FOCK SPACES CORRESPONDING TO POSITIVE DEFINITE LINEAR TRANSFORMATIONS

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Abstract. Suppose $A$ is a positive real linear transformation on a finite dimensional complex inner product space $V$. The reproducing kernel for the Fock space of square integrable holomorphic functions on $V$ relative to the Gaussian measure $d\mu_A(z) = \sqrt{\det A} e^{-\text{Re}(Az,z)} \, dz$ is described in terms of the holomorphic–antiholomorphic decomposition of the linear operator $A$. Moreover, if $A$ commutes with a conjugation on $V$, then a restriction mapping to the real vectors in $V$ is polarized to obtain a Segal–Bargmann transform, which we also study in the Gaussian-measure setting.

Introduction

The classical Segal-Bargmann transform is an integral transform which defines a unitary isomorphism of $L^2(\mathbb{R}^n)$ onto the Hilbert space $F(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$ which are square integrable with respect to the Gaussian measure $\mu = \pi^{-n} e^{-|z|^2} dxdy$, where $dxdy$ stands for the Lebesgue measure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$, see [1, 3, 4, 5, 10, 11]. There have been several generalizations of this transform, based on the heat equation or the representation theory of Lie groups [6, 9, 12]. In particular, it was shown in [9] that the Segal-Bargmann transform is a special case of the restriction principle, i.e., construction of unitary isomorphisms based on the polarization of a restriction map. This principle was first introduced in [9], see also [8], where several examples were explained from that point of view. In short the restriction principle can be explained in the following way. Let $M_C$ be a complex manifold and let $M \subset M_C$ be a totally real submanifold. Let $F = F(M_C)$ be a Hilbert space of holomorphic functions on $M_C$ such that the evaluation maps $F \ni F \mapsto F(z) \in \mathbb{C}$ are continuous for all $z \in M_C$, i.e., $F$ is a reproducing Hilbert space. There exists a function $K : M_C \times M_C \to \mathbb{C}$ holomorphic in the first variable, anti-holomorphic in the second variable, and such that the following hold:

(a) $K(z, w) = \overline{K(w, z)}$ for all $z, w \in M_C$;
(b) If $K_w(z) := K(z, w)$ then $K_w \in F$ and

$$F(w) = (F, K_w), \quad \forall F \in F, z \in M_C.$$  

The function $K$ is the reproducing kernel for the Hilbert space. Let $D : M \to \mathbb{C}^*$ be measurable. Then the restriction map $RF := DF|_M$ is injective. Assume that there is a measure $\mu$ on $M$ such that $RF \in L^2(M, \mu)$ for all $F$ in a dense subset of
Provided \( R \) is closeable, polarizing \( R^* \) we can write
\[ R^* = U|R^*| \]
where \( U : L^2(M, \mu) \to F \) is a unitary isomorphism. Using that \( F \) is a reproducing Hilbert space we get that
\[ Uf(z) = (Uf, \mathcal{K}_z) = (f, U^*\mathcal{K}_z) = \int_M f(m)(U^*\mathcal{K}_z)(m) \, d\mu(m). \]
Thus \( Uf \) is always an integral operator. We notice also that the formula for \( U \) shows that the important object in this analysis is the reproducing kernel \( \mathcal{K}(z, w) \).

The reproducing kernel for the classical Fock space is given by \( \mathcal{K}(z, w) = e^{z\bar{w}} \).

By taking \( D(x) := (2\pi)^{-n/4} e^{-|x|^2} \), which is closely related to the heat kernel, we arrive at the classical Segal–Bargmann transform
\[ Ug(x) = (2/\pi)^{n/4} e^{-(x, x)/2} \int g(y) e^{-(x-y, x-y)} \, dy. \]

The same principle can be used to construct the Hall–transform for compact Lie groups. In [2], Driver and Hall, motivated by application to quantum Yang-Mills theory, introduced a Fock space and Segal–Bargmann transform depending on two parameters \( r, s > 0 \), giving different weights to the \( x \) and \( y \) directions, where \( z = x + iy \in \mathbb{C}^n \) (this was also studied in [12]). Thus \( F \) is now the space of holomorphic functions \( F(z) \) on \( \mathbb{C}^n \) which are square-integrable with respect to the Gaussian measure
\[ dM_{r,s}(z) = \frac{1}{\pi^{n/2} |\pi|^{n/2}} e^{-x^2 - y^2}. \]

In [12] the reproducing kernel and the Segal–Bargmann transform for this space is worked out. This construction has a natural generalization by viewing \( r^{-1} \) and \( s^{-1} \) as the diagonal elements in a positive definite matrix \( A = d(r^{-1}I_n, s^{-1}I_n) \). The measure is then simply
\[ (0.1) \quad dM_{r,s}(z) = \frac{\sqrt{\det(A)}}{\pi^n} e^{-(Az, z)} \, dxdy \]
and this has meaning for any positive definite matrix \( A \).

In this paper we show that (0.1) gives rise to a Fock space \( F_A \) for arbitrary positive matrices \( A \). We find an expression for the reproducing kernel \( K_A(z, w) \). We use the restriction principle to construct a natural generalization of the Segal-Bargmann transform for this space, with a certain natural restriction on \( A \). We study this also in the Gaussian setting, and indicate a generalization to infinite dimensions.

1. The Fock space and the restriction principle

In this section we recall some standard facts about the classical Fock space of holomorphic functions on \( \mathbb{C}^n \). We refer to [5] for details and further information. Let \( \mu \) be the measure \( d\mu = \pi^{-n} e^{-|z|^2} \, dxdy \) and let \( F \) be the classical Fock-space of holomorphic functions \( F : \mathbb{C}^n \to \mathbb{C} \) such that
\[ ||F||^2 := \int |F(z)|^2 \, d\mu(z) < \infty. \]
(Note that the term “Fock space” is also used for the completed symmetric tensor algebra over a Hilbert space, but that is not our usage here.) The space \( F \) is a reproducing Hilbert space with inner product
\[ (F, G) = \int F(z)\overline{G(z)} \, d\mu \]
and reproducing kernel \( K(z, w) = e^{(z, w)} \), where \( \langle z, w \rangle = z\overline{w} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n \). Thus
\[
F(w) = \int F(z)\overline{K(z, w)} \, d\mu = (F, K_w)
\]
where \( K_w(z) = K(z, w) \). The function \( K(z, w) \) is holomorphic in the first variable, anti-holomorphic in the second variable, and \( K(z, w) = \overline{K(w, z)} \). Notice that \( K(z, z) = (K_z, K_z) \). Hence \( \|K_z\| = e^{(z, z)/2} \). Finally the linear space of finite linear combinations \( \sum c_j K_{z_j}, z_j \in \mathbb{C}^n, c_j \in \mathbb{C} \), is dense in \( F \). An orthonormal system in \( F \) is given by the monomials \( e_{\alpha}(z) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}/\sqrt{\alpha_1! \cdots \alpha_n!}, \alpha \in \mathbb{N}_0^n \).

