FANO AND WEAK FANO HESSENBERG VARIETIES

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ABSTRACT. Regular semisimple Hessenberg varieties are smooth subvarieties of the flag variety, and their examples contain the flag variety itself and the permutohedral variety. We give a complete classification of Fano and weak Fano regular semisimple Hessenberg varieties in type A in terms of combinatorics of Hessenberg functions. In particular, we show that if the anti-canonical bundle of a regular semisimple Hessenberg variety is nef, then it is in fact nef and big.

1. INTRODUCTION

Hessenberg varieties in Lie type $A_{n-1}$ are subvarieties of the flag variety of nested linear subspaces of $\mathbb{C}^n$. They were introduced by De Mari-Procesi-Shayman \[10, 9\], and they have been studied from the perspective of geometry, representation theory, and combinatorics. For an $n \times n$ matrix $X$ and a Hessenberg function $h : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$, the Hessenberg variety associated with $X$ and $h$ is given as

$$\text{Hess}(X, h) := \{ V_\bullet \in \text{Fl}(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \text{ for all } 1 \leq i \leq n \},$$

where $\text{Fl}(\mathbb{C}^n)$ is the flag variety of $\mathbb{C}^n$ consisting of sequences $V_\bullet = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ of linear subspaces of $\mathbb{C}^n$ such that $\dim \mathbb{C} V_i = i$ for $1 \leq i \leq n$. If $S$ is an $n \times n$ regular semisimple matrix (i.e. an $n \times n$ matrix with $n$ distinct eigenvalues), then $\text{Hess}(S, h)$ is smooth, which is called a regular semisimple Hessenberg variety. There are two extremal examples of regular semisimple Hessenberg varieties: the flag variety itself and the permutohedral variety which is a toric variety associated with the fan consisting of the collection of Weyl chambers of type $A_{n-1}$. The flag variety is a Fano variety (see \[7, Propositions 1.4.1 and 2.2.8 (iv)\]), whereas the permutohedral variety is not except for very small ranks. However, the permutohedral variety is a weak Fano variety (\[5, 15\]). Here, a complex algebraic variety is said to be Fano (resp. weak Fano) if its anti-canonical bundle is ample (resp. nef and big). In this paper, we give a complete classification of Fano and weak Fano regular semisimple Hessenberg varieties in terms of the combinatorics of the Hessenberg functions.

For each $1 \leq k \leq n-1$, let $h_k : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, n\}$ be the Hessenberg function given by $h_k(i) = k + i$ for $1 \leq i \leq n - k$. This Hessenberg function is called the “$k$-banded form”, and $\text{Hess}(S, h_k)$ are the Hessenberg varieties studied in \[10\]. For example, $h_1$ gives the permutohedral variety $\text{Hess}(S, h_1)$, and $h_{n-1}$ gives the flag variety $\text{Hess}(S, h_{n-1}) = \text{Fl}(\mathbb{C}^n)$. The following theorem characterizes when a regular semisimple Hessenberg variety is Fano.

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Theorem A. Let $X = \text{Hess}(S, h)$ be a regular semisimple Hessenberg variety with $h(i) \geq i + 1$ for all $1 \leq i < n$. Then the following are equivalent:

(i) the anti-canonical bundle of $X$ is ample (that is, $X$ is Fano);
(ii) $h = h_k$ for some $k$ such that $\frac{n-1}{2} \leq k \leq n - 1$.

The permutohedral variety $\text{Hess}(S, h_1)$ is not Fano unless $n \leq 3$, but it is always weak Fano as we explained above. The next theorem characterizes when a regular semisimple Hessenberg variety is weak Fano.

Theorem B. Let $X = \text{Hess}(S, h)$ be a regular semisimple Hessenberg variety with $h(i) \geq i + 1$ for all $1 \leq i < n$. Then the following are equivalent:

(i) the anti-canonical bundle of $X$ is nef;
(ii) the anti-canonical bundle of $X$ is nef and big (that is, $X$ is weak Fano);
(iii) the inequality

\[ h(i) - h(i + 1) + 2 - h^*(n + 1 - i) + h^*(n - i) \geq 0 \]

holds for all $1 \leq i \leq n - 1$, where $h^*$ denotes the transpose of $h$.

For example, $\text{Hess}(S, h)$ for $h = (3, 3, 4, 4)$ is a weak Fano variety since $h = (3, 3, 4, 4)$ satisfies condition (iii) of Theorem B. Similarly, we obtain the following.

Corollary 1.1. For $1 \leq k \leq n-1$, the regular semisimple Hessenberg variety $\text{Hess}(S, h_k)$ with the $k$-banded form $h_k$ is a weak Fano variety.

To give the above classifications, we first compute the anti-canonical bundles of regular semisimple Hessenberg varieties explicitly, and we study their volumes by using the theory of line bundles over Richardson varieties. We note that the method of using Richardson varieties for computations of volumes of line bundles over Hessenberg varieties is motivated by Anderson-Tymoczko \cite{4} and Harada-Horiguchi-Masuda-Park \cite{13}.

Let us see some geometric application of Theorem B. By the Kawamata-Viehweg vanishing \cite{18} Theorem 4.3.1, we see that $H^i(Z, L) = 0$, $i > 0$, for a smooth weak Fano variety $Z$ and a nef line bundle $L$ over $Z$. Hence we obtain the following.

Corollary 1.2. Let $X = \text{Hess}(S, h)$ be a regular semisimple Hessenberg variety with $h(i) \geq i + 1$ for all $1 \leq i < n$. If $h$ satisfies condition (iii) in Theorem B, then the equality

\[ H^i(\text{Hess}(S, h), L_\mu) = 0 \]

holds for all $i > 0$ and dominant integral weights $\mu$; see Section 2.2 and Lemma 3.5 for more details on the line bundle $L_\mu$.

Moreover, if $h$ satisfies condition (iii) in Theorem B, then $\text{Hess}(S, h)$ is a smooth Mori dream space since Theorem B implies that it is a smooth log Fano variety (cf. \cite{20}). Hence, according to Postinghel-Urbinati \cite{21} Theorem 4.9, such $\text{Hess}(S, h)$ admits a Newton-Okounkov body with desirable properties. In particular, it follows by Anderson \cite{3} that there exists a toric degeneration of $\text{Hess}(S, h)$ for which we can apply Harada-Kaveh’s result \cite{14} Theorems A and B to ensure the existence of a completely integrable system on $\text{Hess}(S, h)$. For example, the flag variety $\text{Fl}(\mathbb{C}^n) = \text{Hess}(S, h_{n-1})$ and the permutohedral variety $\text{Hess}(S, h_1)$ have these properties. It would be interesting to find explicit completely integrable systems on $\text{Hess}(S, h_k)$ for $1 \leq k \leq n - 1$. 
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2. Basic definitions and notations

In this section, we recall some basic definitions and notations on Hessenberg varieties, which we will use throughout this paper.

2.1. Regular semisimple Hessenberg varieties. Let \( n \) be a positive integer, and we denote by \([n]\) the set \(\{1, 2, \ldots, n\}\). A function \( h: [n] \to [n] \) is called a Hessenberg function if it satisfies the following conditions:

(i) \( h(1) \leq h(2) \leq \cdots \leq h(n) \),

(ii) \( h(i) \geq i \) for all \( 1 \leq i \leq n \).

We frequently express this function by listing its values as \( h = (h(1), h(2), \ldots, h(n)) \). Also, we may think of it as the boundary path of the configuration of boxes on the square grid of size \( n \) which consists of boxes in the \( i \)-th row and the \( j \)-th column satisfying \( i \leq h(j) \) for \( i, j \in [n] \). For example, if \( n = 5 \) and \( h = (3, 4, 4, 5, 5) \), then the corresponding boundary path is drawn in Figure 1.

![Figure 1](image)

**Figure 1.** The boundary path corresponding to \( h = (3, 4, 4, 5, 5) \).

For an \( n \times n \) matrix \( X \) and a Hessenberg function \( h: [n] \to [n] \), the Hessenberg variety associated with \( X \) and \( h \) is defined to be

\[
\text{Hess}(X, h) := \{ V_{\bullet} \in Fl(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \text{ for all } 1 \leq i \leq n \},
\]

where \( Fl(\mathbb{C}^n) \) is the flag variety of \( \mathbb{C}^n \) consisting of sequences \( V_{\bullet} = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n) \) of linear subspaces of \( \mathbb{C}^n \) such that \( \dim \mathbb{C} V_i = i \) for \( 1 \leq i \leq n \). Let \( S \) be a complex \( n \times n \) regular semisimple matrix (i.e. a complex \( n \times n \) matrix with \( n \) distinct eigenvalues). Then \( \text{Hess}(S, h) \) is called a regular semisimple Hessenberg variety. It is known that \( \text{Hess}(S, h) \) is a smooth projective variety (\[\square\] \[\square\]).

In this paper, we always assume that

\[
h(i) \geq i + 1 \quad (1 \leq i < n)
\]
For example, if \( n \) is dominant integral weight if we can write \( \mu \), we will also denote by \( h \).

Figure 2. Since we are assuming that \( \alpha_{i,j} = x_i - x_j \) be the standard \((i,j)\)-th positive root. A weight \( \mu \) of \( T \) is called a dominant integral weight if we can write \( \mu = \sum_{i=1}^{n-1} a_i \xi_i \) with \( a_i \geq 0 \) for all \( 1 \leq i \leq n - 1 \). We denote by \( P_+ \) the semigroup of the dominant integral weights.

### 2.2. Line bundles over flag varieties

Let \( G = SL_n(\mathbb{C}) \) be the complex special linear group of degree \( n \). Let \( B \subseteq G \) be the Borel subgroup consisting of the upper-triangular matrices, and \( T \subseteq B \) the maximal torus of \( B \) consisting of the diagonal matrices. We may identify the flag variety \( Fl(\mathbb{C}^n) \) with \( G/B \) by sending \( gb \in G/B \) to the flag \( V_i = \sum_{j=1}^i \mathbb{C}g_j \), where \( g_j \) is the \( j \)-th column vector of \( g \). Let \( \mu : T \to \mathbb{C}^\times \) be a weight of \( T \). By composing this with the canonical projection \( B \to T \), we obtain a homomorphism \( \mu : B \to \mathbb{C}^\times \) which we also denote by \( \mu \). Let \( \mathbb{C}^*_\mu = \mathbb{C} \) be the 1-dimensional representation of \( B \) given by \( b \cdot z = \mu(b)z \). We denote by \( \mathbb{C}^*_\mu \) its dual representation. Since the quotient map \( p : G \to G/B \) is a principal \( B \)-bundle, we obtain the associated line bundle over \( Fl(\mathbb{C}^n) = G/B \):

\[
L_\mu := G \times_B \mathbb{C}^*_\mu.
\]

Namely, it is the quotient of the product \( G \times \mathbb{C} \) by the right \( B \)-action given by \((g,z) \cdot b = (gb, \mu^{-1}(b^{-1})z) = (gb, \mu(b)z) \) for \( b \in B \) and \((g,z) \in G \times \mathbb{C} \). For a subvariety \( Z \subseteq G/B \), we will also denote by \( L_\mu \) the restriction of \( L_\mu \) to \( Z \) by abusing notation.

For \( 1 \leq i \leq n \), let \( x_i \) be the weight of \( T \) which sends \( t = \text{diag}(t_1, \ldots, t_n) \in T \) to \( t_i \in \mathbb{C}^\times \). We identify weights of \( T \) as induced homomorphisms \( \text{Lie}(T) \to \text{Lie}(\mathbb{C}^\times) = \mathbb{C} \) to use the additive notation, e.g. \( x_1 + x_2 + \cdots + x_n = 0 \). The standard \( i \)-th fundamental weight of \( T \) is given by \( \xi_i = x_1 + x_2 + \cdots + x_i \) for \( 1 \leq i \leq n - 1 \). Also, for \( 1 \leq i < j \leq n \), let \( \alpha_{i,j} = x_i - x_j \) be the standard \((i,j)\)-th positive root. A weight \( \mu \) of \( T \) is called \( n \), then we have \( h^* = (2,4,5,5,5) \). See Figure 2. Since we are assuming that \( h \) satisfies (2.1), so does \( h^* \), that is, \( h^*(i) \geq i + 1 \) (1 \leq i < n). Set \( \xi_i = \sum_{1 \leq i < j \leq n} \alpha_{i,j} = \sum_{1 \leq i < j \leq n} (x_i - x_j) \).

### 3. Fano Hessenberg varieties

In this section, we describe the anti-canonical bundle of \( Hess(S,h) \) in terms of a line bundle over the flag variety \( Fl(\mathbb{C}^n) \), and give a proof of Theorem A which is stated in Section 1.1.

#### 3.1. The anti-canonical bundles of Hessenberg varieties

For a Hessenberg function \( h : [n] \to [n] \), let \( h^* : [n] \to [n] \) be the transpose of \( h \), that is,

\[
h^*(i) := |\{ k \in [n] \mid n + 1 - i \leq h(k) \}|.
\]

For example, if \( n = 5 \) and \( h = (3,4,4,5,5) \), then we have \( h^* = (2,4,5,5,5) \). See Figure 2. Since we are assuming that \( h \) satisfies (2.1), so does \( h^* \), that is, \( h^*(i) \geq i + 1 \) (1 \leq i < n). Set

\[
\xi_i := \sum_{1 \leq i < j \leq h(i)} \alpha_{i,j} = \sum_{1 \leq i < j \leq h(i)} (x_i - x_j).
\]
Lemma 3.1. The following equality holds:
\[
\xi_h = \sum_{i=1}^{n-1} \left( h(i) - h(i+1) + 2 - h^*(n+1-i) + h^*(n-i) \right) x_i.
\]

Proof. By definition, we have
\[
\xi_h = \sum_{1 \leq k < \ell \leq h(k)} (x_k - x_\ell) = \sum_{1 \leq k < \ell \leq h(k)} x_k - \sum_{1 \leq k < \ell \leq h(k)} x_\ell.
\]
For each \(1 \leq i \leq n\), we count the number of \(x_i\) appearing in the right-most expression. In the former summand, the number of \(x_i\) is \(|\{\ell \in [n] \mid i < \ell \leq h(i)\}|\) which is equal to \(h(i) - i\). In the latter summand, the number of \(x_i\) is \(|\{k \in [n] \mid k < i \leq h(k)\}|\) which is equal to \(h^*(n+1-i) - (n+1-i)\) by the definition of \(h^*\). Thus we obtain
\[
\xi_h = \sum_{i=1}^{n-1} \left( h(i) - h^*(n+1-i) + n + 1 - 2i \right) x_i
\]
\[
= \sum_{i=1}^{n-1} \left( h(i) - h^*(n+1-i) + n - 2i + h^*(1) \right) x_i,
\]
where we used \(x_n = -(x_1 + x_2 + \cdots + x_{n-1})\) for the second equality. Since we have \(x_i = \varpi_i - \varpi_{i-1}\) with the convention \(\varpi_0 = 0\), this means the desired equality. \(\square\)

Proposition 3.2. The anti-canonical bundle of \(\text{Hess}(S, h)\) is isomorphic to \(L_{\xi_h}\).

Proof. Since \(\text{Hess}(S, h)\) is a smooth projective variety which admits a torus action with finite fixed points \([9]\), the higher cohomology groups of the structure sheaf vanish \([8]\). This means that there is a natural isomorphism \(\text{Pic}(\text{Hess}(S, h)) \cong H^2(\text{Hess}(S, h); \mathbb{Z})\) so that algebraic line bundles \(L\) and \(L'\) over \(\text{Hess}(S, h)\) are isomorphic if and only if their first Chern classes coincide (see for instance \([1]\) Corollary 5.3).

For Hessenberg functions \(h: [n] \to [n]\) and \(h': [n] \to [n]\), we say \(h' \subseteq h\) if and only if \(h'(i) \leq h(i)\) for all \(1 \leq i \leq n\). This gives a partial order on the set of Hessenberg functions on \([n]\). We prove the claim by descending induction on \(h\) with respect to the partial order \(\subseteq\) given above.

When \(h = (n, n, \ldots, n)\), we know that \(\text{Hess}(S, h) = FL(\mathbb{C}^n) = G/B\). Writing \(g = \text{Lie}(G)\) and \(b = \text{Lie}(B)\), the tangent bundle of \(G/B\) is isomorphic to the vector bundle \(G \times^B (g/b)\), which is the quotient of the product \(G \times (g/b)\) by the \(B\)-action given by \(b \cdot (g, v) = (gb^{-1}, b \cdot v)\), where \(b \cdot v\) is induced from the adjoint action of \(B\) on \(g\). This means that the anti-canonical bundle of \(FL(\mathbb{C}^n)\) is given by the line bundle \(G \times^B \wedge^N (g/b)\),
where $N = \dim \mathbb{C} Fl(\mathbb{C}^n) = \dim \mathfrak{g}/\mathfrak{b}$. Thus it is isomorphic to $L_\xi$, where $\xi$ is the sum of all positive roots ([7, Proposition 2.2.7 (ii)]):

$$\xi = \sum_{1 \leq i < j \leq n} \alpha_{i,j}.$$ 

This verifies the case of $h = (n, n, \ldots, n)$.

