A CLASS OF FULLY NONLINEAR EQUATIONS ARISING IN CONFORMAL GEOMETRY

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Abstract. In this paper, we consider a class of fully nonlinear equations on closed smooth Riemannian manifolds, which can be viewed as an extension of $\sigma_k$ Yamabe equation. Moreover, we prove local gradient and second derivative estimates for solutions to these equations and establish an existence result associated to them.

1. Introduction

Let $(M, g_0)$ be a smooth, compact Riemannian manifold of dimension $n \geq 3$. The $k$-th elementary symmetric polynomial is denoted by $\sigma_k$:

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}. $$

Let $V$ be a $(0, 2)$ tensor on $(M, g)$. We define $\sigma_k$- curvature of $V$ by

$$\sigma_k(V)$$

where $\sigma_k(V)$ means $\sigma_k$ is applied to the eigenvalues of $g^{-1}V$.

Definition 1.1. We say $V$ is $k$-admissible if $\sigma_1(V) > 0, \cdots, \sigma_k(V) > 0$.

Remark 1.1. $V$ is $k$-admissible if and only if the vector of eigenvalues of $g^{-1}V$, $\lambda = (\lambda_1, \cdots, \lambda_n)$, lies in $\Gamma_k$, i.e. $V \in \Gamma_k$, where $\Gamma_k$ is an open convex symmetric cone with vertex at the origin

$$\Gamma_k = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n| \; \sigma_j(\lambda) > 0, 1 \leq j \leq k\},$$

$1 \leq l \leq n$.

In our paper, we study the problem of prescribing linear combination of $\sigma_k$-curvature of $V$ on $(M, g)$:

$$(1.1) \quad \sigma_k(V) + \alpha(x)\sigma_{k-1}(V) = f(x) \quad 3 \leq k \leq n.$$
where \(\alpha(x) \in C^\infty(M)\) and \(0 < f(x) \in C^\infty(M)\), \(V\) is a \((0, 2)\) symmetric tensor on \((M, g)\). It was introduced in Krylov [Kry95]. The author considered the case with \(\alpha(x) \leq 0\) and studied Dirichlet problem of the following degenerate equation in a \((k - 1)\)-convex domain \(D\) in \(\mathbb{R}^n\),

\[
\sigma_k(D^2u) = \sum_{l=0}^{k-1} \alpha_l(x)\sigma_l(D^2u),
\]

based on his observation that the above equation is elliptic in admissible \(\Gamma_k\) cone if all the coefficient \(\alpha_l(x) \geq 0\). Recently, in [GZ19] the authors studied

\[
\sigma_k(D^2u + uI) + \alpha(x)\sigma_{k-1}(D^2u + uI) = \sum_{l=0}^{k-2} \alpha_l\sigma_l(D^2u + uI) \quad \text{on} \quad S^n,
\]

which arises in the problem of prescribing convex combination of area measures [Sch13]. And they also studied Krylov’ equation. A key new feature they observed is that there is no sign requirement for the coefficient function \(\sigma_{k-1}\). Thus, the proper admissible set of solutions of the equation is \(\Gamma_{k-1}\), not \(\Gamma_k\).

Denote by \(Ric_g\) and \(R_g\) the Ricci and scalar curvatures of \(g\) respectively. If we replace \(V\) with the Schouten tensor

\[
A_g = \frac{1}{n-2} \left( Ric_g - \frac{R_g}{2(n-1)} g \right)
\]

and take \(\alpha(x) = 0\) in (1.1), it turns out to be a prescribing \(\sigma_k\)-scalar curvature equation which was first introduce by [V00] and [V02]:

\[
(1.2) \quad \sigma_k(A_g) = f.
\]

When \(f = const.\), (1.2) is just the \(\sigma_k\)-Yamabe equation. In the past two decades, a great deal of mathematical effort has been devoted to the study of \(\sigma_k\)-Yamabe problem. For \(k = 1\) and \(f = const.\), (1.2) reduce to classical Yamabe problem. It has been solved by [Y60, T68, Au76, S84, E92]. In 2009, a remarkable result was proved by [BM09, KMS09] that the solution set of (1.2) for \(k = 1\) is compact if and only if \(n \leq 24\). For \(k \geq 2\), Viaclovsky [V00] established an important result, that the metric \(g_0\) is a critical point of the functional \(g \mapsto \int_M \sigma_k(A_g) dv_g\) restricted to the set of unit volume metrics if and only if \(\sigma_k(A_{g_0}) = const.\), provided either (i) \(k \neq n/2\) and \((M, g_0)\) is locally conformally flat or (ii) \(k = 2\). In dimension 4, [CGY02a] prove that if the Yamabe constant \(\inf_{\mathcal{G}_0} = \int_M \sigma_1(A_{\tilde{g}})\) and \(\int_M \sigma_2(A_{g_0})\) are both positive, then we can find a conformal metric \(\tilde{g}\) such that \(\sigma_2(A_{\tilde{g}})\) is a positive constant. Later on, Guan-Wang and Li-Li proved that on locally
conformally flat manifolds, the $\sigma_k$-Yamabe equation is solvable for $k < n/2$ in [GW03b] and [LL03] respectively if $\sigma_i(A_{g_0}) > 0$ for $1 \leq i \leq k$; see also [STW07]. In [BV04], the authors present a conformal variational characterization in dimension $n = 2k$ of the equation $\sigma_k(A_g) = \text{constant}$. In [HL04], the authors showed that a metric $g$ on an $n$-dimensional ($n \geq 5$) locally conformally flat manifold with volume one has constant sectional curvature if and only if $g$ is a critical point of functional

$$F_2(g) = \int \sigma_2(A_g) dv_g$$

restricted to the metrics of volume one and $F_2(g) > 0$. In [GV07] Gursky and Vioclovsky solved the prescribing curvature problem for $k > n/2$ in affirmation, provided $\sigma_i(A_{g_0}) > 0$ for $1 \leq i \leq k$ and $(M, g)$ is not conformally equivalent to a unit sphere. In [HLS08], the authors proved some rigidity properties in terms of the curvature $\sigma_k(A_g)$ on closed locally conformally flat manifolds under the assumption that $A_g$ is semi-positive definite for $k \geq 3$ and $\text{Ric}_g$ is semi-positive definite. A remarkable result proved by Trudinger and Wang in [TW10] asserts that the existence theorem and compactness of solutions to the $\sigma_k$-Yamabe problem for $k = n/2$ hold true. For more general class of functions, including quotient curvature, were first considered by Lichnerowicz [L58]. Then in [GLW10], the authors consider the problem

$$\frac{\sigma_2}{\sigma_1}(A_g) = f,$$

where $f$ is a given smooth function. They proved that, let $g_0$ be a positive scalar curvature metric, then in its conformal class there is a conformal metric with $\sigma_2(A_g) = \kappa \sigma_1(A_g)$, for some constant $\kappa$. For the manifolds with boundary, Chen showed the variational property of the k-curvature and solved the k-Yamabe problem on locally conformally manifolds with umbilic boundary in [Chen09]. Related works include [Chen07, HSh11], etc..

For the manifolds with negative curvature, there are also many interesting results. In [GV03a], the authors proved that every compact manifold with negative Ricci curvature is conformal to a metric $g$ with $\det \text{Ric}_g = \text{const}$. In [LiSh05], the authors using a parabolic approach proved that there exists a conformal metric $g$ such that $\sigma_k(-\text{Ric}_g) = \text{const.}$ if $-\text{Ric}_{g_0} \in \Gamma_k$. For the manifolds with totally geodesic boundary, [ShY14] showed that for some positive function $f(x)$ and $h < k$ there exists a conformal metric $g$ such
that
\begin{align}
\frac{\sigma_k}{\sigma_h}(-Ric_g) &= f(x) \\
h_g &= 0
\end{align}
where $h_g$ is the mean curvature. In \cite{Su17}, it was proved that on $\mathbb{R}^n$, there exists a conformal metric $g$ such that
\begin{align}
\frac{\sigma_k}{\sigma_h}(-Ric_g) &= \psi(x),
\end{align}
where $k > h$, $\psi(x)$ is a positive function and $C_1|x|^{-kl} \leq \psi(x) \leq C_2|x|^{-kl}$ for $x$ large and $l > 2$.

A key issue for the study of $\sigma_k$-Yamabe problem is the a priori estimates. $C^1$ and $C^2$ estimates for the equation (1.2) and (1.3) have been studied extensively. See \cite{Chen05, GW03a, GW03b, LL03, STW07, W06} for local interior estimates and \cite{V02} for global estimates on closed manifolds. Another interesting problem is to study the fully nonlinear equation (1.2) on a compact Riemannian manifold $(M^n, g)$ with boundary $\partial M$. In \cite{G07}, Bo Guan studied the existence problem under the Dirichlet boundary condition. The pioneering works on the Dirichlet problems for fully nonlinear elliptic equations are \cite{CNS85, Tr90} etc.. The Neumann problem for (1.2) has been studied by S. Chen \cite{Chen07, Chen09}, Jin-Li-Li \cite{JLL07}, Li-Li \cite{LL06}, He-Sheng \cite{HSh13} and Sheng-Yuan \cite{ShY13}, etc.. They obtain local estimates for solutions and establish some existence results under various conditions.

