Two-particle system in Coulomb potential for twist-deformed space-time

Marcin Daszkiewicz

Institute of Theoretical Physics, University of Wroclaw pl. Maxa Borna 9, 50-206 Wroclaw, Poland

E-mail: marcin@ift.uni.wroc.pl

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Abstract

In this article, we define a two-particle system in Coulomb potential for twist-deformed space-time with spatial directions commuting to time-dependent function $k_f(t)$. Particularly, we provide the proper Hamiltonian function and further, we rewrite it in terms of commutative variables. Besides, we demonstrate, that for small values of deformation parameters, the obtained in the perturbation framework, first-ordered corrections for ground helium energy are equal to zero. In such a way, we show that the nontrivial impact of space-time noncommutativity appears only at the second order of the quantum-mechanical correction expansion.

Keywords: twist deformation, noncommutative space-time, ground Helium state

1. Introduction

The suggestion to use noncommutative coordinates goes back to Heisenberg and was firstly formalized by Snyder in [1]. Recently, there were also found formal arguments based mainly on quantum gravity [2–5] and string theory models [6–8], indicating that space-time at the Planck scale should be noncommutative, i.e., it should have a quantum nature. Consequently, a lot of papers appeared dealing with noncommutative classical and quantum mechanics (see e.g. [9–19]) as well as with field theoretical models (see e.g. [20–33]), in which the quantum space-time is employed.

In accordance with the Hopf-algebraic classification of all deformations of relativistic [34] and nonrelativistic [35] symmetries, one can distinguish three basic types of space-time noncommutativity (see also [36] for details):

1. Canonical ($\theta^{\mu\nu}$-deformed) type of quantum space [37–41]

$$[x_i, x_j] = i\theta^{\mu\nu} \delta_{\mu\nu},$$ (1)

2. Lie-algebraic modification of classical space-time [41–51]

$$[x_i, x_j] = i\theta^{\mu\nu} \delta_{\mu\nu} x_i x_j,$$ (2)

and

3. Quadratic deformation of Minkowski and Galilei spaces [41, 51–53]

$$[x_i, x_j] = i\theta^{\mu\nu} \delta_{\mu\nu} x_i x_j,$$ (3)

with coefficients $\theta^{\mu\nu}$ being constants.

Moreover, it has been demonstrated in [36] that in the case of the so-called N-enlarged Newton-Hooke Hopf algebras $U_{\text{NHN}}^{\text{N}}(NH)$ the twist deformation provides the new space-time noncommutativity of the form$^{1,2,3}$

$$[x_i, x_j] = 0, \quad [x_i, x_j] = i f_1\left(\frac{t}{\tau}\right)\theta_{ij}(x),$$ (4)

with time-dependent functions

$$f_1\left(\frac{t}{\tau}\right) = f\left(\sinh\left(\frac{t}{\tau}\right)\right)$$ or $$f_1\left(\frac{t}{\tau}\right) = f\left(\cosh\left(\frac{t}{\tau}\right)\right),$$

$^{1}$ $x_0 = ct$.

$^{2}$ The discussed space-times have been defined as the quantum representation spaces, so-called Hopf modules (see e.g. [13, 14]), for the quantum $\text{N}$-enlarged Newton-Hooke Hopf algebras.

$^{3}$ The indices $+$ and $-$ in functions $f_1\left(\frac{t}{\tau}\right)$ and $f_2\left(\frac{t}{\tau}\right)$ correspond to $\text{N}$-enlarged Newton-Hooke Hopf algebras $U_{\text{NHN}}^{\text{N}}(NH)$ and $U_{\text{NHN}}^{\text{N}}(NH)$ respectively.
f(t/τ) = f(sin(t/τ)) or f(t/τ) = f(cos(t/τ)).

θ_{ij}(x) \sim \theta_{ij} = \text{const or } \theta_{ij}(x) \sim \theta_{ij}^0 x_i and \tau denoting the time scale parameter—the cosmological constant. Besides, it should be noted that the above mentioned quantum spaces 1), 2) and 3) can be obtained by the proper contraction limit of the commutation relations 4).

