Turbulence without inertia in quantum fluids

Demosthenes Kivotides
Low Temperature Laboratory,
Helsinki University of Technology,
P.O. Box 2200, FIN-02015 HUT, Finland

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Numerical calculations of $^4\text{He} - \text{II}$ hydrodynamics show that a dense tangle of superfluid vortices induces in an initially stationary normal fluid a highly dissipative, complex, vortical flow pattern ("turbulence") with $k^{-2.2}$ energy spectrum scaling and fluctuations Reynolds number of order unity. In this normal fluid flow the effects of mutual friction excitation from the superfluid vortices and those of viscous stresses are of the same order. The results suggest that in previous experiments the dynamics of decaying, high Reynolds number, quantum turbulent flows could only weakly be affected by the quantized vortices. As a consequence, their energy spectra would be (to a very good approximation) the classical, Navier-Stokes type, energy spectra of the normal fluid component.

In quantum fluid turbulence [1], a tangle of quantized vortices interacts via mutual friction forces with thermal excitations (known as normal fluid) of the superfluid ground state. Until recently, theoretical investigations involved only the dynamics of the superfluid vortices [2]. The normal fluid flow was assumed to possess infinite inertia and it was prescribed in a kinematic way. Numerical and computational methods for fully dynamical quantized turbulence calculations were developed for the first time in [3]. Subsequently, the strategy of gradually increasing complexity was adopted in performing a series of three dimensional calculations. First, it was shown in [4] that a superfluid vortex ring induces in an initially stationary normal fluid two coaxial vortices which together with the quantized ring propagate as a triple ring structure. A second calculation [5] allowed a small number of reconnections in order to show that Kelvin waves excited by the latter induced a dramatic increase of kinetic energy dissipation rate in the normal fluid. The present Letter introduces the element of vortex line density characteristic of experiments while keeping the condition of initial stationarity for the normal fluid.

The numerical method is described in detail in the mentioned references. If $S(\xi, t)$ is the three dimensional representation of the vortex tangle (where $\xi$ is the arclength parametrization along the loops), then its motion obeys the equation [2]:

$$\frac{dS}{dt} = V_t = h V_s + h_x S' \times (V_n - V_s) - h_x S' \times (S' \times V_n)$$

(1)

where the superfluid velocity $V_s$ is given by the Biot-Savart integral:

$$V_s(x) \equiv V_t(x) = \frac{\kappa}{4\pi} \int \frac{(S - x) \times dS}{|S - x|^3}$$

(2)

$t$ is time, $x$ is space, $\kappa$ is the quantum of circulation, $V_n$ is the velocity of the normal fluid, $S' = \frac{dS}{d\xi}$ is the unit tangent vector while $h$, $h_x$ and $h_{xx}$ are constants related to mutual friction physics. In writing equation (1) one ignores the inertia of the vortices and the spectral-flow (Kopnin) force that is relevant only for fermionic quantum fluids. The equation includes the Magnus, drag and Iordanskii forces [6]. We call the sum of the Iordanskii and drag forces mutual friction. In the definition of $V_s$ any irrotational ground state motion has been neglected. The motion of the normal fluid is governed by the forced incompressible Navier-Stokes equation:

$$\frac{\partial V_n}{\partial t} + (V_n \cdot \nabla)V_n = -\frac{1}{\rho} \nabla p + \nu \nabla^2 V_n + \frac{1}{\rho_n} F$$

(3)

$$\nabla \cdot V_n = 0$$

(4)

where $\rho_n$ is the density of the normal fluid, $\rho$ is the total density of the fluid, $p$ is the pressure, $\nu = \frac{\mu}{\rho_n}$ is the kinematic viscosity ($\mu$ stands for the dynamic viscosity) and $F$ is the mutual friction force per unit volume. The latter is being calculated from the formula for the sum of the drag and Iordanskii forces per unit vortex length $f$:

$$f = \rho_s kd_{xx} S' \times (S' \times (V_n - V_t)) - \rho_s kd_{xx} S' \times (V_n - V_t)$$

(5)

where $\rho_s$ is the density of the super fluid and the dimensionless term $d_{xx}$ incorporates both the Iordanskii and the corresponding drag force component coefficients. The numerical procedure for obtaining $F$ from $f$ detects first all segments of the vortex tangle inside a numerical grid cell. Then, it numerically integrates the known $f$ over the length of these segments and divides with the cell volume. Grid cells that contain no vortices have zero mutual friction force.

