Relativistic Thermodynamics

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ABSTRACT

The interaction between metric and matter is expressed by Einstein’s field equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$. The model that has been in use for 80 years is a phenomenological expression for $T_{\mu\nu}$ suggested by Tolman. This paper begins with a review of the principal shortcomings of the phenomenological approach and calls for a theory that preserves the formulation of pure gravity as an action principle.

A major part of this paper is an exposition of extended thermodynamics, as an action principle in the spirit of Gibbs’ variational principles. When we accept certain limitations on the allowed types of flow, it leads to a complete integration of General Relativity with Thermodynamics. A generalization that includes general flows is a major goal of current research.

The subject is local thermodynamics and hydrodynamics, including systems with several components and phases, and with a general equation of state.

The essential part of the paper is the lifting of the action principle of thermodynamics to the context of General Relativity. The matter part of the Lagrangian density includes a kinetic part $a la$ Fetter and Walecka and the potential $f + sT + P$, where $f$ is the thermodynamic free energy density and $s$ is the entropy density. The pressure term $P$, in the case that the system is identified with the universe, is the cosmological constant. Extensions to include mixtures, with chemical interactions and phase transformations, are in a stage of development. As an example of astronomical applications we sketch a recent application to Dark Matter in the Milky Way.

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I. Introduction

Extensions of Einstein’s theory of gravitation were sought, and found, from 1916 onwards. This paper deals with the incorporation of thermodynamics into General Relativity, the relativistic extension of thermodynamics.

The prototype of a unified field theory is the unification of the theories of Maxwell, Einstein and Dirac that is defined by a combined Lagrangian. But ‘electrodynamics’ is a vast edifice that includes, besides the elementary interactions of photons and electrons, the whole field of electromagnetic interactions of matter, including most of condensed matter physics. The Maxwell-Einstein system does not encompass electromagnetism in this widest sense; it is principally a relativistic theory of elementary particles. It is significant that all unified theories studied so far are based on action principles, and this even applies to some proposals for combining Thermodynamics with General Relativity (Taub 1954, Bardeen 1970, Schutz 1970). Modern astrophysics is not based on an action principle, and yet Einstein’s action has been invoked many times, after it was used by Eddington in his book (Eddington 1926).

It is evident that a unification of thermodynamics with General Relativity must begin by formulating thermodynamics in terms of an action principle. This is anything but new, for it was the inspiration and guiding light of Gibbs’ famous 300 page paper (Gibbs 1878).

The first aim of this work is to give a formulation of thermodynamics as a Eulerian field theory, where the dynamical variables are densities, pressure, temperature and entropy, all fields over space and time, that will allow for its integration into General Relativity. The formulation proposed in this paper is restricted in some ways; it is a start.

The potential benefits of formulating a fully fledged action principle for thermodynamics extend far beyond the present context; one important advantage is that each successful application places constraints on the Lagrangian that must be respected by subsequent applications. In principle, one should aim at a “Lagrangian for hydrogen” and a “Lagrangian for water” that would attempt to encompass all properties of those substances, in all their phases.

In the strictest sense thermodynamics is the study of equilibria, but of course we shall not be confined to that narrow context. In the first place, to make meaningful contact with General Relativity, we have to deal with a localized thermodynamics of fluids, including hydrodynamics. In the second place we shall have to deal with systems with several components and several phases, with chemical reactions, and with a general class of equations of state.

The focus on an action principle limits the scope of the theory. The most severe limitation is that the flows have to be piecewise irrotational and determined locally by a velocity potential, although limited vorticity can be accommodated in the manner of Onsager. See review by Eyink and Sreenivasan (2006). * Prospects for overcoming this limitation are discussed at the end of this paper. A fixed Lagrangian determines

* Points of vortices are omitted from the spacial manifold and the velocity potential is a section of a fiber bundle over the manifold.
an adiabat and applies to adiabatic fluctuations. That this is useful is due to the fact that some non-adiabatic changes are quasi static; they take place on a very long time scale. Most treatments of continuous, non adiabatic changes assume that the time development proceeds through a sequence of adiabatic equilibria. In the case of the simplest systems the adiabats may be indexed by a single parameter, related to the entropy. A general expression for the Lagrangian, containing a free parameter, then applies to a family of systems interrelated by the exchange of heat. Recognition of this fact, together with the approximate additivity of Lagrangians of composite systems, is a promising starting point for a study of the entropy and the non adiabatic processes of heterogeneous systems. In particular, the action principle provides a convenient framework within which to study entropy, the most difficult concept in thermodynamics. (See Fronsdal 2014.)

