∗-CONFORMAL $\eta$-RICCI SOLITONS IN $\epsilon$-KENMOTSU MANIFOLDS

Abdul Haseeb and Rajendra Prasad

ABSTRACT. We characterize $\epsilon$-Kenmotsu manifolds admitting $\ast$-conformal $\eta$-Ricci solitons. At last, an example of 7-dimension $\epsilon$-Kenmotsu manifold is given.

1. Introduction

In 1993, Bejancu and Duggal [2] introduced the concept of $\epsilon$-Sasakian manifolds. Later, it was shown by Xufeng and Xiaoli [19] that every $\epsilon$-Sasakian manifolds are real hypersurfaces of indefinite Kahlerian manifolds. In 1972, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [14]. We call it Kenmotsu manifold. The concept of an $\epsilon$-Kenmotsu manifold was introduced by De and Sarkar [5] who showed that the existence of new structure on an indefinite metric influences the curvatures. $\epsilon$-Kenmotsu manifolds have also been studied by various authors in several ways to a different extent such as [10, 11, 12, 18] and many others.

In 1982, Hamilton [9] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$ 

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(M, g)$ is a generalization of an Einstein metric such that $\mathcal{L}_V g + 2S + 2\lambda g = 0$, where $S$ is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative operator along the vector field $V$ on $M$ and $\lambda$ is a real number. The Ricci soliton is said to be shrinking, steady or expanding according to $\lambda$ being negative, zero or positive, respectively.

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As a generalization of the Ricci soliton, the notion of \(\eta\)-Ricci soliton was introduced by Cho and Kimura \[4\] and is given by \(\mathcal{L}_V g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0\), where \(\lambda\) and \(\mu\) are real numbers.

In 2004, the concept of conformal Ricci flow which is a variation of the classical Ricci flow equation was introduced by Fischer \[6\]. In the classical Ricci flow equation the unit volume constraint plays an important role, but in the conformal Ricci flow equation, the scalar curvature \(r\) is considered as a constraint. The conformal Ricci flow on \(M\) is defined by the equation \(\[6\]
\[
\frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) = -pg
\]
and \(r = -1\), where \(p\) is a scalar non-dynamical field (time dependent scalar field), \(r\) is the scalar curvature of the manifold and \(n\) is the dimension of manifold.

The conformal Ricci soliton equation and conformal \(\eta\)-Ricci soliton equation are given by \(\[1\]\n\[
\mathcal{L}_V g + 2S = \left(2\lambda - \left(p + \frac{2}{n}\right)\right) g,
\]
\[
\mathcal{L}_V g + 2S + (2\lambda - \left(p + \frac{2}{n}\right)) g + 2\mu \eta \otimes \eta = 0,
\]
respectively, where \(\lambda\) and \(\mu\) are constants.

The notion of \(*\)-Ricci tensor on almost Hermitian manifolds was introduced by Tachibana \[17\]. Later, Hamada \[8\] studied \(*\)-Ricci flat real hypersurfaces of complex space forms and Blair \[3\] defined \(*\)-Ricci tensor in contact metric manifolds given by
\[
S^*(X,Y) = g(Q^*X,Y) = \text{Trace} \{\phi \circ R(X,\phi Y)\},
\]
where \(Q^*\) is the \(*\)-Ricci operator and \(S^*\) is a tensor field of type \((0,2)\).

**Definition 1.1.** \[13\] A Riemannian metric \(g\) on \(M\) is called a \(*\)-Ricci soliton, if
\[
(\mathcal{L}_V g)(X,Y) + 2S^*(X,Y) + 2\lambda g(X,Y) = 0
\]
for all vector fields \(X, Y\) on \(M\) and \(\lambda\) is a constant.

If \(S^*(X,Y) = \lambda g(X,Y) + \mu \eta(X)\eta(Y)\) for all vector fields \(X, Y\) and \(\lambda, \mu\) are smooth functions, then the manifold is called \(*\)-\(\eta\)-Einstein manifold. Further if \(\mu = 0\), that is, \(S^*(X,Y) = \lambda g(X,Y)\) for all vector fields \(X, Y\), then the manifold becomes \(*\)-Einstein.

Recently, the \(*\)-Ricci solitons on almost contact metric manifolds have been studied by various authors such as \[7, 13, 15, 16\] and many others.

