Local existence proofs for the boundary value problem for static spherically symmetric Einstein-Yang-Mills fields with compact gauge groups

Todd A. Oliynyk §

H.P. Künzle ¶

Department of Mathematical Sciences, University of Alberta
Edmonton, Canada T6G 2G1

Abstract

We prove local existence and uniqueness of static spherically symmetric solutions of the Einstein-Yang-Mills equations for an arbitrary compact semisimple gauge group in the so-called regular case. By this we mean the equations obtained when the rotation group acts on the principal bundle on which the Yang-Mills connection takes its values in a particularly simple way (the only one ever considered in the literature). The boundary value problem that results for possible asymptotically flat soliton or black hole solutions is very singular and just establishing that local power series solutions exist at the center and asymptotic solutions at infinity amounts to a nontrivial algebraic problem. We discuss the possible field equations obtained for different group actions and solve the algebraic problem on how the local solutions depend on initial data at the center and at infinity.

1 Introduction

Over the last dozen years much has been learned about the classical interaction of Yang-Mills fields with the gravitational field of Einstein’s general relativity. Most investigations have concentrated on Yang-Mills fields with the gauge group SU(2) starting with Bartnik and Mckinnon’s discovery of globally regular and asymptotically flat numerical solutions. Their global existence was analytically proved and many further properties like stability of these particle-like or soliton solutions and the corresponding black hole solutions were investigated numerically as well as analytically. Moreover, many different matter fields can be minimally coupled to the gravitational and Yang-Mills fields, and corresponding spherically symmetric solutions have been, mostly numerically, but sometimes also analytically studied. We refer for the (hundreds of) references to the review article [26].

Some similar phenomena were found for special models with gauge groups SU(n) for n > 2 [11, 13, 16, 18], and the general static spherically symmetric equations for general compact gauge groups were derived already quite early [1, 3].

For larger gauge groups than SU(2) the notion of spherical symmetry is no longer straightforward enough for a simple ansatz to work. Instead one needs to consider the possible actions of the symme-
try group $SO(3)$ or $SU(2)$ by automorphisms of principal bundles over space-times whose structure group $G$ is the gauge group of the Yang-Mills field. A conjugacy class of such automorphisms is characterized by a generator $\Lambda_0$ which is an element of a Cartan subalgebra $h$ of the complexified Lie algebra $g$ of $G$. Mostly one restricts consideration to fields which are regular at the center or, for black hole fields, to those for which the Yang-Mills-curvature falls off sufficiently fast at infinity. In these cases these are called the regular models. They also correspond to the “no magnetic charge” case in \cite{1}. For these group actions the element $\Lambda_0$ of $h$ must be an $A_1$-vector or defining vector of an $\mathfrak{sl}(2)$-subalgebra of $g$.

That there is a remarkable variety of possible actions was shown by Bartnik \cite{2} for the case where $G$ is any group with Lie algebra $\mathfrak{su}(n)$. More generally, for arbitrary semisimple Lie algebras these $A_1$-vectors were classified by Mal’cev \cite{20} and Dynkin \cite{10} and can now also be obtained more conveniently from the theory of nilpotent orbits \cite{9}.

One of these classes of actions of the symmetry group is somewhat distinguished. It corresponds to a principal $A_1$-vector in Dynkin’s terminology and we will call it a principal action. To our knowledge almost all work for larger gauge groups has been done for this case \cite{13, 16, 19, 22}. For a slightly bigger class of actions, the “generic” class in \cite{7} which we will call regular, the $A_1$-vector lies in the interior of a fundamental Weyl chamber. Brodbeck and Straumann \cite{7, 8} proved that all regular static asymptotically flat solutions are unstable against time dependent perturbations. They were able to do this without establishing existence or any properties of these solutions.

While it is easy to show that, at least for the regular case, some global solutions exist, namely those which arise by scaling from some imbedded $SU(2)$ solutions, only isolated, mostly numerical, results have been obtained about more general global solutions for the principal $SU(n)$ actions for $n = 3, 4, 5$ \cite{13, 14, 16, 19, 22}.

The purpose of this paper is to discuss the classification of all the regular actions of $SU(2)$ by automorphisms of $G$-principal bundles over spherically symmetric static space-times and to analyze the resulting Einstein-Yang-Mills field equations to the extent of establishing that the singular boundary value problem obtained for the globally regular and asymptotically flat solutions is well defined “at both ends”, namely at the center or the black hole horizon and at infinity. The local solutions that we obtain near these points are actually analytic. Consequently, there exists convergent powerseries representations for these solutions at least for small distances from the center, the black hole horizon and infinity. Essentially we generalize the results of \cite{19} from the principal action on $SU(n)$-bundles to regular actions on bundles with (simply connected) semisimple compact structure groups. Although this represents only a first step in an analysis of possible (non scaled) global solutions establishing these local existence theorems is already quite complicated. It is worthwhile to note, that if any of the local solutions can be extended to a global one, then the results of Brodbeck and Straumann \cite{8} apply and show that the solution must be unstable.

For all these regular models it turns out that the Yang-Mills potential can be chosen (i.e. suitably gauged) to depend only on $\ell$ real-valued functions of a radial coordinate $r$ where $\ell$ is the rank of the Lie algebra of the gauge group. In addition, the metric will be given by two more functions of $r$. These $(\ell + 2)$ functions satisfy a nonlinear system of ordinary differential equations which has singularities at $r = 0$, when $r \to \infty$, and at the horizon where $r = r_H$, say. We need to analyze these singularities to determine the “initial conditions” for these functions and the number of free parameters that can be chosen when solving the equations numerically, for example, by the method of shooting to a meeting point. In this paper we will only establish what these parameters are, we will not solve the equations numerically.

There are many models for which the $A_1$-vector $\Lambda_0$ is on the boundary of a Weyl chamber. To our knowledge almost no results have been obtained for them, but we have reason to believe that some of our methods may also be useful for these irregular models.

In section 2 we review the description of the class of static spherically symmetric models and in
section 3, we show, starting from the field equations, that the special class of models we call regular can be reduced to the principal case for imbedded semisimple groups. We discuss the initial value problems somewhat informally in section 4, where we derive the relatively complicated way in which a solution depends on parameters chosen at the endpoints of the \( r \)-interval. In section 5, we extend some elementary facts that are well known for \( SU(2) \)-solutions to general compact \( G \). Finally, the major part of this paper consists of the proofs, divided into section 6 containing algebraic lemmas and the proof of the local existence theorems for the differential equation system in section 7.

2 Classes of spherically symmetric Yang-Mills connections

Since there is no natural action of the symmetry group on the principal bundle we need to consider all possibilities, i.e. all conjugacy classes of actions of \( SO(3) \), or for simplicity, \( SU(2) \) by automorphisms on principal \( G \)-bundles \( P \) over space-time \( M \) which project onto isometries of \( M \) with orbits diffeomorphic to 2-spheres. We assume throughout that \( G \) is a compact semisimple connected and simply connected Lie group.

Then these conjugacy classes are in one-to-one correspondence with integral elements \( \Lambda_0 \) of the closed fundamental Weyl chamber \( W(S) \) belonging to some basis \( S \) of the roots of \( g \) for some chosen Cartan subalgebra \( h \). Here \( g = (g_0)_c \) stands for the complexification of the Lie algebra \( g_0 \) of the structure group \( G \) of \( P \). If \( \{\tau_i\} \) is a standard basis of the Lie algebra \( su(2) \) such that \([\tau_i, \tau_j] = \epsilon_{ij}^k \tau_k \) then \( \Lambda_0 \) may be chosen such that

\[
\Lambda_0 = 2i \lambda(\tau_3)
\]

where \( \lambda \) is the (induced Lie algebra) homomorphism from the isotropy group \( I_{x_0} \) of the \( SU(2) \)-action on \( M \) at \( x_0 \in M \) determined by \( k \cdot u_0 = u_0 \cdot \lambda(k) \forall k \in I_{x_0} \) if \( u_0 \in \pi^{-1}(x_0) \).

Wang’s theorem [17, 27] on connections that are invariant under actions transitive on the base manifold has been adapted to spherically symmetric space-time manifolds by Brodbeck and Straumann [6]. They show that in a Schwarzschild type coordinate system \((t, r, \theta, \phi)\) and the metric

\[
g = -NS^2dt^2 + N^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.1}
\]

a gauge can always be chosen such that the Yang-Mills-connection form is locally given by

\[
A = \tilde{A} + \hat{A}
\]

where \( \hat{A} \) is a 1-form on the quotient space parametrized by the \( r \) and \( t \) coordinates and

\[
\hat{A} = \Lambda_1 d\theta + (\Lambda_2 \sin \theta + \Lambda_3 \cos \theta)d\phi \tag{2.2}
\]

where \( \Lambda_3 = -\frac{i}{2}\Lambda_0 \) is the constant isotropy generator and \( \Lambda_1 \) and \( \Lambda_2 \) are functions of \( r \) and \( t \) that satisfy

\[
[\Lambda_2, \Lambda_3] = \Lambda_1 \quad \text{and} \quad [\Lambda_3, \Lambda_1] = \Lambda_2. \tag{2.3}
\]

Since we only consider static fields we can assume that \( \Lambda_1 \) and \( \Lambda_2 \) depend only on \( r \). Moreover, we will also concentrate on the “magnetic” case and assume that the part \( \hat{A} \) of the gauge potential which contributes “electric” or “Coulomb” terms vanishes, i.e. we put

\[
\hat{A} = 0.
\]

This condition is not as restrictive as it seems. For, as proved in [6], it also follows in the regular case (defined below) if the field is smooth at the center \( r = 0 \) and falls off sufficiently fast at infinity.
So far we still have infinitely many possible actions of $SU(2)$ on the principal bundle, namely one for each element in the intersection $\overline{W(S)} \cap I$ of the fundamental Weyl chamber and the integral lattice $I := \ker(\exp h)$. However, since we want the Yang-Mills-connection to be regular also at a center ($r = 0$, defined as a connected set of fixed points of the $SU(2)$-action on $M$) and/or the Yang-Mills-field to fall off in an asymptotic region (have no magnetic charge according to [7]) we must have

$$[\Omega^0_1, \Omega^0_2] = \Lambda_3$$

and/or

$$[\Omega^\infty_1, \Omega^\infty_2] = \Lambda_3$$

where

$$\Omega_i^{0,\infty} := \lim_{r \to 0,\infty} \Lambda_i(r), \quad (i = 1, 2).$$

In other words, in these limits there must exist a Lie algebra homomorphism of $su(2)$ into $\mathfrak{g}_0$. This is shown most easily by observing that the Einstein equations would otherwise lead to infinite pressure or density at a center.

Since $\Lambda_3$ is constant, however, equations (2.4) represent not only conditions on $\Lambda_1(r)$ and $\Lambda_2(r)$, but also on $\Lambda_3$ and hence on $\Lambda_0$ which must now be the generating (or defining) vector of an $\mathfrak{sl}(2)$ (i.e. $A_1$) subalgebra of $\mathfrak{g}$. (If both limits exist it then also follows that there must be an automorphism of $\mathfrak{g}$ taking $\Omega_i^0$ into $\Omega_i^\infty$.) The set of these so-called $A_1$-vectors, however, is finite (and in one-to-one correspondence with conjugacy classes of $\mathfrak{sl}(2)$ subalgebras). It has been studied and tabulated by Mal’cev [20] and Dynkin [10] and is described by so-called weighted Dynkin diagrams (called characteristics in [10]), where to each simple root in the diagram is associated a number from the set $\{0, 1, 2\}$. (See [9] for a more recent exposition). These numbers represent the values of the simple roots on the generating vector $\Lambda_0$ chosen such that it lies in $\overline{W(S)}$.

Thus these tables serve as a classification of all the spherically symmetric “magnetic” Einstein-Yang-Mills models which are regular at the center and/or obey the standard fall-off conditions at infinity for any given compact gauge group.

### 3 Field equations and reduction of the regular models

The field equations are well known. We state them here in a form following [6] for the static regular case only, where $\Lambda_0$ is an $A_1$-vector. Let the space-time metric $g$ be given by (2.1) and the Yang-Mills-potential $A = \hat{A}$ by (2.2). Define, in addition to $\Lambda_0$, $\Lambda_\pm := \mp \Lambda_1 - i \Lambda_2$

so that the Wang equations (2.3) become

$$[\Lambda_0, \Lambda_\pm] = \pm 2 \Lambda_\pm.$$ 

(3.1)

Then $\Lambda_+(r)$ and $\Lambda_-(r)$ are $\mathfrak{g}$-valued functions, $\Lambda_0$ a (constant) vector in the fundamental Weyl chamber of $\mathfrak{h}$ and $\{\Lambda_0, \Lambda_+, \Lambda_-\}$ is a standard triple in the limit $r \to 0$ or $r \to \infty$ for the Lie algebra $\mathfrak{g}$. Now $\mathfrak{h}$ is the Cartan subalgebra of the complexified Lie algebra $\mathfrak{g}$, i.e. $\mathfrak{h} = \mathfrak{h}_0 \oplus i \mathfrak{h}_0$, where $\mathfrak{h}_0$ is the real Cartan subalgebra of a compact real form $\mathfrak{g}_0$ of $\mathfrak{g}$, and we choose conventions such that the conjugation operator $c : \mathfrak{g} \to \mathfrak{g}$ satisfies $c(X + iY) = X - iY \forall X, Y \in \mathfrak{g}_0$. Then

$$\Lambda_- = -c(\Lambda_+)$$

(3.2)

so that the dependent variables consist only of $N$, $S$ and the components of $\Lambda_+$. 

The field equations now reduce to
\[
m' = (NG + r^{-2}P), \tag{3.3}
\]
\[
S^{-1}S' = 2G, \tag{3.4}
\]
\[
r^{2}N \Lambda'' + 2(m - r^{-1}P) \Lambda' + F = 0, \tag{3.5}
\]
\[
[\Lambda, \Lambda'] - [\Lambda', \Lambda] = 0 \tag{3.6}
\]
where \(\dot{\cdot} := \frac{d}{dr}\) and
\[
N := 1 - \frac{2m}{r}, \quad G := \frac{1}{2}(\Lambda'_+, \Lambda'_-), \quad P := -\frac{1}{2}(\hat{F}, \hat{F}), \tag{3.7}
\]
\[
\hat{F} := \frac{1}{2}(\Lambda_0 - [\Lambda_+, \Lambda_-]),
\[
F := -i[\hat{F}, \Lambda_+]. \tag{3.8}
\]
Here \((, )\) is an invariant inner product on \(\mathfrak{g}\). It is determined up to a factor on each simple component of a semi-simple \(\mathfrak{g}\) and induces a norm \(|.|\) on (the Euclidean) \(\mathfrak{h}\) and therefore its dual. We choose these factors so that \((, )\) is a positive multiple of the Killing form on each simple component. If they are chosen such that the length of the long simple roots are all 1 then the equations will agree with those in [19] for the principal \(SU(n)\) case.