The working fluid is $^4\text{He} - \text{II}$ and so the quantum of circulation has the value $\kappa = 9.97 \cdot 10^{-4} \text{cm}^2/\text{s}$. The calculation is done at $T = 1.3K$ for which the other parameters of the problem have the values: $\nu = 23.30 \cdot 10^{-4} \text{cm}^2/\text{s}$ (with the ratio $\frac{\nu}{\rho} = 0.42$), $\rho_n = 6.5 \cdot 10^{-3} \text{g/cm}^{-3}$,
$\rho_s = 138.6 \cdot 10^{-3} g/cm^{-3}$ (with $\frac{L}{\rho_s} = 21.323$), $h = 0.978$, $h_x = 4.0937 \cdot 10^{-2}$, $h_{xx} = 2.175 \cdot 10^{-2}$, $d_x = -2.045 \cdot 10^{-2}$ and $d_{xx} = 4.270 \cdot 10^{-2}$.

We employ periodic boundary conditions. Both fluids are advanced with the same timestep. Provision has been taken so that the latter resolves adequately the fastest Kelvin waves present in the tangle. This requirement leads to rather constricted time steps, $\Delta t = 0.483 \cdot 10^{-3} s$, which are of the order of the viscous time scale in the normal fluid. The grid size for the Navier-Stokes calculation is $64^3$. In this way the width of a numerical cell is $\Delta x = l_b/64 = 1.56 \cdot 10^{-3} cm$ and this length is used also for discretizing the vortex loops. Here, $l_b = 0.1 cm$ is the size of the system. Initially there are 351 randomly oriented vortex rings with radii between 0.34$l_b$ and 0.45$l_b$ (and therefore with curvatures $c$ between 20$cm^{-1}$ and 30$cm^{-1}$). The initial tangle length is $L = 77.9 cm$ and $N = 99527$ vortex points are used for its discretization.

At the same time, the vortex line density is $\Lambda = L/l_b^3 = 0.779 \cdot 10^5 cm^{-2}$ and the average intervortex spacing is $\delta \approx \Lambda^{-1/2} = 0.0036 cm$ which corresponds to wavenumber $k_0 \approx 277 cm^{-1}$. Our vortex line density is representative of experimental conditions. For example, the value reported in fig.2 of [7] for grid velocity $v_g = 5 cm/s$ and $T = 1.5K$ was $\Lambda \approx 2 \cdot 10^5 cm^{-2}$. Our vortex line density is larger than the value of $\Lambda \approx 0.18 \cdot 10^5 cm^{-2}$ (again for $v_g = 5 cm/s$ and $T = 1.5K$), reported in the same experiment at saturation when the observed classical decay begins.

We calculate the average kinetic energy $E_n$, enstrophy $\Omega$ and helicity $H$ of the normal fluid defined as: $E_n = \frac{1}{\rho_s} \int u \cdot u \, dx$, $\Omega = \frac{1}{\rho_s} \int \omega \cdot \omega \, dx$ and $H = \frac{1}{\rho_s} \int u \cdot \omega \, dx$. In these relations $u$ is the fluctuating part of the normal fluid velocity and $V = l_b^3$ is the system volume. Using $E_n$ we can define a $Re$ number for the normal fluid velocity fluctuations: $Re = ul_b/\nu$ where $u = \sqrt{\frac{2}{\rho_s} E_n}$ is the inten-
sity of the velocity fluctuations. We also calculate the normal fluid velocity spectrum $E_n(k)$ having the property: 
\[ \frac{1}{2\pi} \int u \cdot u \, dx = \int_0^\infty E_n(k) \, dk. \]

