ON THE QUANTIZATION OF ABELIAN GAUGE FIELD THEORIES ON RIEMANN SURFACES

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ABSTRACT

In this paper we quantize the abelian gauge field theories on a Riemann surface $M$ in the Feynman gauge. The fields take their values on any nontrivial line bundle $P(M,U(1))$ with first Chern class $c_1 = 2\pi k$, $k \in \mathbb{Z}$. The point of view adopted here is that of small quantum perturbations $A^{\text{qu}}$ around a classical instantonic solution $A^{\text{cl}} \in P(M,U(1))$. The explicit form of the instantonic fields $A^{\text{cl}}$ and of the propagator $\langle A^{\text{qu}}A^{\text{qu}} \rangle$ is derived. The case in which the theory interacts only with an external current $J$ is completely solved evaluating the generating functional $Z[J]$. Finally, we consider the Schwinger model, or two dimensional quantum electrodynamics. We show that it is possible to integrate out from the path integral the nonphysical gauge degrees of freedom. The upshot is a nonlocal field theory of fermions describing the dynamics of the electrons on a Riemann surface. The fermions interact through a potential whose short distance behavior is explicitly derived. In particular, it turns out that this behavior can only depend on zero modes and in the case of the sphere, where there are not zero modes, the interactions between the electrons switch off at very high energies yielding a free field theory.

July 1993

1 This work is carried out in the framework of the EC Research Programme “Gauge theories, applied supersymmetry and quantum gravity”.
1. INTRODUCTION

One of the main motivations for studying the gauge field theories in two dimensions is that they provide a good laboratory in order to illustrate some important properties which are present also in the more realistic four dimensional models, like anomalies, confinement and dynamical symmetry breaking \[1\]. Moreover, following the recent developments in string theories and topological field theories, there is also an interest in quantizing the two dimensional field theories on a manifold and in particular on the closed and orientable Riemann surfaces \[2\], \[3\], \[4\], \[5\], \[6\], \[7\], \[8\], \[9\]. Unfortunately, most of the literature in string theory deals with the pure Yang-Mills field theory, in which the observables are metric independent objects like the Wilson loops and, more in general, our knowledge about the interacting case is confined until now to very simple topologies. For example the Schwinger model or two dimensional quantum electrodynamics $QED_2$ has been understood only in the flat case \[10\], on the cylinder \[11\], on the sphere \[12\] and on the torus \[13\], \[14\].

To overcome at least in part these limitations, we propose a perturbative approach which is able to quantize the interacting abelian gauge fields on a Riemann surface, even in the presence of nontrivial topological sectors. This approach, which is the main result of this paper, allows for example the quantization of models containing massless scalar field theories and fermions. The quantization of other abelian gauge field theories, in which for instance the matter fields are massive, is however restricted by the fact that the propagators of these fields are not known on a Riemann surface. Another important feature of the formalism presented here, is that it is very explicit. For this reason, also phenomenological considerations concerning the effects of a nontrivial gravitational background on field theories are possible (on this topic see as an introduction refs. \[15\]). The search of these effects is in fact the second aim of this paper. To this purpose we study the case of the chiral Schwinger model \[16\], comparing the high energy behaviors of the chiral fermions on a Riemann surface with the analogous results obtained considering other topologies.

The approach followed here to quantize the abelian gauge field theories is a generalization of \[17\], where the propagators and the vertices of the Schwinger model in the Lorentz gauge were firstly computed on a Riemann surface giving a way of deriving also the higher order contributions to the correlation functions. Despite of this success, there are in ref. \[17\] some problems in treating models which contain scalar fields and, more important,
the case of nontrivial topological sectors was not considered. To overcome these difficulties, we exploit here the Feynman gauge instead of the Lorentz gauge. It is important to stress at this point that the Lorentz and Feynman gauges are probably the only gauges in which the equations of motion satisfied by the propagator of the gauge fields become simple enough to be solved. The disadvantage of the Lorentz gauge is however the fact that the propagator acts only in the space of the transverse degrees of freedom. For this reason one has always to separate the transverse and longitudinal components of the gauge fields operating a different perturbative treatment for both of them. This is an unnecessary complication for example when the matter fields are massless scalars. In this case, in fact, the unphysical longitudinal components of the gauge fields remain coupled to the matter fields at any perturbative order and it is difficult to show that they do not contribute to the physical amplitudes. Fortunately, this problem is not present in the Feynman gauge where the propagator is orthogonal only with respect to the harmonic part of the gauge fields. The zero modes are also unphysical but they represent only a discrete number of degrees of freedom which is easy to cope with.

Another progress made with respect to ref. [17] is that we consider in this work also gauge fields belonging to nontrivial line bundles. These line bundles are characterized by a nonvanishing value of the first Chern class $c_1$. Of course, nontrivial line bundles with $c_1 \neq 0$ are not visible within the frame of perturbation theory. Nevertheless we are still allowed to consider small perturbations $A^\text{qu}_\mu$ around a classical gauge field $A^\text{cl}_\mu$ with $c_1 \neq 0$. This is the point of view adopted here. Despite of the fact that the fields $A^\text{cl}_\mu$ cannot be globally defined on $M$, we have not found any problem in fixing the Lorentz gauge $\partial^\mu A_\mu = 0$ in a global way, basically because this gauge fixing does not depend on the fields themselves but on their derivatives. Moreover, the instantonic solutions $A^\text{cl}_\mu$ which we explicitly construct here, satisfy exactly the two dimensional Maxwell equations in the Feynman gauge. It is remarkable that a classical field $A^\text{cl}_\mu$ of this kind decouples from the kinetic part of the action and appears only in the interaction with the matter fields. This fact allows the computation of the propagator of the gauge fields $A^\text{qu}_\mu$ as in the case of the trivial topological sector and, consequently, the perturbative approach explained in ref. [17] becomes realizable also for nontrivial line bundles.

After these improvements with respect to ref. [17], it is possible to study the effects of the gravitational background provided by a Riemann surface on a wide range of models containing massless fermions and bosons. In this paper, we concentrate on the chiral
Schwinger model which describes the quantum electrodynamics of massless electrons in two dimensions (for this reason we will call it also QED$_2$). This choice is motivated by the physical relevance of the Schwinger model in physics [1], [10]. The usual approach to the Schwinger model is to integrate in the path integral the fermionic degrees of freedom first. In this way, one obtains an effective theory of massive gauge fields when nontrivial topological sectors are absent [8], [18]. However, in the case of nontrivial topological sectors, there is, at least to our knowledge, no known way of integrating over the fermionic degrees of freedom on a Riemann surface. For this reason, we follow here another strategy, eliminating first the gauge fields from the path integral. The idea behind this strategy is that the physical fields in the Schwinger model are the fermions and not the gauge fields. The latter have in fact only a discrete number of degrees of freedom, provided by the zero modes and by the instantonic part $A^{\text{cl}}$. After the integration over the quantum part of the gauge fields $A^{\text{qu}}$, we obtain indeed the effective field theory of the massless electrons. The electrons still remain minimally coupled to the instantonic and harmonic components of the gauge fields, but these appear now only as external fields. All the properties of the theory which do not depend on the nontrivial topological sectors, are instead concentrated in a nonlocal term, describing the self-interactions between the fermions. One of these properties, which is physically very important, is the high energy behavior of the electrons. This behavior is clearly not influenced by the presence of external instantonic fields because it is entirely dominated by the electromagnetic forces at very short distances. As an upshot, we compute here explicitly the asymptotic form of the potential governing the interactions between the electrons at very high energies. As we show, it strongly depends on the topology. On a sphere, for example, the potential vanishes at very high energies, which is a rather surprising result. On a Riemann surface with genus $g \geq 1$, instead, the potential does apparently not vanish at short distances due to the presence of zero modes. Unfortunately, we are not able to estimate this zero mode contribution exactly, but from the physical point of view, it would be surprising that the short distance behavior of the particles is determined by the zero modes. Thus, we believe that also on a Riemann surface of genus $g \geq 1$ the electrons become free at high energies as it happens on the sphere. Finally, the behavior of the potential on a Riemann surface is completely different from that of the flat case, where the manifold has for instance the topology of a disk. In this sense, it would be very interesting to study the Schwinger model also on the Euclidean space $\mathbb{R}^2$. However, the procedure of integrating in the gauge degrees of freedom followed here is no longer possible on $\mathbb{R}^2$ since there are problems in constructing the propagator of
the gauge fields as we pointed out in ref. [17]. To conclude, we mention that the effective theory of the electrons found here resembles very much the nonlocal generalization of the Thirring model [19] in the sense of ref. [20] and we believe therefore that it describes an integrable model.