View \( \mathbb{R}^n \subset \mathbb{C}^n \) as a totally real submanifold of \( \mathbb{C}^n \). We will now recall the construction of the classical Segal-Bargmann transform using the restriction principle, see [9]. For constructing a restriction map as explained in the introduction we need to choose the function \( D(x) \). One motivation for the choice of \( D \) is the heat kernel, but another one, more closely related to representation theory, is that the restriction map should commute with the action of \( \mathbb{R}^n \) on the Fock space and \( L^2(\mathbb{R}^n) \). Indeed, take
\[
T(x)F(z) = m(x, z)F(z - x)
\]
for \( F \) in \( F \) where \( m(x, z) \) has properties sufficient to make \( x \mapsto T(x) \) a unitary representation of \( \mathbb{R}^n \) on \( F \). Namely, \( m \) is a multiplier, i.e., \( m(x, z)m(y, z - x) = m(x + y, z); z \mapsto m(x, z) \) is holomorphic in \( z \) for each \( x \); and \( |m(x, z)| = \left| \frac{d\mu(z-x)}{d\mu(z)} \right|^2 = e^{\text{Re}z-x-\|x\|^2/2} \). Note \( m(x, z) := e^{z-x-\|x\|^2/2} \) has these properties.

Set \( D(x) = (2\pi)^{-n/4}m(0, x) = (2\pi)^{-n/4}e^{-\|x\|^2/2} \) and define \( R : F \to C^\infty(\mathbb{R}^n) \) by
\[
RF(x) := D(x)F(x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}F(x).
\]

Then
\[
RT(y)F(x) = (2\pi)^{-n/4}e^{-\|x\|^2/2}T(y)F(x)
= (2\pi)^{-n/4}e^{-\|x\|^2/2}e^{y-x-\|y\|^2/2}F(x - y)
= (2\pi)^{-n/4}e^{\|y\|^2/2}F(x - y)
= RF(x - y).
\]

As \( \mathbb{R}^n \) is a totally real submanifold of \( \mathbb{C}^n \), it follows that \( R \) is injective. Furthermore the holomorphic polynomials \( p(z) = \sum a_\alpha z^\alpha \) are dense in \( F \) and obviously \( Rp \in L^2(\mathbb{R}^n) \). Hence all the Hermite functions \( h_\alpha(x) = (-1)^{|\alpha|} \left( D^{\alpha}e^{-\|x\|^2} \right) e^{\|x\|^2/2} \) are in the image of \( R \); so \( \text{Im}(R) \) is dense in \( L^2(\mathbb{R}^n) \) and \( R \) is a densely defined operator from \( F \) into \( L^2(\mathbb{R}^n) \). It follows easily from the fact that the maps \( F \mapsto F(z) \) are continuous, that \( R \) is a closed operator. Hence \( R \) has an adjoint \( R^* : L^2(\mathbb{R}^n) \to F \). For \( z, w \in \mathbb{C}^n \), let \( (z, w) = \sum z_j w_j \). Then:
\[
R^*g(z) = (R^*g, K_z)
= (g, RK_z)
= (2\pi)^{-n/4} \int g(y)e^{-\|y\|^2/2}e^{z \cdot y} \, dy
= (2\pi)^{-n/4}e^{(z, z)/2} \int g(y)e^{-(z - y \cdot z - y)/2} \, dy
= (2\pi)^{n/4}e^{(z, z)/2}g \star p(z)
\]
where \( p(z) = (2\pi)^{-n/2}e^{-(z,z)/2} \) is holomorphic. Hence
\[
(1.1) \quad RR^*g(x) = g * p(x).
\]
As \( p \in L^1(\mathbb{R}^n) \), it follows that \( ||RR^*|| \leq ||p||_1 \); so \( RR^* \) is continuous.

Thus

**Lemma 1.1.** The maps \( R \) and \( R^* \) are continuous.

Let \( p_t(x) = (2\pi)^{-n/2}e^{-(x,x)/2t} \) be the heat kernel on \( \mathbb{R}^n \). Then \( (p_t)_{t>0} \) is a convolution semigroup and \( p = p_1 \). Hence \( \sqrt{RR^*} = p_{1/2} \) or
\[
RUg(x) = |R^*|g(x) = p_{1/2} * g(x) = \pi^{-n/2} \int g(y)e^{-(x-y,x-y)}dy.
\]
It follows that
\[
Ug(x) = (2/\pi)^{n/4}e^{(x,x)/2} \int g(y)e^{-(x-y,x-y)}dy
\]
for \( x \in \mathbb{R}^n \). But the function on the right hand side is holomorphic in \( x \). Analytic continuation gives the following theorem.

**Theorem 1.2.** The map \( U : L^2(\mathbb{R}^n) \to F \) given by
\[
Ug(z) = (2/\pi)^{n/4} \int g(y) \exp(- (y,y) + 2(z,y) - (z,z)/2) dy
\]
is a unitary isomorphism. \( U \) is called the Segal–Bargmann transform.

2. Twisted Fock spaces

Let \( V \simeq \mathbb{C}^n \) be a finite dimensional complex vector space of complex dimension \( n \) and let \( \langle \cdot, \cdot \rangle \) be a complex inner product. As before we will sometimes write \( \langle z, w \rangle = z \cdot w \). We will also consider \( V \) as a real vector space with real inner product defined by \( \langle z, w \rangle = \text{Re}\langle z, w \rangle \). Notice that \( \langle z, z \rangle = \langle z, z \rangle \) for all \( z \in \mathbb{C}^n \). Let \( J \) be the real linear transformation of \( V \) given by \( Jz = iz \). Note that \( J^* = -J = J^{-1} \) and thus \( J \) is a skew symmetric real linear transformation. Fix a real linear transformation \( A \). Then \( A = H + K \) where
\[
H := \frac{A + J^{-1}AJ}{2} \quad \text{and} \quad K := \frac{A - J^{-1}AJ}{2}.
\]
Note that \( HJ = \frac{1}{2}(AJ - J^{-1}A) = \frac{1}{2}J(J^{-1}AJ + A) = JH \) and \( KJ = \frac{1}{2}(AJ + J^{-1}A) = \frac{1}{2}J(J^{-1}AJ - A) = -JK \). Furthermore \( H \) is complex linear and \( K \) is conjugate linear. We assume that \( A \) is symmetric and positive definite.