Note that \(G \geq 0\) and also \(P \geq 0\). This follows from (3.2), \(c(\hat{F}) = \hat{F}\), and the fact that \(\langle X | Y \rangle := -(c(X), Y)\) is a Hermitian inner product on \(\mathfrak{g}\) (cf. (6.1)). Energy density, radial and tangential pressure are then given by

\[
4\pi e = r^{-2}(NG + r^{-2}P), \quad 4\pi p_r = r^{-2}(NG - r^{-2}P), \quad 4\pi p_\theta = r^{-4}P. \tag{3.9}
\]

We now choose a Chevalley-Weyl basis of \(\mathfrak{g}\) using mostly the notation of [12]. Let \(R\) be the set of roots in \(\mathfrak{h}^*\), \(S = \{\alpha_1, \ldots, \alpha_\ell\}\) a base of \(R\) (\(\ell\) being the rank of \(\mathfrak{g}\)), define
\[
\langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{|\beta|^2},
\]
\[
(t_\alpha, X) := \alpha(X) \forall X \in \mathfrak{h},
\]
and
\[
\mathfrak{h}_\alpha := \frac{2t_\alpha}{|\alpha|^2}.
\]

Then \(\{\mathfrak{h}_i := \mathfrak{h}_\alpha, \mathfrak{e}_\alpha, \mathfrak{e}_{-\alpha} | i = 1, \ldots, \ell, \alpha \in R\}\) is a basis of \(\mathfrak{g}\) corresponding to the decomposition
\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})
\]
\((R^+\) being the set of positive roots with respect to the base \(S)\) for which we choose the conventions
\[
[\mathfrak{e}_\alpha, \mathfrak{e}_{-\alpha}] = \mathfrak{h}_\alpha, \quad [\mathfrak{e}_{-\alpha}, \mathfrak{e}_{-\beta}] = -[\mathfrak{e}_\alpha, \mathfrak{e}_\beta], \quad (\mathfrak{e}_\alpha, \mathfrak{e}_{-\alpha}) = \frac{2}{|\alpha|^2}. \tag{3.10}
\]

Now it follows directly [10] from the defining relations
\[
[\mathfrak{e}_0, \mathfrak{e}_\pm] = \pm 2\mathfrak{e}_\pm, \quad [\mathfrak{e}_+, \mathfrak{e}_-] = \mathfrak{e}_0
\]
of an \( \mathfrak{sl}(2) \)-subalgebra \( \text{span}\{e_0, e_{\pm}\} \) of \( \mathfrak{g} \), with the help of
\[
[\mathfrak{h}, e_\alpha] = \alpha(\mathfrak{h}) e_\alpha,
\]
that \( e_0 \) can only be an \( A_1 \)-vector provided
\[
\alpha(e_0) = 2 \quad \text{for some } \alpha \in R.
\]
Thus, if we let
\[
\Lambda_0 = \sum_{i=1}^\ell \lambda_i h_i \in \mathfrak{h}.
\]
then equations (3.1) imply that
\[
\Lambda_+(r) = \sum_{\alpha \in S_\lambda} w_\alpha(r) e_\alpha
\]
where
\[
S_\lambda := \{ \alpha \in R \mid \alpha(\Lambda_0) = 2 \}
\]
is a set of roots depending only on the homomorphism \( \lambda \) or, equivalently, on the coefficients \( \lambda_i \) in (3.11).
Similarly we have
\[
\Lambda_-(r) = \sum_{\alpha \in S_\lambda} v_\alpha(r) e_{-\alpha},
\]
but by (3.2) and the fact that complex conjugation maps
\[
c : h_i \mapsto -h_i, e_\alpha \mapsto -e_{-\alpha}
\]
it follows that
\[
v_\alpha(r) = \overline{w}_\alpha(r).
\]
Our system is thus determined once the two real functions \( m(r) \) and \( S(r) \) and the complex functions \( w_\alpha(r) \) for all \( \alpha \in S_\lambda \) are known.
If we now substitute (3.13) into equations (3.3) to (3.8) we need to calculate the Lie brackets between the various \( e_\alpha \) for \( \alpha \in S_\lambda \). In general, this may produce many more equations than dependent variables. On the other hand the Yang-Mills-potential \( \hat{A} \) determined by \( \Lambda_+ \) still contains some gauge freedom. It is not known, at present, whether there is any systematic method to solve this system of equations.
However, as Brodbeck and Straumann [7] have observed, there are special symmetry actions for which this system of equations is much simpler, in fact, very similar to the principal \( SU(n) \) case. This happens when \( \Lambda_0 \) is a vector in the open fundamental Weyl chamber of \( \mathfrak{h} \). They call these models generic, but since, as we will see, they are really a small minority of all possible ones we will call them regular.
In the following \( \Lambda_0 \) is not required to be an \( A_1 \)-vector.

**Theorem 1 (Brodbeck/Straumann [7]).** If \( \Lambda_0 \) is in the open Weyl chamber \( W(S) \) then the set \( S_\lambda \) is a \( \Pi \)-system, i.e. satisfies
(i) if $\alpha, \beta \in S_\lambda$ then $\alpha - \beta \not\in R$,

(ii) $S_\lambda$ is linearly independent

and is therefore the base of a root system $R_\lambda$ which generates a Lie subalgebra $g_\lambda$ of $g$ spanned by

$\{h_\alpha, e_\alpha, e_{-\alpha} | \alpha \in R_\lambda\}$.

Moreover, if $h_\lambda := \text{span}\{h_\alpha | \alpha \in S_\lambda\}$ and $h_\lambda^\perp := \bigcap_{\alpha \in S_\lambda} \ker \alpha$ then

$$b = b_\lambda^\perp + h_\lambda^\perp$$

with $A_0^\perp = \sum_{\alpha \in R_\lambda^+} h_\alpha$.

If $\Lambda_0$ is an $A_1$-vector then $\Lambda_0^\perp = 0$ (but $h_\lambda^\perp$ need not be trivial).

In particular, $A_0^\perp$ is twice the lowest weight vector of $h_\lambda$ and we have by the definitions of $S_\lambda$ and $h_\lambda^\perp$

$$\alpha(A_0^\perp) = 2 \quad \text{and} \quad \alpha(A_0^\perp) = 0 \quad \forall \alpha \in S_\lambda.$$

We will from now on only consider the regular case.

First, $\Lambda_+$ can be treated as a $g_\lambda$-valued function,

$$\Lambda_+(r) = \sum_{j=1}^{\ell_\lambda} w_j(r)\tilde{e}_j$$

where now $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{\ell_\lambda}\}$ is the base of $S_\lambda$ and $\tilde{e}_j := e_{\tilde{\alpha}_j}$. Moreover, $A_0^\perp = \sum_{j=1}^{\ell_\lambda} \lambda_j^\perp \tilde{h}_j$ with $\tilde{h}_j := h_{\tilde{\alpha}_j}$.

Then, by (3.7) noting also that $\tilde{\alpha}_j(A_0^\perp) = 0$,

$$\hat{F} = \frac{i}{2} \left( \sum_{j=1}^{\ell_\lambda} \lambda_j^\perp \tilde{h}_j + A_0^\perp - \left[ \sum_{j=1}^{\ell_\lambda} w_j \tilde{e}_j, \sum_{k=1}^{\ell_\lambda} \tilde{w}_k \tilde{e}_k \right] \right) = \frac{i}{2} \left( \sum_{j=1}^{\ell_\lambda} (\lambda_j^\perp - |w_j|^2) \tilde{h}_j + A_0^\perp \right)$$

(3.14)

where (3.10) was used and the fact that differences of two simple roots are not roots which implies that

$$[e_\alpha, e_{-\beta}] = 0 \quad \forall \alpha, \beta \in S_\lambda, \alpha \neq \beta.$$  

(3.15)

Substituting this expression into (3.8) gives

$$F = \frac{i}{2} \sum_{j,k=1}^{\ell_\lambda} w_j (\tilde{\alpha}_j, \tilde{\alpha}_k)(\lambda_k^\perp - |w_k|^2) \tilde{e}_j + \frac{1}{2} \sum_{j=1}^{\ell_\lambda} w_j [A_0^\perp, \tilde{e}_j].$$

(3.16)

But since $[A_0^\perp, \tilde{e}_k] = \tilde{\alpha}_k(A_0^\perp)\tilde{e}_k = 0$, in view of the definition of $S_\lambda^+$, the last term vanishes. Equation (3.5) therefore becomes

$$r^2 N w_j'' + 2(m - r^{-1} P) w_j' + \frac{1}{2} \sum_{k=1}^{\ell_\lambda} w_j c_{jk} (\lambda_k^\perp - |w_k|^2) = 0$$

where we have introduced

$$c_{jk} := (\tilde{\alpha}_j, \tilde{\alpha}_k)$$
for the Cartan matrix of \( g_\Lambda \) and where now

\[
P = \frac{1}{8} \sum_{j,k=1}^{\ell \Lambda} (\lambda_j^w - |w_j|^2) h_{jk}(\lambda_k^w - |w_k|^2) + |\Lambda_0^j|^2 \quad \text{with} \quad h_{jk} := \frac{2(\tilde{\alpha}_j, \tilde{\alpha}_k)}{|\tilde{\alpha}_j|^2},
\]

(3.17)

\[
G = \sum_{j=1}^{\ell \Lambda} |w'_j|^2
\]

(3.18)

Finally, (3.6) simply becomes in view of (3.10)

\[
\sum_{j,k=1}^{\ell \Lambda} (w_j w'_k - w'_j w_k) [e_{\tilde{\alpha}_j}, e_{-\tilde{\alpha}_k}] = \sum_{j=1}^{\ell \Lambda} (w_j w'_j - w'_j w_j) h_j = 0
\]

(3.19)

so that the phase of \( w_j \) is constant and may be chosen to be zero by a gauge transformation. One can thus assume that the \( w_j(r) \) are real-valued functions.

It remains to determine the subalgebra \( g_\Lambda \) for a given \( A_1 \)-vector \( \Lambda_0 \) in the open fundamental Weyl chamber.

First, we note that for a semisimple group for which the Cartan subalgebra \( h \) splits into an orthogonal sum \( h = \bigoplus h_k \) the decomposition in Theorem 2 splits into corresponding decompositions of each of the \( h_k \). So we need only investigate the regular actions of simple Lie groups.

Now the \( A_1 \)-vector in the Cartan subalgebra \( h \) of an semisimple Lie algebra \( g \) is uniquely given by the numbers

\[
\chi = (\chi_1, \ldots, \chi_{\ell}) := (\alpha_1(\Lambda_0), \ldots, \alpha_{\ell}(\Lambda_0)),
\]

(3.20)

called the *characteristic* in [11]. It is known [2, 10] that if \( \Lambda_0 \) is in the closed fundamental Weyl chamber then \( \chi_k \in \{0, 1, 2\} \), and all possible such characteristic have been found and tabulated. It is clear from the definition of \( S_\chi \) in (3.12) that \( \chi_k = 2 \forall k \) for \( h_\Lambda \). Such \( A_1 \)-vectors define principal \( A_1 \)-subalgebras and thus principal actions of \( SU(2) \) on the bundle. We now have

**Theorem 2.**

(i) The possible regular \( A_1 \)-subalgebras of simple Lie algebras consist of the principal subalgebras of all Lie algebras \( A_\ell, B_\ell, C_\ell, D_\ell, E_\ell, F_4 \) and \( G_2 \) and of those subalgebras of \( A_\ell = \mathfrak{so}(\ell+1) \) with even \( \ell \) corresponding to partitions \( \ell+1-k \) for any integer \( k = 1, \ldots, \ell/2 \) or, equivalently, characteristic \((22..211..112..22)\) (2k ‘1’s in the middle, 2’s in all other positions).

(ii) The Lie algebra \( \mathfrak{g}_\Lambda \) is equal to \( g \) in the principal case, and for \( A_\ell \) with even \( \ell \) equal to \( A_{\ell-1} \) for \( k = 1 \) and to \( A_{\ell-k} \oplus A_{\ell-1} \) for \( k = 2, \ldots, \ell/2 \).

(iii) In the principal case \( h_\Lambda = h \). For all \( A_1 \)-subalgebras of \( A_\ell \) with even \( \ell \) the orthogonal space \( h_\Lambda^\perp \) is one-dimensional.

**Proof.** Part (i) follows quite easily from the discussion and the tables in [11] (Sections 5.3 and 4.4). For part (ii) that \( h_\Lambda = h \) in the principal case is obvious. To compute \( S_\chi \) for a given \( \ell = 2m \) and given \( k > 0 \) note that all positive roots of \( A_\ell \) are of the form \( \sum_{p=1}^{k} \alpha_p \) for \( 1 \leq j \leq k \leq 2m \) so that using that \( \alpha_i(\Lambda_0) = 2 \) for \( i = 1, \ldots, m-k \) and \( i = m+k+1, \ldots, 2m \) and \( \alpha_i(\Lambda_0) = 1 \) otherwise one sees that

\[
S_\chi = \bigcup_{i=1}^{m-k} \alpha_i \cup \bigcup_{j=1}^{2k-1} (\alpha_{m-k+j} + \alpha_{m-k+j+1}) \cup \bigcup_{i=m+k+1}^{2m} \alpha_i.
\]

(3.21)
Recalling that for $A_\ell$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

it is seen immediately that the Cartan matrix for $S_\lambda$ is the one for $A_{\ell-1}$ if $k = 1$ while it takes a simple reordering of the roots to verify the statement in (ii) for $k = 2, \ldots, \ell/2$.

(iii) That $h^1 = h$ in the principal case is obvious from the definition and that $\dim S^1 = 1$ follows from the observation that $\alpha(X) = 0 \forall \alpha \in S_\lambda$ amounts to $2m - 1$ linearly independent equations according to (3.21).

In summary, we have shown that all regular models can be reduced to those with the principal action for semisimple gauge groups. Also the term $\Lambda_0^+ \perp 0$ occurring in (3.14), (3.16) and (3.17) can now be dropped.

4 Constructing local solutions regular at the center, horizon or at infinity

So far we have shown that the static spherically symmetric and magnetic EYM equations for the regular action reduce to those for the principal action for semi-simple gauge groups. (We now drop the index $\lambda$ from $g, h$, etc.) They consist of (3.4), which can be integrated easily once the other equations are solved, and

$$m' = (NG + r^{-2} P),$$

$$r^2 Nw''_j + 2(m - r^{-1} P)w'_j + \frac{1}{2} \sum_{k=1}^{\ell} w_j c_{jk}(\lambda_k - w_k^2) = 0$$

where the $w_k$ are real-valued functions of $r$, $(c_{ij}) := (\langle \alpha_j, \alpha_k \rangle)$ is the Cartan matrix of the reduced structure group, and

$$P = \frac{1}{8} \sum_{j,k=1}^{\ell} (\lambda_j - w_j^2) h_{jk}(\lambda_k - w_k^2) \quad (4.3)$$

$$G = \sum_{j=1}^{\ell} \frac{w_j^2}{|\alpha_j|^2} \quad (4.4)$$

$$h_{jk} = \frac{2c_{jk}}{|\alpha_j|^2} \quad (4.5)$$

$$\lambda_j = 2 \sum_{k=1}^{\ell} (c^{-1})_{jk} \quad (4.6)$$

The expressions for components $\lambda_k$ of the $A_1$-vector $\Lambda_0$ follow from (3.24) and the fact that for the principal action $\chi_k = 2 \forall k$.

