Vortices & Fractons

Darshil Doshi\textsuperscript{1} and Andrey Gromov\textsuperscript{1}

\textsuperscript{1}Brown Theoretical Physics Center & Department of Physics, Brown University, Providence, Rhode Island 02912, USA

(Dated: May 8, 2020)

We discuss a simple and experimentally available realization of fracton physics. We note that superfluid vortices form a Hamiltonian system that conserves total dipole moment and trace of the quadrupole moment of vorticity; thereby establishing a relation to a traceless scalar charge theory in two spatial dimensions. Next we consider the limit where the number of vortices is large and show that emergent vortex hydrodynamics also conserves these moments. Finally, we show the motion of vortices and of fractons on curved manifolds, which can be realized as curved $^4$He films.

Introduction. — Fracton phases of matter are characterized by the presence of immobile or partially mobile local excitations. The constraints on excitation mobility stem from the conservation laws of multipole moments of charge density \cite{1, 2, 3}. Phases that support fracton excitations were first discovered in exactly solvable quantum lattice models \cite{4-6}. One systematic approach to characterization and classification of fracton phases is based on tensor \cite{1, 2, 7-9, 50-52} and multipole gauge theories (MGT) \cite{3, 10}. Recent years have witnessed a significant interest in development and classification of phases of quantum matter supporting fracton excitations \cite{11-38}, with possible applications ranging from quantum memory to quantum elasticity and quantum gravity. For recent reviews see \cite{39, 40}. Despite substantial theoretical progress there are few proposals for experimental realization of the fracton physics \cite{34, 41}.

One prominent, but down-to-earth example of excitations with restricted mobility is crystalline defects \cite{42-49}. There, dislocations satisfy the glide constraint that forces them to move along their Burgers vector, while disclinations are immobile.

In this Note we point out that fracton physics is exhibited by superfluid vortices that have been experimentally observed for many decades. We show that vortices in two spatial dimensions share the mobility constraints with the traceless scalar charge theory (TSCT). We review the Hamiltonian formulation of the vortex dynamics and show that it manifestly conserves dipole and (trace of) quadrupole moments of vorticity. In superfluids, the vorticity of individual vortices is quantized and locally conserved, which leads to identification of vorticity with the scalar charge. These conservation laws imply that isolated vortices are immobile, while vortex dipoles move perpendicular to their dipole moment. Both vortices and their dipoles can be readily created and studied experimentally in superfluid He \cite{53}, BECs \cite{54, 55}, polaron superfluids \cite{56, 57} and non-linear media \cite{58}. We then consider a hydrodynamic limit where the number of vortices becomes large; and collective, hydrodynamic description is applied to the vortices themselves. Remarkably, the resulting hydrodynamics admits a Hamiltonian formulation with Poisson brackets realizing the classical $w_{\infty}$ algebra. We show that vortex hydrodynamics is also equivalent to scalar charge theory and provide a microscopic collective field theory expression for the rank-2 symmetric current. Finally, we discuss the behavior of vortices and fractons on curved manifolds, which can be realized as curved $^4$He films.

Vortices. — We consider a two dimensional incompressible ideal fluid. It is described by the Euler equations

$$\left(\partial_0 + u_i \partial_i\right) u_j = -\partial_j P,$$

where $P$ is the pressure and $u_i$ is the velocity field. The combination $\partial_0 + u_i \partial_i$ is known as material derivative. The incompressibility condition implies that $\partial_i u_i = 0$. Taking curl of (1) we obtain the Helmholtz equation

$$\left(\partial_0 + u_i \partial_i\right) \omega = 0,$$

where $\omega = \epsilon^{ij} \partial_i u_j$ is the vorticity. Eq.(2) admits solutions where the vorticity is concentrated in a finite number of point vortices. The complex velocity field $u_z = u_1 + i u_2$ takes form

$$u_z(z) = -i \sum_{\alpha=1}^{N} \frac{\gamma_{\alpha}}{z - z_{\alpha}(t)}, \quad \omega(z) = \sum_{\alpha=1}^{N} \gamma_{\alpha} \delta^2(z - z_{\alpha}(t)),$$

where $z_{\alpha}(t) = x_{\alpha}^1(t) + ix_{\alpha}^2(t)$ (we will switch between complex and Cartesian coordinates at will) is time-dependent position of the $\alpha$-th vortex and $2\pi \gamma_{\alpha}$ is its circulation; while $\gamma = |\gamma_{\alpha}|$ is the vortex strength. We have assumed that vorticity is quantized in the units of $\gamma$, which is the case in the superfluids \cite{53}.