**Lemma 2.1.** The complex linear transformation \( H \) is self adjoint, positive with respect to the inner product \( \langle \cdot, \cdot \rangle \), and invertible.

**Proof.** Since \( A \) is positive and invertible as a real linear transformation, we have \( (Az, z) > 0 \) for all \( z \neq 0 \). But \( J \) is real linear and skew symmetric. Hence \( (JAJ^{-1}z, z) > 0 \) for all \( z \neq 0 \). In particular \( H = \frac{1}{2}(A + JAJ^{-1}) \) is complex linear, symmetric with respect to the real inner product \( \langle \cdot, \cdot \rangle \), and positive. We know \( (Hv, w) = (v, Hw) \). Thus \( \text{Re}(Hv, w) = \text{Re}(v, Hw) \). From this we obtain
\[
\text{Re}(Hiv, w) = \text{Re}(iv, Hw).
\]
This implies $\text{Im}(Hv, w) = \text{Im}(v, Hw)$. Putting these together gives $\langle Hv, w \rangle = \langle v, Hw \rangle$. Hence $H$ is complex self adjoint and $\langle H z, z \rangle > 0$ for $z \neq 0$. □

**Lemma 2.2.** Let $w \in V$. Then $\langle Aw, w \rangle = \langle Aw, w \rangle + i\text{Im}(Kw, w)$ and $\langle Aw, w \rangle = \langle Kw, w \rangle + (Kw, w)$.

**Proof.** Let $w \in V$. Then

$$\langle Aw, w \rangle = \langle Kw, w \rangle = \langle Kw, w \rangle + i\text{Im}(Kw, w)$$

This implies the first statement. Taking the real part in the second line gives the second claim, which also follows directly from bilinearity of $(\cdot, \cdot)$. □

Denote by $\det_V$ the determinant of a $\mathbb{R}$-linear map on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Let $d\mu_A(z) = \pi^{-n}\sqrt{\det_V A}e^{-(Az, z)}dxdy$ and let $F_A$ be the space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$ such that

$$||F||_A^2 := \int |F(z)|^2 d\mu_A < \infty.$$

Our normalization of $d\mu$ is chosen so that $||1||_A = 1$. Just as in the classical case one can show that $F_A$ is a reproducing Hilbert space, but this will also follow from the following Lemma. We notice that all the holomorphic polynomials $p(z)$ are in $F$. To simplify the notation, we let $T_1 = H^{-1/2}$. Then $T_1$ is symmetric, positive definite and complex linear. Let $c_A = \sqrt{\det_V (A^{1/2}T_1)} = (\det_V (A)/\det_V (H))^{1/4}$.

**Lemma 2.3.** Let $F : V \to \mathbb{C}$ be holomorphic. Then $F \in F_A$ if and only if $F \circ T_1 \in F$ and the map $\Psi : F \to F_A$ given by

$$\Psi(F)(w) := c_A \exp \left(-\frac{(K T_1 w, T_1 w)/2}{2}\right) F(T_1 w)$$

is a unitary isomorphism. In particular

$$\Psi^* F(w) = \Psi^{-1} F(w) = c_A^{-1} \exp \left(\frac{(K w, w)/2}{2}\right) F(\sqrt{H} w).$$

**Proof.** Let $F : V \to \mathbb{C}$. Then $F$ is holomorphic if and only if $F \circ T_1$ is holomorphic as $T_1$ is complex linear and invertible. Moreover, we also have:

$$||\Psi F||^2 = \pi^{-n} \int |\Psi F(w)|^2 e^{-\langle w, w \rangle} dw$$

$$= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle Kw, w \rangle} e^{-\langle \sqrt{H} w, \sqrt{H} w \rangle} dw$$

$$= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle Kw, w \rangle} e^{-\langle (H + K) w, w \rangle} dw$$

$$= \pi^{-n} \sqrt{\det_V A} \int |F(w)|^2 e^{-\langle Aw, w \rangle} dw$$

$$= ||F||_A^2$$

and thus, by polarization, $\Psi$ is unitary. □
Theorem 2.4. The space $F_A$ is a reproducing Hilbert space with reproducing kernel $K_A(z, w) = c_A^{-2}e^{\frac{1}{2}((Kz, z) - (Hw, w))}e^{\frac{1}{2}(Kw, w)}$.

Proof. By Lemma 2.3 we get
$$c_A \exp(-\frac{1}{2}(K_{T_1w, T_1w})/2)F(T_1w) = \Psi(F)(w) = (\Psi(F), K_w)_{F_A} = (F, \Psi^*(K_w))_{F}.$$ Hence
$$K_A(z, w) = c_A^{-1}\exp(\frac{1}{2}(Kw, w)/2)\Psi^*(K_{\sqrt{T}w}) = c_A^{-2}e^{\frac{1}{2}((Kz, z) - (Hw, w))}e^{\frac{1}{2}(Kw, w)}.$$}

3. The Restriction Map

We assume as before that $A > 0$. We notice that Lemma 2.3 gives a unitary isomorphism $\Psi^*U : L^2(\mathbb{R}^d) \rightarrow F_A$, where $U$ is the classical Segal-Bargmann transform. But this is not the natural transform that we are looking for. As $H$ is positive definite there is an orthonormal basis $e_1, \ldots, e_n$ of $V$ and positive numbers $\lambda_j > 0$ such that $He_j = \lambda_j e_j$. Let $V_R := \sum \mathbb{R}e_j$. Set $\sigma(\sum a_i e_i) = \sum \overline{a}_i e_i$. Then $\sigma$ is a conjugation with $V_R = \{z : \sigma z = z\}$. We say that a vector is real if it belongs to $V_R$. As $He_j = \lambda_j e_j$ with $\lambda_j \in \mathbb{R}$ it follows that $HV_R \subseteq V_R$. We denote by det the determinant of a $\mathbb{R}$-linear map of $V_R$.

Lemma 3.1. $\langle Kz, w \rangle = \langle Kw, z \rangle$.

