Pion propagation in real time field theory at finite temperature

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We describe how the thermal counterpart of a vacuum two-point function may be obtained in the real time formalism in a simple way by using directly the 2 × 2 matrices that different elements acquire in this formalism. Using this procedure we calculate the analytic (single component) thermal amplitude for the pion pole term in the ensemble average of two axial-vector currents to two loops in chiral perturbation theory. The general expressions obtained for the effective mass and decay constants of the pion are evaluated in the chiral and the nonrelativistic limits. We also investigate the effect of massive states on these effective parameters.

I. INTRODUCTION

The real time thermal field theory is apparently complicated by the fact that all two-point functions in this formalism assume the form of 2 × 2 matrices [1]. These matrices, however, have simple structures: If we factor out certain matrices depending on the distribution function only, these become diagonal, each with essentially a single independent element with proper analytic properties. But in actual computations one tends to ignore the matrix structure, starting instead with the so-called physical 11-element, encountering though summation over indices at all interaction vertices in a Feynman graph. Such a procedure encounters ill-defined products of components of the matrix propagator, that must be combined together to get a well-defined quantity. Further the 11-component does not have a simple analytic structure.

In this work we show that it is both simple and elegant to work with the matrix amplitudes. All one has to do is to write out the usual vacuum amplitude. The thermal matrix amplitude is then obtained by replacing its elements like the propagator, the self-energy and the vertices by the corresponding matrices. Factorizing these matrices as mentioned above, we immediately get the analytic amplitude representing the dynamics of the system in the heat bath.

Here we apply this procedure to calculate the pion pole term in the two-point function of the axial-vector currents to two loops in chiral perturbation theory [2]. This problem was studied earlier by several authors [3–5], in particular, by Toublan [6]. After obtaining the analytic amplitude, we follow him to find the pion pole position and the residue. We then find these pole parameters in the chiral limit, in agreement with his results. We also evaluate them in the nonrelativistic region and consider the effect of massive states on them.

In Sec. II we write down the effective chiral Lagrangian to fourth order, needed to obtain all the required vertices. In the next Sec. III, we obtain the vacuum amplitude from all the Feynman graphs up to two loops contributing to the pion pole term of the two-point function. The corresponding thermal amplitude is obtained in Sec.IV, from which we derive the effective parameters, namely the pion mass and the decay constants at finite temperature. These expressions are evaluated analytically in Sec. V in the high and low temperature limits. We examine the contribution of the massive states in Sec. VI. Finally we bring out the main features of our work in Sec. VII.

Appendix A constitutes an essential part of this work. Reviewing briefly the real time thermal field theory, we discuss here at length how the vacuum amplitude for an individual Feynman graph may be converted into its thermal counterpart. In Appendix B we write the integrals appearing in the nonfactorizable amplitudes. In the last Appendix C we collect the results for the relevant integrals in the high and the low temperature region.

II. CHIRAL PERTURBATION THEORY

We consider the QCD Lagrangian for the doublet of light quarks, u and d. In the absence of their masses, it has chiral symmetry, being invariant under $SU(2)_R \times SU(2)_L$. This symmetry is supposed to be broken spontaneously to $SU(2)_V$ of ordinary isospin, generating the massless pions as the Goldstone bosons.

In the physical case of non-zero quark masses, chiral symmetry is also broken explicitly to the same isospin subgroup, if we neglect the mass difference of u and d quarks. The pions become the pseudo-Goldstone bosons acquiring mass
$M$, given by $M^2 = 2m_q B$ to lowest order, where $B$ is related to the quark condensate in vacuum, whose dynamical generation leads to the spontaneous symmetry breaking.

As already stated, we are interested in the pion pole term in the two point function of the axial-vector currents,

$$A^a_\mu = \overline{\tau}_i \gamma_\mu \gamma_5 \frac{\tau^a}{2} q \quad a = 1, 2, 3,$$

evaluated in chiral perturbation theory, the effective theory of QCD at low energy. Such functions are best calculated in the external field method, in which one introduces in the original QCD Lagrangian an external field $a^e_\mu(x)$ coupled to $A^a_\mu(x)$ as well as a field $v^a_\mu(x)$ for the vector currents [2]. The global chiral symmetry is then promoted to a local one, with appropriate transformation properties of the external fields.

In the effective theory, the pion fields may be collected in the form of an unitary matrix,

$$\langle U(x) = e^{i\phi^a(x)\tau^a/F},$$

where $\tau^a$ are the Pauli matrices. The constant $F$ may be identified as the pion decay constant in the chiral limit. The local symmetry requires us to replace the ordinary derivative by the covariant one,

$$D_\mu U = \partial_\mu U - i\{a_\mu, U\}, \quad (2.1)$$

where, for our purpose, we retain only the external field $a_\mu(x)$. The two-point function of $A_\mu(x)$ is now obtained as the coefficient of the quadratic term in $a_\mu(x)$ in the perturbative evaluation of the generating functional with the effective Lagrangian.

As a non-renormalizable theory, the effective Lagrangian consists of a series of terms with an increasing number of derivatives and/or quark mass factors,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \cdots.$$ 

The leading term is given by

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \{ \langle D_\mu U^\dagger D^\mu U \rangle + M^2 \langle U + U^\dagger \rangle \}, \quad (2.2)$$

where $\langle A \rangle$ denotes the trace of the $2\times2$ matrix $A$. Here the first term is invariant under the chiral transformations. The second term represents explicit symmetry breaking due to the quark mass term in the QCD Lagrangian.

The next, non-leading piece in $\mathcal{L}_{\text{eff}}$ is [2,7]

$$\mathcal{L}^{(4)} = \frac{1}{4} l_1 \langle D_\mu U^\dagger D^\mu U \rangle^2 + \frac{1}{4} l_2 \langle D_\mu U^\dagger D^\mu U \rangle \langle D_\rho U^\dagger D^\rho U \rangle + \frac{1}{8} l_4 M^2 \langle D_\mu U^\dagger D^\mu U \rangle \langle U + U^\dagger \rangle + \frac{1}{16} (l_3 + l_4) M^4 \langle U + U^\dagger \rangle^2. \quad (2.3)$$

It provides counterterms necessary to renormalize the one-loop graphs with vertices from $\mathcal{L}^{(2)}$. Thus the bare coupling constants $l_1, \ldots, l_4$ contain a pole at $d = 4$ in dimensional regularization. The coefficients of these poles may be determined by evaluating all the one-loop graphs. Alternatively, these may be obtained directly by calculating the short distance behaviour of the generating functional to one loop [2]. Adopting the notation introduced in this Ref., the renormalized coupling constants $\tilde{l}_1, \ldots, \tilde{l}_4$ are defined by,

$$l_i = \gamma_i \left( \lambda + \frac{1}{32\pi^2} \gamma_i \right), \quad (2.4)$$

with

$$\gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2.$$ 

The pole is contained in $\lambda$,

$$\lambda = \frac{M^{d-4}}{(4\pi)^2} \left( \frac{1}{d-4} - \frac{1}{2} \left[ \ln 4\pi + \Gamma'(1) + 1 \right] + O(d-4) \right). \quad (2.5)$$

Up to the factor $\gamma_i/32\pi^2$, the constants $\tilde{l}_i$ are running coupling constants at the scale $M$. The $M$-dependence of $\tilde{l}_i$ can be made explicit by relating these to $l_i^\prime$, the renormalized coupling constants at any other scale $\mu$,

$$l_i^\prime = \frac{\gamma_i}{32\pi^2} \left( \tilde{l}_i + \ln \frac{M^2}{\mu^2} \right). \quad (2.6)$$

In Sec.V we shall use this equation to show the finiteness of the pion pole parameters in the chiral limit.

The renormalization of the two-loop graphs would require vertices from the next higher piece, $\mathcal{L}^{(6)}$ in $\mathcal{L}_{\text{eff}}$. But we do not need it, as we are not interested in the vacuum amplitudes, but only in their temperature dependent parts.
III. VACUUM AMPLITUDE

Here we obtain the pion pole contribution to the vacuum two-point function of the axial-vector current,

$$\delta^{ab} T_{\mu\nu}(q) = i \int d^4 x e^{iqx} \langle 0 | T A^a_{\mu}(x) A^b_{\nu}(0) | 0 \rangle \pi \text{pole},$$

(3.1)

by calculating all the Feynman graphs up to two loops with the interaction vertices given by $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$. These graphs are conveniently divided into four groups, shown in Figs. 1-4.

