Duality between coherent quantum phase slip and Josephson junctions in nanosheets by the dual Hamiltonian method

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The duality between coherent quantum phase slip and Josephson junction in nanosheets was investigated by dual Hamiltonian method. This is equivalent to the duality between superconductivity and superinsulator in the 2 + 1d at zero temperature. This method proved to be reliable within the Villain approximation. The possibility of the dual Ginzburg Landau theory, which is the phenomenology of superinsulators and the dual BCS theory, which is a microscopic theory, also shown.

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1. INTRODUCTION

In recent decades, two types of phenomena, which are considered to be dual states in superconductivity, have attracted attention. One of them is a phenomenon called coherent quantum phase slip, which is mainly known as a dual phenomenon of a Josephson junction (JJ) in a one-dimensional superconducting system. The other is a phenomenon called superinsulator, which is mainly known as a dual phenomenon to superconductors. A theoretical study of coherent quantum phase slip was submitted by E. Mooij et al. As flux quanta through nanosuperconducting wires, and an experimental demonstration of coherent quantum phase slip was performed by Astafiev et al. which embedded in a large superconducting loop with InOx Achieved in wire. On the other hand, the concept of superinsulator was first conceived in 1978 by ’t Hooft as a theory explaining quark confinement. The superinsulator in the case of condensed matter was rediscovered by Diamantini et al. in 1996 as a periodic mixed Chern-Simons Abelian gauge theory describing charge-vortex coupling equivalent to planar Josephson junction arrays in a self-dual approximation. Then in 1998, A Kramer and S Doniach also rediscovered superinsulators from a phenomenological model in which vortices had a finite mass and traveled in a dissipative environment. Further, in 2012, M. Yoneda et al. rediscovered superinsulators from superinsulator / superconductor / superinsulator junction that is as the dual junction to Josephson junction. Experimentally, TiN, InOx and NbTiN films are currently known as observed super insulating materials. Recently, superinsulators have attracted attention as a powerful desktop environment for QCD phenomena. For realizing a single-color version of quantum chromodynamics and elucidating quark confinement and asymptotic freedom. In the previous paper, we introduced two Hamiltonians dual to each other for 1d nanowire-based Josephson junction and quantum phase slip junction (QPSJ), and applied dual conditions between the current and voltage of the electric circuit. A general theory to construct a dual system called dual Hamiltonian method was proposed. Furthermore, a rough discussion on the superconductivity-superinsulator transition in a 1 + 1d system at zero temperature from these two Hamiltonians was conducted. Our results show that in the 1 + 1d system at zero temperature, coherent quantum phase slip and superinsulator are completely equivalent phenomena. The main purpose of this work is to extend the theory of 1 + 1d systems on nanowires shown in the previous paper to the theory of 2 + 1d systems on nanosheets. The rest of this paper is organized as follows: In Section 2, we use the dual Hamiltonian method for between the JJ model and the QPSJ model on the nanosheet at zero temperature, to derive a between the phase and the amplitude dual relations, and a dual relations between the various constants. In Section 3, based on the nonlinear Legendre transformation between the Lagrangians and the Hamiltonians with canonical conjugate variables of infinite order in a compact lattice space, and between the phase and the amplitude relationship derived in the previous section, we show that there is exact duality between the JJ model and the QPSJ model. In addition, the phase diagram between the JJ state (superconducting state) and the QPSJ state (superinsulating state) was discussed. In Section 4, we prove the validity of the results of the previous section by deriving a dual transformation from the anisotropic XY (AXY) model to the gauged dual anisotropic XY (DAXY) model by the Villain approximation in the 2 + 1d system. In Section 5, contrary to Section 4, we derive a dual transformation from the DAXY model to the AXY model with gauge by the Villain approximation in 2 + 1d system. In Section 6, Dual Ginzburg-Landau (DGL) theory was derived from the mean field approximation of the gauged QPSJ model, and the possibility of confinement of electric flux in the superinsulator was shown. In section 7, we calculated two loop corrections to the mean field approximation for the critical value of the QPS amplitude. For Summary and discussion in Section 8, summarizing the conclusions of this paper, discussing whether the minimum unit of charge confinement in a super insulator is 2e or e, finally, the possibility of dual BCS theory is described. In
Appendix A, the numerical evaluation of the anisotropic massless lattice Green function is shown at the origin of $x=0$, which is necessary for loop correction. In Appendix B, the effective energy approach for the QPSJ model was explained.

2. DUAL HAMILTONIAN METHOD BETWEEN JJ MODEL AND QPSJ MODEL ON THE NANOSHEET AT ZERO TEMPERATURE

Consider a checkered nanosheet consisting of superconductors and superinsulators as shown in FIG. 1. Here, a black square unit is a superconductor, and a white square unit is a superinsulator. In such a 2$d$ checked nanosheet,

![checkered nanosheet](image)

FIG. 1: The checkered nanosheet by the superconductors and superinsulators.

When the superconductor region acts strongly, the JJ state becomes strong, and the Hamiltonian in this case is as follows:

$$H_{JJ} = E_c \sum_{x=1}^{M} [N_\theta (x) ]^2 + E_d \sum_{x=1}^{M} \sum_{j=1}^{2} [1 - \cos \nabla_j \theta (x) ],$$  \hspace{1cm} (1)

where $N_\theta (x) \equiv N_\theta (x, \tau)$ means the particles number of the Cooper pair that is canonical conjugate to the phase $\theta$ of the Cooper pair, and the space difference of the phase $\theta (x) \equiv \theta (x, \tau)$ is defined by $\nabla_j \theta (x, \tau) \equiv \theta (x, \tau) - \theta (x-ae, \tau)$. In addition, $\tau \equiv h/\beta$ $(\beta \equiv 1/k_BT$ is the reverse temperature), $x = (x_1, x_2)$ and $M \equiv L^2/a^2$ $(L$ and $a$ are space length and lattice spacing, respectively) are the imaginary time, the lattice points and the total number of lattices in the two-dimensional lattice space, respectively. $E_c \equiv (2e)^2/2C$ and $E_d \equiv \Phi_0 I_c/2\pi$ are charging energy per Cooper pair and Josephson energy, respectively, where $C, I_c$ and $\Phi_0 = h/2e$ are the capacitance, the critical current and the magnetic flux-quantum, respectively. From the Hamiltonian of Eq. (1), Josephson’s equations are as follows:

$$V (t) = \frac{\hbar}{2e} \frac{\partial \theta (x)}{\partial t} = \frac{2N_\theta (x)}{2e} E_c,$$

$$I_j (x) = -\frac{2\pi}{\Phi_0} E_j \sin \nabla_j \theta (x).$$  \hspace{1cm} (2)

Where, $V (x)$ and $I_j (x)$ $(j=1, 2)$ are the voltage and the $j$ components of current in $JJ$, respectively. On the other hand, when the superinsulator region acts strongly, the QPSJ state becomes strong, and the Hamiltonian in this case is as follows:

$$H_{QPSJ} = E_L \sum_{x=1}^{M} [\tilde{N}_\theta (x) ]^2 + E_S \sum_{j=1}^{M} \sum_{x=1}^{M} [1 - \cos \nabla_j \tilde{\theta} (x) ],$$  \hspace{1cm} (3)

where, $\tilde{N}_\theta (x) \equiv \tilde{N}_\theta (x, \tau)$ means the magnetic flux quantum numbers $\tilde{\theta} (x) \equiv \tilde{\theta} (x, \tau)$ of the magnetic flux quantum field. $E_L \equiv \Phi_0^2/2L$ and $E_S \equiv 2eV_c/2\pi$ are inductive energy per magnetic flux quantum and the QPS amplitude, respectively, where $V_c$ is the critical voltage. From the Hamiltonian of Eq. (3), the dual Josephson equations are as follows:

$$\tilde{V} (x) = \frac{\hbar}{\Phi_0} \frac{\partial \tilde{\theta} (x)}{\partial \tau} = \frac{2\tilde{N}_\theta (x)}{\Phi_0} E_L,$$

$$\tilde{I}_j (x) = \frac{2\pi}{e} E_S \sin \nabla_j \tilde{\theta} (x).$$  \hspace{1cm} (4)

Where, $\tilde{V} (x)$ and $\tilde{I}_j (x)$ $(j=1, 2)$ are the dual voltage and the $j$ components of dual current in QPSJ, respectively. As the first step of the dual Hamiltonian method, two dual conditions between equation Eq.(2) and the dual equations of Eq.(4) are assumed as follows:

$$V (x) \equiv \tilde{I} (x), \; I (x) \equiv \tilde{V} (x).$$  \hspace{1cm} (5)

where, $I \equiv \sqrt{I_1^2 + I_2^2}$ and $\tilde{I} \equiv \sqrt{\tilde{I}_1^2 + \tilde{I}_2^2}$ are intensity of current in $JJ$ and intensity of the dual current in $QPSJ$, respectively. When the condition of Eq.(5) is imposed between Eq. (3) and (4), the following two relational expressions between phase and number of particles between dual systems are derived as shown below. One of them is the relationship between the phase $\theta$ of the Cooper pair and the magnetic flux quantum numbers $\tilde{N}_\theta$, and the other is the relationship between the phase $\tilde{\theta}$ of the magnetic flux quantum and the number $N_\theta$ of the Cooper pair, as follows:

$$N_\theta (x) = \frac{1}{2\pi} \sqrt{\sum_{j=1}^{2} \sin^2 \nabla_j \theta (x)},$$

$$\tilde{N}_\theta (x) = \frac{1}{2\pi} \sqrt{\sum_{j=1}^{2} \sin^2 \nabla_j \tilde{\theta} (x)}.$$  \hspace{1cm} (6)

In Eq.(6), the first equation is shown that the Cooper pair number is proportional to the strength of the flux quantum current, and the next equation is shown that the flux quantum number is proportional to the strength of the Cooper pair current. Since the Cooper pair current and the flux quantum current form a closed loop, the number of Cooper pairs and the flux quantum number can be considered as the winding number of the current loop formed from each dual current. If the relationships described in Eq.(6) are satisfied, the relationship between the QPS amplitude and charging energy per single-charge, and the relationship between Josephson
energy and inductive energy per magnetic flux-quantum, are as follows:

$$E_S = \frac{E_c}{2\pi^2}, \quad E_J = \frac{E_L}{2\pi^2}.$$ (7)