The notion of \(*\)-conformal \(\eta\)-Ricci soliton is defined as follows:
\[
\mathcal{L}_V g + 2S^* + \left(2\lambda - \left(p + \frac{2}{n}\right)\right) g + 2\mu \eta \otimes \eta = 0,
\]
where \(\mathcal{L}_V\) is the Lie derivative along the vector field \(V\), \(S^*\) is the \(*\)-Ricci tensor and \(\lambda, \mu\) are constants.

In the present paper we study \(*\)-conformal \(\eta\)-Ricci solitons in an \(\epsilon\)-Kenmotsu manifold satisfying certain curvature conditions.
2. Preliminaries

An $n$-dimensional smooth manifold $(M, g)$ is said to be an $\epsilon$-almost contact metric manifold [2], if it admits a $(1,1)$ tensor field $\phi$, a structure vector field $\xi$, a 1-form $\eta$ and an indefinite metric $g$ such that

\begin{align*}
\phi^2 X &= -X + \eta(X)\xi, \quad \eta(\xi) = 1, \\
g(\xi, \xi) &= \epsilon, \quad \eta(X) = \epsilon g(X, \xi), \\
g(\phi X, \phi Y) &= g(X, Y) - \epsilon \eta(X)\eta(Y)
\end{align*}

for all vector fields $X, Y$ on $M$, where $\epsilon$ is $1$ or $-1$ according as $\xi$ is spacelike or timelike vector fields and rank $\phi$ is $(n-1)$. If $d\eta(X, Y) = g(X, \phi Y)$ for every $X, Y \in \chi(M)$, then we say that $M$ is an $\epsilon$-contact metric manifold. Also, we have $\phi \xi = 0, \eta(\phi X) = 0$. If an $\epsilon$-contact metric manifold satisfies

$$(\nabla_X \phi)(Y) = -g(X, \phi Y)\xi - \epsilon \eta(Y)\phi X,$$

where $\nabla$ denotes the Levi-Civita connection with respect to $g$, then $M$ is called an $\epsilon$-Kenmotsu manifold [5].

An $\epsilon$-almost contact metric manifold is an $\epsilon$-Kenmotsu, if and only if

$$\nabla_X \xi = \epsilon(X - \eta(X)\xi).$$

Moreover, the curvature tensor $R$, the Ricci tensor $S$ and the Ricci operator $Q$ in an $\epsilon$-Kenmotsu manifold $M$ with respect to the Levi-Civita connection satisfies

\begin{align*}
(\nabla_X \eta)Y &= g(X, Y) - \epsilon \eta(X)\eta(Y), \\
R(X, Y)\xi &= \eta(X)Y - \eta(Y)X, \\
R(\xi, X)Y &= \eta(Y)X - \epsilon \eta(X)\xi, \\
R(\xi, \xi)X &= -R(X, \xi)\xi = X - \eta(X)\xi, \\
\eta(R(X, Y)Z) &= \epsilon(g(X, Z)\eta(Y) - g(Y, Z)\eta(X)), \\
S(X, \xi) &= -(n-1)\eta(X), \quad S(\xi, \xi) = -(n-1), \\
Q\xi &= -\epsilon(n-1)\xi
\end{align*}

for any $X, Y, Z$ on $M$, where $g(QX, Y) = S(X, Y)$. We note that if $\epsilon = 1$ and the structure vector field $\xi$ is spacelike, then an $\epsilon$-Kenmotsu manifold is usual Kenmotsu manifold.

**Definition 2.1.** An $\epsilon$-Kenmotsu manifold $M$ is said to be $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions on $M$. If $b = 0$ (resp., $a = 0$), then the manifold is called Einstein (resp., special type of an $\eta$-Einstein) manifold.

**Definition 2.2.** The concircular curvature tensor $C$ in an $n$-dimensional $\epsilon$-Kenmotsu manifold $M$ is defined by [20]

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

where $R$ is the Riemannian curvature tensor and $r$ is the scalar curvature of the manifold.
Lemma 2.1. In an \( n \)-dimensional \( \epsilon \)-Kenmotsu manifold, we have
\[
\bar{R}(X, Y, \phi Z, \phi W) = \bar{R}(X, Y, Z, W) + \epsilon \Phi(X, Z)\Phi(Y, W) - \epsilon \Phi(Y, Z)\Phi(X, W) + \epsilon g(Y, Z)g(X, W) - \epsilon g(X, Z)g(Y, W)
\]
for any \( X, Y, Z, W \) on \( M \), where \( \bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W) \) and \( \Phi \) is the fundamental 2-form of \( M \) defined by \( \Phi(X, Y) = g(X, \phi Y) \).

