STRUCTURE OF TREES WITH RESPECT TO NODAL VERTEX SETS

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Abstract. Let \( T \) be a tree with a given adjacency eigenvalue \( \lambda \). In this paper, by using the \( \lambda \)-minimal trees, we determine the structure of trees with a given multiplicity of the eigenvalue \( \lambda \). Furthermore, we consider the relationship between the structure of trees and the eigensystem of a given Laplacian eigenvalue.

1. Introduction

All graphs in this paper are finite and simple, unless noted otherwise. We denote by \([n]\) the set \{1,\ldots,n\}. For any integer \( n \in \mathbb{N} \), we denote by \( M_n(\mathbb{R}) \) and \( \text{Sym}_n(\mathbb{R}) \) the set of all \( n \times n \) real matrices and all \( n \times n \) symmetric matrices, respectively. The complete graph on \( n \) vertices is denoted by \( K_n \). For any graph \( G \), a vertex \( v \in V(G) \), and a subgraph \( H \) of \( G \), we denote by \( N_H(v) \) the set \{\( u : u \in V(H), uv \in E(G) \)\}. The set of all eigenvalues of \( G \) is denoted by \( \text{Spec}(G) \).

Let \( T \) be a tree. We denote by \( \rho(T) \) the spectral radius (the largest eigenvalue) of \( T \). We say a vector \( x \in \mathbb{R}^n \) is nowhere-zero, if for every \( i \in [n] \), \( x_i \neq 0 \). For any integers \( i, e \), denotes the vector with a 1 in the \( i \)th coordinate and 0’s elsewhere. Let \( \theta \) be an algebraic integer with the monic minimal polynomial \( f(x) \). The norm of \( \theta \) is the product of the conjugates of \( \theta \) and we denote by \( \text{Norm}(\theta) \). We say \( \theta \) is a totally real algebraic integer if every root of \( f(x) \) is real. We denote by \( \text{TRAI} \), the set of all totally real algebraic integers. We say \( \theta \) is a totally positive algebraic integer if every root of \( f(x) \) is a positive real number. We denote by \( \text{TPAI} \), the set of all totally positive algebraic integers. The following theorem states that every element of \( \text{TRAI} \) is a tree eigenvalue.

Theorem 1. [16, Theorem 1] Every totally real algebraic integer is an eigenvalue of some finite tree.

In this paper, we define the \( \lambda \)-minimal (resp., \( \mu \)-minimal for a Laplacian eigenvalue) trees and by using them, we determine the structure of trees with eigenvalue \( \lambda \) (resp. \( \mu \)) of a given multiplicity. There are many papers on considering the structure of trees with a given adjacency eigenvalue \( \lambda \). These papers consider the structure of trees and study the relationship between the structure of trees and their eigenvectors and eigenvalues. In these works, the authors consider the zero coordinates of eigenvectors and their corresponding vertices. See [4,5], [8], [11–14], and [17].

In this paper, we consider the structure of acyclic matrices and prove results above in a unified and new manner. Also, as a consequence, we obtain some results on the relationship between the structure of trees and the Laplacian eigenvectors. In Section 2, we consider the relationship between the multiplicity of an eigenvalue \( \lambda \) of an acyclic matrix and its structure. In Sections 3, we consider the \( \lambda \)-minimal trees for the adjacency eigenvalues of trees. In Section 4, we define minimal cut-trees for the Laplacian eigenvalues.

2. The Multiplicity and Nodal Vertex Sets

Suppose that \( M = [m_{ij}] \in \text{Sym}_n(\mathbb{R}) \). We denote by \( G_M \) the simple graph on \( n \) vertices such that for every \( i, j \in V(G_M) \), \( i \) and \( j \) are adjacent if and only if \( m_{ij} \neq 0 \). Whenever \( G_M \) is a tree

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Lemma 4. \[ x \in i_c \]

Suppose that \( \theta \) is an eigenvalue of \( A \), then there exists a \( \theta \)-eigenvector \( x \) such that for every \( u \notin N_\theta(A) \), \( x_u \neq 0 \).

Proof. Suppose that \( m_\theta(A) = k \) and \( x_1, \ldots, x_k \) are \( k \) orthogonal \( \theta \)-eigenvectors. If \( \alpha, \beta \in \mathbb{R}^n \), then we can choose \( c \in \mathbb{R} \) such that for the vector \( y = \alpha + c\beta \), \( y_i = 0 \) if and only if \( \alpha_i = \beta_i = 0 \), \( i \in [n] \). Therefore, we choose some real numbers \( c_1, \ldots, c_k \) and put \( x = \sum_{i=1}^k c_i x_i \) such that \( x_u \neq 0 \) if \( u \notin N_\theta(A) \).

Lemma 4. [2, Lemma 9] Let \( H \) and \( L \) be two symmetric matrices with row and column indices \( I \) and \( J \), respectively. Suppose that \( m_H(\lambda) = 1 \) and \( G \) is the symmetric matrix given below,

\[
G = \begin{pmatrix} I & H & x & A \\ v & x^T & \alpha & y \\ J & A^T & y & L \end{pmatrix}
\]

for a matrix \( A \) and a vector \( x \). If \( A^T \alpha = 0 \) and \( x^T \alpha \neq 0 \), for a \( \lambda \)-eigenvector \( \alpha \) of \( H \), then \( m_G(-\varepsilon)(\lambda) = m_G(\lambda) + 1 \).

