Normal hyperbolicity and unbounded critical manifolds

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Abstract
This work is motivated by mathematical questions arising in differential equation models for autocatalytic reactions. We extend the local theory of singularities in fast–slow polynomial vector fields to classes of unbounded manifolds which lose normal hyperbolicity due to an alignment of the tangent and normal bundles. A projective transformation is used to localize the unbounded problem. Then the blow-up method is employed to characterize the loss of normal hyperbolicity for the transformed slow manifolds. Our analysis yields a rigorous scaling law for all unbounded manifolds which exhibit a power-law decay for the alignment with a fast subsystem domain. Furthermore, the proof also provides a technical extension of the blow-up method itself by augmenting the analysis with an optimality criterion for the blow-up exponents.

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1. Introduction

The motivation of this work is several models of singularly perturbed differential equations arising in applications [9, 10]. In [21] Merkin et al propose to study a prototypical autocatalytic system given by four reactions

\[ P \rightarrow Y, \quad Y \rightarrow X, \quad Y + 2X \rightarrow 3X, \quad X \rightarrow Z, \]

where \( X, Y \) are the two main reactants, \( P \) is a constant ‘pool’-chemical and \( Z \) is the product. Then it can be shown, using standard mass-action kinetics and non-dimensionalization [24],
that (1) leads to a two-dimensional (2D) system of ordinary differential equations (ODEs) given by

\[
\begin{align*}
\epsilon \dot{x} &= yx^2 + y - x, \\
\dot{y} &= \xi - yx^2 - y,
\end{align*}
\]

where \(\dot{}\) denotes derivative with respect to time \(\tau\), the phase space variables \(x, y\) are dimensionless concentrations associated to \(X, Y\) respectively and \(\epsilon, \xi\) are parameters; we note that the parameter \(\epsilon\) is defined by the ratio of reaction rates for \(Y \rightarrow X\) and \(X \rightarrow C\) [24, p 6192].

Furthermore, note that the nonlinear term arises due to the autocatalytic reaction part \(Y + 2X \rightarrow 3X\). It has been proven in [10] that the 2D-autocatalator (2) can exhibit an attracting relaxation-oscillation periodic orbit for certain ranges of the parameters; see also figure 1(a). In [24] Petrov et al generalized (1) by including a further reactant

\[
P \rightarrow Y, \quad P + Z \rightarrow Y + Z, \quad Y \rightarrow X, \quad X \rightarrow Z, \quad Y + 2X \rightarrow 3X, \quad Z \rightarrow W.
\]

(3)

As before, it is straightforward [24] to derive from (3) the ODEs

\[
\begin{align*}
\epsilon \dot{x} &= yx^2 + y - x, \\
\dot{y} &= \xi(x + z) - yx^2 - y, \\
\dot{z} &= x - z,
\end{align*}
\]

(4)

where \((\xi, \kappa, \epsilon)\) are parameters and \((x, y, z) \in (\mathbb{R}^*_+)^3\) are the concentrations. Numerical studies [9, 22, 23] have shown that periodic, mixed-mode and chaotic oscillations exist for the three-dimensional (3D) autocatalator (4).

For both models (2), (4) it is often assumed that the ratio of time scales \(\epsilon > 0\) is a sufficiently small parameter; we shall also denote this assumption by \(0 < \epsilon \ll 1\). In this case it follows that both autocatalator models are fast–slow (or singularly perturbed) ODEs. We use the notation \(p\) to denote a point in the phase space of concentrations for the autocatalator models i.e. \(p = (x, y) \in (\mathbb{R}^*_+)^2\) or \(p = (x, y, z) \in (\mathbb{R}^*_+)^3\). The \(x\)-nullcline, or critical manifold, for both models is given by

\[
C_0 = \{p : yx^2 + y - x = 0\} = \left\{p : y = \frac{x}{1 + x^2}\right\}.
\]

(5)

Note that \(C_0\) is an unbounded smooth manifold. The angle between the tangent spaces \(T_pC_0\) and the hyperplanes \(\{p : y = \text{const.}\}\) decays to zero as \(x \to +\infty\); see figure 1(b). This
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alignment, which is expected to imply a loss of normal hyperbolicity in the system, already causes substantial difficulties in the rigorous analysis of the dynamics of the 2D autocatalator model [10]. The global return mechanism induced by the unbounded part of \( C_0 \) also plays a key role for the complex oscillatory patterns of the 3D autocatalator [9, p 98] which have been observed in numerical simulations.

In a completely different context a model with similar properties to the autocatalator was proposed by Rankin et al [25] as a caricature system for effects in aircraft ground dynamics

\[
\begin{aligned}
\epsilon \dot{x} &= y + (x - \xi) \exp \left( \frac{x}{\kappa} \right), \\
\dot{y} &= \nu - x,
\end{aligned}
\]

(6)

where \((\xi, \kappa, \nu, \epsilon)\) are parameters, \(0 < \epsilon \ll 1\) and \(C_0 = \{ y = (\xi - x) \exp(x/\kappa) \}\) also aligns with \( \{ y = 0 \} \) as \( x \to -\infty \) if \( \kappa > 0 \) or as \( x \to +\infty \) if \( \kappa < 0 \).

Given these examples, it is desirable to build a general theory of fast-slow systems with unbounded critical manifolds. The main contributions of this work, stated here in a non-technical form, are the following.

- We study a general class of critical manifolds which may have an arbitrary power-law decay \( y \sim 1/x^s \) (as \( x \to \infty \)) for the alignment with a fast subsystem domain. This includes the autocatalytic critical manifolds as special cases and answers open questions arising from various numerical studies.
- Using the blow-up method we give a rigorous proof when normal hyperbolicity for a perturbation of the critical manifold fails i.e. which is the largest region up to which a slow manifold, obtained as a perturbation of the critical manifold, is normally hyperbolic. The relevant scaling law turns out to be given by

\[
(x, y) \sim \left( \mathcal{O}(\epsilon^{-1/(s+1)}), \mathcal{O}(\epsilon^{s/(s+1)}) \right), \quad \text{as } \epsilon \to 0,
\]

(7)

where \(-s\) is the power law exponent for the asymptotic decay of the critical manifold. A precise statement of the result is given in theorem 8.1.
- On a technical level we contribute to a further development of the blow-up method by augmenting it with an ‘optimality-criterion’ of blow-up coefficients which have to be chosen in the analysis.
- Furthermore, the result answers an open question posed by Guckenheimer and Scheper [9] on the global return mechanism for the autocatalator model and provides a more general theoretical approach to parts of the work by Gucwa and Szmolyan [10]. The scaling (7) also has immediate consequences for the asymptotics of oscillatory patterns.

The paper is structured as follows: in section 2 we introduce the notation and the required background from fast-slow systems. In section 3 a formal asymptotic argument is given to illustrate the important scaling properties and the system is localized via a projective transformation. The local system is desingularized via blow-up in section 4. The charts and transition functions between the two relevant charts are calculated. The dynamics in each chart are analysed in sections 5 and 6, respectively. The optimality criterion for the blow-up coefficients is proven in section 7. The main scaling result is summarized in section 8 and its implications are discussed.

