IMPACT PARAMETER DEPENDENT QUARK DISTRIBUTIONS OF THE PION

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We present first results for the impact parameter dependent quark distribution in the pion obtained in transverse lattice gauge theory, and discuss recent predictions from other models.
1 Introduction

First-principles QCD calculations of the structure of hadrons are available from lattice gauge theory. In particular, the low moments of traditional lightcone momentum fraction $x$ dependent structure functions have been calculated [1]. Such moments are however consistent with a wide variety of functional forms for the structure functions themselves. The latter can be estimated directly in QCD-based models such as truncated Dyson-Schwinger equations [2], coarse transverse lattice gauge theory [3], chiral quark models [4, 5] and instanton liquid [6]. Typically, at low resolution scales these models disagree in detail about the shape of quark distributions, although the differences are mollified by evolution to higher scales, where experimental data can be compared with. More general off-forward matrix elements, which are coherent functions also of transverse parton momenta that interpolate between traditional structure functions and form factors, have also begun to be studied in these models. In particular, the traditional structure functions may be generalised to ones depending also on the transverse impact parameter $b$ of the struck quark, which retain a probabilistic interpretation. Their experimental measurement will provide much stronger constraints on theory than hitherto available.

We present new results for impact parameter dependent valence quark distribution functions of the pion, obtained via the transverse lattice method. We find that the $x - b$ dependence does not factorize, as is sometimes assumed in phenomenological models. In particular, at large lightcone momentum fraction $x$, the quark distribution is dominated by small impact parameters $b < 2/3 \text{ fm}$, while at small $x$ the opposite is true. The result at large $x$ is shown to be consistent with a recent construction of generalised parton distributions using double distributions [7], with chiral NJL quark models using the Pauli-Villars regulator [8], and with a simple two-body formula with sharp fall-off in transverse space. Experimental measurement of the small-$b$ dependence could serve to distinguish between these and other models, clarifying the behaviour of the integrated distribution deduced previously from pion-nucleon Drell-Yan scattering.

2 transverse lattice results

The transverse lattice formulation of gauge theory represents the physical gluonic degrees of freedom by gauge-covariant links of colour flux on a lattice transverse to a null-plane of quantisation [9]. In a recent paper [10], van de Sande and the author applied Discrete Light Cone Quantisation (DLCQ) [11] to this formulation in order to estimate the lightcone wavefunctions of the pion. Lightcone wavefunctions provide a fundamental starting point for the investigation of hadronic observables represented as matrix elements of currents. The computation performed in ref. [10] was done to leading order of the $1/N_c$ expansion and used a coarse transverse lattice of spacing $a \sim 2/3 \text{ fm}$. This generated a series of gauge-invariant effective interactions, consistent with residual Lorentz symmetries, the low energy truncation of which left a set of couplings to be determined. The couplings were fixed by requiring finiteness, optimisation of full Lorentz covariance, and phenomenological fits to masses $M_\pi$, $M_\rho$, the decay constant $f_\pi$, and string tension $\sqrt{\sigma}$.

We introduce lightcone coordinates \( \{ z^+ = (z^0 + z^3)/\sqrt{2}, z^- = (z^0 - z^3)/\sqrt{2}, z = (z^1, z^2) \} \). At lightcone time $z^+ = 0$, we will write a hadron state in the impact-parameter representation \( |P^+, b_0\rangle \). \( P^+ = (P^0 + P^3)/\sqrt{2} \) is the total hadron lightcone momentum, $b_0$ is the transverse position of the hadron, assumed spinless. In this representation, a generalised quark distribution may be defined in lightcone gauge as the matrix element [12]

\[
\mathcal{I}(x, \xi, b) = \frac{1}{\sqrt{1 - \xi^2}} \langle P_\text{out}^{+}, b^\text{out}_0 | \int \frac{dz^-}{4\pi} e^{i x P^+ z^-} \bar{q}(0, -z^-/2, b) \gamma^+ q(0, z^-/2, b) | P_\text{in}^{+}, b^\text{in}_0 \rangle ,
\]

where

\[
\xi = \frac{(P_\text{in} - P_\text{out})^+}{(P_\text{in} + P_\text{out})^+} , \quad b^\text{out}_0 = -\frac{\xi}{1 - \xi} b , \quad b^\text{in}_0 = \frac{\xi}{1 + \xi} b ,
\]

