Abstract

A central question in federated learning (FL) is how to design optimization algorithms that minimize the communication cost of training a model over heterogeneous data distributed across many clients. A popular technique for reducing communication is the use of local steps, where clients take multiple optimization steps over local data before communicating with the server (e.g., FedAvg, SCAFFOLD). This contrasts with centralized methods, where clients take one optimization step per communication round (e.g., minibatch SGD). A recent lower bound on the communication complexity of first-order methods shows that centralized methods are optimal over highly-heterogeneous data, whereas local methods are optimal over purely homogeneous data (Woodworth et al., 2020b). For intermediate heterogeneity levels, no algorithm is known to match the lower bound. In this paper, we propose a multistage optimization scheme that nearly matches the lower bound across all heterogeneity levels. The idea is to first run a local method up to a heterogeneity-induced error floor; next, we switch to a centralized method for the remaining steps. Our analysis may help explain empirically-successful stepsize decay methods in FL (Charles & Konečný, 2020). We demonstrate the scheme’s practical utility in image classification tasks.

1. Introduction

In federated learning (FL) (McMahan et al., 2017; Kairouz et al., 2019; Li et al., 2020), distributed clients interact with a central server to learn a model without directly sharing their data with the server. The training objective is of the form:

$$\min_x \left[ F(x) = \frac{1}{N} \sum_{i=1}^{N} F_i(x) \right]$$ (1)

where $x$ is the model parameter, $i$ indexes the clients (or devices), $N$ is the number of clients, and $F_i(x)$ is a (strongly convex) loss that only depends on that client’s data. Typical FL deployments have two properties that make optimizing Eq. (1) challenging: (i) Data heterogeneity: Let $F_i(x) = \mathbb{E}_{z_i \sim P_i}[f(x; z_i)]$ for some loss function $f$, where $z_i$ is the client’s data and $P_i$’s capture the heterogeneity of the clients’ data distributions. Formally, define client heterogeneity $\zeta$ as

$$\zeta^2 := \sup_{x} \frac{1}{N} \sum_{i=1}^{N} \|\nabla F(x) - \nabla F_i(x)\|^2,$$ (2)

which captures how different the local client gradients are from the overall global gradient (in the worst-case) (Woodworth et al., 2020a). As in (Koloskova et al., 2020; Karimireddy et al., 2020b; Woodworth et al., 2020a), we define $\zeta^2 := \frac{1}{N} \sum_{i=1}^{N} \|\nabla F_i(x)\|^2$ as the heterogeneity at the optimum $x^*$. (ii) Communication cost: In many deployments of FL, clients may have limited bandwidths (e.g., mobile devices) and hence high communicate cost. Due to these two challenges, most federated optimization algorithms alternate between local rounds of computation, where clients process only their own data to save communication, and global rounds, where clients synchronize with the central server to resolve their disagreements due to heterogeneity.

A large number of federated optimization algorithms navigate this trade-off between reducing communication and resolving data heterogeneity by modifying the amount and the nature of local and global computation (McMahan et al., 2017; Li et al., 2018; Wang et al., 2019b;a; Li et al., 2019; Karimireddy et al., 2020b;a; Al-Shedivat et al., 2020; Reddi et al., 2020; Charles & Konečný, 2020). A recent work (Woodworth et al., 2020a) showed a lower bound for the...
convergence rate (with respect to number of communication rounds $R$) of any first-order distributed optimization algorithms for strongly-convex, smooth objectives as a function of client data heterogeneity $\zeta_\star$ (Section 2, Eq. (7)). This lower bound has highlighted two gaps in the literature: (1) Today, no single optimization algorithm has been shown to match the lower bound of (Woodworth et al., 2020a) for all values of $\zeta_\star$. (2) Although different baseline optimization algorithms match the lower bound in different heterogeneity regimes, it remains unknown if any algorithm can match the lower bound when $0 < \zeta_\star^2 < \beta^{3/2} \Delta / \mu^{1/2}$, where $\Delta$ is the initial distance to the optimum, and the functions $F_i$ are $\beta$-smooth and $\mu$-strongly convex. This a substantial regime: $\Delta$ can be sizable if one starts optimization far away from the optimum. In addition, the difference between upper bounds and lower bounds in this regime is significant, as we discuss in Section § 2. To explain these two gaps more precisely, we start by examining several baseline optimization algorithms.

To begin, consider the simple baseline of Minibatch SGD (Woodworth et al., 2020a). Here, in each global communication round, the central server sends each device the current model. Then the each device calculates a single local stochastic gradient at this model over a minibatch of size $K$ and sends it to the server. The server averages these to make a global minibatch gradient over $NK$ samples and performs a model update. Further, this approach can be accelerated in the Nesterov sense (Ghadimi & Lan, 2012). Woodworth et al. (2020a) recently showed that Accelerated Minibatch SGD is optimal in the highly-heterogeneous regimes when $\zeta_\star^2 > \beta^{3/2} \Delta / \mu^{1/2}$. However, in other regimes, (Accelerated) Minibatch SGD may be suboptimal as it does not exploit local computation to reduce communication cost.

A popular alternative approach that uses local computation is called Federated Averaging (FedAvg) (McMahan et al., 2017)\textsuperscript{1}. Here each client $i$ independently runs SGD on its own objective $F_i$ for $K$ iterations (local updates), and periodically communicates the most recent iterate to the server, which averages the updates. Compared to Accelerated Minibatch SGD, this approach allows clients to learn from local data in between communication rounds. However, on strongly convex objectives, FedAvg has only been shown to achieve the lower bound (Woodworth et al., 2020a) when the data is purely homogeneous $\zeta_\star^2 = \zeta^2 = 0$. In fact, FedAvg’s known rates can only outperform Accelerated Minibatch SGD’s rates when heterogeneity is almost insignificant: $\zeta^2 \leq \mu \epsilon$, where $\epsilon$ is the target sub-optimality gap for Eq. (1) (Woodworth et al., 2020a). Further, there is an algorithm specific lower bound for FedAvg that shows that improving the analysis of FedAvg will not bring us meaningfully closer to optimal rates or even allow FedAvg to beat Accelerated Minibatch SGD when heterogeneity is non-negligible (Woodworth et al., 2020a).

Several approaches have been proposed to improve FedAvg via cross-device variance reduction (Karimireddy et al., 2020b; Gorbunov et al., 2020) and adaptivity and momentum (Reddi et al., 2020). Empirically, these variants of FedAvg outperform baselines like (Accelerated) Minibatch SGD. However, none of their existing analyses show any order-wise improvements over FedAvg.

In summary, currently there are only two baseline optimization algorithms that theoretically match the lower bound of (Woodworth et al., 2020a): FedAvg when $\zeta^2 = 0$ (which implies $\zeta_\star^2 = 0$) and Accelerated Minibatch SGD when $\zeta_\star^2 > \beta^{3/2} \Delta / \mu^{1/2}$. Several natural questions follow. (1) Does there exist an algorithm that achieves the lower bound when $0 < \zeta_\star^2 < \beta^{3/2} \Delta / \mu^{1/2}$? In fact, (Woodworth et al., 2020a) posed a weaker open question, asking whether it was possible to improve on Accelerated Minibatch SGD in the regime where data heterogeneity was bounded but not insignificant. (2) If so, can the same algorithm simultaneously achieve optimal convergence rates across all values of $\zeta_\star$?

\textbf{This Work.} In this work, we answer both questions in the affirmative by drawing inspiration from the extensive literature on \textit{multistage algorithms}, which schedule the learning rate to decay over the course of optimization. Such algorithms have been used to achieve provably fast convergence rates in the convex optimization setting (Aybat et al., 2019; Fallah et al., 2020). In the federated setting, multistage algorithms have been heuristically used to improve practical performance (Reddi et al., 2020; Charles & Konečnỳ, 2020).

Our work, however, exploits a key insight: in federated learning over heterogeneous data, multistage algorithms should tune not only stepsizes, but also the \textit{dependence on local computation}. That is, far from the optimum, local computation can be beneficial for reducing computation; near the optimum, it is more important to communicate frequently to resolve heterogeneity. This intuition has been explored in the homogeneous setting (Wang & Joshi, 2019), where the theoretical benefits do not appear to give order-wise convergence improvements.

