Linearly Coupled Directed Percolation in the Strong Coupling Regime

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Abstract

We consider directed percolation processes for particle types $A$ and $B$ coupled unidirectionally by a transmutation reaction $A \rightarrow B$. It is shown that the strong coupling regime of this recently introduced problem defines a universality class with upper critical dimension $d_c = 6$. Exact expressions are derived for the scaling dimensions in the inactive phase above $d = 4$. Below $d = 4$ the interactions of the normal directed percolation also get relevant.

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It is a well known fact that the phase diagram (or more generally, the parameter space) of homogeneous media may contain critical submanifolds. Near such a critical manifold some local restoring forces vanish and the system becomes “soft” with respect to some external forces. As a consequence next-neighbour interactions and long-range correlations begin to play an important role. If in addition a random force is acting in the system, then a complex behaviour may result. The most important experimental aspects of critical phenomena are scale invariance and universality.

The universality classes may be subdivided into categories like “Critical Statics”, “Critical Dynamics”, “Reaction-Diffusion” and “Self-Organized”. Dozens of universality classes have been identified. Formally there are infinitely many universality classes - most of them without a plausible physical interpretation, of course.

The directed percolation (DP) universality class is one of the simplest and most important in the reaction-diffusion type category. It occurs in contexts like birth-death processes, population growth and chemical reactions. It also may be given a purely static interpretation and it formally coincides with Reggeon field theory. This field theory contains nonlinearities of lowest (=cubic) order, and the general expectation is that it thus describes various processes with an active state for a one component order parameter. Essential preconditions are that the particles diffuse and that there are no conservation laws.

Recently Täuber et al. introduced a model describing DP processes for $N$ particle types $A, B, ...$ coupled by transmutation reactions. For 2 particle species the model may
be defined by the reaction equations

\[
\begin{align*}
A & \rightarrow 2A, \quad A \rightarrow 0, \quad 2A \rightarrow 0, \quad A_n \xrightarrow{\lambda} A_{n+e}, \\
B & \rightarrow 2B, \quad B \rightarrow 0, \quad 2B \rightarrow 0, \quad B_n \xrightarrow{\lambda} B_{n+e}, \\
A & \xrightarrow{\sigma} B.
\end{align*}
\] (1)

The particles may branch, decay and annihilate with some given rates. Together with the diffusion reaction (rate \(\lambda\)) this yields normal directed percolation. The two DP processes are coupled by the transmutation reaction (rate \(\sigma\)). An interesting interpretation of the reactions (1) arises when one considers the particle species as particles in different layers on a substrate. The reactions (1) then are a model for the adsorption and desorption of particles.\[5\]

A field theory corresponding to the process (1) may be derived by standard techniques\[6\] \[7\] \[8\], the action functional reads\[5\]

\[
S = \int d^dx dt \left\{ -\phi \left[ \partial_t + r_0^{(\phi)} - \nabla^2 \right] \phi - \widetilde{\psi} \left[ \partial_t + r_0^{(\psi)} - \nabla^2 \right] \psi + \sigma \psi \widetilde{\phi} \right\} + \int d^dx dt \left\{ \frac{g_\phi}{2\sqrt{K_d}} \phi \widetilde{\phi}^2 - \frac{g_\psi}{2\sqrt{K_d}} \psi \widetilde{\phi}^2 \right\}.
\] (2)

Here \(\phi\) corresponds to \(A\), \(\psi\) corresponds to \(B\), and quartic terms that are irrelevant in the renormalization group sense have been dropped. The constant \(K_d\) is \(2^{1-d} \pi^{-d/2}/\Gamma(d/2)\). To reach the critical point the parameters \(r_0^{(\phi)}\) and \(r_0^{(\psi)}\) (or their renormalized counterparts) have to be adjusted to 0. Here we shall only consider the cases \(r_0^{(\phi)} = r_0^{(\psi)}\) or \(r_0^{(\phi)} = 0\).

For a transmutation rate \(\sigma = 0\) \(S\) decomposes into uncoupled \(A\) and \(B\) DP action functionals. In fact, it follows from the reaction equations (1) or the Langevin equations corresponding to \(S\), that response functions of the type \(\langle \phi^m \phi^n \rangle\) and \(\langle \widetilde{\psi}^m \widetilde{\psi}^n \rangle\) are \(\sigma\)-independent. For \(A\) the reaction \(A \rightarrow B\) has the same effect as \(A \rightarrow 0\), and for \(B\) \(\sigma\) has no effect as long as there are no \(A\) particles. The coupling is only visible in the response functions \(\langle \widetilde{\psi}^m \widetilde{\psi}^n \rangle\). This implies that the critical exponents \(\beta\), \(\nu\) and \(z\) retain their DP values.\[5\] It is also of interest to note that \(g_\phi\) and \(u_\phi\) may be given the same value by rescaling \(\phi\) or \(\widetilde{\phi}\), making the action invariant under the transformation \(\{\phi \leftrightarrow -\widetilde{\phi}, \; t \leftrightarrow -t\}\) in the \(\sigma = 0\) case. It is equally well possible to fix \(g_\phi = 1\) and to get a renormalization group fixed point \(g_\phi = O(1)\) and \(u_\phi = O(4 - d)\) instead of \(g_\phi = \sqrt{O(4 - d)}\) and \(u_\phi = \sqrt{O(4 - d)}\), and analogously for \(g_\psi\) and \(u_\psi\).

