New Progress in Classic Area: Polynomial Root-squaring and Root-finding

Victor Y. Pan

Department of Computer Science
Lehman College of the City University of New York
Bronx, NY 10468 USA and
Ph.D. Programs in Mathematics and Computer Science
The Graduate Center of the City University of New York
New York, NY 10036 USA
victor.pan@lehman.cuny.edu
http://comet.lehman.cuny.edu/vpan/

Abstract. The DLG root-squaring iterations, due to Dandelin 1826 and rediscovered by Lobachevsky 1834 and Gräffe 1837, have been the main approach to root-finding for a univariate polynomial \( p(x) \) in the 19th century and beyond, but not so nowadays because these iterations are prone to severe numerical stability problems. Trying to avoid these problems we have found simple but novel reduction of the iterations applied for Newton’s inverse ratio \(-\frac{p'(x)}{p(x)}\) to approximation of the power sums of the zeros of \( p(x) \) and its reverse polynomial. The resulting polynomial root-finders can be devised and performed independently of DLG iterations – based on Newton’s identities or Cauchy integrals. In the former case the computation involve a set of leading or tailing coefficients of an input polynomial. In the latter case we must scale the variable and increase the arithmetic computational cost to ensure numerical stability. Nevertheless the cost is still manageable, at least for fast root-refinement, and the algorithms can be applied to a black box polynomial \( p(x) \) – given by a black box for the evaluation of the ratio \( \frac{p'(x)}{p(x)} \) rather than by its coefficients. This enables important computational benefits, including (i) efficient recursive as well as concurrent approximation of a set of zeros of \( p(x) \) or even all of its zeros, (ii) acceleration where an input polynomial can be evaluated fast, and (iii) extension to approximation of the eigenvalues of a matrix or a polynomial matrix, being efficient if the matrix can be inverted fast, e.g., is data sparse. We also recall our recent fast algorithms for approximation of the root radii, that is, the distances to the roots from the origin or any complex value to the zeros of \( p(x) \), and propose to apply it for fast black box initialization of polynomial root-finding by means of functional iterations such as Newton’s, Ehrlich’s, and Weierstrass’s.

Keywords: symbolic-numeric computing, computer algebra, polynomial computations, root-finding, matrix eigenvalues

2020 Math. Subject Classification: 65H05, 26C10, 30C15, 65H17
1 Introduction: The State of the Art, Related Work, and Our Progress

1.1. Polynomial root-finding. The venerated problem of univariate polynomial root-finding, that is, approximation of the roots of the equation \( p(x) = 0 \) or just roots, aka zeros of \( p(x) \), for a \( d \)th degree polynomial

\[
p(x) := \sum_{i=0}^{d} p_i x^i = p_d \prod_{j=1}^{d} (x - x_j), \ p_d \neq 0, \tag{1}
\]

has been central for Mathematics and Computational Mathematics for four millennia, from Sumerian times to about 1850 [18]. Nowadays it is still highly important in various areas of symbolic and numerical computing [20, Introduction], [31].

1.2. Classic root-squaring and applications to root-finding. The DLG root-squaring iterations by Dandelin 1826, Lobachevsky 1834, Gräffe 1837 [9],

\[
p_0(x) := p(x), \ p_{h+1}(x) := (-1)^d p_h(\sqrt{x}) p_h(\sqrt{-x}), \ h = 0, 1, \ldots, \tag{2}
\]

\[
p_h(x) := \sum_{i=0}^{d} p_i^{(h)} x^i = \prod_{j=1}^{d} (x - x_j^{(h)}), \ x_j^{(h)} = x_j^{2h}, \ j = 1, \ldots, d; \ h = 0, 1, \ldots, \tag{3}
\]

have dominated polynomial root-finding in the 19-th century and beyond. The iterations were performed by hand, and people, called “calculateurs” or “computers”, were paid for this work [22].

Vieta’s formulas express the ratios \((-1)^j \frac{p_{d-j+1}^{(h)}}{p_{d-j}^{(h)}}\) as the \(j\)th elementary symmetric polynomials in the roots, and hence

\[
- \frac{p_{d-j+1}^{(h)}}{p_{d-j}^{(h)}} \approx x_j^{(h)} = x_j^{2h} \tag{4}
\]

if \(2^h\) is large enough and if

\[
\max_{i > j} |x_i| < |x_j| < \min_{i < j} |x_i| \text{ where } |x_0| = +\infty \text{ and } |x_{d+1}| = 0. \tag{5}
\]

Thus the ratios \(- \frac{p_{d-j+1}^{(h)}}{p_{d-j}^{(h)}}\) approximate the zeros \(x_j^{(h)} = x_j^{2h}\) of the polynomial \(p_h(x)\), and we can readily recover the root radii \(|x_j|\), and then the roots \(x_j\) if they are real. Various complex polynomial root-finders extensively involve root radii (see [26, 27, 31]) but also apply DLG root-squaring to strengthening the isolation of the unit circle \(\{x : |x| = 1\}\) from the zeros of \(p(x)\) (see, e.g., [27, 4], and [31, Secs. 5.1.4, 5.3, 5.6, 5.7, 6.3, 6.6, 10.5]) and to root-finding over finite fields [7].

1 To simplify subsequent notation we let \(p(x)\) be monic.
We focus on approximation (4) for $j = d$ and $j = 1$:

\[ x_d^{(h)} \approx -\frac{p_{(h)}}{p_0^{(h)}} \quad \text{and} \quad x_1^{(h)} \approx -\frac{p_{d-1}^{(h)}}{p_d^{(h)}}. \tag{6} \]

### 1.3. Recovery of complex zeros $x_j$ of $p(x)$ from the zero $x_j^{(h)}$ of $p_h(x)$

We can recover $x_j$ from $y_j := x_j^{(h)} = x_j^{2^h}$ as follows: check whether $p(y_j^{1/2^h}) = 0$ for each of $2^h$ candidates $y_j^{1/2^h}$ and weed out up to $2^h - 1$ “false” candidates. $2^h$ tests are expensive if $2^h$ is large, but by extending the descending process of [25] we can manage with only $2h$ tests if we recursively apply the following recipe for $i = h - 1, \ldots, 1, 0$: given a zero $x_j^{(i+1)}$ of $p_{i+1}(x)$, select one of the two candidates $\pm (x_j^{(i+1)})^{1/2}$ for being a zero of $p_i(x)$; then decrease $i$ by 1 and repeat until $i$ vanishes.

### 1.4. FG iterations

By following Fiedler [5], Gemingani extends DLG iterations in [6] to approximate the zeros $x_j$ themselves rather than their high powers $2^j$ and thus to get rid of the descending process for non-real zeros $x_j$. He substitutes $2x$ for $2$ on the left sides of expressions of [5] for a fixed polynomial $q(x)$, arrives at Fiedler/Gemingani’s (FG) iterations:

\[ q_0(x) := q(x), \quad 2\sqrt{x}q_{h+1}(x) := q_h(\sqrt{x})p_h(\sqrt{x}) - q_h(\sqrt{x})p_h(\sqrt{x}), \tag{7} \]

for $h = 0, 1, \ldots$, writes

\[ q(x) := xp'(x) \quad \text{and} \quad q_h(x) := \sum_{i=0}^{d} q_i^{(h)} x^i, \quad h = 0, 1, \ldots \quad (\text{cf. (3)}), \tag{8} \]

and proves the following result (see [6, Thm. 2]).