2
and $q$ ($\bar{q}$) is an (anti)quark field. It can be made gauge-invariant by insertion of a light-like Wilson line and is not to be confused with so-called $k_T$-unintegrated parton distributions, which we do not discuss here. These are qualitatively different and have problems with gauge invariance [12]. We concentrate on the case $\xi = 0$. This is the most physically intuitive since $\mathcal{I}$ is then simply the probability of a quark carrying fraction $x$ of the lightcone momentum $P^+$ when at transverse position $b$ [13, 14]. It is also the most easily investigated in the context of DLCQ, which discretizes lightcone momentum fractions. At non-zero $\xi$, one would have to extrapolate the DLCQ cutoff in lightcone wavefunctions first, before using them to compute observables like Eq. (1), rather than the easier task of extrapolating the observables themselves.

As well as the DLCQ regulator, which we extrapolate in observables, the transverse lattice theory discretizes the transverse co-ordinate $z$ on a square lattice of spacing $a \sim 2/3\text{fm}$, held fixed, and decomposes the hadron state into its parton constituents. As well as the quarks $q$ localised at transverse sites, other partons consist of gauge potentials $A_\pm$, which are eliminated as dynamical degrees of freedom by lightcone gauge choice $A_- = 0$, and $N_c \times N_c$ matrix link-fields $M_\nu(z^-,z)$, which carry color flux from site $z$ to $z + a \vec{r}$, where $\vec{r}$ is a unit vector in direction $r \in \{1, 2\}$. $M_\nu(z^-,z)$ is the oppositely oriented link and we label the transverse orientation of link fields by indices $\lambda_j \in \{\pm 1, \pm 2\}$. By convention, we will associate the center of the link, $z + 0.5a \vec{r}$, as the transverse position of the link-field parton. In the large-$N_c$ limit, a meson state is then written as a superposition Fock states consisting of a $q$-$\bar{q}$ pair joined by various configurations of a string of link fields (to ensure residual transverse gauge invariance in the lightcone gauge).

In ref. [10], the partonic decomposition was done for a hadron state $|P^+, \mathbf{P}\rangle$ fully in momentum space. In particular, the wavefunctions $\psi_n(x_i, h, h', \lambda_j)$ were computed\(^1\) for the pion in the case

$$|P^+, \mathbf{P}\rangle = 0 = \sum_{n=2}^{\infty} \int [dx]_n \sum_{z_1, z_2, h, h'} \left[ \sum_{\lambda_j} \psi_n(x_1, h, h', \lambda_j) \langle x_1, h, z_1| (x_2, \lambda_1, z_2) \rangle \cdots \langle x_{n-1}, h', z_{n-1}| (x_n, h', z_n) \rangle \right],$$

(3)

where $h, h'$ denote quark and anti-quark helicities respectively, $x_i$ is the fraction of $P^+$ carried by the $i^{th}$ parton,

$$\int [dx]_n = \int dx_1 \cdots dx_n \delta \left( \sum_{i=1}^n x_i - 1 \right)$$

(4)

and $\left[ \sum_{\lambda_j} \right]$ indicates that the sum over orientations of links must form an unbroken chain on the transverse lattice between quark and anti-quark. Note that once the transverse positions $z_1$ and $z_n$ of the quark and antiquark are specified, it is sufficient to just enumerate the sequence of link orientations $\lambda_j$ rather than their actual postions $z_j$.

