RATIONAL CONNECTIVITY AND
ANALYTIC CONTRACTIBILITY

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Abstract. Let $k$ be an algebraically closed field of characteristic 0, and let $f : X \to Y$ be a morphism of smooth projective varieties over the ring $k((t))$ of formal Laurent series. We prove that if a general geometric fiber of $f$ is rationally connected, then the Berkovich analytifications of $X$ and $Y$ are homotopy equivalent. Two important consequences of this result are that the homotopy type of the Berkovich analytification of any smooth projective variety $X$ over $k((t))$ is a birational invariant of $X$, and that the Berkovich analytification of a rationally connected smooth projective variety over $k((t))$ is contractible.

1. Introduction

Unless otherwise specified, we work over a field $k$ which is algebraically closed of characteristic 0. Given a $k$-scheme $S$, we say that $X \to S$ is a variety over $S$ if $X$ is an integral, finite type scheme over $S$. We use the term generic point to mean the point of the variety $X$ corresponding to its field of rational functions, while a general point refers to any closed point chosen inside some open set. If $T$ is a second $S$-scheme, then a general $T$-point of $X$ refers to any $T$-valued point $T \to X$ whose image is a general point.

1.1. A suggestive dictionary.

Let $K$ be a topological field, topologically complete with respect to a fixed norm $|−| : K \to \mathbb{R}_{\geq 0}$. Assume that this norm satisfies the ultrametric triangle inequality $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in K$. We call a field $K$ complete with respect to such a norm a non-archimedean field.

In close analogy with the operation of complex analytification, Berkovich showed how to associate a well behaved topological space $X_{an}$ to each finite type $K$-scheme $X$ [Ber90, §3.4]. The space $X_{an}$ is called the Berkovich analytification of $X$. Berkovich observed that there exists a partial dictionary between algebro-geometric properties of $X$ and topological properties of $X_{an}$. Namely:

Theorem [Ber90, Theorem 3.4.8]. Let $X$ be a scheme locally of finite type over $K$. Then:

(i) $X$ is separated $\iff X_{an}$ is Hausdorff.
(ii) $X$ is proper $\iff X_{an}$ is Hausdorff and compact.
(iii) $X$ is connected $\iff X_{an}$ is pathwise connected.
(iv) The algebraic dimension of $X = \text{the topological dimension of } X_{an}$.

Naturally, one would like to continue Berkovich’s dictionary, and so we ask:

Question 1.1.1. Which other algebro-geometric properties of $X$ correspond to topological properties of $X_{an}$?
One fundamental invariant of a topological space is its homotopy type. Our main theorem in the present article provides a tool for relating the birational geometry of $K$-varieties to the homotopy types of their Berkovich analytifications, in the special case that $K = k((t))$, the field of formal Laurent series over an algebraically closed field $k$ of characteristic 0:

**Theorem 1.1.2.** Let $k$ be an algebraically closed field of characteristic 0, and let $f : X \to Y$ be a surjective map of smooth projective varieties over the nonarchimedean field $K = k((t))$. Suppose that for a general geometric point $q$ of $Y$ the fiber $X_q$ is rationally connected. Then the Berkovich spaces $X^\text{an}$ and $Y^\text{an}$ are homotopy equivalent.

As immediate corollaries, we have

**Corollary 1.1.3.** If $Y$ is a smooth $k((t))$-variety and $f : X \to Y$ is a $\mathbb{P}^n$-bundle over $Y$, then the induced map $f^\text{an} : X^\text{an} \to Y^\text{an}$ is a homotopy equivalence.

**Corollary 1.1.4.** Let $X$ be a smooth projective $k((t))$-variety. If $X$ is rationally connected, then its Berkovich analytification $X^\text{an}$ has the homotopy type of a point, i.e., is contractible.

Berkovich showed that for projective space $\mathbb{P}^n_K$ of any dimension $n$, over any non-archimedean field $K$, the analytification $(\mathbb{P}^n_K)^\text{an}$ is contractible [Ber90, Theorem 6.1.5]. The simplest example of a rationally connected variety is projective $n$-space, so Corollary 1.1.4 can be seen as a broad extension of Berkovich’s contractibility result in the special case that $K = k((t))$, and Corollary 1.1.3 can be seen as an extension of Berkovich’s result to the relative setting.

Finally, given $X$ and $Y$ smooth, projective and birational, we can choose a common resolution of singularities $Z$ of both $X$ and $Y$, so that Theorem 1.1.2 also implies

**Corollary 1.1.5.** Let $X$ and $Y$ be smooth projective $k((t))$-varieties. If $X$ and $Y$ are birationally equivalent, then their Berkovich analytifications $X^\text{an}$ and $Y^\text{an}$ are homotopy equivalent.

Note that being rationally connected is a birational invariant. Recent work of Mustaţă and Nicaise [MN13] and Nicaise and Xu [NX13] explores connections between the minimal model program (MMP) and the geometry of Berkovich spaces. We adopt many of their techniques in the proof of Theorem 1.1.2. We also use recent results by de Fernex, Kollár, and Xu on the homotopy type of the dual complex of a resolution of singularities [dFKX12]. It is our use of these results that forces us to work over the non-archimedean field $k((t))$, with $k$ algebraically closed of characteristic 0. It remains an open question whether or not a counterpart to Theorem 1.1.2 or any of its Corollaries 1.1.3 through 1.1.5 holds over more general non-archimedean fields.

1.2. Organization of the paper.

In Section 2 we briefly review Berkovich analytifications and their skeleta. We use results of [MN13] and [NX13] to construct a skeleton $\text{Sk}(\mathcal{X}) \subset X^\text{an}$, and we reduce the proof of Theorem 1.1.2 to a verification that the map $f : X \to Y$ appearing in the statement of the theorem induces a homotopy equivalence $\text{Sk}(\mathcal{X}) \simeq \text{Sk}(\mathcal{Y})$.

In Section 3 we review rational connectivity of algebraic varieties in both the absolute and relative settings.

The main technical contribution of the paper occurs in Section 4. Here we show that if $\mathcal{X}$ is an snc $k[[t]]$-model of a rationally connected variety, then we can construct a degenerating family of rationally connected varieties over an algebraic curve, such that the dual complex of the special
fiber of this degenerating family is homeomorphic to our skeleton $\text{Sk}(\mathcal{X})$. We actually need a relative version of this statement in order to prove Theorem 1.1.2, and it is the relative version that we prove in Section 4. This puts us in a situation where the results of [dFKX12] become applicable.

