On the Second Boundary Value Problem for Monge–Ampère Type Equations and Geometric Optics

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Abstract

In this paper, we prove the existence of classical solutions to second boundary value problems for generated prescribed Jacobian equations, as recently developed by the second author, thereby obtaining extensions of classical solvability of optimal transportation problems to problems arising in near field geometric optics. Our results depend in particular on a priori second derivative estimates recently established by the authors under weak co-dimension one convexity hypotheses on the associated matrix functions with respect to the gradient variables, \((A3w)\). We also avoid domain deformations by using the convexity theory of generating functions to construct unique initial solutions for our homotopy family, thereby enabling application of the degree theory for nonlinear oblique boundary value problems.

1. Introduction

Let \(\Omega\) be a domain in \(n\) dimensional Euclidean space \(\mathbb{R}^n\), and \(Y\) be a mapping from \(\Omega \times \mathbb{R} \times \mathbb{R}^n\) into \(\mathbb{R}^n\). The prescribed Jacobian equation (PJE) has the following form:

\[
\det DY(\cdot, u, Du) = \psi(\cdot, u, Du),
\]

where \(\psi\) is a given scalar function on \(\Omega \times \mathbb{R} \times \mathbb{R}^n\) and \(Du\) is the gradient vector of the function \(u : \Omega \to \mathbb{R}\). We are concerned here with mappings \(Y\) which can be generated by a smooth generating function \(g\) defined on domains \(\Gamma \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\), which embrace applications in geometric optics and optimal transportation \([7,19]\). In the general set up, we assume \(g \in C^4(\Gamma)\), where \(\Gamma\) has the property that the projections

\[
I(x, y) = \{z \in \mathbb{R} | (x, y, z) \in \Gamma\}
\]

are open intervals. Denoting

\[
\mathcal{U} = \{(x, g(x, y, z), g_x(x, y, z)) | (x, y, z) \in \Gamma\},
\]

(1.2)
we then have the following conditions:

**A1:** For each \((x, u, p) \in \mathcal{U}\), there exists a unique point \((x, y, z) \in \Gamma\) satisfying

\[
g(x, y, z) = u, \quad g_x(x, y, z) = p;
\]

**A2:** \(g_z < 0\), \(\det E \neq 0\), in \(\Gamma\), where \(E\) is the \(n \times n\) matrix given by

\[
E = [E_{i, j}] = g_{x,y} - (g_z)^{-1} g_{x,z} \otimes g_y.
\]

The sign of \(g_z\) in A2 can be changed as we wish. Here we fix the sign of \(g_z\) to be negative in accordance with [7,19]. By defining \(Y(x, u, p) = y\) and \(Z(x, u, p) = z\) in A1, the mapping \(Y\) together with the dual function \(Z\) are generated by equations

\[
g(x, Y, Z) = u, \quad g_x(x, Y, Z) = p. \tag{1.3}
\]

Since the Jacobian determinant of the mapping \((y, z) \to (g_x, g)(y, z)\) is \(g_z \det E\), \(\neq 0\) by A2, the functions \(Y\) and \(Z\) are \(C^3\) smooth. By differentiating (1.3) with respect to \(p\), we have \(Y_p = E^{-1}\). Also, by differentiating (1.3) for \(p = Du\), with respect to \(x\), we obtain the generated prescribed Jacobian equation (GPJE),

\[
\mathcal{F}[u] := \det[D^2u - g_{xx}(\cdot, Y(\cdot, u, Du), Z(\cdot, u, Du))] = \det E(\cdot, Y, Z)\psi(\cdot, u, Du), \tag{1.4}
\]

when the one-jet \(J_1[u](\Omega) := \{(x, u, Du)\mid x \in \Omega\} \subset \mathcal{U}\), which can also be calculated from equation (1.1) directly. As usual, we shall denote

\[
A(\cdot, u, p) = g_{xx}(\cdot, Y(\cdot, u, p), Z(\cdot, u, p)), \quad B(\cdot, u, p) = \det E(\cdot, Y(\cdot, u, p), Z(\cdot, u, p))\psi(\cdot, u, p). \tag{1.5}
\]

Then a function \(u \in C^2(\Omega)\) is elliptic (degenerate elliptic) for equation (1.4), whenever \(D^2u - A(\cdot, u, Du) > 0\), \((\geq 0)\), which implies the right hand side \(B(\cdot, u, Du) > 0\), \((\geq 0)\). We refer the reader to [19] for more background material about generated prescribed Jacobian equations.

The second boundary value problem for equation (1.1) is to prescribe the image

\[
Tu(\Omega) := Y(\cdot, u, Du)(\Omega) = \Omega^*, \tag{1.6}
\]

where \(\Omega^* \subset \mathbb{R}^n\) is a target domain. For applications to geometric optics, the function \(\psi\) is separable in the sense that

\[
|\psi|(x, u, p) = \frac{f(x)}{f^* \circ Y(x, u, p)} \tag{1.7}
\]

for positive intensities \(f \in L^1(\Omega)\) and \(f^* \in L^1(\Omega^*)\). Then a necessary condition for the existence of an elliptic solution with the mapping \(Tu\) being a diffeomorphism, to the second boundary value problem (1.4), (1.6), is the conservation of energy

\[
\int_{\Omega} f = \int_{\Omega^*} f^*. \tag{1.8}
\]
We shall assume $f$ and $f^*$ are both smooth and have positive lower bounds and upper bounds. Note that in optimal transportation \cite{16,23}, $f$ and $f^*$ are densities, and the condition (1.8) is called the mass balance condition.

The strict monotonicity property of the generating function $g$ with respect to $z$, enables us to define a dual generating function $g^*$,

$$g(x, y, g^*(x, y, u)) = u,$$

with $(x, y, u) \in \Gamma^* := \{(x, y, g(x, y, z)) | (x, y, z) \in \Gamma\}$, $g_x^* = -g_x/g_z$, $g_y^* = -g_y/g_z$ and $g_{z}^* = 1/g_z$, which leads to a dual condition to A1, namely, $A1^*$: The mapping $Q := -g_y/g_z$ is one-to-one in $x$, for all $(x, y, z) \in \Gamma$.

We assume also certain conditions on the generating function $g$ which are expressed in terms of the matrix $A$. Extending the necessary assumption $A3w$ for regularity in optimal transportation in \cite{15,17,23}, we assume the following regular condition for the matrix function $A$ with respect to $p$, which we formulate together with its strict version \cite{16}:

$A3w$ ($A3$): The matrix function $A$ is regular (strictly regular) in $U$, that is $A$ is co-dimension one convex (strictly co-dimension one convex) with respect to $p$ in the sense that,

$$A_{ij}^{kl} \xi_i \xi_j \eta_k \eta_l := (D_{pk} p_i A_{ij}) \xi_i \xi_j \eta_k \eta_l \geq 0, \ (> 0)$$

in $U$, for all $\xi, \eta \in \mathbb{R}^n$ such that $\xi \cdot \eta = 0$.

We also need a monotonicity condition on the matrix $A$ with respect to $u$, namely $A4w$ or $A4^w$.

$A4w$ ($A4^w$): The matrix $A$ is monotone increasing (decreasing) with respect to $u$ in $U$, that is,

$$D_u A_{ij} \xi_i \xi_j \geq 0, \ (\leq 0)$$

in $U$, for all $\xi \in \mathbb{R}^n$.

As in \cite{7,19}, we need to impose convexity assumptions on the set $U$ when we apply conditions $A3w$ and $A4w$ (or $A4^w$). For globally smooth solutions these conditions will be assured by the necessary convexity assumptions on our domains and we may restrict our consideration accordingly. Thus we need only assume that the product $\tilde{\Omega} \times \tilde{\Omega}^*$ lies in the projection of $\Gamma$ on $\mathbb{R}^n \times \mathbb{R}^n$ with $U$ replaced by

$$U(\Omega, \Omega^*) = \{(x, g(x, y, z), g_x(x, y, z)) | x \in \tilde{\Omega}, y \in \tilde{\Omega}^*, z \in I(x, y)\}$$

in conditions $A3w$, $A3$, $A4w$ and $A4^w$. Note that there is no loss of generality in maintaining $A1$, $A2$ and $A1^*$ as $\Gamma$ can be redefined so its projection on $\mathbb{R}^n \times \mathbb{R}^n$ is close to $\tilde{\Omega} \times \tilde{\Omega}^*$.

We next have the following condition to guarantee the appropriate controls on $J_1[u]$, which is a refinement of condition G5 in \cite{19} (see also \cite{20}); namely, writing $J(x, y) = g(x, y, \cdot) I(x, y)$, we assume:
A5: There exists an infinite open interval $J_0$ and a positive constant $K_0$, such that $J_0 \subset J(x, y)$ and

$$|g(x, y, z)| < K_0,$$

for all $x \in \overline{\Omega}$, $y \in \overline{\Omega}^*$, $g(x, y, z) \in J_0$.

