STABLE MANIFOLD MARKET SEQUENCES

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Abstract. In this article, we construct examples of discrete-time, dynamic, partial equilibrium, single product, competition market sequences, namely, \( \{m_t\}_{t=0}^{\infty} \), in which, potentially active firms are countably infinite, the inverse demand function is linear, and the initial market \( m_0 \) is null. For Cournot markets, in which, the number of firms is defined exogenously, as a finite positive integer, namely \( n : n > 3 \), the long term behavior of the quantity supplied, into the market, by Cournot firms is not well explored and is unknown. In this article, we conjecture, that in all such cases, the Cournot equilibrium, provided that it exists, is unreachable. We construct Cournot market sequences, which might be viewed, as appropriate resource tools, through which, the “unreachability” of Cournot equilibrium points is being resolved. Our construction guidelines are, the stable manifolds of Cournot equilibrium points. Moreover, if the number of active firms, increases to infinity and the marginal costs of all active firms are identical, the aggregate market supply, increases to a competitive limit and each firm, at infinity, faces a market price equal to its marginal cost. Hence, the market sequence approaches a perfectly competitive equilibrium. In the case, where marginal costs are not identical, we show, that there exists a market sequence, \( \{m_t\}_{t=0}^{\infty} \), which approaches an infinite dimensional Cournot equilibrium point. In addition, we construct a sequence of Cournot market sequences, namely, \( \{m_{it}\}_{i=0}^{\infty}, i \geq 1 \), which, for each, \( i \), approaches an imperfectly competitive equilibrium. The sequence of the equilibrium points and the double sequence, \( \{m_{it}\} \), both approach, the equilibrium, at infinity, of the market sequence, \( \{m_t\} \).

1. Introduction and basic assumptions. A significant amount of literature has been studying the behavior of markets, with a large or an infinite number of competing firms. A question of paramount importance is, whether or not, a sequence of imperfectly competitive markets, for example Cournot markets, converges to a perfectly competitive equilibrium. See [7], [8], [13], [15], [16], [17], [18], [21], and [22]. In this article we construct examples of discrete-time, dynamic, partial equilibrium, single product, competition market sequences, namely, \( \{m_t\}_{t=0}^{\infty} \), in which, potentially active firms are countably infinite and the inverse demand function is linear. More precisely, we assume that the clearing price of the market is, \( p = \max\{0, A - bD\} \), \( A, b > 0 \). For each, time stage, \( t \), the cost function of each firm, \( j \), is linear, with no fixed costs, namely, \( C_j(t, q) = c_j(t)q \), where, for each level

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of production, \( q \), as time, \( t \), increases, cost decreases. Moreover, the time varying, marginal cost sequence, \( \{c_j(t)\} \), is chosen such that, in each subsequent time stage, at most one “new” firm becomes active. Thereafter, for the construction of a market sequence, we use as guidelines, the stable manifolds of Cournot equilibrium points. For a given time stage, \( t \), the aggregate market quantity, supplied by the set of active firms, is equal to the corresponding stable manifold level. If a “new” firm becomes active, in a future stage, the amount it supplies into the market, shifts the aggregate market supply, from the existing stable manifold level to a higher stable manifold level. As a result, the market price decreases. Also, we prove, that in each time stage, \( t \), the actual profit of each active firm is strictly positive. Moreover, if the number of active firms, increases to infinity and the marginal costs of all active firms are identical, the aggregate market supply, increases to a competitive limit and each firm, at infinity, faces a market price equal to its marginal cost. Hence, the market sequence approaches a perfectly competitive equilibrium. In the case, where marginal costs are not identical, there exists a market sequence, which approaches a Cournot equilibrium point (with infinitely many components, all positive). In addition, we construct a sequence of Cournot market sequences, namely, \( \{m_{it}\}_{i\geq0} \), \( i \geq 1 \). For each, \( i \), the market sequence, \( \{m_{it}\}_{i\geq0} \), contains a finite number of active firms and as \( t \) increases to infinity, approaches an imperfectly competitive equilibrium point. Also, for each, \( t \), the market, \( m_{it} \), as \( i \) increases to infinity, approaches the market, \( m_t \). The limit of the sequence, formed by the equilibrium points of the Cournot markets, \( \{m_{it}\}_{i\geq0} \), for fixed, \( i \), as well as, the double limit of the sequence \( \{m_{it}\} \), is equal to the equilibrium, at infinity, of the market sequence \( \{m_t\} \). For each \( t \geq 0 \), a market, \( m_t \), is represented by the infinite supply vector, \( (y_1(t), \ldots, y_n(t), \ldots) \). The initial market, \( m_0 \), is null. Firms are free to leave the market, enter the market when profitable and compete on the production of a single product. For each \( t \), the short-run equilibrium occurs, if, \( D_t = S_t \), where, \( S_t = \sum_{j=0}^{\infty} y_j(t) \), is aggregate market supply.

Our work here, is a continuation of our work in [12], where among other things, we established, that the Cournot competition model, with variable marginal costs and fixed number of firms \( n \), is represented by the system of \( n \) equations:

\[
y_j(t+1) = \max \left\{ 0, -\frac{\phi(n,t)}{2} - \frac{y_j(t)}{2} + \frac{A - c_j(t+1)}{2b} \right\}, \quad t \geq 0, \quad 1 \leq j \leq n, \quad (1.0.1)
\]

where, \( \phi(n,t) \), is the aggregate supply, at time \( t \). In general, the Cournot model is not linear. Assume, for just a moment, that, \( c_j(t+1) = c_j \). If the Cournot equilibrium exists, (meaning all components are positive), then a strictly “positive” solution of System (1.0.1), satisfies the system of linear equations,

\[
y_j(t+1) = -\frac{\phi(n,t)}{2} - \frac{y_j(t)}{2} + \frac{A - c_j}{2b}, \quad t \geq 0, \quad 1 \leq j \leq n. \quad (1.0.2)
\]

However, there exists a solution of System (1.0.1), which still describes the behavior of the Cournot firms, \( \{1, \ldots, n\} \), which is not strictly positive and does not satisfy (1.0.2). System (1.0.2), represents the system of linearized equations of (1.0.1), with constant marginal costs, about the Cournot equilibrium. If the number of firms is greater than three, the eigenvalues of the characteristic equation of (1.0.2), lie inside the unit disc, except for one eigenvalue with magnitude greater than one. This implies that, the Cournot equilibrium is unstable. More precisely, is a saddle point. This instability of the Cournot equilibrium creates severe problems, in the sense that, the market sequence described by (1.0.1), might not reach an equilibrium. Our
study in this article is focused, on the construction of adjusted market sequences, \( \{m_t\} \), which approach an equilibrium point. The paper is organized as follows:

In Section 2, we set up the problem. We choose a family of marginal cost sequences, \( \{c_j(t)\}_{t=0}^{\infty} \), for which, during each time stage, \( t \), at most one “new” firm becomes active and the aggregate market supply, \( S_t \), is bounded away from infinity. We present a system of infinitely many non-autonomous difference equations together with its equilibrium points.

In Section 3, we prove, in the symmetric case, where the marginal costs of all active firms are identical, that the market described by the system of infinitely many equations, presented in Section 2, does not reach an equilibrium point. In addition, we prove that within a finite number of stages, all active firms exit the market and the market becomes null. Moreover, in all future time stages no firm is profitable.

In Section 4, motivated by the results of Section 3, we adjust the market sequence, by modifying the system of equations, presented in Section 2. The modified system incorporates our central hypothesis, called the Stable Manifold Hypothesis. This is the key idea of our paper. The new modified system of equations contains the same equilibrium points, as the system presented in Section 2.

In Section 5, we present sufficient conditions, in terms of the parameter \( A \) and the marginal cost sequence, \( \{c_j\} \), for the existence of a market sequence, \( \{m_t\} \), in which all firms become active sequentially. We prove that, if the marginal costs of all firms are identical, then the limiting market is perfectly competitive. In the case, where the marginal costs are different, we prove, that the limiting market, contains infinitely many firms, each of which produces a strictly positive equilibrium quantity. If the sum of all equilibrium quantities is equal to the aggregate supply limit, then the equilibrium of the limiting market is a Cournot equilibrium. We extend and generalize our results, by constructing market sequences, which contain both active and nonactive firms. The set of active firms, might be quite arbitrary, depending on the choice of appropriate subsequences of the marginal cost sequence. The number of nonactive firms might be finite or infinite. The limiting market, in the general case, has similar characteristics, as in the case, where all firms become active. We also present a numerical example, in which the marginal cost sequence is not constant, which approaches a Cournot equilibrium point with infinitely many firms. For comparison purposes, we also present a numerical solution of a market sequence, in which the Stable Manifold Hypothesis is not incorporated. In this example, firms exit the market within a finite number of stages.

In Section 6, we present a sequence of Cournot market sequences, namely \( \{m_{it}\}_{t=0}^{\infty}, i \geq 1 \). For each, \( i \), the corresponding market sequence, approaches an imperfectly competitive Cournot equilibrium. Also, for each, \( t \), as \( i \) increases to infinity, the market, \( m_{it} \), converges to the market, \( m_t \), constructed in Section 5. We also prove, that the limit of the sequence of Cournot equilibrium points, as well as, the double limit of the sequence, \( \{m_{it}\}_{t=0}^{\infty}, i \geq 1 \), both approach the limit, at infinity, of the market sequence, \( \{m_t\} \), constructed in Section 5.

The proofs of all theorems, are presented, in the Appendix at the end of the paper.

