Simultaneous Resolvability in Families of Corona Product Graphs

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Abstract

Let \( G \) be a graph family defined on a common vertex set \( V \) and let \( d \) be a distance defined on every graph \( G \in G \). A set \( S \subset V \) is said to be a simultaneous metric generator for \( G \) if for every \( G \in G \) and every pair of different vertices \( u, v \in V \) there exists \( s \in S \) such that \( d(s, u) \neq d(s, v) \). The simultaneous metric dimension of \( G \) is the smallest integer \( k \) such that there is a simultaneous metric generator for \( G \) of cardinality \( k \). We study the simultaneous metric dimension of families composed by corona product graphs. Specifically, we focus on the case of two particular distances defined on every \( G \in G \), namely, the geodesic distance \( d_G \) and the distance \( d_{G,2} : V \times V \rightarrow \mathbb{N} \cup \{0\} \) defined as \( d_{G,2}(x, y) = \min\{d_G(x, y), 2\} \).

1 Introduction

A generator of a metric space \((X, d)\) is a set \( S \subset X \) of points in the space with the property that every point of \( X \) is uniquely determined by the distances from the elements of \( S \). Given a simple and connected graph \( G = (V, E) \), we consider the function \( d_G : V \times V \rightarrow \mathbb{N} \cup \{0\} \), where \( d_G(x, y) \) is the length of a shortest path between \( u \) and \( v \) and \( \mathbb{N} \) is the set of positive integers. Then \((V, d_G)\) is a metric space since \( d_G \) satisfies (i) \( d_G(x, x) = 0 \) for all \( x \in V \), (ii) \( d_G(x, y) = d_G(y, x) \) for all \( x, y \in V \) and (iii) \( d_G(x, y) \leq d_G(x, z) + d_G(z, y) \) for all \( x, y, z \in V \). A vertex \( v \in V \) is said to distinguish two vertices \( x \) and \( y \) if \( d_G(v, x) \neq d_G(v, y) \). A set \( S \subset V \) is said to be a metric generator for \( G \) if any pair of vertices of \( G \) is distinguished by some element of \( S \). A minimum cardinality metric generator is called a metric basis, and its cardinality the metric dimension of \( G \), denoted by \( \dim(G) \).

The notion of metric dimension of a graph was introduced by Slater in [21], where metric generators were called locating sets. Harary and Melter independently introduced the same concept in [13], where metric generators were called resolving sets.
The concept of adjacency generator\(^1\) was introduced by Jannesari and Omoomi in [15] as a tool to study the metric dimension of lexicographic product graphs. A set \(S \subset V\) of vertices in a graph \(G = (V, E)\) is said to be an adjacency generator for \(G\) if for every two vertices \(x, y \in V - S\) there exists \(s \in S\) such that \(s\) is adjacent to exactly one of \(x\) and \(y\). A minimum cardinality adjacency generator is called an adjacency basis of \(G\), and its cardinality the adjacency dimension of \(G\), denoted by \(\dim_A(G)\).

Since any adjacency basis is a metric generator, \(\dim(G) \leq \dim_A(G)\). Besides, for any connected graph \(G\) of diameter at most two, \(\dim_A(G) = \dim(G)\). Moreover, \(S\) is an adjacency generator for \(G\) if and only if \(S\) is an adjacency generator for its complement \(\overline{G}\). This is justified by the fact that given an adjacency generator \(S\) for \(G\), it holds that for every \(x, y \in V - S\) there exists \(s \in S\) such that \(s\) is adjacent to exactly one of \(x\) and \(y\), and this property holds in \(\overline{G}\). Thus, \(\dim_A(G) = \dim_A(\overline{G})\).

This concept has been studied further by Fernau and Rodríguez-Veláquez in [8,9] where they showed that the metric dimension of the corona product of a graph of order \(n\) and some non-trivial graph \(H\) equals \(n\) times the adjacency dimension of \(H\). As a consequence of this strong relation they showed that the problem of computing the adjacency dimension is NP-hard.

As pointed out in [8,9], any adjacency generator of a graph \(G = (V, E)\) is also a metric generator in a suitably chosen metric space. Given a positive integer \(t\), we define the distance function \(d_{G,t} : V \times V \rightarrow \mathbb{N} \cup \{0\}\), where

\[
d_{G,t}(x, y) = \min\{d_G(x, y), t\}.
\]

Then any metric generator for \((V, d_{G,t})\) is a metric generator for \((V, d_{G,t+1})\) and, as a consequence, the metric dimension of \((V, d_{G,t+1})\) is less than or equal to the metric dimension of \((V, d_{G,t})\). In particular, the metric dimension of \((V, d_{G,1})\) is equal to \(|V| - 1\), the metric dimension of \((V, d_{G,2})\) is equal to \(\dim_A(G)\) and, if \(G\) has diameter \(D(G)\), then \(d_{G,D(G)} = d_G\) and so the metric dimension of \((V, d_{G,D(G)})\) is equal to \(\dim(G)\). Notice that when using the metric \(d_{G,t}\) the concept of metric generator needs not be restricted to the case of connected graphs\(^2\).

Let \(\mathcal{G} = \{G_1, G_2, ..., G_k\}\) be a family of (not necessarily edge-disjoint) connected graphs \(G_i = (V, E_i)\) with common vertex set \(V\) (the union of whose edge sets is not necessarily the complete graph). Ramírez-Cruz, Oellermann and Rodríguez-Veláquez defined in [19,20] a simultaneous metric generator for \(\mathcal{G}\) as a set \(S \subset V\) such that \(S\) is simultaneously a metric generator for each \(G_i\). A smallest simultaneous metric generator for \(\mathcal{G}\) is a simultaneous metric basis of \(\mathcal{G}\), and its cardinality the simultaneous metric dimension of \(\mathcal{G}\), is denoted by \(\text{Sd}G(\mathcal{G})\) or explicitly by \(\text{Sd}(G_1, G_2, ..., G_k)\). By analogy, we defined in [18] the concept of simultaneous adjacency generator for \(\mathcal{G}\), simultaneous adjacency basis of \(\mathcal{G}\) and the simultaneous adjacency dimension of \(\mathcal{G}\), denoted by \(\text{Sd}_A(\mathcal{G})\) or explicitly by \(\text{Sd}_A(G_1, G_2, ..., G_k)\). For instance, the set \(\{1, 3, 6, 7, 8\}\) is a simultaneous adjacency basis of the family

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\(^1\)Adjacency generators were called adjacency resolving sets in [15].

\(^2\)For any pair of vertices \(x, y\) belonging to different connected components of \(G\) we can assume that \(d_G(x, y) = \infty\) and so \(d_{G,t}(x, y) = t\) for any \(t\) greater than or equal to the maximum diameter of a connected component of \(G\).
$G = \{G_1, G_2, G_3\}$ shown in Figure 1, while the set $\{1, 6, 7, 8\}$ is a simultaneous metric basis, so $\text{Sd}_A(G) = 5$ and $\text{Sd}(G) = 4$.

Figure 1: The set $\{1, 3, 6, 7, 8\}$ is a simultaneous adjacency basis of $\{G_1, G_2, G_3\}$, whereas $\{1, 6, 7, 8\}$ is a simultaneous metric basis.

The study of simultaneous parameters in graphs was introduced by Brigham and Dutton in [3], where they studied simultaneous domination. This should not be confused with studies on families sharing a constant value on a parameter, for instance the study presented in [14], where several graph families such that all members have a constant metric dimension are studied, enforcing no constraints regarding whether all members share a metric basis or not. In particular, the study of the simultaneous metric dimension was introduced in [19, 20], where the authors obtained sharp bounds for this invariant for general families of graphs and gave closed formulae or tight bounds for the simultaneous metric dimension of several specific graph families. For a given graph $G$ they described a process for obtaining a lower bound on the maximum number of graphs in a family containing $G$ that has simultaneous metric dimension equal to $\dim(G)$. Moreover, it was shown that the problem of finding the simultaneous metric dimension of families of trees is NP-hard, even when the metric dimension of individual trees can be efficiently computed. This suggests the usefulness of finding the simultaneous metric dimension for special classes of graphs or obtaining good bounds on this invariant. In this paper, we obtain closed formulae for the simultaneous metric (adjacency) dimension of corona product graphs. In particular, we show that the simultaneous adjacency dimension is an important tool for the study of the simultaneous metric dimension of corona product graphs.