In this section we will discuss the general problem of finding solutions that are regular at the center or at the horizon and have an appropriate fall off as $r \to \infty$. Proofs will be given later in sections 3 and 6.

Equations (4.1) and (4.2) are very similar to the corresponding ones in the principal $SU(n)$ case analyzed in detail in [19]. So we can expect most of those results to generalize. First of all, when the dependent variables $m$ and $w_k$ are expanded in power series in terms of $r$ at $r = 0$, in terms of $r - r_H$ at $r = r_H$, and in terms of $r^{-1}$ at infinity (under the assumption that all the quantities are
finite in these limits) then \((4.1)\) and \((4.2)\) yield a system of algebraic equations. For example, at \(r = 0\) with \(f(r) = \sum_{k=0}^{\infty} f_k r^k\) we find

\[
m_{k+1} = \frac{1}{k+1} \left( g_k + p_{k+2} - 2 \sum_{h=2}^{k-2} m_{k-h} g_h \right),
\]

(4.7)

\[
\ell \sum_{j=1}^{\ell} \left( A_{ij} - k(k+1) \delta_{ij} \right) w_{j,k+1} = b_{i,k}
\]

(4.8)

for \(k = 0, 1, 2, \ldots\) where

\[
A_{ij} := w_{i,0} c_{ij} w_{j,0}
\]

(4.9)

and the \(b_{i,k}\) are complicated expressions involving lower order terms. For the lowest order terms we find

\[
m_0 = m_1 = m_2 = 0, \quad w_{1,0}^2 = \lambda_i, \quad w_{i,1} = 0.
\]

(4.10)

That \(r = 0\) is a singular point for the system \((4.1),(4.2)\) manifests itself in the fact that the initial data at \(r = 0\) for regular solutions are not simply the values of the functions \(m, w_i\) and \(w_i'\) but that some of these values are restricted like in \((4.10)\) and some higher order coefficients in the power series for the \(w_i\) remain arbitrary, namely for those orders \(k\) for which the matrix \(A = (A_{ij})\) has eigenvalue \(k(k+1)\). It now turns out that the eigenvalues of \(A\) are precisely of this form for certain integer values of \(k\). In fact, for the simple Lie algebras we can calculate the spectrum directly from the Cartan matrix and find the values given in Table 1. The proof for the classical Lie algebras of arbitrary rank follows from the properties of the root system and results at the end of section 6.

Table 1: The eigenvalues of the coefficient matrix \(A\) for the simple Lie algebras are given by the set \(\text{spec}(A) = \{ k(k+1) | k \in \mathcal{E} \}\). For the classical Lie algebras the table entry gives \(k_j\) for \(j = 1, 2, \ldots = \ell = \text{rank}(g)\). Note that \(k = 1\) belongs to \(\mathcal{E}\) for all Lie algebras.

| Lie algebra | \(\mathcal{E}\) |
|-------------|----------------|
| \(A_\ell\)  | \(j\)          |
| \(B_\ell\)  | \(2j - 1\)     |
| \(C_\ell\)  | \(2j - 1\)     |
| \(D_\ell\)  | \[
\begin{aligned}
2j - 1 & \quad \text{if } j \leq (\ell + 2)/2 \\
\ell - 1 & \quad \text{if } j = (\ell + 2)/2 \\
2j - 3 & \quad \text{if } j > (\ell + 2)/2
\end{aligned}
\] |
| \(E_6\)     | \(1, 4, 5, 7, 8, 11\) |
| \(E_7\)     | \(1, 5, 7, 9, 11, 13, 17\) |
| \(E_8\)     | \(1, 7, 11, 13, 17, 19, 23, 29\) |
| \(F_4\)     | \(1, 5, 7, 11\) |
| \(G_2\)     | \(1, 5\) |

The eigenspaces for the simple Lie algebras are all onedimensional except for \(D_\ell\) where certain 'middle' eigenvalues occur twice. For semisimple Lie algebras the matrix \(A\) will be a direct sum of those for the simple components and thus may have multiple eigenvalues.
It is now clear that a formal power series solution of equations (4.1) and (4.2) is well defined and contains $\ell$ free parameters provided equation (4.8) can be solved, i.e. provided the vector $b_k := (b_{1,k}, \ldots, b_{\ell,k})$ lies in the left kernel of $(\Lambda - k(k+1)I)$. Since $b_k$ is a very complicated expression this is cumbersome to prove in general. In [19] the proof for $G = SU(n)$ was achieved using properties of a class of orthogonal polynomials, an approach that does not easily generalize to other groups. In sections 6 and 7 we present a proof that depends directly on the root structure of the Lie algebra $g$ treated as an $\mathfrak{sl}(2, \mathbb{C})$-module.

The structure of the recursion relations for the power series of regular solutions in $r^{-1}$ at infinity is very similar to the one at $r = 0$. At the remaining singular point of (4.2), namely at a regular horizon ($N(r_H) = 0, N'(r_H) > 0$), however, the only conditions on initial values turn out to be some inequalities.

Calculating the formal power series is indeed necessary to start off numerical integration when searching for global regular solutions. For an existence and uniqueness proof, however, it is more convenient to recast the equations in a form to which the following (slight generalization of a) theorem by Breitenlohner, Forgács and Maison [16] applies.

**Theorem 3.** The system of differential equations

$$
\frac{du_i}{dt} = t^{\mu_i} f_i(t, u, v) \quad i = 1, \ldots, m \\
\frac{dv_j}{dt} = -h_j(u)v_j + t^{\nu_j} g_j(t, u, v) \quad j = 1, \ldots, n
$$

where $\mu_i, \nu_j$ are integers greater than 1, $f_i$ and $g_j$ analytic functions in a neighborhood of $(0, c_0, 0) \in \mathbb{R}^{1+m+n}$, and $h_j : \mathbb{R}^m \to \mathbb{R}$ functions, positive in a neighborhood of $c_0 \in \mathbb{R}^m$, has a unique analytic solution $t \mapsto (u_i(t), v_j(t))$ such that

$$u_i(t) = c_i + O(t^{\mu_i}) \quad \text{and} \quad v_j(t) = O(t^{\nu_j})$$

for $|t| < R$ for some $R > 0$ if $|c - c_0|$ is small enough. Moreover, the solution depends analytically on the parameters $c_i$.

**Proof.** By the standard method solving the differential equation with initial data is replaced by finding a fixed point for the map $T : (u, v) \mapsto (\tilde{u}, \tilde{v})$ with

$$
\tilde{u}_i(t) = c_i + \int_0^t t^{\mu_i-1} f_i[\tau, u(\tau), v(\tau)] \, d\tau \\
\tilde{v}_j(t) = t^{\nu_j} \int_0^t \tau^{\nu_j-1} g_j[\tau, u(\tau), v(\tau)] \, d\tau
$$

where $\kappa_j := h_j(c)$ and $\hat{g}(j)(t, u, v) := g_j(t, u, v) - t^{-\nu_j}[h_j(u) - h_j(c)]v_j$. To show that $T$ is a contracting map on a suitable Banach space one can use a method very similar to the one in [19].

To bring the system (4.1) and (4.2) into a form that satisfies the hypotheses of theorem 3 it is necessary to make a suitable transformation of the variables $m$ and $w_j$. The proofs that this can be done are basically equivalent to showing that the formal power series exist and are given in section 6. We then have

**Theorem 4.** The system (4.1) and (4.2) has an analytic solution for small $r$ of the form

$$w_i(r) = w_{i,0} + \sum_{j=1}^\ell C_{ij} r^{\kappa_j+1} u_j(r), \quad i = 1, \ldots, \ell$$
where \( C = (C_{ij}) \) is a nonsingular matrix whose \( j \)-th column is an eigenvector to eigenvalue \( k_j(k_j + 1) \) of the matrix \( A \). The solution is uniquely determined by the initial values \( u_j(0) = \beta_j \) for arbitrary \( \beta \). The function \( m(r) \) is then determined and satisfies \( m(r) = O(r^3) \) for small \( r \).

Note that the \( w_{i,0} \) are determined up to the sign by (4.10). From (7.18), we see that solutions from theorem 4 satisfy \( P = O(r^4) \) and \( G = O(r^2) \). It follows that for these solutions all physical quantities such as the pressure and mass density are finite at \( r = 0 \).

The situation is rather similar for solutions analytic in \( r^{-1} \) near infinity. We have, with the same matrix \( C \),

**Theorem 5.** The system (4.1) and (4.2) has an analytic solution for small \( z = r^{-1} \) of the form

\[
\begin{align*}
w_i(r) &= w_{i,\infty} + \sum_{j=1}^{\ell} C_{ij} r^{-k_j} u_j(r^{-1}), & i = 1, \ldots, \ell, \\
m(r) &= m_{\infty} + O(r^{-1})
\end{align*}
\]

The solution is uniquely determined by the initial values \( u_j(0) = \alpha_j \) and \( m_{\infty} \) for arbitrary \( \alpha_j \) and \( m_{\infty} \).

Again \( w_{i,\infty} \) is determined up to the sign by \( w_{i,\infty}^2 = \lambda_i \). An overall sign in \( w_i(r) \) does not affect the Yang-Mills field nor the geometry and physics. But \( w_{i,0} \) and \( w_{i,\infty} \) may have the same or different signs for global solutions.

Finally we have the corresponding theorem for local solutions near a regular horizon.

**Theorem 6.** The system (4.1) and (4.2) has a solution analytic in \( t = r - r_H \) for small \( t \) at a regular horizon, i.e. where \( N(r_H) = 0 \) and \( N'(r_H) > 0 \). The solution is uniquely determined by the values of \( w_j(r_H) \) which must be chosen such that

\[
N'(r_H) = \frac{1}{r_H} - \frac{2}{r_H^2} P(r_H) > 0
\]

or, equivalently,

\[
2P(r_H) = \frac{1}{4} \sum_{i,j=1}^{\ell} (\lambda_i - w_i(r_H)) h_{ij} (\lambda_j - w_j(r_H)) < r_H^2.
\]

### 5 Elementary properties and scaled solutions

The observation already made in 3 for \( SU(2) \) and generalized to \( SU(n) \) that global solutions, if they exist, must be bounded by their values at infinity (or zero) is easily extended to the regular case for arbitrary \( G \).

**Theorem 7.** If a solution \((m, w_1, \ldots, w_\ell)\) is defined and \( C^2 \) in the connected outer domain \( D := \{ r| 0 \leq r_H \leq r < \infty \} \) (where \( N(r) > 0 \)) and if

\[
m(r) = m_{\infty} + O(1/r) \quad \text{and} \quad w_j(r) = w_{j,\infty} + O(1/r) \quad \text{as} \quad r \to \infty
\]

then

\[
w_j(r)^2 \leq w_{j,\infty}^2 = \lambda_j \quad \forall \ r \in \overset{\circ}{D}.
\]

Moreover, if \( G \) is a simple group and \( w_j(r_1) = w_{j,\infty} \) for some \( j \) and for some \( r_1 \in \overset{\circ}{D} \) then \( w_j(r) = w_{j,\infty} \forall \ r \in D \), \( m = \text{const} \), the Yang-Mills field vanishes, and the metric is the Schwarzschild one.

If \( G \) is semisimple and \( w_j(r_1) = w_{j,\infty} \) for some \( j \) and for some \( r_1 \in \overset{\circ}{D} \) then the field equations reduce to those of the subgroup of \( G \) obtained by deleting from the Cartan subalgebra \( h \) of \( g \) the simple component in which \( h_j \) lies.
Proof. Let \( v_j := w_j^2 \). Then \( v_j(r) \geq 0 \ \forall \ r \) and (5.2) gives
\[
2r^2 N v_j'' - r^2 N v_j' + 4(m - r^{-1} P) v_j' + 2v_j^2 \sum_{k=1}^\ell c_{jk} (\lambda_k - v_k) = 0. \tag{5.1}
\]
Let \( V_j := \sup_{r \in D} v_j(r) \). Then \( V_j > 0 \) because the asymptotic value of \( v_j(r) \) is \( \lambda_j > 0 \). Now assume that \( v_j(r_j) = V_j \) for some \( r_j \in D \). Then \( v_j(r_j) \) is an absolute maximum so that \( v_j'(r_j) = 0 \) and \( v_j''(r_j) \leq 0 \). It follows from (5.1) that \( \sum_{j=1}^\ell c_{ij} (\lambda_j - v_j(r_j)) \geq 0 \) which in view of (4.6) is equivalent to
\[
\sum_{j=1}^\ell c_{ij} v_j(r_j) \leq 2 \ \forall \ i
\]
or
\[
v_i(r_i) \leq 1 + \frac{1}{2} \sum_{j \neq i} (-c_{ij}) v_j(r_i) \leq 1 + \frac{1}{2} \sum_{j \neq i} (-c_{ij}) \sup_{r \in D} v_j(r)
\]
whence \( V_i \leq 1 + \frac{1}{2} \sum_{j \neq i} (-c_{ij}) V_j \) or
\[
\sum_{j=1}^\ell c_{ij} V_j \leq 2 \ \forall \ i.
\]
(Note that for all Cartan matrices \( c_{ii} = 2 \) and \( c_{ij} \leq 0 \) if \( i \neq j \).)

This last set of inequalities, however, can be multiplied with the inverse Cartan matrix since the latter has only positive entries. Using (4.6) again then gives \( 0 \leq V_j \leq \lambda_j \ \forall \ j \), thus \( V_j = \lambda_j \) since \( \lambda_j \) is the asymptotic value.

Suppose now that \( v_i(r_i) = \lambda_i \) for some \( i \) and some \( r_i \in D \). Then we find from (5.1)
\[
r_* N(r_*) v_i''(r_*) = \lambda_i \sum_{j \neq i} (-c_{ij})(\lambda_j - v_j(r_*)) \geq 0
\]
which contradicts that \( v_i(r) \) has a maximum at \( r_* \) unless \( v_j''(r_*) = 0 \) and, in the case of a simple Lie algebra, the neighboring \( v_j \) also assume their maximal values. (For a simple Lie algebra there is a \( c_{ij} < 0 \) for some \( j \neq i \) for any \( i \).) It follows that all \( v_j(r_*) = \lambda_j \) for all \( j \) for which the root \( \alpha_j \) is in the same simple component of \( \mathfrak{h}^* \). However, if all \( v_j(r_*) = \lambda_j \) and thus \( v_j'(r_*) = 0 \ \forall \ j \) then the initial conditions for the differential equations (5.1) are all trivial and since \( r_* \) is not a singular point it follows by the uniqueness of the solution that it must by the one for which \( v_j(r) \equiv \lambda_j \). It then also follows from (4.1) that \( m = \text{const} \) so that the Yang-Mills field vanishes and the geometry is the one of the Schwarzschild solution.

