A simple derivation of level spacing of quasinormal frequencies for a black hole with a deficit solid angle and quintessence-like matter

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In this paper, we investigate analytically the level space of the imaginary part of quasinormal frequencies for a black hole with a deficit solid angle and quintessence-like matter by the Padmanabhan’s method [1]. Padmanabhan presented a method to study analytically the imaginary part of quasinormal frequencies for a class of spherically symmetric spacetimes including Schwarzschild-de Sitter black holes which has an evenly spaced structure. The results show that the level space of scalar and gravitational quasinormal frequencies for this kind of black holes only depend on the surface gravity of black-hole horizon in the range of $-1 < w < -\frac{1}{4}$, respectively. We also extend the range of $w$ to $w \leq -1$, the results of which are similar to that in $-1 < w < -\frac{1}{4}$ case. Particularly, a black hole with a deficit solid angle in accelerating universe will be a Schwarzschild-de Sitter black hole, fixing $w = -1$ and $\epsilon^2 = 0$. And a black hole with a deficit solid angle in the accelerating universe will be a Schwarzschild black hole, when $\rho_0 = 0$ and $\epsilon^2 = 0$. In this paper, $w$ is the parameter of state equation, $\epsilon^2$ is a parameter relating to a deficit solid angle and $\rho_0$ is the density of static spherically symmetrical quintessence-like matter at $r = 1$.

I. INTRODUCTION

Recently, an elegant work on the quasinormal modes (QNMs) of a black hole was done by Padmanabhan [1], who studied analytically their role in response of the black hole to external perturbation. Since the gravitational radiation excited by the black hole oscillation is dominated by its QNMs, one can determine the parameters of a black hole by analyzing the QNMs in its gravitational radiation. So, besides their importance in the analysis of the stability of the black hole, QNMs are important in the search for black holes and their gravitational radiation. Many physicists believe that figure of QNMs is a unique fingerprint in directly identifying the existence of a black hole. It was found that the structure of the spectrum of QNMs (corresponding to quasinormal frequencies) consists of the real part and the imaginary part independent of initial conditions. For example, quasinormal frequencies of Schwarzschild black hole are

$$\omega_n = i\kappa(n + \frac{1}{2}) + \frac{\ln 3}{2\pi}\kappa + O(n^{\frac{3}{2}}).$$ \hspace{1cm} (1)

where $\kappa$ is the surface gravity. The real part of quasinormal frequencies which relates to black hole area quantization have been studied widely [2-4]. Meanwhile, since it is difficult to have a physical understanding of the constant spacing of quasinormal frequencies, the imaginary part was out of researchers’ visions until recent years. Padmanabhan [1] presented a new analytical method (Born approximation) which can reproduce the structure of the imaginary part of quasinormal frequencies with evenly spacing in a general class of spherically symmetric spacetimes for large $n$. And this derivation can not give the real part of quasinormal frequencies. According to the Born approximation, he came to a significant conclusion that thermodynamics of the black hole horizon has an effect on the imaginary part. Then, Padmanabhan and his collaborator [5], using this method, discussed the structure of the imaginary part of frequencies for Schwarzschild-de Sitter black holes. They proved that although this spacetimes has two horizons, the black hole horizon and the cosmological horizon, the imaginary part is only related to the surface gravity of the black hole horizon. And they explained appropriately why quasinormal modes vanish in pure de-Sitter spacetime.

The phase transition in the early universe could have produced different kinds of topological defects, whose cosmological implications are very important [6-8]. The global monopole, which has divergent mass in flat spacetime, is one of the most interesting defects. When one considers gravity, the linearly divergent mass of the global monopole has an effect analogous to that of a deficit solid angle plus a tiny mass at the origin. It has been shown that this effective mass is actually negative [9, 10]. Barriola and Vilenkin point out that the metric of global monopole with a large positive mass $M$, describes a black hole of mass $M$ carrying a global monopole charge. Such a black hole can be formed when a global monopole is swallowed by a Schwarzschild black hole [11]. On the other hand, current observations [12-15] (cosmic microwave background, Type Ia Supernovae, baryon acoustic oscillation, integrated Sachs-Wolfe effect correlations, etc.) show that there exists a spatially homogeneous and gravitationally repulsive energy component referred to as dark energy in our universe [16-18]. One of dark energy candidates is scalar-field dark energy
models such as quintessence \((-1 < w < -\frac{1}{3})\) or phantom \((w < -1)\), in which \(w = \frac{\kappa}{2}\) is the parameter of state equation. The solutions have been found for a global monopole surrounded by the static spherically-symmetric quintessence-like matter \([11, 20]\). When such a global monopole is swallowed by an ordinary black hole, a black hole with quintessence-like matter and a deficit solid angle can be formed \([11]\). Therefore, it is worth further investigating level spacing of quasinormal frequencies of this kind of black holes analytically. Padmanabhan has pointed out that the quantities of a physical system with constant spacing \([1, 2]\) are very interesting when they have constant spacing \([1, 2]\).

