ON SOLVING ILL-CONDITIONED LINEAR SYSTEMS

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ABSTRACT. This paper presents the first results to combine two theoretically sound methods (spectral projection and multigrid methods) together to attack ill-conditioned linear systems. Our preliminary results show that the proposed algorithm applied to a Krylov subspace method takes much fewer iterations for solving an ill-conditioned problem downloaded from a popular online sparse matrix collection.

1. INTRODUCTION

It is well-known that robustness and efficiency of iterative methods are affected by the condition number of a linear system. When a linear system has a large condition number, usually due to eigenvalues that are close to the origin of the spectrum domain, iterative methods tend to take many iterations before a convergence criterion is satisfied. Sometimes, iterative methods will fail to converge within a reasonable computer elapsed time, or even do not converge at all, if the condition number is too large. Unstable linear systems, or systems with large condition numbers, are called ill-conditioned. For an ill-conditioned linear system, slight changes in the coefficient matrix or the right-hand-side cause large changes in the solution. Typically, roundoff error in the computer arithmetics can cause instability when attempts are made to solve an ill-conditioned system either directly or iteratively on a computer.

It is widely recognized that linear systems resulting from discretizing ill-posed integral equations of the first kind are highly ill-conditioned. This is because the eigenvalues for the first kind integral equations with continuous or weakly singular kernels have an accumulation point at zero. Integral equations of the first kind are frequently seen in statistics, such as unbiased estimation, estimating a prior distribution on a parameter given the marginal distribution of the data and the likelihood, and similar tests for normal theory problems. They also arise from indirect measurements and nondestructive testing in inverse problems. Other ill-conditioned linear systems can be seen in training of neural networks, seismic analysis, Cauchy problem for parabolic equations, and multiphase flow of chemicals. For pertinent references of ill-conditioned linear systems, one can see, for example, Engl [16] and Groetsch [21].

Solving these ill-conditioned linear algebra problems becomes a long-standing bottleneck for advancing the use of iterative methods. The convergence of iterative methods for ill-conditioned problems, however, can be improved by using preconditioning. Development of preconditioning techniques is therefore a very active research area. A preconditioning strategy that deflates few isolated external eigenvalues was first introduced by Nicolaides [26], and investigated by several others [19,25,35,38]. The deflation strategy is an action that removes the influence of a subspace of the eigenspace on the iterative process. A common way to deflate an eigenspace is to construct a proper projector $P$ as a preconditioner and solve

$$PAx = Pb, \quad P, A \in \mathbb{C}^{N \times N}. \quad (1)$$

The deflation projector $P$, orthogonal the matrix $A$ and the vector $b$ against some subspace, is defined by

$$P = I - AZ(Z^HAZ)^{-1}Z^H, \quad Z \in \mathbb{C}^{N \times m}, \quad (2)$$
where $Z$ is a matrix of deflation subspace, i.e., the space to be projected out of the residual, and $I$ is the identity matrix of appropriate size $[19, 29]$. We assume that (1) $m \ll N$ and (2) $Z$ has rank $m$. A deflated $N \times N$ system (1) has an eigensystem different from that of $Ax = b$. Suppose that $A$ is diagonalizable, and set $Z = [v_1, \ldots, v_m]$, whose columns are eigenvectors of $A$ associated with eigenvalues $\lambda_1, \ldots, \lambda_m$. Then the spectrum $\sigma(PA)$ would contain the same eigenvalues of $A$, except $\lambda_1, \ldots, \lambda_m$. Usually, eigenvectors are not easily available. This motivates us to develop an efficient and robust algorithm for finding an approximate deflation subspace, other than using the exact eigenvectors to construct the deflation projector $P$.

