Modified Maxwell equations from CPT-even Lorentz violation with Minimum Length

T. Prudêncio¹*, L. S. Amorim²†, H. Belich²‡, H. L. C. Louzada²§

¹Coordination of Science & Technology (CCCT-BICT), Federal University of Maranhão (UFMA), 65080-805, São Luís-MA, Brazil. and

²Departamento de Física e Química, Federal University of Espírito Santo (UFES), 29060-900, Vitória, ES, Brazil.

(Dated: June 15, 2017)

Here we discuss the presence of CPT-even Lorentz violation (LV) in the presence of a deformed Heisenberg algebra that leads to a minimum length (ML). We consider the case of a Maxwell lagrangian modified by the presence of a $K_F$ CPT-even LV theory and ML. We then derive a set of modified Maxwell equations in the cases of LV and ML and only ML. We verified that in the case of electromagnetic waves in the vacuum the presence of ML does not change the consequences of LV. On the other hand, in a material media ML changes the whole set of equations that can lead to important effects with respect to the usual equations. We also considered the more general case including LV and the modified equations in terms of matter fields. We then derived the refractive index as a function of the matter fields depending on LV and ML, and in particular we showed the behaviour of the refractive index with respect to the non-commutative parameter.

I. INTRODUCTION

Despite the great success of Standard Model (SM) in describing through field theory the regimes that unifies weak and electromagnetic interactions, SM has clear limitations concerning regimes of unification in the Planck era, mainly related to the presence of

* email: prudencio.thiago@ufma.br, thprudencio@gmail.com
† email: lesamorim@gmail.com
‡ email: belichjr@gmail.com
§ email: haofisica@bol.com.br.
gravity. The interest in investigating the physics beyond SM has been increased with the need of understanding the problem of Dark Matter. The regimes of interaction between the dark and non-dark sectors, can induce the detection of a weak fifth force. This was tested investigating anomalous decays of an excited state of $^8$Be [1]. Also, we have the unbalance between matter-antimatter that has not been clarified by the SM [2-8].

Investigating the physics beyond SM, Kostelecký and Samuel [9] proposed a new regime where spontaneous violation of symmetry occurs through non-scalar fields based on string field theory, leading to a vacuum with a tensor nature. A consistent description of fluctuations around this new vacuum is obtained if the components of the background field are constant, and by the fact that the minimum in the the background not represented by a scalar, and consequently Lorentz symmetry is spontaneously broken [10].

This extension SM by Lorentz symmetry violation (LV) has been considered for fields that belong to a more fundamental theory, which may induce the spontaneous violation of Lorentz symmetry based on a specific potential. It is worth mentioning that this extension of SM keeps the gauge invariance, the conservation of energy and momentum and the covariance under observer rotations and boosts, where this extension is called as the Standard Model Extension (SME) [11, 12]. In this context, it is well-known that the presence of terms that violate the Lorentz symmetry imposes at least one privileged direction in the spacetime. In recent decades, studies of the Lorentz violation (LV) have been made in several branches of physics [13-36]. The LV has been investigated in two major scenarios proposed: spontaneous Lorentz symmetry violation (SLV) caused by a tensorial background treated above, and the breaking made by generalization of uncertainty principle - the non-commutative geometry.

On the other hand the proposal of noncommutative geometry was developed in 1980 by A. Connes [37] and it was realized that the non-commutative geometry would be a scheme to extend the standard model in several ways [38]. In the 90s the proposal appears naturally in the context of string theory [39, 40]. In this way we may obtain an effective theory describing scenarios in string theory whose in the low energy limit is reduced to a known physical theory.

Noncommutative geometry also appear in a condensed matter context as an effective theory that describes the electron in a two-dimensional surface attached to a strong magnetic field. This effective theory describes the Quantum Hall Effect. The electron would be trapped in the lowest Landau levels and presents the Hall conductance in $e^2/h$ units [41].
The effect of non-commutativity could be tested for instance in an hydrogen atom, one of the simplest quantum systems that allows theoretical predictions and experimental verifications of high accuracy \[42\]. There are many papers where the energy spectrum of the hydrogen atom in the presence minimum length is calculated \[43–45\], some of which have divergences in levels \(s(n = 1)\) \[44\].

