Flavour-mixing gauge field theory of massive Majorana neutrinos

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Abstract
A gauge-field theory for massive neutral particles is developed on the basis of the real four-component Majorana equation. By use of its spin operator, a purely imaginary representation of the SU(2) algebra can be defined, which gives a covariant derivative that is real. Such a coupling to the gauge field preserves the real nature of the Majorana equation even when including interactions. As the associated isospin is four-dimensional, this procedure introduces four intrinsic degrees of freedom to the Majorana field, which may be related to four flavours. The main aim is to describe here the mathematical possibility for coupling Majorana particles with a gauge field which resembles that of the weak interaction. By adding a fourth member to the family, flavour could become a dynamic trait of the neutral Majorana particles, and thus lead to a dynamic understanding of mixing.
1 Introduction

According to the canonical standard model of elementary particle physics, the leptons and quarks come in three flavours, are massless and thus obey chiral symmetry, but then they can acquire mass through the Higgs mechanism (see, e.g., the text books [1, 2, 3]). The Dirac equation [4] is fundamental in all this and well understood, however the nature of the neutrinos involved remains less clear. Are they Dirac fermions or Majorana [5] particles? The physics of neutrinos and its application to particle astrophysics remains a very active research area. In the past neutrinos were often described by the massless Weyl [6] equations involving only two-component Pauli [7] spinors. However, since convincing empirical evidence [8] for the finite neutrino masses and their associated oscillations [9, 10] had been found in the past decades, massive neutrinos have been discussed, and furthermore another very massive neutrino species or sterile ones have been considered in four-neutrino models [11], for example to explain the light neutrinos masses by the see-saw mechanism [9]. The tensions with the three-neutrino paradigm have recently been discussed by Kayser [12], who also reviewed the key arguments in support for neutrinos being Majorana fermions [13]. Therefore the Majorana equation with various mass terms has found strong attention, either in its complex two-component (see the recent review by Dreiner et al. [14]) or real four-component form, and been used in modern quantum field theory for the description of massive neutrinos. Thus theoretical reasoning and new empirical results from laboratory as well as cosmology suggest the possible existence of a fourth neutrino flavour. The state of affairs (as of 2006) and the research perspectives are described in the comprehensive review by Mohapatra and Smirnov [15].

The purpose of the present paper is to show that for massive neutral Majorana particles (perhaps representing the observed neutrinos) a gauge-field theory based on the real four-component Majorana equation is feasible. Using its spin operator, which obeys the angular momentum algebra like SU(2) but is purely imaginary, we can define an appropriate symmetry group which gives a connection that is real, and thus the coupling to the gauge field (also real) does keep the real nature of the Majorana equation even when including interactions. As the isospin is four-dimensional, this procedure introduces four intrinsic degrees of freedom to the Majorana field, the interpretation of which remains open to speculation. Here the main aim is to describe the mathematical possibility for coupling the Majorana particles with a gauge field, which resembles that of the weak interactions between leptons and quarks [16, 17, 18]. Although the neutrinos are presently known to come in three flavours, we will here argue that, by adding a fourth member to the family, flavour could become a dynamic trait of the neutral Majorana particles and be related to the gauge theory described subsequently.

The paper is organized as follows. We discuss the relevant key aspects of the Majorana equation (its eigenfunctions are given in the appendix) in the next section, which is more of a tutorial nature. Then the real Lorentz transformation is given, in which the purely imaginary Majorana spin operator occurs
prominently. In our opinion, it strongly suggests itself as the adequate isospin for the SU(2)-like algebra adopted in the gauge symmetry considered in the subsequent section. When assuming four flavours, the resulting gauge theory provides a dynamic picture of flavour mixing, i.e. quantum flavour dynamics, and likely also new insights into neutrino oscillations.

2 The Majorana equation

In this introductory section, we consider the Dirac equation in its Majorana representation. We use standard symbols, notations and definitions, and conventionally units of $\hbar = c = 1$, with the four-momentum operator denoted as $P_\mu = (E, -\vec{p}) = i\partial_\mu = i(\partial/\partial t, \partial/\partial \vec{x})$, which acts on the spinor wave function $\psi(x, t)$. The particle mass is $m$. The four-vector $\gamma^\mu$ consists of the four Dirac gamma matrices that come in various representations. As first found out by Majorana [5], there exists a purely imaginary representation such that $\gamma^\mu = i\bar{\gamma}^\mu$, which makes the Dirac equation real. We simply refer to it as Majorana equation in the remainder of this work. This real equation reads

$$\bar{\gamma}^\mu \partial_\mu \psi + m\psi = 0. \tag{1}$$

The Majorana equation involves the below defined $4 \times 4$ matrices. Throughout the text we will use top-barred symbols to indicate real $4 \times 4$ matrices. The gamma matrices can after [19] be defined as

$$\bar{\gamma}^\mu = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\alpha \\ -\alpha & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}. \tag{2}$$

The gamma matrices mutually anticommute and obey: $\bar{\gamma}^\mu \bar{\gamma}^\nu + \bar{\gamma}^\nu \bar{\gamma}^\mu = -2g^{\mu\nu}$, with euclidian metric $g^{\mu\nu}$. The three associated real $2 \times 2$ matrices read

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Note that $\gamma = \beta\alpha$, and that all three matrices mutually anti-commute with each other, just like the Pauli matrices do. All what we need in the following are the algebraic properties of these $2 \times 2$ matrices involved, like $\alpha\beta + \beta\alpha = 0$, $\alpha\gamma + \gamma\alpha = 0$, and $\beta\gamma + \gamma\beta = 0$, and that $\alpha^2 = \beta^2 = 1$ and $\gamma^2 = -1$. The three Pauli matrices have their standard form, and are given by $\sigma_x = \beta$, $\sigma_y = i\gamma$, and $\sigma_z = \alpha$.

The four-component Majorana equation [1] is a direct consequence of, and fully equivalent to, the two-component complex Majorana equation. It involves only the Pauli matrix operators acting on a two-component spinor $\phi$, but introduces subtle complications that are caused by the spin-flip operator $\tau$, and reads as follows:

$$i\left(\frac{\partial}{\partial t} + \sigma \cdot \frac{\partial}{\partial \vec{x}}\right)\phi(x, t) = m\tau\phi(x, t). \tag{4}$$
It can be derived without invoking the Dirac equation in the first place, as was demonstrated by Marsch [19, 20] recently. As shown years ago by Case [21], this equation can be also derived from Dirac’s equation in its chiral form.