2. Setting up the problem. We focus on the construction of market sequences, which converge to an equilibrium point. We assume a countably infinite number of Cournot firms. The inverse demand function is given by: \( p = \max\{0, A - bD\} \), \( A, b > 0 \). The initial market is null. We need a certain restriction, with respect
to the rate, at which new firms, enter the market. In the case where all firms become active, during time stage, \( t = 1 \), each firm, \( j \), supplies the market with the monopoly quantity, \( \frac{A - c_j(t)}{2b} \), \( (c_j(1) \text{ denotes the marginal cost of firm } j \text{ at time stage } 1) \), the aggregate supply increases to infinity and the market price becomes zero. This forces all firms to exit the market, during the next stage. Clearly, if this phenomenon repeats infinitely many times, the market sequence does not approach an equilibrium point. As a matter of fact, the only known case, for which the market sequence approaches an equilibrium point, is the case of a duopoly, with constant marginal costs. For the behavior of the duopoly, with variable marginal costs, see [12]. In this article, we assume, that the rate at which “new” firms enter the market (number of firms per unit of time) is at most one. More precisely, we assume infinitely many technologies, which vary continuously, with respect to time.

We define: \( c_j(t) = f_j(t), \ t \in \{0, 1, \ldots, \} \), where \( \{f_j(t)\} \), is a family of continuously differentiable functions, with the following properties:

\[
f_j(t) \leq 0, \ \forall \ t \in [0, \infty), \ f_j(t) \geq A, \ \forall \ t \in [0, j), \ f_j(t) = c_j \in (0, A), \forall t \in [j, \infty).
\]

Thus, our marginal cost hypothesis is:

\[
(H_1): (c_j(t) \geq A, \ j > t \geq 0), \ (c_j(t) = c_j \in (0, A), \ t \geq j \geq 1). \tag{2.0.1}
\]

The first condition, \( c_j(t) \geq A, j > t \geq 0 \), of our marginal cost Hypothesis, \( H_1 \), implies that, for each time stage, \( t \geq 0 \), the firms with indices, \( j > t \), do not enter the market, \( m_t \), with inverse linear demand function, \( p(t) = A - bD_t, \ 0 < D_t < \frac{A}{v}, A, b > 0 \), because the corresponding marginal cost is greater than or equal to \( A \), and so, greater than the corresponding perceived market price. Thus, during each time stage, \( t \), we find that, \( y_j(t) = 0, \forall j > t \), and so, the set of indices of active firms, is a subset of \( \{1, \ldots, t\} \). Equivalently, a firm, \( j, j \geq 1 \), is not active in the markets, \( m_0, \ldots, m_{j-1} \), that is, \( y_j(t) = 0, \forall t \in \{0, \ldots, j-1\} \). The second condition, \( c_j(t) = c_j \in (0, A), \forall t \geq j \geq 1 \), implies, that the marginal cost of each firm, \( j \), is a positive constant, for all time stages, \( t \geq j \). Although, firm 1, does become active, during, time stage, \( t = 1 \), the general statement is, that a firm, \( j \), might become active during a time stage, \( t \), only if, \( t \geq j \). At each time stage, at most one “new” firm, becomes active. That is, during stage 1, firm 1 only, becomes active, in stage 2, firm 1 and firm 2 only, might become active, and so on. The aggregate supply, \( S_t \), at each time stage, \( t \), is bounded away from infinity, and is equal to, \( \sum_{j=1}^{n} y_j(t) \), where \( y_j(t) \geq 0, j \in \{1, \ldots, t\} \).

The supply vector sequence, \( \{y_1(t), \ldots, y_j(t), \ldots\} \), which is associated with the set of Cournot firms, namely, \( \{1, \ldots, j\} \), is a solution of the system of best response equations:

\[
y_j(t+1) = \max \left\{ 0, -\frac{S_t}{2} + \frac{y_j(t)}{2} + \frac{A - c_j(t+1)}{2b} \right\}, \ t \geq 0, \ j \geq 1. \tag{2.0.2}
\]

In view of \( H_1 \), System (2.0.2), becomes:

\[
y_j(t+1) = \left\{ \begin{array}{ll}
\max \left\{ 0, \frac{y_j(t)}{2} - \frac{S_t}{2} + \frac{A - c_j}{2b} \right\}, & t = j - 1, j, \ldots, \forall j \geq 1, \\
0, & t = -1, \ldots, j - 2, \forall j \geq 1.
\end{array} \right. \tag{2.0.3}
\]

System (2.0.3), is well defined, in terms of aggregate supply being bounded away from infinity. However, this does not imply that the market sequence, \( \{m_t\} \), approaches an equilibrium point. Actually, we conjecture that, solutions of System
(2.0.3) do not approach an equilibrium point, except for the case, where the number of firms that become active and remain active, for all future stages, is either exactly one or exactly two. In fact, we prove this conjecture, in the case, where \( c_j = c, \forall j \geq 1 \).

2.1. Equilibrium points. System (2.0.3), possesses infinitely many equilibrium points. Each one of them, is an infinite supply vector, \((\bar{y}_1, \ldots, \bar{y}_j, \ldots)\), with an infinite number of zero values and a positive finite number of nonzero values. More precisely, there exists a set of positive integers namely, \( M = \{l_1, \ldots, l_N\} \), such that, the positive values of the equilibrium are: \( \bar{y}_j = \frac{A + \sum_{k=1}^{N} c_{ik} - (N+1)c_j}{(N+1)b} > 0, \forall j \in \{1, \ldots, N\} \). The zero values of the equilibrium are defined as follows: \( \bar{y}_k = 0, k \notin M \).

The aggregate supply and the corresponding market price are, \( \bar{S} = \frac{N(A - \bar{c}(M))}{(N+1)b} \), and \( \bar{p} = \frac{A + N\bar{c}(M)}{N+1} \), respectively. \( (N\bar{c}(M) = \sum_{j=1}^{N} c_j) \).

Furthermore, in the limiting case, \( t = \infty \), there exist three types of equilibrium points:

1) A “positive equilibrium”, of the form, \((\bar{y}_1, \ldots, \bar{y}_j, \ldots)\), where \( \bar{y}_j = \frac{\bar{c} - c_j}{b} > 0, \forall j \geq 1 \), \( \lim_{t \to \infty} S_t = \frac{A - \bar{c}}{b} > 0 \), \( \lim_{t \to \infty} p(t) = \lim_{j \to \infty} c_j = \bar{c} \in (0, A) \), where \( p(t) \), denotes the market price at each stage \( t \). The supply vector, \((\bar{y}_1, \ldots, \bar{y}_j, \ldots)\), is an equilibrium solution of the equation:

\[
\frac{w_j(t + 1)}{2} = \max \left\{ 0, \frac{A - \bar{c}}{2b} + \frac{w_j(t)}{2} + \frac{A - c_j}{2b} \right\}, t = 0, 1, \ldots, \forall j \geq 1.
\]

At the equilibrium, the quantity supplied by each firm, \( j \), is equal to \( \frac{\bar{c} - c_j}{b} \). The aggregate supply and market price are, \( \frac{A - \bar{c}}{b} \) and \( \bar{c} \), respectively. In this case, the equilibrium vector, \((\bar{y}_1, \ldots, \bar{y}_j, \ldots)\), is a Cournot equilibrium, only if, \( \sum_{j=1}^{\infty} \frac{\bar{c} - c_j}{b} = \frac{A - \bar{c}}{b} \).

2) An equilibrium, with infinitely many nonzero values. The number of zero values might be either finite or infinite. More precisely, there exists an infinite set of positive integers namely, \( M = \{l_1, \ldots, l_n, \ldots\} \), such that, the positive values of the equilibrium are: \( \bar{y}_j = \frac{\bar{c} - c_j}{b} > 0, \forall j \in N \). The zero values of the equilibrium are defined as follows: \( \bar{y}_k = 0, k \notin M \). Moreover, \( \lim_{t \to \infty} S_t = \frac{A - \bar{c}}{b} > 0 \), \( \lim_{t \to \infty} p(t) = \lim_{j \to \infty} c_j = \bar{c} \in (0, A) \), where \( p(t) \), denotes the market price at each stage \( t \).

3) The zero equilibrium point, for which we have \( \lim_{t \to \infty} S_t = \frac{A - \bar{c}}{b} > 0 \), \( \lim_{t \to \infty} p(t) = \bar{c} = c_j, \forall j \geq 1 \). This equilibrium point is perfectly competitive.

3. The symmetric case. The next theorem verifies our conjecture about the global behavior of solutions of System (2.0.3), in the symmetric case.

Theorem 3.1. Consider the market sequence, which corresponds to the system of equations, presented in (2.0.3). Also, assume that, for each \( j \geq 1 \), \( c_j = c \in (0, A) \). Assume that the initial market is null. Then the following statements are true:

(i) The re-scaled \((r_t)\) infinite supply vectors, from \( t = 1 \) to \( t = 8 \) are:

\[
(1, 0)_{r_1}, (2, 1, 0)_{r_2}, (3, 2, 1, 0)_{r_3}, (5, 4, 3, 2, 0)_{r_4}, (7, 6, 5, 4, 2, 0)_{r_5}, (15, 14, 13, 12, 10, 8, 0)_{r_6}, (7, 6, 5, 4, 2, 0)_{r_7}, (111, 110, 109, 108, 106, 104, 104, 104, 0)_{r_8}.
\]

where \((x_1, \ldots, x_n, \ldots)_{r_t} = \frac{A-c}{2b}(x_1, \ldots, x_n, \ldots), t \geq 1 \leq t \leq 8 \) and \( \bar{0} = (0, \ldots, 0, \ldots) \).

(ii) For each \( t \geq 0 \),

\[
(y_1(2t + 9), \ldots, y_i(2t + 9), \ldots) = 0 \text{ and } (y_1(2t + 10), \ldots, y_j(2t + 10), \ldots) = (1_{2t+10}, 0)_{r_1},
\]
where $\bar{1}_{2t+10}$ is a finite $(2t+10)$-dimensional vector consisting from $2t+10$ 1’s. Furthermore, the market aggregate supply satisfies: $S_{2t+9} = 0, \forall t \geq 0$ and $S_{2t+10} \uparrow \infty$. For the market price, we have, $p(2t+9) = A, \forall t \geq 0$ and $p(2t+10) \downarrow 0$. Moreover, for all $t \geq 9$, no firm is profitable.