Summary

Section II reviews the phenomenological approach to Relativistic Astrophysics and highlights its most severe shortcomings, thus providing the principal motivation for the creation of Relativistic Thermodynamics.

Section III is a summary of recent progress towards an action principle formulation of thermodynamics, limited to potential flow, and Section IV lifts this formulation of thermodynamics to complete integration with General Relativity. At the end is a sample application of the theory to a problem of galactic dynamics.

Section V gives a glimpse of a theory, under development, that would circumvent the limitation to irrotational flows.

II. The phenomenological approach

For the application of General Relativity to astrophysical and astronomical problems Tolman (1934) proposed the following modification of pure gravitational dynamics,

\[ G_{\mu\nu} = 8\pi T_{\mu\nu}, \quad T_{\mu\nu} = (\rho + p)U_\mu U_\nu - p g_{\mu\nu}. \]  

(2.1)

Here \( \rho \) and \( p \) are scalar fields and \( U \) is a vector field. The only \( a \ priori \) restriction imposed on these fields is the normalization condition

\[ g^{\mu\nu}U_\mu U_\nu = 1. \]  

(2.2)

The principal difficulty that we have with this suggestion is that it does not include any dynamics for the matter sector, except for the normalization condition (2.2) that is appropriate for the dynamics of non interacting particles.

The left side of the equation satisfies the Bianchi identity; therefore consistency demands that the right hand side satisfy the ‘Bianchi constraint’,

\[ T^{\mu\nu} U_\nu = 0. \]  

(2.3)
Tensor fields that satisfy this constraint are the energy momentum tensors of relativistic field theories, but here $T$ is purely phenomenological, with no inherent dynamics, and the constraint (2.3) is imposed by the coupling to the metric.

An analogy will clarify this difficulty. Maxwellan electrodynamics has

$$\partial_{\mu} F^{\mu\nu} = J^{\nu}.$$ 

The left side is identically divergence less and for this reason the current must satisfy the constraint $J^{\nu}_{\nu} = 0$. The suggestion that this condition be imposed by hand is unheard of, and quite inconsistent with standard perturbation theory. A field theory of matter is qualified to interact with the vector potential only if it can offer a current that is conserved by virtue of the matter field equations. And a field theory of matter is qualified to interact with the metric only if it can present a tensor that satisfies the Bianchi constraint by virtue of the matter field equations.

Compared to this fundamental shortcoming, the next problem may seem to be a minor one: the model offers no conserved mass density, no conserved current. But since the intention is to provide a relativistic generalization of hydrodynamics, where the equation of continuity is one of only two fundamental relations, it is a defect that should not be ignored. In fact, Weinberg (1972), and also Misner, Thorne and Wheeler (1972), have proposed to improve the prospects of the theory by introducing a second density $\sigma$ with the defining property that the current $\sigma U^\mu$ is conserved. It seems that this suggestion has not been taken up in astrophysical applications.

The fact that Eq. (2.1) has been widely used in spite of these well known difficulties must be ascribed to a belief that it may not be possible to improve on it. * We shall see, in Section IV, that it is certainly possible to do so in certain cases, and in Section VI, that the prospects for a more general solution of the problem are not all negative.

In as much as we are concerned with the relation between thermodynamics and General Relativity it is natural to call attention to a recent paper by Verlinde (2009). This paper proposes an interesting connection between gravity and entropy, but it does not develop a concrete approach to the problem of what is to be done with the matter contribution to Einstein’s equations.

III. Thermodynamics

III.1. The energy functional of a simple system

The dynamical variables of a unary, homogeneous system are the temperature, the pressure, the entropy and the volume.