**Theorem 1.** Let

\[ |x_d| < \min_{j<d} |x_j|, \tag{9} \]

Then

\[ \frac{q_0^{(h)}}{p_1^{(h)}} = \frac{q_h(0)}{p'_h(0)} = x_d \left( 1 + O\left( \frac{x_d}{x_{d-1}} \right) \right)^2 \quad \text{for} \quad h = 0, 1, \ldots \]

### 1.5. Complexity and numerical stability

One can reduce every DLG or FG iteration to one or two polynomial multiplications, respectively, and then perform it by using $O(d \log(d))$ arithmetic operations (cf., e.g., [25, Sec. 2.4]), but the coefficients of the polynomials $p_h(x)$ change their absolute values dramatically and unevenly as $h$ grows, implying severe numerical instability already for rather small integers $h$. Realistically, one can safely apply at most order of $\log(\log(d))$ DLG iterations or resort to randomized renormalization of [22], the

\[ \text{See other extensions of DLG iterations in [2] and [19].} \]
only known stabilization recipe, based on using polar representation of complex numbers \( x = |x| \exp(2\pi \sqrt{-1}) \); back and forth transition to computations, making them not competitive.

1.6. Our progress (outline). Given a function \( f(x) \) and its derivative define Newton’s inverse ratio

\[
\frac{1}{N(f(x))} := -\frac{f'(x)}{f(x)} \tag{10}
\]

and notice that the two ratios in (6) are equal to the values at 0 of Newton’s inverse ratios for \( f(x) \) denoting \( p_k(x) \) and its reverse polynomial

\[
p_{\text{rev}}(x) := x^d p\left(\frac{1}{x}\right) = \sum_{i=0}^{d} p_i x^{d-i}, \quad p_{\text{rev}}(x) = p_0 \prod_{j=1}^{d} \left(x - \frac{1}{x_j}\right) \text{ if } p_0 \neq 0, \tag{11}
\]

obtained from \( p(x) \) by reversing the order of its coefficients. We combine Vieta’s formula with a well-known expression for Newton’s inverse ratio through the zeros of \( p(x) \) (cf. (17)) and immediately deduce that the two ratios in (6) are the \( 2^h \)th power sums of the zeros of \( p(x) \) and of their reciprocals (see (24)). These power sums are close to the \( 2^h \)th powers of the extremal zeros of \( p(x) \) if assumptions (5) hold for \( j = 1 \) and \( j = d \) and if the integer \( 2^h \) is sufficiently large.

Therefore, we can implicitly perform \( h \) DLG iterations by computing these power sums with all cited benefits and applications and with the hope to avoid problems of numerical stability, at least for larger \( h \) than in the DLG case.

Furthermore, we need neither FG iterations nor descending process to approximate non-real zeros of \( p(x) \). We just compute the ratio of the \( h \)th and \((h + 1)\)st power sums for a sufficiently large integer \( h \) because these ratios converge fast to the two extremal (that is, absolutely largest and absolutely smallest) zeros of \( p(x) \).

For power sum computation, we can apply the known algorithms based on Newton’s identities or Cauchy integrals. In the latter case we scale the variable to ensure numerical stability at the price of some increase of arithmetic cost, but the algorithms can be applied to a black box polynomial \( p(x) \) – given by a black box for the evaluation of the ratio \( \frac{p'(x)}{p(x)} \) rather than by its coefficients, and this enables some important computational benefits:

(i) We can recursively extend this approximation of a single zero of \( p(x) \) to approximation of many or all its zeros

\footnote{Root-finding for a black box polynomial with the goal of minimizing the number of evaluations of the ratio \( \frac{p'(x)}{p(x)} \) or \( \frac{p(x)}{p'(x)} \) has been a well-known research subject for years but remained in stalemate since the paper \([15]\) of 2016, which covered the history and the State of the Art of that time. The papers \([30]\), \([16]\), and \([31]\) have reported new significant progress. Moreover, \([31]\) is a comprehensive study of black box polynomial root-finders, which covers subdivision and various functional iterations: Newton’s, Schröder’s, Weierstrass’s, aka Durand – Kerner’s, and Ehrlich’s, aka Aberth’s.}
(ii) The black box algorithms run fast for polynomials that can be evaluated fast, e.g., are sparse, shifted sparse or Mandelbrot’s, and can be readily extended to efficient eigen-solving for matrices that can be inverted fast.

(iii) The algorithms allow numerically safe Taylor’s shifts and scaling of the variable $x$ (see [30]), that is, can be applied to the polynomial $t(x) = p\left(\frac{x-c}{\rho}\right)$ for any pair of complex $c$ and positive $\rho$. This enables efficient refinement of approximate zeros of $p(x)$ by using our algorithms as well as $k$-fold parallel acceleration of root-finding by using $k$ processors and concurrent application of our root-finders with shifts to $k$ distinct centers $c_1, \ldots, c_k$.

Clearly, Newton’s ratio $-p(x)/p'(x)$ vanishes at the zeros of $p(x)$. In Sec. 3 we combine this simple observation with our algorithms of [31, Secs. 6.2 and 6.3] for root radii of a black box polynomial to compute non-costly initialization of functional iterations such as Newton’s, Ehrlich’s (aka Aberth’s), and Weierstrass’s (aka Durand-Kerner’s) for black box polynomial root-finding.

1.7. Further acceleration of root-finding: some recipes and an algorithm. (i) We propose heuristic acceleration of our algorithms based on performing fixed or random rotations of the unit disc, but so far (ii) an alternative black box polynomial root-finder in [8] seems to accelerate computation of the $2^k$th power of the two extremal zeros of $p(x)$ by a factor of $2^k/k$ versus our algorithms based on power sum approximation. The paper [8] extends DLG iterations for Newton’s inverse ratios based on equation

$$\frac{p_{h+1}(x)}{p_h(x)} = \frac{1}{2\sqrt{x}} \left( \frac{p_h(\sqrt{x})}{p_h(\sqrt{-x})} - \frac{p_h(\sqrt{-x})}{p_h(\sqrt{x})} \right), \quad h = 0, 1, \ldots, \quad (12)$$

appeared as Eqn. (78) in a revision of [31] of January 2022. The paper [8] ingeniously performs these iterations by using polar representation of complex numbers $x$, in which $\sqrt{x}$ and $\sqrt{-x}$ are computed very fast. As in [22], application of polar representation enabled numerical stabilization, but the resulting algorithm of [8] is deterministic, fast, and can be applied to a black box polynomial $p(x)$.

Authors’ extensive tests in the Graduate Center of the City University of New York have demonstrated the efficiency of that algorithm for approximation of the extremal root radii of the test polynomials of MPSolve – the package of polynomial root-finding subroutines, currently of user’s choice.

(iii) To extend [8] to approximation of the complex extremal zeros of $p(x)$ themselves rather than their high powers, one can apply the descending process of Sec. 1.3 or can try to extend the algorithm of [8] to FG iterations for Newton’s inverse ratios:

$$\frac{q_{h+1}(x)}{p_{h+1}(x)} = \frac{1}{2\sqrt{x}} \left( \frac{q_h(\sqrt{x})}{q_h(\sqrt{-x})} - \frac{q_h(\sqrt{-x})}{q_h(\sqrt{x})} \right), \quad h = 0, 1, \ldots, \quad (13)$$

(iv) Would application of polar representation be as efficient in the case of original DLG and FG iterations applied to a polynomial $p(x)$ rather than to the inverse Newton’s ratio? Probably not so, but it is interesting to investigate this.

1.8. Organization of the paper. We devote the next section to some background material. In Sec. 3 we extend our recent algorithms for root-radii
approximation to a new variant of Lehmer’s root-finder and to fast initialization of functional iterations for polynomial root-finding. In Sec. 4 we reduce approximation of zeros of a polynomial to approximation of the power sums of its zeros. In Secs. 5 and 6 we approximate the power sums of the zeros of a polynomial in two ways – by using Newton’s identities and by means of approximation of Cauchy integrals. In Sec. 7 we extend root-finding from the absolutely smallest and largest zero of \( p(x) \) to all its zeros. In Sec. 8 we specify some benefits of black box polynomial root-finders. In Sec. 9 we estimate extremal root radii, partly relaxing the root separation assumptions (5). We devote short Sec. 10 to conclusions.