Remark 1. In the proof of [2, Lemma 8], it is shown that for every \( \lambda \)-eigenvector \( \xi \) of \( G \), \( \xi = 0 \).

Proposition 5. [2, Corollary 10] Let \( H \) and \( L \) be two vertex disjoint (weighted) graphs such that \( m_H(\theta) = 1 \), for some \( \theta \in \mathbb{R} \). Suppose that \( G \) is a graph formed by joining an arbitrary vertex \( v \in V(L) \) to some arbitrary vertices of \( H \). If for a \( \theta \)-eigenvector \( x \) of \( H \), \( \sum_{u \in N_H(v)} x_u \neq 0 \), then \( m_{L-v}(\theta) = m_{G}(\theta) \).

\[ \begin{array}{c}
L - v: \\
L \\
H
\end{array} \]

Figure 1. The graph \( G \) of Proposition 5.

The following elementary equation is important for us in the sequel. For any matrix \( M = [m_{ij}] \in \text{Sym}_n(\mathbb{R}) \) with an eigenvalue \( \theta \) and a \( \theta \)-eigenvector \( x \), for every \( i \in [n] \), we have

\[
(\theta - m_{ii})x_i = \sum_{j \neq i} m_{ij} x_j = \sum_{j \in E(G_M)} m_{ij} x_j.
\]
Connection Between Two Matrices: Let \( A = [a_{ij}] \in \text{Sym}_n(\mathbb{R}) \) and \( B = [b_{ij}] \in \text{Sym}_m(\mathbb{R}) \) for some \( m, n \in \mathbb{N} \). We define a matrix \( C \) as a connection between \( A \) and \( B \) for some \( r \in [n], s \in [m] \) and a nonzero real number \( \omega \), as follows:

\[
C = \begin{pmatrix} r & \vdots & \vdots & \omega & \vdots & \vdots \\ \vdots & A & \vdots & \omega & \vdots & \vdots \\ \vdots & \vdots & \vdots & \omega & \vdots & \vdots \\ 0 & \vdots & \vdots & 0 & \vdots & \vdots \\ n + s & \vdots & \vdots & \omega & \vdots & B \\
\end{pmatrix}
\]

\[
c_{ij} = \begin{cases} a_{ij} & i, j \in [n], \\ b_{(i-n)(j-n)} & n + 1 \leq i, j \leq n + m, \\ \omega & i = r, j = s + n, \\ \omega & j = r, i = s + n, \\ 0 & \text{otherwise.} \\ \end{cases}
\]

Acyclic Irreducible Matrix Construction: Suppose that \( \theta \in \mathbb{R} \). First, we define three subsets of symmetric matrices:

- \( \mathcal{Z}_\theta = \{ A : G_A \text{ is a tree, } m_A(\theta) = 0 \} \subset \bigcup_{n \geq 1} \text{Sym}_n(\mathbb{R}) \),
- \( \mathcal{Min}_\theta = \{ A : G_A \text{ is a tree and } A \text{ has a nowhere-zero } \theta\text{-eigenvector} \} \subset \bigcup_{n \geq 1} \text{Sym}_n(\mathbb{R}) \),
- \( \mathcal{Lin}_\theta = \{ A = [a]_{1 \times 1} : a \in \mathbb{R} \} \).

By Proposition 2 every element of \( \mathcal{Min}_\theta \) has the eigenvalue \( \theta \) with the multiplicity 1. Now, we make an acyclic irreducible matrix by connecting some elements of \( \mathcal{Z}_\theta \cup \mathcal{Min}_\theta \cup \mathcal{Lin}_\theta \) such that

1. the resulting matrix is acyclic and irreducible,
2. for every selected matrices \( m, m' \in \mathcal{Z}_\theta \cup \mathcal{Min}_\theta \), we do not connect them,
3. every chosen element of \( \mathcal{Lin}_\theta \) (as a linking vertex) is connected to at least two elements of \( \mathcal{Min}_\theta \).

We define the set \( A_\theta \) as

\[
A_\theta := \{ P(\bigoplus_{i=1}^{k} A_i)P^T : P \text{ is a permutation matrix, } k \in \mathbb{N}, \text{ and } A_i \text{ is obtained by the construction above} \},
\]

where,

\[
\bigoplus_{i=1}^{k} A_i := \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}
\]

In the following schematic figure, \( m_i \in \mathcal{Z}_\theta \cup \mathcal{Min}_\theta \), \( i \in [n] \).
Lemma 6. Every \( m \in \mathfrak{M}_{\theta} \) has a unique representation (up to permutation) in \( A_{\theta} \).

Proof. By Proposition 2, \( m \) doesn’t have any eigenvector with a zero entry. Suppose that \( \xi \) is a \( \theta \)-eigenvector of \( m \). There exists a pendant vertex of \( T_m \), say \( u \), such that \( \xi(u) \neq 0 \). Now, we can choose the subtree \( T' \) of \( T_m \) containing \( u \) such that \( m|_{T'} \in A_{\theta} \) and \( T' \) is adjacent with exactly one vertex of \( T_m - T' \), say \( v \). By Remark 1, \( \xi_v = 0 \), a contradiction. Hence, \( m \) has a unique representation in \( A_{\theta} \).

\[ \square \]

The following theorem of Fiedler describes a relation between the multiplicity of an eigenvalue and the linking vertices.

