Volatility density estimation by multiplicative deconvolution

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ABSTRACT
We study the non-parametric estimation of an unknown stationary density \( f_V \) of an unobserved strictly stationary volatility process \((V_t)_{t \geq 0}\) on \( \mathbb{R}_+^2 := (0, \infty)^2 \) based on discrete-time observations in a stochastic volatility model. We identify the underlying multiplicative measurement error model and build an estimator based on the estimation of the Mellin transform of the scaled, integrated volatility process and a spectral cut-off regularisation of the inverse of the Mellin transform. We prove that the proposed estimator leads to a consistent estimation strategy. A fully data-driven choice of \( k \in \mathbb{R}_+^2 \) is proposed and upper bounds for the mean integrated squared risk are provided. Throughout our study, regularity properties of the volatility process are necessary for the analysis of the estimator. These assumptions are fulfilled by several examples of volatility processes which are listed and used in a simulation study to illustrate a reasonable behaviour of the proposed estimator.

KEYWORDS
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Stochastic volatility model, Non-parametric statistics, Multiplicative measurement errors, Mellin transform, Adaptivity

1. Introduction

In this work, we are interested in estimating the unknown stationary density \( f_V : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+ \) of an unobserved, strictly stationary volatility process \((V_t)_{t \geq 0}\), \( V_t = (V_{t,1}, V_{t,2})^T \) in a stochastic volatility model with discrete-time observations. More precisely, we assume that we have access to the discrete-time observations \( Z_{\Delta}, \ldots, Z_{\Delta n}, n \in \mathbb{N}, \Delta \in (0,1), \) of the solution \((Z_t)_{t \geq 0}\) of the stochastic differential equation

\[
dZ_t = \Sigma_t dW_t, \quad \Sigma_t := \begin{pmatrix} \sqrt{V_{t,1}} & 0 \\ 0 & \sqrt{V_{t,2}} \end{pmatrix}, \quad Z_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.1}
\]

where \((W_t)_{t \geq 0}, W_t = (W_{t,1}, W_{t,2})^T\) is a standard Brownian motion on \( \mathbb{R}^2 \), stochastically independent of \((V_t)_{t \geq 0}\).

In the non-parametric literature, the stochastic volatility model has been intensively studied in the earlier 2000s. Introduced by [19] as a natural expansion of the constant...
volatility model studied by [4], the interpretation of the volatility as a stochastic process itself enabled the theory to explain in-practice-observed phenomenons, as pointed out by [22].

The stochastic volatility model has been intensively studied by the authors of [14], [15] and [16] developing limit theorems of the empirical distribution, studying parameter estimation and including the model in a hidden markov model framework. Later on, non-parametric estimators have been studied for instance by [9] and [26] where [9] considered a regression-type estimation problem while [26] considered the point-wise estimation of the stationary density of the volatility process. Both, [26] and [9] studied kernel estimators and univariate volatility processes. The generalisation of [26] for multivariate volatility processes was done by [25] with an isotropic choice of the bandwidth, while a different structure of multivariate volatility processes had been considered in [26]. A penalised projection estimator of the stationary density was studied in [10]. Assuming that the volatility process is an diffusion process [11] proposed a penalised projection estimator for the volatility and drift coefficients in a stochastic volatility model.

Frequently, the mentioned authors built their non-parametric estimators on a log-transformation of the data in order to rewrite the estimation problem into an additive deconvolution problem and use standard deconvolution estimators. This was a common strategy in the non-parametric literature to adress multiplicative errors. In contrary to this strategy, [3] studied the multiplicative measurement error model directly by using the Mellin transform to solve the underlying multiplicative convolution. [3] proposed a kernel density estimator and studied its pointwise risk. Based on this work, [7] constructed a spectral cut-off estimator in the multiplicative measurement error model with global risk. [5] then generalised the results of [7], which are stated for univariate variables, for multivariate density estimation under multiplicative measurement errors with anisotropic choice of the smoothing parameter. Based on the results of [5], we will consider a multivariate stochastic volatility model, similar to the one considered in [25], and propose an anisotropic non-parametric estimator of the stationary density exploiting the rich theory of Mellin transforms.

Our approach differs in the following way from the existing literature. Instead of using a log-transformation of the data, we adress the multiplicative deconvolution problem directly. Despite the fact that this seems to be more natural, we are additionally able to identify and study the underlying inverse problem in a more convenient way, as done in [5] and state more general results. Indeed, our results include the log transformation approach as a special case, as pointed out by [3] and [7]. In contrary to [25], we study an anisotropic choice of the smoothing parameter which in general leads to a more flexible estimator, compare [12] and [5].

The paper is structured as follows. In Section 1 we introduce the bivariate stochastic volatility model, identify the underlying multiplicative deconvolution problem and collect the regularity assumptions on the volatility process \((V_t)_{t \geq 0}\). In Section 2 we introduce the Mellin transform and build an estimator based on the estimation of the Mellin transform of the scaled, integrated volatility process and a spectral cut-off regularisation of the inverse Mellin transform. We measure the performance of our estimator in terms of the mean integrated squared error and provide upper bounds for arbitrary choices of \(k \in \mathbb{R}_+\). We then propose a fully data-driven choice of \(k \in \mathbb{R}_+^2\), based only on the observations \(Z_{\Delta}, \ldots, Z_{n\Delta}\) and bound the risk of the resulting data-driven density estimator. Several examples of volatility processes are then studied in Section 3.1 and used in a simulation study to show reasonable the performance of the proposed estimation strategy. More general results for the density estimation in a
multiplicative measurement error model with stationary data are stated in Section 4, which are needed in the proofs of the results of Section 2. The proof of Section 1.2 and 4 are collected in the Appendix 5.

**Stochastic volatility model** In this paper, we consider the following version of a multivariate stochastic volatility model, motivated by [13], which has also been considered by [25]. For a strictly stationary unobserved Markov process \((V_t)_{t \geq 0}\), we consider the solution \((Z_t)_{t \geq 0}\) of the stochastic differential equation (1.1) where \((W_t)_{t \geq 0}\) is a standard 2-dimensional Brownian motion, stochastically independent of the process \((V_t)_{t \geq 0}\). Then motivated by the work [17], respectively [10], we study the scaled increments of our discrete-time sample \((Z_\Delta)_{j \in [n]}\) for \(\Delta \in (0,1)\) and \([n] := [1,n] \cap \mathbb{N}\). More precisely, let \(D_j := \Delta^{-1/2}(Z_{\Delta j} - Z_{\Delta (j-1)})\), understood componentwise for \(j \in [n]\), then conditioned on \((V_t)_{t \geq 0}\) we have

\[
D_j = \frac{1}{\sqrt{\Delta}} \left( \begin{array}{c} \int_{(j-1)\Delta}^{j\Delta} V_{t,1} dW_{t,1} \\ \int_{(j-1)\Delta}^{j\Delta} V_{t,2} dW_{t,2} \end{array} \right) \sim N(0,\Sigma_V), \quad \Sigma_V := \left( \begin{array}{cc} V_{j,1} & 0 \\ 0 & V_{j,2} \end{array} \right)
\]

exploiting the independence of \((V_t)_{t \geq 0}\) and \((W_t)_{t \geq 0}\), where \(V_{j,\ell} := \Delta^{-1} \int_{(j-1)\Delta}^{j\Delta} V_{s,\ell} ds\), \(\ell \in \{1,2\}\). As a direct consequence, we write

\[
Y_j := \left( \begin{array}{c} Y_{j,1} \\ Y_{j,2} \end{array} \right) := \left( \begin{array}{c} D_{j,1}^2 \\ D_{j,2}^2 \end{array} \right) = \left( \begin{array}{c} V_{j,1} U_{j,1} \\ V_{j,2} U_{j,2} \end{array} \right) := \left( \begin{array}{c} X_{j,1} U_{j,1} \\ X_{j,2} U_{j,2} \end{array} \right) := X_j U_j
\]

where \(X_j\) and \(U_j\) a stochastically independent and \((U_j)_{j \in [n]}\) is an i.i.d. (independent, identically distributed) sequence with \(U_{1,1},U_{1,2} \overset{i.i.d.}{\sim} \chi_1^2 = \Gamma(1/2,1/2)\). In other words, the stochastic volatility model can be expressed as a multiplicative measurement error model with \(\chi^2\)-distributed, respectively Gamma distributed noise. While the authors from [17], [10], [26] and [25] used a log-transformation of the data, we will instead exploit the theory of multivariate Mellin transform and their use in non-parametric density estimation introduced in [3] to build a multiplicative deconvolution density estimator.

**Assumption on the volatility process \((V_t)_{t \geq 0}\)** Throughout this paper, we will need to assume some regularity of the volatility process \((V_t)_{t \geq 0}\) to ensure the well-definedness of the upcoming objects and to deduce consistency of our proposed estimation strategy. As usual in non-parametric approaches, we aim to consider an ensemble of assumptions which can be proven for a wide class of examples of volatility processes. To motivate that these assumptions are not restrictive, we will show in Section 3.1 a number of examples of frequently studied volatility processes. Now let us assume that the discrete-time sample \((Z_\Delta)_{j \in [n]}\) is drawn from a process \((Z_t)_{t \geq 0}\) solving (1.1) where

\begin{itemize}
  \item \((A_0)\) \((W_t)_{t \geq 0}\) is a two-dimensional Brownian motion, independent of the process \((V_t)_{t \geq 0}\) on \(\mathbb{R}_+^2\),
  \item \((A_1)\) \((V_t)_{t \geq 0}\) is a time-homogeneous Markov process, with continuous sample paths, strictly stationary and ergodic. The stationary distribution of \((V_t)_{t \geq 0}\) admits a density \(f_V\) with respect to the Lebesgue measure on \(\mathbb{R}_+^2\),
\end{itemize}
(A₂) \((V_t)_{t \geq 0}\) is \(\beta\)-mixing, with \(\int_{\mathbb{R}^+} \beta_V(s) ds < \infty\), where

\[
\beta_V(s) = TV(P(V_t, V_s), \mathbb{P}^0 \otimes \mathbb{P}^{V_0}), \quad s \in \mathbb{R}^+,
\]

where TV is the total variation distance.

For the estimation we will be in need of the following additional assumption

(A₃) There exists a constant \(c > 0\) such that \(\mathbb{E}(|\log(X_{1,1}) - \log(V_{0,1})| + |\log(X_{1,2}) - \log(V_{0,2})|) \leq c\sqrt{\Delta}\).

In Section 3.1, we will deliver examples of volatility processes \((V_t)_{t \geq 0}\) which satisfy (A₀)-(A₃). While assumptions (A₀)-(A₂) are widely considered in the literature and proven for several diffusion processes, assumption (A₃) is of rather technical nature. A practical proposition in the univariate case was proposed by [10]. Here, we want to state a bivariate counterpart. The proof of Proposition 1.1 can be found in Section 5.2. Here, we denote for \(a \in \mathbb{R}^2\) the Euclidean norm by \(|a|_{\mathbb{R}^2}^2 := a_1^2 + a_2^2\) and for a matrix \(A \in \mathbb{R}^{(2,2)}\) the Frobenius norm by \(|A|_{F}^2 := A_{1,1}^2 + A_{1,2}^2 + A_{2,1}^2 + A_{2,2}^2\). Furthermore, for \(p \in \mathbb{N}_0\) we denote by \(C^p(D)\) the set of all \(p\)-times continuously differentiable functions on \(D \subseteq \mathbb{R}^2\).