2 Background: Basic definitions, auxiliary results, and some applications to root-finding

2.1 Definitions

– We write “roots” for “the roots of the equation \( p(x) = 0 \)” and enumerate them in non-decreasing order of their absolute values: \( |x_1| \geq |x_2| \geq \cdots \geq |x_d| \).

– \( D(c, \rho) \), \( C(c, \rho) \), and \( A(c, \rho, \rho') \) denote a disc, a circle (circumference), and an annulus on the complex plane, respectively:

\[
D(c, \rho) := \{ x : |x - c| \leq \rho \}, \quad C(c, \rho) := \{ x : |x - c| = \rho \},
\]

\[
A(c, \rho, \rho') := \{ x : \rho \leq |x - c| \leq \rho' \}.
\]

– A disc \( D = D(c, \rho) \) and its boundary circle \( C = C(c, \rho) \) are \( \theta \)-isolated for \( \theta \geq 1 \) if the annulus \( A(c, \rho/\theta, \theta \rho) \) contains no roots. Supremum of such values \( \theta \) is the isolation of \( D \) and \( C \). “Isolated” stands for “\( \theta \)-isolated” where \( \theta - 1 \) exceeds a positive constant.

– Write \( \zeta \) and \( \zeta_q \) for a primitive \( q \)th root of unity

\[
\zeta := \zeta_q := \exp \left( \frac{2\pi \sqrt{-1}}{q} \right). \tag{14}
\]

– For a complex \( c \) and a positive \( \rho \), Taylor’s shift of the variable, or translation, together with scaling,

\[
x \iff y = \frac{x - c}{\rho}, \tag{15}
\]

map polynomials, discs and circles:

\[
p(x) \iff t_{c,\rho}(y) = p\left( \frac{x - c}{\rho} \right), \tag{16}
\]

\[
D(c, \rho) := \{ x : |x - c| \leq \rho \} \iff D(0, 1),
\]

\[
C(c, \rho) := \{ x : |x - c| = \rho \} \iff C(0, 1).
\]
2.2 Auxiliary results

Substitute factorization (1) into the ratio \(-\frac{p'(x)}{p(x)}\) and obtain the following well-known equation,
\[-\frac{p'(x)}{p(x)} = \sum_{j=1}^{d} \frac{1}{x_j - x}, \tag{17}\]
which implies that
\[-\frac{p'(0)}{p(0)} = \sum_{j=1}^{d} \frac{1}{x_j} \quad \text{and} \quad -\frac{p_{rev}'(0)}{p_{rev}(0)} = \sum_{j=1}^{d} x_j. \tag{18}\]

We can approximate the value \(p'(c)\) for a black box polynomial \(p(x)\) based on the equation \(p'(c) = \lim_{x \to c} \frac{p(x)}{p(x)}\) and can evaluate \(p'(c)\) by applying the algorithm that supports the following result.

**Theorem 2.** Given a black box function \(f(x)\) over a field \(K\) of constants that has a derivative and a straight line algorithm (that is, one with no branching) that evaluates \(f(x)\) at a point \(x\) by using \(A\) additions and subtractions, \(S\) scalar multiplications (that is, multiplications by elements of the field \(K\)), and \(M\) other multiplications and divisions, one can extend this algorithm to the evaluation at \(x\) of both \(f(x)\) and \(f'(x)\) by using \(2A + M\) additions and subtractions, \(2S\) scalar multiplications, and \(3M\) other multiplications and divisions.

**Proof.** See [14] or [3, Thm. 2] for a constructive proof of this theorem for any function \(f(x_1, \ldots, x_s)\) that has partial derivatives in all its \(s\) variables \(x_1, \ldots, x_s\).

2.3 Some known root radii estimates and algorithms

Given the coefficients of \(p(x)\) one can readily compute the following narrow range for the extremal root radii \(|x_d|\) and \(|x_1|\) (cf. [10, Sec. 6.4], [21,36]) and can closely approximate all the \(d\) root radii at a low Boolean cost:
\[-\frac{1}{d} \hat{r}_+ \leq |x_1| < 2\hat{r}_+, \quad \frac{1}{2} \hat{r}_- \leq |x_d| \leq d \hat{r}_- \tag{19}\]
where
\[r_- := \min_{i \geq 1} \left| \frac{p_0}{p_i} \right|^\dagger \quad \text{and} \quad r_+ := \max_{i \geq 1} \left| \frac{p_{d-i}}{p_d} \right|^\dagger. \tag{20}\]

In particular, for \(i = 1\) Eqns. (19) and (20) together imply that
\[r_d \leq d \left| \frac{p_0}{p_1} \right| = d \left| \frac{p(0)}{p'(0)} \right|. \tag{21}\]

**Theorem 3.** [12, Prop. 3] Given the coefficients of a polynomial \(p(x)\), one can approximate all the \(d\) root radii \(|x_1|, \ldots, |x_d|\) within the relative error bound \(4d\) at a Boolean cost in \(O(d \log(||p||_\infty))\).

\(^4\) A variation of this theorem was first outlined and briefly analyzed in [35] for fast approximation of a single root radius and then extended in [23, Sec. 4] (cf. also [24, Sec. 5]) to approximation of all root radii of a polynomial.
Corollary 1. Given the coefficients of a polynomial \( p = p(x) \) and a positive \( \Delta \), one can approximate all the \( d \) root radii \(|x_1|, \ldots, |x_d|\) within a relative error bound \( \Delta \) in \( \frac{1}{d^{\phi}} \) at a Boolean cost in \( O(d \log(||p||_\infty) + d^2 \log^2(d)) \).

The algorithm supporting the corollary computes at a low cost \( d \) concentric narrow annuli covering the circles \( C(0, |x_j|) \) for \( j = 1, \ldots, d \) (some annuli may pairwise overlap), whose union \( U \) contains all \( d \) roots. [12] strengthens the benefits of having such annuli by also computing the \( d \) root radii from a fixed positive \( c \) and \( c \sqrt{-1} \). In particular the search of the roots, that is, the zeros of \( p(x) \), is reduced to the intersection of the three unions.

All these algorithms and estimates for root radii, and even for the extremal root radii, only apply to a polynomial \( p(x) \) given with its coefficients, but the randomized algorithms of [31] Sec. 6.2 fast approximate the \( j \)th smallest root radius for any \( j \) of a black box polynomial (slightly faster for \( j = 1 \) and \( j = d \)). [31] Sec. 6.3] extends these algorithms to approximation of all \( d \) root radii of a black box polynomial at the price of slow down only by a factor of \( \log(d) \); even for all root radii the algorithm of [31] Sec. 6.3] evaluates the ratio \( \frac{p'(x)}{p(x)} \) at the number of points \( x \) nearly linear in \( d \).

Theorem 4. Suppose that a black box polynomial \( p(x) \) of a degree \( d \) has precisely \( m \) zeros, counted with their multiplicities, that lie in isolated unit disc \( D(0, 1) \). Then for fixed positive \( \phi, \epsilon = 1/2^\phi, \) and \( v > 1 \), the randomized algorithm of [31], Secs. 6.2 and 6.3] performs with a probability of error at most \( 1/2^v \), evaluates the ratio \( p'(x)/p(x) \) at \( O(vm \log(b/\phi)) \) points, and for any fixed \( j, 1 \leq j \leq m \), approximates the \( j \)th smallest root radius \( |x_j| \) within the relative error \( 2^\phi \) or determines that \( |x_j| \leq \epsilon \). At the price of increasing the number of evaluation points by a factor of \( O(\log(m)) \) the algorithm outputs such approximations to all \( m \) root radii.

This black box algorithm can be readily extended to approximate the distances from any fixed complex center \( c \) to all zeros of a polynomial lying in a disc \( D(c, \rho) \) for any fixed positive \( \rho \) as well as the distances from a fixed center \( c \) to all zeros of \( p(x) \). This implies further benefits – see [29] and [31] Secs. 10.6 and 10.7 ].

