LIMITS FOR EMBEDDING DISTRIBUTIONS

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Abstract. In this paper, we first establish a central limit theorem which is new in probability, then we find and prove that, under some conditions, the embedding distributions of $H$-linear family of graphs with spiders are asymptotic normal distributions. As corollaries, the asymptotic normality for the embedding distributions of path-like sequence of graphs with spiders and the genus distributions of ladder-like sequence of graphs are given. We also prove that the limit of Euler-genus distributions is the same as that of crosscap-number distributions. The results here can been seen a version of central limit theorem in topological graph theory.

1. Introduction

A graph is a pair $G = (V, E)$, where $V = V(G)$ is the set of vertices, and $E = E(G)$ is the set of edges. In topological graph theory, a graph is permitted to have both loops and multiple edges. A surface $S$ is a compact connected 2-dimensional manifold without boundary. The orientable surface $O_k (k \geq 0)$ can be obtained from a sphere with $k$ handles attached, where $k$ is called the genus of $O_k$, and the non-orientable surface $N_j (j \geq 1)$ with $j$ crosscaps, where $j$ is called the crosscap-number of $N_j$. The Euler-genus $\gamma^E$ of a surface $S$ is given by

$$\gamma^E = \begin{cases} 2k, & \text{if } S = O_k, \\ j, & \text{if } S = N_j. \end{cases}$$

We use $S_i$ to denote the surface $S$ with Euler-genus $i$, for $i \geq 0$.

A graph $G$ is embeddable into a surface $S$ if it can be drawn in the surface such that any edge does not pass through any vertex and any two edges do not cross each other. If $G$ is embedded on the surface $S$, then the components of $S - G$ are the faces of the embedding. A graph embedding is called a 2-cell cellular embedding if any simple closed curve in that face can be continuously deformed or contracted in that face to a single point. All graph embeddings in the paper are 2-cell cellular embeddings.

A rotation at a vertex $v$ of a graph $G$ is a cyclic ordering of the edge-ends incident at $v$. A (pure) rotation system $\rho$ of a graph $G$ is an assignment of a rotation at every vertex of $G$. A general rotation system for a graph $G$ is a pair $(\rho, \lambda)$, where $\rho$ is a rotation system and $\lambda$ is a map on $E(G)$ with values in $\{0, 1\}$. If $\lambda(e) = 1$, then the edge $e$ is said to be twisted; otherwise $\lambda(e) = 0$, and we call the edge $e$ untwisted. If $\lambda(e) = 0$, for all $e \in E(G)$, then the general rotation system $(\rho, \lambda)$ is a pure rotation system. It is well-known that any graph embedding can be described by a general rotation system. Let $T$ be a spanning tree of $G$, a $T$-rotation system $(\rho, \lambda)$ of $G$ is a general rotation system $(\rho, \lambda)$ such that $\lambda(e) = 0$, for every edge $e \in E(T)$. For a

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fixed spanning tree $T$, two embeddings of $G$ are considered to be equivalent if their $T$-rotation systems are combinatorially equivalent. It is known that there is a sequence of vertex-flips that transforms a general rotation system into a $T$-rotation system.

The number of (distinct) cellular embeddings of a graph $G$ on the surfaces $O_k$, $N_j$, and $S_i$ are denoted by $\gamma_k(G)$, $\tilde{\gamma}_j(G)$, and $\varepsilon_i(G)$, respectively. By the genus distribution of a graph $G$ we mean the sequence

$$\gamma_0(G), \gamma_1(G), \gamma_2(G), \cdots,$$

and the genus polynomial of $G$ is

$$\Gamma_G(x) = \sum_{k=0}^{\infty} \gamma_k(G)x^k.$$ 

Similarly, we have the crosscap-number distribution $\{\tilde{\gamma}_i(G)\}_{n=1}^{\infty}$ and the Euler-genus distribution $\{\varepsilon_i(G)\}_{n=1}^{\infty}$. The crosscap-number polynomial $\tilde{\Gamma}_G(x)$ and the Euler-genus polynomial $\varepsilon_G(x)$ of $G$ are defined analogously.

For a deeper discussion of the above concepts, we may refer the reader to [3, 4]. The following assumption will be needed throughout the paper. When we say embedding distribution of a graph $G$, we mean its genus distribution, crosscap-number distribution or Euler-genus distribution.

Usually, the embedding distribution of a graph $G$ with tractable size can be calculated explicitly, we still concern the global feature of the embedding distribution of $G$. For example: 

(1) Log-concavity. For this aspect, we refer to [10, 12, 13, 14, 29] etc. 

(2) Average genus, average crosscap-number and average Euler-genus. The average genus of graph $G$ is given by

$$\gamma_{avg}(G) = \frac{\Gamma_G'(1)}{\Gamma_G(1)} = \sum_{k=0}^{\infty} \frac{k \cdot \gamma_k(G)}{\Gamma_G(1)},$$

the average crosscap-number $\tilde{\gamma}_{avg}(G)$ and average Euler-genus $\varepsilon_{avg}(G)$ of a graph $G$ is similarly defined. The study of average genus, average crosscap-number and average Euler-genus received many attentions in topological graph theory. For researches on this aspect, one can see [2, 26, 33] etc. There is also a notation of variance. For example, see that in [30]. The variance of the genus distribution of the graph $G$ is given by

$$\gamma_{var}(G) = \sum_{k=0}^{\infty} (k - \gamma_{avg}(G))^2 \frac{\gamma_k(G)}{\Gamma_G(1)}.$$ 

We define the variance of crosscap-number distribution $\tilde{\gamma}_{var}(G)$ and of Euler-genus distribution $\varepsilon_{var}(G)$ similarly.

The motivation of this article is as follows. Let $\{G^n_i\}_{n=1}^{\infty}$ be a sequence of linear family of graphs with spiders whose definition is given in subsection 3.1, and we denote the embedding distribution of graph $G^n_i$ by $\{p_i(n)\}_{i=0}^{\infty}$. The normalized sequence of $\{p_i(n)\}_{i=0}^{\infty}$ is

$$\frac{p_i(n)}{\sum_{k=0}^{\infty} p_k(n)}, \quad i = 0, 1, \cdots.$$

Then, the above sequence is a distribution in probability, we denote it by $F_n$. One problem appears, when $n$ is big enough, whether the distribution $F_n$ will look like some well-known distribution in probability. If the answer is yes, then it demonstrates the outline of embedding distribution for graph $G^n_i$ when $n$ is big enough. In the point of mathematics, this is to seek the limit for $F_n$ or the embedding distribution of graph $G^n_i$. 
In this paper, we make researches on the embedding distributions which are closely related to the above problem. Under some weak conditions, we show the embedding distributions (genus, crosscap-number or Euler-genus distributions) of $G_n^0$ are asymptotically normal distribution when $n$ tends to infinity. We say the embedding distributions (genus, crosscap-number or Euler-genus distributions) of $G_n^0$ are asymptotically normal distribution with mean $\mu_n$ and variance $\sigma_n^2$ if

$$\lim_{n \to \infty} \sup_x \left| \sum_{i \leq x, x \in \mu_n} p_i(n) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0.$$  

Since normal distributions have many very good properties, the genus distributions (crosscap-number or Euler-genus distributions) of $G_n^0$ also have many good properties when $n$ is big enough. Such as: (1) Symmetry, normal distributions are symmetric around their mean. (2) Normal distributions are defined by two parameters, the mean $\mu$ and the standard deviation $\sigma$. Approximately 95% of the area of a normal distribution is within two standard deviations of the mean. This implies that the genus distributions of $G_n^0$ are mainly concentrated on the interval $(\gamma_{avg}(G_n^0)-2\sqrt{\gamma_{var}(G_n^0)}, \gamma_{avg}(G_n^0)+2\sqrt{\gamma_{var}(G_n^0)})$ when $n$ is big enough. Similar results also hold for crosscap-number and Euler-genus distributions. We also show that the genus distributions (crosscap-number or Euler-genus distributions) of $G_n^0$ are not always asymptotically normal distribution. This is the first time someone prove the embedding distributions of some families of graphs are asymptotically normal.

In Section 2, we establish a central limit theorem which is also new in probability. In Section 3, we apply this central limit theorem to the embedding distributions of $G_n^0$ and give their limits. In Section 4, some examples are demonstrated.

