On Noncommutative and semi-Riemannian Geometry

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Abstract
We introduce the notion of a semi-Riemannian spectral triple which generalizes the notion of spectral triple and allows for a treatment of semi-Riemannian manifolds within a noncommutative setting. It turns out that the relevant spaces in noncommutative semi-Riemannian geometry are not Hilbert spaces any more but Krein spaces, and Dirac operators are Krein-selfadjoint. We show that the noncommutative tori can be endowed with a semi-Riemannian structure in this way. For the noncommutative tori as well as for semi-Riemannian spin manifolds the dimension, the signature of the metric, and the integral of a function can be recovered from the spectral data.

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1 Introduction

The Gel’fand-Naimark theorem states that any unital commutative $C^*$-algebra can be realized as an algebra of continuous functions on a compact Hausdorff space. In noncommutative geometry one thinks of a noncommutative $C^*$-algebra as an algebra of functions on some "virtual" space and tries to imitate geometrical constructions which work for the case of commutative algebras. Connes functional analytic approach (see [3]) to noncommutative geometry starts with the observation that the metric information of a compact Riemannian spin manifold $M$ is
encoded in the triple \((C^\infty(M),\mathcal{D},L^2(M,S))\), where \(\mathcal{D}\) is the Dirac operator and \(L^2(M,S)\) is the Hilbert space of square integrable sections of the spinor bundle. The algebra \(C^\infty(M)\) is realized as a \(*\)-algebra of bounded operators on \(L^2(M,S)\). The space of characters of \(C^\infty(M)\) is canonically isomorphic to the set of points of \(M\) and the Riemannian distance between to point \(p\) and \(q\) can be recovered from the equation

\[
d(p,q) = \sup |f(p) - f(q)|; \ f \in C^\infty(M), \ ||[\mathcal{D},f]|| \leq 1.
\]

The noncommutative generalization of the object \((C^\infty(M),\mathcal{D},L^2(M,S))\) is the so-called spectral triple, which we can think of as a generalization of the theory of compact Riemannian manifolds. For a further introduction to noncommutative geometry we would like to refer the reader to [9], [16], [22], [17] and the references therein.

Recently there have been attempts to get analogues of spectral triples which allow for a treatment of non-compact manifolds (see [20]) and of globally hyperbolic Lorentzian manifolds (see [18, 15, 14, 13]). Such a treatment seems necessary if one wants to study physical models, which are defined on spaces with Lorentzian rather than Riemannian metrics. The idea in [11], [15], [14] and [13] is to foliate the spacetime into Cauchy surfaces and to treat the Cauchy surfaces as Riemannian manifolds. Whereas this approach seems promising for the study of evolution equations in physics, its dependence on the foliation and the restriction to Lorentzian signatures is disturbing from the mathematical point of view.

In this paper we suggest a notion of semi-Riemannian spectral triple, which allows to treat compact semi-Riemannian manifolds (of arbitrary signature) within noncommutative geometry. Such a triple \((\mathcal{A}, \mathcal{D}, \mathcal{H})\) consists of an involutive algebra \(\mathcal{A}\) of bounded operators acting on a Krein space \(\mathcal{H}\) and a Krein-selfadjoint operator \(\mathcal{D}\). An important role is played by the fundamental symmetries of the Krein space. These are operators \(\mathcal{J} : \mathcal{H} \rightarrow \mathcal{H}\) with \(\mathcal{J}^2 = 1\) such that \((\cdot, \mathcal{J}\cdot) = (\mathcal{J}\cdot, \cdot)\) is a positive definite scalar product turning \(\mathcal{H}\) into a Hilbert space. They can be used to obtain ordinary spectral triples from semi-Riemannian spectral triples in a similar way as this is done in physics by "Wick rotation", which is used to pass to Riemannian signatures of the metric. For example if \(M\) is a Lorentzian spin manifold, \(\mathcal{A} = C^\infty_0(M)\) and \(\mathcal{D}\) is the Dirac operator which acts on the Krein space of square integrable sections \(\mathcal{H}\) of the spinor bundle, the triple \((\mathcal{A}, \mathcal{D}, \mathcal{H})\) is a semi-Riemannian spectral triple. For a special class of fundamental symmetries the operator \(\frac{1}{2}((\mathcal{J}\mathcal{D})^2 + (D\mathcal{J})^2)\) is a Laplace-type operator with respect to a Riemannian metric. We can think of this metric as a Wick rotated form of the
Lorentzian metric. We use this to show that one can define a notion of dimension for semi-Riemannian spectral triples. In the commutative case and for the non-commutative semi-Riemannian torus we show that there is a canonical notion of integration and one can recover the signature of the metric from the spectral data.

In sections 2-5 we review the basic notions and results on spectral triples, Krein spaces and Dirac operators on semi-Riemannian manifolds. Sections 6 and 7 contain the main results of this paper.

2 Spectral triples

Definition 2.1. A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) consists of a unital \(*\)-algebra \(\mathcal{A}\) of bounded operators on a separable Hilbert space \(\mathcal{H}\) and a selfadjoint operator \(D\) on \(\mathcal{H}\), such that the commutator \([D,a]\) is bounded for all \(a \in \mathcal{A}\). A spectral triple is said to be even if there exists an operator \(\chi = \chi^*, \chi^2 = 1\) on the Hilbert space such that

\[
\chi a = a \chi, \quad \forall a \in \mathcal{A}, \tag{2}
\]

\[
D \chi = -\chi D. \tag{3}
\]

For a compact operator \(a\) denote by \(\mu_k(a)\) the ordered sequence of its singular values, i.e. \(\mu_k(a)\) are the eigenvalues of \(|a|\) such that \(\mu_1(a) \geq \mu_2(a) \geq \ldots\), with each eigenvalue repeated according to its multiplicity. The characteristic sequence of \(a\) is defined by \(\sigma_k(a) := \sum_{i=1}^k \mu_i(a)\). Let \(p \geq 1\) be a real number. A compact operator \(a\) is said to be in \(L^p\) if

\[
\sup_{n \geq 1} \frac{\sigma_n(a)}{n^{(p-1)/p}} < \infty \quad \text{for} \quad p > 1, \tag{4}
\]

\[
\sup_{n \geq 2} \frac{\sigma_n(a)}{\ln n} < \infty \quad \text{for} \quad p = 1. \tag{5}
\]

The spaces \(L^{p+}\) are two 2-sided ideals in \(B(\mathcal{H})\). Note that if \(a \in L^p\), then \(|a|^p \in L^1\).

Let now \(l^\infty(\mathbb{N})\) be the von Neumann algebra of bounded sequences. If a state \(\omega\) on \(l^\infty(\mathbb{N})\) satisfies the conditions

- \(\lim_{n \to \infty} x_n = x \Rightarrow \omega((x_n)) = x\),
- \(\omega((x_{2n})) = \omega((x_n))\),
we say that \( \omega \) is in \( \Gamma_s(l^\infty) \). The set \( \Gamma_s(l^\infty) \) turns out to be non-empty ([8]). For each positive \( a \in L^{1+} \) and each state \( \omega \in \Gamma_s(l^\infty) \) we define \( \text{Tr}_\omega(a) := \omega(x(a)) \), where \( x(a)_n = \frac{\sigma_n(a)}{\ln n} \) for \( n \geq 2 \) and \( x(a)_1 = 0 \). It can be shown that for each \( \omega \) the map \( a \rightarrow \text{Tr}_\omega(a) \) extends to a finite trace on \( L^{1+} \) and to a singular trace on \( B(H) \) (see [8, 3]).

**Definition 2.2.** Let \( p \geq 1 \) be a real number. A spectral triple is called \( p^+ \)-summable if \( (1 + D^2)^{-p/2} \) is in \( L^{p^+} \).

In case a spectral triple is \( p^+ \)-summable the map \( a \rightarrow \text{Tr}_\omega(a(1 + D^2)^{-p/2}) \) is well defined on the algebra \( A_D \) generated by \( A \) and \([D,A]\). It can be shown that if \( A_D \) is contained in the domain of smoothness of the derivation \( \delta(\cdot) := [[D,\cdot],\cdot] \), this map is a trace (see [3]). A differential operator on a Riemannian manifold is said to be of Dirac type if it is of first order and the principal symbol \( \sigma \) of \( D \) satisfies the relation

\[
\sigma(\xi)^2 = g(\xi,\xi) \text{id}_{E_p} \quad \forall \xi \in T^*_p M, p \in M. \tag{6}
\]

The geometry of a compact Riemannian spin manifold can be encoded in a spectral triple (see [8]).

**Theorem 2.3 (Connes).** Let \( M \) be a compact Riemannian manifold of dimension \( n \) and \( E \) a hermitian vector bundle over \( M \) of rank \( k \). Let \( H \) be the Hilbert space of square integrable sections of \( E \) and let \( A \) be \( C^\infty(M) \) which acts on \( H \) by multiplication. Assume that \( D \) is a symmetric differential operator of Dirac type on \( E \). Then \((A,H,D)\) is an \( n^+ \)-summable spectral triple. As a compact space \( M \) is the spectrum of the \( C^* \)-algebra, which is the norm closure of \( A \). The geodesic distance on \( M \) is given by

\[
d(p, q) = \sup \, |f(p) - f(q)|; \quad f \in A, \quad \|[D,f]\| \leq 1. \tag{7}
\]

Furthermore for \( f \in C^\infty(M) \) we have

\[
\int_M f \sqrt{|g|} dx = c(n,k) \text{Tr}_\omega(f(1 + |D|^2)^{-n/2}), \tag{8}
\]

where \( c(n,k) = 2^{n-1} \pi^{n/2} k^{-1} n \Gamma(n/2) \).
3 Differential calculus and spectral triples

Let \( \mathcal{A} \) be a unital algebra. Denote by \( \overline{\mathcal{A}} \) the vector space \( \mathcal{A}/(\mathbb{C}1) \) and define \( \Omega^n \mathcal{A} := \mathcal{A} \otimes \overline{\mathcal{A}}^n \). We write \( (a_0, a_1, \ldots, a_n) \) for the image of \( a_0 \otimes \ldots \otimes a_n \) in \( \Omega^n \mathcal{A} \). On \( \Omega \mathcal{A} := \bigoplus_{n=0}^{\infty} \Omega^n \mathcal{A} \) one now defines an operator \( d \) of degree one and a product by

\[
d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n),
\]
\[
(a_0, \ldots, a_n)(a_{n+1}, \ldots, a_k) = \sum_{i=0}^{n} (-1)^{n-i}(a_0, \ldots, a_i a_{i+1}, \ldots, a_k).
\]

This determines a differential algebra structure on \( \Omega \mathcal{A} \). If \( \mathcal{A} \) is a star algebra, one makes \( \Omega \mathcal{A} \) a star algebra by \( (a_0, \ldots, a_n)^* := (-1)^n(a_0^*, \ldots, a_n^*) \cdot a_0^* \). The pair \( (\Omega \mathcal{A}, d) \) is called the universal differential envelope of \( \mathcal{A} \).

A spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) gives rise to a *-representation of \( \Omega \mathcal{A} \) on \( \mathcal{H} \) by the map

\[
\pi : \Omega \mathcal{A} \to \mathcal{B}(\mathcal{H}),
\]
\[
\pi((a_0, a_1, \ldots, a_n)) := a_0[D, a_1] \cdots [D, a_n], \quad a_j \in \mathcal{A}.
\]

Let \( j_0 \) be the graded two-sided ideal \( j_0 := \bigoplus_n j_0^n \) given by

\[
j_0^n := \{ \omega \in \Omega^n \mathcal{A}; \pi(\omega) = 0 \}.
\]

In general, \( j_0 \) is not a differential ideal. That is why it is not possible to define the space of forms to be the image \( \pi(\Omega \mathcal{A}) \). However \( j := j_0 + dj_0 \) is a graded differential two-sided ideal.

**Definition 3.1.** The graded differential algebra of Connes forms over \( \mathcal{A} \) is defined by

\[
\Omega_D \mathcal{A} := \Omega \mathcal{A}/j \cong \bigoplus_n \pi(\Omega^n \mathcal{A})/\pi(dj_0 \cap \Omega^n \mathcal{A}).
\]

**Example 3.2.** The space of one-forms \( \Omega_D^1 \mathcal{A} \cong \pi(\Omega^1 \mathcal{A}) \) is the space of bounded operators of the form

\[
\omega_1 = \sum_k a_0^k [D, a_1^k], \quad a_i^k \in \mathcal{A}.
\]
Proposition 3.3. Let \((A, \mathcal{H}, D)\) be as in Theorem 2.3. As graded differential algebras \(\Omega_D A\) and \(\Gamma(\Lambda M)\) are isomorphic.

See [6], p.552 or [16], section 7.2.1 for a proof.

If the spectral triple \((A, \mathcal{H}, D)\) is \(n^+\)-summable the map

\[ w_1 \times w_2 \rightarrow \langle w_1, w_2 \rangle := \text{Tr}_\omega(w_1^* w_2 (1 + |D|^2)^{-n/2}) \]  

(14)

defines for each \(\omega\) a scalar product on the space of one-forms. In the case of Proposition 3.3 this scalar product coincides up to a scalar factor with the metric-induced scalar product on the space of one-forms.

4 Krein Spaces

4.1 Fundamentals

Let \(V\) be a vector space over \(\mathbb{C}\). An indefinite inner product on \(V\) is a map \((\cdot, \cdot) : V \times V \rightarrow \mathbb{C}\) which satisfies

\[(v, \lambda w_1 + \mu w_2) = \lambda(v, w_1) + \mu(v, w_2)\]
\[(v_1, v_2) = (v_2, v_1).\]

The indefinite inner product is said to be non-degenerate, if

\[(v, w) = 0, \quad \forall v \in V \Rightarrow w = 0.\]

A non-degenerated indefinite inner product space \(V\) is called decomposable if it can be written as the direct sum of orthogonal subspaces \(V^+\) and \(V^-\) such that the inner product is positive definite on \(V^+\) and negative definite on \(V^-\). The inner product then defines a norm on these subspaces. \(V^+\) and \(V^-\) are called intrinsically complete if they are complete in these norms. A non-degenerate indefinite inner product space which is decomposable such that the subspaces \(V^+\) and \(V^-\) are intrinsically complete is called a Krein space. For every decomposition \(V = V^+ \oplus V^-\) the operator \(\mathcal{J} = \text{id} \oplus -\text{id}\) defines a positive definite inner product (the \(\mathcal{J}\)-inner product) by \((\cdot, \cdot)_3 := (\cdot, \mathcal{J}\cdot)\). Such an operator \(\mathcal{J}\) is called a fundamental symmetry. It turns out that if \(V\) is a Krein space each fundamental symmetry makes \(V\) a Hilbert space. Furthermore two Hilbert space norms associated to different fundamental symmetries are equivalent. The topology induced by these norms is called the strong topology on \(V\). The theory of Krein spaces can be found in [4]. For the sake of completeness we will review in the following the main properties of linear operators on Krein spaces.
4.2 Operators on Krein spaces

If $A$ is a linear operator on a Krein space $V$ we say that $A$ is densely defined if the domain of definition $\mathcal{D}(A)$ of $A$ is strongly dense in $V$. Let $A$ be a densely defined operator on a Krein space $V$. We may define the Krein adjoint $A^+$ in the following way. Let $\mathcal{D}(A^+)$ be the set of vectors $v$, such that there is a vector $v^+$ with

$$(v, Aw) = (v^+, w) \quad \forall w \in \mathcal{D}(A).$$

We set $A^+ v := v^+$. A densely defined operator is called Krein-selfadjoint if $A = A^+$. An operator is called closed if its graph is closed in the strong topology, i.e. if the operator is closed as an operator on the Hilbert space associated to one (and hence to all) of the fundamental symmetries. If the closure of the graph of an operator $A$ in the strong topology is an operator graph, then $A$ is called closeable, the closure $\overline{A}$ is the operator associated with the closure of the operator graph. It turns out that a densely defined operator $A$ is closeable if and only if $A^+$ is densely defined. The closure of $A$ is then given by $\overline{A} = A^{++} := (A^+)^+$. Clearly, a Krein-selfadjoint operator is always closed. A densely defined operator is called essentially Krein-selfadjoint if it is closable and its closure is Krein-selfadjoint.

Note that for any fundamental symmetry we have the equality $A^+ = \mathcal{J} A^* \mathcal{J}$, if the star denotes the adjoint in the Hilbert space defined by the $\mathcal{J}$-inner product. Therefore, given a fundamental symmetry $\mathcal{J}$ and a Krein-selfadjoint operator $A$, the operators $\mathcal{J} A$ and $A^* \mathcal{J}$ are selfadjoint as operators in the Hilbert space induced by the $\mathcal{J}$-inner product. The symmetric operators $\text{Re}(\frac{1}{2}(A + A^*))$ and $\text{Re}(\frac{i}{2}(A - A^*))$ are called real and imaginary parts of $A$. The sum of the squares of these operators is formally given by $(A)^2 := \frac{1}{2}(A^* A + AA^*)$. It is natural to define the $\mathcal{J}$-modulus of $A$ as its square root. For a special class of fundamental symmetries this can be done straightforwardly. We have

**Proposition 4.1.** Let $A$ be a Krein-selfadjoint operator on a Krein space $V$. Let $\mathcal{J}$ be a fundamental symmetry, such that $\text{dom}(A) \cap \mathcal{J} \text{dom}(A)$ is dense in $V$. Let $\langle \cdot, \cdot \rangle$ be the scalar product associated with $\mathcal{J}$. Then the quadratic form

$$q(\phi_1, \phi_2) := \frac{1}{2}(\langle A\phi_1, A\phi_2 \rangle + \langle A^* \phi_1, A^* \phi_2 \rangle)$$

on $\text{dom}(A) \cap \mathcal{J} \text{dom}(A)$ is closed and the unique selfadjoint operator $(A)^{1/2}$ associated with this form commutes with $\mathcal{J}$. Therefore, it is Krein-selfadjoint. Moreover $\text{dom}((A)^{1/2}) = \text{dom}(A) \cap \mathcal{J} \text{dom}(A)$. 

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Proof. Since both $A$ and $A^*$ are closed, the quadratic form is closed as well. We repeat the
construction of the selfadjoint operator associated with this form (see [19], Theorem VIII.15). Denote $W = \text{dom}(A) \cap \overline{\text{dom}(A)}$. The pairing of the scalar product yields an inclusion of spaces $W \subset V \subset W^*$, where $W^*$ is the dual space of $W$. We define the operator $B : W \to W^*$ by $[B \phi](\psi) := q(\psi, \phi) + \langle \psi, \phi \rangle$. $B$ is in isometric isomorphism. With $\text{dom}(B) := \{ \psi \in W; B \psi \in V \}$ the operator $B := B|_{\text{dom}(B)} : \text{dom}(B) \to V$ is selfadjoint and $(A)_{3} = B - 1$. By construction $\tilde{J}$ restricts on $W$ to a norm preserving isomorphism and its adjoint map $\tilde{J}^* : W^* \to W^*$ is the continuous extension of $J$. From the definition of $\tilde{B}$ we get immediately $\tilde{B}\tilde{J} = \tilde{J}^* \tilde{B}$. Therefore, $\text{dom}((A)_{J}) = \text{dom}(B)$ is invariant under the action of $\tilde{J}$ and furthermore $(A)_{\tilde{J}}$ and $\tilde{J}$ commute. The form domain of $(A)_{\tilde{J}}$ is $\text{dom}((A)^{1/2})$, and we conclude that $\text{dom}((A)_{J}^{1/2}) = W$. 

We therefore define

**Definition 4.2.** Let $A$ be a Krein-selfadjoint operator on a Krein space $V$ and suppose that $\tilde{J}$ is a fundamental symmetry such that $\text{dom}(A) \cap \overline{\text{dom}(A)}$ is dense in $V$. Then the $\tilde{J}$-modulus $[A]_{\tilde{J}}$ of $A$ is the Krein-selfadjoint operator $(A)^{1/2}$ constructed above.