Remarkably, the vortex coordinates $x_{\alpha}^\alpha(t)$ form a Hamiltonian system \cite{59}

$$H = -2\pi \sum_{\alpha<\beta} \gamma_{\alpha} \gamma_{\beta} \ln |x_{\alpha}^\alpha - x_{\beta}^\beta|,$$

$$\{x_{\alpha}^\alpha, -2\pi \gamma_{\beta} x_{\beta}^\beta\} = \delta^{\alpha\beta},$$
where $\alpha, \beta = 1, 2, \ldots, N$ label the vortex strength. We refer the reader to [60, 61] for an in-depth review of the vortex systems.

Dynamical system (4)-(5) also describes charged particles moving in a strong magnetic field, in the limit of infinite cyclotron frequency, or equivalently, on the lowest Landau level. Consequently, all our results apply verbatim to plasma in strong magnetic field.

In dealing with (4)-(5) it is useful to use the complex coordinates $z_\alpha$. In complex notations the only non-trivial Poisson brackets take the form
\[
\{z_\alpha, \bar{z}_\beta\} = i(\pi \gamma_\alpha)^{-1}\delta_{\alpha\beta}.
\]
(6)

The equations of motion are
\[
\ddot{z}_\alpha = -i \sum_{\beta=1, \beta \neq \alpha}^{N} \frac{\gamma_\beta}{z_\alpha - \bar{z}_\beta}.
\]
(7)

It is worth emphasizing that $H$ is not just the potential energy. Due to the non-trivial commutations relations between $z_\alpha$ and $\bar{z}_\alpha$, $H$ can be viewed as kinetic energy. 

Conservation laws. — Hamiltonian $H$ is translation and rotation invariant. The corresponding integrals of motion are known as [62] impulse, $P_i$ and angular impulse, $L$ [61, 63]. They are given by
\[
P_i = -2\pi \epsilon_{ij} \sum_\alpha \gamma_\alpha x_i^\alpha, \quad L = 2\pi \sum_\alpha \gamma_\alpha x_i^\alpha x_j^\alpha \delta_{ij}.
\]
(8)

We recognize in Eqs. (8) that impulse is related to the dipole moment of vorticity (also known as center of circulation), while angular impulse corresponds the trace of the quadrupole moment of vorticity, $Q_{ij}$ (also known as moment of circulation), according to
\[
P_i = -\epsilon_{ij} D_j, \quad L = \delta_{ij} Q_{ij}.
\]
(9)

Together, the quantities $P_i, L, D_i, Q_{ij}$ form a multipole algebra [64]
\[
\{L, P_i\} = 2\epsilon_{ij} P_j, \quad \{P_i, D_j\} = -\delta_{ij} \Gamma, \quad \{L, D_i\} = -2\epsilon_{ij} D_j, \quad \{P_i, \delta_{jk} Q_{jk}\} = -2D_i,
\]
(10)

where we have introduced the total vortex strength
\[
\Gamma = \sum_{\alpha=1}^{N} \gamma_\alpha.
\]
(11)

Thus, the vortices are equivalent to a traceless scalar charge theory; where total charge, dipole, and trace of the quadrupole are conserved [1]. Isolated charges are immobile; while isolated dipoles move perpendicular to their dipole moment. Vortices and vortex dipoles are experimentally available with the present day technology.

Mobility constraints. — Conservation laws (8) imply that motion of many vortices is constrained to preserve the dipole and quadrupole moment. Moreover, since the conserved quantities $(H, D_i D^i, \delta_{ij} Q_{ij})$ are in involution, the problem of $N$ vortices is integrable for $N \leq 3$ [65].

Other typical cases are chaotic [61]. We discuss the “fractonic” motion of vortices next.

A single or well-isolated, vortex is immobile. Analogously to fractons, the mass of an isolated vortex is not well-defined. A broad class of definitions [66] leads to the diverging mass, which agrees with fracton ideas [67].

Dipole consisting of two vortices with opposite vorticities moves in a straight line perpendicular to its dipole moment. At low temperatures vorticity neutral systems “condense” into a gas of neutral dipoles [68]. The dipole of two vortices of the same vorticities moves in a closed orbit around their “center of vorticity”, while keeping the distance between the two vortices constant. Motion of dipole is illustrated in Fig. 1. Relative distances can only change if the number of vortices is $N \geq 3$ [61].

The quadrupole of two vortex-dipoles exhibits a variety of complex dynamics. One common type of interactions (particularly at low temperature) is scattering between two dipoles as shown in Fig. 1. As a result of scatter-
ing a vortex dipole makes a $\pi/2$ turn, which agrees with phenomenology of TSCT.

