How (not) to teach Lorentz covariance of the Dirac equation

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Received 27 November 2013, revised 17 January 2014
Accepted for publication 3 February 2014
Published 5 March 2014

Abstract
In the textbook proofs of the Lorentz covariance of the Dirac equation, one treats the wave function as a spinor and gamma matrices as scalars, leading to a quite complicated formalism with several pedagogic drawbacks. As an alternative, I propose to teach the Dirac equation and its Lorentz covariance by using a much simpler, but physically equivalent formalism, in which these drawbacks do not appear. In this alternative formalism, the wave function transforms as a scalar and gamma matrices as components of a vector, such that the standard physically relevant bilinear combinations do not change their transformation properties. The alternative formalism allows also a natural construction of some additional non-standard bilinear combinations with well-defined transformation properties.

Keywords: Dirac equation, Lorentz covariance, scalar wave function

1. Introduction

I like to ask tricky questions. For a warm up, here is a simple one appropriate to undergraduate students. Let $x$ be the position operator and $|\psi\rangle$ the quantum state. Which one of the two changes with time? No doubt, many students will recall how these quantities appear in the Schrödinger equation, which will lead them to the answer that $|\psi\rangle$ changes with time, while $x$ does not. Certainly not a wrong answer, but there is a much better one: it depends on the picture.

In the Schrödinger picture $|\psi\rangle$ changes with time and $x$ does not, while in the Heisenberg picture $x$ changes with time and $|\psi\rangle$ does not. This is consistent because neither $x$ nor $|\psi\rangle$ is a physical quantity by itself, while physical quantities do not depend on the picture.

Good! Now after this warm up, here is a tricky question that I really wanted to ask. The question is appropriate to graduate students, their teachers, and even experienced experts in quantum field theory and particle physics. In the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

(1)
which of the two quantities, $\gamma^\mu$ and $\psi$, changes under a Lorentz transformation? With very rare exceptions, almost everybody (assuming that they know what they are talking about) will answer that $\psi$ changes and $\gamma^\mu$ does not. Yet, quite analogously to the warm-up question above, that is not the best answer. A much better answer is that it depends on the picture too. In the standard picture known to everybody $\psi$ transforms and $\gamma^\mu$ does not, but there is also an alternative picture in which $\gamma^\mu$ transforms and $\psi$ does not. Neither $\gamma^\mu$ nor $\psi$ is a physical quantity by itself, while physical quantities do not depend on the picture.

Nevertheless, almost nobody will tell you about this alternative picture of $\gamma^\mu$ and $\psi$. Why? Because that is not how they are taught. The purpose of the present paper is to teach you that. And not only because the alternative picture exists, but also because it is much simpler.

Fortunately, there is a relatively small community of physicists who are more likely to tell you about the alternative picture, or at least recognize immediately that it exists if you point it out to them. These are people who work with spinors in curved spacetime. They know that the picture in which $\psi$ transforms and $\gamma^\mu$ does not is not appropriate for general coordinate transformations, of which Lorentz transformations are nothing but a special case. Therefore, to deal with spinors in curved spacetime, they must first ‘unlearn’ what they have learned about spinors in flat spacetime, and then learn again how to think about them in a different way. In this new picture, it turns out [1–3] that $\gamma^\mu$ transforms as a vector (which should not be surprising given the index $\mu$ it carries), while $\psi$ does not transform because it is a scalar (which should also not be surprising given that it does not carry any vector index at all).

Unfortunately, the treatment of spinors in curved spacetime requires some advanced concepts such as tetrads (called also vierbeins), with which people working in flat spacetime are usually not familiar. Hence, it is not so simple to convey the idea to the flat-spacetime people by using the techniques developed by the curved-spacetime people. Therefore, in this paper I develop a much simpler way to explain the alternative picture of spinors in flat spacetime. (A simple remark that $\gamma^\mu$ and $\psi$ in flat spacetime may transform as a vector and a scalar, respectively, can also be found in [4].) After the readers see this alternative picture, it is my hope that at least some of them will say: Wow, that’s so simple, why didn’t they teach us spinors that way from the start?

