Classification of two-dimensional smooth projective algebraic semigroups

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Introduction:
In this article, we consider algebraic varieties and schemes defined over the field $\mathbb{C}$ of complex numbers. An algebraic semigroup is an algebraic variety endowed with an associative composition law which is a morphism of varieties. If in addition there is a neutral element, then we obtain the notion of an algebraic monoid.

The theory of linear (or equivalently, affine) algebraic monoids has been chiefly developed by Putcha and Renner (see the books [11] and [12]). Also, the structure of complete algebraic monoids is given by a result of Mumford and Ramanujam ([8] Chapter 2, Section 4, Appendix): if a complete irreducible variety $X$ has a composition law with a neutral element, then $X$ is an abelian variety with group law $\mu$.

In this article, we address the classification of algebraic semigroups and we consider the case of smooth projective surfaces. The case of curves is treated in [3], which also obtains a general structure theorem for complete algebraic semigroups. Using that theorem, we can reduce our classification problem to the following one: determine all tuples $(S, C, \pi, \sigma)$, where $\pi$ is a morphism from such a surface $S$ to a curve $C$, with a section $\sigma$.

We then show that, in most cases, the algebraic semigroup structures on $S$ can be reduced to the minimal model of $S$. Then in Theorem 2.1 we give a full description of those surfaces $S$, which has at least one non-trivial algebraic semigroup structure, when their Kodaira dimension satisfies $\kappa(S) = -\infty$. In the third part, we consider the case $\kappa(S) = 0$, and in Theorem 3.3 and Theorem 3.4 we solve this classification problem when $S$ is bielliptic or abelian; in Theorem 3.14, we show that there are no non-trivial algebraic semigroup structures on an Enriques surface or a general $K3$ surface. For the case $\kappa(S) = 1$, in Theorem 4.2 we give a description of one special type of elliptic surfaces which also admit non-trivial algebraic semigroup laws.

For a given surface $S$, it is an interesting problem to describe all algebraic
semigroup structures on it and determine the dimension of this moduli. In this article, we solve this problem by using a result of [3], Section 4.5, Remark 16: the families of algebraic semigroup laws on a given variety $X$ are parametrized by a quasi-projective scheme $SL(S)$.

Moreover, given an algebraic semigroup structure $\mu_0$ on $S$ and the associated contraction $S \xrightarrow{\sigma} C$, the connected component of $\mu_0$ in $SL(S)$ is isomorphic to

$$\text{Mor}_\pi(C, S) \times A$$

where $\text{Mor}_\pi(C, S)$ is the scheme of sections of $\pi$ and $A$ is an abelian variety which is the smallest two sided ideal of $(S, \mu_0)$. Since $\text{Mor}_\pi(C, S)$ can be viewed as an open subscheme of $Hilb(S)$ by assigning a section to its image in $S$, we can use the local study of $Hilb(S)$ to determine the local dimension of $\text{Mor}_\pi(C, S)$ at each section.

By using the above ”deformation method ”, we solve the problem for those surfaces satisfying $\kappa(S) = 0$ or 1. In Theorem 5.2, we apply the function field version of Mordell-Weil theorem to solve the problem in the case ”$\kappa(S) = 2$”.

Throughout this article, we use the books [5] and [2] as general references for surfaces.

1 Definitions and Rough classification

**Definition 1.1.** An abstract semigroup is a set $S$ equipped with an associative composition law $\mu : S \times S \to S$. When $S$ is a variety and $\mu$ is a morphism, we say that $(S, \mu)$ is an algebraic semigroup.

**Theorem 1.2.** (M.Brion [3], Section 4.3, Theorem 6.)

Let $S$ be a complete variety, and $\mu : S \times S \to S$ a morphism. Then $(S, \mu)$ is an algebraic semigroup if and only if there exist complete varieties $X$, $Y$, an abelian variety $(A, +)$ and morphisms $(\sigma, \pi)$ making the following diagram commutative:

$$\begin{array}{ccc}
X \times A \times Y & \xrightarrow{\sigma} & S \\
\downarrow \alpha & & \\
X \times A \times Y & \xrightarrow{\pi} & S
\end{array}$$

and satisfying $\mu(s_1, s_2) = \sigma(\nu(\pi(s_1), \pi(s_2)))$, where

$$\nu((x_1, a_1, y_1), (x_2, a_2, y_2)) = (x_1, a_1 + a_2, y_2).$$

In particular, $\sigma$ is a closed immersion and a section of $\pi$.

**Notation:** We call $X \times A \times Y$ the kernel of $(S, \mu)$ and $A$ the associated abelian variety of $(S, \mu)$. 

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1.1 First steps in classification

Note that \( \dim S = 2 \) and \( \sigma : X \times A \times Y \hookrightarrow S \) is a closed immersion, so \( \dim X + \dim Y + \dim A \leq 2 \).

Then we list all the possibilities as follows:

1) \( \dim X = \dim Y = 0, \dim A = 2 \). Then \((S, \mu) = (A, +)\) is an abelian surface.

2) \( \dim X = \dim Y = 0, \dim A = 1 \). Then \((A, +)\) is an elliptic curve and we have the following commutative diagram:

\[
\begin{array}{ccc}
(A, +) & \overset{\sigma}{\longrightarrow} & (S, \mu) \\
\downarrow{\mu} & & \downarrow{\pi} \\
(A, +) & & (S, \mu)
\end{array}
\]

where the semigroup law is given by the formula:

\[
\mu(s_1, s_2) = \sigma(\pi(s_1) + \pi(s_2)).
\]

3) \( \dim X = \dim Y = \dim A = 0 \) and \( \mu \) is constant.

4) \( \dim X = 1, \dim Y = 0, \dim A = 0 \) and \( \mu(s_1, s_2) = \sigma(\pi(s_1)) \).

5) (similar to 4)) \( \dim X = 0, \dim Y = 1, \dim A = 0 \) and \( \mu(s_1, s_2) = \sigma(\pi(s_2)) \).

6) \( \dim X = 1, \dim Y = 0, \dim A = 1 \). Then \( S = X \times A \), and the semigroup law is given by \( \mu((x_1, a_1), (x_2, a_2)) = (x_1, a_1 + a_2) \).

7) (similar to 6)) \( \dim X = 0, \dim Y = 1, \dim A = 1 \). Then \( S = Y \times A \) and the semigroup law is given by \( \mu((y_1, a_1), (y_2, a_2)) = (y_2, a_1 + a_2) \).

8) \( \dim X = 2, \dim Y = 0, \dim A = 0 \). Then \( S = X \), and the semigroup law is given by \( \mu(s_1, s_2) = s_1 \).

9) (similar to 8)) \( \dim X = 0, \dim Y = 2, \dim A = 0 \). Then \( S = X \), and the semigroup law is given by \( \mu(s_1, s_2) = s_2 \).

10) \( \dim X = \dim Y = 1 \). Then \( S = X \times Y \) and the semigroup law is given by \( \mu((x_1, y_1), (x_2, y_2)) = (x_1, y_2) \).

Remark 1):

We call cases 1) 3) 6) 7) 8) 9) 10) trivial and they will not be considered any more.

In cases 2) 4) 5), we observe that \( S \) always maps to a curve \( C \) via some
morphism \( \pi : S \rightarrow C \) with a section \( \sigma : C \rightarrow S \), and there is always an algebraic semigroup law \( \tilde{\mu} \) on \( C \), such that the semigroup law \( \mu \) on \( S \) is given by:

\[
\mu(s_1, s_2) = \sigma(\tilde{\mu}(\pi(s_1), \pi(s_2))).
\]

To be concrete, in case 2), \((C, \tilde{\mu}) = (A, +); \) in case 4), \( C = X \), and for all \( x_1, x_2 \in X \), \( \tilde{\mu}(x_1, x_2) = x_1 \); in case 5), \( C = Y \), and for all \( y_1, y_2 \in Y \), \( \tilde{\mu}(y_1, y_2) = y_2 \). We call the semigroup laws on \( S \) in these cases nontrivial.

In the following, we deal with non-trivial algebraic semigroup structures, and we denote the associated contraction morphism \( \pi \) and its section \( \sigma \), by the following diagram \( S \xrightarrow{\pi} \xleftarrow{\sigma} C \).

**Lemma 1.3.** If \( S \xrightarrow{\pi} \xleftarrow{\sigma} C \), then \( \pi_* \mathcal{O}_S = \mathcal{O}_C \).

**Proof.** [3], Section 4.3, Lemma 1.

From this lemma, we deduce that \( \pi \) is a fibration, (which means every fibre of \( \pi \) is connected), and the projective curve \( C \) is normal, hence smooth. Now in order to solve the problem, our task becomes the following one: classify all \( S \xrightarrow{\pi} \xleftarrow{\sigma} C \), where \( \pi \) is a fibration and \( \sigma \) is a section of \( \pi \).

### 1.2 Reduction of the problem to the minimal model

In this subsection, we keep the assumption that \( S \xrightarrow{\pi} \xleftarrow{\sigma} C \). When \( g(C) \geq 1 \), we prove that \( S \) has a non-trivial algebraic semigroup structure if and only if its minimal model has one. This will reduce our problem to the case that \( S \) is minimal. In Proposition 1.4, we consider the blowing-up of a point on \( S \), and we show that any non-trivial algebraic semigroup structure can be lifted to this blowing-up. Then we give an example to show that in some cases, after blowing-down an exceptional rational curve on \( S \), the non-trivial semigroup structures are not preserved. In Proposition 1.5, we show that, if \( g(C) \geq 1 \), any non-trivial algebraic semigroup structure is preserved after contracting an exceptional rational curve.

The main results of this subsection are Proposition 1.4 and Proposition 1.5.

**Proposition 1.4.** Assume that the non-trivial semigroup law \( \mu \) on \( S \) is given by the formula \((1)\) in Remark 1). Let \( \varphi : S' \rightarrow S \) denote the blowing-up of an arbitrary point \( P \) on \( S \). Then \( S' \) has a unique non-trivial algebraic semigroup structure such that \( \varphi \) is a homomorphism.

**Proof.** First, we construct a morphism from \( S' \) to \( C \), and a section of this morphism; then we construct a non-trivial algebraic semigroup structure on \( S' \), and verify that \( \varphi \) is a homomorphism.

Composing \( \pi \) with \( \varphi \), we get a morphism from \( S' \) to \( C \), denoted by \( \pi' = \pi \circ \varphi : S' \rightarrow C \).
We now construct a section of $\pi'$. Since $\sigma$ maps $C$ isomorphically to its image $\sigma(C)$, in order to construct a morphism from $C$ to $S'$, we consider the strict transform of $\sigma(C)$, and denote it by $C'$.

Then $\varphi_{|C'} : C' \to \sigma(C)$ is the blowing-up of the point $P$ on $\sigma(C)$. Since $\sigma(C)$ is smooth, $\varphi_{|C'}$ is an isomorphism. Let $\sigma' = \varphi_{|C'}^{-1} \circ \sigma$, then $\sigma'$ is a morphism from $C$ to $S'$. Next let us verify that $\sigma'$ is a section of $\pi'$:

$$\pi' \circ \sigma' = \pi \circ \varphi \circ \sigma' = \pi \circ \sigma = id_C.$$  \hspace{1cm} (2)

Thus we construct a semigroup law $\mu'$ on $S'$ by using the formula (1) in Remark 1. We now verify that $\varphi'$ is a homomorphism of semigroups. Let $s'_1, s'_2 \in S'$, then

$$\mu(\varphi(s'_1), \varphi(s'_2)) = \sigma(\tilde{\mu} \circ \varphi(\tilde{s}'_1), \pi \circ \varphi(\tilde{s}'_2))) = \sigma(\tilde{\mu}(\pi'(s'_1), \pi'(s'_2)))$$  \hspace{1cm} (3)

and we also have the following equations:

$$\varphi(\mu'(s'_1, s'_2)) = \varphi'(\sigma(\tilde{\mu}(\pi'(s'_1), \pi'(s'_2)))) = \sigma(\tilde{\mu}(\pi'(s'_1), \pi(s'_2))).$$  \hspace{1cm} (4)

So we get $\mu(\varphi(s'_1), \varphi(s'_2)) = \varphi(\mu'(s'_1, s'_2))$, which means that $\varphi$ is a homomorphism of semigroups.