**Statistical mechanics.**— Although many-vortex dynamics is chaotic, for certain vortex configurations the relative positions of vortices are completely frozen. Such configurations are called vortex crystals or vortex equilibria [69]. The examples include $N$ identical collinear vortices situated in the roots of $N$-th Hermite polynomial as well as Adler-Moser polynomials, identical vortices located in the vertices of a regular $N$-polygon, etc. There are many other examples (see [69] for a review). Vortex crystals can move as rigid objects in which case they are referred to as relative equilibria, or can be stationary. Such configurations explore a very small fraction of the phase space. This is immediately obvious since for a vortex system phase space coincides with the configuration space. Vortex crystals emerge experimentally after relaxation of highly turbulent two dimensional flows [70, 71].

It is tempting to compare vortex crystals to the Hilbert space fragmentation seen in quantum dipole conserving systems [72–74]. There, Hilbert space “shatters” into many disconnected subspaces, within each such subspace either integrability or thermalization is possible.

Mobility constraints combined with the phase space reduction lead to an exotic statistical mechanics of vortices [75, 76]. In particular, above critical energies vortices experience “negative temperature” [75, 77], which follows from the structure of the phase space [78]. At negative temperature the vortices of the same vorticity tend to clump together, which nicely corresponds to gravitational attraction of fractons discussed in [21]. Vortex crystals may be an obstruction to ergodicity: Clusters of vortices take a very long time to merge [76]. To the best of our knowledge, the ergodicity of vortex system is still an open problem [79].

**Vortex hydrodynamics.**— Next we would like to consider a limit where the number of vortices is very large. Due to chaotic behavior and strong interactions between the vortices, this limit admits a description in terms of an emergent hydrodynamics [80]. We will show that in hydrodynamic limit the dipole and trace of the quadrupole moments are conserved. These conservation laws will be made manifest by re-writing the continuity equation in the rank-2 form

$$\dot{\rho} + \partial_i \partial_j J_{ij} = 0, \quad \text{Tr}(J_{ij}) = 0,$$

where $J_{ij}$ is the symmetric, traceless rank-2 tensor. Here $\rho = (2\pi \gamma)^{-1} \omega$ is the local vortex density. This form of continuity equation implies that the dipole moment and the trace of the quadrupole moment are conserved

$$\partial_0 D_k = \partial_0 \int x_k \rho = \int x_k \partial_i \partial_j J_{ij} = 0,$$

$$\partial_0 \text{Tr}(Q_{ij}) = \partial_0 \int x^2 \rho = \int \text{Tr}(J_{ij}) = 0.$$  

We would like emphasize one subtle difference between traditional TSCT and vortices: The former is non-chiral, while the latter is chiral. In TSCT a dipole moves perpendicular to its dipole moment; while for a vortex dipole, the dipole moment and the direction of motion form a right pair.

Vortex hydrodynamics for the chiral flow (i.e. when all vortices are of the same vorticity, $\gamma_0 = \gamma$) was derived by Wiegmann-Abanov in [80]. The continuum limit of the vortex Hamiltonian (4) is [81]

$$H_{WA} = \frac{1}{2} \int [\alpha - \eta^2 (\partial_i \ln \rho)(\partial_i \ln \rho)] \, d^2 r,$$

where $v_i$ is the vortex velocity and $\eta = \frac{\gamma^2}{\pi}$. Vortex fluid is incompressible $\partial_i v_i = 0$ and $v_i$ is completely determined by the density through [80]

$$\epsilon_{ij} \partial_i v_j = 2 \pi \gamma \rho + \eta \Delta \ln \rho.$$  

The Poisson brackets form the classical $w_\infty$ algebra

$$\{\rho(x), \rho(x') \} = \epsilon_{rs} \partial_s \partial_r \rho(x) \delta(x - x'),$$

where $\partial_i = \frac{\partial}{\partial x_i}$. Brackets between velocity and density are deduced from (16)

$$\{ v_k(x), \rho(x') \} = -\partial_k (\rho(x) \delta(x - x')) - \eta \epsilon_{ij} \partial_j \left[ \frac{1}{\rho} \epsilon_{rs} \partial'_r \partial_s (\rho(x) \delta(x - x')) \right].$$