Let us first derive the familiar one loop results. The vertex correction graphs (b) and (c) of Fig. 1 modify the residue of the free amplitude of graph (a) to give

$$T^{[1a,b,c]}_{\mu\nu}(q) = q_\mu q_\nu F^2 \{ 1 + 4 \eta (3l_4 - 2J)/3 \} i\Delta(q).$$

(3.2)

where $\eta = \frac{M^2}{F^2}$ is an expansion parameter and $\Delta(q)$ is the free pion propagator, $\Delta(q) = i/(q^2 - M^2 + i\epsilon)$. $J$ is a divergent one-loop integral,

$$J(M) = \frac{1}{M^2} \int \frac{d^4k}{(2\pi)^4} \Delta(k) \equiv 2\lambda,$$

(3.3)
with \( \lambda \) given by Eq.(2.5). To include the self-energy graphs, it is convenient to introduce here the well-known Dyson-Schwinger equation for the complete propagator \( \Delta'(q) \),
\[
\Delta'(q) = \Delta(q) + \Delta(q)(-i\Sigma(q))\Delta'(q)
\]
where the self-energy part \( \Sigma \) of graphs (d) and (e) of Fig. 1 is given by
\[
\Sigma(q) = -2\eta(3l_4 - J)(q^2 - M^2)/3 + F^2\eta^2(4l_3 + J)/2 .
\]
Eq.(3.4) may be solved by iteration,
\[
\Delta'(q) = \Delta(q) + \Delta(q)(-i\Sigma(q))\Delta(q) + \Delta(q)(-i\Sigma(q)\Delta(q))^2 + \cdots
\]
or in closed form,
\[
\Delta'(q) = \frac{\Delta(q)}{1 + i\Sigma(q)\Delta(q)} .
\]
The self-energy correction is now included in Eq.(3.2) by replacing \( \Delta(q) \) with \( \Delta'(q) \). Thus we get the one-loop result for the pion pole,
\[
T_{\mu\nu}(q) = -q_\mu q_\nu \frac{F_\pi^2}{q^2 - M_\pi^2 + i\epsilon},
\]
with
\[
M_\pi^2 = M^2 \{1 + 2\eta(l_4 + J/4)\} = M^2(1 - \eta\bar{l}_3/32\pi^2),
\]
\[
F_\pi = F \{1 + \eta(l_4 - J)\} = F(1 + \eta\bar{l}_4/16\pi^2),
\]
on using Eq.(2.4).

We now include the two-loop graphs. First consider those of Figs. 1, 2 and 3, that are actually products of two one-loop parts. The total contribution of all these graphs may be put in the form
\[
T^{(1+2+3)}_{\mu\nu}(q) = q_\mu q_\nu F^2 \sum_{n=1}^3 \{ \gamma_n i\Delta(q) + \sigma_n M^2 \Delta^2(q) \} + 8F^3\eta^2(l_1 + 2l_3)(q_\mu J_{\nu\lambda} + q_\nu J_{\mu\lambda})q^\lambda i\Delta(q) ,
\]
where we show separately the sums of contributions of all graphs in each of Figs.1, 2 and 3. Thus the sum of graphs in Fig.1 is given by the \( n = 1 \) term \(^1\). Note here that the two loop graphs from (f) to (l), being iterations of graphs from (b) to (e), are automatically included in Eq.(3.7). But we prefer to write them explicitly in Eq.(3.10), getting
\[
\gamma_1 = 1 + 2\eta(l_4 - J) - 4\eta^2(l_4 - J)(3l_4 - J)/3 ,
\]
\[
\sigma_1 = \eta(4l_3 + J)/2 - 2\eta^2 J(4l_3 + J)/3 ,
\]
where we cancel factors as, \( (q^2 - m^2)\Delta^2(q) = i\Delta(q) \), that can also be justified at finite temperatures. In this way we get directly the shifts in the residue and the pole position from graphs of Fig.1 as \( F^2(\gamma_1 - 1) \) and \( M^2\sigma_1/\gamma_1 \) respectively. Next, the graphs of Fig. 2 contain only the vertices of \( L^{(2)} \) and their sum is given by the \( n = 2 \) term, with
\[
\gamma_2 = \eta^2 J(8J + 3J')/3 , \quad \sigma_2 = -\eta^2 J(3J + 2J')/8
\]
where we encounter a new divergent integral related to the earlier one as,
\[
J'(M) = i \int \frac{d^4k}{(2\pi)^4} \Delta^2(k) = -\frac{\partial}{\partial M^2}(M^2 J) = -2\lambda - \frac{1}{16\pi^2} .
\]

\(^1\)A piece, namely \(-q_\mu q_\nu F^2(2\pi)\eta^2(J + 4l_3)^2M^4i\Delta^3(q)\), is omitted here, as it is automatically included when we put the \( n = 1 \) term in the form of a simple pole.
Lastly, the sum of graphs of Fig. 3 with vertices from $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ is given by the term $n = 3$ with
\[
\gamma_3 = \eta^2 \{ (36l_1 + 12l_2 - 25l_4)J + 12l_3 J' + 12l_5^2 \} / 3, \\
\sigma_3 = -\eta^2 \{ (36l_1 + 12l_2 + 16l_3 - 3l_4)J + 3l_3 J' + 24(l_1 + 2l_2)q^\lambda q^\sigma J_{\lambda\sigma}/M^2 \} / 3, 
\]
(3.14)

\[
\text{together with the remaining term in Eq.(3.10), where we have still another divergent integral,}
\]
\[
J_{\mu\nu}(M) = \frac{1}{M^4} \int \frac{d^4k}{(2\pi)^4} k_\mu k_\nu \Delta(k). 
\]
(3.15)

Actually this term is also proportional to $q_\mu q_\nu$, once the integral is evaluated. But in view of its extension to finite temperature, we keep it as such.

Finally we have the amplitude from the nonfactorizable two-loop graphs of Fig. 4,
\[
T_\mu^{(4)}(q) = -\frac{2i}{9F^2} \{ q_\mu \Delta(q) \Gamma_\mu(q) + \Gamma_\mu(q) \Delta(q) q_\nu \} + \frac{1}{18F^2} q_\mu q_\nu \Delta(q) \Sigma(q) \Delta(q),
\]
(3.16)
where the vertex function $\Gamma_\mu(q)$ of graph (a) is
\[
\Gamma_\mu(q) = i \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (2q_\mu - 3k_1\mu - 3k_2\mu) f \Delta(k_1) \Delta(k_2) \Delta(q - k_1 - k_2),
\]
(3.17)
and the self-energy function $\Sigma(q)$ of graph (b) is
\[
\Sigma(q) = -i \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} (3M^4 + 2f^2) \Delta(k_1) \Delta(k_2) \Delta(q - k_1 - k_2),
\]
(3.18)
with $f$ standing for the function,
\[
f(q, k_1, k_2) = k_1^2 + k_2^2 + 4k_1k_2 + M^2 + 2q \cdot (k_1 + k_2) - 2q^2.
\]
The order of the factors in Eq.(3.16) is in anticipation of their matrix structures of the thermal amplitudes in the next section.

