THE PRECISE RANGE OF INDICES FOR
THE RH\(_r\)- AND A\(_p\)- WEIGHT CLASSES

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Abstract. For \(w \in A_p(RH_r)\) we determine the precise range of indices so that
\(w \in RH_r(A_p)\), the precise range of \(q < p\) for which \(w \in A_q\), and the precise range of
\(\tau > 1\) for which \(w^\tau \in A_p\).

1. Introduction. The weight classes which we will study in this paper control
weighted norm inequalities for the Hardy-Littlewood maximal operator, singular
integral operators, etc., and are defined as follows. A weight \(w\), i.e., \(w : \mathbb{R}^n \to \mathbb{R}^+\), is
in the reverse Hölder class \(RH_r\), \(1 < r < \infty\), provided
\[
\left( \frac{1}{|Q|} \int_Q w^r \right)^{1/r} \leq c \left( \frac{1}{|Q|} \int_Q w \right),
\]
and \(w \in A_p\) iff
\[
\frac{1}{|I|} \int_I w \left( \frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p'-1} \leq c \quad \text{(see [2])},
\]
where \(Q\) is an arbitrary cube in \(\mathbb{R}^n\). These classes are related, and we have \(A_\infty = \cup_{p<\infty} A_p = \cup_{1<r} RH_r\).

The purpose of this paper is the following.

1. For \(w \in A_p\) find the precise range of \(r\)'s such that \(w \in RH_r\), the precise range
of \(q < p\) for which \(w \in A_q\), and the precise range of \(\tau > 1\) such that \(w^\tau \in A_p\).

2. For \(w \in RH_r\) find the precise range of \(p\)'s such that \(w \in A_p\), and the precise
range of \(q > r\) for which \(w \in RH_q\).

The higher integrability of \(w \in RH_r\) is due to Gehring [3], i.e., there exists
\(p > r\) such that \(w \in RH_p\). Closely related to this is the property that \(w \in A_p\)
implies the existence of \(q < p\) such that \(w \in A_q\). The exact range of higher order
integrability for \(w \in RH_r\) has been found recently by Kinnunen [4] for \(n = 1\).
The sharp upper index is given as the solution to \(c^{\frac{p-r}{r} - (p')^r = 1}\), where \(c\) is the
constant in \(\frac{1}{|I|} \int_I w^r \leq c \left( \frac{1}{|I|} \int_I w \right)^r\). Our upper bound has a different form and
depends upon appropriate factorizations of \(w\). Apart from factorizations our proofs
will be along the lines of [4].

In sections 2 and 3 our analysis will be in \(\mathbb{R}\), and \(\mathbb{R}^n\)-versions of these results
will be taken up in section 4.

2. The classes \(A_1\) and \(RH_\infty\). Throughout \(I\) will denote an arbitrary
interval in \(\mathbb{R}\). For \(w \in A_1\), we let \(A_1(w)\) be the inf of all constants \(c\) for which

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\[ \frac{1}{|I|} \int_I w \leq c \inf_I w, \text{ and for } w \in RH_\infty, \text{ we denote by } RH_\infty(w) \text{ the inf of all constants } c \text{ such that } \sup_I w \leq c \frac{1}{|I|} \int_I w \text{ for all intervals } I. \text{ The classes } A_1 \text{ and } RH_\infty \text{ generate } A_\infty \text{ in the sense that } A_\infty = A_1 \cdot RH_\infty, \text{ and } RH_\infty \text{ plays the same role relative to } RH_r \text{ as } A_1 \text{ does to } A_p[1]. \]

**Theorem 1.** Let \( w \in A_1 \) and let \( c = A_1(w) \). Then \( w \in RH_r \) for all \( 1 < r < \frac{c}{c-1} \). The range of \( r \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** This is one of the results in Kinnunen’s thesis [4]. The proof consists by showing first that for \( \lambda \geq \inf_I w \equiv \lambda_I \)

\[ \int_{\{x \in I : w(x) > \lambda\}} w \, dx \leq c\lambda \{x \in I : w(x) > \lambda\} \]

where \( c = A_1(w) \). This is a reverse Chebyshev inequality and is in fact equivalent with \( w \in A_1 \) [4]. Next multiply this inequality by \( \lambda', r > -1 \) and integrate with respect to \( \lambda \) from \( \lambda_I \) to \( \infty \). We shall be more precise in section 4 when we treat the \( \mathbb{R}^n \) case using the same type of distributional inequality. The weight \( x^{1/c-1}, c > 1 \) shows that the range of \( r \) is best possible on \( \mathbb{R}_+ \).

