UNCERTAINTY RELATIONS FOR TWO OBSERVABLES COUPLED WITH THE THIRD ONE

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Abstract
A new lower boundary for the product of variances of two observables is obtained in the case, when these observables are entangled with the third one. This boundary can be higher than the Robertson–Schrödinger one. The special case of the two-dimensional pure Gaussian state is considered as an example.

1 Introduction
90 years ago, Heisenberg [1] introduced (although in an approximate form) the famous “uncertainty relation” (UR)

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$ (1)

that was soon proven rigorously in the frameworks of the wave function description of quantum systems by Kennard [2] and Weyl [3]. A few years later, Robertson [4] and Schrödinger [5] proved a more general inequality

$$\sigma_A \sigma_B \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2$$ (2)

for arbitrary Hermitian operators $\hat{A}$ and $\hat{B}$. According to them (and following Heisenberg’s idea [1]), the “uncertainty” of a quantity $A$ was defined as the square root of its variance (or mean squared deviation):

$$\Delta A \equiv \sqrt{\sigma_A}, \quad \sigma_A \equiv \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2.$$ (3)

It is known that inequality (1) is saturated (becomes the equality) for the wave functions describing shifted ground states of the harmonic oscillator [6], called nowadays after Glauber [7] as “coherent states”. But what can we say about those quantum states, which possess the left-hand sides of inequalities (1) or (2) bigger than the right-hand ones? One of possible answers is that, probably, one should add some extra terms to the right-hand sides of (1) or (2) in such cases, taking into account some additional parameters or specific properties of concrete quantum systems under consideration. The first step in this direction was made by Robertson and Schrödinger in the same papers [4,5]. Namely, they obtained a more precise version of (2), taking into account the average value of the anticommutator $\{\hat{A}, \hat{B}\}$:

$$\sigma_A \sigma_B \geq \sigma_{AB}^2 + \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2 \equiv \left| \langle (\delta \hat{A})(\delta \hat{B}) \rangle \right|^2 \equiv G_{AB}^2,$$ (4)

where

$$\sigma_{AB} \equiv \frac{1}{2} \langle \hat{A} \hat{B} + \hat{B} \hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \equiv \frac{1}{2} \left\langle \{\delta \hat{A}, \delta \hat{B}\} \right\rangle, \quad \delta \hat{A} \equiv \hat{A} - \langle \hat{A} \rangle.$$ (5)

The special case of (4) is the following generalization of (1) for the coordinate and momentum operators:

$$\sigma_p \sigma_x - \sigma_{xp}^2 \geq \frac{\hbar^2}{4}.$$ (6)

The equality takes place for all Gaussian wave functions, as was discovered for the first time by Kennard [2].

Inequality (6) can be rewritten in the form [8]

$$\sigma_p \sigma_x \geq \frac{\hbar^2}{4(1 - r^2)}, \quad r = \frac{\sigma_{xp}}{\sqrt{\sigma_p \sigma_x}}.$$ (7)
which emphasizes the role of the “correlation coefficient” \( r \) as an additional parameter, responsible for the increase of product \( \sigma_p \sigma_x \). One could treat the relation (7) as though an “effective Planck constant” \( \hbar \left( 1 - r^2 \right)^{-1/2} \) is occurring instead of the usual constant \( \hbar \). Such an interpretation of inequalities (6) and (7) was discussed, e.g., in Refs. [9,10]. The explicit form of “correlated coherent states”, saturating inequality (6), is as follows [8],

\[
\psi(x) = \left( 2\pi \sigma_x \right)^{-1/4} \exp \left[ -\frac{x^2}{4\sigma_x} \left( 1 - \frac{ir}{\sqrt{1 - r^2}} \right) + \frac{\alpha x}{\sqrt{\sigma_x}} - \frac{1}{2} \left( \alpha^2 + |\alpha|^2 \right) \right].
\] (8)

Inequalities (4) and (7) explain the increase of the uncertainty product \( \sigma_A \sigma_B \) due to the existence of some “intrinsic” restrictions in the quantum system under investigation (the nonzero correlation coefficient). However, this product can increase also due to some “extrinsic” constraints, if the system interacts with other systems (“environment”). For example, in the case of an equilibrium state of a harmonic oscillator with frequency \( \omega \) at temperature \( T \), the uncertainty product equals \( \sigma_p \sigma_x = \frac{\hbar}{2 \coth \left( \frac{\hbar \omega}{2k_B T} \right)} \) \((k_B \text{ is the Boltzmann constant})\). In the high-temperature case \( k_B T \gg \hbar \omega \), the right-hand side of this equality is so large, that inequality (1) becomes practically useless.