Furthermore, inductance and capacitance are related to the critical current and the critical voltage, respectively, as follows:

$$L = \frac{\Phi_0}{2\pi I_c}, \quad C = \frac{2e}{2\pi V_c},$$ (8)

The Lagrangians at zero temperature in Eq.(1) and Eq.(3) are as follows:

$$L_{JJ} = -\sum_{x=1}^{M} \left\{ \frac{E_0^J}{2} (\nabla_r \theta(x))^2 + E_J \sum_{j=1}^{2} \left[ 1 - \cos \nabla_r \theta(x) \right] \right\},$$ (9)

$$L_{QPS} = -\sum_{x=1}^{M} \left\{ \frac{E_0^Q}{2} (\nabla_r \theta(x))^2 + E_Q \sum_{j=1}^{2} \left[ 1 - \cos \nabla_r \theta(x) \right] \right\},$$ (10)

where $E_0^J$ and $E_0^Q$ can be considered as the imaginary time components of $JJ$ and $QPS$, respectively, and it is defined as follows:

$$E_0^J \equiv \frac{h^2}{2a_0^2 E_c}, \quad E_0^Q \equiv \frac{h^2}{2a_0^2 E_L},$$ (11)

and $a_0 \equiv \tau_{\text{max}}/M_r$ is an imaginary time spacing of time dimension, $\tau_{\text{max}}$ and $M_r$ are an imaginary time length and an imaginary time division number, respectively. Eq.(8) and (11) can be summarized as a relation between $JJ$ energy and $QPS$ energy as follows:

$$E_0^J \equiv \frac{1}{4\pi^2 E_S'}, \quad E_0^Q \equiv \frac{1}{4\pi^2 E_J'},$$ (12)

where $E_J' \equiv E_{J,a_0/h}, \quad E_0^J \equiv E_J^0 a_0/h, \quad E'_{S} \equiv E_{S,a_0/h}$ and $E_0^Q \equiv E_{Q,a_0/h}$ represent the nondimensional energies, respectively. The first terms of Eq.(9) and (10) are expressed in a quadratic form for the imaginary time difference of each phase, but these second terms are expressed in a cosine form for the spatial difference of each phase. Here, the cosine form also introduce approximately to the first term of Eq.(9) and (10) in consideration of the periodicity in the lattice space as follows:

$$L_{AXY} = -\sum_{x=1}^{M} \left\{ E_0^J \left[ 1 - \cos \nabla_r \theta(x) \right] + E_J \sum_{j=1}^{2} \left[ 1 - \cos \nabla_r \theta(x) \right] \right\},$$ (13)

$$L_{DAXY} = -\sum_{x=1}^{M} \left\{ E_0^Q \left[ 1 - \cos \nabla_r \theta(x) \right] + E_Q \sum_{j=1}^{2} \left[ 1 - \cos \nabla_r \theta(x) \right] \right\},$$ (14)

where, $L_{AXY}$ and $L_{DAXY}$ are equivalent to $L_{JJ}$ and $L_{QPS}$, respectively, and anisotropic $XY$ model (AXY) and dual anisotropic $XY$ model (DAXY) model, respectively. From the Lagrangian in Eq.(13), the partition function of $JJ$ state (superconducting state) at zero temperature is as follows:

$$Z_{AXY} \equiv \exp \left\{ -\left( E_0^J + E_J' \right) M M_r \right\} Z'_{AXY},$$ (15)

$$Z'_{AXY} \equiv \int D\theta \exp \left\{ E_0^J \cos \nabla_r \theta(x) + E_J' \sum_{j=1}^{2} \cos \nabla_r \theta(x) \right\}.$$ (16)

The partition functions $Z_{AXY}$ and $Z_{DAXY}$ in Eq.(15) and (16) are the starting points for the analysis in the following sections.

### 3. Dual Transformation Between the JJ Model and the QPSJ Model on the Nanosheet

This section describes the nonlinear Legendre transformation between the Hamiltonian and the Lagrangian with canonical conjugate variables of the infinite order in a compact $2+1d$ lattice space on a nanosheet, and we show that there is exact duality between the $JJ$ model and the $QPSJ$ model. For the partition functions $Z_{JJ}$ in Lagrangian of Eq.(8), the auxiliary field $N(x)$ by Hubbard-Stratonovich transformation is introduced as follows:

$$Z_{JJ} = \int D\theta \exp \left\{ \int \left[ iN(x) \nabla_r \theta(x) - E_0^J N(x)^2 - E_J' \sum_{j=1}^{2} \left[ 1 - \cos \nabla_r \theta(x) \right] \right] \right\},$$ (17)

It can be seen that the auxiliary field $N(x)$ is the same as the number $N_0(x)$ of the cooper pairs introduced in Eq.(8). The canonical conjugate momentum $p_{\theta}(x)$ with respect to $\theta(x)$ in the Lagrangian of the lattice space of Eq.(8) is defined as the following equation.

$$i p_{\theta}(x) \equiv i h N(x) = -a_0 E_0^J \nabla_r \theta(x).$$ (18)

On the other hand, for the partition functions $Z_{AXY}$ in Eq.(15), the auxiliary field $N(x)$ by Hubbard-Stratonovich transformation is introduced as follows:

$$Z_{AXY} = \int D\theta \exp \left\{ \int 2N(x) \sin(\nabla_r \theta(x)/2) - E_0^J N(x)^2 - E_J' \sum_{j=1}^{2} \left[ 1 - \cos \nabla_r \theta(x) \right] \right\},$$ (19)

where, $E_0^J$ and $E_J'$ are the critical current and the critical voltage, respectively, and the auxiliary field $N(x)$ is expressed as the number of cooper pairs introduced in Eq.(15).
The canonical conjugate momentum $p_\theta(x)$ with respect to $\theta(x)$ in the Lagrangian of the compact lattice space of Eq. (14) is defined as the following equation:

$$i\eta_\theta(x) \equiv i\hbar N_\theta(x) = -2\alpha_0 E_j^0 \sin \left[ \nabla, \theta(x)/2 \right].$$  \hspace{1cm} (20)

In Eq. (20), if the linear approximation of $\sin(\nabla, \theta/2) \approx \nabla, \theta/2$ holds, it matches the canonical conjugate momentum in Eq. (15). Therefore, the contents of the curly bracket of the exponential function between Eq. (13) and (14) can be considered as the nonlinear Legendre transformation introduced by the canonical conjugate momentum of Eq. (18). Similarly, for the partition function $Z_{\text{DAXY}}$ in Eq. (14), the auxiliary field $\tilde{N}(x)$ by Hubbard-Stratonovich transformation is introduced as follows:

$$Z_{\text{DAXY}} = \int d\theta d\tilde{N} \exp \left\{ i\tilde{N}(x) \sin \left[ \nabla, \theta(x)/2 \right] - E_L\tilde{N}(x)^2 \right\} - E_S^x \sum_{j=1}^2 \left[ 1 - \cos \nabla, \tilde{\theta}(x) \right],$$  \hspace{1cm} (21)

It can be seen that the auxiliary field $\tilde{N}(x)$ is the same as the magnetic flux quantum numbers $\tilde{N}_\theta(x)$ in Eq. (3). The canonical conjugate momentum $\tilde{p}_\theta(x)$ with respect to $\tilde{\theta}(x)$ in the Lagrangian of the compact lattice space of Eq. (14) is defined as the following equation:

$$i\tilde{p}_\theta(x) \equiv i\hbar \tilde{N}_\theta(x) = -2\alpha_0 E_S^0 \sin \left[ \nabla, \tilde{\theta}(x)/2 \right].$$  \hspace{1cm} (22)

Therefore, the contents of the curly bracket of the exponential function between Eq. (14) and (21) can also be considered as the nonlinear Legendre transformation introduced by the canonical conjugate momentum of Eq. (22). Since Eq. (20) and (22) are a half-angle relation, a half-angle version of Eq. (3) is introduced as follows:

$$N_\theta(x) = \frac{1}{2\pi} \sum_{j=1}^2 \sin^2 \left[ \nabla, \tilde{\theta}(x)/2 \right],$$

$$\tilde{N}_\theta(x) = \frac{1}{2\pi} \sum_{j=1}^2 \sin^2 \left[ \nabla, \tilde{\theta}(x)/2 \right].$$  \hspace{1cm} (23)

Eq. (23) is equivalent to Eq. (3) within the range of linear approximation. From Eq. (23), (21) and (22), the relationship between the phase $\theta(x)$ of the Cooper pair and the phase $\tilde{\theta}(x)$ of the magnetic flux quantum field is as follows:

$$N_\theta^2(x) = \frac{4(\alpha_0 E_j^0)^2}{\hbar^2} \sin^2 \left[ \nabla, \theta(x)/2 \right] = \frac{1}{\pi} \sum_{j=1}^2 \sin^2 \left[ \nabla, \tilde{\theta}(x)/2 \right],$$

$$\tilde{N}_\theta^2(x) = \frac{4(\alpha_0 E_j^0)^2}{\hbar^2} \sin^2 \left[ \nabla, \tilde{\theta}(x)/2 \right] = \frac{1}{\pi} \sum_{j=1}^2 \sin^2 \left[ \nabla, \theta(x)/2 \right].$$  \hspace{1cm} (24)

Using Eq. (3), (11) and (24), one obtains the following relation:

$$E_L \sum_x \left[ N_\theta(x)^2 \right] = -E_S \sum_x \left[ 1 - \cos \nabla, \theta(x) \right] = E_S \sum_x \sum_{j=1}^2 \left[ 1 - \cos \nabla, \tilde{\theta}(x) \right].$$  \hspace{1cm} (25)

