SIMILARITY SOLUTIONS OF
A MULTIDIMENSIONAL REPLICATOR DYNAMICS
INTEGRODIFFERENTIAL EQUATION

VASSILIS G. PAPANICOLAOU and KYRIAKI VASILAKOPOULOU

Department of Mathematics
National Technical University of Athens
Zografou Campus
157 80 Athens, Greece

(Communicated by Athanasios Yannacopoulos)

ABSTRACT. We consider a nonlinear degenerate parabolic equation containing
a nonlocal term, where the spatial variable \( x \) belongs to \( \mathbb{R}^d \), \( d \geq 2 \). The
equation serves as a replicator dynamics model where the set of strategies
is \( \mathbb{R}^d \) (hence a continuum). In our model the payoff operator (which is the
continuous analog of the payoff matrix) is nonsymmetric and, also, evolves
with time. We are interested in solutions \( u(t,x) \) of our equation which are
positive and their integral (with respect to \( x \)) over the whole space \( \mathbb{R}^d \) is 1,
for any \( t > 0 \). These solutions, being probability densities, can serve as time-
evolving mixed strategies of a player. We show that for our model there is an
one-parameter family of self-similar such solutions \( u(t,x) \), all approaching the
Dirac delta function \( \delta(x) \) as \( t \to 0^+ \). The present work extends our earlier
work [11] which dealt with the case \( d = 1 \).

1. Introduction - The replicator dynamics equation. The replicator dynamics
models are popular in evolutionary game theory. They have significant applications
in economics, population biology, as well as in other areas of science [3], [4],
[12], [13].

Replicator dynamics has been studied extensively in the finite-dimensional case:
Let \( A = (a_{ij}) \) be an \( m \times m \) negative matrix. The typical replicator dynamics
equation is [3]

\[
\frac{\partial u}{\partial t} = \left[ Au - (u, Au) \right] u = (Au)u - (u, Au)u
\]

(1)

where the subscript \( t \) in \( u_t \) denotes derivative with respect to the time variable
\( t \), \( (u, Au) \) denotes the usual inner product, i.e., the dot product of the vectors \( u \)
and \( Au \), and \( (Au)u \) is the vector whose \( i \)-th component is the product of the \( i \)-th
components of \( (Au) \) and \( u \), i.e. the “pointwise product” of two vectors.

The matrix \( A \) is called the payoff matrix, while \( S = \{1, 2, \ldots, m\} \) is the strategy
space and the vector

\[
u = (u_1(t), u_2(t), \ldots, u_m(t))^T
\]
is a probability (mass) on \( S \), meaning that
\[
u_i(t) \geq 0, \quad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} \nu_i(t) = 1
\]
(notice that if the above conditions are satisfied for \( t = 0 \), then they are satisfied for all \( t \geq 0 \), under the flow (1)). The vector \( \nu \) represents the mixed strategy of one member of the population, i.e., one player against the rest of the population. The dependence of \( \nu \) in \( t \) allows the player to update his strategy, in order to increase his payoff.

Infinite dimensional versions of this evolutionary strategy models have been proposed, e.g., in [1] and [8] (see also [9] and the survey [3]) in connection to certain economic and biological applications. For instance, there are situations where (pure) strategies correspond to geographical points, and hence it is natural to model the set of strategies by a continuum. However, the abstract form of the proposed equations does not allow one to obtain much insight, for example on the form of solutions.

In order to make some progress in this direction, the works [6] and [10] initiated the study of the case where \( S \) is the set \( \mathbb{R}^d \), \( d \geq 1 \), and the payoff operator \( A \) is the Laplacian operator \( \Delta \). Then the evolution law (1) becomes
\[
u_t = [\Delta \nu - (\nu, \Delta \nu)] \nu, \quad (2)
\]
where \((\cdot, \cdot)\) denotes the usual inner product of the (real) Hilbert space \( L^2(\mathbb{R}^d) \) of the square-integrable functions defined on \( \mathbb{R}^d \), namely
\[
(u, v) := \int_{\mathbb{R}^d} u(x)v(x)dx.
\]
(3)

References [6] and [10] deal only with the special problem of constructing an one-parameter family of self-similar solutions for (2), namely, solutions \( \nu \) of the form
\[
u(t, x) = t^{-\kappa} g_d(\rho t^{-\lambda}), \quad \text{where} \quad \rho := |x| = \sqrt{x_1^2 + \cdots + x_d^2}.
\]
(4)
A peculiar feature of these solutions is that all of them are probability densities on \( \mathbb{R}^d \), for all \( t > 0 \), and approach the Dirac delta function \( \delta(x) \) as \( t \to 0^+ \). It is worth mentioning that if we consider the equation (2) in a bounded domain \( \Omega \) of \( \mathbb{R}^d \) with Dirichlet boundary conditions and an initial condition \( \nu(0, x) \) which is a probability density and does not vanish inside \( \Omega \), then it has been very recently established [5], [7] that \( \nu(t, x) \) exists for all \( t > 0 \) and approaches, as \( t \to \infty \) the equilibrium solution which is also a probability density (notice that there is exactly one such equilibrium solution). Of course, in the case where the domain is the whole \( \mathbb{R}^d \) such equilibrium solutions do not exist, and this is one reason why self-similar solutions become important.

One criticism towards (2) is that the Laplacian operator \( \Delta \) is a symmetric operator and, also, time-independent. A payoff operator \( A \) which is symmetric with respect to the inner product \((\cdot, \cdot)\) corresponds to the case of a partnership game, where interests of both players coincide (see, e.g., [3]). These are unrealistic features for a payoff operator in a replicator dynamics model. For this reason, in our recent work [11], we considered a nonsymmetric and time-dependent payoff operator, namely
\[
A \nu = A(t) \nu = \frac{\partial^2 \nu}{\partial x^2} + \alpha t^{-2/3} \frac{\partial \nu}{\partial x},
\]
(5)
where $\alpha > 0$ is a parameter. Then, in view of (5), the replicator dynamics equation (1) becomes

$$u_t = \left[ u_{xx} + \alpha t^{-2/3} xu_x - \int_{-\infty}^{\infty} (u(t) + \alpha t^{-2/3} xu_x) \, dx \right] u, \quad t > 0, \quad x \in \mathbb{R}, \quad (6)$$

with $u = u(t, x)$. In [11] it was shown that (5) possesses an one-parameter family of self-similar solutions, all approaching the Dirac delta function $\delta(x)$ as $t \to 0^+$. Viewed as functions of $x$, all those solutions are probability densities on $\mathbb{R}^d$, for each $t > 0$.

In the present work we study the $d$-dimensional version of (6), $d \geq 2$, where the strategy space is $S = \mathbb{R}^d$, while the operator $A$ is

$$Au(t, x) := \Delta u(t, x) + \alpha t^\gamma x \cdot \nabla u(t, x) \quad (7)$$

where $\alpha > 0$ is an arbitrary but fixed constant, $\gamma$ is a specific constant (as we will see $\gamma = -2/d$), $t > 0$, $x \in \mathbb{R}^d$, while the operators $\Delta$ and $\nabla$ are acting on the $x$ variable. In this case, the corresponding replicator dynamics problem (1) takes the form

$$u_t = \left[ \Delta u + \alpha t^\gamma \int_{\mathbb{R}^d} (u \Delta u + \alpha u t^\gamma x \cdot \nabla u) \, dx \right] u, \quad t > 0, \quad x \in \mathbb{R}^d, \quad (8)$$

with

$$\int_{\mathbb{R}^d} u(0, x) \, dx = 1 \quad \text{and} \quad u(0, x) \geq 0, \quad x \in \mathbb{R}^d. \quad (9)$$

Incidentally, let us remark that (8), as well as (6) and (2), can be viewed as nonlinear (and nonlocal) Fokker-Planck equations.

The main result of this article is the construction of an one-parameter family of self-similar solutions for (8)-(9), i.e. solutions $u$ of the form (4). As in the previous cases, all these solutions are probability densities on $\mathbb{R}^d$, for all $t > 0$, and approach the Dirac delta function $\delta(x)$ as $t \to 0^+$.