The equilibrium state of a harmonic oscillator is a mixed quantum states, described by means of the statistical operator (density matrix) \( \hat{\rho} \). The degree of mixing is frequently characterized by the difference \( 1 - \mu \), where \( \mu \equiv \text{Tr}(\hat{\rho}^2) \) is the “quantum purity”. It is known that for any quantum state described by means of a Gaussian density matrix or the Wigner function (in particular, for the equilibrium state), the following equality holds for systems with one degree of freedom (see, e.g., [11]):

\[
\sqrt{\sigma_p \sigma_x} - \sigma_{xp}^2 = \frac{\hbar}{2\mu}.
\] (9)

The generalized “purity bounded uncertainty relation” for mixed quantum states can be written in the form

\[
\sqrt{\sigma_p \sigma_x} - \sigma_{xp}^2 \geq \frac{\hbar}{2} \Phi(\mu),
\] (10)

where \( \Phi(\mu) \) is a monotonous function of \( \mu \), satisfying the relations \( \Phi(1) = 1 \leq \Phi(\mu) \leq \mu^{-1} \) for \( 0 < \mu \leq 1 \). Its explicit form turned out rather complicated, but it can be described with a good accuracy by a simple approximate formula [12,13]

\[
\tilde{\Phi}(\mu) = 4 + \sqrt{16 + 9\mu^2} = \frac{9\mu}{9\mu}.
\] (11)

In particular, the following asymptotical formula holds for \( \mu \ll 1 \) (its leading term was obtained for the first time by Bastiaans [14]):

\[
\tilde{\Phi}(\mu) = 8 \frac{\mu}{9\mu} \left( 1 + \frac{9}{64} \mu^2 + \ldots \right),
\] (12)

so that \( |\Phi(\mu) - 8/(9\mu)| < 0.01 \) for \( \mu \leq 0.25 \). Both formulas, (11) and (12), show that Gaussian states do not minimize the precise uncertainty relation for mixed states. The minimum value is achieved for some diagonal mixtures of finite numbers of the Fock states of the harmonic oscillator.

Inequality (10) can be considered as some kind of “coarse-grained” relations, since it hides all details of the interaction (entanglement) between the system under study and the “environment”. Our goal is to derive a new inequality, where some of these details appear explicitly.

2 The new inequality and its illustration

The main idea is to start from some inequality related to three observables and find its consequences with respect to admissible values of the product of two selected variances. A general scheme of obtaining the uncertainty relations for several observables in terms of covariances was given by Robertson in 1934 [15]. Let us remind it.
Consider $N$ arbitrary operators $\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_N$, and construct the operator $\hat{f} = \sum_{j=1}^{N} \alpha_j (\hat{z}_j - \langle \hat{z}_j \rangle)$, where $\alpha_j$ are arbitrary complex numbers. The inequalities, which can be interpreted as generalized uncertainty relations, are the consequences of the fundamental inequality $\langle \hat{f} | \hat{f} \rangle \geq 0$, that must be satisfied for any pure or mixed quantum state (the symbol $\hat{f}^\dagger$ means the Hermitian conjugated operator). In the explicit form, this inequality is the condition of positive semi-definiteness of the quadratic form $\alpha^*_j F_{jm} \alpha_m$, whose coefficients $F_{jm} = \left\langle \left( \hat{z}_j - \langle \hat{z}_j \rangle \right)^* \left( \hat{z}_m - \langle \hat{z}_m \rangle \right) \right\rangle$ form the Hermitian matrix $F = \| F_{jm} \|$. One has only to use the known conditions of the positive semi-definiteness of Hermitian quadratic forms to write down the explicit inequalities for the elements of matrix $F$. All such inequalities can be considered as generalizations of inequality (2) to the case of more than two operators. Many of them can be found in the review [12].

If all operators $\hat{z}_j$ are Hermitian, then it is convenient to split matrix $F$ as $F = X + iY$, where $X$ and $Y$ are real (and positive), while $\xi$ may be an arbitrary complex number, satisfying the restriction $\xi = \sqrt{X_{11}X_{22}} \geq 0$. It suits quite well for our purposes, since it contains all elements of matrices $X$ and $Y$. Its explicit form in the special case of $N = 3$ reads

$$X_{11}X_{22}X_{33} \geq X_{11} \left( X_{23}^2 + Y_{23}^2 \right) + X_{22} \left( X_{13}^2 + Y_{13}^2 \right) + X_{33} \left( X_{12}^2 + Y_{12}^2 \right) + 2 \left( X_{12}Y_{23}Y_{31} + X_{23}Y_{31}Y_{12} + X_{31}Y_{12}Y_{23} - X_{12}X_{23}X_{31} \right),$$

(14)

Formula (14) was obtained by Synge [16] (without any reference to Robertson’s paper). Recently, it was re-derived in Ref. [17].