**Proposition 1.1.** Suppose the volatility process \((V_t)_{t \geq 0}\) satisfies (either) one of the following conditions

(i) \(V_t = (\exp(Z_{t,1}), \exp(Z_{t,2}))^T, t \geq 0, \) where \((Z_t)_{t \geq 0}\) is a strictly stationary and ergodic diffusion process on \(\mathbb{R}^2\) satisfying \(dZ_t = \tilde{b}(Z_t) + \tilde{a}(Z_t) d\tilde{W}_t, (\tilde{W}_t)_{t \geq 0}\) standard Brownian motion on \(\mathbb{R}^2\) such that there exists \(\bar{L} > 0\) with

\[
||\tilde{a}(x)||_{\mathbb{F}} + ||\tilde{b}(x)||_{\mathbb{R}^2} \leq \bar{L}(1 + |x|_{\mathbb{R}^2})
\]

for all \(x \in \mathbb{R}^2, \tilde{b}, \tilde{a}, i,j \in C^0(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)\) for \(i,j \in [2]\) and \(\mathbb{E}(|Z_0|_{\mathbb{R}^2}^2) < \infty\) or

(ii) or \((V_t)_{t \geq 0}\) is a strictly stationary and ergodic diffusion process on \(\mathbb{R}^2\) satisfying \(dV_t = b(V_t) + a(V_t) d\tilde{W}_t\) such that there exists \(L > 0\) with

\[
||a(x)||_{\mathbb{F}} + ||b(x)||_{\mathbb{R}^2} \leq \bar{L}(1 + |x|_{\mathbb{R}^2})
\]

for all \(x \in \mathbb{R}^2, b_i, a_{i,j} \in C^0(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)\) for \(i,j \in [2]\) and additionally let \(\mathbb{E}(|V_0|_{\mathbb{R}^2}^2) < \infty\) for \(i \in [2]\) hold true.

Then \((V_t)_{t \geq 0}\) satisfies (A₃).

After this brief introduction to the stochastic volatility model, let us propose a non-parametric density estimator based on a multiplicative deconvolution.

### 2. Stochastic volatility density estimation

In this section we introduce the Mellin transform and start to collect some of its major properties, which are stated in [5]. We then propose our estimator.
Notations and definitions of the Mellin transform

For two vectors \( \mathbf{u} = (u_1, u_2)^T, \mathbf{v} = (v_1, v_2)^T \in \mathbb{R}^2 \) and a scalar \( \lambda \in \mathbb{R} \) we define the componentwise multiplication \( \mathbf{u} \mathbf{v} := \mathbf{u} \cdot \mathbf{v} := (u_1 v_1, u_2 v_2)^T \) and denote by \( \lambda \mathbf{u} \) the usual scalar multiplication. Further, if \( v_1, v_2 \neq 0 \) we define the multivariate power by \( \mathbf{u}^\mathbf{v} := v_1^{u_1} v_2^{u_2} \). Additionally, we define the componentwise division by \( \mathbf{u}/\mathbf{v} := (u_1/v_1, u_2/v_2)^T \). We denote the usual Euclidean scalar product and norm on \( \mathbb{R}^2 \) by \( \langle \mathbf{u}, \mathbf{v} \rangle_{\mathbb{R}^2} := \sum_{i=1}^2 u_i v_i \) and \( |\mathbf{u}|_{\mathbb{R}^2} := \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle_{\mathbb{R}^2}} \). Moreover, we set \( \mathbf{1} := (1, 1)^T \in \mathbb{R}_+, \) respectively \( \mathbf{0} := (0, 0)^T \).

For a positive random vector \( \mathbf{Z} \) with \( \mathbb{E}(\mathbf{Z}^{-1}) = \mathbb{E}(Z_1^{-1}Z_2^{-1}) < \infty, \) \( c \in \mathbb{R}^2 \), we define the Mellin transform \( \mathcal{M}_c[\mathbf{Z}] \) of \( \mathbf{Z} \) as the function

\[
\mathcal{M}_c[\mathbf{Z}]: \mathbb{R}^2 \to \mathbb{C}, \quad t \mapsto \mathcal{M}_c[\mathbf{Z}](t) := \mathbb{E}(\mathbf{Z}^{c^{-1}+it}).
\]

As a consequence the convolution theorem for the Mellin transform holds true, that is for \( UV \) independent with \( \mathbb{E}(U(V)^{c^{-1}}) < \infty \),

\[
\mathcal{M}_c[UV](t) = \mathcal{M}_c[U](t) \mathcal{M}_c[V](t), \quad t \in \mathbb{R}^2.
\]

If \( \mathbf{Z} \) emits a Lebesgue density \( h: \mathbb{R}^2_+ \to \mathbb{R}_+ \), then we can write \( \mathcal{M}_c[\mathbf{Z}](t) = \int_{\mathbb{R}^2_+} x^{c^{-1}+it} h(x) dx, \ t \in \mathbb{R}^2 \).

Motivated by this, we define the set \( \mathcal{L}^1(\mathbb{R}^2_+, x^{c^{-1}}) := \{ h: \mathbb{R}^2_+ \to \mathbb{C} : \|h\|_{\mathcal{L}^1(\mathbb{R}^2_+, x^{c^{-1}})} := \int_{\mathbb{R}^2_+} |h(x)| x^{c^{-1}} dx < \infty \} \). Then we can generalise the notion of the Mellin transform for \( \mathcal{L}^1(\mathbb{R}^2_+, x^{c^{-1}}) \) function. Indeed, for \( h \in \mathcal{L}^1(\mathbb{R}^2_+, x^{c^{-1}}) \) we define the Mellin transform of \( h \) at the development point \( c \in \mathbb{R}^2 \) as the function \( \mathcal{M}_c[h]: \mathbb{R}^2 \to \mathbb{C} \) by

\[
\mathcal{M}_c[h](t) := \int_{\mathbb{R}^2_+} x^{c^{-1}+it} h(x) dx, \quad t \in \mathbb{R}^2.
\]

In analogy to the Fourier transform, one can define the Mellin transform for square integrable functions. We define the weighted norm by \( \|h\|_{\mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1})}^2 := \int_{\mathbb{R}^2_+} |h(x)|^2 x^{2c-1} dx \) for a measurable function \( h: \mathbb{R}^2_+ \to \mathbb{C} \) and denote by \( \mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) the set of all complex-valued, measurable functions with finite \( \|\|_{\mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1})} \)-norm and by \( \langle h_1, h_2 \rangle_{\mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1})} := \int_{\mathbb{R}^2_+} h_1(x) h_2(x) x^{2c-1} dx \) for \( h_1, h_2 \in \mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) the corresponding weighted scalar product. Similarly, we define \( \mathcal{L}^2(\mathbb{R}^2) := \{ H: \mathbb{R}^2 \to \mathbb{C} \text{ measurable} : \|H\|_{\mathcal{L}^2(\mathbb{R}^2)} := \int_{\mathbb{R}^2} |H(x)|^2 H(x) dx < \infty \} \).

We are then able to define the Mellin transform as the isomorphism \( \mathcal{M}_c : \mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1}) \to \mathcal{L}^2(\mathbb{R}^2) \). For a precise definition of the multivariate Mellin transform and its connection to the Fourier transform, we refer to [5]. Nevertheless, if \( h \in \mathcal{L}^1(\mathbb{R}^2_+, x^{c^{-1}}) \cap \mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) both notions coincide. By abuse of notation we will denote by \( \mathcal{M}_c[h] \) both notions, for \( h \in \mathcal{L}^1(\mathbb{R}^2_+, x^{c^{-1}}) \), respectively \( h \in \mathcal{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) of the Mellin transform. For a more detailed collection of the properties of the Mellin transform we refer to Section 4 respectively [5].

**Estimation strategy**

For \( k \in \mathbb{R}^2_+ \) we define the hypercuboid \( [-k, k] := [-k_1, k_1] \times [-k_2, k_2] \). Then based on the work of [5], we define for any \( c \in \mathbb{R}^2 \) with \( \mathbb{E}(Y^{c^{-1}}_1) < \infty \)
and $k \in \mathbb{R}^+_1$, $\Delta \in (0, 1)$ the estimator $\hat{f}_{\Delta,k}$ by

$$\hat{f}_{\Delta,k}(x) := \frac{1}{4\pi^2} \int_{[-k,k]} x^{-c-it} \frac{\hat{M}_c(t)}{M_c(c)} dt, \quad x, k \in \mathbb{R}^+_1,$$

where $g : \mathbb{R}^+_1 \to \mathbb{R}^+_1$ is the density of $U_1$ and $\hat{M}_c(t) := n^{-1} \sum_{j \in [n]} Y_j e^{-1-it}$, $t \in \mathbb{R}^2$, is the empirical Mellin transform of the sample $(Y_j)_{j \in [n]}$. Here, the moment assumption is trivially fulfilled for the case $c = 1$. We will mainly focus on this special case in this section while theoretical results for general choices of $c \in \mathbb{R}$ are given in Section 4. Let us assume that $f_V \in L^2(\mathbb{R}_+^2, x^1)$. By construction (2.2), we have $\hat{f}_{\Delta,k} \in L^2(\mathbb{R}_+^2, x^1)$ for any $k \in \mathbb{R}_+^2$. Furthermore, we define the approximation $f_{V,k} \in L^2(\mathbb{R}_+^2, x^1)$ by

$$f_{V,k}(x) := \frac{1}{4\pi^2} \int_{[-k,k]} x^{-1-it} M_1[f_V](t) dt, \quad x, k \in \mathbb{R}^+_1.$$

We can now show the following risk bound for the family of estimators $(\hat{f}_{\Delta,k})_{k \in \mathbb{R}_+^2}$ presented in (2.2), implying that for a suitable choice of the cut-off parameter $k \in \mathbb{R}_+^2$ a consistent estimator can be achieved.

**Theorem 2.1** (Upper bound of the risk). Let $f_V \in L^2(\mathbb{R}_+^2, x^1)$ and assumptions (A0) – (A3) hold true. Then, for any $k \in \mathbb{R}_+^2$ and $\Delta \in (0, 1)$,

$$\mathbb{E}_f^H(\|f_V - \hat{f}_{\Delta,k}\|_{x^1}^2) \leq \|f_V - f_{V,k}\|_{x^1}^2 + c\Delta k_1^3 k_2^3 + \frac{1}{4\pi^2 n} \int_{[-k,k]} |M_1[g](t)|^{-2} dt

+ \frac{k_1 k_2}{n\Delta} \int_{\mathbb{R}_+^2} \beta_V(s) ds$$

where $c$ is defined in assumption (A3).

While the squared bias term $\|f_V - f_{V,k}\|_{x^1}^2$ and $(4\pi^2 n)^{-1} \int_{[-k,k]} |M_1[g](t)|^{-2} dt$ already arise in [5] in the multiplicative deconvolution setting for i.i.d. observations, the remaining two summands in the upper bound of Theorem 2.1 are specific to the stochastic volatility model.

More precisely, the last summand $k_1 k_2 n^{-1} \int_{\mathbb{R}_+^2} \beta_V(s) ds$ is an additional variance part due to the underlying dependency of the observations $(X_j)_{j \in [n]}$, compare Proposition 4.2 respectively [8] for a similar arising term in context of survival function estimation under dependency. The second summand, $c\Delta k_1^3 k_2^3$, on the other hand, is an additional bias term due to the fact, that the distributions of $X_1$ and $V_0$ differ.

It is interesting here that the additional bias term is decreasing for smaller values of $\Delta$ while the additional variance term is increasing for fixed values of $n \in \mathbb{N}$. The latter effect is natural, since for fixed $n \in \mathbb{N}$, the time interval $[0, n\Delta]$, where we discretely derive our observations from, is vanishing. Therefore, a choice of $\Delta$ with respect to $n \in \mathbb{N}$ is non-trivial. We will now focus on the variance term $(4\pi^2 n)^{-1} \int_{[-k,k]} |M_1[g](t)|^{-2} dt$. For $(2\pi)^{-2} \int_{[-k,k]} |M_1[g](t)|^{-2} dt$, in the stochastic volatility model, we have $M_1[g](t) = 2^{it+it_2} \pi^{-1} \Gamma(1/2 + it_1) \Gamma(1/2 + it_2)$ leading to

$$|M_1[g](t)|^{-2} = \frac{\cosh^2(\pi t_1) \cosh^2(\pi t_2)}{(2\pi)^2},$$

where $\cosh(t) := \exp(t) + \exp(-t)/2$. 


using the multiplication theorem of the Γ-function. This is an example of super smooth error densities considered for instance in [3] and [6]. This implies the following corollary whose proof is omitted.

