MEASURING THE “NON-STOPPING TIMENESS” OF ENDS OF PREVISIBLE SETS

Ching-Tang Wu*, Ju-Yi Yen and Marc Yor

Abstract. In this paper, we propose some “measurements” of the “non-stopping timeness” of ends $G$ of previsible sets, such that $G$ avoids stopping times, in an ambient filtration. We then study several explicit examples, involving last passage times of some remarkable martingales.

1. INTRODUCTION: ABOUT ENDS OF PREVISIBLE SETS

In this paper, we are interested in random times $G$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ as ends of $(\mathcal{F}_t)$-previsible sets $\Gamma$, that is,

\begin{equation}
G \equiv G^\Gamma = \sup \{ t : (t, \omega) \in \Gamma \}.
\end{equation}

For simplicity, we shall make the following assumptions:

(C) All $(\mathcal{F}_t, P)$-martingales are continuous;

(A) For any $(\mathcal{F}_t)$-stopping time $T$, $P(G = T) = 0$.

To such a random time, one associates the Azéma supermartingale

$Z^G_t = P(G > t | \mathcal{F}_t),$

which, under (C) and (A), admits a continuous version as shown by the following theorem.

**Theorem 1.1.** Under (C) and (A), there exists a unique positive local martingale $(N_t, t \geq 0)$, with $N_0 = 1$, such that

$Z^G_t = P(G > t | \mathcal{F}_t) = \frac{N_t}{S_t},$

where $S_t := \sup_{s \leq t} N_s$ for $t \geq 0$. 

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Proof. See [8]: page 16, Proposition 1.3.

Note that since $G < \infty$ a.s., $N_t \rightarrow 0$ a.s. We note further that $\log(S_\infty)$ is distributed exponentially, since by Doob’s maximal identity

$$
\log(S_\infty) \overset{(\text{law})}{=} \log\left(\frac{1}{U}\right),
$$

where $U$ is uniform on $[0, 1]$. Then, the additive decomposition of the supermartingale $N_t/S_t$ is given by

$$
\frac{N_t}{S_t} = 1 + \int_0^t dN_u/S_u - \log(S_t) = E[\log(S_\infty)|\mathcal{F}_t] - \log(S_t).
$$

Note that the martingale $E[\log(S_\infty)|\mathcal{F}_t]$ belongs to $BMO$ since from (2),

$$
E[\log(S_\infty) - \log(S_t)|\mathcal{F}_t] \leq 1.
$$

In a number of questions, it is very interesting to consider the smallest filtration $(\mathcal{F}_t')_{t \geq 0}$, which contains $(\mathcal{F}_t)$, and makes $G$ a stopping time; this filtration is usually denoted $(\mathcal{F}_t^G)_{t \geq 0}$. One of the interests of $(Z_t^G)$ is that it allows to write any $(\mathcal{F}_t)$-martingale as a semimartingale in $(\mathcal{F}_t^G)_{t \geq 0}$; see e.g. [2, 3, 8, 9], for both general formulae and many examples.

Recently, it has been understood that Black-Scholes like formulae are closely related with certain such $G$’s, thus throwing a new light on a cornerstone of mathematical finance, see, e.g. [6, 7]. In the present paper, with (A) as our essential hypothesis, we would like to measure “how much $G$ differs from an $(\mathcal{F}_t)$ stopping time”. The remainder of this paper consists in two sections. In Section 2, we propose several criterions to measure the NST ($\equiv$ Non Stopping Timeness) of $G$’s which satisfy (C) and (A). In Section 3, we compute explicitly this function $m_G$ for various examples, where $G$ is the last passage time at a level of a martingale which converges to 0, as $t \rightarrow \infty$.

