LOCALIZATION AND STATIONARY PHASE APPROXIMATION ON SUPERMANIFOLDS

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Abstract. Given an odd vector field $Q$ on a supermanifold $M$ and a $Q$-invariant density $\mu$ on $M$, under certain compactness conditions on $Q$, the value of the integral $\int_M \mu$ is determined by the value of $\mu$ on any neighborhood of the vanishing locus $N$ of $Q$. We present a formula for the integral in the case where $N$ is a subsupermanifold which is appropriately non-degenerate with respect to $Q$.

In the process, we discuss the linear algebra necessary to express our result in a coordinate independent way. We also extend stationary phase approximation and the Morse-Bott Lemma to supermanifolds.

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1. Introduction

The localization phenomenon in the form discussed in this paper, although not in the language of super geometry, first appeared in the work of Duistermaat and Heckman [DH82]. They compute the volume of a symplectic manifold with a Hamiltonian circle action in terms of data at the fixed
points of the action. The ideas were extended to localization in equivariant cohomology by Berline and Vergne [BV82] and independently by Atiyah and Bott [AB84].

The study of supermanifolds originated in physics in order to understand supersymmetry and in particular to make supersymmetry manifest on the level of classical physics. The path integral quantization in this case involves an integral over an infinite-dimensional supermanifold. Witten [Wit82] was the first to apply the localization techniques to supersymmetric theories in order to reduce infinite dimensional path integrals to finite dimensional ones. Localization turned out to be a very powerful tool for studying non-perturbative aspects of supersymmetric field theories. For a survey of recent applications of localization techniques to supersymmetric quantum field theories see [PZB+16].

In order to isolate the essential features of localization, Schwarz and Zaboronsky [SZ97] analyzed the case of general finite dimensional supermanifolds. They show that given an odd vector field $Q$ on a supermanifold $M$ such that $[Q,Q]$ comes from an action of a compact torus, the integral $\int_M \mu$ depends only on the value of $\mu$ on any open neighborhood of the vanishing space $N$ of $Q$. Moreover, in the case where $N$ is an appropriately non-degenerate discrete subsupermanifold with respect to $Q$, Schwarz and Zaboronsky derive a formula for the integral $\int_M \mu$ in terms of the restriction to $N$ of $\mu$ and the action of the Lie derivative $L_Q$ on the normal bundle of $N$. In this paper we extend their result to the case where $N$ is a non-degenerate subsupermanifold which need not be discrete (Theorem 5.2.2).

We summarize the arguments of Schwarz and Zaboronsky from [SZ97]. Under the assumptions that $Q^2$ comes from an action of a compact torus, the authors construct an odd function $\sigma$ having the properties that $Q^2 \sigma = 0$ and $Q \sigma$ is invertible on the complement of $N$. Then, if the support of $\mu$ is disjoint from $N$ we have

$$\int_M \mu = \int_M L_Q (\frac{\sigma}{Q \sigma} \mu) = 0$$

where the last equality follows from the invariance of integration under diffeomorphisms. More generally, as we will review in §5, given an open neighborhood $U \supset N$ there is a $Q$-invariant even function $g$ which equals 1 in a neighborhood of $N$ and vanishes outside of $U$. By the previous argument, for any such $g$ we have

$$\int_M \mu = \int_M g \mu + \int_M (1-g) \mu = \int_M g \mu.$$

To compute the actual value of the integral, the authors consider the function

$$Z(\lambda) := \int_M \mu e^{i \lambda Q \sigma}$$

for $\lambda \in \mathbb{R}_{\geq 0}$. Since

$$\frac{d}{d\lambda} Z(\lambda) = i \int_M \mu Q \sigma e^{i \lambda Q \sigma} = i \int_M L_Q (\mu \sigma e^{i \lambda Q \sigma}) = 0,$$

the function $Z(\lambda)$ is constant. The stationary phase approximation computes the asymptotic behavior of $Z(\lambda)$ as $\lambda \to \infty$ in terms of the local data of $\mu$ and $Q \sigma$ on the critical subsupermanifold of $Q \sigma$ in case the critical subsupermanifold is non-degenerate. Since $Z(\lambda)$ does not depend on $\lambda$, the limit $\lim_{\lambda \to \infty} Z(\lambda)$ equals the desired integral $\int_M \mu = Z(0)$.

In this paper, we first work out an expression for stationary phase approximation on supermanifolds, i.e. assuming $Q \sigma$ has non-degenerate critical subsupermanifold, we compute the asymptotic behavior of $Z(\lambda)$ in terms of the Hessian of $Q \sigma$ and restriction of $\mu$ to the critical subsupermanifold (Theorem 4.2.2). To do this, we prove a generalization of the Morse-Bott Lemma to supermanifolds

\footnote{It will be made precise what we mean by the vanishing space in §5.1.}
We then show that if the vanishing space of $Q$ is a non-degenerate\footnote{The vanishing locus $N$ of a vector field $Q$ is non-degenerate if it is a subsupermanifold and the restriction of the Lie derivative $L_Q$ to the normal bundle $\nu_N$ is an automorphism.} subsupermanifold $N$, then $N$ is also the non-degenerate critical subsupermanifold of $Q\sigma$ and we compute the Hessian of $Q\sigma$ in terms of the Lie derivative $L_Q$. In particular, we end up with a formula (Theorem 5.2.2) for $\int_M \mu$ in terms of $Q$ and $\mu$ not depending on the auxiliary function $\sigma$. This has been done in [SZ97] in the case where $N$ consists of isolated points.