Above in (4) we defined an important operator that is not of pure algebraic nature but involves the complex-conjugation operator named as $C$. This antihermitian operator called $\tau$ is defined as $\tau = \sigma_3 C$, and obeys $\tau^{-1} = -\tau$. The operation of $\tau$ on the spin vector $\sigma$ leads to its inversion, i.e., the operation $\tau \sigma \tau^{-1} = -\sigma$ yields a spin flip. We also have $\tau i = -i \tau$ because of the action of $C$. Therefore, $\tau$ anti-commutes with the momentum four-vector operator $P^\mu$. Moreover, a simple phase factor like $\exp(i\theta)$ (with some angle $\theta$) does not commute with the mass term on the right-hand side of (4). Consequently, that equation can neither describe electromagnetic interactions nor most other complex gauge-field couplings in which the imaginary unit $i$ appears explicitly.

To avoid the complications introduced by the operator $\tau$, we prefer to work with the real four-dimensional Majorana equation (1), which we quote again but now in conventional Hamiltonian form as used by Majorana [5] himself as follows

$$\left(\frac{\partial}{\partial t} + \bar{\alpha} \cdot \frac{\partial}{\partial x}\right) \psi(x, t) = m\bar{\beta} \psi(x, t),$$

which formally resembles the complex two-component version (4). This last equation is as usually obtained by multiplying the manifestly covariant form (1) from the left by $\bar{\gamma}^0 = \bar{\beta}$ and defining $\bar{\alpha}^\mu = -\bar{\gamma}^0 \gamma^\mu = (1, \bar{\alpha})$. The real matrix three-vector $\bar{\alpha}$ is then given as

$$\bar{\alpha} = \left(\begin{array}{c} \beta & 0 & 0 \\ 0 & \alpha & \beta \end{array}\right), \left(\begin{array}{c} 0 & \gamma \\ -\gamma & 0 \end{array}\right), \left(\begin{array}{c} \alpha & 0 \\ 0 & \alpha \end{array}\right),$$

and is symmetric, which means equal to its transposed matrix, $\bar{\alpha} = \bar{\alpha}^T$. The eigenfunctions of the Majorana equations (4) and (5) are briefly discussed in the appendix Section 6 where it is shown that the free real Majorana can be decomposed into left- and right-helical particle and antiparticle components. It is chirally irreducible, though, unless one considers complex solutions.

At this point, we quote the Lagrangian density of the Majorana field, which is obtained by inserting the Majorana gamma matrices into the general Dirac Lagrange density. Note that the factor $i$ is important here to ensure that we are dealing with hermitian operators having real eigenvalues. Yet the factor $i$ is irrelevant, and does not appear, in the Majorana equation (5) itself. The final result is (where the superscript $T$ denotes the transposed spinor) given by

$$L_M = i\psi^T(\bar{\alpha}^\mu \partial_\mu - \bar{\beta}m)\psi.$$ 

Its variation with respect to $\psi^T$ yields the above equation of motion (5) for the Majorana field. It should here be stressed again that the Majorana equation has two degrees of freedom less than the complex Dirac equation in its standard form and describes particle and antiparticles of opposite mean helicity. So using the real Dirac, i.e. the Majorana, equation implies and automatically ensures this reduction of the degrees of freedom (see the appendices Section 7 and Section 8). Throughout the remainder of our paper we will stay with the real description.
3 Symmetries and Lorentz transformation in Majorana representation

Now let us briefly discuss the symmetries of the Majorana equation (5). To prepare this we recall that there is an important symmetry operator, namely the chiral matrix operator $\gamma^5$, which (see for example [2]) is defined as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\bar{\gamma}^5$. Inserting the gamma matrices of equation (2) we obtain,

$$\bar{\gamma}^5 = \bar{\alpha}_x\bar{\alpha}_y\bar{\alpha}_z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (8)$$

Like the other gamma matrices, $\gamma^5$ is purely imaginary. As a result one can not, by standard projection, obtain real right- or left-chiral components of the real eigen-spinor solutions of the Majorana equation (5) as given by (42) and (43) in the appendix Section 7.

Obviously, the chirality operator $\bar{\gamma}^5$ by its definition (8) commutes with $\bar{\alpha}$.

But we require an operator like the $\tau$ of Section 2 which anticommutes with the alphas. The product $\bar{\beta}\bar{\gamma}^5 = \delta = \bar{\beta}\bar{\alpha}_x\bar{\alpha}_y\bar{\alpha}_z$ has that desired property and thus corresponds to $\tau$. So we define:

$$\bar{\delta} = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad (9)$$

the square of which is $-1$ (like the square of $\tau$), and which anticommutes with $\beta$ as well.

Concerning the symmetries of the Majorana equation, we consider in particular chirality conjugation $C$, parity $P$, and time reversal $T$ operations. Generally speaking the Majorana equation is invariant under the symmetry operation $O$, if the spinor $\psi^O = O\psi$ also fulfils that equation. Therefore, when applying the operation $O$ from the left and its inverse $O^{-1}$ from the right, whereby the unit operator is given by the decomposition $OO^{-1} = 1$, we obtain the result

$$\left(O \frac{\partial}{\partial t} O^{-1} + O(\bar{\alpha} \cdot \frac{\partial}{\partial x}) O^{-1} - mO\bar{\beta}O^{-1}\right) O\psi(x, t) = 0. \quad (11)$$

We conventionally define the time and space coordinate inversion operations $T$ and $P$ on a spinor $\psi$ by

$$T\psi(x, t) = \psi(x, -t), \quad (12)$$
$$P\psi(x, t) = \psi(-x, t), \quad (13)$$

and also recall the complex conjugation operation $C$, which gives $CiC^{-1} = -i$, and but which here has no role to play as everything is real. With these preparations in mind, it is easy to see which operators provide the requested symmetry operations. We compose them in the Table 1. To complete the


Table 1: Symmetry operations

| Operation     | Time reversal | Parity | Chirality conjugation |
|---------------|---------------|--------|-----------------------|
| Operator      | $\bar{\delta}T$ | $\beta P$ | $\bar{\delta}$ |

operator algebra, it is important to note the following commutation relations. Of course, the coordinate reversal operators $T$ and $P$ commute with $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\delta}$.

