Second order asymptotic efficiency for a Poisson process

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10 June, 2016
Rennes
Outline

First Order Estimation
  Statement of the Problem
  Lower bound

Second Order Estimation
  Classes of As. Efficient Estimators
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  Sketch of the Proof
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In non-parametric estimation the unknown object is a function.
Inhomogeneous Poisson process

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- We observe a periodic Poisson process with a known period $\tau$
  \[ X^T = \{ X(t), \ t \in [0, T] \}, \ T = n\tau. \]
In non-parametric estimation the unknown object is a function.

We observe a periodic Poisson process with a known period \( \tau \)

\[
X^T = \{X(t), \ t \in [0, T]\}, \ T = n\tau.
\]

\( X(0) = 0 \), has independent increments and there exists a positive, increasing function \( \Lambda(t) \) s.t. for all \( t \in [0, T] \)

\[
P(X(t) = k) = \frac{[\Lambda(t)]^k}{k!} e^{-\Lambda(t)}, \ k = 0, 1, \ldots.
\]
Trajectory of a Poisson process
• In the definition

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We consider the case where \( \Lambda(\cdot) \) is absolutely continuous with \( \Lambda(t) = \int_0^t \lambda(s) \, ds \).
Models description

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- We consider the case were \( \Lambda(\cdot) \) is absolutely continuous

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- The positive function \( \lambda(\cdot) \) is called the intensity function and the periodicity of a Poisson process means the periodicity of its intensity function

\[ \lambda(t) = \lambda(t + k\tau), \quad t \in [0, \tau], \quad k \in \mathbb{Z}_+. \]
Mean and the Intensity functions

- With the notations

\[ X_j(t) = X((j - 1)\tau + t) - X((j - 1)\tau), \quad t \in [0, \tau], \]

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- We would like to have Hájek-Le Cam type lower bounds for function estimation

\[
\lim_{\delta \downarrow 0} \lim_{n \to +\infty} \sup_{|\theta - \theta_0| \leq \delta} n\mathbb{E}_{\theta}(\bar{\theta}_n - \theta)^2 \geq \frac{1}{I(\theta_0)}.
\]
Mean function estimation

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• To assess the quality of an estimator we use the MISE

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E_{\Lambda} ||\bar{\Lambda}_n - \Lambda||^2 = E_{\Lambda} \int_0^\tau (\bar{\Lambda}_n(t) - \Lambda(t))^2 dt.
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\]

- The simplest estimator is the \textit{empirical mean function}

\[
\hat{\Lambda}_n(t) = \frac{1}{n} \sum_{j=1}^{n} X_j(t), \ t \in [0, \tau].
\]
The basic equality for the EMF

- The following basic equality for the EMF implies two things

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- Can we have better rate of convergence or smaller asymptotic variance for an estimator?
As. efficiency of the EMF

- The EMF is asymptotically efficient among all estimators \( \bar{\Lambda}_n(t) \),
As. efficiency of the EMF

- The EMF is asymptotically efficient among all estimators $\tilde{\Lambda}_n(t)$,
- Kutoyants’ result-for all estimators $\tilde{\Lambda}_n(t)$

$$\lim_{\delta \downarrow 0} \lim_{n \to +\infty} \sup_{\Lambda \in V_{\delta}} \mathbb{E}_\Lambda \| \sqrt{n}(\tilde{\Lambda}_n - \Lambda) \|^2 \geq \int_0^\tau \Lambda^*(t) dt,$$

with $V_{\delta} = \{ \Lambda : \sup_{0 \leq t \leq \tau} |\Lambda(t) - \Lambda^*(t)| \leq \delta \}, \delta > 0.$
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- Reformulation

$$\lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}} \left( E_{\Lambda} \|\sqrt{n}(\bar{\Lambda}_n - \Lambda)\|^2 - \int_0^\tau \Lambda(t)dt \right) \geq 0.$$
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- $\mathcal{F} \subset L_2[0, \tau]$ is a sufficiently “rich”, bounded set.
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- $\mathcal{F} \subset L_2[0, \tau]$ is a sufficiently “rich”, bounded set.
- Can we have other asymptotically efficient estimators?
Existence of other as. efficient estimators depends on the regularity conditions imposed on unknown $\Lambda(\cdot)$.
Efficient estimators

