma' = L \Sigma h^\dagger = h \Sigma R^\dagger \]  
(81)

Note that as we have mentioned before, the parameters \( \vec{\alpha}' \) depend on the coset parameters \( \vec{\zeta} \), the transformation law for \( \Sigma \) here is non-linear because the factor \( h \) depends on \( \vec{\alpha}' \). However, the combination \( U = \Sigma^2 \) will have a simple transformation law,
\[ U' = LUR^\dagger. \]  
(82)
Since $L, R$ are independent of the coset parameters $\xi$, this transformation law is linear and will be useful for constructing Lagrangian.

We are interested in the cases where spontaneous symmetry breaking is generated by the scalar fields. Some of these scalar fields become the massless Goldstone bosons, like the pions in the $\sigma$-model and others remain massive, like $\sigma$-field. Consider the scalar fields in the $\sigma$-model, where we will use the notation,

$$\phi(x) = \begin{pmatrix} \pi_1(x) \\ \pi_2(x) \\ \pi_3(x) \\ \pi_4(x) \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

The vacuum expectation value which gives the classical ground state is of the form,

$$\langle \phi \rangle_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}$$

Suppose we make a very general assumption that from a given point in $\phi$-space we can reach any other point by some group transformation. (Space is transitive). Then the general field configuration can be written as

$$\phi(x) = g \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{pmatrix}$$

for some $g \in G$.

Remark: Strictly speaking we should use the representation matrices $D(g)$ of the group element $g$ rather than $g$ itself. However for simplicity of notation, $g$ here is a shorthand for $D(g)$.

From the coset decomposition in Eq(64), we can write

$$g = e^{i\pi \cdot A} e^{i\alpha \cdot V}$$

Here we have chosen coset parameters to be the pion fields. Then we can write the scalar fields as

$$\phi(x) = g \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{pmatrix} = e^{i\pi \cdot A} e^{i\alpha \cdot V} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{pmatrix} = e^{i\pi \cdot A} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{pmatrix}$$

where we have used the fact that the vector $(0, 0, 0, \sigma)$ is proportional to the vacuum configuration $(0, 0, 0, v)$ and is invariant under the subgroup $H = \{e^{i\pi \cdot V}\}$.

Since the Goldstone bosons are identified as the coset parameters, they transform the same way as $\xi$ in Eq(70), i.e.

$$g_1 e^{i\pi \cdot A} = e^{i\pi' \cdot A} e^{i\alpha' \cdot V}$$

where

$$\pi' = \pi (\pi, g_1) \quad \alpha' = \alpha (\pi, g_1)$$
For the case \( g = h \in H \), we have from Eq(75),
\[
\pi_j' = D_{ji}(h) \pi_i
\]  
(90)

Remark: The scalar fields here have property that after separating out the Goldstone bosons, the remainders are proportional to the vacuum expectation value and is then invariant under the subgroup \( H \). This is true only for scalar fields in the vector representation in \( O(n) \) or \( SU(n) \) groups and is not true for scalars in more general representation. In more general cases, we can separate out the Goldstone bosons by writing \( \phi(x) \) as
\[
\phi(x) = e^{i \vec{\pi} \cdot \vec{A}} \chi(x)
\]  
(91)

where \( \chi(x) \) contains all the massive fields. Under the action of group element \( g_1 \), we have
\[
g_1 \phi = g_1 e^{i \vec{\pi} \cdot \vec{A}} \chi(x) = e^{i \vec{\pi}' \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}} \chi(x) = e^{i \vec{\pi}' \cdot \vec{A}} \chi'(x)
\]  
(92)

where
\[
\chi'(x) = e^{i \vec{\alpha} \cdot \vec{V}} \chi(x)
\]  
(93)

To see how the Goldstone bosons transform under the axial transformation, we set \( L = R^t = e^{i \vec{\theta} \cdot \vec{A}} \), in Eq(81) to get
\[
e^{i 2 \vec{\pi} \cdot \vec{A}} = e^{i \vec{\theta} \cdot \vec{A}} e^{i \vec{\alpha} \cdot \vec{V}} e^{i \vec{\theta} \cdot \vec{A}}
\]  
(94)

