Research Article

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Resolving resolution dimensions in triangulated categories

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Abstract: Let $\mathcal{T}$ be a triangulated category with a proper class $\xi$ of triangles and $\mathcal{X}$ be a subcategory of $\mathcal{T}$. We first introduce the notion of $\mathcal{X}$-resolution dimensions for a resolving subcategory of $\mathcal{T}$ and then give some descriptions of objects having finite $\mathcal{X}$-resolution dimensions. In particular, we obtain Auslander-Buchweitz approximations for these objects. As applications, we construct adjoint pairs for two kinds of inclusion functors and characterize objects having finite $\mathcal{X}$-resolution dimensions in terms of a notion of $\xi$-cellular towers. We also construct a new resolving subcategory from a given resolving subcategory and reformulate some known results.

Keywords: triangulated categories, a proper class of triangles, resolving resolution dimensions, resolving subcategories, Auslander-Buchweitz approximations

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1 Introduction

Approximation theory is the main part of relative homological algebra and representation theory of algebras, and its starting point is to approximate arbitrary objects by a class of suitable subcategories. In particular, resolving subcategories play important roles in approximation theory (e.g., [1–3]). As an important example of resolving subcategories, Auslander and Buchweitz [4] studied the approximation theory of the subcategory consisting of maximal Cohen-Macaulay modules over an artin algebra, and Hernández et al. [5] developed an analogous theory for triangulated categories. Using the approximation triangles established by Hernández et al. [5, Theorem 5.4], Di and Wang [6] constructed additive functors (adjoint pairs) between additive quotient categories. On the other hand, Zhu [7] studied the resolution dimension with respect to a resolving subcategory in an abelian category, and Huang [8] introduced relative preresorlving subcategories in an abelian category and defined homological dimensions relative to these subcategories, which generalized many known results (see [4,9,10]).

In analogy to relative homological algebra in abelian categories, Beligiannis [11] developed a relative version of homological algebra in a triangulated category $\mathcal{T}$, that is, a pair $(\mathcal{T}, \xi)$, in which $\xi$ is a proper class of triangles (see Definition 2.4). Under this notion, a triangulated category is just equipped with a proper class consisting of all triangles. However, there are lots of non-trivial cases, for example, let $\mathcal{T}$ be a compactly generated triangulated category, then the class $\xi$ consisting of pure triangles is a proper class ([12]), and the pair $(\mathcal{T}, \xi)$ is no longer triangulated in general. Later on, this theory has been paid more attentions and developed (e.g., [13–17]). It is natural to ask how the approximation theory acts on this relative setting of triangulated categories. In [18], Ma et al., introduced the notions of (pre)resolving...
subcategories and homological dimensions relative to these subcategories in this relative setting, which
gives a parallel theory analogy to that of abelian categories [8]. In this paper, we devote to further studying
relative homological dimensions in triangulated categories with respect to a resolving subcategory. The
paper is organized as follows:

In Section 2, we give some terminology and some preliminary results.
In Section 3, some homological properties of resolving subcategories are obtained. In particular, we
obtain Auslander-Buchweitz approximation triangles (see Proposition 3.10) for objects having finite resol-
veng resolution dimensions. Our main result is the following:

**Theorem.** Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$, a $\mathcal{Ext}$-injective $\xi$-cogenerator of $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands. For any $M \in \mathcal{T}$, if $M \in \hat{\mathcal{X}}$, then the following statements are equivalent:

1. $X \text{- res.dim } M \leq m$.
2. $\mathcal{Q}^n(M) \in X$ for all $n \geq m$.
3. $\mathcal{O}^n_M(M) \in X$ for all $n \geq m$.
4. $\mathcal{Ext}_n^\mathcal{H}(M, H) = 0$ for all $n > m$ and all $H \in \mathcal{H}$.
5. $\mathcal{Ext}_n^\mathcal{H}(M, L) = 0$ for all $n > m$ and all $L \in \mathcal{H}$.
6. $M$ admits a right $X$-approximation $\varphi : X \to M$, where $\varphi$ is $\xi$-proper epic, such that $K = \text{Hoker } \varphi$ satisfying $\mathcal{H}$-res.dim $K \leq m - 1$.
7. There are two triangles

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in $\xi$ such that $X_M$ and $X^M$ are in $\mathcal{X}$ and $\mathcal{H}$-res.dim $W_M \leq m - 1$, $\mathcal{H}$-res.dim $W^M = X$-res.dim $W^M \leq m$.

In Section 4, we will further study objects having finite resolution dimensions with respect to a resolv-
ing subcategory $\mathcal{X}$. We first construct adjoint pairs for two kinds of inclusion functors. Then we charac-
terize objects having finite resolution dimensions in terms of a notion of $\xi$-cellular towers.

As an application, in Section 5, given a resolving subcategory $\mathcal{X}$ of $\mathcal{T}$, we construct a new resolving
subcategory $\mathcal{GP}_\mathcal{H}(\xi)$ with a $\mathcal{Ext}$-injective $\xi$-cogenerator $\mathcal{X} \cap \ ^\perp \mathcal{X}$, which generalizes the Gorenstein projective subcategory $\mathcal{GP}(\xi)$ given by Asadollahi and Salarian [13]. Applying the obtained results to $\mathcal{GP}_\mathcal{H}(\xi)$, we
generalize some known results in [13–15].

Throughout this paper, all subcategories are full, additive, and closed under isomorphisms.

## 2 Preliminaries

Let $\mathcal{T}$ be an additive category and $\Sigma : \mathcal{T} \to \mathcal{T}$ an additive functor. One defines the category $\text{Diag}(\mathcal{T}, \Sigma)$ as follows:

- An object of $\text{Diag}(\mathcal{T}, \Sigma)$ is a diagram in $\mathcal{T}$ of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$.
- A morphism in $\text{Diag}(\mathcal{T}, \Sigma)$ between $X_1 \xrightarrow{u_1} Y_1 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} \Sigma X_1$, $i = 1, 2$, is a triple $(a, \beta, \gamma)$ of morphisms in $\mathcal{T}$ such that the following diagram:

$$
\begin{array}{c}
X_1 \xrightarrow{u_1} Y_1 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} \Sigma X_1 \\
\downarrow \alpha \downarrow \beta \downarrow \gamma \downarrow \Sigma \alpha \\
X_2 \xrightarrow{u_2} Y_2 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} \Sigma X_2
\end{array}
$$

commutes.
A triangulated category is a triple \((\mathcal{T}, \Sigma, \Delta)\), where \(\mathcal{T}\) is an additive category and \(\Sigma: \mathcal{T} \rightarrow \mathcal{T}\) is an autoequivalence of \(\mathcal{T}\) (called suspension functor), and \(\Delta\) is a full subcategory of \(\text{Diag}(\mathcal{T}, \Sigma)\) which is closed under isomorphisms and satisfies the axioms \((T_1)-(T_4)\) in [11, Section 2.1] (also see [19]), where the axiom \((T_4)\) is called the octahedral axiom. The elements in \(\Delta\) are called triangles.

The following result is well known, which is an efficient tool in studying triangulated categories.

**Remark 2.1.** [11, Proposition 2.1] Let \(\mathcal{T}\) be an additive category and \(\Sigma: \mathcal{T} \rightarrow \mathcal{T}\) an autoequivalence of \(\mathcal{T}\), and \(\Delta\) a full subcategory of \(\text{Diag}(\mathcal{T}, \Sigma)\) which is closed under isomorphisms. Suppose that the triple \((\mathcal{T}, \Sigma, \Delta)\) satisfies all the axioms of a triangulated category except possibly of the octahedral axiom. Then, the following statements are equivalent:

1. **Octahedral axiom.** For any two morphisms \(u: X \rightarrow Y\) and \(v: Y \rightarrow Z\), there exists a commutative diagram

   \[
   \begin{array}{ccc}
   X & \overset{u}{\rightarrow} & Y & \overset{u'}{\rightarrow} & Z' & \overset{u''}{\rightarrow} & \Sigma X \\
   \downarrow & & \downarrow \alpha & & \downarrow \Sigma u & & \downarrow \\
   X & \overset{v}{\rightarrow} & Z & \overset{v'}{\rightarrow} & Y' & \overset{v''}{\rightarrow} & \Sigma X \\
   \downarrow & & \downarrow \beta & & \downarrow \Sigma v & & \downarrow \\
   Y & \overset{\Sigma u}{\rightarrow} & Z & \overset{\Sigma v}{\rightarrow} & Y' & \overset{\Sigma v''}{\rightarrow} & \Sigma Y \\
   \downarrow & & \downarrow \gamma & & \downarrow \Sigma u & & \downarrow \\
   0 & \overset{0}{\rightarrow} & \Sigma Z' & \overset{(\Sigma u')v''}{\rightarrow} & \Sigma Z'' & \overset{0}{\rightarrow} & 0,
   \end{array}
   \]

   in which all rows and the third column are triangles in \(\Delta\).

2. **Base change.** For any triangle \(X \overset{u}{\rightarrow} Y \overset{v}{\rightarrow} Z \overset{w}{\rightarrow} \Sigma X\) in \(\Delta\) and any morphism \(\alpha: Z' \rightarrow Z\), there exists the following commutative diagram:

   \[
   \begin{array}{ccc}
   0 & \overset{0}{\rightarrow} & X' & \overset{0}{\rightarrow} & X' & \overset{0}{\rightarrow} & 0 \\
   \downarrow & & \downarrow \beta' & & \downarrow \beta' & & \downarrow \\
   X & \overset{u'}{\rightarrow} & Y' & \overset{v'}{\rightarrow} & Z' & \overset{w'}{\rightarrow} & \Sigma X \\
   \downarrow & & \downarrow \alpha' & & \downarrow \alpha' & & \downarrow \\
   X & \overset{u}{\rightarrow} & Y & \overset{v}{\rightarrow} & Z & \overset{w}{\rightarrow} & \Sigma X \\
   \downarrow & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \\
   0 & \overset{0}{\rightarrow} & \Sigma X' & \overset{0}{\rightarrow} & \Sigma X' & \overset{0}{\rightarrow} & 0,
   \end{array}
   \]

   in which all rows and columns are triangles in \(\Delta\).

3. **Cobase change.** For any triangle \(X \overset{u}{\rightarrow} Y \overset{v}{\rightarrow} Z \overset{w}{\rightarrow} \Sigma X\) in \(\Delta\) and any morphism \(\beta: X \rightarrow X'\), there exists the following commutative diagram:

   \[
   \begin{array}{ccc}
   0 & \overset{0}{\rightarrow} & \Sigma^{-1}Z' & \overset{0}{\rightarrow} & \Sigma^{-1}Z' & \overset{0}{\rightarrow} & 0 \\
   \downarrow & & \downarrow -\Sigma^{-1}\gamma & & \downarrow -\Sigma^{-1}\gamma & & \downarrow \\
   \Sigma^{-1}Z & \overset{0}{\rightarrow} & X & \overset{u}{\rightarrow} & Y & \overset{v}{\rightarrow} & Z \\
   \downarrow & & \downarrow \beta & & \downarrow \beta' & & \downarrow \\
   \Sigma^{-1}Z & \overset{0}{\rightarrow} & X' & \overset{u'}{\rightarrow} & Y' & \overset{v'}{\rightarrow} & Z \\
   \downarrow & & \downarrow \alpha & & \downarrow \alpha' & & \downarrow \\
   0 & \overset{0}{\rightarrow} & Z' & \overset{0}{\rightarrow} & Z' & \overset{0}{\rightarrow} & 0,
   \end{array}
   \]

   in which all rows and columns are triangles in \(\Delta\).

