Note on Custodial Symmetry in the Two-Higgs-Doublet Model

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We present a simple and transparent method to study custodial symmetry in the Two-Higgs-Doublet Model. The method allows to formulate the basis independent, sufficient and necessary, conditions for the custodial symmetry of the scalar potential. The relation between the custodial transformation and CP is discussed and clarified.

1. INTRODUCTION

The \( \rho \) parameter is experimentally measured as \[ \rho = 1.0008^{+0.0017}_{-0.0007}, \]
where at tree-level \( \rho \equiv m_W^2 / \cos^2(\theta_W) / m_Z^2 \) with \( \theta_W \) the Weinberg angle and \( m_W \) and \( m_Z \) the electroweak gauge boson masses.

In the Standard Model (SM) with a Higgs sector consisting of one Higgs doublet \( \phi \) there is an extra symmetry of the Higgs potential

\[ V_{SM} = -\lambda (\phi^\dagger \phi)^4 + \mu (\phi^\dagger \phi)^2. \]

which is responsible for \( \rho \approx 1 \); because of its role in insuring small corrections to \( \rho \) this symmetry of \( V_{SM} \) is commonly called a custodial symmetry (CS) \[2\]. Its form can be made manifest by decomposing the complex Higgs-doublet into the real components \( \phi_1, \phi_2, \phi_3, \phi_4 \),

\[ \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} \phi_1 + i \phi_2 \\ \phi_3 + i \phi_4 \end{pmatrix}. \]

We then find immediately \( (\phi^\dagger \phi) = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 \), hence, the potential is invariant under \( SO(4) \sim SU(2)_L \times SU(2)_R, \) \((\sim \) means that both sides have the same Lie algebra), and the CS is then the diagonal \( SU(2) \) subgroup.

In order to study the CS it is convenient to introduce the following matrix

\[ M \equiv (i \sigma^2 \phi^*, \ \phi) = \begin{pmatrix} \phi^0^* & \phi^+ \\ -\phi^- & \phi^0 \end{pmatrix}. \]

Then \( M \) transforms under \( SU(2)_L \times SU(2)_R \) as

\[ M \rightarrow LMR^\dagger, \]

where \( L, R \in SU(2)_{L,R} \) respectively. In the SM one can assume without loss of generality that the vacuum expectation value of \( \phi \) is real, whence, after spontaneous symmetry breaking (SSB), \( \langle M \rangle \propto 1_2 \), so \( SU(2)_L \times SU(2)_R \) is broken to \( SU(2)_{diag} \) (the diagonal subgroup) and it is the invariance of \( |D_\mu \phi|^2 \) with respect to \( SU(2)_{diag} \) that insures \( \rho = 1 \) at tree-level \[2, 3\]. The term custodial symmetry is reserved in the literature for the \( SU(2)_{diag} \) transformations under which both would-be Goldstone bosons and the corresponding gauge bosons transform as triplets. In this work we will also refer to custodial transformations (CT) as those generated by the full group \( SU(2)_L \times SU(2)_R \). We will always assume that the vacuum respects the diagonal \( SU(2) \), the CS.

Now we can write

\[ (\phi^\dagger \phi) = \frac{1}{2} \text{Tr}(M^\dagger M) \]

which is manifestly invariant under \[10\] so that \( V_{SM} \) will be CT invariant. It is easy to see, however, that the CS is explicitly violated in the SM by both the hypercharge gauge interactions and the Yukawa couplings; so the CS is but an approximate symmetry in the SM. Note that when the CS-violating coefficients are set to zero all massive vector bosons are mass degenerate (corresponding to \( \rho = 1 \)) to all orders in perturbation theory \[2, 3\].

It is easy to see that, even at tree-level, \( \rho = 1 \) cannot be realized naturally in an extended scalar sector unless the scalar multiplets belong to a specific set of isospin representations. The singlets and isodoublets are the simplest of these “\( \rho \)-safe” representations; hereafter we will focus on isodoublet extensions of the SM. It is well known \[2, 4, 5\] that even for two scalar isodoublets, in general, there exist potentially large radiative corrections to \( \rho - 1 \) proportional to the squares of the scalar masses. A remedy has also been proposed \[2, 4, 5\] through two generalizations of the CT to the two-Higgs-doublet model (THDM) \[3\].

\[ ^1 \] Recently the subject has been resurrected in \[6\].
In this note we will revisit this issue and derive basis-independent conditions for the CS for the THDM potential using both the conventional approach and the bilinear formalism of [4, 5], which allows to illustrate the CT in a transparent way. We also discuss the relation between CT and CP invariance.