In Ref. [6] the vertex and self-energy integrals have been cast in a particularly convenient form using the symmetries of the integrands under the interchange of the integration variables. Thus if one defines
\[
K(q) = \frac{i}{M^2} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \Delta(k_1) \Delta(k_2) \Delta(q - k_1 - k_2),
\]
(3.19)
\[
K_{\mu\nu}(q) = \frac{i}{M^2} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} k_{1\mu} k_{1\nu} \Delta(k_1) \Delta(k_2) \Delta(q - k_1 - k_2),
\]
(3.20)
they may be written as
\[
\Gamma_\mu(q) = -2F^4 \eta^2 [q_\mu (3J^2 - K) + 9K_{\mu\rho} q^\rho],
\]
(3.21)
\[
\Sigma(q) = -F^4 \eta^2 [4(9J^2 - 4K)(q^2 - M^2) + 3M^2(8J^2 - K) + 72q^\rho q^\sigma K_{\rho\sigma}] .
\]
(3.22)
Then Eq.(3.16) simplifies to
\[
T_\mu^{(4)} = q_\mu q_\nu F^2 \eta^2 \left\{ \frac{2}{3} J^2 i \Delta(q) - \frac{1}{6} (8J^2 - K + 24q^\rho q^\sigma K_{\rho\sigma}/M^2) M^2 \Delta^2(q) \right\} + 4F^2 \eta^2 (q_\mu K_{\nu\lambda} + q_\nu K_{\mu\lambda}) q^\lambda i \Delta(q).
\]
(3.23)

The sum of amplitudes (3.10) and (3.23), along with the one from the tree graphs with a single insertion of vertices from $\mathcal{L}^{(6)}$ (not calculated above) would give the complete, renormalized vacuum amplitude. One may then extend the one-loop results (3.8, 3.9) for the pion pole parameters to two-loops. Instead, however, we turn to the corresponding thermal amplitudes to find the temperature dependence of these parameters.
IV. THERMAL AMPLITUDE

The thermal (ensemble averaged) two-point function of the axial-vector current is a $2 \times 2$ matrix in the real time formalism, whose $ij$-th element is

$$
(T^{\beta}_{\mu\nu})_{ij} = i \int d^{4}x e^{i q \cdot x} \text{Tr}[\rho T_{c} A_{\mu}^{i}(\phi_{i}(x)) A_{\nu}^{j}(\phi_{j}(x))]/\text{Tr}\rho|_{\pi \text{ pole}}, \quad \rho = e^{-\beta H},
$$

(4.1)

where $T_{c}$ denotes time ordering with respect to the time contour of Fig.9 in Appendix A. There we discuss at length how to obtain the matrix amplitude for an individual graph from its vacuum amplitude. To summarize, all we need is to replace the loop integrals $(J, J', J_{\mu\nu})$ encountered in the vacuum amplitudes by $(J^{\beta}, J'^{\beta}, J_{\mu\nu}^{\beta})$, where, in effect, the vacuum pion propagator is replaced by the 11- or 22-component of the thermal propagator. Further, the elements of the vacuum theory, namely, $(\Delta, \Sigma, \Gamma)$ need be replaced by the matrices $(\Delta, \Sigma, \Gamma)$ and also $\Delta^{2}$ by $\Delta\tau\Delta$, where the matrices are given in Appendix A. Having obtained the matrix amplitude, we put in the factorized forms for all the matrices to get an equation among diagonal matrices, each of whose 11- and 22-components are identical up to complex conjugation and possibly a $(-)$ sign. Thus we leave behind the matrix structure and work with the (single component) analytic amplitude $T_{\mu\nu}^{\beta}(q)$. Almost repeating Eqs. (3.10) and (3.23) we write it as the sum of

$$
T_{\mu\nu}^{\beta} = q_{\mu} q_{\nu} F^{2} 
\sum_{n=1}^{3} \left\{ \gamma_{n}^{\beta} \Delta(q) + \sigma_{n}^{\beta} M^{2} \Delta^{2}(q) \right\} + 8 F^{2} \eta^{2} (1 + 2 \beta) (q_{\mu} J_{\nu\lambda}^{\beta} + q_{\nu} J_{\mu\lambda}^{\beta}) \Delta(q),
$$

(4.2)

where $\gamma_{n}^{\beta}$ and $\sigma_{n}^{\beta}$ are obtained from $\gamma$ and $\sigma$ of Eqs. (3.11), (3.12) and (3.14) after replacing $J$'s by $J^{\beta}$'s, and

$$
T_{\mu\nu}^{(4)} = q_{\mu} q_{\nu} F^{2} \eta^{2} \left[ \frac{2}{3} (J^{\beta})^{2} \Delta(q) - \frac{1}{6} \left( 8 (J^{\beta})^{2} - K^{\beta} + 24 q^{2} q^{2} K_{\rho\sigma}/M^{2} \right) M^{2} \Delta^{2}(q) \right] + 4 F^{2} \eta^{2} (q_{\mu} K_{\nu\lambda}^{\beta} + q_{\nu} K_{\mu\lambda}^{\beta}) \Delta(q).
$$

(4.3)

where, as in Eqs.(3.21) and (3.22) for the vacuum case, we have expressed $\Gamma$ and $\Sigma$ in terms of $K^{\beta}$, $K_{\mu\nu}^{\beta}$ (and $J^{\beta}$) with

$$
K^{\beta}(q) = \frac{i}{M^{2}} \int \frac{d^{4}k_{1}}{(2\pi)^{4}} \frac{d^{4}k_{2}}{(2\pi)^{4}} \Delta_{11}(k_{1}) \Delta_{11}(k_{2}) \Delta_{11}(q - k_{1} - k_{2}),
$$

(4.4)

and

$$
K_{\mu\nu}^{\beta}(q) = \frac{i}{M^{4}} \int \frac{d^{4}k_{1}}{(2\pi)^{4}} \frac{d^{4}k_{2}}{(2\pi)^{4}} \delta_{1\mu} \delta_{1\nu} \Delta_{11}(k_{1}) \Delta_{11}(k_{2}) \Delta_{11}(q - k_{1} - k_{2}).
$$

(4.5)

We must point out here that Eqs.(4.4) and (4.5) hold only for the real parts. (The imaginary parts of both sides differ by the factor $(1 + 2\beta(|q_{0}|)^{-1}$, as follows from Eqs.(A.27) and (A.31).) Since, however, we are interested only in the real parts of the pole parameters, our imprecise notation will lead to no error.

As already stated, the vacuum part of this thermal amplitude is of no interest to us here, beyond the one loop results given in Eqs.(3.8) and (3.9). In the $\beta$-dependent part, we have to isolate the finite terms from the divergent ones. To this end, we separate the $\Delta_{11}$ or $\Delta_{22}$ into its vacuum and thermal parts in the expressions for $J^{\beta}$'s and $K^{\beta}$'s. In the case of $J^{\beta}$'s we have simply,

$$
J^{\beta} = J + J', \quad J'^{\beta} = J' + J', \quad J_{\mu\nu}^{\beta} = J_{\mu\nu} + J_{\mu\nu}',
$$

where

$$
(J, J', J_{\mu\nu}') = \int \frac{d^{4}k}{(2\pi)^{4}} n(|k_{0}|) \left( \frac{1}{M^{2}} - \frac{\partial}{\partial M^{2}} \frac{k_{\mu} k_{\nu}}{M^{4}} \right) \delta(k^{2} - M^{2}).
$$

(4.6)

$K^{\beta}$ splits as

$$
K^{\beta}(q) = K(q) + K^{\beta}(q)|_{\text{div}} + \overline{K}' + \overline{K}(q).
$$

(4.7)

Here the second term gives the $\beta$-dependent divergent pieces, all proportional to $\lambda$. The finite, temperature dependent part can be expressed partly in terms of $J$'s defined above and the remainder as certain $q$-dependent integrals, constituting the third and the fourth term respectively. A similar decomposition holds for $K_{\mu\nu}^{\beta}$,

$$
K_{\mu\nu}^{\beta}(q) = K_{\mu\nu}(q) + K_{\mu\nu}^{\beta}(q)|_{\text{div}} + \overline{K}'_{\mu\nu} + \overline{K}_{\mu\nu}(q).
$$

(4.8)
All the pieces in Eqs.(4.7) and (4.8) are displayed in Appendix B.

It is now easy to see that all the $\beta$-dependent divergent pieces cancel out. We then get the complete, renormalized thermal amplitude as,

$$\mathcal{T}_{\mu\nu}(q) = \mathcal{T}_{\mu\nu}'(q) + \mathcal{T}_{\mu\nu}(q),$$

where $\mathcal{T}_{\mu\nu}'$ is the vacuum amplitude without the free pole term, which is put in the $\beta$-dependent piece $\mathcal{T}_{\mu\nu}$

$$\mathcal{T}_{\mu\nu}(q) = q_\mu q_\nu F^2 \left\{ (1 - 2\mathcal{J}\eta + An^2)i\Delta(q) + \left\{ \frac{1}{2} \mathcal{J}\eta - \left( B - \frac{\mathcal{K}(q)}{6} + \frac{g^4 q^6 S_{\lambda\sigma}(q)}{M^2} \right) \eta^2 \right\} M^2 \Delta^2(q) \right\},$$

$$+ F^2 \left\{ q_\mu S_{\nu\lambda}(q) + q_\nu S_{\mu\lambda}(q) \right\} q^\lambda \eta^2 i\Delta(q),$$