3 Some applications of root radii computation to black box polynomial root-finding

Recall Lehmer’s celebrated polynomial root-finder and devise its new black box variant based on our black box algorithms for extremal root radii and on the following two observations: (i) Newton’s ratio \(-p(x)/p'(x)\) vanishes at the zeros of \( p(x) \) and (ii) the circle \( C(0, r_d) \) contains such zeros or a zero.
Algorithm 5 [Lehmer-Newton’s root-finder.]

**INPUT:** a black box polynomial \( p(x) \) of a degree \( d \) and a positive tolerance value \( \epsilon \).

**OUTPUT:** a complex \( z \), approximating a zero of \( p(x) \) within \( \epsilon \).

**COMPUTATIONS:**
1. Compute an upper bound \( r'_d = d |\frac{p(0)}{p'(0)}| \) on the smallest root radius \( r_d = |x_d| \) and output \( z = 0 \) if \( r'_d \leq \epsilon \).
2. Otherwise compute a refined upper bound \( r''_d \) on \( r_d \) by applying a root radius algorithm of [21, Sec 6.2]. Output \( z = 0 \) if \( r''_d \leq \epsilon \).
3. Otherwise compute the values of \(-\frac{p(x)}{p'(x)}\) (Newton’s ratios) at \( q \) equally space points of the circle \( C(0, r'_d) \) for a sufficiently large \( q \).
4. Choose a point \( c \) at which the value of the ratio is absolutely smallest If \( d |\frac{p(c)}{p'(c)}| \leq \epsilon \), than output \( z = c \).
5. Otherwise shift the origin into this point and go to stage 2.

**Remark 1.** Between stages 3 and 4, one can apply a root-refiner at \( c \), e.g., Newton’s iterations.

If approximations \( r'_j \) have been computed to \( w \) root radii \( r_j \), for \( j = 1, \ldots, w \), then one can concurrently apply stage 2 above to the \( d \) circles \( C(0, r'_j) \), for \( j = 1, \ldots, d \), and then can use the \( w \) computed approximations to the \( w \) zeros of \( p(x) \) to initialize various functional iterations for root-finding, e.g., Newton’s or Schröder’s. For \( w = d \) there are more option of functional iterations, e.g., Ehrlich’s (aka Aberth’s) or Weierstrass’s (aka Durand-Kerner’s) (see [17] and [20] for a variety of such iterations).

4 Extremal roots from the power sums of the roots

**Theorem 6.** [See Remark 2]\ Let \( f_k(z) := f \cdot \prod_{j=1}^{w} (z - z_j^k) \) and \( \hat{f}_k(z) := \hat{f} \cdot \prod_{j=1}^{w} (z - z_j^{-k}) \), for two nonzero scalars \( f \) and \( \hat{f} \), so that \( \hat{f}(z) \) is a scaled reverse polynomial of \( f(z) \). Then

\[
-\frac{f'_k(0)}{f_k(0)} = \sum_{j=1}^{w} z_j^{-k} \quad \text{and} \quad -\frac{\hat{f}'_k(0)}{\hat{f}_k(0)} = \sum_{j=1}^{w} z_j^k \quad \text{for} \quad k = 0, 1, \ldots . \quad (22)
\]

**Proof.** Substitute \( p(x) := f_k(z) \) into Eqn. (18).

**Remark 2.** Thm. 6 does not involve DLG iterations, but let \( \hat{f}_k(z) := p_{h, \text{rev}}(x) \) denote the reverse polynomial of \( p_h(x) \), write

\[
z := x, \quad k = 2^h, \quad \text{and} \quad f_k(z) := p_h(x), \quad (23)
\]

apply Thm. 6 and obtain

\[
-\frac{p_{d-1}^{(h)}}{p_d^{(h)}} = -\frac{p_h'(0)}{p_h(0)} = \sum_{j=1}^{d} x_j^{-k}, \quad -\frac{p_1^{(h)}}{p_0^{(h)}} = -\frac{p_{h, \text{rev}}'(0)}{p_{h, \text{rev}}(0)} = \sum_{j=1}^{d} x_j^k \quad \text{for} \quad k = 2^h. \quad (24)
\]
Now reduce approximation of the absolutely smallest (resp. largest) zero of a polynomial to approximation of two consecutive sufficiently large power sums of its zeros (resp. reciprocals of its zeros) and bound the approximation errors. Such a bound is proportional to the $k$th power of the ratio $\frac{z_w}{z_{w-m}}$ or $\frac{z_{m+1}}{z_1}$ and thus decreases fast as the ratio decreases. The ratio can be computed at a low cost by means of the fast algorithms that support Cor. 4 and Thm 4 and is small if we begin with an approximation to a zero of $p(x)$ and apply our algorithms to refine it fast.

**Corollary 2.** For polynomials $f_i(z)$ and $\hat{f}_i(z)$ of Thm. 4, $i = k, k+1$, and two integers $k \geq 0$ and $m$, $1 \leq m \leq w$, it holds that

(i) $-\frac{f_i(0)}{f_k(0)} = (1 + \Delta_{w,k,m})mz_w^k$ and $\frac{f_{k+1}(0)f_k(0)}{f_{k+1}(0)f_k(0)} = (1 + \gamma_{w,k,m})z_w$ where

$$1 + \gamma_{w,k,m} = \frac{1 + \Delta_{w,k+1,m}}{1 + \Delta_{w,k,m}}, \quad |\Delta_{w,i,m}| \leq \frac{w - m}{m} |\frac{z_w}{z_{w-m}}|^i, \quad i = k, k+1, \quad (25)$$

if

$$|z_w| = |z_{w-1}| = \cdots = |z_{w-m+1}| < |z_{w-m}| = \min_{j \geq w-m} \frac{z_j}{|z_j|}. \quad (26)$$

(ii) $-\frac{\hat{f}_i(0)}{\hat{f}_k(0)} = (1 + \Delta_{1,k,m})mz_1^k$ and $\frac{\hat{f}_{k+1}(0)\hat{f}_k(0)}{\hat{f}_{k+1}(0)\hat{f}_k(0)} = (1 + \gamma_{1,k,m})z_1$ where

$$1 + \gamma_{1,k,m} = \frac{1 + \Delta_{1,k+1,m}}{1 + \Delta_{1,k,m}}, \quad |\Delta_{1,i,m}| \leq \frac{w - m}{m} |\frac{z_{m+1}}{z_1}|^i, \quad i = k, k+1 \quad (27)$$

if

$$|z_1| = |z_2| = \cdots = |z_m| > |z_{m+1}| = \max_{j > m} \frac{z_j}{|z_j|}. \quad (28)$$

**Corollary 3.** Under the assumptions of Cor. 2

(i) let (27) hold and let $\Delta_{w,m} \geq \max \frac{z_{w-m}}{z_w}$. Then

$$\gamma_{w,k,m} \leq \frac{2 \Delta_{w,m}^k}{1 - \Delta_{w,m}^k}, \quad (29)$$

$$\gamma_{w,k,m} \leq \epsilon = 1/2b$$ for $k \geq (b + 2) \log_{\Delta_{w,m}} \Delta_{w,m}^k$ if $2 \Delta_{w,m}^k \leq 1. \quad (30)$$

Likewise, (ii) let (28) hold and let $\Delta_{1,m} \geq \max \frac{z_{m+1}}{z_1}$. Then

$$\gamma_{1,k,m} \leq \frac{2 \Delta_{1,m}^k}{1 - \Delta_{1,m}^k}, \quad (31)$$

$$\gamma_{1,k,m} \leq \epsilon = 1/2b$$ for $k \geq (b + 2) \log_{\Delta_{1,m}} \Delta_{1,m}^k$ if $2 \Delta_{1,m}^k \leq 1. \quad (32)$$

**Proof.** First deduce bound (29) from (20); from this obtain that $\gamma_{w,k,m} \leq 4 \Delta_{w,m}^k$ if $2 \Delta_{w,m}^k \leq 1$. Now prove claim (i) by taking binary logarithms on both sides. Similarly prove claim (ii).
5 Approximation in Classic Area: Polynomial Root-squaring and Root-finding