The laws of classical, equilibrium thermodynamics are expressed in terms of potentials $U,F,H$ or $G$. Each is defined as a function of specific variables (the natural variables), namely

$$U(S,V), \quad F(T,V) \quad H(S,P) \quad G(P,T).$$

* Applications are too numerous to be quoted; but some relatively recent extensions of Tolman’s suggestion can be found in Israel and Stewart (1979), Andreassen (2011), Okabe et al (2011).
According to Gibbs (1878), who quotes Massieu (1876), a thermodynamic system is completely defined by any one of these functions; an explicit expression for a potential in terms of its natural variables is called a fundamental relation. The four potentials are related by Legendre transformations and the fundamental relations for a given system are equivalent.

The most fundamental concept introduced by Gibbs is the representation of the states of a system in terms of surfaces in a Euclidean space with coordinates $V, T, P, S$. Gibbs based the entire theory on a principle of minimal energy and on a subsidiary principle of maximum entropy. The energy is subject to independent variations of 2 of the variables, the other 2 held fixed. The Gibbsean surfaces are 2-dimensional surfaces in four dimensions, determined by the variational equations. Physical configurations are points on this surface.

The action is thus a function of the four independent variables. To define the system one specifies the potentials and obtains 2 independent relations among the variables that define the Gibbsean two dimensional surface.

The relations obtained by variation of the generic action must have universal validity and the following definition of the energy functional satisfies that criterion,

$$E = F(V, T) + ST + VP.$$ 

According to Gibbs, in the description of adiabatic processes the energy is to be varied with entropy and pressure held fixed. Independent variation of $V$ and $T$ give the standard relations

$$\frac{\partial F}{\partial T} + S = 0, \quad \frac{\partial F}{\partial V} + P = 0.$$ 

The system is thus defined by the free energy and the values chosen for $P$ and $S$. To encode the properties of the system in the expression for the free energy $F$, rather than $U$ or $G$, is often the preferred choice, see for example Rowlinson (1959).

### III.2. Localization

The local extrapolation of thermodynamics seeks to promote these relations to field equations that describe spatial but, in the first instance, time independent configurations. Until further notice we consider systems with one component and a single phase. The function $F$ and the variable $S$ are interpreted as specific densities, the variable $V$ as a specific volume. The mass density is $\rho = 1/V$ and densities $f, s$ are defined by *

$$f(\rho, T) = \rho F(\rho, T), \quad s = \rho S.$$ 

The energy density, now including the kinetic energy, is

$$h = \rho \bar{v}^2/2 + f(\rho, T) + sT,$$

(3.1)  

* The representation of the entropy density $s$ as a product $\rho S$ is meant to imply that the specific entropy density $S$, rather than $s$, is independent of the other dynamical variables.
with $\rho, S, T$ and $P$ treated as independent variables. The vector field $\vec{v}$ is interpreted as a flow velocity.

We are going to extend the action principle to incorporate the hydrodynamic equation of motion for the velocity field, as well as the equation of continuity. We begin by introducing the Poisson bracket; for two functionals $f$ and $g$,

$$\{f, g\} = \int d^3x \left( \frac{\partial f}{\partial v_i} \frac{\partial g}{\partial \rho} - f, g \right).$$  \hspace{1cm} (3.2)

Then the two fundamental equations of hydrodynamics take the form

$$\dot{\rho} = \{H, \rho\} = -\vec{\nabla} \cdot (\rho \vec{v}) \hspace{1cm} (3.3)$$

and

$$\dot{\vec{v}} = \{H, \vec{v}\} = -\vec{\nabla} \frac{\partial h}{\partial \rho}. \hspace{1cm} (3.4)$$

The first equation is the equation of continuity and the second equation will be related to the Bernoulli equation. The energy functional as defined by Eq.(3.1) is thus a fully fledged Hamiltonian, although the temperature does not (yet) have a canonical conjugate field.