The paper is organized as follows. In section 2, I review the standard way of teaching Lorentz covariance of the Dirac equation and discuss some pedagogical drawbacks of such teaching. In section 3, I formulate the alternative picture for the Dirac equation which avoids these pedagogical drawbacks, and show that this alternative picture is much simpler and yet physically equivalent to the standard one. This alternative picture can be taught even without ever referring to the standard one, as I outline in section 4. The conclusions are drawn in section 5.

## 2. Standard teaching and its drawbacks

### 2.1. Elements of the standard teaching of the Lorentz covariance of the Dirac equation

Let

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$  \hspace{1cm} (2)$$

be the Dirac equation in the Lorentz system $\mathcal{S}$, where $\gamma^\mu$ are the standard Dirac matrices [5] obeying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$  \hspace{1cm} (3)$$

and $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric. Let

$$(i\gamma^\mu \partial_\mu^\prime - m)\psi^\prime = 0$$  \hspace{1cm} (4)$$
be the Dirac equation in another Lorentz system $S'$. (Here $\psi = \psi(x)$ and $\psi' = \psi'(x')$, but I do not write the $x$-dependence explicitly.) The Lorentz transformation of spacetime coordinates can be written in the form
\[ x' = \Lambda_{\mu}^{\nu}x^\nu, \]
(5)
or more compactly in the matrix form $x' = \Lambda x$. The inverse of it is $x = \Lambda^{-1}x'$, which in the component form reads
\[ x^\mu = (\Lambda^{-1})^\mu_\nu x'^\nu. \]
(6)
Equation (5) implies
\[ \frac{\partial x'^\mu}{\partial x^\nu} = \Lambda_{\mu}^{\nu}. \]
(7)
Since
\[ \frac{\partial}{\partial x^\nu} = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial}{\partial x'^\mu}, \]
(8)
(7) implies $\partial'_\mu = \Lambda_{\mu}^\nu \partial_\nu$, which we write as
\[ \partial'_\mu = \Lambda_{\mu}^\alpha \partial_\alpha. \]
(9)
Therefore (4) can be written as
\[ (i\gamma^\mu \Lambda_{\mu}^\nu \partial_\nu - m)\psi' = 0. \]
(10)
Next write
\[ \psi' = S\psi, \]
(11)
where $S$ is some $x$-independent matrix the properties of which need to be determined. For that purpose one multiplies (2) (with $\mu \rightarrow \alpha$) with $S$ from the left and inserts $1 = S^{-1}S$, so
\[ i\gamma^\nu S^{-1}\partial_\nu S\psi - mS\psi = 0. \]
(12)
Using (11), this can be written as
\[ (i\gamma^\nu S^{-1}\partial_\nu - m)\psi' = 0. \]
(13)
Comparing (13) with (10), one obtains
\[ S\gamma^\mu S^{-1} = \gamma^\mu \Lambda_{\mu}^\nu. \]
(14)
From this equation one can determine $S$ as a function of $\Lambda_{\mu}^\nu$, but the procedure is quite complicated (see e.g. [5]), so I omit it. But even without the explicit expression for $S$ as a function of $\Lambda_{\mu}^\nu$, it is clear that the inverse Lorentz transformation must correspond to the inverse $S$. Therefore (14) can also be written as
\[ S^{-1}\gamma^\nu S = \gamma^\mu (\Lambda^{-1})_{\mu}^\nu. \]
(15)
Using (14), it is possible to prove that
\[ S^{-1} = \gamma^0 S\gamma^0. \]
(16)
(Unfortunately, I am not aware of any simple proof of (16). The simplest proof I am aware of, but is still quite involved, is presented in [6].) By multiplying (16) with $\gamma^0$ from the left and using $\gamma^0\gamma^0 = 1$ (which follows directly from (3)), one gets a very useful form of (16)
\[ \gamma^0 S^{-1} = S\gamma^0. \]
(17)
An important consequence of (16) is that $S^{-1} \neq S'$, i.e. $S'S \neq 1$. Therefore
\[ (\psi'^\dagger\psi')' = \psi'^\dagger S'S\psi \neq \psi'^\dagger\psi, \]
(18)
which shows that $\psi^\dagger \psi$ does not transform as a scalar. On the other hand, defining