Remark. Regarding asymptotics, the notation \( \mathcal{O}(\cdot) \) is used in the usual way, i.e. \( v(z) = \mathcal{O}(w(z)) \) as \( z \to z^* \) if and only if \( \lim \sup_{z \to z^*} |v(z)/w(z)| < \infty \). The shorthand \( w(z) \asymp w(z) \) is used whenever \( w(z) = \mathcal{O}(v(z)) \) and \( v(z) = \mathcal{O}(w(z)) \) hold simultaneously. Furthermore, \( v(z) \ll w(z) \) indicates that \( \lim_{z \to z^*} |v(z)/w(z)| = 0 \).
2. Background and notation

In this paper, we restrict our attention to the analysis of family of planar fast–slow systems given by

\[
\begin{align*}
\frac{dx}{dt} &= x' = f(x, y), \\
\frac{dy}{dt} &= y' = \epsilon g(x, y),
\end{align*}
\]

where \((x, y) \in \mathbb{R}^2, 0 < \epsilon \ll 1\) and \(f : \mathbb{R}^2 \to \mathbb{R}, g : \mathbb{R}^2 \to \mathbb{R}\) are assumed to be sufficiently smooth. We also mention already that we shall assume a suitable non-degeneracy condition on \(g\) below i.e. \(g(x, y) \neq 0\) in a suitable region of phase space.

**Remark.** By focusing on planar systems we cover, under suitable non-degeneracy hypotheses, the case of the 2D autocatalator mode. Regarding the 3D autocatalator, one expects that centre manifold techniques, similar to the generalization for folded singularities from three to arbitrary finite-dimensional fast–slow systems [30], can be applied to make the results we develop applicable.

We are going to choose particular forms of \(f, g\) below but introduce some general terminology beforehand; for more detailed reviews/introductions to fast–slow systems see [3, 13, 14]. The critical set of (8) is given by

\[ C_0 := \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}. \]

We are going to assume that \(C_0\) defines a smooth manifold. \(C_0\) is called normally hyperbolic at a point \((x^*, y^*) \in C_0\) if \(\partial_x f(x^*, y^*) \neq 0\). We will assume that \(C_0\) is normally hyperbolic at every point in \(\mathbb{R}^2\) but \(C_0\) will be unbounded. Let us point out that there has been a lot of work on the geometric theory of planar fast–slow systems recently by De Maesschalck and Dumortier, see e.g. [18–20], particularly in the context of local singularities, periodic orbits and Liénard systems.

**Remark.** The general definition of normal hyperbolicity of an invariant manifold \([6, 11, 31]\) requires the splitting of tangent and normal dynamics. Recall that a compact manifold \(M \subset \mathbb{R}^N\) is normally hyperbolic for a vector field \(F : \mathbb{R}^N \to \mathbb{R}^N\) if there exists a continuous splitting of the tangent bundle

\[ T_M \mathbb{R}^N = N^u \oplus T_M \oplus N^s, \]

where the linearization \(DF\) expands \(N^u\) and contracts \(N^s\) more sharply than \(T_M\). In \(\mathbb{R}^2\), we obviously just need one of the normal bundles if \(\text{dim}(M) = 1\). It is important to point out that we do restrict ourselves to perturbations of normally hyperbolic manifolds in the class of fast–slow vector fields of the form (8) being interested in loss of normal hyperbolicity of invariant manifolds within this class of perturbed vector fields where the perturbation is given by \(\epsilon g(x, y)\).

The implicit function theorem yields that \(C_0\) is locally a graph. In our case, the parametrization will be global so that

\[ C_0 = \{(x, y) \in \mathbb{R}^2 : x = h(y)\}. \]

for some smooth function \(h : \mathbb{R} \to \mathbb{R}\). The singular limit \(\epsilon \to 0\) of (8) defines a differential algebraic equation, also called the fast subsystem

\[
\begin{align*}
x' &= f(x, y), \\
y' &= 0.
\end{align*}
\]
Changing to the slow time scale $\tau = \epsilon t$ in (8) and taking the singular limit $\epsilon = 0$ yields the slow subsystem defined on $C_0$
\[
0 = 0 = f(x, y),
\]
\[
\frac{dy}{d\tau} = \dot{y} = g(x, y).
\]
(10)
The slow subsystem can be written as $\dot{y} = g(h(y), y)$. If $h$ is invertible, the critical manifold can also be defined via $C_0 = \{(x, y) \in \mathbb{R}^2 : y = h^{-1}(x) =: c(x)\}$. Implicit differentiation with respect to $\tau$ yields $\dot{y} = c'(x)\dot{x}$ so that the slow subsystem is $c'(x)\dot{x} = g(x, c(x))$.

Fenichel’s theorem [7, 13] implies that any compact submanifold $M_0 \subset C_0$ persists as a slow manifold $M_\epsilon$ for $0 < \epsilon \ll 1$. Furthermore, $M_\epsilon$ is locally invariant and diffeomorphic to $M_0$. The flow on $M_\epsilon$ converges to the slow flow as $\epsilon \to 0$. For any fixed small $\epsilon > 0$ we can define $M_\epsilon$ and then extend it under the flow of (8) but this extension may no longer be normally hyperbolic as an invariant manifold for the full system (8). To understand this effect we propose to study the family of model systems given by
\[
f(x, y, \epsilon) = 1 - x sy
\]
for $s \in \mathbb{N}$. (11)

Furthermore, we are going to assume that
\[
g(x, y) = \mu
\]
for some $\mu \neq 0$. (12)
The assumption (12) of the slow variable dynamics is motivated by the fact that if $g(x, y) \neq 0$ for a suitable region in phase space, then we may re-scale time to obtain (12) [2, section 1.5]. The choice (11) for the fast vector field is motivated by the asymptotic expansion of $C_0$ for the autocatalator models (2),(4) as $x \to \infty$. In fact, returning to the chemical interpretation the general model class corresponds to autocatalytic reaction steps of the form $Y + (s + 1)X \to (s + 2)X$.

The smooth unbounded critical manifold for (11) is given by
\[
C_0 := \{(x, y) \in \mathbb{R}^2 : y = \frac{1}{x^s} = c(x)\}.
\]
(13)
As $|x| \to \infty$ the tangent space $T_{(x,y)}C_0$ starts to align with the x-axis since $c(x) \to 0$ and $c'(x) \to 0$ as $|x| \to \infty$. By reflection symmetry we restrict our focus to the positive part $C_0 \cap (\mathbb{R}^+_x)^2$ from now on i.e. the intersection with the non-negative quadrant will be understood. Note that the model class (11) covers already a wide variety of power-law decay rates. Since we are only interested in the unbounded part of $C_0$ we shall restrict the dynamics to
\[
H^\sigma := \{(x, y) \in \mathbb{R}^2 : x > \sigma\}
\]
for a suitable fixed $\sigma > 0$ with $\sigma \asymp \mathcal{O}(1)$ as $\epsilon \to 0$; see also figure 2(a). Since
\[
\partial_x f(x^*, y^*) = -s(x^*)^{s-1}y^* \neq 0 \text{ for } (x^*, y^*) \in C_0
\]
it follows that any compact submanifold $M_0 \subset C_0$ is normally hyperbolic and perturbs, by Fenichel’s theorem, to a normally hyperbolic slow manifold for $\epsilon > 0$ sufficiently small. However, for fixed small $\epsilon > 0$ we do not know yet when the extension of $M_\epsilon$ under the flow starts to deviate from the critical manifold approximation.

