Principle study of image reconstruction algorithms in muon tomography

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ABSTRACT: Muon tomography, as a novel method of radiography, utilizes the multiple Coulomb scattering property of muons from cosmic rays to discriminate materials. Researchers around the world have been developing various detection systems and image reconstruction algorithms for muon tomography applications, such as in the fields of nuclear reactor monitoring and cargo inspection for contrabands. This paper studies the principle of image reconstruction in muon tomography. The implementation and comparison of some popular algorithms with our simulation dataset are presented, together with ideas regarding potential improvements in image quality and material discrimination performance.

KEYWORDS: Computerized Tomography (CT) and Computed Radiography (CR); Analysis and statistical methods; Simulation methods and programs; Search for radioactive and fissile materials

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1 Introduction

Muons dominate the composition of cosmic rays reaching Earth’s surface, the vertical intensity of which is approximately $70 \text{ m}^{-2} \text{s}^{-1} \text{ sr}^{-1}$ at sea level according to the literature [1]. Researchers have studied in detail the properties of cosmic muons, including their angular and energy distributions, through theories and experiments [2, 3]. According to the scattering theory from Rossi [4], charged particles traversing a plate medium by multiple Coulomb scattering (MCS) have projected scattering angles and lateral displacements that approximately follow a Gaussian distribution. Derived from the marginal distribution, the mean value of the projected scattering angles is 0, and the variance is evaluated by the well-known Rossi formula (eq. (1.1))

$$
\sigma_{\theta}^2 = \frac{E_z^2}{2p^2v^2} \cdot \frac{x}{X_0}.
$$

(1.1)
In the formula, $\sigma^2_{\theta}$ is the variance of the scattering angles, $E_s$ is a constant with a value of 21 MeV, $X_0$ represents the radiation length of materials, $x$ is the thickness of the plate, and $p$ and $v$ are the momentum and velocity of muons, respectively. MSC theory has been developed continuously in later research [4–8]. A correction term in the formula obtained by fitting the results from Moliere’s theory was first suggested by Highland and further improved by Lynch and Dahl, with which the relatively simple and accurate formula was proposed (eq. (1.2)). Frühwirth and Regler introduced a Gaussian mixture model to describe both the core and the tail of the scattering angle distribution

$$
\sigma^2_{\theta} = \left( \frac{13.6 \text{ MeV}}{pv} \right)^2 \cdot \frac{x}{X_0} \left( 1 + 0.088 \log_{10} \frac{x}{X_0} \right)^2.
$$

(1.2)

Based on MCS theory, the concept of muon tomodigraphy was first developed at the Los Alamos National Laboratory (LANL) [9, 10] and then adopted by other groups around the world [11–14]. The scattering density is defined on the basis of the Rossi formula (eq. (1.1)) for material discrimination in muon tomodigraphy. With a known thickness of muon penetration, the scattering density ($\lambda$) is a value that depends on the radiation length and muon momentum, as shown in eq. (1.3)

$$
\lambda \triangleq \frac{\sigma^2_{\theta}}{x} = \left( \frac{15 \text{ MeV}}{pv} \right)^2 \cdot \frac{1}{X_0}.
$$

(1.3)

The main challenge in the design of a detection system is the acquisition of muon trajectories before and after target penetration. Existing systems consist of multiple layers of large two-dimensional position-sensitive detectors on both sides of the sample area. For the balance between position precision and economic cost, drift chambers, multigap resistive plate chambers (MRPCs), gas electron multipliers (GEMs) and plastic scintillating fibers are selected to form the detection system in the experiments [10–14]. For example, the Tsinghua University Muon Tomography Facility (TUMUTY) was built with 6 layers of MRPC detectors, and the incoming and outgoing directions of muons are calculated by fitting 3 interacting points of the top and bottom detector groups [12]. Proposals of measuring muon momentum by the time-of-flight (TOF) or MCS in dense detectors have also been surveyed in the literature [15, 16].

Muon tomodigraphy has the advantage of not requiring complex particle sources, unlike X-ray or proton CT. However, the low flux of cosmic muons limits the amount of observed data, which is a significant challenge in muon tomodigraphy. Although the discrimination of high-atomic-number materials such as fissile materials proves to be feasible, there is still not much progress for common materials with low or medium atomic numbers. In addition to the development of detection systems, image reconstruction algorithms for the muon tomodigraphy are effective approaches for investigating the possibility of discriminating these materials.