Motivated by Krylov \cite{Kry95} and Guan-Zhang \cite{GZ19}, it is natural to study the equation of type (1.1) with $V = A_g$ and $-Ric_g$ which can be viewed as extensions of prescribing $\sigma_k$ curvature problem in conformal geometry. Let us first consider $V = -\frac{Ric_g}{n-2}$ and study:
\begin{align}
\sigma_k\left(-\frac{Ric_g}{n-2}\right) + \alpha(x)\sigma_{k-1}\left(-\frac{Ric_g}{n-2}\right) = f(x), \quad 3 \leq k \leq n.
\end{align}

**Theorem 1.2.** Let $(M, g_0)$ be a smooth closed Riemannian manifold of dimension $n \geq 3$. Assume that $-\text{Ric}_{g_0}$ is $k$-admissible for $k \geq 3$. Then there exists a conformal metric $g$ satisfies equation (1.3)
if either

**Case (A)** $0 \geq \alpha \in C^\infty(M)$, and $0 < f(x) \in C^\infty(M)$,
or
Case (B) \(0 > \alpha \in C^\infty(M), \text{ and } f(x) = 0\).

To prove Theorem 1.2 we will establish the following estimates:

**Theorem 1.3.** Let \((M, g_0)\) be a smooth closed Riemannian manifold of dimension \(n \geq 3\).

**Case (A)** Assume \(f \geq \theta > 0\). Let \(u\) be a solution of (1.5), \(g = \exp(2u)g_0\) and \(-\text{Ric}_g \in \Gamma_{k-1}\). Then there exists constant \(C\), depending on \(||\alpha||_{C^2(B_r)}, ||f||_{C^2(B_r)}, \sup_{B_r}|u|, \theta, g_0\), such that

\[
(1.6) \quad \sup_{B_r^2} |\nabla u|^2 + |\nabla^2 u| \leq C.
\]

**Case (B)** Assume \(\alpha < 0, f \equiv 0\). Let \(u\) be a solution of (1.5), \(g = \exp(-2u)g_0\) and \(-\text{Ric}_g \in \Gamma_k\). Then there exists constant \(C\), depending on \(||\alpha||_{C^2(B_r)}, \sup_{B_r}|u|, g_0\), such that

\[
(1.7) \quad \sup_{B_r^2} |\nabla u|^2 + |\nabla^2 u| \leq C.
\]

Then let us replace \(V\) with Schouten tensor \(A_g\) in (1.1) and consider:

\[
(1.8) \quad \sigma_k(A_g) + \alpha(x)\sigma_{k-1}(A_g) = f(x), \quad 3 \leq k \leq n.
\]

We will establish the following a priori estimates.

**Theorem 1.4.** Let \((M, g_0)\) be a smooth closed Riemannian manifold of dimension \(n \geq 3\). Assume that Schouten tensor \(A_{g_0}\) is \(k-1\)-admissible for \(k \geq 3\), and \(f(x) \geq \theta > 0\). Let \(u\) be a solution of (1.8), \(g = \exp(-2u)g_0\) and \(A_g \in \Gamma_{k-1}\). Then there exists constant \(C\), depending on \(||\alpha||_{C^2(B_r)}, ||f||_{C^2(B_r)}, \sup_{B_r}|u|, \theta, g_0\), such that

\[
(1.9) \quad \sup_{B_r^2} |\nabla u|^2 + |\nabla^2 u| \leq C.
\]

Then naturally the following interesting problem would be proposed.

**Problem 1.5.** Since on the locally conformally flat manifolds, \(C^0\) estimates and the existence result for \(\sigma_k\)-Yamabe problem hold true, is the existence result also true for the equation (1.8) on the locally conformally flat manifolds?

The present paper is built up as follows. In Sect. 2 we start with some background. We will prove \(C^0\) estimates in Sect. 3. The \(C^1\) and \(C^2\) estimates are treated in Sect. 4. Theorem 1.2 is proved in Sect. 5.
2. Preliminaries

We first recall the following Newton-Maclaurin inequality.

Lemma 2.1. ([Tr90] LT94) Let $\lambda \in \mathbb{R}^n$. For $0 \leq l < k \leq n$, $r > s \geq 0$, $k \geq r, l \geq s$,

\begin{equation}
(1) \quad k(n - l + 1)\sigma_{l-1}(\lambda)\sigma_k(\lambda) \leq l(n - k + 1)\sigma_l(\lambda)\sigma_{k-1}(\lambda).
\end{equation}

\begin{equation}
(2) \quad \left[\frac{\sigma_k(\lambda)/C^k_n}{\sigma_l(\lambda)/C^l_n}\right]^{\frac{1}{k-l}} \leq \left[\frac{\sigma_r(\lambda)/C^r_n}{\sigma_s(\lambda)/C^s_n}\right]^{\frac{1}{r-s}}, \text{ for } \lambda \in \Gamma_k.
\end{equation}

Let $g = \exp(2u)g_0$. Then

\[ \text{Ric}_g = (n - 2) \left( -\nabla^2 u - \frac{1}{n - 2} \Delta u g_0 - |\nabla u|^2 g_0 + du \otimes du + \frac{\text{Ric}_{g_0}}{n - 2} \right). \]

Denote

\[ U = \nabla^2 u + \frac{1}{n - 2} \Delta u g_0 + |\nabla u|^2 g_0 - du \otimes du - t \frac{\text{Ric}_{g_0}}{n - 2} + \frac{1 - t}{n} g_0. \]

We will use the above conformal translation later.

Case (A) We first assume $f > 0$. Consider a family of equations

\[ G(U) = \frac{\sigma_k(tU + (1-t)trU g_0)}{\sigma_{k-1}(tU + (1-t)trU g_0)} \]

\begin{align*}
&= \frac{(1-t)\sigma_k(e) + tf(x) \exp(2ku)}{\sigma_{k-1}(tU + (1-t)trU g_0)} \\
&= -t\alpha \exp(2u),
\end{align*}

where $t \in [0,1], e = (1, \cdots, 1)$. Then we will prove that the equations we discussed are elliptic and concave.

Proposition 2.2. ([GZ19]) Let $\eta = (\eta_1, \cdots, \eta_n), e = (1, \cdots, 1)$. Denote by $\text{tr}\eta$ the function $\sum_{i=1}^n \eta_i$. Then the operator

\[ G(\eta) := \frac{\sigma_k(t\eta + (1-t)\text{tr}\eta \cdot e)}{\sigma_{k-1}(t\eta + (1-t)\text{tr}\eta \cdot e)} - h(x) \frac{1}{\sigma_{k-1}(t\eta + (1-t)\text{tr}\eta \cdot e)} \]

is elliptic and concave if $\eta \in \Gamma_{k-1}, 0 < h(x) \in C^\infty(M)$. 

Proof.

(1) Set $\mu = t\eta + (1 - t)tr\eta \cdot e$, $G_k(\eta) = \frac{\sigma_k(t\eta + (1 - t)tr\eta \cdot e)}{\sigma_{k-1}(t\eta + (1 - t)tr\eta \cdot e)}$. By direct calculation, we obtain

$$
\sigma_{k-1}(\mu|\i)\sigma_{k-1}(\mu) - \sigma_k(\mu)\sigma_{k-2}(\mu|\i)
= \sigma_{k-1}(\mu|\i)(\sigma_{k-1}(\mu|\i) + \mu_i\sigma_{k-2}(\mu|\i)) - (\sigma_k(\mu|\i) + \mu_i\sigma_{k-1}(\mu|\i))\sigma_{k-2}(\mu|\i)
= \sigma_{k-1}(\mu|\i)\sigma_{k-1}(\mu|\i) - \sigma_k(\mu|\i)\sigma_{k-2}(\mu|\i)
\geq (1 - \frac{(k - 1)(n - k)}{k(n - k + 1)})\sigma_{k-1}^2(\mu|\i)
$$

(2.3) \[ \frac{n}{k(n - k + 1)}\sigma_{k-1}^2(\mu|\i) \geq 0, \]

where we have used Lemma 2.1. Thus

$$
\frac{\partial G_k}{\partial \eta_j} = (t\delta_{ij} + (1 - t))\frac{\partial G_k}{\partial \mu_j}
= (t\delta_{ij} + (1 - t))\sigma_{k-1}(\mu|\i)\sigma_{k-1}(\mu) - \sigma_k(\mu|\i)\sigma_{k-2}(\mu|\i)
\geq (1 - \frac{(k - 1)(n - k)}{k(n - k + 1)})\sigma_{k-1}^2(\mu|\i)
$$

(2.4)

Now let $G_0(\eta) = -\frac{1}{\sigma_{k-1}(t\eta + (1 - t)tr\eta \cdot e)}$. It is easy to see that

$$
\frac{\partial G_0}{\partial \eta_j} \geq 0,
$$

(2.5) From (2.4) and (2.5), we know that $\frac{\partial G}{\partial \eta_j} = \frac{\partial G_k}{\partial \eta_j} + h(x)\frac{\partial G_0}{\partial \eta_j} > 0$ and the operator $G$ is elliptic.