$$\tilde{\psi} = \psi^\dagger \gamma^0$$

and using (17) one obtains

$$(\tilde{\psi} \psi)' = \psi^\dagger S^\dagger \gamma_0 S \psi = \psi^\dagger \gamma^0 S^{-1} S \psi = \psi^\dagger \gamma^0 \psi = \tilde{\psi} \psi,$$

which shows that $\tilde{\psi} \psi = \psi^\dagger \gamma^0 \psi$ transforms as a scalar.

In a similar way one finds

$$(\tilde{\psi} \gamma^\mu \psi)' = \psi^\dagger S^\dagger \gamma_0 \gamma^\mu S \psi = \psi^\dagger \gamma^0 \gamma^{-1} \gamma^\mu S \psi.$$

Using (15), it can be written as

$$(\tilde{\psi} \gamma^\mu \psi)' = (\Lambda^{-1})^\mu_v \psi^\dagger \gamma_0 \gamma^v \psi = (\Lambda^{-1})^\mu_v \tilde{\psi} \gamma^v \psi,$$

so comparing it with (6) one concludes that $\tilde{\psi} \gamma^\mu \psi$ transforms as a vector.

2.2. The drawbacks of standard teaching

From section 2.1 one can see that the Lorentz covariance of the Dirac equation is quite complicated. For comparison, Lorentz covariance of the Maxwell equations is much simpler. If possible, it would certainly be desirable to have a simpler formulation of the Lorentz covariance for the Dirac equation.

Moreover, there is something potentially confusing about the standard teaching outlined in section 2.1. The notation $\gamma^\mu$ suggests that this object also might transform as a vector. Why then, is $\gamma^\mu$ in (4) not replaced by $\gamma'^\mu$? Most textbooks which discuss Lorentz covariance of the Dirac equation, including those by the authors of [6–9], do not attempt to answer that question. From the pedagogical point of view, this is certainly not the best way to teach Lorentz covariance of the Dirac equation.

In some textbooks, including those by the authors of [5, 10–12], a somewhat better approach is exploited. They note that $\gamma'^\mu$ is not equal to $\gamma^\mu$, but explain that they are related by a unitary transformation. Consequently, their argument goes, the $\gamma^\mu$ can be fixed and viewed as objects that do not transform under Lorentz transformations. Nevertheless, from the pedagogical point of view, such an approach is also not completely satisfying. A unitary transformation which transforms $\gamma'^\mu$ back to $\gamma^\mu$ should affect also the spinor $\psi$. So, how would $\psi$ transform if one did not choose to transform $\gamma'^\mu$ back to $\gamma^\mu$? In that case, would $\psi$ still transform as a spinor? Or would it perhaps become a scalar? The textbooks above say nothing about that, so it is also not the perfect way to teach Lorentz covariance of the Dirac equation.

The two approaches above have in common that they first introduce the Dirac equation, and then show that $\psi$ transforms in a specific way, known as the transformation of spinors. As an alternative, some textbooks, including those by the authors of [13–15], choose a reversed pedagogy. They first introduce the concept of spinors as abstract algebraic objects (that even do not need to depend on $x$), and then introduce the Dirac equation as an application of spinor mathematics to physics. No doubt, such an approach offers a much deeper mathematical understanding of spinors. In particular, their transformation properties are obtained without referring to the Dirac equation. In addition, the mathematical origin of $\gamma'^\mu$ matrices is explained, from which it becomes clear why they are fixed matrices which do not transform. Nevertheless, even that mathematically more sophisticated approach is not perfect from the pedagogical point of view, precisely because it is mathematically sophisticated. Namely, the mathematical sophistication makes the theory even more complicated, which many practically oriented physicists view as an unnecessary distraction from their true goal—learning physics.
Finally, there are many textbooks which study the Dirac equation but do not really attempt to prove its Lorentz covariance. Such books may be excellent for teaching what they really want to teach (e.g. how to calculate the scattering amplitude for elementary particles), but in the context of teaching Lorentz covariance of the Dirac equation they do not deserve to be mentioned.

