Discrete Anderson Speckle

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When a disordered array of coupled waveguides is illuminated with an extended coherent optical field, discrete speckle develops: partially coherent light with a granular intensity distribution on the lattice sites. The same paradigm applies to a variety of other settings in photonics, such as imperfectly coupled resonators or fibers with randomly coupled cores. Through numerical simulations and analytical modeling, we uncover a set of surprising features that characterize discrete speckle in one- and two-dimensional lattices known to exhibit transverse Anderson localization. Firstly, the fingerprint of localization is embedded in the fluctuations of the discrete speckle and is revealed in the narrowing of the spatial coherence function. Secondly, the transverse coherence length (or speckle grain size) is frozen during propagation. Thirdly, the axial coherence depth is independent of the axial position, thereby resulting in a coherence voxel of fixed volume independently of position. We take these unique features collectively to define a new regime that we call discrete Anderson speckle.

Speckle, the granular spatial intensity pattern imbibed to a coherent optical field after traversing a disordered medium or reflecting from a rough surface, has been studied for decades extending back to the invention of the laser \cite{1,2} – and was known even earlier in radio waves \cite{3,4}. It is a universal phenomenon associated with the interference of random waves. An archetypical arrangement is shown in Fig. [1](a) where a coherent wave traverses a thin phase screen and the random phase is converted into random intensity upon free-space propagation, which we refer to hereon as conventional speckle. Indeed, the propagation of light in random media or scattering from rough surfaces is critical to practical applications in bio-imaging \cite{5}, subsurface exploration \cite{6}, and astronomical observations through turbulent atmospheres \cite{7}. As such, the study of speckle has recently become of central importance in extracting information from – or transmitting it through – complex turbid media \cite{8,13}.

In a multiplicity of contexts, light may be confined to propagate on the sites of a discrete lattice, such as those defined by coupled photorefractive \cite{15}, semiconductor \cite{16}, or fs laser written silica \cite{17} waveguide arrays, random fiber cores \cite{18}, coupled optical resonators \cite{19} or photonic-crystal waveguides \cite{20}. Whether classical \cite{15,20} or quantum light \cite{21,24} is utilized, propagation of an extended coherent field along a disordered photonic lattice produces discrete speckle on the lattice sites [Fig. 1(b)] – in contrast to conventional continuous speckle. One feature arising from the interference between randomly scattered waves in an otherwise periodic potential is Anderson localization \cite{25,26}, which is manifested in the lack of diffusion of the wave function. Optics has enabled direct observation of so-called transverse localization \cite{27} in coupled waveguide arrays on a transversely disordered lattice \cite{15,18,28}, among other realizations \cite{29}. Usually in such experiments, only a single waveguide is excited and spatially non-stationary discrete speckle develops. The typical measure of localization in this scenario is the spatial width of the ensemble-averaged intensity distribution of transmitted light \cite{28}. If instead the waveguides are illuminated by extended coherent light, a configuration that has not been thoroughly investigated heretofore \cite{16,30}, a discrete speckle pattern with spatially invariant statistics develops that apparently masks the localization signature.

In this paper, we investigate numerically and analytically the statistical properties of discrete speckle in one- and two-dimensional (1D and 2D) disordered Anderson lattices upon extended illumination [Fig. 1(b)]. We show that the fingerprint of localization is embedded in the fluctuations of the emerging light and is thus revealed in the coherence function. We uncover a surprising phenomenon: the transverse coherence width associated with an extended coherent field is determined by the localization length resulting from a single-site excitation. Consequently, beyond a critical distance, the transverse speckle grain size ‘freezes’ upon subsequent propagation along the lattice [Fig. 1(d)]. Furthermore, the axial coherence depth is independent of axial position, leading to a coherence ‘voxel’ of fixed volume independent of position. We take these features collectively to define a new regime that we call discrete Anderson speckle’. Our findings are in contradistinction to the familiar characteristics of conventional speckle \cite{31}, wherein the transverse coherence length grows with the free-space propagation distance [Fig. 1(c)], as dictated by the van Cittert-Zernike theorem \cite{32}.