Throughout this paper, \(\mathcal{T} = (\mathcal{T}, \Sigma, \Delta)\) is a triangulated category.
Definition 2.2. [11] A triangle
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \]
is called split if it is isomorphic to the triangle
\[ X \xrightarrow{(1,0)} X \oplus Z \xrightarrow{(0,1)} Z \xrightarrow{0} \Sigma X. \]

We use \( \Delta_0 \) to denote the full subcategory of \( \Delta \) consisting of all split triangles.

Definition 2.3. [11] Let \( \xi \) be a class of triangles in \( \mathcal{T} \).
1. \( \xi \) is said to be closed under base change (resp. cobase change) if for any triangle
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \]
in \( \xi \) and any morphism \( \alpha : Z' \longrightarrow Z \) (resp. \( \beta : X \longrightarrow X' \)) as in Remark 2.1(2) (resp. Remark 2.1(3)), the triangle
\[ X \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X \] (resp. \( X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} \Sigma X' \))
is in \( \xi \).
2. \( \xi \) is said to be closed under suspension if for any triangle
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \]
in \( \xi \) and any \( i \in \mathbb{Z} \) (the set of all integers), the triangle
\[ \Sigma^i X \xrightarrow{(-1)^i \Sigma u} \Sigma^i Y \xrightarrow{(-1)^i \Sigma v} \Sigma^i Z \xrightarrow{(-1)^i \Sigma w} \Sigma^{i+1} X \]
is in \( \xi \).
3. \( \xi \) is called saturated if in the situation of base change as in Remark 2.1(2), whenever the third vertical and the second horizontal triangles are in \( \xi \), then the triangle
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \]
is in \( \xi \).

Definition 2.4. [11] A class \( \xi \) of triangles in \( \mathcal{T} \) is called proper if the following conditions are satisfied.
1. \( \xi \) is closed under isomorphisms, finite coproducts and \( \Delta_0 \subseteq \xi \).
2. \( \xi \) is closed under suspensions and is saturated.
3. \( \xi \) is closed under base and cobase change.

Throughout this paper, we always assume that \( \xi \) is a proper class of triangles in \( \mathcal{T} \).

Definition 2.5. [11] Let
\[ X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \]
be a triangle in \( \xi \). Then, the morphism \( u \) (resp. \( v \)) is called \( \xi \)-proper monic (resp. \( \xi \)-proper epic), and \( u \) (resp. \( v \)) is called the hokernel of \( v \) (resp. the hocokernel of \( u \)).

We use Hoker \( v \) to denote the hokernel of \( v : Y \longrightarrow Z \). Dually, we use Hocok \( u \) to denote the hocokernel of \( u : X \longrightarrow Y \). For any triangle,
\[ X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \]
in \( \xi \). We say that \( X \) is closed under \( \xi \)-extensions if \( X, Z \in X \), it holds that \( Y \in X \). We say that \( X \) is closed under hokernels of \( \xi \)-proper epimorphisms (resp. hocokernels of \( \xi \)-proper monomorphisms) if \( Y, Z \in X \) (resp. \( X, Y \in X \)), it holds that \( X \in X \) (resp. \( Z \in X \)).
Definition 2.6. (see [11, 4.1]) An object $P$ (resp. $I$) in $\mathcal{T}$ is called $\xi$-projective (resp. $\xi$-injective) if for any triangle $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ in $\xi$, the induced complex
\[
\begin{align*}
0 \rightarrow \text{Hom}_\mathcal{T}(P, X) \rightarrow \text{Hom}_\mathcal{T}(P, Y) \rightarrow \text{Hom}_\mathcal{T}(P, Z) &\rightarrow 0 \\
(\text{resp. } 0 \rightarrow \text{Hom}_\mathcal{T}(Z, I) \rightarrow \text{Hom}_\mathcal{T}(Y, I) \rightarrow \text{Hom}_\mathcal{T}(X, I) &\rightarrow 0)
\end{align*}
\]
is exact. We use $\mathcal{P}(\xi)$ (resp. $\mathcal{I}(\xi)$) to denote the full subcategory of $\mathcal{T}$ consisting of $\xi$-projective (resp. $\xi$-injective) objects.

We say that $\mathcal{T}$ has enough $\xi$-projective objects if for any object $M \in \mathcal{T}$, there exists a triangle $K \rightarrow P \rightarrow M \rightarrow \Sigma K$ in $\xi$ with $P \in \mathcal{P}(\xi)$. Dually, we say that $\mathcal{T}$ has enough $\xi$-injective objects if for any object $M \in \mathcal{T}$, there exists a triangle $M \rightarrow I \rightarrow K \rightarrow \Sigma M$ in $\xi$ with $I \in \mathcal{I}(\xi)$.

Remark 2.7. $\mathcal{P}(\xi)$ is closed under direct summands, hocokernels of $\xi$-proper epimorphisms, and $\xi$-extensions. Dually, $\mathcal{I}(\xi)$ is closed under direct summands, hocokernels of $\xi$-proper monomorphisms, and $\xi$-extensions.

Definition 2.8. Let $\mathcal{E}$ be a subcategory of $\mathcal{T}$.

1. A triangle
\[
X \rightarrow Y \rightarrow Z \rightarrow \Sigma X
\]

in $\xi$ is called $\text{Hom}_\mathcal{T}(\mathcal{E}, -)$-exact (resp. $\text{Hom}_\mathcal{T}(\mathcal{E}, -)$-exact) if for any object $E \in \mathcal{E}$, the induced complex
\[
\begin{align*}
0 \rightarrow \text{Hom}_\mathcal{T}(E, X) \rightarrow \text{Hom}_\mathcal{T}(E, Y) \rightarrow \text{Hom}_\mathcal{T}(E, Z) &\rightarrow 0 \\
(\text{resp. } 0 \rightarrow \text{Hom}_\mathcal{T}(Z, E) \rightarrow \text{Hom}_\mathcal{T}(Y, E) \rightarrow \text{Hom}_\mathcal{T}(X, E) &\rightarrow 0)
\end{align*}
\]
is exact.

2. [13] A $\xi$-exact complex is a complex
\[
\cdots \rightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \rightarrow \cdots
\]

in $\mathcal{T}$ such that for any $n \in \mathbb{Z}$, there exists a triangle
\[
K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}
\]
in $\xi$ and the differential $d_n$ is defined as $d_n = g_{n-1}f_n$. A $\xi$-exact complex as (2.1) is called $\text{Hom}_\mathcal{T}(\mathcal{E}, -)$-exact (resp. $\text{Hom}_\mathcal{T}(\mathcal{E}, -)$-exact) if the triangle (2.2) is $\text{Hom}_\mathcal{T}(\mathcal{E}, -)$-exact (resp. $\text{Hom}_\mathcal{T}(\mathcal{E}, -)$-exact) for any $n \in \mathbb{Z}$.

Asadollahi and Salarian [13] introduced the notion of $\xi$-Gorenstein projective objects.

Definition 2.9. [13, Definition 3.6] Let $\mathcal{T}$ be a triangulated category with enough $\xi$-projective objects and $X$ an object in $\mathcal{T}$. A complete $\xi$-projective resolution is a $\text{Hom}_\mathcal{T}(\mathcal{P}(\xi), \mathcal{E})$-exact $\xi$-exact complex
\[
\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots
\]
in $\mathcal{T}$ with all $P_i$-projective objects. The objects $K_n$ as in (2.2) are called $\xi$-Gorenstein projective objects. We use $\mathcal{GP}(\xi)$ to denote the full subcategory of $\mathcal{T}$ consisting of all $\xi$-Gorenstein projective objects.

Throughout this paper, we always assume that $\mathcal{T}$ is a triangulated category with enough $\xi$-projective objects and $\xi$-injective objects.

Let $M$ be an object in $\mathcal{T}$. Beligiannis [11] defined the $\xi$-extension groups $\xi \text{Ext}^n_\xi(-, M)$ to be the $n$th right $\xi$-derived functor of the functor $\text{Hom}_\mathcal{T}(-, M)$, that is,
\[
\xi \text{Ext}^n_\xi(-, M) = R^n_\xi \text{Hom}_\mathcal{T}(-, M).
\]
Remark 2.10. Let
\[ X \to Y \to Z \to \Sigma X \]
be a triangle in \( \xi \). By [11, Corollary 4.12], there exists a long exact sequence
\[
0 \to \xi t^0_\xi(Z, M) \to \xi t^0_\xi(Y, M) \to \xi t^0_\xi(X, M) \to \\
\xi t^1_\xi(Z, M) \to \xi t^1_\xi(Y, M) \to \xi t^1_\xi(X, M) \to \cdots
\]
of “\( \xi t \)” functor. If \( T \) has enough \( \xi \)-injective objects and \( N \) is an object in \( T \), then there exists a long exact sequence
\[
0 \to \xi t^0_\xi(N, X) \to \xi t^0_\xi(N, Y) \to \xi t^0_\xi(N, Z) \to \\
\xi t^1_\xi(N, X) \to \xi t^1_\xi(N, Y) \to \xi t^1_\xi(N, Z) \to \cdots
\]
of “\( \xi t \)” functor.

Following Remark 2.10, we usually use the strategy of “dimension shifting,” which is an important tool in relative homological theory of triangulated categories.

Now, we set
\[
\mathcal{X} = \{ M \in T | \xi t^0_\xi(X, M) = 0 \text{ for all } X \in \mathcal{X} \},
\]
\[
\mathcal{X}^+ = \{ M \in T | \xi t^0_\xi(M, X) = 0 \text{ for all } X \in \mathcal{X} \}.
\]

For two subcategories \( \mathcal{H} \) and \( \mathcal{X} \) of \( T \), we say \( \mathcal{H} \perp \mathcal{X} \) if \( \mathcal{H} \subseteq \mathcal{X} \) (equivalently, \( \mathcal{X} \subseteq \mathcal{H}^\perp \)).

Taking \( C = E = \mathcal{P}(\xi) \) in [18, Definitions 3.1 and 3.2], we have the following definitions.

**Definition 2.11.** (cf. [18, Definition 3.1]) Let \( \mathcal{H} \) and \( \mathcal{X} \) be two subcategories of \( T \) with \( \mathcal{H} \subseteq \mathcal{X} \). Then, \( \mathcal{H} \) is called a \( \xi \)-cogenerator of \( \mathcal{X} \) if for any object \( X \) in \( \mathcal{X} \), there exists a triangle
\[
X \to H \to Z \to \Sigma X
\]
in \( \xi \) with \( H \) an object in \( \mathcal{H} \) and \( Z \) an object in \( \mathcal{X} \). In particular, a \( \xi \)-cogenerator \( \mathcal{H} \) is called \( \xi t \)-injective if \( \mathcal{X} \perp \mathcal{H} \).

**Definition 2.12.** (cf. [18, Definition 3.2]) Let \( T \) be a triangulated category with enough \( \xi \)-projective objects and \( \mathcal{X} \) a subcategory of \( T \). Then, \( \mathcal{X} \) is called a resolving subcategory of \( T \) if the following conditions are satisfied.
(1) \( \mathcal{P}(\xi) \subseteq \mathcal{X} \).
(2) \( \mathcal{X} \) is closed under \( \xi \)-extensions.
(3) \( \mathcal{X} \) is closed under hokernels of \( \xi \)-proper epimorphisms.

3 Resolution dimensions with respect to a resolving subcategory

Taking \( E = \mathcal{P}(\xi) \) in [18, Definition 3.5], we first have the following definition.