### III.3. The Lagrangian

Going a step further, we pass to a Lagrangian formulation. As far as was known until recently (but see Section V), this can be done only if the velocity field is irrotational, expressible as

$$\vec{v} = -\text{grad} \Phi.$$

The correct form of the kinetic part of the action for hydrodynamics was presented by Fetter and Walecka (1980). Generalizing, we define an action and a Lagrangian density by

$$A = \int dt \int_\Sigma (d^3x \mathcal{L} - VP), \quad \mathcal{L} = \rho \dot{\Phi} - h,$$

$$h = \rho \vec{v}^2/2 + \phi + f(\rho, T) + sT. \hspace{1cm} (3.5)$$

We have included the gravitational potential $\phi$ of a fixed gravitational field. This makes $\Phi$ canonically conjugate to the density $\rho$. The equations of hydrodynamics are now variational equations. Variation of $\Phi$ and $\rho$ gives the Hamiltonian equations of motion (3.3-4) in the form

$$\dot{\rho} = -\frac{\partial h}{\partial \Phi}, \quad \dot{\Phi} = \frac{\partial h}{\partial \rho}.$$

The scope of the theory is revealed by listing the Euler Lagrange equations:
• Variation of the field $T$ gives the adiabatic relation for densities,

$$\frac{\partial f}{\partial T} + s = 0.$$  \hfill (3.6)

For a one component system, in the case that the specific entropy density $S$ is uniform, this is the usual polytropic condition. In the case of multi component systems it has a significance that by far exceeds what is usually recognized.

• Variation of the velocity potential gives the equation of continuity, Eq.(3.3).

• Local variation of $\rho$ (variations that vanish at the boundary of the domain $\Sigma$), with $S$ and $P$ held fixed, gives

$$\frac{h}{\rho} + \rho \frac{\partial (f/\rho)}{\partial \rho} = \text{(constant)}.$$  

The local thermodynamic pressure is defined by

$$\rho^2 \frac{\partial (f/\rho)}{\partial \rho} - p = 0$$

and the last equation reduces to the hydrostatic condition.

$$\frac{\nabla h + p}{\rho} = 0.$$  \hfill (3.7)

Alternatively, the Euler Lagrange equation that comes from variation of $\rho$ can be written

$$\dot{\Phi} - \frac{v^2}{2} = \phi + q,$$

where $q$ is the chemical potential. Taking the gradient gives

$$\frac{D\vec{v}}{Dt} = -\nabla \phi - \nabla q.$$  \hfill (3.8)

The function $f + sT$ is a function of three independent variables and

$$\nabla (f + sT) = \frac{\partial (f + sT)}{\partial T} \nabla T + \frac{\partial (f + sT)}{\partial \rho} \nabla \rho + \frac{\partial (f + sT)}{\partial S} \nabla S.$$  

The first term vanishes by the equation of motion and the third term vanishes if $S$ is uniform, hence in that case

$$\nabla p = \rho \nabla \frac{\partial (f + sT)}{\partial \rho} = \rho \nabla q.$$  

Consequently, “Newton’s equation” for the acceleration (the Bernoulli equation)

$$\rho \frac{D\vec{v}}{Dt} = -\rho \nabla \phi - \nabla p,$$  

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is the Euler Lagrange equation obtained by variation of the density. In the static case, this reduces to the hydrostatic condition, 
\[ \rho \vec{\nabla} \phi + \vec{\nabla} p = 0. \]

It is important to remember that this much used equation is consistent with thermodynamics only when the specific entropy is uniform.

- A complimentary variation of the density, with the mass fixed but the volume not, gives the result that the thermodynamic pressure must coincide with the pressure \( P \) at the boundary. In the interior the thermodynamic pressure is balanced by external forces.

This thermodynamic action principle, based on the hydrodynamic Lagrangian of Fetter and Walecka, and on Gibbs’ principle of minimum energy, allows for incorporation into the framework of General Relativity, as shall be shown in Part IV of this paper. That is not the only advantage of this formulation of thermodynamics. The following sections give a very short account of some other applications and generalizations.

Fig.1. An attempt to present the Gibbsean view. The blue lines represent adiabatic families of configurations, each characterized by a fixed value of the entropy. The red line is the manifold of adiabatic equilibria; a process of slow dissipation can best be understood when it takes place along this line.

