Abstract

Recently proposed quasi-classical $\bar{\partial}$-dressing method provides a systematic approach to study the weakly dispersive limit of integrable systems. We apply the quasi-classical $\bar{\partial}$-dressing method to describe dispersive corrections of any order. We show how to calculate the $\bar{\partial}$ problems at any order for a rather general class of integrable systems, presenting explicit results for the KP hierarchy case. We demonstrate the stability of the method at each order. We construct an infinite set of commuting flows at first order which allow a description analogous to the zero order (purely dispersionless) case, highlighting a Whitham type structure. Obstacles for the construction of the higher order dispersive corrections are also discussed.

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1 Introduction

During the last decade a great interest has been focused on the dispersionless limit of integrable nonlinear dispersive systems. They arise in various contexts of physics and mathematical physics: topological field theory and
strings, matrix models, interface dynamics, nonlinear optics, conformal mappings theory\cite{1}-\cite{19}. Several methods and approaches have been used in order to study the features of this type of systems, like the quasi-classical Lax pair representation with its close relationship with the Whitham universal hierarchy\cite{4}-\cite{7}; the hodograph transformations to calculate particular and interesting classes of exact solutions (rarefaction and shock waves)\cite{9}; the quasi-classical version of the inverse scattering method allowing to analyze various 1+1-dimensional systems\cite{10}. Moreover, the recent formulation of the quasi-classical $\bar{\partial}$-dressing method\cite{20}-\cite{22} introduces a more general and systematic approach to multidimensional integrable systems with weak dispersion, which preserves the power of the standard method. It allows to build integrable systems and solutions simultaneously; that is a non trivial fact, because of the difficulty to obtain exact explicit solutions\cite{23}. In addition, new interesting interrelations have been revealed, like an intriguing connection with the quasi-conformal mapping theory, a strong similarity with the theory of semi-classical approximation to quantum mechanics\cite{24} and geometric asymptotics methods to calculate wave corrections to geometrical optics\cite{25}. While the method has been rather widely discussed for purely dispersionless limit of some famous integrable hierarchies (KP, mKP, 2DTL)\cite{20, 21, 22, 26, 27} the study of the dispersive corrections is open yet. The latter is an interesting and challenging question in the context where one would like to be able to investigate the properties of the full dispersive system through an approximation theory.

A main goal of the present paper is to extend the quasi-classical $\bar{\partial}$-dressing method to higher order dispersive corrections, starting an investigation of the general properties at the dispersive orders of an integrable hierarchy. We will show that the quasi-classical $\bar{\partial}$-dressing method works as effectively as in the pure dispersionless case, and the first order admits a construction parallel to leading order case. We present all calculations for the weakly dispersive KP hierarchy as illustrative example. In this paper we do not consider the problem of construction of bounded solutions. So, we do not care about secular terms which usually appear in asymptotic expansion\cite{28}.

We briefly present the standard $\bar{\partial}$-method in the Sec.2 and the quasi-classical $\bar{\partial}$-method in the Sec.3. In the Sec.4 we calculate the quasi-classical $\bar{\partial}$-problems at any order in the dispersive parameter, giving the explicit formulas at fourth order. In the Sec.5 we recall the method at the first order\cite{20}, and prove a theorem allowing us to build an infinite set of flows reproducing the first corrections to the dKP equation. In the Sec.6 we present the gener-
alization of the \( \bar{\partial} \)-dressing method for higher orders. The Sec. 7 contains some considerations about the possibility to describe the first order introducing a certain generalization of the Whitham hierarchy. Finally, we present some concluding remarks.

2 The standard \( \bar{\partial} \)-dressing method.

The standard \( \bar{\partial} \)-dressing method is a powerful procedure allowing to construct multidimensional integrable hierarchies of equations and their solutions. In this section we outline the main ideas of the method\cite{29, 30}. The \( \bar{\partial} \)-dressing method is based on the nonlocal \( \bar{\partial} \)-problem

\[
\frac{\partial \chi(z, \bar{z}; t)}{\partial \bar{z}} = \int_{\mathbb{C}} d\mu \wedge d\bar{\mu} \chi(\mu, \bar{\mu}; t) g(\mu, t) R_0(\mu, \bar{\mu}; z, \bar{z}) g^{-1}(z, t),
\]

where \( z \) is the spectral parameter and \( \bar{z} \) its complex conjugate, \( t = (t_1, t_2, t_3, \ldots) \) the vector of the times, \( \chi(z, \bar{z}; t) \) a complex-valued function on the complex plane \( \mathbb{C} \) and \( R_0(\mu, \bar{\mu}; z, \bar{z}) \) is the \( \bar{\partial} \)-data.

The \( \bar{\partial} \)-equation (1) encodes all informations about integrable hierarchies. It is assumed that the problem (1) is uniquely solvable. Usually, one consider its solutions with canonical normalization, i.e.

\[
\chi \to 1 + \frac{\chi_1}{z} + \frac{\chi_2}{z^2} + \ldots,
\]

as \( z \to \infty \).

The particular form of the function \( g(z, t) \) sets an integrable hierarchy. Specifically, in order to describe the KP hierarchy one has to take

\[
g(z, t) = e^{S_0(z, t)}, \quad S_0(z, t) = \sum_{k=1}^{\infty} z^k t_k.
\]