Finally the uniqueness of the semigroup law making $\varphi$ a homomorphism is due to the fact that $S' \to S$ is birational.

We already know that blowing-up of a point on $S$ preserves a non-trivial algebraic semigroup structure, what happens if we perform a blowing-down? The next example shows that the non-trivial semigroup structures are not necessarily preserved.

**Example 1.** Consider the blowing-down morphism:

$$\varphi : \text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{P}^2.$$  

We show that there exist non-trivial algebraic semigroup structures on $\text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$, but every algebraic semigroup structure on $\mathbb{P}^2$ is trivial. Note that $\text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$ is ruled over $\mathbb{P}^1$ and the ruling morphism $\pi$ has sections. For each section, we can define a semigroup law on $\text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$ as in Remark 1). For example, recall the projection morphism $\sigma : \mathbb{P}^1 \to \text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$. We can thus define a non-trivial semigroup law $\mu$ on $\text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1))$ by $\mu(x_1, x_2) = \sigma \circ \pi(x_1)$. Also recall that for an arbitrary smooth projective curve $C$, there are only constant morphisms from $\mathbb{P}^2$ to $C$, which means that there will be no retraction from $\mathbb{P}^2$ to a curve. It follows that the only possible algebraic semigroup structures on $\mathbb{P}^2$ are trivial.
Proposition 1.5. Let $(S, \mu)$ satisfy the same assumptions as in Proposition 1.4.

Let $\varphi : S \to \tilde{S}$ denote the blowing-down of an exceptional rational curve $E$ on $S$. Furthermore, we require that $\pi(E)$ is a point $P$. Then there exists a unique non-trivial semigroup law $(\tilde{S}, \tilde{\mu})$ such that $\varphi$ is a homomorphism.

Now let us begin the proof of Proposition 1.5.

**Proof.** First we construct a morphism from $C$ to $\tilde{S}$, then we construct a contraction of this morphism. After defining a non-trivial semigroup structure on $\tilde{S}$, we verify that $\varphi$ is a homomorphism.

Composing $\varphi$ with $\sigma$, we get a morphism $\tilde{\sigma} = \varphi \circ \sigma : C \to \tilde{S}$.

Since $\varphi : S - E \to \tilde{S} - P$ is an isomorphism, there is a set-theoretical map $\tilde{\pi}$ from $\tilde{S}$ to $C$ satisfying $\pi = \tilde{\pi} \circ \varphi$.

Let us verify that $\tilde{\pi}$ is a morphism. It is easy to see $\tilde{\pi}$ is continuous, because the preimage of any finite set is closed in $\tilde{S}$. We now define $\tilde{\pi}^\# : O_C \to \tilde{\pi}_* O_{\tilde{S}}$. For any open subset $U$ of $C$, there is a ring homomorphism $\pi^\# U : O_C(U) \to O_S(\pi^{-1}(U))$. Since $\varphi_* O_S = O_{\tilde{S}}$, there is an isomorphism $\tilde{\pi}^\#_{\pi^{-1}(U)} : O_{\tilde{S}}(\tilde{\pi}^{-1}(U)) \to O_S(\pi^{-1}(U))$. We define $\tilde{\pi}^\#_{\pi^{-1}(U)} : O_C(U) \to O_S(\tilde{\pi}^{-1}(U))$ by $\tilde{\pi}^\#_{\pi^{-1}(U)} = \varphi^{-1}_{\pi^{-1}} \circ \pi^\# U$. Then $(\tilde{\pi}, \tilde{\pi}^\#)$ is a morphism of ringed spaces, so we get a morphism $\tilde{\pi}$ from $\tilde{S}$ to $C$.

Now we verify $\tilde{\sigma}$ is a section of $\tilde{\pi}$:

$$\tilde{\pi} \circ \tilde{\sigma} = \tilde{\pi} \circ \varphi \circ \sigma = \pi \circ \sigma = id_C.$$  \hspace{1cm} (5)

Thus we have constructed a non-trivial algebraic semigroup structure $\tilde{\mu}$ on $\tilde{S}$ as in Remark 1).

Let us verify that $\varphi$ is a homomorphism of semigroups: for any $s_1, s_2 \in S$,

$$\varphi(\mu(s_1, s_2)) = \varphi(\tilde{\mu}(\pi(s_1), \pi(s_2))) = \varphi(\tilde{\mu}(\tilde{\pi}(s_1), \tilde{\pi}(s_2))).$$  \hspace{1cm} (6)

and

$$\tilde{\mu}(\varphi(s_1), \varphi(s_2)) = \tilde{\sigma} \circ \tilde{\mu}(\tilde{\pi} \circ \varphi(s_1), \tilde{\pi} \circ \varphi(s_2)).$$  \hspace{1cm} (7)

Since

$$\tilde{\pi} \circ \varphi = \pi \hspace{1cm} \text{(8)}$$

and

$$\tilde{\sigma} = \varphi \circ \sigma \hspace{1cm} \text{(9)}$$

we have $\mu(\varphi(s_1), \varphi(s_2)) = \varphi(\tilde{\mu}(s_1, s_2))$, which means that $\varphi$ is a homomorphism.

Finally, $S$ and $\tilde{S}$ are birationally equivalent, so there is a unique non-trivial algebraic semigroup structure on $\tilde{S}$ such that $\varphi$ is a homomorphism. \hfill \qed

In view of the last two propositions, when $g(C) \geq 1$, we can can reduce our problem to the case where $S$ is minimal, the case $g(C) = 0$ need further study. But whatever in what follows, we always assume $S$ is minimal.
2 The case $\kappa(S) = -\infty$

In this section, we always assume $\kappa(S) = -\infty$, and our aim is to prove:

**Theorem 2.1** (Main theorem).

If $S \xrightarrow{\pi} C$.

1) If $g(C) \geq 1$, then $\pi$ is a ruling morphism.

2) If $g(C) = 0$ and the general fibre of $\pi$ is rational, then $S \cong \text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-d))$ for some $d \neq 1$ and $\pi$ is the ruling morphism.

3) If $g(C) = 0$, i.e. $C \cong \mathbb{P}^1$ and the general fibre of $\pi$ is not rational, then $S \cong \mathbb{P}^1 \times X$, where $X$ is a projective smooth curve satisfying $g(X) \geq 1$ and $\pi = \text{pr}_1$ is the first projection to $\mathbb{P}^1$.

The parts 1) and 2) of Theorem 2.1 are more or less trivial. In order to prove 3), we analyze the cone of curves of $S$, $\text{NE}(S)$. To be more precise, we show that $\text{NE}(S)$ is two-dimensional, and give an explicit description of the extremal rays of this convex cone. Before proving Theorem 2.1, we need some general facts about ruled surfaces.

**Definition 2.2.** (definition and notation)

a) (definition of normalised sheaf).

If $\pi : S \to C$ is a ruled surface, then there exists a rank two locally free sheaf $\mathcal{E}$ on $C$ s.t. $S \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, $H^0(\mathcal{E}) \neq 0$ and for an arbitrary invertible sheaf $\mathcal{L}$ with $\deg(\mathcal{L}) < 0$ on $C$, $H^0(\mathcal{E} \otimes \mathcal{L}) = 0$ (the existence of such $\mathcal{E}$ can be found in [5], Chapter 5, Proposition 2.8). We call such $\mathcal{E}$ normalised.

b) (definition of invariant $e$)

For any normalised sheaf $\mathcal{E}$, we define $e$ by

$$e = -\deg(\mathcal{E}) = -\deg(\bigwedge^2 \mathcal{E}). \quad (10)$$

This number is a well-defined invariant of $S$, not depending on the $\mathcal{E}$ we choose.

(The proof that $e$ is a well-defined invariant, can be found in [5], Chapter 5, Proposition 2.8).

c) (some notation)

In $\text{Pic}(S)$, we let $C_0$ denote the linear equivalence class of $\mathcal{O}_{\mathbb{P}^1(\mathcal{E})}(1)$. And we let $f$ denote the numerical equivalence class of any general fibre of $\pi$. 

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We quote the following lemmas without proof, they can be found in the book [5], Proposition 2.6, Proposition 2.20 and Proposition 2.21. We will use these lemmas to describe $NE(S)$.

**Lemma 2.3.** Let $S \simeq \mathbb{P}_C(E)$ be a ruled surface over $C$, for some normalised sheaf $E$. Then there is a one-to-one correspondence between sections of $\pi$ and surjections $E \to \mathcal{L} \to 0$, where $\mathcal{L}$ is an invertible sheaf on $C$. There exists a section $\sigma$ such that $\sigma(C) = \mathcal{O}_{\mathbb{P}_C(E)}(1)$ in $\text{Pic}(S)$ ($\sigma$ may not be unique), $\sigma$ is called a canonical section of $\pi$.

**Lemma 2.4.** If $\pi : S \to C$ is a ruled surface, then
a) $\text{Pic}(S) = \mathbb{Z}C_0 \oplus \pi^*\text{Pic}(C)$.

b) The self-intersection number satisfies $C_0^2 = -e$.

c) The canonical line bundle of $S$ satisfies $K_S \sim_{\text{num}} -2C_0 + (2g(C) - 2 - e)f$.

**Lemma 2.5.** If $\pi : S \to C$ is a ruled surface, with $e \geq 0$ then:

a) For any irreducible curve $Y$ on $S$, $Y$ is numerically equivalent to $aC_0 + bf$, for some $a, b \in \mathbb{Z}$. If $Y$ is not numerically equivalent to $C_0$ or $f$, then $a > 0, b \geq ae$.

b) A divisor $D \sim_{\text{num}} aC_0 + bf$ is ample if and only if $a > 0$ and $b > ae$.

**Lemma 2.6.** If $\pi : S \to C$ is a ruled surface, with $e < 0$ then:

a) For any irreducible curve $Y$ on $S$, $Y$ is numerically equivalent to $aC_0 + bf$ for some $a, b \in \mathbb{Z}$. If $Y$ is not numerically equivalent to $C_0$ or $f$, then either $a = 1, b > 0$ or $a \geq 2, 2b \geq ae$.

b) A divisor $D \sim_{\text{num}} aC_0 + bf$ is ample if and only if $a > 0, 2b > ae$.

The next lemma is a version of the “Rigidity lemma”. (See Lemma 1.15, [4]) For convenience, we give a detailed proof in the following.

**Lemma 2.7.** (Rigidity lemma)
Assume that there are two fibrations $\pi_1 : S \to C_1$, $\pi_2 : S \to C_2$ where $C_1$, $C_2$ are smooth curves. If $\pi_2$ contracts one fibre $F_1$ of $\pi_1$, then it contracts all fibres of $\pi_1$ and there exists an isomorphism $\theta : C_1 \to C_2$, s.t. $\pi_2 = \theta \circ \pi_1$, which means $\pi_1$ and $\pi_2$ are isomorphic as fibrations.

**Proof.** First we prove that $\pi_2$ contracts all fibres of $\pi_1$. Observe that $\pi_2$ contracts $F_1$ if and only if $\pi_2^*(F_1) \sim_{\text{num}} 0$. Since the fibres of $\pi_1$ are parameterized by $C_1$, they are all numerically equivalent. So for an arbitrary fibre $F$ of $\pi_1$, we have $\pi_2^*(F) \sim_{\text{num}} 0$, which implies that $\pi_2$ also contracts $F$.