These findings have their foundation in the different beam propagation dynamics that distinguish discrete lattices from continuous media. Nevertheless, despite the distinctions between conventional and discrete Anderson speckle, both phenomena have a common feature: each system contains a single realization of a random function of the transverse coordinate. In conventional speckle the randomness is confined to the thin screen, while in discrete Anderson speckle it extends axially without change. Our results help elucidate the ultimate resolution limits of imaging through an Anderson lattice \cite{18}, introduce new strategies for engineering the spatial optical coherence of a beam of light \cite{33}, and indicate the potential for tuning higher-order field statistics beyond the Gaussian limits.

Previous investigations of electromagnetic-wave propagation through random media have studied the dimensionless conductance, which is proportional to the transmittance \cite{34,35}. In such systems, disorder – and hence localization – is primarily axial instead of transverse. In case of the 1D and 2D photonic systems examined here, the situation is quite distinct since the disorder is transverse and back-scattering is not allowed, so that the transmittance is always unity (in the absence of absorption) and the localization is observed in a plane transverse to the propagation axis.

I. DISCRETE OPTICAL LATTICE MODEL

Field propagation along a 1D lattice of parallel waveguides with evanescent nearest-neighbor-only coupling [Fig. 2] is given by the coupled equations \cite{29}

\[
\frac{dE_x(z)}{dz} + \beta_x E_x + C_{x-1}E_{x-1} + C_{x+1}E_{x+1} = 0,
\]

\[
(1)
\]
where $E_z(z)$ is the complex optical field in the $x^{th}$ waveguide ($x = -N, \ldots, N$) at axial position $z$, $\beta_x$ is the propagation constant of waveguide $x$, and $C_{x, x+1}$ is the coupling coefficient between adjacent waveguides $x$ and $x + 1$. The evolution of the input field $E_i(x_i)$ to the output $E_o(x_o)$ at $z$ may be written as $E_o(x_o) = \sum_i h(x_o, x_i)E_i(x_i)$, where $h(x_o, x_i)$ represents the system’s impulse response function after propagating an axial distance $z$ (see the Supplement). The point spread function (PSF) $|h(x_o, x_i)|^2$ is the ensemble average. In general, similar behavior is observed in 2D lattices [Fig. 3(a)]. We define the localization length $\sigma_z$ as the root-mean-square width of the mean PSF. As shown in the insets of Fig. 2(b) and in Fig. 3(b), $\sigma_z$ decreases monotonically with increasing $\Delta C$ at fixed distance $z$ in 1D and 2D lattices. On the other hand, $\sigma_z$ typically increases with $z$ at fixed $\Delta C$ until it saturates, a signature of localization, which happens earlier for large $\Delta C$ [Fig. 2(b), inset]. For later reference, we note that for short propagation distances at intermediate disorder levels, features of both localized and ballistic states coexist.

Disorder Classes

We consider two classes of disorder. The first, diagonal disorder [21], is characterized by constant $C_{x, x+1} = C$ and random $\beta_x$ having a uniform probability distribution of mean $\bar{\beta}$ and half width $\Delta \bar{\beta}$. The second class, off-diagonal disorder [22], is characterized by fixed $\beta_x = \tilde{\beta}$ and random $C_{x, x+1}$ having a uniform probability distribution of mean $\bar{C}$ and half width $\Delta \bar{C}$. Both disorder classes exhibit similar behavior in our investigations; we thus report here results for off-diagonal disorder and relegate those for diagonal disorder to the Supplement. The findings of this study are presented in terms of dimensionless variables by writing the coupling coefficients in units of their average $\bar{C}$, and the distance $z$ in units of the coupling length $\ell = 1/\bar{C}$. Throughout, $\Delta C$ ranges from 0 to 1. Lattice sizes are chosen large enough so that all the central results in this paper are independent of lattice size $N_l = 2N + 1$. Further details are provided in the Supplement.

