Exponential laws for weighted function spaces and regularity of weighted mapping groups

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Abstract

Let $E$ be a locally convex space, $U \subseteq \mathbb{R}^n$ as well as $V \subseteq \mathbb{R}^m$ be open and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. Locally convex spaces $C^{k,l}(U \times V, E)$ of functions with different degrees of differentiability in the $U$- and $V$-variable were recently studied by H. Alzaareer, who established an exponential law of the form $C^{k,l}(U \times V, E) \sim C^k(U, C^l(V, E))$. We establish an analogous exponential law $C^{k,l}_{W_1 \otimes W_2}(U \times V, E) \sim C^k_{W_1}(U, C^l_{W_2}(V, E))$ for suitable spaces of weighted $C^{k,l}$-maps, as well as an analogue for spaces of weighted continuous functions on locally compact spaces. The results entail that certain Lie groups $C^l_{W_1}(U, H)$ of weighted mappings introduced by B. Walter are $C^k$-regular, for each $C^k$-regular Lie group $H$ modeled on a locally convex space and a suitable set of weights $W$.

1 Introduction

For many purposes in Mathematical Analysis it is very natural to control the growth of a continuous (or differentiable) function by means of weight functions. Prime examples are the Schwartz spaces of rapidly decreasing smooth functions in the theory of (tempered) distributions.

Spaces of weighted continuous functions on topological spaces were introduced and studied by L. Nachbin [15] and W. H. Summers [20] (in the scalar-valued case), K.-D. Bierstedt [3] and Prolla J.B. [18] (in the vector-valued case), and many others. Many results concerning spaces of weighted differentiable functions can be found in [6] and [21]. The aim of this article is the development of an Exponential Law for spaces of weighted differentiable functions with values in locally convex spaces, which says that under some conditions we have

$$C^k_{W_1}(U, C^l_{W_2}(V, E)) \sim C^k_{W_1 \otimes W_2}(U \times V, E),$$

where $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ are open subsets, $W_1, W_2$ are sets of weights on $U$ and $V$, respectively, and $k, l \in \mathbb{N}_0 \cup \{\infty\}$.

Starting with a simpler situation, we consider a subspace $C^k_W(X, E)$ of $C(X, E)$, where $X$ is a Hausdorff topological space, $E$ is a Hausdorff locally convex space and $W$ is a set of functions $f : X \to [0, \infty]$, called weights, and endow it with a locally convex topology (see Definition 2.11 for details). Using the Exponential Law for spaces of continuous functions (Proposition 2.12 as discussed in [9, Appendix B]), we prove Theorem 2.21 which states:
Theorem (Exponential Law for spaces of weighted continuous functions). Let \(X_1, X_2\) be locally compact spaces and \(E\) be a Hausdorff locally convex space. Let \(W_1\) and \(W_2\) be sets of weights on \(X_1\) and \(X_2\), respectively, such that

(i) \(W_1, W_2\) satisfy the o-condition,

(ii) all weights \(f \in W_1, g \in W_2\) are bounded on compact subsets of \(X_1\) and \(X_2\), respectively,

(iii) for each compact subset \(K \subseteq X_1\) there exists a weight \(f \in W_1\) such that \(\inf_{x \in K} f(x) > 0\), and likewise for \(W_2\).

Then the linear map

\[ \Psi : C^{W_1}(X_1, C^{W_2}(X_2, E)) \to C^{W}(X_1 \times X_2, E), \quad \gamma \mapsto \gamma^\wedge, \]

where \(W = W_1 \otimes W_2\), is a homeomorphism.

Further, after recalling the concept of differentiability in locally convex spaces, we pass on to spaces of weighted differentiable functions introduced in Definitions 3.6 and 3.19. Using the Exponential Law for spaces of differentiable functions (see Proposition 3.20, as proven in [1, Theorem 94]), we prove the following Theorem 3.37:

Theorem (Exponential Law for spaces of weighted differentiable functions). Let \(E\) be a Hausdorff locally convex space, \(U \subseteq \mathbb{R}^n\) and \(V \subseteq \mathbb{R}^m\) be open subsets, and \(k, l \in \mathbb{N}_0 \cup \{\infty\}\). For the set of weights \(W_1 \subseteq C^k(U, [0, \infty[)\) on \(U\) we assume that

(i) \(W_1\) satisfies the o-condition,

(ii) for each \(f \in W_1\) and \(\alpha \in \mathbb{N}_0^n\) with \(|\alpha| \leq k\) there exists \(g \in W_1\) such that

\[ \left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq g(x) \]

for all \(x \in U\),

and likewise for the set of weights \(W_2 \subseteq C^l(V, [0, \infty[)\) on \(V\). Then the linear map

\[ \Psi : C^{W_1}(U, C^{W_2}(V, E)) \to C^{W}(U \times V, E), \quad \gamma \mapsto \gamma^\wedge, \]

where \(W = W_1 \otimes W_2\), is a homeomorphism.

In the proof, we crucially use that the space \(C^{k,l}_c(U \times V, E)\) is dense in \(C^{k,l}_W(U \times V, E)\) (see Proposition 3.20). The proof of this proposition varies the proof of the density of \(C^\infty_c(U, \mathbb{R})\) in the space \(C^k(U, \mathbb{R})\) by H.G. Garnir, M. De Wilde and J. Schmets in [6]. An immediate consequence (Corollary 3.38) of the Exponential Law is that we have

\[ C^{W_1}(U, C^{W_2}(V, E)) \cong C^{W_2}(V, C^{W_1}(U, E)) \]

for all \(k, l \in \mathbb{N}_0 \cup \{\infty\}\) and suitable sets of weights \(W_1, W_2\). Moreover, after some modifications in the proof of Proposition 3.20 we obtain that
$C^k_{W_1}(U, C^l_{W_2}(K, E)) \cong C^l_{W_2}(K, C^k_{W_1}(U, E))$, where $U \subseteq \mathbb{R}^n$ is open, $K \subseteq \mathbb{R}^m$ is convex and compact, and $W_1, W_2$ are appropriate sets of weights on $U$ and $K$ (see Corollary 3.41). As a special case, constructing a set of weights $W_1$ on a subset $U \subseteq \mathbb{R}^n$ such that

$C^k_{W_1}(U, E) = C^k(U, E)$

as topological vector spaces for each $k \in \mathbb{N}_0 \cup \{\infty\}$, we get the results

$C^k(U, C^l_{W_2}(V, E)) \cong C^l_{W_2}(V, C^k(U, E))$

and

$C^k(K, C^l_{W_2}(V, E)) \cong C^l_{W_2}(V, C^k(K, E))$, where $U \subseteq \mathbb{R}^n$ is open, $K \subseteq \mathbb{R}^m$ is convex and compact, $W_2$ is a suitable set of weights on an open subset $V \subseteq \mathbb{R}^m$, and $l \in \mathbb{N}_0 \cup \{\infty\}$ (see Remark 3.42 for details).

The last section deals with regularity of Lie groups of weighted Lie group-valued functions. Precisely, applying Corollary 3.41, we show in Theorem 4.5:

**Theorem.** Let $E$ be a Hausdorff locally convex space and $H$ be a $C^k$-regular Lie group modeled on $E$, with $k \in \mathbb{N}_0 \cup \{\infty\}$. Let $W \subseteq C^{\ell}(U, [0, \infty[)$ be a set of weights on an open subset $U \subseteq \mathbb{R}^n$ such that

(i) $1_U \in W$,

(ii) $W$ satisfies the $o$-condition,

(iii) for each $f \in W$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$ there exists $g \in W$ such that

$\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq g(x)$

for all $x \in U$.

Then the Lie group $G := C^{\ell}_{W}(U, H)$ is $C^k$-regular for each $l \in \mathbb{N}_0 \cup \{\infty\}$.

For the theory of Lie groups modeled on locally convex spaces, the reader is referred to [14], [16] and [10]. It is known (cf. [14] and [7]) that if $H$ is a locally convex Lie group and $K$ is a smooth compact manifold, then $C^l(K, H)$ is a locally convex Lie group modeled on the space $C^l(K, L(H))$ for each $l \in \mathbb{N}_0 \cup \{\infty\}$. In Definition 4.2, we recall the concept of $C^k$-regularity of Lie groups, which goes back to J. Milnor (see [14]), who works with $C^\infty$-regularity (simply called regularity). If $H$ is a $C^k$-regular Lie group, then the Lie group $C^k(K, H)$ is $C^k$-regular for each $l \in \mathbb{N}_0 \cup \{\infty\}$ (as proven by H. Glöckner in [8]). The construction of Lie groups of weighted Lie group-valued functions is discussed in [21] (generalizing the seminal work of H. Boseck, G. Czichowski and K.-P. Rudolph [4]). Moreover, B. Walter shows in [21] that if $U$ is an open subset of a normed space, $1_U \in W$ and $H$ is a Banach Lie group, then the Lie group $C^{\ell}_{W}(U, H)$ is regular.
Exponential laws for function spaces related to infinite-dimensional Lie groups have also been established in the recent work [13] by A.Kriegl, P.W.Michor and A.Rainer, in the setting of convenient differential calculus. For the most part, the results are complementary. Taking $U = \mathbb{R}^n$, $V = \mathbb{R}^m$, $k = l = \infty$ and $W_1, W_2$ as the respective sets of all squares of polynomial functions in Theorem 3.37, we obtain an exponential law $S(\mathbb{R}^{n+m}, E) \cong S(\mathbb{R}^n(S(\mathbb{R}^m, E))$ for Schwartz spaces of vector-valued rapidly decreasing smooth functions, which (as a bornological isomorphism) is also covered by [13].

The results presented here are based on the author’s master’s thesis [17] advised by Helge Glöckner (Paderborn).

All of the topological vector spaces will be $K$-vector spaces, with $K \in \{\mathbb{R}, \mathbb{C}\}$. Further, we denote the set of all compact subsets of a topological space $X$ by $\mathcal{K}(X)$, and the set of all continuous seminorms on a locally convex space $E$ will be denoted by $\mathcal{P}_E$.

2 Spaces of weighted continuous functions and the Exponential Law

In this section we present the construction of the space of weighted continuous functions with values in a locally convex space and the corresponding topology. We study some important properties of such spaces and finally we establish an Exponential Law for spaces of weighted continuous functions.

**Definition 2.1.** Let $X$ be a Hausdorff topological space and $E$ be a Hausdorff locally convex space. We denote by $\mathcal{W}$ a nonempty set of maps $f : X \to [0, \infty]$ such that for each $x \in X$ there exists $f_x \in \mathcal{W}$ with $f_x(x) > 0$ (and call the elements of $\mathcal{W}$ weights).

For a continuous function $\gamma : X \to E$, a seminorm $q \in \mathcal{P}_E$ and $f \in \mathcal{W}$ we define

$$\|\gamma\|_{f,q} := \sup_{x \in X} f(x)q(\gamma(x)) \in [0, \infty].$$

Furthermore, we define the vector space of *weighted continuous functions*

$$\mathcal{C}_\mathcal{W}(X, E) := \left\{ \gamma \in C(X, E) : (\forall f \in \mathcal{W}) (\forall q \in \mathcal{P}_E) \|\gamma\|_{f,q} < \infty \right\}$$

and endow it with the locally convex topology induced by the seminorms

$$\|\|_{f,q} : \mathcal{C}_\mathcal{W}(X, E) \to [0, \infty].$$

For a subset $U \subseteq X$ we write

$$\mathcal{C}_\mathcal{W}(U, E) := \mathcal{C}_\mathcal{W}|_U (U, E),$$

where

$$\mathcal{W}|_U := \{f|_U : f \in \mathcal{W}\}.$$

Finally, for a subset $V \subseteq E$ we define

$$\mathcal{C}_\mathcal{W}(X, V) := \{ \gamma \in \mathcal{C}_\mathcal{W}(X, E) : \gamma(X) \subseteq V \}.$$
Remark 2.2. The point evaluation
\[ \text{ev}_x : C_w(X, E) \to E, \quad \text{ev}_x(\gamma) := \gamma(x) \]
is continuous for all \( x \in X \), since
\[ q(\gamma(x)) = \frac{1}{f_\gamma(x)} f_\gamma(x) q(\gamma(x)) = \frac{1}{f_\gamma(x)} \| \gamma \|_{f_\gamma, q}, \]
for each seminorm \( q \in \mathcal{P}_E \) and a certain weight \( f_\gamma \in \mathcal{W} \) with \( f_\gamma(x) > 0 \). Thus, the topology on \( C_w(X, E) \) is Hausdorff.

Remark 2.3. Let \( X \) be a Hausdorff topological space and \( E \) be a Hausdorff locally convex space. If \( \mathcal{W}_1 \) is a set of weights on \( X \), then for the set of weights
\[ \mathcal{W}_2 := \left\{ \sum_{i=1}^n r_i f_i : f_i \in \mathcal{W}_1, r_i > 0 \text{ for } i \in \{1, \ldots, n\}, n \in \mathbb{N} \right\} \]
on \( X \) we have
\[ C_{\mathcal{W}_1}(X, E) = C_{\mathcal{W}_2}(X, E) \]
as topological vector spaces. In fact, since \( \mathcal{W}_1 \subseteq \mathcal{W}_2 \), we have \( C_{\mathcal{W}_2}(X, E) \subseteq C_{\mathcal{W}_1}(X, E) \) and the inclusion map \( C_{\mathcal{W}_2}(X, E) \to C_{\mathcal{W}_1}(X, E) \) is continuous. Conversely, if \( \gamma \in C_{\mathcal{W}_1}(X, E) \), \( f \in \mathcal{W}_2 \) (that is \( f = r_1 f_1 + \cdots + r_n f_n \) for some weights \( f_1, \ldots, f_n \in \mathcal{W}_1 \) and \( r_1, \ldots, r_n > 0 \)) and \( q \in \mathcal{P}_E \), then
\[ \| \gamma \|_{f, q} = \| \gamma \|_{r_1 f_1 + \cdots + r_n f_n, q} \leq r_1 \| \gamma \|_{f_1, q} + \cdots + r_n \| \gamma \|_{f_n, q} < \infty, \]
thus \( \gamma \in C_{\mathcal{W}_2}(X, E) \) and the inclusion map \( C_{\mathcal{W}_1}(X, E) \to C_{\mathcal{W}_2}(X, E) \) is continuous.

Therefore, we can always assume that for a set of weights \( \mathcal{W} \) on \( X \) we have
\[ (\forall f_1, \ldots, f_n \in \mathcal{W}) (\forall r_1, \ldots, r_n > 0) r_1 f_1 + \cdots + r_n f_n \in \mathcal{W}. \]

Remark 2.4. Recall that a basis for the compact-open topology on \( C(X, Y) \), where \( X, Y \) are Hausdorff topological spaces, is given by the sets
\[ [K_1, U_1] \cap \ldots \cap [K_n, U_n], \]
where \( n \in \mathbb{N} \), \( K_1, \ldots, K_n \in \mathcal{K}(X) \), \( U_1, \ldots, U_n \subseteq Y \) are open sets and
\[ [K_i, U_i] := \{ \gamma \in C(X, Y) : \gamma(K_i) \subseteq U_i \} \]
for each \( i \in \{1, \ldots, n\} \). We always endow the space \( C(X, Y) \) with the compact-open topology. Further, if \( E \) is a Hausdorff locally convex space, then the compact-open topology on \( C(X, E) \) coincides with the locally convex topology induced by the seminorms
\[ \| \cdot \|_{K, q} : C(X, E) \to [0, \infty[, \quad \| \cdot \|_{K, q} := \sup_{x \in K} q(\gamma(x)), \]
where \( K \in \mathcal{K}(X) \) and \( q \in \mathcal{P}_E \). (This topology is known as the topology of uniform convergence on compact sets. For details, see, for example, [5].)

Now we can easily show the continuity of the following inclusion map:
Lemma 2.5. Let $X$ be a Hausdorff topological space, $E$ be a Hausdorff locally convex space and $W$ be a set of weights on $X$. Assume that for each compact subset $K \subseteq X$ there exists a weight $f_K \in W$ such that $\inf_{x \in K} f_K(x) > 0$. Then the inclusion map

$$i : C_W(X, E) \rightarrow C(X, E),$$

is linear and continuous.

Proof. The linearity of the inclusion map $i$ is clear. Now, for a compact subset $K \subseteq X$ we have $\varepsilon := \inf_{x \in K} f_K(x) > 0$ for a suitable weight $f_K \in W$. Then

$$\varepsilon \mathbb{1}_K \leq f_K,$$

and hence for a seminorm $q \in \mathcal{P}_E$ we get

$$\|\gamma\|_{K,q} \overset{\text{def}}{=} \sup_{x \in K} q(\gamma(x)) = \sup_{x \in X} K(x)q(\gamma(x)) \leq \frac{1}{\varepsilon} \sup_{x \in X} f_K(x)q(\gamma(x)) = \frac{1}{\varepsilon} \|\gamma\|_{f_K,q},$$

for each $\gamma \in C_W(X, E)$. Thus the map $i$ is continuous.

Remark 2.6. If all given weights on $X$ are continuous, then the condition in Lemma 2.5 is satisfied, that is, for each compact subset $K \subseteq X$ there is a weight $f_K \in W$ such that $\inf_{x \in K} f_K(x) > 0$. In fact, for each weight $f \in W$ we define the set

$$U_f := \{x \in X : f(x) > 0\},$$

which is an open subset of $X$, since $f$ is continuous. By definition of $W$, for each $x \in K$ there is a weight $f_x \in W$ such that $f_x(x) > 0$, thus $x \in U_{f_x}$. Since $K$ is compact, there exist $x_1, \ldots, x_n \in K$ such that

$$K \subseteq \bigcup_{i=1}^n U_{f_{x_i}},$$

that is, for each $x \in K$ we have $x \in U_{f_{x_i}}$ for some $i \in \{1, \ldots, n\}$. We set

$$f_K := f_{x_1} + \cdots + f_{x_n},$$

which is a weight on $X$ (see Remark 2.3). Then we see that

$$(\forall x \in K)(\exists i \in \{1, \ldots, n\}) f_K(x) \geq f_{x_i}(x) > 0,$$

and, using that the minimum is attained as $K$ is compact, we obtain

$$\inf_{x \in K} f_K(x) > 0,$$

as required.

Also superposition operators $C_W(X, \lambda)$ are continuous.
Lemma 2.7. Let $E$, $F$ be Hausdorff locally convex spaces and $\lambda : E \to F$ be a continuous linear function. Let $X$ be a Hausdorff topological space and $W$ be a set of weights on $X$. If $\gamma \in C_W(X, E)$, then

$$\lambda \circ \gamma \in C_W(X, F)$$

and the map

$$C_W(X, \lambda) : C_W(X, E) \to C_W(X, F), \quad \gamma \mapsto \lambda \circ \gamma$$

is continuous and linear.

Proof. If $q \in P_F$, then $q \circ \lambda \in P_E$. Therefore, for a weight $f \in W$ we see that

$$\|\lambda \circ \gamma\|_{f, q} \overset{\text{def}}{=} \sup_{x \in X} f(x) q(\lambda(\gamma(x))) = \|\gamma\|_{f \circ q \circ \lambda} < \infty,$$

since $\gamma \in C_W(X, E)$. Hence $\lambda \circ \gamma \in C_W(X, F)$.

The linearity of the map $C_W(X, \lambda)$ is clear, and we have

$$\|C_W(X, \lambda)(\gamma)\|_{f, q} = \|\lambda \circ \gamma\|_{f, q} = \|\gamma\|_{f \circ q \circ \lambda},$$

thus the continuity follows. \qed

Remark 2.8. We recall that a topological space $X$ is called a $k$-space if each subset $A \subseteq X$ is closed if and only if $A \cap K$ is closed in $K$ for each subset $K \in \mathcal{K}(X)$. In this case, a function $\gamma : X \to Y$ to a topological space $Y$ is continuous if and only if $\gamma|_K$ is continuous for each $K \in \mathcal{K}(X)$. For example, each locally convex space and each metrizable space is a $k$-space. (Details can be found in [12].)

We show that in the following case the space $C_W(X, E)$ is complete.

Proposition 2.9. Let $X$ be a $k$-space and $E$ be a complete Hausdorff locally convex space. If $W$ is a set of weights on $X$ such that for each compact set $K \subseteq X$ there exists a weight $f_K \in W$ such that $\inf_{x \in K} f_K(x) > 0$, then the space $C_W(X, E)$ is complete.