Following a standard procedure\cite{30}, it is possible to construct an infinite set of linear operators \( M_i \), providing an infinite set of linear equations

\[
M_i \chi = 0,
\]

which are automatically compatible. Equations (4) provide us with the corresponding integrable hierarchy. For example, the pair of operators (Lax pair)
\[ M_2 = \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} - u(x, y, t), \]
\[ M_3 = \frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3} - 3 \frac{u}{2} \frac{\partial}{\partial x} - 3 \frac{u_x}{2} - 3 \frac{u_{-1}^1}{4} u_y, \]

with the notations

\[
t_1 = x; \quad t_2 = y; \quad t_3 = t; \quad u_{-t} = \frac{\partial u}{\partial t}; \quad \partial_{x}^{-1} = \int_{-\infty}^{x} dx,
\]

provides us with the KP equation (sometimes called “dispersionfull” KP)

\[
\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{\partial^3 u}{\partial x^3} + \frac{3}{2} \frac{u}{2} \frac{\partial u}{\partial x} \right) + \frac{3}{4} \frac{\partial^2 u}{\partial y^2}.
\]

\[
(5)
\]

\[ \bar{\partial}-\text{dressing method allow us to construct a wide class of exact solutions of the KP equation and the whole KP hierarchy.} \]

3 Quasi-classical \( \bar{\partial} \)-dressing method.

3.1 Dispersionless limit.

In order to illustrate the quasi-classical \( \bar{\partial} \) formalism, we, following Ref. [20], introduce slow variables \( t_i = \frac{T_i}{\epsilon} \), \( \partial_{t_i} = \epsilon \partial_T \), and assume that \( g(z, T; \epsilon) = e^{-\tilde{S}(z, \bar{z}; T) / \epsilon} \), where \( \epsilon \) is a small dispersive parameter. We are looking for solutions of the \( \bar{\partial} \)-problem in the form

\[
\chi \left( z, \bar{z}; \frac{T}{\epsilon} \right) = \tilde{\chi}(z, \bar{z}; T; \epsilon) \exp \left( \frac{\tilde{S}(z, \bar{z}; T)}{\epsilon} \right),
\]

\[
(6)
\]

where \( \tilde{\chi}(z, \bar{z}; T; \epsilon) = \sum_{n=0}^{\infty} \varphi_n(z, \bar{z}; T) \epsilon^n \) is an asymptotic expansion. The function \( \tilde{S} \) has the following expansion

\[
\tilde{S} \rightarrow \frac{\tilde{S}_1}{z} + \frac{\tilde{S}_2}{z^2} + \frac{\tilde{S}_3}{z^3} + \ldots, \quad z \rightarrow \infty.
\]

\[
(7)
\]
Let us consider the $\bar{\partial}$-kernel of the following, quite general form:

$$R_0(\mu, \bar{\mu}; z, \bar{z}) = \frac{i}{2} \sum_{k=0}^{\infty} (-1)^k e^{k-1} \delta^{(k)}(\mu - z) \Gamma_k(z, \bar{z}). \quad (8)$$

The derivative $\delta$-functions $\delta^{(k,n)}$ are defined, in the standard way, as the distributions such that

$$-\frac{1}{2i} \int_{\mathcal{C}} d\mu \wedge d\bar{\mu} \delta^{(k,n)}(\mu - z) f(\mu, \bar{\mu}) = (-1)^{k+n} \frac{\partial^{k+n} f(z, \bar{z})}{\partial z^k \partial \bar{z}^n}, \quad (9)$$

and $\delta^{(k)} := \delta^{(k,0)}$.

Introducing

$$S(z, \bar{z}; T) = S_0(z; T) + \tilde{S}(z, \bar{z}; T), \quad (10)$$

one has, in the KP case,

$$S = \sum_{k=1}^{\infty} z^k T_k + \sum_{j=1}^{\infty} \frac{\tilde{S}_j}{z^j}, \quad z \to \infty. \quad (11)$$

A straightforward calculation for the problem (11) with the kernel (8) gives the dispersionless (classical) limit of $\bar{\partial}$-problem

$$\frac{\partial S}{\partial \bar{z}} = W \left( z, \bar{z}, \frac{\partial S}{\partial z} \right), \quad (12)$$

where

$$W = \sum_{k=0}^{\infty} (-1)^k \left( \frac{\partial S}{\partial z} \right)^k \Gamma_k. \quad (13)$$

In this limit the role of the nonlocal linear $\bar{\partial}$-problem (11) is held by the nonlinear $\bar{\partial}$-equation (12), that is a local nonlinear equation of Hamilton-Jacobi type. The latter encodes all informations about the integrable systems, like in the dispersionfull case. Nevertheless, there are substantial technical differences in the dressing procedure. In particular, the quasi-classical $\bar{\partial}$-method is based crucially on the Beltrami equation's properties. We present them in the next subsection.
3.2 The Beltrami equation \((BE)\).

The Beltrami equation \((BE)\) has the form

\[
\frac{\partial \Psi}{\partial \bar{z}} = \Omega(z, \bar{z}) \frac{\partial \Psi}{\partial z},
\]

where \(z \in \mathbb{C}\). Under certain conditions, it has the following properties (see e.g. [32]):

1. If \(\Omega\) satisfies \(|\Omega| < k < 1\), then the only solution of \(BE\) such that \(\frac{\partial \Psi}{\partial \bar{z}}\) is locally \(L^p\) for some \(p > 2\), and such that \(\Psi\) vanishes at some point of the extended plane \(\mathbb{C}^*\) is \(\Psi \equiv 0\) (Vekua’s theorem).

2. If \(\Psi_1, \Psi_2, \ldots, \Psi_N\) are solutions of \(BE\), a differentiable function \(f(\Psi_1, \Psi_2, \ldots, \Psi_N)\) with arbitrary \(N\) is a solution of \(BE\) too.

3. The solutions of the \(BE\) under the previous conditions give quasi-conformal maps with the complex dilatation \(\Omega(z, \bar{z})[33]\).

It is assumed, in all our further constructions, that these properties of \(BE\) are satisfied.