In this paper, through the scattering amplitude in Born approximation \([1]\), we study the level spacing of scalar and gravitational quasinormal frequencies of in the background of a black hole with a deficit solid angle surrounded by quintessence-like matter, respectively. We can show that there exist only two cases for \(w < -\frac{1}{3}\). In the \(w = -\frac{2k-1}{3}\) \((k \text{ is positive integer})\) case, the metric function \(f(r)\) has three zero points for \(-\infty < r < +\infty\). In \(w \neq -\frac{2k-1}{3}\) case, \(f(r)\) has only two zero points. Therefore, the spacetimes have two horizons, the black hole horizon and the cosmological horizon. The analytical results show that the imaginary value of scalar and gravitational quasinormal frequencies only depend on the surface gravity of the black hole horizon, which has an equally spaced structure. We also consider \(w \leq -1\) case, and obtain similar results to those in \(-1 < w < -\frac{1}{3}\) case.

II. A BLACK HOLE WITH A DEFICIT SOLID ANGLE AND QUINTESSENCE-LIKE MATTER

To be specific, we shall work within a particular model in unit \(c = 1\), where a global \(O(3)\) symmetry is broken down to \(U(1)\). The Lagrangian density is

\[
\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_{\mu} \phi^a \partial_{\nu} \phi^a - \frac{\lambda^2}{4} (\phi^a \phi^a - \sigma_0^2)^2, \tag{2}
\]

where \(\phi^a\) is triplet of scalar fields, and the isovector index \(a = 1, 2, 3\). The hedgehog configuration describing a global monopole is

\[
\phi^a = \sigma_0 q^a \frac{\tilde{r}^a}{\tilde{r}}, \quad \text{with} \quad x^a x^a = \tilde{r}^2. \tag{3}
\]

so that we shall actually have a monopole solution if \(q \to 1\) at spatial infinity and \(q \to 0\) near the origin. The solutions for a global monopole surrounded by the static spherically-symmetric quintessence-like matter are as follows \([19]\):

\[
ds^2 = (1 - \frac{2G\sigma_0 m}{r} - \epsilon^2 + \frac{\rho_0}{3w} r^{-3w-1}) dt^2 - \frac{1}{1 - \frac{2G\sigma_0 m}{r} - \epsilon^2 + \frac{\rho_0}{3w} r^{-3w-1}} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{4}
\]

where \(\epsilon \equiv \sqrt{8\pi G\sigma_0^2}\) is a dimensionless parameter of a deficit solid angle, \(w < -\frac{1}{3}\), \(m \approx \frac{8\pi \rho_0}{3w}\) and \(\rho_0\) is the density of static spherically symmetrical quintessence-like matter at \(r = 1\).

When such a global monopole is swallowed by an ordinary black hole with mass \(M\), a black hole with a deficit solid angle surrounded by quintessence-like matter can be formed:

\[
ds^2 = f(r) dt^2 - \frac{1}{f(r)} dr^2 - r^2 d\Omega^2, \tag{5}
\]

where \(f(r) = (1 - \frac{2M}{r} - \epsilon^2 + \frac{\rho_0}{3w} r^{-3w-1})\), \(M = G\sigma_0 (\tilde{M} - m)\) is the dimensionless parameter of mass of a black hole with a deficit solid angle surrounded by quintessence-like matter. The necessary condition is

\[
\rho_0 < (1 - \epsilon^2) \frac{(1 - \epsilon^2)(3|w| - 1)}{6M|w|^3}. \tag{6}
\]

for the existence of a black hole. For example, we consider \(w = -\frac{1}{2}\) case. If \(\rho_0 < (1 - \epsilon^2)\sqrt{\frac{1 - \epsilon^2}{6M}}\), there exists the solution of a black hole with a deficit solid angle surrounded by quintessence-like matter. And if \(\rho_0 \geq (1 - \epsilon^2)\sqrt{\frac{1 - \epsilon^2}{6M}}\), there exists the naked singularity solution.