Theorem 7. [5, Theorem 2.4] Let \( \theta \in \mathbb{R} \) and \( A = (a_{ik}) \) be an \( n \times n \) acyclic matrix, let \( y = (y_i) \) be a \( \lambda \)-eigenvector of \( A \). If there are not two indices \( i, k \) such that \( a_{ik} \neq 0 \) and \( y_i = y_k = 0 \), then

\[
m_\theta(A) = c + \sum_{k=3}^{n-1} (k - 2)s_k,
\]

where \( c \) is the number of components of \( G_A \) and \( s_k, (k = 3, \ldots, n-1) \), is the number of those indices \( j \) for which \( y_j = 0 \) and \( a_{jl} \neq 0 \) for exactly \( k \) indices \( l \neq j \). In other words, \( s_k \), is the number of vertices of \( G_A \) corresponding to zero coordinates of \( y \) and having degree \( k \).

We generalize Theorem 7 for all cases of eigenvectors.

Theorem 8. Let \( \theta \in \mathbb{R} \) and \( A \) be an acyclic matrix. If \( F = (V(G_A) \setminus N_\theta^\circ(A), E(G_A) \setminus E_\theta^\circ(A)) \), then

i. \( A \in A_\theta \),
ii. \( m_\theta(A) = c + \sum_{u \in \partial N_\theta(A)} (d_F(u) - 2) \), where \( c \) is the number of components of \( F \),
iii. \( m_\theta(A) = |\{ \text{components of } (F - \partial N_\theta(A)) \} | - |\partial N_\theta(A)| \).

Proof. i.,ii.: Since the statements for \( A \) and \( PAP^T \), for any permutation matrix \( P \), are equivalent, without loss of generality suppose that \( A = \bigoplus_{i=1}^k A_i \) for some integer \( k \) and irreducible acyclic matrices \( A_1, \ldots, A_k \).
First, if $m_\theta(A) = 0$, then $A_i \in \mathcal{F}_\theta$, $i \in [k]$. Therefore, $A \in \mathcal{A}_\theta$, $F$ is empty, and $m_\theta(A) = 0$.

Now, suppose that $m_\theta(A) > 0$ and by Lemma 3, $x$ is a $\theta$-eigenvector such that for every $u \notin N_\theta(A)$, $x_u \neq 0$.

We prove theorem by induction on $|\partial N_\theta(A)|$. If $|\partial N_\theta(A)| = 0$, then for every $i \in [k]$, $x_{|A_i} = 0$ or $x_{|A_i}$ is nowhere-zero. Assume that, without loss of generality, $m_\theta(A_1) = \cdots = m_\theta(A_k) = 1$ and $m_\theta(A_{k+1}) = \cdots = m_\theta(A_{k+1}) = 0$, for some $t \in [k]$. Hence, $m_\theta(A) = t$, $A_1, \ldots, A_k \in \mathcal{Min}_\theta$, and $A_{t+1}, \ldots, A_k \in \mathcal{F}_\theta$.

So, by Lemma 6, $A = \bigoplus_{i=1}^k A_i \in \mathcal{A}_\theta$, $F = G \bigoplus_{i=1}^k A_i$, and $c = t$. Thus we have $m_\theta(A) = t = c + 0 = c + \sum_{u \in \partial N_\theta(A)} (d_F(u) - 2)$.

As the induction hypothesis, assume that the statements are true if $|\partial N_\theta(A)| = r - 1$ and we have $|\partial N_\theta(A)| = r$.

Suppose that, without loss of generality, $m_\theta(A_1) > 0$ and $|\partial N_\theta(A_1)| > 0$. Since $A_1$ is irreducible and acyclic, there exists an acyclic irreducible submatrix $B_1$ of $A_1$ such that $x_{B_1}$ is nowhere-zero and (from irreducibility of $A_1$) the vertices of $T_{B_1}$ have exactly one neighbor in $\partial N_\theta(A_1)$, say $v$. Suppose that $C_1$ is the matrix that is obtained from $A_1$ by deleting the rows and columns corresponding to $B_1$ and $v$. By the induction hypothesis, $C_1 \in \mathcal{A}_\theta$ and assume that $v_1, \ldots, v_s \in \mathcal{Lin}_\theta, u_1, \ldots, u_p \in \mathcal{F}_\theta$, and $w_1, \ldots, w_q \in \mathcal{Min}_\theta$ are the submatrices of $C_1$ such that are connected to $v$. By (1), at least one of the neighbors of $v$, other than $B_1$, is an element of $\mathcal{Min}_\theta$. Therefore, $A_1, A \in \mathcal{A}_\theta$.

Assume that $F' = (V(G_{C_1}) \setminus N_\theta^0(C_1), E(G_{C_1}) \setminus \mathcal{E}_\theta^0(C_1))$, $F_i = (V(G_{A_i}) \setminus N_\theta^0(A_i), E(G_{A_i}) \setminus \mathcal{E}_\theta^0(A_i))$, and $c_i$ is the number of components of $F_i$, for $i \in [k]$. By Proposition 5, $m_\theta(A_1) = m_\theta(C_1)$.