3.3 Zero order.

The zero order case has been widely discussed, for different integrable hierarchies in various papers [20, 21, 22, 26, 27]. Here, we outline the procedure for the KP hierarchy. The first crucial observation is that the symmetries of the \(\bar{\partial}\)-problem at leading order in \(\epsilon\), for any time \(T_j\), i.e. \(\delta S = \frac{\partial S}{\partial T_j} \delta T_j\), are given by the \(BE\)

\[
\frac{\partial}{\partial \bar{z}} \left( \frac{\partial S}{\partial T_j} \right) = W' \frac{\partial}{\partial z} \left( \frac{\partial S}{\partial T_j} \right).
\]

(15)

We assume that the complex dilatation \(W'\) has good properties required for the application of the Vekua’s theorem. From (11) it follows that

\[
\frac{\partial S}{\partial T_j} = z^j + \frac{1}{z} \frac{\partial S_1}{\partial T_j} + \frac{1}{z^2} \frac{\partial S_2}{\partial T_j} + \ldots, \quad z \to \infty.
\]

(16)

Using the \(BE\) properties, one gets the following equations in time variables [20, 21].
\[
\frac{\partial S}{\partial y} - \left( \frac{\partial S}{\partial x} \right)^2 - u_0(x, y, t) = 0, \tag{17}
\]
\[
\frac{\partial S}{\partial t} - \left( \frac{\partial S}{\partial x} \right)^3 - \frac{3}{2} u_0 \frac{\partial S}{\partial x} - V_0(x, y, t) = 0, \tag{18}
\]

where the notation

\[ T_1 = x; \quad T_2 = y; \quad T_3 = t, \tag{19} \]

is adopted and

\[ u_0 = -2 \partial \tilde{S}_1 \partial x; \quad \frac{\partial V_0}{\partial x} = \frac{3}{4} \frac{\partial u_0}{\partial y}. \tag{20} \]

Indeed, due to the property 2 of BE, the left hand sides of equations (17) and (18) are solutions of BE and since they vanish at \( z \to \infty \), then, by virtue of the Vekua’s theorem, they vanish identically on whole complex plane. Equations (17) and (18) are automatically compatible and by expansion in \( 1/z \)-power, according to the standard \( \partial \)-dressing procedure, one obtains the well known dispersionless KP (dKP) equation (or Zabolotskaya-Khokhlov equation)

\[
\frac{\partial^2 u_0}{\partial t \partial x} = \frac{3}{2} \frac{\partial}{\partial x} \left( u_0 \frac{\partial u_0}{\partial x} \right) + \frac{3}{4} \frac{\partial^2 u_0}{\partial y^2}. \tag{21}
\]

Usually the dKP equation and its dispersive corrections are obtained, by a slow times limit, from dispersionfull KP equation (5), inserting the following asymptotic expansion in terms of a small dispersive parameter

\[ u \left( \frac{T}{\varepsilon} \right) = \sum_{n=0}^{\infty} u_n(T) \varepsilon^n. \tag{22} \]

We stress that the formal expansion (22) implies, in general, the appearance of secular terms. Usually one imposes additional conditions on equations (17) and (18) to ensure the uniform validity of this expansion (23). In this paper we do not care about boundedness of solutions (22) and, consequently, about secular terms.
Actually, as it is well known, equations (17) and (18) are the first of an infinite set of equations in all time variables $T_n$, with $n \in \mathbb{N} \setminus \{0\}$, reproducing the whole KP hierarchy. Assuming

$$W(z, \bar{z}, T) = \theta(r - |z|)V \left( z, \bar{z}, \frac{\partial S}{\partial z} \right),$$

with $r > 0$, the function $S$ is an analytic function outside the circle $\mathcal{D} = \{ z \in \mathbb{C} \mid |z| < r \}$. That is in agreement with the asymptotic behavior (11). At last, the dKP hierarchy can be written in compact form as follows

$$\frac{\partial S}{\partial T_n}(z, T) = \Omega_n(p(z, T), T); \ n \geq 1,$$

where $z \in \mathbb{C} \setminus \mathcal{D}$, and

$$p := \frac{\partial S}{\partial x}.$$  

Taking into account that

$$p = z + \sum_{j=1}^{\infty} \frac{1}{z^j} \frac{\partial S_j}{\partial x},$$

$$\frac{\partial S}{\partial T_n} = z^n + O \left( \frac{1}{z} \right); \ \ \ z \to \infty,$$

then

$$\Omega_n(p, T) - (\mathcal{Z}_0^n(p, T))_+ = O \left( \frac{1}{z} \right); \ \ \ z \to \infty,$$

where $\mathcal{Z}_0$ denotes the expansion for $z$ obtained by inversion of equation (26), and the symbol $(\cdot)_+$ means the polynomial part of the expansion. The left hand sides of equations (28) are solutions of $BE$, so they vanish identically, i.e.

$$\Omega_n(p, T) = (\mathcal{Z}_0^n(p, T))_+.$$

Then, $\Omega_n$ can be connected to a suitable expansion series in terms of the $p$ variable. Assuming

$$\mathcal{Z}_0(p, T) = p + \sum_{j=1}^{\infty} \frac{a_j(T)}{p^j},$$

then

$$\mathcal{Z}_0^n(p, T) = p + \sum_{j=1}^{\infty} \frac{a_j(T)}{p^j},$$
one has
\[ \Omega_1 = p; \quad \Omega_2 = p^2 + 2a_1; \quad \Omega_3 = p^3 + 3a_1p + 3a_2, \quad (31) \]
that reproduces equations (17) and (18), by identifications
\[ u_0 = 2a_1; \quad V_0 = 3a_2. \quad (32) \]
The set of flows \( \Omega_n \) represents a set of quasi-conformal maps for which the
dispersionless integrable hierarchy describes a class of integrable deformations\[21\].

4 Quasi-classical \( \bar{\partial} \)-dressing method at any order.

In this section we calculate the corrections at any order for the kernel (8).
\[ R_0(\mu, \bar{\mu}; z, \bar{z}) = \frac{i}{2} \sum_{k=0}^{\infty} (-1)^k \epsilon^{k-1} \delta^{(k)}(\mu - z)\Gamma_k(z, \bar{z}). \]

Let us observe that the independence of \( \Gamma_{k,n} \) on the integration variables
\( \mu \) and \( \bar{\mu} \), does not reduce the generality of the kernel (8). Indeed, by the
delta function properties, it provides the same result, up to a redefinition, as
for the kernels where \( \Gamma_{k,n} \) depends on integration variables \( \mu \) and \( \bar{\mu} \).

