Geometric Algorithms for Minimal Enclosing Discs in Strictly Convex Normed Planes*

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With the geometric background provided by Alonso, Martini, and Spirova [2], we show the validity of the Elzinga–Hearn algorithm and the Shamos–Hoey algorithm for solving the minimal enclosing disc problem in strictly convex normed planes.

Keywords. minimal enclosing disc, norm-acute triangle, norm-obtuse triangle, strictly convex normed space, Voronoi diagram

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1 Introduction

In 1857, Sylvester [16] posed the minimal enclosing disc problem, which asks for the smallest disc which covers a given finite point set in the Euclidean plane. A natural extension of this is obtained if one replaces the family of Euclidean discs by the family $F$ of homothetic images of a given non-empty, convex, compact set $B \subset \mathbb{R}^2$ which is centrally symmetric with respect to its interior point $0 = (0,0)$ (the origin). The family $F$ becomes the family of balls with respect to a suitable norm $\|\cdot\|$ on $\mathbb{R}^2$. Clearly, $B$ and $\|\cdot\|$ are combined by the two relations $B = \{x \mid \|x\| \leq 1\}$ and, for $x \in \mathbb{R}^2$, $\|x\| = \inf \{\lambda > 0 \mid x \in \lambda B\}$. We write $B(x,\lambda) = \lambda B + x$ and $S(x,\lambda) = \lambda bd(B) + x$ for the disc (i.e., a ball in two dimensions) and the circle centered at $x$ with radius $\lambda$, respectively. The pair $(\mathbb{R}^2,\|\cdot\|)$ is called a normed plane. For a compact set $P \subset \mathbb{R}^2$, the minimal enclosing disc problem is posed by

$$\inf_{x \in \mathbb{R}^2} \max_{p \in P} \|x - p\|. \quad (1)$$

The existence of solutions can be shown by standard compactness arguments. We denote the solution set of (1) by MEDC($P$), i.e., MEDC($P$) is the set of centers of discs that have smallest possible radius and contain $P$. The optimal value of (1), i.e., the corresponding radius, is denoted by MER($P$). In general, the minimal enclosing disc problem is not uniquely solvable, as depicted in Figure 1.

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Figure 1. Minimal enclosing discs need not to be unique. The dots mark the points of $P$, and the figure shows two minimal enclosing discs with respect to the $\ell_\infty$-norm.

For the Euclidean norm, two algorithmic approaches are known which rely on simple geometric concepts; that of Elzinga and Hearn [8] and that of Shamos and Hoey [15]. These are the notions of obtuseness, rightness, and acuteness of triangles as well as the notion of Voronoi diagrams. The purpose of this article is to show how these concepts can be generalized to a wider class of norms (see Section 2) and how the corresponding algorithms can be proved to be valid (see Section 3).

2 Strictly convex norms, triangles, and Voronoi diagrams

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is called strictly convex if $\|x + y\| < 2$ whenever $\|x\| = \|y\| = 1$. Geometrically this means that the boundary of the unit ball does not contain any non-degenerate line segments. Throughout this article, we shall work in strictly convex normed planes, that is, pairs $(\mathbb{R}^2, \|\cdot\|)$ where $\|\cdot\|$ is a strictly convex norm. At first, let us fix the notation for some geometric entities. The straight line through $x$ and $y$ is denoted by $\langle x, y \rangle = \{\lambda x + (1 - \lambda)y \mid \lambda \in \mathbb{R}\}$.

By the term triangle, we understand a set $P \subset \mathbb{R}^2$ of cardinality $\text{card}(P) = 3$. If $x$ satisfies the equation
$$\|x - p_1\| = \|x - p_2\| = \|x - p_3\|,$$
then $x$ is called a circumcenter of the triangle $\{p_1, p_2, p_3\}$. In that case, the disc $B(x, \|x - p_1\|)$ is called a circumdisc of $P$. Note that in strictly convex normed planes, the number of circumcenters of each triangle is either 0 or 1 [13, Proposition 14.8].