II. DISCRETE ANDERSON SPECKLE: TRANSVERSE COHERENCE

A. Anderson Localization

To set the stage for examining transverse coherence of discrete speckle in Anderson lattices upon uniform illumination, we first describe briefly the results of single-waveguide excitation. When disorder is absent ($\Delta C = 0$), ballistic spread leads to an extended output state [Fig. 2(a)]. Progressively introducing disorder into the lattice results in a gradual transition to an exponentially localized state [Fig. 2(b)] manifested in the pronounced confinement of the mean PSF $\langle |h(x_o, 0)|^2 \rangle$ around the excitation waveguide, where $\langle \cdot \rangle$ is the ensemble average. In general, similar behavior is observed in 2D lattices [Fig. 3(a)]. We define the localization length $\sigma_z$ as the root-mean-square width of the mean PSF. As shown in the insets of Fig. 2(b) and in Fig. 3(b), $\sigma_z$ decreases monotonically with increasing $\Delta C$ at fixed distance $z$ in 1D and 2D lattices. On the other hand, $\sigma_z$ typically increases with $z$ at fixed $\Delta C$ until it saturates, a signature of localization, which happens earlier for large $\Delta C$ [Fig. 2(b), inset]. For later reference, we note that for short propagation distances at intermediate disorder levels, features of both localized and ballistic states coexist.

Figure 2 | Anderson localization and discrete speckle in 1D waveguide lattices.
(a) PSF $I(x_o) = |h(x_o, 0)|^2$ at $z = 10$ for a 1D periodic array for single-waveguide excitation at $x_i = 0$. Inset is a schematic of the configuration. (b) Mean PSF $I(x_o) = \langle |h(x_o, 0)|^2 \rangle$ for disordered 1D arrays. Insets show the localization length $\sigma_z$ as a function of $\Delta C$ (for fixed $z = 10$) and of $z$ (for fixed values of $\Delta C$). For the values of $\sigma_z$ in the insets, 21 points for $\Delta C$ and 200 for $z$ are chosen. (c) Realizations of discrete speckle at various disorder levels ($z = 10$) for extended uniform coherent input light. The dotted lines are ensemble averages. We use $N_l = 151$ throughout.
We now move on to our investigation of the global statistics of light in Anderson lattices by examining the case of coherent extended uniform illumination. For a 1D array, $E_l(x_l) = 1$ and the output field is $E_o(x_o) = \sum_i h(x_o, x_i)$, which is a random function of $x_o$ in the case of a disordered lattice; a similar relation holds for 2D arrays. In the absence of disorder, the extended intensity distribution is invariant with respect to propagation [Fig. 3(c) for 2D]. Upon introducing disorder, this uniform distribution transitions into a granular intensity pattern $I(x_o) = \langle |E_o(x_o)|^2 \rangle$ defined on the lattice sites – which we call discrete speckle. Examples of individual realizations for 1D and 2D lattices are shown in Fig. 2(c) and Fig. 3(c), respectively. Several characteristics are immediately apparent in these results. First, with increasing disorder, the grain size – which is related to the transverse spatial coherence width – decreases. On the other hand, the speckle contrast $c$ – defined as the ratio of the standard deviation in the speckle intensity $c_{\lambda}$ to its mean intensity $I_o$, $c = c_{\lambda}/I_o$ – increases with disorder. These observations are tell-tale signs of a decrease in the transverse coherence width with increasing disorder. Indeed, these characteristics are shared with conventional speckle.

Despite the spatially varying intensity distribution $|E_o(x_o)|^2$ in the individual realizations for extended input, the statistical homogeneity of this discrete speckle is clear in the uniform distribution obtained upon averaging multiple realizations $\langle |E_o(x_o)|^2 \rangle$ [the dotted lines in Fig. 2(c)]. The coherence function at a pair of positions $x_o$ and $x_o + x$ in 1D is therefore a function of only the separation $x$, \begin{equation}
G^{(1)}(x_o, x_o + x) = G^{(1)}(0, x) = \langle E^*_o(0) E_o(x) \rangle = \sum_{x', x''} (h^*(0, x') h(x', x'')).
\end{equation}

Its normalized version is the complex degree of coherence $g^{(1)}(x) = G^{(1)}(0, x)/\sqrt{G^{(1)}(0, 0) G^{(1)}(x, x)}$, with $0 \leq |g^{(1)}(x)| \leq 1$. In 2D discrete speckle, we similarly write the complex degree of coherence $g^{(1)}(r)$ as a function of the radial separation distance $r$ shown in Fig. 3(c). For later reference (see Section 5, Analytical Model), we note that transverse spatial invariance results in the double summation in Eq. 2, separating over the two impulse response functions, such that $G^{(1)}(0, x) = \langle \eta \sum_{\eta'} h(x, x') \rangle$, where $\eta = \sum_{\eta'} h^*(0, x')$ is a zero-mean, complex random variable.