3. Asymptotic expansion and projective transformation

A good intuitive understanding of the problem is gained by considering one of the simplest possible systems in the model class (11) given by $s = 1$ and $\mu = 1$ so that
\[
\epsilon \dot{x} = 1 - yx,
\]
\[
\dot{y} = 1.
\]
(14)
From the viewpoint of formal asymptotics we can simply calculate a slow manifold $\mathcal{M}_\epsilon$ by making the ansatz

$$x(y) = x_0(y) + \epsilon x_1(y) + \epsilon^2 x_2(y) + \cdots. \tag{15}$$

Inserting (15) into (14) and collecting terms of different orders in $\epsilon$ yields

$$\mathcal{O}(1): \quad 0 = 1 - yx_0(y),$$

$$\mathcal{O}(\epsilon^k): \quad \frac{dx_{k-1}}{dy} = -yx_k(y), \tag{16}$$

for $k \in \mathbb{N}$. A direct calculation using (16) gives the formal asymptotics of the slow manifold

$$x(y) = \frac{1}{y} - \sum_{k=1}^K (2k - 1) \frac{\epsilon^k}{y^{2k+1}} + \mathcal{O}(\epsilon^{K+1}) \quad \text{as } \epsilon \to 0. \tag{17}$$

The series (17) fails to be asymptotic when $1/y \sim \epsilon^k/y^{2k+1}$ as $\epsilon \to 0$ or $y \sim \mathcal{O}(\sqrt{\epsilon})$. Since the critical manifold of (14) is given as the graph of $y = 1/x$ we expect that normal hyperbolicity of the associated slow manifold breaks down for $x \sim \mathcal{O}(\epsilon^{-1/2})$. The example (14) suggests that, depending on the asymptotics of the critical manifold as $x \to \infty$, there is an $\epsilon$-dependent scaling law that governs within which regime we can use the critical manifold approximation provided by Fenichel’s theorem. To give a rigorous proof of the asymptotic scaling we would like to desingularize the problem. In [10] the autocatalator is considered and phase space is re-scaled by an $\epsilon$-dependent transformation; see also further remarks in section 8. Here we use a different approach by first completely localizing the problem; see also [29]. Consider a projective coordinate transformation

$$\rho : \mathcal{H}^1 \to (\mathbb{R}^2 - \mathcal{H}^1) \quad \rho(x, y) = \left(\frac{1}{x}, y\right), \tag{18}$$

where $\sigma = 1$ is chosen for computational convenience for the domain $\mathcal{H}^\sigma$. All the following results are easily checked to be independent of a fixed chosen $\sigma > 0$ independent of $\epsilon$; see also figure 2(a). A direct calculation yields the following result:

**Lemma 3.1.** Applying the coordinate transformation (18) yields a vector field defined on $(\mathbb{R}^2 - \mathcal{H}^1)$ and given by

$$\epsilon \dot{v} = (yv^{2-\epsilon} - v^2),$$

$$\dot{y} = \mu. \tag{19}$$
The vector field (19) can be desingularized, similar to the desingularization for folded singularities [28], via multiplication by $v^s$ and a time re-scaling which leads to
\[
\begin{align*}
\epsilon \dot{v} &= v^2 (y - v'), \\
\dot{y} &= \mu v', \\
\epsilon' &= 0.
\end{align*}
\] (20)

Note that the transformation reverses time on orbits for $v < 0$ if $s$ is odd but that we always restrict to the non-negative quadrant which implies that we do not have to consider this issue here. After the transformation the original critical manifold $C_0 = \{y = 1/x'\}$ is given by
\[
\rho(C_0) = \{(v, y) \in \mathbb{R}^2 - \mathcal{H}^1 : y = v'\} =: S_0.
\]
Observe that $S_0$ is normally hyperbolic except at the degenerate singularity $(v, y) = (0, 0)$. The key point of the projective transformation $\rho$ is that it localized the approach towards a singular point to calculate the breakdown of normal hyperbolicity; in fact, it is not the singularity arising from infinity we are interested in but the scaling of the associated slow manifold $S_\epsilon$ in a sufficiently small ball excluding $(v, y) = (0, 0)$. Note that the case $s = 1$ yields the classical transcritical structure for $S_0$ with $f(v, y) = v(y - v)$; see [16, 27] and references therein. To desingularize (19) one can try to apply a blow-up transformation. The blow-up method [4] was introduced into fast–slow systems theory in the work of Dumortier and Roussarie [5]. For additional background on the blow-up method for fast–slow local singularities see [17].

4. Desingularization via blow-up

Re-writing (20) on the fast time scale and augmenting the system by $\epsilon' = 0$ leads to the ODEs
\[
\begin{align*}
v' &= v^2 (y - v'), \\
y' &= \epsilon \mu v', \\
\epsilon' &= 0,
\end{align*}
\] (21)
which can also be viewed as a vector field $X : \mathbb{R}^3 \to T\mathbb{R}^3 \cong \mathbb{R}^3$; the natural identification of $\mathbb{R}^3$ and its tangent bundle will be made from now on. Note that the equilibrium point $(v, y, \epsilon) = (0, 0, 0)$ of (21) is not hyperbolic. The blow-up method provides a geometric way to desingularize this point. We shall briefly recall the main elements of the method; readers familiar with the method may proceed to lemma 4.2.

