A LIE ALGEBRA THAT CAN BE WRITTEN AS A SUM OF TWO NILPOTENT SUBALGEBRAS, IS SOLVABLE

P.A. ZUSMANOVICH

In 1963 O. Kegel raised the following question: is a Lie ring written as a sum of two nilpotent subrings solvable? Recently, Kostrikin [1] brought a renewed attention to this question in the case of finite-dimensional algebras over a field. The question is easily solved in the affirmative in the case of characteristic zero (cf. [1] or [2]). The purpose of this note is to prove the following theorem.

**Theorem.** Over a field of characteristic \( p > 5 \), a finite-dimensional Lie algebra written as a sum of two nilpotent subalgebras, is solvable.

In [1], [3], [4], [5], a similar statement is proved under additional restrictions on the nilpotency index of one summand (with fewer restrictions on the characteristic of the ground field). Note that the theorem is no longer true when \( p = 2 \) (an appropriate counter example has been constructed in [4]). We make an essential use of Weisfeiler’s results [6] on Lie algebras with a solvable maximal subalgebra which dictates a restriction upon the characteristic of the ground field.

We turn to the proof of the theorem. We may assume the ground field \( K \) to be algebraically closed. Let \( L \) be a counter example to the theorem having the least possible dimension, let \( L = A + B \) be the corresponding decomposition into a sum of nilpotent subalgebras, where \( \dim A \leq \dim B \). The following lemma can be easily deduced from the minimality of the counter example (cf. [1]).

**Lemma 1.**

(i) \( L \) is semisimple.

(ii) If \( L_0 \) is a proper subalgebra of \( L \) containing \( A \) (or \( B \)), then \( L_0 \) is solvable. In particular, \( L \) possesses a solvable maximal subalgebra.

The following result has been proved in [6]: a maximal solvable subalgebra in a simple Lie algebra determines a long filtration in it. A closer analysis of this proof enables us to replace the simplicity condition by semisimplicity. Indeed, a semisimple Lie algebra with a solvable maximal subalgebra possesses a unique ideal by Block’s theorem [7]. The proof of Theorem 1.2.2 in [6] may be repeated for such algebras almost verbatim. But the proof of Theorem 1.5.1 relies only on the conclusion of Theorem 1.2.2 and never makes use of the simplicity of the algebra.

Let \( L_0 \) be a maximal solvable subalgebra of \( L \) containing \( A \) (it exists due to Lemma 1). By the above, we can apply Theorem 2.1.3 in [6] which states that for the filtration \( L \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots \), determined by the subalgebra \( L_0 \), the associated graded algebra will have the form

\[
gr L = G = S \otimes O_m + D,
\]

where \( S \) is a simple Lie algebra isomorphic to \( sl_2(K) \) or to the Zassenhaus algebra \( W_1(n) \), \( O_m \) is the algebra of truncated polynomials in \( m \) variables, \( D \) is a solvable subalgebra of \( W_m = Der(O_m) \) such that \( O_m \) contains no \( D \)-invariant ideals. The grading is given as

\[\text{Math. Notes 50 (1991), 909-912.}\]
follows:

\( G_i = \langle e_i \rangle \otimes O_m, \quad i \neq 0 \)

\( G_0 = \langle e_0 \rangle + D, \)

where \( S = \bigoplus_{i \geq -1} \langle e_i \rangle \) is the standard grading. Henceforth, this particular grading is meant each time we refer to \( S \otimes O_m + D \) as a graded algebra.

If \( D = 0 \), then \( L \) is simple and, by Corollary 2.1.4 in [6], \( L = S \). The impossibility of representing it as a sum of two nilpotent subalgebras can be easily verified in this case, for instance, by means of an argument which we will apply below in the general case. So we assume henceforth that \( D \neq 0 \). Furthermore, in the case of \( S = sl_2(K) \) the remaining argument is either the same as that for \( S = W_1(n) \), or much simpler. Thus, we put henceforth \( S = W_1(n) \).

It has been shown in [8] that each \( \{ G_i \} \)-deformation of the algebra \( G \) determined by the grading (1) contains an ideal which is a deformation of the algebra \( W_1(n) \otimes O_m \). We will provide a cohomological proof of a somewhat more general result using definitions, notation, and facts from [9].