We have carried out an extensive computational exploration of the coherence properties of light propagating in Anderson lattices. Figures 4(a) and 5(a) depict the magnitudes of $g^{(1)}(x)$ and $g^{(1)}(r)$ for 1D and 2D lattices, respectively, revealing a non-zero pedestal $|g^{(1)}(\infty)|$ riding on which is a finite-width distribution. This pedestal $|g^{(1)}(\infty)|$ signifies the survival of long-range transverse order; that is, some level of transverse correlation is maintained regardless of the separation between the pair of waveguides. Indeed, $|g^{(1)}(\infty)|$ decreases monotonically with $\Delta C$ until it vanishes altogether at a threshold $\Delta C$ value [Fig. 4(b) for 1D and Fig. 5(b) for 2D].

It is useful at this point to compare the coherence of discrete speckle described above to that of conventional speckle produced in the arrangement shown in Fig. 1(a). The random component of the screen phase $\phi$ is typically a Gaussian process with zero mean, variance $\sigma_\phi^2$, and spatially invariant transverse correlation of width $x_c$, which we take as a transverse unit length in analogy to the unit separation between the waveguides on a lattice. During propagation along $z$, the field passes through two regimes. In the first regime where $z < 2N_c x_c^2/\lambda$ ($N_c$ is the size of the illuminating beam in units of $x_c$ and $\lambda$ is the wavelength), the coherence properties do not change with $z$. Interestingly, the coherence function $g^{(1)}(z)$ for conventional speckle contains a pedestal associated with the specular component of the field when the thin phase screen has small $\sigma_\phi^2$ [31], in analogy to the pedestal resulting from ballistic propagation in its discrete counterpart for small $\Delta C$ [Figs. 4(a) and 5(a)]. Conventional speckle, the pedestal height drops gradually with increased $\sigma_\phi^2$ for fixed $z$ [similarly to the behavior of $|g^{(1)}(\infty)|$ with $\Delta C$ in Figs. 4(b) and 5(b)], and gradually vanishes as the field leaves this regime, i.e., $z > 2N_c x_c^2/\lambda$. In the far field, $g^{(1)}(z)$ becomes the Fourier transform of the illumination spot and the grain size increases continuously with $z$ in accordance with the van Cittert-Zernike theorem [Fig. 1(c)].

A distinction between ‘near-’ and ‘far-field’ may be similarly made for discrete speckle based on the disappearance of the pedestal $g^{(1)}(\infty)$. For small distances, $g^{(1)}(\infty)$ is non-zero and the discrete speckle undergoes dynamical changes upon propagation as shown in Fig. 4(c)-(d). However, for a given disorder level $\Delta C$, the pedestal vanishes after some distance $z > 5 \Delta C$ [Fig. 4(e)].
Figure 4 | Transverse coherence for 1D discrete Anderson speckle. (a) Magnitude of $g^{(1)}(x)$ for 1D arrays for various disorder levels $\Delta C$ at propagation distance $z = 10$. (b) The long-range-order coherence pedestal $|g^{(1)}(\infty)|$ as a function of $\Delta C$ at $z = 10$. The circles in (b) correspond to the same values of $\Delta C$ in (a). (c,d) The magnitude of $g^{(1)}(x)$ at various $z$ for (c) $\Delta C = 0.2$ and (d) $\Delta C = 0.4$. The pedestal decreases with $z$ and $g^{(1)}(x)$ becomes stationary with respect to further propagation. (e) $|g^{(1)}(\infty)|$ as a function of $z$ at various $\Delta C$. (f) Transverse coherence width $\sigma_c$ as a function of $\Delta C$ at $z = 20$. All areas shaded in gray, and also the dashed arrows, indicate the onset of the discrete Anderson speckle (DAS) regime.