Consider the manifold $\tilde{B} := S^2 \times [0, r_0]$ for some $r_0 > 0$ where $S^2 \subset \mathbb{R}^3$ denotes the unit sphere. Denote the coordinates on $\tilde{B}$ by $(\tilde{v}, \tilde{y}, \tilde{\epsilon})$ where $(\tilde{v}, \tilde{y}, \tilde{\epsilon}) \in S^2$. The geometric idea of the blow-up technique is to ‘insert’ the sphere $S^2 \times \{\tilde{r} = 0\}$ at the non-hyperbolic equilibrium point. In this process, one wants to leave the qualitative dynamics away from $(v, y, \epsilon) = (0, 0, 0)$ unchanged, which is going to correspond to the region $S^2 \times \{\tilde{r} > 0\}$; see also figure 3 and/or [17, figure 2]. A general weighted blow-up transformation $\Phi : \tilde{B} \to \mathbb{R}^3$ is then given by
\[
v = \tilde{r}^{\alpha_v} \tilde{v}, \quad y = \tilde{r}^{\alpha_y} \tilde{y}, \quad \epsilon = \tilde{r}^{\alpha_\epsilon} \tilde{\epsilon},
\] (22)
where we choose the coefficients as
\[
(\alpha_v, \alpha_y, \alpha_\epsilon) = (1, s, s + 1).
\] (23)
The choice of $(\alpha_v, \alpha_y, \alpha_\epsilon)$ can be motivated by using the Newton polygon [1, 4]; we shall not detail the method for choosing the coefficients here. Via $\Phi$ a vector field $\tilde{X}$ is induced on the space $\tilde{B}$ by requiring $\Phi_* : T\tilde{B} \to T\mathbb{R}^3 \cong \mathbb{R}^3$
is the usual push-forward; see also figure 3. The next step is to introduce suitable charts for the manifold \( \overline{B} \) to simplify the calculations. In particular, the goal is to introduce charts \( \kappa_i : \overline{B} \rightarrow \mathbb{R}^3 \), where \( i \in \mathbb{N} \) is a label, such that applying the coordinate transformation \( \kappa_j \circ \Phi^{-1} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) yields an elegant computable way to analyse the blown-up vector field \( \tilde{X} \) in local coordinates; see figure 3. Recall that we restrict to the manifold \( C_0 \) in the positive quadrant. Let \( B_2 \) := \( \overline{B} \cap \{ \tilde{y} > 0 \} \) and \( B_1 \) := \( \overline{B} \cap \{ \tilde{\epsilon} > 0 \} \) and consider two charts
\[
\kappa_1 : B_1 \rightarrow \mathbb{R}^3 \quad \text{and} \quad \kappa_2 : B_2 \rightarrow \mathbb{R}^3,
\]
where \((v_1, r_1, \epsilon_1)\) and \((v_2, y_2, r_2)\) denote the respective coordinates for \( B_1 \) and \( B_2 \). We define the chart maps \( \kappa_1 \) and \( \kappa_2 \) by
\[
\begin{align*}
v_1 &= \tilde{v} \tilde{y}^{-1/s}, & r_1 &= \tilde{r} \tilde{y}^{1/s}, & \epsilon_1 &= \tilde{\epsilon} \tilde{y}^{-(s+1)/s}, \\
v_2 &= \tilde{v} \tilde{\epsilon}^{-1/(s+1)}, & y_2 &= \tilde{y} \tilde{\epsilon}^{-s/(s+1)}, & r_2 &= \tilde{r} \tilde{\epsilon}^{1/(s+1)}.
\end{align*}
\]
(24)
This choice is not obvious so we briefly explain the motivation. Previous blow-up calculations, see for example [5, 16, 17], suggest that it is desirable to define coordinate charts by conditions such as ‘\( \tilde{y} = 1 \)’ or ‘\( \tilde{\epsilon} = 1 \)’. More precisely, we want the coordinate change maps \( \Phi \circ \kappa_i^{-1} \) to be given by
\[
\Phi \circ \kappa_i^{-1}((v_1, y_1, \epsilon_1)) = (r_1 v_1, r_1^{1/s} \epsilon_1) \quad \text{or} \quad \Phi \circ \kappa_2^{-1}((v_2, y_2, \epsilon_2)) = (r_2 v_2, r_2^{1/s} y_2, r_2^{1/(s+1)}).
\]
(25)
Hence, we may think formally of setting \( \tilde{y} = 1 \) or \( \tilde{\epsilon} = 1 \) in the original blow-up map \( \Phi \) and then re-labelling the coordinates on the right-hand sides in (22) to arrive at the desired maps (25). Of course, starting from the definitions (24) does not require these formal considerations and we may just check the next result; see also figure 3.

**Lemma 4.1.** The transitions functions \( \kappa_{12} \) and \( \kappa_{21} \) between charts are
\[
\begin{align*}
v_2 &= v_1 e_1^{1/(s+1)}, & y_2 &= e_1^{1/s} e_1^{1/(s+1)}, & r_2 &= r_1 e_1^{1/(s+1)}, \\
v_1 &= v_2 y_2^{1/s}, & r_1 &= r_2 y_2^{1/s}, & \epsilon_1 &= y_2^{-(s+1)/s}.
\end{align*}
\]
(26)
**Proof.** The proof is a calculation based upon using the definitions (24). For example, we find
\[
v_2 = \tilde{v} \tilde{\epsilon}^{-1/(s+1)} = v_1 \tilde{y}^{1/s} \tilde{e}^{-1/(s+1)} = v_1 \tilde{y}^{1/s} \epsilon_1^{1/(s+1)} \tilde{y}^{-1/s} = v_1 \epsilon_1^{-1/(s+1)}.
\]
The calculation for the other transition functions is similar. \( \square \)
Lemma 4.2. The desingularized vector field in \( \kappa_1 \) is given by
\[
\begin{align*}
v'_1 &= sv_1^2 (1 - v_1) - \mu \epsilon_1 v_1^{s+1}, \\
r'_1 &= \mu r_1 \epsilon_1 v_1^s, \\
\epsilon'_1 &= -(s + 1) \mu \epsilon_1^2 v_1^s.
\end{align*}
\] (27)

The desingularized vector field in \( \kappa_2 \) is given by
\[
\begin{align*}
v'_2 &= v_2^2 (y_2 - v_1), \\
y'_2 &= \mu v_2^2.
\end{align*}
\] (28)

Proof of Lemma 4.2. The required calculations follow by direct differentiation, for example, one has
\[
r'_1 = (\mu/s) r_1^{s+2} \epsilon_1 v_1^s \quad \text{and} \quad v' = r'_1 v_1 + r_1 v'_1,
\]
which imply upon algebraic manipulation that
\[
v'_1 = r_1^{s+1} \left( v_1^2 (1 - v_1) - \frac{\mu}{s} \epsilon_1 v_1^{s+1} \right).
\]

Similarly, one obtains the equation for \( \epsilon_1 \) given by
\[
\epsilon'_1 = -\frac{(s + 1)}{s} \mu r_1^{s+1} \epsilon_1^2 v_1^s.
\]

A division by the common factor \( r_1^{s+1} \) and a time re-scaling yield (27). In \( \kappa_2 \) the transformation is just a re-scaling \((v, y) = (\epsilon_1^{1/(s+1)} v_2, \epsilon_1^{1/(s+1)} y_2)\) and a time rescaling \( t \to t/\epsilon \) gives (28).

5. Dynamics in the first chart

We can continue the critical manifold \( S_0 = \{(v, y) \in (\mathbb{R}^2)^+ : y = v^s\} \) into the chart \( \kappa_1 \). Using the definition of the blow-up map and Lemma (4.1) imply the relevant continuation result:

Lemma 5.1. \((\kappa_1 \circ \Phi^{-1})(S_0) = \{(v_1, y_1) \in (\mathbb{R}^2)^+ : v_1 = 1\} \Rightarrow S_{1,a}\).