\textbf{Contributions.} In this paper, we first show that multistage optimization, when applied to baseline federated minimization algorithms, achieves comparable or better worst-case convergence rates than known baselines (including minibatch SGD and FedAvg) in all heterogeneity regimes. In doing so, multistage algorithms resolve an open problem from (Woodworth et al., 2020a), which asked if one can design an algorithm that combines the advantages of both local SGD and minibatch SGD and enjoys guarantees that dom-

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\textsuperscript{1}This approach and its variants has also been referred to as Local Stochastic Gradient Descent (Stich, 2018; Khaled et al., 2020; Woodworth et al., 2020b) and LocalUpdate (Charles & Konečnỳ, 2020) in literature, and it has its origin in Parallelized Stochastic Gradient Descent (Zinkevich et al., 2010).
We bound the initial suboptimality gap as \( \Delta \). We show these theoretical gains by first analyzing an algorithm that (nearly) achieves the lower bound: a simple two-stage combination of Accelerated Minibatch SGD and FedAvg. Loosely, we first run FedAvg up to an error floor that depends on client heterogeneity; then, we use Accelerated Minibatch SGD to complete the optimization. Further, we show that similar gains can be obtained by a more general multistage algorithm which gradually moves from the FedAvg update to Minibatch SGD update. This multistage has the added benefit of not requiring the knowledge of the noise bound \( \sigma^2 \), the heterogeneity \( \zeta^2 \), or the initial distance to the optimum \( \Delta \). This intuition is not specific to the combination of Accelerated Minibatch SGD and FedAvg. We demonstrate the generality of multistaging by applying it to other pairs of FL optimization algorithms to obtain the same nearly-optimal worst-case convergence rate (but possibly much better average-case convergence rate as shown in our experiments in § 4).

Our theoretical results shine some light on the reason behind the empirical success of stepsize scheduling in FL (Charles & Konečný, 2020; Reddi et al., 2020). The connections between these works and ours is explored in § 5.

Finally, we demonstrate the practical value of these theoretical insights through experiments involving logistic regression on the MNIST dataset (LeCun et al., 2010).

### 2. Model and Related Work

As stated in § 1, \( N \) is the number of clients and heterogeneity \( \zeta \) (resp. heterogeneity at optimum \( \zeta_* \)) is defined in (resp. below) Eq. (2). We assume that each \( F_i \) is \( \beta \)-smooth and \( \mu \)-strongly convex; we let \( \kappa = \beta/\mu \) denote the condition number. We assume bounded variance of stochastic gradients on each machine:

\[
E_{z_i}[\|\nabla f(x; z_i) - \nabla F_i(x)\|^2] \leq \sigma^2.
\]

We bound the initial suboptimality gap as \( \Delta \geq E_{x_0}[F(x_0)] - F(x^*) \), where \( x_0 \) is the initial point, \( x^* \) is the global minimizer, and the expectation is taken over any potential randomness of the initialization. Optimization proceeds in rounds; \( K \) is the number of client iterations per round, and \( R \) is the number of rounds. In some cases, clients take minibatches of size \( B \) in their local steps; unless specified otherwise, \( B = 1 \). Our goal is to minimize the optimization error after \( R \) rounds (or to minimize round complexity, the number of rounds \( R \) to a given error \( \epsilon \)). Here \( R \) is the standard proxy for the communication cost. We use the notation \( \tilde{O} \) to hide polylogarithmic factors.

### 2.1. Related Work

To date, the fastest (worst-case) convergence rate for our setting is achieved either by FedAvg (McMahan et al., 2017) or Minibatch SGD in different heterogeneity regimes.

**FedAvg.** Convergence properties of Federated Averaging for convex minimization were first studied in the homogeneous client setting (Stich, 2018; Wang & Joshi, 2018; Woodworth et al., 2020b). These rates were later extended to the heterogeneous client setting (Khaled et al., 2020; Karimireddy et al., 2020b; Woodworth et al., 2020a), including an accompanying lower bound (Woodworth et al., 2020a).

The fastest known rate for the worst-case optimization error of FedAvg is due to (Gorbunov et al., 2020) (Table 1):

\[
\tilde{O}(\kappa \Delta \exp(-\frac{KR}{\kappa}) + \frac{\sigma^2}{\mu NKR} + \frac{\kappa \sigma^2}{\mu KR^2} + \frac{\kappa \zeta^2}{\mu R^2}).
\] (4)

If we take enough local steps \( K \), this error is dominated by the last term, \( \frac{\kappa \zeta^2}{\mu R^2} \). If heterogeneity is low (e.g., \( \zeta^2 = 0 \)), then for large enough \( K \), only one round of communication is needed. This explains why FedAvg outperforms Accelerated Minibatch SGD in the low-heterogeneity regime.

**Accelerated Minibatch SGD.** In the other extreme of highly heterogeneous clients Accelerated Minibatch SGD is optimal. It has a convergence rate (Ghadimi & Lan, 2012; Woodworth et al., 2020a) of

\[
\mathcal{O}(\Delta \exp\left(-\frac{R}{\sqrt{\kappa}}\right) + \frac{\sigma^2}{\mu NKR}).
\] (5)

Recall the FedAvg upper bound in Eq 4. Consider the noiseless case: \( \sigma^2 = 0 \). By solving for the round complexity needed for FedAvg’s upper bound and Accelerated Minibatch SGD’s upper bound to reach \( \epsilon \) accuracy and comparing them, we see FedAvg’s communication complexity is not less than that of the basic minibatch algorithms unless \( \zeta^2 \leq \mu \rho \log(1/\epsilon) \), i.e., heterogeneity must scale with \( \epsilon \), and this is unrealistic.

This is not a product of the analysis of FedAvg being loose. Woodworth et al. (2020a) also showed a lower bound for FedAvg of

\[
\Omega(\min\{\Delta \exp(-\frac{R}{\sqrt{\kappa}}), \frac{\kappa \zeta^2}{\mu R^2}\} + \frac{\sigma^2}{\mu NKR} + \min\{\Delta, \frac{\kappa \sigma^2}{\mu KR^2}\}).
\] (6)

Considering the noiseless case, Accelerated Minibatch SGD has a convergence rate of \( \Delta \exp(-R/\sqrt{\kappa}) \) and FedAvg has the lower bound \( \min\{\Delta \exp(-R/\kappa), \frac{\kappa \zeta^2}{\mu R^2}\} \). \( \Delta \exp(-R/\kappa) \) is always worse than \( \Delta \exp(-R/\sqrt{\kappa}) \), so FedAvg can only ever outperform Accelerated Minibatch...
Reducing the Communication Cost of Federated Learning through Multistage Optimization

Table 1. Error rates under the strongly convex conditions. Here we hide constants and log terms. ASG denotes the constant stepsize variant of Accelerated Minibatch SGD that we specify in Theorem 4. M-FedAvg is the multistage variant of FedAvg we introduce in Theorem 5. M-ASG is a previously known multistage accelerated stochastic gradient method (Aybat et al., 2019). When applicable, _K_ and _B_ are taken large enough to make the dominating terms in _R_ and _B_ be folded into _σ^2_ everywhere to simplify notation. “X → Y” denotes first running “X” and using the output as input to “Y”. All upper bounds require knowledge of at least two of _σ^2_, _Δ_, _ζ^2_, except for Minibatch SGD and the FedAvg → M-ASG rate. All rates achieved by FedAvg → X for some algorithm X are also achieved by SCAFFOLD → X. The two SCAFFOLD rates are achieved by different stepsize choices.

| Method/Analysis | _E_F(\hat{x}) - F(z^*) ≤ O(·) |
|-----------------|----------------------------------|
| **Minibatch Methods** | |
| Minibatch SGD (SGD) (Woodworth et al., 2020a) | \(\frac{\beta \Delta}{\mu} \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R} \), |
| Minibatch AC-SA (AC-SA) (Woodworth et al., 2020a) | \(\Delta \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R} \) [\(R ≥ Ω(\sqrt{\frac{1}{\mu}})\)] |
| **Federated Algorithms** | |
| SCAFFOLD (Karimireddy et al., 2020b) | \((\beta \Delta + \frac{\mu \zeta^2}{\beta^2}) \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R}\), |
| FedAvg (Gorbunov et al., 2020) | \(\kappa \Delta \exp\left(-\frac{KR}{\kappa}\right) + \frac{\sigma^2}{\mu N K R} + \frac{\beta \sigma^2}{\mu K R^2} + \frac{\beta^2 \sigma^2}{\mu^2 R^2}\), |
| SCAFFOLD (Theorem 7) | \(\kappa \Delta \exp\left(-\frac{KR}{\kappa}\right) + \frac{\sigma^2}{\mu N K R} + \frac{\beta \sigma^2}{\mu K R^2} + \frac{\beta^2 \sigma^2}{\mu^2 R^2}\), |
| FedAvg → SGD (Theorem 1) | \(\min\{\kappa \Delta, \frac{\zeta^2 \sigma^2}{\mu N K R}\} \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R}\), |
| FedAvg → ASG (Theorem 2) | \(\min\{\Delta, \frac{\beta \zeta^2}{\mu N K R}\} \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R} + \frac{\beta \sigma^2}{\mu K R^2} + \frac{\beta^2 \sigma^2}{\mu^2 R^2}\), |
| M-FedAvg → M-ASG (Theorem 3) | \(\min\{\Delta, \frac{\beta \zeta^2}{\mu N K R}\} + \kappa \Delta \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R}\), |
| Lower bound (Woodworth et al., 2020a) | \(\min\{\frac{\Delta}{\sqrt{\kappa}} \frac{\mu \zeta^2}{\beta^2} \} \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R}\), |

SGD when \(\Delta \exp\left(-\frac{R}{\sqrt{\mu}}\right)\) is bigger than \(\kappa \zeta^2 \mu R^2\) to reach \(\epsilon\) accuracy. Solving for \(R\), this means that FedAvg’s lower bound can only result in lower communication complexity than Accelerated Minibatch SGD when \(\zeta^2 \leq \mu \epsilon\). So even in the absolute best case scenario, where the lower bound of (Woodworth et al., 2020a) is the convergence rate of FedAvg, heterogeneity still must scale with \(\epsilon\) for it to outperform minibatch algorithms.