We can write

\( \{ x, y \} = [x, y] + \sum_{s \geq 1} \psi_s(x, y) = [x, y] + \psi(x, y), \)

where \( [\cdot, \cdot] \) and \( \{ \cdot, \cdot \} \) are multiplications in the algebras \( G \) and \( L \), respectively, and

\( \psi_s \in C^2_s(G, G) = \{ \psi \in C^2(G, G) \mid \psi(G_i, G_j) \subseteq G_{i+j+s} \}. \)

The Jacobi identity implies that

\( d\psi_s + \sum_{i+j=s} \psi_i \ast \psi_j = 0, \)

where \( d \) is the coboundary operator, and \( \ast \) is defined as follows:

\( \varphi \ast \psi(x, y, z) = \varphi(\psi((x, y), z)) + \varphi(\psi(z, x), y) + \varphi(\psi(y, z), x). \)

We want to show that, up to coboundaries, \( \psi_s = 0 \) for \( 1 \leq s < p \). Suppose that, by means of a coboundary change, we have already achieved the equalities \( \psi_s = 0, \ 1 \leq s < k < p \). By virtue of (2), \( \psi_k \in Z^2_k(G, G) \). Defining the action of \( G \) on \( C^2(G, G) \) in the standard way, it is easy to see that \( Z^2_k(G, G) \) is invariant relative to the action of \( G_0 \).

Consider the torus \( e_0 \otimes \langle 1 \rangle \) in \( G \). The root subspaces relative to the action of this torus in \( G \) are

\( \widehat{G}_i = \langle e_i, e_{i+p}, e_{i+2p}, \ldots, e_{i+p^n-p} \rangle \otimes O_m, \quad i \in \mathbb{Z}_p, i \neq 0, \)

\( \widehat{G}_0 = \langle e_0, e_p, e_{2p}, \ldots, e_{p^n-p} \rangle \otimes O_m + D. \)

By the above, each cocycle in \( Z^2_k(G, G) \) is cohomologically equivalent to some cocycle in \( Z^2_k(G, G) \) invariant relative to the action of this torus, and we may put

\( \psi_k(\widehat{G}_i, \widehat{G}_j) \subseteq \widehat{G}_{i+j}. \)

But since \( \psi_k \in Z^2_k(G, G) \) and \( k < p \), we have \( \psi_k = 0 \). Thus,

\( \psi_k(G_i, G_j) \subseteq \bigoplus_{s \geq p} G_{i+j+s}. \)
We will view the algebra $L$ as the vector space $W_1(n) \otimes O_m + D$ with multiplication $\{ \cdot, \cdot \}$. On the subalgebra $B$ an induced filtration can be constructed: $B_i = B \cap L_i$. We define a homomorphism

$$\varphi : grB \rightarrow grL, \quad x + B_{i+1} \mapsto x + L_{i+1}, \quad x \in B_i.$$  

It is easily seen that $Ker \varphi = 0$ and, therefore, we may identify $grB$ with a subalgebra of $grL$.

**Lemma 2.**

(i) $grB$ is a homogeneous nilpotent subalgebra of $grL$.

(ii) $(grB)_{-1} = e_{-1} \otimes O_m$.

(iii) $pr_{DgrB} = D$, where the left-hand side of the equality denotes the projection of $grB$ onto $D$ (in the algebra $grL$).

**Proof.** (i) is obvious.

(ii) Since $L = A + B$ and $A \subseteq L_0$, we have

$$(grB)_{-1} = (grL)_{-1} = e_{-1} \otimes O_m.$$  

(iii) Clearly, $pr_{DgrB}$ is a subalgebra of $D$. Therefore, $W_1(n) \otimes O_m + pr_{DgrB}$ is a subalgebra of $W_1(n) \otimes O_m + D$ containing $grB$ and closed under the multiplication $\{ \cdot, \cdot \}$ (this follows from (3)). So there exists a subalgebra $M$ of $L$ such that $B \subseteq M \subseteq L$ and $grM = W_1(n) \otimes O_m + pr_{DgrB}$. If $pr_{DgrB} \neq D$, then $M \neq L$ and, by Lemma 2, $M$ is solvable, whence $grM$ is solvable, which is impossible. \hfill \Box

**Lemma 3.** Suppose that $N$ is a subalgebra of $W_1(n) \otimes O_m + D$ (relative to the usual multiplication $[\cdot, \cdot]$), and $O_m$ contains no proper $D$-invariant ideals. If $N$ satisfies the conclusions of Lemma 2, then

(i) $N_0 \simeq D$.

