MAPS OF DEGREE ONE, RELATIVE LS CATEGORY AND HIGHER TOPOLOGICAL COMPLEXITIES

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Abstract. In this paper, we introduce relative LS category of a map and study some of its properties. Then we introduce ‘higher topological complexity’ of a map, a homotopy invariant. We give a cohomological lower bound and compare it with previously known ‘topological complexity’ of a map. Moreover, we study the relation between Lusternik-Schnirelmann category and topological complexity of two closed oriented manifolds connected by a degree one map.

1. Introduction

The concept of “sectional category” of a fibration \( p: E \to B \) was introduced by Schwarz in \cite{Sch66} under the name “genus”. James \cite{Jam78} proposes to replace the overworked term “genus” by “sectional category”. This invariant \( \text{secat}(p) \) is one less than the Schwarz genus of \( p \) in \cite{Sch66}. In particular, \( \text{secat}(p) = 0 \) if and only if \( p \) has a section. Sectional category of \( p \) is the minimum cardinality of the open coverings of \( B \) such that on each open set in the covering there is a homotopy section of \( p \). If no such integer exists then it is convention that \( \text{secat}(p) = \infty \). If \( E \) is contractible and \( p \) is surjective then \( \text{secat}(p) \) is equal to the classical Lusternik-Schnirelmann category (in short LS category throughout this paper) of \( B \). Several applications and properties of sectional category and LS-category can be found in \cite{CLOT03}. There is a generalization of LS category called category of a map \( f: X \to Y \).

We recall that LS category of \( f \) is the minimum number \( \text{cat}(f) \) such that \( X \) can be covered by \( \text{cat}(f) + 1 \) many open subsets and the restriction of \( f \) on each of these open subsets is null-homotopic. In particular, \( \text{cat}(i) = \text{cat}(X) \). It is a homotopy invariant and

\[
\text{cat}(f \times g) \leq \text{cat}(f) + \text{cat}(g)
\]

see \cite{CLOT03, Sta02}. In this paper, we, in particular introduce relative LS category of a map on a pair of spaces and study some of its properties.

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Another well known particular case of sectional category is Farber’s topological complexity \[\text{Far03}\]. He introduced topological complexity of a configuration space to understand the navigational complexity of the motion of a robot. We denote the closed unit interval \([0,1]\) by \(I\). Let \(X\) be a path connected Hausdorff space and \(X^{[0,1]}\) the free path space in \(X\) equipped with compact-open topology and

\[
\pi: X^I \to X \times X
\]

the free path fibration defined by \(\pi(\alpha) = (\alpha(0), \alpha(1))\) for \(\alpha \in X^I\). Then \(\text{secat}(\pi)\) is equal to Farber’s topological complexity, denoted by \(\text{TC}(X)\).

So, we have close relatives: topological complexity and Lusternik–Schnirelmann category. Both are numerical invariants and are special cases of Schwarz genus (sectional category). Thus, since we work with the pair (LS category of spaces, LS category of maps) and in view of parallelism between LS and TC, it seems reasonable to loop the presentation and introduce TC of mappings.

A. Dranishnikov enquired an appropriate definition for topological complexity of a map in his talk at the CIEM, Castro Urdiales in 2014. Let \(f: X \to Y\) be a map. Given \(y_1, y_2 \in \text{Im} f\), find a continuous motion planner \(\alpha\) in \(X\) such that \(f(\alpha(0)) = y_1\) and \(f(\alpha(1)) = y_2\). To answer this, we introduce a definition of “(higher) topological complexity” of a map.

We note that there are several notions of topological complexity of a map, but most of them lacks topologist’s basic demand namely homotopy invariance. The basic goal to define map from one state \(X\) to another state \(Y\) is to study properties of one from another. For example, if \(f: X \to Y\) and \(f\) is nice enough then the complexity of the state \(X\) could be approximate by the complexity of the state \(Y\), and this expectation is very natural. On the other hand, topologists expect that if \(X\) and \(Y\) are less complex then complexity of \(f\) has be less. In particular, if \(X\) and \(Y\) are contractible, that is there is no obstruction in both the state, then from topological point of view, complexity of a map from \(X\) to \(Y\) should be trivial. But an Example in page 4 of \[\text{Pav19}\] contradicts this topological expectation. So, we believe our definition is more appropriate according to topological point of view.

Recently Murillo and Wu \[\text{MW19}\] gave a definition of ‘topological complexity’ of a (work) map which is a homotopy invariant. They gave a cohomological lower bound for this invariant. However, our approach is totally different to define ‘topological complexity’ of a map as well as we introduce higher topological complexity of a map.

The paper is organize as follows. In Section 2 we recall two versions of relative category and introduce the concept of relative LS category of a map which is denoted by \(\text{relcat}(f)\). We also study several properties of this invariant which generalizes some of the properties of category of a map. We show that

\[
\text{relcat}(f \times g) \leq \text{relcat}(f) + \text{relcat}(g)
\]
and give a cohomological lower bound of relcat(f) in Proposition 2.8 and Theorem 2.10 respectively. We ask the Question 2.9 following Ganea conjecture.

In Section 3, we introduce the concept of ‘higher topological complexity’, denoted by TC_n(f), of a map f: X → Y. We show that it is a homotopy invariant and the following holds.

$$\text{TC}_n(f) \leq \min\{\text{TC}_n(X), \text{TC}_n(Y)\}$$

and

$$\text{cat}(f) \leq \text{TC}_n(f) \leq \text{TC}_{n+1}(f) \leq \text{cat}(f^{n+1}) \leq (n + 1) \text{cat}(f)$$

for all n ≥ 2. That is growth of TC_n(f) is linear. The works of [Pav19, RD18] and [MW19] study ‘topological complexity’ of a map. However, we discuss higher analogue of it. We compare our definition with the previous ones.

Section 4 is dedicated to answer the following open problem for a broader class of manifolds. This question was asked by Rudyak [Rud99] and now it is consider as a conjecture, see [Dra15, DS19].

1.1. Question. Given two closed manifolds M and N, assume that there exists a map f: M → N of degree ±1. Is it true that cat(M) ≥ cat(N)?

1.2. Remark. Since we speak about the degree of a map M → N, the manifolds M and N (like in 1.1, say) must be assumed to be oriented. So, in future we will not mention “oriented” in such situations.

We replace cat by TC in Question 1.1 and ask a similar question in Section 5. We give affirmative answer to this question too for several category of manifolds and when the dimensions are less than 4. In this paper, the word “space” means “topological space” unless some other is said explicitly, and all maps of spaces are assumed to be continuous.

2. Two Versions of Relative Category

In this section, we introduce relative category of a map and study some of its properties. Let cat X denote the LS category of a space X. Note that we use the normalized category, that is cat of the point is zero. Many topological constructions exploit relative version, informally saying to use an object “modulo subspaces”. Here we use two following definitions.

2.1. Definition ([CLOT03 Definition 7.1]). Let (X, A) be a pair of topological spaces. The relative category cat(X, A) is the least non-negative integer k such that X can be cover by open sets V_0, V_1, ..., V_k with A ⊆ V_0 and, for i ∈ {1, ..., k} the set V_i is contractible in X, and there exists a homotopy H: (V_0 × [0, 1], A × [0, 1]) → (X, A) with H(−, 1) is the inclusion map V_0 ↪ X and H(V_0, 0) ⊆ A.