A set of hadron states boosted to general transverse momentum $\mathbf{P}$ can be obtained by applying the Poincaré generators $\mathbf{M}^+ = (M^+)^{1, 2}$. This gives for each parton Fock state

$$\exp \left[ -i \mathbf{P}^+ / P^+ \right] \langle (x_1, h, z_1); (x_2, \lambda_1, z_2); \cdots; (x_{n-1}, \lambda_{n-2}, z_{n-1}); (x_n, h', z_n) \rangle$$

$$= \exp [i \mathbf{P} \cdot \mathbf{c}] \langle (x_1, h, z_1); (x_2, \lambda_1, z_2); \cdots; (x_{n-1}, \lambda_{n-2}, z_{n-1}); (x_n, h', z_n) \rangle,$$

(5)

$$\mathbf{c} = (c^1, c^2) = \sum_{i=1}^{n} x_i \mathbf{z}_i.$$  

(6)

Before investigating the interplay of transverse and longitudinal structure, we check the special case of the electromagnetic form factor

$$\langle P^+_{\text{in}}; \mathbf{P}_{\text{in}} | j_{\mu}^{\text{em}} | P^+_{\text{out}}; \mathbf{P}_{\text{out}} \rangle = F^\prime(Q^2) (P^+_{\text{in}} + P^+_{\text{out}})^\mu$$

(7)

\(^1\)The raw DLCQ data for wavefunctions $\psi_n$ found in ref.[10] are available from the internet URL http://www.geneva.edu/~bovdcs/four/
where \( Q = P_{out} - P_m \). Using the decomposition (3), boosts (5), and wavefunctions \( \psi_n \), \( F(Q^2 = -t) \) may be extracted most simply from the \( \mu = + \) component of the matrix element (7) in the case of purely transverse momentum transfer \( Q \) (Drell-Yan frame)

\[
F(Q^2) = \sum_{n=2}^{\infty} \int [dx]_n \sum_{\mathbf{z}_1, \mathbf{z}_n, h, h'} \mathbb{S}_{\mathbf{\lambda}}^{\mathbf{\lambda}} \exp \{ iQ \cdot c \} |\psi_n(x, h, h', \lambda_j)|^2. \tag{8}
\]

Figure 1 shows that calculations of the \( \psi_n \) truncated at \( n = 3 \) \[15\] and \( n = 5 \) \[10\] give a form factor converging close to the experimental one (it should be noted that the latter suffers from uncertainties due to the extrapolation to the pion pole). A cutoff on \( n \) cuts off the maximum quark-antiquark separation. With the \( n = 5 \) truncation, the maximum quark antiquark separation is \( 3a \approx 2 \) fm, which explains why the calculated charge radius \( r_n = \sqrt{6dF(-t)/dt}\big|_{t=0} \approx 0.59 \) fm is slightly less than the experimental value \( 0.663 \) fm.

The traditional quark distribution function \( V(x) = \sum_\mathbf{b} I(x, 0, \mathbf{b}) \) was already given in ref.[10]

\[
V(x) = \frac{(1 - x)^{0.33}}{x^{0.5}} \{ 0.33 - 1.1\sqrt{x} + 2x \} \tag{9}
\]

and is shown in Figure 2(a) at a transverse resolution scale of 0.5GeV, set roughly by the lattice spacing \( a \). As manifestly required of many-body gauge theory lightcone wavefunctions, it satisfies the usual normalisation and sum rules, rising Regge behaviour at small \( x \) and vanishing at \( x \rightarrow 1 \). The latter two properties are finite energy conditions \[17\].

To investigate simultaneously the transverse and longitudinal structure we can use the impact parameter dependent parton distributions \( I(x, 0, \mathbf{b}) \). We must first translate results to hadron impact parameter space in the transverse directions. Integrating \( P \) over the Brillouin zone, we obtain a hadron state localised at transverse position \( \mathbf{0} \)

\[
|P^+, b_0 = 0\rangle = \sum_{n=2}^{\infty} \int [dx]_n \sum_{\mathbf{z}_1, \mathbf{z}_n, h, h'} \mathbb{S}_{\mathbf{\lambda}}^{\mathbf{\lambda}} \frac{a^2 \sin c^2 \pi/a \sin c^2 \pi/a}{c^2} \psi_n(x, h, h', \lambda_j)
\]

\[
\times \langle x_1, h, \mathbf{z}_1; (x_2, \lambda_1, \mathbf{z}_2); \cdots; (x_{n-1}, \lambda_{n-2}, \mathbf{z}_{n-1}); (x_n, h', \mathbf{z}_n) \rangle. \tag{10}
\]