Let me end this section with an exercise for the reader. Take three books which study the Dirac equation, not all of which are mentioned in the list of references for this paper. For each of the three books, answer the following questions: are spinors introduced before or after introducing the Dirac equation? Is Lorentz covariance of the Dirac equation proved? Is it explicitly stated that $\gamma^\mu$ does not transform under Lorentz transformations? If yes, how is it explained? Is it written down explicitly how $\psi$ transforms under Lorentz transformations? If yes, is that transformation law derived?

3. Two pictures for the Dirac equation

In section 2, I have studied the standard picture for the Dirac equation, in which the wave function $\psi$ transforms as a spinor under Lorentz transformations of spacetime coordinates, while the gamma matrices $\gamma^\mu$ do not transform at all. Since the transforming quantity transforms as a spinor, I refer to this picture as the spinor picture.

Here I introduce a different picture for the Dirac equation, in which the wave function does not transform under Lorentz transformations of spacetime coordinates, while the gamma matrices transform as components of a vector. Since the transforming quantity transforms as a vector, I refer to this picture as the vector picture. To distinguish the wave function and gamma matrices in the vector picture from those in the spinor picture, those in the vector picture are denoted by $\Psi$ and $\Gamma^\mu$ respectively.

The Dirac equation (2) in the vector picture reads
\[(i\Gamma^\mu \partial_\mu - m)\Psi = 0.\]
(23)

Since I postulate that $\Gamma^\mu$ transforms as a vector, (5) implies that it transforms according to
\[\Gamma'^\mu = \Lambda^\mu_\nu \Gamma'^\nu.\]
(24)

Likewise, postulating that $\Psi$ is a scalar means
\[\Psi' = \Psi.\]
(25)

Since $\Gamma^\mu$ is a vector and $\Psi$ is a scalar, the Lorentz covariance of (23) is quite trivial. Nevertheless, for the sake of completeness, let me present the proof explicitly. Equation (24) can be inverted as
\[\Gamma'^\mu = (\Lambda^{-1})^\mu_\nu \Gamma'^\nu.\]
(26)

This together with (25) and (9) gives
\[\Gamma'^\mu \partial'_\mu \Psi' = (\Lambda^{-1})^\mu_\nu \Gamma'^\nu \Lambda^\alpha_\mu \partial_\alpha \Psi = \Lambda^\alpha_\mu (\Lambda^{-1})^\mu_\nu \Gamma'^\nu \partial_\alpha \Psi = (\Lambda \Lambda^{-1})^\mu_\nu \Gamma'^\nu \partial_\alpha \Psi = \delta^\alpha_\nu \Gamma'^\nu \partial_\alpha \Psi = \Gamma'^\mu \partial_\mu \Psi.\]
(27)

This means that $\Gamma'^\mu \partial'_\mu \Psi' = \Gamma^\mu \partial_\mu \Psi$, which shows that (23) is Lorentz covariant. Note that this simple proof does not depend on equation (14).