**Corollary 2.2.** Let \( f_V \in L^2(\mathbb{R}_+^2, x^1) \) and assumptions \((A_0) - (A_3)\) hold true. Then, for any \( k \in \mathbb{R}_+^2 \),

\[
E^n_{f_V}(\|f_V - \hat{f}_{\Delta,k}\|_{x^2}^2) \leq \|f_V - f_{V,k}\|_{x^2}^2 + c\Delta k^3 + \frac{e^{\pi(k_1 + k_2)}}{n} + \frac{k_1k_2}{n\Delta} \int_{\mathbb{R}_+^2} \beta_V(s)ds
\]

where \( c \) is defined in assumption \((A_3)\). Now for any \( \Delta = \Delta_n \to 0 \) with \( n\Delta_n \to \infty \) as \( n \to \infty \) we can find a sequence \((k_n)_{n \in \mathbb{N}} \) with \( k_n \to \infty \), such that

\[
E^n_{f_V}(\|f_V - \hat{f}_{\Delta,k_n}\|_{x^2}^2) \to 0,
\]

implying that \( \|f_V - \hat{f}_{\Delta,k_n}\|_{x^2}^2 \to 0 \) in probability.

Although, Corollary 2.2 implies the existence of \((k_n, \Delta_n)_{n \in \mathbb{N}} \) such that \( \hat{f}_{k_n,\Delta_n} \) is a consistent estimator of \( f_V \), a choice of \( k_n \in \mathbb{R}_+^2 \) which minimises the risk would still depend on the decay of the squared bias term \( \|f_V - f_{V,k}\|_{x^2}^2 \) which, without further assumptions, is unknown. We therefore propose in the next paragraph a fully data-driven estimator based on the model selection approach presented in [5] with small assumptions, is unknown. We therefore propose in the next paragraph a fully data-driven estimator based on the model selection approach presented in [5] with small adjustments inspired by the work of [12]. In Section 3.1 we will study examples of volatility processes and deduce their expected rate.

**Data-driven choice of \( k \in \mathbb{R}_+^2 \).** First we reduce the space of possible cut-off parameters to \( \mathcal{K}_n := \{ k \in [\log(n)] \times [\log(n)] : \exp(\pi(k_1 + k_2)) \leq n \} \). This reduction is rather natural, since any choice of \( k_n \), leading to a consistent estimator, implies that \( \exp(\pi(k_{n,1} + k_{n,2}))n^{-1} \) goes to 0 for \( n \to \infty \). We define the model selection method for \( \chi > 0 \) by

\[
\hat{k} := \arg\min_{k \in \mathcal{K}_n} -\|\hat{f}_{\Delta,k}\|_{x^2}^2 + \text{pen}(k), \quad \text{pen}(k) := \chi k_1k_2 \exp(\pi(k_1 + k_2))n^{-1}. \tag{2.3}
\]

In comparison to the model selection in [5], the penalty term \( \text{pen}(k) \) overestimates the variance. This is a frequently observed necessity when it comes to deconvolution estimators with super smooth error densities, compare [12].

**Theorem 2.3 (Data-driven choice of \( k \)).** Let \( f_V \in L^2(\mathbb{R}_+^2, x^1) \) and \((A_0) - (A_3)\) hold true. Then there exists \( \chi_0 \in \mathbb{R}_+ \) such that for all \( \chi \geq \chi_0 \),

\[
E^n_{f_V}(\|\hat{f}_{\Delta,k} - f_V\|_{x^2}^2) \leq C \inf_{k \in \mathcal{K}_n} (\|f_V - f_{V,k}\|_{x^2}^2 + \text{pen}(k)) + C(\epsilon) \Delta \log^6(n) + \frac{C(g)}{n} + \frac{C(\beta_V) \log^2(n)}{n\Delta}
\]

where \( C(\epsilon), C(g) \) and \( C(\beta_V) \) are positive constants only depending on \((A_3)\), \( g \) and \( \beta_V \). Then balancing \( \Delta \) with respect to \( n \in \mathbb{N} \) leads to \( \Delta := \Delta_n = (n^{1/2} \log^2(n))^{-1} \).
implying

$$\mathbb{E}^n_{f_V} (\| \tilde{f}_{\Delta_n} - f_V \|_{x^2}^2) \leq C \inf_{k \in \mathbb{K}_n} (\| f_V - f_{V,k} \|_{x^2}^2) + C(c, g, \beta_V) \frac{\log^4(n)}{n}.$$ 

We end this section by giving a short discussion for which values of $c$ the stated Theorem 2.1 and 2.3 can be generalised.

**Remark 2.4.** For values of $c \neq 1$ additional assumptions on the moments of $X$ and $U$ are needed, compare Proposition 4.2 and Theorem 4.3. Since in the stochastic volatility model the distribution of $U_1$ is known to follow a $\chi^2$-distribution in each direction, we can deduce restrictions on $c$ to ensure that the estimator is well-defined.

Then from $E_g(U_1^{2(c-1)}) < \infty$, we deduce $c > 3/4$ in order to generalise the result of Theorem 2.1. This excludes the case of an unweighted $L^2$ risk which corresponds to the case of $c = (1/2, 1/2)^T$. To generalise Theorem 2.3 we need that $E(U_4^{4(c-1)}) < \infty$, leading to $c > 7/8$.

### 3. Examples for volatility processes and numerical studies

The following section is separated into two parts. In the first a collection of examples of volatility processes, fulfilling $(A_0)$-$(A_3)$, is given with a study of the upcoming bias terms and the rates of the fully data-driven anisotropic estimators $\tilde{f}_{\Delta_n}$. In the second part, we will briefly illustrate the expected behaviour of the proposed estimator via a Monte-Carlo simulation study.

#### 3.1. Collection of volatility processes

**Exponential of a bivariate Ornstein-Uhlenbeck process** Let $(Z_t)_{t \geq 0}$ be the stationary solution of the stochastic differential equation

$$dZ_t = \begin{pmatrix} -9 & 1 \\ 0 & -7 \end{pmatrix} Z_t dt + \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} d\tilde{W}_t, \tag{3.1}$$

where $(\tilde{W}_t)_{t \geq 0}$ is a standard Brownian motion. In this situation, the invariant density $f_Z$ of the process $(Z_t)_{t \geq 0}$ is given by $f_Z \sim N(0, \Sigma)$ where

$$\Sigma = \frac{1}{7} \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

and the process $(Z_t)_{t \geq 0}$ is $\beta$-mixing with exponential decay, compare Section 5.2 Examples. Then, $V_t := (\exp(Z_{t,1}), \exp(Z_{t,2}))^T$ is stationary, $\beta$-mixing with density $f_V$ given by

$$f_V(x) = \sqrt{\frac{7}{2\pi x_1 x_2}} \exp \left( - \log^2(x_1) + \log(x_1) \log(x_2) - 2 \log^2(x_2) \right), \quad x \in \mathbb{R}_+^2.$$
In other words, \( V_0 \) follows a bivariate log normal distribution. Further, exploiting Proposition 1.1, we have that (A0)-(A3) are fulfilled. By [5], the Mellin transform of \( f_V \) is then given by

\[
\mathcal{M}_1[f_V](t) = \exp \left( -\frac{1}{2} t^T \Sigma t \right), \quad t \in \mathbb{R}^2.
\]

From this we can deduce that \( \|f_V - f_{V,k}\|_{\mathbb{R}^2}^2 \leq L(e^{-\frac{1}{2}k_1^2} + e^{-\frac{1}{2}k_2^2}) \) for some numerical constant \( L > 0 \). Then, a direct calculus and Theorem 2.3 implies for this particular case of \( (V_t)_{t \geq 0} \)

\[
\mathbb{E}^n_{f_V} (\|\hat{f}_{\Delta, k} - f_V\|_{\mathbb{R}^2}^2) \leq C(V, g) \frac{\log^4(n)}{n}.
\]

**Bivariate Cox-Ingersoll-Ross process** Let \( \rho_1, \rho_2 \in \mathbb{N} \) and set \( \rho = (\rho_1, \rho_2)^T \). Then we define the bivariate Cox-Ingersoll-Ross process with independent coordinates as the strictly stationary solution of

\[
dV_t = (2\rho - V_t)dt + \sqrt{2} \begin{pmatrix} \sqrt{V_{t,1}} & 0 \\ 0 & \sqrt{V_{t,2}} \end{pmatrix} d\tilde{W}_t,
\]

(3.2)

in other words, each variate is a Cox-Ingersoll-Ross process. From the univariate case, we deduce that the process fulfills (A1). To see (A2), one exploits that we can construct an univariate, and thus also a bivariate, CIR process using the sums of the squared coordinates of a multivariate Ornstein-Uhlenbeck processes without drift, which is \( \beta \)-mixing with exponential decay. Thus (A2) holds true. In this situation, the invariant density \( f_V \) is given by a Gamma distribution

\[
f_V(x) = \frac{x^{\rho_1-1}}{\Gamma(\rho_1)\Gamma(\rho_2)} \exp(-x_1 - x_2) 1_{\mathbb{R}^2_+}(x), \quad x \in \mathbb{R}^2.
\]

Exploiting Proposition 5.5, for \( i \in [2] \) we have \( \mathbb{E}(\sup_{t \in [0,\Delta]} V_{t,i}^{-2}) \leq 2\mathbb{E}(V_{0,i}^{-2}) < \infty \) for \( \rho_1, \rho_2 \geq 3 \), which together with Proposition 1.1 implies (A3). The Mellin transform of \( f_V \) is given by

\[
\mathcal{M}_1[f_V](t) = \frac{\Gamma(\rho_1 + it_1)\Gamma(\rho_2 + it_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}, \quad t \in \mathbb{R}^2.
\]

Thus applying the Stirling inequality for Gamma functions, compare [1] Corollary 1.4.4., \( \|f_V - f_{V,k}\|_{\mathbb{R}^2}^2 \leq L(\rho_1, \rho_2)(k_1^{2(\rho_1+1)}e^{-k_1^2} + k_2^{2(\rho_2+1)}e^{-k_2^2}) \) for a constant \( L(\rho_1, \rho_2) \) dependent on \( \rho_1, \rho_2 \). Then, we derive

\[
\mathbb{E}^n_{f_V} (\|\hat{f}_{\Delta, k} - f_V\|_{\mathbb{R}^2}^2) \leq C(V, g) \frac{\log^2(\rho_1\vee\rho_2+1)(n)}{n}.
\]

**Exponential of a bivariate Cox-Ingersoll-Ross process** We consider \( V_t := (\exp(Z_{t,1}), \exp(Z_{t,2}))^T \) where \( (Z_t)_{t \geq 0} \) is an bivariate Cox-Ingersoll Ross process with \( \rho \in \mathbb{N}^2 \). Instantly, the properties (A1) and (A2) are given. The invariant density \( f_V \)
is here given by the density of a Log-Gamma distribution that is,
\[
f_V(x) = \frac{\log^{\rho_1-1}(x_1) \log^{\rho_2-1}(x_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} x^{-2} 1_{(1,\infty) \times}(x), \quad x \in \mathbb{R}^2_+.
\]

For (A_3) we again use Proposition 1.1. The corresponding Mellin transform is then given by
\[
\mathcal{M}_1[f_V](t) = (1-it)^{-\rho}, \quad t \in \mathbb{R}^2,
\]
with \(\|f_V - f_{V,k}\|_{x^2}^2 \leq L(\rho_1, \rho_2)(k_1^{-2\rho_1+1} + k_2^{-2\rho_2+1})\) for \(k \in \mathbb{R}_+^2\). Further, we have
\[
\mathbb{E}^n_{f_V}(\|\hat{f}_{\Delta, k} - f_V\|_{x^2}^2) \leq C(V, g) \log(n)^{-2(\rho_1 \wedge \rho_2)+1}.
\]

**Comment** Based on Theorem 2.3, it was clear that the rate of the fully data-driven estimator cannot achieve a rate better than \(\log^3(n)/\sqrt{n}\). In the case of the exponential of an Ornstein-Uhlenbeck process, we have seen that the fast decay of \(\mathcal{M}_1[f_V]\) implies that the estimator achieves this rate, while for the CIR processes, a slight disgression of the rate is observed.