2. SEVERAL POSSIBLE "NST" CRITERIONS

Consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, an $(\mathcal{F}_t)$-previsible set $\Gamma$ and a random time $G$ given by (1). Our aim is to discuss the difference between $G$ and an $(\mathcal{F}_t)$-stopping time. A natural question is to consider the function

$$
m_G(t) = E\left[\left(1_{G \geq t} - P(G > t|\mathcal{F}_t)^2\right)\right].
$$

If $G$ is an $(\mathcal{F}_t)$-stopping time, the Azéma supermartingale $Z_t^G = P(G \geq t|\mathcal{F}_t)$ is identically equal to $1_{G \geq t}$. Thus, $m_G(t) = 0$ for all $t$. If $G$ is not an $(\mathcal{F}_t)$-stopping time, a simple but useful remark is

$$
m_G(t) = E\left[Z_t^G (1 - Z_t^G)\right].
$$
Instead of considering the “full” function \((m_G(t), t \geq 0)\), we may consider only:

\[
m^*_G = \sup_{t \geq 0} m_G(t)
\]

as a “global” measurement of the NST of \(G\).

Here are two other, a priori natural, measurements of the NST of \(G\):

\[
m^{**}_G = \sup_{t \geq 0} \left( Z^G_t (1 - Z^G_t) \right)
\]

and

\[
\tilde{m}_G = \sup_{T \geq 0} E \left[ Z^G_T (1 - Z^G_T) \right]
\]

where \(T\) runs over all \((\mathcal{F}_t)\) stopping times.

However, we cannot expect to learn very much from \(m^{**}_G\) and \(\tilde{m}_G\), since it is easily shown the following result.

**Lemma 2.2.**

\[
m^{**}_G = \tilde{m}_G = \frac{1}{4}.
\]

**Proof.**

(i) The fact that \(m^{**}_G = 1/4\) follows immediately from

\[
\sup_{x \in [0,1]} x(1-x) = \frac{1}{4},
\]

and the fact that, a.s., the range of the process \((Z^G_t, t \geq 0)\) is \([0,1]\) since \(Z^G_0 = 1\),
\(Z^G_\infty = 0\), and \((Z^G_t, t \geq 0)\) is continuous.

(ii) Let us consider \(T_a = \inf\{t : Z^G_t = a\}\), for \(0 < a < 1\). Then

\[
Z^G_t (1 - Z^G_t) \big|_{t = T_a} = a(1-a).
\]

Hence,

\[
\sup_{a \in [0,1]} E \left[ Z^G_{T_a} (1 - Z^G_{T_a}) \right] = \sup_{a \in [0,1]} (a(1-a)) = \frac{1}{4}.
\]

An immediate result is that \(1/4\) is an upper bound of \(m_G\) due to the definition. Moreover, there are some other measurements which have been investigated in a number of literatures.

**Remark 2.3.**

(1) (The optional stopping time discrepancy \(\mu_G\)) It has been shown in [4], of stopping times, among random times, as the times \(\tau\) such that for every bounded martingale \((M_t)_{t \geq 0}\) one has
where, under our hypothesis (C), we may define $F_\tau = \sigma\{H_\tau; H$ previsible$.\}$

Thus, another measurement of the NST of $G$ is

$$\mu_G = \sup_{M_\infty \in L^2(F_\infty)} E^F [\left(M_\infty - E^F[M_\infty|F_\tau]\right)^2].$$

(2) (Distance from stopping times) We introduce

$$\nu_G = \inf_{T \geq 0} E[|G - T|],$$

where $T$ runs over all $(F_t)$ stopping times. However, this quantity may be infinite as $G$ may have infinite expectation. We note that this distance was precisely computed by du Toit-Peskir-Shiryaev in the example of [1]. A more adequate distance may be:

$$\nu_G' = \inf_{T \geq 0} \left( E \left[ \frac{|G - T|}{1 + |G - T|} \right] \right)$$

In this paper we concentrate uniquely on the study of $(m_G(t), t \geq 0)$ using the technique of Azéma supermartingale and enlargement of filtration.