The Lagrangian applies to adiabatic changes only. Relaxation is non adiabatic and irreversible. What makes it possible to talk about adiabatic dynamics is the fact (the assumption) that relaxation takes place on a much longer time scale. The variational equations determine the temperature as a function of the other variables, but there is no equation for the time derivative \( \dot{T} \). Over short time intervals the temperature follows the variations of the density, over long time intervals it is determined by the heat equation. To go further we should incorporate the heat equation into our framework, which is difficult since this equation describes irreversible phenomena and because the process involves external agents.
III. 4. Generalizations

Here we describe a fairly general situation that can be encompassed by the theory in the present stage of development.

Consider a system described by two independently conserved densities, two independent, irrotational vector fields, temperature, entropy and external pressure, and confined to a domain Σ in space. We suppose that we have an expression for the free energy density \( f(\rho_1, \rho_2, T) \) and propose the following Lagrangian density

\[
\mathcal{L} = \sum_j \rho_j (\dot{\Phi}_j - \vec{v}_j^2 / 2 - \phi) - f(\rho_1, \rho_2, T) - T \sum_j \rho_j S_j - \frac{\hat{a}}{3} T^4. \tag{3.9}
\]

We have included the Newtonian potential \( \phi \), in recognition of the fact that the \( \rho \)'s are interpreted as mass densities, and the Stefan-Boltzmann pressure, reserving comments for Section IV. A velocity potential \( \Phi_j \) is canonically conjugate to the conserved density \( \rho_j, j = 1, 2 \). (Fronsdal 2014.) The central problem is the determination of the total free energy density. The most primitive guess and a good approximation in some cases is obtained by summation over the components,

\[
f(\rho_1, \rho_2, \ldots; T) = \sum_i f_i(\rho_i, T). \tag{3.10}
\]

Corrections, in the form of interaction terms, are necessary and will be discussed below.

If a chemical reaction is taking place only the total density is conserved and the first sum is reduced to one term. (The case when more than 2 components are present, and one or more chemical reactions are taking place will not be discussed here. See Fronsdal 2014.) The specific densities \( S_1, S_2 \) are usually taken to be uniform; that is, spatially constants. The appearance of an arbitrary constant in any adiabatic Lagrangian is expected, for it must be possible to add heat and this must be reflected in the equations of motion. But the appearance of \( n \) arbitrary constants (one for each component) requires that we specify the path that the system will follow in this \( n \)-dimensional “entropy space”. In certain circumstances the Gibbs-Dalton hypothesis may serve to fix the ratios, but a deeper analysis is required to account for properties of real gases. For a system with chemical reactions, as in the case of the molecular dissociation of hydrogen, the addition of heat to the compound system leads along a path along which \( T(S_1 - S_2) \) is fixed. With this assumption the Saha equation (Saha 1921) may be derived by variation of the Lagrangian with respect to the densities. In the case of a saturated van der Waals gas, the path is defined by the Maxwell rule of equal areas, derived from the subsidiary principle of maximal entropy, with the result that \( T(S_2 - S_2) = \epsilon \), the evaporation energy. In this case too, non adiabatic (but reversible) changes are taking place in which the entropy is changing and both cases provide some insight into the use and the concept of entropy. The representation of the total entropy density as a linear function of the mass densities is an attempt to model the entropy of mixtures.
Example, hydrogen atmospheres

Hydrogen is present in most stars, the molecular form at low temperatures and the atomic form at higher temperatures. Here we consider the phenomenon of molecular dissociation, below the ionization temperature. The Hamiltonian density, in a first approximation, is

\[ h = \rho (\lambda + \vec{v}^2/2 + \phi) + \epsilon \rho_2 + T \sum_i R_i \rho_i \ln \frac{\rho}{T n_i} + \sum_i \rho_i S_i T. \]

The index is 1 for the molecular component and 2 for the atomic component, \( \epsilon \) is the molecular binding energy. Variation of the densities, with \( \rho = \rho_1 + \rho_2 \) fixed, leads to the following equation for the concentration \( r = \rho_2/\rho \) at fixed density

\[ \frac{r^2}{1 - r} = \frac{1}{e \rho} T^{.5} e^{-\epsilon/R T}. \]

This is Saha’s equation. It was derived under the assumption that \( S_1 - S_2 = \epsilon/T \).