The function \(y(r)\) is given by

\[
y(r) = f(r) = 1 - \frac{2M}{r} - \epsilon^2 + \frac{\rho_0}{3w} r^{-3w-1}, \tag{7}
\]

Next, we discuss an equation as follows

\[
y(r) = 0. \tag{8}
\]

Obviously, the roots of \(f(r) = 0\) are identical to those of Eq. \([5]\). Then, we will discuss the real roots of \(y(r) = 0\). In the range of \(0 < r < +\infty\), there is only one stationary point at \(\tilde{r} = (\frac{\rho_0}{3w})^{\frac{1}{3w}}\). From the necessary condition \([6]\), we have \(y(\tilde{r}) > 0\). Since \(y'(\tilde{r}) = (3w + 1)\rho_0 < 0\) (the prime denotes the derivative with respect to \(r\)) and \(y(0) = -2M\), we conclude that the equation \(y(r) = 0\) has and only has two positive real roots. That means the spacetime has two horizons, the black hole horizon \((r = r_b)\) and the cosmological horizon \((r = r_c)\), where \(r_b < r_c\). Next, we will consider whether there exist negative real roots of \(y(r) = 0\). If \(r\) is real and negative, we have

\[
r^{-3w} = (-1)^{-3w} |r|^{-3w} = [\cos (-3w) \pi + i \sin (-3w) \pi] |r|^{-3w}, \tag{9}
\]

Therefore, Eq. \([5]\) can be reduced to

\[
(-1)^{-3w-1} \frac{\rho_0}{3w} |r|^{-3w} + (1 - \epsilon^2)|r| + 2M = 0. \tag{10}
\]

Using a similar method, we find that there is a negative root \(i \epsilon < w < -\frac{1}{3}\). But in \(w < -\frac{2k-1}{3}\) case, the real roots are not negative for Eq. \([5]\).
III. QUASINORMAL MODES

A. The $-1 < w < -\frac{1}{3}$ Case

Now, we consider concretely the behaviors of scalar perturbations in a black hole with quintessence-like matter and a deficit solid angle. The propagation of a massless scalar field is described by the Klein-Gordon equation

$$\nabla_\mu \nabla^\mu \Phi = 0 \quad (\mu = 0, 1, 2, 3). \quad (11)$$

Then we separate variables by setting

$$\Phi(t, r, \theta, \phi) = \frac{1}{r} \psi(r) Y_{lm}(\theta, \phi) e^{i\omega t}, \quad (12)$$

where $Y_{lm}(\theta, \phi)$ are the usual spherical harmonics. Submitting Eqs. (12) to (11), we obtain

$$\frac{d^2 \psi(r)}{dr^2} + \left( \omega^2 - V_s \right) \psi(r) = 0, \quad (13)$$

where $r_s$ is the tortoise coordinate

$$r_s = \int \frac{1}{f(r)} dr = \frac{1}{2\kappa_b} \ln \left[ \frac{r}{r_b} - 1 \right] - \frac{1}{2\kappa_c} \ln \left[ 1 - \frac{r}{r_c} \right], \quad (14)$$

and $V_s$ is the effective potential

$$V_s = f(r) \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \frac{\rho_0(3w+1)}{3w} r^{-3w-3} \right], \quad (15)$$

where $\kappa_b$ and $\kappa_c$ represent the surface gravity of the black hole horizon and the surface gravity of cosmological horizon, respectively.

For gravitational perturbations, the metric function is expressed as

$$g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}, \quad (16)$$

where $g_{\mu\nu}$ is the background metric, and $h_{\mu\nu}$ is a small perturbation. Here, We adopt the canonical form for $h_{\mu\nu}$ in classical Regge-Wheeler gauge [22]

$$h_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & h_0(r) \\
0 & 0 & 0 & h_1(r) \\
h_0(r) & h_1(r) & 0 & 0 \\
h_0(r) & 0 & 0 & 0
\end{pmatrix} e^{-i\omega t} \left( \sin \theta \frac{\partial}{\partial \theta} \right) P_l(\cos \theta), \quad (17)$$

Introducing $Q(r) = \frac{1-2M}{r^2} + \frac{\rho_0}{r^3} r^{-3w-1} h_1(r)$, we obtain

$$\frac{d^2 Q(r)}{dr^2} + \left( \omega^2 - V_g \right) Q(r) = 0, \quad (18)$$

where $V_g$ is the effective potential

$$V_g = f(r) \left[ \frac{l(l+1)}{r^2} - \frac{6M}{r^3} + \frac{\rho_0(3w+1)}{3w} r^{-3w-3} \right], \quad (19)$$