In Section 5, we establish a relative version of [dFKX12, Theorem 41], and use it to complete the proof of Theorem 1.1.2.

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2. Skeleton in Berkovich spaces

2.1. The Berkovich analytification.

Let $K$ be a topological field, complete with respect to a continuous, non-archimedean norm $|\cdot| : K \to \mathbb{R}_{\geq 0}$. Assume that this norm is discrete, in the sense that its value group $|K^\times|$ is a discrete subgroup of $\mathbb{R}_{\geq 0}$. Let $R = \{a \in K : |a| \leq 1\}$ denote the ring of integers in $K$, with unique maximal ideal $m = \{a \in K : |a| < 1\}$, and denote its residue field $k \overset{\text{def}}{=} R/m$.

Let $X$ be a proper $K$-variety, and let $X^\text{an}$ denote its Berkovich analytification. A variety for us will always be separated, and hence its Berkovich analytification will always be a Hausdorff topological space. In this article, we consider $X^\text{an}$ solely as a topological space, completely ignoring the analytic structure sheaf on $X^\text{an}$.

Let us briefly recall that if $U = \text{Spec} A$ is an affine local chart on $X$, then $U^\text{an}$ is an open subset of $X^\text{an}$ given by

$$U^\text{an} = \left\{\begin{array}{ll}
\text{multiplicative seminorms} & \text{that restrict to the norm } |\cdot| \text{ on } K \subset A \\
|\cdot|_x : A \to \mathbb{R}_{\geq 0} & \text{such that } |\cdot|_x \text{ on } K \subset A
\end{array}\right\}$$

Each function $f \in A$ determines a map $\text{ev}_f : U^\text{an} \to \mathbb{R}_{\geq 0}$ that takes $|\cdot|_x \mapsto |f|_x$. The topology on $U^\text{an}$ is the coarsest topology that renders each of these maps continuous. We glue $X^\text{an}$ from these topological spaces $U^\text{an}$, and the resulting topological space is independent of the particular affine cover we use.

2.2. Skeletons.

The Berkovich analytification tends to be large and complicated. For example, the point set underlying the Berkovich analytification of a curve has a natural structure of an infinite graph. One powerful technique for understanding a given Berkovich analytification is to produce a smaller space inside $X^\text{an}$, called a skeleton, which reflects much of the geometry of the ambient space. Specifically, this skeleton is supposed to be a finite simplicial complex onto which $X^\text{an}$ deformation retracts. In the case of curves, the skeleton turns out to be a finite graph.

A general technique for producing a skeleton is to choose a model $\mathcal{X}$ of $X$ over the local ring $R$ of integers in $K$, such that the special fiber $\mathcal{X}_k$ is a reduced simple normal crossing divisor in $\mathcal{X}$. Each divisor corresponds to a point in $X^\text{an}$, and the full dual complex of $\mathcal{X}_k$ embeds as a closed subspace of $X^\text{an}$. Note though that because this construction requires the choice of a model $\mathcal{X}$, it is not canonical.

It is often desirable to produce a skeleton that is canonical in the sense that it can be recovered uniquely from $X$. One construction which produces such a skeleton is due to Mustață and Nicaise.
Following work of Kontsevich and Soibelman on mirror symmetry, their construction uses weight functions coming from pluricanonical forms on $X$, and Nicaise and Xu have used this construction to study degenerations of Calabi-Yau varieties. This construction is in many ways analogous to the theory of minimal models. If $X$ is of general type, and if the canonical divisor $K_X$ is nef, one expects to be able to produce a minimal model $\mathcal{X}$ over $R$, with $K_X$ nef. This model will be unique up to simple birational modifications called flops. In the case where $X$ is a Calabi-Yau variety, Nicaise and Xu show that this skeleton is given by the dual complex of a good minimal dlt model of $X$, if one exists.

On the other hand, a rationally connected variety has no pluricanonical forms. When one runs the minimal model program on a rationally connected variety $X$, the end result is a Mori fiber space. In contrast to the general type case, there may be many different ways of expressing $X$ as birational to a Mori fiber space. Thus from the perspective of minimal model theory, one should not expect there to be a distinguished way of building a skeleton for a rationally connected $X$.

### 2.3. Construction of a non-canonical skeleton.

Let $X$ be a proper $K$-variety. An $R$-model of $X$ is any separated flat $R$-scheme $\mathcal{X}$ with generic fiber $\mathcal{X}_K \cong X$. A proper snc-model of $X$ is any proper $R$-model $\mathcal{X}$ of $X$ such that:

1. $\mathcal{X}$ is normal;
2. The special fiber $\mathcal{X}_k$ is a strict normal crossing divisor in $X$.

Assume that our proper $K$-variety $X$ is regular, connected, and that it has a proper snc-model $\mathcal{X}_k$. As Mustață and Nicaise explain in [MN13, §3], even when the special fiber $\mathcal{X}_k$ is not reduced, so that the $m$-adic completion $\mathcal{X}$ is not pluristable, we can still construct a cell complex $D(\mathcal{X}_k) \subset \mathcal{X}^{an}$ from $\mathcal{X}_k$ as follows.

Points in $D(\mathcal{X}_k)$ are pairs $(\xi, x)$ where $\xi$ is any generic point of the intersection of finitely many irreducible components of $\mathcal{X}_k$, and where $x = (x_1, \ldots, x_m)$ is any point in the topological space $D^\circ(\xi) \subset \mathbb{R}^m_{>0}$ cut out by the equation

$$\sum_{\text{irreducible components } V_i \text{ of } \mathcal{X}_k \text{ containing } \xi} x_i = 1$$

Here $m$ is the number of irreducible components of $\mathcal{X}_k$ containing $\xi$. We can attach these spaces $D^\circ(\xi)$ to one another via boundary relations induced by the specialization relations between the generic points $\xi$ (see [Nic11, §3] for details). Note that the resulting cell complex $D(\mathcal{X}_k)$ coincides with the cell complex $\Delta(\text{supp}(\mathcal{X}_k))$ that de Fernex, Kollár, and Xu construct in [dFKX12, §2].