In §2 and §3, we will briefly review constructions of super linear algebra and supermanifolds that will be necessary in later chapters. For a more elaborate introduction, see [DM99, Lei80, Man88]. In §4 we will discuss the Morse-Bott Lemma and stationary phase approximation in the setting of supermanifolds. In §5, we state and prove the localization theorem.
where \( \theta_I := \prod_{i \in I} \theta^i \) with the product taken with increasing order of index and \( a_I \in \mathbb{R} \). Note that \((\theta^i)^2 = 0\) for all \( i \) and \( \theta^i \theta^j = -\theta^j \theta^i \).

### 2.2. Modules.

Let \( A \) be a super algebra. A left \( A \)-module \( M \) is a super vector space with the action given by an even morphism of super vector spaces

\[
A \otimes M \to M
\]

satisfying

\[
1.m = m \quad \forall m \in M
\]

\[
a.(b.m) = (ab).m \quad \forall a,b \in A, \quad \forall m \in M.
\]

Right \( A \)-modules are defined analogously. If \( A \) is commutative then a left module structure on \( M \) defines a right module structure via

\[
m.a = (-1)^{p(a)p(m)}a.m
\]

We define the parity reversal functor \( \Pi \) on left \( A \)-modules in the following way: Let \( \tilde{\Pi} \) be a free left \( A \)-module generated by an odd element \( \pi \). We give it a right action of \( A \) by

\[
(a.\pi)b = (-1)^{p(b)ab}\pi
\]

For any left \( A \)-module \( M \) we define

\[
\Pi M := \tilde{\Pi} \otimes_A M.
\]

As an ungraded vector space, \( \Pi M \) is isomorphic to \( M \) but the parity of homogeneous elements is flipped. We use the notation

\[
\Pi m := \pi \otimes m
\]

where \( m \in M \). The \( A \) action on \( \Pi M \) is

\[
a.(\Pi m) = (-1)^{p(a)\Pi(a.m)}.
\]

A homomorphism between two right \( A \)-modules \( N, M \) is a map

\[
f : M \to N
\]

such that

\[
f(m.a) = f(m).a
\]

A homomorphism \( f \) between two left \( A \)-modules \( M, N \) is such that

\[
f(am) = (-1)^{p(a)p(f)}af(m).
\]

One can check that when \( A \) is commutative, the notions of morphisms between \( M \) and \( N \) as left and right \( A \)-modules coincide. The vector space \( \text{Hom}_A(M, N) \) is naturally graded by whether a homomorphism preserves the grading or reverses it.

Assume from now on that \( A \) is a commutative super algebra. A free module \( M \) over \( A \) has dimension \( p|q \) if it is freely generated by \( p \) even elements and \( q \) odd elements. Let \( M \) be a free module and consider a basis \( \{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\} \) of \( M \) where \( e_1, \ldots, e_p \) are even and \( e_{p+1}, \ldots, e_{p+q} \) are odd. An element \( m \in M \) can be given by right coordinates \( \{m_i\} \):

\[
m = \sum_i e_i m_i.
\]

We will briefly develop the conventions we will use when describing operations on modules via matrices in a fixed bases. We will denote column vectors by \( |\cdot\rangle \), i.e. for \( m \in M \), \( |m\rangle \) is the column vector of right coordinates of \( m \). Let \( M, N \) be free modules with bases \( \beta = \{e_i\} \) and \( \gamma = \{f_j\} \) respectively. Given a morphism \( F : M \to N \), define the matrix \( F^\gamma_\beta \) by

\[
F(e_i) = \sum_j f_j \cdot [F^\gamma_\beta]_{ij}.
\]
We will omit explicit reference to the bases when there is no ambiguity. It is straightforward to check that 

$$|Fm⟩ = F|m⟩.$$ 

The matrix of $F$ is naturally written in block form. If $M$ and $N$ have dimensions $p|q$ and $p'|q'$ respectively then

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A \in \text{Mat}(p' \times p)$, $B \in \text{Mat}(q' \times p)$, $C \in \text{Mat}(p' \times q)$ and $D \in \text{Mat}(q' \times q)$. If $F$ is an even homomorphism, then entries of $A$ and $D$ are even while entries of $B$ and $C$ are odd.