Substituting the expansion (6) into equation (1), one obtains, in direct
way, the quasiclassical \( \bar{\partial} \)-problem at any order. For this purpose, it is conve-
nient to use the Faà de Bruno polynomials\[31\] defined as follows
\[ h_{n}[g(x)] = \partial_x h_{n-1}[g(x)] + h_1[g(x)]h_{n-1}[g(x)], \quad (33) \]
with \( h_0[g(x)] = 1 \) and \( h_1[g(x)] = h[g(x)] = \partial_x g(x), \partial_x \equiv \frac{\partial}{\partial x} \). Let us note
that \( h_{n}[g(x)] = h_n[g(x)] \left( \partial_x g, \partial_x^2 g, \ldots, \partial_x^n g \right) \), in other words, it depends
on the derivatives of \( g \) until \( n - th \) order.
In our case we have
\[ h_i[k \log S(z, \bar{z})], \quad (34) \]
for some $\tilde{k} \in \mathbb{N}$. Observing that

$$h_l[\tilde{k} \log S(z, \bar{z})] = \sum_{l'=1}^l C_{l,l'} S^{-l'},$$  \hspace{1cm} (35)

we denote

$$\mathcal{H}_{s,l} = \frac{C_{s,l}}{l! \left( \binom{\tilde{k}}{l} \right)},$$  \hspace{1cm} (36)

where $\tilde{k}$ is an arbitrary integer larger than $l$. We note that the quantity (36) does not depend on $\tilde{k}$. Now, we introduce the operator $\hat{W}_{p,q}$

$$\hat{W}_{p,q} = W_{p,q} \partial_z^p,$$

$$W_{p,q} = \sum_{k=0}^{\infty} (-1)^k \binom{k}{p} \mathcal{H}_{k-p,k-q} \Gamma_{k,n}.$$

In particular, we have

$$W = \hat{W}_{0,0} = \sum_{k=0}^{\infty} (-1)^k \mathcal{H}_{k,k} \Gamma_{k,n},$$

$$\mathcal{H}_{k,k} = \left( \frac{\partial S}{\partial z} \right)^k.$$

So, we have all ingredients to write the quasiclassical $\bar{\partial}$-problem at $n$th order:

$$\frac{\partial S}{\partial \bar{z}}(z, \bar{z}) = W(z, \bar{z}, \frac{\partial S}{\partial z}),$$  \hspace{1cm} (37)

$$\frac{\partial \varphi_0}{\partial \bar{z}} = W' \frac{\partial \varphi_0}{\partial \bar{z}} + \frac{1}{2} W'' \frac{\partial^2 S}{\partial z^2} \varphi_0,$$  \hspace{1cm} (38)

$$\ldots$$

$$\frac{\partial \varphi_n}{\partial \bar{z}} = W' \frac{\partial \varphi_n}{\partial \bar{z}} + \frac{1}{2} W'' \frac{\partial^2 S}{\partial z^2} \varphi_n + \sum_{q=1}^{n+1} \hat{W}_{p,q} \varphi_{n-q+1},$$  \hspace{1cm} (39)
where

\[ W^{(i)} = \frac{\partial^i}{\partial \xi^i} W(z, \bar{z}, \xi). \]

It is interesting to observe that equation \(37\) is a Hamilton-Jacobi type equation, the first order correction is a homogeneous linear equation, while the next orders are linear equations, but nonhomogeneous ones. In agreement with the normalization condition \(2\) we have

\[
\begin{align*}
\varphi_0 &\rightarrow 1 + \frac{\varphi_{0,1}}{z} + \frac{\varphi_{0,2}}{z^2} + \frac{\varphi_{0,3}}{z^3} + \ldots, \\
\varphi_1 &\rightarrow \frac{\varphi_{1,1}}{z} + \frac{\varphi_{1,2}}{z^2} + \frac{\varphi_{1,3}}{z^3} + \ldots, \\
\varphi_n &\rightarrow \frac{\varphi_{n,1}}{z} + \frac{\varphi_{n,2}}{z^2} + \frac{\varphi_{n,3}}{z^3} + \ldots; \quad n \in \mathbb{N} \setminus \{0\}.
\end{align*}
\]

The explicit problem at fourth order, i.e. \(n = 3\) is

\[
\begin{align*}
\frac{\partial S}{\partial \bar{z}} &= W \left( z, \bar{z}, \frac{\partial S}{\partial z} \right), \quad \text{(40)} \\
D_0 \varphi_0 &= 0, \quad \text{(41)} \\
D_0 \varphi_1 &= D_1 \varphi_0, \quad \text{(42)} \\
D_0 \varphi_2 &= D_1 \varphi_1 + D_2 \varphi_0, \quad \text{(43)} \\
D_0 \varphi_3 &= D_1 \varphi_2 + D_2 \varphi_1 + D_3 \varphi_0, \quad \text{(44)}
\end{align*}
\]

