Quiver $\mathcal{D}$-Modules and the Riemann-Hilbert Correspondence

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Abstract

In this paper, we show that every regular singular $\mathcal{D}$-module in $\mathbb{C}^n$ whose singular locus is a normal crossing is isomorphic to a quiver $\mathcal{D}$-module – a $\mathcal{D}$-module whose definition is based on certain representations of the hypercube quiver. To be more precise we give an equivalence of the respective categories. Our definition of quiver $\mathcal{D}$-modules is based on the one of Khoroshkin and Varchenko. To prove the equivalence, we use an equivalence by Galligo, Granger and Maisonobe for regular singular $\mathcal{D}$-modules whose singular locus is a normal crossing which involves the classical Riemann-Hilbert correspondence.

The classical version of the Riemann-Hilbert correspondence, as it was proven independently by Masaki Kashiwara [Kas84] and Zoghman Mebkhout [Meb84] in 1980, yields an equivalence $\text{Mod}_{\text{rh}}(\mathcal{D}_X) \xrightarrow{\cong} \text{Perv}(X)$ between regular holonomic $\mathcal{D}_X$-modules and perverse sheaves on $X$. In dimension one locally at 0 other equivalences are known which one might obtain from the classical version: The equivalence of the category $\text{Mod}_{\text{rh}}(\mathcal{D})$ of regular holonomic $\mathcal{D}$-modules with the category $\mathcal{C}_1$ and the category $\text{Qu}_\Sigma^1$, the categories of finite quiver representations $\text{E}uv\Sigma^1$, respectively (see [Mal91], [Bjö93] or [Dim04]). In higher dimension André Galligo, Michel Granger and Philippe Maisonobe proved that in the case of a normal crossing divisor in dimension $n$, the category of perverse sheaves with respect to the induced normal crossing stratification is equivalent to the category $\mathcal{C}_n$ (the generalization of $\mathcal{C}_1$). This means, using the Riemann-Hilbert correspondence, that the category of regular holonomic $\mathcal{D}$-modules whose singular locus is a normal crossing is equivalent to $\mathcal{C}_n$ (see [GGM85a] and [GGM85b]). However, it is not that easy to assign a $\mathcal{D}$-module to a given quiver representation with respect to this equivalence concretely.

A contribution to the question of how to assign to a quiver representation a $\mathcal{D}$-module comes from Sergei Khoroshkin and Alexander Varchenko [KV06]. To a given hyperplane arrangement in $\mathbb{C}^n$, they associate a quiver. And to each finite representation over $\mathbb{C}$ of such a quiver, they associate a $\mathcal{D}$-module in a rather intuitive way. This yields a functor $E$ from the category of representations over these quivers into the category of holonomic $\mathcal{D}$-modules. Using this definition in dimension one, one sees that this gives a functor from $\text{Qu}_\Sigma^1$ to $\text{Mod}_{\text{rh}}(\mathcal{D})$ one can use as quasi-inverse for the equivalence above. In particular one sees that every regular holonomic $\mathcal{D}$-module in dimension one locally at 0 is isomorphic to a quiver $\mathcal{D}$-module. This makes their construction very promising for higher dimensions.

The main idea of our work is to use this construction of quiver $\mathcal{D}$-modules by Khoroshkin and Varchenko in the case of a normal crossing hyperplane arrangement and to combine it with the
theorem of Galligo, Granger and Maisonobe. In Section 1 we give some general statements on representations of the hypercube quiver. In Section 2 we define the category of quiver $\mathcal{O}$-modules and give their main properties. In our Main Theorem 2.7 we give the link between this category and the category $\mathcal{Q}_{u_1}$ using the theorem of Galligo, Granger and Maisonobe.

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1 Finite representations of the hypercube quiver

In the following let us consider finite representations over $\mathbb{C}$ of the following quiver:

Let $n \in \mathbb{N}^+$ and let $\mathcal{P}(\{1, \ldots, n\})$ denote the power set of $\{1, \ldots, n\}$. The quiver consists of $2^n$ vertices which we denote by $I \in \mathcal{P}(\{1, \ldots, n\})$, and $n2^n$ oriented edges

$$I \rightarrow I \cup \{i\}$$

for $i \in \{1, \ldots, n\} \setminus I$. This gives us a kind of hypercube quiver by imaging that the vertices of the quiver lie on the vertices of a $n$-dimensional hypercube, and we have two edges exactly for every edge of the hypercube.

1.1 Definitions and basic properties

In the following we are going to define three standard categories of hypercube quiver representations, denoted $\mathcal{Q}_{u_1}$, $\mathcal{C}_n$ and $\mathcal{Q}_{u_1}^{\Sigma_1}$. Let us start with the definition of $\mathcal{Q}_{u_1}$. This is basically just the category of finite representations over $\mathbb{C}$ of the above hypercube quiver.

**Definition 1.1.** The category $\mathcal{Q}_{u_1}$ for $n \in \mathbb{N}^+$ is defined as follows:

- The objects consist of $2^n$ finitely generated $\mathbb{C}$-vector spaces denoted $V_I$ where $I \in \mathcal{P}(\{1, \ldots, n\})$, equipped with $n2^n$ linear maps $u_{I,i}$ and $y_{I,i}$ for $i \in \{1, \ldots, n\} \setminus I$,

$$V_I \xleftarrow{\ u_{I,i}\ } \ xrightarrow{\ y_{I,i}\ } V_{I \cup \{i\}}$$

and they satisfy the following commutativity conditions for $i, j \in \{1, \ldots, n\} \setminus I$:

$$u_{I \cup \{i\},j} \circ u_{I,i} = u_{I \cup \{j\},i} \circ u_{I,j} \quad \quad y_{I,i} \circ y_{I \cup \{i\},j} = y_{I,j} \circ y_{I \cup \{j\},i}$$

$$y_{I \cup \{i\},j} \circ u_{I \cup \{j\},i} = u_{I,i} \circ y_{I,j}$$

- A morphism between two objects $(V_I, u_{I,i}, y_{I,i})$ and $(V'_I, u'_{I,i}, y'_{I,i})$ is given by $2^n$ linear maps $h_I: V_I \rightarrow V'_I$ such that $u'_{I,i} \circ h_I = h_{I \cup \{i\}} \circ u_{I,i}$ and $h_I \circ y_{I,i} = y'_{I,i} \circ h_{I \cup \{i\}}$ for $i \in \{1, \ldots, n\} \setminus I$. 


Now, let us define the categories $C_n$ and $\mathcal{Qu}_n^{\Sigma_1}$. They are full subcategories of $\mathcal{Qu}_n$ fulfilling an additional constraint on their objects.

**Definition 1.2.** The category $C_n$ is the full subcategory of $\mathcal{Qu}_n$ such that every object $(V_I, u_{I,i}, y_{I,i})$ additionally fulfills that $u_{I,i} \circ y_{I,i} + \text{Id}_{V(I,i)}$ and $y_{I,i} \circ u_{I,i} + \text{Id}_{V_I}$ are invertible.

**Definition 1.3.** The category $\mathcal{Qu}_n^{\Sigma_1}$ is the full subcategory of $\mathcal{Qu}_n$ such that every object $(V_I, u_{I,i}, y_{I,i})$ additionally fulfills that $\text{Spec}(u_{I,i} \circ y_{I,i})$, $\text{Spec}(y_{I,i} \circ u_{I,i})$ is invertible, and $\text{Spec}(u_{I,i} \circ y_{I,i}) \subseteq \Sigma_1 := \Sigma + 1$ where

$$\Sigma := \left\{ \alpha \in \mathbb{C} \mid -1 \leq \text{Re}(\alpha) \leq 0, \text{Im}(\alpha) = \begin{cases} \geq 0 & \text{if } \text{Re}(\alpha) = -1, \\ < 0 & \text{if } \text{Re}(\alpha) = 0, \\ \text{arbitrary} & \text{otherwise.} \end{cases} \right\}$$

We note that $u_{I,i} \circ y_{I,i} + \text{Id}_{V(I,i)}$ is invertible iff $y_{I,i} \circ u_{I,i} + \text{Id}_{V_I}$ is invertible, and $\text{Spec}(u_{I,i} \circ y_{I,i}) \subseteq \Sigma_1$ iff $\text{Spec}(y_{I,i} \circ u_{I,i}) \subseteq \Sigma_1$.

The last topic in this subsection is a dualizing functor acting on our quiver categories.

**Definition/Proposition 1.4.** The contravariant functor $D : \mathcal{Qu}_n \to \mathcal{Qu}_n$ is defined on objects $(V_I, u_{I,i}, y_{I,i})$ of $\mathcal{Qu}_n$ by

$$D \left( V_I \leftrightarrow \frac{u_{I,i}}{y_{I,i}} V_{I \cup \{i\}} \right) := V_I^* \left( \frac{y_{I,i}^*}{u_{I,i}^*} \right) V_{I \cup \{i\}}^*$$

where $V_I^*$ is the dual vector space of $V_I$ and $u_{I,i}^*$, $y_{I,i}^*$ are the dual/transpose maps of $u_{I,i}$, $y_{I,i}$. Let $(h_I)$ denote a morphism in $\mathcal{Qu}_n$. Then we set

$$D((h_I)) := (h_I^*)$$

where $h_I^*$ is the dual map of $h_I$. This yields an equivalence of categories where $D$ is its own quasi-inverse, and it also establishes an equivalence from $\mathcal{Qu}_n^{\Sigma_1}$ to $\mathcal{Qu}_n^{\Sigma_1}$, and from $C_n$ to $C_n$ as well.

### 1.2 An equivalence of categories

The goal of this subsection is to prove in Theorem 1.6 an equivalence (or rather an isomorphism) of the categories $C_n$ and $\mathcal{Qu}_n^{\Sigma_1}$. This is a principal component of the present work. We use the following pair of functors:

**Definition 1.5.** The covariant functors $Q : \mathcal{Qu}_n^{\Sigma_1} \to C_n$ and $G : C_n \to \mathcal{Qu}_n^{\Sigma_1}$ are defined by: Let $(V_I, u_{I,i}, c_{I,i})$ denote an object in $\mathcal{Qu}_n^{\Sigma_1}$ and let $(h_I)$ denote a morphism in $\mathcal{Qu}_n^{\Sigma_1}$. We set

$$Q((V_I, u_{I,i}, c_{I,i})) := (V_I, u_{I,i}, y_{I,i}) \text{ and } Q((h_I)) := (h_I)$$

where $y_{I,i} := \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (c_{I,i} \circ u_{I,i})^{k-1} \circ c_{I,i} = c_{I,i} \circ \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (u_{I,i} \circ c_{I,i})^{k-1}$. 

3
Let \((V, u_I, i, w_I, i)\) denote an object in \(C\) and let \((h_I)\) denote a morphism in \(C\). Then, set

\[ G((V, u_I, i, w_I, i)) := (V, u_I, i, x_I, i) \] and \( G((h_I)) := (h_I) \).

The map \(x_I, i\) is given as follows: Let \(s_I, i: V_I \rightarrow V_I\) denote the unique linear map with eigenvalues in \(\Sigma_1\) such that

\[ e^{2\pi i s_I, i} = w_I, i \circ u_I, i + \text{Id}_{V_I} \] and set \(x_I, i := \left( \sum_{k=1}^{\infty} \frac{(2\pi i)^k k^{-1}}{k!} s^k_{I, i} \right)^{-1} \circ w_I, i\).

**Theorem 1.6.** The category \(C\) is isomorphic to the category \(\text{Qui}^{\Sigma_1}_n\) using the pair of covariant functors \(Q: \text{Qui}^{\Sigma_1}_n \rightarrow C\) and \(G: C \rightarrow \text{Qui}^{\Sigma_1}_n\).

Note that \(i\) in \(2\pi i\) is the imaginary unit. Before proving the theorem, we verify two helpful statements from matrix analysis.

**Proposition 1.7.** Let \(E\) denote a finite dimensional \(\mathbb{C}\)-vector space and let \(f: E \rightarrow E\) denote an invertible linear map. Then there exists a unique linear map \(g: E \rightarrow E\) with \(\text{Spec}(g) \subset \Sigma_1\) such that

\[ f = e^{2\pi i g}. \]

**Proof.** We choose the branch of the logarithm defined on \(\mathbb{C} \setminus \mathbb{R}_{\geq 0}\) with image contained in \(\{\alpha \in \mathbb{C} \mid 0 < \text{Im}(\alpha) < 2\pi\}\), and use a unique extension to \(\mathbb{C} \setminus \{0\}\) with image in \(2\pi i \Sigma_1\). Note that every complex number has a unique representative in this strip up to \(\pm 2\pi i \mathbb{N}\). The existence and uniqueness of \(2\pi i g\) follows now with the aid of [HJ91, Corollary 6.2.12] which deals with finding a matrix \(A\) for a given invertible matrix \(B\) such that \(e^A = B\).