**Definition 3.1.** Let \( \mathcal{X} \) be a subcategory of \( T \) and \( M \) an object in \( T \). The \( \mathcal{X} \)-resolution dimension of \( M \), written \( \mathcal{X} \)-res.dim \( M \), is defined by
\[
\mathcal{X} \text{-res.dim } M = \inf \{ n \geq 0 | \text{ there exists a } \xi \text{-exact complex } 0 \to X_n \to \cdots \to X_i \to \cdots \to X_0 \to M \to 0 \text{ in } T \text{ with all } X_i \text{ objects in } \mathcal{X} \}.
\]
For a \( \xi \)-exact complex
\[
\ldots \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \xrightarrow{f_0} M \xrightarrow{0},
\]
with all $X_i \in X$. The Hoker $f_{n-1}$ is called an $n$th $\xi$-$X$-syzygy of $M$, denoted by $\Omega^n_\xi(M)$. In case for $X = \mathcal{P}(\xi)$, we write $\xi$-pd $M = X$-res.dim $M$ and $\Omega^n_\xi(M) = \Omega^n_{\mathcal{P}(\xi)}(M)$. In case for $X = \mathcal{G}(\xi)$, $X$-res.dim $M$ coincides with $\xi$-$\mathcal{G}$pd $M$ defined in [13] as $\xi$-Gorenstein projective dimension. We use $\mathcal{X}$ to denote the full subcategory of $\mathcal{T}$ whose objects have finite $\mathcal{X}$-resolution dimension.

**Lemma 3.2.** Let $\mathcal{T}$ be a triangulated category and $X$ a resolving subcategory of $\mathcal{T}$. For any object $M \in \mathcal{T}$, if

$$0 \to X_n \to \cdots \to X_i \to X_0 \to M \to 0$$

and

$$0 \to Y_n \to \cdots \to Y_i \to Y_0 \to M \to 0$$

are $\xi$-exact complexes with all $X_i$ and $Y_i$ in $X$ for $0 \leq i \leq n - 1$, then $X_n \in X$ if and only if $Y_n \in X$.

**Proof.** For $M \in \mathcal{T}$, there exists a $\xi$-exact complex

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

with $P_i \in \mathcal{P}(\xi)$ for $0 \leq i \leq n - 1$.

Consider the following triangle:

$$K^n_1 \to X_0 \to M \to \Sigma K^n_1$$

in $\xi$. As a similar argument to that of [11, Proposition 4.11], we get the following $\xi$-exact complex

$$0 \to K_n \to X_n \oplus P_{n-1} \to X_{n-1} \oplus P_{n-2} \to \cdots \to X_0 \oplus P_1 \to X_1 \oplus P_0 \to X_0 \to 0.$$  

Similarly, we have the following $\xi$-exact complex

$$0 \to K_n \to Y_n \oplus P_{n-1} \to Y_{n-1} \oplus P_{n-2} \to \cdots \to Y_2 \oplus P_1 \to Y_1 \oplus P_0 \to Y_0 \to 0.$$  

Set

$$X = \text{Hoker}(X_{n-1} \oplus P_{n-2} \to X_{n-2} \oplus P_{n-3})$$

and

$$Y = \text{Hoker}(Y_{n-1} \oplus P_{n-2} \to Y_{n-2} \oplus P_{n-3}).$$

Since $X$ is resolving, we have that $X$ and $Y$ are objects in $X$. Consider the following triangles:

$$K_n \to X_n \oplus P_{n-1} \to X \to \Sigma K_n$$

and

$$K_n \to Y_n \oplus P_{n-1} \to Y \to \Sigma K_n$$

in $\xi$, we have that $X_n \oplus P_{n-1} \in X$ if and only if $K_n \in X$ if and only if $Y_n \oplus P_{n-1} \in X$.

But from the following triangles in $\xi$

$$X_n \oplus X_{n-1} \to X_{n-1} \oplus P_{n-2} \to \Sigma X_n$$

and

$$Y_n \oplus Y_{n-1} \to Y_{n-1} \oplus P_{n-2} \to \Sigma Y_n,$$

we have that $X_n \in X$ if and only if $X_n \oplus P_{n-1} \in X$, and $Y_n \in X$ if and only if $Y_n \oplus P_{n-1} \in X$. Thus, $X_n \in X$ if and only if $Y_n \in X$.  

Using the above, we can get:

**Proposition 3.3.** Let $X$ be a resolving subcategory of $\mathcal{T}$ and $M \in \mathcal{T}$. Then, the following statements are equivalent:

1. $X$-res.dim $M \leq m$.
2. $\Omega^n_\mathcal{P}(M) \in X$ for $n \geq m$.
3. $\Omega^n_\mathcal{G}(M) \in X$ for $n \geq m$.

**Proof.** Apply Lemma 3.2.
Now we can compare resolution dimensions in a given triangle in $\xi$ as follows.

**Proposition 3.4.** Let $X$ be a resolving subcategory of $T$, and let 
$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$
be a triangle in $\xi$. Then, we have the following statements:
1. $X$-res.$\dim B \leq \max\{X$-res.$\dim A, X$-res.$\dim C\}$.
2. $X$-res.$\dim A \leq \max\{X$-res.$\dim B, X$-res.$\dim C - 1\}$.
3. $X$-res.$\dim C \leq \max\{X$-res.$\dim A + 1, X$-res.$\dim B\}$.

**Proof.** For any $A \in T$, if $X$-res.$\dim A = m$, by Proposition 3.3, we have the following $\xi$-exact complex
$$0 \longrightarrow P_m^A \longrightarrow P_{m-1}^A \longrightarrow \cdots \longrightarrow P_1^A \longrightarrow P_0^A \longrightarrow A \longrightarrow 0$$
in $T$ with $P_i^A \in P(\xi)$ for $0 \leq i \leq m - 1$ and $P_m^A \in X$.

(1) Assume $X$-res.$\dim A = m$ and $X$-res.$\dim C = n$. We proceed it by induction on $m$ and $n$. The case $m = n = 0$ is trivial. Without loss of generality, we assume $m \leq n$, then we can let $P_i^A = 0$ for $i > m$. As a similar argument to that of [11, Proposition 4.11], we get the following $\xi$-exact complex:
$$0 \longrightarrow P_n^A \oplus P_n^C \longrightarrow P_{n-1}^A \oplus P_{n-1}^C \longrightarrow \cdots \longrightarrow P_0^A \oplus P_0^C \longrightarrow B \longrightarrow 0$$
in $T$. Thus, $X$-res.$\dim B \leq n$ and the desired assertion are obtained.

(2) Assume $X$-res.$\dim B = m$ and $X$-res.$\dim C = n$. We proceed it by induction on $m$ and $n$. The case $m = n = 0$ is trivial. Without loss of generality, we assume $m \leq n - 1$, then we can let $P_i^B = 0$ for $i > m$. By [18, Theorem 3.7], there exist a $\xi$-exact complex
$$0 \longrightarrow P_n^C \oplus P_{n-1}^B \longrightarrow P_{n-1}^C \oplus P_{n-2}^B \longrightarrow \cdots \longrightarrow P_1^C \oplus P_1^B \longrightarrow K \longrightarrow A \longrightarrow 0$$
and a triangle
$$K \longrightarrow P_1^C \oplus P_0^B \longrightarrow P_0^C \longrightarrow K[1]$$
in $\xi$, it follows that $K \in P(\xi)$ by Remark 2.7. Thus, $X$-res.$\dim A \leq n - 1$ and the desired assertion is obtained.

(3) Assume $X$-res.$\dim A = m$ and $X$-res.$\dim B = n$. We proceed it by induction on $m$ and $n$. The case $m = n = 0$ is trivial. Without loss of generality, we assume $m + 1 \leq n$, then we can let $P_i^A = 0$ for $i > m$. By [18, Theorem 3.8], we have the following $\xi$-exact complex
$$0 \longrightarrow P_n^B \oplus P_{n-1}^A \longrightarrow \cdots \longrightarrow P_2^B \oplus P_1^A \longrightarrow P_1^B \oplus P_0^A \longrightarrow P_0^B \longrightarrow C \longrightarrow 0$$
in $T$, thus $X$-res.$\dim A \leq n$ and the desired assertion is obtained. \hfill $\Box$

As direct results, we have the following closure properties for the subcategory $\tilde{X}$.

**Remark 3.5.** If $X$ is a resolving subcategory of $T$, then $\tilde{X}$ is closed under hokernels of $\xi$-proper epimorphisms, hokernels of $\xi$-proper monomorphisms, and $\xi$-extensions.

**Corollary 3.6.** Let $X$ be a resolving subcategory of $T$, and let 
$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$
be a triangle in $\xi$. Then, we have the following statements:
1. (cf. [18, Proposition 3.11]) Assume that $C$ is an object in $X$. Then, $X$-res.$\dim A = X$-res.$\dim B$.
2. Assume that $B$ is an object in $X$. Then, either $A \in X$ or else $X$-res.$\dim A = X$-res.$\dim C - 1$.
3. (cf. [18, Proposition 3.13]) Assume that $A$ is an object in $X$ and neither $B$ nor $C$ in $X$. Then, $X$-res.$\dim B = X$-res.$\dim C$. 


Proposition 3.7. Let $\mathcal{H}$ and $\mathcal{X}$ be two subcategories of $\mathcal{T}$ with $\mathcal{H} \subseteq \mathcal{X}$.

(1) $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{X}}$.

(2) If $\mathcal{X}$ is resolving, then for any $M \in \widehat{\mathcal{H}}$, $\mathcal{H}$-res.dim $M = \mathcal{X}$-res.dim $M$ if and only if $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.

In particular, if $\mathcal{X} \perp \mathcal{H}$, and $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands, then $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.

Proof.

(1) It is clear.

(2) $(\Rightarrow)$ Clearly, $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap \mathcal{X}$. Let $M \in \widehat{\mathcal{H}} \cap \mathcal{X}$. By the assumption, we have $\mathcal{H}$-res.dim $M = \mathcal{X}$-res.dim $M = 0$, then $M \in \mathcal{H}$, so $\widehat{\mathcal{H}} \cap \mathcal{X} \subseteq \mathcal{H}$. Thus, $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.

$(\Leftarrow)$ Let $M \in \widehat{\mathcal{H}}$. Suppose $\mathcal{H}$-res.dim $M = n$ and $\mathcal{X}$-res.dim $M = m$. Clearly, $m \leq n$. Consider the following $\xi$-exact complexes:

$$0 \rightarrow H_n \rightarrow \cdots \rightarrow H_0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow X_m \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0$$

with $H_i \in \mathcal{H}$ and $X_j \in \mathcal{X}$ for all $0 \leq i \leq n$ and $0 \leq j \leq m$. Since $\mathcal{H} \subseteq \mathcal{X}$, we have $\Omega^n_{\mathcal{H}}(M) \in \mathcal{X}$ by Lemma 3.2. Then, $\Omega^m_{\mathcal{H}}(M) \in \widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$, and thus, $\mathcal{H}$-res.dim $M \leq m$ and the desired equality is obtained.

Now, we assume that $\mathcal{X} \perp \mathcal{H}$ and $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands. Clearly, $\mathcal{H} \subseteq \widehat{\mathcal{H}} \cap \mathcal{X}$. Conversely, let $M \in \widehat{\mathcal{H}} \cap \mathcal{X}$. There exists a $\xi$-exact complex

$$0 \rightarrow H_n \rightarrow H_{n-1} \rightarrow \cdots \rightarrow H_0 \rightarrow M \rightarrow 0.$$

Set $K_i = \text{Hoker}(H_i \rightarrow H_{i-1})$ for $0 \leq i \leq n-2$, where $H_{-1} = M$. Since $\mathcal{X}$ is resolving, we have $K_i \in \mathcal{X}$, and hence, $K_i \in \widehat{\mathcal{H}} \cap \mathcal{X}$. Consider the following triangle:

$$H_n \rightarrow H_{n-1} \rightarrow K_{n-2} \rightarrow \Sigma H_n$$

in $\mathcal{X}$. Since $\text{ext}^1_{\mathcal{X}}(K_{n-2}, H_n) = 0$ by the assumption that $\mathcal{X} \perp \mathcal{H}$, we have that the triangle (1) is split. It follows that $H_{n-1} \cong H_n \oplus K_{n-2}$ and there exists a triangle

$$K_{n-2} \rightarrow H_{n-1} \rightarrow H_n \rightarrow 0 \Sigma K_{n-2}$$

in $\mathcal{X}$. Since $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands by assumption, we have $K_{n-2} \in \mathcal{H}$. Repeating this process, we can obtain each $K_i \in \mathcal{H}$, hence, $M \in \mathcal{H}$ and $\widehat{\mathcal{H}} \cap \mathcal{X} \subseteq \mathcal{H}$. Thus, $\widehat{\mathcal{H}} \cap \mathcal{X} = \mathcal{H}$.