### 4.3 Ideals of operators on Krein spaces

Since all $\tilde{J}$-inner products define equivalent norms, properties of operators like boundedness and compactness, which depend only on the topological structure of the Hilbert space, carry over to Krein spaces without change. The algebra of bounded operators in a Krein space $V$ will be denoted by $B(V)$. Each fundamental symmetry $\tilde{J}$ defines a norm on $B(V)$ by $||a||_{\tilde{J}} := \sup_{v} \frac{||\tilde{J}v||}{||v||}$, where $||v||_{\tilde{J}}^2 = (v, \tilde{J}v)$. The norms on $B(V)$ induced by different fundamental symmetries are equivalent. We choose a fundamental symmetry $\tilde{J}$ and view $V$ as a Hilbert space with the $\tilde{J}$-inner product. Since $\mathcal{L}^{p+}$ are ideals in $B(V)$, we have $B^{-1} \mathcal{L}^{p+} B = \mathcal{L}^{p+}$ for any invertible operator in $B(V)$. Therefore, the definition of $\mathcal{L}^{p+}$ does not depend on the choice of scalar product and consequently it is independent of the chosen fundamental symmetry. The same argument applies to the Dixmier traces. Let $\omega \in \Gamma_{\omega}(l^{\infty})$ be fixed. Then for any $a \in L^{1+}$ and any invertible operator in $B(V)$ we have $\text{Tr}_{\omega}(B^{-1}aB) = \text{Tr}_{\omega}(a)$. Therefore, the Dixmier trace does not depend on the choice of fundamental symmetry. We conclude that both $\mathcal{L}^{p+}$ and $\text{Tr}_{\omega}$ make sense on Krein spaces without referring to a particular fundamental symmetry.
5 Clifford algebras and the Dirac operator

5.1 Clifford algebras and the spinor modules

Let \( q_{n,k} \) be the quadratic form
\[
q(x) = -x_1^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2
\]
on \( \mathbb{R}^n \). The Clifford algebra \( \text{Cl}_{n,k} \) is the algebra generated by the symbols \( c(x) \) with \( x \in \mathbb{R}^n \) and the relations
\[
x \rightarrow c(x) \text{ is linear,}
\]
\[
c(x)^2 = q_{n,k}(x).1.
\]

Let \( \text{Cl}_{n,k}^c \) be the complexification of \( \text{Cl}_{n,k} \) endowed with the antilinear involution \( +^\dagger \) defined by \( c(v)^+ = (-1)^k c(v) \). For \( n \) even the algebra \( \text{Cl}_{n,k}^c \) is isomorphic to the matrix algebra \( \text{Mat}_C(2^{\frac{n}{2}}) \), for \( n \) odd it is isomorphic to \( \text{Mat}_C(2^{\frac{n}{2}}) \oplus \text{Mat}_C(2^{\frac{n}{2}}) \).

Let \( \sigma_1, \sigma_2, \sigma_3 \) be the Pauli matrices and define
\[
\tau(j) = \begin{cases} 
i & \text{for } j \leq k \\ 1 & \text{for } j > k. \end{cases}
\]

For even \( n \) we define the isomorphism \( \Phi_{n,k} : \text{Cl}_{n,k}^c \rightarrow \text{Mat}_C(2^{\frac{n}{2}}) \) by
\[
\Phi_{n,k}(c(x_{2j+1})) := \tau(2j + 1) \cdot \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3} \otimes \sigma_1 \otimes 1 \otimes \ldots \otimes 1,
\]
\[
\Phi_{n,k}(c(x_{2j})) := \tau(2j) \cdot \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3} \otimes \sigma_2 \otimes 1 \otimes \ldots \otimes 1.
\]

Whereas for odd \( n = 2m + 1 \) we define \( \Phi_{n,k} : \text{Cl}_{n,k}^c \rightarrow \text{Mat}_C(2^{\frac{2m}{2}}) \oplus \text{Mat}_C(2^{\frac{2m}{2}}) \) by
\[
\Phi_{n,k}(c(x_j)) := \begin{cases} 
\Phi_{2m,k}(c(x_j)) \oplus \Phi_{2m,k}(c(x_j)) & \text{for } 1 \leq j \leq 2m, \\
\tau(j) \{(\sigma_3 \otimes \ldots \otimes \sigma_3) \oplus (-\sigma_3 \otimes \ldots \otimes \sigma_3)\} & \text{for } j = 2m + 1.
\end{cases}
\]

For even \( n \) the isomorphism \( \Phi_{n,k} \) gives an irreducible representation of \( \text{Cl}_{n,k}^c \) on \( \Delta_{n,k} := \mathbb{C}^{2^{\frac{n}{2}}} \), whereas for \( n \) odd we obtain an irreducible representation on \( \Delta_{n,k} := \mathbb{C}^{2^{\frac{2m}{2}}} \) by restricting \( \Phi_{n,k} \) to the first component. The restrictions of these representations to the group \( \text{Spin}(n, k) \subset \text{Cl}_{n,k} \) are the well known spinor representations on \( \Delta_{n,k} \). In the following we write \( \gamma(v) \) for the image
of $c(v)$ under this representation. In case $n$ is even we define the grading operator $\chi := i^{\frac{n(n-1)}{2}} + k \gamma(x_1) \cdots \gamma(x_n)$. We have

$$\chi^2 = 1,$$

$$\chi \gamma(v) + \gamma(v) \chi = 0. \quad (19)$$

There is no analogue to this operator in the odd dimensional case. There is a natural non-degenerate indefinite inner product on the modules $\Delta_{n,k}$ given by

$$(u, v) = i^{\frac{k(k+1)}{2}} \langle \gamma(x_1) \cdots \gamma(x_k) u, v \rangle_{\mathbb{C}^2[n^2]} . \quad (21)$$

This indefinite inner product is invariant under the action of the group $\text{Spin}(n, k)^+$ which is the double covering group of $\text{SO}(n, k)^+$. Furthermore the Krein-adjoint $\Phi_{n,k}(x)^+$ of $\Phi_{n,k}(x)$ is given by $\Phi_{n,k}(x^+)$. If $n$ is even, one gets for the grading operator $\chi^+ = (-1)^k \chi$. Up to a factor this inner product is uniquely determined by these properties.

### 5.2 Fundamental symmetries of the spinor modules

Let now $n$ and $k$ be fixed and denote by $g$ the unique bilinear form on $\mathbb{R}^n$ such that $g(v, v) = q_{n,k}(v)$. A spacelike reflection is linear map $r : \mathbb{R}^n \to \mathbb{R}^n$ with $r^2 = 1$, $g(ru, rv) = g(u, v)$ for all $u, v \in \mathbb{R}^n$ such that $g(\cdot, r\cdot)$ is a positive definite inner product. Each such reflection determines a splitting $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ into $g$-orthogonal eigenspaces of $r$ for eigenvalue $-1$ and $+1$. Clearly, $g$ is negative definite on the first and positive definite on the second summand. Conversely for each splitting of $\mathbb{R}^n$ into a direct sum of $g$-orthogonal subspaces such that $g$ is negative or positive definite on the summands determines a spacelike reflection.

To each such spacelike reflection we can associate a fundamental symmetry of the Krein space $\Delta_{n,k}$. We choose an oriented orthonormal basis $(e_1, \ldots, e_k)$ in the eigenspace for eigenvalue $-1$. Then the operator $\mathfrak{J}_r := i^{\frac{k(k+1)}{2}} \gamma(e_1) \cdots \gamma(e_k)$ is a fundamental symmetry of $\Delta_{n,k}$ and we have $\mathfrak{J}_r \gamma(v) \mathfrak{J}_r = (-1)^k \gamma(rv)$. In general, not all the fundamental symmetries are of this form. The following criterion will turn out to be useful.

**Proposition 5.1.** Let $\mathfrak{J}$ be a fundamental symmetry of the Krein space $\Delta_{n,k}$ such that for each $v \in \mathbb{R}^n \subset \mathbb{C}l_{n,k}^+$ the matrix

$$(\mathfrak{J} \gamma(v))^2 + (\gamma(v) \mathfrak{J})^2$$

of $c(v)$ under this representation. In case $n$ is even we define the grading operator $\chi := i^{\frac{n(n-1)}{2}} + k \gamma(x_1) \cdots \gamma(x_n)$. We have

$$\chi^2 = 1,$$

$$\chi \gamma(v) + \gamma(v) \chi = 0. \quad (19)$$

There is no analogue to this operator in the odd dimensional case. There is a natural non-degenerate indefinite inner product on the modules $\Delta_{n,k}$ given by

$$(u, v) = i^{\frac{k(k+1)}{2}} \langle \gamma(x_1) \cdots \gamma(x_k) u, v \rangle_{\mathbb{C}^2[n^2]} . \quad (21)$$

This indefinite inner product is invariant under the action of the group $\text{Spin}(n, k)^+$ which is the double covering group of $\text{SO}(n, k)^+$. Furthermore the Krein-adjoint $\Phi_{n,k}(x)^+$ of $\Phi_{n,k}(x)$ is given by $\Phi_{n,k}(x^+)$. If $n$ is even, one gets for the grading operator $\chi^+ = (-1)^k \chi$. Up to a factor this inner product is uniquely determined by these properties.

### 5.2 Fundamental symmetries of the spinor modules

Let now $n$ and $k$ be fixed and denote by $g$ the unique bilinear form on $\mathbb{R}^n$ such that $g(v, v) = q_{n,k}(v)$. A spacelike reflection is linear map $r : \mathbb{R}^n \to \mathbb{R}^n$ with $r^2 = 1$, $g(ru, rv) = g(u, v)$ for all $u, v \in \mathbb{R}^n$ such that $g(\cdot, r\cdot)$ is a positive definite inner product. Each such reflection determines a splitting $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^{n-k}$ into $g$-orthogonal eigenspaces of $r$ for eigenvalue $-1$ and $+1$. Clearly, $g$ is negative definite on the first and positive definite on the second summand. Conversely for each splitting of $\mathbb{R}^n$ into a direct sum of $g$-orthogonal subspaces such that $g$ is negative or positive definite on the summands determines a spacelike reflection.

To each such spacelike reflection we can associate a fundamental symmetry of the Krein space $\Delta_{n,k}$. We choose an oriented orthonormal basis $(e_1, \ldots, e_k)$ in the eigenspace for eigenvalue $-1$. Then the operator $\mathfrak{J}_r := i^{\frac{k(k+1)}{2}} \gamma(e_1) \cdots \gamma(e_k)$ is a fundamental symmetry of $\Delta_{n,k}$ and we have $\mathfrak{J}_r \gamma(v) \mathfrak{J}_r = (-1)^k \gamma(rv)$. In general, not all the fundamental symmetries are of this form. The following criterion will turn out to be useful.

