On $f$-Symmetries of the Independence Polynomial

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Abstract
An independent set in a graph is a set of pairwise non-adjacent vertices, and $\alpha(G)$ is the size of a maximum independent set in the graph $G$.

If $s_k$ is the number of independent sets of cardinality $k$ in $G$, then

$$I(G; x) = s_0 + s_1 x + s_2 x^2 + ... + s_{\alpha} x^{\alpha}, \alpha = \alpha(G),$$

is called the independence polynomial of $G$ (I. Gutman and F. Harary, 1983).

If $s_{\alpha-i} = f(i) \cdot s_i$ holds for every $i \in \{0,1,\ldots,\lceil \alpha/2 \rceil \}$, then $I(G; x)$ is called $f$-symmetric ($f$-palindromic). If $f(i) = 1, i \in \{0,1,\ldots,\lceil \alpha/2 \rceil \}$, then $I(G; x)$ is called symmetric (palindromic).

The corona of the graphs $G$ and $H$ is the graph $G \circ H$ obtained by joining each vertex of $G$ to all the vertices of a copy of $H$.

In this paper we show that if $H$ is a graph with $p$ vertices, $q$ edges, and $\alpha(H) = 2$, then $I(G \circ H; x)$ is $f$-symmetric, where

$$f(i) = \left( \frac{p(p-1)}{2} - q \right)^{\frac{\alpha-i}{2}}, 0 \leq i \leq \alpha = \alpha(G \circ H).$$

In particular, if $H = K_r - e, r \geq 2$, we show that $I(G \circ H; x)$ is symmetric and unimodal, with a unique mode. This finding generalizes results due to Stevanović [22] and Mandrescu [21] claiming that $I(G \circ (K_2 - e); x) = I(G \circ 2K_1; x)$ is symmetric and unimodal for every graph $G$.

Keywords: independent set, independence polynomial, symmetric polynomial, palindromic polynomial.

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1 Introduction

Throughout this paper $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subset V(G)$, then $G[X]$ is the subgraph of $G$ spanned by $X$. 
By $G - W$ we mean the subgraph $G[V - W]$, if $W \subseteq V(G)$. We also denote by $G - F$ the subgraph of $G$ obtained by deleting the edges of $F$, for $F \subseteq E(G)$, and we write shortly $G - e$, whenever $F = \{e\}$.

The neighborhood of a vertex $v \in V$ is the set

$$N_G(v) = \{w : w \in V \text{ and } vw \in E\},$$

and

$$N_G[v] = N_G(v) \cup \{v\}.$$  

If there is no ambiguity on $G$, we use $N(v)$ and $N[v]$, respectively.

$K_n, P_n, C_n$ denote respectively, the complete graph on $n \geq 1$ vertices, the chordless path on $n \geq 1$ vertices, and the chordless cycle on $n \geq 3$ vertices.

The disjoint union of the graphs $G_1, G_2$ is the graph $G = G_1 \cup G_2$ having as vertex set the disjoint union of $V(G_1), V(G_2)$, and as edge set the disjoint union of $E(G_1), E(G_2)$. In particular, $nG$ denotes the disjoint union of $n \geq 1$ copies of the graph $G$. The Zykov sum of the disjoint graphs $G_1, G_2$ is the graph $G_1 + G_2$ with $V(G_1) \cup V(G_2)$ as a vertex set and

$$E(G_1) \cup E(G_2) \cup \{v_1v_2 : v_1 \in V(G_1), v_2 \in V(G_2)\}$$

as an edge set.

The corona of the graphs $G$ and $H$ is the graph $G \circ H$ obtained from $G$ and $|V(G)|$ copies of $H$, such that each vertex of $G$ is joined to all vertices of a copy of $H$.

An independent set in $G$ is a set of pairwise non-adjacent vertices. An independent set of maximum size is a maximum independent set of $G$, and the independence number $\alpha(G)$ is the cardinality of a maximum independent set in $G$.

Let $s_k$ be the number of independent sets of size $k$ in a graph $G$. The polynomial

$$I(G; x) = s_0 + s_1x + s_2x^2 + \ldots + s_\alpha x^\alpha, \quad \alpha = \alpha(G),$$

is called the independence polynomial of $G$ [1]. For a survey on independence polynomials of graphs see [15]. Some basic procedures to compute the independence polynomial of a graph are recalled in the following.