Proof. Note that $\sigma K$ is complex linear. Since $J^* = -J$, $K = \frac{1}{2}(A - JAJ^{-1})$ is real symmetric. Thus $\langle Kw, z \rangle = \langle w, Kz \rangle = \langle Kz, w \rangle$. Also note $iKz, w = \langle JKz, w \rangle = -(KJz, w) - (Jz, Kw) = -(iz, Kw)$. Hence $Re(iKz, w) = -Re(iz, Kw)$. So $-Im(Kz, w) = Im(z, Kw)$. This gives $Im(Kw, z) = Im(Kz, w)$. Hence $\langle Kz, w \rangle = \langle Kw, z \rangle$.

Lemma 3.2. $(\sigma K)^* = K \sigma$.

Proof. We have $\langle \sigma z, \sigma w \rangle = \langle w, z \rangle$. Hence
$$\langle \sigma Kz, w \rangle = \langle \sigma w, \sigma^2 Kz \rangle = \langle \sigma w, Kz \rangle = \langle z, K \sigma w \rangle.$$}

Corollary 3.3. If $x, y \in V_R$, then $\langle Hx, y \rangle$ is real and $\langle Ax, y \rangle = \langle Ay, x \rangle$.

Proof. Clearly $\langle \cdot, \cdot \rangle$ is real on $V_R \times V_R$. Since $HV_R \subseteq V_R$, we see $\langle Hx, y \rangle$ is real. Next, $\langle Ax, y \rangle = \langle Hx, y \rangle + \langle Kx, y \rangle$. The term $\langle Hx, y \rangle$ equals $\langle Hy, x \rangle$ because $\langle Hx, y \rangle$ is real and $H$ is self-adjoint. On the other hand, $\langle Kx, y \rangle = \langle Ky, x \rangle$ by Lemma 3.1. So $\langle Ax, y \rangle = \langle Ay, x \rangle$.

Lemma 3.4. Define $m : V_R \times V \rightarrow \mathbb{C}$ by $m(x, z) = e^{\langle Hz, x \rangle}e^{\frac{1}{2}(Kz, z)}e^{-\langle Ax, x \rangle/2}$. Then $m$ is a multiplier. Moreover, if $T_xF(z) := m(x, z)(F(z) - x)$, then $x \mapsto T_x$ is a representation of the abelian group $V_R$ on $F_A$. It is unitary if $KV_R \subseteq V_R$. 


Proof: We first show \( m \) is a multiplier:

\[
\begin{align*}
    m(x, z) m(y, z - x) &= e^{(H z, x)} e^{(K \bar{z}, x)} e^{-(A x, z)/2} e^{(H (z - x), y)} e^{(K \bar{z} - x, y)} e^{-(A y, y)/2} \\
    &= e^{(H z, x + y)} e^{(K \bar{z}, x + y)} e^{-(H x, y)} e^{-(K x, y)} e^{-(A x, z)/2} e^{-(A y, y)/2} \\
    &= e^{(H z, x + y)} e^{(K \bar{z}, x + y)} e^{-(A x, z)/2 - (A y, y)/2} \\
    &= e^{(H z, x + y)} e^{(K \bar{z}, x + y)} e^{-(A(x + y), x + y)/2} \\
    &= m(x + y, z).
\end{align*}
\]

Since \( m \) is a multiplier, we have \( T_x T_y = T_{x + y} \). For each \( T_x \) to be unitary, we need \( |m(x, z)| = e^{(Az, x) - (Ax, x)/2} \). But

\[
|m(x, z)| = e^{(H z, x)} e^{(K \bar{z}, x)} e^{-(Ax, x)/2} = e^{(Ax, x) - (Ax, x)/2} e^{(K \bar{z} - K z, x)}.
\]

Thus \( T_x \) is unitary for all \( x \) if and only if the real part of every vector \( K \bar{z} - K z \) is 0. Since \( \bar{z} - z \) runs over \( iV \), \( V \), \( T_x \) is unitary for all \( x \) if and only if \( K(iV) \subset iV \), which is equivalent to \( K(V) \subset V \).

\( \square \)

Notice that \( \det_V H = (\det H)^2 \). To simplify some calculations later on we define

\[
c := (2\pi)^{-n/4} \left( \frac{\det V A}{\det H} \right)^{1/4}.
\]

We remark for further reference:

**Lemma 3.5.** \( c^{-2} A^2 = \frac{\sqrt{\det H}}{(2\pi)^{n/2}} \) and \( c^{-1} \sqrt{\frac{\det H}{\pi^{n/2}}} = \left( \frac{2}{\pi} \right)^{n/4} \frac{(\det H)^{3/4}}{(\det V A)^{1/4}} \).

Let \( D(x) = cm(x, 0) = ce^{-(Ax, x)/2} \) and define \( R : F_A \to C^\infty(V) \) by \( RF(x) := D(x)F(x) \). Extending the bilinear form \( x \mapsto (Ax, x) \) to a complex bilinear form \( \langle z, z \rangle_A \) on \( V \) shows that \( D \) has a holomorphic extension to \( V \).

**Lemma 3.6.** The restriction map \( R \) intertwines the action of \( V \) on \( F_A \) and the left regular action \( L \) on functions on \( V \).

**Proof.** We have

\[
R(T_y F)(x) = cm(x, 0)T_y F(x) = cm(x, 0)m(y, x)F(x - y) = cm(x, 0)m(-y, -x)F(x - y) = cm(x - y, 0)F(x - y) = Ly RF(x).
\]

\( \square \)

## 4. The Generalized Segal–Bargmann Transform

As for the classical space, \( R \) is a densely defined, closed operator. It also has dense image in \( L^2(V) \). To see this, let \( \{ h_\alpha \}_\alpha \) be the orthonormal basis of \( L^2(V) \) given by the Hermite functions. Then \( \{ \det(A) \chi h_\alpha(\sqrt{A}z) \}_\alpha \) is an orthonormal basis of \( L^2(V) \) which is contained in the image of \( R \). It follows again that \( R \) has an adjoint and

\[
R^* h(z) = (R^* h, K_A z) = (h, RK_A z)
\]
where $K_{A,z}(w) = K_A(w,z) = c_A^{-2} e^{\frac{1}{2} \langle Kw, w \rangle} e^{\langle Hw, z \rangle} e^{\frac{1}{2} \langle Kz, z \rangle}$. Thus

$$R^* h(z) = c \int h(x) e^{-\langle Ax, x \rangle/2} K_A(x, z) \, dx$$

$$= c_A^{-2} e \int h(x) e^{-\langle Ax, x \rangle/2} e^{\frac{1}{2} \langle Kz, z \rangle} e^{\langle Hx, z \rangle} e^{\frac{1}{2} \langle Kx, x \rangle} \, dx$$

$$= c_A^{-1} e e^{\frac{1}{2} \langle Kz, z \rangle} e^{\langle Hx, z \rangle} $$

$$= c_A^{-2} e e^{\frac{1}{2} \langle Kz, z \rangle} e^{\langle Hx, z \rangle} $$

$$= c_A^{-2} e e^{\frac{1}{2} \langle Kz, z \rangle} e^{\langle Hx, z \rangle} $$

for $\langle z, Hx \rangle = \langle Hz, z \rangle = \langle x, Hz \rangle$ and $\langle z, Hx \rangle = \langle z, Hz \rangle$. Thus we finally arrive at

$$(4.1) R^* h(z) = c_A^{-2} e e^{\frac{1}{2} \langle Kz, z \rangle} e^{\langle Hx, z \rangle} * h(z).$$