## 2. A Central Limit Theorem

For a non-negative integer sequence $\{p_i(n)\}_{i=0}^{\infty}$, let $P_n(x) = \sum_{i=0}^{\infty} p_i(n)x^i, x \in \mathbb{R}$. In this section, we always assume $P_n(x)$ satisfies a $k^{th}$-order homogeneous linear recurrence relation

$$P_n(x) = b_1(x)P_{n-1}(x) + b_2(x)P_{n-2}(x) + \cdots + b_k(x)P_{n-k}(x),  \tag{2.1}$$

where $b_j(x) (1 \leq j \leq k)$ are polynomials with integer coefficients. We define a polynomial associated with (2.1)

$$F(x, \lambda) = \lambda^k - b_1(x)\lambda^{k-1} - b_2(x)\lambda^{k-2} - \cdots - b_{k-1}(x)\lambda - b_k(x). \tag{2.2}$$

Obviously, for any $x \in \mathbb{R}$, there exist some $r = r(x) \in \mathbb{N}$, $m_1(x), \cdots, m_r(x) \in \mathbb{N}$ with $\sum_{i=1}^{r} m_i(x) = k$, and numbers $\lambda_i(x), i = 1, \cdots, r$ with $|\lambda_1(x)| \geq |\lambda_2(x)| \geq |\lambda_3(x)| \geq \cdots |\lambda_r(x)|$ such that

$$F(x, \lambda) = (\lambda - \lambda_1(x))^{m_1(x)}(\lambda - \lambda_2(x))^{m_2(x)} \cdots (\lambda - \lambda_r(x))^{m_r(x)}. \tag{2.3}$$

And the general solution to (2.1) is given by

$$P_n(x) = \sum_{i=1}^{r} \lambda_i^n(x)(a_{i,0}(x) + a_{i,1}(x)n + \cdots + a_{i,m_i(x)-1}(x)n^{m_i(x)-1}). \tag{2.4}$$

Let

$$e = \frac{\lambda_1(1)}{D}, \quad v = \frac{-\left(\lambda_1(1)\right)^2 + D \cdot \lambda_1''(1) + D \cdot \lambda_1'(1)}{D^2}. \tag{2.5}$$
where $D = \lambda_1(1)$. For any $n \in \mathbb{N}$, let $X_n$ be a random variable with distribution
\[
P(X_n = i) = \frac{p_i(n)}{P_n(1)}, \quad i = 0, 1, \ldots,
\]

The remainder of this section is devoted to the proof of the following theorem.

**Theorem 2.1.** Let $P_n(x) = \sum_{i=0}^{\infty} p_i(n)x^i, n \geq k + 1$ be polynomials satisfying (2.1). At $x = 1$, suppose the multiplicity of maximal root for polynomial (2.2) is 1. Then the following results hold depending on the value of $v$.

**Case I:** $v > 0$. The law of $X_n$ is asymptotically normal with mean $e \cdot n$ and variance $v \cdot n$ when $n$ tends to infinity. That is,
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{1}{P_n(1)} \sum_{\lambda_1 + \cdots + \lambda_r = n} \sum_{0 \leq i \leq x^{\delta+n}} p_i(n) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \right| = 0.
\]
In particular, we have
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \frac{1}{P_n(1)} \left| \sum_{\lambda_1 + \cdots + \lambda_r = n} \sum_{0 \leq i \leq x^{\delta+n}} p_i(n) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \right| = 0.
\]

**Case II:** $v < 0$. This case is impossible to appear.

**Case III:** $v = 0$. For any $\alpha > \frac{1}{\delta}$, the law of $X_n$ is asymptotically one-point distribution concentrated at 0. In more accurate words, the following holds.
\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{X_n - e \cdot n}{n^{\alpha}} \leq x \right) = \begin{cases} 1, & \text{if } x \geq 0, \\ 0, & \text{else}. \end{cases}
\]
Furthermore, if all these functions $b_1(x), \cdots, b_k(x)$ are constant, then the limits of the law of $X_n$ is a discrete distribution. That is, for some $\kappa \in \mathbb{N}$ and $\omega_j, j = 0, \cdots, \kappa$ with
\[
\sum_{j=1}^{\kappa} \omega_j = 1, \quad \text{we have}
\]
\[
\lim_{n \to \infty} \mathbb{P}(X_n = j) = \omega_j, j = 0, \cdots, \kappa.
\]

**Proof.** One arrives at that
\[
D = \lambda_1(1) > |\lambda_2(1)| \geq |\lambda_3(1)| \geq \cdots |\lambda_r(1)| \quad \text{and} \quad m_1(1) = 1.
\]
By [21] and the smooth of $F$, for $i = 1, \cdots, r$, $\lambda_i(x)$ are continuous. Thus, for some $\delta > 0$, we have
\[
\lambda_1(x) > |\lambda_2(x)| \geq |\lambda_3(x)| \geq \cdots |\lambda_r(x)|, \quad \forall x \in (1 - \delta, 1 + \delta).
\]
For $m_1(x)$ and $\lambda_1(x)$, we have the following fact.

Fact: for some $\delta > 0$, we have $\lambda_1(x)$ is smooth on $(1 - \delta, 1 + \delta)$ and
\[
m_1(x) = 1, \quad \forall x \in (1 - \delta, 1 + \delta).
\]
One easily sees that
\[
F(1, \lambda) = (\lambda - \lambda_1(1))(\lambda - \lambda_2(1))^{m_2(1)} \cdots (\lambda - \lambda_r(1))^{m_r(1)}.
\]
and
\[
\frac{\partial F(x, \lambda)}{\partial \lambda} \bigg|_{x=1, \lambda = \lambda_1(1)} \neq 0.
\]
Actually, $\lambda_1(x)$ can be seen an implicit function decided by

$$F(x, \lambda) = 0.$$ 

By the smooth of $F$ and the implicit function theorem, $\lambda_1(x)$ is smooth on $(1 - \varepsilon, 1 + \varepsilon)$ for some $\varepsilon > 0$. By the smooth of $F$ and (2.10), for some $\varepsilon > 0$, we have

$$\frac{\partial F(x, \lambda)}{\partial \lambda} \neq 0, \quad \forall x \in (1 - \varepsilon, 1 + \varepsilon), \forall \lambda \in (\lambda_1(1) - \varepsilon, \lambda_1(1) + \varepsilon)$$

which yields the desired result (2.9).

Combining (2.7), the general solution to (2.1) is given by

$$P_n(x) = a(x)\lambda_1^n(x) + \sum_{i=2}^{r} \lambda_i^n(x) \left( a_i,0(x) + a_i,1(x)n + \cdots + a_i,m_i(x)-1(x)n^{m_i(x)-1} \right).$$

(2.11)

We consider the following three different cases.

**Case I:** $v > 0$. Let $Y_n = X_n - en\sqrt{vn}$ and $\phi_{Y_n}(t) = Ee^{itY_n}$ be the characteristic function of $Y_n$, where $i$ is a complex number with $i^2 = -1$.

In order to prove

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| P(Y_n \leq x) - \int_{-\infty}^{x} e^{-u^2/2} du \right| = 0,$$

by the continuity theory (Chapter 15) for characteristic function in probability, we only need to prove

$$\lim_{n \to \infty} \phi_{Y_n}(t) = \lim_{n \to \infty} Ee^{it\sqrt{v_n}X_n} = \int_{\mathbb{R}} e^{it\sqrt{2}u} e^{-u^2/2} du = e^{-t^2/2}, \quad \forall t.$$

(2.12)

We will give a proof of this.

