Non-existence of \(n\)-dimensional \(T\)-embedded discs in \(\mathbb{R}^{2n}\)

Gordana Stojanovic and Serge Tabachnikov *

*Department of Mathematics, Brown University
Providence, RI 02912, USA

and

Department of Mathematics, Penn State University
University Park, PA 16802, USA

A number of recent papers concerned various non-degeneracy conditions on embedding and immersions of smooth manifolds in affine and projective spaces defined in terms of mutual positions of the tangent spaces at distinct points, see [1, 2, 3, 4, 6, 7, 9]. Following Ghomi [1], a \(C^1\)-embedded manifold \(M^n \subset \mathbb{R}^N\) is called \(T\)-embedded if the tangent spaces to \(M\) at distinct points do not intersect. For example, the cubic curve \((x, x^2, x^3)\) is a \(T\)-embedding of \(\mathbb{R}\) to \(\mathbb{R}^3\), and the direct product of such curves gives a \(T\)-embedding of \(\mathbb{R}^n\) to \(\mathbb{R}^{3n}\).

A \(T\)-embedding \(M^n \to \mathbb{R}^N\) induces a topological embedding of the tangent bundle \(TM \to \mathbb{R}^N\), hence \(N \geq 2n\). One of the results in [1] is that no closed manifold \(M^n\) admits \(T\)-embeddings to \(\mathbb{R}^{2n}\). In this note we strengthen this result as follows.

**Theorem 1** There exist no \(C^2\)-smooth \(T\)-embedded discs \(D^n\) in \(\mathbb{R}^{2n}\).

**Proof.** Arguing by contradiction, assume that such a disc \(D^n\) exists. Choose the tangent space at the origin and its orthogonal complement as coordinate \(n\)-dimensional spaces. Making \(D\) smaller, if necessary, assume that the disc is the graph of a (germ of a) \(C^2\) smooth map \(f : \mathbb{R}^n \to \mathbb{R}^n\). Let \(U \subset \mathbb{R}^n\) be the domain of \(f\).

*Partially supported by NSF*
Let \( z = (u, f(u)) \in D \) where \( u \in U \). The tangent space \( T_zD \) is given by a linear equation \( y = A(u)x - b(u) \) where \( A(u) \) is an \( n \times n \) matrix and \( b(u) \) is a vector in \( \mathbb{R}^n \), both depending on \( u \). In terms of \( f \), they have the following expressions. Let \( f_1, ..., f_n \) be the components of \( f \).

**Lemma 1.1** One has:

\[
A_{ij} = \frac{\partial f_i}{\partial u_j}, \quad b_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial u_k} u_k - f_i.
\]

**Proof.** The first statement is obvious, and the second follows from the fact that the space \( y = A(u)x - b(u) \) passes through the point \( z = (u, f(u)) \). \( \square \)

One has the next characterization of \( t \)-discs.

**Lemma 1.2** For all \( u \neq v \in U \), the vector \( b(u) - b(v) \) does not belong to \( \text{Im}(A(u) - A(v)) \).

**Proof.** The spaces \( y = A(u)x - b(u) \) and \( y = A(v)x - b(v) \) intersect if and only if \( b(u) - b(v) \in \text{Im}(A(u) - A(v)) \). \( \square \)

**Lemma 1.3** If \( u \neq v \) then \( b(u) \neq b(v) \) and \( A(u) - A(v) \) is degenerate.

**Proof.** The first claim follows from the fact that zero vector lies in any subspace, contradicting Lemma 1.2. If \( A(u) - A(v) \) is nondegenerate then it is surjective, again contradicting Lemma 1.2. \( \square \)

Now we compute the Jacobian of the map \( b : U \to \mathbb{R}^n \). Denote by \( E \) the Euler vector field in \( \mathbb{R}^n \):

\[
E = \sum_{k=1}^{n} u_k \frac{\partial}{\partial u_k}.
\]

**Lemma 1.4** One has:

\[
\frac{\partial b_i}{\partial u_j} = \sum_{k} \frac{\partial^2 f_i}{\partial u_j \partial u_k} u_k = E(A_{ij}).
\]

**Proof.** This follows from Lemma 1. \( \square \)
Lemma 1.5 For all $u \in U$, the Jacobian $Jb$ of the map $b$ is degenerate.

