Synchronization for a Fuzzy Cellular Neural Networks with Mixed Time Delays

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Abstract. This paper investigates the general decay synchronization (GDS) for fuzzy cellular neural networks (FCNNs). Compared with previous research results, constant time delays and distributed time delays are taken into consideration. By using the Lyapunov function method and some inequality techniques, several sufficient conditions are derived on the GDS for FCNNs. Ultimately, a numerical example is also carried out to validate the practicability and feasibility of our proposed results.

Keywords: Fuzzy cellular neural network; General decay synchronization; General activation functions; Mixed Time delay.

1. Introduction
In the past two decades, cellular neural networks (CNN) have been successfully applied to various fields such as in combinatorial optimization, associative memory, image processing, signal processing, and other areas [1-3]. However, in the hardware implementation of CNNs, such as in mathematical modeling of real world problems, the uncertainty or vagueness is unavoidable. In order to take vagueness into consideration, Yang and Chua in [5, 6] further pioneered the so-called fuzzy cellular neural networks (FCNNs), which has shown that fuzzy cellular neural network (FCNN) integrates fuzzy logic into the structure of traditional CNN and maintains local connectedness among cells. Thus, it is of great importance to analyze the dynamical analysis of FCNNs both theoretical and applied points of view. Until now, there exists an extensive literature concerning the imperative results on synchronization of FCNNs have been developed, see [6-12]. On the other hand, the time delays inevitably exist in natural and man-made systems and cannot be neglected. For example, in the signal communication among the neurons of FCNNs, time delays are always ubiquitous because of the limited switching speed of the amplifiers and the traffic congestions in signal propagation. There has been a lot of literature related to the study of fuzzy cellular neural networks with time delays [4-12] and the references cited therein.

It is worth noting that the convergence rate or time of the system is not easy to be obtained in many practical cases. For example, consider equation \( \dot{y}(x) = -\frac{1}{2} y'(x), x \geq 0. \) Although, we can find that the equation is asymptotically stable, but we cannot in position to estimate the convergent rat of the solution of it [13]. However, the general decay synchronization can overcome this problem.

For work on the GDS, there are some published papers[12-15]. Besides, hitherto, research on GDS for fuzzy cellular neural networks with mixed time delays, there is no theoretical result published. Therefore, inspired by the above descriptions, analysis and reasons, in this paper we study the following n-dimensional FCNNs with mixed time delays.
where \( i \in \Pi \equiv \{1, 2, \ldots, n\}, n \geq 2, \sigma > 0 \) corresponds to the transmission delay. The explanation of \( x_i(t) \) and other parameters with symbols given in [12], we hence omit it here.

The remaining of this paper is organized in following parts. In section 2, we introduce some basic notations model descriptions, definitions and useful lemmas. Main results are drawn in section 3. Next, a numerical example are offered in section 4. Finally, we will give a conclusions about what we study in this paper, what we had in this paper and future work.

2. Preliminaries

In this paper, we always use \( \Pi \equiv \{1, 2, \ldots, n\} \) and \( R^+ = [0, +\infty) \), unless otherwise stated. The initial conditions associated with system (1) are given by

\[
\chi_i(s) = \varphi_i(s), \quad s \in [-\sigma, 0], i = 1, 2, \ldots, n,
\]

where \( \varphi_i(s) = (\varphi_1(s), \varphi_2(s), \ldots, \varphi_n(s)) \in C([-\sigma, 0], R^n) \), which denotes the Banach space of all continuous functions mapping \([-\sigma, 0]\) into \( R^n \) with p-norm (\( p \geq 1 \) is a positive integer) defined by

\[
\|\varphi\|_p = \left[ \sup_{s \in [-\sigma, 0]} \left| \sum_{i=1}^{n} \varphi_i(s) \right|^p \right]^{\frac{1}{p}}.
\]