Let $a_n = \frac{1}{\sqrt{vn}}$, $b_n = \sqrt{\frac{e^2n}{v}}$ and $y = e^{an}$. By these definitions, one easily sees that

$$\frac{n\lambda_1(1)}{D} a_n - b_n = ne\sqrt{\frac{e^2n}{v}} = 0,
\frac{n\lambda_1(1)D + \lambda_1''(1)D - \lambda'(1)^2}{2D^2} a_n^2 = n \cdot \frac{v}{2} \cdot \frac{1}{vn} = \frac{1}{2}$$

(2.13)

By Taylor formula, we have

$$\ln \frac{\lambda_1(y)}{D} = \frac{\lambda_1(1)}{D}(y - 1) + \frac{1}{2D^2} \left[ \lambda_1''(1)D - (\lambda_1'(1))^2 \right] (y - 1)^2 + o((y - 1)^2),$$

and

$$y = 1 + a_n t + a_n^2 t^2 + o(a_n^2 t^2).$$
Therefore, by \( \lim_{n \to \infty} y = 1 \), it holds that
\[
\ln \frac{\lambda_1(y)}{D} = \frac{\lambda_1'(1)}{D}(a_n t + \frac{1}{2}a_n^2 t^2 + o(a_n^2 t^2)) \\
+ \frac{1}{2D^2} [\lambda_1''(1)D - (\lambda_1'(1))^2] (a_n t + \frac{1}{2}a_n^2 t^2 + o(a_n^2 t^2))^2 \\
+ o((a_n t + \frac{1}{2}a_n^2 t^2 + o(a_n^2 t^2))^2).
\]

where in the last equality, we have used (2.5).

In the above equality, we replace \( e \) by \( y \), and get
\[
\ln \frac{\lambda_1(y)}{D} = \frac{\lambda_1'(1)}{D}(a_n t + \frac{1}{2}a_n^2 t^2 + \frac{1}{2D^2} [\lambda_1''(1)D - (\lambda_1'(1))^2] a_n^2 t^2 + o(\frac{1}{n}).
\]

By Taylor formula, we have
\[
\lim_{n \to \infty} E e^{\frac{\lambda_1(y) - \lambda_1(1)}{\sqrt{\frac{\lambda_1(1)}{y}}} - \frac{\lambda_1(1)}{y} t} = \lim_{n \to \infty} P_n(e^{a_n t}e^{-b_n t}) = \lim_{n \to \infty} \frac{P_n(y)e^{-b_n t}}{a(1)D^n}
\]

where in the last equality, we have used (2.5).

In the above equality, we replace \( t \) by \( it \). Then, we get
\[
\lim_{n \to \infty} E e^{it \frac{\lambda_1(y) - \lambda_1(1)}{\sqrt{\frac{\lambda_1(1)}{y}}} - \frac{\lambda_1(1)}{y} t} = e^{-\frac{t^2}{2}}
\]

which finishes the proof of (2.12).

**Case II:** \( v < 0 \). Let \( Y_n = \frac{X_n - vn}{\sqrt{\frac{\lambda_1(1)}{y}}} \) and \( \phi_{Y_n}(t) = E e^{itY_n} \). Set \( a_n = \frac{1}{\sqrt{-vn}} \), \( b_n = \sqrt{\frac{\lambda_1(1)}{y}} \) and \( y = e^{a_n t} \).

By Taylor formula, we have
\[
\ln \frac{\lambda_1(y)}{D} = \frac{\lambda_1'(1)}{D}(a_n t + \frac{1}{2}a_n^2 t^2) + \frac{1}{2D^2} [\lambda_1''(1)D - (\lambda_1'(1))^2] a_n^2 t^2 + o(\frac{1}{n}).
\]

Then, by (2.7), we have
\[
\lim_{n \to \infty} E e^{\frac{\lambda_1(y) - \lambda_1(1)}{\sqrt{\frac{\lambda_1(1)}{y}}} - \frac{\lambda_1(1)}{y} t} = \lim_{n \to \infty} \frac{P_n(e^{a_n t}e^{-b_n t})}{a(1)D^n} = \lim_{n \to \infty} \frac{P_n(y)e^{-b_n t}}{a(1)D^n}
\]

where in the last equality, we have used (2.5).

In the above equality, we replace \( t \) by \( it \) and get
\[
\lim_{n \to \infty} E e^{it \frac{\lambda_1(y) - \lambda_1(1)}{\sqrt{\frac{\lambda_1(1)}{y}}} - \frac{\lambda_1(1)}{y} t} = e^{-\frac{t^2}{2}}.
\]

On the other hand, by the properties of characteristic function, one sees
\[
|E e^{it \frac{\lambda_1(y) - \lambda_1(1)}{\sqrt{\frac{\lambda_1(1)}{y}}}}| \leq 1
\]
which contradicts with (2.13) and gives the desired result.

**Case III:** \( v = 0 \). Let \( Y_n = X_n - cn, a_n = n^{-\alpha}, b_n = cn^{1-\alpha} \) and \( y = e^{an}t \).

By Taylor formula and \( v = 0 \), we have

\[
\ln \left( \frac{\lambda_i(y)}{D} \right) = \frac{\lambda_i'(1)}{D} (y - 1) + \frac{1}{2D^2} [\lambda_i''(1)D - (\lambda_i'(1))^2] (y - 1)^2 + \mathcal{O}(y, y^3)
\]

\[
= \frac{\lambda_i'(1)}{D} (a_n t + \frac{1}{2} a_n^2 t^2) + \frac{1}{2D^2} [\lambda_i''(1)D - (\lambda_i'(1))^2] a_n^2 t^2 + \mathcal{O}(n^{-3\alpha})
\]

\[
= \frac{\lambda_i'(1)}{D} a_n t + \mathcal{O}(n^{-3\alpha}) = e^{an} t + \mathcal{O}(n^{-3\alpha}).
\]

Therefore,

\[
\lim_{n \to \infty} \exp \left\{ n \ln \left( \frac{\lambda_i(y)}{D} \right) - b_n t \right\} = \lim_{n \to \infty} \exp \left\{ n \ln \frac{en^{-\alpha} t + n \cdot \mathcal{O}(n^{-3\alpha}) - en^{1-\alpha} t}{-b_n t} \right\} = 1.
\]

Then, by (2.7), we have

\[
\lim_{n \to \infty} \mathbb{E} e^{\frac{X_n - cn}{an}} = \lim_{n \to \infty} \mathbb{E} e^{\frac{a_n}{an}X_n - \frac{b_n}{an} t} = \lim_{n \to \infty} \frac{P_n(y) e^{-b_n t}}{a(1) D^n} = \lim_{n \to \infty} \frac{P_n(y) e^{-b_n t}}{a(1) D^n}
\]

\[
= \lim_{n \to \infty} \frac{\lambda_i'(y) a(y)}{a(1) D^n} e^{-b_n t} = \lim_{n \to \infty} \lambda_i'(y) D e^{-b_n t} = \exp \left\{ n \ln \frac{\lambda_i(y)}{D} - b_n t \right\}.
\]

In the above equality, we replace \( n \) by \( i t \) and get

\[
\lim_{n \to \infty} \mathbb{E} e^{\frac{X_n - cn}{an}} = 1
\]

which yields the desired result.

Now we give a proof of (2.10). Noting (2.11), we have

\[
P_n(x) = a(x) \lambda_i^n + \sum_{i=2}^{r} \left( a_{i,0}(x) + a_{i,1}(x) n + \cdots + a_{i,m_i(x)-1}(x) n^{m_i(x)-1} \right) \lambda_i^n,
\]

where \( m_i(x) = m_i, i = 2, \cdots, r \) are constants. Since \( P_n(x) \) is a polynomial of \( x \) and \( \lambda_i, i = 1, \cdots, r \) are constant, \( a(x) \) is also a polynomial of \( x \). Assume

\[
a(x) = \sum_{j=0}^{\kappa} c_j x^j.
\]

then,

\[
\lim_{n \to \infty} \mathbb{E} e^{\frac{X_n}{an}} = \lim_{n \to \infty} \frac{P_n(e^t)}{P_n(1)} = \lim_{n \to \infty} \frac{a(e^t) D^n}{a(1) D^n} = \frac{a(e^t)}{a(1)} = \sum_{j=0}^{\kappa} \omega_j e^{jt},
\]

where \( \omega_j = \frac{c_j}{a(1)} \). In the above equality, we replace \( n \) by \( it \) and get

\[
(2.15) \quad \lim_{n \to \infty} \mathbb{E} e^{\frac{X_n}{an}} = \sum_{j=0}^{\kappa} \omega_j e^{jt}.
\]

By the continuity theory ([8], Chapter 15]) for characteristic function in probability, we obtain the desired result.