A possible way to explore the implementation of noncommutatives theories is by the deformation of the Heisenberg algebra. In particular, a modified Heisenberg algebra is achieved by adding certain small corrections to the canonical commutation relations, as shown by A. Kempf and contributors \[46–50\], to the minimum uncertainty in the position measurement, \(\delta x_0\), called minimum length. The existence of this minimum length was also suggested by quantum gravity and string theory \[51–53\].

Recently, Quesne and Tkachuk have introduced a Lorentz covariant deformed algebra that describes a quantized \(D + 1\)-dimensional \[54, 55\], it is given by the following generalized commutation relations:

\[
\begin{align*}
{[X, P]} &= -i\hbar \frac{1}{\beta} \left[ \frac{1}{\beta} - P_\rho P^\rho \right] g^{\mu\nu} \beta P^\mu P^\nu, \quad (1) \\
{[P, P]} &= 0, \quad (2) \\
{[X, X]} &= i\hbar \frac{(2 - \beta') (2 + \beta') P_\rho P^\rho (P^\mu X^\nu - P^\nu X^\mu)}{(\frac{1}{\beta} - P_\rho P^\rho)} \quad (3)
\end{align*}
\]

where \(\mu, \nu, \rho = 0, 1, \cdots, D\), \(g_{\mu\nu} = g^{\mu\nu} = diag(1, -1, -1, \cdots, -1)\), \(\beta\) and \(\beta'\) are deformation parameters, and we suppose \(\beta, \beta' > 0\). From uncertain relation we conclude that the minimum length (ML) is

\[
(\delta X^i)_0 = \hbar \beta \sqrt{\left( \frac{1}{\beta} - \langle (P^0)^2 \rangle \right) \left[ 1 - \frac{1}{\beta} \right]} \quad \forall i \in \{1, \cdots, D\}.
\]

A algebra representation \[56\] that satisfies (1) in the first order in \(\beta, \beta'\) is given by:

\[
\begin{align*}
X^\mu &= x^\mu - \beta \frac{1}{2} \left( 1 - \frac{\beta'}{2} \right) (x^\mu p_\rho p^\rho + p_\rho p^\rho x^\mu), \quad \forall i \in \{1, \cdots, D\} \\
P^\mu &= \frac{2 \beta - P^\rho p_\rho}{\beta'} \beta' p^\mu,
\end{align*}
\]

where \(x^\mu\) and \(p^\mu = i\hbar \partial^\mu\) are the position and momentum operators. Particular cases are achieved for \(\beta' = 2\beta\) and \(\beta' = \beta\) the Quesne-Tkachuk algebra is simplified.
Here we investigate the scenario of anisotropy in polarized electromagnetic waves generated by the presence of a Lorentz symmetry breaking tensor \((K_F)_{\mu\nu\kappa\lambda}\) which appears in Standard Model Extended (SME) in the CPT-even gauge sector \cite{57,58}, and in the presence of a minimum length provided by a non-commutative structure of a Heisenberg algebra. The effects of these anisotropies in the nature of vacuum polarized electromagnetic waves is then discussed.

The structure of this paper is the following: Sections II we present our modified electromagnetism with minimum length (ML) in the CPT-even gauge sector LV. In section III, discuss the effect of the ML in the set of Maxwell equations. In the section IV, we derive the set of Maxwell equations and Matter fields in the presence of ML in the CPT-even gauge LV. Section V, we derive the matter fields in the Fourier transformed space and obtain the corresponding refractive index in the presence of LV and ML. Finally, in section VI, we address our conclusions.

II. CPT-EVEN GAUGE SECTOR WITH MINIMUM LENGTH

A CPT-even gauge sector of SME can be described by the following lagrangean \cite{59}

\[
\mathcal{L}_{2N} = -\frac{1}{4\mu_0} \left( F_{\mu\nu} F^{\mu\nu} - (K_F)_{\mu\nu\kappa\lambda} F^{\mu\nu} F^{\kappa\lambda} \right) - A_\mu J^\mu, \tag{6}
\]

where \((K_F)_{\mu\nu\kappa\lambda}\) is a tensor with non-dimensional and renormalizable coupling tensor responsible by LV, whose symmetries are the same as the Riemann tensor and the double trace vanishes, i.e.,

\[
(K_F)_{\mu\kappa\lambda} = -(K_F)_{\nu\mu\kappa}, \quad (K_F)_{\mu\nu\kappa\lambda} = -(K_F)_{\mu\nu\lambda\kappa}, \quad (K_F)_{\mu\nu\kappa\lambda} = (K_F)_{\kappa\lambda\mu\nu}; \tag{7}
\]

\[
(K_F)_{\mu\nu\kappa\lambda} + (K_F)_{\mu\kappa\lambda\nu} + (K_F)_{\mu\lambda\nu\kappa} = 0, \tag{8}
\]

\[
(K_F)^{\mu\nu}_{\mu\nu} = 0. \tag{9}
\]