Analysing the flow (27) gives more information about \( S_{1,a} \). The vector field (27) has two invariant subspaces \( \epsilon_1 = 0 \) and \( r_1 = 0 \). In the first subspace the flow is
\[
\begin{align*}
v'_1 &= sv_1^2 (1 - v_1), \\
r'_1 &= 0,
\end{align*}
\] (29)

which has a line of equilibrium points \( S_{1,a} \) which are attracting in the \( v_1 \)-direction. There is also a line of equilibria at \( v_1 = 0 \) which we do not have to consider. In the second invariant subspace the flow is
\[
\begin{align*}
v'_1 &= sv_1^2 (1 - v_1) - \mu \epsilon_1 v_1^{s+1}, \\
\epsilon'_1 &= -(s + 1) \mu \epsilon_1^2 v_1^s.
\end{align*}
\] (30)

Linearizing around \((v_1, \epsilon_1) = (1, 0)\) provides the important local dynamics:

Lemma 5.2. The vector field (30) has a centre-stable equilibrium at \((v_1, \epsilon_1) = (1, 0)\), \( p_{1,a} \) with eigenvalues \(-1\) and \(-s^2\) and associated eigenvectors \((-\mu/s^2, 1)^T\) and \((1, 0)^T\).

Lemma 5.3. Locally near \( p_{1,a} \), the equilibrium \( p_{1,a} \) has a one-dimensional centre manifold
\[
\mathcal{N}_{1,a} = \{v_1 = 1 - \epsilon_1 \frac{\mu}{s} + c_{11} \epsilon_1^2 + \mathcal{O}(\epsilon_1^3) : = n_{1,a}(\epsilon_1) + \mathcal{O}(\epsilon_1^3)\},
\]
where \( c_{11} = -(1 + s + s^2) \mu^2/(2s^3) \).
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Proof of lemma 5.3. Existence is guaranteed by the centre manifold theorem [8]. Using a translation \( V_1 = v_1 - 1 \) moves \( p_{1,a} \) to the origin. A further linear coordinate change

\[
\begin{pmatrix}
 V_1 \\
 \epsilon_1
\end{pmatrix} = \begin{pmatrix}
 -\mu s^2 & 1 \\
 1 & 0
\end{pmatrix} \begin{pmatrix}
 z_1 \\
 z_2
\end{pmatrix}
\]

transforms (30) into

\[
\begin{align*}
 z_1' &= 0 - \mu (1 + s) z_1^2 + O(3), \\
 z_2' &= -s^2 - \mu^2 \left( 1 + \frac{1}{2s} + \frac{1}{2s^2} \right) z_1^2 + 2\mu z_1 z_2 - \left( \frac{3s^2}{2} + \frac{s^3}{2} \right) z_2^2 + O(3),
\end{align*}
\]

where \( O(3) = O(z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3) \). Making the ansatz \( z_2 = c_{11} z_1^2 + O(z_1^3) \) and substituting into the invariance equation [8] for the centre manifold at \( z = (0,0) \) implies the condition

\[
0 = \left( c_{11} s + \mu^2 + \frac{\mu^2}{2s} + \frac{\mu^2}{2s^2} \right) z_1^2 + O(z_1^3).
\]

Therefore, \( c_{11} = -(1 + s + s^2)\mu^2/(2s^3) \) and transforming back to \((v_1, \epsilon_1)\) yields the result. \( \square \)

As before, other equilibria of (30) will not be of relevance here. Lemma 5.2 and the centre manifold theorem [8] imply the next result:

**Lemma 5.4.** There exists a centre-stable manifold \( M_{1,a} \) for (27) at \( p_{1,a} \) containing \( S_{1,a} \) and \( N_{1,a} \). Furthermore, \( M_{1,a} \) is locally given as a graph of a map \( y_1 = h_{1,a}(r_1, \epsilon_1) \).

From the flow on the centre manifold we see that there are two cases based upon the sign of \( \mu \). If \( \mu > 0 \) then trajectories in the centre manifold flow away from the sphere \( S^2 \) while for \( \mu < 0 \), trajectories in \( M_{1,a} \) flow from the chart \( \kappa_1 \) onto the sphere. We are only going to deal with the case \( \mu < 0 \) from now on. The case \( \mu > 0 \) can be obtained from \( \mu < 0 \) by a time reversal of the original problem (21). Figure 4(a) provides a sketch of the results and the notation for the relevant dynamical objects.

6. Dynamics in the second chart

Now the critical manifold in the chart \( \kappa_1 \) has to be continued into the chart \( \kappa_2 \). This transition is studied using the continuation \( M_{2,a} \) of \( M_{1,a} \) under the flow in \( \kappa_2 \); see also figure 4(b).
Lemma 6.1. The curve $N_{1,a} \subset M_{1,a}$ transforms as

$$
\kappa_{12}(N_{1,a}) = \left\{ (v_2, y_2) \in \mathbb{R}^2 : v_2 = 1 - \frac{\mu}{y_2} + \frac{c_{11}}{y_2^{2(\kappa+1)/\kappa}} + O\left(\frac{1}{y_2^{\kappa+1}}\right) \text{ as } y_2 \to +\infty \right\}.
$$

We introduce the notation $N_{2,a} := \kappa_{12}(N_{1,a})$ and note that $N_{2,a}$ can be extended under the flow (28) in chart $\kappa_2$ for initial conditions with $v_2 \approx 1$ and arbitrarily large $y_2 > 0$. To emphasize that this extension is a trajectory we are going to denote it by $n_{2,a}(t) = (n_{v_2,a}(t), n_{y_2,a}(t))^T \in \mathbb{R}^2$. A case distinction based upon the value of $s$ will be necessary. For $s = 2k$ with $k \in \mathbb{N}$ the differential equation is

$$
v'_2 = v^2_2(y_2 - v^{2k}_2),
$$

$$
y'_2 = \mu v^{2k}_2,
$$

so that $v'_2 < 0$ and $v'_2 < 0$ for $v_2 < v^{2k}_2$ imply that $n_{v_2,a}(t) \to 0$ for $t \to +\infty$. The variational equation is defined via the time-dependent matrix

$$
A(t) := D_{(v_2, y_2)} \left( \frac{v^2_2(y_2 - v^{2k}_2)}{\mu v^{2k}_2} \right)|_{n_{2,a}(t)} = \begin{pmatrix} 2n_{v_2,a}(t)n_{y_2,a}(t) - (2k + 2)n_{v_2,a}(t)^{2k+1}/2k\mu \, n_{v_2,a}(t)^{2k-1} & 1 \\ n_{v_2,a}(t)^2 & 0 \end{pmatrix},
$$

which shows that the variational equation becomes asymptotically autonomous in forward time [26] and $A(t)$ approaches a double zero eigenvalue when $t \to +\infty$. Hence $M_{2,a}$ is not everywhere normally hyperbolic in $\kappa_2$. A possible alternative argument for $k > 1$, leading to the same conclusion, involves desingularizing (31) using division by $v^2_2$ and observing that the resulting vector field is parallel on the $y_2$-axis with no flow in the $y_2$-component. However, with this argument the case $k = 0$ is special but can be treated using standard results about the resulting Riccati equation which also appears in the blow-up analysis of the fold point [15].