Suppose by induction that the anti-canonical bundle of $\text{Hess}(S, h)$ is isomorphic to $L_\xi h$ for a Hessenberg function $h: [n] \to [n]$ satisfying condition (2.1). Let $p := \min\{j \in [n] \mid h(j) \geq j + 2\}$ if exists, and $h' \subset h$ the Hessenberg function given by

$$h'(i) = \begin{cases} h(i) & \text{if } i \neq p, \\ h(i) - 1 & \text{if } i = p. \end{cases}$$

Then $h'$ also satisfies condition (2.1), and $\text{Hess}(S, h')$ is a nonsingular subvariety of $\text{Hess}(S, h)$ with codimension 1. By [2, Lemma 5.2 and the proof of Lemma 8.11], the normal bundle of $\text{Hess}(S, h')$ in $\text{Hess}(S, h)$ is isomorphic to $L_{\alpha_{p,h}(p)}$. Denote by $\omega_h$ and $\omega_{h'}$ the canonical bundles of $\text{Hess}(S, h)$ and $\text{Hess}(S, h')$, respectively. Then the adjunction formula tells us that

$$\omega_{h'} \cong \omega_h|_{\text{Hess}(S,h')} \otimes L_{\alpha_{p,h}(p)}.$$ 

By the induction hypothesis, the dual of the line bundle in the right-hand side is isomorphic to $L_\xi h \otimes L^{-\alpha_{p,h}(p)} \cong L_\xi h'$, as desired. □

For $w \in \mathcal{S}_n$, we denote by

$$X_w \subseteq G/B \quad (\text{resp., } \Omega_w \subseteq G/B)$$

the Schubert variety (resp., the dual Schubert variety) associated with $w$, that is,

$$X_w = \overline{BwB/B} \quad (\text{resp., } \Omega_w = \overline{B^-wB/B}),$$

where $B^- \subseteq G$ is the Borel subgroup of lower-triangular matrices. Then we have

$$\dim_{\mathbb{C}}(X_w) = \dim_{\mathbb{C}}(G/B) - \dim_{\mathbb{C}}(\Omega_w) = \ell(w).$$

Let $s_1, s_2, \ldots, s_{n-1} \in \mathcal{S}_n$ be the simple transpositions, and $e \in \mathcal{S}_n$ the identity element. We call each $X_{s_i}$ ($1 \leq i \leq n - 1$) a Schubert curve.

Lemma 3.3. ([7 Proposition 1.4.3]) Let $\mu = \sum_{i=1}^{n-1} a_i \omega_i$ ($a_i \in \mathbb{Z}$), and $L_\mu$ the corresponding line bundle over $\text{Fl}(\mathbb{C}^n)$. Then

$$a_i = \int_{X_{s_i}} c_1(L_\mu) \quad (1 \leq i \leq n - 1).$$

For a complex algebraic variety $Y$, a line bundle $L$ over $Y$ is called nef (or numerically effective) if the intersection number with an arbitrary irreducible curve in $Y$ is non-negative. $L$ is called ample if the global sections of $L^{\otimes m}$ give an embedding of $Y$ into a projective space for a large enough integer $m > 0$. Note that if $L$ is ample, then $L$ is nef ([13, Example 1.4.5]). For the definitions of Fano varieties and weak Fano varieties, we refer Section [1].

Lemma 3.4. $\text{Hess}(S, h)$ contains all the Schubert curves.
Lemma 3.7. Let \( \mu \) be a weight of \( T \). Then the following hold.

1. \( L_\mu \) is ample on \( \text{Hess}(S, h) \) if and only if \( \mu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \omega_i \).
2. \( L_\mu \) is nef on \( \text{Hess}(S, h) \) if and only if \( \mu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \omega_i \).

Proof. (1) If \( \mu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \omega_i \), then we see by \([7\) the proof of Proposition 1.4.1\)] that \( L_\mu \) is ample on \( F_l(\mathbb{C}^n) \). Hence \( L_\mu \) is ample on \( \text{Hess}(S, h) \) by definition. Conversely, assume that \( L_\mu \) is ample on \( \text{Hess}(S, h) \). We know from Lemma 3.4 that \( \text{Hess}(S, h) \) contains all the Schubert curves \( X_{s_i} \). Hence, for all \( 1 \leq i \leq n-1 \), we have

\[
\int_{X_{s_i}} c_1(L_\mu) > 0
\]

by the Nakai-Moishezon-Kleiman criterion (see \([18\) Theorem 1.2.23\]). By Lemma 3.3, this means that \( \mu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \omega_i \).

(2) If \( \mu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \omega_i \), then we see that \( L_\mu \) is a nef line bundle over \( F_l(\mathbb{C}^n) \) (see \([18\) Example 1.4.4 (i)\)] and \([7\) the proof of Proposition 1.4.1\]). Hence \( L_\mu \) is nef on \( \text{Hess}(S, h) \) by definition. Conversely, suppose that \( L_\mu \) is nef on \( \text{Hess}(S, h) \). Then we have

\[
\int_{X_{s_i}} c_1(L_\mu) \geq 0
\]

for all \( 1 \leq i \leq n-1 \) by Lemma 3.4. Thus Lemma 3.3 shows that \( \mu \in \sum_{i=1}^{n-1} \mathbb{Z}_{\geq 0} \omega_i \). \( \square \)

Proposition 3.6 and Lemma 3.5 now implies the following claim which establishes the equivalence of (i) and (iii) in Theorem B stated in Section 4.

Proposition 3.6. The anti-canonical bundle of \( \text{Hess}(S, h) \) is nef if and only if the inequality

\[
d_i := h(i) - h(i+1) + 2 - h^*(n+1-i) + h^*(n-i) \geq 0
\]

holds for all \( 1 \leq i \leq n-1 \).

Motivated by Proposition 3.6, we say a Hessenberg function \( h: [n] \to [n] \) is nef if it satisfies the inequality \( 3.1 \) for all \( 1 \leq i \leq n-1 \). This definition implies the following property which we will use in Section 5.

Lemma 3.7. Suppose that \( h \) is nef. Then the following hold for all \( 1 \leq i \leq n-2 \).

1. If \( h(i) = h(i+1) < n \), then \( h(i+1) < h(i+2) \).
2. If \( h^*(i) = h^*(i+1) < n \), then \( h^*(i+1) < h^*(i+2) \).

\( \text{In fact, } X_{s_k} \text{ is the connected component of } \text{Hess}(S, h^{(k)}) \text{ containing the } T\text{-fixed point } s_k B/B. \)
Proof. (1) Suppose for a contradiction that \( h(i + 1) = h(i + 2) \). Then we have \( h(i) = h(i + 1) = h(i + 2) \) by the assumption, which implies that \( h^*(j + 1) > h^*(j) + 2 \) for \( j = n - h(i) \) since \( h^* \) is the transpose of \( h \). However, this is impossible since we have \( d_{h(i)} \geq 0 \). The same argument works for (2) by replacing \( h \) with \( h^* \).

3.2. Proof of Theorem A. In this subsection, we prove Theorem A which is stated in Section 1. For this purpose, we prepare the following two lemmas.

Lemma 3.8. Assume that \( \text{Hess}(S, h) \) is Fano. Then the inequalities \( h(i + 1) \leq h(i) + 1 \) and \( h^*(i + 1) \leq h^*(i) + 1 \) hold for all \( 1 \leq i \leq n - 1 \).

Proof. Since the anti-canonical bundle of \( \text{Hess}(S, h) \) is ample by the assumption, Proposition 3.2 and Lemma 3.5 (1) imply that the coefficients of \( \xi_h \) with respect to the fundamental weights \( \varpi_i \) must be positive, that is,

\[
\begin{align*}
h(i) - h(i + 1) + 2 + h^*(n - i) - h^*(n + 1 - i) &> 0
\end{align*}
\]

by Lemma 3.1. In the left-hand side, we know that \( h(i) - h(i + 1) \) and \( h^*(n - i) - h^*(n + 1 - i) \) are both less than or equal to 0. Thus the inequality means that we must have \(-1 \leq h(i) - h(i + 1) \) and \(-1 \leq h^*(n - i) - h^*(n + 1 - i) \) for all \( 1 \leq i \leq n - 1 \), which implies the desired inequalities.

Lemma 3.9. Assume that \( \text{Hess}(S, h) \) is Fano. If \( h(i + 1) = h(i) \), then \( h(i) = n \).

Proof. Suppose for a contradiction that we have \( h(i) = h(i + 1) < n \) for some \( i \in [n - 1] \). Writing \( j = n - h(i) \), this means that \( h^*(j + 1) \geq h^*(j) + 2 \), which contradicts Lemma 3.8.

We now give a proof of Theorem A.

Proof of Theorem A. Assume that \( \text{Hess}(S, h) \) is Fano. Then Lemmas 3.8 and 3.9 imply that \( h \) must be of the form \( (k + 1, k + 2, ..., n - 1, n, ..., n) \) for some \( 1 \leq k \leq n - 1 \). This means that

\[
\xi_h = \sum_{i=1}^{k} \varpi_i + \sum_{i=n-k}^{n-1} \varpi_i.
\]

Since \( L_{\xi_h} \) is ample by the assumption, Lemma 3.5 (1) now implies that all coefficients of \( \xi_h \) with respect to the fundamental weights must be positive. Hence it follows that \( k \geq n - k - 1 \) by (3.2), which is equivalent to \( \frac{n-1}{2} \leq k \leq n - 1 \).

If \( h = (k + 1, k + 2, ..., n - 1, n, ..., n) \) for some \( \frac{n-1}{2} \leq k \leq n - 1 \), then one can directly verify that (3.2) holds. Thus, the coefficients of \( \xi_h \) with respect to the fundamental weights are positive. This means by Lemma 3.5 (1) that \( L_{\xi_h} \) is ample on \( \text{Hess}(S, h) \), which implies that \( \text{Hess}(S, h) \) is Fano.

4. Relation with Richardson varieties

In this section, we study bigness of the anti-canonical bundles of regular semisimple Hessenberg varieties via positivity of line bundles over Richardson varieties.
4.1. Richardson varieties. Denote by $\leq$ the Bruhat order on $\mathfrak{S}_n$, and by $\prec$ a cover in the Bruhat order, that is, $u \prec v$ if and only if $u < v$ and $\ell(v) = \ell(u) + 1$. For $v, w \in \mathfrak{S}_n$ such that $v \leq w$, the subvariety

$$X_w^v := X_w \cap \Omega_v \subseteq G/B$$

is called a Richardson variety, where $X_w$ is a Schubert variety, and $\Omega_v$ is a dual Schubert variety (see Section 3.1). The Richardson variety $X_w^v$ is irreducible, and we have

$$\dim(\mathbb{C}(X_w^v)) = \ell(w) - \ell(v).$$

4.2. Bigness of line bundles over Richardson varieties. A line bundle $L$ over a normal projective variety $Y$ is said to be big if the Iitaka dimension $\kappa(Y, L)$ takes the maximum possible value, i.e. the dimension of $Y$ ([18, Definition 2.2.1]). It is equivalent to the inequality

$$\limsup_{m \to \infty} \frac{h^0(X, L^m)}{m^d} > 0,$$

where $d = \dim \mathbb{C} Y$. In this subsection, we study big line bundles over Richardson varieties, which come from line bundles over the flag variety. Our main reference is [17].

Recall from Section 2.2 that each weight $\mu$ of $T$ defines a line bundle $L_\mu$ over $X_w^v \subseteq G/B$. In Corollary 4.5, we give a necessary and sufficient condition for $L_\mu$ to be a big line bundle over $X_w^v$ under the assumption that $\mu \in P^+$, which is a straightforward consequence of [17]. Fix a parabolic subgroup $B \subseteq P \subseteq G$. Let $\mathfrak{S}_P \subset \mathfrak{S}_n$ denote the corresponding parabolic subgroup, and

$$\pi_P: \mathfrak{S}_n \twoheadrightarrow \mathfrak{S}_n/\mathfrak{S}_P$$

the canonical projection onto the set of left cosets. We use the $P$-Bruhat order on $\mathfrak{S}_n$, which is a specific lift of the Bruhat order on $\mathfrak{S}_n/\mathfrak{S}_P$ (see [6, Ch. 2] and [23, Ch. 4] for references on the Bruhat order on $\mathfrak{S}_n/\mathfrak{S}_P$).

**Definition 4.1** (see [17, Sect. 2]). The $P$-Bruhat order $\leq_P$ on $\mathfrak{S}_n$ is defined by: $v \leq_P w$ if and only if there is a chain

$$v = u_0 \prec u_1 \prec u_2 \prec \cdots \prec u_k = w$$

such that

$$\pi_P(u_0) \prec \pi_P(u_1) \prec \pi_P(u_2) \prec \cdots \prec \pi_P(u_k)$$

in the Bruhat order on $\mathfrak{S}_n/\mathfrak{S}_P$.

**Example 4.2.** Let $n = 3$, and take a parabolic subgroup $B \subseteq P \subseteq G$ such that $\mathfrak{S}_P$ is generated by $s_1$. Since

$$\pi_P(e) = \pi_P(s_1) < \pi_P(s_2) = \pi_P(s_2s_1) < \pi_P(s_1s_2) = \pi_P(s_1s_2s_1),$$

the $P$-Bruhat order $\leq_P$ on $\mathfrak{S}_3$ is given by the following:

$$e <_P s_2 <_P s_1s_2,$$

$$s_1 <_P s_2s_1 <_P s_1s_2s_1,$$

$$s_1 <_P s_1s_2.$$
By abuse of notation, let \( \pi_P : G/B \to G/P \) denote the canonical projection. For \( v, w \in S_n \) such that \( v \leq w \), we set \( \Pi_v^w := \pi_P(X_v^w) \subseteq G/P \).

This variety \( \Pi_v^w \) is called a projected Richardson variety. The projected Richardson variety was studied by Lusztig \[19\] and Rietsch \[22\] in the context of total positivity, and by Goodearl-Yakimov \[12\] in the context of Poisson geometry. In order to study big line bundles over \( X_v^w \), we use the following relations (Propositions 4.3, 4.4) between \( X_v^w \) and \( \Pi_v^w \), which are given in \[17\].

**Proposition 4.3** (see the proof of \[17\] Theorem 4.5). For \( v, w \in S_n \) such that \( v \leq w \) and an ample line bundle \( L \) over \( G/P \), the map \( \pi_P^* : H^0(\Pi_v^w, L) \to H^0(X_v^w, \pi_P^* L) \) is a \( \mathbb{C} \)-linear isomorphism.

**Proposition 4.4** (see \[17\] Sect. 3). For \( v, w \in S_n \) such that \( v \leq w \), the morphism \( \pi_P : X_v^w \to \Pi_v^w \) is birational if and only if \( v \leq_P w \). In addition, this is equivalent to \( \dim_{\mathbb{C}}(X_v^w) = \dim_{\mathbb{C}}(\Pi_v^w) \).

For \( \mu \in P_+ \), let \( S_\mu \subseteq S_n \) be the parabolic subgroup generated by \( \{s_i \mid 1 \leq i \leq n-1, \ s_i(\mu) = \mu\} \).

The equality \( s_i(\mu) = \mu \) is equivalent to the condition that \( \mu_i = 0 \) when we write \( \mu = \sum_{j=1}^{n-1} \mu_j w_j \). We denote by \( B \subseteq P_\mu \subseteq G \) the unique parabolic subgroup such that \( S_\mu = S_{P_\mu} \).

**Corollary 4.5.** For \( \mu \in P_+ \) and \( v, w \in S_n \) such that \( v \leq w \), the line bundle \( L_\mu \) over \( X_v^w \) is big if and only if \( v \leq_P w \).

**Proof.** By the definition of \( P_\mu \), the line bundle \( L_\mu \) over \( G/B \) is the pull-back of the ample line bundle \( L_\mu \) over \( G/P_\mu \) (see \[16\] Sect. II.4.4]). Thus \( L_\mu \) on \( \Pi_v^w \subseteq G/P_\mu \) is big since it is ample. Since we have \( H^0(\Pi_v^w, L_\mu^\otimes k) \simeq H^0(X_v^w, L_\mu^\otimes k) \) for all \( k \in \mathbb{Z}_{>0} \) by Proposition 4.3, this and the definition of big line bundles imply that the line bundle \( L_\mu \) over \( X_v^w \) is big if and only if \( \dim_{\mathbb{C}}(X_v^w) = \dim_{\mathbb{C}}(\Pi_v^w) \). Since this is equivalent to \( v \leq_P w \) by Proposition 4.4, we obtain the assertion. \( \square \)

### 4.3. Hessenberg varieties and Richardson varieties

Anderson-Tymoczko \[4\] introduced a permutation associated with a Hessenberg function to express the cohomology classes of Hessenberg varieties in terms of Schubert classes. We use a similar but slightly different notation.

