Forbidden patterns in financial time series

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The existence of forbidden patterns, i.e., certain missing sequences in a given time series, is a recently proposed instrument of potential application in the study of time series. Forbidden patterns are related to the permutation entropy, which has the basic properties of classic chaos indicators, such as Lyapunov exponent or Kolmogorov entropy, thus allowing to separate deterministic (usually chaotic) from random series; however, it requires less values of the series to be calculated, and it is suitable for using with small datasets. In this Letter, the appearance of forbidden patterns is studied in different economical indicators like stock indices (Dow Jones Industrial Average and Nasdaq Composite), NYSE stocks (IBM and Boeing) and others (10-year Bond interest rate), to find evidences of deterministic behavior in their evolutions. Moreover, the rate of appearance of the forbidden patterns is calculated, and some considerations about the underlying dynamics are suggested.

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Extracting information from real time series has been a hot topic during the last decades [1, 2, 3, 4, 5, 6, 7, 10, 11, 12]. From the point of view of time series analysis two main goals arise when facing a real evolution of a certain variable: first, identifying the underlying nature of the phenomenon represented by the sequence of observations and, second, trying to predict the evolution of the variable. Both of these goals, identification and forecasting, require the treatment of the time series, usually by combining different tools. Statistical methods in order to obtain a model of the mean process have been the classical approach, leading to autoregressive, integrated and moving average models [3, 8]. Characterization of non-linear time series, mainly chaotic, has also attracted the interest of the scientific community [8, 9], where phase space reconstruction, spectral analysis or wavelets methods have been revealed to be good indicators of the underlying dynamics of a real time series.

Recently the study of the order patterns has been proposed as a technique of evaluating the determinism of a given time series [10–12]. Consider a discrete information source emitting a series of observable values \(x_1, x_2, \ldots, x_N\), ordered by time; it is possible to split the data in overlapping sets of length \(d\), and study their order patterns, e.g., \(x_1 < x_2 < \ldots < x_d\). Every group of \(d\) adjacent values form a certain permutation \(\Pi\), which is one of the \(d!\) possible permutations [12]. The basis of the topological permutation entropy [13] is to define, for every group of \(d\) adjacent values within a discrete dataset, the corresponding permutation pattern, and study the overall statistics of these patterns [10, 11]. For a pattern dimension of, e.g., \(d = 3\), if \(x_2 < x_1 < x_3\), the resulting pattern would be \(\Pi = (2, 1, 3)\), i.e., the lower value of the series is the second, followed by the first, while the last value is the highest. In principle, the dimension \(d\) could be any integer higher than one and several values of \(d\) could be defined within the same dataset. We just have to take into account that the higher the dimension considered, the larger the quantity of data needed, so in this paper we consider \(3 < d < 7\), which are high enough values to distinguish the existence (or not) of forbidden patterns.

When analyzing a time series of length \(N\), we obtain \(N - d + 1\) overlapping groups of adjacent values, each one with a corresponding order pattern. If the series has a random behavior, any permutation can appear, and therefore no pattern is forbidden. Moreover, their probability distribution should be flat since any permutation has the same probability of occurrence when the dataset is long enough to exclude statistical fluctuations. Nevertheless, when the series corresponds to a chaotic variable there are some patterns that cannot be encountered in the data due to the underlying deterministic structure: they are the so-called forbidden patterns. It has been demonstrated that most chaotic systems exhibit forbidden patterns, and that in many cases (ergodic finite-alphabet information sources) the measure of the number of this patterns is related to other classic metric entropy rates (e.g., the Lyapunov exponent) [12].

In the current Letter we study the existence of forbidden patterns in economical time series. Specifically, we analyzed the appearance (or absence) of this patterns in several financial indicators. We have seen that, despite quantitative differences, the order pattern analysis reveals the existence of forbidden order patterns in all time series analyzed here, which indicates an underlying deterministic behavior. Furthermore, we have followed the evolution of the forbidden patterns, which allows to identify periods of time where noise or randomness is overtaking the deterministic behavior of the financial indicators.