Proof. Let $(\gamma_a)_{a \in A}$ be a Cauchy net in $C_W(X, E)$. Since the inclusion map $i : C_W(X, E) \to C(X, E)$ is continuous (see Lemma 2.3), $(\gamma_a)_{a \in A}$ is a Cauchy net in $C(X, E)$. But the space $C(X, E)$ is complete, by [12, Thm.12], whence $(\gamma_a)_{a \in A}$ converges to a $\gamma \in C(X, E)$. It remains to show that $\gamma \in C_W(X, E)$ and the Cauchy net $(\gamma_a)_{a \in A}$ converges to $\gamma$ in $C_W(X, E)$. To this end, let $f \in W$, $q \in P_E$ and $\varepsilon > 0$. There exists an index $a_\varepsilon \in A$ such that

$$(\forall a_1, a_2 \geq a_\varepsilon) \|\gamma_{a_1} - \gamma_{a_2}\|_{f, q} \leq \varepsilon,$$

that is

$$(\forall a_1, a_2 \geq a_\varepsilon, x \in X) f(x) q(\gamma_{a_1}(x) - \gamma_{a_2}(x)) \leq \varepsilon,$$

by definition of the seminorm $\|\gamma\|_{f, q}$. Passing to the limit in $a_2$, we obtain

$$(\forall a_1 \geq a_\varepsilon, x \in X) f(x) q(\gamma_{a_1}(x) - \gamma(x)) \leq \varepsilon. \quad (1)$$

Thus we see that for all $x \in X$
\begin{align*}
f(x)q(\gamma(x)) &= f(x)q(\gamma_0(x) + \gamma(x) - \gamma_0(x)) \\
&\leq f(x)q(\gamma_0(x)) + f(x)q(\gamma(x) - \gamma_0(x)) \\
&\leq f(x)q(\gamma_0(x)) + \varepsilon,
\end{align*}

hence
\[\|\gamma\|_{f,q} \leq \|\gamma_0\|_{f,q} + \varepsilon < \infty.\]

Thus \(\gamma \in C_W(X, E)\) and \(1\) shows that \((\gamma_a)_{a \in A}\) converges to \(\gamma\) in \(C_W(X, E)\), as required.

We construct a set of weights for maps on products using sets of weights on the two factors.

**Definition 2.10.** Let \(X_1\) and \(X_2\) be Hausdorff topological spaces and \(W_1, W_2\) be sets of weights on \(X_1\) and \(X_2\), respectively. For \(f_1 \in W_1\) and \(f_2 \in W_2\) we define the map
\[f_1 \otimes f_2 : X_1 \times X_2 \to [0, \infty[, \quad (x_1, x_2) \mapsto f_1(x_1)f_2(x_2)\]
and obtain a set of weights on \(X_1 \times X_2\) via
\[W_1 \otimes W_2 := \{f_1 \otimes f_2 : f_1 \in W_1, f_2 \in W_2\}.\]

The following lemma will be useful.

**Lemma 2.11.** Let \(E, F\) be Hausdorff locally convex spaces and \(\lambda : E \to F\) be a continuous, linear and injective function. If for each seminorm \(q_1 \in \mathcal{P}_E\) there exists a seminorm \(q_2 \in \mathcal{P}_F\) such that \(q_2(\lambda(x)) = q_1(x)\) for all \(x \in E\), then \(\lambda\) is a topological embedding.

**Proof.** We need to show that the map
\[\left(\lambda^{\lambda(E)}\right)^{-1} : \lambda(E) \to E\]
is continuous. Let \(q_1 \in \mathcal{P}_E\). By assumption, there exists a seminorm \(q_2 \in \mathcal{P}_F\) such that \(q_1 = q_2 \circ \lambda\). Then
\[q_1(\lambda^{-1}(y)) = q_2(\lambda(\lambda^{-1}(y))) = q_2(y),\]
for all \(y \in \lambda(E)\). Hence \(\lambda\) is a topological embedding.

Before proving the first part of the Exponential Law for spaces of weighted continuous functions, let us recall the Exponential Law for spaces of continuous functions, which can be found in [9, Proposition B.15].

**Proposition 2.12** (Exponential Law for spaces of continuous functions). Let \(X_1, X_2, Y\) be Hausdorff topological spaces. If \(\gamma : X_1 \times X_2 \to Y\) is a continuous map, then also the map
\[\gamma^\gamma : X_1 \to C(X_2, Y), \quad x \mapsto \gamma^\gamma(x) := \gamma(x, \bullet)\]
is continuous. Moreover, the map

$$\Phi : C(X_1 \times X_2, Y) \to C(X_1, C(X_2, Y)), \quad \gamma \mapsto \gamma^\wedge$$

is a topological embedding.

If $X_2$ is locally compact or $X_1 \times X_2$ is a $k$-space, then $\Phi$ is a homeomorphism.

**Theorem 2.13.** Let $E$ be a Hausdorff locally convex space and $X_1$, $X_2$ be Hausdorff topological spaces such that $X_2$ is locally compact or $X_1 \times X_2$ is a $k$-space. Let $W_1$ and $W_2$ be sets of weights on $X_1$ and $X_2$, respectively. We assume that for each compact subset $K \subseteq X_1$ there exists a weight $f \in W_1$ such that $\inf_{x \in K} f(x) > 0$, and likewise for $W_2$. If $\gamma \in C_{W_1}(X_1, C_{W_2}(X_2, E))$, then $\gamma^\wedge \in C_{W}(X_1 \times X_2, E)$, where $\gamma^\wedge$ is the map

$$\gamma^\wedge : X_1 \times X_2 \to E, \quad \gamma^\wedge(x_1, x_2) := \gamma(x_1)(x_2)$$

and $W = W_1 \otimes W_2$. Furthermore, the map

$$\Psi : C_{W_1}(X_1, C_{W_2}(X_2, E)) \to C_{W}(X_1 \times X_2, E), \quad \gamma \mapsto \gamma^\wedge$$

is linear and a topological embedding.

**Proof.** By Lemma 2.7, the map

$$C_{W_1}(X_1, i) : C_{W_1}(X_1, C_{W_2}(X_2, E)) \to C_{W_1}(X_1, C(X_2, E))$$

is continuous and linear, where $i$ is the continuous and linear inclusion map

$$i : C_{W_2}(X_2, E) \to C(X_2, E),$$

as in Lemma 2.5. Also the inclusion map

$$j : C_{W_1}(X_1, C(X_2, E)) \to C(X_1, C(X_2, E))$$

is continuous and linear, by Lemma 2.5. We assumed that $X_1 \times X_2$ is a $k$-space or $X_2$ is locally compact, thus the map $\Phi$ in Proposition 2.12 is a homeomorphism. Hence, using the inverse map

$$\Phi^{-1} : C(X_1, C(X_2, E)) \to C(X_1 \times X_2, E), \quad \gamma \mapsto \gamma^\wedge$$

we set

$$\Theta := \Phi^{-1} \circ j \circ C_{W_1}(X_1, i)$$

and obtain the continuous linear map

$$\Theta : C_{W_1}(X_1, C_{W_2}(X_2, E)) \to C(X_1 \times X_2, E), \quad \gamma \mapsto \gamma^\wedge.$$
Since $\gamma(x_1) \in C_{W_2}(X_2, E)$, we see that
\[
\sup_{x_2 \in X_2} f_2(x_2)q(\gamma(x_1)(x_2)) = \|\gamma(x_1)\|_{f_2,q} < \infty,
\]
and hence
\[
\sup_{x_1 \in X_1} f_1(x_1) \sup_{x_2 \in X_2} f_2(x_2)q(\gamma(x_1)(x_2)) = \|\gamma\|_{f_1,\|f_2\|_{f_2,q}} < \infty,
\]
since $\gamma \in C_{W_1}(X_1, C_{W_2}(X_2, E))$, whence $\gamma^\wedge \in C_{W}(X_1 \times X_2, E)$, as required. Therefore, we have $\Theta(C_{W_1}(X_1, C_{W_2}(X_2, E))) \subseteq C_{W}(X_1 \times X_2, E)$ and the map
\[
\Psi := \Theta \quad \begin{array}{|c|}
\hline
C_{W}(X_1 \times X_2, E) \\
\hline
\end{array} 
\quad \begin{array}{|c|}
\hline
C_{W_1}(X_1, C_{W_2}(X_2, E)) \\
\hline
\end{array}
\quad \gamma \mapsto \gamma^\wedge
\]
is continuous, linear and, obviously, injective. Thus, $\Psi$ is a topological embedding, by Lemma 2.11, since
\[
\|\Psi(\gamma)\|_{f,q} = \|\gamma^\wedge\|_{f,q} = \|\gamma\|_{f_1,\|f_2\|_{f_2,q}},
\]
for each $f_1 \in W_1$, $f_2 \in W_2$, $q \in P_E$ and $\gamma \in C_{W_1}(X_1, C_{W_2}(X_2, E))$.

We want to find conditions ensuring that the topological embedding defined in the preceding theorem will be bijective. We will work with spaces of continuous compactly supported functions:

**Definition 2.14.** Let $X$ be a Hausdorff topological space and $E$ be a Hausdorff locally convex space. For a compact subset $K \subseteq X$ we define the space
\[
C_K(X, E) := \{\gamma \in C(X, E) : \text{supp}(\gamma) \subseteq K\}
\]
(whence $\text{supp}(\gamma)$ denotes the support of $\gamma$) and endow it with the locally convex topology defined by the seminorms
\[
\|\cdot\|_q : C_K(X, E) \to [0, \infty[, \quad \|\gamma\|_q := \sup_{x \in K} q(\gamma(x)),
\]
where $q \in P_E$.

Additionally, we define the space
\[
C_c(X, E) := \{\gamma \in C(X, E) : \text{supp}(\gamma) \text{ is compact}\}.
\]
We obviously have

\[ C_c(X, E) = \bigcup_{K \in K(X)} C_K(X, E). \]

Applying the classical Exponential Law (Proposition 2.12), we get the following result:

**Lemma 2.15.** Let \( X_1, X_2 \) be Hausdorff topological spaces and \( E \) be a Hausdorff locally convex space. If \( \gamma \in C_{K_1 \times K_2}(X_1 \times X_2, E) \) for some compact subsets \( K_1 \subseteq X_1, K_2 \subseteq X_2 \), then

\[ \gamma_x := \gamma(x, \bullet) \in C_{K_2}(X_2, E) \]

for all \( x \in X_1 \) and

\[ \gamma^\vee \in C_{K_1}(X_1, C_{K_2}(X_2, E)), \]

where \( \gamma^\vee \) is the map

\[ \gamma^\vee : X_1 \to C_{K_2}(X_2, E), \quad \gamma^\vee(x) := \gamma_x. \]

**Proof.** The map \( \gamma_{x_1} \) is continuous for each \( x_1 \in X_1 \). Further, if \( x_2 \in X_2 \setminus K_2 \), then we have

\[ \gamma_{x_1}(x_2) = \gamma(x_1, x_2) = 0, \]

whence \( \text{supp}(\gamma_{x_1}) \subseteq K_2 \). Thus we have

\[ \gamma_{x_1} \in C_{K_2}(X_2, E). \]

That is, \( \gamma^\vee(X_1) \subseteq C_{K_2}(X_2, E) \) and using Proposition 2.12 we see that

\[ \gamma^\vee : X_1 \to C_{K_2}(X_2, E), \quad \gamma^\vee(x) := \gamma_x \]

is continuous. For \( x_1 \in X_1 \setminus K_1 \) and \( x_2 \in X_2 \) we have

\[ \gamma^\vee(x_1)(x_2) = \gamma(x_1, x_2) = 0, \]

whence \( \text{supp}(\gamma^\vee) \subseteq K_1 \), and therefore \( \gamma^\vee \in C_{K_1}(X_1, C_{K_2}(X_2, E)) \), as required.

The following lemma will be helpful for an important result.

**Lemma 2.16.** Let \( X \) be a Hausdorff topological space, \( E \) be a Hausdorff locally convex space and \( K \subseteq X \) be a compact subset. If \( W \) is a set of weights on \( X \) such that each weight \( f \in W \) is bounded on \( K \), then

\[ C_K(X, E) \subseteq C_W(X, E) \]

and the inclusion map

\[ i : C_K(X, E) \to C_W(X, E) \]

is continuous and linear.
Proof. Let $\gamma \in C_K(X, E)$, $f \in W$ and $q \in \mathcal{P}_E$. Since $f$ is bounded on $K$, we set

$$r := \sup_{x \in K} f(x) < \infty,$$

and obtain

$$\|\gamma\|_{f,q} \overset{\text{def}}{=} \sup_{x \in X} f(x) q(\gamma(x)) = \sup_{x \in K} f(x) q(\gamma(x)) \leq r \|\gamma\|_q < \infty,$$

whence $\gamma \in C_W(X, E)$.

The inclusion map $i : C_K(X, E) \to C_W(X, E)$ is obviously linear and since

$$\|\gamma\|_{f,q} \leq r \|\gamma\|_q,$$

the map $i$ is continuous.

Proposition 2.17. Let $X_1, X_2$ be Hausdorff topological spaces, $E$ be a Hausdorff locally convex space and $W_1, W_2$ be sets of weights on $X_1$ and $X_2$, respectively. If each of the following conditions is satisfied:

(i) $X_2$ is locally compact or $X_1 \times X_2$ is a $k$-space,

(ii) all weights $f \in W_1$, $g \in W_2$ are bounded on compact subsets of $X_1$ and $X_2$, respectively,

(iii) for each compact subset $K \subseteq X_1$ there exists a weight $f \in W_1$ such that $\inf_{x \in K} f(x) > 0$, and likewise for $W_2$,

then $C_c(X_1 \times X_2, E) \subseteq \text{im}(\Psi)$, where $\Psi$ is the topological embedding

$$\Psi : C_{W_1}(X_1, C_{W_2}(X_2, E)) \to C_W(X_1 \times X_2, E), \quad \gamma \mapsto \gamma^\wedge$$

defined in Theorem 2.13.

Proof. Let $\gamma \in C_c(X_1 \times X_2, E)$. We show that $\gamma^\wedge \in C_{W_1}(X_1, C_{W_2}(X_2, E))$. (Then $\Psi(\gamma^\wedge) = (\gamma^\wedge)^\wedge = \gamma$, and the proof is finished.) Consider the projections

$$\pi_1 : X_1 \times X_2 \to X_1, \quad \pi_2 : X_1 \times X_2 \to X_2$$

onto the first and second component, respectively. For $K := \text{supp}(\gamma) \subseteq X_1 \times X_2$ we define $K_1 := \pi_1(K)$ and $K_2 := \pi_2(K)$. Since $K$ is compact and the projection maps are continuous, the sets $K_1$ and $K_2$ are compact and we have $K \subseteq K_1 \times K_2$, whence $\gamma \in C_{K_1 \times K_2}(X_1 \times X_2, E)$. From Lemma 2.15 we conclude that $\gamma^\wedge \in C_{K_1}(X_1, C_{K_2}(X_2, E))$. The inclusion map

$$i : C_{K_2}(X, E) \to C_{W_2}(X_2, E)$$

is continuous and linear by Lemma 2.16, thus we have

$$\gamma^\wedge = i \circ \gamma^\wedge \in C_{K_1}(X_1, C_{W_2}(X_2, E)) \subseteq C_{W_1}(X_1, C_{W_2}(X_2, E)),$$

as required.

We introduce the following notation:
Definition 2.18. Let $X$ be a Hausdorff topological space. For two functions $f, g : X \to [0, \infty]$ we write

$$f = o(g)$$

if for each $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subseteq X$ such that $f(x) \leq \varepsilon g(x)$ for all $x \in X \setminus K_\varepsilon$.

We say that a set of weights $W$ on $X$ satisfies the $o$-condition if for each weight $f \in W$ there exists a weight $g \in W$ such that $f = o(g)$.

Lemma 2.19. Let $X_1, X_2$ be Hausdorff topological spaces. If the sets of weights $W_1$ and $W_2$ on $X_1$ and $X_2$, respectively, satisfy the $o$-condition, then also the set of weights $W_1 \otimes W_2$ on $X_1 \times X_2$ satisfies the $o$-condition.

Proof. Let $f \in W_1 \otimes W_2$, that is $f = f_1 \otimes f_2$, where $f_1 \in W_1$ and $f_2 \in W_2$. By assumption, $f_1 = o(g_1)$ and $f_2 = o(g_2)$ for some weights $g_1 \in W_1$, $g_2 \in W_2$. By Remark 2.3, we can define the weights $h_1 := f_1 + g_1 \in W_1$, $h_2 := f_2 + g_2 \in W_2$, so that

$$f_1 \leq h_1, \quad f_1 = o(h_1),$$

$$f_2 \leq h_2, \quad f_2 = o(h_2).$$

Now, given $\varepsilon > 0$, there exist compact subsets $K_1 \subseteq X_1, K_2 \subseteq X_2$ such that for $x_1 \in X_1 \setminus K_1, x_2 \in X_2$

$$f_1(x_1)f_2(x_2) \leq \varepsilon h_1(x_1)f_2(x_2) \leq \varepsilon h_1(x_1)h_2(x_2),$$

and for $x_1 \in X_1, x_2 \in X_2 \setminus K_2$

$$f_1(x_1)f_2(x_2) \leq f_1(x_1)\varepsilon h_2(x_2) \leq h_1(x_1)\varepsilon h_2(x_2).$$

Hence, since the subset $K_1 \times K_2 \subseteq X_1 \times X_2$ is compact, we have

$$f = o(h),$$

where $h := h_1 \otimes h_2 \in W_1 \otimes W_2$. 

The next lemma enables us to prove the Exponential Law for spaces of weighted continuous functions.

Lemma 2.20. Let $X$ be a Hausdorff locally compact space and $E$ be a Hausdorff locally convex space. If $W$ is a set of weights on $X$ such that

(i) each weight $f \in W$ is bounded on compact subsets of $X$,

(ii) the $o$-condition is satisfied,

then $C_c(X, E)$ is dense in $C_W(X, E)$.

Proof. Using Lemma 2.16 we conclude that $C_c(X, E) \subseteq C_W(X, E)$. By assumption, $W$ satisfies the $o$-condition, thus for a weight $f \in W$ there is a weight $g \in W$ such that $f = o(g)$. Let $\gamma \in C_W(X, E)$, $q \in P_E$ and $\varepsilon > 0$. We choose a $\delta > 0$ such that

$$\delta \|\gamma\|_{g,q} < \varepsilon. \tag{2}$$
Since \( f = o(g) \), there is a compact subset \( K_\delta \subseteq X \) with

\[
 f(x) \leq \delta g(x)
\]  

(3)

for all \( x \in X \setminus K_\delta \). The space \( X \) is assumed locally compact, thus there is an open subset \( U_\delta \subseteq X \) such that \( \overline{U_\delta} \) is compact and \( K_\delta \subseteq U_\delta \). By Urysohn’s Lemma, there is a continuous function \( h : X \to [0,1] \) such that \( h|_{K_\delta} = 1 \) and \( h|_{X \setminus U_\delta} = 0 \). We set

\[
 \eta := h \cdot \gamma.
\]

Then \( \eta \in C_c(X, E) \) because \( \text{supp}(\eta) \subseteq U_\delta \). Further,

\[
 f(x)q(\eta(x) - \gamma(x)) = 0
\]

for all \( x \in K_\delta \). Now, if \( x \in X \setminus K_\delta \), then we have

\[
 f(x)q(\gamma(x)) = f(x)q(h(x)\gamma(x) - \gamma(x)) = f(x) |h(x) - 1| q(\gamma(x)) \leq 1
\]

using (3). Therefore

\[
 \|\eta - \gamma\|_{q,g} \leq \delta \|\gamma\|_{g,q} < \varepsilon,
\]

by the choice of \( \delta \) in (2). Thus \( C_c(X, E) \) is dense in \( C_W(X, E) \).

Theorem 2.21 (Exponential Law for spaces of weighted continuous functions). Let \( X_1, X_2 \) be locally compact spaces and \( E \) be a Hausdorff locally convex space. Let \( W_1 \) and \( W_2 \) be sets of weights on \( X_1 \) and \( X_2 \), respectively, such that

(i) \( W_1, W_2 \) satisfy the \( o \)-condition,

(ii) all weights \( f \in W_1, g \in W_2 \) are bounded on compact subsets of \( X_1 \) and \( X_2 \), respectively,

(iii) for each compact subset \( K \subseteq X_1 \) there exists a weight \( f \in W_1 \) such that \( \inf_{x \in K} f(x) > 0 \), and likewise for \( W_2 \).

Then the linear map

\[
 \Psi : C_{W_1}(X_1, C_{W_2}(X_2, E)) \to C_W(X_1 \times X_2, E), \quad \gamma \mapsto \gamma^\wedge,
\]

where \( W = W_1 \otimes W_2 \), is a homeomorphism.

