Quantum computing and polynomial equations over the finite field $\mathbb{Z}_2$

Christopher M. Dawson,1,2∗∗ Henry L. Haselgrove,1,3 † Andrew P. Hines,1,2 ‡ Duncan Mortimer,1,2 † Michael A. Nielsen,1,4 ‡ and Tobias J. Osborne5‡

1School of Physical Sciences, The University of Queensland, Brisbane, Queensland 4072, Australia
2Centre for Quantum Computer Technology, The University of Queensland, Brisbane, Queensland 4072, Australia
3Information Sciences Laboratory, Defence Science and Technology Organisation, Edinburgh 5111, Australia
4School of Information Technology and Electrical Engineering, The University of Queensland, Brisbane, Queensland 4072, Australia
5School of Mathematics, University of Bristol, University Walk, Bristol BS8 1TW, United Kingdom

(Dated: April 1, 2022)

What is the computational power of a quantum computer? We show that determining the output of a quantum computation is equivalent to counting the number of solutions to an easily computed set of polynomials defined over the finite field $\mathbb{Z}_2$. This connection allows simple proofs to be given for two known relationships between quantum and classical complexity classes, namely $\text{BQP} \subseteq \text{PP}$ and $\text{BQP} \subseteq \text{PSPACE}$.

I. INTRODUCTION

Quantum computers have stimulated great interest due to their promise of being able to solve problems considered infeasible on conventional classical computers.1,2 This interest has led to rapid developments in physics, mathematics, and computer science.3,4 One of the central open problems in quantum computation is to precisely characterize the power of quantum computers, i.e., what problems they can and cannot solve efficiently.

In this paper we show that determining the output of a quantum computation is equivalent to counting the number of solutions to certain sets of polynomial equations over the finite field $\mathbb{Z}_2$. Equivalently, in the language of algebraic geometry, this means counting the number of points in an algebraic variety. The proof combines Feynman’s sum-over-paths formulation of quantum mechanics5,6 with a description of quantum computing in terms of a universal set of quantum gates specially chosen to make the sum-over-paths take a simple form.

This reformulation of quantum computation is interesting for several reasons. First, it reveals a connection between quantum computation and one of the central problems in algebraic geometry. Indeed, much of the development of modern algebraic geometry7,8 has been driven by the problem of counting the points in an algebraic variety, e.g., this problem gave rise to the well-known Weil conjectures9. Second, it reveals a connection between quantum computation and computational complexity. In particular, computational complexity theorists have shown that the problem of counting solutions to polynomials over finite fields is $\#\text{P}$-complete10,11.

As a consequence of this connection, our result has as a corollary a simple proof of one of the sharpest known results relating quantum and classical complexity classes12,13. $\text{BQP} \subseteq \text{PP}$, where $\text{BQP}$ is, informally, the class of decision problems efficiently solvable on a quantum computer. The complexity class $\text{P}^\#\text{P}$ is, in turn, a subset of the well-known complexity class $\text{PSPACE}$ of problems requiring polynomial space (but possible exponential time) to solve on a classical computer. Proving $\text{P} \neq \text{PSPACE}$ would represent a major breakthrough in classical complexity theory. A consequence of the result $\text{BQP} \subseteq \text{PSPACE}$ is that any proof that quantum computers are more efficient than classical computers will imply $\text{P} \neq \text{PSPACE}$, and thus would have major implications for classical computational complexity.

Our techniques also imply a simple proof of a result even sharper than $\text{BQP} \subseteq \text{P}^\#\text{P}$, namely $\text{BQP} \subseteq \text{PP}$. To our knowledge, this is the sharpest known relation between $\text{BQP}$ and a natural classical complexity class12.

As described above, our approach is based on the sum-over-paths formulation of quantum mechanics. Interestingly, the papers just mentioned12,13,14 all use variants of the sum-over-paths formulation to obtain their relations between $\text{BQP}$ and various classical complexity classes. This is also true of the paper by Knill and Laflamme15 which connects quantum computation to the problem of estimating quadratic weight enumerators. What all these papers share in common is that in evalu-

1 For a general overview of computational complexity theory, see12. For definitions and references on all the complexity classes we consider here, and many others, an excellent reference is Aaronson’s “Complexity Zoo”13.