Thus,

$$m_\theta(A) = \sum_{i=1}^{k} m_\theta(A_i) = \sum_{i=2}^{k} m_\theta(A_i) + m_\theta(C_1)$$

$$= \sum_{i=2}^{k} m_\theta(A_i) + (c_1 + (q - 1)) + \sum_{u \in \partial N_\theta(C_1)} (d_{F'}(u) - 2)$$

$$= \sum_{i=2}^{k} m_\theta(A_i) + c_1 + (d_{F_i}(v) - 2) + \sum_{u \in \partial N_\theta(A_i)} (d_{F_i}(u) - 2)$$
\[ \begin{align*}
&= \sum_{i=2}^{k} m_{\theta}(A_i) + c_1 + \sum_{u \in \partial N_{\theta}(A_1)} (d_{F_1}(u) - 2) \\
&= \sum_{i=1}^{k} (c_i + \sum_{u \in \partial N_{\theta}(A_i)} (d_{F_i}(u) - 2)) \\
&= c + \sum_{u \in \partial N_{\theta}(A)} (d_{F}(u) - 2).
\end{align*} \]

iii.: If \( m_{\theta}(A) = 0 \) or \( |\partial N_{\theta}(A)| = 0 \), then the statement is true. Suppose that \( m_{\theta}(A) > 0 \), \(|\{\text{componets of } (F - \partial N_{\theta}(A)) \}\} = p \), and \( |\partial N_{\theta}(A)| = q \). Similar to the proof of i.,ii., by Proposition 5 and induction on \( q \), we have
\[ m_{\theta}(A_1) = m_{\theta}(C_1) \] and
\[ m_{\theta}(A) = \sum_{i=1}^{k} m_{\theta}(A_i) = \sum_{i=2}^{k} m_{\theta}(A_i) + m_{\theta}(C_1) = (p - 1) - (q - 1) = p - q = |\{\text{componets of } (F - \partial N_{\theta}(A)) \}|- |\partial N_{\theta}(A)|. \]

\[ \Box \]

Remark 2. The formula iii. in Theorem 8, is a corollary of [17, Theorem 2].

Remark 3. Suppose that \( T \) is a tree and \( m_{T}(\lambda) = k \) for given \( \lambda, k \). It can be easily seen that \( m_{T - N_{\lambda}(A_T)}(\lambda) = k \) and its \( \lambda \)-eigenvectors are \( \lambda \)-eigenvectors of \( T \) by removing the entries corresponding to \( N_{\lambda}(A_T) \). Also, in Theorem 8, if we have the eigenvectors of \( m_i \), \( i \in [n] \), we can obtain the eigenvectors of the resulting matrix easily.

Example 1. For the tree \( T \) that is shown in Figure 4, we have \( m_{T}(\sqrt{2}) = 3 \) and it has the decomposition as below.

![Figure 4. The structure of a tree with a \( \sqrt{2} \)-eigenvector.](image)

**Parter-Weiner vertices:** Suppose that \( T \) is a tree with a given eigenvalue \( \lambda \). The vertex \( v \) of \( T \) is called

- **Parter-Wiener vertex** if by removing \( v \), the multiplicity of \( \lambda \) increases by 1,
- **downer vertex** if by removing \( v \), the multiplicity of \( \lambda \) decreases by 1,
- **neutral vertex** if by removing \( v \), the multiplicity of \( \lambda \) does not change.

Remark 4. It is easy to see that the linking vertices (\( \partial N_{\lambda}(A) \)) in Theorem 8 are Parter-Weiner vertices. The vertices (indices) in \( N_{\lambda}(A) \) (the indices of elements of \( 3_{\lambda} \)) are neutral vertices and the vertices of the elements of \( \mathfrak{Min}_{\lambda} \) are downer vertices.
3. Minimal Trees with Respect to the Adjacency Eigenvalues

In the sequel of the paper, for trees we denote by $Z_{\lambda}$ the set $\{T : T$ is a tree, $m_T(\lambda) = 0\}$ and denote by $T_{\min, \lambda}$ the set $\{T : T$ is a tree with a nowhere-zero $\lambda$-eigenvector$\}$.

**Definition 1.** Let $\lambda \in \text{TRA}_1$. A tree $T$ is a $\lambda$-minimal tree, if it has a nowhere-zero $\lambda$-eigenvector.

Such trees in Definition 1, were defined in [6] as minimal trees and in [17] as $\lambda$-primes.

**Remark 5.** By Proposition 2, for a $\lambda$-minimal tree we have $m_T(\lambda) = 1$.

**Lemma 9.** For every $\lambda \in \text{TRA}_1$, we have $T_{\min, \lambda} \neq \emptyset$.

**Proof.** Suppose that $T$ is a tree such that $m_T(\lambda) > 0$. By removing the vertices of $N_{\lambda}(T)$ we have some components that have multiplicity 1 and they are minimal. □

By Perron-Frobenius theorem we have that every tree $T$ is an element of $T_{\min, \rho(T)}$ and we pose the following conjecture.

**Conjecture 1.** Let $\theta \in \text{TRA}_1$ with the minimal polynomial $f(x)$. If $\lambda = \max_{|z| = 0} f(z)$, then there exists a tree $T$ such that $\rho(T) = \lambda$.