**Proof.** By using equations \((2.3)-(2.4)\), \((2.6)\) and the expression of the curvature tensor \( \bar{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \) in \( \bar{R}(X, Y, \phi Z, \phi W) = g(\bar{R}(X, Y)\phi Z, \phi W) \), after straightforward calculations \((2.13)\) follows. \( \square \)

Lemma 2.2. In an \( n \)-dimensional \( \epsilon \)-Kenmotsu manifold the \( \ast \)-Ricci tensor is given by
\[
S^\ast(Y, Z) = S(Y, Z) + \epsilon(n - 2)g(Y, Z) + \eta(Y)\eta(Z)
\]
for any \( Y, Z \) on \( M \).

**Proof.** Let \( \{e_i\}, i = 1, 2, \ldots, n \) be an orthonormal basis of the tangent space at each point of the manifold. Therefore from \((2.13)\) and \((1.1)\), we have
\[
S^\ast(Y, Z) = \sum_{i=1}^{n} \bar{R}(e_i, Y, \phi Z, \phi e_i)
\]
\[
= \sum_{i=1}^{n} [\bar{R}(e_i, Y, Z, e_i) + \epsilon\Phi(e_i, Z)\Phi(Y, e_i) - \epsilon\Phi(Y, Z)\Phi(e_i, e_i) + \epsilon g(Y, Z)g(e_i, e_i) - \epsilon g(e_i, Z)g(Y, e_i)].
\]
By using \((2.3)\) and \( \Phi(X, Y) = g(X, \phi Y) \) in the above equation, \((2.14)\) follows. \( \square \)

3. \( \ast \)-conformal \( \eta \)-Ricci solitons in \( \epsilon \)-Kenmotsu manifolds

Let an \( n \)-dimensional \( \epsilon \)-Kenmotsu manifold admits \( \ast \)-conformal \( \eta \)-Ricci soliton. Then \((1.2)\) holds and thus we have
\[
(L_\xi g)(Y, Z) + 2S^\ast(Y, Z) + \left[2\lambda - \left(p + \frac{2}{n}\right)\right]g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.
\]
In an \( \epsilon \)-Kenmotsu manifold, we have \((1.1)\)
\[
(L_\xi g)(Y, Z) = 2\epsilon(g(Y, Z) - \epsilon\eta(Y)\eta(Z)).
\]
Therefore, from \((3.1)\) and \((3.2)\), we find
\[
S^\ast(Y, Z) = -\left[\epsilon + \lambda - \frac{1}{2}\left(p + \frac{2}{n}\right)\right]g(Y, Z) + (1 - \mu)\eta(Y)\eta(Z).
\]
By using \((3.3)\), \((3.1)\) takes the form
\[
S(Y, Z) = -\left[\epsilon\mu - \epsilon - \lambda - \frac{1}{2}\left(p + \frac{2}{n}\right)\right]g(Y, Z) - \mu\eta(Y)\eta(Z).
\]
which is of the form
\[ S(Y, Z) = A g(Y, Z) + B \eta(Y) \eta(Z), \]
where \( A = -[n \epsilon - \epsilon + \lambda - \frac{1}{2}(p + \frac{2}{n})] \) and \( B = -\mu \). Taking \( Z = \xi \) in (3.4), we find
\[ S(Y, \xi) = -\left[ n - 1 + \epsilon \lambda + \mu - \frac{\epsilon}{2} \left( p + \frac{2}{n} \right) \right] \eta(Y). \]
From equations (2.10) and (3.5), we obtain
\[ \lambda + \epsilon \mu = \frac{1}{2} \left( p + \frac{2}{n} \right). \]
Thus we have the following:

**Theorem 3.1.** If an \( n \)-dimensional \( \epsilon \)-Kenmotsu manifold admits \(*\)-conformal \( \eta \)-Ricci soliton, then the manifold is an \( \eta \)-Einstein manifold of the form (3.4) and the scalars \( \lambda \) and \( \mu \) are related by
\[ \lambda + \epsilon \mu = \frac{1}{2} \left( p + \frac{2}{n} \right). \]
Now we consider an \( \epsilon \)-Kenmotsu manifold admitting \(*\)-conformal \( \eta \)-Ricci soliton and have Codazzi type of Ricci tensor and cyclic parallel Ricci tensor.

