THE SINGULAR SUPPORTS OF IC SHEAVES ON QUASIMAPS' SPACES ARE IRREDUCIBLE

MICHAEL FINKELBERG, ALEXANDER KUZNETSOV, AND IVAN MIRKOVIĆ

1. Introduction

1.1. Let $C$ be a smooth projective curve of genus 0. Let $B$ be the variety of complete flags in an $n$-dimensional vector space $V$. Given an $(n-1)$-tuple $\alpha \in \mathbb{N}[I]$ of positive integers one can consider the space $Q_\alpha$ of algebraic maps of degree $\alpha$ from $C$ to $B$. This space is noncompact. Some remarkable compactifications $Q_\alpha^D$ (Quasimaps), $Q_\alpha^L$ (Quasiflags) of $Q_\alpha$ were constructed by Drinfeld and Laumon respectively. In [Ku] it was proved that the natural map $\pi : Q_\alpha^L \to Q_\alpha^D$ is a small resolution of singularities. The aim of the present note is to study the singular support of the Goresky-MacPherson sheaf $IC_\alpha$ on the Quasimaps' space $Q_\alpha^D$.

Namely, we prove that this singular support $SS(IC_\alpha)$ is irreducible. The proof is based on the factorization property of Quasimaps' space and on the detailed analysis of Laumon’s resolution $\pi : Q_\alpha^L \to Q_\alpha^D$.

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This note is a sequel to [Ku] and [FK]. In fact, the local geometry of $Q_\alpha^D$ was the subject of [Ku]; the global geometry of $Q_\alpha^D$ was the subject of [FK], while the microlocal geometry of $Q_\alpha^D$ is the subject of the present work. We will freely refer the reader to [Ku] and [FK].

2. Reductions of the main theorem

2.1. Notations.

2.1.1. We choose a basis $\{v_1, \ldots, v_n\}$ in $V$. This choice defines a Cartan subgroup $H \subset G = SL(V) = SL_n$ of matrices diagonal with respect to this basis, and a Borel subgroup $B \subset G$ of matrices upper triangular with respect to this basis. We have $B = G/B$.

Let $I = \{1, \ldots, n-1\}$ be the set of simple coroots of $G = SL_n$. Let $R^+$ denote the set of positive coroots, and let $2p = \sum_{\theta \in R^+} \theta$. For $\alpha = \sum_{i \in I} a_i \in \mathbb{N}[I]$ we set $|\alpha| := \sum a_i$. Let $X$ be the lattice of weights of $G, H$. Let $X^+ \subset X$ be the set of dominant (with respect to $B$) weights. For $\lambda \in X^+$ let $V_\lambda$ denote the irreducible representation of $G$ with the highest weight $\lambda$.

Recall the notations of [Ku] concerning Kostant’s partition function. For $\gamma \in \mathbb{N}[I]$ a Kostant partition of $\gamma$ is a decomposition of $\gamma$ into a sum of positive coroots with multiplicities. The set of Kostant partitions of $\gamma$ is denoted by $\mathcal{K}(\gamma)$.

There is a natural bijection between the set of pairs $1 \leq q \leq p \leq n-1$ and $R^+$, namely, $(p, q)$ corresponds to $i_q + i_{q+1} + \ldots + i_p$. Thus a Kostant partition $\kappa$ is given by a collection of nonnegative integers $(\kappa_{p,q}), 1 \leq q \leq p \leq n-1$. Following loc. cit. (9) we define a collection $\mu(\kappa)$ as follows: $\mu_{p,q} = \sum_{r \leq q \leq p \leq s} K_{s,r}$.

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Recall that for $\gamma \in \mathbb{N}[I]$ we denote by $\Gamma(\gamma)$ the set of all partitions of $\gamma$, i.e. multisubsets (subsets with multiplicities) $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ of $\mathbb{N}[I]$ with $\sum_{r=1}^k \gamma_r = \gamma, \ \gamma_r > 0$ (see e.g. [Ku], 1.3).

The configuration space of colored effective divisors of multidegree $\gamma$ (the set of colors is $I$) is denoted by $C_\gamma$. The diagonal stratification $C_\gamma = \bigsqcup_{\Gamma \in \Gamma(\gamma)} C_\gamma^\Gamma$ was introduced e.g. in loc. cit. Recall that for $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ we have $\dim C_\gamma^\Gamma = k$.