Proof. Lemma 1.4 implies that
\[
Jb = \lim_{\varepsilon \to 0} \frac{A(u + \varepsilon u) - A(u)}{\varepsilon}.
\]
By Lemma 1.3 with $v = u + \varepsilon u$, the numerator is a degenerate matrix for all $\varepsilon$, and so is its quotient by $\varepsilon$. Thus $Jb$ is a limit of degenerate matrices. Since determinant is a continuous function, the limit also has zero determinant and therefore is degenerate. $\square$

Finally, we arrive at a contradiction. By Lemma 1.3, the map $b$ is one-to-one, and by the invariance of domain theorem, its image has positive measure. By Lemma 1.5, every value of $b$ is singular, and by Sard’s Lemma its image has zero measure. This completes the proof of Theorem 1. $\square$

According to Lemma 1.3, the $n$-parameter family of $n \times n$ matrices $A(u), u \in D^n$ enjoys the property that $A(u) - A(v)$ is degenerate for all $u \neq v$. If $n = 2$, such families can be explicitly described. Assume that not all matrices $A(u)$ are zero.

Theorem 2 The family $A(u)$ consists either of the matrices with a fixed 1-dimensional image or with a fixed 1-dimensional kernel.

Proof. Let $M_2$ be the space of linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. One has a non-degenerate quadratic form in $M_2$ given by the determinant of a matrix; this form has signature $(2, 2)$. Consider the respective dot product.

Let $V \subset M_2$ be the linear span of the family $A(u)$.

Lemma 2.1 The subspace $V$ is isotropic.

Proof. It suffices to prove that $A(u) \cdot A(v) = 0$ for all $u, v$. If $u = v$, this means precisely that $A(u)$ is degenerate. For $u \neq v$, the matrix $A(u) - A(v)$ is degenerate, hence $(A(u) - A(v)) \cdot (A(u) - A(v)) = 0$. Using bilinearity of the dot product, it follows that $A(u) \cdot A(v) = 0$. $\square$

Since the dot product is non-degenerate, an isotropic subspace is at most 2-dimensional.
Lemma 2.2 A 2-dimensional isotropic subspace in $M_2$ consists either of the matrices with a fixed 1-dimensional image or with a fixed 1-dimensional kernel.

Proof. Let $A \in V$ be a non-zero matrix. Choose a basis in the target space $\mathbb{R}^2$ in such a way that $\text{Im} \ A$ is orthogonal to the column vector $(0, 1)$. Then

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with $a^2 + b^2 \neq 0$. Let $B \in V$ be another matrix, not proportional to $A$. Then $A \cdot B = 0$, and hence

$$B = \begin{pmatrix} c & d \\ at & bt \end{pmatrix}$$

for some real $c, d, t$. If $t = 0$ then $(c, d)$ is not proportional to $(a, b)$, and the space $V$ consists of matrices with zero second row. This is the first case of the lemma: the matrices have a fixed image spanned by the column vector $(1, 0)$.

Otherwise, $t \neq 0$. Since $B$ is degenerate, one has: $(c, d) = s(a, b)$ for some real $s$. Then

$$\frac{B - sA}{t} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix},$$

and the space $V$ consists of matrices with a fixed kernel spanned by the column vector $(-b, a)$. $\square$

Lemma 2.2 obviously implies Theorem 2. $\square$

For $n = 2$, Theorem 2 implies the claim of Theorem 1. Indeed, assume that the Jacobi matrix $Jf$ has a fixed 1-dimensional kernel, say, spanned by vector $\xi$. Then the map $f$ has zero directional derivative along $\xi$, and the tangent planes to the graph of $f$ are the same along this direction. Hence this graph is not $T$-embedded. Likewise, if $Jf$ has a fixed 1-dimensional image then the transpose matrix has a fixed kernel, say, $\eta$. This implies that the function $f(u) \cdot \eta$ has zero differential, and hence the image of $f$ is 1-dimensional. It follows that the graph of $f$ belongs to a 3-dimensional space and therefore is not $T$-embedded.