2.2. Definition ([LM15 Definition 2.1, 2.2]).
(1) Let \((X, A)\) be a pair of topological spaces. An open subset \(U\) of \(X\) is called relative categorical if there is a homotopy \(H: U \times [0, 1] \to X\) such that \(H(\cdot, 1)\) is the inclusion \(U \hookrightarrow X\) and \(H(U, 0) \subseteq A\).

(2) Relative category of the pair \((X, A)\), denoted by \(\text{relcat}(X, A)\), is the least integer \(n\) such that \(X\) can be cover by \(n + 1\) many relative categorical sets. Otherwise, we say this is infinity.

Note that \(\text{cat}(X, x_0) = \text{cat} X = \text{relcat}(X, x_0)\) where \(x_0 \in X\). The following example shows that Definition 2.1 and Definition 2.2 are different. In general.

2.3. Example. Let \(X = S^1 \times S^1\) be the product of two circles and \(A = \{(x, x) \in S^1 \times S^1\}\) the diagonal. We prove that \(\text{relcat}(X, A) = 1\) and \(\text{cat}(X, A) = 2\).

First, we show that \(\text{cat}(X, A) = 2\). Let \(\text{cl}(Z)\) denote the cup-length of a space \(Z\). The Puppe sequence of the cofibration \(A \to X \to X/A\) yields an epimorphism (in fact, an isomorphism) \(H^2(X/A) \to H^2(X)\). Since \(\text{cl}(X) = 2\), we conclude that \(\text{cl}(X/A) \geq 2\) (in fact, \(\text{cl}(X/A) = 2\)). Hence \(\text{cat}(X, A) \geq 2\) by [CLOT03, Proposition 7.4].

Let \(V_0\) be a tubular open neighborhood of \(A\). Then \(X - V_0\) is a closed annulus in \(X\), and it can be covered by two contractible open sets \(V_1\) and \(V_2\) in \(X\). So, \(\text{cat}(X, A) \leq 2\), and thus \(\text{cat}(X, A) = 2\).

We show that \(\text{relcat}(X, A) = 1\). Let \(L_1\) and \(L_2\) be two disjoint circle in \(X\) both parallel to \(A\). Here we consider \(X\) as the quotient space obtained by identifying parallel edges of a square. Let \(U_i = X - L_i\) for \(i = 1, 2\). Then \(U_i\) is homeomorphic to an annulus containing the image of \(A\) as the angular circle. So \(U_i\) is contractible to \(A\) for \(i = 1, 2\). Hence \(\text{relcat}(X, A) = 1\).

Therefore, the relative category in Definition 2.1 and 2.2 are different. □

Now we are in a position to introduce the concept of relative category of a map. Then we study their several properties in the remaining of this section.

2.4. Definition. Let \(g: (B, C) \to (X, A)\) be a map of pairs \((C\) could be empty subset\). Then \(g\) is called relatively inessential map if there is a homotopy \(H: B \times [0, 1] \to X\) such that \(H(\cdot, 1)\) is \(g\), \(H(B, 0) \subseteq A\), \(H(C, t) \subseteq A\) for all \(t \in [0, 1]\).

2.5. Definition. Let \(f: (B, C) \to (X, A)\) be a map of pairs. Then \(\text{relcat}(f) \leq \ell\) if \(B\) has an open cover by \(B_0, \ldots, B_\ell\) such that \(f|_{(B_i, B_i \cap C)}\) is relatively inessential. Minimum \(\ell\) with this property is called relative-category of \(f\). Otherwise we say it is infinity.

Clearly, if \(C = \emptyset\) or \(C = \{pt\}\) and \(A = \{pt\}\) then \(\text{relcat}(f)\) turns into known invariant \(\text{cat}(f)\) for \(f: B \to X\), see [CLOT03, Exercise 1.16].

2.6. Theorem. The number \(\text{relcat}(f)\) is a homotopy invariant.

Proof. Let \(f, g: (B, C) \to (X, A)\) be homotopic maps relative to \(C\). Then there is a homotopy \(H: B \times [0, 1] \to X\) such that \(H(b, 0) = f(b), H(b, 1) = g(b)\). Then \(\text{relcat}(f) \leq \ell\) if \(\text{relcat}(g) \leq \ell\) for all \(\ell\).
\( g(b) \) for all \( b \in B \), and \( H(C,t) \subseteq A \) for all \( t \in [0,1] \). Let \( B_i \) be an open subset of \( B \) such that the map
\[
g|_{(B_i,B_i \cap C)} : (B_i,B_i \cap C) \to (X,A)
\]
is relatively inessential. Now we use the homotopy \( H \) to conclude that
\[
f|_{(B_i,B_i \cap C)} : (B_i,B_i \cap C) \to (X,A)
\]
is relatively inessential. Therefore, \( \text{relcat}(f) \leq \text{relcat}(g) \). Similarly, the opposite inequality holds. \( \square \)

Note that for identity map, \( \text{id} : (X,A) \to (X,A) \), \( \text{relcat}(\text{id}) = \text{relcat}(X,A) \).

Moreover, we have the following from Definition 2.2 and 2.5.

2.7. Proposition. Let \( f : (B,C) \to (X,A) \) and \( g : (Y,D) \to (B,C) \) be two maps. Then

(i) \( \text{relcat}(f) \leq \min\{\text{relcat}(B,C), \text{relcat}(X,A)\} \).
(ii) \( \text{relcat}(f \circ g) \leq \min\{\text{relcat}(f), \text{relcat}(g)\} \).

2.8. Proposition. Let \( f : (B,C) \to (X,A) \) and \( g : (E,F) \to (Y,D) \) be two maps. Then \( \text{relcat}(f \times g) \leq \text{relcat}(f) + \text{relcat}(g) \).

Proof. The proof is similar to the proof of \( \text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y) \) \cite{CLOT03} Theorem 1.37 with some modification. \( \square \)

We can ask the following question generalizing Ganea conjecture \cite{Gan71}.

2.9. Question. Let \( pt \) be a point in \( S^n \). What are the pair of spaces \( (X,Y) \) so that
\[
\text{relcat}(X \times S^n, Y \times \{pt\}) = \text{relcat}(X,Y) + 1?
\]

We note that when \( Y \) is a point in \( X \), then Question 2.9 is known as Ganea conjecture, and counter example of this conjecture was first given by Iwase \cite{Iwa98}.

In the rest of this section we give a cohomological lower bound of \( \text{relcat}(f) \). If \( R \) be a commutative ring, then the \emph{nilpotency index} of \( R \) is the non-negative integer \( n \) such that \( R^n \neq 0 \) but \( R^{n+1} = 0 \) and it is denoted by \( \text{nil} R \). Let \( f : (B,C) \to (X,A) \) be a map of pairs, and let \( f^* : H^*(X,A) \to H^*(B,C) \) be the induced homomorphism.