Assume that \( R^n \) be the space of n-dimensional real column vectors. For any \( \|x\| = (x_1, x_2, \ldots, x_n) \in R^n \), \( \|x\| \) denotes a vector norm defined by

\[
\|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}
\]

In the paper, we consider system (1) as the drive system, the response system is given as follows

\[
\dot{y}_i(t) = -c_iy_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t - \sigma)) + \sum_{j=1}^{n} d_{ij} \int_{-\sigma}^{0} f_j(y_j(s))ds + \nabla a_{ij} f_j(y_j(t - \sigma)) + \nabla b_{ij} f_j(y_j(t - \sigma)) + \nabla c_i y_i + \nabla S_i y_i + I_i + u_i(t),
\]

where \( u_i(t) \) is the controller to be designed.

Let \( e_i(t) = y_i(t) - \chi_i(t) \), then from (1) and (4), the error dynamical system is expressed as

\[
\dot{e}_i(t) = -c_i e_i(t) + \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{f}_j(e_j(t)) + \sum_{j=1}^{n} \tilde{b}_{ij} \tilde{f}_j(e_j(t - \sigma)) + \sum_{j=1}^{n} \tilde{d}_{ij} \int_{-\sigma}^{0} \tilde{f}_j(e_j(s))ds + \nabla \tilde{a}_{ij} \tilde{f}_j(e_j(t - \sigma)) + \nabla \tilde{b}_{ij} \tilde{f}_j(e_j(t - \sigma)) + u_i(t),
\]

where
\[
\tilde{f}_j(e_j(t)) = f_j(y_j(t)) - f_j(x_j(t)), \quad \tilde{f}_j(e_j(t - \sigma)) = f_j(y_j(t - \sigma)) - f_j(x_j(t - \sigma)).
\]

Throughout this paper, we assume that the following assumptions are satisfied.

\textbf{H} \_1 : The activation functions \( f_i(u) \) are continuous and there exist real constants \( L_j > 0 \) and positive real numbers \( \overline{L}_j > 0 \) such that,

\[
L_j \leq \frac{f_j(u_1) - f_j(u_2)}{u_1 - u_2} \leq \overline{L}_j,
\]

for all \( u_1, u_2 \in \mathbb{R}, u_1 \neq u_2 \) and \( j \in \mathbb{n} \).

For further study, we need the following definitions and lemmas. First, we will give the definitions of \( \psi^- \) type function and GDS.

\textbf{Definition 1.} \((\cite{14,15})\). A function \( \psi : \mathbb{R}^+ \to [1, +\infty) \) is said to be \( \psi^- \) type function if it satisfies the following conditions

1) It is differentiable and nondecreasing;

2) \( \psi(0) = 1 \) and \( \psi(+\infty) = +\infty \);

3) \( ~t \, \psi(t) = \psi(t)/\psi(t) \) is nondecreasing and \( \psi^* = \sup_{t \geq 0} \tilde{\psi}(t) < +\infty \), where \( \psi(t) \) is the time derivative of \( \psi(t) \).

For any \( t, s \geq 0 \), \( \psi(t+s) \leq \psi(t) \psi(s) \).

It is not difficult to check that functions \( \psi(t) = e^{\alpha t} \) and \( \psi(t) = (1+t)^{\alpha} \) for any \( \alpha > 0 \) satisfy the above four conditions, thus can be seen as \( \psi^- \) type functions.

\textbf{Definition 2.} \((\cite{14,15})\). The drive-response systems (1) and (4) are said to be general decay synchronized if there exist a constant \( \varepsilon > 0 \) and a \( \psi^- \) type function \( \psi \) such that for any solutions \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1) and \( y(t) = (y_1(t), y_2(t), \ldots, y_n(t)) \) of system (4), one has

\[
\limsup_{t \to +\infty} \frac{\log \| y(t) - x(t) \|}{\log \psi(t)} \leq \varepsilon,
\]

where \( \varepsilon > 0 \) can be seen the convergence rate as synchronization error approaches zero.