Theorem 4 shows among other things that for a given semisimple gauge group \( G \) and a given group action (characterized by \( \Lambda_0 \)) there may be special solutions that reduce the YM-connection to a subgroup of \( G \) that is a product of some of the simple factors of \( G \). Somewhat similarly, since the group \( SU(2) \) can be isomorphically imbedded in every compact (simply connected) semisimple or simple Lie group the Bartnik-Mckinmon solution [3] can be obtained as a special solution for all the models considered here. The following special BM-solution for arbitrary compact \( G \) was already obtained in [5].

Consider the gauge group \( G \) and the symmetry group action (characterized by \( \Lambda_0 \)) fixed and such that \( \Lambda_0 \) is regular so that the field equations are given by (3.4), (4.1) and (4.2). Select any \( \Omega_+ \).
such that the set \( \{ \Lambda_0, \Omega_+, \Omega_- \} \) is a standard triple with \( c(\Omega_+) = -\Omega_- \) and let \( \Lambda_+(r) = u(r)\Omega_+ \) or, equivalently, \( u_1(r) = u_{1,\infty}u(r) \). Then the field equations become

\[
\begin{align*}
m' &= g_0(Nu'^2 + \frac{1}{2}r^{-2}(1 - u^2)^2), \\
r^2Nu'' + (2m - g_0r^{-1}(1 - u^2)^2)u' + g_0u(1 - u^2) &= 0, \\
S^{-1}S' &= 2g_0r^{-1}u'^2.
\end{align*}
\]

where \( g_0 = \frac{1}{4} \sum_{i,j} \lambda_i h_{ij} \). By introducing a new radial variable \( x := r g_0^{-1/2} \) one sees easily that (5.2)-(5.4) reduce to the well studied equations for the \( SU(2) \)-Einstein-Yang-Mills fields.

Since \( \Lambda_0 \) fixes the conjugacy class of the symmetry group action on the bundle different choices of \( \Omega_+ \) will lead to isomorphic gauge connections, namely reductions of the \( G \)-connection to an \( SU(2) \)-connection on the principal bundle for the particular space-time. They are thus physically equivalent.

In view of the existence theorems for the \( G = SU(2) \) case [4, 24, 25] it now follows that the system (4.1) and (4.2) always admits some global solutions

**Theorem 8.** There exists a countably infinite family of globally regular solutions of the Einstein-Yang-Mills-equations for any simply connected compact semisimple gauge group \( G \) on a static spherically symmetric asymptotically flat space-time diffeomorphic to \( \mathbb{R}^4 \). Similarly, for any \( r_H > 0 \) there exists an infinite family of asymptotically flat black hole solutions with black hole radius \( r_H \).

### 6 The Lie algebra \( g \) as an \( sl(2, \mathbb{C}) \) submodule

In this section we collect all of the algebraic results needed to prove theorems 4 and 5. Introduce a non-degenerate Hermitian inner product \( \langle \ | \rangle : g \times g \rightarrow \mathbb{C} \) by

\[
\langle X|Y \rangle := -\langle c(X), Y \rangle \quad \forall \ X, Y \in g ,
\]

recalling that \( c : g \rightarrow g \) is the conjugation operator determined by the compact real form \( g_0 \). Then \( \langle \ | \rangle \) restricts to a real positive definite inner product on \( g_0 \). From the invariance properties of \( \langle \ , \ \rangle \) it follows that \( \langle \ | \rangle \) satisfies

\[
\begin{align*}
\langle X|Y \rangle &= \langle Y|X \rangle , \\
\langle c(X)c(Y) \rangle &= \langle X|Y \rangle , \\
\langle [X,c(Y)]|Z \rangle &= \langle X|[Y,Z] \rangle
\end{align*}
\]

for all \( X, Y, Z \in g \). Treating \( g \) as a \( \mathbb{R} \)-linear space by restricting scalar multiplication to multiplication by reals, we can introduce a positive definite inner product \( \langle \langle \ | \rangle \rangle : g \times g \rightarrow \mathbb{R} \) on \( g \) defined by

\[
\langle \langle X|Y \rangle \rangle := \text{Re} \langle X|Y \rangle \quad \forall \ X, Y \in g .
\]

Let \( \| \cdot \| \) denote the norm induced on \( g \) by \( \langle \langle \ | \rangle \rangle \), i.e.

\[
\| X \| = \sqrt{\langle \langle X|X \rangle \rangle} \quad \forall \ X \in g .
\]

From the above properties satisfied by \( \langle \ | \rangle \), it straightforward to verify that \( \langle \langle \ | \rangle \rangle \) satisfies

\[
\begin{align*}
\langle \langle X|Y \rangle \rangle &= \langle \langle Y|X \rangle \rangle , \\
\langle \langle c(X)c(Y) \rangle \rangle &= \langle \langle X|Y \rangle \rangle , \\
\langle \langle [X,c(Y)]|Z \rangle \rangle &= \langle \langle X|[Y,Z] \rangle \rangle
\end{align*}
\]
for all \(X, Y, Z \in g\).

Let \(\Omega_+ , \Omega_- \in g\) be two vectors such that

\[
[\Lambda_0, \Omega_{\pm}] = \pm 2\Omega_{\pm} , \quad [\Omega_+, \Omega_-] = \Lambda_0 \quad \text{and} \quad c(\Omega_+) = -\Omega_- .
\]

Then \(\text{span}_C \{\Lambda_0, \Omega_+, \Omega_-\} \cong \mathfrak{sl}(2, \mathbb{C})\). The dot notation will often be used to denote the adjoint action of \(\text{span}_C \{\Lambda_0, \Omega_+, \Omega_-\}\) on \(g\), i.e.

\[
X.Y := \text{ad}(X)(Y) \quad \forall X \in \text{span}_C \{\Lambda_0, \Omega_+, \Omega_-\}, \ Y \in g .
\]

Because \(\Lambda_0\) is a semisimple element, \(\text{ad}(\Lambda_0)\) is diagonalizable and it follows from \(\mathfrak{sl}(2)-\)representation theory \([12]\) that the eigenvalues are integers. Let \(V_n\) denote the eigenspaces of \(\text{ad}(\Lambda_0)\), i.e.

\[
V_n := \{ X \in g \mid \Lambda_0.X = nX \} \quad n \in \mathbb{Z} .
\]

It also follows from \(\mathfrak{sl}(2, \mathbb{C})-\)representation theory that if \(X \in g\) is a highest weight vector of the adjoint representation of \(\text{span}_C \{\Lambda_0, \Omega_+, \Omega_-\}\) with weight \(n\), and we define \(X_{-1} = 0\), \(X_0 = X\) and \(X_j = (1/j!)\Omega_-^j.X_0\) \((j \geq 0)\), then

\[
\Lambda_0.X_j = (n - 2j)X_j ,
\]

\[
\Omega_- .X_j = (j + 1)X_{j+1} ,
\]

\[
\Omega_+ .X_j = (n - j + 1)X_{j-1} \quad (j \geq 0) .
\]

**Proposition 1.** There exists \(M\) highest weight vectors \(\xi^1, \xi^2, \ldots, \xi^M\) for the adjoint representation of \(\text{span}_C \{\Lambda_0, \Omega_+, \Omega_-\}\) on \(g\) that satisfy

(i) the \(\xi^i\) have weights \(2k_j\) where \(j = 1, 2, \ldots, M\) and \(1 = k_1 \leq k_2 \leq \cdots \leq k_M\),

(ii) if \(V(\xi^j)\) denotes the irreducible submodule of \(g\) generated by \(\xi^j\), then the sum \(\sum_{j=1}^M V(\xi^j)\) is direct,

(iii) if \(\xi^j_l = (1/l!)\Omega_-^l .\xi^j\) then

\[
c(\xi^j_l) = (-1)^j \xi^j_{2k_j - l} ,
\]

(iv) \(M = |S_\lambda|\) and the set \(\{\xi^j_{k_{j-1}} \mid j = 1, 2, \ldots M\}\) forms a basis for \(V_2\) over \(\mathbb{C}\).

**Proof.** (i) and (ii): The conjugation operator \(c\) satisfies

\[
c([X, Y]) = [c(X), c(Y)] \quad \forall X, Y \in g .
\]

Because \(\Lambda_0 \in \mathfrak{h}_0\),

\[
c(\Lambda_0) = -\Lambda_0 .
\]

Using \((3.3)\), \((6.6)\), and \((6.7)\), it is easy to see that

\[
c \circ \text{ad}(\Omega_+)^n = (-1)^n \text{ad}(\Omega_+)^n \circ c \quad \text{for every } n \in \mathbb{Z}_{\geq 0} \quad \text{and} \quad c \circ \text{ad}(\Lambda_0) = -\text{ad}(\Lambda_0) \circ c .
\]

As usual, define the Casimir operator \(C\) by

\[
C = \frac{1}{2} \text{ad}(\Lambda_0)^2 + \text{ad}(\Omega_+)\text{ad}(\Omega_-) + \text{ad}(\Omega_-)\text{ad}(\Omega_+) .
\]
Then \( \mathfrak{g} \) can be decomposed as follows

\[
\mathfrak{g} = \bigoplus_p V(s_p, v^p)
\]

(6.9)

where \( V(s_p, v^p) \) is a highest weight module generated by the highest weight vector \( v^p \) of weight \( s_p \), and it has the property

\[
\mathcal{C}_{V(s_p, v^p)} = \left( \frac{1}{2} s_p^2 + s_p \right) \text{id}_{V(s_p, v^p)} \quad \forall \ p .
\]

(6.10)

From (6.8) it follows that \( \mathcal{C} \circ c = c \circ \mathcal{C} \). Using this result and (6.10), we see that

\[
c(V(s_p, v^p)) \subseteq V(s_p, v^p) \quad \forall \ p .
\]

(6.11)

Let \( \{ s_{p_1}, s_{p_2}, \ldots, s_{p_M} \} \) be the set of weights from the decomposition (6.9) that are even and greater than zero. We will assume that they are ordered so that \( s_{p_1} \leq s_{p_2} \leq \ldots \leq s_{p_M} \). Define \( k_j \) as \( s_{p_j} / 2 \). Then the \( k_j \) are positive integers that satisfy \( k_1 \leq k_2 \leq \ldots \leq k_M \). Note that \( k_1 = 1 \) because \( \Omega_+ \) is a highest weight vector with weight 2. To simplify notation, set \( V^j := v^{p_j} \). As before with highest weight vectors (see (6.4)), we let \( v^j = (1/!)\Omega^j \). Define

\[
\xi^j = \begin{cases} 
iv^j + c(iv^j) & \text{if } c(iv^j) = -c(v^j) \\
v^j + c(v^j) & \text{otherwise}
\end{cases}
\]

(6.12)

for \( j = 1, 2, \ldots, M \). Then straightforward calculation using (6.8) and (6.4) shows that \( \Lambda_0, \xi^j = 2k_j \xi^j \) and \( \Omega_+, \xi^j = 0 \) for \( j = 1, 2, \ldots, M \). This implies that the \( \xi^j \) are all highest weight vectors of weight \( 2k_j \). Let \( V(\xi^j) \) denote the irreducible submodule generated by \( \xi^j \). From (6.11) and (6.12) it is clear that \( \xi^j \subseteq V(2k_j, V^j) \) and hence \( V(\xi^j) = V(2k_j, V^j) \). Thus the decomposition (6.9) shows that the sum \( \sum_{j=1}^{M} V(\xi^j) \) is direct.

(iii): The relationship (6.3) follows from (6.8), (6.12), and (6.4).

(iv): Because the numbers \( 2k_1, 2k_2, \ldots, 2k_M \) exhaust all the positive even weights and the sum \( \sum_{j=1}^{M} V(\xi^j) \) is direct, it follows from \( \mathfrak{sl}(2, \mathbb{C}) \)-representation theory that \( \{ \xi^j \mid j = 1, 2, \ldots, M \} \) is a basis over \( \mathbb{C} \) for \( V_2 \). But \( \{ e_\alpha \mid \alpha \in S_\lambda \} \) is also a basis over \( \mathbb{C} \) for \( V_2 \). Therefore we must have \( M = |S_\lambda| \). \( \square \)

Define an \( \mathbb{R} \)-linear operator \( A : \mathfrak{g} \rightarrow \mathfrak{g} \) by

\[
A = \frac{1}{2} \text{ad}(\Omega_+) \circ (\text{ad}(\Omega_-) + \text{ad}(\Omega_+) \circ c).
\]

(6.13)

Proposition 2. The \( \mathbb{R} \)-linear operator \( A \) is symmetric with respect to the inner product \( \langle | \rangle \), i.e. \( \langle A(X)|Y \rangle = \langle X|A(Y) \rangle \forall X, Y \in \mathfrak{g} \).

Proof. From (3.2) and the properties (1.3) of the inner product \( \langle | \rangle \), it is not hard to show that

\[
\langle \Omega_+, \Omega_-, X \rangle = \langle X|\Omega_+, \Omega_- \rangle \quad \text{and} \quad \langle \Omega_+, \Omega_+, c(X) \rangle = \langle X|\Omega_+, \Omega_+, c(Y) \rangle
\]

for every \( X, Y \in \mathfrak{g} \). From the definition of \( A \), it is then obvious that \( \langle A(X)|Y \rangle = \langle A|A(Y) \rangle \) for every \( X, Y \in \mathfrak{g} \). \( \square \)

Lemma 1.

\[
A(V_2) \subseteq V_2
\]

(6.14)

Proof. It follows from \( \mathfrak{sl}(2, \mathbb{C}) \)-representation theory that \( \Omega_+, V_n \subseteq V_{n+2} \). From (6.8) it is clear that \( c(V_n) \subseteq V_n \). Thus \( \Omega_+, \Omega_- V_2 \subseteq V_2 \) and \( \Omega_+, \Omega_+, c(V_2) \subseteq V_2 \) which implies that \( A(V_2) \subseteq V_2 \). \( \square \)
This proposition shows that $A$ restricts to an operator on $V_2$. We denote this operator by

$$A_2 := A|_{V_2}. \quad (6.15)$$

Label the integers $k_j$ from proposition 3 as follows

$$1 = k_{J_1} = k_{J_1+1} = \cdots = k_{J_1+m_1-1} < k_{J_2} = k_{J_2+1} = \cdots = k_{J_2+m_2-1} < \cdots < k_{J_I} = k_{J_I+1} = \cdots = k_{J_I+m_I-1},$$

where $J_1 = 1$, $J_l + m_l = J_{l+1}$ for $l = 1, 2, \ldots, I$ and $J_{I+1} = M - 1$. Define

$$k_l := k_{J_l} \quad l = 1, 2, \ldots, I. \quad (6.16)$$

The set $\{\xi^l_{k_l-1} \mid j = 1, 2, \ldots M\}$ forms a basis over $\mathbb{C}$ of $V_2$ by proposition 3 (iv). Therefore the set of vectors $\{X^l_s, Y^l_s \mid l = 1, 2, \ldots, I; s = 0, 1, \ldots, m_l - 1\}$ where

$$X^l_s := \begin{cases} \xi^{l+s}_{k_l-1} & \text{if } k_l \text{ is odd} \\ \iota^{l+s}_{k_l-1} & \text{if } k_l \text{ is even} \end{cases} \quad \text{and} \quad Y^l_s := iX^l_s, \quad (6.17)$$

forms a basis of $V_2$ over $\mathbb{R}$. By proposition 3 we know that $A$ is symmetric and therefore diagonalizable. This forces $A_2$ to also be diagonalizable. The next lemma shows that $\{X^l_s, Y^l_s \mid l = 1, 2, \ldots, I; s = 0, 1, \ldots, m_l - 1\}$ is in fact an eigenbasis of $A_2$.