**Proposition 10.** Let $\lambda \in \text{TRA}_1$. If $\lambda = 0$, then $T_{\min, \lambda} = \{K_1\}$ and $|T_{\min, \lambda}| = +\infty$ otherwise.

**Proof.** First, suppose that $\lambda = 0$. By (1), it is easy to see that the only $T \in T_{\min, 0}$ is $K_1$.

Now, suppose that $\lambda \neq 0$. We construct from every element of $T \in T_{\min, \lambda}$, another element $T' \in T_{\min, \lambda}$ such that $|V(T)| < |V(T')|$. Assume that $T \in T_{\min, \lambda}$ with a pendant vertex $v$ and its neighbor $u$. Suppose that $x$ is the $\lambda$-eigenvector such that $x_v = 1$. Hence, $x_u = \lambda$.

![Figure 5. A $\lambda$-eigenvector of $T$.](image)

For an integer $k > 1$, we construct the tree $T'$ as following: we use $2k - 1$ copies $T_1, \ldots, T_k, T'_1, \ldots, T'_{k-1}$ of $T$ and identify all of the copies of vertex $v$, as shown below. Also, we connect to every copy $u'_i$ of $u$, $i \in [k - 1]$, two new pendant vertices. Now, we assign to $T_1, \ldots, T_k$ the vector $x$ and to $T'_1, \ldots, T'_{k-1}$ except $v$, the vector $-x$, and assign $-1$ to new pendant vertices. By (1), it is easy to check that this vector is a nowhere-zero $\lambda$-eigenvector and hence, $T' \in T_{\min, \lambda}$.

![Figure 6. A $\lambda$-eigenvector of $T'$.](image)
In the following figure, there is another construction of a \( \lambda \)-minimal tree from \( T \) of Proposition 10.

\[
\begin{align*}
T_1 & \quad \lambda \\
T_2 & \quad \lambda \\
T_3 & \quad \lambda \\
T' & \quad \lambda \\
T_4 & \quad \lambda
\end{align*}
\]

**Figure 7.** A \( \lambda \)-minimal tree and a \( \lambda \)-eigenvector.

Suppose that \( k \in \mathbb{N} \) and \( \lambda_1, \ldots, \lambda_k \in \text{TRAI} \). It is easy to make a tree \( T \) such that \( \lambda_1, \ldots, \lambda_k \in \text{Spec}(T) \). We pose the following conjecture about the minimal trees of given eigenvalues.

**Conjecture 2.** Suppose that \( k \in \mathbb{N} \) and \( \lambda_1, \ldots, \lambda_k \in \text{TRAI} \setminus \{0\} \).

\begin{itemize}
  \item \( T_{\text{min}, \lambda_1} \not\subseteq T_{\text{min}, \lambda_2} \),
  \item \( \bigcap_{i=1}^{k} T_{\text{min}, \lambda_i} \neq \emptyset \).
\end{itemize}

4. **Minimal Trees with Respect to the Laplacian Eigenvalues**

In this section, we consider the structure of trees with respect to the Laplacian eigenvectors. First, we recall and generalize two theorems on the integer Laplacian eigenvalues.

**Proposition 11.** \([9, \text{Proposition 6}]\) Let \( G \) be a graph with \( n \) vertices and \( t \geq 1 \) spanning trees. If \( \mu \) is a positive integer eigenvalue of \( L_G \), then \( \mu \mid nt \). If \( G \) is Laplacian integral, then \( \mu^m \mid nt \), where \( m = m_{L_G}(\mu) \).

A generalization of Proposition 11 is the following proposition.

**Proposition 12.** Let \( G \) be a connected graph on \( n \) vertices and \( t \) spanning trees. If \( \mu \) is a nonzero eigenvalue of \( L_G \), then \((\text{Norm}(\mu))^m \mid nt \), where \( m = m_{L_G}(\mu) \).

**Proof.** Suppose that \( \psi(x) \) is the characteristic polynomial of \( L_G \). So, \( \psi(x) = x \prod_i f_i(x)^{m_i} = x(x^{n-1} - \cdots + (-1)^{n-1}nt) \), where \( \{f_i(x)\} \) are the minimal polynomials of the nonzero eigenvalues of \( L_G \). Hence \( \prod_i f_i(0)^{m_i} = nt \). Therefore, \((\text{Norm}(\mu))^m \mid nt \). \( \square \)

Another well-known theorem on the multiplicity of integer Laplacian eigenvalues of trees is the following theorem.

**Theorem 13.** \([10, \text{Theorem 2.1}]\) Suppose \( T \) is a tree on \( n \) vertices. If \( \mu > 1 \) is an integer eigenvalue of \( L_T \) with corresponding eigenvector \( u \), then

\begin{itemize}
  \item i. \( \mu \mid n \);
  \item ii. no coordinate of \( u \) is zero;
  \item iii. \( m_{L_T}(\mu) = 1 \).
\end{itemize}

We prove the following generalization of Theorem 13.
Theorem 14. Let \( T \) be a tree on \( n \) vertices. If \( \mu \) is an eigenvalue of \( L_T \) and \( \text{Norm}(\mu) > 1 \), then

i. \( \text{Norm}(\mu)|n; \)
ii. no coordinate of a \( \mu \)-eigenvector is zero;
iii. \( m_{L_T}(\mu) = 1 \).