where
\begin{align*}
D_0 &= \frac{\partial}{\partial \bar{z}} - W \frac{\partial}{\partial z} - \frac{1}{2} W'' \frac{\partial^2 S}{\partial z^2}, \\
D_1 &= \frac{1}{2} W'' \frac{\partial^2 S}{\partial z^2} + \frac{1}{2} W'' \frac{\partial^2 S}{\partial z \partial \bar{z}} + \frac{1}{6} W''' \frac{\partial^3 S}{\partial z^3} + \frac{1}{8} W^{(4)} \left( \frac{\partial^2 S}{\partial z^2} \right)^2, \\
D_2 &= \frac{1}{6} W''' \frac{\partial^3 S}{\partial z^3} + \frac{1}{4} W^{(4)} \frac{\partial^2 S}{\partial z^2} \frac{\partial^2 S}{\partial z \partial \bar{z}} + \left( \frac{1}{6} W^{(4)} \frac{\partial^3 S}{\partial z^3} + \frac{1}{8} W^{(5)} \left( \frac{\partial^2 S}{\partial z^2} \right)^2 \right) \frac{\partial}{\partial z} + \\
&\quad + \frac{1}{24} W^{(4)} \frac{\partial^4 S}{\partial z^4} + \frac{1}{12} W^{(5)} \frac{\partial^2 S \partial^3 S}{\partial z^2 \partial z^3} + \frac{1}{48} W^{(6)} \left( \frac{\partial^2 S}{\partial z^2} \right)^3, \\
D_3 &= \frac{1}{24} W^{(4)} \frac{\partial^4 S}{\partial z^4} + \frac{1}{12} W^{(5)} \frac{\partial^2 S \partial^3 S}{\partial z^2 \partial z^3} + \left( \frac{1}{12} W^{(5)} \frac{\partial^3 S}{\partial z^3} + \frac{1}{16} W^{(6)} \left( \frac{\partial^2 S}{\partial z^2} \right)^2 \right) \frac{\partial^2}{\partial z^2} + \\
&\quad + \left( \frac{1}{24} W^{(5)} \frac{\partial^4 S}{\partial z^4} + \frac{1}{12} W^{(6)} \frac{\partial^2 S \partial^3 S}{\partial z^2 \partial z^3} + \frac{1}{48} W^{(7)} \left( \frac{\partial^2 S}{\partial z^2} \right)^3 \right) \frac{\partial}{\partial z} + \\
&\quad + \frac{1}{120} W^{(5)} \frac{\partial^5 S}{\partial z^5} + \frac{1}{72} W^{(6)} \left( \frac{\partial^3 S}{\partial z^3} \right)^2 + \frac{1}{48} W^{(6)} \frac{\partial^2 S \partial^4 S}{\partial z^2 \partial z^4} + \frac{1}{48} W^{(7)} \left( \frac{\partial^2 S}{\partial z^2} \right)^2 \frac{\partial^3 S}{\partial z^3} + \\
&\quad + \frac{1}{384} W^{(8)} \left( \frac{\partial^2 S}{\partial z^2} \right)^4.
\end{align*}

We stress the strong analogy between equations (40)-(44) and the Hamilton-Jacobi and higher transport equations which arise in the semi-classical approximation [24] and theory of geometric asymptotics [25]. Anyway, the main difference with the cited approaches is that in our case the role of the phase function is played by the function $S$ that is a complex valued one rather than a real valued one.

## 5 First order contribution.

The $BE$'s properties are very useful for the study of higher order corrections too [20]. This analysis has been initiated in the paper [20]. Some results of [20] are presented here for completeness. Fixed an arbitrary solution $S(z, \bar{z}, T)$ of equation (40), let $\varphi_0$ and $L \varphi_0$ be two solutions of equation (41), where $L$ is a suitable linear operator depending on time variables. The ratio $L \varphi_0/\varphi_0$
satisfies the BE
\[
\frac{\partial}{\partial \bar{z}} \left( \frac{L \phi_0}{\phi_0} \right) = W' \frac{\partial}{\partial z} \left( \frac{L \phi_0}{\phi_0} \right). 
\] (45)

Then, choosing \( L \) such that
\[
L \phi_0 \to 0; \; z \to \infty
\] (46)
as a result of the Vekua’s theorem, one gets
\[
L \phi_0 = 0; \; \forall z \in \mathbb{C}. 
\] (47)

In particular, for the first equation of the KP hierarchy we have
\[
L_1 \phi_0 \equiv \left( \frac{\partial}{\partial y} - 2 \frac{\partial S}{\partial x} \frac{\partial}{\partial x} - \frac{\partial^2 S}{\partial x^2} - u_1(x, y, t) \right) \phi_0 = 0, 
\] (48)
\[
L_2 \phi_0 \equiv \left[ \frac{\partial}{\partial t} - \left( 3 \left( \frac{\partial S}{\partial x} \right)^2 + \frac{3}{2} u_0 \right) \frac{\partial}{\partial x} - 3 \frac{\partial S}{\partial x} \frac{\partial^2 S}{\partial x \partial x^2} - 3 \frac{\partial S}{\partial x} u_1 + \frac{3}{4} \frac{\partial u_0}{\partial y} - V_1 \right] \phi_0 = 0, 
\] (49)
where
\[
u_1 = -2 \frac{\partial \phi_0}{\partial x}; \quad \frac{\partial V_1}{\partial x} = \frac{3}{4} \frac{\partial u_1}{\partial y}, 
\] (50)
are extracted from the condition (46) and \( u_0 \) is an arbitrary solution of equation (21).

Setting to zero the \( 1/z \)-power expansion’s coefficients in (48) and (49), we find the first dispersive correction to dKP equation
\[
\frac{\partial^2 u_1}{\partial t \partial x} = 3 \frac{\partial^2}{\partial x^2} (u_0 u_1) + 3 \frac{\partial^2 u_1}{\partial y^2}. 
\] (51)

Let us note that \( u_1 \) satisfies the equation defining the symmetries of dKP equation (21). Indeed, considering equation (21) for \( u_0 + \delta u_0 \), one, obviously, gets the equation
\[
\frac{\partial^2 (\delta u)}{\partial t \partial x} = S_0 (\delta u), 
\] (52)
where

\[ S_0(\cdot) = \frac{3}{2} \partial^2_x (u_0') + \frac{3}{4} \partial^2_y (\cdot), \quad (53) \]

that coincide with (51).

Now we will present some new results concerning the first order corrections.