We now give the following definition.

Definition 3.8. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ and $M$ an object in $\mathcal{T}$. A $\xi$-proper epimorphism $X \rightarrow M$ is said to be a right $\mathcal{X}$-approximation of $M$ if $\text{Hom}_{\mathcal{T}}(\bar{X}, X) \rightarrow \text{Hom}_{\mathcal{T}}(\bar{X}, M)$ is exact for any $\bar{X} \in \mathcal{X}$. In this case, there is a triangle $K \rightarrow X \rightarrow M \rightarrow \Sigma K$ in $\mathcal{X}$.

We need the following easy and useful observation.

Lemma 3.9. Let $\mathcal{H}$ and $\mathcal{X}$ be two subcategories of $\mathcal{T}$.

(1) If $\mathcal{X} \perp \mathcal{H}$, then $\mathcal{X} \perp \widehat{\mathcal{H}}$. In particular, if $\mathcal{H} \perp \mathcal{H}$, then $\mathcal{H} \perp \widehat{\mathcal{H}}$.

(2) If $M \in \downarrow \mathcal{H}$, then $M \in \downarrow \widehat{\mathcal{H}}$.

Proof. Apply Remark 2.10.
The following is an analogous theory of Auslander-Buchweitz approximations (see \cite{4,5}).

**Proposition 3.10.** Let \( \mathcal{X} \) be a subcategory of \( \mathcal{T} \) closed under \( \xi \)-extensions, and let \( \mathcal{H} \) be a subcategory of \( \mathcal{T} \) such that \( \mathcal{H} \) is a \( \xi \)-cogenerator of \( \mathcal{X} \). Then, for each \( M \in \mathcal{T} \) with \( \mathcal{X} \)-res.dim \( M = n < \infty \), there exist two triangles

\[
K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K \tag{2}
\]

and

\[
M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M \tag{3}
\]

in \( \xi \), where \( X, X' \in \mathcal{X} \), \( \mathcal{H} \)-res.dim \( K \leq n - 1 \) and \( \mathcal{H} \)-res.dim \( W \leq n \) (if \( n = 0 \), this should be interpreted as \( K = 0 \)).

**Proof.** We proceed by induction on \( n \). The case for \( n = 0 \) is trivial. If \( n = 1 \), there exists a triangle

\[
X_0 \longrightarrow X_1 \longrightarrow M \longrightarrow \Sigma X_0 \tag{4}
\]

in \( \xi \) with \( X_0, X_1 \in \mathcal{X} \). Since \( \mathcal{H} \) is a \( \xi \)-cogenerator of \( \mathcal{X} \), there is a triangle

\[
X_0 \longrightarrow H \longrightarrow X' \longrightarrow \Sigma X_0
\]

in \( \xi \) with \( H \in \mathcal{H} \) and \( X' \in \mathcal{X} \). Applying cobase change for the triangle (4) along the morphism \( X_0 \longrightarrow H \), we get the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \Sigma^{-1} X'_1 \longrightarrow \Sigma^{-1} X'_1 \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-1} M \longrightarrow X_1 \longrightarrow X_0 \longrightarrow M \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-1} M \longrightarrow H \quad \longrightarrow \quad X'_0 \longrightarrow M \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \quad \longrightarrow \quad X'_1 \quad \longrightarrow \quad X'_1 \quad \longrightarrow \quad 0.
\end{array}
\]

Since \( \xi \) is closed under cobase changes, we obtain that the triangle

\[
H \longrightarrow X'_0 \longrightarrow M \longrightarrow \Sigma H \tag{5}
\]

is in \( \xi \) with \( \mathcal{H} \text{-res.dim} \ H = 0 \). Note that \( \alpha' \ u = \alpha \) is \( \xi \)-proper epic, so we have that \( \alpha' \) is \( \xi \)-proper epic by \cite[Proposition 2.7]{16}; hence, the triangle

\[
X_0 \longrightarrow X'_0 \longrightarrow X'_1 \longrightarrow \Sigma X_0
\]

is in \( \xi \). Since \( \mathcal{X} \) is closed under \( \xi \)-extensions by assumption, we have \( X'_0 \in \mathcal{X} \). So, (5) is the first desired triangle.

For \( X'_0 \), there is a triangle

\[
X'_0 \longrightarrow H_0 \longrightarrow X''_0 \longrightarrow \Sigma X'_0
\]

in \( \xi \) with \( H_0 \in \mathcal{H} \) and \( X''_0 \in \mathcal{X} \). Applying cobase change for the triangle (5) along the morphism \( X'_0 \longrightarrow H_0 \), we get the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \Sigma^{-1} X''_0 \longrightarrow \Sigma^{-1} X''_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
H \quad \longrightarrow \quad X'_0 \quad \longrightarrow \quad M \quad \longrightarrow \quad \Sigma H \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
H \quad \longrightarrow \quad H_0 \quad \longrightarrow \quad U \quad \longrightarrow \quad \Sigma H \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \quad \longrightarrow \quad X''_0 \quad \longrightarrow \quad X''_0 \quad \longrightarrow \quad 0.
\end{array}
\]
Note that $u' = \beta u$ is $\xi$-proper monic by [16, Proposition 2.6], so the third horizontal triangle is in $\xi$. Since $y'y = y$ is $\xi$-proper epic, $y'$ is $\xi$-proper epic by [16, Proposition 2.7]. So the triangle

$$M \longrightarrow U \longrightarrow X_0^\sigma \longrightarrow \Sigma M$$

is in $\xi$ with $\mathcal{H}$-res.dim $U \leq 1$ and $X_0^\sigma \in \mathcal{X}$, which is the second desired triangle.

Now suppose $n \geq 2$. Then, there is a triangle

$$K' \longrightarrow X_0 \longrightarrow M \longrightarrow \Sigma K'$$

in $\xi$ with $\mathcal{X}$-res.dim $K' \leq n - 1$ and $X_0 \in \mathcal{X}$. For $K'$, by the induction hypothesis, we get a triangle

$$K' \longrightarrow K \longrightarrow X_2 \longrightarrow \Sigma K'$$

in $\xi$ with $\mathcal{H}$-res.dim $K \leq n - 1$ and $X_2 \in \mathcal{X}$. Applying cobase change for the triangle (7) along the morphism $K' \longrightarrow K$, we get the following commutative diagram:

```
0 \longrightarrow \Sigma^{-1}X_2 \longrightarrow \Sigma^{-1}X_2 \longrightarrow 0
\downarrow \quad \downarrow \quad \downarrow
\Sigma^{-1}M \longrightarrow K' \longrightarrow X_0 \longrightarrow M
\quad \downarrow \quad \downarrow \quad \downarrow
\Sigma^{-1}M \longrightarrow K \longrightarrow X \longrightarrow M
\quad \downarrow \quad \downarrow \quad \downarrow
0 \longrightarrow X_2 \longrightarrow X_2 \longrightarrow 0.
```

Note that $\lambda'\kappa = \lambda$ is $\xi$-proper epic, then $\lambda'$ is $\xi$-proper epic by [16, Proposition 2.7], so the triangle

$$X_0 \longrightarrow X \longrightarrow X_2 \longrightarrow \Sigma X_0$$

is in $\xi$. It follows that $X \in \mathcal{X}$ from the assumption that $\mathcal{X}$ is closed under $\xi$-extensions. Since $\xi$ is closed under cobase changes, we obtain the first desired triangle

$$K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K$$

in $\xi$ with $\mathcal{H}$-res.dim $K \leq n - 1$ and $X \in \mathcal{X}$.

For $X$, since $\mathcal{H}$ is a $\xi$-cogenerator of $\mathcal{X}$, we get the following triangle

$$X \longrightarrow H_1 \longrightarrow X' \longrightarrow \Sigma X$$

in $\xi$ with $H_1 \in \mathcal{H}$ and $X' \in \mathcal{X}$.

Applying cobase change for the triangle (8) along the morphism $X \longrightarrow H_1$, we get the following commutative diagram:

```
0 \longrightarrow \Sigma^{-1}X' \longrightarrow \Sigma^{-1}X' \longrightarrow 0
\downarrow \quad \downarrow \quad \downarrow
K \longrightarrow X \longrightarrow M \longrightarrow \Sigma K
\quad \downarrow \quad \downarrow \quad \downarrow
K \longrightarrow H_1 \longrightarrow W \longrightarrow \Sigma K
\quad \downarrow \quad \downarrow \quad \downarrow
0 \longrightarrow X' \longrightarrow X' \longrightarrow 0.
```

As a similar argument to that of the diagram (6), we obtain that the triangles

$$K \longrightarrow H_1 \longrightarrow W \longrightarrow \Sigma K$$

and

$$M \longrightarrow W \longrightarrow X' \longrightarrow \Sigma M$$

are in $\xi$. Thus, (9) is the second desired triangle in $\xi$ with $\mathcal{H}$-res.dim $W \leq n$ and $X' \in \mathcal{X}$. 
In particular, suppose $X \perp \mathcal{H}$, by Lemma 3.9, we have $X \perp \widehat{\mathcal{H}}$. Then, $\xi\mathcal{T}(X, K) = 0$ for any $X \in X$, it follows that $\text{Hom}_\mathcal{T}(X, X) \rightarrow \text{Hom}_\mathcal{T}(X, M) \rightarrow 0$ is exact. Thus, the $\xi$-proper epimorphism $X \rightarrow M$ is a right $X$-approximation of $M$.

\textbf{Proposition 3.11.} Keep the notion as Proposition 3.10. Assume $M \in \hat{X}$ with $X$-res.dim $M = n < \infty$.

(1) If $X$ is resolving, then in the triangles (2) and (3), we have $\mathcal{H}$-res.dim $K = n - 1$ and $\mathcal{H}$-res.dim $W = X$-res.dim $W = n$.

In particular, if $X \perp \mathcal{H}$, then the $\xi$-proper epimorphism $X \rightarrow M$ in the triangle (2) is a right $X$-approximation of $M$, such that $\mathcal{H}$-res.dim $K = n - 1$ (if $n = 0$, it should be interpreted $K = 0$).

(2) If $X \perp \mathcal{H}$ and $X$ is resolving, then there is a triangle

$$M \rightarrow M' \rightarrow X \rightarrow \Sigma M$$

in $\xi$ with $M' \in X^c$, $X \in X$ and $X$-res.dim $M = X$-res.dim $M'$.

(3) (a) Let $\omega_{\mathcal{H}} = \mathcal{H}^c \cap \mathcal{H}$. If $\omega_{\mathcal{H}}$ is a $\xi$-cogenerator of $\mathcal{H}$ and $\mathcal{H}$ is closed under $\xi$-extensions, then $X \perp \omega_{\mathcal{H}}$ if and only if $X \perp (\mathcal{H}^c \cap \widehat{\mathcal{H}})$.

(b) If $X$ is a resolving and $\omega_X = X \cap X^c$ is a $\xi$-cogenerator of $X$ and $M \in X^c$, then $X$-res.dim $M = \omega_X$-res.dim $M$.