**Definition 4.6.** For a Hessenberg function \( h : [n] \to [n] \), we define \( w_h \in S_n \) as follows: let \( w_h(1) = h(1) \), and take \( w_h(i) \) to be the \( (n + 1 - h(i)) \)-th largest element of \( [n] \setminus \{w_h(1), \ldots, w_h(i-1)\} \).
For example, if \( n = 5 \) and \( h = (3, 4, 5, 5, 5) \) as in Figure 1 then \( w_h = 3 4 2 5 1 \) in one-line notation. The positions of 1’s of the permutation matrix associated with \( w_h \) are depicted as the dots in Figure 3.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

**Figure 3.** The positions of 1’s of the permutation matrix associated with \( w_h \) for \( h = (3, 4, 5, 5, 5) \).

**Remark 4.7.** The permutation \( w(h) := (w_0 w_h)^{-1} \) is precisely the one which was considered in [4].

Let \([\text{Hess}(S, h)] \in H^*(Fl(\mathbb{C}^n))\) be the cohomology class of \( \text{Hess}(S, h) \). We have the following formula\(^2\) for \([\text{Hess}(S, h)]\) in terms of products of Schubert classes due to [4, Corollary 3.3 and equation (14)]:

\[
[\text{Hess}(S, h)] = \sum_{u \in \mathfrak{S}_n; \ell(u) + \ell(w_h) = \ell(u w_h)} [\Omega_u][\Omega_{w_0 u w_h}].
\]

Using this formula, we deduce a sufficient condition for the anti-canonical bundle \( L_{\xi_h} \) of \( \text{Hess}(S, h) \) to be big when it is assumed to be nef.

**Proposition 4.8.** Assume that \( \xi_h \in P_+ \), that is, \( L_{\xi_h} \) is a nef line bundle over \( G/B \). If there exists \( u \in \mathfrak{S}_n \) such that

\[
\ell(u) + \ell(w_h) = \ell(u w_h), \quad u \leq_{\xi_h} u w_h,
\]

then \( L_{\xi_h} \) is a big line bundle over \( \text{Hess}(S, h) \).

**Proof.** According to [18, Theorem 2.2.16], it suffices to prove that

\[
\int_{\text{Hess}(S, h)} c_1(L_{\xi_h})^d > 0,
\]

where \( d = \dim_{\mathbb{C}} \text{Hess}(S, h) \). By multiplying the class \([\text{Hess}(S, h)] \in H^*(Fl(\mathbb{C}^n))\), we may express the integral on \( \text{Hess}(S, h) \) as an integral on \( Fl(\mathbb{C}^n) \):

\[
\int_{\text{Hess}(S, h)} c_1(L_{\xi_h})^d = \int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\text{Hess}(S, h)].
\]

Combining this with (4.1), we obtain

\[
\int_{\text{Hess}(S, h)} c_1(L_{\xi_h})^d = \sum_{u \in \mathfrak{S}_n; \ell(u) + \ell(w_h) = \ell(u w_h)} \int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\Omega_u][\Omega_{w_0 u w_h}].
\]

\(^2\)In [4], it was described in terms of the permutation \( w(h) \) which is explained in Remark 4.7.
We claim that each summand in the right-hand side is non-negative. This is because we may expand the product $[\Omega_u][\Omega_{uwu}]$ as a non-negative sum of the (dual) Schubert classes by Kleiman’s transversality theorem ([7, Sect. 1.3]):

$$[\Omega_u][\Omega_{uwu}] = \sum_{v \in S_n} c_v [\Omega_v] \quad (c_v \geq 0).$$

Hence each integral in the right-hand side of (4.3) is expressed as

$$\int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\Omega_u][\Omega_{uwu}] = \sum_{v \in S_n} c_v \int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\Omega_v] = \sum_{v \in S_n} c_v \int_{\Omega_v} c_1(L_{\xi_h})^d.$$

Since $L_{\xi_h}$ is nef on $\Omega_v$, this is a non-negative integer, as claimed above. Thus it suffices to find a permutation $u \in S_n$ in (4.3) such that

$$\int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\Omega_u][\Omega_{uwu}] > 0.$$

Now, take $u \in S_n$ which satisfies the assumption (4.2). Then the integral

$$\int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\Omega_u][\Omega_{uwu}]$$

appears as a summand in (4.3). Note that the second condition of (4.2) implies that $u \preceq uw_h$ in the Bruhat order. Since $[\Omega_{uwu}] = [X_{uwu}]$ by [11, Lemma 3 in Sect. 10.2], we have

$$[\Omega_u][\Omega_{uwu}] = [\Omega_u][X_{uwu}] = [X_u^{uw}],$$

where the second equality follows from $u \preceq uw_h$ and [7, Sect. 1.3]. Hence it follows that

$$\int_{Fl(\mathbb{C}^n)} c_1(L_{\xi_h})^d[\Omega_u][\Omega_{uwu}] = \int_{X_u^{uw}} c_1(L_{\xi_h})^d,$$

where we note that $d = \dim_{\mathbb{C}}(\text{Hess}(S, h)) = \dim_{\mathbb{C}}(X_u^{uw})$. Since $u \preceq P_{\xi_h} w_h$, we see by Corollary 1.5 that $L_{\xi_h}$ on $X_u^{uw}$ is nef and big, which implies that

$$\int_{X_u^{uw}} c_1(L_{\xi_h})^d > 0$$

by [18, Theorem 2.2.16]. From this and the argument above, it follows that

$$\int_{\text{Hess}(S, h)} c_1(L_{\xi_h})^d > 0.$$

Let $\mathcal{S}_P \subset \mathcal{S}_n$ be a parabolic subgroup as in Section 4.2. Note that for $w \in \mathcal{S}_n$, there is a unique factorization

$$w = w^P w_P$$

with $w^P \in \mathcal{S}_P$ and $w_P \in \mathcal{S}_P$, where $\mathcal{S}_P$ is the set of minimal length representatives for $\mathcal{S}_n/\mathcal{S}_P$ (cf. [7, Sect. 1.2]). In the next section, we will use the following claim to find the desired $u \in \mathcal{S}_n$ in the previous proposition.

**Lemma 4.9.** If $u \in \mathcal{S}_P$ and $u_P = (uw_h)^P$, then we have $u \preceq P_{\xi_h} uw_h$. 


Proposition 2.5. Hence there exists a chain 

\[ e = u_0 < u_1 < u_2 < \cdots < u_k = (uw_h)^P \]

of permutations \( u_0, \ldots, u_k \in \mathfrak{S}_n \) such that 

\[ \pi_P(u_0) < \pi_P(u_1) < \pi_P(u_2) < \cdots < \pi_P(u_k). \]

It follows that \( u_i \in \mathfrak{S}_P \) for all \( 0 \leq i \leq k \) by induction on \( i \). We prove this as follows. Since \( u_0 = e \in \mathfrak{S}_P \), we have \( \ell(\pi_P(u_0)) = \ell(u_0) \), and hence we obtain that 

\[ \ell(\pi_P(u_1)) - \ell(\pi_P(u_0)) \leq \ell(u_1) - \ell(u_0) = 1. \]

Since \( \pi_P(u_0) < \pi_P(u_1) \), it also follows that \( \ell(\pi_P(u_1)) - \ell(\pi_P(u_0)) \geq 1 \), and hence that 

\[ \ell(\pi_P(u_1)) - \ell(\pi_P(u_0)) = 1. \]

Thus we obtain \( \ell(\pi_P(u_1)) = \ell(u_1) \) by \( \ell(\pi_P(u_0)) = \ell(u_0) \), and this means that \( u_1 \in \mathfrak{S}_P \). Continuing this argument, we have \( u_i \in \mathfrak{S}_P \) for all \( 0 \leq i \leq k \). From these, it follows that 

\[
\begin{align*}
    u_P &= u_0u_P < u_1u_P < u_2u_P < \cdots < u_ku_P = (uw_h)^P u_P, \\
    \pi_P(u_0u_P) < \pi_P(u_1u_P) < \pi_P(u_2u_P) < \cdots < \pi_P(u_ku_P).
\end{align*}
\]

The left-most permutation is \( u \) by the assumption \( u \in \mathfrak{S}_P \), and the right-most permutation is \( uw_h \) by the assumption \( u_P = (uw_h)_P \). Thus we have proved \( u \leq_P uw_h \). \qed

5. Weak Fano Hessenberg varieties

In this section, we prove Theorem B which is stated in Section 1. We first prepare some notations and lemmas in Sections 5.1 and 5.2. A proof of Theorem B is given in Section 5.3. To exhibit our argument, we provide a pair of running examples for \( n = 20 \) and \( n = 19 \) in Section 5.4, which we will refer repeatedly. Throughout this section, we always assume that \( h \) is nef, that is, we assume that 

\[ h(i) - h(i + 1) + 2 - h^*(n + 1 - i) + h^*(n - i) \geq 0 \]

for all \( 1 \leq i \leq n - 1 \).

5.1. Preliminary notations. Let \( h: [n] \to [n] \) be a nef Hessenberg function satisfying the assumption (2.1), that is, \( h(i) \geq i + 1 \) for \( 1 \leq i < n \). The weight \( \xi_h \in \mathcal{P}_+ \) of the anti-canonical bundle \( L_{\xi_h} \) of Hess(\( S, h \)) defines a parabolic subgroup \( \mathfrak{S}_{P_{\xi_h}} \subseteq \mathfrak{S}_n \) as in Section 4.2. This subgroup is generated by the simple transpositions \( s_i \) satisfying \( s_i(\xi_h) = \xi_h \), that is, \( d_i = 0 \) when we write \( \xi_h = \sum_{i=1}^{n-1} d_i \Xi_i \). Let us describe this more explicitly in what follows. For \( 1 \leq i \leq n - 1 \) such that \( s_i(\xi_h) = \xi_h \), we set 

\[
\begin{align*}
    k_i &= \max \{ k \geq 0 \mid s_{i+1}(\xi_h) = \cdots = s_{i+k}(\xi_h) = \xi_h \}, \\
    k_{i, -} &= \max \{ k \geq 0 \mid s_{i-k}(\xi_h) = \cdots = s_{i-1}(\xi_h) = s_i(\xi_h) = \xi_h \}, \\
    J_i &= \{ i - k_{i, -}, \ldots, i - 1, i, i + 1, \ldots, i + k_i + 1 \}.
\end{align*}
\]

Noticing that \( J_{i-k_{i, -}} = \cdots = J_i = \cdots = J_{i+k_i} \), let \( J := \{ J_i \mid 1 \leq i \leq n - 1, s_i(\xi_h) = \xi_h \} \). For example, \( J = \{ J_9, J_{10} \} \) in the running example in Section 5.4. The parabolic subgroup \( \mathfrak{S}_{P_{\xi_h}} \) is now given by \( \prod_{J_i \in J} \mathfrak{S}_{J_i} \subseteq \mathfrak{S}_n \), where each \( \mathfrak{S}_{J_i} \) is the permutation
group on $J_i$ which is regarded as a subgroup of $S_n$ in the natural way. For simplicity, we denote this parabolic subgroup by $G_{J_i}$, that is,

$$G_{J_i} = \prod_{J_i \in J} G_{J_i} = G_{P_n} \subseteq S_n.$$  

When $J = \emptyset$, we mean that $G_J$ is the trivial subgroup consisting of the identity element. For example, if $n = 7$ and $h = (2, 4, 5, 6, 7, 7, 7)$, then $\xi_h = 0\varpi_1 + 0\varpi_2 + \varpi_3 + 0\varpi_4 + \varpi_5 + \varpi_6$ so that $G_J \cong S_3 \times S_2$. When we indicate the dependence of $J_i$ and $J$ on the Hessenberg function $h$, we will also denote them by $J_i(h)$ and $J(h)$, respectively.

Recall from Section 4.3 that for a permutation $w \in S_n$, there is a unique factorization

$$w = w^J w_J$$

with $w^J \in S^J$ and $w_J \in G_J$, where $S^J$ is the set of minimal length representatives for $S_n / G_J$. Note that $w_J$ encodes the order of the numbers of $w$ on each $J_i$ in one-line notation. More specifically, for $v, w \in S_n$, the equality $v_J = w_J$ is equivalent to the condition that the following statement holds for each $J_i \in J$:

$$v(j_1) < v(j_2) \text{ if and only if } w(j_1) < w(j_2) \text{ for } j_1, j_2 \in J_i.$$

For $2 \leq i \leq n$, we say that $h$ is **stable at** $i$ if

$$h(i) = h(i - 1).$$

For example, $h$ is stable at $i = 3$ in the running example in Section 5.4.

For $1 \leq i \leq n - 1$, we consider the following two conditions

$$(5.1) \quad \{j \geq i \mid h(j + 1) = h(j) + 2\} = \emptyset \text{ or } \min\{j \geq i \mid h(j + 1) = h(j)\} < \min\{j \geq i \mid h(j + 1) = h(j) + 2\},$$

and

$$(5.2) \quad \{j \geq i \mid h(j + 1) = h(j) + 2\} \neq \emptyset \text{ and } \min\{j \geq i \mid h(j + 1) = h(j)\} > \min\{j \geq i \mid h(j + 1) = h(j) + 2\}.$$

Noticing that either (5.1) or (5.2) holds for all $1 \leq i \leq n - 1$, we set

$$i^{(+)} = \begin{cases} i, & \text{if } i \text{ satisfies (5.1)}, \\ w_h^{-1}(h(\hat{i}) + 1), & \text{if } i \text{ satisfies (5.2)}, \end{cases}$$

for $1 \leq i \leq n - 1$, where $\hat{i} := \min\{j \geq i \mid h(j + 1) = h(j) + 2\}$. See Figure 4. Note that $i \leq i^{(+)} \leq n - 1$ since $n = w_h^{-1}(1)$. It is obvious that the (+)-operation will be trivial after repeating it on $i$ sufficiently many times $i \mapsto i^{(+)} \mapsto (i^{(+)})^{(+)} \mapsto \cdots$, and we denote by $i^{(+\infty)}(\leq n - 1)$ the limit of this sequence.

For $1 \leq i \leq n - 1$, we define

$$(5.3) \quad L_i := \min\{j \geq i^{(+\infty)} \mid h(j) = h(j + 1)\} + 1,$$

where the set appearing in the right-hand side is non-empty since we have $h(n - 1) = h(n)$. For example, $L_4 = 6$ and $L_6 = 15$ in the running example of $n = 20$ in Section 6.3. When we indicate the dependence of these operations on the Hessenberg function $h$, we will also denote them by $i^{(+h)}$ and $L_i(h)$, respectively.
Lemma 5.1. Let $h: [n] \to [n]$ be a nef Hessenberg function. For $1 \leq i \leq n - 1$ and $i + 1 \leq j < L_i$, we have

$$w_h(i) < w_h(j).$$

Proof. We first consider the case $i = i^{(+)}$. In this case, we have

$$h(k) = h(k - 1) + 1 \quad (i + 1 \leq k < L_i),$$

which means that $w_h(k) = h(k)$ for $i + 1 \leq k < L_i$ by the definition of $w_h$. Taking this equality in the case $k = j$, we obtain

$$w_h(i) \leq h(i) \leq h(j) = w_h(j),$$

where the first equality follows from the definition of $w_h$. Since $i \neq j$, we obtain the desired claim in this case.

We next consider the case $i < i^{(+)}$. In this case, it is clear that

$$(5.4) \quad w_h(i) < w_h(k) \quad (i + 1 \leq k \leq i^{(+)}),$$

by the definition of $i^{(+)}$, and the maximality of $w_h(k)$. Let us prove that we can extend the range of $k$ as

$$(5.5) \quad w_h(i) < w_h(k) \quad (i + 1 \leq k \leq (i^{(+)})^{(+)})$$

We take cases. If $i^{(+)} = (i^{(+)})^{(+)},$ then $(5.5)$ is the same as $(5.4)$. If $i^{(+)} < (i^{(+)})^{(+)},$ then we have

$$(5.6) \quad w_h(i^{(+)}) < w_h(k) \quad (i^{(+)} + 1 \leq k \leq (i^{(+)})^{(+)})$$

as we obtained $(5.4)$. Combining $(5.4)$ and $(5.6)$, we obtain $(5.5)$ in this case as well. By continuing this argument, we see that

$$(5.7) \quad w_h(i) < w_h(k) \quad (i + 1 \leq k \leq i^{(+\infty)}).$$

Hence we assume $i^{(+\infty)} + 1 \leq j$ in the following. Then, since $(i^{(+\infty)})^{(+) = i^{(+)}}$, the same argument as in the case $i = i^{(+)}$ implies that $w_h(i^{(+\infty)}) < w_h(j)$. Combining this with $(5.1)$, we obtain $w_h(i) < w_h(j)$. □
Lemma 5.2. If $s_1(\xi_h) = \xi_h$, then
$$h(1) < h(2) < \cdots < h(k_1 + 2).$$
In particular, $w_h(k) = h(k)$ for $k \in J_1$.