**Theorem 1** Let \( A_n \) and \( C_n \) be the differentiable functions defined by

\[
\begin{align*}
A_n &= A_n(p, T) = \left( \frac{d (Z_0^n)}{dp} \right), \\
C_n &= C_n(p, \partial_x p, T) = \frac{1}{2} \frac{dA_n}{dx}(p, T),
\end{align*}
\]

where \( Z_0 \) is given by (30), \( B_n = B_n(p, T) \) an arbitrary differentiable function, \( \varphi_0 \) some solution of equation (41) and the linear operator \( L^{(n)} \) is given by

\[
L^{(n)} = \frac{\partial}{\partial T_n} - A_n \frac{\partial}{\partial x} - B_n - C_n. \quad (54)
\]

Then \( L^{(n)} \varphi_0 \) is also the solution of equation (41).

**Proof.** By the use of equation (40) and \( p = \frac{\partial S}{\partial x}(z, T) \) it’s immediate to verify that

\[
\begin{align*}
\frac{\partial A_n}{\partial z} &= W' \frac{\partial A_n}{\partial p} \frac{\partial}{\partial z} \frac{\partial S}{\partial x} = W' \frac{\partial A_n}{\partial z}, \\
\frac{\partial B_n}{\partial z} &= W' \frac{\partial B_n}{\partial p} \frac{\partial}{\partial z} \frac{\partial S}{\partial x} = W' \frac{\partial B_n}{\partial z}, \\
\frac{\partial C_n}{\partial z} &= W' \frac{\partial C_n}{\partial p} \frac{\partial}{\partial z} \frac{\partial S}{\partial x} + W'' \frac{\partial C_n}{\partial (\partial_x p)} \frac{\partial^2 S}{\partial x^2} + W'' \frac{\partial C_n}{\partial (\partial_x p)} \left( \frac{\partial}{\partial z} \frac{\partial S}{\partial x} \right)^2. \quad (57)
\end{align*}
\]

Differentiating equation (24) with respect to \( z \), we get

\[
\frac{\partial}{\partial z} \left( \frac{\partial S}{\partial T_n} \right) = A_n \frac{\partial}{\partial z} \left( \frac{\partial S}{\partial x} \right). \quad (58)
\]
Moreover, the definition of the function $C_n$

$$C_n = \frac{1}{2} \frac{dA_n}{dx} = \frac{1}{2} \frac{\partial A_n}{\partial p} \frac{\partial p}{\partial x} + \frac{1}{2} \frac{\partial A_n}{\partial x} , \quad (59)$$

implies that

$$\frac{\partial C_n}{\partial (\partial_x p)} = \frac{1}{2} \frac{\partial A_n}{\partial p} . \quad (60)$$

Calculating $D_0 (L^{(n)} \varphi_0)$, where $D_0$ is given in equation (45), and exploiting equations (40), (41) and the set of equalities (55)-(58), (60), one finds

$$D_0 (L^{(n)} \varphi_0) = 0. \quad (61)$$

This completes the proof. □

Based on the theorem (1), we choose, in particular,

$$B_n = \left( \eta(p, T) \frac{d(Z^n_0)}{dp} \right)_+ \quad (62)$$

where $\eta$ is a series defined by

$$\eta(p, T) = \sum_{j=1}^{\infty} b_j(T) \frac{1}{p^j} . \quad (63)$$

A choice of the coefficients in equation (30) and (63) in such way that $L \varphi_0 \to 0$ for $z \to \infty$, together with the previous arguments on the $BE$, gives us the following set of linear problems in time variables

$$L^{(n)} \varphi_0 = 0. \quad (64)$$

These equations can be rearranged by analogy with the leading order case, in the form

$$\frac{\partial \log \varphi_0}{\partial T_n}(z, T) = \Lambda_n (q(z, T), p(z, T), \partial_x p(z, T), T) ; \quad z \in \mathbb{C} \setminus D, \quad (65)$$

where
\[ \Lambda_n = \left( \mathcal{L}_1 \frac{d(Z^n_0)}{dp} + \frac{1}{2} \frac{d}{dx} \frac{d(Z^n_0)}{dp} \right) +, \quad (66) \]

\[ \mathcal{L}_1 := q + \eta(p, T), \quad q := \frac{\partial \log \varphi_0}{\partial x}. \]

Taking \( n = 1, 2, 3 \)

\[ \begin{align*}
\Lambda_1 &= q, \\
\Lambda_2 &= 2pq + 2b_1 + \partial_x p, \\
\Lambda_3 &= 3p^2q + 3a_1q + 3b_1p + 3p\partial_x p + \frac{3}{2} \partial_x a_1 + 3b_2,
\end{align*} \]

we reproduce equations (48) and (49) and, consequently, the first order dispersive correction to the dKP equation by the identifications

\[ \begin{align*}
u_0 &= 2a_1; \\
V_0 &= 3a_2; \\
u_1 &= 2b_1; \\
V_1 &= 3b_2.
\]

6 Higher order contributions.

Unlike the first order, the higher order corrections are characterized by non-homogeneous equations. Here we will show how the \( \bar{\partial} \)-dressing procedure works in these cases, discussing explicitly the KP equation.

Let us consider a set of functions \( \varphi_0, \varphi_1, \ldots, \varphi_n \) satisfying the \( \bar{\partial} \) problem at \((n+1)-th\) order and \( n+1 \) linear operators in time variables \( K^{(0)}, K^{(1)}, \ldots, K^{(n)} \) such that the quantity \( \sum_{m=0}^{n} K^{(m)} \varphi_{n-m} \) satisfies equation (41). Then the ratio

\[ \frac{\sum_{m=0}^{n} K^{(m)} \varphi_{n-m}}{\varphi_0} \]

is a solution of \( BE \) with complex dilatation \( W' \).
Using the same arguments as in the first order case, choosing $K^{(j)}$, $j = 0, \ldots, n$, in such a way that $\sum_{m=0}^{n} K^{(m)} \varphi_{n-m} \to 0$ for $z \to \infty$, we get the linear equations

$$\sum_{m=0}^{n} K^{(m)} \varphi_{n-m} = 0; \quad \forall z \in \mathbb{C}. \quad (68)$$