In the case of exponential of a CIR process, the rate is of logarithmic decay which is typical for super smooth errors and densities with polynomial decaying Mellin transform.

### 3.2. Numerical simulation

We illustrate the performance of the estimator \(\hat{f}_{\Delta, k}\), defined in (2.2) and (2.3), using a Monte-Carlo simulation. To do so, we sample for \(\Delta = 0.01\) fixed and varying sample sizes \(n \in \mathbb{N}\) from an exponential Ornstein-Uhlenbeck process, \((V_t)_{t \in [0,\Delta n]}\), and calculate the scaled integrated volatilities \((V_j)_{j \in [n]}\). Here, the sampling from the process and the calculation of the upcoming integral are solved by numerical discretisation.

In Figure 1, we compare the estimator \(\hat{f}_{\Delta, k}\) in the volatility model with the estimator \(\hat{f}_k\) of [5], based on the direct observation \((V_j)_{j \in [n]}\), that is without noise.
Figure 1.: Estimators \( \hat{f}_k \) (left) and \( \hat{f}_{k,\Delta} \) (right) depicted for 50 Monte-Carlo simulations with \( n = 5000 \) based on \((\mathbf{V}_j)_{j \in \mathbb{N}}\), respectively based on \((\mathbf{Y}_j)_{j \in \mathbb{N}}\). Top plots: the true density (left) and the pointwise median of the estimators (right). Bottom plots: sections for \( x = 0.54 \) (right) and \( y = 0.54 \) (left) with true density \( f \) (black curve) and pointwise empirical median (red curve) of the 50 estimates.

In Figure 1, one sees the impact of the noise on the performance of the estimator. Focusing on the pointwise median, the remaining bias is clearly observable which is consistent with the theory. Figure 2 illustrates the improvement of the behavior of the estimator for increasing sample size.

Figure 2.: Estimator \( \hat{f}_{k,\Delta} \) depicted for 50 Monte-Carlo simulations with \( n = 5000 \) (left) and \( n = 20000 \) (right) based on \((\mathbf{Y}_j)_{j \in \mathbb{N}}\). Top plots: true density (left) and the pointwise median of the estimators (right). Bottom plots: sections for \( x = 0.54 \) (right) and \( y = 0.54 \) (left) with true density \( f \) (black curve) and pointwise empirical median (red curve) of the 50 estimates.

Comment. The simulation study implies the reasonable behavior of the estimator. For increasing sample size the error of the estimator is decaying. Furthermore, it seems that the underlying dependence has a negligible effect on the rate compared to the super smooth error densities. This observation is consistent with the theoretical results of Theorem 2.1 and 2.3.
4. Multiplicative measurement error model for stationary processes

In the following section we consider the estimation of the density \( f : \mathbb{R}^2_+ \to \mathbb{R}_+ \) of a positive, bivariate random vector \( X = (X_1, X_2)^T \) based on a strictly stationary sample of \( X \) under multiplicative measurement errors, that is, we consider the observations

\[
Y_i := \begin{pmatrix} Y_{i,1} \\ Y_{i,2} \end{pmatrix} = \begin{pmatrix} X_{i,1}U_{i,1} \\ X_{i,2}U_{i,2} \end{pmatrix} = X_iU_i, \quad i \in [n],
\]

where \((X_i)_{i \in [n]}\) are sampled from a strictly stationary process with stationary density given by \( f \) and \((U_i)_{i \in [n]}\) is an i.i.d. sequence drawn from the error density \( g : \mathbb{R}^2_+ \to \mathbb{R}_+ \). To do so, we will borrow ideas from [5] and [8]. In comparison to [5] and [8], where smooth error densities has been considered, our main focus will lie on super smooth error densities. Before building our estimator, let us briefly summarise main properties of the Mellin transform presented in [5].

**The Mellin transform** Let \( c \in \mathbb{R}^2_+ \). For two functions \( h_1, h_2 \in \mathbb{L}^1(\mathbb{R}^2_+, x^{c-1}) \) we define the multiplicative convolution \( h_1 \ast h_2 \) of \( h_1 \) and \( h_2 \) by

\[
(h_1 \ast h_2)(y) = \int_{\mathbb{R}^2_+} h_1(y/x)h_2(x)x^{-1}dx, \quad y \in \mathbb{R}^2. \quad (4.1)
\]

It can be shown, \( h_1 \ast h_2 \) is well-defined, \( h_1 \ast h_2 = h_2 \ast h_1 \) and \( h_1 \ast h_2 \in \mathbb{L}^1(\mathbb{R}^2_+, x^{c-1}) \). A proof sketch of this property and the following results can be found in [5]. If additionally \( h_1 \in \mathbb{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) then \( h_1 \ast h_2 \in \mathbb{L}^2(\mathbb{R}^2_+, x^{2c-1}) \). One key property of the Mellin transform, which makes it so appealing for the use of multiplicative deconvolution, is the so-called convolution theorem, that is, for \( h_1, h_2 \in \mathbb{L}^1(\mathbb{R}^2_+, x^{c-1}) \),

\[
\mathcal{M}_c[h_1 \ast h_2](t) = \mathcal{M}_c[h_1](t)\mathcal{M}_c[h_2](t), \quad t \in \mathbb{R}^2. \quad (4.2)
\]

By construction, the operator \( \mathcal{M}_c : \mathbb{L}^2(\mathbb{R}^2_+, x^{2c-1}) \to \mathbb{L}^2(\mathbb{R}^2) \) is an isomorphism. Denoting by \( \mathcal{M}_c^{-1} : \mathbb{L}^2(\mathbb{R}^2) \to \mathbb{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) its inverse, we can state that if additional to \( H \in \mathbb{L}^2(\mathbb{R}^2) \), \( H \in \mathbb{L}^1(\mathbb{R}^2) \) holds true, then \( \mathcal{M}_c^{-1}[H] \) can be expressed explicitly by

\[
\mathcal{M}_c^{-1}[H](x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} x^{-c-it}H(t)dt, \quad \text{for any } x \in \mathbb{R}^2_+. \quad (4.3)
\]

Furthermore, we can directly show that a Plancherel-type equation, respectively Parseval-type equation, is valid for the Mellin transform. For all \( h_1, h_2 \in \mathbb{L}^2(\mathbb{R}^2_+, x^{2c-1}) \) holds

\[
\langle h_1, h_2 \rangle_{x^{2c-1}} = \frac{1}{4\pi^2} \langle \mathcal{M}_c[h_1], \mathcal{M}_c[h_2] \rangle_{\mathbb{R}^2}, \quad (4.4)
\]

and thus \( \|h_1\|_{x^{2c-1}}^2 = (4\pi)^{-1}\|\mathcal{M}_c[h_1]\|_{\mathbb{R}^2}^2 \).
Estimation strategy Let $c \in \mathbb{R}^2$ and $\mathbb{E}_{f_Y}(Y_i^{c-1}) < \infty$ and $f \in L^2(\mathbb{R}_+, x^{2c-1})$. Then, we define for the spectral cut-off estimator $\hat{f}_k$, studied in [5],

$$
\hat{f}_k(x) = \frac{1}{4\pi^2} \int_{[-k,k]} x^{c-it} \frac{\widehat{M}_c(t)}{M_c[g](t)} dt, \quad \text{with } \widehat{M}_c(t) := \frac{1}{n} \sum_{j=1}^{n} Y_j^{c-1+it}, \quad (4.5)
$$

for $x \in \mathbb{R}_+^2$. To ensure that the estimator is well-defined, we assume that $g$ fulfills

$$
\forall t \in \mathbb{R}^2: M_c[g](t) \neq 0 \text{ and } \forall k \in \mathbb{R}_+^2: \int_{[-k,k]} |M_c[g](t)|^{-2} dt < \infty. \quad ([G0])
$$

Remark 4.1. Assumption [G0] is not unusual in context of deconvolution problems, compare [21]. Examples of multivariate density, which fulfills the assumption [G0] are presented in [5].

The following proposition is a generalisation of the results in [5] for strictly stationary data. Its proof is postponed to Appendix 5.4.

(i) Technische Einführung der Mellin transformierter für Multivariate

Proposition 4.2 (Upper bound of the risk). Let $f \in L^2(\mathbb{R}_+, x^{2c-1})$, $\mu_Y := \mathbb{E}_{f_Y}(Y_1^{2c-2}) < \infty$ and $g$ fulfill [G0]. Then, for any $k \in \mathbb{R}_+$,

$$
\mathbb{E}_{f_Y}^n(\|f - \hat{f}_k\|_{x^{2c-1}}^2) \leq \|f - f_k\|_{x^{2c-1}}^2 + \frac{\mu_Y \Lambda_g(k)}{n} + \frac{1}{4\pi^2} \int_{[-k,k]} \text{Var}_{fY} (\widehat{M}_X(t)) dt \quad (4.6)
$$

where $\Lambda_g(k) := (4\pi^2)^{-1} \int_{[-k,k]} |M_c[g](t)|^{-2} dt$ and $\widehat{M}_X(t) := n^{-1} \sum_{j=1}^{n} X_j^{c-1+it}$. 

Let us comment on the bound in Proposition 4.2. The upper bound of the risk, for $k \in \mathbb{R}_+$, consists of the usual squared bias term $\|f - f_k\|_{x^{2c-1}}^2$ and a decomposition of the variance term $\mathbb{E}_{f_Y}^n(\|f_k - f_k\|_{x^{2c-1}}^2)$. We decomposed it into an inverse problem term $\mu_Y \Delta_g(k)n^{-1}$, which also appears in [5] and an dependency term $(4\pi^2)^{-1} \int_{[-k,k]} \text{Var}_{fY} (\widehat{M}_X(t)) dt$, which is consistent with the result of [8]. Nevertheless, it is clear to seen that the squared bias term is decreasing for $k \in \mathbb{R}_+$ componentwise increasing, while the variance term is increasing. A choice of $k \in \mathbb{R}_+$ is without further information of the unknown density non-trivial. Therefore we propose in the next paragraph a fully data-driven choice of $k \in \mathbb{R}_+$, that is a choice, which is only dependent on the sample $(Y_j)_{j=1}^{n}$ without further knowledge about the density $f$.

Data-driven choice of $k \in \mathbb{R}_+$. We restrict ourselves to the case $P^{U_1} = \Gamma^{(1/2,1/2)}$, motivated by the stochastic volatility model, for a simple display of results. First we reduce the space of possible cut-off parameters to $\mathcal{K}_n := \{k \in [\log(n)] \times [\log(n)] : \Lambda_g(k) \leq n\}$. Then we define the model selection method for $\chi > 0$ and
\[\hat{\mu}_Y := n^{-1} \sum_{j \in [n]} Y_j^{2(c-1)} \text{ by}\]

\[\hat{\kappa} := \arg \min_{k \in K_n} -\|\hat{f}_k\|_{2c-1}^2 + \pen(k), \quad \pen(k) := \chi \hat{\mu}_Y k_1 \Lambda_g(k) \frac{\Lambda_g(k)}{n}. \quad (4.7)\]

Compared to the penalty in [7], we see that the term \(\pen(k) := \mathbb{E}(\pen(k)) = \chi \hat{\mu}_Y k_1 \Lambda_g(k) n^{-1}\) is not of the same order of the variance term. This overestimation of the variance term for supersmooth error densities is commonly found in the deconvolution literature.