(4) Suppose that $\mathcal{H}$ and $X$ are resolving. If $\omega_{\mathcal{H}} = \mathcal{H} \cap \mathcal{H}^c$ is a $\xi$-cogenerator of $\mathcal{H}$ and $X \perp \omega_{\mathcal{H}}$, then $M$ admits a right $X$-approximation $X' \rightarrow M$ such that $K'' \rightarrow X' \rightarrow M \rightarrow \Sigma K''$ is a triangle in $\xi$, where $\mathcal{H}$-res.dim $K'' = n - 1$. In fact, we have $\omega_{\mathcal{H}}$-res.dim $K'' = n - 1$.

\textbf{Proof.}

(1) Suppose $X$ is resolving. Applying Corollary 3.6(2) to the triangle (2) yields that $X$-res.dim $K = n - 1$. On the other hand, since $\mathcal{H} \subseteq X$, we have $n - 1 = X$-res.dim $K \leq \mathcal{H}$-res.dim $K \leq n - 1$. Thus, $\mathcal{H}$-res.dim $K = n - 1$.

Moreover, applying Corollary 3.6(1) to the triangle (3) implies $X$-res.dim $W = X$-res.dim $M = n$. So, $n = X$-res.dim $W \leq \mathcal{H}$-res.dim $W \leq n$. Hence, $\mathcal{H}$-res.dim $W = X$-res.dim $W = n$.

The last assertion follows from the above argument and Proposition 3.10.

(2) Since $X \perp \mathcal{H}$, we have $X \perp \widehat{\mathcal{H}}$ by Lemma 3.9, and so the result immediately follows from (1).

(3) (a) Suppose $X \perp (\mathcal{H}^c \cap \widehat{\mathcal{H}})$. Clearly, $\omega_{\mathcal{H}} = \mathcal{H}^c \cap \mathcal{H} \subseteq \mathcal{H}^c \cap \widehat{\mathcal{H}} \subseteq X^c$, that is, $X \perp \omega_{\mathcal{H}}$.

(=) Suppose $X \perp \omega_{\mathcal{H}}$. Let $L \in \mathcal{H}^c \cap \widehat{\mathcal{H}}$. By Proposition 3.10, there exists a triangle

$$K' \rightarrow H_0 \rightarrow L \rightarrow \Sigma K'$$

in $\xi$ with $H_0 \in \mathcal{H}$ and $\omega_{\mathcal{H}}$-res.dim $K' \leq \mathcal{H}$-res.dim $L - 1 < \infty$. Note that $K' \in \mathcal{H}^c$ by Lemma 3.9, so $L \in \mathcal{H}^c$ implies $H_0 \in \mathcal{H}^c$. Then, $H_0 \in \omega_{\mathcal{H}}$, and so, $L \in \omega_{\mathcal{H}}$. Since $X \perp \omega_{\mathcal{H}}$, we have $L \in X^c$ by Lemma 3.9. Thus, $X \perp (\mathcal{H}^c \cap \widehat{\mathcal{H}})$.

(b) Suppose $X$-res.dim $M = n$, by (1), there exists a triangle

$$K \rightarrow X_0 \rightarrow M \rightarrow \Sigma K$$

in $\xi$ with $X_0 \in \mathcal{X}$ and $\omega_X$-res.dim $K = n - 1$. Note that $M \in X^c$ and $K \in X^c$, so $X_0 \in X^c$, and hence, $X_0 \in \omega_X$. It follows that $\omega_X$-res.dim $M \leq n$. But $n = X$-res.dim $M \leq \omega_X$-res.dim $M \leq n$, thus $X$-res.dim $M = \omega_X$-res.dim $M$.

(4) Suppose $X$-res.dim $M = n$, by (1), there exists a triangle

$$K \rightarrow X_0 \rightarrow M \rightarrow \Sigma K$$

(10)

in $\xi$ with $X_0 \in \mathcal{X}$ and $\mathcal{H}$-res.dim $K = n - 1$. By (2), there is a triangle

$$K \rightarrow K'' \rightarrow H \rightarrow \Sigma K$$
in $\xi$ with $H \in \mathcal{H}$, $K'' \in \mathcal{H}^+$ and $\mathcal{H}$-res.$\dim K'' = \mathcal{H}$-res.$\dim K$. Then, $K'' \in \mathcal{H}^+ \cap \mathcal{H}$. Applying cobase change for the triangle (10) along the morphism $K \rightarrow K''$, we get the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow \Sigma^{-1} H \rightarrow \Sigma^{-1} H \rightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-1} M \rightarrow K \rightarrow X_0 \rightarrow M \\
\downarrow \quad \downarrow \quad \downarrow \\
\Sigma^{-1} M \rightarrow K'' \rightarrow X' \rightarrow M \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow H \rightarrow H \rightarrow 0.
\end{array}
\]

One can see that the triangle

\[
K'' \rightarrow X' \rightarrow M \rightarrow \Sigma K''
\]

is in $\xi$ and $X' \in \mathcal{X}$. Note that $X \perp \omega_{\mathcal{H}}$, so $X \perp \mathcal{H}^+ \cap \mathcal{H}$ by (3)(a). Then, $\xi \text{xt}_1(\mathcal{X}, K'') = 0$ for any $\mathcal{X} \in \mathcal{X}$, and so, $\text{Hom}_\mathcal{H}(\mathcal{X}, X') \rightarrow \text{Hom}_\mathcal{H}(\mathcal{X}, M) \rightarrow 0$ is exact. Thus, the $\xi$-proper epimorphism $X' \rightarrow M$ is a right $\mathcal{X}$-approximation of $M$ and $\mathcal{H}$-res.$\dim K'' = n - 1$ in the triangle (11). Note that $K'' \in \mathcal{H}^+$, so we have $\omega_{\mathcal{H}}$-res.$\dim K'' = \mathcal{H}$-res.$\dim K'' = n - 1$ by (3)(b).

\[\square\]

**Lemma 3.12.** Let $\mathcal{H}$ be a subcategory of $\mathcal{T}$ with $\mathcal{H} \perp \mathcal{H}$. Assume that $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands. Then, $\mathcal{H} = \mathcal{H}^+ \cap \mathcal{H}$.

**Proof.** Clearly, $\mathcal{H} \subseteq \mathcal{H}^+ \cap \mathcal{H}$.

Conversely, let $M \in \mathcal{H}^+ \cap \mathcal{H}$. Consider the following $\xi$-exact complex:

\[
0 \rightarrow H_n \rightarrow H_{n-1} \rightarrow \cdots \rightarrow H_0 \rightarrow M \rightarrow 0.
\]

Set $K_i = \text{Hoker}(H_i \rightarrow H_{i-1})$ for $0 \leq i \leq n - 2$, where $H_1 = M$. Then, $M \in \mathcal{H}^+$ yields $K_i \in \mathcal{H}^+$, and so the triangle

\[
H_n \rightarrow H_{n-1} \rightarrow K_{n-2} \rightarrow \Sigma H_n
\]

is split. It follows that $H_{n-1} \cong H_n \oplus K_{n-2}$ and there exists a triangle

\[
K_{n-2} \rightarrow H_{n-1} \rightarrow H_n \rightarrow \Sigma K_{n-2}
\]

in $\xi$. Since $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands by assumption, we have $K_{n-2} \in \mathcal{H}$. Repeating this process, we can obtain $K_i \in \mathcal{H}$, hence $M \in \mathcal{H}$ and $\mathcal{H}^+ \cap \mathcal{H} \subseteq \mathcal{H}$. Thus, $\mathcal{H}^+ \cap \mathcal{H} = \mathcal{H}$.

\[\square\]

**Proposition 3.13.** Let $\mathcal{X}$ be a resolving subcategory and $\mathcal{H}$ a $\xi$-injective $\xi$-cogenerator of $\mathcal{X}$. Assume that $\mathcal{H}$ is closed under hokernels of $\xi$-proper epimorphisms or closed under direct summands. Then, $\mathcal{X} = \mathcal{X} \cap ^+ \mathcal{H} = \mathcal{X} \cap \mathcal{H}^+$.

**Proof.** Clearly, $\mathcal{X} \subseteq \mathcal{X} \cap ^+ \mathcal{H}$ and $\mathcal{X} \cap ^+ \mathcal{H} \subseteq \mathcal{X} \cap ^+ \mathcal{H}$.

Now, let $M \in \mathcal{X} \cap ^+ \mathcal{H}$. Then, by Lemma 3.9, we have $M \in \mathcal{X} \cap ^+ \mathcal{H}$, and hence, $\mathcal{X} \cap ^+ \mathcal{H} \subseteq \mathcal{X} \cap ^+ \mathcal{H}$. On the other hand, by Proposition 3.10, there is a triangle

\[
K \rightarrow \mathcal{X} \rightarrow M \rightarrow \Sigma K
\]

in $\xi$ with $X \in \mathcal{X}$ and $\mathcal{X}$-res.$\dim K < \infty$. Note that $M \in \mathcal{H}^+$ implies $K \in \mathcal{H}^+$, and hence, $K \in \mathcal{H}^+ \cap \mathcal{H}^+ = \mathcal{H}$ by Lemma 3.12. Note that $\xi \text{xt}_1(M, K) = 0$, so the triangle (12) is split; hence, $X \cong K \oplus M$. Consider the following triangle
\[ M \rightarrow X \rightarrow K \rightarrow \Sigma M \]

in \( \xi \). It follows that \( M \in \mathcal{X} \) from the assumption that \( \mathcal{X} \) is resolving. Thus, \( \mathcal{X} \cap \mathcal{H} \subseteq \mathcal{X} \).

\[
\Box
\]

Our main result is the following.

**Theorem 3.14.** Let \( \mathcal{X} \) be a resolving subcategory of \( \mathcal{T} \) and \( \mathcal{H} \) a \( \xi \text{-inj} \) \( \xi \text{-cogenerator of } \mathcal{X} \). Assume that \( \mathcal{H} \) is closed under hokernels of \( \xi \text{-proper epimorphisms} \) or closed under direct summands. For any \( M \in \mathcal{T} \), if \( M \in \widehat{\mathcal{X}} \), then the following statements are equivalent:

1. \( \mathcal{X} \text{-} \text{res.dim } M \leq m \).
2. \( \Omega^n(M) \in \mathcal{X} \) for all \( n \geq m \).
3. \( \Omega^n_M(M) \in \mathcal{X} \) for all \( n \geq m \).
4. \( \xi \text{-} \text{ext}^n_M(M, H) = 0 \) for all \( n > m \) and all \( H \in \mathcal{H} \).
5. \( \xi \text{-} \text{ext}^n_M(M, L) = 0 \) for all \( n > m \) and all \( L \in \widehat{\mathcal{H}} \).
6. \( M \) admits a right \( \mathcal{X} \text{-approximation } \varphi : X \rightarrow M \), where \( \varphi \) is \( \xi \text{-proper epic} \), such that \( K = \text{Hoker } \varphi \) satisfying \( \mathcal{H} \text{-res.dim } K \leq m - 1 \).
7. There are two triangles

\[ W_M \rightarrow X_M \rightarrow M \rightarrow \Sigma W_M \]

and

\[ M \rightarrow W^M \rightarrow X^M \rightarrow \Sigma M \]

in \( \xi \) such that \( X^M \in \mathcal{X} \) and \( \mathcal{H} \text{-res.dim } W_M \leq m - 1 \), \( \mathcal{H} \text{-res.dim } W^M = \mathcal{X} \text{-res.dim } W^M \leq m \).