We are interested in computing the equation of motion for the density $\rho$

$$\dot{\rho}(x) = \{ H_{WA}, \rho(x) \}.$$  

Direct calculation gives the continuity equation

$$\dot{\rho} + \partial_k j_k = 0 \iff \dot{\rho} + v_k \partial_k \rho = D_0 \rho = 0,$$

where $j_k = \rho v_k$. This is consistent with Helmholtz equation (2). The consistency is non-trivial since (2) includes the material derivative with $u_i$, while the material derivative contains $v_i$ in (20). The equivalence of (2) and (20) is established using the relation between $u_i$ and $v_i$ [80]

$$v_i = u_i - \frac{\gamma}{4} \epsilon_{ij} \partial_j \ln \rho.$$  

Using the identity

$$2 \pi \gamma u_i \rho = \epsilon_{ik} \left[ \partial_j (u_j u_k) - \frac{1}{2} \partial_k (u_j u_j) \right]$$

with either (2) or (20) we find

$$j_i = \partial_j J_{ij}, \quad J_{ij} = \frac{1}{2 \pi \gamma} \left( \epsilon_{ik} u_j u_k - \frac{1}{2} \epsilon_{ij} u^2 \right) - \frac{\gamma}{4} \epsilon_{ij} \rho.$$
The anti-symmetric part of $J_{ij}$ drops out from (12). In the chiral case an equivalent relation was derived in [82].

Emergent hydrodynamics for vortices of positive and negative vorticity was developed by Yu-Bradley [83]. The conservation of the impulse and angular impulse holds in their model as well [84]. We will discuss an independent collective field theory derivation of the rank-2 conservation law (12) for arbitrary number of vortices next.

Collective field theory of vortices.— We now turn to the collective form of (7). Presence of positive and negative vortices is assumed.

Density and current fields are defined as follows

$$
\rho(z) = \frac{1}{\gamma} \sum_{\alpha} \gamma_{\alpha} \delta(z-z_{\alpha}) , \\
J_{z}(z) = \rho(z)v(z) = \frac{1}{\gamma} \sum_{\alpha} \gamma_{\alpha} \hat{z}_{\alpha} \delta(z-z_{\alpha}) .
$$

(24)

We will need the complex notation $j_{z} = j_{1} - ij_{2}$ and the \delta-function identity

$$
\partial_{z} \frac{1}{\bar{z}} = \partial_{\bar{z}} \frac{1}{\bar{z}} = \pi \delta(z) .
$$

(25)

Time derivative of the density is given by

$$
\dot{\rho} = - \frac{1}{\gamma} \sum_{\alpha=1}^{N} \left[ \gamma_{\alpha} \hat{z}_{\alpha} \partial_{\bar{z}} \delta(z-z_{\alpha}) + \gamma_{\alpha} \hat{z}_{\alpha} \partial_{z} \delta(z-z_{\alpha}) \right] .
$$

(26)

Using (7) this is transformed into the

$$
\dot{\rho} + \partial_{\bar{z}} \partial_{z} J_{\bar{z}z} + \partial_{z} \partial_{\bar{z}} J_{zz} = 0 ,
$$

(27)

where we have introduced a traceless symmetric tensor current

$$
J_{zz} = \frac{1}{2\pi i\gamma} \left( \sum_{\alpha} \frac{\gamma_{\alpha}}{z-z_{\alpha}} \right)^{2} - \partial_{z} \sum_{\alpha} \frac{\gamma_{\alpha}^{2}}{z-z_{\alpha}} ,
$$

(28)

and $J_{\bar{z}z} = J_{zz}$. It is crucial that in (28) the second order poles cancel. In Cartesian components the symmetric tensor current is given by

$$
J_{ij} = \frac{1}{2\pi}\left( \epsilon_{ik}u_{j}u_{k} - \frac{1}{2} \epsilon_{ij}u^{2} \right) - \frac{\gamma}{4} \epsilon_{ij} n ,
$$

(29)

$$
n = (2\pi \gamma)^{-1} \sum_{\alpha} \gamma_{\alpha} \delta(z-z_{\alpha}) .
$$

(30)

where we introduced the vortex number density $n(z)$. This is the central result of the present work: The continuity equation takes form (12). The above derivation is general and applies to hydro with vortices of both kinds present. In particular, it applies to the case when total vorticity is 0.

Curved space.— Symmetric tensor gauge theories do not remain gauge invariant on a curved space [43]. Furthermore, the conservation law of dipole moment cannot remain unchanged on a curved space. Below, we show that, on a curved space, the dynamics of vortices and the mobility constraints change. Vortices on a curved space have been studied in [85–87] and can be experimentally realized in thin $^4$He films. Vortex hydrodynamics of chiral flows was generalized to curved spaces in [82]. Vortex problem on a surface of a sphere is also relevant for geophysical and atmospheric applications. The Helmholtz equation on a curved surface takes form [82]

$$
\dot{\rho} + u_{i} \nabla_{i} \left( \rho + \frac{s}{4\pi R} \right) = 0 ,
$$

(31)

where $\nabla_{i}$ is a covariant derivative, $R$ is Ricci curvature and $s - \frac{1}{2}$ is the geometric spin of a vortex. Eq.(31) also takes form (12) with slightly modified $J_{ij}$ [82]

$$
J_{ij}(R) = J_{ij} \bigg|_{R=0} + \left[ \frac{1}{2\pi\gamma} \nabla_{i}u_{j} + \frac{\gamma}{2} \epsilon_{ij} \left( \rho - \frac{s}{4\pi R} \right) \right] .
$$