Then $\pi_2$ factors through $\pi_1$ set-theoretically, i.e. there exists a map $\theta$ s.t. $\pi_2 = \theta \circ \pi_1$.

We now show that $\theta$ is a morphism.

Pick an arbitrary open set $U$ of $C_2$, we have $\pi_2^{-1}(U) = \pi_1^{-1} \circ \theta^{-1}(U)$. Since
\(\pi_1\) is surjective, \(\pi_1(\pi_2^{-1}(U)) = \theta^{-1}(U)\). Because \(\pi_1\) is flat, it is an open map. Since \(\pi_1\) maps the open set \(\pi_2^{-1}(U)\) onto \(\theta^{-1}(U)\), so \(\theta^{-1}(U)\) is open in \(C_1\). Now we have proved that \(\theta\) is continuous.

Then we construct \(\theta^\sharp : \mathcal{O}_{C_2} \rightarrow \theta_*\mathcal{O}_{C_1}\). Since both \(\pi_1\) have connected fibres, \(\pi_1^*\mathcal{O}_S = \mathcal{O}_{C_1}\). So for an arbitrary open subset \(U\) of \(C_2\), both \(\pi_1^1 : \mathcal{O}_{C_2}(U) \rightarrow \mathcal{O}_S(\pi_2^{-1}(U))\) and \(\pi_2^1 : \mathcal{O}_{C_1}(\theta^{-1}(U)) \rightarrow \mathcal{O}_S(\pi_2^{-1}(U))\) are isomorphisms. We define \(\theta_U^\sharp : \mathcal{O}_{C_2}(U) \rightarrow \mathcal{O}_{C_1}(\theta^{-1}(U))\) by \(\theta_U^\sharp = \pi_2^\sharp \circ \pi_1^\sharp\), then \((\theta, \theta^\sharp)\) is a morphism of ringed spaces. Now we have proved that \(\theta\) is a morphism of varieties.

Then \(\theta\) is a finite morphism between smooth projective curves. Since both \(\pi_1\) have connected fibres, \(\text{deg}(\theta) = 1\), which implies \(\theta\) is an isomorphism. \(\square\)

Recall the following result from the book [1], Theorem 18.4, Chapter 3:

**Theorem 2.8.** (Iitaka’s conjecture \(C_{2,1}\)). Let \(\varphi : S \rightarrow C\) be a fibration, then the following inequality holds for any general fibre \(S_c\):

\[
\kappa(S_c) + \kappa(C) \leq \kappa(S).
\]  

(11)

Now let us begin the proof of Theorem 2.8.

**Proof.**

1) Assume that \(g(C) \geq 1\). Then for any general fibre \(F_0\) of \(\pi\), by Theorem 2.1, \(\kappa(F_0) + \kappa(C) \leq \kappa(S) = -\infty\). Then \(\kappa(F_0) = -\infty\), which means \(F_0\) is smooth rational, so \(\pi\) is a ruling morphism.

2) If \(g(C) = 0\) and general fibres of \(\pi\) are smooth rational curves, then \(S\) is ruled over \(C \simeq \mathbb{P}^1\), which implies that \(S\) is rational. Since \(S\) is minimal, \(S\) is \(\mathbb{P}^2\) or \(S \simeq \text{Proj}_d(\mathcal{O} \oplus \mathcal{O}(d))\) for some \(d \neq 1\). But every morphism from \(\mathbb{P}^2\) to \(\mathbb{P}^1\) is constant, so there will be no sections of \(\pi\). So \(S \not\simeq \mathbb{P}^2\) is impossible, and we get \(S \simeq \text{Proj}_d(\mathcal{O} \oplus \mathcal{O}(d))\) for some \(d \neq 1\).

3) In this paragraph, we show that \(S\) is ruled over some smooth curve \(X\) and introduce some notation. Since \(\kappa(S) = -\infty\) and \(S\) is minimal, \(S\) is a ruled surface over some projective smooth curve \(X\). We denote the ruling morphism by \(\varphi : S \rightarrow X\), and write \(S\) as \(\mathbb{P}_X(\mathcal{E})\) for some normalised sheaf \(\mathcal{E}\) on \(X\). Using the notation of Definition 2.3, we denote the canonical section of \(\varphi\) by \(\tau\). Then we have the following diagram involving two morphisms with sections:

\[
\begin{array}{ccc}
S & \xrightarrow{\tau} & X \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & & \end{array}
\]

We now show that \(NE(S)\) is closed. Let \(F_0\) be a general fibre of \(\pi\) and denote its numerical equivalence class by \(f_0\). We show that \(F_0\) can’t
be contracted by \( \varphi \). Otherwise, by the “Rigidity lemma”, \( \varphi \) will factor through \( \pi \). Then \( \varphi \), \( \pi \) will be isomorphic as fibrations, the general fibres of \( \pi \) will be rational, which contradicts our assumptions of 3). So \( f_0 \) is not a multiple of \( f \), which implies that the extremal lines generated by them are different. Since the second Betti-number, \( b_2(S) = 2 \), the cone \( NE(S) \) is two-dimensional, and its boundary \( \partial NE(S) \) consists of two extremal lines. Since the self-intersection numbers \( f_0^2 \) and \( f^2 \) are zero, by the “cone theorem for surfaces”(Lemma 6.2, Chapter 6, [4]), we know that both \( f_0 \) and \( f \) lie in \( \partial NE(S) \). It follows that \( \partial NE(S) \) consists exactly of the two extremal lines generated by \( f_0 \) and \( f \). Since \( f_0 \) and \( f \) belong to \( NE(S) \), \( \partial NE(S) \subset NE(S) \), which means \( NE(S) \) is closed.

In the following, we aim to prove that the intersection number \( f_0 \cdot f = 1 \). The two cases \( e < 0 \), \( e \geq 0 \) will be dealt with separately.

a) The case \( e < 0 \):

First, we determine the ample cone and the nef cone of \( S \).

By Lemma 2.6 we have

\[ \text{Amp}(S) = \{ aC_0 + bf | a > 0, \quad 2b > ae \}. \]

Taking the closure of \( \text{Amp}(S) \), we get the nef cone:

\[ \text{Nef}(S) = \{ aC_0 + bf | a \geq 0, \quad 2b \geq ae \}. \]

Then, we determine the numerical equivalence class of \( f_0 \).

Let \( f_0 = a_1 C_0 + b_1 f \), where \( a_1 = 1 \), \( b_1 \geq 0 \) or \( a_1 \geq 2 \), \( 2b_1 \geq a_1 e \).

If \( a_1 = 1 \) and \( f_0 = C_0 + b_1 f \), \( b_1 \geq 0 \), then \( f_0 \in \text{Amp}(S) \). This contradicts \( f_0^2 = 0 \).

So \( f_0 = a_1 C_0 + b_1 f \), where \( a_1 \geq 2 \), \( 2b_1 \geq a_1 e \).

Let us determine the numbers \( a_1 \) and \( b_1 \) by calculating the intersection numbers \( f_0^2 \) and \( f_0 \cdot \sigma(\mathbb{P}^1) \).

\[ 0 = f_0^2 = a_1^2 C_0^2 + 2a_1 b_1 = 2a_1 b_1 - e a_1^2 \]  \hspace{1cm} (12)

we get \( 2b_1 = a_1 e \), which implies \( f_0 = \frac{a_1}{2}(2C_0 + ef) \).

Let \( d = (2C_0 + ef) \cdot \sigma(\mathbb{P}^1) \), since \( f_0 \cdot \sigma(\mathbb{P}^1) = 1 \), \( a_1 d = 2 \). Note that \( a_1 \geq 2 \), so \( a_1 = 2 \), \( b_1 = e \). Hence

\[ f_0 = 2C_0 + ef. \]  \hspace{1cm} (13)

Then we show that \( f_0 \cdot f = 1 \).

If \( \sigma(\mathbb{P}^1) = f \), then \( f_0 \cdot f = f_0 \cdot \sigma(\mathbb{P}^1) = 1 \), we are done.

Our aim is to exclude the case \( \sigma(\mathbb{P}^1) \neq f \). Since \( f_0 \cdot \sigma(\mathbb{P}^1) = 1 \), \( \sigma(\mathbb{P}^1) \) is not a multiple of \( f_0 \). If \( \sigma(\mathbb{P}^1) \neq f \), by Lemma 2.6 \( \sigma(\mathbb{P}^1) = a_2 C_0 + b_2 f \), where \( a_2 = 1 \), \( b_2 \geq 0 \) or \( a_2 \geq 2 \), \( 2b_2 > a_2 e \), in both cases, \((a_2, b_2)\) satisfies \( a_2 > 0 \), \( 2b_2 > a_2 e \), which means \( \sigma(\mathbb{P}^1) \) lies in \( \text{Amp}(S) \).
Consider the canonical divisor $K_S$:
\[
K_S = -2C_0 + (2g(X) - 2 - e)f
\]
\[
= (-2C_0 - ef) + (2g(X) - 2)f
\]
\[
= -f_0 + (2g(X) - 2)f.
\]

So by Riemann-Roch theorem:
\[
g(\sigma(P^1)) = 1 + \frac{1}{2}(\sigma(P^1)^2 + \sigma(P^1) \cdot K_S)
\]
\[
= 1 + \frac{1}{2}\sigma(P^1)^2 + \frac{1}{2}\{-f_0 + (2g(X) - 2)f\} \cdot \sigma(P^1).
\]

Recall a result of Nagata \[10\], Theorem 1:
“Let $S$ be a $P^1$-bundle over a smooth curve $X$ of genus $g(X)$, then the invariant $e \geq -g(X)$.”

So $g(X) \geq -e > 0$. Since $\sigma(P^1)$ is ample, $\sigma(P^1)^2 > 0$ and $\sigma(P^1) \cdot f > 0$. We have
\[
g(\sigma(P^1)) > 1 - \frac{1}{2}f_0 \cdot \sigma(P^1) = 1 - \frac{1}{2} > 0
\]
which is impossible.

So finally we get $\sigma(P^1) = f$ and $f \cdot f_0 = \sigma(P^1) \cdot f_0 = 1$

b) The case $e \geq 0$

We now show $f_0 = C_0$.

By Lemma 2.5 a), we have $f_0 = a_1C_0 + b_1f$ where $a_1 \geq 0$ and $b_1 \geq a_1e$ or $a_1 = 1, b_1 = 0$.

If $b_1 > a_1e$. Then by Lemma 2.6 b), $f_0$ is ample, which is impossible.

So $b_1 = a_1e$, $f_0 = a_1(C_0 + ef)$. Since $f_0^2 = 0, a_1^2(C_0^2 + 2e) = a_1^2e = 0$.

So $b_1 = a_1e$ must be zero, and $f_0 = a_1C_0$. But $\sigma(P^1) \cdot f_0 = 1$, so $a_1\sigma(P^1) \cdot C_0 = 1$, which implies that $a_1$ must be 1. So $f_0 = C_0$ and $f_0 \cdot f = 1$.

In conclusion of our analysis of cases a) and b), we get that $f_0 \cdot f = 1$.

Define $\alpha : S \to P^1 \times X$ by $\alpha = (\pi, \varphi)$, we show $\alpha$ is an isomorphism.

Assume there is an irreducible curve $C'$ on $S$ contracted by $\alpha$, then $C'$ will be contracted by both $\pi$ and $\varphi$. Since every fibre of $\varphi$ is integral, $C'$ will be some fibre $F'$ of $\varphi$. But we already know that no fibre of $\varphi$ can be contracted by $\pi$. So $\alpha$ contracts no reducible curves on $S$. This implies that $\alpha$ is a quasi-finite morphism. Since $\alpha$ is also projective, it is a finite morphism.