Proposition 2.1 ([12, Lemma 2.1.1.1]). Let $p_1, p_2 \in \mathbb{R}^2$ be distinct points in a strictly convex normed plane. The bisector
$$\text{bis}(p_1, p_2) = \{x \in \mathbb{R}^2 \mid \|x - p_1\| = \|x - p_2\|\}$$
is homeomorphic to a straight line.

Definition 2.2 (see [15, p. 159]). Let $P \subset \mathbb{R}^2$ be a given finite point set. The farthest-point Voronoi region of $p \in P$ is defined as
$$\{y \in \mathbb{R}^2 \mid \|y - p\| \geq \|y - q\| \forall q \in P \setminus \{p\}\}.$$
The collection of all farthest-point Voronoi regions is said to be the farthest-point Voronoi diagram.

By Proposition 2.1, bisectors do not have interior points in strictly convex normed planes. Therefore, the boundary of a farthest-point Voronoi region consists of pieces of curves without endpoints (loci of points belonging to exactly two farthest-point Voronoi regions) and their endpoints (points belonging to at least three farthest-point Voronoi regions). The former are called edges, and the latter are called vertices of the diagram. For given \( x \in \mathbb{R}^2 \) and an arbitrary finite set \( P \subset \mathbb{R}^2 \), one can easily verify the equality \( \sup \{ \| y - x \| \mid y \in \text{conv}(P) \} = \sup \{ \| p - x \| \mid p \in P \} \). Due to this fact and the strict convexity of the norm, the farthest points of \( x \) among the points of a finite set \( P \) are necessarily extreme points of \( \text{conv}(P) \). In particular, the farthest-point Voronoi region of \( p \in P \) is empty, if \( p \) is not an extreme point of \( \text{conv}(P) \).

The next two lemmas describe how one half of the bisector can be parametrized by the distance from the two sites generating the bisector.

**Lemma 2.3.** Let \( \varphi : [0, \infty) \rightarrow [\mu_0, \infty) \) be a continuous bijection. Then \( \varphi \) is strictly increasing.

**Lemma 2.4** ([17, Lemma 2.2]). Let \( p_1, p_2 \in \mathbb{R}^2 \). Furthermore, let \( H^+ \) be one of the closed half planes bounded by the straight line \( \langle p_1, p_2 \rangle \), define \( \text{bis}^+(p_1, p_2) = \text{bis}(p_1, p_2) \cap H^+ \), and let \( \gamma : [0, \infty) \rightarrow \text{bis}^+(p_1, p_2) \) be a homeomorphism. Then the mapping \( \varphi : [0, \infty) \rightarrow [\frac{1}{2} \| p_1 - p_2 \| , \infty) \), \( \varphi(t) = \| p_1 - \gamma(t) \| , \) is strictly increasing.

**Proof.** Obviously, \( \varphi \) is continuous. Let us assume that \( \varphi \) is not injective. Then there exist two distinct points \( x, y \in \text{bis}^+(p_1, p_2) \) such that \( \lambda := \| p_1 - x \| = \| p_1 - y \| \). By [1, Corollary 3.1(a)], \( S(p_1, \lambda) \cap S(p_2, \lambda) \) contains the whole segment \( [x, y] \). This contradicts the strict convexity. The mapping \( \varphi \) is also surjective. Indeed, for \( \mu \in [\frac{1}{2} \| p_1 - p_2 \| , \infty) \), the intersection \( S(p_1, \mu) \cap S(p_2, \mu) \cap H^+ \) is a singleton (see [13, Proposition 14.3]) which, by definition, belongs to \( \text{bis}^+(p_1, p_2) \). We have \( \gamma(0) = \frac{1}{2}(p_1 + p_2) \), and thus \( \varphi(0) = \frac{1}{2} \| p_1 - p_2 \| \). By Lemma 2.3, the assertion follows.

For the sake of completeness, we cite two propositions each of which is crucial both for Theorem 2.7 and the understanding of the algorithms in Section 3.

**Proposition 2.5** ([3, Lemma 1.2]). Let \( (\mathbb{R}^2, \| \cdot \|) \) be a normed plane. The norm is strictly convex if and only if MEDC(P) is a singleton for every compact set \( P \subset \mathbb{R}^2 \).

