Perfect 2-colorings of the generalized Petersen graph $GP(n, 3)$

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Abstract

In this paper we enumerate the parameter matrices of all perfect 2-colorings of the generalized Petersen graphs $GP(n, 3)$, where $n \geq 7$. We also give some basic results for $GP(n, k)$.

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1. Introduction

The theory of error-correcting codes has always been a popular subject in group theory, combinatorial configuration, covering problems and even diophantine number theory. So, mathematicians always show a lot of interest in this historical research field. The problem of finding all perfect codes was begun by M. Golay in 1949. Perfect code is originally a topic in the theory of error-correcting codes. All perfect codes are known to be completely regular, which were introduced by Delsarte in 1973. A set of vertices, say C, of a simple graph is called completely regular code with covering radius $\rho$, if the distance partition of the vertex set with respect C is equitable. Therefore, the problem of existence of equitable partitions in graph is of great importance in graph theory. There is another term for this concept in the literature as ”perfect $m$-coloring”.

As explained above, enumerating parameter matrices in graphs is a key problem to find perfect
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codes in graphs. For example, by the results of this paper, we can easily conclude that the graph $GP(9,3)$ has just two nontrivial completely regular codes with the size of 9, which neither of them is perfect. There has always been a notably interest in enumerating parameter matrices of some popular families of graphs ”johnson graphs”, ”hypercube graphs” and recently ”generalized petersen graphs” (see [1, 2, 3, 4, 5, 6, 7, 8, 9]).

In this article, all parameter matrices of $GP(n, 3)$ are enumerated.

2. Definition and Concepts

In this section, some basic definitions and concepts are given.

Definition 2.1. The generalized petersen graph $GP(n, k)$, also denoted $P(n, k)$, for $n \geq 3$ and $1 \leq k < \frac{n}{2}$, is a connected cubic graph that has vertices, respectively, edges given by

$$V(GP(n, k)) = \{a_i, b_i : 0 \leq i \leq n-1\},$$

$$E(GP(n, k)) = \{a_ia_{i+1}, a_ib_i, b_ib_{i+k} : 0 \leq i \leq n-1\},$$

These graphs were introduced by Coxeter (1950) and named by Watkins (1969). $GP(n, k)$ is isomorphic to $GP(n, n - k)$. It is why we consider $k < \frac{n}{2}$, with no restriction of generality.

Definition 2.2. For a graph $G$ and an integer $m$, we call a mapping $T : V(G) \rightarrow \{1, \ldots, m\}$ a perfect $m$-coloring with matrix $A = (a_{ij})_{i,j\in\{1,\ldots,m\}}$, if it is surjective, and for all $i, j$, for every vertex of color $i$, the number of its neighbors of color $j$ is equal to $a_{ij}$. We call the matrix $A$ the parameter matrix of a perfect coloring. In the case $m = 2$, the first color is called white, and the second color black.

Remark 2.1. In this paper, we consider all perfect 2-colorings, up to renaming the colors; i.e, we identify the perfect 2-coloring with the matrix

$$\begin{bmatrix}
a_{22} & a_{21} \\
a_{12} & a_{11}
\end{bmatrix},$$

obtained by switching the colors with the original coloring.

3. The Existence of Perfect 2-Colorings of $GP(n, 3)$

In this section, we first give some results covering necessary conditions for the existence of perfect 2-colorings of $GP(n, k)$ graphs with a given parameter matrix $A = (a_{ij})_{i,j=1,2}$, and then we enumerate the parameters of all perfect 2-colorings of $GP(n, 3)$.

The first and perhaps the simplest necessary condition for the existence of a perfect 2-colorings of $GP(n, k)$ with the matrix $\begin{bmatrix}a_{11} & a_{12} \\
a_{21} & a_{22}\end{bmatrix}$ is

$$a_{11} + a_{12} = a_{21} + a_{22} = 3.$$
Also, it is clear that neither $a_{12}$ nor $a_{21}$ cannot be equal to zero, otherwise white and black vertices of $GP(n, k)$ would not be adjacent, which is impossible, as the graph is connected.

By the presented conditions, a parameter matrix of a perfect 2-coloring of $GP(n, k)$ must be one of the following matrices:

$$
A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}.
$$

The next proposition provides a formula for calculating the number of white vertices in a perfect 2-coloring (see [4]).

**Proposition 3.1.** If $W$ is the set of white vertices in a perfect 2-coloring of a graph $G$ with matrix $A = (a_{ij})_{i,j=1,2}$, then

$$|W| = |V(G)| \frac{a_{21}}{a_{12} + a_{21}}.$$  

Now, we are ready to enumerate the parameter matrices of all perfect 2-colorings of $GP(n, 3)$. In [1], the parameter matrices of all perfect 2-colorings of $GP(n, k)$ with the matrices of $A_1$ and $A_6$ are enumerated. So, we just present theorems in order to enumerate parameter matrices corresponding to perfect 2-colorings of $GP(n, 3)$ with the matrices $A_2, A_3, A_4,$ and $A_5$.