**Definition 3.1.** An \( \epsilon \)-Kenmotsu manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor \( S \) of type \((0, 2)\) is non-zero and satisfies the following condition
\[ (\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \]
for all \( X, Y, Z \in \chi(M) \).

Taking covariant derivative of (3.4) and making use of (2.6), we find
\[ (\nabla_X S)(Y, Z) = -\mu [g(X, Y) \eta(Z) + g(X, Z) \eta(Y) - 2g(Y) \eta(X) \eta(Y) \eta(Z)]. \]
If the Ricci tensor \( S \) is of Codazzi type, then we have from (3.4) and (3.5) that
\[ \mu [g(X, Z) \eta(Y) - g(Y, Z) \eta(X)] = 0 \]
from which it follows that either \( \mu = 0 \) or \( g(X, Z) \eta(Y) - g(Y, Z) \eta(X) = 0 \). Therefore,
(i) if \( \mu = 0 \), then the \(*\)-conformal \( \eta \)-Ricci soliton becomes \(*\)-conformal Ricci soliton. Hence we state the following:

**Theorem 3.2.** A \(*\)-conformal \( \eta \)-Ricci soliton in an \( \epsilon \)-Kenmotsu manifold whose Ricci tensor is of Codazzi type becomes a \(*\)-conformal Ricci soliton.

Again for \( \mu = 0 \), (3.4) becomes \( S(Y, Z) = -\epsilon(n - 1)g(Y, Z) \). Therefore the manifold becomes an Einstein manifold. Also it is known that a 3-dimensional Einstein manifold is a manifold of constant curvature \[20\]. Thus we have:

**Corollary 3.1.** An \( \epsilon \)-Kenmotsu manifold whose Ricci tensor is of Codazzi-type admitting \(*\)-conformal Ricci solitons is a manifold of constant curvature.

(ii) If \( g(X, Z) \eta(Y) - g(Y, Z) \eta(X) = 0 \), then we replace \( Y = \xi \) in the foregoing equation, we obtain \( g(X, Z) = \epsilon \eta(X) \eta(Z) \) which by substituting \( X \) by \( QX \) turns to \( S(X, Z) = -(n - 1) \eta(X) \eta(Z) \). Thus we have the following:
Theorem 3.3. If an $n$-dimensional $\epsilon$-Kenmotsu manifold admits $\ast$-conformal $\eta$-Ricci soliton and the manifold has Ricci tensor of Codazzi type, then the manifold is a special type of an $\eta$-Einstein manifold.

Definition 3.2. An $\epsilon$-Kenmotsu manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor $S$ of type $(0, 2)$ is non-zero and satisfies the following condition
\[(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0\]
for all $X, Y, Z \in \chi(M)$.

Let an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $\ast$-conformal $\eta$-Ricci soliton and the manifold has cyclic parallel Ricci tensor, then (3.9) holds. By virtue of (3.8), we have
\[\mu[g(X, Y)\eta(Z) + g(Y, Z)\eta(X) + g(Z, X)\eta(Y) - 6\epsilon\eta(X)\eta(Y)\eta(Z)] = 0\]
which by putting $Z = \xi$ reduces to
\[\mu[g(X, Y) - \epsilon\eta(X)\eta(Y)] = 0 \implies \mu g(\phi X, \phi Y) = 0\]
from which it follows that $\mu = 0$ and $g(\phi X, \phi Y) \neq 0$. Thus we have the following:

Theorem 3.4. A $\ast$-conformal $\eta$-Ricci soliton in an $\epsilon$-Kenmotsu manifold whose Ricci tensor is cyclic parallel becomes a $\ast$-conformal Ricci soliton.

Now by considering (3.6) along with $\mu = 0$, we get from (3.4) that
\[(3.12)\]
\[S(Y, Z) = -\epsilon(n - 1)g(Y, Z)\]
Thus we have the following:

Corollary 3.2. If an $n$-dimensional $\epsilon$-Kenmotsu manifold admits $\ast$-conformal $\eta$-Ricci soliton and the manifold has a cyclic parallel Ricci tensor, then the manifold is an Einstein manifold of the form (3.12).