2.1.2. For the definition of Laumon’s Quasiflags’ space $Q^L_\alpha$ the reader may consult [La] 4.2, or [Ku] 1.4. It is the space of complete flags of locally free subsheaves $0 \subset E_1 \subset \cdots \subset E_{n-1} \subset V \otimes \mathcal{O}_C =: V$ such that $\text{rank}(E_k) = k$, and $\text{deg}(E_k) = -a_k$.

It is known to be a smooth projective variety of dimension $2|\alpha| + \dim B$.

2.1.3. For the definition of Drinfeld’s Quasimaps’ space $Q^D_\alpha$ the reader may consult [Ku] 1.2. It is the space of collections of invertible subsheaves $L_\lambda \subset V_\lambda \otimes \mathcal{O}_C$ for each dominant weight $\lambda \in X^+$ satisfying Plücker relations, and such that $\text{deg} L_\lambda = -\langle \lambda, \alpha \rangle$.

It is known to be a (singular, in general) projective variety of dimension $2|\alpha| + \dim B$.

The open subspace $Q_\alpha \subset Q^D_\alpha$ of genuine maps is formed by the collections of line subbundles (as opposed to invertible subsheaves) $L_\lambda \subset V_\lambda \otimes \mathcal{O}_C$. In fact, it is an open stratum of the stratification by the type of degeneration of $Q^D_\alpha$ introduced in [Ku] 1.3:

$$Q^D_\alpha = \bigsqcup_{\Gamma \in \Gamma(\alpha - \beta)} \mathcal{D}_\alpha^\beta, \Gamma$$

We have $\mathcal{D}_\alpha^\beta = Q_\alpha$, and $\mathcal{D}_\alpha^\beta, \Gamma = Q_\beta \times C_\Gamma^\beta$ (see loc. cit. 1.3.5).

The space $Q^D_\alpha$ is naturally embedded into the product of projective spaces

$$\mathbb{P}_\alpha = \prod_{1 \leq p \leq n-1} \mathbb{P} (\text{Hom}(\mathcal{O}_C(-\langle \omega_p, \alpha \rangle), V_{\omega_p} \otimes \mathcal{O}_C))$$

and is closed in it (see loc. cit. 1.2.5). Here $\omega_p$ stands for the fundamental weight dual to the coroot $i_p$.

The fundamental representation $V_{\omega_p}$ equals $\Lambda^p V$.

2.2. We will study the characteristic cycle of the Goresky-MacPherson perverse sheaf (or the corresponding regular holonomic $D$-module) $IC_\alpha$ on $Q^D_\alpha$. As $Q^D_\alpha$ is embedded into the smooth space $\mathbb{P}_\alpha$, we will view this characteristic cycle $SS(IC_\alpha)$ as a Lagrangian cycle in the cotangent bundle $T^* \mathbb{P}_\alpha$. A priori we have the following equality:

$$SS(IC_\alpha) = T^*_{\mathbb{P}_\alpha} \mathbb{P}_\alpha + \sum_{\beta < \alpha} m_\beta^\alpha T^*_{\mathcal{D}_\alpha^\beta, \Gamma} \mathbb{P}_\alpha,$$

where $m_\beta^\alpha$ counts the closures of conormal bundles with multiplicities.

**Theorem.** $SS(IC_\alpha) = T^*_{\mathbb{P}_\alpha} \mathbb{P}_\alpha$ is irreducible.

In the following subsections we will reduce the Theorem to a statement about geometry of Laumon’s resolution.
2.3. We fix a coordinate \( z \) on \( C \) identifying it with the standard \( \mathbb{P}^1 \). We denote by \( Q_\infty^\infty \subset Q_\alpha^D \) the open subspace formed by quasimaps which are genuine maps in a neighbourhood of the point \( \infty \in C \). In other words, \((L_\lambda \subset V_\lambda \otimes \mathcal{O}_C)_{\lambda \in X^+} \in Q_\infty^\infty \) iff for each \( \lambda \) the invertible subsheaf \( L_\lambda \subset V_\lambda \otimes \mathcal{O}_C \) is a line subbundle in some neighbourhood of \( \infty \in C \).