We then write this lagrangean in the presence of the minimum length (4), i.e.,

\[
x^\mu \rightarrow X^\mu = x^\mu, \tag{10}
\]

\[
\partial^\mu \rightarrow \nabla^\mu = (1 + \beta \hbar^2 \Box) \partial^\mu,
\]

where \(\Box = \partial_\mu \partial^\mu\). Neglecting terms \(O(2)\) in \(\beta\), we obtain

\[
\mathcal{L}_{2NM} = -\frac{1}{4\mu_0} \left( F_{\mu\nu} F^{\mu\nu} - (K_F)_{\mu\nu\kappa\lambda} F^{\mu\nu} F^{\kappa\lambda} \right) - \frac{\beta \hbar^2}{2\mu_0} \left( F_{\mu\nu} \Box F^{\mu\nu} - (K_F)_{\mu\nu\kappa\lambda} F^{\mu\nu} \Box F^{\kappa\lambda} \right) - A_\mu J^\mu.
\]
or equivalent
\[ \mathcal{L}_{2NM} = \mathcal{L}_{2N} - \frac{\beta \hbar^2}{2\mu_0} \left( F_{\mu\nu} \Box F^{\mu\nu} - (K_F)_{\mu\nu\kappa\lambda} F^{\mu\nu} \Box F^{\kappa\lambda} \right). \]  

(11)

This will lead to the following equations
\[ (1 + 2\beta \hbar^2 \Box) \left[ \partial_\nu F^{\nu\mu} - (K_F)^{\mu\rho\phi} \partial_\nu F^{\rho\phi} \right] = \mu_0 J^\mu. \]  

(12)

A parametrization of this theory is described in [60, 61], where 19 independent components of \((K_F)\) are described in terms of four matrices \(3 \times 3\), named: \((\kappa_{DE}), (\kappa_{HB}), (\kappa_{DB})\) e \((\kappa_{HE})\). These component relations are given by
\[ (\kappa_{DE})^{jk} = -2(K_F)^{0j0k}, \quad (\kappa_{HB})^{jk} = \frac{1}{2} \varepsilon^{jlp} \varepsilon^{klm} (K_F)^{pqlm}, \]  

(13)
\[ (\kappa_{DB})^{jk} = - (\kappa_{HE})^{kj} = \varepsilon^{kpq} (K_F)^{0j0q}, \]  

(14)

from which we see that \((\kappa_{DE})\) and \((\kappa_{HB})\) are symmetric, while \((\kappa_{DB})\) is not. We then take \(\mu = 0\), in (12), leading to the following modified Gauss law
\[ (1 + 2\beta \hbar^2 \Box) \left[ \partial_i E^i + (\kappa_{DE})_{ij} \partial_j E^j + c(\kappa_{DB})_{ik} \partial_k B_k \right] = \frac{\rho}{\varepsilon_0}. \]  

(15)

Taking \(\mu = i\), we obtain the modified Ampère-Maxwell law
\[ (1 + 2\beta \hbar^2 \Box) \left[ -\partial_t E^i/c^2 + \varepsilon_{ijk} \partial_j B^k - (\kappa_{DE})_{ij} \partial_j E^j/c^2 + (\kappa_{DB})_{ik} \partial_k B_k/c^2 + \partial_t \varepsilon_{ijk}(K_{DB})_{mk} \partial_p B^p/c \right] = \mu_0 J^i. \]  

(16)

In a vector form, these equations are written as
\[ (1 + 2\beta \hbar^2 \Box) \left[ \nabla \cdot \mathbf{E} + (\kappa_{DE} \cdot \nabla) \cdot \mathbf{E} + c(\kappa_{DB} \cdot \nabla) \cdot \mathbf{B} \right] = \frac{\rho}{\varepsilon_0}. \]  