(4.10)

where the tensor $S_{\mu\nu}$ is given by

$$S_{\mu\nu}(q) = l\mathcal{J}_{\mu\nu} + 4\mathcal{K}_{\mu\nu}(q),$$

(4.11)

and the ($\beta$-dependent) constants $A$ and $B$ are built out of $\mathcal{J}$ and $\mathcal{J}'$,

$$A = \mathcal{J}(2\mathcal{J} + l') + \mathcal{J}' \left( \mathcal{J} - \frac{l_3}{16\pi^2} \right), \quad B = \mathcal{J}(\frac{10}{8} \mathcal{J} + l') + \frac{\mathcal{J}'}{4} \left( \mathcal{J} - \frac{l_3}{16\pi^2} \right).$$

(4.12)

The three combinations of coupling constants introduced above are,

$$l = \frac{1}{12\pi^2} \left( \frac{7_1 + 4l_2}{48\pi^2} \right), \quad l' = \frac{1}{48\pi^2} \left( 6l_1 + 4l_2 - 6l_4 - \frac{7}{3} \right), \quad l'' = \frac{1}{48\pi^2} \left( 6l_1 + 4l_2 - 6l_3 - \frac{55}{12} \right).$$

(4.13)

In Eq.(4.10) we have left out regular, non-pole pieces arising out of explicit factors of $q^2$. We remove further non-pole pieces by expanding the $q$-dependent functions in $q_0$ in the neighbourhood of the pole $q_0^2 = \tilde{q}^2 + M^2 = \omega^2$ at fixed $\tilde{q}$. Thus

$$\mathcal{K}(q) = \mathcal{K}^{(0)}(\omega) + (q_0^2 - \omega^2)\mathcal{K}^{(1)}(\omega) + \cdots, \quad \mathcal{K}^{(1)}(\omega) = \frac{1}{2\omega} \frac{\partial}{\partial q_0} \mathcal{K}(q)|_{q_0 = \omega},$$

(4.14)

and similarly for $S_{\mu\nu}(q)$. The resulting expression may be put in the form

$$\mathcal{T}_{\mu\nu}(q) = -\frac{f_{\mu}(q)f_{\nu}(q)}{q_0^2 - \Omega^2(\omega)},$$

(4.15)

where

$$\Omega^2(\omega) = \tilde{q}^2 + M^2 \left\{ 1 + \frac{1}{2} \mathcal{J}\eta - \left( B - \frac{\mathcal{K}^{(0)}}{6} + \frac{M^2 (q^4 S_{\lambda\sigma})^{(0)} - \mathcal{J}^2}{6} \right) \eta^2 \right\},$$

$$f_{\mu}(q) = F \left\{ q_\mu \left\{ 1 - \mathcal{J}\eta + \frac{1}{2} \left( A - \frac{M^2}{6} \mathcal{K}^{(1)} - (q^4 S_{\lambda\sigma})^{(1)} - \mathcal{J}^2 \right) \eta^2 \right\} + (S_{\lambda\sigma} q^\lambda)^{(0)} \eta^2 \right\}. \quad (4.16)$$

Note that $f_{\mu}(q)$ is not proportional to $q_\mu$ due to presence of the last term. This is due to the lack of Lorentz invariance in a medium, which serves as the preferred frame of reference. Thus, as in nonrelativistic systems [8], we have here two different $F$’s, the temporal and the spatial $[9]$.

$$f_0(q) = q_0 F^0(q), \quad f_i(q) = q_i F^i(q)$$

(4.17)

One may now find the thermal dispersion curve for the pion and its decay 'constants' at different values of $|\tilde{q}|$.

Instead, however, we set $\tilde{q} = 0$ and find the effective mass and the decay constants as a function of temperature. Converting the parameters $F$ and $M$ to the physical values by Eqs.(3.8) and (3.9) and using the result (B.9) of Appendix B, we finally get them as

$$M_\pi^2(T) = \frac{1}{2} \left\{ 1 + \frac{M_\pi^2}{2F_\pi^2} \mathcal{J} - \frac{M_\pi^4}{F_\pi^4} \left\{ l'' \mathcal{J} + \frac{11}{8} \mathcal{J}^2 + \frac{1}{4} \mathcal{J}\mathcal{J}' - \frac{\mathcal{K}}{6} + l\mathcal{J}_{00} + 4\mathcal{K}_{00} \right\} \right\},$$

(4.18)

$$F^i = F_\pi \left\{ 1 - \frac{M_\pi^2}{F_\pi^2} \mathcal{J} + \frac{M_\pi^4}{F_\pi^4} \left\{ (\mathcal{J} + l') \mathcal{J}' + \frac{M_\pi \partial K}{12} + l\mathcal{J}_{00} + 4\mathcal{K}_{00} - 2M_\pi \frac{\partial \mathcal{K}_{00}}{\partial q_0} \right\} \right\},$$

(4.19)

$$\left( F^i(T) - F^i(0) \right)/F_\pi = \frac{M_\pi^2}{3F_\pi^2} \left\{ -12M_\pi C - l\mathcal{J} + 4\mathcal{J}^2 - \mathcal{K} + l\mathcal{J}_{00} + 4\mathcal{K}_{00} \right\},$$

(4.20)
where \( l'' \) is given by
\[
l'' = \frac{1}{192\pi^2} \left( 24\vec{t}_1 + 16\vec{t}_2 - 27\vec{t}_3 - 24\vec{t}_4 - \frac{55}{3} \right),
\]
and \( C \) is the coefficient of the linear term in the expansion of \( \overline{K}_{0i}(q) \) around \( \vec{q} = 0 \),
\[
\overline{K}_{0i}(q) = C(q_0)q_i + \cdots.
\]
Note that all the quantities \( J, K \text{ etc.} \) are now functions of \( M_\pi \). These results agree with Ref. [6] except for the definition of \( l'' \) [10].

**V. EVALUATION**

We now need the values of the coupling constants, \( \vec{t}_i, i = 1, \cdots, 4 \). They were already determined in the original work [2], but all of them are not accurate enough [11]. The best values obtained so far follows from matching the dispersion theoretic phenomenological representation for \( \pi\pi \) scattering amplitude to its two-loop evaluation in chiral perturbation theory [12],

\[
\vec{t}_1 = -0.4 \pm 0.6, \quad \vec{t}_2 = 4.3 \pm 0.1, \quad \text{(two loop)}
\]

while the values relevant in the context of one-loop approximation are,

\[
\vec{t}_1 = -1.9 \pm 0.2, \quad \vec{t}_2 = 5.25 \pm 0.04. \quad \text{(one loop)}
\]

The difference in the two sets of values are attributed to the infrared singularities that can be better dealt with in the two-loop matching than in the case of one-loop. The original crude estimate of \( \vec{t}_3 \) [2],

\[
\vec{t}_3 = 2.9 \pm 2.4, \quad \text{(one loop)}
\]

has not been improved further. Finally the two-loop estimate of \( \vec{t}_4 \) [12],

\[
\vec{t}_4 = 4.4 \pm 0.2, \quad \text{(two loop)}
\]

does not differ much from the original one-loop estimate [2],

\[
\vec{t}_4 = 4.3 \pm 0.9, \quad \text{(one loop)}
\]

as the infrared singularities are weakly present here.

It should be noted here that in our two-loop calculation of the pion pole in the axial-vector Green’s function at finite temperature, it is actually the scattering amplitude in vacuum to one loop that enters its temperature dependent part. It is thus appropriate to use the one-loop estimate of the coupling constants in the present context.

We now evaluate the pion pole parameters in two regions of temperature. First consider the so-called high temperature limit, \( T \gg M_\pi \). To remain within the domain of the low temperature expansion, this limit is implemented not by letting \( T \) increase, but instead by holding \( T \) fixed and sending \( M_\pi \) to zero. The value of \( M_\pi \) is determined by the quark masses. Thus the high temperature limit is equivalent to the chiral limit of QCD theory.