Note that we can assume that $J_0 = (m_0, \infty)$ for some constant $m_0 \geq -\infty$ or $J_0 = (-\infty, M_0)$, for a constant $M_0$. The situation when $J_0$ is finite will be considered at the end of our existence proof.

Finally, to complete our hypotheses, we adopt some domain convexity definitions from [14,18], which extend the corresponding conditions for optimal transportation in [23]. It will be convenient to express these more generally in terms of the mapping $Y$ generated by $g$.

The $C^2$ domain $\Omega$ is $Y$-convex (uniformly $Y$-convex) with respect to $\Omega^* \times J$, where $J$ is an open interval in $J(\Omega, \Omega^*)$, if it is connected and

$$[D_ig_j(x) - D_pk A_{ij}(x, u, p) \gamma_k(x)] \tau_i \tau_j \geq 0, (\delta_0)$$

for all $x \in \partial \Omega$, $u \in J$, $Y(x, u, p) \in \Omega^*$, unit outer normal $\gamma$ and unit tangent vector $\tau$, (for some constant $\delta_0 > 0$).

The domain $\Omega^*$ is $Y^*$-convex (uniformly $Y^*$-convex) with respect to $\Omega \times J$ if the images

$$\mathcal{P}(x, u, \Omega^*) = \{p \in \mathbb{R}^n | (x, u, p) \in \mathcal{U}, Y(x, u, p) \in \Omega^*\}$$

are convex for all $(x, u) \in \Omega \times J$, (uniformly convex for all $x \in \overline{\Omega}$, $u \in \overline{J}$).

The lack of symmetry between our formulations is caused by using the $u$ variable in both cases. We will discuss them further, including their relationship with the $g$-convexity notion introduced in [19], in conjunction with the application of our estimates from [7] in Section 3.

We can now state our main theorem for the second boundary value problem (1.4), (1.6).

**Theorem 1.1.** Let $g \in C^4(\Gamma)$ be a generating function satisfying conditions A1, A2, A1*, A3w, A5 and either A4w, A4w* or A3, with $C^4$ bounded domains $\Omega$, $\Omega^*$ in $\mathbb{R}^n$ which are respectively uniformly $Y$-convex and uniformly $Y^*$-convex with respect to each other and any interval $J \subset J_0$. Suppose also the function $\psi$ satisfies (1.7), (1.8). Then there exists an elliptic solution $u \in C^3(\overline{\Omega})$ of the second boundary value problem (1.4), (1.6), whose range lies in $J_0$. Furthermore, the mapping $Tu$ is a $C^2$ smooth diffeomorphism from $\overline{\Omega}$ to $\overline{\Omega}^*$.

Note that by varying $m_0$ or $M_0$ we obtain the existence of an infinite number of solutions. We also note that elliptic solutions of (1.4) will be $g$-convex in the sense of [19] under appropriate domain convexity conditions and this property is also crucial in our proof; (see Section 2).

We remark that the a priori second order derivative estimates up to the boundary and the existence of classical solutions for the second boundary value problem for (far field) geometric optics problems are raised in [5] in the context of far field
reflector antenna problems. Such problems are solved in the broader context of optimal transportation in [23]. In this paper, we consider more general situations of second boundary value problems for generated prescribed Jacobian equations, which embrace those examples in near field optics problems in [7, 14, 19]. Moreover, we can avoid the $c$-boundedness of domains as in [23] or the $Y$-boundedness as in [14, 17], since both the second derivative estimates in [7] and the continuity method used in Section 3 do not depend on such conditions. Based on the second derivative estimate, Corollary 3.1 in [8], we also have the existence of the second boundary value problem for more general augmented Hessian equations; (Remark 3.4).

This paper is organised as follows. In Section 2, we construct a uniformly $g$-convex function which approximately satisfies the boundary condition (1.6). The construction is realised by extension and mollification of an initial construction of a uniformly elliptic function and uses the convexity theory of generating functions developed in [19, 20]. In Section 3, we start from the uniformly $g$-convex function constructed in Section 2 to prove the existence result, Theorem 1.1, by using a more elaborate version of the degree argument employed in [14], which does not require domain deformation. We also give a more precise version of Theorem 1.1, which permits the interval $J_0$ to be finite; (Remark 3.2). Finally in Section 4, we consider applications to problems in near field geometric optics, including more precise versions of the flat target cases in [14].

2. Construction of Uniformly $g$-Convex Function

In this section, we shall construct a uniformly $g$-convex function approximately satisfying the second boundary condition (1.6), in preparation for the homotopy argument in Section 3.

2.1. Initial Construction

We first recall some convexity notions with respect to the generating function in [19] and then construct an initial uniformly elliptic function $u^0$ whose $Tu^0$ mapping over $\Omega$ is a subset of $\Omega^*$. We recall from [19] that a function $u \in C^0(\Omega)$ is $g$-convex, if for each $x_0 \in \Omega$, there exists $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0) = \cap_{x \in \Omega} I(x, y_0)$ such that $u(x_0) = g(x_0, y_0, z_0)$ and $u(x) \geq g(x, y_0, z_0)$ for all $x \in \Omega$. If $u(x) > g(x, y_0, z_0)$ for all $x \neq x_0$, then we call $u$ strictly $g$-convex. If a $g$-convex function $u$ is differentiable at $x_0$, then $y_0 = Tu(x_0) = Y(x_0, u(x_0), Du(x_0))$, while if $u$ is twice differentiable at $x_0$, then

$$D^2u(x_0) \geq g_{xx}(x_0, y_0, z_0).$$

(2.1)

A function $u \in C^2(\Omega)$ satisfying (2.1) for all $x_0 \in \Omega$ is called locally $g$-convex in $\Omega$. Moreover, the inequality (2.1) implies that a locally $g$-convex function $u$ of (1.4) is automatically degenerate elliptic. We call a $g$-convex function $u \in C^2(\Omega)$ uniformly $g$-convex if the inequality (2.1) is strict, that is $u$ is also elliptic. Correspondingly, the function $g_0 = g(\cdot, y_0, z_0)$ is called a $g$-affine function, which is a $g$-support of $u$ at $x_0$ if $u(x_0) = g_0(x_0)$ and $u(x) \geq g_0(x)$ for all $x \in \Omega$. 
We also recall the corresponding notion of $g$-convexity for a domain $\Omega$ in [19]. A domain $\Omega$ is $g$-convex (uniformly $g$-convex) with respect to $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0)$, if the image $Q_0(\Omega) := -g_y/g_z(\cdot, y_0, z_0)(\Omega)$ is convex (uniformly convex) in $\mathbb{R}^n$. From Lemma 2.4 in [19], under the assumptions that $A_1, A_2$ and $A_1^*$ hold in $\mathcal{U}$, a $C^2$ (connected) domain $\Omega$ being $g$-convex (uniformly $g$-convex) with respect to $y_0, z_0$ is equivalent to

$$[D_i y_j(x) - g_{ij,p_k}(x, y_0, z_0)g_k(x)]\tau_i \tau_j \geq 0, \quad (> 0) \quad (2.2)$$

for all $x \in \partial \Omega$, unit outer normal $\gamma$ and unit tangent vector $\tau$. Then we see that a $C^2$ domain $\Omega$ is $Y$-convex (uniformly $Y$-convex) with respect to $\Omega^* \times J$ if $\Omega$ is $g$-convex (uniformly $g$-convex) respect to $y_0$ and $z_0$ for all $y_0 \in \Omega^*$ and $z_0 = g^*(x, y_0, u_0)$ for all $x \in \Omega$ and $u_0 \in J$. Conversely if $\Omega$ is $Y$-convex (uniformly $Y$-convex) with respect to $\Omega^* \times J$, then $\Omega$ is $g$-convex (uniformly $g$-convex) with respect to $y_0$ and $z_0$ for all $y_0 \in \Omega^*$ and $z_0$ satisfying $g(\cdot, y_0, z_0)(\Omega) \subset J$. We also recall that for generating functions, the notions of $Y^*$-convexity are equivalent to $g^*$-convexity.