(2) Denote by $F(\eta)$ the operator $(\sigma_{k-1}(t\eta + (1 - t)tr\eta \cdot e))^{-1}$. Then $-G_0 = F^{-k+1}. \quad \text{Let } \mu = t\eta + (1 - t)tr\eta \cdot e. \text{ By direct calculation}

$$
\frac{\partial^2 G_0}{\partial \mu_i \partial \mu_j}
= -(-k + 1)(-k)F^{-k-1}\frac{\partial F}{\partial \mu_i} \frac{\partial F}{\partial \mu_j} - (-k + 1) \frac{\partial^2 F}{\partial \mu_i \partial \mu_j} \leq 0.
$$

(2.6)

Note that

$$
\frac{\partial^2 G_0}{\partial \eta_p \partial \eta_q}
= \frac{\partial^2 G_0}{\partial \mu_i \partial \mu_j}(t\delta_{pi} + (1 - t))(t\delta_{qj} + (1 - t)).
$$

(2.7)
Combining with (2.6) and (2.7) we obtain that the matrix \( \frac{\partial^2 G_0}{\partial \eta_i \partial \eta_j} \) is negative semi-definite and \( G_0 \) is concave. Similarly, by the concavity of \( \frac{\sigma_k}{\sigma_{k-1}} \), it is not difficult to derive the concavity of \( G_k \). Then \( G = G_k + h(x)G_0 \) is concave. \( \square \)

Now we will state the a priori estimates for the solutions of (2.1), the proof of which will be given in Sect 3.1, Sect 4.1 and Sect 4.2, respectively.

**Lemma 2.3.** Assume \( f > 0, -\text{Ric}_{g_0} \in \Gamma_k \). Let \( u \) be a solution of (2.1) and \( U \in \Gamma_{k-1} \). Then there exists constant \( C \), depending on \( \alpha, f, g_0 \), such that

\[
(2.8) \quad \sup_M |u| \leq C.
\]

**Lemma 2.4.** Assume \( f \geq \theta > 0 \). Let \( u \) be a solution of (2.1) and \( U \in \Gamma_{k-1} \). Then there exists constant \( C \), depending on \( ||\alpha||_{C^2(B_r)}, ||f||_{C^2(B_r)}, \sup_{B_r} |u|, \theta, g_0 \), such that

\[
(2.9) \quad \sup_{B_{\frac{r}{2}}} |\nabla u|^2 \leq C.
\]

**Lemma 2.5.** Assume \( f \geq \theta > 0 \). Let \( u \) be a solution of (2.1) and \( U \in \Gamma_{k-1} \). Then there exists constant \( C \), depending on \( ||\alpha||_{C^2(B_r)}, ||f||_{C^2(B_r)}, \sup_{B_r} |u|, \sup_{B_r} |\nabla u|, \theta, g_0 \), such that

\[
(2.10) \quad \sup_{B_{\frac{r}{2}}} |\nabla^2 u| \leq C.
\]

**Case (B)** Now we assume \( f = 0 \) and \( \alpha < 0 \). Consider a family of equations

\[
G(U) = \frac{\sigma_k(tU + (1-t)\text{tr}UG_0)}{\sigma_{k-1}(tU + (1-t)\text{tr}UG_0)}
\]

\[
= (1-t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha \exp(2u),
\]

where \( t \in [0, 1], e = (1, \cdots, 1) \). The following are the a priori estimates for the solutions of (2.11), the proof of which will be given in Sect 3.2, Sect 4.3 and Sect 4.4.

**Lemma 2.6.** Assume \( f > 0 \). Let \( u \) be a solution of (2.11) and \( U \in \Gamma_k \). Then there exists constant \( C \), depending on \( \alpha, g_0 \), such that

\[
(2.12) \quad \sup_M |u| \leq C.
\]

**Lemma 2.7.** Assume \( \alpha < 0 \). Let \( u \) be a solution of (2.11) and \( U \in \Gamma_k \). Then there exists constant \( C \), depending on \( ||\alpha||_{C^2(B_r)}, \sup_{B_r} |u|, g_0 \), such that

\[
(2.13) \quad \sup_{B_{\frac{r}{2}}} |\nabla u|^2 \leq C.
\]
Lemma 2.8. Assume \( \alpha < 0 \). Let \( u \) be a solution of (2.11) and \( U \in \Gamma_k \). Then there exists constant \( C \), depending on \( ||\alpha||_{C^2(\mathbb{R}^n)} \), \( \sup_{B_r} |u| \), \( \sup_{B_r} |\nabla u| \), \( g_0 \), such that

\[
(2.14) \quad \sup_{B_{2r}} |\nabla^2 u| \leq C.
\]

Case(C) Let \( \tilde{g} = \exp(-2u)g_0 \). Then

\[
A_{\tilde{g}} = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g_0 + A_{g_0}.
\]

Now denote

\[
W = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g_0 + A_{g_0}.
\]

We consider the following:

\[
G(W) = \frac{\sigma_k(W)}{\sigma_{k-1}(W)} - \frac{\exp(-2k\theta f)}{\sigma_{k-1}(W)}
\]

\[
(2.15)
\]

\[
= -\alpha(x) \exp(-2u).
\]

We will prove the following theorem, the proof of which will be given in Sect 4.5.

Lemma 2.9. (Theorem 1.4) Assume \( f(x) > \theta > 0 \). Let \( u \) be a solution of (2.15) and \( W \in \Gamma_{k-1} \). Then there exists constant \( C \), depending on \( ||\alpha||_{C^2(\mathbb{R}^n)} \), \( ||f||_{C^2(\mathbb{R}^n)} \), \( \sup_{B_r} |u| \), \( \theta \), \( g_0 \), such that

\[
(2.16) \quad \sup_{B_{2r}} |\nabla u|^2 + |\nabla^2 u| \leq C.
\]

3. \( C^0 \) estimates

3.1. Proof of Lemma 2.3. Consider

\[
\frac{\sigma_k(tU + (1-t)trUg_0)}{\sigma_{k-1}(tU + (1-t)trUg_0)} + t\alpha \exp(2u)
\]

\[
(3.1)
\]

\[
= \frac{((1-t)\sigma_k(e) + tf(x)) \exp(2ku)}{\sigma_{k-1}(tU + (1-t)trUg_0)}
\]

where \( t \in [0,1] \), \( e = (1, \ldots, 1) \),

\[
U = \nabla^2 u + \frac{1}{n-2} \Delta u g_0 + |\nabla u|^2 g_0 - du \otimes du - \frac{1}{n-2} Ric_{g_0} + \frac{1-t}{n} g_0.
\]
Let $u$ be the solution of equation (3.1). Suppose the maximum point of $u$ is attained at $x_1$. Note that $\frac{\sigma_k}{\sigma_{k-1}}$ is concave in $\Gamma_{k-1}$ (see [HS99]). Thus

$$\frac{\sigma_k}{\sigma_{k-1}}(A + B) \geq \frac{\sigma_k}{\sigma_{k-1}}(A) + \frac{\sigma_k}{\sigma_{k-1}}(B) \geq \frac{\sigma_k}{\sigma_{k-1}}(B)$$

if $A$ is positive definite and $B \in \Gamma_{k-1}$. Since $\nabla^2u(x_1)$ is negative definite and $\nabla u(x_1) = 0$, thus at this point

$$\frac{\sigma_k(tB_{g_0} + (1-t)trB_{g_0}g_0)}{\sigma_{k-1}(tB_{g_0} + (1-t)trB_{g_0}g_0)} \geq \frac{\sigma_k(tU + (1-t)trUg_0)}{\sigma_{k-1}(tU + (1-t)trUg_0)},$$

where $B_{g_0} = -t\frac{Ric_{g_0}}{n-2} + \frac{1-t}{n}g_0$. Besides,

$$\frac{1}{\sigma_{k-1}(tB_{g_0} + (1-t)trB_{g_0}g_0)} \leq \frac{1}{\sigma_{k-1}(tU + (1-t)trUg_0)}.$$ 

Therefore,

$$C + C \exp(2u(x_1)) \geq C \exp(2ku(x_1))$$

and we have

$$u(x_1) \leq C,$$

which means

$$\sup_{x \in M} u(x) \leq C.$$ 

Similarly, calculate at the minimal point of $u$, we obtain

$$\inf_{x \in M} u(x) \geq -C.$$ 

### 3.2. Proof of Lemma 2.6

Let consider

$$G(U) = \frac{\sigma_k(tU + (1-t)trUg_0)}{\sigma_{k-1}(tU + (1-t)trUg_0)}$$

(3.2)

$$= ((1-t)\frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha) \exp(2u),$$

where $t \in [0, 1], e = (1, \cdots, 1),

$$U = \nabla^2u + \frac{1}{n-2}\Delta u g_0 + |\nabla u|^2g_0 - du \otimes du - t\frac{Ric_{g_0}}{n-2} + \frac{1-t}{n}g_0.$$ 

Assume $x_1$ is the maximum point of $u$. Then

$$C \geq C \exp(2u(x_1))$$

and we have

$$u(x_1) \leq C,$$
which means
\[
\sup_{x \in M} u(x) \leq C.
\]
Similarly, calculate at the minimal point of \(u\), and we find
\[
\inf_{x \in M} u(x) \geq -C.
\]

