ON THE UNIVERSAL STRING THEORY

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ABSTRACT

Very recently Berkovits and Vafa have argued that the $N=0$ string is a particular choice of background of the $N=1$ string. Under the assumption that the physical states of the $N=0$ string theory came essentially from the matter degrees of freedom, they proved that the amplitudes of both string theories agree. They also conjectured that this should persist whatever the form of the physical states. The aim of this note is to prove that both theories have the same spectrum of physical states without making any assumption on the form of the physical states. We also notice in passing that this result is reminiscent of a well-known fact in the theory of induced representations and we explore what repercussions this may have in the search for the universal string theory.

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Introduction

In a very recent paper Berkovits and Vafa [1] have argued that \( N=0 \) string theory can be thought of as a special background configuration of \( N=1 \) string theory and that, in turn, any \( N=1 \) string theory can be seen as a special background configuration of \( N=2 \) string theory. More concretely, what was shown in [1] is that the amplitudes of the \( N=0 \) string theory could be recovered from amplitudes of the \( N=1 \) string theory, and similarly for the other embedding. This was proven under the assumption that the physical states were given essentially by excitations of the matter degrees of freedom. This condition, while it is certainly true for critical string theory (away from zero momentum) is not generally true, as evinced for instance, by the rich spectrum of the noncritical string theories with \( c < 1 \).

In the first footnote of [1], it is already pointed out that their results should still hold true regardless of the form of the physical states. It is the purpose of this note to elaborate on this footnote and to prove that for the first of these embeddings, the physical spectrum of both string theories agree. We find it convenient to paraphrase the results of [1] in the language of BRST cohomology. Let us consider the case of \( N=0 \) strings.

Let \( T_m(z) \) denote the energy-momentum tensor of any CFT with \( c=26 \). Let \( \mathcal{H} \) denote the Hilbert space of this CFT. The physical states of the string with background \( \mathcal{H} \) are given by the BRST cohomology \( H_{N=0}(\mathcal{H}) \).Consider now the CFT defined by a fermionic BC system \((b_1,c_1)\) of weights \((\frac{3}{2},-\frac{1}{2})\) and let \( \mathcal{F} \) denote its Hilbert space. It carries a representation of the Virasoro algebra with \( c=-11 \). More is true however and, as shown in [1], \( \mathcal{H} \otimes \mathcal{F} \) is the Hilbert space of an \( N=1 \) superconformal field theory (sCFT) with \( \hat{c}=10 \) or, equivalently, \( c=15 \). This means that we can consider \( \mathcal{H} \otimes \mathcal{F} \) as a possible background for an \( N=1 \) string theory, whose physical states will be given by the BRST cohomology \( H_{N=1}(\mathcal{H} \otimes \mathcal{F}) \). In this language, the embedding of string theories of [1] simply translates to

\[
H_{N=1}(\mathcal{H} \otimes \mathcal{F}) \cong H_{N=0}(\mathcal{H}),
\]

where we mean an isomorphism as rings. This point may need some elaboration. BRST cohomology has a natural ring structure: to every BRST cocycle there corresponds a BRST invariant operator and the OPE of such operators induce a multiplication on the physical states. String amplitudes can in principle be computed from a knowledge of the operator products of the physical operators, hence if two string theories have isomorphic BRST cohomology rings, they are equivalent in the sense that all amplitudes will coincide. The equality between the amplitudes was proven in [1] for those \( \mathcal{H} \) such that all the cohomology is generated by “matter” excitations. What we would like to show in this note, is that (1) is valid for all \( \mathcal{H} \). We will only partially succeed. We will prove that (1) is true at the level of vector spaces.

The idea of the proof is very simple. We approximate the computation of \( H_{N=1}(\mathcal{H} \otimes \mathcal{F}) \) by a spectral sequence whose second term is precisely \( H_{N=0}(\mathcal{H}) \) and we show that this approximation is actually exact. The spectral sequence that will be useful in this case arises out of a very obvious filtration of the complex computing \( H_{N=1}(\mathcal{H} \otimes \mathcal{F}) \). This makes it possible to generalize this result to other embeddings of string theories. In an effort to keep this note as short as possible, though, we will only discuss the embedding (1) and leave the more general results to a lengthier and more detailed forthcoming publication.
The isomorphism (1) is reminiscent of a well-known fact in the theory of induced representations. Suppose that $h \subset g$ are Lie algebras. Then there is a way to induce a representation of $g$ from a representation of $h$, which goes by the name of the induced module construction. Indeed, if $V$ is any representation of $h$ then it is a module over the universal enveloping algebra $U(h)$. We can then “extend the scalars” and define

$$W \equiv U(g) \otimes_{U(h)} V.$$  

(2)

$W$ is naturally a left $U(g)$-module and hence a representation of $g$. It is then a classic result (see, for example, [2]) that in Lie algebra homology

$$H_*(g; W) \cong H_*(h; V).$$  

(3)

To determine whether (1) is a semi-infinite instance of (3) is beyond the scope of the present paper, but we will explore this analogy further in the concluding section to see what it can tell us about the search for the universal string theory.