- Existence of other as. efficient estimators depends on the regularity conditions imposed on unknown $\Lambda(\cdot)$.
- In other words, it depends on the choice of the set $\mathcal{F}$ in

$$\lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}} \left( \mathbb{E}_{\Lambda} \left\| \sqrt{n}(\tilde{\Lambda}_n - \Lambda) \right\|^2 - \int_0^T \Lambda(t) dt \right) \geq 0.$$
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  \lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}} \left( E_{\Lambda} \left| \sqrt{n}(\bar{\Lambda}_n - \Lambda) \right|^2 - \int_{0}^{T} \Lambda(t) dt \right) \geq 0.
  \]
- Demanding existence of derivatives of higher order of the unknown function, we can enlarge the class of as. efficient estimators.
First results

- At first, consider the $L_2$ ball with a center $\Lambda^*$

$$\mathcal{B}(R) = \{\Lambda : ||\Lambda - \Lambda^*||^2 \leq R, \Lambda^*(\tau) = \Lambda(\tau)\}.$$
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- Consider a kernel-type estimator

$$\tilde{\Lambda}_n(t) = \int_0^\tau K_n(s - t)(\hat{\Lambda}_n(s) - \Lambda^*(s))ds + \Lambda^*(t).$$
First results

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- Kernels satisfy

  \[ K_n(u) \geq 0, u \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right], \quad \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} K_n(u)du = 1, \quad n \in \mathbb{N}, \]

  and we continue them $\tau$ periodically on the whole real line $\mathbb{R}$

  \[ K_n(u) = K_n(-u), \quad K_n(u) = K_n(u + k\tau), \quad u \in \left[-\frac{\tau}{2}, \frac{\tau}{2}\right], \quad k \in \mathbb{Z}. \]
Kernel-type estimator

- Consider the trigonometric basis in $L_2[0, \tau]$

\[
\phi_1(t) = \sqrt{\frac{1}{\tau}}, \quad \phi_{2l}(t) = \sqrt{\frac{2}{\tau}} \cos \frac{2\pi l}{\tau} t, \quad \phi_{2l+1}(t) = \sqrt{\frac{2}{\tau}} \sin \frac{2\pi l}{\tau} t.
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\]

- Coefficients of the kernel-type estimator w.r.t. this basis

\[
\tilde{\Lambda}_{1,n} = \hat{\Lambda}_{1,n}, \quad \tilde{\Lambda}_{2l,n} = \sqrt{\frac{\tau}{2}} K_{2l,n}(\hat{\Lambda}_{2l,n} - \Lambda^*_{2l}) + \Lambda^*_{2l},
\]

\[
\tilde{\Lambda}_{2l+1,n} = \sqrt{\frac{\tau}{2}} K_{2l,n}(\hat{\Lambda}_{2l+1,n} - \Lambda^*_{2l+1}) + \Lambda^*_{2l+1}, \quad l \in \mathcal{N},
\]

where \( \hat{\Lambda}_{l,n} \) are the Fourier coefficients of the EMF.
Efficiency over a ball

- A kernel-type estimator

\[
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A kernel-type estimator

\[ \tilde{\Lambda}_n(t) = \int_0^T K_n(s - t)(\hat{\Lambda}_n(s) - \Lambda_*(s))ds + \Lambda_*(t). \]

with a kernel satisfying the condition

\[ n \sup_{l \geq 1} \left| \sqrt{\frac{T}{2} K_{2l,n} - 1} \right|^2 \longrightarrow 0, \]
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\[ n \sup_{l \geq 1} \left| \frac{n}{2} K_{2l,n} - 1 \right|^2 \longrightarrow 0, \]

- is asymptotically efficient over a ball

\[ \lim_{n \rightarrow +\infty} \sup_{\Lambda \in \mathcal{B}(R)} \left( E_{\Lambda} \left| \sqrt{n}(\tilde{\Lambda}_n - \Lambda) \right|^2 - \int_0^\tau \Lambda(t)dt \right) = 0. \]
Efficiency over a compact set