The values of the relevant integrals in the chiral limit are given in the Appendix C. The contributing terms are only \( \eta^2 \mathbf{J}^2 \) and the combination,

\[
\eta^2 (l \mathbf{J}_{00} + 4 \overline{K}_{00}) = \frac{T^4}{36F_4} (Z(T) + c),
\]

where

\[
Z(T) = \ln \frac{M}{T} + \frac{1}{10}(\vec{t}_1 + 4\vec{t}_2),
\]

\[
c = -\frac{7}{15} - \ln 2 + 1 - I_1 + I_2 = 0.30,
\]

as obtained from (4.13) and (C.1-2). Here \( Z(T) \) has actually no logarithmic singularity in the chiral limit, as can be seen by shifting the renormalization scale of the coupling constants with Eq.(2.6) from \( M \) to any other value \( \mu \).
We thus get the results for the pion mass and decay constants to two loops at finite temperature in the chiral limit as

\[
\frac{M_\pi^2(T)}{M_\pi^2} = 1 + \frac{T^2}{24F^2} - \frac{T^4}{36F^4} \left( Z(T) + c + \frac{11}{32} \right)
\]

\[
= 1 + \frac{T^2}{4F^2} - \frac{T^4}{36F^4} \ln \frac{\Lambda_M}{T} \tag{5.2}
\]

\[
\frac{F_\pi(T)}{F_\pi} = 1 - \frac{T^2}{12F^2} + \frac{T^4}{72F^4} \left( Z(T) + c + \frac{1}{4} \right)
\]

\[
= 1 - \frac{T^2}{12F^2} + \frac{T^4}{72F^4} \ln \frac{\Lambda_F}{T} \tag{5.3}
\]

\[
\frac{F_\pi(T) - F_\pi^*(T)}{F_\pi} = \frac{T^4}{27F^4} \left( Z(T) + c + \frac{1}{4} \right)
\]

\[
= \frac{T^4}{27F^4} \ln \frac{\Lambda_\Delta}{T} \tag{5.4}
\]

where the logarithmic scales are

\[
\Lambda_M = 1.8 \text{ GeV}, \quad \Lambda_F = \Lambda_\Delta = 1.6 \text{ GeV},
\]

in agreement with Ref. [6]. Note that \( \Lambda_F \) is associated with \( F_\pi(T) \) and not its square, as in this Ref. (Had we chosen the two-loop coupling constants, we would get somewhat smaller values for the \( \Lambda \)'s, namely, \( \Lambda_M = 1.4 \text{ GeV}, \Lambda_F = \Lambda_\Delta = 1.3 \text{ GeV} \).)

Next consider the low temperature limit, \( T \ll M_\pi \). We shall express all quantities in terms of the dimensionless ratio

\[
\tau = \frac{T}{M_\pi}
\]

times possibly a power of temperature.

The leading behaviour of all the pole parameters in the low temperature region is given essentially by that of \( \bar{\mathcal{J}} \), as seen from Eqs.(C.4-6). Thus we get

\[
\frac{M_\pi^2(T)}{M_\pi^2} \bigg|_{\tau} = 1 + \left[ \frac{M_\pi^2}{2F_\pi^2} - \frac{5g_1 M_\pi^4}{24\pi^2 F_\pi^4} \right] \bar{\mathcal{J}} \bigg|_{\tau} \tag{5.5}
\]

\[
\frac{F_\pi(T)}{F_\pi} \bigg|_{\tau} = 1 - \left[ \frac{M_\pi^2}{F_\pi^2} - \frac{5g_2 M_\pi^4}{48\pi^2 F_\pi^4} \right] \bar{\mathcal{J}} \bigg|_{\tau} \tag{5.6}
\]

\[
\frac{F_\pi(T) - F_\pi^*(T)}{F_\pi} \bigg|_{\tau} = \frac{g_3 M_\pi^4}{12\pi^2 F_\pi^4} \bar{\mathcal{J}} \bigg|_{\tau} \tag{5.7}
\]

where

\[
g_1 = 7 + 2\bar{\mathcal{J}}_1 - 27\bar{\mathcal{J}}_2 + 3\bar{\mathcal{J}}_3 - \frac{9}{4} \bar{\mathcal{J}}_4
\]

\[
g_2 = 7 + 2\bar{\mathcal{J}}_1 - \frac{9}{4} \bar{\mathcal{J}}_2 + \frac{3}{10} \bar{\mathcal{J}}_3
\]

\[
g_3 = 7 + 4\bar{\mathcal{J}}_1 + \frac{4}{3} \bar{\mathcal{J}}_2
\]

and \( \bar{\mathcal{J}} \big|_{\tau} \) is given by Eq.(C.4).

It is interesting to note that the two-loop contributions to the effective parameters are always of opposite sign compared to that of one-loop, when it contributes. Figs. 5, 6 and 7 show the temperature dependence of these parameters. Besides the pion interactions giving these results, massive states are also expected to contribute. But as we show below, this contribution turns out to be negligibly small up to \( T \simeq 100 \text{ MeV} \).
FIG. 5. Thermal pion mass squared in chiral (continuous curve) and nonrelativistic (dashed one) limits.

FIG. 6. 'Temporal' type of thermal pion decay constant in the two limits as in Fig. 5.

FIG. 7. Difference of temporal and spatial types of pion decay constant in the two limits as in Figs. 5 and 6.

VI. MASSIVE STATES

So far we considered only the pionic interactions in obtaining the pion pole parameters. We now examine the leading contribution from the state(s) that are massive in the chiral limit. Here the $\rho$ meson appears to be the most important such state contributing through the graphs of Fig.8.

FIG. 8. Graphs with $\rho$ propagator contributing to the pion pole.
The additional pieces of chiral Lagrangian needed for evaluation of the graphs may be obtained from Ref. [13,14]. Their construction follows from noting that the non-Goldstone fields transform only under the unbroken isospin subgroup \( SU(2)_V \), with its group parameters dependent on the Goldstone fields. Accordingly we have to replace the ordinary derivative on the \( \rho \) meson field \( \rho_\mu(x) \) by the covariant one,

\[
\nabla_\mu \rho_\nu = \partial_\mu \rho_\nu + [\Gamma_\mu, \rho_\nu].
\]

Here the connection \( \Gamma_\mu \) is given by

\[
\Gamma_\mu = \frac{1}{2} u^\dagger (\partial_\mu - ia_\mu) u + \frac{1}{2} u (\partial_\mu + ia_\mu) u^\dagger, \quad u^2 = U,
\]

where, as in Eq.(2.1), we have set the external vector field to zero. One need also construct appropriate variables from \( D_\mu U \) and \( F^{\mu\nu}_{R,L} \), the right- and the left-handed field strengths built out of the external vector and the axial-vector potentials, that transform only under \( SU(2)_V \), namely

\[
\rho_\mu = i u^\dagger D_\mu U u^\dagger = u_\mu^\dagger
\]

and

\[
f^{\mu\nu}_\pm = \pm u^\dagger F^{\mu\nu}_\pm u + u F^{\mu\nu}_\pm u^\dagger.
\]

In terms of the above variables, one can write the required pieces of the chirally invariant Lagrangian as [13,14],

\[
\mathcal{L}_{\text{int}}(\rho) = \frac{1}{2\sqrt{2}m_\rho} (F_\rho (\rho_{\mu\nu} f^{\mu\nu}_R + i G_\rho [\rho_{\mu\nu} [u^\mu, u^\nu]]), \tag{6.1}
\]

where \( \rho_{\mu\nu} = \nabla_\mu \rho_\nu - \nabla_\nu \rho_\mu \). From the decay rates \( \Gamma(\rho^0 \to e^+ e^-) \) and \( \Gamma(\rho \to 2\pi) \), one gets respectively \( F_\rho = 154 \text{ MeV} \) and \( G_\rho = 69 \text{ MeV} \) [13].

It is now simple to calculate the vacuum amplitudes of graphs in Fig. 8, which may then be converted to their thermal counterparts, following discussions in Appendix A. Now the \( \rho \) particles are rather scarce in the medium for temperatures up to about 100 MeV [7]. Accordingly we drop the \( T \)-dependent term in the \( \rho \) propagator. Further, as the momenta involved are small compared to \( m_\rho \), we keep only the leading term in the expansion of the vacuum \( \rho \) propagator in inverse powers of \( m_\rho^2 \). The amplitudes thus simplified can be expressed through \( J^3 \) and \( J^3_{\mu\nu} \) only.

