THE ALGEBRA OF BOUNDED LINEAR OPERATORS ON $\ell_p \oplus \ell_q$
HAS INFINITELY MANY CLOSED IDEALS

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Abstract. We prove that in the reflexive range $1 < p < q < \infty$, the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ of all bounded linear operators on $\ell_p \oplus \ell_q$ has infinitely many closed ideals. This solves a problem raised by A. Pietsch [4, Problem 5.3.3] in his book, ‘Operator ideals’.

1. Introduction

Classifying the closed ideals of the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ of bounded linear operators on $\ell_p \oplus \ell_q$ has a long history. There were several results proved in the 1970s, and the reader is referred to the book by Pietsch [4, Chapter 5] for details. In particular, [4, Theorem 5.3.2] asserts that for $1 \leq p < q < \infty$ there are exactly two maximal ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$. These are the closures of the ideals of operators factoring through $\ell_p$ and $\ell_q$, respectively. [4, Theorem 5.3.2] also establishes a one-to-one correspondence between the set of all other closed, proper ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$ and the set of all closed ideals of $\mathcal{L}(\ell_p, \ell_q)$. Here an ideal of $\mathcal{L}(\ell_p, \ell_q)$ is a subspace $J$ of $\mathcal{L}(\ell_p, \ell_q)$ with the property that $AB \in J$ whenever $A \in \mathcal{L}(\ell_q), T \in J$ and $B \in \mathcal{L}(\ell_p)$.

Pietsch raises the following problem.

Problem ([4, Problem 5.3.3]). For $1 \leq p < q$ does $\mathcal{L}(\ell_p, \ell_q)$ have infinitely many closed ideals?

Since we are dealing with Banach spaces with bases, it is clear that the compact operators $K = K(\ell_p, \ell_q)$ are the smallest non-trivial (i.e., non-zero) closed ideal. Since the formal inclusion map $I_{p,q}: \ell_p \to \ell_q$ is not compact, $K$ is a proper ideal. For anyone well versed in basis techniques, it is a not too difficult exercise that every operator in $\mathcal{L}(\ell_p, \ell_q)$ is strictly singular, and that every non-compact operator factors $I_{p,q}$. It follows that $\{0\} \subseteq K \subseteq J_{p,q} \subset S = \mathcal{L}(\ell_p, \ell_q)$, where $J_{p,q}$ is the closure of the ideal of operators factoring through $I_{p,q}, S = S(\ell_p, \ell_q)$ is the ideal of strictly singular operators, and moreover any other closed ideal of $\mathcal{L}(\ell_p, \ell_q)$ must contain $J_{p,q}$. It is, however, not obvious that $J_{p,q}$ is proper. This was shown for $1 < p < q < \infty$ by V. D. Milman [3] who first proved that $I_{p,q}$ is finitely strictly singular, and then exhibited an operator in $\mathcal{L}(\ell_p, \ell_q)$ that is not finitely strictly singular. (Definitions will be given in Section 2 below.) The next significant result was proved by B. Sari, N. Tomczak-Jaegermann, V.G. Troitsky and the first named author. In [7] they showed that for $1 < p < 2 < q < \infty$, the ideal $FS$ of finitely strictly singular operators and the ideal $J_{12}$ generated by operators factoring through $\ell_2$ are proper, distinct, and distinct from the ones above. So in this case $\mathcal{L}(\ell_p, \ell_q)$ has at least four non-trivial, proper closed ideals. Later the first named author [5] found two more ideals, again in the range $1 < p < 2 < q < \infty$, namely the ideals generated by operators factoring through the formal inclusion

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maps $I_{p,2}$ and $I_{2,q}$, respectively. These new ideals lie between $I_{p,q}$ and $\mathcal{FS} \cap \mathcal{J}^{l_2}$, and hence the fact they are incomparable shows that $\mathcal{J}^{l_2} \neq \mathcal{FS} \cap \mathcal{J}^{l_2}$, and thus there are at least seven non-trivial, proper, closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ when $1 < p < 2 < q < \infty$.

The main result of this paper is a solution of Pietsch’s question in the reflexive range.

**Theorem A.** For all $1 < p < q < \infty$ there is a chain of size the continuum consisting of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ that lie between the ideals $\mathcal{J}^{l_2}$ and $\mathcal{FS}$.

Moreover, we obtain the following refinement.

**Theorem B.** For all $1 < p < 2 < q < \infty$ there is a chain of size the continuum consisting of closed ideals in $\mathcal{L}(\ell_p, \ell_q)$ that lie between $\mathcal{J}^{l_2}$ and $\mathcal{F} S$.

We note that these results provide further examples of separable reflexive Banach spaces $X$ such that $\mathcal{L}(X)$ has continuum many closed ideals. The first such examples were given by A. Pietsch [5].

The paper is organized as follows. In Section 2 we introduce definitions, notations and certain complemented subspaces of $\ell_p$ that will later lead to new ideals. A crucial rôle is played here by H. P. Rosenthal’s famous $X_p$ spaces, which we recall in some detail. We shall use $\ell_p$-direct sums of finite-dimensional versions of $X_p$. The so called lower fundamental function (defined below) of these direct sums will be computed at the end of Section 2. In Section 3 we prove a key lemma that will be at the heart of the proof of our main results. The latter will be presented in Section 4. We conclude with a list of open problems in Section 5.

2. **Definitions and known results**

In this section we introduce a large number of definitions as well as some preliminary results. This long section is divided into digestable chunks.

2.1. **$\ell_p$ spaces.** For $1 \leq p \leq \infty$ we denote by $p'$ the conjugate index of $p$. So we have $\frac{1}{p} + \frac{1}{p'} = 1$. We will denote by $\{e_{p,i} : 1 \leq i < \infty\}$ the unit vector basis of $\ell_p$ when $1 \leq p < \infty$ and of $c_0$ when $p = \infty$. For $n \in \mathbb{N}$, the unit vector basis of $\ell^n_p$ will be denoted by $\{e_{p,j}^{(n)} : 1 \leq j \leq n\}$.

Fix $p \in (1, \infty)$ and define $Z_p$ to be the $\ell_p$-direct sum $Z_p = (\bigoplus_{n=1}^{\infty} \ell^n_2)_{\ell_p}$. This has canonical unit vector basis $\{e_{p,j}^{(n)} : n \in \mathbb{N}, 1 \leq j \leq n\}$. By Kintchine’s inequality, the spaces $\ell^n_2$, $n \in \mathbb{N}$, are uniformly complemented in $\ell_p$. It follows that $Z_p$ is also complemented in $\ell_p$, and hence isomorphic to it by Pelczyński’s Decomposition Theorem. We fix once and for all an isomorphism $U_p : Z_p \to \ell_p$. Although the spaces $Z_p$ and $\ell_p$ are isomorphic, their canonical unit vector bases $\{e_{p,j}^{(n)} : n \in \mathbb{N}, 1 \leq j \leq n\}$ and $\{e_{p,i} : 1 \leq i < \infty\}$, respectively, are very different when $p \neq 2$. This is an example of the following convention that we adopt throughout this paper. For us a Banach space means a Banach space together with a fixed basis (which will always be normalized and 1-unconditional). We will use different notation for the same space if we consider more than one basis.

2.2. **Operator ideals.** Let $X$ and $Y$ be Banach spaces. We denote by $\mathcal{L}(X,Y)$ the space of all bounded, linear operators from $X$ to $Y$. We write $\mathcal{L}(X)$ for $\mathcal{L}(X,X)$, and we let $I_X \in \mathcal{L}(X)$ stand for the identity operator on $X$. If $X$ and $Y$ have fixed bases $(x_i)$ and $(y_i)$, respectively, such that $(y_i)$ is dominated by $(x_i)$, then we write $I_{X,Y}$ for the formal inclusion $X \to Y$ defined by

$$I_{X,Y} \left( \sum_i a_i x_i \right) = \sum_i a_i y_i ,$$

for all $a_i \in \mathbb{C}$.
which is well defined and bounded. When $X$ (respectively, $Y$) is $\ell_p$, $1 \leq p < \infty$, then we write $I_{p,Y}$ (respectively, $I_{X,p}$) instead of $I_{X,Y}$. Similarly, we write $I_{\infty,Y}$ instead of $I_{\infty,Y}$, etc.

By an ideal of $\mathcal{L}(X,Y)$ we mean a subspace $\mathcal{J}$ of $\mathcal{L}(X,Y)$ satisfying the ideal property: $ATB \in \mathcal{J}$ for all $A \in \mathcal{L}(Y), T \in \mathcal{J}$ and $B \in \mathcal{L}(X)$. When $X = Y$ this coincides with usual algebraic notion of an ideal of the algebra $\mathcal{L}(X)$. A closed ideal is an ideal that is closed in the operator norm. Clearly, the closure of an ideal is a closed ideal.

Our notion of ideal is equivalent to Pietsch’s notion of operator ideal [4]. The latter is a functor that assigns to each pair $(V,W)$ of Banach spaces a subspace $\mathcal{J}(V,W)$ of $\mathcal{L}(V,W)$ such that for all Banach spaces $U,V,W,Z$ and for all $A \in \mathcal{L}(W,Z), T \in \mathcal{J}(V,W), B \in \mathcal{L}(U,V)$, we have $ATB \in \mathcal{J}(U,Z)$. This is called a closed operator ideal if $\mathcal{J}(V,W)$ is a closed subspace of $\mathcal{L}(V,W)$ for all $V,W$. Given a (closed) operator ideal $\mathcal{J}$, it is clear that $\mathcal{J}(V,W)$ is then an ideal (respectively, closed ideal), in the above sense, of $\mathcal{L}(V,W)$ for all spaces $V,W$. Conversely, given a (closed) ideal $\mathcal{J}$ of $\mathcal{L}(X,Y)$ in the above sense, the functor that assigns to $(V,W)$ (the closure of) the set of all finite sums of operators of the form $ATB$ with $A \in \mathcal{L}(Y,W), T \in \mathcal{J}$ and $B \in \mathcal{L}(V,X)$ is a (closed) operator ideal in the sense of Pietsch for which $\mathcal{J}(X,Y) = \mathcal{J}$.