4. $\ast$-conformal $\eta$-Ricci solitons in $\epsilon$-Kenmotsu manifolds satisfying $R(\xi, X) \cdot S = 0$

Let an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $\ast$-conformal $\eta$-Ricci soliton satisfies $R(\xi, X) \cdot S = 0$. Then we have
\[(4.1)\]
\[S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0\]
for all $X, Y, Z \in \chi(M)$. By using (2.7) in (4.1), we have
\[S(\eta(Y)X - \epsilon g(X, Y)\xi, Z) + S(Y, \eta(Z)X - \epsilon g(X, Z)\xi) = 0\]
which by taking $Z = \xi$ and using (3.5) takes the form
\[(4.2) S(X, Y) = -\epsilon(n - 1 + \epsilon\lambda + \mu - \frac{\epsilon}{2}(p + \frac{2}{n}))g(X, Y)\].
Now from (5.4) and (4.2), we obtain

$$\mu(g(X,Y) - \epsilon \eta(X)\eta(Y)) = 0 \implies \mu g(\phi X, \phi Y) = 0$$

from which it follows that $\mu = 0$ and $g(\phi X, \phi Y) \neq 0$. Thus we have the following:

**Theorem 4.1.** A $\ast$-conformal $\eta$-Ricci soliton in an $\epsilon$-Kenmotsu manifold satisfying $R(\xi, X) \cdot S = 0$ becomes a $\ast$-conformal Ricci soliton.

By virtue of (3.4), (4.2) becomes

$$S(X, Y) = -\epsilon(n - 1)g(X, Y).$$

Thus we have the following:

**Corollary 4.1.** If an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $\ast$-conformal $\eta$-Ricci soliton satisfies $R(\xi, X) \cdot S = 0$, then the manifold is an Einstein manifold of the form (4.3).

5. $\ast$-conformal $\eta$-Ricci solitons in $\epsilon$-Kenmotsu manifolds satisfying $S(\xi, X) \cdot R = 0$

Let an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $\ast$-conformal $\eta$-Ricci soliton satisfies $S(\xi, X) \cdot R = 0$. Then we have

$$(X \wedge S\xi)R(U, V)Z + R((X \wedge S\xi)U, V)Z + R(U, (X \wedge S\xi)V)Z$$

$$+ R(U, V)(X \wedge S\xi)Z = 0$$

for any $X, U, V, Z \in \chi(M)$, where the endomorphism $X \wedge S\xi$ is defined by

$$(X \wedge S\xi)Z = S(\xi, Z)X - S(X, Z)\xi.$$

Using definition (5.3), (5.4) becomes

$$S(\xi, R(U, V)Z)X - S(X, R(U, V)Z)\xi + S(\xi, U)R(X, V)Z - S(X, U)R(\xi, V)Z$$

$$+ S(\xi, V)R(U, X)Z - S(S(X, V)R(U, \xi)Z + S(\xi, Z)R(U, V)X - S(X, Z)R(U, V)\xi = 0.$$

Taking the inner product of above equation with $\xi$, we have

$$S(\xi, R(U, V)Z)\eta(X) - S(X, R(U, V)Z) + S(\xi, U)\eta(R(X, V)Z)$$

$$- S(X, U)\eta(R(\xi, V)Z) + S(\xi, V)\eta(R(U, X)Z) - S(X, V)\eta(R(U, \xi)Z)$$

$$+ S(\xi, Z)\eta(R(U, V)X) - S(X, Z)\eta(R(U, V)\xi = 0,$$

$\epsilon \neq 0$, which by putting $V = Z = \xi$ and using (2.6) - (2.8) reduces to

$$S(X, U) - \eta(U)S(X, \xi) - \eta(X)S(\xi, U) + S(\xi, \xi)\eta(X)\eta(U)$$

$$+ cS(\xi, \xi)g(X, U) - S(\xi, \xi)\eta(X)\eta(U) = 0.$$

In view of (5.5), (5.4) takes the form

$$(5.5) \quad S(X, U) = \epsilon \left(n - 1 + \epsilon \lambda + \mu - \frac{\epsilon}{2} \left(p + \frac{2}{n}\right) \right) g(X, U)$$

$$- 2 \left(n - 1 + \epsilon \lambda + \mu - \frac{\epsilon}{2} \left(p + \frac{2}{n}\right) \right) \eta(X)\eta(U).$$
Now taking $X = U = \xi$ in (5.5) and using (2.11), we find
\[(5.6)\]
\[\lambda = 1 \frac{1}{2} \left( p + \frac{2}{n} \right) \epsilon \mu.\]

Thus we can state the following:

**Theorem 5.1.** If an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $*$-conformal $\eta$-Ricci soliton satisfies $S(\xi, X) \cdot R = 0$, then the scalars $\lambda$ and $\mu$ are related by $\lambda + \epsilon \mu = 1 \frac{1}{2} \left( p + \frac{2}{n} \right)$.