**Proposition 5.1.** Let $\mathfrak{J}$ be a fundamental symmetry of the Krein space $\Delta_{n,k}$ such that for each $v \in \mathbb{R}^n \subset \mathbb{C}l_{n,k}^+$ the matrix

$$(\mathfrak{J} \gamma(v))^2 + (\gamma(v) \mathfrak{J})^2$$
is proportional to the identity. If \( n \) is even assume furthermore that \( \mathcal{J} \) commutes or anticommutes with the grading operator \( \chi \). Then there is a spacelike reflection \( r \) such that \( \mathcal{J} = \mathcal{J}_r \).

**Proof.** By assumption \( h(u, v) = (-1)^k \{ \mathcal{J} \gamma(u) \mathcal{J}, \gamma(v) \} = \{ \gamma(u)^*, \gamma(v) \} \) is a real valued bilinear form on \( \mathbb{R}^n \), where \( \{ \cdot, \cdot \} \) denotes the anti-commutator and the * is the adjoint in the \( \mathcal{J} \)-scalar product. Clearly, \( h(v, v) \geq 0 \) for all \( v \in \mathbb{R}^n \). Therefore, there exists a matrix \( a \in \text{End}(\mathbb{R}^n) \) such that

\[
\{ \gamma(u)^*, \gamma(v) \} = \{ \gamma(a u), \gamma(v) \}
\]  

(22)

for all \( u, v \in \mathbb{R}^n \). As a consequence \( \delta(u) = \gamma(u)^* - \gamma(a u) \) anti-commutes with all elements \( \gamma(v) \). In the odd dimensional case there is no such matrix other than 0 and in the even dimensional case \( \delta \) must be a multiple of the grading operator. Therefore, \( \gamma^*(v) = (-1)^k \mathcal{J} \gamma(v) \mathcal{J} \) can be written as a sum \( \gamma(a v) + \lambda(v) \chi \), where \( \lambda \) is a linear form on \( \mathbb{R}^n \). From \( \gamma(v)^* = \gamma(v) \) and \( \mathcal{J} \chi \mathcal{J} = \pm \chi \) we get \( a^2 = 1 \). For eigenvectors \( av = \pm v \) of \( a \) one gets from the equation \( (\gamma(v)^*)^2 = \gamma(v)^2 \) that \( \lambda(v)^2 = 0 \). Hence, \( \lambda = 0 \). We showed that for \( n \) even or odd we always have \( \mathcal{J} \gamma(v) \mathcal{J} = (-1)^k \gamma(av) \), for a reflection \( a \). The bilinear form \( h \) is \( h(u, v) = (u, av) \) and since it is positive semi-definite and \( a \) has trivial kernel, it is positive definite. Therefore, \( a \) is a spacelike reflection and consequently \( \mathcal{J} a \gamma(v) \mathcal{J}_a = (-1)^k \gamma(av) \). It remains to show that \( \mathcal{J}_a = \mathcal{J} \). From the above relation one gets \( \mathcal{J} \mathcal{J}_a \gamma(v) \mathcal{J}_a \mathcal{J} = \gamma(v) \), and therefore, \( \mathcal{J} \mathcal{J}_a \) commutes with all \( \gamma(v) \) and has to be a multiple of the identity. Hence, \( \mathcal{J} = z \mathcal{J}_a \) for some complex number \( z \). From \( \mathcal{J}^2 = \mathcal{J}_a^2 = 1 \), \( \mathcal{J}^+_a = \mathcal{J}_a \) and \( \mathcal{J}^+ = \mathcal{J} \) we get \( z = \pm 1 \). Since the fundamental symmetries both give rise to positive definite scalar products on \( \Delta_{n,k} \), we conclude that \( z = 1 \) and \( \mathcal{J} = \mathcal{J}_a \). \( \Box \)

### 5.3 Semi-Riemannian geometry and the Dirac operator

Let \( M \) be a smooth \( n \)-dimensional manifold. A semi-Riemannian metric \( g \) on \( M \) is a smooth section in the bundle \( T^* M \otimes T^* M \), such that for all \( x \in M \) the bilinear form \( g_x \) on \( T_x^* M \times T_x^* M \) is non-degenerate. If \( g_x(v, v) = q_{n,k}(v) \) for a special choice of basis we say that \( g_x \) has signature \( (n, k) \). If \( g_x \) has signature \( (n, k) \) for all \( x \in M \) the metric is called semi-Riemannian. If the signature is \( (n, 0) \) then the metric is called Riemannian, in case the signature is \( (n, 1) \) the metric is called Lorentzian. A vector field \( \xi \) is called timelike (spacelike, lightlike) if \( g(\xi, \xi) < 0, ~ (> 0, = 0) \). The metric can be used to identify \( T^* M \) and \( TM \) and
therefore, $g$ can be regarded as a section in $TM \otimes TM$ inducing a scalar product on $T^*_x M$. See [1] or [4] for elementary properties of semi-Riemannian manifolds.

If $(M, g)$ is a semi-Riemannian metric of signature $(n, k)$, then the tangent-bundle $TM$ can be split into an orthogonal direct sum $TM = F^k_1 \oplus F^{n-k}_2$, where $g$ is negative definite on $F^k_1$ and positive definite on $F^{n-k}_2$. For such a splitting we can define a map $r : TM \to TM$ by $r(x, k_1 \oplus k_2) := (x, -k_1 \oplus k_2)$. Then the metric $g^r$ defined by $g^r(a, b) := g(a, rb)$ is positive definite. Conversely suppose there is an endomorphism of vector bundles $r : TM \to TM$ with $g(r\cdot, r\cdot) = g$, $r^2 = \text{id}$ and such that $g^r := g(\cdot, r\cdot)$ is positive definite. Then there is a splitting such that $r(x, k_1 \oplus k_2) = (x, -k_1 \oplus k_2)$. We call such maps spacelike reflections. Obviously, $r : TM \to TM$ is a spacelike reflection, if the restrictions of $r$ to the fibres $T_x M$ are spacelike reflections in the sense of the last section. In the following we call $g^r$ the Riemannian metric associated with $r$.

In case the bundle $TM$, $(F^k_1, F^{n-k}_2)$ is orientable the manifold is called orientable (time-orientable, space-orientable). Assume we are given an orientable, time-orientable semi-Riemannian manifold $(M, g)$ of signature $(n, k)$. Then the bundle of oriented orthonormal frames is an $\text{SO}(n, k)^+$-principal bundle.

We saw that the metric information of a Riemannian manifold can be encoded in a spectral triple, where $D$ was any Dirac type operator on some hermitian vector bundle $E$. In the case of semi-Riemannian manifolds there arises a major problem. Namely that Dirac type operators are not selfadjoint any more. We will see however that there exists a Krein space structure on the space of sections of $E$ such that there are Krein-selfadjoint Dirac type operators. Assume now that $M$ is an orientable time-orientable semi-Riemannian manifold. Let $E$ be a vector bundle over $M$ and assume that $D$ is of Dirac type. This means that $D$ is a first order differential operator and the principal symbol $\sigma$ of $D$ satisfies the relation

$$\sigma(\xi)^2 = g(\xi, \xi) \text{id}_{E_p}, \quad \forall \xi \in T^*_p M, p \in M. \quad (23)$$

Therefore, $\gamma := \sigma$ satisfies the Clifford relations, which makes $E$ a module for the Clifford algebra bundle. Let $r : TM \to TM$ be a spacelike reflection and identify $TM$ with $T^* M$ using the metric. Let $T^* M = F^k_1 \oplus F^{n-k}_2$ be the splitting such that $r(x, k_1 \oplus k_2) = (x, -k_1 \oplus k_2)$. Then there is a hermitian structure $\langle \cdot, \cdot \rangle$ on $E$ such that $\sigma_x(\xi)$ is anti-symmetric if $\xi \in F^k_1$ and symmetric if $\xi \in F^{n-k}_2$. Let $e_1, \ldots, e_k$ be a local oriented orthonormal frame for $F^k_1$ and define $\mathcal{J} := i^{k(k+1)/2} \gamma(e_1) \cdots \gamma(e_k)$. $\mathcal{J}$ is independent of the choice of frames and the indefinite inner product $\langle \cdot, \cdot \rangle_x := \langle \cdot, \mathcal{J}(x)\cdot \rangle_x$ on $E_x$ is non-degenerate. It makes $E$ an non-degenerate indefinite inner product bundle. Moreover, $i^k \sigma_x$ is symmetric with
respect to this indefinite inner product. The space of square integrable sections of $E$ is a Krein space endowed with the indefinite inner product structure

$$(f, g) := \int_M (f_x, g_x)_x \sqrt{|g|} dx.$$  \hspace{1cm} (24)

To each spacelike reflection $r'$ we can associate a fundamental symmetry $\mathcal{J}_{r'}$ of this Krein space by $\mathcal{J}_{r'} := i^{k(k+1)/2} \gamma(e_1) \cdots \gamma(e_k)$, where $e_1, \ldots, e_k$ is a local oriented orthonormal frame for $F_{1}^{k}$. We conclude that for a time-orientable orientable semi-Riemannian manifold there exists a Dirac type operator $D$ on some non-degenerate indefinite inner product vector bundle $E$ such that $i^kD$ is symmetric with respect to this inner product. The following theorem was proved for Dirac operators on spin manifolds in [2]. For the sake of completeness and since the original proof is in german, we give a proof here.

**Theorem 5.2 (Baum, 1981).** Let $E$ be a non-degenerate indefinite inner product vector bundle over an orientable time-orientable semi-Riemannian manifold $M^{n,k}$. Let $D : \Gamma_0(E) \to \Gamma_0(E)$ be a symmetric differential operator such that $i^kD$ is of Dirac type. If there exists a spacelike reflection $r$ such that the Riemannian metric associated with this reflection is complete, then $D$ is essentially Krein-selfadjoint. In particular, if $M$ is compact then $D$ is always essentially Krein-selfadjoint.

**Proof.** Let $\mathcal{J}$ be the fundamental symmetry associated with the splitting and let $L^2(E)$ be the Hilbert space of sections which are square integrable with respect to the positive definite inner product induced by $\mathcal{J}$. We denote this scalar product in the following by $\langle \cdot, \cdot \rangle$. It is clearly sufficient to show that $P = \mathcal{J}D$ is essentially selfadjoint in $L^2(E)$. Note that $P$ is a first order differential operator which is symmetric in $L^2(E)$. Therefore, it is closeable. The proof consists of two steps. Let $\text{dom}_0(P^*)$ be the intersection of $\text{dom}(P^*)$ with the space of compactly supported square integrable section. We first show that $\text{dom}_0(P^*) \subset \text{dom}(\mathcal{P})$. In the second step we show that $\text{dom}_0(P^*)$ is dense in the Hilbert space $\text{dom}(P^*)$ endowed with the scalar product $\langle x, y \rangle_{P^*} := \langle x, y \rangle + \langle P^* x, P^* y \rangle$. The combination of these results shows that $\text{dom}(\mathcal{P})$ is dense in the Hilbert space $\text{dom}(P^*)$, and therefore, $P$ is essentially selfadjoint.