The results of Theorem 3.1 suggest, that the market described by System (2.0.3), needs some type of treatment, in terms of sustainability. For six subsequent stages, the number of active firms increases, from one to six. In each one of these stages, the market is profitable for each active firm. If, $t = 7$, firm 6, exits the market, while firm 7, is not able to become active. For the remaining five firms, the market is profitable. In addition, in this particular stage, supply decreases rapidly and this makes the market price increase to such a level, that during the next stage, $t = 8$, the supply levels soar, and as a result, the price plunges, forcing all eight active firms, to exit the market, during time stage, $t = 9$. In all future stages, the results are unrealistic. However, this type of situation is typical in Cournot markets, in the case where the number of firms increases, above three. If there is enough cooperation or communication between the first ten firms, the results could be different. For example, if the first ten firms, cooperate and unite, they might enter, time stage, $t = 10$, as a monopoly, which maximizes its profit for all ten firms and then distributes an equal share. However, the type of behavior, in which all firms are forced to exit the market, will still appear, within a finite number of future stages, implying that the convergence to an equilibrium point, needs a different type of treatment.

4. Stable manifold hypothesis. In this section, we state our central hypothesis, which we call: **Stable Manifold Hypothesis**. This hypothesis is based on our conjecture, that market sequences, which are initially null and are described by System (2.0.3), do not approach an equilibrium. As mentioned, Monopoly and Duopoly are both exempted, from this conjecture. The hypothesis, which we present in this section, appropriately adjusts markets by modifying System (2.0.3). The modified system has the same equilibrium points and the corresponding market sequence approaches an equilibrium point.

Our study, is related to the “sustainability” problem, that appears in Cournot markets. More precisely, if $n > 3$, the Cournot equilibrium, of the $n$-firm Cournot market,

$$
y_j(t + 1) = \max \left\{ 0, -\frac{\phi(n, t)}{2} + \frac{y_j(t)}{2} + \frac{A - c_j}{2b} \right\}, \quad t \geq 0, \quad 1 \leq j \leq n,
$$

where $\phi(n, t)$, is the aggregate supply and, $c_j$, is the marginal cost of each firm $j$, is an unstable saddle point (see [12], [19], [20]). The stable manifold theory predicts, that there exists, a set of initial conditions called the stable manifold, (see [5]), such that, a solution with initial conditions, within the stable manifold set, converges to the Cournot equilibrium. On the other hand, if for some $t$, the supply values, $\{y_1(t), \ldots, y_n(t)\}$, do not belong in the stable manifold set, the supply behavior of Cournot firms, in the long run, is unknown. In fact, extensive computer observations suggest, that in the majority of cases, firms are likely to exit the market, within a finite number of time stages, and the corresponding supply vector does not approach an equilibrium point. This particular behavior appears in System (2.0.3), as well.

Our goal is, to modify System (2.0.3), such that, during each time stage $t$, the corresponding set of active firms, supplies the market at the stable manifold supply
level. This is a specific aggregate supply level, which depends on the number of active firms and is known before hand. In fact, for a given time stage and a given set, namely \( M = \{l_1, \ldots, l_N\} \), of active firms, the stable manifold supply level is equal to: \( Q_M = \frac{N(1-\bar{c}(M))}{(N+1)b} \), where \( N\bar{c}(M) = \sum_{j=1}^N c_j \). Equivalently, one may see, that the stable manifold supply level is, the sum of the equilibrium values of the Cournot market, which contains, as many firms, as set \( M \) suggests. Let us present a simple example. During time stage, \( t = 1 \), the supply vector of System (2.0.3), is \( (y_1(1), \ldots, y_j(1), \ldots) = (\frac{A-c_1}{2b}, 0) \), where 0 is the zero vector. It is easy to see that, \( y_1(1) = \frac{A-c_1}{2b} = \bar{Q}_{M_1} \), where \( M_1 = \{1\} \). In other words, firm 1, supplies market, \( m_1 \), at the stable manifold supply level, \( \bar{Q}_{M_1} \). If, no new firm becomes active, in the future, the stable manifold supply level, remains the same, for all subsequent time stages. If this happens, then the market remains, a firm 1, monopoly. Next, assume, that firm 2, becomes active at stage, \( t = 2 \). No other firm, becomes active at stage, \( t = 2 \). Also, firm 1, the monopolist, keeps supplying the market, with the same amount, as in stage 1. Thus, the index set of active firms, of market, \( m_2 \), is equal to \( M_2 = \{1, 2\} \). The corresponding stable manifold supply level is, \( \bar{Q}_{M_2} = \frac{2(1-\bar{c}(M_2))}{2b} = \frac{2A-c_1-c_2}{2b} \). In view of (2.0.3), it is easy to see that, the amount of best response of firm 2, is equal to, \( \frac{A+c_1-2c_2}{2b} \). Thus, provided that, \( c_2 < \frac{A+c_1}{2} \), firm 2 becomes active and \( y_2(2) = \frac{A+c_1}{2} \).

Firm 1, supplies market, \( m_2 \), with, \( y_1(2) = \frac{A-c_1}{2b} \). The aggregate supply is, \( S_2 = y_2(1) + y_2(2) > \bar{Q}_{M_2} \). Thus, the aggregate supply level of the two firms, is greater than the corresponding stable manifold level. This slight discrepancy, might result into a divergent market sequence. If no other firm was to become active in the market and if \( c_1 < \frac{A+c_2}{2} \), one may easily see, that both firms remain active, for all future time stages and the market approaches a Cournot equilibrium. However, if this does not happen and another firm, firm 3, for example, becomes active during time stage, \( t = 3 \), we conjecture, that the corresponding market sequence does not approach an equilibrium point. For the symmetric case, this conjecture was established in Theorem 3.1. The question now is the following: **What type of adjustment in firm behavior is sufficient, such that, the market sequences approaches an equilibrium?** As far, as the presented example is concerned, the answer that we provide in this article, suggests, that firm 2, which is “new” in the market, must supply, \( Q_{M_2} - \bar{Q}_{M_1} > 0 \), such that the aggregate supply, \( S_2 \), shifts, from the stable manifold level, \( \bar{Q}_{M_1} \), to a higher stable manifold level, \( \bar{Q}_{M_2} = \bar{Q}_{M_1} + (\bar{Q}_{M_2} - \bar{Q}_{M_1}) \).

We are now ready to state our central hypothesis: The Stable Manifold Hypothesis.

\((H)\): Assume that during time stage, \( t \geq 1 \), the index set of active firms, of market, \( m_t \), is equal to, \( M_t = \{l_1, \ldots, l_N\} \) and the aggregate supply is equal to \( S_t \).

Set \( \bar{Q}_{M_t \cup \{t+1\}} = \frac{(N+1)(A-\bar{c}(M_t \cup \{t+1\}))}{(N+2)b} \) \( (4.0.2) \), where \( (N+1)\bar{c}(M_t \cup \{t+1\}) = \sum_{k=1}^N c_k + c_{t+1} \). Then the amount supplied by firm, \( t+1 \), into market, \( m_{t+1} \), is equal to, \( \max\{0, \bar{Q}_{M_t \cup \{t+1\}} - S_t\} \).

**Remark 4.1.** Note that Hypothesis \((H)\), accounts only for firms, that become active in the market for the first time. That is, a firm which is already active, supplies the market through its best response function. On the other hand, if a “new” firm, with index, \( t + 1 \), believes that market, \( m_{t+1} \), is profitable, it supplies
into the market, the amount \( \tilde{Q}_{M_t \cup \{ t+1 \}} - S_t \). However, if in addition, \( S_t = Q_{M_t} \), (which is not part of our Hypothesis) then the amount supplied, by the new firm, is equal to, \( \tilde{Q}_{M_t \cup \{ t+1 \}} - Q_{M_t} \). Moreover, if all firms, which were active in market, \( m_t \), remain active in market, \( m_{t+1} \), (this is not part of our Hypothesis either) they supply into the market, the amount, \( \tilde{Q}_{M_t} \), and so, the aggregate supply of the market, from time stage, \( t \), to the next one, \( t + 1 \), rises from the stable manifold level, \( \tilde{Q}_{M_t} \), to the higher stable manifold level, \( \tilde{Q}_{M_{t+1}} \), where, \( M_{t+1} = M_t \cup \{ t+1 \} \).

It is also important to mention that, our Hypothesis, is not a sufficient condition, for a “new” firm to become active. That is, the quantity, \( \max \{ 0, \tilde{Q}_{M_t \cup \{ t+1 \}} - S_t \} \), might as well be zero. In fact, the set of active firms, during each time stage, \( t \), will be determined by the choice of the marginal cost sequence, \( \{ c_j \} \).

A hypothesis, which is equivalent with \( (H_2) \), in the sense, that results into the same additional amount of supply, is the Equilibrium Locus Hypothesis. That is, we assume that each “new” firm calculates the quantity to be supplied, by using a substitute of a supply function, which is the equilibrium locus. It is an amazing fact, that, by using the equilibrium locus and calculating the market price, for which, demand equals “supply”, the equilibrium locus quantity coincides, with the stable manifold supply level and guides the new firm to supply the market, with the same amount, as \( (H_2) \) suggests. From this point of view our hypothesis would be as follows:

\( (H_2) ': \) Assume that during time stage, \( t \geq 1 \), the index set of active firms, of market, \( m_t \), is equal with \( M_t = \{ l_1, \ldots, l_N \} \). Set \( (N+1)c(M_t \cup \{ t+1 \}) = \sum_{k=1}^N c_k l_k + c_{t+1} \). Then the amount supplied by firm, \( t + 1 \), into market, \( m_{t+1} \), is equal to, \( \max \{ 0, S_{t+1} - S_t \} \), where, \( S_{t+1} = \frac{(N+1)(p(t+1) - c(M_t \cup \{ t+1 \}))}{b} \), is the aggregate supply expected by firm, \( t+1 \), in market, \( m_{t+1} \), and \( p(t+1) \), is determined by the equation \( S_{t+1} = D_{t+1} \).