**Lower Bound.** The lower bound for first-order distributed algorithms in the heterogeneous setting, from (Woodworth et al., 2020a), is (also shown in Table 1):

\[
Ω\left(\min\left\{\frac{\Delta}{\sqrt{\kappa}} \frac{\mu \zeta^2}{\beta^2} \right\} \exp\left(-\frac{R}{\sqrt{\mu}}\right) + \frac{\sigma^2}{\mu N K R}\right). \quad (7)
\]

Note the exponential difference between the lower bound in Eq. (7) and the tightest-known \(\zeta\)-dependent upper bound (i.e., FedAvg, Eq. (4)) with respect to their dependency on heterogeneity \(\zeta^2\) and rounds \(R\). In the upper bound (Eq. (4)), \(\zeta^2\) is scaled by \(1/R^2\), whereas in the lower bound (Eq. (7)), it is scaled by a term decaying exponentially fast in \(R\). A natural question is whether this gap can be closed.

**Closing the Gap.** Currently, no algorithm is known to close this gap, or even achieve comparable or faster rates than both Accelerated Minibatch SGD and FedAvg across all heterogeneity regimes. This may be partially because prior theoretical analysis for FL algorithms focused mainly on three hyperparameter regimes: (i) local update stepsizes scale as \(1/K\) (e.g., S-Local-SVRG or FedAdam), (ii) local update stepsizes are small or zero (e.g., SCAFFOLD), and (iii) local update stepsizes do not depend on \(K\) (e.g., FedAvg). Intuitively, regimes (i) and (ii) do not take sufficient advantage of local steps to excel in the moderate-to-low heterogeneity regime, whereas algorithms in regime (iii) may converge too quickly to local optima, leading to a severe dependency on heterogeneity. The only work outside these regimes (to our knowledge) applies only to quadratic objective functions (Charles & Konečný, 2020) and does not achieve the FedAvg rate, possibly due to insufficiently aggressive stepsize decay.

In this paper, we show that this theoretical gap can be closed by carefully combining existing methods (in stepsizes which have known analyses) in stages. This multi-stage approach sequentially leverages the large local stepsizes of (iii) and the small local stepsizes of (i) and (ii), thus getting the benefits of both. In the next section, we explain how to achieve these gains through a simple multistage federated minimization framework.

**Multistage Algorithms.** Multistage algorithms have been
employed in the classical convex optimization setting to obtain optimal rates (Aybat et al., 2019; Fallah et al., 2020; Ghadimi & Lan, 2012; Woodworth et al., 2020a). The general idea is to exponentially decrease the learning rate of some algorithm (for example, Accelerated SGD) in stages to optimally balance the optimizer’s error due to bias and variance. Our federated setting differs from the prior multistage optimization in the following ways: (1) there is optimization error due to client drift (Reddi et al., 2020; Karimireddy et al., 2020b) in addition to error due to bias and variance; this additional type of error cannot be accounted for by existing multistage algorithms, (2) our objective is to minimize communication rounds rather than iteration complexity, and (3) we “multistage” or schedule more hyperparameters than just a learning rate.

In (Wang & Joshi, 2019), a strategy with decreasing number of local updates was used to transition from FedAvg to SGD, so as to speed up distributed optimization. This strategy is conceptually similar to the learning rate decay studied in our work. However: (1) their work focuses on reducing wall-clock time in the homogeneous client setting whereas we focus on reducing communication cost among heterogeneous clients. (2) (Wang & Joshi, 2019) does not show order-wise improvements in convergence time. In the homogeneous setting, later work showed that simple baselines (e.g., Accelerated Minibatch SGD and accelerated SGD on just one machine) already achieve optimal convergence rates (Woodworth et al., 2021) without any multistaging.

### 3. Multistage Federated Minimization

In this section, we introduce a framework for multistage federated minimization algorithms in the strongly convex setting. The framework (specified in Algorithm 1) proceeds over $S$ stages; between stages, we can change both the optimization algorithm and the associated hyperparameters (e.g., stepsizes). Each stage $s \leq S$ requires us to choose a local (client) optimization algorithm (e.g., FedAvg), a set of inner (client) and outer (server) stepsizes, a duration in number of rounds $R$, a number of local client steps $K$, and other hyperparameters (e.g., momentum, weights for averaging client updates).

**Step 1: Local updates:** For a given stage $s \leq S$, we iterate over rounds $r \in \{1, \ldots, R\}$. In a round $r$, client $i$ first initializes their iterate as $x_{i,0}^{(s,r)} = x^{(s,r)}$, and then runs $K$ local update steps of its local optimizer using the local stepsize $\eta_{i}^{(s)}$. Next we give the different update rules used by different algorithm at client $i$. Here $\hat{\nabla} F_i(x)$ denotes the possibly random gradient evaluated at $x$.

- FedAvg and (Accelerated) Minibatch SGD at step $k$, uses the update rule:
  \[
  x_{i,k+1}^{(s,r)} = x_{i,k}^{(s,r)} - \eta_{i}^{(s)} \hat{\nabla} F_i(x_{i,k}^{(s,r)})
  \] (8)

However, (Accelerated) Minibatch SGD sets the local stepsize as $\eta_{i}^{(s)} = 0$. This implies that $x_{i,k}^{(s,r)} = x^{(s,r)}$ for all $0 \leq k \leq K$.

- Client variance-reduced methods like SS-Local-SGD (Gorbunov et al., 2020) which is a stateless variant of SCAFFOLD) at step $k$ uses the udpate rule:
  \[
  x_{i,k+1}^{(s,r)} = x_{i,k}^{(s,r)} - \eta_{i}^{(s)}(\hat{\nabla} F_i(x_{i,k}^{(s,r)}) + \hat{\nabla} F(x^{(s,r)}) - \hat{\nabla} F(x^{(s,r)}))
  \]

**Step 2: Global round update:** All the clients send back their effective updates to the server. Then the server take a uniform average of the updates of all the $N$ clients to compute the pseudo-gradient $G_{i}^{(s,r)}$. This is abstracply by the step $G_{i}^{(s,r)} = \text{ClientOPT}(\eta_{i}^{(s,r)}, x^{(s,r)})$, where

\[
\text{ClientOPT}(\eta_{i}^{(s,r)}, x^{(s,r)}) = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} \hat{\nabla} F_i(x_{i,k}^{(s,r)})
\]

For (Accelerated) Minibatch SGD ClientOPT since we set $\eta_{i}^{(s,r)} = 0$, then $x_{i,k}^{(s,r)} = x^{(s,r)}$. Therefore, we can simplify ClientOPT as a simple re-scaled minibatched stochastic gradient of the global function, with a batchsize of $K$, i.e.

\[
\text{ClientOPT}(\eta_{i}^{(s,r)} = 0, x^{(s,r)}) = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{K} \hat{\nabla} F_i(x^{(s,r)})
\]
Finally, the server takes a (accelerated) stochastic pseudo-gradient descent step using its round iterate history, and the computed pseudo-gradient $g_i^{(s,r)}$ and global stepsizes $\eta_i^{(s,r)}$.

**Step 3: Stage update:** Once $R$ rounds are complete, the server completes the stage by taking a weighted average of its global iterates with weights $\{\{w_k^{(s,r)}\}_{r=1}^{R}\}_{k=1}^{K}$.

$$\text{StageWeightedAverage}\left(\{\{w_k^{(s,r)}\}_{r=1}^{R}\}_{k=1}^{K}\right) = \frac{1}{W^{(s)}} \sum_{r=1}^{R} \sum_{k=1}^{K} w_k^{(s,r)} \left( \frac{1}{N} \sum_{i=1}^{N} x_i^{(s,r)} \right),$$

where $W^{(s)} = \sum_{r=1}^{R} \sum_{k=1}^{K} w_k^{(s,r)}$. Although this weighted sum is commonly used to facilitate theoretical analysis (Karimireddy et al., 2020b; Gorbunov et al., 2020), it is often unnecessary in practice (i.e., we can just output the last iterate). The weights are determined by strong convexity $\mu$ and stepsizes, and do not require further tuning.

### 3.1. Two stage algorithms

We begin with a two-stage instantiation of Algorithm 1. The main idea is to first run FedAvg for a constant fraction of the round budget $R$, which reduces error to the “heterogeneity floor”. Then we use a minibatch algorithm to complete convergence.