2.10. Theorem. \( \text{nil} \text{Im}(f^*) \leq \text{relcat}(f) \).

Proof. Let \( \text{relcat}(f) = k \) and \( g_i : (B_i,B_i \cap C) \to (X,A) \) relatively inessential map for \( i = 0,1,\ldots,k \) such that \( g_i \) is a restriction of \( f \) and \( B_0,B_1,\ldots,B_k \) covers \( B \). For the triple \( (B,B_i,B_i \cap C) \), we have the following long exact sequence
\[
\cdots \to H^*(B,B_i) \xrightarrow{q_i^*} H^*(B,B_i \cap C) \xrightarrow{i_i^*} H^*(B_i,B_i \cap C) \to \cdots.
\]
Also we have the following commutative diagrams induced from natural inclusions and some restrictions of $f$.

$$
\begin{align*}
&H^*(X, A) \\
\xymatrix{ H^*(B, B_i \cap C) \ar[rr]^{g^*_i} \ar[rruu]^{f^*_i} & & H^*(B_i, B_i \cap C) }
\end{align*}

$\text{and}$

$$
\begin{align*}
&H^*(B, C) \\
\xymatrix{ H^*(X, A) \ar[rr]^{f^*} \ar[rruu]^{f^*_i} & & H^*(B, B_i \cap C) }
\end{align*}

Suppose $\beta_0, \beta_1, \ldots, \beta_k$ belong to $\text{Im}(f)$. Then $\beta_i = f^*(\alpha_i)$ for some $\alpha_i \in H^*(X, A)$ and $i \in \{0, \ldots, k\}$. Since $g_i$ is inessential, then $g^*$ is trivial. So $\iota^*_i(f^*_i(\alpha_i)) = 0 \in H^*(B_i, B_i \cap C)$. Thus $\bar{f}^*_i(\alpha_i) = q^*_i(\gamma_i)$ for some $\gamma_i \in H^*(B, B_i)$ and $i \in \{0, \ldots, k\}$. By relative cohomology product rule, we get $\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_k \in H^*(B, B) = 0$, and $f_0^*(\alpha_0) \cup f_1^*(\alpha_1) \cup \cdots \cup f_k^*(\alpha_k) \in H^*(B, C)$. So using [Pav19, Proposition 3.19] and the map $q: (B, C) \to (B, B)$ we can get

$$
\begin{align*}
\beta_0 \cup \beta_1 \cup \cdots \cup \beta_k &= f^*(\alpha_0) \cup f^*(\alpha_1) \cup \cdots \cup f^*(\alpha_k) \\
&= f^*(\alpha_0) \cup f^*(\alpha_1) \cup \cdots \cup f^*(\alpha_k) \\
&= q^*(\gamma_0) \cup q^*(\gamma_1) \cup \cdots \cup q^*(\gamma_k) \\
&= q^*(\gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_k) \\
&= 0.
\end{align*}
$$

This proves the conclusion. \qed

3. Topological complexity of a map

In different context ‘topological complexity of a map’ has been studied in the work of [Pav19], [RD18] and [MW19]. In this section, we introduce the concept of ‘higher topological complexity’ of a map in a different way and show that it is a homotopy invariant. Then we compare it with the previous ones and study some properties of this new invariant. Consider the free path fibration

$$
\pi: X^I \to X \times X, \quad \pi(\alpha) = (\alpha(0), \alpha(1)), \quad \alpha: I \to X.
$$

Fatber [Far03] defined the topological complexity $\text{TC}(X)$ of a space $X$ as the sectional category of $\pi$, and showed a nice application of this notion to robot motion planning, [Far08]. Later Rudyak [Rud10], see also [BGRT14] introduced the “higher analogues” of topological complexity (also related to robotics, by the way). Let us recall the definition.
Given a space $X$, consider the fibration

$$\pi_n : X^I \to X^n, \tag{3.2}$$

$$\pi_n(\alpha) = \left( \alpha(0), \alpha \left( \frac{1}{n-1} \right), \ldots, \alpha \left( \frac{n-2}{n-1} \right), \alpha(1) \right)$$

where $\alpha \in X^I$.

3.1. **Definition.** A higher, or sequential topological complexity of order $n$ of a space $X$ (denoted by $\text{TC}_n(X)$) is the sectional category of $\pi_n$. So, $\text{TC}_n(X) = \text{secat}(\pi_n)$.

Note that $\text{TC}_2$ coincides with the invariant $\text{TC}$ introduced by Farber.

Recall that the LS category of a map $f$ have two important properties: $\text{cat}(f)$ is a homotopy invariant of $f$, and $\text{cat}(\text{id}_X) = \text{cat}(X)$.

Since $\text{cat}$ is a closed relative of $\text{TC}$, it seems reasonable and useful to introduce topological complexity $\text{TC}(f)$ of a map $f$ having, in particular, properties that are similar to two above mentioned ones.

Let $\Delta_n : X \to X^n$ be the diagonal of the product $X^n$ for $n \geq 2$.

3.2. **Definition.** Let $X$ and $Y$ be path connected spaces, and $f : X \to Y$ be a map. Let $\phi_n := f \times \cdots \times f$ (n-times) and

$$\overline{\phi}_n : (X^n, \Delta_n(X)) \to (Y^n, \Delta_n(Y))$$

be the induced maps of pairs. We define the $n$-th (higher) topological complexity of $f$, denoted by $\text{TC}_n(f)$, as $\text{TC}_n(f) := \text{relcat} \overline{\phi}_n$.

Now one can see that $\text{TC}_n$ turns out to be a functor on the category of topological spaces.

Previously we had three following version of ‘topological complexity’ of a map: the first one that is given by Pavešić, [Pav19], we denote it by $\text{TC}^{\text{Pav}}$, the second one that is given by Rami and Derfoufi, [RD18], we denote it by $\text{TC}^{\text{RD}}$, and third one is by Murillo and Wu [MW19], we denoted by $\text{TC}^{\text{MW}}$. Here we have the equalities

$$\text{TC}^{\text{Pav}}(\text{id}_X) = \text{TC}(X) = \text{TC}^{\text{RD}}(\text{id}_X) = \text{TC}^{\text{MW}}(\text{id}_X).$$

But neither $\text{TC}^{\text{Pav}}$ nor $\text{TC}^{\text{RD}}$ are homotopy invariant, see Example 3.3 and 3.4. Next we compare them with our definition of $\text{TC}_2(f)$.

First we recall the definition of topological complexity $\text{TC}^{\text{Pav}}(f)$ of a map $f$. Let $q : E \to B$ be a surjective map. The sectional number, denoted by $\text{sec}(q)$, of $q$ is the smallest length $n$ of the filtrations

$$\emptyset = V_0 \subset V_1 \subset \cdots \subset V_n = B$$

such that there is a section of $q^{-1}(V_i - V_{i-1}) \to V_i - V_{i-1}$ for $i = 1, \ldots, n$.