In this paper, we categorize algorithms for muon tomodigraphy into two stages: muon trajectory estimation and image reconstruction. Since the exact trajectories of muons inside imaging objects are unknown, estimation algorithms are applied to the measurements, which include the simple straight line path (SLP), the point of closest approach (PoCA) [10] and the most likely path (MLP) from MCS theory [17–19]. Tomography results are usually described as three-dimensional voxelized images in which each voxel contains some unique material properties. In muon tomodigraphy, the discriminative
property of materials is the scattering density estimated from the variance of scattering angles in each voxel. The second stage of muon tomography aims to determine the scattering density of each voxel from accumulated muon trajectories. Several algorithms have been proposed for the second stage, including assigning the total scattering angles to the PoCA [10], statistical reconstruction based on MCS theory [20], and the algebraic reconstruction technique algorithm (ART) in X-ray CT [21]. Universal algorithms that improve the radiological image qualities have also been tested, such as total variation (TV) regularization [22].

In section 2, we review the details of image reconstruction algorithms in muon tomography. Acquisition of the simulation data and implementation of the algorithms are presented in section 3, and the test results are shown and discussed in section 4. The final section summarizes these algorithms and discusses inspirations from our test results. Future improvements to the algorithms will make muon tomography more practical and compatible for industrial applications.

2 Algorithms

Muon tomography involves accumulating scattering angles at different locations on an imaging object and then generating a map of scattering densities based on their variance calculation. Figure 1 shows the radiation lengths and scattering densities of some materials and demonstrates the discrimination capability of muon tomography.

Figure 1. Radiation lengths and scattering densities of materials at 4 GeV.

The procedures for image reconstruction in muon tomography can be divided into two stages: the estimation of muon trajectories inside the imaging object and the allocation of MCS to voxels. We review the existing algorithms in each stage in this section.
2.1 Trajectory estimation

In muon tomography, the detection system is designed to measure incident and exiting muon trajectories as accurately as possible. However, muon trajectories passing through the interior of imaging objects remain a “black box”. Although the actual trajectories are quite complicated and unpredictable, some estimation algorithms have been developed based on the small scattering angle property of MCS.

2.1.1 Straight line path (SLP)

Usually, the imaging objects are thin compared with the high penetrating ability of cosmic muons, and the distribution of scattering angles centers at 0. One obvious method for obtaining the trajectory is to connect the entry and exit points directly with a straight line. In voxelized space, the SLP can be efficiently calculated using the Siddon algorithm [23, 24].

This algorithm is sufficiently accurate and widely used for X-ray CT, but the divergence of trajectories is not negligible in charged-particle-related tomography. Other estimation algorithms taking scattering into account show superior imaging quality when applied in muon tomography.

2.1.2 Point of closest approach (PoCA)

As the name suggests, the PoCA is the point on the closest approach from one line to the other. In two-dimensional space, the PoCA is simply the intersection of two lines. Since skew lines in three-dimensional space do not always intersect, the closest approach is the line segment simultaneously perpendicular to both lines. In practice, the PoCA is defined as the midpoint of the closest approach, which can be obtained by solving the following equations

\[
\begin{align*}
    P_1 &= p_1 + t_1 d_1 \\
    P_2 &= p_2 + t_2 d_2 \\
    (P_1 - P_2) \cdot d_1 &= 0 \\
    (P_1 - P_2) \cdot d_2 &= 0
\end{align*}
\]

(2.1)

In the equation, \( P = p + td \) is the parametric equation of points on straight lines, with \( p_1, d_1 \) and \( p_2, d_2 \) being the known points and directions of the incident and exiting muon trajectories, respectively.

After the derivation of the PoCA, the estimated trajectory is a polyline composed of the entry point, the PoCA and the exit point. This algorithm offers a better approximation than does the SLP, but a portion of muon trajectories whose PoCAs are not inside the scattering medium will be discarded.

2.1.3 Most Likely Path (MLP)

Initially developed in proton tomography research, the MLP combines MCS theory and the maximum likelihood estimation to find particle trajectories. Given the information of the entry and exit points, the likelihood of a muon passing a specific point was formulated by Schulte [17]

\[
L(Y|Y_0) = L(Y_1|Y)L(Y|Y_0).
\]

(2.2)
Figure 2. Illustration of trajectory estimation algorithms.