In this paper we shall only deal with closed ideals. Recall that $T \in \mathcal{L}(X,Y)$ is strictly singular if it not an isomorphism on any infinite-dimensional subspace of $X$, and $T$ is finitely strictly singular if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for every subspace $E$ of $X$ with dim $E \geq n$ there exists $x \in E$ such that $\|Tx\| < \varepsilon \|x\|$. We denote by $\mathcal{K}(X,Y) \subseteq \mathcal{FS}(X,Y) \subseteq \mathcal{S}(X,Y)$ the ideals of, respectively, compact, finitely strictly singular and strictly singular operators. When $X = Y$ these become $\mathcal{K}(X) \subseteq \mathcal{FS}(X) \subseteq \mathcal{S}(X)$. It is not hard to see that these are all closed operator ideals.

For an operator $T: U \to V$ between Banach spaces $U$ and $V$, we let $\mathcal{J}^T = \mathcal{J}^T(X,Y)$ be the closed ideal of $\mathcal{L}(X,Y)$ generated by operators factoring through $T$. Thus $\mathcal{J}^T$ is the closure in $\mathcal{L}(X,Y)$ of all finite sums of operators of the form $ATB$, where $A \in \mathcal{L}(V,Y)$ and $B \in \mathcal{L}(X,U)$. When $U = V$ and $T = I_U$ then we write $\mathcal{J}^U$ instead of $\mathcal{J}^{I_U}$.

2.3. Current state of affairs. We recall that for $1 \leq p < q < \infty$ the algebra $\mathcal{L}(\ell_p \oplus \ell_q)$ has exactly two maximal ideals, the closures of the ideals of operators factoring through $\ell_p$ and $\ell_q$, respectively, and the set of all non-maximal closed ideals of $\mathcal{L}(\ell_p \oplus \ell_q)$ is in a one-to-one, inclusion-preserving correspondence with the set of proper, closed ideals of $\mathcal{L}(\ell_p, \ell_q)$. These results are stated in [4] Theorem 5.3.2. We also refer the reader to Section 2 of [8].

Using the notation established in [7,2], the following diagrams summarize what we know about non-trivial closed ideals of $\mathcal{L}(\ell_p, \ell_q)$. When $1 < p < q < \infty$ we have

$$\mathcal{K} \subseteq \mathcal{J}^{I_p, q} \subset \mathcal{FS} \subseteq \mathcal{L}(\ell_p, \ell_q).$$

Recall that the last two inclusions are due to V. D. Milman [3]. Also, if $\mathcal{J}$ is a proper, closed ideal distinct from $\mathcal{K}$, then $\mathcal{J}$ contains $\mathcal{J}^{I_p, q}$.

Combining the main results of [7] and [8], we obtain the following picture in the range $1 < p < 2 < q < \infty$.

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| \mathcal{K} | \mathcal{J}^{I_p, q} | \mathcal{FS} | \mathcal{FS} \cap \mathcal{J}^{I_p, q} | \mathcal{FS} \cup \mathcal{J}^{I_p, q} | \mathcal{L}(\ell_p, \ell_q) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                  |                 |                 |                 |                 |                 |
|                  | \mathcal{J}^{I_p, q} | \mathcal{FS} | \mathcal{FS} \cap \mathcal{J}^{I_p, q} | \mathcal{FS} \cup \mathcal{J}^{I_p, q} | \mathcal{L}(\ell_p, \ell_q) |
|                  | \mathcal{J}^{I_p, q} | \mathcal{FS} | \mathcal{FS} \cap \mathcal{J}^{I_p, q} | \mathcal{FS} \cup \mathcal{J}^{I_p, q} | \mathcal{L}(\ell_p, \ell_q) |
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Here arrows stand for inclusion. It is not known whether the ideal $\mathcal{FS} \vee J^{\ell_2}$, the smallest closed ideal containing $\mathcal{FS}$ and $J^{\ell_2}$, is proper. All other inclusions are strict. The ideals $\mathcal{K}$ and $J^{\ell_2\vee}$ are the smallest, respectively second smallest ideals. In [7] it was also shown that any ideal containing an operator not in $\mathcal{FS}$ must contain $J^{\ell_2}$. It follows that there is no ideal between $\mathcal{FS} \cap J^{\ell_2}$ and $J^{\ell_2}$, and that $\mathcal{FS} \vee J^{\ell_2}$ is the only immediate successor of $\mathcal{FS}$. Two ideals connected by $\mathcal{I}$ are incomparable. It is not known whether $J^{\ell_2\vee}$ is a proper subset of $J^{\ell_2\vee} \cap J^{\ell_2\vee'}$.

2.4. Rosenthal’s $X_p$ spaces. In 1970 H. P. Rosenthal published an influential paper [6] with important consequences both for the theory of $L_p$ spaces, and for probability theory. This paper grew out from his study of sequences of independent random variables with mean zero in $L_p = L_p[0, 1]$ for $1 < p < \infty$. It led to the introduction of the spaces $X_p$ which we now describe. Let $2 < p < \infty$ and $w = (w_n)_{n=1}^\infty$ be a sequence in $(0, 1]$. The space $X_{p,w}$ is the completion of the space $c_{00}$ of finite scalar sequences with respect to the norm

$$\|(a_n)\|_{p,w} = \left( \sum |a_n|^p \right)^{\frac{1}{p}} \vee \left( \sum w_n^p |a_n|^2 \right)^{\frac{1}{2}}.$$ Rosenthal proved the following [6, Theorem 4]. Let $(f_n)$ be a sequence of independent, symmetric, 3-valued random variables, and let $Y_p$ be the closed linear span of $(f_n)$ in $L_p$ for $1 < p < \infty$. Then $Y_p$ is $K_p$-complemented in $L_p$, where $K_p$ is a constant depending only on $p$. Moreover, if $p > 2$, then $Y_p$ is isomorphic to $X_{p,w}$, where $w_n = \|f_n\|_{L_2}/\|f_n\|_{L_p}$, and $Y_{p'}$ is isomorphic to $X_{p',w}$. More precisely, the proof shows that if $1 < p < \infty$ and we assume that $\|f_n\|_{L_p} = 1$ for all $n \in \mathbb{N}$ (as we clearly may), then there is a projection $P_p: L_p \to Y_p$ given by $P_pf = \sum a_nf_n$, where $a_n = \int_0^1 f(x)f_n(x) \, dx \cdot \|f_n\|_{L_2}^{-2}$ for all $n \in \mathbb{N}$, which satisfies

\begin{align*}
(1) \quad \|(a_n)\|_{p,w} \leq \|P_pf\|_{L_p} \leq K_p\|(a_n)\|_{p,w} & \leq K_p\|f\|_{L_p} & \text{if } 2 < p < \infty, \\
(2) \quad \frac{1}{K_p}\|(a_n)\|_{p',w} \leq \|P_pf\|_{L_p} \leq \|(a_n)\|_{p',w} \leq K_p\|f\|_{L_p} & \text{if } 1 < p < 2. \end{align*}

Here in the case $2 < p < \infty$ we have $w_n = \|f_n\|_{L_2}$ for all $n \in \mathbb{N}$, whereas if $1 < p < 2$, then $w_n = \|f_n\|_{L_2}^{-1}$, and $\|(a_n)\|_{p',w}$ denotes the norm of $\sum a_ne_n^*$ in the dual space $X_{p',w}^*$, and $(e_n^*)$ is the sequence biorthogonal to the unit vector basis $(e_n)$ of $X_{p',w}$. Note that $P_2$ is simply the orthogonal projection of $L_2$ onto $Y_2$ and $K_2 = 1$; for $2 < p < \infty$ we obtain $P_p$ as the restriction to $L_p$ of $P_2$, and for $1 < p < 2$ we have $P_p = P_{p'}$ and $K_p = K_{p'}$. It follows from [6, Lemma 2] that in all cases the sequence $(f_n)$ is a $1$-unconditional basis of $Y_p$.

Let $2 < p < \infty$. It is easy to see that $X_{p,w}$ is isomorphic to one of the spaces $\ell_2, \ell_p$ and $\ell_2 \oplus \ell_p$ unless $(w_n)$ satisfies

\begin{equation}
(3) \quad \liminf w_n = 0 \quad \text{and} \quad \sum_{n, w_n < \varepsilon} w_n^{2p} = \infty \text{ for all } \varepsilon > 0. \end{equation}

Rosenthal proved that if sequences $(w_n)$ and $(w'_n)$ both satisfy (3), then the corresponding spaces $X_{p,w}$ and $X_{p,w'}$ are isomorphic and distinct from any of the spaces $\ell_2, \ell_p$ and $\ell_2 \oplus \ell_p$.

In this paper we shall use finite-dimensional versions of Rosenthal’s $X_p$ spaces, and we will only need the result about the existence of well-isomorphic and well-complemented copies in $L_p$. We begin with some definitions.
Given $2 < p < \infty$, $0 < w \leq 1$ and $n \in \mathbb{N}$, we denote by $E_{p,w}^{(n)}$ the Banach space $(\mathbb{R}^n, \| \cdot \|_{p,w})$, where
\[
\left\| (a_j)_{j=1}^n \right\|_{p,w} = \left( \sum_{j=1}^n |a_j|^p \right)^{\frac{1}{p}} \lor \left( \sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}.
\]

We write $\{e_j^{(n)} : 1 \leq j \leq n\}$ for the unit vector basis of $E_{p,w}^{(n)}$, and we denote by $\{e_j^{(n)*} : 1 \leq j \leq n\}$ the unit vector basis of the dual space $(E_{p,w}^{(n)})^*$, which is biorthogonal to the unit vector basis of $E_{p,w}^{(n)}$.