(17)
\[ (1 + 2\beta \hbar^2 \Box) \left[ -\partial_t \mathbf{E}/c^2 + \nabla \times \mathbf{B} - \kappa_{DE} \cdot \partial_t \mathbf{E}/c^2 + \kappa_{DB} \cdot \partial_t \mathbf{B}/c + \nabla \times (\kappa_{HB} \cdot \mathbf{B}) + \nabla \times (\kappa_{DB} \cdot \mathbf{E})/c \right] = \mu_0 \mathbf{J}. \]  

(18)

In the vacuum \((J^\mu = 0)\) the minimum length modifies Gauss and Ampère-Maxwell laws by the presence of a global factor resulting from \((1 + 2\beta \hbar^2 \Box)\). The dispersion relation will furnish the modes \(p^2 = |\mathbf{p}|^2 + 1/2\beta\) and the particle mass relation \(m = 1/2\sqrt{\beta}c\). We conclude that for the CPT-even gauge sector of SME, the presence of SLV and minimum length are independent effects, i.e., even for an electrodynamics without LV there is a massive pole resulting from non-commutativity [62].
III. MAXWELL EQUATIONS CHANGED BY MINIMUM LENGTH

Let us consider the whole set of Maxwell equations

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \]  
\[ \nabla \cdot \mathbf{B} = 0, \]  
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \]  
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \]

The presence of a minimum length transformation due to non-commutativity

\[ \nabla \rightarrow (1 + \beta \hbar^2 \Box) \nabla \]  
\[ \frac{\partial}{\partial t} \rightarrow (1 + \beta \hbar^2 \Box) \frac{\partial}{\partial t} \]

will lead to the following modified Maxwell equations

\[ (1 + \beta \hbar^2 \Box) \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \]  
\[ (1 + \beta \hbar^2 \Box) \nabla \cdot \mathbf{B} = 0, \]  
\[ (1 + \beta \hbar^2 \Box) \nabla \times \mathbf{E} = -(1 + \beta \hbar^2 \Box) \frac{\partial \mathbf{B}}{\partial t}, \]  
\[ (1 + \beta \hbar^2 \Box) \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 (1 + \beta \hbar^2 \Box) \frac{\partial \mathbf{E}}{\partial t}, \]

Consequently, electromagnetic waves in the presence of a minimum length will remain the same as if this type of commutativity was not present. The only distinction is in the presence of a source terms, eqs. (25) and (28). In this case, as usual, let us split the current density in terms of free, polarization and magnetization contributions

\[ \mathbf{J} = \mathbf{J}_f + \mathbf{J}_P + \mathbf{J}_M, \]

and the charge density in terms of free and polarization terms

\[ \rho = \rho_f + \rho_P, \]
where the material media has associated electric polarization $\mathbf{P}$ and magnetization $\mathbf{M}$, with corresponding definitions

$$\rho_{\mathbf{P}} = -\nabla \cdot \mathbf{P}$$  \hspace{1cm} (31)  

$$\mathbf{J}_{\mathbf{P}} = \frac{\partial \mathbf{P}}{\partial t}$$  \hspace{1cm} (32)  

$$\mathbf{J}_{\mathbf{M}} = \nabla \times \mathbf{M}.$$  \hspace{1cm} (33)

where Applying a divergent $\nabla \cdot$ in eq. (28),

$$\nabla \cdot (1 + \beta \hbar^2 \Box) \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{J} + \mu_0 \varepsilon_0 \nabla \cdot (1 + \beta \hbar^2 \Box) \frac{\partial \mathbf{E}}{\partial t},$$  \hspace{1cm} (34)

and taking into account eq. (25), we have the validity of continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0,$$  \hspace{1cm} (35)

and also for the electric polarization

$$\nabla \cdot \mathbf{J}_{\mathbf{P}} + \frac{\partial \rho_{\mathbf{P}}}{\partial t} = 0.$$  \hspace{1cm} (36)

Thus, the charge density can be separated in the free part $\rho_f$ and the part depending on polarization, and the current density has a contribution due to free contributions $J_f$, polarization and magnetization, as given by

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_{\mathbf{P}} + \mathbf{J}_{\mathbf{M}}.$$  \hspace{1cm} (37)