The material presented in this paper is divided as follows. In Section 2 we show how the introduction of the Feynman gauge simplifies the quantization of the abelian gauge field theories on a manifold with conformally flat metric. In Section 3 we restrict ourselves to the case of a general closed and orientable Riemann surface. An instantonic solution $A^{\text{cl}}$ of the field equations in the Feynman gauge is explicitly derived. The first Chern class of the connection $A^{\text{cl}}$ is $c_1 = 2\pi k$. We show that $A^{\text{cl}}$ is necessarily multivalued, but this multivaluedness can be reabsorbed performing a gauge transformation provided $k$ is an integer. In Section 4 we quantize the abelian gauge fields $A^{\text{qu}}$ representing small perturbations around the instantonic solution $A^{\text{cl}}$. Following ref. [17], we derive the explicit expression of the propagator of the $A^{\text{qu}}$. The relevant properties of the propagator, like orthogonality with respect to the space of the flat connections and singlevaluedness around the nontrivial homology cycles, are proven. In this way, we can exactly solve the pure abelian gauge field theory coupled to an external current computing the generating functional of the correlation functions on any nontrivial line bundle. In Section 5 we treat the quantum electrodynamics (QED$_2$) on a Riemann surface. The knowledge of the propagator and the fact that it is orthogonal to the zero modes allows the integration over the gauge fields $A^{\text{qu}}$ in the path integral. As a consequence, we obtain the effective action of electrons mentioned above. Finally, in Section 6 we evaluate the relativistic potential governing the forces between the electrons at very short distances. Finally, we discuss in the Conclusions the possible extensions of the results presented here.
2. THE GAUGE FIXING CONDITION ON A RIEMANN SURFACE

In this Section we consider the Maxwell Field Theory (MFT) on a Riemann surface \( M \) coupled to an external source \( J^\alpha \). This is a pure gauge field theory with \( U(1) \) gauge group of symmetry and the following action:

\[
S_{\text{free}} = \int_M d^2 z \sqrt{g} \left( \frac{1}{4} F_{\alpha \beta} F^{\alpha \beta} + J_\alpha A^\alpha \right) \tag{2.1}
\]

where \( g = \det |g_{\alpha \beta}| \). \( M \) is now a general, closed and orientable Riemann surface of genus \( g \), provided with an Euclidean metric \( g_{\alpha \beta} \). A covering of \( M \) is given by a system of open sets \( \{ U_i \} \) parametrized by the local complex coordinates \( z^{(i)} \) and \( \bar{z}^{(i)} \). In the following we will drop the indices \( i \) corresponding to the local patches of the covering. Greek indices will denote complex indices. For instance \( A_\alpha(z, \bar{z}) \equiv (A_z(z, \bar{z}), A_{\bar{z}}(z, \bar{z})) \). Moreover, let \( \omega_i(z) dz, i = 1, \ldots, g \), be a set of holomorphic differentials, normalized in the following way with respect to the canonical basis of homology cycles \( A_i \) and \( B_i \):

\[
\oint_{A_i} \omega_j dz = 0 \quad \oint_{B_i} \omega_j dz = \Omega_{ij} \tag{2.2}
\]

\( \Omega_{ij} \) is called the period matrix. The tensor \( F_{\alpha \beta} \) represents the usual field strength:

\[
F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \quad F^{\alpha \beta} = g^{\alpha \gamma} g^{\beta \delta} F_{\gamma \delta} \tag{2.3}
\]

It is easy to check that the only nonvanishing components of the tensor \( F_{\alpha \beta} \) are \( F_{z \bar{z}} = -F_{\bar{z} z} \). The action (2.1) is what we need to compute the propagator of the gauge fields.

As a first step in order to quantize the MFT we have to choose a gauge fixing. A convenient choice is the set of covariant gauges defined by the condition:

\[
g^{\alpha \beta} \partial_\alpha A_\beta = 0 \tag{2.4}
\]

We notice here that for a general choice of the metric the usual splitting of the fields in transverse and longitudinal components does not make sense. The gauge fixed partition function in the Euclidean space looks as follows:

\[
Z_0[J] = \int DA_z DA_{\bar{z}} \exp \left[ - (S_{\text{free}} + S_{\text{gf}}) \right] \tag{2.5}
\]
where:

\[
S_{gf} = \frac{1}{2\lambda} \int_M d^2z g^{\alpha\beta} g^{\gamma\delta} \partial_\alpha A_\gamma \partial_\beta A_\delta \tag{2.6}
\]

We ignore for the moment the Faddeev–Popov term containing the decoupled ghost action.

Due to the presence of the metric, the computation of the partition function \( Z_0[J] \) and therefore of the propagator becomes involved. Still we can simplify \( S_{\text{free}} \) using the following observation:

\[
\frac{1}{4} g^{\alpha\beta} g^{\gamma\delta} F_{\alpha\gamma} F_{\beta\delta} = \frac{1}{2} g^{-1} F_{z\bar{z}}^2 \tag{2.7}
\]

where \( g^{-1} = g^{zz} g^{\bar{z}\bar{z}} - g^{z\bar{z}} g^{\bar{z}z} \). In this way:

\[
S_{\text{free}} = \frac{1}{2} \int_M d^2z g^{-\frac{1}{2}} F_{z\bar{z}}^2 \tag{2.8}
\]

However, the gauge fixing part of the action \( S_{gf} \) remains complicated. For this reason, we exploit the fact that on a two dimensional manifold it is always possible to choose a conformally flat metric of the kind:

\[
g_{zz} = g_{\bar{z}\bar{z}} = 0 \quad g_{z\bar{z}} = g_{\bar{z}z} \quad g^{z\bar{z}} = 1 \tag{2.9}
\]

In this metric \( \sqrt{g} = g_{z\bar{z}} \) and the gauge fixing condition of eq. (2.4) becomes independent on \( g_{\alpha\beta} \):

\[
\partial_z A_{\bar{z}} + \partial_{\bar{z}} A_z = 0 \tag{2.10}
\]

Moreover, the transversal and longitudinal components of the fields can be defined through the Hodge decomposition, as we will see in the next Section.