3. A STUDY OF SEVERAL INTERESTING EXAMPLES OF FUNCTIONS $m_G(t)$

3.1. Some general formulae

We shall compute $(m_G(t), t \geq 0)$ in some particular cases where

$$G = G_K = \sup\{t \geq 0: M_t = K\}, \quad K \leq 1,$$

with $M_0 = 1, M_t \geq 0$, a continuous local martingale such that $M_t \to 0$. We recall that (see, e.g. [2, 8]):

$$Z_t = P(G_K \geq t|F_t) = 1 \wedge \left( \frac{M_t}{K} \right).$$

Thus

$$(8) \quad m_K(t) = E[Z_t(1 - Z_t)] = \frac{1}{K^2} E \left[ M_t(K - M_t)^+ \right].$$

Consider the particular case $M_t = \epsilon_t = \exp(B_t - t/2)$, with $(B_t)$ a standard Brownian motion, and $G_K = \sup\{t : \epsilon_t = K\}$ for $K \leq 1$.

From formula (8), we deduce:
\[ m_K(t) = \frac{1}{K^2} E \left[ \xi_t (K - \xi_t)^+ \right] \]
\[ = \frac{1}{K^2} E \left[ \left( K - \exp \left( B_t + \frac{t}{2} \right) \right)^+ \right] \text{ (by Cameron-Martin)} \]
\[ = \frac{1}{K^2} \left\{ K P \left( \exp \left( B_t + \frac{t}{2} \right) < K \right) - E \left[ 1_{(\exp(B_t + \frac{t}{2}) < K)} \exp \left( B_t + \frac{t}{2} \right) \right] \right\}. \]

Set \( K = e^l \), we have
\[ m_K(t) = e^{-l} P \left( B_t + \frac{t}{2} < l \right) - e^l e^{-2l} P \left( B_t + \frac{3t}{2} < l \right) \]
\[ = \left( e^{-l} - e^{l-2l} \right) P \left( B_1 < -\frac{3\sqrt{t}}{2} + \frac{l}{\sqrt{t}} \right) \]
\[ + e^{-l} P \left( -\frac{3\sqrt{t}}{2} + \frac{l}{\sqrt{t}} < B_1 < -\frac{\sqrt{t}}{2} + \frac{l}{\sqrt{t}} \right). \]

In particular,
\[ m_1(t) = (1 - e^l) P \left( B_1 < -\frac{3\sqrt{t}}{2} \right) + P \left( -\frac{3\sqrt{t}}{2} < B_1 < -\frac{\sqrt{t}}{2} \right). \]

Figure 1 presents the graphs of \( m_K(t) \) for some \( K \)'s.
3.2. The case \( G = G_{\gamma T} = \sup\{u \leq T : B_u = a\} \)

For fixed time \( T \) and \( a \in \mathbb{R} \), the associated Azéma supermartingale is of the form

\[
Z_t = \Phi \left( \frac{|B_t - a|}{\sqrt{T - t}} \right) 1_{\{t < T\}}
\]

(see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]), where \( \Phi(x) = \sqrt{\frac{2}{\pi}} \int_{x}^{\infty} e^{-u^2/2} \, du \). Then for \( t < T \), using change of variables we have

\[
m_{G}^{a,T}(t) = E \left[ \Phi \left( \frac{\sqrt{T} B_1 - a}{\sqrt{T-t}} \right) \left( 1 - \Phi \left( \frac{\sqrt{T} B_1 - a}{\sqrt{T-t}} \right) \right) \right] = m^{a/\sqrt{T}} \left( \frac{T - t}{t} \right),
\]

where

\[
m^D(c) = \frac{c}{\sqrt{2\pi}} \int_{0}^{\infty} \Phi(y)(1 - \Phi(y)) \left( \exp \left( -\frac{(cy + D \sqrt{c^2 + 1})^2}{2} \right) \right) \, dy.
\]

Hence

\[
m_{G}^{a,T} := \sup_{0 \leq t \leq T} m_{G}^{a,T}(t) = \sup_{c \geq 0} m^{a/\sqrt{T}}(c).
\]

Remark 3.4.

