Conformal Scalar Curvature Equation on $S^n$:
Functions With Two Close Critical Points
(Twin Pseudo-Peaks)

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Abstract

By using the Lyapunov-Schmidt reduction method without perturbation, we consider existence results for the conformal scalar curvature on $S^n$ ($n \geq 3$) when the prescribed function (after being projected to $\mathbb{R}^n$) has two close critical points, which have the same value (positive), equal “flatness” (‘twin’; flatness $< n - 2$), and exhibit maximal behavior in certain directions (‘pseudo-peaks’). The proof relies on a balance between the two main contributions to the reduced functional - one from the critical points and the other from the interaction of the two bubbles.

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1. Introduction.

As a counterpart of the Yamabe problem \([7] [8] [19] [33] [37] [39]\) (cf. also \([2]\)) , the prescribed scalar curvature problem in \(S^n (n \geq 3)\) asks for a positive solution \(U\) to the nonlinear partial differential equation

\[
\Delta U - \tilde{c}_n n (n-1) U + (\tilde{c}_n K)^{\frac{n+2}{n}} = 0 \quad \text{in} \quad S^n \quad (U > 0),
\]

where \(K\) is a prescribed function on \(S^n\). Here \(\tilde{c}_n = (n-2)/[4(n-1)]\). See \(\S 1\) d for the rather standard notations we use. Also known as the Nirenberg/ Kazdan - Warner problem \([36]\) , it can be compared to the classical Minkowski problem on prescribing Gaussian curvature for convex compact surfaces in \(\mathbb{R}^3\). The hallmark of equation (1.1) is the critical Sobolev exponent: the injection \(H^{1,2}(S^n) \hookrightarrow L^{\frac{2n}{n-2}}(S^n)\) is not compact, typified by blow - up gathering at critical point(s) of \(K\). Close to half a century (cf. an early work in 1972 by Dimitri Koutroufiotis \([20]\), whose thesis adviser is Louis Nirenberg) , equation (1.1) serves as a vehicle for sophisticated techniques in nonlinear partial differential equations to be deployed and developed. It can also be branched out to complete manifolds, CR manifolds, \(Q\) - curvature, as well as related to mean field equations. See some recent works \([1] [14] [12] [13] [15] [21] [30] [31] [32] [27] [34] [35] [40]\) on the topic, and the references therein. In general, existence results involve symmetry on \(K\), or local conditions on the critical points of \(K\) together with index inequality(ies). The following result provides a good picture \{see \([1] [11]\), and in particular \([14]\) regarding \((iv)\) below\}. Assume the following \((i)-(iv)\).

\begin{enumerate}[(i)]
  \item \(K\) is a smooth Morse function \[namely, all its critical points (collected in the set denoted by \(\text{Crt}\)) are non-degenerate].
  \item \(\Delta K(x_c) \neq 0\) for all \(x_c \in \text{Crt}\).
  \item \[\sum_{x_c \in \text{Crt} <} (-1)^{\text{Ind}(K, x_c)} \neq (-1)^n\], here \(\text{Crt} < = \{ x_c \in \text{Crt} | \Delta K(x_c) < 0 \}\).
  \item \(K\) is “sufficiently” close to a positive constant.
\end{enumerate}

Then equation (1.1) has a positive solution (precisely, see Theorem 7.1 in \([1]\), pp. 103). Recall that the index of a non - degenerate critical point is the number of negative eigenvalues of the Hessian matrix at that point.

We observe that in case \(\text{Crt} <\) contains only one point, say at the north pole \(N\), then it must be the peak (maximal point), and hence (together with the non - degenerate condition)

\[\text{Ind}(K, N) = n \implies \sum_{x_c \in \text{Crt} <} (-1)^{\text{Ind}(K, x_c)} = (-1)^{\text{Ind}(K, N)} = (-1)^n.\]

Moreover, via the Kazdan - Warner (Pohozaev) identity, if \(K\) is strictly decreasing from \(N\) to \(S\), measured via the geodesics, then equation (1.1) does not have any positive solution at all (cf. also \([18]\)).
Motivated by this, attention is given to the situation where \( \text{Crt}_c \) contains at least two points. Cf. \([6\) \] \([4\) \] \([10\) \] \([41\) \] (a discussion on the existence and non-existence results can be found in §1c). Thus in this article we consider juxtaposed ‘twin’ pseudo-peaks (described in §1a). We note that “side-by-side” is a kind of symmetry condition. To state the local conditions on the Taylor expansions at the two pseudo-peaks, we introduce the stereographic projection \( \hat{\mathcal{P}} \) [see (4.15)] which sends the north pole to infinity. Equation (1.1) is transformed to

\[
\Delta v + (\hat{c}_n K) v^{\frac{n+2}{2}} = 0 \quad \text{in} \quad \mathbb{R}^n \quad (v > 0).
\]

(1.2)

See, for examples, \([24\) \] \([38\) \]. Here

\[
v(y) := U(\hat{\mathcal{P}}^{-1}(y)) \cdot \left(\frac{2}{1 + |y|^2}\right)^{\frac{n-2}{2}} \quad \text{and} \quad K(y) = K(x) \quad \text{for} \quad y = \hat{\mathcal{P}}(x) \in \mathbb{R}^n.
\]

(1.3)

(Note that \(\max K\) is not affected by whichever point we choose as the north pole.)

§1a. Close ‘twin’ pseudo-peaks and their key parameters. Consider two critical points \(q_1\) and \(q_2\) of \(K\). Via a translation, one may assume without loss of generality that

\[
q_1 = 0.
\]

(1.4)

Let \(\gamma\) denote the distance (or gap) between the two critical points. Moreover, in this article, we always assume that \(q_1\) and \(q_2\) are close, namely,

\[
\gamma := |q_2| = O(1).
\]

(1.5)

The two critical points are symmetric (or ‘twin’) in the following sense [(1.6) & (1.8)]:

\[
K(0) = K(q_2) > 0
\]

(1.6)

[after a rescaling, we may accept without loss of generality that

\[
(\hat{c}_n \cdot K)(0) = (\hat{c}_n \cdot K)(q_2) = n(n - 2).
\]

(1.7)

in conjunction with (similarity on the Taylor expansions)

\[
(\hat{c}_n \cdot K)(y) = n(n - 2) + P_j^\ell (y - q_j) + R_j^{\ell + 1} (y - q_j) \quad \text{for} \quad y \in B_{q_j}(\rho)
\]

(1.8)

and \(j = 1, 2\). Here

\[
\rho = \bar{h} \cdot \gamma \quad (\bar{h} \text{ is a fixed number less than half}),
\]

(1.9)(i)

\(P_j^\ell\) is a homogeneous polynomial of degree \(\ell \geq 2\) (\(\ell\) - the flatness, is the same for \(j = 1, 2\), and

\[
(1.9)_{(ii)}
\]
(1.9) The remainder in the Taylor expansion, satisfying
\[ |R_{j}^{\ell+1}(y)| \leq C_{R_1} \cdot |y|^{\ell+1} \quad \text{for } y \in B_o(\rho), \]
\[ |R_{j}^{\ell+1}(y)| \leq C_{R_2} \cdot |y - q_j|^{\ell+1} \quad \text{for } y \in B_{q_j}(\rho). \]
Here \( C_{R_1} \) and \( C_{R_2} \) are positive constants. (1.8) implies that
\[ |(\tilde{c}_n \cdot K)(y) - n(n-2)| \leq C_{P_1} \cdot |y|^{\ell} \quad \text{for } y \in B_o(\rho), \]
\[ |(\tilde{c}_n \cdot K)(y) - n(n-2)| \leq C_{P_2} \cdot |y - q_j|^{\ell} \quad \text{for } y \in B_{q_j}(\rho). \]
Here the positive constant \( C_{P_j} \) (\( j = 1, 2 \)) is linked to the sum of the absolute values of the coefficients of \( P_j \). Assume that
\[ \ell \text{ is even, and let } h_{\ell} = \frac{1}{2} \cdot \ell. \]
Hence
\[ \Delta^{(h_{\ell})} P_j^{\ell}(\tilde{y}) = \Delta \left( \cdots \Delta \left[ \Delta \left[ \Delta \left[ \Delta \left[ \Delta \left[ P_j^{\ell}(\tilde{y}) \right] \right] \right] \right] \right) = \varpi_j \quad (j = 1, 2) \]
is a number. Here \( \tilde{y} = y - q_j \). The key condition for the critical points \( q_1 \) and \( q_2 \) to be called pseudo-peaks is the following:
\[ \varpi_j < 0 \quad \text{for } j = 1, 2. \]
We add the following “symmetry” condition as well:
\[ \frac{1}{C_p} \cdot |\varpi_1| \leq |\varpi_2| \leq C_p \cdot |\varpi_1|, \]
and
\[ \varpi_1 \geq -C_\omega. \]
In (1.14) and (1.15), \( C_p \) and \( C_\omega \) are positive constants.

**Main Theorem 1.16.** For \( 6 \leq n < 10 \), let
\[ \ell \in [2, n-2) \]
be an even integer \( K \in C^{\ell+1}(S^n) \), and \( K \) the projection of \( K \) to \( \mathbb{R}^n \) via (1.3). Assume that
\[ |K| \leq \bar{C}_b \quad \text{in } \mathbb{R}^n, \]
and \( K \) has twin pseudo-peaks in the sense of (1.7), (1.8) and (1.13), located at \( q_1 = 0 \) and \( q_2 \in \mathbb{R}^n \). Under the conditions in (1.10), (1.11), (1.14) and (1.15), there is a positive constant \( \gamma_o \) so that if
\[ |q_2| \leq \gamma_o, \]
then equation (1.1) has a positive \( C^2 \)-solution. Moreover, \( \gamma_o \) depends only on \( n, \ell, \bar{C}_b, \) and the parameters of the twin pseudo-peaks (namely, \( h, C_{R_1}, C_{R_2}, C_{P_1}, C_{P_2}, C_p \) and \( C_\omega \)).
Remarks.

(1) To gain an idea on the dependence of $\gamma_0$ on $C_\omega$ [appeared in (1.14)], we have

$$\gamma_0 \approx \frac{c_\mu}{C_\omega^\frac{1}{2}}.$$  

Here the small positive number $c_\mu$ depends on the other parameters in Theorem 1.16. See §5c.

(2) With the help of Theorem 1.16, one can consider multiple solutions for well-separated multiple twin pseudo-peaks.

(3) There is no condition on other critical points.

(4) Dimension restriction ($n = 6, 7, 8 & 9$) mainly due to the process when key information are extracted out of the reduced functional (refer to Proposition 4.1).

§1b. Lyapunov-Schmidt reduction method without perturbation. Organization. As described in [1], the elegant Lyapunov-Schmidt reduction method is considered on those $K$ which is a perturbation of a positive constant, that is (after a rescaling),

$$\left(\tilde{c}_n K\right) = n(n - 2) + \varepsilon \cdot (\tilde{c}_n H).$$

Here $\varepsilon$ is “small enough.” A new insight is introduced in [40], where Wei and Yan bring home to the point that when a large number of standard bubbles are arranged near the critical points of $K$, one can still apply the Lyapunov-Schmidt reduction method, this time without the requirement on $\varepsilon$ being close to zero (see also an earlier work of Yan [42]). Thus the number of bubbles replaces the parameter $\varepsilon$.

In this article, we show that by “planting” one bubble each near one of the twin pseudo-peaks, the Lyapunov-Schmidt reduction method is also applicable without the need for $K$ being close to a constant (§2 & §3). In this case the “gap” $\gamma$ take the place of the parameter $\varepsilon$. Moreover, we show that the reduced functional has two main contributions (Proposition 4.1; cf. also [10]), one from the critical point (§4), and the other from the interaction with the other bubble (§2b). By properly balancing these two effects, we show that equation (1.2) has a solution if the peaks are close enough (§5). This solution can be transferred back to $S^n$ via (1.3) as a solution of (1.1). Moreover, as the two bubbles are highly concentrated near the twin pseudo-peaks, other critical points (if any) do not contribute to the consideration. This is in harmony with a theme in [23] (cf. also [22]) that concentration can be put to good use to find solutions of equation (1.1).
§ 1 c. Comparison with some related existence and non-existence results. Our result should be compared to [10], in which the authors use a version of Lyapunov-Schmidt reduction method for $\varepsilon$ small enough, when $H$ in (1.19) has two critical points [among other possible critical point(s)], say at $q'_1 = 0$ and $q'_2 = (q'_{2|1}, \ldots, q'_{2|n}) \in \mathbb{R}^n$ (not necessarily close), which satisfy

\begin{equation}
H(y) = H(0) + \left( a_1 |y_1|^{\beta_1} + \cdots + a_n |y_n|^{\beta_1} \right) + O\left(|y_i|^{\beta_1+\sigma_1}\right) \quad \text{for} \quad y \in B_o(\rho),
\end{equation}

\begin{equation}
H(y) = H(q'_2) + \left( b_1 |y_1 - q'_{2|1}|^{\beta_2} + \cdots + b_n |y_n - q'_{2|n}|^{\beta_2} \right)
+ O\left(|y_i - q'_{2|l}|^{\beta_2+\sigma_2}\right) \quad \text{for} \quad y \in B_{q'_2}(\rho) \quad \left[ \rho < \frac{|q'_2|}{2} \right],
\end{equation}

where $\beta_1, \beta_2 \in (0, n-2)$,

\[ a_i \neq 0, \; b_i \neq 0, \; \sum_{i=1}^{n} a_i < 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i < 0 \quad (i = 1, 2, \ldots, n), \]

then for $\varepsilon$ in (1.19) small enough, equation (1.2) has a (two peaks) solution (see Theorem 1.1 in [10] for the precise description). In the above, $\sigma_1, \sigma_2 \in (0, 1)$ are fixed numbers. Besides the requirement on $\varepsilon$ being small enough, we note that in (1.20), there is no cross over terms like $y_1 \times y_2 \cdots$, which is allowed in our Main Theorem 1.16.

In [41], a counterpart to the situation above is considered. There Yan studies the case when $K$ has a pair of strictly local maximum points at $m_1$ and $m_2$, whose distance $|m_1 - m_2|$ is very large [flatness of these two local maxima is in the range $(n-2, n)$]. See Theorem 1.1 in [41] for the complete statements.

On the other hand, a non-existence result obtained by Bianchi in [5] suggests that for certain "very sharp" twin peaks with flatness lesser than or equal to $n-2$, equation (1.1) has no positive solution. For details, see [4] [5]. Cf. also [27]. Thus the smallness of $\gamma$ in the Main Theorem cannot be totally removed.

§ 1 d. General conditions, assumptions and conventions. Throughout this work,

\begin{equation}
S^n = \{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_{n+1}^2 = 1 \} \quad (n \geq 3),
\end{equation}

with the induced metric $g_1$. $\Delta_1$ is the Laplace-Beltrami operator associated with $g_1$ on $S^n$. Likewise, $\Delta$ is the Laplace-Beltrami operator associated with Euclidean metric $g_o$ on $\mathbb{R}^n$, with coordinates $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Moreover, the norm $\| \|$ and the inner product $\langle , \rangle$ are defined via Euclidean metric $g_o$ on $\mathbb{R}^n$. 

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•1 As mentioned earlier, $\tilde{c}_n = \frac{n-2}{4(n-1)}$. We observe the practice on using ‘$C$’, possibly with sub-indices, to denote various positive constants, which may be rendered differently from line to line according to contents. Whilst we use ‘$\tilde{c}$’ or ‘$\tilde{C}$’, possibly with sub-indices, to denote a fixed positive constant which always keeps the same value as it is first defined.

•2 Denote by $B_y(r)$ the open ball in $(\mathbb{R}^n, g_o)$ with center at $y$ and radius $r > 0$, and $\partial B_y(r)$ its boundary. Whenever there is no risk of misunderstanding, we suppress $dy$ from the integral expressions on domains in $\mathbb{R}^n$.

§ 1 e. e-Appendix. Some of the preparatory estimates are situational modifications of well-established arguments. We gather those details in the e-Appendix, which is presented from pp. 36 onward.

§ 2. The Lyapunov-Schmidt reduction scheme sans perturbation: the case of two bubbles.

Equation (1.2) is naturally associated with the Hilbert space

\[ (2.1) \quad D^{1,2} = D^{1,2}(\mathbb{R}^n) := \left\{ f \in L^{\frac{2n}{n-2}}(\mathbb{R}^n) \cap W^{1,2}_{loc}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \langle \nabla f, \nabla f \rangle < \infty \right\}. \]

The inner product is defined by

\[ (2.2) \quad \langle f, \psi \rangle_{\nabla} := \int_{\mathbb{R}^n} \langle \nabla f, \nabla \psi \rangle \quad \text{for} \quad f, \psi \in D^{1,2}, \quad \text{and} \quad ||f||_{\nabla}^2 := \langle f, f \rangle_{\nabla}. \]

The functional corresponding to (1.2) is given by

\[ (2.3) \quad I(f) = \frac{1}{2} \int_{\mathbb{R}^n} \langle \nabla f, \nabla f \rangle - \left( \frac{n-2}{2n} \right) \cdot \int_{\mathbb{R}^n} \langle \tilde{c}_n \cdot K \rangle f_{+}^{\frac{2n}{n-2}} \quad \text{for} \quad f \in D^{1,2}. \]

Here $f_+$ denotes the positive part of $f$. See Part I [24] on the regularity of the critical points of (2.3). Cf. also [9] in relation to equation (1.1). Let

\[ (2.4) \quad (\tilde{c}_n \cdot K) = n(n-2) + (\tilde{c}_n \cdot H) \iff (\tilde{c}_n \cdot H) = (\tilde{c}_n \cdot K) - n(n-2). \]

Accordingly, I can be split into two parts

\[ (2.5) \quad I(f) = I_o(f) + G(f), \]

where

\[ (2.6) \quad I_o(f) = \frac{1}{2} \int_{\mathbb{R}^n} \langle \nabla f, \nabla f \rangle - n(n-2) \cdot \left( \frac{n-2}{2n} \right) \int_{\mathbb{R}^n} f_{+}^{\frac{2n}{n-2}}, \]

and

\[ (2.7) \quad G(f) = -\frac{n-2}{2n} \cdot \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) f_{+}^{\frac{2n}{n-2}} \quad \text{for} \quad f \in D^{1,2}. \]
Here we pay special attention on the negative sign in $G(f)$. One of the key themes in this article is to expound the interaction between $I_0'$ and $G'$.

Let us present the following flow chart to guide our discussion.

$$I(f) = I_0(f) + G(f) \quad \text{for} \quad f \in D^{1,2}.$$  

"Pseudo" Kernel of $I_0'$:  
$$Z_{\sigma} = \{ z_{\sigma} = V_{\lambda_1}, \xi_1 + V_{\lambda_2}, \xi_2 \} \quad \text{[Refer to (2.11)].}$$  

Tangent space, cf. (3.2).  

Write $D^{1,2} = T_{z_{\sigma}} Z \oplus \perp_{z_{\sigma}}$.  

Projection unto the "normal".  

The auxiliary equation. "Small" solution: $w_{z_{\sigma}} \in \perp_{\sigma}$.  

(Cancellation along the normal directions.)

Finite dimension functional: $(\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)$.  

(Critical point $\tilde{z}_{\sigma}$ of the reduced functional.)

(Full functional.)

$(\tilde{z}_{\sigma} + w_{\tilde{z}_{\sigma}}) = V_{\tilde{\lambda}_1}, \tilde{\xi}_1 + V_{\tilde{\lambda}_2}, \tilde{\xi}_2 + w_{\tilde{z}_{\sigma}}$ is a solution of equation (1.2).  

(Refer to Lemma 3.44.)

Flow Chart of the Lyapunov-Schmidt reduction scheme without perturbation.

§ 2a. First order property - interaction between two 'well-separated' bubbles. For $f \in D^{1,2}$, a calculation using (2.6) shows that the Fréchet derivative of $I_0$ at $f$ is given by

$$I_0'(f)[h] = \int_{\mathbb{R}^n} \left[ \langle \nabla f, \nabla h \rangle - n(n - 2) f_{+}^{n-2} \cdot h \right] \quad \text{for} \quad h \in D^{1,2}.$$
The kernel of $I'_o$ consists of functions of the type (see [9])

$$V_{\lambda, \xi}(y) = \left(\frac{\lambda}{\lambda^2 + |y - \xi|^2}\right)^{\frac{n-2}{2}}$$

for $(\lambda, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n$.

which satisfies the equation

$$\Delta V_{\lambda, \xi}(y) + n(n-2)[V_{\lambda, \xi}(y)]^{\frac{n-2}{n}} = 0 \quad \text{in} \quad \mathbb{R}^n.$$

We consider juxtaposition of two bubbles

$$z_\sigma = V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2} \quad \text{for} \quad (\lambda_1, \lambda_2; \xi_1, \xi_2) \in (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n).$$

§ 2 b. Unit and restrictions. In the following we assume that

$$\bar{C}^{-1} \cdot \lambda_2 < \lambda_1 < \bar{C} \cdot \lambda_2, \quad |\xi_1| < \bar{c} \cdot \lambda \quad \text{and} \quad |\xi_2 - q_2| < \bar{c} \cdot \lambda.$$

Here $\bar{C} (> 1)$ and $\bar{c} (\approx 0^+)$ are positive constants (to be more precisely described in § 5).