Since $deg(\alpha) = f_0 \cdot f = 1$, $\alpha$ is an isomorphism. So we get $S \simeq P^1 \times X$ and $\pi = pr_1$. 

\[\square\]
For a given surface $S$, it is natural to ask how many algebraic semi-group laws exist on it? In the next theorem we solve this problem for minimal rational surfaces.

**Theorem 2.9.** If $S$ is a minimal rational surface, then there are finitely many algebraic semigroup laws on $S$ modulo $\text{Aut}(S)$.

**Proof.** Let $(S, \mu)$ be a non-trivial algebraic semigroup law, we denote its associated contraction by the following diagram: $S \xrightarrow{\mu} C$, where $C$ is the kernel of $\mu$. Since $(S, \mu)$ is non-trivial, $C \cong \mathbb{P}^1$ and $\pi$ is a ruling morphism. So $S \cong \text{Proj}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-d))$ for some $d \neq 1$. Then sections of the ruling morphism $\pi$ are in one-to-one correspondence with surjections $\mathcal{O} \oplus \mathcal{O}(-d) \xrightarrow{p} \mathcal{L} \rightarrow 0$, where $\mathcal{L}$ is an invertible sheaf on $\mathbb{P}^1$. Consider the kernel of $p$, we denote it by $N$, then $N$ is also an invertible sheaf. Now there is an exact sequence $0 \rightarrow N \rightarrow \mathcal{O} \oplus \mathcal{O}(-d) \xrightarrow{p} \mathcal{L} \rightarrow 0$. Observe that $\text{deg}(N) \leq 0$, we let $N = \mathcal{O}(-d_2)$ and $\mathcal{L} = \mathcal{O}(d_1)$, where $d_2 \geq 0$ and both $d_i$ are integers.

**Case 1),** $d_2 > 0$.

We consider the long-exact sequence:

$0 \rightarrow H^0(\mathbb{P}^1, \mathcal{O}(-d_2)) \rightarrow H^0(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(-d_2)) \xrightarrow{p_1} H^0(\mathbb{P}^1, \mathcal{L})$.

Since $\dim H^0(\mathbb{P}^1, \mathcal{O}(-d_2)) = 1$, $p_1$ is injective. So $\dim H^0(\mathbb{P}^1, \mathcal{L}) \geq 1$, which implies that $\text{deg}(\mathcal{L}) \geq 0$. Consider

$$
\text{Ext}^1_{\mathbb{P}^1}(N, \mathcal{L}) = \text{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(-d_2), \mathcal{O}(d_1)) = \text{Ext}^1_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}(d_1 + d_2)) = H^1(\mathbb{P}^1, \mathcal{O}(d_1 + d_2)) = H^0(\mathbb{P}^1, \mathcal{O}(-2 - d_1 - d_2)).
$$

Since $d_1 = \text{deg}(\mathcal{L}) \geq 0$, $d_2 > 0$, $\dim H^0(\mathbb{P}^1, \mathcal{O}(-d_1 - d_2)) = 0$. So any extension of $N$ by $\mathcal{L}$ is trivial, which implies $N \oplus \mathcal{L} \cong \mathcal{O} \oplus \mathcal{O}(-d)$. Since $\mathcal{O} \oplus \mathcal{O}(-d)$ is normalised, $\text{deg}(\mathcal{L}) \leq 0$, but we already know that $\text{deg}(\mathcal{L}) \geq 0$, so $\mathcal{L} \cong \mathcal{O}$. Observe that

$$
\Lambda^2(N \oplus \mathcal{L}) = N \otimes \mathcal{L} = \mathcal{O}(-d).
$$

So $N \cong \mathcal{O}(-d)$ and there exists a commutative diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}(-d) & \rightarrow & \mathcal{O} \oplus \mathcal{O}(-d) & \rightarrow & \mathcal{O} & \rightarrow & 0 \\
\downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \downarrow & \cong & \\
0 & \rightarrow & N & \rightarrow & \mathcal{O} \oplus \mathcal{O}(-d) & \rightarrow & \mathcal{L} & \rightarrow & 0
\end{array}
\]
where $\theta$ is an automorphism.

Case 2), $d_2 = 0$.

We have $N \simeq \mathcal{O}$ and $\mathcal{L} = \mathcal{O}(-d)$. Consider the surjection

$$\mathcal{O} \oplus \mathcal{O}(-d) \xrightarrow{p} \mathcal{L} \xrightarrow{} 0,$$

then $p \in \text{Hom}(\mathcal{O} \oplus \mathcal{O}(-d), \mathcal{O}(-d)) = \text{Hom}(\mathcal{O}(-d), \mathcal{O}(-d)) = \mathbb{C}$, so $p$ is a multiplication by some non-zero scalar $\lambda \in \mathbb{C}$. Consider the isomorphism $\alpha = (id, \times \lambda) : \mathcal{O} \oplus \mathcal{O}(-d) \to \mathcal{O} \oplus \mathcal{O}(-d)$, then it fits into the following commutative diagram:

$$\begin{array}{ccc}
\mathcal{O} \oplus \mathcal{O}(-d) & \xrightarrow{p} & \mathcal{L} \\
\downarrow{\alpha} & & \downarrow{\simeq} \\
\mathcal{O} \oplus \mathcal{O}(-d) & \xrightarrow{p_2} & \mathcal{O}(-d)
\end{array}$$

where $p_2$ is the second projection.

In conclusion, modulo $\text{Aut}(\mathcal{O} \oplus \mathcal{O}(-d))$, there are finitely many surjections $\mathcal{O} \oplus \mathcal{O}(-d) \to \mathcal{L}$. So modulo $\text{Aut}(S)$, there are finitely many sections of $\pi$. $\square$

3  Classification in the case $\kappa(S) = 0$

In this section, we assume that the Kodaira dimension of $S$ equals zero.

First we state a classification theorem, which is part of “Enriques Classification Theorem”. We recall some useful notations: $p_g = \dim H^0(S, \mathcal{O}_S(K_S))$, $q = \dim H^1(S, \mathcal{O}_S)$.

**Theorem 3.1. (Classification theorem)**

If $\kappa(S) = 0$, then $S$ is one of the following surfaces:

1) $K3$ surface, $p_g = 1, q = 0, K_S = 0$.

2) Enriques surface, $p_g = 0, q = 0, 2K_S = 0$.

3) Bielliptic surface.

This means there are two elliptic curves $E$, $F$ and a finite group $G$ of translations of $E$ acting also on $F$ such that $F/G \simeq \mathbb{P}^1$, and $S \simeq (E \times F)/G$. In this case, $p_g = 0, q = 1$.

4) Abelian surface. In this case $p_g = 1, q = 2$.

**Proof.** [2], Chapter 8, Theorem 2.

Observe that in cases 1) and 2), $q = 0$; in cases 3) and 4), $q > 0$. We consider the case $q > 0$ first, then in the second part of this section, we consider the case $q = 0$. 13
3.1 The case $q > 0$

First let us recall the classification of bielliptic surfaces in the following lemma. Our main results of this subsection are Theorem 3.3 and Theorem 3.4.

**Lemma 3.2.** Keep the notation of Theorem 3.1, then every bielliptic surface is of one of the following types:

1) $(E \times F)/G$, $G = \mathbb{Z}/2\mathbb{Z}$, acting on $F$ by $x \mapsto -x$.

2) $(E \times F)/G$, $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, acting on $F$ by $x \mapsto -x, x \mapsto x + \varepsilon$, where $\varepsilon \in \mathbb{F}_2$.

3) $(E \times F_i)/G$, where $F_i = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z})$, and $G = \mathbb{Z}/4\mathbb{Z}$, acting on $F$ by $x \mapsto ix$.

4) $(E \times F_i)/G$, $G = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, acting on $F$ by $x \mapsto ix, x \mapsto x + \frac{1+i}{2}$.

5) $(E \times F_\rho)/G$, where $F_\rho = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}_\rho)$, $\rho$ is a primitive cubic root of identity, $G = \mathbb{Z}/3\mathbb{Z}$ acting on $F$ by $x \mapsto \rho x$.

6) $(E \times F_\rho)/G$, and $G = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, acting on $F$ by $x \mapsto \rho x, x \mapsto x + \frac{1}{3}$.

7) $(E \times F_\rho)/G$, and $G = \mathbb{Z}/6\mathbb{Z}$, acting on $F$ by $x \mapsto -\rho x$.

**Proof.** [2], Chapter 6, Proposition 6.20.

**Theorem 3.3.** Keep the notation of Lemma 3.2. If $S$ is bielliptic, and $S \cong \mathbb{C} \times C$, then:

a) $E/G \simeq C$.

b) the action of $G$ on $F$ has a fixed point $P$.

c) $S$ is of type 1), 3), 5) or 7). Moreover, $\pi = p_1$ is the first projection and $\sigma : E/G \to S \simeq (E \times F)/G$ is of the form $\sigma(x) = (x, P)$.

**Theorem 3.4.** Let $S$ be an abelian surface, and $S \cong \mathbb{C} \times C$. Then $S \cong C \times E$ for some elliptic curve $E$, $\pi = p_1$ is the first projection and $\sigma : C \to C \times E$, is defined by $x \mapsto (x, y_0)$ for some $y_0 \in E$.

In the following, we analyze the Albanese variety $Alb(S)$ and show that $C$ is elliptic. Then we use this result to prove Theorem 3.4. For Theorem 3.3, we transfer our problem to the existence of $G$-equivariant morphisms from $E$ to $F$. Then by some detailed calculations, we find that the action of $G$ on $F$ must have a fixed point.

In the following lemma, we show that $C$ is elliptic.

Before starting the proof, we recall the definition and the universal property
Let $X$ be a smooth projective variety. There exists an abelian variety $A$ and a morphism $\alpha_X : X \to A$ with the following universal property: for any complex torus $T$ and any morphism $f : X \to T$, there exists a unique morphism $\tilde{f} : A \to T$, such that $f \circ \alpha = f$. The abelian variety $A$ is called the Albanese variety of $X$, and written by $Alb(X)$; the morphism $\alpha_X$ is called the Albanese morphism.

**Lemma 3.5.** If $q(S) > 0$ and there exists a curve $C$ s.t. $S \subset C$, then $C$ must be elliptic.

**Proof.** By Theorem 3.1, $S$ is bielliptic or abelian.

a) The case when $S$ is abelian. Firstly, $C$ is not rational, because an abelian variety contains no rational curve. Now we prove $g(C) \leq 1$.

Recall that for an arbitrary smooth variety $X$, $Alb(X)$, as a group, is generated by the image $\alpha(X)$.

So the surjective morphism $\pi : S \to C$ induces a surjective morphism $\tilde{\pi} : Alb(S) \to Alb(C)$ such that the following diagram:

$$
\begin{array}{ccc}
S & \xrightarrow{\pi} & C \\
\downarrow{\alpha_S} & & \downarrow{\alpha_C} \\
Alb(S) & \xrightarrow{\tilde{\pi}} & Alb(C)
\end{array}
$$

is commutative.

Since $\alpha_S$ is an isomorphism, $\tilde{\pi} \circ \alpha_S$ is surjective. Note that it factors through the curve $C$, so $g(C) = \dim(Alb(C)) \leq 1$.

We conclude that $C$ must be elliptic.

b) The case $S$ is bielliptic.

Using the notation of Theorem 3.1, we write $S$ as $(E \times F)/G$.