Suppose that we want to deflate a set of eigenvalues of $A$ enclosed in a circle $\Gamma$ that is centered at the origin with the radius $r$. Without loss of generality, let this set of eigenvalues be \{\lambda_1, \cdots, \lambda_k\}. Let the subspace spanned by the corresponding eigenvectors of \{\lambda_1, \cdots, \lambda_k\} be $Z_k = \text{Span}\{v_1, \cdots, v_k\}$. Then the deflation subspace matrix $Z$ in (2) obtained by randomly selecting $m$ vectors from $Z_k$ can be written as a contour integral $[30]$

$$Z = \frac{1}{2\pi\sqrt{-1}} \oint_{\Gamma} (zI - A)^{-1}Y \, dz,$$

where $Y$ is a random matrix of size $N \times m$. If the above contour integral is approximated by a Gaussian quadrature, we have

$$Z = \sum_{i=1}^{q} \omega_i (z_i I - A)^{-1}Y,$$  \hspace{1cm} (4)

where $\omega_i$ are the weights, $z_i$ are the Gaussian points, and $q$ is the number of Gaussian points on $\Gamma$ for the quadrature. It is worth noting that (4) is required to solve $q$ shifted linear systems $(z_i I - A)X = Y$, $i = 1, \cdots, q$. Using (4) for the deflation projector $P$ in (2), the preconditioned linear system (1) is no longer severely ill-conditioned.

We remark that the construction of a deflation subspace matrix $Z$ through (4) is motivated by the works in [27, 32, 33, 36].

2. Methodology

We consider the solution of the linear system

$$Ax = b$$  \hspace{1cm} (5)

by a Krylov subspace method, where we assume that $A \in \mathbb{C}^{N \times N}$ is nonsingular and $b \in \mathbb{C}^N$. Let an initial guess $x_0 \in \mathbb{C}^N$ be given along with its residual $r_0 = b - Ax_0$. A Krylov subspace method recursively constructs an approximate solution, $x_j$, such that

$$x_j \in x_0 + K_j(A, r_0) \equiv x_0 + \text{span}\{r_0, Ar_0, \ldots, A^{j-1}r_0\},$$

and its residual $r_j = b - Ax_j$ satisfies some desired conditions. It is well-known that the convergence rate of a Krylov subspace method depends on the eigenvalue distribution of the coefficient matrix $A$. A variety of error bounds on $r_j$ exist in the literature. Let us take GMRES [31] as an example.

2.1. GMRES. In GMRES, the residual $r_j$ is required to satisfy the condition

$$\|r_j\|_2 = \min_{\xi \in x_0 + K_j(A, r_0)} \|b - A\xi\|_2,$$

namely, the approximate solution $x_j$ obtained at iteration $j$ of GMRES is optimal in terms of residual norm. In the case where $A$ is diagonalizable, an upper bound on $\|r_j\|_2$ is provided by the following result.
Theorem 1. ([29, Corollary 6.33]) Suppose that \( A \) can be decomposed as
\[
A = V \Lambda V^{-1}
\]
with \( \Lambda \) being the diagonal matrix of eigenvalues. Let \( E(c, d, a) \) denote the ellipse in the complex plane with center \( c \), focal distance \( d \), and semi-major axis \( a \) (see Fig. 1(a)). If all the eigenvalues of \( A \) are located in \( E(c, d, a) \) that excludes the origin of the complex plane, then
\[
\|r_j\|_2 \leq \kappa_2(V) \frac{C_j(\frac{d}{a})}{|C_j(\frac{d}{a})|}\|r_0\|_2
\]
where \( \kappa_2(V) = \|V\|_2\|V^{-1}\|_2 \) and \( C_j \) is the Chebyshev polynomial of degree \( j \).

**Figure 1.** (a) A schematic ellipse in the complex plane with center \( c \), focal distance \( d \), and semi-major axis \( a \). (b) Eigenvalue distribution of the test matrix \( bcsstm27 \).

An explicit expression of \( \frac{C_j(\frac{d}{a})}{C_j(\frac{d}{a})} \) can be found on p.207 of [29], and under some additional assumptions on \( E(c, d, a) \) (say, the ellipse in Fig. 1(a))
\[
\frac{C_j(\frac{d}{a})}{C_j(\frac{d}{a})} \approx \left( \frac{a + \sqrt{a^2 - d^2}}{c + \sqrt{c^2 - d^2}} \right)^j \equiv \delta^j.
\]

The upper bound in (7) therefore contains two factors: the condition number \( \kappa_2(V) \) of the eigenvector matrix \( V \) and the scalar \( \delta \) determined by the distribution of the eigenvalues of \( A \). If \( A \) is nearly normal and has a spectrum \( \sigma(A) \) which is clustered around 1, we would have \( \kappa_2(V) \approx 1 \) and \( \delta < 1 \). In this case, \( \|r_j\|_2 \) decays exponentially in a rate of power \( \delta^j \), resulting in a fast convergence of GMRES. The error bound (7) does not tell the whole story, however, because the convergence rate can also be enhanced if the eigenvalues of \( A \) are clustered [37].