By (2.10), the condition \( v > 0 \) is necessary to ensure the asymptotic normality. In the end of this section, we give a remark here to explain the \( e, v \) appeared in Theorem 2.1.
Remark 2.2. In the special case, when \( P_n(x) = a(x)\lambda^n_1(x) \), it holds that

\[
\mathbb{E}X_n = \frac{P'_n(1)}{P_n(1)} = e \cdot n + O(1)
\]

and

\[
\text{Var}(X_n) = \frac{P''_n(1) + P'_n(1)}{P_n(1)} \cdot \left( \frac{P'_n(1)}{P_n(1)} \right)^2 = v \cdot n + O(1).
\]

Since \( \lambda_1(x) > \lambda_i(x), i = 2, \cdots, k \), we can expect that for the \( P_n(x) \) in (2.11) and \( P_n(x) = a(x)\lambda^n_1(x) \), they have the same asymptotic mean and variance when \( n \) tends to infinity.

3. The limits for embedding distributions

In this section, we consider the limit for embedding distributions of \( H \)-linear family of graphs with spiders \( \{G_n^n\}_{n=1}^\infty \). In subsection 3.1, we give a definition of \( G^n_n \). In subsection 3.2, we briefly describe the production matrix. Then we give the limit of embedding distribution for graph \( G^n_n \) in subsection 3.3. Finally, we demonstrate the relation between the limit of crosscap-number distributions and Euler-genus distributions in subsection 3.4.

3.1. \( H \)-linear family of graphs with spiders. The definition of \( H \)-linear family of graphs with spiders, gave by Chen and Gross [6], is a generalization of \( H \)-linear family of graphs introduced by Stahl [27]. Suppose \( H \) is a connected graph. Let \( U = \{u_1, \cdots, u_s\} \) and \( V = \{v_1, \cdots, v_s\} \) be two disjoint subsets of \( V(H) \). For \( i = 1, 2, \cdots \), suppose \( H_i \) is a copy of \( H \) and let \( f_i : H \to H_i \) be an isomorphism. For each \( i \geq 1 \) and \( 1 \leq j \leq s \), we set \( u_{i,j} = f_i(u_j) \) and \( v_{i,j} = f_i(v_j) \). An \( H \)-linear family of graphs, denoted by \( \mathcal{G} = \{G^n_n\}_{n=1}^\infty \), is defined as follows:

- \( G_1 = H_1 \).
- \( G_n \) is constructed by \( G_{n-1} \) and \( H_n \) be amalgamating the vertex \( v_{n-1,j} \) of \( G_{n-1} \) with the vertex \( u_{n,j} \) of \( H_n \) for \( j = 1, \cdots, s \).

Figure 3.1 shows an example for the graphs \( H_1, H_2 \) and \( G_2 \).

Now, we introduce the definition of \( H \)-linear family of graphs with spiders. For \( 1 \leq j \leq s \), let \((J_j, t_{j,i'})\) and \((\overline{J}_j, t_{j,i'})\) be graphs in which \( \{t_{j,i'}\}, \{t_{j,i'}\} \), respectively, are sets of root-vertices. For \( 1 \leq i \leq s \), \( I_i \) and \( \overline{I}_i \) are subsets of \( \{1, \cdots, s\} \). A graph \( \{G^n_n\}_{n=1}^\infty \) is constructed from \( G_n \) by amalgamating the vertex \( u_{1,i} \) of \( G_n \) with the vertex \( t_{j,i'} \) of \( J_j \) for \( j \in \overline{I}_i \), and amalgamating the vertex \( v_{n,i} \) of \( G_n \) with \( t_{j,i'} \) of \( \overline{J}_j \) for \( j \in I_i \). The graphs \((J_j, t_{j,i'})\) and \((\overline{J}_j, t_{j,i'})\) are called spiders for the sequence \( \mathcal{G} \). The resulting sequence of graphs is said to be an \( H \)-linear family of graphs.
with spiders and is denoted by \( G^o \). We call \( G^o \) ring-like, if there is a spider among \( J_1, J_2, \ldots, J_s \) that coincide with a spider among \( J_1, J_2, \ldots, J_s \). The graphs in Figure 3.2 demonstrate an example of ring-like, in which \( s = 1 \) and \( J_1 = J_1 \).

3.2. Production matrix. By permutation-partition pairs, Stahl [27] showed that the calculation of genus polynomial of \( G_n \) can be converted to a (transfer) matrix method. Such matrices are also called production matrices [11] or transfer matrix [20] (using different techniques and methods). Here we briefly describe the production matrix of \( G \) (or \( G^o \)). For more details on this, see [20, 11] etc. We refer to [3] for face-tracing algorithm.

We suppose that there are \( k \) embedding types for the graph \( G_n \) with roots \( u_{1,1}, u_{1,2}, \ldots, u_{1,s}, v_{n,1}, v_{n,2}, \ldots, v_{n,s} \). For \( 1 \leq j \leq k \), let \( \gamma^j_i(G_n) \) be the number of embeddings of \( G_n \) in \( O_k \) with type \( j \) and

\[
\Gamma^j_{G_n}(x) = \sum_{i \geq 0} \gamma^j_i(G_n)x^i.
\]

From the definition \( H \)-linear family of graphs and face-tracing algorithm, we obtain

\[
(\Gamma^1_{G_n}(x), \Gamma^2_{G_n}(x), \ldots, \Gamma^k_{G_n}(x))^T = M(x) \cdot (\Gamma^1_{G_{n-1}}(x), \Gamma^2_{G_{n-1}}(x), \ldots, \Gamma^k_{G_{n-1}}(x))^T.
\]

(3.1)

where \( \alpha^T \) is the transpose of the vector \( \alpha \) and

\[
M(x) = \begin{bmatrix} m_{1,1}(x) & m_{1,2}(x) & \cdots & m_{1,k}(x) \\ m_{2,1}(x) & m_{2,2}(x) & \cdots & m_{2,k}(x) \\ \vdots & \vdots & \ddots & \vdots \\ m_{k,1}(x) & m_{k,2}(x) & \cdots & m_{k,k}(x) \end{bmatrix}
\]

is the transfer matrix [20] or production matrix [11] of genus distribution of \( G \) (or \( G^o \)). In [11], the authors showed that the production matrix \( M(x) \) can be calculated with a computer program.

Let

\[
V_{G_n}(x) = (\Gamma^1_{G_n}(x), \Gamma^2_{G_n}(x), \ldots, \Gamma^k_{G_n}(x))^T,
\]

another property for the genus polynomial of \( H \)-linear sequence with spiders is that there exists a \( k \)-dimensional row-vector

\[
V = (v_1(x), \ldots, v_k(x))
\]

such that \( \Gamma^j_{G_n}(x) = V \cdot V_{G_n}(x) \). Note that if there are no spiders, that is \( G^o = G \), then \( V = (1, 1, \cdots, 1) \).

For the case of Euler-genus polynomials, there also has the production matrix of Euler-genus distribution of \( G \) (or \( G^o \)). We take [4] as an example of this.

**Figure 3.2.** Using a spider to construct \( G^o \)
Example 3.1. Suppose $P_n$ is the path graph on $n$ vertices. An ladder graph $L_n$ is obtained by taking the graphical cartesian product of the path graph $P_n$ with $P_2$, i.e. $L_n = P_n \square P_2$. From Section 3 in [4], the $2 \times 2$ production matrix of Euler-genus distribution of the ladder graph is given by

$$M(x) = \begin{bmatrix} 2 & 4 \\ 2x + 4x^2 & 4x \end{bmatrix}.$$ 

It follows that

$$M(1) = \begin{bmatrix} 2 & 4 \\ 6 & 4 \end{bmatrix}.$$ 

The maximum genus, maximum non-orientable genus and maximum Euler-genus of a graph $G$, denoted by $\gamma_{\max}(G)$, $\tilde{\gamma}_{\max}(G)$ and $\varepsilon_{\max}(G)$ respectively, are given by $\gamma_{\max}(G) = \max\{i|\gamma_i(G) > 0\}$, $\tilde{\gamma}_{\max}(G) = \max\{i|\tilde{\gamma}_i(G) > 0\}$ and $\varepsilon_{\max}(G) = \max\{i|\varepsilon_i(G) > 0\}$ respectively. One can see that $\varepsilon_{\max}(G) = \max\{2\gamma_{\max}(G), \tilde{\gamma}_{\max}(G)\}$. A cactus graph, also called a cactus tree, is a connected graph in which any two graph cycles have no vertex in common. Recall that a graph $G$ with orientable maximum genus 0 if and only if $G$ is the cactus graph, and a graph $H$ of maximum Euler-genus 0 if and only if $H$ is homeomorphic to the path graph $P_n$ on $n$ vertices for $n > 1$.