For scalar and gravitational perturbations, we can show that the effective potentials $V_s$ and $V_g$ vanish at two horizons, which correspond to $r_s \to -\infty$ ($r = r_b$) and $r_s \to +\infty$ ($r = r_c$). So, the wave functions as solutions for Eq. (13) and Eq. (17) can be plane wave as follows

$$\psi \sim \begin{cases}
e^{-i\omega r_s} & r_s \to -\infty, \\
e^{-i\omega r_s} & r_s \to +\infty.
\end{cases} \quad (20)$$

and

$$Q \sim \begin{cases}
e^{-i\omega r_s} & r_s \to -\infty, \\
e^{-i\omega r_s} & r_s \to +\infty.
\end{cases} \quad (21)$$

where $\omega$ denotes quasinormal frequencies, the imaginary part of which are identical to thepoles of the scattering amplitude $S(\omega)$ in momentum space.

The scattering amplitude in the Born approximation, as the Fourier transform of potential $V(x)$ in momentum space, is

$$S(q) = \int dx V(x)e^{-iqx}, \quad (22)$$

where $q = kr - k_b$ is the momentum transfer. In one dimension, we take $k_b = -k_1$, then $q = -2k_1$. Thus, the scattering amplitude can be written as

$$S(\omega) = \int_{-\infty}^{+\infty} dr \psi(r(r_s)) e^{i2\omega r_s}, \quad (23)$$

Submitting Eqs. (14)-(15) to Eq. (22), we get

$$S(\omega) = \int_{r_b}^{r_c} dr \left[ \frac{l(l+1)}{r^2} + \frac{2M}{r^3} + \frac{\rho_0(3w+1)}{3w} r^{-3w-3} \right]$$

$$\times \left( \frac{r}{r_b} - 1 \right) \frac{\omega}{\kappa_b} \left( 1 - \frac{r}{r_c} \right)^{-1} \frac{\omega}{\kappa_c}$$

$$= -6MI_3 + l(l+1)I_2 + \frac{\rho_0(3w+1)}{3w} I_{3w+3}, \quad (24)$$

where

$$I_N = \int_{r_b}^{r_c} dr r^{-N} \left( \frac{r}{r_b} - 1 \right) \frac{\omega}{\kappa_b} \left( 1 - \frac{r}{r_c} \right)^{-1} \frac{\omega}{\kappa_c}$$

$$\times F_1 \left( 1 + i\frac{\omega}{\kappa_b}, N, 2 + i\omega \left[ \frac{1}{r_b} - \frac{1}{r_c} \right] \right) \frac{1}{\Gamma(1+i\frac{\omega}{\kappa_b})} \Gamma(1+i\frac{\omega}{\kappa_b}) \frac{1}{\Gamma(2+i\omega \left[ \frac{1}{r_b} - \frac{1}{r_c} \right])} (N = 3w+3, 3, 3), \quad (25)$$

In the expression of $I_N$, the ratio of $\Gamma(1+i\frac{\omega}{\kappa_b})$ doesn’t have poles anywhere [22]. Although the factor $\Gamma(1+i\frac{\omega}{\kappa_b})$ has poles at $Im(\omega_n) = -n\kappa_c (n \gg 1)$, we know that $-n\kappa_c$ is not the imaginary value of quasinormal frequencies from the definition of quasinormal modes. Therefore, we infer that the pole structure of scattering amplitude is given by

$$S(\omega) \propto \Gamma(1+i\frac{\omega}{\kappa_b}), \quad (26)$$
It means that the imaginary part of quasinormal frequencies for scalar and gravitational perturbations is given by

$$Im(\omega_n) = n \kappa_b \quad (n \gg 1).$$

(27)

It is clear that the level spacing of the imaginary parts of scalar and gravitational quasinormal frequencies are determined by the surface gravity at black hole horizon in $-1 < w < -\frac{1}{3}$ case, and it is equally spaced.

We have used the first Born approximation to obtain the QNMs Spectrum for the black hole with a deficit solid angle and quintessence-like matter. The approximation gives the correct level spacing for the imaginary values of the QN frequencies. As we know, QN frequencies include the real and imaginary parts. However, the analytical expression of the real part is difficult to obtain. The numerical studies of the real part can be carried out by WKB-like approximation techniques. By numerical calculations, we attain that $\frac{Im(\omega_{n+1})}{Im(\omega_n)} \approx 1 + \frac{1}{3} \quad (n \gg 1)$ for scalar and gravitational perturbations in the $-1 < w < -\frac{1}{3}$ case showed in Tables 1-2, which is consistent with the result of the formula (27). Next, we will discuss QNMs of this kind of black holes in the range of $w \leq -1$.