Let us first consider in (11) the time reversal, $O = T = \bar{\delta}T$. Apparently, it affects the mass term via $\bar{\beta}$ as well as $\bar{\alpha}$, where the signs are reversed, and also changes the sign of the first term. Therefore, also $\psi^T = \bar{\delta}\psi(x, -t)$ solves the Majorana equation (5). The parity operation $O = P = \beta P$ commutes with the mass term and does not affect the first term in (5), and it also leaves the momentum term invariant since $\alpha$ and $x$ both change signs together. Therefore, also $\psi^P = \beta\psi(-x, t)$ solves the Majorana equation. Finally, we consider chirality conjugation defined as $O = C = \bar{\delta}$. It changes the signs of the alpha and beta terms in (5) with no net effect on the time-derivate term. Therefore, $\psi^C = \bar{\delta}\psi(x, t)$ does not solve the Majorana equation, but in fact its chirality-conjugated version which is only derived in the appendix Section 8.

In conclusion, the symmetry operations of Table 1 work in a transparent way on the Majorana equation.

To obtain the spin operator of the Majorana field, we now consider the Lorentz group generators, which in its four-component spinor representation are known [5] to be given by the commutator $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, which yields in Majorana representation with the gamma matrices as given in (2) the subsequent matrix tensor:

$$\sigma^{\mu\nu} = \begin{pmatrix}
0 & i\bar{\alpha}_x & i\bar{\alpha}_y & i\bar{\alpha}_z \\
-i\bar{\alpha}_x & 0 & \Sigma_x & -\Sigma_y \\
-i\bar{\alpha}_y & -\Sigma_z & 0 & \Sigma_x \\
-i\bar{\alpha}_z & \Sigma_y & -\Sigma_x & 0
\end{pmatrix}.$$

(14)

Note that these tensor elements are $4 \times 4$ matrices. Thus we obtained by definition the spin operator $\Sigma$ of the Majorana field, which unlike the spin operators in the Dirac or Weyl representation, is not diagonal and purely imaginary. It reads:

$$\Sigma = i\Sigma = i\begin{pmatrix}
0 & -\beta \\
\beta & 0
\end{pmatrix}, \begin{pmatrix}
\gamma & 0 \\
0 & \gamma
\end{pmatrix}, \begin{pmatrix}
0 & -\alpha \\
\alpha & 0
\end{pmatrix}.$$

(15)

The spin operator does not commute with $\bar{\alpha}$ but with $\bar{\beta}$. Therefore its eigenfunctions are not eigenfunctions of the Majorana equation. Remember that $\bar{\alpha}$ in turn anticommutes with $\bar{\beta}$. Note further that the spin operator is of course hermitian, and as a result we have $\Sigma^T = -\Sigma$, whereas $\bar{\alpha}^T = \bar{\alpha}$. We also recall that the above spin matrices obey the same relations like the Pauli matrices, i.e. $[\Sigma_x, \Sigma_y] = 2i\Sigma_z$, where indices can be permuted in a cyclic way. So the Majorana spin operator can as usually be defined as $S_M = \frac{1}{2}\Sigma$, which forms the SU(2) Lie algebra, also generating the familiar SU(2) Lie group of
the weak interactions. Finally, by using the $\bar{\alpha}$ and $\bar{\Sigma}$ matrices, the Lorentz transformation operator $\Lambda(\vartheta, \varphi)$ turns out to be real, and the related matrix can then be written as:

$$S(\Lambda) = \exp\left(\frac{1}{2} \bar{\alpha} \cdot \vartheta + \frac{1}{2} \bar{\Sigma} \cdot \varphi\right),$$

where the vector angle $\vartheta$ refers to genuine Lorentz transformations and $\varphi$ to proper spatial rotations. Note that $S(\Lambda)^T \beta S(\Lambda) = \bar{\beta}$, a key feature which ensures Lorentz invariance of the Lagrangian, which is obvious for the mass term in particular in the Lagrange density (7).

### 4 Gauge field theory of flavour dynamics

We are now coming to the main theme of this paper, which is the question of Majorana (neutrino) gauge-field generated dynamics. Let us first consider an Abelian gauge field $A_\mu(x)$ like in electrodynamics. Conventionally, this is introduced into the field equation by replacing, according to the minimal coupling principle, the time-space derivative $\partial_\mu$ by the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu.$$ (17)

Concerning, the Majorana equation (12), this coupling causes a serious problem as the corresponding complex phase, like $\exp(ie\lambda(x))$, where $\lambda(x)$ is a function of space-time and $e$ the coupling constant (charge), in the wavefunction does not commute with the operator $\tau$. Fortunately, there is no such problem with (5), other than the solution spinor now has to be complex, which is not what we were aiming for to begin with. However, there is a remedy for getting rid of the unwanted imaginary unit, if we can find a non-Abelian symmetry group [22] the matrix representation of which is purely imaginary (and so electromagnetic interaction is excluded at the outset). The spin algebra as given by the Majorana spin operator according to (15) provides just the desired representation, which yet is not identical with the fundamental unitary representation of SU(2).

Therefore we will choose (with some coupling constant $g$) the connection involving the un-normed isospin operator defined by the matrix three-vector $S = i\bar{S}$ as follows:

$$D_\mu = \partial_\mu + igS \cdot A_\mu = \partial_\mu - g\bar{S} \cdot A_\mu,$$ (18)

which is real and thus still permits real solutions to be obtained for the interacting Majorana field and its spinors. The real matrix isospin three-vector reads explicitly:

$$S = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.$$ (19)
The spin matrix algebra leads for any real three-vectors $A$ and $B$ to the useful relation:

$$(S \cdot A)(\bar{S} \cdot B) = -A \cdot B + \bar{S} \cdot (A \times B).$$

(20)

As the Majorana matrix representation is four-dimensional, we have thus introduced in the Majorana field four new internal degrees of freedom related to the gauge field, which of course requires a physical interpretation. We return to this issue in the discussion section and here proceed formally. Note that $S$ acts on a four-component spinor

$$\Psi = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix}.$$  

(21)