(1) For \( a \in \mathbb{R} \), \( m_{G}^{a,T} = m_{G}^{-a,T} \), since \( m^D(c) = m^{-D}(c) \).

(2) \( m_{G}^{0,T} \) is independent of \( T \), since

\[
m_{G}^{0,T} = \sup_{c \geq 0} \frac{2c}{\sqrt{2\pi}} \int_{0}^{\infty} \Phi(y)(1 - \Phi(y)) \exp \left( -\frac{c^2y^2}{2} \right) \, dy
\]

is independent of \( T \).

(3) the value of \( m_{G}^{a,T} \) depends only on \( D := a/\sqrt{T} \), e.g., \((a, T) = (1, 1) \) and \((a, T) = (1/2, 1/4) \) have the same \( m_{G}^{a,T} \) value, since \( D = 1 \) in both cases.

Remark 3.5. Table 1 gives the values of \( m_{G}^{a,T} \) for some \( D \).

Table 1. The values of \( m_{G}^{a,T} \) for some \( D \)

| \( D \) | 0     | 0.1   | 0.2   | 0.3   | 0.4   |
|-------|-------|-------|-------|-------|-------|
| \( m_{G}^{0,T} \)| 0.17548 | 0.175531 | 0.173103 | 0.220612 | 0.244867 |
| \( m_{G}^{0.5,T} \)| 0.5    | 0.6    | 0.7    | 1     | 1.1   |
| \( m_{G}^{0.25,T} \)| 0.249704 | 0.24059 | 0.218382 | 0.132556 | 0.105833 |
| \( m_{G}^{1.2,T} \)| 1.2    | 1.5    | 2      | 3      | 5     |
| \( m_{G}^{0.08,T} \)| 0.0840563 | 0.0416004 | 0.0122678 | 0.000653202 | 1.30174 \times 10^{-7} |
In fact, if $D$ satisfies $\Phi(D) = \frac{1}{2}$ (i.e., $D$ around 0.47693627), then $m_G^{a,T} = \frac{1}{4}$ and the maximum occurs at $t = 0$. The same as $m_G^{\ast\ast}$ and $\tilde{m}_G$.

Figures 2–4 present the graphs $m^D(c)$ for some $D$. The horizontal axis is the value of $c = \sqrt{\frac{T - t}{t}}$ and the vertical axis is the value of $m^D(c)$, and its maximum is exactly $m_G^{a,T}$.

![Figure 2](image1)

**Fig. 2.** $D = 0$: \(\cdot\); $D = 0.1$: \(-\); $D = 0.2$: \(\cdot\).

![Figure 3](image2)

**Fig. 3.** $D = 0.2$: \(-\); $D = 0.3$: \(-\); $D = 0.4$: \(\cdot\).

### 3.3. The case $G = G_{T_a} = \sup\{t < T_a : B_t = 0\}$

Here, we denote $T_a = \inf\{u : B_u = a\}$, for $a > 0$; and $S_t = \sup_{0 \leq u \leq t} B_u$. The corresponding Azéma supermartingale is given by

$$Z_t = 1 - \frac{1}{a} B_{t \wedge T_a}^+,$$

see, e.g., Table (1α) of Progressive Enlargements, p. 32 of [8]. Thus, we obtain:
\[
m_G(t) = E \left[ \left( \frac{1}{a} B_t^{+1} \right) \left( 1 - \frac{1}{a} B_t^{+1} \right) \right] \\
= \frac{1}{a^2} E \left[ 1_{(t<T_a)} 1_{(B_t>0)} B_t(a-B_t) \right] \\
= \frac{1}{a^2} E \left[ 1_{(S_t<a)} 1_{(B_t>0)} B_t(a-B_t) \right].
\]

Let \( \varphi(x) = E \left[ 1_{(S_1<x)} 1_{(B_1>0)} B_1(x-B_1) \right] \); then

\[
m_G(t) = t \frac{1}{a^2} \varphi \left( \frac{a}{\sqrt{t}} \right).
\]