**Intuition:** We start by giving some intuition as to why such a strategy might be successful. Figure 1, shows a toy example that conveys the main idea. Consider two strongly convex client objectives: $F_1(x) = \frac{1}{2}(x-1)^2$ and $F_2(x) = (x+1)^2$. The global objective $F(x) = \frac{F_1(x) + F_2(x)}{2}$ is their average. Figure 1 (top) plots the objectives, and Figure 1 (bottom) displays their gradients.

Far away from the optimum, due to strong convexity, all client gradients point towards the optimal solution; specifically, this occurs when $x \in (-\infty, -1] \cup [1, \infty)$. In this regime, clients can use local steps (e.g., FedAvg) to save on communication while still ensuring that per-client global updates are consistent with each other. On the other hand, close to the optimum, i.e., when $x \in (-1, 1)$, some client gradients may point away from the optimum. This suggests that clients should not take local steps to avoid driving the global solution away from the optimum. Therefore, when we are close to the optimum (where “closeness” is determined by the clients’ data heterogeneity), we use an algorithm without local steps (e.g., Minibatch SGD), since such algorithms are less affected by client gradient disagreement. This intuition holds more generally for other problems because we have assumed strong convexity of our objective functions.

Our observation that client gradients will disagree as one approaches the optimal solution has been observed in practice even for objective functions that are not convex. (Charles et al., 2021) observe that over the course of FedAvg execution on neural network StackOverflow next word prediction, client gradients become more orthogonal from each other, i.e. client gradients have different directions.

Our technique parallels multistage algorithms for obtaining optimal dependence on noise in convex optimization (Aybat et al., 2019). There, the idea is to start with a large stepsize to reduce the bias term (i.e. the exponential decay term) in the convergence rate, then decay the stepsize to reduce the noise term until convergence. It also has similarities to the approach of (Wang & Joshi, 2019), which reduces the number of local steps over the course of training.

In this subsection, we find that two stages suffice to achieve low round complexities if $\Delta, \zeta^2, \sigma^2$ are known. In the next subsection, we show how to achieve low round complexities when $\Delta, \zeta^2, \sigma^2$ are unknown, by using more than two stages. We first show how to combine FedAvg with Minibatch SGD to obtain exponential gains in round complexity (as a function of heterogeneity $\zeta$) compared to either baseline.

**Theorem 1.** Run Algorithm 1 over $S = 2$ stages. In the first
stage, use FedAvg as for \( R/2 \) rounds under the following parameter settings: \( \eta = \eta_0^{(1,r)} = \eta_0^{(2,r)} = \mathcal{O}(\min\{\frac{\beta}{\mu}, \frac{1}{\mu K R^2}\}) \).

\( S = 1, \gamma^{(1)} = 0, \) and \( w_k^{(1,r)} = (1 - \mu)\gamma^{K+k+1} \). In the second stage, run Minibatch SGD (so \( \gamma^{(2)} = 0 \)) for \( R/2 \) rounds under the following settings:

1. If \( R > \frac{4\beta}{\mu} \) and \( r < \left\lfloor \frac{R}{2} \right\rfloor \):
   \( \eta_0^{(1,r)} = 0, \eta_0^{(2,r)} = \eta_0 = \frac{1}{4\gamma}, w_k^{(2,r)} = 0 \)

2. If \( R > \frac{4\beta}{\mu} \) and \( r \geq \left\lfloor R/2 \right\rfloor \):
   \( \eta_0^{(2,r)} = 0, \eta_0 = \eta_0 = \frac{2}{(\mu - R - \frac{1}{2} R)} \),
   \( w_k^{(2,r)} = (\kappa + r - \left\lfloor \frac{R}{2} \right\rfloor)^2 \)

3. If \( R \leq \frac{4\beta}{\mu} \):
   \( \eta_0^{(2,r)} = \frac{1}{4\gamma}, \eta_0^{(2,r)} = 0, \)
   \( w_k^{(2,r)} = (1 - \mu)\eta_0^{(2,r)} \).

Denote the output of this second stage by \( x^{(2)} \). For large enough number of local rounds \( K \), the convergence rate of this two-stage algorithm is

\[
\mathbb{E} F(x^{(2)}) - F(x^*) \leq \mathcal{O}\left(\frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2} \exp\left(-\frac{R}{\kappa}\right) + \frac{\sigma^2}{\mu N K R^3}\right).
\]

**Proof:** See Section A

This algorithm already exponentially improves the dependence of the round complexity on heterogeneity from the previous state of the art of \( \frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2} \) to \( \frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2} \exp(\frac{-R}{\kappa}) \).

Next, we obtain our main result, which tightens the dependency on \( \kappa \) to (nearly) up to exponentially shrinking condition number factors and supposing \( \zeta^2 \) is of the order of \( \zeta^2 \) match the lower bound in Eq. (7) by replacing Minibatch SGD with Accelerated Minibatch SGD.

**Theorem 2.** Run Algorithm 1 over \( S = 2 \) stages. In the first stage, use FedAvg for \( R/2 \) rounds under the parameter settings of Stage 1 from Theorem 1.

In the second stage, run Accelerated Minibatch SGD for \( R/2 \) rounds with the following parameter settings:

\[
\gamma^{(1)} = \frac{1}{\sqrt{\mu_0 K}}, \eta_0^{(1,r)} = \eta_0 = \mathcal{O}(\min\{\frac{1}{\mu K}, \frac{1}{\mu N K^2}\})
\]

\( w_k^{(2,r)} = 0 \) for all \( k, r \) except for \( w_1^{(2,R/2)} = 1 \) (i.e. return the last iterate).

Denote the output of this second stage by \( x^{(2)} \). For large enough number of local rounds \( K \), the expected convergence rate of this two-stage algorithm is

\[
\mathbb{E} F(x^{(2)}) - F(x^*) \leq \mathcal{O}\left(\frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2} \exp\left(-\frac{R}{\sqrt{\kappa}}\right) + \frac{\sigma^2}{\mu N K R^3} + \frac{\beta \sigma^2}{\mu N K R^3}\right).
\]

**Proof:** See Section D

Observe that this rate is strictly better than the FedAvg rate (4) and is also strictly better than the Minibatch Accelerated SGD rate (5) when \( \frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2} \leq \Delta \) (since the Minibatch Accelerated SGD rate requires that \( R \geq \sqrt{\kappa} \)).

In practice, the initialization can be far from the optimum, leading to large \( \Delta \). The same rates can be achieved when using one round of SS-Local-SGD as the local optimization algorithm for both of the above two-stage algorithms.

### 3.2. Multistage algorithms

The previous algorithms require knowledge of several data- and problem-dependent parameters, such as \( \beta, \mu, \sigma^2, \zeta^2 \), and \( \Delta \). Although the FL literature commonly assumes knowledge of such parameters (Woodworth et al., 2020a; Karimireddy et al., 2020b; Gorbunov et al., 2020), this may be unrealistic in practice. In this section, we explore how to use multistage algorithms to remove dependencies on all such parameters except \( \beta \), which can be estimated from line search techniques (Beck & Teboulle, 2009; Schmidt et al., 2015), and \( \mu \), which can be estimated from regularization (which is often used in machine learning problems).

The key idea is to divide our total stages \( S \) into two superstages: the first superstage uses one optimization algorithm (e.g., FedAvg) with one hyperparameter schedule, and the second superstage uses another (e.g., Minibatch SGD).

**Theorem 3.** Run Algorithm 1 over two superstages for a total of \( R \) rounds, with \( R \geq 4\sqrt{\kappa} \). In the first superstage, run FedAvg in the setting of Theorem 5 for \( R/2 \) rounds.

In the second superstage, run Multistage Accelerated Stochastic Gradient (M-ASG) (Aybat et al., 2019) with a minibatch of size \( BNK \), using the following parameters (for legibility, we reset \( s = 1 \) for the second superstage):

\[
\gamma^{(1)} = 0, \eta_{(1)} = \frac{1}{R \beta} \text{ with } R^{(1)} \geq 1,
\]

\[
\gamma^{(2)} = \frac{1}{\sqrt{\mu_{0} K}}, \eta_{(2)} = \mathcal{O}\left(\sqrt{\log(2p+2)}\right) \text{ for any } p \geq 1 \text{ and } s \geq 2
\]

\[
R^{(1)} = \left(\frac{1}{\sqrt{\mu_{0} K}}\right) \left(\frac{1}{\sqrt{\mu_{0} K}}\right) \left(\frac{1}{\sqrt{\mu_{0} K}}\right)
\]

For \( B \) large enough, this achieves expected convergence rate of

\[
\mathbb{E} F(x^{(S)}) - F(x^*) \leq \mathcal{O}\left(\frac{(\kappa \Delta \exp(-R) + \frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2}) \exp\left(-\frac{R}{\sqrt{\kappa}}\right) + \frac{\sigma^2}{\mu N K R^3}}{1 + \sqrt{\mu^2 R^2}}\right).
\]

**Proof:** See Section C

For simplicity, suppose \( \zeta^2 > 0, R \geq \log\left(\frac{\mu^2 R^2}{\mu N K R^3}\right) \) (where \( R \) is large enough for \( \kappa \Delta \exp(-R) \) to match \( \frac{\beta^2 \eta_0^{(2,r)}}{\mu^2 R^2} \); this is a
relatively few number of rounds). Then this multistage algorithm has a better convergence rate with respect to communication cost than both FedAvg and Accelerated Minibatch SGD when \( 0 < \zeta^2 < \frac{\mu^2}{4} \) (note that now the variance is \( \frac{\zeta^2}{2} \) and that both algorithms require \( R \geq \Omega(\sqrt{n}) \)). This is done without knowledge of \( \zeta^2, \sigma^2, \) or \( \Delta \).