Each single component relates to the Majorana equation (1) or (5). For the free field the eigenfunction $\psi$ is given in the appendix Section 6. The extended Majorana Lagrange density, while including the gauge-field interaction, now has two parts which read:

$$L_M + L_1 = i\Psi^T (\bar{\alpha}^\mu (\partial_\mu - g\bar{S} \cdot A_\mu) - \bar{\beta}m)\Psi.$$  

(22)

We can identify in (22) a term associated with the particle flux density. This interaction term can conventionally be written as

$$L_1 = -j^\mu \cdot A_\mu,$$  

(23)

where the real isospin current density generated by the SU(2)-like gauge symmetry is given by

$$j^\mu = gi\Psi^T \bar{\alpha}^\mu \bar{S} \Psi = g\Psi^T \bar{\alpha}^\mu S \Psi.$$  

(24)

The tensor of the gauge field strength is according to standard theory given by the commutator of the connection (18), i.e. we have

$$F_{\mu\nu} = i g [D_\mu, D_\nu] = i\bar{S} \cdot F_{\mu\nu},$$  

(25)

where use has been made of (20). Thus the antisymmetric field tensor (written as a normal three vector) is, again by help of (20), derived in the concise form:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + 2gA_\mu \times A_\nu.$$  

(26)

As is well known from non-Abelian gauge theory, the non-commuting nature of the vector components of the isospin $S$ yields the nonlinear terms in the field tensor (26). The Lagrangian of the present SU(2)-like gauge field (see text books) theory is defined by

$$L_F = -\frac{1}{16} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} (F_{\mu\nu} \cdot F^{\mu\nu}).$$  

(27)
The term on the right of (27) comes from the properties of the isospin matrices in (19), which also obey the relation: \( \text{Tr}(\bar{S}_i S_j) = -4\delta_{i,j} \). This produces the normalization factor 16 in \( L_F \) instead of 2 of the standard SU(2) gauge group in its fundamental representation.

The two combined equations (22) and (27) define the total Lagrangian, \( \mathcal{L} = L_F + L_I + L_M \), and govern the dynamics of the fermionic Majorana field and the associated bosonic gauge fields. Variation of the Yang-Mills [15] action four-integral over \( \mathcal{L} \) with respect to the relevant field variables finally yields the two real coupled partial-differential matrix equations:

\[
(\bar{\gamma}^\mu \partial_\mu + m)\Psi(x) = g\bar{\gamma}^\mu A_\mu(x) \cdot \bar{S}\Psi(x),
\]

\[
\partial_\mu F^{\mu\nu}(x) + 2g(A_\mu(x) \times F^{\mu\nu}(x)) = g\Psi^T(x)\bar{\alpha}^\nu S\Psi(x).
\]

Remember that \( \bar{\gamma}^\mu = \bar{\beta}\bar{\alpha}^\mu \). Both equations (28) and (29) have terms linear in the fields (free fields) and quadratic ones describing the field coupling. The gauge field equation (29) in addition contains a term quadratic in this field, a property leading to nonlinear interaction of the field with itself.

In the next section we will make some drastic approximations to the above complex system of coupled field equations, yet which provide an interesting simplified model which already seems to indicate the possibility of flavour mixing and oscillations at frequencies determined by the effective masses as induced by mean-gauge-field coupling.

5 Flavour mixing and neutrino oscillations

Let us consider a very much simplified version of the general gauge-field model equations (28) and (29). Assume a fixed location and time dependence only, and further consider a mean-field approach to the gauge fields, thereby assuming a static field, with the vector \( A_\mu(x) = (A, 0, 0) \) being constant. Then \( \bar{\gamma}^\mu A_\mu = \bar{\beta}A \). Dimensionally, the coupling constant \( g \) times the field \( A \) must correspond to a mass, and thus we may write \( gA \cdot S \) as a field-related mass matrix, which by help of (19) takes the form:

\[
\bar{M} = \begin{pmatrix}
0 & -M_y & -M_z & -M_x \\
+M_y & 0 & -M_x & +M_z \\
+M_z & +M_x & 0 & -M_y \\
+M_x & -M_z & M_y & 0
\end{pmatrix},
\]

which obeys \( \bar{M}^T = -\bar{M} \), where the superscript \( T \) indicates the transposed matrix, and which reflects the properties of the real Majorana field spin operator. Then the spinor time evolution is given by the mixing equation:

\[
\left( \bar{\beta}(\partial_t + \bar{M}) + m \right) \Psi(t) = 0.
\]
In this simple mean-field model, there are four masses corresponding to the basic mass $m$ and three others (inertia induced by mean-gauge-field coupling). The spinor solution involves oscillations or flavour mixing at the frequencies to be determined from the solution of the eigenvalue problem posed by equation (31).

We can rewrite it as a standard second-order oscillation equation as follows. Multiplying the equations, i.e. their wavefunctions, by $\bar{\beta}$ yields two coupled equations for $\Psi$ and $\bar{\beta}\Psi$, which can be inserted into each other to obtain a coupled oscillator (remember there are four flavour degrees of freedom) equation as follows:

$$\left(\frac{\partial^2}{\partial t^2} + 2\bar{M} \frac{\partial}{\partial t} - (M_x^2 + M_y^2 + M_z^2) + m^2\right) \Psi(t) = 0. \quad (32)$$

Without the field coupling we have independent harmonic oscillations with a phase $mt$, with it we obtain mixing, which is associated with frequency splitting and damping or excitation in dependence upon the mean static gauge-field vector $\bar{M} = (M_x, M_y, M_z)$.