\textbf{H} \_2 : For the functions \( \psi(t), \tilde{\psi}(t) \) given in Definition 1, there exist a function \( \rho(t) \in C(R, R^+) \) and a constant \( \delta \) such that for any \( t \geq 0 \)

\[
\tilde{\psi}(t) \leq 1, \quad \sup_{t \in [0, +\infty)} \int_0^t \psi^s(s) \rho(s) ds < +\infty.
\]  

(6)

Next, we present two useful lemmas. Following lemmas are essential to our later study.

\textbf{Lemma 1.} \((\cite{12})\). Suppose \( x \) and \( y \) are two states of system (1), then the following inequalities hold,

\[
\begin{align*}
\left\| \sum_{j=1}^n \alpha_{ij} f_j(x_j) - \sum_{j=1}^n \alpha_{ij} f_j(y_j) \right\| & \leq \sum_{j=1}^n \| \alpha_{ij} \| \left\| f_j(x_j) - f_j(y_j) \right\|, \\
\left\| \sum_{j=1}^n \beta_{ij} f_j(x_j) - \sum_{j=1}^n \beta_{ij} f_j(y_j) \right\| & \leq \sum_{j=1}^n \| \beta_{ij} \| \left\| f_j(x_j) - f_j(y_j) \right\|.
\end{align*}
\]

\textbf{Lemma 2.} \([12]\). Under the assumption \textbf{H} \_2 \, , assume that the synchronization error \( e(t) = y(t) - x(t) \) of driver-response systems (1) and (4) is satisfied the differential equation \( \dot{e}(t) = g(t, e_i) \), where \( e_i = \)
e(t + s) for \( s \in [-\sigma, 0] \), the function \( g(t, e_i) \) is locally bounded. If there exist a differentiable functional \( V(t, e_i) : \mathbb{R}^+ \times C \to \mathbb{R}^+ \), and positive constants \( \lambda_1, \lambda_2 \) such that for any \( (t, e_i) \in \mathbb{R}^+ \times C \)

\[
(\lambda_1 \|e(t)\|_p)^p \leq V(t, e_i) \text{ and } \left| \frac{dV(t, e_i)}{dt} \right|_{(s)} \leq -\delta V(t, e_i) + \lambda_2 \rho(t),
\]

Where \( x(t) \) and \( y(t) \) are solutions of systems (1) and (4) respectively, \( \delta > 0 \) and \( \rho(t) \) are defined in \( H_2 \). Then the driver-response systems (1) and (4) are general decay synchronized in the sense of Definition 2, and the convergence rate is \( \delta/2 \).

### 3. Main Results

In this section, we will obtain some sufficient condition to insure the GDS of systems (1) and (4). First under assumption \( H_2 \), designing the controller \( u_i(t) \) of response system (4) as follows:

\[
u_{ij}(t) = -\eta_i \|e_j(t)\|_p e_j(t) \quad \text{ for } i \in \Pi
\]

Where \( \eta_i \) for \( i \in \Pi \) is control gains satisfying the following \( H_3 \):

\[
-c_i - \eta_i + \sum_{j=1}^{n} [W_{ij} + \sigma C_{ij}] + \gamma_i + \chi_i < 0.
\]

Where

\[
W_{ij} = \frac{1}{p} \left( |b_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + |a_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + |\alpha_j|^{\rho_{ij}} L_j^{\alpha_{ij}} + |\beta_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} \right)
\]

\[
C_{ij} = \frac{1}{p} \left[ \sum_{j=1,j \neq i}^{n} \sum_{l=1}^{p-1} |d_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + \sum_{j=1,j \neq i}^{n} |d_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} \right],
\]

\[
\chi_i = \frac{1}{p} \left[ \sum_{j=1,j \neq i}^{n} \sum_{l=1}^{p-1} |a_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + \sum_{j=1,j \neq i}^{n} |b_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + \sum_{j=1,j \neq i}^{n} |a_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + \sum_{j=1,j \neq i}^{n} |b_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + \sum_{j=1,j \neq i}^{n} |a_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} + \sum_{j=1,j \neq i}^{n} |b_{ij}|^{\rho_{ij}} L_j^{\alpha_{ij}} \right]
\]

\( \gamma_i \) will be defined in proof of Theorem 1.