The concept of equilibrium locus, serving as the supply function substitute for oligopolies and its relationship with variational conjectures, has been studied by several authors. See [2], [3], and [14]. It is also important to mention, that in [11], the authors argue that the decision rules of a firm, especially those that are to be followed by lower level managers, must be made in advance. In [9], the author argues that firms may use supply functions to determine the quantity as a function of market price. Also, in [10] the author studies imperfect competition markets by utilizing supply functions.

The Stable Manifold Hypothesis, modifies System (2.0.3), as follows:

\[
y_j(t+1) = \begin{cases} 
\max \left\{ 0, \frac{y_j(t)}{2} - \frac{S_t}{2} + \frac{A - c_j}{2b} \right\}, & t = j, j+1, \ldots, \\
\max \left\{ 0, \tilde{Q}_{M_t \cup \{ t+1 \}} - S_t \right\}, & t = j-1, \\
0, & t = -1, \ldots, j-2,
\end{cases} \quad (4.0.3)
\]

where, \( \tilde{Q}_{M_t \cup \{ t+1 \}} \), is the quantity presented in (4.0.2) and \( S_t \) is the aggregate supply at time stage \( t \).

**Theorem 4.1.** Let \( \{ m_t \} \) be the market sequence corresponding to (4.0.3). Assume that:

\( (H_3) : \) The sequence \( \{ M_t \} \), of the sets of indices of active firms, is increasing, \( M_t \subseteq M_{t+1}, \forall t \geq 0 \).

Then, for all \( j \geq 1 \), \( y_j(j) = \tilde{Q}_{M_j} - \tilde{Q}_{M_{j-1}} \), and \( S_j = \tilde{Q}_{M_j} \), where \( M_j = M_{j-1} \) or \( M_j = M_{j-1} \cup \{ j \} \). Moreover, if firm, \( j \), becomes active at stage \( t = j \), then its
supply sequence, \( \{y_j(t)\} \), satisfies the equation
\[
y_j(t + 1) = \begin{cases} 
\frac{y_j(t)}{2} - \frac{Q_{M_t}}{2} + \frac{A - c_j}{2b}, & t = j, j + 1, \ldots, \\
\frac{Q_{M_t}}{2} - \frac{Q_{M_{t-1}}}{2}, & t = j - 1, \\
0, & t = -1, \ldots, j - 2,
\end{cases}
\] (4.0.4)
where \( Q_{M_t} = \frac{N(A - \bar{c}(M_t))}{(N + 1)b} \), \( N = N(t) \), represents the number of active firms, at time stage, \( t \), and \( N(M_t) = \sum_{k \in M_t} c_k \). If firm, \( j \) does not become active, at stage, \( t = j \), then \( y_j(t) = 0 \), for all \( t \geq 0 \).

**Remark 4.2.** Although the hypothesis, on the increasing nature of the sequence, \( \{M_t\} \), might seem abstract or general, when we construct the stable manifold market, we give explicit conditions, in terms of the parameter \( A \) and the marginal cost sequence \( \{c_j\} \), such that this hypothesis is satisfied. We should also point out that this hypothesis is very important. In fact, under this hypothesis, apart from the sequence \( \{A_t\} \), we give explicit conditions, in terms of the parameter \( A \) and the marginal cost sequence \( \{c_j\} \), such that this hypothesis is satisfied. We should also point out that this hypothesis is very important. In fact, under this hypothesis, apart from the obvious fact, that if a firm becomes active, it remains active, it also implies that each set of active firms, is a market, in which, a Cournot equilibrium exists.

**Remark 4.3.** Note that, Theorem 4.1, divides the market sequence, \( \{m_t\} \), into two subsets. The subset of inactive firms and the subset of active firms. Both types of firms, and consequently, the entire market, is represented by the system of equations:
\[
y_j(t + 1) = \begin{cases} 
\max \left\{ 0, \frac{y_j(t)}{2} - \frac{Q_{M_t}}{2} + \frac{A - c_j}{2b} \right\}, & t = j, j + 1, \ldots, \\
\frac{Q_{M_t}}{2} - \frac{Q_{M_{t-1}}}{2}, & t = j - 1, \\
0, & t = -1, \ldots, j - 2.
\end{cases}
\] (4.0.5)

We conjecture that, a solution of System (2.0.3), with zero initial conditions, is not a solution of System (4.0.5) and vice versa. However, one may easily see, that the two systems have the same equilibrium points. Provided that \((H_3)\) holds, we prove in this article, that market sequences, \( \{m_t\}_{t=0}^{\infty} \), \( m_0 = \emptyset \), described by System (4.0.5), approach an equilibrium. System (4.0.5), might be viewed as, the appropriate resource tool, through which, a market sequence, \( \{m_t\} \), approaches the equilibrium points of System (2.0.3).

**Remark 4.4.** In the market sequence which corresponds to (4.0.5), the amount supplied by a “new” firm, during its entry stage, is lower than the amount of the best response, that the firm would supply into the market, without the Stable Manifold Hypothesis. The corresponding profit is lower, as well. However, we conjecture that the foregone profit is compensated by the benefit of being an active firm, for all time stages. Note that, in all market sequences that we construct in this article, the profit of each firm is strictly positive, during each time stage, \( t \).

5. A stable manifold market where all firms become active.

**Theorem 5.1.** Let \( \{c_j\}_{j=1}^{\infty} \), be a marginal cost sequence, for which it holds
\[
\max_{1 \leq k \leq i+1} c_k < \min_{1 \leq k \leq i+1} \frac{A + \sum_{j=1}^{i+1} c_j}{i + 1}, \quad \text{for all } i \geq 1.
\] (5.0.1)

Set \( \bar{c}_t = \sum_{j=1}^{t} c_j \), \( \bar{Q}_t = \frac{t(A - \bar{c}_t)}{(t + 1)b} \), and \( \bar{Q}_0 = 0 \). Consider the market \( \{m_t\}_{t=0}^{\infty} \), corresponding to System (4.0.3). Then, at each stage, \( t \), Hypothesis \((H_3)\) is satisfied.
More precisely, the indices of active firms form the index set, \( M_t = \{1, \ldots, t\} \). Moreover, for each \( j \geq 1 \), System (4.0.3) becomes:

\[
y_j(t + 1) = \begin{cases} 
y_j(t) - \frac{Q_j}{2} + \frac{A - c_j}{2b}, & t = j, j + 1, \ldots, \\
y_j(j - 2), & t = j - 1, \\
0, & t = -1, \ldots, j - 2.
\end{cases}
\]

The aggregate market supply is: \( S_t = \sum_{j=1}^t y_j(t) = \frac{t(A - c_j)}{(t+1)b} \) and the clearing price of the market is given by: \( p(t) = \frac{A + t c_j}{t+1} \). Moreover, at each time stage, \( t \), the actual profit of each firm is strictly positive.

5.1. Preview of the market behavior and limiting results. In this section, we present a preview of the market sequence, \( \{m_t\} \) constructed in Theorem 5.1. The number of active firms increases to infinity. There are no nonactive firms. Every firm becomes active. For each \( j \geq 1 \), firm \( j \), becomes active for the first time, at time \( t = j \), correctly predicts the market price, which is: \( p(j) = \frac{A + j c_j}{j+1} \), where \( j c_j = \sum_{k=1}^j c_k \). For each \( t \), the market, \( m_t \), is at the equilibrium level, that is, \( \frac{t(p(t) - \bar{c})}{b} \), which is also equal to the stable manifold level, that is, \( \frac{t(A - \bar{c})}{b} \). The market price sequence, \( \{p(t)\}_{t=0}^\infty \), decreases to a limit, namely \( \bar{c} \), which is equal to the limit of the sequence, \( \{\bar{c}_t\}_{t=0}^\infty \). For each \( t \), there exists an “implied” inverse supply function, namely \( f(S_t) = \bar{c}_t + \frac{b}{t} : S_t \), which determines the market equilibrium, if \( S_t = D_t \), with slope \( \frac{b}{t} \), which decreases to zero (supply becomes more and more elastic). At infinity, the market reaches an equilibrium, in which both supply and demand are equal to the competitive output level, \( \frac{A - \bar{c}}{b} \).

In the next theorem we present the limiting behavior of the market, \( \{m_t\} \), constructed in Theorem 5.1.

**Theorem 5.2.** Assume that (5.0.1) holds. Consider the market sequence, \( \{m_t\} \), corresponding to System (5.0.2). Then,

\[
\lim_{t \to \infty} y_j(t) = \bar{y}_j = \bar{c} - c_j, \quad \lim_{t \to \infty} S_t = \frac{A - \bar{c}}{b}, \quad \text{and} \quad \lim_{t \to \infty} p(t) = \bar{c},
\]

where \( \bar{c} = \lim_{j \to \infty} \bar{c}_j \in (0, A) \). Moreover, if, \( \lim_{n \to \infty} \frac{n(\bar{c}_t - \bar{c_n})}{b} = \frac{A - \bar{c}}{b} \), or equivalently, \( \sum_{j=1}^\infty \frac{\bar{c}_j - \bar{c_n}}{b} = \frac{A - \bar{c}}{b} \), then the market sequence, \( \{m_t\} \), approaches a Cournot equilibrium.