The next corollary will be auxiliary to prove commutativity conditions later.

**Corollary 1.8.** Let \(E, F\) denote two finite dimensional \(\mathbb{C}\)-vector spaces and let \(\gamma: E \rightarrow F\) denote a linear map. Furthermore, let \(\alpha: E \rightarrow E\) and \(\beta: F \rightarrow F\) denote two linear maps with \(\text{Spec}(\alpha), \text{Spec}(\beta) \subset \Sigma_1\). Then:

\[ \gamma \circ e^{2\pi i \alpha} = e^{2\pi i \beta} \circ \gamma \iff \gamma \circ \alpha = \beta \circ \gamma. \]

**Proof.** We only need to prove the direction “\(\Rightarrow\)”.

(i) Assume that \(\gamma: E \rightarrow F\) is invertible. We receive

\[ \gamma \circ e^{2\pi i \alpha} = e^{2\pi i \beta} \circ \gamma \iff e^{2\pi i \beta} = e^{2\pi i (\gamma \circ \alpha \circ \gamma^{-1})}. \]

The eigenvalues of \(\beta\) and \(\gamma \circ \alpha \circ \gamma^{-1}\) are both contained in \(\Sigma_1\). Using the uniqueness given in Proposition 1.7, we receive the claimed equality.

(ii) To prove the general case, we need the following small statement:

\(\alpha\) preserves a linear subspace \(\bar{E}\) of \(E\), i.e. \(\alpha(\bar{E}) \subset \bar{E}\). \(\iff e^{2\pi i \alpha}\) preserves \(\bar{E}\).
The direction “⇒” is clear. To prove “⇐”, use [HJ91, Corollary 6.2.12] to receive that $\alpha$ is a polynomial in $e^{2\pi i \alpha}$ (the concrete form of the polynomial depends on the map $\alpha$).

(iii) Now, let us prove the general case. Examining $\gamma \circ e^{2\pi i \alpha} = e^{2\pi i \beta} \circ \gamma$, we see that $e^{2\pi i \beta}$ preserves $\text{im}(\gamma)$ and that $e^{2\pi i \alpha}$ preserves $\ker(\gamma)$. By part (ii), $\text{im}(\gamma)$ and $\ker(\gamma)$ are preserved by $\beta$ and $\alpha$, respectively. This gives us well-defined maps

$$e^{2\pi i \alpha}: E/\ker(\gamma) \to E/\ker(\gamma) \quad \text{and} \quad \pi: E/\ker(\gamma) \to E/\ker(\gamma)$$

where $e^{2\pi i \alpha} = e^{2\pi i \alpha}$. Consider the following diagram whose first and last square commute:

One has $\text{Spec}(\pi) \subset \text{Spec}(\alpha) \subset \Sigma_1$. By part (i) we receive now that the commutativity of

$$E/\ker(\gamma) \xrightarrow{\pi} \text{im}(\gamma) \quad \text{implies that} \quad E/\ker(\gamma) \xrightarrow{\pi} \text{im}(\gamma)$$

commutes. This yields the commutativity of the above diagram and proves the claim. \[ \square \]

Proof of Theorem 1.6. For simplicity, we set for a linear map $A$

$$\psi(A) := \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} A^{k-1}.$$ 

We need to check several small statements to obtain the theorem:

(i) Verify that $x_{I,i}$ is well-defined: The map $w_{I,i} \circ u_{I,i} + \text{Id}_{V_I}$ is invertible by assumption. The existence and uniqueness of $s_{I,i}$ follows from Proposition 1.7. As $\psi(0) = 2\pi i \neq 0$ and

$$\psi(\lambda) = e^{2\pi i \lambda} - \frac{1}{\lambda} \neq 0$$

for $\lambda \in \Sigma_1 \setminus \{0\}$, we receive that the eigenvalues of $\psi(s_{I,i})$ are non-zero and therefore $x_{I,i} = \psi(s_{I,i})^{-1} \circ w_{I,i}$ is well-defined.

(ii) We have to check that $Q: Qw_n^{\Sigma_1} \to C_n$ is a well-defined functor:

Let $(V_I, u_{I,i}, c_{I,i})$ denote an object in $Qw_n^{\Sigma_1}$ and let $(V_I, u_{I,i}, y_{I,i})$ denote its image under $Q$. This is indeed an object in $C_n$: The map

$$y_{I,i} \circ u_{I,i} + \text{Id}_{V_I} = \psi(c_{I,i} \circ u_{I,i}) \circ c_{I,i} \circ u_{I,i} + \text{Id}_{V_I} = e^{2\pi i (c_{I,i} \circ u_{I,i})}$$
is obviously invertible. The commutativity conditions in $C_n$ follow by direct computation from those in $Qu^\Sigma_1_n$. Now, let $(h_I)$ denote a morphism from $(V_I, u_{I,i}, c_{I,i})$ to $(V'_I, u'_{I,i}, c'_{I,i})$ in $Qu^\Sigma_1_n$. To prove that $Q((h_I)) = (h_I)$ is a morphism in $C_n$ from $(V_I, u_{I,i}, y_{I,i})$ to $(V'_I, u'_{I,i}, y'_{I,i})$, we need to check the equations

$$u'_{I,i} \circ h_I = h_{I \cup \{i\}} \circ u_{I,i} \quad \text{and} \quad h_I \circ y_{I,i} = y'_{I,i} \circ h_{I \cup \{i\}}.$$ 

Both equations follow directly from the fact that $(h_I)$ is a morphism in $Qu^\Sigma_1_n$.

(iii) We have to check that $G : C_n \to Qu^\Sigma_1_n$ is a well-defined functor:

Let $(V_I, u_{I,i}, w_{I,i})$ denote an object in $C_n$ and let $(V_I, u_{I,i}, x_{I,i})$ denote its image under $G$. $(V_I, u_{I,i}, x_{I,i})$ is indeed an object in $Qu^\Sigma_1_n$:

- We have the equality
  $$x_{I,i} \circ u_{I,i} = \psi(s_{I,i})^{-1} \circ w_{I,i} \circ u_{I,i} = \psi(s_{I,i})^{-1} \circ (e^{2\pi i s_{I,i}} - \text{Id}_{V_I}) = s_{I,i}$$
  which gives us $\text{Spec}(x_{I,i} \circ u_{I,i}) \subset \Sigma_1$.

- To prove the commutativity conditions, we use the following identities in $C_n$:
  $$e^{2\pi is_{I,j}} \circ w_{I,i} = w_{I,i} \circ e^{2\pi is_{I \cup \{i\},j}}$$
  $$u_{I,i} \circ e^{2\pi is_{I,j}} = e^{2\pi is_{I \cup \{i\},j}} \circ u_{I,i}$$
  $$e^{2\pi is_{I,i}} \circ e^{2\pi is_{I,j}} = e^{2\pi is_{I,j}} \circ e^{2\pi is_{I,i}}$$

These yield with the aid of Corollary 1.8:

$$\star \ s_{I,j} \circ w_{I,i} = w_{I,i} \circ s_{I \cup \{i\},j} \implies w_{I,i} \circ \psi(s_{I \cup \{i\},j})^{-1} = \psi(s_{I,j})^{-1} \circ w_{I,i}$$
$$\star \ s_{I,i} \circ s_{I,j} = s_{I,j} \circ s_{I,i} \implies \psi(s_{I,i})^{-1} \circ \psi(s_{I,j})^{-1} = \psi(s_{I,j})^{-1} \circ \psi(s_{I,i})^{-1}$$
$$\star \ u_{I,i} \circ s_{I,j} = s_{I \cup \{i\},j} \circ u_{I,i} \implies \psi(s_{I \cup \{i\},j})^{-1} \circ u_{I,i} = u_{I,i} \circ \psi(s_{I,j})^{-1}$$

The commutativity conditions follow now immediately.

Now, let $(h_I)$ denote a morphism from $(V_I, u_{I,i}, w_{I,i})$ to $(V'_I, u'_{I,i}, w'_{I,i})$ in $C_n$. To prove that $G((h_I)) = (h_I)$ is a morphism from $(V_I, u_{I,i}, x_{I,i})$ to $(V'_I, u'_{I,i}, x'_{I,i})$ in $Qu^\Sigma_1_n$, we need to verify the identities

$$\star \ u'_{I,i} \circ h_I = h_{I \cup \{i\}} \circ u_{I,i}$$
$$\star \ h_I \circ x_{I,i} = x'_{I,i} \circ h_{I \cup \{i\}} \iff h_I \circ \psi(s_{I,i})^{-1} \circ w_{I,i} = \psi(s'_{I,i})^{-1} \circ h_I \circ w_{I,i}.$$ 

The first one follows directly. To prove the second equation we use the equality

$$e^{2\pi is'_{I,i}} \circ h_I = h_I \circ e^{2\pi is_{I,i}}.$$ 

Now, Corollary 1.8 yields

$$s'_{I,i} \circ h_I = h_I \circ s_{I,i} \quad \text{and therefore} \quad h_I \circ \psi(s_{I,i})^{-1} = \psi(s'_{I,i})^{-1} \circ h_I.$$
(iv) We show that \( Q \circ G = \text{Id}_{\mathcal{C}_n} \): For this we need to check for an object \((V_I, u_{I,i}, w_{I,i})\) in \( \mathcal{C}_n \) that \((Q \circ G)((V_I, u_{I,i}, w_{I,i})) = (V_I, u_{I,i}, w_{I,i})\). Let
\[
G((V_I, u_{I,i}, w_{I,i})) =: (V_I, u_{I,i}, x_{I,i}) \quad \text{where} \quad x_{I,i} = \psi(s_{I,i})^{-1} \circ w_{I,i} \quad \text{and} \\
Q((V_I, u_{I,i}, x_{I,i})) =: (V_I, u_{I,i}, y_{I,i}) \quad \text{where} \quad y_{I,i} = \psi(x_{I,i} \circ u_{I,i}) \circ x_{I,i}.
\]
By part (iii) of the proof, \( x_{I,i}, u_{I,i} \) and \( s_{I,i} \) fulfil the equality \( x_{I,i} \circ u_{I,i} = s_{I,i} \). Using the definition of \( x_{I,i} \), this yields
\[
y_{I,i} = \psi(x_{I,i} \circ u_{I,i}) \circ x_{I,i} = \psi(s_{I,i}) \circ \psi(s_{I,i})^{-1} \circ w_{I,i} = w_{I,i}.
\]
(iv) We show that \( G \circ Q = \text{Id}_{\text{Qui}_n^\Sigma_1} \): We need to verify for an object \((V_I, u_{I,i}, c_{I,i})\) in \( \text{Qui}_n^\Sigma_1 \) that \((G \circ Q)((V_I, u_{I,i}, c_{I,i})) = (V_I, u_{I,i}, c_{I,i})\). We set
\[
Q((V_I, u_{I,i}, c_{I,i})) =: (V_I, u_{I,i}, y_{I,i}) \quad \text{where} \quad y_{I,i} = \psi(c_{I,i} \circ u_{I,i}) \circ c_{I,i} \quad \text{and} \\
G((V_I, u_{I,i}, y_{I,i})) =: (V_I, u_{I,i}, x_{I,i}) \quad \text{where} \quad x_{I,i} = \psi(s_{I,i})^{-1} \circ y_{I,i}.
\]
We have the equality
\[
e^{2\pi i (c_{I,i} \circ u_{I,i})} = y_{I,i} \circ u_{I,i} + \text{Id}_{V_I} = e^{2\pi i s_{I,i}}.
\]
The eigenvalues of \( c_{I,i} \circ u_{I,i} \) and \( s_{I,i} \) are contained in \( \Sigma_1 \). Thus, the uniqueness of \( s_{I,i} \) (cf. Proposition 1.7) yields \( c_{I,i} \circ u_{I,i} = s_{I,i} \). Hence,
\[
x_{I,i} = \psi(s_{I,i})^{-1} \circ y_{I,i} = \psi(s_{I,i} \circ u_{I,i})^{-1} \circ \psi(c_{I,i} \circ u_{I,i}) \circ c_{I,i} = c_{I,i}.
\]
All in all, this shows that \( Q : \text{Qui}_n^\Sigma_1 \rightarrow \mathcal{C}_n \) and \( G : \mathcal{C}_n \rightarrow \text{Qui}_n^\Sigma_1 \) are inverse functors to each other and therefore they define an isomorphism between the categories \( \text{Qui}_n^\Sigma_1 \) and \( \mathcal{C}_n \). \( \square \)

2 Quiver \( \mathcal{D} \)-modules in \( \mathbb{C}^n \) whose singular locus is a normal crossing

From now on \( \mathcal{O} = \mathcal{O}_X \) will always denote the sheaf of analytic functions on \( X = \mathbb{C}^n \) for a fixed integer \( n \in \mathbb{N}^+ \), and \( \mathcal{D} = \mathcal{D}_X \) denotes the sheaf of rings of linear partial differential operators with analytic coefficients. Furthermore, we denote by \( z_1, \ldots, z_n \) the coordinates of \( \mathbb{C}^n \) and by \( \partial_i = \frac{\partial}{\partial z_i} \) the \( i \)-th partial derivation operator for \( i = 1, \ldots, n \).