With (2.11), we define

$$\lambda = \sqrt{\lambda_1 \cdot \lambda_2} \quad \text{and} \quad D = \frac{\gamma}{\lambda} \left(= \frac{|q_2|}{\sqrt{\lambda_1 \cdot \lambda_2}}\right).$$

These imply

$$\frac{1}{\sqrt{C}} \cdot \lambda_j \leq \lambda \leq \sqrt{C} \cdot \lambda_j \quad \text{for} \quad j = 1, 2,$$

$$[D - 2 \bar{c}] \leq d := \frac{|\xi_1 - \xi_2|}{\sqrt{\lambda_1 \cdot \lambda_2}} \leq [D + 2 \bar{c}] \quad \text{and} \quad \frac{1}{d} = \frac{1}{D} \cdot \left[1 + O\left(\frac{\bar{c}}{D}\right)\right].$$

§ 2 c. Weak interaction. We know that

$$I'_o(V_{\lambda, \xi}) \equiv 0, \quad \text{but} \quad I'_o(V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2}) \neq 0.$$

In this section we investigate the “interaction” in more detail. From (2.8) and (2.11) we have

$$I'_o(z_\sigma)[h] = n(n-2) \int_{\mathbb{R}^n} \left\{ \left|[V_{\lambda_1, \xi_1}]^{\frac{n+2}{n-2}} + [V_{\lambda_2, \xi_2}]^{\frac{n+2}{n-2}} \right| - [V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2}]^{\frac{n+2}{n-2}} \right\} \cdot h.$$

for $h \in \mathcal{D}^{1, 2}$. 

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Lemma 2.16 (Weak Interaction Lemma). Assume that $n \geq 6$, with the notations and conditions in (2.12) and (2.13), there exists a positive constant $\bar{D}_1 > 1$ such that

\begin{equation}
\text{(2.17)} \quad \text{if} \quad D = \frac{\gamma}{\bar{A}} \geq \bar{D}_1,
\end{equation}

\begin{equation}
\text{(2.18)} \quad \text{then} \quad \| I'_{\sigma}(z_\sigma) \| \leq \bar{C}_1 \cdot \frac{\ln D}{D^{n/2}}.
\end{equation}

In (2.17) and (2.18), the positive constants $\bar{D}_1$ and $\bar{C}_1$ can be precisely determined by $\bar{C}$, $\bar{c}$ [appeared in (2.12)] and $n$, and they are independent on $(\lambda_1, \lambda_2; \xi_1, \xi_2)$ as long as (2.12) is satisfied.

The proof can be seen from the proof of Lemma 2.1 in [28], together with Lemma A.5 in the Appendix.

\section{b. Interaction terms.}

In the following we describe the interaction between two bubbles via (2.15). We first observe that in a small neighborhood of $\xi_1$, $V_{\lambda_2, \xi_2}$ is small when compared to $V_{\lambda_1, \xi_1}$. Precisely, we let

\begin{equation}
\text{(2.19)} \quad \rho_\mu = \mu \cdot |\xi_1 - \xi_2|.
\end{equation}

Here $\mu$ is a chosen small positive number so that

\begin{equation}
\text{(2.20)} \quad \mu \to 0^+ \text{ (slowly) and } \mu^M \cdot D \to \infty \text{ when } D \to \infty.
\end{equation}

Here $M$ is a (fixed) large integer. For most particular purpose one can take

$$\mu = \frac{1}{2} \cdot \frac{1}{D^\epsilon} \quad \text{for} \quad D \gg 1,$$

where $\epsilon < 1$ is any fixed small positive number. Under the conditions in (2.12), we have

\begin{equation}
\text{(2.21)} \quad V_{\lambda_2, \xi_2}(y) = \left(\frac{\lambda_2}{\lambda_2^2 + |y - \xi_2|^2}\right)^\frac{n-2}{2} = \left(\frac{1}{\lambda_1} + \frac{|y - \xi_2|^2}{\lambda_1 \cdot \lambda_2}\right)^\frac{n-2}{2}
\end{equation}

$$= \frac{1}{\lambda_1^{n-2}} \cdot \left(\frac{\lambda_2}{\lambda_1^{\frac{1}{2}}} + \frac{|(y - \xi_1) + (\xi_1 - \xi_2)|^2}{\lambda_1^{1/2} \cdot \lambda_2}\right)^\frac{n-2}{2}$$

$$= \frac{1}{\lambda_1^{n-2}} \cdot \left[\frac{(\xi_1 - \xi_2)^2}{\lambda_1 \cdot \lambda_2} + \frac{|y - \xi_1|^2}{\lambda_1 \cdot \lambda_2} + \frac{2(y - \xi_1) \cdot (\xi_1 - \xi_2)}{\lambda_1 \cdot \lambda_2}\right]^\frac{n-2}{2}
\{ \uparrow = d^2 [\gg 1; \text{ cf. (2.14)}], \text{ dominating term} \}$$
By using the partition $A$ and the inequality \( o \) in the above we apply (2.13), (2.14) and (2.20). Moreover, \( o(1) \to 0 \) as \( D \to \infty \). Compare with

\[
V_{\lambda_1, \xi_1} \approx \frac{1}{\lambda_1^{n/2}} \cdot \frac{1}{D^{n/2}} \cdot \frac{1}{\mu^{n/2}} \quad \text{on} \quad \partial B_{\xi_1}(\rho_\mu).
\]

Let

\[\mathcal{I} := \left( V_{\lambda_1, \xi_1}^{\frac{n-2}{2}} + V_{\lambda_2, \xi_2}^{\frac{n-2}{2}} \right) - (V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2})^{\frac{n-2}{n-2}} ( < 0 ).\]

By using the partition

\[\mathbb{R}^n = B_{\xi_1}(\rho_\mu) \cup B_{\xi_2}(\rho_\mu) \cup \left\{ \mathbb{R}^n \setminus [B_{\xi_1}(\rho_\mu) \cup B_{\xi_2}(\rho_\mu)] \right\},\]

and the inequality (\( A \) and \( B \) are positive numbers)

\[\left( A^{\frac{n-2}{2}} + B^{\frac{n-2}{2}} \right) - (A + B)^{\frac{n-2}{n-2}} = -\frac{n + 2}{n - 2} \cdot A^{\frac{n-2}{n-2}} \cdot B + O(1) \cdot B^{\frac{n-2}{n-2}} \quad \text{for} \quad \frac{B}{A} \text{ small},\]

we obtain (see § A.1 in the e-Appendix for more detail)

\[I_0'(z_o) [\partial_{\lambda_1} V_{\lambda_1, \xi_1}] = n(n - 2) \int_{\mathbb{R}^n} \mathcal{I} \cdot [\partial_{\lambda_1} V_{\lambda_1, \xi_1}]\]

\[= -\bar{C}_1^+(n) \cdot \frac{1}{\lambda_1} \cdot \frac{\lambda_1^{n/2} \cdot \lambda_2^{n/2}}{\gamma^{n/2}} \cdot [1 + o(1)].\]

Here

\[\bar{C}_1^+(n) = n(n - 2) \cdot \omega_n \cdot \frac{n - 2}{2n} .\]

Likewise, we extract the leading term in \( \xi \)-derivatives (refer to § A.1 in the e-Appendix for more of the calculations):

\[I_0'(z_o) [\partial_{\xi_1} V_{\lambda_1, \xi_1}] = n(n - 2) \int_{\mathbb{R}^n} \mathcal{I} \cdot [\partial_{\xi_1} V_{\lambda_1, \xi_1}]\]

\[= \bar{C}_2^+(n) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} \cdot (\xi_1 - \xi_2) \cdot \frac{1}{D^n} \cdot [1 + o(1)].\]

Here \( \bar{C}_2^+(n) \) is a positive constant depending on \( n \) only.
Similarly for the expressions on $\partial \lambda_2 V_{\lambda_2}, \xi_2$ [with the same constant $\bar{C}_1^+(n)$] and $\partial \xi_{21} V_{\lambda_2}, \xi_2$ [with the same constant $\bar{C}_2^+(n)$]. Here conditions (2.12) and (2.17) apply, and $o(1) \to 0$ when $D \to \infty$. We present the rather standard calculations in §A.1 in the e-Appendix, paying special attention of the sign $(+/-)$. Compare also with Lemma B.2 and Lemma B.4 in [10] and formulas 2.119 and 2.206 in [3].

§ 3. Second order property - solving the equation in the perpendicular directions.

As is often the case in mathematics, simplicity is linked with orthogonality, such as the Lagrange multiplier method. Likewise, the Lyapunov-Schmidt reduction method consists of solving the equation $(3.1)$

$$I'(u) \equiv 0 \quad \text{for an unknown } u \in D^{1,2}$$

in two steps, first in the ‘perpendicular’ direction, and then in the ‘horizontal’ direction. Here we first consider the ‘perpendicular’ direction. Given $\mathbf{z}_\sigma$ as in (2.11), let

$$(3.2) \quad \perp_\sigma = \left\{ h \in D^{1,2} \mid \left< h, \partial \lambda_j V_{\lambda_j}, \xi_j \right>_{\nabla \nabla} = \left< h, \partial \xi_{j|k} V_{\lambda_j}, \xi_j \right>_{\nabla \nabla} = 0 \right. \quad \text{for } j = 1, 2; \quad k = 1, 2, \cdots, n \},$$

and

$$(3.3) \quad P_\sigma : D^{1,2} \to \perp_\sigma$$

be the orthogonal projection. Here $\xi_j = \left( \xi_{j1}, \cdots, \xi_{jn} \right) \in \mathbb{R}^n$ for $j = 1, 2$. Fixed a $\mathbf{z}_\sigma$, to solve the auxiliary equation (‘perpendicular’ direction) is to find an unknown $w_{\mathbf{z}_\sigma} \in \perp_\sigma$ in the equation

$$(3.4) \quad P_\sigma \circ I'_\xi (\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma}) = 0 \quad (w_{\mathbf{z}_\sigma} \in \perp_\sigma \text{ is called informally “} \perp-\text{solution”}) .$$

In order to apply the implicit function theorem to solve equation (3.4), we proceed to the second Fréchet derivative of $I_\sigma$, in particular, the “diagonal element”:

$$(3.5) \quad \left( I''_\sigma (\mathbf{z}_\sigma) [f] f \right) = \int_{\mathbb{R}^n} \left[ \left< \nabla f, \nabla f \right> - n (n + 2) \left< V_{\lambda_1} + V_{\lambda_2}, f \right> \right]$$

for $f \in D^{1,2}$. For a proof of the following lemma, see [3], and Lemma 2.5 in [28].

**Lemma 3.6 (Non-degeneracy Lemma).** Assume that $n \geq 6$. Under the conditions in (2.12), there exists a positive constant $\bar{D}_2$ such that

$$(3.7) \quad \text{if } \quad D \geq \bar{D}_2 ,$$

$$(3.8) \quad \text{then } \quad \left| \left( I''_\sigma (\mathbf{z}_\sigma) [f] f \right) \right| \geq \bar{c}_\sigma^2 \| f \|_{\nabla \nabla}^2 \quad \text{for all } f \in \perp_\sigma .$$
Here the constant $\bar{c}_\sigma$ is independent on $(\lambda_1, \lambda_2; \xi_1, \xi_2)$ as long as (2.12) and the condition in (3.7) are fulfilled.

As $\perp_\sigma \subset D^{1,2}$ is itself a complete Hilbert space, we consider the restriction

\[
(3.9) \quad (I''_o(z_\sigma)[f] h) = \int_{\mathbb{R}^n} \left[ \langle \nabla f, \nabla h \rangle - n(n+2)z_\sigma^{\frac{4}{n-2}} \cdot f \cdot h \right] \quad \text{for } f, h \in \perp_\sigma.
\]

Via the Riesz Representation Theorem, we obtain a linear map

\[
(3.10) \quad f \mapsto \left\{ \frac{(I''_o(z_\sigma)[f] h)}{\| h \|_2^2} \right\} \cdot h.
\]

The following result can be seen as a direct consequence of Lemma 3.6. We refer to [1], or [28].

**Lemma 3.11.** Under the conditions in Lemma 3.6, the map given in (3.10) is an isomorphism with uniformly bounded inverse.

With the help of Lemma 3.12 below (a proof can be found in the Appendix), we now show that when the two bubbles are concentrated around the two critical points, one can solve the “perpendicular” direction, just like in the perturbation case [1].

**Lemma 3.12.** Assume that $n \geq 6$, $|H| \leq C_b$, and under the conditions in (1.8), (1.9), (1.10), (1.11) and (2.12). Given a number $m$ so that $m \cdot \ell > 2$, there is a positive number $\bar{D}_3$ such that

\[
\int_{\mathbb{R}^n} |H|^m \cdot \frac{z_\sigma^{2n}}{\lambda^2} \leq \begin{cases} 
C_4 \cdot \lambda^{m \ell} + R & \text{if } m \cdot \ell < n, \\
C_5 \cdot \lambda^{n-o(1)} + R & \text{if } m \cdot \ell = n, \\
C_6 \cdot \lambda^n + R & \text{if } m \cdot \ell > n.
\end{cases}
\]

Here

\[
R = \lambda^2 \cdot \frac{C_7}{D^{n-2}} + \frac{C_8}{D^{n-o(1)}},
\]

and $o(1) \to 0^+$ as $\lambda \to 0$. Moreover, the constants $\bar{D}_3$, $C_4$, $C_5$, $C_6$, $C_7$ and $C_8$ are independent on $(\lambda_1, \lambda_2; \xi_1, \xi_2)$ as long as (2.12) and the conditions in (1.9)\textsubscript{i}, (1.9)\textsubscript{ii}, (1.9)\textsubscript{iii}, (1.10) and (1.11) are fulfilled.
Theorem 3.13 (Existence of small $\perp$–solution.) Assume that $n \geq 6$, $\ell \geq 2$, and the conditions in Lemma 3.12. Then there exist positive numbers $\bar{D}_4$ (relatively “large”) and $\tilde{\epsilon}$ (“small”) such that for each $z_\sigma$ with

$$D \geq \bar{D}_4,$$

the auxiliary equation

$$P_\sigma \circ I'(z_\sigma + w) = 0 \quad (w \in \perp_\sigma) \quad (3.14)$$

has a unique “small” solution $w_{z_\sigma} \in \perp_\sigma$, precisely,

$$\|w_{z_\sigma}\|_\nabla \leq \tilde{\epsilon}. \quad (3.15)$$

Moreover, one can take $\tilde{\epsilon} \to 0^+$ as $D \to \infty$.

The constant $\bar{D}_4$ is independent on $(\lambda_1, \lambda_2; \xi_1, \xi_2)$ as long as (2.12) and the conditions in (1.9) i, (1.9) ii, (1.9) iii (1.10) and (1.11) are fulfilled. In addition, $w_{z_\sigma}$ depends on the parameters $(\lambda_1, \lambda_2; \xi_1, \xi_2)$ of $z_\sigma \left( = V_{\lambda_1}, \xi_1 + V_{\lambda_2}, \xi_2 \right)$ in a $C^1$ manner.

Proof. From (2.5)–(2.7), we have

$$I'(z_\sigma + w)[\bullet] = \int_{\mathbb{R}^n} \left[ \langle \nabla (z_\sigma + w), \nabla [\bullet] \rangle - n (n - 2) (z_\sigma + w)_{n-2}^{\frac{n+2}{n-2}} \cdot [\bullet] \right]$$

$$- \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) (z_\sigma + w)_{n-2}^{\frac{n+2}{n-2}} \cdot [\bullet] \quad \text{for } \bullet \in D^{1,2} \quad (w \in \perp_\sigma).$$

Write

$$P_\sigma \circ I'(z_\sigma + w) = P_\sigma \circ I'_o(z_\sigma) \cdot P_\sigma \circ I''_o(z_\sigma)[w] + P_\sigma \circ G'(z_\sigma + w) + P_\sigma \circ R_{z_\sigma}(w) \quad (3.16)$$

for $\|w\|_\nabla$ small,

where

$$I'_o(z_\sigma)[\bullet] = \int_{\mathbb{R}^n} \left[ \langle \nabla (z_\sigma), \nabla [\bullet] \rangle - n (n - 2) (z_\sigma)_{n-2}^{\frac{n+2}{n-2}} \cdot [\bullet] \right],$$

$$I''_o(z_\sigma)[w \cdot \bullet] = \int_{\mathbb{R}^n} \left[ \langle \nabla w, \nabla [\bullet] \rangle - n (n + 2) z_\sigma^{\frac{1}{n-2}} w \cdot [\bullet] \right],$$

$$G'(z_\sigma + w)[\bullet] = - \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) (z_\sigma + w)_{n-2}^{\frac{n+2}{n-2}} \cdot [\bullet],$$

$$R_{z_\sigma}(w)[\bullet] = \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) (z_\sigma + w)_{n-2}^{\frac{n+2}{n-2}} \cdot [\bullet].$$
and

\( R_{z,\sigma}(w)[\bullet] = \int_{\mathbb{R}^n} \left[ n(n - 2)z_{\sigma}^{\frac{n+2}{n-2}} \cdot [\bullet] - n(n - 2)(z_{\sigma} + w)^{\frac{n+2}{n-2}} \cdot [\bullet] + n(n + 2)z_{\sigma}^{\frac{4}{n-2}} w \cdot [\bullet] \right] \)

\[ = -n(n - 2) \int_{\mathbb{R}^n} \left[ (z_{\sigma} + w)^{\frac{n+2}{n-2}} - z_{\sigma}^{\frac{n+2}{n-2}} - \frac{n + 2}{n - 2} z_{\sigma}^{\frac{4}{n-2}} w \right] \cdot [\bullet]. \]

In order to solve equation (3.14), that is

\[ \mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma})[w] = -\mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma}) - \mathcal{P}_{\sigma} \circ \mathbf{G}'(z_{\sigma} + w) - \mathcal{P}_{\sigma} \circ R_{z,\sigma}(w), \]

"interaction term" \( \uparrow \) depending on \( w \) \( \uparrow \)

we first seek a solution \( w \) to

\[ \mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma})[w] = -\mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma}). \]

\((\star)\) The interaction term. From the Weak Interaction Lemma 2.16,

\[ \| \mathbf{I}'(z_{\sigma}) \| \leq C_1 \cdot \frac{1}{D^{n+2/2 - o(1)}}. \]

Using Lemma 3.11, we can find \( \bar{w}_1 \in \perp_{\sigma} \) such that

\[ \mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma})[\bar{w}_1] = -\mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma}), \]

and

\[ \| \bar{w}_1 \|_{\mathcal{V}} \leq C_2 \cdot \frac{1}{D^{n+2/2 - o(1)}}. \]

In (3.18) and (3.19), \( o(1) \to 0^+ \) as \( D \to \infty \).

\((\star)\) Fixed point. Next [because of the linearity of \( \mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma}) \)], we intend to find \( w_2 \in \perp_{\sigma} \) such that

\[ \mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma})[w_2] = - \left\{ \left[ \mathcal{P}_{\sigma} \circ \mathbf{G}' \right](z_{\sigma} + \bar{w}_1 + w_2) + \mathcal{P}_{\sigma} \circ R_{z,\sigma}(\bar{w}_1 + w_2) \right\}. \]

[Here \( \bar{w}_1 \) appears in (3.19).] That is, we seek a fixed point to

\[ \mathbf{T}(\bullet) := - \left( \mathcal{P}_{\sigma} \circ \mathbf{I}'(z_{\sigma}) \right)^{-1} \left\{ \left[ \mathcal{P}_{\sigma} \circ \mathbf{G}' \right](z_{\sigma} + \bar{w}_1 + \bullet) + \mathcal{P}_{\sigma} \circ R_{z,\sigma}(\bar{w}_1 + \bullet) \right\}. \]

From (3.16) we have

\[ \mathbf{G}'(z_{\sigma} + \bar{w}_1 + w_2)[h] = - \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H)(z_{\sigma} + \bar{w}_1 + w_2)^{\frac{n+2}{n-2}} \cdot h \quad \text{for} \quad h \in \mathcal{D}^{1,2}. \]
Also, Lemma 3.12 implies that

\[
\int_{\mathbb{R}^n} |H|^{\frac{2n}{n+2}} \cdot z^2 = O(\lambda^*) + \lambda^2 \cdot O\left(\frac{1}{D^{n-2}}\right) + O\left(\frac{1}{D^{n-o(1)}}\right).
\]

where

\[
* = \begin{cases} 
\frac{2n}{n+2} \cdot \ell & \text{if } \frac{2n}{n+2} \cdot \ell < n, \\
\frac{n-o(1)}{n+2} & \text{if } \frac{2n}{n+2} \cdot \ell \geq n.
\end{cases}
\]

Here \(o(1) \to 0^+\) as \(\lambda \to 0\) [condition (2.12) applies]. Using the Minkowski inequality and Sobolev inequality we obtain

\[
|G'(z_\sigma + \bar{w}_1 + w_2) [h]| \\
\leq |\tilde{c}_n| \cdot \int_{\mathbb{R}^n} |H| \cdot (z_\sigma + \bar{w}_1 + w_2)^{\frac{n+2}{n-2}} \cdot |h| \\
\leq C \cdot \left\{ \int_{\mathbb{R}^n} |H| \cdot z_\sigma^{\frac{n+2}{n-2}} \cdot |h| + \int_{\mathbb{R}^n} |H| \cdot w_1^{\frac{n+2}{n-2}} \cdot |h| + \int_{\mathbb{R}^n} |H| \cdot w_2^{\frac{n+2}{n-2}} \cdot |h| \right\}
\leq C(n) \cdot \left\{ O(\lambda^*) + \lambda^2 \cdot O\left(\frac{1}{D^{n-2}}\right) + O\left(\frac{1}{D^{n-2}}\right)^{\frac{n+2}{2n}} \right\}
\text{+} \left( |\bar{w}_1|_{\nabla}^{\frac{n+2}{n-2}} + |w_2|_{\nabla}^{\frac{n+2}{n-2}} \right) \cdot \|h\|_{\nabla}.
\]

Likewise \{ from (3.17), see also [28] \},

\[
|R_{z_\sigma} (\bar{w}_1 + w_2) (h)| \leq C(n) \cdot \left\{ \frac{1}{D^{n+o(1)}} + \left( |\bar{w}_1|_{\nabla}^{\frac{n+2}{n-2}} + |w_2|_{\nabla}^{\frac{n+2}{n-2}} \right) \right\} \cdot \|h\|_{\nabla}.
\]

Hence by choosing \(\lambda = \sqrt{\lambda_1 \cdot \lambda_2}\) and \(c\) to be small, we obtain

\[
\|w_2\|_{\nabla} \leq c \implies \|T(h)\| \leq c.
\]

Note that, via (3.19), (3.24) and (3.25), \(c \to 0^+\) as \(\lambda \to 0^+\) and \(D \to \infty\).

