QUANTUM FIELDS IN CURVED SPACETIME:
NON GLOBAL HYPERBOLICITY AND LOCALITY

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Abstract. We briefly review the current status of the algebraic approach to quantum field theory on globally hyperbolic spacetimes, both axiomatic – for general field theories, and constructive – for a linear Klein-Gordon model. We recall the concept of F-locality, introduced in the latter context in BS Kay: Rev. Math. Phys., Special Issue, 167-195 (1992) and explain how it can be formulated at an axiomatic level for a general field theory (as a condition on algebras-with-net-structure) on both globally hyperbolic and non globally hyperbolic spacetimes. We also discuss the current status of the question whether/when algebras satisfying F-locality can exist for the Klein-Gordon model on spacetimes which are chronology violating.

1. Introduction

The extension of the algebraic approach to quantum field theory to the context of a general curved spacetime is of interest for (at least) two reasons. Firstly, quantum field theory in curved spacetime is by now (see e.g. [1, 2]) a well-established theory for a wide range of phenomena involving “particle creation” and “vacuum polarization” effects due to strong gravitational fields. As long as one stays away from scales where the quantum nature of the gravitational field itself becomes important, this theory is expected to provide an excellent approximate description\(^1\) for such phenomena, and has led, amongst other things, to the remarkable prediction by Hawking [3] of black-hole evaporation. One of the features of the subject is that one loses, in general, any single preferred quantum state which may be regarded as a “vacuum” and the concept of “particles” becomes vague and/or observer-dependent. On the other hand, one is interested in expectation values of local observables such as the

\(^1\)To be published in the proceedings of the conference ‘Operator Algebras and Quantum Field Theory’ held at Accademia Nazionale dei Lincei, Rome, Italy, July 1996. (Editors S. Doplicher, R. Longo, J. Roberts, L. Zsido. Publisher [probably] International Press [distributed by the AMS])

\(^2\)In cases where the back-reaction of created particles etc. on the background geometry is expected to be of importance, it is believed that this can be taken into account by arranging that the background metric satisfies a semiclassical version of Einstein’s equation in which the right hand side is taken to be the expectation value in a suitable state of the renormalized stress-energy tensor operator of the relevant quantum fields.
field itself and the stress-energy tensor which have a more objective existence. The algebraic approach to quantum field theory, in which the primary objects are such local observables and the various states of interest may all be treated on an equal footing (as positive linear functionals on the algebra of local observables) seems to be ideally suited (even essential!) to discuss such matters in a precise way, and has been used successfully e.g. in mathematical analyses of questions related to the Hawking effect [4, 5].

Secondly, even if one’s primary interest is in quantum field theory in Minkowski space, one hopes that, by widening the context to a general curved spacetime, one may achieve a deeper appreciation of the theory. In particular, one hopes to attain a more completely local understanding of many features of quantum field theory and to free oneself from any unnecessary reliance on global features such as Poincaré invariance.

The general context adopted in most discussions of quantum theory in curved spacetime involves a quite thoroughgoing abandonment of many of the global features of Minkowski space. It is usual to consider spacetimes with different topologies – e.g. one might study quantum fields on the “timelike cylinder” – i.e. the quotient of Minkowski space by a fixed spacelike translation. (The name refers to the, easily visualized, two-dimensional version.) Further, it is usual to consider spacetimes which have no isometries at all (other than the identity) – e.g. one could obviously perturb the metric on the time-like cylinder so as to remove both its time-translational and space-translational symmetry. However, there is one global feature which is usually retained, namely global hyperbolicity. Here we recall (see e.g. [6, 7]) that a spacetime is said to be globally hyperbolic if it is time-orientable and has a Cauchy surface – i.e. an achronal spacelike hypersurface which is intersected exactly once by every inextendible causal curve. (As a manifold, such a spacetime is then necessarily a product of the Cauchy surface with the real line.) There are of course very good reasons for retaining this feature: Physically, realistic spacetime models – for the universe or for black holes etc. – are usually globally hyperbolic. Mathematically, globally hyperbolic spacetimes retain many of the qualitative features of the causal structure of Minkowski space. Moreover, classical linear hyperbolic equations such as the Klein-Gordon equation

\[(\Box - m^2)\phi = 0 \quad (1.1)\]