Also with the above simplifications an easy solution of the partial differential equations that determine the components of the propagator is not possible unless we choose the following particular values of the parameter \( \lambda \) in eq. (2.6):

a) \( \lambda = 1 \) (Feynman gauge).

b) \( \lambda = 0 \) (Lorentz gauge).

The advantages and drawbacks of the choice of the Landau gauge have already been discussed in the Introduction and in [17]. In order to formulate the interacting MFT in the simplest possible way that easily generalizes to the Yang–Mills field theories, we investigate now the theory in the Feynman gauge.
3. THE CLASSICAL EQUATIONS OF MOTION

After an easy calculation, we find the following expression for the total action \( S = S_{\text{free}} + S_{\text{gf}} \) appearing in eq. (2.5) in the Feynman gauge:

\[
S_{\text{MFT}}(A, J) = \int_M d^2z \left[ g^{zz} (\partial_z A \bar{z} + \partial_{\bar{z}} A z) + J z A \bar{z} + J_{\bar{z}} A z \right]
\]

(3.1)

The classical equations of motion related to this action become now relatively simple:

\[
\partial_z g^{zz} \partial_{\bar{z}} A z = J \bar{z} \quad \quad \partial_{\bar{z}} g^{zz} \partial_z A z = J z
\]

(3.2)

In order to study eq. (3.2), we split the fields using the Hodge decomposition:

\[
A_{\alpha} = \epsilon_{\alpha \beta} \partial^\beta \varphi + \partial_{\alpha} \rho + A_{\alpha}^{\text{har}} + A_{\alpha}^I
\]

(3.3)

where \( \epsilon_{zz} = -\epsilon_{\bar{z}\bar{z}} = ig_{zz} \) is the completely antisymmetric Levi–Civita tensor and \( i^2 = -1 \).

In eq. (3.3) \( \varphi(z, \bar{z}) \) is a real scalar field representing the transverse degree of freedom. As a matter of fact, the components of the gauge fields in \( \varphi \) are \( A^T_z = \partial_z \varphi \) and \( A^T_{\bar{z}} = -\partial_{\bar{z}} \varphi \) and it is easy to check that they satisfy eq. (2.11), so that they are purely transversal. The component of the gauge fields in the real scalar field \( \rho(z, \bar{z}) \) denote instead the longitudinal degrees of freedom. Finally, the \( A_{\alpha}^{\text{har}} \) represent the flat connections in the abelian case. They take into account of the \( g \) zero modes \( \omega_i(z) dz \) discussed in the previous Section. We notice that the zero modes describe nonphysical degrees of freedom and therefore, in the interacting case, they should not propagate within the amplitudes.

The first three terms of the right hand side of eq. (3.3) correspond to the exact, coexact and harmonic forms in which a differential \( A_{\alpha} \) can be decomposed on a Riemann surface. This is the singlevalued part of the gauge fields, while \( A_{\alpha}^I \) is an instantonic and periodically multivalued differential. The multivaluedness occurs when the field \( A_{\alpha}^I \) is transported along a nontrivial homology cycle of the Riemann surface. Remembering that the first Chern class is defined by

\[
c_1 = \int_M d^2z F_{zz}
\]

(3.4)

with \( c_1 = 2\pi k \) and \( k \in \mathbb{Z} \), we can say that \( A_{\alpha}^I \) is a solution of these equations defined on a nontrivial line bundle \( P(M, U(1)) \) characterized by \( c_1 \neq 0 \).
The point of view we will take here in quantizing the Maxwell field theory is that of small perturbations around an instantonic classical field \( A^\text{cl}_\alpha \equiv A^\text{cl}_\alpha \). Let us now compute the explicit form of \( A^\text{cl}_\alpha \). These \( A^\text{cl}_\alpha \) are very similar to the Maxwell connections of ref. [6], apart from the fact that they do not satisfy the pure Maxwell equations, but the gauge fixed Maxwell equations (3.2). In order to compute \( A^\text{cl}_\alpha \), we have to consider the homogeneous classical equations of motion putting \( J_\alpha = 0 \) in eq. (3.2). Due to the fact that the Riemann surface \( M \) is a compact, orientable manifold without boundary, the only possible nontrivial solutions of the homogeneous equations of motion are:

\[
\partial_z A^I_z = \alpha_1 g_{z\bar{z}} \quad \partial_{\bar{z}} A^I_{\bar{z}} = \alpha_2 g_{z\bar{z}} \quad (3.5)
\]

\( \alpha_1 \) and \( \alpha_2 \) being arbitrary constants. However, \( \alpha_1 \) and \( \alpha_2 \) become uniquely determined once we require that \( A^I_\alpha \) satisfies eq. (3.4) and the gauge condition (2.10). As a matter of fact, substituting eq. (3.5) in eq. (3.4), we have:

\[
\alpha_1 - \alpha_2 = \frac{2\pi k}{A} \quad (3.6)
\]

where \( A = \int_M d^2 z g_{z\bar{z}} \) is the total area of the Riemann surface \( M \). Finally, exploiting eq. (2.10) we get:

\[
\alpha_1 + \alpha_2 = 0 \quad (3.7)
\]

From eqs. (3.6) and (3.7) we get the following implicit expression for the instantonic solutions:

\[
\partial_z A^I_z = \frac{\pi k}{A} g_{z\bar{z}} \quad \partial_{\bar{z}} A^I_{\bar{z}} = -\frac{\pi k}{A} g_{z\bar{z}} \quad (3.8)
\]

Let us notice that in this way we have obtained nontrivial instantonic solutions of the Maxwell field theory in the Feynman gauge. Their first Chern class is \( 2\pi k \).

In order to have the explicit expression of \( A^I_\alpha \), we try the ansatz:

\[
A^I_z = \alpha_1 \int_M d^2 w \partial_z \tilde{K}(z, w) g_{w\bar{w}} - \sum_{i,j=1}^g \alpha_1 A \omega_i(z) |\text{Im} \ \Omega_{ij}|^{-1} \int_{z_0}^{z} \tilde{\omega}_j(\tilde{w}) d\tilde{w} \quad (3.9)
\]

\[
A^I_{\bar{z}} = \alpha_2 \int_M d^2 w \partial_{\bar{z}} \tilde{K}(z, w) g_{w\bar{w}} - \sum_{i,j=1}^g \alpha_2 A \tilde{\omega}_i(\tilde{z}) |\text{Im} \ \Omega_{ij}|^{-1} \int_{z_0}^{z} \omega_j(w) dw \quad (3.10)
\]

where

\[
\tilde{K}(z, w) = \log|E(z, w)|^2 + \sum_{i,j=1}^g \left[ \text{Im} \int_w^{z} \omega_j(s) ds \right] |\text{Im} \ \Omega_{ij}|^{-1} \left[ \text{Im} \int_w^{z} \omega_j(s) ds \right] \quad (3.11)
\]
Remembering that
\[ \partial_z \bar{\partial}_\bar{z} \bar{K}(z, w) = \delta^{(2)}_{z, \bar{z}}(z, w) + \omega_i(z) |Im \ \Omega_{ij}|^{-1} \tilde{\omega}_j(\bar{z}) \] (3.12)
we can easily check that the solutions (3.9) and (3.10) satisfy exactly the eqs. (3.8). The fields \( A^I_\alpha \) defined in (3.9) and (3.10) are periodic around the homology cycles \( A_i \) and \( B_i \), \( i = 1, \ldots, g \). Transporting \( n_l \) times \( A^I_\alpha \) around \( B_l \), \( l = 1, \ldots, g \), we get for example:

\[ A^I_\alpha = A^I_\alpha + 2\pi k n_l \sum_{i,j=1}^{g} \bar{\Omega}_{ij}(\Omega - \bar{\Omega})^{-1}_{ji} \omega_i(z) \] (3.13)

\[ A^{\bar{I}}_\alpha = A^{\bar{I}}_\alpha - 2\pi k n_l \sum_{i,j=1}^{g} \Omega_{ij}(\Omega - \bar{\Omega})^{-1}_{ji} \bar{\omega}_i(\bar{z}) \] (3.14)