\((\star)\) Contraction map. Consider \([\bar{w}_1, \text{first appeared in (3.19), and } w_1 \text{ below are to be distinguished}]\)

\[
\|T(w_1) - T(w_2)\|, \quad \text{where } \|w_1\|_{\nabla} \leq c \text{ and } \|w_2\|_{\nabla} \leq c.
\]

Using the inequality ( cf. for example [11], \(n \geq 6\))

\[
\left| a^{\frac{n+2}{2n}} - b^{\frac{n+2}{2n}} \right| \leq C(n) \cdot |a - b| \cdot (a + b)^{\frac{1}{2n}} \quad \text{for } a > 0 \& b > 0,
\]

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we have

\[(3.27)\]

\[
\left| \left[ \mathbf{G}'(\mathbf{z}_\sigma + \bar{w}_1 + w_1) - \mathbf{G}'(\mathbf{z}_\sigma + \bar{w}_1 + w_2) \right]\mathbf{h} \right|
\]

\[
= \tilde{c}_n \cdot \int_{\mathbb{R}^n} |H| \left[ (\mathbf{z}_\sigma + \bar{w}_1 + w_1)^{n+2 \over n} - (\mathbf{z}_\sigma + \bar{w}_1 + w_2)^{n+2 \over n} \right] \cdot \mathbf{h}
\]

\[
\leq C \cdot \int_{\mathbb{R}^n} |H| \cdot |w_1 - w_2| \cdot (\mathbf{z}_\sigma + |\bar{w}_1| + |w_1| + |w_2|)^{n \over n-2} \cdot \mathbf{h}
\]

\[
\leq C(n) \cdot \left[ O(\lambda^{n-o(1)}) + O\left(\frac{1}{D^{n-2}}\right)\right]^2 + \|\bar{w}_1\|^{n \over n-2} + \|w_1\|^{n \over n-2} + \|w_2\|^{n \over n-2} * \|h\| \cdot \|w_1 - w_2\| \quad \text{(taking the condition } \ell \geq 2 \text{ into consideration)}.\]

Here we use Minkowski inequality and Sobolev inequality again.

Finally, for the remainder, we make use of the inequality

\[
\left| \left[ (\mathbf{z}_\sigma + \bar{w}_1 + w_1)^{n+2 \over n} - (\mathbf{z}_\sigma + \bar{w}_1 + w_2)^{n+2 \over n} \right] - \frac{n + 2}{n - 2} \cdot \mathbf{z}_\sigma^{n \over n-2} \cdot [w_1 - w_2] \right|
\]

\[
\leq C(n) \cdot \left\{ \|w_1 - w_2\| \left[ \|\bar{w}_1\|^{4 \over n-2} + \|w_1\|^{4 \over n-2} + \|w_2\|^{4 \over n-2} \right] + \|w_1 - w_2\|^{n+2 \over n-2} \right\}.
\]

This comes from inequalities of the following forms.

\[(3.28)\]

\[
1 - (1 + T)^{n+2 \over n-2} + \frac{n + 2}{n - 2} \cdot T \leq C_1 \cdot |T|^{n+2 \over n-2} \quad \text{(} n \geq 6 \text{, } |T| \text{ is small)},
\]

\[(3.29)\]

\[
\left| a^{n \over n-2} - \left(a + s\right)^{n \over n-2} \right| \leq C_2 \cdot s^{n \over n-2} \quad \text{for } a > 0, \ a + s > 0, \ (n \geq 6).
\]

Here \(C_1\) and \(C_2\) depend on \(n\) only. It follows that

\[(3.30)\]

\[
|R_{z_\sigma}(\bar{w}_1 + w_1) - R_{z_\sigma}(\bar{w}_1 + w_2)|
\]

\[
\leq C(n) \cdot \|w_1 - w_2\| \cdot \left[ \|\bar{w}_1\|^{3 \over n-2} + \|w_1\|^{3 \over n-2} + \|w_2\|^{3 \over n-2} + \|w_1 - w_2\|^{3 \over n-2} \right] \cdot \|h\|.
\]

Hence we can find a positive number \(\gamma < 1\) so that

\[
\|\mathbf{T}(w_1) - \mathbf{T}(w_2)\| \leq \gamma \cdot \|w_1 - w_2\| \quad \text{, where } \|w_1\| \leq c \text{ and } \|w_2\| \leq c.
\]
Via the contraction mapping theorem, (3.21) has a unique fixed point $\bar{w}_2$ with

$$\|\bar{w}_2\|_\nabla \leq c.$$ 

Thus we find a solution

$$w_{z\sigma} = \bar{w}_1 + \bar{w}_2$$

to auxiliary equation (3.14). Moreover,

$$\|w_{z\sigma}\|_\nabla \leq \|\bar{w}_1\|_\nabla + \|\bar{w}_2\|_\nabla \implies \|w_{z\sigma}\|_\nabla = 2 \cdot c.$$ 

Thus we can take $\tilde{\epsilon} = 2 \cdot c$ in (3.15). With this, together with (3.27) and (3.30), we can also show that the small $\perp-$ solution $w_{z\sigma}$ depends on $z_{\sigma}$ in a $C^1$ manner. Cf. [1] and the proof of Proposition 4.2 in [35]. This completes the proof of the theorem.

\[\square\]

§ 3 a. Finite dimension reduction. Let $w_{z\sigma}$ be the unique small $\perp-$ solution of the auxiliary equation (as described in Theorem 3.13). Consider the reduced functional, which depends on $(\lambda_1, \lambda_2; \xi_1, \xi_2) \in (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)$:

\[(3.31)\]

$$I_{R,H}(\lambda_1, \lambda_2; \xi_1, \xi_2) = I_{R,H}(z_{\sigma} + w_{z\sigma}) = \frac{1}{2} \int_{\mathbb{R}^n} \langle \nabla (z_{\sigma} + w_{z\sigma}), \nabla (z_{\sigma} + w_{z\sigma}) \rangle$$

$$- \frac{1}{2} \cdot (n - 2)^2 \int_{\mathbb{R}^n} (z_{\sigma} + w_{z\sigma})_+^{\frac{n}{2} - 2} + \left[ - \frac{n - 2}{2n} \right] \cdot \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H)(z_{\sigma} + w_{z\sigma})_+^{\frac{2n}{2} - 2}.$$ 

This finite dimensional reduced functional forms the main object in our study. We first show its link to the full functional (2.3).

Lemma 3.32. Under the conditions in Theorem 3.13, if $z_{\sigma} = V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2}$ is a critical point of the reduced functional in (3.31), that is

\[(3.33)\]

$$\frac{\partial I_{R}}{\partial \lambda_k} \bigg|_{(\lambda_1, \lambda_2; \xi_1, \xi_2)} = \frac{\partial I_{R}}{\partial \xi_\ell} \bigg|_{(\lambda_1, \lambda_2; \xi_1, \xi_2)} = 0$$

for $k = 1, 2$, and $\ell = 1, \cdots, n$, then $z_{\sigma} + w_{z\sigma}$ is a critical point of (the full functional) $I$, that is,

\[(3.34)\]

$$I'(z_{\sigma} + w_{z\sigma}) = 0.$$ 

Using the smallness of $\|w_{z\sigma}\|_\nabla$ provided by (3.15) when $D$ is large enough, the proof of Lemma 3.44 is similar to the proof of Theorem 2.8 in [28]. There one can also find information on the regularity of the solution $z_{\sigma} + w_{z\sigma}$ in (3.34), as well as the property that it can be transferred back to $S^n$ via (1.3).
§ 3 b. Degree and gradient. It is convenient and natural to work with the coupled quasi-hyperbolic gradient, denoted by \((\lambda \cdot \nabla)\) (introduced in \([28]\)) and defined by

\[
(\lambda \cdot \nabla) T := (\lambda_1 \cdot D_{1o} T, \lambda_1 \cdot D_{11} T, \ldots, \lambda_1 \cdot D_{1n} T; \lambda_2 \cdot D_{2o} T, \lambda_2 \cdot D_{21} T, \ldots, \lambda_2 \cdot D_{2n} T)
\]

for \(T \in C^1(\overline{\Omega})\), where \(\Omega \subset (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)\) is a bounded domain with smooth boundary \(\partial \Omega\), and \(\overline{\Omega} \subset (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)\).

In (3.35),

\[
D_{ko} = \frac{\partial}{\partial \lambda_k} \quad \text{and} \quad D_{k\ell} = \frac{\partial}{\partial \xi_{k|\ell}} \quad \text{for} \quad k = 1, 2 \quad \text{and} \quad \ell = 1, \ldots, n.
\]

As usual

\[
\| (\lambda \cdot \nabla) T \| = \sqrt{ (\lambda_1 \cdot D_{1o} T)^2 + \sum_{\ell=1}^{n} (\lambda_1 \cdot D_{1\ell} T)^2 + (\lambda_2 \cdot D_{2o} T)^2 \ + \sum_{\ell=1}^{n} (\lambda_2 \cdot D_{2\ell} T)^2 }.
\]

The following theorem can be shown by using the homotopy invariance of the degree. See \([28]\)

**Theorem 3.38.** Let \(\Omega\) be as described in the above, and \(\mathcal{F}, \mathcal{G} : (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^{2(n+1)}\) be of class \(C^0(\overline{\Omega})\), which satisfy

\[
\min_{\partial \Omega} \left\{ \| \mathcal{F} \| \right\} > 0 ,
\]

\[
\min_{\partial \Omega} \left\{ \| \mathcal{F} \| \right\} > \max_{\partial \Omega} \left\{ \| \mathcal{F} - \mathcal{G} \| \right\}.
\]

Then we have

\[
\deg \left[ \mathcal{G}, \Omega, \vec{0} \right] = \deg \left[ \mathcal{F}, \Omega, \vec{0} \right].
\]

In particular [under conditions (3.39) and (3.40)], if \(\deg \left[ \mathcal{F}, \Omega, \vec{0} \right] \neq 0\), then there is a point \(p \in \Omega\) such that

\[
\mathcal{G}(p) = \vec{0}.
\]
§ 3 b. Estimates on \( w_{z\sigma} \). In order to extract effectively from the reduced functional from the key information (Proposition 4.1), we need the following estimates, which are shown in [28].

Lemma 3.43. Under the conditions in Theorem 3.13, let \( w_{z\sigma} \) be the unique small \( \perp - \) solution of the auxiliary equation (3.14) [which satisfies (3.15)]. We have

\[
\| w_{z\sigma} \|_{\nabla} \leq \bar{C}_2 \cdot \left\{ \int_{\mathbb{R}^n} |H|^{\frac{2n}{n+2}} \cdot Z \right\}^{\frac{n+2}{2n}} + O \left( \frac{1}{D^{2-o(1)}} \right),
\]

and

\[
\| \lambda_k \cdot D_{k\ell} w_{z\sigma} \|_{\nabla} \leq \bar{C}_3 \cdot \left\{ \int_{\mathbb{R}^n} |H|^{\frac{2n}{n+2}} \cdot Z \right\}^{\frac{n+2}{2n}} + O \left( \frac{1}{D^{2-o(1)}} \right).
\]

Here \( k = 1, 2; \ell = 0, 1, \cdots, n \). In addition, \( \bar{C}_2 \) and \( \bar{C}_3 \) are positive constants independent on \( (\lambda_1, \lambda_2; \xi_1, \xi_2) \) as long as the conditions in Theorem 3.13 are satisfied. [Here \( o(1) \to 0^+ \) as \( D \to \infty \). Cf. (2.18).]

Combining with Lemma 3.12, we obtain the following result.

Lemma 3.44. Under the conditions in Theorem 3.13, assume also that

(3.45) \[ \lambda^\ell \leq \frac{C_0}{D^{n-2}} \quad \text{and} \quad 2 \leq \ell < n-2, \]

then we have

(3.46) \[ \| w_{z\sigma} \|_{\nabla} \leq \bar{C}_4 \cdot \frac{1}{D^{\frac{n+2}{n} - o(1)}} \quad \text{and} \]

(3.47) \[ \| \lambda_k \cdot D_{k\ell} w_{z\sigma} \|_{\nabla} \left( \leq \frac{C}{D^{2+\frac{n+2}{n}}} + \frac{C}{D^4} + \frac{C}{D^{2-o(1)}} \right) \leq \bar{C}_5 \cdot \frac{1}{D^{2-o(1)}}. \]

Here the positive constants \( C, \bar{C}_4 \) and \( \bar{C}_5 \) depend on \( C_0 \), but are independent on \( (\lambda_1, \lambda_2; \xi_1, \xi_2) \) as long as the conditions in Theorem 3.13 are satisfied.

§ 4. Extracting main information from the reduced functional.

In the following we show that the main contributions to reduced functional (3.31) come from the critical points (the twin pseudo-peaks) and the interaction of the two bubbles [cf. (2.23) & (2.25)]. Cf. [10]. In separating the key terms, it is here that the dimension restriction \( n < 10 \) comes in. In the following we denote the reduced functional by \( I_{RH} \), highlighting the dependence on \( H \).
Proposition 4.1. For \( 10 > n \geq 6 \), assume the conditions in Theorem 3.13. There is a positive constant \( \bar{D}_5 \left( \geq \bar{D}_4 \right) \) such that if

\[
D \geq \bar{D}_5,
\]

then we have

\[
\left\| \left( \lambda_1 \cdot \frac{\partial}{\partial \lambda_1} \right) I_{R_H}(z) \right\| - \left\{ C_0^+ \cdot \left[ - \omega_1 \cdot \lambda_1^\ell - C_b^+ \cdot \lambda_1 \cdot \lambda_2 \cdot \frac{n-2}{\gamma n-2} \right] \cdot [1 + o(1)] \right\| \leq o(1) \cdot \frac{1}{D^{n-2}},
\]

(4.2)

\[
\left\| \left( \lambda_1 \cdot \frac{\partial}{\partial \xi_{1j}} \right) I_{R_H}(z) \right\| - \left\{ \tilde{C}_c^+ \cdot \omega_1 \cdot \lambda_1^\ell \cdot \frac{\xi_1}{\lambda_1} + \tilde{C}_d^+ \cdot \left[ \xi_1 - \xi_2 \right] \cdot \frac{1}{\lambda_2} \cdot [1 + o(1)] \right\| \leq o(1) \cdot \frac{1}{D^{n-2}}
\]

(4.3)

for \( j = 1, 2, \ldots, n \). Here (as usual) \( z_\sigma = V_1 \cdot \xi_1 + V_2 \cdot \xi_2 \). In the above, \( o(1) \to 0^+ \) when \( D \to \infty \). Similar estimates hold for derivatives in \( \lambda_2 \) and \( \xi_2 \), with the same positive constants \( \tilde{C}_a^+ \), \( \tilde{C}_b^+ \), \( \tilde{C}_c^+ \) and \( \tilde{C}_d^+ \) (which depend only on \( n \) and \( \ell \)) as in (4.2) and (4.3).

Proof. Arguing as in the proof of Proposition 2.11 in [28], and using (2.23)–(2.25), we have (see § A.2 in the e-Appendix)

(4.4)

\[
\left\| \left( \lambda_1 \cdot \frac{\partial}{\partial \lambda_1} \right) I_{R_H}(z) \right\| - \left\{ \left( \lambda_1 \cdot \frac{\partial}{\partial \lambda_1} \right) G(z_\sigma) - \omega_1 \cdot \frac{(n-2)^2}{2} \cdot \lambda_1 \cdot \lambda_2 \cdot \frac{n-2}{\gamma n-2} \cdot \left[ 1 + o(1) \right] \right\} \leq o(1) \cdot \frac{1}{D^{n-2}},
\]

(4.5)

\[
\left\| \left( \lambda_1 \cdot \frac{\partial}{\partial \xi_{1j}} \right) I_{R_H}(z) \right\| - \left\{ \left( \lambda_1 \cdot \frac{\partial}{\partial \xi_{1j}} \right) G(z_\sigma) + \lambda_1 \cdot \frac{C^+(n)}{\lambda_1 \cdot \lambda_2} \cdot \left[ \xi_1 - \xi_2 \right] \cdot \frac{1}{\lambda_2} \cdot \left[ 1 + o(1) \right] \right\} \leq o(1) \cdot \frac{1}{D^{n-2}}.
\]

Here \( o(1) \to 0 \) as \( D \to \infty \). We continue from (4.4):
(4.6)  
\[
\frac{\partial}{\partial \lambda_1} G(z_\sigma) = -\frac{n - 2}{2n} \cdot \frac{\partial}{\partial \lambda_1} \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot (V_1 + V_2)^{\frac{n}{2n}} \quad (V_1 = V_{\lambda_1}, \xi_1, V_2 = V_{\lambda_2}, \xi_2)
\]

\[
= -\int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot (V_1 + V_2)^{\frac{n+2}{2n}} \cdot \frac{\partial V_1}{\partial \lambda_1}
\]

\[
= -\int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot V_1^{\frac{n+2}{n}} \cdot \frac{\partial V_1}{\partial \lambda_1} \cdot [1 + o(1)] \quad \text{(similar to Weak Interaction Lemma 2.16)}
\]

\[
= -\int_{\mathbb{R}^n} [(\tilde{c}_n \cdot H)] \cdot V_1^{\frac{n+2}{n}} \cdot \left\{ -\frac{n - 2}{2} \cdot \lambda_1^{n-4} \cdot \frac{\lambda_1^2 \cdot \lambda^2 - |y - \xi|^2}{(\lambda^2 + |y - \xi|^2)^2} \right\} dy \cdot [1 + o(1)]
\]

[ direct calculation from (2.9) \uparrow; \ o(1) \to 0^+ \text{ when } D \to \infty ]

\[
= \frac{n - 2}{2} \cdot \int_{\mathbb{R}^n} [(\tilde{c}_n \cdot H)] \cdot V_1^{\frac{n+2}{n}} \cdot \left\{ -\frac{n - 2}{2} \cdot \lambda_1^{n-4} \cdot \frac{2 \lambda_1^2 - \left[ \lambda_1^2 + |y - \xi|^2 \right]}{(\lambda^2 + |y - \xi|^2)^2} \right\} dy \cdot [1 + o(1)]
\]

\[
\quad \ast \left\{ 2 \cdot \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi|^2} \right)^\frac{n}{2} - \frac{1}{\lambda_1} \cdot \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi|^2} \right)^\frac{n-2}{2} \right\} \cdot [1 + o(1)] dy
\]

\[
= \frac{n - 2}{2} \cdot \int_{\mathbb{R}^n} [(\tilde{c}_n \cdot H)] \cdot \left[ 2 \cdot \left( \frac{\lambda_1}{\lambda_1^2 + |y|^2} \right)^\frac{n+2}{2} - 2 \cdot \left( \frac{\lambda_1}{\lambda_1^2 + |y|^2} \right)^n \right] dy \cdot [1 + o(1)]
\]

[ cf. (1.8), (1.10) and (2.4) \uparrow; \uparrow \text{ similar to (A.10)} \downarrow Y = \frac{y}{\lambda} ]

\[
= \frac{n - 2}{2} \cdot \frac{\lambda^2}{\lambda_1} \cdot \int_{\mathbb{R}^n} [P_1(Y)] \cdot \left[ 2 \cdot \left( \frac{1}{1 + |Y|^2} \right)^{n+1} - \left( \frac{1}{1 + |Y|^2} \right)^n \right] \cdot dY \cdot [1 + o(1)]
\]

[ via (1.8) & (1.10) ]

\[
= -\frac{n - 2}{2} \cdot \frac{\lambda^2}{\lambda_1} \cdot \int_{\mathbb{R}^n} [P_1(Y)] \cdot \left[ \left( \frac{1}{1 + |Y|^2} \right)^n - 2 \cdot \left( \frac{1}{1 + |Y|^2} \right)^{n+1} \right] \cdot dY \cdot [1 + o(1)].
\]

Here \( o(1) \to 0 \) as \( D \to \infty \).
Now we apply the following result, whose proof is similar to the proof of Lemma A.6.8 in [26]; see also § A.2 in the e-Appendix.