There is a diagram:

$$
\begin{array}{ccc}
E \times F & \xrightarrow{\varphi} & (E \times F)/G \\
\downarrow{f} & & \downarrow{\pi} \\
C
\end{array}
$$

where $\varphi$ is the quotient morphism. Let $f = \pi \circ \varphi$, we get a morphism from the abelian surface $E \times F$ to $C$. Then $f$ induces a surjective morphism $\tilde{f} : E \times F \to Alb(C)$, such that the diagram:

$$
\begin{array}{ccc}
E \times F & \xrightarrow{f} & C \\
\downarrow{f} & & \downarrow{\alpha_C} \\
Alb(C)
\end{array}
$$

is commutative. Since $f$ factors through the curve $C$, $\dim(Alb(C)) \leq 1$, so $g(C) \leq 1$. 

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If $g(C) = 0$, we consider the cartesian square:

$$
\begin{array}{ccc}
Y & \xrightarrow{\sigma'} & E \times F \\
\downarrow & & \downarrow \varphi \\
C \simeq \mathbb{P}^1 & \xrightarrow{\sigma} & (E \times F)/G
\end{array}
$$

Observe that $G$ acts freely on $E \times F$, the quotient morphism $\varphi : E \times F \to (E \times F)/G$ is étale, so $Y$ is also an étale cover of $\mathbb{P}^1$. Because $\mathbb{P}^1$ is simply-connected, $Y$ must be a disjoint union of finitely many copies of $\mathbb{P}^1$. So there is a closed immersion $\sigma' : \bigsqcup \mathbb{P}^1 \to E \times F$. Since $E \times F$ is an abelian surface, and there are no rational curves on it, we get a contradiction. So $C$ is an elliptic curve and this completes our proof.

Now let us complete the proof of Theorem 3.4.

**Proof.** By the previous lemma, $C$ is an elliptic curve. Pick a point $P_0$ on $C$ as the origin, then $C$ becomes a 1-dimensional algebraic group. So $\pi$ is a homomorphism of abelian varieties. Let $E = \ker(\pi)$, then $S \simeq C \times E$. □

In the rest of this part, we assume $S$ is bielliptic.

**Lemma 3.6.** If $S \xrightarrow{\pi} C$, and $g(C) = 1$, then $C \simeq Alb(S)$ and $\alpha_C \circ \pi = \alpha_S$.

**Proof.** By the universal property of the Albanese variety, $\pi$ and $\varphi$ induce the diagram:

$$
Alb(S) \xrightarrow{\sigma'} Alb(C)
$$

such that $\pi' \circ \sigma' = id_{Alb(C)}$. So $Alb(S) \simeq Alb(C) \times \ker(\pi')$. Since $C$ is elliptic, $\dim Alb(C) = 1$ and $Alb(C) \simeq C$. Observe that $q(S) = \dim Alb(S) = 1$, so $\dim Alb(S) = \dim Alb(C)$. Then $Alb(S) \simeq Alb(C) \simeq C$. □

**Lemma 3.7.** Using the notations of Theorem 3.1, we write $S$ as $(E \times F)/G$. If $S \xrightarrow{\pi} C$, then $C \simeq E/G$ and $\pi = p_1$.

**Proof.** By Lemma 3.5, $C$ is elliptic, then $C$ satisfies all the assumptions of Lemma 3.6, so $C \simeq Alb(S)$. Recall that $G$ is a group of translations of $E$. Also $E/G$ is elliptic, so that $Alb(E/G) = E/G$. By the universal property of the Albanese variety, the first projection $p_1$ induces a morphism $\bar{p}_1$ such that the diagram :

$$
\begin{array}{ccc}
S & \xrightarrow{p_1} & E/G \\
\downarrow \pi & & \downarrow \bar{p}_1 \\
C & & \\
\end{array}
$$

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is commutative.
Note that $\tilde{p}_1$ is a finite morphism between projective curves and $p_1$ has connected fibres, so $\deg(\tilde{p}_1) = 1$. Then $\tilde{p}_1$ is an isomorphism, which completes our proof.

In the following lemma, we transfer our problem to the existence of $G$-morphisms from $E$ to $F$.

**Lemma 3.8.** There exists a section $\sigma$ of $p_1 : S \to E/G$ if and only if there exists a $G$-morphism $h : E \to F$.

**Proof.** The “if” part is trivial, we now prove the “only if” part.

The quotient morphism $\varphi : E \to E/G$ induces a cartesian square:

\[
\begin{array}{ccc}
E \times F & \xrightarrow{\tilde{\varphi}} & (E \times F)/G \\
\downarrow{p'_1} & & \downarrow{p_1} \\
E & \xrightarrow{\varphi} & E/G \\
\end{array}
\]

where $p'_1$ is the first projection, and $\tilde{\varphi}$ is the quotient morphisms.

By the above cartesian square, any section $\sigma$ of $p_1$ induces a section of $p'_1$. We denote it by $\delta : E \to E \times F$, for all $x \in E$, $\delta(x) = (x, h(x))$, where $h = p_2 \circ \delta$.

Now we illustrate all the morphisms in the following commutative diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{\delta} & (E \times F)/G \\
\downarrow{id_E} & & \downarrow{\tilde{\varphi}} \\
E \times F & \xrightarrow{\varphi} & E/G \\
\end{array}
\]

and verify that $h$ is a $G$-morphism. For any $x \in E$ and $g \in G$, we denote the action of $g$ on $x$, by $g \cdot x$. We aim to show that $\delta(g \cdot x) = g \cdot \delta(x)$. It suffices to verify that

\[
\sigma \circ \varphi(g \cdot x) = g \cdot (\sigma \circ \varphi(x)). \tag{21}
\]

Since $G$ acts on $(E \times F)/G$ trivially,

\[
g \cdot (\sigma \circ \varphi(x)) = \sigma \circ \varphi(x). \tag{22}
\]

Observe that $\varphi$ is a quotient morphism, so

\[
\varphi(g \cdot x) = \varphi(x). \tag{23}
\]
\[ \sigma \circ \varphi(g \cdot x) = \sigma \circ \varphi(x) = g \cdot (\sigma \circ \varphi(x)). \]  \hfill (24)

Equation (21) holds, so \( \delta \) is a \( G \)-morphism, hence so is \( h \).

Now we show that the action of \( G \) on \( F \) has a fixed point.

**Lemma 3.9.** If \( p_1 : S \twoheadrightarrow E/G \) has a section \( \sigma \), then the action of \( G \) on \( F \) has a fixed point.

**Proof.** By Lemma 3.8, \( \sigma \) induces a \( G \)-morphism \( h : E \rightarrow F \). We write it as for all \( x \in E \), \( h(x) = Ax + a \), where \( A \) is the linear part of \( h \). Observe that \( G \) acts on \( E \) as translations, we denote its action by: for all \( x \in E \), \( g \cdot x = x + x_g \). And we denote the action of \( G \) on \( F \) by: for all \( y \in F \), \( g \cdot y = l(y) + \varepsilon_g \), where \( l \) is the linear part.

Since \( h(g \cdot x) = g \cdot h(x) \), we have

\[ h(x + x_g) = g \cdot (Ax + a). \]  \hfill (25)

The left hand of (25) equals

\[ h(x + x_g) = A(x + x_g) + a. \]  \hfill (26)

The right hand of (25) equals

\[ g \cdot (Ax + a) = l(Ax + a) + \varepsilon_g \]  \hfill (27)
\[ = (l(Ax) + \varepsilon_g) + (l(a) + \varepsilon_g) - \varepsilon_g \]  \hfill (28)
\[ = g \cdot (Ax) + g \cdot a - \varepsilon_g. \]  \hfill (29)

So

\[ A(x + x_g) + a = g \cdot (Ax) + g \cdot a - \varepsilon_g. \]  \hfill (30)

Let \( x = 0 \) in the above equation (31), we have

\[ A(x_g) + a = (g \cdot 0 - \varepsilon_g) + g \cdot a. \]  \hfill (32)

But

\[ g \cdot 0 = l(0) + \varepsilon_g = \varepsilon_g. \]  \hfill (33)

So

\[ A(x_g) + a = g \cdot a. \]  \hfill (34)

Subtracting equation (34) from equation (31), we get

\[ Ax = g \cdot Ax - \varepsilon_g, \]  \hfill (35)

which implies that

\[ g \cdot Ax = Ax + \varepsilon_g. \]  \hfill (36)
If $h : E \to F$ is constant, i.e. $h$ maps $E$ to a single point $P \in F$, then $h$ is $G$-morphism, i.e. $g \cdot h(x) = h(g \cdot x)$, implies that $g \cdot P = P$, which means the action of $G$ on $F$ has a fixed point.

Otherwise $h$ is surjective. Then its linear part $A$, is a linear automorphism of the complex plane which maps the lattice of $E$ into the lattice of $F$. So by equation (36), for all $y \in F$, we have $g \cdot y = y + \varepsilon$, which means that $G$ acts on $F$ by translations.

Note that for all types of bielliptic surfaces in Lemma 3.2, the action of $G$ on $F$ has a non-trivial linear part. So $h$ is not surjective, hence it is a constant morphism, and the action of $G$ on $F$ has a fixed point.

Observe that, for surfaces of types 2), 4), 6) in Lemma 3.2, the action of $G$ on $F$ has no fixed point on $F$. This completes the proof of Theorem 3.3.

Now recall a known result (see [3], Section 4.5, Remark 16): Consider the functor of composition laws on a variety $S$, i.e. the contravariant functor from schemes to sets given by $T \mapsto \text{Hom}(S \times S \times T, S)$, then the families of algebraic semigroup laws yield a closed subfunctor, and this subfunctor is represented by a closed subscheme $SL(S) \subseteq \text{Hom}(S \times S, S)$.

For a given algebraic law $\mu_{t_0}$, we denote its associated contraction by $S \xrightarrow{\pi} C$ and its associated abelian variety by $A$, then the connected component of $\mu_{t_0}$ in $SL(S)$ is identified with the closed subscheme of $\text{Hom}(C, S) \times A$ consisting of those pairs $(\varphi, g)$ such that $\varphi$ is a section of $\pi : S \to C$. We denote the scheme of sections of $\pi$ by $\text{Mor}_\pi(C, S)$, it is isomorphic to an open subscheme of $\text{Hilb}(S)$ by assigning every section to its image in $S$. By using the local study of $\text{Hilb}(S)$, for any section $\sigma$ of $\pi$, the dimension of the tangent space of $\text{Hilb}(S)$ at $\sigma(C)$ is $h^0(\sigma(C), N_{\sigma(C)/S})$, and the obstruction lies in $H^1(\sigma(C), N_{\sigma(C)/S})$ (where $N_{\sigma(C)/S}$ is the normal bundle of $\sigma(C)$ in $S$). (For the discussion of local study of $\text{Hilb}(S)$, see [9].)

In the following, we want to study the structure of $\text{Mor}_\pi(C, S)$. For $\text{Kod}(S) = 0$, we have the following two theorems.

**Theorem 3.10.** Assume that $S$ is a bielliptic surface, if there is a curve $C$ satisfying $S \xleftarrow{\pi} C$, then $\text{Mor}_\pi(C, S)$ consists of reduced isolated points.

**Theorem 3.11.** If $S$ is an abelian surface, and there is a curve $E$ satisfying $S \xleftarrow{\pi} E$, then $\text{Mor}_\pi(E, S) = S/E = \text{ker}(\pi)$.

We postpone the proof of Theorem 3.10 and Theorem 3.11 to Section 6. There we will determine the structure of $\text{Mor}_\pi(C, S)$ for any smooth elliptic fibration $\pi$ in a more uniform way, see Theorem 6.1. Then Theorem 3.10 and Theorem 3.11 are just direct corollaries of Theorem 6.1.
3.2 The case \( q(S) = 0 \)

The main result of this part is Theorem 3.14.

**Lemma 3.12.** If \( q(S) = 0 \) and there exists \( S \xrightarrow{\pi} C \), then \( C \) is rational and \( \pi \) is an elliptic fibration.