Given $1 < p < 2$, $0 < w \leq 1$ and $n \in \mathbb{N}$, we fix once and for all a sequence $f_j^{(n)} = f_{p,w,j}^{(n)}$, $1 \leq j \leq n$, of independent symmetric, 3-valued random variables with $\| f_j^{(n)} \|_{L_p} = 1$ and $\| f_j^{(n)} \|_{L_2} = \frac{1}{3}$ for $1 \leq j \leq n$. We then define $F_{p,w}^{(n)}$ to be the subspace span of $\{f_j^{(n)} : 1 \leq j \leq n\}$ of $L_p$. It follows from (2) that
\[
\frac{1}{K_p} \left\| \sum_{j=1}^n a_j e_j^{(n)*} \right\|_{L_p} \leq \left( \sum_{j=1}^n |a_j| \right)_{L_p} \leq \left( \sum_{j=1}^n |a_j| \right)_{L_p} \leq k^\frac{1}{2} \wedge \frac{1}{w} k^\frac{1}{2},
\]
where $\{e_j^{(n)*} : 1 \leq j \leq n\}$ is the unit vector basis of the dual space $(F_{p,w}^{(n)})^*$ as defined above. Since the $f_j^{(n)}$ are 3-valued, $F_{p,w}^{(n)}$ is a subspace of the span of indicator functions of $3^n$ pairwise disjoint sets. Thus, we can and will think of $F_{p,w}^{(n)}$ as a subspace of $\ell_p^{k_n}$, where $k_n = 3^n$.

**Proposition 1.** Let $1 < p < 2$, $0 < w \leq 1$ and $n \in \mathbb{N}$. Then

(i) $\{ f_j^{(n)} : 1 \leq j \leq n \}$ is a normalized, 1-unconditional basis of $F_{p,w}^{(n)}$.

(ii) There exists a projection $P_{p,w}^{(n)} : \ell_p^{k_n} \to F_{p,w}^{(n)}$ onto $F_{p,w}^{(n)}$ with $\|P_{p,w}^{(n)}\| \leq K_p$.

(iii) For each $1 \leq k \leq n$ and for every $A \subset \{1, \ldots, n\}$ with $|A| = k$ we have
\[
\frac{1}{K_p} \cdot \left( k^\frac{1}{2} \wedge \frac{1}{w} k^\frac{1}{2} \right) \leq \left\| \sum_{j \in A} f_j^{(n)} \right\|_{L_p} \leq k^\frac{1}{2} \wedge \frac{1}{w} k^\frac{1}{2}.
\]

**Proof.** (i) and (ii) follow from the results of H. P. Rosenthal, [3] Theorem 4] and [3] Lemma 2], that we cited above. By (i) we will have proved (iii) if we show that
\[
\left\| \sum_{j=1}^k e_j^{(n)*} \right\|_{L_p} = k^\frac{1}{2} \wedge \frac{1}{w} k^\frac{1}{2},
\]
where $\{e_j^{(n)*} : 1 \leq j \leq n\}$ is the unit vector basis of $(E_{p,w}^{(n)})^*$ as defined above. Now, by definition, we have
\[
\left\| \sum_{j=1}^k e_j^{(n)*} \right\| = \max \left\{ \sum_{j=1}^k |a_j| : \sum_{j=1}^k |a_j|^p \leq 1 \text{ and } w^\alpha \sum_{j=1}^k |a_j|^2 \leq 1 \right\}.
\]
Then by symmetry of $\| \cdot \|_{p,w}$, the maximum occurs when $a_1 = a_2 = \cdots = a_k = t$, say. So
\[
\left\| \sum_{j=1}^k e_j^{(n)*} \right\| = \max \left\{ kt : kt^p \leq 1 \text{ and } w^\alpha kt^2 \leq 1 \right\} = k^\frac{1}{2} \wedge \frac{1}{w} k^\frac{1}{2},
\]
as claimed. \qed
Remark. We mention two extreme examples. When \( w = 1 \), then \( E_p^{(n)}(v) \cong \ell_2^n \), and when \( w = n^{-\eta} \) with \( \eta = \frac{1}{p} - \frac{1}{2} \), then \( E_p^{(n)}(v) \cong \ell_2^n \). In both cases the formal identity map is an isometric isomorphism. It follows by \( \text{Proposition 1(i)} \) that if \( w = 1 \), then \( F_p \cong \ell_2^n \), and if \( w = n^{-\eta} \), then \( F_p(n) \cong \ell_2^n \). In both cases the formal identity is a \( K_p \)-isomorphism.

2.5. The spaces \( Y_{p,v} \). Fix \( 1 < p < 2 \), and let \( v = (v_n) \) be a decreasing sequence in \((0,1)\). For each \( n \in \mathbb{N} \), let \( F_n \) be the subspace \( E_p^{(n)}(v) \) of \( \ell_p^n \) with basis \( f_j^{(n)} : 1 \leq j \leq n \) as defined in Section 2.4 above. We introduce the space \( Y_{p,v} \) defined to be the \( \ell_p \)-direct sum \( Y_{p,v} = (\bigoplus_{n=1}^{\infty} F_n)_{\ell_p} \). This is a \( K_p \)-complemented subspace of \( \ell_p \).

Indeed, the diagonal operator

\[
P_{p,v} = \text{diag} \left( P_{p,v}^{(n)} \right) : \ell_p \rightarrow \bigoplus_{n=1}^{\infty} \ell_2^n \rightarrow \ell_p \equiv \bigoplus_{n=1}^{\infty} \ell_2^n \]

is a projection onto \( Y_{p,v} \), where \( P_{p,v}^{(n)} \) is the projection given by \( \text{Proposition 1(ii)} \).

Furthermore, \( Y_{p,v} \) is equipped with the normalized, \( 1 \)-unconditional basis \( \{ f_j^{(n)} : n \in \mathbb{N} \}, 1 \leq j \leq n \) : \( j \leq n \}. \) Note that \( Y_{p,v} \), as a complemented subspace of \( \ell_p \), is isomorphic to \( \ell_p \). However, we shall never make this identification, and instead consider \( Y_{p,v} \) as a complemented subspace of \( \ell_p \) with corresponding projection \( P_{p,v} \) fixed as above.

We conclude this section by proving a norm estimate, \( \text{Lemma 3} \) below, on sums of basis vectors of \( Y_{p,v} \). We begin with fixing some notation. Let \( X \) be a Banach space with a fixed basis \( \{ x_i \} \) (finite or infinite). Let \( N = \mathbb{N} \) if \( \dim(X) = \infty \), and \( N = \{ 1,2,\ldots,\dim(X) \} \) otherwise. We define the \textit{fundamental function} \( \varphi_X : N \rightarrow \mathbb{R} \) of \( X \) by setting

\[
\varphi_X(k) = \sup \left\{ \left\| \sum_{A \subseteq N} A x_i \right\| : A \subseteq N, \ |A| \leq k \right\}, \quad k \in N.
\]

We then extend the definition of \( \varphi_X \) to the real interval \( I = \bigcup_{1 \leq k < \dim(X)} [k,k+1] \) by linear interpolation. The fundamental function plays an important rôle in the study of so called greedy bases. Here we shall only need the following facts (see e.g., [1], Section 2).

**Proposition 2.** The functions \( t \mapsto \varphi_X(t) \) and \( t \mapsto \frac{k}{\varphi_X(t)} \), \( t \in I \), are increasing. The concave envelope \( \psi : I \rightarrow \mathbb{R} \) of \( \varphi_X \), i.e., the (pointwise) smallest concave function dominating \( \varphi_X \), satisfies \( \psi(t) \leq 2 \varphi_X(t) \) for all \( t \in I \).

We next introduce the \textit{lower fundamental function} \( \lambda_X : N \rightarrow \mathbb{R} \) of \( X \) defined by

\[
\lambda_X(k) = \inf \left\{ \left\| \sum_{A \subseteq N} A x_i \right\| : A \subseteq N, \ |A| \geq k \right\}, \quad k \in N,
\]

and extend the definition to \( I \) by linear interpolation. It is clear that \( \lambda_X \) is an increasing function on \( I \).

**Example.** \( \text{Proposition 1(iii)} \) shows that

\[
\frac{1}{K_p} \cdot \left( k^\frac{n}{p} \wedge \frac{1}{w} k^\frac{n}{p} \right) \leq \lambda_F(k) \leq \varphi_F(k) \leq k^\frac{n}{p} \wedge \frac{1}{w} k^\frac{n}{p},
\]

where \( F = E_p^{(n)}(v) \) and \( 1 \leq k \leq n \).

We now turn to the estimate on the lower fundamental function of \( Y_{p,v} \), as promised.
Lemma 3. Given $1 < p < 2$, let $v = (v_n)$ be a decreasing sequence in $(0,1]$ such that $v_n \geq n^{-\eta}$ for all $n \in \mathbb{N}$, where $\eta = \frac{1}{p} - \frac{1}{2}$. Then for each $k \in \mathbb{N}$ we have

$$\lambda_{Y_{p,v}}(k) \geq \frac{1}{K_p \sqrt{2}} \frac{1}{v_l} \cdot l, \quad \text{where } l = \left\lceil \sqrt{\frac{\eta}{2}} \right\rceil.$$  

(We put $v_0 = 1$ to cover the case $k = 1$.)

Proof. Let $A$ be a subset of $\{(n,j) : n \in \mathbb{N}, \ 1 \leq j \leq n\}$ with $|A| \geq k$. For each $n \in \mathbb{N}$ set $A_n = A \cap \{(n,j) : 1 \leq j \leq n\}$. By Proposition 1(iii) we can write $\mathbb{N}$ as the union of disjoint sets $L$ and $R$, where $L = \{n \in \mathbb{N} : \lambda_{F_n}(|A_n|) \geq \frac{1}{K_p v_n} |A_n|^{\frac{1}{2}}\}$ and $R = \{n \in \mathbb{N} : \lambda_{F_n}(|A_n|) \geq \frac{1}{K_p v_n} |A_n|^{\frac{1}{2}}\} \setminus L$. Then

$$\|\sum_{(n,j) \in A} f_{(n)}^{(j)}\|_{Y_{p,v}} \geq \left(\sum_{n \in L} \lambda_{F_n}(|A_n|)^{\frac{1}{2}}\right)^{\frac{1}{2}} \geq \frac{1}{K_p} \left(\sum_{n \in L} |A_n| + \sum_{n \in R} \left(\frac{1}{v_n} |A_n|^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}},$$

and hence, using $p < 2$, we obtain

$$K_p \|\sum_{(n,j) \in A} f_{(n)}^{(j)}\|_{Y_{p,v}} \geq \left(\sum_{n \in L} |A_n| + \sum_{n \in R} \left(\frac{1}{v_n} |A_n|^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$  

(5)

Set $l = \left\lceil \sqrt{\frac{\eta}{2}} \right\rceil$. Since $\sum_n |A_n| = |A| \geq k$, either $\sum_{n \in L} |A_n| \geq l^2$ or $\sum_{n \in R} |A_n| \geq l^2$. In the former case, inequality (5) immediately gives

$$K_p \|\sum_{(n,j) \in A} f_{(n)}^{(j)}\|_{Y_{p,v}} \geq l^2 \geq \frac{1}{v_l} \cdot l,$$

where the second inequality follows from the assumption that $v_n \geq n^{-\eta}$ for all $n \in \mathbb{N}$.