**Lemma 4.7 (Reduction Lemma).** In $\mathbb{R}^n$, $n \geq 3$, consider a homogeneous polynomial $Q_\ell$ of even degree $\ell \leq n - 1$. We have

\begin{equation}
\int_{\mathbb{R}^n} Q_\ell(y) \cdot \left(\frac{1}{1 + |y|^2}\right)^n dy = \frac{J_n}{\ell \cdot (\ell - 2) \cdots 2 \cdot 1} \cdot \Delta_h^{[h_\ell]} Q_\ell \quad (h_\ell = \ell/2),
\end{equation}

where

\begin{equation}
J_n = \int_{\mathbb{R}^n} y_1^2 \cdots y_{h_\ell}^2 \cdot \left(\frac{1}{1 + |y|^2}\right)^n dy \quad [y = (y_1, \ldots, y_{h_\ell}, \ldots, y_n)].
\end{equation}

Likewise,

\begin{equation}
\int_{\mathbb{R}^n} Q_\ell(y) \cdot \left(\frac{1}{1 + |y|^2}\right)^{n+1} dy = \frac{J_{n+1}}{\ell \cdot (\ell - 2) \cdots 2 \cdot 1} \cdot \Delta_h^{[h_\ell]} Q_\ell,
\end{equation}

where

\begin{equation}
J_{n+1} = \int_{\mathbb{R}^n} y_1^2 \cdots y_{h_\ell}^2 \cdot \left(\frac{1}{1 + |y|^2}\right)^{n+1} dy \quad (\implies J_n - J_{n+1} > 0).
\end{equation}

**Proof of Proposition 4.1 continues...** It follows from (4.4) and Lemma 4.5 that

\begin{equation}
\left(\lambda_1 \cdot \frac{\partial}{\partial \lambda_1}\right) G_H(z_\sigma) = \bar{C}_1^+ (n) \cdot [\nabla_1] \cdot \lambda_1^\ell \cdot [1 + o(1)]
\end{equation}

where

\begin{equation}
\bar{C}_1^+ (n, \ell) = \frac{n - 2}{2} \cdot \frac{J_n - 2J_{n+1}}{\ell \cdot (\ell - 2) \cdots 2 \cdot 1}.
\end{equation}

To show that

\begin{equation}
J_n - 2J_{n+1} > 0,
\end{equation}

consider the stereographic projection

\begin{equation}
\hat{P} : S^n \setminus \{N\} \to \mathbb{R}^n \\
x \mapsto y = \hat{P}(x), \quad \text{where } y_i = \frac{x_i}{1 - x_{n+1}}, \quad 1 \leq i \leq n.
\end{equation}

Here $x = (x_1, \ldots, x_{n+1}) \in S^n \setminus \{N\}$ and $N = (0, \ldots, 0, 1)$ is the north pole. Conversely,

\begin{equation}
x_i = \frac{2y_i}{1 + r^2}, \quad 1 \leq i \leq n, \quad \text{and} \quad x_{n+1} = \frac{r^2 - 1}{r^2 + 1}, \quad \text{where } r = |y|.
\end{equation}
It is known that $\hat{P}$ is a conformal map between $(S^n \setminus \{N\}, g_1)$ and $(\mathbb{R}^n, g_o)$. The conformal factor is given by

$$g_1(x) = \left[\frac{4}{(1 + r^2)^2}\right] \cdot g_o(y) \quad \text{for} \quad y = \hat{P}(x).$$

Back to checking (4.14):

$$J_n - 2J_{n+1} = \int_{\mathbb{R}^n} y_1^2 \cdots y_n^2 \cdot \left(\frac{1}{1 + r^2}\right)^n \cdot \left[1 - \frac{2}{1 + r^2}\right] \, dy$$

$$= \int_{\mathbb{R}^n} y_1^2 \cdots y_n^2 \cdot \left(\frac{1}{1 + r^2}\right)^n \cdot \left[\frac{r^2 - 1}{r^2 + 1}\right] \, dy$$

$$= \frac{1}{2^n} \cdot \int_{S^n} \left[\frac{x_1^2}{(1 - x_{n+1})^2}\right] \cdots \left[\frac{x_n^2}{(1 - x_{n+1})^2}\right] \cdot x_{n+1} \, dS$$

$$= \frac{1}{2^n} \cdot \int_{S^n_+} \left[\frac{x_1^2}{(1 - x_{n+1})^2}\right] \cdots \left[\frac{x_n^2}{(1 - x_{n+1})^2}\right] \cdot x_{n+1} \, dS$$

$$+ \frac{1}{2^n} \cdot \int_{S^n_-} \left[\frac{x_1^2}{(1 - x_{n+1})^2}\right] \cdots \left[\frac{x_n^2}{(1 - x_{n+1})^2}\right] \cdot x_{n+1} \, dS$$

$$= \frac{1}{2^n} \cdot \int_{S^n_+} \left[\frac{x_1^2}{(1 - x_{n+1})^2}\right] \cdots \left[\frac{x_n^2}{(1 - x_{n+1})^2}\right] \cdot x_{n+1} \, dS$$

$$+ \frac{1}{2^n} \cdot \int_{S^n_-} \left[\frac{x_1^2}{(1 - (-x_{n+1})^2)}\right] \cdots \left[\frac{x_n^2}{(1 - (-x_{n+1})^2)}\right] \cdot [-x_{n+1}] \, dS.$$ 

Here $S^n_+$ is the upper hemisphere, and $S^n_-$ the lower. Consider a fixed point

$$(x_1, \cdots, x_{n+1}) \in S^n_+$$

and its "reflection":

$$(x_1, \cdots, x_{n+1}) \in S^n_-.$$

For $0 < x_{n+1} < 1$, we have

$$\left[\frac{x_1^2}{(1 - x_{n+1})^2}\right] \geq \left[\frac{x_1^2}{(1 - [-x_{n+1})^2]}\right], \cdots, \left[\frac{x_n^2}{(1 - x_{n+1})^2}\right] \geq \left[\frac{x_n^2}{(1 - [-x_{n+1})^2]}\right].$$

Thus (4.14) holds.
For derivative in $\lambda_2$, we have similar expression with the same constant as in (4.13). Likewise (see §A.4 in the e-Appendix),

\begin{equation}
\left( \lambda_1 \cdot \frac{\partial}{\partial \xi_1} \right) G_H(z_\sigma) = \bar{C}_2^+ (n, \ell) \cdot \omega_1 \cdot \lambda_1 \cdot \frac{\xi_1}{\lambda_1} \cdot \frac{\lambda_1^\ell}{\lambda_1} \cdot [1 + o(1)] ,
\end{equation}

with similar expression for derivatives in $\xi_2$. \hfill \Box

§ 5. The target - solving the equations - balancing the local and global contributions.

In view of (4.2) and (4.3), and Theorem 3.38, our attention is drawn to the terms in the brackets in (4.2) and (4.3). Thus consider the map

\begin{equation}
\mathcal{T}(P) = (T_{1_0}, T_{2_0} ; T_{1_1}, \cdots, T_{1_n} ; T_{2_1}, \cdots, T_{2_n}) \in \mathbb{R}^{2(n+1)} ,
\end{equation}

\begin{equation}
P = (\lambda_1, \lambda_2; \xi_{1_1}, \cdots, \xi_{1_n}; \xi_{2_1}, \cdots, \xi_{2_n}) \in (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n) ,
\end{equation}

where

\begin{equation}
T_{1_0}(P) = C_1^+ \cdot |\omega_1| \cdot \lambda_1^\ell - \frac{(\sqrt{\lambda_1 \cdot \lambda_2})^{n-2}}{\gamma^{n-2}} \left( C_1^+ = \frac{\bar{C}_a^+}{C_b^+} \right) ,
\end{equation}

\begin{equation}
T_{2_0}(P) = C_1^+ \cdot |\omega_2| \cdot \lambda_2^\ell - \frac{(\sqrt{\lambda_1 \cdot \lambda_2})^{n-2}}{\gamma^{n-2}} \left( C_2^+ = \frac{\bar{C}_c^+}{C_b^+} \right. \text{ and } C_3^+ = \frac{\bar{C}_d^+}{C_b^+} \right) ,
\end{equation}

\begin{equation}
T_{1_j}(P) = C_2^+ \cdot \omega_1 \cdot \lambda_1^\ell \cdot \frac{\xi_1}{\lambda_1} + C_3^+ \cdot \frac{1}{\lambda_2} \cdot \frac{(\sqrt{\lambda_1 \cdot \lambda_2})^{n}}{\gamma^{n}} \cdot (\xi_1 - \xi_2) , \quad \text{and}
\end{equation}

\begin{equation}
T_{2_j}(P) = C_2^+ \cdot \omega_2 \cdot \lambda_2^\ell \cdot \frac{\xi_2}{\lambda_2} + C_3^+ \cdot \frac{1}{\lambda_1} \cdot \frac{(\sqrt{\lambda_1 \cdot \lambda_2})^{n}}{\gamma^{n}} \cdot (\xi_2 - \xi_1) ,
\end{equation}

To find

\begin{equation}
\lambda_1 = \lambda_1_{r_1} \quad \text{and} \quad \lambda_2 = \lambda_2_{r_2} \quad \text{so that} \quad T_{1_0} = T_{2_0} = 0 ,
\end{equation}

we let

\begin{equation}
\lambda_2 = \alpha \cdot \lambda_1 .
\end{equation}
Here the positive constant $\alpha$ is chosen so that

\begin{equation}
|\omega_1| = |\omega_2| \cdot \alpha^\ell \iff \alpha = \left(\frac{|\omega_1|}{|\omega_2|}\right)^\frac{1}{\ell} \implies \lambda_1 = \left[\gamma^{n-2} \cdot \frac{C_1^+ \cdot |\omega_1|}{\alpha^{n-2}}\right]^{\frac{1}{n-2} - \varepsilon}
\end{equation}

[↑ via (5.2)]
\hspace{2cm}
\implies \lambda_2 = \left(\frac{|\omega_1|}{|\omega_2|}\right)^\frac{1}{\ell} \cdot \lambda_1.

Note that $\lambda_1, \lambda_2 \to 0^+$ as $\gamma \to 0^+$. (5.7) and (5.8) imply that

\begin{equation}
C_p^{-\frac{1}{\ell}} \cdot \lambda_2 < \lambda_1 < C_p^{\frac{1}{\ell}} \cdot \lambda_2
\end{equation}

and

\begin{equation}
\lambda^\ell \leq C_o \cdot \frac{1}{D^{n-2}} \quad [2 \leq \ell < n - 2]
\end{equation}

for $(\lambda_1, \lambda_2; \xi_1, \xi_2)$ close to $(\lambda_{1r}, \lambda_{2r}; 0, p_2)$. Cf. (3.45). Here the positive constant $C_p \approx \frac{\omega_2}{\omega_1}$.

Next, to seek

\begin{equation}
\xi_1_j = \xi_{1jr}\quad \text{and} \quad \xi_2_j = \xi_{2jr}\quad \text{so that} \quad T_{1j} = T_{2j} = 0
\end{equation}

for $j = 1, 2, \cdots, n$, we let

\begin{equation}
\xi_{1jr} = -\beta \cdot (\xi_{2jr} - q_{2j}) \quad \text{for} \quad j = 1, 2, \cdots, n,
\end{equation}

where the positive constant $\beta$ (independent of $j$) is chosen so that

\begin{equation}
\omega_1 \cdot \lambda_1^\ell \cdot \frac{\lambda_2}{\lambda_{1r}} \cdot \beta = \omega_2 \cdot \lambda_2^\ell \cdot \frac{\lambda_1}{\lambda_{2r}} \quad [\implies (5.4) "=\" (5.5)]
\end{equation}

Note that $\beta = O(1)$ via (5.7) and (5.8). Thus we find $\xi_{1jr}$ via (5.4) by writing

\begin{equation}
C_2^+ \cdot \omega_1 \cdot \lambda_1^\ell \cdot \frac{\xi_{1jr}}{\lambda_{1r}} = -C_3^+ \cdot \frac{1}{\lambda_2} \cdot \left(\frac{\sqrt{\lambda_1 \cdot \lambda_2}}{\gamma^n}\right)^n \cdot (\xi_{1jr} - q_{2j} - [\xi_{2jr} - q_{2j}])
\end{equation}

\begin{align*}
\implies C_2^+ \cdot \frac{\left(\frac{\sqrt{\lambda_1 \cdot \lambda_2}}{\gamma^n}\right)^{n-2}}{\lambda_1^\ell} \cdot \frac{\xi_{1jr}}{\lambda_{1r}} &= C_3^+ \cdot \frac{1}{\alpha} \cdot \left(\frac{\sqrt{\lambda_1 \cdot \lambda_2}}{\gamma^n}\right)^n \cdot \left[\frac{\xi_{1jr}}{\lambda_{1r}} \cdot \left(1 + \frac{1}{\beta}\right) - \frac{q_{2j}}{\lambda_{1r}}\right] \\
\implies \frac{\xi_{1jr}}{\lambda_{1r}} \cdot \left\{ C_2^+ \cdot \frac{1}{\lambda_1^\ell} - C_3^+ \cdot \frac{1}{\alpha} \cdot \left(1 + \frac{1}{\beta}\right) \right\} &= -C_3^+ \cdot \frac{1}{\alpha} \cdot \frac{q_{2j}}{\lambda_{1r}}
\end{align*}

\begin{align*}
\uparrow \quad &\text{as $D \to \infty$, $D$ takes the form $\frac{\gamma}{\sqrt{\lambda_1 \cdot \lambda_2}}$}.
\end{align*}
From here we can find $\xi_{1r}$, and hence $\xi_{2r}$. We observe that

$$\begin{align*}
(5.14) \quad \xi_{1r} &= O\left(\frac{1}{D}\right) \cdot \lambda_{1r} = o(1) \cdot \lambda_{1r} \quad \text{and} \quad (\xi_{2r} - q_2) = -\frac{\xi_{1r}}{\beta} = o(1) \cdot \lambda_{1r}, \\
[ o(1) \to 0^+ \text{ as } D \to \infty ], \quad \text{and the solution}
(5.15) \quad P_r := (\lambda_{1r}, \lambda_{2r}, \xi_{11r}, \ldots, \xi_{1n_r}, \xi_{21r}, \ldots, \xi_{2n_r}) \quad \text{to} \quad T = \vec{0} \quad \text{is unique}.
\end{align*}$$

§ 5a. Jacobian matrix. In order to compute the degree of $T$ at the image $\vec{0}$, (in a small neighborhood of $P_r$), let us consider the Jacobian matrix of the map $T$:

$$\begin{pmatrix}
\frac{\partial T_1}{\partial \lambda_1} & \frac{\partial T_1}{\partial \lambda_2} & 0 & \ldots & 0 \\
\frac{\partial T_2}{\partial \lambda_1} & \frac{\partial T_2}{\partial \lambda_2} & 0 & \ldots & 0 \\
\frac{\partial T_1}{\partial \xi_1} & \frac{\partial T_2}{\partial \xi_1} & \frac{\partial T_1}{\xi_1} & \ldots & \frac{\partial T_1}{\xi_1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{\partial T_1}{\partial \xi_{1n_r}} & \frac{\partial T_2}{\partial \xi_{1n_r}} & \frac{\partial T_1}{\xi_{1n_r}} & \ldots & \frac{\partial T_1}{\xi_{1n_r}} \\
\end{pmatrix}$$

At $P_r$, we have [using $(5.2)-(5.5)$ and $T(P_r) = \vec{0}$]

$$\begin{align*}
\frac{\partial T_1}{\partial \lambda_1} \bigg|_{P_r} &= C_1^+ \cdot |\varpi_1| \cdot \ell \cdot (\lambda_{1r})^{\ell-1} - \frac{n-2}{2} \cdot \frac{1}{\lambda_{1r}} \cdot \frac{(\sqrt{\lambda_{1r} \cdot \lambda_{2r}})^{n-2}}{\gamma^{n-2}} \\
&= \left[ \ell - \frac{n-2}{2} \right] \cdot C_1^+ \cdot |\varpi_1| \cdot (\lambda_{1r})^{\ell-1} \\
&\quad (\uparrow \text{‘unit’ for this calculation})
\end{align*}$$

and

$$\begin{align*}
\frac{\partial T_2}{\partial \lambda_2} \bigg|_{P_r} &= C_1^+ \cdot |\varpi_2| \cdot \ell \cdot (\lambda_{2r})^{\ell-1} - \frac{n-2}{2} \cdot \frac{1}{\lambda_{2r}} \cdot \frac{(\sqrt{\lambda_{1r} \cdot \lambda_{2r}})^{n-2}}{\gamma^{n-2}} \\
&= \left[ \ell - \frac{n-2}{2} \right] \cdot C_1^+ \cdot |\varpi_2| \cdot \alpha^{\ell-1} \cdot (\lambda_{1r})^{\ell-1} \quad [ \text{using } \lambda_{2r} = \alpha \cdot \lambda_{1r} ]
\end{align*}$$

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It follows that the Jacobian matrix, evaluated at \( P \), equal to \( \lambda \), in the above, \( I \) is a diagonal matrix with each diagonal entry \( \gamma^{n-2} \).

\[
\begin{align*}
\frac{\partial T_1}{\partial \lambda_2} \bigg|_{P} &= -\frac{n-2}{2} \cdot \frac{1}{\lambda_2} \cdot \left( \sqrt{\lambda_1 \cdot \lambda_2} \right)^{n-2} = -\frac{n-2}{2} \cdot C_1^+ \cdot |\omega_2| \cdot \alpha \cdot (\lambda_1)^\ell - 1, \\
\frac{\partial T_2}{\partial \lambda_1} \bigg|_{P} &= -\frac{n-2}{2} \cdot \frac{1}{\lambda_1} \cdot \left( \sqrt{\lambda_1 \cdot \lambda_2} \right)^{n-2} = -\frac{n-2}{2} \cdot C_1^+ \cdot |\omega_1| \cdot (\lambda_1)^\ell - 1, \\
\frac{\partial T_1}{\partial \xi_j} \bigg|_{P} &= o([\lambda_1]^{\ell - 1}) , \quad \frac{\partial T_2}{\partial \xi_j} \bigg|_{P} = o([\lambda_1]^{\ell - 1}) ,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial T_1}{\partial \xi_j} \bigg|_{P} &= C_2^+ \cdot \omega_1 \cdot (\lambda_1)^\ell - 1 + o([\lambda_1]^{\ell - 1}) \quad \text{[via (5.4) \\ (5.14)]} , \\
\frac{\partial T_2}{\partial \xi_j} \bigg|_{P} &= C_2^+ \cdot \omega_2 \cdot (\lambda_1)^\ell - 1 + o([\lambda_1]^{\ell - 1}) ,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial T_1}{\partial \lambda_1} \bigg|_{P} &= o([\lambda_1]^{\ell - 1}) , \quad \frac{\partial T_1}{\partial \lambda_2} = o([\lambda_1]^{\ell - 1}) , \\
\frac{\partial T_2}{\partial \lambda_1} \bigg|_{P} &= o([\lambda_1]^{\ell - 1}) , \quad \frac{\partial T_2}{\partial \lambda_2} \bigg|_{P} = o([\lambda_1]^{\ell - 1}) \quad (j = 1 , 2 , \cdots n).
\end{align*}
\]

It follows that the Jacobian matrix, evaluated at \( P \), can be written as

(5.16)

\[
\begin{pmatrix}
\ell - \frac{n-2}{2} \cdot C_1^+ \cdot |\omega_1| & -\frac{n-2}{2} \cdot C_1^+ \cdot |\omega_2| \cdot \alpha \cdot (\lambda_1)^\ell - 1 & 0 \cdots & 0 \\
-\frac{n-2}{2} \cdot C_1^+ \cdot |\omega_1| & \ell - \frac{n-2}{2} \cdot C_1^+ \cdot |\omega_2| \cdot \alpha \cdot (\lambda_1)^\ell - 1 & 0 \cdots & 0 \\
o(1) & o(1) & C_2^+ \cdot |\omega_1| \cdot [1 + o(1)] & o(1) \\
o(1) & o(1) \cdots & o(1) \cdots & C_2^+ \cdot |\omega_2| \cdot [1 + o(1)] & o(1) \cdots
\end{pmatrix}
* \mathbf{I}_{\lambda_1^\ell}.
\]

In the above, \( \mathbf{I}_{\lambda_1^\ell} \) is the \( [2(n+1)] \times [2(n+1)] \) diagonal matrix with each diagonal entry equal to \( (\lambda_1)^\ell - 1 \). Focusing on the four terms at the top left hand corner of the matrix in
the Jacobian determinant at $P_\tau$ is given by

\[(5.17)\]

\[
\text{Jacobian Det.} = \left[ (C_1^2)^2 \cdot |w_1| \cdot |w_2| \cdot \alpha^{\ell-1} \cdot \left\{ \ell \cdot \left[ \ell - 2 \cdot \frac{n-2}{2} \right] \right\} \right] \times \\
\times \left\{ \left[ (C_2^n)^2 \cdot |w_1|^n \cdot |w_2|^n \right] \cdot [1 + o(1)] \ast [(\lambda_{1\tau})^{\ell-1}]^{2(n+1)} < 0 \right.,
\]

as $\ell < n - 2$. Here $o(1) \to 0$ as $D \to 0$. Together with (5.15), we conclude that

\[(5.18)\]

\[
\text{Deg} \left( \mathcal{T}, B_{P_\tau}(c \cdot \lambda_{1\tau}), \vec{0} \right) = -1 \quad \text{when } D_\tau = \frac{\gamma}{\lambda_{\tau}} \geq \bar{D}_6,
\]

\[(5.19)\]

where $\lambda_{\tau} = \sqrt{\lambda_{1\tau} \cdot \lambda_{2\tau}}$.