2. Self similar solutions. Consider the problem (8)-(9). We will look for solutions $u(t, x)$ of the form (4). We set $s = r t^{-\lambda}$ (hence $r = s t^\lambda$). Then $u(t, x)$ of (4) can be also written as $u(t, x) = r^{-\lambda} g_d(s)$ (notice that $0 < s < \infty$). It follows that

$$u_t = -\kappa t^{-(\kappa+1)} g_d(s) + t^{-\lambda} g_d''(s) (-\lambda r t^{-(\lambda+1)}) = -t^{-(\kappa+1)} [\kappa g_d(s) + \lambda sg_d'(s)]. \quad (10)$$

Since $u$ of (4) is radial in $x$, we have

$$\Delta u = u_{rr} + \frac{d-1}{r} u_r \quad \text{and} \quad \nabla u = \frac{u_r}{r} x,$$

where

$$u_r = t^{-(\kappa+\lambda)} g_d'(s) \quad (11)$$

and

$$u_{rr} = t^{-(\kappa+2\lambda)} g_d''(s). \quad (12)$$

From the above, we obtain

$$\Delta u = t^{-(\kappa+2\lambda)} \left( g_d''(s) + \frac{d-1}{s} g_d'(s) \right). \quad (13)$$

By (7) we have

$$(u, Au) = \int_{\mathbb{R}^d} u(Au) = \int_{\mathbb{R}^d} u \Delta u + \alpha t^\gamma \int_{\mathbb{R}^d} u x \cdot \nabla u, \quad (14)$$
Finally, assuming that \( \lim \) have \( |\nabla^\alpha| \) outward unit normal vector. An easy calculation gives that for radial functions we

Furthermore, using Green’s first identity, we have

Then, by (15), we have

where \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \), \( \sigma \) is the surface measure on \( S^{d-1} \) (namely the measure on \( S^{d-1} \) induced by the \( d \)-dimensional Lebesgue measure), while \( \sigma_d \) is the total measure of \( S^{d-1} \). Recall that

where \( \Gamma(\cdot) \) is the Gamma function.

Combining (16) with (4), (11), we obtain

Finally, assuming that \( \lim_{s \to \infty} g_d(s)^2 s^d = 0 \), we have

Furthermore, using Green’s first identity, we have

where \( S^{d-1}(R) \) is the sphere of radius \( R \) in \( \mathbb{R}^d \) centered at the origin, and \( \eta \) is its outward unit normal vector. An easy calculation gives that for radial functions we have \( |\nabla u|^2 = u_r^2 \), thus

We set

and

Then, by (18), (19), (20), (21), equation (14) becomes

and by (13), (15), (11), operator \( A \) takes the form

\[
Au = t^{-(\kappa+2\lambda)} \left( g_d''(s) + \frac{d-1}{s} g_d'(s) \right) + \alpha t^\gamma r u_r \\
= t^{-(\kappa+2\lambda)} \left( g_d''(s) + \frac{d-1}{s} g_d'(s) \right) + \alpha t^{\gamma-\kappa} s g_d'(s).
\]
Substituting (4), (10), (22), (23) in (8), we have
\[
\kappa g_d(s) + \lambda s g_d'(s) = -t^{1-\kappa-2\lambda} g_d(s) g''_d(s) - \frac{d-1}{s} t^{1-\kappa-2\lambda} g_d(s) g'_d(s) \\
- \alpha t^{1+\gamma-\kappa} s g_d(s) g'_d(s) - t^{1-2\kappa-2\lambda+\lambda d} g_d(s) g''_d(s) \\
- \frac{\sigma d}{2} t^{1+\gamma+\lambda d-2\kappa} g_d(s) \Lambda_d [g_d]
\] (24)
The only way that (24) is a meaningful equation is that it does not contain \( t \). Therefore we are forced to take
\[
1 - \kappa - 2\lambda = 0, \quad 1 + \gamma - \kappa = 0, \quad 1 - 2\kappa - 2\lambda + \lambda d = 0, \quad 1 + \gamma + \lambda d - 2\kappa = 0.
\]
This gives
\[
\kappa = \frac{d}{d+2}, \quad \lambda = \frac{1}{d+2}, \quad \gamma = -\frac{2}{d+2}.
\] (25)
Hence, by (25), (4) gives
\[
u(t, x) = t^{-\frac{d}{d+2}} g_d(rt^{-\frac{1}{d+2}})
\] (26)
and the operator \( A \) becomes
\[
Au = \Delta u + \alpha t^{-\frac{d}{d+2}} x \cdot \nabla u.
\] (27)
Finally, we notice that, under (26), (9) gives
\[
\int_{\mathbb{R}^d} u(t, x) dx = \int_{S^{d-1}} \int_0^\infty w^{d-1} dv d\sigma \\
= \sigma_d \int_0^\infty t^{-\frac{d}{d+2}} g_d(s)(t^{-\frac{1}{d+2}})^{d-1} t^{\frac{d}{d+2}} ds \\
= \sigma_d \int_0^\infty s^{d-1} g_d(s) ds,
\]
which is independent of \( t \).
Thus, if we set
\[
\sigma_d \int_0^\infty s^{d-1} g_d(s) ds = 1,
\]
then
\[
\int_{\mathbb{R}^d} u(t, x) dx = 1, \text{ for all } t \geq 0.
\]
The following lemma summarizes what we have done so far.

**Lemma 2.1.** Let \( d \geq 2 \). If
\[
u(t, x) = t^{-\kappa} g_d(rt^{-\lambda}) \quad (\text{as usual } r = |x|)
\] (28)
is a probability density in \( x \) and satisfies (8), then we must have
\[
\kappa = \frac{d}{d+2}, \quad \lambda = \frac{1}{d+2}, \quad \gamma = -\frac{2}{d+2},
\] (29)
\[
g_d(s) \geq 0, \quad s \in \mathbb{R},
\] (30)
\[
\sigma_d \int_0^\infty s^{d-1} g_d(s) ds = 1
\] (31)
and
g_d(s)g_d''(s) + \left( \frac{d-1}{s} + \alpha s \right) g_d(s)g_d'(s) + \frac{d}{s+d+2} g_d(s) + \frac{s}{d+2} g_d'(s) \\
+ K_d[g_d]g_d(s) + \frac{d \alpha}{2} \Lambda_d[g_d]g_d(s) = 0, \hspace{1cm} (32)

where

\[ K_d[g_d] = \sigma_d \int_0^\infty s^{d-1} g_d'(s)^2 ds \hspace{1cm} (33) \]

and

\[ \Lambda_d[g_d] = \sigma_d \int_0^\infty s^{d-1} g_d(s)^2 ds. \hspace{1cm} (34) \]

Conversely, if (29)-(34) hold, then \( u(t,x) \) given by (28) is a probability density in \( x \) and satisfies (8).

We need to show that there exist function(s) \( g_d(s) \) satisfying (30), (31), (32), together with (33), (34). In order to do that we must first consider an auxiliary problem.

3. The auxiliary problem. Consider the problem

\[ q(s)q''(s) + \left( \frac{d-1}{s} + \alpha s \right) q(s)q'(s) + \frac{s}{d+2} q'(s) + \mu q(s) = 0, \hspace{0.5cm} s > 0 \hspace{1cm} (35) \]

\[ q(0) = A > 0, \hspace{0.5cm} q'(0) = 0 \hspace{1cm} (36) \]

where \( \alpha > 0 \) is a parameter, \( d \geq 2 \) is a natural number and \( \mu \) is a real parameter satisfying

\[ \mu > \frac{d}{d+2}. \hspace{1cm} (37) \]

We note that the above initial conditions (36) are interpreted in the sense of limits as \( s \to 0^+ \). Equation (35) can be written in the form

\[ q''(s) + \left( \frac{d-1}{s} + \alpha s \right) q'(s) + \frac{s}{d+2} \frac{q'(s)}{q(s)} + \mu = 0, \hspace{0.5cm} s > 0 \hspace{1cm} (38) \]

as long as \( q(s) \neq 0 \). By Proposition 1 of the Appendix we have that there is an \( \epsilon > 0 \) such that (35) and (36) have a unique solution \( q(s) \) for \( s \in [0, \epsilon] \).

**Lemma 3.1.** The solution \( q(s) \) of (35), (36) exists for all \( s > 0 \) and it is a strictly positive function which is decreasing on \((0, +\infty)\). Also,

\[ \lim_{s \to \infty} q(s) = 0, \hspace{0.5cm} \lim_{s \to \infty} q'(s) = 0, \]

\[ \int_0^\infty q'(s)^2 ds < \infty, \hspace{0.5cm} \int_0^\infty q(s) ds < \infty, \hspace{0.5cm} \int_0^\infty q(s)^2 ds < \infty. \]

**Proof.** Let \([0,b)\) with \( 0 < b \leq \infty \) be the maximal existence interval of \( q \). We will prove that \( b = \infty \).

Suppose that \( 0 < b < \infty \). Then, by a well-known theorem in the theory of ordinary differential equations [2],

\[ \lim_{s \to b^-} |q'(s)| + |q(s)| = \infty. \hspace{1cm} (39) \]

**Claim 1.** Suppose that the function \( q \) is positive on some interval \((0, s_1)\) with \( 0 < s_1 < b \). Then \( q' \) must stay negative on \((0, s_1)\).
Suppose that \( q' \) does not stay negative on \((0, s_1)\). Set \( s_2 = \inf\{s \in [0, s_1] : q'(s) \geq 0\} \). If \( s_2 = 0 \), then \( q'(s) > 0 \) for \( s \in (0, r) \), for some \( r > 0 \). Hence (38) implies that \( q''(s) < 0 \) for all \( s \in (0, r) \). Therefore, \( q' \) is decreasing on \((0, r)\) and then \( q'(s) < q'(0) = 0 \) for \( s \in (0, r) \), which is a contradiction. Therefore, \( s_2 > 0 \). Since \( q'(s_2) = 0 \), by (38) we get \( q''(s_2) < 0 \). Furthermore, \( q'(s) < 0 \) for all \( s \in (0, s_2) \), and then \( q''(s_2) \geq 0 \), which is a contradiction. Consequently, \( q' \) remains negative on \((0, s_1)\).