Note that the local $g$-convexity of a function $u$ on a $g$-convex domain $\Omega$ can imply its global $g$-convexity. In particular, if we assume that $g$ satisfies conditions $A_1, A_2, A_1^*$, $A_3w$ and $A_4w$, $\Omega$ is $g$-convex with respect to each point in $(Y, Z)(\cdot, u, Du)(\Omega)$, $u \in C^2(\Omega)$ is locally $g$-convex in $\Omega$, (and $\Gamma$ is sufficiently large), then $u$ is $g$-convex in $\Omega$; see Lemma 2.1 in [19]. We remark also that condition $A_4w$ is removed in [20]. Lemma 2.1, provided that $\Omega$ is $g$-convex with respect to each point $y \in Tu(\Omega)$, $z \in g^*(\cdot, y, u)(\Omega)$ and that the largeness of $\Gamma$ is ensured by assuming $u(\Omega) \subset J(\Omega, Tu(\Omega))$ and $(x, u(x), p) \in \mathcal{U}$ for all $x \in \Omega$ and $p$ in the convex hull of $Du(\Omega)$. As a consequence, elliptic solutions $u$ of the second boundary value problem (1.4), (1.6), satisfying

$$[\inf u - K_0d, \sup u + K_0d] \subset J_0, \quad (2.3)$$

where $d = \text{diam}\Omega$, will be strictly $g$-convex under the hypotheses of Theorem 1.1. More generally, if we strengthen the convexity assumption on $\Omega$ in Theorem 1.1 so that $\Omega$ is $g$-convex with respect to all $y \in \Omega^*$ and $z = g^*(x, y, u)$ for all $x \in \Omega$, $u \in J_0$, then elliptic solutions $u$ of (1.4), (1.6), satisfying $u(\Omega) \subset J_0$ will be strictly $g$-convex.

Let $u \in C^0(\Omega)$ be $g$-convex in $\Omega$. The $g$-normal mapping of $u$ at $x_0 \in \Omega$ is the set

$$Tu(x_0) = \{y_0 \in \Gamma_{\mathcal{L}, \Omega}| u(x) \geq g(x, y_0, g^*(x_0, y_0, u(x_0))) \text{ for all } x \in \Omega\}.$$

For $E \subset \Omega$, we denote $Tu(E) = \cup_{x \in E} Tu(x)$. When $u$ is differentiable, $Tu$ agrees with the previous terminology that $Tu = Y(x, u, Du)$. In general, we only have

$$Tu(x_0) \subset Y(x_0, u(x_0), \partial u(x_0)),$$

where $\partial u$ denotes the subdifferential of $u$. However, if the generating function satisfies the conditions $A_1, A_2, A_1^*, A_3w$, (and again $\Gamma$ is sufficiently large), we then have for $g$-convex $u \in C^0(\Omega)$,

$$Tu(x_0) = Y(x_0, u(x_0), \partial u(x_0)),$$
for any \( x_0 \in \Omega \); see Lemmas 2.2 in \([19,20]\) for detailed statements. The reader can refer to \([6,19,20]\) for the more detailed \( g \)-convexity theory related to generating functions.

Next, we show how to construct a uniformly \( g \)-convex function \( u^0 \) by a smooth perturbation of a \( g \)-affine function \( g_0 = g(\cdot, y_0, z_0) \). Such a construction has already been established in \([7]\). One can refer to Lemma 2.1 in \([7]\) for more details. We just sketch the construction of \( u^0 \) for completeness. Set \( \Gamma(\Omega, \Omega^*) = \{(x, y, z) \in \Gamma \mid x \in \Omega, y \in \Omega^*, z \in I(x, y)\} \), suppose \( g_0 = g(\cdot, y_0, z_0) \) is a \( g \)-affine function on \( \Omega \), for \((y_0, z_0) \in \Omega^* \times I(\Omega, y_0)\). Now let \( u^0 = g_\rho \) be the \( g^* \)-transform, introduced in \([19]\), of the function

\[
v_\rho(y) = z_0 - \sqrt{\rho^2 - |y - y_0|^2}
\]

given by

\[
g_\rho(x) = v^*_\rho(x) = \sup_{y \in B_\rho} g(x, y, v_\rho(y)), \tag{2.4}
\]

where \( B_\rho = B_\rho(y_0) \) and \( \rho \) is sufficiently small to ensure that \( \Gamma_0 = \tilde{\Omega} \times \tilde{B}_\rho \times [z_0 - \rho, z_0] \subset \tilde{\Gamma}(\Omega, \Omega^*) \). Then \( u^0 \) is a uniformly \( g \)-convex function in \( \Omega \), with image

\[
\omega^* := Tu^0(\Omega) \subset B_\rho(y_0) \subset \Omega^*, \tag{2.5}
\]

where \( Tu^0 = Y(\cdot, u^0, Du^0) \) is a diffeomorphism between \( \Omega \) and \( \omega^* \). We can also estimate

\[
g_0 - \sup_{\Gamma_0} |g_y| \rho \leq g_\rho \leq g_0, \tag{2.6}
\]

which shows that \( g_\rho \) converges uniformly to \( g_0 \) as \( \rho \) tends to zero.

We remark that under the hypotheses of Theorem 1.1, we can also determine a suitable \( u^0 \) so that \( Tu^0(\Omega) = B_\rho(y_0) \) for \( \rho \) sufficiently small. This is accomplished by using domain foliation in Section 3, similarly to \([14,23]\).

### 2.2. A Fundamental Geometric Characterization

In this subsection, we derive a geometric property of the uniformly \( Y \)-convex domain \( \Omega \), which will be used in Section 2.3 to extend the initial construction \( u^0 \) in Section 2.1 from \( \Omega \) to a neighbourhood \( \Omega^\delta = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < \delta\} \) for some \( \delta > 0 \).

Suppose that the domains \( \Omega, \Omega^* \) and generating function \( g \) satisfy conditions A1, A2, A1*, A3w, A5, and \( \Omega, \Omega^* \) are respectively uniformly \( Y \)-convex, \( Y^* \)-convex with respect to each other and any interval \( J \subset \subset J_0 \). We denote the unit outer normal of \( \partial \Omega \) by \( \gamma \) and let \( u \in C^2(\tilde{\Omega}) \) be \( g \)-convex and \( g_0 = g(\cdot, y_0, z_0) \) be a \( g \)-affine function defined on \( \tilde{\Omega} \) such that

\[
Tu(\tilde{\Omega}) \cup \{y_0\} \subset \Omega^*, \ u(\tilde{\Omega}), g_0(\tilde{\Omega}) \subset J_0. \tag{2.7}
\]

Letting \( h = u - g_0 \) denote the height of \( u \) above \( g_0 \), we also assume for some boundary point \( x_0 \in \partial \Omega \)

\[
h(x_0) = 0, \ Dh(x_0) = -s\gamma_0, \tag{2.8}
\]

where \( \gamma_0 = \gamma(x_0) \) and \( s \) is a positive constant.
Note that in order to ensure the inclusions, \( u, g_0(\bar{\Omega}) \subset J_0 \), we may assume \([u_0 - K_0d, u_0 + K_0d] \subset J_0 \), where \( u_0 = u(x_0) = g_0(x_0) \).

The following key lemma shows that \( h \) is positive away from \( x_0 \):

\[ \text{Lemma 2.1. Under the above hypotheses, the functions } g_0, u \text{ satisfy} \]
\[ g_0(x) < u(x), \quad \text{for all } x \in \bar{\Omega} \setminus \{x_0\}. \]  

(2.9)

Lemma 2.1 is a consequence of the special case when \( u = g_1 = g(\cdot, y_0, z_0) \) is also \( g \)-affine. The property (2.9) in Lemma 2.1 then asserts that the domain \( \Omega \) lies strictly on one side of the level set of the function \( h = g_1 - g_0 \), passing through \( x_0 \), which is tangential to \( \partial \Omega \) at \( x_0 \) by virtue of (2.8). This may be proved by modification of the proof of the corresponding inequality in the optimal transportation case, namely inequality (7.3) in [23], which originated in [22]. The proof presented here follows the approach in [19–21], using a fundamental differential inequality for the \( h \) function. (Note that [21] should be substituted for reference [21] in [23], and \( c \) should be replaced by \(-c\) in inequality (7.3) in [23] and its subsequent proof.)

**Proof of Lemma 2.1.** First we suppose that \( u \) is uniformly \( g \)-convex and \( g_0 = g(\cdot, y_0, z_0) \) satisfies (2.7) but not necessarily (2.8) and let \( x \) be some point in \( \bar{\Omega} \setminus \{x_0\} \) so that by the uniform \( g \)-convexity of \( \Omega \), the open \( g \)-segment, with respect to \( y_0, z_0 \), joining \( x \) to \( x_0 \) lies in \( \Omega \). Setting \( q_0 = Q(x_0, y_0, z_0) \), \( q = Q(x, y_0, z_0) \), \( q_t = (1 - t)q_0 + tq \), for \( 0 \leq t \leq 1 \) we thus have

\[ x_t := X(q_t, y_0, z_0) \in \Omega. \]

Defining the function \( h_0 \) on \([0, 1]\) by \( h_0(t) = h(x_t) \), we then have from [19], the differential inequality

\[ h''_0 > 0, \]  

(2.10)

whenever \( h_0 = h'_0 = 0 \).

