Rotons and the dispersion of ripplons of \( He \ II \)

I N Adamenko\(^1\), K E Nemchenko\(^1\) and I V Tanatarov\(^2\)

\(^1\) Karazin Kharkov National University, Svobody Sq. 4, Kharkov, 61077, Ukraine
\(^2\) Akhiezer Institute for Theoretical Physics, NSC "Kharkov Institute for Physics and Technology" of NASU, Akademicheskaya St. 1, 61108, Ukraine

E-mail: igor.tanatarov@gmail.com

Abstract. We obtain the dispersion curve for ripplons using a simple analytic model of \( He \ II \). Asymptotes are derived for small frequencies and close to the roton threshold. At large wave vectors the curve is shown to be determined by the rotons’ dispersion, and it approaches the roton threshold with zero derivative, at the peak of inverted parabola. The results are compared with experimental data.

Ripplons, surface modes of superfluid helium, and their interaction with the bulk modes, especially rotons, have been studied both in theory and experiment (see [1, 2, 3] and references cited therein). We investigate the dispersion curve of ripplons in the frame of a simple model of \( He \ II \), which was introduced in [4] and was used in [5, 6, 7] to describe interaction of \( He \ II \) phonons and rotons with solid interfaces. In this model the quantum fluid is described as continuous medium with correlations and the dispersion relation of its bulk excitations enters the equations explicitly. We start with a simple analytical dispersion relation \( \Omega(k) \), which well approximates the dispersion curve of \( He \ II \) in both long wave length and short wave length regions. The surface solution of the equations of the quantum fluid, which corresponds to the ripplon, is obtained explicitly. With the help of boundary conditions at the free surface with tension, an analytic equation for the dispersion relation of the surface mode is derived. It is solved numerically. Expansions are derived both for small frequencies and close to the roton threshold \( \Delta \).

Let us consider the half-space \( z > 0 \) filled by superfluid helium. It obeys the ordinary linearized equations of an ideal liquid, but the relation between the deviations of pressure \( P \) and density \( \rho \), from the respective equilibrium values, is nonlocal [4]. The problem in terms of pressure can be expressed as a nonlocal wave equation which applies in half-space [6]

\[
\Delta P(r, t) = \int_{z_1 > 0} d^3 r_1 h(|r - r_1|)\ddot{P}(r_1, t), \quad x, y, z \in (-\infty, \infty), \quad z \in (0, \infty).
\] (1)

We assume here that the interface is sharp enough to consider that the kernel \( h(r) \) is the same in the presence of the interface as it is in the bulk medium. It is related to the dispersion relation of the bulk fluid \( \Omega(k) \) through its Fourier transform

\[
h(k) = \frac{k^2}{\Omega^2(k)}.
\] (2)

Thus the model allows us to describe a quantum fluid with arbitrary dispersion relation.
We consider the following dispersion relation in the bulk liquid:

\[ \Omega^2(k) = s^2 k^2 \left\{ 1 + 2 \lambda \frac{k^2}{k_g^2} + \frac{k^4}{k_1^4} \right\}. \quad (3) \]

Here \( s \) is sound velocity at zero frequency, \( k_g \) is wave vector that determines the scale of the curve and \( \lambda \) its shape. We adopt the same set of values as in [6]: \( s = 230.7 \text{ m/s}, k_g = 1.9828 \text{ Å}^{-1} \), and \( \lambda = -0.9667 \). For such parameters the relation (3) is a good approximation to the measured non-monicotonic dispersion relation of superfluid helium (see Fig.1). Then the coordinates of the roton minimum are those measured in experiments at SVP [8]: \( k_{rot} = 0.9670 k_g = 1.913 \text{ Å}^{-1} \) and \( \Delta = \hbar \Omega(k_{rot})/k_B = 8.712 \text{ K} \) (\( k_B \) is the Boltzmann constant).

The problem (1) does not have an even solution with respect to \( z \), as was assumed in [1]. Its solution in 1D for arbitrary dispersion in the case where only one root is real, as it is for energies less than \( k_B \Delta \), was derived in [5], and later in [6] generalized to 3D for the dispersion relation (3). It was shown that the general solution can be represented as a sum of basis solutions with given frequency \( \omega \) and the tangential component of wave vector \( \mathbf{k}_r \), which have the form

\[
P_{(k_a, k_\beta, k_\gamma)}(r, t; \mathbf{k}_r, \omega) = P_{0}^{(i)} \left\{ \frac{(k_{a2} - k_{+z})(k_{a2} + k_{-z})}{(k_{a2} - k_{\beta2})(k_{a2} - k_{\gamma2})} e^{ik_{a2}z} + \right.
\left. + \frac{(k_{\beta2} - k_{+z})(k_{\beta2} + k_{-z})}{(k_{\beta2} - k_{\gamma2})(k_{\beta2} - k_{\gamma2})} e^{ik_{\beta2}z} + \frac{(k_{\gamma2} - k_{+z})(k_{\gamma2} + k_{-z})}{(k_{\gamma2} - k_{\beta2})(k_{\gamma2} - k_{\beta2})} e^{ik_{\gamma2}z} \right\} e^{i(\mathbf{k}_r \cdot \mathbf{r} - \omega t)}. \quad (4)\]