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- It belongs to $\Sigma(R)$

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  \[ \Sigma(R) = \{ \Lambda : \| \lambda - \lambda^* \|^2 \leq R, \Lambda^*(\tau) = \Lambda(\tau) \}. \]
- A kernel-type estimator with the kernel satisfying
  \[ n \sup_{l \geq 1} \left| \frac{\sqrt{\frac{\tau}{2}} K_{2l,n} - 1}{\frac{2\pi l}{\tau}} \right|^2 \rightarrow 0, \]
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• is asymptotically efficient over \( \Sigma(R) \)

\[ \lim_{n \to +\infty} \sup_{\Lambda \in \Sigma(R)} \left( E_{\Lambda} \| \sqrt{n} (\tilde{\Lambda}_n - \Lambda) \|^2 - \int_0^\tau \Lambda(t) dt \right) = 0. \]
Example of another as. effective estimator

- Consider a kernel

\[ K(u) \geq 0, \ u \in \left[ -\frac{T}{2}, \frac{T}{2} \right], \int_{-\frac{T}{2}}^{\frac{T}{2}} K(u)\,du = 1, \]

\[ K(u) = K(-u), \ K(u) = K(u + k\tau), \ u \in \left[ -\frac{T}{2}, \frac{T}{2} \right], \ k \in \mathbb{Z}. \]
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\[ K_n(u) = \frac{1}{h_n} K \left( \frac{u}{h_n} \right) \mathbb{1} \left\{ |u| \leq \frac{\tau}{2} h_n \right\} \]

satisfy the previous condition and hence
Example of another as. effective estimator

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satisfy the previous condition and hence

- the corresponding kernel-type estimator

\[ \tilde{\Lambda}_n(t) = \int_0^t K_n(s - t)(\hat{\Lambda}_n(s) - \Lambda_*(s)) ds + \Lambda_*(t). \]

is as. efficient over \( \Sigma(R) \).
Second order efficiency

- How to compare as. efficient estimators?
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- The first step would be to find the rate of convergence in

$$\lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}} \left( E_{\Lambda} \left| \sqrt{n}(\bar{\Lambda}_n - \Lambda) \right|^2 - \int_0^T \Lambda(t) \, dt \right) \geq 0,$$

• that is, the sequence

$$\gamma_n \to +\infty \text{ s.t. } \lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}} \gamma_n \left( E_{\Lambda} \left| \sqrt{n}(\bar{\Lambda}_n - \Lambda) \right|^2 - \int_0^T \Lambda(t) \, dt \right) \geq C.$$
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- Then, to construct an estimator which attains this bound.
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- Then, to construct an estimator which attains this bound.
- Calculate the constant \( C \).
Second order estimation was introduced by [Golubev G.K. and Levit B.Ya., 1996] in the distribution function estimation problem.
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For other models second order efficiency was proved by [Dalalyan A.S. and Kutoyants Yu.A., 2004], [Golubev G.K. and Härdle W., 2000].
Related works

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- Asymptotic efficiency in non-parametric estimation problems was done for the first time in [Pinsker M.S., 1980].
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Asymptotic efficiency in non-parametric estimation problems was done for the first time in [Pinsker M.S., 1980],

where the analogue of the inverse of the Fisher information in non-parametric estimation problem was calculated (Pinsker’s constant).
Main theorems

- Introduce

\[ \mathcal{F}_{m}^{\text{per}}(R, S) = \left\{ \Lambda(\cdot) : \int_{0}^{\tau} [\lambda^{(m-1)}(t)]^2 \, dt \leq R, \, \Lambda(0) = 0, \, \Lambda(\tau) = S \right\} \]

where \( R > 0, \, S > 0, \, m > 1 \), are given constants.
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where \( R > 0, S > 0, m > 1 \), are given constants.

• For all estimators \( \tilde{\Lambda}_n(t) \) of the mean function \( \Lambda(t) \), following lower bound holds

\[
\lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}_m(R, S)} n^{\frac{1}{2m-1}} \left( E_{\Lambda} \| \sqrt{n}(\tilde{\Lambda}_n - \Lambda) \|^2 - \int_0^\tau \Lambda(t) dt \right) \geq -\Pi,
\]
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• For all estimators \( \bar{\Lambda}_n(t) \) of the mean function \( \Lambda(t) \), following lower bound holds

\[
\lim_{n \to +\infty} \sup_{\Lambda \in F_m(R,S)} n^{\frac{1}{2m-1}} \left( E_\Lambda \| \sqrt{n}(\bar{\Lambda}_n - \Lambda) \|^2 - \int_0^\tau \Lambda(t) dt \right) \geq -\Pi,
\]