With regard to condition (3.39), using the regularity of the map $\mathcal{T}$ and (5.11), one deduces that, for $\mu$ to be small enough and $P \in (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)$,

\[(5.20)\]

\[
\| P - P_\tau \| = \mu \cdot \lambda_{1\tau} \quad [B_{P_\tau}(\mu \cdot \lambda_{1\tau}) \subset (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)]
\]

\[
\Rightarrow \| \mathcal{T}(P) \| = \| \mathcal{T}(P) - \mathcal{T}(P_\tau) \| \quad \text{[as } \mathcal{T}(P_\tau) = 0]}
\]

\[
\geq \bar{c}' \cdot (\lambda_{1\tau})^{\ell-1} \cdot (\mu \cdot \lambda_{1\tau})
\]

\[
\geq \bar{c}'' \cdot (\lambda_{\tau})^\ell \geq \bar{c} \cdot (\lambda_{1\tau})^\ell \left[ = O \left( \frac{1}{D_{n-2}} \right); \ (2.14) \ \text{is used} \right],
\]

\[
\lambda_{\tau} = \sqrt{\lambda_{1\tau} \cdot \lambda_{2\tau}} \quad \text{where}
\]

Here $\mu$ and $\bar{c}'$ are independent on $P$ as long as the conditions in (5.18) & (5.20) are satisfied. Note that in $B_{P_\tau}(\mu \cdot \lambda_{1\tau})$, via (5.4) and (5.9), condition (3.45) is satisfied. We summarize the discussion in this subsection in the following.

**Lemma 5.21.** The map $\mathcal{T} : (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n) \to \mathbb{R}^{2(n+1)}$, as defined in (5.1)–(5.5), has an unique point $P_\tau \in (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)$ [given via (5.15)] so that

\[
\mathcal{T}(P_\tau) = \vec{0}.
\]

In addition, for a small enough positive number $\mu$, we have [refer to (5.8)]

\[
\text{Deg} \left( \mathcal{T}, B_{P_\tau}(\mu \cdot \lambda_{1\tau}), \vec{0} \right) = -1,
\]

and

\[
\min_{\partial B_{P_\tau}(\mu \cdot \lambda_{1\tau})} \{ \| \mathcal{T} \| \} \geq \bar{c} \cdot (\lambda)^\ell.
\]

Moreover, condition (2.12) and (3.45) are satisfied by all $P = (\lambda_1, \lambda_2; \xi_1, \xi_2) \in B_{P_\tau}(\mu \cdot \lambda_{1\tau})$. 

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§ 5 b. Proof of the main theorem. In view of (4.2) and (4.3), we consider
\[ \bar{C}_b^+ \cdot \mathcal{T} \]
recall that \( \bar{C}_b^+ = \omega_n \cdot \frac{(n - 2)^2}{2} \), cf. (4.2).

Accordingly, denote the terms within \{ \cdots \} [referring to (4.2) and (4.3)] by
\[ (5.22) \quad \bar{C}_b^+ \cdot \mathcal{T}_{o(1)} \{ = \bar{C}_b^+ \cdot \mathcal{T} \cdot [1 + o(1)] \}. \]

From (5.1)–(5.5) and \( \mathcal{T}(P_\tau) = \vec{0} \), we obtain
\[ (5.23) \quad \left\| \bar{C}_b^+ \left[ \mathcal{T}_{o(1)}(P) - \mathcal{T}(P) \right] \right\| \leq o(1) \cdot \lambda_\tau \equiv o(1) \cdot \frac{1}{\bar{D}_{n-2}} \quad \text{for} \quad P \in B_{P_\tau}(\mu \cdot \lambda_1). \]

Via (4.2) and (4.3), we have
\[ (5.24) \quad \left\| (\lambda \cdot \nabla) I_{R_H}(\zeta_\sigma) - \bar{C}_b^+ \cdot \mathcal{T}_{o(1)} \right\| \leq o(1) \cdot \lambda_\tau \equiv o(1) \cdot (\lambda_\tau)_{\ell} \]
\[ \implies \left\| (\lambda \cdot \nabla) I_{R_H}(\zeta_\sigma) - \bar{C}_b^+ \cdot \mathcal{T}(P) \right\| \leq o(1) \cdot \frac{1}{\bar{D}_{n-2}} \approx o(1) \cdot \lambda_\tau \]
\[ \quad \text{for} \quad P \in B_{P_\tau}(\mu \cdot \lambda_1). \]

In (5.23) and (5.24), \( o(1) \to 0^+ \) as \( \bar{D}_\tau := \frac{\gamma}{\lambda_\tau} \to \infty \), and \( \zeta_\sigma \) corresponds to \( P \) via
\[ z_{\sigma} = V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2} \longleftrightarrow P = (\lambda_1, \lambda_2; \xi_1, \xi_2). \]

Thus there exists a positive constant \( \bar{D}_6 \geq \bar{D}_5 \), which depends on \( n, \ell, \bar{C}_b, h, C_{R_1}, C_{R_2}, C_{P_1}, C_{P_2}, \) and \( C_p \), such that if
\[ (5.25) \quad \bar{D}_\tau \geq \bar{D}_7, \]
then the degree properties [as shown in Theorem 3.38], (5.18), (5.19) and (5.24) imply that
\[ (5.26) \quad \text{Deg} \left( (\lambda \cdot \nabla) I_{R_H}, B_{P_\tau}(\mu \cdot \lambda_1) \right) = \text{Deg} \left( \mathcal{T}, B_{P_\tau}(\mu \cdot \lambda_1) \right) = -1 \quad (\neq 0). \]

Hence the reduced functional has a critical point at \( P_R = (\lambda_1, \lambda_2; \xi_1, \xi_2) \in B_{P_\tau}(\mu \cdot \lambda_1) \). Via Lemma 3.32, equation (1.2) has a solution of the form \( \bar{z}_\sigma + w_{\zeta_\sigma} \), where
\[ \bar{z}_\sigma = V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2}. \]
§ 5 c. Estimate on $\gamma_0$. The following consideration provides an idea on the size of $\gamma_0$ [cf. Remark (1) in § 1 a]. From condition (5.25)

\[
D_\tau = \frac{\gamma}{\sqrt{\lambda_{1r} \cdot \lambda_{2r}}} \geq \tilde{D}_7 \iff \left( \frac{\gamma}{\sqrt{\lambda_{1r} \cdot \lambda_{2r}}} \right)^{n-2} \geq \tilde{D}_7^{n-2}.
\]

\[
\iff C_1^+ \cdot \frac{1}{|\omega_1|} \cdot \frac{1}{\lambda_{1r}^\ell} \geq \tilde{D}_7^{n-2} \quad \text{[via (5.2) with $T(P_\tau) = 0$]}
\]

\[
\iff C \cdot \frac{1}{|\omega_1|} \cdot \frac{1}{|\omega_1|^{\ell(n-2)} - \tau} \cdot \frac{1}{\gamma^{\ell(n-2)} - \tau} \geq \tilde{D}_7^{n-2} \quad \text{[using (5.8)]}
\]

\[
\iff C \cdot \frac{1}{\tilde{D}_7^{\ell(n-2) - \tau}} \cdot \frac{1}{|\omega_1|^{\ell}} \geq \gamma \quad \text{[cf. (1.15)].}
\]

Appendix.

Lemma A.1. For positive numbers $a$, $b$, $\tau$, $\beta$, and $M$, we have the following.

(A.2) \((0 <) \beta \leq 2 \implies \left| (a + b)^\beta - (a^\beta + b^\beta) \right| \leq \bar{C}_\beta \cdot (a \cdot b)^{p\tau} \).

Here $\bar{C}_\beta$ is a positive constant depending on $\beta$, but not on $a$ and $b$.

(A.3) \(M \geq 2 \implies \left| (a + b)^M - (a^M + b^M) \right| \leq \bar{C}_M \cdot (a^{M-1} \cdot b + b^{M-1} \cdot a) \).

Here $\bar{C}_M$ is a positive constant depending on $M$, but not on $a$ and $b$. Moreover,

(A.4) \((0 <) \tau \leq 1 \implies (a + b)^\tau \leq a^\tau + b^\tau \).

Lemma A.5. For $n \geq 6$, let $P$ and $Q$ be positive numbers with

\[ P + Q = n \,.
\]

For $(\lambda_1, \lambda_2; \xi_1, \xi_2) \in (\mathbb{R}^+ \times \mathbb{R}^+) \times (\mathbb{R}^n \times \mathbb{R}^n)$, let

\[
S(\lambda_1, \lambda_2; \xi_1, \xi_2; P, Q) := \int_{\mathbb{R}^n} \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi_1|^2} \right)^P \cdot \left( \frac{\lambda_2}{\lambda_2^2 + |y - \xi_2|^2} \right)^Q \, dy.
\]
Assume that
\[(A.6) \quad \bar{C}^{-1} \lambda_1 \leq \lambda_2 \leq \bar{C} \lambda_1, \quad |\xi_1| \leq \lambda \quad \text{and} \quad |\xi_2 - q_2| \leq \lambda.\]

There is a positive number \(\bar{D}_1\) such that for 
\[(A.7) \quad \text{if} \quad D \geq \bar{D}_1,\]

\[(A.8) \quad \text{then} \quad S(\lambda_1, \xi_1; \lambda_2, \xi_2; P, Q) \leq \begin{cases} \frac{C_3}{D^2P} & \text{if} \quad P < \frac{n}{2}, \\ \frac{C_3}{D} \cdot \ln D & \text{if} \quad P = Q = \frac{n}{2}, \\ \frac{C_3}{D^2Q} & \text{if} \quad P > \frac{n}{2}. \end{cases}\]

Here \(\bar{C}_3\) depends on \(n, \bar{C}, \text{and } Q\). In particular, \(\bar{D}_1\) and \(\bar{C}_3\) do not depend on \((\lambda_1, \lambda_2; \xi_1, \xi_2)\) as long as \((A.6)\) is satisfied.

For the proofs of Lemmas A.1 and A.5, we refer to [28].

Proof of Lemma 3.12. Let \(V_1 = V_{\lambda_1, \xi_1}\) and \(V_2 = V_{\lambda_2, \xi_2}\). Via Lemma A.5, we have
\[(A.9) \quad \left| (V_1 + V_2)^{\frac{n}{n-2}} - \left( V_1^{\frac{2n}{n-2}} + V_2^{\frac{2n}{n-2}} \right) \right| \leq C \cdot \left( V_1^{\frac{n+2}{n-2}} \cdot V_2^{\frac{n+2}{n-2}} \cdot V_1 \right).\]

As
\[(A.10) \quad \frac{1}{(\lambda_1^2 + |y - \xi|^2)^n} = \frac{1}{(\lambda^2 + |y|^2)^n} \cdot \left\{ 1 - n \cdot \frac{|\xi|^2 - 2y \cdot \xi}{\lambda^2 + |y|^2} + \ldots \right\} \]

\[\Rightarrow \quad \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi|^2} \right)^n = \left( \frac{\lambda_1}{\lambda^2 + |y|^2} \right)^n \cdot [1 + o(1)].\]

Here \(o(1) \to 0^+\) as \(\bar{c} \to 0^+\). Consider the partition
\[(A.11) \quad \mathbb{R}^n = B_o(\rho) \cup B_{q_2}(\rho) \cup \{ \mathbb{R}^n \setminus [B_o(\rho) \cup B_{q_2}(\rho)] \} .\]

Here \(\rho\) is the same number as in (1.9)(i). The leading terms in the estimate in Lemma 3.12 can be obtained by integral of the form
In the above we apply (1.10), (1.11) and the following.

The "cross over" terms are estimated as in § (use of (2.21).

Next, we consider the mixed terms that come from the right hand side of (A.9), again making use of (A.14) for the contribution outside $B_0(\rho) \cup B_{q_2}(\rho)$. Observe that

$$\frac{\rho}{\lambda} = \bar{\h} \cdot \bar{\gamma} = \bar{\h} \cdot D \quad \text{[cf. (1.9)(i)]}.$$ 

The "cross over" terms are estimated as in § 2 b [cf. (2.21)], giving rise to

$$\int_{B_0(\rho)} V_2^{n-2} = O\left(\frac{1}{D^n}\right).$$

Next, we consider the mixed terms that come from the right hand side of (A.9), again making use of (2.21).

(A.13) 

$$\int_{\mathbb{R}^n} |H(y)|^m \cdot \left(\frac{\lambda_1}{\lambda^2 + |y - \xi_1|^2}\right)^{n+2} \cdot \left(\frac{\lambda_2}{\lambda^2 + |y - \xi_2|^2}\right)^{n-2}$$

$$\leq C_1(n) \cdot \frac{1}{\lambda_{1}^{2}} \cdot \frac{1}{D_{o}^{n-2}} \cdot \left[1 + o(1)\right] \cdot \int_{B_0(\rho)} r^{m-\ell} \cdot \left(\frac{\lambda_1}{\lambda^2 + |y - \xi_1|^2}\right)^{n+2} dy +$$

$$+ C_2(n) \cdot \frac{1}{\lambda_{2}^{2}} \cdot \frac{1}{D_{o}^{n+2}} \cdot \left[1 + o(1)\right] \cdot \int_{B_{q_2}(\rho)} r^{m-\ell} \cdot \left(\frac{\lambda_2}{\lambda^2 + |y - \xi_2|^2}\right)^{n-2} dy +$$

$$+ O\left(\frac{1}{D^{n-o(1)}}\right) \quad \text{(\uparrow) \text{ see below; shift center to } q_2 \uparrow}$$

$$\leq C_3(n) \cdot \lambda^2 \cdot \frac{1}{D_{o}^{n-2}} \cdot \left[1 + o(1)\right] + O\left(\frac{1}{D^{n-o(1)}}\right).$$

In the above we apply (1.10), (1.11) and the following.

(A.14) 

$$\int_{\mathbb{R}^n \setminus \left[B_0(\rho) \cup B_{q_2}(\rho)\right]} |H|^m \cdot V_1^{n+2} \cdot V_2$$

$$\leq C_1^m \cdot \int_{\left\{\mathbb{R}^n \setminus \left[B_0(\rho) \cup B_{q_2}(\rho)\right]\right\} \cap \{ V_1 > V_2 \}} V_1^{n+2} \cdot V_2 +$$

$$+ C_1^m \cdot \int_{\left\{\mathbb{R}^n \setminus \left[B_0(\rho) \cup B_{q_2}(\rho)\right]\right\} \cap \{ V_2 > V_1 \}} V_1^{n+2} \cdot V_2$$

$$\leq C_1^m \cdot C(n) \cdot \int_\rho \left(\frac{\lambda}{\lambda^2 + r^2}\right)^n r^{n-1} \cdot dr \leq C_1^m \cdot C'(n) \cdot \frac{1}{D^{n-o(1)}}.$$ 

Similarly we estimate the corresponding integral for $V_2^{n+2} \cdot V_1$. □

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e-Appendix to the Article

“Conformal Scalar Curvature Equation on $S^n$ - Functions With Two Close Critical Points (Pseudo Twin-Peaks)”

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In this e-Appendix we follow the notations, conventions, equation numbers, section numbers, lemma, proposition and theorem numbers as used in the main article [29], unless otherwise is specifically mentioned (for instances, those equation numbers starting with ‘A’).

§ A.1. Estimates (2.23) & (2.25).

See also Lemma B.2 and Lemma B.4 in [10] and formulas 2.119 and 2.206 in [3]. For

$$z_\sigma = V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2},$$

where [cf. (2.12)]

(A.1.1) $\bar{C}^{-1} \cdot \lambda_2 < \lambda_1 < \bar{C} \cdot \lambda_2$,  \quad \xi_1 < \bar{c} \cdot \lambda \quad \text{and} \quad |\xi_2 - q_2| < \bar{c} \cdot \lambda,$

let

(A.1.2) $\rho_\mu = \mu \cdot |\xi_1 - \xi_2|.$

Here $\mu$ is a chosen small positive number so that

(A.1.3) $\mu \to 0^+$ (slowly) and $\mu^M \cdot D \to \infty$ when $D \to \infty$.

Here $M$ is a (fixed) large integer. See (A.1.12), (A.1.19) and (A.1.28). For most particular purpose one can take

(A.1.4) $\mu = \frac{1}{2} \cdot \frac{1}{D^\epsilon}$ for $D \gg 1$,

where $\epsilon < 1$ is any fixed small positive number.
Consider the partition
\[(A.1.5) \quad \mathbb{R}^n = B_{\xi_1}(\rho_\mu) \cup B_{\xi_2}(\rho_\mu) \cup \{ \mathbb{R}^n \setminus [B_{\xi_1}(\rho_\mu) \cup B_{\xi_2}(\rho_\mu)] \}.\]
We compare \(V_1\) and \(V_2\) on \(B_{\xi_1}(\rho_\mu)\). In order to do so, we make use of the following inequalities (\(A\) and \(B\) are positive numbers)
\[(A.1.6) \quad \left( \frac{1}{1 + t} \right)^{\frac{n-2}{2}} = 1 - \frac{n - 2}{2} \cdot t \cdot [1 + o(1)] \quad \text{for } t > 0 \text{ small}, \]
\[(A.1.7) \quad (A + B)^{\frac{n+2}{n-2}} = A^{\frac{n+2}{n-2}} + \frac{n + 2}{n - 2} \cdot A^{\frac{4}{n-2}} \cdot B + O(1) \cdot B^{\frac{n+2}{n-2}} \quad \text{for } \frac{B}{A} \text{ small}, \]
where \(o(1) \to 0\) as \(t \to 0\).

It follows from (A.1.7) that
\[(A.1.7') \quad (A + B)^{\frac{n+2}{n-2}} - \left( A^{\frac{n+2}{n-2}} + B^{\frac{n+2}{n-2}} \right) = \frac{n + 2}{n - 2} \cdot A^{\frac{4}{n-2}} \cdot B + O(1) \cdot B^{\frac{n+2}{n-2}} \quad \text{for } \frac{B}{A} \text{ small}. \]

In \(B_{\xi_1}(\rho_\mu)\), we have
\[(A.1.8) \quad V_{\lambda_2, \xi_2}(y) = \left( \frac{\lambda_2}{\lambda_2^2 + |y - \xi_2|^2} \right)^{\frac{n-2}{2}} \]
\[= \left( \frac{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}{\lambda_1 + |y - \xi_2|^2 \lambda_1 \lambda_2} \right)^{\frac{n-2}{2}} = \frac{1}{\lambda_1^{\frac{n-2}{2}}} \cdot \left( \frac{\lambda_2}{\lambda_1} + \frac{|y - \xi_1|^2}{\lambda_1 \cdot \lambda_2} + \frac{1}{\lambda_1 \cdot \lambda_2} \right)^{\frac{n-2}{2}} \]
\[= \frac{1}{\lambda_1^{\frac{n-2}{2}}} \cdot \frac{1}{d^{n-2}} \cdot \left( 1 + \frac{1}{d^2} \cdot \frac{|y - \xi_1|^2}{\lambda_1 \cdot \lambda_2} + \frac{1}{d^2} \cdot \frac{2(y - \xi_1)(\xi_1 - \xi_2)}{\lambda_1 \cdot \lambda_2} \right)^{\frac{n-2}{2}} \]
\[\leftarrow \quad = \quad t \quad \left( \leftarrow \text{small} \right) \quad \rightarrow \]
\[= \frac{1}{\lambda_1^{\frac{n-2}{2}}} \cdot \frac{1}{d^{n-2}} \cdot \left\{ 1 - \frac{n - 2}{2} \cdot t \cdot [1 + o(1)] \right\} \quad \text{[via (A.1.6)]}. \]

Recall that
\[d := \frac{|\xi_1 - \xi_2|}{\sqrt{\lambda_1 \cdot \lambda_2}}.\]
In (A.1.8), \( o(1) \to 0 \) as
\[
(A.1.9) \quad t = \frac{1}{d^2} \left[ \left( \frac{\lambda_2}{\lambda_1} \right) + \frac{|y - \xi_1|^2}{\lambda_1 \cdot \lambda_2} + \frac{2 (y - \xi_1) \cdot (\xi_1 - \xi_2)}{\lambda_1 \cdot \lambda_2} \right] \to 0.
\]

Note that
\[
(A.1.10) \quad \frac{1}{d^2} \cdot \frac{|y - \xi_1|^2}{\lambda_1 \cdot \lambda_2} \leq \frac{1}{d^2} \cdot \frac{\rho^2_\mu}{\lambda_1 \cdot \lambda_2} = \frac{1}{d^2} \cdot \frac{|\xi_1 - \xi_2|^2}{\lambda_1 \cdot \lambda_2} \cdot \mu^2 = \mu^2
\]
and
\[
(A.1.11) \quad \frac{1}{d^2} \cdot \frac{(y - \xi_1) \cdot (\xi_1 - \xi_2)}{\lambda_1 \cdot \lambda_2} \leq \frac{1}{d^2} \cdot \frac{|y - \xi_1|}{\sqrt{\lambda_1 \cdot \lambda_2}} \cdot \frac{|\xi_1 - \xi_2|}{\sqrt{\lambda_1 \cdot \lambda_2}}
\leq \frac{1}{d^2} \cdot \frac{\rho_\mu}{\sqrt{\lambda_1 \cdot \lambda_2}} \cdot d = \mu.
\]

Recall (A.1.3). Hence we obtain
\[
(A.1.12) \quad V_2 = \frac{1}{\lambda_1^{n-2}} \cdot \frac{1}{D^{n-2}} \cdot [1 + o(1)] \quad \text{in} \quad B_{\xi_1}(\rho_\mu) \left( D = \frac{\gamma}{\sqrt{\lambda_1 \cdot \lambda_2}} \right).
\]

Here \( o(1) \to 0 \) as \( D \to \infty \). In the above we apply (A.1.3), (2.13) and (2.14).