The first modified Maxwell equation leads to the following generalization involving a matter field

$$\nabla \cdot \mathbf{D}_\beta = \rho_f$$  \hspace{1cm} (38)

where we have a generalized response $\mathbf{D}$ to the material media

$$\mathbf{D}_\beta = (\varepsilon_0(1 + \beta \hbar^2 \Box) \mathbf{E} + \mathbf{P}) = \mathbf{D} + \beta \hbar^2 \Box \mathbf{E}$$  \hspace{1cm} (39)

On the other hand, we then have

$$(1 + \beta \hbar^2 \Box) \nabla \times \mathbf{B} = \mu_0 \left( \mathbf{J}_f + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} \right) + \mu_0 \varepsilon_0 (1 + \beta \hbar^2 \Box) \frac{\partial \mathbf{E}}{\partial t}.$$  \hspace{1cm} (40)
where we also have a generalized response $H$ to the material media

$$H_{\beta} = \frac{1 + \beta \hbar^2 \Box}{\mu_0} B - M$$

$$= H + \frac{\beta \hbar^2 \Box}{\mu_0} B.$$  \hspace{1cm} (41)

and the generalized equation

$$\nabla \times H_{\beta} = J_f + \frac{\partial D_{\beta}}{\partial t}.$$  \hspace{1cm} (42)

Using the constitutive relation for polarization $P = \varepsilon_0 \chi_e E$ and the generalized one for magnetization

$$M = \chi_{m,\beta} H_{\beta},$$

we then have generalized relations for the fields in material media depending on operator coming from ML

$$D_{\beta} = \varepsilon_0 \left(1 + \beta \hbar^2 \Box + \chi_e\right) E,$$

$$H_{\beta} = \frac{1}{(1 + \chi_{m,\beta}) \mu_0} (1 + \beta \hbar^2 \Box) B.$$  \hspace{1cm} (43)

We then have generalized equations involving matter field in the material media

$$(1 + \beta \hbar^2 \Box) \nabla \cdot D_{\beta} = (1 + \beta \hbar^2 \Box + \chi_e) \rho,$$

$$\nabla \cdot H_{\beta} = 0,$$  \hspace{1cm} (46)

$$(1 + \beta \hbar^2 \Box) \nabla \times D_{\beta} = - \left(1 + \beta \hbar^2 \Box + \chi_e\right) (1 + \chi_{m,\beta}) \varepsilon_0 \mu_0 \frac{\partial H_{\beta}}{\partial t},$$  \hspace{1cm} (47)

$$(1 + \chi_{m,\beta}) \mu_0 \nabla \times H_{\beta} = \mu_0 J + \mu_0 (1 + \beta \hbar^2 \Box) \frac{\partial}{\partial t} (1 + \beta \hbar^2 \Box + \chi_e)^{-1} D_{\beta},$$  \hspace{1cm} (48)

where the equations (38) and (42) are simplest forms of eqs. (46) and (49), when expressed in terms of free charge and current densities.

**IV. MAXWELL EQUATIONS CHANGED BY MINIMUM LENGTH AND LORENTZ VIOLATION**

Let us now consider the equations (17) and (18), using $\beta' = 2\beta$,

$$\left(1 + \beta' \hbar^2 \Box\right) \left[\nabla \cdot E + (\kappa_{DE} \cdot \nabla) \cdot E + c (\kappa_{DB} \cdot \nabla) \cdot B\right] = \frac{\rho}{\varepsilon_0}.$$  \hspace{1cm} (50)
\[
(1 + \beta^2 \Box) \left[ -\partial_t E/c^2 + \nabla \times B - \kappa_{DE} \cdot \partial_t E/c^2 + \kappa_{DB} \cdot \partial_t B/c + \nabla \times (\kappa_{HB} \cdot B) + \nabla \times (\kappa_{DB} \cdot E)/c \right] = \mu_0 J,
\]

that, with the equations

\[
(1 + \beta h^2 \Box) \nabla \cdot B = 0, \quad (52)
\]

\[
(1 + \beta h^2 \Box) \nabla \times E = -(1 + \beta h^2 \Box) \frac{\partial B}{\partial t}, \quad (53)
\]

form the set of modified Maxwell equations with CPT-even LV and minimum length.