where

\[
\Pi = \Pi_m(R, S) = (2m - 1)R \left( \frac{S}{\pi R} \frac{m}{(2m - 1)(m - 1)} \right)^{\frac{2m}{2m-1}},
\]

plays the role of the Pinsker’s constant.
Second order as. efficient estimator

- Consider

\[
\Lambda^*_n(t) = \hat{\Lambda}_0,n\phi_0(t) + \sum_{l=1}^{N_n} \tilde{K}_{l,n} \hat{\Lambda}_{l,n}\phi_l(t),
\]

where \( \{\phi_l\}_{l=0}^{+\infty} \) is the trigonometric cosine basis, \( \hat{\Lambda}_{l,n} \) are the Fourier coefficients of the EMF w.r.t. this basis and

\[
\tilde{K}_{l,n} = \left(1 - \left|\frac{\pi l}{\tau}\right|^m \alpha^*_n\right)_+, \quad \alpha^*_n = \left[\frac{S}{nR\pi} \frac{m}{(2m-1)(m-1)} \right]^{\frac{m}{2m-1}} ,
\]

\[
N_n = \frac{\tau}{\pi} (\alpha^*_n)^{-\frac{1}{m}} \approx C n^{\frac{1}{2m-1}} , \quad x_+ = \max(x, 0), \ x \in \mathbb{R}.
\]
Second order as. efficient estimator

- Consider

\[ \Lambda_n^*(t) = \hat{\Lambda}_{0,n} \phi_0(t) + \sum_{l=1}^{N_n} \tilde{K}_{l,n} \hat{\Lambda}_{l,n} \phi_l(t), \]

where \( \{ \phi_l \}_{l=0}^{+\infty} \) is the trigonometric cosine basis, \( \hat{\Lambda}_{l,n} \) are the Fourier coefficients of the EMF w.r.t. this basis and

\[ \begin{align*}
\tilde{K}_{l,n} &= \left( 1 - \left| \frac{\pi l}{\tau} \right|^m \alpha_n^* \right)_+, \\
\alpha_n^* &= \left[ \frac{S}{nR \pi} \frac{m}{(2m-1)(m-1)} \right]^{\frac{m}{2m-1}}, \\
N_n &= \frac{\tau}{\pi} (\alpha_n^*)^{-\frac{1}{m}} \approx C n^{\frac{1}{2m-1}}, \\
x_+ &= \max(x, 0), \ x \in \mathbb{R}.
\end{align*} \]

- The estimator \( \Lambda_n^*(t) \) attains the lower bound described above, that is,

\[ \lim_{n \to +\infty} \sup_{\Lambda \in \mathcal{F}_m(R,S)} n^{\frac{1}{2m-1}} \left( \mathbb{E}_{\Lambda} \left\| \sqrt{n}(\bar{\Lambda}_n - \Lambda) \right\|^2 - \int_0^\tau \Lambda(t) \, dt \right) = -\Pi. \]
Sketch of the proof

- Proof consists of several steps:
Sketch of the proof

- Proof consists of several steps:

  - Worst Error
  - Ellipsoid
  - maximal mean error
  - Ellipsoid
  - maximal mean
  - Ellipsoid
  - heavy functions of maximizing prior
  - Ellipsoid
  - shrunken prior
First step

Reduce the minimax problem to a Bayes risk maximization problem

\[
\sup_{\Lambda \in \mathcal{F}} \left( m(R, S) \left( E_{\Lambda} ||\bar{\Lambda}_n - \Lambda||^2 - E_{\Lambda} ||\hat{\Lambda}_n - \Lambda||^2 \right) \right) \\
\geq \sup_{Q \in \mathcal{P}} \int_{\mathcal{F}} m(R, S) \left( E_{\Lambda} ||\bar{\Lambda}_n - \Lambda||^2 - E_{\Lambda} ||\hat{\Lambda}_n - \Lambda||^2 \right) dQ.
\]

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10 June, 2016 Rennes
First step