The self-energy graph contributes a \( T \)-dependent amplitude,

\[
F^2 q_\mu q_\nu 24 \frac{G_\rho^2 M_\rho^2}{m_\rho^4} \eta^2 (J - \tau_0) \left( i \Delta(q) + \frac{2}{3} \Delta^2(q) \right), \tag{6.2}
\]

Clearly it vanishes in the chiral (high \( T \)) limit. Also its leading contribution vanishes in the region where pions move nonrelativistically (low \( T \) region). In the same way, the vertex graph contributes a \( T \)-dependent part,

\[
F^2 q_\mu G_\rho M_\rho^2 \eta^2 \{ (-2J_{00} q_\mu q_\nu + q_\mu J_{\nu\lambda} + J_{\mu\nu} J_{\lambda\tau}) q^\lambda \} i \Delta(q), \tag{6.3}
\]

which again vanishes in the chiral limit, but is finite in the low temperature region. Surprisingly, it contributes only to \( F_\pi^\ast(T) \),

\[
\frac{F_\pi^\ast(T)}{F_\pi} \Big|_{(\rho)T} = -8 \frac{M_\rho^4}{F_\pi^4} \frac{F_\rho G_\rho M_\rho^2}{m_\rho^4} J_\tau, \tag{6.4}
\]

This contribution, however, is only about 3% of the pionic contribution in Eq.(5.7).

VII. DISCUSSION

Though the propagators, vertices and self-energies assume the form of \( 2 \times 2 \) matrices in the real time field theory at finite temperature, each of them is essentially given by a single analytic function, as can be seen from an appropriate factorization of these matrices. The same is true of the ensemble average of (the \( T \)-product of) any two operators. Here we show that one can find its thermal \( 2 \times 2 \) matrix amplitude directly from the vacuum amplitude and take advantage
of this factorization to get the analytic (single component) thermal amplitude. Thus compared to the commonly followed practice of considering the 11-element of the thermal matrix, the use of matrices not only simplifies and frees the calculation from ill-defined quantities at intermediate steps, but also yields directly the amplitude with proper analytic properties, not possessed by the 11-element.

In this work we use this matrix method to calculate the thermal pion pole term in the axial-vector two-point function in the framework of chiral perturbation theory. From the analytic amplitude we derive the effective mass and the decay constants of the pion at finite temperature. These are evaluated in two limits, the chiral and the nonrelativistic. The two evaluations agree rather closely up to about $T \sim 100$ MeV. We also examine the contribution of the massive states ($\rho$) to the effective parameters and find it to be negligible.

Finally we compare our work with that of Toublan [6], whom we follow at a number of points. He obtains the thermal amplitude in a somewhat intuitive manner, while we formulate rules to write the matrix amplitude, which leads immediately to the analytic thermal amplitude. These rules, in effect, justify his way of writing the thermal amplitude, as far as its real part is concerned. Our results (4.18-20) for the effective mass and decay constants of the pion agree completely with his, as the term with this coefficient does not contribute in this limit. We also find the effective parameters in the nonrelativistic limit, in which this term does contribute. In his discussion of the effect of massive states on the pion parameters, he considers only the real states in the heat bath and concludes it to be negligible up to $T \sim 100$ MeV. We also consider the effect of the virtual massive states in vacuum and find it too to be negligible.

**APPENDIX A**

The feature of the real time thermal field theory that distinguishes it from the vacuum theory is the time contour in their generating functionals. While it is the infinite real line for the vacuum theory, it must be augmented with a return path for the thermal theory. One example of such a path that we shall use is shown in Fig. 5. The return path may be folded onto the onward path, generating fields with a displaced time argument. Thus compared with a return path for the thermal theory. One example of such a path that we shall use is shown in Fig. 5. The return in their generating functionals. While it is the infinite real line for the vacuum theory, it must be augmented with parameters in the nonrelativistic limit, in which this term does contribute. In his discussion of the effect of massive states on the pion theory, he agrees completely with his, as the term with this coefficient does not contribute in this limit. We also find the effective parameters in the nonrelativistic limit, in which this term does contribute. In his discussion of the effect of massive states on the pion parameters, he considers only the real states in the heat bath and concludes it to be negligible up to $T \sim 100$ MeV. We also consider the effect of the virtual massive states in vacuum and find it too to be negligible.

![Fig. 9. The complex time contour of real time thermal field theory](image)

The two sets of fields make any thermal two-point function a $2 \times 2$ matrix. In particular, the free pion propagator is

$$\Delta^{ab}(x-y) = \delta^{ab} \Delta(x-y) = \left( \begin{array}{cc} \langle T \phi_1^a(x) \phi_1^b(y) \rangle & \langle \phi_2^b(y) \phi_1^a(x) \rangle \\ \langle \phi_2^a(x) \phi_1^b(y) \rangle & \langle T \phi_2^a(x) \phi_2^b(y) \rangle \end{array} \right), \quad (A.3)$$

APPENDIX A
where $\tilde{T}$ denotes anti-time ordering, can be evaluated directly in momentum space as,

$$
\Delta(q) = \left( \begin{array}{cc}
\Delta(q) + 2\pi n(q_0)\delta(q^2 - M^2) & 2\pi n(q_0)\delta(q^2 - M^2)e^{i\beta(q_0)/2} \\
2\pi n(q_0)\delta(q^2 - M^2)e^{-i\beta(q_0)/2} & \Delta^*(q) + 2\pi n(q_0)\delta(q^2 - M^2)
\end{array} \right)
$$

(A.4)

where $n(q_0) = (e^{i\beta(q_0)} - 1)^{-1}$ is the pion distribution function and $\Delta(q) = i/(q^2 - M^2 + i\epsilon)$ is the pion propagator in vacuum. (A boldface letter will always indicate a $2 \times 2$ matrix.)

Given the Feynman amplitude for any two-point function in vacuum, it is simple to write the corresponding thermal matrix amplitude. The correspondence may be found by comparing the Wick contractions for graphs in the two cases, with particular attention to the $-$ sign before the ‘ghost’ Lagrangian. As an example, consider the graph (d) of Fig. 1, for which we show this correspondence in detail. Its vacuum amplitude in coordinate space is obtained from

$$
-F^2 i \int d^4 z \langle 0 | T \partial_\mu \phi^a(x) \partial_\nu \phi^b(y) L_{int}(\phi(z)) | 0 \rangle,
$$

(A.5)

where $L_{int}$ is a piece in $L^{(2)}$,

$$
L_{int}(\phi) = -\frac{1}{4F^2} \left\{ \phi^\dagger \cdot \partial_\mu \phi \cdot \partial^\nu \phi^\dagger - \phi \cdot \partial_\mu \phi \cdot \partial^\nu \phi - \frac{M^2}{4} (\phi^\dagger \cdot \phi)^2 \right\},
$$

(A.6)

the field $\phi$ denoting the pion iso-vector triplet $(\phi^1, \phi^2, \phi^3)$. It gives the amplitude in momentum space,

$$
T_{\mu\nu}^{(d)}(q) = q_\mu q_\nu J(M) M^2 \left\{ \frac{2}{3} i \Delta(q) + \frac{1}{2} M^2 \Delta^2(q) \right\},
$$

(A.7)

where $J(M)$ is defined by Eq.(3.3). The single propagator in this expression arises from the cancellation,

$$
(q^2 - M^2)\Delta^2(q) = i\Delta(q).
$$

(A.8)

We now identify the contractions in (A.5) that produce this result. To focus on the contractions, we omit the derivatives and isospin indices on the pion fields and write schematically a term of the matrix element (A.5) as

$$
\langle 0 | T \phi(x) \phi(y) \phi^4(z) | 0 \rangle
\sim J(M) \Delta(x - z) \Delta(z - y)
\sim J(M) \Delta^2(q),
$$

(A.9)

in momentum space. To get the corresponding thermal matrix amplitude, consider its $ij$-th element,

$$
-F^2 i \int d^4 z \langle T \partial_\mu \phi_i^\dagger(x) \partial_\nu \phi_j(y) \{ L_{int}(\phi_i(z)) - L_{int}(\phi_j(z)) \} \rangle.
$$

(A.10)

Again we write schematically a term of this matrix element and contract its fields as,

$$
\langle T \phi_i(x) \phi_j(y) \{ \phi_i^4(z) - \phi_j^4(z) \} \rangle
\sim J^\beta(M) \langle T \phi_i(x) \phi_j(y) \{ \phi_i^2(z) - \phi_j^2(z) \} \rangle
\sim J^\beta(M) \{ \Delta_{11}(x - z) \Delta_{1j}(z - y) - \Delta_{12}(x - z) \Delta_{2j}(z - y) \}
\sim J^\beta(M) \{ \Delta(q) \tau \Delta(q) \}_{ij},
$$

(A.11)

in momentum space, where we use in the second line the fact that the contractions of two $\phi_1$'s and two $\phi_2$'s at the same point yield the same result,

$$
J^\beta(M) = \frac{1}{M^2} \int \frac{d^4 k}{(2\pi)^4} \Delta(q)_{11,\text{or}22}
$$

(A.12)

and the matrix $\tau$ is

$$
\tau = \left( \begin{array}{cc}
1 & 0 \\
0 & -1
\end{array} \right).
$$

(A.13)
Note that a cancellation similar to Eq.(A.8) for the vacuum case works also here,

\[ (q^2 - M^2) \Delta(q) \tau \Delta(q) = i \Delta(q). \] (A.14)

Comparing the contractions (A.9) and (A.11), we see that the thermal matrix amplitude of graph(1d) can be obtained from the vacuum amplitude (A.7) simply by replacing \( J \) with \( J^\beta \), \( \Delta \) with \( \Delta \) and \( \Delta^2 \) by \( \Delta \tau \Delta \) in it.