4. \(C^1\) and \(C^2\) Estimates

4.1. Proof of Lemma 2.4. Let
\[
U = \nabla^2 u + \frac{1}{n-2} \Delta u g_0 + |\nabla u|^2 g_0 - d u \otimes d u - t \frac{Ric g_0}{n-2} + \frac{1-t}{n} g_0.
\]
We consider the following equations
\[
\frac{\sigma_k(tU + (1-t)trU g_0)}{\sigma_{k-1}(tU + (1-t)trU g_0)} + t\alpha \exp(2u)(x) = \frac{((1-t)\sigma_k(e) + tf(x)) \exp(2ku)}{\sigma_{k-1}(tU + (1-t)trU g_0)},
\]
where \(t \in [0,1], e = (1, \ldots, 1)\). For the convenience of notations, we will denote
\[
G_k(U) = \frac{\sigma_k(tU + (1-t)trU g_0)}{\sigma_{k-1}(tU + (1-t)trU g_0)}, G_0(U) = -\frac{1}{\sigma_{k-1}(tU + (1-t)trU g_0)},
\]
\[
P_k(V) = \frac{\sigma_k(V)}{\sigma_{k-1}(V)}, \quad P_0(V) = -\frac{1}{\sigma_{k-1}(V)},
\]
\[
P(V) = P_k(V) + ((1-t)\sigma_k(e) + tf(x)) \exp(2ku)P_0(V),
\]
where \(V = tU + (1-t)trU g_0\). We further denote by \(\sigma_i^j, P_i^j, P_i^j_0, G_i^j, C_i^j\) and \(C_0^j\) the functions \(\frac{\partial \sigma}{\partial V_i^j}, \frac{\partial P}{\partial V_i^j}, \frac{\partial P}{\partial V_i^j}, \frac{\partial G}{\partial U_{ij}}, \frac{\partial G}{\partial U_{ij}}\) and \(\frac{\partial G}{\partial U_{ij}}\) respectively. By direct calculation, we have
\[
P_i^j V_{ii} = P_i^j V_{ii} + ((1-t)\sigma_k(e) + tf(x)) \exp(2ku)P_0^j V_{ii}
\]
\[
= \frac{\sigma_k^j V_{ii} \sigma_{k-1} - \sigma_k \sigma_{k-1}^j V_{ii}}{\sigma_{k-1}^2} + ((1-t)\sigma_k(e) + tf(x)) \exp(2ku)\frac{\sigma_{k-1}^j V_{ii}}{\sigma_{k-1}^2} - (k-1)((1-t)\sigma_k(e) + tf(x)) \exp(2ku)G_0
\]
\[
\geq P = -t\alpha \exp(2u).
\]
It is obviously that
\[
G_i^j U_{ii} = P_i^j (t\delta_{ij} + (1-t)) U_{ii} = P_i^j V_{jj}.
\]
Moreover, we differentiate the equation (4.1) and obtain
\[
G^{ij}U_{ijp} + \left( (1 - t)\sigma_k(e) + tf(x) \right) \exp(2ku) p G_0 = G^{ij} U_{ijp} + \left( (1 - t)\sigma_k(e) + tf(x) \right) \exp(2ku) p G_0 + \left( (1 - t)\sigma_k(e) + tf(x) \right) \exp(2ku) G_0
\]
(4.4)
\[= -(t\alpha \exp(2u)) p. \]

Set \( \min_M \gamma' > 0, \min_M (\gamma'' - \gamma'^2) > 0 \) and
\[Q = \rho \cdot (1 + \frac{\|\nabla u\|^2}{2}) e^{\gamma(u)} := \rho \cdot K. \]
Here 0 \( \leq \rho \leq 1 \) is a cutoff function depending only on \( r \) such that \( \rho = 1 \) in \( B^2_r \) and \( \rho = 0 \) outside \( B_r \), moreover
\[|\nabla \rho| \leq \frac{C \rho^{1/2}}{r}, \quad |\nabla^2 \rho| \leq \frac{C}{r^2}. \]
Assume that \( \max_M Q = Q(\tilde{x}), U_{ij}(\tilde{x}) \) and hence \( P^{ij}(\tilde{x}) \) and \( G^{ij}(\tilde{x}) \) are diagonal. Then differentiate \( K \) at the point \( \tilde{x} \). By direct calculation, we have
\[K_i(\tilde{x}) = e^{\gamma(u)} \left( 1 + \frac{u_i^2}{2} \right) \gamma' u_i + u_i u_i, \]
and
\[K_{ij}(\tilde{x}) = e^{\gamma(u)} \left( 1 + \frac{u_i^2}{2} \right) \left( \gamma'' u_{ij} + \gamma' u_i u_j + 2u_i u_j \gamma' u_i + u_i u_i + u_i u_i \right), \]
(4.5)

Note that \( \tilde{x} \) is the maximum point of \( Q \), we have
\[0 = Q_i(\tilde{x}) = \rho_i K + \rho K_i. \]
(4.6)
Moreover,
\[Q_{ii} = \rho_{ii} K + 2\rho_i K_i + \rho K_{ii}. \]
(4.7)
If we plug (4.6) into (4.7), we have
\[Q_{ii} = \rho_{ii} K - 2\rho_i \frac{\rho_i}{\rho} K + \rho K_{ii}. \]
(4.8)
Since \( G^{ij} \) is positive definite and \( Q_{ij} \) is negative definite, we find
\[0 \geq \rho e^{-\gamma(u)} \left( G^{ii} + \frac{\sum G_{pp}}{n - 2} g_0^{ii} \right) Q_{ii}(\tilde{x}) \]
\[\geq \rho e^{-\gamma(u)} \left( G^{ii} + \frac{\sum G_{pp}}{n - 2} g_0^{ii} \right) \left( \rho K_{ii} + \rho_{ii} K - 2\rho_i \frac{\rho_i}{\rho} K \right) \]
\[(4.9) \quad \geq e^{-\gamma(u)} \left(G^{ii} + \frac{\sum G^{pp}_{ii}}{n-2} g^{ii}_0\right) \left(\rho^2 K_{ii} - C \rho K\right).
\]

Then inserting (4.5) into (4.9) and using Ricci identity, we obtain

\[
0 \geq \rho^2 \left(G^{ii} + \frac{\sum G^{pp}_{ii}}{n-2} g^{ii}_0\right) \left(u_i u_{iii} + (1 + \frac{u_i^2}{2}) \left(\gamma' \gamma' + \gamma''\right) u_i u_i + 2\gamma' u_i u_{ii} u_i + u_i u_{ii}\right)
- C\rho \sum_i G^{ii}(|\nabla u|^2 + 1)
\]

\[
\geq \rho^2 \left(G^{ii} + \frac{\sum G^{pp}_{ii}}{n-2} g^{ii}_0\right) \left(u_i u_{iii} + (1 + \frac{u_i^2}{2}) \left(\gamma' \gamma' + \gamma''\right) u_i u_i + 2\gamma' u_i u_{ii} u_i + u_i u_{ii}\right)
- C\rho \sum_i G^{ii}(|\nabla u|^2 + 1).
\]

(4.10)

Moreover, recalling the definition of \(U\), we obtain

\[
0 \geq \rho^2 u_i G^{ii} \left(U_{iii} - (u_p^2 - u_i u_i)\right)
+ \rho^2 \gamma' G^{ii} \left(U_{ii} - (u_p^2 - u_i u_i)\right) \left(1 + \frac{u_i^2}{2}\right)
+ \rho^2 G^{ii} \left(1 + \frac{u_i^2}{2}\right) \left(\gamma' \gamma' + \gamma''\right) \left(u_i^2 + \frac{u_p^2}{n-2}\right)
+ 2G^{ii} \left(\gamma' u_i u_{ii} u_i + \frac{\gamma' u_p u_{ip} u_{il}}{n-2}\right) - C\rho \sum_i G^{ii}(|\nabla u|^2 + 1).
\]

(4.11)

Then, combining (4.4), (4.3), (4.1) and (4.6), it is straightforward to show that

\[
0 \geq \rho^2 G^{ii} \left(- u_i u_p u_{ip} + u_i u_i u_i + \gamma'(-u_p^2 + u_i^2)(1 + \frac{u_i^2}{2})\right)
+ \rho^2 G^{ii} \left(1 + \frac{u_i^2}{2}\right) \left(\gamma' \gamma' + \gamma''\right) \left(u_i^2 + \frac{u_p^2}{n-2}\right)
- 2\rho^2 G^{ii} \left(1 + \frac{u_i^2}{2}\right) \left(\gamma' \gamma'\right) \left(u_i^2 + \frac{u_p^2}{n-2}\right)
-C\rho \sum_i G^{ii}(|\nabla u|^3 + 1) - C\rho \sum_i G^{ii}(|\nabla u|^2 + 1) + \rho^2 G_0(|\nabla u|^2 + 1)
\]
\[
\geq \rho^2 G_{ii} \left( \gamma' (1 + \frac{u_i^2}{2}) u_p^2 - \gamma' (1 + \frac{u_i^2}{2}) u_i^2 \right.
\]
\[
+ \gamma' (-u_p^2 + u_i^2) (1 + \frac{u_i^2}{2}) + (1 + \frac{u_i^2}{2}) (-\gamma^2 + \gamma'')(u_i^2 + \frac{u_i^2}{n-2}) \right)
\]
\[
-C \sum_i G_{ii} (\rho^2 |\nabla u|^3 + 1) + \rho^2 G_0 (|\nabla u|^2 + 1)
\]
\[
\geq \rho^2 G_{ii} \left( (1 + \frac{u_i^2}{2}) (-\gamma^2 + \gamma'') \left( \frac{u_i^2}{n-2} \right) \right)
\]
\[
(4.12)
\]
where we have used \(-\gamma^2 + \gamma'' > 0\). Now let us divide the proof into two cases.