2.1 Definitions and basic properties

Let us first define objects of quiver \( \mathcal{D} \)-modules. These are \( \mathcal{D} \)-modules defined in a very natural manner on the basis of certain quiver representations where we use the category \( \text{Qui}_n \) as starting point. Our definition is based on the one in [KV06, Subsection 4.2] in the case of a normal crossing hyperplane arrangement whereas we use analytic \( \mathcal{D} \)-modules instead of algebraic ones.
**Definition 2.1** (Variant of [KV06]). Let \( \mathcal{V}_n = (V_I, B_{I\cup\{i\}}, I, B_{I, I\cup\{i\}}) \) denote an object in the category \( \text{Qui}_n \). We define the associated quiver \( \mathcal{D} \)-module \( E\mathcal{V}_n \) as the quotient of

\[
\bigoplus_{I \in \mathcal{P}(\{1, \ldots, n\})} \left( \mathcal{D} \otimes \mathcal{O}_I \otimes \mathbb{C} V_I \right)
\]

by the subsheaf \( \mathcal{J} \). The sections of \( \mathcal{J} \) over \( U \subset \mathbb{C}^n \), open, are given by \( \mathbb{C} \)-linear combinations of the following elements

\[
a \partial_i \otimes \omega_I \otimes v_I - a \otimes \omega_{I \cup \{i\}} \otimes B_{I \cup \{i\}, I}(v_I) \quad \text{and} \quad
az_i \otimes \omega_{I \cup \{i\}} \otimes v_{I \cup \{i\}} - a \otimes \omega_I \otimes B_{I, I \cup \{i\}, I}(v_{I \cup \{i\}})
\]

where \( I \neq \{1, \ldots, n\}, \ i \notin I, \ a \in \mathcal{D}(U), \ v_J \in V_J \) for \( J \in \mathcal{P}(\{1, \ldots, n\}) \) and \( \mathcal{O}_J := \{ c \omega_J \mid c \in \mathbb{C} \} \) is a 1-dimensional \( \mathbb{C} \)-vector space generated by the element \( \omega_J \). The left \( \mathcal{D} \)-module structure on \( E\mathcal{V}_n \) is given by left multiplication.

\( \mathcal{O}_J \) is used here to clarify to which summand of \( \bigoplus_I \left( \mathcal{D} \otimes \mathcal{O}_I \otimes \mathbb{C} V_I \right) \) an element belongs to. Our next aim is to receive a functor from \( \text{Qui}_n \) into the category of \( \mathcal{D} \)-modules on \( \mathbb{C}^n \).

**Corollary 2.2.** Let \( \mathcal{V}_n = (V_I, B_{I \cup \{i\}}, I, B_{I, I \cup \{i\}}) \) and \( \mathcal{V}'_n = (V'_I, B'_{I \cup \{i\}}, I, B'_{I, I \cup \{i\}}) \) denote two objects in \( \text{Qui}_n \) and let

\[\phi = (h_I): \mathcal{V}_n \to \mathcal{V}'_n\]

denote a morphism from \( \mathcal{V}_n \) to \( \mathcal{V}'_n \). Then \( \phi \) induces naturally a morphism

\[E\phi: E\mathcal{V}_n \to E\mathcal{V}'_n.\]

**Proof.** Consider the following diagram whose rows are exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J} & \longrightarrow & \bigoplus_{I \in \mathcal{P}(\{1, \ldots, n\})} \left( \mathcal{D} \otimes \mathcal{O}_I \otimes \mathbb{C} V_I \right) & \longrightarrow & E\mathcal{V}_n & \longrightarrow & 0 \\
\downarrow h & & \Uparrow \bigoplus_{I \in \mathcal{P}(\{1, \ldots, n\})} (\text{id}_\mathcal{D} \otimes \text{id}_\mathcal{O}_I \otimes h_I) & & \Downarrow & & & \\
0 & \longrightarrow & \mathcal{J}' & \longrightarrow & \bigoplus_{I \in \mathcal{P}(\{1, \ldots, n\})} \left( \mathcal{D} \otimes \mathcal{O}_I \otimes \mathbb{C} V'_I \right) & \longrightarrow & E\mathcal{V}'_n & \longrightarrow & 0
\end{array}
\]

\((h_I)\) induces naturally a \( \mathcal{D} \)-linear map \( \tilde{h}: \mathcal{J} \to \mathcal{J}' \) which fulfils on sections over \( U \subset \mathbb{C}^n \), open,

\[
\tilde{h}(a \partial_i \otimes \omega_I \otimes v_I - a \otimes \omega_{I \cup \{i\}} \otimes B_{I \cup \{i\}, I}(v_I)) =
\]

\[
a \partial_i \otimes \omega_I \otimes h_I(v_I) - a \otimes \omega_{I \cup \{i\}} \otimes h_{I \cup \{i\}}(B_{I \cup \{i\}, I}(v_I)) =
\]

\[
a \partial_i \otimes \omega_I \otimes h_I(v_I) - a \otimes \omega_{I \cup \{i\}} \otimes B'_{I, I \cup \{i\}, I}(h_I(v_I)) \quad \text{and}
\]

\[
a \partial_i \otimes \omega_I \otimes v_I - a \otimes \omega_I \otimes B_{I, I \cup \{i\}}(v_{I \cup \{i\}}) =
\]

\[
a z_i \otimes \omega_{I \cup \{i\}} \otimes h_{I \cup \{i\}}(v_{I \cup \{i\}}) - a \otimes \omega_I \otimes h_I(B_{I, I \cup \{i\}}(v_{I \cup \{i\}})) =
\]

\[
a z_i \otimes \omega_{I \cup \{i\}} \otimes h_{I \cup \{i\}}(v_{I \cup \{i\}}) - a \otimes \omega_I \otimes B'_{I, I \cup \{i\}}(h_I(v_{I \cup \{i\}})).
\]
This makes the square commute as indicated. In particular, it induces in a natural way a \(\mathcal{D}\)-linear morphism from \(EV_n\) to \(EV'_n\).

**Proposition/Definition 2.3.** Let \(\text{Mod}(\mathcal{D})\) denote the category of \(\mathcal{D}\)-modules on \(\mathbb{C}^n\). Then we receive a covariant functor, denoted \(E\), from the category \(\text{Qui}_n\) into the category of \(\mathcal{D}\)-modules

\[
E: \text{Qui}_n \rightarrow \text{Mod}(\mathcal{D}).
\]

\(E\) associates to an object \(V_n\) in \(\text{Qui}_n\) the object \(EV_n\) in \(\text{Mod}(\mathcal{D})\) from Definition 2.1, and it associates to a morphism \(\phi: V_n \rightarrow V'_n\) in \(\text{Qui}_n\) the morphism \(E\phi: EV_n \rightarrow EV'_n\) in \(\text{Mod}(\mathcal{D})\) from Corollary 2.2. The category of quiver \(\mathcal{D}\)-modules is the essential image of the functor \(E\).

**Proof.** Let \((h_I)\) and \((g_I)\) denote two morphisms in \(\text{Qui}_n\) with compatible source and target, respectively. Then \(E\) preserves the composition of morphisms, using Corollary 2.2, as

\[
(Id_\mathcal{D} \otimes Id_{\mathcal{T}_I} \otimes h_I) \circ (Id_\mathcal{D} \otimes Id_{\mathcal{T}_I} \otimes g_I) = (Id_\mathcal{D} \otimes Id_{\mathcal{T}_I} \otimes (h_I \circ g_I)) \quad \text{and} \quad \tilde{h} \circ \tilde{g} = \tilde{h} \circ \tilde{g}.
\]

As \(E\) also preserves the identity morphism, \(E\) is indeed a functor from \(\text{Qui}_n\) to \(\text{Mod}(\mathcal{D})\). \(\square\)

Now, we define the category \(\text{Mod}^S_{\mathcal{D}}(\mathcal{D})\) of regular singular holonomic \(\mathcal{D}\)-modules whose singular locus is contained in the normal crossing. However, we have to mention that this denomination is a little bit sloppy as in fact the objects in \(\text{Mod}^S_{\mathcal{D}}(\mathcal{D})\) need to fulfill a property on their characteristic variety from which the property on the singular locus follows.

**Definition 2.4.** Let \(S := \{z_1, \ldots, z_n = 0\}\) denote the normal crossing in \(\mathbb{C}^n\). \(S\) induces naturally a (Whitney) stratification of \(\mathbb{C}^n\) by \(2^n\) disjoint strata \(X_I \subset \mathbb{C}^n\) which are defined by

\[
\overline{X}_I := \{z_i = 0 \mid i \in I\}, \quad X_I := \overline{X}_I \setminus \left( \bigcup_{J \in \mathcal{P}(\{1, \ldots, n\}) \setminus J \subset X_I} \overline{X}_J \right)
\]

for \(I \in \mathcal{P}(\{1, \ldots, n\})\). This fulfills \(X_{\emptyset} = \mathbb{C}^n \setminus S\) and \(S = \bigcup_{I, \dim X_I < n} X_I\).

The category \(\text{Mod}^S_{\mathcal{D}}(\mathcal{D})\) is then defined to be the category of regular singular holonomic \(\mathcal{D}\)-modules whose characteristic variety is contained in

\[
\bigcup \{T^*_{X_I} \mathbb{C}^n \mid I \in \mathcal{P}(\{1, \ldots, n\})\}
\]

where \(T^*_{X_I} \mathbb{C}^n = \{(z, \xi) = (z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) \in T^* \mathbb{C}^n \mid z \in X_I, \xi(v) = 0 \forall v \in TX_I\} = \{(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) \in T^* \mathbb{C}^n \mid z_i = 0 \iff i \in I, \xi_i = 0 \text{ for } i \notin I\}\).

We note that

\[
\bigcup \{T^*_{X_I} \mathbb{C}^n \mid I \in \mathcal{P}(\{1, \ldots, n\})\} = \Delta_S
\]

where \(\Delta_S := \{(z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) \in T^* \mathbb{C}^n \mid z_i \xi_i = 0 \text{ for all } i \in \{1, \ldots, n\}\}\).
which simplifies Definition 2.4. Thus, \( \mathcal{Mod}^S_{rh}(\mathcal{D}) \) is the category of regular singular holonomic \( \mathcal{D} \)-modules whose characteristic variety is contained in \( \Delta_S \).

Let us explain how the property on the singular locus is implied by this: Let \( \mathcal{M} \) denote a holonomic \( \mathcal{D} \)-module and let \( \text{Char}(\mathcal{M}) \) denote its characteristic variety. The singular locus of \( \mathcal{M} \) is defined as the projection of \( \text{Char}(\mathcal{M}) \setminus T^*_\mathbb{C}^n \mathbb{C}^n \) onto \( \mathbb{C}^n \) where \( T^*_\mathbb{C}^n \mathbb{C}^n \) is the zero section of \( T^* \mathbb{C}^n \). By direct computation one sees that the projection of \( \Delta_S \setminus T^*_\mathbb{C}^n \mathbb{C}^n \) to \( \mathbb{C}^n \) is equal to \( S = \bigcup_{I, \dim X_I < n} X_I \). Thus, the singular locus of \( \mathcal{M} \) is contained in \( S \).

The stratification \( S \) of \( \mathbb{C}^n \) in fact also determines the characteristic variety of the quiver \( \mathcal{D} \)-modules as we will see now.

**Theorem 2.5.** The functor \( E \) maps from the category \( \text{Qui}_n \) into the category \( \mathcal{Mod}^S_{rh}(\mathcal{D}) \).