If there is no such integer then $\text{sec}(p) = \infty$. We note that sectional number and sectional category of $q$ are equal if $q$ is a fibration. Let $X, Y$ be path connected spaces and $f : X \to Y$ a surjective map. Consider the fibration
\( \pi : X^I \to X \times X \) as in (3.1). It induces a continuous map \( \pi_f : X^I \to X \times Y \) by \( \pi_f = (\text{id} \times f) \circ \pi \). Now, the topological complexity \( \text{TC}^{\text{Pav}}(f) \) of \( f \) is defined as the sectional number of \( \pi_f \), that is

\[
(3.3) \quad \text{TC}^{\text{Pav}}(f) := \text{sec}(\pi_f).
\]

We note that [Pav19, Corollary 3.9] says that \( \text{TC}^{\text{Pav}} \) is a fiber homotopy invariant.

Now we recall the topological complexity \( \text{TC}^{\text{RD}}(g) \) of a map \( g \). Let \( Z \) be a path connected space and \( g : Z \to W \) be a map. Then the fiber space \( Z \times_W Z \) is a subset of \( Z \times Z \). Let

\[
\pi^g : \pi^{-1}(Z \times_W Z) \to Z \times_W Z.
\]

be the fibration induced from \( \pi : Z^I \to Z \times Z \) by the inclusion

\[
Z \times_W Z \to Z \times Z.
\]

Then \( \text{TC}^{\text{RD}}(g) \) is the sectional category of \( \pi^g \), that is

\[
(3.4) \quad \text{TC}^{\text{RD}}(g) := \text{secat}(\pi^g).
\]

It turns out that \( \text{TC}^{\text{RD}} \) is also a fiber homotopy invariant [RD18, Corollary 7]. We also note that this definition is a particular case of “relative topological complexity” studied in [Far08].

In contrast to these two definitions of the “topological complexity of a map”, our definition is a homotopy invariant, a topologist primary interest. Let \( X \) be a contractible space and \( Y \) a non-contractible space. Let \( f : X \to Y \) be a surjective fibration, for example \( \exp : [0, 1] \to S^1 \). Then by [Pav19, Proposition 3.2] \( \text{TC}^{\text{Pav}}(f) \geq \text{cat}(Y) \geq 1 \).

But our definition gives \( \text{TC}(f) = 0 \). Therefore these two are different. On the other hand, if \( g \) is a constant map then \( \text{TC}^{\text{RD}}(g) \) is the topological complexity \( \text{TC}(X) \) of \( X \) which is strictly greater than \( \text{cat}(X) \) in general.

3.3. Example. Let \( f, g : [0, 3] \to [0, 2] \) be two continuous functions defined by the following.

\[
f(x) = \begin{cases} 
  x & \text{if } 0 \leq x \leq 1 \\
  1 & \text{if } 1 \leq x \leq 2 \\
  x - 1 & \text{if } 2 \leq x \leq 3 
\end{cases}
\]

and

\[
g(x) = \frac{2x}{3} & \text{if } 0 \leq x \leq 3.
\]

Then \( f \) has no continuous section, and \( g \) has a section. So \( \text{TC}^{\text{Pav}}(f) > 0 \) and \( \text{TC}^{\text{Pav}}(g) = 0 \). But \( f \) and \( g \) are homotopic by \( tf + (1 - t)g, t \in [0, 1] \). Therefore \( \text{TC}^{\text{Pav}} \) is not a homotopy invariant. This is mentioned in [Pav19], but an explicit example was not given.
3.4. Example. In this example we show that $\text{TC}^{\text{RD}}$, “topological complexity of a map” defined in [RD18] is not a homotopy invariant. Let $X$ be a path connected topological space and $CX$ the cone on $X$ with apex $a$. Write $CX = X \times [0,1]/X \times \{1\}$. Define $\iota, c : X \to CX$ where $\iota$ is the inclusion $\iota(x) = (x,0)$ and $c(X) = a$. Then $\text{TC}^{\text{RD}}(\iota) = \text{TC}(X)$ (Farber’s topological complexity).

For the map $\iota$, we have $X \times_{CX} X = \Delta X$. Then $\pi^{-1}(\Delta X)$ is the free loop space on $X$. So constant maps induce a section of $\pi^{\iota}$. Therefore $\text{TC}^{\text{RD}}(\iota) = 1 \neq \text{TC}(X) = \text{TC}^{\text{RD}}(c)$ in general. It remain to note that $\iota$ and $c$ are homotopic.

Now we recall the definition of topological complexity $\text{TC}^{\text{MW}}(f)$ of (work) map. Let $f : X \to Y$ be a continuous map. Then $\text{TC}^{\text{MW}}(f)$ is the least integer $n \leq \infty$ such that there exist open subsets $U_0, \ldots, U_n$ of $X \times X$ on each of which there is a map $s_i : U_i \to X$ satisfying $(f \times f) \circ \Delta_2 \circ s_i \simeq f \times f|_{U_i}$ for $i = 0, \ldots, n$. We also note that Scott defines topological complexity $\text{TC}^S(f)$ of a map in [Sco20, Definition 3.1], and he shows that these two definitions are equivalent.

Let $(a,b) \in U_i$. So $s_i(a,b) \in X$. Then $(f \times f) \circ \Delta_2 \circ s_i(a,b) \in \Delta_2(Y)$. Yet, we cannot conclude that $f \times f|_{U_i} : (U, U \cap \Delta_2(X)) \to (Y, \Delta_2(Y))$ is inessential in the sense of Definition 2.4. However, if we assume $f \times f|_{U_i}$ is inessential, then it is homotopic to a map $U_i \to \Delta_2(Y)$. Thus by [Sco20, Theorem 3.4] we get the following.

$$(3.5) \quad \text{TC}^{\text{MW}}(f) = \text{TC}^S(f) \leq \text{TC}_2(f).$$

We do not know yet if $\text{TC}_2(f) = \text{TC}^{\text{MW}}(f)$. Nevertheless, we give higher analogue of the topological complexity of a map which generalizes higher topological complexity of a space.

In the remaining we study some properties, lower bound and upper bound of $\text{TC}_n(f)$.

3.5. Proposition. We have $\text{TC}_n(f) \leq \min\{\text{TC}_n(X), \text{TC}_n(Y)\}$ for all maps $f : X \to Y$.

Proof. This follows from Proposition 2.7 and [BS18, Proposition 3.7].

We note that the sectional category of a map does not exceed the LS-category of the the codomain. Thus

$$\text{TC}_n(f) \leq \min\{n \text{cat}(X), n \text{cat}(Y)\}.$$ 

A part of the following result generalizes [Rud10, Proposition 3.3].