2. THE CUSTODIAL SYMMETRY

The most general potential for the THDM may be written in terms of the following doublets carrying the same hypercharge:

\[
\phi_i = \left( \phi^+_i, \phi^0_i \right), \quad \phi_2 = \left( \phi^+_2, \phi^0_2 \right),
\]

Then the potential reads [3]

\[
V = m^2_{11}(\phi^+_1 \phi_1) + m^2_{12}(\phi^+_1 \phi_2) - m^2_{12} (\phi^+_2 \phi_2) - \frac{1}{2} \lambda_1 (\phi^+_1 \phi_1)^2 + \frac{1}{2} \lambda_2 (\phi^+_2 \phi_2)^2 + \lambda_3 (\phi^+_1 \phi_1)(\phi^+_2 \phi_2) + \lambda_4 (\phi^+_2 \phi_2)(\phi^+_1 \phi_1) + \frac{1}{2} \lambda_5 (\phi^+_2 \phi_2)^2 + \lambda_5 (\phi^+_2 \phi_2)^2
\]

with \(m^2_{11}, m^2_{22}, \lambda_1, \lambda_2, 3, 4\) real and \(m^2_{12}, \lambda_5, 6, 7\) complex.

For studies of the CS within the THDM it is convenient to introduce the following set of matrices

\[
M_{ij} = \left( \begin{array}{c}
\phi^+_i \\
\phi^0_i
\end{array} \right) \rightarrow \left( \begin{array}{c}
\phi^+_j \\
\phi^0_j
\end{array} \right)
\]

where \(i = 1, 2\) refers to the scalar doublets. It is easy to see that all bilinears \(\phi_i^+ \phi_j\) can be expressed in terms of \(M_{11}\) and \(M_{22}\), or in terms of \(M_{12}\). Therefore the scalar potential (2.2) could also be written using \(M_{11}\) and \(M_{22}\) or \(M_{12}\).

The following two versions of the CT for THDM have been considered in the literature [3]:

- Type I: In this case it is useful to express the potential in terms of \(M_{11}\) and \(M_{22}\). The transformation is a straightforward generalization of [1, 5]:

\[
M_{ii} \xrightarrow{CT_{i}} M'_{ii} = LM_{ii}R^\dagger \quad \text{for} \quad i = 1, 2
\]

- Type II: For this version of the CT, considered in [3], it is convenient to express the potential using \(M_{12}\) only. The corresponding CT reads:

\[
M_{21} \xrightarrow{CT_{ii}} M'_{21} = LM_{21}R^\dagger
\]

where \(L\) and \(R\) belongs to \(SU(2)_L\) and \(SU(2)_R\), respectively. Note that one cannot simultaneously have invariance under \([4, 5]\) and \([2, 3]\) since, for example, the second case mixes \(\phi_1\) and \(\phi_2\) while the first one does not.

It is worth mentioning here that, since both Higgsboson doublets carry the same quantum numbers, physical content of the model we are considering cannot depend on a choice of the basis adopted for the scalar doublet fields \((\phi_1, \phi_2)\). Nevertheless, the form of the Lagrangian obviously changes by a change of basis; also the form of custodial transformation will change if we change the basis. In what follows we will investigate consequences of such a unitary basis transformation:

\[
\phi_i \rightarrow \hat{\phi}_i = \sum_j U_{ij} \phi_j \quad \text{for} \quad i = 1, 2
\]

where \(U \in U(2)\). This rotation implies the following change for \(M_{ij}\)

\[
M_{ij} \equiv (\hat{\phi}_i, \phi_j) \rightarrow \hat{M}_{ij} \equiv (\hat{\phi}_i, \hat{\phi}_j) = \sum_k U^*_{ik} \hat{\phi}_k, \sum_l U_{jl} \phi_l = \frac{1}{2} \sum_{kl} \hat{M}_{kl} \left( \begin{array}{cc}
U^*_{ik} & 0 \\
0 & U_{jl}
\end{array} \right)
\]