2 A stronger relation, $\text{BQP} \subseteq \text{AWPP}$ has been proved by Fortnow and Rogers14. However, as they note, AWPP is a rather artificial complexity class.
II. EXAMPLE OF HOW TO FIND THE POLYNOMIALS

In this section, we provide an example showing how to calculate a transition amplitude for a quantum circuit by counting the number of solutions to sets of polynomial equations. We explain the method with the aid of a simple example quantum circuit, but delay the general proof until the next section.

All our results rely on constructing circuits out of certain specially chosen sets of universal gates. For our initial discussion we will use the Toffoli and Hadamard gates, which have been shown to be universal for quantum computation by Shi \cite{Shi09} and Aharonov \cite{Aharonov90}. Later we'll discuss the general properties of a gate set necessary to make our style of argument work. Recall that a Toffoli gate has action on computational basis states given by \(|x, y, z\rangle \rightarrow |x, y, z \oplus xy\rangle\). The single-qubit Hadamard gate maps \(|0\rangle\) to \((|0\rangle + |1\rangle)/\sqrt{2}\) and \(|1\rangle\) to \((|0\rangle - |1\rangle)/\sqrt{2}\).

Suppose we are given some \(N\)-qubit quantum circuit, constructed from Toffoli and Hadamard gates. A simple example for \(N = 3\) is shown in Fig. 1. Imagine, further, that we wish to calculate the matrix element \(<b|U|a\rangle\), where \(U\) is the unitary action of the quantum circuit, and where \(|a\rangle\) and \(|b\rangle\) are computational basis states (that is, \(|a\rangle = |a_0, a_1, \ldots, a_{N-1}\rangle\), where \((a_0, a_1, \ldots, a_{N-1}) \in \{0, 1\}^N\) is a bit string, and similarly for \(|b\rangle\)).

To determine \(<b|U|a\rangle\) we will define the notion of a set of allowed or admissible classical paths through the circuit, from input \((a_0, a_1, \ldots, a_{N-1})\) to output \((b_0, b_1, \ldots, b_{N-1})\), and the corresponding phases associated with each of those paths. To define the allowed classical paths, we first define what we shall call a classical version of the quantum circuit in Fig. 1. This is a classical circuit which is formed by replacing each of the qubits in Fig. 1 with a classical bit, by replacing the quantum Toffoli gate by a classical Toffoli gate, which takes \((a_1, a_2, a_3)\) to \((a_1, a_2, a_3 \oplus a_1 a_2)\), and by replacing each Hadamard gate by what we shall call a classical Hadamard gate, which, regardless of its input, may output either a zero or a one. To describe this situation we introduce a path variable \(x \in \mathbb{Z}_2\) to denote the output of the Hadamard:

$$a_1 \xrightarrow{H} x.$$  \hspace{1cm} (1)

A classical path is then a sequence of classical bit strings, \(a, a', a'', \ldots\), with the respective bit strings corresponding to the state of the classical circuit after each gate has been applied. A classical path is said to be allowed or admissible if there exists a choice of the path variables

\[^3\] Note that the Hadamard and Toffoli gates have only real matrix elements, so cannot generate a dense subset of the full unitary group. The universality proofs of \cite{Shi09, Aharonov90} use simple encodings to achieve universality.
by the following sum over allowed paths from elements of the polynomial ring \( \mathbb{Z}^2 \) over all (classical) Hadamard gates, of the product of a little mysterious at first; its importance will become path with path variables \( \phi \). Fig. 2 shows all allowed classical paths from the input string \((a_1, a_2, a_3)\), as a function of the path variables \((x_1, \ldots, x_4) \equiv x\). Note that there are \(2^4 = 16\) admissible paths corresponding to all the possible bit assignments of the variables \( x \). The output bit values, denoted \( B_j(x) \), are polynomial functions of the \( x_j \), i.e., elements of the polynomial ring \( \mathbb{Z}_2[x] \). In this example, \( B_1(x) = x_3 \oplus x_2 x_4 \), \( B_2(x) = x_2 \), and \( B_3(x) = x_4 \).