Now, based on the nonlinear controller (8), the following theorem can be obtained.

**Theorem 1.** Suppose \( H_1, H_2 \) and \( H_3 \) hold, then the response network (4) can begeneral decay synchronized with the drive network (1) under the nonlinear controller (8).

**Proof.** Firstly, we construct the following Lyapunov-Krasovskii functional:

\[
V(t) = \sum_{i=1}^{n} |e_i(t)|^p + \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{t-s}^{t} pW_{ij} |e_j(s)|^p ds + \sum_{i=1}^{n} \sum_{j=1}^{n} \int_{t-s}^{t} 2pC_{ij} |e_j(s)|^p ds dt,
\]

Calculating the derivative of \( V(t) \), we get
\[ V(t) = \sum_{i=1}^{n} \text{sign}(e_i(t)) \{ p |e_i(t)|^{p-1} [-c_i e_i(t)] + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j(e_j(t-\sigma)) \] 
\[ + \sum_{j=1}^{n} d_j \int_{t-\sigma}^{t} \tilde{f}_j(e_j(s)) ds + \sum_{j=1}^{n} \alpha_j \tilde{f}_j(e_j(t-\sigma)) + \sum_{j=1}^{n} \beta_j \tilde{f}_j(e_j(t-\sigma)) \] 
\[ - \eta \frac{\|e(t)\|_p}{\|e(t)\|_p + \rho(t)} \} + \sum_{i=1}^{n} \sum_{j=1}^{n} p W_{ij} |e_j(t)|^p - |e_j(t-\sigma)|^p \]
\[ + \sum_{j=1}^{n} \sum_{i=1}^{n} 2 p C_{ij} |e_j(t)|^p - \int_{t-\sigma}^{t} |e_j(t+\tau)|^p d\tau \]
\[ \leq \sum_{i=1}^{n} \{ p |e_i(t)|^{p-1} [-c_i e_i(t)] + \sum_{j=1}^{n} a_{ij} \tilde{f}_j(e_j(t)) + \sum_{j=1}^{n} b_{ij} \tilde{f}_j(e_j(t-\sigma)) \] 
\[ + \sum_{j=1}^{n} d_j \int_{t-\sigma}^{t} \tilde{f}_j(e_j(s)) ds + \sum_{j=1}^{n} \alpha_j \tilde{f}_j(e_j(t-\sigma)) \] 
\[ - \eta \frac{\|e(t)\|_p}{\|e(t)\|_p + \rho(t)} \} + \sum_{i=1}^{n} \sum_{j=1}^{n} p W_{ij} |e_j(t)|^p + \sum_{i=1}^{n} 2 p C_{ij} |e_j(t)|^p \] 
\[ - \sum_{i=1}^{n} \sum_{j=1}^{n} 2 p C_{ij} \int_{t-\sigma}^{t} |e_j(t+\tau)|^p d\tau - \sum_{i=1}^{n} \sum_{j=1}^{n} p W_{ij} |e_j(t-\sigma)|^p \]
\[ \leq \sum_{i=1}^{n} \{ [-p(c_i - \sum_{j=1}^{n} W_{ji} - \sum_{j=1}^{n} 2 \sigma C_{ji} - \gamma_i)] |e_i(t)|^p + \sum_{j=1}^{n} |a_{ij} | p |e_i(t)|^{p-1} \tilde{f}_j(e_j(t)) \] 
\[ + \sum_{j=1}^{n} |b_{ij} | p |e_i(t)|^{p-1} \tilde{f}_j(e_j(t-\sigma)) + \sum_{j=1}^{n} |\alpha_j | p |e_i(t)|^{p-1} \tilde{f}_j(e_j(t-\sigma)) \] 
\[ + \sum_{j=1}^{n} |\beta_j | p |e_i(t)|^{p-1} \tilde{f}_j(e_j(t-\sigma)) \] 
\[ - \frac{p \eta \|e(t)\|_p |e(t)|^p}{\|e(t)\|_p + \rho(t)} \] 
\[ + \sum_{j=1}^{n} d_j p |e_j(t)|^{p-1} \int_{t-\sigma}^{t} \tilde{f}_j(e_j(s)) ds - \sum_{j=1}^{n} p W_{ij} |e_j(t-\sigma)|^p - \sum_{j=1}^{n} 2 p C_{ij} \int_{t-\sigma}^{t} |e_j(s)|^p ds \} \]