**Proof.** If \( q(S) = 0 \), then by Theorem 3.1, \( S \) is K3 or Enriques. In both cases, \( K_S \sim_{\text{num}} 0 \). Pick an arbitrary general fibre \( F \) of \( \pi \). By the genus formula, we have:

\[
g(F) = 1 + \frac{1}{2}(F \cdot K_S + F^2) = 1
\]

so \( \pi \) is an elliptic fibration. On the other hand, \( \pi \) induces a surjective morphism \( \tilde{\pi} : \text{Alb}(S) \to \text{Alb}(C) \). So

\[
g(C) = \dim \text{Alb}(C) \leq \dim \text{Alb}(S) = q(S) = 0.
\]

Then \( C \) is rational.

**Lemma 3.13.** Every elliptic fibration \( \pi : S \to \mathbb{P}^1 \) of an Enriques surface \( S \) has exactly two multiple fibres, \( 2F \) and \( 2F' \).

**Proof.** [1], Chapter 8, Lemma 17.2.

**Theorem 3.14.** If \( S \) is an Enriques surface or a general K3 surface, there is no \( S \xrightarrow{\pi} C \). (Hence there is no non-trivial semigroup structure on \( S \).)

**Proof.** If \( S \) is an Enriques surface and such a diagram exist, then \( C \) is rational, and \( \pi \) is an elliptic fibration. By Lemma 3.11, there exists a multiple fibre \( 2F \). For a general smooth fibre \( F_0 \), \( \sigma(C) \cdot F_0 = 1 \), but under our assumptions, \( \sigma(C) \cdot F_0 = \sigma(C) \cdot 2F \geq 2 \), which is a contradiction.

The case for \( S \) is a general K3 surface, see Proposition 7.1.3, [6]

**Theorem 3.15.** For any K3 surface \( S \) and any fibration \( \pi : S \to C \), \( \text{Mor}_\pi(C, S) \) consists of reduced isolated points.

**Proof.** Since \( q(S) = 0 \), \( C \cong \mathbb{P}^1 \). Let us determine the structure of \( \text{Mor}_\pi(\mathbb{P}^1, S) \). For any section \( \sigma \) of \( \pi \), the tangent space of \( \text{Hilb}(S) \) at \( \sigma(\mathbb{P}^1) \) is \( H^0(\mathbb{P}^1, \mathcal{N}_\sigma(\mathbb{P}^1)/S) \), where \( \mathcal{N}_\sigma(\mathbb{P}^1)/S \) is the normal bundle of \( \sigma(\mathbb{P}^1) \) in \( S \). Since \( \mathbb{P}^1 \) and \( S \) are both smooth, by the adjunction formula, \( \mathcal{N}_\sigma(\mathbb{P}^1)/S = \omega_S^{-1} \mid_{\sigma(\mathbb{P}^1)} \otimes \omega_{\mathbb{P}^1} \). By the definition of K3, \( \omega_S \) is trivial, so \( \mathcal{N}_\sigma(\mathbb{P}^1)/S \) is \( \mathcal{O}(-2) \), and \( \dim H^0(\mathbb{P}^1, \mathcal{N}_\sigma(\mathbb{P}^1)/S) = 0 \). So \( \dim T_{\sigma(\mathbb{P}^1)}(\text{Hilb}(S)) = 0 \), so the dimension of \( \text{Mor}_\pi(\mathbb{P}^1, S) \) at \( \sigma(\mathbb{P}^1) \) is zero.

\[ \Box \]
4 The case $\kappa(S) = 1$

In this section, we always assume $\kappa(S) = 1$. The main result is Theorem 4.2.

Lemma 4.1. There exists an elliptic fibration $\varphi : S \to B$, where $B$ is a smooth projective curve.

Proof. [2], Chapter 9, Proposition 2.

So we have the following diagram:

$$
\begin{array}{ccc}
S & \xrightarrow{\varphi} & C \\
\downarrow & & \downarrow \\
B & \xleftarrow{\sigma} & \end{array}
$$

Remark 2): If $\pi = \varphi$, then $\varphi$ is an elliptic fibration with a section. This implies that the generic fibre $F$ of $\varphi$ is a smooth curve of genus 1 over the functional field of $B$, and the set of its rational points is nonempty. So $F$ is an elliptic curve and it can be imbedded into $\mathbb{P}^2_{k(B)}$ as a cubic curve. For any two minimal surfaces $S_i$ and two elliptic fibrations $\varphi_i : S_i \to B$, if their generic fibres are isomorphic, then the two fibrations $\varphi_i$ are isomorphic. So to give a minimal surface $S$ and an elliptic fibration $\varphi : S \to B$ is equivalent to give a homogeneous cubic equation in $\mathbb{P}^2_{k(B)}$. So in what follows, we focus on the case $\pi \neq \varphi$.

Theorem 4.2. If $g(C) \geq 1$, then $S \simeq B \times C$ and $\varphi$ is the first projection, $\pi$ is the second projection.

The key point to prove Theorem 4.2 is to show that $\sigma$ maps $C$ into a fibre of $\varphi$. First we want to determine the singular fibres of the elliptic fibration $\varphi$. For this, we recall “Kodaira’s table of singular fibres”.

Lemma 4.3. (Kodaira’s table of singular fibres)

Let $f : S \to \Delta$ be an elliptic fibration over the unit disk $\Delta$, such that all fibres $S_x, x \neq 0$, are smooth. We list all possibilities for the central fibre $S_0$:

a) $S_0$ is irreducible. Then $S_0$ is either smooth elliptic, or rational with a node, or rational with a cusp.

b) $S_0$ is reducible. Then every component $C_i$ of $S_0 = \sum n_i C_i$ is a $(-2)$-rational curve.

c) $S_0$ is a multiple fibre, we write it as $S_0 = mS_0'$. Then $S_0'$ is smooth elliptic, or rational with a node, or of type b)

Proof. [1], Chapter 5, Proposition 8.1.

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Lemma 4.4. If \( g(C) \geq 1 \), then \( g(C) \) must be 1. Moreover any fibre of \( \varphi \) is a multiple of a smooth elliptic curve.

Proof. Assume \( g(C) \geq 2 \). For an arbitrary smooth fibre \( F \) of \( \varphi \),

\[
1 = g(F) < g(C)
\]  
so \( \pi \) contracts \( F \). By the “Rigidity lemma”, \( \pi \) will factor through \( \varphi \), which contradicts our assumptions. So \( g(C) = 1 \).

Let \( F_0 \) be a singular fibre. If \( F_0 \) is not a multiple of a smooth elliptic curve, then by “Kodaira’s table of singular fibres”, \( F_0 \) is a sum of irreducible rational curves. Then \( F_0 \) will be contracted by \( \pi \), and by the “Rigidity lemma” again, \( \pi \) will factor through \( \varphi \). This contradicts our assumptions. So every fibre of \( \varphi \) is a smooth elliptic curve, or a multiple of a smooth elliptic curve.

In view of Lemma 4.4, the elliptic fibration \( \varphi \) is “almost smooth”: its singular fibres are only multiples of smooth curves. In the following lemma, we will see that after performing a base change, we can eliminate all the multiple fibres, and get a smooth elliptic fibration.

Lemma 4.5. Let \( \varphi : S \rightarrow B \) be a morphism from a surface onto a smooth curve whose fibres are multiples of smooth curves. Then there are a ramified Galois cover \( q : B' \rightarrow B \) with Galois group \( G \), a surface \( S' \) and a commutative cartesian square:

\[
\begin{array}{ccc}
S' & \xrightarrow{q'} & S \\
\downarrow{p'} & & \downarrow{p} \\
B' & \xrightarrow{q} & B
\end{array}
\]

such that the action of \( G \) on \( B' \) lifts to \( S' \), \( q' \) induces an isomorphism \( S'/G \simeq S \) and \( p' \) is smooth.

Proof. Just use the cyclic covering trick. \( \square \), Chapter 6, Lemma 7.

In some cases, we can even get a trivial elliptic fibration, after performing successive base changes.

Lemma 4.6. Let \( p : S \rightarrow B \) be a smooth morphism from a surface to a curve, and \( F \) a fibre of \( p \). Assume either that \( g(B) = 1 \) and \( g(F) \geq 1 \), or that \( g(F) = 1 \). Then there exists an étale cover \( B' \) of \( B \), such that the fibration \( p' : S' = S \times_B B' \) is trivial, i.e. \( S' \simeq B' \times F \). Furthermore, we can take the cover \( B' \rightarrow B \) to be Galois with group \( G \), say, so that \( S \simeq (B' \times F)/G \).

Proof. \( \square \), Chapter 6, Proposition 8.
Lemma 4.7. Assume that $S$ and $C$ satisfy all the assumptions of Theorem 4.2. Then $\sigma$ maps $C$ into a fibre of $\varphi$.

Proof. If not, $\varphi \circ \sigma : C \to B$ will be a finite morphism between curves. By Hurwitz’s Theorem, $g(B) \leq 1$. Let $\varphi_C = \varphi|_{\sigma(C)} : \sigma(C) \to B$, we now determine the degree of the ramification divisor $R$ of $\varphi_C$.

If $g(B) = 1$, assume that $\varphi$ has a multiple fibre $F_b$, and write $F_b = mF'_b$. Let $\sigma(C)$ intersect with $F'_b$ at a point $P$, then the ramification index of $\varphi_C$ at $P$ satisfies

$$e_P \geq m > 1.$$ (40)

Then we get a ramified morphism $\varphi_C$ between elliptic curves, which is impossible.

So $\varphi : S \to B$ is a smooth elliptic fibration.

Then by Lemma 4.6, there exists an étale cover $B' \to B$ such that $S' = B' \times_B S$ is a trivial elliptic fibration. Then

$$Kod(S') = Kod(B' \times F) = Kod(B') + Kod(F) \quad (41)$$

where $F$ is a general fibre of $\varphi$. Since $B'$ and $F$ are both elliptic curves, $Kod(S') = 0$. But $Kod(S') \geq Kod(S) \geq 1$, which is a contradiction.

So $g(B) = 0$. Assume that at points $b_1, \ldots, b_s$ of $B$, $\varphi$ has multiple fibres $F_i = m_iF'_i (1 \leq i \leq s)$ and each $F'_i$ intersects with $\sigma(C)$ at points $P_{i,1}, \ldots, P_{i,j_i}$ with multiplicities $n_{i,1}, \ldots, n_{i,j_i}$. Then $\deg(R)$ satisfies:

$$\deg(R) \geq \sum_{i=1}^{s} \sum_{j=1}^{j_i} (e_{P_{i,j}} - 1)$$ (42)

$$= \sum_{i=1}^{s} \sum_{k=1}^{j_i} (m_in_{i,k} - 1)$$ (43)

$$= \sum_{i=1}^{s} \{m_i(\sum_{k=1}^{j_i} n_{i,k}) - j_i\}$$ (44)

$$= \sum_{i=1}^{s} (m_iF'_i \cdot \sigma(C) - j_i)$$ (45)

$$= \sum_{i=1}^{s} (F_i \cdot \sigma(C) - j_i)$$ (46)

$$= \sum_{i=1}^{s} (\deg(\varphi_C) - j_i)$$ (47)

Since $\sum_{k=1}^{j_i} n_{i,k} = F'_i \cdot \sigma(C)$ and each $n_{i,j} \geq 1$, so

$$j_i \leq F'_i \cdot \sigma(C) = deg(\varphi_C)/m_i.$$ (48)
So
\[ \text{deg}(R) \geq \sum_{i=1}^{s} (\text{deg}(\varphi_{C}) - j_{i}) \geq \sum_{i=1}^{s} \text{deg}(\varphi_{C})(1 - \frac{1}{m_{i}}). \] (49)

Using Hurwitz’s Theorem, we calculate \( \text{deg}(R) \) as:
\[
\begin{align*}
\text{deg}(R) &= 2g(E) - 2 - \text{deg}(\varphi_{C})(2g(B) - 2) \\
&= 2\text{deg}(\varphi_{C}).
\end{align*}
\] (50) (51) (52)

Now by comparing (49) and (52), we get
\[
\sum_{i=1}^{s} (1 - \frac{1}{m_{i}}) - 2 \leq 0.
\] (53)

Recall Lemma 7.1 in [13]: for any elliptic fibration \( \varphi \), we define an invariant by
\[
\delta(\varphi) = \chi(O_{S}) + 2g(B) - 2 + \sum_{i=1}^{s} (1 - \frac{1}{m_{i}}),
\] (54)

then \( \text{Kod}(S) = 1 \) if and only if
\[
\delta(\varphi) > 0.
\] (55)

In our situation:
\[
\delta(\varphi) = \chi(O_{S}) - 2 + \sum_{i=1}^{s} (1 - \frac{1}{m_{i}}).
\] (56)

By the inequality (53), \( \delta(\varphi) \leq \chi(O_{S}) \).