**Remark 5.1.** Note that, in the symmetric case, if the marginal cost sequence, \( \{c_j\} \) is constant, \( c_j = c, \forall j \), the market sequence, \( \{m_t\} \), approaches a market, in which the aggregate supply is equal to, \( \frac{A - \bar{c}}{b} \), and the marginal cost of each firm is equal to the market price. That is, the market sequence, approaches a perfectly competitive market. Moreover, each firm eventually, becomes infinitesimal, with respect to both quantity supplied and profit. However, we find it amazing, that in the heterogeneous, with respect to marginal costs case, as long as, \( \sum_{j=1}^\infty \frac{\varepsilon - c_j}{b} = \frac{A - \bar{c}}{b} \), the market sequence, \( \{m_t\} \), approaches a Cournot equilibrium, with infinitely many firms. Also, for this market sequence, the following holds: For every \( \varepsilon > 0 \), there exists, \( N = N(\varepsilon) \), such that, for all \( t \geq N \), the aggregate supply, \( \sum_{j=1}^N y_j(t) \), which corresponds to firms, \( 1, \ldots, N \), satisfies: \( \frac{A - \bar{c}}{b} - \varepsilon < \sum_{j=1}^N y_j(t) < \frac{A - \bar{c}}{b} \). In other words, as long as, all firms with index, \( 1, \ldots, N \), become active, they supply almost
the entire market, for all future time stages, while the remaining active firms, supply an arbitrarily small amount. This particular property, is based on the fact that: 
\[ \lim_{n,t \to \infty} \sum_{j=1}^{n} y_j(t) = \lim_{n \to \infty} \lim_{t \to \infty} \sum_{j=1}^{n} y_j(t) = \lim_{t \to \infty} \lim_{n \to \infty} \sum_{j=1}^{n} y_j(t). \]

5.2. Examples of marginal cost sequences and market sequences. Note that, although the sequence \( \{\bar{c}_i\}_{i=1}^{\infty} \), converges to a limit, we have not imposed an explicit requirement, with respect to the convergence of sequence \( \{c_j\}_{j=1}^{\infty} \). Actually, the requirement is, Condition (5.0.1). However, we must admit, that the only examples, that we are able to offer, are related to convergent sequences. It is easy to see that if the sequence \( \{c_j\} \) is periodic, it does not satisfy (5.0.1). We must point out, that periodicity might be possible, if we dynamically enlarge market demand, by converting the constant \( A \), into a time varying sequence, namely \( \{A_i\} \). We do believe that Condition (5.0.1), implies, that the sequence, \( \{c_j\} \), converges to a limit, but we are not able to prove it.

We should notify the reader, that if, the sequence is strictly decreasing, Condition (5.0.1), is not satisfied. On the other hand, if the sequence, \( \{c_j\} \), is increasing, or all of its convergent subsequences are increasing, Condition (5.0.1), may be satisfied. We present here, the existence of an increasing sequence, for which, Condition (5.0.1), is satisfied.

Theorem 5.3. Assume that \( A \in (0, \infty) \). Then there exists a sequence of positive numbers, namely \( \{c_n\}_{n=1}^{\infty} \), such that
\[ 0 < c_n \leq c_{n+1} < \frac{A + n\bar{c}_n}{n+1} < A, \text{ for all } n \geq 1, \] (5.2.1)
where \( n\bar{c}_n = \sum_{j=1}^{n} c_j \). Also, it holds that, \( \lim_{n \to \infty} c_n = \lim_{n \to \infty} \bar{c}_n = \bar{c} \in (0, A) \). Moreover, every positive and increasing sequence, which satisfies Condition (5.2.1), satisfies Condition (5.0.1), as well.

Remark 5.2. Condition (5.2.1), although less general, is much simpler, when compared to Condition (5.0.1). For the remainder of this article, whenever a condition on the marginal cost sequence, \( \{c_j\} \), is needed, we will use the simpler Condition (5.2.1).

In the next example, we present a market sequence, \( \{m_t\} \), which converges to a Cournot equilibrium.

Example 5.1. Consider the marginal cost sequence \( \{c_j\} \), defined as follows: \( c_n = \frac{(n+1)^2-1}{(n+1)^2}, \) \( n \geq 1 \). Clearly, \( \bar{c} = \lim_{n \to \infty} c_n = \lim_{n \to \infty} \bar{c}_n = 1 \). Let \( A = \frac{x^2}{6} \). Then, Condition (5.2.1), is satisfied, as it is equivalent, with \( \sum_{j=1}^{n} \frac{1}{j^2} < \frac{x^2}{6} + \frac{1}{n+1}, n \geq 1 \).

Also, \( \Phi_n = \frac{n\bar{c}-n\bar{c}_n}{6} = \frac{1}{b} \sum_{j=1}^{n} \frac{1}{j^2} \) and \( \lim_{n \to \infty} \Phi_n = \frac{x^2-1}{6} \). The market sequence, \( \{m_t\} \), which is presented in Theorem 5.1, in this particular case, approaches the Cournot equilibrium, \( \left( \frac{1}{36}, \ldots, \frac{1}{(n+1)^2}, \ldots \right) \), the aggregate supply of the limiting Cournot market is equal to, \( \frac{x^2}{6} \), and the corresponding market price is, \( p = 1 \).

In Figure 1 below, we present numerically the subsequence, \( \{m_{1_{\infty=0}}\} \), of a market sequence, \( \{m_{1_{\infty=0}}\} \), similar to the one presented in Example 5.1. In fact, the only difference is, that the value of the parameter, \( A \), is equal to 3.

In Figures 2 and 3 below, assuming, \( A = 3 \) and \( c_n = \frac{(n+1)^2-1}{(n+1)^2}, n \geq 1 \), we present numerically, nine stages of the market sequence, represented by System (2.0.3). The
Figure 1. The variable on the horizontal axis is time. Numerical solution of System (4.0.3), representing a sustainable market, within a time frame from 1 to 20. The lower part of the graph shows the distributions of the amounts supplied into the market by 20 firms, which become active sequentially from stage 1 to stage 20. The values of the parameters are: $A = 3$, $b = 1$, $c_j = 1 - \frac{1}{j(j+1)}$, $j \geq 1$.

Figure 2. The variable on the horizontal axis is time. Numerical Solution of System (2.0.3), which represents a non-sustainable market (The Stable Manifold Hypothesis is not incorporated). Time frame varies from 1 to 9. Firms, 1-6, become active sequentially. Firms 7 and 8 are not visible. They become active only at stage 8. At stage 9, all firms exit the market. The values of the parameters are: $A = 3$, $b = 1$, $c_j = 1 - \frac{1}{j(j+1)}$, $j \geq 1$.

Presentation serves as a comparison tool, because in this case, the Stable Manifold Hypothesis is not incorporated and we conjecture that the market does not approach an equilibrium point. As it turns out, in this case, all active firms exit the market, during time stage, $t = 9$. The market becomes null, at that stage.

5.3. Extensions and generalizations. The market sequence, $\{m_t\}$, which is presented in Theorem 5.1, might be generalized, such that, both subsets, the subset of inactive firms and the subset of active firms, are nonempty. This market sequence,
is similar to the market sequence presented in Theorem 5.1. In fact, the inactive firms, have no effect, with respect to, established market supply or established market price, at any stage. However, in this type of market, there exist, arbitrary lengths of time periods, for which the market, operates with a constant number of firms, until a new firm becomes active. In these time periods, both aggregate supply and market price remain constant, until, a new firm becomes active. At the time stage, in which, a new firm becomes active, the aggregate supply increases and the market price decreases. At infinity, the limiting values of supply and market price, are: \( \frac{A-c}{b} \) and \( \bar{c} \), respectively. Furthermore, since the marginal cost, of each one of the inactive firms, remains the same even at infinity, as it turns out, at infinity, their marginal cost is greater than or equal, to the market price, and so, the limiting market is not profitable for them.

**Theorem 5.4.** Assume that \( \{c_j\}_{j=1}^\infty \) is a marginal cost sequence. Let \( \{l_n\}_{n=1}^\infty \) be a strictly increasing sequence of positive integers, with \( l_1 = 1 \), such that

\[
0 < c_n \leq c_{n+1} < \frac{A + \sum_{j=1}^n c_j}{n+1} < A, \text{ for all } n \geq 1 \quad (5.3.1)
\]

and for each \( q \): \( l_n < q < l_{n+1} \), for some \( n \geq 1 \),

\[
c_q \geq \frac{A + \sum_{j=1}^n c_j}{n+1}. \quad (5.3.2)
\]

Consider the market sequence, \( \{m_t\} \) corresponding to System (4.0.3). For each \( j \geq 1 \), set \( M_l = \{l_1,\ldots,l_j\} \), \( \tilde{c}(M_l) = \sum_{k=1}^j c_k \), \( \tilde{Q}(M_l) = \frac{j(A-\tilde{c}(M_l))}{(j+1)b} \), and \( \tilde{Q}(\emptyset) = 0 \). Then, for all \( t \), Hypothesis (H3) is satisfied. More precisely, for all \( t \in \{l_j,\ldots,l_{j+1} - 1\} \) and for all \( j \geq 1 \), the set of indices of market active firms, form the index set \( M_t = M_{l_j} \). Moreover, the subsystem of (4.0.3), which corresponds to active firms becomes:

\[
y_j(t+1) = \begin{cases} 
y_{j}(t) & \text{if } t < l_j \leq t \leq l_{j+k+1} - 1, \ k \geq 0, \\
\frac{y_{j}(t)}{2} - \frac{\tilde{Q}(M_{l_j+k})}{2b} + \frac{A-c_{l_j}}{2b}, & \text{if } l_{j+k} \leq t \leq l_{j+k+1} - 1, \ k \geq 0, \\
\frac{Q(M_{l_j}) - \tilde{Q}(M_{l_j-1})}{b}, & \text{if } t = l_j - 1, \\
0, & \text{if } t = -1,\ldots,l_j - 2.
\end{cases} \quad (5.3.3)
\]
Also, for all \( t \in \{l_j, \ldots, l_{j+1} - 1\} \) and for all \( j \geq 1 \), the market aggregate supply and price are given by: \( S_t = S_{l_j} \) and \( p(t) = p(l_j) \), respectively. Moreover, at each time stage, \( t \), the actual profit of each active firm is strictly positive.