We now define the phase \( \phi(x) \) of an allowed classical path with path variables \( x \). This definition may appear a little mysterious at first; its importance will become clearer below. We define the phase to be the sum (modulo 2) over all (classical) Hadamard gates, of the product of the input and output bit values of the gate:

\[
\phi(x) = \sum_{\text{Hadamard gates}} \text{(input value)(output value)}. \tag{2}
\]

\(\phi(x)\) is a polynomial function of the \( x_j \). In this example \( \phi(x) = a_1 x_1 \oplus a_2 x_2 \oplus a_3 x_3 \oplus x_4 (a_3 \oplus x_1 x_2) \).

It turns out that the matrix element \( \langle b | U | a \rangle \) is given by the following sum over allowed paths from \( a \) to \( b \):

\[
\langle b | U | a \rangle = \frac{1}{\sqrt{2^h}} \sum_{x : B(x) = b} (-1)^{\phi(x)}. \tag{3}
\]

We will prove this explicitly in the next section; however, we hope this expression is at least plausible to the reader, expressing the transition amplitude as a sum over the allowed paths through the circuit, with amplitudes of the appropriate magnitude and phase. The terms in the sum all have the same absolute value, but vary in sign. We define \( \#(0) \) and \( \#(1) \) to be the number of positive and negative terms in the sum respectively. That is,

\[
\#(0) = | \{ x : B(x) = b \text{ and } \phi(x) = 0 \} |, \tag{4}
\]

and

\[
\#(1) = | \{ x : B(x) = b \text{ and } \phi(x) = 1 \} |. \tag{5}
\]

Eq. (3) can now be written as

\[
\langle b | U | a \rangle = \frac{1}{\sqrt{2^h}} [\#(0) - \#(1)]. \tag{6}
\]

The expressions in Eqs. (4) and (5) each count solutions to a system of \( N + 1 \) polynomials in \( h \) variables over the field \( \mathbb{Z}_2 \). We can make some general remarks about the properties of these polynomials. If we ensure that each Toffoli gate is followed by a Hadamard gate on the target line (if necessary by inserting pairs of Hadamard gates, which act as the identity gate), then the polynomials \( B_j(x) \) will have at most two terms and have order at most two, and \( \phi(x) \) will have at most \( 2h \) terms and order at most three.

As a simple example of this method in action, let’s calculate \( \langle 0 | U | 0 \rangle \) in the example circuit. \( B(x) = 0 \) has two solutions, \( x_1 = 0 \) or 1, \( x_2 = 0 \), \( x_3 = 0 \), \( x_4 = 0 \). For each solution \( \phi(x) = 0 \), so \( \#(0) = 2 \) and \( \#(1) = 0 \). Thus, \( \langle 0 | U | 0 \rangle = 2/\sqrt{16} = 1/2 \), which is easily verified to be the correct amplitude.

### III. General Proof

In this section we prove that the method in the previous section works for any quantum circuit made from Hadamard and Toffoli gates. To do this, we express the unitary action \( U \) of the circuit as a product of gates:

\[
U = U^{(M)} U^{(M-1)} \ldots U^{(2)} U^{(1)}. \tag{7}
\]

Each of the gates \( U^{(m)} \) represents either a Hadamard acting on one qubit, or a Toffoli gate acting on three qubits. Each \( U^{(m)} \) will thus act as an identity on all but one or three of the qubits.