To understand this equation better, we first take the infinitesimal transformation, \( |\vec{\theta}| \ll 1 \),
\[
e^{i 2 \vec{\pi} \cdot \vec{A}} = e^{i 2 \vec{\pi} \cdot \vec{A}} + \left( i \vec{\theta} \cdot \vec{A} \right) e^{i 2 \vec{\pi} \cdot \vec{A}} + e^{i 2 \vec{\pi} \cdot \vec{A}} \left( i \vec{\theta} \cdot \vec{A} \right) + \cdots
\]  
(95)

and then expand this in powers of \( \vec{\pi}' \) s,
\[
1 + 2i \vec{\pi}' \cdot \vec{A} + \cdots = 1 + 2i \vec{\pi} \cdot \vec{A} + 2i \vec{\theta} \cdot \vec{A} + \cdots
\]

Comparing both sides we see that
\[
\vec{\pi}' = \vec{\pi} + \vec{\theta} + \cdots
\]  
(96)

Thus there is an inhomogeneous term in the transformation law for the Goldstone bosons \( \vec{\pi} \) and this is why Goldstone bosons have derivative coupling.

Since the transformation law for \( U = e^{i 2 \vec{\pi} \cdot \vec{A}} \) is linear and simple, it is easier to construct the chirally invariant interaction in terms of \( U \) rather than \( \Sigma \). It is easy to see that the only invariants without derivatives will involve trace of some powers of \( UU^\dagger \), which is just an identity matrix. (This also implies that the Goldstone boson coupling will involve derivatives). Thus the interaction with lowest numbers of derivative is of the form
\[
\mathcal{L} = tr (\partial_\mu U \partial^\mu U)
\]  
(97)
**Covariant derivative:** The parity symmetry in the $\sigma$-model is responsible for getting the simple combination $U(x)$ which transforms linearly. For the more general case where there is no such simplification, to construct invariant terms involving derivatives is quite complicating because the non-linear transformation law will involve Goldstone boson fields which are space-time dependent. This means that we need to construct the covariant derivatives. Furthermore, to exhibit the low energy explicitly we need to couple Goldstone field with derivative to other matter fields. We will now discuss briefly in the simple case of $\sigma$-model.

Write the scalar fields $\phi(x)$ in the form,

$$
\phi(x) = e^{i\vec{A}(x) \cdot \vec{\pi}} \left( \begin{array}{c} 0 \\ 0 \\ \sigma(x) \end{array} \right) = \Sigma(x) \chi(x) 
$$

with $\Sigma(x) = e^{i\vec{A}(x) \cdot \vec{A}}$. As before under the action of group element $g$, we have

$$
g \Sigma(x) = \Sigma'(x) h(x) \quad \text{with} \quad h(x) = e^{i\vec{\alpha} \cdot \vec{V}}
$$

Since $\phi$ transforms linearly, we have

$$
\phi' = g \phi
$$

which implies

$$
\Sigma' \chi' = g \Sigma \chi = \Sigma' h \chi \quad \text{or} \quad \chi' = h \chi
$$

This means that $\chi$ transforms non-linearly because $\vec{\alpha}$ in $h$ depend on $\pi$ fields. Making use of the simplification for the case of chiral symmetry we have, from transformation law,

$$
\partial_\mu \Sigma' = L (\partial_\mu \Sigma h^{-1} + \Sigma \partial_\mu h^{-1}) = (\partial_\mu \Sigma h + \Sigma \partial_\mu h) R^I
$$

and

$$
\Sigma'^{-1} \partial_\mu \Sigma' = h (\Sigma^{-1} \partial_\mu \Sigma) h^{-1} + h\partial_\mu h^{-1} \quad \text{(102)}
$$

$$
\partial_\mu \Sigma'^{-1} = h (\partial_\mu \Sigma \Sigma^{-1}) h^{-1} - h\partial_\mu h^{-1} \quad \text{(103)}
$$