**Proposition 2.6** ([9, Section (1.7)]). Let \( (\mathbb{R}^2, \| \cdot \|) \) be a strictly convex normed plane. Let \( (c, \lambda) \in \mathbb{R}^2 \times [0, +\infty) \) and \( x, y \in B(c, \lambda) \). If \( \| x - y \| = \text{diam}(B(c, \lambda)) = 2\lambda \), then \( \frac{1}{2}(x + y) = c \).

Rademacher and Toeplitz [14, Chapter 16] proved the following theorem for the Euclidean plane. We give an extension for strictly convex normed planes.
Theorem 2.7. Let \( N \geq 2 \), and let \( P = \{p_1, \ldots, p_N\} \) be a finite set in the strictly convex normed plane \((\mathbb{R}^2, \|\cdot\|)\). Further, let \( B(\bar{x}, \bar{\lambda}) \) be the minimal enclosing disc of \( P \). Then \( \text{card}(S(\bar{x}, \lambda) \cap P) \geq 2 \), and every semicircle of \( S(\bar{x}, \lambda) \) (that is, the intersection of the circle with a closed half plane) contains at least one point from \( P \).

**Proof.** Suppose \( \text{card}(S(\bar{x}, \lambda) \cap P) = 0 \). Then \( \|\bar{x} - p_i\| < \lambda \) for all \( i \in \{1, \ldots, N\} \). Hence, the disc centered at \( \bar{x} \) with radius \( \max_{i=1, \ldots, N} \|\bar{x} - p_i\| \) contains \( P \) but has smaller radius. This contradicts \( \lambda = \text{MER}(P) \).

Suppose \( \text{card}(S(\bar{x}, \lambda) \cap P) = 1 \), say \( \|\bar{x} - p_i\| = \lambda > \|\bar{x} - p_i\| \) for all \( i \in \{2, \ldots, N\} \). There exists \( \varepsilon > 0 \) such that \( B(p_i, \varepsilon) \subset B(\bar{x}, \lambda) \) for all \( i \in \{2, \ldots, N\} \). Therefore, \( B(\bar{x} + \varepsilon(p_1 - \bar{x}), \lambda) \) is another disc containing \( P \) and having radius \( \lambda \). Thus we have a contradiction to Proposition 2.5. It follows that \( \text{card}(S(\bar{x}, \lambda) \cap P) \geq 2 \).

Suppose now that there is an arc \( A \) (that is, a connected subset of a circle) containing a semicircle of \( S(\bar{x}, \lambda) \) without points from \( P \) and having endpoints \( p_1, p_2 \in P \). Move the designated minimal enclosing disc center along the bisector \( \text{bis}(p_1, p_2) \) towards \( \frac{1}{2}(p_1 + p_2) \) and keep \( \|\bar{x} - p_1\| \) as the designated minimal enclosing radius until the center reaches \( \frac{1}{2}(p_1 + p_2) \) or a third point from \( P \) hits the boundary. In the language of Lemma 2.4, the center of the designated minimal enclosing disc is a “moving bisector point” \( \bar{x} = \gamma(t) \). As it moves towards the midpoint \( \frac{1}{2}(p_1 + p_2) \), the parameter \( t \) decreases. Thus the distance \( \|p_1 - \gamma(t)\| = \|p_2 - \gamma(t)\| \), which coincides with the designated radius, also decreases. \( \square \)

An illustration of the main steps of the proof of Theorem 2.7 can be found in Figure 2.

![Figure 2](image)

**Figure 2.** An \( \ell^4 \)-norm example for Rademacher and Toeplitz’s theorem. The unfilled dots are the points of \( P \), the filled ones are centers or auxiliary points.

From Theorem 2.7, it follows that the minimal enclosing disc \( B(\bar{x}, \lambda) \) of a finite set \( P \) is a two-point disc, i.e., there are \( p, p' \in P \) such that \( \bar{x} = \frac{1}{2}(p + p') \) and \( \lambda = \frac{1}{2} \|p - p'\| \), or it is the circumscribed circle of at least three points from \( P \).

Alonso, Martini, and Spirova [1, 2] extend the notions of acuteness, rightness, and obtuseness of triangles in the following way to normed planes \((\mathbb{R}^2, \|\cdot\|)\).