This periodicity amounts however to a gauge transformation. In fact, let us set
\[ g(z, \bar{z}) = \exp \left[ 2\pi i k n_l \sum_{i,j=1}^{g} \left( \int_{z_0}^{z} \Omega_{ij}(\Omega - \bar{\Omega})^{-1}_{ji} \omega_i(w)dw - \int_{\bar{z}_0}^{\bar{z}} \bar{\Omega}_{ij}(\bar{\Omega} - \Omega)^{-1}_{ji} \bar{\omega}_i(\bar{w})d\bar{w} \right) \right] \] (3.15)

We claim that \( g(z, \bar{z}) \) represents a singlevalued \( U(1) \) gauge transformation on a Riemann surface corresponding to the flat line bundle given by the holomorphic connections \( A^{\text{har}}_{\alpha,i} = A^I_\alpha - A^{\bar{I}}_\alpha \). As a matter of fact, the connection contained in the exponent of eq. (3.15) is of the form
\[ 2\pi k n_l \sum_{i,j=1}^{g} \bar{\Omega}_{ij}(\bar{\Omega} - \Omega)^{-1}_{ji} \omega_i(z) + \text{c.c.} = \alpha_k n_l \] (3.16)
\( \alpha_i \) being the real harmonic differential with the following holonomies around the homology cycles:
\[ \oint_{A_i} \alpha_j = \delta_{ij} \quad \oint_{B_i} \alpha_j = 0 \]

If we consider the behavior of (3.9) and (3.10) around the homology cycles \( A_i \), we arrive to an analogous result in which the \( \alpha_i \) are replaced by the real harmonic differentials \( \beta_i \). Using eq. (3.16) in eq. (3.15), it is now easy to see that \( g(z, \bar{z}) \) has exactly the form of a good \( U(1) \) gauge transformation according to ref. [21]. The proof that eq. (3.13) and (3.14) represent gauge transformations is concluded noting that these equations can be rewritten in the following form:
\[ A^I_\alpha = A^I_\alpha + ig^{-1} \partial_\alpha g \] (3.17)
Of course, the above gauge transformation is well defined only if $k$ in eq. (3.4) is an integer, otherwise $g(z, \bar{z})$ becomes multivalued. Therefore, we have proven that the gauge connection $A^{I}_{\alpha}$ given in (3.9) and (3.11) are the correct solutions of the equations (3.8). They represent the generalization to a Riemann surface of the connections given in the case of a torus in refs. [6] and [14]. In both cases the connections are periodic around the homology cycles but the periodicity amounts to a gauge transformation.

4. THE PROPAGATOR IN THE FEYNMAN GAUGE

In this Section we construct the propagator

$$G_{\alpha\beta}(z, w) \equiv \langle A^\alpha_{\alpha}(z, \bar{z})A^\beta_{\beta}(w, \bar{w}) \rangle$$ (4.1)

From the action (3.1) it is easy to see that the components $G_{zw}(z, w)$ and $G_{\bar{z}\bar{w}}(z, w)$ vanish identically. The remaining components of the propagator satisfy the following two equations:

$$\partial_{\bar{z}}g^{z\bar{z}}\partial_z G_{zw}(z, w) = \delta_{z\bar{w}}(z, w) + \text{zero modes} \quad (4.2)$$

$$\partial_zg^{z\bar{z}}\partial_{\bar{z}} G_{\bar{z}w}(z, w) = \delta_{z\bar{w}}(z, w) + \text{zero modes} \quad (4.3)$$

Analogous equations are to be solved in the variable $w$. The exact form of the zero mode contribution appearing in the left hand side of eqs. (4.2) and (4.3) will be given below.

In solving eqs. (4.2) and (4.3) we can use the techniques developed in ref. [17] in the case of the Lorentz gauge fixing. The only difference that occurs in the Feynman gauge is that now both the transversal and longitudinal fields are propagated, so that we have two distinct equations of motion in $A_z$ and $A_{\bar{z}}$. In the Lorentz gauge, instead, there is only one equation in $A_z$ or, equivalently, in $A_{\bar{z}}$ (see ref [17] for more details). A detailed account of the way in which the components of the propagator can be found was already provided in ref. [17]. Here we just give the result:

$$G_{zw}(z, w) = -\int_M d^2tg_{t\bar{t}}\partial_zK(z, t)\partial_{\bar{w}}K(w, t)$$ (4.4)

$$G_{\bar{z}\bar{w}}(z, w) = -\int_M d^2tg_{t\bar{t}}\partial_{\bar{z}}K(z, t)\partial_{\bar{w}}K(w, t)$$ (4.5)
where \( K(z, w) \) is the usual scalar Green function defined by the relations:

\[
\partial_{\bar{z}} \partial_{z} K(z, w) = \delta^{(2)}(z, w) - \frac{g_{zz}}{A} \tag{4.6}
\]

\[
\partial_{\bar{z}} \partial_{w} K(w, z) = -\delta^{(2)}(z, w) + \bar{\omega}_{i}(\bar{z}) [\text{Im } \Omega]_{ij}^{-1} \omega_{j}(w) \tag{4.7}
\]

\[
\int_{M} d^{2}t g_{tt} K(z, t) = 0 \tag{4.8}
\]

As it is well known, \( K(z, w) \) has an explicit expression in terms of the Green function (3.11):

\[
K(z, w) = \tilde{K}(z, w) - \frac{1}{A} \int d^{2}t g_{tt} \left( \tilde{K}(z, t) + \tilde{K}(t, w) \right) + \frac{1}{A^{2}} \int \int d^{2}s d^{2}t \ g_{tt} g_{ss} \tilde{K}(t, s) \tag{4.9}
\]

Now we discuss the properties of the Green functions (4.4) and (4.5). In particular, we show that they fulfill all the possible requirements of a physical propagator. First of all, let us check that the components \( G_{zw}(z, w) \) and \( G_{\bar{z}w}(z, w) \) of the propagator given in eq. (4.4) and (4.5) satisfy the correct equations of motion (4.2) and (4.3). Using the properties (4.6) and (4.8) of the scalar Green function \( K(z, w) \) it is possible to show that:

\[
\partial_{\bar{z}} g^{z\bar{z}} \partial_{z} G_{zw}(z, w) = \delta^{(2)}(z, w) - \sum_{i,j=1}^{g} \bar{\omega}_{i}(\bar{z}) [\text{Im } \Omega]_{ij}^{-1} \omega_{j}(w) \tag{4.10}
\]