We now consider the case when $\sum_{n \in R} |A_n| \geq l^2$. Choose $s \in \mathbb{N}$ and $0 \leq s' \leq s$ such that

$$\sum_{n \in R} |A_n| = \sum_{n=1}^{s-1} n + s'. $$

Since $(v_n)$ is decreasing, we have

$$\sum_{n \in R} \frac{1}{v_n} |A_n| \geq \sum_{n=1}^{s-1} \frac{1}{n} + \frac{1}{v_n} s' \geq \frac{1}{v_l} \left(\sum_{n=1}^{s-1} n + s'\right) \geq \frac{1}{v_l} \cdot \frac{l^2}{2},$$

where we used $\sum_{n=1}^{l-1} n \leq \frac{l^2}{2}$. Hence by (5) we obtain

$$K_p \|\sum_{(n,j) \in A} f_{(n)}^{(j)}\|_{Y_{p,v}} \geq \frac{1}{\sqrt{2}} \frac{1}{v_l} \cdot l.$$  

(7)

The claim now follows from (5) and (7) above. \hfill \Box

3. The key lemma

This section is entirely devoted to a result that will play a central rôle in distinguishing closed ideals. It roughly says that if one has a bounded operator and the fundamental function of the domain is asymptotically smaller than the lower fundamental function of the range space, then a large proportion of basis vectors must map to ‘flat’ vectors.
Lemma 4. Let $Y$ be an infinite-dimensional Banach space with a normalized, 1-unconditional basis $(f_j)$. For each $m \in \mathbb{N}$ let $G_m$ be an $m$-dimensional Banach space with a normalized, 1-unconditional basis $\{g_i^{(m)} : 1 \leq i \leq m\}$. Assume that

\[
\lim_{k \to \infty} \sup_{m \geq k} \frac{\varphi_G_m(k)}{k} = 0, \quad \text{and}
\]

\[
\lim_{m \to \infty} \frac{\varphi_G_m(m)}{\lambda_Y(cm)} = 0 \quad \text{for all } c > 0.
\]

If $B_m : G_m \to Y$ is a sequence of operators with $\sup_m \|B_m\| \leq 1$, then

\[
\frac{1}{m} \sum_{i=1}^{m} \|B_m(g_i^{(m)})\|_{\infty} \to 0 \quad \text{as } m \to \infty.
\]

Here $\|y\|_{\infty} = \sup_j |y_j|$ for $y = \sum_j y_j f_j \in Y$.

Remark. Before proving our lemma let us look at the extreme case. Assume that for each $m \in \mathbb{N}$ we are given a linear operator $B_m$ from $\ell^\infty$ to $\ell_1$ with $\|B_m\| \leq 1$. In that special case we can easily deduce our claim from Grothendieck’s inequality. Indeed, fixing $m \in \mathbb{N}$, we can write $B_m$ as a matrix $B_m = (B_m(j,i))$, $i = 1, \ldots, m$, $j \in \mathbb{N}$, with

\[
\sup \left\{ \sum_{i=1}^{m} \sum_{j=1}^{\infty} t_j B_m(j,i) s_i : |s_i|, |t_j| \leq 1 \text{ for } 1 \leq i \leq m, 1 \leq j < \infty \right\} = \|B_m\| \leq 1,
\]

and we then have to show that

\[
\frac{1}{m} \sum_{i=1}^{m} \|B_m(e_{\infty,i}^{(m)})\|_{\infty} = \frac{1}{m} \sum_{i=1}^{m} \max_{j \in \mathbb{N}} |B_m(j,i)| \to 0 \quad \text{as } m \to \infty.
\]

Now Grothendieck’s inequality implies that

\[
\sum_{i=1}^{m} \sum_{j=1}^{\infty} B_m(j,i) y_j, x_i \leq K_G,
\]

whenever $x_i, i = 1, \ldots, m$, and $y_j, j \in \mathbb{N}$, are elements of the unit ball of a Hilbert space $H$, and where $K_G$ denotes the Grothendieck constant. We choose for each $i = 1, \ldots, m$ an integer $j_i \in \mathbb{N}$ such that $|B_m(j_i,i)| = \max_{j \in \mathbb{N}} |B_m(j,i)|$. We then let $H = \ell^\infty_m$ and $x_i = e_{\infty,i}^{(m)}$ for $i = 1, \ldots, m$. For each $j \in \mathbb{N}$ we define a vector $y_j = \sum_{i=1}^{m} \tilde{y}_j(i) e_{\infty,i}^{(m)}$ in $\ell^\infty_m$ as follows.

\[
\tilde{y}_j(i) = \begin{cases} 
\text{sign}(B_m(j,i)) & \text{if } j = j_i, \\
0 & \text{otherwise}.
\end{cases}
\]

Note that $\|y_j\|_{\ell^\infty_m} \leq \sqrt{m}$, and so $y_j = \tilde{y}_j / \sqrt{m} \in B_{\ell^\infty_m}$ for each $j \in \mathbb{N}$. It follows that

\[
K_G \geq \sum_{i=1}^{m} \sum_{j=1}^{\infty} B_m(j,i) y_j, x_i = \sum_{i=1}^{m} \sum_{j=1}^{\infty} B_m(j,i) y_j(i)
\]

\[
= \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |B_m(j_i,i)| = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \max_{j \in \mathbb{N}} |B_m(j,i)|,
\]

which yields our claim in this special case.

Proof of Lemma. Fix $\theta > 0$. By (5) there exists $k_0 \in \mathbb{N}$ such that

\[
\frac{\varphi_G_m(k)}{k} < \frac{\theta}{2} \quad \text{for all } m \geq k \geq k_0.
\]

Fix $g > 0$. By (5) there exists $k_0 \in \mathbb{N}$ such that

\[
\frac{\varphi_G_m(k)}{k} < \frac{\theta}{2} \quad \text{for all } m \geq k \geq k_0.
\]
Set \( c = \varrho k_0^{-1} \). By (9) we may choose \( m_0 \in \mathbb{N} \) such that
\[
m_0 > k_0 \quad \text{and} \quad \frac{\varphi A_n(m)}{\lambda Y(cm)} < \varrho \quad \text{for all} \ m > m_0 .
\]
Now fix \( m > m_0 \) and set
\[
A = \left\{ i \in \{1, 2, \ldots, m\} : \|B_m(g_i^{(m)})\|_\infty \geq \varrho \right\} .
\]
We will show that \(|A| \leq gm\). It will then follow that
\[
\frac{1}{m} \sum_{i=1}^{m} \|B_m(g_i^{(m)})\|_\infty \leq \varrho + \frac{|A|}{m} \leq 2\varrho,
\]
and since \( m > m_0 \) and \( \varrho > 0 \) were arbitrary, the proof of the lemma will be complete. To show that \(|A| \leq gm\) we argue by contradiction, and assume that \(|A| \geq gm\). For each \( i \in A \) fix \( j_i \in \mathbb{N} \) such that
\[
\left[ \left[ B_m(g_i^{(m)}) \right]_{j_i} \right] \geq \varrho .
\]
Here \([y]_j\) denotes, for \( y \in Y \), the \( j \)th coordinate of \( y \) with respect to the basis \((f_j)_j^{\infty} = 1\).
We then set \( \tilde{A} = \{ j_i : i \in A \} \), and for \( j \in \tilde{A} \) we let \( A_j = \{ i \in A : j_i = j \} \). We shall now obtain a number of inequalities that will eventually lead to a contradiction.

Fix \( j \in \tilde{A} \) and for each \( i = 1, \ldots, m \) let \( \varepsilon_i \) be the sign of \( \{B_m(g_i^{(m)})\}_j \). Since \( \|B_m\| \leq 1 \) and \((g_i^{(m)})\) is 1-unconditional, we have
\[
\varphi A_n([A_j]) \geq \left\| \sum_{i \in A_j} \varepsilon_i g_i^{(m)} \right\|_{G_m} \geq \left\| \sum_{i \in A_j} \varepsilon_i B_m(g_i^{(m)}) \right\|_Y
\geq \left[ \sum_{i \in A_j} \varepsilon_i B_m(g_i^{(m)}) \right]_{j} \geq |A_j| \varrho .
\]
Let \( \psi \) be the concave envelope of \( \varphi A_n \). Since \( A \) is the disjoint union of the sets \( A_j \), \( j \in \tilde{A} \), we obtain
\[
|A| = \sum_{j \in \tilde{A}} |A_j| \leq \varrho^{-1} \sum_{j \in \tilde{A}} \varphi A_n([A_j]) \leq \varrho^{-1} \sum_{j \in \tilde{A}} \psi([A_j])
\leq \varrho^{-1} |\tilde{A}| \cdot \psi \left( \frac{|\tilde{A}|}{|A|} \right) \leq 2\varrho^{-1} |\tilde{A}| \cdot \varphi A_n \left( \frac{|\tilde{A}|}{|A|} \right),
\]
where the first inequality of the second line uses the concavity of \( \psi \), and the second inequality uses Proposition 12. Now if \( \frac{|\tilde{A}|}{|A|} > k_0 \), then it follows from the above that
\[
\frac{\varphi A_n(k_0)}{k_0} > \frac{\varrho}{2} \quad \text{which contradicts the choice of} \ k_0 .
\]
Thus
\[
(10) \quad |A| \leq k_0 |\tilde{A}| .
\]
We next fix independent Rademacher random variables \((r_i)_{i \in A}\), and show the estimate
\[
(11) \quad \mathbb{E} \left[ \sum_{i \in A} r_i B_m(g_i^{(m)}) \right]_{j} \geq \varrho \quad \text{for all} \ j \in \tilde{A} .
\]
To see this fix \( j \in \tilde{A} \) and set \( y_i = \left[ B_m(g_i^{(m)}) \right]_{j} \) for \( i \in A \). By definition of \( \tilde{A} \) there is an \( i_0 \in A \) such that \( j_{i_0} = j \), and hence \( |y_{i_0}| \geq \varrho \). Thus
\[
\mathbb{E} \left[ \sum_{i \in A} r_i y_i \right] = \mathbb{E} \left[ \sum_{i \in A} r_{i_0} r_i y_i \right] = \mathbb{E} \left[ y_{i_0} \right] + \sum_{i \in A, j \neq i_0} r_{i_0} r_i y_i
\]
\( \mathbb{L}(F \otimes G) \) has infinitely many closed ideals

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Proof.