Taking into account the previous definitions, we have

\[
(1 + \beta^2 \Box) \left[ \nabla + (\kappa_{DE} \cdot \nabla) \right] \cdot D_{\beta'} = (1 + \beta^2 \Box + \chi_e) \left[ \rho - \frac{(1 + \chi_{m,\beta})}{c} (\kappa_{DB} \cdot \nabla) \cdot H_{\beta'} \right],
\]

\[
\mu_0 (1 + \chi_{m,\beta}) \nabla \cdot H_{\beta} = 0, \quad (54)
\]

\[
(1 + \beta h^2 \Box) \nabla \times D_{\beta} = -\frac{1}{c^2} (1 + \beta h^2 \Box + \chi_e) (1 + \chi_{m,\beta}) \frac{\partial H_{\beta}}{\partial t}, \quad (55)
\]

\[
\frac{1}{c^2} (1 + \beta h^2 \Box + \chi_e) (1 + \chi_{m,\beta'}) \left[ \nabla \times (H_{\beta'} + \kappa_{HB} \cdot H_{\beta'}) + \frac{1}{c} \kappa_{DB} \cdot \partial_t H_{\beta'} \right] = \frac{1}{c^2} (1 + \beta h^2 \Box + \chi_e) J + \frac{1}{c^2} (1 + \beta^2 \Box) \left[ \partial_t (D_{\beta} + \kappa_{DE} \cdot D_{\beta}) - c \nabla \times (\kappa_{DB} \cdot D_{\beta}) \right]
\]

\[
(56)
\]

\[
(57)
\]

We can consider the case where

\[
\kappa_{DB} \cdot D_{\beta} = \kappa_{DE} \cdot D_{\beta} = 0, \quad (58)
\]

\[
\kappa_{HB} \cdot H_{\beta'} = 0, \quad (59)
\]

\[
\kappa_{DB} \cdot \partial_t H_{\beta'} = 0. \quad (60)
\]

that will reduce eq. (57) to the following

\[
(1 + \beta h^2 \Box + \chi_e) (1 + \chi_{m,\beta'}) \nabla \times H_{\beta'} = (1 + \beta h^2 \Box + \chi_e) J + (1 + \beta^2 \Box) \partial_t D_{\beta}
\]

\[
(61)
\]
V. MATTER FIELDS IN THE FOURIER TRANSFORMED SPACE

Taking the generalized form of a matter field in the scenario of LV with ML, in the Fourier transformed space, we have

\[
\begin{align*}
D_\beta(p, \omega = p_0) &= \varepsilon_0 \left( 1 + \beta \hbar^2 p_\mu p^\mu + \chi_e \right) E(p, \omega = p_0), \\
H_\beta(p, \omega = p_0) &= \frac{1}{(1 + \chi_{m,\beta}) \mu_0} \left( 1 + \beta \hbar^2 p_\mu p^\mu \right) B(p, \omega = p_0).
\end{align*}
\] (62) (63)

We then have the corresponding

\[
\begin{align*}
\varepsilon_\beta(p, \omega = p_0) &= \varepsilon_0 \left( 1 + \beta \hbar^2 p_\mu p^\mu + \chi_e \right) \\
\mu_\beta(p, \omega = p_0) &= \frac{(1 + \chi_{m,\beta}) \mu_0}{(1 + \beta \hbar^2 p_\mu p^\mu)}
\end{align*}
\] (64) (65)

The refractive index associated to the material is then given by

\[
n_\beta = \sqrt{\left( 1 + \beta \hbar^2 p_\mu p^\mu + \chi_e \right) \frac{(1 + \chi_{m,\beta})}{(1 + \beta \hbar^2 p_\mu p^\mu)}}
\] (66)

In particular, we display in the figure [I] for \( \hbar^2 = 1, \chi_{m,\beta} = 1, \chi_e = 1, p_\mu p^\mu = 1 \), the behavior of the refractive index as a function of \( \beta \) in the ML for this CPT-even LV scenario with ML. We also have that, in the absence of ML, the refractive index is modified by the presence of a LV encapsulated in the generalized \( \chi_e \) and \( \chi_{m,\beta} \) in the presence of LV. In the limits \( \chi_{m,\beta} << 1, \chi_e << 1 \), the refractive index is also reduced to a ±1, where the −1 corresponds to a metamaterial behaviour.