(32)

Note that the last term in (32) contributes to the equations of motion only when curvature is inhomogeneous. We can draw the following conclusion from (31)-(32). On a surface of constant curvature an isolated vortex remains immobile [86, 88], which is consistent with [43]. A dipole moves along a geodesic that is perpendicular to the dipole moment; which is consistent with the corresponding fracton observations made in [89].

On a surface of variable curvature an isolated vortex does move: the dipole conservation law is broken and fractonic property is lost; in agreement with [43]. The potential force acting on an isolated vortex is obtained by differentiating the Robin function [90]. The dipole moves along a geodesic in the general case [91].

Conclusions.— We have established an equivalence between vortex dynamics in two-dimensional superfluids and traceless scalar charge theory. We have shown that vortices provide a Hamiltonian realization of fracton dynamics for any finite number of vortices and in hydrodynamic limit. Thus superfluid vortices provide a readily available platform for experimental realization of fracton quasiparticles.

Similar conservation laws hold in 3 dimensions for vortex lines. We leave the exploration of higher dimensional case, discussion of more refined probes of fracton dynamics in superfluids and BECs such as role of the trap and finite lifetime, generalization to chiral superfluids such as $^3$He and many other open question to future work. Theory of vortices plays central role in statistical approach to turbulence [75]; where the questions of ergodicity and validity of statistical mechanics are central [79]. It would be very interesting to see if fracton-inspired ideas can lead to new insight into quantum and classical turbulence as well as the problem of quantization of vortex dynamics.
Finally, dynamics of electrons residing in the lowest Landau level is formally identical to that of vortices, consequently we expect applications of fracton inspired ideas to the physics of fractional quantum Hall effect.

Acknowledgments.— We thank A. Abanov, A. Bogatskiy, S. Moroz for comments on the manuscript and A. Bogatskiy for bringing the results of [82] to our attention. A.G. was supported by the Brown University.