Phase transitions, speed of sound and mixtures

For a pure substance some simple phase transitions and the associated dynamics are described by using the free energy of a van der Waals gas. As the temperature is lowered (non adiabatically) to the domain where the density is no longer determined by temperature and pressure the system behaves in some respects like a mixture, but surface tension and gravity leads to separation of the phases. The theory describes the separated equilibrium configurations and adiabatic dynamics but not the process whereby the system gives up or gains heat from its surroundings, nor the process of separation of the phases. In the saturated, separated phase the total Lagrangian is simply the sum of the Lagrangians of the components, and

\[ f(\rho_1, \rho_2, ..., T) = \sum_i f_i(\rho_i, T). \quad (3.10) \]

Variation of the densities gives the pressure and locates the critical point but it does not yield Maxwell’s rule. Maxwell’s rule can be derived from Gibbs complimentary variational principle of maximal entropy.

The question of the validity of Eq.(3.10) is vital in other contexts, such as the propagation of sound and the determination of critical parameters in gaseous mixtures. The latter problem is often formulated in terms of the free energy, while sound speeds are derived from very different considerations. (de la Mora and Puri 1985.) This brings us to what is without a doubt the most important consequence of adopting the action principle for all applications of thermodynamics: Every successful application gives information about the Lagrangian. For example, the study of sound propagation in mixtures at normal temperatures suggests corrections to the formula (3.10) by interaction terms of the very specific form (Fronsdal 2014)

\[ \alpha_{ij}(\rho_i \rho_j)^k, \quad k = .5. \]
The most complete account of critical parameters of mixtures have been obtained by a modification of the van der Waals formula for the pressure, but preliminary results based on the following formula for the free energy are also encouraging,

$$f(\rho_1, \rho_2, \ldots; T) = \sum_i f_i(\rho_i, T) + \alpha_{ij}(\rho_i\rho_j). \tag{3.11}$$

The particular form of the interacting term is suggested by observation of the excess free energy. When more data on sound propagation at low temperatures become available it will be imperative to reconcile observations of critical phenomena with sound speeds in mixtures, using the same expression for the free energy to account for two very different phenomena.

The ultimate goal of thermodynamic phenomenology should be to determine the Lagrangian for each substance, including mixtures of all kinds and including all phases, and accounting for a large family of adiabatic properties.

IV. Thermodynamics and General Relativity

Following Taub (1954), Schutz (1970) and Bardeen (1970), we propose a partial unification of thermodynamics and General Relativity. But unlike previous suggestions this one is defined by a specific Lagrangian density,

$$\mathcal{L}_{tot} = \frac{1}{8\pi G}(R + \Lambda) + \mathcal{L}_{\text{matter}},$$

where the second term is a relativistic extension of the Lagrangian density in Eq.(3.5),

$$\mathcal{L}_{\text{matter}} = \frac{1}{2} \sum_j \rho_j (g^{\mu\nu}\psi_{j,\mu}\psi_{j,\nu} - c^2) - f(\rho_1, \rho_2, \ldots, \rho_n; T) - T \sum_1^n \rho_i S_i + \frac{\hat{a}}{3} T^4. \tag{4.1}\$$

The radiation term may perhaps be included in the free energy, but we prefer to add it explicitly. Eq.(4.1) is expected to be valid for mixtures with no chemical interactions and well away from phase transitions. There has been progress in dealing with chemical interactions, but only in the non relativistic context (Fronsdal 2014). The constant $\Lambda$ is the cosmological constant and in the case of an astrophysical system that extends to infinity it is interpreted as the pressure at the boundary.