VI. CONCLUDING REMARKS

We have considered a CPT-even gauge sector of SME in the presence of a deformed Heisenberg algebra with minimum length (ML). In this scenario, we derived a set of modified Maxwell equations. We conclude that the usual effects of modified electromagnetic waves does not change in the presence of a ML. However, in a material media, the effects of the deformed algebra in the set of Maxwell equation could be verified even in absence of LV. We derived both sets of Modified Maxwell equations in material media, i.e., with and without LV. In particular, we derived the dielectric functions in the Fourier transformed space and derived the refractive index in the presence of LV and ML; finally we showed how the refractive index is related to the non-commutative parameter.
FIG. 1: (Color online) Refractive index as a function of the non-commutative parameter $\beta$.

These results are important aspects for the tests with SME and non-commutativity, in particular, in condensed matter scenarios that could verify the tensors of SME and the parameters of ML, with experimental tests.

**Acknowledgements**

The authors acknowledge the supports by CNPq, CAPES, FAPES and FAPEMA-UNIVERSAL-01401/16 (Brazil).

[1] J. L. Feng et al, Phys. Rev. Lett. 117, 071803 (2016).
[2] J.E. Kim, Phys. Rep. 150, 1 (1987).
[3] H.-Y. Cheng, Phys. Rep. 158, 1 (1988).
[4] J.E. Kim and G. Carosi, Rev. Mod. Phys. 82, 557 (2010).
[5] G. Pignol, Int. J. Mod. Phys. A 30, 1530048 (2015).
[6] Y.V. Stadnik and V. V. Flambaum, Phys. Rev. D 89, 043522 (2014).
[7] B. M. Roberts, Y. V. Stadnik, V. A. Dzuba, V. V. Flambaum, N. Leefer, and D. Budker, Phys. Rev. D 90, 096005 (2014).
[8] M. Pospelov and A. Ritz, Ann. of Phys. 318, 119 (2005).
[9] V. A. Kostelecký and S. Samuel, Phys. Rev. D 39, 683 (1989).
[10] H. Belich, T. Costa-Soares, M. A. Santos and M. T. D. Orlando, Rev. Bras. Ens. Fís. 29, 1
(2007).

[11] D. Colladay and V. A. Kostelecký, Phys. Rev. D 55, 6760 (1997).

[12] D. Colladay and V. A. Kostelecký, Phys. Rev. D 58, 116002 (1998).

[13] H. Belich, T. Costa-Soares, M. M. Ferreira Jr., J. A. Helayêl-Neto, F. M. O. Moucherek, Phys. Rev. D 74, 065009 (2006).

[14] G. Gazzola, H. G. Fargnoli, A. P. Baêta Scarpelli, M. Sampaio, M. C. Nemes, J. Phys. G: Nucl. Part. Phys. 39, 035002 (2012).

[15] H. Belich, L. P. Collato, T. Costa-Soares, J.A. Helayêl-Neto, M.T.D. Orlando, Eur. Phys. J. C 62, 425 (2009).

[16] R. Casana, M. M. Ferreira Jr., V. E. Mouchrek-Santos, E. O. Silva, Phys. Lett. B 746, 171 (2015).

[17] R. Casana, C. F. Farias, M. M. Ferreira, Phys. Rev. D 92, 125024 (2015).

[18] R. Casana, M. M. Ferreira Jr., F. E. P. dos Santos, Phys. Rev. D 90, 105025 (2014).

[19] R. Casana, M. M. Ferreira Jr., E. da Hora, A. B. F. Neves, Eur. Phys. J. C 74, 3064 (2014).

[20] R. Casana, M. M. Ferreira Jr., R. V. Maluf, F. E. P. dos Santos, Phys. Lett. B 726, 815 (2013).

[21] J. B. Araujo, R. Casana, M. M. Ferreira Jr, Phys. Lett. B, 760, 302-308 (2016).

[22] H. Belich, T. Costa-Soares, M.M. Ferreira Jr., J.A. Helayêl-Neto, Eur. Phys. J. C 41, 421 (2005).

[23] M. A. Ajaib, Int. J. Mod. Phys. A 27, 1250139 (2012).

[24] A. G. Grushin, Phys. Rev. D 86, 045001 (2012).

[25] H. Belich, F. J. L. Leal, H. L. C. Louzada, M. T. D. Orlando, Phys. Rev. D 86, 125037 (2012).