Now from equations (5.5) and (5.6), we get
\[(5.7)\]
\[S(X, U) = \epsilon(n - 1)g(X, U) - 2(n - 1)\eta(X)\eta(U).\]

Thus we have the following:

**Corollary 5.1.** If an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $*$-conformal $\eta$-Ricci soliton satisfies $S(\xi, X) \cdot R = 0$, then the manifold is an $\eta$-Einstein manifold of the form (5.7).

### 6. $*$-conformal $\eta$-Ricci solitons in $\epsilon$-Kenmotsu manifolds satisfying $C(\xi, X) \cdot S = 0$

Let an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $*$-conformal $\eta$-Ricci soliton satisfies $C(\xi, X) \cdot S = 0$. Then we have
\[(6.1)\]
\[S(C(\xi, X)Y, Z) + S(Y, C(\xi, X)Z) = 0.\]

From (2.12), we find
\[(6.2)\]
\[C(\xi, X)Y = \left( 1 + \frac{\epsilon r}{n(n - 1)} \right)(\eta(Y)X - \epsilon g(X, Y)\xi).\]

By making use of (6.2) in (6.1), we have
\[
\left( 1 + \frac{\epsilon r}{n(n - 1)} \right) [\eta(Y)S(X, Z) - \epsilon g(X, Y)S(\xi, Z) + \eta(Z)S(X, Y) - \epsilon g(X, Z)S(Y, \xi)] = 0
\]
which by putting $Z = \xi$ and using (2.11), (2.2) and (3.5) reduces to
\[
\left( 1 + \frac{\epsilon r}{n(n - 1)} \right) \left[ S(X, Y) + \epsilon \left( n - 1 + \epsilon \lambda + \mu - \frac{\epsilon}{2} \left( p + \frac{2}{n} \right) \right) g(X, Y) \right] = 0.
\]

Therefore we have either $r = -\epsilon n(n - 1)$, or
\[(6.3)\]
\[S(X, Y) = -\epsilon \left( n - 1 + \epsilon \lambda + \mu - \frac{\epsilon}{2} \left( p + \frac{2}{n} \right) \right) g(X, Y).
\]

From the equations (5.7) and (6.3), we obtain
\[\mu(g(X, Y) - \epsilon \eta(X)\eta(Y)) = 0 \implies \mu g(\phi X, \phi Y) = 0\]
from which it follows that $\mu = 0$ and $g(\phi X, \phi Y) \neq 0$. Thus we have the following:

**Theorem 6.1.** If an $n$-dimensional $\epsilon$-Kenmotsu manifold admitting $*$-conformal $\eta$-Ricci soliton satisfying $C(\xi, X) \cdot S = 0$, then either the scalar curvature is constant or $*$-conformal $\eta$-Ricci soliton on the manifold becomes a $*$-conformal Ricci soliton.
By virtue of \((3.6)\), \((6.3)\) turns to
\[(6.4)\]
\[S(X, Y) = -\epsilon(n - 1)g(X, Y).\]
Thus we have the following:

**Corollary 6.1.** If an \(n\)-dimensional \(\epsilon\)-Kenmotsu manifold admitting \(*\)-conformal \(\eta\)-Ricci soliton satisfies \(C(\xi, X) \cdot S = 0\), then the manifold is an Einstein manifold of the form \((6.4)\).

### 7. \(\phi\)-concircularly flat \(\epsilon\)-Kenmotsu manifolds admitting \(*\)-conformal \(\eta\)-Ricci solitons

**Definition 7.1.** An \(\epsilon\)-Kenmotsu manifold is said to be \(\phi\)-concircularly flat if
\[(7.1)\]
\[\phi^2C(\phi X, \phi Y)\phi Z = 0\]
for all \(X, Y, Z\) on \(M\).