5.1 An overall error bound for approximation of an extremal zero of a black box polynomial

Combine (41) with bounds of Cor. 3 for simplicity assuming that $f(x) = p(x)$ and $v = d$ (extension to $f(x)$ being a factor of $p(x)$ is a straightforward exercise), and obtain

**Theorem 7.** Given two tolerance values $\epsilon = 1/2^b$ and $\epsilon_0 = 1/2^b_0$ and the bounds $\Delta_{d,m}$ and $\Delta_{1,m}$ of Cor. 3 (see Sec. 2.3). Choose integer $k$ to ensure an upper bound $\epsilon = 1/2^b$ on the approximation error of $x_w$ and $x_1$ by the power sums of the zeros of $p(x)$ according to Cor. 3. Compute approximations $s_k'$ and $s_{k+1}'$, to the power sums $s_k$ and $s_{k+1}$ of the zeros of $p(x)$ as well as approximations $s_k'$ and $s_{k+1}'$ to the power sums $s_k$ and $s_{k+1}$ of the zeros of the reverse polynomial $p_{rev}(x)$, respectively, in all cases within a fixed tolerance bound $\epsilon_0$. Then

\[
\left| \frac{s_{k+1}'}{s_k'} - x_d \right| \leq \epsilon + \epsilon_0 \text{ and } \left| \frac{s_{k+1}'}{s_k} - x_1 \right| \leq \epsilon + \epsilon_0.
\]

5.2 Newton’s and Cauchy’s approximation of the power sums

Next we approximate the power sums of the zeros and the reciprocals of the zeros of a polynomial by expressing them first via Newton’s identities and then via Cauchy integrals. Newton’s identities only enable us to compute the power sums for $p(x)$ itself and involve a set of its leading or trailing coefficients. With Cauchy integrals we scale the variable and increase arithmetic cost to avoid numerical stability problems but express the power sums for a black box polynomial $p(x)$ as well as for its factors via the values of Newton’s inverse ratio $-p'(x)/p(x)$ for a black box polynomial $p(x)$, leading to significant computational benefits.

5.3 Newton’s Power Sums Computation

Given the $k + 2$ trailing coefficients $p_0 = 1, p_1, \ldots, p_{k+1}$ of a polynomial $p(x)$, with $p(0) = 1$, which are the $k + 2$ leading coefficients $p'_d, p'_d-1, \ldots, p'_d-k-1$ of the monic reverse polynomial $p_{rev}(x)$, we can compute the $k + 1$ power sums $s'_1, \ldots, s'_{k+1}$ of $p_{rev}(x)$ by solving the triangular Toeplitz linear system of $k + 1$ Newton’s identities, which is equivalent to computing the reciprocal of $p_{rev}(x)$ modulo $x^{k+2}$ [28 Sec. 2.5]:

\[
\begin{align*}
  s'_1 + p'_d &= 0, \\
  p'_d-1 s'_1 + s'_2 + 2p'_d-2 &= 0, \\
  p'_d-2 s'_1 + p'_d-1 s'_2 + s'_3 + 3p'_d-3 &= 0, \\
  &\vdots \\
  s'_1 + \sum_{j=1}^{i-1} p'_d-j s'_{1-j} &= -ip'_d-i, \quad i = 1, \ldots, k + 1.
\end{align*}
\]
We refer to this classic algorithm as Newton’s Power Sums.

**Arithmetic complexity.** We can solve linear system (33) at arithmetic cost of $2(k+1)^2$ by using back substitution but can decrease it to $O(k \log(k))$ with FFT [28, Sec. 2.6].

Given $k+2$ leading coefficients of $p(x)$ we can similarly compute the $k+1$ power sums $s_1, \ldots, s_{k+1}$ of $p(x)$.

### 5.4 Cauchy integral algorithms

#### 1. Integral formula. For a non-negative integer $h$, express the $h$th power sum of the $w$ roots of a polynomial $p(x)$ lying in a domain $\mathcal{D}$ on the complex plane as a Cauchy integral over a boundary $\mathcal{C}$ of this domain:

$$s_h := \sum_{j=1}^{w} x_j^h = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{p'(x)}{p(x)} x^h \, dx. \quad (34)$$

$s_h$ is the power sum of all $d$ zero of $p(x)$ if all of them lie in the domain $\mathcal{D}$.

To express the power sums of the reciprocals of the zeros of $p(x)$, substitute $p_{rev}(x)$ for $p(x)$ into (34).

#### 2. Algorithmic options. We must assume some isolation of the boundary contour $\mathcal{C}$ from the zeros of $p(x)$ for otherwise the output errors of the known algorithms for approximation of the integral (34) are unbounded. Next we recall some algorithmic options.

(i) It is attractive to approximate integral (34) by applying trapezoid rule (see its exponentially convergent adaptive version in the BOOST library) modified for arbitrary precision, although this does not support reduction modulo $x^q - 1$, which would simplify computations in the case where $q \ll d$ and a polynomial $p(x)$ is given with its coefficients.

(ii) Kirrinnis in [13] approximates Cauchy integrals over various smooth contours $\mathcal{C}$.

(iii) By following [35] choose $\mathcal{C} = C(0, 1)$ and approximate integral (34) with finite Cauchy sums $s_{h,q}$ defined as follows.

**Definition 1.** For a polynomial $p(x)$, two integers $q$ and $h$ such that $0 \leq h < q$, and $\zeta_q := \exp \left( \frac{2\pi i}{q} \right)$ of (14), define Cauchy sum

$$s_{h,q} := \frac{1}{q} \sum_{g=0}^{q-1} \zeta_q^{(h+1)g} \frac{p'\left(\zeta_q^g\right)}{p\left(\zeta_q^g\right)}. \quad (35)$$

We proved in [31] and recalled in [11] the following result.

**Theorem 8.** For a polynomial $p(x)$ of (14) and a positive integer $q$, the Cauchy sum $s_{h,q}$ of (35) satisfies $s_{h,q} = \sum_{j=1}^{d} x_j^h \frac{x_j^q}{1-x_j^q}$ for $h = 0, 1, \ldots, q-1$ unless $x_j^q = 1$ for some integer $j$, $1 \leq j \leq d$.

---

This was suggested to us by Oren Bassik of the PhD program in Mathematics of the Graduate Center of CUNY.
extensively applies Cauchy sums $s(h, q)$ to root-finding in the case of small integers $h$. Next we recall some relevant results and algorithms.

6 Cauchy approximation of the power sums of the zeros of $p(x)$ lying in the unit disc $D(0, 1)$

6.1 Computation of Cauchy sums

Algorithm 9 [Cauchy sum computation.]

**INPUT:** $p(x)$, $h$ and $q$ of Def. 1 such that $\prod_{g=0}^{q-1} p(\zeta^g) \neq 0$.

**OUTPUT:** the Cauchy sum $s_{h, q}$ of (35).

**COMPUTATIONS:**
1. Compute the values $r_g := \frac{p'(\zeta^g)}{p(\zeta^g)}$ for $g = 0, 1, \ldots, q-1$.
2. Compute and output $s_{h, q} := \frac{1}{q} \sum_{g=0}^{q-1} \zeta^{(h+1)g} r_g$.

1. **Arithmetic cost.** Alg. 9 evaluates the ratio $\frac{p'(x)}{p(x)}$ at $q$ points at stage 1 and performs $2q$ arithmetic operations at stage 2.

   Given the coefficients of $p(x)$ and $q < d$ we perform stage 1 faster: (i) reduce the polynomials $p(x)$ and $p'(x)$ modulo $x^q - 1$ at the cost of performing $2d - 2q + 1$ subtractions, (ii) evaluate the two resulting polynomials at the $q$th roots of unity, that is, perform discrete Fourier transform (DFT) by using $O(q \log(q))$ arithmetic operations, and (iii) compute the $q$ ratios of the $q$ pairs of the values output by the two DFTs at the $q$th roots of unity.