**Theorem 4.3** (Data-driven choice of \(k\)). Let \(f \in L^2(\mathbb{R}_+^2, x^{2c-1})\) and \(\mathbb{E}(Y^{4(c-1)}) < \infty\). Then there exists \(\chi_0 \in \mathbb{R}_+\) such that for all \(\chi \geq \chi_0\),

\[
\mathbb{E}(\|\hat{f}_k - f\|_{2c-1}^2) \leq 3 \inf_{k \in K_n} (\|f_{k_1} - f\|_{2c-1}^2 + \pen(k)) + C(g) \frac{\mu_X}{n}
+ C(g, \chi) \frac{\mathbb{E}(X^{4(c-1)} \log(n))^2}{\mu_X n} + \int_{[-K_n, K_n]} \text{Var}_f(\mathcal{M}_X(t)) dt
+ C(g, \chi) \frac{\text{Var}(\hat{\mu}_X) \log(n)^2}{\mu_X}
\]

where \(C(g)\), respectively \(C(g, \chi)\) are positive constant only depending on \(g\), respectively \(g\) and \(\chi\) and \(K_n := ([\log(n)], [\log(n)])^T\). Furthermore, we directly deduce that

\[
\mathbb{E}(\|\hat{f}_k - f\|_{2c-1}^2) \leq 11 \inf_{k \in K_n} (\|f - f_k\|_{2c-1}^2 + \pen(k)) + C(g) \frac{\mu_X}{n}
+ C(g, \chi) \frac{\mathbb{E}(X^{4(c-1)} \log(n))^2}{\mu_X n} + \int_{[-K_n, K_n]} \text{Var}_f(\mathcal{M}_X(t)) dt
+ C(g, \chi) \frac{\text{Var}(\hat{\mu}_X) \log(n)^2}{\mu_X}
\]

**Remark 4.4.** It is worth stressing out that the first inequality of Theorem 4.3 still holds true even without the assumption \(f \in L^2(\mathbb{R}_+^2, x^{2c-1})\). Indeed, by the abuse of notation \(\mathcal{M}_c(f)(t) := \mathbb{E}(X^{c-1+it})\), \(t \in \mathbb{R}_+^2\), we can ensure that the functions \(f_k, k \in K_n\), are still well-defined with \(\mathbb{E}_f(\hat{f}_k) = f_k\). In other words, it is sufficient to use the \(L^1\) notion of the Mellin transform since we only consider the distance between our estimator \(\hat{f}_k\) and \(f_{K_n}\) and are therefore not in need of the assumption \(f \in L^2(\mathbb{R}_+, x^{2c-1})\).

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5. Appendix

5.1. Useful inequalities

The following inequality is due to [24], the formulation of the first part can be found for example in [20].

Lemma 5.1. (Talagrand’s inequality) Let $Z_1, \ldots, Z_n$ be independent $\mathbb{Z}$-valued random variables and let $\bar{\nu}_h = n^{-1} \sum_{i=1}^n [\nu_h(Z_i) - \mathbb{E}(\nu_h(Z_i))]$ for $\nu_h$ belonging to a countable class $\{\nu_h, h \in H\}$ of measurable functions. Then, for all $\varepsilon > 0$

$$
\mathbb{E} \left( \sup_{h \in H} |\bar{\nu}_h|^2 - 2(1 + 2\varepsilon)\Psi^2 \right)_{+} \leq C \left[ \frac{\tau}{n} \exp \left( - \frac{K_1 n \Psi^2}{\varepsilon} \right) + \frac{\psi^2}{C^2 n^2} \exp \left( - \frac{C_2 K_2 \sqrt{\varepsilon} n \Psi}{\psi} \right) \right]
$$

(5.1)

with numerical constants $C_\varepsilon := \sqrt{1 + \varepsilon} - 1$ and $C > 0$ and where

$$
\sup_{h \in H} \sup_{z \in \mathbb{Z}} |\nu_h(z)| \leq \psi, \quad \mathbb{E} \left( \sup_{h \in H} |\bar{\nu}_h| \right) \leq \Psi, \quad \sup_{h \in H} \frac{1}{n} \sum_{j=1}^n \mathbb{V} \text{ar}(\nu_h(Z_i)) \leq \tau.
$$

The key statement regarding $\beta$-mixing processes is delivered by the proposed variance bound derived by [2] after Lemma 4.1 of the same work. Their approach is based on the original idea of [27, Theorem 2.1].

Lemma 5.2. Let $(Z_j)_{j \in \mathbb{Z}}$ be a strictly stationary process of real-valued random variables with common marginal distribution $\mathbb{P}$. There exists a sequence $(b_k)_{k \in \mathbb{N}}$ of measurable functions $b_k : \mathbb{R} \to [0, 1]$ with $\mathbb{E}[b_k(Z_0)] = \beta(Z_0, Z_k)$ such that for any measurable function $h$ with $\mathbb{E}[|h(Z_0)|^2] < \infty$ and $b = \sum_{k=1}^\infty (k + 1)^{p-2} b_k : \mathbb{R} \to [0, \infty]$, $p \geq 2$,

$$
\mathbb{V} \text{ar} \left( \sum_{j=1}^n h(Z_j) \right) \leq 4n \mathbb{E}[|h(Z_0)|^2 b(Z_0)]
$$

where we set $b_0 \equiv 1$.

5.2. Proofs of Section 1

Proof of Proposition 1.1. Let us begin with (i). For $i \in [2]$ we have

$$
\log(X_{1,i}) - \log(V_{0,i}) = \log \left( \Delta^{-1} \int_0^{t_i} e^{Z_{t,i} - Z_{0,i}} \, dt \right) \leq \sup_{t \in [0, \Delta]} Z_{t,i} - Z_{0,i}
$$

$$
\log(V_{0,i}) - \log(X_{1,i}) = -\log \left( \Delta^{-1} \int_0^{t_i} e^{Z_{t,i} - Z_{0,i}} \, dt \right) \leq -\inf_{t \in [0, \Delta]} Z_{t,i} - Z_{0,i}
$$

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implying that \( \mathbb{E}(\log|X_{1,i}|) \leq \mathbb{E}(\sup_{t \in [0,\Delta]} |Z_{t,i} - Z_{0,i}|) \). Next since \( Z_{t,i} - Z_{0,i} = \int_0^t b_i(Z_t)dt + \sum_{j=1}^2 \int_0^t a_{j,i}(Z_t)dW_t^j \) we get

\[
\mathbb{E}(\sup_{t \in [0,\Delta]} |Z_{t,i} - Z_{0,i}|) \leq \Delta \mathbb{E}(|b_i(Z_0)|) + \sum_{j=1}^2 \mathbb{E} \left( \int_0^t |a_{j,i}(Z_t)|dW_t^j \right)
\]

\[
\leq c(1 + \mathbb{E}(|Z_0|_2^2))\Delta^{1/2}
\]

using the Jensen inequality and the Burkholder-Davis-Gundy inequality.

For (ii) we first see that for \( x, y \in \mathbb{R}_+ \) with \( x < y \) holds \( |\log(y) - \log(x)| = |\log(y/x) = \log(1 + (y - x)/x) | \leq (y - x)/x = |y - x|/x \) which implies that \( |\log(y) - \log(x)| \leq |y - x|/(x \wedge y) \). Here, \( a \wedge b := \min(a, b) \) for \( a, b \in \mathbb{R} \). We deduce that

\[
\mathbb{E}(|\log|X_{1,i} - \log(V_{0,i})|^2) \leq \mathbb{E}(|X_{1,i} - V_{0,i}|^2)\mathbb{E}((V_{0,i} \wedge X_{1,i})^{-2})
\]

\[
\leq \mathbb{E}(\sup_{t \in [0,\Delta]} |V_{t,i} - V_{0,i}|^2)\mathbb{E}((V_{0,i} \wedge X_{1,i})^{-2})
\]

Since \( X_{1,i} \geq \inf_{t \in [0,\Delta]} V_{t,i} \) we get \( \mathbb{E}((V_{0,i} \wedge X_{1,i})^{-2}) \leq \mathbb{E}(\sup_{t \in [0,\Delta]} (V_{t,i})^{-2}) \). Analogously to (i) we can show that \( \mathbb{E}(\sup_{t \in [0,\Delta]} |V_{t,i} - V_{0,i}|^2) \leq c\Delta(1 + \mathbb{E}(|V_0|_2^2)) \). \( \square \)

5.3. Proof of Section 2.

**Proof of Theorem 2.1.** By a disjoint support argument and the Plancherel equality we have \( \langle f_V - f_{V,k}, f_{V,k} - \hat{f}_{\Delta,k} \rangle_{\mathbb{Z}^2} = 0 \) implying

\[
\|f_V - \hat{f}_{\Delta,k}\|_{\mathbb{Z}^2}^2 = \|f_V - f_{V,k}\|_{\mathbb{Z}^2}^2 + \|f_{V,k} - \hat{f}_{\Delta,k}\|_{\mathbb{Z}^2}^2.
\]

Let us define \( f_{\Delta,k} := \mathbb{E}_{f_V}(\hat{f}_{\Delta,k}) \). Then, \( \mathbb{E}_{f_V}(\langle f_{V,k} - f_{\Delta,k}, f_{\Delta,k} - \hat{f}_{\Delta,k} \rangle_{\mathbb{Z}^2}) = 0 \) and

\[
\mathbb{E}_{f_V}(\|f_V - \hat{f}_{\Delta,k}\|_{\mathbb{Z}^2}^2) = \|f_V - f_{V,k}\|_{\mathbb{Z}^2}^2 + \|f_{V,k} - f_{\Delta,k}\|_{\mathbb{Z}^2}^2 + \mathbb{E}_{f_V}(\|f_{\Delta,k} - \hat{f}_{\Delta,k}\|_{\mathbb{Z}^2}^2).
\]

Following the steps of the proof of Proposition 4.2 we get

\[
\mathbb{E}(\|f_{\Delta,k} - \hat{f}_{\Delta,k}\|_{\mathbb{Z}^2}^2) \leq \frac{\Lambda_g(k)}{n} + \frac{1}{(2\pi)^2} \int_{[-k,k]} \text{Var}(\hat{\mathcal{M}}_X(t))dt
\]

where \( \hat{\mathcal{M}}_X(t) := n^{-1} \sum_{j=1}^n X_j^t \). Now since \( (X_j)_{j \in [n]} \) is a measurable transformation of \( (V_i)_{i \geq 0} \) which fulfills \( (A_1) - (A_2) \) we have \( \beta_{\mathcal{M}}(j) \leq \beta_{V}(\Delta(j - 1)) \) and, with Lemma 5.2 that

\[
\text{Var}(\hat{\mathcal{M}}_X(t)) \leq 4 \sum_{k=1}^{\infty} \beta_{V}(\Delta(k - 1)) \leq 4 \frac{1}{n\Delta} \int_{\mathbb{R}_+} \beta_{V}(s)ds.
\]
Considering the term $\|f_{V,k} - f_{\Delta,k}\|_{x_1}^2$, we get, since $|x^{ij} - y^{ij}| \leq t_1|\log(x_1) - \log(y_1)| + t_2|\log(x_2) - \log(y_2)|$ that

$$\|f_{V,k} - f_{\Delta,k}\|_{x_1}^2 = \frac{1}{(2\pi)^2} \int_{[-k,k]} |\mathbb{E}((V_0)^{ij} - X^{ij}_1)|^2 dt \leq \int_{[-k,k]} t_1^2 \mathbb{E}(|\log(V_{0,1}) - \log(X_{1,1})|^2 + t_2^2 \mathbb{E}(|\log(V_{0,2}) - \log(X_{1,2})|^2 dt \leq k^3 \mathbb{E}(|\log(V_{0,1}) - \log(X_{1,1})|) + |\log(V_{0,2}) - \log(X_{1,2})|^2 \leq c\Delta k^2$$

exploiting $(A_3)$.