\begin{enumerate}
\item[(A1)] \(\frac{\sigma_k}{\sigma_{k-1}} \leq (|\nabla u|)^{\frac{1}{k}}\). Then from equation (4.1) we have
\[
-\rho \frac{\sigma_k}{\sigma_{k-1}} - t \alpha \exp(2u) \rho = \rho (ft + (1-t)\sigma_k(e)) \exp(2ku) G_0.
\]

For \(z \in B_r(\tilde{x})\), we have
\[
(4.13)
\]
\[
\rho G_0(\tilde{x}) \geq -\frac{-t \sup_{B_r} \alpha \exp(2u) - \rho(|\nabla u|)^{\frac{1}{k}}}{\inf_{B_r} \left[ (ft + (1-t)\sigma_k(e)) \exp(2ku) \right]} \geq -C(\rho(|\nabla u|)^{\frac{1}{k}} + 1).
\]
Note that \(\sum_i G_{ii} \geq \frac{n-k+1}{k} (GZ19)\), combining (4.12) and (4.13), we find
\[
\rho |\nabla u|^2 \leq C.
\]

\item[(A2)] \(\frac{\sigma_k}{\sigma_{k-1}} \geq (|\nabla u|)^{\frac{1}{k}}\). By Lemma 2.1 we have
\[
(4.14)
\]
\[
|G_0| \leq (|\nabla u|)^{-\frac{k+1}{k}}.
\]
As the same as before, \(\sum_i G_{ii} \geq \frac{n-k+1}{k} (GZ19)\), (4.14) and (4.12) implies
\[
\rho |\nabla u|^2 \leq C.
\]
\end{enumerate}

4.2. Proof of Lemma 2.5. Let
\[
U = \nabla^2 u + \frac{1}{n-2} \Delta u g_0 + |\nabla u|^2 g_0 - du \otimes du - t \frac{Ric}{n-2} + \frac{1}{n} g_0.
\]
We consider the following equation
\[
(4.15)
\]
\[
\frac{\sigma_k(tU + (1-t)tr U g_0)}{\sigma_{k-1}(tU + (1-t)tr U g_0)} + t \alpha \exp(2u)(x) = \frac{(1-t)\sigma_k(e) + tf(x)}{\sigma_{k-1}(tU + (1-t)tr U g_0)} \exp(2ku),
\]
where \(t \in [0,1], e = (1, \cdots, 1)\). Take the auxiliary function
\[
H(x) = \rho(\Delta u + n|\nabla u|^2) := \rho K.
\]
Here $0 \leq \rho \leq 1$ is a cutoff function depending only on $r$ such that $\rho = 1$ in $B_2$ and $\rho = 0$ outside $B_r$, moreover

$$|\nabla \rho| \leq \frac{C\rho^{1/2}}{r}, \quad |\nabla^2 \rho| \leq \frac{C}{r^2}.$$ 

Assume $x_0$ is the maximum point of $H$. Then at $x_0$,

$$H_i(x_0) = \rho_i K + \rho K_i = \rho_i(u_{kk} + nu^2_k) + \rho(u_{kki} + 2nu_k u_{ki}) = 0. \tag{4.16}$$

Differentiating both sides of (4.16), we have

$$H_{ii}(x_0) = \rho_{ii} K + \rho K_{ii} + 2\rho_i K_i. \tag{4.17}$$

If we plug (4.16) back into (4.17), we obtain

$$H_{ii}(x_0) = \rho_{ii} K + \rho K_{ii} - 2\rho_i \frac{\rho_i}{\rho} K. \tag{4.18}$$

Recall

$$U = \nabla^2 u + \frac{1}{n-2} \Delta u g_0 + |\nabla u|^2 g_0 - du \otimes du - t \frac{Ric g_0}{n-2} + \frac{1-t}{n} g_0.$$ 

We will calculate in the normal coordinates which is centered at $x_0$. We further assume $U_{ij}$ and hence $\frac{\partial G}{\partial U_{ij}}$ are diagonal at the point $x_0$. Since $U \in \Gamma_2$, we have

$$|U_{ij}| \leq C trU, trU > 0.$$ 

Therefore,

$$|u_{ij}| \leq C(\Delta u + 1). \tag{4.19}$$

In view of (4.19), we may assume

$$\Delta u > C.$$ 

For the convenience of notations, we will denote

$$G_k(U) = \frac{\sigma_k(tU + (1-t)trU g_0)}{\sigma_{k-1}(tU + (1-t)trU g_0)}, G_0(U) = -\frac{1}{\sigma_{k-1}(tU + (1-t)trU g_0)},$$

where $e = (1, \cdots, 1)$. We further denote by $G^{ij}, G^{ij}_k$ and $G^{ij}_0$ the functions $\frac{\partial G}{\partial U_{ij}}, \frac{\partial G_k}{\partial U_{ij}}$ and $\frac{\partial G_0}{\partial U_{ij}}$ respectively. We differentiate the equation (4.15) and obtain

$$G^{ij} U_{ijp} + ((1-t)\sigma_k(e) + tf(x)) \exp(2ku)_p G_0$$

$$= G^{ij}_k U_{ijp} + ((1-t)\sigma_k(e) + tf(x)) \exp(2ku)G^{ij}_0 U_{ijp}$$

$$+ (((1-t)\sigma_k(e) + tf(x)) \exp(2ku))_p G_0$$

$$= -(t\alpha \exp(2u))_p. \tag{4.20}$$
Differentiate the equation (4.15) another time and we obtain

\[ G^{ij,rs}U_{ijp}U_{rsp} + G^{ij}U_{ijpp} + 2((1-t)\sigma_k(e) + tf(x))\exp(2ku)pG_0^{ij}U_{ijp} \]

\[ + ((1-t)\sigma_k(e) + tf(x))\exp(2ku)ppG_0 \]

\[ = G_k^{ij,rs}U_{ijp}U_{rsp} + G_k^{ij}U_{ijpp} + (1-t)\sigma_k(e) + tf(x)\exp(2ku)G_k^{ij,rs}U_{ijp}U_{rsp} \]

\[ + ((1-t)\sigma_k(e) + tf(x))\exp(2ku)ppG_0^{ij}U_{ijp} \]

\[ + ((1-t)\sigma_k(e) + tf(x))\exp(2ku)ppG_0 \]

\[ - (t\alpha \exp(2u))pp. \]

Moreover, from (3.10) in [GZ19], we have

\[ (4.22) \]

\[ -G_0^{ij,rs}U_{ijp}U_{rsp} \geq -(1 + \frac{1}{k+1})G_0^{ij}G_0^{rs}U_{ijp}U_{rsp}. \]

From the positivity of \( G^{ij} \) and negativity of \( H_{ij} \), we find

\[ (4.23) \]

\[ 0 \geq \rho(G^{ii} + \frac{\sum G^{pp}}{n-2}g_0^{ii})H_{ii}(x_0) \]

Then plug (4.18) into (4.23), we obtain

\[ 0 \geq \rho^2(G^{ii} + \frac{\sum G^{pp}}{n-2}g_0^{ii})(u_{ppii} + 2nu_pu_{pii} + 2nu_{pi}u_{pi}) - \rho CG^{ii}(\Delta u). \]

Now using Ricci identity yields

\[ 0 \geq \rho^2G^{ii}(U_{iipp} + (u_i)^2)_{pp} - (u_i^2)_{pp} \]

\[ + 2nu_p(U_{ipp} + (u_i^2)p - (u_i^2)p_p) + 2nu_{pi} + \frac{2n}{n-2}u_{iip} - C\Delta u \]

\[ \geq \rho^2G^{ii}(U_{iipp} + 2(-2nu_iu_{iip} + u_i^2) - (-2nu_iu_{iip} + u_i^2) \]

\[ + 2nu_p(U_{ipp} + 2(u_iu_{iip}) - 2(u_iu_{iip})) + 2nu_{pi} + \frac{2n}{n-2}u_{iip} - C\Delta u \]

\[ \geq G^{ii}(\frac{n+2}{n-2}\rho^2(\Delta u)^2 - C\rho\Delta u) + \rho^2G^{ii}U_{iipp} + 2\rho^2u_pG^{ii}U_{iipp} \]
where we have used the assumption that $\Delta u$ is sufficiently large. Then using the concavity of $G_k$, (4.20) and (4.21) we deduce that

$$0 \geq G^{ii} \left( \frac{n+2}{n-2} \rho^2 (\Delta u)^2 - C \rho \Delta u \right)$$

$$- \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) G_0^{-1} G_0^{ij} G_0^{rs} U_{ijp} U_{rsp}$$

$$- 2 \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) p G_0^{ij} U_{ijp}$$

$$- \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) p G_0$$

$$- \rho^2 \left( (t \alpha \exp(2u))_{pp} \right)$$

$$+ 2 \rho^2 u_p \left( - \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) \right) p G_0 - (t \alpha \exp(2u))_p$$

By use of (4.22), it yields

$$0 \geq G^{ii} \left( \frac{n+2}{n-2} \rho^2 (\Delta u)^2 - C \rho \Delta u \right)$$

$$- \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) \left( 1 + \frac{1}{k+1} \right) G_0^{-1} G_0^{ij} G_0^{rs} U_{ijp} U_{rsp}$$

$$- 2 \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) p G_0^{ij} U_{ijp} + C \rho^2 G_0 (\Delta u + 1)$$

$$\geq G^{ii} \left( \frac{n+2}{n-2} \rho^2 (\Delta u)^2 - C \rho \Delta u \right) + C \rho G_0 (\Delta u + 1)$$

$$- \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) \left( 1 + \frac{1}{k+1} \right) G_0^{-1}$$

$$\left( G_0^{ij} U_{ijp} - \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) \left( 1 + \frac{1}{k+1} \right) G_0^{-1} \right)^2$$

$$- \rho^2 \left( ((1-t) \sigma_k(e) + tf(x)) \exp(2ku) \right) \left( 1 + \frac{1}{k+1} \right) G_0^{-1} - C$$

$$(4.24) \geq G^{ii} \left( \frac{n+2}{n-2} \rho^2 (\Delta u)^2 - C \rho \Delta u \right) + C \rho^2 G_0 (\Delta u + 1) - C.$$

Let us divide the proof into two cases.