At each point $(\xi, x)$ in $D(\mathcal{X}_k)$, with $x = (x_1, \ldots, x_m)$, let $V_1, \ldots, V_m$ be the set of irreducible components of $\mathcal{X}_k,_{red}$ containing the generic point $\xi$. For each $1 \leq i \leq m$, let $N_i$ denote the multiplicity of the component $V_i$ in the special fiber $\mathcal{X}_k$. Then the coordinatewise rescaled $m$-tuple

$$v = \left( \frac{x_{V_1}}{N_1}, \ldots, \frac{x_{V_m}}{N_m} \right) \quad \text{in} \quad \mathbb{R}^m_{>0}$$

determines a multiplicative seminorm

$$| - |_v : K(X) \rightarrow \mathbb{R}_{\geq 0}$$
defined as follows. Each rational function $f \in \hat{O}_{X,\xi}$ admits an \textit{admissible decomposition}, meaning a decomposition of the form

$$f = \sum_{u \in \mathbb{Z}_{\geq 0}^m} a_u T^u$$

with $a_u = \begin{cases} \text{a unit in } \hat{O}_{X,\xi} & \text{or 0 otherwise} \end{cases}$

(see [MN13, Proposition 2.4.6] for details). When $f$ is in $O_{X,\xi}$ with an admissible decomposition (2), the seminorm $|−|$ returns

$$|f|_v = \max_{u \in \mathbb{Z}_{\geq 0}^m, a_u \neq 0} \frac{1}{e^{(u,v)}}$$

Here $⟨−,−⟩ : \mathbb{Z}_{\geq 0}^m \times \mathbb{R}_{\geq 0}^m \rightarrow \mathbb{R}_{\geq 0}$ denotes the standard pairing. One checks that this gives a well-defined map $|−|_v : \hat{O}_{X,\xi} \rightarrow \mathbb{R}_{\geq 0}$ that extends to a multiplicative seminorm on $K(X)$, and we take this to be our seminorm (1) (see [MN13, §'s 2.4 & 3.1] for details). The resulting assignment $v \mapsto |−|_v$ induces a closed embedding

$$\text{Sk} : D(\mathcal{X}_k) \longrightarrow X^{\text{an}}$$

(see [MN13, §3.1.3 & Proposition 3.1.4]). We refer to its image as the \textit{skeleton} of $X^{\text{an}}$ associated to $\mathcal{X}$, and denote it $\text{Sk}(\mathcal{X})$.

2.4. \textbf{Retracting to the skeleton over characteristic-0 Laurent series.}

Let $k$ be an algebraically closed field of characteristic 0, and equip the field $K = k((t))$ of formal Laurent series with the norm $|−| : k((t)) \rightarrow \mathbb{R}_{\geq 0}$ defined by

$$\left| \sum_{n=m}^{\infty} a_n t^n \right| = \frac{1}{e^{m}} \quad \text{for} \quad a_m \neq 0$$

This norm is non-archimedean, and $k((t))$ is complete with respect to the topology that it induces. The resulting ring of integers in $k((t))$ is the ring $\mathcal{R} = k[[t]]$ of formal power series, with maximal ideal $m = (t)$ and residue field $\bar{k} = k[[t]]/(t)$.

Let $X$ be an irreducible, smooth projective variety over $k((t))$. For each choice of closed embedding $X \longrightarrow \mathbb{P}^d_{k((t))}$, we obtain a $k[[t]]$-model $\mathcal{X}$ of $X$ as the closure

$$\mathcal{X} \overset{\text{def}}{=} \overline{X}$$

inside $\mathbb{P}^d_{k[[t]]}$. This $k[[t]]$-scheme $\mathcal{X}$ is flat. Thus it is a bona fide proper $k[[t]]$-model of $X$. Moreover, $\mathcal{X}$ is reduced whenever $X$ is, so we can assume without loss of generality that $\mathcal{X}$ is normal.

We have no assurance, a priori, that the proper $k[[t]]$-model $\mathcal{X}$ is an snc-model. However, because we work over a residue field $k$ of characteristic 0, and because $X$ is smooth, we can use Hironaka’s resolution of singularities to replace $\mathcal{X}$ with such a model if need be. After making this replacement, the construction of the previous section provides us with a skeleton

$$\text{Sk}(\mathcal{X}) \subset X^{\text{an}}$$

Nicaias and Xu prove that, at least over $k((t))$, this skeleton is a strong deformation retract of $X^{\text{an}}$ [NX13, Theorem 3.1.3]. Thus we immediately have the following:

\textbf{Lemma 2.4.1.} With $X$ and $\mathcal{X}$ as above, the Berkovich analytification $X^{\text{an}}$ has the same homotopy type as the dual complex $D(\mathcal{X}_k)$. \hfill \Box
Remark 2.4.2. From Lemma 2.4.1 it follows immediately that to prove that the analytification $f^\an : X^\an \to Y^\an$ of a given map $f : X \to Y$ of $k((t))$-varieties, it suffices to show that $f$ induces a homotopy equivalence $D(\mathcal{X}_k) \simeq D(\mathcal{Y}_k)$ between the dual complexes of the special fibers in snc $k[[t]]$-models $\mathcal{X}$ and $\mathcal{Y}$ of $X$ and $Y$, respectively.

One would like to establish homotopy equivalence of $D(\mathcal{X}_k)$ and $D(\mathcal{Y}_k)$ by using techniques from MMP directly on a $k[[t]]$-model $\mathcal{X} \to \mathcal{Y}$ of the map $f : X \to Y$. Unfortunately, the necessary MMP techniques are not established over rings, like $k$, which are not of finite type over a field of characteristic 0. We will have to use a “spreading out” argument in order to transfer our task into a setting where the needed MMP techniques are known.

3. Rationally Connected Varieties

3.1. Definitions and first properties.

The details in this section are well known to experts; we refer to [Deb01] and [Kol96].

One way to distinguish birational classes of varieties is by considering the sets of rational curves they contain. If $X$ is a smooth variety over an algebraically closed field, we say that $X$ is rationally connected if given any two closed points $p$ and $q$ of $X$, there is a map $f : \mathbb{P}^1 \to X$ such that both $p$ and $q$ are in the image of $f$. Projective $n$-space is rationally connected for instance, since any two closed points of $\mathbb{P}^n$ are contained in a line. Many other familiar varieties are also rationally connected, including Grassmanians, toric varieties, and smooth Fano varieties.