Figure 5 | Transverse coherence for 2D discrete Anderson speckle. (a) Magnitude of $g^{(1)}(r)$ for 2D arrays for various disorder levels $\Delta C$ at propagation distance $z = 10$. (b) The long-range-order coherence pedestal $|g^{(1)}(\infty)|$ as a function of $\Delta C$ at $z = 10$. The hexagons in (b) correspond to the same values of $\Delta C$ in (a). (c,d) The magnitude of $g^{(1)}(r)$ at various $z$ for (c) $\Delta C = 0.2$ and (d) $\Delta C = 0.4$. The pedestal height decreases with $z$ and $g^{(1)}(r)$ becomes stationary with respect to further propagation. (e) $|g^{(1)}(\infty)|$ as a function of $z$ at various $\Delta C$. (f) Transverse coherence width $\sigma_c$ as a function of $\Delta C$ at $z = 20$. All areas shaded in gray, and also the dashed arrows, indicate the onset of the discrete Anderson speckle (DAS) regime.

III. DISCRETE ANDERSON LOCALIZATION: AXIAL COHERENCE

Further insight may be drawn from a detailed examination of the axial coherence propagation dynamics. We plot $I(x_0; z) = |E(x_0; z)|^2$ for three realizations at $\Delta C = 0.2, 0.4, 1.0$ in Fig. 7(a). The longitudinal freezing of the transverse discrete speckle is evident for all three cases in the far field, resulting in axial filamentation of the intensity distribution – corresponding to the non-overlapping uncorrelated paths along the disordered lattice mentioned above. Evaluation of the axial coherence function $G^{(1)}(z, \Delta z) = \sum_x |E^*(x_0; z)E(x_0; z + \Delta z)|$ reveals that it is in fact independent of $z$ altogether. The normalized axial degree of coherence $|g^{(1)}(\Delta z)|$ decays with $\Delta z$ at a rate proportional to the disorder level [Fig. 7(b)], so that its FWHM or axial coherence depth $\sigma_a$ drops with disorder [Fig. 7(c)]. This behavior is stationary along $z$. Finally, a unique aspect of the features described in this Section is that they are evident in individual realizations, unlike observations of Anderson localization that necessitate ensemble averaging.

We have found that the transverse coherence width $\sigma_c$ reaches a steady state in the discrete Anderson speckle regime and the statistical homogeneity renders it independent of transverse position $x$. Furthermore, the axial coherence depth $\sigma_a$ for a fixed disorder level is independent of axial position $z$ (and is primarily due to dephasing; see Figs. S3-S5 in the Supplement). By combining these findings concerning transverse and axial coherence in disordered lattices, we conclude that a coherence ‘voxel’ of fixed volume exists everywhere along the lattice in the discrete Anderson speckle regime. The volume of this coherence voxel depends solely on
This equation can be interpreted in light of the Klyshko advanced-wave picture as a cascade of the three steps illustrated in Fig. 8(b).

IV. ANALYTICAL MODEL

We have shown numerically that the fingerprint of localization exists in the fluctuations of the discrete speckle emerging from Anderson lattices for an extended coherent input. It may be initially surprising that a link exists between the localization length (typically associated with a point excitation and averaging over output intensity) and the transverse coherence width (associated with an extended input and averaging over field products for pairs of waveguides); see Fig. 6. Our goal here is to link the extended-illumination scheme that has been our focus [Fig. 1(b)] with the more usual single-waveguide excitation strategy [Fig. 2(a,b)]. To elucidate this link, we adapt to our setting a conceptual scheme from quantum optics known as ‘Klyshko’s advanced-wave picture’ [38][39], which is also used to classical fields. This scheme allows for the identification of correlation functions of an extended field traversing an optical system with the field or intensity of a double-pass configuration (backward then forward) of a point source through the same system.

We start by depicting in Fig. 8(a) the 1D scenario we have investigated in this paper, wherein an extended coherent field traverses a random lattice (ΔC = 1). Averaging the output intensity |Eo(x0)|2 over multiple realizations yields a constant distribution with no localization signature [Fig. 2(c)]. Nevertheless, computing the spatially stationary coherence function G(1)(0, x) by averaging over products of fields from pairs of waveguides separated by x yields a localized function (independently of x0) of width σc.

Referring to Eq. 2 we write G(1)(0, x) as

\[ G^{(1)}(0, x) = \left\{ \sum_{x'} h(x, x') \sum_{x''} h^*(0, x'') \right\} \]

This equation can be interpreted in light of the Klyshko advanced-wave picture as a cascade of the three steps illustrated in Fig. 8(b).