**Proof.** Let \( \mathcal{V}_n = (V_I, B_{I, \cup(j)}, B_{I, I, \cup(j)}) \) denote an object in \( \text{Qui}_n \). We define the good filtration \( F_k E\mathcal{V}_n \) on \( E\mathcal{V}_n \) as the filtration induced by the exact sequence

\[
\mathcal{D} \otimes \left( \bigoplus_J \overline{\Omega}_J \otimes V_J \right) \rightarrow E\mathcal{V}_n \rightarrow 0
\]

using the standard filtration \( F_\bullet \mathcal{D} \) of \( \mathcal{D} \). Recall that \( F_k \mathcal{D} \) is the subsheaf of \( \mathcal{D} \) of differential operators of order at most \( k \in \mathbb{Z} \). Set

\[
gr^F_k E\mathcal{V}_n := F_k E\mathcal{V}_n / F_{k-1} E\mathcal{V}_n \quad \text{and} \quad \text{gr}^F \mathcal{D} := \bigoplus_k \text{gr}^F_k \mathcal{D} \cong \mathcal{O}_{\mathbb{C}^n}[\xi_1, \ldots, \xi_n].
\]

Let \( P \in F_k \mathcal{D}(U) \) for \( U \subset \mathbb{C}^n \), open, and \( v_I \in V_I \) for \( I \in \mathcal{P}(\{1, \ldots, n\}) \). We denote by \( [P \otimes \omega_I \otimes v_I] \) the image of \( P \otimes \omega_I \otimes v_I \in F_k \mathcal{D}(U) \otimes \overline{\Omega}_I \otimes V_I \) in \( F_k E\mathcal{V}_n \). Furthermore, let \( \sigma_k [P \otimes \omega_I \otimes v_I] \) be the image of \( [P \otimes \omega_I \otimes v_I] \) in \( \text{gr}^F_k E\mathcal{V}_n \). We will prove that \( z_I \xi_i \) annihilates \( \text{gr}^F_k E\mathcal{V}_n \) for every \( k \in \mathbb{Z} \) and every \( i \in \{1, \ldots, n\} \). We need to distinguish two cases:

If \( i \notin I \), then

\[
z_i \xi_i \cdot \sigma_k [P \otimes \omega_I \otimes v_I] = \sigma_{k+1} [P z_i \partial_i \otimes \omega_I \otimes v_I] = \sigma_{k+1} [P \otimes \omega_I \otimes B_{I, \cup(i)} B_{I, \cup(i)} \partial_i v_I] = 0.
\]

If \( i \in I \), then

\[
z_i \xi_i \cdot \sigma_k [P \otimes \omega_I \otimes v_I] = \sigma_{k+1} [P \partial_i z_i \otimes \omega_I \otimes v_I] = \sigma_{k+1} [P \otimes \omega_I \otimes B_{I, I, \cup(i)} B_{I, \cup(i)} \partial_i v_I] = 0.
\]

In both cases \( z_i \xi_i \) is an annihilator. Thus, the characteristic variety of \( E\mathcal{V}_n \) is contained in \( \Delta_S \). This also shows us that the dimension of the characteristic variety of \( E\mathcal{V}_n \) is at most \( n = \dim_{\mathbb{C}_X} X \) and therefore \( E\mathcal{V}_n \) is holonomic. As well, we see that \( E\mathcal{V}_n \) is a regular holonomic \( \mathcal{D} \)-module using the fact that \( (z_i \xi_i)^1 \) is an annihilator of \( \text{gr}^F E\mathcal{V}_n = \oplus_k \text{gr}^F_k E\mathcal{V}_n \) (cf. [Kas03, Definition 5.2]).

We note that in [KV06] a similar but slightly different proof of the holonomicity and the statement on the characteristic variety is given.
2.2 An equivalence with regular singular $\mathscr{D}$-modules in $\mathbb{C}^n$ whose singular locus is a normal crossing

Let us clarify some notational facts: Let $\iota: U \hookrightarrow X$ denote the inclusion for an open subset $U$ of $\mathbb{C}^n$. Then $\Gamma_U$ is the functor which maps sheaves on $\mathbb{C}^n$ to sheaves on $\mathbb{C}^n$ defined by

$$\Gamma_U := \iota_* \iota^{-1}.$$ 

Moreover, let

$$\mathbb{C}^n = \prod_{i=1}^n \mathbb{C}_i \quad \text{and} \quad W_i := \mathbb{C}_i \setminus \mathbb{R}_0^+.$$ 

And for $I \in \mathcal{P}(\{1, \ldots, n\})$ set

$$\Lambda_I := \sum_{k \in I} \Gamma_{\mathbb{C}_k} \times \prod_{i=1, i \neq k}^n W_i \mathcal{O} \quad \text{and} \quad \mathcal{O}_I := \frac{\Gamma_{\prod_{i=1}^n W_i \mathcal{O}}}{\Lambda_I}.$$ 

Note that $(\Gamma_{\prod_{i=1}^n W_i \mathcal{O}})_0$ and $\Lambda_{I,0}$ are unitary, commutative rings w.r.t. addition and multiplication of functions. And $\mathcal{O}_{I,0}$ is an abelian additive group and a unitary left $\mathcal{D}_0$-module. But $\mathcal{O}_{I,0}$ is not a ring, as in general the multiplication of functions is not well-defined.

The following theorem of A. Galligo, M. Granger and Ph. Maisonobe will be important for our computations:

**Theorem 2.6** ([GGM85a], [GGM85b]). The contravariant functor $\mathcal{A}$ from $\operatorname{Mod}^S_{\mathcal{D}}(\mathcal{D})$ to $\mathcal{C}_n$

$$\mathcal{A}: \operatorname{Mod}^S_{\mathcal{D}}(\mathcal{D}) \rightarrow \mathcal{C}_n$$

$$\mathcal{M} \mapsto \operatorname{Hom}_{\mathcal{D},0}(\mathcal{M}_0, \mathcal{O}_{I,0}) \xrightarrow{\text{can}_{I,i}} \operatorname{Hom}_{\mathcal{D},0}(\mathcal{M}_0, \mathcal{O}_{I,0})$$

establishes an equivalence of categories. $\text{can}_{I,i}$ is the canonical map or quotient map which sends solutions with values in $\mathcal{O}_{I,0}$ to solutions with values in $\mathcal{O}_{I \cup \{i\},0}$. $\text{var}_{I,i}$ is the variation around $z_i = 0$, i.e. we have

$$\text{var}_{I,i}(F) = M_i F - F \quad \text{for} \quad F \in \operatorname{Hom}_{\mathcal{D},0}(\mathcal{M}_0, \mathcal{O}_{I \cup \{i\},0})$$

where $M_i F$ is the class of a representative of $F$ after analytic continuation around the axis $z_i = 0$. A $\mathcal{D}$-linear morphism $\phi: \mathcal{M} \rightarrow \mathcal{N}$ in $\operatorname{Mod}^S_{\mathcal{D}}(\mathcal{D})$ is mapped under $\mathcal{A}$ to the morphism

$$\left( \operatorname{Hom}_{\mathcal{D},0}(\phi_0, \mathcal{O}_{I,0}) \right) \quad \text{in} \quad \mathcal{C}_n$$

where $\operatorname{Hom}_{\mathcal{D},0}(\phi_0, \mathcal{O}_{I,0}): \operatorname{Hom}_{\mathcal{D},0}(\mathcal{N}_0, \mathcal{O}_{I,0}) \rightarrow \operatorname{Hom}_{\mathcal{D},0}(\mathcal{M}_0, \mathcal{O}_{I,0})$ is given by $g \mapsto g \circ \phi_0$.

In their paper [GGM85a], Galligo, Granger and Maisonobe prove that the category of perverse sheaves in $\mathbb{C}^n$ with respect to the normal crossing stratification $\Xi$ is equivalent to the category of
They establish a functor
\[ \alpha : \text{Perv}(\mathcal{C}^n) \to \mathcal{C}_n. \]

Composing the functor \( \alpha \) with the solution functor \( \text{Sol} \) one receives an equivalence of the categories \( \text{Mod}_{rh}^S(\mathcal{D}) \) and \( \mathcal{C}_n \) whereby the functor \( \mathcal{A} \) is naturally isomorphic to \( \alpha \circ \text{Sol} \) [GGM85b].

Now, we are ready to state and prove the Main theorem:

**Theorem 2.7.** The functors \( \mathcal{A} \circ E \) and \( Q \circ D \) are naturally isomorphic, i.e. the following diagram commutes up to a natural isomorphism

\[
\begin{array}{ccc}
\text{Mod}_{rh}^S(\mathcal{D}) & \xrightarrow{\mathcal{A} \circ E} & \mathcal{C}_n \\
E & \downarrow & \\
\text{Qui}_{n}^{\Sigma_1} & \xrightarrow{D} & \text{Qui}_{n}^{\Sigma_1} \end{array}
\]

In particular, \( E : \text{Qui}_{n}^{\Sigma_1} \to \text{Mod}_{rh}^S(\mathcal{D}) \) is an equivalence of categories with quasi-inverse \( D \circ \mathcal{G} \circ \mathcal{A} \), and \( E \circ D \circ \mathcal{G} \) is a quasi-inverse of \( \mathcal{A} \).

Furthermore, \( E \) is essentially surjective. This means the category of quiver \( \mathcal{D} \)-modules is exactly the category \( \text{Mod}_{rh}^S(\mathcal{D}) \), and every \( \mathcal{D} \)-module in \( \text{Mod}_{rh}^S(\mathcal{D}) \) is in fact isomorphic to a quiver \( \mathcal{D} \)-module as given in Definition 2.1.

In [KV06, Proposition 4.4] an equivalence of categories of quiver \( \mathcal{D} \)-modules in the case of a central arrangement of hyperplanes is also stated. But the essential image of the equivalence is not completely clarified. As domain they use a full subcategory of the category of representations over the quiver corresponding to the arrangement. This subcategory is defined by restricting the eigenvalues of several maps involved in the quiver representation. In the case of a normal crossing, this restriction is much more rigid than our restriction from \( \text{Qui}_n \) to \( \text{Qui}_n^{\Sigma_1} \). This is a strong evidence that the essential image of their equivalence in our setting is not \( \text{Mod}_{rh}^S(\mathcal{D}) \) or \( \text{Mod}_{rh}^S(\mathcal{D}) \).

The main parts of the proof of Theorem 2.7 are accomplished in Proposition 2.10 and Proposition 2.12. Before applying the functor \( \mathcal{A} \) to our quiver \( \mathcal{D} \)-modules we state some properties of \( O_{I,0} \) in Lemma 2.8 and Lemma 2.9 to simplify the arguments later.

**Lemma 2.8.** For \( I \in \mathcal{P}(\{1, \ldots, n\}) \) let \( O_I \) as above. Then,

(i) \( z_j \) acts bijective on \( O_{I,0} \) if and only if \( j \notin I \).

(ii) \( \partial_j \) acts bijective on \( O_{I,0} \) if and only if \( j \in I \).

**Proof.** Let \( j \in \mathcal{P}(\{1, \ldots, n\}) \).

(i) For \( k = 1, \ldots, n \) we use \( Z_k \) as dummy for \( C_k \) or \( W_k \). The inverse of \( z_j \) fulfills that \( \frac{1}{z_j} \in \left( \prod_{i=1}^{n} z_i \right)_0 \) if and only if \( Z_j = W_j \). Thus \( z_j \) acts bijective on \( \left( \prod_{i=1}^{n} z_i \right)_0 \) if and only if \( Z_j = W_j \).

As \( \Lambda_{I,0} = \left( \sum_{k \in I} \prod_{i=1}^{n} z_i \cdot \prod_{i \neq k} W_i \right)_0 \), we immediately see that \( z_j \) acts bijective on \( O_{I,0} \equiv \frac{\left( \prod_{i=1}^{n} W_i \right)_0}{\Lambda_{I,0}} \) if and only if \( j \notin I \).
Let \( F(z_1, \ldots, z_n) \in (\Gamma \prod_{i=1}^n W_i \mathcal{O})_0 \). As \( \prod_{i=1}^n W_i \) is simply connected there exists a function \( F(z_1, \ldots, z_n) \in (\Gamma \prod_{i=1}^n W_i \mathcal{O})_0 \) such that \( \partial_j F = f \). The other primitives of \( f \) w.r.t. \( \partial_j \) are given by \( F(z_1, \ldots, z_j-1, z_{j+1}, \ldots, z_n) \), where \( C \in (\Gamma_c \times \prod_{i=1, i \neq j}^n W_i \mathcal{O})_0 \) as \( C \) does not depend on \( z_j \). Clearly, \( j \in I \) if and only if for any such \( C \) it follows that \( C \in \Lambda_{I,0} \). Now, we see that functions in \( \mathcal{O}_{I,0} \) have a uniquely defined primitive w.r.t. \( \partial_j \) if and only if \( j \in I \) (constants etc. move into the denominator of \( \mathcal{O}_{I,0} \)).