For the general case in Eq. (4), it is clear that we can write

\[
\langle b | U | a \rangle = \sum_{c,d,\ldots,z} \langle b | U^{(M)} | z \rangle \langle z | U^{(M-1)} | y \rangle \ldots \langle d | U^{(2)} | c \rangle \langle c | U^{(1)} | a \rangle \tag{8}
\]

where \( c, \ldots, z \) are each bit strings of length \( N \). Eq. (8) expresses the transition amplitude of the entire circuit from the initial state \( a \) to the final state \( b \) as a sum of amplitudes over all sequences of computational states \( a \rightarrow c \rightarrow d \rightarrow \ldots \rightarrow z \rightarrow b \) through the circuit. For each of these sequences, the corresponding term in the sum is given by a product of contributions from each of the gates \( U^{(m)} \). This is already very close to the “sum over classical paths” approach used in the previous section. To complete the connection, we must first show that the sum in Eq. (8) may be restricted to the set of allowed classical paths as defined in the previous section, and second, that those remaining terms in the sum are equal to the terms in Eq. (8).

Consider a factor \( \langle s | U^{(m)} | r \rangle \) from Eq. (8). Say that \( U^{(m)} \) acts as the identity on qubit \( k \). Then, if \( s_k \neq r_k \), the factor \( \langle s | U^{(m)} | r \rangle \) will be zero. Suppose that this is not the case for any of the qubits on which \( U^{(m)} \) acts as the identity, and that \( U^{(m)} \) is a Toffoli gate acting on qubits \( k_1, k_2 \) and \( k_3 \). From the definition of the Toffoli gate, \( \langle s | U^{(m)} | r \rangle \) will equal one if \( s_{k_1} = r_{k_1} \), \( s_{k_2} = r_{k_2} \), and \( s_{k_3} = r_{k_3} \), and will equal zero otherwise. Alternatively, suppose \( U^{(m)} \) is a Hadamard gate acting on qubit \( k \). Then, from the definition of the gate, \( \langle s | U^{(m)} | r \rangle \) will equal \(-1/\sqrt{2} \) if \( s_k = r_k = 1 \) (that is, if \( s_k r_k = 1 \)), and will equal \( 1/\sqrt{2} \) otherwise.

From these observations, we see that the terms in Eq. (8) are nonzero only when \( a \rightarrow c \rightarrow \ldots \rightarrow z \rightarrow b \)
is an allowed classical path. Furthermore, the absolute value of any nonzero term will be equal to $1/\sqrt{2^m}$, where $h$ is the number of the gates $U^{(m)}$ that are Hadamards. Further, the sign of such a term will be $(-1)^p$, where $p$ is defined by summing over the product of the input and output values to each classical Hadamard gate. Thus, we have proven Eq. \ref{eq:3} and hence Eq. \ref{eq:6}, as required.

IV. APPLICATION TO DECISION PROBLEMS AND COMPUTATIONAL COMPLEXITY

The results described in the previous section have interesting consequences in the special case when the quantum computer is being used to solve a decision problem, i.e., a problem where the answer is either “yes” or “no”. Many interesting problems are either explicitly decision problems, e.g., satisfiability, or can be recast as equivalent decision problems, e.g., factoring and the travelling salesman problem.

A priori it is not obvious that the ability to calculate, even extremely efficiently, the matrix elements of quantum circuits is particularly useful. This is because there are exponentially many matrix elements for a given quantum circuit. It would appear that even a constant-time recipe to calculate matrix elements would, at best, provide an exponential method to estimate the output of a given quantum circuit. However, following \cite{20}, we now show that when a quantum circuit is being used to solve a decision problem, the output of the quantum circuit can be inferred from knowledge of one fixed matrix element.

Consider a quantum circuit $U_x$ for an instance $x$ of an arbitrary decision problem. We suppose for now that the quantum circuit is deterministic, i.e., it gives the correct output with probability 1. We may also suppose, without loss of generality, that the output (“no” or “yes”) is indicated by the value of the first output qubit $Q_1$, as $|0\rangle$ or $|1\rangle$, respectively. The remaining output qubits $Q_2 \cdots Q_N$ are left in a “junk” quantum state $|\phi\rangle$. By adjoining an ancilla qubit $Q_A$ initialised in the state $|0\rangle$ and then applying a CNOT gate on $Q_1$ and $Q_A$, the answer $f(x)$ can be copied into $Q_A$ (see Fig. \ref{fig:3}). Finally, applying the inverse operation $U_x$ to qubits $Q_1 \cdots Q_N$ uncomputes the output state to the initial state $|0\rangle$. The state of the quantum computer is now $|0\rangle_{Q_1 \cdots Q_N} |f(x)\rangle_{Q_A}$. Without loss of generality it is therefore possible to assume that any quantum circuit for an instance $x$ of a decision problem outputs one of only two possible states: $|0\rangle |0\rangle$ or $|0\rangle |1\rangle$. It follows that knowledge of only one matrix element is enough to understand the output of such a quantum circuit.