(A1) $\frac{\Delta u}{\sigma_{k-1}} \leq (\Delta u)^{\frac{1}{2}}$. Then by the equation we have

$$\rho |G_0| = \frac{\rho}{\sigma_{k-1}} \left( \frac{\rho}{\sigma_{k-1}} + t \alpha \sup_{B_r} \exp(2u) \right) \leq C \rho (\Delta u)^{\frac{1}{2}}.$$ (4.25)

Since $\sum G^{ii} \geq \frac{n-k+1}{k}$, (4.24) and (4.25) imply $\rho \Delta u \leq C$. 

By use of (4.22), it yields
\((\text{A2}) \, \frac{\sigma_k}{\sigma_{k-1}} > (\Delta u)^\frac{1}{\gamma} \). Then by Lemma \ref{lem:2.1},

\[(4.26) \quad |G_0| = \frac{1}{\sigma_{k-1}} \leq C \left( \frac{\sigma_{k-1}^{k-1}}{\sigma_k} \right) \leq (\Delta u)^{-\frac{k-1}{\gamma}}.
\]

Now we also derive \(\rho \Delta u \leq C\) from (4.24) and (4.26).

4.3. Proof of Lemma \ref{lem:2.7}. Let

\[
U = \nabla^2 u + \frac{1}{n-2} \Delta u g_0 + |\nabla u|^2 g_0 - du \otimes du - t \frac{Ric_{g_0}}{n-2} + \frac{1-t}{n} g_0.
\]

We consider the following equation

\[(4.27) \quad \frac{\sigma_k(tU + (1-t)tr U g_0)}{\sigma_{k-1}(tU + (1-t)tr U g_0)} = \left( (1-t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha \right) \exp (2u),
\]

where \(t \in [0,1], \, e = (1, \cdots , 1)\). For the convenience of notations, we will denote

\[
P_k(V) = \frac{\sigma_k(V)}{\sigma_{k-1}(V)}, \quad G(U) = \frac{\sigma_k(tU + (1-t)tr U g_0)}{\sigma_{k-1}(tU + (1-t)tr U g_0)},
\]

where \(V = tU + (1-t)tr U g_0\). We further denote by \(\sigma_l^{ij}, \, P_k^{ij}, \, G^{ij}\) the functions \(\frac{\partial \sigma_l}{\partial V_{ij}}, \frac{\partial P_k}{\partial V_{ij}}\) and \(\frac{\partial G}{\partial U_{ij}}\) respectively. By direct calculation

\[
G^{ii} U_{ii} = P_k^{ii} \left( t\delta_{ij} + (1-t) \right) U_{ii} = P_k^{ii} V_{ii} = \frac{\sigma_k^{ii} V_{ii} - \sigma_k^{ii} V_{ii} - \sigma_k \sigma_{k-1}^{ii} V_{ii}}{\sigma_{k-1}^{ii}} = \frac{\sigma_k \sigma_{k-1}}{\sigma_{k-1}^{2}}
\]

\[(4.28) \quad \geq P_k = \left( (1-t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha \right) \exp (2u).
\]

Moreover, we differentiate the equation (4.27) and obtain

\[
G^{ij} U_{ijp} = \left( (1-t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha \right) \exp (2u) p,
\]

\[(4.29) \quad \leq (\{ (1-t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha \} \exp (2u) ) p.
\]

Set \(\min_M \gamma' > 0, \min_M (\gamma'' - \gamma'^2) > 0\), and

\[
Q = \rho \cdot \left( 1 + \frac{|\nabla u|^2}{2} \right) e^{\gamma(u)} := \rho \cdot K.
\]
Here $0 \leq \rho \leq 1$ is a cutoff function depending only on $r$ such that $\rho = 1$ in $B_r$ and $\rho = 0$ outside $B_r$, moreover
\[
|\nabla \rho| \leq \frac{C\rho^{1/2}}{r}, \quad |\nabla^2 \rho| \leq \frac{C}{r^2}.
\]
Assume that $\max_{M} Q = Q(\bar{x})$, $U_{ij}(\bar{x})$ and hence $P^{ij}(\bar{x})$ and $G^{ij}(\bar{x})$ are diagonal. Then differentiating $K$ at the point $\bar{x}$, we have
\[
K_i(\bar{x}) = e^{\gamma(u)} \left( 1 + \frac{u_i^2}{2} \right) \gamma' u_i + u_i u_{ii}
\]
and
\[
K_{ij}(\bar{x}) = e^{\gamma(u)} \left( 1 + \frac{u_i^2}{2} \right) \left( \gamma'^2 u_i u_j + \gamma' u_{ij} + \gamma'' u_i u_j \right) + 2u_i u_j \gamma' u_i + u_{ij} u_{ii} + u_{ii} u_{ij}
\]
Since $\bar{x}$ is the maximum point of $Q$, we have
\[
0 = Q_i(\bar{x}) = \rho_i K + \rho K_i.
\]
Differentiating both sides of (4.31) gives
\[
Q_{ii} = \rho_{ii} K + 2\rho_i K_i + \rho K_{ii}.
\]
Then inserting (4.31) into (4.32) and using the positivity of $G^{ij}$, the negativity of $Q_{ij}$, we deduce that
\[
0 \geq \rho e^{-\gamma(u)} (G^{ii} + \sum_{n-2} G^{pp}_{i} g^{ii}_{0}) Q_{ii}(\bar{x})
\]
\[
\geq \rho e^{-\gamma(u)} (G^{ii} + \sum_{n-2} G^{pp}_{i} g^{ii}_{0}) (\rho_{ii} K + 2\rho_i K_i + \rho K_{ii})
\]
\[
\geq e^{-\gamma(u)} (G^{ii} + \sum_{n-2} G^{pp}_{i} g^{ii}_{0}) (\rho^2 K_{ii} - C \rho K).
\]
Then we plug (4.30) into (4.33) and obtain
\[
0 \geq \rho^2 (G^{ii} + \sum_{n-2} G^{pp}_{i} g^{ii}_{0})
\]
\[
\left( u_{i} u_{ii} + \left( 1 + \frac{u_i^2}{2} \right) \left( \gamma'^2 + \gamma'' \right) u_i u_i + \gamma' u_{ii} \right) + 2\gamma' u_{ii} u_i + u_{ii} u_{ii}
\]
\[
- C \rho \sum_{i} G^{ii} (|\nabla u|^2 + 1).
\]
Moreover, Ricci identity gives
\[
0 \geq \rho^2 (G^{ii} + \sum_{n-2} G^{pp}_{ii}) (u_t u_{tii} + (1 + \frac{u_i^2}{2}) \left( (\gamma' \gamma' + \gamma'') u_t u_i + \gamma' u_{tii} \right) + 2 \gamma' u_t u_{tii} + u_t u_i) \\
- C \rho \sum_i G^{ii} (|\nabla u|^2 + 1).
\]

Using the definition of \( U \), we have
\[
0 \geq \rho^2 u_t G^{ii} \left( U_{ii} - (u_p^2 - u_i u_i) \right) \\
+ \rho^2 \gamma' G^{ii} (U_{ii} - (u_p^2 - u_i u_i)) (1 + \frac{u_i^2}{2}) \\
+ \rho^2 u_t G^{ii} (1 + \frac{u_i^2}{2}) (\gamma' \gamma' + \gamma'') (u_i^2 + \frac{u_p^2}{n-2}) \\
+ 2 G^{ii} \left( \gamma' u_t u_{tii} + \frac{\gamma' u_p u_p u_i}{n-2} \right) \\
- C \rho \sum_i G^{ii} (|\nabla u|^2 + 1).
\]
(4.35)