**Lemma 2.9.** For every \( r \in \{1, \ldots, n\} \) fix a branch of the logarithm on \( \mathbb{C}_r \setminus \mathbb{R}_{\geq 0} \). Let \( M \in \mathbb{N}^+ \) and let \( A \) denote a \( M \times M \)-matrix with values in \( \mathbb{C} \). We set

\[
\gamma_r^A := \exp(A \cdot \ln(z_r)).
\]

\( z_r^A \) is considered as a matrix with entries in \( (\Gamma \prod_{i=1}^n W_i \mathcal{O})_0 \) and all entries of \( z_r^A \) are invertible w.r.t. multiplication of functions in \( (\Gamma \prod_{i=1}^n W_i \mathcal{O})_0 \) for \( t \neq r \). Then:

(i) The matrix \( z_r^A \) is invertible in \( (\Gamma \prod_{i=1}^n W_i \mathcal{O})_0 \) and \( (\Gamma_c \times \prod_{i=1, i \neq t}^n W_i \mathcal{O})_0 \) for \( t \neq r \).

(ii) Let \( I = \{m_1, \ldots, m_{|I|}\} \in \mathcal{P}(\{1, \ldots, n\}) \) and \( \{l_1, \ldots, l_{n-|I|}\} = \{1, \ldots, n\} \setminus I \). Assume we are given pairwise commuting \( M \times M \)-matrices \( A_{m_1}, \ldots, A_{m_{|I|}}, A_{l_1}, \ldots, A_{l_{n-|I|}} \) with values in \( \mathbb{C} \), and the eigenvalues of \( A_{m_1}, \ldots, A_{m_{|I|}} \) lie in \( \Sigma \). Let \( \lambda = (\lambda_1, \ldots, \lambda_M)^T \in \mathbb{C}^M \) and

\[
\tilde{\mathcal{F}} := z_{l_1}^{A_{|I|}} \cdots z_{l_{n-|I|}}^{A_{|I|}} z_{m_1}^{A_{|I|}} \cdots z_{m_{|I|}}^{A_{|I|}}.
\]

Then:

\[
\partial_{m_1}^{-1} \cdots \partial_{m_{|I|}}^{-1} \tilde{\mathcal{F}} \cdot \lambda \in (\Lambda_{I,0})^M \iff \lambda = (0, \ldots, 0)^T
\]

**Proof.** (i) This becomes clear by passing to the Jordan normal form \( J \) of \( A \). Let \( \mu_1, \ldots, \mu_q \in \mathbb{C} \) denote the eigenvalues of \( A \). Then,

\[
\det(\exp(A \cdot \ln(z_r))) = \det(\exp(J \cdot \ln(z_r))) = \prod_{i=1}^q (z_r^{\mu_i})^{p_i} \neq 0
\]

where \( p_1, \ldots, p_q \in \mathbb{N}^+ \). This yields the invertibility of \( \exp(A \cdot \ln(z_r)) \).

(ii) We prove “\( \Rightarrow \)”. For simplicity let \( I = \{1, \ldots, |I|\} \). By part (i), the claim is equivalent to

\[
\partial_1^{-1} z_1^{A_1} \cdots \partial_{|I|}^{-1} z_{|I|}^{A_{|I|}} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} \in (\Lambda_{I,0})^M.
\]

This means, for \( l = 1, \ldots, |I| \), we find \( f_l(z_1, \ldots, z_n) \in \left((\Gamma_c \times \prod_{i=1, i \neq l}^n W_i \mathcal{O})_0 \right)^M \) such that

\[
\partial_1^{-1} z_1^{A_1} \cdots \partial_{|I|}^{-1} z_{|I|}^{A_{|I|}} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_M \end{pmatrix} = \sum_{l=1}^{|I|} f_l(z_1, \ldots, z_n).
\]
Let us apply \(\var|_I \circ \cdots \circ \var_1\) to both sides of the equation where \(\var_1\) was given by \(M_I - \Id\):

- Let us treat the (LHS): We receive

\[
(\var|_I \circ \cdots \circ \var_1)(\text{LHS}) = \var|_I(\partial_I^{-1} z_1^{A_1}) \cdot \cdots \cdot \var|_I(\partial_I^{-1} z_I^{A_I}) \cdot \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_M
\end{pmatrix}.
\]

Let us prove that \(\var|_I(\partial_I^{-1} z_I^{A_I})\) is invertible in \((\Gamma \prod_{l=1}^n W_l \mathcal{O})_0\): We may pass to a single Jordan block \(J_a\) with eigenvalue \(a \in \Sigma\) for our arguments. First, let \(a \neq -1\). Then

\[
\var|_I \left( \partial_I^{-1} \exp(J_a \ln(z_I)) \right) = (J_a + \Id)^{-1} \cdot \exp((J_a + \Id) \ln(z_I)) \cdot (e^{2\pi i (J_a + \Id)} - \Id).
\]

Using part (i) and \(\text{Spec}(J_a + \Id) \subset \mathbb{C} \setminus \mathbb{Z}\), this is an invertible matrix in \((\Gamma \prod_{l=1}^n W_l \mathcal{O})_0\).

Now, let \(a = -1\). The matrix \(\partial_I^{-1} \exp(J_{-1} \ln(z_I))\) is (up to a matrix which is independent of \(z_I\)) an upper-triangular matrix with \(\ln(z_I)\) on the diagonal. Hence, the matrix \(\var|_I \left( \partial_I^{-1} \exp(J_{-1} \ln(z_I)) \right)\) is an upper-triangular matrix with \(2\pi i\) as diagonal entry, and therefore it is invertible in \((\Gamma \prod_{l=1}^n W_l \mathcal{O})_0\).

- Now, consider the (RHS): Using \(M_I f_1 = f_1\) and \(M_I(\sum_{l=1}^I f_l) = M_1 f_1 + M_1(\sum_{l=2}^I f_l)\), we receive

\[
(\var|_I \circ \cdots \circ \var_1)(\text{RHS}) = (\var|_I \circ \cdots \circ \var_2)\left( M_I(\sum_{l=1}^I f_l) - \sum_{l=1}^I f_l \right) =

\[
= (\var|_I \circ \cdots \circ \var_2)\left( M_1(\sum_{l=2}^I f_l) - \sum_{l=2}^I f_l \right) = (\var|_I \circ \cdots \circ \var_1)(\sum_{l=2}^I f_l).
\]

As the variations commute on the left hand side (LHS), we receive furthermore

\[
(\var|_I \circ \cdots \circ \var_1)(\sum_{l=2}^I f_l) = (\var|_I \circ \cdots \circ \var_3 \circ \var_1 \circ \var_2)(\sum_{l=3}^I f_l) =

\[
= (\var|_I \circ \cdots \circ \var_3 \circ \var_1 \circ \var_2)(\sum_{l=3}^I f_l).
\]

Continuing this process, it yields \((\var|_I \circ \cdots \circ \var_1)(\text{RHS}) = (0, \ldots, 0)^T\).

This leads to the equality

\[
\var_1(\partial_I^{-1} z_1^{A_1}) \cdot \cdots \cdot \var|_I(\partial_I^{-1} z_I^{A_I}) \cdot \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_M
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}.
\]

The invertibility of all the matrices \(\var|_I(\partial_I^{-1} z_I^{A_I})\) gives \(\lambda_1 = \cdots = \lambda_M = 0\) as claimed. \(\square\)

Now, let us consider how the quiver representation looks like after applying \(A\) to a quiver \(D\)-module.
Proposition 2.10. Let $\mathcal{V}_n = (V_I, B_{I \cup \{i\}, I}, B_{I \cup \{i\}})$ denote an object in $\mathcal{Q}w^{\Sigma}_n$ with $E\mathcal{V}_n$ as corresponding quiver $\mathcal{D}$-module. Then, for every $I \in \mathcal{P} \{\{1, \ldots, n\}\}$, we are given a canonical isomorphism

$$a : V_I \xrightarrow{\cong} \text{Hom}_{\mathcal{D},0} ((E\mathcal{V}_n)_0, \mathcal{O}_{I,0}).$$

Proof. We abbreviate $\mathcal{V} = \mathcal{V}_n$. The proof of this lemma will be carried out in several steps:

(i) We are given the following natural isomorphism:

$$\text{Hom}_{\mathcal{D},0} ((E\mathcal{V})_0, \mathcal{O}_{I,0}) = \left\{ \phi \in \text{Hom}_{\mathcal{D},0} \left( \bigoplus_J (\mathcal{D}_0 \otimes \overline{\mathcal{D}}_J \otimes V_J), \mathcal{O}_{I,0} \right) \mid \phi(\partial_j \otimes \omega_J \otimes v_J - 1 \otimes \omega_{J \cup \{j\}} \otimes B_{J \cup \{j\}, J} (v_J)) = 0, \phi(z_j \otimes \omega_{J \cup \{j\}} \otimes v_{J \cup \{j\}} - 1 \otimes \omega_J \otimes B_{J \cup \{j\}, J} (v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \ldots, n\}, j \notin J \right\} \cong \left\{ \bigoplus_J \phi^J \in \bigoplus_J \text{Hom}_\mathbb{C} (V_J, \mathcal{O}_{I,0}) \mid \partial_j \cdot \phi^J (v_J) - \phi^J (v_J) = 0, z_j \cdot \phi^J (v_{J \cup \{j\}}) - \phi^J (B_{J \cup \{j\}, J} (v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \ldots, n\}, j \notin J \right\}.$$

(ii) Consider the following system of equations from step (i)

$$\partial_j \cdot \phi^J - \phi^J (v_J) = 0 \quad \text{and} \quad z_j \cdot \phi^J (v_{J \cup \{j\}}) - \phi^J (B_{J \cup \{j\}, J} (v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \ldots, n\}, j \notin J \quad (\ast)$$

where $\phi^J \in \text{Hom}_\mathbb{C} (V_J, \mathcal{O}_{I,0})$ and $J \neq \{1, \ldots, n\}, j \in \{1, \ldots, n\} \setminus J$. We use the following algorithm (ALG) to express step by step every $\phi^K$ uniquely in terms of $\phi^J$ for $K \in \mathcal{P} \{\{1, \ldots, n\}\} \setminus I$:

1. Step: If $K_2 = \emptyset$, skip this step. Otherwise we have for $l_1 \in K_2$:

$$z_{l_1} \cdot \phi^K - \phi^{K \setminus \{l_1\}} \circ B_{K \setminus \{l_1\}, K} = 0 \iff \phi^K = \frac{1}{z_{l_1}} \cdot \left( \phi^{K \setminus \{l_1\}} \circ B_{K \setminus \{l_1\}, K} \right)$$

For $l_2 \in K_2 \setminus \{l_1\}$ use the equation

$$z_{l_2} \cdot \phi^K \setminus \{l_1\} - \phi^{K \setminus \{l_1, l_2\}} \circ B_{K \setminus \{l_1, l_2\}, K \setminus \{l_1\}} = 0$$

to express $\phi^K$ in terms of $\phi^{K \setminus \{l_1, l_2\}}$. Continue until $\phi^K$ is expressed in terms of $\phi^{K_1}$.

2. Step: If $K_3 = \emptyset$, we already expressed $\phi^K$ in terms of $\phi^J$. Otherwise we have for $m_1 \in K_3$:

$$\partial_m \cdot \phi^K_1 - \phi^{K_1 \cup \{m_1\}} \circ B_{K_1 \cup \{m_1\}, K_1} = 0 \iff \phi^K_1 = \partial_m^{-1} \cdot \left( \phi^{K_1 \cup \{m_1\}} \circ B_{K_1 \cup \{m_1\}, K_1} \right)$$. 

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For $m_2 \in K_3 \setminus \{m_1\}$ use the equation
\[
\partial_{m_2} \cdot \phi^{K_1 \cup \{m_1\}}_1 - \phi^{K_1 \cup \{m_1,m_2\}}_1 \circ B_{K_1 \cup \{m_1,m_2\}, K_1 \cup \{m_1\}} = 0
\]
to express $\phi^{K_1}_I$ in terms of $\phi^{K_1 \cup \{m_1,m_2\}}_I$. Continue until $\phi^{K_1}_I$ and therefore $\phi^K$ is expressed in terms of $\phi^I$.

The order in which we solve for $\phi^I$ in (ALG) does not influence the result. This is ensured by the commutativity conditions on the maps $B_{\bullet \bullet}$ and the fact that $z_I, \partial_m$ commute for $l \notin I, m \in I$. Therefore every $\phi^K$ can be uniquely expressed in terms of $\phi^I$.