**Claim 2.** The function \( q \) is positive and decreasing on \([0, b)\) whereas \( q' \) is negative on \((0, b)\).

By Claim 1, it suffices to show that \( q \) is positive on \([0, b)\). Suppose that \( q \) is not positive on \([0, b)\). Let \( s_1 > 0 \) such that \( q(s_1) = 0 \) while \( q(s) > 0 \), for all \( s \in (0, s_1) \). Multiplying both sides of (38) with \( s^{d-1} \), we have

\[
s^{d-1}q''(s) + (d - 1)s^{d-2}q'(s) + \alpha s^d q'(s) + \frac{s^d}{d + 2} \frac{q'(s)}{q(s)} + \mu s^{d-1} = 0. \tag{40}
\]

Integrating (40) from 0 to \( s \in (0, s_1) \), and using the fact that \( q'(0) = 0 \) and \( q(s) > 0 \) for all \( s \in (0, s_1) \), implies that

\[
s^{d-1}q'(s) + \alpha s^d q(s) - \alpha d \int_0^s \xi^{d-1} q(\xi) d\xi + \frac{1}{d + 2} \int_0^s \xi^d \frac{q'(\xi)}{q(\xi)} d\xi + \frac{\mu}{d} s^d = 0 \tag{41}
\]

or

\[
s^{d-1}q'(s) = \alpha d \int_0^s \xi^{d-1} q(\xi) d\xi - \alpha s^{d} q(s) - \frac{s^d}{d + 2} \ln q(s) + \frac{d}{d + 2} \int_0^s \xi^{d-1} \ln q(\xi) d\xi - \frac{\mu}{d} s^d. \tag{42}
\]

Observe that since \( q(s) > 0 \), for all \( s \in (0, s_1) \), then \( q'(s) < 0 \), for all \( s \in (0, s_1) \) (Claim 1) and the function

\[
f(s) := -\frac{1}{d + 2} \ln q(s) \tag{43}
\]

has positive derivative on \((0, s_1)\). Hence, the function \( f \) is increasing in \((0, s_1)\), whereas it tends to \( \infty \) as \( s \to \frac{s_1}{2} \). Then, it is easy to show (see, e.g., Proposition A.2 of [10]), that

\[
\lim_{s \to \frac{s_1}{2}} \left[ s^d f(s) - d \int_0^s \xi^{d-1} f(\xi) d\xi \right] = \infty,
\]

i.e., (recall (43))

\[
\lim_{s \to \frac{s_1}{2}} \left[ -\frac{s^d}{d + 2} \ln q(s) + \frac{d}{d + 2} \int_0^s \xi^{d-1} \ln q(\xi) d\xi \right] = \infty. \tag{44}
\]

Hence, by (42) and (44), we obtain

\[
\lim_{s \to \frac{s_1}{2}} s^{d-1} q'(s) = \infty,
\]

which is impossible, since \( q' \) stays negative in \((0, s_1) \subset (0, b)\). Therefore, such an \( s_1 \) cannot exist. Then \( q(s) > 0 \) for all \( s \in [0, b) \) and \( q'(s) < 0 \) for all \( s \in (0, b) \). Consequently, \( q \) is decreasing in \([0, b)\), therefore \( 0 < q(s) \leq q(0) = A \) for all \( s \in [0, b) \).

Thus,

\[
\lim_{s \to b^-} q(s) \neq \infty,
\]
hence, (39) implies that
\[ \lim_{s \to b^-} |q'(s)| = \infty \]
and since \( q'(s) < 0 \) for all \( s > 0 \), we have
\[ \lim_{s \to b^-} q'(s) = -\infty. \]
Then, also
\[ \lim_{s \to b^-} \inf q''(s) = -\infty \]
which contradicts (38). Thus, (39) is not true. Consequently, \( b = \infty \) and the solution \( q(s) \) of (35), (36) exists for all \( s \geq 0 \). Furthermore, \( q \) is strictly positive and decreasing on \([0, +\infty)\) while \( q' \) is negative on \((0, +\infty)\).

Hence, \( 0 < q(s) < q(0) = A \) for all \( s > 0 \). It follows that
\[ \lim_{s \to \infty} q(s) = L, \]
namely
\[ q(s) = L + o(1), \quad \text{as } s \to \infty, \]  
where \( L \in [0, A) \). Let suppose \( L > 0 \). Then (45) implies that, as \( s \to \infty \),
\[ \ln q(s) = \ln (L + o(1)) = \ln L + o(1)) = \ln L + o(1). \]  
Using (45), (46) in (42), we obtain
\[ s^{d-1}q'(s) = \alpha d \int_0^s \xi^{d-1} (L + o(1)) d\xi - \alpha s^d (L + o(1)) \]
\[ - \frac{s^d}{d+2} (\ln L + o(1)) + \frac{d}{d+2} \int_0^s \xi^{d-1} (\ln L + o(1)) d\xi - \frac{\mu}{d} s^d \]
which implies
\[ s^{d-1}q'(s) = -\frac{\mu}{d} s^d + o(s) \]
i.e.,
\[ q'(s) = -\frac{\mu}{d} s + o(s), \quad \text{as } s \to \infty, \]
which contradicts (45). Therefore, \( L = 0 \), i.e.,
\[ \lim_{s \to \infty} q(s) = 0. \]  
Furthermore, we notice that
\[ \int_0^\infty q'(s) ds = \lim_{s \to \infty} q(s) - q(0) = -A, \]
hence \( q' \in L_1(0, +\infty) \).

Suppose that
\[ \lim_{s \to \infty} \inf q'(s) < 0. \]  
Then, from (47) and (49), there is a sequence \((s_n)\), such that \( s_n \to \infty \), \( q' \) attains a local minimum at \( s_n \) for all \( n \) and
\[ \lim_{n \to \infty} q'(s_n) = -\delta, \quad \text{for some } \delta > 0. \]  
But, since \( q'(s_n) \) is a local minimum for \( q' \), we must have \( q''(s_n) = 0 \), hence (38) gives
\[ \left( \frac{d-1}{s_n} + \alpha s_n \right) q'(s_n) + \frac{s_n}{d+2} q'(s_n) + \mu = 0. \]
Therefore, from the above and (51), we obtain

\[ |q'(s_n)| \leq \frac{\mu(d+2)A}{[\alpha(d+2)q(s_n) + 1]s_n}. \]

Then

\[ \lim_{n \to \infty} q'(s_n) = 0 \]

which contradicts (50). Thus, we have established that

\[ \lim_{s \to \infty} q'(s) = 0. \]

This, together with the fact that \( q' \) is integrable, implies \( q' \in L_2(0, +\infty) \), i.e.,

\[ \int_0^\infty q'(s)^2 ds < \infty. \] (53)

Integrating by parts in (35) from 1 to \( s \) (\( s > 1 \)), we get

\[
q(s)q'(s) - q(1)q'(1) - \int_1^s q'(\xi)^2 d\xi + \frac{d-1}{2}q(s)^2 - \frac{(d-1)q(1)^2}{2} + \frac{d-1}{2} \int_1^s \frac{q(\xi)^2}{2} d\xi + \frac{\alpha}{2} s q(s)^2 - \frac{\alpha q(1)^2}{2} - \frac{\alpha}{2} \int_1^s q(\xi)^2 d\xi + \frac{\alpha q(1)^2}{2} - \frac{\alpha q(1)^2}{2} = 0
\]

or

\[
\left( \mu - \frac{1}{d+2} \right) \int_1^s q(\xi) d\xi = q(1)q'(1) - q(s)q'(s) + \int_1^s q'(\xi)^2 d\xi
\]

\[
+ \frac{(d-1)q(1)^2}{2} - \frac{d-1}{2} \frac{q(s)^2}{2} - \frac{d-1}{2} \int_1^s \frac{q(\xi)^2}{2} d\xi + \frac{\alpha q(1)^2}{2} - \frac{\alpha}{2} s q(s)^2 + \frac{\alpha}{2} \int_1^s q(\xi)^2 d\xi + \frac{\alpha q(1)^2}{2} - \frac{\alpha q(1)^2}{2} - \frac{s}{d+2} q(s).
\]

Therefore,

\[
\left( \mu - \frac{1}{d+2} \right) \int_1^s q(\xi) d\xi \leq \int_1^s q'(\xi)^2 d\xi - q(s)q'(s) + \frac{(d-1)q(1)^2}{2} + \frac{\alpha q(1)^2}{2} + \frac{\alpha}{2} \int_1^s q(\xi)^2 d\xi + \frac{\alpha q(1)^2}{2}.
\]
Especially,
\[
\left(\mu - \frac{1}{d+2}\right) \int_1^s q(\xi) d\xi \leq \int_1^s q'(\xi)^2 d\xi - q(s)q'(s) + \frac{\alpha}{2} \int_1^s q(\xi)^2 d\xi + (d-1)A^2 + \frac{\alpha A^2}{2} + \frac{A}{d+2},
\]
(54)
since 0 < q(s) < A, q'(s) < 0 for all s ∈ (0, +∞).