Let $P : V_R \to V_R$ be positive. Define $\phi_P(x) = \sqrt{\det(P)} (2\pi)^{-n/2} e^{-||x||^2/2}$. For $t > 0$, let $P(t) = P/t$.

**Lemma 4.1.** Let the notation be as above. Then $0 < t \mapsto \phi_{P(t)}$ is a convolution semigroup, i.e., $\phi_{P(t)} = \phi_{P(t)} * \phi_{P(t)}$.

**Proof.** This follows by change of parameters $y = \sqrt{P} x$ from the fact that $\phi_{P(t)}(x) = (2\pi t)^{-n/2} e^{-||x||^2/2t}$ is a convolution semigroup.

We define a unitary operator $W$ on $L^2(V_R)$ by

$$Wf(x) = e^{i \text{Im}(x, Kx)} f(x) = e^{i \text{Im}(x, Ax)} f(x).$$

We know $W = I$ if $KV_R \subseteq V_R$ and this occurs if $A$ leaves $V_R$ invariant.

**Lemma 4.2.** Let $h$ be in the domain of definition of $R^*$. Then $RR^* h = W(\phi_H * h)$.

**Proof.** We notice first that $c_A^{-2} e^2 = (2\pi)^{-n/2} \sqrt{\det H}$ by Lemma 3.3. From (4.1) we then get

$$RR^* h(x) = c e^{\frac{1}{2} \langle Ax, x \rangle} R^* h(x)$$

$$= c_A^{-2} e e^{\frac{1}{2} \langle Ax, x \rangle} e^{\langle Hx, z \rangle} e^{\frac{1}{2} \langle y, Hg \rangle} * h(x)$$

$$= (2\pi)^{-n/2} \sqrt{\det(H)} e^{\frac{1}{2} \langle Ax, x \rangle} e^{\langle x, Ax \rangle} e^{\frac{1}{2} \langle y, Hg \rangle} * h(x)$$

$$= (2\pi)^{-n/2} \sqrt{\det(H)} e^{i \text{Im}(x, Ax)} \int e^{-\frac{1}{2} \langle y, Hg \rangle} h(x - y) \, dy$$

$$= (2\pi)^{-n/2} \sqrt{\det(H)} e^{i \text{Im}(x, Ax)} \int e^{-\frac{1}{2} \langle y, Hg \rangle} h(x - y) \, dy$$

$$= W(\phi_H * h)(x)$$

Lemma 4.1 and Lemma 4.2 leads to the following corollary:
Corollary 4.3. Suppose $AV_R \subseteq V_R$. Then
\[ |R^*|h(x) = \phi_{H(1/2)} * h(x) = \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_R} e^{-\|\sqrt{\pi}y\|^2} h(x-y) \, dy. \]

Theorem 4.4 (The Segal–Bargmann Transform). Suppose $A$ leaves $V_R$ invariant. Then the operator $U_A : L^2(V_R) \to F_A$ defined by
\[ U_A f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{(\det H)^{3/4}}{(\det V_A)^{1/4}} e^{\frac{1}{2}((H z, \bar{z}) + (z, K z))} \int e^{(H(h), z-y)} f(y) \, dy. \]
is a unitary isomorphism. The map $U_A$ is called the generalized Segal–Bargmann transform.

Proof. By polarization we can write $R^* = U |R^*|$ where $U : L^2(V_R) \to F_A$ is a unitary isomorphism. Taking adjoints gives $|R^*|U^* = R$. Hence $RU = |R^*|$. Hence
\[ cm(x)U h(x) = RU h(x) = (|R^*| h)(x) = \frac{\sqrt{\det(H)}}{\pi^{n/2}} \int_{V_R} e^{-\|\sqrt{\pi}y\|^2} h(x-y) \, dy. \]

Since $m(x) = e^{-\frac{1}{2}((x, H x)+(x, K x))}$, we have using Lemma 3.3,
\[ U f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{(\det H)^{3/4}}{(\det V_A)^{1/4}} e^{\frac{1}{2}((H z, \bar{z}) + (z, K z))} \int e^{(x-y, H(x-y))} f(y) \, dy. \]

By holomorphicity, this implies
\[ U f(z) = \left( \frac{2}{\pi} \right)^{n/4} \frac{(\det H)^{3/4}}{(\det V_A)^{1/4}} e^{\frac{1}{2}((H z, \bar{z}) + (z, K z))} \int e^{(H(z-y), \bar{z}-y)} f(y) \, dy \]
is the Bargmann–Segal transform. \qed

5. The Gaussian Formulation

In infinite dimensions, there is no useful notion of Lebesgue measure but Gaussian measure does make sense. So, with a view to extension to infinite dimensions, we will recast our generalized Segal-Bargmann transform using Gaussian measure instead of Lebesgue measure as the background measure on $V_R$. Of course, we have already defined the Fock space $F_A$ using Gaussian measure.