Evidently, \( Q_\infty^\infty \) intersects all the strata \( \mathcal{D}_{\beta, \Gamma} \). Thus it suffices to prove the irreducibility of the singular support of Goresky-MacPherson sheaf of \( Q_\infty^\infty \).

There is a well-defined map of evaluation at \( \infty \in C \):

\[ \Upsilon_\alpha : Q_\alpha^\infty \rightarrow B \]

It is compatible with the stratification of \( Q_\alpha^\infty \) and realizes \( Q_\alpha^\infty \) as a (stratified) fibre bundle over \( B \). In effect, \( G \) acts naturally both on \( Q_\alpha^\infty \) (preserving stratification) and on \( B \); the map \( \Upsilon_\alpha \) is equivariant, and \( B \) is homogeneous. We denote the fiber \( \Upsilon_\alpha^{-1}(B) \) over the point \( B \in B \) by \( Z_\alpha \).

It inherits the stratification

\[ Z_\alpha = \bigcup_{\beta \leq \alpha} Z \mathcal{D}_{\alpha}^{\beta, \Gamma} \]

from \( Q_\alpha^\infty \) and \( Q_\alpha^D \). It is just the transversal intersection of the fiber \( \Upsilon_\alpha^{-1}(B) \) with the stratification of \( Q_\alpha^\infty \). As in \[ K \] 1.3.5 we have \( Z \mathcal{D}_{\alpha}^{\beta, \Gamma} \sim \rightarrow Z_\beta \times (\mathbb{C} - \infty)^{\Gamma_{\beta}} \).

Hence it suffices to prove the irreducibility of the singular support \( SS(IC(Z_\alpha)) \) of Goresky-MacPherson sheaf \( IC(Z_\alpha) \) of \( Z_\alpha \).

2.4. Factorization. The Theorem 6.3 of \[ FM \] admits the following immediate Corollary. Let \((\phi_{\beta}, \gamma_1 x_1, \ldots, \gamma_k x_k) = \phi_{\alpha} \in Z_\beta \times (\mathbb{C} - \infty)^{\alpha - \beta} = Z \mathcal{D}_{\alpha}^{\beta, \Gamma} \subset Z_\alpha \). Consider also the points \((\phi_r, \gamma_r x_r) = \phi_{\gamma_r} \in Z_0 \times (\mathbb{C} - \infty)^{\gamma_r} \subset Z_{\gamma_r}, 1 \leq r \leq k \).

**Proposition.** There is an analytic open neighbourhood \( U_\alpha \) (resp. \( U_{\beta}, U_{\gamma_r}, 1 \leq r \leq k \)) of \( \phi_\alpha \) (resp. \( \phi_\beta, \phi_{\gamma_r}, 1 \leq r \leq k \)) in \( Z_\alpha \) (resp. \( Z_\beta, Z_{\gamma_r}, 1 \leq r \leq k \)) such that

\[ U_\alpha \sim \rightarrow U_{\beta} \times \prod_{1 \leq r \leq k} U_{\gamma_r}. \]

\( \square \)

Recall the nonnegative integers \( m_{\alpha}^{\beta, \Gamma} \) introduced in \[ 2.2 \]. The Proposition implies the following Corollary.

**Corollary.** \( m_{\alpha}^{\beta, \Gamma} = \prod_{1 \leq r \leq k} m_{\gamma_r}^{0, \{ \gamma_r \}} \). \( \square \)

Thus to prove that all the multiplicities \( m_{\alpha}^{\beta, \Gamma} \) vanish, it suffices to check the vanishing of \( m_{\gamma_r}^{0, \{ \gamma_r \}} \) for arbitrary \( \gamma > 0 \).