Y in the equation represents a vector containing the scattering angle and lateral displacement at depth in the medium, and \( Y_0 \) and \( Y_1 \) are the directions and displacements of the entry and exit points, respectively. According to MCS theory, the projected scattering angle and lateral displacement approximately follow a joint Gaussian distribution. The covariance matrix of this distribution is shown in eq. (2.3), with \( \lambda \) and \( z \) as the scattering density and depth in the medium, respectively

\[
\Sigma = \lambda \begin{bmatrix}
z^3 & z^2 \\
\frac{z^2}{3} & \frac{z^2}{2} \\
\frac{z^2}{2} & z
\end{bmatrix}.
\] (2.3)

The expression of the MLP is derived by maximizing the probability of a muon passing point \( Y \), as shown in eq. (2.4), where \( R_0 \) and \( R_1 \) are matrices that translate the entry and exit points to the corresponding depth; the covariance matrices \( \Sigma_0 \) and \( \Sigma_1 \) are also calculated at that depth

\[
Y_{\text{MLP}} = \left( \Sigma_0^{-1} + R_1^T \Sigma_1^{-1} R_1 \right)^{-1} \left( \Sigma_0^{-1} R_0 Y_0 + R_1^T \Sigma_1^{-1} Y_1 \right).
\] (2.4)

The MLP represents the statistically most likely trajectory of muons that enter and exit at the same points and angles. For charged particles whose energy loss in the medium is not negligible, the elements of the covariance matrix become integral expressions to include this factor [19]. Since the prior scattering densities in the object are unknown, a homogeneous medium is often assumed.
2.2 Reconstruction

With trajectory estimation algorithms and the accumulation of cosmic muon measurements, the next stage requires an algorithm generating the distribution of scattering densities inside the imaging object. Researchers have developed various image reconstruction algorithms for muon tomography and compared them with the work of their predecessors in terms of the resolution, noise-to-signal ratio, etc.

2.2.1 Direct allocation

By definition, the scattering density is the variance of the scattering angles scaled by the traversing thickness. Thus, images can be reconstructed after each voxel has enough accumulated scattering angle data. Direct allocation methods analyze the estimated muon trajectories and assign scattering angles directly to some voxels.

The most straightforward strategy is to assign the average scattering angle to each voxel along the path. This strategy may be useful for homogeneous media, but our targets of interest are structures with large scattering density differences. In this case, the most commonly used direct allocation method utilizes the PoCA information and assumes that a single point accounts for the total scattering, while other points on the trajectory have zero scattering angles [10]. This method is often called the PoCA algorithm in the literature and should be distinguished from the PoCA in trajectory estimation. The reconstruction process is described in algorithm 1.

\[
\text{Data: Muon trajectories} \\
\text{Result: Scattering densities of each voxel} \\
\text{Initialize scattering angle container } S_i \text{ for each voxel;} \\
\text{foreach muon trajectory } \mu \text{ do} \\
\quad \text{foreach voxel } v_i \text{ in } \mu \text{ do} \\
\quad \quad \text{if PoCA in } v_i \text{ then} \\
\quad \quad \quad \quad \text{Append } \theta \text{ to } S_i \\
\quad \quad \quad \text{else} \\
\quad \quad \quad \quad \quad \text{Append } 0 \text{ to } S_i \\
\quad \quad \text{end} \\
\quad \text{end} \\
\text{Calculate the variance of } S_i \text{ for each voxel, and } \lambda_i = \frac{\text{Var}[S_i]}{x_i} \\
\]

Algorithm 1. Process of PoCA algorithm.

In addition to the basic PoCA algorithm, researchers have explored some ideas for improvements. For example, considering the measurement uncertainties, the “pitchfork” method is introduced, where muon trajectories are randomly sampled according to the presumed uncertainties and scattering densities are evaluated on this extended dataset [13]. An algorithm utilizing MLP estimation is also examined by assigning the scattering angle to the point along the MLP with the largest distance from the SLP [18].
2.2.2 Maximum likelihood scattering and displacement (MLSD)

Direct allocation methods are easy to implement and efficient in computation, while reconstructed images tend to be noisy due to the oversimplified assumption of scattering and uncertainties in trajectory estimation. Schultz developed an algorithm based on the statistical model of MCS in his dissertation [10] and named it maximum likelihood scattering and displacement. Further improvement of the maximization method in MLSD using the expectation-maximization (EM) algorithm is presented in the article [20].