**Proof.**

1. \( (1) \Leftrightarrow (2) \Leftrightarrow (3) \) It follows from Proposition 3.3.
2. \( (1) \Leftrightarrow (6) \) It follows from Proposition 3.11(1).
3. \( (1) \Leftrightarrow (7) \) It follows from Proposition 3.11(1).
4. \( (1) \Rightarrow (4) \) Suppose \( \mathcal{X} \text{-res.dim } M \leq m \). There is a \( \xi \text{-exact complex} \)

\[ 0 \rightarrow X_m \rightarrow \cdots \rightarrow X_0 \rightarrow M \rightarrow 0 \]

with all \( X_i \in \mathcal{X} \). Since \( \mathcal{H} \) is a \( \xi \text{-inj} \) \( \xi \text{-cogenerator of } \mathcal{X} \), we have \( \xi \text{-ext}^{i-1}(X_i, H) = 0 \) for all \( H \in \mathcal{H} \).

So, \( \xi \text{-ext}^n_M(M, H) \equiv \xi \text{-ext}^{-m}(X_m, H) = 0 \) for \( n > m \).
5. \( (4) \Rightarrow (5) \) It follows from Lemma 3.9.
6. \( (5) \Rightarrow (4) \) It is clear.
7. \( (4) \Rightarrow (1) \) Since \( M \in \widehat{\mathcal{X}} \), by Proposition 3.11(1), there is a triangle \( K \rightarrow X \rightarrow M \rightarrow \Sigma K \) in \( \xi \) with \( \mathcal{H} \text{-res.dim } K < \infty \) and \( X \in \mathcal{X} \). Then, \( \xi \text{-ext}^i(K, H) \equiv \xi \text{-ext}^{-i}(X, H) \) for \( H \in \mathcal{H} \) and \( i \geq 1 \) since \( \xi \text{-ext}^{i-1}(X, H) = 0 \).

So, \( \xi \text{-ext}^{-m}(K, H) = 0 \). Note that \( \mathcal{H} \text{-res.dim } K < \infty \), so we have the following \( \xi \text{-exact complex} \)

\[ 0 \rightarrow H_n \rightarrow \cdots \rightarrow H_0 \rightarrow K \rightarrow 0 \]

with all \( H_i \in \mathcal{H} \). Then,

\[ \xi \text{-ext}^i(\Omega^{-i}_{\mathcal{H}}(K), H) \equiv \xi \text{-ext}^{-i+1}(K, H) = 0 \]

for \( i \geq 1 \) and all \( H \in \mathcal{H} \), which means \( \Omega^{-i}_{\mathcal{H}}(K) \in \mathcal{H} \). Note that \( \mathcal{H} \text{-res.dim } \Omega^{-i}_{\mathcal{H}}(K) < \infty \), hence, \( \Omega^{-i}_{\mathcal{H}}(K) \in \widehat{\mathcal{H}} \cap \mathcal{H} \). It follows that \( \Omega^{-i}_{\mathcal{H}}(K) \in \mathcal{H} \) from Lemma 3.12, so \( \mathcal{H} \text{-res.dim } K \leq m - 1 \). Thus, \( \mathcal{X} \text{-res.dim } M \leq m \).
4 Additive quotient categories and $\xi$-cellular towers with respect to a resolving subcategory

In this section, we will further study objects having finite resolution dimension with respect to a resolving subcategory $X$. We first construct adjoint pairs for two kinds of inclusion functors. Then, we characterize objects having finite resolution dimension in terms of a notion of $\xi$-cellular towers.

4.1 Adjoint pairs

Suppose that $D$ and $X$ are two subcategories of $\mathcal{T}$. Denote by $[D]$ the ideal of $X$ consisting of morphisms factoring through some object in $D$. Thus, we have a quotient category $X/[D]$, which is also an additive category.

**Lemma 4.1.** Let $X$ be a resolving subcategory of $\mathcal{T}$ and $H$ a $\xi$-injective $\xi$-cogenerator of $X$. Assume that $f : X \rightarrow M$ is a morphism in $\mathcal{T}$ with $X \in X$ and $M \in \widehat{X}$, then the following statements are equivalent:

1. $f$ factors through an object in $H$.
2. $f$ factors through an object in $\partial H$.

**Proof.** It suffices to show that (2) $\Rightarrow$ (1). Suppose that $f$ factors through an object $L \in \partial H$. Then, $f = gh$, where $h : X \rightarrow L$ and $g : L \rightarrow M$. Consider the following triangle

$$L' \rightarrow H \rightarrow L \rightarrow \Sigma L'$$

in $\xi$ with $H \in H$ and $L' \in \partial H$. Note that $H$ is a $\xi$-injective $\xi$-cogenerator of $X$, by Lemma 3.9, we have $\xi^2(X, L') = 0$. So, $h$ factors through $H$, it follows that $f$ factors through $H$. $\square$

**Lemma 4.2.** Let $X$ be a resolving subcategory of $\mathcal{T}$ and $H$ a $\xi$-injective $\xi$-cogenerator of $X$, and let $M, N \in \widehat{X}$. Assume that $f : M \rightarrow N$ is a morphism in $\mathcal{T}$, consider two triangles

$$W_M \xrightarrow{a} X_M \xrightarrow{p} M \rightarrow \Sigma W_M \quad \text{and} \quad W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \rightarrow \Sigma W_N$$

in $\xi$ with $X_M, X_N \in X$ and $W_M, W_N \in \widehat{H}$ (see Proposition 3.10), then we have the following statements:

1. There exists a morphism $g : X_M \rightarrow X_N$ such that $ag = fp$.
2. If $g, g' : X_M \rightarrow X_N$ are two morphisms such that $ag = fp$ and $ag' = fp$, then $[g] = [g']$ in $\text{Hom}_{X/[H]}(X_M, X_N)$.

**Proof.**

1. Apply Proposition 3.10.
2. Suppose $g, g' : X_M \rightarrow X_N$ are two morphisms such that $ag = fp$ and $ag' = fp$, then $q(g' - g) = qg' - qg = 0$, and so there exists a morphism $h : X_M \rightarrow W_N$ such that $g' - g = \beta h$, that is, there is a commutative diagram as follows:

$$
\begin{array}{ccc}
W_N & \xrightarrow{\beta} & X_N \\
\downarrow{q} & & \downarrow{q} \\
& N & \xrightarrow{\Sigma} W_N \\
\end{array}
$$

Note that $W_N \in \widehat{H}$, so $g' - g : X_M \rightarrow X_N$ factors through an object in $H$ by Lemma 4.1. Thus, $[g] = [g']$ in $\text{Hom}_{X/[H]}(X_M, X_N)$. $\square$

By Lemma 4.2, there exists a well-defined additive functor
which maps an object \( M \in \mathcal{X} \) to \( X_M \) and a morphism \( f : M \to N \in \Hom_{\mathcal{X}}(M, N) \) to \( [g] \in \Hom_{\mathcal{X}/[\mathcal{H}]}(X_M, X_N) \) as described in Lemma 4.2.

Clearly, we have \( F(H) = 0 \) for any object \( H \in \mathcal{H} \). Hence, \( F \) factors through \( \mathcal{X}/[\mathcal{H}] \). That is, there exists an additive functor \( \mu : \mathcal{X}/[\mathcal{H}] \to \mathcal{X}/[\mathcal{H}] \) making the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{X}/[\mathcal{H}] & \xrightarrow{\mu} & \mathcal{X}/[\mathcal{H}] \\
\pi \downarrow & & \downarrow \mu \\
\mathcal{X}/[\mathcal{H}] & & \\
\end{array}
\]

where \( \pi \) is the canonical quotient functor.

Now we show that the additive functor \( \mu \) defined above and the inclusion functor between additive quotients \( \mathcal{X}/[\mathcal{H}] \) and \( \mathcal{X}/[\mathcal{H}] \) are adjoint.

**Theorem 4.3.** Let \( \mathcal{X} \) be a resolving subcategory of \( \mathcal{T} \) and \( \mathcal{H} \) a \( \xi \)-injective \( \xi \)-cogenerator of \( \mathcal{X} \). Then, the additive functor \( \mu : \mathcal{X}/[\mathcal{H}] \to \mathcal{X}/[\mathcal{H}] \) defined above is right adjoint to the inclusion functor \( \mathcal{X}/[\mathcal{H}] \to \mathcal{X}/[\mathcal{H}] \).

**Proof.** Let \( X \in \mathcal{X} \) and \( N \in \mathcal{X} \). By Proposition 3.10, there is a triangle

\[
W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \to \Sigma W_N
\]

in \( \xi \) with \( W_N \in \mathcal{H} \) and \( X_N \in \mathcal{X} \). Note that the additive map

\[
[q] : \Hom_{\mathcal{X}/[\mathcal{H}]}(X, \mu(N)) \to \Hom_{\mathcal{X}/[\mathcal{H}]}(X, N)
\]

is natural in both \( X \) and \( N \) by Lemma 4.2. We claim that \( [q] \) is an isomorphism.

Indeed, since \( \mathcal{H} \) is a \( \xi \)-injective \( \xi \)-cogenerator of \( \mathcal{X} \), by Lemma 3.9, we have \( \xi \text{-} \text{inj} \mathcal{X}(X, W_N) = 0 \), and hence, \( \Hom_{\mathcal{X}}(X, X_N) \to \Hom_{\mathcal{H}}(X, N) \) is an epimorphism, so \( [q] \) is still an epimorphism.

Now, assume that \( g : X \to X_N \) is a morphism such that \( [q] = [q][q] = [q][q] = [0] \in \Hom_{\mathcal{X}/[\mathcal{H}]}(X, N) \). Then, there exists an object \( H \in \mathcal{H} \) such that \( qg = \text{ts} \) as the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{s} & H \\
\beta \downarrow & & \downarrow \theta \\
W_N \xrightarrow{\beta} X_N \xrightarrow{q} N \to \Sigma W_N.
\end{array}
\]

Note that \( \xi \text{-} \text{inj} \mathcal{X}(H, W_N) = 0 \) by assumption, so there exists a morphism \( \theta : H \to X_N \) such that \( t = \theta \beta \). Since \( q(g - \theta s) = qg - q\theta s = ts - ts = 0 \), so \( g - \theta s \) factors through \( W_N \). By Lemma 4.1, \( g - \theta s \) factors through an object in \( \mathcal{H} \). It follows that \( [g - \theta s] = [0] \in \Hom_{\mathcal{X}/[\mathcal{H}]}(X, N) \). Since \( \theta s = 0 \in \Hom_{\mathcal{X}/[\mathcal{H}]}(X, N) \), we have \( 0 = [g] \in \Hom_{\mathcal{X}/[\mathcal{H}]}(X, N) \). So \( [q] \) is a monomorphism, and thus, \( [q] \) is an isomorphism.

**Corollary 4.4.** Let \( \mathcal{X} \) be a resolving subcategory of \( \mathcal{T} \) and \( \mathcal{H} \) a \( \xi \)-injective \( \xi \)-cogenerator of \( \mathcal{X} \). Assume that \( \mathcal{H} \) is closed under direct summands. For any \( N \in \mathcal{X} \), the following statements are equivalent:

1. \( N \in \mathcal{H} \).
2. There is a triangle

\[
W_N \xrightarrow{} X_N \xrightarrow{q} N \to \Sigma W_N
\]

in \( \xi \) with \( W_N \in \mathcal{H} \) and \( X_N \in \mathcal{X} \) such that \( [q] = [0] \in \Hom_{\mathcal{X}/[\mathcal{H}]}(X, N) \).

**Proof.** The assertion (1) \( \Rightarrow \) (2) follows from Lemma 4.1. It suffices to show (2) \( \Rightarrow \) (1). Note that the adjunction isomorphism established in Theorem 4.3 implies that the additive map
[q] : \text{Hom}_{\mathcal{X}[\mathcal{H}]}(X_N, X_N) \to \text{Hom}_{\mathcal{X}[\mathcal{H}]}(X_N, N)

is isomorphic. Since \([q], [\text{id}_{X_N}] = [q \text{id}_{X_N}] = [q] = [0] \in \text{Hom}_{\mathcal{X}[\mathcal{H}]}(X_N, N) = 0\), so \([\text{id}_{X_N}] = [0] \in \text{Hom}_{\mathcal{X}[\mathcal{H}]}(X_N, X_N)\), and thus, \(\text{id}_{X_N}\) factors through an object \(H \in \mathcal{H}\). It follows that \(X_N\) is a direct summand of \(W_N\). Since \(\mathcal{H}\) is closed under direct summands, we have \(X_N \in \mathcal{H}\). Thus, \(N \in \mathcal{H}\).