**Remark 5.3.** The limiting behavior of the market sequence, \( \{m_t\} \), presented in Theorem 5.4, is similar to the limiting behavior of the market sequence, \( \{m_t\} \), presented in Theorem 5.1. More precisely, for each \( j \geq 1 \),

\[
\lim_{t \to \infty} y_{i,t}(t) = \bar{y}_{i,t}, \quad \lim_{t \to \infty} S_t = \frac{A - \bar{c}}{b}, \quad \text{and} \quad \lim_{t \to \infty} p(t) = \bar{c},
\]

where \( \bar{c} = \lim_{j \to \infty} \bar{c}(M_{l_j}) \in (0, A) \). In the symmetric case, in which the marginal costs of all active firms, are equal, the market sequence, of active firms, converges to a perfectly competitive equilibrium. Moreover, if

\[
\lim_{j \to \infty} \left( A - \frac{j(\bar{c} - \bar{c}(M_{l_j}))}{b} \right) = \frac{A - \bar{c}}{b},
\]

then the submarket sequence, comprised by active firms only, approaches a Cournot equilibrium.

6. Finite number of firms. In this section, we construct a market sequence, \( \{m_t\}_{t=0}^{\infty} \), with a finite number of active firms. More precisely, for an arbitrary, but fixed, positive integer, \( i \geq 1 \), the firms \( 1, \ldots, i \), become active, at the corresponding time stages, while the firms \( i+1, i+2, \ldots \), never become active. The aggregate market supply increases and the market price decreases, from one stage to the next. Moreover, for all \( t > i \), aggregate supply and market price remain constant, and the vector supply sequence, \( \{(y_1(t), \ldots, y_i(t))\}_{t=0}^{\infty} \), which represents the subset of active firms, approaches a Cournot equilibrium.

The next theorem contains all the appropriate details.

**Theorem 6.1.** Assume that \( \{c_j\}_{j=1}^{\infty} \) is a marginal cost sequence, such that

\[
0 < c_i \leq c_{i+1} < \frac{A + ic_i}{i+1} \leq c_q < A, \text{ for all } 1 \leq n \leq i-1 \text{ and } q \geq i, \text{ if } i > 1 \quad (6.0.1)
\]

and

\[
0 < c_1 < \frac{A + c_1}{2} \leq c_q < A, \text{ for all } q > 1, \text{ if } i = 1. \quad (6.0.2)
\]

Consider the market sequence, \( \{m_t\} \) corresponding to System (4.0.3). Set \( \tilde{c}_t = \sum_{j=1}^{c_j} c_j, \tilde{Q}_t = \frac{t(A - \bar{c})}{(t+1)b}, \tilde{Q}_0 = 0 \). Then, for each \( t \), Hypothesis \( (H_3) \) is satisfied. More precisely, at each time stage, \( t \geq 1 \), the indices of active firms form the index set, \( M_t = \{1, \ldots, \min\{i,t\}\} \). Moreover, for each \( j : 1 \leq j \leq i \), System (4.0.3) becomes:

\[
y_{i}(t+1) = \begin{cases} 
-\frac{Q_{\min\{i,t\}}}{2} + \frac{y_{i}(t)}{2} + \frac{A-c_{i}}{2b}, & t = j, j+1, \ldots, \\
\bar{Q}_j - \bar{Q}_{j-1}, & t = j-1, \\
0, & t = -1, \ldots, j-2.
\end{cases} \quad (6.0.3)
\]

For each \( t \geq 1 \), the aggregate supply and the market price are:

\[
S_t = \frac{\min\{i,t\}(A - \bar{c}_{\min\{i,t\}})}{\min\{i,t\}+1} \quad \text{and} \quad p(t) = \frac{A + \min\{i,t\}c_{\min\{i,t\}}}{\min\{i,t\}+1}. \quad (6.0.4)
\]

The market sequence, of active firms, approaches a Cournot equilibrium. More precisely, \( \lim_{t \to \infty} y_j(t) = \bar{y}_j \), where \( \bar{y}_j = \frac{A + i\bar{c}_j - (j+1)c_j}{2(j+1)b} \), \( 1 \leq j \leq i \). Also, \( \lim_{t \to \infty} S_t = \bar{Q}_i \), and \( \lim_{t \to \infty} p(t) = \frac{A + i\bar{c}_j}{i+1} \).
6.1. The stable manifold market as the uniform limit of a sequence of Cournot markets. In this section, we construct a sequence of Cournot markets, namely \( \{m_{it}\} \). For each \( i \), the market sequence, \( \{m_{it}\} \), is presented in Theorem 6.1. We prove that, as \( i \) increases to infinity, the market \( m_{it} \) converges to the market, \( m_t \), described by (5.0.2), uniformly for all \( t \). By that we mean: the quantity supplied, by firm, \( i \), into market, \( m_{it} \), at time stage, \( t \), converges to the amount supplied, by firm \( j \), into market \( m_t \), uniformly for all \( t \). Moreover, in view of Theorem 6.1, as \( t \) increases to infinity, each Cournot market sequence, \( \{m_{it}\}_{t=0}^{\infty} \), converges to an imperfectly competitive Cournot equilibrium. The Cournot equilibrium sequence, \( \{x_{ij}(t)\}_{t=0}^{\infty} \), of the market sequence, \( \{m_{it}\}_{t=0}^{\infty} \), is omitted.

Proof. The proof is based on straightforward algebraic calculations and the details are omitted.

Appendix-Proofs.

The proof of Theorem 3.1.

Proof. The proof is based on straightforward algebraic calculations and the details are omitted.

The proof of Theorem 4.1.

Proof. The proof will be by induction. For \( j = 1 \), we have that \( M_0 = \emptyset, S_0 = \bar{Q}_{M_0} = 0 \), and \( y_1(1) = \frac{A - c_1}{2b} = \bar{Q}_{M_0, \cup(1)} - S_0 > 0 \). Thus, firm 1 becomes active, during time stage, \( t = 1 \), and so, since \( y_1(1) = 0 \), for all \( j > 1 \), \( M_t = \{1\} = M_0 \cup \{1\} \), \( y_1(1) = \bar{Q}_{M_1} - \bar{Q}_{M_0} \), and \( S_1 = \bar{Q}_{M_1} \). Assume that for an arbitrary \( j > 1 \) and
for all \( k \in \{1, \ldots, j\} \), \( y_k(k) = \bar{Q}_{M_k} - \bar{Q}_{M_{k-1}} \), \( S_k = \bar{Q}_{M_k} \) and \( M_k = M_{k-1} \) or \( M_k = M_{k-1} \cup \{k\} \). Set \( M_1 = \{1, \ldots, N\} \subseteq \{1, \ldots, N\}, N \geq 1, t_1 = 1 \). Next, we prove that \( y_j(j+1) = Q_{M_{j+1}} - Q_{M_j} \), \( S_{j+1} = \bar{Q}_{M_{j+1}} \), and \( M_{j+1} = M_j \) or \( M_{j+1} = M_j \cup \{j+1\} \). There exist two cases: \( y_j(j) = 0 \) or \( y_j(j) > 0 \). We present the proof in the case, where \( y_j(j) = Q_{M_j} - Q_{M_{j-1}} > 0 \). The proof in the other case, is similar and the details are omitted. First, observe that if a firm has index, \( k < j \) and \( k \notin M_{j-1} \), then \( y_k(j) = 0 \). From this it follows that, the perceived market price, \( p_k(j, y_k(j)) = 0 = A - bQ_{M_{j-1}} \leq c_k \). Since, \( Q_{M_j} > Q_{M_{j-1}} \), we have \( p_k(j+1) = A - b(Q_{M_j} + y_k(j+1)) < c_k \), and so, the firm, \( k \), does not become active in market, \( M_{j+1} \). Moreover, for each firm with index, \( b > j \geq 1 \), we have \( y_k(j+1) = 0 \). Since, \( M_j \subseteq M_{j+1} \), we see that \( M_{j+1} = M_j \) or \( M_{j+1} = M_j \cup \{j+1\} \). If firm, \( j+1 \), does not find profit in market, \( M_{j+1} \), the result follows. On the other hand, if firm, \( j+1 \), becomes active in market, \( M_{j+1} \), then \( M_{j+1} = M_j \cup \{j+1\} \). Also, the fact that the firm finds the market profitable, implies that, the marginal cost, \( c_j \), of the firm is less than the perceived market price, \( p_j(j+1, y_j(j+1)) \), which equivalently implies that, \( Q_{M_j} > Q_{M_{j+1}} \). This implies that, \( y_{j+1}(j+1) = \max\{0, Q_{M_j} \cup (j+1) - Q_{M_j}\} = Q_{M_{j+1}} - Q_{M_j} > 0 \). Finally, we prove that, \( S_{j+1} = \bar{Q}_{M_{j+1}} \). Indeed, observe that, \( \sum_{k \in M_j} y_k(j+1) = \sum_{k \in M_j} y_k(j) = Q_{M_j} \), and so, \( S_{j+1} = \sum_{k \in M_j} y_k(j+1) + y_{j+1}(j+1) = \bar{Q}_{M_{j+1}} \).

In view of Hypothesis, \( H_3 \), on the increasing nature of the sequence, \( \{M_t\} \), if the firm, \( j \), becomes active at time stage, \( t = j \), then \( y_j(t) > 0 \), for all \( t > j \). Furthermore, \( S_t = \bar{Q}_{M_t} \), for all, \( t \). Hence, the supply sequences of active firms, satisfy (4.0.5).

We also prove that if a firm, \( j \), does not become active, at time stage, \( t = j \), it never becomes active. Indeed, if firm, \( j \), does not find profit in market, \( M_j \), it implies that, \( c_j \geq p_j(j, y_j(j) = 0) = A - bQ_{M_{j-1}} \). Also, clearly, \( Q_{M_j} \geq Q_{M_{j-1}} \), for all \( j \geq 1 \), from which it follows that, \( c_j \geq A - bQ_{M_j} \), for all \( t > j \). Hence, firm, \( j \), does not find profit in the market, for all \( t > j \), which implies that, \( y_j(t) = 0 \), for all \( t \). The proof is complete.