Now let us suppose \( u = g_1 = g(\cdot, y_1, z_1) \) is \( g \)-affine with \( h_0(0) = 0, h'_0(0) > 0 \) and let \( u = g_\rho \) denote the uniformly \( g \)-convex approximation (2.3) to \( g_1 \), with \( y_0, z_0 \) replaced by \( y_1, z_1 \) and set \( \rho = \rho_0 = g_\rho, h_{\rho,0} = (h_\rho)_0. \) Following [20], we suppose \( h(x) = h_0(1) \leq 0 \). Then since we must have \( h_0 > 0 \), for small \( t \), it follows that there exists \( \delta > 0 \) so that \( z_0 - \delta \in \Omega \). \( \Omega \) is uniformly \( g \)-convex with respect to \( y_0, z_\delta \) and, when \( z_0 \) is replaced by \( z_\delta \) in \( g_0 \), the function \( h_{\rho,0} \) takes a zero maximum at some \( t^* \in (0, 1) \), for sufficiently small \( \rho \), with \( h_{\rho,0}(0), h_{\rho,0}(1) < 0 \), which contradicts (2.10). Consequently, using the formula

\[ h'_0(0) = D_\eta h(x_0), \quad \text{where the vector } \eta \text{ is given by} \]

\[ \eta_j = -g_z E^{i,j}(x_0, y_0, z_0)[q_i - (q_0)_i], \]  

(2.11)

where \( E^{i,j} = (D_{pj} Y^i) = E^{-1} \), we obtain \( h(x) > 0 \) for \( x \in \bar{\Omega} \setminus \{x_0\} \), provided \( h(x_0) = 0 \) and \( D_\eta h(x_0) > 0 \). Now from (2.8), we have

\[ D_\eta h(x_0) = g_z E^{i,j}(x_0, y_0, z_0)[q_i - (q_0)_i](y_0)_j > 0, \]  

(2.12)

and we conclude (2.9) since the vector \( E^{-1}(x_0, y_0, z_0)\gamma_0 \) is a positive multiple of the outer normal at \( q_0 \) to the uniformly convex domain \( Q(\cdot, y_0, z_0) \). By replacing \( g \)-convex \( u \) by its \( g \)-support at \( x_0 \), we obtain Lemma 2.1 in its full generality. \[ \square \]
Remark 2.1. When $u$ is uniformly $g$-convex or A3 or A4w hold, we do not need to use the approximation (2.3) in the above proof. This is automatic when $u$ is uniformly $g$-convex while if A3 holds we also have the strict inequality (2.10) when $u$ is only assumed $g$-convex. In the case A4w, we have from [19] the differential inequality

$$h''_0 \geq -K|h'_0|,$$

whenever $h_0 \geq 0$, for some positive constant $K$, and we infer $h(x) > 0$ directly, without adjusting $z_0$, as in [19]. Moreover if the strict version A4 of condition A4w holds, then we have again the strict inequality (2.10) for $g$-convex $u$.

2.3. Extension

In this subsection, we use the property (2.9) to extend our initial construction $u^0$ from $\Omega$ to $\Omega^\delta = \{x \in \mathbb{R}^n | \text{dist}(x, \Omega) < \delta\}$, following the argument in the optimal transportation case [23]. Note that we only need to extend $\Omega$ to a sufficiently small neighbourhood, so $\delta$ can be chosen sufficiently small. We may assume $u^0 \in C^\infty(\bar{\Omega})$ by approximation and make the extension using envelopes of $g$-affine functions. Recall that a $g$-affine function in $\Omega$ has the form $g^0 = g(\cdot, y_0, z_0)$ for $y_0 \in \mathbb{R}^n$, $z_0 \in I(\Omega, y_0)$ and its $g$-normal mapping image $Tg^0(\Omega) = \{y_0\}$. We consider the following admissible set:

$$S = \{g^0(x) | g^0(x) \text{ is } g-\text{affine in } \Omega^\delta, g^0 \leq u^0 \text{ in } \Omega, Tg^0(\Omega) \subset \Omega^*\},$$

and take

$$u_1(x) = \sup_{g^0 \in S} \{u^0, g^0\}, \quad x \in \Omega^\delta. \quad (2.13)$$

Then the following lemma, extending Lemma 7.1 in [23], describes the properties of the function $u_1$:

Lemma 2.2. Assume that the domains $\Omega, \Omega^*$ and generating function $g$ satisfy conditions A1, A2, A1*, A3w, A5, and $\Omega, \Omega^*$ are respectively uniformly $Y$-convex, uniformly $Y^*$-convex with respect to each other and any interval $J \subset \subset J_0$. Then, for sufficiently small $\delta$ and $u^0$ satisfying (2.3), the function $u_1$ is a $g$-convex extension of $u^0$ from $\Omega$ to $\Omega^\delta$, whose $g$-normal image under $u_1$ is $\Omega^*$. Moreover, for any $x \in \Omega^\delta - \bar{\Omega}$, there exist unique points $x_b \in \partial \Omega, y_b = Tu_1(x_b) \in \partial \Omega^*$, such that $Tu_1(\ell_{y_b}) = y_b$, where $\ell_{y_b}$ is the open $g$-segment with respect to $y_b$, $z_0 = g^*(x_b, y_b, u^0(x_b))$, joining $x_b$ to $x$, with the resultant mappings being $C^2$ diffeomorphisms from $\partial \Omega^r$ to $\partial \Omega, \partial \Omega^*$ respectively, for any $r < \delta$.

Proof. We take any $g$-affine function $g^0 = g(x, y, z_0)$ in $S$, with $y \in \Omega^* \setminus \omega^*$. Since $u^0$ is uniformly $g$-convex, by decreasing $z_0$, thereby increasing $g^0$, the graph of $g^0$ will touch $u^0$ from below at a point $x_b \in \partial \Omega$. Accordingly we may assume that the $g$-affine function $\tilde{g} \in S$, given by

$$\tilde{g}(x) := \tilde{g}_{x_b, y_b, z_0}(x) = g(x, y, z),$$
for the same \( y \) in \( g_0 \) and some \( z < z_0 \), touches \( u^0 \) from below at \( x_b \in \partial \Omega \), whence \( z = z_y = g^*(x_b, y_b, u^0_b) \), where \( u^0_b = u^0(x_b) \). Since \( \bar{g} \leq u^0 \) in \( \Omega \), \( \bar{g}(x_b) = u^0_b \), the point \( y \) must lie on \( \ell^*_x \), which is the image under \( Y(x_b, u^0_b, \cdot) \) of the straight line from \( D u^0_b = D u^0(x_b) \) with the slope \( \gamma_0 \), that is,

\[
y = Y(x_b, u^0_b, D u^0_b + s \gamma_0) \in \ell^*_x
\]

for some \( s \geq 0, \gamma_0 = \gamma(x_b) \). Moreover, \( \ell^*_x \) starts at the point \( y_{0,b} = Tu^0(x_b) \). Conversely, for any \( x_b \in \partial \Omega \), \( y \in \ell^*_x \), we have from (2.9) that

\[
\bar{g}(x) = g(x, y, z_y) < u^0(x), \quad \text{for } x \in \bar{\Omega} \setminus \{x_b\}. \tag{2.14}
\]

This proves that \( u_1 \) is indeed a \( g \)-convex extension of \( u^0 \) from \( \Omega \) to \( \Omega^\delta \).

To proceed further, from the uniform \( Y^* \)-convexity of \( \Omega^* \), \( \ell^*_x \) intersects with \( \partial \Omega^* \) at the unique point \( y_b \), and from the uniform \( g \)-convexity of \( u^0 \), \( \ell^*_x \) only intersects with \( \partial \omega^* \) at the initial point \( y_{0,b} \). We then restrict \( \ell^*_x \) to the segment joining \( y_{0,b} \) and \( y_b \). From the argument above, the mapping from \( x_b \) to \( y_b \) is onto \( \partial \Omega^* \). From (2.9), it is also one-to-one as the \( g \)-affine function \( \bar{g} \) cannot meet \( \partial \Omega \) at another point \( x' \). It follows then the mapping from \( x_b \) to \( y_b \) is a \( C^2 \) diffeomorphism from \( \partial \Omega \) to \( \partial \Omega^* \). Next, if \( B_r \) is a sufficiently small exterior tangent ball of \( \Omega \) at \( x_b \), it will also be uniformly \( Y \)-convex. Defining \( z_b = g^*(x_b, y_b, u^0_b) \), we then have from (2.9) again that

\[
g(x, y_b, z_b) \geq g(x, y, z_y) \tag{2.15}
\]

for all \( x \in B_r, y \in \ell^*_x \). Note that \( \gamma_0 \) is now the inner normal at \( x_b \) to \( B_r \). Thus, we have

\[
u_1 = \max_{x_b \in \partial \Omega} \{u^0, \bar{g}_{x_b, y_b, z_b}\}. \tag{2.16}
\]