Recall that \( c = k_0^{-1} \) and \( A \subset \{1, \ldots, m\} \) with \( |A| \geq gm \). So \( |A| \leq m \), and by \((10)\), \( |A| \geq cm \). Thus, the above gives

\[
\varphi \leq \frac{\varphi_{G_1}(|A|)}{\lambda_Y(|A|)} \leq \frac{\varphi_{G_1}(m)}{\lambda_Y(cm)} < \varphi
\]

by the choice of \( m_0 \). This contradiction completes the proof. \( \square \)

4. Proof of the main results

In the previous section we described a situation when images of basis vectors are on average ‘flat’. Here we begin with a calculation (Lemma 5 below) that shows that certain formal inclusion maps reduce the norm of ‘flat’ vectors. We then introduce, when these ideals are distinct from one another. This will lead to a proof of our main result, Theorem A. In the rest of the section we follow a similar strategy and establish Theorem B.

Lemma 5. Given \( 1 < p < 2 \) and \( p < q < \infty \), let \( n \in \mathbb{N} \), \( v \in (0, 1] \), and \( F = F_{p,v}^{(n)} \) with basis \( \{ f_j^{(n)} : 1 \leq j \leq n \} \). Let \( y = \sum_{j=1}^{n} y_j f_j^{(n)} \in F \) with \( \|y\|_F \leq 1 \), and let \( \tilde{y} = \sum_{j=1}^{n} y_j e_j^{(n)} \in L^2 \). If \( \|y\|_{L^\infty} = \max |y_j| \leq \sigma \leq 1 \) and \( v \leq \sigma^{\frac{1}{2} - \frac{1}{p}} \), then

\[
\|\tilde{y}\|_{L^2}^p \leq \max \{ C_{p,K}^p \cdot \sigma^r \cdot \|y\|_F^p \}
\]

where \( C_p \) is the cotype-2 constant of \( L^p \) and \( r = \min \{ \frac{p}{2} - \frac{p}{2}, \frac{q}{2} - \frac{q}{p} \} \).

Here we recall that for \( 1 \leq p \leq 2 \) the Banach space \( L_p[0,1] \), and hence \( L^p \), has cotype 2. This is a consequence of Khintchine’s inequality. See for example [2] Definition 1.6.12.

Proof. If \( \|\tilde{y}\|_{L^2}^p \leq \sqrt{\sigma} \), then

\[
\|\tilde{y}\|_{L^2}^p = \|\tilde{y}\|_{L^2}^{p-p} \cdot \|\tilde{y}\|_{L^2}^p \leq \sigma^{\frac{p}{2} - \frac{1}{2}} \cdot \|\tilde{y}\|_{L^2}^p \leq C_p \cdot \sigma^{\frac{1}{2} - \frac{1}{p}} \cdot \|y\|_F^p ,
\]

where we use the fact, an easy consequence of the definition of cotype, that in \( L^p \) a normalized, 1-unconditional basis \( C_p \)-dominates the unit vector basis of \( L^2 \). So the claim holds in this case.
Now assume that $\|\tilde{y}\|_{\ell_p} > \sqrt{\sigma}$. Set $z_j = \frac{y_j}{\|y\|_{\ell_p}^{1/2}}$ for $1 \leq j \leq n$, and let $\tilde{z} = \sum_{j=1}^{n} z_j e_{2,j}^{(n)}$. Then $\|\tilde{z}\|_{\ell_p} = 1$ and $\langle \tilde{y}, \tilde{z} \rangle = \|\tilde{y}\|_{\ell_p}$. Note also that $|z_j| \leq \sqrt{\sigma}$ for all $j$. So we have

$$\left( \sum_{j=1}^{n} |z_j|^p \right)^{1/p} = \left( \sum_{j=1}^{n} |z_j|^{p-2} \cdot |z_j|^2 \right)^{1/p} \leq \sigma^{\frac{p-2}{p}} \cdot \left( \sum_{j=1}^{n} |z_j|^2 \right)^{1/p} = \sigma^{\frac{p-2}{p}}.$$

On the other hand, we have $v \cdot \|\tilde{z}\|_{\ell_p} = v \leq \sigma^{\frac{p}{2}} - \frac{p}{2}$ by assumption. It follows that $\|z\|_{\ell_p} \leq \sigma^{\frac{p}{2}} - \frac{p}{2}$, where $z = \sum_{j=1}^{n} z_j e_{2,j}^{(n)}$, and $\{e_{2,j}^{(n)} : 1 \leq j \leq n\}$ is the unit vector basis of $E = E_{p,q} = (\mathbb{R}^n, \|\cdot\|_{\ell_p})$. Hence by (4) we have

$$\|\tilde{y}\|_{\ell_p}^2 = \langle \tilde{y}, \tilde{z} \rangle \leq \|(v_j)\|_{\ell_p} \cdot \|\tilde{z}\|_{\ell_p} \leq K_p \cdot \sigma^{\frac{p}{2}} - \frac{p}{2} \cdot \|y\|_F^2.$$

It follows that $\|\tilde{y}\|_{\ell_p} \leq K_p^{\frac{1}{2}} \cdot \sigma^{\frac{p}{2}} - \frac{p}{2} \cdot \|y\|_F^2$, which proves the claim since $\|y\|_F \leq 1$, and so $\|y\|_p \leq \|y\|_F^2$. \qed

Fix $1 < p < 2$ and $p < q < \infty$. Let $v = (v_n)$ be a decreasing sequence in $(0,1]$. In Section 2.3 we introduced the complemented subspace $Y_v = Y_{p,v}$ of $\ell_p$ with corresponding projection $P_v = P_{p,v}$. As already mentioned in the proof above, for each $n \in \mathbb{N}$, the unit vector basis of $\ell_p^n$ is $C_p$-dominated by the normalized, 1-unconditional basis $\{f_j^{(n)} : 1 \leq j \leq n\}$ of $F_n = F_{p,v,n}^{(n)}$. Thus the formal inclusion map

$$I_{Y_v,Z_q} : Y_v = \left( \bigoplus_{n=1}^{\infty} F_n \right)_{\ell_p} \to Z_q = \left( \bigoplus_{n=1}^{\infty} \ell_2^n \right)_{\ell_q}$$

given by $I_{Y_v,Z_q}(f_j^{(n)}) = e_{2,j}^{(n)}$ is well-defined and bounded. This defines the closed ideal $\mathcal{J}^{I_{Y_v,Z_q}}$ of $\mathcal{L}(\ell_p,\ell_q)$ generated by operators factoring through $I_{Y_v,Z_q}$. We also fixed, in Section 2.3, an isomorphism $U_q : Z_q \to \ell_q$. Note that the operator $T_v = U_q \circ I_{Y_v,Z_q} \circ P_v$ belongs to the ideal $\mathcal{J}^{I_{Y_v,Z_q}}$. The next result establishes conditions on two sequences $v$ and $w$ which imply that $T_w \notin \mathcal{J}^{I_{Y_w,Z_q}}$.

**Theorem 6.** Fix $1 < p < 2$ and $p < q < \infty$. Let $v = (v_n)$ and $w = (w_n)$ be decreasing sequences in $(0,1]$. Consider $Y_v$, $I_{Y_v,Z_q}$, and $T_v$ as above. Assume that $v_n \geq n^{-\eta}$ and $w_n \geq n^{-\eta}$ for all $n \in \mathbb{N}$, where $\eta = \frac{1}{p} - \frac{1}{2}$. Further assume that

$$\frac{v_n}{w_n} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad c \in (0,1)$$

(where we simplify notation by letting $v_x = v_{\lfloor x \rfloor}$ for $x \in \mathbb{R}$). Then $T_w \notin \mathcal{J}^{I_{Y_w,Z_q}}$.

**Proof.** For each $n \in \mathbb{N}$ let $F_n = F_{p,v,n}^{(n)}$ and $G_n = F_{p,w,n}^{(n)}$ with unit vector bases $\{f_j^{(n)} : 1 \leq j \leq n\}$ and $\{g_j^{(n)} : 1 \leq j \leq n\}$, respectively. Thus $f_j^{(n)} = f_{p,v,n,j}^{(n)}$ and $g_j^{(n)} = f_{p,w,n,j}^{(n)}$ for $1 \leq j \leq n$ using the notation introduced in Section 2.4. To simplify notation we write $Y = Y_v$, $Z = Z_q$, $U = U_q$ and $T = T_v$. Thus $U : Z \to \ell_q$ is an isomorphism,

$$I_{Y,Z} : Y = \left( \bigoplus_{n=1}^{\infty} F_n \right)_{\ell_p} \to Z = \left( \bigoplus_{n=1}^{\infty} \ell_2^n \right)_{\ell_q}$$

is given by $I_{Y,Z}(f_j^{(n)}) = e_{2,j}^{(n)}$, and $T : \ell_p = (\bigoplus_{n=1}^{\infty} \ell_2^n)_{\ell_p} \to \ell_q$ is the composite $T = U \circ I_{Y_w,Z_q} \circ P_w$. Note that $T(g_j^{(n)}) = U(e_{2,n}^{(n)})$.
We need to show that $T \notin J^{iv, z}$. We achieve this by finding a separating functional $\Phi \in L(\ell_p, \ell_q)^*$ as follows. For each $m \in \mathbb{N}$ we define $\Phi_m \in L(\ell_p, \ell_q)^*$ by setting

$$\Phi_m(V) = \frac{1}{m} \sum_{i=1}^{m} \langle c^{(m)}_{2,i}, U^{-1}V(g^{(m)}_i) \rangle, \quad V \in L(\ell_p, \ell_q).$$