For instance, the second order dispersive corrections to dKP equation are associated with the following pair

$$K^{(0)}_1 \varphi_1 + K^{(1)}_1 \varphi_0 = 0, \quad (69)$$
$$K^{(0)}_2 \varphi_1 + K^{(1)}_2 \varphi_0 = 0, \quad (70)$$

where $K^{(0)}_1 = L_1$ and $K^{(0)}_2 = L_2$ are given by (48) and (49),

$$K^{(1)}_1 = -\frac{\partial^2}{\partial x^2} - u_2, \quad (71)$$
$$K^{(1)}_2 = -3 \frac{\partial S}{\partial x} \frac{\partial^2}{\partial x^2} \left(3 \frac{\partial^2 S}{\partial x^2} + 3 \frac{u_1}{2} \right) \frac{\partial}{\partial x} - \frac{\partial^3 S}{\partial x^3} - \frac{3 \partial S}{2 \partial x} u_2 +$$
$$-\frac{3}{4} \frac{\partial u_1}{\partial x} - V_2, \quad (72)$$

and

$$u_2 = -2 \frac{\partial \varphi_{1,1}}{\partial x}; \quad \frac{\partial V_2}{\partial x} = \frac{3}{4} \frac{\partial u_2}{\partial y}, \quad (73)$$

are deducted similarly to (50).

Equations (69) and (70) are compatible if and only if $u_2$ satisfies the second order dispersive correction to dKP equation

$$\frac{\partial^2 u_2}{\partial t \partial x} = \frac{3}{2} \frac{\partial^2}{\partial x^2} (u_0 u_2) + \frac{3}{4} \frac{\partial^2 u_2}{\partial y^2} + \frac{3}{4} \frac{\partial^2 u_1^2}{\partial x^2} + \frac{1}{4} \frac{\partial^4 u_0}{\partial x^4}. \quad (74)$$

The third order correction corresponds to the compatibility condition for the pair of the following linear problems

$$K^{(0)}_1 \varphi_2 + K^{(1)}_1 \varphi_1 + K^{(2)}_1 \varphi_0 = 0, \quad (75)$$
$$K^{(0)}_2 \varphi_2 + K^{(1)}_2 \varphi_1 + K^{(2)}_2 \varphi_0 = 0, \quad (76)$$
where

\[ K_{1}^{(2)} = -u_3, \]  
\[ K_{2}^{(2)} = -\frac{\partial^3}{\partial x^3} - \frac{3}{2} u_3 \frac{\partial S}{\partial x} - \frac{3}{2} u_2 \frac{\partial}{\partial x} - \frac{3}{4} \frac{\partial u_2}{\partial x} - V_3, \]

with the definitions

\[ u_3 = -2 \frac{\partial \varphi_{2,1}}{\partial x}; \quad \frac{\partial V_3}{\partial x} = \frac{3}{4} \frac{\partial u_3}{\partial y}. \]

The equation for \( u_3 \) is of the form

\[ \frac{\partial^2 u_3}{\partial t \partial x} = \frac{3}{2} \frac{\partial^2}{\partial x^2} (u_0 u_3) + \frac{3}{2} \frac{\partial^2}{\partial x^2} (u_1 u_2) + \frac{3}{4} \frac{\partial^2 u_3}{\partial y^2} + \frac{1}{4} \frac{\partial^4 u_1}{\partial x^4}. \]

A simple observation from equations (69) and (70) allows us to rewrite them in a compact form, suggesting the generalization to any order. In order to realize that, we introduce the following operator

\[ \Delta^k [S; \theta] = \begin{cases} e^{-\theta S} \frac{\partial^k}{\partial x^k} e^{\theta S} & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases} \]

Noting that for \( k \geq 0 \)

\[ \frac{dr}{d\theta} \Delta^k [S; \theta] = \left[ \ldots \left[ \frac{\partial^k}{\partial x^k}, S \right], \ldots, S \right] \] \( r \) brackets

the transport equations can be written as follows

\[ \frac{\partial \varphi_n}{\partial T_k} = \sum_{r=0}^{k-1} \frac{1}{r!} \frac{dr}{d\theta} \Delta^k [S; 0] \varphi_{n-k+r+1} + \frac{k}{2} \sum_{j=0}^{n} \sum_{r=0}^{k-2} u_{j+r} \frac{dr}{d\theta} \Delta^{k-2} [S; 0] \varphi_{n-j} + \]

\[ + \sum_{j=0}^{n} \left( \frac{3}{4} \frac{\partial u_j}{\partial x} + V_{j+1} \right) \Delta^{k-3} [S; 0] \varphi_{n-j}, \]

for \( k = 1, 2, 3. \)

It would be nice to get a similar formula for higher times too.
At last, as we noted above, a solution $u_1$ of the first dispersive equation is defined up to a symmetry of the dKP equation. Using this freedom one can fix a “gauge” putting $u_1 = 0$. Equations for higher corrections imply that the gauge $u_n = 0$ for each odd $n$ is an admissible one. In such a gauge the even order corrective equations take the form

$$\frac{\partial^2 (u_n)}{\partial t \partial x} = S_0 (u_n) + f_n, \quad n \text{ even.}$$

(84)

That is a nonhomogeneous equation with the corresponding homogeneous one given by the symmetry equation. By virtue of the Fredholm’s theorem, the nonhomogeneous term must be orthogonal to the solutions $\delta^* u$ of adjoint homogeneous equation, i.e.

$$\int \delta^* u(P) f_n(P) dP = 0, \quad P = (x, y, t, \ldots).$$

(85)

One can check that for regular solutions of the KP equation, the condition (85) is satisfied. Quite different situation takes place for the singular sector of the dKP equation. For instance, for breaking waves solutions for which $\delta u_0 \equiv \frac{\partial u_0}{\partial x} \to \infty$, the condition (85) breaks. So, there are obstacles to construct the higher order quasi-classical corrections for dKP equation originated from its own singular sector. Such an obstacle is analogous to a typical one appearing in the construction of global asymptotics in the semi-classical approximation to the quantum mechanics[24] and in the study of the wave corrections to geometrical optics[25], because of existence of caustics. In the dispersionless KdV case this kind of problem induces a stratification of the affine space of times providing a classification of the singularities[34]. The problem of obstacles for the construction of the higher order corrections to dKP equation will be considered elsewhere.