In this article, we define a two-particle system in Coulomb potential for twist-deformed space-time [36] with \[\theta_{ij}(x) \sim \theta_{ij}^0\]. Particularly, we provide the proper Hamiltonian function and further, we rewrite it in terms of commutative variables. Besides, we demonstrate that, for small values of deformation parameters, the obtained in the perturbation framework, first-ordered corrections for ground Helium energy are equal to zero. In such a way we show that a nontrivial impact of space-time noncommutativity appears only at the second order of the quantum-mechanical correction expansion.

The paper is organized as follows. In the second section we remind of the basic facts concerning the twisted, nonrelativistic spaces [36]. In the next section, we recall the perturbation calculations of ground energy for a helium atom in the case of classical space-time. In the fourth section, which is devoted to their twisted counterpart, we demonstrate that first-ordered corrections vanish. The discussion and final remarks are presented in the last section.

2. Twist-deformed space-times

In this section we turn closer to the twisted N-enlarged Newton-Hooke spaces equipped with two spatial directions commuting to classical time, i.e., we consider spaces of the form [36]

\[ [t, \xi_1] = [\xi_1, \xi_1] = [\xi_2, \xi_1] = 0, \]
\[ [\xi_1, \xi_2] = if(t); \quad i = 1, 2, 3, \]
(5)
with function \( f(t) \) given for example for the most simple \( N = 1 \) case by

\[ f(t) = f_{\kappa_1}(t) = f_{\kappa_2}(t) = f_{\kappa_3}(t) = f_{\kappa_4}(t) = f_{\kappa_5}(t) = f_{\kappa_6}(t) = f_{\kappa_7}(t) = f_{\kappa_8}(t) = f_{\kappa_9}(t) = f_{\kappa_10}(t), \]
(6)

\[ f(t) = f_{\kappa_1}(t) = f_{\kappa_2}(t) = f_{\kappa_3}(t) = f_{\kappa_4}(t) = f_{\kappa_5}(t) = f_{\kappa_6}(t) = f_{\kappa_7}(t) = f_{\kappa_8}(t) = f_{\kappa_9}(t) = f_{\kappa_10}(t), \]
(7)

\[ f(t) = f_{\kappa_1}(t) = f_{\kappa_2}(t) = f_{\kappa_3}(t) = f_{\kappa_4}(t) = f_{\kappa_5}(t) = f_{\kappa_6}(t) = f_{\kappa_7}(t) = f_{\kappa_8}(t) = f_{\kappa_9}(t) = f_{\kappa_10}(t), \]
(8)

\[ f(t) = f_{\kappa_1}(t) = f_{\kappa_2}(t) = f_{\kappa_3}(t) = f_{\kappa_4}(t) = f_{\kappa_5}(t) = f_{\kappa_6}(t) = f_{\kappa_7}(t) = f_{\kappa_8}(t) = f_{\kappa_9}(t) = f_{\kappa_10}(t), \]
(9)

\[ f(t) = f_{\kappa_1}(t) = f_{\kappa_2}(t) = f_{\kappa_3}(t) = f_{\kappa_4}(t) = f_{\kappa_5}(t) = f_{\kappa_6}(t) = f_{\kappa_7}(t) = f_{\kappa_8}(t) = f_{\kappa_9}(t) = f_{\kappa_10}(t), \]
(10)

\[ f(t) = f_{\kappa_1}(t) = f_{\kappa_2}(t) = f_{\kappa_3}(t) = f_{\kappa_4}(t) = f_{\kappa_5}(t) = f_{\kappa_6}(t) = f_{\kappa_7}(t) = f_{\kappa_8}(t) = f_{\kappa_9}(t) = f_{\kappa_10}(t), \]
(11)