2. **Approximation errors.**

   We can readily deduce from Thm. 8 the following estimates of 35.

   **Theorem 10.** For a polynomial $p(x)$, two integers $h$ and $q$, $0 \leq h < q$, and $\theta > 1$, let the annulus $\{x : \frac{1}{\theta} \leq |x| \leq \theta\}$ contain no zeros of $p(x)$. Then

   $$|s_{h, q} - s_h| \leq \frac{d \theta^h}{\theta^q - 1} \text{ for } h = 0, 1, \ldots, q - 1,$$

   and so

   $$|s_{h, q} - s_h| \leq \epsilon_0 = \frac{1}{2^{b_0}}$$

   for a fixed positive $\epsilon_0$ if

   $$q - h \geq \log_\theta \left(1 + \frac{d}{\epsilon_0}\right) = \log_\theta(1 + b_0 d) \text{ for } h = 0, 1, \ldots, q - 1.$$

   **Remark 3.** 35 is an upper bound on the value $|s_{h, q} - s_h|$; it can be pessimistic. For a heuristic recipe towards its decrease we can rotate the disc $D(0, 1)$ by fixed or random angles and try to deduce from Thm. 8 and/or test empirically whether this can help decrease approximation error.
6.2 Cauchy sums in any disc

Taylor’s shift with scaling enables us to reduce the computation of Cauchy sums $s_{h,q}$ in any disc $D(c, \rho)$ to the case of the unit disc $D(0, 1)$.

**Definition 2.** Apply Taylor’s shifts with scaling (15) to define the Cauchy sums for a polynomial $p$, a positive integer $q$, and the disc $D(c, \rho)$ as follows:

$$s_{h,q}(p, c, \rho) := \frac{p}{q} \sum_{g=0}^{q-1} \zeta^{(h+1)g} \rho^g \frac{p'(c + \rho \zeta^g)}{p(c + \rho \zeta^g)} \text{ for } \zeta \text{ of (15)}$$

and $h = 0, 1, \ldots, q - 1$,

that is, $s_{0,q}(p, c, \rho)$ is the Cauchy sums $s_{0,q}(t, 0, 1)$ for the positive integer $q$, the polynomial $t(y) = p\left(\frac{y - c}{\rho}\right)$, and the unit disc $D(0, 1)$.

Clearly, Taylor’s shift with scaling (15) preserves isolation of a disc but changes the zeros of $p(x)$, their power sums, and Cauchy sums. One can, however, apply transformation $x \mapsto y$ of (15) to map a disc $D(c, \rho)$ into the unit disc $D(0, 1)$, approximate the zeros $y_j$ of $t(y)$ lying in that disc, and then apply converse map $y \mapsto x = c + py$ to approximate the zeros $x_j$ of $p(x)$.

To estimate the errors of the approximation of the power sums (39) where $D = D(c, \rho)$, combine Thm. (10) with equations $x_j = c + \rho y_j$ where $y_j$ denote the zeros of $t(y) = p\left(\frac{y - c}{\rho}\right)$ in the unit disc $D(0, 1)$.

For discs $D(0, \rho)$, centered at the origin, extension of bound (38) is particularly simple: $|s_{h,q} - s_h| \leq \epsilon_0 = 1/2^b_0$ if

$$q - h \geq \log_\theta \left(1 + \frac{d_\theta}{\epsilon_0}\right) \text{ for } h = 0, 1, \ldots, q - 1,$$

and then we immediately extend bound (38) by just replacing $b_0 d$ with $\rho b_0 d$, that is, the bound $|s_{h,q} - s_h| < 1/2^b_0$ of (39) is ensured if

$$q - h \geq \log_2(\rho b_0 d).$$

6.3 Numerical stability problems and a remedy

In the case where the unit disc is isolated and an integer $h$ is large, computation of Cauchy sums $s_{h,q}$ is prone to numerical stability problems, aggravated as $h$ grows larger. Indeed, all the zeros of $p(x)$ lying in the isolated disc $D(0, 1)$ are noticeably exceeded by 1. Therefore, the Cauchy sums and the ratios $|s_{h,q}|/\max_\zeta \frac{p'(\zeta^g)}{p(\zeta^g)}$ converge to 0 exponentially fast as $h$ grows. Already for moderately large integers $h$, the terms $\frac{p'(\zeta^g)}{p(\zeta^g)}$ are nearly annihilated in the computation of $s_{h,q}$ according to (35), and this means numerical stability problems.

A remedy by means of scaling the variable. Given an exponent $h$ and target error tolerance $\epsilon_0 = 1/2^b_0$, scale the variable $x \leftarrow \rho y$ and choose isolation $\theta > 1$ such that $\theta^h = 2$, say. Then choose

$$\theta^h = 2 \text{ and } q \geq h \log_2(1 + d_\theta 2^{b_0 + 1}).$$
Substitute equation $\theta^h = 2$ into (36) and deduce that $|s_{h,q} - s_h| \leq \frac{2d}{q^2 - 1} \leq \frac{1}{2q^2} = \epsilon_0$.

This bound on $q$ tends to be fairly large unless $h$ and/or $b_0$ are nicely bounded, which can be the case if, say, the value $|z_w/z_{w-m}|$ in Cor. $\text{3}$ is small.

7 Approximation of a sequence of the zeros of $p(x)$ with implicit deflation

Let $g_0(x)$ denote a polynomial $g(x)$ whose zero set $\{x_j, j = 1, 2, \ldots, w\}$ is the set of the zeros of $p(x)$ lying in the unit disc $D(0, 1)$; in particular, $g_0(x) = p(x)$ if $w = d$.

Suppose that we have computed, e.g., by applying the algorithms of the previous sections, an approximation $z_1$ to a zero $x_d$ of $g_0(x)$ in $D(0, 1)$ such that $|z_1 - x_d| \leq \epsilon |z_1|$ for a fixed tolerance $\epsilon < 1/4$, say$^6$.

We can deflate the factor $x - z_1$ by approximating the quotient $p(x)/(x - z_1)$ with a $(d - 1)$st degree polynomial and apply to it the same root-finder. Recursively we can approximate all $w$ zeros of $p(x)$ lying in the disc $D(0, 1)$.

Such deflation destroys sparsity of $p(x)$ and blows up its overall coefficient length (e.g., consider $p(x) = x^d + 1$), but we can avoid these problems by applying implicit deflation where we only compute the values of the ratios

$$
\frac{f'(x)}{f(x)} \quad \text{for} \quad f(x) = \frac{p(x)}{g(x)}, \quad g(x) = \prod_{j=1}^{d}(x - z_j),
$$

(43)

for $z_j$ denoting the computed approximations to $x_{d-j+1}$, $j = 1, \ldots, m$, but never compute polynomial coefficients.

Dealing with black box polynomial root-finders we can reduce computation of Newton’s inverse ratio $\frac{f'(x)}{f(x)}$ for $f(x)$ at some set of points $x$ to the same task for $p(x)$ because of the following simple observation.

Theorem 11. Under $\text{[43]}$ it holds that

$$
\frac{f'(x)}{f(x)} = \frac{p'(x)}{p(x)} - \sum_{j=1}^{k} \frac{1}{x - z_j}.
$$

If we can reuse the same points $x$ for approximation of the zeros $z_j$ for all $j$, e.g., if we evaluate Newton’s inverse ratio at the $q$th roots of unity for a fixed integer $q$, we would only need to recursively subtract from the values $\frac{p'(x)}{p(x)}$ the reciprocals $\frac{1}{x-z_j}$, each time performing just a single division (by $\frac{1}{x-z_j}$) and two subtractions.

$^6$ Quite typically, we can very fast refine approximation of the zeros $x_1, x_2, \ldots$ by means of Newton’s or Schröder’s iterations; alternatively, we can apply the algorithms of this paper.
In the particular case where a root-finder is reduced to power sum computation based on Thm 7, we can merely subtract the powers of computed roots from the computed power sums, thus performing a single subtraction instead of two subtractions and division.