Proof. i. It is a corollary of Proposition 12.

ii. Suppose by contradiction that \( \xi \) is a \( \mu \)-eigenvector with some zero coordinates. By deleting the rows and columns corresponding to the zero entries of \( \xi \), we have at least one submatrix \( L \) of \( L_T \) such that

\[
L = L_{T'} + e_i e_i^T,
\]

for a subtree \( T' \) and an index \( i \), such that \( L \) has a nowhere-zero \( \mu \)-eigenvector. Therefore \( \text{Norm}(\mu)|\det(L) \)

But \( \det(L) = \det(L_{T'}) + e_i^T \text{adj}(L_{T'}) e_i = 1 \) (see [15, Problem 3.1]) and hence this is a contradiction.

iii. By previous item and Proposition 2, we have \( m_{L_T}(\mu) = 1 \). \( \square \)

Definition 2. A half-edge is an edge with one end-vertex. A cut-tree is obtained by adding some half-edges to a tree. For an integer \( k \geq 0 \), a \( k \)-cut-tree is a cut-tree with \( k \) half-edges.

Example 2. Every tree is a 0-cut-tree. In the figure below, there are a 1-cut-tree and a 4-cut-tree.

![Figure 8. Two cut-trees.](image)

The adjacency matrix and the Laplacian matrix of the cut-trees above are

\[
A_{\bar{T}_c} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad L_{\bar{T}_c} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}.
\]

Definition 3. Let \( \mu \in \text{TPAI} \). A cut-tree \( \bar{T}_c \) is a \( \mu \)-L-minimal cut-tree, if \( L_{\bar{T}_c} \) has a nowhere-zero \( \mu \)-eigenvector \( x \) such that for every \( uv \in E(\bar{T}_c) \), \( x_u \neq x_v \).

We denote by \( \mathcal{L}_{\text{min},\mu}^k \) the set of all \( \mu \)-L-minimal \( k \)-cut-trees and \( \mathcal{L}_{\text{min},\mu}^k \) := \( \bigcup_{k=0}^{+\infty} \mathcal{L}_{\text{min},\mu}^k \).

Remark 6. For every \( \mu \)-L-minimal cut-tree \( \bar{T}_c \) and for every \( uv \in E(\bar{T}_c) \), we have \( m_{L_{\bar{T}_c-uv}}(\mu) = 0 \).

Lemma 15. Let \( \mu \) be a Laplacian eigenvalue of a tree \( T \). If a \( \mu \)-eigenvector \( \xi \) vanishes at a vertex of \( T \), then \( \mathcal{L}_{\text{min},\mu}^1 \neq \emptyset \).

Proof. Since \( L_T \) is an acyclic matrix, by removing the indices of \( L_T \) that are corresponding to the zero entries of \( \xi \), we obtain at least two Laplacian matrices of two elements of \( \mathcal{L}_{\text{min},\mu}^1 \). \( \square \)

Lemma 16. Let \( s \) be a positive Laplacian eigenvalue of a tree \( T \). If \( s > 1 \), then \( |\mathcal{L}_{\text{min},s}^1| = +\infty \).

Proof. Suppose that \( T \) is an \( s \)-L-minimal tree with a pendant vertex \( v \) and its neighbor \( u \). Assume that \( \xi \) is the \( s \)-eigenvector that \( \xi_v = 1 \) and \( \xi_u = 1 - s \). By (1), it is easy to see that the following construction, as shown in figure below, is a new \( s \)-L-minimal tree.
Conjecture 3. Every totally positive algebraic integer is a Laplacian eigenvalue of some tree.

If \( \text{Norm}(\mu) > 1 \), then \( \mathcal{L}T_{\min,\mu}^1 = \emptyset \), since \( |\det(L_T)| = 1 \), for every \( T_c \in \mathcal{L}T_{\min,\mu}^1 \).

Conjecture 4. Let \( \mu \) be a Laplacian eigenvalue of some tree \( T \) and \( \mu \neq 0 \).

- If \( \text{Norm}(\mu) = 1 \), then \( |\mathcal{L}T_{\min,\mu}^0| = |\mathcal{L}T_{\min,\mu}^1| = +\infty \).
- If \( \text{Norm}(\mu) > 1 \), then \( |\mathcal{L}T_{\min,\mu}^0| = +\infty \).

4.1. Laplacian Tree Structure and Eigenvectors.

Definition 4. Let \( \mu \in \text{TPAI} \). A cut-tree \( T_c \) is a \( \mu \)-L cut-tree, if \( L_T \) has a nowhere-zero \( \mu \)-eigenvector.

We denote by \( \mathcal{L}T^k_\mu \) the set of all \( \mu \)-L \( k \)-cut-trees and \( \mathcal{L}T_\mu := \bigcup_{k=0}^{+\infty} \mathcal{L}T^k_\mu \).

Remark 7. Laplacian Tree Structure. For every \( \mu \)-L cut-tree \( T_c \) and \( \mu \)-eigenvector \( x \), if we remove the edges \( \{uv \in E(T_c) : x_u = x_v\} \), we have some \( \mu \)-L-minimal cut-trees \( 1 \), see Lemma 8].