**Theorem 1.1** [7] (i) $I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x)$;

(ii) $I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1$;

(iii) $I(G; x) = I(G - v; x) + x \cdot I(G - N[v]; x)$ holds for every $v \in V(G)$.

A finite sequence of real numbers $(a_0, a_1, a_2, \ldots, a_n)$ is said to be:

- **unimodal** if there exists an index $k \in \{0, 1, \ldots, n\}$, called the mode of the sequence, such that
  $$a_0 \leq \ldots \leq a_{k-1} \leq a_k \geq a_{k+1} \geq \ldots \geq a_n;$$

- **$f$-symmetric ($f$-palindromic)** if $a_{n-i} = f(i) \cdot a_i$ for all $i \in \{0, \ldots, \lfloor n/2 \rfloor\}$;

- **symmetric (palindromic)** if $a_i = a_{n-i}, i = 0, 1, \ldots, \lfloor n/2 \rfloor$, i.e., $f(i) = 1$ for all $i \in \{0, \ldots, \lfloor n/2 \rfloor\}$.  

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A polynomial is called unimodal (symmetric, f-symmetric) if the sequence of its coefficients is unimodal (symmetric, f-symmetric, respectively). For instance, the independence polynomial:

- \( I(K_{127} + 3K_7; x) = 1 + 148x + 147x^2 + 343x^3 \) is non-unimodal;
- \( I(K_{43} + 3K_7; x) = 1 + 64x + 147x^2 + 343x^3 \) is unimodal and non-symmetric;
- \( I(K_{18} + 3K_3 + 4K_1; x) = 1 + 31x + 33x^2 + 31x^3 + x^4 \) is symmetric and unimodal;
- \( I(K_{52} + 3K_4 + 4K_1; x) = 1 + 68x + 54x^2 + 68x^3 + x^4 \) is symmetric and non-unimodal;
- \( I(P_3 \circ (K_2 \cup K_1); x) = 1 + 12x + 52x^2 + 105x^3 + 104x^4 + 48x^5 + 8x^6 \) is f-symmetric for \( f(i) = 2^{3-i}, 0 \leq i \leq 3 \).

For other examples, see [12, 13, 14, 16, 23, 24]. Alavi, Malde, Schwenk and Erdős proved that for every permutation \( \pi \) of \( \{1, 2, ..., \alpha \} \) there is a graph \( G \) with \( \alpha(G) = \alpha \) such that \( s_{\pi(1)} < s_{\pi(2)} < ... < s_{\pi(\alpha)} \).

Theorem 1.2 [9] \( I(G \circ H; x) = (I(H; x))^n \cdot I \left( G; \frac{x}{I(H; x)} \right) \), where \( n = |V(G)| \).

The symmetry of matching polynomial and characteristic polynomial of a graph were examined in [11], while for independence polynomial we quote [10, 22, 18, 19]. It is known that the product of two unimodal polynomials is not necessarily unimodal.

Theorem 1.3 [2] If \( P \) and \( Q \) are both unimodal and symmetric, then \( P \cdot Q \) is unimodal and symmetric.

However, the above result can not be generalized to the case when \( P \) is unimodal and symmetric, while \( Q \) is unimodal and non-symmetric; e.g.,

\[
\begin{align*}
P &= 1 + x + 3x^2 + x^3 + x^4, \\
Q &= 1 + x + x^2 + x^3 + 2x^4, \text{ while} \\
P \cdot Q &= 1 + 2x + 5x^2 + 6x^3 + 8x^4 + 7x^5 + 8x^6 + 3x^7 + 2x^8.
\end{align*}
\]

It is worth mentioning that one can produce graphs with symmetric independence polynomials in different ways (see, for instance, [3, 13, 22]).

In this paper we prove that if \( H \) is a graph with \( p \) vertices, \( q \) edges, and \( \alpha(H) = 2 \), then \( I(G \circ H; x) \) is f-symmetric, where \( f(i) = \left( \frac{p(p-1)}{2} - q \right)^{\frac{2}{p-1}} \), \( 0 \leq i \leq \alpha = \alpha(G \circ H) \).