**Lemma 2.**

$$A_2(X^l_s) = k_l(k_l + 1)X^l_s \quad \text{and} \quad A_2(Y^l_s) = 0 \quad \text{for } l = 1, 2, \ldots, I \quad \text{and} \quad s = 0, 1, \ldots, m_l - 1. \quad (6.18)$$

**Proof.** Using the formulas (6.4) and proposition 3 (iii) it is easy to show that

$$A(\xi^l_{k_l-1}) = \frac{1}{2}k_l(k_l + 1) (1 + (-1)^{k_l-1}) \xi^l_{k_l-1} \quad \text{and} \quad A(i\xi^l_{k_l-1}) = \frac{i}{2}k_l(k_l + 1) (1 + (-1)^{k_l}) i\xi^l_{k_l-1} \quad \text{for } j = 1, 2, \ldots M. \quad (6.19)$$

An immediate consequence of this lemma is that $\text{spec}(A_2) = \{0\} \cup \{k_l(k_l + 1) \mid j = 1, 2, \ldots I\}$ and $m_j$ is the dimension of the eigenspace corresponding to the eigenvalue $k_j(k_j + 1)$. Note that $I$ is the number of distinct positive eigenvalues of $A_2$.

Define

$$E_0^l = \text{span}_\mathbb{R}\{Y^l_s \mid s = 0, 1, \ldots, m_l - 1\} \quad \text{and} \quad E_+^l = \text{span}_\mathbb{R}\{X^l_s \mid s = 0, 1, \ldots, m_l - 1\}, \quad (6.19)$$

and

$$E_0 = \bigoplus_{l=1}^I E^l_0, \quad E_+ = \bigoplus_{l=1}^I E^l_+. \quad (6.20)$$

Then $E_0 = \ker(A_2)$ and $E_+^l$ is the eigenspace of $A_2$ corresponding to the eigenvalue $k_l(k_l + 1)$. Moreover, using proposition 3 (iv), we see that

$$V_2 = E_0 \oplus E_+. \quad (6.21)$$

**Lemma 3.** Suppose $X \in V_2$. Then $X \in \bigoplus_{q=1}^I E^q_0 \oplus E^q_+$ if and only if $\Omega^k X = 0$. 

Proof. From the formulas (6.4), we get
\[
\Omega_l^{q-1} \cdot \xi_{k_l} = \begin{cases} 0 & \text{if } q > k_l \\ \frac{d(q, k_l)}{d(q, k_l - 1)} & \text{if } q \leq k_l \end{cases},
\]
where \(d(q, r) = \frac{(q + r)!}{(r + 1)!}\). This implies that
\[
\Omega_l^{q-1} \cdot X_p^q = \begin{cases} 0 & \text{if } l > k_q \\ \frac{\beta_q}{d(l, k_q)} \xi_{k_q} & \text{if } l \leq k_q \end{cases},
\]
and
\[
\Omega_l^{q-1} \cdot Y_p^q = \begin{cases} 0 & \text{if } l > k_q \\ \frac{\beta_q}{d(l, k_q)} \xi_{k_q} & \text{if } l \leq k_q \end{cases},
\]
where
\[
\beta_q = \begin{cases} 1 & \text{if } k_q \text{ is odd} \\ i & \text{if } k_q \text{ is even} \end{cases}.
\]

Suppose \(X \in V_2 = \bigoplus_{q=1}^I E_0^q \oplus E_+^q\). Then there exists real constants \(a_{qp}\) and \(b_{qp}\) such that
\[
X = \sum_{q=1}^I \sum_{p=0}^{m_q-1} (a_{qp} Y_p^q + b_{qp} X_p^q).
\]
Suppose \(\Omega_{k_q}^l \cdot X = 0\). Then (6.22), (6.23), and (6.24) imply that
\[
\sum_{q=1}^I \sum_{p=0}^{m_q-1} \left( a_{qp} \beta_q d(l, k_q) \xi_{k_q} + b_{qp} \beta_q d(k_q + 1, k_q) \xi_{k_q} \right) = 0.
\]
But the set of vectors
\[
\left\{ \xi_{k_q} \xi_{k_q}^{l+1}, \xi_{k_q} \xi_{k_q}^{l+1} \mid q = l + 1, l + 2, \ldots, I \right\}
\]
is linearly independent over \(\mathbb{R}\). Therefore \(X = \sum_{q=1}^I \sum_{p=0}^{m_q-1} (a_{qp} Y_p^q + b_{qp} X_p^q)\) which implies that \(X \in \bigoplus_{q=1}^I E_0^q \oplus E_+^q\).
Conversely, suppose \(X \in \bigoplus_{q=1}^I E_0^q \oplus E_+^q\). The \(X\) can be written in the form (6.24) and it is easy using (6.22) and (6.23) to verify that \(\Omega_{k_q}^l \cdot X = 0\).

Lemma 4. Suppose \(X \in V_2\). Then \(X \in \bigoplus_{q=1}^I E_0^q \oplus E_+^q\) if and only if \(\Omega_{k_q}^l \cdot c(X) = 0\).

Proof. Proved in a similar fashion as lemma 3.

Lemma 5. Let \(\tilde{\sim} : \mathbb{Z}_{\geq -1} \to \{1, 2, \ldots, I\}\) be the map defined by
\[
\tilde{\sim} 1 = \tilde{0} = 1 \text{ and } \tilde{s} = \max\{ l \mid k_l \leq s \} \text{ if } s > 0.
\]
Then
(i) \(k_s \leq s\) for every \(s \in \mathbb{Z}_{\geq 0}\).
(ii) \( k_s \leq s < k_{s+1} \) for every \( s \in \{0, 1, \ldots, k_f - 1\} \).

Proof. (i) This is obvious from the definition of \( \sim \).

(ii) From part (i), \( k_s \leq s \). So suppose \( k_{s+1} \leq s \). Then from the definition of \( \sim \) it is clear \( k_{s+1} \leq k_s \). But because \( k_1 < k_2 < \cdots < k_f \), it follows that \( s + 1 \leq s \) which is a contradiction. Thus \( k_{s+1} > s \) and we are done. \( \square \)

Lemma 6. If \( X \in V_2, k_\bar{p} + s < k_{\bar{p}+1} (s \geq 0) \), and \( \Omega_+^{k_\bar{p}+s}.X = 0 \), then \( \Omega_+^{k_\bar{p}}.X = 0 \).

Proof. Assume \( s > 0 \), otherwise we are done. Because \( X \in V_2 \), we have \( \Omega_+^{k_\bar{p}+s-1}.X \in V_2(k_{\bar{p}+s}) \). By assumption \( \Omega_+^{k_\bar{p}+s}.X = 0 \), so

\[
\Omega_+^{k_\bar{p}+s-1}.X \in V_2(k_{\bar{p}+s}) \cap \ker(\text{ad}(\Omega_+)) .
\]

But, if \( n \in Z_{>0} \), then

\[
V_{2n} \cap \ker(\text{ad}(\Omega_+)) \neq \{0\} \iff n \in \{k_1, k_2, \ldots, k_f\} ,
\]

because otherwise \( g \) would contain an irreducible \( \text{span}_{C}\Lambda_0, \Omega_+, \Omega_- \)-submodule with weight \( 2n \in Z_{>0} \{2k_1, 2k_2, \ldots, 2k_f\} \). This is impossible as the set \( \{2k_1, 2k_2, \ldots, 2k_f\} \) exhausts all the positive even weights of the irreducible \( \text{span}_{C}\Lambda_0, \Omega_+, \Omega_- \)-submodules in \( g \). Therefore \( \Omega_+^{k_\bar{p}+s-1}.X = 0 \) as \( k_\bar{p} < k'_s + s < k_{\bar{p}+1} \) implies that \( (k_\bar{p} + s) \) is not in \( \{k_1, k_2, \ldots, k_f\} \). Repeat the above argument with \( s' = s - 1 \) to arrive at \( \Omega_+^{k_\bar{p}+s'-1}.X = \Omega_+^{k_\bar{p}+s-2}.X = 0 \). Continuing in this manner, we find \( \Omega_+^{k_\bar{p}}.X = 0 \). \( \square \)

The next theorem is the key result needed to prove that the EYM equations can be put into a form where theorem \( \square \) applies in a neighborhood of the origin \( r = 0 \).

**Theorem 9.** Suppose \( p \in \{1, 2, \ldots, k_f - 1\} \) and \( Z_0, Z_1, \ldots, Z_{p+1} \in V_2 \) is a sequence of vectors that satisfy \( Z_0 \in E_0^1 \oplus E_1^1 \) and \( Z_{n+1} \in \bigoplus_{q=1}^n E_0^q \oplus E_1^q \) for \( n = 0, 1, \ldots, p \). Then for every \( j \in \{1, 2, \ldots, p + 1\} \), \( s \in \{0, 1, 2, \ldots, j\} \)

(i) \( [c(Z_{j-s}), Z_s, Z_{p+2-j}] \in \bigoplus_{q=1}^{\bar{p}} E_0^q \oplus E_1^q \)

(ii) \( [c(Z_{p+2-j}), Z_{j-s}, Z_s] \in \bigoplus_{q=1}^{\bar{p}} E_0^q \oplus E_1^q \)

Proof. (i) Suppose \( Z_0, Z_1, \ldots, Z_{p+1} \in V_2 \) is a sequence satisfying \( Z_0 \in E_0^1 \oplus E_1^1 \) and \( Z_{n+1} \in \bigoplus_{q=1}^n E_0^q \oplus E_1^q \) for \( n = 0, 1, \ldots, p \). Then

\[
\Omega_+^{k_0(n-1)\sim}.Z_n = \Omega_+^{k_0(n-1)\sim+2}.c(Z_n) = 0 \quad (6.25)
\]

for \( n = 0, 1, 2, \ldots, p + 1 \) by lemmas \( \square \) and \( \square \). Now, if \( j \in \{1, 2, \ldots, p + 1\} \) and \( s \in \{0, 1, \ldots, j\} \), then

\[
\Omega_+^p.[c(Z_{j-s}), Z_s, Z_{p+2-j}] = \sum_{l=0}^{p} \sum_{m=0}^{l} \binom{p}{l} \binom{l}{m} a_{psjlm}
\]

where

\[
a_{psjlm} = [\Omega_+^m, c(Z_{j-s}), Z_s, \Omega_+^{l-m}.Z_s, \Omega_+^{p-l}.Z_{p+2-j}] .
\]

Applying \( \square, \square \) yields \( a_{psjlm} = 0 \) if \( m - 2 \geq k_{(j-s-1)\sim} \) or \( l - m \geq k_{(s-1)\sim} \) or \( p - l \geq k_{(p+1)\sim} \). But because of lemma \( \square \) (i), this implies that \( a_{psjlm} = 0 \) if \( m - 2 \geq j - s - 1 \) or \( l - m \geq s - 1 \) or
p - l \geq p + 1 - j$. It follows that $a_{p,s,l,m} = 0$ unless $l$ and $m$ satisfy $j - 1 < l < m + s - 1 < j$ which is impossible. Therefore $a_{p,s,l,m} = 0$ for all $l$ and $m$. Thus $\Omega^p_+ [[c(Z_{j-s}), Z_s], Z_{p+2-j}] = 0$. But then it follows from lemmas 9 (ii) and 9 that $\Omega^p_+ [[c(Z_{j-s}), Z_s], Z_{p+2-j}] = 0$ and hence $[[c(Z_{j-s}), Z_s], Z_{p+2-j}] \in \bigoplus_{q=1}^p E_q^0 \oplus E_q^+$ by lemma 9.

(ii) It follows from similar arguments that $[[c(Z_{p+2-j}), Z_{j-s}], Z_s] \in \bigoplus_{q=1}^p E_q^0 \oplus E_q^+$. □

It is worthwhile to note that all the above results did not depend on $\Lambda_0$ being regular. However, for what follows we will need $\Lambda_0$ to be regular.

**Proposition 3.** Suppose $\Lambda_0$ is regular. Then $\text{span}_C \{ \xi^1, \xi^2, \ldots, \xi^M \}$ is an Abelian subalgebra of $g_\lambda$ and hence also an Abelian subalgebra of $g$.