The results of Fig.11 show that (due to excitation from mutual friction force) energy is being transferred from the superfluid to the normal fluid with simultaneous decrease in vortex tangle length. The latter does not occur in a monotonous fashion. In particular, around $t = 0.03 \, s$ a local enstrophy maximum (consistently associated with a reduction in the slope of normal kinetic energy growth) is accompanied by rapid vortex length growth. This phenomenon could be explained by noticing that the initial vortex configuration is not the natural state of the system. There must be a transient length increase having to do with the induction and evolution of (reconnection triggered) Kelvin waves along the vortices. Indeed since the time step resolves the evolution of (reconnection triggered) Kelvin waves along the vortices. Moreover, although there are initially 351 loops in the system, at the end of the transient there are approximately 10, a number that remains constant afterwards with $2 - 3$ loops having more than 90% of the tangle length. The computation also shows that (at $t \approx 0.097 \, s$) there is a critical tangle length $L_c = 74.55 \, cm$ and an associated fractal dimension (measured with the box counting algorithm) $D_c = 1.87$ for which the normal fluid energy attains a peak. The fractal scaling was observed over almost two decades from the system size ($l_b = 0.1 \, cm$) to the discretization length along the vortices ($\Delta x = 1.5 \cdot 10^{-3}$). This could mean that as the length increases and the normal fluid volume not in the vicinity of a quantized vortex increases, regions appear in the normal fluid where the viscous action is not counteracted by mutual friction excitation. In these regions energy can only be dissipated into heat. Notice that at this time 86% of the tangle’s length belonged to a single loop. The latter loop was also found to be a fractal with dimension $D_{fr} = 1.84$.

Helicity keeps oscillating around zero. Since helicity is identically zero for two dimensional vortical flow, this diagnostic might be an indication of a tendency of the normal fluid flow to occur (locally) on planes normal to the superfluid vortices and the normal flow vorticity (see [4] for a demonstration of this).

The velocity spectrum at normal kinetic energy peak is found in Fig.2 to scale like $E_n(k) \propto k^{-2.2}$. The end of the energy spectrum coincides with the average intervortex spacing wavenumber (for the same time) $k_\delta \approx 273 \, cm^{-1}$. Comparing with $E_n(k) \propto k^{-5/3}$ of inertial turbulence, we comment that the present flow does not comply with the familiar classical turbulence concept of energy injection at large scales and its subsequent transfer by nonlinear terms towards smaller scales through a local in spectral space cascade. Instead, normal fluid motion is being simultaneously excited and dissipated at all resolved flow scales by mutual friction and viscous forces respectively. The curvature histogram $h(c)$ in Fig.2 (again at energy peak) which indicates that reconnections exit in the initial tangle curvature scales much finer than the resolved velocity scales supports this argument. The acquired energy would tend to remain localized in the respective wavenumber space regime where it first appeared. This is because energy fluxes in wavenumber space are nonlinear effects [4] and the present nonlinearity in the normal flow is weak (since if it was strong we should have seen the formation of an inertial range with high $Re$ number in the normal fluid). It is important not to confuse the present turbulence with the dissipation regime of classical turbulence.

Although viscous effects are strong and the spectrum slope is steep, at the same time the system is forced at all resolvable scales and we do not have an exponential cut-off. In fact, the recorded $-2.2$ scaling exponent is less steep than the $-3$ energy scaling exponent of the direct inertial enstrophy cascade in two dimensional turbulence [10]. Incidentally, the energy flux in the direct enstrophy cascade of the latter case is also zero [10] holding a resemblance to the present case.

From a somewhat different perspective, one notes that in the computation of [2] involving the propagation of a single superfluid ring, the circulation strength $\kappa$ of the induced normal vortices was found to be of order $\kappa$. For $4He - II \kappa$ is of the order of $\nu$. For example, at $T = 2.171$ when $\frac{\kappa}{\rho \nu} = 0.0467$ it is $\frac{\kappa}{\nu} = 5.47$ which is an indicative upper limit for this parameter. For comparison, in classical fluids $\frac{\kappa}{\nu}$ could easily acquire values of the order $10^6$. Because of these, the induced normal flow in [2] was a highly dissipative flow. The present results suggest that the above physical picture is not affected by the much higher vortex line density. It is obvious that vortex tangles much denser than those found in [2] are required in order to achieve induced normal flow Reynolds numbers similar to those found in classical fluid dynamics.