In particular, if \( H = K_r - e \), where \( r \geq 2 \) and \( e \) is an edge of \( K_r \), then \( I(G \circ H; x) \) is symmetric and unimodal with a unique mode. As a consequence, we deduce that \( I(G \circ (K_2 - e); x) = I(G \circ 2K_1; x) \) is both symmetric [22] and unimodal [21] for every graph \( G \).
2 Results

The polynomial \( P(x) \) is symmetric if and only if it equals its reciprocal, i.e.,

\[
P(x) = x^{\deg(P)} \cdot P \left( \frac{1}{x} \right).
\]

We generalize this observation using \( \frac{1}{cx} \) instead of \( \frac{1}{x} \).

**Lemma 2.1** If \( P(x) = \sum_{i=0}^{n} a_i x^i \) is a polynomial of degree \( n \), then

\[
P(x) = c^\frac{n}{2} \cdot x^n \cdot P \left( \frac{1}{cx} \right) \text{ if and only if } a_{n-i} = c^{i-n} a_i, 0 \leq i \leq n.
\]

**Proof.** Since

\[
c^\frac{n}{2} \cdot x^n \cdot P \left( \frac{1}{cx} \right) = c^\frac{n}{2} \cdot x^n \cdot \sum_{i=0}^{n} \frac{a_i}{(cx)^i} = \sum_{i=0}^{n} c^{\frac{n}{2} - i} \cdot a_i \cdot x^{n-i} = \sum_{i=0}^{n} c^{i-n} \cdot a_{n-i} \cdot x^i,
\]

we infer that

\[
P(x) = c^\frac{n}{2} \cdot x^n \cdot P \left( \frac{1}{cx} \right) \iff a_i = c^{i-n} \cdot a_{n-i} \iff a_{n-i} = c^{i-n} \cdot a_i, 0 \leq i \leq n,
\]

and this completes the proof. ■

If \( \frac{a(x)}{b(x)} = \frac{a(f(x))}{b(f(x))} \), then \( f(x) \) is an invariant of the rational function \( \frac{a(x)}{b(x)} \). Actually, from the point of view of Gutman’s formula from Theorem 1.2, we are mostly interested in finding invariants for rational functions of the form \( \frac{x}{b(x)} \).

**Lemma 2.2** The rational function \( \frac{x}{b_0 + b_1 x + b_2 x^2} \) admits only two invariants, namely, \( f_1(x) = x \) and \( f_2(x) = \frac{b_0}{b_2 x} \).

**Proof.** Since

\[
x \cdot \frac{b_0 + b_1 x + b_2 x^2}{b_0 + b_1 f(x) + b_2 f(x)}^2 \iff b_2 x f(x)^2 - (b_0 + b_2 x^2) f(x) + b_0 x = 0,
\]

we get the following solutions: \( f_1(x) = x \) and \( f_2(x) = \frac{b_0}{b_2 x} \). ■

**Theorem 2.3** Let \( G \) be a graph of order \( n \), and \( H \) be a graph of order \( p \) and size \( q \), with \( \alpha(H) = 2 \). Then the polynomial \( I(G \circ H; x) \) is:

(i) \( f \)-symmetric, where \( f(i) = \left( \frac{\alpha(p+1)}{2} - q \right)^{\frac{n}{2} - i}, 0 \leq i \leq \alpha = \alpha(G \circ H); \)