To complete the proof of Lemma 2.2, we need to show that for each \( x \in \Omega^\delta \setminus \Omega \), there exists a unique \( x_b \in \partial \Omega \), where the maximum in (2.16) is attained. For this, we invoke the \( g \)-transform of \( u_1 \),

\[
v^0(y) = \sup_{x \in \Omega^\delta} \{g^*(x, y, u_1(x))\}, \quad y \in \Omega^*
\]

which extends the \( g \)-transform of \( u^0 \) in \( \omega^* \). Moreover, by \( \bar{g} \leq u^0 \) in \( \Omega \), we see that for \( y \in \ell^*_x \), the supremum is attained at \( x_b \). Hence, we have

\[
v^0(y) = g^*(x_b, y, u^0_b), \quad \text{for all } y \in \ell^*_x. \tag{2.17}
\]

One easily verifies that \( v^0 \) is smooth in \( \bar{\Omega}^* \setminus \partial \omega^* \). Using (2.17) and arguing as before, we infer that for any point \( x \in \Omega^\delta \setminus \Omega \), there exists a unique point \( y_b \in \partial \Omega \) such that

\[
g^*_{x, y_b, u_0}(y) \leq v^0(y), \quad \forall y \in \bar{\Omega}^*, \tag{2.18}
\]

for some \( u_0 \). Moreover, \( x \) lies on \( \ell_{y_b} \) which is the image under \( X(y_b, v^0(y_b), \cdot) \) of the straight line segment from \( D v^0(y_b) = D v^0(y_b) + s \gamma^*(y_b) \) with the slope \( \gamma^*(b_b) \), namely,

\[
x = X(y_b, v^0(y_b), D v^0(y_b) + s \gamma^*(y_b)) \in \ell_{y_b}.
\]
where \( \gamma^* \) denotes the unit outer normal to \( \partial \Omega^* \), \( s \in [0, \tilde{s}] \), and \( \tilde{s} \) is a small constant. Note that \( x_b = X(y_b, v^0(y_b), Dv^0(y_b)) \). From (2.18), we see that the maximum in (2.16) is attained at \( x_b, y_b \), so

\[
u_1(x) = \bar{g}_{x,y_b,z_b}(x), \quad x \in \ell_{y_b},
\]

with \( Tu(\ell_{y_b} - \{x_b\}) = y_b, Tu(x_b) = \ell^*_b \). From the obliqueness of \( \ell_{y_b} \) on \( \partial \Omega \), we have that the mapping from \( x \in \Omega^r \) to \( x_b \) is one-to-one for sufficiently small \( r \). This completes the proof of Lemma 2.2. \( \Box \)

### 2.4. Adjustment and Mollification

In this subsection, we will make further adjustment and mollification of the extended function \( u_1 \). From the construction of \( u_1 \) in the previous subsection, we know that \( u_1 \) is smooth in \( \Omega^\delta \setminus \partial \Omega \). Modifying \( u_1 \) in \( \Omega^\delta \setminus \partial \Omega \), by defining

\[
u = \begin{cases} 
\nu, & x \in \Omega, \\
\nu + \epsilon d^2, & x \in \Omega^\delta \setminus \Omega,
\end{cases}
\tag{2.19}
\]

where \( \epsilon \) is small perturbation of \( \Omega^\delta \) containing \( \Omega^* \).

We can now mollify the function \( u \) by

\[
u_\epsilon(x) = \rho \star u = \int_{\mathbb{R}^n} \epsilon^{-n} \rho \left( \frac{x-y}{\epsilon} \right) u(y) dy = \int_{\mathbb{R}^n} \rho(y) u(x-\epsilon y) dy,
\tag{2.21}
\]

where \( \rho \in C^\infty_0(B(1)) \) is any nonnegative symmetric mollifier satisfying \( \int_{B(1)} \rho = 1, \epsilon \) is a positive constant. Taking \( \epsilon < \delta/2 \) sufficiently small and \( x \in \Omega^\delta \), we will show that \( u_\epsilon(x) \) is uniformly \( g \)-convex in \( \Omega^\delta \). Note that the image of the \( g \)-normal mapping of \( u_\epsilon \) in \( \Omega^\delta \) is a small perturbation of \( \Omega^* \). First, we recall some properties of \( u_\epsilon \) from [23],

\[
Du_\epsilon(x) = \int_{\mathbb{R}^n} \rho(y) Du(x-\epsilon y) dy,
\tag{2.22}
\]

\[
D^2u_\epsilon(x) \geq \int_{\mathbb{R}^n} \rho(y) D^2u(x-\epsilon y) dy
\tag{2.23}
\]

where \( P_{x,\epsilon} := \{ y \in \mathbb{R}^n | x - \epsilon y \in \partial \Omega \} \). We then divide \( \Omega^\delta / \Omega^\delta = U_1 \cup U_2 \cup U_3 \) where \( U_1 := \{ x \in \Omega^\delta | \text{dist}(x, \partial \Omega) \geq \epsilon \} , U_2 := \{ x \in \Omega^\delta | \text{dist}(x, \partial \Omega) \in (\epsilon', \epsilon) \} \) and \( U_3 := \{ x \in \Omega^\delta | \text{dist}(x, \partial \Omega) \leq \epsilon' \} \), with \( \epsilon' = (1 - \sigma) \epsilon \) and \( \sigma \in (1/2, 1) \) is a constant close to 1. It is clear that \( u_\epsilon \) is smooth and uniformly \( g \)-convex in \( U_1 \) provided \( \epsilon \) is sufficiently small. Also, by choosing \( \sigma \) sufficiently close to 1, for any \( x \in U_2 \), from (2.21) and (2.22), \( u_\epsilon, Du_\epsilon \) are small perturbations of \( u \) and \( Du \),
respectively. By (2.23), we have \( u_\epsilon \) is smooth and uniformly \( g \)-convex in \( U_2 \). We next check the uniform \( g \)-convexity of \( u_\epsilon \) in \( U_3 \). For any point \( x_0 \in U_3 \), without loss of generality, we choose the nearest point of \( x_0 \) on \( \partial \Omega \) to be the origin, and choose the direction pointing from 0 to \( x_0 \) to be \( e_n \) so that \( \partial \Omega \) is tangent to \( \{ x_n = 0 \} \) and \( x_0 = (0, \ldots, 0, x_0,n) \). We choose a unit vector \( \tau \) tangential to \( \partial \Omega \) at 0. Without loss of generality, we can assume \( \tau = (1, 0, \ldots, 0) \). Then we need to prove

\[
D_{11} u_\epsilon (x_0) - A_{11}(x_0, u_\epsilon (x_0), Du_\epsilon (x_0)) > 0. \tag{2.24}
\]

By the choice of coordinates, \( D_{11} u_\epsilon (x_0) \) is a small perturbation of \( D_1 u(x_0) \). Notice that, from (2.21), \( u_\epsilon(x_0) \) is also a small perturbation of \( u(x_0) \). It now suffices to prove

\[
D_{11} u_\epsilon (x_0) - A_{11}(x_0, u(x_0), D_1 u(x_0), D' u_\epsilon(x_0)) > 0, \tag{2.25}
\]

where \( D' u_\epsilon = (D_2 u_\epsilon, \ldots, D_n u_\epsilon) \). From A3w, \( A_{11} \) is convex with respect to \( D' u_\epsilon \). Therefore, we have

\[
A_{11}(x_0, u(x_0), D_1 u(x_0), D' u_\epsilon(x_0)) \leq \int_{\mathbb{R}^n} \epsilon^{-n} \rho \left( \frac{x_0 - y}{\epsilon} \right) A_{11}(x_0, u(x_0), D_1 u(x_0), D' u(y)) dy. \tag{2.26}
\]

From (2.26), and again from (2.23), we have (2.25) holds. From the property of the second integral in (7.19) in [23], we also have

\[
D_{nn} u_\epsilon (x_0) - A_{nn}(x_0, u_\epsilon(x_0), Du_\epsilon(x_0)) \geq K, \tag{2.27}
\]

for sufficiently large \( K \), provided \( \epsilon \) is sufficiently small. Combining (2.25) and (2.27), we know that \( u_\epsilon \) is uniformly \( g \)-convex in \( U_3 \). Thus, we have proved that the function \( u_\epsilon \) is smooth and uniformly \( g \)-convex in \( \Omega_2^\delta \) for \( \epsilon < \frac{\delta}{2} \) sufficiently small.

Then by appropriate adjustment of the domain \( \Omega \), we have the following lemma, which gives the existence of uniformly \( g \)-convex smooth functions with approximating target domains:

**Lemma 2.3.** Let the domains \( \Omega \), \( \Omega^* \) and the generating function \( g \) satisfy the hypotheses of Theorem 1.1. Then for any \( \epsilon > 0 \), there exists a uniformly \( g^* \)-convex \( C^4 \) approximating domain \( (\Omega^*)^\epsilon \) lying within the distance \( \epsilon \) of \( \Omega^* \), together with a uniformly \( g \)-convex function \( u \in C^4(\Omega) \) satisfying the boundary condition (1.6) for \( (\Omega^*)^\epsilon \).