Because we will be working over $k((t))$, which is not algebraically closed, we will need a slightly more general definition than the one above.

Definition 3.1.1. Assume that $F$ is a field of characteristic 0. A proper scheme $X$ over $F$ is rationally connected if there exists a variety $M$ over $F$, and a morphism $u : M \times \mathbb{P}^1 \to X$ such that the induced map $u^{(2)} : M \times \mathbb{P}^1 \times \mathbb{P}^1 \to X \times_F X$ is dominant.

This definition follows [Deb01] Definition 4.3], and corresponds to the definition of a separably rationally connected scheme in [Kol96] Definition IV.3.2]. The two notions are equivalent in characteristic 0. Note that in Definition 3.1.1 it is equivalent to take $u$ to be a rational map instead of a morphism. When the field $F$ is algebraically closed and $X$ is smooth, this definition is equivalent to the condition that every pair of closed points of $X$ be connected by a rational curve.

Definition 3.1.1 behaves well under birational modification: if $X'$ is birational to $X$, then $X'$ is rationally connected if and only if $X$ is. Thus all rational varieties are rationally connected. Rationally connected varieties also enjoy the additional property that the image of a rationally connected variety is rationally connected:

Proposition 3.1.2. Let $X$ be rationally connected, and let $f : X \to Z$ be a surjective morphism. Then $Z$ is rationally connected.

Proof. We have $u : M \times \mathbb{P}^1 \to X$, with $u^{(2)}$ dominant. Compose to get $f \circ u : M \times \mathbb{P}^1 \to Z$, and observe that surjectivity of $f$ implies that the induced map $M \times \mathbb{P}^1 \times \mathbb{P}^1 \to Z$ is dominant. □

3.2. Rational connectivity in families.

Rational connectivity behaves well in families. If $\pi : Y \to S$ is a proper, smooth map of varieties over an algebraically closed field $F$ of characteristic 0, then the locus of points $s \in S$ with rationally connected fiber $Y_s$ is both open in $S$ and is a countable union of closed subsets of $S$ [Kol96 Corollary IV.3.5.2 & Theorem IV.3.11]. In particular, if $S$ is connected, then every fiber $Y_s$ is rationally connected as soon as one of these fibers is.

When $F$ is algebraically closed of characteristic 0, we say that a morphism $\pi : Y \to S$ of connected $F$-varieties is a rationally connected fibration if $\pi$ is proper, and if the fiber of $\pi$ over a
general point of $S$ is rationally connected. This implies that every fiber contained in the smooth locus of $\pi$ is rationally connected.

**Proposition 3.2.1.** Let $S$ be a variety with generic point $\nu$, and let $\pi : Y \rightarrow S$ be a proper map. If the fiber $Y_\nu$ is rationally connected, then $\pi$ is a rationally connected fibration.

**Proof.** Let $\nu$ be the generic point of $S$. We have a variety $M$ defined over $\nu$ with a map $u : M \times \mathbb{P}^1 \rightarrow Y_\nu$ such that the induced map $u^{(2)} : M \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Y_\nu \times_\nu Y_\nu$ is dominant. The variety $M$ and the map $u$ are defined over some affine open in $S$, so there is an open set $V$ in $S$ such that we have a variety $M_V$ and a rational map $u_V : M_V \times \mathbb{P}^1 \rightarrow Y_V$ such that the induced map $u_V^{(2)}$ is dominant. We can assume $M_V$ is quasiprojective, and then the image of $M_V \times \mathbb{P}^1 \times \mathbb{P}^1$ is a constructible set in $Y_V$.

Thus for a general point $p$ in $V$ the image is dense in the fiber $Y_p$, and therefore $Y_p$ is rationally connected.

\[ \square \]

### 3.3. Rational connectivity and the Minimal Model program.

Loosely speaking, the prevalence of rational curves on a variety measures the failure of its canonical divisor to be positive, and so being rationally connected is a strict condition on the birational geometry of a variety $X$. For example, if $X$ is a smooth Fano variety ($-K_X$ is ample) then we expect to find many rational curves on $X$, and in fact such an $X$ is always rationally connected [Cam92]. While the converse is not true, in a certain sense rationally connected varieties can be built out of Fano varieties using the minimal model program.

The minimal model program is a program for classifying algebraic varieties up to birational equivalence [KM98]. Given a smooth projective variety $X$, the main goal of the MMP is to produce a variety $\tilde{X}$ birational to $X$, with at worst mild singularities, such that the canonical divisor $K_{\tilde{X}}$ is nef (that is, nonnegative on every curve). One calls the variety $\tilde{X}$ a minimal model of $X$.

The strategy for producing a minimal model of $X$ is to systematically eliminate curves $C$ in $X$ satisfying $K_X \cdot C < 0$. This is accomplished by either contracting a divisor $D$ in $X$ covered by such curves, or by applying a birational operation called a flip. Assuming that the procedure of repeatedly applying these two operations terminates after finitely many steps, the end result is either a minimal model or a map $\tilde{X} \rightarrow Z$ of relative Picard number 1, such that $-K_{\tilde{X}}$ is relatively ample. The map $\tilde{X} \rightarrow Z$ is called a Mori fiber space. In general, termination of flips is a difficult open problem in birational geometry. However, Birkar, Cascini, Hacon, and McKernan [BCHM10] establish the existence of minimal models by showing termination of flips for a special MMP called the MMP with scaling.

The operations constituting the MMP are closely related to the existence of rational curves on the variety $X$: if there exists a curve $C$ in $X$ such that $K_X \cdot C < 0$, then there is always a rational curve with this same property. Whenever we contract a divisor in running the MMP, we do so by contracting a family of rational curves that cover the divisor.

Having a covering family of rational curves on a smooth variety $X$ precludes the existence of a section of $nK_X$ for any positive $n$. In fact slightly more is true:

**Proposition 3.3.1.** If $X$ is a smooth and rationally connected projective variety over an algebraically closed field, then $K_X$ is not pseudoeffective.