We may think that the mass term could be replaced with a diagonal mass matrix representing different intrinsic masses of the four flavours, i.e. we may define

$$\bar{m} = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{pmatrix}. \quad (33)$$

However, such a matrix $\bar{m}$ does not commute with $\bar{M}$, and the SU(2) flavour gauge symmetry would be broken by adopting it. Therefore, we will not consider this case any further for the sake of simplicity. But we consider now spatial variations, given again a constant background gauge field, and add the spatial derivative term to describe spatial propagation, and thus obtain:

$$\left(\frac{\partial}{\partial t} + \bar{\alpha} \cdot \frac{\partial}{\partial x}\right) \Psi(x, t) = (\beta m + \bar{M})\Psi(x, t), \quad (34)$$

which differs from the free Majorana field \[9\] essentially by the gauge-field-induced mass mixing term. Considering again $\Psi$ and $\bar{\beta}\Psi$, and by eliminating one them, we obtain

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 2\bar{M} \frac{\partial}{\partial t} + \bar{M}^2 + m^2\right) \Psi(x, t) = 0. \quad (35)$$

Use has been made of the property $(\bar{\alpha} \cdot \mathbf{D})^2 = \mathbf{D}^2$ for any vector $\mathbf{D}$, and that the alphas and beta anticommute. Since the equation is real, we can assume a harmonic wave superposition and make for the solution the ansatz:

$$\Psi(x, t) = \mathcal{C} \cos(Et - \mathbf{p} \cdot \mathbf{x}) + \mathcal{S} \sin(Et - \mathbf{p} \cdot \mathbf{x}), \quad (36)$$

with four-component amplitude or flavour-polarisation spinors $\mathcal{C}$ and $\mathcal{S}$. Insertion of this ansatz yields the coupled set of equations:

$$(-E^2 + \mathbf{p}^2 + m^2 - M^2)\mathcal{C} - 2EM\mathcal{S} = 0, \quad (37)$$
\((-E^2 + p^2 + m^2 - M^2)\mathcal{S} + 2EM\mathcal{C} = 0,\) \hspace{1cm} (38)

where \(M = \sqrt{-\frac{1}{4}\text{Tr}(M^2)} = \sqrt{M^2}.\) For nontrivial solutions to exist the determinant of the system (37) and (38) must vanish, which yields for the energy eigenvalues the four roots:

\[E_{1,2} = \pm M + \sqrt{p^2 + m^2},\] \hspace{1cm} (39)

\[E_{3,4} = \pm M - \sqrt{p^2 + m^2}.\] \hspace{1cm} (40)

For zero gauge field one retains the normal relativistic energies of free particles. Apparently, the constant gauge field yields energy splitting and flavour mixing, as the new eigenstates are obtained by mixing of the free-particle eigenstates. The mixed state is determined by the flavour spinors \(\mathcal{C}\) and \(\mathcal{S},\) which we shall not calculate in detail. Owing to the application of the gauge field the four flavours acquired different energies in dependence on the field strength \(M,\) and thus the degeneracy of the original (just one mass \(m\)) energy spectrum has been lifted.

Finally, we recall that for free Majorana particles of each flavour species the polarization spinors depend on energy and momentum, according to the eigenfunctions given in appendix Section 7. For the free Majorana field, when being written like in (36) in terms of sine and cosine functions, the corresponding four-component spinors \(C\) and \(S\) are determined by the polarization spinor \(\tilde{\alpha},\) i.e. \(C = 1/\sqrt{2}\text{Re}\tilde{\alpha}\) and \(S = 1/\sqrt{2}\text{Im}\tilde{\alpha}\) for particles, and similarly for antiparticles. For them we just have to replace \(\tilde{\alpha}\) by \(\tilde{\beta},\) whose dependence on energy is given by the coefficient \(\varepsilon_{\pm}\) as defined in the appendix Section 7. The corresponding flavour polarization spinors are more complicated, though, and have in principle to be determined by finding for each energy the eigenvectors of the full sixteen-dimensional system given by equations (37) and (38). This tedious calculation shall not be done here.

6 Discussion and conclusion

A gauge theory for massive neutral particles has been developed on the basis of the real four-component Majorana equation. The novel aspect is that by use of its spin operator a purely imaginary representation of the SU(2) algebra can be defined, which provides a covariant derivative or connection to the gauge field that is real. Such a minimal coupling can preserve the real nature of the Majorana equation. The associated isospin is four-dimensional, and thus via this procedure we introduce four intrinsic degrees of freedom to the Majorana field. What could be the nature of these degrees of freedom?

The empirical fact that neutrinos oscillate and thereby change flavour has motivated us to make such a proposal. The main aim was to describe the mathematical feasibility of coupling the real Majorana field with a gauge field, a possibility that was not obvious and, to our best knowledge of the literature, not known before. Adding a fourth member to the flavour family, is here just a
matter of mathematical necessity and imposed by the choice of our isospin, but it remains physically speculative. However, flavour could in this way become a dynamic trait of the neutral Majorana particles, and thus lead to their linkage and a deeper understanding of neutrino mixing.

We know the standard-model fermions come in three flavours, yet the present model would imply another fourth flavour, beyond the electron, muon and tau, and their neutrinos, and perhaps similarly for the quarks as well. The four flavours result here from the dimension of the representation of the gauge group, and as such are a cogent consequence if we accept that present description at all. Its dimension is that of space-time in the covariant Dirac and Majorana equations, and thus given just by the dimension of the gamma matrices. The particle quantum states in these equations are ordered according to past and future (antiparticles and particles) and right-helical or left-helical. The isospin, when being based on the spin operator that normally describes rotations in three-dimensional space, may correspond to the remaining four spatial orientations, which is forward and backward and up and down. This may be considered a simple intuitive interpretation of the physical content of the 16-component spinor $\Psi$, which arranges each flavour family into a quadruplett of real Majorana spinors for neutrinos, or perhaps complex Dirac spinors for charged leptons or quarks. So the sixteen components correspond to the sixteen ($2^4$) signed domains of space-time, and thus the number of flavours must be four but cannot be higher.

The simple static mean-field approximation of the gauge field can produce Majorana-field flavour oscillations and mixing, can lift the mass degeneracy of the four Majorana neutrinos and cause a splitting of their mass spectrum, the size of which is determined by the gauge field strength. So mass is partly acquired here from gauge-field energy. This simple model just illustrates the physical potential of a flavour-mixing gauge theory of massive neutrinos.