Since the ellipse \( E(c, d, a) \) in Theorem 1 is required to include all eigenvalues of \( A \), the outlying eigenvalues may keep the ellipse large, implying a large \( \delta \). To reduce \( \delta \), we therefore wish to remove these outlying eigenvalues from \( \sigma(A) \). Any procedure of doing so is known as deflation. GMRES in combination with deflation is called Deflated GMRES.
2.2. Deflated GMRES. Suppose $x^*$ is the exact solution of (5). Let a so-called deflation-subspace matrix $Z = [z_1, \ldots, z_m] \in \mathbb{C}^{N \times m}$ be given, whose columns are linearly independent. Define the two projectors \[ P \equiv I - AZ(Z^H AZ)^{-1}Z^H \quad \text{and} \quad \tilde{P} \equiv I - Z(Z^H AZ)^{-1}Z^H A, \] where $Z^H AZ$ is assumed to be invertible. It is straightforward to verify that $P^2 = P$, $\tilde{P}^2 = \tilde{P}$ and $PA = \tilde{P}A$.

Using $\tilde{P}$, we split $x^*$ into two parts:

\[ x^* = (I - \tilde{P})x^* + \tilde{P}x^* \equiv x^*_1 + x^*_2. \]

For $x^*_1$, we have

\[ x^*_1 = (I - \tilde{P})x^* = Z(Z^H AZ)^{-1}Z^H Ax^* = Z(Z^H AZ)^{-1}Z^H b. \]

For $x^*_2$, we obtain

\[ x^*_2 = A^{-1}Pb, \]

since $Ax^*_2 = A\tilde{P}x^* = PAx^* = Pb$. Now, if $x^\#$ is a solution of the singular system

\[ PAx = Pb, \] (10)

then

\[ A\tilde{P}x^\# = Pb \iff \tilde{P}x^\# = A^{-1}Pb = x^*_2. \]

Based on the above observation, a Deflated GMRES algorithm is given in Algorithm 1.

**Algorithm 1 Deflated GMRES**

Choose $Z$;

Compute $x^*_1 = Z(Z^H AZ)^{-1}Z^H b$;

Solve $PAx = Pb$ by GMRES to obtain a solution $x^\#$;

Compute $x^*_2 = \tilde{P}x^\#$;

Determine $x^* = x^*_1 + x^*_2$.

Assume that the nonsingular $A \in \mathbb{C}^{N \times N}$ has a decomposition (6) with $V = [v_1, \ldots, v_N]$ and $\Lambda = \text{diag}\{\lambda_1, \ldots, \lambda_N\}$. If we set $Z = [v_1, \ldots, v_m]$ in (9), then the spectrum $\sigma(PA)$ contains all the eigenvalues of $A$ except $\lambda_1, \ldots, \lambda_m$, namely, $\sigma(PA) = \{0, \cdots, 0, \lambda_{m+1}, \cdots, \lambda_N\}$.

Perform a $QR$ factorization on $V$ as follows:

\[ V = QR \equiv [Q_1, Q_2] \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}, \] (11)

where $Q_1 \in \mathbb{C}^{N \times m}$ and $R_{11} \in \mathbb{C}^{m \times m}$. If we set $Z = [v_1, \ldots, v_m]$ and apply GMRES to solve (10), an upper bound on $\|r_j\|$ is given by the following theorem [40].

**Theorem 2.** Suppose that $A$ has a decomposition (6), and suppose GMRES is used to solve (10) with $Z = [v_1, \ldots, v_m]$. If all the eigenvalues $\lambda_{m+1}, \ldots, \lambda_N$ of $A$ are located in an ellipse $E(c, d, a)$ which excludes the origin of the complex plane, then

\[ \|r_j\|_2 \leq \kappa_2(R_{22}) \frac{C_j((\hat{d})^2)}{C_j(\hat{d})^2} \|r_0\|_2. \] (12)
With (8), the upper bound (12) of the residual norm \( \|r_j\|_2 \) of Deflated GMRES is determined by the condition number of \( R_{22} \) (rather than \( V \)), and the scalar \( \delta \) which is determined by the distribution of the undeflated eigenvalues \( \lambda_{m+1}, \ldots, \lambda_N \) of \( A \).