As it was already mentioned in the introduction, in \( \tau \to \infty \) limit the above quantum spaces reproduce the canonical (1), Lie-algebraic (2) as well as quadratic (3) type of space-time noncommutativity, with

\[ f_{\kappa_1}(t) = \kappa_1, \]
(12)
\[ f_{\kappa_2}(t) = \kappa_2 t, \]
(13)
\[ f_{\kappa_3}(t) = \kappa_3 t^2, \]
(14)
\[ f_{\kappa_4}(t) = \kappa_4 t^3, \]
(15)
\[ f_{\kappa_5}(t) = \frac{1}{2} \kappa_5 t^2, \]
(16)
\[ f_{\kappa_6}(t) = \frac{1}{2} \kappa_6 t^3. \]
(17)

Moreover, let us notice that the spaces (5) can be extended to the case of multiparticle systems as follows

\[ [t, \xi_i] = [\xi_i, \xi_i] = [\xi_i, \xi_i] = 0; \quad i = 1, 2, 3, \]
\[ [\xi_i, \xi_j] = if_i(t) \delta_{ij}, \]
(18)

with \( A, B = 1, 2, \ldots, M \). Besides, it should be observed that such an extension (blind in \( A, B \) indexes with canonical deformation (1)). Precisely, in \( \tau \) approaching infinity limit the space (18) with function \( f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t) = f_{\kappa_i}(t), \)

\[ [t, \xi_i] = [\xi_i, \xi_i] = [\xi_i, \xi_i] = 0; \quad i = 1, 2, 3, \]
\[ [\xi_i, \xi_j] = if_i(t) \delta_{ij}, \]
(19)

Obviously, for \( f(t) \) approaching zero all above spaces become classical.

\[ 6 \] It should be noted that modification of the relation (19) (blind in \( A, B \) indexes as well) is in accordance with the formal arguments proposed in [55]. Precisely, the relations (19) are constructed, which adopt so-called braided tensor algebra procedure, dictated by structure of quantum R-matrix for canonical deformation [57, 58]. We would like to mention, however, that in [30] an erroneous conclusion has been stated that is based on such a twisted symmetry the noncommutative quantum field theory (QFT) on the quantum space satisfying the relation in (1), and the usual commutative QFT are identical. This conclusion has been reached by a misuse of the proper transformation properties of the fields in the corresponding noncommutative space-time [59].

\[ 5 \] For such a construction in the case of most simple (canonical) type of space-time noncommutativity see [29].
3. Two-particle ($M = 2$) Coulomb system in commutative space-time

In this chapter we recall the facts concerning the model of two particles mutually interacting and moving in the presence of Coulomb potential. Then, in accordance with \cite{60–68}, the corresponding Hamiltonian function is given by

$$H[p_1, p_2; r_1, r_2] = H_0[p_1, p_2; r_1, r_2] + H_{\text{int}}[\tilde{r}_1, \tilde{r}_2],$$

(20)

where

$$H_0[p_1, p_2; r_1, r_2] = \frac{1}{2}[p_1^2/m_1 + p_2^2/m_2] - \frac{Qq_1}{r_1} - \frac{Qq_2}{r_2},$$

(21)

and

$$H_{\text{int}}[\tilde{r}_1, \tilde{r}_2] = \frac{q_1q_2}{r_{12}}; \quad r_{12} = |\tilde{r}_1 - \tilde{r}_2|,$$

(22)

with $m_1, m_2, q_1, q_2$ and $Q$ denoting the masses and charges of the particles and the charge of the source respectively. It should be also noted, that the first, $H_0$-term in the formula (20) describes the sum of two energy operators for hydrogen atoms, while with the second one, the interaction of the particles is called electrostatic repulsion.