If we define

$$
v_\mu = \frac{1}{2} \left[ \Sigma^{-1} \partial_\mu \Sigma - \partial_\mu \Sigma \Sigma^{-1} \right] \quad \text{(104)}
$$

$$
a_\mu = \frac{1}{2} \left[ \Sigma^{-1} \partial_\mu \Sigma + \partial_\mu \Sigma \Sigma^{-1} \right] \quad \text{(105)}
$$

we get

$$
v'_\mu = hv_\mu h^{-1} + h\partial_\mu h^{-1}
$$

$$
a_\mu = ha_\mu h^{-1}
$$

This means that $v_\mu$ transforms like ”gauge field”, while $a_\mu$ transforms as global adjoint field. Therefore $v_\mu$ can be used to construct the covariant derivative and $a_\mu$ is like a global axial vector field.
**Nucleon Field**  The nucleon field in the linear $\sigma$-model has the transformation properties,

$$ N_L \to N'_L = LN_L, \quad \text{and} \quad N_R \to N'_R = RN_R $$

Thus to couple nucleon fields to the Goldstone bosons, we could write down the following $SU(2) \times SU(2)$ invariant coupling,

$$ \mathcal{L}_{int} = g \left( \overline{N}_L \Sigma N_R + \overline{N}_R \Sigma^\dagger N_L \right). $$

However, this is not of the form of derivative coupling which exhibits the low energy theorem explicitly. For this purpose and general non-linear realization, we define a new nucleon field by,

$$ N = e^{i\vec{\pi} \cdot \vec{A}} \tilde{N} $$

Under the action of the group element $g$, we get

$$ gN = e^{i\vec{\pi} \cdot \vec{A}} \tilde{N} = e^{i\vec{\pi}' \cdot \vec{A}} e^{i\vec{\alpha} \cdot \vec{V}} \tilde{N} = e^{i\vec{\pi}' \cdot \vec{A}} \tilde{N}' $$

where

$$ \tilde{N}' = e^{i\vec{\alpha} \cdot \vec{V}} \tilde{N} = h\tilde{N} $$

Thus $\tilde{N}$ transforms according to the representation of the subgroup $H$ but with the group parameters depend on $\pi$ fields, $\vec{\alpha} \left( \vec{\pi}, g \right)$. Then from the transformation properties given in Eqs(??), we can write down the derivative coupling as

$$ \mathcal{L}_N = \overline{\tilde{N}} \gamma^\mu (i\partial_\mu - v_\mu) \tilde{N} + g \overline{\tilde{N}} \gamma^\mu a_\mu \tilde{N} $$

which will yield the low energy theorem explicitly.

**IV. Examples in the Non-relativistic System**

In the framework of relativistic field theory, e.g. in $SU(2) \times SU(2)$ $\sigma$-model, spontaneous symmetry breaking seems to be put in by hand, i.e. setting the quadratic terms to have negative sign in the scalar potential in order to develop vacuum expectation value. This is rather ad hoc and no physical reason is given for why this is the case. We will now discuss some simple non-relativistic examples of spontaneous symmetry breaking in order to shed some light on this [18].

**A. Infinite range Ising model**

Consider a system of $N$ spins on an one dimensional lattice with Hamiltonian,

$$ H = -\frac{J}{N} \sum_{i<j}^N s_i s_j - B \sum_{i}^N s_i $$

(106)
where $s_i = \pm 1$, $J$ is the coupling constant for spin-spin interaction and $B$ is the external magnetic field. In this Hamiltonian, for calculational simplicity we allow every spin to interact with every other spin, while more realistic situation will be the short range nearest neighbor interaction. But the interest here is to see how the spontaneous symmetry breaking come about and we will ignore this. The partition function is given by

$$Z = \text{Tr} \left( e^{-\beta H} \right) = \sum_{s_i = \pm 1} \exp \left( \frac{\beta J}{2N} \left( \sum_i s_i \right)^2 + \beta B \sum_i s_i \right)$$

(107)