Proof. From the definition of $g_\lambda$, it follows that $\text{span}_C \{ \Lambda_0, \Omega_+, \Omega_- \} \subset g_\lambda$ and $V_2 \subset g_\lambda$. But by proposition 3 $V_2 = \text{span}_C \{ \xi^1_{k_1-1}, \xi^2_{k_2-1}, \ldots, \xi^M_{k_M-1} \}$, and hence

$$\frac{(k_i+1)!}{(2k_i)!} \Omega_+^{k_i-1} \xi^{k_i-1}_i = \xi^i \in g_\lambda$$

for $i = 1, 2, \ldots, M$. Therefore $\text{span}_C \{ \xi^1, \xi^2, \ldots, \xi^M \} \subset g_\lambda$. The $\xi^i$ are highest weight vectors, consequently

$$\text{span}_C \{ \xi^1, \xi^2, \ldots, \xi^M \} \subset g^{\Omega_+}_\lambda$$

where $g^{\Omega_+}_\lambda = \{ X \in g_\lambda \mid [\Omega_+, X] = 0 \}$. Define $V_{\lambda,n} := \{ X \in g_\lambda \mid \Lambda_0, X = nX \}$. By theorem 2 $S_\lambda$ is a base a system of roots of $g_\lambda$ and $\alpha(\Lambda_0) = 2$ for every $\alpha \in S_\lambda$ and hence it follows that $V_{\lambda,2} = V_2$. Using $sl(2, \mathbb{C})$-representation theory, it is not hard to show that $\text{dim}_C g^{\Omega_+}_\lambda = \text{dim}_C V_{\lambda,2}$. But $\text{dim}_C V_2 = |S_\lambda|$, and therefore $\text{dim}_C g^{\Omega_+}_\lambda = |S_\lambda|$. By proposition 3, $|S_\lambda| = M$ and hence we get from (6.26) that

$$\text{span}_C \{ \xi^1, \xi^2, \ldots, \xi^M \} = g^{\Omega_+}_\lambda.$$

(6.27)

Theorem 8 proved that $|S_\lambda| = \text{dim}_C g_\lambda$ which in turn gives, via the above result, $\text{dim}_C g^{\Omega_+}_\lambda = \text{dim}_C g_\lambda$. Applying lemma 2.1.15 of 8 then shows that

$$\text{dim}_C g^{\Omega_+}_\lambda = \min \{ \text{dim}_C g^X_\lambda \mid X \in g_\lambda \}.$$

(6.28)

We can identify $g_\lambda$ with the dual $g^*_\lambda$ using the form $(\cdot, \cdot)$, i.e.

$$\iota : g_{\lambda,0} \rightarrow g^*_{\lambda,0} \quad \iota(X)(\cdot) = (X, \cdot).$$

So if $f \in g^*_\lambda$ and we define $g^f_\lambda = \{ X \in g_\lambda \mid \text{Ad}^*_\lambda(f) = 0 \}$, then it can be shown that

$$g^{\iota(X)}_\lambda = g^X_\lambda \quad \forall X \in g.$$  

(6.29)

Let $G_\lambda$ be a connected complex semisimple Lie group with Lie algebra $g_\lambda$. Then for $f \in g^*_\lambda$, $g^f_\lambda$ is the Lie algebra of coadjoint isotropy group $G_{\lambda,f} = \{ a \in G_\lambda \mid \text{Ad}^*_\lambda(f) = f \}$. But then (6.27), (6.28), (6.29) and a straightforward generalization of theorem 9.3.10 in 21 to complex Lie groups imply that $\text{span}_C \{ \xi^1, \xi^2, \ldots, \xi^M \}$ is an Abelian subalgebra. □

The next theorem is the key result needed to prove that the EYM equations can be put into a form where theorem 3 applies in a neighborhood $r = \infty$. Although this theorem looks very similar to theorem 3, it is more difficult to prove. Similar arguments as in theorem 3 are employed, but these only go part of the way. Proposition 3 is needed to complete the proof.
Theorem 10. Assume that $\Lambda_0$ is regular. Suppose $p \in \{0,1,2,\ldots,k_j\}$ and $Z_0,Z_1,\ldots,Z_p \in V_2$ is a sequence of vectors that satisfy $Z_n \in \bigoplus_{q=1}^n E_0^q \oplus E_+^q$ for $n = 0,1,\ldots,p$. Then for every $j \in \{1,2,\ldots,p\}$, $s \in \{0,1,\ldots,j\}$

\[(i) \quad \{c(Z_{j-s}),Z_s,Z_{p+1-j}\} \in \bigoplus_{q=1}^p E_0^q \oplus E_+^q \]

\[(ii) \quad \{c(Z_{p+1-j}),Z_{j-s},Z_s\} \in \bigoplus_{q=1}^p E_0^q \oplus E_+^q \]

Proof. (i) Suppose $Z_0,Z_1,\ldots,Z_p \in V_2$ is a sequence satisfying $Z_n \in \bigoplus_{q=1}^n E_0^q \oplus E_+^q$ for $n = 0,1,\ldots,p$. Then

$$\Omega^p_+ \cdot Z_n = \Omega^p_+ \cdot c(Z_n) = 0 \quad (6.30)$$

for $n = 0,1,\ldots,p$ by lemmas 3 and 4. Suppose $j \in \{1,2,\ldots,p\}$ and $s \in \{0,1,\ldots,j\}$. Then

$$\Omega^p_+ \cdot [c(Z_{j-s}),Z_s,Z_{p+1-j}] = \sum_{l=0}^p \sum_{m=0}^l \left( \begin{array}{c} p \\ l \end{array} \right) \left( \begin{array}{c} l \\ m \end{array} \right) a_{psjlm} \quad (6.31)$$

where

$$a_{psjlm} = [\Omega^m_+ \cdot c(Z_{j-s}),\Omega^{l-m}_+ \cdot Z_s,\Omega^{p-l}_+ \cdot Z_{p+1-j}] .$$

Applying (6.30) yields $a_{psjlm} = 0$ if $m - 2 \geq k_{(j-s)}$ or $l - m \geq k_s$ or $p - l \geq k_{(p+1-j)}$. But because of lemma 3 (i), this implies that $a_{psjlm} = 0$ if $m - 2 \geq j - s$ or $l - m \geq s$ or $p - l \geq p + 1 - j$. It follows that $a_{psjlm} = 0$ unless $l$ and $m$ satisfy $j - 1 < l < m + s < j + 2$ which implies that $l = j$ and $m + s = j + 1$. Thus the sum (6.31) reduces to

$$\Omega^p_+ \cdot [c(Z_{j-s}),Z_s,Z_{p+1-j}] = \left( \begin{array}{c} p \\ j \end{array} \right) \left( \begin{array}{c} j \\ j + 1 - s \end{array} \right) [X_1,X_2,X_3]$$

where $X_1 = \Omega^{j-s+1}_+ \cdot c(Z_{j-s})$, $X_2 = \Omega^{s-1}_+ \cdot Z_s$, and $X_3 = \Omega^{p-j}_+ \cdot Z_{p+1-j}$. Applying (6.30) then shows that $\Omega_+ X_a = 0$ for $a = 1,2,3$. Because the $X_a$ have even weights,

$$X_1,X_2,X_3 \in \text{span} \{\xi^1,\xi^2,\ldots,\xi^M\} .$$

But $\text{span} \{\xi^1,\xi^2,\ldots,\xi^M\}$ is an Abelian subalgebra by proposition 3, so $[X_1,X_2,X_3] = 0$ which implies that $\Omega^p_+ \cdot [c(Z_{j-s}),Z_s,Z_{p+1-j}] = 0$. We then get via lemma 3 that $\Omega^p_+ \cdot [c(Z_{j-s}),Z_s,Z_{p+1-j}] = 0$ and hence $[c(Z_{j-s}),Z_s,Z_{p+1-j}] \in \bigoplus_{q=1}^p E_0^q \oplus E_+^q$ by lemma 3.

(ii) The proof that $[c(Z_{p+1-j}),Z_{j-s},Z_s] \in \bigoplus_{q=1}^p E_0^q \oplus E_+^q$ is similar to part (i). \qed

Proposition 4. If $\Omega_+ \in \sum_{a \in S_+} \mathbb{R} e_a$ and $\Lambda_0$ is regular, then $E_+ = \sum_{a \in S_+} \mathbb{R} e_a$.

Proof. Introduce a basis $\{Z_j | 1 \leq j \leq M\}$ over $\mathbb{R}$ for $E_+$ by defining

$$Z_j = \begin{cases} \xi^i_{k_j-1} & \text{if } k_j \text{ is odd} \\ i \xi^j_{k_j-1} & \text{if } k_j \text{ is even} \end{cases} \quad 1 \leq j \leq M .$$

Equations (6.4) and proposition 3 (iii) can be used to show that

$$\Omega_+ \cdot c(Z_j) = \Omega_- \cdot Z_j \quad 1 \leq j \leq M . \quad (6.32)$$
By assumption $\Omega_+ = \sum_{\alpha \in S_\lambda} w_\alpha e_\alpha$ for some set of constants $w_\alpha \in \mathbb{R}$. Because $c(\Omega_+) = -\Omega_-$ and $c(e_\alpha) = -e_{-\alpha}$, $\Omega_- = \sum_{\alpha \in S_\lambda} w_\alpha e_{-\alpha}$. Since $Z_j \in V_2$, $Z_j = \sum_{\alpha \in S_\lambda} a_{j\alpha} e_\alpha$ for some set of constants $a_{j\alpha} \in \mathbb{C}$. So then $c(Z_j) = -\sum_{\alpha \in S_\lambda} \overline{a}_{j\alpha} e_{-\alpha}$. Now, since $\Lambda_0$ is regular, equation (3.13) holds. Therefore

$$\Omega_- Z_j = \sum_{\alpha \in S_\lambda} w_\alpha a_{j\alpha} [e_{-\alpha}, e_\alpha] = \sum_{\alpha \in S_\lambda} -w_\alpha a_{j\alpha} h_\alpha,$$

while

$$\Omega_+ c(Z_j) = \sum_{\alpha \in S_\lambda} -w_\alpha \overline{a}_{j\alpha} [e_\alpha, e_{-\alpha}] = \sum_{\alpha \in S_\lambda} -w_\alpha \overline{a}_{j\alpha} h_\alpha.$$

The three results (6.32), (6.33), and (6.34) then yield

$$\sum_{\alpha \in S_\lambda} w_\alpha (a_{j\alpha} - \overline{a}_{j\alpha}) h_\alpha = 0.$$

Since $\Lambda_0$ is regular, it follows that $w_\alpha \neq 0$ for all $\alpha \in S_\lambda$ and the set $\{ h_\alpha \mid \alpha \in S_\lambda \}$ is linearly independent. Thus $a_{j\alpha} - \overline{a}_{j\alpha} = 0$ for all $\alpha \in S_\lambda$ and $j = 1, 2, \ldots, M$. So $Z_j \in \sum_{\alpha \in S_\lambda} \mathbb{R} e_\alpha$ for $j = 1, 2, \ldots, M$ which implies that $E_+ \subset \sum_{\alpha \in S_\lambda} \mathbb{R} e_\alpha$. However, $\dim \mathbb{R}^+ = \dim (\sum_{\alpha \in S_\lambda} \mathbb{R} e_\alpha) = |S_\lambda|$ and therefore $E_+ = \sum_{\alpha \in S_\lambda} \mathbb{R} e_\alpha$. □

Suppose $\Lambda_0$ is regular and $\Omega_+ = \sum_{\alpha \in S_\lambda} w_\alpha e_\alpha$ where $w_\alpha \in \mathbb{R}$ for every $\alpha \in S_\lambda$. Then using (3.10), (3.13), and the fact that $c(e_\alpha) = -e_{-\alpha}$, it is not difficult to show that

$$A_2(e_\alpha) = \sum_{\beta \in S_\lambda} w_\beta \langle \beta, \alpha \rangle w_\alpha e_\beta.$$

This result along with (6.21) and proposition 3 shows that $\{ e_\alpha \mid \alpha \in S_\lambda \}$ can be completed to a basis over $\mathbb{R}$ of $V_2$ so that the matrix of $A_2$ with respect to this basis takes the form

$$[A_2] = \begin{pmatrix} 0 & 0 \\ 0 & [A_{\alpha\beta}] \end{pmatrix},$$

with

$$A_{\alpha\beta} = w_\alpha \langle \alpha, \beta \rangle w_\beta.$$

7 Local Uniqueness and Existence Proofs

In this section we present the proofs of theorems 6, 7, and 8. The proof of theorem 6 is the easiest and does not depend on the results of the section 3. Define

$$\mathcal{E} := \{ k_j \mid j = 1, 2, \ldots I \}$$

with the $k_j$ defined in (6.16) and let

$$\text{pr}_q^d : E_+ \to E_+^q \quad q = 1, 2, \ldots I$$

denote the projection operators between the spaces defined in (6.19) and (6.20). If $a \in \mathbb{R}$, we will use $I_\epsilon(a)$ to denote an interval of radius $\epsilon$ about $a$, i.e.

$$I_\epsilon(a) = (a - \epsilon, a + \epsilon) \, .$$
From proposition 4 and (3.19), we know that the solution $\Lambda_+(r)$ to equation (3.6) is, up to a
gauge transformation, completely characterized by the condition

$$\Lambda_+(r) \in E_+ \quad \forall \ r.$$ (7.1)

As discussed previously, if we can solve the two EYM equations (3.3) and (3.5) for the variables
\[\{\Lambda_+(r), m(r)\}\] the the remaining equation (3.4) can be integrated to yield $S$. Consequently, we are
only interested in the equations (3.3) and (3.5).

**Proof of theorem 4.** The proof of this theorem involves finding a change of variables to put the system
of differential equations (3.3) and (3.5) into a form where theorem 3 applies in a neighborhood of
$r = 0$.

Since $\Lambda_+$ satisfies (7.1), we can introduce new variables \[\{u_{s+1}(r) \mid s \in E\}\] that satisfy

$$\Lambda_+(r) = \Omega_+ + \sum_{s \in E} u_{s+1}(r) r^{s+1}$$ (7.2)

where $\Omega_+ = \Lambda_+(0)$ and $u_{s+1}(r) \in E_+^s$ for all $r$ and $s \in E$. Because $E_+ = \bigoplus_{q=1}^\infty E_+^q$, it is obvious
that this transformation is invertible. Define

$$\chi_{s+1} = \begin{cases} 1 & \text{if } s \in E \\ 0 & \text{otherwise} \end{cases},$$

Then we can write $\Lambda_+(r) = \Omega_+ + \sum_{k=0}^\infty \chi_k u_k(r) r^k$. Substituting this in (3.8) shows that there exists
an integer $N_1$ such that

$$F = - \sum_{k \in E} A_2(u_{k+1}) r^{k+1} + \sum_{k=2}^{N_1} f_k r^k,$$

where

$$f_k = \frac{1}{2} \sum_{j=2}^{k-2} \left\{ [[\Omega_+, c(\chi_j u_j)] + [\Omega_-, \chi_j u_j], \chi_{k-j} u_{k-j}] \\
+ [[\chi_j u_j, c(\chi_{k-j} u_{k-j})], \Omega_+] + \sum_{s=2}^{j-2} [[\chi_s u_s, c(\chi_{k-s} u_{k-s})], \chi_{k-j} u_{k-j}] \right\}.$$

But $A_2(u_{k+1}) = k(k+1)u_{k+1}$ for every $k \in E$ by lemma 3 and hence

$$F = - \sum_{k \in E} k(k+1)u_{k+1} r^{k+1} + \sum_{k=2}^{N_1} f_k r^k. $$ (7.3)