2.5. It remains to prove that the conormal bundle \( T_{\mathbb{P}_{}^0, (\{ \gamma \})}^{*} \subset \mathbb{P}_{}^0 \) to the closed stratum of \( Q_\gamma \) enters the singular support \( SS(IC_\alpha) \) with multiplicity 0. To this end we choose a point \((B, \gamma_0) = \phi \in B \times C = Q_0 \times C_{\{ \gamma_0 \}} = \mathcal{D}_{\gamma_0}^{0, \{ \gamma_0 \}} \subset Q_\gamma \subset \mathbb{P}_{} \). We also choose a sufficiently generic meromorphic function \( f \) on \( \mathbb{P}_{} \) regular around \( \phi \) and vanishing on \( \mathcal{D}_{\gamma_0}^{0, \{ \gamma_0 \}} \). According to the Proposition 8.6.4 of \[ KS \], the multiplicity in question is 0 iff \( \Phi_f (IC_\gamma)_\phi = 0 \), i.e. the stalk of vanishing cycles sheaf at the point \( \phi \) vanishes.

To compute the stalk of vanishing cycles sheaf we use the following argument, borrowed from \[ BF \] §1. As \( \pi : Q_\gamma^L \rightarrow Q_\gamma^D \) is a small resolution of singularities, up to a shift, \( IC_\alpha = \pi_* \mathcal{Q} \). By the proper base change, \( \Phi_f \pi_* \mathcal{Q} = \pi_* \Phi_f \pi_* \mathcal{Q} \). So it suffices to check that \( \Phi_f \pi_* \mathcal{Q}|_{\pi^{-1}(\phi)} = 0 \).
Let us denote the differential of the function $f$ at the point $\phi$ by $\xi$ so that $(\phi, \xi) \in T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$. Then the support of $\Phi_{f^*\overline{Q}|_{\pi^{-1}(\phi)}}$ is \textit{a priori} contained in the microlocal fiber over $(\phi, \xi)$ which we define presently.

2.5.1. \textit{Definition.} Let $\varpi : A \to B$ be a map of smooth varieties. For $a \in A$ let $d_a^*\varpi : T^*_{\varpi(a)} B \to T^*_a A$ denote the codifferential, and let $(b, \eta)$ be a point in $T^* B$. Then the \textit{microlocal fiber} of $\varpi$ over $(b, \eta)$ is defined to be the set of points $a \in \varpi^{-1}(b)$ such that $d_a^*\varpi(\eta) = 0$.

2.5.2. Thus we have reduced the Theorem 2.2 to the following Proposition.

\textbf{Proposition.} For a sufficiently generic $\xi$ such that $(\phi, \xi) \in T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$, the microlocal fiber of Laumon’s resolution $\pi$ over $(\phi, \xi)$ is empty. Equivalently, the cone $\cup_{E_* \in \pi^{-1}(\phi)} \text{Ker}(d^*_{E_*} \pi)$ is a proper subvariety of the fiber of $T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$ at $\phi$.

2.6. \textbf{Piecification of a simple fiber.} The fiber $\pi^{-1}(\phi)$ was called the \textit{simple fiber} in \cite{Ku} §2. It was proved in loc. cit. 2.3.3 that $\pi^{-1}(\phi)$ is a disjoint union of (pseudo)affine spaces $\mathcal{G}(\mu(\kappa))$ where $\kappa$ runs through the set $\mathcal{A}(\gamma)$ of Kostant partitions of $\gamma$ (for the notation $\mu(\kappa)$ see 2.1.1 or \cite{Ku} (9)). Another way to parametrize these pseudoaffine pieces was introduced in \cite{FK} 2.11. Let us recall it here.

We define nonnegative integers $c_p, 1 \leq p \leq n - 1$, so that $\gamma = \sum_{p=1}^{n-1} c_p p$.

2.6.1. \textbf{Definition.} $\mathcal{D}(\gamma)$ is the set of collections of nonnegative integers $(d_{p,q})_{1 \leq q \leq p \leq n - 1}$ such that

a) For any $1 \leq q \leq p \leq r \leq n - 1$ we have $d_{r,q} \leq d_{p,q}$;

b) For any $1 \leq p \leq n - 1$ we have $\sum_{q=1}^{p} d_{p,q} = c_p$.

2.6.2. \textbf{Lemma.} The correspondence $\kappa = (\kappa_{p,q})_{1 \leq q \leq p \leq n - 1} \mapsto (d_{p,q} := \sum_{r=p}^{n-1} \kappa_{r,q})_{1 \leq q \leq p \leq n - 1}$ defines a bijection between $\mathcal{A}(\gamma)$ and $\mathcal{D}(\gamma)$.