(where \(\Box\) denotes the Laplace-Beltrami operator for the spacetime metric, assumed here to have signature \((-+++))\) on such spacetimes admit a well-posed Cauchy problem and have globally well defined advanced and retarded (distributional) fundamental solutions \(\Delta^A\) and \(\Delta^R\) satisfying

\[(\Box_x - m^2)\Delta^{A/R}(x,y) = \delta(x,y)\]

and supported, respectively, on the set of pairs \((x,y)\) for which \(y\) is located to the future/past of \(x\).\(^3\)

\(^3\)Here we adopt the customary loose practice of thinking of (bi)distributions as if they are (bi)scalar functions. We also identify functions with distributions by integrating them with test functions using the natural volume element provided by the (square root of the absolute value of the determinant of the) metric.
Nevertheless, it is of interest to contemplate the removal of this last global condition and to ask whether it is possible (and how one might try) to generalize the algebraic approach to quantum field theory to the general class of not-necessarily globally hyperbolic spacetimes and, in this short article, we shall briefly discuss some recent and current work by the author and collaborators on this problem \[8, 9, 10, 11\]. (For other work on related questions, see e.g. \[12\].) We shall be particularly concerned with spacetimes which are chronology violating\(^4\) (i.e. which have closed timelike curves) a simple example of which is e.g. the \textit{spacelike cylinder} i.e. the quotient of Minkowski space by a fixed \textit{timelike} translation. A subclass of chronology-violating spacetimes, namely those with \textit{compactly generated Cauchy horizons} (a simple two-dimensional example is Misner space – i.e. the quotient of half of two-dimensional Minkowski space by a fixed Lorentz boost – see e.g. \[6, 8\] for details) may, according to the analysis in \[14\] be regarded as models for spacetimes in which time-machines get manufactured and thus our question is relevant to the currently topical question \[14, 15\] of whether it is possible in principle to manufacture a time machine.

The basic geometrical consideration which underlies our subsequent discussion has to do with the elementary result that, in any spacetime, any point has neighbourhoods which are globally hyperbolic in their intrinsic geometry. (We shall refer to them as \textit{GH neighbourhoods}.) Moreover, such GH neighbourhoods may be as small as we like in the sense that any neighbourhood of any point contains a GH neighbourhood of that point. When the full spacetime is globally hyperbolic, one may always find such a small-as-we-like GH neighbourhood around any point with the property\(^5\) that its intrinsic and induced causal structures coincide (i.e. that pairs of points which are timelike [respectively spacelike, null] related in the intrinsic geometry of the neighbourhood will also be timelike [respectively spacelike, null] related globally). In Minkowski space, examples of GH neighbourhoods with this property are the familiar double cones. However, in a non globally hyperbolic spacetime, it may be impossible to find a GH neighbourhood of a given point with equal intrinsic and induced causal structures. Thus, e.g. in the spacelike cylinder, such neighbourhoods will clearly always contain pairs of points which are intrinsically spacelike related but globally timelike related. A key feature of spacetimes with compactly generated Cauchy horizons is that (Proposition 2 of \[9\]) they necessarily have some points (the \textit{base points} of \[9\]) in their Cauchy horizons with the property

\[1.1 \textbf{Property.} \textit{Every GH neighbourhood of }p\textit{ contains a pair of points } (q, r) \textit{ which are intrinsically spacelike related but globally null related.}\]

\[2. \textbf{Nets of Local Algebras on Globally Hyperbolic Spacetimes}\]

One expects a given quantum field theory on a given globally hyperbolic spacetime \((M, g)\) to be describable by a \textit{C}\(^*\) algebra (with identity \(I\)) with net

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\(^4\)We shall, however, continue to assume that our spacetimes are time-orientable. For a discussion of why it is difficult/impossible to quantize a field theory on a non-time-orientable spacetime, see \[8\]. (See also \[13\].)