For $s = 2k + 1$ and $k \in \mathbb{N}$ the vector field is

$$
v'_2 = v^2_2(y_2 - v^{2k+1}_2),
$$

$$
y'_2 = \mu v^{2k+1}_2.
$$

Note that (32) does not have the same nice monotonicity properties as before. To show the convergence towards $v_2 = 0$ define

$$
V(v_2, y_2) := \frac{1}{2k} v^{2k}_2 + \frac{1}{2|\mu|} y^{2}_2 \Rightarrow \frac{dV}{dr} = v^{2k-1}_2 v'_2 + \frac{1}{|\mu|} y_2 y'_2 = -v_2^{2k+1} < 0
$$

showing that $V$ is a Lyapunov function [12] which implies the required convergence. Computing the variational equation yields

$$
A(t) := D_{(v_2, y_2)} \left( \frac{v^2_2(y_2 - v^{2k+1}_2)}{\mu v^{2k+1}_2} \right)|_{n_{2,a}(t)} = \begin{pmatrix} 2n_{v_2,a}(t)n_{y_2,a}(t) - (2k + 3)n_{v_2,a}(t)^{2k+2}/\mu (2k + 1)n_{v_2,a}(t)^{2k} & n_{v_2,a}(t)^2 \\ \mu (2k + 1)n_{v_2,a}(t)^{2k} & 0 \end{pmatrix},
$$

which shows that the variational equation becomes asymptotically autonomous in forward time and $A(t)$ approaches a double zero eigenvalue when $t \to +\infty$. As before, one could also use a suitable division by a power of $v_2$ and observe that the resulting vector field is parallel on the $y_2$-axis.

Lemma 6.2. $M_{2,a}$ is not everywhere a normally hyperbolic manifold inside $\kappa_2$, i.e., there is no hyperbolic splitting of the tangent and normal directions for $M_{2,a}$ as the fast flow aligns with the slow flow on $M_{2,a}$. 

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Lemma 6.2 is just a local restatement of the alignment property of the original slow manifold $C_\epsilon$ with the fast subsystem domains. This confirms the conjectured loss of normal hyperbolicity as $x \to +\infty$. The previous results combine to give the following statement which is a partial version of our main result.

**Proposition 6.3.** Consider the family of planar fast–slow systems
\begin{align}
x' &= 1 - x' y, \\
y' &= \epsilon \mu.
\end{align}
for $s \in \mathbb{N}, \mu \neq 0$ and $\epsilon > 0$ sufficiently small. Let $M_0 \subset C_0$ be a compact submanifold of the critical manifold. Then $M_\epsilon$ extends to a normally hyperbolic manifold of (33) up to a domain of size
\begin{align}
(x, y) &= (O(\epsilon^{-1/(s+1)}), O(\epsilon^{s/(s+1)})) \text{ as } \epsilon \to 0.
\end{align}
Under the blow-up map (22)–(23) the manifold $M_\epsilon$ cannot be extended to a normally hyperbolic slow manifold to a subset of a larger domain.

**Proof.** We work in projective coordinates via $\rho$. We can restrict to pieces of the critical manifold lying in the positive quadrant without loss of generality. $M_\epsilon$ is obtained from $M_0$ by Fenichel’s theorem since $\epsilon > 0$ is sufficiently small. Using the blow-up transformation and lemma 5.4, it follows that $M_\epsilon$ extends to a normally hyperbolic manifold in the chart $\kappa_1$. By lemma 6.2 the extension is not normally hyperbolic in the chart $\kappa_2$. By lemma 4.1 and lemma 4.2 it follows that the neighbourhood of $(v, y) = (0, 0)$ scales as
\begin{align}
(v, y) &= \left(\epsilon^{1/(s+1)} v_2, \epsilon^{s/(s+1)} y_2\right).
\end{align}
By applying a blow-down transformation it follows that $M_\epsilon$ is normally hyperbolic up to a scaling region given by (34) and not normally hyperbolic inside some larger domain. Via $\rho^{-1}$ we get $(x, y) = (O(\epsilon^{-1/(s+1)}), O(\epsilon^{s/(s+1)}))$ as $\epsilon \to 0$ in original coordinates so that the result follows. □

7. Smaller regions

To prove the final result, we aim to strengthen proposition 6.3 as it claims that the slow manifold extends and then normal hyperbolicity breaks down for the scaling (34) using the blow-up (22) and (23). A priori, we could have chosen different exponents for the blow-up which could have allowed us to extend the slow manifold even further. Here we show that the exponents in (34) are indeed optimal in the sense that for any pair $(\alpha_1, \alpha_2), \alpha_i \geq 0$ with $0 < \alpha_1 + \alpha_2$ the manifold $C_\epsilon$ is not normally hyperbolic in the larger region
\begin{align}
(x, y) &= \left(\mathcal{O}(\epsilon^{-(1+\alpha_1)/(s+1)}), \mathcal{O}(\epsilon^{(s+\alpha_2)/(s+1)})\right), \text{ as } \epsilon \to 0.
\end{align}
Note that it suffices to assume that $0 < \alpha_1 + \alpha_2 \ll 1$ so that the region is slightly larger. Consider the modified blow-up
\begin{align}
v &= \tilde{r}^{\alpha_1} \tilde{v}, \quad y = \tilde{r}^{\alpha_2} \tilde{y}, \quad \epsilon = \tilde{r}^{s+1} \tilde{\epsilon}.
\end{align}
Observe that in the chart $\kappa_2$ the blow-up (35) reduces to the re-scaling
\begin{align}
v &= \epsilon^{(1+\alpha_1)/(s+1)} v_2, \quad y = \epsilon^{(s+\alpha_2)/(s+1)} y_2
\end{align}
which yields a strictly smaller scaling region than before.
Lemma 7.1. In the chart \( \kappa_1 \) the vector field is
\[
\begin{align*}
\dot{r}_1 &= \frac{\mu}{s + \alpha_2} r_1^{2 - \alpha_1 + 2 \alpha_2} \epsilon_1 v_1', \\
\dot{v}_1' &= -\frac{\mu (1 + \alpha_1)}{s + \alpha_2} r_1^{1 - \alpha_1 + 2 \alpha_2} \epsilon_1 v_1'^2 + r_1^{\alpha_1 + \alpha_2} (r_1^{\alpha_2} - r_1^{2 \alpha_1} v_1'), \\
\epsilon_1 &= -\frac{(s + 1) \mu}{s + \alpha_2} r_1^{1 - \alpha_1 + 2 \alpha_2} \epsilon_1^2 v_1'^2.
\end{align*}
\]

Proposition 7.2. The extension of the manifold \( S_\epsilon \) in the chart \( \kappa_1 \) for the blow-up (35) is not normally hyperbolic as \( r_1 \to 0 \).