**Definition 2.8.** A triangle with vertices \( p_1, p_2, p_3 \in \mathbb{R}^2 \) is called **norm-acute at** \( p_k \) if

\[
\left\| p_k - \frac{p_1 + p_2}{2} \right\| > \frac{\|p_1 - p_2\|}{2},
\]  

(2)
where \( \{i, j, k\} = \{1, 2, 3\} \). It is called \( \textit{norm-right} \) at \( p_k \) if the inequality in (2) is changed into \( "=" \), and it is called \( \textit{norm-obtuse} \) at \( p_k \) if this inequality is changed into \( "<" \).

Figure 3 shows an example for this classification for the \( \ell^4 \)-norm. The results of [2, Figure 3].

**Figure 3.** Norm-obtuseness, norm-rightness, and norm-acuteness at \( p_3 \), resp., for the \( \ell^4 \)-norm.

Section 3] show that the following definition provides, for normed planes, a subdivision of the family of all triangles into the following subfamilies.

**Definition 2.9 (see [2, Definition 3.1]).** A triangle \( P \subset \mathbb{R}^2 \) is called

(a) \( \textit{norm-obtuse} \) if it is norm-obtuse at one vertex and norm-acute at the other two vertices;

(b) \( \textit{doubly norm-right} \) if it is norm-right at two vertices and norm-acute at the remaining vertex;

(c) \( \textit{norm-right} \) if it is norm-right at exactly one vertex and norm-acute at the other two vertices;

(d) \( \textit{norm-acute} \) if it is norm-acute at all three vertices.

From [2, Lemma 5.1] it follows that doubly norm-right triangles cannot occur in strictly convex normed planes.

## 3 The algorithms

In this section, we show that the algorithms by Shamos and Hoey [15] and Elzinga and Hearn [8], which were designed to solve the Euclidean minimal enclosing disc problem for \( \textit{finite} \) sets \( P \), can be carried over verbatim to strictly convex normed planes. After Chrystal’s algorithm [6], Elzinga and Hearn’s algorithm was the second milestone in tackling the minimal enclosing disc problem for the Euclidean plane. Drezner and Shelah [7] prove its \( \Omega(N^2) \) running time.

**Algorithm 3.1 (Elzinga/Hearn 1972).** \textbf{Require:} \( P \subset \mathbb{R}^2 \), \( \text{card}(P) \geq 2 \)

1. Choose \( p_1, p_2 \in P \), \( p_1 \neq p_2 \)
2. \( \bar{x} \leftarrow \frac{1}{2}(p_1 + p_2), \lambda \leftarrow \frac{1}{2} \|p_1 - p_2\| \)
3. \( \text{if} \ |\| \bar{x} - p \| \leq \lambda \forall p \in P \text{ then} \)
Algorithm 3.1 computes the center $\bar{x}$ and the radius $\bar{\lambda}$ the minimal enclosing ball of the given point set $P$.

Proof. It is easy to see that Algorithm 3.1 only checks two-point discs and circumdiscs determined by points of $P$. Since there are only finitely many such discs, it suffices to show that the considered radii increase with each iteration. Assume there are two chosen points $p_1, p_2$ and we enter step 2. We check if the two-point disc $B(x', \lambda')$ of these two points already contains the whole set $P$. If the answer is affirmative, we are finished since no smaller disc contains $p_1$ and $p_2$. (This is a consequence of Proposition 2.6.) Otherwise there is a point outside $B(x', \lambda')$. We call it $p_3$ and enter step 8 with $p_1$, $p_2$, and $p_3$.

Case 1: The triangle $\{p_1, p_2, p_3\}$ is norm-obtuse at $p_1$, say. The next disc $B(x'', \lambda'')$ under consideration is the two-point disc of $p_2$ and $p_3$. By [2, Table 1], we have $2\lambda'' = ||p_2 - p_3|| > ||p_1 - p_2|| = 2\lambda'$, i.e., $\lambda'' > \lambda'$.

Case 2: The triangle $\{p_1, p_2, p_3\}$ is norm-right at $p_1$, say. The next disc $B(x'', \lambda'')$ under consideration is the two-point disc of $p_2$ and $p_3$. By [2, Table 1], we have

$$2\lambda'' = ||p_2 - p_3|| \geq ||p_1 - p_2|| = 2\lambda'$$

(3)

Proof.