Hence

\[
I(x, 0, \mathbf{b}) = \sum_{n=2}^{\infty} \int [dx]_n \sum_{\mathbf{z}_1, \mathbf{z}_n, h, h'} \mathbb{S}_{\mathbf{\lambda}}^{\mathbf{\lambda}} \frac{a^2 \sin c^2 \pi/a \sin c^2 \pi/a}{c^2} \left[ \frac{\sin c \lambda/a \sin c \lambda/a}{c_1} \right]^2 \frac{\sin c \lambda/a \sin c \lambda/a}{c_2} \frac{\sin c \lambda/a \sin c \lambda/a}{c_2} |\psi_n(x, h, h', \lambda_j)|^2. \tag{11}
\]

Due to the lattice cutoff, this should strictly be interpreted as the probability of finding the quark within one lattice spacing of impact parameter \( \mathbf{b} \). We show results for the first few values of \( b = |\mathbf{b}| \) sampled along lattice axes in fig.2(b)-(c). Fig. 2(b) is the contribution from \( \mathbf{b} = (0, 0) \), Fig. 2(c) is the sum of contributions from \( \mathbf{b} = (a, 0), (0, a), (-a, 0), (0, -a), \) etc.. The equations of these curves are fits to extrapolated DLCQ data

\[
b = 0 : \quad 0.18(1 - 4.3\sqrt{x} + 11x)x^{-0.16}(1 - x)^{0.37}
\]

\[
b = a : \quad 0.11(1 - 1.9\sqrt{x} + 5.5x)x^{-0.78}(1 - x)^{2.0}
\]

\[
b = 2a : \quad 0.04(1 - 4.2\sqrt{x} + 6.2x)x^{-0.76}(1 - x)^{1.94}
\]

\[
b = 3a : \quad 0.005(1 - 4.0\sqrt{x} + 8.2x)x^{-0.66}(1 - x)^{1.2} \tag{12}
\]

The figure clearly shows that the valence region \( x > 0.5 \) is dominated by quarks with impact parameter less than the charge radius \( \sim 2/3 \) fm. It also suggests that the rapid rise at small \( x \) is dominated by larger impact parameters \( b > 2/3 \) fm.
The above results have an intuitive explanation. The transverse lattice formulation is non-perturbatively gauge invariant. As a result of local gauge invariance in the transverse direction — an axial gauge fixing is applied in other directions — quarks and antiquarks must be joined by a string of flux-carrying link variables. Since these link variables are themselves also dynamical parton degrees of freedom carrying a portion of the hadron $P^+$, they cost energy. At large quark $x$, all other partons are forced to carry small $x$ and it is favourable to have the minimum number of them. The suppression of link partons and gauge invariance ensures widely separated quarks are suppressed in this region. We observe that this behaviour sets in quite quickly after $x = 0.5$. The rise at small $x$ is due to higher Fock states [17] containing more link partons, which naturally allows more widely separated quarks to contribute.

The analytic dependence on $b$ is difficult to characterize due to that fact that only a coarse sampling is possible and this represents averages over unit lattice cells centred at $b$. However, we found that the following transverse-continuum two-body form, for quarks at transverse positions $z_1, z_2$, reproduces the qualitative features of the distributions in Fig. 2 for $x > 0.2$,

$$
I(x, 0, b) = \frac{8V(x)}{3\pi \left(1 + \frac{|z_1|^2 + |z_2|^2}{m^2}\right)^3} \left(1 + \frac{x^2}{(1-x)^2}\right).
$$

(13)

We identify $b = |z_1|$ and the positions are constrained by the fixed hadron transverse 'center-of-momentum'

$$
c = xz_1 + (1 - x)z_2 = 0.
$$

(14)

The other $x$-dependent factors in (13) have been chosen simply so that $\int d^2b I(x, 0, b) = V(x)$. It does not completely fit the small-$x$-behaviour, but a two-body formula is not expected to work well in this region. Eq.(13) implies a sharp fall-off of the hadron wavefunction in transverse space at a particular ($x$-dependent) radius.