Note that if we do not make an adjustment of the domain \( \Omega \), we get a uniformly \( g \)-convex function for approximating domains for both \( \Omega \) and \( \Omega^* \).

### 3. Proof of Existence Theorems

In this section, we give the proof of the existence result, Theorem 1.1, utilizing the method of continuity, supplemented by degree theory for nonlinear oblique boundary value problems, as in [1,2,11,12,14]. From Lemma 2.3, we can assume
initially that there exists a uniformly \( g \)-convex function \( u_0 \in C^4(\bar{\Omega}) \) satisfying (1.6), that is \( Tu_0(\Omega) = \Omega^* \). From our construction in Section 2, we can also assume the inclusion (2.3). We will also need that the second boundary value condition (1.6) implies a nonlinear oblique boundary condition for uniformly elliptic functions \( u \), [18]. In particular, letting \( \phi^* \in C^2(\mathbb{R}^n) \) be a defining function for \( \Omega^* \), satisfying \( \phi^* = 0, D\phi^* \neq 0 \) on \( \partial \Omega^*, \phi^* < 0 \) in \( \Omega^*, \phi^* > 0 \) in \( \mathbb{R}^n - \Omega^* \) and setting

\[
G(x, u, p) = \phi^* \circ Y(x, u, p)
\]

for \( (x, u, p) \in \mathcal{U} \), we obtain

\[
G[u] := G(\cdot, u, Du) = 0, \quad \text{on } \partial \Omega,
\]

(3.1)
together with the obliqueness condition

\[
G_p(\cdot, u, Du) \cdot \gamma > 0, \quad \text{on } \partial \Omega.
\]

(3.3)

Furthermore, the \( Y^* \)-convexity (uniform \( Y^* \)-convexity) of \( \Omega^* \) with respect to \( \Omega \times J \) implies that \( G \) is convex (uniformly convex) in \( p \) for \( x \in \partial \Omega, u \in J, Y(x, u, p) \in \partial \Omega^* \), and additionally the uniform \( Y^* \)-convexity of \( \Omega^* \) implies that \( G = \phi^* \circ Y \) is uniformly convex in \( p \) when \( Y \) lies in some neighbourhood \( N^* = \{|\phi^*| < \delta \} \) of \( \partial \Omega^* \), for some \( \delta > 0 \). In particular, these properties are essential for showing that the initial problem in our homotopy family (3.4) is uniquely solvable; (see Lemma 3.1 below). We remark also that the boundary condition (1.6) is implied by (3.2) when \( Tu \) is also one-to-one on \( \bar{\Omega} \) and this would follow from the ellipticity of \( u \) on \( \bar{\Omega} \), together with the \( g \)-convexity of \( \Omega \) with respect to some \( y_0, z_0 \).

We now consider for \( 0 \leq t \leq 1, \tau > 0 \) and \( \epsilon > 0 \) the family of generated prescribed Jacobian equations

\[
|\det DTu| = e^{[\tau(1-t)+\epsilon](u-u_0)}|tf + (1-t) f^* \circ Tu_0|/f^* \circ Tu, \quad \text{in } \Omega
\]

(3.4)

for elliptic solutions \( u \in C^2(\bar{\Omega}) \), with one-jet \( J_1[u](\Omega) \subset \subset \mathcal{U} \), and range \( u(\Omega) \subset \subset J_0 \), where \( Tu = Y(\cdot, u, Du) \). Here we call the solution \( u \) elliptic if \( D^2u > A(\cdot, u, Du) \), so that equation (3.4) can still be written in the forms (1.1) and (1.4); (see equation (3.9)).

**Lemma 3.1.** Under the hypotheses of Theorem 1.1, for sufficiently large \( \tau \), \( u_0 \) is the unique elliptic solution of the second boundary value problem (3.4), (1.6) at \( t = 0 \).

**Proof.** First suppose \( u \in C^2(\bar{\Omega}) \) is an elliptic solution of (3.4), (1.6) for some \( t \in [0, 1] \), so that \( Tu \) is a diffeomorphism from \( \Omega \) to \( \Omega^* \). Multiplying by \( f^* \circ Tu \) and integrating (3.4) over \( \Omega \), we obtain, from the change of variables formula and the conservation of energy (1.8),

\[
t \int_{\Omega} e^{[\tau(1-t)+\epsilon](u-u_0)} - 1 \right) f + (1-t) \int_{\Omega^*} e^{[\tau(1-t)+\epsilon](u-u_0) - (Tu_0)^{-1} - 1} f^* = 0,
\]

(3.5)
which implies that $u = u_0$ at some point in $\Omega$. From A1, A5, and the assumption that $u$ lies in the open interval $J_0$, we have

$$\sup_{\Omega} |Du| \leq K_0,$$  \hfill (3.6)

where $K_0$ is the constant in A5. From (2.3) we then have $u(\tilde{\Omega}) \subset J$ for some fixed $J \subset J_0$, so that, in particular,

$$\sup_{\Omega} |u| \leq M_0$$  \hfill (3.7)

for some fixed constant $M_0$, depending on $u_0$, $K_0d$ and $J_0$. Thus any elliptic solution $u$ of (3.4), (1.6) satisfies a uniform $C^1$ bound

$$|u| : J_0 \backslash \Omega \leq C,$$  \hfill (3.8)

with its one-jet $J_1[u](\tilde{\Omega})$ lying in a fixed set $U_0 \subset U$. Writing equation (3.4) in the Monge–Ampère form,

$$F[u] := \log \det [D^2u - A(\cdot, u, Du)] = [\tau (1-t) + \epsilon] (u - u_0) + \log B_t(\cdot, u, Du),$$  \hfill (3.9)

where

$$B_t = \det E[(tf + (1-t)f^* \circ Tu_0) \det DTu_0]/f^* \circ Y,$$

we can then infer higher order estimates for elliptic solutions $u$, which we will need for our continuity argument.

Returning to the uniqueness assertion in Lemma 3.1 in the case $t = 0$, we consider the function $w$ given by

$$w = e^{-\kappa \phi}(u - u_0),$$  \hfill (3.10)

where $\kappa$ is a positive constant to be fixed later and $\phi \in C^2(\tilde{\Omega})$ is a defining function for $\Omega$, satisfying $\phi = 0$ on $\partial \Omega$, $D\phi = \gamma$ on $\partial \Omega$ and $\phi < 0$ in $\Omega$. Assuming that the function $w$ attains its positive maximum at $x_0 \in \Omega$, then we have

$$Dw(x_0) = 0, \quad \text{and} \quad Lw(x_0) \leq 0,$$  \hfill (3.11)

where $L$ is a linearized operator defined by

$$L := F^{ij}(M[u_0])D_{ij},$$  \hfill (3.12)

with $F^{ij}(M[u_0]) = \frac{\partial F(M[u_0])}{\partial r_{ij}}$, $\{r_{ij}\} = M[u_0] = D^2u_0 - A(x, u_0, Du_0)$. By a direct calculation, we have

$$Lw(x_0) = F^{ij}(M[u_0](x_0))[e^{-\kappa \phi}(x_0)((D_{ij}u_0(x_0) - A_{ij}(x_0, u_0, x_0, Du_0(x_0))$$

$$- (D_{ij}u_0(x_0) - A_{ij}(x_0, u_0, x_0, Du_0(x_0)))$$

$$+ e^{-\kappa \phi}(x_0)[A_{ij}(x_0, u(x_0), Du(x_0)) - A_{ij}(x_0, u_0(x_0), Du_0(x_0))]$$

$$+ 2\kappa D_i \phi(x_0)D_j(u - u_0)(x_0)] + \kappa [-D_{ij} \phi(x_0)$$

$$+ \kappa D_i \phi(x_0)D_j \phi(x_0)]w(x_0)).$$  \hfill (3.13)
Using the concavity of “log det” and equation (3.9) at $t = 0$, we have

$$e^{-\kappa \phi(x_0)} F^{ij}(M[u_0](x_0))[(D_{ij} u(x_0) - A_{ij}(x_0, u(x_0), Du(x_0)))]$$

$$\geq e^{-\kappa \phi(x_0)} (F[u(x_0)] - F[u_0(x_0)])$$

$$= (\tau + \epsilon) w(x_0) + e^{-\kappa \phi(x_0)} [\log B_0(x_0, u(x_0), Du(x_0)) - \log B_0(x_0, u_0(x_0), Du_0(x_0))].$$