4. Experiments

We empirically evaluate multistaging on federated regularized logistic regression. Let \((x_{i,j}, y_{i,j})\) be the \( j \)th datapoint of the \( i \)th client. We minimize Eq. (1) where

\[
F_i(w) = \frac{1}{n_i} \sum_{j=1}^{n_i} -y_{i,j} \log(w^\top x_{i,j}) - (1 - y_{i,j}) \log(1 - w^\top x_{i,j})) + \frac{\mu}{2} \|w\|^2.
\]

**Baselines.** We compare the communication round complexities of four single-stage baselines: FedAvg, Minibatch SGD (SGD), Accelerated Minibatch SGD (ASG), and SCAFFOLD (precisely, SS-Local-SGD, which is a stateless variant of SCAFFOLD). We compare these baselines to our multistage combinations.

**Dataset.** We use the MNIST dataset of handwritten digits from 0-9 (LeCun et al., 2010). We model a federated setting with five clients by partitioning the data into groups. First, we take 500 images from each class (digit) for all experiments, totaling 5,000 images. Each client’s local data is a mixture of data drawn exclusively from two digit classes (leading to heterogeneity), and data sampled uniformly from all classes. We call a federated dataset \( X\% \) homogeneous if first \( X\% \) of each class’s 500 images is shuffled and evenly partitioned to each client. The remaining \((100 - X)\% \) of the dataset is partitioned as follows: client \( i \in \{1, \ldots, 5\} \) receives the remaining non-shuffled data from classes \( 2i - 2 \) and \( 2i - 1 \). For example, in a 50\% homogeneous setup, client 3 has 250 samples from digit 4, 250 samples from digit 5, and 500 samples drawn uniformly from all classes. Note that 100\% homogeneity is not the same thing as setting heterogeneity \( \zeta = 0 \) due to sampling randomness; we use this technique for lack of a better control over \( \zeta \). In all experiments we are concerned with variants of binary classification, so we let all the even classes represent 0’s and the odd classes represent 1’s. We set \( K = 20 \).

**Hyperparameters.** All experiments are initialized at 0 with regularization \( \mu = 0.1 \). We fix the total number of rounds \( R \) (differs across experiments). For algorithms using Nesterov momentum, we calculate the momentum parameter from strong convexity \( \mu \) and the chosen stepsizes. For the two-stage methods, we tune the stage 1 stepsizes \( \eta = \eta^{(1,r)}_l = \eta^{(1,r)}_g \) in the range below:

\[
\{10^{-3}, 10^{-2.5}, 10^{-2}, 10^{-1.5}, 10^{-1}\}
\]

For the second stage we let the stepsizes be \( \eta^{(2,r)}_g = 0, \eta^{(2,r)}_l = \eta/K \). We also tune the percentage of rounds before switching to the second stage in the range as

\[
\{10^{-2}, 10^{-1.625}, 10^{-1.25}, 10^{-0.875}, 10^{-0.5}\}
\]

For multi-stage methods (\( S > 2 \)), the number of total stages is problem-dependent. In superstage 1, we first tune the Stage 1 stepsize \( \eta \) (as well as the fraction of total rounds allocated to Stage 1 in the range (11)). Let \( R^{(1)} \) be the number of rounds in the first stage. For subsequent stages \( s \geq 2 \), we use step sizes \( \eta^{(s,r)}_g = \eta^{(s,r)}_l = \frac{\eta}{2^s} \) and \( R^{(s)} = 2^s R^{(1)} \). Once the stepsize decreases to \( \eta/K \), we enter the second superstage. Since we evaluate only minibatch algorithms in superstage 2, we use the same parameters as superstage 1, except the starting stepsize is \( \eta^{(1,r)}_l = \frac{\eta}{K} \) while \( \eta^{(1,r)}_g = 0 \).

**Results.** Figure 2 compares the convergence of multi-stage algorithms to their single-stage counterparts in the stochastic gradient setting (minibatches are 1% of a client’s data), over \( R = 100 \) rounds. The curves marked “\( X \rightarrow Y \)” indicate a multi-stage algorithm that starts with Algorithm \( X \) in (Super-)Stage 1 and transitions to Algorithm \( Y \) in (Super-)Stage 2. Diamonds indicate stage transitions. Each such curve represents the better curve between the two-stage and the multi-stage (\( S > 2 \)) version. We vary the homogeneity of the dataset from 0% (left) to 100% (right). We observe that in all homogeneity settings, the multi-stage algorithm demonstrates (constant) improvements over either baseline alone. These improvements grow more pronounced as data becomes more homogeneous. We also find that multistage algorithms outperform two-stage algorithms (and both outperform single-stage algorithms). Finally, we also observe that SCAFFOLD→SGD outperforms all other curves, including SCAFFOLD→ASG. This can be explained with the fact that acceleration introduces additional noise terms in the stochastic setting as shown in Theorem 2, which can contribute to error if one does not take \( K \) large enough (we set \( K = 20 \) in all our experiments).

5. Discussion: Stepsize Decay in Practice

In this section, we make the case that our strategy of switching from a local step-utilizing algorithm (like FedAvg) to a centralized algorithm (like Minibatch SGD) is similar to the stepsize decay strategies in (Reddi et al., 2020; Charles & Konečný, 2020), thus potentially explaining the empirical gains they see in their papers. In (Charles & Konečný, 2020), the decay scheme is roughly as follows: if the loss decreases too little, then reduce the outer stepsizes by a factor.
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of 0.9 and the inner stepsize by a factor of 0.1. It is easy to see that in a relatively few number of decay events, their algorithm will essentially have switched from a local step-utilizing algorithm to a centralized algorithm. Therefore, it is plausible that the practical improvements they observe from their decay scheme can largely be explained by our results.

In (Reddi et al., 2020), the definition of inner/outer stepsize is a bit different to ours or (Charles & Konečný, 2020). They define the inner stepsize $\eta_l$ as the stepsize that updates

$$x_{i,k} \leftarrow x_{i,k-1} - \eta_l \nabla F_i(x_{i,k-1})$$

and then the clients send $\delta^r_i := x_{i,K} - x_{i,0}$ to the server, where the server averages $\delta^r := \frac{1}{N} \sum_{i=1}^{N} \delta^r_i$ and then updates the global model with outer stepsize $\eta_g$ as

$$x^r \leftarrow x^{r-1} + \eta_g \delta^r$$

where we leave out the modifications for adaptivity that they introduce for simplicity (where adaptivity is in the coordinate-wise sense like in Adam (Kingma & Ba, 2014)). In their case, they decay their $\eta_l$ with a factor of 0.1 every 500 rounds. In this case as well, we see that the algorithm will switch from a local step-utilizing algorithm to a centralized algorithm with a few decay events. So we may also plausibly conclude that the practical improvements they see from stepsize decay can largely be explained by our results.

6. Conclusion

In this paper, we show that multistage parameter schedules improve FL communication complexity in theory and in practice. Future work includes tightening our upper bounds from $\zeta^2$ to $\zeta^2$ and getting rid of the additional $\kappa \Delta \exp(-R)$ term in Theorem 3.

Acknowledgements

This work is funded by faculty research awards from Google, J.P. Morgan Chase, and the Sloan Foundation. This work is also supported in part by NSF grants IIS-1929955, CCF-2019844 as a part of Institute for Foundations of Machine Learning, and CA-2040675.

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A. Proof of Theorem 1

First, let the total budget for the number of rounds be $2R$. We run the two stages of the algorithm with $R$ rounds.