- Reduce the minimax problem to a Bayes risk maximization problem

\[
\sup_{\Lambda \in \mathcal{F}_m^{(\text{per})} (R,S)} \left( E_{\Lambda} \| \tilde{\Lambda}_n - \Lambda \|^2 - E_{\Lambda} \| \hat{\Lambda}_n - \Lambda \|^2 \right) \geq \\
\sup_{Q \in \mathcal{P}} \int_{\mathcal{F}_m^{(\text{per})} (R,S)} \left( E_{\Lambda} \| \tilde{\Lambda}_n - \Lambda \|^2 - E_{\Lambda} \| \hat{\Lambda}_n - \Lambda \|^2 \right) dQ.
\]
Second step

In the maximization problem replace the set of probabilities \( P(F) \) concentrated on \( \mathcal{F}(\mathbf{r}, \mathbf{s}) \) with the set of probabilities

\[
\sup_{Q \in \mathcal{P}(F)} \int_{\mathcal{F}(\mathbf{r}, \mathbf{s})} m(\mathbf{r}, \mathbf{s}) \left( \mathbb{E} \Lambda || \bar{\Lambda} - \Lambda ||^2 - \mathbb{E} \Lambda || \hat{\Lambda} - \Lambda ||^2 \right) dQ,
\]

by the set of probabilities \( E(F) \) concentrated on \( \mathcal{F}(\mathbf{r}, \mathbf{s}) \) in mean.
Second step

- In the maximization problem replace the set of probabilities $\mathcal{P}(\mathcal{F})$ concentrated on $\mathcal{F}_m^{(per)}(R, S)$

$$
\sup_{Q \in \mathcal{P}(\mathcal{F})} \int_{\mathcal{F}_m^{(per)}(R, S)} \left( E_\Lambda ||\bar{\Lambda}_n - \Lambda||^2 - E_\Lambda ||\hat{\Lambda}_n - \Lambda||^2 \right) dQ,
$$
Second step

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\]

- by the set of probabilities $\mathbb{E}(\mathcal{F})$ concentrated on $\mathcal{F}_m^{(per)}(R, S)$ in mean.
Third step

Replace the ellipsoid by the least favorable parametric family (heavy functions)

\[
\sup_{\Lambda \theta \in F(\text{per})} m(R, S) \int \Theta (E_{\theta} |\bar{\Lambda}_n - \Lambda_{\theta}|^2 - E_{\theta} |\hat{\Lambda}_n - \Lambda_{\theta}|^2) dQ.
\]
Third step

- Replace the ellipsoid by the least favorable parametric family (heavy functions)

\[
\sup_{\Lambda_\theta \in \mathcal{F}_m^{(\text{per})} (R, S)} \int_{\Theta} \left( \mathbf{E}_\theta ||\tilde{\Lambda}_n - \Lambda_\theta||^2 - \mathbf{E}_\theta ||\hat{\Lambda}_n - \Lambda_\theta||^2 \right) dQ.
\]
Fourth step

- Shrink the heavy functions and the least favorable prior distribution to fit the ellipsoid

\[ Q\{\theta: \Lambda_{\theta}/\in F_{\text{per}}(R, S)\} = o(n^{-2}). \]
Fourth step

- Shrink the heavy functions and the least favorable prior distribution to fit the ellipsoid

\[ Q\{\theta : \Lambda_\theta \notin \mathcal{F}_m^{(per)}(R, S)\} = o(n^{-2}). \]
Further work

• What can be done or what had to be done?
Further work

- What can be done or what had to be done?
- The condition $\Lambda(\tau) = S$ in the definition of the set $\mathcal{F}_m^{(per)}(R, S)$ have to be replaced by $\Lambda(\tau) \leq S$. The last one cannot be thrown out since with a notation $\pi_j(t) = X_j(t) - \Lambda(t)$ we get

$$\hat{\Lambda}_n(t) = \Lambda(t) + \frac{1}{n} \sum_{j=1}^{n} \pi_j(t), \text{ data=signal+“noise”}$$

and the variance of the noise is $\frac{1}{n}\Lambda(t)$. (Simultaneous estimation of the function and its variance).
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- Adaptive estimation-construct an estimator that does not depend on $m, S, R$. 
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and the variance of the noise is $\frac{1}{n} \Lambda(t)$. (Simultaneous estimation of the function and its variance).
- Adaptive estimation-construct an estimator that does not depend on $m, S, R$.
- Consider other models or formulate a general result for non-parametric LAN.
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