Now let us determine \( \chi(O_{S}) \). Recall Noether’s Formula:
\[
\chi(O_{S}) = \frac{1}{12} (K_{S}^{2} + \chi_{\text{top}}(S)).
\] (57)

Since \( S \) is minimal and \( \text{Kod}(S) = 1 \), we have \( K_{S}^{2} = 0 \). So
\[
\chi(O_{S}) = \frac{1}{12} \chi_{\text{top}}(S).
\] (58)

Recall that
\[
\chi_{\text{top}}(S) = \chi_{\text{top}}(B)\chi_{\text{top}}(F) + \sum_{i=1}^{s} (\chi_{\text{top}}(F_{i}) - \chi_{\text{top}}(F)),
\] (59)

where \( F_{i} \) are the singular fibres and \( F \) is a general smooth fibre. In our case, \( F_{i} \) is a multiple of a smooth elliptic curve. So \( \chi_{\text{top}}(F_{i}) = \chi_{\text{top}}(F) = 0 \), hence \( \chi_{\text{top}}(S) = 0 \), which implies
\[
\delta(\varphi) \leq 0.
\] (60)
This contradicts the assumption that \( \text{Kod}(S) = 1 \).

In conclusion, the assumption that \( \varphi \circ \sigma : C \to B \) is a finite morphism always leads to contradictions, so \( \sigma \) maps \( E \) into a fibre of \( \varphi \).

Now we begin the proof of Theorem 4.2.

**Proof.** We first consider the case when \( \varphi \) is smooth. First we define \( \alpha = (\pi, \varphi) : S \to C \times B \), we show that \( \alpha \) is an isomorphism. By Lemma 4.7, \( \sigma \) maps \( C \) into a fibre \( F_b \) of \( \varphi \). If \( \varphi \) is smooth, \( F_b \) is integral, so \( \sigma(C) = F_b \). For an arbitrary fibre \( F \) of \( \pi \), since \( \sigma \) is a section of \( \pi \), \( F \cdot F_b = F \cdot \sigma(C) = 1 \). Then \( \alpha \) is birational. If there exists an irreducible curve \( X \) on \( S \) contracted by \( \alpha \), then \( X \) will be a fibre of \( \varphi \), because \( \varphi \) has smooth integral fibres. So \( \pi \) contracts one fibre of \( \varphi \). By the “Rigidity lemma”, \( \pi \) will factor through \( \varphi \), contradicts to our assumptions. So \( \alpha \) is a birational quasi-finite, projective morphism, which is certainly an isomorphism.

Now we deal with the general case, which allows \( \varphi \) has singular fibres. We know that, by Lemma 4.5, there exists \( q : B' \to B \), such that \( \varphi' : S' \simeq B' \times_B S \to B' \) is smooth. Note that \( \varphi \circ \sigma : C \to B \) maps \( C \) to a single point of \( b \in B \). Then there exists a constant morphism \( f : C \to B' \) lifts \( \varphi \circ \sigma \), such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi \circ \sigma} & B' \\
\downarrow{q} & & \downarrow{q} \\
B & & B
\end{array}
\]

is commutative. By the universal property of the fibre product, there exists a morphism \( \sigma' : C \to S' \) satisfying \( q' \circ \sigma' = \sigma \) and \( \varphi' \circ q = f \).

We illustrate all the morphisms in the following diagram:

\[
\begin{array}{ccc}
S' & \xrightarrow{\varphi'} & C \\
\downarrow{q'} & \downarrow{\varphi} & \downarrow{\varphi} \\
B' & \xrightarrow{q} & B
\end{array}
\]

and verify \( \sigma' \) is a section of \( \pi \circ q' \):

\[
(\pi \circ q') \circ \sigma' = \pi \circ (q' \circ \sigma') = \pi \circ \sigma' = \text{id}_C.
\]

(61)

So there is a diagram:

\[
\begin{array}{ccc}
S' & \xrightarrow{\varphi'} & C \\
\downarrow{\varphi'} & \downarrow{f} & \\
B' & & B
\end{array}
\]
such that
\[ \pi' = \pi \circ q' \tag{62} \]
\[ \varphi' \circ \sigma' = f. \tag{63} \]

Then \( \varphi' \circ \sigma'(C) = f(C) = \{b'\} \), which means \( \sigma' \) maps \( C \) into a fibre \( F_y \) of \( \varphi' \). Now \( S' \) is a smooth fibration over \( C \) and \( \sigma' \) maps \( C \) into a fibre \( F_y \) of \( \varphi' \).

According to what we have discussed about our problem in the smooth case, \( \alpha' = (\pi', \varphi') : S' \longrightarrow C \times B' \) is an isomorphism.

Let \( G \) be the Galois group of the covering \( q : B' \longrightarrow B \), then \( S \simeq S'/G \simeq (C \times B')/G \) and the diagram:

\[
\begin{array}{ccc}
S' & \simeq & C \times B' \\
\downarrow \quad q' & & \downarrow \pi \\
S & \simeq & (C \times B')/G \\
\downarrow \quad p_1 & \quad & \downarrow \pi \\
C
\end{array}
\]
is commutative. We denote the action of \( G \) on \( C \times B' \) by
\[ g(x, b') = (\phi_Y(g)(x), gb'). \tag{64} \]

Since \( p_1 \) factors through the quotient morphism \( q' \),
\[ p_1(g(x, b')) = p_1((x, b')). \tag{65} \]
This means that for all \( g \in G \) and for all \( x \in C \), \( \phi_Y(g)(x) = x \). So \( G \) acts trivially on the factor \( C \). Then we have
\[ S \simeq (C \times B')/G \simeq C \times (B'/G) \simeq C \times B. \]

The proof of Theorem 4.2 is completed. \( \square \)

Now we prove a result about sections of non-smooth elliptic fibrations.

**Theorem 4.8.** If there is a curve \( C \) satisfies that \( S \hookrightarrow C \), and \( \pi \) is not a smooth elliptic fibration, then \( \text{Mor}_\pi(C, S) \) consists of reduced isolated points.

**Proof.** Consider the diagram we introduced in Lemma 4.1,

\[
\begin{array}{ccc}
S & \hookrightarrow & C \\
\downarrow \varphi & & \\
B
\end{array}
\]

Then there are three possibilities:
1) \((B, \varphi) \simeq (C, \pi)\).

2) \((B, \varphi) \not\simeq (C, \pi)\) and \(g(C) \geq 1\).

3) \((B, \varphi) \not\simeq (C, \pi)\) and \(g(C) = 0\).

For case 2), by Theorem 4.2, \(S \simeq B \times C\) and \(\pi\) is the second projection, hence it is a smooth elliptic fibration, contradicts to our assumption.

For case 3), \(C \simeq \mathbb{P}^1\). By the “adjunction formula”,
\[
0 = g(C) = 1 + \frac{1}{2}(K_S \cdot C + C^2).
\]
Since \(S\) is an elliptic surface over \(B\) and \(Kod(S) = 1\), \(K_S \sim mF\), where \(F\) is a fibre of \(\varphi\) and \(m\) is positive. So \(K_S\) is nef, which implies that \(K_S \cdot C \geq 0\) and \(C^2 < 0\). So \(deg(N_{C/S}) = C^2 < 0\), which implies that \(\dim H^0(C, N_{C/S}) = 0\).

For case 1), \(\pi\) is an elliptic fibration over \(C\) with a section. Then
\[
K_S \sim^{num} \left(\chi(O_S) + 2g(C) - 2\right)F,
\]
where \(F\) is a fibre of \(\pi\).

So
\[
K_S^{-1} \cdot C = -\left(\chi(O_S) + 2g(C) - 2\right),
\]
\[
deg(N_{C/S}) = deg(K_S^{-1}|C \otimes \omega) = -\chi(O_S).
\]
By Noether’s formula,
\[
\chi(O_S) = \frac{1}{12}(K_S^2 + \chi_{top}(O_S)).
\] (66)

Since \(K_S^2 = 0\),
\[
\chi(O_S) = \frac{1}{12}\chi_{top}(S).
\] (67)

Recall that
\[
\chi_{top}(S) = \chi_{top}(B)\chi_{top}(F) + \sum_{i=1}^{s}(\chi_{top}(F_i) - \chi_{top}(F)),
\]
where \(F_i\) are the singular fibres and \(F\) is a general smooth fibre. If \(\pi\) is not smooth, \(\chi_{top}(S) > 0\), which implies that \(deg(N_{C/S} < 0)\), so \(\dim H^0(C, N_{C/S}) = 0\).

In the above theorem, we miss the case that \(\pi\) is a smooth fibration, we will give an answer to this case in Theorem 6.1. Then we can complete our discussion of \(Mor_{\pi}(C, S)\), when \(\kappa(S) = 1\).
5 The case $\kappa(S) = 2$

In this section, we always assume that $\kappa(S) = 2$. For any non-trivial fibration $\pi : S \rightarrow C$, we want to study $\text{Mor}_\pi(C, S)$ by counting the rational points on the generic fibre of $\pi$. This idea is illustrated in the following lemma and our main result of this section is Theorem 5.2.

**Lemma 5.1.** Consider a fibration $\pi : S \rightarrow C$. Then sections of $\pi$ are in one-to-one correspondence with $k(C)$-rational points of the generic fibre.

**Proof.** Let $E$ denote the generic fibre of $\pi$ and consider the following cartesian square:

$$
\begin{array}{ccc}
E & \xrightarrow{\pi'} & \text{Spec } k(C) \\
\downarrow{\tau} & & \downarrow{\tau} \\
S & \xrightarrow{\pi} & C,
\end{array}
$$

then any section of $\pi$ induces a section of $\pi'$. Conversely, given a section $\sigma'$ of $\pi'$, we pick an arbitrary point $P \in C$, then consider the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec } k(C) & \xrightarrow{\tau \circ \sigma'} & S \\
\downarrow{f} & & \downarrow{\pi} \\
\text{Spec } \mathcal{O}_{C,P} & \xrightarrow{g} & C.
\end{array}
$$

Since $\mathcal{O}_{C,P}$ is a discrete valuation ring and $\pi$ is a projective morphism, by the “Valuative Criterion of Properness”, there is a unique morphism $h : \text{Spec}\mathcal{O}_{C,P} \rightarrow S$ satisfying $h \circ f = \tau \circ \sigma'$ and $\pi \circ f = g$. As $P$ varies along $C$, and $C$ is a projective smooth curve, we get a morphism $\sigma : C \rightarrow S$ satisfying $\pi \circ \sigma = \text{id}_S$.