By substituting (4.28) and (4.29) into (4.35) and using (4.31), we obtain
\[
0 \geq \rho^2 G^{ii} \left( - u_t u_p u_{pl} + u_t u_i u_{il} + \gamma' (-u_p^2 + u_i^2)(1 + \frac{u_i^2}{2}) \right) \\
+ \rho^2 G^{ii} \left( (1 + \frac{u_i^2}{2})(\gamma' \gamma' + \gamma'') (u_i^2 + \frac{u_p^2}{n-2}) \right) \\
- 2 \rho^2 G^{ii} \left( (\gamma' \gamma' + \gamma'') (u_i^2 + \frac{u_p^2}{n-2}) \right) \\
- C \sum_i G^{ii} (\rho^2 |\nabla u|^3 + \rho |\nabla u|^2 + 1) \\
\geq \rho^2 G^{ii} \left( \gamma' (1 + \frac{u_i^2}{2}) u_p^2 - \gamma' (1 + \frac{u_i^2}{2}) u_i^2 \\
+ \gamma' (-u_p^2 + u_i^2)(1 + \frac{u_i^2}{2}) + (1 + \frac{u_i^2}{2})(-\gamma' \gamma' + \gamma'') (u_i^2 + \frac{u_p^2}{n-2}) \right) \\
- C \sum_i G^{ii} (\rho^3 |\nabla u|^3 + \rho |\nabla u|^2 + 1) \\
\geq \rho^2 G^{ii} \left( (1 + \frac{u_i^2}{2})(-\gamma' \gamma' + \gamma'') (\frac{u_p^2}{n-2}) \right) \\
- C \sum_i G^{ii} (\rho^2 |\nabla u|^3 + 1),
\]
(4.36)
where we have used $-\gamma'^2 + \gamma'' > 0$. Thanks to (4.36), (2.13) is proved.

### 4.4. Proof of Lemma 2.8

Let

$$U = \nabla^2 u + \frac{1}{n-2} \triangle u g_0 + |\nabla u|^2 g_0 - du \otimes du - \frac{t \text{Ric}_{g_0}}{n-2} + \frac{1-t}{n} g_0.$$  

We consider the following equation

$$\frac{\sigma_k(tU + (1-t)\text{tr}Ug_0)}{\sigma_{k-1}(tU + (1-t)\text{tr}Ug_0)} = \left((1-t)\frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t\alpha\right) \exp(2u),$$

where $t \in [0,1]$, $e = (1, \cdots, 1)$. Take the auxiliary function

$$H(x) = \rho(\Delta u + n|\nabla u|^2) := \rho K.$$  

Here $0 \leq \rho \leq 1$ is a cutoff function depending only on $r$ such that $\rho = 1$ in $B^*_{2r}$ and $\rho = 0$ outside $B_r$, moreover

$$|\nabla \rho| \leq \frac{C\rho^{1/2}}{r}, \quad |\nabla^2 \rho| \leq \frac{C}{r^2}.$$  

Assume $x_0$ is the maximum point of $H$. We will calculate in the normal coordinates which is centered at $x_0$. We further assume $U_{ij}$ and hence $\frac{\partial G}{\partial U_{ij}}$ are diagonal at the point $x_0$. Then at $x_0$,

$$H_i(x_0) = \rho_t K + \rho K_i = \rho_t(u_{kk} + nu^2_k) + \rho(u_{kk} + 2nu_k u_{ki}) = 0.$$  

Since $U \in \Gamma_2$, we have

$$|U_{ij}| \leq C\text{tr}U, \text{tr}U > 0.$$  

Therefore,

$$|u_{ij}| \leq C(\Delta u + 1).$$

In view of (4.39), without loss of generality, we may assume

$$\Delta u > C.$$  

For the convenience of notations, we will denote

$$G(U) = \frac{\sigma_k(tU + (1-t)\text{tr}Ug_0)}{\sigma_{k-1}(tU + (1-t)\text{tr}Ug_0)},$$
where \( e = (1, \cdots, 1) \). We further denote by \( G^{ij} \) the functions \( \frac{\partial G}{\partial U^{ij}} \). We differentiate the equation (4.37) and obtain

\[
G^{ij}_k U_{ijp} = \left( \left( (1 - t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t \alpha \right) \exp(2u) \right)_p.
\]

(4.40)

Differentiate the equation (4.37) another time and we obtain

\[
G^{ij,rs} U_{ijp} U_{rsp} + G^{ij} U_{ijpp} = \left( \left( (1 - t) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - t \alpha \right) \exp(2u) \right)_{pp}.
\]

(4.41)

By direct calculation, we have

\[
0 \geq \rho \left( G^{ii} + \sum_{n-2} G^{pp} g_0^{ii} \right) H_{ii}(x_0)
\]

\[
= \rho^2 \left( G^{ii} + \sum_{n-2} G^{pp} g_0^{ii} \right) \left( u_{ppii} + 2nu_p u_{pil} + 2nu_i u_{pil} - C \Delta u \right) - \rho CG^{ii} \Delta u,
\]

where we have used Ricci identity. Moreover, from the definition of \( U \) and (4.38), we obtain

\[
0 \geq \rho^2 G^{ii} (U_{iipp} + (u_i^2)_{pp} - (u_i^2)_{pp}) + 2nu_p (U_{iip} + (u_i^2)_{ip} - (u_i^2)_{p}) + 2nu_i (u_{ip} + (u_i^2)_{ip} - (u_i^2)_{p}) - \rho CG^{ii} \Delta u
\]

\[
\geq \rho^2 G^{ii} (U_{iipp} + 2(-2nu_i u_{ip} u_p + u_{ip}^2) - (-2nu_i u_{ip} u_p + u_{ip}^2)
\]

\[
+ 2nu_p (U_{iip} + 2(u_i u_{ip}) - 2(u_i u_{ip})) + 2nu_i (u_{ip} + \frac{2n}{n-2} u_{ip}^2 - C \Delta u) - \rho CG^{ii} \Delta u.
\]

(4.42)

Now substituting (4.40) and (4.41) into (4.42) and using the concavity of \( G \), we have

\[
0 \geq G^{ii} \left( \frac{n+2}{n-2} \rho^2 (\Delta u)^2 - C \rho \Delta u \right) + \rho^2 G^{ii} U_{iipp} + 2\rho^2 u_p G^{ii} U_{iip}
\]

\[
\geq G^{ii} \left( \frac{n+2}{n-2} \rho^2 (\Delta u)^2 - C \rho \Delta u \right).
\]

(4.43)

Now we also derive \( \rho \Delta u \leq C \) from (4.43).

4.5. Proof of Lemma 2.9. Let

\[
W = \nabla^2 u + du \otimes du - \frac{1}{2} |\nabla u|^2 g_0 + A_{g_0}.
\]

(4.44)
We consider the following equation
\[(4.45) \quad \frac{\sigma_k(W)}{\sigma_{k-1}(W)} + \alpha \exp(-2u) = \frac{\varphi(x, u)}{\sigma_{k-1}(W)},\]
where \(t \in [0, 1]\),
\[\varphi(x, u) = \exp(-2ku) f.\]

Take the auxiliary function
\[H(x) = \rho \cdot (\Delta u + |\nabla u|^2) := \rho \cdot K,\]
where \(0 \leq \rho \leq 1\) is a cutoff function depending only on \(r\) such that \(\rho = 1\) in \(B_{\frac{r}{2}}\) and \(\rho = 0\) outside \(B_r\), moreover
\[|\nabla \rho| \leq \frac{C \rho^{1/2}}{r}, \quad |\nabla^2 \rho| \leq \frac{C}{r^2}.\]

Assume \(x_0\) is the maximum point of \(H\). Then at \(x_0\),
\[(4.46) \quad \rho_i K + \rho K_i = 0.\]

We will calculate in the normal coordinates which is centered at \(x_0\). We further assume \(W_{ij}\) and hence \(\frac{\partial G}{\partial W_{ij}}\) are diagonal at the point \(x_0\). Since \(W \in \Gamma_2\), we have
\[|W_{ij}| \leq C tr W, tr W > 0.\]

Therefore,
\[(4.47) \quad |u_{ij}| + |\nabla u|^2 \leq C(\Delta u + 1).\]

In view of (4.47), we may assume
\[\Delta u > C.\]

For the convenience of notations, we will denote
\[G_k(W) = \frac{\sigma_k(W)}{\sigma_{k-1}(W)}, G_0(W) = -\frac{1}{\sigma_{k-1}(W)}.\]

We further denote by \(G^{ij}\), \(G^{ij}_k\) and \(G^{ij}_0\) the functions \(\frac{\partial G}{\partial W_{ij}}, \frac{\partial G}{\partial W_{ij}}\) and \(\frac{\partial G_0}{\partial W_{ij}}\) respectively.