**Remark:** A. Torre pointed out to me that on \( \mathbb{R} \) the weight \( w(x) = |x|^{1/c-1}, c > 1 \) does not provide an example for the precise range since \( A_1(w) \leq 2c \). This is easily seen by considering intervals \( I = [a, b] \) with \( a < 0 < b \) and \( |a| \leq b \). However, if we define for \( w \in A_1 \), \( c_w = \inf c \) for which \( \frac{1}{|I|} \int_I w \leq c \inf_I w \) for all intervals \( I \subset \mathbb{R} \setminus \{0\} \), then \( w \in RH_p \), \( 1 < p < c_w/(c_w - 1) \). To prove this, observe first that \( w \in RH_p(\mathbb{R}_+) \cap RH_p(\mathbb{R}_-) \) for \( 1 < p < c_w/(c_w - 1) \) by Theorem 1. If \( I = [a, b] \) with \( a < 0 < b, |a| \leq b \), and if \( I_0 = [-b, b] \), then

\[
\frac{1}{|I|} \int_I w^p \leq \frac{2}{|I_0|} \int_{-b}^b w^p \leq \frac{1}{b} \int_{-b}^0 w^p + \frac{1}{b} \int_0^b w^p \leq c \left( \frac{1}{b} \int_{-b}^0 w^p \right)^{1/p} + \left( \frac{1}{b} \int_0^b w^p \right)^{1/p} \leq c_p \left( \frac{1}{|I|} \int_I w \right)^{1/p},
\]

for \( 1 < p < c_w/(c_w - 1) \). This range is best possible, as the weight \( w(x) = |x|^{1/c-1}, c > 1 \), shows. The same type of remark applies to the remaining Theorems of this note.

**Theorem 2.** Let \( w \in RH_\infty \) and let \( c = RH_\infty(w) \). Then \( w \in A_p \) for all \( p > c \), and the range of \( p \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** We first show that for \( 0 < \lambda \leq \lambda_I \equiv \sup_I w \)

\[ \lambda \{x \in I : w(x) < \lambda\} \leq c \int_{\{x \in I : w(x) < \lambda\}} w \]

where \( c = RH_\infty(w) \). If \( G \) is an arbitrary open set with \( I \cap \{w < \lambda\} \subset G \subset I \), then we have to show that \( L \leq c \int_G w \), where \( L \) is the left side of the above inequality.
There are two cases. If $|G| = |I|$, then $L \leq \lambda |G| = \lambda |I| \leq c \int_G w = \int_G w$. If $|G| < |I|$, we cover $G$ by a disjoint union of intervals $\{I_j\}$ maximal with respect to $|I_j \setminus G| = 0$. Then $|G| = \sum |I_j|$ and $|\sigma I_j \setminus G| > 0$ for every $\sigma > 1$, where $\sigma I$ is the interval concentric with $I$ and of length $\sigma |I|$. It follows that $\sup w \geq \lambda$ and thus for every $\sigma > 1$

$$\lambda |\sigma I_j| \leq c \int_{\sigma I_j} w.$$ 

We now let $\sigma \searrow 1$ and obtain $L \leq \lambda \sum |I_j| \leq c \sum \int_{I_j} w = c \int_G w$.

Below we need the finiteness of $\int_I 1/w$ and for that purpose we truncate $w$ from below, i.e., for $0 < \beta < \lambda \leq \lambda_I$ let

$$w_\beta(x) = \begin{cases} w(x), & w(x) > \beta \\ \beta, & w(x) \leq \beta. \end{cases}$$

Since $RH_\infty(w_\beta) \leq c$ we have the same inequality (2) with $w$ replaced by $w_\beta$. We multiply this inequality by $\lambda^{-r}$, $r > 2$ and integrate to get

$$\int_\beta^{\lambda_I} \frac{1}{\lambda^{r-1}} \int \chi_{\{w_\beta < \lambda\} \cap I}(x) \, dx \, d\lambda \leq c \int_\beta^\infty \frac{1}{\lambda^r} \int_{\{w_\beta < \lambda\} \cap I} w \, dx \, d\lambda.$$ 