**The proof of Theorem 5.1.**

**Proof.** The proof will be, by induction on the market sequence, \( \{m_t\}_{t=0}^{\infty} \). More precisely, we prove that, for each \( t \geq 0 \), the market, \( m_{t+1} \), contains the firms, \( \{1, \ldots, t + 1\} \), or equivalently, \( M_{t+1} = \{1, \ldots, t + 1\}, y_j(t+1) = \frac{\bar{Q}_t + y_j(t)}{2} + \frac{A - c_j}{2b}, j = 1, \ldots, t, y_{t+1}(t+1) = \bar{Q}_t - \bar{Q}_t, S_{t+1} = \bar{Q}_{t+1}, \) and \( p(t+1) = \frac{A + c_0}{t+1} \). (6.1.2)

Clearly, \( m_0 = \emptyset, M_0 = \emptyset, \) and \( S_0 = \bar{Q}_{M_0} = 0 \). Consider the market \( m_1 \). In view of (4.0.3), \( y_j(1) = 0 \), for all \( j > 1 \) and \( y_1(1) = \max\{0, Q_{M_0} \cup \{1\} - Q_{M_0}\} = Q_{\{1\}} = \frac{A - c_0}{2b} > 0 \). Furthermore, \( M_1 = M_0 \cup \{1\} = \{1\}, Q_{\{1\}} = \bar{Q}_1 - \bar{Q}_0, S_1 = y_1(1) = \bar{Q}_1 \), and \( p(1) = \frac{A + c_0}{2} \). This proves the first step of induction.

By the induction hypothesis, assume that, for some \( T > 0 \), each market, within the market sequence, \( \{m_t\}_{t=0}^{T} \), contains the firms, with indices, \( \{1, \ldots, t + 1\} \), and (6.1.2), (6.1.3), both hold. In particular, if \( t = T \), for each \( j : 1 \leq j \leq T \), we have \( y_j(T+1) = \frac{\bar{Q}_T + y_j(T)}{2} + \frac{A - c_j}{2b}, y_{T+1}(T+1) = \bar{Q}_{T+1} - \bar{Q}_T, S_{T+1} = \frac{(T+1)(A - c_{T+1})}{(T+2)b} = \bar{Q}_{T+1}, \) and \( p(T+1) = \frac{A(T+1) + c_1}{T+2} \).
During the next time stage, $T + 2$, the market price, perceived by each firm, $j : 1 \leq j \leq T + 1$, is equal to, $p_j(T + 2) = A - b(Q_{t+1} - y_j(T + 1) + y_j(T + 2))$. Firm $j$, sees that, if the aggregate supply, at time stage, $T + 2$, is equal to, $Q_{t+1}$, then the corresponding price, namely $\frac{A + (T + 1)\overline{c_t}}{T + 2}$, is lower than the perceived price, $p_j(T + 2, y_j(T + 2) = 0)$. Thus, $c_j < \frac{A + (T + 1)\overline{c_t}}{T + 2}$, implies that, $c_j < p_j(T + 2, y_j(T + 2) = 0$, from which it follows, that the firm, $j$, can supply into the market, a positive amount, $y_j(T + 2)$, with profit. However, $c_j < \frac{A + (T + 1)\overline{c_t}}{T + 2}$, is implied by (5.0.1), and so, each firm, $j$, active in market, $m_{T+1}$, is also active in market, $m_{T+2}$. Hence, $M_{T+1} = \{1, \ldots, T + 1\} \subseteq M_{T+2}$. Thus, (4.0.3), implies that, for all, $j : 1 \leq j \leq T + 1$, $y_j(T + 2) = -\frac{Q_{T+1}}{2} + \frac{y_j(T + 1) + A - c_j}{2b}$. Moreover, it is easy to see that, the aggregate supply, during, time stage, $T + 2$, which corresponds to firms, $\{1, \ldots, T + 1\}$, is $\sum_{j=1}^{T+1} y_j(T + 2) = \sum_{j=1}^{T+1} y_j(T + 1) = Q_{T+1}$.

The market price, during time stage, $T + 2$, perceived by firm, $T + 2$, is equal to, $p_{T+2}(T + 2) = A - b(Q_{T+1} + y_{T+2}(T + 2))$. The perceived price, $p_{T+2}(T + 2, Y_{T+2}(T + 2) = 0)$, is exactly equal to the actual market price, $p(T + 1)$, during time stage, $T + 1$. If its marginal cost, $c_{T+2}$, is less than $p(T + 1)$, the firm can supply into the market, a positive amount, $y_{T+2}(T + 2)$, with a profit. However, the inequality, $c_{T+2} < p(T + 1)$, is implied by Condition (5.0.1), and so, the firm becomes active, during time stage, $T + 2$. Since, for each firm, with index, $j$, greater than $T + 2$, we have, $y_j(T + 2) = 0$, we see that, $M_{T+2} = M_{T+1} \cup \{T + 2\} = \{1, \ldots, T + 1, T + 2\}$. From System (4.0.3), we have, $y_{T+2} = \max\{0, Q_{M_{T+1} \cup \{T + 2\}} - Q_{M_{T+1}}\}$. Since, $M_{T+1} \cup \{T + 1\} = M_{T+2} = \{1, \ldots, T + 2\}$ and $y_{T+2}(T + 2) > 0$, we find that, $y_{T+2} = \bar{Q}_{T+2} - \bar{Q}_{T+1}$. Hence, $S_{T+2} = \sum_{j=1}^{T+2} y_j(T + 2) = \sum_{j=1}^{T+1} y_j(T + 1) + y_{T+2}(T + 2) = \bar{Q}_{T+1} + \bar{Q}_{T+2} - \bar{Q}_{T+1} = \bar{Q}_{T+2}$ and $y_{T+1}(T + 2) = A - bD_{T+2} = \frac{A + (T + 2)\overline{c_t}}{T + 3}$. Finally, the result, that the actual profit of each firm is strictly positive, follows from (5.0.1), together with the fact that, during each time stage, $t$, the market price is, $p(t) = \frac{A + \overline{c_t}}{t + 1}$. The proof is complete.

The proof of Theorem 5.2.

Proof. First, we establish that, for each $j \geq 1$, $\lim_{t \to \infty} y_j(t) = \bar{y}_j = \frac{\bar{c} - \overline{c_t}}{b}$. Assume that, $j$, is arbitrary, however, fixed positive integer. In view of Theorem 5.1, $S_t = \bar{Q}_t$ and $p_t = \frac{A + \overline{c_t}}{t + 1}$, for all $t$. In view of (5.0.1), it is easy to see that, $\{\bar{Q}_t\}$, is an increasing sequence, which is bounded from above, by a positive constant. Thus, increases to a positive limit. From this and since, $\bar{Q}_t = \frac{b(A - \overline{c_t})}{(t + 1)b}$, we find that, the limit of the sequence, $\bar{c}_t$, also exists. Assume that, $\lim_{t \to \infty} \bar{c}_t = \bar{c}$. Then, $\lim_{t \to \infty} S_t = \lim_{t \to \infty} \bar{Q}_t = \frac{A - \bar{c}}{b}$. Note that, $\bar{c} > 0$, because otherwise, in view of (5.0.1), the entire sequence, $\{c_j\}$, would be identically equal to zero, which is a contradiction. We also prove that, $\bar{c} \neq A$. Indeed, in view of (5.0.1), the price sequence, $\{p(t)\}_{t=0}^{\infty}$, is decreasing and its limit, is equal to $\bar{c}$. Since, $p(0) < A$, it follows that, $\bar{c} \in (0, A)$. Clearly, for each $j, t, 0 \leq y_j(t) \leq S_t \leq \frac{A - \bar{c}}{b}$, and so, all supply sequences are bounded from above. In fact, the double sequence, $\{y_j(t)\}$, is uniformly bounded from above, by a positive constant. Set $s_j = \limsup_{t \to \infty} y_j(t)$ and $i_j = \liminf_{t \to \infty} y_j(t)$. From Eq. (5.0.2), we have

$$s_j \leq \frac{s_j}{2} - \lim_{t \to \infty} \frac{\bar{Q}_t}{2} + \frac{A - c_j}{2b} \quad \text{and} \quad i_j \geq \frac{i_j}{2} - \lim_{t \to \infty} \frac{\bar{Q}_t}{2} + \frac{A - c_j}{2b},$$

from which it follows, that $s_j \leq \frac{\bar{c} - \overline{c_t}}{b} \leq i_j$, and so, $\lim_{t \to \infty} y_j(t) = \frac{\bar{c} - \overline{c_t}}{b}$. 


Next, we prove that, the sequence, \( \{ \Phi_n \} \), where \( \Phi(n) = \frac{n(\bar{c} - \bar{c}_n)}{b} \), converges. Indeed, in view of (5.0.1), we see that the sequence, \( \{ \Phi_n \} \), is increasing. Moreover, is bounded from above, by the positive constant, \( \frac{A - \bar{c}}{b} \). Now, assume that \( \lim_{n \to \infty} \Phi_n = \frac{A - \bar{c}}{b} \). Then, clearly \( \sum_{j=1}^{\infty} \bar{y}_j = \sum_{j=1}^{\infty} \frac{\bar{c} - \bar{c}_j}{b} = \frac{A - \bar{c}}{b} \). Next, consider the market, consisting from infinitely many firms, 1, \ldots, \( j \), \ldots, such that, the infinite supply vector, is \( (\bar{y}_1, \ldots, \bar{y}_j, \ldots) \), where \( \bar{y}_j = \frac{\bar{c} - c_j}{b} \). Then, the aggregate supply is equal to, \( S = \frac{A - \bar{c}}{b} \), and the corresponding market price is \( p = \bar{c} \). Thus, the best response function implies, that during the next stage, each firm \( j \) supplies the market, with the same amount, \( \bar{y}_j \). Hence, the market sequence, \( \{ m_t \} \), approaches a Cournot equilibrium, with infinitely many firms.

The proof of Theorem 5.3.