Note that above the sum stands in front of a product of matrices, so that the elements of summed matrices are correlated. From (2.7) we can determine the form of the CT in the new basis; for example, for type I (2.4) we obtain,

\[
\hat{M}_{ij} \xrightarrow{CT_{i}} \frac{1}{2} \sum_k \sum_{op} L \hat{M}_{op} \left[ \left( \begin{array}{cc}
U_{ok} & 0 \\
0 & U_{pk}
\end{array} \right) R^\dagger \left( \begin{array}{cc}
U^*_{ik} & 0 \\
0 & U_{jk}
\end{array} \right) \right]
\]

3. CT IN TERMS OF GAUGE INVARIANT BILINEAR

Due to gauge invariance the doublets in the potential (2.4) can only appear in bilinear form, that is, in terms of \((\phi_i, \phi_j)\). It turns out that it is very convenient to discuss CS using the bilinears instead of the fields themselves. There are just four independent bilinears that can be combined as follows [2, 3]:

\[
K_0 = \phi^+_1 \phi_1 + \phi^+_2 \phi_2, \quad K = \phi^+_1 \phi_1 + \phi^+_2 \phi_2
\]

Any choice of \(K_0 \geq 0\) and \(K^2 \leq K^2_0\) fix the doublets modulo a gauge transformation. The potential (2.2) can be expressed in terms of \(K_0\) and \(K\) in a very compact and suggestive way

\[
V = \xi_0 K_0 + \xi^T K + \eta_{00} K_0^2 + 2K_0 \eta^T K + K^T E K
\]
with the following real parameters

\[ \begin{align*}
\xi_0, & \quad \eta_{00}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \lambda_3 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \\
E = E^T = & \begin{pmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{pmatrix}
\end{align*} \tag{3.3} \]

which have the following expressions in terms of the original parameters in \( 2.2 \):

\[ \begin{align*}
\xi_0 &= \frac{1}{2}(m_{11}^2 + m_{22}^2), \quad \xi = \frac{1}{2} \begin{pmatrix} -2\text{Re}(m_{12}^2) \\ 2\text{Im}(m_{12}^2) \\ m_{11}^2 - m_{22}^2 \end{pmatrix}, \\
\eta_{00} &= \frac{1}{8}(\lambda_1 + \lambda_2) + \frac{1}{4}\lambda_3, \quad \eta = \frac{1}{4} \begin{pmatrix} \text{Re}(\lambda_6 + \lambda_7) \\ -\text{Im}(\lambda_6 + \lambda_7) \\ \frac{1}{2}(\lambda_1 - \lambda_2) \end{pmatrix}, \\
E &= \frac{1}{4} \begin{pmatrix} \lambda_4 + \text{Re}(\lambda_5) & -\text{Im}(\lambda_5) & \text{Re}(\lambda_6 - \lambda_7) \\ -\text{Im}(\lambda_5) & \lambda_5 - \text{Re}(\lambda_5) & -\text{Im}(\lambda_6 - \lambda_7) \\ \text{Re}(\lambda_6 - \lambda_7) & -\text{Im}(\lambda_6 - \lambda_7) & \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 \end{pmatrix} \tag{3.4} \end{align*} \]

It is easy to see that the transformations of \( K_0 \) and \( K \) under a change of basis \( 2.0 \) read

\[ \begin{align*}
K_0 \to K_0' &= K_0, \\
K \to K' &= R(U)K,
\end{align*} \tag{3.5} \]

where the matrix \( R(U) \in SO(3) \) is given in terms of \( U \) through

\[ U^\dagger \sigma^a U = R_{ab}(U) \sigma^b. \tag{3.6} \]

The effect of a change of basis \( 3.5 \) corresponds to the following change of the potential parameters

\[ \begin{align*}
\xi \to \xi' &= R(U)^\dagger \xi, \\
\eta \to \eta' &= R(U)^\dagger \eta, \\
E \to E' &= R(U)^\dagger ER(U).
\end{align*} \tag{3.7} \]

Now we turn to the description of the CS in terms of these parameters.