Proof. Step 1. First we assume that the space \( E \) is complete. Since \( X \) is a \( k \)-space, being locally compact, we conclude from Proposition 2.9 that \( C_{W_1}(X_2, E) \) is complete, whence also \( C_{W_1}(X_1, C_{W_2}(X_2, E)) \) is complete. By Theorem 2.13 the map \( \Psi \) is a topological embedding, thus \( \text{im}(\Psi) \) is complete, hence it is closed in \( C_W(X_1 \times X_2, E) \). We know by Proposition 2.17 that \( C_c(X_1 \times X_2, E) \subseteq \text{im}(\Psi) \). Moreover, by Lemma 2.20 the space \( C_c(X_1 \times X_2, E) \) is dense in \( C_W(X_1 \times X_2, E) \) (because \( X_1 \times X_2 \) is locally compact and \( W = W_1 \otimes W_2 \) satisfies the \( o \)-condition,
by Lemma 2.19. Thus \( \text{im}(\Psi) \) is dense in \( C_W(X_1 \times X_2, E) \). Consequently, we have

\[
\text{im}(\Psi) = \text{im}(\Psi) = C_W(X_1 \times X_2, E),
\]

which shows that the topological embedding \( \Psi \) is surjective, hence a homeomorphism.

Step 2. Now we show the surjectivity of \( \Psi \) in the general case. To this end, let \( \gamma \in C_W(X_1 \times X_2, E) \subseteq C_W(X_1 \times X_2, \tilde{E}) \), where \( \tilde{E} \) is the completion of \( E \). The map

\[
\tilde{\Psi} : C_W^1(X_1, C_W(X_2, \tilde{E})) \to C_W(X_1 \times X_2, \tilde{E}), \quad \eta \mapsto \eta^\wedge
\]

is bijective, by Step 1, thus there exists \( \eta \in C_W^1(X_1, C_W(X_2, \tilde{E})) \) such that

\[
\tilde{\Psi}(\eta) = \eta^\wedge = \gamma.
\]

But this means that

\[
\eta(x_1)(x_2) = \eta^\wedge(x_1, x_2) = \gamma(x_1, x_2) \in E
\]

for all \((x_1, x_2) \in X_1 \times X_2\), whence \( \eta \in C_W^1(X_1, C_W(X_2, E)) \). Therefore, we have \( \Psi(\eta) = \gamma \), and \( \Psi \) is surjective, as asserted.

3 Spaces of weighted differentiable functions and the Exponential Law

In this section we establish an Exponential Law for spaces of weighted differentiable functions. First of all we recall some concepts of differentiability of maps between locally convex spaces (the calculus of maps defined on open subsets of locally convex spaces goes back to A.Bastiani [2] and is also known as Keller’s \( C^k_c \)-calculus [11]). Since we are also interested in differentiability on sets which are not necessarily open (for example, on the interval \([0, 1]\)), we recall the following concept (cf. [10]):

**Definition 3.1.** Let \( E \) be a Hausdorff locally convex space. We call a subset \( U \subseteq E \) locally convex if each element \( x \in U \) has a convex neighborhood \( V \subseteq U \).

Note that each open set and each convex set with nonempty interior satisfies this condition.

**Definition 3.2.** Let \( E, F \) be Hausdorff locally convex spaces

(a) Let \( \gamma : U \to F \) be a map on an open subset \( U \subseteq E \). For \( x \in U \) and \( h \in E \) the derivative of \( \gamma \) at \( x \) in the direction \( h \) is defined as

\[
D_h \gamma(x) := d^{(1)} \gamma(x, h) := d\gamma(x, h) := \lim_{t \to 0} \frac{\gamma(x + th) - \gamma(x)}{t},
\]

whenever the limit exists.

For \( k \in \mathbb{N} \) the map \( \gamma \) is called a \( C^k \)-map if \( \gamma \) is continuous, the iterated directional derivatives
\[
\begin{align*}
d^{(j)}\gamma(x, h_1, \ldots, h_j) &:= (D_{h_j} \cdots D_{h_1}\gamma)(x) \\
&\text{exist for all } j \in \mathbb{N} \text{ with } j \leq k, (x, h_1, \ldots, h_j) \in U \times E^j, \text{ and define}
\end{align*}
\]
continuous functions
\[
\begin{align*}
d^{(j)}\gamma : U \times E^j &\to F.
\end{align*}
\]
If \( \gamma \) is \( C^k \) for each \( k \in \mathbb{N} \), then \( \gamma \) is called \( C^\infty \) or \( \text{smooth} \); continuous maps are called \( C^0 \) and \( d^{(0)}\gamma := \gamma \).

(b) Let \( U \subseteq E \) be a locally convex subset with dense interior. A continuous map \( \gamma : U \to F \) is called a \( C^k \)-map (for \( k \in \mathbb{N}_0 \cup \{\infty\} \)) if \( \gamma|_{U^b} : U^b \to F \) is \( C^k \) and for each \( j \in \mathbb{N}_0 \) with \( j \leq k \) the map \( d^{(j)}\gamma|_{U^b} : U^b \times E^j \to F \) admits a (unique) continuous extension \( d^{(j)}\gamma : U \times E^j \to F \).

In \([1]\), a notion of differentiability on products of locally convex subsets is presented:

**Definition 3.3.** Let \( E_1, \ldots, E_n \) and \( F \) be Hausdorff locally convex spaces.

(a) Let \( U_i \subseteq E_i \) be an open subset for each \( i \in \{1, \ldots, n\} \) and \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). A continuous function \( \gamma : U_1 \times \cdots \times U_n \to F \) is called a \( C^\alpha \)-map if for all \( \beta \in \mathbb{N}^n_0 \) with \( \beta \leq \alpha \) the iterated directional derivatives
\[
\begin{align*}
d^\alpha \gamma(x, h_1, \ldots, h_n) &:= (\hat{D}_1 \cdots \hat{D}_n\gamma)(x),
\end{align*}
\]
where
\[
\begin{align*}
\hat{D}_i\gamma(x) &:= (D_{(h_i)^*_a} \cdots D_{(h_i)^*_1}\gamma)(x)
\end{align*}
\]
eexist for all \( x := (x_1, \ldots, x_n) \in U_1 \times \cdots \times U_n, h_i := ((h_i)_1, \ldots, (h_i)_a) \in E_i^{(h_i)} \), \( h_i^*_a := ((h_i)_1^*, \ldots, (h_i)_a^*) \in ([0]^{1-a} \times E_i \times [0]^{n-1})^{(h_i)} \) and the maps
\[
\begin{align*}
d^\alpha \gamma : U_1 \times \cdots \times U_n \times E_1^{(h_1)} \times \cdots \times E_n^{(h_n)} &\to F
\end{align*}
\]
are continuous.

(b) Let \( U_i \subseteq E_i \) be a locally convex subset with dense interior for each \( i \in \{1, \ldots, n\} \) and \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \). A continuous map \( \gamma : U_1 \times \cdots \times U_n \to F \) is called a \( C^\alpha \)-map if \( \gamma|_{U^{(h)}_1 \times \cdots \times U^{(h)}_n} : U^{(h)}_1 \times \cdots \times U^{(h)}_n \to F \) is a \( C^\alpha \)-map and for all \( \beta \in \mathbb{N}^n_0 \) with \( \beta \leq \alpha \) the maps
\[
\begin{align*}
d^\alpha \gamma|_{U^{(h)}_1 \times \cdots \times U^{(h)}_n} : U^{(h)}_1 \times \cdots \times U^{(h)}_n \times E_1^{(h_1)} \times \cdots \times E_n^{(h_n)} &\to F
\end{align*}
\]
extend to (unique) continuous maps
\[
\begin{align*}
d^\alpha \gamma : U_1 \times \cdots \times U_n \times E_1^{(h_1)} \times \cdots \times E_n^{(h_n)} &\to F.
\end{align*}
\]
Now, consider the following concept of differentiability for functions defined on subsets of \( \mathbb{R}^n \):

**Definition 3.4.** Let \( E \) be a Hausdorff locally convex space.

(a) Let \( U \subseteq \mathbb{R}^n \) be an open subset, \( \gamma : U \to E \) be a continuous function, \( x \in U \) and \( i \in \{1, \ldots, n\} \). We define the **partial derivative of** \( \gamma \) **with respect to** the \( i \)-th variable via

\[
\frac{\partial 1}{\partial x_i} \gamma(x) := \frac{\partial}{\partial x_i} \gamma(x) := \lim_{t \to 0} \frac{\gamma(x + te_i) - \gamma(x)}{t},
\]

whenever the limit exists. We call \( \gamma \) **partially** \( C^1 \) if \( \gamma \) is continuous, the partial derivative \( \frac{\partial}{\partial x_i} \gamma(x) \) exists for each \( x \in U \) and \( i \in \{1, \ldots, n\} \) and each of the maps

\[
\frac{\partial}{\partial x_i} \gamma : U \to E
\]
is continuous. For \( k \in \mathbb{N} \cup \{\infty\} \) we call \( \gamma \) a **partially** \( C^k \)-map if \( \gamma \) is partially \( C^1 \) and each of the partial derivatives \( \frac{\partial}{\partial x_i} \gamma \) is partially \( C^{k-1} \). In this case we denote the higher partial derivatives of \( \gamma \) by

\[
\frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) := \frac{\partial^\alpha_1}{\partial x_1} \cdots \frac{\partial^\alpha_n}{\partial x_n} \gamma(x)
\]
for \( x \in U \) and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \), which define continuous functions

\[
\frac{\partial^\alpha}{\partial x^\alpha} \gamma : U \to E.
\]

Finally, we call continuous maps **partially** \( C^0 \).

(b) If \( U \subseteq \mathbb{R}^n \) is a locally convex subset with dense interior, then we call \( \gamma \) a **partially** \( C^k \)-map if \( \gamma \) is partially \( C^1 \) and each of the partial derivatives \( \frac{\partial}{\partial x_i} \gamma \) is partially \( C^{k-1} \). In this case we denote the higher partial derivatives of \( \gamma \) by

\[
\frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) := \frac{\partial^\alpha_1}{\partial x_1} \cdots \frac{\partial^\alpha_n}{\partial x_n} \gamma(x)
\]
for \( x \in U \) and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \), which define continuous functions

\[
\frac{\partial^\alpha}{\partial x^\alpha} \gamma : U \to E.
\]

Finally, we call continuous maps **partially** \( C^0 \).

**Remark 3.5.** If \( E \) is a Hausdorff locally convex space, \( U := U_1 \times \cdots \times U_n \subseteq \mathbb{R}^n \) is a locally convex subset with dense interior, and \( k \in \mathbb{N}_0 \cup \{\infty\} \), then for a continuous function \( \gamma : U \to E \) the following assertions are equivalent:

(i) \( \gamma \) is partially \( C^k \),

(ii) \( \gamma \) is \( C^\alpha \) for each \( \alpha \in (\mathbb{N}_0 \cup \{\infty\})^n \) with \( |\alpha| \leq k \),

(iii) \( \gamma \) is \( C^k \).

Moreover, the topology on the space \( C^k(U, E) \) (consisting of all \( C^k \)-functions \( \gamma : U \to E \)) defined in \( \Pi \) coincides with the locally convex topology defined by the seminorms

\[
\|\cdot\|_{K,\alpha,q} : C^k(U, E) \to [0, \infty], \quad \|\gamma\|_{K,\alpha,q} := \sup_{x \in K} \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right),
\]
where \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \), \( q \in \mathcal{P}_E \) and \( K \subseteq U \) is compact.
We pass on to the definition of the space of weighted differentiable functions:

**Definition 3.6.** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ be a locally convex subset with dense interior and $\gamma : U \to E$ be partially $C^k$ with $k \in \mathbb{N}_0 \cup \{\infty\}$. Let $W$ be a set of weights on $U$. For a weight $f \in W$, a seminorm $q \in \mathcal{P}_E$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ we define

$$\|\gamma\|_{f,\alpha,q} := \sup_{x \in U} f(x)q\left(\frac{\partial^\alpha}{\partial x^\alpha}\gamma(x)\right) \in [0, \infty].$$

Further, we define the vector space of weighted $C^k$-functions

$$C^k_W(U, E) := \{\gamma \in C^k(U, E) : (\forall f \in W)(\forall q \in \mathcal{P}_E)(\forall \alpha \in \mathbb{N}_0^n, |\alpha| \leq k) \|\gamma\|_{f,\alpha,q} < \infty\}$$

and endow it with the locally convex topology induced by the seminorms

$$\|\cdot\|_{f,\alpha,q} : C^k_W(U, E) \to [0, \infty]$$

where $f \in W$, $q \in \mathcal{P}_E$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$.

For a subset $V \subseteq E$ we additionally define

$$C^k_W(U, V) := \{\gamma \in C^k_W(U, E) : \gamma(U) \subseteq V\}.$$

**Remark 3.7.** We recall that for a set $X$, the initial topology on $X$ with respect to the family $(\gamma_i)_{i \in I}$ of mappings $\gamma_i : X \to Y_i$ into topological spaces $Y_i$ is defined as the coarsest topology making each $\gamma_i$ continuous.

**Remark 3.8.** The topology on $C^k_W(U, E)$ is Hausdorff, since the point evaluation map $ev_x : C^k_W(U, E) \to E, \gamma \mapsto \gamma(x)$ is continuous for each $x \in U$, this can be shown as in Remark 2.2.

Further, the topology on $C^k_W(U, E)$ for $k \in \mathbb{N}$ is initial with respect to the maps

$$\frac{\partial^\alpha}{\partial x^\alpha} : C^k_W(U, E) \to C^k_W(U, E)$$

for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Moreover, using the transitivity of initial topologies (cf. [10]), one can show that this topology is initial with respect to the maps

$$j : C^k_W(U, E) \to C^k_W(U, E), \frac{\partial}{\partial x_i} : C^k_W(U, E) \to C^{k-1}_W(U, E)$$

where $j$ is the inclusion map and $i \in \{1, \ldots, n\}$. That is, the map

$$\Theta : C^k_W(U, E) \to C^k_W(U, E) \times C^{k-1}_W(U, E)^n, \quad \gamma \mapsto (\gamma, \frac{\partial}{\partial x_1}\gamma, \ldots, \frac{\partial}{\partial x_n}\gamma)$$

is a (linear) topological embedding.

Further, the topology on $C^\infty_W(U, E)$ is initial with respect to the inclusion maps

$$C^\infty_W(U, E) \to C^k_W(U, E).$$
for \( k \in \mathbb{N}_0 \); we even have

\[
C^\infty_W(U, E) = \lim_{\leftarrow} C^k_W(U, E).
\]

The following lemma can be proven similarly to Lemma 2.5.

**Lemma 3.9.** Let \( E \) be a Hausdorff locally convex space and \( U \subseteq \mathbb{R}^n \) be a locally convex subset with dense interior. Let \( W \) be a set of weights on \( U \) such that for each compact subset \( K \subseteq U \) there is a weight \( f_K \in W \) with \( \inf_{x \in K} f_K(x) > 0 \). Then the inclusion map

\[
i : C^k_W(U, E) \to C^k(U, E)
\]

is linear and continuous for each \( k \in \mathbb{N}_0 \cup \{\infty\} \).

The next two lemmas can be proven with simple induction.

**Lemma 3.10.** Let \( F \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \) be a locally convex subset with dense interior and \( \gamma : U \to E \) be a partially \( C^k \)-map for \( k \in \mathbb{N} \cup \{\infty\} \). If \( E \) is a subspace of \( F \), then \( \gamma \in C^k(U, E) \) if and only if

\[
\frac{\partial^\alpha}{\partial x^\alpha}(U) \subseteq E \text{ for each } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq k.
\]

**Lemma 3.11.** Let \( E, F \) be Hausdorff locally convex spaces and \( \lambda : E \to F \) be a continuous linear map. Let \( U \subseteq \mathbb{R}^n \) be a locally convex subset with dense interior and \( \gamma : U \to E \) be a partially \( C^k \)-map with \( k \in \mathbb{N} \cup \{\infty\} \). The map

\[
\lambda \circ \gamma : U \to F
\]

is partially \( C^k \) and

\[
\frac{\partial^\alpha}{\partial x^\alpha}(\lambda \circ \gamma) = \lambda \circ \frac{\partial^\alpha}{\partial x^\alpha} \gamma
\]

for each \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \).

The continuity of superposition operators \( C^k_W(U, \lambda) \) can be proven similarly to Lemma 2.7 using the preceding fact.

**Lemma 3.12.** Let \( E, F \) be Hausdorff locally convex spaces and \( \lambda : E \to F \) be a continuous linear function. Let \( U \subseteq \mathbb{R}^n \) be a locally convex subset with dense interior and \( W \) be a set of weights on \( U \). If \( \gamma \in C^k_W(U, E) \), then

\[
\lambda \circ \gamma \in C^k_W(U, F)
\]

for each \( k \in \mathbb{N}_0 \cup \{\infty\} \).

Moreover, the map

\[
C^k_W(U, \lambda) : C^k_W(U, E) \to C^k_W(U, F), \quad \gamma \mapsto \lambda \circ \gamma
\]

is continuous and linear.

For further work we recall a notion of integrability of curves with values in a locally convex space. Many important results (for example, the corresponding Fundamental Theorem of Calculus) can be found in [10].

**Remark 3.13.** Let \( E \) be a Hausdorff locally convex space, \( \gamma : I \to E \) be a continuous curve on an interval \( I \subseteq \mathbb{R} \) and \( a, b \in I \). If there exists an element \( z \in E \) such that
\[ \lambda(z) = \int_a^b \lambda(\gamma(t)) \, dt \]

for each \( \lambda \in E' \), then \( z \) is called the weak integral of \( \gamma \) from \( a \) to \( b \) and we write

\[ \int_a^b \gamma(t) \, dt : = z. \]

Note that, since the space \( E' \) separates points on \( E \) by the Hahn-Banach Theorem, the weak integral is uniquely determined if it exists. Further, it is known that the weak integral of a curve \( \gamma : I \to E \) always exists if \( E \) is sequentially complete (see, for example [10]). Since every locally convex space \( F \) can be completed, the weak integral \( z : = \int_a^b \eta(t) \, dt \) of a continuous curve \( \eta : I \to F \) always exists in the completion \( \tilde{F} \) of \( F \). One can show that the weak integral \( w : = \int_a^b \eta(t) \, dt \) in \( F \) exists if and only if \( z \in F \), in which case we have \( z = w \).

We will deduce the completeness of \( C^k_W(U, E) \) from the next proposition:

**Proposition 3.14.** Let \( E \) be a Hausdorff locally convex space and \( U \subseteq \mathbb{R}^n \) be a locally convex subset with dense interior. Let \( W \) be a set of weights on \( U \) such that for each compact subset \( K \subseteq U \) there is a weight \( f_K \in W \) with \( \inf_{x \in K} f_K(x) > 0 \). Then the map

\[ \Theta : C^k_W(U, E) \to C_W(U, E) \times C^{k-1}_W(U, E)^n, \quad \gamma \mapsto (\gamma, \frac{\partial}{\partial x_1} \gamma, \ldots, \frac{\partial}{\partial x_n} \gamma) \]

is a linear topological embedding with closed image, for each \( k \in \mathbb{N} \).

**Proof.** We know that \( \Theta \) is a topological embedding, by Remark 3.8, and it is easy to show that \( \Theta \) is linear.

Now, let \((\gamma_a)_{a \in A}\) be a net in \( C^k_W(U, E) \) such that \((\gamma_a)_{a \in A}\) converges to \( \gamma \) in \( C_W(U, E) \) and each of the nets \((\frac{\partial}{\partial x_i} \gamma_a)_{a \in A}\) converges to \( \gamma^i \) in \( C^{k-1}_W(U, E) \), for each \( i \in \{1, \ldots, n\} \). We have to show that \((\gamma, \gamma^1, \ldots, \gamma^n) \in \text{im}(\Theta) \), that is, \( \gamma \in C^k_W(U, E) \) and \( \frac{\partial}{\partial x_i} \gamma = \gamma^i \).