The following two basic properties are followed by the definition of $H$-linear family of graphs with spiders.

Proposition 3.1. For $1 \leq j \leq k$, $\sum_{i=1}^{k} m_{ij}(1)$ are all the same constant $D$. Moreover, for any $n \geq 2$, we have $\frac{P_n(1)}{P_{n-1}(1)} = D$, where $P_n(x) = \Gamma_{G_n}(x)$ or $E_{G_n}(x)$.

Remark 3.2. In the proof of Theorem 3.5 below, we will see $D$ have the same meaning as that appeared in Section 2, so we use the same notation.

Proposition 3.3. Suppose $M(x)$ is the production matrix of genus distribution (Euler-genus distribution) of $G$, then $M(x)$ is a constant if and only if the maximum genus (maximum Euler-genus) of $G_n$ equals 0, $\forall n \in \mathbb{N}$.

3.3. The limits for embedding distributions of $H$-linear families of graphs with spiders. A square matrix $A = (a_{i,j})_{i,j=1}^{k}$ is said to be non-negative if

$$a_{i,j} \geq 0, \quad \forall i, j = 1, \cdots, k.$$ 

Let $A$ be a non-negative $k \times k$ matrix with maximal eigenvalue $r$ and suppose that $A$ has exactly $h$ eigenvalues of modulus $r$. The number $h$ is called the index of imprimitivity of $A$. If $h = 1$, the matrix $A$ is said to be primitive; otherwise, it is imprimitive. A square matrix $A = (a_{i,j})_{i,j=1}^{k}$ is said to a stochastic matrix if

$$\sum_{i=1}^{k} a_{ij} = 1, j = 1, \cdots, k.$$ 

The following property of primitive stochastic matrix can be found in Proposition 9.2 in [1].

Proposition 3.4. Every eigenvalue $\lambda$ of a stochastic matrix $A$ satisfies $|\lambda| \leq 1$. Furthermore, if the stochastic matrix $A$ is primitive, then all other eigenvalues of modulus are less than 1, and algebraic multiplicity of 1 is one.
Theorem 3.5. Consider the sequence of graphs $G^o = \{G^o_n\}_{n=1}^{\infty}$. Let $\{p_i(n), i = 0, 1, \ldots, \}$ be the genus polynomial (Euler-genus polynomial) of graph $G^o_n$, $P_n(x)$ be the genus polynomial (Euler-genus polynomial) of $G^o_n$ and $M(x)$ be the production matrix for $G^o$. If the matrix $M(1)$ is primitive, then the results of Theorem 2.1 hold. Furthermore, if $M(x)$ is a constant, then the limit of the law for embedding distributions of $G^o_n$ is a discrete distribution.

Proof. Suppose that the characteristic polynomial of the production matrix $M(x)$ is

$$F(x, \lambda) = \lambda^k - b_1(x)\lambda^{k-1} - b_2(x)\lambda^{k-2} - \cdots - b_{k-1}(x)\lambda - b_k(x),$$

where $b_j(x)(1 \leq j \leq k)$ are polynomials with integer coefficients. We also assume $F(x, \lambda) = (\lambda - \lambda_1(x))^{m_1(x)} \cdots (\lambda - \lambda_r(x))^{m_r(x)}$. Then, by the results in [6] ([4]), the sequence of genus polynomials (Euler-genus polynomial) of $H$-linear family of graphs with spiders $G^o_n$ satisfy the following $k^{th}$-order linear recursion

$$P_n(x) = b_1(x)P_{n-1}(x) + b_2(x)P_{n-2}(x) + \cdots + b_k(x)P_{n-k}(x).$$

By Proposition 3.1 and Proposition 3.4 if $M(1)$ is primitive, we have

$$D = \lambda_1(1) > |\lambda_2(1)| \geq |\lambda_3(1)| \geq \cdots |\lambda_r(1)|$$

and $m_1(1) = 1$.

For any $n \in \mathbb{N}$, we denote the embedding distribution of graph $G^o_n$ by $\{p_i(n), i = 0, 1, 2, \ldots, \}$ and let $X_n$ be a random variable with distribution

$$\mathbb{P}(X_n = i) = \frac{p_i(n)}{P_n(1)}, \quad i = 0, 1, \cdots,$$

and

$$e = \frac{\lambda'_1(1)}{D}, \quad v = \frac{-\left(\lambda'_1(1)\right)^2 + D \cdot \lambda''_1(1) + D \cdot \lambda'_1(1)}{D^2}.$$

So following the lines in the proof of Theorem 2.1 we finish our proof. Furthermore if $M(x)$ is a constant, then all these functions $b_1(x), \ldots, b_k(x)$ are constant. Noting the case III in Theorem 2.1 this theorem follows. □

The primitive of the matrix $M(1)$ is very important in our proof. For this, we give the following example.

Example 3.2. Let

$$M(x) = \begin{bmatrix} x + 1 & 0 \\ 0 & 2x \end{bmatrix}.$$ 

$M(1)$ is imprimitive. By calculation, we obtain

$$\lambda_1(x) = \frac{3x + 1 + |x - 1|}{2}, \quad \lambda_2(x) = \frac{3x + 1 - |x - 1|}{2}.$$ 

In this case, we even don’t have the differentiability of $\lambda_1(x)$ at $x = 1$.

Remark 3.6. As pointed by Stahl [27], in all known cases, the production matrix $M(x)$ for the genus distributions of any $H$-linear family of graphs is primitive at $x = 1$. Currently, we don’t know whether this is true for general (or most) linear families of graphs.
In the rest of this subsection, we apply Theorem 3.5 to path-like and ladder-like sequences of graphs.

A vertex with degree 1 is called a pendant vertex, and the edge incident with that vertex is called a pendant edge. If a pendant vertex \( u \) of a graph \( G \) is chosen to be a root, then the vertex \( u \) is called a pendant root. Suppose \((H,u,v)\) is a connected graph with two pendant roots \( u,v \). For \( i=1,2,\ldots,n \), let \((H_i,u_i,v_i)\) be a copy of \((H,u,v)\). By the way in subsection 3.1, we construct a \((H,u,v)\)-linear family of graphs and a \((H,u,v)\)-linear family of graphs with spiders \((J_1,t_{1,v})\) and \((J_1,t_{1,v'})\), where \((J_1,t_{1,v})\) and \((J_1,t_{1,v'})\) be two connected graphs with roots \( t_{1,v},\bar{t}_{1,v'} \) respectively. For the \((H,u,v)\)-linear family of graphs with spiders or not, they have the same production matrix. Therefore, we use the same notation \( \{P^H_n\}_{n=1}^{\infty} \) to denote them. For convenience, we call the graph \( P^H_n \) path-like. Figure 3.3 demonstrate the graphs \( H \) and \( P^H_1 \), the shadow part of \( H \) can be any connected graph.

![Graph H (left), and path-like graph P^H_n (right)](image)

**Corollary 3.7.** The genus distributions (Euler-genus distributions) of the path-like sequence of graphs \( \{P^H_n\}_{n=1}^{\infty} \) with spiders are asymptotic normal distribution if the maximum genus (maximum Euler-genus) of \((H,u,v)\) is greater than 0.

**Proof.** Let \((H,u,v)\) be a graph with two pendant roots \( u,v \). We introduce the following two partial genus distributions (partial Euler-genus distributions) for \((H,u,v)\). Let \( d_i(H) \) be the number of embeddings of \((H,u,v)\) in the surface \( O_i \) \((S_i)\) such that \( u,v \) lie on different face-boundary walks. In this case, we say that the embedding has type \( d \). Similarly, let \( s_i(H) \) be the number of embeddings of \((H,u,v)\) in \( O_i \) \((S_i)\) such that \( u,v \) lie on the same face-boundary walk, and we call the embedding has type \( s \). The two partial genus polynomials (partial Euler-genus polynomials) of \((H,u,v)\) are given by \( D_H(x) = \sum_{i \geq 0} d_i(H)x^i \), and \( S_H(x) = \sum_{i \geq 0} s_i(H)x^i \). Clearly,

\[
P_H(x) = D_H(x) + S_H(x),
\]

where \( P_H(x) \) is the genus polynomial (Euler-genus polynomial) of \((H,u,v)\). By face-tracing and Euler formula, we have the following claim.