In the left side of the above inequality we interchange the order of integration and obtain

$$\int_I \int_\beta^{\lambda_I} \frac{1}{\lambda^{r-1}} \, d\lambda \, dx = \frac{1}{r-2} \int_I \left( \frac{1}{w_\beta^r} - \frac{1}{\lambda_I^r} \right)$$

$$= \frac{1}{r-2} \int_I \frac{dx}{w_\beta^r} - \frac{1}{r-2} \frac{|I|}{\lambda_I^r}.$$ 

Similarly, the right side equals

$$\int_I \int_\beta^\infty w(x) \frac{1}{\lambda^r} \, d\lambda \, dx = \frac{1}{r-1} \int_I \frac{dx}{w_\beta^r}.$$ 

Hence

$$\int_I \frac{dx}{w_\beta^r} \leq c \frac{r-2}{r-1} \int_I \frac{dx}{w_\beta^r} + \frac{|I|}{\lambda_I^r}.$$ 

Next choose $r > 2$ so that $c(r-2) < r-1$. Since the integrals involved are $< \infty$ we see that

$$C \int_I \frac{dx}{w^r_\beta} \leq \frac{|I|}{\lambda^r_I}.$$
where \( \alpha = r - 2 \). We let \( \beta \searrow 0 \) and get that \( w^{-\alpha} \in A_1 \). This, of course, immediately implies that \( w \in A_{1+1/\alpha} \). Also observe that \( c\alpha < \alpha + 1 \) is equivalent with \( p = 1 + 1/\alpha > c \).

The weight \( w(x) = x^{c-1} \), \( c > 1 \) has \( RH_{\infty}(w) = c \) on \( \mathbb{R}_+ \) and is in \( A_p \) for \( p > c \), but not in \( A_c \).

**Remark:** (i) It is easy to see that (2) is actually equivalent with \( w \in RH_{\infty} \). (ii) For an example of the best range on \( \mathbb{R} \) proceed as in the remark after Theorem 1.

3. The classes \( A_p \) and \( RH_r \). In this section we use Theorems 1, 2 for the extension to the classes \( A_p \) and \( RH_r \), and, as we shall see, the range of the indices will be governed by factorizations of the weight. In [1] we have shown that \( w \in RH_r \) iff \( w = w_0w_1 \) where \( w_0 \in RH_{\infty} \) and \( w_1 \in RH_r \cap A_1 \). But then \( w_1 \in A_1 \), and thus \( w \in RH_r \) iff \( w = uv^{1/r} \) with \( u \in RH_{\infty} \) and \( v \in A_1 \). We shall also use the factorization [2] for \( A_p \), i.e., \( w \in A_p \) iff \( w = uv^{1/p} \) with \( u, v \in A_1 \).

**Theorem 3.** Let \( w = uv^{1/r} \) be in \( RH_r \) with \( u \in RH_{\infty} \) and \( v \in A_1 \), and let \( c_1 = RH_{\infty}(u) \). Then \( w \in A_p \) for all \( p > c_1 \). This range of \( p \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** Let \( p > c_1 \). Then

\[
\frac{1}{|I|} \int_I w^{1/r} \left( \frac{1}{|I|} \int_I (uv^{1/r})^{1-p'} \right)^{p-1} \leq \sup_I u \frac{1}{|I|} \int_I v^{1/r} \left( \frac{1}{|I|} \int_I u^{1-p} \right)^{p-1} \leq \sup_I \frac{1}{|I|} \int_I v^{1/r} \leq C,
\]

since \( u \in A_p, p > c_1 \) by Theorem 2.

The weight \( w(x) = x^{c_1-1} \) which is in \( RH_{\infty} \subset RH_r \) shows that the range of \( p \)'s is best possible on \( \mathbb{R}_+ \).

The next result will give us the precise range of higher integrability of \( w \in RH_r \).

**Theorem 4.** Let \( w = uv^{1/r} \) be in \( RH_r \) with \( u \in RH_{\infty} \) and \( v \in A_1 \). If \( c_2 = A_1(v) \), then \( w \in RH_p \) for all \( r \leq p < c_2r/(c_2 - 1) \). The range of \( p \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** Let \( p \) satisfy the above inequality, and then choose \( q > 1 \) such that

\[
p < \frac{c_2r}{q(c_2 - 1)}.
\]

Since \( 1 \leq \frac{pq}{r} < \frac{c_2}{c_2 - 1} \) by Theorem 1, \( v \in RH_{qp/r} \). This and Hölder’s inequality gives us

\[
\frac{1}{|I|} \int_I u^p = \left( \frac{1}{|I|} \int_I u^{q/p} \right)^{1/q} \left( \frac{1}{|I|} \int_I v_{qp/r} \right)^{1/q} \leq \sup_I u^p \cdot c \left( \frac{1}{|I|} \int_I v \right)^{p/r} \leq c' \sup_I u^p \cdot \inf_I v^{p/r} \leq c'' \left( \frac{1}{|I|} \int_I w^{1/r} \right)^p,
\]
since \( w \in RH_{\infty} \).