If we suppose that
\[
\int_1^\infty q(\xi) d\xi = \infty,
\]
(55)
then, by (47), (52), (53) and (54), we get
\[
\int_1^\infty q(\xi)^2 d\xi = \infty.
\]
Setting
\[
M := \int_1^\infty q'(\xi)^2 d\xi + \frac{(d-1)A^2}{2} + \frac{\alpha A^2}{2} + \frac{A}{d+2} < \infty,
\]
and using the fact that q(\xi) > 0 for all \xi > 0, formula (54) gives
\[
\mu - \frac{1}{d+2} \leq \frac{M - q(s)q'(s) + \frac{\alpha}{2} \int_1^s q(\xi)^2 d\xi}{\int_1^s q(\xi) d\xi},
\]
(56)
Under (55), L'Hospital's Rule and the fact that q(s)q'(s) is bounded yield
\[
\lim_{s \to \infty} \frac{M - q(s)q'(s) + \frac{\alpha}{2} \int_1^s q(\xi)^2 d\xi}{\int_1^s q(\xi) d\xi} \leq \lim_{s \to \infty} \frac{\frac{\alpha}{2} q(s)^2}{q(s)} = \frac{\alpha}{2},
\]
and (56) implies \(\mu - \frac{1}{d+2} \leq 0\), which is a contradiction, since \(\mu > \frac{d}{d+2}\) and \(d \geq 2\).

Consequently,
\[
\int_1^\infty q(\xi) d\xi < \infty,
\]
thus \(q\) is integrable on \([1, +\infty)\), so \(q\) is integrable on \([0, +\infty)\), i.e.,
\[
\int_0^\infty q(\xi) d\xi < \infty.
\]
(57)
The function \(q\) is strictly positive and decreasing on \((0, +\infty)\) with \(0 < q(s) \leq A\), for all \(s \geq 0\). Since
\[
\lim_{s \to +\infty} q(s) = 0,
\]
there is a \(s_0 > 0\) such that \(0 < q^2(s) < q(s)\), for all \(s \geq s_0\)
\[
0 < \int_{s_0}^\infty q(s)^2 ds < \int_{s_0}^\infty q(s) ds.
\]
But \(q \in L_1(0, +\infty)\), thus
\[
\int_0^\infty q(s)^2 ds < \infty.
\]
(58)
The proof of this key lemma is now complete. \qed
Lemma 3.2. Let \( q(s) \) be the solution of the problem (35), (36). Then

\[
\begin{align*}
(i) & \quad \int_0^\infty s^d q'(s)ds = -d \int_0^\infty s^{d-1}q(s)ds < \infty, \\
(ii) & \quad \lim_{s \to \infty} s^d q(s) = 0, \\
(iii) & \quad \int_0^\infty s^d q'(s)^2 ds < \infty, \\
(iv) & \quad \int_0^\infty s^{d-1}q(s)^2 ds < \infty, \\
(v) & \quad \lim_{s \to \infty} s^d q(s) = 0, \\
(vi) & \quad \int_0^\infty s^d q'(s)^2 ds < \infty.
\end{align*}
\]

Proof. First we notice that, since by Lemma 3.1 \( q' \) is negative and \( \lim_{s \to \infty} q'(s) = 0 \), (iii) follows immediately from (i). We will use an inductive argument to show that for each \( n \in \{1, 2, \ldots, d\} \) we have

\[
\begin{align*}
(i') & \quad \int_0^\infty s^n q'(s)ds = -n \int_0^\infty s^{n-1}q(s)ds < \infty \\
(ii') & \quad \lim_{s \to \infty} s^n q(s) = 0 \\
(iii') & \quad \int_0^\infty s^n q'(s)^2 ds < \infty
\end{align*}
\]

For \( n = 1 \), by Lemma 3.1 we have

\[\int_0^\infty q(s)ds < \infty \quad \text{and} \quad \lim_{s \to \infty} q(s) = 0\]

whereas \( q(s) > 0 \), for all \( s \in [0, +\infty) \) and \( q \) is decreasing on \( [0, +\infty) \). By a standard result of calculus we conclude that

\[\lim_{s \to \infty} sq(s) = 0.\]  

Hence,

\[\int_0^\infty sq'(s)ds = \lim_{s \to \infty} sq(s) - \int_0^\infty q(s)ds = - \int_0^\infty q(s)ds < \infty.\]

Thus (i'), (ii') are valid for \( n = 1 \) and then (iii') is valid for \( n = 1 \), i.e.,

\[\int_0^\infty sq'(s)^2 ds < \infty.\]

Next, we fix \( n \in \{2, 3, \ldots, d\} \) and suppose that (i'), (ii') hold for \( k \in \{1, 2, \ldots, n-1\} \), i.e., for all \( k \in \{1, 2, \ldots, n-1\} \) we have

\[\int_0^\infty s^k q'(s)ds = -k \int_0^\infty s^{k-1}q(s)ds < \infty, \quad \lim_{s \to \infty} s^k q(s) = 0, \quad \int_0^\infty s^k q'(s)^2 ds < \infty.\]
In particular, for \( k = n - 1 \) hold
\[
\begin{align*}
(a) & \quad \int_0^\infty s^{n-1}q(s)ds = -(n-1) \int_0^\infty s^{n-2}q(s)ds < \infty, \quad (66) \\
(b) & \quad \lim_{s \to \infty} s^{n-1}q(s) = 0, \quad (67) \\
(c) & \quad \int_0^\infty s^{n-1}q'(s)^2ds < \infty. \quad (68)
\end{align*}
\]

We will proof that (i'), (ii') also hold for \( k = n \). Multiplying both sides of (35) with \( s^{n-1} \), integrating from \( 0 \) to \( s > 0 \) and using \( q'(0) = 0 \), we obtain
\[
\begin{align*}
s^{n-1}q(s)q'(s) - \int_0^s \xi^{n-1}q'((\xi))d\xi + (d-n)\int_0^s \xi^{n-2}q(\xi)q'(\xi)d\xi + \frac{\alpha}{2}s^nq(s)^2 \\
- \frac{\alpha n}{2} \int_0^s \xi^{n-1}q(\xi)d\xi + \frac{s^n}{d+2}q(s) - \frac{n}{d+2} \int_0^s \xi^{n-1}q(\xi)d\xi \\
+ \mu \int_0^s \xi^{n-1}q(\xi)d\xi = 0
\end{align*}
\]
or
\[
\begin{align*}
\left(\mu - \frac{n}{d+2}\right) \int_0^s \xi^{n-1}q(\xi)d\xi = \int_0^s \xi^{n-1}q'(\xi)d\xi - s^{n-1}q(s)q'(s) \\
- (d-n) \int_0^s \xi^{n-2}q(\xi)q'(\xi)d\xi - \frac{\alpha}{2}s^nq(s)^2 \\
+ \frac{\alpha n}{2} \int_0^s \xi^{n-1}q(\xi)^2d\xi - \frac{s^n}{d+2}q(s).
\end{align*}
\]