As in (2.22),
\[
(A.1.13) \quad I = \left\{ \left( V_{\xi_1, \xi_1}^{n-2} + V_{\xi_2, \xi_2}^{n-2} \right) - (V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2})^{n-2} \right\} ( < 0 )
\]

According to the partition in (A.1.5), we break down the integral as
\[
(A.1.14) \quad \int_{\mathbb{R}^n} I \cdot [\partial_{\lambda_1} V_1] = \int_{B_{\xi_1}(\rho_\mu)} I \cdot [\partial_{\lambda_1} V_1] + \int_{B_{\xi_2}(\rho_\mu)} I \cdot [\partial_{\lambda_1} V_1]
\]
\[
+ \int_{\mathbb{R}^n \setminus [B_{\xi_1}(\rho_\mu) \cup B_{\xi_2}(\rho_\mu)]} I \cdot [\partial_{\lambda_1} V_1].
\]

A direct calculation shows that
\[
(A.1.15) \quad \frac{\partial V_{\lambda_1, \xi_1}}{\partial \lambda_1} = \frac{\partial}{\partial \lambda_1} \left[ \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi_1|^2} \right)^{\frac{n-2}{2}} \right]
\]
\[
= - \frac{n - 2}{2} \cdot \lambda_1^{\frac{n-4}{2}} \cdot \frac{(\lambda_1^2 - |y - \xi_1|^2)}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n-2}{2}}} \implies \left| \frac{\partial V_{\lambda_1, \xi_1}}{\partial \lambda_1} \right| \leq \frac{n - 2}{2} \cdot \frac{1}{\lambda_1} \cdot V_{\lambda_1, \xi_1}.
\]
In the following, we play careful attention on the \(( \pm )\) sign. Applying (A.1.7'), we have

(A.1.16)

\[
\int_{B_{\xi_1}(\rho_n)} \mathcal{I} \cdot [\partial_{\lambda_1} V_1] \left( = \int_{B_{\xi_1}(\rho_n)} \left\{ \begin{array}{c}
\left( V_1^{\frac{n+2}{n-2}} + V_2^{\frac{n+2}{n-2}} \right) - (V_1 + V_2)^{\frac{n+2}{n-2}} \end{array} \right\} \cdot [\partial_{\lambda_1} V_1] \right)
\]

\[
= \int_{B_{\xi_1}(\rho_n)} \left\{ -\frac{n+2}{n-2} \cdot V_1^{\frac{1}{n-2}} \cdot V_2 + O(1) \cdot V_2^{\frac{n+2}{n-2}} \right\} \cdot [\partial_{\lambda_1} V_1] \quad (V_1 = V_{\lambda_1}, \xi_1, \quad V_2 = V_{\lambda_2}, \xi_2)
\]

\[
= + \frac{n+2}{2} \cdot \int_{B_{\xi_1}(\rho_n)} V_1^{\frac{n-2}{n-2}} \cdot \left( \frac{n-2}{\gamma^{n-2}} \cdot [1 + o(1)] \right) \cdot \left\{ \begin{array}{c}
\frac{\lambda_1^{\frac{n-4}{2}}}{\gamma^{n-2}} \cdot \left( \frac{\lambda_1^2 - |y - \xi_1|^2}{\gamma^2} \right) \end{array} \right\}
\]

\[
+ O(1) \cdot \int_{B_{\xi_1}(\rho_n)} V_2^{\frac{n+2}{n-2}} \cdot [\partial_{\lambda_1} V_1]
\]

\[
= \omega_n \cdot \frac{n+2}{2} \cdot \frac{\lambda_1^{\frac{n-2}{2}} \cdot \lambda_2^{\frac{n-2}{2}}}{\gamma^{n-2}} \cdot 1 \cdot \int_{0}^{\rho_n} \left( \begin{array}{c}
\frac{\lambda_1}{\gamma^2 + r^2} \cdot \left( \frac{\lambda_1^2 - r^2}{(\lambda_1^2 + r^2)^\frac{2}{n}} \right)
\end{array} \right) \cdot r^{n-1} \cdot dr \cdot [1 + o(1)]
\]

\[
\int_{0}^{\rho_n} \left( \begin{array}{c}
\frac{\lambda_1}{\gamma^2 + r^2} \cdot \left( \frac{\lambda_1^2 - r^2}{(\lambda_1^2 + r^2)^\frac{2}{n}} \right)
\end{array} \right) \cdot r^{n-1} \cdot dr
\]

\[
= \int_{0}^{\frac{\pi}{2}} \left( \begin{array}{c}
[\sin \theta]^n \cdot [\cos \theta]^3 - [\sin \theta]^{n+1} \cdot [\cos \theta]
\end{array} \right) d\theta
\]

\[
= \int_{0}^{\frac{\pi}{2}} \left( \begin{array}{c}
[\sin \theta]^n \cdot [\cos \theta] - 2[\sin \theta]^{n+1} \cdot [\cos \theta]
\end{array} \right) d\theta
\]

\[
= [1 + o(1)] \cdot \int_{0}^{\frac{\pi}{2}} \left\{ [\sin \theta]^{n-1} - 2[\sin \theta]^{n+1} \right\} d[\sin \theta] \quad \left( \frac{\rho_n}{\lambda_1} \gg 1 \right)
\]

\[
= [1 + o(1)] \cdot \left( \frac{1}{n} - \frac{2}{n+2} \right) = [1 + o(1)] \cdot (-1) \cdot \frac{n-2}{n(n+2)}
\]

\[
\uparrow \text{-ve}.
\]
It follows that

\[ (A.1.18) \quad \int_{B_{\xi_{\rho_\mu}}(\rho_\mu)} \mathbb{T} [\partial_{\lambda_1} V_1] = - \left( \omega_n \cdot \frac{n - 2}{2n} \cdot \frac{\lambda_1^{\frac{n-2}{2}} \cdot \lambda_2^{\frac{n-2}{2}}}{\gamma^{n-2}} \cdot \frac{1}{\lambda_1} \cdot [1 + o(1)] + \mathbf{II} \right). \]

Here \( o(1) \rightarrow 0 \) as \( D \rightarrow \infty \) and

\[ (A.1.19) \quad \frac{\rho_\mu}{\lambda_1} \rightarrow \infty. \]

In (A.1.18), we recognize the leading term in (2.23) after multiplying by the number \( n(n - 2) \).

As for the remainder, we utilize (A.1.8), (A.1.15), and continue with

\[ (A.1.20) \]

\[ |\mathbf{II}| \leq C \cdot \frac{1}{\lambda_1} \cdot \int_{B_{\xi_{\rho_\mu}}(\rho_\mu)} V_2^{\frac{n+2}{n-2}} \cdot V_1 \quad \left( \left| \frac{\partial V_1}{\partial \lambda_1} \right| \leq \frac{n - 2}{2} \cdot \frac{1}{\lambda_1} \cdot V_1 \right) \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_1^{\frac{n+2}{n-2}}} \cdot \frac{1}{d^{n+2}} \cdot [1 + o(1)] \cdot \int_0^{\rho_\mu} \left( \frac{\lambda_1}{\lambda_1^2 + r^2} \right)^{\frac{n-2}{2}} \cdot r^{n-1} \cdot dr. \]

Let \( r = \lambda_1 \cdot \tan \theta \). We come to

\[ (A.1.21) \]

\[ \int_0^{\rho_\mu} \left( \frac{\lambda_1}{\lambda_1^2 + r^2} \right)^{\frac{n-2}{2}} \cdot r^{n-1} \cdot dr \]

\[ = \int_0^{\arctan \left( \frac{\rho_\mu}{\lambda_1} \right)} \frac{\lambda_1^{\frac{n-2}{2}} \cdot \lambda_1 \cdot \lambda_1^{n-1}}{\lambda_1^{n-2}} \cdot \left[ \sin \theta \right]^{n-1} \cdot \left[ \cos \theta \right]^{-1} \cdot \left[ \cos \theta \right]^{n-2} d \theta \]

\[ \leq C \cdot \lambda_1^{\frac{n+2}{2}} \cdot \int_0^{\arctan \left( \frac{\rho_\mu}{\lambda_1} \right)} \frac{d \left[ \cos \theta \right]}{\left[ \cos \theta \right]^3} \]

\[ \leq \lambda_1^{\frac{n+2}{2}} \cdot \left| \frac{1}{\left[ \cos \theta \right]^2} \right|_{\arctan \left( \frac{\rho_\mu}{\lambda_1} \right)} \]

\[ \leq C \cdot \lambda_1^{\frac{n+2}{2}} \cdot \left| \frac{1}{\cos^2 \left( \arctan \left( \frac{\rho_\mu}{\lambda_1} \right) \right)} - 1 \right| \cdot \left| \arctan \left( \frac{\rho_\mu}{\lambda_1} \right) \approx \frac{\pi}{2} \right| \]

\[ \leq C \cdot \lambda_1^{\frac{n+2}{2}} \cdot \left( \tan \left( \arctan \left( \frac{\rho_\mu}{\lambda_1} \right) \right) \right)^2 \left| \sin \left( \arctan \left( \frac{\rho_\mu}{\lambda_1} \right) \right) \right| \approx 1 \]

\[ \leq C \cdot \lambda_1^{\frac{n+2}{2}} \cdot \rho_\mu^2 \]
\[ |\mathbf{II}| = O(\mu^2) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{D^n} \quad \text{[recall that } \rho_\mu = \mu \cdot |\xi_1 - \xi_2| \text{].} \]

Here we apply (2.14).
In a similar way, we consider the integral on the other ball.

\[ \int_{B_{R_2}(\rho \mu)} \mathcal{I} \cdot [\partial_{\lambda_1} V_1] \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \int_{B_{R_2}(\rho \mu)} \frac{dx}{\lambda_2^{n-2}} \cdot V_2 \cdot V_1 + II' \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{\lambda_2^{n-2}} \cdot \frac{1}{d^{n-2}} \right)^2 \cdot \int_0^{\rho \mu} \left( \frac{\lambda_2}{\lambda_2^2 + r^2} \right)^2 r^{n-1} \cdot dr + II' \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{d^{n-2}} \right)^2 \cdot \int_0^{\arctan \left( \frac{\rho \mu}{\lambda_2} \right)} \frac{1}{\lambda_2^2} \cdot \lambda_2^{n-1} \cdot \frac{1}{\lambda_2^4} \cdot \frac{1}{\cos \theta} \cdot \frac{1}{\cos \theta^2} \cdot d \theta + II' \]

\[ \leq - C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{d^{n-2}} \right)^2 \cdot \int_0^{\arctan \left( \frac{\rho \mu}{\lambda_2} \right)} \frac{1}{\cos \theta} \cdot \frac{1}{\cos \theta^{n-3}} \cdot d \cos \theta + II' \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{d^{n-2}} \right)^2 \cdot \left[ \frac{1}{\cos \arctan \left( \frac{\rho \mu}{\lambda_2} \right)} - 1 \right] + II' \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{d^{n-2}} \right)^2 \cdot \left[ \frac{\sin \arctan \left( \frac{\rho \mu}{\lambda_2} \right)}{\cos \arctan \left( \frac{\rho \mu}{\lambda_2} \right)} \right] + II' \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{d^{n-2}} \right)^2 \cdot \left[ \tan \arctan \left( \frac{\rho \mu}{\lambda_2} \right) \right] + II' \]

\[ \leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{d^{n-2}} \right)^2 \cdot \left( \frac{\rho \mu}{\lambda_2} \right)^{n-4} + II' \]

\[ \leq O(\mu^{n-4}) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{d^n} + II' \]
\[ \leq o(1) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{D^n} + II' \quad \quad [o(1) \to 0^+ \text{ as } D \to \infty]. \]

In (A.1.22),
\begin{equation}
(A.1.23)
\begin{align*}
II' &= O(1) \cdot \int_{B_{\xi_2} (\rho_\mu)} V_1^{\frac{n+2}{n-2}} \cdot [\partial_{\lambda_1} V_1] \leq C \cdot \int_{B_{\xi_2} (\rho_\mu)} V_1^{\frac{n+2}{n-2}} \cdot \frac{V_1}{\lambda_1} \quad \text{[via (A.1.15)]} \\
\Rightarrow \quad |II'| &\leq C(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{\lambda_2} \cdot \frac{1}{d^{n-2}} \right)^{\frac{2n}{n-2}} \cdot \rho_\mu^n = C(n) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{d^{2n}} \cdot \left( \frac{\rho_\mu}{\lambda_2} \right)^n \\
&= O(\mu^n) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{d^n} = o(1) \cdot \frac{1}{\lambda_1} \cdot \frac{1}{D^n}.
\end{align*}
\end{equation}

**Outside** \( B_{\xi_1} (\rho_\mu) \cup B_{\xi_2} (\rho_\mu) \). Recall the following inequalities.
\begin{equation}
(A.1.24) \quad \left| (a + b)^{\frac{n+2}{n-2}} - \left( a^{\frac{n+2}{n-2}} + b^{\frac{n+2}{n-2}} \right) \right| \leq C(n) \cdot \left[ a^{\frac{1}{n-2}} + b^{\frac{1}{n-2}} \right] \cdot \min \{ a, b \}.
\end{equation}

and
\begin{equation}
(A.1.25) \quad \left| (a + b)^\alpha - \left( a^{\alpha + \frac{2}{\alpha - 2}} \cdot a^{\alpha - 2} \cdot b \right) \right| \leq C(n, \alpha) \cdot \left[ b^{\alpha} + a^{\alpha - 2} \cdot \min \{ a^2, b^2 \} \right].
\end{equation}

\begin{equation}
(A.1.26) \quad M \geq 2 \quad \Rightarrow \quad \left| (a + b)^M - (a^M + b^M) \right| \leq C_M \cdot \left( a^{M-1} \cdot b + b^{M-1} \cdot a \right).
\end{equation}

Here \( a \) and \( b \) are positive numbers, and \( \alpha \) is a fixed positive number. See (1.1) and (1.2) in [29]. Cf. also [1] and [41]. We continue with
\begin{equation}
(A.1.27)
\begin{align*}
&\left| \int_{\mathbb{R}^n \setminus \{ B_{\xi_1} (\rho_\mu) \cup B_{\xi_2} (\rho_\mu) \}} \left( V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2} \right)^{\frac{n+2}{n-2}} - \left( V_{\lambda_1, \xi_1}^{\frac{n+2}{n-2}} + V_{\lambda_2, \xi_2}^{\frac{n+2}{n-2}} \right) \cdot [\partial_{\lambda_1} V_1] \right| \\
&\leq \int_{\{ \mathbb{R}^n \setminus \{ B_{\xi_1} (\rho_\mu) \cup B_{\xi_2} (\rho_\mu) \} \} \cap \{ V_1 > V_2 \}} \left( V_1^{\frac{4}{n-2}} + V_2^{\frac{4}{n-2}} \right) \cdot V_2 \cdot \frac{V_1}{\lambda_1} + \quad \text{[via (A.1.15)]} \\
&\quad + \int_{\{ \mathbb{R}^n \setminus \{ B_{\xi_1} (\rho_\mu) \cup B_{\xi_2} (\rho_\mu) \} \} \cap \{ V_2 > V_1 \}} \left( V_1^{\frac{4}{n-2}} + V_2^{\frac{4}{n-2}} \right) \cdot V_1 \cdot \frac{V_1}{\lambda_1} \\
&\leq C(n) \cdot \frac{1}{\lambda} \int_{\rho_\mu}^\infty \left( \frac{\lambda}{\lambda^2 + r^2} \right)^n r^{n-1} \cdot dr
\end{align*}
\end{equation}
\[
\begin{align*}
C(n) \cdot \frac{1}{\lambda} \int_{\arctan\left(\frac{\omega}{\lambda}\right)}^{\pi} \frac{\lambda^n \cdot \lambda^{n-1} \cdot \tan \theta \cdot \left[ \cos \theta \right]^{n-1} \cdot \left[ \cos \theta \right]^{2(n-1)}}{\lambda^{2n}} d\theta \\
\leq C(n) \cdot \frac{1}{\lambda} \left[ - \int_{\arctan\left(\frac{\omega}{\lambda}\right)}^{\pi} \left[ \cos \theta \right]^{n-1} d\cos \theta \right] \leq C(n) \cdot \frac{1}{\lambda} \left[ - \left[ \cos \theta \right]^{n} \right]_{\arctan\left(\frac{\omega}{\lambda}\right)}^{\pi} \\
\approx C(n) \cdot \frac{1}{\lambda} \cdot \frac{1}{\left[ \tan \theta \right]^{n}} \leq C(n) \cdot \frac{1}{\lambda} \cdot \frac{1}{\left(\frac{\omega}{\lambda}\right)^{n}} = C(n) \cdot \frac{1}{\lambda} \cdot \frac{1}{d^{n}} \cdot \frac{1}{\mu^{n}} \\
\leq o(1) \cdot \frac{1}{\lambda} \cdot \frac{1}{D^{n-o(1)}}.
\end{align*}
\]

One is led to
\[
(A.1.28) \quad D^{o(1)} \cdot \mu^{n} \rightarrow \infty \quad \implies \quad \frac{1}{D^{o(1)}} \cdot \frac{1}{\mu^{n}} \rightarrow 0.
\]

Hence we conclude that
\[
(A.1.29) \quad \int_{\mathbb{R}^n} \left\{ \left( V^{n+2}_{\lambda_1, \xi_1} + V^{n+2}_{\lambda_2, \xi_2} \right) - \left( V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2} \right)^{n+2} \right\} \cdot \left[ \partial_{\lambda_1} V_{1} \right] \\
= - \left( \omega_n \cdot \frac{n-2}{2n} \right) \cdot \frac{\lambda_1^{n-2} \cdot \lambda_2^{n-2}}{\gamma^{n-2}} \cdot \frac{1}{\lambda_1} \cdot \left[ 1 + o(1) \right] + \frac{1}{\lambda_1} \cdot O\left( \frac{1}{D^{n-o(1)}} \right).
\]

Here \( o(1) \rightarrow 0 \) as \( D \rightarrow \infty \), using the property of \( \mu \) described in (A.1.3). Note that
\[
\frac{\lambda_1^{n-2} \cdot \lambda_2^{n-2}}{\gamma^{n-2}} = \frac{1}{D^{n-2}} \quad [\text{cf. (2.13)}].
\]
Extracting the leading term in $\xi$-derivatives.

\[(A.1.30) \quad \frac{\partial V_{\lambda_1, \xi_1}}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} \left[ \left( \frac{\lambda}{\lambda_1^2 + |y - \xi_1|^2} \right)^{\frac{\lambda_1^2}{2}} \right] \]

\[= - \frac{n-2}{2} \cdot \lambda_1^{\frac{n-2}{2}} \cdot \frac{2(\xi_1 - y_1)}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n}{2}}} \]

\[\implies \left| \frac{\partial V_{\lambda_1, \xi_1}}{\partial \xi_1} \right| = \frac{n-2}{2} \cdot \lambda_1^{\frac{n-2}{2}} \cdot \frac{1}{\lambda_1} \cdot \frac{2 \lambda_1 \cdot |\xi_1 - y_1|}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n}{2}}} \leq \frac{n-2}{2} \cdot \frac{1}{\lambda_1} \cdot V_{\lambda_1, \xi_1} . \]

We apply the same partition as in (A.1.5), and also (A.1.13) for the definition of $I$. Following (A.1.16) and using (A.1.30), we have

\[(A.1.31) \quad \int_{B_{\xi_1}(\rho_\mu)} I \cdot [\partial_{\xi_1} V_1] = + \frac{n+2}{2} \cdot \int_{B_{\xi_1}(\rho_\mu)} V_1^{\frac{n}{n-2}} \cdot V_2 \cdot \left\{ \lambda_1^{\frac{n-2}{2}} \cdot \frac{2(\xi_1 - y_1)}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n}{2}}} \right\} + \text{III} . \]

\text{III} \ is \ estimated \ as \ in \ (A.1.21). \ Recall \ (A.1.14) \ and \ (A.1.15). \ In \ particular,

\[I = - I . \]

Consider the main term in (A.1.31):

\[(A.1.32) \quad + \frac{n+2}{2} \cdot \int_{B_{\xi_1}(\rho_\mu)} V_1^{\frac{n}{n-2}} \cdot V_2 \cdot \left\{ \lambda_1^{\frac{n-2}{2}} \cdot \frac{2(\xi_1 - y_1)}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n}{2}}} \right\} \]

\[= + \frac{n+2}{2} \cdot \int_{B_{\xi_1}(\rho_\mu)} V_1^{\frac{n}{n-2}} \cdot \left( \frac{1}{\lambda_1^{\frac{n-2}{2}}} \cdot \frac{1}{d^{n-2}} \right) \cdot \left\{ 1 - \frac{n-2}{2} \cdot t \cdot [1 + o(1)] \right\} \times \]

\[\times \left\{ \lambda_1^{\frac{n-2}{2}} \cdot \frac{2(\xi_1 - y_1)}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n}{2}}} \right\} \]

\[\text{[via } (A.1.8) \text{]} \]

\[\text{(} \downarrow \text{ take note of the sign) } \]

\[= - \frac{n+2}{2} \cdot \frac{n-2}{2} \cdot \frac{1}{d^{n-2}} \cdot \int_{B_{\xi_1}(\rho_\mu)} V_1^{\frac{n}{n-2}} \cdot t \cdot \left\{ \frac{2(\xi_1 - y_1)}{(\lambda_1^2 + |y - \xi_1|^2)^{\frac{n}{2}}} \right\} \cdot [1 + o(1)] \]

\[\text{[see } (A.1.33) \text{ below] .} \]

Here we make use of the “symmetry”
\[ \int_{B_{\epsilon_1}(\rho_\mu)} V_1^{n-2} \cdot \frac{(y_1 - \xi_1)}{\left(\lambda_1^2 + |y - \xi_1|^2\right)^\frac{3}{2}} \, dy = \int_{B_{\rho}(\rho_\mu)} \left(\frac{\lambda_1}{\lambda_1^2 + |\bar{y}|^2}\right)^2 \cdot \frac{1}{\left(\lambda_1^2 + |\bar{y}|^2\right)^\frac{3}{2}} \cdot \bar{y}_1 \, d\bar{y} = 0, \]

where \( \bar{y} = y - \xi_1 \).  