**First step:** Note that since $P$ is symmetric, the operator $P^*$ is a closed extension of $P$, and furthermore the adjoint operator $P' : \mathcal{D}'(E) \to \mathcal{D}'(E)$ is the continuous extension of $P$ and of $P^*$ to the space of distributions. Assume that $f$ is in $\text{dom}_0(P^*)$. Then both $f$ and $g = P^*f$ have compact support. Clearly, $f$ is a weak
solution to the equation $Pf = g$, hence, it is also a strong solution (see e.g. [21], Prop. 7.4), i.e. there is a sequence $f_n$ converging to $f$ in the $L^2$-sense such that $Pf_n$ converges to $g$ also in the $L^2$-sense. Therefore, $f$ is in $\text{dom}(P)$.

**Second step:** Assume that $f \in \text{dom}(P^*)$. We will construct a sequence $f_n$ in $\text{dom}_0(P^*)$ such that $f_n \to f$ and $Pf_n \to P^*f$ in the $L^2$-sense. Fix an $x_0 \in M$ and let $\text{dist}(x)$ be a regularized distance function from $x_0$ in the complete Riemannian metric associated with the splitting. Choose a function $\chi \in C_0^\infty(\mathbb{R})$ with $0 \leq \chi \leq 1$, $\chi(t) = 0$ for $t \geq 2$, $\chi(t) = 1$ for $t \leq 1$, and $|\chi'| \leq 2$. We set $\chi_n(x) := \chi(\frac{1}{n}\text{dist}(x))$. By completeness of the manifold, all $\chi_n$ are compactly supported. We define the sequence $f_n := \chi_nf$ and clearly, $f_n \in \text{dom}_0(P^*)$. Denoting by $\sigma$ the principal symbol of $D$, we have $P'f_n = -i\overline{\sigma}(d\chi_n)f + \chi_nP'f$. Clearly, $\chi_nP'f \to P'f$ in the $L^2$-sense. For the first summand we have the estimate

$$
\|\overline{\sigma}(d\chi_n)f\|^2 \leq \int_{B_{2n} - B_n} \frac{4}{n^2} \|f\|^2,
$$

(25)

where $B_r$ denotes the metric ball with radius $r$ centred at $x_0$. Since the right hand side vanishes in the limit $n \to \infty$, we conclude that $f_n \to f$ and $Pf_n \to P^*f$ in the $L^2$-sense. Therefore, $\text{dom}_0(P^*)$ is dense in the Hilbert space $\text{dom}(P^*)$.

**Example 5.3.** A spin structure on a time-oriented oriented semi-Riemannian manifold $M^{n,k}$ is a Spin$(n, k)^+$-principal bundle $P$ over $M^{n,k}$ together with a smooth covering $\eta$ from $P$ onto the bundle $Q$ of oriented orthonormal frames, such that the following diagram is commutative.

$$
\begin{array}{ccc}
P \times \text{Spin}(n, k)^+ & \longrightarrow & P \\
\downarrow \eta \times \lambda & & \downarrow \eta \\
Q \times \text{SO}(n, k)^+ & \longrightarrow & Q \\
\end{array}
\quad (26)
$$

Here $\lambda$ denotes the covering map $\text{Spin}(n, k)^+ \to \text{SO}(n, k)^+$. The spinor bundle $S$ associated with a Spin structure is the associated bundle $P \times_\pi \Delta_{n,k}$, where $\pi$ denotes the representation of Spin$(n, k)^+$ on $\Delta_{n,k}$. Let $\nabla : \Gamma(S) \to \Gamma(S) \otimes \Lambda^1$ be the Levi-Civita connection on the spinor bundle. The Dirac operator $\slashed{D}$ is defined by $-i\gamma \circ \nabla$, where $\gamma$ denotes the action of covector fields on sections of the spinor bundle by Clifford multiplication. $\slashed{D}$ is clearly of Dirac type and it was shown in [3] (see also [3]) that the space of square integrable sections of $S$ is a Krein space such that $i^k\slashed{D}$ is symmetric.
6 Semi-Riemannian spectral triples

**Definition 6.1.** A semi-Riemannian spectral triple is a tuple \((\mathcal{A}, D, \mathcal{H})\), where \(\mathcal{A}\) is a \(\ast\)-algebra of bounded operators on a Krein space \(\mathcal{H}\) such that \(a^* = a^+\), and \(D\) is a Krein-selfadjoint operator on \(\mathcal{H}\), such that the commutators \([D, a]\) are bounded for all \(a \in \mathcal{A}\). A semi-Riemannian spectral triple is called even if there is a distinguished operator \(\chi\), anticommuting with \(D\) and commuting with \(\mathcal{A}\) and with \(\chi^2 = 1\) and \(\chi^+ = \pm \chi\). If such an operator does not exist we say the spectral triple is odd and set by definition \(\chi = 1\). We call \(\chi\) the grading operator.

For a semi-Riemannian spectral triple we can repeat the construction of differential forms almost unchanged. Denote again the universal differential envelope of \(\mathcal{A}\) by \((\Omega \mathcal{A}, d)\). Clearly, the map

\[
\pi : \Omega \mathcal{A} \to B(\mathcal{H}),
\]

\[
\pi((a_0, a_1, \ldots, a_n)) := a_0[D, a_1] \cdots [D, a_n], \quad a_j \in \mathcal{A}
\]

is a representation of \(\Omega \mathcal{A}\) on \(\mathcal{H}\) such that \(\pi(a^*) = \pi(a)^+\) for all \(a \in \Omega \mathcal{A}\), where \(+\) denotes the Krein adjoint. We define the graded two sided ideal \(j_0 \oplus j_0^+\) by

\[
j_0^n := \{ \omega \in \Omega^n \mathcal{A}; \pi(\omega) = 0 \},
\]

(27)

and as in the case of spectral triples \(j = j_0 + dj_0\) is a graded two-sided differential ideal. We define \(\Omega_D \mathcal{A} := \Omega \mathcal{A}/j\). Clearly, \(\Omega_D^n \mathcal{A} \simeq \pi(\Omega^n \mathcal{A})/\pi(dj_0 \cap \Omega^n \mathcal{A})\).

**Example 6.2.** Suppose that \(M^{n,k}\) is a compact time-orientable orientable semi-Riemannian spin manifold with spinor bundle \(E\). Let \(\mathcal{H}\) be the Krein space of square integrable sections of \(E\) and let \(D = \frac{i}{\sqrt{2}} \mathcal{D}\), where \(\mathcal{D}\) is the Dirac operator. Then the triple \((C^\infty(M), D, \mathcal{H})\) is a semi-Riemannian spectral triple, and in the same way as this is done for the Riemannian case one shows that as a graded differential algebra \(\Lambda_D C^\infty(M)\) is canonically isomorphic to the algebra of differential forms on \(M\). We call this triple the canonical triple associated with \(M\).

For Riemannian spin manifolds the differential structure is encoded in the Dirac operator. For example the space of smooth sections of the spinor bundle coincides with the space \(\bigcap_n \text{dom}(D^n)\). This is essentially due to the ellipticity of the Dirac operator. In the semi-Riemannian case the Dirac operator is not elliptic any more and sections in \(\bigcap_n \text{dom}(D^n)\) may be singular in the lightlike directions. We will circumvent this problem by introducing the notion of a smooth semi-Riemannian spectral triple. Let \((\mathcal{A}, D, \mathcal{H})\) be a semi-Riemannian spectral triple.
and suppose there is a fundamental symmetry, such that \( \text{dom}(D) \cap \mathcal{F} \text{dom}(D) \) is dense in \( \mathcal{H} \). Then the operator \( \Delta_3 := ([D]^2 + 1)^{1/2} \) is a self-adjoint operator on the Hilbert space \( \mathcal{H} \) with scalar product \( (\cdot, \cdot) \). Let \( \mathcal{H}_3^s \) be the closure of \( \bigcap_n \text{dom}(\Delta_3^n) \) in the norm \( \|\psi\|_s := \|\Delta_3^s \psi\| \). We have for \( s > 0 \) the equality \( \mathcal{H}_3^s = \text{dom}(\Delta_3^n) \), and the indefinite inner product on \( \mathcal{H} \) can be used to identify \( \mathcal{H}_3^s \) with the topological dual of \( \mathcal{H}_3^{-s} \). We define \( \mathcal{H}_3^{\infty} := \bigcap_s \mathcal{H}_3^s \) and \( \mathcal{H}_3^{-\infty} := \bigcup_s \mathcal{H}_3^s \). A map \( a : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty} \) is said to be in \( \text{op}^r\mathcal{H}^s \) if it continuously maps \( \mathcal{H}^s \) to \( \mathcal{H}^{s-r} \).

Clearly, \( \Delta_3 \in \text{op}^1\mathcal{H}^0 \) and \( \mathcal{F} \in \text{op}^0\mathcal{H}^0 \). We introduce an equivalence relation on the set of fundamental symmetries \( \mathcal{F} \) such that \( \text{dom}(D) \cap \mathcal{F} \text{dom}(D) \) is dense in \( \mathcal{H} \) in the following way. We say that \( \mathcal{F}_1 \sim \mathcal{F}_2 \) if \( \mathcal{H}_{3,1}^s = \mathcal{H}_{3,2}^s \) as topological vector spaces. If we are dealing with a distinguished equivalence class, we will leave away the index \( \mathcal{F} \) and write e.g \( \text{op}^r\mathcal{F} \) for \( \text{op}^r\mathcal{H}^s \) and \( \mathcal{F} \) for \( \mathcal{F}_3^s \), since these objects clearly depend only on the equivalence class of \( \mathcal{F} \). The spaces \( \text{op}^r\mathcal{F} \) have been introduced in [7] in the context of spectral triples.