Proof. Suppose that there exists $1 \leq q \leq k_1 + 1$ such that $h(q) = h(q + 1)$. Then the definition of $k_1$ implies that $s_q(\xi_h) = \xi_h$. Hence we have
\begin{equation}
(5.8) \quad h^*(n + 1 - q) = h^*(n - q) + 2.
\end{equation}
This in fact implies
$$h^*(n) = h^*(n - 1) + 2$$
as follows. If $q = 1$, then the claim is obvious. If not, then let $q' := n + 1 - h^*(n + 1 - q)$. We then have $h(q') = h(q' + 1)$ by (5.8), and $1 \leq q' < q$ by
$$q' = n + 1 - h^*(n + 1 - q) < n + 1 - (n + 1 - q) = q.$$
This means that $q' \leq k_1 + 1$, and hence we have
$$h^*(n + 1 - q') = h^*(n - q') + 2$$
as above. By continuing this argument, it follows that $h^*(n) = h^*(n - 1) + 2$, as claimed above. However, this implies that $h(1) = h(2) = 1$, which contradicts the definition of a Hessenberg function. \hfill \Box

5.2. Principle of similar shapes. For each $1 \leq i \leq n$, let
$$D(i) := n - h^*(n + 1 - i).$$
This measures the horizontal distance between the left-side wall and the boundary of $h$ on the $i$-th row; see Figure 5. For example, $D(11) = 3$ and $D(13) = 6$ in the running example for $n = 20$ in Section 5.4.

![Figure 5](image_url)

**Figure 5.** The pictorial meaning of $D(i)$.

Lemma 5.3. $D(i) < w_h^{-1}(j)$ for $1 \leq i \leq j \leq n$. 
Proof. If $D(i) = 0$, then the claim is obvious. Thus we may assume $D(i) \geq 1$. It suffices to show that we have $w_i(l) < i$ for all $1 \leq l \leq D(i)$. Suppose that $1 \leq l \leq D(i)$. Since we are assuming $D(i) \geq 1$, it is clear that we have $h(l) < i$. This implies that $w_i(l) \leq h(l) < i$, as desired. □

Suppose that $s_i(\xi_h) = \xi_h$. Then we have

$$h(i) - h(i + 1) + 2 - h^*(n + 1 - i) + h^*(n - i) = 0,$$

which is equivalent to

$$(5.9) \quad h(i) - h(i + 1) + 2 + D(i) - D(i + 1) = 0.$$  

This condition and Lemma 5.7 impose a strong restriction on the shape of $h$ as we observe in what follows. We first consider the following relation among positions of $1 \leq i < n$ for which we have $h(i + 1) = h(i)$.

**Lemma 5.4.** Let $h: [n] \to [n]$ be a nef Hessenberg function, and suppose that $s_i(\xi_h) = \xi_h$. If $h(i + 1) = h(i)$, then $h(k + 2) = h(k + 1)$, where $k = D(i)$.

Proof. If $h(i + 1) = h(i)$, then we have $D(i + 1) - D(i) = 2$ since $s_i(\xi_h) = \xi_h$. This means that $h(k + 2) = h(k + 1)$ by the pictorial meaning of $k = D(i)$. □

As the converse of Lemma 5.4, we obtain the following.

**Lemma 5.5.** Let $h: [n] \to [n]$ be a nef Hessenberg function, and $k = D(i)$ for some $1 \leq i < n$. If $k \geq 1$ and $h(k + 2) = h(k + 1)$, then we have

$$(1) \quad D(i + 1) = D(i) + 2, \quad \text{or (2) } D(i + 2) = D(i + 1) + 2 = D(i) + 2.$$  

In case (1), if $s_i(\xi_h) = \xi_h$ in addition, then $h(i + 1) = h(i)$. In case (2), if $s_{i+1}(\xi_h) = \xi_h$ in addition, then $h(i + 2) = h(i + 1)$.

Proof. The former claim follows by the pictorial meaning of $D(i)$ and Lemma 3.7 (2). The latter claim is a direct consequence of (5.9). □

We next consider the following relation among positions of $1 \leq i < n$ for which we have $h(i + 1) = h(i) + 2$.

**Lemma 5.6.** Let $h: [n] \to [n]$ be a nef Hessenberg function, and suppose that $s_i(\xi_h) = \xi_h$. If $D(i) \geq 1$ and $h(i + 1) = h(i) + 2$, then $h(k + 1) = h(k) + 2$, where $k = D(i)$.

Proof. Note first that $i > 1$ since $D(1) = 0$. Since $s_i(\xi_h) = \xi_h$, the assumption $h(i + 1) = h(i) + 2$ means that $D(i) = D(i + 1)$. Corollary 5.1 now implies that we must have $D(i - 1) < D(i) = D(i + 1) < D(i + 2)$, which means that $h(k + 1) = h(k) + 2$ by the pictorial meaning of $k = D(i)$. □

As the converse of Lemma 5.6, we have the following claim.

**Lemma 5.7.** Let $h: [n] \to [n]$ be a nef Hessenberg function, and $k = D(i)$ for some $1 \leq i \leq n$. If $k \geq 1$ and $h(k + 1) = h(k) + 2$, then we have either

$$(1) \quad D(i) = D(i + 1), \quad \text{or (2) } D(i - 1) = D(i).$$  

In case (1), if $s_i(\xi_h) = \xi_h$ in addition, then $h(i + 1) = h(i) + 2$. In case (2), if $s_{i-1}(\xi_h) = \xi_h$ in addition, then $h(i) = h(i - 1) + 2$.

Proof. The former claim is obvious by the pictorial meaning of $D(i)$. The latter claim follows immediately by (5.9). □
Let \( I = [a - 1, b] \subseteq [n] \) for some \( 1 < a < b \leq n \), and suppose that \( s_i(\xi_h) = \xi_h \) for all \( a - 1 \leq i \leq b - 1 \). Then the four lemmas above imply that if \( D(a) \geq 1 \), \( h(a) \neq h(a-1)+2 \), \( h(b) \neq h(b-1)+2 \), then the information in what order the positions \( i \) satisfying \( h(i+1) = h(i) \) and the positions \( j \) satisfying \( h(j+1) = h(j)+2 \) appear must be the same for the intervals \( [a, b] \) and \([D(a), D(b)]\). Here, we need \( h(a) \neq h(a-1)+2 \) because of Lemma 5.6. We also assume \( h(b) \neq h(b-1)+2 \) because of Lemma 5.6.

We call this the **principle of similar shapes** on \([a, b]\) and \([D(a), D(b)]\). We use the word “similar” because we ignore the information how the positions \( k \) satisfying \( h(k+1) = h(k)+1 \) appear when we consider this principle. For example, if we take \([a, b] = \{10, 11, 12, 13\}\) in the running example for \( n = 20 \), then the shape of \( h \) on \([a, b]\) and that of \( h \) on \([D(a), D(b)]\) = \(\{1, 2, 3, 4, 5, 6\}\) are similar in this sense.

**Remark 5.8.** If \( a = h(1) \), then we have \( D(a) = 0 \). In this case, however, the principle of similar shapes on the intervals \([h(1), b]\) and \([1, D(b)]\) is valid if \( s_i(\xi_h) = \xi_h \) for all \( h(1)-1 \leq i \leq b-1 \) and \( h(b) \neq h(b-1)+2 \) without the assumptions \( D(a) \geq 1 \) and \( h(a) \neq h(a-1)+2 \). This follows because we have \( h(h(1)+1) \neq h(h(1))+2 \) by \( s_h(1)(\xi_h) = \xi_h \) and \( h^*(n+1-h(1)) > h^*(n-h(1)) \). We need to treat this case as well later.

**Lemma 5.9.** Let \( h: [n] \to [n] \) be a nef Hessenberg function. If \([i-1, L_i] \subseteq J_k \) for some \( 1 \leq k \leq n-1 \) and \( h(i) \neq h(i-1)+2 \), then \( D(i^{(+)}) = D(i^{(+)}) \).

**Proof.** To begin with, we show that \( D(i) \geq 1 \). If \( D(i) = 0 \), then we also have \( D(i-1) = 0 \), which means that \( h^*(n+1-i) = h^*(n+1-(i-1)) = n \). Since we have \([i-1, i] \subset [i-1, L_i] \subseteq J_k \), this implies that \( h(i) = h(i-1) + 2 \), which is a contradiction to our assumption.

Let us prove that \( D(i^{(+)}) = D(i^{(+)}) \). We first consider the case that \( i \) satisfies condition \([5.1]\). In this case, we have \( i^{(+)}/i \) by definition, which means that

\[
\begin{align*}
  h(j) &= h(j-1) + 1 \quad (i+1 \leq j < L_i), \\
  h(L_i) &= h(L_i-1).
\end{align*}
\]

The assumptions \([i-1, L_i] \subseteq J_k \) and \( h(i) \neq h(i-1)+2 \) mean that we may apply the principle of similar shapes to \([i, L_i] \) and \([D(i), D(L_i)]\), and then the above equalities imply that

\[
\begin{align*}
  h(l) &= h(l-1) + 1 \quad \text{for all } l < m, \\
  h(m) &= h(m-1),
\end{align*}
\]

where \( m = D(L_i-1)+2 \) by Lemma 5.4. This means that \( D(i) \) satisfies condition \([5.1]\) after replacing \( i \) by \( D(i) \). Thus it follows that \( D(i^{(+)}) = D(i^{(+)}) \), as desired.

Next, we consider the case that \( i \) satisfies condition \([5.2]\). By the assumption, we have \([i-1, i^{(+)}/i] \subset [i-1, L_i] \subseteq J_k \), and hence we may apply the principle of similar shapes on \([i, i^{(+)}/i] \) and \([D(i), D(i^{(+)}/i)]\) in a way similar to above, and we see that \( D(i) \) satisfies condition \([5.2]\) as well. Noticing this, it is straightforward to verify \( D(i) = D(i^{(+)}/i) \). Hence the desired claim \( D(i^{(+)}) = D(i^{(+)}/i) \) is equivalent to

\[
D(w^{-1}_h(h(i)+1)) = w^{-1}_h(h(D(i))+1).
\]
In addition, we have $h(\hat{i}) \neq h(\hat{i} - 1) + 2$ since the equality $h(\hat{i}) = h(\hat{i} - 1) + 2$ implies with the minimality of $\hat{i}$ ($\geq i$) that $h(i) = h(i - 1) + 2$, which contradicts our assumption. Thus we may assume $i = \hat{i}$, that is, $h(i + 1) = h(i) + 2$, to prove $D(i^{(+)}) = D(i^{(+)})$ in what follows. Notice that

$$h(D(i) + 1) = h(D(i)) + 2,$$

which follows by $h(i + 1) = h(i) + 2$ and Lemma 5.6. Also, since we have $h(i^{(+)}) = h(i^{(+)} - 1)$, it follows that

$$h(D(i^{(+)}) = h(D(i^{(+)}) - 1)$$

by Lemma 5.4 and $D(i^{(+)} - 1) = D(i^{(+)} - 2)$. See Figure 6 which visualizes the equalities (5.10) and (5.11). From these, it suffices to show that $w_h$ takes $h(D(i)) + 1$ as its value at $D(i^{(+)})$.

![Figure 6. The picture of (5.10) and (5.11).](image)

By the principle of similar shapes on $[i, i^{(+)}) \subseteq J_k$ and $[D(i), D(i^{(+)})]$, we see that (5.12) still hold when we replace $i$ and $i^{(+)}$ by $D(i)$ and $D(i^{(+)})$, respectively. Namely, we have

$$T[D(i), k] > S[D(i), k]$$

for each $D(i) < k < D(i^{(+)})$, and

$$T[D(i), D(i^{(+)})] = S[D(i), D(i^{(+)})].$$

This in particular implies that $w_h(D(i^{(+)}) = h(D(i)) + 1$ by the maximality of $w_h$, as desired.

**Lemma 5.10.** Let $h: [n] \to [n]$ be a nef Hessenberg function. If $[i - 1, L_i] \subseteq J_k$ for some $1 \leq k \leq n - 1$ and $h(i) \neq h(i - 1) + 2$, then $D(L_i) = L_{D(i)}$. 

□
Proof. By the assumption, the previous lemma shows that $D(i^{(+)}) = D(i^{(+)})$. By taking $D(i^{(+)}) = D(i^{(+)})$. Note that we have $[i^{(+)}, L_i] \subseteq [i - 1, L_i] \subseteq J_k$, and $h(i^{(+)}) \neq h(i^{(+)})$. Here, the latter claim follows because if $i = i^{(+)})$, then the claim is precisely the assumption $h(i) \neq h(i - 1) + 2$, and if $i < i^{(+)})$, then $h$ is stable at $i^{(+)})$, that is, $h(i^{(+)}) = h(i^{(+)}) - 1$, which implies the claim. Thus we obtain $D(i^{(+)}) = D(i^{(+)})$ by the previous lemma. Combining this with the previous equality above, we obtain

$$D(i^{(+)}) = D(i^{(+)}) + 1.$$ 

By continuing this process sufficiently many times, we obtain

$$D(i^{(+)}) = D(i^{(+)}) + 1.$$ 

We also have

$$h(i^{(+)}) \neq h(i^{(+)}) - 1 + 2$$

by an argument similar to that above. Thus, by $D(i^{(+)})$ and $[i^{(+)}, L_i] \subseteq [i - 1, L_i] \subseteq J_k$, we may assume $i = i^{(+)})$ to prove $D(L_i) = L_D(i)$ in what follows.

Since we have $i = i^{(+)})$, we know that $i$ satisfies condition (5.1), which means that we have

$$h(j) = h(j - 1) + 1 \quad (i + 1 \leq j < L_i),$$

and

$$h(L_i) = h(L_i - 1).$$

Because of (5.13), we also have $D(i) = D(i^{(+)})$. Thus $D(i)$ also satisfies condition (5.1), which means that we have

$$h(k) = h(k - 1) + 1 \quad (D(i) + 1 \leq k < L_D(i)),$$

and

$$h(L_D(i)) = h(L_D(i) - 1).$$

Thus, by $[i - 1, L_i] \subseteq J_k$ and Lemma 5.1, it follows that

$$D(L_i - 1) + 1 = L_D(i) - 1.$$ 

Since we have $h(L_i) = h(L_i - 1)$ and $[L_i - 1, L_i] \subseteq [i - 1, L_i] \subseteq J_k$, it follows that $D(L_i - 1) = D(L_i) = 2$ by (5.9). Combining this with the above equality, we obtain $D(L_i) = L_D(i)$, as desired. \qed

5.3. **Proof of Theorem B.** Let $h: [n] \to [n]$ be a nef Hessenberg function satisfying the assumption (2.1), that is, $h(i) \geq i + 1$ for $1 \leq i < n$. In this subsection, we give a proof of Theorem B which is stated in Section 1.

We already established the equivalence of (i) and (iii) in Theorem B by Proposition 3.4. Recalling that the anti-canonical bundle of Hess$(S, h)$ is isomorphic to $L_{\xi}$ by Proposition 5.2, it suffices to prove that if $L_{\xi}$ on Hess$(S, h)$ is nef, then it is in fact big. By Proposition 4.8 and Lemma 4.9 together with the notations given in Section 6.1, it is enough to show that there exists $u \in \Theta_S$ such that $\ell(u) + \ell(w_h) = \ell(uw_h)$ and $u_{\gamma} = (uw_h)_\gamma$.

Our proof is induction on $n$. To control induction, we require two additional conditions as seen below. Namely, we prove the following, where we say that $h$ is **strictly increasing** on an interval $[a, b] \subseteq [n]$ ($a < b$) if

$$h(a) < h(a + 1) < \cdots < h(b).$$
Theorem 5.11. Let $h: [n] \to [n]$ be a nef Hessenberg function satisfying (2.1). Then there exists $u \in \mathcal{G}_J$ such that the following conditions hold:

(i) $\ell(uw_h) = \ell(u) + \ell(w_h)$;
(ii) $(uw_h)_j = u_j$;
(iii) if $s_i(\xi_n) = \xi_i$ and $h$ is strictly increasing on $J_i$, then $u(j) = j$ for all $j \in J_i$;
(iv) for $1 \leq i \leq n - 1$ and $i + 1 \leq j < L_i$, we have $uw_h(i) < uw_h(j)$.