It is well known, that if one neglects $H_{\text{int}}$-term then the eigenvectors and eigenvalues of $H$-operator

$$H\psi = H_0\psi = E\psi,$$

(23)

are given by\footnote{We put $Q = Ze$. $q_1 = q_2 = e$ as well as $m_1 = m_2 = m_e$ where $e$ and $m_e$ are the charge and mass of the electron.}

$$\psi = \psi_{(n_1,l_1,m_1)}\psi_{(n_2,l_2,m_2)},$$

(24)

$$E = E_{\text{helium}} = -Z^2\frac{m_e^4}{2}\left[\frac{1}{n_1} + \frac{1}{n_2}\right]$$

$$= -Z^2E_{\text{helium}}\left[\frac{1}{n_1} + \frac{1}{n_2}\right],$$

(25)

where

$$\left[\frac{\hat{p}_1^2}{2m_1} - \frac{e}{r_1}\right]\psi_{(n_1,l_1,m_1)} = E_{\text{helium}}\psi_{(n_1,l_1,m_1)}; \quad i = 1, 2,$$

(26)

and where $n, l, m$ denote main, orbital and azimuthal quantum number respectively. However, in order to find the whole spectrum of the model, i.e., in order to solve the following equation

$$H\psi = (H_0 + H_{\text{int}})\psi_{\text{tot}} = E_{\text{tot}}\psi_{\text{tot}},$$

(27)

with $H_{\text{int}} \neq 0$, one can use the perturbation expansion method \cite{63}. Then, the first-order corrections to the energy levels (25) look as follows

$$E^{(1)} = (\psi, H_{\text{int}}\psi),$$

(28)

while

$$\psi_{\text{tot}} = \psi + \lambda_1\psi^{(1)} + \lambda_2\psi^{(2)} + \cdots,$$

(29)

$$E_{\text{tot}} = E + \lambda_1E^{(1)} + \lambda_2E^{(2)} + \cdots,$$

(30)

Particularly, for ground state energy described by the set $n = 1, l = m = 0$ we have\footnote{The more precise outcomes were obtained in the framework of so-called variational method \cite{60}, where $E_{\text{helium}} = -78.95\text{ eV}$ and $\Delta E = 0.08\text{ eV}$. Hence, it seems quite sensible to analyze the noncommutative correction using variational approach. However, the main aim of this article concerns the exploration of perturbation scheme, and just for this reason, the studies in variational groundwork are omitted and postponed for further investigation.}

$$\psi(r_1, r_2) = \psi_{100}(r_1)\psi_{100}(r_2) = \frac{1}{\pi}\left[\frac{Ze}{a_0}\right]^3e^{-Z(r_1 + r_2)/a_0},$$

(31)

and, by simple integration we obtain

$$E^{(1)} = E_{100} = \frac{5Z}{4}E_{\text{he}},$$

(32)

Consequently, the total energy of the system is given by

$$E_{\text{ground}} = -2Z^2E_{\text{he}} + \frac{5Z}{4}E_{\text{he}},$$

(33)

while in the case of helium atom ($Z = 2$), it is equal to\footnote{We define the Hamiltonian operator by replacement in the formula (20) the classical operators $(x, p)$ by their noncommutative counterparts $(\hat{x}, \hat{p})$.}

$$E_{\text{helium}} = -8E_{\text{he}} + \frac{10}{4}E_{\text{he}} = -108.84\text{ eV} + 34.01\text{ eV} = -74.83\text{ eV},$$

(34)

However, unfortunately, it should be also noted, that the above result is different than the experimental one, in accordance with

$$E_{\text{experimental}} = -79.03\text{ eV},$$

(35)

and finally, we get\footnote{We define the Hamiltonian operator by replacement in the formula (20) the classical operators $(x, p)$ by their noncommutative counterparts $(\hat{x}, \hat{p})$.}

$$\Delta E = E_{\text{helium}} - E_{\text{experimental}} = -74.83\text{ eV} + 79.03\text{ eV} = 4.20\text{ eV},$$

(36)

4. Two-particle ($M = 2$) Coulomb system in twisted space-time

Let us now turn to the main aim of our investigations, i.e., to the construction of two-particle Coulomb system for quantum space-times (18). In the first step of our construction, we extend the twisted space to the whole algebra of momentum and position operators as follows

$$[\hat{x}_A, \hat{x}_B] = \int f_\alpha(t)\delta_{AB}, \quad [\hat{p}_A, \hat{p}_B] = 0 = [\hat{x}_A, \hat{p}_B],$$

$$[\hat{x}_A, \hat{p}_B] = i\hbar\delta_{AB}.$$  

(37)

One can check that the above relations satisfy the Jacobi identity and for deformation parameter $\kappa_a$ approaching zero become classical.