Recall that

\[ \text{(A.1.33)} \]

\[ \text{(A.1.34)} \]

\[
\mathbf{t} = \frac{1}{d_1^2} \cdot \left(\frac{\lambda_2}{\lambda_1}\right) + \frac{1}{d_2^2} \cdot \frac{|y - \xi_1|^2}{\lambda_1 \cdot \lambda_2} + \frac{1}{d_2^2} \cdot \frac{2(y - \xi_1) \cdot (\xi_1 - \xi_2)}{\lambda_1 \cdot \lambda_2}.
\]

Likewise, the integrations involving the first two terms in (A.1.34) also yield nothing. Finally, we come down to the integral

\[
\int_{B_{\epsilon_1}(\rho_\mu)} V_1^{n-2} \cdot \frac{1}{\left(\lambda_1^2 + |y - \xi_1|^2\right)^\frac{3}{2}} \cdot (y_1 - \xi_1) \cdot (y - \xi_1)_j = 0 \quad \text{for} \quad j \neq 1.
\]

Via symmetry again,

\[
\int_{B_{\epsilon_1}(\rho_\mu)} V_1^{n-2} \cdot \frac{1}{\left(\lambda_1^2 + |y - \xi_1|^2\right)^\frac{3}{2}} \cdot (y_1 - \xi_1)_j = 0 \quad \text{for} \quad j \neq 1.
\]

Thus we are left with

\[ \text{(A.1.35)} \]

\[
\frac{n + 2 - n - 2}{d_1^2} \cdot \frac{1}{\lambda_1 \cdot \lambda_2} \cdot \left[ \int_{B_{\epsilon_1}(\rho_\mu)} V_1^{n-2} \cdot \frac{1}{\left(\lambda_1^2 + |y - \xi_1|^2\right)^\frac{3}{2}} \cdot (y_1 - \xi_1)^2 \, dy \right] \cdot (\xi_1_1 - \xi_2,)
\]

\[
= C(n) \cdot [1 + o(1)] \cdot \frac{1}{d_1^2} \cdot \frac{1}{\lambda_1 \cdot \lambda_2} \cdot (\xi_1_1 - \xi_2) = O\left(\frac{1}{\lambda_1} \cdot \frac{1}{D^{n-1}}\right),
\]

where

\[
C(n) = (n + 2)(n - 2) \cdot \int_{\mathbb{R}^n} \frac{Y_1^2}{(1 + |Y|^2)^\frac{3}{2} + 2} \, dY > 0,
\]

via the changes of variables \( \bar{y} = y - \xi_1 \) and \( Y = \frac{\bar{y}}{\lambda_1} \).

Here \( o(1) \to 0 \) as \( D \to \infty \) and \( \mu \cdot D \to \infty \).
We estimate the integral on the other ball via (A.1.36)

\[
\left| \int_{B_{\xi_2}(\rho_\mu)} V_2^{\frac{n}{n-2}} \cdot V_1 \cdot [\partial_{\xi_1} V_1] \right|
\]

\[
= \frac{n-2}{2} \cdot \left| \int_{B_{\xi_2}(\rho_\mu)} V_2^{\frac{n}{n-2}} \cdot \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi_1|^2} \right)^{\frac{n-2}{2}} \cdot \left\{ \frac{1}{\lambda_1} \cdot \frac{2 \lambda_1^2}{\left( \lambda_1^2 + |y - \xi_1|^2 \right)^{\frac{n}{2}}} \right\} \cdot (\xi_{11} - y_1) \right|
\]

\[
= (n - 2) \cdot \frac{1}{\lambda_1} \cdot \left| \int_{B_{\xi_2}(\rho_\mu)} V_2^{\frac{n}{n-2}} \cdot \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi_1|^2} \right)^{\frac{n-2}{2}} \cdot \left[ \left( \frac{\lambda_1}{\lambda_1^2 + |y - \xi_1|^2} \right)^{\frac{n-2}{2}} \right] \cdot (\xi_{11} - y_1) \right|
\]

\[
= C^+(n) \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{\lambda_2^{\frac{n-2}{2}}} \cdot \frac{1}{d^{n-2}} \right) \cdot \left( \frac{1}{\lambda_2^2} \cdot \frac{1}{d^n} \right) \cdot [1 + o(1)] \times \left| \int_{B_{\xi_2}(\rho_\mu)} \left( \frac{\lambda_2}{\lambda_2^2 + |y - \xi_2|^2} \right)^2 \cdot [ (\xi_{11} - \xi_{21}) + (\xi_{21} - y_1)] \right| + \text{III}'
\]

[as in (A.1.22)]

\[
\leq C \cdot \frac{1}{\lambda_1} \cdot \left( \frac{1}{\lambda_2^{\frac{n-2}{2}}} \cdot \frac{1}{d^{n-2}} \right) \cdot \left( \frac{1}{\lambda_2^2} \cdot \frac{1}{d^n} \right) \cdot [1 + o(1)] \cdot \left| \int_0^{\rho_\mu} \left( \frac{\lambda_2}{\lambda_2^2 + r^2} \right)^2 \cdot r^{n-1} dr \right| \cdot |\xi_{11} - \xi_{21}| + \text{III}'
\]

[cf. (A.1.22)]

\[
\leq C \cdot \frac{1}{\lambda_1} \cdot \frac{1}{\lambda_2} \cdot \frac{1}{d^{n+2}} \cdot [1 + o(1)] \times |\xi_{11} - \xi_{21}| + \text{III}'
\]

\[
= C \cdot \frac{1}{\lambda_1} \cdot \frac{1}{d^{n+1}} + \text{III}'.
\]

The estimates of III' and the remaining parts are similar to the case with derivative on \(\lambda_1\), using (A.1.30).
§ A.2. Proof of (4.4) & (4.5).

We modify the argument presented in the proof of Proposition A.5.27 in the e-Appendix of [28]. We begin with (3.31). Refer to the notations in (3.37).

\[ I_R(z_\sigma) = I(z_\sigma + w_{z_\sigma}) \]

\[ = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} \langle \nabla (z_\sigma + w_{z_\sigma}), \nabla (z_\sigma + w_{z_\sigma}) \rangle - \frac{1}{2} \cdot (n - 2)^2 \cdot [z + w_{z_\sigma}]^\frac{2n}{n-2} \right\} \]

\[ + \left[ - \frac{n - 2}{2n} \right] \cdot \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot [z_\sigma + w_{z_\sigma}]^\frac{2n}{n-2} \]

\[ \implies D_{k_1} I_R(z_\sigma) = \int_{\mathbb{R}^n} \left\{ \langle \nabla [z_\sigma + w_{z_\sigma}], \nabla (D_{k_1} z_\sigma + D_{k_1} w_{z_\sigma}) \rangle \right\} \]

\[ - n (n - 2) \cdot (D_{k_1} z_\sigma + D_{k_1} w_{z_\sigma}) \cdot [z_\sigma + w_{z_\sigma}]^\frac{n+2}{n-2} \]

\[ + \left[ - \frac{n - 2}{2n} \right] \cdot \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot (D_{k_1} z_\sigma + D_{k_1} w_{z_\sigma}) \cdot [z_\sigma + w_{z_\sigma}]^\frac{n+2}{n-2} \]

\[ \implies D_{k_1} I_R(z_\sigma) = \int_{\mathbb{R}^n} \left\{ \langle \nabla z_\sigma, \nabla D_{k_1} z_\sigma \rangle - n (n - 2)(D_{k_1} z_\sigma) z_\sigma^\frac{n+2}{n-2} \right\} \]

\[ \leftarrow ( = I'_o(z_\sigma) [D_{k_1} z_\sigma] ) \rightarrow \]

\[ + \int_{\mathbb{R}^n} \langle \nabla z_\sigma, \nabla D_{k_1} w_{z_\sigma} \rangle + \int_{\mathbb{R}^n} \langle \nabla w_{z_\sigma}, \nabla D_{k_1} z_\sigma \rangle \]

\[ + \int_{\mathbb{R}^n} \langle \nabla w_{z_\sigma}, \nabla D_{k_1} w_{z_\sigma} \rangle \]

\[ - n(n - 2) \int_{\mathbb{R}^n} [D_{k_1} z_\sigma] \cdot \left[ (z_\sigma + w_{z_\sigma})^\frac{n+2}{n-2} - z_\sigma^\frac{n+2}{n-2} \right] \]

\[ - n (n - 2) \int_{\mathbb{R}^n} [D_{k_1} w_{z_\sigma}] \cdot [z_\sigma + w_{z_\sigma}]^\frac{n+2}{n-2} \]

\[ - \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot [D_{k_1} z_\sigma] \cdot z_\sigma^\frac{n+2}{n-2} \]

\[ \leftarrow (\text{key term} = D_{k_1} [G(z_\sigma)]) \rightarrow \]

\[ - \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot [D_{k_1} z_\sigma] \cdot \left[ (z_\sigma + w_{z_\sigma})^\frac{n+2}{n-2} - z_\sigma^\frac{n+2}{n-2} \right] \]
\[-\int_{\mathbb{R}^n} (\tilde{c}_{\eta} \cdot H) \cdot [D_{k_t} w_{\mathbf{z}_\sigma}] \cdot [\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma}]^\frac{n+2}{n-2}\]

\[= D_{k_t} [G(\mathbf{z}_\sigma)] + I'_o(\mathbf{z}_\sigma) [D_{k_t} \mathbf{z}_\sigma] \]

\[+ \int_{\mathbb{R}^n} \left\{ \langle \nabla w_{\mathbf{z}_\sigma}, \nabla D_{k_t} \mathbf{z}_\sigma \rangle - n (n+2) \cdot \mathbf{z}_\sigma^\frac{n+2}{n-2} [D_{k_t} \mathbf{z}_\sigma] \cdot w_{\mathbf{z}_\sigma} \right\} \]

\[\left(\leftarrow \quad = (I''_o(\mathbf{z}_\sigma) [D_{k_t} \mathbf{z}_\sigma] w_{\mathbf{z}_\sigma}) \quad \rightarrow \right)\]

\[- n (n-2) \int_{\mathbb{R}^n} [D_{k_t} w_{\mathbf{z}_\sigma}] \cdot (\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma})^\frac{n+2}{n-2} - \mathbf{z}_\sigma^\frac{n+2}{n-2} - \left(\frac{n+2}{n-2}\right) \mathbf{z}_\sigma^\frac{1}{n-2} w_{\mathbf{z}_\sigma} \]

\[-n(n-2) \int_{\mathbb{R}^n} [D_{k_t} w_{\mathbf{z}_\sigma}] (\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma})^\frac{n+2}{n-2} - \int_{\mathbb{R}^n} \langle [\Delta \mathbf{z}_\sigma] \cdot D_{k_t} w_{\mathbf{z}_\sigma} \]

\[+ n (n+2) \cdot \int_{\mathbb{R}^n} \mathbf{z}_\sigma^\frac{1}{n-2} [D_{k_t} w_{\mathbf{z}_\sigma}] \cdot w_{\mathbf{z}_\sigma} \quad \left\{ \uparrow \Delta \mathbf{z}_\sigma = -n (n-2) \left[ V_1^\frac{n+2}{n-2} + V_2^\frac{n+2}{n-2} \right] \right\} \]

\[+ \int_{\mathbb{R}^n} \left\{ \langle \nabla w_{\mathbf{z}_\sigma}, \nabla D_{k_t} w_{\mathbf{z}_\sigma} \rangle - n (n+2) \cdot \mathbf{z}_\sigma^\frac{n+2}{n-2} [D_{k_t} w_{\mathbf{z}_\sigma}] \cdot w_{\mathbf{z}_\sigma} \right\} \]

\[\left(\leftarrow \quad = (I''_o(\mathbf{z}_\sigma) [D_{k_t} w_{\mathbf{z}_\sigma}] w_{\mathbf{z}_\sigma}) \quad \rightarrow \right)\]

\[- \int_{\mathbb{R}^n} (\tilde{c}_{\eta} \cdot H) \cdot [D_{k_t} \mathbf{z}_\sigma] \cdot (\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma})^\frac{n+2}{n-2} - \mathbf{z}_\sigma^\frac{n+2}{n-2} \]

\[-\int_{\mathbb{R}^n} (\tilde{c}_{\eta} \cdot H) \cdot [D_{k_t} w_{\mathbf{z}_\sigma}] \cdot [\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma}]^\frac{n+2}{n-2} \]

\[= [D_{k_t} G(\mathbf{z}_\sigma)] + I'_o(\mathbf{z}_\sigma) [D_{k_t} \mathbf{z}_\sigma] \quad \leftarrow \text{refer to (2.23 and 2.25)} \]

\[+ (I''_o(\mathbf{z}_\sigma) [D_{k_t} \mathbf{z}_\sigma] w_{\mathbf{z}_\sigma}) + (I''_o(\mathbf{z}_\sigma) [D_{k_t} w_{\mathbf{z}_\sigma}] w_{\mathbf{z}_\sigma}) + \]

\[+ I + II + III + IV + V, \]

where

\[I = -n (n-2) \int_{\mathbb{R}^n} [D_{k_t} \mathbf{z}_\sigma] \cdot (\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma})^\frac{n+2}{n-2} - \mathbf{z}_\sigma^\frac{n+2}{n-2} - \left(\frac{n+2}{n-2}\right) \mathbf{z}_\sigma^\frac{1}{n-2} w_{\mathbf{z}_\sigma}, \]

\[\text{and} \]

\[II = -n (n-2) \int_{\mathbb{R}^n} [D_{k_t} w_{\mathbf{z}_\sigma}] \cdot (\mathbf{z}_\sigma + w_{\mathbf{z}_\sigma})^\frac{n+2}{n-2} - \mathbf{z}_\sigma^\frac{n+2}{n-2} - \left(\frac{n+2}{n-2}\right) \mathbf{z}_\sigma^\frac{1}{n-2} w_{\mathbf{z}_\sigma}, \]

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In the following we specify

\[
III = - \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot [D_{k_{\ell}} z_{\sigma}] \cdot \left( [z_{\sigma} + w_{z_{\sigma}}]_{n/2}^{n+2} - z_{\sigma} \right),
\]

\[
IV = - \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) \cdot [D_{k_{\ell}} w_{z_{\sigma}}] \cdot [z_{\sigma} + w_{z_{\sigma}}]_{n/2}^{n+2},
\]

\[
V = - n (n + 2) \int_{\mathbb{R}^n} [D_{k_{\ell}} w_{z_{\sigma}}] \cdot \left( z_{\sigma}^{n+2} - \left( V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2} \right) \right).
\]

For the term

\[
[D_{k_{\ell}} G(z_{\sigma})] = - \frac{n - 2}{2 n} \cdot D_{k_{\ell}} \int_{\mathbb{R}^n} (\tilde{c}_n \cdot H) z_{\sigma}^{n/2},
\]

see (4.6), (4.12) and (4.14). As for the term,

\[
I_o(z_{\sigma}) [D_{k_{\ell}} z_{\sigma}] = n (n - 2) \int_{\mathbb{R}^n} \left( (V_{\lambda_1}^{n/2} + V_{\lambda_2}^{n/2}) - (V_{\lambda_1, \xi_1} + V_{\lambda_2, \xi_2})^{n+2} \right) [D_{k_{\ell}} z_{\sigma}],
\]

refer to (2.23) and (2.25).

In the following we take it that [see (3.45), and compare with (2.14)]

\[
\lambda^\ell \leq \frac{C_o}{D^{n-2}} \quad \text{and} \quad 2 \leq \ell < n - 2.
\]

Refer to (3.45) in Lemma 3.44. From Lemma A.5.5 in the e-Appendix [28], and Lemma 3.44, we have

\[
(A.2.1) \quad |(I_o''(z_{\sigma}) [D_{k_{\ell}} z_{\sigma}] w_{z_{\sigma}})| = O\left( \frac{1}{d^{2-o(1)}} \right) \cdot \max\left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\} \cdot \|w_{z_{\sigma}}\|_\nabla
\]

\[
\leq O\left( \frac{1}{d^{2-o} + 2} \right) \cdot \max\left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\}.
\]

Using the uniform bound on the operator \( I_o''(z_{\sigma}) \), one has

\[
(A.2.2) \quad |(I_o''(z_f) [w_{z_f}] (\lambda_k \cdot D_{k_{\ell}} w_{z_f})| \leq C \cdot \|w_{z_f}\|_\nabla \cdot \|\lambda_k \cdot \nabla w_{z_f}\|_\nabla \approx O\left( \frac{1}{d^{2-o} + 2} \right).
\]

In the following we specify \( n \geq 6 \). Argue as in the proof of Proposition A.5.27 in the e-Appendix of [28], we obtain

\[
|\lambda \cdot I| = - n (n - 2) \int_{\mathbb{R}^n} [\lambda \cdot D_{k_{\ell}} z_{\sigma}] \cdot \left( [z_{\sigma} + w_{z_{\sigma}}]_{n/2}^{n+2} - z_{\sigma} \right)_{n/2}^{n+2} - \left( \frac{n + 2}{n - 2} \right) z_{\sigma}^{n/2} \cdot w_{z_{\sigma}},
\]

(A.2.3) \quad \therefore \quad |\lambda \cdot I| \leq C \cdot \|w_{z_{\sigma}}\|_{\nabla}^{n+2} \approx O\left( \frac{1}{d^{n+2-o} + 2} \right).
\]

Recall that \( \lambda = \sqrt{\lambda_1 \cdot \lambda_2} \). Recall also (2.13). Likewise

\[
(A.2.4) \quad |\lambda \cdot I| \leq C \cdot \|w_{z_{\sigma}}\|_{\nabla}^{n+2} \cdot \|w_{z_{\sigma}}\|_{\nabla} \approx O\left( \frac{1}{d^{n+2} + 2} \right).
\]

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The following (highly coupled) terms are fine.

\[ (A.2.5) \quad |\mathbf{III}| \leq C \left( \int_{\mathbb{R}^n} |c_n \cdot H \frac{2n}{n+2} \cdot z_{\sigma}^2 \frac{\alpha}{\alpha+2} \cdot \|w_{z_s}\|_\nabla \right) \cdot \left( \max \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\} \right) \]

\[ + C \left( \int_{\mathbb{R}^n} |c_n \cdot H \frac{2n}{n+2} \cdot z_{\sigma}^2 \frac{\alpha}{\alpha+2} \cdot \|w_{z_s}\|_\nabla \right) \cdot \left( \max \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\} \right) \]

\[ \leq C \cdot \frac{1}{d^{\frac{n+2}{4} - o(1)}} \left( \max \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\} \right). \]

\[ (A.2.6) \quad |\mathbf{IV}| \leq C \cdot \left( \int_{\mathbb{R}^n} |c_n \cdot H \frac{2n}{n+2} \cdot z_{\sigma}^2 \frac{\alpha}{\alpha+2} \cdot \|\nabla w_{z_s}\|_\nabla \right) + C \cdot \|\nabla w_{z_s}\|_\nabla \cdot \|w_{z_s}\|_\nabla^2 \]

\[ \leq C \cdot \frac{1}{d^{\frac{n+2}{4} + o(1)}} \left( \max \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\} \right). \]

\[ (A.2.7) \quad |\mathbf{V}| \leq C \cdot \frac{1}{d^{\frac{n+2}{4} - o(1)}} \cdot \|\nabla w_{z_s}\|_\nabla \leq C \cdot \frac{1}{d^{\frac{n+2}{4} - o(1)}} \cdot \left( \max \left\{ \frac{1}{\lambda_1}, \frac{1}{\lambda_2} \right\} \right). \]

Moreover

\[ (A.2.8) \quad \frac{n+2}{2} + 2 > n - 2 \iff 10 > n \iff n = 6, 7, 8 & 9, \]

\[ (A.2.9) \quad \frac{n+2}{2} \cdot \frac{n+2}{n-2} > n - 1 \iff 10n > n^2 \iff n = 6, 7, 8 & 9. \]

Combining the estimates we obtain (4.4) and (4.5), where the condition 10 > n ≥ 6 is used.

\[ \square \]
§ A.3. Proof of the Reduction Lemma 4.7.

We largely follow [26].