**Proof.** By [Deb01] Ex 4.7, Cor 4.11, there is a family of rational curves on $X$ such that

1. $-K_X \cdot C \leq -2$ for any curve $C$ in the family.
(2) For a general point \( p \in X \), there is a curve \( C \) in the family passing through \( p \).

Choose any ample divisor \( A \) on \( X \). Then for sufficiently small \( \lambda > 0 \), \( \lambda A \cdot C < 2 \). Thus \((K_X + \lambda A) \cdot C < 0\). Since given any closed subset of \( X \) we can choose \( C \) to not be contained in that set, we must have that no multiple of \((K_X + \lambda A)\) has a section. \( \square \)

As a consequence, by \([\text{BCHM10}, \text{Cor 1.3.3}]\), running any MMP with scaling on a smooth rationally connected \( X \) produces a Mori fiber space \( \tilde{X} \to Z \). The base \( Z \) is the image of a rationally connected variety, so it too is rationally connected. If we repeatedly run an MMP with scaling starting with a rationally connected variety, we produce a long sequence of divisorial contractions, flips, and Mori fibrations terminating at a single point.

4. SPREADING OUT OVER A CURVE

4.1. Rationally connected spreading out.

Throughout the remainder of this article, we fix an algebraically closed field \( k \) of characteristic 0, and let \( K = k((t)) \) with ring of integers \( R = k[[t]] \).

The present section is devoted to a proof that if \( f : X \to Y \) is a surjective morphism of smooth projective \( k((t)) \)-varieties, then we can spread any snc \( k[[t]] \)-model \( \mathcal{X} \to \mathcal{Y} \) of this morphism out to a morphism over a smooth curve. Moreover, we can do so in a way that preserves the morphism’s general relative smoothness and relative rational connectivity when these properties hold over general points of \( Y \).

We begin by establishing a non-relative version of spreading out that replaces \( k[[t]] \) with a finite type \( k \)-variety \( S \).

Proposition 4.1.1. Let \( X \) be a projective, rationally connected \( k((t)) \)-variety, and let \( \mathcal{X} \) be a projective \( k[[t]] \)-model of \( X \). Then there exists a finite type \( k \)-variety \( S \), a dominant morphism \( \text{Spec } k[[t]] \to S \), and a projective variety \( V \) over \( S \), such that

\[
\mathcal{X} \cong \text{Spec } k[[t]] \times_S V
\]

and such that the fiber \( V_q \) over a general point \( q \) in \( S \) is rationally connected.

Furthermore, if \( k[[t]] \to S' \) is a dominant map into any finite type \( k \)-variety, not necessarily the variety \( S \) provided by the previous paragraph, and if \( V' \) is a projective \( S' \)-variety with \( \mathcal{X} \cong \text{Spec } k[[t]] \times_{S'} V' \), then the fiber \( V'_q \) over a general point \( q \) in \( S' \) is again rationally connected.

Proof. We first show that we can construct varieties \( S \) and \( V \) with the required properties.

As in the proof of \([\text{MN13}, \text{Proposition 5.1.2}]\), we use the spreading out techniques from \([\text{Gro66, \S8 & Théorème 11.6.1}]\) to obtain:

- An integrally closed, finite-type \( k \)-subalgebra \( A \subset k[[t]] \);
- An affine open neighborhood \( \text{Spec } A_0 \subset \text{Spec } A \) containing the image of the closed point under the map \( \text{Spec } k[[t]] \to \text{Spec } A \);
- A flat projective family \( V \to \text{Spec } A_0 \) such that \( \mathcal{X} \cong \text{Spec } k[[t]] \times_{A_0} V \).

Our assumption that \( X \) is rationally connected means that there exists a \( k((t)) \)-variety \( M \) and a map \( f : M \times \mathbb{P}^1 \to X \) such that the induced map

\[
f^{(2)} : M \times \mathbb{P}^1 \times \mathbb{P}^1 \to X \times_X X_{k((t))}
\]
is dominant. Choose some flat $k[[t]]$ model $\mathcal{M}$ of $M$. We can run the spreading out for $\mathcal{X}$ and $\mathcal{M}$ simultaneously, by taking a simultaneous extension of the relevant subrings inside $k[[t]]$. Thus, changing our choice of $A \subset k[[t]]$ and $A_0$ if need be, we also obtain:

- A flat family $N \longrightarrow \text{Spec } A_0$ such that $\mathcal{M} \cong \text{Spec } k[[t]] \times_{A_0} N$.

Finally, let $\Gamma$ be the graph of $f$ in $(M \times \mathbb{P}^1) \times_{k((t))} X$, and let $\mathcal{G} \subset (\mathcal{M} \times \mathbb{P}^1) \times_{k[[t]]} \mathcal{X}$ be a flat $k[[t]]$-model for $\Gamma$. By [Gro66, Lemme 8.5.2.1, Théorème 8.10.5, & Théorème 11.6.1], combined with the arguments of the previous paragraph, we can run spreading out for $\mathcal{X}$, $\mathcal{M}$, and $\mathcal{G}$ simultaneously. Changing our choice of $A \subset k[[t]]$ and $A_0$ once again if need be, we obtain:

- A closed subscheme $G \subset (N \times \mathbb{P}^1) \times_{A_0} V$, flat over $A_0$, satisfying $\mathcal{G} \cong \text{Spec } k[[t]] \times_{A_0} G$.