Throughout the paper, we will use the notation $K_n$, $K_{r,s}$, $C_n$, $N_n$ and $P_n$ for complete graphs, complete bipartite graphs, cycle graphs, empty graphs and path graphs of order $n$, respectively. We use the notation $u \sim v$ if $u$ and $v$ are adjacent and $G \cong H$ if $G$ and $H$ are isomorphic graphs. For a vertex $v$ of a graph $G$, $N_G(v)$ will denote the set of neighbours or open neighbourhood of $v$ in $G$, i.e. $N_G(v) = \{u \in V(G) : u \sim v\}$. The closed neighbourhood, denoted by $N_G[v]$, equals $N_G(v) \cup \{v\}$. If there is no ambiguity, we will simple write $N(v)$ or $N[v]$. Two vertices $x, y \in V(G)$
are twins in $G$ if $N_G[x] = N_G[y]$ or $N_G(x) = N_G(y)$. If $N_G[x] = N_G[y]$, they are said to be true twins, whereas if $N_G(x) = N_G(y)$ they are said to be false twins. We also define $\delta(v) = |N(v)|$ as the degree of vertex $v$, as well as $\delta(G) = \min_{v \in V(G)} \{\delta(v)\}$ and $\Delta(G) = \max_{v \in V(G)} \{\delta(v)\}$. The subgraph induced by a set $S$ of vertices will be denoted by $\langle S \rangle$, the diameter of a graph will be denoted by $D(G)$ and the girth by $g(G)$. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2 The simultaneous adjacency dimension: basic bounds and tools

The following general bounds on $Sd_A(G)$ were discussed in [18].

Remark 1. [18] For any family $G = \{G_1, G_2, \ldots, G_t\}$ of connected graphs on a common vertex set $V$, the following results hold:

(i) $Sd_A(G) \geq \max_{i \in \{1, \ldots, k\}} \{\dim_A(G_i)\}$.

(ii) $Sd_A(G) \geq Sd(G)$.

(iii) $Sd_A(G) \leq |V| - 1$.

It was also shown in [18] that if $G$ is graph family defined on a common vertex set $V$, such that for every pair of different vertices $u, v \in V$ there exists a graph $G \in G$ where $u$ and $v$ are twins, then $Sd_A(G) = |V| - 1$. In particular, any family $G$ containing a complete graph or an empty graph satisfies $Sd_A(G) = |V| - 1$. Moreover, since a graph and its complement have the same adjacency generators, we have that $Sd_A(G) = Sd_A(\overline{G}) = Sd_A(G \cup \overline{G})$, where $\overline{G} = \{\overline{G} : G \in G\}$.

Let $G = (V, E)$ be a graph and let $\text{Perm}(V)$ be the set of all permutations of $V$. Given a subset $X \subseteq V$, the stabilizer of $X$ is the set of permutations

$$S(X) = \{f \in \text{Perm}(V) : f(x) = x, \text{ for every } x \in X\}.$$  

As usual, we denote by $f(X)$ the image of a subset $X$ under $f$, i.e., $f(X) = \{f(x) : x \in X\}$.

Let $G = (V, E)$ be a graph and let $B \subset V$ be a nonempty set. For any permutation $f \in S(B)$ of $V$ we say that a graph $G' = (V, E')$ belongs to the family $\mathcal{G}_{B,f}(G)$ if and only if $N_{G'}(x) = f(N_G(x))$ for every $x \in B$. We define the subgraph $\langle B_G \rangle_w = (N_G[B], E_w)$ of $G$, weakly induced by $B$, where $N_G[B] = \cup_{x \in B} N_G[x]$ and $E_w$ is the set of all edges having at least one vertex in $B$. It was shown in [18] that $\langle B_G \rangle_w \cong \langle B_G \rangle_w$ for any $f \in S(B)$ and any graph $G' \in \mathcal{G}_{B,f}(G)$. We define the graph family $\mathcal{G}_B(G)$, associated to $B$, as

$$\mathcal{G}_B(G) = \bigcup_{f \in S(B)} \mathcal{G}_{B,f}(G).$$

The following result shows that, given a graph $G$ and an adjacency basis $B$ of $G$, we can obtaining large families of graphs having $B$ as a simultaneous adjacency generator.
Figure 2: A subfamily $\mathcal{H}$ of $G_{B}(C_{8})$, where $B = \{1, 3, 7\}$. For every $H_{i} \in \mathcal{H}$, $\dim_{A}(H_{i}) = \dim_{A}(C_{8}) = 3$. Moreover, $B$ is a simultaneous adjacency basis of $\mathcal{H}$, so $Sd_{A}(\mathcal{H}) = 3$.

**Theorem 2.** [18] Any adjacency basis $B$ of a graph $G$ is a simultaneous adjacency generator for any family of graphs $\mathcal{H} \subseteq G_{B}(G)$. Moreover, if $G \in \mathcal{H}$, then

$$Sd_{A}(\mathcal{H}) = \dim_{A}(G).$$

To illustrate this, Figure 2 shows a graph family $\mathcal{H} = \{H_{1}, H_{2}, H_{3}, H_{4}\} \subseteq G_{B}(C_{8})$, where $B = \{1, 3, 7\}$ and $Sd_{A}(\mathcal{H}) = \dim_{A}(C_{8})$.

3 Results on families of corona product graphs

Let $G$ be a graph of order $n$ and $H$ be a graph. The *corona product* of $G$ and $H$, denoted by $G \odot H$, was defined in [10] as the graph obtained from $G$ and $H$ by taking one copy of $G$ and $n$ copies of $H$ and joining by an edge each vertex from the $i$-th copy of $H$ with the $i$-th vertex of $G$. The reader is referred to [1, 2, 4–12, 16, 17, 22–25] for some known results on corona product graphs.

In order to present our results on the simultaneous metric (adjacency) dimension of graph families composed by corona product graphs, we need to introduce some additional notation. For a family $\mathcal{G}$ of connected non-trivial graphs defined on a common vertex set $V$ and a family $\mathcal{H}$ of non-trivial graphs defined on a common vertex set $V'$, we define the family

$$\mathcal{G} \odot \mathcal{H} = \{G \odot H : G \in \mathcal{G} \text{ and } H \in \mathcal{H}\}.$$
In particular, if $\mathcal{G} = \{G\}$, we will use the notation $G \odot \mathcal{H}$, whereas if $\mathcal{H} = \{H\}$, we will use the notation $\mathcal{G} \odot H$.

Given $G \in \mathcal{G}$ and $H \in \mathcal{H}$, we denote by $H_i = (V'_i, E_i)$ the subgraph of $G \odot H$ corresponding to the $i$-th copy of $H$. Notice that for any $i \in V$ the graph $H_i$, which is isomorphic to $H$, does not depend on $G$. Hence, the graphs in $\mathcal{G} \odot \mathcal{H}$ are defined on the vertex set $V \cup \left( \bigcup_{i \in V} V'_i \right)$. Analogously, for every $i \in V$ we define the graph family

$$\mathcal{H}_i = \{H_i = (V''_i, E_i) : H \in \mathcal{H}\}.$$ 

Also, given a set $W \subset V'$ and $i \in V$, we denote by $W_i$ the subset of $V''_i$ corresponding to $W$. To clarify this notation, Figure 3 shows the graph $C_4 \odot (K_1 \cup K_2)$. In the figure, $V = \{1, 2, 3, 4\}$ and $V' = \{a, b, c\}$, whereas $V''_i = \{a_i, b_i, c_i\}$ for $i \in \{1, 2, 3, 4\}$.