Eq. (25) means that the charging energy by the charge $2e$ in JJ is equal to the $QPSJ$, that is, the condensation energy of the magnetic flux. Eq. (23) means that the charging energy by the flux quantum $\Phi_0$ in the $QPSJ$ is equal to the JJ energy, that is, the condensation energy of the Cooper pair. By establishing the relationship of Eq. (23) and (22), between the Hamiltonian Eq. (4) and (5), and between the Lagrangian Eq. (13) and (14), it shows that each of them is self-dual. Thus, from the canonical conjugate momentum of Eq. (21) and (22), and the relationship between the phase and amplitude of Eq. (1) derived in the previous section, an exact duality between the JJ and $QPSJ$ models has been proven. From the Josephson’s equations Eq. (2) and dual Josephson’s equations Eq. (1), the electrical conductance $G$ derived as follows:

$$G = G_Q \frac{dN_\theta(x)}{d\tilde{N}_\theta(x)} = G_Q \frac{E_J \tilde{N}_\theta(x)}{E_S N_\theta(x)},$$  \hspace{1cm} (27)

where $G_Q \equiv (2e)^2/h$ is the quantum conductance. From the Eq. (27), the following elliptical orbit can be drawn:

$$g^{-1} \eta_\theta^2 + g \tilde{N}_\theta^2 = \frac{1}{\eta},$$

$$g \equiv (E_J/E_S)^{1/2}, \quad \eta \equiv \sqrt{E_J E_S}/\Gamma,$$  \hspace{1cm} (28)

where, $\Gamma$ is an arbitrary integral constant with a energy dimension. From Eq. (27), we can draw an elliptical orbit as shown in FIG. 2. FIG. 2 (a) means a JJ junction state (superconducting state) for $E_J \gg E_S$, FIG. 2 (b) means a Quantum Hall state (Bose semiconductor state) for $E_J = E_S$, and FIG. 2 (c) means a $QPSJ$ junction state (superinsulating state) for $E_J \ll E_S$. The results shown in FIG 2 are very similar to those in the reference. Compare the result of Eq. (27) with the result of the quantum Hall effect shown below.

$$G = G_Q \nu,$$  \hspace{1cm} (29)

The $\nu$ is the Landau level occupancy defined as:

$$\nu \equiv \frac{N_c}{N_\theta},$$  \hspace{1cm} (30)

where, $N_c$ and $N_\theta$ are the electron number and the number of magnetic fluxes (vortex number), respectively. In our model, the correspondence is $N_c \rightarrow N_\theta$ and $N_\theta \rightarrow \tilde{N}_\theta$. When the Hall conductivity of the quantum Hall effect in the bulk is derived by the Kubo formula, the Landau level occupancy $\nu$ can also be expressed as the topological quantum number called the Chern number. From Eq. (23), (27), (29) and (30), it can be seen that the quantum Hall state is obtained if the following relational expression is satisfied:

$$E_J \left[ 1 - \cos \theta(x) \right] = E_S \left[ 1 - \cos \tilde{\theta}(x) \right].$$  \hspace{1cm} (31)
Schematic phase diagram with $\eta > 0$, insulator phase and a topological metal phase exist. If $\nu$ is represented by the Chern number, it is expected that in the vicinity of the region where Landau level occupancy is high, $E_J \ll E_S$. On the other hand, if $\eta > g$, and if $E_J = E_S$, $n_{\eta}$, and if $\eta > 1$, $g \equiv (E_J/E_S)^{1/2} \geq 1$, it means a Bose metal [25]. FIG. 3 shows a schematic phase diagram of $g$ versus $\eta$.

FIG. 2: Elliptic orbit with $N_0$ vs $\tilde{N}_0$

Eq. (31) means that the energy of the Josephson junction is equal to the energy of the quantum phase slip. In the vicinity of the region where Landau level occupancy $\nu$ is represented by the Chern number, it is expected that not only the quantum Hall phase but also a topological insulator phase and a topological metal phase exist. If $\eta > 1$, $g \equiv (E_J/E_S)^{1/2} \leq 1$, it means a Bose insulator [25] and if $\eta > 1$, $g \equiv (E_J/E_S)^{1/2} \geq 1$, it means a Bose metal [25].

4. DUAL TRANSFORMATION FROM THE AXY MODEL TO THE GAUGED DAXY MODEL BY VILLAIN APPROXIMATION IN 2 + 1d SYSTEM

In this section, we find the dual transformation between the AXY model and the DAXY model from the Villain approximation. First, apply the Villain approximation to $Z'_{AXY}$ introduced in Eq. (32) as follows:

$$Z_{QV} \equiv R_Q \int D\theta \exp \left\{ \sum_{a,E} \left( - \frac{(E_a-E)}{2} \right) \left( \nabla \theta - 2\pi n_a \right) \right\}$$

where $Z_{QV}$ is the Villain approximations of the partition function $Z'_{AXY}$, and $R_Q \equiv [R_v(E)^2 R_v(E')^2]_{MM}$ is Villain's normalization parameter, where $R_v(E) \equiv \sqrt{2\pi E} I_0(E)$, and $(E_v) \equiv -2\ln[I_0(E)/I_1(E)]$. $I_0(E)$ and $I_1(E)$ represent modified Bessel functions of order zero and order one, respectively. The summation symbols

$$\sum_{n_{\eta}} \sum_{x,\tau} \sum_{j=0}^{\infty} \sum_{\nu_j(x,\tau) = -\infty}$$

are used for the integer fields $n_{\eta_j} (x, \tau)$. The partition function in Eq. (32) is equivalent to the Euclidean version of the quantum vortex dynamics in a film of superfluid helium introduced by Kleiner [26]. For Eq. (32), the following identities associated with the Jacobi theta function are used:

$$\sum_{n_{\eta}} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{2\pi n} \exp \left[ \frac{-b^2}{2E} + ib\theta \right] \right\} = \sum_{n_{\eta}} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{2\pi n} \exp \left[ \frac{-b^2}{2E} + ib\theta \right] \right\}$$

As a result, Eq. (32) can be rewritten as follows:

$$Z_{QV} = C_{QV} \sum_{j=0}^{\infty} \delta_{\nu_j,0} \sum_{x,\tau} \left[ -\frac{1}{2(E_j^2)} v^2 + \frac{1}{2(E_j^2)} \sum_{j=1}^{\infty} \frac{b_j^2}{n_{\eta}} \right]$$

where $C_{QV}$ is normalization parameter defined by $[I_0(E)^2 I_0(E')^2]_{MM}$. $b_j (x)$ is auxiliary magnetic fields with integer values. Dual integer vector potentials ($i = 0, 1, 2$) is introduced as follows [25]:

$$b_i (x) = \varepsilon_{ijk} \nabla_j \tilde{a}_k (x) = (\nabla \times \tilde{a})(x),$$

where $\varepsilon_{ijk}$ is the Levi-Civita symbol in 3d. By using the dual transformations of Eq. (35), the following Eq. (36):

$$Z_{QV} = \sum_{\{b_i\}} \delta_{\nu_j,0} \sum_{x,\tau} \left[ -\frac{1}{2(E_j^2)} v^2 + \frac{1}{2(E_j^2)} \sum_{j=1}^{\infty} \frac{b_j^2}{n_{\eta}} \right]$$

Using Poisson's formula of the following Eq. (37) for equation (36),

$$\sum_{\{\tilde{a}_j\} = \infty} f (\tilde{a}_j) = \int D\tilde{a}_j f (\tilde{a}_j) \sum_{\{\tilde{a}_j\} = \infty} \delta_{\nu_j,0} \sum_{x,\tau} \left[ 2\pi \sum_{j=1}^{\infty} \tilde{a}_j \right]$$

FIG. 3: Schematic phase diagram for $g$-$\eta$

Schematic phase diagram with $g$ on the horizontal axis and $\eta$ on the vertical axis.
then Eq. (38) as follows:

$$Z_{QV} = C_{QV} \int D\tilde{\omega} \sum_{\{\tilde{\alpha}_i\}} \delta_{\tilde{\omega},\tilde{\alpha}_0} \exp \left\{ -\frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}_0 \right)^2 \right\} \prod_{i=1}^{2} \left( 1 - \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}_i \right)^2 + i2\pi \sum_{j=0}^{2} \tilde{I}_j \tilde{\alpha}_j \right), \tag{38}$$

Poisson's formula of Eq. (33) converts the integer valued vector potentials $\tilde{\alpha}_i$ to the continuous valued vector potentials $\tilde{\alpha}'_i$. According to the quantum vortex dynamics, its Euclidean Lagrangian density is as follows:

$$L_{QV}(\tilde{x}) = \frac{1}{2(E^2)_v} \sum_{i=1}^{2} \tilde{\beta}_i^2(x) + \frac{1}{2(E^2)_v} \tilde{\beta}_0^2(x) - i2\pi \sum_{j=0}^{2} \tilde{I}_j(x) \tilde{\alpha}_j'(x), \tag{39}$$

where $\tilde{\beta}_i (i = 1, 2)$ and $\tilde{\beta}_0$ can be considered as a dual electric field and a dual magnetic field in a $2 + 1d$ dual electromagnetism, respectively, and are defined as follows:

$$\tilde{\beta}_0(x) \equiv \nabla_1 \tilde{\alpha}_2(x) - \nabla_2 \tilde{\alpha}_1(x),$$

$$\tilde{\beta}_1(x) \equiv \nabla_2 \tilde{\alpha}_0(x) - \nabla_0 \tilde{\alpha}_2(x),$$

$$\tilde{\beta}_2(x) \equiv \nabla_0 \tilde{\alpha}_1(x) - \nabla_1 \tilde{\alpha}_0(x), \tag{40}$$

If the 1,2 components $\tilde{e}_1$ and $\tilde{e}_2$ of the dual electric field are set as $\tilde{e}_1 \equiv \tilde{\beta}_2$, and $\tilde{e}_2 \equiv -\tilde{\beta}_1$, respectively, the dual Maxwell's equations from the Lagrangian of Eq. (39) are as follows:

$$\frac{1}{(E^2)_v} \left( \nabla_1 \tilde{e}_1(x) + \nabla_2 \tilde{e}_2(x) \right) = i2\pi \tilde{l}_0(x),$$

$$\frac{1}{(E^2)_v} \nabla_1 \tilde{\beta}_0(x) - \frac{1}{(E^2)_v} \nabla_0 \tilde{e}_1(x) = i2\pi \tilde{l}_1(x),$$

$$-\frac{1}{(E^2)_v} \nabla_0 \tilde{e}_2(x) + \frac{1}{(E^2)_v} \nabla_1 \tilde{\beta}_0(x) = i2\pi \tilde{l}_2(x). \tag{41}$$