Let \(M\) be an \(n\)-dimensional \(\phi\)-concircularly flat \(\epsilon\)-Kenmotsu manifold admitting \(*\)-conformal \(\eta\)-Ricci soliton. Therefore from \((7.1)\), it follows that
\[(7.2)\]
\[g(C(\phi X, \phi Y)\phi Z, \phi W) = 0.\]
In view of \((2.12)\), \((7.2)\) turns to
\[(7.3)\]
\[g[R(\phi X, \phi Y)\phi Z, \phi W] = \frac{r}{n(n - 1)}[g(\phi Y, \phi Z)g(\phi X, \phi W)\]
\[\quad - g(\phi X, \phi Z)g(\phi Y, \phi W)].\]
Let \(\{e_1, e_2, \ldots, e_{n-1}, \xi\}\) be a local orthonormal basis of the vector fields on \(M\). Using that \(\{\phi e_1, \phi e_2, \ldots, \phi e_{n-1}, \xi\}\) is also a local orthonormal basis. If we put \(X = W = e_i\) in \((7.3)\) and sum up with respect to \(i\) \((1 \leq i \leq n - 1)\), then we have
\[(7.4)\]
\[\sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] = \frac{r}{n(n - 1)} \sum_{i=1}^{n-1} [g(\phi Y, \phi Z)g(\phi e_i, \phi e_i)\]
\[\quad - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)].\]
It can be easily verified that
\[(7.5)\]
\[\sum_{i=1}^{n-1} g[R(\phi e_i, \phi Y)\phi Z, \phi e_i] = S(\phi Y, \phi Z) + \epsilon g(\phi Y, \phi Z),\]
\[(7.6)\]
\[\sum_{i=1}^{n-1} g(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = g(\phi Y, \phi Z),\]
\[(7.7)\]
\[\sum_{i=1}^{n-1} g(\phi e_i, \phi e_i) = (n - 1).\]
By using \((7.5)\)–\((7.7)\) in \((7.4)\), we obtain
\[(7.8)\]
\[S(\phi Y, \phi Z) = \left[\frac{r(n - 2)}{n(n - 1)} - \epsilon\right]g(\phi Y, \phi Z).\]
By virtue of (3.4) and (5.6), we find

\begin{equation}
S(\phi Y, \phi Z) = \epsilon (\mu - n + 1) g(\phi Y, \phi Z),
\end{equation}

\begin{equation}
r = \sum_{i=1}^{n} S(e_i, e_i) = \epsilon (n\mu - \mu - n^2 + n).
\end{equation}

By using (7.9) and (7.10), (7.8) gives

\begin{equation}
S(\phi Y, \phi Z) = 0 \text{ from which it follows that } \mu = 0 \text{ and } \epsilon (n - 1) g(\phi Y, \phi Z) \neq 0. \text{ Thus we have the following:}
\end{equation}

**Theorem 7.1.** A $\ast$-conformal $\eta$-Ricci soliton in $\phi$-concircularly flat $\epsilon$-Kenmotsu manifolds becomes a $\ast$-conformal Ricci soliton.

Now by using (5.6) along with $\mu = 0$ in (3.4), we obtain

\begin{equation}
S(X, Y) = -\epsilon (n - 1) g(X, Y).
\end{equation}

Thus we have the following:

**Corollary 7.1.** If an $n$-dimensional $\phi$-concircularly flat $\epsilon$-Kenmotsu manifold admits $\ast$-conformal $\eta$-Ricci soliton, then the manifold is an Einstein manifold of the form (7.11).

**Example 7.1.** We consider the 7-dimensional manifold $M = \{(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7\}$, where $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ are the standard coordinates in $\mathbb{R}^7$. Let $e_1, e_2, e_3, e_4, e_5, e_6$ and $e_7$ be the vector fields on $M$ given by

\begin{align*}
e_1 &= z \frac{\partial}{\partial x_1}, \quad e_2 = z \frac{\partial}{\partial x_2}, \quad e_3 = z \frac{\partial}{\partial x_3}, \quad e_4 = z \frac{\partial}{\partial x_4}, \\
e_5 &= z \frac{\partial}{\partial x_5}, \quad e_6 = z \frac{\partial}{\partial x_6}, \quad e_7 = -\epsilon z \frac{\partial}{\partial x_7} = \xi.
\end{align*}

Let $g$ be the indefinite Riemannian metric defined by $g(e_i, e_j) = 0$, $i \neq j$, $i, j = 1, 2, 3, 4, 5, 6, 7$ and $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = g(e_6, e_6) = g(e_7, e_7) = 1.$

Let $\eta$ be the 1-form on $M$ defined by $\eta(X) = \epsilon g(X, e_7) = \epsilon g(X, \xi)$ for all $X \in \chi(M)$.