![Figure 3: The graph $G \odot H$, where $G \cong C_4$ and $H \cong K_1 \cup K_2$.](image)

### 3.1 Simultaneous metric dimension

We first introduce a useful relation between the metric generators of two corona product graphs with a common second factor, which allows to determine the simultaneous metric dimension of several families of corona product graphs through the study of the metric dimension of a specific corona product graph.

**Theorem 3.** Let $G_1$ and $G_2$ be two connected non-trivial graphs on a common vertex set and let $H$ be a non-trivial graph. Then any metric generator for $G_1 \odot H$ is a metric generator for $G_2 \odot H$.

**Proof.** Let $V$ be the vertex set of $G_1$ and $G_2$ and let $V'$ be the vertex set of $H$. We claim that any metric generator $B$ for $G_1 \odot H$ is a metric generator for $G_2 \odot H$. 

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To see this, we differentiate the following three cases for two different vertices \( x, y \in V(G_2 \odot H) - B \).

1. \( x, y \in V'_i \). Since no vertex belonging to \( B - V'_i \) distinguishes the pair \( x, y \) in \( G_1 \odot H \), there must exist \( u \in V'_i \cap B \) which distinguishes them. This vertex \( u \) also distinguishes \( x \) and \( y \) in \( G_2 \odot H \).

2. Either \( x \in V'_i \) and \( y \in V'_j \) or \( x = i \) and \( y \in V'_j \), where \( i \neq j \). For these two possibilities we take \( u \in B \cap V'_i \) and we conclude that \( d_{G_2 \odot H}(x, u) \leq 2 \neq 3 \leq d_{G_2 \odot H}(y, u) \).

3. \( x = i \) and \( y = j \). In this case for \( u \in B \cap V'_i \) we have \( d_{G_2 \odot H}(x, u) = 1 \neq 2 \leq d_{G_2 \odot H}(y, u) \).

In conclusion, \( B \) is a metric generator for \( G_2 \odot H \). \(\square\)

The following result is a direct consequence of Theorem 3.

**Corollary 4.** Let \( \mathcal{G} \) be a family of connected non-trivial graphs on a common vertex set and let \( \mathcal{H} \) be a family of non-trivial graphs on a common vertex set. Then, for any \( G \in \mathcal{G} \),

\[
Sd(\mathcal{G} \odot \mathcal{H}) = Sd(G \odot \mathcal{H}).
\]

The following result, obtained in \([8]\), provides a strong link between the metric dimension of the corona product of two graphs and the adjacency dimension of the second graph involved in the product operation.

**Theorem 5.** \([8]\) For any connected graph \( G \) of order \( n \geq 2 \) and any non-trivial graph \( H \),

\[
\dim(G \odot H) = n \cdot \dim_A(H).
\]

We now present a generalisation of Theorem 5 to deal with graph families.

**Theorem 6.** For any family \( \mathcal{G} \) composed by connected non-trivial graphs on a common vertex set \( V \) and any family \( \mathcal{H} \) composed by non-trivial graphs on a common vertex set,

\[
Sd(\mathcal{G} \odot \mathcal{H}) = |V| \cdot Sd_A(\mathcal{H}).
\]

**Proof.** Throughout the proof we consider two arbitrary graphs \( G \in \mathcal{G} \) and \( H \in \mathcal{H} \). Let \( B \) be a simultaneous metric basis of \( \mathcal{G} \odot \mathcal{H} \) and let \( B_i = B \cap V'_i \). Clearly, \( B_i \cap B_j = \emptyset \) for every \( i \neq j \). Since no pair of vertices \( x, y \in H_i \) is distinguished by any vertex \( v \in B_j, i \neq j \), we have that \( B_i \) is an adjacency generator for \( H_i \). Hence, the set \( B' \subset V' \) corresponding to \( B_i \subset V'_i \) is an adjacency generator for \( H \) and, since \( B' \) does not depend on the election of \( H \), it is a simultaneous adjacency generator for \( \mathcal{H} \) and, as a result,

\[
Sd(\mathcal{G} \odot \mathcal{H}) = |B| \geq \sum_{i \in V} |B_i| = |V||B'| \geq |V| \cdot Sd_A(\mathcal{H}).
\]
Now, let $W$ be a simultaneous adjacency basis of $\mathcal{H}$ and let $W_i = W \cap V'_i$. By analogy to the proof of Theorem 5 we see that $S = \bigcup_{i \in V} W_i$ is a metric generator for $G \odot H$. Since $S$ does not depend on the election of $G$ and $H$, it is a simultaneous metric generator for $G \odot H$ and so
\[
Sd(\mathcal{G} \odot \mathcal{H}) \leq |S| = \sum_{i \in V} |W_i| = |V| \cdot Sd_A(\mathcal{H}).
\]
Therefore, the equality holds.

The following result is a direct consequence of Theorems 2 and 6.

**Proposition 7.** Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$. Let $H$ be a non-trivial graph and let $B$ be an adjacency basis of $H$. Then, for every $\mathcal{H} \subseteq \mathcal{G}_B(H)$ such that $H \in \mathcal{H}$,
\[
Sd(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \dim_A(H).
\]

### 3.2 Simultaneous adjacency dimension

Given a family $\mathcal{G}$ of connected non-trivial graphs on a common vertex set $V$ and a family $\mathcal{H}$ of non-trivial graphs on a common vertex set, Remark 1 and Theorem 6 lead to

\[
Sd_A(\mathcal{G} \odot \mathcal{H}) \geq Sd(\mathcal{G} \odot \mathcal{H}) = |V| \cdot Sd_A(\mathcal{H}).
\]

Therefore, there exists an integer $f(\mathcal{G}, \mathcal{H}) \geq 0$ such that
\[
Sd_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot Sd_A(\mathcal{H}) + f(\mathcal{G}, \mathcal{H}).
\]

If $\mathcal{G} = \{G\}$ or $\mathcal{H} = \{H\}$, we will use the notations $f(\mathcal{G}, \mathcal{H})$, $f(\mathcal{G}, H)$ or $f(G, H)$, as convenient. It is easy to check that for any simultaneous adjacency basis $W$ of $\mathcal{H}$ and any $i \in V$, the set $(V - \{i\}) \cup \left( \bigcup_{j \in V} W_j \right)$ is a simultaneous adjacency generator for $\mathcal{G} \odot \mathcal{H}$, where $W_j$ is the subset of $V'_j$ corresponding to $W \subset V'$. Hence,
\[
0 \leq f(\mathcal{G}, \mathcal{H}) \leq |V| - 1.
\]

From now on, our goal is to determine the value of $f(\mathcal{G}, \mathcal{H})$ under different sets of conditions. We begin by pointing out a useful fact which we will use throughout the remainder of this section. Let $B$ be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$, and let $B_i = B \cap V'_i$. The following observation is a consequence of the fact that for any graph $G \odot H \in \mathcal{G} \odot \mathcal{H}$ and $i \in V$, no vertex in $B - B_i$ is able to distinguish two vertices in $V'_i$.

**Remark 8.** Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$ and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set $V'$. Let $B$ be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$ and let $B_i = B \cap V'_i$ for every $i \in V$. Then, $B_i$ is a simultaneous adjacency generator for $\mathcal{H}_i$. 

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Now, consider the following known result where $f(G, H) = 0$.