Let us conclude with two examples motivated by the following erroneous attempt to prove Theorem 1: if there exists a $T$-embedded disc $D^n \subset \mathbb{R}^{2n}$...
then its tangent spaces provide a foliation \( \mathcal{F} \) of a domain in \( \mathbb{R}^{2n} \) by \( n \)-dimensional affine subspaces. Then \( D^n \) is everywhere tangent to the leaves of this \( n \)-dimensional foliation and therefore must lie within a leaf. The mistake in this argument is that, no matter how smooth the embedding is, the foliation \( \mathcal{F} \) is not differentiable. This phenomenon is illustrated in the following example.

**Example 1** Let \( \gamma \) be a smooth plane curve with positive curvature and free from vertices (extrema of curvature). Then, by the classical Kneser theorem (1912), the osculating circles to \( \gamma \) are pairwise disjoint and nested as illustrated in figure 1; see, e.g., [5, 8]. These osculating circles foliate the annulus \( A \) between the largest and smallest of them. Denote this foliation by \( \mathcal{F} \). Then \( \mathcal{F} \) is not \( C^1 \), namely, one has the following result.

![Figure 1: Osculating circles of a spiral](image)

**Proposition 1.1** Let \( f : A \rightarrow \mathbb{R} \) be a differentiable function, constant on the leaves of \( \mathcal{F} \). Then \( f \) is constant in \( A \).

**Proof.** Since \( f \) is constant on the leaves of \( \mathcal{F} \), the differential \( df \) vanishes on any vector tangent to any leaf. Since \( \gamma \) is everywhere tangent to the leaves, \( df \) is zero on the tangent vectors to \( \gamma \). Hence \( f \) is constant on \( \gamma \). But \( A \) is the union of the leaves of \( \mathcal{F} \) through the points of \( \gamma \), hence \( f \) is constant in \( A \). \( \square \)

One also wonders whether \( \mathbb{R}^{2n} \) can be foliated by non-parallel affine \( n \)-dimensional subspaces (clearly impossible for \( n = 1 \).
Example 2 The following construction gives a foliation of $\mathbb{R}^4$ by pairwise non-parallel 2-dimensional affine subspaces. Start with partitioning 3-dimensional space into the vertical $z$-axis and the hyperboloids of 1 sheet

$$x^2 + y^2 = t(z^2 + 1), \quad t > 0$$

(when $t = 0$, one has the $z$-axis). Each hyperboloid is foliated by lines, and thus $\mathbb{R}^3$ gets foliated by lines; these lines are pairwise skew. Multiply this foliation by $\mathbb{R}^1$ to obtain the desired example.

References

[1] M. Ghomi. Tangent bundle embeddings of manifolds in Euclidean space. Preprint.

[2] M. Ghomi. Nonexistence of skew loops on ellipsoids. Preprint.

[3] M. Ghomi, B. Solomon. Skew loops and quadratic surfaces. Comment. Math. Helv., 77 (2002) 767–782.

[4] M. Ghomi, S. Tabachnikov. Totally skew embeddings of manifolds. Preprint.

[5] H. Guggenheimer. Differential geometry. McGraw-Hill, 1963.

[6] J.-P. Sha, B. Solomon. No skew branes on non-degenerate hyperquadrics. Preprint.

[7] S. Tabachnikov. On skew loops, skew branes and quadratic hypersurfaces. Moscow Math. J. 3 (2003), 681–690.

[8] S. Tabachnikov. Parameterized curves, Minkowski caustics, Minkowski vertices and conservative line fields. L’Enseign. Math. 43 (1997), 3–26

[9] Y.-Q. Wu. Knots and links without parallel tangents. Bull. London Math. Soc. 34 (2002), 681–690.