For Eq. (39), in order to integrate out for continuous valued vector potentials $\tilde{\alpha}'_j$, the partition function when the axial gauge-fixing condition $\alpha'_0 = 0$ is imposed is as follows:

$$Z_{QV} = C_{QV} \int D\tilde{\omega} \sum_{\{\tilde{\alpha}'_i\}} \delta_{\tilde{\omega},\tilde{\alpha}'_0} \exp \left\{ -\frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_0 \right)^2 \right\} \prod_{i=1}^{2} \left( 1 - \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_i \right)^2 \right), \tag{42}$$

where, the orthogonal symbol $\perp$ as a superscript indicates that only the components $\tilde{\alpha}^1_1$ and $\tilde{\alpha}^2_2$ exist. And, the metric $g^{ab}$ is defined as follows:

$$g^{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{(E^2)_v}{(E^2)_v},$$

where $\gamma$ is an anisotropic parameter in the JJ model. Integrating over the continuous valued gauge fields $\tilde{\alpha}'_1$ and $\tilde{\alpha}'_2$ for Eq. (42) yields following the partition function:

$$Z_{QV} = C_{QV} \sum_{\{\tilde{\alpha}'_i\}} \delta_{\tilde{\omega},\tilde{\alpha}'_0} \exp \left\{ -\frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_0 \right)^2 \right\} \frac{\gamma}{\gamma} \cos \left( \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_1 \right)^2 \right), \tag{44}$$

where $C'_{QV} \equiv \frac{C_{QV}}{(\det(-\eta_{ab}\nabla_a\nabla_b))^{\frac{1}{2}}[\det(-\nabla_0\nabla_0)]^{\frac{1}{2}}}$, the anisotropic massless lattice potential (or lattice Green function), $V_0(x)$ is defined as:

$$V_0(x) \equiv \frac{1}{g^{ab}\nabla_a \nabla_b}, \tag{45}$$

Now, from this lattice potential, the "split lattice potential" $V'_0(x)$ obtained by dividing the "core lattice potential" $V_0(0)$ and its the "split difference operator" $\nabla'_i$ are introduced as follows:

$$V'_0(x) \equiv \frac{1}{g^{ab}\nabla_a \nabla_b}, \tag{46}$$

Thus, we have the following identity:

$$\exp \sum_{x,j} \left[ -2\pi \left( E^2_j \right)_v \tilde{I}_j(x) \tilde{V}_j'(x - x') \tilde{I}_j'(x') - 2\pi \left( E^2_j \right)_v \tilde{I}_j(x) \tilde{V}_j(x - x') \tilde{I}_j'(x') \right]$$

$$-C'_{QV} \int D\tilde{\omega} \exp \left\{ -\frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_0 \right)^2 + \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_i \right)^2 \right\}, \tag{47}$$

where $C'_{QV} \equiv \frac{C_{QV}}{(\det(-\nabla_0\nabla_0))^{\frac{1}{2}}[\det(-\nabla_0\nabla_0)]^{\frac{1}{2}}}$. Using Eq. (40) and (41) in Eq. (44) gives:

$$Z_{QV} = C'_{QV} \int D\tilde{\omega} \exp \left\{ -\frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_0 \right)^2 + \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_i \right)^2 \right\} \sum_{\{\tilde{\alpha}'_i\}} \delta_{\tilde{\omega},\tilde{\alpha}'_0} \exp \left\{ -2\pi \tilde{l}_0(x) \left( E^2_j \right)_v \tilde{I}_j(x) - 2\pi \tilde{l}_0(x) \left( E^2_j \right)_v \tilde{I}_j(x) + i2\pi \sum_{j=0}^{2} \tilde{I}_j \tilde{\alpha}'_j \right\}, \tag{48}$$

where $C'_{QV} \equiv \frac{C_{QV}}{(\det(-\nabla_0\nabla_0))^{\frac{1}{2}}[\det(-\nabla_0\nabla_0)]^{\frac{1}{2}}}$, and the metric $f^{ab}$ are defined as:

$$f^{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

Further, from Eq. (13), (22) and (34), the following identity can also be obtained:

$$\sum_{\{\tilde{\alpha}'_i\}} \delta_{\tilde{\omega},\tilde{\alpha}'_0} \exp \left\{ -2\pi \tilde{l}_0(x) \left( E^2_j \right)_v \tilde{I}_j(x) - 2\pi \tilde{l}_0(x) \left( E^2_j \right)_v \tilde{I}_j(x) + i2\pi \sum_{j=0}^{2} \tilde{I}_j \tilde{\alpha}'_j \right\}$$

$$-\int D\tilde{\omega} \exp(-\frac{1}{2}(\nabla \times \tilde{\alpha}'_0)^2) \left( 1 - \cos \left( \nabla_0 \tilde{\alpha}'_0 \right) \right) \frac{\gamma}{\gamma} \cos \left( \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_0 \right)^2 \right) \gamma \cos \left( \frac{1}{2(E^2)_v} \left( \nabla \times \tilde{\alpha}'_1 \right)^2 \right), \tag{50}$$

As an analogy to the relational expression between the JJ energy and the QPSJ energy in Eq. (13), the following relational expression is obtained from Eq. (50):

$$\left( E^2_S \right)_v = \frac{1}{4\pi^2 \tilde{l}_0(0) \left( E^2_j \right)_v}, \quad \left( E^2_S \right)_v = \frac{1}{4\pi^2 \tilde{l}_0(0) \left( E^2_j \right)_v}. \tag{51}$$
Eq. (12) and (23) differ only in the coefficients of the core lattice potential $V_0(0)$, so we can conclude that this is a reasonable result. Using Eq. (50) and (23) in Eq. (48) gives:

$$Z_{QV} = C_{QV} \int D\tilde{\alpha} \exp \left\{ -\frac{1}{2} \left( \tilde{\alpha} \cdot \frac{\partial}{\partial \tilde{\alpha}} \right) \left( \tilde{\alpha} \times \tilde{\alpha} \right) \right\} D\tilde{\alpha} \exp(-1) = \sum_x \left\{ (E_0^\alpha)^4 \left[ 1 - \cos \left( \tilde{\alpha}_0 \tilde{\alpha} \right) \right] + (E_0^\alpha)^2 \sum_{j=1}^2 \left[ 1 - \cos \left( \tilde{\alpha}_0 \tilde{\alpha}_j \right) \right] \right\} ,$$

(52)

Eq. (23) represents the gauge-coupled QPSJ partition function by the dual gauge field $\tilde{\alpha}_x$. For the coefficient $(E_0^\alpha)^2$ of the dual gauge field energy, use the relationship of Eq. (23) and (38), next, when these non-dimensional energy constants are converted to the original energy dimension, it will be as follows:

$$Z_{QV} = C_{QV} \int D\tilde{\alpha} \exp \left\{ -\frac{1}{2} \left( \tilde{\alpha} \cdot \frac{\partial}{\partial \tilde{\alpha}} \right) \left( \tilde{\alpha} \times \tilde{\alpha} \right) \right\} D\tilde{\alpha} \exp(-1) \exp \left\{ \frac{1}{2} \sum_{x} \left( E_0^\alpha \left[ 1 - \cos \left( \tilde{\alpha}_0 \tilde{\alpha} \right) \right] + (E_0)^2 \sum_{j=1}^2 \left[ 1 - \cos \left( \tilde{\alpha}_0 \tilde{\alpha}_j \right) \right] \right) \right\} ,$$

(53)

Further, by scaling 2 to the non-dimensional dual gauge field $\tilde{\alpha}_x$, we introduce a new dual gauge field $\tilde{\alpha}_x$ as follows:

$$2 \tilde{\alpha}_x(x) \equiv \tilde{\alpha}_x(x) ,$$

(54)

By the transformation of Eq. (14) and Eq. (53) is transformed as follows:

$$Z_{QV} = C_{QV} \int D\alpha \exp \left\{ \frac{1}{2} \sum_{x} \left( E_0 \left[ 1 - \cos \left( \alpha_0 \alpha_x \right) \right] + (E_0)^2 \sum_{j=1}^2 \left[ 1 - \cos \left( \alpha_0 \alpha_j \right) \right] \right) \right\} D\alpha \exp(-1) \exp \left\{ \frac{1}{2} \sum_{x} \left( E_0 \left[ 1 - \cos \left( \alpha_0 \alpha_x \right) \right] + (E_0)^2 \sum_{j=1}^2 \left[ 1 - \cos \left( \alpha_0 \alpha_j \right) \right] \right) \right\} ,$$

(55)

where, $\mu_0 = C/V_0(0)$, $q_0 = 2\pi/2e = \Phi_0/h$ represents a unit magnetic charge. Eq. (23) shows that the pure AXY model (J) model without gauge coupling) has been dual transformed to the gauged DAXY model (gauged QPSJ model) by the Villain approximation in the 2 + 1d system. In other words, the gauge QPSJ model is a frozen lattice dual superconductor[23][22][14], for the JJ model without gauge coupling.