**Proof of Theorem 2.3.** Since $\mathbb{E}^n_{f_{V}}(\|f_{V} - \hat{f}_{\Delta,k}\|_{x_1}^2) = \|f_{V} - f_{V,K_n}\|_{x_1}^2 + \mathbb{E}^n_{f_{V}}(\|f_{V,K_n} - \hat{f}_{\Delta,k}\|_{x_1}^2)$ it follows

$$\mathbb{E}^n_{f_{V}}(\|f_{V} - \hat{f}_{\Delta,k}\|_{x_1}^2) \leq \|f_{V} - f_{V,K_n}\|_{x_1}^2 + 2\|f_{V,K_n} - f_{\Delta,K_n}\|_{x_1}^2 + 2\mathbb{E}^n_{f_{V}}(\|f_{\Delta,K_n} - \hat{f}_{\Delta,k}\|_{x_1}^2).$$

For the last summand we can apply the result of Theorem 4.3 with $f_{K_n} := f_{\Delta,K_n}$ and $\hat{f}_k := \hat{f}_{\Delta,k}$ to get

$$\mathbb{E}^n_{f_{V}}(\|f_{\Delta,K_n} - \hat{f}_{\Delta,k}\|_{x_1}^2) \leq \inf_{k \in K_n} \mathbb{E}^n_{f_{V}}(\|f_{\Delta,K_n} - f_{\Delta,k}\|_{x_1}^2 + \text{pen}(k)) + \frac{C(g)}{n} + \int_{[-K_n,K_n]} \text{Var}^n_{f_{V}}(\hat{\mathcal{M}}_X(t)) dt.$$

Here, several summands in the bound of Theorem 4.3 can be omitted since the penalty term $\text{pen}(k)$ has not to be estimated in the case of $c = (1,1)^T$. Now since

$$\|f_{\Delta,K_n} - f_{\Delta,k}\|_{x_1}^2 \leq 3(\|f_{\Delta,K_n} - f_{V,K_n}\|_{x_1}^2 + \|f_{V,K_n} - f_{\Delta,k}\|_{x_1}^2 + \|f_{V,k} - f_{\Delta,k}\|_{x_1}^2) \leq 6\|f_{\Delta,K_n} - f_{V,K_n}\|_{x_1}^2 + 3\|f_{V,K_n} - f_{V,k}\|_{x_1}^2$$

we get

$$\mathbb{E}(\|f_{V} - \hat{f}_{\Delta,k}\|_{x_1}^2) \leq C \left( \inf_{k \in K_n} \mathbb{E}(\|f_{V} - f_{V,k}\|_{x_1}^2 + \text{pen}(k)) + \|f_{\Delta,K_n} - f_{V,K_n}\|_{x_1}^2 \right) + \frac{C(g)}{n} + 8 \int_{[-K_n,K_n]} \text{Var}^n_{f_{V}}(\hat{\mathcal{M}}_X(t)) dt \leq C \inf_{k \in K_n} \mathbb{E}(\|f_{V} - f_{V,k}\|_{x_1}^2 + \text{pen}(k)) + C \Delta \log(n)^6 + \frac{C(g)}{n} + \frac{C(\beta_V \log^2(n))}{n\Delta}$$

following the proof steps of Theorem 2.1 and using that $K_n^1 = K_{n,1}K_{n,2} \leq \log^2(n)$. \qed
5.4. Proof of Section 4

**Proof of Proposition 4.2.** The Plancherel equation (4.4) implies \( \langle f_k, f - f_k \rangle_{e^{2n-1}} = \langle M_c[f_k], M_c[f - f_k] \rangle_{e^2} = 0 \) since \( M_c[f_k] \) and \( M_c[f - f_k] \) have disjoint support. Thus

\[
\mathbb{E}_{f_Y}^n(\|\hat{f}_k - f\|_{e^{2n-1}}^2) = \|f - f_k\|_{e^{2n-1}}^2 + \mathbb{E}_{f_Y}^n(\|\hat{f}_k - f\|_{e^{2n-1}}^2) = \|f - f_k\|_{e^{2n-1}}^2 + \frac{1}{(2\pi)^2} \int_{[-k,k]} \text{Var}_{f_Y}^n(M(t)) \frac{|M_c[g](t)|^2}{|M_c[g](t)|^2} dt
\]

by application of the Parseval equality (4.4) and the Fubini-Tonelli theorem. Since for any \( t \in \mathbb{R}^2 \)

\[
\text{Var}_{f_Y}^n(M(t)) = \text{Var}_{f_Y}^n(M_c(t) - M_c[g](t)) + \text{Var}_{f_Y}^n(M_c(t)) + |M_c[g](t)|^2 \text{Var}_{f_Y}^n(M(t)),
\]

we decompose the variance term \( \mathbb{E}_{f_Y}^n(\|\hat{f}_k - f\|_{e^{2n-1}}^2) \) into

\[
\int_{[-k,k]} \text{Var}_{f_Y}^n(M(t)) \frac{|M_c[g](t)|^2}{|M_c[g](t)|^2} dt = \int_{[-k,k]} \text{Var}_{f_Y}^n(M_c(t) - M_c[g](t)) \frac{|M_c[g](t)|^2}{|M_c[g](t)|^2} dt + \int_{[-k,k]} \text{Var}_{f_Y}^n(M(t)) dt.
\]

Now, the equality \( \mathbb{E}_{f_Y}^n(X_j^{c-1+it}X_j^{c-1-it}(U_j^{c-1+it} - M_c[g](t))(U_j^{c-1-it} - M_c[g](t)) = \delta_{j,j'}\mathbb{E}_f(X_j^{2(c-1)}|\text{Var}_g(U_j^{c-1+it}) \text{ Var}_g(U_j^{c-1-it})) \) implies for the first summand

\[
\int_{[-k,k]} \text{Var}_{f_Y}^n(M_c(t) - M_c[g](t)) \frac{|M_c[g](t)|^2}{|M_c[g](t)|^2} dt = \int_{[-k,k]} \frac{\mathbb{E}_f(X_j^{2(c-1)}|\text{Var}_g(U_j^{c-1+it}))}{n|M_c[g](t)|^2} dt \leq \frac{(2\pi)^2 \mu_Y \Lambda_g(k)}{n}.
\]

**Proof of Theorem 4.3.** Let \( k \in K \) and let us keep in mind that \([-k',k'] = \text{supp}(M_c[f_k])\), for \( k' \in K \). Further we choose \( K_n \in (\mathbb{N}^*)^2 \) such that \( K_{1,n} = K_{2,n} := [\log(n)] \). Then for all \( k' \in K_n \) holds \([-k',k'] \subseteq [-K_n,K_n] \). Further, we have for any \( k' \in K_n \) that \( \|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 = \|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 \) implying with (4.7)

\[
\|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 + \text{pen}(k) \leq \|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 + \text{pen}(k).
\]

Now for every \( k' \in K_n \) we have

\[
\|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 = \|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 + \|\hat{f}_k - \hat{f}_k\|_{e^{2n-1}}^2 + 2\langle \hat{f}_k - \hat{f}_k, \hat{f}_k - \hat{f}_k \rangle_{e^{2n-1}}
\]

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which implies that

\[
\|\hat{f}_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 - \|f_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 \\
= \|\hat{f}_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 - \|\hat{f}_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 + 2\langle \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\rangle_{\mathcal{A}_2;\mathcal{K}} \\
\leq \text{pen}(k) - \text{pen}(\hat{k}) + 2\langle \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\rangle_{\mathcal{A}_2;\mathcal{K}}. \tag{5.2}
\]

Since \((\hat{f}_k - f_k, \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n})_{\mathcal{A}_2;\mathcal{K}} = \|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 + \|f_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}} - \|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2.\) we get with (5.2)

\[
\|\hat{f}_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 \leq \|f_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 - \|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 + 2\langle \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\rangle_{\mathcal{A}_2;\mathcal{K}} + \text{pen}(k) + 2\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 - \text{pen}(\hat{k}). \tag{5.3}
\]

Let us study the term \(|2\langle \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\rangle_{\mathcal{A}_2;\mathcal{K}}|\). We remind that \(k' \in K_n\)

\[
\|\hat{f}_{k'} - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |1_{[-k', k']}(t)| |\mathcal{M}_c[f\gamma](t) - \mathcal{M}_c(t)|^2 |d(t) dt.
\]

Setting \(A^* := [-\hat{k}, \hat{k}] \cup [-k, k]\) we have \(\mathcal{M}_c[f_k - f_k] = \mathcal{M}_c[f](1_{[-\hat{k}, \hat{k}]} - 1_{[-k, k]})\) and \(\text{supp}(\mathcal{M}_c[f_k - f_k]) \subseteq A^* \subseteq [-K_n, K_n]\). The Cauchy Schwarz inequality and the inequality \(2ab \leq a^2 + b^2\), for \(a, b \in \mathbb{R}\), implies

\[
|2\langle \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\rangle_{\mathcal{A}_2;\mathcal{K}}| = \frac{2}{(2\pi)^2} \left| \int_{A^*} \mathcal{M}_c[f_k - f_k](t) \frac{\mathcal{M}_c[-t] - \mathcal{M}_c(f\gamma)(-t)}{|\mathcal{M}_c(f\gamma)(t)|^2} dt \right| \\
\leq \frac{1}{4} \|f_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 + \frac{4}{(2\pi)^2} \int_{\mathbb{R}^2} 1_{A^*}(t) |\mathcal{M}_c[f\gamma](t) - \mathcal{M}_c(t)|^2 |d(t) dt \\
\leq \frac{\|f_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2}{2} + \frac{\|f_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2}{2} + 4\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 + 4\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2,
\]

using that \(1_{A^*} \leq 1_{A_k} + 1_{A_k}\). Thus

\[
|2\langle \hat{f}_k - f_k, \hat{f}_k - f_{K_n} - f_{K_n}\rangle_{\mathcal{A}_2;\mathcal{K}}| \leq \frac{\|f_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2}{2} + \frac{\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2}{2} + 4\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 \\
+ \frac{7}{2}\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2,
\]

which implies with (5.3)

\[
\|\hat{f}_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 \leq 3\|f_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 + 6\|f_k - f_{K_n}\|_{\mathcal{A}_2;\mathcal{K}}^2 + 2\text{pen}(k) \\
+ 11\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2 - 2\text{pen}(\hat{k}).
\]

Since \(E_{f\gamma}(\text{pen}(k)) = \text{pen}(k)\) and as \(\chi_0 \geq 6\) we get now, \(6E_{f\gamma}(\|\hat{f}_k - f_k\|_{\mathcal{A}_2;\mathcal{K}}^2) \leq \)
\[ \frac{6\nu \Lambda_n(k)}{n} + \int_{[-k, k]} \Var_f^n(\hat{M}_X(t))\,dt \leq \text{pen}(k) + \int_{[-k, k]} \Var_f^n(\hat{M}_X(t))\,dt \] and

\[ \mathbb{E}_f^n(\|\hat{f}_k - f_{k_n}\|_{\infty}^2) \leq 3 (\|f_{k_n} - f_k\|_{\infty}^2 + \text{pen}(k)) + \int_{[-K_n, K_n]} \Var_f^n(\hat{M}_X(t))\,dt \]

+ \int_{[-k, k]} \Var_f^n(\hat{M}_X(t))\,dt \]

+ \mathbb{E}_f^n(\{(\text{pen}(k) - 2\text{pen}(k))\}^+).

The theorem follows by applying the following two Lemmas and taking the infimum over \( k \in K_n \).

**Lemma 5.3.** Under the assumptions of Theorem 4.3 we get

\[ \mathbb{E}_f^n(\|\hat{f}_k - f_{k_n}\|_{\infty}^2 - \frac{1}{12} \text{pen}(k)) \leq C(g) \left( \frac{\mu_X}{n} + \frac{\mathbb{E}_f(X_1^{4(c-1)})}{\mu_X} \log^2(n) \right) \]

+ \frac{\Var_f^n(\hat{\mu}_X) \log(n)^2}{\mu_X} + \int_{[-K_n, K_n]} \Var_f^n(\hat{M}_X(t))\,dt .

where \( C(g) > 0 \) is a positive constant only depending on \( g \).

**Lemma 5.4.** Under the assumptions of Theorem 4.3 we get

\[ \mathbb{E}_f^n(\{(\text{pen}(k) - 2\text{pen}(k))\}^+) \leq C(\chi, \mu_X, \mathbb{E}_f(X_1^{4(c-1)})) \log(n)^2(n^{-1} + \Var_f^n(\hat{\mu}_X)) \]

where \( C(\chi, \mu_X, \mathbb{E}_f(X_1^{4(c-1)})) > 0 \) is a constant dependent on \( \chi, \mu_X, \mathbb{E}_f(X_1^{4(c-1)}) \).