2.3. Spectral Projector and Construction of \( Z \). Spectral projector is described in detail in §3.1.3-§3.1.4 of [30]. Other reference includes [6,15,24]. Let \( A = VJV^{-1} \) be the Jordan canonical decomposition of \( A \) where

\[
V = [v_1, v_2, \ldots, v_N] \quad \text{and} \quad J = \text{diag}\{J_{N_1}(\lambda_1), J_{N_2}(\lambda_2), \ldots, J_{N_d}(\lambda_d)\}.
\]

The eigenvalues \( \lambda_i \) in \( J \) are not necessarily distinct and can be repeated according to their multiplicities, and the diagonal block \( J_{N_i}(\lambda_i) \) in \( J \) is an \( N_i \times N_i \) Jordan block associated with the eigenvalue \( \lambda_i \).

Let \( \Gamma \) be a given positively oriented simple closed curve in the complex plane. Without loss of generality, let the set of eigenvalues of \( A \) enclosed by \( \Gamma \) be \( \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \). In other words, the eigenvalues \( \lambda_{k+1}, \ldots, \lambda_d \) lie outside the region enclosed by \( \Gamma \). Set \( s \equiv N_1 + N_2 + \ldots + N_k \), the number of eigenvalues inside \( \Gamma \) with multiplicity taken into account. Then the residue

\[
P_{\Gamma} = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma} (zI - A)^{-1} dz
\]
defines a projection operator onto the space \( \sum_{i=1}^{k} \text{Null}(A - \lambda_i I)^{l_i} \) where \( l_i \) is the index of \( \lambda_i \), namely,

\[
\text{Range}(P_{\Gamma}) = \text{span}\{v_1, v_2, \ldots, v_s\}.
\]

In particular, if \( A \) has a diagonal decomposition (6), \( P_{\Gamma} \) is a projector onto the sum \( \sum_{i=1}^{k} \mathbb{E}_{\lambda_i} \) of the \( \lambda_i \)-eigenspace \( \mathbb{E}_{\lambda_i} \) of \( A \).

Pick a random matrix \( Y \in \mathbb{C}^{N \times s} \) and set

\[
Z = P_{\Gamma}Y = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma} (zI - A)^{-1}Y dz
\]
in (9). Then we almost surely have \( \sigma(PA) = \{0, \cdots, 0, \lambda_{k+1}, \cdots, \lambda_d\} \). Therefore all the eigenvalues of \( A \) inside \( \Gamma \) are removed from the spectrum of \( PA \).

2.4. Numerical Examples. In this subsection, we demonstrate the effect of the deflation-subspace matrix \( Z \) defined by (13) applied to the solution of the following two test data downloaded from The University of Florida Sparse Matrix Collection\(^1\):

(a) \texttt{bcsstm27} from a mass matrix buckling problem. \texttt{bcsstm27} is a 1224 \times 1224 real symmetric and indefinite matrix \( A \) with 56,126 nonzero entries. As the right-hand side in (5), we set \( b = A1 \) where \( 1 = [1,1,\ldots,1]^T \). A spectral plot for \texttt{bcsstm27} is in Figure 1(b).

(b) \texttt{mahindas} from an economic problem. \texttt{mahindas} is a 1258 \times 1258 real unsymmetric matrix \( A \) with 7,682 nonzero entries. Again, we set \( b = A1 \) as the right-hand side in (5). A spectral plot for \texttt{mahindas} is in Figure 2(a).

All the computations were done in Matlab Version 7.1 on a Windows 7 machine with a Pentium 4 processor. An ILU preconditioner generated by the Matlab function \([L, U, P] = \text{luinc}(A, 0')\) was used for \texttt{mahindas}, namely, instead of solving (5), we solved

\[
\tilde{A}x = \tilde{b}
\]

where \( \tilde{A} = L^{-1}PAU^{-1} \) and \( \tilde{b} = L^{-1}Pb \), and accordingly the \( A \) and \( b \) in (10) were replaced with \( \tilde{A} \) and \( \tilde{b} \) respectively. Since the \( U \) factor obtained from \texttt{luinc} had some zeros along its main diagonal,

\(^1\text{http://www.cise.ufl.edu/research/sparse/matrices/}\)
we replaced those zeros by 1 so that $U$ was invertible. A spectral plot for $\tilde{A}$ is given in Figure 2(b). On the other hand, we did not use any preconditioner for $bcsstm27$.