3.6. Proposition. Let $f : X \to Y$ be a map. Then

$$\text{cat}(f) \leq \text{TC}_n(f) \leq \text{TC}_{n+1}(f) \leq \text{cat}(f^{n+1}) \leq (n+1) \text{cat}(f)$$

for all $n$. 
Proof. Let $x \in X$ and

$$\phi_{n+1}|_{(B, B \cap \Delta_{n+1}(X))}: (B, B \cap \Delta_{n+1}(X)) \to (Y^{n+1}, \Delta_{n+1}(Y))$$

inefficient for some open subset $B \subset X^{n+1}$. Let $A = X^n \times x$, $\phi'_n$ be the restriction of $\phi_{n+1}$ on $A$, and $pr_x: Y^{n+1} \to Y^n \times f(x)$ the projection. Then $\phi'_n$ is $\phi_n$ and the following composition

$$pr_x \circ \phi'_n: (A \cap B, A \cap B \cap (\Delta_n(X), x)) \to (Y^n \times f(x), (\Delta_n(Y), f(x)))$$

is inefficient. This proves the second inequality. The first inequality follows from similar arguments. The third inequality follows from their definitions. The last inequality follows from (1.1). \qed

We remark that (3.5), Proposition 3.5 and 3.6 imply [Sco20] Proposition 3.8 in particular. Moreover, growth of $TC_n(f)$ is linear like $TC_n(X)$. Let $X, Y$ be a non-contractible space and $f: X \to Y$ constant map. Then $TC_n(f) = 1$, but $\text{cat}(X) \geq 2$ and $\text{cat}(Y) \geq 2$. So we cannot say $\text{cat}(X)$ or $\text{cat}(Y)$ is a lower bound for $TC_n(f)$ in general. However we can give the following cohomological lower bound by Theorem 2.10.

3.7. Proposition. Let $f: X \to Y$ and $\overline{\phi}_n$ be as in Definition 3.2. Then $\text{nil Im}(\overline{\phi}_n) \leq TC_n(f)$ for any $n$.

The map $\Delta_2: X \to X \times X$ induces a map $\Delta_2: H^*(X) \otimes H^*(X) \to H^*(X)$ defined by the cup product. If $f: X \to Y$ is a map, then $\ker \Delta_2 \subseteq \text{Im}(f \times f)^*$. Therefore we get the following inequalities.

3.8. Corollary. $\text{nil ker } U|_{\text{Im}(f \times f)^*} \leq \text{nil Im}(f \times f)^*$, $\text{TC}^{MW}(f) \leq TC_2(f)$.

3.9. Example. Let $L(p; q_1, \ldots, q_n)$ be the generalized lens space and $S^{2n+1} \xrightarrow{f} L(p; q_1, \ldots, q_n)$ be the corresponding orbit map. So $\deg(f) = p$. That is $f$ is essential. Thus $1 \leq TC_2(f)$. On the other hand $TC_2(f) \leq TC(S^{2n+1}) = 1$ by Proposition 3.5 and [Far15] Theorem 8. So $TC_2(f) = 1$ and it does not depend on the degree of $f$.

3.10. Example. Let $n \geq 3$ be odd and $\mathbb{R} P^n \xrightarrow{f} \mathbb{R} P^n/\mathbb{R} P^{n-1} \cong S^n$ the natural quotient map. So $g$ is essential which implies that $1 \leq TC_2(f)$. On the other hand, $TC_2(g) \leq TC(S^n)$ by Proposition 3.5. Thus $TC_2(g) = 1$. Whereas the results in [MW19] cannot determine $\text{TC}^{MW}(g)$. However, we can have

$$1 \leq \text{TC}^{MW}(g) \leq TC_2(g) = 1.$$

4. Maps of Degree 1 and Lusternik–Schnirelmann category

In this section, we give affirmative answer of Question 1.1 in a broader category, other than the class of manifolds discussed in [Rud17]. Please, pay attention to Remark 1.2. For future references, we note the following 2 known facts.
4.1. Proposition. If $M$ be a closed manifold with $\text{cat } M = 1$ then $M$ is a homotopy sphere.

Proof. Recall that $\pi_1(X)$ is free for all CW spaces (not necessarily manifold) $X$ with $\text{cat } X = 1$ (i.e. the so-called co-$H$-spaces), see [CLOT03, Ex.1.21] or [Ark11, Prop. 2.4.3]. For simply connected closed manifold $M^n$ we claim that $H^i(M) = 0$ for $i \neq 0, n$ (and hence $M$ is a homotopy sphere). Indeed, if $a \in H^k(M), a \neq 0$ for some $k \neq 0, n$ then there is a Poincaré dual class $b \in H^{n-k}(M)$ with $a \bowtie b \neq 0$. Hence cup-length of $M$ is at least 2, and so $\text{cat } M \geq 2$, a contradiction. If $M$ is not simply connected then $\pi_1(M)$ is non-trivial free group, and so $H^1(M)$ is a non-zero free abelian group, and so $H^1(M) \neq 0$. Hence, asserting as above, we see that cup-length of $M$ is at least 2. Thus, $\text{cat } M \geq 2$, a contradiction.

4.2. Proposition. Let $f: M \to N$ be a map of degree $\pm 1$. Then $f_*: H_*(M) \to H_*(N)$ is an epimorphism and $f^*: H^*(N) \to H^*(M)$ is a monomorphism. Furthermore, $f_*: \pi_1(M) \to \pi_1(N)$ is an epimorphism.

Proof. For $H_*, H^*$ see [Rud98, Theorem V.2.13], cf. also [Dyer69]. For $\pi_1$ see [DR09, Prop 5.1].

4.3. Corollary. Let $f: M \to N$ be a map of degree $\pm 1$. If $M$ is a homotopy sphere then $N$ is also.

We recall the notion of category weight, see [Rud96, Rud99, Str97].

4.4. Definition. Given a connected CW space $X$ and spectrum (cohomology theory) $E^*$, the category weight of $u \in E^*(X)$, denoted by $\text{wgt } u$, is defined by

$$\text{wgt } u \geq k \text{ if and only if } \varphi^*(u) = 0 \text{ for all } \varphi: A \to X \text{ with } \text{cat } \varphi > k.$$ 

Another definition looks as follows: $\text{wgt } u$ is the greater integer $k$ such that $p_{k-1}^*(u) = 0$. Here $p_{k-1}: G_{k-1}(X) \to X$ is the $(k-1)$th Ganea fibration, [CLOT03, Definition 1.51]. For the proof of equivalence of these two definitions, see Section 2.7 in [CLOT03, Lemma 8.21]

4.5. Remark. The origin of notion of category weight goes back to Fadell and Husseini [FH92]. However, their definition has a disadvantage that it is not a homotopy invariant.

4.6. Proposition. Category weight has the following properties.

1. $\text{wgt } u \leq \text{cat } X \leq \dim X$ for all $u \in E^*(X)$;
2. $\text{wgt}(f^*(u)) \geq \text{wgt } u$ for any $u \in E^*(X)$ and $f: Y \to X$ with $f^*(u) \neq 0$;
3. If $E^*$ is ring spectrum then $\text{wgt}(u \bowtie v) \geq \text{wgt } u + \text{wgt } v$ for $u, v \in E^*(X)$;
4. If $u \in H^s(K(\pi, 1); R)$ then $\text{wgt } u \geq s$.

Proof. See [Rud99, Theorem 1.8] or [CLOT03, Prop. 8.22]
4.7. **Definition** ([Rud99]). Let $X$ and $E$ be as in Definition 4.4. An element $u \in E^*$ is a detecting element for $X$ is $\text{wgt } u = \text{cat}(X)$.

4.8. **Proposition** ([Rud17, Section 4]). Let $f : M \to N$ be a map of degree $\pm 1$. If dimension of $M, N$ is less than 5, then $\text{cat}(M) \geq \text{cat}(N)$.