In order to illustrate the absence of certain order patterns in deterministic time series, we plot in Fig.4 the number of existing forbidden patterns in a random time
series (a) and an equivalent series generated by a logistic map (b), which is obtained from $x_{n+1} = 4x_n(1-x_n)$ and $0 \leq x_0 \leq 1$. At first sight, we can see how the logistic function generates a higher number of forbidden patterns, revealing its chaotic behavior. Moreover, the existence of this kind of patterns is intrinsic to its nature (i.e., its construction); adding samples to the series does not help in lowering their number. On the other hand, the random case shows forbidden patterns due only to statistical limitations, and their number goes to zero for long enough time series. From the example above, we can construct a rule for time series characterization: if the number of forbidden patterns is greater than the quantity encountered in an equivalent random series (at least an order of magnitude above), that series has a deterministic structure.

As we see in Fig. 1-(a) it is important to define long enough series to avoid statistical distortions at the output. For a pattern dimension $d$ and a time series of length $N$, the number of possible patterns is $d!$, while the number of groups of data of dimension $d$ is $N - d + 1$. To guarantee that every pattern appears at least once in the dataset, we must ensure that $N - d + 1 > d!$, and therefore that $N > d! + d - 1$. For that reason, we have chosen a lower bound of $N > (d+1)!$, which avoids problems of undersampling.

![Figure 1](image1.png)

**FIG. 1:** Number of forbidden patterns $n(d,N)$ for: Dow Jones Industrial Average (a), Nasdaq Composite (b), NYSE IBM (c), NYSE Boeing (d), 10-Years Bond interest rate (e). Three different values of the pattern length $d$ are computed, $d = \{4, 5, 6\}$.

We have studied the appearance of forbidden patterns in real the time series of different financial indicators, namely: Dow Jones Industrial Average [14] and Nasdaq Composite [15] (US stocks indices), IBM and Boeing [16] (NYSE stocks), and the 10-Year U.S. Bond rate [17]. As in previous examples we have computed the $d = \{4, 5, 6\}$ ordinal patterns for different lengths of the time series. Results are shown in Figs. 2-(a)-(b)-(c)-(d)-(e), while Fig. 1-(a) corresponds to an equivalent time series that has been randomly generated.

At first sight it is clear that in all cases the number of forbidden patterns are much higher (two orders of magnitude) than the expected in the random case. This fact reveals the existence of a deterministic component in the evolution of all time series which coexists with stochastic fluctuations. As shown in [12], the combination of a deterministic and noisy signal introduces a decay in the number of forbidden patterns when increasing the length of the dataset, which is the case of the financial time series. Nevertheless, it has been demonstrated that the number of forbidden patterns is at least one order of magnitude higher than in the random case, even for
In this sense, we obtain in all cases a number of forbidden patterns that is at least two orders of magnitude from that of the random case, indicating the existence of driving deterministic forces. From (a) to (e) we have ordered the financial indices according to their number of forbidden patterns, e.g., from the least deterministic to the most deterministic series. In this way, we can infer what time series are more influenced by random fluctuations, which is a valuable information for the development of economic models that predict the evolution of the financial time series. Moreover, the presence of forbidden patterns seems to be related with the market operations involving the first four series: their mean trade volume is ordered from high to low (respectively 2.5 B$, 2 B$, 5 M$ and 4 M$), as if a great number of operations results in a more random behavior. It is worth mentioning that the decay in the number of forbidden patterns with the series length decreases with the total number of forbidden patterns, since it is somehow related: the lower the dependence on the time series length, the higher the deterministic part of the series. Finally we must remark that the results obtained for the 10-year Bond rate. The high number of forbidden patterns and the fact that they are nearly independent from the length of the time series indicates the high deterministic behavior of this particular financial series.

At this point, let us move our attention to the probability distribution function of the order patterns within the time series. When dealing with random time series, every permutation pattern should have the same probability to appear, and therefore, when $N \to \infty$ their probability distribution should be a flat function. In limited (random) time series the probability should follow a Poisson distribution, centered at $n_{\text{mean}} = T_p/N_p$, where $N_p$ is the total number of sequences and $T_p$ is the number of possible patterns. On the other side, we have seen that deterministic time series have certain forbidden patterns, which may lead to different probability distributions. Particularly, in cases where noise is mixed with a deterministic signal, the analysis of the probability distribution can show interesting results, specially if it moves away from the poissonian distribution.