**Theorem 9.** [8] Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a non-trivial graph. If there exists an adjacency basis $S$ of $H$, which is also a dominating set, and if for every $v \in V(H) - S$, it is satisfied that $S \not\subseteq N_H(v)$, then

$$\dim_A(G \odot H) = n \cdot \dim_A(H).$$

As the next result shows, Theorem 9 can be generalised to the case of families of the form $G \odot H$. To that end, recall the notion of simultaneous domination which, as we mentioned previously, was introduced in [3]. On a graph family $G$, defined on a common vertex set $V$, a set $M \subseteq V$ is a simultaneous dominating set if it is a dominating set of every graph $G \in G$.

**Theorem 10.** Let $G$ be a family of connected non-trivial graphs on a common vertex set $V$ and let $H$ be a family of non-trivial graphs on a common vertex set $V'$. If there exists a simultaneous adjacency basis $B$ of $H$ which is also a simultaneous dominating set and satisfies $B \not\subseteq N_H(v)$ for every $H \in H$ and every $v \in V'$, then

$$\text{Sd}_A(G \odot H) = |V| \cdot \text{Sd}_A(H).$$

*Proof.* By (1) we only need to show that $\text{Sd}_A(G \odot H) \leq |V| \cdot \text{Sd}_A(H)$. To this end, assume that $B$ is a simultaneous adjacency basis of $H$ which is a simultaneous dominating set of $H$ and satisfies $B \not\subseteq N_H(v)$ for every $H \in H$ and every $v \in V'$. Consider an arbitrary graph $G \odot H \in G \odot H$ and let $B_i = B \cap V'_i$, for every $i \in V$. By analogy to the proof of Theorem 9 we see that $S = \bigcup_{i \in V} B_i$ is an adjacency generator for $G \odot H$ and, since $S$ does not depend on the election of $G$ and $H$, it is a simultaneous adjacency generator for $G \odot H$. Thus, $\text{Sd}_A(G \odot H) \leq |S| = |V| \cdot \text{Sd}_A(H)$, and the equality holds. \qed

In order to analyse special cases of Theorem 10, we introduce the following auxiliary results.

**Lemma 11.** [18] Let $G$ be a connected graph. If $D(G) \geq 6$ or $G \in \{P_n, C_n\}$ for $n \geq 7$, or $G$ is a graph of girth $g(G) \geq 5$ and minimum degree $\delta(G) \geq 3$ then for every adjacency generator $B$ for $G$ and every $v \in V(G)$, $B \not\subseteq N_G(v)$.

**Lemma 12.** [18] Let $P_n$ and $C_n$ be a path and a cycle graph of order $n \geq 7$. If $n \not\equiv 1 \pmod{5}$ and $n \not\equiv 3 \pmod{5}$, then there exist adjacency bases of $P_n$ and $C_n$ that are dominating sets.

**Lemma 13.** [15] For any integer $n \geq 4$,

$$\dim_A(C_n) = \dim_A(P_n) = \left\lfloor \frac{2n + 2}{5} \right\rfloor.$$
**Proposition 14.** Let \( G \) be a family of connected non-trivial graphs on a common vertex set \( V \). Let \( P_n \) be a path graph of order \( n \geq 7 \) such that \( n \not\equiv 1 \mod 5 \) and \( n \not\equiv 3 \mod 5 \), and let \( C_n \) be the cycle graph obtained from \( P_n \) by joining its leaves by an edge. Let \( B \) be an adjacency basis of \( P_n \) and \( C_n \) which is also a dominating set of both. Then, for every \( H \subseteq G_B(P_n) \cup G_B(C_n) \) such that \( P_n \in H \) or \( C_n \in H \),

\[
Sd_A(G \odot H) = |V| \cdot \left\lceil \frac{2n + 2}{5} \right\rceil.
\]

*Proof.* The existence of \( B \) is a consequence of Lemma 12. Since \( P_n \in H \) or \( C_n \in H \), by Theorem 2 we deduce that \( B \) is a simultaneous adjacency basis of \( H \). Let \( V' = V(P_n) = V(C_n) \). By the definition of \( G_B \), we have that \( \bigcup_{v \in B} N_H(v) = \bigcup_{v \in B} N_{P_n}(v) = V' \) or \( \bigcup_{v \in B} N_H(v) = \bigcup_{v \in B} N_{C_n}(v) = V' \) for every \( H \in H \), so \( B \) is a dominating set of every \( H \in H \). Moreover, by Lemma 11, we have that \( B \not\subseteq N_{P_n}(v) \) and \( B \not\subseteq N_{C_n}(v) \) for every \( v \in V' \). Furthermore, by the definition of \( G_B \), we have that \( B \cap N_H(v) = B \cap N_{P_n}(v) \) or \( B \cap N_H(v) = B \cap N_{C_n}(v) \) for every \( H \in H \) and every \( v \in V' \), so \( B \not\subseteq N_H(v) \) for every \( H \in H \) and every \( v \in V' \). In consequence, the result follows from Lemma 13 and Theorems 2 and 10. \( \square \)

In order to show some cases where \( f(G, H) = |V| - 1 \), we present the following result.

**Theorem 15.** Let \( G \) be a family of connected non-trivial graphs on a common vertex set \( V \) and let \( H \) be a family of non-trivial graphs on a common vertex set. If for every simultaneous adjacency basis \( B \) of \( H \) there exists \( H \in H \) where \( B \) is not a dominating set, then

\[
Sd_A(G \odot H) = |V| \cdot Sd_A(H) + |V| - 1.
\]

*Proof.* By (2) and (3) we have that \( Sd_A(G \odot H) \leq |V| \cdot Sd_A(H) + |V| - 1 \). It remains to prove that \( Sd_A(G \odot H) \geq |V| \cdot Sd_A(H) + |V| - 1 \).

Let \( U \) be a simultaneous adjacency basis of \( G \odot H \), let \( U_i = U \cap V'_i \) and let \( U_0 = U \cap V \). By Remark 8, \( U_i \) is a simultaneous adjacency generator for \( H_i \) for every \( i \in V \). Consider the partition \( \{V_1, V_2\} \) of \( V \) defined as

\[
V_1 = \{i \in V : |U_i| = Sd_A(H)\} \quad \text{and} \quad V_2 = \{i \in V : |U_i| \geq Sd_A(H) + 1\}.
\]

For any \( i, j \in V_1 \), \( i \neq j \), we have that there exist a graph \( H \in H \) and two vertices \( x \in V'_i - U_i \) and \( y \in V'_j - U_j \) such that \( U_i \cap N_H(x) = \emptyset \) and \( U_j \cap N_H(y) = \emptyset \). Thus, \( i \in U \) or \( j \in U \) and so \( |U_0| \geq |V_1| - 1 \). In conclusion,

\[
Sd_A(G \odot H) = |U_0| + \sum_{i \in V_1} |U_i| + \sum_{i \in V_2} |U_i|
\]

\[
\geq (|V_1| - 1) + |V_1| \cdot Sd_A(H) + |V_2| \cdot (Sd_A(H) + 1)
\]

\[
= |V| \cdot Sd_A(H) + |V| - 1.
\]

Therefore, the result follows. \( \square \)
Now we treat some specific families for which the previous results hold. We first introduce an auxiliary result.

**Lemma 16.** Let $P_n$ and $C_n$ be a path and a cycle graph of order $n \geq 7$. If $n \equiv 1 \mod 5$ or $n \equiv 3 \mod 5$, then no adjacency basis of $P_n$ or $C_n$ is a dominating set.

**Proof.** In $C_n$, consider an adjacency basis $B$ and a path $v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}$, where the subscripts are taken modulo $n$. If $v_i, v_{i+2} \in B$ and $v_{i+1} \notin B$, then $\{v_{i+1}\}$ is said to be a 1-gap of $B$. Likewise, if $v_i, v_{i+3} \in B$ and $v_{i+1}, v_{i+2} \notin B$, then $\{v_{i+1}, v_{i+2}\}$ is said to be a 2-gap of $B$ and if $v_i, v_{i+4} \in B$ and $v_{i+1}, v_{i+2}, v_{i+3} \notin B$, then $\{v_{i+1}, v_{i+2}, v_{i+3}\}$ is said to be a 3-gap of $B$. Since $B$ is an adjacency basis of $C_n$, it has no gaps of size 4 or larger and it has at most one 3-gap. Moreover, every 2- or 3-gap must be neighboured by two 1-gaps and the number of gaps of either size is at most $\dim_A(C_n)$.