Lemma A.3.1 (Reduction Lemma). In $\mathbb{R}^n$, $n \geq 3$, consider a homogeneous polynomial $Q_\ell$ of even degree $\ell \leq n - 1$. We have

\[(A.3.2) \quad \int_{\mathbb{R}^n} Q_\ell(y) \cdot \left(\frac{1}{1 + |y|^2}\right)^n \, dy = \frac{J_n}{\ell \cdot (\ell - 2) \cdots 2 \cdot 1} \cdot [\Delta_\ell^{(h_\ell)} Q_\ell] \quad (h_\ell = \ell/2),\]

where

\[(A.3.3) \quad J_n = \int_{\mathbb{R}^n} y_1^2 \cdots y_{h_\ell}^2 \cdot \left(\frac{1}{1 + |y|^2}\right)^n \, dy \quad [y = (y_1, \cdots, y_\ell, \cdots, y_n)].\]

Likewise,

\[(A.3.4) \quad \int_{\mathbb{R}^n} Q_\ell(y) \cdot \left(\frac{1}{1 + |y|^2}\right)^{n+1} \, dy = \frac{J_{n+1}}{\ell \cdot (\ell - 2) \cdots 2 \cdot 1} \cdot [\Delta_\ell^{(h_\ell)} Q_\ell],\]

where

\[(A.3.5) \quad J_{n+1} = \int_{\mathbb{R}^n} y_1^2 \cdots y_{h_\ell}^2 \cdot \left(\frac{1}{1 + |y|^2}\right)^{n+1} \, dy \quad (\implies J_{n+1} < J_n).\]

Here\[
\Delta_\ell^{(h_\ell)} Q_\ell(y) = \Delta_\ell (\cdots [\Delta_\ell [\Delta_\ell Q_\ell(y)]]).
\]

Proof. We observe that, as $\ell \leq n - 1$, the integrals in (A.3.2) and (A.3.3) are absolutely convergent. Keeping the notation $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, consider a typical term in $Q_\ell$:

\[(A.3.6) \quad y_{1}^{\alpha_1} \cdot y_{1}^{\alpha_2} \cdots y_{\ell}^{\alpha_\ell}, \quad \text{where } \alpha_j \geq 0 \quad \text{and } \sum_{j=1}^{n} \alpha_j = \ell \ (\leq n - 1).\]

If one of the indices (say, $\alpha_j$) is an odd natural number, via symmetry, we have

\[(A.3.7) \quad \int_{\mathbb{R}^n} \left[ y_{1}^{\alpha_1} \cdot y_{1}^{\alpha_2} \cdots y_{\ell}^{\alpha_\ell} \cdots y_{n}^{\alpha_n} \right] \cdot \left(\frac{1}{1 + |y|^2}\right)^n \, dy = 0 \quad (\alpha_j \text{ is odd}).\]

Direct calculation also shows that in this situation\[
\Delta_\ell^{(h_\ell)} \left[ y_{1}^{\alpha_1} \cdot y_{1}^{\alpha_2} \cdots y_{\ell}^{\alpha_\ell} \cdots y_{n}^{\alpha_n} \right] = 0 \quad (\alpha_j \text{ is odd}).
\]
Via the vanishing formula (A.3.7) and the reduction formula (A.3.9), we have

the simplest case that can happen to the integral in (A.3.2) is

We seek to reduce other even multi-index cases to that in (A.3.8). As

is even if each $\alpha_j$ ($1 \leq j \leq n$) is either an even natural number or zero. With respect to this, the simplest case that can happen to the integral in (A.3.2) is

\[
A.3.8 \quad J := \int_{\mathbb{R}^n} [y_1^{\alpha_1} \cdots y_n^{\alpha_n}] \cdot \left(\frac{1}{1 + r^2}\right)^n \, dy \quad (r = |y|).
\]

We seek to reduce other even multi-index cases to that in (A.3.8). As $\ell \leq n - 1$, we consider the following.

\[
y_1^{\alpha_1 + 2} \cdots y_n^{\alpha_n}, \quad \text{where} \quad \alpha = (k + 2, \alpha_2, \cdots, \alpha_{n-1}, 0) \quad \text{is even}.
\]

Here $k \geq 2$ is an even number. By using Fubini’s theorem and integration by parts formula, we obtain the following reduction formula.

\[
A.3.9 \quad \int_{\mathbb{R}^n} [y_1^{\alpha_1} \cdots y_n^{\alpha_n}] \cdot \left(\frac{1}{1 + r^2}\right)^n \, dy = (k + 1) \int_{\mathbb{R}^n} y_1^{k+2} \cdots y_n^{\alpha_n} \cdot \left(\frac{1}{1 + r^2}\right)^n \, dy
\]

for $2 \leq k \leq n - 3$.

See § A.3 below.

In view of (A.3.9), we introduce another notation. For an integer $m \geq 0$, define

\[
A.3.10 \quad m!_{-2} = 1 \quad \text{if} \quad m = 0 \quad \text{or} \quad 2; \quad m!_{-2} = 0 \quad \text{if} \quad m \quad \text{is odd};
\]

\[
m!_{-2} = (m - 1)(m - 3)(m - 5) \cdots 3 \cdot 1 \quad \text{if} \quad m \geq 4 \quad \text{is even}.
\]

Via the vanishing formula (A.3.7) and the reduction formula (A.3.9), we have

\[
A.3.11 \quad \int_{\mathbb{R}^n} [y_1^{\alpha_1} \cdots y_n^{\alpha_n}] \cdot \left(\frac{1}{1 + r^2}\right)^n \, dy = (\alpha_1)!_{-2} \times \cdots \times (\alpha_n)!_{-2} \cdot J.
\]

On the other side, calculation shows that

\[
A.3.12 \quad B := \Delta_{o}^{(h_{\ell})} \left[ y_1^2 \cdots y_n^2 \right] = \ell (\ell - 2)(\ell - 4) \cdots 2 \cdot 1.
\]

Claim. Let $\alpha_2, \cdots, \alpha_{n-1}$ be even natural numbers or zero, and

\[
A.3.13 \quad \ell = (k + 2) + \alpha_2 + \cdots + \alpha_{n-1}, \quad \text{where} \quad k \geq 2 \quad \text{is an even integer}.
\]

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Then
\[(A.3.14) \quad \Delta_o^{(h)} \{ y_{1_1}^{k+2} \cdot [ y_{1_2}^{o_2} \cdot \cdots y_{n-1_1}^{o_{n-1}} ] \} = (k + 1) \cdot \Delta_o^{(h)} \{ y_{1_1}^k \cdot y_{1_n}^2 \cdot [ y_{1_2}^{o_2} \cdot \cdots y_{n-1_1}^{o_{n-1}} ] \} . \]

Refer to §A.3.

Thus using \((A.3.14)\) repeatedly, we are led to
\[(A.3.15) \quad \Delta_o^{(h)} \{ y_{1_1}^{\alpha_1} \cdot \cdots y_{1_n}^{\alpha_n} \} = (\alpha_1)!_{-2} \times \cdots \times (\alpha_n)!_{-2} \cdot B . \]

Using the linearity of the operations, together with \((A.3.11)\) and \((A.3.15)\), we obtain
\[(A.3.16) \quad \int_{\mathbb{R}^n} Q_\ell(y) \cdot \left( \frac{1}{1 + |y|^2} \right)^n dy = \frac{J}{B} \cdot [ \Delta_o^{(h)} Q_\ell ] . \]

In particular, we establish \((A.3.2)\). The case for \( n + 1 \) is a direct modification of the above argument. \( \square \)

**Verification of \((A.3.9)\) and \((A.3.14)\)**

Refer to \((A.3.9)\) for the notation we use.

\[
\int_{\mathbb{R}^n} (\text{terms without } y_1 & y_n) \cdot y_1^{k+2} \cdot \left( \frac{1}{1 + r^2} \right)^n dy \quad \text{(absolute convergence)}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 & y_n) \times \quad \text{(Fubini’s Theorem)}
\]

\[
\times \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1^{k+2} \cdot \left( \frac{1}{1 + [ y_2^2 + \cdots + y_{n-1_1}^2 ] + y_1^2 + y_n^2 } \right)^n dy_1 dy_n \right] dy_2 \cdots dy_{n-1}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 & y_n) \times \n
\times \left[ \int_{0}^{2\pi} \int_{0}^{\rho^{k+2}} \left( \frac{1}{1 + [ y_2^2 + \cdots + y_{n-1_1}^2 ] + \rho^2 } \right)^n \cdot \rho d\theta d\rho \right] dy_2 \cdots dy_{n-1}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 & y_n) \times \n
\times \int_{0}^{2\pi} \rho^{k+3} (\sin^{k+2} \theta) \cdot \left( \frac{1}{1 + [ y_2^2 + \cdots + y_{n-1_1}^2 ] + \rho^2 } \right)^n \cdot d\theta d\rho \right] dy_2 \cdots dy_{n-1}
\]

\[
= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (\text{terms without } y_1 & y_n) \times
\]

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\[
\times \left[ \int_0^\infty \rho^{k+3} \left( \frac{1}{1 + [y_{i_2}^2 + \cdots + y_{i_{n-1}}^2] + \rho^2} \right)^n \cdot \left\{ \int_0^{2\pi} \sin^{k+2} \theta \, d\theta \right\} \, d\rho \right] dy_{i_2} \cdots dy_{i_{n-1}}.
\]

Here \((\text{terms without } y_{i_1} \text{ & } y_{i_n})\) is a polynomial on \(y_{i_2}, \ldots, y_{i_{n-1}}\), having sufficiently low degree so that the integral is absolutely convergent. Likewise,

\[
\int_\infty^- \cdots \int_\infty^- (\text{terms without } y_{i_1} \text{ & } y_{i_n}) \times
\]

\[
\left[ \int^-\infty \int_-\infty y_{i_1}^k y_{i_n}^2 \cdot \left( \frac{1}{1 + [y_{i_2}^2 + \cdots + y_{i_{n-1}}^2] + y_{i_1}^2 + y_{i_n}^2} \right)^n \right] dy_{i_1} \cdots dy_{i_n}
\] = \[ \int^-\infty \cdots \int^-\infty (\text{terms without } y_{i_1} \text{ & } y_{i_n}) \times
\]

\[
\left[ \int_0^\infty \rho^{k+3} \left( \frac{1}{1 + [y_{i_2}^2 + \cdots + y_{i_{n-1}}^2] + \rho^2} \right)^n \cdot \left\{ \int_0^{2\pi} (\sin^k \theta) (\cos^2 \theta) \, d\theta \right\} \, d\rho \right] dy_{i_2} \cdots dy_{i_{n-1}}.
\]

A direct calculation using integration by parts shows that

\[
\int_0^{2\pi} (\sin^{k+2} \theta) \, d\theta = -\int_0^{2\pi} (\sin^{k+1} \theta) \, d[\cos \theta] = (k + 1) \int_0^{2\pi} (\sin^k \theta) (\cos^2 \theta) \, d\theta.
\]

Hence we deduce (A.3.9). To show (A.3.14), let \([\text{refer to (A.3.14)}]\)

\[(A.3.17) \quad \ell = \alpha_2 + \cdots + \alpha_{n-1}.
\]

We demonstrate how to use induction on \(\ell\) to prove the assertion. Recall that

\((I)\) When \(\ell = 0\), the term specified in (A.3.14) is a constant. We have

\[
\Delta^{(k+2)} \partial y_{i_1}^k = (k + 2) (k + 1) \cdots 3 \cdot 2 \cdot 1;
\]

\[
\Delta_\partial \{ (k + 1) y_{i_1}^k y_{i_n}^2 \} = (k + 1) k (k - 1) y_{i_1}^{k-2} y_{i_n}^2 + 2 (k + 1) y_{i_1}^k;
\]

\[
\Delta^{(2)}_\partial \{ (k + 1) y_{i_1}^k y_{i_n}^2 \} = (k + 1) k (k - 1) (k - 2) (k - 3) y_{i_1}^{k-4} y_{i_n}^2 + 2 \times 2 (k + 1) k (k - 1) y_{i_1}^{k-2},
\]

\]
\[ \Delta_o^{(h \ell - 2)} \left\{ (k + 1) y_{l_1}^k y_{l_n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot y_{l_1}^2 y_{l_n}^2 + \\
+ 2 \left[ h \ell - 2 \right] (k + 1) k (k - 1) \cdots 5 \cdot y_{l_1}^4 ; \]
\[ \Delta_o^{(h \ell - 1)} \left\{ (k + 1) y_{l_1}^k y_{l_n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot [y_{l_1}^2 + y_{l_n}^2] + \\
+ 2 \left[ h \ell - 2 \right] (k + 1) k (k - 1) \cdots 5 \cdot 4 \cdot 3 y_{l_1}^2 , \]
\[ \Delta_o^{h \ell} \left\{ (k + 1) y_{l_1}^k y_{l_n}^2 \right\} = (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 \cdot 4 + \\
+ [ \ell - 4] (k + 1) k (k - 1) \cdots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\
= (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 [\ell - 4 + 4] \\
= (k + 2) (k + 1) k (k - 1) (k - 2) (k - 3) \cdots 3 \cdot 2 \cdot 1 \\
\text{(as } \ell = k + 2 \text{ in this case).} \]

Hence the case \( \ell = 0 \) is settled.

(II) As an induction hypothesis, suppose that

\[ (A.3.18) \ \Delta_o^{(h \ell)} \left\{ y_{l_1}^{k+2} \cdot [\cdots \text{degree } = \ell \cdots ] \right\} = \Delta_o^{(h \ell)} \left\{ (k + 1) y_{l_1}^k y_{l_n}^2 \times [\cdots \text{degree } = \ell \cdots ] \right\} \]

holds for \( \ell = (k + 2) + \ell \), where \( k \geq 2 \) (variable), but \( \ell > 0 \) (fixed). We continue to use the notations above and there is no \( y_{l_1} \) or \( y_{l_n} \) inside the homogeneous polynomial denoted by \([\cdots \text{degree } = \ell \cdots ]\). Let us go on to show

\[ (A.3.19) \ \Delta_o^{(h \ell)} \left\{ y_{l_1}^{k+2} \cdot [\cdots \text{degree } = \ell + 2 \cdots ] \right\} = \Delta_o^{(h \ell)} \left\{ (k + 1) y_{l_1}^k y_{l_n}^2 \cdot [\cdots \text{degree } = \ell + 2 \cdots ] \right\} , \]

where \( k \geq 2 \) is even. Let us find the first Laplacians:

\[ (A.3.20) \ \Delta_o \left\{ y_{l_1}^{k+2} \cdot [\cdots \text{degree } = \ell + 2 \cdots ] \right\} = (k + 2) (k + 1) y_{l_1}^k \cdot [\cdots \text{degree } = \ell + 2 \cdots ] + y_{l_1}^{k+2} \cdot \Delta_o \left\{ [\cdots \text{degree } = \ell + 2 \cdots ] \right\} = k (k + 1) y_{l_1}^k \cdot [\cdots \text{degree } = \ell + 2 \cdots ] + 2 (k + 1) y_{l_1}^k \cdot [\cdots \text{degree } = \ell + 2 \cdots ] + y_{l_1}^{k+2} \cdot \Delta_o \left\{ [\cdots \text{degree } = \ell + 2 \cdots ] \right\} \]
\( (\leftarrow \uparrow \text{ degree } = \ell \rightarrow ) ; \)

\[
\Delta_o \left\{ (k+1) y_i^k y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} \\
= (k+1) k (k-1) y_i^{k-2} y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \\
+ 2 (k+1) y_i^k \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \\
+ (k+1) y_i^k y_n^2 \cdot \{ \Delta_o \cdots \text{ degree } = \ell + 2 \cdots \} \\
\text{ ( } \leftarrow \uparrow \text{ degree } = \ell \rightarrow ) ,
\]

[ observe that \((k+2)(k+1) - 2(k+1) = k(k+1)\).]

Via the induction hypothesis, the last two terms in the respective expressions are equal. After simplification, to verify (A.3.19), it suffices to show that

\[
\Delta_o^{(h-1)} \left\{ y_i^k \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} \\
= \Delta_o^{(h-1)} \left\{ (k-1) y_i^{k-2} y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} .
\]

Applying \(\Delta_o\) on the terms

\[
\left\{ y_i^k \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} \text{ and } \left\{ (k-1) y_i^{k-2} y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\},
\]

using similar calculation and cancelation as in (A.3.20), we come down gradually to verify (A.3.21)

\[
\Delta_o^{(h-\frac{k+2}{2})} \left\{ y_i^4 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} \\
= \Delta_o^{(h-\frac{k+2}{2})} \left\{ 3 y_i^2 y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} .
\]

Note that

\[
\ell = (k+2) + (\ell + 2) \implies \frac{\ell}{2} - \frac{k-2}{2} = \frac{4 + (\ell + 2)}{2}.
\]

Apply the Laplacian on the two terms inside the brackets in (A.3.21) and obtain

\[
\Delta_o \left\{ y_i^4 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\} = 4 \cdot 3 y_i^2 y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] + \\
+ y_i^4 \cdot \{ \Delta_o \cdots \text{ degree } = \ell + 2 \cdots \} ; \\
\text{ ( } \leftarrow \uparrow \text{ degree } = \ell \rightarrow )
\]

\[
\Delta_o \left\{ 3 y_i^2 y_n^2 \cdot [\cdots \text{ degree } = \ell + 2 \cdots] \right\}
\]

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\[
\begin{align*}
&= 3 \cdot 2 \cdot 1 \cdot [y_{i_1}^2 + y_{i_2}^2] \cdot [\cdots \text{degree} = \ell + 2 \cdots] + \\
&\quad + 3 y_{i_1}^2 y_{i_2}^2 \cdot \{ \Delta \cdot [\cdots \text{degree} = \ell + 2 \cdots] \}
\end{align*}
\]

Again we apply the induction hypothesis to cancel the last term in each expression above. Apply the Laplacian again and obtain
(A.3.22) \[ \Delta_o \left\{ 4 \cdot 3 \ y^2_i \cdot \{ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \right\} \]
\[ = 4 \cdot 3 \cdot 2 \cdot \{ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} + 4 \cdot 3 \cdot y^2_i \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \; ; \]

(A.3.23) \[ \Delta_o \left\{ 3 \cdot 2 \ [y^2_i + y^2_n] \cdot \{ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \right\} \]
\[ = 3 \cdot 2 \cdot 4 \cdot \{ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \]
\[ + 3 \cdot 2 \cdot [y^2_i + y^2_n] \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \; . \]

As \[ 4 \cdot 3 \cdot y^2_i \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \]
\[ = 2 \cdot 3 \cdot y^2_i \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} + \]
\[ + 2 \cdot 3 \cdot y^2_i \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \; , \]
\[ = 3 \cdot 2 \cdot [y^2_i + y^2_n] \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \]
\[ = 3 \cdot 2 \cdot y^2_i \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \]
\[ + 3 \cdot 2 \cdot y^2_n \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \; , \]
and \[ \Delta_o \left( \frac{2 + (\lambda + 2)}{2} \right) \left[ y^2_i \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \right] \]
\[ = \Delta_o \left( \frac{2 + (\lambda + 2)}{2} \right) \left[ y^2_n \cdot \{ \Delta_o \ [ \cdots \ \text{degree} = \lambda + 2 \ \cdots \} \} \right] \; (\leftarrow \text{equal to a number}) , \]
we apply the remaining order of Laplacian on (A.3.22) and (A.3.23), yielding the same numbers. Hence we verify (A.3.20), and so (A.3.21). This completes the induction step. \( \square \)
§ A. 4. Interaction terms (4.14).

We begin with
\[
\frac{\partial}{\partial \xi_j} \int_{\mathbb{R}^n} (\bar{c}_n \cdot H). (V_1 + V_2)^{\frac{n+2}{n-2}} = \frac{2n}{n-2} \cdot \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) (V_1 + V_2)^{\frac{n+2}{n-2}} \cdot \frac{\partial V_1}{\partial \xi_j} 
\]
\[
= \frac{2n}{n-2} \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) V_1^{\frac{n+2}{n-2}} \cdot \frac{\partial V_1}{\partial \xi_j} \right] \cdot [1 + o(1)] 
\]

[similar to Weak Interaction Lemma 2.16; \(\uparrow \quad \downarrow\) cf. (A.1.30)]
\[
= \frac{n-2}{2} \cdot \frac{2n}{n-2} \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) V_1^{\frac{n+2}{n-2}} \cdot \left\{ - \frac{n+2}{2} \cdot \frac{2(\xi_{1j} - y_j)}{(\lambda_1^2 + |y| \xi_1^2)^{\frac{n+2}{2}}} \right\} \right] \cdot [1 + o(1)] 
\]
\[
= -2n \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) V_1^{\frac{n+2}{n-2}} \cdot \left\{ \frac{(\xi_{1j} - y_j)}{(\lambda_1^2 + |y| \xi_1^2)} \right\} \right] \cdot [1 + o(1)] 
\]
\[
= -2n \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) \left( \frac{\lambda_1}{\lambda_1^2 + |y|^2} \right)^n \cdot \left\{ \frac{1}{(\lambda_1^2 + |y|^2)} \right\} \right] \cdot [1 + o(1)] 
\]

(via symmetry)
\[
= -2n \cdot \frac{\xi_{1j}}{\lambda_1} \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) \left( \frac{\lambda_1}{\lambda_1^2 + |y|^2} \right)^n \cdot \left\{ \frac{\lambda_1}{(\lambda_1^2 + |y|^2)} \right\} \right] \cdot [1 + o(1)] 
\]

It follows that
\[
- \frac{n-2}{2n} \left( \lambda_1 \cdot \frac{\partial}{\partial \xi_j} \right) \int_{\mathbb{R}^n} (\bar{c}_n \cdot H). (V_1 + V_2)^{\frac{n+2}{n-2}} 
\]
\[
= (n-2) \cdot \lambda_1 \cdot \frac{\xi_{1j}}{\lambda_1} \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) \left( \frac{\lambda_1}{\lambda_1^2 + |y|^2} \right)^{n+1} \right] \cdot [1 + o(1)] 
\]
\[
= (n-2) \cdot \lambda_1 \cdot \frac{\xi_{1j}}{\lambda_1} \cdot \frac{\lambda^\ell}{\lambda_1} \left[ \int_{\mathbb{R}^n} (\bar{c}_n \cdot H) (Y) \left( \frac{1}{1 + |Y|^2} \right)^{n+1} \right] \cdot [1 + o(1)] 
\]
\[
= C_2^+ (n, \ell) \cdot \bar{w}_1 \cdot \lambda_1 \cdot \frac{\xi_{1j}}{\lambda_1} \cdot \frac{\lambda^\ell}{\lambda_1} \cdot [1 + o(1)] \quad \text{[similar to (4.6); } Y = \lambda_1^{-1} y \text{].} 
\]
Here we use (1.8) and (1.10).
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