As before, $V$ is a finite-dimensional complex vector space with Hermitian inner-product $\langle \cdot, \cdot \rangle$, and $A : V \to V$ is a real-linear map which is symmetric, positive-definite with respect to the real inner-product $\langle \cdot, \cdot \rangle = \text{Re} \langle \cdot, \cdot \rangle$, i.e. $(A z, z) > 0$ for all $z \in V$ except $z = 0$. We assume, furthermore, that there is a real subspace $V_R$ for which $V = V_R + iV_R$, the inner-product $\langle \cdot, \cdot \rangle$ is real-valued on $V_R$ and $A(V_R) \subseteq V_R$. As usual, $A$ is the sum
\[ A = H + K \]
where $H = (A - iA_i)/2$ is complex-linear on $V$ and $K = (A + iA_i)/2$ is complex-conjugate-linear. The real subspaces $V_R$ and $iV_R$ are $\langle \cdot, \cdot \rangle$-orthogonal because for any $x, y \in V_R$ we have $\langle x, iy \rangle = \text{Re} \langle x, iy \rangle = -\text{Re} \langle i(x, y) \rangle$, since $\langle x, y \rangle$ is real, by
hypothesis. Since $A$ preserves $V_\mathbb{R}$ and is symmetric, it also preserves the orthogonal complement $iV_\mathbb{R}$. Thus $A$ has the block diagonal form

$$A = \begin{bmatrix} R & 0 \\ 0 & T \end{bmatrix} = d(X, Y)$$

Here, and henceforth, we use the notation $d(X, Y)$ to mean the real-linear map $V \to V$ given by $a \mapsto Xa$ and $ia \mapsto iYa$ for all $a \in V_\mathbb{R}$, where $X, Y$ are real-linear operators on $V_\mathbb{R}$. Note that $d(X, Y)$ is complex-linear if and only if $X = Y$ and is complex-conjugate-linear if and only if $Y = -X$. The operator $d(X, X)$ is the unique complex-linear map $V \to V$ which restricts to $X$ on $V_\mathbb{R}$, and we will denote it by $X_V$:

$$X_V = \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix}$$

The hypothesis that $A$ is symmetric and positive-definite (by which we mean $A > 0$, not just $A \geq 0$) means that $R$ and $T$ are symmetric, positive definite on $V_\mathbb{R}$. Consequently, the real-linear operator $S$ on $V_\mathbb{R}$ given by

$$S = 2(R^{-1} + T^{-1})^{-1}$$

is also symmetric, positive-definite.

The operators $H$ and $K$ on $V$ are given by

$$H = \frac{1}{2} (R_V + T_V), \quad K = d \left( \frac{1}{2} (R - T), \frac{1}{2} (T - R) \right)$$

Using the conjugation map

$$\sigma : V \to V : a + ib \mapsto a - ib$$

for $a, b \in V_\mathbb{R}$ we can also write $K$ as

$$K = \frac{1}{2} (R_V - T_V) \sigma$$

Now consider the holomorphic functions $\rho_T$ and $\rho_S$ on $V$ given by

$$\rho_T(z) = \frac{(\det T)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{4} \langle T_V z, \bar{z} \rangle}$$

$$\rho_S(z) = \frac{(\det S)^{1/2}}{(2\pi)^{n/2}} e^{-\frac{1}{4} \langle S_V z, \bar{z} \rangle}$$

where $n = \text{dim } V_\mathbb{R}$. Restricted to $V_\mathbb{R}$, these are density functions for Gaussian probability measures.

The **Segal-Bargmann transform** in this setting is given by the map

$$S_A : L^2(V_\mathbb{R}, \rho_S(x) dx) \to \mathbf{F}_A : f \mapsto S_A f$$

where

$$S_A f(z) = \int_{V_\mathbb{R}} f(x) \rho_T(z - x) dx = \int_{V_\mathbb{R}} f(x) c(x, z) \rho_S(x) dx$$

where the generalized “coherent state” function $c$ is specified, for $x \in V_\mathbb{R}$ and $z \in V$, by

$$c(x, z) = \frac{\rho_T(x - z)}{\rho_S(x)}$$
It is possible to take (5.4) as the starting point, with \( f \in L^2(V_R, \rho_S(x)dx) \) and prove that: (i) \( S_A f(z) \) is well-defined, (ii) \( S_A f \) is in \( F_A \), (iii) \( S_A \) is a unitary isomorphism onto \( F_A \). However, we shall not work out everything in this approach since we have essentially proven all this in the preceding sections. Full details of a direct approach would be obtained by generalizing the procedure used in [12]. In the present discussion we shall work out only some of the properties of \( S_A \).

**Lemma 5.1.** Let \( w, z \in V \). Then:

1. The function \( x \mapsto c(x, z) \) belongs to \( L^2(V_R, \rho_S(x)dx) \), thereby ensuring that the integral (5.3) defining \( S_A f(z) \) is well-defined;
2. The \( S_A \)-transform of \( c(\cdot, w) \) is \( K_A(\cdot, \bar{w}) \):
   \[
   [S_A c(\cdot, w)](z) = K_A(z, \bar{w})
   \]
   and so, in particular,
   \[
   K_A(z, w) = \int_{V_R} \frac{\rho_R(x-z)\rho_R(x-\bar{w})}{\rho_S(x)} \, dx
   \]
3. The transform \( S_A \) preserves inner-products on the linear span of the functions \( c(\cdot, w) \):
   \[
   \langle c(\cdot, w), c(\cdot, z) \rangle_{L^2(V_R, \rho_S(x)dx)} = K_A(w, z) = \langle K_A(\cdot, \bar{w}), K_A(\cdot, \bar{z}) \rangle_{F_A}
   \]

**Proof.** (i) is equivalent to finiteness of \( \int_{V_R} \frac{|c(x-z)|^2}{\rho_S(x)} \, dx \), which is equivalent to positivity of the operator \( 2T - S \). To see that \( 2T - S \) is positive observe that
\[
2T - S = 2T[(R^{-1} + T^{-1}) - T^{-1}](R^{-1} + T^{-1})^{-1} = 2TR^{-1} + 1 - TR^{-1}S = 2(T^{-1} + T^{-1}RT^{-1})^{-1}
\]
and in this last line \( T^{-1} > 0 \) (being the inverse of \( T > 0 \)) and \( (T^{-1}RT^{-1}x, x) = (RT^{-1}x, T^{-1}x) \geq 0 \) by positivity of \( R \). Thus \( 2T - S \) is positive, being twice the inverse of the positive operator \( T^{-1} + T^{-1}RT^{-1} \).