Numerical solutions with deflated restarted GMRES of the linear systems resulted from the discretization of the two dimensional steady-state convection-diffusion equation

$$- [u_{xx} + u_{yy} + Re \left( p(x, y)u_x + q(x, y)u_y \right)] = f(x, y), \quad (x, y) \in [0, 1]^2 \quad (14)$$

with Dirichlet boundary conditions were studied in depth in [7]. In [7], two types of deflation-subspace matrix $Z$ are used: eigenvectors obtained from the Matlab function $eig$, and algebraic subdomain vectors. The $Z$ of algebraic subdomain vectors works well for the fluid flow problem (14), but not for other problems. Accurately calculating eigenvalues of large linear systems, on the other hand, is very time-consuming. Therefore deflation with the $Z$ of true eigenvectors is not practicable. Numerical experiments in [7] show that eigenvalues close to the origin hamper the convergence of a Krylov subspace method. Hence, deflation of these eigenvalues is very beneficial.

Based on this observation, we chose in our experiments the $\Gamma$ in (13) to be a circle $D(c, r)$ with the center $c$ near the origin. For the $Y$ in (13), we picked a random $Y \in \mathbb{R}^{N \times m}$ with $m$ not less than the exact number $s$ of eigenvalues inside $\Gamma$. We remark that an efficient stochastic estimation method of $s$ has been developed in [20]. Moreover, we computed the integral in (13) by the Legendre-Gauss quadrature

$$Z = \frac{r}{2} \int_{-1}^{1} e^{\pi \theta \sqrt{-1}} \left( (c + re^{\pi \theta \sqrt{-1}} I - A)^{-1} Y \right) d\theta \approx \frac{r}{2} \sum_{k=1}^{q} \omega_k e^{\pi \theta_k \sqrt{-1}} \left( (c + re^{\pi \theta_k \sqrt{-1}} I - A)^{-1} Y \right), \quad (15)$$

where $\omega_k$ and $\theta_k$ are the Legendre-Gauss weights and nodes on the interval $[-1, 1]$ with truncation order $q$. In (15), there are $mq$ linear systems $((c + re^{\pi \theta_k \sqrt{-1}} I - A)x = y_j$ to solve. We solved each of them by BiCG with the stopping tolerance $tol = 10^{-10}$ and the maximum number of iterations $maxit = N$.

In our experiments, we performed the following three computations:

#1 Solve (5) without any deflation.
#2 Compute $Z$ through (15). Perform $QR$ factorization on $Z$: $Z = QR$ where $Q \in \mathbb{C}^{N \times m}$ and $R \in \mathbb{C}^{m \times m}$. Then set $Z = Q$ which is the $Z$ in (9). Then solve (10).
#3 Use the Matlab function $eig$ to compute the eigenvectors $v_1, v_2, \ldots, v_s$ of $A$ whose associated eigenvalues lying inside $\Gamma$. Pick an $M \in \mathbb{R}^{s \times m}$ randomly, and set $Z = [v_1, v_2, \ldots, v_s]M$. 

![Figure 2. (a) Eigenvalue distribution of the test matrix $mahindas$. (b) Eigenvalue distribution of the ILU(0)-preconditioned $mahindas$.](image-url)
Table 1. A comparison of solving (5) and (10) by BiCG. For mahindas, a ILU(0) preconditioner was applied. Γ is a circle with center c and radius r. The q in (15) is $q = 2^7$.

| Matrix    | Circle Γ $(c,r)$ | #eig in Γ $m$ | Computation #1 | Computation #2 | Computation #3 |
|-----------|-----------------|---------------|----------------|----------------|----------------|
| bcsstm27  | $(0, 5)$        | 363           | 1224000        | $4.0 \times 10^{-6}$ | 763             | $8.9 \times 10^{-8}$ | 266 | $9.8 \times 10^{-8}$ |
| mahindas  | $(-1, 1)$       | 400           | 1258000        | 1.3            | $3.937 \times 10^{-6}$ | $5.0 \times 10^{-8}$ | 1555 | $4.7 \times 10^{-8}$ |

Perform QR factorization on $Z$: $Z = QR$ where $Q \in \mathbb{C}^{N \times m}$ and $R \in \mathbb{C}^{m \times m}$. Then set $Z = Q$ which is the $Z$ in (9). Then solve (10).