Using the identity for the Gaussian integral,

$$\int_{-\infty}^{+\infty} dx e^{-ax^2 + bx} = \sqrt{\frac{\pi}{a}} e^{-\frac{b^2}{4a}}$$

we can write the partition function as

$$Z = \sum_{s_i = \pm 1} \sqrt{\frac{N \beta J}{2 \pi}} \int_{-\infty}^{+\infty} dx \exp \left[ - \frac{N \beta J}{2} x^2 + \beta (Jx + B) \left( \sum_i s_i \right) \right]$$

(108)

Now we can sum over each $s_i$ independently,

$$\sum_{s_i = \pm 1} \exp \left( \beta (Jx + B) \left( \sum_i s_i \right) \right) = \exp \{ N \log [2 \cosh \beta (Jx + B)] \}$$

and

$$Z = \sqrt{\frac{N \beta J}{2 \pi}} \int_{-\infty}^{+\infty} dx \exp \left( - \frac{N \beta J}{2} x^2 + N \log [2 \cosh \beta (Jx + B)] \right)$$

(109)

From the partition function $Z$, we can compute the average spin,

$$S = \frac{1}{N} < \sum_i s_i > = - \frac{1}{\beta N} \frac{\partial}{\partial B} \ln Z.$$  

(110)

If $S \neq 0$ in the limit the external field vanishes, $B \to 0$, then we have spontaneous symmetry breaking. Since we are interested in the case where $N$ is very large, we can use saddle point method to compute $Z$. Write

$$Z = \sqrt{\frac{N \beta J}{2 \pi}} \int_{-\infty}^{+\infty} dx \exp \{ -N \beta f(x) \}$$

(111)

where

$$f(x) = \frac{Jx^2}{2} - \frac{1}{\beta} \log [2 \cosh \beta (Jx + B)]$$

(112)

The minimum of $f(x)$ is given by

$$f'(x) = 0, \quad \Rightarrow \quad x = \tanh \beta (Jx + B)$$

(113)
Let $x_i, i = 0, 1, 2, \cdots$ be the solutions of this transcendental equation, then

$$Z = \sqrt{\frac{N\beta J}{2\pi}} \sum_i \exp\{-N\beta f(x_i)\} \sqrt{\frac{2\pi}{Nf''(x_i)}} \quad (114)$$

Suppose $x_0$ is the smallest of these solutions, it will dominate the partition function for large $N$. Then the average magnetization is then

$$S = -\frac{1}{\beta N} \frac{\partial}{\partial B} \ln Z = \tanh \beta (Jx_0 + B) = x_0 \quad (115)$$

where we have used the equation satisfied by $x_0$. Thus the minimum of $f(x)$ will correspond to the average magnetization.

To study spontaneous symmetry breaking, we set $B = 0$, in Eq(113), and get

$$x = \tanh \beta Jx \quad (116)$$

It turns out that this equation has only the trivial solution, $x = 0$ if $\beta J < 1$ and non-trivial solution exists only for $\beta J > 1$. To understand this feature, we expand $f(x)$ in powers of $x$ for the case $B = 0$,

$$f(x) = \frac{Jx^2}{2} - \frac{1}{\beta} \log \left[1 + \frac{1}{2}(\beta Jx)^2 + \frac{1}{4!}(\beta Jx)^4 + \cdots\right] \quad (117)$$

Thus $\beta J > 1$ corresponds to negative quadratic term, which is the familiar situation in the scalar potential in the $\sigma$-model and the like. In terms of temperature this condition, we have

$$J > kT. \quad (118)$$

Here $J$ is the coupling which wants to align the spins in the same direction while the effect of temperature is to randomize spins. Thus the condition in Eq(118) simply means that the interaction of spins has to overcome the thermalization in order to produce significant spin alignment. The temperature $T_c = \frac{J}{k}$ is usually called the critical temperature and spontaneous symmetry breaking is possible only for $T < T_c$. In this simple example, the non-zero magnetization $S$ breaks the symmetry, $s_i \rightarrow -s_i$, and originates from the competition between spin-spin interaction which aligns the spin and the thermalization which tends to destroy the alignment.