We note that if $f : M \to N$ is a degree one map and $\text{cat}(N) \leq 2$, then $\text{cat}(M) \geq \text{cat}(N)$ in any dimension.

The following propositions give an effect of having detecting element after taking finite product with spheres.

4.9. **Proposition.** Let $g : M \to N \times S^{n_1} \times \cdots \times S^{n_\ell}$ be a map of degree $\pm 1$ for some $\ell \geq 0$.

(i) If $N$ possesses a detecting element in $H^*$ then
\[ \text{cat}(M) \geq \text{cat}(N \times S^{n_1} \times \cdots \times S^{n_\ell}). \]

(ii) If $M$ and $N$ are stable parallelizable manifold and $N$ is a $(q - 1)$-connected manifold with $q \text{cat}(N) = \dim N \geq 4$ then
\[ \text{cat}(M) \geq \text{cat}(N \times S^{n_1} \times \cdots \times S^{n_\ell}). \]

**Proof.** For sake of simplicity, put $N^* := N \times S^{n_1} \times \cdots \times S^{n_\ell}$.

(i) If $N$ possesses a detecting element in $H^*$, then by [Rud99, Corollary 2.3] $N^*$ possesses a detecting element, say, $u$. Since $\deg g = \pm 1$ we conclude that $g^* : H^*(N^*) \to H^*(M)$ is a monomorphism by [Rud17, Lemma 2.2]. Thus, by Prop. 4.6 we have
\[ \text{cat}(M) \geq \text{wgt } g^* u \geq \text{wgt } u = \text{cat}(N^*). \]

(ii) If $M, N$ are stable parallelizable manifold then $M, N$ is $S$-orientable with respect to the sphere spectrum $S$. Recall that $N$ is a $(q - 1)$-connected manifold with $q \text{cat}(N) = \dim N \geq 4$. Hence by [Rud99, Theorem 3.1], $N$ possesses a detecting element $v \in E^*(N)$ for some spectrum $E$. So, $N^*$ has a detecting element for $E$ by [Rud99, Lemma 2.2]. Since every spectrum is a module spectrum over $S$, we conclude that $g^* : E^*(N^*) \to E^*(M)$ is a monomorphism by [Rud98, Theorem V.2.13]. Now the proof can be completed as the previous part.

4.10. **Proposition.** Let $f : M \to N$ be a map of degree $\pm 1$ and suppose that
\[ \text{cat}(M \times S^{n_1} \times \cdots \times S^{n_k}) \geq \text{cat}(N \times S^{n_1} \times \cdots \times S^{n_k}), \quad k \geq 1. \]

If either $N$ possesses a detecting element or $\dim N \leq 2 \text{cat}(N) - 3$ then $\text{cat}(M) \geq \text{cat}(N)$.

**Proof.** If $N$ possesses a detecting element, then by [Rud99, Corollary 2.3] we have $\text{cat}(N \times S^{n_1} \times \cdots \times S^{n_k}) = \text{cat}(N) + k$.

Also, if $\dim N \leq 2 \text{cat}(N) + 3$, then by [Rud99, Theorem 3.8], we have $\text{cat}(N \times S^{n_1} \times \cdots \times S^{n_k}) = \text{cat}(N) + k$. 

Since \( \text{cat}(X \times Y) \leq \text{cat}(X) + \text{cat}(Y) \) for all \( X, Y \), we have
\[
\text{cat}(M \times S^{m_1} \times \cdots \times S^{m_k}) \leq \text{cat}(M) + k.
\]
This implies the conclusion. \( \square \)

Now, let \( N = (N^{2n}, \omega) \) be a closed symplectic manifold with the closed non-degenerate symplectic 2-form \( \omega \). Let \([\omega] \in H^2(N; \mathbb{R})\) be the de Rham cohomology class of \( \omega \).

**4.11. Proposition.** Let \( f : M \to N \) be a map of degree \( \pm 1 \) and \( N \) is a symplectic manifold as above. In addition, assume that \( N \) is simply connected. Then \( \text{cat}(M) \geq \text{cat}(N) \).

**Proof.** By the cup-length theory for LS category, we have \( \text{cat}(N) \geq n \) as \([\omega]^n \neq 0\). Furthermore, \( \text{cat}(N) \leq n \) as \( N \) is simply connected of dimension \( 2n \). Hence, \( \text{cat}(N) = n \). Since \( f^* : H^*(N; \mathbb{R}) \to H^*(M; \mathbb{R}) \) is injective by [Rud17, Lemma 2.2], we have \( f^*[\omega]^n \neq 0 \). Therefore, \( \text{cat}(M) \geq n \). \( \square \)

This result can be improved as follows.

**4.12. Proposition.** Let \( f : M \to N \) be a map of degree \( \pm 1 \) and \( (N^{2n}, \omega) \) a symplectic manifold such that the de Rham cohomology class \([\omega] \neq 0\) vanishes on the image of \( \psi_N : \pi_2(N) \to H_2(N) \) where \( \psi \) is the Hurewicz homomorphism. Then \( \text{cat}(M) = \text{cat}(N) = 2n \).

**Proof.** Since \([\omega] \) vanishes on \( \psi \), there exist a map \( g : N \to K(\pi_1(N); 1) \) and cohomology class \( \omega_g \in H^2(K(\pi_1(N); 1)) \) such that \( g^*(\omega_g) = [\omega] \), see [CLOT03, Lemma 8.18]. Note that \( \text{wgt}(\omega_g) \geq 2 \), and so \( \text{wgt}(\omega) \geq 2 \), by Propositions 4.6(2,4).

Since \( \omega \) is a symplectic form, \([\omega^n] \neq 0\), and so \( g^*(\omega^n_g) \neq 0 \). Hence
\[
\text{cat}(N) \geq \text{wgt}(g^*(\omega^n_g)) \geq n \text{wgt}(g^*(\omega_g)) \geq 2n.
\]
So, \( \text{cat}(N) = 2n \). Furthermore \( \text{wgt}(f^*[\omega^n]) \neq 0 \) because \( f^* \) is monic (\( \text{deg} f = \pm 1 \)). Recall that \( \text{wgt}(\omega) \geq 2 \). Thus, by Prop. 4.6
\[
\text{cat}(M) \geq \text{wgt} f^*[\omega^n] \geq \text{wgt}[\omega^n] = n \text{wgt}[\omega] \geq 2n.
\]
\( \square \)

**4.13. Proposition.** Let \( f : M \to N \) be a map of degree \( \pm 1 \). Suppose that \( \dim M = 5 = \dim N \), and \( \text{cat}(N) = 5 \). Then \( \text{cat}(M) \geq \text{cat}(N) \).