### B. The $w \leq -1$ Case

In the $w \leq -1$ case, there are two subcases. From the analysis of the real roots of $f(r) = 0$, we know that there are only two positive roots in $w \neq -\frac{2k+1}{3}$ subcase, similar to that in the range of $-1 < w < -\frac{1}{3}$. In $w = -\frac{2k+1}{3}$ subcase, there are two positive and one negative real roots of $f(r) = 0$. The tortoise coordinate $r_*$ is

$$r_* = \frac{1}{2b} \ln \frac{r}{r_b - 1} - \frac{1}{2\kappa_b} \ln \frac{1 - r}{r_c} + \frac{1}{f'(r_h)} \ln \frac{1 + r}{r_h},$$

(28)

where $r_h$ is the negative root, which satisfies

$$\frac{\rho_0}{3w} \frac{r_c^{2k+1} + r_b^{2k+1} - 2r_n^{2k+1}}{(r_c + r_b - 2r_h) = 0,}$$

(29)

According to Abel theorem, we can not find the exact form by the finite algebra operation, except for $k = 1$. For $k = 1$, we obtain the relation as follows

$$r_h = -(r_c + r_b).$$

(30)

which is similar to that in Schwarzschild-de Sitter spacetime. By the way, if $\rho_0 = 0$ (there is no quintessence) and $\epsilon^2 = 0, f(r)$ is the same as the metric of Schwarzschild black hole. Submitting Eqs. (15) and (28) to Eq. (22), we get

$$S(\omega) = \int^{r_c}_{r_h} dr \frac{l(l+1)}{r^2} \left[ 2M \frac{3w}{3w} r^{-3w-3} \right] \times \left( \frac{r}{r_h} - 1 \right)^{-1} \left( 1 - \frac{r}{r_c} \right)^{-1} \left( 1 + \frac{r}{r_h} \right)^{\frac{1}{f'(r_h)}} \frac{1}{r_h}$$

$$= 2MI_3 + l(l+1)I_2 + \frac{\rho_0(3w+1)}{3w} I_{3w+3},$$

(31)

Submitting Eqs. (18) and (28) to Eq. (22), we get

$$S(\omega) = \int^{r_c}_{r_h} dr \frac{l(l+1)}{r^2} \left[ -6M \frac{3w}{3w} r^{-3w-3} \right] \times \left( \frac{r}{r_h} - 1 \right)^{-1} \left( 1 - \frac{r}{r_c} \right)^{-1} \left( 1 + \frac{r}{r_h} \right)^{\frac{1}{f'(r_h)}} \frac{1}{r_h}$$

$$= -6MI_3 + l(l+1)I_2 + \frac{\rho_0(3w+1)}{3w} I_{3w+3},$$

(32)

where

$$I_N = \int^{r_c}_{r_h} dr r^{-N} \left( \frac{r}{r_h} - 1 \right)^{-1} \left( 1 - \frac{r}{r_c} \right)^{-1} \left( 1 + \frac{r}{r_h} \right)^{\frac{1}{f'(r_h)}} \frac{1}{r_h}$$

$$= \frac{r_c - r_h}{r_h} \left( \frac{r_c - r_h}{r_h} \right)^{-1} \left( \frac{r_h + |r|}{r_h} \right)^{\frac{1}{f'(r_h)}} \frac{1}{r_h} N$$

$$\times \left( \frac{r_c - r_h}{r_h} \right)^{-1} \left( \frac{r_h + |r|}{r_h} \right)^{\frac{1}{f'(r_h)}} \frac{1}{r_h}$$

$$\times \left( 1 + i \omega \kappa_b \right) \left( 1 - i \omega \kappa_b \right) \frac{\Gamma(1 + 3w)}{\Gamma(2 + 3w)},$$

(33)

The ratio of $\frac{r_c - r_h}{r_h} \left( \frac{r_c - r_h}{r_h} \right)^{-1} \left( \frac{r_h + |r|}{r_h} \right)^{\frac{1}{f'(r_h)}} \frac{1}{r_h}$ also doesn’t have poles anywhere. It is easy to find that the poles of scattering amplitude are only determined by the poles of the function $\Gamma(1 + i \omega \kappa_b)$. Thus, the imaginary part of scalar and gravitational quasinormal frequencies in $w = -\frac{2k+1}{3}$ case are also given by

$$Im(\omega_n) = n \kappa_b \quad (n \gg 1).$$

(34)

The level spacing of the imaginary part of frequencies for scalar and gravitational perturbations only depends on the surface gravity of the black hole horizon in $w = -\frac{2k+1}{3}$ case, which is supported by our numerical results listed in Tables 3-4.