By the mean value theorem, we have

$$A_{ij}(x_0, u(x_0), Du(x_0)) - A_{ij}(x_0, u_0(x_0), Du_0(x_0))$$

$$= D_u A_{ij}(x_0, \hat{u}(x_0), Du(x_0))(u - u_0(x_0))$$

$$+ D_p A_{ij}(x_0, u_0(x_0), \hat{p}(x_0)) D_k(u - u_0(x_0)), \quad (3.15)$$

for all $i, j = 1, \ldots, n$, where $\hat{u} = (1 - \theta_1) u + \theta_1 u_0$, $\hat{p} = (1 - \theta_2) Du + \theta_2 Du_0$, for some $0 < \theta_1, \theta_2 < 1$. Similarly, again by the mean value theorem, we have

$$\log B_0(x_0, u(x_0), Du(x_0)) - \log B_0(x_0, u_0(x_0), Du_0(x_0))$$

$$= D_u (\log B_0)(x_0, \tilde{u}(x_0), Du(x_0))(u - u_0(x_0))$$

$$+ D_p (\log B_0)(x_0, u_0(x_0), \tilde{p}(x_0)) D_k(u - u_0(x_0)), \quad (3.16)$$

where $\tilde{u} = (1 - \xi_1) u + \xi_1 u_0$, $\tilde{p} = (1 - \xi_2) Du + \xi_2 Du_0$, for some $0 < \xi_1, \xi_2 < 1$. From the equality in (3.11), we have

$$D_i(u - u_0)(x_0) = \kappa(u - u_0)(x_0) D_i \phi(x_0), \quad \text{for } i = 1, \ldots, n. \quad (3.17)$$

Substituting (3.14), (3.15), (3.16) into (3.13) and using (3.17), we get

$$L w(x_0) \geq (\tau + \epsilon - C) w(x_0), \quad (3.18)$$

where the constant $C$ depends on $F^{ij}(M[u_0])$, $D_u A$, $D_P A$, $D_u B_0$, $D_P B_0$, $B_0$, $M_0$, $K_0$, $\phi$ and $\kappa$. By choosing $\tau$ sufficiently large such that $\tau + \epsilon > C$, we get $L w(x_0) > 0$. It follows that the function $w$ cannot take a positive maximum in $\Omega$.

Accordingly, we suppose that $w$ takes a positive maximum at a point $x_0 \in \partial \Omega$, whence

$$\beta_0 \cdot D w(x_0) = e^{-\kappa \phi(x_0)} [\beta_0 \cdot D(u - u_0)(x_0) - \kappa \beta_0 \cdot \gamma(x_0)(u - u_0)(x_0)] \geq 0, \quad (3.19)$$

where $\beta_0 = G_p(J_1[u_0](x_0))$ and from the obliqueness of $G$ with respect to $u_0$, we have $\beta_0 \cdot \gamma(x_0) > 0$. Now we extend $G$ so that $G \in C^1(\partial \Omega \times J \times \mathbb{R}^n)$ is convex in $p$ and agrees with (3.1) for $|\phi^* \circ Y| < \delta/2$. From the convexity of $G$ and the boundary condition (3.2), we then have

$$G(x_0, u_0(x_0), Du(x_0)) \geq \beta_0 \cdot D(u - u_0)(x_0) \geq \kappa \beta_0 \cdot \gamma(x_0)(u - u_0)(x_0). \quad (3.20)$$

Now for sufficiently large $\kappa$, depending on $G_u$, $M_0$, $K_0$ and $\beta_0 \cdot \gamma(x_0)$, we have from (3.2) again,

$$G(x_0, u_0(x_0), Du(x_0)) < \kappa \beta_0 \cdot \gamma(x_0)(u - u_0)(x_0). \quad (3.21)$$

Consequently we must have $u \leq u_0$ in $\Omega$ and we immediately conclude $u = u_0$ in $\Omega$ from (3.5). \qed
Now we can complete the proof of Theorem 1.1. With $\tau$ fixed, in accordance with Lemma 3.1 and $\epsilon$ sufficiently small, say $\epsilon < 1$, we first note from (3.8) and $J_1[u](\Omega) \subset U_0$ that $|\det E|$ and $B_t$ will have uniform positive lower bounds for elliptic solutions of (3.4), (1.6). Invoking the alternative conditions $A4w$ or $A4*w$ or $A3$, we then have uniform global second derivative estimates from [7], (in the first two cases), and [14], (in the last case). From the Hölder estimates for second derivatives [13] and the linear theory [3], we have uniform estimates in the spaces $C^{4,\alpha}(\Omega)$ for $\alpha < 1$, provided $\Omega$ is smooth enough, say $C^5$. Accordingly there exists a bounded open set $\bar{\Omega}$ in $C^{4,\alpha}(\Omega)$ such that the operators $F$ and $G$ are respectively elliptic and oblique with respect to all $u \in \bar{\Omega}$ and the boundary value problems (3.4), (1.6) have no elliptic solutions in $\partial \Omega$. Furthermore the set $\bar{\Omega}$ can be chosen so that $u_0 \in \bar{\Omega}$ and $Tu$ is one-to-one on $\bar{\Omega}$ for all $u \in \bar{\Omega}$. This latter property implies that condition (1.6) is in fact equivalent to the oblique condition (3.2) for our solutions, so that we then conclude the solvability of the boundary value problem (3.4), (1.6), at $t = 1$, from the degree theory for oblique boundary value problems, explicitly from Case (ii) of Theorem 10.23, (with $k = m_1 = 1$), in [2] or from assertions (a) and (d) of Corollary 2.1 in [12]. For this, as well as the uniqueness of our initial solution $u_0$, we also need to observe, from the proof of Lemma 3.1, that the linearized operator $L$, associated with the boundary value problem (3.9), (3.2) at $t = 0$ and $u = u_0$, is one-to-one, whence by virtue of the Schauder theory [3], $L$ is an isomorphism from $C^{4,\alpha}(\Omega)$ to $C^{2,\alpha}(\Omega) \times C^{3,\alpha}(\partial \Omega)$. Note that the treatment of the second order case in [12], based on the Dirichlet problem case in [11], is somewhat simpler than the general theory in [2] and the degree constructed there is homotopy invariant whereas in the generality of quasilinear Fredholm operators in [2], the sign of the degree may change.

Finally we complete the proof of Theorem 1.1 by sending $\epsilon$ to 0 in (3.4) and subsequently in our approximations $\Omega_\epsilon^*$, using again our a priori solution bounds.

**Remark 3.1.** To prove the existence theorem for classical solutions in the optimal transportation problem [23], there are two different approaches using the method of continuity. The first approach, (in Section 5 in [23]), is based on domain deformation, which requires an appropriate foliation which follows from a global barrier condition, (see (1.21) or (5.7) in [23]). The second approach, (in Section 7 in [23]), is based on a direct construction of uniformly elliptic functions with approximating target domains, which can be applied without domain deformation. Our proof here, in the more general geometric optics setting, utilizes the second approach in [23] without the domain variation and global barrier. Since we have also obtained the second derivative estimate in [7] without the global barrier condition, we can completely avoid the global barrier condition for the existence result, Theorem 1.1.

**Remark 3.2.** As remarked in Section 1, there clearly exist an infinite number of solutions in Theorem 1.1. By inspection of our proof in Sections 2 and 3, we may also assume that the interval $J_0$ in condition A5 is finite, provided there exists a $g$-affine function $g_0$ satisfying (2.3) and $Tg_0 \in \Omega^*$. Then we have a more precise result under the hypotheses of Theorem 1.1, namely there exists an elliptic solution $u \in C^3(\Omega)$ of the second boundary value problem (1.4), (1.6) whose graph intersects that of $g_0$. 

Remark 3.3. The supplementary conditions, A4w or $A4^*w$ or A3, in Theorem 1.1 are only used to guarantee global second derivative bounds so it would be interesting to prove such bounds just under conditions A1, A2, $A1^*$ and A3w. Such bounds were originally proved in [23] for general Monge–Ampère type equations also under the addition of the global barrier condition (see equation (3.3) in [23]), which was subsequently removed in [7] for optimal transportation equations and, more generally, generated prescribed Jacobian equations, satisfying A4w or $A4^*w$. We remark also that the alternative duality method proposed in [23], Theorem 3.2, for the case when $A$ depends only on $p$, is not valid, although the case when $n = 2$ still follows directly without using our construction in [7]. Similarly the foreshadowed alternative use of duality at the end of [7] is only valid for $n = 2$. We are grateful to Philippe Delanöe for pointing out this problem to us.