**Proof of Lemma 5.3.** First we see that

\[ \mathbb{E}_f^n(\|\hat{f}_k - f_{k_n}\|_{\infty}^2 - \frac{1}{12} \text{pen}(k)) \leq \mathbb{E}_f^n(\max_{k \in K_n} (\|\hat{f}_k - f_{k_n}\|_{\infty}^2 - \frac{1}{12} \text{pen}(k))) . \]

Defining \( B_k := \{ h \in S_k : \|h\|_{\infty} = 1 \} \) and \( \bar{v}_h := (\hat{f}_k - f_{k_n})_{\infty}, h \in B_k \), we have \( \|\hat{f}_k - f_{k_n}\|_{\infty} = \sup_{h \in B_k} \bar{v}_h \). Further, we decompose \( \bar{v}_h \) into \( \bar{v}_h = \bar{v}_{h, \text{in}} + \bar{v}_{h, \text{de}} \), where

\[ \bar{v}_{h, \text{in}} := \frac{1}{n} \sum_{j \in [n]} (v_h(Y_j) - \mathbb{E}_f(\nu_h(Y_j))), \quad \bar{v}_{h, \text{de}} := \frac{1}{4\pi^2} \int_{[-k, k]} \frac{Y_j^{c-1+|t|}}{\mathcal{M}_c[g](t)} \mathcal{M}_c[h](-t)\,dt . \]

and \( \bar{v}_{h, \text{de}} = n^{-1} \sum_{j \in [n]} \mathbb{E}_f(\nu_h(Y_j)) - \mathbb{E}_f(\nu_h(Y_j)). \) Thus

\[ \mathbb{E}_f^n(\|\hat{f}_k - f_{k_n}\|_{\infty}^2 - \frac{1}{12} \text{pen}(k)) \leq 2\mathbb{E}_f^n(\max_{k \in K_n} (\bar{v}_{h, \text{in}}^2 + \frac{1}{24} \text{pen}(k))) + \]

+ \mathbb{E}_f^n(\max_{k \in K_n} \bar{v}_{h, \text{de}}^2) =: I_1 + I_2 .

\[ I_1 + I_2 \]
For the term $I_2$: $\mathbb{E}_n[\mathbf{Y}_j^{c-1+it}] - \mathbb{E}_n(\mathbf{Y}_j^{c-1+it}) = \mathcal{M}_c[g](t)(\mathbf{X}_j^{c-1+it} - \mathbb{E}_f(\mathbf{X}_j^{c-1+it}))$

implies for any $k \in \mathcal{K}_n$ and any $h \in B_k$,

$$|\varphi_{h,de}| = \frac{|(\hat{M}_X - \mathbb{E}_f(\hat{M}_X), \mathcal{M}_c[h]|_{\mathbb{R}^2})|}{4\pi^2} \leq \frac{\|\mathcal{M}_c[h]\|_{\mathbb{R}^2}}{2\pi},$$

using the Cauchy-Schwarz inequality and $\|\mathcal{M}_c[h]\|_{\mathbb{R}^2} = 2\pi\|h\|_{\mathbb{R}^2} \leq 2\pi$. Thus

$$\mathbb{E}_n|\varphi_{h,de}^2| \leq \frac{1}{(2\pi)^2} \int_{-\mathcal{K}_n, \mathcal{K}_n} \mathcal{V}(\hat{M}_X(t))dt.$$ 

Next for $I_1$, we decompose the process again to be able to apply the Talagrand inequality. To do so, let us define $\chi \mu \hat{X} := \chi \mu \hat{X} k^1 \Lambda g(k)n^{-1}$. Then,

$$\left(\sup_{h \in B_k} \varphi_{h,an}^2 - \frac{1}{24} \mathcal{P}(k)\right) = \left(\sup_{h \in B_k} \varphi_{h,an}^2 - \frac{1}{36} \mathcal{P}(k)\right) + \frac{1}{24}(\frac{\mathcal{P}(k)}{3}) - \mathcal{P}(k) + \mathcal{P}(k).$$

For the second summand, let us define $\Omega X := \{|\hat{X} - \mu X| \leq \mu X/2\}$. Then on $\Omega X$ we have $\hat{X} \leq 3\mu X/2$ and thus

$$\mathbb{E}_n\left(\max_{k \in \mathcal{K}_n} (\mathcal{P}(k) - \mathcal{P}(k))^+ \right) \leq \chi(\mathcal{K}_n)^1 \mu \mathbb{E}_n\left(\frac{2}{3} \mathcal{P}(k) - \mathcal{P}(k)\right) + 1\Omega \chi$$

$$= C(\chi, \mu U) \log^2(n) \mathbb{E}_n\left((\hat{X} - \mu X)_{\Omega \chi} \right) \leq C(\chi, \mu U, \mu X) \log^2(n) \mathcal{V}(\hat{X})$$, since $|\hat{X} - \mu X|_{\Omega \chi} \leq 2|\hat{X} - \mu X|_{\Omega \chi}$. For the first summand we see

$$\mathbb{E}_n\left(\max_{k \in \mathcal{K}_n} (\sup_{h \in B_k} \varphi_{h,an}^2 - \frac{1}{36} \mathcal{P}(k))^+\right) = \mathbb{E}_n\left(\mathbb{E}_X\left(\max_{k \in \mathcal{K}_n} (\sup_{h \in B_k} \varphi_{h,an}^2 - \frac{1}{36} \mathcal{P}(k))^+\right)\right).$$

Thus we start by considering the inner conditional expectation to bound the term. By the construction of $\varphi_{h,an}$, its summands conditioned on $\sigma(X_i, i \geq 0)$ are independent but not identically distributed. We are aiming to apply the Talagrand inequality, Lemma[5.1]. We therefore split, for a sequence $(c_n)_{n \in \mathbb{N}}$ specified afterwards, the process again in the following way

$$\varphi_{h,1} := n^{-1} \sum_{j \in [n]} \varphi_{h}(Y_j) \mathbb{1}_{(0,c_n]}(Y_j^{c-1}) - \mathbb{E}_X(\varphi_{h}(Y_j) \mathbb{1}_{(0,c_n]}(Y_j^{c-1}))$$

and

$$\varphi_{h,2} := n^{-1} \sum_{j \in [n]} \varphi_{h}(Y_j) \mathbb{1}_{(c_n,\infty)}(Y_j^{c-1}) - \mathbb{E}_X(\varphi_{h}(Y_j) \mathbb{1}_{(c_n,\infty)}(Y_j^{c-1}))$$

to get

$$\mathbb{E}_X\left(\max_{k \in \mathcal{K}_n} \mathbb{E}_{\varphi_{h,an}^2 - \frac{1}{36} \mathcal{P}(k))^+\right) \leq 2\mathbb{E}_X\left(\max_{k \in \mathcal{K}_n} \mathbb{E}_{\varphi_{h,an}^2 - \frac{1}{36} \mathcal{P}(k))^+\right) + 2\mathbb{E}_X\left(\max_{k \in \mathcal{K}_n} \mathbb{E}_{\varphi_{h,an}^2 - \frac{1}{36} \mathcal{P}(k))^+\right)^2 =: M_1 + M_2$$

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where we will now consider the two summands $M_1, M_2$ separately. To bound the $M_1$ term we will use the Talagrand inequality \[\mathbb{E}_X(\sup_{t \in B_k} |\tilde{\nu}_{h,1}|^2 - \frac{1}{n^2} \tilde{\mu}_{\nu}(k))_+\], which will be used to show the claim. We want to emphasize that we are able to apply the Talagrand inequality on the sets $B_k$ since $B_k$ has a dense countable subset and due to continuity arguments. Further, we see that the random variables $\nu_h(Y_j)^1_{(0,c_n)}(Y_j^{c-1}) - \mathbb{E}_X(\nu_h(Y_j)^1_{(0,c_n)}(Y_j^{c-1}))$, $j \in [n]$, are conditioned on $\sigma(X_i, i \geq 0)$, centered and independent but not identically distributed. In order to apply Talagrand’s inequality, we need to find the constants $\Psi, \psi, \tau$ such that

$$\sup_{h \in B_k} \sup_{y \in \mathbb{R}^n} |\nu_h(y)|^1_{(0,c_n)}(y^{c-1}) \leq \psi; \quad \mathbb{E}_X(\sup_{h \in B_k} |\tilde{\nu}_{h,1}|) \leq \Psi; \quad \sup_{h \in B_k} \frac{1}{n} \sum_{j \in [n]} \mathbb{V}ar_X(\nu_h(Y_j)^1_{(0,c_n)}(Y_j^{c-1})) \leq \tau.$$ 

We start with $\Psi^2$. Let us define $\tilde{M}_c(t) := n^{-1} \sum_{j \in [n]} Y_j^{c-1+it}1_{(0,c_n)}(Y_j^{c-1})$ as an unbiased estimator of $\mathcal{M}_c[fY 1_{(0,c_n)}(y^{c-1})](t)$ and

$$\tilde{f}_k(x) := \frac{1}{(2\pi)^2} \int_{[-k,k]} \frac{x-e^{-it}}{\mathcal{M}_c[g](t)} dt$$

where $n^{-1} \sum_{j \in [n]} \nu_h(Y_j)^1_{(0,c_n)}(Y_j^{c-1}) = \langle \tilde{f}_k, h \rangle_{x^{2e-1}}$. Thus, we have for any $h \in B_k$ that $\tilde{\nu}_{h,1} = \langle h, \tilde{f}_k - \mathbb{E}_X(\tilde{f}_k) \rangle_{x^{2e-1}} \leq \|h\|_{x^{2e-1}}^2 \|\tilde{f}_k - \mathbb{E}_X(\tilde{f}_k)\|_{x^{2e-1}}^2$. Since $\|h\|_{x^{2e-1}} \leq 1$, we get

$$\mathbb{E}_X(\sup_{h \in B_k} \tilde{\nu}_{h,1}^2) \leq \mathbb{E}_X(\|\tilde{f}_k - \mathbb{E}_X(\tilde{f}_k)\|^2) = \frac{1}{2\pi} \int_{[-k,k]} E_X(\langle \tilde{M}(t) - \mathbb{E}_X(\tilde{M}(t)) \rangle^2) dt.$$ 

Now since $Y_j^{c-1+it}1_{(0,c_n)}(Y_j^{c-1}) - \mathbb{E}_X(Y_j^{c-1+it}1_{(0,c_n)}(Y_j^{c-1}))$ are independent conditioned on $\sigma(X_i : i \geq 0)$ we obtain

$$\mathbb{E}_X(\|\tilde{M}(t) - \mathbb{E}_X(\tilde{M}(t))\|^2) \leq \frac{1}{n^2} \sum_{j \in [n]} \mathbb{E}_X(Y_j^{2(c-1)}1_{(0,c_n)}(Y_j^{c-1})) = \frac{\mu_U}{n} \tilde{\mu}_X,$$ 

which motivates the choice $\mathbb{E}_X(\sup_{h \in B_k} \tilde{\nu}_{h,1}^2) \leq \mu_U \tilde{\mu}_X \Lambda_\nu(k) n^{-1} =: \Psi^2$.

Next we consider $\psi$. Let $y \in \mathbb{R}^n$ and $h \in B_k$. Then using the Cauchy-Schwarz inequality, $|\nu_h(y)|1_{(0,c_n)}(y^{c-1})| = (2\pi)^{-2} c_n^2 \int_{[-k,k]} y \mathcal{M}_c[h](t) dt \leq (2\pi)^{-2} c_n^2 \int_{[-k,k]} \mathcal{M}_c[g](t)^2 dt \leq c_n^2 \Lambda_\nu(k) =: \psi^2$ since $|y^t| = 1$ for all $t \in \mathbb{R}^2$.