**Proof.** By [Rud99, Corollary 3.3 (ii)], there is a detecting element for \( N \). Now the result follows from [Rud99, Theorem 1.8 (iii)]. \( \square \)

**4.14. Proposition.** Let \( f : M \to N \) be a map of degree \( \pm 1 \). Assume that \( N \) a simply connected manifold, and dimension of \( M, N \) is equal to 5 or 6. Then \( \text{cat}(M) \geq \text{cat}(N) \).
Proof. First, suppose that \( \dim N = 5 \). Then \( \cat(N) \leq 2 \) by [CLOT03, Theorem 1.50]. The case \( \cat(N) = 1 \) is obvious. Furthermore, if \( \cat(N) = 2 \), then \( \cat(M) \geq 2 \). Indeed, by way of contradiction, suppose that \( \cat(M) = 1 \). Then, by Prop. 4.11 \( M \) is homotopy equivalent to the sphere \( S^5 \). Now, the condition \( \deg f = \pm 1 \) implies that \( N \) is a homotopy sphere by Corollary 4.3 and thus \( \cat(N) = 1 \). This is a contradiction.

Let \( \dim N = 6 \). Then \( \cat(N) \leq 3 \). If \( \cat(N) \leq 2 \), then argument is similar to the previous paragraph. Now suppose that \( \cat(N) = 3 \), and so \( 2 \cat(N) = 6 \geq 4 \). It follow from [Rud99, Corollary 3.1] that \( N \) has a detecting element \( u \). Moreover, we know that \( f^*(u) \neq 0 \) because \( \deg f = \pm 1 \). This implies that \( \wgt f^*(u) \geq \wgt u = 6 \) by Prop. 4.6 and thus \( \cat(M) = 6 \).

\[ \square \]

4.15. Proposition. Let \( f : M \to L_{p}^{2n+1} \) be a map of degree \( \pm 1 \) where \( L_{p}^{2n+1} \) is a lens space. Then \( \cat(M) \geq \cat(L_{p}^{2n+1}) \).

Proof. We know that
\[
H^*(L_{p}^{2n+1}; \mathbb{Z}/p) = \mathbb{Z}/p[u, \beta u] u^2 = 0, (\beta u)^{n+1} = 0
\]
where \( \beta \) is the Bockstein homomorphism mod \( p \). Furthermore, \( \wgt u = 2 \). [FH92, Rud99]. So, \( \wgt (u \sim (\beta u)^{n}) \geq 2n + 1 \) by Prop. 4.6(3). Hence, \( \wgt (u \sim (\beta u)^{n}) = 2n + 1 \) by Prop. 4.6(1). Therefore, \( f^*(u \sim (\beta u)^{n}) \neq 0 \) by [Rud17, Lemma 2.2].

Thus, \( \cat(L_{p}^{2n+1}) \geq \wgt (f^*(u \sim (\beta u)^{n}) = 2n + 1 \) by Prop. 4.6 \[ \square \]

4.16. Proposition. Let \( f : M \to N \) be a map of degree \( \pm 1 \) and \( N \) is a sphere bundle over sphere with a cross-section. Then \( \cat(M) \geq \cat(N) \).

Proof. Let \( q : N \to S^n \) be a sphere bundle with fiber \( S^m \). So, by (3.3) in [JW54], \( N \) has a cell complex structure given by \( N = e^0 \cup e^m \cup e^n \cup e^{m+n} \). Since \( N \) has a cross-section, \( e^0 \cup e^m \cup e^n \) is homeomorphic to \( S^m \vee S^n \), see discussion after (3.3) in [JW54]. Thus \( \cat(N) \leq 2 \). So by the remark after Proposition 4.8 the conclusion follows.

\[ \square \]

We recall the definition of toric manifold briefly following [DJ91]. A toric manifold is a \( 2n \)-dimensional smooth manifold equipped with an \( n \)-dimensional torus action such that the action is locally resemble the \( T^n \)-action on \( \mathbb{C}^n \) and the orbit space has the structure of an \( n \)-dimensional simple polytope.

4.17. Proposition. Let \( f : M \to N \) be a map of degree \( \pm 1 \) and \( N \) a toric manifold. Then \( \cat(M) \geq \cat(N) \).

Proof. Let \( q : N \to P \) be the orbit map where \( \dim(N) = 2n \) and \( P \) is an \( n \)-dimensional polytope. Let \( v \) be a vertex of \( P \). So there is unique \( u \) many codimension one faces \( F_1, \ldots, F_n \) of \( P \) such that \( v = F_1 \cap \cdots \cap F_n \). From the discussion just above [DJ91, Proposition 3.10], we get that the codimension-2 submanifolds \( q^{-1}(F_1), \ldots, q^{-1}(F_n) \) intersects transversely and \( q^{-1}(v) = q^{-1}(F_1) \cap \cdots \cap q^{-1}(F_n) \) is a point. Let \( x_i \) be the Poincaré dual of \( q^{-1}(F_i) \). So \( x_1, x_2, \ldots, x_n \in H^2(N; \mathbb{Z}) \) such that \( x_1 \sim \cdots \sim x_n \) is nonzero in \( H^*(N; \mathbb{Z}) \).
This implies that cup-length of \( N \) is at least \( n \). Also \( N \) is simply connected by [DJ91, Cor. 3.9]. Now, the proposition follows from [Rud17, Proposition 3.3]. □

This result can be extended to the class of locally standard torus manifolds [MP06, Section 4] where the orbit space may not be a simple polytope always.

4.18. **Proposition.** Let \( f: M \to N \) be a map of degree \( \pm 1 \) and \( N \) is a 2\( n \)-dimensional locally standard torus manifold. If the orbit space \( N/T^n \) is simply connected and contains a boundary of a simple polytope, then \( \text{cat}(M) \geq \text{cat}(N) \).

**Proof.** In the proof of [BS15, Lemma 4.2], the authors show that there are cohomology classes \( x_1, x_2, \ldots, x_n \in H^2(N; \mathbb{Z}) \) such that \( x_1 \smile \cdots \smile x_n \) is nonzero in \( H^*(N; \mathbb{Z}) \). Also, \( \text{cat}(N) = n \) by [BS15, Theorem 4.3]. Then the result follows from [Rud17, Proposition 3.3]. □

4.19. **Remark.** Dranishnikov and Scott [DS19] proved the inequality \( \text{cat}(M) \geq \text{cat}(N) \) in case of the existence of a normal map \( M \to N \) of degree 1, provided that \( \text{dim} N \leq 2r \text{cat} N - 3 \). Here \( M \) and \( N \) are closed manifolds and the \( N \) is \((r - 1)\)-connected, \( r \geq 1 \).

### 5. Maps of Degree 1 and Topological Complexity

In this section, we compare the topological complexity of two orientable manifolds when there is a degree one map between them. We recall that \( \text{TC}_2(X) \) of Definition 3.1 is Farber’s \( \text{TC}(X) \). The following proposition is proved in [LM15].

5.1. **Proposition.** Let \( \Delta X \) be the diagonal space of the product \( X \times X \). Then \( \text{TC}(X) = \text{relcat}(X \times X, \Delta X) \) if \( \text{TC}(X) \) is finite.

Analogously to the Question 1.1, one may ask the following.

5.2. **Question.** Let \( n \geq 2 \) and \( M, N \) two closed orientable manifolds such that there exists a map \( f: M \to N \) of degree \( \pm 1 \). Is it true that \( \text{TC}_n(M) \geq \text{TC}_n(N) \)?