We now differentiate the following cases for $C_n$:

1. $n = 5k + 1$, $k \geq 2$. In this case, by Lemma 13, $\dim_A(C_n) = 2k$, and thus $n - \dim_A(C_n) = 3k + 1$. Since any 2-gap must be neighboured by two 1-gaps, any adjacency basis $B$ of $C_n$ has at most $k$ 2-gaps. Now, assume that $B$ has no 3-gaps. Then $|V(C_n) - B| = 3k < 3k + 1 = n - |B|$, which is a contradiction. Thus, any adjacency basis of $C_n$ has a 3-gap, i.e. it is not a dominating set.

2. $n = 5k + 3$, $k \geq 1$. In this case, by Lemma 13, $\dim_A(C_n) = 2k + 1$, and thus $n - \dim_A(C_n) = 3k + 2$. As in the previous case, any adjacency basis $B$ of $C_n$ has at most $k$ 2-gaps. Now assume that $B$ has no 3-gaps. Then $|V(C_n) - B| = 3k + 1 < 3k + 2 = n - |B|$, which is a contradiction. Thus, any adjacency basis of $C_n$ has a 3-gap, i.e. it is not a dominating set.

By the set of cases above, the result holds for $C_n$.

Now, let $C'_n$ be the cycle obtained from $P_n$ by joining its leaves $v_1$ and $v_n$ by an edge. Let $V = V(P_n) = V(C'_n)$ and let $B$ be an adjacency basis of $P_n$. Since for two different vertices $x, y \in V$, $d_{C'_n}(x, y) \neq d_{P_n}(x, y)$ if and only if $x, y \in \{v_1, v_n\}$, if $v_1, v_n \in B$ or $v_1, v_n \notin B$, then $B$ is an adjacency basis of $C'_n$. Moreover, some vertex $w \in V - B$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so $B$ is not a dominating set of $P_n$. We now treat the case where $v_1 \in B$ and $v_n \notin B$. If $v_{n-1} \notin B$ then $B$ is not a dominating set of $P_n$. If $v_{n-1} \in B$ and $v_2 \notin B$, we have that $d_{C'_n}(v_2, v_{n-1}) = d_{P_n}(v_2, v_{n-1}) = 2 \neq 1 = d_{P_n}(v_n, v_{n-1}) = d_{C'_n}(v_n, v_{n-1})$, whereas for any other pair of different vertices $x, y \in V - B$ there exists $z \in B$ such that $d_{C'_n}(x, z) = d_{P_n}(x, z) \neq d_{P_n}(y, z) = d_{C'_n}(y, z)$, so $B$ is an adjacency basis of $C'_n$ where $\{v_n\}$ is a 1-gap. In consequence, some vertex $w \in V - (B \cup \{v_n\})$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so $B$ is not a dominating set of $P_n$. Finally, if $v_2, v_{n-1} \in B$, then for any pair of different vertices $x, y \in V - B$ there exists $z \in B - \{v_1\}$ such that $d_{C'_n}(x, z) = d_{P_n}(x, z) \neq d_{P_n}(y, z) = d_{C'_n}(y, z)$, so $B$ is an adjacency basis of $C'_n$ where $\{v_n\}$ is a 1-gap. As in the previous case, some vertex $w \in V - (B \cup \{v_n\})$ satisfies $B \cap N_{P_n}(w) = B \cap N_{C'_n}(w) = \emptyset$, so $B$ is not a dominating set of $P_n$. The proof is complete. \(\square\)

Lemma 16 allows us to give the following result.
Proposition 17. Let $G$ be a family of connected non-trivial graphs on a common vertex set $V$. Let $P_n$ be a path graph of order $n \geq 7$, $n \equiv 1 \mod 5$ or $n \equiv 3 \mod 5$, and let $C_n$ be the cycle graph obtained from $P_n$ by joining its leaves by an edge. Let $B$ be a simultaneous adjacency basis of $\{P_n, C_n\}$. Then, for every family $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ such that $\mathcal{H}_1$ is composed by paths, $\mathcal{H}_1 \subseteq \mathcal{G}_B(P_n)$, $P_n \in \mathcal{H}_1$, $\mathcal{H}_2$ is composed by cycles $\mathcal{H}_2 \subseteq \mathcal{G}_B(C_n)$, and $C_n \in \mathcal{H}_2$,

$$\text{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \left(\left\lfloor \frac{2n+2}{5} \right\rfloor + 1 \right) - 1.$$ 

Proof. Note that $B$ is an adjacency basis of both $P_n$ and $C_n$. Since $P_n \in \mathcal{H}_1$ and $C_n \in \mathcal{H}_2$, we have that $B$ is a simultaneous adjacency basis of $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ by Theorem 2. Moreover, since every $H \in \mathcal{H}_1$ is a path graph and every $H \in \mathcal{H}_2$ is a cycle we have that $\dim_A(H) = \text{Sd}_A(H)$ for every $H \in \mathcal{H}$, so every simultaneous adjacency basis of $\mathcal{H}$ is an adjacency basis of every $H \in \mathcal{H}$ and, by Lemma 16, is not a dominating set of $H$. Thus, the result follows from Theorem 15. 

It is worth noting that for a path graph $P_n$ and a cycle graph $C_n$, $n \geq 7$, $n \equiv 1 \mod 5$ or $n \equiv 3 \mod 5$, and an adjacency basis $B$ of both, the family $\mathcal{G}_B(P_n)$ contains $(n - \left\lfloor \frac{2n+2}{5} \right\rfloor)!$ path graphs, whereas the family $\mathcal{G}_B(C_n)$ contains $(n - \left\lfloor \frac{2n+2}{5} \right\rfloor)!$ cycle graphs.

Proposition 18. Let $G$ be a family of connected non-trivial graphs on a common vertex set $V$ and let $\mathcal{H} = \{N_t \cup H_1, N_t \cup H_2, \ldots, N_t \cup H_k\}$, where $N_t$ is an empty graph of order $t \geq 1$ and $H_1, H_2, \ldots, H_k$ are connected non-trivial graphs on a common vertex set. Then,

$$\text{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \text{Sd}_A(\mathcal{H}) + |V| - 1.$$ 

Proof. Consider that the common vertex set of $\mathcal{H}$ has the form $V' = V(N_t) \cup V''$, where $V(N_t)$ and $V''$ are disjoint. Let $B$ be a simultaneous adjacency basis of $\mathcal{H}$, and let $B'' = B \cap V''$. Consider an arbitrary graph $N_t \cup H \in \mathcal{H}$. The vertices of $N_t$ are false twins, so $V(N_t) \subseteq B$ if and only if there exists $v \in V''$ such that $B \cap N_H(v) = \emptyset$. If such $v$ exists, it is not dominated by $B$, so the result follows from Theorem 15. Otherwise, $V(N_t) - B = \{v'\}$ and $B \cap N_H(v') = \emptyset$, so the result follows from Theorem 15. 

As usual, given a graph $G$, we denote by $\gamma(G)$ the domination number of $G$.

Theorem 19. [8] Let $G$ be a connected graph of order $n \geq 2$ and let $H$ be a non-trivial graph. If there exists an adjacency basis of $H$, which is also a dominating set and if, for any adjacency basis $S$ of $H$, there exists $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then

$$\dim_A(G \odot H) = n \cdot \dim_A(H) + \gamma(G).$$

The simultaneous domination number of a family $\mathcal{G}$, which we will denote as $S\gamma(\mathcal{G})$, is the minimum cardinality of a simultaneous dominating set. The next result is a generalisation of Theorem 19 to the case of $\mathcal{G} \odot \mathcal{H}$. 

