Dispersion Relations
in Gauge Theories with Confinement

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Abstract

The analytic structure of physical amplitudes is considered for gauge theories with confinement of excitations corresponding to the elementary fields. Confinement is defined in terms of the BRST algebra. BRST-invariant, local, composite fields are introduced, which interpolate between physical asymptotic states. It is shown that the singularities of physical amplitudes are the same as in an effective theory with only physical fields. In particular, there are no structure singularities (anomalous thresholds) associated with confined constituents, like quarks and gluons. The old proofs of dispersion relations for hadronic amplitudes remain valid in QCD.

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It is the purpose of this talk, to give a survey of the problems involved in the derivation of analytic properties of physical amplitudes in gauge theories with confinement. This report is restricted to a brief resumé of the essential points discussed in the talk.  

Dispersion relations for amplitudes describing reactions between hadrons, and for form factors describing the structure of particles, have long played an important rôle in particle physics [3, 4, 5, 6]. Analytic properties of Green’s functions are fundamental for proving many important results in quantum field theory. With physical amplitudes being boundary values of holomorphic functions of several complex variables, appropriate different amplitudes can be associated with the same analytic function.  

In the past, analytic properties have been studied within the framework of a local field theory of hadrons, formulated in a state space of definite metric [7, 8, 9, 10]. Heisenberg fields associated with hadrons were introduced. They interpolate between asymptotic states of non-interacting physical particles, and they commute or anti-commute at space-like separations. For the Fourier transforms of retarded and advanced products of these operators, the local commutativity gives rise to tubes (wedges) \( W^\pm \) of holomorphy, while the boundary values at real point are tempered distributions. As a consequence of spectral conditions resulting from lower bounds for the spectrum of the energy-momentum operator, one obtains real domains \( R \), where retarded and advanced amplitudes coincide as distributions. Under these conditions, the Edge of the Wedge Theorem [7] is applicable. It provides an analytic function in the union of the wedges \( W = W^+ \cup W^- \) and a finite, complex neighborhood \( N(R) \) of the real domain \( R : W \cup N(R) \). The task is then

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\(^3\)For more detailed discussions, I refer to the report covering my talk at the 1994 ICMP in Paris [1], and to the articles [2].
to construct the *Envelope of Holomorphy* $E(W \cup N(R))$ of this basic region of analyticity [11, 7, 9]. The envelope is the largest domain into which all functions, which are regular in the basic region $W \cup N(R)$, can be continued. It is characteristic for the input, and additional information is needed in order to get larger regions of analyticity. More detailed use of unitarity, would be required, and of specific features of the spectrum, but these conditions are difficult to implement [12].

The limitations of general derivations can usually be understood as being related to singularities which describe structure due to unphysical constituents of the physical particles involved in the amplitude (unphysical anomalous thresholds). The simple spectral conditions used are not sufficient to eliminate these constituents.

It is sometimes helpful to turn the problem around, and ask for singularities of an amplitude, which are definitely expected on the basis of the spectrum of the theory. Here weak coupling perturbation expansions of the effective hadronic theory can be very useful. Even though these expansions may not provide a reasonable approximation of the amplitude itself, they can be a guide to the location of singularities [13].

There are many details involved in the derivation of dispersion relations and related analytic properties of Green’s functions [7, 9, 14, 15, 16], but the path via the Edge of the Wedge Theorem, and the Envelope of Holomorphy, is the essence of the problem. Although the theory of functions with several complex variables is the natural framework for obtaining regions of analyticity, in special cases, like those involving only one complex four-vector, methods from the theory of differential equations and of distributions can be used in order to obtain the analytic domain corresponding to the envelope discussed above [3, 4].
In this talk, we are interested in the analytic structure of physical amplitudes in gauge theories with confinement. We will often use the language of QCD. In these theories, the spectrum is not related to the elementary Heisenberg fields appearing in the initial formulation. Rather, the elementary fields correspond to unphysical, confined excitations. In the effective hadron theories discussed above, the derivation of dispersion relations, and of other analytic properties, is quite rigorous, given the basic axioms of quantum field theory in a state space with definite metric. In contrast, for gauge theories with confinement, we will have to use several features of the theory, which have not been proven rigorously in the non-perturbative framework. The required assumptions will be discussed in the following. They are mainly concerned with the definition of confinement with the help of the BRST-algebra, and with the construction of local, BRST-invariant physical fields as composites of confined Heisenberg fields.