Also, if we join by an edge a vertex of a \( \mu \)-L cut-tree and a vertex of another \( \mu \)-L cut-tree, we will obtain a new \( \mu \)-L cut-tree.

Suppose that \( T \) is a tree, \( \mu \in \mathbb{R} \), and \( m_{L_T}(\mu) = k \). By Theorem 8, if we remove \( \partial N_\mu(L_T) \) from \( L_T \) we have Laplacian matrices of some \( \mu \)-L cut-trees such as \( L_{T_c} \in \mathcal{M}_{\min} \) and some cut-trees such as \( T_c \), where \( L_{T_c} \in \mathcal{T}_\mu \).

Example 3. Suppose that \( k \in \mathbb{N} \), \( \lambda \in \text{TRAI} \), and \( \mu \in \text{TPAI} \). We characterize all trees \( T \) such that they have the minimum vertices and

- \( m_T(\lambda) = k \);
- \( m_{L_T}(\mu) = k \).

If \( k = 1 \), then every \( \lambda \)-minimal tree and every \( \mu \)-minimal tree with minimum vertices is a solution. If \( k > 1 \), by Theorem 8, a solution is a tree \( T \) with minimum vertices has only one linking vertex and \( k+1 \) \( \lambda \)-minimal trees (and \( k+1 \) \( \mu \)-minimal cut-trees) with minimum vertices that connected to the linking vertex.

![Figure 9. An s-L-minimal tree and an s-eigenvector.](image-url)
Therefore, the roots of \( \phi_{S} \), are the roots of \( \frac{\phi_{S}(\lambda)}{\phi_{S-v_{1}}(\lambda)} = \lambda_{i} \), where \( \lambda_{i} \) is a root of \( \phi_{T}(\lambda) \), for every \( i \in [n] \).

### Problem 1
Suppose that \( \lambda \in \text{TRAI} \) and \( \mu \) is a Laplacian eigenvalue of a tree.
- What is the size of \( \{ T : T \in \mathcal{T}_{\text{min}, \lambda} \text{ with minimum order} \} \)?
- If \( \text{Norm}(\mu) > 1 \), what is the size of \( \{ T : T \in \mathcal{L}T_{\text{min}, \mu}^{0} \text{ with minimum order} \} \)?

#### 4.2. Totally Minimal Trees
We say a tree \( T \) is totally minimal if for every \( \lambda \in \text{Spec}(T) \), \( T \in \mathcal{T}_{\text{min}, \lambda} \). By Theorem 8, it is easy to see that \( T \) is totally minimal if and only if for every \( v \in V(T) \), \( \text{Spec}(T) \cap \text{Spec}(T - v) = \emptyset \).

Suppose that \( G \) is a graph of order \( n \), \( V(G) = \{ u_{1}, \ldots, u_{n} \} \), and \( H \) is a rooted graph with the root \( v_{1} \). The rooted product \( G \circ H \) is the graph that obtained from \( G \) and \( n \) copy \( H_{1}, \ldots, H_{n} \) of \( H \) by identifying the root \( v_{1} \) of \( H_{i} \) with the vertex \( u_{i} \) of \( G \):

\[
V(G \circ H) = V(G) \times V(H), (u_{i}, v_{j}) \sim (u_{k}, v_{l}) \iff (i = j = 1 \text{ and } u_{i} \sim u_{k}) \text{ or } (i = k \text{ and } v_{j} \sim v_{l}).
\]

As a corollary of [7, Theorem 2.1] we have the following theorem.

**Theorem 17.** [7, see Theorem 2.1] Suppose that \( G \) is a graph. If \( H \) is a rooted graph, then

\[
(2) \quad \phi_{G \circ H}(\lambda) = |\phi_{H}(\lambda)|I - \phi_{H - v}(\lambda)A_{G}|
\]

**Theorem 18.** Let \( T \) be a totally minimal tree. The following trees are totally minimal:
- i. \( P_{p-1} \), for any prime number \( p \),
- ii. \( T \circ P_{2} \),
- iii. \( T \circ P_{4} \), where the root of \( P_{4} \) is of degree 2.

**Proof.** i. If \( V(P_{p-1}) = \{ v_{1}, \ldots, v_{p-1} \} \), then \( \xi_{k} = (\sin(\frac{k\pi}{p}), \ldots, \sin(\frac{k(p-1)\pi}{p})) \) is a \( \lambda_{k} \)-eigenvector for \( \lambda_{k} = 2 \cos(\frac{\pi}{p}) \), \( k \in [p - 1] \) [3, Section 1.4]. Hence, we have \( \sin(\frac{\pi}{p}) \neq 0 \), for \( i, k \in [p - 1] \).

ii., iii.: Suppose that \( T \) is of order \( n \), \( V(T) = \{ u_{1}, \ldots, u_{n} \} \), \( S \) is a totally minimal tree, \( V(S) = \{ v_{1}, \ldots, v_{m} \} \), and \( v_{1} \) is the root of \( S \). By Proposition 2, \( \phi_{S}(\lambda) \) and \( \phi_{S-v_{1}}(\lambda) \) do not have a common root, hence from equation (2), \( \phi_{T \circ S}(\lambda) \) and \( \phi_{S-v_{1}}(\lambda) \) do not have a common root. By Theorem 17

\[
\phi_{T \circ S}(\lambda) = (\phi_{S-v_{1}}(\lambda))^{n}\left[ \phi_{S}(\lambda) - A_{T} \right] = (\phi_{S-v_{1}}(\lambda))^{n}\phi_{T}(\frac{\phi_{S}(\lambda)}{\phi_{S-v_{1}}(\lambda)}).
\]