A useful corollary of this construction is that to determine the output of the quantum circuit it is actually sufficient to determine the sign of a single amplitude. To see this, suppose in the construction above that we had applied a NOT gate and a Hadamard first to the ancilla qubit, so it was in the state $(|0\rangle - |1\rangle)/\sqrt{2}$. Applying $U_x$, the CNOT and $U_x$, as was done in the earlier construction, followed by another Hadamard to the ancilla qubit, gives as output $(-1)^{f(x)} |0\rangle |0\rangle$. It follows that knowledge of just the sign of the amplitude determines $f(x)$. In consequence of this observation, through the sequel we will change our notation somewhat, and denote the relevant matrix element as $|0\rangle U_x |0\rangle$.

In making this argument, we have assumed that the quantum circuit solves the decision problem with probability 1, i.e., deterministically. However, following the lines of \cite{21}, a more elaborate calculation based on the same idea shows that a similar result holds even for a quantum circuit which only outputs the correct answer probabilistically, provided the probability is bounded below by some appropriate constant, say $3/4$. Thus, to determine the output of a quantum circuit which solves some decision problem it suffices to determine the sign of a single matrix element $|0\rangle U_x |0\rangle$.

In the context of quantum computational complexity our results provide as an immediate corollary the result $\mathsf{BQP} \subseteq \mathsf{P}^\#\mathsf{P}$, due to Bernstein and Vazirani \cite{14}. We now outline the proof. Recall that the class $\mathsf{P}^\#\mathsf{P}$ is defined to be the class of decision problems decidable in polynomial time with the aid of an oracle which computes, at unit cost, the solution to a problem complete for $\mathsf{#P}$. To see that $\mathsf{BQP} \subseteq \mathsf{P}^\#\mathsf{P}$, we suppose our oracle $O(p)$ accepts as input a set of polynomials $p \in \mathbb{Z}_2[x]$, and returns the number of solutions of the corresponding set of equations over $\mathbb{Z}_2$. Supplied with such an oracle it is clear from the results of Sections \ref{section:1} and \ref{section:2} that we can determine the amplitudes at the output of a uniformly generated quantum circuit of polynomial size, and thus decide any language in $\mathsf{BQP}$. Because the problem of counting solutions to polynomial equations over $\mathbb{Z}_2$ is clearly in $\mathsf{#P}$, we obtain the inclusion $\mathsf{BQP} \subseteq \mathsf{P}^\#\mathsf{P}$.

Our techniques can also be used to prove the stronger inclusion $\mathsf{BQP} \subseteq \mathsf{PP}$, due to Adleman, Demarrais, and Huang \cite{15}. To see this, recall the definition of $\mathsf{PP}$ due to Fenner, Fortnow and Kurtz \cite{22}. First, define a GapP.

\footnote{The original definition is due to \cite{22}, but is more unwieldy in the present context.}
function to be a function expressible as the difference of two \#P functions. Then the class \( \text{PP} \) consists of all those languages \( L \) such that for some \( \text{GapP} \) function \( f \), and for all \( x \), either (a) if \( x \) is in \( L \), then \( f(x) > 0 \); or (b) if \( x \) is not in \( L \) then \( f(x) < 0 \).

The proof that \( \text{BQP} \) is a subset of \( \text{PP} \) is now trivial. The transition amplitude \( \langle 0|U_x|0 \rangle \) may be written as

\[
\langle 0|U_x|0 \rangle = \frac{\#(0) - \#(1)}{\sqrt{2^n}},
\]

which makes it a difference of two \#P functions, \#(0) and \#(1), and thus the amplitude \( \langle 0|U_x|0 \rangle \) is a \( \text{GapP} \) function. The result follows using [22]'s definition of \( \text{PP} \).