\(^5\)Note however, that, even when the full spacetime is globally hyperbolic, it is not true that every GH neighbourhood has this property. See e.g. the discussion of the "helical strip" example in \[8\].
structure, $\mathcal{A}(M, g)$. To say it has a net structure means there is a preferred assignment of a subalgebra $\mathcal{A}(M, g)(\mathcal{O})$ to each open set $\mathcal{O}$ with compact closure in $M$ — satisfying the properties:

(1) **(Isotony.)** If $\mathcal{O}_1$ is a subset of $\mathcal{O}_2$, then $\mathcal{A}(M, g)(\mathcal{O}_1)$ is a subalgebra of $\mathcal{A}(M, g)(\mathcal{O}_2)$.

(2) **(Spacelike Commutativity.)** If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike related, then $\mathcal{A}(M, g)(\mathcal{O}_1)$ and $\mathcal{A}(M, g)(\mathcal{O}_2)$ commute.

In one axiomatic approach [16] (which we shall adopt below “by default”) one expects the local algebras $\mathcal{A}(M, g)(\mathcal{O})$ to be $\mathcal{W}^*$ algebras (specifically hyperfinite type III$_1$ factors and satisfying the condition of having trivial “algebra at a point”) and the total algebra to equal their $\mathcal{C}^*$ inductive limit. In this approach, the physically interesting states would be the locally norm al states. Alternatively, and essentially equivalently, one can adopt a point of view where the local algebras are (smaller) $\mathcal{C}^*$ algebras if one imposes suitable conditions on the set of physically relevant states [17, 18]. One advantage of the former approach is that the $\mathcal{A}(M, g)(\mathcal{O})$ are then expected [19, 5] to be large enough to contain operators which would correspond to the (say exponentiated) smeared (renormalized) stress-energy tensor operator.

We remark that, given any subspacetime $(N, g)$ of $(M, g)$, one can define the subalgebra (with its inherited net structure) $\mathcal{A}(M, g; N)$. In our first mentioned (default) approach:

2.1 **Definition.** $\mathcal{A}(M, g; N)$ is defined to be the $\mathcal{C}^*$ completion of the set of all $\mathcal{A}(M, g)(\mathcal{O})$ for which the closure of $\mathcal{O}$ is contained in $N$. (And, for such $\mathcal{O}$, $\mathcal{A}(M, g; N)(\mathcal{O})$ is defined to equal $\mathcal{A}(M, g)(\mathcal{O})$.)

In the case of a linear field theory [5], it is the structure referred to in the alternative point of view above which is closer to what is immediately constructable: Given the covariant Klein-Gordon equation (1.1) on a globally hyperbolic spacetime $(M, g)$, there is a natural [5, 20] “minimal” $\mathcal{C}^*$ algebra $\mathcal{B}(M, g)$ generated by operators $W(F)$, $F \in C_0^\infty(M)$ (formally related to the smeared Hermitian quantum field $\phi(F) = \int_M \phi(x)F(x)|g|^{1/2}d^4x$ by $W(F) = \exp(-i\phi(F)))$ satisfying

$$W((\Box - m^2)F) = 0$$

(2.1)

and

$$W(F_1)W(F_2) = \exp(-i\Delta(F_1, F_2)/2)W(F_1 + F_2)$$

(2.2)

where $\Delta = \Delta^A - \Delta^R$ is the advanced minus retarded fundamental (antisymmetric distributional bi-)solution to Equation (1.1). This algebra acquires a net structure by defining the local algebra $\mathcal{B}(M, g)(\mathcal{O})$ to be the closed linear span of the $W(F)$ for $F$ supported in $\mathcal{O}$.