First, a point excitation at x0 = 0 propagates backward through the system h to the x’ plane, as dictated by the conjugation operation. Second, the output field from this backward propagation is spatially averaged over x’ to yield the complex random variable η = \( \sum_{x'} h^*(-0, x') \), which is then equally distributed over points x” in the input plane for a second pass forward through the same realization of the system h. Third, the uniform extended field of amplitude η propagates forward through h to produce an output random field \( \hat{E}(x) = η \sum_{x''} h(x, x'') \). Ensemble averaging results in \( \langle \hat{E}(x) \rangle = G^{(1)}(0, x) \) per Eq. 2 and Eq. 3.

Let us examine the third step in this cascade, the forward pass. Each waveguide at position x” is fed with a noisy field having complex random amplitude η with zero mean. The ensemble average of the output field in the x-plane contributed by each waveguide is \( \langle ηh(x, x'') \rangle \). While the ensemble average of η and \( \langle h(x, x'') \rangle \) (for high disorder levels) is each zero, the average of their product need not be so since both random variables are generated by the same realization of the disordered lattice. Indeed, since η is generated by the random lattice environment in the vicinity of x = 0 in the Anderson localization limit, then it correlates only with h in the same vicinity, while remaining uncorrelated \( \langle ηh(x, x'') \rangle \sim 0 \) when h is evaluated away from the origin, as shown in Fig. 8(b). Consequently, only a few waveguides in the vicinity of x” = 0 contribute to the forward pass. Since h produces a localized output for a point excitation, the few-waveguide excitation here results in a slightly broader localized spot whose width is σc (resulting from the convolution of the impulse response function with the width of the distribution in Fig. 8(b)). We have thus established on these grounds that σc is intimately linked with the localization length σa, but is expected to be slightly larger – as was shown numerically in Fig. 6.

We next proceed to an analytical model of discrete Anderson speckle based on modal analysis. Using the eigenmodes of the lattice coupling matrix, we justify (1) the freezing of the trans-
verse coherence width $\sigma_c$ (and hence the speckle grain size) once the discrete Anderson speckle regime is reached, and (2) the independence of the axial coherence depth $\sigma_a$ from axial position $z$.

### A. Origin of the freezing of the transverse coherence width

We analyze the propagation of the field along an Anderson lattice in terms of the eigenmodes and eigenvalues of the Hermitian coupling matrix $\hat{H}$ that is defined by the equation of dynamics in Eq. 4 by writing

$$\frac{d\mathbf{E}(z)}{dz} + \hat{H}\mathbf{E} = 0,$$

where $\mathbf{E}$ is a vector of length $2N + 1$ containing the field amplitudes in the waveguides, and $\hat{H}$ is a real symmetric (and hence Hermitian) matrix with the wave numbers along the diagonal and coupling coefficients off the diagonal. If the eigenmodes and eigenvalues of $\hat{H}$ are $\phi_n(x)$ and $b_n$, respectively, then since $h = e^{i\hat{H}z}$, the eigenvalue problem is defined for the impulse response function as

$$\sum_{x'} h(x, x'; z) \phi_n(x') = e^{i b_n z} \phi_n(x),$$

such that the impulse response function may be expressed as

$$h(x_0, x_1; z) = \sum_n e^{i b_n z} \phi_n(x_0) \phi_n(x_1).$$

We have made use of the fact that the eigenmodes are real since $\hat{H}$ is real and symmetric. Using this definition, we recast the joint transverse-axial coherence function in terms of $\phi_n(x)$ and $b_n$,

$$G^{(1)}(0, x; z, z + \Delta z) = \sum_{x', x''} \langle h^*(0, x'; z) h(x, x''; z + \Delta z) \rangle$$

$$= \sum_{x, x'} \sum_{n, m} \langle \phi_n(0) \phi_n(x') \phi_m(x) \phi_m(x'') e^{i [b_n - b_m] z - i b_n \Delta z} \rangle.$$  