Täuber et al.\[5\] examined the model (2) for \(\sigma \neq 0\) and noticed that additional cubic interactions \(\int d^dx dt \left\{ \frac{s_1}{2} \phi \widetilde{\phi}^2 + \frac{s_2}{4} \psi \widetilde{\phi}^2 + \frac{s_3}{4} \psi^2 \widetilde{\phi} + s_4 \phi \psi \widetilde{\phi} \right\}\) are generated in the loop expansion and must be taken into account in a renormalization group calculation. The complete model displays two fixed lines in the space of the coupling constants, one of them stable. This result was confirmed and generalized by Janssen\[9\], who also showed that the parameter \(\sigma\) carries the same scaling dimension as \(r_0 - r_{0c}\). At any rate, \(\sigma\) is strongly relevant, and a quick crossover to a strong coupling regime will occur.

Here we are interested in this strong coupling regime and now set \(\sigma = 1\). For \(\sigma \neq 0\) there is a “rapidity” symmetry\[9\]

\[
\phi \leftrightarrow -\widetilde{\psi}, \; \widetilde{\phi} \leftrightarrow -\psi, \; t \leftrightarrow -t, \; g_\phi \leftrightarrow g_\psi, \; u_\phi \leftrightarrow u_\psi.
\] (3)
Table 1: Field exponents and dimensions of possibly relevant terms (for the $u_i = 0$ model). The asterisk denotes an exponent $\geq 1$.

In a formal sense the term *unidirectional coupling* thus is somewhat misleading. To simplify the calculation we rescale the fields to get $g = g_\phi = g_\psi$.

**Dimensional analysis**

The goal now is a 2-loop renormalization group (RG) calculation for the model (2) in the $\sigma = 1$ case. A first and elementary step of any RG calculation is a dimensional analysis of the action integral. From this one obtains the critical dimension of the model and the canonical wave vector dimensions of the fields, coordinates and coupling constants.

There are some choices here, and it is of interest to formalize the (simple) procedure. The condition that the terms of the action integral be dimensionless leads to linear equations for the canonical wave vector dimensions of the fields and coordinates. Let us call the number of fields plus the number of coordinates the *model order* of the field theory (6 in the case of Eq.(2)). The determination of the model order minus 1 canonical dimensions of the fields and coordinates and of one coupling constant (or the critical dimension instead) requires at least model order linearly independent terms from the action integral. For $\varepsilon = 0$ the linear equations may also be interpreted as equations for a hyperplane in a model order dimensional exponent space. The integer valued coordinates of this space are the exponents occurring in the terms of the action integral, the canonical wave vector dimensions are the components of the normal vector of this hyperplane. Selecting the terms of a critical model thus amounts to determining a hyperplane in a model order dimensional space. The normal vector of this hyperplane (1, 2; 2, 2, 4, 4) in the case of Eq.(2)) together with the critical dimension, is a good signature for a critical model.$^1$ The terms in the halfspace containing the origin are relevant, the terms in the other halfspace are irrelevant. The procedure of selecting a critical hyperplane may also be captured in a computer program. [10]

For the action integral (2) the result for a hyperplane containing the $\psi\tilde{\psi}$-term is a critical dimension $d_c = 6$, and with $\varepsilon = d_c - d$,

$$[g] = \varepsilon/2, \quad [u_i] = -2 + \varepsilon/2, \quad [\omega] = 2, \quad [\phi] = [\tilde{\psi}] = 2 - \varepsilon/2, \quad [\tilde{\phi}] = [\psi] = 4 - \varepsilon/2.$$  

Since it is crucial for the consistency of the model it must be checked now that the action (2) with $u_i = 0$ and without the $s_i$ terms contains all terms relevant or marginal near $d = 6$, that may be generated from other terms present originally, even irrelevant ones. The essential fact to note to this purpose is that any vertex with an external $\phi$ at least also