For \( x \in U^o \), \( e_i \in \mathbb{R}^n \) and \( 0 < t \leq 1 \) with \( K_{x,t,i} := \{x + tue_i : u \in [0,1]\} \subseteq U^o \), we have

\[ \gamma(x + te_i) - \gamma(x) = \lim_{a \in A} \frac{\gamma_a(x + te_i) - \gamma_a(x)}{t}. \]

We apply the Mean Value Theorem (cf. [10]) and obtain

\[ \lim_{a \in A} \frac{\gamma_a(x + te_i) - \gamma_a(x)}{t} = \lim_{a \in A} \frac{1}{t} \int_0^1 d\gamma_a(x + tue_i, te_i) \, du = \lim_{a \in A} \frac{1}{t} \int_0^1 td\gamma_a(x + tue_i, e_i) \, du = \lim_{a \in A} \frac{1}{t} \int_0^1 \frac{\partial}{\partial x_i} \gamma_a(x + tue_i) \, du. \]
By the hypothesis on $W$, the net \( \left( \frac{\partial}{\partial x_i} \gamma \right)_{a \in A} \) converges to $\gamma^i$ uniformly on the segment $K_{x,t,i}$ (see Lemma 3.9), hence we can use the fact, that weak integrals and uniform limits can be interchanged (see [10]) and get

\[
\lim_{a \in A} \int_0^1 \frac{\partial}{\partial x_i} \gamma_a(x + tue_i) du = \int_0^1 \gamma^i(x + tue_i) du.
\]

The map

\[(t,u) \mapsto \gamma^i(x + tue_i)\]

is continuous, whence the parameter-dependent integral

\[t \mapsto \int_0^1 \gamma^i(x + tue_i) du\]

is continuous (also at $t = 0$, see [10]) and we have

\[
\lim_{t \to 0} \frac{\gamma(x + tue_i) - \gamma(x)}{t} = \lim_{t \to 0} \int_0^1 \gamma^i(x + tue_i) du = \int_0^1 \gamma^i(x) du = \gamma^i(x).
\]

Thus we see that $\gamma_i|_{U^\circ}$ is partially $C^1$ with partial derivatives $\frac{\partial}{\partial x_i} (\gamma_i|_{U^\circ}) = \gamma^i|_{U^\circ}$ for each $i \in \{1, \ldots, n\}$. But $\gamma_i|_{U^\circ}$ continuously extends to the map $\gamma^i : U \to E$. Hence $\frac{\partial}{\partial x_i} (\gamma|_{U^\circ})$ continuously extends to

\[
\frac{\partial}{\partial x_i} \gamma : U \to E, \quad x \mapsto \gamma^i(x),
\]

whence $\gamma$ is partially $C^1$. Moreover, each of the maps $\frac{\partial}{\partial x_i} \gamma = \gamma^i$ is partially $C^{k-1}$, thus $\gamma$ is partially $C^k$.

It remains to show that $\gamma \in C^k_W(U, E)$. To this end, let $f \in W$, $q \in P_E$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. If $|\alpha| = 0$, then we have

\[
\|\gamma\|_{f,\alpha,q} = \|\gamma\|_{f,q} < \infty.
\]

Otherwise, we write $\alpha = \beta + e_i$ for suitable $\beta \in \mathbb{N}_0^n$ and $i \in \{1, \ldots, n\}$ and obtain

\[
\|\gamma\|_{f,\alpha,q} \overset{\text{def}}{=} \sup_{x \in U} f(x) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right) = \sup_{x \in U} f(x) q \left( \frac{\partial^\beta}{\partial x^\beta} \frac{\partial}{\partial x_i} \gamma(x) \right) = \left\| \frac{\partial}{\partial x_i} \gamma \right\|_{f,\beta,q} < \infty,
\]

since $\frac{\partial}{\partial x_i} \gamma = \gamma^i \in C^{k-1}_W(U, E)$, and the assertion is proven.

\textbf{Proposition 3.15.} Let $E$ be a complete Hausdorff locally convex space and $U \subseteq \mathbb{R}^n$ be a locally convex subset with dense interior. If $W$ is a set of weights on $U$ such that for each compact subset $K \subseteq U$ there is a weight $f_K \in W$ with $\inf_{x \in K} f_K(x) > 0$, then the space $C^k_W(U, E)$ is complete for each $k \in \mathbb{N}_0 \cup \{\infty\}$.
Proof. First we proof the assertion for \( k < \infty \) by induction.

The case \( k = 0 \). Follows from Proposition 2.9 since every metrizable space is a \( k \)-space.

Induction step. By the induction hypothesis, the space \( C^k_W(U, E) \times C^{k-1}_W(U, E)^n \) is complete. From Proposition 3.14 we conclude that \( C^k_W(U, E) \cong \text{im}(\Theta) \) is complete, since \( \text{im}(\Theta) \) is closed (hence complete) in \( C^k_W(U, E) \times C^{k-1}_W(U, E)^n \).

Finally, from Remark 3.8 we deduce that \( C^\infty_W(U, E) \) is complete, since projective limits of complete topological vector spaces are complete (see [19, 5.3]).

Definition 3.16. Let \( E \) be a Hausdorff locally convex space and \( k, l \in \mathbb{N}_0 \cup \{\infty\} \).

(a) Let \( U \subseteq \mathbb{R}^n \), \( V \subseteq \mathbb{R}^m \) be open subsets. We call a map \( \gamma : U \times V \to E \) a partially \( C^{k,l} \)-map, if for each \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \), \( \beta \in \mathbb{N}_0^m \) with \( |\beta| \leq l \) the partial derivative

\[
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y)
\]

exists in \( E \) for all \( (x, y) \in U \times V \), and the map

\[
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma : U \times V \to E.
\]

is continuous.

(b) If \( U \subseteq \mathbb{R}^n \), \( V \subseteq \mathbb{R}^m \) are locally convex with dense interior, then we say that a continuous map \( \gamma : U \times V \to E \) is partially \( C^{k,l} \) if \( \gamma\big|_{U \times V} \) is partially \( C^{k,l} \) and for each \( \alpha, \beta \) as in (a) the map

\[
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \left( \gamma\big|_{U \times V} \right) : U^\alpha \times V^\beta \to E
\]

admits a (unique) continuous extension

\[
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma : U \times V \to E.
\]

Remark 3.17. A continuous map \( \gamma : U \times V \to E \) as in Definition 3.16 is partially \( C^{k,l} \) if and only if \( \gamma \) is \( C^{(\delta, \varepsilon)} \) for all \( \delta \in (\mathbb{N}_0 \cup \{\infty\})^n \) with \( |\delta| \leq k \) and \( \varepsilon \in (\mathbb{N}_0 \cup \{\infty\})^m \) with \( |\varepsilon| \leq l \). Thus, using the Schwarz’ Theorem for \( C^\alpha \)-mappings ([1, Prop.69]) we see that

\[
\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) = \frac{\partial^\beta}{\partial y^\beta} \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x, y)
\]

for all \( (x, y) \in U \times V \). Moreover, \( \gamma : U \times V \to E \) is partially \( C^{k,l} \) if and only if the map

\[
V \times U \to E, \quad (y, x) \mapsto \gamma(x, y)
\]

is partially \( C^{l,k} \).
The topology on $C^{k,l}(U \times V, E)$ (the space of all partially $C^{k,l}$-maps $\gamma : U \times V \to E$) defined in Remark 3.18 coincides with the locally convex topology defined by the seminorms

$$\|\|_{K,(\alpha,\beta),q} : C^{k,l}(U \times V, E) \to [0, \infty [ ,$$

$$\|\gamma\|_{K,(\alpha,\beta),q} := \sup_{(x,y) \in K} q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x,y) \right)$$

with $\alpha \in \mathbb{N}_0^k$, $\beta \in \mathbb{N}_0^m$ such that $|\alpha| \leq k$, $|\beta| \leq l$, $K \subseteq U \times V$ a compact subset and $q \in \mathcal{P}_E$.

**Remark 3.18 (Product Rule).** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior, and $k,l \in \mathbb{N}_0 \cup \{\infty\}$. We can show the following Product Rules for differentiable functions:

(i) If the maps $f : U \to \mathbb{R}$ and $\gamma : U \to E$ are partially $C^k$, then the map $f \cdot \gamma$ is partially $C^k$ and the partial derivatives are given by

$$\frac{\partial^\alpha}{\partial x^\alpha} (f \cdot \gamma)(x) = \sum_{\tau \leq \alpha} \left( \begin{array}{c} \alpha \\ \tau \end{array} \right) \frac{\partial^{\alpha - \tau}}{\partial x^{\alpha - \tau}} f(x) \frac{\partial^\tau}{\partial x^\tau} \gamma(x)$$

for each $x \in U$ and $\alpha \in \mathbb{N}_0^k$ with $|\alpha| \leq k$.

(ii) If the maps $f : U \times V \to \mathbb{R}$ and $\gamma : U \times V \to E$ are partially $C^{k,l}$, then the map $f \cdot \gamma : U \times V \to E$ is partially $C^{k,l}$ with partial derivatives

$$\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (f \cdot \gamma)(x,y) = \sum_{\tau \leq \alpha \leq \beta} \sum_{\kappa \leq \beta} \left( \begin{array}{c} \alpha \\ \tau \end{array} \right) \left( \begin{array}{c} \beta \\ \kappa \end{array} \right) \frac{\partial^{\alpha - \tau}}{\partial x^{\alpha - \tau}} f(x,y) \frac{\partial^{\beta - \kappa}}{\partial y^{\beta - \kappa}} \gamma(x,y)$$

for $\alpha \in \mathbb{N}_0^k$ with $|\alpha| \leq k$, $\beta \in \mathbb{N}_0^m$ with $|\beta| \leq l$ and $(x,y) \in U \times V$.

**Definition 3.19.** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior, and $k,l \in \mathbb{N}_0 \cup \{\infty\}$. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be sets of weights on $U$ and $V$, respectively, and $\mathcal{W} := \mathcal{W}_1 \otimes \mathcal{W}_2$. For $\gamma \in C^{k,l}(U \times V, E)$, $f = f_1 \otimes f_2 \in \mathcal{W}$, $q \in \mathcal{P}_E$ and $\alpha \in \mathbb{N}_0^k$ with $|\alpha| \leq k$, $\beta \in \mathbb{N}_0^m$ with $|\beta| \leq l$ we define

$$\|\gamma\|_{f,(\alpha,\beta),q} := \sup_{(x,y) \in U \times V} f_1(x)f_2(y)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x,y) \right) \in [0, \infty]$$

and endow the vector space

$$C^{k,l}_{\mathcal{W}}(U \times V, E) := \{ \gamma \in C^{k,l}(U \times V, E) : (\forall f \in \mathcal{W})(\forall q \in \mathcal{P}_E)$$

$$(\forall \alpha \in \mathbb{N}_0^k, |\alpha| \leq k)(\forall \beta \in \mathbb{N}_0^m, |\beta| \leq l) \|\gamma\|_{f,(\alpha,\beta),q} < \infty \}$$

with the locally convex topology defined by the seminorms

$$\|\|_{f,(\alpha,\beta),q} : C^{k,l}_{\mathcal{W}}(U \times V, E) \to [0, \infty [ .$$
Using the fact, that metrizable spaces are $k$-spaces, we can rephrase the Exponential Law for spaces of differentiable functions (which is proven in [1, Theorem 94]) as follows:

**Proposition 3.20 (Exponential Law for spaces of differentiable functions).** Let $E$ be a Hausdorff locally convex space and $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior. If $\gamma \in C_{k,l}^{k,l}(U \times V, E)$ for some $k, l \in \mathbb{N}_0 \cup \{\infty\}$, then

$$
\gamma_x := \gamma(x, \bullet) : V \to E, \quad \gamma_x(y) := \gamma(x, y)
$$

is partially $C^l$ for each $x \in U$, and

$$
\gamma^\gamma : U \to C^l(V, E), \quad \gamma^\gamma(x) := \gamma_x
$$

is partially $C^k$.

Moreover, the linear map

$$
\Phi : C_{k,l}^{k,l}(U \times V, E) \to C^k(U, C^l(V, E)), \quad \gamma \mapsto \gamma^\gamma
$$

is a homeomorphism.

Now we obtain the following intermediate result:

**Theorem 3.21.** Let $E$ be a Hausdorff locally convex space and $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior. Let $W_1$ and $W_2$ be sets of weights on $U$ and $V$, respectively. We assume that for each compact subset $K \subseteq U$ there is a weight $f_K \in W_1$ such that $\inf_{x \in K} f_K(x) > 0$, and likewise for $W_2$. If $\gamma \in C_{W_1}^k(U, C_{W_2}^l(V, E))$ for some $k, l \in \mathbb{N}_0 \cup \{\infty\}$, then

$$
\gamma^\wedge \in C_{W_1}^{k,l}(U \times V, E),
$$

where $\gamma^\wedge$ is the map

$$
\gamma^\wedge : U \times V \to E, \quad \gamma^\wedge(x, y) := \gamma(x)(y)
$$

and $W = W_1 \otimes W_2$.

Moreover, the map

$$
\Psi : C_{W_1}^k(U, C_{W_2}^l(V, E)) \to C_{W_1}^{k,l}(U \times V, E), \quad \gamma \mapsto \gamma^\wedge
$$

is a topological embedding.

**Proof.** We know by Lemma 3.9 that the inclusion maps

$$
i : C_{W_1}^k(U, C^l(V, E)) \to C^k(U, C^l(V, E))$$

and

$$
j : C_{W_2}^l(V, E) \to C^l(V, E)
$$

are continuous and linear. Thus, by Lemma 3.12 the map

$$
C_{W_1}^k(U, j) : C_{W_1}^k(U, C_{W_2}^l(V, E)) \to C_{W_1}^k(U, C^l(V, E))
$$

is continuous and linear. Now, we use the inverse map
\( \Phi^{-1} : C^k(U, C^l(V, E)) \to C^{k,l}(U \times V, E), \quad \gamma \mapsto \gamma^\wedge \)

of the homeomorphism \( \Phi \) from Proposition 3.20 and define the continuous linear map

\[
\Theta := \Phi^{-1} \circ i \circ C^k_{W_1}(U, j) : C^k_{W_1}(U, C^l_{W_2}(V, E)) \to C^{k,l}(U \times V, E)
\]

\[
\gamma \mapsto \gamma^\wedge.
\]

To show that \( \Theta(\gamma) = \gamma^\wedge \in C^{k,l}_W(U \times V, E) \) for \( \gamma \in C^{k,l}_W(U, C^l_{W_2}(V, E)) \), let \( q \in \mathcal{P}_E \), \( f \in W \) (that is \( f = f_1 \otimes f_2 \) for some weights \( f_1 \in W_1, f_2 \in W_2 \)) and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k, \beta \in \mathbb{N}_0^n \) with \( |\beta| \leq l \). We have

\[
\|\gamma^\wedge\|_{f,\alpha,\beta,q} \overset{\text{def}}{=} \sup\limits_{(x,y) \in U \times V} f_1(x)f_2(y)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma^\wedge(x, y) \right)
\]

\[
= \sup\limits_{x \in U} f_1(x) \sup\limits_{y \in V} f_2(y)q \left( \frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma^\wedge(x, \bullet) \right)(y) \right)
\]

\[
= \sup\limits_{x \in U} f_1(x) \sup\limits_{y \in V} f_2(y)q \left( \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \right). \tag{4}
\]

We have \( \frac{\partial^\beta}{\partial y^\beta} \gamma(x) \in C^l_{W_2}(V, E) \) for each \( x \in U \), hence

\[
\sup\limits_{y \in V} f_2(y)q \left( \frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right)(y) \right) = \left\| \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right\|_{f_2,\beta,q} < \infty. \tag{5}
\]

Applying (5) to (4), we obtain

\[
\|\gamma^\wedge\|_{f,\alpha,\beta,q} = \sup\limits_{x \in U} f_1(x) \sup\limits_{y \in V} f_2(y)q \left( \frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right)(y) \right)
\]

\[
= \sup\limits_{x \in U} f_1(x) \left\| \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right\|_{f_2,\beta,q}
\]

\[
= \|\gamma\|_{f_1,\alpha,\|\|f_2,\beta,q} < \infty,
\]

whence \( \gamma^\wedge \in C^{k,l}_W(U \times V, E) \).

Thus, we can define the map

\[
\Psi := \Theta \mid_{C^{k,l}_W(U \times V, E)} : C^k_{W_1}(U, C^l_{W_2}(V, E)) \to C^{k,l}_W(U \times V, E), \quad \gamma \mapsto \gamma^\wedge,
\]

which is continuous, linear and injective, by construction. Moreover, for each \( f_1 \in W_1, f_2 \in W_2, q \in \mathcal{P}_E \), and \( \alpha, \beta \) as above, we have

\[
\|\Psi(\gamma)\|_{f_1 \otimes f_2,\alpha,\beta,q} \overset{\text{def}}{=} \|\gamma^\wedge\|_{f_1 \otimes f_2,\alpha,\beta,q} = \|\gamma\|_{f_1,\alpha,\|\|f_2,\beta,q}
\]

for all \( \gamma \in C^k_{W_1}(U, C^l_{W_2}(V, E)) \), by (4). Thus, by Lemma 2.11, the map \( \Psi \) is a topological embedding, which completes the proof. \( \square \)
For the further work, we need spaces of differentiable maps with compact support.

**Definition 3.22.** Let $E$ be a Hausdorff locally convex space and $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior. For a compact subset $K \subseteq U$ and $k \in \mathbb{N}_0 \cup \{\infty\}$ we define the space

$$C^k_K(U, E) := \{ \gamma \in C^k(U, E) : \text{supp}(\gamma) \subseteq K \}$$

and endow it with the locally convex topology defined by the seminorms

$$\|\cdot\|_{\alpha, q} : C^k_K(U, E) \to [0, \infty[, \quad \|\gamma\|_{\alpha, q} := \sup_{x \in K} q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \gamma(x) \right)$$

with $q \in \mathcal{P}_E$ and $\alpha \in \mathbb{N}^n_0$ such that $|\alpha| \leq k$.

Similarly, for a compact subset $L \subseteq U \times V$ and $k, l \in \mathbb{N}_0 \cup \{\infty\}$, the space

$$C^{k,l}_L(U \times V, E) := \{ \gamma \in C^{k,l}(U \times V, E) : \text{supp}(\gamma) \subseteq L \}$$

is endowed with the locally convex topology defined by the seminorms

$$\|\cdot\|_{(\alpha, \beta), q} : C^{k,l}_L(U \times V, E) \to [0, \infty[, \quad \|\gamma\|_{(\alpha, \beta), q} := \sup_{(x, y) \in L} q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\beta}}{\partial y^{\beta}} \gamma(x, y) \right)$$

with $q \in \mathcal{P}_E$, $\alpha \in \mathbb{N}^n_0$ such that $|\alpha| \leq k$ and $\beta \in \mathbb{N}^m_0$ such that $|\beta| \leq l$.

Additionally, we define the spaces

$$C^k(U, E) := \{ \gamma \in C^k(U, E) : \text{supp}(\gamma) \subseteq U \text{ is compact} \} = \bigcup \{ C^k_K(U, E) : K \subseteq U \text{ is compact} \}.$$

and

$$C^{k,l}(U \times V, E) := \{ \gamma \in C^{k,l}(U \times V, E) : \text{supp}(\gamma) \subseteq U \times V \text{ is compact} \} = \bigcup \{ C^{k,l}_L(U \times V, E) : L \subseteq U \times V \text{ is compact} \}.$$

**Lemma 3.23.** Let $E$ be a Hausdorff locally convex space and $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior. Let $K_1 \subseteq U$ and $K_2 \subseteq V$ be compact subsets. If $\gamma \in C^{k,l}_{K_1 \times K_2}(U \times V, E)$, then

$$\gamma_x := \gamma(x, \bullet) \in C^l_{K_2}(V, E)$$

for each $x \in U$, and

$$\gamma^y \in C^k_{K_1}(U, C^l_{K_2}(V, E))$$

holds for the map

$$\gamma^y : U \to C^l_{K_2}(V, E), \quad x \mapsto \gamma_x.$$
Proof. We know from Proposition 3.20 that the map \( \gamma_x \) is partially \( C^l \) for each \( x \in U \). If \( y \in V \setminus K_2 \), then

\[
\gamma_x(y) = \gamma(x, y) = 0,
\]
whence \( \text{supp}(\gamma_x) \subseteq K_2 \). Thus we have

\[
\gamma_x \in C^l_{K_2}(V, E).
\]

Further, the map

\[
\gamma^\vee: U \to C^l(V, E), \quad x \mapsto \gamma_x
\]
is partially \( C^k \) (by Proposition 3.20). But we have

\[
\frac{\partial^\alpha}{\partial x^\alpha} \gamma^\vee(x) = \frac{\partial^\alpha}{\partial x^\alpha} \gamma_x \in C^l_{K_2}(V, E)
\]
for each \( x \in U \) and \( \alpha \in \mathbb{N}^n_0 \) with \( |\alpha| \leq k \), whence \( \gamma^\vee \in C^k(U, C^l_{K_2}(V, E)) \), by Lemma 3.10. Moreover, if \( x \in U \setminus K_1 \) and \( y \in V \), then we have

\[
\gamma^\vee(x)(y) = \gamma(x, y) = 0,
\]
hence \( \gamma^\vee \in C^k_{K_1}(U, C^l_{K_2}(V, E)) \), as asserted.

The next two lemmas can be proven similarly to Lemma 2.16.

Lemma 3.24. Let \( E \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \) be a locally convex subset with dense interior and \( K \subseteq U \) be compact. If \( W \) is a set of weights on \( U \) such that each weight \( f \in W \) is bounded on \( K \), then

\[
C^k_K(U, E) \subseteq C^k_W(U, E)
\]
and the inclusion map

\[
i: C^k_K(U, E) \to C^k_W(U, E)
\]
is continuous and linear for each \( k \in \mathbb{N}_0 \cup \{\infty\} \).