V. SAMPLING METHODS FOR SIMULATING QUANTUM CIRCUITS

An interesting question to ask in the light of our results is whether they can provide a means of speeding up the simulation of quantum computers by classical means. An obvious technique for doing this is to use Monte Carlo sampling to estimate the number of solutions to the equations of interest. Unfortunately, this technique fails to work in general.

To see why sampling fails, let’s look at how such an algorithm might work. We suppose we have worked out the polynomials \( B_j(x_1, x_2, \ldots, x_h) \), where \( j \) ranges over the \( N \) output qubits, and there are \( h \) Hadamard gates. For a decision problem, an admissible path \( x \) for the quantum circuit is one for which \( B_j(x) = 0 \), \( \forall j = 1, \ldots, N \), corresponding to calculating the matrix element \( (0|U_x|0) \). As before we define

\[
\#(0) = |\{x | B(x) = 0 \text{ and } \phi(x) = 0\}|,
\]

and

\[
\#(1) = |\{x | B(x) = 0 \text{ and } \phi(x) = 1\}|,
\]

so that

\[
(0|U_x|0) = \frac{1}{\sqrt{2^n}}[\#(0) - \#(1)].
\]

To estimate \( (0|U_x|0) \) we need to estimate both \#(0) and \#(1). \textit{A priori} it is possible that \#(0) and \#(1) scale as \( 2^h \), so to obtain an estimate of \( (0|U_x|0) \) accurate to some constant precision requires a number of trials exponential in \( h \). It follows from these observations that Monte Carlo sampling is not, in general, an efficient way of estimating amplitudes for quantum computation.

VI. OTHER GATE SETS

What other universal gate sets might be amenable to the path-sums approach? Might it be possible to find universal sets giving rise to sets of polynomial equations for which it is possible to efficiently estimate the number of solutions? To answer these questions, note that the key property of the gate set used in the path-sum approach is that the amplitudes in each gate are balanced, i.e., the non-zero elements of the gate all have the same absolute value. For example, the non-zero elements of the Hadamard gate all have absolute value \( 1/\sqrt{2} \), while the non-zero elements of the Toffoli gate all have absolute value 1. Given any universal set of balanced gates, it is not difficult to write down a sum-over-paths formulation along similar lines to that done for the Hadamard-Toffoli set in Sections III and IV.

Rather than a general discussion, we will simply give a single informative example of the results obtained when this approach is followed, basing our discussion on the universal set consisting of \( T = \left( \begin{array}{cc} 1 & 0 \\ 0 & e^{\pi i/4} \end{array} \right) \), \( H \), and \( \text{cnot} \). Applying the procedure described in Sections III and IV to a quantum circuit composed only of \( T \), \( H \), and \( \text{cnot} \) shows that that an amplitude \( \langle b|U|a \rangle \) can be found according to the following recipe. First, define a classical circuit corresponding to the quantum circuit by replacing all qubits by bits, quantum \( \text{cnots} \) by classical \( \text{cnots} \), \( T \) gates by identity operations, and Hadamard gates by classical Hadamard gates, as earlier. Notions of path variables and admissible paths are defined as before; note that the polynomials \( B_j(x) \) are now linear in the path variables \( x \), due to the linearity of the gates appearing in the classical circuit. We can thus write the transition amplitude

\[
\langle b|U|a \rangle = \sum_{x : B_j(x) = b} \Phi(x),
\]

where the phase factor \( \Phi(x) \) depends on the value of the path variables \( x \). Following an argument analogous to that in Sections III and IV we find that these phases can be written in the form \( \exp(i\pi \phi(x)/4) \), where \( \phi(x) \) is a polynomial in the path variables \( x \); it is a polynomial in mixed arithmetic involving multiplication over \( Z_2 \) and addition over \( Z_8 \). Because the \( B_j \) are linear, it is possible to eliminate variables, moving to a new set of unconstrained variables \( y \in Z_2 \times Z_8 \), and writing the transition amplitude as

\[
\langle b|U|a \rangle = \sum_y \exp(i\pi \phi(y)/4),
\]

where \( \phi(y) \) is again a polynomial in mixed arithmetic involving multiplication over \( Z_2 \) and addition over \( Z_8 \); with a little work, it can easily be verified that \( \phi \) is of order at most two. It follows that all we need do is count the number of solutions to the eight equations \( \phi(y) = 0, \phi(y) = 1, \ldots, \phi(y) = 7 \), in order to determine such a transition amplitude.