7 A Whitham type structure.

It is well known that the dispersionless limit of some integrable hierarchies admits a symplectic structure[4]-[7]. In particular, one can see that starting with the function $S(z, T)$ outside the circle $D$ it is possible to introduce a 2-form closed $\omega_0$ in terms of the p-variable such that

$$\omega_0 = d\Omega_n(p, T) \wedge dT_n = dZ_0(p, T) \wedge d\tilde{\mathcal{M}}_0(p, T),$$

(86)

19
where the sum over index \( n \) is assumed. Here the last member contains the pair of Darboux coordinates \( Z_0 \) and \( \tilde{M}_0(p, T) = M_0(Z_0(p, T), T) = M_0(z, T) \) and the function

\[
M_0(z, T) := \frac{\partial S}{\partial z}(z, T)
\]

is usually called Orlov’s function \cite{35}.

From (86), it follows that

\[
1 = \{Z_0, \tilde{M}_0\}_p, \tag{88}
\]

\[
\frac{\partial Z_0}{\partial T_n} = \{\Omega_n, Z_0\}_p, \tag{89}
\]

\[
\frac{\partial \tilde{M}_0}{\partial T_n} = \{\Omega_n, \tilde{M}_0\}_p. \tag{90}
\]

where the Poisson bracket is defined as follows

\[
\{f(\alpha, \beta), g(\alpha, \beta)\}_{\alpha, \beta} = \frac{\partial f}{\partial \alpha} \frac{\partial g}{\partial \beta} - \frac{\partial g}{\partial \alpha} \frac{\partial f}{\partial \beta}. \tag{91}
\]

Equation (88) is usually referred to as string equation \cite{6} and equation (89) is the dispersionless Lax pair. Moreover, equation (86) implies the zero curvature condition

\[
\omega_0 \wedge \omega_0 = 0. \tag{92}
\]

That is another way to write the compatibility condition between equations (89) or (90) ; indeed, expanding the total differentials, one gets from equation (92) the set of equations

\[
\frac{\partial \Omega_n}{\partial T_m} - \frac{\partial \Omega_m}{\partial T_n} + \{\Omega_n, \Omega_m\}_p, \tag{93}
\]

that are the universal Whitham hierarchy equations \cite{4}. In particular, for \( n = 2 \) and \( m = 3 \) one obtains equation (21).

The construction of the infinite set of flows in equation (65), allows us to describe the first correction in a way analogous to the zero curvature condition (92). Let us start with the total differential of \( \log \phi_0 \)

\[
d \log \varphi_0(z, T) = \Lambda_n(g(z, T), p(z, T), \partial_z p(z, T), T) dT_n + M_1(z, T) dz. \tag{94}
\]
The function
\[ \mathcal{M}_1(z, T) = \frac{\partial \log \varphi_0(z, T)}{\partial z} \] (95)
is an analogous of the Orlov's function. Differentiating equation (94), one can introduce the 2-form
\[ \omega_1 := d\Lambda_n (q(z, T), p(z, T), \partial_x p(z, T), T) \wedge dT_n = dz \wedge d\mathcal{M}_1(z, T). \] (96)

The equation (86) allows us to identify a Whitham type structure defined by the equation
\[ \omega_1 \wedge \omega_1 = 0, \] (97)
which can be given explicitly
\[
\frac{\partial \Lambda_n}{\partial T_m} - \frac{\partial \Lambda_m}{\partial T_n} + \{\Lambda_n, \Lambda_m\}_q, x + \{\Lambda_n, \Lambda_m\}_q, p \frac{\partial p}{\partial x} + \{\Lambda_n, \Lambda_m\}_q, \partial_x p \frac{\partial^2 p}{\partial x^2} = G_n(\Lambda_m) - G_m(\Lambda_n), \] (98)
where \( G_n \) is defined as
\[
G_n(f) = \left( \frac{d\Omega_n}{dx} \frac{\partial}{\partial p} + \frac{d^2\Omega_n}{dx^2} \frac{\partial}{\partial x (\partial_x p)} \right) f. \] (99)

Let us note that for \( n = 2 \) and \( m = 3 \), equation (98) gives, of course equation (51).

8 Concluding remarks and perspectives.

In this paper we demonstrated how to generalize the quasi-classical \( \bar{\partial} \)-dressing method at any dispersive order. In particular, the formula (65) suggests that a regular structure survives at higher orders, allowing to define a potential encoding all informations about the hierarchy at the first dispersive order, just like in the purely dispersionless case. Moreover the paper provides us with several intriguing observations which could be subjects of future study. In particular, it would be useful to analyze the connection between the standard theory of the asymptotics approximation and the quasi-classical \( \bar{\partial} \)-dressing.
approach. This could set a relationship between the source of the obstacles in
the construction of the corrections, discussed in the section (6), and the cau-
stics problem. As far as concerning possible applications it would be of interest
to consider the KdV limit of the KP hierarchy to establish a connection with
the Dubrovin-Zhang theory[3]. Another intriguing matter of study and appli-
cation is associated with a possible physical interpretation of the solutions of
dispersionless systems as describing integrable dynamics of interfaces[17]. In
this context, one should investigate a quasiconformal dynamics described by
the quasiclassical $\bar{\partial}$ problem and the corrections (37-39). One could analyze
the singular behavior of interfaces’ evolution, by introducing, for instance,
a small “dispersive” parameter. For such a purpose, a deeper study of the
singular sector order by order will be also necessary.

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