Since $\|\Phi_m\| \leq\|U^{-1}\|$ for all $m$, the sequence $(\Phi_m)$ has a $\omega^*$-accumulation point $\Phi$ in $L(\ell_p, \ell_q)^*$. Note that $\Phi_m(T) = 1$ for all $m$, and hence $\Phi(T) = 1$. The proof will be complete if we can show that $J^{iv, z}$ is contained in the kernel of $\Phi$. To see this, fix $A \in L(Z, \ell_q)$ and $B \in L(\ell_p, Y)$ with $\|A\| \leq 1$ and $\|B\| \leq 1$. It is sufficient to show that

$$\Phi_m(AI_{Y,Z}B) \to 0 \quad \text{as} \quad m \to \infty.$$  

Let $B_m: G_m \to Y$ denote the restriction to $G_m$ of $B$. We shall use Lemma 2 to show that

$$\frac{1}{m} \sum_{i=1}^{m} \|B_m(g^{(m)}_i)\|_\infty \to 0 \quad \text{as} \quad m \to \infty.$$  

Recall that for $y = \sum_{n=1}^{\infty} \sum_{j=1}^{n} y_{n,j} f^{(n)}_j$ in $Y$ we let $\|y\|_\infty = \sup_{n \in \mathbb{N}, 1 \leq j \leq n} |y_{n,j}|$. By Proposition 1(iii) we have $\varphi_{G_m}(k) \leq k^{\frac{1}{p}}$ for all $1 \leq k \leq m$, and so condition (8) of Lemma 2 certainly holds. Now by Lemma 3 we have

$$\lambda Y(k) \geq \frac{1}{3K_p} \cdot \frac{1}{v^{\frac{3}{2k}}} \cdot \sqrt{k}$$

for all large $k$. On the other hand, by Proposition 1(iii) we have $\varphi_{G_m}(m) \leq \frac{1}{m^{\frac{1}{p}}} m^{\frac{1}{q}}$ for all $m$. So for any $c > 0$ and for all large $m \in \mathbb{N}$, it follows that

$$\frac{\varphi_{G_m}(m)}{\lambda Y(cm)} \leq C \cdot \frac{v^{\frac{3}{2k}}}{u_m},$$

where $C$ and $c'$ are constants depending only on $c$ (and $p$). Thus, by assumption (12), condition (9) also holds, and Lemma 4 applies. This completes the proof of (13). To see (13), fix $g \in (0, 1)$ and choose $n_0 \in \mathbb{N}$ such that $v_n \leq g^{\frac{1}{p}} \cdot \frac{1}{n}$ for all $n \geq n_0$. This is possible, since by (12) we have $v_n \to 0$ as $n \to \infty$. Note that

$$|\Phi_m(AI_{Y,Z}B)| = \frac{1}{m} \left| \sum_{i=1}^{m} \left( A^* (U^{-1})^* c^{(m)}_{2,i}, I_{Y,Z} B_m(g^{(m)}_i) \right) \right|$$

(15)

$$\leq \frac{1}{m} \sum_{i=1}^{m} \|U^{-1}\| \cdot \|I_{Y,Z} B_m(g^{(m)}_i)\|_Z.$$  

Fix $1 \leq i \leq m$, and set $x = B_m(g^{(m)}_i)$. Write $x = \sum_{n=1}^{\infty} \sum_{j=1}^{n} x_{n,j} f^{(n)}_j$, and let $\sigma = g \vee \|x\|_\infty$. Note that $v_n \leq \sigma^{\frac{1}{p}} \cdot \frac{1}{n}$ for all $n \geq n_0$. Hence by Lemma 5 we have

$$\left( \sum_{j=1}^{n} |x_{n,j}|^2 \right)^{\frac{1}{2}} \leq M \cdot \sigma^r \cdot \left( \sum_{j=1}^{n} |x_{n,j} f^{(n)}_j| \right)^{\frac{1}{r}}$$

for all $n \geq n_0$, where $M = \max\{C_p^{\sigma}, K_p^{\sigma}\}$ and $r = \min\{\frac{q}{2} - \frac{p}{2}, \frac{q}{2} - \frac{p}{p}\}$. It follows that

$$\|I_{Y,Z} B_m(g^{(m)}_i)\|_Z = \|I_{Y,Z}(x)\|_Z = \left( \sum_{n=1}^{\infty} \left( \sum_{j=1}^{n} |x_{n,j}|^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{q}}.$$
\[ \begin{align*}
&\leq \left( \sum_{n=1}^{n_0} \sum_{j=1}^{n} |x_{n,j}|^2 \right)^{\frac{1}{2}} + M^{\frac{1}{p}} \cdot \sigma_{\varphi}^{\frac{1}{q}} \cdot \left( \sum_{n>n_0} \left\| \sum_{j=1}^{m} x_{n,j} f_j^{(n)} \right\|_{F_n}^p \right)^{\frac{1}{p}} \\
&\leq \|x\|_\infty \cdot \left( \sum_{n=1}^{n_0} n^{\frac{2}{q}} \right)^{\frac{1}{2}} + M^{\frac{1}{p}} \cdot \sigma_{\varphi}^{\frac{1}{q}} \cdot \|x\|_\infty^{\frac{1}{q}} \leq \|x\|_\infty \cdot N + M^{\frac{1}{p}} \cdot \sigma_{\varphi}^{\frac{1}{q}} \\
&= N \cdot \|B_m(g_{i_1}^{(m)})\|_\infty + M^{\frac{1}{p}} \cdot \sigma_{\varphi}^{\frac{1}{q}},
\end{align*} \]
where we put
\[ N = \left( \sum_{n=1}^{n_0} n^{\frac{2}{q}} \right)^{\frac{1}{2}}. \]
Hence, using (15) and putting \( \sigma_{i_1}^{(m)} = \varphi \lor \|B_m(g_{i_1}^{(m)})\|_\infty \) for all \( m \in \mathbb{N} \) and \( 1 \leq i \leq m \), we obtain
\[ |U^{-1}|^{-1} \cdot |\Phi_m(A_{12}B)| \leq N \cdot \frac{1}{m} \sum_{i=1}^{m} \|B_m(g_{i_1}^{(m)})\|_\infty + M^{\frac{1}{p}} \cdot \left( \frac{1}{m} \sum_{i=1}^{m} \sigma_{i_1}^{(m)} \right)^{\frac{1}{q}} \]
\[ \leq N \cdot \frac{1}{m} \sum_{i=1}^{m} \|B_m(g_{i_1}^{(m)})\|_\infty + M^{\frac{1}{p}} \cdot \left( \frac{1}{m} \sum_{i=1}^{m} \sigma_{i_1}^{(m)} \right)^{\frac{1}{q}}. \]
To see the second inequality note that \( \frac{1}{q} < \frac{1}{2} \), and so the function \( t \mapsto t^{\frac{1}{q}} \) is concave. Now it follows from (13) that
\[ \limsup_m |\Phi_m(A_{12}B)| \leq |U^{-1}| \cdot M^{\frac{1}{p}} \cdot \varphi^{\frac{1}{q}}. \]
Since \( \varphi > 0 \) was arbitrary, the proof of (13), and hence of the theorem is complete. \( \square \)

**Corollary 7.** Let \( 1 < p < 2 \) and \( p < q < \infty \). Let \( \varphi = (\varphi_n) \) and \( \psi = (\psi_n) \) be decreasing sequences in \((0,1]\) bounded below by \( n^{-\eta} \), \( \eta = \frac{1}{p} - \frac{1}{2} \), and satisfying condition (12). Then \( I_{12}^{v_n,\varphi} \subseteq I_{12}^{v_n,\psi} \).

**Proof.** It follows from (12) that \( v_n \leq \psi_n \). Hence, using the notation of the proof of Theorem 4 the basis \( (f_j^{(n)}) \) of \( F_n \) dominates the basis \( (g_j^{(n)}) \) of \( G_n \) for all large \( n \). Indeed, this follows from (12). Thus, \( I_{12}^{v_n,\varphi} \) factors through \( I_{12}^{v_n,\psi} \) via the formal inclusion map \( I_{12}^{v_n,\psi} \), and thus \( I_{12}^{v_n,\varphi} \subseteq I_{12}^{v_n,\psi} \). The claim now follows immediately from Theorem 6 since \( T_n \in \mathcal{F}_{12}^{v_n,\psi} \).

Before proving Theorem A we need to show that certain maps are finitely strictly singular.

**Proposition 8.** Fix \( 1 < p < 2 \) and \( p < q < \infty \). Let \( \varphi = (\varphi_n) \) be a decreasing sequence in \((0,1]\) bounded below by \( n^{-\eta} \), \( \eta = \frac{1}{p} - \frac{1}{2} \), such that \( \varphi_n \to 0 \) as \( n \to \infty \). Set \( Y = Y_{p,\varphi} \). Then the formal inclusion maps \( I_{12}^{v_n,\varphi} \) and \( I_{12}^{v_n,\varphi} \) are finitely strictly singular.

**Proof.** For each \( n \in \mathbb{N} \) let \( F_n = F_{p,\varphi_n}^{(n)} \) and \( E_n = (\mathbb{R}^n, \|\cdot\|_{p,\varphi_n}) \) with unit vector bases \( \{f_j^{(n)} : 1 \leq j \leq n\} \) and \( \{e_j^{(n)} : 1 \leq j \leq n\} \), respectively. We first prove that \( I_{12}^{v_n,\varphi} \) is finitely strictly singular. Fix \( \varepsilon > 0 \). Choose \( \varphi \in (0,1) \) such that \( \varphi + M^{\frac{1}{p}} \cdot \varphi^{\frac{1}{q}} < \varepsilon \), where, as before, \( M = \max\{C_{p,\varphi}, C_{p,\varphi}^2\} \), \( r = \min\{\frac{\varphi}{p}, \frac{\varphi}{p}, \frac{\varphi}{p}\} \) and \( C_p \) is the type-2 constant of \( \ell_p \). Next fix \( n_0 \in \mathbb{N} \) such that \( n_0 \leq \varphi^{\frac{1}{q}} \) for all \( n \geq n_0 \). Set \( N = \left( \sum_{n=1}^{n_0} n^{\frac{2}{q}} \right)^{\frac{1}{2}}. \) Finally, choose \( d \in \mathbb{N} \) such that \( K_{12}^{d} \cdot 2^{\frac{d}{2}} \cdot N < \varphi \).