**Definition 6.3.** A smooth semi-Riemannian spectral triple is a semi-Riemannian spectral triple \((A, D, \mathcal{H})\) together with a distinguished non-empty equivalence class of fundamental symmetries \( \mathcal{F} \), such that \( D \in \text{op}^1\mathcal{F} \). We say a fundamental symmetry is smooth if \( \mathcal{F} \in \mathcal{F} \).

**Example 6.4.** Let \( M^{n,k} \) be a compact orientable time-orientable semi-Riemannian spin manifold and let \((A, D, \mathcal{H})\) be its canonical semi-Riemannian spectral triple. Let \( E \) be the spinor bundle. For each spacelike reflection \( r \) we constructed in the previous section a fundamental symmetry \( \mathcal{F}_r \) of the Krein space \( \mathcal{H} \). The fundamental symmetries of the form \( \mathcal{F}_r \) belong to one and the same equivalence class and therefore define a smooth semi-Riemannian spectral triple. This can most easily be seen using the calculus of pseudodifferential operators. Note that \( \Delta_3 \) is an elliptic classical pseudodifferential operator of order 1. Therefore, \( \mathcal{H}_3^s \) coincides with the Sobolev space \( H_s(M, E) \) of sections of \( E \) and \( \mathcal{H}^{\infty} \) coincides with the space of smooth sections \( \Gamma(E) \). In the following we will think of the canonical triple associated with \( M \) as a smooth semi-Riemannian spectral triple with the above smooth structure.

Suppose that \((A, D, \mathcal{H})\) is a smooth semi-Riemannian spectral triple. Let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be smooth fundamental symmetries. Then \( \Delta_3^{1} \) is in \( \text{op}^{-1} \) and therefore, \( \Delta_3^{1} \Delta_3^{-1} \) is bounded. As a consequence \( \Delta_3^{-1} \) is in \( \mathcal{L}^{p+} \) if and only if \( \Delta_3^{1} \) is in \( \mathcal{L}^{p+} \).

**Definition 6.5.** We say a smooth semi-Riemannian spectral triple \((A, D, \mathcal{H})\) is \( p^+ \)-summable if for one (and hence for all) smooth fundamental symmetries \( \mathcal{F} \) the operator \( \Delta_3^{-1} \) is in \( \mathcal{L}^{p+} \).
For the canonical triple associated with a semi-Riemannian spin manifold we have a distinguished set of fundamental symmetries, namely those which are of the form $\mathcal{J} = \mathcal{J}_r$ for some spacelike reflection $r$. We may ask now if there is an analogue of this set in the general case.

**Definition 6.6.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a smooth semi-Riemannian spectral triple. We say a fundamental symmetry $\mathcal{J}$ is admissible if

1. $\mathcal{J}$ is smooth,
2. $\mathcal{J}\chi\mathcal{J} = \chi^+$,
3. $\mathcal{J}$ commutes with all elements of $\mathcal{A}$,
4. $\mathcal{J}\pi(\Omega^p\mathcal{A})\mathcal{J} = \pi(\Omega^p\mathcal{A})$.
5. $\mathcal{J}\pi(d_j \cap \Omega^p\mathcal{A})\mathcal{J} = \pi(d_j \cap \Omega^p\mathcal{A})$ if $p \geq 2$.

If $\mathcal{J}$ is admissible and $^*$ denotes the adjoint in the $\mathcal{J}$-inner product, then the above conditions imply that $a^* = a^+$ for all $a \in \mathcal{A}$, $\chi^* = \chi$ and that $^*$ leaves the spaces $\Omega^p_D\mathcal{A}$ invariant. The following theorem shows that in the case of a canonical triple associated with a spin manifold the set of admissible fundamental symmetries is canonically isomorphic to the set of spacelike reflections.

**Theorem 6.7.** Suppose that $M^{n,k}$ is an orientable time-orientable compact semi-Riemannian spin manifold and let $(\mathcal{A}, D, \mathcal{H})$ be its canonical smooth triple with grading operator $\chi$. Then the set of admissible fundamental symmetries coincides with the set

$$\{\mathcal{J}_r : r \text{ is a spacelike reflection}\}.$$

**Proof.** We first show that $\mathcal{J}_r$ is admissible. Clearly, $\mathcal{J}_r$ commutes with all elements of $\mathcal{A}$ and $\mathcal{J}\chi\mathcal{J} = \chi^+$. Moreover $\mathcal{J}_r$ is smooth by construction (see Example 6.4). We need to show that the sets $\pi(\Omega^p\mathcal{A})$ and $\pi(d_j \cap \Omega^p\mathcal{A})$ are invariant under conjugation by $\mathcal{J}_r$. We denote by $\gamma$ the principal symbol of $D$. Since $[D, f] = -(i)^{k+1}\gamma(df)$, the space $\pi(\Omega^p\mathcal{A})$ is the set of operators of the form $\sum f^j(1)^j \gamma(v_1^j) \cdots \gamma(v_p^j)$, where $f^j \in C^\infty(M)$ and $v_1^j, \ldots, v_p^j \in \Gamma(T^*M)$. Since $\mathcal{J}_r \gamma(v)\mathcal{J}_r = (-1)^k \gamma(rv)$, this space is invariant under conjugation by $\mathcal{J}_r$. The proof of Proposition 7.2.2 in [16] shows that $\pi(d_j \cap \Omega^p\mathcal{A})$ coincides with the set of operators of the form $\sum f^j \gamma(v_1^j) \cdots \gamma(v_{p-2}^j)$, where $f^j \in \mathcal{A}$, $v_1^j, \ldots, v_{k}^j \in T^*M$. 17
This set is also invariant under conjugation by $\mathfrak{J}_r$, and we conclude that $\mathfrak{J}_r$ is admissible. Suppose now we have another admissible fundamental symmetry $\mathfrak{J}$ in $[\mathfrak{J}_r]$. Since $\mathfrak{J} \in \text{op}^0$, it acts continuously on $\Gamma(E)$ and since $\mathfrak{J}$ commutes with $\mathcal{A}$, it leaves the fibres invariant. It follows that $\mathfrak{J}$ is a smooth endomorphism of the spinor bundle. For a point $x \in M$ we denote by $\mathfrak{J}(x)$ the restriction of $\mathfrak{J}$ to the fibre at $x$. Since $\mathfrak{J}$ is admissible, $\mathfrak{J} \cdot \mathfrak{J}$ must leave the space of one-forms invariant. This implies that for all $v \in T^*_x M$ the matrix $\mathfrak{J}(x) \gamma(v) \mathfrak{J}(x)$ is again of the form $\gamma(u)$ for some $u \in T^*_x M$. By Prop. 5.1 there exists a spacelike reflection on the fibre at $x$ inducing $\mathfrak{J}(x)$. Therefore, there is spacelike reflection $r$ such that $\mathfrak{J} = \mathfrak{J}_r$.

**Theorem 6.8.** Let $M^{n,k}$ be a compact orientable time-orientable semi-Riemannian spin manifold and let $(\mathcal{A}, D, \mathcal{H})$ be the canonical smooth semi-Riemannian spectral triple associated with $M$. Then $(\mathcal{A}, D, \mathcal{H})$ is $n^+$-summable and for each $f \in C^\infty(M)$ and each admissible fundamental symmetry $\mathfrak{J}$ we have

$$\int_M f = c(n) \text{Tr}_\omega(f \Delta^{-n}_3),$$

where integration is taken with respect to the semi-Riemannian volume form $\sqrt{|g|}$ and $c(n) = 2^{-n-[n/2]-1} \pi^{n/2} n \Gamma(n/2)$. Moreover with the same $f$ and $\mathfrak{J}$

$$\text{Tr}_\omega(f D^2 \Delta^{-n-2}_3) = (-1)^k \frac{n-2k}{n} \text{Tr}_\omega(f \Delta^{-n}_3).$$

**Proof.** Let $g^r$ be the Riemannian metric associated with a spacelike reflection. By construction the metric volume form of $g^r$ coincides with the metric volume form of the semi-Riemannian metric. Now the principal symbol $\sigma_1$ of $\Delta_3$ is given by $\sigma_1(k) = \sqrt{g^r(k,k)}$ for covectors $k \in T^* M$. Connes trace formula gives equation 28. What is left is to show that equation 29 holds. The operator $D^2 \Delta^{-n-2}_3$ is a classical pseudodifferential operator of order $-n$ and its principal symbol $\sigma_2$ is given by $\sigma_2(k) = (-1)^k g(k,k) g^r(k,k)^{-n/2-1}$. Therefore, the principal symbol of $f D^2 \Delta^{-n-2}_3$ is $f \sigma_2$. In order to calculate the relevant Dixmier trace we have to integrate this symbol over the cosphere bundle in some Riemannian metric. The result will be independent of the chosen Riemannian metric. In case $\mathfrak{J} = \mathfrak{J}_r$ we use $g^r$ to integrate. On the cosphere bundle $\sigma_2$ restricts to $g(k,k)$. Therefore,

$$\text{Tr}_\omega(f D^2 \Delta^{-n-2}_3) = \frac{1}{c(n)} (-1)^k \text{Vol}(S^{n-1})^{-1} \int_{S^* M} f \cdot g.$$

18
For local integration we can choose an oriented orthonormal frame $k_1, \ldots, k_n$ such that $g(k_i, k_i) = -1$ for $i = 1 \ldots k$ and $g(k_i, k_i) = 1$ for $i = k + 1 \ldots n$. This shows that

$$\int_{S^n} f \cdot g = (-1)^k \left( \int_M f \right) \cdot \int_{S^{n-1}} (-\xi_1^2 - \ldots - \xi_k^2 + \xi_{k+1}^2 + \ldots + \xi_n)$$

$$= (-1)^k \text{Vol}(S^{n-1}) \frac{n-2k}{n} \int_M f,$$

which concludes the proof. 

Equation 29 shows that one can indeed recover the signature from the spectral data and that the notion of integration is independent of the chosen admissible fundamental symmetry.

The conditions for a fundamental symmetry to be admissible are in a sense minimal and it is not clear at this point that one does not need further conditions in order to get a sensible noncommutative geometry. For example one may require in addition that the set $\mathcal{A} \cup [D, \mathcal{A}]$ is contained in the domain of smoothness of the derivation $\delta_3(\cdot) = [\Delta_3, \cdot]$. This is clearly true for admissible fundamental symmetries in the case of a canonical spectral triple associated with a manifold. In the general case however we can not expect this to hold. We think it is also worth noting that for the classical situation there exist a number of equivalent definitions of admissibility. For example one has

**Proposition 6.9.** Let $(\mathcal{A}, D, \mathcal{H})$ be as in Theorem 5.8. Assume that $\mathcal{J}$ is a smooth fundamental symmetry that commutes with all elements of $\mathcal{A}$ and $\mathcal{J}\chi = \chi^+$. Then $\mathcal{A} \cup [D, \mathcal{A}]$ is contained in the domain of the derivation $\delta_3$ if and only if $\mathcal{J}$ is admissible.