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Theorem 20. Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$ and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set $V'$. If there exists a simultaneous adjacency basis of $\mathcal{H}$ which is also a simultaneous dominating set, and for every simultaneous adjacency basis $B$ of $\mathcal{H}$ there exist $H \in \mathcal{H}$ and $v \in V' - B$ such that $B \subseteq N_H(v)$, then

$$\text{Sd}_A(\mathcal{G} \odot \mathcal{H}) = |V| \cdot \text{Sd}_A(\mathcal{H}) + S_\gamma(\mathcal{G}).$$

Proof. We first address the proof of $\text{Sd}_A(\mathcal{G} \odot \mathcal{H}) \geq |V| \cdot \text{Sd}_A(\mathcal{H}) + S_\gamma(\mathcal{G})$. Let $U$ be a simultaneous adjacency basis of $\mathcal{G} \odot \mathcal{H}$, let $U_i = U \cap V_i'$, and let $U_0 = U \cap V$. By Remark 8, $U_i$ is a simultaneous adjacency basis of $\mathcal{H}_i$ for every $i \in V$. Consider the partition $\{V_1, V_2\}$ of $V$ defined as

$$V_1 = \{i \in V : |U_i| = \text{Sd}_A(\mathcal{H})\} \text{ and } V_2 = \{i \in V : |U_i| \geq \text{Sd}_A(\mathcal{H}) + 1\}.$$

For every $i \in V_1$, the set $U_i$ is a simultaneous adjacency basis of $\mathcal{H}_i$, so there exist $H \in \mathcal{H}$ and $x \in V_i'$ such that $U_i \subseteq N_H(x)$, causing $i$ and $x$ not to be distinguished by any $y \in U_i$ in any graph belonging to $\mathcal{G} \odot H$. Thus, either $i \in U_0$ or for every $G \in \mathcal{G}$ there exists $z \in U_0$ such that $d_{G \odot H_2}(i, z) = 1 \neq 2 = d_{G \odot H_2}(x, z)$. In consequence, $V_2 \cup U_0$ must be a simultaneous dominating set of $\mathcal{G}$, so $|V_2 \cup U_0| \geq S_\gamma(\mathcal{G})$. Finally,

$$\text{Sd}_A(\mathcal{G} \odot \mathcal{H}) = \sum_{i \in V_1} |U_i| + \sum_{i \in V_2} |U_i| + |U_0|$$

$$\geq \sum_{i \in V_1} \text{Sd}_A(\mathcal{H}) + \sum_{i \in V_2} (\text{Sd}_A(\mathcal{H}) + 1) + |U_0|$$

$$= |V| \cdot \text{Sd}_A(\mathcal{H}) + |V_2| + |U_0|$$

$$\geq |V| \cdot \text{Sd}_A(\mathcal{H}) + |V_2 \cup U_0|$$

$$\geq |V| \cdot \text{Sd}_A(\mathcal{H}) + S_\gamma(\mathcal{G}).$$

Now, let $W$ be a simultaneous adjacency basis of $\mathcal{H}$ which is also a simultaneous dominating set of $\mathcal{H}$. Consider an arbitrary graph $G \odot H \in \mathcal{G} \odot \mathcal{H}$, and let $W_i = W \cap V_i'$. By analogy to the proof of Theorem 19, we have that $S = M \cup \left( \bigcup_{i \in V} W_i \right)$, where $M$ is a minimum simultaneous dominating set of $\mathcal{G}$, is an adjacency generator for $G \odot H$. Since $S$ does not depend on the election of $G$ and $H$, it is a simultaneous adjacency generator for $\mathcal{G} \odot \mathcal{H}$. Thus, $\text{Sd}_A(\mathcal{G} \odot \mathcal{H}) \leq |S| = |V| \cdot \text{Sd}_A(\mathcal{H}) + S_\gamma(\mathcal{G})$, so the equality holds.

Several specific families for which the previous result holds will be described in Theorem 28 and Propositions 29 and 30. Now, in order to present our next result, we need some additional definitions. Let $v \in V(G)$ be a vertex of a graph $G$ and let $G - v$ be the graph obtained by removing from $G$ the vertex $v$ and all its incident edges. Consider the following auxiliary domination parameter, which is defined in [8]:

$$\gamma'(G) = \min_{v \in V(G)} \{\gamma(G - v)\}$$

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Theorem 21. [8] Let $H$ be a non-trivial graph such that some of its adjacency bases are also dominating sets, and some are not. If there exists an adjacency basis $S'$ of $H$ such that for every $v \in V(H) - S'$ it is satisfied that $S' \not\subseteq N_H(v)$, and for any adjacency basis $S$ of $H$ which is also a dominating set, there exists some $v \in V(H) - S$ such that $S \subseteq N_H(v)$, then for any connected non-trivial graph $G$

$$\dim_A(G \circ H) = n \cdot \dim_A(H) + \gamma'(G).$$

The following result is a generalisation of Theorem 21 to the case of $G \circ \mathcal{H}$.

Theorem 22. Let $G$ be a connected graph of order $n \geq 2$ and let $\mathcal{H}$ be a family of non-trivial graphs on a common vertex set $V'$ such that some of its simultaneous adjacency bases are also simultaneous dominating sets, and some are not. If there exists a simultaneous adjacency basis $B'$ of $\mathcal{H}$ such that $B' \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V' - B'$, and for every simultaneous adjacency basis $B$ of $H$ which is also a simultaneous dominating set there exist $H' \in \mathcal{H}$ and $w \in V' - B$ such that $B \subseteq N_{H'}(w)$, then

$$\text{Sd}_A(G \circ \mathcal{H}) = n \cdot \text{Sd}_A(\mathcal{H}) + \gamma'(G).$$

Proof. In the family $G \circ \mathcal{H}$, we have that $V = V(G)$. We first address the proof of $\text{Sd}_A(G \circ \mathcal{H}) \geq n \cdot \text{Sd}_A(\mathcal{H}) + \gamma'(G)$. Let $U$ be a simultaneous adjacency basis of $G \circ \mathcal{H}$, let $U_i = U \cap V'_i$, and let $U_0 = B \cap V$. By Remark 8, $U_i$ is a simultaneous adjacency generator for $\mathcal{H}_i$ for every $i \in V$. Consider the partition $\{V_1, V_2, V_3\}$ of $V$, where $V_1$ contains the vertices $i \in V$ such that $U_i$ is a simultaneous adjacency basis of $\mathcal{H}_i$ but is not a simultaneous dominating set, $V_2$ contains the vertices $i \in V$ such that $U_i$ is a simultaneous adjacency basis and a simultaneous dominating set of $\mathcal{H}_i$, and $V_3$ is composed by the vertices $i \in V$ such that $U_i$ is not a simultaneous adjacency basis of $\mathcal{H}_i$.