Now, it remains to compute the function \( \varphi \). We note that

\[
\varphi(x) = E \left[ B_1^+(x-B_1)^+ \right] - E \left[ 1_{(S_1>x)} B_1^+(x-B_1)^+ \right].
\]

We shall take advantage of the very useful formula:

\[
P(S_1 > x|B_1 = a) = \exp(-2x(x-a)), \quad x \geq a > 0,
\]

see, e.g., [5], p.425. Thus, we find

\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x dy \, y(x-y) \left( \exp \left( -\frac{y^2}{2} \right) - \exp \left( -\frac{1}{2} (2x-y)^2 \right) \right)
\]

Thus,

\[
\frac{\varphi(x)}{x^2} = \frac{x}{\sqrt{2\pi}} \int_0^1 du \, u(1-u) \left( \exp \left( -\frac{x^2u^2}{2} \right) - \exp \left( -\frac{x^2}{2} (2-u)^2 \right) \right).
\]
Note that the value of $\sup_{t>0} m_G(t) = \sup_{x>0} \frac{\varphi(x)}{x^2}$ is independent of the value of $a$, since $m_G(t)$ depends only on the value of $x := a/\sqrt{t}$.

3.4. The case $G = \mathcal{L}_a = \sup\{u : R_u = a\}$

We have

$$Z_t = 1 \wedge \left(\frac{a}{R_t}\right)^{2\mu},$$

see, e.g., Table (1α) of Progressive Enlargements, p.32 of [8]. Here, $(R_u)$ is a Bessel process of index $\mu$ starting at 0, i.e., $R$ is a $d$-dimensional Bessel process with $d = 2(\mu + 1)$. Thus,

$$m_G(t) = E \left[ \left(1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}\right) \left(1 - 1 \wedge \left(\frac{a}{R_t}\right)^{2\mu}\right) \right]$$

$$= E \left[ 1_{\left(\frac{a}{\sqrt{t}R_1} < 1\right)} \left(\frac{a}{\sqrt{t}R_1}\right)^{2\mu} \left(1 - \left(\frac{a}{\sqrt{t}R_1}\right)^{2\mu}\right) \right].$$

Using the fact that $R_1^2 \overset{\text{(law)}}{=} 2\gamma_{d/2}$, where $\gamma_{d/2}$ has a gamma law with parameter $(d/2, 1)$, we get

$$m_G(t) = \varphi_{\mu} \left(\frac{a^2}{2t}\right),$$

where

$$\varphi_{\mu}(z) = \frac{1}{\Gamma(\mu + 1)} \left\{ z^\mu e^{-z} - z^{2\mu} \int_z^\infty \frac{du}{u^\mu e^{-u}} \right\}.$$
Figure 5 presents the graphs of $\varphi_\mu$ for $\mu = 1/2, 1, 3/2, 5/2, 7/2, 9/2, 11/2$ and $13/2$. We also approximate $z_\mu$, the unique positive real number which achieves the max of $\varphi_\mu$. This will give us the value $m_\mu \overset{\text{def}}{=} m^*_\mu$, for these $G = L_\alpha$ (note that, for a given $\mu$, the value does not depend on $\alpha$; this is because of the scaling property).

It is not difficult to show that: $z_\mu$ is the unique solution of

$$(E_\mu): \quad \frac{1}{2z} = \int_0^\infty \frac{dh}{(1 + h)^\mu} e^{-hz}$$

and also

$$m_\mu = \frac{1}{\Gamma(\mu + 1)} e^{-z_\mu} \frac{(z_\mu)^\mu}{2}$$

Note that

$$m_\mu \leq m'_\mu \overset{\text{def}}{=} \frac{1}{\Gamma(\mu + 1)} \sup_{z \geq 0} \left( e^{-z} \frac{z^\mu}{2} \right).$$

Figure 6 presents the graphs of $m_\mu$ and $m'_\mu$.

![Graphs of $m_\mu$ and $m'_\mu$.](image)

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