2.6.3. Using the above Lemma we can rewrite the parametrization of the pseudoaffine pieces of the simple fiber as follows:

$$\pi^{-1}(\phi) = \bigcup_{\mathfrak{d} \in \mathcal{D}(\gamma)} \mathcal{G}(\mathfrak{d})$$

In these terms the dimension formula of \cite{Ku} 2.3.3 reads as follows: for $\mathfrak{d} = (d_{p,q})_{1 \leq q \leq p \leq n - 1}$ we have $\dim \mathcal{G}(\mathfrak{d}) = \sum_{1 \leq q \leq p \leq n - 1} d_{p,q}$.

Note also that $\sum_{1 \leq q \leq p \leq n - 1} d_{p,q} = \sum_{1 \leq p \leq n - 1} c_p = |\gamma|$.

2.7. \textbf{Proposition.} For arbitrary $\mathfrak{d} = (d_{p,q})_{1 \leq q \leq p \leq n - 1} \in \mathcal{D}(\gamma)$ and arbitrary quasiflag $E_* \in \mathcal{G}(\mathfrak{d}) \subset \pi^{-1}(\phi)$ we have $\dim \text{Ker}(d^*_{E_*} \pi) < \sum_{1 \leq p \leq n - 1} d_{p,q} + \sum_{1 \leq q \leq p \leq n - 1} d_{p,q} - 1$.

This Proposition implies the Proposition 2.5.2 straightforwardly. In effect, $\text{codim} \text{Ker}(d^*_{E_*} \pi) = \dim \overline{Q} - \dim \text{Ker}(d^*_{E_*} \pi) > 2|\gamma| + \dim B - \sum_{1 \leq p \leq n - 1} d_{p,q} - \sum_{1 \leq q \leq p \leq n - 1} d_{p,q} + 1 = \dim B + 1 + \sum_{1 \leq q \leq p \leq n - 1} d_{p,q}$.

Hence the subspace $\text{Ker}(d^*_{E_*} \pi) \subset T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$ has codimension greater than $\dim B + 1 + \sum_{1 \leq q \leq p \leq n - 1} d_{p,q}$.

Recall that $\dim \overline{Q}^{0,1}(\gamma) = \dim B + 1$. Hence the codimension of $\text{Ker}(d^*_{E_*} \pi) \cap T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$ in the fiber of $T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$ at $\phi$ is greater than $\sum_{1 \leq q \leq p \leq n - 1} d_{p,q} = \dim \mathcal{G}(\mathfrak{d})$. Hence the cone $\cup_{E_* \in \mathcal{G}(\mathfrak{d})} \text{Ker}(d^*_{E_*} \pi)$ is a proper subvariety of the fiber of $T^*_{\overline{D}_\gamma(\gamma)} \mathbb{P}_\gamma$ at $\phi$. 


3.1.2. Consider a point $L$ representing of $\Omega$ in coherent sheaves on $C$. The union of these proper subvarieties over $I$ is again a proper subvariety of the fiber of $T^*_{D^\gamma_1(\gamma)\mathbb{P}\gamma}$ at $\phi$ which concludes the proof of the Proposition 2.5.2.

2.8. Fixed points. It remains to prove the Proposition 2.7. To this end recall that the Cartan group $H$ acts on $V$ and hence on $Q^L_\alpha$. The group $\mathbb{C}^*$ of dilations of $C = \mathbb{P}^1$ preserving 0 and $\infty$ also acts on $Q^L_\alpha$ commuting with the action of $H$. Hence we obtain the action of a torus $T := H \times \mathbb{C}^*$ on $Q^L_\alpha$.

It preserves the simple fiber $\pi^{-1}(\phi)$ and its pseudoaffine pieces $\mathcal{G}(\delta)$, $\delta \in D(\gamma)$, for evident reasons. It was proved in [FK] 2.12 that each piece $\mathcal{G}(\delta)$, $\delta = (d_{p,q})_{1 \leq q \leq \gamma - 1}$ contains exactly one $T$-fixed point $\delta(\delta) = (E_1, \ldots, E_{n-1})$. Here

$$
E_1 = E_{1,1}, \\
E_2 = E_{2,1} \oplus E_{2,2}, \\
\vdots \\
E_{n-1} = E_{n-1,1} \oplus E_{n-1,2} \oplus \ldots \oplus E_{n-1,n-1}
$$

and $E_{p,q} = \mathcal{O}(-d_{p,q}) \subset \mathcal{O}_{V}$ with quotient sheaf $\frac{\mathcal{O}}{\mathcal{O}(-d_{p,q})}$ concentrated at $0 \in C$.