One then restricts attention to the set of states, $\omega$, whose (unsmeread) symmetrzed two-point functions have short-distance singularities with the Hadamard form:

$$\omega(\phi(x)\phi(y) + \phi(y)\phi(x)) = (1/2\pi^2)(u/\sigma + v \ln |\sigma| + w)$$

where $\sigma$ denotes the square of the interval between $x$ and $y$, $u$ and $v$ are smooth two-point functions determined by the local geometry and $w$ is a smooth two-point function which depends on the state. (Here, a suitable principle part
prescription is understood in giving this formula meaning for smeared fields – see [5] for details.) We remark that (as conjectured in [21]) it is now known that this local Hadamard condition on a two-point function (combined with the restrictions on the two-point function due to positivity) prevents it from having “non-local spacelike singularities”. This is one result of a recent mathematical development [22] due to Radzikowski which, importing concepts from microlocal analysis [23], characterizes Hadamard two-point functions in terms of their wave-front sets [22]. This development promises to have far-reaching applications beyond this immediate problem. (See also [9, 24, 25].)

It is also now known [18] that (as conjectured in [19]) the quasifree Hadamard states are locally quasiequivalent. One may thus enlarge each of the local algebras \( B(M, g)(O) \) to a well-defined \( W^* \) algebra \( A(M, g)(O) \) (by taking the double commutant of each \( B(M, g)(O) \) in the GNS representation of any quasifree Hadamard state, and regarding the result as an abstract \( W^* \) algebra) and define \( A(M, g) \) to be the \( C^* \) inductive limit of all of these, thus obtaining a candidate algebra-with-net-structure in the sense of the (default) axiomatic approach discussed above. We refer to [18] (see also [26]) for the proof of this local quasiequivalence result, and for partial results towards establishing further expected properties of the resulting net of local algebras.

Before turning to discuss whether/how one might generalize the above axioms/constructions to non globally hyperbolic spacetimes, we draw attention to a locality property (pointed out in [8]) inherent in the construction of \( B(M, g) \) which will be of relevance to that discussion: Namely, it possesses:

2.2 The F-Locality Property (Klein-Gordon Version). Any neighbourhood of any point \( p \) in \((M, g)\) contains a \( \text{GH} \) neighbourhood \( N \) of \( p \) such that the map which sends the element denoted “\( W(F) \)” in \( B(M, g)(N) \) to the element denoted “\( W(F) \)” in \( B(N, g) \) (where \( F \) ranges over \( C^\infty_0(N) \)) extends to an isomorphism.

(Here, we make the obvious adaptation from the discussion in [8] which was couched in terms of a slightly different technical framework based on * algebras of smeared fields.)

To see that this property holds, choose \( N \) to have the same induced and intrinsic causal structure (see Section 1). The isomorphism is then clear since the advanced minus retarded fundamental solution for \((M, g)\), when restricted to \( N \), will coincide with the intrinsic advanced minus retarded fundamental solution for \((N, g)\).

We remark here that the algebra-with-net-structure \( A(M, g) \) constructed, as sketched above, for the Klein-Gordon model will inherit an obvious version of this property. This may be best expressed by slightly changing viewpoint and thinking of the resulting property as a property of the map

\[(M, g) \mapsto A(M, g)\]

from the set of all globally hyperbolic spacetimes to the set of algebras-with-net-structure:

2.3 The F-Locality Property (Generalizable Formulation). Given any globally hyperbolic spacetime \((M, g)\), every neighbourhood of every point \( p \) in
$M$ contains a GH neighbourhood $N$ of $p$ such that there is a net-structure-preserving isomorphism between $\mathcal{A}(M, g; N)$ and $\mathcal{A}(N, g)$. (Here, $\mathcal{A}(M, g; N)$ is defined as in Definition 2.1.)

It seems reasonable to expect that (as we indicate by the above name) this property should hold quite generally – i.e. for the (default) algebras-with-net-structure of not-necessarily-linear quantum field theories on globally hyperbolic spacetimes.