So clearly (\ref{algr}) implies that $\phi^I \in \text{Hom}_{\mathbb{C}}(V_I, \mathcal{O}_{I,0})$ fulfills the system
\[
\begin{align*}
&z_I \partial_t \cdot \phi^I - \phi^I \circ (B_{I, I \cup \{l\}} B_{I \cup \{l\}, I}) = 0 \\
&z_m \partial_m \cdot \phi^I - \phi^I \circ (B_{I, I \setminus \{m\}} B_{I \setminus \{m\}, I} - \text{Id}) = 0
\end{align*}
\]
of $n$ equations where $l \notin I, m \in I$. On the other hand (\ref{algr}) is likewise implied by (\ref{algs}) using (ALG) as definition of $\phi^K$ for all $K \in \mathcal{P} \{\{1, \ldots, n\}\} \setminus I$. This shows us that in fact

\[
\text{Hom}_{\mathbb{C}}((EV)_0, \mathcal{O}_{I,0}) \cong \left\{ \phi^I \in \text{Hom}_{\mathbb{C}}(V_I, \mathcal{O}_{I,0}) \mid z_I \partial_t \cdot \phi^I(v_I) - \phi^I(B_{I, I \cup \{l\}} B_{I \cup \{l\}, I}(v_I)) = 0, \\
z_m \partial_m \cdot \phi^I(v_I) - \phi^I((B_{I, I \setminus \{m\}} B_{I \setminus \{m\}, I} - \text{Id})(v_I)) = 0 \text{ for } l \notin I, m \in I \right\}. \tag{1}
\]

(iii) The dimension of $\text{Hom}_{\mathbb{C}}((EV)_0, \mathcal{O}_{I,0})$ over $\mathbb{C}$ is finite (see [GGM85b]). We use the following proposition of [GGM85b] to give an upper bound hereof:

Let $z_I^* = (z_1, \ldots, z_n, \xi_1, \ldots, \xi_n) \in \mathbb{T}^n \mathbb{C}^n$ verifying $z_i \xi_i = 0$ for all $i$, and $z_i = 0 \Leftrightarrow i \in I$ and $\xi_i \neq 0 \Leftrightarrow i \in I$. Then $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}}((EV)_0, \mathcal{O}_{I,0}) = \text{mult}_{z_I^*} EV$.

We use the definition of multiplicity as given in [GM93, Chapter V]. Supplementary, we use [Ser75, Subsection II.B)4]. As the definition becomes clear during the following computations, we do not repeat it here.

We use the good filtration on $EV$ from the proof of Theorem 2.5. Its sections over $U \subset \mathbb{C}^n$, open, are given by

\[
F_k E\mathcal{V}(U) = \frac{F_k \mathcal{D}(U) \otimes \left( \bigoplus_J \mathcal{M}_J \otimes V_J \right)}{(F_k \mathcal{D}(U) \otimes \left( \bigoplus_J \mathcal{M}_J \otimes V_J \right) \cap \mathcal{J}(U))}
\]

for $k \in \mathbb{N}_0$, and for $k \in \mathbb{Z} \setminus \mathbb{N}_0$ we have $F_k E\mathcal{V} = 0$. As before let

\[
\text{gr}^F_0 E\mathcal{V} = F_k E\mathcal{V}/F_{k-1} E\mathcal{V} \quad \text{and} \quad \text{gr}^F E\mathcal{V} = \bigoplus_{k \in \mathbb{N}_0} \text{gr}^F_k E\mathcal{V}.
\]

Let $k \in \mathbb{N}_0$. Fix a point $\tilde{z}_I^* = (\tilde{z}_I^1, \ldots, \tilde{z}_I^n, \tilde{\xi}_I^1, \ldots, \tilde{\xi}_I^n) =: (\tilde{z}_I, \tilde{\xi}_I)$ where $\tilde{z}_i = 0 \Leftrightarrow i \in I$ and $\tilde{\xi}_i \neq 0 \Leftrightarrow i \in I$. Consider the stalk of $\text{gr}^F E\mathcal{V}$ at $\tilde{z}_I$. Set

\[
M := (\text{gr}^F E\mathcal{V})_{\tilde{z}_I}.
\]
denote by \([P \otimes \omega_{K \cup \{i\}} \otimes v_{K \cup \{i\}}]\) the image of \(P \otimes \omega_{K \cup \{i\}} \otimes v_{K \cup \{i\}} \in F_k \bar{O}_{\bar{z}_i} \otimes \bar{O}_{K \cup \{i\}} \otimes V_{K \cup \{i\}}\) in \((F_k EV)_{\bar{z}_i}\). We have the following identity in \((F_k EV)_{\bar{z}_i}\):

\[
[P \otimes \omega_{K \cup \{i\}} \otimes v_{K \cup \{i\}}] = [z_i^{-1} P \otimes \omega_K \otimes B_{K, K \cup \{i\}}(v_{K \cup \{i\}})]
\]

This allows us to “eliminate” all summands \([F_k \bar{O}_{\bar{z}_i} \otimes \bar{O}_J \otimes V_J]\) in \((F_k EV)_{\bar{z}_i}\) with \(J \setminus I \neq \emptyset\). Hence, we may assume that

\[
(F_k EV)_{\bar{z}_i} = \frac{F_k \bar{O}_{\bar{z}_i} \otimes (\bigoplus_{J \subseteq I} \bar{O}_J \otimes V_J)}{(F_k \bar{O}_{\bar{z}_i} \otimes (\bigoplus_{J \subseteq I} \bar{O}_J \otimes V_J)) \cap \mathcal{J}_{\bar{z}_i}} \quad \text{or} \quad EV = \frac{\mathcal{O} \otimes (\bigoplus_{J \subseteq I} \bar{O}_J \otimes V_J)}{(\mathcal{O} \otimes (\bigoplus_{J \subseteq I} \bar{O}_J \otimes V_J)) \cap \mathcal{J}}.
\]

We have \(\text{gr}^F \mathcal{O}_{\bar{z}_i} \cong \mathcal{O}_{\bar{z}_i}[\xi_1, \ldots, \xi_n]\) and \(M\) is a finitely generated \(\mathcal{O}_{\bar{z}_i}[\xi_1, \ldots, \xi_n]\)-module. We denote by \(\text{Max}\) the maximal ideal of the local ring \(\mathcal{O}_{\bar{z}_i}\). Let

\[
Q_{\xi_i} := \text{Max} + (\xi_1 - \tilde{\xi}_1, \ldots, \xi_n - \tilde{\xi}_n).
\]

This defines a maximal ideal in \(\mathcal{O}_{\bar{z}_i}[\xi_1, \ldots, \xi_n]\). Thus, \(M/Q_{\xi_i}M\) is a finitely generated \(\mathcal{O}_{\bar{z}_i}[\xi_1, \ldots, \xi_n]/Q_{\xi_i}\)-vector space. Therefore, there exists a polynomial \(P_{M,Q_{\xi_i}}(N)\), called Hilbert-Samuel polynomial, and an integer \(N_0 \in \mathbb{N}\) such that

\[
P_{M,Q_{\xi_i}}(N) = \text{length}(M/Q_{\xi_i}M)^N \quad \text{for all} \quad N \geq N_0.
\]

The highest degree term of \(P\) has the form \(e d^d\) where \(e \in \mathbb{N}\), \(d \in \mathbb{N}\) and by definition

\[
e = \text{mult}_{\bar{z}_i} EV.
\]

Applying [Ser75, Proposition 11a) in Subsection II.B)4], we receive

\[
P_{M,Q_{\xi_i}}(N) = P_{T^{-1}M,T^{-1}Q_{\xi_i}}(N) \quad \text{for} \quad T := \mathcal{O}_{\bar{z}_i}[\xi_1, \ldots, \xi_n] \setminus Q_{\xi_i}.
\]

So we need to consider the localisation of \(M\) at \(T\):

\[
T^{-1}M = \bigoplus_{k \in \mathbb{N}_0} T^{-1}(\text{gr}^F_k EV)_{\bar{z}_i}
\]

Let \([P \otimes \omega_K \otimes v_K]\) denote the image of \(P \otimes \omega_K \otimes v_K \in F_k \bar{O}_{\bar{z}_i} \otimes \bar{O}_K \otimes V_K\) in \((\text{gr}^F_k EV)_{\bar{z}_i}\) for \(K \subseteq I\). For every \(i \in I \setminus K\) we have the following identity in \((\text{gr}^F_{k+1} EV)_{\bar{z}_i}\):

\[
\xi_i \cdot [P \otimes \omega_K \otimes v_K] = [P \otimes \omega_{K \cup \{i\}} \otimes B_{K \cup \{i\}, K}(v_K)] = 0
\]

Consider this identity in \(T^{-1}M\): The map \(\xi_i \cdot _{-}: T^{-1}(\text{gr}^F_k EV)_{\bar{z}_i} \to T^{-1}(\text{gr}^F_{k+1} EV)_{\bar{z}_i}\) is
bijection for $i \in I$, as $\tilde{\xi}_i \neq 0$ for $i \in I$. Therefore, $\frac{[\mathcal{P} \otimes \omega_I \otimes V_\alpha]}{I}$ = 0 and we may assume that

$$T^{-1}(F_k EV)_{\tilde{z}_I} = \frac{T^{-1}F_k \mathcal{D}_{\tilde{z}_I} \otimes \bar{\Omega}_I \otimes V_I}{(T^{-1}F_k \mathcal{D}_{\tilde{z}_I} \otimes \bar{\Omega}_I \otimes V_I) \cap T^{-1}J_{\tilde{z}_I}}.$$  

Using [Ser75, Proposition 11a)] the other way round, we may assume that

$$(F_k EV)_{\tilde{z}_I} = \frac{F_k \mathcal{D}_{\tilde{z}_I} \otimes \bar{\Omega}_I \otimes V_I}{(F_k \mathcal{D}_{\tilde{z}_I} \otimes \bar{\Omega}_I \otimes V_I) \cap J_{\tilde{z}_I}} \quad \text{or} \quad EV = \frac{\mathcal{D} \otimes \bar{\Omega}_I \otimes V_I}{(\mathcal{D} \otimes \bar{\Omega}_I \otimes V_I) \cap J}.$$  

For simplicity let $I = \{1, \ldots, |I|\}$ for the moment. Set $n_I := \dim_{\mathbb{C}}(V_I)$. Consider the following exact sequence of holonomic $\mathcal{D}$-modules

$$0 \longrightarrow \ker(\pi) \longrightarrow \widetilde{\mathcal{N}} \overset{\pi}{\longrightarrow} EV \longrightarrow 0$$

where $\widetilde{\mathcal{N}} := \mathcal{D} \otimes V_I/(z_1 \otimes V_I, \ldots, z_{|I|} \otimes V_I, \partial_{|I|+1} \otimes V_I, \ldots, \partial_n \otimes V_I) \cong \bigoplus_{n_I \text{-times}} \mathcal{D}/(z_1, \ldots, z_{|I|}, \partial_{|I|+1}, \ldots, \partial_n) \cong \bigoplus_{n_I \text{-times}} \mathcal{N}.$

This sequence yields $\text{mult}_{\tilde{z}_I} EV \leq \text{mult}_{\tilde{z}_I} \mathcal{N}$. Furthermore, $\text{mult}_{\tilde{z}_I} \mathcal{N} = n_I \cdot \text{mult}_{\tilde{z}_I} \mathcal{N}$. So let us compute $\text{mult}_{\tilde{z}_I} \mathcal{N}$ where $\tilde{z}_I^* = (0, \ldots, 0, \tilde{z}_{|I|+1}, \ldots, \tilde{z}_n, \tilde{\xi}_1, \ldots, \tilde{\xi}_I, 0, \ldots, 0) =: (\tilde{z}_I, \tilde{\xi}_I)$ with $\tilde{z}_{|I|+1}, \ldots, \tilde{z}_n, \tilde{\xi}_1, \ldots, \tilde{\xi}_I \neq 0$.