We differentiate the equation (2.15) and obtain
\[G^{ij}W_{ijp} + \varphi(x, u)_p G_0 = G^{ij}_k W_{ijp} + \varphi(x, u) G^{ij}_0 W_{ijp} + \varphi(x, u)_p G_0\]
\[(4.48) = -(\alpha \exp(-2u))_p.\]

Differentiate the equation another time and we obtain
\[G^{ij,rs}W_{ijp} W_{rsp} + G^{ij} W_{ijpp}\]
Moreover, from (3.10) in [GZ19], we have

$$\tag{4.49} -G_0^{i,j}W_{ij,p}W_{rs,p} \geq -(1 + \frac{1}{k+1})G_0^{-1}G_0^{i,j}G_0^{r,s}W_{ij,p}W_{rs,p}.$$  

Besides, from $G^{ii} \geq \frac{2-k+1}{k}$ (see [GZ19]) we have

$$\tag{4.50} -C \geq -CG^{ii}.$$  

In view of (4.47), we may assume $\Delta u > C$. Using (4.46), we have

$$0 \geq \rho G^{ii}H_{ii}(x_0) = \rho G^{ii}(\rho_i K + 2\rho_i K_i + \rho K_{ii}) \geq \rho^2 G^{ii}(u_{ppii} + 2u_{p}u_{p}u_{ii} + 2u_{p}^{2}u_{pi}) - C \rho G^{ii}\Delta u.$$  

Then by use of Ricci identity and the definition of $W$ we obtain

$$0 \geq \rho^2 G^{ii}(u_{iipp} + 2u_{p}u_{iipp} + 2u_{p}^{2}u_{pi} - C\Delta u) - C \rho G^{ii}\Delta u \geq \rho^2 G^{ii}(W_{iipp} - (u_{i}^{2})_{pp} + \frac{(u_{i}^{2})_{p}{2} + 2u_{p}(W_{iip} - (u_{i}^{2})_{p}}) + 2u_{p}^{2} - C\Delta u) - C \rho G^{ii}\Delta u.$$  

From the concavity of $G_{k}$, (4.48) and (4.49) we deduce

$$0 \geq \rho^2 G^{ii}(W_{iipp} - 2(-2u_{i}u_{i}u_{p} + u_{ip}^{2}) + (-2u_{i}u_{i}u_{p} + u_{ip}^{2}) + 2u_{p}(W_{iip} - (u_{i}^{2})_{p}) + 2u_{p}^{2} - C\Delta u) - C \rho G^{ii}\Delta u \geq \rho^2 G^{ii}(\Delta u)^{2} + \rho^2 G^{ii}W_{iipp} + 2\rho^2 u_{p}G^{ii}W_{iip} - C \rho G^{ii}\Delta u \geq \rho^2 G^{ii}(\Delta u)^{2} - C \rho G^{ii}\Delta u$$

$$-\rho^2 \varphi(x, u)G^{i,j}G^{i,j, r,s}W_{i, j, p}W_{r, s, p},$$

$$-2\rho^2 \varphi(x, u)G^{i,j}W_{i, j, p}$$

$$-\rho^2 \varphi(x, u)G^{i,j}G_{0}^{i,j}W_{i, j, p},$$

$$-\rho^2 \varphi(x, u)G_{0}^{i,j}G_{0}^{i,j} + \rho^2(\alpha \exp(-2u))_{pp}$$

$$+ 2\rho^2 u_{p}(-\varphi(x, u)pG_{0} - (\alpha \exp(-2u))(x))_{p})$$
Moreover, using (4.50) we have

\[ 0 \geq \rho^2 G^{ii}((\Delta u)^2) - C \rho G^{ii} \Delta u \]
\[ - \rho^2 \varphi(x, u) (1 + \frac{1}{p+1}) G_0^{-1} G_0^{ij} G_0^{rs} W_{ijp} W_{rsp} \]
\[ - 2 \rho^2 \varphi(x, u) G_0^{ij} W_{ijp} \]
\[ + C \rho^2 G_0(\Delta u + 1). \]

\[ \geq G^{ii} (\rho^2(\Delta u)^2 - \rho \Delta u) + C \rho^2 G_0(\Delta u) \]
\[ - \rho^2 \varphi(x, u) (1 + \frac{1}{k+1}) G_0^{-1} (G_0^{ij} W_{ijp} - \frac{\varphi(x, u)_p}{-\varphi(x, u)(1 + \frac{1}{k+1}) G_0^{-1}})^2 \]
\[ - \frac{\rho^2 \varphi(x, u)^2_p}{-\varphi(x, u)(1 + \frac{1}{k+1}) G_0^{-1}} \]

(4.52) \[ \geq G^{ii} (\rho^2(\Delta u)^2 - C \rho \Delta u) + C \rho^2 G_0(\Delta u + 1). \]

Let us divide the proof into two cases.

(1) \[ \frac{\sigma_k}{\sigma_{k-1}} \leq (\Delta u)^{\frac{1}{k}}. \] Then

(4.53) \[ \rho G_0 \geq \frac{\rho}{\sigma_{k-1}} \geq \frac{\rho(-\frac{\sigma_k}{\sigma_{k-1}} - \sup_{B_r} \varphi(x, u(x))}{\inf_{B_r} \varphi(x, u(x))} \geq -C \rho(\Delta u)^{\frac{1}{k}}. \]

Note that \( \sum_i G^{ii} \geq \frac{\rho}{\sigma_{k-1}}, \) thus (4.52) and (4.53) imply \( \rho \Delta u \leq C. \)

(2) \[ \frac{\sigma_k}{\sigma_{k-1}} > (\Delta u)^{\frac{1}{k}}. \] Then by Lemma 2.1

(4.54) \[ |G_0| = \frac{1}{\sigma_{k-1}} \leq C \left( \frac{\sigma_{k-1}}{\sigma_k} \right)^{k-1} \leq (\Delta u)^{\frac{k-1}{k}}. \]

Now we also derive \( \rho \Delta u \leq C \) from (4.52) and (4.54).

5. Proof of Theorem 1.2

Case (A) \( f > 0, \alpha \leq 0. \) Consider the equation

\[ \sigma_k(t \eta + (1 - t)tr \eta \cdot e) + t \alpha \exp(2u) \sigma_{k-1}(t \eta + (1 - t)tr \eta \cdot e) \]
\[ = (1 - t)(\exp 2u)^k \sigma_k(e) + t \exp(2ku) f(x), \]

where \( \eta \) are the eigenvalues of \( g_0^{-1}(\nabla^2 u + \frac{1}{n-2} \Delta u g_0 + |\nabla u|^2 g_0 - du \otimes du - t \frac{Ric}{n-2} + \frac{1-t}{n} g_0) \)
and \( \eta = (\eta_1, \cdots, \eta_n). \)
When $t = 0$, (5.1) becomes
\begin{equation}
\frac{2n-2}{n-2} \Delta u + (n-1)|\nabla u|^2 + 1 = \exp 2u.
\end{equation}
(5.2)

Assume $x$ and $y$ are the maximum and minimum points of $u$ respectively. Then by (5.2),
\begin{equation}
1 \geq \exp(2u(x))
\end{equation}
and
\begin{equation}
1 \leq \exp(2u(y)).
\end{equation}

Thus $u \equiv 0$. In other words, $u = 0$ is the unique solution for (5.1) at $t = 0$. Let $F = G_k + ((1 - t)\sigma_k(e) + tf) \exp(2ku)G_0 + t\alpha \exp(2u) := G + t\alpha \exp(2u)$ and $u_s$ be the variation of $u$ such that $u' = \varphi$ at $s = 0$. Then
\begin{equation}
F' = G^{ij} \varphi_{ij} + 1\text{st derivatives in } \varphi
\end{equation}
(5.3)
\begin{equation}
- \left( -2t\alpha \exp(2u) - 2k((1 - t)\sigma_k(e) + tf)G_0 \exp(2ku) \right) \varphi.
\end{equation}

Thus linearized operator is invertible since $\alpha \leq 0$ and $G_0 < 0$. (See Theorem 6.14 of [GT].) Therefore the degree is nonzero. (The Leray-Schauder degree is defined in [Li89].) Thus after establishing the a priori estimates Lemma 2.3, Lemma 2.4, and Lemma 2.5 we know that (5.1) is uniformly elliptic. In particular, from [Eva82] and [Kry83], $u \in C^{2,\alpha}$, and the Schauder estimates give classical regularity. Lastly, by homotopy-invariance we obtain a solution at $t = 1$.

**Case (B) $f = 0, \alpha < 0$.**

Consider the equation
\begin{equation}
\frac{\sigma_k}{\sigma_k - 1} (t\eta + (1 - t)\text{tr} \eta \cdot e)
\end{equation}
\begin{equation}
= \left( (1 - t)\frac{\sigma_k(e)}{\sigma_k - 1(e)} - t\alpha \right) \exp 2u,
\end{equation}
(5.4)

where $\eta_i$ are the eigenvalues of $g_0^{-1}(\nabla^2 u - du \otimes du + \frac{1}{2} |\nabla u|^2 g_0 - tA_{g_0} + \frac{1}{n} g_0)$ and $\eta = (\eta_1, \cdots, \eta_n)$.

When $t = 0$, (5.4) becomes
\begin{equation}
\frac{2n-2}{n-2} \Delta u + (n-1)|\nabla u|^2 + 1 = \exp 2u.
\end{equation}
(5.5)
Assume $x$ and $y$ are the maximum and minimum points of $u$ respectively. Then by (5.5),
\[
1 \geq \exp(2u(x))
\]
and
\[
1 \leq \exp(2u(y)).
\]
Thus $u \equiv 0$ is the unique solution for (5.4) at $t = 0$. Besides, the linearized operator is invertible. Therefore the degree is nonzero. (The Leray-Schauder degree is defined in [Li89].) The a priori estimates Lemma 2.6, Lemma 2.7 and Lemma 2.8 imply that (5.4) is uniformly elliptic. In particular, from [Eva82] and [Kry83], $u \in C^{2,\alpha}$, and the Schauder estimates give classical regularity. Lastly, by homotopy-invariance we obtain a solution at $t = 1$.

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