For $\tau$ we use the crude bound $\tau = n\psi^2$. Hence, we have $\frac{n\Psi^2}{\tau} = 1$ and $\frac{n\psi}{\psi} = \frac{\sqrt{\sigma_0 \sigma_X n}}{c_n}$.
and get
\[
\mathbb{E}_{|X}(\sup_{h \in B_k} \tilde{\nu}_{h,1}^2 - 2(1 + 2\varepsilon)\mu_U \tilde{\mu}_X \frac{\Lambda_g(k)}{n}) \leq \frac{C_g}{n} \left( \frac{C_g}{n} \tilde{\mu}_X \mu_U \Lambda_g(k) \exp(-K_1\varepsilon) + \frac{\Lambda_g(k)c_n^2}{n} \exp(-K_2\sqrt{\varepsilon} \sqrt{\mu_U \tilde{\mu}_X nc_n^{-1}}) \right).
\]

Choosing now \( \varepsilon = 4\alpha_1\alpha_2 k^2 / K_1 \) we get applying assumption \([G1]\) for \( k \geq k_g \) that \( \Lambda_g(k) \exp(-K_1\varepsilon) \leq C_g k^{2\gamma} \exp(-\alpha^T k) \) which is summable over \( \mathbb{N}^2 \). Next for \( k^2 \geq k_g \) we get \( C\sqrt{\varepsilon} \geq \varepsilon / 2 \) and choosing \( c_n := \sqrt{n\mu_U \tilde{\mu}_X 2K_2 / K_1} \) leading to
\[
\mathbb{E}_{|X}(\sup_{h \in B_k} \tilde{\nu}_{h,1}^2 - 2(1 + 2\varepsilon)\mu_U \tilde{\mu}_X \frac{\Lambda_g(k)}{n}) \leq \frac{C_g}{n} \tilde{\mu}_X \mu_U \left( k^{2\gamma} e^{-\alpha^T k} + \Lambda_g(k)e^{-K_1\varepsilon} \right) \leq \frac{C_g}{n} \tilde{\mu}_X \mu_U k^{2\gamma} e^{-\alpha^T k}.
\]

Hence, there exists a \( \chi_0 > 0 \) such that for all \( \chi > \chi_0 \) holds \( \frac{1}{T^2} \text{pen}(k) \geq (2 + 4\alpha_1\alpha_2 k^2 / K_1)\mu_U \tilde{\mu}_X \Lambda_g(k)n^{-1} \) implying
\[
\sum_{k \leq K_n} \mathbb{E}_{|X}(\sup_{t \in B_k} \tilde{\nu}_{h,1}^2 - \frac{1}{T^2} \text{pen}(k)) + \frac{C_g}{n} \tilde{\mu}_X \mu_U \frac{k^{2\gamma} \exp(-\alpha^T t)}{n} \leq \frac{C(g)\tilde{\mu}_X}{n}.
\]

Now, we consider \( M_2 \). Let us define \( \mathbf{f}_k := \mathbf{f}_k - \mathbf{f}_k \). Then from \( \nu_{h,2} = \nu_{h,1} - \nu_{h,1} \) we deduce \( \nu_{h,2} = (\mathbf{f}_k - \mathbb{E}_{|X}(\mathbf{f}_k), h)^2_{x_2^{n-1}} \leq ||\mathbf{f}_k - \mathbb{E}_{|X}(\mathbf{f}_k)||_{x_2^{n-1}}^2 \) for any \( h \in B_k \). Further,
\[
\max_{k \in K_n} ||\mathbf{f}_k - \mathbb{E}_{|X}(\mathbf{f}_k)||_{x_2^{n-1}}^2 \leq \sum_{k \in K_n} ||\mathbf{f}_k - \mathbb{E}_{|X}(\mathbf{f}_k)||_{x_2^{n-1}}^2
\]
and for each summand \( k \in K_n \) we have
\[
\mathbb{E}_{|X}(||\mathbf{f}_k - \mathbb{E}_{|X}(\mathbf{f}_k)||_{x_2^{n-1}}^2) = \frac{1}{(2\pi)^2} \int_{[-k,k]} \frac{\text{Var}_{|X}(\tilde{\mu}(t) - \tilde{\mu}(t))}{M_c(t)} dt \leq \frac{1}{n^2} \sum_{j=1}^{n} \mathbb{E}_{|X}(Y_j^{2(c-1)} 1_{(c,\infty)}(Y_j)) \Lambda_g(k).
\]

Let us define the event \( \Xi_X := \{ \tilde{\mu}_X \geq \mu_X / 2 \} \). Then, we have
\[
\frac{1}{n^2} \sum_{j \in [n]} \mathbb{E}_{|X}(Y_j^{2(c-1)} 1_{(c,\infty)}(Y_j)) \Lambda_g(k) \leq \frac{C_g}{nc_n} \sum_{j \in [n]} X_j^{2(p+1)(c-1)} \Lambda_g(k) \leq \frac{C_g}{n^2} \mathbb{E}_{|X}(U_j^{4(c-1)}) \sum_{j=1}^{n} X_j^{4(c-1)}.
\]

where on \( \Xi_X \) we can state that \( c_n^{-p} = C(g)n^{-p/2} \tilde{\mu}_X^{-p/2} \leq C(g)\mu_X^{-p/2} n^{-p/2} \). Then \( p = 2 \) leads to
\[
\mathbb{E}_{|X}(||\mathbf{f}_K - \mathbb{E}_{|X}(\mathbf{f}_K)||_{x_2^{n-1}}^2) 1_{\Xi_X} \leq \frac{C(g)}{n^\delta} \mathbb{E}_{|X}(U_1^{4(c-1)}) \sum_{j=1}^{n} X_j^{4(c-1)}.
\]
On the other hand,
\[
\frac{1}{n^2} \sum_{j=1}^{n} E_X (Y_j^{2(c-1)} 1_{(c,\infty)} (Y_j^{c-1})) \Lambda_g(k) 1_{x} \leq \frac{\mu_Y}{2} 1_{x} \leq \frac{\mu_Y}{2} 1_{x}. 
\]

Using now that \(|K_n| \leq \log(n)^2\) we get
\[
M_2 \leq C(g) \frac{\log(n)^2}{\mu_X} \left( \frac{E_f(X_1^{4(c-1)})}{n} + \text{Var}_f(\hat{\mu}_X) \right). 
\]

These three bounds imply
\[
E_f^n \left( \| \hat{f}_k - f_k \|_{a_w}^2 - \frac{1}{12} \text{pen}(\hat{k}) \right) \leq C(g) \left( \frac{\mu_X}{n} + \frac{E_f(X_1^{4(c-1)}) \log(n)^2}{\mu_X n} + \frac{\text{Var}_f(\hat{\mu}_X) (\log(n)^2)}{\mu_X} \right).
\]

\begin{proof}[Proof of Lemma 5.4] Let us define \(\Omega := \{ |\hat{\mu}_Y - \mu_Y| \leq \mu_Y / 2 \}\). Then on \(\Omega\) we have \(2\hat{\mu}_Y \geq \mu_Y\), respectively
\[
E_f^n ((\text{pen}(k) - 2\text{pen}(\hat{k}))_+) = \chi E_f^n \left( \frac{k_1}{\mu_Y} \Lambda_g(\hat{k}) (\mu_Y - 2\hat{\mu}_Y)_+ \right) 
\leq 2\chi K_n^{1/2} \text{Var}_f(\hat{\mu}_Y) \leq 2\chi \log(n)^2 \frac{\text{Var}_f(\hat{\mu}_Y)}{\mu_Y}.
\]

Now in analogy to the proof of 4.2 we get
\[
\text{Var}_f^n(\hat{\mu}_Y) = \frac{E_f(X_1^{4(c-1)})}{n} + \text{Var}_f^n(\hat{\mu}_X). 
\]
\end{proof}

\begin{thebibliography}{9}
\bibitem{Andrews} George E. Andrews, Richard Askey, and Ranjan Roy. \emph{Special Functions}. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1999.
\bibitem{Asin} Nicolas Asin and Jan Johannes. Adaptive nonparametric estimation in the presence of dependence. \emph{Journal of Nonparametric Statistics}, 29(4):694–730, 2017.
\bibitem{Belomestny} Denis Belomestny and Alexander Goldenshluger. Nonparametric density estimation from observations with multiplicative measurement errors. In Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, volume 56, pages 36–67. Institut Henri Poincaré, 2020.
\bibitem{Black} Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. \emph{The Journal of Political Economy}, 81(3):637–654, 1973.
\end{thebibliography}
[5] Sergio Brenner Miguel. Anisotropic spectral cut-off estimation under multiplicative measurement errors. *Journal of Multivariate Analysis*, 190:Paper No. 104990, 18 pp., 2022.

[6] Sergio Brenner Miguel, Fabienne Comte, and Jan Johannes. Linear functional estimation under multiplicative measurement errors. *arXiv e-prints*, page arXiv:2111.14920, November 2021.

[7] Sergio Brenner Miguel, Fabienne Comte, and Jan Johannes. Spectral cut-off regularisation for density estimation under multiplicative measurement errors. *Electronic Journal of Statistics*, 15(1):3551–3573, 2021.

[8] Sergio Brenner Miguel and Nathawut Phandoidaen. Multiplicative deconvolution in survival analysis under dependency. *Statistics*, 56(2):297–328, 2022.

[9] Fabienne Comte. Kernel deconvolution of stochastic volatility models. *Journal of Time Series Analysis*, 25(4):563–582, 2004.

[10] Fabienne Comte and Valentine Genon-Catalot. Penalized projection estimator for volatility density. *Scandinavian journal of statistics*, 33(4):875–893, 2006.

[11] Fabienne Comte, Valentine Genon-Catalot, and Yves Rozenholc. Nonparametric estimation for a stochastic volatility model. *Finance and Stochastics*, 14(1):49–80, 2010.

[12] Fabienne Comte and Claire Lacour. Anisotropic adaptive kernel deconvolution. *Annales de l’I.H.P. Probabilités et statistiques*, 49(2):569–609, 2013.

[13] Jon Danielsson. Multivariate stochastic volatility models: estimation and a comparison with garch models. *Journal of Empirical Finance*, 1998.

[14] Valentine Genon-Catalot, Thierry Jeantheau, and Catherine Laredo. Limit theorems for discretely observed stochastic volatility models. *Bernoulli*, pages 283–303, 1998.

[15] Valentine Genon-Catalot, Thierry Jeantheau, and Catherine Laredo. Parameter estimation for discretely observed stochastic volatility models. *Bernoulli*, pages 855–872, 1999.

[16] Valentine Genon-Catalot, Thierry Jeantheau, and Catherine Laredo. Stochastic volatility models as hidden markov models and statistical applications. *Bernoulli*, pages 1051–1079, 2000.

[17] Valentine Genon-Catalot, Thierry Jeantheau, and Catherine Laredo. Conditional likelihood estimators for hidden markov models and stochastic volatility models. *Scandinavian journal of statistics*, 30(2):297–316, 2003.

[18] Arnaud Gloter. Discrete sampling of an integrated diffusion process and parameter estimation of the diffusion coefficient. *ESAIM: Probability and Statistics*, 4:205–227, 2000.

[19] John Hull and Alan White. The pricing of options on assets with stochastic volatilities. *The journal of finance*, 42(2):281–300, 1987.

[20] Thierry Klein and Emmanuel Rio. Concentration around the mean for maxima of empirical processes. *The Annals of Probability*, 33(3):1060–1077, 2005.

[21] Alexander Meister. Density deconvolution. In *Deconvolution Problems in Nonparametric Statistics*, pages 5–105. Springer, 2009.

[22] Eric Renault and Nizar Touzi. Option hedging and implied volatilities in a stochastic volatility model 1. *Mathematical Finance*, 6(3):279–302, 1996.

[23] Emeline Schmisser. Penalized nonparametric drift estimation for a multidimensional diffusion process. *Statistics*, 47(1):61–84, 2013.

[24] Michel Talagrand. New concentration inequalities in product spaces. *Inventiones mathematicae*, 126:505–563, 1996.

[25] Bert Van Es and Peter Spreij. Estimation of a multivariate stochastic volatility density by kernel deconvolution. *Journal of multivariate analysis*, 102(3):683–697, 2011.

[26] Bert Van Es, Peter Spreij, and Harry Van Zanten. Nonparametric volatility density estimation. *Bernoulli*, 9(3):451–465, 2003.

[27] Gabrielle Viennet. Inequalities for absolutely regular sequences: application to density estimation. *Probab. Theory Related Fields*, 107(4):467–492, 1997.