Since \( q(s) > 0 \) for all \( s \geq 0 \) and \( q'(s) < 0 \) for all \( s > 0 \), from (69) we have
\[
\begin{align*}
\left(\mu - \frac{n}{d+2}\right) \int_0^s \xi^{n-1}q(\xi)d\xi &\leq \int_0^s \xi^{n-1}q'(\xi)d\xi - s^{n-1}q(s)q'(s) \\
- (d-n) \int_0^s \xi^{n-2}q(\xi)q'(\xi)d\xi \\
+ \frac{\alpha n}{2} \int_0^s \xi^{n-1}q(\xi)^2d\xi.
\end{align*}
\]
From (67) and the fact that \( \lim_{s \to \infty} q'(s) = 0 \) we get
\[
\lim_{s \to \infty} s^{n-1}q(s)q'(s) = 0. \quad (71)
\]
If \( n \geq 3 \), our hypothesis enables us to apply (i') for \( k = n - 2 \) and using the fact that \( q'(s) < 0 \), for all \( s > 0 \) and \( q(s) \leq A \), for all \( s \geq 0 \), we have
\[
\int_0^\infty \xi^{n-2}|q'(\xi)|d\xi = -\int_0^\infty \xi^{n-2}q'(\xi)d\xi < \infty
\]
and
\[
(d-n)\int_0^\infty \xi^{n-2}q(\xi)|q'(\xi)|d\xi \leq A(d-n)\int_0^\infty \xi^{n-2}|q'(\xi)|d\xi < \infty.
\]
Thus,
\[
- (d-n)\int_0^\infty \xi^{n-2}q(\xi)q'(\xi)d\xi \leq -A(d-n)\int_0^\infty \xi^{n-2}q'(\xi)d\xi < \infty. \quad (72)
\]
For $n = 2$, the above inequality also holds. Indeed, for $s > 0$ we have
\[
\int_0^s q(\xi) |q'(\xi)| \, d\xi = - \int_0^s q(\xi)q'q(\xi) \, d\xi = - \frac{q(s)^2}{2} + \frac{q(0)^2}{2} = \frac{A^2}{2} - \frac{q(s)^2}{2}
\]
and then
\[
\int_0^\infty q(\xi) |q'(\xi)| \, d\xi = \frac{A^2}{2} - \frac{1}{2} \lim_{s \to \infty} q(s)^2 = \frac{A^2}{2} < A^2 = A \int_0^\infty |q'(\xi)| \, d\xi < \infty.
\]
Suppose that
\[
\int_0^\infty \xi^{n-1} q(\xi) \, d\xi = \infty,
\]
then (68), (70), (71) and (72) imply that
\[
\int_0^\infty \xi^{-1} q(\xi)^2 \, d\xi = \infty.
\]
We set
\[
T := \int_0^\infty \xi^{n-1} q'(\xi)^2 \, d\xi - (d - n) \int_0^\infty \xi^{n-2} q(\xi)q'(\xi) \, d\xi < \infty.
\]
Then, the inequality (70) can be written
\[
\left(\mu - \frac{n}{d + 2}\right) \int_0^s \xi^{n-1} q(\xi) \, d\xi \leq T - s^{n-1} q(s)q'(s) + \frac{an}{2} \int_0^s \xi^{n-1} q(\xi)^2 \, d\xi.
\]
Furthermore, $\xi^{d-1} q(\xi) > 0$ for all $\xi > 0$ and the above inequality gives
\[
\mu - \frac{n}{d + 2} \leq \frac{T - s^{n-1} q(s)q'(s) + \frac{an}{2} \int_0^s \xi^{n-1} q(\xi)^2 \, d\xi}{\int_0^s \xi^{n-1} q(\xi) \, d\xi}.
\]
Hence, under (73), L’Hospital’s Rule together with the fact that $s^{n-1} q(s)q'(s)$ is bounded yield
\[
\lim_{s \to \infty} \frac{T - s^{n-1} q(s)q'(s) + \frac{an}{2} \int_0^s \xi^{n-1} q(\xi)^2 \, d\xi}{\int_0^s \xi^{n-1} q(\xi) \, d\xi} \leq \lim_{s \to \infty} \frac{\frac{an}{2} s^{n-1} q(s)^2}{s^{n-1} q(s)} = \frac{an}{2} \lim_{s \to \infty} q(s) = 0.
\]
But, then, from (74) we have $\mu - \frac{n}{d + 2} \leq 0$, which contradicts the fact that $\mu > \frac{d}{d\pi^2} \geq \frac{n}{d\pi^2}$. Consequently,
\[
\int_0^\infty \xi^{n-1} q(\xi) \, d\xi < \infty.
\]
Since $q$ is decreasing and positive on $[0, +\infty)$ and $\lim_{s \to \infty} q(s) = 0$, then
\[
\int_0^\infty s^{n-1} q(s)^2 \, ds < \infty.
\]
Hence, (75), (76) hold for $n = d$, i.e.,
\[
\int_0^\infty s^{d-1} q(s) \, ds < \infty,
\]
and
\[
\int_0^\infty s^{d-1} q'(s) \, ds < \infty,
\]
and this establishes (iv).
By (69) we have
\[
\frac{\alpha}{2} s^n q(s)^2 + \frac{s^n}{d+2} q(s) = -\left( \mu - \frac{n}{d+2} \right) \int_0^s \xi^{n-1} q(\xi)d\xi + \int_0^s \xi^{n-1} q(\xi)^2d\xi
- s^{n-1} q(s)q'(s) - (d-n) \int_0^s \xi^{n-2} q(\xi)q'(\xi)d\xi
+ \frac{\alpha n}{2} \int_0^s \xi^{n-1} q(\xi)^2d\xi,
\]
and from (68), (71), (72), (75), (76) we obtain
\[
\lim_{s \to \infty} \left[ \frac{\alpha}{2} s^n q(s)^2 + \frac{s^n}{d+2} q(s) \right] = N \in \mathbb{R}.
\]
If we suppose that \( N \neq 0 \), then the above limit tells us that \( (s) := \alpha(d + 2)s^{n-1}q(s)^2 + 2s^{n-1}q(s) \), is asymptotic to \( 2(d+2)Ns^{-1} \), which is impossible because \( \mathbb{N}(s) \) is integrable. Hence,
\[
\lim_{s \to \infty} \left[ \frac{\alpha}{2} s^n q(s)^2 + \frac{s^n}{d+2} q(s) \right] = 0.
\]
Since \( q(s) > 0 \), for all \( s \geq 0 \) then
\[
0 \leq \frac{s^n}{d+2} q(s) \leq \frac{\alpha}{2} s^n q(s)^2 + \frac{s^n}{d+2} q(s)
\]
and thus
\[
\lim_{s \to \infty} s^n q(s) = 0
\]
which means that (ii’) holds for \( n \). So (ii’) holds for all \( n \in \{1, 2, ..., d\} \). Hence,
\[
\left| \int_0^\infty s^n q'(s)ds \right| = \left| \lim_{s \to \infty} s^n q(s) - n \int_0^\infty s^{n-1} q(s)ds \right| = -n \int_0^\infty s^{n-1} q(s)ds < \infty.
\]
Thus, the (i’) holds for all \( n \in \{1, 2, ..., d\} \), so the (iii’) holds for all \( n \in \{1, 2, ..., d\} \). Hence, for \( n = d \) we get
\[
\int_0^\infty s^d q'(s)ds = -d \int_0^\infty s^{d-1}q(s)ds < \infty, \quad \lim_{s \to \infty} s^d q(s) = 0, \quad \int_0^\infty s^d q'(s)^2ds < \infty,
\]
\[
\int_0^\infty s^{d-1}q(s)^2ds < \infty \quad \text{and} \quad \lim_{s \to \infty} \left[ \frac{s^d q(s)}{d+2} + \frac{\alpha}{2} s^d q(s)^2 \right] = 0.
\]
Then,
\[
\int_0^\infty \left[ \frac{s^d q'(s)}{d+2} + \alpha s^d q'(s)q(s) \right] ds = \lim_{s \to \infty} \left[ \frac{s^d q(s)}{d+2} + \frac{\alpha}{2} s^d q(s)^2 \right]
- \frac{d}{d+2} \int_0^\infty s^{d-1}q(s)ds
- \frac{\alpha d}{2} \int_0^\infty s^{d-1}q(s)^2ds.
\]
consequently,
\[
- \int_0^\infty \left[ \frac{s^d q'(s)}{d+2} + \alpha s^d q'(s)q(s) \right] ds = \frac{d}{d+2} \int_0^\infty s^{d-1}q(s)ds + \frac{\alpha d}{2} \int_0^\infty s^{d-1}q(s)^2ds,
\]
which is (vi).
Corollary 1. If \( q(s) \) satisfies (35) and (36), then
\[
\left( \mu - \frac{d}{d+2} \right) \int_0^\infty s^{d-1} q(s)ds = \int_0^\infty s^{d-1} q'(s)^2 ds + \frac{\alpha d}{2} \int_0^\infty s^{d-1} q(s)^2 ds. \tag{77}
\]

Proof. For \( n = d \), (69) yields
\[
\left( \mu - \frac{d}{d+2} \right) \int_0^s \xi^{d-1} q(\xi)d\xi = \int_0^s \xi^{d-1} q'(\xi)^2 d\xi - s^{d-1} q(s)q'(s) - \frac{\alpha}{2} s^d q(s)^2
\]
and using (63), we obtain
\[
\left( \mu - \frac{d}{d+2} \right) \int_0^\infty s^{d-1} q(s)ds = \int_0^\infty s^{d-1} q'(s)^2 ds + \frac{\alpha d}{2} \int_0^\infty s^{d-1} q(s)^2 ds.
\]

4. The construction of the self-similar solutions.

Lemma 4.1. Let \( q(s) \) be the solution of the problem (35), (36). Then
\[
(i) \quad \|q'\|_\infty \leq \mu \sqrt{\frac{(d+2)A}{1 + \alpha (d+2)A}}, \quad \text{where} \quad \|\cdot\|_\infty \quad \text{denotes the sup-norm}, \tag{78}
\]
\[
(ii) \quad \int_0^\infty s^{d-1} q(s)ds \geq \frac{A^{1+\frac{\alpha}{2}}}{\mu^d(d+1)} \left[ \frac{1 + \alpha (d+2)A}{d+2} \right]^d, \tag{79}
\]
\[
(iii) \quad \int_0^\infty s^{d-1} q(s)^2 ds \geq \frac{2A^{2+\frac{\alpha}{2}}}{\mu^d(d+1)(d+2)} \left[ \frac{1 + \alpha (d+2)A}{d+2} \right]^d. \tag{80}
\]