Lemma 3.25. Let \( E \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \) and \( V \subseteq \mathbb{R}^m \) be locally convex subsets with dense interior, and \( k, l \in \mathbb{N}_0 \cup \{\infty\} \). Let \( W_1 \) and \( W_2 \) be sets of weights on \( U \) and \( V \), respectively, consisting of functions that are bounded on compact sets. Then

\[
C^{k,l}_K(U \times V, E) \subseteq C^{k,l}_{W_1 \otimes W_2}(U \times V, E)
\]
for each compact subset \( K \subseteq U \times V \), and the inclusion map

\[
i: C^{k,l}_K(U \times V, E) \to C^{k,l}_{W_1 \otimes W_2}(U \times V, E)
\]
is continuous and linear.

The next proposition will be essential for the proof of the Exponential Law:

Proposition 3.26. Let \( E \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \) and \( V \subseteq \mathbb{R}^m \) be open subsets, and \( k, l \in \mathbb{N}_0 \cup \{\infty\} \). Assume that for a set of weights \( W_1 \subseteq C^k(U, [0, \infty]) \) on \( U \) we have:
(i) \( W_1 \) satisfies the \( o \)-condition,

(ii) for each \( f \in W_1 \) and \( \alpha \in \mathbb{N}_0^k \) with \( |\alpha| \leq k \) there exists \( g \in W_1 \) such that

\[
\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq g(x)
\]

for all \( x \in U \),

and likewise for a set of weights \( W_2 \subseteq C^1(V, [0, \infty]) \) on \( V \). Then the space \( C^k_{\mathcal{W}}(U \times V, E) \) is dense in \( C^k_{\mathcal{W}}(U \times V, E) \), where \( W = W_1 \otimes W_2 \).

The proof (which can be found after Lemma 3.28) varies the proof of the density of \( C^\infty_c(U, \mathbb{R}) \) in the space \( C^\infty_{\mathcal{W}}(U, \mathbb{R}) \) by H.G. Garnir, M. De Wilde and J. Schmets in [6]. In addition, several auxiliary results will be necessary. We introduce some useful constructions.

**Lemma 3.27.** Let \( E \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \), \( V \subseteq \mathbb{R}^m \) be locally convex subsets with dense interior. Let \( W_1 \) and \( W_2 \) be sets of weights on \( U \) and \( V \), respectively. For a \( C^k_{\mathcal{W}} \)-map \( \gamma : U \times V \to E \) (with \( k, l \in \mathbb{N}_0 \cup \{\infty\} \)) we define the maps

\[
\gamma_{\beta,y,f_2} := f_2(y) \frac{\partial^\beta}{\partial y^\beta} \gamma(\bullet, y) : U \to E
\]

for some weight \( f_2 \in W_2 \), \( \beta \in \mathbb{N}_0^m \) with \( |\beta| \leq l \) and \( y \in V \), and

\[
\gamma_{\alpha,x,f_1} := f_1(x) \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x, \bullet) : V \to E
\]

for some weight \( f_1 \in W_1 \), \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \) and \( x \in U \).

If \( \gamma \in C^k_{\mathcal{W}_1 \otimes \mathcal{W}_2}(U \times V, E) \), then \( \gamma_{\beta,y,f_2} \in C^k_{\mathcal{W}_1}(U, E) \) and \( \gamma_{\alpha,x,f_1} \in C^k_{\mathcal{W}_2}(V, E) \).

**Proof.** We prove the assertion for the map \( \gamma_{\beta,y,f_2} \); the proof for \( \gamma_{\alpha,x,f_1} \) will be similar. From [1] Lemma 28 follows that the map \( \frac{\partial^\beta}{\partial y^\beta} \gamma(\bullet, y) \) is partially \( C^k \), whence also the map \( f_2(y) \frac{\partial^\beta}{\partial y^\beta} \gamma(\bullet, y) = \gamma_{\beta,y,f_2} \) is partially \( C^k \). Now, if \( f_1 \in W_1 \), \( q \in \mathcal{P}_E \), and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \), then we have

\[
\| \gamma_{\beta,y,f_2} \|_{f_1,\alpha,q} \overset{def}{=} \sup_{x \in U} f_1(x) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma_{\beta,y,f_2}(x) \right)
\]

\[
= \sup_{x \in U} f_1(x) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \left( f_2(y) \frac{\partial^\beta}{\partial y^\beta} \gamma(\bullet, y) \right) (x) \right)
\]

\[
= \sup_{x \in U} f_1(x) f_2(y) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \right) = \| \gamma \|_{f_1 \otimes f_2, \alpha, \beta, q} < \infty.
\]

Hence \( \gamma_{\beta,y,f_2} \in C^k_{\mathcal{W}_1}(U, E) \).

The next result will be very helpful:

**Lemma 3.28.** Let \( f : U \to \mathbb{R} \) be a map on an open subset \( U \subseteq \mathbb{R}^n \). If \( f = o(1) \), then \( f(x) \to 0 \) as \( \| x \|_\infty \to \infty \) or \( x \to \varnothing \in \partial U \).
Proof. First, let \((x_m)_{m \in \mathbb{N}}\) be a sequence in \(U\) such that \(\|x_m\|_{\infty} \to \infty\) as \(m \to \infty\), and let \(\varepsilon > 0\). Since \(f = o(1)\), there is a compact subset \(K_{\varepsilon} \subseteq U\) such that \(|f(x)| \leq \varepsilon\) for each \(x \in U \setminus K_{\varepsilon}\). Further, there exists \(N \in \mathbb{N}\) such that for each \(m \geq N\) we have \(\|x_m\|_{\infty} > \max_{x \in K_{\varepsilon}} \|x\|_{\infty}\), that is \(x_m \in U \setminus K_{\varepsilon}\). Thus \(|f(x_m)| \leq \varepsilon\) for all \(m \geq N\), as required.

Now let \((x_m)_{m \in \mathbb{N}}\) be a sequence in \(U\) which converges to some \(x \in \partial U\), \(\varepsilon > 0\) and \(K_{\varepsilon}\) as above. The set \(\mathbb{R}^n \setminus K_{\varepsilon}\) is open in \(\mathbb{R}^n\) and \(x \in \mathbb{R}^n \setminus K_{\varepsilon}\), thus there exists \(N \in \mathbb{N}\) such that for all \(m \geq N\) we have \(x_m \in \mathbb{R}^n \setminus K_{\varepsilon}\) (more precisely, we have \(x_m \in U \setminus K_{\varepsilon}\), whence \(f(x_m) \leq \varepsilon\) for all \(m \geq N\).

Let us consider some cases, in which the products of weights and weighted maps tend to zero.

**Remark 3.29.** We recall from Definition \ref{def:weights} that if the set of weights \(W\) on an open subset \(U \subseteq \mathbb{R}^n\) satisfies the \(o\)-condition, then for each weight \(f \in W\) there is a weight \(g \in W\) such that \(f = o(g)\). (We can always assume that \(g(x) = 0\) only if \(f(x) = 0\) and set \(\frac{f(x)}{g(x)} := 0\) in this case.) Therefore, we have \(\frac{f}{g} = o(1)\), which yields that

\[
\frac{f(x)}{g(x)} \to 0
\]

as \(\|x\|_{\infty} \to \infty\) or \(x \to \partial U\) (by Lemma \ref{lemma:product1}). Thus for each \(\gamma \in C^k_{00}(U, E)\) and \(\alpha \in \mathbb{N}^n_0\) with \(|\alpha| \leq k\) we have

\[
f(x) \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \to 0
\]

as \(\|x\|_{\infty} \to \infty\) or \(x \to \partial U\). In fact, if \(q \in \mathcal{P}_E\) and \(f = o(g)\), then

\[
f(x)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right) = \frac{f(x)}{g(x)} g(x) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right) \leq \frac{f(x)}{g(x)} \|\gamma\|_{g,\alpha,q} \to 0.
\]

If, additionally, \(W \subseteq C^k(U, [0, \infty[)\) and for each weight \(f \in W\) and \(\alpha \in \mathbb{N}^n_0\) with \(|\alpha| \leq k\) there exists a weight \(h_{f,\alpha} \in W\) such that

\[
\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq h_{f,\alpha}(x)
\]

for all \(x \in U\), then for all \(\beta, \tau \in \mathbb{N}^n_0\) with \(|\beta| + |\tau| \leq k\) we have

\[
\frac{\partial^\beta}{\partial x^\beta} \left( f \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma \right)(x) \to 0
\]

as \(\|x\|_{\infty} \to \infty\) or \(x \to \partial U\). To see this, we use the Product Rule (see Remark \ref{remark:product}(i)) and obtain
\[ q \left( \frac{\partial^\beta}{\partial x^\beta} \left( f \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma \right) (x) \right) = q \left( \sum_{\kappa \leq \beta} \left( \frac{\beta - \kappa}{\kappa} \right) \frac{\partial^{\beta - \kappa}}{\partial x^{\beta - \kappa}} f(x) \frac{\partial^{\kappa + \tau}}{\partial x^{\kappa + \tau}} \gamma(x) \right) \]
\[ \leq \sum_{\kappa \leq \beta} \left( \frac{\beta}{\kappa} \right) \left| \frac{\partial^{\beta - \kappa}}{\partial x^{\beta - \kappa}} f(x) \right| q \left( \frac{\partial^{\kappa + \tau}}{\partial x^{\kappa + \tau}} \gamma(x) \right) \]
\[ \leq \sum_{\kappa \leq \beta} \left( \frac{\beta}{\kappa} \right) h_{f, \beta - \kappa}(x) q \left( \frac{\partial^{\kappa + \tau}}{\partial x^{\kappa + \tau}} \gamma(x) \right) \rightarrow 0, \]
by (7).

**Lemma 3.30.** Let \( E \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \) and \( V \subseteq \mathbb{R}^m \) be locally convex subsets with dense interior, and \( k, l \in \mathbb{N}_0 \cup \{\infty\} \). Let \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) be sets of weights on \( U \) and \( V \), respectively, and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \), \( \beta \in \mathbb{N}_0^m \) with \( |\beta| \leq l \).

(i) If \( U \) is open and \( \mathcal{W}_1 \) satisfies the \( \alpha \)-condition, then for each \( f_1 \in \mathcal{W}_1 \), \( f_2 \in \mathcal{W}_2 \) and \( \gamma \in C_{\mathcal{W}_1 \otimes \mathcal{W}_2}^k(U \times V, E) \) we have

\[ f_1(x) f_2(y) \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \rightarrow 0 \]
uniformly in \( y \), as \( \|x\|_\infty \rightarrow \infty \) or \( x \rightarrow \bar{x} \in \partial U \).

(ii) If \( V \) is open and \( \mathcal{W}_2 \) satisfies the \( \alpha \)-condition, then for each \( f_1 \in \mathcal{W}_1 \), \( f_2 \in \mathcal{W}_2 \) and \( \gamma \in C_{\mathcal{W}_1 \otimes \mathcal{W}_2}^k(U \times V, E) \) we have

\[ f_1(x) f_2(y) \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \rightarrow 0 \]
uniformly in \( x \), as \( \|y\|_\infty \rightarrow \infty \) or \( y \rightarrow \bar{y} \in \partial V \).

**Proof.** Assume that \( U \) is open and \( \mathcal{W}_1 \) satisfies the \( \alpha \)-condition. Then there exists a weight \( g_1 \in \mathcal{W}_1 \) such that \( f_1 = o(g_1) \). For a seminorm \( q \in \mathcal{P}_E \), \( x \in U \) and \( y \in V \) we have

\[ f_1(x) f_2(y) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \right) = \frac{f_1(x)}{g_1(x)} f_2(y) q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \right) \]
\[ \leq \frac{f_1(x)}{g_1(x)} \|g_1 \otimes f_2, (\alpha, \beta), q \| \rightarrow 0 \]
as \( \|x\|_\infty \rightarrow \infty \) or \( x \rightarrow \bar{x} \in \partial U \), see Remark 3.29. Thus (i) holds. To prove (ii), we proceed similarly.

We will also need the following facts concerning differentiable extensions of mappings:

**Lemma 3.31.** Let \( E \) be a Hausdorff locally convex space and \( \gamma : [a, b] \rightarrow E \) be a continuous curve. If \( \gamma \mid_{[a, b]} \) is \( C^1 \) and the derivative \( \left( \gamma \mid_{[a, b]} \right)' \) admits a continuous extension \( \eta : [a, b] \rightarrow E \), then \( \gamma \) is \( C^1 \) and \( \gamma' = \eta \).
Proof. For each $t \in [a, b]$ the curve $\gamma_{|[t,b]}$ is $C^1$, thus

$$\gamma(t) = \gamma(b) - \int_t^b \gamma'(s)ds = \gamma(b) - \int_t^b \eta(s)ds,$$

by the first part of the Fundamental Theorem of Calculus (see [10]). By Remark 3.13, the weak integral $\int_a^b \eta(s)ds$ exists in the completion $\tilde{E}$ of $E$, thus we can define the continuous curve

$$\xi : [a, b] \to \tilde{E} \quad \xi(t) := \gamma(b) - \int_t^b \eta(s)ds.$$

Since $\gamma$ is continuous and $\gamma_{|[a,b]} = \xi_{|[a,b]}$ (by (8)), we conclude that $\gamma = \xi$. Hence $\xi([a,b]) \subseteq E$, that is $\int_a^b \eta(s)ds$ exists in $E$. The second part of the Fundamental Theorem of Calculus (can also be found in [10]) yields that $\gamma$ is $C^1$ and $\gamma' = \eta$, as required.

Proposition 3.32. Let $E$ be a Hausdorff locally convex space and $\gamma : U \to E$ be a partially $C^k$-map on an open subset $U \subseteq \mathbb{R}^n$ for $k \in \mathbb{N} \cup \{\infty\}$. Assume that $\gamma$ admits the continuous extension $\eta : \mathbb{R}^n \to E$, $\eta(x) := \left\{ \begin{array}{ll} \gamma(x) & x \in U \\ 0 & x \notin U \end{array} \right.$ (9) and the partial derivatives $\frac{\partial^\alpha}{\partial x^\alpha} \gamma$ admit the continuous extensions

$$\eta_{\alpha} : \mathbb{R}^n \to E, \quad \eta_{\alpha}(x) := \left\{ \begin{array}{ll} \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) & x \in U \\ 0 & x \notin U \end{array} \right.$$

(10)

for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$. Then the map $\eta$ is partially $C^k$ with the partial derivatives $\frac{\partial^\alpha}{\partial x^\alpha} \eta = \eta_{\alpha}$ for all $\alpha$.

Proof. We may assume that $k$ is finite and prove the assertion by induction.

The case $k = 1$.

Step 1: $n = 1$. Assume that $\gamma : U \to E$ is a $C^1$-map on an open subset $U \subseteq \mathbb{R}$ such that there are continuous extensions

$$\eta : \mathbb{R} \to E, \quad \eta(t) := \left\{ \begin{array}{ll} \gamma(t) & t \in U \\ 0 & t \notin U \end{array} \right.$$

of $\gamma$ and

$$\eta_1 : \mathbb{R} \to E, \quad \eta_1(t) := \left\{ \begin{array}{ll} \gamma'(t) & t \in U \\ 0 & t \notin U \end{array} \right.$$

of the derivative $\gamma'$. For $t \in U \cup (\mathbb{R} \setminus \overline{U})$ we obviously have

$$\eta'(t) = \eta_1(t).$$

Now, we want to show that the right derivative $\eta'_+(t) := \lim_{s \to 0^+} \frac{\eta(t+s)-\eta(t)}{s}$ exists for each $t \in \partial U$ and we have

$$\eta'_+(t) = \eta_1(t) = 0.$$

(11)
The proof for the left derivative \( \eta'_1(t) := \lim_{s \to 0^-} \frac{\eta(t+s) - \eta(t)}{s} \) is similar.

Let \( (t_m)_{m \in \mathbb{N}} \subseteq ]t, \infty[ \) be a sequence such that \( t_m \to t \) as \( m \to \infty \). To prove (11), it suffices to show that

\[
\lim_{m \to \infty} q \left( \frac{\eta(t_m) - \eta(t)}{t_m - t} \right) = 0
\]

for each seminorm \( q \in \mathcal{P}_E \).

If \( t_m \notin U \) for all \( m \in \mathbb{N} \), then the assertion is clear. Otherwise, for \( m \in \mathbb{N} \) with \( t_m \in U \) we define

\[
s_m := \min \{ s \in [t, t_m [ : [t, t_m] \subseteq U \},
\]

and obtain

\[
q \left( \frac{\eta(t_m) - \eta(t)}{t_m - t} \right) = q \left( \frac{\eta(t_m)}{t_m - t} \right) = \frac{1}{t_m - t} q (\eta(t_m)) \leq \frac{1}{t_m - s_m} q (\eta(t_m)) = \frac{1}{t_m - s_m} q (\eta(t_m) - \eta(s_m)).
\]

In the last step we used the fact that \( s_m \notin U \), whence \( \eta(s_m) = 0 \). Since \( ]s_m, t_m[ \subseteq U \), we have

\[
\eta |_{]s_m, t_m[} = \gamma |_{]s_m, t_m[},
\]

thus \( \eta |_{]s_m, t_m[} \) is \( C^1 \) and \( \left( \eta |_{]s_m, t_m[} \right)' = \left( \gamma |_{]s_m, t_m[} \right)' \) admits the continuous extension \( \eta |_{]s_m, t_m[} \). Lemma 3.3.1 yields that \( \eta |_{]s_m, t_m[} \) is a \( C^1 \)-curve with the derivative \( \left( \eta |_{]s_m, t_m[} \right)' = \eta |_{]s_m, t_m[} \). Using the first part of the Fundamental Theorem of Calculus (see (10)), we obtain

\[
\frac{1}{t_m - s_m} q (\eta(t_m) - \eta(s_m)) = \frac{1}{t_m - s_m} q \left( \int_{s_m}^{t_m} \eta_1(u) du \right).
\]

Further, we have

\[
\frac{1}{t_m - s_m} q \left( \int_{s_m}^{t_m} \eta_1(u) du \right) \leq \frac{t_m - s_m}{t_m - s_m} \min_{u \in [s_m, t_m]} q(\eta_1(u)) = q(\eta_1(S_m)),
\]

for a suitable \( S_m \in ]s_m, t_m[ \subseteq ]t, t_m[ \) (see again (10)). But for \( m \to \infty \) we have \( S_m \to t \), whence

\[
\lim_{m \to \infty} q(\eta_1(S_m)) = q(\eta_1(t)) = 0,
\]

as required. Consequently, the map \( \eta \) is \( C^1 \) and \( \eta' = \eta_1 \).

Step 2: \( n \in \mathbb{N} \). Now, let \( U \subseteq \mathbb{R}^n \) be an open subset and \( \gamma : U \to E \) be partially \( C^1 \) such that \( \gamma \) admits the continuous extension

\[
\eta : \mathbb{R}^n \to E, \quad \eta(x) := \begin{cases} 
\gamma(x) & x \in U \\
0 & x \notin U 
\end{cases}
\]

[32]
and for each $i \in \{1, \ldots, n\}$ the partial derivative $\frac{\partial}{\partial x_i} \gamma$ admits the continuous extension

$$
\eta_{e_i} : \mathbb{R}^n \to E, \quad \eta_{e_i}(x) := \begin{cases} 
\frac{\partial}{\partial x_i} \gamma(x) & x \in U \\
0 & x \notin U.
\end{cases}
$$

For $x \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$ we define the continuous maps

$$
\xi : \mathbb{R} \to E, \quad \xi(t) := \eta(x + te_i)
$$

and

$$
\xi_i : \mathbb{R} \to E, \quad \xi_i(t) := \eta_{e_i}(x + te_i).
$$

Further, we define the set

$$
V := \{ t \in \mathbb{R} : x + te_i \in U \},
$$

which is open in $\mathbb{R}$. For $t \in V$ we obtain

$$
\xi(t) \overset{def}{=} \eta(x + te_i) = \gamma(x + te_i),
$$

thus $\xi|_V$ is $C^1$ with the derivative

$$
(\xi|_V)'(t) = \frac{\partial}{\partial x_i} \gamma(x + te_i) = \eta_{e_i}(x + te_i) = \xi_i(t).
$$

If $t \notin V$, then we have

$$
\xi(t) \overset{def}{=} \eta(x + te_i) = 0
$$

and

$$
\xi_i(t) \overset{def}{=} \eta_{e_i}(x + te_i) = 0.
$$

Thus, using the result of the first step of the proof, we know that $\xi$ is a $C^1$-map with the derivative $\xi' = \xi_i$. Hence we have

$$
\lim_{t \to 0} \frac{\eta(x + te_i) - \eta(x)}{t} = \lim_{t \to 0} \frac{\xi(t) - \xi(0)}{t} = \xi'(0) = \xi_i(0) = \eta_{e_i}(x).
$$

Therefore, the map $\eta$ is partially $C^1$ with $\frac{\partial}{\partial x_i} \eta = \eta_{e_i}$ for each $i \in \{1, \ldots, n\}$.