[1] M. Pretko, Physical Review B 95, 115139 (2017).
[2] M. Pretko, Physical Review B 96, 035119 (2017).
[3] A. Gromov, Physical Review X 9, 031035 (2019).
[4] C. Chamon, Physical review letters 94, 040402 (2005).
[5] J. Haah, Physical Review A 83, 042330 (2011).
[6] S. Vijay, J. Haah, and L. Fu, Physical Review B 92, 235136 (2015).
[7] C. Xu, arXiv preprint cond-mat/0602443 (2006).
[8] C. Xu and P. Hofava, Physical Review D 81, 104033 (2010).
[9] A. Rasmussen, Y.-Z. You, and C. Xu, arXiv preprint arXiv:1601.08235 (2016).
[10] D. Bulmash and M. Barkeshli, arXiv preprint arXiv:1806.01855 (2018).
[11] A. Prem, M. Pretko, and R. Nandkishore, arXiv preprint arXiv:1709.09673 (2017).
[12] A. Prem, S.-J. Huang, H. Song, and M. Hermele, arXiv preprint arXiv:1806.04687 (2018).
[13] A. Prem, S. Vijay, Y.-Z. Chou, M. Pretko, and R. M. Nandkishore, arXiv preprint arXiv:1806.04148 (2018).
[14] K. Slagle, A. Prem, and M. Pretko, arXiv preprint arXiv:1807.00827 (2018).
[15] H. Song, A. Prem, S.-J. Huang, and M. A. Martin-Delgado, Phys. Rev. B 99, 155118 (2019).
[16] K. Slagle and Y. B. Kim, Physical Review B 96, 165106 (2017).
[17] K. Slagle and Y. B. Kim, Physical Review B 96, 195139 (2017).
[18] Y. B. K. Kevin Slagle, arXiv preprint arXiv:1712.04511 (2017).
[19] W. Shirley, K. Slagle, and X. Chen, arXiv preprint arXiv:1806.08679 (2018).
[20] K. Slagle, D. Aasen, and D. Williamson, arXiv preprint arXiv:1812.01613 (2018).
[21] M. Pretko, Physical Review D 96, 024051 (2017).
[22] M. Pretko, Physical Review B 96, 125151 (2017).
[23] T. Devakul, S. Parameswaran, and S. Sondhi, Physical Review B 97, 041110 (2018).
[24] T. Devakul, Y. You, F. Burnell, and S. Sondhi, arXiv preprint arXiv:1805.04097 (2018).
[25] Y. You, T. Devakul, F. Burnell, and S. Sondhi, Physical Review B 98, 035112 (2018).
[26] Y. You, T. Devakul, F. Burnell, and S. Sondhi, arXiv preprint arXiv:1805.09800 (2018).
[27] Y. You, T. Devakul, S. Sondhi, and F. Burnell, arXiv preprint arXiv:1904.11530 (2019).
[28] Z. Weinstein, E. Cobanera, G. Ortiz, and Z. Nussinov, arXiv preprint arXiv:1812.04561 (2018).
[29] J. Wang, K. Xu, and S.-T. Yau, arXiv preprint arXiv:1911.01804 (2019).
[30] D. Aasen, D. Bulmash, A. Prem, K. Slagle, and D. J. Williamson, arXiv preprint arXiv:2002.05166 (2020).
[31] H. Ma, E. Lake, X. Chen, and M. Hermele, Physical Review B 95, 245126 (2017).
[32] H. Ma, A. Schmitz, S. Parameswaran, M. Hermele, and R. M. Nandkishore, Physical Review B 97, 125101 (2018).
[33] H. Yan, arXiv preprint arXiv:1807.05942 (2018).
[34] H. Yan, O. Benton, L. D. Jaubert, and N. Shannon, arXiv preprint arXiv:1902.10934 (2019).
[35] H. Yan et al., Physical Review B 99, 155126 (2019).
[36] A. Schmitz, H. Ma, R. M. Nandkishore, and S. Parameswaran, Physical Review B 97, 134426 (2018).
[37] H. Ma and M. Pretko, Physical Review B 98, 125105 (2018).
[38] A. Gromov, A. Lucas, and R. M. Nandkishore, arXiv preprint arXiv:2003.09429 (2020).
[39] R. M. Nandkishore and M. Hermele, arXiv preprint arXiv:1803.11196 (2018).
[40] M. Pretko, X. Chen, and Y. You, arXiv preprint arXiv:2001.01722 (2020).
[41] J. Sous and M. Pretko, arXiv preprint arXiv:1904.08424 (2019).
[42] M. Pretko and L. Radzihovsky, Physical Review Letters 120, 195301 (2018).
[43] A. Gromov, Phys. Rev. Lett. 122, 076403 (2019).
[44] L. Radzihovsky and M. Hermele, Physical Review Letters 124, 050402 (2020).
[45] A. Gromov and P. Surówka, arXiv preprint arXiv:1808.06984 (2019).
[46] M. Pretko and L. Radzihovsky, arXiv preprint arXiv:1808.05616 (2018).
[47] A. Kumar and A. C. Potter, arXiv preprint arXiv:1808.05621 (2018).
[48] M. Pretko, Z. Zhai, and L. Radzihovsky, arXiv preprint arXiv:1907.12577 (2019).
[49] A. Gromov, arXiv preprint arXiv:2002.11817 (2020).
[50] H. Kleinert, Physics Letters A 91, 295 (1982).
[51] H. Kleinert, Physics Letters A 96, 302 (1983).
[52] H. Kleinert, Physics Letters A 97, 51 (1983).
[53] R. J. Donnelly, Quantized vortices in helium II, Vol. 2 (Cambridge University Press, 1991).
[54] T. Neely, E. Samson, A. Bradley, M. Davis, and B. P. Anderson, Physical review letters 104, 160401 (2010).
[55] D. Freilich, D. Bianchi, A. Kaufman, T. Langin, and D. Hall, Science 329, 1182 (2010).
[56] D. Sanvitto, S. Pigeon, A. Amo, D. Ballarini, M. De Giorgi, I. Carusotto, R. Hivet, F. Pisanello, V. Sala, P. Guimaraes, et al., Nature photonics 5, 610 (2011).
[57] G. Nardin, G. Grosso, Y. Léger, B. Piętka, F. Morier-Genoud, and B. Deveaud-Pledran, Nature Physics 7, 635 (2011).
[58] A. Mamaev, M. Saffman, and A. Zozulya, Physical review letters 77, 4544 (1996).
[59] C. Lin, Proceedings of the National Academy of Sciences of the United States of America 27, 570 (1941).
[60] P. K. Newton, The N-vortex problem: analytical techniques, Vol. 145 (Springer Science & Business Media, 2013).
[61] H. Aref, Journal of mathematical Physics 48, 065401 (2007).
[62] These are also known as center of circulation and moment of circulation.
It is important to note that unlike in the case of theories studied in [3], there are no additional internal symmetries responsible for the conservation of dipole and quadrupole tensors. These conservations, instead, stem from spatial symmetries and non-commutativity.