Next, we compare additive quotients \(\widehat{\mathcal{H}}/\mathcal{X}\) and \(\widetilde{\mathcal{X}}/\mathcal{X}\).

**Lemma 4.5.** Let \(\mathcal{X}\) be a resolving subcategory of \(\mathcal{T}\) and \(\mathcal{H}\) a \(\text{Ext}\)-injective \(\xi\)-cogenerator of \(\mathcal{X}\), and let \(M, N \in \mathcal{X}\). Assume that \(f : M \to N\) is a morphism in \(\mathcal{T}\), consider two triangles

\[
M \overset{s}{\to} W_M \overset{l}{\to} X_M \to \Sigma M \quad \text{and} \quad N \overset{l}{\to} W_N \overset{r}{\to} X_N \to \Sigma N
\]

in \(\xi\) with \(X_M, X_N \in \mathcal{X}\) and \(W_M, W_N \in \mathcal{H}\) (see Proposition 3.10), then, we have the following statements:

1. There exists a morphism \(g : W_M \to W_N\) such that \(gs = tf\).
2. If \(g, g' : W_M \to W_N\) are two morphisms such that \(gs = tf\) and \(g's = tf\), then \([g] = [g']\) in \(\text{Hom}_{\mathcal{X}[\mathcal{H}]}(X_M, X_N)\).

**Proof.**

1. Since \(\mathcal{X} \perp \mathcal{H}\) by assumption, we have \(\text{Ext}(X_M, W_N) = 0\) by Lemma 3.9. So, there exists a morphism \(g : W_M \to W_N\) such that \(gs = tf\).
2. Suppose \(g, g' : W_M \to W_N\) are two morphisms such that \(gs = tf\) and \(g's = tf\), then \((g' - g)s = g's - gs = 0\), and so there exists a morphism \(h' : X_M \to W_N\) such that \(g' - g = h'\), that is, there is a commutative diagram as follows:

\[
\begin{array}{ccc}
M & \overset{s}{\to} & W_M & \overset{l}{\to} & X_M & \to & \Sigma M \\
\downarrow{g'} & & \downarrow{g} & & \downarrow{h} & & \\
W_N & & & & & \end{array}
\]

Note that \(X_M \in \mathcal{X}\), so \(g' - g : W_M \to W_N\) factors through an object in \(\mathcal{X}\). Thus, \([g] = [g']\) in \(\text{Hom}_{\mathcal{X}[\mathcal{H}]}(W_M, W_N)\).

By Lemma 4.5, there exists a well-defined additive functor

\[G : \widetilde{\mathcal{X}} \to \widehat{\mathcal{H}}/\mathcal{X},\]

which maps an object \(M \in \mathcal{X}\) to \(W_M\) and a morphism \(f : M \to N\) to \([g] \in \text{Hom}_{\widehat{\mathcal{H}}/\mathcal{X}}(W_M, W_N)\) as described in Lemma 4.5.

Clearly, we have \(G(X) = 0\) for any object \(X \in \mathcal{X}\). Hence, \(G\) factors through \(\widetilde{\mathcal{X}}/\mathcal{X}\). That is, there exists an additive functor \(\eta : \widetilde{\mathcal{X}}/\mathcal{X} \to \widehat{\mathcal{H}}/\mathcal{X}\) making the following diagram commutes

\[
\begin{array}{ccc}
\widetilde{\mathcal{X}} & \xrightarrow{\pi} & \widetilde{\mathcal{X}}/\mathcal{X} \\
\downarrow{G} & & \downarrow{\eta} \\
\widehat{\mathcal{H}}/\mathcal{X} \\
\end{array}
\]

where \(\eta\) is the canonical quotient functor.

Now we show that the additive functor \(\eta\) defined above and the inclusion functor between additive quotients \(\widehat{\mathcal{H}}/\mathcal{X}\) and \(\widetilde{\mathcal{X}}/\mathcal{X}\) are adjoint.

**Theorem 4.6.** Let \(\mathcal{X}\) be a resolving subcategory of \(\mathcal{T}\) and \(\mathcal{H}\) a \(\text{Ext}\)-injective \(\xi\)-cogenerator of \(\mathcal{X}\). Then, the additive functor \(\eta : \widetilde{\mathcal{X}}/\mathcal{X} \to \widehat{\mathcal{H}}/\mathcal{X}\) defined above is left adjoint to the inclusion functor \(\widehat{\mathcal{H}}/\mathcal{X} \to \widetilde{\mathcal{X}}/\mathcal{X}\).
Proof. Let $K$ be an object in $\overline{\mathcal{H}}$ and $M$ an object in $\mathcal{X}$. By Proposition 3.10, there is a triangle
\[ M \xrightarrow{s} W^M \xrightarrow{i} X^M \xrightarrow{} \Sigma M \]
in $\xi$ with $W^M \in \overline{\mathcal{H}}$ and $X^M \in \mathcal{X}$. Note that the additive map
\[ [s]^* : \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(\eta(M), K) \rightarrow \text{Hom}_{\mathcal{X}/\mathcal{X}}(M, K) \]
is natural in both $M$ and $K$ by Lemma 4.5. We claim that $[s]^*$ is an isomorphism.

Indeed, since $\mathcal{H}$ is a $\xi\text{-}t$-injective cogenerator of $\mathcal{X}$, by Lemma 3.9, we have $\xi\text{-}t(X^M, K) = 0$, and hence, $\text{Hom}_\mathcal{H}(W^M, K) \rightarrow \text{Hom}_\mathcal{H}(M, K)$ is an epimorphism, so $[s]^*$ is still an epimorphism.

Now, assume that $g : W^M \rightarrow K$ is a morphism such that $[gs] = [g][s] = [s][g] = [0] \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(M, K)$. Then, there exists an object $X \in \mathcal{X}$ such that $gs = kv$. Since $\mathcal{H}$ is a $\xi\text{-}t$-injective $\xi$-cogenerator of $\mathcal{X}$, there exists a triangle
\[ X \rightarrow H \rightarrow X' \rightarrow \Sigma X \]
in $\xi$ with $H \in \mathcal{H}$ and $X' \in \mathcal{X}$. Note that $\xi\text{-}t(X^M, H) = 0$ and $\xi\text{-}t(X', K) = 0$, so we get the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{s} & W^M & \xrightarrow{i} & X^M & \xrightarrow{} & \Sigma M \\
\downarrow{v} & & \downarrow{v'} & & \downarrow{w'} & & \\
X & \xrightarrow{k} & H & \rightarrow & X' & \rightarrow & \Sigma X \\
\downarrow{w} & & \downarrow{v''} & & & & \\
K & & & & & & \\
\end{array}
\]

It follows that $[v''v'] = [0] \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(W^M, K)$ as $H \in \mathcal{X}$. Since $v''v's = kv = gs \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(M, K)$, by Lemma 4.5(2), we have $[g] = [v''v'] \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(W^M, K)$, and hence, $[g] = 0$. So $[s]^*$ is a monomorphism, and thus, $[s]^*$ is an isomorphism.

\[ \square \]

Corollary 4.7. Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$ and $\mathcal{H}$ a $\xi\text{-}t$-injective $\xi$-cogenerator of $\mathcal{X}$. Assume that $\mathcal{X}$ is closed under direct summands. For any $N \in \mathcal{X}$, the following statements are equivalent:

(1) $N \in \mathcal{X}$.

(2) There is a triangle
\[ N \xrightarrow{s} W^N \rightarrow X^N \rightarrow \Sigma N \]
in $\xi$ with $W^N \in \overline{\mathcal{H}}$ and $X^N \in \mathcal{X}$ such that $[s] = [0] \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(N, W^N)$.

Proof. The assertion (1) $\Rightarrow$ (2) is obvious. It suffices to show (2) $\Rightarrow$ (1). Note that the adjunction isomorphism established in Theorem 4.6 implies that the additive map
\[ [s]^* : \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(W^N, W^N) \rightarrow \text{Hom}_{\mathcal{X}/\mathcal{X}}(N, W^N) \]
is isomorphic. Since $[s]^*[\text{id}_{W^N}] = [\text{id}_{W^N}s] = [s] = [0] \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(N, W^N) = 0$, so $[\text{id}_{W^N}] = [0] \in \text{Hom}_{\overline{\mathcal{H}}/\mathcal{X}}(W^N, W^N)$, and thus, $\text{id}_{W^N}$ factors through an object $X' \in \mathcal{X}$. It follows that $W^N$ is a direct summand of $X'$. Since $\mathcal{X}$ is closed under direct summands, we have $W^N \in \mathcal{X}$. Thus, $N \in \mathcal{X}$. \[ \square \]
4.2 A characterization of finite resolution dimension via $\xi$-cellular towers

For $M \in \widehat{\mathcal{A}}$, there exists a triangle
\[
K_1 \xrightarrow{f_0} X_0 \xrightarrow{g_0} M \xrightarrow{h_0} \Sigma K_1
\]
(13)
in $\xi$ with $X_0 \in \mathcal{X}$ and $K_1 \in \widehat{\mathcal{X}}$. Similarly, there exists a triangle
\[
K_2 \xrightarrow{f_1} X_1 \xrightarrow{g_1} K_1 \xrightarrow{h_1} \Sigma K_2
\]
in $\xi$ with $X_1 \in \mathcal{X}$ and $K_2 \in \widehat{\mathcal{X}}$. Continuing the above procedure for $K_n$, there exists a triangle
\[
K_{n+1} \xrightarrow{f_n} X_n \xrightarrow{g_n} K_n \xrightarrow{h_n} \Sigma K_{n+1}
\]
in $\xi$ with $X_n \in \mathcal{X}$ and $K_{n+1} \in \widehat{\mathcal{X}}$.

Applying cobase change for the triangle (13) along the morphism $h_1 : K_1 \to \Sigma K_2$, we get the following commutative diagram:

\[
\begin{array}{c}
\Sigma^{-1} M \xrightarrow{u_1} K_1 \xrightarrow{f_0} X_0 \xrightarrow{g_0} M \\
\Sigma^{-1} M \xrightarrow{u_2} \Sigma K_2 \xrightarrow{u_2} C_2 \xrightarrow{v_2} M \, ,
\end{array}
\]

where the triangle
\[
\Sigma K_2 \xrightarrow{u_2} C_2 \xrightarrow{v_2} M \to \Sigma^2 K_2
\]
is in $\xi$. Next consider the triangle (14) along the morphism $-\Sigma h_2 : \Sigma K_2 \to \Sigma^2 K_3$, we get the following commutative diagram:

\[
\begin{array}{c}
\Sigma^{-1} M \xrightarrow{u_2} \Sigma K_2 \xrightarrow{u_2} C_2 \xrightarrow{v_2} M \\
\Sigma^{-1} M \xrightarrow{u_3} \Sigma^2 K_3 \xrightarrow{u_3} C_3 \xrightarrow{v_3} M \, ,
\end{array}
\]

where the triangle $\Sigma^2 K_3 \xrightarrow{u_3} C_3 \xrightarrow{v_3} M \to \Sigma^3 K_3$ is in $\xi$.