Eq.(4.1) is a simple extension of the non relativistic Lagrangian to which it reduces when in the kinetic term one expands

$$\psi_i = c^2 t + \Phi_i$$

* The simplest example of this theory was proposed by Fronsdal (2007a, 2008).
and takes the limit as \(c\) tends to infinity. Of the dynamical metric there remains, to this order, only the Newtonian potential \(\phi\). Up to terms of order \(1/c^2\),

\[
\frac{1}{2}(g^{\mu\nu}\psi_{i,\mu}\psi_{i,\nu} - c^2) = \frac{i}{2c^2}(1 - 2\phi/c^2)(\dot{\psi}_i)^2 - \frac{1}{2}(\nabla\psi_i)^2 - \frac{1}{2}c^2 = \dot{\Phi}_j - \frac{1}{2}(\nabla\Phi_i)^2 - \phi.
\]

The action is relativistically invariant (invariant under diffeomorphisms) and it follows that the Bianchi constraint is satisfied by virtue of the equations of motion.

The matter energy momentum tensor is

\[
T_{\mu\nu} = \sum_i \psi_{i,\mu}\psi_{i,\nu} - \mathcal{L} g_{\mu\nu}.
\]

Hence \(\mathcal{L}\), on shell, is the pressure, just as it is in the non relativistic theory. The expression is similar to that of Tolman (1934), with some differences:

1. Any number of densities and vector fields can be accommodated, with any equation of state.
2. The charges that are conserved in the non relativistic theory remain conserved in General Relativity,

\[
\partial_\mu J^\mu_j = 0, \quad J^\mu_j = \sqrt{-g}g^{\mu\nu}\psi_{j,\nu}.
\]

3. The Lagrangian is determined by the non relativistic free energy and the interpretation is the same as in the non relativistic theory. Simple chemical reactions and changes of phase can be described precisely as in non relativistic thermodynamics.
4. All the equations of ordinary thermodynamics and hydrodynamics are recovered in the non relativistic limit.
5. The simplest examples (ideal gas spheres, radiation neglected) lead to equations of motion that are almost identical to those used in simple stellar models.
6. The black body radiation pressure and energy is included by simply adding the term \(T^4/3\) to the Lagrangian density. This is appropriate in a region where radiation is believed to be in equilibrium with matter, as in the photosphere of the Sun. It effectively raises or lowers the effective adiabatic index towards the asymptotic limit \(n = 3\) as the temperature goes to infinity. It is an improvement of the approximation introduced by Eddington (1926) who postulated that the ratio \(p_{\text{gas}}/p_{\text{radiation}}\) is uniform.
7. The temperature is an independent field, fixed by the equation of motion, Eq.(3.2). This is important in the case that radiation is taken into account for this affects the polytropic relation. It leads to a correct inclusion of radiation pressure and energy.
8. As to the interpretation of the variables it should be noted that the pressure is the on-shell value of the Lagrangian density, so that the term \(-pg_{\mu\nu}\) in the energy momentum tensor is the same in both theories, but the dynamics is in the expression for the Lagrangian and it is lost when the Lagrangian is replaced by its value on shell. As to the other term, it can be compared to Tolman’s expression only in the case of a simple system. There is no conserved energy density in General Relativity, but in the non relativistic limit

\[
\rho\psi\dot{\psi} = \psi \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = h + \mathcal{L},
\]
and on shell it is \( h + p \), where \( h \) is the Hamiltonian. Hence Tolman’s \( \rho \) corresponds to the Hamilton density. An essential difference is that Tolman’s formula deals with on shell quantities that obey no dynamics besides what is implied by the Bianchi identities.

9. The entropy enters in precisely the same way as in the classical theory. Any advance in the conceptual or operational meaning of entropy is transferred to the relativistic theory, without additional complications.

10. Since the mass flow current is conserved it can be coupled to the electromagnetic potential and the theory then becomes a model of relativistic plasma. This also allows to make contact with the Reissner Nordstrom solution of Einstein’s equation (Fronsdal 2007b).

The non relativistic theory of binary mixtures is easily lifted to the relativistic context, but a relativistic theory of more complicated mixtures has not yet been developed.

The important problem of rotational and vortex motion in General Relativity must await the development of a variational approach that allows for irrotational flows, see the last section.