**Claim:** If the graph \((P^H_{n-1}, u_1, v_{n-1})\) and \((H_n, u_n, v_n)\) embed on surfaces \( O_i \) \((S_i)\) and \( O_j \) \((S_j)\), respectively, then the graph \( P^H_n \) embeds on \( O_{i+j} \) \((S_{i+j})\).

By using the claim above, we will build recurrence formulas for the partial genus polynomials of \( P^H_n \). There are four cases.

**Case 1:** If both the embeddings of \( P^H_{n-1} \) and \((H_n, u_n, v_n)\) have type \( d \), then the embedding of \( P^H_n \) has type \( d \). This case contributes to \( D_{P^H_n}(x) \) the term \( D_{P^H_{n-1}}(x)D_{H_n}(x) \).

**Case 2:** If the embeddings of \( P^H_{n-1} \) and \((H_n, u_n, v_n)\) have type \( d \) and \( s \), respectively, then the embedding of \( P^H_n \) has type \( d \). This case contributes to \( D_{P^H_n}(x) \) the term \( D_{P^H_{n-1}}(x)S_{H_n}(x) \).

**Case 3:** If the embedding of \( P^H_{n-1} \) has type \( s \) and the embedding of \((H_n, u_n, v_n)\) has type \( d \), then the embedding of \( P^H_n \) has type \( d \), this case contributes to \( D_{P^H_n}(x) \) the term \( S_{P^H_{n-1}}(x)D_{H_n}(x) \).
Case 4: If both the embeddings of $P_{n-1}^H$ and $(H_n, u_n, v_n)$ have type $s$, then the embedding of $P_n^H$ has type $s$. This case contributes to $S_{P_n^H}(x)$ the term $S_{P_{n-1}^H}(x)S_{H_n}(x)$.

The following linear recurrence system of equations summarizes the four cases above.

\begin{equation}
D_{P_n^H}(x) = (D_{H_n}(x) + S_{H_n}(x))D_{P_{n-1}^H}(x) + D_{H_n}(x)S_{P_{n-1}^H}(x),
\end{equation}

\begin{equation}
S_{P_n^H}(x) = S_{H_n}(x)S_{P_{n-1}^H}(x).
\end{equation}

Rewriting the equations above as

\[
\begin{pmatrix}
D_{P_n^H}(x) \\
S_{P_n^H}(x)
\end{pmatrix} =
\begin{pmatrix}
D_{H_n}(x) + S_{H_n}(x) & D_{H_n}(x) \\
0 & S_{H_n}(x)
\end{pmatrix}
\begin{pmatrix}
D_{P_{n-1}^H}(x) \\
S_{P_{n-1}^H}(x)
\end{pmatrix}.
\]

Since $(H_n, u_n, v_n)$ be a copy of $(H, u, v)$, thus the production matrix $M(x)$ of the genus (Euler-genus) distributions of $\{P_n^H\}^\infty_{n=1}$ is

\[
\begin{pmatrix}
D_{H}(x) + S_{H}(x) & D_{H}(x) \\
0 & S_{H}(x)
\end{pmatrix}.
\]

For simplicity of writing, we use $D(x)$, $S(x)$ and $P(x)$ to denote $D_H(x)$, $S_H(x)$ and $P_H(x)$, respectively. Obviously, the eigenvalues of matrix $M(x)$ are given by

$\lambda_1(x) = D(x) + S(x)$, \quad $\lambda_2(x) = S(x)$.

Since the graph $H$ is connected, $S(x) = 0$ is impossible. We make the following discussions on the $D(x)$

If $D(x) = 0$, then $H$ is the path graph $P_m$ on $m$ vertices ($m \geq 2$). In this case, the maximum genus (maximum Euler-genus) of $(H, u, v)$ is equal 0.

Now we consider the case $D(x) \neq 0$. Under this situation, $\lambda_1(1) > \lambda_2(1)$ and the matrix $M(1)$ is primitive. By direct calculation, we get

$\epsilon = \frac{D'(1) + S'(1)}{D(1) + S(1)}$

$v = \frac{-(D'(1) + S'(1))^2 + (D(1) + S(1))\left(D''(1) + S''(1) + D'(1) + S'(1)\right)}{(D(1) + S(1))^2}$.

We assume $P(x) = \sum_m c_m x^m$. By Cauchy-Schwarz inequality,

\[
\left(\sum_m mc_m\right)^2 \leq \sum_m m^2 c_m \cdot \sum_m c_m = \left(\sum_m (m^2 - m)c_m + \sum_m mc_m\right) \cdot \sum_m c_m,
\]

which implies $P'(1)^2 \leq (P''(1) + P'(1)) \cdot P(1)$. By this inequality and (3.3), one easily sees that $v \geq 0$. Therefore, $v = 0$ is equivalent to the above Cauchy-Schwarz inequality becomes equality, that is

\[
\left(\sum_m mc_m\right)^2 = \sum_m m^2 c_m \cdot \sum_m c_m.
\]

Since $P(x) \neq 0$, we have $c_m > 0$ for some $m \geq 0$. The above equality is also equivalent to that for some $x \in \mathbb{R}$

\[
m\sqrt{c_m} + x\sqrt{c_m} = 0, \quad \forall m \geq 0.
\]
Therefore, \( v = 0 \) if and only if \( \gamma_{\text{max}}(H) = \gamma_{\text{min}}(H) \) (\( \varepsilon_{\text{max}}(H) = \varepsilon_{\text{min}}(H) \)). Noting that, a known fact in the topological graph theory says that \( \gamma_{\text{max}}(H) = \gamma_{\text{min}}(H) \) (\( \varepsilon_{\text{max}}(H) = \varepsilon_{\text{min}}(H) \)) implies that \( H \) is the cactus graph (\( H \) is the path graph), we complete our proof.

\[
\begin{align*}
\text{Figure 3.4. Graph } H \text{ (left), and ladder-like graph } L_n^H \text{ (right)}
\end{align*}
\]

Given any graph \((H, u, v)\) whose root-vertices \(u\) and \(v\) are both 1-valent, as in Figure 3.4, we construct a ladder-like sequence of graphs \(\{(L_n^H, u_n, v_n)\}_{n=1}^\infty\). The shadow part of \(H\) can be any connected graph, and the ladder-like sequences are a special case of \(H\)-linear family of graphs.

**Corollary 3.8.** The genus distributions of the ladder-like sequence of graphs \(\{L_n^H\}_{n=1}^\infty\) are asymptotic normal distribution.

**Proof.** By [7], the production matrix for genus distributions of the ladder-like sequence of graphs \(L_4^H, L_2^H, L_3^H, \ldots\) is

\[
M(x) = p(x) \begin{bmatrix} 4x & 2x & 0 \\ 0 & 0 & 0 \\ 0 & 2x & 4x \end{bmatrix} + q(x) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 4 \\ 4x & 2x & 0 \end{bmatrix},
\]

where \(p(x), q(x) \in \mathbb{Z}(x)\) are the partial genus polynomials for \(H\). Also by [7], \(q(x) = 0\) is impossible.

Obviously, \(D = 4(p(1) + q(1))\) and the eigenvalues of matrix \(M(1)\) are given by

\[
\lambda_1 = 4(p(1) + q(1)), \quad \lambda_2 = 4p(1) - 2q(1), \quad \lambda_3 = 0.
\]

By \(p(1) \geq 0\) and \(q(1) > 0\), one easily sees that

\[
\lambda_1 > |\lambda_2|, \quad \lambda_1 > |\lambda_3|
\]

and the matrix \(M(1)\) is primitive.