**The Complexes**

Since we want to prove a result about the equality between two cohomology spaces, we start by describing the complexes that compute them.

As in the introduction, let $T_m(z)$ denote the energy momentum tensor of a CFT with $c=26$. We will let $\mathcal{H}$ denote the Hilbert space of this CFT. To compute $H_{N=0}(\mathcal{H})$ we introduce fermionic ghosts $(b, c)$ of weights $(2, -1)$. The BRST current is defined by

$$J_{BRST}^{N=0} = T_m c + bc \partial c$$

and its charge by

$$Q_{N=0} = \oint_{C_0} \frac{dz}{2\pi i} J_{BRST}^{N=0}(z).$$

(5)

Because $c = 26$ it follows that $Q_{N=0}^2 = 0$ whence its cohomology $H_{N=0}(\mathcal{H})$ is well-defined. It is known in the mathematical literature as the semi-infinite cohomology of the Virasoro algebra relative its center and with coefficients in the representation $\mathcal{H}$.

We now introduce another fermionic BC system $(b_1, c_1)$ of weights $(\frac{3}{2}, -\frac{1}{2})$ and we call its Hilbert space $\mathcal{F}$. It carries a representation of the Virasoro algebra with $c=-11$. The tensor product $\mathcal{H} \otimes \mathcal{F}$ carries therefore a representation of the Virasoro algebra with $c=15$. One of the remarkable results of [1] is that it carries a representation of the $N=1$ (super)Virasoro algebra as well. Indeed, $T$ and $G$ defined by

$$T \equiv T_m - \frac{3}{2} b_1 \partial c_1 - \frac{1}{2} \partial b_1 c_1 + \frac{1}{2} \partial^2 (c_1 \partial c_1)$$

$$G \equiv b_1 + c_1 T_m + c_1 \partial c_1 b_1 + \frac{5}{2} \partial^2 c_1$$

(6)

(7)

satisfy the OPEs defining the $N=1$ superconformal algebra with $c = 15$. In other words, the tensor product $\mathcal{H} \otimes \mathcal{F}$ can be understood as the Hilbert space of an $N=1$ sCFT with $c=15$. 

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Now, given any sCFT with $c=15$ and Hilbert space $S$, we can define its BRST cohomology $H_{N=1}(S)$ as follows. We introduce in addition to the fermionic ghosts $(b, c)$ discussed above, a bosonic BC system $(\beta, \gamma)$ of weights $(\frac{3}{2}, -\frac{1}{2})$ and we define the BRST current by

$$J_{BRST}^{N=1} = Tc + G\gamma + bc\partial c - b\gamma^2 - c\beta\partial\gamma + \frac{1}{2}\partial c\beta\gamma .$$

(8)

Again because of $c=15$, its charge

$$Q_{N=1} = \oint_{C_0} \frac{dz}{2\pi i} J_{BRST}^{N=1}(z)$$

(9)

squares to zero and its cohomology—denoted by $H_{N=1}(S)$—can be defined.

In particular when $S = \mathcal{H} \otimes \mathcal{F}$, the BRST current is given by (8) where $T$ and $G$ are given by (6) and (7) respectively. Explicitly, we find

$$J_{BRST}^{N=1} = T_m c + bc\partial c - b\gamma^2 + b_1\gamma - \frac{3}{2}b_1\partial c_1 c - b_1\partial c_1 c_1\gamma - \frac{3}{2}c\beta\partial\gamma + T_m c_1\gamma$$

$$-\frac{1}{2}\partial b_1 c_1 c - \frac{1}{2}c\partial\beta\gamma + \frac{5}{2}\partial^2 c_1\gamma - \frac{1}{2}\partial(\partial^2 c_1 c_1)c .$$

(10)

It is this explicit form of $J_{BRST}^{N=1}$ that we shall exploit to prove the isomorphism (1). We will see, however, that it is rather the overall structure of $J_{BRST}^{N=1}$ that plays a role and this makes possible the generalizations alluded to in the introduction.