**Proof.** By assumption $\mathcal{J}$ is a smooth endomorphism of the spinor bundle. Let $\mathcal{J}(x)$ be the restriction to the fibre at $x$. Denote by $\sigma$ the principal symbol of $D$. The principal symbol $A$ of the second order pseudodifferential operator $\Delta_3^2$ is given by $A_x(v) = \frac{1}{2}(3\sigma_x(v)3\sigma_x(v) + \sigma_x(v)3\sigma_x(v)3) \cdot \int T_0^*M$. The principal symbol of $\Delta_3$ is $A^{1/2}$. Assume now that $[\Delta_3, a]$ is bounded for all $a$ in $\mathcal{A} \cup [D, \mathcal{A}]$. Then the principal symbol of the first order operator $[\Delta_3, a]$ must vanish. This implies that $A^{1/2}_x(v)$ commutes with all $\sigma_x(u); u \in T_x^*M$. Since the Clifford action is irreducible, $A_x(v)$ is a multiple of the identity and by Prop. 5.1 we have $\mathcal{J} = \mathcal{J}^r$ for some spacelike reflection $r$. 

\[ \square \]
7 The noncommutative tori

Definition 7.1. Let $\theta$ be a pre-symplectic form on $\mathbb{R}^n$. We denote by $A_\theta$ the unital $C^*$-algebra generated by symbols $u(y)$, $y \in \mathbb{Z}^n$ and relations

\begin{align*}
u(y)^* &= u(y)^{-1} \
u(y_1)u(y_2) &= e^{i\pi\theta(y_1,y_2)}u(y_1 + y_2).
\end{align*}

Let $S(\mathbb{Z}^n)$ be the Schwarz space over $\mathbb{Z}^n$, i.e. the space of functions on $\mathbb{Z}^n$ with

\[
sup_{y \in \mathbb{Z}^n} (1 + |y|^2)^p|a(y)|^2 < \infty \quad \forall p \in \mathbb{N}.
\]

The rotation algebra $A_\theta$ is defined by

\[
A_\theta := \left\{ a = \sum_{y \in \mathbb{Z}^n} a(y)u(y) ; a \in S(\mathbb{Z}^n) \right\}.
\]

It is well known that the linear functional $\tau : A_\theta \to \mathbb{C}$ defined by

\[
\tau \left( \sum_{y \in \mathbb{Z}^n} a(y)u(y) \right) := a(0),
\]

is a faithful tracial state over $A_\theta$. In particular we have $\tau(a^*a) = \sum_{y \in \mathbb{Z}^n} |a(y)|^2$.

Note that $A_\theta$ is generated by the elements $u_k := u(e_k)$, where $e_k$ are the basis elements in $\mathbb{Z}^n$. They satisfy the relations

\begin{align*}
u_k^* &= u_k^{-1} \
u_ku_i &= e^{2\pi i\theta_{ik}}u_iu_k
\end{align*}

Definition 7.2. The basic derivations $\delta_1, \ldots, \delta_n$ on $A_\theta$ are defined by

\[
\delta_j \left( \sum_{y \in \mathbb{Z}^n} a(y)u(y) \right) := 2\pi i \sum_{y \in \mathbb{Z}^n} y_ja(y)u(y).
\]

One checks easily that these are indeed derivations.
Let $\mathcal{H}_\tau$ be the GNS-Hilbert space of the state $\tau$. Since $\tau$ is faithful, $\mathcal{H}_\tau$ coincides with the closure of $\mathcal{A}_\theta$ in the norm $\|a\|_\tau^2 = \tau(a^*a)$. The basic derivations extend to closed skew-adjoint operators on $\mathcal{H}_\tau$. Denote by $\mathbb{R}^{n,k}$ the vector space $\mathbb{R}^n$ endowed with the indefinite metric $q_{n,k}$ and let $\text{Cl}_{n,k}^e$ be the corresponding Clifford algebra. Let $\Delta_{n,k}$ be the natural Clifford module for $\text{Cl}_{n,k}^e$. Denote by $\gamma_i$ the representation of $\mathbb{R}^n \subset \text{Cl}_{n,k}^e$ on $\Delta_{n,k}$. We choose a basis $\{e_i\}$ in $\mathbb{R}^n$ such that the $\gamma_i := \gamma(e_i)$ satisfy $\gamma_i^2 = -1$ for $i = 1, \ldots, k$ and $\gamma_i^2 = +1$ for $i = k + 1, \ldots, n$. We have

**Proposition 7.3.** Let $\mathcal{H} = \mathcal{H}_\tau \otimes \Delta_{n,k}$ and let $D$ be the closure of the operator

$$D_0 := i^{k-1} \left( \sum_{i=1}^{n} \gamma_i \delta_i \right)$$

on $\mathcal{H}$ with domain $\text{dom}(D_0) = \mathcal{A}_\theta \otimes \Delta_{n,k}$. Then $\mathcal{H}$ is a Krein space with the indefinite inner product defined by

$$(\psi_1 \otimes v_1, \psi_2 \otimes v_2) := \langle \psi_1, \psi_2 \rangle_{\mathcal{H}_\tau}(v_1, v_2)_{\Delta_{n,k}},$$

and $(\mathcal{A}_\theta, \mathcal{H}, D)$ is a semi-Riemannian spectral triple. If $n$ is even the triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ is even.

**Proof.** Since $\Delta_{n,k}$ is finite dimensional and decomposable, each decomposition of $\Delta_{n,k} = V^+ \oplus V^-$ into positive and negative definite subspaces gives rise to a decomposition $\mathcal{H} = \mathcal{H}_\tau \otimes V^+ \oplus \mathcal{H}_\tau \otimes V^-$. Clearly, the subspaces are intrinsically complete. Therefore, $\mathcal{H}$ is a Krein space. Next we show that $D_0$ is essentially Krein-selfadjoint on $\mathcal{H}$. Clearly, $\mathfrak{J} := i^{k(k+1)/2} \gamma_1 \cdots \gamma_k$ is a fundamental symmetry of $\mathcal{H}$ and it is enough to show that the symmetric operator $\mathfrak{J} D_0$ is essentially selfadjoint on $\mathcal{H}$ endowed with the scalar product induced by $\mathfrak{J}$. The vectors $u(y) \in \mathcal{A}_\theta \subset \mathcal{H}_\tau$ form a total set in $\mathcal{H}_\tau$, and it is easy to see that the vectors of the form $u(y) \otimes \psi$ are analytic for $\mathfrak{J} D_0$. By Nelson’s theorem $\mathfrak{J} D_0$ is essentially selfadjoint on $\mathcal{A}_\theta \otimes \Delta_{n,k} \subset \mathcal{H}$ and therefore, $D_0$ is essentially Krein-selfadjoint. For all $a \in \mathcal{A}_\theta$ we have $[D, a] = i^{k-1} \sum_i \gamma_i \delta_i$, which is clearly a bounded operator. Hence, $(\mathcal{A}_\theta, \mathcal{H}, D)$ is a semi Riemannian spectral triple. For even $n$ this triple is even and the grading operator $\chi$ is just the grading operator in $\text{Cl}_{n,k}^e$ acting on the second tensor factor. In case $n$ is odd the triple is odd and we set $\chi = 1$.

In the following we will need the image of the universal differential forms and the junk forms under the representation $\pi : \Omega \mathcal{A}_\theta \rightarrow \mathcal{B}(\mathcal{H})$ associated with $(\mathcal{A}_\theta, \mathcal{H}, D)$.
Lemma 7.4. For the above defined semi-Riemannian spectral triple we have

\[
\pi(\Omega^m A_\theta) = \left\{ \sum_j a^j \gamma(v_1^j) \cdots \gamma(v_m^j) ; \ a^j \in A_\theta, v_i^j \in \Delta_{n,k} \right\}, \quad (42)
\]

\[
\pi(dj_0 \cap \Omega^m A_\theta) = \left\{ \sum_j a^j \gamma(v_1^j) \cdots \gamma(v_{m-2}^j) ; \ a^j \in A_\theta, v_i^j \in \Delta_{n,k} \right\}. \quad (43)
\]

Proof. The first equation follows from the relation \([D,a] = i^{k-1} \sum_i \gamma_i \delta_i(a)\). It remains to show that the second equation holds. Let \(\omega\) be the \((m-1)\)-form \((f_0 df_0 - df_0 f_0) df_1 \cdots df_{m-2}\) with \(f_0 = u_1\). We have \(\delta_i f_0 = 2\pi i \delta_{i_0} f_0\). A short calculation shows that

\[
\pi(\omega) = 0
\]

and therefore, the form

\[
\pi(d\omega) = -8\pi^2 \gamma_0^2 f_0^2 [D, f_1] \cdots [D, f_m]
\]

is an element of \(\pi(dj_0 \cap \Omega^m A_\theta)\). The \(A_\theta\)-module generated by this form is the set \(A^m\) of elements of the form \(\sum_j a^j \gamma(v_1^j) \cdots \gamma(v_{m-2}^j)\) with \(a^j \in A_\theta\) and \(v_1^j, \ldots, v_{k-2}^j \in \Delta_{n,k}\). Therefore, \(A^m \subset \pi(dj_0 \cap \Omega^m A_\theta)\). In case \(m - 1 \geq n\) this shows that \(\pi(\Omega^m A_\theta) = \pi(dj_0 \cap \Omega^m A_\theta)\) and the above formula is a consequence of this. We treat the case \(m \leq n\). Suppose that \(\omega = \sum_j f_0^j df_1^j \cdots df_{m-1}^j\) and that

\[
\pi(\omega) = \gamma_{\mu_1} \cdots \gamma_{\mu_{m-1}} \sum_j f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}} f_{m-1}^j = 0. \quad (45)
\]

This implies that

\[
\sum_j f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}} f_{m-1}^j = 0, \quad (46)
\]

where the square bracket indicates the complete anti-symmetrization of the indices. If we apply \(\delta_{\mu_0}\) to the left of this equation and anti-symmetrize in all indices we obtain

\[
\sum_j \delta_{\mu_0} f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}} f_{m-1}^j = 0. \quad (47)
\]

Since

\[
\pi(d\omega) = \gamma_{\mu_0} \cdots \gamma_{\mu_{m-1}} \sum_j \delta_{\mu_0} f_0^j \delta_{\mu_1} f_1^j \cdots \delta_{\mu_{m-1}} f_{m-1}^j,
\]

we finally obtain \(\pi(d\omega) \in A^m\). \(\square\)
The above lemma implies that \( \Omega_D \mathcal{A}_\theta \cong \bigoplus_m \mathcal{A}_\theta \otimes \Lambda^m \mathbb{R}^n \) and the differential is given by \( d(a \otimes e_1 \wedge \ldots \wedge e_k) = \sum_i \delta_i(a) \otimes e_i \wedge e_1 \wedge \ldots \wedge e_k \). Here \( e_i \) is a distinguished basis in \( \mathbb{R}^n \).