1: Choose $p_3 \in P$ such that $\|\bar{x} - p_3\| > \bar{\lambda}$
2: Go to 8
3: Choose $p_4 \in \{p_1, p_2, p_3\}$ such that $\|p_4 - p_i\| = \max_{i=1,2,3} \|p_4 - p_i\|
4: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}
5: \phantom{4: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} p_6 \leftarrow \text{the point among } \{p_1, p_2, p_3\} \setminus \{p_5\} \text{ in the half plane bounded by } \langle p_5, \bar{x} \rangle \text{ opposite to } p_4
6: \phantom{4: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} \text{else}
7: \phantom{6: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} \phantom{\text{else}} \text{Choose } p_6 \in \{p_1, p_2, p_3\} \setminus \{p_5\}
8: \phantom{4: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} \phantom{6: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} \phantom{\text{else}} \phantom{\text{else}} \text{Go to 8}
9: \phantom{4: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} \phantom{6: \text{if } p_4 \not\in \{p_5, \bar{x}\} \text{ then}} \phantom{\text{else}} \phantom{\text{else}} \{p_1, p_2, p_3\} \leftarrow \{p_4, p_5, p_6\}
10: \text{Return } (\bar{x}, \bar{\lambda})
with equality if the triangle \( \{p_1, p_2, p_3\} \) is isosceles with \( \|p_2 - p_3\| = \|p_1 - p_2\| > \|p_1 - p_3\| \) or if the triangle is equilateral. By [2, Proposition 3.3.(b)], the latter case only occurs in normed planes where the discs are parallelograms. The former case is impossible in strictly convex normed planes because

\[
\left\| \frac{p_1 + p_2}{2} - p_1 \right\| = \left\| \frac{p_2 - p_3}{2} \right\|
\]

\[
> \left\| \frac{1}{2} \left( \frac{p_1 + p_2}{2} - p_1 \right) + \frac{1}{2} \left( p_2 - p_3 \right) \right\| = \frac{\|p_2 - p_1\|}{2}
\]

would yield a contradiction to the assumption that \( \{p_1, p_2, p_3\} \) is isosceles. It follows that the inequality in (3) is strict.

Case 3: The triangle \( \{p_1, p_2, p_3\} \) is norm-acute. In step 12 and step 13, the circumdisc \( B(x'', \lambda'') \) of \( \{p_1, p_2, p_3\} \) is constructed. Note that the feasibility of step 12 follows from the fact that norm-acute triangles in strictly convex normed planes have exactly one circumcircle, see [13, Proposition 14 8.] and [2, Theorem 6.1]. By applying Proposition 2.6 twice, we conclude that \( \lambda'' > \lambda' \) since

\[2\lambda'' = \text{diam}(B(x'', \lambda'')) > \|p_1 - p_2\| = \text{diam}(B(x', \lambda')) = 2\lambda'.\]

If \( B(x'', \lambda'') \) contains the whole set \( P \), it is already the solution since no smaller disc contains \( \{p_1, p_2, p_3\} \). (This is a consequence of [2, Theorem 6.4].) Otherwise there is an outside point \( p_4 \in P \). We cannot have

\[\|p_4 - p_1\| = \|p_4 - p_2\| = \|p_4 - p_3\|\]
in step 19, since otherwise \( p_4 = x'' \), which is particularly not outside \( B(x'', \lambda'') \). In other words, there are at most two points among \( p_1, p_2, p_3 \) which are candidates for \( p_5 \). The choice of \( p_6 \) in step 21 is possible by [2, Theorem 6.3], which says in particular that \( x'' \) lies in the interior of the convex hull of \( \{p_1, p_2, p_3\} \). Consequently, the straight line through \( p_5 \) and the \( x'' \) does not pass through any point from \( \{p_1, p_2, p_3\} \setminus \{p_5\} \). If \( p_4 \) lies on this straight line, we will show that it is irrelevant which of the two points from \( \{p_1, p_2, p_3\} \setminus \{p_5\} \) is chosen as \( p_6 \).