The non-trivial aspects of (23), however, are (i) to find out how $\Psi$ and $\Gamma^\mu$ are related to $\psi$ and $\gamma^\mu$, and (ii) to prove that (23) is equivalent to (2). This is what I do next. (In particular, unlike the proof of Lorentz covariance in (27), the proof of equivalence of the two pictures will depend on (14).)
As the transformation properties of $\Psi$, $\Gamma^\mu$, $\psi$ and $\gamma^\mu$ are defined, to establish the general relation between them it is sufficient to specify the relation in one particular Lorentz system of coordinates. For convenience it can be chosen to be the laboratory system $S_{\text{lab}}$, in which I choose

$$\Gamma^\mu_{\text{lab}} = \gamma^\mu, \quad \Psi = \psi_{\text{lab}}. \tag{28}$$

In this sense the laboratory system can be thought of as a ‘preferred’ system of coordinates, but it does not ruin the Lorentz covariance of the vector picture, because the only purpose of the ‘preferred’ system is to establish the relation between the two pictures. Indeed, to use the analogy from the introduction, this is very much analogous to the fact that the operators and states in the Heisenberg picture coincide with those in the Schrödinger picture at one particular ‘initial’ value of time $t_0$, but it does not ruin the fact that each of the pictures by itself is invariant under time translations. Moreover, I show in the appendix how $\Psi$ and $\Gamma^\mu$ can be defined in a more mathematically elegant way, without explicitly referring to any particular system of coordinates.

Now, the rest of the analysis is straightforward. Equation (23) in the system $S_{\text{lab}}$ is

$$(i\Gamma^\mu_{\text{lab}}\partial^\mu_{\text{lab}} - m)\psi_{\text{lab}} = 0. \tag{29}$$

Using (28), this can be written as

$$(i\gamma^\mu\partial^\mu_{\text{lab}} - m)\psi_{\text{lab}} = 0. \tag{30}$$

Now take $\Lambda$ to be the Lorentz transformation that connects the system $S$ with the system $S_{\text{lab}}$, so that (11) and (9) become

$$\psi_{\text{lab}} = S\psi, \tag{31}$$
$$\partial^\mu_{\text{lab}} = \Lambda^\mu_\alpha \partial^\alpha. \tag{32}$$

In this way (30) can be written as

$$(i\gamma^\mu \Lambda^\mu_\alpha \partial^\alpha - m)S\psi = 0, \tag{33}$$

which after using (14) becomes

$$iS\gamma^\mu S^{-1} \partial_{\text{lab}} \psi - mS\psi = 0. \tag{34}$$

Hence, by multiplying with $S^{-1}$ from the left one finally obtains

$$(i\gamma^\mu \partial^\mu_{\text{lab}} - m)\psi = 0. \tag{35}$$

In this way, from the Dirac equation in the vector picture (23) I have derived the Dirac equation in the spinor picture (35). The derivation can also be inverted step by step, implying that starting from the Dirac equation in the spinor picture (35) one can derive the Dirac equation in the vector picture (23). This proves that the two pictures are equivalent.

It is also instructive to see how some bilinear combinations of $\Psi$ are related to those of $\psi$. I first define

$$\tilde{\Psi} \equiv \Psi^\dagger \gamma^0, \tag{36}$$

which is clearly a scalar. Therefore, it is obvious that $\tilde{\Psi} \Psi$ is a scalar and $\tilde{\Psi} \Gamma^\mu \Psi$ a vector, so it does not need to be proved. What needs to be proved is that they are equal to $\psi \psi$ and $\psi \gamma^\mu \psi$, respectively. For $\tilde{\Psi} \Psi$, I perform a straightforward proof

$$\tilde{\Psi} \Psi = \psi^\dagger \gamma^0 \psi = \psi_{\text{lab}}^\dagger \gamma^0 \psi_{\text{lab}} = \psi_{\text{lab}} \psi_{\text{lab}} = (\psi \psi)_{\text{lab}} = \tilde{\Psi} \psi, \tag{37}$$

where in the last equality I have used the fact that $\tilde{\Psi} \psi$ is a scalar. For $\tilde{\Psi} \Gamma^\mu \Psi$ the simplest way is to use a trick. Since it is known that both $\Psi \Gamma^\mu \Psi$ and $\psi \gamma^\mu \psi$ transform as vectors, it is
sufficient to show that they are equal in one particular Lorentz system. But they are obviously equal in the laboratory system, because in that system \( \Psi = \hat{\Psi} \), \( \Gamma^\mu = \gamma^\mu \) and \( \Psi = \psi \). Therefore \( \Psi \Gamma^\mu \Psi \) is equal to \( \hat{\Psi} \gamma^\mu \hat{\psi} \) in all Lorentz systems, which finishes the proof.