The Spectral Sequence and the Calculation

The isomorphism we are after essentially boils down to a cancellation of the degrees of freedom of the fermionic BC system $(b_1, c_1)$ and of the bosonic ghosts $(\beta, \gamma)$, which is reminiscent of the quartet mechanism of Kugo and Ojima. In fact, the original quartet mechanism applied to abelian constraints. In other words, it would correspond to the BRST differential associated to the current$^1$ $J^{(0)} = b_1\gamma$. Interestingly enough, a glance at the explicit expression (10) for the $N=1$ BRST current shows that such a term is indeed present. The idea behind the proof of (1) is then to isolate this term by breaking up $J_{BRST}^{N=1}$ into terms of different degrees:

$$J_{BRST}^{N=1} = J^{(0)} + J^{(1)} + \cdots$$

(11)

in such a way that we can approximate the BRST cohomology of $J_{BRST}^{N=1}$ by computing the cohomology induced by the $J^{(i)}$. The gadget that takes care of organizing this data is called a spectral sequence and in the particular case that we are considering, it will be the spectral sequence associated to a filtered complex.

We choose to define the following degrees for our operators:

$$\deg c = -\deg b = 1 , \quad \deg c_1 = \deg \gamma = -\deg b_1 = -\deg \beta = 2 ,$$

(12)

and $\deg T_m = 0$. Notice that this is compatible with the operator product algebra. Decomposing $J_{BRST}^{N=1}$ under this grading we find that $J_{BRST}^{N=1} = \sum_{i=0}^{5} J^{(i)}$ where $J^{(i)}$ has degree $i$.

$^1$ The reason for the superscript will become obvious shortly.
and they are given by

\begin{align*}
J^{(0)} &= b_1 \gamma \\
J^{(1)} &= T_m c + bc \partial c - \frac{3}{2} b_1 \partial c_1 c - \frac{3}{2} \beta \partial \gamma c - \frac{1}{2} \partial b_1 c_1 c - \frac{1}{2} \partial \beta \gamma c \\
J^{(2)} &= 0 \\
J^{(3)} &= -b_1^2 - b_1 \partial c_1 c_1 \gamma + T_m c_1 \gamma \\
J^{(4)} &= \frac{5}{2} \partial^2 c_1 \gamma \\
J^{(5)} &= -\frac{1}{2} \partial (\partial^2 c_1 c_1) c.
\end{align*}

Let $Q_{N=1} = \sum_{i=0}^5 Q_{(i)}$ denote the decomposition of the charge. We can also break up the equation $Q_{N=1}^2 = 0$ into components of different degrees. Because deg is compatible with the operator product algebra, each graded component has to be zero separately and this means, in particular, that $Q_{(0)}^2 = 0$, $[Q_{(0)}, Q_{(1)}] = 0$, and $Q_{(1)}^2 = 0$. Among other things it makes sense to talk about the cohomology of $Q_{(0)}$.

The above decomposition of $Q_{N=1}$ makes the BRST complex for the $N=1$ string with that particular background into a filtered complex. Standard techniques in homological algebra now apply \cite{3}. In particular, there exists a spectral sequence converging to the cohomology of $Q_{N=1}$ and whose first term is the cohomology of $Q_{(0)}$. The space that $Q_{(0)}$ acts on is the tensor product of the Hilbert spaces of the “matter” sCFT $\mathcal{H} \otimes \mathcal{F}$ and the ghost sCFT $\mathcal{A}_{b,c} \otimes \mathcal{S}_{\beta,\gamma}$. But since $Q_{(0)}$ does not depend on $(b, c)$ nor on $T_m$ it effectively acts only on $\mathcal{F} \otimes \mathcal{S}_{\beta,\gamma}$. Fortunately the cohomology of $Q_{(0)}$ in this space has already been computed in \cite{4}. In fact, the Lemma in section 3 of \cite{4} says that there is only one nontrivial cocycle and this is the projective invariant vacuum. Therefore we find that the first term in the spectral sequence—the cohomology of $Q_{(0)}$—is isomorphic to $\mathcal{A}_{b,c} \otimes \mathcal{H}$.

The next term in the spectral sequence is the cohomology of $Q_{(1)}$ computed in the cohomology space of $Q_{(0)}$, that is, on $\mathcal{A}_{b,c} \otimes \mathcal{H}$. But on this space, $Q_{(1)}$ reduces to $Q_{N=0}$ and hence the second term in the spectral sequence is $H_{N=0} (\mathcal{H})$.