Proof. Suppose first that \( s \alpha_1 - \alpha_2 \leq 0 \). Since \( 0 < \alpha_1 + \alpha_2 \leq 1 \) we can desingularize the vector field (36) by \( r_1^{1 - \alpha_1 + 2 \alpha_2} \) as \( 1 - \alpha_2 + s + s \alpha_1 > 0 \). Considering the invariant subspace defined by \( \epsilon_1 = 0 \) of the desingularized vector field yields the equation
\[
\dot{v}_1' = r_1^{2 + 2 \alpha_2 - 2 \alpha_1} v_1'^2 (r_1^{\alpha_1} - r_1^{2 \alpha_1} v_1')
\]
which has a line of equilibria given by \( v_1 = r_1^{(\alpha_1 - \alpha_2)/s} \) corresponding to the critical manifold \( S_0 \). Linearizing (37) around this line gives the variational equation
\[
V' = r_1^{\alpha_1 + 2 \alpha_2 - 3 \alpha_1} (2 r_1^{(\alpha_1 - \alpha_2)/s} - (s + 1) r_1^{\alpha_1 - \alpha_2} r_1^{(\alpha_1 - \alpha_2)/(s+1)}) V
\]
which reduces to \( V' = 0 \cdot V \) as \( r_1 \to 0 \). Hence, in this case, the extension of \( S_\epsilon \) is not normally hyperbolic in all of \( \kappa_1 \). In the case \( s \alpha_1 - \alpha_2 > 0 \) we can desingularize by \( r_1^\alpha \) where \( \alpha^* = 1 + s + \min(\alpha_1 + \alpha_2, s \alpha_1 - \alpha_2) \) in which case again a multiplicative factor containing \( r_1 \) appears in the variational equation.

The main point of the previous proof is that we cannot remove the \( r_1 \)-dependence in the chart \( \kappa_1 \) which means that there is no centre manifold at the point \( p_{1,0} \) extending up to the sphere \( S^2 \), as previously for the blow-up with \( \alpha_1 = 0 = \alpha_2 \).

8. The main result and conclusions

Finally, we can combine the previous result to obtain the main result about loss of normal hyperbolicity and scaling. Recall again that we always restrict the dynamics to \( (\mathbb{R}^n_0)^2 \cap \mathcal{H}^1 \) which is a subset in the non-negative quadrant bounded away from \( x = 0 \).

Theorem 8.1. Consider the family of planar fast–slow systems
\[
\begin{align*}
x' &= 1 - x^s y, \\
y' &= \epsilon \mu.
\end{align*}
\]
for \( s \in \mathbb{N}, \mu \neq 0 \) and \( \epsilon > 0 \) sufficiently small. Let \( M_0 \subset C_0 \) be a compact submanifold of the critical manifold. Then \( M_\epsilon \) extends to a normally hyperbolic manifold of (38) up to a domain of size
\[
(x, y) \simeq (\mathcal{O}(\epsilon^{-1/(s+1)}), \mathcal{O}(\epsilon^{s/(s+1)})), \quad \text{as } \epsilon \to 0.
\]
under the flow of (38). \( M_\epsilon \) is not normally hyperbolic for any larger domain i.e. there does not exist a normally hyperbolic invariant manifold \( M_\epsilon \) in a domain strictly larger than (39) as \( \epsilon \to 0 \) such that \( M_\epsilon \) is strictly contained in \( M_\epsilon \).
Proof. Apply propositions 6.3 and 7.2. □

It is expected that theorem 8.1 still holds if we allow for higher-order terms in the fast variable, which do not influence the asymptotic decay of the critical manifold. However, if we want to go beyond the breakdown regime of normal hyperbolicity and analyse trajectories in arbitrary neighbourhoods of \((x, y) = (\pm \infty, 0)\) then we conjecture that higher-order terms could play a role as demonstrated in [16].

Theorem 8.1 can be applied to the 2D autocatalator. Indeed, we directly find the asymptotics of the critical manifold \(C_0\), given in (5), is given by \(y \sim 1/x\) as \(x \to +\infty\). Furthermore, if \(\xi\) is fixed in the 2D autocatalator (2) then we have that on \(C_0 = \{y = x/(1+x^2)\}\) the slow component of the vector field satisfies

\[
g(x, y)|_{(x, y) \in C_0} = (\xi - yx^2 - y)|_{(x, y) \in C_0} = \xi - \frac{x^3}{1+x^2} - \frac{x}{1+x^2} \sim \frac{x^3}{1+x^2} - \xi - \frac{x}{1+x^2}
\]

as \(x \to +\infty\) so that the condition \(g(x, y) \neq 0\) holds as \(x \to +\infty\). It is very interesting to point out the relation between the approach by Gucwa and Szmolyan [10] and the general theorem 8.1. The scaling proposed in [10, p 787] to capture the large relaxation-type oscillation for the autocatalator is

\[
X := \epsilon x, \quad Y := y, \quad T := \tau \frac{\epsilon^2}{\epsilon^2}.
\]

This yields the system

\[
\begin{align*}
\frac{dX}{dT} &= YX^2 + \epsilon^2Y - \epsilon X, \\
\frac{dY}{dT} &= -YX^2 + \xi \epsilon^2 - \epsilon^2Y.
\end{align*}
\]

The phase portrait of (41), after taking the limit \(\epsilon \to 0\), is shown in figure 5(b). For \(\epsilon = 0\), the two coordinate axes are equilibria of (41). It turns out that two consecutive blow-ups are needed in [10]. First, the \(Y\)-axis is blown-up to a cylinder and from this cylinder one can make a fast jump to the \(X\)-axis, say to a point \(p^*\); see [10, figure 8] or figure 5(c). However, then one encounters a non-hyperbolic point, denoted \(q\) in the notation of [10], when trying to return to the attracting part of the slow manifold.

The relaxation-type periodic orbit of the 2D autocatalator model jumps towards \(x \to +\infty\). As shown in figure 5(a) the periodic orbit jumps near a fold point of the critical manifold; see also figure 5(c). However, it has a travel distance and only turns around for \(x \sim O(\epsilon^{-1})\), which is what motivated the scaling (40). In the blown-up space of [10] this occurs near \(p^*\). Theorem 8.1 states that the slow manifold loses normal hyperbolicity when \(x \sim O(\epsilon^{-1/2})\). This scaling does allow the possibility of global crossings of the critical manifold when \(x \sim O(\epsilon^{-1})\). Furthermore, theorem 8.1 shows that we cannot have a normally hyperbolic manifold which facilitates the passage from a region near \(p^*\) back to a neighbourhood near the attracting part of \(C_0\). This comparison yields two important practical conclusions.