3 Discussion

Generalised quark distributions of the pion have recently been analysed in other models. Mukherjee et al. [7] construct generalised parton distributions from double distributions, which satisfy general reduction, spectral, and polynomiality conditions. (Their construction has been criticised on grounds of positivity [18], but this does not affect the discussion here.) We could not obtain the impact parameter dependent distribution in closed form from their formulas, but one may deduce its transform

$$
\int d^2b \exp\{ib\cdot q\} I(x, 0, b) \propto \frac{(1-x)}{\sqrt{x}^\lambda} A(\lambda)\rho^\lambda
$$

(15)

$$
\lambda = \frac{16m^2x}{ts^2(1-x)} \left(1 + \frac{s^2(x-0.5)^2}{x(1-x)}\right)
$$

(16)

$$
A(\lambda) = \frac{\lambda - 2}{\lambda(1+\lambda)^2} - \frac{4 + \lambda}{(1+\lambda)^5/2} \log \left[\frac{1 + \sqrt{1 + \lambda}}{\sqrt{\lambda}}\right]
$$

(17)

where $t = -q^2 < 0$, while $m = 0.46$ GeV and $s = 0.81$ are parameters they adjusted for a fit to the pion form factor. As $x \to 1$, the expression (15) becomes $t$-independent, i.e. dominated by the contribution from $b = 0$. This is consistent with the general behaviour expected as $x \to 1$ [13] and the transverse lattice result. The result is independent of the tunable parameters in the construction and any possible modifications at small $x$ necessary to account for positivity [7].

Generalised parton distributions have been calculated for the pion in chiral NJL models using the Pauli-Villars regulator [8]. This respects symmetries of the model and accurately fits a wide range of experimental data. It has the striking result that, at the cut-off scale of the model $\sim 0.3 - 0.4$ GeV,
the quark distribution function \( V(x) \) is constant in the chiral limit \([4]\). This is, in fact, not far from the transverse lattice result, which is associated with a slightly higher resolution scale. The authors of ref.[8] find that the transformed impact parameter dependent distribution (15) rises at large \( x \) for large \( -t \), consistent with the transverse lattice result. It is significant that the results in the NJL model and transverse lattice are so similar. The NJL model treats chiral symmetry transparently, yet is an effective quark theory containing no explicit gluonic degrees of freedom. The transverse lattice construction generically (explicitly) breaks chiral symmetry, which one tries to minimize by tuning of couplings, yet incorporates explicitly the confining effects of multiple gluonic degrees of freedom.

A new approach using a spectral quark model has also been used to obtain the same constant valence lightcone wavefunction of the pion in ref.[19]. It would be interesting to know the result this model gives for the impact parameter dependence. Results from truncated Dyson-Schwinger equations [20], the Instanton Liquid [21], and other models [22] are also available for the traditional distribution that is integrated over \( b \). The detailed shapes, even for the integrated distributions \( V(x) \), generally differ amongst all the models considered in this paper, although this could in part be due to different renormalisation scales and schemes.

Existing data on \( V(x) \) from pion-nucleon Drell-Yan scattering [23] largely agrees with the NJL model (Pauli-Villars regulated) and the transverse lattice, but shows some differences in the valence region (large \( x \)) of other models, notably the otherwise reliable Dyson-Schwinger equations [20] and the Instanton Liquid [21]. Experimental measurement of impact parameter dependent distributions would throw light on this puzzle. In principle they can be found from wide-angle real Compton scattering off pions. Data collected above a certain large transverse momentum transfer \( \sqrt{-t} \) could be compared, as a function of \( x \), with the full valence quark distribution function \( V(x) \) of the pion, which corresponds to \( t = 0 \). The transverse lattice result would predict that they are the same at large \( x \).

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Figure 1: Pion form factor $F(-t)$. Solid curves are the transverse lattice result from eq.(8), gray curve for wavefunctions calculated within a truncation at $n = 3$, black curve for $n = 5$. The experimental data points are from ref.[16].
Figure 2: Impact parameter dependent valence quark distributions of the pion. (a) The full distribution $V(x)$, which sums over all impact parameters $b$. (b)-(e) The sum of contributions at impact parameters $b$ along lattice axes such that $b = |b| = 0, a, 2a, 3a$ respectively.