Define

$$v_{s+1} = u_{s+1}' \quad \forall \ s \in E.$$

Using (7.2), (7.3) and (7.4), the EYM equation (3.5) can be written as

$$r \sum_{k \in E} v_{k+1} r^{k+1} = -\frac{2}{r} \sum_{k \in E} (k+1)v_{k+1} r^{k+1} + \sum_{k \in E} \frac{k(k+1)}{r} \left( \frac{1}{N} - 1 \right) u_{k+1} r^{k+1}$$

$$- \frac{2}{rN} \left( m - \frac{1}{r} \right) \sum_{k \in E} (v_{k+1} r^{k+1} + (k+1)u_{k+1} r^k) - \frac{1}{N} \sum_{k=4}^{N_1} f_k r^{k-1}. $$ (7.5)
Applying the projections \( pr_k^+ \) for every \( k \in \mathcal{E} \) to equation (7.3) yields

\[
rv_{k+1}' = -2(k+1)v_{k+1} - \frac{2}{rN} \left( m - \frac{1}{r}P \right) v_{k+1} + \frac{k(k+1)}{r} \left( \frac{1}{N} - 1 \right) u_{k+1}
\]

\[
- \frac{2}{r^2N} \left( m - \frac{1}{r}P \right) (k+1)u_{k+1}r^k - \frac{1}{r^{k+1}N} \sum_{s=2}^{N_1-2} pr_k^+ (f_{s+2}) r^{s+1} \quad \forall \ k \in \mathcal{E}.
\]

(7.6)

The last term in (7.4) is the main obstruction to putting the equation into a form where theorem 3 applies. It seems to contain terms of order \( r^{-s} \) \( (s > 0) \). However, as we shall now see, the results of section 6 can be used to show that

\[
\frac{1}{r^{k+1}N} \sum_{s=0}^{N_1-2} pr_k^+ (f_{s+2}) r^{s+1} = \frac{1}{N} \sum_{s=k}^{N_1-2} pr_k^+ (f_{s+2}) r^{s-k}.
\]

(7.7)

Namely, by using proposition 4, we can show that \( f_k \in E_+ \) for all \( k \). From the definition of the \( u_{s+1} \) it is clear that \( \chi_{s+1} u_{s+1} \in \bigoplus_{q=1}^s E_+^q \) for \( 0 \leq s \leq k_f \), and so it follows from theorem 3 by letting \( Z_0 = \Omega_+ \) and \( Z_{k+1} = \chi_k u_{k+1} \) for \( k \geq 0 \) that \( f_{s+2} \in \bigoplus_{q=1}^s E_+^q \). Consequently, for every \( k \in \mathcal{E} \)

\[
pr_k^+ (f_{s+2}) = 0 \quad \text{if} \quad s < k,
\]

because \( k \in \mathcal{E} \) implies that \( k = k \hat{k} \) and hence it follows for \( s < k = k \hat{s} \) that \( \hat{s} < \hat{k} \). This proves (7.7).

Therefore we can rewrite (7.6) as

\[
rv_{k+1}' = -2(k+1)v_{k+1} - \frac{2}{rN} \left( m - \frac{1}{r}P \right) v_{k+1} + \frac{k(k+1)}{r} \left( \frac{1}{N} - 1 \right) u_{k+1}
\]

\[
- \frac{2}{r^2N} \left( m - \frac{1}{r}P \right) (k+1)u_{k+1}r^k - \frac{1}{r^{k+1}N} \sum_{s=k}^{N_1-2} pr_k^+ (f_{s+3}) r^{s-k}
\]

\[
+ \left( 1 - \frac{1}{N} \right) pr_k^+ (f_{k+2}) - pr_k^+ (f_{k+2}) \quad \forall \ k \in \mathcal{E}.
\]

(7.8)

Using the properties (3.3) of \( \langle \cdot | \cdot \rangle \) and the fact that \( A_2(u_2) = 2u_2 \), it can be shown that there exists analytic functions

\[
\hat{P} : E_+ \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{and} \quad \hat{G} : E_+ \times E_+ \times \mathbb{R} \rightarrow \mathbb{R},
\]

such that

\[
P = r^4 \| u_2 \|^2 + r^5 \hat{P}(u, r) \quad \text{and} \quad G = r^2 2 \| u_2 \|^2 + r^3 \hat{G}(u, v, r),
\]

(7.9)

where \( u = \sum_{s \in \mathcal{E}} u_{s+1}, \ v = \sum_{s \in \mathcal{E}} v_{s+1}, \) and \( \| \cdot \| \) is defined by (6.2). Introduce a new “mass” variable \( \mu \) by

\[
\mu = \frac{1}{r^3} (m - r^3 \| u_2 \|^2).
\]

(7.10)

Recall that \( k_1 = 1 \), so 1 is always in \( \mathcal{E} \) and hence \( u_2 \) is always defined. We can then write the EYM equation (3.3) as

\[
rm' = -3\mu + r \left\{ \hat{P}(u, r) + \hat{G}(u, v, r) - 2\langle u_2 | v_2 \rangle - 2r (\mu + \| u_2 \|^2) (2 \| u_2 \|^2 + r \hat{G}(u, v, r)) \right\}.
\]

(7.11)
Introduce one last change of variables via
\[ \hat{v}^{k+1} = v^k + \frac{1}{2(k+1)^2} p^k (f_{k+2}) . \] (7.12)

Fix \( X \in E_+ \) and define \( \hat{v} = \sum_{s \in E} \hat{v}_s \). Then using (7.8), (7.10) and (7.12), it can be shown that there exists a neighborhood of \( N_X \) of \( X \) in \( E_+ \), an \( \epsilon > 0 \), and a sequence of analytic maps
\[ G_k : N_X \times E_+ \times I_\epsilon(0) \times I_\epsilon(0) \rightarrow E_0^k \quad \forall k \in E , \]
such that
\[ r \hat{v}'^{k+1} = -2(k+1) \hat{v}^{k+1} + r G_k(u, \hat{v}, \mu, r) \quad \forall k \in E . \] (7.13)

Also from (7.4), (7.11) and (7.12), it is not difficult to show that there exists analytic maps
\[ H_k : E_+ \times E_+ \rightarrow E_+^k \quad \forall k \in E \quad \text{and} \quad K : E_+ \times E_+ \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} , \]
such that
\[ ru'^{k+1} = r H_k(u, \hat{v}) \quad \forall k \in E , \]
\[ r \mu' = -3 \mu + r K(u, \hat{v}, \mu, r) . \] (7.14)

The system of differential equations (7.13), (7.14) and (7.15) are in the form for which theorem 3 applies. Applying this theorem shows that for fixed \( X \in E_+ \) there exist a unique solution
\[ \{ u_{s+1}(r, Y) \}, \{ \hat{v}_{s+1}(r, Y) \}, \{ \mu(r, Y) \} \]
to this system of differential equations that is analytic in a neighborhood of \( (r, Y) = (0, X) \) and that satisfies
\[ u_s(r, Y) = Y_s + O(r) \quad \forall s \in E , \] (7.16)
\[ \hat{v}_s(r, Y) = O(r) \quad \forall s \in E , \] (7.17)
\[ \mu(r, Y) = O(r) , \]
where \( Y_s = p^s_k (Y) \). From equation (7.10), it is then clear that mass \( m \) satisfies
\[ m(r) = O(r^3) . \] (7.18)

Also from (7.9), (7.12), (7.16) and (7.17) it is not difficult to see that
\[ P = O(r^4) \quad \text{and} \quad G = O(r^2) . \] (7.19)

From the results of the previous section there exists an orthonormal basis \( \{ f_j \} \) for \( E_+ \) consisting of eigenvectors for \( A_2 \), i.e., \( A_2(f_j) = k_j(k_j + 1)f_j \). Thus we can introduce new variables \( \{ \hat{u}_j(r) \} \) via
\[ \sum_{s \in E} u_{s+1}(r) r^{s+1} = \sum_{j=1}^M \hat{u}_j(r) r^{k_j+1} f_j . \] (7.20)

From proposition 3 we know that \( M = |S_\lambda| \). So we can write \( S_\lambda = \{ \sigma_j \} \) and we get from proposition 4 that \( \{ e_{\sigma_j} \} \) is also a basis for \( E_+ \). Therefore there exists a real non-singular matrix \( C_{ij} \) such that
\[ f_j = \sum_{k=1}^M C_{kj} e_{\sigma_k} . \] (7.21)
Expand $\Omega_+$ and $\Lambda_+$ in the basis $\{ e_{\alpha_j} | j = 1, 2, \ldots M \}$ as follows

$$
\Omega_+ = \sum_{j=1}^{M} w_{j,0} e_{\alpha_j} \quad \text{and} \quad \Lambda_+(r) = \sum_{j=1}^{M} w_j(r) e_{\alpha_j} .
$$

(7.21)

Then results (7.2), (7.19), (7.20), and (7.21) imply that

$$
w_i(r) = w_i,0 + \sum_{j=1}^{M} C_{ij} \hat{u}_j(r) r^{k_j+1} \quad i = 1, 2, \ldots M ,$$

while from (7.16) and (7.19) it is clear that

$$
\hat{u}_j(r, Y) = \beta_j(Y) + O(\frac{1}{r}) \quad j = 1, 2, \ldots , M ,
$$

where $\beta_j(Y) = \langle \langle f_j | Y \rangle \rangle$.

Proof of theorem 5. The proof of this theorem involves finding a change of variables to put the system of differential equations (3.3) and (3.5) into a form where theorem 3 applies in a neighborhood of $z = 0$ where 

$$z = \frac{1}{r}.$$ 

This proof is similar to the proof of theorem 4 with the exception that theorem 10 is needed instead of theorem 9.

Since $\Lambda_+$ satisfies (7.1), we can introduce new variables $\{ u_s(z) | s \in E \}$ that satisfy

$$
\Lambda_+(z) = \Omega_+ + \sum_{s \in E} u_s(z) z^s
$$

(7.22)

where $\Omega_+ = \Lambda_+|_{z=0}$ and $u_s(z) \in E_+^s$ for all $z$ and $s \in E$. Because $E_+ = \bigoplus_{q=1}^{q} E_+^q$, it is obvious that this transformation is invertible. Define

$$
\chi_s = \begin{cases} 
1 & \text{if } s \in E \\
0 & \text{otherwise}
\end{cases} .
$$

Then we can write $\Lambda_+(z) = \Omega_+ + \sum_{k=0}^{\infty} \chi_k u_k(z) z^k$. Substituting this in (3.5) shows that there exists an integer $N_1$ such that

$$
\mathcal{F} = -\sum_{k \in E} A_2(u_k) z^k + \sum_{k=1}^{N_1} f_k z^k .
$$

where

$$
f_k = \frac{1}{2} \sum_{j=1}^{k-1} \left\{ \left[ \Omega_+, c(\chi_j u_j) \right] + \left[ \Omega_-, \chi_j u_j, \chi_{k-j} u_{k-j} \right] \right. \\
+ \left[ \left[ \chi_j u_j, c(\chi_{k-j} u_{k-j}) \right], \Omega_+ \right] + \sum_{s=1}^{j-1} \left[ \left[ \chi_s u_s, c(\chi_{j-s} u_{j-s}) \right], \chi_{k-j} u_{k-j} \right] \left. \right\} .
$$

But $A_2(u_k) = k(k+1) u_k$ for every $k \in E$ by lemma 2 and hence

$$
\mathcal{F} = -\sum_{k \in E} k(k+1) u_k z^k + \sum_{k=1}^{N_1} f_k z^k .
$$

(7.23)

Define

$$
v_s = \hat{u}_s \quad \forall s \in E .
$$

(7.24)
where \( \frac{d}{dx}(\cdot) \). Using (7.24), (7.23) and (7.22), the EYM equation (3.3) can be written as

\[
\sum_{k \in E} \frac{\partial}{\partial z} z^{k+1} = \sum_{k \in E} -2(k+1)u_k z^k + \sum_{k \in E} \left\{ \frac{2}{z} \left( 1 - \frac{1}{N} \right) v_k + \frac{1}{z^2} \left( \frac{1}{N} - 1 - 2mz \right) k(k-1)u_k \right. \\
+ \frac{4m}{z} \left( \frac{1}{N} - 1 \right) ku_k + \frac{2}{N} (2m - z^2 P) v_k - \frac{z P}{N} ku_k \right\} z^{k+1} + \sum_{k \in E} 2mk(k+1)u_k z^k \\
- \frac{1}{N} \sum_{k=1}^{N_1} f_k z^{k-1}
\]

(7.25)

Applying the projections \( pr_+^k \) for every \( k \in E \) to equation (7.24) yields

\[
z \hat{v}_k = -2(k+1)v_k + z \left\{ \frac{2}{z} \left( 1 - \frac{1}{N} \right) v_k + \frac{1}{z^2} \left( \frac{1}{N} - 1 - 2mz \right) k(k-1)u_k \right. \\
+ \frac{4m}{z} \left( \frac{1}{N} - 1 \right) ku_k + \frac{2}{N} (2m - z^2 P) v_k - \frac{z P}{N} ku_k \right\} z^{k+1} + \sum_{k \in E} 2mk(k+1)u_k z^k \\
- \frac{1}{z^k N} \sum_{s=0}^{N_1-1} pr_+^k (f_{s+1}) z^s \quad \forall k \in E.
\]

(7.26)

The last term in (7.26) is the main obstruction to putting this equation into a form where theorem 3 applies. It seems to contain terms of order \( z^{-s} \) \((s > 0)\). But this is not the case as the results of section 3 can be used to show that

\[
\frac{1}{z^k N} \sum_{s=0}^{N_1-1} pr_+^k (f_{s+1}) z^s = \frac{1}{N} \sum_{s=k}^{N_1-1} pr_+^k (f_{s+1}) z^{s-k}.
\]

Namely, using proposition 4, it can be shown that \( f_k \in E_+ \) for all \( k \). From the definition of the \( u_s \) it is obvious that \( \chi_s u_s \in \bigoplus_{q=1}^s E^q_+ \) for \( 1 \leq s \leq k \), and therefore by letting \( Z_0 = \Omega_+ \) and \( Z_k = \chi_k u_k \) for \( k \geq 1 \) we get \( f_{s+1} \in \bigoplus_{q=1}^s E^q_+ \) via theorem 10. Consequently, for every \( k \in E \)

\[
pr_+^k (f_{s+1}) = 0 \quad \text{if } s < k,
\]

because \( k \in E \) implies that \( k = k_{\tilde{k}} \) and hence it follows for \( s < k = k_{\tilde{k}} \) that \( \tilde{s} < \tilde{k} \). Therefore we can rewrite (7.26) as

\[
z \hat{v}_k = -2(k+1)v_k + z \left\{ \frac{2}{z} \left( 1 - \frac{1}{N} \right) v_k + \frac{1}{z^2} \left( \frac{1}{N} - 1 - 3mz \right) k(k-1)u_k \right. \\
+ \frac{4m}{z} \left( \frac{1}{N} - 1 \right) ku_k + \frac{2}{N} (2m - z^2 P) v_k - \frac{z P}{N} ku_k \right\} z^{k+1} + \sum_{s=k+1}^{N_1-1} pr_+^k (f_{s+1}) z^{s-k-1} \\
+ \left( 1 - \frac{1}{N} \right) pr_+^k (f_{k+1}) \right\} + 2mk(k+1)u_k - pr_+^k (f_{k+1}) \quad \forall k \in E.
\]

(7.27)