2.8.1. Now the $T$-action contracts $\mathcal{G}(\delta)$ to $\delta(\delta)$. Since the map $\pi$ is $T$-equivariant, and the dimension of $\text{Ker}(d_E \pi)$ is lower semicontinuous, the Proposition 2.7 follows from the next one.

**Key Proposition.** For arbitrary $\delta = (d_{p,q})_{1 \leq q \leq \gamma - 1} \in D(\gamma)$ ($\gamma \neq 0$) we have $\dim \text{Ker}(d_{\delta(\delta)} \pi) < \sum_{1 \leq p \leq \gamma - 1} d_{p,p} + \sum_{1 \leq q \leq \gamma - 1} d_{p,q} - 1$.

The proof will be given in the next section.

2.8.2. **Remark.** In general, the pieces $\mathcal{G}(\delta)$ of the simple fiber are not equisingular, i.e. $\dim \text{Ker}(d_E \pi)$ is not constant along a piece. The simplest example occurs for $G = SL_3$, $\gamma = 2i_1 + 2i_2$. Then the simple fiber is a singular 2-dimensional quadric. Its singular point is the fixed point of the 1-dimensional piece $\mathcal{G}(\delta)$ where $d_{1,1} = 2, d_{2,1} = d_{2,2} = 1$. At this point we have $\dim \text{Ker}(d_{\delta(\delta)} \pi) = 3$ while at the other points in this piece we have $\dim \text{Ker}(d_E \pi) = 2$.

3. The proof of the Key Proposition

3.1. **Tangent spaces.** Let $\Omega$ be the following quiver: $\Omega = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n - 1$. Thus the set of vertices coincides with $I$. A quasiflag $(E_1 \rightarrow E_2 \rightarrow \ldots \rightarrow E_{n-1} \subset \mathcal{V}) \in Q^L_\alpha$ may be viewed as a representation of $\Omega$ in the category of coherent sheaves on $C$. If we denote the quotient sheaf $\mathcal{V}/E_p$ by $Q_p$, $1 \leq p \leq n - 1$, we have another representation of $\Omega$ in coherent sheaves on $C$, namely,

$$
Q_\bullet := (Q_1 \rightarrow Q_2 \rightarrow \ldots \rightarrow Q_{n-1})
$$

3.1.1. **Exercise.** $T_{E_\bullet} Q^L_\gamma = \text{Hom}_{Q_\bullet}(E_\bullet, Q_\bullet)$ where $\text{Hom}_{Q_\bullet}(?, ?)$ stands for the morphisms in the category of representations of $\Omega$ in coherent sheaves on $C$.

3.1.2. **Exercise.** $T_{L_\bullet} \mathbb{P}_\gamma = \prod_{p=1}^{n-1} \text{Hom}(L_p, V_{\omega_p} \otimes \mathcal{O}_C/L_p)$.
3.1.3. Recall that for \( E_\bullet \in Q^L_\gamma \) we have \( \pi(E_\bullet) = \mathcal{L}_\bullet \in \mathbb{P}_\gamma \) where \( \mathcal{L}_p = \Lambda^p E_p \) for \( 1 \leq p \leq n - 1 \).

Exercise. For \( h_\bullet = (h_1, \ldots, h_{n-1}) \in T_{E_\bullet} Q^L_\gamma \) we have \( d_{E_\bullet} \pi(h_\bullet) = (\Lambda^1 h_1, \Lambda^2 h_2, \ldots, \Lambda^{n-1} h_{n-1}) \in T_{E_\bullet} \mathbb{P}_\gamma \).