Since we require a covariant formulation of the theory, we must use a quantization in a state space $\mathcal{V}$ of indefinite metric \cite{17}. This space contains quanta like ghosts and longitudinal and space-like gluons, which are unphysical even in the weak coupling limit. We use the BRST-algebra \cite{18} in order to define an invariant, physical state space $\mathcal{H}$ with positive definite metric as a cohomology of a nilpotent BRST-operator $Q$. The assumptions involved here are the non-perturbative existence of a BRST-operator, and its completeness \cite{19,20}. The latter notion implies that all states $\Psi \in \mathcal{V}$, which satisfy $Q\Psi = 0$, and which have zero norm, are of the form $\Psi = Q\Phi$, $\Phi \in \mathcal{V}$. This means, that states with zero ghost number, containing ghost-antighost pairs, are eliminated. They have indefinite metric and would make it impossible to define a physical state space with definite norm. There are arguments for completeness, but I do not know of a general proof in four-dimensional
gauge theories like QCD. In certain string theories, completeness has been proven explicitly, but these are more simple structures.

Already in weak-coupling perturbation theory, the ghosts and the transverse and time-like gluons are eliminated from the physical state space $\mathcal{H}$ in a kinematical fashion. They are not color singlet states, but form quartet representations of the BRST-algebra. In the full theory, we expect that also quarks and transverse gluons are confined, and do not appear as elements of $\mathcal{H}$, at least at zero temperature. They also form quartet representations, together with other unphysical states. With certain limitations concerning the number of flavors (less than ten for QCD), we have given arguments that, for dynamical reasons, transverse gluons cannot be elements of the cohomology space $\mathcal{H}$ \cite{20}. Some more preliminary methods also exclude quarks \cite{21}. Our arguments for confinement are based upon superconvergence relations for the gluon propagator \cite{22,23}, and they involve renormalization group methods. These arguments are valid for zero temperature. At finite temperatures, a new dimensionful parameter is present, and there may be de-confinement. If our methods are applied to certain $N = 1$ SUSY models \cite{24}, as far as the number of flavors is concerned, they agree with results obtained on the basis of duality and holomorphy of the superpotential \cite{25}. An approximately linear quark-antiquark potential is obtained on the basis of superconvergence, with the same restrictions for the number of flavors \cite{26,27}.

Confinement of quarks and gluons does not necessarily imply the existence of massive states in $\mathcal{H}$, which can be interpreted as hadrons. But, possibly with further restrictions of the number of flavors, the existence of hadrons may be a reasonable assumption. $N = 1$ SUSY models are encouraging in this respect. For our purpose, we assume that the BRST singlet states, which span $\mathcal{H}$, are hadrons.
As we have mentioned, local, interpolating Heisenberg fields are the basis for obtaining analytic properties. The support of retarded and advanced amplitudes implies Fourier transforms, which are analytic in wedges containing no real points. In QCD, we need to construct local, composite operators, which interpolate between non-interacting, asymptotic hadron fields. These local Heisenberg fields of hadrons must be BRST invariant operators, constructed from the elementary fields associated with confined quarks and gluons. The construction of local composite operators has been studied extensively in quantum field theory \[28\], in particular as leading terms of operator product expansions \[29\]. These have been derived in renormalized, weak coupling perturbation theory, but here we use the composite fields in a non-perturbative framework. It is important to realize, that the local character of these fields is related to the center-of-mass motion of the constituents, and does it not imply a point-like structure of the bound system. In quantum field theory, the extended distribution of a particle, viewed as a composite of other particles, is described by the anomalous thresholds (structure singularities) of form factors, scattering amplitudes, and other Green’s functions \[30, 31, 32, 33\]. These singularities are particularly prominent for loosely bound systems, like the deuteron, for example, where the range of the wave function is much larger than the size of the pion cloud.