Due to that full GMRES is too expensive for us to use in terms of time and storage, rather than use GMRES, we employed BiCG as the Krylov solver in the solution of (5) and (10). The initial guesses for BiCG were $x = 0$, and the stopping criteria were $\|b - Ax\|_2/\|b\|_2 < 10^{-7}$ for (5) and $\|Pb - PAx\|_2/\|Pb\|_2 < 10^{-7}$ for (10) respectively.

Numerical results are summarized in Table 1. In this table, the column titled with “#eig in Γ” is a column of numbers of eigenvalues of $A$ inside Γ. The columns titled with “#iter” are columns of numbers of iterations, and the columns with “Err” are columns of true relative errors $\|b - Ax\|_2/\|b\|_2$ or $\|Pb - PAx\|_2/\|Pb\|_2$.

In these two experiments, BiCG essentially did not converge. With an appropriate eigenvalue-deflation, however, the situation was changed significantly. The most expensive part in the proposed method is clearly the computation of the $Z$ in (15). In next subsection, we describe the state-of-the-art parallel multigrid methods which may be applied to the computation.

2.5. Multigrid. We can formulate either geometric multigrid [1, 2, 5, 9, 17, 18, 23, 39] or algebraic multigrid [34] using the same notation level to level using the abstract multigrid approach developed in [3, 8, 10, 11, 13, 13].

Assuming the cost of the smoother (or rougher) on each level is $O(N_j)$, $j = 1, \cdots, k$, Algorithm MGC with $p$ recursions to solve problems on level $k - 1$ has complexity

$$W_{MGC}(N_k) = \begin{cases} O(N_k) & \text{if } 1 \leq p \leq \sigma \\ O(N_k \log N_k) & \text{if } p = \sigma \\ O(N_k^{\log p}) & \text{if } p > \sigma. \end{cases}$$

(16)

Under the right circumstances, multigrid is of optimal order as a solver.

Consider the example (14) in §2.4. A simple geometric multigrid approximation to (14) produces a very good solution in 4 V Cycles or 2 W cycles using the deflated GMRES as the rougher. Each V or W Cycle is $O(N_k)$. Hence, we have an optimal order solver for (14), which would not be the case if we used BiCG or deflated GMRES on a single grid.

High performance computing versions of multigrid based on using hardware acceleration with memory caches was extensively studied in the early 2000's [14].

Parallelization of Algorithm MGC is straightforward [12].

- For geometric multigrid, on each level $j$, data is split using a domain decomposition paradigm. Parallel smoothers (roughers) are used. The convergence rate degrades from the standard serial theoretical rate, but not by a lot, and scaling is good given sufficient data.
- For algebraic multigrid, the algorithms can be either straightforward (e.g., Ruge-Studen [28] or Beck [4]) to quite complicated (e.g., AMGe [22]). Solutions have existed for a number of years, so it is a matter of choosing an existing implementation. In some cases, using a tool like METIS or ParMETIS is sufficient to create a domain decomposition-like
system based on graph connections in $A_j$, which reduces parallelization back to something similar to the geometric case.

In many cases, the complexity of this type of parallel multigrid for $P$ processors becomes

$$W_{MGC,P}(N_k) = W_{MGC}(N_k) \log P/P.$$  

(17)

3. Conclusions and Future Work

The novelties of this research include (i) we incorporate the delation projector $P$ with the $Z$ described in (2) and (4) into Krylov subspace methods to enhance the stability and accelerate the convergence of the iterative methods for solving ill-conditioned linear algebraic systems, and (ii) we will also implement robust and efficient parallel multigrid methods for solving (4) and realize a software package for a wide variety of applications.

To our best knowledge, the constructions of most, if not all, deflation subspace matrices $Z$ in the literature are problem dependent. Further, some of them are ad-hoc, e.g., the algebraic subdomain deflation in [19]. The method proposed here is problem independent.

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