(ii) symmetric if and only if \( H = K_r - e \) for some \( r \geq 2 \).
Proof. (i) Since \( \alpha(H) = 2 \), one can write \( I(H; x) = 1 + px + mx^2 \), where \( m = \frac{p(p-1)}{2} - q \). By Lemma 2.2, the function \( g(x) = (mx)^{-1} \) is the only non-trivial invariant of \( \frac{x}{I(H; x)} \). Thus we get
\[
I(H; g(x)) = 1 + p \cdot g(x) + m \cdot (g(x))^2 = (mx)^{-1} \cdot I(H; x).
\]
According to Lemma 2.2, it follows
\[
I(G \circ H; g(x)) = (I(H; g(x)))^n \cdot I(G; \frac{g(x)}{I(H; g(x))}) = (mx^2)^{-n} \cdot (I(H; x))^n \cdot I(G; \frac{x}{I(H; x)}) = (mx^2)^{-n} \cdot I(G \circ H; x).
\]
Consequently, we have \( I(G \circ H; x) = m^n \cdot x^{2n} \cdot I(G \circ H; \frac{1}{mx}) \). Since every \( v \in V(G) \) is joined in \( G \circ H \) to a copy of \( H \), it follows that each independent set \( S \) of \( G \circ H \) containing a pair of non-adjacent vertices from every copy of \( H \) is a maximum independent in \( G \circ H \), i.e., \( \alpha(G \circ H) = 2n \). Lemma 2.1 implies that \( I(G \circ H; x) \) is \( f \)-symmetric, where
\[
f(i) = \left( \frac{p(p-1)}{2} - q \right)^{n-i}, 0 \leq i \leq 2n = \deg(I(G \circ H; x)).
\]

(ii) The polynomial \( I(G \circ H; x) \) is symmetric if and only if \( f(i) = 1, 0 \leq i \leq \alpha \). By part (i), it means \( \frac{p(p-1)}{2} - q = 1 \), and this holds if and only if \( H = K_p - e \), where \( e \) is some edge of \( K_p \). □

It is worth noticing that \( K_2 - e = 2K_1 \), and this leads to the following.

Corollary 2.4 [22] The polynomial \( I(G \circ 2K_1; x) \) is symmetric for every graph \( G \).

Recall that a graph \( G \) is perfect if \( \chi(H) = \omega(H) \) for every induced subgraph \( H \) of \( G \), where \( \chi(H) \) denotes the chromatic number of \( H \) [4].

Proposition 2.5 [17] If \( G \) is a perfect graph with \( \alpha(G) = \alpha \) and \( \omega = \omega(G) \), then
\[
s_{\lceil \omega \alpha - 1 \rceil/\omega + 1} \geq \ldots \geq s_{\alpha - 1} \geq s_\alpha.
\]

The Strong Perfect Graph Theorem, due to Chudnovsky et al., [5], [6], asserts that a graph \( G \) is perfect if and only if it contains no odd hole (i.e., \( C_{2n+1}, n \geq 2 \)) and no odd antihole (i.e., \( C_{2n+1}, n \geq 2 \)) as an induced subgraph.

Proposition 2.6 If \( G \) is a perfect graph of order \( n \), then the coefficients \( s_i \) of the polynomial \( I(G \circ (K_p \cup K_q); x) \) satisfy the following:
\[
s_{\lceil (2n\omega - 1) / (\omega + 1) \rceil} \geq \ldots \geq s_{2n-1} \geq s_{2n} \quad \text{and} \quad s_0 \leq \ldots \leq s_{t-1} \leq s_t,
\]
where \( t = 2n - \left( \lceil (2n\omega - 1) / (\omega + 1) \rceil - 2n \right) \omega = \max \{ \omega(G), p, q \} \).
Hence we obtain that
\[ \left( G \circ (K_p \cup K_q) \right) = 2n, \quad \omega = \omega \left( G \circ (K_p \cup K_q) \right) = \max \{ \omega (G) \cdot p, q \}. \]

Since \( G \circ (K_p \cup K_q) \) has no odd hole and no odd antihole as an induced subgraph, Strong Perfect Graph Theorem assures that \( G \circ (K_p \cup K_q) \) is perfect. According to Proposition 2.7 it follows that
\[ s_{[(2n\omega - 1)/(\omega + 1)]} \geq \cdots \geq s_{2n-1} \geq s_{2n}. \]

By Theorem 2.3 we have that
\[ s_{2n-i} = (pq)^{n-i} \cdot s_i, 0 \leq i \leq n. \]

Since \( s_{2n-i} \geq s_{2n-i+1} \), for \( 2n - i \geq \lceil (2n\omega - 1)/(\omega + 1) \rceil \), we obtain
\[ (pq)^{n-i} \cdot s_i \geq (pq)^{n-i+1} \cdot s_{i-1} \iff s_i \geq pq \cdot s_{i-1} \iff s_{i-1} \leq s_i, \]
for \( 1 \leq i \leq 2n - \lceil (2n\omega - 1)/(\omega + 1) \rceil \), as claimed. ■

It is easy to see that the sum of two symmetric and unimodal polynomials is not necessarily symmetric and/or unimodal.