All the other $Q_{(i>1)}$ are automatically zero since they involve the $(\beta, \gamma)$ and $(b_1, c_1)$ fields in one way or another. Therefore we conclude that the spectral sequence degenerates at the second term yielding the isomorphism (1). Notice that the proof does not rely on the particular form of $T_m$ hence it is true for arbitrary $N=0$ string backgrounds. The proof has one shortcoming, though. It does not respect the ring structure. And although it seems rather plausible that (1) does indeed hold as rings, we cannot unfortunately conclude this from our proof.
On the Concept of a Universal String Theory

As mentioned in the introduction, the isomorphism (1) is reminiscent of the analogous result in the theory of Lie algebra homology by which to compute the homology of a Lie algebra \( h \) with coefficients in a representation \( V \), one can proceed directly or, alternatively, one can first embed \( h \) in \( g \) and then compute the homology of \( g \) with coefficients in the representation induced by \( V \)—the point being that all the homological information of the induced representation is contained already in \( V \).

In our case, the Virasoro algebra plays the role of \( h \), the \( N=1 \) supervirasoro algebra plays the role of \( g \) and tensoring with the CFT of the \((b_1,c_1)\) system seems to play the role of the induced module construction (2). It is not inconceivable that this analogy contains some element of truth and that this construction is in fact a semi-infinite version of the induced module construction, but we will not attempt to elucidate this point here and now.\(^2\) Nevertheless, let us pursue the analogy a little bit farther to see what it implies on the existence of the universal string theory.

What seems to emerge is that there is not a unique universal string theory, but rather that there is probably a universal string theory in each hierarchy of embeddings we can consider. For example, in the Lie algebra case, there are many chains of embeddings into which, say, \( sl_2 \) fits. For example we can take the following

\[
sl_2 \subset sl_3 \subset sl_4 \subset \cdots
\]  

(13).

The functorial nature of the induced module construction guarantees that the two ways we can induce a representation in \( sl_4 \), say, starting from a representation of \( sl_2 \)—that is, either by considering the embedding \( sl_2 \subset sl_4 \) directly, or else by going through \( sl_3 \)—are actually equivalent. Therefore we can take the inductive limit of (13)—call it \( sl_{\infty} \)—and induce a module from a module in \( sl_2 \). However notice that depending on how we actually embed \( sl_n \) in \( sl_{n+1} \) we will get a different \( sl_{\infty} \) and thus a different “universal” algebra. Worse yet, we could make use of the accidental isomorphism of complex Lie algebras \( sl_2 \cong so_3 \) to consider other chains of embeddings like

\[
sl_2 \cong so_3 \subset so_5 \subset so_7 \subset \cdots,
\]

(14)

which would again lead to a different universal object.

Similarly in string theory, we could choose as in [1] to embed the \( N=0 \) Virasoro algebra in the \( N=1 \) superVirasoro algebra and this in turn inside the \( N=2 \) superVirasoro algebra, but one could equally well have started by embedding it in, say, \( W_3 \) and pursue another chain of embeddings.\(^3\) Presumably both directions would yield different universal string theories. \(^{2}\) Although see [5] for a semi-infinite version of (3). It would be interesting to see if both constructions coincide for this case.

\(^3\) Note, however, that there is no chain of embeddings associated to the \( W_n \) algebras, but rather (at least in the classical case) an inverse system of reductions: \( W_n \leftarrow W_{n+1} \). (This was exploited in [6] to define the notion of a universal \( W \)-algebra.) But we mean any chain of embeddings of increasingly more symmetrical CFTs, the first two of which have as chiral algebras the Virasoro algebra and \( W_3 \) respectively.
Why one would choose one direction in which to embed a particular string theory over any other seems to be purely a phenomenological matter; but then again, the choice of vacuum seems to have the final word on the phenomenology. It would seem therefore that we need some more clues to narrow the search for the universal string theory.

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REFERENCES

[1] N. Berkovits and C. Vafa, On the Uniqueness of String Theory, hep-th/9310170.
[2] A.W. Knapp, Lie Groups, Lie Algebras, and Cohomology, Mathematical Notes, PUP (1988) and in particular section VI.2
[3] For an introduction to spectral sequences try R. Bott and L. Tu, Differential Forms in Algebraic Topology, Springer Verlag, 1982.
[4] J.M. Figueroa-O’Farrill, Nucl. Phys. B343 (1990) 450.
[5] A. A. Voronov, Semi-infinite Homological Algebra, Preprint 1993.
[6] J.M. Figueroa-O’Farrill and E. Ramos, J. Math. Phys. 33 (1992) 833 (Erratum: ibid. 34 (1993) 887).