Now let \( H \) be a subspace of \( Y \) of dimension at least \( 2d^2 \). Then by a result of V. D. Milman [3] (see also [2] Lemma 3.4), there exists \( x = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} f_{n,j}^{(n)} \in \mathcal{F}_{12}^{v_n,\varphi} \).
that \( Y \) needs to obtain a series of estimates as in the proof of Theorem 6. We first observe

Now fix \( q \) and since \( I \),

Hence by Lemma 3, assuming as we may that \( \|x\|_Y = 1 \), we have

\[
1 = \|x\|_Y \geq \|x\|_\infty \cdot \lambda_Y (2d^2) \geq \|x\|_\infty \cdot \frac{1}{K_p \sqrt{2}} \cdot \frac{d}{v_d}.
\]

Set \( \sigma = \varrho \vee \|x\|_\infty \). As in the proof of Theorem 6, we obtain

\[
\|I_{Y,Z}(x)\| \leq N \cdot \|x\|_\infty + M^\frac{\varrho}{\sigma} \cdot \sigma^2.
\]

By (16) and by the choice of \( d \), we have \( N \cdot \|x\|_\infty \leq K_p \cdot \frac{2\mu}{d} \cdot N < \varrho \). In particular, \( \sigma = \varrho \), and hence the above gives

\[
\|I_{Y,Z}(x)\| \leq \varrho + M^\frac{\varrho}{\sigma} \cdot \varrho^2 < \varepsilon
\]

by the choice of \( \varrho \).

The proof for \( I_{Z_p,Y^*} \) is similar. One first needs to prove a dual version of Lemma 5 which is easier since in \( E_q \), we have an explicit formula for the norm, and then one needs to obtain a series of estimates as in the proof of Theorem 6. We first observe that \( Y^* \) is isomorphic to \( W = (\bigoplus_{n=1}^{\infty} E_n)_{\ell_2^p} \) by (3), and so it is sufficient to show that the formal inclusion map \( I_{Z_p,W} \) is finitely strictly singular. So let us fix \( \varepsilon > 0 \), and then choose \( \varrho > 0 \) such that \( \varrho + \varrho 2^{1-p} \leq \varepsilon \). We may and shall assume that \( p < q \leq 2 \). Indeed, given \( p < q_1 < q_2 \), we have \( I_{Z_{q_1},W} = I_{Z_{q_1},W} \circ I_{Z_{q_2},Z_{q_1}} \). Now choose \( n_0 \in \mathbb{N} \) such that \( v_n < \varrho^\frac{1-p}{p} \) for all \( n \geq n_0 \). Set \( N = \left( \sum_{n=1}^{n_0} n^p \right)^\frac{1}{p} \) and choose \( d \in \mathbb{N} \) with \( d^\frac{1}{p} \cdot N < \varrho \).

Given a subspace \( H \) of \( Z_{q_1} \) of dimension at least \( d \), use Milman's lemma again to find \( x = \sum_{n=1}^{\infty} \sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \in H \) with \( \|x\|_{Z_{q_1}} = 1 \) such that \( |x_{n,i}| = \|x\|_\infty = \sup_{n \in \mathbb{N}, 1 \leq j \leq n} |x_{n,j}| \) for at least \( 2 \leq d' \). Since \( 2 \leq q' \), we have

\[
1 = \|x\|_{Z_{q'}} \geq \|x\|_{\ell_2^p} \geq \|x\|_\infty \cdot d^\frac{1}{p}, \text{ and so } \|x\|_\infty \leq d^\frac{1}{p} \cdot \varrho.
\]

Now fix \( n \in \mathbb{N} \) with \( n \geq n_0 \). On the one hand, we have

\[
\sum_{j=1}^{n} |x_{n,j}|^p = \sum_{j=1}^{n} |x_{n,j}|^{p' - q'} |x_{n,j}|^q' \leq \varrho^{p' - q'} \cdot \left( \sum_{j=1}^{n} |x_{n,j}|^2 \right)^\frac{q'}{2},
\]

where we used \( \|x\|_\infty < \varrho \) and that \( 2 \leq q' \). On the other hand, by the choice of \( n_0 \), and since \( q' < p' \), we have

\[
e^{\frac{q'}{2}} \left( \sum_{j=1}^{n} |x_{n,j}|^2 \right)^\frac{q'}{2} \leq \varrho^{p' - q'} \cdot \left( \sum_{j=1}^{n} |x_{n,j}|^2 \right)^\frac{q'}{2}.
\]

The previous two inequalities imply that

\[
\left\| \sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \right\|_{E_n} \leq \varrho^{p' - q'} \cdot \left\| \sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \right\|_{E_n}\]

We deduce the following estimates.

\[
\|I_{Z_{q'},W}(x)\| = \left( \sum_{n=1}^{\infty} \left\| \sum_{j=1}^{n} x_{n,j} e_{2,j}^{(n)} \right\|_{E_n} \right)^\frac{p'}{p}.
\]
\[ \mathcal{L}(\ell_p \oplus \ell_q) \text{ has infinitely many closed ideals} \]

We are now ready to prove our main result.

Proof of Theorem 11 We first consider the case \( 1 < p < 2 \) and \( p < q < \infty \). Put \( \eta = \frac{1}{p} - \frac{1}{2} \), and define \( f : \mathbb{N} \to \mathbb{R} \) by setting \( f(n) = n^{-\eta} \) for each \( n \in \mathbb{N} \). For an infinite set \( M \subset \mathbb{N} \) we define a decreasing sequence \( w_M = (w_M(n))_{n=1}^{\infty} \) in \((0,1]\) as follows. Let \( m_1 < m_2 < \ldots \) be the elements of \( M \). We set \( w_M(1) = 1 \), \( w_M(2^{m_k}) = f(2^k) \) for each \( k \in \mathbb{N} \), and then extend the definition of \( w_M \) to the rest of \( \mathbb{N} \) by linear interpolation. It is clear that \( w_M(n) \geq n^{-\eta} \) for all \( n \in \mathbb{N} \). We will show that for infinite sets \( M \subset N \subset \mathbb{N} \) with \( N \setminus M \) also infinite, the sequences \( v = w_N \) and \( w = w_M \) satisfy condition (12) of Theorem 6. Hence, by Corollary 7 we will have \( J^{I_{Y_{\mathcal{M}}}, x_0} \subset J^{I_{Y_{\mathcal{M}}}, x_0} \). Let us first explain how we complete the proof of our main theorem from here. We fix a chain \( \mathcal{C} \) of size the continuum consisting of infinite subsets of \( \mathbb{N} \) with any two having infinite difference. For \( M \in \mathcal{C} \) put \( Y_M = Y_{p,w_M} \). By above, the closed ideals \( J^{I_{Y_{\mathcal{M}}}, x_0} \), \( M \in \mathcal{C} \), are pairwise distinct and comparable. Moreover, for each \( M \in \mathcal{C} \), the operator \( I_{Y_{\mathcal{M}}, x_0} \) is finitely strictly singular by Proposition 8 and is clearly not compact. Hence the ideal \( J^{I_{Y_{\mathcal{M}}}, x_0} \) lies between \( J^{I_{Y_{\mathcal{M}}}, w} \) and \( FS \).

Using the same \( p \) and \( q \), we now consider ideals in \( \mathcal{L}(\ell_{p'}, \ell_{p'}) \). For each \( M \in \mathcal{C} \), we have \( I_{Y_{\mathcal{M}}}, Y_{\mathcal{M}} = I_{Y_{\mathcal{M}}, x_0} \), and hence by simple duality we have

\[ J^{I_{Y_{\mathcal{M}}}, x_0}(\ell_{p'}, \ell_{p'}) = \{ T^* : T \in J^{I_{Y_{\mathcal{M}}}, x_0} \} \]

Thus \( \{ J^{I_{Y_{\mathcal{M}}}, x_0} : M \in \mathcal{C} \} \) is a chain of closed ideals in \( \mathcal{L}(\ell_{p'}, \ell_{p'}) \) of size the continuum. By Proposition 8, the operators \( I_{Y_{\mathcal{M}}}, Y_{\mathcal{M}} \), \( M \in \mathcal{C} \), are finitely strictly singular, and clearly not compact, so these ideals also lie between \( J^{I_{Y_{\mathcal{M}}}, w} \) and \( FS \).

Since \( q' < p' \) and \( 2 < p' \), we have covered all remaining cases.

Let us now return to our claim: for infinite sets \( M \subset N \subset \mathbb{N} \) with \( N \setminus M \) also infinite, the sequences \( v = w_N \) and \( w = w_M \) satisfy condition (12) of Theorem 6.

Fix \( l \in \mathbb{N} \). We will show that for all sufficiently large \( n \in \mathbb{N} \), we have \( w_N(n^{\frac{1}{2}}) \leq 2^{-n^l} \), which proves the claim. Since \( N \setminus M \) is infinite, there exists \( k_0 \in \mathbb{N} \) such that for all \( n \geq m_{k_0} \) we have \( |N \cap \{1, \ldots, n\}| \geq |M \cap \{1, \ldots, n\}| + l + 2 \). Fix \( n > 2^{3^{m_{k_0}+1}} \). This defines \( k > k_0 \) such that \( 2^{3^m} \leq n < 2^{3^{m+1}} \). It follows that

\[ w_M(n) \geq w_M(2^{3^{m+1}}) = f(2^{k+1}) \]

Next define \( k' \in \mathbb{N} \) by \( n_{k'} \leq m_k - 1 < n_{k'+1} \). By the choice of \( k_0 \), and since \( m_k - 1 \geq m_{k_0} \), we have

\[ k' = |N \cap \{1, \ldots, m_k - 1\}| \geq |M \cap \{1, \ldots, m_k - 1\}| + l + 2 = k + l + 1 \]

It follows that

\[ w_N(n^{\frac{1}{2}}) \leq w_N(2^{3^{m_{k'}-1}}) \leq w_N(2^{3^{m_{k'}}}) = f(2^{k'}) \leq f(2^{k+l+1}) \]
Putting together (18) and (19) yields
\[ \frac{w_N(n^2)}{w_M(n)} \leq f(2^{k+1}) = 2^{-np}. \]
This holds for any \( n \geq m_{k_0} \), so the proof of our claim is complete. \( \square \)

We conclude this section with a proof of Theorem 9. This will be very similar to the general case but simpler. We shall still rely on our key lemma from Section 3.