For that reason, let \( p_5 := p_1 \) be a farthest point to \( p_1 \) among \( \{p_1, p_2, p_3\} \). Without loss of generality, \( p_6 := p_2 \). We allow \( p_6 \) to be a farthest point to \( p_4 \) among \( \{p_1, p_2, p_3\} \), i.e., \( \|p_4 - p_6\| = \|p_4 - p_5\| \), and we allow \( p_4 \in \langle p_5, x'' \rangle \).

Case 3.1: The triangle \( \{p_4, p_5, p_6\} \) is norm-obtuse at \( p_5 \). This would imply that \( \|p_4 - p_5\| < \|p_4 - p_6\| \), see [2, Table 1]. This is a contradiction to our assumptions.

Case 3.2: The triangle \( \{p_4, p_5, p_6\} \) is norm-obtuse at \( p_6 \). Then \( x''' = \frac{1}{2}(p_4 + p_5) \) is the new center and \( \lambda''' = \frac{7}{4} \|p_4 - p_5\| \) is the new radius. By norm-obtuseness at \( p_6 \), we have \( \|x''' - p_6\| < \|x''' - p_5\| \) and, consequently, \( x''' \in \text{bis}(p_5, p_6) \). If we assume \( \lambda''' = \lambda'' \), then \( x''' \) does not lie on \( \langle p_5, x'' \rangle \) since otherwise \( p_4 \in B(x'', \lambda'') \). Hence \( x''' \) lies in the interior of the shaded sector in Figure 4. Furthermore, we have \( \|x''' - p_5\| = \lambda''' \leq \lambda'' \).
and thus the interior of the shaded sector cuts $B(p_5, \lambda'')$. Especially, the intersection of $\text{bis}(p_5, p_6)$ and $B(p_5, \lambda'')$ contains a point $y$. But $y$ lies “strictly afterwards” $x''$ on the bisector of $p_5$ and $p_6$ (in the sense of Lemma 2.4), i.e., $\lambda'' \geq \|y - p_5\| > \|x'' - p_5\| = \lambda''$. This is a contradiction.

Figure 4. Proof of Theorem 3.2: Case 3.
Case 3.3: In any other case, $x'''' \in \text{bis}(p_5, p_6)$. If we assume $\lambda'''' \geq \lambda'''$, then $x'''' \in \text{conv}(\{p_5, p_6, x''''\})$, see [13, Proposition 18] and Lemma 2.4. The straight line through $p_4$ and $x''''$ separates $\text{conv}(\{p_5, p_6, x''''\})$ into two parts, namely $\text{conv}(\{s, p_6, x''''\})$ and $\text{conv}(\{p_5, s, x''''\})$ as depicted in Figure 5. Note that although it is the case in Figure 5, it is not clear whether the part of $\text{bis}(p_5, p_6)$ between $x''''$ and $\frac{1}{2}(p_5 + p_6)$ is fully contained in one of the sets $\text{conv}(\{s, p_6, x''''\})$ and $\text{conv}(\{p_5, s, x''''\})$.

Case 3.3.1: If $x'''' \in \text{conv}(\{p_5, s, x''''\})$, then $\text{conv}(\{p_4, x'''', p_6\}) \subset \text{conv}(\{p_4, x'''', p_6\})$. Now [13, Corollary 28] yields
\[
2\lambda'''' \geq \|p_4 - x''''\| + \|p_6 - x''''\| > \|p_4 - x''''\| + \|p_6 - x''''\| > 2\lambda'',
\]
a contradiction to the assumption.

Case 3.3.2: If $x'''' \in \text{conv}(\{s, p_6, x''''\})$, then $p_4, x''', x''''$ and $p_6$ form in this order a convex quadrangle. Now [13, Proposition 7] yields
\[
\|x''' - p_4\| + \|x'''' - p_6\| < \|x'''' - p_4\| + \|x'''' - p_6\| \implies \|x''' - p_4\| < \|x'''' - p_6\| = \lambda'',
\]
a contradiction.

\[\square\]