7 Appendix I: Eigenfunctions of the free Majorana equation

Let us discuss briefly the eigenfunctions of the free Majorana field, which obeys equation (3) or (4) and can be decomposed into its particle and antiparticle components, and according to the recent papers of Marsch [19, 20] be written

$$\psi(x, t) = \psi_P(x, t) + \psi_A(x, t).$$  \hspace{1cm} (41)

These two contributions can be expressed, in terms of the complex four-component polarization spinors to be defined below, separately as follows:

$$\psi_P(x, t) = \exp(-iEt + ip \cdot x)a(p)\tilde{\alpha}(p) + \exp(iEt - ip \cdot x)a^*(p)\tilde{\alpha}^*(p),$$  \hspace{1cm} (42)

$$\psi_A(x, t) = \exp(-iEt + ip \cdot x)b(p)\tilde{\beta}(p) + \exp(iEt - ip \cdot x)b^*(p)\tilde{\beta}^*(p).$$  \hspace{1cm} (43)
Apparently, \( \psi_{p,A} = \psi_{p,A}^\dagger \), and thus the wave functions are real. The polarization spinors \( \tilde{\alpha} \) and \( \tilde{\beta} \) are the two eigenfunctions of the helicity operator \( \boldsymbol{\Sigma} \cdot \mathbf{p} \) (as defined in Section 3) with eigenvalues +1 and -1.

\[
(\boldsymbol{\Sigma} \cdot \mathbf{p}) \tilde{\alpha}(\mathbf{p}) = +\tilde{\alpha}(\mathbf{p}).
\]

\[
(\boldsymbol{\Sigma} \cdot \mathbf{p}) \tilde{\beta}(\mathbf{p}) = -\tilde{\beta}(\mathbf{p}).
\]

After quantization the complex Fourier amplitudes \( a^*(\mathbf{p}) \) and \( b^*(\mathbf{p}) \) turn into creation operators of particles and antiparticles, which on average (in the sense of a quantum-field expectation value the Majorana field) have opposite helicities. For the sake of completeness we give here the full four-component polarization spinors, which in terms of the angles of the momentum unit vector \( \mathbf{p} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) and of the module \( p \) read as follows:

\[
\tilde{\alpha}(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} (+\varepsilon_+(p) \cos \frac{\phi}{2} e^{-\frac{i}{2} \phi} + \varepsilon_-(p) \sin \frac{\phi}{2} e^{\frac{i}{2} \phi}) (1 - i) \\ (+\varepsilon_+(p) \sin \frac{\phi}{2} e^{\frac{i}{2} \phi} - \varepsilon_-(p) \cos \frac{\phi}{2} e^{-\frac{i}{2} \phi}) (1 - i) \\ (+\varepsilon_+(p) \cos \frac{\phi}{2} e^{-\frac{i}{2} \phi} - \varepsilon_-(p) \sin \frac{\phi}{2} e^{\frac{i}{2} \phi}) (1 + i) \\ (+\varepsilon_-(p) \sin \frac{\phi}{2} e^{\frac{i}{2} \phi} + \varepsilon_-(p) \cos \frac{\phi}{2} e^{-\frac{i}{2} \phi}) (1 + i) \end{pmatrix}.
\]

\[
\tilde{\beta}(\mathbf{p}) = \frac{1}{2} \begin{pmatrix} (-\varepsilon_-(p) \sin \frac{\phi}{2} e^{-\frac{i}{2} \phi} + \varepsilon_+(p) \cos \frac{\phi}{2} e^{\frac{i}{2} \phi}) (1 - i) \\ (-\varepsilon_-(p) \cos \frac{\phi}{2} e^{\frac{i}{2} \phi} + \varepsilon_+(p) \sin \frac{\phi}{2} e^{-\frac{i}{2} \phi}) (1 - i) \\ (-\varepsilon_-(p) \sin \frac{\phi}{2} e^{\frac{i}{2} \phi} - \varepsilon_+(p) \cos \frac{\phi}{2} e^{-\frac{i}{2} \phi}) (1 + i) \\ (+\varepsilon_-(p) \cos \frac{\phi}{2} e^{-\frac{i}{2} \phi} - \varepsilon_+(p) \sin \frac{\phi}{2} e^{\frac{i}{2} \phi}) (1 + i) \end{pmatrix}.
\]

The squares of which add up to unity, \( \varepsilon_+^2 + \varepsilon_-^2 = 1 \). The relativistic energy of a particle is \( E(\mathbf{p}) = \sqrt{m^2 + p^2} \). A useful property of the epsilons is the obvious relation \( (E \pm p) \varepsilon_{\mp} = m \varepsilon_{\pm} \). Using their properties, one can readily show that \( \tilde{\alpha}^\dagger \tilde{\alpha} = 1 \) and \( \tilde{\beta}^\dagger \tilde{\beta} = 1 \), respectively, \( \tilde{\alpha}^\dagger \tilde{\beta} = 0 \) and \( \tilde{\beta}^\dagger \tilde{\alpha} = 0 \). If \( m = 0 \), then \( \varepsilon_+ = 1 \) and \( \varepsilon_- = 0 \). It should be emphasized that the above spinors \( \psi_A \) and \( \psi_P \) always are a superposition of both helicity states, as it is required for a massive relativistic particle. The eigenvalue equation of the helicity operator (based on the spin operator, see Section 3) in Fourier space reads for the complex conjugated polarization spinor:

\[
(\boldsymbol{\Sigma} \cdot \mathbf{p}) \tilde{\alpha}^*(\mathbf{p}) = -\tilde{\alpha}^*(\mathbf{p}).
\]

Similarly, for \( \tilde{\beta}^*(\mathbf{p}) \), which has the opposite positive helicity, we obtain:

\[
(\boldsymbol{\Sigma} \cdot \mathbf{p}) \tilde{\beta}^*(\mathbf{p}) = +\tilde{\beta}^*(\mathbf{p}).
\]

By multiplying out explicitly all possible scalar products involving the two complex conjugated polarization vectors \( \tilde{\alpha}^*(\mathbf{p}) \) and \( \tilde{\beta}^*(\mathbf{p}) \), one finds further that
\[ \alpha^\dagger \alpha^* = (\alpha^T \alpha)^* = 0 \quad \text{and} \quad \beta^\dagger \beta^* = (\beta^T \beta)^* = 0, \]
\[ \text{but} \quad \beta^\dagger \alpha^* = (\beta^T \alpha)^* = i(\epsilon_2^+ - \epsilon_2^-), \]
and similarly \[ \beta^\dagger \beta^* = (\beta^T \beta)^* = 0, \]
but \[ \alpha^\dagger \beta^* = (\beta^T \alpha)^* = i(\epsilon_2^+ - \epsilon_2^-), \]
Consequently, there are four independent real polarization vectors spanning the full Hilbert space of the Majorana equation. The eigenvalues \[ \pm E(p) \]
are two-fold degenerate, and their subspaces are spanned by \( \tilde{\alpha}(p) \) and \( \alpha^*(p) \), respectively \( \tilde{\beta}(p) \) and \( \beta^*(p) \). Both helicities are needed because of the finite mass of the relativistic Majorana particles. However, the degeneracy concerning the replacement of the spin \( \Sigma \) by its negative (spin reversal) has been lifted, because the real Majorana equation does not obey chirality conjugation (see again Section 3).