From now on we fix \( 1 < p < 2 < q < \infty \). As usual, for a decreasing sequence \( v = (v_n) \) in \( (0, 1] \) we consider the complemented subspace \( Y_v = Y_{p,v} \) of \( \ell_p \) with corresponding projection \( P_v = P_{p,v} \) as introduced in Section 2. Since \( 2 < q \), the formal inclusion \( I_{Z_q,q}: Z_q = (\bigoplus_{n=1}^\infty \ell_2^n)_{\ell_q} \to \ell_q = (\bigoplus_{n=1}^\infty \ell_q^n)_{\ell_q} \) given by \( I_{Z_q,q}(e^{(n)}_q) = e^{(n)}_{q,j} \) is bounded, and hence, so is the formal inclusion \( I_{Y_v,q} = I_{Z_q,q} \circ I_{Y_v,q} \). We shall consider the closed ideal \( \mathcal{J}^{I_{Y_v,q}} \) which contains the operator \( S_v = I_{Y_v,q} \circ P_v \).

As before, we first distinguish ideals corresponding to different sequences.

**Theorem 9.** Let \( v \) and \( w \) be as in Theorem 9. Then \( S_v \notin \mathcal{J}^{I_{Y_v,q}} \).

**Proof.** We follow closely the proof of Theorem 9. For each \( n \in \mathbb{N} \) let \( F_n = F_{p,v,n}^{(n)} \) and \( G_n = P_{p,v,n}^{(n)} \) with bases \( \{ f^{(n)}_{j,n} : 1 \leq j \leq n \} \) and \( \{ g^{(n)}_{j,n} : 1 \leq j \leq n \} \), respectively. To simplify notation we write \( Y = Y_v \) and \( S = S_v \). Thus
\[ I_{Y_v,q}: Y = \left( \bigoplus_{n=1}^\infty F_n \right)_{\ell_p} \to \ell_q = \left( \bigoplus_{n=1}^\infty e^{(n)}_q \right)_{\ell_q} \]
is given by \( I_{Y_v,q}(f^{(n)}_{j,n}) = e^{(n)}_{q,j} \), and \( S: \ell_p = (\bigoplus_{n=m}^{\infty} \ell_p^n)_{\ell_p} \to \ell_q \) is the composite \( S = I_{Y_v,q} \circ P_v \). Note that \( S(g^{(n)}_{j,n}) = e^{(n)}_{q,j} \).

For each \( m \in \mathbb{N} \) we define \( \Phi_m \in \mathcal{L}(\ell_p, \ell_q)^* \) by setting
\[ \Phi_m(V) = \frac{1}{m} \sum_{i=1}^m \langle e^{(m)}_{q,j} , V(g^{(m)}_{j,n}) \rangle , \quad V \in \mathcal{L}(\ell_p, \ell_q). \]
Since \( \| \Phi_m \| \leq 1 \) for all \( m \), the sequence \( (\Phi_m) \) has a \( \omega^* \)-accumulation point \( \Phi \) in the unit ball of \( \mathcal{L}(\ell_p, \ell_q)^* \). Note that \( \Phi_m(S) = 1 \) for all \( m \), and hence \( \Phi(S) = 1 \). The proof will be complete if we can show that \( \mathcal{J}^{I_{Y_v,q}} \) is contained in the kernel of \( \Phi \). To see this, fix \( A \in \mathcal{L}(\ell_q) \) and \( B \in \mathcal{L}(\ell_p, \ell_q) \) with \( \| A \| \leq 1 \) and \( \| B \| \leq 1 \). It is sufficient to show that
\[ \Phi_m(A I_{Y_v,q} B) \to 0 \quad \text{as} \quad m \to \infty. \]
Let \( B_m : G_m \to Y \) denote the restriction to \( G_m \) of \( B \). Exactly as in Theorem 9 we use Lemma 4 to show that
\[ \frac{1}{m} \sum_{i=1}^m \| B_m(g^{(m)}_{j,n}) \|_\infty \to 0 \quad \text{as} \quad m \to \infty, \]
and obtain the estimate
\[ |\Phi_m(A I_{Y_v,q} B)| \leq \frac{1}{m} \sum_{i=1}^m \| I_{Y_v,q} B_m(g^{(m)}_{j,n}) \|_{\ell_q}. \]
At this point we depart from the proof of Theorem 9 as the argument becomes simpler. Let \( x = \sum_{n=1}^\infty \sum_{j=1}^n x_{n,j} f^{(n)}_{j} \in Y \) with \( \| x \|_Y \leq 1 \). Then for each \( n \in \mathbb{N} \) we have
\[ \sum_{j=1}^n |x_{n,j}|^q \leq \sum_{j=1}^n |x_{n,j}|^{q-2} \cdot |x_{n,j}|^2 \leq C_p^2 \cdot \| x \|_\infty^{q-2} \cdot \left( \sum_{j=1}^n |x_{n,j} f^{(n)}_{j} | \right)^2. \]
\[ \mathcal{L}(\ell_p \oplus \ell_q) \text{ has infinitely many closed ideals} \]

\[ \leq C_p^2 \cdot \|x\|_{\infty}^{-2} \cdot \left\| \sum_{j=1}^{n} x_{n,j} f_j^{(n)} \right\|_{F_n}^p. \]

It follows that

\[ \|I_{Y,q}(x)\|_{\ell_q} = \sum_{n=1}^{\infty} \sum_{j=1}^{n} |x_{n,j}|^q \leq C_p^2 \cdot \|x\|_{\infty}^{-2} \cdot \sum_{n=1}^{\infty} \left\| \sum_{j=1}^{n} x_{n,j} f_j^{(n)} \right\|_{F_n}^p \]

\[ = C_p^2 \cdot \|x\|_{\infty}^{-2} \cdot \|x\|_{Y}^p \leq C_p^2 \cdot \|x\|_{\infty}^{-2}. \]

Applying this to (22) we obtain

\[ |\Phi_m(\mathcal{A}I_{Y,q}B)| \leq C_p^2 \cdot \frac{1}{m} \sum_{i=1}^{m} \left\| B_m \left( g_i^{(m)} \right) \right\|_{1-\frac{2}{q}} \leq C_p^2 \cdot \left( \frac{1}{m} \sum_{i=1}^{m} \left\| B_m \left( g_i^{(m)} \right) \right\|_{\infty} \right)^{1-\frac{2}{q}}, \]

using the fact that the function \( t \mapsto t^{1-\frac{2}{q}} \) is concave. Now we use (21) to deduce (20), as required. \( \square \)

**Proof of Theorem A**. The proof of Theorem A provides a continuum size chain \( \mathcal{C} \) of infinite subsets of \( \mathbb{N} \), and corresponding sequences \( w_M, M \in \mathcal{C} \). In this case, it leads to spaces \( Y_M = Y_{p,w_M} \) and closed ideals \( \mathcal{J}^{Y_M,s} \). The proof of Theorem A shows that if \( M, N \in \mathcal{C} \) and \( M \subset N \), then \( v = w_N \) and \( w = w_M \) satisfy the hypotheses in Theorem A. In particular, the unit vector basis of \( Y_N \) dominates the unit vector basis of \( Y_M \), and so \( \mathcal{J}^{Y_N,s} \subset \mathcal{J}^{Y_M,s} \). Moreover, by Theorem A this inclusion is strict. Thus, the family \( \{ \mathcal{J}^{Y_M,s} : M \in \mathcal{C} \} \) of closed ideals of \( \mathcal{L}(\ell_p,\ell_q) \) is a chain and has size the continuum.

Finally, for each \( M \in \mathcal{C} \), the operator \( I_{Y_M,q} \) factors through the formal inclusion \( I_{2,q}: \ell_2 = \left( \bigoplus_{n=1}^{\infty} \ell_2^n \right)_{\ell_2} \rightarrow \ell_q = \left( \bigoplus_{n=1}^{\infty} \ell_q^n \right)_{\ell_q} \) via the formal inclusion \( I_{Y_2} \). Thus \( \mathcal{J}^{Y_M,s} \) is contained in \( \mathcal{J}^{I_{2,s}} \), and contains \( \mathcal{J}^{I_{s,s}} \) since \( I_{Y_M,q} \) is not compact. \( \square \)

5. Open problems

There are a number of natural questions that remain or arise after our work. The first aim would be to answer Pietsch’s question in the range \( 1 \leq p < q \).

**Problem 10.** Given \( 1 \leq p < q < \infty \), are there infinitely many closed ideals in \( \mathcal{L}(\ell_1,\ell_q) \) or in \( \mathcal{L}(\ell_p,\ell_0) \)?

Even in the reflexive range we do not know the exact number of closed ideals.

**Problem 11.** Given \( 1 < p < q < \infty \), find the cardinality of the set of closed ideals in \( \mathcal{L}(\ell_p,\ell_q) \).

We now know this is at least \( c \). On the other hand, it is clear that the cardinality of \( \mathcal{L}(\ell_p,\ell_q) \) is \( c \), and hence there can be at most \( 2^c \) closed ideals. Of course, one could pose the same problem with \( \ell_1 \) replacing \( \ell_p \), etc.

Note that in the case \( 1 < p < 2 < q < \infty \) we constructed two continuum chains of closed ideals. Are these equal? More generally, we could ask the following question about the lattice structure of closed ideals in \( \mathcal{L}(\ell_p,\ell_q) \).

**Problem 12.** Do the closed ideals of \( \mathcal{L}(\ell_p,\ell_q) \), after ignoring a finite number of them, form a chain?

So far all our new ideals are generated by a single operator. Note that if \( T \) is a compact operator of infinite rank, then \( \mathcal{K} = \mathcal{J}^T \). It is then natural to ask the following.

**Problem 13.** Is \( \mathcal{F}S \) generated by one operator? Are all closed ideals of \( \mathcal{L}(\ell_p,\ell_q) \) generated by one operator?
Another candidate of a closed ideal, not representable by a single operator could be the closure of the union of one of the chains we defined in the previous section.

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