**Induction step.** We assume that $\gamma : U \to E$ is a partially $C^k$ map on an open subset $U \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}$ and $k \geq 2$, and that the continuous extensions $\eta$ and $\eta_{e_{\alpha}}$ defined in 9 and 10 exist for all $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq k$. Since $\gamma$ is partially $C^1$, the map $\eta$ is partially $C^1$ with the partial derivatives $\frac{\partial}{\partial x_i} \eta = \eta_{e_i}$ for all $i \in \{1, \ldots, n\}$. By assumption, there exist continuous maps

$$
\eta_{\beta + e_i} : \mathbb{R}^n \to E, \quad \eta_{\beta + e_i}(x) := \begin{cases} 
\frac{\partial^{\beta + e_i}}{\partial x^{\beta + e_i}} \gamma(x) & x \in U \\
0 & x \notin U
\end{cases}
$$

with $\beta \in \mathbb{N}^n_0$ and $\beta \leq k$. Then we have $\eta_{\beta + e_i}(x) = \eta_{e_i}(x)$ for all $x \in \overline{U}$. Consequently, $\eta_{\beta + e_i}$ is a partially $C^k$ map on $U$. Thus, by induction, $\eta_{\beta + e_i}$ is a partially $C^k$ map on $U$. Therefore, the map $\eta$ is partially $C^k$ with $\frac{\partial}{\partial x_i} \eta = \eta_{e_i}$ for each $i \in \{1, \ldots, n\}$. Hence we have

$$
\lim_{t \to 0} \frac{\eta(x + te_i) - \eta(x)}{t} = \lim_{t \to 0} \frac{\xi(t) - \xi(0)}{t} = \xi'(0) = \xi_i(0) = \eta_{e_i}(x).
$$

Therefore, the map $\eta$ is partially $C^k$ with $\frac{\partial}{\partial x_i} \eta = \eta_{e_i}$ for each $i \in \{1, \ldots, n\}$. Hence we have

$$
\lim_{t \to 0} \frac{\eta(x + te_i) - \eta(x)}{t} = \lim_{t \to 0} \frac{\xi(t) - \xi(0)}{t} = \xi'(0) = \xi_i(0) = \eta_{e_i}(x).
$$
for each $i \in \{1, \ldots, n\}$ and $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq k - 1$ (we may assume that $|\alpha| \neq 0$). Since each $\frac{\partial^\beta}{\partial x^\beta} \gamma$ is partially $C^{k-1}$ (by definition) with the continuous extension $\eta_{c_{i}}$, and the extensions $\eta_{c_{i}+e_{i}}$ of its partial derivatives, we conclude that $\eta_{c_{i}}$ is partially $C^{k-1}$ with $\frac{\partial^\beta}{\partial x^\beta} \eta_{c_{i}} = \eta_{c_{i}+e_{i}}$, by induction. Thus, $\eta$ is partially $C^k$, since $\eta$ is partially $C^1$ and each $\eta_{c_{i}} = \frac{\partial^\beta}{\partial x^\beta} \eta$ is partially $C^{k-1}$, and we have
\[
\frac{\partial^\alpha}{\partial x^\alpha} \eta = \frac{\partial^{\beta+e_{i}}}{\partial x^{\beta+e_{i}}} \eta = \frac{\partial^{\beta}}{\partial x^{\beta}} \frac{\partial}{\partial x_{i}} \eta = \frac{\partial^{\beta}}{\partial x^{\beta}} \eta_{c_{i}} = \eta_{c_{i}+e_{i}} = \eta_{c_{i}}
\]
for each $\alpha \in \mathbb{N}_0^n$ with $0 < |\alpha| \leq k$. □

**Lemma 3.33.** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ be an open subset and $k \in \mathbb{N}$. Let $\gamma : \mathbb{R}^n \rightarrow E$ be a partially $C^k$-map such that for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ and $y \notin U$ we have $\gamma(y) = 0 = \frac{\partial^\alpha}{\partial x^\alpha} \gamma(y)$. For each seminorm $q \in \mathcal{P}_E$, $x \in U$ and $\overline{y} \in \partial U$ there exists $\xi \in [0, 1]$ such that
\[
q(\gamma(x)) \leq \frac{1}{(k-1)!} \|x - \overline{y}\|_\infty^k \sum_{|\alpha|=k} q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x + \xi(x - \overline{y})) \right).
\]

**Proof.** We define the curve
\[
h : [0, 1] \rightarrow E, \quad h(t) := \gamma(\overline{y} + t(x - \overline{y})),
\]
which is $C^k$ with derivatives
\[
h^{(j)}(t) = \sum_{i_1, \ldots, i_j=1}^n (x_{i_1} - \overline{y}_{i_1}) \cdots (x_{i_j} - \overline{y}_{i_j}) \frac{\partial^j}{\partial x_{i_1} \cdots \partial x_{i_j}} \gamma(\overline{y} + t(x - \overline{y})) \quad (12)
\]
for each $j \leq k$. (We have $h^{(j)}(t) = d^{(j)} \gamma(\overline{y} + t(x - \overline{y}), x - \overline{y}, \ldots, x - \overline{y})$.) Thus we have
\[
h(1) = \gamma(x) \quad \text{and} \quad h^{(j)}(0) = 0 \quad (13)
\]
for all $j \in \mathbb{N}_0$ such that $j \leq k$. By Hahn-Banach, for each $z \in E$ there is $\lambda \in E'$ such that $\lambda(z) = q(z)$ and $|\lambda(z')| \leq q(z')$ for all $z' \in E$. Therefore, with $z := h(1)$ we have
\[
q(h(1)) = (\lambda \circ h)(1) = \sum_{j=0}^{k-1} \frac{1}{j!} (\lambda \circ h)^{(j)}(0) + \int_0^1 \frac{1}{(k-1)!} (\lambda \circ h)^{(k)}(s)ds.
\]
(cf. Taylor’s Theorem in [22, 10.15]). Applying Lemma 3.11 we get
\[
(\lambda \circ h)^{(j)}(0) = \lambda(h^{(j)}(0)) = 0, \quad (14)
\]
by (13), thus we have
\[
q(h(1)) = \int_0^1 \frac{1}{(k-1)!} (\lambda(h^{(k)}(s))ds.
\]
The Mean Value Theorem yields $\xi \in [0, 1]$ such that
\[
\int_0^1 \frac{(1 - s)^{k-1}}{(k-1)!} \lambda(h^{(k)}(s)) ds = \frac{(1 - \xi)^{k-1}}{(k-1)!} \lambda(h^{(k)}(\xi)) \leq \frac{1}{(k-1)!} \lambda(h^{(k)}(\xi)),
\]
whence we have
\[
q(h(1)) \leq \frac{1}{(k-1)!} \lambda(h^{(k)}(\xi)) \leq \frac{1}{(k-1)!} q(h^{(k)}(\xi)).
\]
Finally, we apply (12) and get
\[
q(\gamma(x)) = q(h(1)) \leq \frac{1}{(k-1)!} \|x - \bar{x}\|_\infty \sum_{i_1, \ldots, i_k=1}^n |x_{i_1} - \bar{x}_{i_1}| \cdots |x_{i_k} - \bar{x}_{i_k}| q \left( \frac{\partial^k}{\partial x_{i_k} \cdots \partial x_{i_1}} \gamma(\overline{x} + \xi(x - \overline{x})) \right)
\]
\[
\leq \frac{1}{(k-1)!} \|x - \bar{x}\|_\infty \sum_{i_1, \ldots, i_k=1}^n q \left( \frac{\partial^k}{\partial x_{i_k} \cdots \partial x_{i_1}} \gamma(\overline{x} + \xi(x - \overline{x})) \right)
\]
\[
= \frac{1}{(k-1)!} \|x - \bar{x}\|_\infty \sum_{|\alpha|=k} |\alpha| q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(\overline{x} + \xi(x - \overline{x})) \right),
\]
as asserted. \qed

Further we will work with some so-called mollifiers.

**Remark 3.34.** We recall that a mollifier is a function
\[
\rho_\varepsilon : \mathbb{R}^n \to \mathbb{R}, \quad \rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \rho \left( \frac{x}{\varepsilon} \right),
\]
where \(\varepsilon > 0\) and

(i) \(\rho \in C^\infty(\mathbb{R}^n, \mathbb{R})\),

(ii) \(\rho(x) \geq 0\) for all \(x \in \mathbb{R}^n\),

(iii) \(\text{supp}(\rho) \subseteq B_1(0)\),

(iv) \(\int_{\mathbb{R}^n} \rho(x) d\lambda_n(x) = 1\).

It is known that (i), (ii) and (iv) also hold for \(\rho_\varepsilon\) and

\[
\text{supp}(\rho_\varepsilon) \subseteq B_\varepsilon(0).
\]

**Lemma 3.35.** Let \(B \subseteq \mathbb{R}^n\) be a Borel set and \(\varepsilon > 0\). For the map
\[
g : \mathbb{R}^n \to [0, 1], \quad g(x) := (I_B * \rho_\varepsilon)(x),
\]
where \(*\) means the convolution, we have
\[
\left\| \frac{\partial^\alpha}{\partial x^\alpha} g \right\|_{L^1} \leq \frac{1}{\varepsilon^{|\alpha|}} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \rho \right\|_{L^1}
\]
for each \(\alpha \in \mathbb{N}_0^n\).
We define the map \( \frac{\partial^\alpha}{\partial x^\alpha} g = \frac{\partial^\alpha}{\partial x^\alpha} (I_B * \rho_\varepsilon) = I_B * \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon, \) and for \( x \in \mathbb{R}^n \) we have

\[
\left| \left( I_B * \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon \right)(x) \right| = \left| \int_{\mathbb{R}^n} I_B(x-y) \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon(y) d\lambda_n(y) \right| \\
\leq \int_{\mathbb{R}^n} \left| I_B(x-y) \right| \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon(y) \right| d\lambda_n(y) \\
\leq \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon(y) \right| d\lambda_n(y).
\]

Using the definition of \( \rho_\varepsilon \), we obtain

\[
\frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon(y) = \frac{1}{\varepsilon^n} \frac{1}{|\alpha|!} \frac{\partial^\alpha}{\partial x^\alpha} \rho \left( \frac{y}{\varepsilon} \right),
\]

whence

\[
\int_{\mathbb{R}^n} \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon(y) \right| d\lambda_n(y) = \frac{1}{\varepsilon^n} \frac{1}{|\alpha|!} \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho \left( \frac{y}{\varepsilon} \right) \right| d\lambda_n(y).
\]

After the substitution \( u = \frac{y}{\varepsilon} \) we get

\[
\frac{1}{\varepsilon^n} \frac{1}{|\alpha|!} \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho \left( \frac{y}{\varepsilon} \right) \right| d\lambda_n(y) = \frac{1}{\varepsilon^n} \frac{1}{|\alpha|!} \int_{\mathbb{R}^n} \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho(u) \right| d\lambda_n(u) \leq \frac{1}{\varepsilon^n} \frac{1}{|\alpha|!} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \rho \right\|_{L^1}.
\]

Therefore, we have

\[
\left\| \frac{\partial^\alpha}{\partial x^\alpha} g \right\| = \sup_{x \in \mathbb{R}^n} \left| \left( I_B * \frac{\partial^\alpha}{\partial x^\alpha} \rho_\varepsilon \right)(x) \right| \leq \frac{1}{\varepsilon^n} \frac{1}{|\alpha|!} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \rho \right\|_{L^1},
\]

as asserted. \( \square \)

Now, we have all the tools together to prove Proposition 3.26.

Proof of Proposition 3.26. Step 1. First we show that the set

\[
M := \left\{ \eta \in C^k_{W}(U \times V, E) : (\exists r > 0) (\forall(x, y) \in U \times V) \right. \\
\left. \| (x, y) \|_\infty > r \Rightarrow \eta(x, y) = 0 \right\}
\]

is dense in \( C^k_{W}(U \times V, E) \). To this end, let \( \gamma \in C^k_{W}(U \times V, E) \) and \( r_1 > 0 \).

Step 1.1. We define the map

\[
\eta_{r_1} : U \times V \to [0, 1], \quad (x, y) \mapsto \left( I_{B_{r_1 + \frac{1}{2} |\alpha|!}} * \rho_{\varepsilon, \frac{1}{2} |\alpha|!} \right)(x),
\]

using the function \( I_{B_{r_1 + \frac{1}{2} |\alpha|!}} : \mathbb{R}^n \to \{0, 1\} \) and the mollifier \( \rho_{\varepsilon, \frac{1}{2} |\alpha|!} : \mathbb{R}^n \to \mathbb{R} \). Note that \( \eta_{r_1}(x, y) = 0 \) if \( \|x\|_\infty > r_1 + 1 \). Since \( \eta_{r_1} \) is partially \( C^\infty \), the map \( \eta_{r_1} \cdot \gamma \)
is partially $C^{k,l}$ (see Remark 3.18 (ii)). To see that $\eta_{r_1} \cdot \gamma \in C^{k,l}_W(U \times V, E)$, let $f = f_1 \otimes f_2 \in W$, $q \in P_E$, $\alpha, \beta \in \mathbb{N}_0^m$ with $|\alpha| \leq k$, $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq l$ and $(x, y) \in U \times V$. We have

$$f_1(x)f_2(y)q\left(\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta}(\eta_{r_1} \cdot \gamma)(x, y)\right)$$

$$= f_1(x)f_2(y)q\left(\sum_{\tau \leq \alpha} \frac{\partial^\alpha}{\partial x^\alpha} \eta_{r_1}(x, y) \frac{\partial^\tau}{\partial x^\tau} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y)\right)$$

$$\leq \sum_{\tau \leq \alpha} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \eta_{r_1}(x, y) \right\| f_1(x)f_2(y)q\left(\frac{\partial^\tau}{\partial x^\tau} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y)\right),$$

because $\frac{\partial^\alpha}{\partial y^\beta} \eta_{r_1}(x, y) = 0$ for all $\kappa \in \mathbb{N}_0^m$ with $\kappa < \beta$.

From Lemma 3.35 it follows that

$$\left\| \frac{\partial^\alpha}{\partial x^\alpha} \eta_{r_1}(x, y) \right\| \leq 2^{|\alpha|-|\tau|} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \right\|_{L_1} =: C_1(\alpha, \tau),$$

whence

$$\|\eta_{r_1} \cdot \gamma\|_{f, (\alpha, \beta), q} \overset{def}{=} \sup_{(x, y) \in U \times V} f_1(x)f_2(y)q\left(\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta}(\eta_{r_1} \cdot \gamma)(x, y)\right)$$

$$\leq \sum_{\tau \leq \alpha} C_1(\alpha, \tau) \|\gamma\|_{f, (\tau, \beta), q} < \infty.$$

Thus $\eta_{r_1} \cdot \gamma \in C^{k,l}_W(U \times V, E)$. Now, we want to show that

$$\|\gamma - \eta_{r_1} \cdot \gamma\|_{f, (\alpha, \beta), q} \rightarrow 0$$

as $r_1 \rightarrow \infty$. The proof is by contradiction. If (14) is false, then for some $f \in W$, $q \in P_E$ and $\alpha, \beta$ as above there exist $\delta > 0$ and a sequence $(s_j)_{j \in \mathbb{N}}$ consisting of values of $r_1$ such that $s_j \rightarrow \infty$ as $j \rightarrow \infty$ and

$$\|\gamma - \eta_{s_j} \cdot \gamma\|_{f, (\alpha, \beta), q} > \delta$$

for each $j \in \mathbb{N}$. Since the seminorm $\|\cdot\|_{f, (\alpha, \beta), q}$ is defined as a supremum, there exist $(x_j, y_j) \in U \times V$ such that

$$f_1(x_j)f_2(y_j)q\left(\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta}(\gamma - \eta_{s_j} \cdot \gamma)(x_j, y_j)\right) > \delta.$$  

(15)

By construction, $\eta_{s_j}(x, y) = 1$ if $\|x\|_\infty \leq s_j$, whence

$$q\left(\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta}(\gamma - \eta_{s_j} \cdot \gamma)(x, y)\right) = 0$$

in this case. Thus, we must have $\|x\|_\infty > s_j$, and hence $\|x\|_\infty \rightarrow \infty$ as $j \rightarrow \infty$. Further, we have
Now, we define the map
\( f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (\gamma - \eta_{s_j}) \right)(x_j, y_j) \)
\( \leq f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x_j, y_j) \right) \)
\( + f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (\eta_{s_j} \cdot \gamma) \right)(x_j, y_j) \)
\( \leq f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x_j, y_j) \right) \)
\( + \sum_{\tau \leq \alpha} \binom{\alpha}{\tau} C_1(\alpha, \tau)f_1(x_j)f_2(y_j)q \left( \frac{\partial^\tau}{\partial x^\tau} \frac{\partial^\beta}{\partial y^\beta} \gamma(x_j, y_j) \right) . \)

Since \( U \) is open and \( W_1 \) satisfies the \( o \)-condition, Lemma 3.30(i) yields
\( f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x_j, y_j) \right) \rightarrow 0 \)
and
\( f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x_j, y_j) \right) \rightarrow 0, \)
uniformly in \( y_j \), as \( j \rightarrow \infty \). This contradicts \( \text{L} \), as required.

**Step 2.** Now, we define the map
\( \zeta_{r_1} : U \times V \rightarrow [0, 1], \quad (x, y) \mapsto \left( \mathbb{1}_{B_{r_1} \times \{0\}} * \rho_{2^k} \right)(y), \)
using the function \( \mathbb{1}_{B_{r_1} \times \{0\}} : \mathbb{R}^m \rightarrow [0, 1] \) and the mollifier \( \rho_{2^k} : \mathbb{R}^m \rightarrow \mathbb{R} \).

Thus \( \zeta_{r_1}(x, y) = 0 \) if \( \|y\|_\infty > r_1 + 1 \). Proceeding similarly to Step 1 (and using that \( V \) is open and \( W_2 \) satisfies the \( o \)-condition), we show that
\( \| \gamma - \zeta_{r_1} \cdot \gamma \|_{f, (\alpha, \beta), q} \rightarrow 0 \)
as \( r_1 \rightarrow \infty \), for each weight \( f = f_1 \otimes f_2 \in W, q \in \mathcal{P}_E, \alpha \in \mathbb{N}_0^m \) with \( |\alpha| \leq k \) and \( \beta \in \mathbb{N}_0^m \) with \( |\beta| \leq t \).

**Conclusion of Step 1.** In Step 1.1, we approximated \( \gamma \in C^k_W(U \times V, E) \) with a function \( \eta \in C^{k-1}_W(U \times V, E) \) such that \( \eta(x, y) = 0 \) if \( \|x\|_\infty > r \) for some \( r > 0 \). Further, from Step 1.2 we conclude that \( \eta \) can be approximated with a function \( \zeta \in C^k_W(U \times V, E) \) such that \( \zeta(x, y) = 0 \) if \( \|x\|_\infty > r \) or \( \|y\|_\infty > r \) for some \( r > 0 \). That is \( \zeta \in M \), whence \( M \) is dense in \( C^k_W(U \times V, E) \), as asserted.

**Step 2.** Now, let \( \gamma \in M \), that is, \( \gamma \in C^k_W(U \times V, E) \) and \( \gamma(x, y) = 0 \) if \( \|x\|_\infty > r \) for some \( r > 0 \).