In the first stage, we run Algorithm 1 with

$$
\eta = \eta_g^{(s,r)} = \eta_i^{(s,r)} = \min\{\frac{1}{4\beta}, \log(\max\{2, \min\{\frac{2\|x^{(0)} - x^*\|^2 \mu^2 R^2}{\sigma^2}, \frac{6\beta K(\sigma^2 + 2K\zeta^2)}{\mu K R}\}})\}.
$$

$S = 1$, $\gamma^{(s)} = 0$, $w_k^{(s,r)} = (1 - \mu\eta)^{rK + k + 1}$. Note this is just running FedAvg (Gorbunov et al., 2020) with the above stepsizes. Therefore, from the proof of Gorbunov et al. (2020)(Eq.82), we know that we get the following convergence rate for the output of the algorithm

$$
\mathbb{E} F(x^{(1)}) - F(x^*) \leq 8\beta \exp(-\frac{K R}{8}) \mathbb{E} \|x^{(0)} - x^*\|^2 + \frac{2(\Phi + 1)\sigma^2}{\mu N K R} + \frac{6\beta(\Phi^2 + 1)\sigma^2}{\mu^2 K R^2} + \frac{12\beta(\Phi^2 + 1)\zeta^2}{\mu^2 R^2}
$$

(15)

where $\Phi = \log(\max\{2, \min\{\frac{2\|x^{(0)} - x^*\|^2 \mu^2 R^2}{\sigma^2}, \frac{6\beta K(\sigma^2 + 2K\zeta^2)}{\mu K R}\}})$.

In the second stage, we run Algorithm 1 on $x^{(1)}$ with $\beta^{(s)} = 0$ and

1. If $R > \frac{4\beta}{\mu}$ and $r < \lfloor \frac{R}{2} \rfloor$: $\eta_h^{(2,r)} = 0$, $\eta_g^{(2,r)} = \eta_i^{(2,r)} = \frac{1}{4\beta}$, $w_k^{(2,r)} = 0$

2. If $\frac{4\beta}{\mu}$ and $r \geq \lfloor \frac{R}{2} \rfloor$: $\eta_h^{(2,r)} = \eta_g = \frac{2}{\mu(2K - \lceil R - \frac{rK + k + 1}{2} \rceil)}$, $w_k^{(2,r)} = (1 - \mu\eta)^{-rK + k + 1} = (1 + \mu\eta)^{-rK + k + 1}$

3. If $R \leq \frac{4\beta}{\mu}$: $\eta_h^{(2,r)} = \frac{1}{4\beta}$, $w_k^{(2,r)} = (1 - \mu\eta)^{-rK + k + 1}$.

Note that this is running Minibatch SGD with the above stepsize schedule. From the proof from Woodworth et al. (2020a), the output of this algorithm is

$$
\mathbb{E} F(x^{(2)}) - F(x^*) \leq 128\kappa(\mathbb{E} F(x^{(1)}) - F(x^*)) \exp(-\frac{R}{8\kappa}) + \frac{72\sigma^2}{\mu N K R}
$$

(16)

Plugging in (15), we get the claim.

B. Proof of Theorem 2

First, let the total budget for the number of rounds be $2R$. We run the two stages of the algorithm with $R$ rounds.

In the first stage, we run Algorithm 1 with

$$
\eta = \eta_g^{(s,r)} = \eta_i^{(s,r)} = \min\{\frac{1}{4\beta}, \log(\max\{2, \min\{\frac{2\|x^{(0)} - x^*\|^2 \mu^2 R^2}{\sigma^2}, \frac{6\beta K(\sigma^2 + 2K\zeta^2)}{\mu K R}\}})\}.
$$

$S = 1$, $\gamma^{(s)} = 0$, $w_k^{(s,r)} = (1 - \mu\eta)^{rK + k + 1}$. Note this is just running FedAvg (Gorbunov et al., 2020) with the above stepsizes. Therefore, from the proof of Gorbunov et al. (2020), we know that we get the following convergence rate for the output of the algorithm

$$
\mathbb{E} F(x^{(1)}) - F(x^*) \leq 8\beta \exp(-\frac{K R}{8}) \mathbb{E} \|x^{(0)} - x^*\|^2 + \frac{2(\Phi + 1)\sigma^2}{\mu N K R} + \frac{6\beta(\Phi^2 + 1)\sigma^2}{\mu^2 K R^2} + \frac{12\beta(\Phi^2 + 1)\zeta^2}{\mu^2 R^2}
$$

(17)

where $\Phi = \log(\max\{2, \min\{\frac{2\|x^{(0)} - x^*\|^2 \mu^2 R^2}{\sigma^2}, \frac{6\beta K(\sigma^2 + 2K\zeta^2)}{\mu K R}\}})$.

Next we run Algorithm 1 on $x^{(1)}$ with $\eta_i^{(s,r)} = 0$ and

$$
\eta_g^{(s,r)} = \eta_i = \min\{\frac{1}{K\beta}, \log(\max\{2, \min\{\frac{2\Delta_N K R}{\sigma^2}, \frac{2\Delta_i \mu^2 N K R^3}{\beta\sigma^2}\}\})\}
$$

(18)
\[ \gamma(s) = \frac{1 - \sqrt{\frac{\mu K}{\sigma^2}}}{1 + \sqrt{\frac{\mu K}{\sigma^2}}}, \quad S = 1, \text{ and } w_k^{(s,r)} = 0 \text{ for all } k, r, s \text{ except for } w_1^{(1,R_1)} = 1 \text{ and } \Delta_1 \leq 8\beta\exp(-\frac{5R}{8})\mathbb{E}\|x^{(0)} - x^*\|^2 + \frac{2(\Phi + 1)\sigma^2}{\mu NKR^2} + \frac{6\beta(\Phi^2 + 1)\sigma^2}{\mu^2 K R^2} + \frac{12\beta(\Phi^2 + 1)\sigma^2}{\mu^2 K R^2}. \]

From Theorem 4 we know that
\[ \mathbb{E} F(x^{(2)}) - F(x^*) \leq 2\Delta_1 \exp\left(-\frac{R}{\sqrt{K}}\right) + \frac{\sigma^2(\Phi^{1/2} + 1)}{\mu NKR} + \frac{\beta\sigma^2(\Phi^{3/2} + 1)}{\mu^2 K^3} \]

where \( \Phi_2 = \log(\max\{2, \min\{\frac{2\Delta_1 \mu K}{\sigma^2}, \frac{2\Delta_2 \mu^2 K^3}{\beta\sigma^2}\}\}) \).

Plugging in \( \Delta_1 \) gets us the desired conclusion.

C. Theorem 4

**Theorem 4.** Consider Nesterov’s Accelerated Stochastic gradient method with the iteration sequence
\[ x_{k+1/2} = (1 + \gamma)x_k - \gamma x_{k-1}, \quad x_{k+1} = x_{k+1/2} - \alpha \nabla F(x_{k+1/2}) \]

with \( \gamma = \frac{1 - \sqrt{\mu K}}{1 + \sqrt{\mu K}} \), then
\[ \mathbb{E} F(x_K) - F(x^*) \leq 2\Delta \exp\left(-\frac{K}{\sqrt{\mu}}\right) + \frac{\sigma^2(\Phi^{1/2} + 1)}{\mu K} + \frac{\beta\sigma^2(\Phi^{3/2} + 1)}{\mu^2 K^3} \]

**Proof.** From Aybat et al. (2019) (Eq.13), we know that
\[ \mathbb{E} V_{k+1} \leq (1 - \sqrt{\alpha \mu})(\mathbb{E} V_k) + \frac{\sigma^2 \alpha}{2}(1 + \alpha \beta) \]

where \( F(x_k) - F(x^*) \leq V_k \leq 2(F(x_k) - F(x^*)) \).

If we unroll this, we get
\[ \mathbb{E} V_K \leq (1 - \sqrt{\alpha \mu})^K \mathbb{E} V_0 + \frac{\sigma^2 \alpha^{1/2}}{2\mu^{1/2}} + \frac{\sigma^2 \alpha^{3/2} \beta}{2\mu^{1/2}} \leq \exp\left(-\sqrt{\alpha \mu} K\right) \mathbb{E} V_0 + \frac{\sigma^2 \alpha^{1/2}}{2\mu^{1/2}} + \frac{\sigma^2 \alpha^{3/2} \beta}{2\mu^{1/2}} \]

Now if
\[ \alpha = \min\{1, \frac{\log(\max\{2, \min\{\frac{2\Delta_1 \mu K}{\sigma^2}, \frac{2\Delta_2 \mu^2 K^3}{\beta\sigma^2}\}\})}{\mu K^2}\} \]

observe that if \( \frac{1}{\beta} \leq \log(\max\{\frac{2\Delta_1 \mu K}{\sigma^2}, \frac{2\Delta_2 \mu^2 K^3}{\beta\sigma^2}\}) \) we know that letting \( \Phi = \log(\max\{2, \min\{\frac{2\Delta_1 \mu K}{\sigma^2}, \frac{2\Delta_2 \mu^2 K^3}{\beta\sigma^2}\}\}) \)
\[ \mathbb{E} V_K \leq \mathbb{E} V_0 \exp\left(-\frac{K}{\sqrt{\mu}}\right) + \frac{\sigma^2 \Phi^{1/2}}{2\mu K} + \frac{\beta\sigma^2 \Phi^{3/2}}{2\mu^2 K^3} \]

and if \( \frac{1}{\beta} > \log(\max\{\frac{2\Delta_1 \mu K}{\sigma^2}, \frac{2\Delta_2 \mu^2 K^3}{\beta\sigma^2}\}) \)
\[ \mathbb{E} V_K \leq \frac{\sigma^2}{\mu K} + \frac{\beta\sigma^2}{\mu^2 K^3} + \frac{\sigma^2 \Phi^{1/2}}{2\mu K} + \frac{\beta\sigma^2 \Phi^{3/2}}{2\mu^2 K^3} \]

where we used that \( \mathbb{E} V_0 \leq 2\Delta \).