Where \( \gamma_i \) is defined as
\[ \gamma_i = \begin{cases} a_{ii} \bar{L}_i, & a_{ii} \geq 0 \\ a_{ii} \bar{L}_i, & a_{ii} < 0 \end{cases} \]

Using (H1) and applying the fact that \( p a_1 a_2 \cdots a_p < a_1^p + a_2^p + \cdots + a_p^p \), Where \( a_i > 0 \).

For \( i = 1, 2, 3, \ldots, p \), we have
\[ \sum_{j=1}^{n} p |a_{ij} | |e_i(t)|^{p-1} \tilde{f}_j(e_j(t))| \leq \sum_{j=1}^{n} \sum_{i=1}^{n} |a_{ij} | p^{\alpha_i} L_j^p |e_i(t)|^p + \sum_{j=1}^{n} |a_{ij} | p^{\alpha_i} L_j^p |e_j(t)|^p \]

and
\[
\sum_{j=1}^{n} p\beta_j \left| e_j(t) \right|^{p-1} \left| \tilde{f}_j(e_j(t) - \tau_j(t)) \right| \leq \sum_{j=1}^{n} \sum_{i=1}^{p-1} \beta_j^{p_\lambda_j} \left| L_{j}^{\rho_\lambda_j} e_j(t) \right|^p + \sum_{j=1}^{n} \left| \beta_j^{p_\gamma_j} \left| L_{j}^{\rho_\gamma_j} e_j(t - \tau_j(t)) \right|^p \right.
\]

Where \( L_j = \max \{L_j, L_j'\} \), \( \lambda_j, \alpha_j, \theta_j \) are positive constants satisfy \( \sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \alpha_j = \sum_{j=1}^{n} \theta_j = 1 \). Similarly, we have
\[
\sum_{j=1}^{n} p\alpha_j \left| e_j(t) \right|^{p-1} \left| \tilde{f}_j(e_j(t) - \tau_j(t)) \right| \leq \sum_{j=1}^{n} \sum_{i=1}^{p-1} \alpha_j^{p_\lambda_j} \left| L_{j}^{\rho_\lambda_j} e_j(t) \right|^p + \sum_{j=1}^{n} \left| \alpha_j^{p_\gamma_j} \left| L_{j}^{\rho_\gamma_j} e_j(t - \tau_j(t)) \right|^p \right.
\]

Introducing above four inequalities to the derivative of \( V(t) \), we get
\[
\dot{V}(t) \leq \sum_{i=1}^{n} - p[c_i + \eta_i - \sum_{j=1}^{n} W_{ji} - \sum_{j=1}^{n} 2\sigma C_{ji} - \gamma_i - \chi_i] \left| e_i(t) \right|^p
\]
\[
+ \sum_{i=1}^{n} p\eta_i \left| e_i(t) \right|^p - \sum_{i=1}^{n} \frac{p\eta_i \left\| \tilde{f}(e_i(t)) \right\|^p}{\left\| e_i(t) \right\|^p + \rho(t)} - \sum_{i=1}^{n} \sum_{j=1}^{n} pC_{ij} \int_{-\tau}^{0} \left| e_j(s) \right|^p ds
\]
\[
\leq \sum_{i=1}^{n} - A \left| e_i(t) \right|^p + \max_{j \in \mathbb{R}} \left\{ \eta_j \right\} \frac{p\left\| \tilde{f}(e_i(t)) \right\|^p}{\left\| e_i(t) \right\|^p + \rho(t)} - \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \int_{-\tau}^{0} \left| e_j(s) \right|^p ds
\]