Remarks:(1)From the partition function in Eq(114) we see that the probability to find the system to have $x_i$ is given by the Boltzmann factor

$$P(x_i) = \frac{\exp(-\beta N f(x_i))}{\sum_i \exp(-\beta N f(x_i))} \quad (119)$$

If $x_0$ is the absolute minimum for $f(x)$, then in the thermodynamic limit $N \rightarrow \infty$, we have $P(x_0) \rightarrow 1$ and the probability for all the other $x_i$ will be
(2) For the case $T$ is near $T_c$, the minimum of $f(x)$ is located at small values of $x$. Thus we can expand $f(x)$ in power series,

$$
f(x) = \frac{1}{2\beta_c} \left( 1 - \frac{\beta}{\beta_c} \right) x^2 + \frac{1}{12 \beta_c^3} x^4 + \cdots \tag{120}$$

The minimum is then

$$
x_0 = \sqrt{\frac{3\beta_c^3}{\beta^3} \left( 1 - \frac{\beta}{\beta_c} \right)} = \sqrt{\frac{3T^3}{T_c^3} \left( 1 - \frac{T}{T_c} \right)} \tag{121}$$

This means that near the critical temperature $T \to T_0$, the dependence of average magnetization on $(T - T_0)$ is non-analytic. This is a typical behavior of physical quantities near the critical point.

B. Superfluid

The superfluid $He^4$ provides a simple example of Goldstone excitation where the excitation energy, $\varepsilon(k)$ goes to zero when the wave number $k \to 0$. The helium atoms are tightly bounded and the long-distance attractive force between atoms are very weak while the short distance is strongly repulsive. Thus a system of helium atoms can be described as a gas of weakly interacting bosons with Hamiltonian,

$$
H = -\frac{1}{2m} \int d^3x \nabla^2 \psi + \frac{1}{2} \int d^3x d^3y \psi^\dagger(x) \psi^\dagger(y) v(x-y) \psi(x) \psi(y) \tag{122}
$$

Here $v(x)$ is the potential describes the effective interaction between helium atoms and $\psi(x)$ is the field operator for the helium atom and satisfies the commutation relation,

$$
[\psi(x), \psi^\dagger(y)] = \delta^3(x-y)
$$

We will assume that the system is in a large box of volume $\Omega$ with periodical boundary condition. Clearly, this Hamiltonian is invariant under the transformation,

$$
\psi(x) \to \psi'(x) = e^{i\alpha} \psi(x) \tag{123}
$$

This is just a $U(1)$ symmetry which says that the number of He atoms is conserved. The conserved charge is just the number operator,

$$
Q = \int d^3x \psi^\dagger(x) \psi(x) \tag{124}
$$

with the commutation relations,

$$
[Q, \psi(x)] = \psi(x), \quad [Q, \psi^\dagger(x)] = -\psi^\dagger(x). \tag{125}
$$
We can expand $\psi(x)$ in plane waves,

$$\psi(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} a_{\vec{k}} e^{i\vec{k} \cdot \vec{x}}$$  \hspace{1cm} (126)$$

where $a_{\vec{k}}$ and $a_{\vec{k}}^\dagger$ are the usual creation and annihilation operators satisfying the commutation relations,

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = 0, \quad [a_{\vec{k}}, a_{\vec{k}'}] = \delta_{\vec{k}, \vec{k}'}$$  \hspace{1cm} (127)$$

The Hamilton is then of the form,

$$H = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} a_{\vec{k}}^\dagger a_{\vec{k}} + \frac{1}{2\Omega} \sum_{\vec{k}_i} \bar{v}(\vec{k}_1 - \vec{k}_3) \delta_{k_1 + k_2, k_3 + k_4} a_{\vec{k}_1}^\dagger a_{\vec{k}_2} a_{\vec{k}_3} a_{\vec{k}_4}$$  \hspace{1cm} (128)$$

where

$$\bar{v} (\vec{k}) = \int d^3x e^{i\vec{k} \cdot \vec{x}} v(x)$$  \hspace{1cm} (129)$$