(ii) is the result of a lengthy calculation which, despite an unpromising start, leads from complicated expressions to simple ones. To avoid writing a lot of complex conjugates we shall use the symmetric complex bilinear pairing \( v \cdot w = \langle v, \bar{w} \rangle \) for \( v, w \in V \), writing \( v^2 \) for \( vv \). More seriously, we shall denote the complex-linear operator \( T_V \) which restricts to \( T \) on \( V_R \) simply by \( T \). It is readily checked that \( T \) continues to be symmetric in the sense that \( Tv \cdot w = v \cdot Tw \) for all \( v, w \in V \). We start with
\[
a \overset{\text{def}}{=} [S_A c(\cdot, w)](z) = \int_{V_R} \frac{\rho_R(x-w)}{\rho_S(x)} \rho_R(z-x) \, dx
\]
\[
= (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_R} e^{-\frac{1}{2}[T(x-w) \cdot (x-w) + T(x-z) \cdot (x-z) - Sx \cdot x]} \, dx
\]
\[
= (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V_R} e^{-\frac{1}{2}[(2T-S)x \cdot x - 2Tx \cdot (w+z) + Tw \cdot w + Tz \cdot z]} \, dx
\]
Recall from the proof of (i) that $2T - S > 0$. For notational simplicity let $L = (2T - S)^{1/2}$ and $M = L^{-1}T$. Then

$$a = (2\pi)^{-n/2} \frac{\det T}{(\det S)^{1/2}} \int_{V} e^{-\frac{1}{2}(Lx - MW(z))^2} dx e^{-\frac{1}{2}[Tw \cdot w + Tw \cdot z - M(w + z) \cdot M(w + z)]}$$

To simplify the last exponent observe that

$$Tw \cdot w - MW \cdot MW = T \cdot w - T \cdot L^{-2}Tw$$

$$= T \cdot w - T \cdot (2T - S)^{-1}T w$$

$$= T \cdot w - \frac{1}{2}T \cdot w \cdot (T^{-1} + T^{-1}RT^{-1})T w$$

$$= T \cdot w - \frac{1}{2}T \cdot w \cdot (w + T^{-1}Rw)$$

$$= \frac{1}{2}(Tw \cdot w - Rw \cdot w)$$

$$= -\langle K\bar{w}, \bar{w} \rangle \text{ by (5.5)}$$

The same holds with $z$ in place of $w$. For the “cross term” we have

$$MW \cdot Mz = T \cdot L^{-2}Tz$$

$$= T \cdot (2T - S)^{-1}Tz$$

$$= \frac{1}{2}T \cdot w \cdot (T^{-1} + T^{-1}RT^{-1})Tz$$

$$= 2w \cdot Hz$$

Putting everything together we have

$$[SAC(\cdot, w)](z) = \frac{\det T}{(\det S)^{1/2}(\det L)} e^{\frac{1}{2}(K\bar{w}, \bar{w})} e^{\langle Hw, \bar{z} \rangle} e^{\frac{1}{2}(K\bar{z}, \bar{z})}$$

In Lemma 6.2 below we prove that

$$\frac{\det T}{(\det S)^{1/2}(\det L)} = \left(\frac{\det V(A)}{\det V(H)}\right)^{-1/2} = c_A^{-2}$$

So

$$[SAC(\cdot, w)](z) = K_A(w, \bar{z})$$

For (iii), we have first:

$$\langle c(\cdot, w), c(\cdot, z) \rangle_{L^2(\rho_S(x)dx)} = [SAC(\cdot, w)](\bar{z}) = K_A(\bar{z}, w) = K_A(w, z)$$

The second equality in (iii) follows from the fact that $K_A$ is a reproducing kernel.

6. THE EVALUATION MAP AND DETERMINANT RELATIONS

Recall the reproducing kernel

$$K_A(z, w) = c_A^{-2} e^{\frac{1}{2}(z, Kz) + \frac{1}{4}(Kw, w) + \langle Hw, w \rangle}$$
where
\[ c_A^{-2} = \left( \frac{\det V^H}{\det V^A} \right)^2 \]

Being a reproducing kernel for \( F_A \) means
\[ f(w) = (f, K_A(\cdot, w)) = \pi^{-n}(\det A)^{1/2} \int_V f(z)K_A(w, z) \, |dz| \]

where \( |dz| = dx dy \) signifies integration with respect to Lebesgue measure on the real inner-product space \( V \). Thus we have

**Proposition 6.1.** For any \( z \in V \), evaluation map
\[ \delta_z : F_A \to \mathbb{C} : f \mapsto f(z) \]
is bounded linear functional with norm
\[ \| \delta_z \| = K_A(z, z)^{1/2} = c_A^{-1} e^{(Az, z)} \]

**Proof.** We have
\[ \| \delta_z f \| = \| f(z) \| = |(f, K_A(\cdot, z))| \leq \| f \|_{F_A} K_A(z, z)^{1/2} \]
because, again by the reproducing kernel property we have
\[ \| K_A(\cdot, z) \|_{F_A}^2 = (K_A(\cdot, z), K_A(\cdot, z))_{F_A} = K_A(z, z) \]

This last calculation also shows that the inequality in (6.3) is an equality of \( f = K_A(\cdot, z) \) and thereby shows that \( \| \delta_z \| \) is actually equal to \( K_A(z, z)^{1/2} \). The latter is readily checked to be equal to \( c_A^{-1} e^{(Az, z)} \). \( \square \)

Next we make two observations about the constant \( c_A \), the first of which has already been used.

**Lemma 6.2.** For the constant \( c_A \) we have
\[ c_A^{-2} = \left( \frac{\det V^H}{\det V^A} \right)^2 = \frac{\det T}{(\det S)^{1/2} \det L} \]
where, as before, \( L = (2T - S)^{1/2} \) and \( S = 2(R^{-1} + T^{-1})^{-1} \).

**Proof.** Recall from (5.13) that \( 2T - S = TR^{-1}S \). Note also that
\[ S^{-1} = \frac{1}{2}(R^{-1} + T^{-1}) = R^{-1} \frac{R + T}{2} T^{-1} = R^{-1} H |V_R|^T T^{-1} \]

So
\[ \left( \frac{\det V^A}{\det V^H} \right)^{1/2} \frac{\det T}{(\det S)^{1/2} \det L} = \frac{(\det R)^{1/2}(\det T)^{1/2}}{(\det S)^{1/2} \det R \det T} \frac{\det T}{(\det S)^{1/2} \det T^{1/2} \det R^{-1/2} \det S^{1/2}} = 1 \]

which implies the desired result. \( \square \)

Next we prove a determinant relation which implies \( c_A \geq 1 \):