Proof. Since \( q' \) is negative on \((0, +\infty)\) with
\[
q'(0) = 0 = \lim_{s \to +\infty} q'(s) \quad \text{and} \quad \|q'\|_\infty = \sup\{ |q'(s)| : s \geq 0 \} = \sup\{ -q'(s) : s \geq 0 \},
\]
it follows that \( q' \) attains its absolute minimum at some \( s_m \in (0, +\infty) \). Hence,
\[
\|q'\|_\infty = |q'(s_m)| = -q'(s_m). \tag{81}
\]
Also, \( q''(s_m) = 0 \), thus (38) gives
\[
q'(s_m) = \frac{-\mu}{s_m^{d-1} + \left[ \alpha + \frac{1}{(d+2)q(s_m)} \right] s_m}. \tag{82}
\]

But \( q \) is decreasing in \([0, +\infty)\), \( q(0) = A, \ s_m \in (0, +\infty) \) and \( d \geq 2 \), therefore
\[
\left[ \frac{1}{(d+2)A} + \alpha \right] s_m < \frac{d-1}{s_m} + \left[ \frac{1}{(d+2)A} + \alpha \right] s_m
\]
\[
\leq \frac{d-1}{s_m} + \left[ \frac{1}{(d+2)q(s_m)} + \alpha \right] s_m.
\]
Furthermore, \( \frac{d-1}{s_m} > 0 \), \( \left[ \frac{1}{(d+2)A} + \alpha \right] s_m > 0 \), and \( \mu > 0 \), hence by (82) we obtain
\[
-q'(s_m) < \frac{\mu}{\left[ \frac{1}{(d+2)A} + \alpha \right] s_m}.
\]
The above inequality and (81) imply
\[ \|q'\|_\infty < \frac{\mu(d + 2)A}{1 + \alpha(d + 2)A} s_m. \] (83)

Meanwhile, since \( q(s) > 0, q'(s) < 0 \), for all \( s > 0 \), (38) yields
\[ q''(s) + \mu = -\left( \frac{d - 1}{s} + \alpha s \right) q'(s) - \frac{s}{d + 2} \frac{q'(s)}{q(s)} \geq 0 \text{ for all } s > 0. \]

Therefore, \( q''(s) \geq -\mu \) for all \( s > 0 \). Consequently, \( q'(s) \geq -\mu s \) for all \( s \geq 0 \), and hence
\[ \|q'\|_{\infty} = -q'(s_m) \leq \mu s_m. \] (84)

By combining (83), (84), we obtain
\[ \|q'\|_{\infty} \leq \min\left\{ \mu s_m, \frac{\mu(d + 2)A}{1 + \alpha(d + 2)A} s_m \right\}. \]

But, no matter what \( s_m \) is, the quantity \( \min\{\mu s_m, \mu(d + 2)A [1 + \alpha(d + 2)A]^{-1} s_m^{-1}\} \) (since the first is increasing in \( s_m \) while the second is decreasing) is always at most \( S \), where
\[ S = \mu s_0 = \frac{\mu(d + 2)A}{1 + \alpha(d + 2)A} s_0, \text{ for some } s_0 > 0. \]

Then,
\[ s_0 = \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}} \quad \text{and} \quad S = \mu \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}}. \]

Thus,
\[ \|q'\|_{\infty} \leq \mu \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}}, \]

which is (78).

Also, \( -\|q'\|_{\infty} \leq q'(s) \), for all \( s \geq 0 \), therefore
\[ q(s) \geq q(0) - \|q'\|_{\infty} s, \text{ for all } s \geq 0. \]

Using (78) together with the fact that \( q(0) = A \), from the above inequality we obtain
\[ q(s) \geq A - s \mu \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}} \text{ for all } s \geq 0, \]

especially for \( s \in [0, \mu^{-1} \sqrt{1 + \alpha(d + 2)A} A(d + 2)^{-1}] \), (since \( q(s) > 0 \), for all \( s \geq 0 \)).

Consequently,
\[ \int_0^\infty s^{d - 1} q(s) ds \geq \int_0^{\mu \sqrt{\frac{1 + \alpha(d + 2)A}{d + 2}} A} s^{d - 1} q(s) ds \geq \int_0^{\mu \sqrt{\frac{1 + \alpha(d + 2)A}{d + 2}} A} s^{d - 1} \left[ A - \mu s \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}} \right] ds. \]

Thus,
\[ \int_0^\infty s^{d - 1} q(s) ds \geq \frac{A^{1 + \frac{d}{2}}}{\mu^d(d + 1)} \left[ \sqrt{\frac{1 + \alpha(d + 2)A}{d + 2}} \right]^d. \]
and this establishes (79).

Furthermore, for \( s \in [0, \mu^{-1} \sqrt{1 + \alpha(d + 2)A} A(d + 2)^{-1}] \), we have

\[
\int_0^\infty s^{d-1} q(s)^2 ds \geq \int_0^{1} s^{d-1} q(s)^2 ds \\
\geq \int_0^{1} (\frac{A}{s} - s \mu \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}})^2 ds.
\]

Thus,

\[
\int_0^\infty s^{d-1} q(s)^2 ds \geq \frac{2A^{2+4}}{\mu^2 d(d+1)(d+2)} \left[ \frac{1 + \alpha(d+2)A}{d+2} \right]^d,
\]

which is (80).

**Lemma 4.2.** If \( q \) is the solution of (35) - (36), then

\[
\frac{q(1) e^{\alpha(d+2)q(1)}}{s \mu(d+2)e^{\frac{\mu(d+2)}{d}}} \leq q(s) e^{\alpha(d+2)q(s)} \leq A d e^{\alpha(d+2)q(s)} e^{\frac{\mu(d+2)}{d+2}}
\]

(85)

for all \( s \geq 1 \).

**Proof.** The function \( q \) is positive and decreasing on \([0, +\infty)\), hence \( 0 < q(s) \leq q(0) = A \) for all \( s \geq 0 \). We set

\[
F(s) := -\frac{d}{d+2} \int_0^s \xi^{d-1} \ln q(\xi) d\xi, \quad s \geq 0.
\]

The equation (41), by (86), becomes

\[
s^{d-1} q(s) + \alpha s^d q(s) - \alpha d \int_0^s \xi^{d-1} q(\xi) d\xi - \frac{s}{d} F(s) + F(s) + \frac{\mu}{d} s^d = 0
\]

or

\[
s F'(s) - dF(s) = ds^{d-1} q(s) + \alpha ds^d q(s) - \alpha d \int_0^s \xi^{d-1} q(\xi) d\xi + \mu s^d,
\]

therefore

\[
\left( \frac{F(s)}{s^d} \right)' = \frac{dq'(s)}{s^d} + \left( \frac{\alpha d}{s^d} \int_0^s \xi^{d-1} q(\xi) d\xi \right)'+ \frac{\mu}{d}
\]

(88)

Integrating the above equation from 1 to \( s \), for \( s \geq 1 \), we obtain

\[
\frac{F(s)}{s^d} = F(1) + d \int_1^s \frac{q'(\xi)}{\xi^2} d\xi + \frac{\alpha d}{s^d} \int \xi^{d-1} q(\xi) d\xi - \alpha d \int_0^1 \xi^{d-1} q(\xi) d\xi + \mu \ln s. \quad (89)
\]

Furthermore, (87) implies

\[
\frac{F(s)}{s^d} = -\frac{\ln q(s)}{d+2} - \frac{q'(s)}{s} - \alpha q(s) + \frac{\alpha d}{s^d} \int_0^s \xi^{d-1} q(\xi) d\xi - \frac{\mu}{d}.
\]

(90)

From (89) and (90), we have

\[
d \int_1^s \frac{q'(\xi)}{\xi^2} d\xi = \alpha d \int_0^1 \xi^{d-1} q(\xi) d\xi - F(1) - \mu \ln s - \frac{\ln q(s)}{d+2} - \frac{q'(s)}{s} - \alpha q(s) - \frac{\mu}{d}.
\]

(91)
Since \( q \) is decreasing in \([0, +\infty)\), then for \( 0 \leq \xi \leq 1 \) we have

\[
\int_0^1 \xi^{d-1} \ln q(1)d\xi \leq -\frac{d+2}{d} F(1) \leq \int_0^1 \xi^{d-1} \ln A d\xi \leq \ln A,
\]
i.e.,

\[
\ln q(1) \leq -(d+2)F(1) \leq \ln A
\]
and furthermore

\[
\int_0^1 q(1)\xi^{d-1}d\xi \leq \int_0^1 \xi^{d-1}q(\xi)d\xi \leq \int_0^1 A\xi^{d-1}d\xi,
\]
i.e.,

\[
q(1) \leq d\int_0^1 \xi^{d-1}q(\xi)d\xi \leq A.
\]