We find, that local, composite fields are a common feature of causal field theories. We assume that possible embeddings of QCD into more comprehensive schemes are not important for confinement, and for scattering processes well below the Planck mass. If local field theory is considered as a low energy limit of string theory, we may perhaps expect deviations from microscopic causality at very small distances, and corresponding corrections to dispersion relations \[10\].
The other important aspect of the composite hadron fields is their BRST-invariance. If applied to an invariant ground state $\Psi_0$ with $Q\Psi_0 = 0$, these fields generate states $\Psi$, which again satisfy $Q\Psi = 0$. Hence the states $\Psi$ are also representatives of physical states. As has been described in \cite{1,2}, if we consider matrix elements of products of BRST-invariant Heisenberg fields between physical states, any decomposition with respect to a complete set of intermediate states in $\mathcal{V}$ requires only a subset of these states which form a complete set in $\mathcal{H}$. These features are most important for the unitarity of the $S$-matrix and for dispersion relations. They imply that only hadronic states play a rôle as absorptive thresholds in various channels of a hadronic amplitude. It follows, that the spectral conditions for hadronic amplitudes are the same as in the old, effective theory. Our definition of confinement implies, that in the collision of hadrons only hadrons are produced as final states.

The construction of local, interpolating hadron fields is not unique. There are equivalence classes of different fields, which have the same asymptotic fields and give rise to the same $S$-matrix. This is a consequence of Borcher’s theorem \cite{35}, which we use here in the physical state space with positive definite metric, although it can be generalized to indefinite metric spaces.

Some of the properties of hadronic amplitudes in QCD, which we have discussed here, may appear to be straightforward, once physical amplitudes are expressed in terms of BRST-invariant, local operator fields. There are, however, many subtle points due to the indefinite metric of the full state space $\mathcal{V}$ \cite{19,36,20}. There is no simple projection into the invariant space $\mathcal{H}$. Unphysical states in $\mathcal{V}$ may well have components in the physical space.

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\[4^4\text{See }\cite{34}\text{ for a discussion of classical, local and gauge invariant composite fields in QCD. I would like to thank Professor Divakaran for bringing this paper to my attention.}\]
or they may acquire such components after a Lorentz transformation. An unphysical state is recognized only by the fact that there exists an equivalence transformation, which removes the component in the physical space without changing the observables of the theory. A detailed discussion of unitarity and of the spectral conditions is therefore indicated [1, 17].

The features of hadronic fields described above apply to absorptive thresholds in a given channel of physical amplitudes. It remains to discuss anomalous thresholds or structure singularities, which have been described earlier for the case of observable constituents. The important question is, whether there are singularities of hadronic amplitudes, which are related to the quark-gluon structure of the hadrons. From the BRST-invariance of the hadron fields, we infer implicitly, that such singularities cannot be present, but a more explicit understanding is desirable. We have shown in [33], that anomalous thresholds are due to poles and absorptive branch points of other hadronic amplitudes, which are related to the one under consideration by analytic continuation into appropriate lower Riemann sheets. In this way, in a non-perturbative manner, we relate structure singularities to absorptive thresholds, and these are only hadronic if quarks and gluons are confined. As a consequence, there are no anomalous thresholds associated with confined quarks and gluons.

We have already mentioned the example of the deuteron as a composite system with loose binding and observable constituents. The form factor of the deuteron is dominated by anomalous thresholds well below the pion branch points. The resulting distribution is as expected on the basis of the Schrödinger wave function. For hadrons, which may be considered as loosely bound systems of heavy quarks, the situation with respect to the quark-gluon structure is completely different. Here the constituents are confined,
and there are no anomalous thresholds describing a long-range quark structure, which one may expect from a constituent quark model on the basis of the Schrödinger wave function. However, there is no problem in obtaining a large mean-square radius with an appropriate form of the discontinuities associated with hadronic thresholds [2, 37, 1]. Where applicable, also hadronic anomalous thresholds may contribute to an extended distribution.

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