The likelihood of scattering densities $\lambda$ in the imaging object with $M$ measurement data $D$ can be expressed as eq. (2.5). Each $D_i$ is a vector composed of the projected scattering angle and lateral displacement measured by the detection system. The probability term $P(D_i|\lambda)$ in the production is derived from the Gaussian approximation of MCS, as shown in eq. (2.6) with the same expression for the covariance matrix in the MLP algorithm (eq. (2.3))

$$P(D_i|\lambda) = \frac{1}{2\pi|\Sigma_i|^1/2} \exp \left( -\frac{1}{2} D_i^T \Sigma_i^{-1} D_i \right).$$

EM iteration is used to solve the maximum likelihood problem in MLSD. The algorithm introduces the hidden data $H$, the element $H_{ij}$ of which is defined as the projected scattering angle and lateral displacement inside the $j$th voxel for the $i$th muon. The values of the hidden data can be estimated from the scattering density and path length in each voxel. Eq. (2.7) and eq. (2.8) show the likelihood function of hidden data given the current estimation of the scattering densities

$$\log P(H|\lambda) = \log \left[ \prod_{i \leq M, j \leq N} P(H_{ij}|\lambda) \right] = \sum_{i \leq M, j \leq N} \log P(H_{ij}|\lambda)$$

$$P(H_{ij}|\lambda) = \frac{1}{2\pi|\Sigma_{ij}|^{1/2}} \exp \left( -\frac{1}{2} H_{ij}^T \Sigma_{ij}^{-1} H_{ij} \right).$$

An auxiliary function $Q(\lambda; \lambda^{(n)})$ is defined in Schultz’s paper [20] as eq. (2.9). The final step of each iteration involves calculating the new estimation of the scattering densities that maximizes this auxiliary function (eq. (2.10))

$$Q(\lambda; \lambda^{(n)}) = \mathbb{E}_{H|D, \lambda^{(n)}}[\log P(H|\lambda)]$$

$$\lambda^{(n+1)} = \arg \max_{\lambda} Q(\lambda; \lambda^{(n)}).$$

MLSD requires much more computational power to reconstruct images than do the direct allocation methods, but this iterative approach is robust and converges to images with better quality.

2.2.3 Maximum a posteriori (MAP) and regularization

The maximum a posteriori is another estimation procedure frequently used in radiological tomography. Similar to MLSD, given the observed data, the probability of the scattering density is expressed according to the Bayesian theorem

$$P(\lambda|D) = \frac{P(D|\lambda)P(\lambda)}{P(D)}.$$
Then, to maximize this probability, we obtain the objective function for the scattering density

$$\lambda = \arg \max_{\lambda \geq 0} P(\lambda | D).$$  \hspace{1cm} (2.12)

Calculating the logarithm of the probability, the objective function is expressed in the following form. If the prior of the scattering density distribution $P(\lambda)$ is flat, we have the maximum likelihood previously discussed. Otherwise, the prior term can be written as a function $U(\lambda)$ scaled by $\beta$, which penalizes the log-likelihood term

$$\lambda = \arg \max_{\lambda \geq 0} [L(D|\lambda) + \log P(\lambda)]
= \arg \max_{\lambda \geq 0} [L(D|\lambda) - \beta U(\lambda)].$$  \hspace{1cm} (2.13)

The MAP estimator is often referred to as regularization in statistical image reconstruction. Many ideas on how to formulate the regularization term have been explored in the X-ray CT community [25], which may be borrowed and applied in muon tomography for various purposes. The most popular regularization term is the Markov random field model-based priors, which assume that material properties are related only to neighboring voxels. Thus, the regularization terms are often expressed in the following equation

$$U(\lambda) = \sum_j \sum_{m \in W_j} w_{jm} \phi(\lambda_j - \lambda_m).$$  \hspace{1cm} (2.14)

$W_j$ is the set of neighboring voxels of voxel $j$, and function $\phi(\Delta)$ represents the relation that two neighboring voxels should follow in prior knowledge. Common choices for the function $\phi(\Delta)$ include the quadratic function and compressed sensing based on the $l_p$ norm, as shown in the following equations

$$\phi(\Delta) = \frac{\Delta^2}{2}$$  \hspace{1cm} (2.15)

$$\phi(\Delta) = \left( \sum_m |\Delta_m|^p \right)^{1/p}.$$  \hspace{1cm} (2.16)

These regularization terms ensure edge preservation and noise reduction during the maximization process. There are some successful applications in muon tomography that result in better-reconstructed images than those of the standard MLSD [22].