So altogether, we have that
\[ \mathbb{E} F(x_K) - F(x^*) \leq 2\Delta \exp\left(-\frac{K}{\sqrt{\mu}}\right) + \frac{\sigma^2(\Phi^{1/2} + 1)}{\mu K} + \frac{\beta\sigma^2(\Phi^{3/2} + 1)}{\mu^2 K^3} \]
D. Proof of Theorem 3

Let \( x^{(s')} \) be the output of multistage FedAvg from the first half of the rounds (\( s' \) is the index of the final stage). Then we know that the output of this has guarantee (from Thm 5)

\[
\mathbb{E} F(x^{(s')}) - F(x^*) \leq O(\kappa \Delta \exp\left(-\frac{KR}{\kappa}\right) + \frac{\beta \sigma^2}{\mu^2 BNKR} + \frac{\beta^2 \sigma^2}{\mu^3 BKR^2} + \frac{\beta^2 \zeta^2}{\mu^3 R^2})
\]  (28)

Where in this convergence rate \( K \) is limited to be at most \( \kappa \). Next, observe that the second half of the rounds are running M-ASG from (Aybat et al., 2019). That has a guarantee of (Corollary 3.8), (with no limit on \( K \))

\[
\mathbb{E} F(x^{(S)}) - F(x^*) \leq O(\Delta_1 \exp\left(-\frac{R}{\sqrt{K}}\right) + \frac{\sigma^2}{\mu BNKR})
\]  (29)

where

\[
\Delta_1 \leq O(\kappa \Delta \exp\left(-\frac{KR}{\kappa}\right) + \frac{\beta \sigma^2}{\mu^2 BNKR} + \frac{\beta^2 \sigma^2}{\mu^3 BKR^2} + \frac{\beta^2 \zeta^2}{\mu^3 R^2})
\]  (30)

Which gives the theorem statement without limits in \( K \) if we choose \( B \) large enough for the variance terms to match \( \max\{\kappa \Delta \exp\left(-\frac{KR}{\kappa}\right), \frac{\beta^2 \zeta^2}{\mu^3 R^2}\} \).

E. Proof of Theorem 5

Theorem 5. Instantiate Algorithm 1 as

\[
\eta_g^{(1,r)} = \eta_l^{(1,r)} = \frac{1}{4\beta}, \quad R^{(1)} K \geq 1
\]

\[
\eta_g^{(1,r)} = \eta_l^{(1,r)} = \frac{1}{2\beta^{2s+2}}, \quad R^{(s)} K = \lceil p2^{s+2}\kappa \log(2) \rceil
\]

where \( p \geq 3 \) is some arbitrary positive number. Also set \( \gamma^{(s)} = 0 \). Let \textit{ClientOPT} be one round of either SS-Local-SGD or FedAvg, except each local step is minibatched with a minibatch of size \( B \), and have \textit{StageWeightedAverage} simply return the last iterate. Furthermore, require that \( K \leq \lceil p\kappa \log(2) \rceil \) and \((p\kappa \log(2)) \mod K = 0\). Then if \( R^{(1)} K = R/C \) for some constant and \( KR \geq 2\kappa \),

\[
\mathbb{E} F(x^{(S)}) - F(x^*) \leq O(\kappa \Delta \exp\left(-\frac{KR}{\kappa}\right) + \frac{\beta \sigma^2}{\mu^2 BNKR} + \frac{\beta^2 \sigma^2}{\mu^3 BKR^2} + \frac{\beta^2 \zeta^2}{\mu^3 R^2})
\]  (31)

Proof. We start by proving that

\[
\text{err}_s \leq \frac{\exp\left(-\left(R_1 K/4(4\kappa)\right)\right)}{2p^{(s-1)}} \text{err}_0 + \frac{\sigma^2}{2s\mu \beta N} + \frac{6}{2^{2s+2}\mu \beta} (K \sigma^2 + 2K^2 \zeta^2)
\]  (32)

Where \( \text{err}_s = \mathbb{E} \| x^{(s,R_s)} - x^* \|^2 \). We prove this by induction on \( s \). Recall that from Theorem 6 we have

\[
\mathbb{E} \| x^{(s,r)} - x^* \|^2 \leq (1 - \mu \eta)^r K \mathbb{E} \| x^{(s,0)} - x^* \|^2 + \frac{\eta \sigma^2}{\mu N} + 6\eta \eta^2 (K \sigma^2 + 2K^2 \zeta^2)
\]  (33)

Starting with the base case, with \( \eta = \frac{1}{4\beta} \), \( R_1 K \geq 1 \), we have

\[
\text{err}_1 \leq \frac{\exp\left(-\left(R_1 K/4(4\kappa)\right)\right)}{4\kappa} \text{err}_0 + \frac{\sigma^2}{4\kappa \beta \mu N} + \frac{3}{2\beta \mu} (K \sigma^2 + 2K^2 \zeta^2)
\]  (34)

Now let us assume the result holds for \( s \) and prove it for \( s + 1 \). Using Theorem 6 for stage \( s + 1 \) with \( \eta = \frac{1}{2^{2s+3} \beta} \) and \( R_{s+1} K = \lceil p2^{s+3}\kappa \log(2) \rceil \) gets us

\[
\text{err}_{s+1} \leq \exp\left(-p \log(2)\right) \text{err}_s + \frac{\sigma^2}{2^{2s+3} \mu \beta N} + \frac{6}{2^{2(s+3)} \mu \beta} (K \sigma^2 + 2K^2 \zeta^2)
\]  (35)
Now using the induction hypothesis, we know
\[
err_s \leq \frac{\exp(-(R^{(1)}K)/4\kappa)}{2^{p(s-1)}} err_0 + \frac{\sigma^2}{2^{s+1}\mu\beta N} + \frac{6}{2^{2s+2}\mu\beta}(K\sigma^2 + 2K^2\zeta^2)
\]  (36)

And substituting it back into (32) while having \( p \geq 3 \) gets us
\[
err_{s+1} \leq \frac{\exp(-(R^{(1)}K)/4\kappa)}{2^{ps}} err_0 + \frac{\sigma^2}{2^{s+1}\mu\beta N} + \frac{6}{2^{2s+2}2^{s+1}\mu\beta}(K\sigma^2 + 2K^2\zeta^2)
\]  (37)

Using the following lemma, we can translate this into a convergence rate:

**Lemma 1.** Let \( R^{(1)}K \geq 1 \) and \( R^{(s)}K = [p2^{s+2}\kappa \log(2)] \). Then we know that \( 2^k \geq \Theta(1)(\frac{RK - R^{(1)}K}{p\kappa}) \).

**Proof.** Comes from a simple unrolling of the stage lengths; similar to the proof in (Fallah et al., 2020).

So if we plug in \( R^{(1)}K = \frac{R}{\kappa} \) for some constant \( C \geq 2 \) and \( KR \geq 2\kappa \), we get the convergence rate in the theorem statement by applying smoothness.

**F. Proof of Theorem 6**

**Theorem 6.** Suppose we ran Algorithm 1 for one stage with stepsizes \( \eta^{(s,r)}_i = \eta^{(s,r)}_r = \eta \), with \( \gamma^{(s)} = 0 \) and return the last iterate. Furthermore, require that \( \eta \leq \frac{1}{\beta} \). Then
\[
E\|x^{(s,r)} - x^*\|^2 \leq (1 - \mu\eta)^rK E\|x^{(s,0)} - x^*\|^2 + \frac{\eta\sigma^2}{\mu N} + \frac{6\kappa(K\eta^2\sigma^2 + 2K^2\eta^2\zeta^2)}{\mu N}
\]  (38)

From Gorbunov et al. (2020) (Eq.82), we have the following for FedAvg:
\[
E\|x^{t+1} - x^*\|^2 \leq (1 - \mu\eta)E\|x^t - x^*\|^2 + 2\beta\eta E\|V_t\| + \eta^2\sigma^2 \quad (39)
\]

where \( V_t = \frac{1}{N} \sum_{i=1}^N E\|x_i^t - x^*\|^2 \), where \( t = Kr + k \) where \( r \) is current round, \( k \) is step on current round. \( x_i^t \) is client \( i \)'s iterate at the \( t \)-th step. \( x^t = \frac{1}{N} \sum_{i=1}^N x_i^t \), and \( \eta \) is both the inner stepsize and outer stepsize (i.e. \( \eta^{(r,s)}_i = \eta, \eta^{(r,s)}_r = \eta \)).