In these two equations and there after we have, for notational simplicity, neglected the vector symbol for the wave vectors $\vec{k}_i$'s. Since $v(x)$ is real we have $\bar{v}(\vec{k}) = \bar{v}(-\vec{k})$. For the trivial case where there is no interaction, $v(x) = 0$, the ground state is just the one in which all particles are in the $\vec{k} = 0$ state,

$$|\Psi_0 >_{\bar{v}=0} = \left(\frac{a_{\vec{0}}^\dagger}{\sqrt{N}}\right)^N |0> \quad \text{where} \quad a_{\vec{0}}|0> = 0 \quad \forall \vec{k}$$  \hspace{1cm} (130)$$

It is clear that if the interaction is small enough, in the ground state and low-lying excited states, most of the particles will be in the $\vec{k} = 0$ state, i.e.

$$<n_0 >> <n_k > \quad \text{with} \quad k \neq 0.$$  \hspace{1cm} (131)$$

where $n_k = a_{\vec{k}}^\dagger a_{\vec{k}}$. We are interested in the cases where $N$, the total number of particles, is very large. Thus $n_0 \sim N$ is very large. From the properties of the creation and annihilation operators,

$$a_{\vec{0}}|n_0> = \sqrt{n_0}|n_0 - 1>, \quad a_{\vec{0}}^\dagger|n_0> = \sqrt{n_0 + 1}|n_0 + 1>$$  \hspace{1cm} (132)$$

we will make the assumption that the matrix elements of $a_{\vec{0}}$ are of order $\sqrt{n_0}$ and $a_{\vec{0}}^\dagger$ of order $\sqrt{n_0 + 1}$. Thus in the limit $n_0 \sim N \to \infty$, commutator of $a_{\vec{0}}$ and $a_{\vec{0}}^\dagger$, is of order unity while $a_{\vec{0}}, a_{\vec{0}}^\dagger$ are of order $\sqrt{N}$,

$$[a_{\vec{0}}, a_{\vec{0}}^\dagger] = 1 \ll a_{\vec{0}} \quad \text{or} \quad a_{\vec{0}}^\dagger \sqrt{n_0}$$  \hspace{1cm} (133)$$

Thus we can neglect the commutator. Since $a_{\vec{0}}$, and $a_{\vec{0}}^\dagger$ commute with all the other operators,

$$[a_{\vec{0}}, a_{\vec{k}}] = [a_{\vec{0}}^\dagger, a_{\vec{k}}] = 0 \quad \text{for} \quad k \neq 0.$$
We can then take \( a_0 \) and \( a_0^\dagger \) to be c-numbers (Schur’s lemma),

\[
a_0 = a_0^\dagger = \sqrt{n_0}
\]

Thus we will replace \( a_0 \) and \( a_0^\dagger \) by \( \sqrt{n_0} \). Then the coefficients of terms quadratic in \( a_k \) and \( a_k^\dagger \), \( k \neq 0 \), in the interaction will be of order \( n_0 \) and those of quartic term is of order 1. Therefore we can make the approximation that neglect the quartic terms and get the Hamiltonian in the form,

\[
H = \sum_{k \neq 0} \frac{\hbar^2 k^2}{2m} a_k^\dagger a_k + \frac{n_0}{2\Omega} \sum_{k \neq 0} [\tilde{v} (k) (a_k^\dagger a_{-k}^\dagger + a_k a_{-k})] + \frac{n_0^2}{2\Omega} \tilde{v} (0)
\]

or

\[
H = \sum_{k \neq 0} \omega_k a_k^\dagger a_k + \frac{n_0}{2\Omega} \sum_{k \neq 0} \tilde{v} (k) (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) + \frac{N^2}{2\Omega} \tilde{v} (0)
\]

where

\[
\omega_k = \frac{\hbar^2 k^2}{2m} + \frac{n_0}{\Omega} \tilde{v} (k)
\]

and we have used

\[
N^2 = \left( n_0 + \sum_{k \neq 0} a_k^\dagger a_k \right) = n_0^2 + 2n_0 \sum_{k \neq 0} a_k^\dagger a_k
\]