Here \( P_{0}^{(i)} = P_{(k_a, k_\beta, k_\gamma)}(0, 0; \mathbf{k}_r, \omega) \); \( k_{a2}(\omega, \mathbf{k}_r \); \( k_{\beta2}(\omega, \mathbf{k}_r \); \) and \( k_{\gamma2}(\omega, \mathbf{k}_r \) are any three of the six roots of the equation \( \Omega^2(k^2) = \omega^2 \) with respect to \( k_z \); \( k_{\pm z} \) are defined by \( k_{\pm z}^2 = k_{\pm 2}^2 - k_{r}^2 \), where \( k_{z} = k_{g} \left( \sqrt{1 + \lambda} \pm i \sqrt{1 - \lambda} \right) / \sqrt{2} \) are the poles of \( h(k) \) in the half-plane \( Re k > 0 \), and the condition that \( k_{+ z} = k_{- z}^* \in \mathbb{C}_+ \) (\( \mathbb{C}_+ \) is the upper complex half-plane).

We are interested in ripplons, and so in solutions which decay away from the interface and correspond to surface waves. Also here we restrict ourselves to consideration of the solutions with energies lower than \( k_B \Delta \). In this case of the three solutions \( k_{\pm 2}^2 \) for \( i = 1, 2, 3 \) of the equation \( \Omega^2(k^2) = \omega^2 \) with \( \Omega^2(k) \) from (3), which is cubic with respect to \( k^2 \), one is real and two are complex. We denote the real solution, which corresponds to a phonon, by \( k_{+ 2}^2 \) and the complex ones, corresponding to rotons, by \( k_{\pm 2}^2 \) and \( k_{\pm 3}^2 \). As the coefficients of the equation are real, \( k_{\pm 2}^2 = k_{\pm 3}^* \).

Of the four corresponding roots \( \pm k_{\pm z} \) for \( i = 2, 3 \) two give damped waves, while the other two give exponentially unbounded waves. If \( k_{\pm 2}^2 < k_{\pm 3}^2 \), then \( k_{\pm 2}^2 = k_{\pm 3}^2 - k_{r}^2 > 0 \) and the two roots \( \pm k_{\pm z} \) give running waves. So only in case \( k_{\pm 2}^2 > k_{\pm 3}^2 \) there are three roots \( k_{\pm z} \) which can be taken as \( k_{a2}, k_{\beta2} \) and \( k_{\gamma2} \) in the solution (4), so that it corresponds to a surface wave. Those are

\[
k_{a2} = k_{12} = i \sqrt{k_{+ 2}^2 - k_{- 1}^2}, \quad k_{\beta2, \gamma2} = k_{2, 3 2} = \sqrt{k_{2, 3 2}^2 - k_{- 2}^2}. \quad (5)\]

For \( k_{2, 3 2} \) the branches in \( \mathbb{C}_+ \) are selected. Then it can be shown that \( k_{2z}(\omega = 0) = k_{+ z}, k_{3z}(\omega = 0) = -k_{- z} \) and \( k_{2z} = -k_{3z}^* \).

Thus we have constructed a solution \( P(r, t; \mathbf{k}_r, \omega) \), that corresponds to a ripplon with given frequency \( \omega \) and wave vector \( \mathbf{k}_r \), in the form (4) with \( k_{a, \beta, \gamma}(\mathbf{k}_r, \omega) \) chosen as described above. This solution consists of three damped waves. The first, with \( k_{a2} = k_{12} \), corresponds to the phonon branch of the bulk helium. It would be the only summand if the dispersion relation of the fluid was linear. The two other summands correspond to the two roton waves. For frequencies less than \( \Omega_{rot} = \Omega(k_{rot}) \) their wave vectors are complex and they do not correspond to propagating quasiparticles. At small frequencies these summands are small, but as \( \omega \) approaches \( \Omega_{rot} \), their amplitude increases and they become of the order of the conventional part.
The velocity of a continuous medium $v$ is found from (4) and the usual relation $\dot{v} = -\nabla P/\rho_0$. The boundary condition is obtained in the standard way of the theory of continuous medium for a free surface with tension $\tau$ (see for example [1]). For the solution of the form $\sim \exp[i(k, r - \omega t)]$ it can be put down in the form