$^1$The normal vector of the critical hyperplane for $\sigma = 0$ is (1, 2; 2, 2, 2, 2).
\[ \Gamma_{\psi,\bar{\psi}} = \begin{array}{c} \text{Diagram A} \\
\end{array} - \frac{g^2}{2} + \frac{g^4}{2} \]

\[ + g^4 \]

\[ \begin{array}{c} \text{Diagram B} \\
\end{array} \]

\[ + \frac{g^4}{2} \]

\[ + g^4 \]

\[ \begin{array}{c} \text{Diagram C} \\
\end{array} + O(g^6) \]

\[ \Gamma_{\psi,\bar{\psi}} = g^2 \phi \]

\[ = g^2 \bar{\psi} \]

\[ = g^2 \phi \]

\[ \sim B + C \]

\[ \text{Figure 1: Contributions to the } \Gamma_{\sigma} \text{ vertex up to 2 loop order. On the top right there are also shown the graphs for the interactions. The heavy line symbolizes the } \psi, \text{ the normal line the } \phi \text{ field. The dot denotes the } \Gamma_{\sigma} \text{ vertex in tree approximation.} \]

contains one external \( \tilde{\phi} \), and vice versa for \( \psi \): an external \( \tilde{\psi} \) implies at least one external \( \psi \). This comes about because in the direction of increasing time a \( \phi \) line may be converted to \( \tilde{\phi} \), but it cannot disappear. And in the direction of decreasing time a \( \psi \) line may be converted to \( \tilde{\psi} \), but cannot disappear (see also Fig. (1)). In Table (1) we have listed the \( \phi \) and \( \tilde{\phi} \) exponents of potentially dangerous terms for combinations of \( \phi \) and \( \tilde{\psi} \) exponents.

The consistency of the action (2) follows from the fact that the canonical dimensions of the field monomials are \( \geq 8 \equiv d_c + [\omega] \), with the exception of the \( r_0 \) terms that must be adjusted to 0 anyway.

**RG calculation**

It follows from the dimensional analysis that the \( u_i \) couplings are strongly irrelevant and may be dropped, at least close to \( d = 6 \). The Feynman graphs for a 2-loop RG calculation for the simplified action (2) are depicted in Fig. (1). It is only the \( \Gamma_{\sigma} \) vertex that gets renormalized. To generate a contribution to the \( g_{\phi} \) interaction or to the \( \phi \) propagator would require an irrelevant \( \phi \tilde{\phi}^m \tilde{\psi}^n \) vertex in the left corner of the time ordered diagram, and analogously for the \( g_{\psi} \) interaction (rapidity inversion symmetry). Consequently only one independent \( Z \) factor \( Z_{\sigma} = (Z_{\psi} Z_{\tilde{\phi}})^{1/2} \) is required. For the individual fields this leads to

\[ Z_{\psi} = Z_{\tilde{\phi}} = Z_{\sigma}, \quad Z_{\tilde{\psi}} = Z_{\phi} = Z_{\sigma}^{-1}. \]

The renormalized vertex function \( \Gamma_{\sigma} \) then is

\[ \Gamma^{(R)}_{\sigma} (k^2, \omega, \tau, g_R, \mu) = Z_{\sigma} \Gamma_{\sigma} (k^2, \omega, \tau, g), \]

where \( \tau = r_0^{(\phi)} - r_0^{(\bar{\psi})} \equiv \tau_R \) and \( g_R = Z_{\sigma}^{-1/2} g \mu^{-\epsilon/2} \) is the dimensionless renormalized coupling constant. The condition that \( g_R \) reaches a fixed point asymptotically requires \( Z_{\sigma} \sim \mu^{-\epsilon} \).

The evaluation of the graphs of Fig. (1) with standard techniques \( [4] \) yields

\[ A = \mu^{-\epsilon} \left( \frac{1}{4\epsilon} + \frac{1}{16} \ln (4) - \frac{3}{16} + O(\epsilon) \right), \]

\[ B = \mu^{-2\epsilon} \left( \frac{1}{32\epsilon^2} + \frac{12 \ln (4) - 6 \ln (3) - 23}{384\epsilon} + O(1) \right), \]

\[ C = \mu^{-2\epsilon} \left( \frac{1}{96\epsilon} + \frac{\ln (4) - \ln (3)}{32\epsilon} + O(1) \right). \]
Using an external frequency $-i\omega = \mu^2$ and minimal subtraction we then find the stable RG fixed point

$$(g_R^2)_{\text{MinSub}}^* = 8\varepsilon + (4 + 40 \ln (4/3)) \varepsilon^2 + O(\varepsilon^3).$$

(5)