**Custodial transformation of type I**

For the CT of type I \( 2.3 \) it is convenient to express \( K_0, K \) in terms of \( M_i \) with \( i = 1, 2 \):

\[ \begin{align*}
K_0 &= \frac{1}{2} \text{Tr}(M_{11}^\dagger M_{11} + M_{22}^\dagger M_{22}), \\
K_1 &= \text{Tr}(M_{11}^\dagger M_{22}), \\
K_2 &= -(i) \text{Tr}(M_{11}^\dagger M_{12}^\dagger M_{22}), \\
K_3 &= \frac{1}{2} \text{Tr}(M_{11}^\dagger M_{11} - M_{22}^\dagger M_{22}).
\end{align*} \tag{3.8} \]

whence the type I CT corresponds to

\[ \begin{align*}
\text{CT}_I : \quad & K_{0,1,3} \to K_{0,1,3}, \\
& K_2 \to -(i)\text{Tr}\left[M_{11}(R^\dagger \tau_3 R)M_{22}^\dagger\right]. \tag{3.9} \end{align*} \]

The invariance of the potential \( 5.2 \) under \( 3.9 \) restricts its parameters as follows

\[ \begin{align*}
\eta &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.10} \end{align*} \]

where the dots denote arbitrary entries. The THDM will be symmetric under this CT if and only if there exists a basis rotation after which the potential parameters take the form \( 3.10 \).

Using \( 3.4 \) we can express the potential \( 3.2 \) with parameters \( 3.10 \) in terms of the original doubles:

\[ \begin{align*}
V &= m_{11}^2(\phi_1^\dagger \phi_1) + m_{22}^2(\phi_2^\dagger \phi_2) \\
&\quad - \text{Re}(m_{12}^2)((\phi_1^\dagger \phi_2) + (\phi_2^\dagger \phi_1)) \\
&\quad + \frac{1}{4}((\lambda_1 + \lambda_2)(\phi_1^\dagger \phi_1)^2 + (\phi_2^\dagger \phi_2)^2) + \lambda_3(\phi_1^\dagger \phi_1)(\phi_2^\dagger \phi_2) \\
&\quad + \frac{1}{2}(\lambda_4 + \text{Re}(\lambda_5))(\phi_1^\dagger \phi_2) + (\phi_2^\dagger \phi_1) \\
&\quad + \text{Re}(\lambda_6)((\phi_1^\dagger \phi_2) + (\phi_2^\dagger \phi_1))(\phi_1^\dagger \phi_1) \\
&\quad + \text{Re}(\lambda_7)((\phi_1^\dagger \phi_2) + (\phi_2^\dagger \phi_1))(\phi_2^\dagger \phi_2) . \tag{3.11} \end{align*} \]

This potential is invariant under the CT of type I and matches the expression in \( \frac{2}{2} \).

**Custodial transformation of type II**

In this case it is useful to express \( K_0, K \) in terms of \( M_{21} \) only:

\[ \begin{align*}
K_0 &= \text{Tr}(M_{21}^\dagger M_{21}), \\
K_1 &= 2\text{Re}(\text{det}M_{21}), \\
K_2 &= -2\text{Im}(\text{det}M_{21}), \\
K_3 &= -\text{Tr}(M_{21}^\dagger \tau_3 M_{21}). \tag{3.12} \end{align*} \]

which then transform as follows:

\[ \begin{align*}
\text{CT}_{II} : \quad & K_{0,1,2} \to K_{0,1,2}, \\
& K_3 \to -\text{Tr}\left[M_{21}(R^\dagger \tau_3 R)M_{21}^\dagger\right]. \tag{3.13} \end{align*} \]

It follows that in order for the potential to be invariant under this CT the parameters take the form

\[ \begin{align*}
\eta &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \xi &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.14} \end{align*} \]
Again we can use (3.4) to write the potential in terms of the two doublets as presented in [3]:

\[
V = \frac{1}{2} \left( m_1^2 + m_2^2 \right) \left( (\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) \right) - m_{12}^2 (\phi_1^\dagger \phi_2) - (m_{12}^2)^* (\phi_1 \phi_1^\dagger) + \frac{1}{4} \left( \lambda_1 + \lambda_2 + 2 \lambda_3 \right) \left( (\phi_1^\dagger \phi_1) + (\phi_2^\dagger \phi_2) \right)^2 + \lambda_4 (\phi_1^\dagger \phi_2) (\phi_2^\dagger \phi_1) + \frac{1}{2} \left[ \lambda_5 (\phi_1^\dagger \phi_2)^2 + \lambda_6 (\phi_2^\dagger \phi_1)^2 \right] + (\text{Im}(\lambda_6) + \text{Im}(\lambda_7)) \left[ (\phi_1^\dagger \phi_1) - (\phi_2^\dagger \phi_2) \right] + (\text{Im}(\lambda_6) + \text{Im}(\lambda_7)) \left[ (\phi_1^\dagger \phi_2) - (\phi_2^\dagger \phi_1) \right].
\]

(3.15)

Note that the parameters are in general complex in this case.