$$v_z = \frac{i\omega}{\sigma k^2} P \bigg|_{z=0}$$

The solution $P$ from (4), with the correct wave vectors as chosen in (5), and the corresponding $v_z$ is substituted into the boundary condition (6) for $z = 0$, and after some transformations we obtain the equation for the dispersion relation of ripplons $\omega(k_r)$. Passing to the dimensionless variables

$$\chi = \omega/s k_g, \quad \kappa = k_r/k_g, \quad \kappa_{iz} = k_{iz}/k_g,$$

we can write it as an equation for $\chi(\kappa)$:

$$\chi^2 = Z_\sigma \kappa^2 \left\{ \tilde{k}_{1z}(\kappa, \chi) - i \left[ \kappa_{2z}(\kappa, \chi) - n_{2z}^0(\kappa) \right] - i \left[ \kappa_{3z}(\kappa, \chi) - n_{3z}^0(\kappa) \right] \right\}$$

Here $\tilde{k}_{1z} = -i\kappa_{1z} = \sqrt{k^2 - k_1^2/k_g^2}$ is real; $n_{2,3}^0(\kappa, \chi = 0)$; and $Z_\sigma = \sigma k_g/\rho_0 s^2$ is an analogue of acoustic impedance for the problem with the free surface with surface tension. Due to relation $\kappa_{2z} = -\kappa_{3z}$, the equation (8) is real. It can be solved numerically in the full range $\chi \in (0, \chi_{rot})$ (where $\chi_{rot} = \Omega_{rot}/s k_g$ is the dimensionless frequency of roton minimum).

**Figure 1.** Line 1 (thick) is the numerical solution for $\omega(k_r)$ in Eq. (8); lines 2-5 are consecutive analytic approximations of $\omega(k_r)$, in which consecutive summands in the expansion by $k_r$ (9) are retained. Line 2 is the hydrodynamic law $\sim k_r^{3/2}$; 3 takes into account compressibility, 4 and 5 incorporate the roton corrections. The points • are experimental data [3]. Line 6 is a spline of experimental data for bulk modes [8], 7 the approximation (3); level $\Delta$ is shown by horizontal line 8. The region close to $\Delta$ is shown in more detail in the inlaid graph, where line 9 (dashed) is the asymptote (10).

The important result is that the curve approaches the line $\chi = \chi_{rot}$ at $\kappa < \kappa_{rot} = k_{rot}/k_g$. The numerical solution gives $\kappa_c \approx 0.578$ and so $k_c = k_g \kappa_c \approx 1.15 \text{Å}^{-1}$.

At small frequencies $\tilde{k}_{1z} \approx \kappa$ and in the zero approximation we obtain from Eq. (8) the classical hydrodynamic relation $\chi^2 = Z_\sigma \kappa^3$. The next approximation can be obtained if we search for $\chi^2$ in the form of a series in $\kappa$. Then we get

$$\chi^2 = Z_\sigma \kappa^3 \left\{ 1 + \frac{Z_\sigma}{2} \kappa + \left( \frac{Z_\sigma^2}{8} - \beta Z_\sigma \right) \kappa^2 + \beta Z_\sigma^2 \kappa^3 + O(\kappa^4) \right\}.$$  

Here $\beta = -2 Im \{\partial \kappa_{2z}/\partial \chi^2\}_{\chi=0, \kappa=0}$ and for the adopted set of parameters $\beta \approx 1.81$. The first two corrections are due to taking into account the compressibility of the fluid. They are also
present in the case of linear dispersion of the bulk medium. The next summands, proportional to $\beta$, represent the roton corrections, that appear due to the roton summands $\kappa_{2,3,z}$ in Eq.(8). The nonlinearity of the phonon dispersion gives corrections $\sim \kappa^4$ and of higher order. Comparing with the exact numerical solution, we see that the series (9) with summands up to $\sim \kappa^3$ quite well approximates the solution quantitatively up to the region near $\chi \approx \chi_{\text{rot}}$.