The $\varepsilon$-expansion (5) does not seem to converge for $\varepsilon = 1$, and it seems to be impossible to extract a numerical value for $g_R^2$ for higher $\varepsilon$ values. However, the critical exponents may be given in closed form. From Eq.(4) and the $\mu$-independence of the unrenormalized vertex function $\Gamma_{\sigma}$ it follows

$$\Gamma_{\sigma}^{(R)}(\frac{k}{\mu_0}, \frac{\omega}{\mu_0^2}, \frac{\tau_R}{\mu_0^4}; g_R(\mu_0)) = Z_{\sigma}^{-1}(\mu) \Gamma_{\sigma}^{(R)}(\frac{k}{\mu}, \frac{\omega}{\mu^2}, \frac{\tau_R}{\mu^4}; g_R(\mu)) \sim k^\varepsilon.$$  

(6)

This finally leads to the scaling equivalence relations for the response function,

$$\langle \tilde{\psi}\phi \rangle(k, \omega, \tau) = \langle \tilde{\psi}\psi \rangle \Gamma_{\sigma} \langle \tilde{\phi}\phi \rangle \sim 1/k^{4-\varepsilon} \sim 1/\omega^{2-\varepsilon/2} \sim 1/\tau^{2-\varepsilon/2}.$$  

This exact result is valid for $d > 4$ and in the inactive phase $\tau \geq 0$.

It was argued above that the response functions $\langle \tilde{\phi}^m\phi^n \rangle$ and $\langle \tilde{\psi}^m\psi^n \rangle$ are independent of $\sigma$, and one might think that it is inconsistent with this normal DP behaviour to have nontrivial $Z$ factors and anomalous dimensions for $\phi, \tilde{\phi}, \psi$ and $\tilde{\psi}$. Actually this is not the case, because general response functions of this type cannot be generated from the action with $u_i = 0$.

**Summary**

It was shown that the problem of unidirectionally coupled directed percolation may be solved with RG techniques above $d = 4$ by expanding around $d_c = 6$. In the inactive phase ($\tau \geq 0$) the scaling dimensions may be given exactly.

In principle the solution $g = O(\varepsilon), u_i = 0$ found for $4 < d < d_c$ may extended continuously to $d < 4$ with $g = O((4 - d)^0)$ and $u_i = O(4 - d)$. In this way the $\langle \tilde{\phi}^m\phi^n \rangle$ and $\langle \tilde{\psi}^m\psi^n \rangle$ response functions can retain their normal DP behaviour. This also agrees with the fact that the scaling relations $\phi \sim \tilde{\psi} \sim k^2$ and $\tilde{\phi} \sim \psi \sim k^{4-\varepsilon}$ coincide with the uncoupled DP scaling relations at $d = 4$.

The situation below $d = 4$ is more complicated for the $\langle \tilde{\psi}\phi \rangle$ response function, because here also the $u_i$ interactions and possibly the mixed interactions considered by Täuber et al. get relevant. Täuber et al. derive a critical exponent $\beta_2 = 1/2 - (4-d)/8 + O(4-d)^2$ describing the expectation value of $\phi$ in the active phase in an expansion around $d = 4$. This procedure was criticized by Janssen in that it relies on the assumption that an exponentiation of logarithms were possible. At any rate, for a finite coupling $\sigma$ the diagrams of the type shown in Fig.11 must be taken into account - all of them are IR singular already below $d_c = 6$. The situation also gets more involved in the active phase, where the irrelevant coupling constant $u_\psi$ plays an important role, limiting the growth of $\psi$: $u_\psi$ is a “dangerous irrelevant parameter” above $d = 4$, that becomes a normal relevant parameter below $d = 4$.

That the 1-loop diagram “A” of Fig.11 poses a problem for an RG calculation and an expansion around $d = 4$ was already noticed by Goldschmidt (diagram “c” of this...
letter) and Täuber et al.\cite{5}. An inspection of the diagrams for the model with \( u_i = 0 \) indeed shows that \( \beta_2 \) is non-classical (\( \neq 1/2 \)) already below \( d = 6 \).

There is some similarity with the critical dynamics of the Heisenberg ferromagnet (model \( J \))\cite{12}, where the dynamics is nonclassical below \( d = 6 \), while the statics is nonclassical below \( d = 4 \). Likewise the static critical exponents of model \( J \) are independent of the dynamics. An essential difference is that the model \( J \) dynamics adds new coordinates and fields, leaving room for a new critical exponent. In contrast, the scaling dimensions of the coupled DP problem are completely determined by normal DP, and one would expect \( \beta_2 = \beta \).

A dimensional analysis for the general coupled DP problem with \( n \) particle species (or \( n \) layers) shows that here also only two nonlinearities of the type displayed in Fig. (1) are relevant near the critical dimension, which now is \( d_c(n) = 2(n + 1) \). These nonlinearities act in the first and the last layer, and below \( d_c(n) \) more and more other nonlinearities get relevant. This indicates that the RG in the form used here may not be the appropriate technique to solve the general case. As a first step it would be of interest to fully understand the \( n = 2 \) case in the active phase and below \( d = 4 \).

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