8 Appendix II: Two Majorana equations

As was mentioned already in the main text, the two-component complex and four-component real Majorana equations are fully equivalent. However, as discussed by Marsch [19, 20], there is a second chirality-conjugated equation obtained by operating with \( \tau \) on (4), which yields
\[ i \left( \frac{\partial}{\partial t} - \sigma \cdot \frac{\partial}{\partial x} \right) \chi(x, t) = -m \tau \chi(x, t), \]
with \( \chi = \tau \phi \), which is the left-chiral counterpart to the right-chiral field \( \phi \). This equation can be obtained by simply inverting the sign of the Pauli matrices in (4), including of course of \( \tau \) as it contains \( \sigma_y \). The \( \pm \) signs in front of the Pauli matrices reflect the reducibility of the Lorentz group, which can in fact be decomposed into its left- and right-chiral components. In the chiral or Weyl representation the charge conjugation operator is given by \( C = \gamma \tau C = \gamma \tau \), which when being squared equals unity. So its eigenvalues are \( \pm \), and its eigenfunctions obey \( C \psi^\pm = \pm \psi^\pm \). The eigenfunction with positive eigenvalue obeys the Dirac equation and in Weyl representation is given by the spinor:
\[ \psi^C = \left( \begin{array}{c} \phi \\ \tau \phi \end{array} \right) = \left( \begin{array}{c} \phi \\ \chi \end{array} \right). \]
Insertion of this spinor yields for the two two-component spinors \( \phi \) and \( \chi \) the twin Majorana equations (4) and (51), which are coupled through the restricting condition \( \chi = \tau \phi \), which according to (52) guarantees chirality self-conjugation.

Correspondingly, we obtain a second real four-component Majorana equation by simply taking the negatives of the matrices \( \alpha, \beta, \gamma \) of equation (3). This choice transposes the vector \( \vec{\alpha} \) of (6) to its negative and also gives a negative \( \vec{\beta} \), so that we obtain the second real Majorana equation as
\[ \left( \frac{\partial}{\partial t} - \vec{\alpha} \cdot \frac{\partial}{\partial x} \right) \tilde{\psi}(x, t) = -m \vec{\beta} \tilde{\psi}(x, t). \]
The solution is according to the Table 1 given by the chirally conjugated spinor \( \tilde{\psi}(x, t) = \delta \tilde{\psi}(x, t) \), where \( \psi \) solves the Majorana equation (5). Note that both
Majorana equations obey the parity and time-reversal symmetry, but individually break by construction chirality conjugation which links them together. Concerning the spin operator \( \gamma^5 \), it will also be transposed for the second conjugated Majorana field and turn into the negative of the primary field, i.e. \( \tilde{\mathbf{S}} = -\mathbf{S} \). However, the connection to the gauge field after \((18)\) will not change, as once we have made our choice of the symmetry group their generators are given and do not undergo the discussed space-time symmetry operations.

Returning to the twin two-component Majorana equations \((4)\) and \((51)\), we recall that in the chiral Weyl representation of the Dirac equation the right- and left-chiral field are obtained by projection with the help of the chirality operator \( \gamma^5 \). Correspondingly, the Dirac field can be decomposed into its right- (index \( R \)) and left-chiral (index \( L \)) components, such that

\[
\psi_R = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_R \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \phi_L \end{pmatrix}, \tag{54}
\]

where we introduced the front factor for normalization, assuming that the functions \( \psi_R \) and \( \psi_L \) are individually normalized to unity. Above we used the symbols \( \phi_R = \phi \) and \( \phi_L = \chi \), with the chirality self-conjugation constraint that \( \chi = \tau \phi \) to be kept in mind. Therefore the mass term mixing the two chiral components, such that in the Lagrange density we have \( m\bar{\psi}\psi = m/2(\phi^\dagger \chi + \chi^\dagger \phi) \), can formally be made diagonal by exploiting the above condition. The price to be paid for this decoupling is that the mass term becomes instead of trivial multiplication a nontrivial operator involving with \( \tau \) also the inconvenient complex-conjugation operation \( C \). As the result we get the Lagrange density

\[
\mathcal{L} = \frac{1}{2} \left( \phi_R^\dagger (i\sigma_\mu \partial_\mu - m\tau_R) \phi_R + \phi_L^\dagger (i\sigma_\mu \partial_\mu - m\tau_L) \phi_L \right). \tag{55}
\]

It can be shown (see the paper by Pal \[23\]) to be hermitian, while the two terms being their hermitian conjugates, which can be validated by using the relations \( \phi_R = -\tau \phi_L \), respectively \( \phi_L = \tau \phi_R \), which are constitutive for the complex two-component Majorana field. Here we defined for the sake of formal symmetry the symbols: \( \tau_R = \tau, \tau_L = -\tau \), with \( \tau = \sigma_x \mathbb{C} \), and \( \sigma_R^\mu = (1, -\sigma) \) and \( \sigma_L^\mu = (1, \sigma) \). However, with \( \sigma^\mu = \sigma_R^\mu \) we may also rewrite \((55)\) in terms of \( \phi = \phi_R \) as follows

\[
\mathcal{L} = i \text{Im} \left( \phi^\dagger (i\sigma^\mu \partial_\mu - m\tau) \phi \right), \tag{56}
\]

which emphasizes that there is only a single two-component complex Majorana field.