This example illustrates a number of interesting features. First, the polynomial \( \phi \) is of order two, as compared with the order three polynomials that arose with the Hadamard-Toffoli gate set. Second, the way in which
we do arithmetic is substantially more complex than in the Hadamard-Toffoli case. Third, we now have only to count solutions to a single polynomial equation, instead of a set of simultaneous polynomial equations. There three features illustrate a general fact: choosing different sets of universal gates gives rise to sets of polynomial equations with different structures. It is an interesting problem to find gate sets giving rise to particularly nice sets of equations in the sum-over-paths approach.

We mention one final variation on our result that seems worth pursuing. This is a sum-over-paths approach to the Heisenberg representation of quantum computation [24], i.e., using the stabiliser formalism [3, 25]. In this formalism the $|0\rangle$ state of $N$ qubits is described as the simultaneous +1 eigenspace of the stabilizer generators $S_0 = (Z_1, \ldots, Z_N)$. Subsequent evolution through a quantum circuit $U_x$ is described by conjugating each of the stabilizer generators by the unitary $U_x$. It is well known that the effect of Clifford group gates such as the Hadamard and CNOT is to take products of Pauli matrices (like $Z_1, \ldots, Z_N$) to other products of Pauli matrices under conjugation, and for a circuit entirely made up of Clifford group gates this enables us to easily determine the final state at the end of the computation. However, the Clifford group gates are not universal on their own. To get a universal set, we need to add another gate, such as the $T$ gate, which when acting on $X$, $Y$, and $Z$ induces the following transformations:

$$TXY = \frac{X + Y}{\sqrt{2}}, \quad TYT = \frac{-X + Y}{\sqrt{2}},$$

and $$TZT = Z.$$

It is now possible to apply a sum-over-paths approach to each of the stabilizer generators, with the $T$ gate playing a similar role to that played by $H$ in the sum-over-paths approach based on quantum states. In this case each “path” is a sequence $S_k$ of sets of stabiliser generators. A “matrix element” is the amplitude to induce a transition between the initial set of (Pauli) stabiliser generators $S_0$ to some final set of (Pauli) stabiliser generators $S_{m-1}$. Thus determining the stabilizer generators at the end of a computation is equivalent to counting the number of solutions to certain sets of polynomial equations. Of course, even if that could be done, there would still remain the difficulty of working out the measurement statistics resulting from a measurement of the final state output at the end of the circuit. Nonetheless, this alternative description may provide a different insight into the complexity of quantum circuit simulation.

VII. CONCLUSION

In conclusion, we have shown that quantum computation is intimately connected with the problem of counting the number of solutions to sets of polynomial equations, i.e., to counting points on an algebraic variety. This is a well-known problem in mathematics, and one of the central problems of algebraic geometry; even for the case where there is just a single polynomial, this problem is connected to deep topics such as the Weil conjectures. In the context of computational complexity, it is known that the problem of counting solutions to polynomials over finite fields is $\#P$-complete. It is possible that better understanding the structure of the polynomial equations associated with quantum computations may result in further insight into the computational power of quantum computers.

Acknowledgements

MAN thanks Richard Cleve, Manny Knill and Mike Mosca for encouraging and informative discussions about the subject of this paper. MAN is especially grateful to Richard Cleve for alerting him to the existence of the work by Ehrenfeucht and Karpinski, and for discussions on the universality of Hadamard and Toffoli. TJO thanks Wim van Dam, Andreas Winter, and Simone Severini for encouraging discussions.

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