5. DUAL TRANSFORMATION FROM THE DAXY MODEL TO THE GAUGED AXY MODEL BY VILLAIN APPROXIMATION IN 2+1 D SYSTEM

In this section, contrary to the previous section, we show the dual transformation from DAXY model to AXY model by Villain approximation. First, apply the Villain approximation to $Z_{DAXY}$ introduced in Eq. (10) as follows:

$$Z_{QV} = C_{QV} \int D\alpha \exp \left\{ -\frac{1}{2} \left( \alpha \cdot \frac{\partial}{\partial \alpha} \right) \left( \alpha \times \alpha \right) \right\} D\alpha \exp(-1) \exp \left\{ -\frac{1}{2} \sum_{x} \left( E_0 \left[ 1 - \cos \left( \alpha_0 \alpha_x \right) \right] + (E_0)^2 \sum_{j=1}^2 \left[ 1 - \cos \left( \alpha_0 \alpha_j \right) \right] \right) \right\} ,$$

(56)

where $R_{QV} \equiv \left\{ R_0 \left( E_0 \right)^2 \right\}^{\text{MMJ}}$, and $Z_{QV}$ is Villain approximations of the partition function $Z_{DAXY}$. Using the Jacobi theta function of Eq. (33) for Eq. (50) can be rewritten as follows:

$$Z_{QV} = C_{QV} \sum_{\{b\}} \delta_{c_i,0} \exp \left\{ \int \left\{ -\frac{1}{2} \left( E_0 \right)^2 \sum_{x=1}^2 \left( \alpha_0 \alpha_x \right) + \frac{1}{4} \sum_{y=1}^2 \left( \alpha_0 \alpha_y \right) \right\} \right\} ,$$

(57)

where $C_{QV} = \left\{ I_0 \left( E_0 \right)^2 \right\}^{\text{MMJ}}$, and $\tilde{b}_x(x)$ is auxiliary dual magnetic fields with integer values. Integer vector potentials $\alpha_i(x)(x = 0, 1, 2)$ is introduced as follows:

$$\tilde{b}_x(x) = \varepsilon_{ij} \nabla_j \alpha_i(x) = (\nabla \times \mathbf{a})_x(x) ,$$

(58)

By using the dual transformations of Eq. (57), the following Eq. (60):

$$Z_{QV} = C_{QV} \sum_{\{b\}} \delta_{c_i,0} \exp \left\{ \int \left\{ -\frac{1}{2 \left( E_0 \right)^2} \sum_{x=1}^2 \left( \nabla \times \alpha_x \right) \right\} \right\} ,$$

(59)

Using Poisson’s formula of the following Eq. (67) for Eq. (59):

$$Z_{QV} = C_{QV} \sum_{\{b\}} \delta_{c_i,0} \exp \left\{ \int \left\{ -\frac{1}{2 \left( E_0 \right)^2} \sum_{x=1}^2 \left( \nabla \times \alpha_x \right) \right\} \right\} ,$$

(60)

Its Euclidean Lagrangian density is as follows:

$$L_{QV}(x) = \frac{1}{2 \left( E_0 \right)^2} \sum_{x=1}^2 \beta_0^2(x) + \frac{1}{2 \left( E_0 \right)^2} \sum_{x=1}^2 \beta_0^2(x) + i 2 \pi \sum_{x=1}^2 j (x) \alpha_j (x) ,$$

(61)

where, $\beta_i(i = 1, 2)$ and $\beta_0$ can be considered as a electric field and a magnetic field in a 2 + 1d electromagnetism, respectively, and are defined as follows:

$$\beta_0(x) \equiv \nabla_1 \alpha_2(x) - \nabla_2 \alpha_1(x) ,$$

$$\beta_1(x) \equiv \nabla_2 \alpha_0(x) - \nabla_0 \alpha_2(x) ,$$

$$\beta_2(x) \equiv \nabla_0 \alpha_1(x) - \nabla_1 \alpha_0(x) ,$$

(62)

If the 1,2 components $e_1$ and $e_2$ of the electric field are set as $e_1 \equiv \beta_2$, and $e_2 \equiv -\beta_1$, respectively, the Maxwell’s equations from Lagrangian of Eq. (62) are as follows:

$$\frac{1}{\left( E_0 \right)^2} \nabla_1 e_1(x) + \nabla_2 e_2(x) = i 2 \pi \delta_0(x) ,$$

$$\frac{1}{\left( E_0 \right)^2} \nabla_2 e_0(x) - \frac{1}{\left( E_0 \right)^2} \nabla_0 e_1(x) = i 2 \pi \delta_1(x) ,$$

$$- \frac{1}{\left( E_0 \right)^2} \nabla_0 e_2(x) - \frac{1}{\left( E_0 \right)^2} \nabla_2 e_0(x) = i 2 \pi \delta_2(x) ,$$

(63)

Integrating over the continuous valued gauge fields $\alpha_1$ and $\alpha_2$ for Eq. (62) yields following the partition function:

$$Z_{QV} = C_{QV} \sum_{\{c\}} \delta_{c_i,0} \exp \left\{ \int \left\{ -2 \pi^2 \left( E_0 \right)^2 \sum_{x=1}^2 \left[ \beta_0(x) \beta_0(x') \right] \right\} \right\} ,$$

(64)
where \( C'_{GQV} \equiv C_{GQV} \left[ \det \left( -\gamma^{ab} \nabla_a \nabla_b \right) \right] ^{1/2} \left[ \det -\nabla \nabla \right] ^{1/2} \), the anisotropic massless lattice potential (or lattice Green function) \( \tilde{V}_0 (x) \) is defined as:

\[
\tilde{V}_0 (x) = \frac{-1}{g^{ab} \nabla_a \nabla_b} (x),
\]

From this lattice potential, we introduce a "split lattice potential" \( \tilde{V}'_0 (x) = \tilde{V}_0 (x) - \tilde{V}_0 (0) \delta_x 0 \) obtained by dividing the "core lattice potential" \( \tilde{V}_0 (0) \) and its "split difference operator" \( \nabla' \) are introduced as follows:

\[
\nabla' = \frac{\nabla}{\sqrt{1 - \tilde{V}_0 (0) \left( -g^{ab} \nabla_a \nabla_b \right)}}
\]

and use the same relational expression as Eq. (57) to (59) in the previous section, so the partition function \( Z_{DQV} \) as follows:

\[
Z_{DQV} = C_{DQV} \left[ D \phi', \exp \left\{ \frac{1}{2} \int \left( \phi' (\nabla' \phi') - \phi' \phi' \right) d^4 \tau \right\} \right]
\]

\[
\times \sum \left\{ \left( E^0 \right)_x \left[ 1 - \cos (\nabla' \phi' - 2 \text{vol}) \right] + (E_x) \sum \left[ 1 - \cos (\nabla' \phi' - 2 \text{vol}) \right] \right\},
\]

where \( C'_{DQV} \equiv C_{DQV} \left[ \det \left( -\nabla' \nabla' \right) \right] ^{1/2} \left[ \det -\nabla \nabla \right] ^{1/2} \), and the metric \( f^{ab} \) are defined as:

\[
\left( E^0 \right)_x \equiv \frac{1}{4 \pi^2 \tilde{V}_0 (0) \left( E^0 \right)_x}, (E^x)_x \equiv \frac{1}{4 \pi^2 \tilde{V}_0 (0) \left( E^0 \right)_x}
\]

If \( \tilde{V}_0 (0) \equiv \tilde{V}_0 (0) \), Eq. (63) and (65) are completely equivalent. For the coefficient \( (E^0) \), of the gauge field energy, use the relationship of Eq. (63) and (65), next, when these non-dimensional energy constants are converted to the original energy dimension, it will be as follows:

\[
Z_{GQV} = C_{GQV} \left[ D \phi', \exp \left\{ \frac{1}{2} \int \left( \phi' (\nabla' \phi') - \phi' \phi' \right) d^4 \tau \right\} \right]
\]

\[
\times \sum \left\{ \left( E^0 \right)_x \left[ 1 - \cos (\nabla' \phi' - 2 \text{vol}) \right] + (E_x) \sum \left[ 1 - \cos (\nabla' \phi' - 2 \text{vol}) \right] \right\},
\]

Further, by scaling \( \Phi_0 \) to the non-dimensional gauge field \( \alpha' \), we introduce a new gauge field \( \alpha' \) as follows:

\[
\Phi_0 \alpha' (x) \equiv \alpha' (x),
\]

Eq. (63) is transformed as follows:

\[
Z_{GQV} = C_{GQV} \left[ D \phi', \exp \left\{ \frac{1}{2} \int \left( \phi' (\nabla' \phi') - \phi' \phi' \right) d^4 \tau \right\} \right]
\]

\[
\times \sum \left\{ \left( E^0 \right)_x \left[ 1 - \cos (\nabla' \phi' - 2 \text{vol}) \right] + (E_x) \sum \left[ 1 - \cos (\nabla' \phi' - 2 \text{vol}) \right] \right\},
\]

where, \( \mu \equiv L / \tilde{V}_0 (0) \), \( 2 \equiv 2e / \hbar = 2 \pi / \Phi_0 \) represents a unit Cooper pair charge, which is twice the unit charge \( q \equiv e / \hbar = 2 \pi / 2 \Phi_0 \). Eq. (70) shows that the pure DAXY model (JJ model without gauge coupling) has been dual transformed to the gauged AXY model (gauged JJ model) by the Villain approximation in the 2 + 1d system. In other words, the gauge JJ model is a frozen lattice dual superconductor for the QPS model without gauge coupling.

6. MEAN FIELD ANALYSIS OF THE GAUGED QPSJ MODEL ON THE NANOSHEET

In this section, we introduce the mean field approximation to the partition function of Eq. (53) and discuss its phase transition. From Eq. (53), the partition function excluding the constant part is newly defined as \( Z_{QPSJ} \) of gauged QPSJ model, and using unit vectors of real two components \( \hat{U}_i = \left[ \cos \theta, \sin \theta \right] \), rewrite as follows:

\[
Z_{QPSJ} = \int \left[ D \phi, \exp \left\{ \frac{1}{2} \int \left( \phi (\nabla \phi) - \phi \phi \right) d^4 \tau \right\} \right]
\]

\[
\times \sum \left\{ \left( E^0 \right)_x \left[ 1 - \cos (\nabla \phi - 2 \text{vol}) \right] + (E_x) \sum \left[ 1 - \cos (\nabla \phi - 2 \text{vol}) \right] \right\},
\]

where \( \mu \equiv h \mu' / a_0 \), the lattice difference operator \( \tilde{R} \) is defined as:

\[
\tilde{R} = 1 + \frac{1}{2d} \left( \sum_{i=1}^{2d} \tilde{D}_i \tilde{D}_i + \gamma \tilde{D}_i \tilde{D}_i \right), \quad \gamma \equiv \frac{E^0}{E^x}
\]

where \( d \equiv 2 + \tilde{\gamma} \) is anisotropic dimensional constants of gauged QPSJ, \( \tilde{R} \) and \( \tilde{D} \) are forward and backward covariant lattice derivatives, respectively, defined for example for a complex field \( \tilde{U} = \tilde{U}_i + i \tilde{U}_j \) as follows:

\[
\tilde{D}_i \tilde{U} (x, \tau) \equiv \tilde{U} (x + 1, \tau) e^{-\eta \tilde{R} \tilde{U}} - \tilde{U} (x, \tau), \quad \tilde{D}_i \tilde{U} (x, \tau) \equiv \tilde{U} (x, \tau) - \tilde{U} (x - 1, \tau) e^{\eta \tilde{R} \tilde{U}}
\]

the same applies to \( \tilde{D}_x \) and \( \tilde{Z} \). In Eq. (72), we introduce two sets of real two component fields \( \psi \) and \( \psi \) (\( l = 1, 2 \)) which satisfy the following identity:

\[
\int_{-\infty}^{\infty} d\psi \int_{-\infty}^{\infty} d\psi \exp \left\{ -\psi (\tilde{U}_1 - \tilde{U}_1) \right\} = 1,
\]

\[
Z_{QPSJ} = \int \left[ D \phi, \exp \left\{ \frac{1}{2} \int \left( \phi (\nabla \phi) - \phi \phi \right) d^4 \tau \right\} \right]
\]

\[
\times \sum \left\{ \left( E^0 \right)_x \left[ 1 - \cos (\nabla \phi - 2 \text{vol}) \right] + (E_x) \sum \left[ 1 - \cos (\nabla \phi - 2 \text{vol}) \right] \right\},
\]