Let \( 0 < \alpha < 1 \) and consider the weight \( w(x) = x^{-\alpha} \) on \( \mathbb{R}_+ \). Then \( w \in RH_r \) for \( 1 < r < 1/\alpha \). We fix such an \( r \) and write \( w = v^{1/r} \), \( v = x^{-\alpha r} \). This is the factorization \( w = uv^{1/r} \) with \( u \equiv 1 \). Since \( A_1(v) = c_2 = 1/(1-\alpha r) \) we see that \( c_2 r/(c_2-1) = 1/\alpha \) which is the precise upper bound of higher integrability for this weight.

For the next Theorem we need the fact [1] that \( v \in A_1 \) implies that \( (1/v)^{\gamma} \in RH_{\infty} \) for every \( \gamma > 0 \).

**Theorem 5.** Let \( w = uv^{1-p} \) be in \( A_p \) with \( u, v \in A_1 \), and let \( c = A_1(u) \). Then \( w \in RH_r \) for all \( 1 < r < c/(c-1) \). The range of \( r \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** We use Theorem 1 and observe that

\[
\frac{1}{|I|} \int_I w^r = \frac{1}{|I|} \int_I u^r \frac{1}{v^{r(p-1)}} \leq \sup_I \frac{1}{v^{r(p-1)}} \left( \frac{1}{|I|} \int_I u \right)^r 
\]

\[
C A_1(u)^r \sup_I \frac{1}{v^{r(p-1)}} \cdot (\inf_I u)^r 
\]

\[
C A_1(u)^r \cdot c^r \left( \frac{1}{|I|} \int_I \frac{1}{v^{p-1}} \cdot \inf_I u \right)^r 
\]

\[
C A_1(u)^r c^r \left( \frac{1}{|I|} \int_I w \right)^r .
\]

The example in Theorem 1 shows that the range of \( r \)'s is best possible on \( \mathbb{R}_+ \).

**Corollary.** Let \( w = uv^{1-p} \) be in \( A_p \) with \( u, v \in A_1 \). If \( c = \max\{A_1(u), A_1(v)\} \), then \( w^r \in A_p \) for \( 1 \leq r < c/(c-1) \) and the range of \( r \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** From Theorem 5 we have that \( w \) and \( w^{1-\tau} \) are in \( RH_r \) for \( 1 \leq \tau < c/(c-1) \) and hence \( w^r \) is in \( A_p \). The weight \( w(x) = x^{-\alpha}, 0 < \alpha < 1 \) is in \( A_1 \subset A_p \) on \( \mathbb{R}_+ \), and thus we can take \( u = x^{-\alpha}, v \equiv 1 \). Then \( c = 1/(1-\alpha) \) and thus \( c/(c-1) = 1/\alpha \), which is best possible.

We will now use Theorem 5 to get the exact range on \( q < p \) such that \( w \in A_p \) implies \( w \in A_q \).

**Theorem 6.** Let \( w = uv^{1-p} \) be in \( A_p \) with \( u, v \in A_1 \) and let \( c_* = A_1(v) \). Then \( w \in A_q \) for all \( q \) satisfying

\[
\frac{(p-1)(c_* - 1)}{c_*} + 1 < q \leq p.
\]

The range of \( q \)'s is best possible on \( \mathbb{R}_+ \).