3.2. From now on we fix \( \gamma > 0, \delta \in \mathcal{D}(\gamma), \delta(\delta) =: E_\bullet \). To unburrden the notations we will denote the tangent space \( T_{E_\bullet} Q^L_\gamma \) by \( T \). Since \( Q^L_\gamma \) is a smooth \((2|\gamma| + \dim B)\)-dimensional variety it suffices to find a subspace \( N \subset T \) of dimension

\[
2|\gamma| + \dim B - \sum_{1 \leq p \leq n-1} d_{p,p} - \sum_{1 \leq q \leq n-1} d_{p,q} + 1 = \sum_{1 \leq q < p \leq n-1} (d_{p,q} + 1) + 1
\]

such that \( d_{E_\bullet} \pi |_N \) is injective.

3.3. Let \( N_0 = \bigoplus_{n-1 \geq p > q \geq 1} \text{Hom}(\mathcal{O}(-d_{p,q}), \mathcal{O}) \). We have \( \dim N_0 = \sum_{n-1 \geq p > q \geq 1} (d_{p,q} + 1) \).

Recall that we have canonically \( T = \text{Hom}(E_\bullet, Q_\bullet) \), where

\[
Q_p = \mathcal{V}/E_p = \left( \bigoplus_{q=1}^p \left( \frac{\mathcal{O}}{\mathcal{O}(-d_{p,q})} \right) v_q \right) \oplus \left( \bigoplus_{q=p+1}^n \mathcal{O} v_q \right).
\]

3.4. Let us define a map \( \nu_0 : N_0 \rightarrow T \) assigning to an element \( (f_{p,q}) \in N_0 \) a morphism \( \nu_0(f_{p,q}) := F \in \text{Hom}_i(E_\bullet, Q_\bullet) \) of graded coherent sheaves, where \( F|_{E_{p,q}} = \bigoplus_{r=p+1}^n F_{r,q}^{r,s} \) and

\[
F_{r,s}^{r,s} : E_{p,q} \rightarrow \mathcal{O}_{v_r} \subset Q_p \quad \text{is defined as the composition} \quad E_{p,q} \subset E_{r,q} = \mathcal{O}(-d_{r,q}) \xrightarrow{f_{r,q}} \mathcal{O}_{v_r}
\]

3.5. Lemma. The map \( F : E_\bullet \rightarrow Q_\bullet \) is a morphism of representations of the quiver \( \Omega \).

Proof. We need to check the commutativity of the following diagram

\[
\begin{array}{ccc}
E_p & \longrightarrow & E_{p'} \\
F \downarrow & & \downarrow F \\
Q_p & \longrightarrow & Q_{p'}
\end{array}
\]

Since \( E_p \) and \( Q_{p'} \) are canonically decomposed into the direct sum it suffices to note that for any \( q \leq p \leq p' < r \) the following diagram

\[
\begin{array}{ccc}
E_{p,q} & \longrightarrow & E_{p',q} & \longrightarrow & E_{r,q} \\
F_{r,s}^{r,s} \downarrow & & F_{r,s}^{r,s} \downarrow & & f_{r,q} \downarrow \\
\mathcal{O}_{v_r} & \longrightarrow & \mathcal{O}_{v_r} & \longrightarrow & \mathcal{O}_{v_r}
\end{array}
\]

commutes and for any \( q \leq p < r \leq p' \) the following diagram

\[
\begin{array}{ccc}
E_{p,q} & \longrightarrow & E_{r,q} & \longrightarrow & E_{p',q} \\
F_{r,q} \downarrow & & 0 \downarrow & & 0 \\
\mathcal{O}_{v_r} & \longrightarrow & \mathcal{O}_{v_r} & \longrightarrow & \left( \frac{\mathcal{O}}{\mathcal{O}(-d_{r,q})} \right) v_r
\end{array}
\]

commutes as well. \( \square \)
3.6. Let \( N_1 = \mathbb{C} \). Let \( p_0 = \min\{1 \leq p \leq n - 1 \mid d_{p, p} > 0\} \) and pick a non-zero element \( f \in \text{Hom}(\mathcal{O}(-d_{p, p}), \Omega_{\nu - d_{p, p}}) \). Define the map \( \nu : N_1 \to T \) by assigning to \( 1 \in N_1 \) the element \( \tilde{f} \in \text{Hom}_\Omega(E_\bullet, Q_\bullet) \) defined on \( E_{p, p} \) as the composition

\[
E_{p, p} = \mathcal{O}(-d_{p, p}) \xrightarrow{f} \frac{\mathcal{O}}{\mathcal{O}(-d_{p, p})} v_p \subset Q_p
\]

and with all other components equal to zero.