**Lemma 2.7** Let \( p(x) \) and \( q(x) \) be polynomials of degree \( r \) and \( r-1 \) respectively, for some \( r \geq 2 \), and let \( p(0) \neq 0 \) and \( q(0) = 0 \). If \( p(x) \) and \( q(x) \) are symmetric and unimodal, then so is \( p(x) + q(x) \). Moreover, if the mode of \( p(x) \) or \( q(x) \) is unique, then the mode of \( p(x) + q(x) \) is unique as well.

**Proof.** The symmetry and unimodality of \( p(x) + q(x) \) were proved in [3].

Assume that the mode of \( p(x) = a_0 + \cdots + a_{s-1}x^{s-1} + a_s x^s + a_{s+1}x^{s+1} + \cdots + a_q x^r \) is equal to \( s \) and is unique, i.e., \( a_0 \leq a_1 \leq \cdots \leq a_{s-1} < a_s > a_{s+1} \geq \cdots \geq a_q \), where \( r = 2s \). The polynomial \( q(x) \) is symmetric, unimodal, deg \( q = r - 1 \), and \( q(0) = 0 \), that is \( q(x) = b_1 x + \cdots + b_1 x^{r-1} \) satisfies \( b_1 \leq \cdots \leq b_{s-1} \leq b_s > b_{s+1} \geq b_{s+2} \). Then \( p(x) + q(x) = a_0 + (a_1 + b_1) x + \cdots + (a_{r-1} + b_{r-1}) x^{r-1} + a_q x^r \) is symmetric, unimodal, and \( a_{s-1} + b_{s-1} < a_s + b_s \), i.e., its mode is equal to \( s \) and it is unique.

Similarly, one can show that the mode of \( p(x) + q(x) \) is unique, whenever the mode of \( q(x) \) is unique. ■

**Lemma 2.8** If \( a > 1 \), and \( P = 1 + s_1 x + \cdots + s_{n-1} x^{n-1} + s_n x^n + s_{n-1} x^{n+1} + \cdots + x^{2n} \) is symmetric, unimodal with a unique mode, then \( Q = (1 + ax + x^2) \cdot P \) is symmetric and unimodal with a unique mode, equal to \( n + 1 \).

**Proof.** The symmetry of \( Q \) follows from Theorem 2.3. The coefficients of \( x^n \), \( x^{n+1} \), \( x^{n+2} \) in \( Q \) are respectively, \( t_n = t_{n+2} = s_n + a s_{n-1} + s_{n-2} \) and \( t_{n+1} = 2s_{n-1} + a s_n \). Hence we obtain that
\[ t_{n+1} - t_n = t_{n+1} - t_{n+2} = (s_{n-1} - s_{n-2}) + (a - 1) (s_n - s_{n-1}) > 0, \]
which implies that the mode of \( Q \) is equal to \( n + 1 \) and it is unique. ■
Theorem 2.9 If $H = K_r - e, r \geq 2$, then the polynomial $I(G \circ H; x)$ is unimodal and symmetric for every graph $G$. Moreover, the mode of $I(G \circ H; x)$ is unique and equal to the order of $G$.

Proof. The polynomial $I(G \circ H; x)$ is symmetric, according to Theorem 2.3(ii).

We show, by induction on the order $n = |V(G)|$ of $G$, that $I(G \circ H; x)$ is unimodal and its mode is unique and equal to $n$.

If $n = 1$, then $G = K_1$ and $I(G \circ H; x) = I(K_{r+1} - e; x) = 1 + (r + 1)x + x^2$, which is clearly unimodal and the mode is unique and equal to $n$.