**Proof.** Since \( w^{1-\tau} = vu^{1-\tau} \) is in \( A_p \), we get from Theorem 5 that \( w^{1-\tau} \in RH_r \) for \( 1 < r < c_*/(c_*-1) \). Hence since \( w \in A_p \)

\[
\frac{1}{|I|} \int_I w \left( \frac{1}{|I|} \int_I w^{1-\tau} \right)^{(p-1)/r} \leq c \frac{1}{|I|} \int_I w \left( \frac{1}{|I|} \int_I w^{1-\tau} \right)^{p-1} \leq C < \infty.
\]
Thus \( w \in A_q, \quad q = 1 + (p - 1)/r, \) i.e., \((p - 1)(c_* - 1)/c_* + 1 < q \leq p.\)

We will now show that this range is best possible. Let on \( \mathbb{R}_+, \) \( w(x) = x \) and fix \( p_0 > 2. \) Then \( w \in A_{p_0} \) and \( w = v^{1-p_0} \) with \( v = x^{1-p_0} \in A_1. \) Since \( A_1(v) = 1/(2-p_0) = c_* \) the lower bound of the range of \( q \)'s given above is 2. It is known [2] that \( w \in A_q \) exactly for \( q > 2.\)

4. Extensions to \( \mathbb{R}^n. \) The results in \( \mathbb{R}^n, n > 1, \) are somewhat different from the \( n = 1 \) versions, and the reason is that we do not know whether the covering of an open set used in Theorem 2, i.e., \( G \subset \bigcup I_j, \) where the \( I_j \)'s are disjoint, \( |I_j \setminus G| = 0, \) and for \( \sigma > 1, \) \( |\sigma I_j \setminus G| > 0, \) has a corresponding analogue in \( \mathbb{R}^n. \) To avoid this difficulty, we consider for \( k \geq 3 \) the classes \( A_{1,k} \) and \( RH^c_{\sigma,k} \) defined as follows. We say that \( w \in A_{1,k} \) iff \( \frac{1}{|Q|} \int_Q w \leq c'_k \inf_{kQ} w, \) and we say that \( w \in RH^c_{\sigma,k} \)

iff \( \sup_{kQ} w \leq c''_k \frac{1}{|Q|} \int_Q w, \) where \( Q \) is the generic notation of a cube in \( \mathbb{R}^n \) and \( kQ \) is the cube concentric with \( Q \) having side-length \( k \times \) the side-length of \( Q. \)

It is easily seen that \( A_{1,k} = A_1 \) and \( RH^c_{\sigma,k} = RH^c_{\sigma}. \) The first equality is obvious, and in the second use the fact that \( w \in RH^c_{\sigma} \) is doubling. It is also clear that \( A_1(w) \leq c'_k \) and \( RH^c_{\sigma}(w) \leq c''_k. \)

**Theorem 7.** Let \( w \in A_{1,k} \) with constant \( c'_k. \) Then \( w \in RH_r \) for \( 1 < r < c'_k/(c'_k - 1). \)

**Proof.** This is the \( n \)-dimensional version of Theorem 1, and we follow the proof in [4]. We first establish

\[
\int_{\{x \in Q : w(x) > \lambda\}} w \leq c'_k |\{x \in Q : w(x) > \lambda\}|
\]

for every \( \lambda \geq \inf_Q w = \lambda_Q. \) To prove this, let \( G \) be an arbitrary open set with \( \{w > \lambda\} \cap Q \subset G \subset Q. \) We need to show that

\[
\int_G w \leq c'_k |G|.
\]

This follows if \( |Q| = |G|. \) If \( |G| < |Q|, \) we let \( \Delta \) be the collection of the dyadic cubes in \( Q \) generated by \( Q. \) For each \( x \in G, \) let \( \tilde{Q}_1(x) \in \Delta \) be the cube of least side-length containing \( x \) such that \( |\tilde{Q}_1 \setminus G| > 0. \) The next generation dyadic subcube of \( \tilde{Q}_1 \) containing \( x, \) say \( Q_1(x) \) has the property that \( |Q_1(x) \setminus G| = 0. \) Since we are dealing with dyadic cubes, we can in this way write \( G \subset \bigcup Q_j, \) where the \( Q_j \)'s are non-overlapping, \( |Q_j \setminus G| = 0, \) and \( |kQ_j \setminus G| > 0 \) for each \( j. \) Thus \( \inf_{kQ_j} w \leq \lambda. \)

From this we see that

\[
\int_G w \leq \sum_j \int_{Q_j} w \leq c'_k \lambda \sum_j |Q_j| = c'_k \lambda |G|.
\]