**Perfect 2-colorings of $GP(n, 3)$ with the matrix $A_2$:**

In this part, we show that the graphs $GP(n, 3)$ have no perfect 2-colorings with the matrix $A_2$.

**Theorem 3.1.** The graphs $GP(n, 3)$ have no perfect 2-colorings with the matrix $A_2$.

**Proof.** At first, we claim that for each perfect 2-coloring, say $T$, of $GP(n, 3)$ with the matrix $A_2$, there are no consecutive vertices $a_i$ and $a_{i+1}$, such that $T(a_i) = T(a_{i+1}) = 2$. To prove it, suppose contrary to our claim, without loss of generality, there is a perfect 2-coloring, say $T$, of $GP(n, 3)$ with the matrix $A_2$, such that $T(a_1) = T(a_2) = 2$. It immediately gives $T(b_1) = T(b_2) = T(a_0) = T(a_3) = 1$ and then $T(b_0) = T(b_3) = T(a_4) = T(b_4) = T(b_5) = 1$. Now, from $T(a_3) = T(a_4) = T(b_4) = 1$, we have $T(a_5) = 2$. Next, from $T(a_4) = T(b_5) = 1$ and $T(a_5) = 2$, we get $T(a_6) = 2$. It gives $T(b_6) = 1$ which is a contradiction with $T(a_3) = T(b_3) = T(b_0) = 1$.

Now, to prove the theorem, suppose the assertion is false. Therefore, there is a perfect 2-coloring, say $T$, of $GP(n, 3)$ with the matrix $A_2$. By symmetry, with no loss of generality, we can assume $T(a_0) = T(b_0) = 1$ and $T(a_1) = 2$. By the above claim, we have $T(a_2) = 1$. Now, from $T(a_0) = T(a_2) = 1$ and $T(a_1) = 2$, it follows that $T(b_1) = 2$. This immediately gives $T(b_2) = T(a_3) = T(b_4) = 1$. Next, by $T(b_4) = 1$ and $T(b_1) = 2$, we get $T(a_4) = 1$ and, in consequence, $T(b_3) = T(a_5) = 2$. Again, by using the above claim, we get $T(a_6) = 1$ and then $T(b_6) = 1$, which is a contradiction with $T(a_3) = T(b_0) = 1$ and $T(b_3) = 2$. 

\[\square\]
Perfect 2-colorings of \( GP(n,3) \) with the matrix \( A_3 \):

We will show that the graphs \( GP(2m,3) \) have a perfect coloring with the matrix \( A_3 \) and the graphs \( GP(2m + 1,3) \) have no perfect 2-colorings with the matrix \( A_3 \).

**Theorem 3.2.** All of the graphs \( GP(n,3) \), where \( n \) is even, have a perfect 2-coloring with the matrix \( A_3 \). Also, there are no perfect 2-colorings of \( GP(n,3) \), where \( n \) is odd, with the matrix \( A_3 \).

**Proof.** To prove the first part, consider the mapping \( T : V(GP(2m,3)) \to \{1, 2\} \) by

\[
T(a_{2i}) = T(b_{2i}) = 1, \\
T(a_{2i+1}) = T(b_{2i+1}) = 2.
\]

for \( i \geq 0 \). It can be easily seen that the given mapping is a perfect 2-coloring of \( GP(2m,3) \) with the matrix \( A_3 \).

To prove the second part, contrary to our claim, suppose there is a perfect 2-coloring, say \( T \), of \( GP(n,3) \), where \( n \) is odd, with the matrix \( A_3 \). Now, we use lemma in ([1], Lemma 3.4).

**Lemma 3.1.** [1] For each perfect 2-coloring \( T \) of \( GP(n,k) \), where \( k \) is a positive even integer or \( 4 \nmid n \), with the matrix \( A_3 \), there are two vertices \( a_i \) and \( b_i \), for some \( 0 \leq i \leq n - 1 \), such that \( T(a_i) = T(b_i) \).

By above Lemma, with no loss of generality, we can assume \( T(a_0) = T(b_0) = 1 \). By knowing that the given mapping in the first part is not a perfect 2-coloring with the matrix \( A_3 \), where \( n \) is odd, we should have two cases below.