Next, we define the proper Hamiltonian operator in a standard way by\footnote{We define the Hamiltonian operator by replacement in the formula (20) the classical operators $(x, p)$ by their noncommutative counterparts $(\hat{x}, \hat{p})$.}

$$\hat{H} = \frac{1}{2m_e}[\hat{p}_1^2 + \hat{p}_2^2] - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{r_{12}},$$

(38)

$$a_0 = 0.5\text{ Å}$$ denotes the Bohr radius.

$$E_{\text{he}} = \frac{1}{2a_0} = 13.605\text{ eV}.$$
and in order to perform the basic analyze of the system, we represent the noncommutative variables \((\hat{x}_i, \hat{p}_j)\) by the classical ones \((x_i, p_j)\) as (see e.g. [64–66])
\[
\hat{x}_A = x_A - f_{\kappa}(t)/(4\hbar)p_{2A} \tag{39}
\]
\[
\hat{x}_A = x_A + f_{\kappa}(t)/(4\hbar)p_{1A}, \tag{40}
\]
\[
\hat{x}_A = x_A, \quad \hat{p}_A = p_A, \tag{41}
\]
where
\[
[x_{AB}, x_{BC}] = 0 = [p_{A}, p_{B}], \quad [x_{AB}, p_{BC}] = i\hbar\delta_{AB}. \tag{42}
\]

Then, the Hamiltonian (38) takes the form
\[
\hat{H}(t) = \frac{1}{2m_e}[\hat{p}^2 + \hat{p}_z^2] - Ze^2 \sum_{A=1}^{2} [(x_{A} - f_{\kappa}(t)/(4\hbar)p_{2A})^2
+ (x_{A} + f_{\kappa}(t)/(4\hbar)p_{1A})^2 + x_{A}^2]^{1/2}
+ e^2[(x_1 - f_{\kappa}(t)/(4\hbar)p_{21}) - x_1
+ f_{\kappa}(t)/(4\hbar)p_{22})^2
+ (x_2 + f_{\kappa}(t)/(4\hbar)p_{11}) - x_2 - f_{\kappa}(t)/(4\hbar)p_{12})^2
+ (x_3 - x_3^2)^2]^{1/2},
\]
while for small values of deformation function \(f_{\kappa}(t)\), it looks as follows
\[
\hat{H}(t) = H_0 + \hat{H}_{\text{int}}(t) = \sum_{A=1}^{2} \frac{\hat{p}^2}{2m_e} - \sum_{A=1}^{2} \frac{Ze^2}{r_A}
+ \frac{e^2}{r_1^2} \sum_{A=1}^{2} \frac{Ze^2 f_{\kappa}(t)}{4\hbar r_A} L_{3A}
+ \frac{e^2 f_{\kappa}(t)}{4\hbar r_1^2} [L_{31} + L_{32}] - \frac{e^2}{4\hbar r_1^2} [G_{12} + G_{21}]
+ \mathcal{O}(f_{\kappa}^2(t)), \tag{44}
\]
with \(L_{3A} = x_{AB}p_{2A} - x_{2A}p_{1A}, \quad G_{AB} = x_{1B}p_{2A} - x_{2B}p_{1A}, \quad \) and with symbol \(\mathcal{O}(f_{\kappa}^2(t))\) denoting the new, deformed interaction vertex, which defines the total spectrum of the model by
\[
\hat{H}(t)\psi(t) = E(t)\psi(t). \tag{45}
\]