Remark 3.3. In fact, the shaded sector in Figure 4 has only one connected component. This follows from a sharpening of [13, Lemma 18], which reads as follows. Suppose the unit circle of a normed plane ($R^2, \|\cdot\|$) does not contain a line segment parallel to the straight line through $p$ and $q$. Then, for every point $z \in \text{bis}(p, q)$, the following relation holds:
\[
\text{bis}(p, q) \setminus \{z\} \subset \{z + \alpha(z - p) + \mu(z - q) \mid \alpha \mu > 0\}.
\]
For the proof of this statement it suffices to assume the existence of a point $w \in \text{bis}(p, q)$ which is distinct from $z$ and lies, say, in $\{\alpha z + (1 - \alpha)p \mid \alpha \geq 1\}$. Since $z, w \in \text{bis}(p, q)$, we have $\|z - p\| = \|z - q\|$ and $\|w - p\| = \|w - q\|$. The collinearity of the points $p, z, w$ yields $\|w - p\| = \|w - z\| = \|z - q\|$. Substituting the term on the left-hand side and the second one on the right-hand side, we obtain $\|w - q\| = \|w - z\| + \|z - q\|$. By [13, Lemma 1], we conclude that the unit circle contains a line segment parallel to the straight line through $p$ and $q$, a contradiction.

Remark 3.4. For practical purposes, Algorithm 3.1 requires an additional subroutine which computes the circumdisc of a given triangle. Assuming that this computation can be done in constant time, the running time of Algorithm 3.1 will still show an $\Omega(N^2)$ behaviour like in the Euclidean case.
Figure 5. Proof of Theorem 3.2: Case 3.3.

In the early years of Computational Geometry, Shamos and Hoey [15] proposed an algorithm for the minimal enclosing disc problem which is based on the construction of farthest-point Voronoi diagrams. For that purpose, they use a divide-and-conquer technique to obtain $O(N \log N)$ running time. For wider classes of norms, constructions of Voronoi diagrams and respective running time results are included in the papers of Lee [10, 11], Chew and Drysdale [5], and Chazelle and Edelsbrunner [4]. The simple structure of farthest-point Voronoi diagrams enables an $O(N)$ search for the optimal disc once the diagram is constructed.

**Algorithm 3.5** (Shamos/Hoey 1975). **Require**: $N \geq 2$, $P = \{p_1, \ldots, p_N\} \subset \mathbb{R}^2$

1: Construct the farthest-point Voronoi diagram with respect to $p_1, \ldots, p_N$
2: for each edge of the diagram do
3: Determine the distance between the two defining points
4: end for
5: Find the maximum among these distances
6: if the two-point disc of the corresponding points contains $p_1, \ldots, p_N$ then
7: return the center and the radius of this disc
8: else
9: for each vertex of the diagram do
10: Compute its distance to one of its defining points
11: end for
12: Find the minimum among these distances
13: return the corresponding vertex of the Voronoi diagram and the minimum distance
14: end if
Theorem 3.6. Algorithm 3.5 computes the center and the radius of the minimal enclosing ball of the given point set $P$.

Proof. Let $B(\bar{x}, \bar{\lambda})$ be the minimal enclosing disc of $P$. By Theorem 2.7, $S(\bar{x}, \bar{\lambda}) \cap P$ contains at least two points. If it contains exactly two points $p_1, p_2$, the center $\bar{x}$ belongs to the farthest-point Voronoi regions of $p_1$ and $p_2$ but not to any other farthest-point Voronoi region, i.e., $\bar{x}$ lies on the edge of the diagram that belongs to $p_1$ and $p_2$. Taking Theorem 2.7 into account, it follows that $\bar{x} = \frac{1}{2}(p_1 + p_2)$. Hence

$$\|p_1 - p_2\| = \text{diam}(B(\bar{x}, \bar{\lambda})) > \|p - p'\| \text{ for all } p, p' \in P \setminus \{p_1, p_2\}. \quad (4)$$

If $S(\bar{x}, \bar{\lambda}) \cap P$ contains at least three points, then $\bar{x}$ lies in the intersection of at least three farthest-point Voronoi regions, i.e., $\bar{x}$ is a vertex of the diagram. In step 5, we are looking for the maximum distance of pairs of points which determine edges of the diagram. Then, by (4), the two-point disc of the corresponding points realizing this maximum is the minimal enclosing disc of $P$ if it contains $P$. If this is not the case, the center of the minimal enclosing disc has to be a vertex of the diagram. Clearly, each disc, which is centered at a vertex of the diagram and contains the (at least three) points that determine the farthest-point Voronoi regions to which the vertex belongs, contains $P$. Thus it suffices to find the smallest disc belonging, in the sense just explained, to a vertex, see step 12. \[\square\]