Let $\phi$ be the (1, 1)-tensor field on $M$ defined by

\begin{align*}
\phi e_1 &= e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = e_4, \quad \phi e_4 = -e_3, \quad \phi e_5 = e_6, \quad \phi e_6 = -e_5, \quad \phi e_7 = 0.
\end{align*}

The linearity property of $\phi$ and $g$ yields

\begin{align*}
\eta(e_7) &= 1, \quad \phi^2 X = -X + \eta(X) \xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X) \eta(Y) \text{ for all } X, Y \in \chi(M). \text{ Thus for } e_7 = \xi, \text{ the structure } (\phi, \xi, \eta, g, \epsilon) \text{ defines an indefinite almost contact metric structure on } M. \text{ Now, by direct computations, we obtain}
\end{align*}

\begin{align*}
[e_1, e_2] &= [e_1, e_3] = [e_1, e_4] = [e_1, e_5] = [e_1, e_6] = [e_2, e_3] = [e_2, e_4] = [e_2, e_5] = 0, \\
[e_2, e_6] &= [e_3, e_4] = [e_3, e_5] = [e_3, e_6] = [e_4, e_5] = [e_4, e_6] = [e_5, e_6] = 0, \\
[e_1, e_7] &= \epsilon e_1, [e_2, e_7] = \epsilon e_2, [e_3, e_7] = \epsilon e_3, [e_4, e_7] = \epsilon e_4, [e_5, e_7] = \epsilon e_5, [e_6, e_7] = \epsilon e_6.
\end{align*}
The Riemannian connection $\nabla$ of the metric $g$ is given by
\[
2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y)
- g([X, Y], Z) + g([Z, X], Y) + g(Z, [X, Y]),
\]
which is known as Koszul’s formula. Using Koszul’s formula, we can easily calculate
\[
\nabla e_i e_1 = -e_7, \quad \nabla e_i e_2 = 0, \quad \nabla e_i e_3 = 0, \quad \nabla e_i e_4 = 0, \quad \nabla e_i e_5 = 0, \quad \nabla e_i e_6 = 0, \quad \nabla e_i e_7 = \epsilon e_i,
\]
Using the above relations, for any vector field $X$ on $M$, it follows that
\[
\nabla_X \xi = \epsilon (X - \eta(X) \xi)
\]
for any $\xi \in \chi(M)$. Hence the manifold $M$ under the consideration is an $\epsilon$-Kenmotsu manifold of dimension seven. From the above results, it is not difficult to find
\[
R(X, Y) Z = -\epsilon (g(Y, Z) X - g(X, Z) Y)
\]
from which it follows that $S(Y, Z) = -6\epsilon g(Y, Z)$ and hence $r = -42\epsilon$. From the equation (3.6), we have
\[
\sum_{i=1}^{7} e_i S(e_i, e_i) = -\left[6\epsilon + \lambda - \frac{1}{2}\left(p + \frac{2}{7}\right)\right] \sum_{i=1}^{7} e_i g(Y, Z) - \sum_{i=1}^{7} e_i \mu \eta(Y) \eta(Z)
\]
where $e_i = g(e_i, e_i)$. This implies
\[
\lambda + \frac{1}{7} \epsilon \mu = \frac{1}{2}\left(p + \frac{2}{7}\right).
\]
From equations (3.6) and (7.12), we obtain $\mu = 0$. Therefore, the data $(g, \xi, \lambda, \mu)$ for $\lambda = \frac{1}{2}(p + \frac{2}{7})$ and $\mu = 0$ defines a $\epsilon$-conformal Ricci soliton on the manifold $(M, \phi, \xi, \eta, g, \epsilon)$.

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Department of Mathematics (Received 16 05 2020)
Faculty of Science, Jazan University
Jazan, Saudi Arabia
malikhaseeb80@gmail.com
haseeb@jazanu.edu.sa

Department of Mathematics and Astronomy
University of Lucknow
Lucknow, India
rp.nampur@rediffmail.com