I have shown above that

\[
\Psi \Psi = \hat{\Psi} \hat{\psi}, \quad \Psi \Gamma^\mu \Psi = \hat{\Psi} \gamma^\mu \hat{\psi}. \tag{38}
\]

The same can be shown for other similar bilinear combinations of \( \psi \). Since physical quantities are expressed in terms of such bilinear combinations, it shows that the two pictures for the Dirac equation are physically equivalent.

The advantage of the vector picture is that the proof of its Lorentz covariance is much simpler. However, that is not the only advantage. The vector picture allows a simple and natural construction of some additional bilinear combinations with well-defined transformation properties, which in the spinor picture cannot be constructed so naturally. The two most interesting combinations are

\[
\rho = \Psi^\dagger \Psi, \tag{39}
\]
\[
j_\mu = \frac{i}{2} \Psi^\dagger \gamma^\mu \Psi, \tag{40}
\]

where \( A \hat{\delta}_\mu B \equiv A (\hat{\delta}_\mu B) - (\hat{\delta}_\mu A) B \). Clearly, (39) transforms as a scalar and (40) transforms as a vector. Their possible physical interpretation is discussed in [16–18].

For someone who has never heard about the vector picture (which refers to the large majority of physicists at the time of writing this paper), it may be hard to believe that (39) and (40) transform as a scalar and a vector, respectively. In particular, is not the claim that (39) transforms as a scalar in contradiction with the fact that (18) does not transform as a scalar? Let me show that there is no contradiction, by attempting to find a contradiction and seeing how exactly the attempt fails. Similarly to (37), one obtains

\[
\Psi^\dagger \Psi = \Psi^\dagger \gamma^0 \gamma^0 \Psi = \psi^\dagger_{\text{lab}} \gamma^0 \gamma^0 \psi_{\text{lab}} = \tilde{\psi}_{\text{lab}} \gamma^0 \psi_{\text{lab}}. \tag{41}
\]

Naively one might think that the last quantity \( \tilde{\psi}_{\text{lab}} \gamma^0 \psi_{\text{lab}} \) transforms as a time-component of a vector, which would contradict the claim that \( \Psi^\dagger \Psi \) transforms as a scalar. But in fact \( \tilde{\psi}_{\text{lab}} \gamma^0 \psi_{\text{lab}} \) does not transform as a time-component of a vector, because

\[
\tilde{\psi}_{\text{lab}} \gamma^0 \psi_{\text{lab}} \neq \hat{\psi} \gamma^0 \psi. \tag{42}
\]

The two quantities in (42) are equal only if \( \psi \) is evaluated in the laboratory system, but in general they are different. Therefore there is no contradiction between the facts that \( \Psi^\dagger \Psi \) and \( \psi^\dagger \psi = \tilde{\psi} \gamma^0 \psi \) transform as a scalar and a time-component of a vector, respectively. These two quantities coincide in one Lorentz system (chosen to be the laboratory one), but in other Lorentz systems they are different quantities. A more formal demonstration of this fact is given also in the appendix.