The result of equation \((55)\) can now be directly transferred to the real Majorana equation, following the same mathematics that lead to equation \((5)\), respectively \((53)\). By decomposing the two parts of \((55)\) into their real and imaginary parts and writing the density in terms of four-component real spinors after \[20\], the resulting alpha and beta matrices will of course change correspondingly, as we obtained \( \tilde{\alpha}_R = \tilde{\alpha} \) and \( \tilde{\alpha}_L = -\tilde{\alpha} \), respectively, \( \tilde{\beta}_R = \tilde{\beta} \) and \( \tilde{\beta}_L = -\tilde{\beta} \). By definition, we obtain \( \tilde{\sigma}_R^\mu = (1, \tilde{\alpha}) \) and \( \tilde{\sigma}_L^\mu = (1, -\tilde{\alpha}) \). Consequently, \( \tilde{\alpha}_R^\mu = \tilde{\alpha}_L^\mu \).
which shows how chirality and space-time are intrinsically connected. We can then write the combined Majorana Lagrange density as

$$L = \frac{i}{2} \left( \bar{\Psi}_R^T (\bar{\alpha}_\mu \partial_\mu - \beta_R m) \Psi_R + \bar{\Psi}_L^T (\bar{\alpha}_\mu \partial_\mu - \beta_L m) \Psi_L \right).$$

(57)

This equation shows a formal symmetry between the left-chiral and right-chiral components of the Majorana fields which at first sight seem to be independent. However, they are not, as we have $\psi_L = \bar{\delta} \psi_R$, and vice versa $\psi_R = -\bar{\delta} \psi_L$. Also recall that $\{\delta, \beta\} = 0$ and $\{\delta, \alpha\} = 0$. Note that by its definition $\delta^T = -\delta$ and $\delta^2 = -1$. So by inserting $\psi_L = \bar{\delta} \psi_R$ into the above Lagrangian, the second term turns out to be exactly equal to the first. So there is only a single real Majorana field the Lagrangian of which was already given in (7). Therefore, considering the real Majorana field involving real four-component spinors, we believe, is advantageous over the complex Majorana field involving two-component complex spinors and requires to take care of the mathematical subtlety and complication of the operator $\tau$.

9 Appendix III: Chiral decomposition of the spin operator

Concerning the possible choice of the gauge symmetry group, traditionally $\text{SU}(N)$ is used with the prominent $\text{SU}(2)$ and $\text{SU}(3)$ Lie groups employed in their fundamental representations for the weak and strong interactions. However, the real $\text{SU}(2)$ representation chosen in (19) is special in so far as it corresponds to the physical quantity angular momentum of the Majorana field. So there is no need to use left- or right-chiral Weyl fields, if the neutrino is assumed to be a massive Majorana particle, which by its very nature comes as a left-helical particle and right-helical antiparticle. The Majorana spin operator $S_M$ was adopted as isospin in the present work. It can be written in terms of the Pauli matrices in the form:

$$S_M = \frac{1}{2} \left( \begin{pmatrix} 0 & -i\sigma_x \\ i\sigma_x & 0 \end{pmatrix}, \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, \begin{pmatrix} 0 & -i\sigma_z \\ i\sigma_z & 0 \end{pmatrix} \right).$$

(58)

We like to discuss then how to couple Dirac fermions (if also carrying the charge $g$) to the same gauge field as discussed in Section 3. The procedure how to do this is not at all obvious. Like in the weak interactions, perhaps a projection onto chiral eigenfunctions is needed for Dirac fermions. This is rather speculative and needs further investigations. As is well known the spin operator of the Dirac equation is given, both in Dirac and Weyl representation, by the expression:

$$S_D = \frac{1}{2} \left( \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}, \begin{pmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{pmatrix}, \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \right).$$

(59)

which is reducible and fully described by the fundamental $\text{SU}(2)$ group as defined by the Pauli matrices. Here we want to mention that the spin can be
decomposed by means of the chiral operator $\gamma^5$ and its related projection operators $P_{R,L} = 1/2(1 \pm \gamma^5)$. They are idempotent, i.e. $P_{R,L}^2 = P_{R,L}$, provide a decomposition of unity: $P_R + P_L = 1$, and are orthogonal by construction: $P_R P_L = 0$. Furthermore, $\gamma^5$ commutes with the spin operator, which is derived from the commutator of two gamma matrices after (14). In fact it commutes with $S$ in any representation. Therefore, we can write for the spin operator $S$, which obeys $S \times S = iS$, formally

$$ S = (P_R + P_L)S = S_R + S_L, \quad (60) $$

and consequently obtain $S_{R,L} \times S_{R,L} = iS_{R,L}$, and $[S_R, S_L] = 0$. For the Majorana field the spin operator (13) is purely imaginary and the projected components become complex, which is not helpful. In the chiral Weyl representation we simply get $S_{R,L} = P_{R,L}\sigma/2$, which is irreducible and simplest. So the question comes up which the adequate representation is to be used for the SU(2) symmetry group.

If we assume that any Dirac fermion is like a Majorana fermion endowed with four internal flavour degrees of freedom, then its interaction is mediated by the same gauge field $A_\mu$ considered for the Majorana fermion. We may take the isospin to be given consistently by the $S$ of (19), and the coupling to the field be given by the same connection (18). However, we may prefer $S = S_D$ to be used in the covariant derivative, i.e. favour the connection

$$ D_\mu = \partial_\mu + igS_D \cdot A_\mu, \quad (61) $$

in the Dirac equation. Formally, such a gauge-field interaction model seems to resemble the weak interaction theory [16, 17, 18] of the standard model, yet there are three major differences. First, when accepting the connection (61) for Dirac fermions, there is no restriction of the gauge-field-coupling to the left-chiral fermion field components only. Such a constraint need not be put anyway on a massive Majorana field, since it is chirally irreducible and by definition reduced to two degrees of freedom in comparison with a Dirac fermion. Secondly, both fermion species can be massive (with a single mass for all flavours), implying though that chiral symmetry is not obeyed. Thirdly, by construction of such a model, it will lead to lepton mixing and gauge-field mediated interactions among all fermions with different flavour, which are here assembled into unconstrained quadruplets, but not into doublets or singlets defined by chiral projection.

Therefore, concerning possible choices of the isospin operator, we could make use of the individual spins as derived from the genuine but different gamma matrices in the Majorana and Dirac representations. Correspondingly, the covariant derivative would employ the respective spin operator of the field considered, which is then used as isospin providing the coupling to the common gauge field. This appears to be mathematically feasible, yet the physical implications remain unclear. To discuss these issues in depth is beyond the scope and intention of the present paper.
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