### IV. CONCLUSION

It is known that QNMs’ frequencies of black holes including Schwarzschild and Schwarzschild-de Sitter are equally spaced, with the level spacing depending only on the surface gravity. In this paper, we generalize this result to a new class of spacetimes and provide the imaginary parts of scalar and gravitational quasinormal frequencies by Padmanabhan’s method. In the view of extensibility and simplicity, Padmanabhan’s analysis is an elegant method. Furthermore, these analyses show that the result is closely related to the thermal nature of horizons and is the consequence of the exponential redshift of the wave modes close the horizon. In $w = -\frac{2k+1}{3}$ case, there are three real roots of $f(r) = 0$, one of which is negative. In $w \neq -\frac{2k+1}{3}$ case, there are two positive real roots. However, the imaginary part of quasinormal frequencies of a black hole with a deficit solid angle for scalar and gravitational perturbations in these two cases are only determined by the surface gravity at black-hole horizon, which are equally spaced. In particular, a black hole with a deficit solid angle in an accelerating universe will be a Schwarzschild-de Sitter black hole, fixing $w = -1$ and $\epsilon^2 = 0$. And a black hole with a deficit solid angle in accelerating universe will be a Schwarzschild black hole, when $\rho_0 = 0$ and $\epsilon^2 = 0$.

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TABLE I: Scalar QN frequencies of a black hole with a deficit solid angle and quintessence-like matter for $M = 1$, $\epsilon^2 = 0.001$, $\rho_0 = 0.01$, $l = 3$ and $w = -\frac{2}{3}$.

| $n$ | $\omega_n$          |
|-----|---------------------|
| 3   | 0.58889-0.67804i    |
| 4   | 0.55178-0.88128i    |
| 5   | 0.50894-1.08641i    |
| 6   | 0.45997-1.29294i    |
| 7   | 0.40455-1.50083i    |
| 8   | 0.34249-1.71015i    |
| 9   | 0.27373-1.92109i    |
| 10  | 0.19828-2.13380i    |

TABLE II: Gravitational QN frequencies of a black hole with a deficit solid angle and quintessence-like matter for $M = 1$, $\epsilon^2 = 0.001$, $\rho_0 = 0.01$, $l = 3$ and $w = -\frac{2}{3}$.

| $n$ | $\omega_n$          |
|-----|---------------------|
| 3   | 0.50553-0.65551i    |
| 4   | 0.46253-0.85301i    |
| 5   | 0.41235-1.05270i    |
| 6   | 0.35457-1.25427i    |
| 7   | 0.28895-1.45781i    |
| 8   | 0.21542-1.66350i    |
| 9   | 0.13402-1.87158i    |
| 10  | 0.04486-2.08225i    |

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TABLE III: Scalar QN frequencies of a black hole with a deficit solid angle and quintessence-like matter for $M = 1$, $\epsilon^2 = 0.001$, $\rho_0 = 0.01$, $l = 3$ and $w = -\frac{4}{3}$.

| n  | $\omega_n$       |
|-----|------------------|
| 3   | 0.56082-0.63141i |
| 4   | 0.52960-0.81378i |
| 5   | 0.49091-0.99737i |
| 6   | 0.44485-1.18239i |
| 7   | 0.39153-1.36903i |
| 8   | 0.33109-1.55749i |
| 9   | 0.26368-1.74794i |
| 10  | 0.18946-1.94053i |

TABLE IV: Gravitational QN frequencies of a black hole with a deficit solid angle and quintessence-like matter for $M = 1$, $\epsilon^2 = 0.001$, $\rho_0 = 0.01$, $l = 3$ and $w = -\frac{4}{3}$.

| n  | $\omega_n$       |
|-----|------------------|
| 3   | 0.48784-0.61217i |
| 4   | 0.45249-0.78844i |
| 5   | 0.40824-0.96629i |
| 6   | 0.35535-1.14609i |
| 7   | 0.29408-1.32814i |
| 8   | 0.22470-1.51269i |
| 9   | 0.14750-1.69993i |
| 10  | 0.06263-1.89005i |