The problem of 4 vortices with vanishing charge and dipole moment is integrable.

D. Thouless and J. Anglin, Physical review letters 99, 105301 (2007).

We note that there are other ways to define vortex mass that leads to vanishing [92] or to finite result [93]. This definition results in an answer given in terms of kelvon excitation energy.

R. H. Kraichnan and D. Montgomery, Reports on Progress in Physics 43, 547 (1980).

T. Lundgren and Y. Pointin, Journal of statistical physics 17, 323 (1977).

D. Montgomery and G. Joyce, The Physics of Fluids 17, 1139 (1974).

Namely, from the fact that the phase space is finite.

G. L. Eyink and K. R. Sreenivasan, Reviews of modern physics 78, 87 (2006).

P. Wiegmann and A. G. Abanov, Physical review letters 113, 034501 (2014).

We assume that the argument of ln $\rho$ is multiplied by an appropriate power of $\ell$. $\ell$ is the length-scale of the patches that we average over when taking the hydrodynamic limit.

A. Bogatskiy, Journal of Physics A: Mathematical and Theoretical 52, 475501 (2019).

We assume that the argument of ln $\rho$ is multiplied by an appropriate power of $\ell$. $\ell$ is the length-scale of the patches that we average over when taking the hydrodynamic limit.

X. Yu and A. S. Bradley, Physical review letters 119, 185301 (2017).

G. L. Eyink and K. R. Sreenivasan, Reviews of modern physics 78, 87 (2006).

P. Wiegmann and A. G. Abanov, Physical review letters 113, 034501 (2014).

We assume that the argument of ln $\rho$ is multiplied by an appropriate power of $\ell$. $\ell$ is the length-scale of the patches that we average over when taking the hydrodynamic limit.

D. Hally, Vortex motion in thin films, Ph.D. thesis, University of British Columbia (1980).

Yu-Bradley hydrodynamics involves two fluids and, consequently, two densities: number and charge density. The dipole moment of the charge density is conserved, while the dipole moment of the number density is not.

D. Hally, Journal of Mathematical Physics 21, 211 (1980).

D. G. Dritschel and S. Boatto, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 471, 20140890 (2015).

K. Slagle, A. Prem, and M. Pretko, Annals of Physics 410, 167910 (2019).

S. Boatto and J. Koiller, arXiv preprint arXiv:0802.4313 (2008).

J. Koiller and S. Boatto, in AIP Conference Proceedings, Vol. 1130 (American Institute of Physics, 2009) pp. 77–88.

G. Baym and E. Chandler, Journal of Low Temperature Physics 50, 57 (1983).

T. Simula, Physical Review A 97, 023609 (2018).
Chiral traceless scalar charge theory

In this Appendix we describe the traceless scalar charge theory that satisfies Eq.(12) and its chiral counterpart. The theory involves a single complex scalar field $\Phi$ and is invariant under the following transformations

$$\Phi' = e^{if(x)}\Phi, \quad f(x) = \lambda + \lambda_k x_k + \zeta |x|^2, \quad (33)$$

where parameters $\lambda, \lambda_k, \zeta$ are arbitrary. The corresponding conservations laws are that of charge, dipole moment and trace of the quadrupole moment. The Lagrangian invariant under these transformations can be found as follows. First, we define the invariant derivative operators

$$D_I(\Phi) = \sigma_I^{ij} (\partial_i \Phi \partial_j \Phi - \Phi \partial_i \partial_j \Phi), \quad (34)$$

where $\sigma_I$ are the Pauli matrices and $I = 1, 3$. Under the (33) we have

$$\delta D_I(\Phi) = \sigma_I^{ij} \partial_i \partial_j f(x) = 0. \quad (35)$$

The invariant Lagrangian is constructed using these derivatives

$$\mathcal{L} = \dot{\Phi}^* \dot{\Phi} + g_1 |D_1(\Phi)|^2 + g_2 |D_2(\Phi)|^2 + g_1^l \text{Re} [(\Phi^*)^2 D_1(\Phi)] + g_2^l \text{Re} [(\Phi^*)^2 D_2(\Phi)] + \mu |\Phi|^2 + \ldots, \quad (36)$$

where ... stands for the higher order terms and the "interesting" terms are the first three. The theory is invariant under $C_4$, but not $SO(2)$ for the generic values of $g_1, g_1^l$. It is $SO(2)$ invariant in the case $g_1 = g_2$ and $g_1^l = g_2^l$. The global symmetry (33) leads to the conservation law

$$\dot{\rho} + \partial_i \partial_j J^{ij} = 0, \quad \text{Tr}(J_{ij}) = 0. \quad (37)$$