If \( \text{dim} M, N = 1 \) then it is clear. In the remaining, we show that this question has affirmative answer when dimension of \( M, N \) are 2 and 3. Note that if the degree of the map in Question 5.2 is more than one then the question has negative answer, for example one can take degree \( k \) map from 3-sphere to lens space. We also analyse this question on several types of oriented manifolds.

5.3. **Definition.** Given a space \( X \), a commutative ring \( R \), and a positive integer \( n \), we say that \( u \in H^*(X^n; R) \) is a zero-divisor class of grade \( n \) if \( d_n u = 0 \). Here \( d_n : X \to X^n \) is the diagonal map. A zero-divisors-cup-length of grade \( n \) of \( H^*(X; R) \), denoted by \( \text{zcl}_n(X) = \text{zcl}_n^R(X) \) is the maximal \( k \)
such that \( u_1 \sim \cdots \sim u_k \neq 0 \) provided each \( u_i \) is a zero-divisor class of grade \( n \).

We note that the cup product in \( H^*(X; R) \) yields a homomorphism

\[
(H^*(X; R))^\otimes n \longrightarrow H^*(X^n; R) \xrightarrow{d_n^*} H^*(X; R).
\]

We also denote the above composition by \( d_n^* \).

5.4. **Theorem.** For any commutative ring \( R \) we have the inequality \( TC_n(X) \geq \text{zcl}^R_n(X) \).

**Proof.** This theorem follows from \cite{Sch66} Theorem 4] if we replace the fibration \( p: E \to B \) in the \cite{Sch66} Theorem 4 by the fibration \( \pi_n: X^I \to X^n \) in (3.2).

5.5. **Proposition.** Let \( f: M \to N \) is a map of degree \( \pm 1 \). Then for any commutative ring \( R \) we have \( \text{zcl}_n(M) \geq \text{zcl}_n(N) \). In particular, if \( TC_n(N) = \text{zcl}_n(N) \) then \( TC_n(M) \geq TC_n(N) \).

**Proof.** Let \( \text{zcl}_n(N) = k \). Consider elements \( u_1, \ldots, u_k \in H^*(N^n; R) \) with \( d^*_n(u_i) = 0 \) for \( i = 1, \ldots, k \) and such that \( u_1 \sim \cdots \sim u_k \neq 0 \). Now we have \( f^*(u_1) \sim \cdots \sim f^*(u_k) \neq 0 \) by \cite{Rud17} Lemma 2.2. So the result follows.

5.6. **Proposition.** Let \( M \to S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m}, k_i > 0 \) be a map degree \( \pm 1 \). Then

\[
TC_n(M) \geq TC_n(S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m}).
\]

**Proof.** Put \( N = S^{k_1} \times S^{k_2} \times \cdots \times S^{k_m} \). Then, by \cite{BGRT14} Cor. 3.12], we have \( TC_n(N) = m(n-1) + \ell \) where \( \ell \) is the number of even dimensional spheres. (Note that the invariant \( \text{cl}(X, n) \) in \cite{BGRT14} is exactly our invariant \( \text{zcl}_n(X) = \text{zcl}^R_n(X) \)). By \cite{BGRT14} Theorem 3.10 we have \( \text{zcl}_n(S^{2k+1}) = n-1 \) and \( \text{zcl}_n(S^{2k}) = n \), and, moreover, the same theorem yields the inequality \( \text{zcl}_n(N) \geq m(n-1) + \ell = TC_n(N) \) by induction. So, \( TC_n(N) = \text{zcl}_n(N) \) by Theorem 5.4. Thus, by Prop. 5.5 we have \( TC_n(M) \geq \text{zcl}_n(M) \geq \text{zcl}_n(N) = TC_n(N) \).

5.7. **Proposition.** Let \( M \) be a stably parallelizable 14-dimensional. Consider the exceptional Lie group \( G_2 \), see e.g. \cite{Ada96}. If \( f: M \to G_2 \) is a map of degree \( \pm 1 \) then \( TC_n(M) \geq TC_n(G_2) \).

**Proof.** We have \( \text{cat}(M) \geq \text{cat}(G_2) \) by \cite{Rud17} Prop. 5.4. Furthermore, \cite{BGRT14} Theorem 3.5] says that \( TC_n(G_2) = (\text{cat}(G_2))^{n-1} \). So proposition follows from the inequality \( (\text{cat}(X))^{n-1} \leq TC_n(X) \) for all \( X \), \cite{BGRT14} Cor. 3.3].

5.8. **Proposition.** Let \( M \) be a closed connected smooth manifold and \( f: M \to SO(n) \) is a degree one map, then \( TC(M) \geq TC(SO(n)) \) if \( n \leq 9 \).
Proof. We have cat$(M) \geq$ cat$(SO(n))$ if $n \leq 9$ by [Rud17, Theorem 5.5]. Furthermore, [BGRT14, Theorem 3.5] says that TC$(G) = \text{cat}(G)$ for a connected Lie group. So proposition follows from the inequality $(\text{cat}(X))^{n-1} \leq \text{TC}_n(X)$. □

5.9. Proposition. If $f : M \to N$ is a map of degree $\pm 1$ and TC$(N) = 2$, then TC$(M) \geq 2$.

Proof. If TC$(M) = 1$ then $M$ must be an odd-dimensional homotopy sphere, see [GLO13]. Thus, $N$ must be an odd-dimensional homotopy sphere by Prop. 4.3. □

5.10. Proposition. If $f : M \to N$ is a map of degree $\pm 1$ and dim $N = 2$, then TC$(M) \geq$ TC$(N)$.

Proof. Since $N$ is an oriented 2-dimensional manifold, it satisfies the hypothesis of Proposition 5.5. So we are done. □

5.11. Theorem. If $f : M \to N$ has degree $\pm 1$, TC$(N) = 3$ or 4 and $M$ is not a rational sphere, then TC$(M) \geq 4$.

Proof. Since $M$ is not a rational sphere, then there is a non-zero cohomology class $\alpha \in H^i(M; \mathbb{R})$ for some $0 < i < \text{dim} M$. Then there is a Poincaré dual, denoted by $\alpha'$, of $\alpha$ such that $\alpha \sim \alpha' \neq 0$ in $H^{\text{dim} M}(M; \mathbb{R})$. Let $a := 1 \otimes \alpha - \alpha \otimes 1$ and $b := 1 \otimes \alpha' - \alpha' \otimes 1$. So $a, b \in zcl_2(M)$ and $a \sim a \sim b \sim b$ is non-zero in the ideal of the zero-divisors. Thus TC$(M) \geq 4$. □

5.12. Corollary. Let $M, N$ be two 3-dimensional oriented manifolds. Suppose that TC$(N) = 3$. If $f : M \to N$ has degree $\pm 1$, then TC$(M) \geq 3$.

Proof. By way of contradiction, suppose that TC$(M) \leq 2$. Than cat $M \leq 2$. If cat $M = 1$ then $M$ is a homotopy sphere, and hence $N$ is a homotopy sphere by Prop. 4.1. This contradicts the condition TC$(N) = 3$. Furthermore, if cat $N = 2$ then $\pi_1(M)$ is a free group by [DKR08]. Hence, $M$ is not a rational homotopy sphere. This contradicts Theorem 5.11. □

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