It is clear that there exists analytic functions

\[
\hat{P} : E_+ \times \mathbb{R} \to \mathbb{R} \quad \text{and} \quad \hat{G} : E_+ \times E_+ \times \mathbb{R} \to \mathbb{R},
\]

such that

\[
P = \hat{P}(u, z) \quad \text{and} \quad G = z^4 \hat{G}(u, v, z),
\]
where \( u = \sum_{s \in E} u_s \) and \( v = \sum_{s \in E} v_s \). The EYM equation (3.3) can then be written as
\[
zm^0 = z\left[(2 \alpha^{-1} \beta^{-1}) z^2 \hat{G}(u, v, z) - P(u, z)\right].
\] (7.28)

Introduce one last change of variables via
\[
\hat{v}_k = v_k + \frac{1}{2(k+1)} \left[ \text{pr}^+_k (f_{k+1}) - km u_k \right].
\] (7.29)

Fix \( a > 0 \) and define \( \hat{v} = \sum_{s \in E} \hat{v}_s \). Then using (7.27) and (7.28), it can be shown that there exists an analytic map
\[
\hat{G}_k : E_+ \times E_+ \times I(a) \times I(0) \rightarrow E_0^k \quad \forall k \in E,
\]
such that
\[
z \hat{v}_k = -2(k+1) \hat{v}_k + z \hat{G}_k(u, \hat{v}, m, z) \quad \forall k \in E. \tag{7.30}
\]

Also from (7.24), (7.28) and (7.29), it is not hard to show that there exists an analytic map
\[
\hat{H}_k : E_+ \times E_+ \times R \rightarrow E_+^k \quad \forall k \in E \quad \text{and} \quad K : E_+ \times E_+ \times R \times R \rightarrow R, \tag{7.31}
\]
such that
\[
z \hat{u}_k = z \hat{H}_k(u, \hat{v}, m) \quad \forall k \in E, \tag{7.32}
\]
\[
z \hat{m} = z K(u, \hat{v}, m, z). \tag{7.33}
\]

The system of differential equations (7.30), (7.32) and (7.33) are in the form for which theorem 4 applies. Applying this theorem shows that for fixed \( (X, a) \in E_+ \times (0, \infty) \) there exist a unique solution \( \{u_k(z, Y, m_\infty), \hat{v}_k(z, Y, m_\infty), m(z, Y, m_\infty)\} \) to this system of differential equations that is analytic in a neighborhood of \( (z, Y, m_\infty) = (0, X, a) \) and satisfies
\[
u_s(z, Y, m_\infty) = Y_s + O(z) \quad \forall s \in E, \tag{7.34}
\]
\[
\hat{v}_s(z, Y, m_\infty) = O(z) \quad \forall s \in E, \tag{7.35}
\]
\[
m(z, Y, m_\infty) = m_\infty + O(z),
\]
where \( Y_s = \text{pr}^+_s (Y) \). Let \( \{f_j | j = 1, 2, \ldots, M\} \) and \( \{e_m | j = 1, 2, \ldots, M\} \) be the same basis for \( E_+ \) as introduced in the proof of theorem 4. Then we can introduce new variables \( \{\hat{u}_j(z) | j = 1, 2, \ldots, M\} \) via
\[
\sum_{s \in E} u_s(z) z^s = \sum_{j=1}^M \hat{u}_j(z) z^{k_j} f_j. \tag{7.35}
\]

Expand \( \Omega_+ \) and \( \Lambda_+ \) in the basis \( \{e_{m_j} | j = 1, 2, \ldots, M\} \) as follows
\[
\Omega_+ = \sum_{j=1}^M w_{j, \infty} e_{m_j} \quad \text{and} \quad \Lambda_+(z) = \sum_{j=1}^M w_j(z) e_{m_j}. \tag{7.36}
\]

Results (7.32), (7.35), (7.20), and (7.36) then imply that
\[
w_j(z) = w_{i, \infty} + \sum_{j=1}^M C_{ij} \hat{u}_j(z) z^{k_j} \quad i = 1, 2, \ldots, M,
\]
while from (7.34) and (7.35) it is clear that
\[
\hat{u}_j(z, Y, m_\infty) = \alpha_j(Y) + O(z) \quad j = 1, 2, \ldots, M,
\]
where \( \alpha_j(Y) = \langle f_j | Y \rangle \). \( \square \)
Proof of theorem 6. The proof of this theorem involves finding a change of variables to put the system of differential equations (3.3) and (3.5) into a form where theorem 3 applies in a neighborhood of \( r = r_H \).

Note that although we use the space \( E_+ \) which was defined in section 6, this proof does not depend on the results of section 6. Indeed, \( E_+ \) can be replaced by \( \sum_{\alpha \in S} \Re e_{\alpha} \) everywhere in the proof below and one does not have to know that \( E_+ = \sum_{\alpha \in S} \Re e_{\alpha} \), which is the content of proposition 4. The notation \( E_+ \) is used for convenience.

Introduce new variables \( t, \mu, \) and \( v \) via

\[
t = r - r_H, \quad N = t(\mu + \nu), \quad v = (\mu + \nu)\Lambda_+. \tag{7.37}
\]

where \( \nu \) is a constant. Then

\[
t \frac{d\Lambda_+}{dt} = t \left( \frac{v}{\mu + \nu} \right), \tag{7.38}
\]

and it is clear that there exists analytic maps

\[
\hat{F}: E_+ \to E_+ \quad \text{and} \quad \hat{P}: E_+ \to \mathbb{R}
\]

such that

\[
\hat{F}(\Lambda_+) = \mathcal{F} \quad \text{and} \quad \hat{P}(\Lambda_+) = P.
\]

Assume \(|\nu| > 0\). Define an analytic map

\[
\hat{G}: E_+ \times I_{|\nu|}(0) \to \mathbb{R}
\]

by

\[
\hat{G}(X, a) = \frac{1}{2(a + \nu)^2} \|X\|^2.
\]

Then

\[
G = \hat{G}(v, \mu).
\]

Using these new variables, we can write the EYM equations (3.3) and (3.5) as

\[
t \frac{d\mu}{dt} = - (\mu + \nu) + \frac{1}{r_H} \hat{P}(\Lambda_+) + t \left[ \frac{1}{t} \left( \frac{1}{t + r_H} - \frac{1}{r_H} \right) \right]
\]

\[
- \frac{2}{t} \left( \frac{1}{(t + r_H)^2} - \frac{1}{r_H^2} \right) \hat{P}(\Lambda_+) + \left( \frac{\mu + \nu}{t + r_H} \right) \left( 1 + 2 \hat{G}(v, \mu) \right), \tag{7.39}
\]

and

\[
t \frac{dv}{dt} = -v - \frac{1}{(t + r_H)^2} \hat{F}(\Lambda_+) - t \left( \frac{2\hat{G}(v, \mu)}{t + r_H} \right) v, \tag{7.40}
\]

respectively. Introduce two new variables \( \hat{\mu} \) and \( \hat{v} \) via

\[
\hat{\mu} = \mu + \nu - \frac{1}{r_H} \hat{P}(\Lambda_+) + \frac{2}{r_H^2} \hat{P}(\Lambda_+), \tag{7.41}
\]

\[
\hat{v} = \frac{1}{r_H} \hat{F}(\Lambda_+). \tag{7.42}
\]
Define an analytic map
\[ \gamma : E_+ \times \mathbb{R} \rightarrow \mathbb{R} \]
by
\[ \gamma(X, a) = a - \nu + \frac{1}{r_H} - \frac{2}{r_H^3} \hat{P}(X) . \]

Fix a vector \( Z \in E_+ \) that satisfies \( \| \frac{1}{r_H} - \frac{2}{r_H^3} \hat{P}(Z) \| > 0 \). Then if we set
\[ \nu = \frac{1}{r_H} - \frac{2}{r_H^3} \hat{P}(Z) , \]
we get \( \gamma(Y, 0) = 0 \). So we can define an open neighborhood \( D \) of \( (Z, 0) \in E_+ \times \mathbb{R} \) by
\[ D = \{ (X, a) \mid \| \gamma(X, a) \| < \| \nu \| \} . \]

Then from (7.38), (7.40), (7.41), and (7.42), it is not hard to show that there exists an \( \varepsilon > 0 \) and analytic maps
\[ G : E_+ \times D \rightarrow \mathbb{R} , \]
\[ H : E_+ \times D \times I_\varepsilon(0) \rightarrow \mathbb{R} , \]
\[ K : E_+ \times D \times I_\varepsilon(0) \rightarrow \mathbb{R} , \]
such that
\[ td\Lambda_+ + dt = tG(\hat{v}, \Lambda_+, \hat{\mu}) , \tag{7.43} \]
\[ td\hat{v} = -\hat{v} + tH(\hat{v}, \Lambda_+, \hat{\mu}, t) , \tag{7.44} \]
\[ td\hat{\mu} = -\hat{\mu} + tK(\hat{v}, \Lambda_+, \hat{\mu}, t) . \tag{7.45} \]

The system of differential equations (7.43), (7.44) and (7.45) is in the form for which theorem 3 applies. Applying this theorem shows that exists a unique solution \( \{ \Lambda_+(t, Y), \hat{v}(t, Y), \hat{\mu}(t, Y) \} \) to this system of differential equations that is analytic in a neighborhood of \( (t, Y) = (0, Z) \) and that satisfies
\[ \Lambda_+(t, Y) = Z + O(t) , \tag{7.46} \]
\[ \hat{v}(t, Y) = O(t) , \]
\[ \hat{\mu}(t, Y) = O(t) . \tag{7.47} \]

Expand \( Z \) and \( \Lambda_+ \) in the basis \( \{ e_\alpha \mid \alpha \in S_\lambda \} \) as follows
\[ Z = \sum_{\alpha \in S_\lambda} w_{\alpha, r_H} e_\alpha \text{ and } \Lambda_+(t) = \sum_{\alpha \in S_\lambda} w_\alpha(t) e_\alpha . \]

Then equation (7.44) shows that
\[ w_{\alpha}(t, Z) = w_{\alpha, r_H} + O(t) \quad \forall \alpha \in S_\lambda . \]

It also not difficult to show that equations (7.37), (7.44), and (7.47) imply that
\[ N(t, Z) = \nu t + O(t^2) . \]

From this it follows immediately that
\[ N(r_H) = 0 \quad \text{and} \quad N'(r_H) = \nu . \]

\[ \square \]
References

[1] R. Bartnik, *The spherically symmetric Einstein Yang-Mills equations*, Relativity Today (Z. Perjés, ed.), 1989, Tihany, Nova Science Pub., Commack NY, 1992, pp. 221–240.

[2] R. Bartnik, *The structure of spherically symmetric su(n) Yang-Mills fields*, J. Math. Phys. 38 (1997), 3623–3638.

[3] R. Bartnik and J. McKinnon, *Particlelike solutions of the Einstein-Yang-Mills equations*, Phys. Rev. Lett. 61 (1988), 141–144.

[4] P. Breitenlohner, P. Forgács, and D. Maison, *Static spherically symmetric solutions of the Einstein-Yang-Mills equations*, Comm. Math. Phys. 163 (1994), 141–172.

[5] O. Brodbeck, *Gravitierende Eichsolitonen und schwarze Löcher mit Yang-Mills-Haar für beliebige Eichgruppen*, Ph.D. thesis, Universität Zürich, 1995.

[6] O. Brodbeck and N. Straumann, *A generalized Birkhoff theorem for the Einstein-Yang-Mills system*, J. Math. Phys. 34 (1993), 2412–2423.

[7] O. Brodbeck and N. Straumann, *Selfgravitating Yang-Mills solitons and their Chern-Simons numbers*, J. Math. Phys. 35 (1994), 899–919.

[8] O. Brodbeck and N. Straumann, *Instability proof for Einstein-Yang-Mills solitons and black holes with arbitrary gauge groups*, J. Math. Phys. 37 (1996), 1414–1433. (gr-qc/9411058)

[9] D.H. Collingwood and W.M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold, New York, 1993.

[10] E.B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. (2) 6 (1957), 111–244.

[11] D.V. Gal’tsov and M.S. Volkov, *Charged non-Abelian su(3) Einstein-Yang-Mills black holes*, Phys. Lett. B 274 (1992), 173–178.

[12] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer New York, 1972.

[13] B. Kleihaus, J. Kunz, and A. Sood, *SU(3) Einstein-Yang-Mills sphalerons and black holes*, Phys. Lett. B 354 (1995), 240–246. (hep-th/9504053)

[14] B. Kleihaus, J. Kunz, and A. Sood, *Sequences of Einstein-Yang-Mills-dilaton black holes*, Phys. Rev. D (3) 54 (1996), 5070–5092. (hep-th/9605109)

[15] B. Kleihaus, J. Kunz, and A. Sood, *Charged SU(N) Einstein-Yang-Mills black holes*, Phys. Lett. B 419 (1998), 284-293. (hep-th/9705179)

[16] B. Kleihaus, J. Kunz, A. Sood, and M. Wirschins, *Sequences of globally regular and black hole solutions in SU(4) Einstein-Yang-Mills theory*, Phys. Rev. D (3) 58 (1998), 084006. (hep-th/9802143)

[17] S. Kobayashi and K. Nomizu, *Foundations of differential geometry I*, Interscience, Wiley, New York, 1963.

[18] H.P. Künzle, *SU(n)-Einstein-Yang-Mills fields with spherical symmetry*, Classical Quantum Gravity 8 (1991), 2283–2297.
[19] H.P. Künzle, *Analysis of the static spherically symmetric SU(n)-Einstein-Yang-Mills equations*, Comm. Math. Phys. **162** (1994), 371–397.

[20] A.I. Mal’cev, *Commutative subalgebras of semisimple Lie algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **9** (1945), 291–300.

[21] J.E. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry*, Springer-Verlag, 1994.

[22] N.E. Mavromatos and E. Winstanley, *Existence theorems for hairy black holes in su(N) Einstein-Yang-Mills theories*, J. Math. Phys. **39** (1998), 4849–4873. (gr-qc/9712049) [k6299]

[23] R.V. Moody and A. Pianzola, *Lie algebras with triangular decompositions*, Wiley, New York, 1995.

[24] J.A. Smoller, A.G. Wasserman, S.-T. Yau, *Existence of black hole solutions for the Einstein-Yang/Mills equations*, Comm. Math. Phys. **154** (1993), 377-401.

[25] J.A. Smoller, A.G. Wasserman, S.-T. Yau, and J.B. McLeod, *Smooth static solutions of the Einstein/Yang-Mills equations*, Comm. Math. Phys. **143** (1991), 115–147.

[26] M.S. Volkov and D.V. Gal’tsov, *Gravitating non-abelian solitons and black holes with Yang-Mills fields*, Phys. Rep. (1998). (hep-th/9810070)

[27] H.C. Wang, *On invariant connections over a principal bundle*, Nagoya Math. J. **13** (1958), 1–19.