Overall, the present data lead to a number of conclusions about high Reynolds number quantum turbulence. First, one can conclude that in fully developed $4He - II$ turbulence the normal fluid inertia is due to imposed pressure gradients and external stirring (e.g. by towing a grid) rather than to excitation from superfluid vortices.
Definitely, the present calculation does not exclude the possibility that at high $Re$ number the normal flow might introduce a kind of organization in the vortex tangle, for example bundles of aligned quantized vortices functioning like classical ones at large enough scales. Then, the superfluid vortex tangle could in turn stir vigorously the normal fluid (at these large scales). However, even if something like this does happen, it ought to bridge the gap between the normal fluid $Re$ number of order 1 of the present calculation and the normal fluid $Re$ numbers $10^3 < Re < 2 \cdot 10^5$ of the experiment of \[7\] (with analogous to ours vortex line density and temperature) or $Re = 1.4 \cdot 10^5$ in the experiment of \[11\]. By the same token, the assumption of \[11\] that the two fluids have comparable vorticities seems unlikely.

Second, there are cases where the action of the mutual friction force could cause an initially laminar normal flow to become unstable and subsequently turbulent \[12\]. This computation suggests that the ensuing normal fluid turbulence would still be maintained by interaction of the normal fluid Reynolds stresses with normal fluid mean velocity gradients induced by the instability \[13\] and not by (the meager) direct energy input from the superfluid vortices.

Third, (and in agreement with the discussion of \[14\]), in high $Re$ number quantum flows of previous experiments \[1, 7, 11\] the observed energy spectra should have been to a very good approximation the unforced Navier-Stokes spectra of the normal fluid. This conclusion could be reached as follows: (a) in the \[2\] experiment, the normal fluid spectrum is the dominant one. To prove this, we first notice that the superfluid kinetic energy in the latter experiment should have been of the order of magnitude of the present one since the peak vortex line densities are close. In the present calculation however, the superfluid energy is of the order of magnitude of the normal fluid energy since the latter is due entirely to the presence of the quantized vortices. By (safely) extrapolating from the current data, we conclude that even if all initial tangle length was instantly transformed to normal fluid energy, we could not match normal turbulent Reynolds numbers of order $10^4$ typical of experiments. Hence, statement (a) follows. In order for the dominant normal fluid spectrum to be also of the unforced Navier-Stokes type, it must be true that: (b) the magnitude of the mutual friction force is much smaller than the magnitude of the normal fluid inertial terms. This appears to be the case in \[2\] since if the mutual friction force was comparable to the inertial terms one would have observed a vigorous energy transfer from the normal fluid to the superfluid. In stating this, we take into account that according to proposition (a) the normal fluid energy exceeds significantly the superfluid energy. This energy transfer would have resulted in an equally vigorous (by orders of magnitude) growth of the vortex tangle length. Yet, the results of \[2\] show that exactly the opposite happens (the length decreases). Therefore, in the experiment of \[2\] mutual friction effects do not scale with inertial normal fluid effects and the quantum fluid spectrum would be approximately that of the unforced Navier-Stokes dynamics of the normal flow.

The above conclusions could have been modified in case the turbulence in \[2\] was not decaying. A key variable is vortex line density and the latter might not attain in \[2\] high values because the normal fluid fluctuations (responsible for its growth at the initial rapid transient in the experiment) decay fast. In case a stationary normal fluid turbulence could be established via a sort of forcing, it would be possible for the vortex tangle to reach a kind of equilibrium with the normal turbulence characterized by a vortex line density corresponding to superfluid energies comparable to those of the normal fluid. Notice that simply increasing the turbulence Reynolds number in \[2\] might not have the latter effect since in this case the increase in superfluid energy (denser tangle) would come together with high values of normal fluid turbulence and the imbalance between the two could be preserved. The associated complexity of such quantum flows makes unlikely their calculation before the satisfactory resolution of a number of computational issues.

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