References

[1] J. Alonso, H. Martini, and M. Spirova, Minimal enclosing discs, circumcircles, and circumcenters in normed planes (Part I), Comput. Geom. 45 (2012), no. 5-6, pp. 258–274, doi: 10.1016/j.comgeo.2012.01.007.

[2] , Minimal enclosing discs, circumcircles, and circumcenters in normed planes (Part II), Comput. Geom. 45 (2012), no. 7, pp. 350–369, doi: 10.1016/j.comgeo.2012.02.003.

[3] D. Amir and Z. Ziegler, Relative Chebyshev centers in normed linear spaces, Part I, J. Approx. Theory 29 (1980), no. 3, pp. 235–252, doi: 10.1016/0021-9045(80)90129-X.

[4] B. Chazelle and H. Edelsbrunner, An improved algorithm for constructing $k$th-order Voronoi diagrams, IEEE Trans. Comput. C-36 (1987), no. 11, pp. 1349–1354, doi: 10.1109/TC.1987.5009474.

[5] L. P. Chew and R. L. S. Drysdale, Voronoi diagrams based on convex distance functions, Proceedings of the First Annual Symposium on Computational Geometry (J. O’Rourke, ed.), ACM, 1985, pp. 235–244, doi: 10.1145/323233.323264.

[6] G. Chrystal, On the problem to construct the minimum circle enclosing $n$ given points in a plane, Proc. Edinburgh Math. Soc. 3 (1885), pp. 30–33.
[7] Z. Drezner and S. Shelah, *On the complexity of the Elzinga–Hearn algorithm for the 1-center problem*, Math. Oper. Res. **12** (1987), no. 2, pp. 255–261, doi: 10.1287/moor.12.2.255.

[8] J. Elzinga and D. W. Hearn, *Geometrical solutions for some minimax location problems*, Transportation Sci. **6** (1972), no. 4, pp. 379–394, doi: 10.1287/trsc.6.4.379.

[9] P. Gritzmann and V. Klee, *Inner and outer j-radii of convex bodies in finite-dimensional normed spaces*, Discrete Comput. Geom. **7** (1992), no. 1, pp. 255–280, doi: 10.1007/BF02187841.

[10] D. T. Lee, *Two-dimensional Voronoi diagrams in the L_p-metric*, J. Assoc. Comput. Mach. **27** (1980), no. 4, pp. 604–618, doi: 10.1145/322217.322219.

[11] , *On k-nearest neighbor Voronoi diagrams in the plane*, IEEE Trans. Comput. **31** (1982), no. 6, pp. 478–487, doi: 10.1109/TC.1982.1676031.

[12] L. Ma, *Bisectors and Voronoi Diagrams for Convex Distance Functions*, Ph.D. thesis, Fernuniversität Hagen, 2000.

[13] H. Martini, K. Swanepoel, and G. Weiβ, *The geometry of Minkowski spaces – a survey, Part I*, Expo. Math. **19** (2001), no. 2, pp. 97–142, doi: 10.1016/S0723-0869(01)80025-6.

[14] H. Rademacher and O. Toeplitz, *The Enjoyment of Math*, 7th ed., Princeton University Press, Princeton, 1994.

[15] M. I. Shamos and D. Hoey, *Closest-point problems*, 16th Annual Symposium on Foundations of Computer Science (Berkeley, Calif., 1975), IEEE Comput. Soc., 1975, pp. 151–162, doi: 10.1109/SFCS.1975.8.

[16] J. J. Sylvester, *A question in the geometry of situation*, Q. J. Math. **1** (1857), p. 79.

[17] J. Väisälä, *Slopes of bisectors in normed planes*, Beitr. Algebra Geom. **54** (2013), no. 1, pp. 225–235, doi: 10.1007/s13366-012-0106-6.

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