### 2.3 Summary of algorithms

Algorithms for the two stages in muon tomography have been continuously developed. Any combination of algorithms within the two stages produces a complete pipeline of reconstruction. As shown in table 1, the MLP has not yet been extensively applied in image reconstruction, which may be a future approach to achieving improvements.
### Table 1. Summary of algorithms for muon tomography.

| Trajectory Estimation | Reconstruction |
|-----------------------|----------------|
| SLP                   | Direct allocation |
| PoCA                  | MLSD            |
| MLP                   | MAP             |

3 Simulation and test

#### 3.1 Monte Carlo simulation of muon scattering

Monte Carlo (MC) simulation is a powerful approach to testing experimental concepts virtually based on physical models. CERN’s GEANT4 toolkit [26] is one of the most popular and flexible MC platforms which enables us to generate a dataset for testing muon tomography algorithms. The available MCS models in GEANT4’s electromagnetic processes include the Wentzel model, the Urban model, and the single scattering (SS) model. To validate these models with empirical formulas (eq. (1.1), eq. (1.2)) used in experiments, scattering angles of muons with a fixed energy of 4 GeV penetrating lead plates of different thicknesses are simulated with the MCS models mentioned above. The standard deviations of the simulated scattering angles are drawn alongside the curves predicted by the empirical formulas in figure 3. Single scattering should be the most accurate method because it models each occurrence of Coulomb scattering, although the computation time will be unacceptable when the penetrated medium is thick. The Wentzel model agrees with eq. (1.2) well and is chosen for the generation of the simulation dataset.

![Figure 3. Comparison of GEANT4 MCS models.](image)

The geometry setup of the simulation is shown in figure 4. Two layers of position-sensitive detectors (green) are placed on each side of the sample area. The sample area has a length, width...
and height of 1 meter. A water container (blue) is placed in the middle with two metal cubes (red) inside. The smaller cube is made of lead, and the other cube is made of iron. The remaining space is filled with air. Cosmic muons are generated by the CRY [27] code and transported through the system by GEANT4. The interaction points of muons in the detectors are recorded event-by-event without position uncertainty as the simulation dataset for later image reconstruction tests.

Figure 4. Geometry of the MC simulation.

3.2 Implementation of algorithms

The review of different algorithms in the previous section suggests that image reconstruction consists of two stages. Therefore, abstract classes for trajectory estimation and reconstruction are included in our software implementation design, which will enhance the flexibility and modularity of the code. The details of the software design are presented in figure 5. “Application” uses “MutoConfig” and “MutoFile” to handle the user’s configuration and the data input and output. “MutoVTrackEstimation” is the interface for trajectory estimation, and “MutoVReconstruction” is the interface for the reconstruction algorithms. The “MutoSiddon” and “MutoMLP” classes implement the trajectory estimation interface for the straight line and the most likely path. For reconstruction, the “MutoPoCA” class means direct allocation with the PoCA trajectory, and the “MutoMLSDEM” class defines both the MLSD and MAP algorithms.
Our code of muon tomography is open source and can be downloaded online.\textsuperscript{1} Currently, direct allocation with the PoCA trajectory, MLSD and MAP algorithms have been implemented and tested with the simulation dataset. The results are shown and discussed in the following section.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{muon_tomography_software_design.png}
\caption{Design of muon tomography software.}
\end{figure}

4 Results

4.1 Trajectory estimation precision

As discussed in section 2, there exist 3 major algorithms for charged particle trajectory estimation. We simulated muons with a fixed energy of 4 GeV scattered inside an aluminum plate of 800 mm thickness and recorded their positions during the simulation as the “true” trajectories. Figure 6 shows an example of the actual path of a muon with the results estimated by different algorithms.