Further, by scaling \( \Phi_0 \) to the non-dimensional gauge field \( \alpha' \), we introduce a new gauge field \( \alpha' \) as follows:

\[
\Phi_0 \alpha' (x) \equiv \alpha' (x),
\]
where we have used that the functional integrals of $\tilde{\theta}$ is given as follows:

$$\prod_{x} \int d\tilde{\theta} \exp \left\{ \sum_{x} \sum_{\alpha} \tilde{\psi}_{\alpha} \tilde{U}_{\alpha} \right\} = \exp \sum_{x} \left\{ \ln I_{0}(\tilde{\psi}^{i}) \right\}, \quad (77)$$

where $I_{0}(\tilde{\psi}^{i}) = (\tilde{\psi}^{i} + \tilde{\psi}^{i})$ is the modified Bessel functions of integer 0th order. In Eq. (76), performing the integrals over fields, we obtain the partition function by the complex field $\tilde{\psi} = \tilde{\psi}_{1} + i\tilde{\psi}_{2}$ and $\tilde{\psi}^{*} = \tilde{\psi}_{1} - i\tilde{\psi}_{2}$.

$$Z_{DGL} = \int D\tilde{\alpha} \prod_{x} \left( \int -\infty^{\infty} dE_{sd} \right) \exp \left\{ -F' \left( \tilde{\psi}, \tilde{\psi}^{*}, \tilde{\alpha} \right) \right\}$$

$$F' \left( \tilde{\psi}, \tilde{\psi}^{*}, \tilde{\alpha} \right) = \sum_{x} \left( \frac{1}{2\mu_{F}} \hat{\psi}^{i} \hat{\psi}_{i} \hat{\alpha}_{i} \right)_{L} \left( \hat{\psi}^{i} \hat{\psi}_{i} \right) + \frac{1}{2 \text{Fr}^{2}} \left( \frac{1}{E_{sd} - 1} \right) \left( \hat{\psi}^{i} \hat{\psi}_{i} \right), \quad (78)$$

In Eq. (78), since $\tilde{\psi}$ and $\tilde{\psi}^{*}$ can be regarded as the order parameter of superinsulator (i.e., disorder parameter of superconductor), the non-dimensional free energy $F'(\tilde{\psi}, \tilde{\psi}^{*}, \tilde{\alpha})$ can be Landau expansion of terms up to $|\tilde{\psi}|^{4}$, $|D_{\psi}\tilde{\psi}|^{2}$ and $|D_{\psi}\tilde{\psi}|^{2}$ as follows:

$$F'_{DGL}(\tilde{\psi}, \tilde{\psi}^{*}, \tilde{\alpha}) = \sum_{x} \left( \frac{1}{2\mu_{F}} \hat{\psi}^{i} \hat{\psi}_{i} \hat{\alpha}_{i} \right)_{L} \left( \hat{\psi}^{i} \hat{\psi}_{i} \right) + \frac{1}{2 \text{Fr}^{2}} \left( \frac{1}{E_{sd} - 1} \right) \left( \hat{\psi}^{i} \hat{\psi}_{i} \right), \quad (79)$$

$F'_{DGL}$ is non-dimensional dual Ginzburg-Landau (DGL) energy of superinsulator or QPSJ model on the nanosheet in $d = 2 + \tilde{\gamma}$ dimension at zero temperature. Therefore, the critical values $E_{S}^{DGL}$ according to the mean field approximation of QPS amplitude $E_{S}$ are as follows:

$$E_{S}^{DGL} \equiv \frac{1}{d} = 1 + \frac{1}{2 + \tilde{\gamma}}, \quad (80)$$

As a continuous limit of Eq. (79), the free energy of DGL is as follows:

$$F_{DGL}(\tilde{\psi}, \tilde{\psi}^{*}, \tilde{\alpha}) = \int d\tilde{\psi} \int d\tilde{\psi}^{*} \left( \frac{1}{2\mu_{F}} \hat{\psi}^{i} \hat{\psi}_{i} \hat{\alpha}_{i} \right)_{L} \left( \hat{\psi}^{i} \hat{\psi}_{i} \right) + \frac{1}{2 \text{Fr}^{2}} \left( \frac{1}{E_{sd} - 1} \right) \left( \hat{\psi}^{i} \hat{\psi}_{i} \right)$$

$$= \frac{1}{2 \text{Fr}^{2}} \left( \frac{1}{E_{sd} - 1} \right) \left( \hat{\psi}^{i} \hat{\psi}_{i} \right) \xi \equiv \frac{E_{S}^{DGL}}{E_{S}^{DGL}} \xi, \quad \beta \equiv \frac{h}{4 \text{m}_{0} a^{2}}, \quad (81)$$

where, $\hat{\psi}^{i}$ is a pseudo mass of magnetic flux having a dimension of $[J \cdot s^{2}]$, and $\hat{\alpha}_{i}$ is a dual vector potentials having a dimension of $[C/m]$. Therefore, the order parameter $\tilde{\psi}$ of the superinsulator is gauge coupled to the $U(1)$ dual gauge field $A_{\mu}$ by unit magnetic charge $q_{m} = 2\pi/2e = \Phi_{0}/h$, and when $\tilde{\psi}$ is in the condensed state, as shown in Fig. 4, the pair of the Cooper pair and the anti-Cooper pair is confined within the superinsulator by the electric flux tubes. It shows the confinement of electric flux in the superinsulator between the Cooper pair and the anti-Cooper pair. Whether this confinement picture, in which the charge of $2e$ for a superinsulator is the smallest unit,
ii) intermediate between type I and type II

$$\tilde{\kappa} = \frac{1}{\sqrt{2}}, \quad \tilde{\sigma}_{SN} = 0.$$ 

iii) type II superinsulator

$$\tilde{\kappa} > \frac{1}{\sqrt{2}}, \quad \tilde{\sigma}_{SN} < 0.$$ 

From the analogy with the mixed state of the type I superconductor, the possibility of the existence of the mixed state in the case of the type II superinsulator is expected, and from the analogy with the Abrikosov magnetic flux lattice of the type II superconductor, the existence of an electric flux lattice is also expected in the case of type II superinsulator on nanosheet.

7. TWO LOOP CORRECTIONS FOR THE MEAN FIELD APPROXIMATION OF QPS AMPLITUDE

The conclusions of this paper are summarized below. First, using the dual Hamiltonian method, the phase and amplitude relation between the JJ and QPSJ models without gauge coupling on 2 + 1 d nanosheets at zero temperature, and derived the relation of various constants. Furthermore, the exact duality between the JJ model and the QPSJ model based on the non-linear Legendre transformation between the Lagrangian and the Hamiltonian using canonical conjugate variables of infinite order in a compact 2 + 1 d lattice space was demonstrated. A dual transformation from the AXY model to the gauged DAXY model by the Villain approximation in the 2 + 1 d system was derived. In 2 + 1 d, there are two main differences between the dual transformation by the dual Hamiltonian method and the dual transformation by the Villain approximation. One is that Eq.(12) and (51) differ only in the core potential $V_0(0)$. Another difference is that in the case of the dual transformation by the Villain approximation, there is a gauge coupling by the dual gauge field $\tilde{\alpha}'_i$, but in the dual transformation by the dual Hamiltonian method means that there is no gauge coupling by the dual gauge field. The gauge coupling by the dual gauge field $\tilde{\alpha}'_i$ is gauge coupled with the $U(1)$ dual gauge field by the unit magnetic charge $\tilde{q}_m \equiv 2\pi /2e$ by the scaling of 2e introduced by Eq.(54), and as shown in Fig 4, a picture was derived that the electric flux of the Cooper relationship with charging energy $E'_c$:

$$E'_c = 2\pi^2 V_0(0) (E'_S)_c.$$ (89)

Note that $V_0 (0)$ in this relational equation is a massless lattice Green function having an anisotropy parameter $\gamma$ in the JJ model defined by Eq.(11). From Eq.(89), we have plotted in Fig 6 the critical value $(E'_c)_c$ of the charging energy $E'_c$ for various anisotropy parameter $\gamma$ in JJ models as a function of $\tilde{\gamma}$.

FIG. 6: The critical value of the charging energy for various anisotropy parameter $\gamma$ in JJ models as a function of $\tilde{\gamma}$.