Let $S \subset \text{Spec } A$ denote the closure of the image of the map $\text{Spec } k[[t]] \longrightarrow \text{Spec } A_0$, so that the map $\text{Spec } k[[t]] \longrightarrow S$ is dominant. We wish to show that over the generic point $\eta \in S$, the fiber $G_{\eta}$ is the graph of a rational map

$$\phi : N_{\eta} \times \mathbb{P}^1 \longrightarrow V_{\eta}$$

Thus, we must show that the map $\mathcal{G}_{\eta} \longrightarrow N_{\eta} \times \mathbb{P}^1$ is birational. To this end, note that we have a commutative diagram of function fields:

$$\begin{array}{ccc}
k(G_{\eta}) & \longrightarrow & k(\Gamma) \\
\uparrow & & \uparrow \\
k(N_{\eta} \times \mathbb{P}^1) & \longrightarrow & k(M \times \mathbb{P}^1) \\
\uparrow & & \uparrow \\
k(S) & \longrightarrow & k((t))
\end{array}$$

The bottom square is co-Cartesian, as is the large square. Hence the top square is also co-Cartesian. Since $k(\Gamma)$ has degree 1 over $k(M \times \mathbb{P}^1)$, this implies that $k(\mathcal{G}_{\eta})$ has degree 1 over $k(N_{\eta} \times \mathbb{P}^1)$. This gives us an open subset $U \subset N_{\eta} \times \mathbb{P}^1$ equipped with a morphism $\phi : U \longrightarrow V_{\eta}$. Note that $\phi$ induces a morphism

$$\phi^{(2)} : U \times_U N_{\eta} \longrightarrow V_{\eta} \times_{k(S)} V_{\eta}$$

Because both $f^{(2)} : M \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow X \times_{k((t))} X$ and $\text{Spec } k[[t]] \longrightarrow S$ are dominant, this morphism $\phi^{(2)}$ is also dominant.

By resolving singularities of $N_{\eta}$, we can choose $N'_{\eta}$ so that the rational map

$$\phi' : N'_{\eta} \times \mathbb{P}^1 \longrightarrow V_{\eta}$$

induced by $\phi$ is defined away from a codimension-2 subset. Let $N''_{\eta}$ be the complement of the image of this set in $N'_{\eta}$. Thus we have a map $\phi' : N''_{\eta} \times \mathbb{P}^1 \longrightarrow V_{\eta}$ such that the induced map

$$\phi''^{(2)} : N''_{\eta} \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow V_{\eta} \times_{k(S)} V_{\eta}$$

is dominant, since this map agrees with $\phi^{(2)}$ on an open set. Hence $V_{\eta}$ is rationally connected, and thus by Proposition 3.2.1 the map $V \longrightarrow S$ is a rationally connected fibration.

Now suppose we have a finite type $k$-variety $V'$ over a finite type $k$-variety $S'$, and suppose given a dominant map $\text{Spec } k[[t]] \longrightarrow S'$ such that

$$\mathcal{X} \cong \text{Spec } k[[t]] \times S'$$
We may assume that $S'$ is affine, so that $S' = \text{Spec } A'$, where $A' \subset k[[t]]$. We can then run the above construction, choosing $A$ so that it is an extension of $A'$ inside $k[[t]]$. We obtain $V$ and $S$ satisfying the proposition, along with a morphism $S \rightarrow S'$ that commutes with the morphisms from $\text{Spec } k[[t]]$.

After replacing $A$ with another extension inside $k[[t]]$ if necessary, we can assume that $V' \times_S S'$ is isomorphic to $V$ over $S'$, by [Gro66 Cor 8.8.2.5], since $V' \times_S S'$ and $V$ become isomorphic over $k[[t]]$. Because the map $\text{Spec } k[[t]] \rightarrow S'$ is dominant, so is $S \rightarrow S'$. Thus the fiber of $V'$ over a general point in $S'$ is the image of the fiber over some general point in $S$. Since the image of a rationally connected variety is connected, the proposition follows.

**Remark 4.1.2.** The previous Proposition 4.1.1 allows us to spread a rationally connected snc $k[[t]]$-model to a model over a finite type $k$-variety. In order to apply the techniques of [JFKX12], we need to spread our $k[[t]]$-model not over just any finite type $k$-variety, but over a smooth curve. Theorem 4.1.3 accomplishes this for not only a rationally connected snc $k[[t]]$-model, but for a rationally connected fibration of snc $k[[t]]$-models.

**Theorem 4.1.3.** Let $f : X \rightarrow Y$ be a surjective morphism of smooth projective varieties over $k(t)$. Assume that for a general point $q$ of $Y$ such that the fiber $X_q$ is smooth and rationally connected. If $f_{k[[t]]} : \mathcal{X} \rightarrow \mathcal{Y}$ is a map of snc $k[[t]]$-models that extends $f$, then there exists a pointed affine curve $C \ni p$ over $k$, along with a map $g_C : V_C \rightarrow W_C$ of smooth $C$-varieties satisfying:

1. The special fibers $V_p$ and $W_p$ are snc inside $V_C$ and $W_C$.
2. The dual complexes of the special fibers $V_p$ and $W_p$ are identified with those of $\mathcal{X}$ and $\mathcal{Y}$, respectively.
3. The fiber of $g_C$ over a general point $q$ of $W_p$ is rationally connected.

**Proof.** As above, we can choose an affine $k$-variety $S$, along with varieties $V$, $W$, and a $\Theta \subset V \times_S W$ such that $V$, $W$, and $\Theta$ pull back to $\mathcal{X}$, $\mathcal{Y}$, and the graph $\Gamma$, respectively. We may assume $\Theta$ is isomorphic to $V$ over $S$, so that $\Theta$ gives the graph of a morphism $g : V \rightarrow W$. Let $A$ denote the coordinate ring of $S$.

The construction of the curve $C$ and flat families $V_C \rightarrow C$ and $W_C \rightarrow C$ proceeds exactly as in the proof of [AMN13] Proposition 5.1.2: Given a positive integer $n$, we use Greenberg approximation [Gro66] to find a morphism

$$A \rightarrow \Theta^h_{k,[t]}$$

to the Henselization of the stalk at the origin in $\mathbb{A}^1_k$, such that the composite of (4) with the projection

$$\Theta^h_{k,[t]} \cong k[[t]] \rightarrow k[[t]]/(t^n)$$

factors through a morphism $A \rightarrow k[[t]]/(t^n)$. Because the Hensilization is the stalk for the étale topology, we can find a smooth curve $C$ and a morphism

$$C \rightarrow \text{Spec } A$$

through which the morphism dual to (4) factors. If we define $V_C$ and $W_C$ to be the pullbacks of $V$ and $W$ along this morphism (5), then it follows immediately from Proposition 4.1.1 that

$$\text{Spec } k[[t]]/(t^n) \times V_C \cong \text{Spec } k[[t]]/(t^n) \times \mathcal{X} \cong \text{Spec } k[[t]]/(t^n) \times V$$

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and similarly for the triple $W_C$, $\mathcal{Y}$, and $W$. Since the dual complexes only depend on the special fibers, the second statement of the proposition is immediate.