4. Teaching only the vector picture

We have seen that there are two equivalent pictures for the Dirac equation: the standard spinor picture and the alternative vector picture. We have also seen that the alternative vector picture is much simpler. Therefore, in this section I propose an alternative way to teach the Dirac equation, by teaching only the vector picture. Of course, a drawback of such teaching would be a clash with most of the existing literature, which could create confusion. Nevertheless, given the advantages of such teaching, I believe it is worthwhile at least to outline how such teaching might look. So this is what I do in what follows.
In an attempt to linearize the Klein–Gordon equation
\[(\partial_\mu \partial^\mu + m^2)\Psi = 0,\] (43)
one obtains the Dirac equation
\[(i\Gamma_\mu \partial_\mu - m)\Psi = 0,\] (44)
where
\[\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}.\] (45)
Here \(\Gamma^\mu\) transforms as a vector and \(\Psi\) as a scalar under Lorentz transformations of spacetime coordinates, so the Lorentz covariance of (44) is obvious. However, (45) suggests that \(\Gamma^\mu\) should be \(n \times n\) matrices, so \(\Psi\) should be an \(n\)-component column. It turns out that the smallest possible value of \(n\) is 4, so one fixes \(n = 4\). One special choice for \(\Gamma^\mu\) satisfying (45) are the standard Dirac matrices \(\gamma^\mu\). (In the rest of the paper they are denoted by \(\gamma^\mu\), but here I modify the notation by putting the bar over \(\mu\) which reminds us that \(\bar{\mu}\) is not a vector index. Namely, the \(\gamma^\mu\) are fixed matrices which do not transform under Lorentz transformations, which is why the notation \(\gamma^{\bar{\mu}}\) is better than \(\gamma^\mu\).) Thus one may determine the vector \(\Gamma^\mu\) in any Lorentz system by choosing one particular Lorentz system, say the laboratory one, in which
\[\Gamma^\mu_{\text{lab}} = \gamma^\mu.\] (46)
Defining
\[\bar{\Psi} = \Psi \gamma^0,\] (47)
the Dirac equation (44) implies that the Dirac vector current
\[j^\mu_{\text{Dirac}} = \bar{\Psi} \Gamma^\mu \Psi\] (48)
is conserved
\[\partial_\mu j^\mu_{\text{Dirac}} = 0.\] (49)
Similarly, the Klein–Gordon equation (43) implies that the Klein–Gordon vector current
\[j_\mu = \frac{1}{2} \Psi^\dagger \overleftrightarrow{\partial_\mu} \Psi\] (50)
is conserved too, i.e.
\[\partial_\mu j_\mu = 0.\] (51)
In most physical applications only the Dirac current is relevant, but in some applications the Klein–Gordon current may be relevant as well. Similarly, one can construct two bilinear scalars \(\bar{\Psi} \Psi\) and \(\Psi^\dagger \Psi\). In most physical applications only \(\bar{\Psi} \Psi\) is relevant, but in some applications \(\Psi^\dagger \Psi\) may be of interest as well.

5. Conclusion

In this paper, I have identified some pedagogical drawbacks in the standard approaches to teaching Lorentz covariance of the Dirac equation, including the fact that the proof of Lorentz covariance is quite complicated. To avoid these drawbacks, I have proposed an alternative way to teach Lorentz covariance of the Dirac equation, by introducing a new formalism. The proposed formalism is inspired by the treatment of spinors in curved spacetime, but is in fact much simpler than that (because it is formulated in flat spacetime) and logically independent of it. The main idea of the formalism is to perform a transformation from the standard \(\psi\) and \(\gamma^\mu\), which transform as a spinor and a scalar, respectively, to new quantities \(\Psi\) and \(\Gamma^\mu\).
which transform as a scalar and a vector, respectively. I have shown that the two formalisms are physically equivalent, but that the new formalism is much simpler and can be taught even without referring to the standard formalism. In addition, the new formalism allows a natural construction of some non-standard bilinear combinations with well-defined transformation properties, such as the vector Klein–Gordon current and the scalar $\Psi^\dagger \Psi$.

**Acknowledgments**

The author is grateful to B Klajn and B Nižić for useful and encouraging comments on the paper, and to S Dowker for drawing attention to some relevant references. This work was supported by the Ministry of Science of the Republic of Croatia under contract no 098-0982930-2864.