In two spatial dimensions it is possible to add a parity-breaking term to the Lagrangian (36). Indeed, consider

$$\mathcal{L}_{\text{odd}} = ie^{kl} \sigma_I^{ij} (\partial_I \Phi^* \partial_k \Phi^* - \Phi^* \partial_I \partial_k \Phi^*) (\partial_j \Phi \partial_i \Phi - \Phi \partial_j \partial_i \Phi), \quad (38)$$

where the overall factor of $i$ is required because the term is imaginary. This term breaks parity due to the presence of the $e$-tensor. This theory is invariant under (33). Invariance under the quadratic symmetry follows from symmetry of the invariant derivative operators

$$\delta \mathcal{L}_{\text{odd}} = ie^{kl} (\partial_I \Phi^* \partial_j \Phi^* - \Phi^* \partial_I \partial_j \Phi^*) \partial_I \partial_j f + \text{c.c.} = ie^{kl} (\partial_I \Phi^* \partial_j \Phi^* - \Phi^* \partial_I \partial_j \Phi^*) + \text{c.c.} = 0. \quad (39)$$

Charged particles in the Lowest Landau Level

Here we show that the dynamics governing point vortices also describes a 2D system of charged particles in the presence of a perpendicular magnetic field, in the lowest Landau level limit. The Lagrangian for such a system is

$$\mathcal{L} = \sum_l \frac{m}{2} \dot{x}_l^i \dot{x}_l^i + \frac{B}{2} \sum_l e^l \epsilon_{ij} x_l^i \dot{x}_l^j - U \sum_{i<j} e^l e^J \ln |x^I - x^J|. \quad (40)$$

The last term denotes the Coulomb interactions between charged particles. Projecting this onto the LLL is equivalent to taking the limit $m \to 0$. In this limit, the Lagrangian and the corresponding Hamiltonian are

$$\mathcal{L} = \frac{B}{2} \sum_l e^l \epsilon_{ij} x_l^i \dot{x}_l^j - H \quad (41)$$

$$H = \frac{U}{2} \sum_l e^l e^J \ln |x^I - x^J|. \quad (42)$$

Note that the Hamiltonian is the same as that in Eq.(4), with the vortex strengths $\gamma_\alpha$ replaced by the charges $e^I$. The conjugate momenta are given by

$$p_l^I = \frac{\partial \mathcal{L}}{\partial \dot{x}_l^I} = -Be^l \epsilon_{ij} x_l^j. \quad (43)$$
The dynamics can then be described by the Poisson bracket

$$\{ x'_I, -B e^J x'_J \} = \delta^{IJ},$$

(44)

which is the same as the one in Eq.(5); and implies dynamics similar to point vortices.

The first term in the action is invariant under an infinite set of area-preserving diffeomorphisms; but the Coulomb interaction reduces the set of symmetries to (global) translations and rotations. The corresponding Noether charges, namely, total momentum and angular momentum are related to electric dipole moment and trace of quadrupole moment respectively.

\[
\begin{align*}
\text{momentum:} & \quad P_i = -B \sum_I e^I \epsilon_{ij} x'_j = -B \epsilon_{ij} D_j \\
\text{angular momentum:} & \quad L = \sum_I \epsilon_{ij} x'_i p'_j = B \sum_I \delta_{ij} x'_i x'_i = \delta_{ij} Q_{ij}
\end{align*}
\]

(45)

(46)

We note that these conserved charges are analogous to those obtained for point vortices in Eq.(8).

**Derivation of the symmetric tensor current from Yu-Bradley hydrodynamics**

Here, we derive the traceless and symmetric tensor current described Eq.(29) for the collective field theory describing binary vortex fluid. We start with the current

$$j = \rho v = \rho u - \frac{i \gamma}{2} \partial_z n,$$

(47)

where \(n\) in the last term is the number density defined in Eq.(30). The last term has been derived in [80]. In Cartesian coordinates, we can see that the current indeed has the form (23).

$$j_i = \rho u_i - \frac{\gamma}{4} \epsilon_{ij} \partial_j n = \partial_j \left[ \epsilon_{ik} u_j u_k - \frac{1}{2} \epsilon_{ij} u^2 - \frac{\gamma}{4} \epsilon_{ij} n \right]$$

(48)

We’ve used the identity (22) in the second equation. The quantity inside the square bracket can then be identified with the tensor current \(J_{ij}\),

$$J_{ij} = \epsilon_{ik} u_j u_k - \frac{1}{2} \epsilon_{ij} u^2 - \frac{\gamma}{4} \epsilon_{ij} n.$$

(49)