A little reflection on other graphs of Figs. 1-3 will convince us that the thermal amplitudes of all these graphs may be obtained from their vacuum amplitudes by the replacements just stated above, together with \( J' \) and \( J_{\mu \nu} \) by \( J'^\beta \) and \( J'^\beta_{\mu \nu} \) respectively, where

\[ J'^\beta(M) = i \int \frac{d^4k}{(2\pi)^4} (\Delta(k) \tau \Delta(k))_{11 \text{ or } 22} \]
\[ = -\frac{\partial}{\partial M^2}(M^2 J^\beta), \] (A.15)

and

\[ J'^\beta_{\mu \nu}(M) = \frac{1}{M^4} \int \frac{d^4k}{(2\pi)^4} k_{\mu} k_{\nu} \Delta(k)_{11 \text{ or } 22} \] (A.16)

In Eq.(A.15) we use the so-called mass-derivative formula for the matrix propagator [15],

\[ (\Delta \tau)^2 = i \frac{\partial}{\partial M^2}(\Delta \tau), \] (A.17)

which is the thermal extension of the trivial relation \( \Delta^2(q) = i\partial \Delta / \partial M^2 \) for the vacuum propagator.

So long we constructed thermal amplitudes with the matrix propagator only, inserting \( \tau \) explicitly to account for the \((-\) sign from the 'ghost' Lagrangian. We now introduce two kinds of parts of graphs namely, the self-energy and the (two-point) vertices, where it is convenient to include the effect of the associated \((-\) sign\(s\) in their definitions. Thus writing the \( ij\)-th component of the matrix amplitude of Fig.4b, again schematically and contracting the fields, we get

\[ \langle T \phi_i(x) \{ \phi_i^1(u) - \phi_i^2(u) \} \{ \phi_j^1(v) - \phi_j^2(v) \} \phi_j(y) \rangle \]
\[ \sim (\Delta(x - u) \Sigma(u - v) \Delta(v - y))_{ij}, \] (A.18)

absorbing the \((-\) signs in the definition of \( \Sigma \),

\[ \Sigma = \begin{pmatrix} s_{11} & -s_{12} \\ -s_{21} & s_{22} \end{pmatrix}, \quad s_{ij} = (\Delta_{ij})^3 \] (A.19)

Likewise, for the graph of Fig. 4a we write

\[ \langle T \phi_i^2(x) \{ \phi_i^1(u) - \phi_i^2(u) \} \phi_j(y) \rangle \sim (\Gamma^{(1)} \Delta)_{ij}, \quad \Gamma^{(1)} = \begin{pmatrix} s_{11} & -s_{12} \\ s_{21} & -s_{22} \end{pmatrix}, \] (A.20)

and

\[ \langle T \phi_i(x) \{ \phi_i^1(u) - \phi_i^2(u) \} \phi_j^3(y) \rangle \sim (\Delta \Gamma^{(2)})_{ij}, \quad \Gamma^{(2)} = \begin{pmatrix} s_{11} & s_{12} \\ -s_{21} & -s_{22} \end{pmatrix}. \] (A.21)

The matrix amplitudes are greatly simplified by factoring out matrices involving only the pion distribution function. Thus the free propagator given by Eq.(A.4) can be factored as

\[ \Delta(q) = U(q) \begin{pmatrix} \Delta(q) & 0 \\ 0 & \Delta'(q) \end{pmatrix} U(q), \quad U(q) = \begin{pmatrix} \sqrt{1 + n} & \sqrt{n} \\ \sqrt{n} & \sqrt{1 + n} \end{pmatrix}. \] (A.22)

Also the full propagator \( \Delta' \) and the two-point function \( T_{\mu \nu} \) admit similar factorizations,

\[ \Delta'(q) = U(q) \begin{pmatrix} \Delta'(q) & 0 \\ 0 & \Delta'(q) \end{pmatrix} U(q), \quad T_{\mu \nu}(q) = U(q) \begin{pmatrix} T_{\mu \nu}(q) & 0 \\ 0 & T_{\mu \nu}(q) \end{pmatrix} U(q), \] (A.23)
as is suggested by evaluation of our graphs. More rigorously, these follow from their spectral representations.

To derive a similar factorization of the self-energy part $\Sigma$, we look at the Dyson-Schwinger equation for the full propagator,

$$\Delta' = \Delta + \Delta(-i\Sigma)\Delta'.$$

(A.24)

Inserting the factorizations for $\Delta'$ and $\Delta$, we infer that $\Sigma$ must have the factorized form

$$\Sigma(q) = U^{-1} \begin{pmatrix} \Sigma^\beta(q) & 0 \\ 0 & -\Sigma^{\beta*}(q) \end{pmatrix} U^{-1}.$$  

(A.25)

It immediately follows that

$$\Sigma_{22} = -\Sigma_{11}, \quad \Sigma_{21} = \Sigma_{12}.$$  

(A.26)

Further we can get the function $\Sigma^\beta$ entirely from $\Sigma_{11}$,

$$Re \Sigma^\beta = Re \Sigma_{11}, \quad Im \Sigma^\beta = \frac{1}{1 + 2n} Im \Sigma_{11}.$$  

(A.27)

In the same way the relations,

$$T \sim -i \Gamma^{(1)} \Delta \sim \Delta(-i \Gamma^{(2)})$$

(A.28)

give us the factorizations,

$$\Gamma^{(1)} = U \begin{pmatrix} \Gamma^\beta & 0 \\ 0 & \Gamma^{\beta*} \end{pmatrix} U^{-1}, \quad \Gamma^{(2)} = U^{-1} \begin{pmatrix} \Gamma^\beta & 0 \\ 0 & \Gamma^{\beta*} \end{pmatrix} U.$$  

(A.29)

We see that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ differ by a $-$ sign in the off-diagonal elements, which is of no consequence to us. We thus omit the superscripts to write the relations given by Eq. (A.29) as,

$$\Gamma_{22} = \Gamma_{11}^*, \quad \Gamma_{21} = \Gamma_{12}.$$  

(A.30)

Again the 11 element determines the function $\Gamma^\beta$ completely,

$$Re \Gamma^\beta = Re \Gamma_{11}, \quad Im \Gamma^\beta = \frac{1}{1 + 2n} Im \Gamma_{11}.$$  

(A.31)

**APPENDIX B**

The $\beta$-dependent, divergent and finite parts of $K^\beta(q)$ and $K_{\mu\nu}^\beta$ have been obtained in Ref. [6], which we reproduce here for completeness. The divergent parts reside only in terms linear in the distribution function, where we need the vacuum integrals,

$$L(p) = i \int \frac{d^4k}{(2\pi)^4} \Delta(k)\Delta(p-k) = -2\lambda + \frac{1}{16\pi^2} + R(p),$$

and

$$L_{\mu\nu}(p) = \frac{i}{M^2} \int \frac{d^4k}{(2\pi)^4} k_{\mu}k_{\nu}\Delta(k)\Delta(p-k)$$

$$= \frac{\lambda}{6M^2} \{p^2 - 6M^2\}g_{\mu\nu} - 4p_\mu p_\nu$$

$$- \frac{1}{2(24\pi M^2)^2} \{20p_\mu p_\nu - 2p^2 g_{\mu\nu} + (4p_\mu p_\nu - (p^2 - 4M^2)g_{\mu\nu} - 4M^2 p_\mu p_\nu/p^2) R(p)\}$$