(ii) $N \subseteq (e_{-1}, e_0) \otimes O_m + D$.

(iii) $D$ consists of nilpotent (viewed as derivations of $O_m$) elements.

**Proof.** Put $F = \{ f \in O_m \mid e_0 \otimes f \in N \}$. We choose an arbitrary element $d \in D$ and find $g \in O_m$ such that $d + e_0 \otimes g \in N$. Then for each $f \in F$

$$e_0 \otimes d(f) = [e_0 \otimes f, d + e_0 \otimes g] \in N.$$  

Therefore, $D(F) \subseteq F$. But then $FO_m$ is a $D$-invariant ideal of $O_m$, whence $FO_m = 0$ or $O_m$. Thus, either $F = 0$, or $F$ contains a polynomial $f$ with a nonzero constant term. In the latter case the equality

$$ad(e_0 \otimes f)^p(e_{-1} \otimes 1) = e_{-1} \otimes f^p = e_{-1} \otimes 1$$  

leads to a contradiction with the nilpotency of $N$. Thus $F = 0$. The map $e_0 \otimes f + d \mapsto d$ is the isomorphism required in (i).

Let $e_i \otimes g \in N$, $i > 0$. Commuting this element as many times as necessary with $e_{-1} \otimes 1$, we obtain the element $e_0 \otimes g \in N$, whence $g = 0$, which proves (ii).

Choose again an arbitrary element $d \in D$, find $g \in O_m$ such that $d + e_0 \otimes g \in N$, and note that the action of $ad(d + e_0 \otimes g)$ on $e_{-1} \otimes O_m$ is determined by the action of the operator $d + R_g$ on $O_m$, where $R_g$ is the multiplication by the element $g$ in $O_m$. The nilpotency of $N$ implies that $(d + R_g)^p = 0$ for some $k$. Jacobson’s formula and the last equality imply that $db^k = 0$, which proves (iii). \hfill \Box

By Lemmas 2 and 3, $\dim B = \dim grB \leq p^m + \dim D$. According to our initial assumption, $\dim L \leq 2 \dim B$, whence $p^{n+m} + \dim D < 2p^m + 2 \dim D$ and $\dim D \geq p^m$. The proof of the theorem is concluded by
Lemma 4. Suppose $D$ is a subalgebra of $W_m$ consisting of nilpotent elements, and $O_m$ contains no $D$-invariant ideals. Then $\dim D < p^m$.

Proof. We will perform induction on $m$. For $m = 1$ the statement of the lemma is obvious (each such subalgebra is one-dimensional). Put $Z = \{z \in Z(D) \mid z^p = 0\}$. We have

$$D(Z(O_m)) \subseteq Z(D(O_m)) \subseteq Z(O_m).$$

Arguing like in the proof of Lemma 3, we deduce that either $Z(O_m) = 0$ or $Z(O_m)$ contains a polynomial with a nonzero constant term. But the former is impossible because $D$ consists of nilpotent elements, so $Z \neq 0$. Therefore, $Z \nsubseteq (W_m)_0$, where $(W_m)_0$ is the zeroth term in the standard filtration (otherwise $Z(O_m)$ is contained in the maximal ideal of $O_m$). Choose $z \in Z$, $z \notin (W_m)_0$. It follows from Demushkin’s results [10] that $z$ is conjugate to the element $\partial/\partial x_1$ in $W_m$. Therefore, we may assume that

$$D \subseteq C_{W_m}(\partial/\partial x_1) = \{f \partial/\partial x_1 \mid f \in O[x_2, \ldots, x_m] + W_{m-1}(x_2, \ldots, x_m)\}.$$ 