Proof. We establish (5.2.1), with the use of induction. First, choose \( 0 < c_1 < A \). Observe that, \( 0 < c_1 < A + c_1 < A \), and so, by choosing \( c_2 \) such that, \( 0 < c_1 \leq c_2 < \frac{A + c_1}{2} < A \), we see that (5.2.1) holds, when \( n = 1 \).

Next, assume that, (5.2.1) holds, for some \( n > 1 \). Then, \( 0 < c_j \leq c_{j+1} < \frac{A + n c_j}{n+1} < A \), for all \( j = 1, \ldots, n \). Clearly, \( 0 < c_{n+1} < \frac{A + (n+1) c_{n+1}}{n+2} < A \). By choosing, \( c_{n+2} \), such that, \( 0 < c_{n+1} \leq c_{n+2} < \frac{A + (n+1) c_{n+1}}{n+2} < A \), we see that, (5.2.1) holds, when \( n = n + 1 \), and by induction, the result follows. The proof is complete.

The proof of Theorem 5.4.

Proof. The proof is similar to the proof of Theorem 5.1 and the details are omitted.

The proof of Theorem 6.1.

Proof. The proof is similar to the proof of Theorem 5.1 and the details are omitted.

The proof of Theorem 6.2. The definition below is on the convergence of a double sequence and will be useful in the sequel.

Claim 6.1. Let \( \{ y_j(t) \}_{j \geq 1, t \geq 0} \) be a double sequence of nonnegative real numbers. Then the double limit of the sequence is equal with zero, that is

\[ \lim_{j,t \to \infty} y_j(t) = 0, \]

if and only if, for every \( \epsilon > 0 \), there exists a positive integer \( N \), (which depends only on \( \epsilon \)) such that:

\[ 0 \leq y_j(t) < \epsilon, \quad \text{for all } j \geq N \text{ and } t \geq N. \]

Theorem 6.3. Let \( \{ c_j \}_{j \geq 1} \) be a marginal cost sequence, for which, (5.2.1), holds. Set \( \bar{c}_t = \frac{\sum_{j=1}^{t} c_j}{t} \), \( \bar{Q}_t = \frac{t(A - \bar{c}_t)}{b(t+1)} \), and \( \bar{Q}_0 = 0 \). For each \( j \geq 1 \), let \( \{ y_j(t) \}_{t \geq 0} \) be the solution of (5.0.2). Then

\[ \lim_{j,t \to \infty} y_j(t) = \lim_{j \to \infty} \lim_{t \to \infty} y_j(t) = \lim_{t \to \infty} \lim_{j \to \infty} y_j(t) = 0. \]
Proof. The proof that both iterated limits exist and are equal with zero, is straightforward and the details are omitted. We establish that the double limit is equal with zero. In view of (5.0.2), for all \( t \geq 0 \) and for all \( j > t \), \( y_j(t) = 0 < \epsilon \), for every positive real number \( \epsilon \). It suffices to prove that for every \( \epsilon > 0 \), there exists \( t_0 \geq 0 \), such that \( 0 < y_j(t) < \epsilon \), for all \( t \geq j \geq t_0 \). Equivalently, we establish that

\[
0 < y_t(t + n) < \epsilon + \frac{\epsilon}{b}, \quad \text{for all \( t \geq t_0 \) and for all \( n \geq 0 \).} \quad (6.1.4)
\]

Note that, Eq. (5.0.2) implies, \( \lim_{t \to \infty} y_t(t) = \lim_{t \to \infty} (\bar{Q}_t - \bar{Q}_{t-1}) = 0 \). Also, since \( Q_t \uparrow \frac{\bar{A} - \bar{c}}{b} \) and in view of (5.2.1), there exists, \( t_0 \geq 0 \), such that for all \( t \geq t_0 \),

\[
0 < y_t(t) < \epsilon, \text{ \( \bar{c} - \bar{c}_0(t) = c_0 < \bar{c} \), and } \frac{\bar{A} - \bar{c}}{b} - \epsilon < \bar{Q}_{t+n} < \frac{\bar{A} - \bar{c}}{b}, \text{ for all } n \geq 0. \]

For the remainder of the proof assume that \( t \geq t_0 \). Clearly, when \( n = 0 \), (6.1.4) holds. When \( n = 1 \), in view of (5.0.2), we have

\[
0 < y_t(t + 1) < \frac{\epsilon}{2} - \frac{A - \bar{c}}{2b} + \frac{\epsilon}{2b} + \frac{A - \bar{c}}{2b} + \frac{\epsilon}{2b} = \epsilon + \frac{\epsilon}{2b} < \epsilon + \frac{\epsilon}{b}.
\]

Also,

\[
0 < y_t(t + 2) < \frac{\epsilon}{2} + \frac{\epsilon}{4b} - \frac{A - \bar{c}}{2b} + \frac{\epsilon}{2b} + \frac{A - \bar{c}}{2b} + \frac{\epsilon}{2b} = \epsilon + \frac{\epsilon}{b} \cdot \left( \frac{1}{2} + \frac{1}{4} \right) < \epsilon + \frac{\epsilon}{b}.
\]

Inductively, for all \( n \geq 0 \),

\[
0 < y_t(t + n) < \frac{\epsilon}{2} + \frac{\epsilon}{b} \cdot \left( \frac{1}{2^2} + \ldots + \frac{1}{2^n} \right) - \frac{A - \bar{c}}{2b} + \frac{\epsilon}{2b} + \frac{A - \bar{c}}{2b} + \frac{\epsilon}{2b} = \epsilon + \frac{\epsilon}{b} \cdot \left( \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} \right) < \epsilon + \frac{\epsilon}{b},
\]

which establishes (6.1.4) and proves the result. The proof is complete. □

Remark 6.1. Note that the result of Theorem 6.3, together with, \( \lim_{j \to \infty} \lim_{t \to \infty} y_j(t) = \lim_{t \to \infty} \frac{\epsilon - c_j}{b} = 0 \), implies that, for every \( \epsilon > 0 \), there exists a positive integer, \( J = J(\epsilon) \), and a positive integer \( T = T(\epsilon) \), both sufficiently large, such that, \( -2\epsilon < y_j(t) - \frac{c_j - \epsilon}{b} < 2\epsilon \), for all \( j \geq J \) and \( t \geq T \). Since, there is only a finite number of \( j \)'s, \( j < J \), which are not taken into account, in the inequality above, for each of which, \( \lim_{t \to \infty} y_j(t) = \frac{\epsilon - c_j}{b} \), the convergence of the sequence, \( \{y_j(t)\} \), as \( t \to \infty \), is uniform, for all \( j \). The converse is also true. If, for each \( j \), the limit of the sequence, \( \{y_j(t)\} \), as \( t \) increases to infinity exists and the convergence is uniform, for all \( j \), and if, the iterated limit, first as \( t \) increases to infinity and then as \( j \) increases to infinity exists, then, the double limit of the sequence, \( \{y_j(t)\} \), exists and its value is equal with the value of the iterated limit.

The proof of the Theorem. The proof of (i) is along the lines of the proof of Theorem 5.1 and the details are omitted. Next, we establish the uniform convergence result. Observe that, for all \( i,j,t \), \( |x_{ij}(t) - y_j(t)| \) is either equal with zero or equal with \( y_j(t) \), or less than or equal to \( |Q_i - Q_j| \). Since, the values of the sequence \( \{Q_n\} \) increase to a positive limit, and in view of the result of Theorem 6.3 and Remark 6.1, which imply that the sequence \( \{y_j(t)\} \) converges uniformly for all \( j \), for a given \( \epsilon > 0 \) and for all \( j \geq 1 \), there exists a positive integer \( i_0 \), which only depends on \( \epsilon \), for which \( |x_{ij}(t) - y_j(t)| < \epsilon \), for all \( i,t \geq i_0 \). Furthermore, for all \( t < i_0 \), for all \( i \geq i_0 \) and for all \( j \), we see that \( |x_{ij}(t) - y_j(t)| = 0 < \epsilon \), from which the uniform convergence result follows. Next, observe that, for every \( \epsilon > 0 \), there exists \( j_0 \), such
that for all $i,j \geq j_0$, $0 < \frac{A+i\varepsilon_i-(i+1)\varepsilon_i}{(i+1)b} - \frac{\bar{c} - \varepsilon_j}{b} = \frac{A-i\varepsilon_i-(i+1)\varepsilon_i}{(i+1)b} < \epsilon$ and $0 < \frac{\bar{c} - \varepsilon_j}{b} < \epsilon$, from which it follows that $\|((x_{i1}, \ldots, x_{it}, 0, \ldots, 0, \ldots) - (y_{i1}, \ldots, y_{jt}, \ldots))\|_\infty < \epsilon$.

Finally, we prove that, as both $i$ and $t$ increase to infinity, the market, $m_{it}$, converges to the equilibrium of the market $m_t$. First, we establish that for each $j$, the double limit $\lim_{t \to \infty} x_{ij}(t) = \frac{\bar{c} - \varepsilon_j}{b}$. Indeed, for each $j$, the sequence, $\{x_{ij}(t)\}$, as $i$ increases to infinity, converges uniformly to, $y_j(t)$, and the limit of the sequence, $\{y_j(t)\}$, as $t$ increases to infinity, is equal with $\frac{\bar{c} - \varepsilon_j}{b}$. In view of Remark 6.1, the result follows. Also, in each market, $m_{it}$, during an arbitrary stage $t$, the aggregate supply and the corresponding price are: $\bar{Q}_{it} = \frac{\min_{t,i}(A - \bar{C}_{ij}(t))}{\min_{t,i}(b^{-1} + 1)}$ and $\bar{p}(i,t) = \frac{A + \min_{t,i}(b^{-1} + 1)}{\min_{t,i}(b^{-1} + 1)}$, respectively, from which it follows that, $\lim_{i,t \to \infty} \bar{Q}_{it} = \frac{A - \bar{c}}{b}$ and $\lim_{i,t \to \infty} \bar{p}(i,t) = \bar{c}$. The proof is complete.

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