Each spacelike reflection in \( \mathbb{R}^{n,k} \) induces a fundamental symmetry \( \tilde{\mathcal{J}}_r \) of \( \Delta_{n,k} \) and clearly, \( \tilde{\mathcal{J}}_r := \text{id} \otimes \tilde{\mathcal{J}}_r \) is a fundamental symmetry of \( \mathcal{H} \). All these fundamental symmetries are in fact equivalent and hence induce the same smooth structure on \((\mathcal{A}_\theta, \mathcal{H}, D)\).

**Proposition 7.5.** Let \( r_1 \) and \( r_2 \) be two spacelike reflections of \( \mathbb{R}^{n,k} \). Then \( \tilde{\mathcal{J}}_{r_1} \sim \tilde{\mathcal{J}}_{r_2} \), i.e. \( \text{dom}(\Delta_{\tilde{\mathcal{J}}_{r_1}}^s) \) and \( \text{dom}(\Delta_{\tilde{\mathcal{J}}_{r_2}}^s) \) are equal and carry the same topology for all \( s \in \mathbb{R} \). Moreover \( \mathcal{H}^\infty = \mathcal{A}_r \otimes \Delta_{n,k} \).

**Proof.** Let \( E \) be the spinor bundle on the commutative torus \( T^{n,k} \) with the flat semi-Riemannian metric of signature \((n, k)\). Let \( \mathcal{H}_c \) be the Krein space of square integrable sections of \( E \). The map \( W : \mathcal{H}_c \to L^2(T^{n,k}) \) defined by \( W(u(y)) = e^{2\pi i(y,x)} \) is unitary and satisfies \( W\delta_i W^{-1} = \partial_i \). The map \( U := W \otimes \text{id} \) is an isometric isomorphism of the Krein spaces \( \mathcal{H} \) and \( \mathcal{H}_c \) and \( UDU^{-1} \) coincides with \( i^k \mathcal{D} \), where \( \mathcal{D} \) is the Dirac operator on the torus. Furthermore \( U\tilde{\mathcal{J}}_r U^{-1} \) are admissible fundamental symmetries of the canonical spectral triple associated with \( T^{n,k} \). As a consequence the operators \( U\Delta_{\tilde{\mathcal{J}}_r} U^{-1} \) are classical pseudodifferential operators of first order and hence, \( \text{dom}(\Delta_{\tilde{\mathcal{J}}_r}^s) = U^{-1}H_s(E) \) for \( s > 0 \) where \( H_s(E) \) is the space of Sobolev sections of order \( s \) of \( E \). Therefore, \( \tilde{\mathcal{J}}_{r_1} \sim \tilde{\mathcal{J}}_{r_2} \). The equation \( \mathcal{H}^\infty = \mathcal{A}_r \otimes \Delta_{n,k} \) follows from \( W^{-1}C^\infty(T^{n,k}) = \mathcal{A}_\theta \subset \mathcal{H}_r \), which is easy to check. \( \square \)

We view in the following \((\mathcal{A}_\theta, \mathcal{H}, D)\) as a smooth semi-Riemannian spectral triple with the above defined smooth structure and refer to it as the noncommutative semi-Riemannian torus \( T^{n,k}_\theta \). For simplicity we restrict our considerations to the case where the algebra \( \mathcal{A}_\theta \) has trivial center.

**Theorem 7.6.** Suppose that \( \mathcal{A}_\theta \) has trivial center. Then the set of admissible fundamental symmetries of \((\mathcal{A}_\theta, \mathcal{H}, D)\) coincides with the set

\[ \{ \mathcal{J}_r : r \text{ is a spacelike reflection of } \Delta_{n,k} \} \]

**Proof.** Let \( r \) be a spacelike reflection of \( \Delta_{n,k} \). By construction \( \mathcal{J}_r \) is smooth. We first show that \( \mathcal{J}_r \) is admissible. Clearly, \( \mathcal{J}_r \) commutes with all elements of \( \mathcal{A}_\theta \) and \( \mathcal{J}_r \hat{\chi} = \hat{\chi}^+ \). Lemma 7.4 shows that indeed \( \mathcal{J}_r \pi(j \cap \Omega^p \mathcal{A}) = \pi(j \cap \Omega^p \mathcal{A}) \) and \( \mathcal{J}_r \pi(\Omega \mathcal{A}) = \pi(\Omega \mathcal{A}) \). Therefore, \( \mathcal{J}_r \) is admissible. Now suppose conversely that \( \mathcal{J}_r \) is an admissible fundamental symmetry. Since \( \mathcal{J}_r \) commutes with \( \mathcal{A}_\theta \), we
Denote by $\chi$ a map $r$. Hence, there exists a spacelike reflection $\gamma$ which acts on $H$ from the right. The space $\pi(\Omega^1 A_\theta)$ is invariant under conjugation by $\gamma$. Therefore, the matrices $\gamma_i \gamma$ must commute with all elements of $A_{opp}$ and therefore have entries in the center of $A_{opp}$, which is trivial. Hence, the vector space spanned by the $\gamma_i$ is invariant under conjugation by $\gamma$. In the same way as in the proof of Prop. 5.1 one checks that the $\gamma_i$ are hermitian and whenever the limit exists.

\[ \text{Theorem 7.7. Suppose that } A_\theta \text{ has trivial center. The smooth semi-Riemannian spectral triple } (A_\theta, H, D) \text{ is } n^+ - \text{summable and for all } a \in A_\theta \text{ and each admissible fundamental symmetry } \gamma \text{ we have} \]

\[ \begin{align*}
\text{Tr}_\omega(a \Delta_3^{-n}) &= \frac{1}{c(n)} \tau(a), \\
\text{Tr}_\omega(a D^2 \Delta_3^{-n-2}) &= (-1)^k \frac{n - 2k}{n} \text{Tr}_\omega(a \Delta_3^{-n}).
\end{align*} \]  

\[ (49) \quad (50) \]

\[ \text{Proof. Let } E \text{ be the spinor bundle on the commutative torus } T^{n,k} \text{ with the flat semi-Riemannian metric of signature } (n, k). \text{ Let } H_c \text{ be the Krein space of square integrable sections of } E. \text{ In the proof of Prop. 7.3 we constructed an isomorphism of Krein spaces } U : H \rightarrow H_c \text{ such that } U D U^{-1} = i^k \mathcal{P}, \text{ where } \mathcal{P} \text{ is the Dirac operator on } T^{n,k}. \text{ Moreover the } U \gamma_i U^{-1} \text{ are admissible fundamental symmetries of the canonical spectral triple associated with } T^{n,k}. \text{ Therefore, by theorem 6.8 } \text{Tr}_\omega(\Delta_3^{-n}) = c(n)^{-1} \text{ and } \text{Tr}_\omega(D^2 \Delta_3^{-n-2}) = (-1)^k \frac{n - 2k}{n} \text{Tr}_\omega(\Delta_3^{-n}). \text{ For all admissible fundamental symmetries. The proof is finished if we can show that } \text{Tr}_\omega(u(y) \Delta_3^{-n}) = 0 \text{ and } \text{Tr}_\omega(u(y) D^2 \Delta_3^{-n-2}) = 0 \text{ whenever } y \neq 0. \text{ Let } \{\psi_i\} \text{ be an orthonormal basis in } \Delta_{n,k}. \text{ Then the elements } \phi_{y,i} := u(y) \otimes \psi_i \in H_\tau \text{ form an orthonormal basis in } H \text{ and they are eigenvectors of } \Delta_3^{-n} \text{ and of } D^2. \text{ By Lemma 7.17 in [3] we have } \text{Tr}_\omega(u(y) \Delta_3^{-n}) = \lim_{y \rightarrow 0^+} \text{Tr}(E_p u(y) \Delta_3^{-n}), \text{ where } E_p \text{ is the orthogonal projector onto the subspace generated by the first } n \text{ eigenvectors of } \Delta_3^{-1} \text{ and whenever the limit exists. But since } \langle \phi_{y,i}, u(y') \phi_{y,i} \rangle_3 = 0 \text{ for all } y' \neq 0, \]

\[ \text{we conclude that } \gamma_i \gamma = 0 \text{ for all } i. \]
we get $\text{Tr}(E_y u(y') \Delta_3^{-n}) = 0$ and consequently $\text{Tr}_\omega(u(y') \Delta_3^{-n}) = 0$. The same argument gives $\text{Tr}_\omega(u(y') D^2 \Delta_3^{-n-2}) = 0$.

8 Outlook

We showed that it is possible to extract the dimension, the signature and a notion of integration from the spectral data of a semi-Riemannian manifold. It would certainly be interesting if one could obtain the Einstein-Hilbert action in a similar way as in the Riemannian case (see [12, 11]). This can probably not be done straightforwardly, but may require some averaging of expressions of the form $\text{Wres}(D^2 \Delta_3^{-n})$ over the set of admissible fundamental symmetries.

Another interesting question is, which further conditions on the admissible fundamental symmetries are necessary in the general situation to guarantee that the functionals $\text{Tr}_\omega(\cdot \Delta_3^{-n})$ and $\text{Tr}_\omega(\cdot D^2 \Delta_3^{-n-2})$ on the algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ do not depend on the choice of $\mathfrak{J}$.

As far as the noncommutative tori are concerned we believe that an analogue of Theorem 7.7 holds in case the center of $\mathcal{A}_\theta$ is not trivial. One should be able to proof this in a similar way as we did it here for the case of a trivial center.

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