Now, consider the flat locus of $g : V \rightarrow W$. This is an open set, so its image in $S$ is open. Let $T \subset S$ be the complement. Now, because the map Spec $k[[t]] \rightarrow S$ is dominant, it does not factor through $T$. Thus there is some $n \geq 2$ such that the induced map from Spec $k[t]/(t^n)$ does not factor through $T$. Thus we may assume that $C$ is not contained in $T$.

The requirement that $\mathcal{X}$ and $\mathcal{Y}$ be snc over $k[[t]]$ includes the requirement that they are both regular. Because the Zariski tangent space at points in the special fiber depends only on the fiber over Spec $k(t)$, this implies that $V_C$ and $W_C$ are both smooth in a neighborhood of their respective special fibers. Localizing if necessary, we may therefore choose $C$ so that $V_C$ and $W_C$ are smooth $[MN13$, Proposition 5.1.2.(6)]. In particular, this implies that a general fiber of the morphism from $V \rightarrow W$ is smooth.

Let $K$ be an algebraic closure of $K = k((t))$, and consider the closed points of $Y_K$. By hypothesis, we know that there is an open set of $Y_K$ where the fiber of $f_K : X_K \rightarrow Y_K$ is smooth and therefore also rationally connected. Now, since the $\overline{K}$ points are dense in $Y_K$, their images are dense in $W$. Each $\overline{K}$ point $\overline{q}$ is defined over a finite extension of $k((t))$ of the form $k((t^{1/n}))$. We can therefore apply Proposition 4.1.1 to see that the fiber of $V \rightarrow W$ over the image of $\overline{q}$ is rationally connected. Applying Proposition 3.2.1, we have that a dense set of closed points inside the closure of the image of $\overline{q}$ is rationally connected, so we must have that there is no proper closed set of $W$ containing all the closed points with rationally connected fibers. Hence these points are dense.

Note that irreducibility of $Y$ implies that $W$ is irreducible.

Because rational connectedness is both open and closed in smooth families, we have that for points in the flat locus of $g$, the fiber of $g$ is rationally connected whenever it is smooth. Since both $V_C$ and $W_C$ are smooth, and $C$ is not contained in $T$, this means that over a general point of $W_C$, the fiber of $g_C$ is rationally connected.

\[ \Box \]

5. Homotopy type of the dual complex

5.1. Singularities of the MMP.

De Fernex, Kollár, and Xu have shown recently that running the MMP on certain varieties with snc divisors preserves the homotopy type of the dual complex $[dFKX12]$. As a result they are able to show the following

\begin{theorem} \[ dFKX12 \] Thm 1 \end{theorem}

Let $X$ be an isolated log terminal singularity. Then for any log resolution, the dual complex is contractible.

Applying their techniques to the case of a degeneration of rational varieties, they are able to show

\begin{theorem} \[ dFKX12 \] Thm 4 \end{theorem}

Let $X \rightarrow C$ be a degeneration of rationally connected varieties over a curve, with special fiber $X_0$. Assume $X$ is smooth and that $X_0$ is snc. Then the dual complex of $X_0$ is contractible.

Log terminal singularities are a natural class of singularities arising in the MMP. Examples of log terminal singularities are provided by quotient singularities and Gorenstein rational singularities. We will need to introduce some related notions for pairs $(X, \Delta)$ so that we may apply the techniques of $[dFKX12]$. The standard reference is $[KM98]$.

Let $X$ be a normal variety over an algebraically closed field of characteristic 0, and let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is Cartier. Let $f : Y \rightarrow X$ be a log resolution of $(X, \Delta)$. This means that:

(i) The variety $Y$ is smooth;
(ii) The map \( f \) is proper and birational;

(iii) If we let \( \{ E_i \} \) be the set whose elements are all components of the exceptional divisors of \( f \) and all components of the strict transform of \( \Delta \), then the \( E_i \) form a simple normal crossing divisor.

We can compare the divisor \( K_X + \Delta \) on \( X \) with its counterpart \( K_Y + \sum E_i \) in \( Y \). These must agree away from the \( E_i \), and so we have

\[
K_Y + \sum E_i = f^*(K_X + \Delta) + \sum a_i E_i
\]

where the pullback of \( K_X + \Delta \) is a well defined \( \mathbb{Q} \)-divisor, and where the \( a_i \) are rational numbers, called the log discrepancies. If every one of the log discrepancies satisfies \( a_i \geq 0 \), then we say that \( (X, \Delta) \) is log canonical, or lc. If each of the log discrepancies satisfies \( a_i > 0 \), then we say that \( (X, \Delta) \) is Kawamata log terminal, or klt. We say that \( X \) itself is log terminal if the pair \((X, 0)\) is klt.

A subvariety \( V \subset X \) is called a log canonical center if it is of the form \( f(E_i) \) for some \( E_i \) with vanishing log discrepancy: \( a_i = 0 \). The pair \((X, \Delta)\) is called divisorial log terminal, or dlt, if for every log canonical center \( V \) of \((X, \Delta)\), there is a neighborhood of the generic point of \( V \) in which \((X, \Delta)\) becomes snc. The related notion of quotient-dlt, or qdlt, is less well known, so we refer to [dFKX12] Section 5. The pair \((X, \Delta)\) is qdlt if every codimension-\(d\), log canonical center of \((X, \Delta)\) is the intersection of \( d \) components of \( \Delta \), each with coefficient 1. The name “quotient dlt” comes from the fact that a qdlt singularity is characterized by being locally a quotient of a dlt singularity by an abelian group [dFKX12] Prop 34.

These singularity classes obey the following logical dependencies:

\[
\text{klt} \Rightarrow \text{dlt} \Rightarrow \text{qdlt} \Rightarrow \text{lc}
\]

**Example 5.1.3.** Let \( X = \mathbb{A}^2 \) and let \( \Delta \) be the coordinate axes both with coefficient 1. Then \((X, \Delta)\) is dlt, qdlt, and lc, but not klt, and the log canonical centers are \( V(x) \), \( V(y) \), and \( V(x, y) \). Let \( X' \) be the affine quadric cone and \( \Delta' \) the union of two lines on this cone. Then \((X', \Delta')\) is the quotient of \((X, \Delta)\) by a \( \mathbb{Z}/2 \) action which preserves the components of \( \Delta \), so \((X', \Delta')\) is qdlt and lc. However, \((X', \Delta')\) is not dlt (nor klt) because it is not snc in a neighborhood of the vertex of the cone, which is a log canonical center.