We use the good filtration $F_\bullet \mathcal{N}$ on $\mathcal{N}$ which is induced by the standard filtration $F_\bullet \mathcal{D}$ of $\mathcal{D}$. So, we consider $(\text{gr}^F \mathcal{N})_{\tilde{z}_I} \cong \mathbb{C}\{z_{|I|+1} - \tilde{z}_{|I|+1}, \ldots, z_n - \tilde{z}_n\}[[\xi_1, \ldots, \xi_{|I|}]$ as a module over $(\text{gr}^F \mathcal{D})_{\tilde{z}_I} \cong \mathbb{C}\{z_1, \ldots, z_{|I|}, \tilde{z}_{|I|+1} - \tilde{z}_{|I|+1}, \ldots, z_n - \tilde{z}_n\}[[\xi_1, \ldots, \xi_n]]$. Let $\text{Max}$ be the maximal ideal of $\mathbb{C}\{z_1, \ldots, z_{|I|}, \tilde{z}_{|I|+1} - \tilde{z}_{|I|+1}, \ldots, z_n - \tilde{z}_n\}$. We need to compute the multiplicity of $(\text{gr}^F \mathcal{N})_{\tilde{z}_I}$ with respect to the maximal ideal $\text{Max} + (\xi_1 - \tilde{\xi}_1, \ldots, \xi_{|I|} - \tilde{\xi}_{|I|}, \tilde{\xi}_{|I|+1}, \ldots, \tilde{\xi}_n)$ of $(\text{gr}^F \mathcal{D})_{\tilde{z}_I}$. A shift of coordinates gives us that we equivalently have to treat

$$\mathbb{C}\{z_{|I|+1}, \ldots, z_n\}[[\xi_1, \ldots, \xi_{|I|}]$$

as a module over $\mathbb{C}\{z_1, \ldots, z_{|I|}\}[[\xi_1, \ldots, \xi_n]]$, and compute its multiplicity with respect to the maximal ideal

$$Q := (z_1, \ldots, z_{|I|}, \xi_1 - \tilde{\xi}_1, \ldots, \xi_{|I|} - \tilde{\xi}_{|I|}, \tilde{\xi}_{|I|+1}, \ldots, \tilde{\xi}_n).$$

So we have to compute

$$\text{length} \left( \frac{\mathbb{C}\{z_{|I|+1}, \ldots, z_n\}[[\xi_1, \ldots, \xi_{|I|}]}{(z_{|I|+1}, \ldots, z_n, \xi_1 - \xi_1, \ldots, \xi_{|I|} - \xi_{|I|})^N \cdot \mathbb{C}\{z_{|I|+1}, \ldots, z_n\}[[\xi_1, \ldots, \xi_{|I|}]}} \right).$$

But this is the number of monomials of degree less than $N$ in $\mathbb{C}\{z_{|I|+1}, \ldots, z_n\}[[\xi_1, \ldots, \xi_{|I|}]$ which is equal to $\binom{N-1+|I|}{N-1}$. This shows us that $\text{mult}_{\tilde{z}_I} \mathcal{N} = 1$ and $\text{mult}_{\tilde{z}_I} EV \leq n_I$.

(iv) Now, we construct the canonical isomorphism $\eta_I$ from $V_I^*$ into (1). For this purpose let $\alpha \in V_I^*$. We define $\eta_I(\alpha)$ as follows:
Let \( \{m_1, \ldots, m_{|I|}\} = I, \{l_1, \ldots, l_{n-|I|}\} = \{1, \ldots, n\} \setminus I \). For a moment fix a basis of \( V_I \) and denote it by \( v_{l_1}, \ldots, v_{l_{n-|I|}} \). In abuse of notation we denote the maps corresponding to the linear maps \( \{\text{linear maps} \} \) by the same symbols. We set 

\[
\mathcal{F} := B_{I,I\cup\{l\}} \cdot z_{l_{n-|I|}} \cdot \ldots \cdot B_{I,I\backslash\{m_{|I|}\}} \cdot z_{m_{|I|}} - \text{Id}
\]

and

\[
\eta_I(\alpha) := \alpha \cdot \mathcal{F}.
\]

One verifies directly that \( \eta_I(\alpha) \) is indeed an element in (1) by plugging it into (\(*\)).

We need to verify that this construction of \( \eta_I \) is independent of the choice of basis of \( V_I \). So, let \( \tilde{v}_{l_1}, \ldots, \tilde{v}_{l_{n-|I|}} \) denote another basis of \( V_I \). Let \( \tilde{B}_{I,I\cup\{l\}}, \tilde{B}_{I,I\backslash\{m\}} \) and \( \tilde{\alpha} \) denote the matrices corresponding to the linear maps \( \{\text{linear maps} \} \) and \( \alpha \) w.r.t. this new basis. Let \( R \) denote the matrix of the change of coordinates from \( \{v_{l_1}, \ldots, v_{l_{n-|I|}}\} \) to \( \{\tilde{v}_{l_1}, \ldots, \tilde{v}_{l_{n-|I|}}\} \). Let \( v_I \in V_I \).

We denote by \( v_I \) in abuse of notation the vector w.r.t. the basis \( \{v_{l_1}, \ldots, v_{l_{n-|I|}}\} \) and by \( \tilde{v}_I \) the vector w.r.t. the basis \( \{\tilde{v}_{l_1}, \ldots, \tilde{v}_{l_{n-|I|}}\} \). We receive

\[
\eta_I(\alpha)(v_I) = \alpha \cdot z_{l_{n-|I|}} \cdot \ldots \cdot z_{l_{n-|I|}} \cdot B_{I,I\cup\{l\}} \cdot B_{I,I\backslash\{m\}} - \text{Id} \cdot \ldots \cdot B_{I,I\backslash\{m_{|I|}\}} - \text{Id} \cdot v_I =
\]

\[
= \tilde{\alpha} R R^{-1} z_{l_{n-|I|}} \cdot \ldots \cdot R^{-1} z_{l_{n-|I|}} \cdot B_{I,I\cup\{l\}} \cdot B_{I,I\backslash\{m\}} - \text{Id} \cdot \ldots \cdot B_{I,I\backslash\{m_{|I|}\}} - \text{Id} \cdot \ldots \cdot R R^{-1} \tilde{v}_I =
\]

\[
= \eta_I(\tilde{\alpha})(\tilde{v}_I).
\]

Hence, our construction is independent of the choice of basis of \( V_I \).

Now, we want to check that \( \eta_I \) is injective. So assume that \( \eta_I(\alpha) \) is the zero mapping. As \( \partial_m \) acts bijective on \( O_{I,0} \) for \( m \in I \) (see Lemma 2.8), this is equivalent to

\[
\partial_m^{-1} \ldots \partial_m^{-1} z_{l_{n-|I|}} \cdot \ldots \cdot z_{l_{n-|I|}} \cdot B_{I,I\cup\{l\}} \cdot B_{I,I\backslash\{m\}} - \text{Id} \cdot \ldots \cdot B_{I,I\backslash\{m_{|I|}\}} - \text{Id} \cdot \alpha^T = \begin{pmatrix} 0 \\ \vdots \end{pmatrix}.
\]

The eigenvalues of \( B_{I,I\backslash\{m\}} - \text{Id} \) are contained in \( \Sigma \) for \( m \in I \), as \( \mathcal{V} \) is an object in \( Quiv_n \Sigma \). Using Lemma 2.9, we receive \( \alpha \equiv 0 \) and \( \eta_I \) is injective. As \( \text{dim}_\mathbb{C} \text{Hom}_{\mathcal{O}_0}(\mathcal{E}V_0, O_{I,0}) \leq n_I \) by part (iv), we immediately receive the bijectivity of \( \eta_I \) as claimed.

Composing the isomorphism from part (i) with (ALG), we receive a natural isomorphism from (1) into \( \text{Hom}_{\mathcal{O}_0}(\mathcal{E}V_0, O_{I,0}) \). Composing this with the isomorphism \( \eta_I \) from \( V_I^* \) into (1), this gives us the canonical isomorphism

\[
\alpha: V_I^* \xrightarrow{\cong} \text{Hom}_{\mathcal{O}_0}(\mathcal{E}V_0, O_{I,0}).
\]

The following statement on the matrix polynomial will be used in the proof of Proposition 2.12.
Corollary 2.11. Let $A$ denote a square matrix with entries in $\mathbb{C}$ and let $i \in \{1, \ldots, n\}$. We fix a branch of the logarithm defined on $\mathbb{C}_i \setminus \mathbb{R}_0^k$ and let $z_i^A = \exp(A \cdot \ln(z_i))$ as before. Set
\[
\varphi_A(z_i) := \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} \cdot \ln(z_i)^{k+1} \quad \text{and} \quad \psi(A) := \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} A^{k-1}.
\]
Then
\[
M_i \varphi_A(z_i) - \varphi_A(z_i) = \psi(A) \cdot z_i^A.
\]

Proof. We have
\[
M_i \varphi_A(z_i) = \sum_{k=0}^{\infty} \frac{A^k}{(k+1)!} \cdot (\ln(z_i) + 2\pi i)^{k+1}.
\]
Furthermore direct computation yields
\[
A \cdot (M_i \varphi_A(z_i) - \varphi_A(z_i)) = (M_i \varphi_A(z_i) - \varphi_A(z_i)) \cdot A = \psi(A) \cdot z_i^A \cdot A = A \cdot \psi(A) \cdot z_i^A.
\]
We may assume for our arguments that $A = J_a$ where $J_a$ is a single Jordan block with eigenvalue $a \in \mathbb{C}$. If $a \neq 0$, our claim follows directly. So assume $a = 0$. The above equation shows us however that $M_i \varphi_A(z_i) - \varphi_A(z_i)$ and $\psi(A) \cdot z_i^A$ coincide up to a possible difference in the entry in the upper-left corner. The entry of $M_i \varphi_A(z_i) - \varphi_A(z_i)$ in the upper-left corner is $\ln(z_i) + 2\pi i - \ln(z_i) = 2\pi i$. The first column of $z_i^A$ is $(1, 0, \ldots, 0)^T$ and the entry in the upper-left corner of $\psi(A)$ is $2\pi i$. Hence, the entry in the upper-left corner of $\psi(A) \cdot z_i^A$ is $2\pi i$ as well which proves the claim.

In the following we prove that the quiver representation one receives after applying $\mathcal{A}$ to a quiver $\mathcal{D}$-module is determined in a simple manner by the starting quiver representation. To do so, we “extend” the canonical isomorphism $a$ from Proposition 2.10 to the whole quiver representation.

Proposition 2.12. Let $V_n = (V_I, B_{I, J}, I, B_{I, I \cup \{i\}})$ be an object in $\text{Quiv}_n^{\Sigma_1}$ and $E V_n$ the corresponding quiver $\mathcal{D}$-module. The image of $E V_n$ under the functor $\mathcal{A}$ is canonically isomorphic to
\[
V^*_I \xleftarrow{u_{I,i}} W_{I,i} \xrightarrow{V^*_I}
\]
where
\[
u_{I,i} = B^*_{I, I \cup \{i\}} \quad \text{and} \quad W_{I,i} = B^*_{I, I \cup \{i\}} \circ \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} (B^*_{I, I \cup \{i\}} \circ B^*_{I, I \cup \{i\}})^{k-1}.
\]

Proof. Let $n_l = \dim_{\mathbb{C}} V_I$ as before. Set $B_{K,L} := B_{K,L} \circ B_{L,K}$ if $K, L \in P(\{1, \ldots, n\})$ are adjacent, i.e. $K = L \cup \{l\}$ or $L = K \cup \{k\}$. The image of $(V_I, B_{I, J}, I, B_{I, I \cup \{i\}})$ under $\mathcal{A} \circ E$ is given by
\[
(\text{Hom}_{\mathcal{D}_0}((E V_n)_0, \mathcal{O}_{I,0}), \text{can}_{I,i}, \text{var}_{I,i}).
\]
First, we reperform the first steps of the proof of Proposition 2.10. Then we compute can and var.