Remark 5.12. In addition to the original conditions (i) and (ii), we require two additional conditions (iii) and (iv) by the following reasons. If $h$ is strictly increasing on some $J_i$, then $w_h$ is also strictly increasing on $J_i$. Hence, by condition (ii), it is natural to seek for $u \in \mathcal{G}_J$ under condition (iii) on $J_i$, that is, $u$ is the identity on $J_i$. This condition will be used in the proof of Lemma 5.14. Condition (iv) is inspired by Lemma 5.1. This condition will be used to control the positions of $(h(1) - k)^{(+)}$ and $L_{h(1) - k}$ for $0 \leq k \leq k_{h(1)}$, in the proofs of Lemmas 5.25 and 5.28.

We proceed by induction on $n$. When $n = 2$, then the assumption (2.1) implies that we must have $h = (2, 2)$, which shows that $J = \emptyset$, so that the assertion is obvious by taking $u = e$. Let $n \geq 3$. If $h(1) = n$, then we must have $h(i) = n$ for all $1 \leq i \leq n$. In this case, we have $J = \emptyset$, and the assertion is obvious. Hence we may assume $h(1) < n$ in what follows so that $s_{h(1)} \in \mathcal{G}_n$ makes sense.

Define a function $h': \{1, 2, \ldots, n - 1\} \to \{1, 2, \ldots, n - 1\}$ by

$$h'(i) := h(i + 1) - 1 \quad (1 \leq i \leq n - 1).$$

Then it is a Hessenberg function, and it is obtained from $h$ by removing all the boxes in the $h(1)$-st row and those in the 1-st column. See the running example in Section 5.4. Note that we have

$$h^*(i) = \begin{cases} h^*(i) & \text{(if } 1 \leq i < n + 1 - h(1)), \\ h^*(i) - 1 (= n - 1) & \text{(if } n + 1 - h(1) \leq i \leq n - 1). \end{cases}$$

By the definition of $w_h$ and $w_{h'}$, we see that

$$w_h(i) = \begin{cases} h(1) & \text{(if } i = 1), \\ w_{h'}(i - 1) & \text{(if } w_{h'}(i - 1) < h(1)), \\ w_{h'}(i - 1) + 1 & \text{(if } w_{h'}(i - 1) \geq h(1)), \end{cases}$$

which implies that

$$\ell(w_h) = \ell(w_{h'}) + (h(1) - 1).$$

Recalling that we are assuming $h(1) < n$, we have the following.

Lemma 5.13. Write $\xi_h = \sum_{i=1}^{n-1} d_i \varpi_i$. Then $\xi_{h'}$ can be written as follows:

$$\xi_{h'} = \varpi_{h(1)-1} + \sum_{i=1}^{n-2} d_{i+1} \varpi_i.$$

In particular, $h'$ is also nef.

Proof. Writing $\xi_{h'} = \sum_{i=1}^{n-2} d_i \varpi_i$, we have

$$d'_i = h'(i) - h'(i + 1) + 2 - h^*(n - i) + h^*(n - 1 - i).$$
By the definition of $h'$ and the description of $h^{**}$ above, we can rewrite this as follows. If $i < h(1) - 1$ or $i > h(1) - 1$, then $d_i'$ is equal to
\[ h(i + 1) - h(i + 2) + 2 - h^{*}(n - i) + h^{*}(n - 1 - i), \]
which is $d_{i+1}$. If $i = h(1) - 1 \ (\leq n - 2)$, then $d_i'$ is equal to
\[ h(i + 1) - h(i + 2) + 2 - h^{*}(n - i) + h^{*}(n - 1 - i) + 1, \]
which is $d_{h(1)} + 1$. This proves the claim. \qed

By Lemma 5.13 if $s_i(\xi_{h'}) = \xi_{h'}$ for some $1 \leq i \leq n - 2$, then $s_{i+1}(\xi_h) = \xi_h$. This means that under the injective group homomorphism $\iota: \mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_n$ given by $s_i \mapsto s_{i+1}$ for $1 \leq i \leq n - 2$, we have
\[ \mathfrak{S}_{J} \hookrightarrow \mathfrak{S}_{J'}, \]
where $J = J(h)$ and $J' = J(h')$. By induction hypothesis, there exists $u' \in \mathfrak{S}_{J'}$ such that conditions (i)–(iv) hold. We denote by $\bar{u}' \in \mathfrak{S}_{J}$ the image of $u'$ under the embedding (5.16). Namely, we have
\[ (5.17) \quad \bar{u}'(i) = \begin{cases} 1 & \text{(if } i = 1), \\ u'(i - 1) + 1 & \text{(if } i > 1), \end{cases} \]
which implies that
\[ (5.18) \quad u'(i) = \bar{u}'(i + 1) - 1 \quad (1 \leq i \leq n - 1). \]
Since $h$ is nef, it follows by Lemma 5.13 that $s_{h(1)-1}(\xi_{h'}) \neq \xi_{h'}$, and hence that $s_{h(1)-1} \notin \mathfrak{S}_{J'}$. We observe that condition (iii) for $u'$ ensures the following property of its image $\bar{u}'$ in $\mathfrak{S}_{J}$.

**Lemma 5.14.** The equality $\bar{u}'(k) = k$ holds for $1 \leq k \leq h(1)$.

**Proof.** Since we have $\bar{u}'(1) = 1$ by definition, it suffices to prove that
\[ (5.19) \quad u'(k) = k \quad (1 \leq k \leq h(1) - 1). \]
Since $s_{h(1)-1}(\xi_{h'}) \neq \xi_{h'}$, we know that each $J'_i$ is contained in either $\{1, 2, \ldots, h(1) - 1\}$ or $\{h(1), h(1) + 1, \ldots, n - 1\}$, where $J' = J_i(h')$. If there are no $J'_i$ such that $J'_i \subseteq \{1, 2, \ldots, h(1) - 1\}$, then the claim is obvious since $u' \in \mathfrak{S}_{J'}$. If there exists $1 \leq i \leq h(1) - 1$ such that $J'_i \subseteq \{1, 2, \ldots, h(1) - 1\}$, then we have $h^{**}(n - j) = n - 1$ for all $j \in J'_i$ since $h(1) - 1 \leq h(2) - 1 = h'(1)$. Hence the equalities
\[ h'(j) - h'(j + 1) + 2 - h^{**}(n - j) + h^{**}(n - 1 - j) = 0 \quad (i - k_i' - 1 \leq j < i + k_i' + 1) \]
now imply that $h'(j + 1) = h'(j) + 2$ for $i - k_i' - 1 \leq j < i + k_i' + 1$, where $k_i'$ and $k_i$ are $k_{i-}$ and $k_i$ for $h'$, respectively. In particular, we have
\[ h'(i - k_i') < \cdots < h'(i) < \cdots < h'(i + k_i') + 1. \]
Hence we see by condition (iii) for $u'$ that $u'(j) = j \ (j \in J'_i)$. Since this holds for all $J'_i \subseteq \{1, 2, \ldots, h(1) - 1\}$, (5.19) follows from $u' \in \mathfrak{S}_{J'}$. \qed

The relation (5.14) between $w_h$ and $w_{h'}$ implies the following relation between $\bar{u}'w_h$ and $u'w_{h'}$ in a similar form by Lemma 5.13.
Corollary 5.15. The following equalities hold:

\[ \tilde{u}'w_h(i) = \begin{cases} 
    h(1) & \text{(if } i = 1), \\
    u'(w_{h'}(i - 1) & \text{(if } u'w_{h'}(i - 1) < h(1)), \\
    u'w_{h'}(i - 1) + 1 & \text{(if } u'w_{h'}(i - 1) \geq h(1)). 
\end{cases} \]

Proof. We compute the values \( \tilde{u}'w_h(i) \) for \( 1 \leq i \leq n \). If \( i = 1 \), then \( \tilde{u}'w_h(i) = \tilde{u}'(h(1)) = h(1) \) by \((5.14)\) and Lemma 5.14. Hence we may assume \( i > 1 \) in the following.

If \( u'w_{h'}(i - 1) < h(1) \), then by \((5.18)\) we have \( \tilde{u}'(w_{h'}(i - 1)) + 1 \leq h(1) \) so that \( w_{h'}(i - 1) < h(1) \) by Lemma 5.14. Thus \((5.14)\) and Lemma 5.14 again show that \( \tilde{u}'w_h(i) = \tilde{u}'w_{h'}(i - 1) = w_{h'}(i - 1) \).

But this is further equal to \( u'w_{h'}(i - 1) \) by \((5.19)\).

If \( u'w_{h'}(i - 1) \geq h(1) \), then \( \tilde{u}'(w_{h'}(i - 1) + 1) \geq h(1) \) by \((5.18)\). Hence it follows that \( w_{h'}(i - 1) + 1 \geq h(1) + 1 \) by Lemma 5.14, and hence that \( w_{h'}(i - 1) \geq h(1) \).

Thus, by \((5.14)\) and \((5.18)\), we have \( \tilde{u}'w_h(i) = \tilde{u}'(w_{h'}(i - 1) + 1) = u'w_{h'}(i - 1) + 1 \), as desired.

\[ \square \]

Lemma 5.16. The equality \( \ell(\tilde{u}'w_h) = \ell(\tilde{u}') + \ell(w_h) \) holds.

Proof. The claim follows from the following direct computations:

\[ \ell(\tilde{u}'w_h) = \ell(u'w_{h'}) + (h(1) - 1) \quad \text{(by Corollary 5.15)} \]

\[ = \ell(u') + \ell(w_{h'}) + (h(1) - 1) \quad \text{(by condition (i) for } u') \]

\[ = \ell(u') + \ell(w_h) \quad \text{(by 5.15)}. \]

\[ \square \]

To construct \( u \in \mathfrak{S}_J \) which satisfies conditions (i)-(iv), we now take cases.

Case 1: \( s_{h(1)}(\xi_h) \neq \xi_h \).

In this case, we set \( u := \tilde{u}' \in \mathfrak{S}_J \). Then condition (i) holds for \( u \) by Lemma 5.16. The assumption \( s_{h(1)}(\xi_h) \neq \xi_h \) implies the following assertions on \( J'_i \) by Lemma 5.13. If \( s_i(\xi_h) = \xi_{h'} \), then \( J'_i = \{ j - 1 \mid j \in J_{i+1} \} \setminus \{0\} \). If \( s_1(\xi_h) = \xi_h \), then \( J'_1 \) is defined if and only if \( k_1 > 0 \). In this case, \( J_1 = \{ j + 1 \mid j \in J'_1 \} \cup \{1\} \). We will use this observation to prove conditions (ii)-(iv) in the following.

Proposition 5.17. Condition (ii) holds for \( u \). That is, the equality \( (uw_h)_J = u_J \) holds.

Proof. Take \( 1 \leq i \leq n - 1 \) such that \( s_i(\xi_h) = \xi_h \). It suffices to prove for \( j_1, j_2 \in J_i \) that \( uw_h(j_1) < uw_h(j_2) \) if and only if \( u(j_1) < u(j_2) \), as we observed in the beginning of Section 5.11.

First, we consider the case \( 1 \notin J_i \). In this case, we have \( j_1, j_2 \geq 2 \), and hence Corollary 5.15 implies that \( uw_h(j_1) < uw_h(j_2) \) if and only if \( u'(j_1 - 1) < u'(j_2 - 1) \). In addition, since \( u = \tilde{u}' \), we have \( u(j_1) < u(j_2) \) if and only if \( u'(j_1 - 1) < u'(j_2 - 1) \) by \((5.18)\). From these and condition (ii) for \( u' \), we conclude the assertion.
We next consider the case $1 \in J_i$. In this case, we have $J_i = J_1 = \{1, 2, \ldots, k_1 + 2\}$, and the same argument as above implies that for $j_1, j_2 \in J_1 \setminus \{1\}$, we have $uw_h(j_1) < uw_h(j_2)$ if and only if $u(j_1) < u(j_2)$. Since $u = \bar{u}'$, it follows from \((5.17)\) that
\[
u(1) = 1 < u(j)
\]
for all $j \in J_1 \setminus \{1\}$. Thus it suffices to prove that $uw_h(1) < uw_h(j)$ for all $j \in J_1 \setminus \{1\}$.

By Lemma \(5.2\) we deduce for $j \in J_1 \setminus \{1\}$ that $h(1) < h(j) = w_h(j)$. Since this means $h(1) < \bar{u}'w_h(j)$ by Lemma \(5.14\), it follows by Lemma \(5.14\) again that
\[
uw_h(1) = u(h(1)) = h(1) < \bar{u}'w_h(j) = uw_h(j),
\]
as desired. \(\Box\)

**Proposition 5.18.** Condition (iii) holds for $u$. That is, if $s_i(\xi_h) = \xi_h$ and $h$ is strictly increasing on $J_i$, then $u(j) = j$ for all $j \in J_i$.

**Proof.** Note that $i \neq h(1)$ since we are assuming $s_{h(1)}(\xi_h) \neq \xi_h$ in Case 1. We first consider the case $1 \notin J_i$. In this case, we have $i \geq 2$, and the assumption $s_i(\xi_h) = \xi_h$ implies $s_{i-1}(\xi_{h'}) = \xi_{h'}$ by Lemma \(5.13\) since $i \neq h(1)$. In addition, we have
\[
k_{i-1} = k_{i-1}, k_{i-1} = k_i, J_{i-1} = \{j - 1 \mid j \in J_i\}.
\]
Hence the assumption
\[
h(i - k_{i-1}) < \cdots < h(i) < \cdots < h(i + k_i + 1)
\]
means by the definition of $h'$ that
\[
h'(i - 1 - k_{i-1}) < \cdots < h'(i - 1) < \cdots < h'(i - 1 + k_{i-1} + 1).
\]
Thus, by condition (iii) for $u'$, we obtain
\[
u(j) = \bar{u}'(j) = u'(j) - 1 + 1 = (j - 1) + 1 = j
\]
for all $j \in J_i$, where the second equality follows from \((5.17)\) and $j \geq 2$.

We next consider the case $1 \in J_i$. In this case, we have $J_i = J_1 = \{1, 2, \ldots, k_1 + 2\}$. If $k_1 = 0$, then the claim is obvious since we have $u(j) = \bar{u}'(j) = j$ for $1 \leq j \leq 2$ \((\leq h(1))\) by Lemma \(5.14\). Hence we may assume that $k_1 \geq 1$. This means that $s_2(\xi_h) = \xi_h$, and hence that $s_1(\xi_{h'}) = \xi_{h'}$ by Lemma \(5.13\) since the assumption $s_{h(1)}(\xi_h) \neq \xi_h$ implies $h(1) \neq 2$. In particular, $J'_1$ is defined, and we have $J'_1 = \{1, 2, \ldots, k'_1 + 2\} = \{1, 2, \ldots, k_1 + 1\}$. Thus, by an argument similar to that above, we obtain $u(j) = j$ for $j \in J_i \setminus \{1\}$. Since we also have $u(1) = \bar{u}'(1) = 1$ by \((5.17)\), it follows that $u(j) = j$ for all $j \in J_i$. \(\Box\)

**Proposition 5.19.** Condition (iv) holds for $u$. That is, for $1 \leq i \leq n - 1$ and $i + 1 \leq j < L_i$, we have $uw_h(i) < uw_h(j)$.

**Proof.** If $i \neq 1$, then we have $L_i = L'_{i-1} + 1$, where $L_i = L_i(h)$ and $L'_{i-1} = L_{i-1}(h')$. Hence the assumption $i + 1 \leq j < L_i$ means that $(i - 1) + 1 \leq j - 1 < L'_{i-1}$, and we see by condition (iv) for $u'$ that
\[
uw_{h'}(i - 1) < uw_{h'}(j - 1),
\]
which implies by Corollary \(5.15\) that $\bar{u}'w_h(i) < \bar{u}'w_h(j)$.
If \(i = 1\), then the assumption \(i + 1 \leq j < L_i\) implies that \(h(1) = w_h(1) < w_h(j)\) by Lemma 5.1. Hence it follows from Lemma 5.14 that

\[ uw_h(1) = h(1) < uw_h(j), \]

as desired. \(\square\)

**Case 2:** \(s_{h(1)}(\xi_h) = \xi_h\).

Let us write \(p_{h(-)} := k_{h(1)}\) and \(p_h := k_{h(1)}\) for simplicity, that is,

\[ J_{h(1)} = \{h(1) - p_{h(-)}, \ldots, h(1) - 1, h(1), h(1) + 1, \ldots, h(1) + p_h + 1\}. \]

We first consider the orders of the numbers of \(\bar{u}'\) and \(\bar{u}'w_h\) on \(J_{h(1)}\) in one-line notation. We set \(J^-_{h(1)} := \{h(1) - p_{h(-)}, \ldots, h(1) - 1, h(1)\}\) and \(J^+_{h(1)} := \{h(1) + 1, h(1) + 2, \ldots, h(1) + p_h + 1\}\) so that

\[ J_{h(1)} = J^-_{h(1)} \sqcup J^+_{h(1)}. \]

Since \(\bar{u}' \in \mathcal{S}_J\), it preserves the subset \(J_{h(1)}\) of \([n]\). By Lemma 5.14, it also preserves the smaller subset \(J^-_{h(1)}\). Since \(J^-_{h(1)}\) is the complement of \(J^+_{h(1)}\) in \(J_{h(1)}\), we see that

\[ \bar{u}'(J^-_{h(1)}) = J^-_{h(1)}, \quad \bar{u}'(J^+_{h(1)}) = J^+_{h(1)}. \]

For example, we have \(J^-_{h(1)} = \{7, 8, 9\}\) and \(J^+_{h(1)} = \{10, 11, \ldots, 14\}\) in the running example of \(n = 20\).