It is inappropriate to rank the precision of the trajectory estimation algorithms by several examples, such as figure 6 due to the stochastic MCS processes. For each simulation event, we sample the actual path of the muon inside the aluminum plate every 10 mm. The estimated trajectories are calculated for the corresponding depths. Then, differences between the simulated and estimated trajectories at each depth are evaluated for this event. We choose the root-mean-square (RMS) of these differences at each depth (eq. (4.1)) as the metric for comparing the algorithms

\[
\text{RMS}_j = \sqrt{\frac{\sum_{i=1}^{n} (d_{ij} - \hat{d}_{ij})^2}{n}}.
\] \hspace{1cm} (4.1)

\textsuperscript{1}https://github.com/nmtzwh/MuonTomography
Figure 6. An example of the simulated trajectory and estimated results.

$j$ is the index for the depth, $i$ is the index for simulation events, with a total number of events $n$, and $d$ and $\hat{d}$ represent the lateral displacement for the simulated and estimated trajectories, respectively.

From the dataset of 10000 simulated muons, we calculate the RMS values at each depth for the 3 algorithms, as shown in figure 7. Since estimations start with the entry and exit data, the deviations between the true trajectories and estimated trajectories are small at both ends but reach their maximum in the middle of the depth axis.

Figure 7. Precision of the trajectory estimation algorithms.

Our results show that among these algorithms, the MLP produces the most accurate estimations regarding divergences from simulated trajectories, which has been verified in other articles [17, 19]. The application of the MLP in both proton and muon tomography has been studied, showing better-reconstructed images than achieved by other trajectory estimation algorithms [18, 28]. For the accurate representation of the unknown trajectories, replacing the SLP or PoCA algorithms with the MLP in the existing reconstruction procedures is a promising strategy for improving the image quality or material discrimination in muon tomography.
4.2 Reconstructed images

Image reconstruction algorithms are tested on the simulation dataset generated with the geometry shown in figure 4. For comparison, the PoCA trajectory estimation is used in all three reconstruction algorithms, including direct allocation to the PoCA (referred to as “PoCA” in this section for simplicity), MLSD and MAP. In our tests, the reconstructed scattering densities from the cosmic muons did not meet the values predicted from theory, e.g. in figure 1. The following section discusses the solution to this problem and compares the reconstruction algorithms in terms of image quality.

4.2.1 Influence of muon momentum

Muon tomography relies on the differences in the scattering density of materials. However, the Rossi formula (eq. (1.1)) suggests that the muon momentum is also an essential factor for the determination of scattering densities. Cosmic muons have kinetic energy covering a wide range. Incident muons of low kinetic energy tend to have larger scattering angles in the medium.

Since the dataset is generated from MC simulation, we have full control of the incident muons and their properties. The dataset is divided into two groups on the basis of the median muon kinetic energy of 2 GeV, and processes of image reconstruction are separately accomplished with the PoCA algorithm. The results (figure 8) indicate that image reconstructed from the high energy group is clearer and smoother, while variation in scattering densities is a severe problem for muons of low kinetic energy.

![Figure 8](image_url)

(a) $E_k \leq 2\text{GeV}$

(b) $E_k > 2\text{GeV}$

**Figure 8.** Horizontal projection (y-x) of PoCA results from two groups of muons.

For material discrimination, representative values of the scattering density should be evaluated in the reconstruction algorithms. In our test, merely calculating the variance of all scattering angles leads to much larger values than those shown in figure 10. We propose a method for scaling and standardizing the evaluation of the scattering density.
First, we define the standard scattering density for muons of momentum \( p_0 \) as \( \lambda_0 \). Then, each scattering angle is scaled to its equivalent with the standard muon momentum using \( \theta = \frac{p\cos\gamma}{p_0\cos\gamma_0} \). The exact momentums of incident muons are normally unknown in real tomography systems, so we cannot directly scale the measured scattering angles. The relation between the measured scattering density and the standard \( \lambda_0 \) is derived in eq. (4.2). The expectation value of \( \mathbb{E}\left[\frac{1}{(pv)^2}\right] \) in the equation can be estimated from the momentum distribution of cosmic muons obtained by theoretical calculation, Monte Carlo simulation or experimental measurement [1, 3, 27]

\[
\sigma^2_\theta = x\lambda = \mathbb{E}[\theta^2] = \mathbb{E}\left[\left(\frac{p_0v_0}{pv}\right)^2\right] = (p_0v_0)^2\mathbb{E}\left[\frac{1}{(pv)^2}\right] \sigma^2_0
= (p_0v_0)^2\mathbb{E}\left[\frac{1}{(pv)^2}\right] \cdot x\lambda_0.
\]