The rest of the argument is exactly the same as in [4].
Remark: The best range of Theorem 7 is not as exact as the range of Theorem 1, but it is so within $\epsilon > 0$. We work on $\mathbb{R}_+$, and there $w \in A^1_k$ means: For every $I \subset \mathbb{R}_+$, 

$$\frac{1}{|I|} \int_I w \leq c'_{k, \inf_{kI \cap \mathbb{R}_+} w, k > 1}.$$ 

Then by Theorem 7, $w \in RH_r(\mathbb{R}_+), 1 < r < c'_k/(c'_k - 1)$. Let now $w(x) = x^{-\gamma}, 0 < \gamma < 1$, and let $c_{k, \gamma} = \inf c'_k$ for which the above displayed inequality holds. We claim that 

$$c_{k, \gamma} = \frac{1}{1 - \gamma} \left( \frac{k + 1}{2} \right)^{\gamma}.$$ 

If $I = [\alpha, \beta], \alpha \geq 0$, then the right endpoint of $kI$ is $b = \alpha + (\beta - \alpha)(k + 1)/2$. If we let $J = [0, \beta]$, then 

$$\frac{1}{|I|} \int_I x^{-\gamma} \leq \frac{1}{|J|} \int_J x^{-\gamma} = \frac{1}{1 - \gamma} \left( \frac{k + 1}{2} \right)^{\gamma} b_1^{-\gamma},$$ 

where $b_1 = \beta(k + 1)/2 = \inf_{kJ \cap \mathbb{R}_+} w$. Since $k > 1, b_1 > b$, and thus $b_1^{-\gamma} < b^{-\gamma}$, and our claim follows.

Next we note that 

$$\frac{c_{k, \gamma}}{c_{k, \gamma} - 1} = \frac{(k + 1)^{\gamma}}{(k + 1)^{\gamma} - 2^{\gamma}(1 - \gamma)} < \frac{1}{\gamma}.$$ 

Let now $\epsilon > 0$. Since, as $\gamma \nearrow 1$, $1/(\gamma + \epsilon) \to 1/(1 + \epsilon)$, and since $c_{k, \gamma}/(c_{k, \gamma} - 1) > 1$, there is $0 < \gamma = \gamma_\epsilon < 1$ such that 

$$\frac{1}{\gamma + \epsilon} < \frac{c_{k, \gamma}}{c_{k, \gamma} - 1} < \frac{1}{\gamma}.$$ 

Finally, note that $x^{-\gamma} \in RH_r(\mathbb{R}_+)$ for $1 < r < 1/\gamma$.

**Theorem 8.** Let $w \in RH_{\infty,k}$ for some $k \geq 3$ with constant $c''_k$. Then $w \in A_p$ for all $p > c''_k$.

**Proof.** The proof is the same as the proof of Theorem 2 using the dyadic covering of Theorem 7.

Remark: Again the best range of $p$'s in Theorem 8 is not as precise as the range in Theorem 2, but it is so within $\epsilon > 0$. As before, on $\mathbb{R}_+$, $w \in RH_{\infty,k}$ means 

$$\sup_{kI \cap \mathbb{R}_+} w \leq c''_k \frac{1}{|I|} \int_I w, k > 1.$$ 

Let $w(x) = x^r, r > 0$, and let $c_{k, r} = \inf c''_k$ appearing above. We claim that 

$$c_{k, r} = (r + 1) \left( \frac{k + 1}{2} \right)^r.$$ 

To see this, let $I = [\alpha, \beta], \alpha > 0$. Then the right endpoint of $kI$ is $b = \alpha + (\beta - \alpha)(k + 1)/2$. Since $x^r$ increases, 

$$\frac{1}{|I|} \int_I w \geq \frac{1}{\beta} \int_0^\beta w = \frac{1}{r + 1} \left( \frac{2}{k + 1} \right)^r b_1^r,$$ 

where $b_1 = \beta(k + 1)/2 = \inf_{kJ \cap \mathbb{R}_+} w$. Since $k > 1, b_1 > b$, and thus $b_1^{-r} < b^{-r}$,
where \( b_1 = \beta(k + 1)/2 > b = \sup w \) on \( kI \). Let \( \epsilon > 0 \) be given. Then there is \( 0 < r = r_\epsilon \) such that \( r + 1 + \epsilon > c_{k,r} > r + 1 \). The right inequality is obvious for any \( r > 0 \), and the left inequality follows from the fact that as \( r \to 0 \), \( c_{k,r} \to 1 \) and \( r + 1 + \epsilon \to 1 + \epsilon \). Finally note that \( x^r \in A_p(\mathbb{R}^+) \) precisely when \( p > r + 1 \).

**Remark:** The remaining Theorems in \( \mathbb{R}^n \) corresponding to the Theorems 3, 4, 5, 6 are the same with the proper change of constants.

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