We now use condition (ii) for \(u'\) to study the orders of the numbers of \(\bar{u}'\) and \(\bar{u}'w_h\) on \(J_{h(1)}\). Since \(J'_{h(1)} = \{h(1), h(1) + 1, \ldots, h(1) + p_h\}\) if \(p_h > 0\), we have

\[ \bar{u}'(j_1) < \bar{u}'(j_2) \text{ if and only if } \bar{u}'w_h(j_1) < \bar{u}'w_h(j_2) \quad (j_1, j_2 \in J_{h(1)}^+) \]

from condition (ii) for \(u'\) on \(J'_{h(1)}\) (cf. the proof of Proposition 5.17). Similarly (but from a slightly complicated argument as we explain below), it follows that

\[ \bar{u}'(j_1) < \bar{u}'(j_2) \text{ if and only if } \bar{u}'w_h(j_1) < \bar{u}'w_h(j_2) \quad (j_1, j_2 \in J_{h(1)}^+). \]

We prove this as follows. By Lemma 5.14, it suffices to show

\[ \bar{u}'w_h(h(1) - p_{h(-)}) < \cdots < \bar{u}'w_h(h(1) - 1) < \bar{u}'w_h(h(1)). \]

We may assume that \(p_{h(-)} \geq 1\) since if not, then there is nothing to prove. Recall from the definition of \(J_{h(1)}\) that we have \(s_{h(1) - p_{h(-)}'} \in \mathcal{S}_J\). We now take cases. If \(h(1) - p_{h(-)} \geq 2\), then the property \(s_{h(1) - p_{h(-)}'} \in \mathcal{S}_J\) implies that \(s_{h(1) - p_{h(-)} - 1} \in \mathcal{S}_{J'}\). This means that \(J'_{h(1) - p_{h(-)} - 1} = \{h(1) - p_{h(-)} - 1, \ldots, h(1) - 2, h(1) - 1\}\) is defined, where we used \(s_{h(1) - 1} \notin \mathcal{S}_{J'}\). Thus Lemma 5.14 and condition (ii) for \(u'\) on \(J'_{h(1) - p_{h(-)} - 1}\) imply (5.23) in this case. If \(h(1) - p_{h(-)} = 1\), then we have \(h(1) < h(2)\) by Lemma 5.22 and this implies that \(w_h(1) = h(1) < h(2) = w_h(2)\). Hence Lemma 5.14 shows that \(\bar{u}'w_h(1) < \bar{u}'w_h(2)\). This means that if \(h(1) = 2\), then we already have (5.23), and hence we may assume \(h(1) \geq 3\) in what follows. We then have \(h(1) - p_{h(-)} = 1 < 2 < h(1)\), and hence the definition of \(J_{h(1)}\) implies that \(s_2(\xi_h) = \xi_h\), which means that \(s_1(\xi_{h'}) = \xi_{h'}\) so that \(J' = \{1, 2, \ldots, h(1) - 2, h(1) - 1\}\) is defined. Thus the same argument as above implies that

\[ \bar{u}'w_h(2) < \bar{u}'w_h(3) < \cdots < \bar{u}'w_h(h(1) - 1) < \bar{u}'w_h(h(1)). \]
Combining this with \( \overline{u}'w_h(1) < \overline{u}'w_h(2) \) proved above, we obtain (5.23) in this case. Hence (5.22) follows.

Recall that we seek for a permutation \( u \in \mathcal{S}_J \) which satisfies \((uw_h)_J = u_J\). We observed in (5.21) and (5.22) above that the numbers of \( \overline{u} \) and \( \overline{u}'w_h \) on \( J^+_h(1) \) are ordered in the same way. For \( \overline{u}' \), we also have the following property on the whole \( J_h(1) \):

\[
(5.24) \quad \overline{u}'(j_1) < \overline{u}'(j_2) \quad (j_1 \in J^-_h(1), \ j_2 \in J^+_h(1))
\]

by (5.20). If (5.24) is also satisfied for \( \overline{u}'w_h \) (after replacing \( \overline{u}' \) by \( \overline{u}'w_h \)), then we may take \( u \) to be \( \overline{u}' \) as we will see in Case 2-a below, but this is not the case in general. To find the desired permutation \( u \in \mathcal{S}_J \), we encode the information how the numbers of \( \overline{u}'w_h \) on the whole \( J_h(1) \) are ordered; in other words, how (5.24) is violated for \( \overline{u}'w_h \) on \( J_h(1) \). Recalling the inequalities (5.23) for \( \overline{u}'w_h \) on \( J^-_h(1) \), we set \( r_k := \overline{u}'w_h(h(1) - k) \) for \( 0 \leq k \leq p_h, - \), that is, we have

\[
(5.25) \quad r_{p_h, -} < \cdots < r_1 < r_0
\]

in one-line notation of \( \overline{u}'w_h \) on \( J^-_h(1) \). We define \( 0 \leq m_{p_h, -} \leq \cdots \leq m_1 \leq m_0 \leq p_h + 1 \) and \( 1 \leq q_1, q_2, \ldots, q_{m_0} \leq p_h + 1 \) by

\[
(5.26) \quad \{1 \leq q \leq p_h + 1 \mid \overline{u}'w_h(h(1) + q) < r_k\} = \{q_1, q_2, \ldots, q_{m_k}\}
\]

for \( 0 \leq k \leq p_h, - \), and by

\[
\overline{u}'w_h(h(1) + q_1) < \overline{u}'w_h(h(1) + q_2) < \cdots < \overline{u}'w_h(h(1) + q_{m_0}).
\]

Here, we mean \( \{q_1, q_2, \ldots, q_{m_k}\} = \emptyset \) when \( m_k = 0 \). The definition (5.26) is well-defined because of (5.25). Let

\[
\Delta_k := m_k - m_{k+1} = |\{1 \leq q \leq p_h + 1 \mid r_{k+1} < \overline{u}'w_h(h(1) + q) < r_k\}|
\]

for \( 0 \leq k \leq p_h, - \), where we take \( r_{p_h, -+1} = m_{p_h, -+1} = 0 \) as conventions so that \( \Delta_{p_h, -} = m_{p_h, -} \). In the running example of \( n = 20 \), we have \( m_2 = 0, \ m_1 = 1, \) and \( m_0 = 2 \) (see also Figure 13).

**Case 2-a:** \( \Delta_k = 0 \) for all \( 0 \leq k \leq p_h, - \).

In this case, the inequalities (5.24) are also satisfied for \( \overline{u}'w_h \) on the whole \( J_h(1) \) by the definition of \( \Delta_k \), and this leads us to set \( u := \overline{u}' \in \mathcal{S}_J \). Then, as in Case 1, condition (i) follows by Lemma 5.16 and condition (iv) follows by the same arguments as that in the proof of Proposition 5.19.

**Proposition 5.20.** The equality \((uw_h)_J = u_J\) holds, that is, condition (ii) holds for \( u \).

**Proof.** Take \( 1 \leq i \leq n - 1 \) such that \( s_i(\xi_h) = \xi_{h} \). It suffices to prove that for \( j_1, j_2 \in J_i \), \( uw_h(j_1) < uw_h(j_2) \) if and only if \( u(j_1) < u(j_2) \). If \( h(1) \notin J_i \), then the proof of Proposition 5.17 in Case 1 implies the assertion (see also the paragraph before Proposition 5.17).

Hence we may assume that \( i = h(1) \). If \( j_1, j_2 \leq h(1) \) or \( j_1, j_2 \geq h(1) + 1 \), then we have \( uw_h(j_1) < uw_h(j_2) \) if and only if \( u(j_1) < u(j_2) \) because of (5.21) and (5.22). Hence it is enough to consider the case \( j_1 \leq h(1) \) and \( h(1) + 1 \leq j_2 \). In this case, we have

\[
u(j_1) < u(j_2)
\]
Case 2-b: \( \Delta_k \geq 1 \) for some \( 0 \leq k \leq p_h, - \)

In this case, the property \( \text{(5.24)} \) does not hold for \( \bar{u}'w_h \) on \( J_{h(1)} \) (after replacing \( \bar{u}' \) by \( \bar{u}'w_h \)) by the definition of \( \Delta_k \), which means that the numbers of \( \bar{u}' \) on \( J_{h(1)} \) and the numbers of \( \bar{u}'w_h \) on \( J_{h(1)} \) are in different orders. Hence we cannot take \( u \) to be \( \bar{u}' \) to have the desired property \( u_j = (uw_h)_j \). To resolve this, we define a certain permutation \( v \in \mathfrak{S}_J \) which makes the order of the numbers of \( u := uv' \) on \( J_{h(1)} \) is the same as the order of the numbers of \( \bar{u}'w_h \) on \( J_{h(1)} \). It is not immediately clear that this implies \( u_j = (uw_h)_j \) since \( \bar{u}'w_h \) is changed as \( uw_h \) simultaneously, but we prove that the equality in fact follows.

To find such \( v \in \mathfrak{S}_J \), we focus on the numbers of \( \bar{u}' \) on \( J_{h(1)} \) which are given by

\[
\bar{u}'(h(1) - k) = h(1) - k \quad (0 \leq k \leq p_h, -)
\]

by Lemma \( 5.14 \). We set

\[
M := \max\{0 \leq k \leq p_h, - \mid \Delta_k = m_k - m_{k+1} \geq 1\}
\]

so that we have

\[
(5.27) \quad 0 = m_{p_h, -} = \cdots = m_{M+1} < m_M \leq \cdots \leq m_1 \leq m_0 \ (\leq p_h + 1)
\]
by definition. Let
\[ v_k := s_{h(1) - k + m_k - 1} \cdots s_{h(1) - k} s_{h(1) - k - 1} \quad (0 \leq k \leq M). \]
Note that \( v_k \) is a cyclic permutation of length \( m_k \) which is visualized in Figure 7. Hence

\[ h(1) - k \leftrightarrow h(1) - k + 1 \leftrightarrow \cdots \leftrightarrow h(1) - k + m_k \]

**Figure 7.** The cyclic permutation \( v_k \).

we have \( \ell(v_M) \leq \cdots \leq \ell(v_1) \leq \ell(v_0) \). Now we set
\[ u := v_M \cdots v_1 v_0 \bar{u}'. \]
Note that each \( v_k \) (0 \( \leq k \leq M \)) is a permutation on \( J_{h(1)} \subseteq [n] \) since we have \( h(1) - p_{h,-} \leq h(1) - k \) and \( h(1) - k + m_k \leq h(1) + p_k + 1 \) in Figure 7. Thus it follows that \( u = v_M \cdots v_1 v_0 \bar{u}' \in \mathcal{S}_f \). In the running example of \( n = 20 \), we have \( J_{h(1)} = \{7, 8, \ldots, 14\} \) and \( M = 1 < 2 = p_{h,-} \), and it follows that \( u = v_1 v_0 \bar{u}' = (s_{10} s_9) \bar{u}' \). Figures 13 and 14 for this example visualize the idea of the definition of \( u \) which we stated at the beginning of Case 2-b: we defined \( u = v_M \cdots v_1 v_0 \bar{u}' \) so that the order of the numbers of \( u \) on \( J_{h(1)} \) is the same as the order of the numbers of \( \bar{u}'w_h \) on \( J_{h(1)} \). To see this in general, we prepare the following lemma.

**Lemma 5.22.** For 1 \( \leq j_1 < j_2 \leq n \), we have \( v_M \cdots v_1 v_0(j_1) > v_M \cdots v_1 v_0(j_2) \) if and only if \( j_1 = h(1) - k \) and \( j_2 = h(1) + l \) for some \( 0 \leq k \leq M \) and \( 1 \leq l \leq m_k \). In particular, the permutation \( v_M \cdots v_1 v_0 \) preserves the order of the numbers in \( [n] \setminus \{h(1) - k \mid 0 \leq k \leq M\} \).

**Proof.** We first assume that \( v_M \cdots v_1 v_0(j_1) > v_M \cdots v_1 v_0(j_2) \). Then there exists \( 0 \leq k \leq M \) such that the following inequalities hold:

\[ (5.28) \]
\[ v_{k-1} \cdots v_0(j_1) < v_{k-1} \cdots v_0(j_2), \]
\[ v_k v_{k-1} \cdots v_0(j_1) > v_k v_{k-1} \cdots v_0(j_2). \]

Then, by the definition of \( v_k \), it follows that
\[ v_{k-1} \cdots v_0(j_1) = h(1) - k, \]
\[ v_k v_{k-1} \cdots v_0(j_1) = h(1) - k + m_k. \]
The first equality implies that \( j_1 = h(1) - k \) since the permutations \( v_0, v_1, \ldots, v_{k-1} \) preserve \( h(1) - k \). The definition of \( v_k \) and inequalities \( (5.28) \) also imply that
\[ h(1) - (k - 1) \leq v_{k-1} \cdots v_0(j_2) \leq h(1) - k + m_k. \]
By the definition of \( v_{k-1} \) and \( h(1) - k + m_k < h(1) - (k - 1) + m_{k-1} \), this means that
\[ h(1) - (k - 2) \leq v_{k-2} \cdots v_0(j_2) \leq h(1) - (k - 1) + m_k. \]
By continuing this argument, we obtain that
\[ h(1) + 1 \leq j_2 \leq h(1) + m_k. \]
Thus we have \( j_2 = h(1) + l \) for some \( 1 \leq l \leq m_k \).
Conversely, assume that \( j_1 = h(1) - k \) and \( j_2 = h(1) + l \) for some \( 0 \leq k \leq M \) and \( 1 \leq l \leq m_k \). By reversing the argument above, we see that

\[
v_k v_{k-1} \cdots v_0(j_1) = h(1) - k + m_k,
\]

\[
h(1) - (k - 1) \leq v_{k-1} \cdots v_0(j_2) \leq h(1) - k + m_k.
\]

In particular, it follows that

\[
v_k v_{k-1} \cdots v_0(j_2) = v_{k-1} \cdots v_0(j_2) - 1 < h(1) - k + m_k = v_k v_{k-1} \cdots v_0(j_1).
\]

Hence we see by (5.27) that

\[
v_M \cdots v_1 v_0(j_2) < h(1) - k + m_k = v_M \cdots v_1 v_0(j_1).
\]

Since the definition of \( q_1, q_2, \ldots, q_{m_0} \) implies that \( \bar{a}' w_h(h(1) + q_l) \) is the \( l \)-th smallest number in \( \bar{a}' w_h(J_{h(1)}^+) \) for \( 1 \leq l \leq m_0 \), we see by (5.20) and (5.21) that \( \bar{a}'(h(1) + q_l) \) is the \( l \)-th smallest number in \( \bar{a}'(J_{h(1)}^+) = J_{h(1)}^+ \), which implies that

\[
\bar{a}'(h(1) + q_l) = h(1) + l \quad \text{for} \ 1 \leq l \leq m_0.
\]

Combining this with Lemmas 5.14 and 5.22 it follows that for \( j_1, j_2 \in [n] \), we have \( \bar{a}'(j_1) < \bar{a}'(j_2) \) and \( u(j_1) > u(j_2) \) if and only if \( j_1 = h(1) - k \) and \( j_2 = h(1) + q_l \) for some \( 0 \leq k \leq M \) and \( 1 \leq l \leq m_k \). Hence, by (5.21) and (5.22), the definition of \( q_1, q_2, \ldots, q_{m_0} \) implies that

\[
u(j_1) < u(j_2) \quad \text{if and only if} \quad \bar{a}' w_h(j_1) < \bar{a}' w_h(j_2) \quad \text{for} \ j_1, j_2 \in J_{h(1)}.
\]

see Figures 13 and 14 for the pictorial meaning of this argument. Indeed, we defined the permutation \( u \in \mathfrak{S}_J \) so that this holds as we claimed above. We will prove that the latter inequality is in fact equivalent to \( w_h(j_1) < w_h(j_2) \) to see condition (ii) for \( u \).

Our first aim is to prove condition (i) for \( u \). For this purpose, we make a few observations in what follows.