**Case 1.** For some positive integer \( i \), \( T(a_i) = T(b_i) = T(b_{i+1}) = 1 \) and \( T(a_{i+1}) = 2 \). It immediately gives \( T(a_{i+2}) = T(a_{i+3}) = T(b_{i+2}) = 1 \). From \( T(a_{i+2}) = T(b_i) = 1 \), we deduce that \( T(b_{i+3}) = 2 \) and then \( T(a_{i+4}) = 1 \). Next, from \( T(b_{i+1}) = T(a_{i+4}) = 1 \), we have \( T(b_{i+4}) = T(a_{i+5}) = 2 \). Now, from \( T(b_{i+1}) = T(a_{i+3}) = 1 \), and \( T(b_{i+3}) = 2 \), it follows that \( T(b_{i+6}) = 2 \), and then from \( T(a_{i+5}) = 2 \), we get \( T(a_{i+6}) = 1 \) and \( T(b_{i+5}) = 2 \). Using this argument, for \( j \geq 0 \), we have

\[
T(a_{10j+i}) = T(b_{10j+i}) = T(b_{10j+i+1}) = T(b_{10j+i+2}) = T(a_{10j+i+3}) = T(a_{10j+i+4}) = T(a_{10j+i+6}) = T(a_{10j+i+7}) = T(b_{10j+i+8}) = T(b_{10j+i+9}) = 1.
\]

and

\[
T(a_{10j+i+1}) = T(a_{10j+i+2}) = T(b_{10j+i+3}) = T(b_{10j+i+4}) = T(a_{10j+i+5}) = T(b_{10j+i+5}) = T(b_{10j+i+6}) = T(b_{10j+i+7}) = T(a_{10j+i+8}) = T(a_{10j+i+9}) = 2.
\]

It gives \( n = 10m \) which contradicts \( n \) is odd.

**Case 2.** For some positive integer \( i \), \( T(a_i) = T(b_i) = T(a_{i+2}) = 1 \) and \( T(a_{i+1}) = T(b_{i+1}) = T(b_{i+2}) = 2 \). It immediately gives \( T(a_{i+3}) = 1 \) and then \( T(a_{i+4}) = T(b_{i+3}) = 2 \). From \( T(a_{i+4}) = T(b_{i+1}) = 2 \), we have \( T(b_{i+4}) = 1 \) and then we deduce that \( T(a_{i+5}) = 2 \) and \( T(b_{i+5}) = T(a_{i+6}) = 1 \). From \( T(a_{i+3}) = T(b_i) = 1 \) and \( T(b_{i+3}) = 2 \), we get \( T(b_{i+6}) = 2 \). Now, from \( T(a_{i+5}) = T(b_{i+6}) = 2 \) and \( T(a_{i+6}) = 1 \), we have \( T(a_{i+7}) = 1 \) and then \( T(b_{i+7}) = 2 \) which is a contradiction of \( T(b_{i+4}) = 1 \) and \( T(a_{i+4}) = T(b_{i+1}) = T(b_{i+7}) = 2 \). \( \square \)
Perfect 2-colorings of $GP(n,3)$ with the matrix $A_4$:

We show that just the graphs $GP(4m,3)$ among the graphs $GP(n,3)$ have a perfect 2-coloring with the matrix $A_4$.

**Theorem 3.3.** All the graphs $GP(n,3)$, where $4 \mid n$, have a perfect 2-coloring with the matrix $A_4$. Also, there are no perfect 2-coloring of $GP(n,3)$, where $4 \nmid n$, with this matrix.

**Proof.** For the first part, consider the mapping $T : V(GP(4m,3)) \to \{1,2\}$ by

$$T(a_{4i}) = T(b_{4i+2}) = 1,$$

$$T(b_{4i}) = T(a_{4i+1}) = T(b_{4i+1}) = T(a_{4i+2}) = T(a_{4i+3}) = T(b_{4i+3}) = 2.$$  

for $i \geq 0$. It can be easily checked that the given mapping is a perfect 2-coloring with the matrix $A_4$.

To prove the second part, contrary to our claim, suppose that there is a perfect 2-coloring of $GP(n,3)$ with the matrix $A_4$, say $T$. With no restriction of generality, let $T(a_0) = 1$. It follows that $T(a_1) = T(b_0) = T(a_{n-1}) = T(b_{n-1}) = 2$. From $T(a_1) = 2$ and $T(a_0) = 1$ we get $T(b_1) = T(a_2) = 2$. Now, we should have two cases below.

**Case 1.** $T(b_2) = 1$. It immediately gives

$$T(a_{4i}) = T(b_{4i+2}) = 1,$$

$$T(b_{4i}) = T(a_{4i+1}) = T(b_{4i+1}) = T(a_{4i+2}) = T(a_{4i+3}) = T(b_{4i+3}) = 2.$$  

for $i \geq 0$. It clearly gives $n = 4m$ which is a contradiction of $4 \nmid n$.