If $i, j \in V_1$, then there exist a graph $H \in \mathcal{H}$ and two vertices $v_i \in V'_i - U_i$ and $v_j \in V'_j - U_j$ such that $U_i \cap N_H(v_i) = \emptyset$ and $U_j \cap N_H(v_j) = \emptyset$. Thus, $i \in U_0$ or $j \in U_0$, so $|U_0 \cap V_1| \geq |V_1| - 1$. If $i \in V_2$, then there exist $H \in \mathcal{H}$ and $x \in V'_i$ such that $U_i \subseteq N_H(x)$. In consequence, the pair $i, x$ is not distinguished by any $y \in U_i$, so either $i \in U_0$ or there exists $z \in U_0$ such that $d_{G \circ \mathcal{H}_2}(i, z) = 1 \neq 2 = d_{G \circ \mathcal{H}_2}(x, z)$. Therefore, at most one vertex of $G$ is not dominated by $U_0 \cup V_3$, so $|U_0 \cup V_3| \geq \gamma'(G)$. Finally,

$$\text{Sd}_A(G \circ \mathcal{H}) = \sum_{i \in V_1 \cup V_2} |U_i| + \sum_{i \in V_3} |U_i| + |U_0|
\geq \sum_{i \in V_1 \cup V_2} \text{Sd}_A(\mathcal{H}) + \sum_{i \in V_3} (\text{Sd}_A(\mathcal{H}) + 1) + |U_0|
= n \cdot \text{Sd}_A(\mathcal{H}) + |V_3| + |U_0|
\geq n \cdot \text{Sd}_A(\mathcal{H}) + |V_3 \cup U_0|
\geq n \cdot \text{Sd}_A(\mathcal{H}) + \gamma'(G).$$

Now, let $W'$ be a simultaneous adjacency basis of $\mathcal{H}$ such that $W' \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V - W'$, and assume that for any simultaneous adjacency
basis \( W \) of \( \mathcal{H} \) which is also a simultaneous dominating set there exist \( H' \in \mathcal{H} \) and \( w \in V - W \) such that \( W \subseteq N_{H'}(w) \). Let \( W'' \) be one of such simultaneous adjacency bases of \( \mathcal{H} \). Consider an arbitrary graph \( G \circ H \in G \circ \mathcal{H} \), let \( W'_i = W' \cap V'_i \) and \( W''_i = W'' \cap V'_i \). Additionally, let \( M \) be a minimum dominating set of \( G - n \), assuming without loss of generality that \( \gamma'(G) = \gamma(G - n) \), and let \( S = M \cup W'' \cup \bigcup_{i \in V - \{n\}} W''_i \). By analogy to the proof of Theorem 21, we have that \( S \) is an adjacency generator for \( G \circ H \). Since \( S \) does not depend on the election of \( G \) and \( H \), it is a simultaneous adjacency generator for \( G \circ \mathcal{H} \). Thus, \( \text{Sd}_A(G \circ \mathcal{H}) \leq |S| = n \cdot \text{Sd}_A(\mathcal{H}) + \gamma'(G) \), so the equality holds.

Consider the family \( \{P_5, C_5\} \), where \( C_5 \) is obtained from \( P_5 \) by joining its leaves with an edge. Assume that \( V(P_5) = V(C_5) = \{v_1, v_2, v_3, v_4, v_5\} \), \( E(P_5) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\} \) and \( E(C_5) = E(P_5) \cup \{v_1v_5\} \). We have that the set \( \{v_2, v_4\} \) is the sole simultaneous adjacency basis which is also a simultaneous dominating set and \( v_3 \) satisfies \( \{v_2, v_3\} \subseteq N_{P_5}(v_3) \) and \( \{v_2, v_4\} \subseteq N_{C_5}(v_3) \). Moreover, the set \( \{v_1, v_5\} \) (as well as \( \{v_2, v_3\} \) and \( \{v_3, v_4\} \)) is a simultaneous adjacency basis such that every vertex \( v_i \) satisfies \( N_{P_5}(v_i) \not\subseteq \{v_1, v_5\} \) and \( N_{C_5}(v_i) \not\subseteq \{v_1, v_5\} \). These facts allow us to obtain examples where Theorem 22 applies. For instance, for any connected graph \( G \) of order \( n \geq 2 \), we have that \( \text{Sd}_A(G \circ \{P_5, C_5\}) = 2n + \gamma'(G) \).

The case where the second factor is a family of join graphs

Given two vertex-disjoint graphs \( G = (V_1, E_1) \) and \( H = (V_2, E_2) \), the join of \( G \) and \( H \), denoted by \( G + H \), is the graph with vertex set \( V(G + H) = V_1 \cup V_2 \) and edge set \( E(G + H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\} \). For two graph families \( \mathcal{G} \) and \( \mathcal{H} \), defined on common vertex sets \( V_1 \) and \( V_2 \), respectively, such that \( V_1 \cap V_2 = \emptyset \), we define the family

\[
\mathcal{G} + \mathcal{H} = \{G + H : G \in \mathcal{G}, H \in \mathcal{H}\}.
\]

In particular, if \( \mathcal{G} = \{G\} \) we will use the notation \( G + \mathcal{H} \). To begin our presentation, we introduce the following auxiliary result.

**Lemma 23.** Let \( \mathcal{G} \) and \( \mathcal{H} \) be two families of non-trivial graphs on common vertex sets \( V_1 \) and \( V_2 \), respectively. Then, every simultaneous adjacency basis of \( \mathcal{G} + \mathcal{H} \) is a simultaneous dominating set of \( \mathcal{G} + \mathcal{H} \).

**Proof.** Let \( B \) be a simultaneous adjacency basis of \( \mathcal{G} + \mathcal{H} \), let \( W_1 = B \cap V_1 \) and \( W_2 = B \cap V_2 \). Since no pair of different vertices \( u, v \in V_2 - W_2 \) is distinguished in any \( G + H \in \mathcal{G} + \mathcal{H} \) by any vertex from \( W_1 \), we have that \( W_2 \) is a simultaneous adjacency generator for \( \mathcal{H} \) and, in consequence, \( W_2 \neq \emptyset \). By an analogous reasoning we can see that \( W_1 \) is a simultaneous adjacency generator for \( \mathcal{G} \) and, in consequence, \( W_1 \neq \emptyset \). Moreover, every vertex in \( V_1 \) is dominated by every vertex in \( W_2 \), whereas every vertex in \( V_2 \) is dominated by every vertex in \( W_1 \), so \( B \) is a dominating set for every \( G + H \in \mathcal{G} + \mathcal{H} \). \( \square \)
The following result, presented in [18], characterizes a large number of families of the form $\mathcal{G} + \mathcal{H}$ whose simultaneous adjacency bases are formed by the union of simultaneous adjacency bases of $\mathcal{G}$ and $\mathcal{H}$.

**Theorem 24.** [18] Let $\mathcal{G}$ and $\mathcal{H}$ be two families of non-trivial graphs on common vertex sets $V_1$ and $V_2$, respectively. If there exists a simultaneous adjacency basis $B$ of $\mathcal{G}$ such that for every $G \in \mathcal{G}$ and every $v \in V_1$, $B \not\subseteq N_G(v)$, then

$$Sd_A(\mathcal{G} + \mathcal{H}) = Sd_A(\mathcal{G}) + Sd_A(\mathcal{H}).$$

As discussed in the proof of Theorem 24, every simultaneous adjacency basis of a family $\mathcal{G} + \mathcal{H}$ satisfying the premises of the theorem is the union of a simultaneous adjacency basis of $\mathcal{H}$ and a simultaneous adjacency basis $B$ of $\mathcal{G}$ such that $B \not\subseteq N_G(v)$ for every $G \in \mathcal{G}$ and every $v \in V_1$.

**Theorem 25.** Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$, and let $\mathcal{H}$ and $\mathcal{H}'$ be families of non-trivial graphs on common vertex sets $V'_1$ and $V'_2$, respectively. If there exist a simultaneous adjacency basis $B$ of $\mathcal{H}$ that satisfies $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'_1$, and a simultaneous adjacency basis $B'$ of $\mathcal{H}'$ that satisfies $B' \not\subseteq N_{H'}(v')$ for every $H' \in \mathcal{H}'$ and every $v' \in V'_2$, then

$$Sd_A(\mathcal{G} \odot (\mathcal{H} + \mathcal{H}')) = |V| \cdot Sd_A(\mathcal{H}) + |V| \cdot Sd_A(\mathcal{H}').$$

**Proof.** Let $B$ and $B'$ be simultaneous adjacency bases of $\mathcal{H}$ and $\mathcal{H}'$, respectively, that satisfy the premises of the theorem, and let $S = B \cup B'$. As shown in the proof of Theorem 24, $S$ is a simultaneous adjacency basis of $\mathcal{H} + \mathcal{H}'$. Moreover, since $B \not\subseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'_1$, and $B' \not\subseteq N_{H'}(v')$ for every $H' \in \mathcal{H}'$ and every $v' \in V'_2$, we have that $S \not\subseteq N_{H+H'}(x)$ for every $H + H' \in \mathcal{H} + \mathcal{H}'$ and every $x \in V'_1 \cup V'_2$. Finally, by Lemma 23, we have that $S$ is a simultaneous dominating set of $\mathcal{H} + \mathcal{H}'$, so $Sd_A(\mathcal{G} \odot (\mathcal{H} + \mathcal{H}')) = |V| \cdot Sd_A(\mathcal{H} + \mathcal{H}') = |V| \cdot Sd_A(\mathcal{H}) + |V| \cdot Sd_A(\mathcal{H}')$ by Theorems 10 and 24.