This would work if we approximate power sums with Cauchy sums where pairwise ratios of root radii are never large, but would not work if these ratios vary much because in that case we should apply scaling to avoid numerical stability problems and cannot reuse evaluation points.

Based on recursive application of Thm. 7 we can output successive approximations to the \( j \)th absolutely smallest or absolutely largest zeros of \( p(x) \) for \( j = 1, 2, \ldots \), but can similarly peel off recursively the zeros closest to a fixed complex point \( c \) by applying implicit deflation to the polynomial \( t_c(x) = p(x - c) \).

We can alternatively apply this recipe concurrently to reverse polynomials of \( t_{c_h}(x) = p(x - c_h) \) for \( h = 1, 2, \ldots \) to approximate a set of zeros of \( p(x) \) closest to the points \( c_1, c_1, \ldots \).

It is not simple to estimate the integer parameters \( h \) and \( q \) at all steps of recursive deflation a priori, but we can estimate them by action provided that their growth does not require to exceed fixed tolerance bounds on the computational precision and arithmetic cost.

8 Some benefits of application of black box polynomial root-finders

Black box polynomial root-finders avoid numerical stability problems caused by Taylor’s shifts and scaling, which are basic tools of various popular root-finders, e.g., subdivision root-finders. This implies first two important benefits listed below. The algorithms have further benefits because they amount to the evaluation of Newton’s inverse ratio \( \frac{p'(x)}{p(x)} \) at sufficiently many points.

1. **Relaxation of assumption** (26). Assumptions (26) and even (5) hold with probability 1 for a polynomial \( t_{c,\rho}(x) \) replacing \( p(x) \) if all zeros \( x_j \) of \( p(x) \) are distinct and if \( c \) is a random variable sampled from any fixed disc \( D \), say, \( D = D(0, 1) \), under the uniform or standard Gaussian normal probability distribution on that disc.

2. **Concurrent root-finding on \( m \) processors** by means of simultaneously computing the DLG or FG sequences for the polynomials \( t_{c_i,\rho_i}(x) \) for fixed or random sets \( c_i \) and \( \rho_i, i = 1, \ldots, m \).

3. **Extension of approximation of a single zero \( x_d \) to recursive approximation of \( k \) absolutely largest or smallest zeros** for a fixed \( k > 1 \) (that is, of all zeros for \( k = d \)) by means of implicit deflation (see Sec. 7).

4. **Acceleration** where \( p(x) \) can be evaluated fast, e.g., is a sparse, shifted sparse, or Mandelbrot’s polynomial, and

5. **Numerically stable evaluation** of polynomial \( p(x) \) given in Bernstein’s or Chebyshev’s bases [31, Sec. 1.2].
Our black box algorithms can be readily extended to *eigen-solving for a matrix or a polynomial matrix*, and this is efficient where a matrix or a polynomial matrix can be inverted fast, e.g., is quasiseparable or has small displacement rank. This extension relies on the equations

$$
\frac{t'(x)}{t(x)} = \text{trace}((xI - T)^{-1})
$$

where $T$ is a matrix with characteristic polynomial $t(x) = \text{det}(xI - T)$ (implied by Eqn. (17)) and

$$
\frac{t'(x)}{t(x)} = \text{trace}(T^{-1}(x)T'(x))
$$

where $T(x)$ is a matrix polynomial and $t(x) = \text{det}(T(x))$ [1] Eqn. (5). Thus our algorithms approximate the eigenvalues of the matrix $T$ or matrix polynomial $T(x)$, which are the zeros of the characteristic polynomial of $T$ or $T(x)$, without computing the coefficients of that polynomial.

### 9 Estimation of extremal root radii

Suppose that we do not apply the known algorithms and just rely on the following well-known bounds on the extremal root radii:

$$
|x_d| \leq d \left| \frac{p(0)}{p'(0)} \right| \quad \text{and} \quad |x_1| \geq \left| \frac{p'_{\text{rev}}(0)}{d_{\text{rev}}(0)} \right|
$$

By extending these bounds to the polynomial $p_k(x)$ of Eqn. (2) we obtain that

$$
|x_d|^{2^k} \leq d \left| \left( \frac{p_k(0)}{p_k'(0)} \right) \right| \quad \text{and} \quad |x_1|^{2^k} \geq \frac{1}{d} \left| \left( \frac{p_{k,\text{rev}}(0)}{p_{k,\text{rev}}'(0)} \right) \right|
$$

Under bounds (26) and (28) these estimates become sharp as $k$ increases, by virtue of Thm. 6; next we argue informally that they themselves tend to be sharp with a high probability under random root models. Indeed,

$$
\frac{1}{|x_d|} \leq \frac{1}{d} \left| \frac{p'(c)}{p(c)} \right| = \frac{1}{d} \sum_{j=1}^{d} \frac{1}{c - x_j}
$$

by virtue of (44), and so the approximation to the root radius $|x_d|$ is poor if and only if severe cancellation occurs in the summation of the $d$ zeros of $p(x)$, and similarly for the approximation of $r_1(c, p)$. Such a cancellation only occurs for a narrow class of polynomials $p(x)$ or, formally, with a low probability under random root models.

Next we prove, however, that estimates (44) and (45) are extremely poor for worst case inputs.

**Theorem 12.** The ratios $|\frac{p(0)}{p'(0)}|$ and $|\frac{p_{\text{rev}}(0)}{p_{\text{rev}}'(0)}|$ are infinite for $p(x) = x^d - h^d$ and $h \neq 0$, while $|x_d| = |x_1| = |h|$.

**Proof.** Observe that the zeros $x_j = h \exp\left(\frac{(j - 1)\sqrt{-1}}{2e}d\right)$ of $p(x) = x^d - h^d$ for $j = 1, 2, \ldots, d$ are the $d$th roots of unity up to scaling by $h$. 

Clearly, the problem persists for the root radius \( r_d(w, p) \) where \( p'(w) \) and \( p'_{rev}(w) \) vanish; rotation of the variable \( p(x) \leftarrow t(x) = p(ax) \) for \( |a| = 1 \) does not fix it but shifts \( p(x) \leftarrow t(x) = p(x - c) \) for \( c \neq 0 \) can fix it, thus enhancing the power of estimates (44) and (45).

10 Conclusions

The classical DLG root-squaring iterations as well as more recent FG iterations of 2001 have been considered purely theoretical constructs whose application to root-finding has been blocked by severe problems of numerical stability.

By combining Vieta’s formulae with well-known equation (17) for a polynomial \( p(x) \) we linked DLG iterations with approximation of the power sums of the zeros of \( p(x) \) and of its reverse polynomial and then further observed that we can ignore DLG techniques and instead directly reduce root-finding to approximation of the high power sums.

These computations can be performed for a black box polynomial \( p(x) \) given by a subroutine for the evaluation of Newton’s inverse ratio \( \frac{-p'(x)}{p(x)} \) at reasonably many points, which leads to a number of additional important benefits, listed in Sec. 8.

We supply some details for approximation of the power sums by means of specific discretization of Cauchy integrals representing these sums, which can be alternatively approximated based on the exponentially convergent adaptive version of trapezoid rule in the BOOST library modified for arbitrary precision.

We comment on numerical stabilization of these computations based on scaling the variable and supply some relevant estimates.

Alternative computation of the power sums based on Newton’s identities is quite promising but can only be applied where sufficiently many leading or trailing coefficients of \( p(x) \) or of its factor are available.

The number of required coefficients decreases where we approximate an extremal zero of \( p(x) \) whose absolute value is not close to the absolute values of any other zero. In that case approximation of Cauchy integrals representing the power sums can also be performed at a lower cost.

Further formal and empirical study should help decide which of variations of these algorithms are more efficient and for which input classes.

In Secs. 1.6 and 1.7 we have already listed some tentative research directions and challenges.