An analogous equation can be derived for the component \( G_{\bar{z}w}(z, w) \) of the propagator. In this way we have determined the contribution of the zero modes in the right hand sides of eqs. (4.2) and (4.3). Moreover, the propagator is singlevalued along the nontrivial homology cycles of the Riemann surface:

\[
\oint_{\gamma} dz G_{zw}(z, w) = \oint_{\gamma} dw G_{zw}(z, w) = \oint_{\gamma} d\bar{z} G_{\bar{z}w}(z, w) = \oint_{\gamma} d\bar{w} G_{\bar{z}w}(z, w) = 0 \tag{4.11}
\]

where \( \gamma = A_{i}, B_{i}, i = 1, \ldots, g \). This is a consequence of the singlevaluedness of the scalar Green function (4.9). Eq. (4.11) guarantees that the fields \( A_{\alpha}^{\text{qu}} \) are singlevalued quantum perturbations on \( M \). Finally, \( G_{\alpha\beta}(z, w) \) is orthogonal to the space of harmonic connections \( A_{\alpha}^{\text{har}} \). This is a trivial consequence of the fact that

\[
\int d^{2}z \partial_{z} K(z, t) \bar{\omega}_{i}(\bar{z}) = \int d^{2}z \partial_{\bar{z}} K(z, t) \omega_{i}(z) = 0
\]
for $i = 1, \ldots, g$. Due to the above equation, it is easy to prove that for any harmonic differential $A_{\alpha}^{\text{har}}$ we have:

$$\int d^2z A_{\bar{z}}^{\text{har}} \int d^2w G_{zw}(z, w)J_w = \int d^2z A_{\bar{z}}^{\text{har}} \int d^2w G_{\bar{z}w}(z, w)J_{\bar{w}} = 0 \quad (4.12)$$

for $i = 1, \ldots, g$. Eq. (4.12) implies that $G_{\alpha\beta}(z, w)$ does not propagate the zero modes inside of the amplitudes. This can be explicitly seen considering the solutions of the classical equations of motion (3.2):

$$A_w(w, \bar{w}) = \int d^2z G_{zw}(z, w)J_{\bar{z}} \quad A_{\bar{w}}(w, \bar{w}) = \int d^2z G_{\bar{z}w}(z, w)J_z \quad (4.13)$$

At this point, we split the current $J_{\alpha}$ as follows:

$$J_z(z, \bar{z}) = \partial_z \chi(z, \bar{z}) + ia_i [\text{Im} \Omega^{-1}]_{ij} \omega_j(z) \quad (4.14)$$

where

$$a_i = -i \int d^2z \bar{\omega}_i(\bar{z})J_z(z, \bar{z}) \quad (4.15)$$

and $\chi(z, \bar{z})$ is a complex scalar field. The component $J_{\bar{z}}$ can be found taking the complex conjugate in the right hand side of eq. (4.14). This decomposition is equivalent to the Hodge decomposition (3.3) apart from the absence of the instantonic fields. Now we insert the gauge field $A_{\bar{w}}$ given in eqs. (4.13) in the original equations of motion (3.2). Using eq. (4.10) we get:

$$\partial_w g^{w\bar{w}} \partial_w \int d^2z G_{\bar{z}w}(z, w)J_z(z, \bar{z}) = \partial_w \chi(w, \bar{w}) \quad (4.16)$$

$$\partial_{\bar{w}} g^{w\bar{w}} \partial_{\bar{w}} \int d^2z G_{zw}(z, w)J_{\bar{z}}(z, \bar{z}) = \partial_{\bar{w}} \chi(w, \bar{w}) \quad (4.17)$$

showing that the zero modes are not propagated by the propagator.

A last property of $G_{\alpha\beta}(z, w)$ comes from the fact that the Hodge decomposition (3.3) is not invertible if $\varphi$ and $\rho$ are constants. This implies the following condition on $\varphi$ and $\rho$:

$$\int_M d^2z \sqrt{g} \varphi(z, \bar{z}) = \int_M d^2z \sqrt{g} \rho(z, \bar{z}) = 0$$

In agreement, the biharmonic Green function

$$G(z, w) = \int_M d^2z g_{tt}K(z,t)K(w,t) \quad (4.18)$$
from which the propagator $G_{\alpha\beta}(z,w)$ can be evaluated, should satisfy the relations:

$$\int_M d^2z \sqrt{g} G(z,w) = \int_M d^2w \sqrt{g} G(z,w) = 0$$

Exploiting eq. (4.8) it is easy to see that the above equations hold.

### 5. TWO DIMENSIONAL QUANTUM ELECTRODYNAMICS ON A RIEMANN SURFACE

Until now, we have investigated the classical MFT in the presence of an external current. In this section we consider the case in which the gauge fields are allowed to interact with other matter fields. In particular, we treat the chiral Schwinger model [16] discussed also in [17] in the case of trivial line bundles. The action of the model is given by:

$$S_{QED_2}[A, \bar{\psi}, \psi, \bar{\xi}, \xi] = S_{MFT}(A, J = 0) + \int_M d^2z \left[ \bar{\psi}_\theta (\partial_z + A_z) \psi_\theta + \bar{\psi}_\bar{\theta} (\partial_{\bar{z}} + A_{\bar{z}}) \psi_{\bar{\theta}} + J_z A_{\bar{z}} + J_{\bar{z}} A_z + g_{\theta\bar{\theta}} (\xi_\theta \psi_{\bar{\theta}} + \xi_{\bar{\theta}} \psi_\theta) + g_{\theta\bar{\theta}} (\bar{\xi}_\bar{\theta} \bar{\psi}_{\bar{\theta}} + \bar{\xi}_\theta \bar{\psi}_\theta) \right]$$

(5.1)

where $\xi, \bar{\xi}$ are the external currents related to the fields $\psi, \bar{\psi}$ respectively and $g_{\theta\bar{\theta}} = \sqrt{g_{zz}}$. For simplicity, we assume that the coupling constant between gauge fields and spinors is equal to one. Moreover, we do not need the external currents associated to the gauge fields, so that they are set to zero. The spinor indices in eq. (5.1) are denoted with $\theta, \bar{\theta}$. We assume that the “physical” boundary conditions for $\psi$ and $\bar{\psi}$ when transported along the homology cycles on $M$ are given by the even spin structure $s = \left[ \begin{array}{c} \bar{a}_0 \\ \bar{b}_0 \end{array} \right]$. $\bar{a}_0$ and $\bar{b}_0$ are two vectors of dimension $g$ whose elements are half integers such that $4 \bar{a}_0 \cdot \bar{b}_0 = 0 \mod 2$ [22]. The matter currents will be denoted as follows:

$$J^m_z = \bar{\psi}_\theta \psi_\theta \quad J^m_{\bar{z}} = \bar{\psi}_{\bar{\theta}} \psi_{\bar{\theta}}$$

(5.2)