**Lemma 5.23.** \( \ell(u) = \ell(\bar{u}') + (m_0 + m_1 + \cdots + m_M) \).

**Proof.** In one-line notation of \( \bar{u}' \), the numbers \( h(1) - M, h(1) - M + 1, \ldots, h(1) \) appear before the numbers \( h(1) + 1, h(1) + 2, \ldots, h(1) + m_0 \) by Lemma 5.14. Hence the assertion follows by Lemma 5.22. \( \square \)

Notice that

\[
D(h(1) - k) = 0 \quad (0 \leq k \leq p_{h,-}).
\]

Since \( s_{h(1)-k}(\xi_h) = \xi_h \) for all \( 1 \leq k \leq p_{h,-} \), this means from (5.31) that

\[
h(h(1) - k) = h(h(1)) - 2k \quad (0 \leq k \leq p_{h,-}).
\]

This leads us to define \( t_1, t_2, \ldots, t_{p_{h,-}} > 0 \) by

\[
(h(1) - k)^{(+)} = h(1) + t_k \quad (1 \leq k \leq p_{h,-}),
\]

which is equivalent to

\[
w_h(h(1) + t_k) = h(h(1) - k) + 1 = h(h(1)) - (2k - 1) \quad (1 \leq k \leq p_{h,-})
\]

by (5.31). See Figure 8. Since we have \( h(h(1)) - 1 > h(h(1)) - 3 > \cdots > h(h(1)) - (2p_{h,-} - 1) \), the maximality of the values of \( w_h \) implies that

\[
t_1 < t_2 < \cdots < t_{p_{h,-}}.
\]
Also, (5.33) implies that $h$ is stable at $h(1) + t_k$:
\[
h(h(1) + t_k - 1) = h(h(1) + t_k) \quad (1 \leq k \leq p_h, -).
\]
For example, we have $h(1) + t_1 = 11$ and $h(1) + t_2 = 13$ in the running example of $n = 20$ in Section 5.4.

![Figure 8. The definition of $t_k$.](image)

We use condition (iv) for $u'$ to prove the following lemma.

**Lemma 5.24.** The set $J_{h(1)}$ does not contain 1. In particular, we have $h(1) - p_{h,-} > 1$.

**Proof.** If $1 \in J_{h(1)}$, then we have $J_{h(1)} = J_1$ and $s_1(\xi_h) = \xi_h$. Lemma 5.2 then implies that
\[
h(1) < h(2) < \cdots < h(h(1)) < \cdots < h(h(1) + p_h + 1).
\]
Since $h$ is stable at $L_i$ by definition, this means that
\[
h(1) + p_h + 1 < L_i
\]
for all $1 \leq i \leq h(1)$. Hence condition (iv) for $u'$ implies (as in the proof of Proposition 5.19) that
\[
\bar{u}'w_h(i) < \bar{u}'w_h(j)
\]
for all $1 \leq i \leq h(1)$ and $i + 1 \leq j \leq h(1) + p_h + 1$. This means by definition that $\Delta_k = m_k - m_{k+1} = 0$ for all $k$, which is a contradiction. □

We note by (5.27) that the set in (5.26) is non-empty for $0 \leq k \leq M$:
\[
\{q_1, q_2, \ldots, q_{m_k}\} \neq \emptyset \quad (0 \leq k \leq M).
\]

**Lemma 5.25.** For $1 \leq k \leq M$, the following equality holds:
\[
w_{h}^{-1}(h(1) - k) = D(h(1) + t_k).
\]

**Proof.** We first show that $[h(1), h(1) + t_k] \subseteq J_{h(1)}$. By the previous lemma, we have $h(1) - (k+1) \geq 1$. Hence it follows from condition (iv) for $u'$ with $i = h(1) - (k+1) \geq 1$ that
\[
u'w_{h'}(h(1) - (k + 1)) < u'w_{h'}(j - 1)
\]
for $h(1) - k + 1 \leq j < i^{(+)'} + 1 (\leq L'_i)$, where we mean $i^{(+)'} = i^{(+)h'}$ and $L'_i = L_i(h')$.

Since we have $i^{(+)'} + 1 = (i + 1)^{(+)h'} = h(1) + t_k$ by the definition of $i = h(1) - (k + 1)$, this inequality and Corollary \[5.13\] imply that

$$\bar{u}'w_h(h(1) - k) < \bar{u}'w_h(j)$$

for $h(1) - k + 1 \leq j < h(1) + t_k$. This means that $h(1) + q_m$ cannot belong to the interval $[h(1) - k + 1, h(1) + t_k - 1]$ for any $1 \leq m \leq m_k$ because of \[5.29\], where we know that $\{q_1, q_2, \ldots, q_{m_k}\} \neq \emptyset$ as pointed out above. Since we have $h(1) + q_m \in J^+_{h(1)}$ by definition, this means that

$$(5.35) \quad h(1) + t_k \leq h(1) + q_m \leq h(1) + p_h + 1$$

for $1 \leq m \leq m_k$. From this, we obtain

$$(5.36) \quad [h(1), h(1) + t_k] \subseteq J_{h(1)}.$$  

This in fact implies the claim of this lemma as we prove in what follows.

We start with the case $k = 1$. Since the definition of $t_1$ means that $w_h(h(1) + t_1) = w_h(h(1)) - 1$ which is smaller than $w_h(h(1))$ by 1, we must have

$$S[h(1), j] \leq T[h(1), j] \quad \text{for } h(1) < j < h(1) + t_1,$$

$$S[h(1), h(1) + t_1] = T[h(1), h(1) + t_1] + 1,$$

where $S[k, \ell]$ and $T[k, \ell]$ are the numbers defined in the proof of Lemma \[5.9\]. This is because, if this does not hold, then $w_h(h(1) + t_1) \neq w_h(h(1)) - 1$ by the maximality of the values of $w_h$. The principle of similar shapes in the sense of Remark \[5.8\] now implies that the same holds on $[1, D(h(1) + t_1)]$. Namely,

$$S[1, j] \leq T[1, j] \quad \text{for } 1 < j < D(h(1) + t_1),$$

$$S[1, D(h(1) + t_1)] = T[1, D(h(1) + t_1)] + 1.$$

This means that

$$w_h(D(h(1) + t_1)) = w_h(1) - 1 = h(1) - 1.$$  

That is, $D(h(1) + t_1) = w_h^{-1}(h(1) - 1)$, as desired.

For $k \geq 2$, we can argue in a similar way on $[h(1) + t_{k-1}, h(1) + t_k]$ with the principle of similar shapes. \[\square\]

**Proposition 5.26.** Condition (i) holds for $u$. That is, the equality $\ell(uw_h) = \ell(u) + \ell(w_h)$ holds.

**Proof.** By Lemmas \[5.16\] and \[5.23\] it suffices to show

$$\ell(uw_h) = \ell(\bar{u}'w_h) + (m_0 + m_1 + \cdots + m_M).$$

Since $u = v_M \cdots v_1v_0\bar{u}'$, Lemma \[5.22\] implies that it is enough to prove

$$(\bar{u}'w_h)^{-1}(h(1) - k) < (\bar{u}'w_h)^{-1}(h(1) + m)$$

for $0 \leq k \leq M$ and $1 \leq m \leq m_k$.

When $k = 0$, we have $(\bar{u}'w_h)^{-1}(h(1)) = 1$ by Lemma \[5.14\]. In particular, we see that

$$(\bar{u}'w_h)^{-1}(h(1)) < (\bar{u}'w_h)^{-1}(h(1) + m)$$
for $1 \leq m \leq m_0$, as desired. Hence we may assume that $1 \leq k \leq M$ in the following. By Lemma 5.25 we have

$$w_h^{-1}(h(1) - k) = D(h(1) + t_k).$$

This means from Lemma 5.3 that

$$w_h^{-1}(h(1) - k) < w_h^{-1}(j) \quad \text{for all } j \geq h(1) + t_k. \quad (5.37)$$

Recalling that we are assuming $1 \leq k \leq M$, we know from (5.35) that

$$h(1) + t_k \leq h(1) + q_m \quad (1 \leq m \leq m_k). \quad (5.38)$$

Thus, for $1 \leq m \leq m_k$, we have

$$\bar{u}'w_h^{-1}(h(1) - k) = w_h^{-1}(h(1) - k) \quad \text{(by Lemma 5.14)}$$

$$< w_h^{-1}(h(1) + q_m) \quad \text{(by (5.37) and (5.38))}$$

$$= (\bar{u}'w_h)^{-1}(h(1) + m) \quad \text{(by (5.29) and } m \leq m_k \leq m_0), \quad \text{as desired}. \square$$

Our next aim is to prove condition (ii) for $u$. To give a proof, we need to know that $\bar{u}'w_h(J_{h(1)})$ does not contain $h(1) - k$ for any $0 \leq k \leq M$, which will be proved in Proposition 5.29. We prepare two lemmas for this.

**Lemma 5.27.** The following hold:

$$(\bar{u}'w_h)^{-1}(h(1)) < (\bar{u}'w_h)^{-1}(h(1) - 1) < \cdots < (\bar{u}'w_h)^{-1}(h(1) - M) \leq h(1) - p_{h,-}. \quad (5.39)$$

**Proof.** Note that the left-most number is equal to 1. This means that if $M = 0$, then there is nothing to prove. Hence we may assume $M \geq 1$ in the following. For $1 \leq k \leq M$, we have

$$(\bar{u}'w_h)^{-1}(h(1) - k) = w_h^{-1}(h(1) - k) \quad \text{(by Lemma 5.14)}$$

$$= D(h(1) + t_k) \quad \text{(by Lemma 5.25),} \quad (5.39)$$

which implies that

$$(\bar{u}'w_h)^{-1}(h(1) - 1) < (\bar{u}'w_h)^{-1}(h(1) - 2) < \cdots < (\bar{u}'w_h)^{-1}(h(1) - M).$$

Next, we write

$$r := (\bar{u}'w_h)^{-1}(h(1) - M),$$

and prove the right-most inequality $r \leq h(1) - p_{h,-}$ in the claim. The same computation as above shows that

$$r = w_h^{-1}(h(1) - M) = D(h(1) + t_M) < h(1) + t_M,$$

where the left-most equality shows that $w_h(r) < h(1)$, which implies that $h$ must be stable at $r$ by the definition of $w_h(r)$. Since $h$ is not stable at $k$ for all $h(1) - p_{h,-} < k \leq h(1)$ by (5.31), it suffices to show that

$$r \notin \{h(1) + 1, h(1) + 2, \ldots, h(1) + t_M\}. \quad (5.40)$$

Recall from (5.33) that

$$w_h(h(1) + t_M) = h(h(1)) - (2M - 1) = h(h(1) - M) + 1,$$

which is depicted in Figure. Now, $h$ is stable at both $r$ and $h(1) + t_M$ where $w_h$ takes the values $h(1) - M$ and $h(h(1) - M) + 1$, respectively. Since $h$ is a Hessenberg function,
the former value $h(1) - M$ is less than the latter value $h(h(1) - M) + 1$. Thus, by the maximality of the values of $w_h$, it follows that

$$r \notin \{h(1) - M + 1, h(1) - M + 2, \ldots, h(1) + t_M\}$$

(see Figure 9), which implies (5.40), as desired. □

As the following lemma indicates, condition (iv) ensures that $L_{h(i) - k} \in J_{h(i)}$ for $0 \leq k \leq M$.

**Lemma 5.28.** For $0 \leq k \leq M$, we have

$$L_{h(i) - k} \leq h(i) + p_h + 1.$$  

**Proof.** We know from (5.34) that $\{q_1, q_2, \ldots, q_{m_k}\} \neq \emptyset$ in (5.20). Hence it suffices to show for $q \in \{q_1, q_2, \ldots, q_{m_k}\}$ that

(5.41)  

$$L_{h(i) - k} \leq h(i) + q.$$  

We first note that $h(i) - k \geq 1$. Since $q \in \{q_1, q_2, \ldots, q_{m_k}\}$, we have

$$u'w_h(h(1) + q) < u'w_h(h(1) - k).$$

Condition (iv) for $u'$ with $i = h(1) - k$ now implies that

$$L_{h(i) - k} \leq h(i) + q$$

(cf. the proof of Proposition 5.19). □

In the proof of the next proposition, we will implicitly use the fact that if $h(i + 1) = h(i) + 2$, then

$$w_h(i^{(+)}) > 1.$$  

This is because $w_h(i^{(+)}) = h(i) + 1 > 1$ by the definition of $(+)$-operation.

**Proposition 5.29.** The set $J_{h(i)}$ does not contain $(\bar{u}'w_h)^{-1}(h(1) - k)$ for any $0 \leq k \leq M$.  

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{The value $w_h(h(1) + t_M)$.
\end{figure}
Proof. The claim for \( k = 0 \) follows from Lemma [5.24]. Suppose \((\bar{u}'w_h)^{-1}(h(1)−k) \in J_{h(1)}\) for some \( 1 \leq k \leq M \), and we will deduce a contradiction. Lemma [5.27] implies that we must have \( k = M \) and
\[
(\bar{u}'w_h)^{-1}(h(1)−M) = h(1)−p_{h−}.\tag{5.42}
\]
By (5.39), this means that
\[
D(h(1)+t_M) = h(1)−p_{h−},\tag{5.43}
\]
which implies that
\[
L_D(h(1)+t_M) = L_{h(1)−p_{h−}}.\tag{5.44}
\]

The equality \( h(1)+t_M = (h(1)−M)^{(+)\)} \) from (5.32) implies that \( L_{h(1)+t_M} = L_{h(1)−M} \) by the definition (5.3). This means that \([h(1)+t_M−1, L_{h(1)+t_M}] \subseteq J_{h(1)}\) by Lemma [5.28].

In addition, we have \( h(h(1)+t_M) \neq h(h(1)+t_M−1)+2 \) since \( h \) is stable at \( h(1)+t_M \) by definition. Thus we deduce from Lemma [5.10] that
\[
D(L_{h(1)+t_M}) = L_{h(1)−p_{h−}},
\]
where the second equality follows from (5.44). Since we have \( (h(1)−p_{h−})^{(+)\} < L_{h(1)−p_{h−}} \) by the definition (5.3), this implies that
\[
(h(1)−p_{h−})^{(+)\} < L_{h(1)−p_{h−}} = D(L_{h(1)+t_M}) < L_{h(1)+t_M}.
\]
By (5.32) and the definition (5.3) again, we deduce by this that \( h(1)+t_{p_{h−}} < L_{h(1)−M} \).

Thus, by Lemma [5.28] we obtain
\[
h(1)+t_{p_{h−}} < h(1)+p_{h+}.
\]
This means that (5.36) holds for \( 1 \leq k \leq p_{h−} \). Hence by the argument after (5.36), the claim of Lemma [5.25] holds for \( 1 \leq k \leq p_{h−} \). This implies that (5.39) holds for \( 1 \leq k \leq p_{h−} \), and thus we obtain
\[
(\bar{u}'w_h)^{-1}(h(1)−M) \leq (\bar{u}'w_h)^{-1}(h(1)−p_{h−}) \tag{5.39}
\]
(cf. the inequalities below (5.39)). Since (5.39) for \( k = p_{h−} \) is now valid, the latter half of the proof of Lemma [5.27] proves
\[
(\bar{u}'w_h)^{-1}(h(1)−p_{h−}) \leq h(1)−p_{h−}
\]
by replacing \( M \) by \( p_{h−} \) in the argument. From the last two inequalities and (5.42), it now follows that \( (\bar{u}'w_h)^{-1}(h(1)−M) = (\bar{u}'w_h)^{-1}(h(1)−p_{h−}) \), and hence \( M = p_{h−} \).

We now deduce a contradiction by using \( M = p_{h−} \). Let \( i = h(1)−M \). Then (5.43) now says that
\[
D(i^{(+)\}) = i
\]
(see Figure [10]). By Lemma [5.28] we know that
\[
[i, i^{(+)\}+1] \subseteq J_{h(1)}
\]
since \( i^{(+)\}+1 \leq L_i = L_{h(1)−M} \). This means that \( s_{i^{(+)\}} \in \mathcal{S}_J \), and hence we have
\[
h(i^{(+)\}+1) = h(i^{(+)\}) + 2
\]
by the case (1) of Lemma [5.7] (see Figure [10]). This means from the definition of the \((+)\)-operation that
\[
i^{(+)\} < (i^{(+)\})^{(+)\}.
\]
The previous proposition gives us information on the positions where \( \overline{u}'w_h \) takes values less than \( h(1) \). In contrast, the next lemma gives us lower bounds for the positions where \( \overline{u}'w_h \) takes values greater than \( h(1) \). We will use both properties to prove condition (ii) for \( u \).