Where
\[
A = c_i + \eta_i - \sum_{j=1}^{n} \left[ W_{ji} - 2\sigma C_{ji} \right] - \gamma_i - \chi_i > 0.
\]

Also by \( \eta = \max_{i \in \mathbb{R}} \left\{ \eta_i \right\} > 0 \) and by using the inequality \( 0 \leq ab/(a + b) \leq a \) for any \( a > 0, b > 0 \), we have
\[
\dot{V}(t) \leq \sum_{i=1}^{n} - A \left| e_i(t) \right|^p + p \eta \rho(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \int_{-\tau}^{0} \left| e_j(s) \right|^p ds
\]
\[
(11)
\]

Next, there exists a constant \( \chi > 1 \), we can obtain that
\[
\sum_{i=1}^{n} \left| e_i(t) \right|^p \leq V(t) \leq \chi \sum_{i=1}^{n} \left| e_i(t) \right|^p + \frac{\chi}{A} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \int_{-\tau}^{0} \left| e_j(s) \right|^p ds
\]
\[
(12)
\]

Where \( A = \min_{i \in \mathbb{R}} \left\{ A_i \right\} \).

Now taking a \( \delta \) such that \( \delta < A \), then from the inequalities (11) and (12), we get
\[
\frac{d}{dt} V(t) + \delta \dot{V}(t) \leq \sum_{i=1}^{n} - A \left| e_i(t) \right|^p + p \eta \rho(t) - \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \int_{-\tau}^{0} \left| e_j(s) \right|^p ds
\]
\[
+ \delta \left[ \chi \sum_{i=1}^{n} \left| e_i(t) \right|^p + \frac{\chi}{A} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} \int_{-\tau}^{0} \left| e_j(s) \right|^p ds \right]
\]
\[ \leq (\delta X - A) \sum_{i=1}^{n} |e_i(t)|^p + \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \frac{\delta X}{A} - 1 \right) C_{ij} \int_{t-j}^{t} |e_i(s)|^p s + p \eta \rho(t) \leq p \eta \rho(t), \]

which means that

\[ \dot{V}(t) + \delta V(t) \leq p \eta \rho(t). \] (13)

Then, from Lemma 2, the drive-response systems (1) and (4) achieve GDS under the adaptive nonlinear controller (8). The convergence rate of \( e(t) \) approaching zero is \( \delta/2 \). The proof is completed.

**Remark 1.** The function \( \psi \) is used as the decay function, so \( \psi \)-type stability is also said to be stability with general decay rate. When \( \psi(t) = e^{\alpha t}, \psi(t) = (1+t)^{\alpha} \) and \( \psi(t) = 1 + \alpha \log(1+t) \) for any \( \alpha > 0, \psi \)-type stability may be specialized as exponential synchronization, polynomial synchronization and logarithmic synchronization. In this paper, if the \( \rho(t) \) in controller (8) is equal to zero, then the GDS can be specialized as exponential synchronization.

### 4. Numerical Simulations

In this section, one example is given to illustrate the effectiveness of our results obtained in this paper.