Therefore, the roots of \( \phi_{T \circ S}(\lambda) \), are the roots of \( \frac{\phi_{S}(\lambda)}{\phi_{S-v_{1}}(\lambda)} = \lambda_{i} \), where \( \lambda_{i} \) is a root of \( \phi_{T}(\lambda) \), for every \( i \in [n] \).
It is sufficient to show that for any root $\theta$ of $\phi_{T\circ S}(\lambda)$ and any vertex $u = (u_i, v_j)$ of $T \circ S$, $\phi_{T\circ S-u}(\theta) \neq 0$. If $j = 1$, 

$$\phi_{T\circ S-u}(\theta) = \phi_{(T-u_i)\circ S}(\theta) \cdot \phi_{S-v_1}(\theta) = (\phi_{S-v_1}(\theta))^n \phi_{T-u_i}(\frac{\phi_S(\theta)}{\phi_{S-v_1}(\theta)}) \neq 0.$$ 

Hence, any $\theta$-eigenvector does not vanish at $u$.

If $j > 1$ and an eigenvalue vanishes at $u$, then by (1), it vanishes at $(u_i, v_1)$. Hence by the previous case we have a contradiction. □

We end this section with the following problem concerning totally minimal trees.

**Problem 2.** Which trees are totally minimal?

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**Appendix**

| $\lambda = 1$ | $\lambda = \sqrt{2}$ |
| --- | --- |
| **$n$** | **Tree** | **$n$** | **Tree** |
| 2 | ![Tree](image) | 3 | ![Tree](image) |
| 6 | ![Tree](image) | 9 | ![Tree](image) |
| 9 | ![Tree](image) | 9 | ![Tree](image) |
| 9 | ![Tree](image) | 9 | ![Tree](image) |
| 10 | ![Tree](image) | 9 | ![Tree](image) |
| 10 | ![Tree](image) | 9 | ![Tree](image) |

Table 1. Minimal trees of $\lambda = 1$ and $\lambda = \sqrt{2}$ with at most 10 vertices.
Table 2. Some $\mu$-L-minimal 0, 1-cut-trees.

| Table 3. 2-L-minimal 0-cut-trees with at most 12 vertices. |
References

[1] A. Bahmani, D. Kiani, Graph reduction techniques and the multiplicity of the Laplacian eigenvalues, Linear Algebra Appl., 503:215–232, 2016.
[2] A. Bahmani, D. Kiani, On the multiplicity of the adjacency eigenvalues of graphs, Linear Algebra Appl., 477:1–20, 2015.
[3] A.E. Brouwer, W.H. Haemers, Spectra of Graphs, Universitext, Springer, New York, 2012.
[4] Y. Z. Fan, S. C. Gong, Y. Wang, Y.B. Gao, On trees with exactly one characteristic element, Linear Algebra Appl. 421.2-3, 233-242, 2007.
[5] M. Fiedler, Eigenvectors of acyclic matrices, Czechoslovak Mathematical Journal 25.4, 607-618, 1975.
[6] L. L. Gardner, K. L. Krystina, The multiplicity of eigenvalues in the adjacency matrix of a tree, MIGHTY XXXV, Illinois State University, September 27-28, 2002.
[7] C.D. Godsil, B.D. McKay, A new graph product and its spectrum, Bulletin of the Australian Mathematical Society, 18(1), 21-28, 1978.
[8] S.C. Gong, Y. Z. Fan, The property of maximal eigenvectors of trees, Linear and Multilinear Algebra 58.1 : 105-111, 2010.
[9] R. Grone, R. Merris, The Laplacian spectrum of a graph II, SIAM Journal on discrete Mathematics 7, 221-229, 1994.
[10] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11, 218-238, 1990.
[11] C.R. Johnson, A. Leal Duarte, P.R. McMichael, Changes in vertex status and the fundamental decomposition of a tree relative to a multiple (Parter) eigenvalue, Disc. App. Math. 2017.
[12] C. R. Johnson, A. Leal Duarte, C. M. Saiago, The Parter-Wiener theorem: refinement and generalization, SIAM Journal of Matrix Analysis and Applications 25: 311330, 2003.
[13] C. R. Johnson, A. Leal Duarte, C. M. Saiago, D. Sher, Eigenvalues, multiplicities and graphs, Contemporary Mathematics 419 : 167, 2006.
[14] K. H. Monfared, B. L. Shader, The nowhere-zero eigenbasis problem for a graph, Linear Algebra Appl. 505 : 296-312, 2016.
[15] V.V. Prasolov, Problems and theorems in linear algebra, Vol. 134. American Mathematical Soc., 1994.
[16] J. Salez, Every totally real algebraic integer is a tree eigenvalue, Journal of Combinatorial Theory, Series B 111 : 249-250, 2015.
[17] J. Salez, Spectral atoms of unimodular random trees, [hal-01374519v1]. arXiv preprint arXiv:1609.09374, 2016.