### 3. The F-Locality Condition

In attempting to extend the algebra construction(s), sketched above for the Klein-Gordon field, to the case of a general non globally hyperbolic spacetime, one immediately encounters the difficulty that there may be no globally defined advanced-minus-retarded fundamental solution and thus no obvious replacement for the relations (2.2). (As far as axioms are concerned, notice e.g. that, on many spacetimes – the spacelike cylinder being one example – there are no spacelike related pairs of points and the Spacelike Commutativity Axiom would, in consequence, be empty of content.) The basic idea, suggested in [8], for overcoming this problem is to replace the global relation (2.2) by a suitable local remnant. There is considerable leeway [8] as to how this might be achieved, but one simple proposal (for a necessary condition) is to demand that the relations (2.2) continue to hold on “sufficiently small finite neighbourhoods” of each point. More precisely, one demands of any candidate replacement $\mathcal{B}(M, g)$ for the minimal C*-algebra that it should still contain elements $W(F)$, $F \in \mathcal{C}_0^\infty(M)$ which (whatever other relations they may satisfy) must satisfy, in addition to the relations (2.1) above, the F-Locality Condition (cf. the above F-Locality Property): Every point $p \in M$ should have a GH neighbourhood $N$ such that the $W(F)$, $F \in \mathcal{C}_0^\infty(N)$ satisfy Relation (2.2) above with $\triangle$ taken to be the intrinsic advanced minus retarded fundamental solution on $(N, g)$. In other words, defining, for any open set $\mathcal{O}$, the local algebra $\mathcal{B}(M, g)(\mathcal{O})$ to be the closed linear span of the $W(F)$ for $F$ supported in $\mathcal{O}$:

#### 3.1 The F-Locality Condition (Klein-Gordon Version).

Every point $p \in M$ must have a GH neighbourhood $N$ such that there is a net-structure-preserving isomorphism between $\mathcal{A}(M, g; N)$ (defined as in Definition 2.1) and $\mathcal{A}(N, g)$.

We remark here that one could generalize this proposal to a general axiomatic framework by assuming we are given a map

$$(N, g) \mapsto \mathcal{A}(N, g)$$

from globally hyperbolic spacetimes to (default) algebras-with-net-structure and demanding, for any proposed algebra-with-net-structure $\mathcal{A}(M, g)$ on a given non globally hyperbolic spacetime $(M, g)$ that it satisfy (cf. Property 2.3 above):

#### 3.2 The F-Locality Condition (Generalizable Formulation).

Every point $p \in M$ must have a GH neighbourhood $N$ such that there is a net-structure-preserving isomorphism between $\mathcal{A}(M, g; N)$ (defined as in Definition 2.1) and $\mathcal{A}(N, g)$. 
4. Difficulties with Chronology-Violating Spacetimes

Returning to the framework discussed above for the Klein-Gordon equation, and focussing now on chronology violating spacetimes (see Section 1) we discuss what is now known about whether/when (quite apart from what might be a full set of necessary and sufficient conditions) there can exist any algebras $\mathcal{B}(M, g)$ at all which satisfy the F-Locality Condition stated above.

For a spacetime $(M, g)$ with compactly generated Cauchy horizon, it is now known quite generally [9] that, with the mild technical assumption that the algebra admits a state $\omega$ in which the non-exponentiated smeared two-point function $\omega(\phi(F_1)\phi(F_2))$ exists$^6$ and is distributional, then there is no algebra $\mathcal{B}(M, g)$ which satisfies F-locality. In fact, this no-go result must hold in any spacetime which contains a point $p$ satisfying Property 1.1. (Or the more general property which results if one replaces the word “spacelike” there by “non-null”.)