Here we immediately see the advantage of the bilinear formalism: while the potentials in conventional notation, (3.11) and (3.13), look quite different, in terms of bilinears $K_0$ $K$ they are very similar (compare (3.10) with (3.14)). In the next section we will show that both potentials are related by a simple basis transformation.

A. Equivalence of the two types of custodial transformation

Here we will show that type I (2.4) and type II (2.5) CT are equivalent, that is, these are the same transformation expressed in different bases.

In the bilinear formalism this is evident: the parameters (3.10) and (3.14), corresponding to type I and II CT are related by a change of basis (3.7) with

\[
R_{I \rightarrow II}^{(1)} = \begin{pmatrix} \sin \alpha & 0 & -\cos \alpha \\ \cos \alpha & 0 & \sin \alpha \\ 0 & -1 & 0 \end{pmatrix} \quad \text{or} \quad R_{I \rightarrow II}^{(2)} = \begin{pmatrix} \sin \alpha & 0 & \cos \alpha \\ \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \end{pmatrix}.
\]

(3.16)

Then solving the equation (3.9) one finds that the corresponding $U$ is

\[
U_{I \rightarrow II} = \frac{e^{i\gamma}}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi} & \pm ie^{-i\varphi} \\ e^{i\varphi} & \mp ie^{i\varphi} \end{pmatrix}
\]

(3.17)

where $\gamma$ is an undetermined phase (the method involving bilinears is not sensitive to an overall phase, which can always be absorbed by a hypercharge transformation of the doublets) and upper and lower signs correspond to $R_{I \rightarrow II}^{(1)}$ with $\varphi = -\alpha/2 + \pi/2$, and $R_{I \rightarrow II}^{(2)}$ with $\varphi = -\alpha/2$, respectively. The rotation $U_{I \rightarrow II}$ can, of course, also be obtained from (2.5) by requiring that the $\phi_1^I$ transform according to (2.4) whenever the $\phi_1^I$ transform according to (2.4).

We close this section with a comment on the equivalence of these two types of CT in a realistic theory that also contains fermions. For the clarity of the argument, let us first assume that Yukawa interactions are absent. Then consider the following two versions (I and II) of THDM: I) A model with the potential $V_I(\phi)$ and II) a model with the potential $V_{II}(\phi) := V_I(U^{\dagger} \phi)$ (equivalent to $V_I(\phi)$, just written in a different basis); the potentials (3.11) and (3.14) serve as an illustration of these two versions, as they are related by the unitary transformation (3.17). As long as Yukawa interactions are not present these two models are physically identical, they differ only by a field redefinition. Now let us switch on Yukawa interactions of the same form in both Lagrangians, $\bar{\nu} \bar{\nu} \Gamma Y \psi$, so that $L_I(\phi) = \cdots - V_I(\phi) + \bar{\nu} \bar{\nu} \Gamma Y \psi + \cdots$ while $L_{II}(\phi) = \cdots - V_I(U^{\dagger} \phi) + \bar{\nu} \bar{\nu} \Gamma Y \psi + \cdots$. Obviously, now $L_I$ and $L_{II}$ are no longer equivalent; they differ by the potentials. However it is interesting to realize that where the difference between them is located is a matter of basis choice: the basis change $\phi \rightarrow U^{-1} \phi$ performed upon $L_{II}$ would shift the difference from the potentials to the Yukawa interactions. It is then interesting to note that in the perturbative expansion only those processes are sensitive to the difference between the two versions that incorporate both Yukawa couplings and couplings that emerge from scalar potentials (so e.g. scalar masses). For instance, the vector-boson vacuum polarizations would be exactly the same in both models at 1-loop (but not in higher orders).