These currents do not contain zero modes with respect to the operators $\partial_\alpha$, since the even spin structures do not admit holomorphic sections. Since we are considering small perturbations around the instantonic solutions of the equations of motion $A^I_{\alpha}$, we split in
eq. (5.1) the gauge fields into a quantum part and a classical and topologically nontrivial contribution given by eqs. (3.8):

$$A_z = A^\text{qu}_\alpha + A^I_\alpha$$  \hspace{1cm} (5.3)

where \(A^\text{qu}_\alpha = \epsilon_{\alpha\beta} \partial^\beta \varphi + \partial^\alpha \rho + A^\text{har}_\alpha\). The advantage of the splitting (5.3) is that \(A^I_\alpha\) satisfies the classical equations of motion and eq. (3.8). Hence, apart from a constant that can be factored out in the path integral, it is easy to see that the only dependence on \(A^I_\alpha\) of (3.1) is in the term with the external currents which has explicitly been neglected in eq. (5.1). Moreover, also the harmonic components of the gauge fields do not contribute to \(S_{\text{MFT}}(A^\text{qu}, J = 0)\). As a consequence, the generating functional \(Z[\bar{\xi}, \xi]\) of QED\(_2\) becomes:

\[
Z_{\text{QED}}[\bar{\xi}, \xi] = \int D\bar{A}^\text{qu}_\alpha D\bar{\psi} D\psi \prod_{i=1}^g d\theta_i d\phi_i \exp \left\{ - \left[ S_{\text{MFT}}(\bar{A}^\text{qu}, J = 0) + \int_M d^2z A^\text{qu}_\alpha J^m,\alpha \right. \right.
\]

\[
+ \left. \int_M d^2z \left( A^\text{har}_\alpha J^m,\alpha + \bar{\psi}_\theta (\partial z + A^I_z) \psi_\theta + \bar{\psi}_\bar{\theta} (\partial \bar{z} + A^I_{\bar{z}}) \psi_\bar{\theta} + \bar{\psi}^\theta \xi_\bar{\theta} + \psi^\bar{\theta} \xi_\theta + \psi^\bar{\theta} \bar{\xi}_\theta + \psi^\theta \bar{\xi}_\bar{\theta} \right) \right]\}
\]

where \(\bar{A}^\text{qu} = A^\text{qu} - A^\text{har}\). In the above equations we have parametrized the space of flat connections in the usual way [21]:

$$A^\text{har}_z dz + A^\text{har}_{\bar{z}} d\bar{z} = 2\pi i (\phi + \Omega \theta) \cdot (\Omega - \Omega)^{-1} \cdot \bar{\omega}(z) dz + \text{c.c}$$

Therefore, the integration over the parameters \(\theta_i\) and \(\phi_i\) is a sum over the flat connections. We note that the only dependence on \(A^\text{har}_\alpha\) is in the interaction term \(\int_M d^2z A^\text{har}_\alpha J^m,\alpha\). Moreover, it is important to stress that the fields \(A^I_\alpha\) are singlevalued on \(M\), apart from a gauge transformation which can be reabsorbed by a gauge transformation of the spinor fields in such a way that the whole action in eq. (5.4) remains singlevalued.

The functional \(Z_{\text{QED}}[\bar{\xi}, \xi]\) can be further simplified. One way to do this, is to perform an integration over \(\bar{\psi}\) and \(\psi\). However, this yields a free theory of massive vector fields\(^2\), for which it is not easy to compute the amplitudes on a Riemannn surface. Most important, these amplitudes have no direct physical significance. In fact, the physical fields are represented by the fermions, while the gauge fields are not observable and have only a discrete number of degrees of freedom in two dimensions. From this point of view, it

\(^2\) Let us notice that it is not clear if the Schwinger model on a manifold is simply a covariantized version of the model in the flat case, see e.g. [18] and [23].
seems preferable to eliminate the gauge fields from eq. (5.4). To this purpose, we have to evaluate the following path integral:

\[ \tilde{Z}^{\text{qu}}[J^m] = \int D\tilde{A}^{\text{qu}} \exp \left[ - \left( S_{\text{MFT}}(\tilde{A}^{\text{qu}}, J = 0) + \int_M \tilde{A}_\alpha^{\text{qu}} J^{m,\alpha} \right) \right] \] (5.5)

We note that \( \tilde{Z}^{\text{qu}}[J^m] \) contains only a sum over exact and coexact differentials, since the harmonic components have already been extracted. Hence, the operators \( \partial_\alpha \) in the exponent of eq. (5.5) act on the space of differentials which is orthogonal to the harmonic components and we are free to integrate by parts. Using standard techniques we perform in eq. (5.5) the change of variables:

\[ \tilde{A}_\alpha^{\text{qu}} = \tilde{A}_\alpha^{\text{qu}} + \frac{1}{2} \int d^2w g_{w\bar{w}} G_{\beta\alpha}(z, w) J^{m,\beta}(w, \bar{w}) \] (5.6)

where \( \alpha = z, \bar{z} \) and \( \beta = w, \bar{w} \). Again, \( \tilde{A}_\alpha^{\text{qu}} \) is still orthogonal to the space of the harmonic differentials because of the properties of the propagator explained in the previous section. Substituting (5.6) in eq. (5.5), we obtain

\[ Z_{\text{QED}}[\bar{\xi}, \xi] = \int D\bar{\psi} D\psi \prod_{i=1}^g d\theta_i d\phi_i \exp \left\{ - \left[ \int_M d^2z \left[ \bar{\psi}_\theta (\partial_\bar{z} + A_\bar{z}) \psi_\theta + \bar{\psi}_\theta \xi_\theta + \psi_\theta \bar{\xi}_\theta \right] 
\right. 
\left. + \int_M d^2z g_{z\bar{z}} A_\alpha^{\text{har}} J^{m,\alpha} + \frac{1}{4} \int_M d^2z d^2z' (J^m_m(z, \bar{z}) G_{z'\bar{z}}(z, z') J^m_m(z', \bar{z}')) + c.c. \right\} \] (5.7)

Therefore, the generating functional of QED on a Riemann surface \( M \) can be expressed as a theory of fermions with a self-interacting potential which is given by the two point function of the gauge fields. The latter is explicitly given in eqs. (4.4) and (4.5), so that it is possible, at least in principle, to compute the behavior of this potential at very high energies, i.e. in the limit \( z \to z' \).

Notice that the formula (5.7) has been obtained only because the propagator \( G_{\alpha\beta}(z, z') \) on a Riemann surface does exist and it is explicitly known. In this respect, difficulties may arise when \( M \) is a noncompact manifold, the complex plane included. In this case, in fact, the propagator of the gauge fields satisfying the physical boundary conditions is not easy to construct \[17\]. The problem is that the biharmonic Green function (4.18) with the desired boundary conditions from which we can derive \( G_{\alpha\beta}(z, w) \) as explained in \[17\] does not exist on noncompact manifolds \[24\].
6. HIGH ENERGY BEHAVIOR OF QUANTUM ELECTRODYNAMICS

In this section we compute the behavior of the potential $G_{\alpha\beta}(z, w)$ appearing in eq. (5.7) at short distances. First of all, we study the flat case, namely a disk $B$ of unitary radius. On the disk, the propagator of the gauge fields is given by the derivatives of the following biharmonic Green function:

$$G(z, w) = \frac{1}{2} |z - w|^2 \log \left[ \frac{|z - w|^2}{(1 - \bar{w}z)(1 - w\bar{z})} \right] + \frac{1}{2} (|z|^2 - 1)(|w|^2 - 1)$$

satisfying the biharmonic equation

$$(\partial_z \partial_w)^2 G(z, w) = \delta^{(2)}(z, w)$$

on $B$. For example, $G_{zw}(z, w) \equiv \partial_z \partial_w G(z, w)$ reads as follows:

$$G_{zw}(z, w) = \frac{1}{2} \left[ -\left( \frac{\bar{z} - \bar{w}}{z - w} \right) + \bar{z} \left( \frac{\bar{z} - \bar{w}}{1 - w\bar{z}} \right) + \bar{w} \left( \frac{\bar{z} - \bar{w}}{1 - \bar{w}z} \right) + \bar{z}\bar{w} \right]$$

(6.2)

Analogously, one can compute $G_{\bar{z}\bar{w}}(z, w) \equiv \partial_{\bar{z}} \partial_{\bar{w}} G(z, w)$. The correct boundary conditions of the fields at the boundary $\partial B$ of $B$ are:

$$A_{\bar{z}} = A_{\bar{w}} = 0 \quad z, \bar{z} \in \partial B$$

(6.3)