Remark 3.4. We may also consider the second boundary value problem (1.6) for more general fully nonlinear, augmented Hessian equations of the form,

$$F[D^2u - A(\cdot, u, Du)] = B(\cdot, u, Du),$$  \hfill (3.22)

where $F$ is an increasing function on the positive cone of $n \times n$ symmetric matrices, $A$ given by (1.5) is defined through a $C^4$ generating function $g$ and $B$ is a positive function in $C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$. The main examples here are the Hessian quotients, $F_{n,k}$ given by

$$F_{n,k} = \frac{\det}{S_k},$$  \hfill (3.23)

where the $k$-Hessian $S_k(r), 0 < k < n$, is the sum of the principal $k \times k$ minors of the matrix $r$, which were considered in the optimal transportation case, when $g$ is given by a cost function, by von Nessi [24]. Global second derivative estimates for elliptic solutions, in the case (3.23) and more generally were proved by us in [8], Corollary 3.1, under conditions A1, A2, $A3w$ and A4w together with $B$ independent of $p$ satisfying the monotonicity $B_u \geq 0$ and the existence of an elliptic subsolution $u \in C^2(\bar{\Omega})$. For such general operators, we can also assume condition A5 to guarantee the gradient estimate when the range of the solution $u$ lies in $J_0$. If we have appropriate solution estimates, we can readily prove the classical existence result following the steps in Sections 2 and 3 of the current paper. In general, the obstacle for the maximum solution estimate arises from the lack of the structures (1.7) and (1.8). However when $B(x, \cdot)(J_0) = (0, \infty)$ for all $x \in \Omega$, we can obtain the solution estimate by modification of Section 5.2 in [24], and thus conclude the classical existence result.

4. Applications in Geometric Optics

In this section we treat the application of Theorem 1.1 to some examples in geometric optics developed in [7] in conjunction with our global second derivative bounds. In particular we consider the reflection and refraction of parallel light beams to targets, which are graphs over orthogonal hyperplanes. Our concern, as in the case of flat targets in [14] is with globally smooth solutions and the reader
is referred to [6,19,20] for local regularity considerations as well as [4,9,10] for more general targets. As in [7], we consider parallel beams in \( \mathbb{R}^{n+1} \), directed in the direction of \( e_{n+1} \), through a domain \( \Omega \subset \mathbb{R}^n \times \{0\} \), illuminating targets which are graphs over domains \( \Omega^* \subset \mathbb{R}^n \times \{0\} \). The targets are allowed to be either flat or non-flat.

4.1. Reflection

Let \( D \) be a domain in \( \mathbb{R}^n \times \mathbb{R}^n \), containing \( \bar{\Omega} \times \bar{\Omega}^* \), and consider the generating function

\[
g(x, y, z) = \Phi(y) - \frac{1}{2z} |x - y|^2,
\]

(4.1)

defined for \( (x, y) \in D \) and \( z < 0 \) where \( \Phi \) is a smooth function on \( \mathbb{R}^n \). Here we have replaced \( z \) by \( -1/z \) in [7] to conform with the subsequent refraction examples. From our calculations in Section 4.2(i) of [7], we see that \( g \) satisfies conditions A1, A2, A1*, A4w on \( \Gamma \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \), which can be defined through its dual set

\[
\Gamma^* = \{(x, y, u) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} | (x, y) \in D, u \in J(x, y)\},
\]

where

\[
J(x, y) = (\Phi(y) + (x - y) \cdot D\Phi(y), \infty).
\]

(4.2)

In the corresponding reflection problem we seek a reflecting surface \( R \) as a graph \( \{(x, u(x)) | x \in \Omega\} \), so that light with intensity \( f \) on \( \Omega \) is mapped under the reflection mapping \( Tu \) to intensity \( f^* \) on \( \Omega^* \), where \( f \) and \( f^* \) satisfy the conservation of energy condition (1.8). This leads to solving the boundary value problem, (1.4), (1.6) and the condition \( u(x) \in J(x, y) \) for all \( x \in \Omega, y \in \Omega^* \), arising from (4.2), is equivalent to the reflector \( R \) lying above the tangent hyperplane to the target \( T \) at any \( y \in \Omega^* \), which is clearly necessary when a ray through \( x \in \Omega \) illuminates \( T \) from above at \( y \in \Omega^* \). Next, from (4.11) in [7], the matrix \( A \) is given by

\[
A(x, u, p) = \frac{1}{Z(x, u, p)} I
\]

(4.3)

so that A3w is satisfied if and only if the function \( 1/Z \) is locally convex in \( p \), where \( Z \) denotes the dual function. Finally we have, again from the calculations in [7], that condition A5 is satisfied for \( J_0 = (m_0, \infty) \), where \( m_0 \) is given by

\[
m_0 = \sup_{x \in \Omega, y \in \Omega^*} [\Phi(y) + (x - y) \cdot D\Phi(y)],
\]

and \( K_0 \) given by

\[
K_0 = \sup_{\Omega^*} (\sqrt{1 + |D\Phi|^2} + |D\Phi|).
\]
4.2. Refraction

We consider refraction from media $I$ to media $II$, through a surface interface $R = \{(x, u(x)) | x \in \Omega\}$, with respective refraction indices $n_1, n_2 > 0$ and set $\kappa = n_1/n_2$. For $\kappa \neq 1$, we consider now generating functions

$$g(x, y, z) = \Phi(y) - \frac{1}{|\kappa^2 - 1|}(\kappa z + \sqrt{z^2 + (\kappa^2 - 1)|x - y|^2}),$$

(4.4)

where again $(x, y) \in D$, $z > \kappa'|x - y|$ for $0 < \kappa < 1$, $> 0$ for $\kappa > 1$, where $\kappa' = \sqrt{|\kappa^2 - 1|}$, and $\Phi$ is a smooth function on $\mathbb{R}^n$. From our calculations in Section 4.2(ii) of [7], we then obtain that $g$ satisfies conditions A1, A2 and A1* as above, with, in place of (4.2),

$$J(x, y) = (-\infty, \Phi(y) + (x - y) \cdot D\Phi(y)) \cap (-\infty, \Phi(y) - \frac{\min\{\kappa, 1\}}{\kappa'}|x - y|),$$

(4.5)

with condition A4w satisfied for $\kappa < 1$ and condition A4*w for $\kappa > 1$. Furthermore, from (4.19) and (4.22) in [7], the matrices $A$ are given by

$$A(x, u, p) = [\text{sign}(1 - \kappa^2)]\sqrt{1 + (1 - \kappa^2)|p|^2} Z(x, u, p) [I + (1 - \kappa^2)p \otimes p],$$

(4.6)

so that condition A3w is satisfied if and only if the function

$$p \mapsto \frac{(1 - \kappa^2)\sqrt{1 + (1 - \kappa^2)|p|^2}}{Z(x, u, p)}$$

is locally convex. Finally to complete our hypotheses for the application of Theorem 1.1, we obtain, again from the calculations in [7], that condition A5 is satisfied for $J_0 = (-\infty, M_0)$, where $M_0$ is given by

$$M_0 = \inf_{x \in \Omega, y \in \Omega^*} \min\{\Phi(y) + (x - y) \cdot D\Phi(y), \Phi(y) - \frac{\min\{\kappa, 1\}}{\kappa'}(1 + \delta)|x - y|\},$$

where $\delta > 0$ for $\kappa < 1$, $\delta = 0$ if $\kappa > 1$, and $K_0 = 2/\kappa \kappa' \delta$ for $\kappa < 1$, $K_0 = 1/\kappa'$ for $\kappa > 1$. Note that in our refraction model, the constraint $u(x) \in J(x, y)$ for all $x \in \Omega$, $y \in \Omega^*$ implies the reflector $R$ lies below the tangent hyperplane to the target $T$ at any $y \in \Omega^*$, which is clearly necessary when a ray through $x \in \Omega$ illuminates $T$ from below at $y \in \Omega^*$.

4.3. Flat Targets

When the target is flat, that is $\Phi = \text{constant}$, our models reduce to those considered in [14] and condition A3 holds, as is seen readily from the formulae for the dual function $Z$, namely

$$Z(x, u, p) = \frac{2(\Phi - u)}{1 - |p|^2}, \quad u > \Phi, \quad |p| < 1.$$
in the case of reflection, and
\[
Z(x,u,p) = \frac{|1 - \kappa^2|(|\Phi - u|\sqrt{1 + (1 - \kappa^2)|p|^2})}{1 + \kappa \sqrt{1 + (1 - \kappa^2)|p|^2}}, \quad u < \Phi, \quad (\kappa^2 - 1)|p|^2 < 1
\]
in the case of refraction. Now applying Theorem 1.1, it follows that the barrier condition (28) in the hypotheses of Theorems 5.1 and 5.2 in [14] can be removed and moreover Remark 3.2 provides more information about the overall solution set, and is also applicable to Theorem 1.2 in [14].

4.4. Reverse Ellipticity

From Theorem 1.1, we also obtain classical solutions to the above reflector problems satisfying the reverse ellipticity condition, \(D^2 u < A(\cdot, u, Du)\). Here again conditions A1, A2, A1*, A4w or A4*w, A5 are satisfied as in Sections 4.1 and 4.2 but we must replace \(Z\) by \(-Z\) or equivalently convexity by concavity to ensure condition A3w. In these models we have chosen at the outset the ellipticity, or equivalently the support from below by focusing quadric surfaces corresponding to our \(g\)-affine functions, in order to embrace the flat target cases in Section 4.3.

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Compliance with Ethical Standards

Conflict of interest All the authors declare that they have no conflict of interest.

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