**Theorem 5.2.** If there is a fibration $\pi : S \rightarrow C$ and $S$ is not a product $C \times C'$ for any smooth curve $C'$, then there are only finitely many sections of $\pi$.

**Proof.** If there are infinitely many sections of $\pi$, we consider the generic fibre $F$ of $\pi$, then by Lemma 5.1, there are infinitely many rational points on $F$. Now recall a theorem of Manin:(See Theorem 3, [7])

**Theorem:** Let $K$ be a regular extension of the field $k$ of characteristic zero and let $C$ be a curve of genus bigger or equal than 2 defined over $K$. If the set of points of $C$ defined over $K$ is infinite, then there is a curve $C'$ which is birationally equivalent to $C$ over $K$ and defined over $k$. All the points of $C_K$ except a finite number are images of points of $C'_k$.

Since $\kappa(S) = 2$, the arithmetic genus of every fibre of $\pi$ is bigger or equal than 2, we can apply this theorem to $F$ and in our case, $k = \mathbb{C}$ and $K = k(C)$,
then there is a curve $C'$ defined over $\mathbb{C}$ and a birational map $\theta : S \to C \times C'$.
Since $\kappa(S) = 2$ and $S$ is minimal, by a corollary of “Castelnuovo’s Theorem”,
$\theta$ is defined everywhere and an isomorphism (See Theorem 19, Chapter 5, [2]). This contradicts our assumption.

6 Sections of elliptic fibration with smooth fibre

Now we fix a smooth elliptic fibration $\pi : S \to C$ and we want to study $\text{Mor}_\pi(C, S)$. By Lemma 4.6, there is an étale cover $D \to C$ satisfying that $S \times_C D$ is a trivial elliptic fibration. So in what follows, we always assume that $S \cong (D \times E)/G$ and consider the elliptic fibration $\pi : S \to D/G$.

**Theorem 6.1.** Assume that $S \cong (D \times E)/G$ where

1) $D$ and $E$ are projective smooth curves, $g(D) \geq 1$ and $g(E) = 1$.
2) $G$ is a finite group, which acts faithfully on $D$ and $E$, and freely on $D$ and $D \times E$.

Now consider $\pi : S \to D/G$, where $\pi$ is the first projection.
Then:
1) $\text{Mor}_\pi(D/G, S) \cong E/G$ if every element of $G$ acts on $E$ only as translations.
2) $\text{Mor}_\pi(D/G, S)$ consists of reduced isolated points if some element $g (\neq \text{id}_G)$ of $G$ has fixed points on $E$.

*Proof.* First we prove claim 1).
We view $\text{Mor}_\pi(D/G, S)$ as an open subscheme of $Hilb(S)$ and by our assumption for all $g \in G$ and $y \in E$, $g \cdot y = y + y_g$, where $y_g$ is a constant.
Note that $E$ is an algebraic group, we want to define its action on $Hilb(S)$ and aim to show that $\text{Mor}_\pi(D/G, S)$, as an open subscheme of $Hilb(S)$, consists of only one orbit.
Now for all $y_0 \in E$, we define $t_{y_0} : D \times E \to D \times E$ as $t_{y_0}(x, y) = (x, y + y_0)$. Since $G$ acts on $E$ as translations and $E$ is an abelian group, $t_{y_0}$ is a $G$-morphism. So it induces an automorphism $\alpha_{y_0}$ of the quotient $(D \times E)/G$, i.e. $\alpha_{y_0} \in \text{Aut}(S)$.
Then we define the action of $E$ on $Hilb(S)$ by
$$\alpha : E \times Hilb(S) \to Hilb(S),$$
$$\alpha(y_0, P) = \alpha_{y_0}(P),$$
where $P$ is any closed subvariety of $S$.
For a section $\sigma$, we want to show that $\pi \circ \alpha_{y_0} \circ \sigma = \text{id}_{D/G}$, then sections of $\pi$ can move by the action of $E$. Observe that $\sigma$ induces a $G$-morphism $f_\sigma : D \to E$, so for all $\bar{x} \in D/G$, $\sigma(\bar{x}) = (x, f_\sigma(x))$, then
$$\pi \circ \alpha_{y_0} \circ \sigma(\bar{x}) = \pi \circ \alpha_{y_0}(x, f_\sigma(x)) = \pi(x, f_\sigma(x) + y_0) = \bar{x}.$$
This implies that the orbit of $\sigma(D/G)$, we denote it by $O$, is contained in $Mor_\pi(D/G, S)$.

Now let us determine the isotropy group of $\sigma(D/G)$. If $\alpha_{y_0} \circ \sigma = \sigma$, then $y_0 = y_g$ for some $g \in G$. Since $G$ acts faithfully on $E$, the isotropy group is isomorphic to $G$ and $O \simeq E/G$.

Note that $\pi$ is a smooth elliptic fibration with a section $\sigma$. Then $\chi(O_S) = 0$, and $K_S \sim^{num} (\chi(O_S) + 2g(D/G) - 2)F \sim^{num} (2g(D/G) - 2)F$, where $F$ is a fibre of $\pi$. So $deg(N_{\sigma(D/G)/S}) = deg(K_S^{-1}\sigma(D/G) \otimes \omega_{\sigma(D/G)}) = 0$, which implies that $\dim H^0(\sigma(D/G), N_{\sigma(D/G)/S}) \leq 1$. Then $\dim Mor_\pi(D/G, S) \leq 1$.

But we already have a closed immersion:

$$E/G \hookrightarrow Mor_\pi(D/G, S),$$

so $Mor_\pi(D/G, S)$ is locally isomorphic to $E/G$, hence smooth everywhere as $\sigma$ varies, which implies $Mor_\pi(D/G, S) \simeq E/G$.

We now prove claim 2), our aim is to calculate $K_S|_{\sigma(D/G)}$.

First, we have the following commutative diagram:

$$\begin{array}{ccc}
D & \xrightarrow{f} & D \times E \\
\downarrow{k} & & \downarrow{h} \\
D/G & \xrightarrow{\sigma} & (D \times E)/G
\end{array}$$

where $k$, $h$ are quotient morphisms and $\sigma$ induces a morphism $f_\sigma : D \longrightarrow E$ satisfying that for all $x \in D$, $f(x) = (x, f_\sigma(x))$.

So $k^* \circ \sigma^*(K_S) = f^* \circ h^*(K_S)$, we denote this sheaf by $\mathcal{L}$, then the sheaf we wanted, $K_S|_{\sigma(D/G)} = \sigma^*(K_S) = (k_* \mathcal{L})^G$.

Since $G$ acts freely on $D \times E$, $h$ is étale so the differential morphism

$$dh : h^*\Omega_S \longrightarrow \Omega_{D \times E}$$

is an isomorphism, furthermore, $dh$ is a $G$–morphism.

Since $\Omega_{D \times E} \simeq p^*\Omega_D \oplus q^*\Omega_E$, so

$$h^*K_S \simeq^G p^*\Omega_D \otimes q^*\Omega_E$$

where "$\simeq^G$" means isomorphic as $G$–sheaves.

Then apply $f^*$ to both sides, we obtain

$$\mathcal{L} = f^* \circ h^*K_S \simeq^G f^*(p^*\Omega_D \otimes q^*\Omega_E) = \Omega_D \otimes_{\mathcal{O}_D} f^*_\sigma\Omega_E.$$

Then

$$\sigma^*K_S \simeq (k_*\mathcal{L})^G \simeq k_*(\Omega_D \otimes_{\mathcal{O}_D} f^*_\sigma\Omega_E)^G.$$

Since $k : D \longrightarrow D/G$ is étale, $\Omega_D \simeq^G k^*\Omega_{D/G}$.

By the projection formula, $k_* (\Omega_D \otimes_{\mathcal{O}_D} f^*_\sigma\Omega_E) \simeq \Omega_{D/G} \otimes_{\mathcal{O}_{D/G}} k_* f^*_\sigma\Omega_E$, so

$$\sigma^*K_S \simeq (\Omega_{D/G} \otimes_{\mathcal{O}_{D/G}} k_* f^*_\sigma\Omega_E)^G.$$
Theorem 6.2. Assume $G$ acts freely on $X$, and $Y = X/G$. Then for all characters $\alpha : G \to \mathbb{C}^*$, $L_\alpha$ is an invertible sheaf, and the multiplication in $\pi_*O_X$ induces an isomorphism $L_\alpha \otimes L_\beta \simeq L_{\alpha\beta}$. The assignment $\alpha \mapsto L_\alpha$ defines an isomorphism $\hat{G} \simeq \ker(Pic Y \to Pic X)$.

Now apply Theorem 6.2 to $X = D$ and $Y = D/G$, then $L_{\chi^{-1}} = L_\chi$. By the adjunction formula:

$$N_{\sigma(D/G)/S} \simeq \sigma^*K_S^{-1} \otimes \Omega_{D/G} \simeq \Omega_{D/G}^{-1} \otimes L_\chi \otimes \Omega_{D/G} \simeq L_\chi.$$  

By our assumption, some element $g \neq id_G$ of $G$ has fixed points on $E$, since $G$ acts on $E$ faithfully, the lineart part of $g$ is non-trivial, which implies that $\chi \neq 1$. By Theorem 6.2, $L_\chi \neq O_{D/G} \in Pic (D/G)$, i.e. $N_{\sigma(D/G)/S} \simeq L_\chi$ is not trivial. By the proof of claim 1), we know that $\deg N_{\sigma(D/G)/S} = 0$. If $H^0(D/G, N_{\sigma(D/G)/S}) \neq 0$, then $N_{\sigma(D/G)/S}$ is linearly equivalent to an effective divisor of degree 0, hence $N_{\sigma(D/G)/S}$ is trivial, we get a contradiction. So $H^0(D/G, N_{\sigma(D/G)/S}) = 0$, and this completes our proof.

Now it is easy to see that Theorem 3.10 and Theorem 3.11 are just direct corollaries of Theorem 6.1. Furthermore, Theorem 6.1 completes our study of $Mor_\pi(C, S)$, when $\kappa(S) = 1$.  

Since $E$ is an abelian variety, $\Omega_E \simeq O_E \otimes C H^0(E, \Omega_E)$, so

$$f^*_g \Omega_E \simeq^G f^*_g O_E \otimes C H^0(E, \Omega_E) \simeq O_D \otimes C H^0(E, \Omega_E).$$

Now we have

$$\sigma^*K_S \simeq (\Omega_{D/G} \otimes O_{D/G} \otimes L_\chi \otimes \Omega_{D/G} \otimes C H^0(E, \Omega_E))^G.$$ 

Since $\text{rank } \Omega_{D/G} = 1$ and $\dim H^0(E, \Omega_E) = 1$, we can denote a $G$-invariant element of $\Omega_{D/G} \otimes O_{D/G} \otimes L_\chi \otimes \Omega_{D/G} \otimes C H^0(E, \Omega_E)$ by a pure tensor $a \otimes b \otimes c$. Then for all $g \in G$, consider the action of $g$ on $a \otimes b \otimes c$:

$$g(a \otimes b \otimes c) = a \otimes gb \otimes gc = a \otimes b \otimes c.$$ 

Since $G$ acts on $H^0(E, \Omega_E)$ by a character $\chi \in \text{Hom}(G, \mathbb{C}^*)$, i.e. $gc = \chi(g)c$, then

$$a \otimes gb \otimes gc = a \otimes gb \otimes \chi(g)c = a \otimes b \otimes c.$$ 

So $gb = \chi(g)^{-1}b$, let $L_\alpha = \{a \in k_*O_D|ga = \alpha(g)a\}$, then

$$\sigma^*K_S \simeq \Omega_{D/G} \otimes O_{D/G} \otimes L_\chi \otimes \Omega_{D/G} \simeq \Omega_{D/G} \otimes L_\chi.$$
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