**Appendix. Formal transformation theory between the spinor and vector pictures of the Dirac equation**

By starting from the standard spinor picture described in section 2.1, in this section I re-derive the main results of section 3 by developing the formal transformation theory between the spinor and vector pictures without explicitly referring to any particular system of coordinates.

Starting from (11), consider the transformation

$$ (S^{-1}\psi)' = SS^{-1}\psi = \psi. \tag{A.1} $$

This shows that $S^{-1}\psi$ transforms as a scalar, so I define the scalar

$$ \Psi = S^{-1}\psi. \tag{A.2} $$

Since $\Psi$ is a scalar, the quantity

$$ \tilde{\Psi} = \Psi^\dagger \gamma^0 \tag{A.3} $$

is also a scalar. Using (A.2), (A.3) can also be written as

$$ \tilde{\Psi} = \psi^\dagger (S^{-1})^\dagger \gamma^0. \tag{A.4} $$

I want to define a quantity $\Gamma^\mu$ by requiring that

$$ \tilde{\Psi} \Gamma^\mu \Psi = \tilde{\psi} \gamma^\mu \psi. \tag{A.5} $$

The right-hand side of (A.5) is a vector while on the left-hand side $\tilde{\Psi}$ and $\Psi$ are scalars, which implies that $\Gamma^\mu$ is a vector. What I need is a relation between $\Gamma^\mu$ and $\gamma^\mu$. For that purpose, I write

$$ \tilde{\Psi} \Gamma^\mu \Psi = \psi^\dagger (S^{-1})^\dagger \gamma^0 \Gamma^\mu S^{-1} \psi $$

$$ = \psi^\dagger (S^{-1})^\dagger \gamma^0 S^{-1} \Gamma^\mu S^{-1} \psi $$

$$ = \psi^\dagger (S^{-1})^\dagger \gamma^0 S^{-1} \Gamma^\mu S^{-1} \psi $$

$$ = \psi^\dagger \gamma^0 \Gamma^\mu S^{-1} \psi $$

$$ = \psi^\dagger \gamma^0 \Gamma^\mu S^{-1} \psi = \psi S \Gamma^\mu S^{-1} \psi. \tag{A.6} $$

where in the third line I have used (17). The comparison with (A.5) shows that

$$ \gamma^\mu = S \Gamma^\mu S^{-1}. \tag{A.7} $$

which is equivalent to

$$ \Gamma^\mu = S^{-1} \gamma^\mu S. \tag{A.8} $$
Now multiply the Dirac equation \((i\gamma^\mu \partial_\mu - m)\psi = 0\) with \(S^{-1}\) from the left and insert \(1 = SS^{-1}\) to obtain
\[
iS^{-1}\gamma^\mu S\partial_\mu S^{-1}\psi - mS^{-1}\psi = 0. \tag{A.9}\]

Using (A.8) and (A.2), this can be written as
\[
(i\Gamma^\mu \partial_\mu - m)\Psi = 0, \tag{A.10}\]
which is manifestly Lorentz covariant.

Now let me check that \(\bar{\Psi}\Psi = \bar{\psi}\psi\). Similarly to (A.6), one obtains
\[
\bar{\Psi}\Psi = \bar{\psi}^\dagger (S^{-1})^\dagger \gamma^0 S^{-1}\psi = \bar{\psi}^\dagger (S^{-1})^\dagger \gamma^0 \psi
= \bar{\psi}^\dagger (SS^{-1})^\dagger \gamma^0 \psi = \bar{\psi}^\dagger \gamma^0 \psi = \bar{\psi}\psi. \tag{A.11}\]

Finally, let me demonstrate that the scalar nature of \(\Psi^\dagger\Psi\) is not in contradiction with the fact that \(\psi^\dagger\psi\) is not a scalar. This is seen from
\[
\psi^\dagger\Psi = \psi^\dagger (S^{-1})^\dagger S^{-1}\psi \neq \psi^\dagger \psi, \tag{A.12}\]
which is a consequence of the fact that \(S\) is not unitary due to (16).

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