Inequality (14) has the form $X_{11}X_{22}X_{33} \geq aX_{11} + bX_{22} + c$, where coefficients $a$, $b$ and $c$ do not contain variances $X_{11}$ and $X_{22}$. Moreover $a$ and $b$ are non-negative. Due to the standard arithmetic-geometric inequality, we have $aX_{11} + bX_{22} \geq 2\sqrt{ab}X_{11}X_{22}$. This means that $X_{33}\xi^2 - 2\sqrt{ab}\xi - c \geq 0$, where $\xi = \sqrt{X_{11}X_{22}} \geq 0$. Consequently, $\xi$ must be greater than the biggest root of the quadratic polynomial in the left-hand side of this inequality: $X_{33}\xi \geq \sqrt{ab} + \sqrt{ab} + cX_{33}$. Thus we arrive at the inequality

$$\Delta z_1 \Delta z_2 \geq \sqrt{G_{12}^2 + \Omega^2 + 2\Gamma},$$

(15)

where

$$G_{jk}^2 = X_{jk}^2 + Y_{jk}^2, \quad \Omega = |G_{13}G_{23}| / X_{33},$$

$$\Gamma = |X_{12} (Y_{23}Y_{31} - X_{23}X_{31}) + Y_{12} (X_{23}Y_{31} + Y_{23}X_{31})| / X_{33}.$$

(16)

(17)

If the observables $z_1$ and $z_2$ are totally independent from $z_3$, then $Y_{13} = Y_{23} = X_{13} = X_{23} = 0$, and (15) is reduced to the Schrödinger–Robertson inequality (1).

If $[\hat{z}_1, \hat{z}_2] = [\hat{z}_2, \hat{z}_3] = 0$ (for example, $z_1 = x$, $z_2 = p_x$ and $z_3 = y$), then

$$\Delta z_1 \Delta z_2 \geq \sqrt{Y_{12}^2 + (X_{12} - X_{13}X_{23}/X_{33})^2 + |X_{13}X_{23}| / X_{33}}.$$

(18)

The right-hand side of this inequality is bigger than the Robertson bound $|Y_{12}|$, if there exist correlations in the pairs $(z_1, z_3)$ and $(z_2, z_3)$, characterized by nonzero values of the covariances $X_{13}$ and $X_{23}$.

To illustrate inequality (18), let us consider a special two-variable pure Gaussian state, described by the wave function

$$\psi(x, y) = \mathcal{N} \exp \left( -\frac{a}{2} x^2 - bx y - \frac{c}{2} y^2 \right),$$

(19)

where $\mathcal{N}$ is the normalization factor. To simplify the following formulas, let us assume that coefficients $a$ and $c$ are real (and positive), while $b$ may be an arbitrary complex number, satisfying the restriction $D \equiv ac - |\text{Re}(b)|^2 > 0$. It is easy to calculate all necessary variances and covariances:

$$X_{11} \equiv \langle x^2 \rangle = \frac{c}{2D}, \quad X_{22} \equiv \langle y^2 \rangle = \frac{a^2}{2D} \left( ac - |\text{Re}(b)|^2 + |\text{Im}(b)|^2 \right),$$

3
\[X_{12} \equiv \frac{1}{2} \langle \hat{x} \hat{p}_x + \hat{p}_x \hat{x} \rangle = \frac{\hbar}{2D} \text{Re}(b) \text{Im}(b), \quad Y_{12} = \frac{\hbar}{2},\]

\[X_{33} \equiv \langle y^2 \rangle = \frac{a}{2D}, \quad X_{13} \equiv \langle xy \rangle = -\frac{\text{Re}(b)}{2D}, \quad X_{23} \equiv \langle p_x y \rangle = -\frac{a \hbar}{2D} \text{Im}(b).\]

In this case we have \(X_{12} = X_{13}X_{23}/X_{33}\), so that (18) can be written as (here \(p \equiv p_x\)) \(\Delta x \Delta p \geq \hbar/2 + |\sigma_{xp}|\). The right-hand side of this inequality is certainly bigger than the Robertson–Schrödinger lower boundary \([(\hbar/2)^2 + \sigma_{xp}^2]^{1/2}\). This happens, because the quantum state describing the \(x\)-subsystem is mixed. Its density matrix \(\rho(x, x') = \int \psi(x, y)\psi^*(x', y) dy\) has the purity \(\mu = \{(ac - |\text{Re}(b)|^2) / (ac + |\text{Im}(b)|^2)\}^{1/2}\). The equality in (18) is achieved for the states (19) with \(|\text{Re}(b)| = |\text{Im}(b)|\). If \(b \neq 0\), then the function (19) cannot be written as a product of some function of \(x\) by some function of \(y\). In other words, this function describes an entangled state with respect to variables \(x\) and \(y\). Due to this entanglement, the uncertainty product \(\Delta x \Delta p\) turns out bigger than the Robertson–Schrödinger boundary.

## 3 Conclusion

The main results of this paper are inequalities (15) and (18), which show how the entanglement of the system under study with other degrees of freedom results in the increase of the minimal value of the uncertainty product with respect to the selected system observables. We have found also an example of quantum states which saturate the new inequality (18). The weakness of inequality (18) is that it reduces to the Robertson–Schrödinger lower boundary (14), if one of covariances, \(X_{13}\) or \(X_{23}\), equals zero, although the uncertainty product can exceed the boundary (14) in such cases. Probably, more general and more strict inequalities can be found, if one applies the scheme of section 2 to systems of more than three observables. This subject is under investigation now.

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