From Woodworth et al. (2020a) we know that for FedAvg
\[
V_t \leq 3K\eta^2\sigma^2 + 6K^2\eta^2\zeta^2 \quad (40)
\]

So together,
\[
E\|x^{t+1} - x^*\|^2 \leq (1 - \mu\eta)E\|x^t - x^*\|^2 + 6\beta\eta(K\eta^2\sigma^2 + 2K^2\eta^2\zeta^2) + \frac{\eta^2\sigma^2}{\mu N} \quad (41)
\]

And unrolling this,
\[
E\|x^T - x^*\|^2 \leq (1 - \mu\eta)^T E\|x^0 - x^*\|^2 + \frac{\eta\sigma^2}{\mu N} + 6\kappa(K\eta^2\sigma^2 + 2K^2\eta^2\zeta^2) \quad (42)
\]

**G. Using SS-Local-SGD instead of FedAvg**

In this section we show how one can use SS-Local-SGD instead of FedAvg as ClientOPT.

**Theorem 7.** Let \( V_t = \frac{1}{N} \sum_{i=1}^N E\|x_i^t - x^*\|^2 \), where \( t = Kr + k \) where \( r \) is current round, \( k \) is step on current round, \( x_i^t \) is client \( i \)'s iterate at the \( t \)-th step. Then SS-Local-SGD has
\[
V_t \leq 48K^2\eta^2\zeta^2 + 9\eta^2\sigma^2 \quad (43)
\]
which implies Algorithm Y in (Super-)Stage 2. Each such curve represents the better (smaller) curve between the two-stage and the (right). We observe that in all homogeneity settings, the multi-stage algorithm demonstrates constant-order improvements R (Gorbunov et al., 2020) to get a convergence rate that differs from (15) by only constants in the terms. Thus we can use this can be substituted into Theorem 6 to get the multistage rate for Theorem 3, or it can be substituted into Theorem G.3 in (Gorbunov et al., 2020) to get a convergence rate that differs from (15) by only constants in the terms. Thus we can use SS-Local-SGD instead of Local-SGD in any of the theorems presented in the main paper with only minor constant changes.

So altogether,

$$V_t \leq \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \|x_i^t - x_j^t\|^2 \quad (44)$$

By Jensen’s. Let $g_i(x_i^t)$ be the SS-Local-SGD local update for client $i$ at iteration $t$. Let $E_t$ denote expectation conditioned on everything up to the $t-1$’th iteration. Let $y^t$ be the last synchronized iterate before iteration $t$. In the following we use the fact that the control iterates are computed with minibatches of size $K$ at the clients.

$$\mathbb{E} \|x_i^t - x_j^t\|^2$$

$$\leq \mathbb{E} \|x_i^{t-1} - x_j^{t-1} - \eta \mathbb{E} \sum_{i=1}^{N} \mathbb{E} \sum_{j=1}^{N} \mathbb{E} \|x_i^t - x_j^t\|^2$$

$$= \mathbb{E} \|x_i^{t-1} - x_j^{t-1} - \eta(\nabla F(y^t) - \nabla F(y^{t-1}) - \nabla F_i(x_i^{t-1})) + \eta(\nabla F(y^t) - \nabla F_j(y^{t-1}) + \nabla F_j(x_j^{t-1}))\|^2 + \eta^2 \sigma^2$$

$$= \mathbb{E} \|x_i^{t-1} - x_j^{t-1} - \nabla F_i(x_i^{t-1}) + \nabla F_j(x_j^{t-1}) - \eta(\nabla F(y^t) - \nabla F(y^{t-1}) + \nabla F_j(x_j^{t-1}))\|^2 + \eta^2 \sigma^2$$

$$\leq (1 + \frac{1}{K-1}) \mathbb{E} \|x_i^{t-1} - x_j^{t-1} - \nabla F_i(x_i^{t-1}) + \nabla F_j(x_j^{t-1})\|^2 + \eta^2 K \mathbb{E} \|\nabla F_i(x_i^{t-1}) - \nabla F_i(x_i^{t-1}) - \nabla F_j(x_j^{t-1}) + \nabla F_j(x_j^{t-1})\|^2$$

$$+ \nabla F(y^{t-1}) - \nabla F_i(y^{t-1}) - \nabla F(y^{t-1}) + \nabla F_j(y^{t-1})\|^2 + 3\eta^2 \sigma^2$$

$$\leq (1 + \frac{1}{K-1}) \mathbb{E} \|x_i^{t-1} - x_j^{t-1}\|^2 + 16K\eta^2 \zeta^2 + 3\eta^2 \sigma^2$$

So altogether,

$$\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \|x_i^t - x_j^t\|^2 \leq \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} (1 + \frac{1}{K-1}) \mathbb{E} \|x_i^{t-1} - x_j^{t-1}\|^2 + 16K\eta^2 \zeta^2 + 3\eta^2 \sigma^2 \quad (45)$$

Unrolling this until the last time the iterates were synchronized, we get

$$\frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{E} \|x_i^t - x_j^t\|^2 \leq 3K(16K\eta^2 \zeta^2 + 3\eta^2 \sigma^2) \quad (46)$$

Which implies

$$V_t \leq 48K^2\eta^2 \zeta^2 + 9\eta^2 \sigma^2 \quad (47)$$

This can be substituted into Theorem 6 to get the multistage rate for Theorem 3, or it can be substituted into Theorem G.3 in (Gorbunov et al., 2020) to get a convergence rate that differs from (15) by only constants in the terms. Thus we can use SS-Local-SGD instead of Local-SGD in any of the theorems presented in the main paper with only minor constant changes.

H. Additional Experimental Results

Figure 3 compares the convergence of multi-stage algorithms to their single-stage counterparts in the deterministic setting (full batch gradients), over $R = 200$ rounds. This is different from Fig. 2, which uses stochastic gradients. The curves marked “X → Y” indicate a multi-stage algorithm that starts with Algorithm X in (Super-)Stage 1 and transitions to Algorithm Y in (Super-)Stage 2. Each such curve represents the better (smaller) curve between the two-stage and the multi-stage ($S > 2$) version of the listed algorithms. We vary the homogeneity of the dataset from 0% (left) to 100% (right). We observe that in all homogeneity settings, the multi-stage algorithm demonstrates constant-order improvements
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Figure 3. All gradients are deterministic. A plot’s title denotes how homogeneous the data is across the clients, as described in Section 4. “X→Y” denotes a multistage algorithm with X as the first stage and Y as the second stage. Across all heterogeneity levels, SS-Local-SGD (with no scheduling) is the best, followed by FedAvg→ASG. Diamonds denote when stage transitions occur.

Figure 4. Stochastic gradients are computed with 1% of the data. Each datapoint is the average of 1000 runs. Plot titles denote data homogeneity (§ 4). “X→Y” denotes a multistage algorithm with X as the first superstage and Y as the second superstage. Across all heterogeneity levels, the multistage algorithms perform the best. Markers denote when a stage transition occurs.

in convergence rates over either baseline alone. These improvements are most visible in the early stages of training, and they grow more pronounced as data becomes more homogeneous. We also find that for the multi-stage curves in this plot, the two-stage version had better performance than the multi-stage version. This may be because without noise, we can aggressively set our Minibatch SGD step size without using multi-staging, so only two stages suffice.

Figure 4 compares the convergence of our two super-stage algorithms with multistage variants of SGD, ASG, FedAvg, and M-SCAFFOLD. For M-FedAvg and M-SCAFFOLD, we simply run the algorithms as we run the multi-stage methods as described in Section 4, except the first super-stage lasts the entire run. For M-ASG and M-SGD, we run the algorithms the same way we run the multi-stage methods as described in Section 4, except the second super-stage lasts the entire run and the initial stepsize is \( \eta \). We find here that our algorithms with two superstages perform best.

Figure 5 gives hyperparameter ablation studies for the SCAFFOLD→SGD experiment at 100% shuffled dataset across clients. We are generally able to take larger initial stepsizes with multi-staging, which is what gives us the gain we observe.
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Figure 5. Stochastic gradients are computed with 1% of the data. Each datapoint is the average of 1000 runs. These are ablation studies of initial stage length and initial stepsize for SCAFFOLD→SGD, when the data is shuffled across all clients completely. No stage refers to simply running SCAFFOLD for a fixed step size. One super stage refers to running SCAFFOLD in a multistage manner (i.e. the first super stage lasts the whole run). Two stage refers to running SCAFFOLD with the initial learning rate specified and switching to SGD after the initial stage length with learning rate $\eta/K$. Two super stage is run as specified in Sec 4. Overall, multistaging appears to allow us to take larger initial step sizes, which may be what is giving us our gain.