Next, in order to solve the equation (44) with respect the energy \(E(t)\), we assume that
\[
\psi(t) = \psi + \lambda \psi^{(1)}(t) + \lambda^2 \psi^{(2)}(t) + \cdots, \tag{46}
\]
\[
E(t) = E + \lambda E^{(1)}(t) + \lambda^2 E^{(2)}(t) + \cdots, \tag{47}
\]
where \(E\) and \(\psi\) are both given by the formula (23); then, we have
\[
E^{(1)}(t) = (\psi, \hat{H}_{\text{int}}(t)\psi). \tag{48}
\]
Further, using the result (32) we get
\[
E^{(1)}(t) = \frac{5Z}{4}E_H + \frac{e^2 f_{\kappa}(t)}{4\hbar} \sum_{i=1}^{4} I_i, \tag{49}
\]
what leads to the total \(f_{\kappa}(t)\)-deformed ground energy of the system of the form\(^\text{12}\)
\[
E_{\text{ground,}\kappa} = - 2Z^2E_H + \frac{5Z}{4} E_H + \frac{e^2 f_{\kappa}(t)}{4\hbar} \sum_{i=1}^{4} I_i; \tag{50}
\]
\[
I_A = - Z \int d\Gamma \psi^* [L_{3A}/r_1^3]\psi; \quad A = 1, 2, \tag{51}
\]
\[
I_A = \int d\Gamma \psi^* [(L_{31} + L_{32})/r_1^3]\psi, \tag{52}
\]
\[
I_A = - \int d\Gamma \psi^* [(G_{12} + G_{21})/r_1^3]\psi. \tag{53}
\]

It should be noted, however, that three first integrals \(I_1, I_2\) and \(I_3\) vanish since \(L_{3A} \psi(r_1, r_2) = 0\). Similarly, by direct noncommutative calculation one can also check, that the last one \(I_4\) is equal to zero as well\(^\text{13,14}\). Hence, we have
\[
E_{\text{corrections,}\kappa} = \frac{e^2 f_{\kappa}(t)}{4\hbar} \sum_{i=1}^{4} I_i = 0, \tag{54}
\]
and, consequently
\[
E_{\text{Helium,}\kappa} = E_{\text{Helium}} = - 74.83 \text{ eV}. \tag{55}
\]

Obviously, for function \(f_{\kappa}(t) = 0\) the above formulas become undeformed.

5. Final remarks

In this article we provide a two-particle system in Coulomb potential defined on twist-deformed space-time (18) with \(A, B = 1, 2\). Particularly, we demonstrate that for small values of deformation function \(f_{\kappa}(t)\), the obtained first-order corrections for ground helium energy vanish. In such a way, we show that the impact of noncommutativity appears just at the second order of the quantum-mechanical perturbation. It should be noted, however, that such an expectation requires the additional investigation, which seems to be very difficult from a technical point of view, and for this reason, it is postponed for further publication\(^\text{15}\).

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\(^{12}\) \(d\Gamma = \prod_{i=1}^{3} \cos \theta_i d\theta_i d\phi_i d^2 r_i d\omega_i\).

\(^{13}\) \(p_1 \psi(r, r_2) = - i\hbar \sum_{\alpha, \beta} \psi(r, r_2)\).

\(^{14}\) The vanishing of \(I_3\) term is with accordance with fact that since tensors \(G_{12}\) and \(G_{21}\) are linear in momentum operator, the last integral remains complex.

\(^{15}\) In the case of the second-order corrections, there appear the objects of the following form
\[
(\psi_{1,2,3,\alpha,\beta,\gamma}\psi_{1,2,3,\alpha,\beta,\gamma}) \hat{H}_{\text{int}}(t)\psi_{1,2,3,\alpha,\beta,\gamma})
\]
where \(\psi_{1,2,3,\alpha,\beta,\gamma}\psi_{1,2,3,\alpha,\beta,\gamma}\) are different than ground state. Unfortunately, their calculation in our case seems to be quite complicated from a technical point of view, and for this reason, it is postponed for further investigation.
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