**Example, dark matter**

An interesting equation of state for stellar interiors, first proposed by Stoner (1930) and Chandrasekhar (1935), is based on a degenerate Fermi gas. A variant was used in a celebrated paper by Oppenheimer and Volkov (1939) and an account of it may be found in Landau and Lifshitz (1958). Another variant of this equation of state was used recently to model the dark matter in the Milky Way. (Fronsdal and Wilcox 2012)

Let \( f \) be the internal energy density. Little or no attention will be given to the temperature; it may be assumed, either, that it is constant, or that the entropy is zero; in either case \( f \) is effectively a function of the density. Variation of the action with respect to the density gives, in the case of a stationary, spherically symmetric metric,

\[
\frac{1}{2}(c^2 g^{00} - 1) =: \phi = \frac{df}{d\rho}.
\]

Taking the gradient leads to the usual hydrostatic conditions relating \( \rho \) \( \text{grad} \phi \) to the gradient of the pressure

\[
p = \rho \frac{\partial f}{\partial \rho} - f.
\]

That is, the gravitational potential is identified with the chemical potential. We note that

\[
\rho = \frac{dp}{d\phi}.
\]

Examples studied by Fronsdal and Wilcox (2012) have (omitting constants) \( \rho(\phi) = \sinh^4 \phi \). The non relativistic approximation leads to the the Laplace equation, \( \Delta \phi = \rho \),

\[
r^{-2} \partial_r r^2 \partial_r \phi = \sinh^4 \phi,
\]

a “sinh-Emden equation” that is solved exactly by

\[
\phi(r) = \ln(1 + b/r).
\]
It can be used to account for the observed rotation curve of visible stars in the Milky Way. It predicts nearly constant orbital velocities up to the distance \( r = b \). The solution is regular all the way to the center and reveals a small core that resembles a black hole. A more elaborate model involving a mixture of two components gives an equation of state that can account for the high gravitational fields observed at about \( 1^{16} \) cm from the center (Fronsdal and Wilcox 2003). The presence of a non vanishing mass density appears to prevent the formation of black holes but to permit very massive accumulations of dark matter. (The equations of state used in this example are not applicable to standard, visible matter.)

V. General velocity fields.

It has seemed to be very difficult to construct an action principle for the case of general, rotational velocity fields. In the non relativistic theory of fluid motion this is reflected in the fact that the only popular approach is based on the Navier Stokes equation. But recently it has been found that something can be done in special cases. Cylindrical Couette flow is one such case. The key to finding a solution is to represent the flow velocity as a time derivative, \( \vec{v} = (d/dt)\vec{X} \). The field \( \vec{X} \) is the field that is usually denoted \( \vec{x} \) in the Lagrangian version (an alternative to the Eulerian version) of fluid mechanics.

The (non relativistic) Lagrangian density for Couette flow is

\[
L = \rho \left( \rho(\Phi + \dot{\vec{X}}^2/2 + \dot{\vec{X}} \cdot \vec{\nabla}\Phi - (\vec{\nabla}\Phi)^2/2 - \phi \right) - f - sT.
\]

The conserved flow is

\[
\rho \vec{v}, \quad \vec{v} = \dot{\vec{X}} - \vec{\nabla}\Phi.
\]

The principal advantage of this model is that it offers a first integral, in the form of the Hamilton, and an associated expression for energy flow. The concept of a conserved energy is not a feature of the Navier Stokes approach and we are not comfortable with the practice of postulating an ad hoc expression for “energy”.

The decomposition of the flow velocity into two parts of which one is potential and the other is solid-body flow is precisely the same as in the traditional treatment with the Navier Stokes equation.

The equations of motion give a good account of the laminar flow observed at low rotation speeds. This analysis is independent of viscosity but not incompatible with it. The more interesting (but more difficult) problem of the many types of flow that are observed at high rotation speeds have not yet been analysed from this point of view.

This simple example demonstrates that vorticity is not incompatible with Lagrangian structure. Fundamental questions remain to be answered and a promising relativistic version is yet to be found.

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