**Lemma 5.30.** For \( 0 \leq k \leq M \) and \( 1 \leq l \leq m_k \), the following holds:

\[
L_{(\overline{u}'w_h)^{-1}(h(1)-k)} \leq (\overline{u}'w_h)^{-1}(h(1)+l).
\]

**Proof.** By (5.29), our claim is the same as

\[
L_{(\overline{u}'w_h)^{-1}(h(1)-k)} \leq w_h^{-1}(h(1)+q)
\]

for \( q \in \{q_1, q_2, \ldots, q_{m_k}\} \). We first consider the case \( k \geq 1 \). In this case, we know from (5.39) that

\[
(\overline{u}'w_h)^{-1}(h(1)-k) = D(h(1)+t_k).
\]

Hence, letting \( r := h(1) + t_k \), what we need to prove is

\[
(5.46) \quad L_{D(r)} \leq w_h^{-1}(h(1)+q).
\]
We know from Lemma 5.28 and (5.32) that $[r - 1, L_r] = [r - 1, L_{h(1) - k}] \subseteq J_{h(1)}$, and that $h$ is stable at $r$ so that we have $L_{D(r)} = D(L_r)$ by Lemma 5.10. Since $L_r = L_{h(1) - k} \leq h(1) + q$ by (5.32) and (5.41), it follows that

$$L_{D(r)} = D(L_r) \leq D(h(1) + q) \leq w^{-1}_h(h(1) + q),$$

where the right-most inequality follows from Lemma 5.3. This is exactly (5.46), as desired.

We next consider the case $k = 0$. In this case, we have $(\bar{u}'w_h)^{-1}(h(1)) = 1$ by Lemma 5.28 and hence what we need to prove is

$$L_1 \leq w^{-1}_h(h(1) + q)$$

for $q \in \{q_1, q_2, \ldots, q_{m_0}\}$ as above. Recall that we have $[h(1), L_{h(1)}] \subseteq J_{h(1)}$ from Lemma 5.28. Thus if we have

$$L_1 = D(L_{h(1)}),$$

then the argument used in the case $k \geq 1$ (i.e. the argument proving (5.47)) works to conclude (5.48) in this case as well. Hence let us prove (5.49) in what follows.

If $h(1) < h(2)$, then we have $D(h(1)) = 0$ and $D(h(1) + 1) = 1$. Since $s_{h(1)}(\xi_h) = \xi_h$, this and (5.3) means that $h(1) + 1 = h(1) + 1$. Hence we obtain $L_{h(1)} = L_{h(1) + 1}$ by the definition (5.3). This implies that $D(L_{h(1)}) = D(L_{h(1) + 1})$, and hence (5.49) follows from Lemma 5.10 in this case.

If $h(1) = h(2)$, then we have $L_1 = 2$. Also, it follows that $D(h(1) + 1) = D(h(1)) + 2$ in this case, and hence that $h$ is stable at $h(1) + 1$ by (5.9) since $s_{h(1)}(\xi_h) = \xi_h$. Thus we obtain $L_{h(1)} = h(1) + 1$ by the definition (5.3), and the right-hand side of (5.49) is equal to $D(h(1) + 1) = 2$ which agrees with the left-hand side $L_1 = 2$ so that (5.49) follows in this case as well.

We now prove condition (ii) for $u$ by using Proposition 5.29 and Lemma 5.30 as declared.

**Proposition 5.31.** Condition (ii) holds for $u$, that is, the equality $(uw_h)_J = u_J$ holds.

**Proof.** Take $1 \leq i \leq n - 1$ such that $s_i(\xi_h) = \xi_h$. It suffices to prove that for $j_1, j_2 \in J_i$, $uw_{h}(j_1) < uw_{h}(j_2)$ if and only if $u(j_1) < u(j_2)$.

We first consider the case $h(1) \in J_i$. Recall from (5.30) that we have $u(j_1) < u(j_2)$ if and only if $\bar{u}'w_{h}(j_1) < \bar{u}'w_{h}(j_2)$. Now, Proposition 5.29 shows that $\bar{u}'w_{h}(j_1)$ does not contain the numbers $h(1) - k$ for $0 \leq k \leq M$. This means that the order of the numbers in $\bar{u}'w_{h}(J_i)$ are the same as that of $uw_{h}(J_i)$ by Lemma 5.22. Hence it follows that $uw_{h}(j_1) < uw_{h}(j_2)$ if and only if $u(j_1) < u(j_2)$, as desired.

We next consider the case $h(1) \notin J_i$. In this case, the proof of Proposition 5.17 implies that $\bar{u}'w_{h}(j_1) < \bar{u}'w_{h}(j_2)$ if and only if $\bar{u}'(j_1) < \bar{u}'(j_2)$. We need to prove that the permutation $v_M \cdots v_1 v_0$ preserves this equivalence. Since $J_i \cap J_{h(1)} = \emptyset$ in this case, Lemma 5.14 implies that $\bar{u}'(J_i)$ does not contain $h(1) - k$ for any $0 \leq k \leq M$. This means that $\bar{u}'(j_1) < \bar{u}'(j_2)$ if and only if $u(j_1) < u(j_2)$ by Lemma 5.22. Hence if $\bar{u}'w_{h}(J_i)$ also does not contain $h(1) - k$ for any $0 \leq k \leq M$, then the assertion follows immediately. Thus we may assume that

$$h(1) - k \in \bar{u}'w_{h}(J_i) \quad \text{for some } 0 \leq k \leq M.$$
in the following. This means that $(\bar{u}'w_h)(J_i)$ contains some numbers which are increased by $v_M \cdots v_0$. Under this assumption, let us prove that
\begin{equation}
(5.51) \quad (v_{k-1} \cdots v_0) \bar{u}'w_h(j) \not\in \{(h(1) - k) + 1, (h(1) - k) + 2, \ldots, h(1) - k + m_k\}
\end{equation}
for all $j \in J_i$ and $0 \leq k \leq M$ satisfying (5.50).

Equivalently, if $k$ satisfies (5.50), then the set $(v_{k-1} \cdots v_0) \bar{u}'w_h(J_i)$ does not contain any number $\ell$ satisfying $h(1) - k < \ell \leq v_k(h(1) - k)$ so that $v_k$ preserves the order of the numbers in $(v_{k-1} \cdots v_0) \bar{u}'w_h(J_i)$. Hence if (5.51) is proved, then it follows that $v_M \cdots v_0$ preserves the order of the numbers in $\bar{u}'w_h(J_i)$, and we obtain the assertion of this proposition.

We prove (5.51) by using Lemma 5.30 in what follows. By (5.50) and Lemma 5.27, we know that $J_i$ lies left to $J_{h(1)}$ in the standard listing of $[n]$, where it holds that $J_i \cap J_{h(1)} = \emptyset$. In particular, we have $J_i \subseteq \{1, h(1) - 1\}$. This implies that
\begin{equation}
(5.52) \quad h(j + 1) = h(j) + 2 \quad (i - k_{i-} \leq j \leq i + k_i)
\end{equation}
by (5.9) since $D(a) = 0$ for $1 \leq a \leq h(1)$. See Figure 11. We claim that

\begin{equation}
(5.53) \quad (\bar{u}'w_h)^{-1}(h(1) - k) = i - k_{i-}.
\end{equation}

To see this, we take cases. If $k \geq 1$, then we see from Lemma 5.14 that
\begin{equation}
(\bar{u}'w_h)^{-1}(h(1) - k) = w_h^{-1}(h(1) - k),
\end{equation}
and we know that $h$ is stable at $w_h^{-1}(h(1) - k)$ since $h(1) - k < h(1)$. Thus (5.52) (see Figure 11) now implies that $w_h^{-1}(h(1) - k) = i - k_{i-}$ since $i - k_{i-}$ is the unique position in $J_i$ where $h$ can be stable. Hence we obtain (5.53) in this case. If $k = 0$, then (5.50) means that $1 \in J_i$ since $h(1) = \bar{u}'w_h(1)$ by Lemma 5.14, and hence we have $(\bar{u}'w_h)^{-1}(h(1) - k) = 1 = i - k_{i-}$. Namely, (5.53) holds in this case as well.

From (5.52), it also follows that $i + k_i + 1 < L_{i-k_i-}$. This is because $L_{i-k_i-}$ is greater than $i - k_{i-}$ and $h$ must be stable at $L_{i-k_i-}$. Now, (5.53) means that
\begin{equation}
i + k_i + 1 < L_{i-k_i-} = L_{(\bar{u}'w_h)^{-1}(h(1) - k)}.
\end{equation}
Hence Lemma 5.30 now implies that
\begin{equation}
(\bar{u}'w_h)^{-1}(h(1) + l) \not\in J_i \quad \text{for } 1 \leq l \leq m_k.
\end{equation}
We can rewrite this as
\[(5.54)\quad \bar{u}'w_h(j) \notin \{h(1) + 1, h(1) + 2, \ldots, h(1) + m_k\} \quad \text{for } j \in J_i.
\]
If \(k = 0\), then this is precisely \((5.51)\). Hence we may assume that \(k \geq 1\). Then \((5.54)\) implies that
\[(5.55)\quad v_0(\bar{u}'w_h(j)) \notin \{h(1), h(1) + 1, \ldots, h(1) - 1 + m_k\}
\]
since \(h(1) + m_k \leq h(1) + m_0\). If \(k = 1\), then this proves \((5.51)\). Hence we may assume that \(k \geq 2\). Then \((5.55)\) implies that
\[v_1v_0(\bar{u}'w_h(j)) \notin \{h(1) - 1, h(1), \ldots, h(1) - 2 + m_k\}
\]
since we have \(h(1) - 1 + m_k \leq h(1) - 1 + m_1\). It is clear that we can continue this argument to see \((5.51)\).

**Proposition 5.32.** Condition (iii) holds for \(u\). That is, if \(s_i(\xi_h) = \xi_h\) and \(h\) is strictly increasing on \(J_i\), then \(u(j) = j\) for all \(j \in J_i\).

**Proof.** To begin with, we show that \(J_{h(1)}\) does not satisfy the assumption of condition (iii). For that purpose, assume that the assumption of condition (iii) holds on \(J_{h(1)}\), that is,
\[h(h(1) - p_{h,-}) < h(h(1) - p_{h,-} + 1) < \cdots < h(h(1) + p_{h}) < h(h(1) + p_{h} + 1).
\]
This means that
\[h(1) + p_h + 1 < L_{h(1) - k} \quad (0 \leq k \leq p_{h,-})\]
since \(h(1) - k < L_{h(1) - k}\) and \(h\) must be stable at \(L_{h(1) - k}\), but this contradicts Lemma 5.28.

Assume that the assumption of condition (iii) holds on \(J_i\) (\(\neq J_{h(1)}\)). The same argument as that in the proof of Proposition 5.18 shows that
\[(5.56)\quad \bar{u}'(j) = j \quad \text{for } j \in J_i.
\]
Now recall that \(u = v_M \cdots v_1 v_0 \bar{u}'\) by definition, and that \(v_M \cdots v_1 v_0 \in S_J\) is in fact a permutation on \(J_{h(1)}\) by definition. Since \(J_i \cap J_{h(1)} = \emptyset\), it follows that \(v_M \cdots v_1 v_0\) is trivial on \(J_i\). This and \((5.56)\) mean that \(u(j) = j\) for \(j \in J_i\).

**Proposition 5.33.** Condition (iv) holds for \(u\). That is, for \(1 \leq i \leq n - 1\) and \(i + 1 \leq j < L_i\), we have \(uw_h(i) < uw_h(j)\).

**Proof.** The same argument as that in the proof of Proposition 5.19 shows that
\[\bar{u}'w_h(i) < \bar{u}'w_h(j) \quad \text{for } 1 \leq i \leq n - 1 \text{ and } i + 1 \leq j < L_i.
\]
We now prove that \(uw_h(i) < uw_h(j)\). Assume for a contradiction that this does not hold. Since \(uw_h = v_M \cdots v_1 v_0 \bar{u}'w_h\), we see by Lemma 5.22 that \(\bar{u}'w_h(i) = h(1) - k\) and \(\bar{u}'w_h(j) = h(1) + l\) for some \(0 \leq k \leq M\) and \(1 \leq l \leq m_k\), which implies that
\[i = (\bar{u}'w_h)^{-1}(h(1) - k),
\[j = (\bar{u}'w_h)^{-1}(h(1) + l).
\]
Hence it follows from Lemma 5.30 that \(L_i \leq j\) which contradicts the assumption \(j < L_i\) of condition (iv), as desired.
5.4. A pair of illustrating examples. In this subsection, we give an example of a pair of nef Hessenberg functions which illustrate the argument in Case 2-b in Section 5.3. Let \( n = 20 \), and \( h : [20] \to [20] \) the Hessenberg function depicted in Figure 12, that is,
\[
h = (9, 10, 11, 12, 12, 13, 15, 17, 18, 18, 19, 20, 20, 20, 20, 20, 20, 20, 20).
\]
Let \( h' : [19] \to [19] \) be the Hessenberg function given by \( h'(i) := h(i+1) - 1 \) for \( 1 \leq i \leq 19 \) as in Section 5.3 (see Figure 12):
\[
h' = (9, 9, 10, 11, 11, 12, 14, 16, 17, 18, 18, 19, 19, 19, 19, 19, 19, 19, 19).
\]

The weights of anti-canonical bundles of \( \text{Hess}(S, h) \) and \( \text{Hess}(S, h') \) are given by
\[
\xi_h = \varpi_1 + 2\varpi_2 + \varpi_3 + \varpi_4 + 2\varpi_5 + \varpi_6 + 2\varpi_{14} + \varpi_{15} + 2\varpi_{16} + \varpi_{17},
\]
\[
\xi_{h'} = 2\varpi_1 + \varpi_2 + \varpi_3 + 2\varpi_4 + \varpi_5 + \varpi_8 + 2\varpi_{13} + \varpi_{14} + 2\varpi_{15} + \varpi_{16},
\]
where \( \varpi_8 = \varpi_{h(1)} - 1 \) (cf. Lemma 5.13). In particular, this provides a pair of examples in Case 2-b since the coefficient of \( \varpi_9 = \varpi_{h(1)} \) in \( \xi_h \) is 0. From this, we see that
\[
J = \{J_9, J_{19}\} \text{ with } J_9 = \{7, 8, \ldots, 14\}, J_{19} = \{18, 19, 20\},
\]
and that
\[
J' = \{J'_7, J'_9, J'_{18}\} \text{ with } J'_7 = \{6, 7, 8\}, J'_9 = \{9, 10, \ldots, 13\}, J'_{18} = \{17, 18, 19\}.
\]
In one-line notation, we have
\[
w_h = 9 \ 8 \ 10 \ 11 \ 7 \ 12 \ 14 \ 16 \ 17 \ 15 \ 18 \ 19 \ 6 \ 5 \ 4 \ 3 \ 2 \ 1 \ (\text{see Figure 12}). \]
(see Figure 12). Take \( u' \in \mathfrak{S}_y \) as
\[
u' = 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 13 \ 10 \ 12 \ 9 \ 11 \ 14 \ 15 \ 16 \ 19 \ 18 \ 17.
\]
where we emphasized the positions of \( J'_{7}, J'_{9}, J'_{18} \) by enclosing the numbers of \( w_{k'} \) on \( J' \). Since these two equalities imply that

\[
u'w_{h'} = 13 8 10 12 7 9 14 16 19 15 18 11 17 6 5 4 3 2 1,
\]
it is now straightforward to verify that \( u' \) satisfies conditions (i)-(iv) directly. We also have

\[
w_{h} = 9 10 8 11 12 7 13 15 17 18 16 19 14 20 6 5 4 3 2 1,
\]
and hence it follows from (5.17) that

\[
\bar{u}' = 1 2 3 4 5 6 7 8 9 14 11 13 10 12 15 16 17 20 19 18,
\]
\[
\bar{u}'w_{h} = 9 14 8 11 13 7 10 15 17 20 16 19 12 18 6 5 4 3 2 1,
\]

where we emphasized \( J_{9} = J_{h(1)} \) and \( J_{19} \) by enclosing the numbers of \( w_{h} \) on \( J \). From this, we see that \( M = 1 < 2 = p_{h,-} \), and that \( m_{0} = 2, m_{1} = 1, m_{2} = 0 \) (see Figure 13).

According to the definition of \( u \) in Case 2-b, we let \( u := v_{1}v_{0}\bar{u}' = (s_{8})(s_{10}s_{9})\bar{u}' \in \mathcal{S}_{J} \).

It then follows that

\[
u = 1 2 3 4 5 6 7 9 11 14 10 13 8 12 15 16 17 20 19 18
\]

The modification from \( \bar{u}' \) to \( u = v_{1}v_{0}\bar{u}' \) makes the order of the numbers of \( u \) on \( J_{h(1)} \) be the same as the order of the numbers of \( \bar{u}'w_{h} \) on \( J_{h(1)} \). We visualize this in Figure 14.

Now \( uw_{h} \) is given by

\[
w_{h} = 11 14 9 10 13 7 8 15 17 20 16 19 12 18 6 5 4 3 2 1,
\]
and one can verify that \( u \) satisfies conditions (i)-(iv).
Figure 14. The positions of 1’s for $\bar{u}'$ and $u$.

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