(i) First, note that the natural isomorphism we gave for $\text{Hom}_{\mathcal{D}_0}((E V_n)_0, \mathcal{O}_{I,0})$ in part (i) of the proof of Proposition 2.10 is compatible with the canonical map and the variation. Therefore,
it extends to the entire object \( (\text{Hom}_{\mathcal{B}} ((EV_n)_0, \mathcal{O}_{I,0}), \text{can}_{I,i}, \text{var}_{I,i}) \). We receive:

\[
\text{Hom}_{\mathcal{B}} ((EV_n)_0, \mathcal{O}_{I,0}) \xrightarrow{\text{can}_{I,i}} \text{Hom}_{\mathcal{B}} ((EV_n)_0, \mathcal{O}_{I \cup \{i\},0}) \cong \\
\left\{ \sum_{j} \phi^j_I \in \bigoplus_{j} \text{Hom}_{\mathcal{C}} (V_J, \mathcal{O}_{I,0}) \mid \partial_j \cdot \phi^j_I (v_J) - \phi^{J \cup \{j\}}_I (B_{J \cup \{j\},J}(v_J)) = 0, \\
z_j \cdot \phi^j_I (v_{J \cup \{j\}}) - \phi^j_I (B_{J \cup \{j\},J}(v_{J \cup \{j\}})) = 0 \text{ for } J \neq \{1, \ldots, n\}, j \in \{1, \ldots, n\} \setminus J \right\}
\]

(ii) Let us fix \( I \in \mathcal{P} \{1, \ldots, n\} \setminus \{1, \ldots, n\} \), \( i \in \{1, \ldots, n\} \setminus I \) temporarily. We may consider only the behaviour under the canonical map and the variation of the two pairs

\[
\left( \begin{array}{c}
\phi^j_I \\
\phi^I_{I \cup \{i\}}
\end{array} \right) \leftrightarrow \left( \begin{array}{c}
\phi^I_{I \cup \{i\}} \\
\phi^I_{I \cup \{i\}}
\end{array} \right).
\]

For any \( K \in \mathcal{P} \{1, \ldots, n\} \), \( \phi^K_I \leftrightarrow \phi^K_{I \cup \{i\}} \) will follow their behaviour under these two maps. This can be seen by adapting (ALG) in the following way: We only use equations of \( \text{Hom}_{\mathcal{C}} \) which involve \( z_j \) for \( j \notin I \cup \{i\} \), or \( \partial_k \) for \( k \in I \). That way we express \( \phi^K_I \) in terms of \( \phi^I_i \) if \( i \notin K \), or in terms of \( \phi^I_{I \cup \{i\}} \) if \( i \in K \). This expression is unique with the same arguments as for (ALG). One observes that \( \phi^K_I, \phi^K_{I \cup \{i\}} \) are build up from \( \phi^I_i, \phi^I_{I \cup \{i\}} \) if \( i \notin K \) (from \( \phi^I_{I \cup \{i\}}, \phi^I_{I \cup \{i\}} \) if \( i \in K \)) in a completely identical manner. This ensures that they behave in the same way under the canonical map and the variation.

(iii) Using the isomorphisms \( \eta_I \) and \( \eta_{I \cup \{i\}} \) from part (iv) of the proof of Proposition 2.10, we can uniquely identify \( \phi^j_I \) and \( \phi^j_{I \cup \{i\}} \) with elements of \( V^*_I \) and \( V^*_{I \cup \{i\}} \), respectively. After a choice of basis of \( V_I \) and \( V_{I \cup \{i\}} \), we may write for some \( \alpha_I \in V^*_I \) and \( \alpha_{I \cup \{i\}} \in V^*_{I \cup \{i\}} \) (we omit set braces for singletons in the following)

\[
\phi^j_I = \eta_I (\alpha_I) = \alpha_I \cdot \mathcal{F}_I \\
\phi^j_{I \cup \{i\}} = \eta_{I \cup \{i\}} (\alpha_{I \cup \{i\}}) = \alpha_{I \cup \{i\}} \cdot \mathcal{F}_{I \cup \{i\}}
\]

\[
\mathcal{F}_I = \sum_{\ell=1}^{s_I} z_{\ell} \cdot \mathcal{F}_I = \eta_I (\alpha_I) = \alpha_I \cdot \mathcal{F}_I
\]

\[
\mathcal{F}_{I \cup \{i\}} = \sum_{\ell=1}^{s_{I \cup \{i\}}} z_{\ell} \cdot \mathcal{F}_{I \cup \{i\}} = \eta_{I \cup \{i\}} (\alpha_{I \cup \{i\}}) = \alpha_{I \cup \{i\}} \cdot \mathcal{F}_{I \cup \{i\}}
\]

where \( \{i, \ell_2, \ldots, l_{n-|I|}\} = \{1, \ldots, n\} \setminus I \), \( \{m_1, \ldots, m_{|I|}\} = I \). This description of \( \phi^*: \) is independent of the choice of basis as we showed in part (iv) of the proof of Proposition 2.10.
Let us give some helpful identities for the computations. We have for \( i, l \notin I, l \neq i, m \in I \):

\[
B_{I, I \cup i} \cdot B_{I, I \cup i} \setminus m = B_{I, \setminus m} \cdot B_{I, I \cup i} \\
B_{I, I \cup i} \cdot B_{I, I \cup i, \setminus i, l} = B_{I, I \cup i} \cdot B_{I, I \cup i} = B_{I, I \cup i} \cdot B_{I, I \cup i} \setminus m = B_{I, I \cup i} \setminus m \cdot B_{I, I \cup i}
\]

(iv) We claimed that the canonical map from \( \begin{pmatrix} \phi^I_l \\ \phi^I_{I \cup i} \end{pmatrix} \) to \( \begin{pmatrix} \phi^I_{I \cup i} \\ \phi^I_{I \cup i} \end{pmatrix} \) is given by \( B^*_{I, I \cup i} \). This means we have to check that the assignment

\[
\alpha_I \mapsto \alpha_{I \cup i} := \alpha_I \cdot B_{I, I \cup i}
\]
describes the canonical map. This follows by direct computations:

\[
\eta_{I \cup i} (\alpha_I \cdot B_{I, I \cup i}) = \alpha_I \cdot B_{I, I \cup i} \cdot \mathcal{F}_{I \cup i} = z_i^{-1} \cdot \alpha_I \cdot \mathcal{F}_I \cdot B_{I, I \cup i} = z_i^{-1} \cdot \phi^I_I \cdot B_{I, I \cup i} = \phi^I_{I \cup i}
\]

and therefore

\[
\partial_i^{-1} \cdot \eta_{I \cup i} (\alpha_I \cdot B_{I, I \cup i}) \cdot B_{I, I \cup i} = \partial_i^{-1} z_i^{-1} \cdot \alpha_I \cdot \mathcal{F}_I \cdot B_{I, I \cup i} \cdot B_{I, I \cup i} = \alpha_I \cdot \mathcal{F}_I = \phi^I_I
\]

With the same arguments as before, one can show that the description of the canonical map is independent of the choice of basis.

(v) We are left with the computation of the variation \( M_i \phi^I_{I \cup i} - \phi^I_{I \cup i} \) and \( M_i \phi^I_{I \cup i} - \phi^I_{I \cup i} \). In particular, we need to check that the assignment

\[
\alpha_{I \cup i} \mapsto \alpha_I := \alpha_{I \cup i} \cdot \left( \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} \cdot (B_{I, I \cup i} B_{I, I \cup i} \setminus i)^k \right) \cdot B_{I, I \cup i} = \alpha_{I \cup i} \cdot \psi(B_{I, I \cup i} \setminus i) \cdot B_{I, I \cup i}
\]
describes the variation. For \( \phi^I_{I \cup i} \) the correctness follows by direct computation:

\[
M_i \phi^I_{I \cup i} - \phi^I_{I \cup i} = \alpha_{I \cup i} \cdot (e^{2\pi i B_{I, I \cup i} \setminus I} - \text{Id}) \cdot \mathcal{F}_{I \cup i} = \eta_{I \cup i} \cdot \Theta_{I, i} \cdot B_{I, I \cup i} \cdot \mathcal{F}_{I \cup i} = \alpha_{I \cup i} \cdot \Theta_{I, i} \cdot \mathcal{F}_I \cdot B_{I, I \cup i} = z_i^{-1} \cdot \eta_{I \cup i} (\alpha_{I \cup i} \cdot \Theta_{I, i}) \cdot B_{I, I \cup i}
\]

Now, let us compute \( M_i \phi^I_{I \cup i} - \phi^I_{I \cup i} \). We use the identity \( \mathcal{F}_{I \cup i} \cdot B_{I, I \cup i} = z_i^{-1} \cdot B_{I, I \cup i} \cdot \mathcal{F}_I \) to rearrange \( \phi^I_{I \cup i} \). We receive

\[
\phi^I_{I \cup i} = \alpha_{I \cup i} \cdot B_{I, I \cup i} \cdot \left( \partial_i^{-1} z_i^{-1} \cdot B_{I, I \cup i} \right) \cdot z_i \cdot z_i \cdot z_i \cdot \ldots \cdot z_i \cdot z_i \cdot z_i \cdot \ldots \cdot z_i \cdot B_{I, I \cup i} \cdot \text{Id}
\]

where \( \partial_i^{-1} z_i^{-1} \cdot B_{I, I \cup i} = \sum_{k=0}^{\infty} \frac{B_{I, I \cup i}^k \ln(z_i)^{k+1}}{k!} \cdot \frac{1}{k+1} =: \varphi_{B_{I, I \cup i}}(z_i) \).

Corollary 2.11 yields

\[
M_i \varphi_{B_{I, I \cup i}}(z_i) - \varphi_{B_{I, I \cup i}}(z_i) = \left( \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} \cdot B_{I, I \cup i}^{k-1} \right) \cdot B_{I, I \cup i}.
\]
This gives us
\[
M_i \phi_{I,ki} - \phi_{I,ki} = \\
= \alpha_{I,ki} B_{I,ki,I} \left( \sum_{k=1}^{\infty} \frac{(2\pi i)^k}{k!} \cdot B_{I,ki,I}^k \right) z_1^{B_{I,ki,I}} z_{I,ki,I}^{B_{I,ki,I}} \cdots z_{i,n-|I|}^{B_{I,ki,I}} - \text{Id} \\
= \alpha_{I,ki} \cdot \Theta_{I,ki} \cdot \mathcal{F}_I = \eta_I (\alpha_{I,ki} \cdot \Theta_{I,ki}).
\]

Once more, note that these computations are independent of the choice of basis. \(\square\)

Now, we have collected all the important pieces for the proof of our Main Theorem 2.7:

**Proof of Theorem 2.7.** We need to examine if the family of isomorphisms from Proposition 2.12 is natural: The isomorphism we gave in part (i) of the proof of Proposition 2.12 is natural. So let \(\mathcal{V} = (V_J, B_{I,\cup j,I}, J, B_{J,\cup j,I})\) and \(\tilde{\mathcal{V}} = (\tilde{V}_J, \tilde{B}_{I,\cup j,I}, J, \tilde{B}_{J,\cup j,I})\) denote two objects in \(Q\mathcal{U}_I^{\Sigma_1}\) and let \(\tau = (h_J)\) denote a morphism from \(\mathcal{V}\) to \(\tilde{\mathcal{V}}\). We need to check that the diagram

\[
\begin{array}{cccc}
\{ \oplus \phi^J_{I,\cup j(I)} \in \oplus J \text{Hom}_C(V_J, O_{I,0}) | \ldots \} & \quad & \{ \oplus \phi^J_{I,\cup j(I)} \in \oplus J \text{Hom}_C(\tilde{V}_J, O_{I,0}) | \ldots \} \\
\oplus J \text{can}^J_{I,\cup j(I)} & \quad \quad & \oplus J \text{can}^J_{I,\cup j(I)} \\
\{ \oplus \phi^J_{I,\cup j(I)} \in \oplus J \text{Hom}_C(V_J, O_{I,0}) | \ldots \} & \quad & \{ \oplus \phi^J_{I,\cup j(I)} \in \oplus J \text{Hom}_C(\tilde{V}_J, O_{I,0}) | \ldots \} \\
\text{Hom}_C(V_J, \mathbb{C}) & \quad & \text{Hom}_C(\tilde{V}_J, \mathbb{C}) \\
B^*_{I,\cup j(I)} | \psi(B^*_{I,\cup j(I)}) \circ B^*_{I,\cup j(I),I} & \quad & B^*_{I,\cup j(I)} | \psi(B^*_{I,\cup j(I)}) \circ B^*_{I,\cup j(I),I} \\
\text{Hom}_C(V_J, \mathbb{C}) & \quad & \text{Hom}_C(\tilde{V}_J, \mathbb{C}) \\
(h_J^*) & \quad & (h_J^*) \\
\end{array}
\]

commutes. The properties indicated by “...” may be found in part (i) of the proof of Proposition 2.12. The morphisms in the horizontal rows are given by

\[
\begin{align*}
\text{Hom}_C((h_J, O_{I,0}) & : \bigoplus J \text{Hom}_C(\tilde{V}_J, O_{I,0}) \to \bigoplus J \text{Hom}_C(V_J, O_{I,0}), \\
\alpha_I & \to \alpha_J \circ h_I \\
\end{align*}
\]

The isomorphisms from the lower row into the upper row are given by \((\text{ALG})\) composed with \((\eta_I)\) and \((\tilde{\eta}_I)\), respectively. The commutativity of the diagram follows now easily using the commutativity conditions of the morphism \((h_I)\) with the \(B_{\bullet,\bullet}\) and \(B_{\bullet,\bullet}\)-maps. Hence, the diagram of Theorem 2.7 commutes up to a natural isomorphism. The remaining claims follow directly from that. \(\square\)
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