4.2.2 Comparing reconstructed images

Images reconstructed with the PoCA, MLSD and MAP algorithms are projected to both the horizontal and vertical planes and displayed in figure 9. The horizontal plane corresponds to the y-x plane in figure 4, and the vertical plane is chosen as the z-x plane. The results indicate that muon tomography has a much better resolution in the horizontal plane than in the vertical plane because two-dimensional detectors are placed horizontally in the simulation (see figure 4) and the PoCA trajectory estimation induces larger uncertainty along the z-axis. Although all algorithms are sufficient to discover the scattering density differences among the metal cubes and the water container, the image reconstructed by the PoCA algorithm shows more noise than the other two statistical methods. The image from the MAP algorithm preserves sharper edges than that of the MLSD after the same number of iterations, which is a reasonable result of regularization.

Image qualities are not always related to material discrimination abilities. For example, the regularization method may produce smoother images with clearer edges but reduce the differences in the quantitative property (scattering density in this case) among materials. Thus, reconstructed scattering density values are statistically analyzed within regions of the two metal cubes and shown as histograms in figure 10. For the PoCA algorithm, there are large scattering density values in voxels that belong to neither of the metal cubes, which directly introduces noise observed in the reconstructed image. Part of the voxels in the iron cube render smaller scattering density values in the MAP algorithm with the \( l_1 \) norm. The lead cube in all situations has voxels that infiltrate the region belonging to iron materials. Overall, MLSD exhibits the best performance in terms of material discrimination but not the best image quality compared with that of the MAP.

5 Conclusions

The theories and algorithms of image reconstruction in muon tomography are reviewed in this paper. We divide the procedures of image reconstruction into two stages, upon which most of the state-of-the-art algorithms have been developed, as listed in table 1.

For the representation of simulated muon scattering data, physics models of MCS in the GEANT4 toolkit are tested with theoretical formulas (figure 3). These tests suggest that the Wentzel model better agrees with MCS theory and should be used in the simulation.
Figure 9. Comparison of reconstruction results. From left to right: PoCA algorithm, MLSD after 50 iterations, and MAP with $l_1$ norm after 50 iterations. From top to bottom: horizontal plane (y-x) and vertical plane (z-x) views.

Figure 10. Comparison of scattering densities in different materials. From left to right: PoCA algorithm, MLSD after 50 iterations, and MAP with $l_1$ norm after 50 iterations.

Trajectory estimation algorithms are described in detail, and their precision is evaluated for the simulated trajectories. The MLP is the most accurate trajectory estimation algorithm (figure 7). However, the application of the MLP in reconstruction remains a topic to be explored.

Popular algorithms for muon tomography are implemented and tested on our simulation dataset, including the PoCA, MLSD and MAP algorithms. The correction of the scattering density considering the muon momentum should be applied in the PoCA algorithm to achieve values comparable to the theoretical predictions. With more complicated algorithms that will bias the scattering density values in the future, an experimental calibration process for materials of interest is required. Our results of dividing muons by a threshold of 2 GeV (figure 8) indicate that momentum measurement of the incident muons would significantly improve the reconstructed images.
Though simple in design and programming, the PoCA algorithm exhibits stable performance and has been widely used as the benchmark algorithm in experiments presented in the literature. Statistical methods such as the MLSD and MAP are superior in terms of image qualities and material discrimination capabilities at the expense of the computing time and complexity in choosing the parameters, such as the regularization functions and iteration steps. There are new promising ideas in the active research field of X-ray CT that will benefit muon tomography as well. For instance, regularization strategies in low-dose X-ray CT should be encouraging for muon tomography suffering from low counting statistics.

Despite continuous improvements in image reconstruction algorithms, whether it is practical and competitive for muon tomography to be applied to industrial applications such as cargo inspection or nuclear material discrimination depends on the limits induced by the cosmic muon flux, the measuring time, the detector resolution and the scattering density differences of the target materials. Thus, a theoretical framework of muon tomography, which predicts the optimal imaging or material discrimination capability for specific applications, is also essential for future algorithm development.

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