Since \( q'(s) \leq 0 \), for all \( s \in [0, +\infty) \) and \( -q'(s) \leq ||q'||_{\infty} \) for all \( s \in [0, +\infty) \), from (78) we have

\[
q'(s) \geq -||q'||_{\infty} \geq -\mu \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}}, \text{ for all } s \geq 0.
\]
Hence, for \( s > 0 \) we get

\[
-\frac{\mu}{s} \frac{(d+2)A}{1+\alpha(d+2)A} \leq \frac{q'(s)}{s} \leq 0 \leq -\frac{q'(s)}{s} \leq \frac{\mu}{s} \frac{(d+2)A}{1+\alpha(d+2)A}.
\]

For \( \xi \in [1, s] \) with \( s \geq 1 \) we have

\[
0 \geq \int_1^s \frac{q'(\xi)}{\xi^2}d\xi \geq \int_1^s q'(\xi)d\xi \geq \int_0^{\infty} q'(\xi)d\xi = \lim_{s \to \infty} q(s) - q(0) = -A,
\]
i.e.,

\[
0 \geq d\int_1^s \frac{q'(\xi)}{\xi^2}d\xi \geq -Ad.
\]

By (95) and (91), we obtain

\[
0 \leq -\alpha d\int_0^1 \xi^{d-1}q(\xi)d\xi + F(1) + \mu \ln s + \frac{\ln q(s)}{d+2} + \frac{q'(s)}{s} + \alpha q(s) + \frac{\mu}{d} \leq Ad
\]
or

\[
\alpha d\int_0^1 \xi^{d-1}q(\xi)d\xi - F(1) - \mu \ln s - \frac{q'(s)}{s} - \frac{\mu}{d} \leq \frac{\ln q(s)}{d+2} + \alpha q(s)
\]

\[
\leq Ad + \alpha d\int_0^1 \xi^{d-1}q(\xi)d\xi - F(1) - \mu \ln s - \frac{q'(s)}{s} - \frac{\mu}{d}.
\]

Then, the above inequality combined with (92) and (93), gives

\[
\alpha(d+2)q(1) + \ln q(1) + \ln s^{-\mu(d+2)} - (d+2)\frac{q'(s)}{s} - \frac{(d+2)\mu}{d} \\
\leq \ln q(s) + \alpha(d+2)q(s) \\
\leq (d+\alpha)(d+2)A + \ln A^d + \ln s^{-\mu(d+2)} \\
- (d+2)\frac{q'(s)}{s} - \frac{(d+2)\mu}{d}.
\]
and by (94) we obtain

\[
\alpha(d + 2)q(1) + \ln q(1) + \ln s^{-\mu(d + 2)} - \frac{(d + 2)\mu}{s} \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}} - \frac{(d + 2)\mu}{d}
\]

\[
\leq \ln q(s) + \alpha(d + 2)q(s)
\]

\[
\leq (d + \alpha)(d + 2)A + \ln A^d + \ln s^{-\mu(d + 2)}
\]

\[
+ \frac{(d + 2)\mu}{s} \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}} - \frac{(d + 2)\mu}{d}.
\]

Therefore,

\[
q(1)e^{\alpha(d + 2)q(1)} \leq q(s)e^{\alpha(d + 2)q(s)}
\]

\[
\leq \frac{A^d e^{(d + \alpha)(d + 2)A} e^{\mu(d + 2)} \sqrt{\frac{(d + 2)A}{1 + \alpha(d + 2)A}}}{s^\mu(d + 2)e^{\mu(d + 2)}}
\]

and this establishes (85).

**Corollary 2.** If \(q(s)\) satisfies (35) and (36), then

1. \(\lim_{A \to \infty} \int_0^\infty s^{d-1} q(s)^2 ds = \infty\) (97)
2. \(\lim_{A \to \infty} \int_0^\infty s^{d-1} q(s) ds = \infty\), (98)
3. \(\lim_{A \to 0^+} \int_0^\infty s^{d-1} q(s) ds = 0\), (99)
4. \(\lim_{A \to 0^+} \int_0^\infty s^{d-1} q'(s)^2 ds = 0\), (100)
5. \(\lim_{A \to 0^+} \int_0^\infty s^{d-1} q(s)^2 ds = 0\). (101)

**Proof.** By (80) we have

\[
\int_0^\infty s^{d-1} q(s)^2 ds \geq \frac{2A^{2+\frac{d}{2}}}{\mu^d d(d + 1)(d + 2)} \left[ \frac{1 + \alpha(d + 2)A}{d + 2} \right]^d,
\]

and since

\[
\lim_{A \to \infty} \frac{2A^{2+\frac{d}{2}}}{\mu^d d(d + 1)(d + 2)} \left[ \frac{1 + \alpha(d + 2)A}{d + 2} \right]^d = \infty,
\]

then

\[
\lim_{A \to \infty} \int_0^\infty s^{d-1} q(s)^2 ds = \infty
\]

and this establishes (97).

Furthermore, by (79), we have

\[
\int_0^\infty s^{d-1} q(s) ds \geq \frac{A^{1+\frac{d}{2}}}{\mu^d d(d + 1)} \left[ \frac{1 + \alpha(d + 2)A}{d + 2} \right]^d.
\]
Since
\[
\lim_{A \to \infty} \frac{A^{1+\frac{d}{d(d+1)}}}{\mu^d(d+1)} \left[ \frac{1 + \alpha(d+2)A}{d+2} \right]^d = \infty
\]
then
\[
\lim_{A \to \infty} \int_0^\infty s^{d-1} q(s) ds = \infty
\]
and this establishes (98).

Next, for \(0 \leq \xi \leq 1\) we get
\[
\int_0^1 \xi^{d-1} q(\xi) d\xi \leq \int_0^1 q(\xi) d\xi \leq \int_0^1 q(0) d\xi = \int_0^1 A d\xi = A,
\]
hence
\[
\int_0^1 \xi^{d-1} q(\xi) d\xi \leq A. \tag{102}
\]
From (85), for \(s \geq 1\) we have
\[
q(s) e^{\alpha(d+2)q(s)} \leq \frac{A^d e^{(d+\alpha)(d+2)} A e^{\mu(d+2)}}{s^{\mu(d+2)} e^{\mu(d+2)}} \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}}
\]
and \(0 < q(s) \leq q(0) = A\), for all \(s \geq 0\), hence \(0 < \alpha(d+2)q(s) \leq \alpha(d+2)A\), for all \(s \geq 0\). Combining the above, for all \(s \geq 1\) we obtain
\[
q(s) < q(s) e^{\alpha(d+2)q(s)} \leq \frac{A^d e^{(d+\alpha)(d+2)} A e^{\mu(d+2)}}{s^{\mu(d+2)} e^{\mu(d+2)}} \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}} \leq \frac{A^d e^{(d+\alpha)(d+2)} A e^{\mu(d+2)}}{s^{\mu(d+2)} e^{\mu(d+2)}} \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}}.
\]
Hence,
\[
\int_1^s \xi^{d-1} q(\xi) d\xi \leq \frac{A^d e^{(d+\alpha)(d+2)} A e^{\mu(d+2)}}{e^{\alpha(d+2)}} \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}} \int_1^s \frac{1}{\xi^{\mu(d+2)-d+1}} d\xi. \tag{103}
\]
Since \(\mu > \frac{d}{d+2}\), then \(\mu(d+2) - d + 1 > 1\), and therefore
\[
\int_1^s \frac{1}{\xi^{\mu(d+2)-d+1}} d\xi < \infty.
\]
Also,
\[
\lim_{A \to 0^+} \frac{A^d e^{(d+\alpha)(d+2)} A e^{\mu(d+2)}}{e^{\alpha(d+2)}} \sqrt{\frac{(d+2)A}{1+\alpha(d+2)A}} = 0,
\]
and then by (103) we have
\[
\lim_{A \to 0^+} \int_1^s \xi^{d-1} q(\xi) d\xi = 0. \tag{104}
\]
Combining (102), (104), we get
\[
\lim_{A \to 0^+} \int_0^\infty s^{d-1} q(s) ds = 0
\]
and this establishes (99).
Furthermore, 
\[ 0 \leq \int_0^\infty s^{d-1}q'(s)^2 ds \leq \int_0^\infty s^{d-1}q'(s)^2 ds + \frac{\alpha d}{2} \int_0^\infty s^{d-1}q(s)^2 ds, \]
hence, from (77) we have 
\[ 0 \leq \int_0^\infty s^{d-1}q'(s)^2 ds \leq \left( \mu - \frac{d}{d+2} \right) \int_0^\infty s^{d-1}q(s) ds. \]
From (99) and the above inequality, we obtain 
\[ \lim_{A \to 0^+} \int_0^\infty s^{d-1}q'(s)^2 ds = 0, \]
which is (100).
Finally, 
\[ 0 \leq \frac{\alpha d}{2} \int_0^\infty s^{d-1}q(s)^2 ds \leq \int_0^\infty s^{d-1}q'(s)^2 ds + \frac{\alpha d}{2} \int_0^\infty s^{d-1}q(s)^2 ds, \]
hence, from (77) we get 
\[ 0 \leq \frac{\alpha d}{2} \int_0^\infty s^{d-1}q(s)^2 ds \leq \left( \mu - \frac{d}{d+2} \right) \int_0^\infty s^{d-1}q(s) ds. \]
From (99) and the above inequality, we obtain 
\[ \lim_{A \to 0^+} \int_0^\infty s^{d-1}q(s)^2 ds = 0 \]
and this establishes (101). □

Corollary 3. Let \( q(s) \) satisfy (35) and (36), in particular \( q(0) = A \). Then, as a function of \( A \), the quantity 
\[ I(A) = \int_0^\infty s^{d-1}q(s) ds \] (105)
is continuous in \((0, +\infty)\).