The first important property of these types of singularities is that each class is preserved under running a \( K_X + \Delta \) MMP. More precisely, if \( g : X \dasharrow X' \) is a divisorial contraction or a flip for the pair \((X, \Delta)\), which is klt/dlt/qdlt/lc, then the pair \((X', g_* \Delta)\) is also klt/dlt/qdlt/lc.

### 5.2. Proof of the Main Theorem.

Before giving a proof of Theorem 1.1.2 we need to establish an analogous result for a morphism of varieties over a curve.

**Theorem 5.2.1.** Let \( C \) be an affine curve over an algebraically closed field \( k \) of characteristic 0, and let \( p \) be a \( k \)-valued point of \( C \). Suppose given projective \( C \)-varieties

\[
\pi_X : X \longrightarrow C \quad \text{and} \quad \pi_Y : Y \longrightarrow C
\]

and let \( \Delta_X \) and \( \Delta_Y \) be the reduced fibers of \( X \) and \( Y \) over \( p \). Assume that \((X, \Delta_X)\) and \((Y, \Delta_Y)\) are both snc pairs, and let \( f : X \longrightarrow Y \) be a surjective morphism over \( C \), such that for a general point \( q \in Y \), the fiber \( X_q = f^{-1}(q) \) is rationally connected. Since \( X \) is smooth, we may assume that \( X_q \) is smooth.

Then the dual complex of \((X, \Delta_X)\) is homotopy equivalent to the dual complex of \((Y, \Delta_Y)\).
The proof of this theorem is adapted directly from \cite{dFKX12}. The key point is that under these hypotheses, every birational step of the MMP induces a specific type of homotopy equivalence, called a collapse, on the dual complex. It then suffices to check that Mori fiber spaces also preserve the dual complex.

**Proof.** Choose a small $\varepsilon > 0$, and consider the pair $(X, \Delta_X - \varepsilon \pi^{-1}(p))$. This is a klt pair, so by \cite{BCHM10} we can run MMP with scaling for this pair. Because $\varepsilon \pi^{-1}(p)$ is numerically trivial, the MMP with scaling is also a $K_X + \Delta_X$ MMP, and it preserves the property of being dlt.

Since $X_q$ is smooth and rationally connected, $K_{X_Y}$ cannot be pseudoeffective, as otherwise for a general fiber $X_q$, the canonical divisor $K_{X_q}$ would be pseudoeffective, contradicting Proposition \ref{prop:3.3.1}.

Thus running a relative MMP with scaling over $Y$ eventually terminates in a Mori fiber space $X' \to Y$ \cite{BCHM10} Cor 1.3.3]. Let $\Delta'_{X'}$ be the reduced special fiber of $X'$, which is the strict transform of $\Delta_X$. Then the pair $(X', \Delta'_{X'})$ is also dlt, since it is the result of running a $K_{X'} + \Delta_{X'}$ MMP. By \cite{dFKX12} Cor 22], the dual complex of $(X, \Delta_X)$ collapses to the dual complex of $(X', \Delta'_{X'})$. These complexes are thus homotopy equivalent.

Let $\Delta_Z$ be the reduced special fiber of $\pi_Z : Z \to C$. The pair $(Z, \Delta_Z)$ may no longer be dlt, but it is q-dlt by \cite{dFKX12} Prop 40]. Because the morphism $X' \to Z$ has relative Picard number 1, we can identify the dual complex of $(X', \Delta'_{X'})$ with that of $(Z, \Delta_Z)$.

Now, let $\phi : Z' \to Z$ be a log resolution of singularities of $Z$. This map induces maps

$$f_{Z'} : Z' \to Y \quad \text{and} \quad \pi_{Z'} : Z' \to C$$

By \cite{dFKX12} Theorem 3, the dual complex of the special fiber $(Z', \Delta'_{Z'})$ is homotopy equivalent to the dual complex of $(Z, \Delta_Z)$. Moreover, if we consider a general point $q$ of $Y$, then there is a dominant rational map on the fibers $X_q \to Z'_{q}$. This implies that $Z'_{q}$ is rationally connected.

The variety $Z'$ satisfies the hypotheses of the theorem. If the dimension of a general fiber is 0, then since $Z' \to Y$ is a resolution of singularities, and $(Y, \Delta_Y)$ is dlt, by \cite{dFKX12} Theorem 3] the dual complexes of $(Z', \Delta'_{Z'})$ and $(Y, \Delta_Y)$ are homotopy equivalent. Otherwise we may proceed by induction, since $\dim Z' < \dim X$. \hfill $\square$

**Proof of Thm \ref{thm:1.1.2}** Let $R$ be the spectrum of the power series ring $k[[t]]$. Using resolution of singularities, extend the map $f : X \to Y$ to a map on snc-models $f_R : \mathcal{X} \to \mathcal{Y}$ over $R$. By Cor \ref{cor:2.4.1} it suffices to show that the dual complexes $D(\mathcal{X}_k)$ and $D(\mathcal{Y}_k)$ are homotopy equivalent.

If $f_R, \mathcal{X}$, and $\mathcal{Y}$ are pulled back from a curve, then the theorem follows immediately by Theorem \ref{thm:5.2.1}.

Otherwise, by Proposition \ref{prop:4.1.3} we can produce a pointed $k$-curve $C \ni p$, along with a map of smooth varieties $g_C : V_C \to W_C$ over $C$, such that:

- The fiber over a general point of $W$ is smooth and rationally connected;
- The special fiber $W_p$ is snc inside $W$ and is isomorphic to $\mathcal{Y}_k$;
- The special fiber $V_p$ is snc inside $V$ and is isomorphic to $\mathcal{X}_k$.

Then by Theorem \ref{thm:5.2.1} the dual complexes $D(V_p)$ and $D(W_p)$ are homotopy equivalent. Thus $D(\mathcal{X}_k)$ and $D(\mathcal{Y}_k)$ are homotopy equivalent and the theorem follows. \hfill $\square$

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