8. SUMMARY AND DISCUSSION

thermodynamic critical dual magnetic field $\tilde{H}_c$, the penetration depth $\tilde{\lambda}$ and coherent length $\tilde{\xi}$ from mean field analysis, all of which depended on the difference between QPS amplitude $E'_S$ and its mean field critical value $E'_{SM}$. Since the mean field approximation is a very rough approximation, in this section we will add a one loop + two loop fluctuation to the mean field. Appendix B, the mean field approximation + one loop + two loop free energy is shown. The critical value $(E'_S)_{2loop}$ of the QPS amplitude $E'_S$ in the mean field approximation + one loop + two loop is given to Eq.(B-14) as a function of the anisotropy parameter $\tilde{\gamma}$, and the results are plotted in Fig 5. Similarly, the results are plotted in Fig 5 for the tricritical point $(E'_S)^{tr}$ of the QPS amplitude given in Eq.(B-15). QPS amplitude $E'_S$ has the following

FIG. 5: The critical value and tricritical point for the QPS amplitude in the mean field approximation + one loop + two loop.
pair and the anti-Cooper pair in units of $2e$ was confined in the superinsulator. In the following, we consider whether the superinsulator’s picture of “confinement in 2e units, that is, confinement by a pair of Cooper and anti-Cooper pair” is correct or incorrect. In the case of the magnetic flux confinement for the superconductor, the magnetic flux quantum $\Phi_0 = h/2e$, which is the minimum unit of magnetic flux is confined. However, if Eq.(55) is correct, in the confinement of charges in the superinsulator, $2e = h/\Phi_0$, which is twice the elementary charge $e = h/(2\Phi_0)$, which is the minimum unit of charge is confined. This is clearly inconsistent with the superconducting case. Therefore, the case of a superinsulator also, it should be considered correct that confinement occurs in units of the elementary charge $e = h/(2\Phi_0)$, which is the minimum unit of charge. In other words, since the scaling by $2e$ in Eq. (91) in Section 4 was completely artificial, we could change it to scaling by $e$ as follows:

$$e\alpha^i (x) \equiv \tilde{a}_i (x),$$

(90)

By the transformation of Eq.(100), Eq.(53) is transformed as follows:

$$Z_{QV} = C_{QV} \left(\prod_{x} e^{\frac{-i}{h} \int_{x} \left[ \frac{1}{2m_c} \left( \partial^2 + \theta \right) \psi \right] \right) \left\{ \prod_{x} e^{\frac{-i}{h} \int_{x} \left[ \frac{1}{2m_c} \left( \partial^2 + \theta \right) \tilde{\psi} \right] \right\},$$

(91)

When Eq.(7) is compared with Eq.(57), the coupling magnetic charge is $2q_m = 2\Phi_0/h = 2\pi/e$, which is twice the unit magnetic charge $q_m = \Phi_0/h = 2\pi/2e$. FIG.7 differs from FIG.4, it shows the confinement of the electric flux between the positive and the negative elementary charges in the superinsulator. To obtain the result of Eq. (91), $E_c$ introduced in Eq. (11) is not $(2e)^2/2C$ but $e^2/2C$. In this case, the DGL free energy for Eq. (11) is as follows:

$$F_{DGL}(\phi, \tilde{\phi}, \tilde{A}) = \int d^3x \left[ \frac{1}{2m} \left( \partial^2 + \theta \right) \phi \right] + \frac{1}{2m_c} \sum_{i=1}^{n} \left( -c \right) \tilde{A}_i$$

$$+ \frac{1}{2m_c} \left( \tilde{A}_i - 2\Phi_0 \tilde{A}_i \right) \tilde{\phi}^{2} + \frac{1}{2m_c} \left( \tilde{A}_i - 2\Phi_0 \tilde{A}_i \right) \tilde{\phi}^{2}$$

(92)

where, $m_{phi}$ is the effective mass of the flux pair (vortex pair in the same rotation direction = superinsulator), and $\Phi$ is the wave function of the flux pair. On the other hand, the GL free energy for Eq. (7) is as follows:

$$F_{GL}(\phi, \tilde{\phi}, \tilde{A}) = \int d^3x \left[ \frac{1}{2m} \left( \partial^2 + \theta \right) \phi \right] + \frac{1}{2m_c} \sum_{i=1}^{n} \left( -c \right) \tilde{A}_i$$

$$+ \frac{1}{2m_c} \left( \tilde{A}_i - 2\Phi_0 \tilde{A}_i \right) \tilde{\phi}^{2} + \frac{1}{2m_c} \left( \tilde{A}_i - 2\Phi_0 \tilde{A}_i \right) \tilde{\phi}^{2}$$

$$\varepsilon = \frac{E_{MF}^{L}}{E'_{J}},$$

(93)

where, $m_c$ is the effective mass of the Cooper pair, and $\psi$ is the wave function of the Cooper pair. It is known that the microscopic theory of superconductivity with respect to the GL theory can be described by the following BCS Hamiltonian $H_{BCS}$.

$$H_{BCS} = \sum_{\sigma = \uparrow, \downarrow} \int d^3x \tilde{\phi}_{\sigma}^{\dagger} (x) \left\{ \frac{1}{2m_c} \left[ \partial_\mu A_\mu \right] \right\}^2 \tilde{\phi}_{\sigma} (x)$$

$$- \frac{|\tilde{q}|}{2} \sum_{\sigma, \sigma'} \int d^3x \phi_{\sigma}^{\dagger} (x) \phi_{\sigma'} (x) \phi_{\sigma'} (x) \phi_{\sigma} (x),$$

(94)

where, $m_c$ is the effective mass of the electron, and $\phi_{\sigma} (x)$ is the electron field having a spin subscript $\sigma$, and is a fermion satisfying the anti-commutation relation

$$\{ \phi_{\sigma} (x), \phi_{\sigma'}^{\dagger} (x') \} = \frac{\delta_{\sigma\sigma'} \delta^3 (x - x')}.$$}

From Eq.(93) and (94), in microscopic theory the matter field $\phi_{\sigma}$ is a fermion field, which is coupling to the gauge field by an elementary charge $e$, on the other hand, for the GL theory, the matter field $\psi$ is the boson field, which is coupling to the gauge field by $2\Phi_0$. Therefore, from the analogy between the BCS theory and the GL theory of superconductivity, it is expected that the microscopic theory of superinsulators with respect to DGL theory in Eq.(92) can be described by the following dual BCS Hamiltonian:

$$H_{DGL} = \sum_{\sigma = \uparrow, \downarrow} \int d^3x \tilde{\phi}_{\sigma}^{\dagger} (x) \left\{ \frac{1}{2m_c} \left[ \partial_\mu A_\mu \right] \right\}^2 \tilde{\phi}_{\sigma} (x)$$

$$- \frac{|\tilde{q}|}{2} \sum_{\sigma, \sigma'} \int d^3x \phi_{\sigma}^{\dagger} (x) \phi_{\sigma'} (x) \phi_{\sigma'} (x) \phi_{\sigma} (x),$$

(95)

where, $m_{phi}$ is the magnetic flux (vortex) quantum, and $\tilde{\phi}_{\sigma}$ is the magnetic flux quantum field having the pseudo spin subscript $\tilde{\sigma}$, and is a fermion satisfying the anti-commutation relation

$$\{ \tilde{\phi}_{\sigma} (x), \tilde{\phi}_{\sigma'}^{\dagger} (x') \} = \frac{\delta_{\tilde{\sigma}\tilde{\sigma'}} \delta^3 (x - x')}.$$}

FIG. 7: Schematic diagram of superinsulator by elementary charge $e$ on nanosheet at zero temperature.

9. ACKNOWLEDGMENTS

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Appendix A: Anisotropic lattice Green function

\[ \mathcal{V}_m(0) \] at the source \( x = 0 \)

Perform numerical evaluation of Anisotropic massive lattice Green function (lattice potential) \( \mathcal{V}_m(0) \) at the origin \( x = 0 \).

\[ \mathcal{V}_m(0) = \sum_{n=0}^{\infty} \frac{h_n}{m^2 + 2d} \] (A1)

where, \( h_n \) is the anisotropic hopping coefficient of the anisotropic massive lattice Green function \( \mathcal{V}_m(0) \) at the origin \( x = 0 \), and is introduced as follows:

\[ h_n \equiv n! \sum_{j=0,2,4}^{n} \left( \frac{\gamma^{n-j}}{(2d-1)!} \right)^j H_j, \] (A2)

where \( H_n \) are the isotropic hopping coefficients, for example, in the case of 2d, \( H_0 = 1, H_2 = 4, H_6 = 36 \), \( H_8 = 400 \), and in the case of 3d, \( H_0 = 1, H_2 = 6, H_6 = 90, H_8 = 1860 \). TABLE I lists examples of \( \gamma = 0.1, 0.2, ..., 0.8, 0.9, 1.0 \). The asymptotic behavior at the anisotropic parameter \( \gamma = 0 \) to 1 in the case of \( 2 < d \leq 3 \).

| \( \gamma \) | \( h_2 \) | \( h_4 \) | \( h_6 \) | \( h_8 \) | \( h_{10} \) |
|---|---|---|---|---|---|
| 0 | 4 | 36 | 400 | 4900 | 63504 |
| 0.1 | 4.02 | 36.003 | 410.83602 | 5125.514241 | 67964.55133 |
| 0.2 | 4.08 | 36.048 | 443.77728 | 5820.335539 | 81960.10908 |
| 0.3 | 4.18 | 36.243 | 500.13058 | 7040.109553 | 107964.55133 |
| 0.4 | 4.32 | 36.768 | 582.09792 | 8880.292915 | 147593.9992 |
| 0.5 | 4.5 | 37.875 | 692.8125 | 11480.27344 | 207665.9648 |
| 0.6 | 4.72 | 39.888 | 836.38912 | 15029.23717 | 294849.9426 |
| 0.7 | 4.98 | 43.203 | 1017.98898 | 19773.88112 | 419126.4121 |
| 0.8 | 5.28 | 48.288 | 1243.89888 | 26028.09681 | 593959.5603 |
| 0.9 | 5.62 | 55.683 | 1521.62482 | 34184.79254 | 837254.033 |
| 1.0 | 6 | 66 | 1860 | 44730 | 1172556 |

small \( m \) of the anisotropic massive lattice Green function \( \mathcal{V}_m(0) \) in the case of \( 2 < d \leq 3 \) is as follows:

\[ \mathcal{V}_m(0) = \sum_{n=0,2,4}^{\infty} \frac{\Delta h_n}{(m^2 + 2d)^n} \] (A3)

\[ \Delta h_n \equiv n! \sum_{j=0,2,4}^{n} \frac{\gamma^{n-j}}{(2d-1)!} \] (A3)

TABLE II shows asymptotic anisotropic hopping coefficient \( \Delta h_n \) up to 10 for the anisotropic parameter \( \gamma = 0 \) to 1 in the case of \( 2 < d \leq 3 \). From Eq. (A3) and TABLE II, when the value of the anisotropic massless lattice Green function \( \mathcal{V}_0(0) \) at the origin \( x = 0 \) is evaluated as the sum of the power series up to \( n = 10 \), it becomes as shown in the following TABLE III and Fig 6:

![Table III](image)

Table III: Asymptotic massless lattice Green function \( \mathcal{V}_0(0) \) at the origin \( x = 0 \) by the sum of the power series up to \( n = 10 \), for the anisotropic parameter \( \gamma = 0 \) to 1 in the case of \( 2 < d \leq 3 \).
The starting point is the order parameter representation of superinsulator by the partition function of Eq.(A1). Where, to simplify the problem, we deal with the case where there is no coupling of gauge fields. First, to derive the 1-loop effective field theory, the free correlation function in the presence of a non-vanishing background field is shown below:

\[
G^{-1}_{\psi}(x_1, x_2)_{ab} \equiv V^2_{ab}(x_1, x_2)_{ab} = E' \frac{\delta^2 \hat{F}'}{\delta \bar{\psi}_a(x_1) \delta \bar{\psi}_b(x_2)} \mid \tilde{\alpha}_l, \quad (B1)
\]

where \(\alpha_l\) is an expectation value of \(\bar{\psi}_l(x)\), and in general, all \(n\)-th order one-particle irreducible graphs involving the vertices can be computed as follows:

\[
V^2_{ab}(x_1, x_2 \ldots, x_n)_{ab} \equiv E' \frac{\delta^2 \hat{F}'}{\delta \bar{\psi}_a(x_1) \cdots \delta \bar{\psi}_a(x_n) \delta \bar{\psi}_b(x_2) \cdots \delta \bar{\psi}_b(x_n)} \mid \tilde{\alpha}_l, \quad (B2)
\]

The 2×2 matrix \(G^{-1}_{\psi}(x_1, x_2)_{ab}\) defined in Eq.(A1) is divided into longitudinal \(G^{-1}_{\psi}(x, y)\) and transverse \(G^{-1}_{\psi}(x, y)\) parts, which parallel and orthogonal to the expected value of the field \(\alpha_l\) as follows:

\[
G^{-1}_{\psi}(x_1, x_2)_{ab} = \frac{1}{2E'Sd} \delta_{ab} \delta_{x_1, x_2} \left( 1 + \frac{1}{2d} \eta_{zz} \right) (x_1, x_2)_{ab} \equiv \delta_{x_1, x_2} \left( 1 + \frac{1}{2d} \eta_{zz} \right) (x_1, x_2)_{ab},
\]

\[
G^{-1}_{\psi}(x, y) = \frac{1}{2E'Sd} \left( \frac{1}{2d} \bar{\psi}_s \bar{\psi}_s (x, y) \right) \equiv \frac{1}{2E'Sd} \left( \frac{1}{2d} \bar{\psi}_s \bar{\psi}_s (x, y) \right) \equiv (B3)
\]

where \(\eta_{zz}\) is the 2×2 matrix \(P^L_{ab}\) and \(P^T_{ab}\) are projection matrices longitudinal and transverse respectively:

\[
\eta_{zz} = P^L_{ab} \eta_L + P^T_{ab} \eta_T, \quad P^L_{ab} \equiv \delta_{ab} \frac{\tilde{\alpha}_a \tilde{\alpha}_b}{|\tilde{\alpha}|^2}, \quad P^T_{ab} \equiv \frac{\tilde{\alpha}_a \tilde{\alpha}_b}{|\tilde{\alpha}|^2}.
\]

By integrating over all quadratic fluctuations in the partition function of Eq.(78), the one-loop effective energy is as follows:

\[
F_{\text{1-loop}} = \frac{1}{2MM} \left( Tr \log \left( -\frac{1}{2d} g^{\mu \nu} \nabla_\mu \nabla_\nu \right) + Tr \log \left( \frac{m^2}{2d} - \frac{1 - m^2}{2d} \frac{1}{2d} g^{\mu \nu} \nabla_\mu \nabla_\nu \right) \right),
\]

\[
m^2 \equiv 4d - (2d)^2 E_S \left[ \left( 1 - \left( \frac{I_1(E_S)}{I_0(E_S)} \right)^2 \right) \right],
\]

where, \(I_0 (E)\) and \(I_1 (E)\) are order zero and order one the modified Bessel functions, respectively. The first trace means zero mass fluctuations (Goldstone modes), and the second trace represents massive fluctuations and can be calculated by hopping expansion as follows:

\[
F_{\text{1-loop}} = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{h_n}{n!} \left( \left( \frac{\partial u}{\partial d} \right)^n + \left( \frac{\partial v}{\partial d} \right)^n \right),
\]

where, \(\eta_{an}\) is the anisotropic hopping coefficient of the anisotropic massive lattice Green function \(V_{mn}(0)\) at the source \(x = 0\) shown in Appendix A. Expressing 1 loop free energy by hopping expansion of Eq.(B6) is as follows:

\[
F_{\text{1-loop}} = -\frac{1}{4d} \left( \left| q \right|^2 \left[ \frac{1}{2} \left( \frac{\partial u}{\partial d} \right)^2 + 6 \left( \frac{\partial u}{\partial d} \right)^2 + 6 \left( \frac{\partial v}{\partial d} \right)^2 \right] \right),
\]

\[
F_{\text{1-loop}} = -\frac{1}{2} \left( \frac{1}{3!} \right) \sum_{x} \left\{ \left( 4 \bar{Q}(\alpha) \right)^2 + 8 \bar{Q}(\alpha) \left[ 18 \bar{G}(\alpha) \bar{G}(\alpha) + 36 \bar{G}(\alpha) \right] + 36 \left( \frac{4 \bar{Q}}{3} \right) \left[ \left( \bar{G}(\alpha) \right)^2 + 6 \left( \bar{G}(\alpha) \right)^2 \right] \right\},
\]

where the dotted accent for \(\hat{Q} (|\tilde{\alpha}|)\) is defined by the modified derivative i.e., \(\hat{Q} \equiv (1/2|\tilde{\alpha}|) dQ/d|\tilde{\alpha}|\), and \(\bar{G}(x)_{ab}\) is defined as:

\[
\bar{G}(x)_{ab} \equiv \left( \bar{\psi}_a(x) \bar{\psi}_b(x) \right) = \left( 1 + \frac{1}{2d} g^{\mu \nu} \nabla_\mu \nabla_\nu \right) G(x)_{ab},
\]

\[
\bar{G}(x) \equiv \bar{\psi}_a \bar{\psi}_b, \quad \bar{Q}(\alpha) = \sum_{\alpha} \bar{G}(0)_{\alpha}, \quad (B10)
\]
where, the trace refers only to the index of the $2 \times 2$ matrix $\hat{G}(x)_{ab}$. To calculate the free energy of the two-loop correction, introduce the hopping expansion of $\hat{G}(x)_{ab}$, as follows:

$$\hat{G}(x,y)_{ab} = \frac{b}{d} \sum_{n=0}^{\infty} \left( \frac{by}{d} \right)^n h(x,y)^{n+1},$$

$$h(x,y) \equiv 2d\delta_{xy} + g^\mu\nu \nabla_\mu \nabla_\nu,$$  \hspace{1cm} (B11)

For $b \ll \bar{d}$, i.e., at the limit of small $E'_{S}$, the free energy due to the mean field $+ \text{one-loop} + \text{two-loops}$ in the up to order $b^4$ is as follows:

$$F' = \frac{\bar{\psi}^2}{4b} - \log I(a) - \frac{b^2}{2d} (\bar{\psi}^2 + \eta_\gamma^2) \left( 1 + \frac{1}{2d} \right)$$

$$\frac{b^3}{6d^3} \left[ \left( \bar{\psi}^2 + \eta_\gamma^2 \right)^2 - \frac{b}{2d} \left( \frac{3\bar{\psi}^4}{4d} + 3\eta_\gamma^4 \right) \left( 1 - \frac{1}{2d} \right) h_4 \right]$$

$$\frac{b^2}{5d^2} \left[ (3\bar{\psi}^4 - 4\eta_\gamma^4 + 2\bar{\psi}^2 \eta_\gamma^2) + 3d \eta_\gamma^2 \right] + 0 \left( b^5 \right),$$

$$h_2 = 2 \tilde{\gamma} (\gamma - \tilde{\gamma}) \ , \ h_4 = 6\gamma^4 + (24D - 12)\gamma^2 + (6 - 24D) \gamma,$$  \hspace{1cm} (B12)

The result of finding the minimum value $\tilde{\psi}_0$ of $\tilde{\psi}$ according to Eq. (B12) is as follows:

$$\tilde{\psi}_0 = \sqrt{8 \left( 1 - \frac{1}{b} \right) + 32\Delta_2 \left( 1 - 64\Delta_1 \right)}$$

$$\Delta_1 = \frac{b^5 \left( 58b^2\Delta - 612\Delta + 51h_4 \right) + 68b^3\Delta \left( 2\Delta + h_2 \right)}{4096b^6},$$

$$\Delta_2 = \frac{3b^2\Delta - 60b^3\Delta^2 \left( 2\Delta + h_2 \right) - 3b^4 \left( 34b_2^2 - 60b + 5h_4 \right)}{768b^6},$$  \hspace{1cm} (B13)

Therefore, for the mean field $+ \text{one loop} + \text{two loop}$ corrections of $E'_{S}$, the critical point $(E'_{S})_{c}^{2\text{loop}}$ and the triple critical point $(E'_{S})_{c}^{3\text{loop}}$ are respectively as follows:

$$\left( E'_{S} \right)_{c}^{2\text{loop}} = \frac{1}{(D + \tilde{\gamma})} \left[ 1 - \left( 120D^4 + 8D^3 \left( 55\tilde{\gamma}^2 + 30\tilde{\gamma} + 8 \right) + 4D \left( 60\tilde{\gamma}^3 + 120\tilde{\gamma}^2 - 58\tilde{\gamma} - 45 + 2\tilde{\gamma} (105\tilde{\gamma}^2 - 58\tilde{\gamma} - 45) \right) / 192(D + \tilde{\gamma}) \right]^{-1},$$  \hspace{1cm} (B14)

$$\left( E'_{S} \right)_{c}^{3\text{loop}} = \frac{1}{\sqrt{5800b_2^2 - 612D + 51h_4}} - \frac{34(b_2 + h_4)}{5800b_2^2 - 612D + 51h_4},$$  \hspace{1cm} (B15)

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