Continuing in this manner, we obtain the following commutative diagram:

\[
\begin{array}{c}
K_1 \xrightarrow{f_0} X_0 \xrightarrow{g_0} M \xrightarrow{h_0} \Sigma K_1 \\
\Sigma K_2 \xrightarrow{u_2} C_2 \xrightarrow{v_2} M \to \Sigma^2 K_2 \\
\Sigma^2 K_3 \xrightarrow{u_3} C_3 \xrightarrow{v_3} M \to \Sigma^3 K_3 \\
\Sigma^{n-1} K_n \xrightarrow{u_n} C_n \xrightarrow{v_n} M \to \Sigma^n K_n
\end{array}
\]

where all the horizontal triangles are in $\xi$. 

Set $C_0 = 0$ and $C_1 = X_0$. The above construction produces a tower
$$0 \longrightarrow C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} \cdots \longrightarrow C_{n-1} \xrightarrow{\gamma_{n-1}} C_n \cdots,$$
which we call the $\xi$-cellular tower of $M$ with respect to $X$.

According to the above construction, one can obtain the following result by Proposition 3.3.

**Theorem 4.8.** Let $X$ be a resolving subcategory of $\mathcal{T}$. For any $M \in \mathcal{T}$, if $M \in \hat{X}$, then the following statements are equivalent:
1. $X$-res.dim $M \leq n$.
2. For each $i > 0$, the morphisms $\nu_{n+1} : C_{n+i} \to M$ of the $\xi$-cellular tower of $M$ with respect to $X$ constructed above are isomorphisms.

**5 Applications**

In this section, we will construct a new resolving subcategory from a given resolving subcategory, which generalizes the notion of $\xi$-Gorenstein projective objects given by Asadollahi and Salarian [13]. By applying the previous results to this subcategory, we obtain some known results in [13–15].

**Definition 5.1.** Let $X$ be a subcategory of $\mathcal{T}$ and $M$ an object in $\mathcal{T}$. A complete $P(\xi)$-$X$-resolution of $M$ is a Hom$_{\mathcal{F}}(-, X)$-exact $\xi$-exact complex
$$
\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots
$$
in $\mathcal{T}$ with all $P_i \in P(\xi)$, $X^i \in X \cap \perp X$ such that both
$$K_i \longrightarrow P_0 \longrightarrow M \longrightarrow \Sigma K_i \quad \text{and} \quad M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M$$
are corresponding triangles in $\xi$. The $GP_X(\xi)$-Gorenstein category is defined as
$$GP_X(\xi) = \{ M \in \mathcal{T} | M \text{ admits a complete } P(\xi)X \text{-resolution} \}.$$

**Remark 5.2.**
1. Since $X$ is a resolving subcategory of $\mathcal{T}$, we have $P(\xi) \subseteq X$, so $P(\xi) \subseteq X \cap \perp X$. Then, we have $K_i \in GP_X(\xi)$.
2. If $M \in GP_X(\xi)$, then $\xi xt^0_\xi(M, X) = \text{Hom}_\mathcal{T}(M, X)$ and $\xi xt^1_\xi(M, X) = 0$ for any $X \in X$. In fact, the following $\xi$-exact complex:
$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$
is a $\xi$-projective resolution of $M$ (see [11]), which is Hom$_{\mathcal{F}}(-, X)$-exact.

Evidently, $M \in GP_X(\xi)$ if and only if $\xi xt^0_\xi(M, X) = \text{Hom}_\mathcal{T}(M, X)$ and $\xi xt^1_\xi(M, X) = 0$ for any $X \in X$, and $M$ admits a Hom$_{\mathcal{F}}(-, X)$-exact $\xi$-exact complex
$$0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow \cdots$$
with $X^i \in X \cap \perp X$.
3. If $X = P(\xi)$, then we have $X \cap \perp X = P(\xi)$ by Lemma 3.12, and thus, $GP_X(\xi)$ coincides with $GP(\xi)$ defined in [13].
Theorem 5.3. Let $X$ be a resolving subcategory of $\mathcal{T}$. Then, $\mathcal{GP}_X(\xi)$ is a resolving subcategory of $\mathcal{T}$.

Proof. Let $P$ be a $\xi$-projective object. Consider the following $\xi$-exact complex:

$$
\cdots \to 0 \to P \xrightarrow{id} P \xrightarrow{0} 0 \to \cdots
$$

in $\mathcal{T}$. Clearly, it is $\text{Hom}_\mathcal{T}(\xi, X)$-exact. In particular,

$$
0 \to P \xrightarrow{id} P \xrightarrow{0} 0 \quad \text{and} \quad P \xrightarrow{id} P \xrightarrow{0} 0 \xrightarrow{0} \Sigma P
$$

are corresponding triangles in $\xi$. Since $P \in X \cap ^+X \cap \mathcal{GP}_X(\xi)$. We have $\mathcal{P}(\xi) \subseteq \mathcal{GP}_X(\xi)$.

As a similar argument to the proof of [18, Theorem 4.3(1)], we obtain that $\mathcal{GP}_X(\xi)$ is closed under $\xi$-extensions and hokernels of $\xi$-proper epimorphisms. Thus, $\mathcal{GP}_X(\xi)$ is a resolving subcategory of $\mathcal{T}$. □

Lemma 5.4. Let $X$ be a resolving subcategory of $\mathcal{T}$ satisfying $X \cap ^+X \subseteq \mathcal{GP}_X(\xi)$. Then, $X \cap ^+X$ is a $\xi$-injective $\xi$-cogenerator of $\mathcal{GP}_X(\xi)$ and is closed under hokernels of $\xi$-proper epimorphisms.

Proof. Let $M \in \mathcal{GP}_X(\xi)$. There is a $\text{Hom}_\mathcal{T}(\xi, X)$-exact triangle

$$
M \longrightarrow X^0 \longrightarrow K^1 \longrightarrow \Sigma M
$$

in $\xi$ with $X^0 \in X \cap ^+X \subseteq \mathcal{GP}_X(\xi)$. For any $\tilde{X} \in X$, applying the functor $\text{Hom}_\mathcal{T}(\xi, \tilde{X})$ to the triangle (15) yields the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \text{Hom}_\mathcal{T}(K^1, \tilde{X}) & \longrightarrow & \text{Hom}_\mathcal{T}(X^0, \tilde{X}) & \longrightarrow & \text{Hom}_\mathcal{T}(M, \tilde{X}) & \longrightarrow & 0 \\
\downarrow & & \cong & & \downarrow & & \cong & & \\
0 & \longrightarrow & \xi \text{xt}^1_\xi(K^1, \tilde{X}) & \longrightarrow & \xi \text{xt}^0_\xi(X^0, \tilde{X}) & \longrightarrow & \xi \text{xt}^0_\xi(M, \tilde{X}) & \longrightarrow & \xi \text{xt}^1_\xi(K^1, \tilde{X}) & \longrightarrow & \xi \text{xt}^1_\xi(X^0, \tilde{X})(= 0),
\end{array}
$$

where the two isomorphisms follow from the assumption that $X^0, M \in \mathcal{GP}_X(\xi)$ and Remark 5.2(2). It follows that \(\xi \text{xt}^1_\xi(K^1, \tilde{X}) = 0\) and \(\xi \text{xt}^0_\xi(K^1, \tilde{X}) \cong \text{Hom}_\mathcal{T}(K^1, \tilde{X})\), so $K^1 \in \mathcal{GP}_X(\xi)$ by Remark 5.2(2), then $X \cap ^+X$ is a $\xi$-cogenerator of $\mathcal{GP}_X(\xi)$. Obviously, $X \cap ^+X$ is a $\xi$-injective $\xi$-cogenerator of $\mathcal{GP}_X(\xi)$.

It is obvious that $X \cap ^+X$ is closed under hokernels of $\xi$-proper epimorphisms. □

As an application of Theorem 3.14, we have:

Proposition 5.5. Let $X$ be a resolving subcategory of $\mathcal{T}$ satisfying $X \cap ^+X \subseteq \mathcal{GP}_X(\xi)$ and $M \in \mathcal{T}$. If $M \in \mathcal{GP}_X(\xi)$, then the following statements are equivalent:

1. $\mathcal{GP}_X(\xi)$-$\text{res.dim} M \leq m$.
2. $\Omega^n(M) \in \mathcal{GP}_X(\xi)$ for all $n \geq m$.
3. $L^\xi(M) \subseteq \mathcal{GP}_X(\xi)$ for all $n \geq m$.
4. $\text{xt}^n_\xi(M, H) = 0$ for all $n > m$ and all $H \in X \cap ^+X$.
5. $\text{xt}^n_\xi(M, L) = 0$ for all $n > m$ and all $L \in X \cap ^+X$.
6. $M$ admits a right $\mathcal{GP}_X(\xi)$-approximation $\varphi : X \to M$, where $\varphi$ is $\xi$-proper epic, such that $K = \text{Hoker} \varphi$ satisfying $\mathcal{H}$-$\text{res.dim} K \leq m - 1$.
7. There are two triangles

$$
W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M
$$

and

$$
M \longrightarrow W_M \longrightarrow X_M \longrightarrow \Sigma M
$$

in $\xi$ such that $X_M, X_M \in \mathcal{GP}_X(\xi)$ and $X \cap ^+X$-$\text{res.dim} W_M \leq m - 1$, $X \cap ^+X$-$\text{res.dim} W_M = \mathcal{GP}_X(\xi)$-$\text{res.dim} W_M \leq m$. 

Immediately, we have:

**Corollary 5.6.** Let $\mathcal{T}$ be a triangulated category and $M \in \mathcal{T}$. If $M \in G\mathcal{P}(\xi)$, then the following statements are equivalent:

1. $G\mathcal{P}(\xi)\text{-res.dim } M \leq m$.
2. $\Omega^n(M) \in G\mathcal{P}(\xi)$ for all $n \geq m$.
3. $\Omega^G_{\mathcal{P}(\xi)}(M) \in G\mathcal{P}(\xi)$ for all $n \geq m$.
4. $\xi t^H(M, H) = 0$ for all $n > m$ and all $P \in \mathcal{P}(\xi)$.
5. $\xi t^H(M, L) = 0$ for all $n > m$ and all $L \in \mathcal{P}(\xi)$.
6. $M$ admits a $G\mathcal{P}(\xi)$-approximation $\varphi : X \to M$, where $\varphi$ is $\xi$-proper epic, such that $K = \text{Hoker } \varphi$ satisfying $\xi$-pd $K \leq m - 1$.
7. There are two triangles

$$W_M \longrightarrow X_M \longrightarrow M \longrightarrow \Sigma W_M$$

and

$$M \longrightarrow W^M \longrightarrow X^M \longrightarrow \Sigma M$$

in $\xi$ such that $X_M$ and $X^M$ are in $\mathcal{X}$ and $\xi$-pd $W_M \leq m - 1$, $\xi$-pd $W^M = G\mathcal{P}(\xi)\text{-res.dim } W^M \leq m$.

**Remark 5.7.** As in Corollary 5.6, (1) $\iff$ (2) $\iff$ (6) is [13, Theorem 4.6 (ii) $\iff$ (iii) $\iff$ (iv)], (1) $\iff$ (5) is [13, Proposition 3.19]. (1) $\iff$ (4) is [14, Remark 2.14].

Following Theorems 4.8 and 5.3, we have the following result, which is a generalization of [15, Proposition 5.1].

**Proposition 5.8.** Let $\mathcal{X}$ be a resolving subcategory of $\mathcal{T}$. For any $M \in \mathcal{T}$, if $M \in G\mathcal{P}_{\mathcal{X}}(\xi)$, then the following statements are equivalent:

1. $G\mathcal{P}_{\mathcal{X}}(\xi)\text{-res.dim } M \leq n$.
2. For each $i > 0$, the morphisms $v_{n+i} : c_{n+i} \to M$ of the $\xi$-cellular tower of $M$ with respect to $G\mathcal{P}_{\mathcal{X}}(\xi)$ constructed above are isomorphisms.

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