With the help of a computer, one arrives at that

\[
e = \frac{3 (p'(1) + q'(1)) + 3p(1) + q(1)}{3p(1) + q(1)},
\]

\[
v = \frac{4}{27(p(1) + q(1))} \left[ 2q(1)^2 - 27(p'(1) + q'(1))^2 
+ 9q(1) \left( 3p''(1) + 3q''(1) + 3q'(1) + 7p'(1) \right) 
+ p(1) \left( 27p''(1) + 27p'(1) + 27q''(1) - 9q'(1) + 14q(1) \right) \right].
\]

In the rest of this corollary, we will prove \(v > 0\). Assume

\[
p(x) = \sum_m a_m x^m, \quad q(x) = \sum_m b_m x^m.
\]

Using Cauchy-Schwarz inequality again, we see that \(p'(1)^2 \leq (p''(1) + p'(1)) \cdot p(1)\) and \(q'(1)^2 \leq (q''(1) + q'(1)) \cdot q(1)\). Therefore, in order to prove \(v > 0\), it is suffice to show that

\[
54p'(1)q'(1) + 9p(1)q'(1) < 2q(1)^2 + 9q(1) \left( 3p''(1) + 7p'(1) \right) + p(1) \left( 27q''(1) + 14q(1) \right).
\]
which is equivalent to
\[
\sum_{i,j} \left[ 54ija_ib_j + 9a_ijb_j \right] < 2q(1)^2 + \sum_{i,j} \left[ 27(i^2 - i) + 63i + 27(j^2 - j) + 14 \right] a_ib_j.
\]

The above inequality is due to
\[
2q(1)^2 + \sum_{i,j} \left[ 27(i - j)^2 + 36(i - j) + 14 \right] a_ib_j = 2q(1)^2 + \sum_{i,j} \left( 27(i - j + 2 \cdot \frac{2}{3})^2 + 2 \right) a_ib_j > 0.
\]

We complete the proof of \( v > 0 \).

By Theorem 3.9, the genus distributions of the ladder-like sequence \( \{L_n^H\}_{n=1}^\infty \) are asymptotic normal distribution with mean \( c \cdot n \) and variance \( v \cdot n \).

\[\square\]

3.4. Limits for crosscap-number distributions of graphs. We demonstrate the relationship between the limits of crosscap-number distributions and Euler-genus distributions.

**Theorem 3.9.** Let \( a_n = \frac{\gamma_n}{\varepsilon_n} \), \( b_n = 1 - a_n \). If \( \lim_{n \to \infty} a_n = 0 \), we have

\[
\lim \sup_{n \to \infty} \frac{1}{x} \left| \sum_{0 \leq i \leq x} \varepsilon_i(G_n^o) - \sum_{0 \leq i \leq x} \hat{\gamma}_i(G_n^o) \right| = 0,
\]

which implies that the limits of crosscap-number distributions are the same as that of Euler-genus distributions.

**Proof.** Since \( a_n = \frac{\gamma_n}{\varepsilon_n} \), then \( b_n = 1 - a_n = \frac{\gamma_n}{\varepsilon_n} \). Let \( U_n, \hat{U}_n, W_n \) be three random variables with distributions given by the genus, crosscap-number and Euler-genus respectively, that is

\[
\mathbb{P}(U_n = i) = \frac{\gamma_i(G_n^o)}{\varepsilon_n(1)}, \quad \mathbb{P}(\hat{U}_n = i) = \frac{\hat{\gamma}_i(G_n^o)}{\varepsilon_n(1)}, \quad \mathbb{P}(W_n = i) = \frac{\varepsilon_i(G_n^o)}{\varepsilon_n(1)}, \quad i = 0, 1, \ldots.
\]

Since

\[
\varepsilon_n(1) = \varepsilon_n(1) + \hat{\gamma}_i(G_n^o),
\]

for any \( i \in \mathbb{N} \), we have

\[
\mathbb{E}_n(1) \cdot \mathbb{P}(W_n = i) = \varepsilon_i(G_n^o) = \gamma_i(G_n^o) + \hat{\gamma}_i(G_n^o)
\]

\[
= \mathbb{P}(U_n = \frac{i}{2}) \cdot \varepsilon_n(1) + \mathbb{P}(\hat{U}_n = i) \cdot \hat{\gamma}_n(1),
\]

here \( \gamma_i(G_n^o) \) is defined as 0 when \( i \) is an odd number. Therefore, it holds that

\[
\mathbb{P}(W_n = i) = a_n \mathbb{P}(2U_n = i) + b_n \mathbb{P}(\hat{U}_n = i)
\]

By this, for any \( x \geq 0 \), we have

\[
\mathbb{P}(W_n \leq x) - \mathbb{P}(\hat{U}_n \leq x) = a_n \mathbb{P}(2U_n \leq x) + (b_n - 1) \mathbb{P}(\hat{U}_n \leq x).
\]

Thus

\[
\left| \mathbb{P}(W_n \leq x) - \mathbb{P}(\hat{U}_n \leq x) \right| \leq a_n + (1 - b_n).
\]

By the definitions of \( W_n, \hat{U}_n \), the above inequality implies (3.6).

\[\square\]

By (3.6), once the limits of Euler-genus distributions is obtained, the limits of crosscap-number distributions is also known.

With the same method as that in Theorem 3.9, we obtain the following corollary.
Corollary 3.10. Let \( \{G_n\}_{n=1}^\infty \) be any sequence of graphs, which is not required to be \( H \)-linear family of graphs with spiders. If \( \lim_{n \to \infty} \frac{\varepsilon(G_n(1))}{\varepsilon(G_n(1))} = 0 \) or \( \lim_{n \to \infty} \beta(G_n) = \infty \), we have
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \sum_{0 \leq i \leq x} \frac{\varepsilon_i(G_n)}{\varepsilon(G_n)} - \sum_{0 \leq i \leq x} \frac{\gamma_i(G_n)}{\Gamma(G_n(1))} \right| = 0.
\]

4. More examples and some research problems

The graphs, which have explicit formulas for their embedding distributions, mainly are linear families of graphs with spiders, see [4, 6, 8, 11, 29] for details. There are many linear families of graphs with spiders which satisfy the conditions of Theorem 2.1 or Theorem 3.5. However, we give a few examples to demonstrate the theorems, including the famous Möbius ladders, Ringel ladders and Circular ladders.

Example 4.1. Let \( Y_n \) be the iterated-claw graph of Figure 4.1.

\[
\begin{align*}
\text{Figure 4.1. The claw } Y_1 \text{ (left), and the iterated-claw graph } Y_n \text{ (right).}
\end{align*}
\]

The genus polynomial for the iterated-claw graph \( Y_n \) [4] is given by
\[
\Gamma_{Y_n}(x) = 20x\Gamma_{Y_{n-1}}(x) - 8(-3x + 8x^2)\Gamma_{Y_{n-2}}(x) - 384x^3\Gamma_{Y_{n-3}}(x).
\]

Since
\[
F(x, \lambda) = \lambda^3 - 20x\lambda^2 + 8(8x^2 - 3x)\lambda + 384x^3,
\]
\[
F(1, \lambda) = (\lambda - 16)(\lambda - 2(1 - \sqrt{7}))(\lambda - 2(\sqrt{7} + 1)),
\]
the conditions of Theorem 2.1 hold for \( P_n(x) = \Gamma_{Y_n}(x) \) and \( \lambda_1(1) = D = 16 \). By the derivative rule of implicit function and with the help of Maple, one sees
\[
e = \frac{6}{7}, \quad v = \frac{8}{147} > 0.
\]

For the constants \( e, v \) given above, we have
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{1}{\Gamma_{Y_n}(1)} \sum_{0 \leq i \leq x} \gamma_i(Y_n) - \int_{-\infty}^{\infty} e^{-\frac{1}{2}u} du \right| = 0.
\]

By [4], the Euler-genus polynomials of \( Y_n \) satisfy the following three-order recurrence relation
\[
\mathcal{E}_{Y_n}(x) = 2(3x + 28x^2)\mathcal{E}_{Y_{n-1}}(x) - 16(-3x^2 - 12x^3 + 4x^4)\mathcal{E}_{Y_{n-2}}(x) - 3072x^6\mathcal{E}_{Y_{n-3}}(x).
\]

Set \( P_n(x) = \mathcal{E}_{Y_n}(x) \) and
\[
F(x, \lambda) = \lambda^3 - 2(3x + 28x^2)\lambda^2 + 16(-3x^2 - 12x^3 + 4x^4)\lambda + 3072x^6.
\]

Since \( F(1, \lambda) = (\lambda - 64)(\lambda - 6)(\lambda + 8) \), the conditions of Theorem 2.1 hold and \( D = \lambda_1(1) = 64 \). By the derivative rule of implicit function and with the help of Maple, one sees
\[
e = \frac{160}{87}, \quad v = \frac{269092}{1975509} > 0.
\]
For the constants $e, v$ given above, the Euler-genus distributions of $Y_n$ are asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$. By Theorem 3.3 the crosscap-number distributions of $Y_n$ are also asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$.