In this way, the spurious harmonic gauge transformation typical of the covariant gauge fixing (2.10) are eliminated. It is easy to show that the propagator (6.2) vanishes at the boundary in $z$ and $w$ separately according to eq. (6.3). Moreover, if (6.3) is satisfied, the Hodge decomposition (3.3) is still valid (see e.g. [8]). As previously remarked, there are no harmonic components in this case. We remember also that in order to derive eq. (5.7) we used the freedom of doing partial integrations in the action (5.1). All the possible boundary terms that can be generated on $B$ in this way are however killed by the boundary conditions (6.3), so that eq. (5.7) holds also on a disk. Now we compute the short distance behavior of $G_{zw}(z, w)$. Setting $z - w = \rho e^{i\theta}$ with $\rho \to 0$, we get from eq. (6.2):

$$\lim_{\substack{z \to w \\ \bar{z} \to \bar{w}}} G_{zw}(z, w) = -\frac{1}{2} e^{-2i\theta} + \frac{1}{2} \bar{w}^2$$

(6.4)
A complete different result arises in the case of a sphere $S^2$ with metric $g_{z\bar{z}}dzd\bar{z} = \frac{dzd\bar{z}}{(1+z\bar{z})^2}$.

On the sphere, the function $K(z,w)$ of eq. (4.9) becomes:

$$K(z,w) = \log \left[ \frac{|z-w|^2}{(1+z\bar{z})(1+w\bar{w})} \right]$$  \hspace{1cm} (6.5)

In order to find the short distance behavior of $G_{zw}(z,w)$ on $S^2$ we have to compute:

$$G_{zw}(z,z) = \int_{S^2} d^2t \left( \partial_z G(z,t) \right)^2 g_{t\bar{t}}$$  \hspace{1cm} (6.6)

It is now possible to prove the following identity:

$$(\partial_z K(z,t))^2 = - (\partial_z + g_{z\bar{z}}\partial_z g^{z\bar{z}})\partial_z K(z,t)$$

Exploiting the above equation and the fact that $\int_{S^2} d^2t \partial_z G(z,t)g_{t\bar{t}} = 0$, we find:

$$G_{zw}(z,z) = 0$$  \hspace{1cm} (6.7)

This result is profoundly different with respect to that of eq. (6.4) and shows how the topology can influence the behavior of the fermions at high energy. In particular, eq. (6.7) shows that at short distances the electrons do not feel any interaction on the sphere.

What happens in the case of a Riemann surface? We expect a result similar to that of the sphere, with the only difference that now the right hand side of eq. (6.7) will be not zero due to the presence of the zero modes. The evaluation of

$$G_{zz}(z,z) = \int_\mathcal{M} d^2t (\partial_z K(z,t))^2 g_{t\bar{t}}$$  \hspace{1cm} (6.8)

requires an equation that expresses the function $(\partial_z K(z,t))^2$ in terms of linear combinations of $K(z,w)$ and its derivatives as we did in the case of the sphere. An equation of this kind has been derived in ref. [25]. Here we will only show that the integral (6.8) is proportional to a zero mode following ref. [26]. First of all, instead of $\partial_z K(z,t)$ we consider the tensor:

$$m_z(z,t) = \partial_z K(z,t) + \sum_{i,j=1}^g \omega_i(z) |\text{Im } \Omega_{ij}|^{-1} \int_{\bar{s}_0}^{\bar{s}} \bar{\omega}_j(\bar{s})d\bar{s}$$  \hspace{1cm} (6.9)
This tensor is multivalued on the Riemann surface \( M \) but has the advantage that it is easier to handle than \( \partial_{\bar{z}} K(z,t) \). It is possible to see that in terms of \( m_{z}(z,t) \) eq. (6.8) becomes:

\[
\int d^{2}tg_{ti}(\partial_{\bar{z}} K(z,t))^2 = \int d^{2}tg_{ti}m_{z}^{2}(z,t) - A \left( \sum_{i,j=1}^{g} \omega_{i}(z) |\text{Im } \Omega_{ij}|^{-1} \int_{\xi_{0}}^{\bar{z}} \bar{\omega}_{j}(\bar{s})d\bar{s} \right)^2
\]

(6.10)

Here we have exploited the fact that from eq. (4.8) it descends that:

\[
\int d^{2}tg_{ti}\partial_{\bar{z}} K(z,t) = 0
\]

(6.11)

Now we evaluate \( \int d^{2}tg_{ti}m_{z}^{2}(z,t) \). To do this, we expand \( m_{z}(z,t) \) around the singularity at \( z = t \):

\[
m_{z}(z,t) \sim \frac{1}{z-t} - \Omega_{z}(z) + \omega_{z}(z) + O(z-t)
\]

(6.12)

where

\[
\Omega_{z}(z) = \frac{1}{A} \int_{M} d^{2}s g_{ss}\partial_{z} \tilde{K}(z,s)
\]

(6.13)

and \( \omega_{z}(z) \) is an irrelevant zero mode. The expansion (6.12) is motivated by the fact that \( m_{z}(z,t) \) satisfies the following equation:

\[
\partial_{\bar{z}}m_{z}(z,t) = \delta^{(2)}_{zz}(z,t) - \frac{g_{zz}}{A} + \sum_{i,j=1}^{g} \omega_{i}(z) |\text{Im } \Omega_{ij}|^{-1} \bar{\omega}_{j}(\bar{z})
\]

(6.14)

In fact, applying the operator \( \partial_{\bar{z}} \) to the right hand side of equations (6.12), we obtain a perfect agreement with eq. (6.14). At this point, it is possible to estimate also the expansion of \( m_{z}^{2}(z,t) \) at the point \( z = t \):

\[
m_{z}^{2}(z,t) \sim \frac{1}{(z-t)^{2}} - 2\frac{(\Omega_{z}(z) - \omega_{z}(z))}{z-t} + \text{zero modes}
\]

(6.15)

The zero modes in eq. (6.15) are multivalued, due to the fact that \( m_{z}(z,t) \) is multivalued on the Riemann surface. Looking at eq. (6.13), we try the following ansatz for \( m_{z}^{2}(z,t) \):

\[
m_{z}^{2}(z,t) = \partial^{2}_{\bar{z}} K(z,t) - 2\partial_{\bar{z}} K(z,t)(\Omega_{z}(z) - \omega_{z}(z)) + \psi_{zz}(z,t) +
\]

\[
\left( \sum_{i,j=1}^{g} \omega_{i}(z) |\text{Im } \Omega_{ij}|^{-1} \int_{\xi_{0}}^{\bar{z}} \bar{\omega}_{j}(\bar{s})d\bar{s} \right)^{2}
\]

(6.16)
With this ansatz, at least the pole structure of $m^2_z(z,t)$ is reconstructed. In fact, if we expand the right hand side of eq. (6.16) at $z = t$, the result is in agreement with eq. (6.15).