Proof. Let \( q(s) = q(s; A) \) be the unique solution of the problem (35), (36). The function \( q(s; A) \) is continuous in \( A \), for \( A > 0 \). Then the function \( \bar{q}(s; A) = s^{d-1}q(s; A) \) is also continuous in \( A \), for \( A > 0 \). For fixed \( A_1, A_2 \) with \( 0 < A_1 < A_2 < +\infty \), from Lemma 4.2, the monotonicity of \( q \) and the condition \( \mu > \frac{d}{d+2} \) imply that the family 
\( \{ \bar{q}(\cdot; A) : A \in [A_1, A_2] \} \) is dominated by the integrable function \( h(s), \tau \geq 0 \), where 
\[ h(s) = \begin{cases} \frac{A_2^\gamma A_1^{d+2}}{(d+2)^{d+2}} & 0 \leq s \leq 1 \\ A_2^\gamma A_1^{d+2} & s > 1 \end{cases} \]
\[ \exp\left( (d+2)\mu \sqrt{\frac{(d+2)A_2}{1+\alpha(d+2)A_1}} \right), \quad s > 1. \]
Hence, the continuity of \( I(A) \) follows by invoking the Dominated Convergence Theorem. □

Theorem 4.3. Let \( d \geq 2 \) and \( \gamma = -\frac{2}{d+2} \). Then, for each number \( \beta \in (0, +\infty) \) there is a self-similar solution of (8) (which is equivalent to (1)), namely a solution \( u \) of the form \( u(t, x) = t^{-\frac{\gamma}{d+d}} g_d(rx - \frac{\gamma}{d+d}) \), where \( r = |x| = \sqrt{x_1^2 + \cdots + x_d^2} \) and \( g_d(s) \) satisfies (30), (31), (32), (33) and (34), such that 
\[ \beta = K_d[g_d] + \frac{\alpha d}{2} \Lambda_d[g_d], \]
where \( K_d[g_d] \) and \( \Lambda_d[g_d] \) are given by (20) and (21), respectively.
Proof. Let $q(s) = q(s; A)$ be the unique solution of the problem (35) - (36) with
\[ \mu = \beta + \frac{d}{d+2}; \]
that is
\[ q''(s) + \left( \frac{d-1}{s} + \alpha s \right) q'(s) + \frac{s}{d+2} q'(s) + \mu q(s) = 0, \quad s > 0, \]
\[ q(0) = A > 0, \quad q'(0) = 0, \]
and set
\[ Q(A) := \sigma_d \int_0^\infty s^{d-1} q'(s; A)^2 ds + \frac{\alpha d}{2} \sigma_d \int_0^\infty s^{d-1} q(s; A)^2 ds. \]
Then, by (77), we have
\[ Q(A) = \beta \sigma_d \int_0^\infty s^{d-1} q(s; A) ds = \beta \sigma_d I(A), \]
(recall (105)). Hence, Corollary 3, tells us that $Q(A)$ is continuous on $(0, +\infty)$.

Also, by (98) and (99) of Corollary 2, we have
\[ \lim_{A \to 0^+} Q(A) = 0 \quad \text{and} \quad \lim_{A \to +\infty} Q(A) = \infty. \]
Thus, $Q(A)$ takes every value between 0 and $+\infty$. In particular, for each number
\[ \beta \in (0, +\infty) \]
there is an $A = A_\beta$ such that
\[ Q(A_\beta) = \beta. \]

Set $g_d(s) := q(s; A_\beta)$. Then,
\[ K_d[g_d] + \frac{\alpha d}{2} A_d[g_d] = \sigma_d \int_0^\infty s^{d-1} g_d'(s)^2 ds + \frac{\alpha d}{2} \sigma_d \int_0^\infty s^{d-1} g(s)^2 ds = Q(A_\beta) = \beta, \]
hence $g_d(s)$ satisfies (32), (33), (34).

Furthermore, by (77) of Corollary 1
\[ \sigma_d \int_0^\infty s^{d-1} g_d(s) ds = \sigma_d \int_0^\infty q(s; A_\beta) ds \]
\[ = \frac{1}{\beta} \left[ \sigma_d \int_0^\infty s^{d-1} q'(s)^2 ds + \frac{\alpha d}{2} \sigma_d \int_0^\infty s^{d-1} q(s)^2 ds \right] \]
\[ = \frac{1}{\beta} Q(A_\beta) = 1, \]
and, hence, $g_d$ also satisfies (31).

Appendix.

Proposition 1. There is an $\epsilon > 0$ such that (35) and (36) has a unique solution
$q(s)$ for $s \in [0, \epsilon]$.

Proof. By multiplying both sides of (38) by $s^{d-1}$, then integrating from 0 to $s > 0$
and using the initial condition $q'(0) = 0$, we get the equivalent problem
\[ s^{d-1} q'(s) + \alpha s^d q(s) - \alpha d \int_0^s \xi^{d-1} q(\xi) d\xi + \frac{1}{d+2} \int_0^s \xi^d q'(\xi) q(\xi) d\xi + \frac{\mu}{d} s^d = 0, \]
\[ q(0) = A. \]
We set \( w = q' \) and we obtain the system

\[
q(s) = \int_0^s w(\xi) d\xi + A, \tag{106}
\]

\[
w(s) = \frac{\alpha d}{s^{d-1}} \int_0^s \xi^{d-1} q(\xi) d\xi - \frac{1}{(d + 2) s^{d-1}} \int_0^s \xi^{d-1} \frac{w(\xi)}{q(\xi)} d\xi - \alpha sq(s) - \frac{\mu}{d}s, \tag{107}
\]

which is equivalent to (35) and (36).

We will proof that for some \( \epsilon > 0 \), the above system possesses a unique solution \((w(s), q(s))\), for \( s \in [0, \epsilon] \).

Fix an \( \epsilon > 0 \) and consider the linear space \( C[0, \epsilon] \) of all real-valued continuous functions defined on the interval \( [0, \epsilon] \) which is a Banach space under the supnorm \( \| \cdot \|_\infty \). Furthermore, \( C[0, \epsilon] \times C[0, \epsilon] \) is also a Banach space with the norm

\[
\| (u, v) \| = \| u \|_\infty + \| v \|_\infty. \tag{108}
\]

We set

\[
Y_\epsilon = \left\{ (q, w) \in C[0, \epsilon] \times C[0, \epsilon] : \| q - A \|_\infty \leq \frac{A}{2}, \| w \|_\infty \leq 1 \right\}.
\]

Then, \( Y_\epsilon \) is a closed subset of \( C[0, \epsilon] \times C[0, \epsilon] \) and thus, it is a complete metric space with respect to the metric induced by the norm (108).

Now, we define the mapping \( \Phi_\epsilon : Y_\epsilon \to C[0, \epsilon] \times C[0, \epsilon] \) by

\[
\Phi_\epsilon[q, w](s) := \left( \int_0^s w(\xi) d\xi + A, \frac{\alpha d}{s^{d-1}} \int_0^s \xi^{d-1} q(\xi) d\xi - \frac{1}{(d + 2) s^{d-1}} \int_0^s \xi^{d-1} \frac{w(\xi)}{q(\xi)} d\xi - \alpha sq(s) - \frac{\mu}{d}s \right).
\]

It is straightforward to proof that by taking \( \epsilon \) sufficiently small, the mapping \( \Phi_\epsilon \) can make a contraction from \( Y_\epsilon \) into itself. Therefore, \( \Phi_\epsilon \) admits a unique fixed point which is the solution to the system (106) and (107) on the interval \( [0, \epsilon] \). \( \square \)

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Received November 2015; revised January 2016.

E-mail address: papanico@math.ntua.gr
E-mail address: kiriakivasilakopoulou@gmail.com