The proof relies on a micro-local version ([23] Volume IV) of the Propagation of Singularities Theorem applied to distributional bisolutions to the Klein-Gordon equation. Roughly speaking, this tells us that a bisolution which is singular for pairs of points $(q, q')$ where $q$ is fixed, say, and $q'$ ranges over a portion of a null geodesic, must remain singular when $q'$ is allowed to range over the full null geodesic. (In the theorem, what actually propagates [along bicharacteristics] is the “wave front set” which consists of [pairs of] points in the cotangent space to $M$ representing pairs of “points-together-with-codirections” at which the distributional bisolution fails to be smooth. See [23] Volume IV.) One focusses on the quantity $\omega(\phi(F_1)\phi(F_2) - \phi(F_2)\phi(F_1))$. Regarded as an (antisymmetric) distributional bisolution to the Klein-Gordon equation, if one assumes that F-locality holds, this must clearly coincide, in some GH neighbourhood $(N, g)$ of $p$ with $i$ times the intrinsic advanced minus retarded fundamental (antisymmetric distributional bi-)solution $\triangle_{(N, g)}$. It is well-known that such a fundamental solution will be singular for nearby intrinsically null related pairs of points in $(N, g)$ but smooth for intrinsically non null related such pairs (in fact it is zero if they are spacelike related!) and, in particular, it will be smooth (zero) for the pair $(q, r)$ of points which we know $N$ must contain by Property 1.1 which are intrinsically non null related but globally null related. However, since this pair is globally null related, we know that we can propagate the singularity from pairs $(q, q')$ – where $q'$ is close to $q$ and ranges over a portion of the null geodesic joining $q$ to $r$ – to the pair $(q, r)$ thus obtaining a contradiction. (See [9] for a full discussion and for other related theorems which, quite independently from any question as to what conditions a field algebra should satisfy, rule out the existence of any everywhere (“weakly” [9]) locally Hadamard distributional bisolution on $(M, g)$ and imply that expectation values in Hadamard states [defined in the “initial globally hyperbolic region”] of the renormalized stress-energy tensor must necessarily become singular at any base point. See also [10].)

Turning to consider chronology-violating spacetimes which do not contain any points satisfying Property 1.1, the situation for F-locality seems to be...
rather delicate. In fact, it was pointed out in [8] that, in the case that \((M, g)\)
is the (four-dimensional) spacelike cylinder (see Section 1) and one specializes
to the massless Klein-Gordon equation, one can construct an F-local algebra
\(\mathcal{B}(M, g)\). (In the language of [8], for this field theory model, this spacetime is
F-quantum compatible): One simply follows the usual construction (see Section
2) used in the globally hyperbolic case, replacing \(\triangle\) in Equation (2.2) by
the result of “wrapping” the Minkowski space fundamental solution around
the cylinder. (i.e. thinking of the spacetime as consisting of equivalence classes \([x]\)
of points in Minkowski space where “\(x\) is equivalent to \(y\)” means that, in some
inertial coordinate system, their space coordinates coincide, while their time
coordinates differ by multiples of some fixed time interval, and with a notation
that treats distributions as if they are functions, one takes
\[
\triangle([x], [y]) = \sum_{y \in [y]} \triangle_{\text{Minkowski}}(x, y).
\]

One easily sees that this produces an F-local theory because, in four dimensions,
the fundamental solution to the Klein-Gordon equation vanishes for timelike
related as well as for spacelike related pairs of points. It is also not difficult to
convince oneself that the construction can be modified to cope with the massless
wave equation on the two-dimensional spacelike cylinder. However, it was left
as an open question in [8] whether any F-local algebra exists on the spacelike
cylinder in the case of the massive Klein-Gordon equation. Clearly, the above
wrapping procedure will now fail because the massive Minkowski fundamental
solution does not vanish for timelike related pairs of points. (Notice however
that it is smooth for such pairs and, in consequence, the wrapping procedure
will give a theory which satisfies the weaker condition of F-locality modulo \(C^\infty\)
as defined in Condition 4.1 below.)

It was shown by Fewster and Higuchi [27] that, perhaps surprisingly, one can
nevertheless construct F-local algebras on the spacelike cylinder in this massive
Klein-Gordon case. The constructions of [27] rely heavily, however, on the
translational invariance of the spacelike cylinder and it is thus not clear from
this work to what extent this result is “generic”. In particular, one can ask (for
a given field theory) whether the property of being F-quantum compatible is
stable under perturbations in the metric. Obviously, in the case of a (confor-
mally coupled) massless model, one will have stability of any F-quantum com-
patible spacetime in this sense under conformally flat perturbations. In some
F-quantum compatible examples (such as conformally coupled massless fields
on compactified Minkowski space or on the two-dimensional null strip – see [9])
arbitrarily small non-conformal perturbations of the metric will, however, lead
to the existence of points satisfying Property 1.1 and hence these models will
obviously be unstable under unrestricted perturbations by the no-go proof of
[9] as sketched above. On the other hand, sufficiently small perturbations of the
metric on the spacelike cylinder will not lead to the existence of such points and
a separate analysis is required: The next simplest case to examine is perhaps
that of massive fields on the spacelike cylinder under conformal perturbations
of the metric. In recent work by Fewster, Higuchi and the author [11], we have
found that, in two dimensions, there is a large class of one-parameter fami-
lies of such perturbations of the spacelike cylinder under which its F-quantum
compatibility for the massive Klein-Gordon model is unstable. (See also [28].) However, it has proven more difficult to establish such an instability result in higher dimensions. We have, however, now found [11] a small number of one-parameter families of conformal perturbations of the spacelike cylinder – by perturbations which respect its space translational invariance but break its time translational invariance – for which one can show that (for arbitrarily small values of the parameter) there does not exist any space-translationally invariant F-local field algebra for the massive Klein Gordon equation.