If $n = 2$, then either $G = 2K_1$ and

\[ I(G \circ H; x) = I(2K_1 \circ H; x) = (1 + (r + 1)x + x^2)^2 = 1 + (2r + 2)x + (r^2 + 2r + 3)x^2 + (2r + 2)x^3 + x^4, \]

or $G = K_2 = \{(v_1, v_2), \{v_1 v_2\}\}$ and

\[ I(G \circ H; x) = I(K_2 \circ H - v_1; x) + x \cdot I(K_2 \circ H - N[v_1]; x) = I(H; x) \cdot I(K_{r+1}; x) + x \cdot I(H; x) = (1 + (r + 1)x + x^2)(1 + (r + 2)x + x^2) = 1 + (2r + 3)x + (r^2 + 3r + 4)x^2 + (2r + 3)x^3 + x^4. \]

In both cases, $I(G \circ H; x)$ is clearly unimodal and the mode is unique and equals $n$.

Let $G$ be a graph of order $n \geq 3$.

Clearly, if $E = \emptyset$, then $I(G \circ H; x) = (1 + (r + 1)x + x^2)^n$, which is unimodal, according to Theorem 1.3 and its mode is unique and equal to $n$, by Lemma 2.8.

Suppose that $E \neq \emptyset$, and let $v \in V$ be with $N_G(v) = \{u_i : 1 \leq i \leq k\}$.

Applying Theorem 2.3 we obtain

\[ I(G \circ H; x) = I(G \circ H - v; x) + x \cdot I(G \circ H - N_G[H \setminus v]; x) = p(x) + q(x), \]

where

\[ p(x) = I(G \circ H - v; x) \text{ and } q(x) = x \cdot I(G \circ H - N_G[H \setminus v]; x). \]

Claim 1. $p(x)$ is symmetric, unimodal with a unique mode, equal to $n$.

First, $p(x) = I(H; x) \cdot I((G - v) \circ H; x)$, because $G \circ H - v$ is the disjoint union of $H$ and $(G - v) \circ H$.

The graph $G \circ H$ has a unique maximum independent set, namely the set $S$ containing the non-adjacent vertices from each $H$. Hence, we get that $\alpha(G \circ H) = \alpha(G \circ H - v) = 2n$, since $S \cap V(G) = \emptyset$, and $v \in V(G)$. According to Theorem 2.3(ii), the polynomial $I((G - v) \circ H; x)$ is symmetric, and by induction hypothesis, $I((G - v) \circ H; x)$ is also unimodal and its mode is unique and equal to $n - 1$. According to Lemma 2.8 $p(x)$ is symmetric and unimodal with a unique mode, equal to $n$.

Claim 2. $q(x)$ is symmetric and unimodal.

Since $G \circ H - N_G[H \setminus v]$ consists of the disjoint union of $kH$ and $(G - N_G[v]) \circ H$, we obtain that $q(x) = x \cdot I((H; x))^k \cdot I((G - N_G[v]) \circ H; x)$.

Further, one can see that $\alpha(G \circ H - N_G[H \setminus v]) = |S - \{a_1, a_2\}| = \alpha(G \circ H) - 2$, where $\{a_1, a_2\} = S \cap N_G[H \setminus v]$. The symmetry of $I((G - N_G[v]) \circ H; x)$ follows from
Theorem 2.3 (ii). By induction hypothesis, $I((G - N_G[v]) \circ H; x)$ is unimodal with a unique mode. Lemma 2.8 ensures that $(I(H; x))^k \bullet I((G - N_G[v]) \circ H; x)$ is symmetric and unimodal, with a unique mode.

**Claim 3.** $I(G \circ H; x)$ is symmetric, unimodal and its mode is unique and equals $n$.

Since $\deg p = \deg q + 1 \geq 2$, and $p(0) = 1$, while $q(0) = 0$, we finally obtain that $I(G \circ H; x) = p(x) + q(x)$ is symmetric and unimodal with a unique mode, according to Lemma 2.7. 

Since $K_2 - e = 2K_1$, we obtain the following.

**Corollary 2.10** \([21]\) The polynomial $I(G \circ 2K_1; x)$ is unimodal for every graph $G$.

### 3 Conclusions

In this paper we started investigating higher symmetries of polynomials with emphasis on independence polynomials of graphs. This new paradigm already showed its usefulness in revealing a new family of graphs with symmetric independence polynomials. We conclude with the following.

**Conjecture 3.1** $I(G \circ H; x)$ is symmetric for every graph $G$ if and only if $H = K_r - e$ for some $r \geq 2$.

**Problem 3.2** Describe the set of invariants of a given rational function $\frac{a(x)}{b(x)}$.

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