Figure 3 shows the results for the financial series studied here. We can see how, in all series, the probability distribution show heavy tails, which indicates the heterogeneity in the probability of appearance (i.e., non-poissonian distribution). This fact distinguishes again the financial time series from the random ones, and shows that the study of the probability distribution function of the order patterns could be a good indicator of the deterministic nature of the series. Since series of Fig. 3 are ordered from low to high deterministic behavior, it is interesting to note that the number of patterns that appear only once or twice increases with the deterministic of the series.

Finally, we will have a look at the time evolution of the forbidden patterns in order to see if their number shifts in time. Fig. 4 represents the evolution of the Nasdaq index from September, 28th 2001 to April, 27th 2006. (upper plot), together with the evolution of their forbidden patterns (bottom plot). In the latter, each point of the plot represents the number of forbidden patterns of dimension $d = 5$ for a time window of 200 samples. We can observe how the number of forbidden patterns do not have a constant rate, on the contrary, it fluctuates in time. Its number is a good indicator of the randomness of the system at a precise moment. In this way, a decrease of the rate of forbidden patterns reflects that random forces are increasing (red ellipses in Fig. 4), leading to an enhance of the unpredictability of the system. On the contrary, when forbidden patterns increase (blue ellipses in Fig. 4), randomness is decreasing, which would indicate that the evolution of the time series would be more predictable. Note that we are not obtaining information about how the system is going to evolve, since we do not have a model describing the time series, but if we had it, the time periods where the number of forbidden patterns increase, would be the most suitable to apply it. Similar results, not shown here, are obtained for the rest of the financial time series.

In summary, we have studied the existence of forbid-
**FIG. 4:** Evolution of the Nasdaq Index and the corresponding forbidden patterns (F.P.) of order 5 \((d = 5)\), for a time window of 200 samples. Blue and red ellipses indicate the existence of relative maxima/minima, which is related, respectively, with the increase and decrease of the predictability of the system.

We have seen that a high number of forbidden patterns, two orders of magnitude higher than in random series, reveals an underlying deterministic behavior in the series, which indicates that predicting models could be suitable to forecast the behavior of financial time series. Furthermore, we have shown that the forbidden pattern rate could be an appropriate tool to quantify the randomness of certain time periods within the financial series, and therefore evaluate the uncertainty of the market. To our knowledge, these are the first results of the forbidden pattern evaluation in real time series, which also open interesting questions to be addressed in the future. We believe that the application of this technique to real time series of different nature, e.g., biological or medical datasets, will show promising results.

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[1] G. Box and G. Jenkins, “Time series analysis: Forecasting and control”, San Francisco: Holden-Day, (1970)

[2] J.M. Gottman, “Time Series Analysis” Cambridge University Press, Cambridge (1981).

[3] P. J. Brockwell and R. A. Davis. “Time Series: Theory and Methods”, Springer-Verlag, New York, USA, (1987).

[4] D.C. Montgomery, L.A. Johnson, and J.S. Gardiner, “Forecasting and Time Series Analysis”, New York: McGraw Hill, Inc., New York (1990).

[5] J. Eatwell, “Time Series and Statistics” W.W. Norton., New York (1991).

[6] J.D. Hamilton, “Time Series Analysis”, Princeton University Press (1994).

[7] “Handbook of Time Series Analysis”, Eds. B. Schelter, M. Winterhalder and Jens Timmer, Wiley-WHC (2007).

[8] H.D.I. Abarbanel, R. Brown, J.J. Sidorowich and L.Sh. Tsimring, “The analysis of observed chaotic data in physical systems”, Rev. Mod. Phys., bf 65, 1331 (1993)

[9] H. Kantz and T. Schreiber, “Nonlinear Time Series Analysis”, Cambridge University Press (2004).

[10] J.M. Amigó, M.B. Kennel and L. Kocarev, Physica D 210, 77 (2005).

[11] J.M. Amigó, L. Kocarev and J. Szczepanski, Phys. Lett. A 355, 27 (2006).

[12] J. M. Amigó, S. Zambrano, M.A.F. Sanjuán, “True and false forbidden patterns in deterministic and random dynamics”, Europhys. Lett., 79, 50001 (2007).

[13] C. Bandt, B. Pompe, Phys. Rev. Lett. 88, 174102 (2002).

[14] http://www.dj.com

[15] http://www.nasdaq.com

[16] http://www.nyse.com

[17] http://finance.yahoo.com/q?s=%5ETNX