In Sec. 8 we specify some simple but promising applications of our fast computation of root radii for a black box polynomial to a variation of Lehmer’s root-finder and black box initialization of polynomial root-finding by means of functional iterations.

The algorithm of [8] for DLG iterations seems to run faster than the other alternatives, and one should test if this indeed so, and also should test its extension of Sec. 1.3 to approximation of complex zeros of \( p(x) \) by means of descending process. Alternatively, one can try to extend [8] to FG iterations.
Acknowledgements: This research has been supported by NSF Grants CCF–1563942 and CCF–1733834 and by PSC CUNY Award 63677-00-51.

References

1. Bini, D.A., Noferini, V.: Solving polynomial eigenvalue problems by means of the Ehrlich-Aberth method. Linear Algebra and its Applications, 439(4), 1130-1149 (2013) DOI:10.1016/J.LAA.2013.02.024
2. Bini, D., Pan, V. Y.: Graeffe’s, Chebyshev–like, and Cardinal’s Processes for Splitting a Polynomial into Factors, J. of Complexity, 12, 492–511 (1996)
3. Baur, W., Strassen, V.: On the Complexity of Partial Derivatives. Theoretical Computer Science 22, 317–330 (1983)
4. Becker, R., Sagraloff, M., Sharma, V., Yap, C.: A near-optimal subdivision algorithm for complex root isolation based on the Pellet test and Newton iteration. J. Symbolic Computation 86, 51–96 (2018) doi: 10.1016/j.jsc.2017.03.009
5. Fiedler, M.: Über das Gräffesche verfahren. Czechoslovak Math. J., 5(80): 506-516 (1955)
6. Gemignani, L.: A generalized Graeffe’s iteration for evaluating polynomials and rational functions, ISSAC’01, 143–149 (2001)
7. Grenet, B., van der Hoeven, J., Lecerf,G.: Deterministic root-finding over finite fields using Graeffe transforms. Applicable Algebra in Engineering, Communication and Computing 27, 237–257 (2015)
8. Go, S., Soto, P.: New Progress in Classic Area: Polynomial Root-squaring and Root-finding II, preprint (2022)
9. Householder, A.S.: Dandelin, Lobachevskii, or Graeffe? Amer. Math. Monthly 66, 464–466 (1959) doi: 10.2307/2310626
10. Henrici, P.: Applied and Computational Complex Analysis. Vol. 1: Power Series, Integration, Conformal Mapping, Location of Zeros. Wiley, NY (1974)
11. Imbach, R., Pan, V.Y.: New progress in univariate polynomial root-finding, In: Procs. of ACM-SIGSAM ISSAC 2020, pages 249 – 256, July 20-23, 2020, Kalamata, Greece, ACM Press (2020). ACM ISBN 978-1-4503-7100-1/20/07
12. Imbach, R., Pan, V.Y.: Root radii and subdivision for polynomial root-finding, In: Computer Algebra in Scientific Computing (CASC’21), Springer Nature Switzerland AG 2021, F. Boulier et al. (Eds.): LNCS 12865, 1—21 (2021) Also arXiv:2102.10821 (22 Feb 2021)
13. Kirrinnis, P.: Fast computation of contour integrals of rational functions. Journal of Complexity 16, 181–212 (2000)
14. Linnaeus, S.: Taylor expansion of the accumulated rounding errors. BIT 16, 146–160 (1976)
15. Louis, A., Vempala, S. S.: Accelerated Newton iteration: roots of black box polynomials and matrix eigenvalues. 2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS), Volume: 1, Pages: 732-740 (2016) DOI Bookmark: 10.1109/FOCS.2016.83. Also arXiv:1511.03186 January 2016
16. Luan, Q., Pan, V. Y., Kim, W., Zaderman, V., Faster Numerical Univariate Polynomial Root-Finding by Means of Subdivision Iterations, In: Computer Algebra in Scientific Computing, Springer Nature Switzerland AG 2020, F. Boulier et al. (Eds.): LNCS 12291, 431-446 (2020)
17. McNamee, J.M.: Numerical Methods for Roots of Polynomials, Part I, XIX+354 pages. Elsevier (2007) ISBN: 044452729X; ISBN13: 9780444527295
18. Merzbach, U.C., Boyer, C.B.: A History of Mathematics. Wiley, New York (2011), fifth edition; doi: 10.1177/027046769201200316
19. Mourrain, B., Pan, V. Y.: Lifting/Descending Processes for Polynomial Zeros and Applications, J. of Complexity, 16, 1, 265–273 (2000)
20. McNamee, J.M., Pan, V. Y.: Numerical Methods for Roots of Polynomials, Part 2 (XXII + 718 pages), Elsevier (2013).
21. Mignotte, M., Stefanescu, D.: Polynomials: An Algorithmic Approach, Springer (1999)
22. Malajovich, G., Zubelli, J.P.: On the geometry of Graeffe iteration. Journal of Complexity 17(3), 541–573 (2001); DOI: 10.1006/jcom.2001.0585
23. Pan, V. Y.: Sequential and parallel complexity of approximate evaluation of polynomial zeros. Computers and Mathematics (with Applications), 14, 8, 591–622 (1987)
24. Pan, V. Y.: Fast and efficient parallel evaluation of the zeros of a polynomial having only real zeros. Computers and Mathematics (with Applications), 17, 11, 1475–1480 (1989)
25. Pan, V.Y.: Optimal and nearly optimal algorithms for approximating polynomial zeros. Computers and Mathematics (with Applications), 31, 12, 97–138 (1996).
26. Pan, V.Y.: Solving a polynomial equation: Some history and recent progress. SIAM Review 39(2), 187–220 (1997) doi: 10.1137/S0036144595288554
27. Pan, V.Y.: Approximation of complex polynomial zeros: modified quadtree (Weyl’s) construction and improved Newton’s iteration. J. Complexity 16(1), 213–264 (2000) doi: 10.1006/jcom.1999.
28. Pan, V.Y.: Structured Matrices and Polynomials: Unified Superfast Algorithms. Birkhäuser/Springer, Boston/New York (2001) doi: 10.1007/978-1-4612-0129-8
29. Pan, V.Y.: Simple and nearly optimal polynomial root-finding by means of root radii approximation. In: Kotsireas, I.S., Martinez-Moro, E. (eds.) Springer Proc. in Math. and Statistics, Ch. 23: Applications of Computer Algebra 198 AG. Springer International Publishing (2017). Chapter doi: 10.1007/978-3-319-56932-1
30. Pan, V.Y.: Acceleration of subdivision root-finding for sparse polynomials. In: Computer Algebra in Scientific Computing, Springer Nature Switzerland AG 2020, F. Boulier et al. (Eds.): LNCS 12291, 461–477 (2020)
31. Pan, V.Y.: New progress in polynomial root-finding. arXiv 1805.12042, last revised in 2022.
32. Pan, V.Y., Zhao, L.: Real root isolation by means of root radii approximation. In: Proceedings of the 17th International Workshop on Computer Algebra in Scientific Computing (CASC’2015), (V. P. Gerdt, W. Koepf, and E. V. Vorozhtsov, editors), Lecture Notes in Computer Science 9301, 347–358, Springer, Heidelberg, (2015) doi: 10.1007/978-3-319-24021-3 26
33. Renegar, J.: On the worst-case arithmetic complexity of approximating zeros of polynomials. J. Complexity. 3(2), 90–113 (1987) doi: 10.1016/0885-064X(87)90022-7
34. Reineke, B.: Diverging orbits for the Ehrlich-Aberth and the Weierstrass root finders. arXiv 2011.01660 (November 20, 2020)
35. Schönhage, A.: The fundamental theorem of algebra in terms of computational complexity. Math. Dept., University of Tübingen, Tübingen, Germany (1982)
36. Yap, C.-K.: Fundamental Problems of Algorithmic Algebra. Oxford University Press (2000). ISBN 978-0195125160