**Example 4.2.** Let $G_n = P_n \square P_3$ be the grid graph \[17\]. The genus polynomials for the grid graph $G_n$ are given by the recursion

$$
\Gamma_{G_n}(x) = (1 + 30x)\Gamma_{G_{n-1}}(x) - 42(-x + 4x^2)\Gamma_{G_{n-2}}(x) - 72(x^2 + 14x^3)\Gamma_{G_{n-3}}(x) + 1728x^4\Gamma_{G_{n-4}}(x).
$$

With the help of Maple, the conditions of Theorem 2.1 hold for $P_n(x) = \Gamma_{G_n}(x)$ and

$$D = \lambda_1(1) = 24, \ e = \frac{34}{41}, \ v = \frac{4816}{68921} > 0. \ $$

The Euler-genus polynomials for the grid graphs $G_n$ \[4\] satisfy the recursion

$$
\mathcal{E}_{G_n}(x) = (1 + 11x + 84x^2)\mathcal{E}_{G_{n-1}}(x) + 12x^2(7 + 30x - 28x^2)\mathcal{E}_{G_{n-2}}(x) - 288x^3(1 + 4x + 32x^2)\mathcal{E}_{G_{n-3}}(x) + 2768x^6\mathcal{E}_{G_{n-4}}(x).
$$

With the help of Maple, the conditions of Theorem 2.1 hold for $P_n(x) = \mathcal{E}_{G_n}(x)$ and

$$D = \lambda_1(1) = 96, \ e = \frac{5488}{3037}, \ v = \frac{4819233780}{28011371653} > 0. \ $$

By the discussions above, the embedding distributions of the grid graph $G_n$ are asymptotic normal distributions.

**Example 4.3.** Suppose that $n$ is a positive integer. Let $C_n$ be the cycle graph on $n$ vertices. The Ringel ladder $R_n$ is obtained by adding an edge joining the two vertices of the leftmost edge and rightmost edge of the ladder graph $L_n$. A circular ladder $CL_n$ is the graphical Cartesian product $C_n \square P_2$. The Möbius ladder $ML_n$ is formed from an $2n$-cycle by adding edges connecting opposite pairs of vertices in the cycle, i.e., the Möbius ladder can be described as a circular ladder with a half-twist. It is known in \[6\] that the Ringel ladder, circular ladder, and Möbius ladder are ring-like families of graphs. Let $H_n$ be the Ringel ladder $R_n$, circular ladder $CL_n$, or Möbius ladder $ML_n$.

By Theorem 3.1 in \[6\], the genus polynomials for $H_n$ \[6\] satisfy the recursion

$$
\Gamma_{H_n}(x) = 4\Gamma_{H_{n-1}}(x) + (-5 + 20x)\Gamma_{H_{n-2}}(x) + (56x + 2)\Gamma_{H_{n-3}}(x) - 4(32x - 11)x\Gamma_{H_{n-4}}(x) + 8(28x - 1) x\Gamma_{H_{n-5}}(x) + 32(8x - 3)x^2\Gamma_{H_{n-6}}(x) - 256x^3\Gamma_{H_{n-7}}(x).
$$

Since

$$
F(x, \lambda) = \lambda^7 - 4\lambda^6 - (-5 + 20x)\lambda^5 - (-56x + 2)\lambda^4 + 4(32x - 11)x\lambda^3 - 8(28x - 1)x\lambda^2 - 32(8x - 3)x^2\lambda + 256x^3 \ \text{and} \ \lambda_1(x) = \sqrt{8x + 1} + 1, \ D = \lambda_1(1) = 4.
$$

With the help of Maple, one easily sees

$$e = \frac{1}{3}, \ v = \frac{2}{27} > 0. \ $$
By Theorem 2.1 for the constants $e, v$ given above, we have

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \frac{1}{\Gamma_{H_n}(1)} \sum_{0 \leq i \leq x \sqrt{n + e - n}} \gamma_i(H_n) - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du \right| = 0.$$ 

The Euler-genus polynomials for $H_n$ are given by the recursion

$$\mathcal{E}_{H_n}(x) = (12x + 4)\mathcal{E}_{H_{n-1}}(x) + (-12x^2 - 34x - 5)\mathcal{E}_{H_{n-2}}(x)$$

$$+ (-240x^3 - 20x^2 + 26x + 2)\mathcal{E}_{H_{n-3}}(x)$$

$$+ 4(80x^3 + 128x^2 + 14x - 1)x\mathcal{E}_{H_{n-4}}(x)$$

$$+ 16(112x^3 + 8x^2 - 14x - 1)x^2\mathcal{E}_{H_{n-5}}(x)$$

$$- 128(8x^3 + 18x + 3)x^4\mathcal{E}_{H_{n-6}}(x)$$

$$- 2048(2x + 1)x^6\mathcal{E}_{H_{n-7}}(x).$$

Since

$$F(x, \lambda) = \lambda^7 - (12x + 4)\lambda^6 - (-12x^2 - 34x - 5)\lambda^5 - (-240x^3 - 20x^2 + 26x + 2)\lambda^4$$

$$- 4(80x^3 + 128x^2 + 14x - 1)x\lambda^3 - 16(112x^3 + 8x^2 - 14x - 1)x^2\lambda^2$$

$$+ 128(8x^3 + 18x + 3)x^4\lambda + 2048(2x + 1)x^6,$$

$$F(1, \lambda) = (\lambda + 2)^2(\lambda - 3)(\lambda - 4)(\lambda - 8)(\lambda - \frac{1}{2}(5 - \sqrt{89}))(\lambda - \frac{1}{2}(5 + \sqrt{89})),$$

the conditions of Theorem 2.1 hold for $P_n(x) = \mathcal{E}_{H_n}(x)$ and $D = \lambda_1(1) = 8$. With the help of Maple, we have

$$\lambda_1(x) = \sqrt{20x^2 + 4x + 1 + 2x + 1}, \quad e = \frac{4}{5}, \quad v = \frac{22}{125} > 0.$$ 

By Theorem 2.1 for the constants $e, v$ given above, the Euler-genus distributions of $H_n$ is asymptotic normal distribution with mean $e \cdot n$ and variance $v \cdot n$.

4.1. Some researches problems. In the end of this paper, we demonstrate some research problems.

**Question 4.1.** In our paper, the limits of the embedding distributions for graphs are normal distributions or some discrete distributions. Can we prove that the limit of the embedding distributions for any $H$-linear family of graphs is a normal distribution or a discrete distribution. Furthermore, if the maximum genus (Euler-genus) $\varepsilon_{\max}(G)$ of $H$ is greater than 0, do we have the limit for the genus distributions (Euler-genus distribution) for any $H$-linear family of graphs is a normal distribution?

**Question 4.2.** Suppose that $\{G_n\}_{n=1}^{\infty}$ is a family of graphs with $\beta(G_n) \to \infty$ (orientable maximum genus $\gamma_M(G_n) \to \infty$). Are the crosscap-number distributions (genus distributions) of $G_n$ asymptotic normal.

A bouquet of circles $B_n$ is define as a graph with one vertex and $n$ edges. A dipole $D_n$ is a graph with two vertices joining by $n$ multiple edges. In [15], Gross, Robbins, and Tucker obtained a second-order recursion for the genus distributions of $B_n$. The genus distributions of $D_n$ were obtained by Rieper in [25], and independently by Kwak and Lee in [16]. The wheel $W_n$ is a graph formed by connecting a single vertex to each of the vertices of a $n$-cycle. The genus distribution of $W_n$ was obtained in [5] by Chen, Gross and Mansour. The following special case of question 4.2 is may not hard.
Question 4.3. Are the embedding distributions of $B_n$, $D_n$ and $W_n$ asymptotic normal?

The following question is for the complete graph $K_n$ on $n$ vertices and complete bipartite graph $K_{m,n}$.

Question 4.4. Are the embedding distributions of $K_n$, and $K_{m,n}$ asymptotic normal?

Another question is the following.

Question 4.5. Is the embedding distribution of a random graph on $n$ vertices asymptotic normal?

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