Note that this Hamiltonian does not conserve the particle number but it conserves the momentum because the removal of \( k = 0 \) mode effects the particle number but not the momentum. Since this Hamiltonian contains only quadratic terms, we can solve this by Bogoliubov transformation as follows. Define the quasi-particle operators by

\[
\alpha_k = \cosh \theta_k \ a_k + \sinh \theta_k \ a_{-k}^\dagger
\]

then we have

\[
\alpha_k^\dagger = \cosh \theta_k \ a_k^\dagger + \sinh \theta_k \ a_{-k}
\]

where \( \theta_k \) is an arbitrary parameter at our disposal. We now write the Hamiltonian in terms of the quasi particle operators by inverting the relations in Eq(??),

\[
a_k = \cosh \theta_k \ \alpha_k - \sinh \theta_k \ \alpha_{-k}^\dagger, \quad a_{-k}^\dagger = -\sinh \theta_k \ \alpha_k + \cosh \theta_k \ \alpha_{-k}^\dagger
\]

and choose the parameter \( \theta_k \) so that the coefficient of the non-diagonal terms \( (\alpha_k^\dagger \alpha_{-k} + \alpha_k \alpha_{-k}) \) is zero. The computation is straightforward and the result is

\[
\tanh 2\theta_k = \frac{\tilde{v}(0)}{\omega_k}
\]
and the Hamiltonian is

\[ H = \sum_{k \neq 0} \varepsilon_k \alpha_k^\dagger \alpha_k + \frac{N^2 \nu (0)}{2 \Omega} + \frac{1}{2} \sum_{k \neq 0} (\varepsilon_k - \omega_k) \]  

(144)

where

\[ \varepsilon_k = \sqrt{\omega_k^2 - \left( \frac{n_0 \nu}{\Omega} \right)^2} = \sqrt{\left( \frac{\hbar^2 k^2}{2m} \right) + 2 \left( \frac{\hbar^2 k^2}{2m} \right) \left( \frac{n_0 \nu (k)}{\Omega} \right)} \]  

(145)

This is just the Hamiltonian for the uncoupled harmonic oscillators and the eigenvalues are

\[ E = \sum_k n_k \varepsilon_k \]  

(146)

The quai-particle energy excitation has the property that

\[ \varepsilon_k \to 0, \quad \text{as} \quad k \to 0 \]  

(147)

which is just the Goldstone excitation. Clearly, the ground state \(|\Psi_0 >\) is the one which is annihilated by all quasi particle operators \(\alpha_k\),

\[ \alpha_k |\Psi_0 > = 0 \quad \forall k \]  

(148)

and the excited states are of the form,

\[ \left( \alpha_{k_1}^\dagger \right)^{n_1} \left( \alpha_{k_2}^\dagger \right)^{n_2} \cdots |\Psi_0 > \]  

(149)

It is straightforward to show that the quasi particle ground state can be written in terms of the original creation operators \(a_k^\dagger\) as

\[ |\Psi_0 > = \sqrt{Z} \exp{\left\{ - \sum_{k_i} \tanh \theta_{k_i} a_{k_i}^\dagger a_{-k_i} \right\} |0 > \} \]  

(150)

where

\[ Z = \prod_{k_i} \left( 1 - \tanh^2 \theta_{k_i} \right) \]  

(151)

This shows that the quasi particle ground state is a complicate combination of the vacuum, 2 particle states, 4 particle states, \cdots etc. To elucidate the Goldstone theorem, we note that the vacuum expectation value of \(\psi (0)\) in the ground state is non-zero,

\[ < \Psi_0 |\psi (0) |\Psi_0 > = \frac{1}{\sqrt{\Omega}} < \Psi_0 |a_0 |\Psi_0 > = \sqrt{\frac{n_0}{\Omega}} \neq 0. \]  

(152)

where we have used the fact that \(< \Psi_0 |\alpha_k |\Psi_0 > = 0, \text{ for } k \neq 0.\) from the momentum conservation. From the commutation relation in Eq(125) we see that this is the symmetry breaking condition which implies that

\[ Q |\Psi_0 > \neq 0. \]  

(153)

The quasi particle excitation which has the property that its energy \(\varepsilon_k\) goes to zero in the limit \(k \to 0\), is the Goldstone excitation implies by the Goldstone’s theorem.
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