The equivalence (by a basis transformation) between the two types of the transformations that we have found above clearly shows a need for a basis independent formulation of an invariance under the CT. That issue is discussed in the next section.

B. Basis independent conditions for CS

The two types of CT considered above are related by a change of basis and are therefore equivalent, but it would clearly be desirable to have a basis-independent set of conditions which insure that a scalar potential is invariant under CT. In terms of the bilinear coefficients in (3.2) these conditions are the following

\[
E \cdot v = 0 \quad \text{and} \quad \xi \cdot v = \eta \cdot v = 0
\]

(3.18)

for some $v \neq 0$. In order to prove the assertion we note that these conditions are basis independent since the first one is equivalent to requiring $\det(E) = 0$. This means that if the conditions (3.18) are satisfied in one basis, they are satisfied in any basis. Therefore it is sufficient to show that (3.18) are necessary and sufficient conditions for a custodial symmetry in a specific basis, for instance the one defined by the parameters (3.14).

First we have to show that (3.14) imply (3.18), which is immediate: $v = (0, 0, 1)$ is the zero eigenvector of $E$. 

in \((3.14)\), and \(v\) is indeed orthogonal to \(\xi, \eta\). Now, assume \((3.18)\), then, since \(E\) is symmetric and has one zero eigenvalue, we can choose a basis where \(E = \text{diag}(E_1, E_2, 0)\), so that we can take \(v = (0, 0, 1)\), and this will be orthogonal to \(\xi, \eta\) only if both these vectors take the form \((\cdot, \cdot, 0)\); it follows that there is a basis where \((3.18)\) imply \((3.14)\).

4. CT VERSUS CP SYMMETRY

In the bilinear formalism it is easily seen that custodial symmetry and CP symmetry are closely related. First we recall the CP transformation of the doublets, which is defined by

\[
\text{CP} : \quad \varphi_i(x) \rightarrow \varphi_i^*(x'), \quad i = 1, 2. \tag{4.1}
\]

Here we have explicitly written the argument of the fields, since the argument is changed under the CP transformation, that is, we have \(x' = (x^0, -x)\). Applying \((4.1)\) to the bilinears \((3.1)\) we see that a CP transformation is a reflection on the 1–3 plane – in addition to the parity transformation for the field argument \([10]\):

\[
\text{CP} : \quad K_{0,1,2}(x) \rightarrow K_{0,1,2}(x'), \quad K_2(x) \rightarrow -K_2(x') \tag{4.2}
\]

We recognize that like in the type I of the CT \([59]\) only the bilinear \(K_2\) transforms nontrivially under CP. We can now easily give the Higgs potential, invariant under \((4.2)\), which has to have the following parameters \([10]\):

\[
\xi = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ \cdot \\ \cdot \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.3}
\]

By a comparison with \((3.10)\) we find that the only difference is the central entry of the matrix \(E\). We thus can state that any Higgs potential, invariant under custodial symmetry, is automatically invariant under the CP transformation. Note that the opposite is in general not true.

This result holds in any basis, since we can for any custodial symmetric model – by a change of basis – go to the parameterization \((3.10)\). We can also find this result by a comparison of the basis-independent conditions given for the CP transformation in \([10]\) and for the custodial symmetry given in \((3.18)\).

It is worth noticing that the requirement of breaking \(SU(2)_L \times SU(2)_R\) down to \(SU(2)\) implies \(v_i = v_i^*\), in other words the possibility of spontaneous CP violation is also eliminated by the requirement of invariance under the CT.

5. COMMENTS AND CONCLUSIONS

The custodial symmetry in the SM is respected by the Higgs potential implying no corrections to the \(\rho\) parameter which grow as \(\alpha m_h^2\). However in the Two-Higgs-Doublet Model the potential in general does not respect this symmetry. In this paper, employing the bilinear formalism, we have formulated basis independent conditions \((3.18)\) which allow for an easy verification of the custodial symmetry of a scalar potential. We have also shown that, as long as Yukawa interactions are irrelevant, two types of custodial symmetry for THDM discussed in the literature are equivalent; they just differ by a choice of basis. We have also clarified relations between the custodial symmetry and CP; it has been shown that any potential which is symmetric under custodial symmetry is also invariant under CP.

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