**Case 2.** $T(b_2) = 2$. It immediately gives $T(a_3) = 1$ and $T(b_3) = T(a_4) = 2$. From $T(a_4) = 2$ and $T(a_3) = 1$, we get $T(b_4) = T(a_5) = 2$. Then, from $T(a_2) = T(b_2) = T(b_{n-1}) = 2$, we have $T(b_5) = 1$. So, we immediately conclude that $T(a_6) = 2$. Now, from $T(b_0) = T(b_3) = 2$ and $T(a_3) = 1$, we have $T(b_6) = 2$. It gives $T(a_7) = 1$ and then $T(b_7) = 2$ which is a contradiction of $T(b_0) = T(b_4) = T(a_4) = 2$.  

Perfect 2-colorings of $GP(n,3)$ with the matrix $A_5$

Here, we show that just the graphs $GP(5m,3)$, where $m \in \mathbb{N}$, among the graphs $GP(n,3)$ have a perfect 2-coloring with the matrix $A_5$.

**Theorem 3.4.** The graphs $GP(5m,5t+2)$ and $GP(5m,5t+3)$, where $t \geq 0$, have a perfect 2-coloring with the matrix $A_5$. $GP(n,k)$ graphs for $n$ such that $5 \nmid n$, have no perfect colorings with the matrix $A_5$.

**Proof.** For the first part, consider the mapping $T : V(GP(5m,5t+2)) \to \{1,2\}$ by

$$T(a_{5i}) = T(a_{5i+2}) = T(a_{5i+3}) = T(b_{5i}) = T(b_{5i+1}) = T(b_{5i+4}) = 2,$$

$$T(a_{5i+1}) = T(a_{5i+4}) = T(b_{5i+2}) = T(b_{5i+3}) = 1,$$

for $i \geq 0$. It can be easily checked that the given mapping gives a perfect 2-coloring with the matrix $A_5$. The mapping $T : V(GP(5m,5t+3)) \to \{1,2\}$ by the exactly above definition is also a perfect 2-coloring with the matrix $A_5$. Moreover, the second part can be proved by Proposition 3.1.  

\[\square\]
Remark 3.1. There is no information for the cases $GP(5m, 5t)$, $GP(5m, 5t + 1)$ and $GP(5m, 5t + 4)$ in Theorem 3.4. So, we leave these cases as an open problem.

Finally, we summerize the obtained results from enumerating the parameter matrices of $GP(n, k)$ in the following table.

|   | $GP(n, 2)$ | $GP(n, 3)$ | $GP(n, k)$ |
|---|-----------|-----------|-----------|
| $A_1$ | all graphs | all graphs | all graphs |
| $A_2$ | just $GP(3m, 2)$ | no graphs | ? |
| $A_3$ | no graphs | just $GP(2m, 3)$ | ? |
| $A_4$ | no graphs | just $GP(4m, 3)$ | ? |
| $A_5$ | just $GP(5m, 2)$ | just $GP(5m, 3)$ | ? |
| $A_6$ | no graphs | just $GP(2m, 3)$ | just $GP(2m, 2t + 1)$ |

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References

[1] A. Mehdi and K. Hamed, Perfect 2-colorings of generalized Petersen graphs, *Proc. Indian Acad. Sci.* **126** (3) (2016), 289–294.

[2] A. Mehdi, K. Hamed, and S. Sajjad, Perfect 3-colorings of $GP(5, 2)$, $GP(6, 2)$, and $GP(7, 2)$, *Journal of the Indonesian Mathematical Society* **24** (2) 2018, 47–53.

[3] A.M. Hadi and K. Hamed, Perfect 2-colorings of Platonic graphs, *Proc. Iranian Journal of Nonlinear Analysis and Application* **8** (2) (2017), 29–35.

[4] S.V. Avgustinovich and I. Yu. Mogilnykh, Perfect 2-colorings of Johnson graphs $J(6, 3)$ and $J(7, 3)$, *Lecture Notes in Computer Science* **5228** (2008), 11–19.

[5] S.V. Avgustinovich and I. Yu. Mogilnykh, Perfect colorings of the Johnson graphs $J(8, 3)$ and $J(8, 4)$ with two colors, *Journal of Applied and Industrial Mathematics* **5** (2011), 19–30.

[6] D.G. Fon-Der-Flaass, A bound on correlation immunity, *Siberian Electronic Mathematical Reports Journal* **4** (2007), 133–135.

[7] D.G. Fon-Der-Flaass, Perfect 2-colorings of a hypercube, *Siberian Mathematical Journal*, **4** (2007), 923–930.
[8] D.G. Fon-der-Flaass, Perfect 2-colorings of a 12-dimensional Cube that achieve a bound of correlation immunity, *Siberian Mathematical Journal* **4** (2007), 292–295.

[9] A.L. Gavrilyuk and S.V. Goryainov, On perfect 2-colorings of Johnson graphs $J(v, 3)$, *Journal of Combinatorial Designs* **21** (2013), 232–252.

[10] C. Godsil, Compact graphs and equitable partitions, *Linear Algebra and Its Application* **255** (1997), 259–266