**Example 1.** For \( n = 2 \), we consider the following chaotic fuzzy neural network system with time-varying delays

\[ \begin{align*}
\dot{x}_i(t) &= -c_i x_i(t) + \sum_{j=1}^{2} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{2} b_{ij} f_j(x_j(t-\sigma)) + \sum_{j=1}^{2} d_{ij} \int_{t-\sigma}^{t} f_j(x_j(s)) ds \\
&+ \sum_{j=1}^{2} a_{ij} f_j(x_j(t-\sigma)) + \sum_{j=1}^{2} b_{ij} f_j(x_j(t-\sigma)) + I_i,
\end{align*} \] (14)

where \( f_i(u) = f_2(u) = \tanh(1.1u) \). The parameters of system (14) are assumed that \( c_1 = c_2 = 1, a_{11} = 1.75, a_{12} = -0.1, a_{21} = -2.05, a_{22} = 0.4, b_{11} = -1.71, b_{12} = -0.59, b_{21} = -2.48, b_{22} = 0.4, d_{11} = 0.5, d_{12} = -0.2, d_{21} = 0, d_{22} = 0.35, \alpha_{11} = -0.88, \alpha_{12} = -0.41, \alpha_{21} = -0.49, \alpha_{22} = -1.37, \beta_{11} = -0.61, \beta_{12} = -0.24, \beta_{21} = -0.51, \beta_{22} = -0.9, \sigma = 1 \) and \( I_i = 0 \) for \( i = 1, 2 \).

The numerical simulation of system (14) with initial values \( x_1(s) = 0.55 \) and \( x_2(s) = 0.45 \) for \( s \in [-1, 0] \) is represented in Fig. 1, we can see that it has a chaotic attractor.

![Figure 1. The transient behavior of delayed fuzzy neural network system (14).](image-url)
The corresponding response system is described by

$$
\dot{y}_i(t) = -c_i y_i(t) + \sum_{j=1}^{2} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{2} b_{ij} f_j(y_j(t - \sigma)) + \sum_{j=1}^{2} d_{ij} \int_{t-\sigma}^{t} f_j(y_j(s)) ds \\
+ \sum_{j=1}^{2} a_{ij}^\theta f_j(y_j(t - \sigma)) + \sum_{j=1}^{2} \beta_{ij} f_j(y_j(t - \sigma)) + I_i + u_i(t),
$$

(15)

Where $c_i, a_{ij}, b_{ij}, d_{ij}, \alpha_{ij}, \beta_{ij}, f_j(t), \sigma$ and $I_i$ are the same as in system (14), and the nonlinear controller $u_i(t)$ is designed as follows

$$
u_i(t) = -\frac{\eta_i \|e(t)\|^\rho e_i(t)}{(e_i(t) + \rho(t))}, i = 1, 2,
$$

(16)

It is not difficult to check that $L_1 = L_2 = 1$. Thus, the assumptions $H_1$, $H_2$ are satisfied. Letting $\rho(t) = e^{-0.1t}$, $\psi(t) = e^t$ and choosing $\eta_1 = 5.5$, $\eta_2 = 6$. Then, the condition $H_3$ of Theorem 1 is satisfied. Therefore, according to the Theorem 1, the drive response systems (14) and (15) can be achieved GDS under the controller (16). The time evolution of synchronization errors between systems (14) and (15) are demonstrated in Fig. 2, where the initial values of response system (15) are chosen $y_i(s) = 0.5$ and $y_j(s) = 0.1$ for $s \in [-1, 0]$. The synchronization curves between systems (14) and (15) are shown in Fig. 3.
5. Conclusion

In this paper, the GDS problem for a class of fuzzy cellular neural networks (FCNNs) with general activation functions and mixed time delays is discussed and derived some simple sufficient conditions on the general decay synchronization of the drive-response systems (1) and (4) by constructing suitable Lyapunov function, applying the novel analysis methods and the method given in [14,15]. In addition, one numerical simulation example is given to validate the correctness of the theoretical findings. Since, in this paper we consider both the constant time delay and distributed time delay. Hence, the established results in this paper can be seen as the improvement and extension of the previously known works. Recently, fractional-order neural networks have been taken into account in some papers. Hence, we have an interesting topic deserve further investigation, such as the dynamic analysis of fractional-order neural networks with delays. We leave this topic for our future work.

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