It remains a zero mode contribution $\psi_{zz}(z,t)$, which satisfies the equation $\partial_z \psi_{zz}(z,t) = 0$ and cannot be determined. We notice also that in eq. (6.16) the quantity $\partial^2_z K(z,t)$ is not a true tensor, because we do not have used the covariant derivatives $\nabla_z$. A true tensor with a double pole in $z = t$ is $\nabla^2_z K(z,t)$, but we do not need it, because the expression of $m^2_z(z,t)$ contained in (6.16) should be inserted in eq. (6.10) and, when integrated, it yields:

$$\int_M d^2t g_{tt} \nabla^2_z K(z,t) = \int_M d^2t g_{tt} \partial^2_z K(z,t) = 0$$

Using the above identity together with eqs. (4.8) and (6.11), we arrive at the following result:

$$\int_M d^2t g_{tt} m^2_z(z,t) = \int_M d^2t g_{tt} \psi_{zz}(z,t) + A \left( \sum_{i,j=1}^g \omega_i(z) |\text{Im } \Omega_{ij}|^{-1} \int_{z_0}^{z} \bar{\omega}_j(s) d\bar{s} \right)^2$$

Substituting this equation in (6.10) we get:

$$\int_M d^2t g_{tt} (\partial_z K(z,t))^2 = \int_M d^2t g_{tt} \psi_{zz}(z,t) - A \left( \sum_{i,j=1}^g \omega_i(z) |\text{Im } \Omega_{ij}|^{-1} \int_{z_0}^{z} \bar{\omega}_j(s) d\bar{s} \right)^2$$

As a consequence of the fact that the left hand side of this equation is singlevalued on the Riemann surface $M$, also the zero mode $\psi_{zz}(z,t)$ should be singlevalued. Therefore, it must be a linear combination of the $3g - 3$ solutions $\psi_{s,zz}(z,t)$ of the equation $\partial_z \psi_{s,zz}(z,t) = 0$ that are allowed on $M$. This linear combination is of the following kind:

$$\psi_{zz}(z,t) = \sum_{s=1}^{3g-3} f_s(t) \psi_{s,zz}(z)$$

where the functions $f_s(t)$, which depend only on the variable $t$, are until now undetermined. Inserting the above expression in eq. (6.17), we have however still a little simplification:

$$\int_M d^2t g_{tt} (\partial_z K(z,t))^2 = \sum_{s=1}^{3g-3} \psi_{s,zz}(z) c_s$$

(6.18)

with $c_s = \int_M d^2t g_{tt} f_s(t)$. In this way, remembering eq. (6.8), we have shown that on a Riemann surface the asymptotic form at very short distances of the potential $G(z,w)$
governing the behavior of the electrons is completely determined by the zero modes of the kind $\psi_{s,zz}(z)$. Unfortunately, we could not derive the coefficients $c_s$ and, even using the more sophisticated methods of ref. [25], it seems not possible to obtain their explicit form. However, general physical considerations would suggest that $c_s = 0$ for $s = 1, \ldots, 3g - 3$. In fact, zero modes are not expected to contribute to observable effects.

7. CONCLUSIONS

In this paper we have quantized the abelian gauge field theories on a Riemann surface for any nontrivial line bundle $P(M, U(1))$. Despite of the fact that we applied the formalism developed here only to the Schwinger model, the method is valid for any gauge field theory with $U(1)$ gauge group of symmetry. The only restriction is that the explicit form of the propagators of the matter fields should be known. For example, this is not the case of the massive fermions of scalar fields. Another technical difficulty is that perturbation theory on a manifold is intrinsically more complicated than in the flat case. As a matter of fact, the flat connections $A^\text{har}_\alpha$ should be treated as external fields. They generate in this way new Feynmann graphs and there is the problem of integrating over the moduli space of flat connections in the path integral. This difficulty is not present if we consider the Schwinger model on a nontrivial line bundle. If $A^1_\alpha = 0$, in fact, the current $J^m_\alpha$ of eq. (5.2) is conserved: $\partial_z J^m_z + \partial_{\bar{z}} J^m_{\bar{z}} = 0$. Hence, $J^m_\alpha$ is a purely transversal vector and, using the orthogonality properties of the Hodge decomposition, we have in eq. (5.7):

$$\int_M d^2z A^\text{har}_\alpha J^m_\alpha = 0$$

As a consequence, the integrand in eq. (5.7) is independent of $\theta_i$ and $\phi_i$, so that the integration in these variables can be factored out yielding the following generating functional:

$$Z_{\text{QED}}[\bar{\xi}, \xi, k = 0] = \int D\bar{\psi}D\psi \exp \left\{ - \int_M d^2z \left[ \bar{\psi}\theta(\partial_{\bar{z}} + A^I_{\bar{z}})\psi + \bar{\psi}\xi_{\theta} + \psi\theta\xi_{\bar{\theta}} \right] + \frac{1}{4} \int_M d^2z d^2z' \left( \bar{\psi}\theta G_{\bar{z}'z}(z, z')\psi_{\bar{\theta}} \right) + c.c. \right\} \quad (7.1)$$

Let us remember that eq. (5.7) can be extended also to the case in which the spinor fields are replaced by more general $b - c$ systems of integer or half-integer conformal weights. In
this case, however, the matter fields have zero modes on a Riemann surface and eq. (5.7) cannot be simplified to eq. (7.1) even on a trivial line bundle.

Another improvement with respect to ref. [17] is that here we succeeded in integrating over the nonphysical gauge degrees of freedom in the path integral of the Schwinger model. In this way, we have obtained an effective field theory describing the dynamics of the two dimensional electrons on a Riemann surface. There are many interesting aspects of this theory which should be better understood. For example, the effective action:

\[
S_{\text{eff}} = \int_M d^2z \left[ \bar{\psi}_\theta (\partial_\bar{z} + A_\bar{z}) \psi_\theta + \bar{\psi}^\theta \xi_\theta + \psi^\theta \xi_\theta \right] + \int_M d^2z d^2z' \bar{\psi}_\theta \psi_\theta G_{z'z}(z, z') \bar{\psi}_\bar{\theta} \psi_{\bar{\theta}} + \text{c.c}
\]

where \( A_\bar{z} = A^I_\bar{z} + A^\text{har}_\bar{z} \), should describe an integrable model, since the Schwinger model is integrable. As a matter of fact, the above action can be interpreted as the action of a generalized version of the Thirring model. Nevertheless, the integrability of \( S_{\text{eff}} \) on a Riemann surface is not clear a priori. Moreover, we still do not know the explicit form of the zero modes in the fermionic sector when \( c_1 \neq 0 \). Since the main subject of this paper is the quantization of the gauge field theories, we will answer these questions elsewhere. Nonetheless, interesting physical informations concerning the high energy behavior of the electrons on a Riemann surface have already been extracted from eqs. (6.4), (6.7) and (6.18). Unfortunately, we were not able to estimate the important zero mode contribution appearing in the asymptotic expression of the potential \( G_{zw}(z, w) \) at short distances of eq. (6.18). However, we believe that it is possible to obtain its explicit form representing the Riemann surface as an algebraic curve and exploiting the methods developed in refs. [27].

8. ACKNOWLEDGEMENTS

The final version of this paper has been greatly benefitted by fruitful conversations with C. M. Hull, M. Mintchev, O. A. Solov’ev and A. Wipf. In particular I am grateful to C. M. Hull who suggested the use of the Feynman gauge and O. A. Solov’ev, who drawed my attention to ref. [25]. I am also grateful to M. B. Green for giving me the possibility of talking at Queen Mary college on the subject of \( QED_2 \) on curved space–times. Finally, I would like to thank S. Theisen and J. Wess for their kind hospitality at the Ludwig Maximilian University of Munich and for their interest in my work. This work was supported by a grant of Consiglio Nazionale delle Ricerche, P.le A. Moro 7, Italy.
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