One possible conclusion to all of this is that, “generically”, F-locality rules out essentially all chronology violating spacetimes and one might be tempted to draw the moral that imposing the usual “globally hyperbolic” rules in the small more or less enforces global hyperbolicity in the large! (Note also/cf. the way in which non-time orientable spacetimes are ruled out in [8, 13].) Notice however that this conclusion could be different if, in making precise what one means by “the usual rules in the small”, one were to adopt a weaker notion than F-locality. We remark in this connection that one could e.g. replace F-locality by the property of F-locality modulo $C^\infty$.

### 4.1 The F-Locality Modulo $C^\infty$ Condition (Klein-Gordon Version)

Every point $p \in M$ should have a GH neighbourhood $N$ such that $\mathcal{B}(M,g)(N)$ is isomorphic (with “$W(F)$” mapping to “$W(F)$”) to some modified $\mathcal{B}(N,g)$, where one is allowed to change the rules for quantizing on $(N,g)$ by replacing $\triangle$ (i.e. the advanced minus retarded fundamental solution on $(N,g)$) in Equation (2.2) by $\triangle + F$ where $F$ is a smooth antisymmetric bisolution to the Klein-Gordon equation on $(N,g)$.

It may be interesting to ask whether/how one can provide a “generalizable formulation” of this condition in the spirit of Property 2.3 and Condition 3.2.

It is easy to see that this relaxation of the F-locality condition would not help in the case of spacetimes with compactly generated Cauchy horizons (or of any spacetime containing a point $p$ for which Property 1.1 [generalized as above] holds) since the proof (sketched above) of the no-go theorem of [9] will clearly still go through. But, it is easy to see that small perturbations of the spacelike cylinder will continue to admit algebras $\mathcal{B}(M,g)$ satisfying this condition even for the massive Klein-Gordon equation. To see this, it suffices to unwrap the spacetime to obtain a periodic perturbation of Minkowski space. As long as this is globally hyperbolic, one can consider its advanced minus retarded fundamental solution $\triangle_{\text{periodic}}$ and construct $\mathcal{B}(M,g)$ by following the rules for globally hyperbolic spacetimes, taking $\triangle$ in Equation (2.2) to be the result of wrapping (in the sense discussed above) $\triangle_{\text{periodic}}$ around the spacetime. Note however that this construction would violate the axiom of spacelike commutativity even in arbitrarily small (GH) neighbourhoods.

In conclusion, one can ask: What do we learn from all this concerning the physically possible states of the world? Our tentative answer would be: If we restrict our attention to those states which admit a description in terms of a classical spacetime with a net of local algebras, and insofar as such nets satisfy similar “laws in the small” to the familiar “globally hyperbolic” laws, then either chronology violating spacetimes are ruled out, or if they can be realized physically, then one would in principle be able to detect this fact locally by
observing local violations of spacelike commutativity. (In the case of spacetimes with compactly generated Cauchy horizons, or more generally of any spacetime containing a point $p$ for which Property 1.1 [generalized as above] holds, the further no-go theorems of [9] which concern the singularity in the stress-energy tensor rule such spacetimes out also in the sense that they cannot arise as semiclassical solutions to Einstein’s equations.)

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