**Step 2.1.** If \( U = \mathbb{R}^n \), then we jump to Step 2.2. Otherwise, for \( \varepsilon > 0 \) we define the set
\( U_\varepsilon := \{ x \in U : d(x, \partial U) \geq \varepsilon \} . \)

Further, for \( r_2 > 0 \) we define the map
\[ \eta_{r_2} : U \times V \to [0, 1], \quad (x, y) \mapsto \left( \mathbb{I}_{U \times V} \ast \rho_{\eta_{r_2}} \right)(x), \]

using the functions \( \mathbb{I}_{U \times V} : \mathbb{R}^n \to \{0, 1\} \) and \( \rho_{\eta_{r_2}} : \mathbb{R}^n \to \mathbb{R} \). (Note that \( \eta_{r_2}(x, y) = 0 \) if \( d(x, \partial U) < \frac{1}{r_2} \).) Then \( \eta_{r_2} \) is partially \( C^\infty \), whence the map \( \eta_{r_2} \cdot \gamma \) is partially \( C^{k,l} \) (see Remark 3.18 (ii)). We want to show that

\[ \| \gamma - \eta_{r_2} \cdot \gamma \|_{f,(\alpha,\beta),q} \to 0 \]  \hfill (16)

as \( r_2 \to \infty \), for each weight \( f = f_1 \otimes f_2 \in \mathcal{W} \), seminorm \( q \in \mathcal{P}_E \) and \( \alpha \in \mathbb{N}_0^k \) with \( |\alpha| \leq k , \beta \in \mathbb{N}_0^m \) with \( |\beta| \leq l \). The proof is by contradiction. If \( \| \gamma - \eta_{r_2} \cdot \gamma \|_{f,(\alpha,\beta),q} \to 0 \) is wrong, then we find \( \delta > 0 \) and a sequence \( (s_j)_{j \in \mathbb{N}} \) of values of \( r_j \) such that \( s_j \to \infty \) as \( j \to \infty \) and for each \( j \in \mathbb{N} \) we have

\[ \| \gamma - \eta_{s_j} \cdot \gamma \|_{f,(\alpha,\beta),q} > \delta. \]

By definition of the seminorm \( \| \cdot \|_{f,(\alpha,\beta),q} \) as a supremum, there exist \((x_j, y_j) \in U \times V\) such that

\[ f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (\gamma - \eta_{s_j} \cdot \gamma)(x_j, y_j) \right) > \delta. \]  \hfill (17)

Since \( \gamma \in M \), we have \( (\gamma - \eta_{s_j} \cdot \gamma)(x, y) = 0 \) for all \((x, y) \in U \times V\) with \( \|(x, y)\|_\infty > r \), therefore we must have \((x_j, y_j) \in B_r(0)\). Thus, after passage to a subsequence, we may assume \((x_j, y_j) \to (\overline{x}, \overline{y}) \in U \times V\) as \( j \to \infty \). Further, we have \( \eta_{s_j}(x, y) = 1 \) for all \( x \in U \setminus \overline{\partial}_j \), and hence \( (\gamma - \eta_{s_j} \cdot \gamma)(x, y) = 0 \) in this case. Thus we conclude that \( d(x_j, \partial U) < \frac{1}{s_j} \), whence \( d(\overline{x}, \partial U) = 0 \), that is \( \overline{x} \in \partial U \).

We have

\[ f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (\gamma - \eta_{s_j} \cdot \gamma)(x_j, y_j) \right) \]

\[ \leq f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (\gamma)(x_j, y_j) \right) \]

\[ + f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} (\eta_{s_j} \cdot \gamma)(x_j, y_j) \right) \]  \hfill (18)

If \( j \to \infty \), then in \( \mathbb{L} \) we have

\[ f_1(x_j)f_2(y_j)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x_j, y_j) \right) \to 0 \]

uniformly in \( y_j \), by Lemma 3.30 (i), since \( U \) is an open subset and \( \mathcal{W}_1 \) satisfies the \( \alpha \)-condition. We apply the Product Rule to \( \mathbb{L} \) and obtain
\[ f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^3}{\partial y^3} (\eta, \gamma)(x_j, y_j) \right) \]
\[ = f_1(x_j)f_2(y_j)q \left( \sum_{\tau \leq \alpha} \left( \frac{\partial^{\alpha-\tau}}{\partial x^{\alpha-\tau}} \eta_{\tau}(x_j, y_j) \frac{\partial^{\tau}}{\partial x^{\tau}} \frac{\partial^3}{\partial y^3} \gamma(x_j, y_j) \right) \right) \]
\[ \leq \sum_{\tau \leq \alpha} \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \right)^{\alpha} \eta_{\tau}(x_j, y_j) \left( f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\tau}}{\partial x^{\tau}} \frac{\partial^3}{\partial y^3} \gamma(x_j, y_j) \right) \right) , \]
using that \( \frac{\partial^{\beta-\kappa}}{\partial y^{\beta-\kappa}} \eta_{\kappa}(x_j, y_j) = 0 \) for all \( \kappa \in \mathbb{N}_0^{\beta} \) with \( \kappa < \beta \).
From Lemma 3.35 we conclude
\[ f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^3}{\partial y^3} (\eta, \gamma)(x_j, y_j) \right) \]
\[ = \sum_{\tau \leq \alpha} \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \right)^{\alpha} C_2(\alpha, \tau) s_j^{[|\alpha| - |\tau|]} \left( f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\tau}}{\partial x^{\tau}} \frac{\partial^3}{\partial y^3} \gamma(x_j, y_j) \right) \right) \]
with
\[ C_2(\alpha, \tau) := 4^{[|\alpha| - |\tau|]} \left\| \frac{\partial^{\alpha-\tau}}{\partial x^{\alpha-\tau}} \right\|_{L^1} . \]
Now, we want to show that the term
\[ s_j^{[|\alpha| - |\tau|]} f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^3}{\partial y^3} (\eta, \gamma)(x_j, y_j) \right) \]
(20)
tends to 0 as \( j \to \infty \), for all \( \tau \leq \alpha \) (which yields the desired contradiction). If \( \tau = \alpha \), then in (20) we have
\[ s_j^{[|\alpha| - |\tau|]} f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^3}{\partial y^3} \gamma(x_j, y_j) \right) \to 0 \]
uniformly in \( y_j \), as \( j \to \infty \), by Lemma 3.33 (i). Otherwise, we rewrite (20) as
\[ s_j^{[|\alpha| - |\tau|]} f_1(x_j)f_2(y_j)q \left( \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^3}{\partial y^3} \gamma(x_j, y_j) \right) \]
\[ = q \left( f_1(x_j) \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left( f_2(y_j) \frac{\partial^3}{\partial y^3} \gamma(\cdot, y_j) \right)(x_j) \right) \]
\[ = q \left( f_1(x_j) \frac{\partial^{\alpha}}{\partial x^{\alpha}} \gamma_{\beta, y_j}(x_j) \right) , \]
where \( \gamma_{\beta, y_j} \) is the map defined in Lemma 3.27 We have \( \gamma_{\beta, y_j} \in C_{W_1}(U, E) \subseteq C_{\mathcal{W}_1}(U, E) \), whence
\[ \frac{\partial^{\tau}}{\partial x^{\tau}} \gamma_{\beta, y_j} \in C_{\mathcal{W}_1}^{[|\alpha| - |\tau|]}(U, E) . \]

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Using Remark 3.29 and Lemma 3.30 (i), we see that for each \( \pi \in \mathbb{N}_0^n \) with \( |\pi| \leq |\alpha| - |\tau| \) we have
\[
f_1(x_j) \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2}(x_j) \to 0
\]
and
\[
\frac{\partial^\tau}{\partial x^\tau} \left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right)(x_j) \to 0
\]
uniformly in \( y_j \), as \( j \to \infty \). Hence, Proposition 3.32 implies that the extension
\[
\left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right) : \mathbb{R}^n \to E, \quad x \mapsto \left\{ \begin{array}{ll} (f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2}) (x) & x \in U \\ 0 & x \notin U \end{array} \right.
\]
is partially \( C^{|\alpha|-|\tau|} \) with partial derivatives
\[
\frac{\partial^\tau}{\partial x^\tau} \left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right) : \mathbb{R}^n \to E, \quad x \mapsto \left\{ \begin{array}{ll} \frac{\partial^\tau}{\partial x^\tau} \left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right) (x) & x \in U \\ 0 & x \notin U. \end{array} \right.
\]
Since \( d(x_j, \partial U) < \frac{1}{s_j} \) for each \( j \in \mathbb{N} \), there exists \( \tau_j \in \partial U \) with \( d_j := d(x_j, \tau_j) < \frac{1}{s_j} \), and \( \tau_j \to \tau \) as \( j \to \infty \). Lemma 3.33 yields \( \xi_j \in [0,1] \) such that
\[
\left| \frac{s_j |\alpha|-|\tau|}{(|\alpha| - |\tau| - 1)!} \sum_{|\alpha|=|\beta|+|\gamma|} q \left( \frac{\partial^\kappa}{\partial x^\kappa} \left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right) (x_j) \right) \right|
\leq \frac{1}{(|\alpha| - |\tau| - 1)!} \sum_{|\alpha|=|\beta|+|\gamma|} q \left( \frac{\partial^\kappa}{\partial x^\kappa} \left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right) (x_j) \right)
\]
with \( x_j' := \tau_j - \xi_j(x_j - \tau_j) \). Thus we have
\[
q \left( \frac{\partial^\kappa}{\partial x^\kappa} \left( f_1 \cdot \frac{\partial^\tau}{\partial x^\tau} \gamma_{\beta,y_j,f_2} \right) (x_j') \right) \to 0
\]
if \( j \to \infty \) (i.e., \( x_j, \tau_j \to \tau \), whence \( x_j' \to \tau \)).

Altogether, we have
\[
f_1(x_j) f_2(y_j) q \left( \frac{\partial^\kappa}{\partial x^\kappa} \frac{\partial^\beta}{\partial y^\beta} (\gamma - \eta_{y_j} \cdot \gamma)(x_j, y_j) \right) \to 0
\]
as \( j \to \infty \), which contradicts (14), as desired.

Step 2.2. If \( V = \mathbb{R}^m \), then we go over to the conclusion of Step 2. Otherwise, we define the set \( V_{\xi} \) and the partially \( C^\infty \)-map
\[
\zeta_{r_2} : U \times V \to [0,1], \quad (x, y) \mapsto \left( 1_{V_{\xi} x} * \rho_{1_{\tau_2}} \right)(y)
\]
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as in Step 2.1 (note that $\zeta_{r_2}(x, y) = 0$ if $d(y, \partial V) < \frac{1}{2r_2}$) and proceed similarly. This yields

$$\|\gamma - \zeta_{r_2} \cdot \gamma\|_{f,(\alpha,\beta),q} \to 0$$

as $r_2 \to \infty$, for each weight $f = f_1 \otimes f_2 \in \mathcal{W}$, seminorm $q \in \mathcal{P}_E$ and $\alpha \in \mathbb{N}_0^m$ with $|\alpha| \leq k$, $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq l$.

Conclusion of Step 2. We conclude that the space $C^k_{\mathbb{R}}(U \times V, E)$ is dense in $M$.

In fact, if $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$, then the assertion is obviously true. Consider the case $U \subsetneq \mathbb{R}^n$ and $V \subsetneq \mathbb{R}^m$. In Step 2.1 we approximated $\gamma \in M$ with a partially $C^k_{\mathbb{R}}$-map $\eta$ such that

$$\text{supp}(\eta) \subseteq \{(x, y) \in U \times V : (\exists r, s > 0) \quad \|\gamma(r, y)\|_{\infty} \leq r \text{ and } d((x, y), \partial U) \geq s\}.$$ 

In Step 2.2 we showed that $\eta$ can be approximated with a partially $C^k_{\mathbb{R}}$-map $\zeta$ such that

$$\text{supp}(\zeta) \subseteq \{(x, y) \in U \times V : (\exists r, s > 0) \quad \|\zeta(r, y)\|_{\infty} \leq r \text{ and } d((x, y), \partial(U \times V)) \geq s\} =: K.$$ 

Furthermore, we have

$$K = \{(x, y) \in \overline{U} \times \overline{V} : (\exists r, s > 0) \quad \|\gamma(r, y)\|_{\infty} \leq r \text{ and } d((x, y), \partial(U \times V)) \geq s\}. \quad (21)$$

(In fact, for $(x, y) \in (\overline{U} \times \overline{V}) \setminus (U \times V) = \overline{U} \times \overline{V} \setminus (U \times V) = \partial(U \times V)$ we have $d((x, y), \partial(U \times V)) = 0$, hence $(x, y)$ is not an element of the right-hand side of (21).) Thus $K$ is a closed subset of $(\overline{U} \times \overline{V}) \cap B_r(0)$, hence $K$ is bounded and consequently compact. Therefore $\zeta \in C^k_{\mathbb{R}}(U \times V, E)$.

If $U \subseteq \mathbb{R}^n$ and $V = \mathbb{R}^m$, then Step 2.1 yields an approximation of $\gamma \in M$ with a partially $C^k_{\mathbb{R}}$-map $\eta$ such that

$$\text{supp}(\eta) \subseteq \{(x, y) \in U \times \mathbb{R}^m : (\exists r, s > 0) \quad \|\gamma(r, y)\|_{\infty} \leq r \text{ and } d((x, y), \partial U) \geq s\} = \{(x, y) \in \overline{U} \times \mathbb{R}^m : (\exists r, s > 0) \quad \|\gamma(r, y)\|_{\infty} \leq r \text{ and } d((x, y), \partial U) \geq s\}.$$ 

(We see again that if $(x, y) \in (\overline{U} \times \mathbb{R}^m) \setminus (U \times \mathbb{R}^m) = \overline{U} \times \mathbb{R}^m \setminus (U \times \mathbb{R}^m) = \partial(U \times \mathbb{R}^m) = \partial U \times \mathbb{R}^m) \cup (\overline{U} \times \partial \mathbb{R}^m) = \partial U \times \mathbb{R}^m$, then $d((x, y), \partial U) = 0$.) Thus $\text{supp}(\eta)$ is a subset of a closed subset of $(\overline{U} \times \mathbb{R}^m) \cap B_r(0)$, and we have $\eta \in C^k_{\mathbb{R}}(U \times \mathbb{R}^m, E)$. In the case $U = \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$, the argumentation is similar, using Step 2.2. \hfill \Box

**Proposition 3.36.** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be locally convex subsets with dense interior, and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. For the set of weights $\mathcal{W}_k$ on $U$ we assume that
(i) each weight is bounded on compact subsets of $U$,

(ii) for each compact subset $K \subseteq U$ there exists a weight $f_K \in W_1$ such that $\inf_{x \in K} f_K(x) > 0$,

and likewise for the set of weights $W_2$ on $V$. Then we have $C^k_c(U \times V, E) \subseteq \text{im}(\Psi)$, where $\Psi$ is the topological embedding

$$\Psi : C^k_{W_1}(U, C^l_{W_2}(V, E)) \to C^k_c(U \times V, E), \quad \gamma \mapsto \gamma^\wedge$$

defined in Theorem 3.21.

Proof. Let $\gamma \in C^k_c(U \times V, E)$. We need to show that $\gamma^\wedge \in C^k_{W_1}(U, C^l_{W_2}(V, E))$, then by construction of $\Psi$ we have $\Psi(\gamma^\wedge) = (\gamma^\wedge)^\wedge = \gamma$, as required.

Let $\pi_1 : U \times V \to U$, $\pi_2 : U \times V \to V$ be the coordinate projections onto the first and second component, respectively. We set $K := \text{supp}(\gamma)$ and $K_1 := \pi_1(K)$, $K_2 := \pi_2(K)$. Since $K$ is compact, so is the set $K_1 \times K_2$ and $K \subseteq K_1 \times K_2$, whence $\gamma \in C^k_{K_1 \times K_2}(U \times V, E)$. Therefore, we have $\gamma^\wedge \in C^k_{K_1}(U, C^l_{K_2}(V, E))$, by Lemma 3.23. We know that $C^l_{K_2}(V, E) \subseteq C^l_{W_2}(V, E)$ and the inclusion map $i : C^l_{K_2}(V, E) \to C^l_{W_2}(V, E)$ is continuous and linear, by Lemma 3.24. Thus $\gamma^\wedge(x) = \gamma_x \in C^l_{W_2}(V, E)$ and $\gamma^\wedge = i \circ \gamma^\wedge : U \to C^l_{W_2}(V, E)$ is partially $C^k$, by Lemma 3.11. Hence $\gamma^\wedge \in C^k_{K_1}(U, C^l_{W_2}(V, E)) \subseteq C^k_{W_1}(U, C^l_{W_2}(V, E))$, as desired.

Now, we prove the Exponential Law for spaces of weighted differentiable functions with values in locally convex spaces.

**Theorem 3.37 (Exponential Law for spaces of weighted differentiable functions).** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, and $k, l \in \mathbb{N}_0 \cup \{ \infty \}$. For the set of weights $W_1 \subseteq C^k(U, [0, \infty])$ on $U$ we assume that

(i) $W_1$ satisfies the $o$-condition,

(ii) for each $f \in W_1$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ there exists $g \in W_1$ such that

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq g(x)$$

for all $x \in U$,

and likewise for the set of weights $W_2 \subseteq C^l(V, [0, \infty])$ on $V$. Then the linear map

$$\Psi : C^k_{W_1}(U, C^l_{W_2}(V, E)) \to C^k_{W}(U \times V, E), \quad \gamma \mapsto \gamma^\wedge,$$

where $W = W_1 \otimes W_2$, is a homeomorphism.
**Proof.** Step 1. First we assume that the space $E$ is complete. By Proposition 3.15 the space $C(W_1, (V, E))$ is complete, whence also the space $C(W_1, (V, E))$ is complete (we recall that the condition in Proposition 3.15 is satisfied, since the weights are assumed continuous, see Remark 2.6). The map $\Psi$ is a topological embedding, by Theorem 3.21 thus $\text{im}(\Psi)$ is complete, hence it is closed in $C^{k,l}_W(U \times V, E)$. By Proposition 3.36 we know that $C^{k,l}_W(U \times V, E) \subseteq \text{im}(\Psi)$, and in Proposition 3.26 we have shown that $C^{k,l}_W(U \times V, E)$ is dense in $C^{k,l}_W(U \times V, E)$.

Thus $\text{im}(\Psi)$ is dense in $C^{k,l}_W(U \times V, E)$, whence $\text{im}(\Psi) = C^{k,l}_W(U \times V, E)$, which shows that $\Psi$ is surjective, hence a homeomorphism (being a topological embedding).

Step 2. Now we show the surjectivity of $\Psi$ in the general case. To this end, let $\tilde{E}$ be the completion of $E$ and $\gamma \in C^{k,l}_W(U \times V, \tilde{E})$. In Step 1 we have shown that the map

$$\tilde{\Psi} : C^{k}_W(U, C^{l}_W(V, \tilde{E})) \to C^{k,l}_W(U \times V, \tilde{E}), \quad \eta \mapsto \eta^\wedge,$$

is a homeomorphism. Hence there exists $\eta \in C^{k}_W(U, C^{l}_W(V, \tilde{E}))$ with

$$\tilde{\Psi}(\eta) = \eta^\wedge = \gamma.$$  (22)

If we can show that $\eta \in C^{k}_W(U, C^{l}_W(V, E))$, then we have $\Psi(\eta) = \gamma$, and the assertion is proven. It suffices to prove that

$$\frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial^\alpha}{\partial x^\alpha} \eta(x) \right)(y) \in E$$  (23)

for all $x \in U$, $y \in V$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k$ and $\beta \in \mathbb{N}_0^m$ with $|\beta| \leq l$. From Lemma 3.10 we then conclude that

$$\frac{\partial^\alpha}{\partial x^\alpha} \eta(x) \in C^{l}_W(V, E)$$

for all $x \in U$ and $\alpha$ as above, and using the lemma again, we get

$$\eta \in C^{k}_W(U, C^{l}_W(V, E)),$$

as required. But using (22), we have

$$\frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial^\alpha}{\partial x^\alpha} \eta(x) \right)(y) = \frac{\partial^\beta}{\partial y^\beta} \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x, \cdot) \right)(y) = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} \gamma(x, y) \in E,$$

thus (23) holds and the proof is finished.  \hfill \Box

A useful consequence of the Exponential Law is the following:

**Corollary 3.38.** Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open subsets, and $k, l \in \mathbb{N}_0 \cup \{\infty\}$. For the set of weights $W_1 \subseteq C^k(U, [0, \infty])$ on $U$ we assume that

(i) $W_1$ satisfies the o-condition,
(ii) for each \( f \in \mathcal{W}_1 \) and \( \alpha \in \mathbb{N}_0^n \) with \( |\alpha| \leq k \) there exists \( g \in \mathcal{W}_1 \) such that

\[
\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq g(x)
\]

for all \( x \in U \),

and likewise for the set of weights \( \mathcal{W}_2 \subseteq \mathcal{C}^d(V,[0,\infty[) \) on \( V \). Then the linear map

\[ \Psi : \mathcal{C}^k_{\mathcal{W}_1}(U, \mathcal{C}^d_{\mathcal{W}_2}(V,E)) \to \mathcal{C}^d_{\mathcal{W}_2}(V, \mathcal{C}^k_{\mathcal{W}_1}(U,E)), \]

\[ \Psi(\gamma) := (y \mapsto (x \mapsto \gamma(x)(y))) \]

is a homeomorphism.