Here $W_{m-1}(x_2, \ldots, x_m)$ is the Lie algebra of derivations of the algebra $O[x_2, \ldots, x_m]$ of truncated polynomials in the variables $x_2, \ldots, x_m$. It is easily seen that the first term is an ideal in this centralizer, so $pr_{W_{m-1}}D$ is a subalgebra of $W_{m-1}$ consisting of nilpotent elements. If $I$ is a $pr_{W_{m-1}}D$-invariant ideal in $O[x_2, \ldots, x_m]$, then $(x_1)I$ is a $D$-invariant ideal in $O_m$. Therefore, $I$ is a trivial ideal and we may apply the induction hypothesis:

$$\dim D \leq p^{m-1} + \dim pr_{W_{m-1}}D < p^{m-1} + p^{m-1} < p^m.$$ 

□

Remark. Of course, Lemma 3 is far from being the best result in this direction, but, apparently, it suffices for our purposes.

In conclusion, we make several remarks concerning the possibility of obtaining the converse to the theorem. As has been noted in [2], the equality $L = H + L^2$, where $H$ is the Cartan subalgebra of a Lie algebra $L$, provides a decomposition into a sum of two nilpotent subalgebras for Lie algebras with a nilpotent commutant. As has been shown in [11], this class of algebras coincides with the class of supersolvable Lie algebras. The semidirect sum $L + V$, where $L$ is a nilpotent Lie algebra and $V$ is a faithful irreducible $L$-module, provides an example of a nonsupersolvable Lie algebra with such a decomposition. On the other hand, if $L$ is the two-dimensional non-abelian Lie algebra and $V$ is the irreducible $p$-dimensional $L$-module, then we obtain an example of a solvable Lie algebra for which such a decomposition is impossible. Thus, the class of Lie algebras that can be represented as sums of two nilpotent subalgebras contains the class of supersolvable algebras and is contained in the class of solvable algebras, both inclusions being strict. It would be interesting to provide a description of this class.

The author is grateful to A.S. Dzhumadil’daev for his attention and help in this work.

Literature cited

[1] A.I. Kostrikin, A solvability criterion for a finite-dimensional Lie algebra, Vestn. Mosk. Univ., Mat., Mekh., No. 2, 5–8 (1982).
[2] M. Goto, Note on a characterization of solvable Lie algebras, J. Sci. Hiroshima Univ., Ser. A-1, 26, No. 1, 1–2 (1962).
[3] C. Pillen, Die Summe einer abelschen und einer nilpotenten Lie-Algebra ist auflösbar, Result. Math., 11, Nos. 1-2, 117–121 (1987).
[4] A.P. Petravchuk, Lie algebras decomposable into sums of an abelian and a nilpotent subalgebra, Ukr. Mat. Zh., 40, No. 3, 385-388 (1988)
A.LIE ALGEBRA THAT CAN BE WRITTEN AS A SUM OF TWO NILPOTENT SUBALGEBRAS

[5] A.G. Gein and F.L. Tolstov, On solvability of algebras decomposable into sums of two nilpotent subalgebras, Izv. Vuzov, Matematika (1989).

[6] B. Weisfeiler, On subalgebras of simple Lie algebras of characteristic $p > 0$, Trans. Amer. Math. Soc., 286, No. 2, 471–503 (1984).

[7] R.E. Block, Determination of the differentably simple rings with a minimal ideal, Ann. Math., 90, No. 3, 433–459 (1969).

[8] M.I. Kuznetsov, Modular simple Lie algebras with a solvable maximal subalgebra, Mat. Sb., 101, No. 1, 77–86 (1976).

[9] A.S. Dzhumadil’daev, Cohomology of Lie algebras of positive characteristic and their applications, Doctoral dissertation, Alma-Ata (1988).

[10] S.P. Demushkin, Cartan subalgebras in the simple Lie $p$-algebras $W_n$ and $S_n$, Sib. Mat. Zh., 11, No. 2, 310–325 (1970).

[11] A.S. Dzhumadil’daev, Irreducible representations of strongly solvable Lie algebras over a field of positive characteristic, Mat. Sb., 123, No. 2, 212–229 (1984).

Institute of Mathematics and Mechanics, Academy of Sciences of the Kazakh SSR.