(B.1)

with

$$R(p) = -\frac{1}{16\pi^2} \int_0^1 dx \ln(1 - p^2x(1-x)/M^2)$$

(B.2)
Then the terms linear in \(n\) are given by

\[
K^{\beta}(q)_{n} = \frac{3}{M^2} \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - M^2)n(|k_0|)L(q - k),
\]

\[
= -6\lambda\mathcal{J} - \frac{3}{16\pi^2} \mathcal{J} + \frac{3}{M^2} \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - M^2)n(|k_0|)R(q - k)
\] (B.3)

and

\[
K^{\beta}_{\mu\nu}(q)_{n} = \frac{1}{M^2} \int \frac{d^4k}{(2\pi)^3} k_\mu k_\nu \delta(k^2 - M^2)n(|k_0|)L(q - k) + \frac{2}{M^2} \int \frac{d^4k}{(2\pi)^3} k_\mu k_\nu \delta(k^2 - M^2)n(|k_0|)L_{\mu\nu}(q - k)
\]

\[
= -\frac{\lambda}{3} \{10\mathcal{J}_{\mu\nu} + \{5g_{\mu\nu} + (4q_\mu q_\nu - q^2 g_{\mu\nu})/M^2\}\mathcal{J} / 12(2\pi)^2 \}
\]

\[
+ \frac{1}{6M^4} \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - M^2)n(|k_0|)R(q - k) \cdot
\]

\[
[4q_\mu q_\nu - 4(q_\mu k_\nu + q_\nu k_\mu) + 10k_\mu k_\nu - g_{\mu\nu}(q^2 - 3M^2 - 2q \cdot k) - 4M^2(q - k)_\mu(q - k)_\nu/(q - k)^2]
\] (B.4)

Next consider the terms quadratic in \(n\). Introducing the integrals,

\[
Q(p) = \int \frac{d^4k}{(2\pi)^3} \frac{\delta(k^2 - M^2)n(k)}{(p - k)^2 - M^2}, \quad Q_\mu(p) = \int \frac{d^4k}{(2\pi)^3} k_\mu \frac{\delta(k^2 - M^2)n(k)}{(p - k)^2 - M^2},
\] (B.5)

we can write them as

\[
K^{\beta}(q)_{n^2} = -\frac{3}{M^2} \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - M^2)n(|k_0|)Q(q - k)
\] (B.6)

and

\[
K^{\beta}_{\mu\nu}(q)_{n^2} = \frac{1}{M^2} \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - M^2)n(|k_0|)\left\{\{q_\mu q_\nu + 4k_\mu k_\nu - 2(q_\mu k_\nu + q_\nu k_\mu)\}Q(q - k) + k_\mu Q_\nu(q - k) + k_\nu Q_\mu(q - k)\right\}
\] (B.7)

These terms as well as the last terms in Eqs.(B.3) and (B.4) belong to \(\mathcal{K}(q)\) and \(\mathcal{K}_{\mu\nu}(q)\) in the notation of Eqs. (4.7) and (4.8).

Note that \(Q_i\) and \(Q_0\) are not independent, but are related by

\[
Q_i(p) = \frac{p_i}{|p|^2} \left\{2p_0Q_0(p) - p^2Q(p) + \mathcal{J}\right\}.
\] (B.8)

Using this relation one can relate \(\mathcal{K}_{ij}\) and \(\mathcal{K}_{00}\) at the pole, \(q_0 = M\) with \(\bar{q} = 0\),

\[
\mathcal{K}_{ij} = \frac{\delta_{ij}}{3} (\mathcal{K}_{00} - \mathcal{K} + \mathcal{J}^2)
\] (B.9)

Being interested in the real parts, we shall not consider terms cubic in \(n\), which are imaginary.

APPENDIX C

Here we write the chiral and the nonrelativistic limits of the integrals occurring in expressions for the effective pion parameters. The chiral limits were obtained in Ref. [6],

\[
\eta J |_x = \frac{T^2}{12F^2}, \quad \frac{MT}{4\pi F^2}, \quad \eta^2 J_{00} |_x = \frac{\pi^2 T^4}{30F^4}
\] (C.1)

\(^2\)In Ref. [6] the integral \(I_1\) is evaluated by expressing it in terms of derivatives of Zeta and Gamma functions. But since the integral \(I_2\) has to be evaluated numerically anyway, we can do the same for \(I_1\) also and get identical result.
\[ \eta^2 \mathcal{R}_0 |_\chi = \frac{T^4}{144 F^4} \left( \ln \frac{M}{T} - \ln 2 + 1 - I_1 + I_2 \right) \] (C.2)

where

\[ I_1 = \frac{15}{2 \pi^4} \int_0^\infty \frac{dx}{x^3 \ln x} \ln x e^{x} - 1 = 0.60 , \]
\[ I_2 = \frac{18}{\pi^2} \int_0^\infty \frac{dx}{x^3} \ln x e^{x} - 1 \int_0^1 \frac{d\alpha}{e^{\alpha x}} \left\{ (1 + \alpha^2) \ln \frac{1 + \alpha}{1 - \alpha} + \alpha \ln \frac{1 - \alpha^2}{\alpha^2 x^2} \right\} = 1.05 \] (C.3)

The integrals for \( \eta^2 \mathcal{R} |_\chi \) and \( \eta^2 \partial \mathcal{R} |_{\partial q_0} |_\chi \) vanish, while those for \( \eta^2 \partial \mathcal{R} |_{\partial q_0} |_\chi \) and \( \eta^2 C |_\chi \) are finite in the chiral limit. (Actually the terms linear and quadratic in \( n \) in each of the later two quantities have singular pieces separately in the this limit, but they cancel out in their respective sums.)

Next we calculate the integrals in the low temperature region, where \( \tau \equiv T/M \pi \ll 1 \). Keeping only the leading terms, we get

\[ J |_\tau = J_{00} |_\tau = \left( \frac{\tau}{2 \pi} \right)^{3/2} e^{-\tau^{1/2}} \] (C.4)
\[ \frac{1}{3} \mathcal{K} |_\tau = \frac{1}{3} \mathcal{K}_{00} |_\tau = \frac{1}{2} M \pi \partial \mathcal{R}_{00} |_{\partial q_0} |_\tau = M \pi C |_\tau = \frac{1}{16 \pi^2} J |_\tau \] (C.5)
\[ M \pi \partial \mathcal{K} |_{\partial q_0} |_\tau = O(\tau^{5/2} e^{-\tau^{1/2}}) \] (C.6)

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[1] G.W. Semenoff and H. Umezawa, Nucl. Phys. B220, 196 (1983); A.J. Niemi and G.W. Semenoff, Ann. Phys. (N.Y.) 152, 105 (1984). For a review see N.P. Landsmann and Ch. G. van Weert, Phys. Rep. 145, 141 (1987).
[2] J. Gasser and H. Leutwyler, Ann. Phys, 158, 142 (1984); Nucl. Phys. B 250, 465 (1985).
[3] J. Gasser and H. Leutwyler, Phys. Lett. B 184, 83 (1987); J. Goity and H. Leutwyler, Phys. Lett. 228, 517 (1989).
[4] A. Schenk, Phys. Rev. D 47, 5138 (1993).
[5] C. Song, Phys. Rev. D 49, 1556 (1994).
[6] D. Toublan, Phys. Rev. 56, 5629 (1997).
[7] P. Gerber and H. Leutwyler, Nucl. Phys. B 321, 387 (1989).
[8] H. Leutwyler, Phys. Rev. D49, 3033 (1994).
[9] R. D. Pisarski and M. Tytgat, Phys. Rev. D54, 2989 (1996).
[10] In the expression for \( 192 \pi/2 \), Toublan [6] finds the coefficients of \( l^3 \) and \( l^4 \) to be \(-15\) and \(-12\) respectively, instead of \(-27\) and \(-24\) found by us.
[11] G. Ecker, in Chiral Dynamics: Theory and experiment by A.M. Bernstein, D. Drechsel and Th. Walcher (Eds.), Springer (1997).
[12] G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B 603, 125 (2001).
[13] G. Ecker, J. Gasser, A. Pich, E. de Rafael, Nucl. Phys. B 321, 311 (1989).
[14] G. Ecker, J. Gasser, H. Leutwyler, A. Pich, E. de Rafael, Phys. Lett. B 223, 425 (1989).
[15] Y. Fujimoto, H. Matsumoto, H. Umezawa and I. Ojima, Phys. Rev. D 30, 1400 (1984).
[16] R.L. Kobes and G.W. Semenoff, Nucl. Phys. B260, 714 (1985).