**Proof.** By Theorem 3.37, the map

\[ \Psi_1 : \mathcal{C}^k_{\mathcal{W}_1}(U, \mathcal{C}^d_{\mathcal{W}_2}(V,E)) \to \mathcal{C}^d_{\mathcal{W}_2 \otimes \mathcal{W}_1}(U \times V,E) \]

is a homeomorphism, and obviously so is

\[ \Psi_2 : \mathcal{C}^d_{\mathcal{W}_2 \otimes \mathcal{W}_1}(U \times V,E) \to \mathcal{C}^k_{\mathcal{W}_1}(V \times U,E), \]

\[ \Psi_2((x,y) \mapsto \gamma(x)(y)) := ((y,x) \mapsto \gamma(x)(y)). \]

Finally, using the homeomorphism

\[ \Psi_3 : \mathcal{C}^d_{\mathcal{W}_2 \otimes \mathcal{W}_1}(V \times U,E) \to \mathcal{C}^k_{\mathcal{W}_2}(V, \mathcal{C}^k_{\mathcal{W}_1}(U,E)), \]

we construct

\[ \Psi := \Psi_3 \circ \Psi_2 \circ \Psi_1 : \mathcal{C}^k_{\mathcal{W}_1}(U, \mathcal{C}^d_{\mathcal{W}_2}(V,E)) \to \mathcal{C}^k_{\mathcal{W}_2}(V, \mathcal{C}^k_{\mathcal{W}_1}(U,E)), \]

and see that

\[ \Psi(\gamma) = (\Psi_3 \circ \Psi_2 \circ \Psi_1)(x \mapsto (y \mapsto \gamma(x)(y))) \]

\[ = (\Psi_3 \circ \Psi_2)((x,y) \mapsto \gamma(x)(y)) \]

\[ = \Psi_3((y,x) \mapsto \gamma(x)(y)) = (y \mapsto (x \mapsto \gamma(x)(y))), \]

and \( \Psi \) is a homeomorphism, by construction.

Using the fact that a convex, compact subset \( K \subseteq \mathbb{R}^m \) with nonempty interior has dense interior (whence differentiability is defined on such sets), we get corresponding results for spaces of weighted \( C^{k,l} \)-functions on products of open and compact subsets, in particular:

**Proposition 3.39.** Let \( E \) be a Hausdorff locally convex space, \( U \subseteq \mathbb{R}^n \) be an open subset, \( K \subseteq \mathbb{R}^m \) be a convex, compact subset (with \( K^\circ \neq \emptyset \)) and \( k,l \in \mathbb{N}_0 \cup \{\infty\} \). Let \( \mathcal{W}_1 \subseteq \mathcal{C}^k(U,[0,\infty[ \) be a set of weights on \( U \) as in Theorem 3.37 and \( \mathcal{W}_2 \) be an arbitrary set of weights on \( K \). Then

(i) the space \( \mathcal{C}^{k,l}_c(U \times K,E) \) is dense in \( \mathcal{C}^{k,l}_{\mathcal{W}_1 \otimes \mathcal{W}_2}(U \times K,E) \)

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and

(ii) the space $C_{c}^{l,k}(K \times U, E)$ is dense in $C_{W_2 \otimes W_1}^{l,k}(K \times U, E)$.

Proof. To prove (i), we use the same arguments as in Step 1.1 in the proof of Proposition 3.26 and approximate $\gamma \in C_{W_1 \otimes W_2}^{k,l}(U \times K, E)$ with a function $\eta \in C_{W_1 \otimes W_2}^{k,l}(U \times K, E)$ such that $\eta(x,y) = 0$ if $\|x\|_{\infty} > r$ for some $r > 0$. If $U = \mathbb{R}^n$, then the proof is finished. Otherwise, we use Step 2.1 and approximate $\eta$ with a partially $C_{c}^{l,k}$-function $\zeta$ such that $\text{supp}(\zeta) \subseteq \{(x,y) \in U \times K : (\exists r, s > 0) (\|x\|_{\infty} \leq r \text{ and } d(x, \partial U) \geq s) \} = \{(x,y) \in U \times K : (\exists r, s > 0) (\|x\|_{\infty} \leq r \text{ and } d(x, \partial U) \geq s) \}$. This is a compact subset (see the conclusion of Step 2 for details), whence $\zeta \in C_{c}^{l,k}(U \times K, E)$, as required.

To prove (ii), we proceed similarly, using Step 1.2 and Step 2.2 in the proof of Proposition 3.26.

Theorem 3.40. Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ be an open subset and $K \subseteq \mathbb{R}^m$ be a convex, compact subset (with $K^\circ \neq \emptyset$). Let $k, l \in \mathbb{N}_0 \cup \{\infty\}$, $W_1 \subseteq C_{c}^{k}(U, [0, \infty[)$ be a set of weights on $U$ as in Theorem 3.37 and $W_2$ be a set of weights on $K$ such that

(i) there is a weight $h_K \in W_2$ such that $\inf_{y \in K} h_K(y) > 0$,

(ii) each weight $h \in W_2$ is bounded on $K$.

Then each of the linear maps

$$\Psi_1 : C_{W_1}^{k}(U, C_{W_2}^{l}(K, E)) \rightarrow C_{W_1 \otimes W_2}^{k,l}(U \times K, E), \quad \gamma \mapsto \gamma \wedge,$$

and

$$\Psi_2 : C_{W_2}^{l}(K, C_{W_1}^{k}(U, E)) \rightarrow C_{W_2 \otimes W_1}^{l,k}(K \times U, E), \quad \gamma \mapsto \gamma \wedge$$

is a homeomorphism.

Proof. The proposition can be proven similarly to Theorem 3.37 using Proposition 3.39 instead of Proposition 3.26.

Corollary 3.41. Let $E$ be a Hausdorff locally convex space, $U \subseteq \mathbb{R}^n$ be an open subset and $K \subseteq \mathbb{R}^m$ be a convex, compact subset (with $K^\circ \neq \emptyset$). Let $k, l \in \mathbb{N}_0 \cup \{\infty\}$, and $W_1, W_2$ be sets of weights on $U$ and $K$, respectively, as is Theorem 3.40.

Then the linear map

$$\Psi : C_{W_1}^{k}(U, C_{W_2}^{l}(K, E)) \rightarrow C_{W_2}^{l}(K, C_{W_1}^{k}(U, E)), \quad \Psi(\gamma) := (y \mapsto (x \mapsto \gamma(x)(y)))$$

is a homeomorphism.
Proof. The assertion can be proven similarly to Corollary 3.38 using Theorem 3.40 for the construction of the homeomorphism $\Psi$. □

Remark 3.42. Setting

$$W_1 := C^\infty_c(U, [0, \infty [),$$
for a locally convex subset $U \subseteq \mathbb{R}^n$, we obtain

$$C^k_{W_1}(U, E) = C^k(U, E)$$
as topological vector spaces, for all $k \in \mathbb{N}_0 \cup \{\infty\}$. In fact, since each $f \in W_1$ is continuous, the condition in Lemma 3.9 is satisfied (see Remark 2.6), whence the inclusion map

$$C^k_{W_1}(U, E) \to C^k(U, E)$$
is continuous. Conversely, if $\gamma \in C^k(U, E)$, $f \in W_1$, $\alpha \in \mathbb{N}^n_0$ with $|\alpha| \leq k$ and $q \in \mathcal{P}_E$, then

$$\|\gamma\|_{f, \alpha, q} \overset{def}{=} \sup_{x \in U} f(x)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right) = \sup_{x \in \text{supp}(f)} f(x)q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right) \leq r \sup_{x \in \text{supp}(f)} q \left( \frac{\partial^\alpha}{\partial x^\alpha} \gamma(x) \right) = r \|\gamma\|_{\text{supp}(f), \alpha, q},$$
where

$$r := \max_{x \in \text{supp}(f)} f(x) < \infty.$$Thus $\gamma \in C^k_{W_1}(U, E)$ and the inclusion map

$$C^k(U, E) \to C^k_{W_1}(U, E)$$
is continuous. Therefore, using Corollary 3.38 we know that

$$C^k(U, C^l_{W_2}(V, E)) \cong C^l_{W_2}(V, C^k(U, E))$$
for open subsets $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$, a suitable set of weights $W_2$ on $V$ and $k, l \in \mathbb{N}_0 \cup \{\infty\}$.

Moreover, using Corollary 3.41 we have

$$C^k(K, C^l_{W_2}(V, E)) \cong C^l_{W_2}(V, C^k(K, E))$$
where $K \subseteq \mathbb{R}^n$ is a convex, compact subset with $K^\circ \neq \emptyset$ and $V$, $W_2$ as above.
4 Regularity of the Lie group $C^l_W(U, H)$

In this section we show that the Exponential Law for spaces of weighted differentiable functions (in particular, Corollary 3.41) can be used to prove $C^k$-regularity of certain Lie groups. Throughout the section, we will use the following definitions:

**Definition 4.1.** A Lie group modeled on a Hausdorff locally convex space $E$ is a group $G$, equipped with a $C^\infty$-manifold structure modeled on $E$ which turns the group multiplication

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh$$

and the inversion

$$\iota : G \to G, \quad g \mapsto g^{-1}$$

into smooth maps.

We always denote by $e_G$ the neutral element of $G$ and by $g := L(G) := T_{e_G}(G)$ the Lie algebra of $G$.

**Definition 4.2.** Let $G$ be a Lie group modeled on a Hausdorff locally convex space $E$ and $g \in G$. Using the tangent map of the left translation $\lambda_g : G \to G, x \mapsto gx$, the product of $g$ and $v \in T_x(G)$ for some $x \in G$ is defined as

$$g.v := (T_x\lambda_g)(v) \in T_{gx}(G).$$

The Lie group $G$ is called $C^k$-semiregular (with $k \in \mathbb{N}_0 \cup \{\infty\}$) if for each $C^k$-curve $\gamma : [0, 1] \to g$ there exists a (unique) $C^{k+1}$-curve $\text{Evol}_G(\gamma) := \eta : [0, 1] \to G$ such that

$$\eta(0) = e_G \text{ and } \eta' = \eta \cdot \gamma.$$  

The Lie group $G$ is called $C^k$-regular if $G$ is $C^k$-semiregular and the map

$$\text{evol}_G : C^k([0, 1], G) \to G, \quad \gamma \mapsto \text{Evol}_G(\gamma)(1)$$

is smooth.

**Remark 4.3.** From [8, Lemma 3.1] follows, that a Lie group is $C^k$-regular if and only if $G$ is $C^k$-semiregular and the map

$$\text{Evol}_G : C^k([0, 1], G) \to C^{k+1}([0, 1], G), \quad \gamma \mapsto \text{Evol}_G(\gamma)$$

is smooth.

For the theory of locally convex Lie groups the reader is referred to [14], [10] and [10]. Recall that if $H$ is a locally convex Lie group, then we can turn $C^l([0, 1], H)$ into a Lie group modeled on $C^l([0, 1], L(H))$. Moreover, if $H$ is $C^k$-regular, then so is $C^l([0, 1], H)$ (more general, the last facts are true for Lie groups $C^l(K, H)$, where $K$ is a compact manifold, see [8]). By [8, Rem. 13.4], each $C^k$-regular Lie group $G$ is $C^l$-regular, for $l \geq k$. Thus $C^\infty$-regularity, simply called regularity, going back to Milnor ([14]), is the weakest concept. Constructions of Lie groups of weighted smooth Lie group-valued mappings can be found in [4]. In [21]
B. Walter describes the construction of Lie groups $C^\nu_W(U, H)$ modeled on the space $C^\nu_W(U, L(H))$, where $U$ is an open subset of a finite-dimensional space $X$ and $W$ is a set of weights on $U$ such that the o-condition is satisfied and $1_U \in W$. (In [21], the author works with spaces $C^\nu_W(U, H)^*$, but these spaces coincide with $C^\nu_W(U, H)$ if the o-condition is satisfied by $W$.) Moreover, the author shows that if $U$ is an open subset of a normed space, $1_U \in W$ and $H$ is a Banach Lie group, then the Lie group $C^\nu_W(U, H)$ is regular.

From [3], we derive the following fact:

**Lemma 4.4.** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, $\Omega \subseteq C^k([0, 1], \mathfrak{g})$ be an open 0-neighborhood and the map

$$\Theta : \Omega \to C^{k+1}([0, 1], G)$$

be smooth. If there exists a family $(\alpha_j)_{j \in J}$ of smooth homomorphisms

$$\alpha_j : G \to G_j$$

to Lie groups $G_j$ such that

$$\alpha_j \circ \Theta(\gamma) = \text{Evol}_{G_j}(T_{\epsilon \alpha_j} \circ \alpha_j \circ \gamma)$$

for each $\gamma \in \Omega$ and the maps $T_{\epsilon \alpha_j} \alpha_j$ separate points on $\mathfrak{g}$, then $G$ is $C^k$-regular and $\Theta = \text{Evol}_G$.

Now, we show the $C^k$-regularity of the Lie group $C^\nu_W(U, H)$, checking that in the following situation all the hypotheses of Lemma 4.4 are satisfied.

**Theorem 4.5.** Let $E$ be a Hausdorff locally convex space and $H$ be a $C^k$-regular Lie group modeled on $E$, with $k \in \mathbb{N}_0 \cup \{\infty\}$. Let $W \subseteq C^\nu(U, [0, \infty))$ be a set of weights on an open subset $U \subseteq \mathbb{R}^n$ such that

(i) $1_U \in W$,

(ii) $W$ satisfies the o-condition,

(iii) for each $f \in W$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$ there exists $g \in W$ such that

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} f(x) \right| \leq g(x)$$

for all $x \in U$.

Then the Lie group $G := C^\nu_W(U, H)$ is $C^k$-regular for each $l \in \mathbb{N}_0 \cup \{\infty\}$.

**Proof.** Let $\mathfrak{h}$ be the Lie algebra of $H$. For an open $\epsilon_H$-neighborhood $V \subseteq H$ and an open 0-neighborhood $W \subseteq \mathfrak{h}$ there exists a $C^\infty$-diffeomorphism $\phi : V \to W$ such that

$$d\phi|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}} \quad \text{and} \quad \phi(\epsilon_H) = 0.$$

Consider the evaluation maps

$$\varepsilon_x : G = C^\nu_W(U, H) \to H, \quad \gamma \mapsto \gamma(x)$$
and

\[ \text{ev}_x : \mathfrak{g} \cong C^l_W(U, \mathfrak{h}) \to \mathfrak{h}, \quad \gamma \mapsto \gamma(x) \]

for \( x \in U \). We have a commutative diagram

\[
\begin{array}{ccc}
C^l_W(U, V) & \xrightarrow{C^l_W(U, \phi)} & C^l_W(U, W) \\
\varepsilon_x |_{C^l_W(U, V)} & & \ev_x |_{C^l_W(U, W)} \\
\phi & & W
\end{array}
\]

where

\[ C^l_W(U, \phi) : C^l_W(U, V) \to C^l_W(U, W) \]

is a \( C^\infty \)-diffeomorphism and \( \text{ev}_x \) is continuous and linear, hence \( C^\infty \) (cf. [10]). Therefore

\[ \varepsilon_x |_{C^l_W(U, V)} = \phi^{-1} \circ \ev_x |_{C^l_W(U, W)} \circ C^l_W(U, \phi) \]

and \( \varepsilon_x \) is \( C^\infty \). Moreover, the diagram

\[
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{dC^l_W(U, \phi)} & C^l_W(U, \mathfrak{h}) \\
\varepsilon_x |_{C^l_W(U, V)} & & \ev_x \\
\phi & & \mathfrak{h}
\end{array}
\]

commutes, thus we identify each \( T_{\varepsilon_x} \varepsilon_x \) with \( \ev_x \) (as \( dC^l_W(U, \phi) |_{\mathfrak{g}} \) is an isomorphism), which separates points on \( \mathfrak{g} \).

The Lie group \( H \) was assumed \( C^k \)-regular, thus the map

\[ \text{Evol}_H : C^k([0,1], \mathfrak{h}) \to C^{k+1}([0,1], H) \]

is \( C^\infty \) (by Remark 4.3), and we have \( \text{Evol}_H(0)(t) = e_H \) for each \( t \in [0,1] \).

Therefore, there exists an open 0-neighborhood \( Q \) in \( C^k([0,1], \mathfrak{h}) \) such that

\[ \text{Evol}_H(Q) \subseteq C^{k+1}([0,1], V). \]

Setting

\[ \Omega := \Psi_1^{-1}(C^l_W(U, Q)), \]

where

\[ \Psi_1 : C^k([0,1], C^l_W(U, \mathfrak{h})) \to C^l_W(U, C^k([0,1], \mathfrak{h})) \]

is the homeomorphism from Corollary 5.4, we obtain an open 0-neighborhood \( \Omega \) in \( C^k([0,1], C^l_W(U, \mathfrak{h})) \). We now construct the \( C^\infty \)-map

\[ \Theta : \Omega \to C^{k+1}([0,1], C^l_W(U, H)) \]
as follows. The map

$$\Theta_1 := \Psi_1|_{\Omega} : \Omega \to C^d_W(U, Q)$$

is a restriction of a continuous and linear map, hence $C^\infty$. Further, the maps $C^{k+1}([0, 1], \phi)$ and $\text{Evol}_H|_Q$ are $C^\infty$, whence the map

$$C^d_W(U, C^{k+1}([0, 1], \phi) \circ \text{Evol}_H|_Q) : C^d_W(U, Q) \to C^d_W(U, C^{k+1}([0, 1], W))$$

is $C^\infty$ (see [21]), and we set

$$\Theta_2 := i \circ C^d_W(U, C^{k+1}([0, 1], \phi) \circ \text{Evol}_H|_Q) : C^d_W(U, Q) \to C^d_W(U, C^{k+1}([0, 1], h)).$$

with the inclusion map $i : C^d_W(U, C^{k+1}([0, 1], W)) \to C^d_W(U, C^{k+1}([0, 1], h)).$ By Corollary 3.4.1 the flip

$$\Psi_2 : C^d_W(U, C^{k+1}([0, 1], h)) \to C^{k+1}([0, 1], C^d_W(U, h))$$

is $C^\infty$ (being continuous and linear). Now, we have $\text{im}(\Psi_2 \circ \Theta_2 \circ \Theta_1) \subseteq C^{k+1}([0, 1], C^d_W(U, h))$, which is open in $C^{k+1}([0, 1], C^d_W(U, h))$, thus we obtain the $C^\infty$-map

$$\Theta_3 := (\Psi_2 \circ \Theta_2 \circ \Theta_1)|_{C^{k+1}([0, 1], C^d_W(U, W))} : \Omega \to C^{k+1}([0, 1], C^d_W(U, W)).$$

With the $C^\infty$-diffeomorphism

$$C^{k+1}([0, 1], C^d_W(U, \phi)^{-1}) : C^{k+1}([0, 1], C^d_W(U, W)) \to C^{k+1}([0, 1], C^d_W(U, V))$$

and the inclusion map $j : C^{k+1}([0, 1], C^d_W(U, V)) \to C^{k+1}([0, 1], C^d_W(U, H))$ we get

$$\Theta_4 := j \circ C^{k+1}([0, 1], C^d_W(U, \phi)^{-1}) : C^{k+1}([0, 1], C^d_W(U, W)) \to C^{k+1}([0, 1], C^d_W(U, H))$$

and finally construct the $C^\infty$-map

$$\Theta := \Theta_4 \circ \Theta_3 : \Omega \to C^{k+1}([0, 1], C^d_W(U, H)).$$

It remains to show that

$$\varepsilon_x \circ \Theta(\gamma) = \text{Evol}_H(\text{ev}_x \circ \gamma)$$

for each $\gamma \in \Omega$ and $x \in U$. By construction of $\Theta$ we have
\[ \Theta(\gamma) = (\Theta_4 \circ \Theta_3)(t \mapsto (x \mapsto \gamma(t)(x))) \]
\[ = (\Theta_4 \circ \Psi_2 \circ \Theta_2 \circ \Theta_1)(t \mapsto (x \mapsto \gamma(t)(x))) \]
\[ = (\Theta_4 \circ \Psi_2 \circ \Theta_2)(x \mapsto (t \mapsto \gamma(t)(x))) \]
\[ = (\Theta_4 \circ \Psi_2)(x \mapsto (s \mapsto \phi(\text{Evol}_H(t \mapsto \gamma(t)(x))(s)))) \]
\[ = \Theta_4(s \mapsto (x \mapsto \phi(\text{Evol}_H(t \mapsto \gamma(t)(x))(s)))) \]
\[ = (s \mapsto (x \mapsto \text{Evol}_H(t \mapsto \gamma(t)(x))(s))), \]

hence for \( s \in [0, 1] \) we see that

\[ (\varepsilon_x \circ \Theta(\gamma))(s) = \text{Evol}_H(t \mapsto \gamma(t)(x))(s) = \text{Evol}_H(\text{ev}_x \circ \gamma)(s), \]

as required, and we obtain the desired result using Lemma 4.4.

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