3.7. Let \( \mathcal{M}(r, d; \mathcal{V}) \) denote the space of rank \( r \) and degree \( d \) subsheaves in \( \mathcal{V} \).

3.7.1. Let \( E \subset \mathcal{V} \) be a rank \( k \) and degree \( d \) subsheaf in the vector bundle \( \mathcal{V} \). Let \( \mathcal{V}/E = \mathcal{T} \oplus \mathcal{F} \) be a decomposition of the quotient sheaf into the sum of the torsion \( \mathcal{T} \) and a locally free sheaf \( \mathcal{F} \). Consider the map \( \det : \mathcal{M}(r, d; \mathcal{V}) \to \mathcal{M}(1, d; \Lambda^k \mathcal{V}) \) sending \( E \) to \( \Lambda^k E \). Then the restriction of its differential \( d_E \det : T_E \mathcal{M}(r, d; \mathcal{V}) = \text{Hom}(E, \mathcal{V}/E) \to \text{Hom}(\Lambda^k E, \Lambda^k \mathcal{V}/\Lambda^k E) = T_{\Lambda^k E} \mathcal{M}(1, d; \Lambda^k \mathcal{V}) \) to the subspace \( \text{Hom}(E, \mathcal{F}) \subset \text{Hom}(E, \mathcal{V}/E) \) factors as \( \text{Hom}(E, \mathcal{F}) \cong \text{Hom}(\Lambda^k E, \Lambda^k \mathcal{V}/\Lambda^k E) \subset \text{Hom}(\Lambda^k E, \Lambda^k \mathcal{V}/\Lambda^k E) \). Therefore it is injective.

3.7.2. Let \( E = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{O}(-d) \) be a subsheaf in \( \mathcal{V} = \mathcal{O}^{\oplus n} \). Then the restriction of differential \( d_E \det \) to the subspace \( \text{Hom}(E, \mathcal{T}) \subset \text{Hom}(E, \mathcal{V}/E) \) is injective.

This immediately follows from the following fact. Let \( \hat{E} = \mathcal{O}^{\oplus r} \subset \mathcal{V} \) be the normalization of \( E \) in \( \mathcal{V} \), that is, the maximal vector subbundle \( \hat{E} \subset \mathcal{V} \) such that \( \hat{E}/E \) is torsion. Then \( \mathcal{T} = \hat{E}/E \cong \Lambda^k \hat{E}\Lambda^k E \subset \Lambda^k \mathcal{V}/\Lambda^k E \).

3.7.3. Clearly, the subsheaves \( \Lambda^k - 1 E \otimes \mathcal{F} \subset \Lambda^k \mathcal{V}/\Lambda^k E \) and \( \mathcal{T} \subset \Lambda^k \mathcal{V}/\Lambda^k E \) do not intersect.

3.7.4. It follows from 3.7.1 and 3.7.2 that the composition \( d_{E, \pi} \circ (\nu_0 \oplus \nu_1) : N_0 \oplus N_1 \to T_{\pi(E_\bullet)} \mathcal{P}_\gamma \) is injective, hence \( N := (\nu_0 \oplus \nu_1)(N_0 \oplus N_1) \subset T_{\pi(E_\bullet)} \mathcal{P}_\gamma \) enjoys the desired property. Namely, \( d_{E, \pi}|_N \) is injective, and \( \dim N = \sum_{1 \leq q \leq n-1} (d_{p, q} + 1) + 1 \).

This completes the proof of the Key Proposition 2.8.1 along with the Main Theorem 2.2. \( \square \)

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Independent Moscow University, 11 Bolshoj Vlasievskij pereulok, Moscow 121002 Russia

E-mail address: fnklberg@main.mccme.rssi.ru

Independent Moscow University, 11 Bolshoj Vlasievskij pereulok, Moscow 121002 Russia

E-mail address: sasha@ium.ips.ras.ru

Dept. of Mathematics and Statistics, University of Massachusetts at Amherst, Amherst MA 01003-4515, USA

E-mail address: mirkovic@math.umass.edu