**Lemma 6.3.** If \( R \) and \( T \) are positive definite \( n \times n \) matrices (symmetric if real) then
\[ \sqrt{\det R \det T} \leq \det \left( \frac{R + T}{2} \right) \]
with equality if and only if \( R = T \).
Proof. Note first that the matrix

\[ D = R^{-1/2}TR^{-1/2} \]

is positive definite because \((R^{-1/2}TR^{-1/2}, x) = (TR^{-1/2}x, R^{-1/2}x) \geq 0\) since \(T > 0\), with equality if and only if \(R^{-1/2}x = 0\) if and only if \(x = 0\). So \(D = (R^{-1/2}TR^{-1/2})^{1/4}\) makes sense and is also positive definite (and is symmetric if we are working with reals). We have then

\[
\begin{align*}
\det R \det T & = \frac{\det R \det (R^{1/2}D^4R^{1/2})}{\det (R^{1/2}(1+D^4)R^{1/2})^2} \\
& = \left[ \det \left( \frac{D^2 + D^{-2}}{2} \right) \right]^{-2} \\
& = \left[ \det \left( I + \left( \frac{1}{\sqrt{2}} D - \frac{1}{\sqrt{2}} D^{-1} \right)^2 \right) \right]^{-2}
\end{align*}
\]

To summarize:

\[ \frac{\det R \det T}{\det (R+T)^2} = \left[ \det \left( I + \left( \frac{1}{\sqrt{2}} D - \frac{1}{\sqrt{2}} D^{-1} \right)^2 \right) \right]^{-2} \]

where \(D = (R^{-1/2}TR^{-1/2})^{1/4}\). Diagonalizing \(D\) makes it apparent that this last term is \(\leq 1\) with equality if and only if \(D = D^{-1}\), which is equivalent to \(D^4 = I\) which holds if and only if \(R = T\). \(\Box\)

As consequence we have for \(c_A\):

\[
c_A = \left( \frac{\det V A}{\det V H} \right)^{1/4} = \left( \frac{\det R \det T}{\det (R+T)^2} \right)^{1/4} = \left( \frac{\sqrt{\det R \det T}}{\det (R+T)} \right)^{1/2}
\]

and so

\[ c_A^{-2} = \frac{\det R+T}{\sqrt{\det R \det T}} \geq 1 \]

with equality if and only if \(R = T\).

When extending this theory to infinite-dimensions we have to note that in order to retain a meaningful notion of evaluation \(\delta_z : f \mapsto f(z)\), the constant \(c_A^{-1}\) which appears in the norm \(\|\delta_z\|\) given in (6.2) must be finite. The expression for \(c_A^{-2}\) obtained from (6.6) gives a more explicit condition on \(R\) and \(T\) for this finiteness to hold.

If \(R\) and \(T\) are both scalar operators, say \(R = rI\) and \(T = tT\), then (6.7) shows that \(c_A^{-1}\) equals \([(r+t)/(2\sqrt{rt})]^{n/2}\) which is bounded as \(n \to \infty\) if and only if \(r = t\). This observation was made in [12].

7. Remarks on extension to infinite dimensions

The Gaussian formulation permits extension to the infinite-dimensional situation, at least with some conditions placed on \(A\) so as to make such an extension reasonable. Suppose then that \(V\) is an infinite-dimensional separable complex Hilbert space, \(V_R\), a real subspace on which the inner-product is real-valued, and \(A : V \to V\) a bounded symmetric, positive-definite real-linear operator carrying \(V_R\) into itself. The operators \(R, T, S, H\) and \(K\) are defined as before. Assume that \(R\)
and $T$ commute and that there is an orthonormal basis $e_1, e_2, \ldots$ of $V_{\mathbb{R}}$ consisting of simultaneous eigenvectors of $R$ and $T$ (greater generality may be possible but we discuss only this case). Let $V_0$ be the complex linear span of $e_1, \ldots, e_n$, and $V_{n, \mathbb{R}}$ the real linear span of $e_1, \ldots, e_n$. Then $A$ restricts to an operator $A_n$ on $V_n$, and we have similarly restrictions $H_n, K_n$ on $V_n$ and $R_n, T_n, S_n$ on $V_{n, \mathbb{R}}$. The unitary transform $S_A$ may be obtained as a limit of the finite-dimensional transforms $S_{A_n}$.

The Gaussian kernels $\rho_S$ and $\rho_T$ do not make sense anymore, and nor does the coherent state $c$, but the Gaussian measures $d\gamma_S(x) = \rho_S(x)dx$ and $\mu_A$ do have meaningful analogs. There is a probability space $V_{\mathbb{R}}^\prime$, with a $\sigma$–algebra $\mathcal{F}$ on which there is a measure $\gamma_S$, and there is a linear map $V_{\mathbb{R}} \to L^2(V_{\mathbb{R}}^\prime, \gamma_A) : x \mapsto G(x) = (x, \cdot)$, such that the $\sigma$–algebra $\mathcal{F}$ is generated by the random variables $G(x)$, and each $G(x)$ is (real) Gaussian with mean 0 and variance $(S^{-1}x, x)$. Similarly, there is probability space $V^\prime$, with a $\sigma$–algebra $\mathcal{F}_1$ on which there is a measure $\mu_A$, and there is a real-linear map $V \to L^2(V^\prime, \mu_A) : z \mapsto G_1(z) = (z, \cdot)$, such that the $\sigma$–algebra $\mathcal{F}_1$ is generated by the random variables $G_1(z)$, and each $G_1(z)$ is (real) Gaussian with mean 0 and variance $\frac{1}{2}(A^{-1}z, z)$. Then for each $z \in V$, written as $z = a + ib$ with $a, b \in V_{\mathbb{R}}$, we have the complex-valued random variable on $V^\prime$ given by

$$\tilde{z} = G_1(a) + iG_1(b)$$

Suppose $g$ is a holomorphic function of $n$ complex variables such that

$$\int_V |g(\hat{e}_1, \ldots, \hat{e}_n)|^2 \, d\mu_A < \infty.$$ 

Define $\mathbb{F}_A$ to be the closed linear span of all functions of the type $g(\hat{e}_1, \ldots, \hat{e}_n)$ in $L^2(\mu_A)$ for all $n \geq 1$. We may then define $S_A$ of a function $f(G(e_1), \ldots, G(e_n))$ to be $(S_A f)(\hat{e}_1, \ldots, \hat{e}_n)$, and then extend $S_A$ be continuity to all of $L^2(\gamma_S)$. In writing $(S_A, f)$ we have identified $V_n$ with $\mathbb{C}^n$ and $V_{n, \mathbb{R}}$ with $\mathbb{R}^n$ using the basis $e_1, \ldots, e_n$.

A potentially significant application of the infinite-dimensional case would be to situations where $V_{\mathbb{R}}$ is a path space and $A$ is arises from a suitable differential operator. For the “classical case” where $R = T = tI$ for some $t > 0$, this leads to the Hall transform $\mathbb{R}$ for Lie groups as well as the path-space version on Lie groups considered in [7].

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