- The second blow-up at the point \(q\) in [10] cannot be avoided and corresponds to the scaling region, where the repelling part of the critical manifold \(C_0\) no longer perturbs to a normally hyperbolic repelling slow manifold at order \((x, y) \sim (O(\epsilon^{-1/2}), O(\epsilon^{1/2}))\).
- The travel distance in the fast direction of order \(x \sim O(1/\epsilon)\) after the jump at the fold point in the autocatalator is crucial. If we would move the fold closer to the \(x\)-axis, then a fast jump could potentially land in a region which is strictly smaller than \((x, y) \sim (O(\epsilon^{-1/2}), O(\epsilon^{1/2}))\). Hence, the oscillations in 2D autocatalator depend crucially on the location of the fold point relative to the asymptotics of \(C_0\) as \(x \to +\infty\).
Figure 5. (a) Numerical illustration for the 2D autocatalator with $\xi = 1.1$ and $\epsilon = 0.01$. The scaling regimes are also illustrated using vertical dashed lines. (b) Sketch of the phase portrait for the system (41) when $\epsilon = 0$. The flow is directed from the equilibria on the $Y$-axis to the equilibria on the $X$-axis. (c) Blow-up of the system from (b) where the $Y$-axis has been blown up to a cylinder. We only show a projection of this cylinder corresponding to a view with $\epsilon = 0$. One the cylinder, the fold point and the critical manifold appear (cf (a)). For details of the blow-up consider [10] from which the figures (b) and (c) have been adapted.

For the model (6) by Rankin et al the decay rate of the critical manifold is exponential and hence faster than any polynomial. In this context, theorem 8.1 implies that a compact submanifold of the critical manifold will loose normal hyperbolicity for arbitrarily large choices of $s \in \mathbb{N}$ in (39). Hence, normal hyperbolicity is lost very close to the origin, in comparison to the power-law decay case.

As an important conclusion from theorem 8.1 one may also understand the behaviour of large amplitude oscillations (LAOs) induced by a global return mechanism with the asymptotic dynamics given by (38) i.e. those LAOs which cross the critical manifold. For example, given a general autocatalytic reaction mechanism $Y + (s + 1)X \rightarrow (s + 2)X$ then the lower bound for the crossing is $O(\epsilon^{-1/(s+1)})$ for the fast component amplitude. Since $s$ is usually known and $\epsilon$ can be computed from the reaction rates the result has quite general applicability.

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References

[1] Berglund N and Kunz H 1999 Memory effects and scaling laws in slowly driven systems J. Phys. A: Math. Gen. 32 15–39
[2] Chicone C 2010 Ordinary Differential Equations with Applications (Texts in Applied Mathematics) 2nd edn (Berlin: Springer)
[3] Desroches M, Guckenheimer J, Kuehn C, Krauskopf B, Osinga H and Wechselberger M 2012 Mixed-mode oscillations with multiple time scales SIAM Rev. 54 211–88
[4] Dumortier F 1993 Techniques in the theory of local bifurcations: blow-up, normal forms, nilpotent bifurcations, singular perturbations Bifurcations and Periodic Orbits of Vector Fields ed D Schmidmiak (Dordrecht: Kluwer) pp 19–73
[5] Dumortier F and Roussarie R 1996 Canard Cycles and Center Manifolds Mem. Am. Math. Soc. 121
[6] Fenichel N 1971 Persistence and smoothness of invariant manifolds for flows Indiana Univ. Math. J. 21 193–225
[7] Fenichel N 1979 Geometric singular perturbation theory for ordinary differential equations J. Diff. Equ. 31 53–98
[8] Guckenheimer J and Holmes P 1983 Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields (New York: Springer)
[9] Guckenheimer J and Schep 2011 A geometric model for mixed-mode oscillations in a chemical system SIAM J. Appl. Dyn. Sys. 10 92–128
[10] Gedeon T 2009 Geometric singular perturbation analysis of an autocatalator model Discrete Continuous Dyn. Syst- S 2 783–806
[11] Hirsch M W, Pugh C C and Shub M 1977 Invariant Manifolds (Berlin: Springer)
[12] Hirsch M W, Smale S and Devaney R 2003 Differential Equations, Dynamical Systems, and an Introduction to Chaos 2nd edn (New York: Academic)
[13] Jones C K R T 1995 Geometric singular perturbation theory Dynamical Systems (Montecatini Terme, 1994) (Lecture Notes in Mathematics vol 1609) (Berlin: Springer) pp 44–118
[14] Kaper T J 1999 An introduction to geometric methods and dynamical systems theory for singular perturbation problems Analyzing Multiscale Phenomena Using Singular Perturbation Methods ed J Cronin and R E O’Malley (Berlin: Springer) pp 85–131
[15] Krupa M and Szmolyan P 2001 Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions SIAM J. Math. Anal. 33 286–314
[16] Krupa M and Szmolyan P 2001 Extending slow manifolds near transcritical and pitchfork singularities Nonlinearity 14 1473–91
[17] Krupa M and Szmolyan P 2001 Geometric analysis of the singularly perturbed fold Multiple-Time-Scale Dynamical Systems (CMA Volumes in Mathematics and its Applications vol 122) (New York: Springer) pp 89–116
[18] De Maesschalck P and Dumortier F 2008 Canard cycles in the presence of slow dynamics with singularities Proc. R. Soc. Edinburgh A 138 265–99
[19] De Maesschalck P and Dumortier F 2011 Bifurcations of multiple relaxation oscillations in polynomial Liénard equations Proc. Am. Math. Soc. 139 2073–85
[20] De Maesschalck P and Dumortier F 2011 Classical Liénard equations of degree $n \geq 6$ can have $\frac{n+1}{2} + 2$ limit cycles J. Diff. Equ. 250 2162–76
[21] Merkin J H, Needham D J and Scott S K 1986 Oscillatory chemical reactions in closed vessels Proc. R. Soc. Lond. A 406 299–323
[22] Milik A and Szmolyan P 2001 Multiple time scales and canards in a chemical oscillator Multiple Time Scale Dynamical Systems ed C K R T Jones vol 122 (Berlin: Springer) pp 117–40
[23] Milik A, Szmolyan P, Loeffelmann H and Groeller E 1998 Geometry of mixed-mode oscillations in the 3-d autocatalator Int. J. Bifurcation Chaos 8 505–19
[24] Petrov V, Scott S K and Showalter K 1992 Mixed-mode oscillations in chemical systems J. Chem. Phys. 97 6191–8
[25] Rankin J, Desroches M, Krauskopf B and Lowenberg M 2011 Canard cycles in aircraft ground dynamics Nonlinear Dyn. 66 681–8
[26] Rasmussen M 2007 Attractivity and Bifurcation for Nonautonomous Dynamical Systems (Berlin: Springer)
[27] Schecter S 1985 Persistent unstable equilibria and closed orbits of a singularly perturbed equation J. Diff. Equ. 60 131–41
[28] Szmolyan P and Wechselberger M 2001 Canards in $\mathbb{R}^3$ J. Diff. Equ. 177 419–53
[29] Wechselberger M 2002 Extending Melnikov-theory to invariant manifolds on non-compact domains Dyn. Syst.: Int. J. 17 215–33
[30] Wechselberger M 2012 A propos de canards (apropos canards) Trans. Am. Math. Soc. 364 3289–309
[31] Wiggins S 1994 Normally Hyperbolic Invariant Manifolds in Dynamical Systems (Berlin: Springer)