The following result is a direct consequence of Lemma 11 and Theorem 25.

**Proposition 26.** Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$. Let $\mathcal{H}$ be a graph family on a common vertex set $V'_1$ of cardinality $|V'_1| \geq 7$ such that every $H \in \mathcal{H}$ is a path graph, a cycle graph, $D(H) \geq 6$, or $g(H) \geq 5$ and $\delta(H) \geq 3$. Let $\mathcal{H}'$ be a graph family on a common vertex set $V'_2$ of cardinality $|V'_2| \geq 7$ satisfying the same conditions as $\mathcal{H}$. Then,

$$Sd_A(\mathcal{G} \odot (\mathcal{H} + \mathcal{H}')) = |V| \cdot Sd_A(\mathcal{H}) + |V| \cdot Sd_A(\mathcal{H}').$$

In addition, following a reasoning analogous to that of the proofs of Propositions 14 and 17, we obtain the following result as a consequence of Lemma 11 and Theorems 2 and 25.
Proposition 27. Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$. Let $H$ be a graph of order $n \geq 7$ which is a path graph, or a cycle graph, or satisfies $D(H) \geq 6$, or $g(H) \geq 5$ and $\delta(H) \geq 3$. Let $H'$ be a graph of order $n' \geq 7$ that satisfies the same conditions as $H$. Let $B$ and $B'$ be adjacency bases of $H$ and $H'$, respectively. Then, for any pair of families $\mathcal{H} \subseteq \mathcal{G}_B(H)$ and $\mathcal{H}' \subseteq \mathcal{G}_{B'}(H')$ such that $H \in \mathcal{H}$ and $H' \in \mathcal{H}'$,

$$\text{Sd}_A(\mathcal{G} \cap (\mathcal{H} + \mathcal{H}')) = |V| \cdot \dim_A(H) + |V| \cdot \dim_A(H').$$

Theorem 28. Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$, and let $\mathcal{H}$ and $\mathcal{H}'$ be families of non-trivial graphs on common vertex sets $V_1'$ and $V_2'$, respectively. If there exists a simultaneous adjacency basis $B$ of $\mathcal{H}$ that satisfies $B \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V_1'$, and for every simultaneous adjacency basis $B'$ of $\mathcal{H}'$ there exist $H' \in \mathcal{H}$ and $v' \in V_2'$ such that $B' \subseteq N_{H'}(v')$, then

$$\text{Sd}_A(\mathcal{G} \cap (\mathcal{H} + \mathcal{H}')) = |V| \cdot \text{Sd}_A(\mathcal{H}) + |V| \cdot \text{Sd}_A(\mathcal{H}') + S_\gamma(\mathcal{G}).$$

Proof. Let $S$ be a simultaneous adjacency basis of $\mathcal{H} + \mathcal{H}'$, let $W = S \cap V_1'$ and let $W' = S \cap V_2'$. As discussed in the proof of Theorem 24, $W$ and $W'$ are simultaneous adjacency bases of $\mathcal{H}$ and $\mathcal{H}'$, respectively. Since there exist $H' \in \mathcal{H}$ and $v' \in V_2'$ such that $W' \subseteq N_{H'}(v')$, we have that $S \subseteq N_{H+H'}(v')$ for any $H \in \mathcal{H}$ by the definition of the join operation. Moreover, by Lemma 23, $S$ is a simultaneous dominating set of $\mathcal{H} + \mathcal{H}'$, so $\text{Sd}_A(\mathcal{G} \cap (\mathcal{H} + \mathcal{H}')) = |V| \cdot \text{Sd}_A(\mathcal{H} + \mathcal{H}') + S_\gamma(\mathcal{G}) = |V| \cdot \text{Sd}_A(\mathcal{H}) + |V| \cdot \text{Sd}_A(\mathcal{H}') + S_\gamma(\mathcal{G})$ by Theorems 20 and 24.

The following results are particular cases of Theorem 28.

Proposition 29. Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$. Let $\mathcal{H}$ be a graph family on a common vertex set $V'$ of cardinality $|V'| \geq 7$ such that every $H \in \mathcal{H}$ is a path graph, a cycle graph, $D(H) \geq 6$, or $g(H) \geq 5$ and $\delta(H) \geq 3$. Let $K_t$ be a complete graph of order $t \geq 2$. Then,

$$\text{Sd}_A(\mathcal{G} \cap (K_t + \mathcal{H})) = |V| \cdot \text{Sd}_A(\mathcal{H}) + |V| \cdot (t - 1) + S_\gamma(\mathcal{G}).$$

Proof. By Theorem 24, $\text{Sd}_A(K_t + \mathcal{H}) = \text{Sd}_A(\mathcal{H}) + t - 1$. Moreover, by Lemma 11, every simultaneous adjacency basis $B$ of $\mathcal{H}$ satisfies $B \nsubseteq N_H(v)$ for every $H \in \mathcal{H}$ and every $v \in V'$. Furthermore, every adjacency basis of $K_t$ has the form $B' = V(K_t) - \{v\}$, where $v$ is an arbitrary vertex of $K_t$. Clearly, $B' \subseteq N_{K_t}(v)$, so the result follows from Theorem 28.

Following a reasoning analogous to that of the proofs of Propositions 14 and 17, we obtain the following result as a consequence of Lemma 11 and Theorems 2, 24 and 28.

Proposition 30. Let $\mathcal{G}$ be a family of connected non-trivial graphs on a common vertex set $V$. Let $H$ be a graph of order $n \geq 7$ which is a path graph, or a cycle graph, or satisfies $D(H) \geq 6$, or $g(H) \geq 5$ and $\delta(H) \geq 3$. Let $K_t$ be a complete graph of order $t \geq 1$. Let $B$ be an adjacency basis of $H$. Then, for any family $\mathcal{H} \subseteq \mathcal{G}_B(H)$ such that $H \in \mathcal{H}$,

$$\text{Sd}_A(\mathcal{G} \cap (K_t + \mathcal{H})) = |V| \cdot \dim_A(H) + |V| \cdot (t - 1) + S_\gamma(\mathcal{G}).$$
As an example of the previous result, consider an arbitrary family $\mathcal{G}$ composed by connected non-trivial graphs on a common vertex set $V$, a complete graph $K_t$ of order $t \geq 2$, a path graph $P_n$ of order $n \geq 7$, and the cycle graph $C_n$ obtained from $P_n$ by joining its leaves by an edge. For any simultaneous adjacency basis $B$ of $\{P_n, C_n\}$ and any family $\mathcal{H} \in \mathcal{G}_B(P_n) \cup \mathcal{G}_B(C_n)$ such that $P_n \in \mathcal{H}$ or $C_n \in \mathcal{H}$, we have that

$$\text{Sd}_A(\mathcal{G} \circ (K_t + \mathcal{H})) = |V| \cdot \left(\left\lfloor \frac{2n + 2}{5} \right\rfloor + t - 1 \right) + S_\gamma(\mathcal{G}).$$

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