The freezing of the speckle grain size in the discrete Anderson speckle regime is realized at large propagation distances $z$ when the following condition is satisfied: $\text{Std} \{ b_n \} \gtrsim 2\pi$; here $\text{Std} \{ \cdot \}$ is the standard deviation. We expect that $\text{Std} \{ b_n \}$ is proportional to $\Delta C$, such that the distance $z$ that satisfies this condition is inversely proportional to $\Delta C$. In the case of off-diagonal disorder, which we have considered here, the eigenvalue $b_0$ is excluded from this condition since it remains deterministic with value 0 [40]. This exclusion is not required in the case of diagonal disorder which is described in the Supplement. We have found numerically that
this limit in lattices with off-diagonal disorder is attained when $\Delta z > 5$, which we have taken to define the discrete Anderson speckle regime.

When the condition $\text{Std}\{b_n\} z \geq 2\pi$ is met, the difference $b_n - b_m$ when $n \neq m$ has same order of magnitude as this standard deviation, but is equal to zero when $n = m$, therefore implying that upon ensemble averaging, the impact of the exponential term in Eq. (7) in $e^{i(b_n-b_m)z} \rightarrow \delta_{n,m}$. Thus, setting $\Delta z = 0$ in the axial regime where $\Delta z > 5$, Eq. (7) reduces to

$$G^{(1)}(0, x; z, z) = \sum_{x', x} \sum_n \langle \phi_n(0) \phi_n(x) \phi_n(x') \phi_n(x''') \rangle. \tag{8}$$

This equation implies that in the discrete Anderson speckle regime the transverse coherence is a function of the separation $x$ but not the axial distance $z$, as demonstrated numerically in Fig. 4.

B. Independence of axial coherence depth from axial position

In considering the axial coherence along the lattice, we make use of the transverse stationarity of the lattice and consider a single lattice site $x$ in Eq. (7) whereupon the axial coherence function is

$$G^{(1)}(x, x; z, z + \Delta z) = \sum_{x', x} \sum_n \langle \phi_n(x) \phi_n(x') \phi_n(x) \phi_n(x') e^{i(b_n-b_m)z} e^{-ib_n \Delta z} \rangle. \tag{9}$$

By taking a spatial average over $x$, we obtain a simplified relation

$$\sum_x G^{(1)}(x, x; z, z + \Delta z) = \sum_{x', x} \sum_n \langle \phi_n(x') \phi_n(x') e^{-ib_n \Delta z} \rangle, \tag{10}$$

in which we used $\sum_x \phi_n(x') \phi_n(x) = \delta_{n,m}$. Consequently, the axial coherence function averaged over the transverse coordinate is altogether independent from $z$. However, since $G^{(1)}(x, x; z, z + \Delta z)$ is stationary in $x$, its statistical properties are the same as those of $\sum_x G^{(1)}(x, x; z, z + \Delta z)$. Therefore, the axial coherence function is independent of $z$ and, as a result its width $\sigma_z$ is also independent of $z$ and relies only on $\Delta z$ — as demonstrated numerically in Fig. 7.

V. CONCLUSION

We have investigated the evolution of a set of mutually coherent waves traveling through 1D and 2D disordered lattices of coupled waveguides. The emerging wave forms discrete speckle that is statistically homogeneous with random intensity distribution on the lattice sites. The disordered lattice structure that results in Anderson localization when a single waveguide is excited exhibits in the case of an extended excitation a complete freezing of the discrete speckle grain size after reaching a steady-state, unlike the usual growth observed in conventional speckle — a regime we refer to as discrete Anderson speckle. Moreover, axial and transverse coherence are independent of position, resulting in a coherence voxel of fixed volume independent of its transverse and axial position on the lattice. These results are applicable to a broad host of photonic systems in which disorder may impact coupling between discrete elements [15–23]. While we have studied second-order field correlations on a discrete lattice, the new behavior reported here signposts important vistas to be investigated in the context of higher-order correlations and photon statistics [41].

Finally, the correspondence between the propagation of light and that of a quantum particle on discrete lattices [29] has led to recently fruitful exchanges between optical and condensed matter physics [42–48]. Our result, therefore, points to new regimes that may be investigated in other physical systems, ranging from Bose-Einstein condensates [49] to acoustic lattices [50], where Anderson localization takes place owing to interference of random waves.

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See Supplement for supporting content.

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