[26] R. Casana, M. M. Ferreira Jr., E. Passos, F. E. P. dos Santos, E. O. Silva, Phys. Rev. D 87, 047701 (2013).

[27] H. Belich, L. P. Colatto, T. Costa-Soares, J. A. Helayêl-Neto and M. T. D. Orlando, Eur. Phys. J. C 62, 425 (2009).

[28] H. Belich, E. O. Silva, M. M. Ferreira Jr. and M. T. D. Orlando, Phys. Rev. D 83, 125025 (2011).

[29] L. R. Ribeiro, E. Passos, C. Furtado and J. R. Nascimento, Int. J. Mod. Phys. A 30, 1550072 (2015).

[30] K. Bakke and H. Belich, J. Phys. G: Nucl. Part. Phys. 40, 065002 (2013).
[31] K. Bakke and H. Belich, J. Phys. G: Nucl. Part. Phys. 39, 085001 (2012).
[32] K. Bakke and H. Belich, Spontaneous Lorentz symmetry violation and low energy scenarios (LAMBERT Academic Publishing, Saarbrücken, 2015).
[33] L. R. Ribeiro, E. Passos and C. Furtado, J. Phys. G: Nucl. Part. Phys. 39, 105004 (2012).
[34] Q. G. Bailey and V. A. Kostelecký, Phys. Rev. D 74, 045001 (2006).
[35] V. A. Kostelecký and J. D. Tasson, Phys. Rev. Lett. 102, 010402 (2009).
[36] V. A. Kostelecký, Phys. Rev. D 69, 105009 (2004).
[37] A. Connes, C* algebras and differential geometry, Compt. Rend. Acad. Sci. (Ser. I Math), A290:599-609 (1980).
[38] Michael R. Douglas, Nikita A. Nekrasov, Rev.Mod.Phys.73:977-1029, (2001).
[39] A. Connes, Michael R. Douglas and Albert Schwarz, JHEP 02:003 (1998).
[40] Michael R. Douglas, Christopher M. Hull, JHEP 02:008 (1998).
[41] Quantum Field Theory in Condensed Matter Physics, Naoto Nagaosa and S. Heusler (1999).
[42] S. G. Karshenboim, Phys. Rep. 422, 1 (2005).
[43] F. Brau, J. Phys. A 32, 7691 (1999).
[44] S. Benczik, L. N. Chang, D. Minic and T. Takeuchi, Phys. Rev. A 72, 012104 (2005).
[45] R. Akhoury, Y.-P. Yao, Phys. Lett. B 572, 37 (2003).
[46] A. Kempf, J. Math Phys. 35, 4483 (1994).
[47] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52, 1108 (1995).
[48] H. Hinrichsen and A. Kemph, J. Math Phys. 37, 2121 (1996).
[49] A. Kempf, J. Math Phys. 38, 1347 (1997).
[50] A. Kempf, J. Phys. A 30, 2093 (1997).
[51] D. J. Gross and P.F. Mende, Nucl. Phys. B 303, 407 (1988).
[52] M. Maggiore, Phys. Lett. B 304, 65 (1993).
[53] E. Witten, Phys. Today 49, 24 (1996).
[54] C. Quesne and V. M. Tkachuk, J. Phys. A: Math. Gen. 39, 10909 (2006).
[55] C. Quesne and V. M. Tkachuk, Czech. J. Phys. 56, 1269 (2006).
[56] V. M. Tkachuk, J. Phys. Stud. 11, 41 (2007).
[57] G. Betschart, E. Kant, and F. R. Klinkhamer, Nucl. Phys. B, 815, 198-214 (2009).
[58] E. Kant, F.R. Klinkhamer, M. Schreck, Phys. Lett. B, 682, 316–321 (2009).
[59] T. Prudêncio, H. Belich, Advances in High Energy Physics, v. 2017, p. 3050724 (2017).
[60] V. A. Kostelecký and M. Mewes, Phys. Rev. Lett. 87, 251304 (2001).

[61] V. A. Kostelecky and Matthew Mewes, Phys. Rev. D 66, 056005 (2002).

[62] S. K. Moayedi, M. R. Setare, B. Khosropour, Advances in High Energy Physics Volume 2013, 657870 (2013),