In this region the behavior of the curve $\chi(\kappa)$ is determined by the roton summands $\kappa_{2,3,z}$. At $\chi \approx \chi_{\text{rot}}$ the roton dispersion can be approximated by $\Omega(\kappa_{2,3}) \approx \Omega_{\text{rot}} + (\kappa_{2,3} - \kappa_{\text{rot}})^2/2m$, where $m$ is the "roton mass". So for $\chi \to \chi_{\text{rot}}-0$ we get $\kappa_{2,3,z} \approx \pm \kappa_{\text{rot}} z + i\sqrt{2\mu(\chi_{\text{rot}}-\chi)} \cdot \kappa_{\text{rot}}/\kappa_{\text{rot}} z$. Here $\mu = ms/hk_y$ and $\kappa_{\text{rot}} z = \sqrt{\kappa_{\text{rot}}^2 - \kappa^2} \in \mathbb{R}$. Then searching for the solution in the form $\chi = \chi_{\text{rot}} - \gamma(\kappa_c - \kappa)^2$, expanding all the summands in Eq. (8) by $(\kappa_c - \kappa)$ and keeping the summands up to the first order of smallness, we obtain the asymptote

$$\chi_{\text{rot}} - \chi(\kappa) \approx (\kappa_c - \kappa)^2 \frac{1}{8\mu} \left(1 - \frac{\kappa_c^2}{\kappa_{\text{rot}}^2}\right) \left[\frac{\kappa_c}{\kappa_{1c}} + 2\frac{\chi_{\text{rot}}^2}{Z\kappa_{\text{rot}}^2} - \kappa_c - \frac{\kappa_{1c} - \chi_{\text{rot}}^2/\kappa_{\text{rot}}^2}{(\kappa_c^2 + \lambda)^2 - \kappa^2 + 1}\right]^{-2},$$

where $\kappa_{1c} = \kappa_{1c}(\kappa_c, \chi_{\text{rot}})$. So the curve $\chi(\kappa)$ approaches the line $\chi = \chi_{\text{rot}}$ at the peak of the inverted parabola, with zero derivative. This result was obtained qualitatively in theory from general considerations in [2], as one of the possible variants.

The approximation (3) deviates from the real dispersion of helium in two points (see Fig.1). It is normal at small frequencies, while at SVP the dispersion relation of $He\ II$ is anomalous, and it reduces the "roton mass" by factor $\sim 1.6$. We could use a better approximation, instead of (3), adding summands of higher order by $k$ to bring the curve $\Omega(k)$ closer to that measured. This would result (see [5]) in small corrections to the existing roots $k_1(\omega)$ and the appearance of additional complex roots with large absolute values. The additional roots would give small additions to the solution (4). At small frequencies, from the expansion (9), we see that a correction to the phonon dispersion can give only summands $\sim k^4$ and of higher order, and to the roton dispersion of order $k^3$ and higher. From the other side, at frequencies close to $\Delta$, the ripplon dispersion is determined by the asymptotic form of $\kappa_{2,3,z}$, which would shift with change of the "roton mass". Therefore, we expect the first effect of using better approximation of $\Omega(k)$ would give slight stretching of the curve $\chi(\kappa)$ along the $\kappa$ axis near the roton frequency. However, we have grounds to expect these changes to be small corrections to the curve obtained in this communication.

Summarizing, in the frame of the model of superfluid helium as a continuous medium with correlations we have derived the equation for the dispersion relation of ripplons (8) in elementary functions. Its numerical solution is in good agreement with the experimental data (see Fig.1). It is well approximated by the analytical expressions for the asymptotes both at small frequencies (9) and near the roton minimum (10). We see that there is need for more experimental data for further analysis, and hope this paper stimulates new experiments on measuring ripplon dispersion at high momenta.

We are grateful to Adrian Wyatt for many useful discussions and to EPSRC of the UK (grant EP/F 019157/1) for support of this work.

References
[1] Pitaevskii L and Stringari S Phys. Rev. B 45 13133-5 (1992)
[2] Lastri A, Dalfovo F, Pitaevskii L and Stringari S J. Low Temp. Phys. 98 Nos. 3/4 227-50 (1995)
[3] Lauter H J, Godfrin H, Frank V L P and Leiderer P Phys. Rev. Lett. 68 2484 (1992)
[4] Adamenko I N, Nemchenko K E and Tanatarov I V Phys. Rev. B 67 104513 (2003)
[5] Adamenko I N, Nemchenko K E and Tanatarov I V Fiz. Nizk. Temp. 32 No.3 255-68 (2006)
[6] Adamenko I N, Nemchenko K E and Tanatarov I V to be published in Phys. Rev. B 77 (2008)
[7] Adamenko I N, Nemchenko K E, Tanatarov I V and Wyatt A F G J. Phys.: Cond. Mat. 20 245103 (2008)
[8] Donnelly R J, Donnelly J A and Hills R N J. Low Temp. Phys. 44 471-489 (1981)