If $β = \{e_i\}$ and $β' = \{e'_i\}$ are two bases, the coordinate change matrix from basis $β$ to basis $β'$ is the matrix $G_{β}^{β'}$ such that 

$$e_i = \sum_j e'_j \cdot [G_{β}^{β'}]_{ij}.$$ 

If $|m⟩$ is the vector representation of $m \in M$ in the basis $β$, then $G_{β}^{β'}|m⟩$ is the vector representation of $m$ in the basis $β'$.

For $M$ an $A$-module, the dual module $M^*$ is defined by 

$$M^* := \text{Hom}_A(M, A).$$

A basis $β = \{e_i\}$ of $M$ defines a dual basis $β^* = \{e^*_i\}$ of $M^*$ by 

$$e^*_i(e_j) = δ^*_i.\]

A morphism $F : M \rightarrow N$ induces a morphism 

$$F^* : N^* \rightarrow M^*$$

by the formula

$$F^*(n^*)(m) := (-1)^{p(n^*)p(F)} n^*(F(m))$$

for all $n^* \in N^*, m \in M$ homogeneous. To state what the matrix representation of the dual homomorphism is, we need to introduce the super analog of the transpose of a matrix.

For a matrix

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

corresponding to a homogeneous morphism as in equation (2.1) we define the supertranspose by 

$$F^{st} = \begin{cases} 
(A^t & C^t) & p(F) = 0 \\
(-B^t & D^t) & p(F) = 1.
\end{cases}$$

To be precise, the supertranspose is defined on a matrix consisting of elements of $A$ only after we know the dimensions of the free modules it is meant to act on. For free modules $M, N$ with fixed bases and a homomorphism $F : M \rightarrow N$, the matrix of the dual homomorphism $F^*$ in the dual bases is given by the supertranspose, i.e.,

$$(F^*)_{β^*}^{β} = (G_{β}^{β'})^{st}.$$
2.3. The Berezinian. The Berezinian is the super analog of the determinant. It is defined on even automorphisms of free $A$-modules. Let $F$ be an even automorphism of a free module $M$. Given a basis of $M$, the matrix representation of $F$ can be written in block form as

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$ 

The Berezinian of $F$ is defined by

$$\text{Ber}(F) = \det(A - BD^{-1}C) \det(D)^{-1}.$$ 

The expression for the Berezinian does not depend on the chosen basis [DM99 § 1.10].

Classically, one can define determinant of an automorphism by its action on the determinant line which is the top exterior power. In the same spirit, we can define the Berezinian line of a module by prescribing how an automorphism of the module acts on this line. Let $M$ be a free $A$-module of dimension $p|q$. The Berezinian line of $M$, denoted by $\text{Ber}M$ has dimension $1|0$ if $q$ is even and $0|1$ if $q$ is odd. A basis $\beta = \{e_1, \ldots, e_{p+q}\}$ of $M$ defines a basis element $b_\beta$ of $\text{Ber}M$ and for an even automorphism $G$ of $M$, the elements $b_\beta$ and $b_{G\beta}$ satisfy the relation

$$b_{G\beta} = \text{Ber}(G)b_\beta.$$ 

An even isomorphism $F : N \rightarrow M$ induces a map $\text{Ber}F : \text{Ber}N \rightarrow \text{Ber}M$ by

$$b_\beta \mapsto b_{F\beta}.$$ 

One can check that for an even automorphism $F : M \rightarrow M$,

$$\text{Ber}(F) = \text{Ber}(F^\star).$$ 

Also, if we denote by $\Pi F : \Pi M \rightarrow \Pi M$ the automorphism of $\Pi M$ induced by $F$, then

$$\text{Ber}(\Pi F) = \text{Ber}(F)^{-1}.$$ 

It follows that there are natural isomorphisms between $\text{Ber}(M)^\star$, $\text{Ber}(M^\star)$ and $\text{Ber}(\Pi M)$. For a basis $\beta$, these isomorphisms identify $(b_\beta)^\star$, $b_\beta^\star$ and $b_{\Pi\beta}$ where $\Pi\beta = \{\Pi e_{p+1}, \ldots, \Pi e_{p+q}, \Pi e_1, \ldots, \Pi e_p\}$ if $\beta = \{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}\}$.

2.4. Orientation. Let $A$ be a commutative super algebra over $\mathbb{R}$.

**Definition 2.4.1.** A sign homomorphism is a group homomorphism

$$\text{sgn} : A^\times \rightarrow \{\pm 1\}$$ 

extending the usual sign on $\mathbb{R}^\times$ where $A^\times$ denotes the invertible elements of $A$.

We refer to elements in $\text{sgn}^{-1}(+1)$ as positive and those in $\text{sgn}^{-1}(-1)$ as negative. Let $M$ be a free $p|q$ module over $A$. There are multiple, non-equivalent notions of orientation on $M$ coming from even and odd parts of $M$.

**Definition 2.4.2.** For $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} \in GL(p|q, A)$, we define the $(i, j)$ orientation of $D$ by

$$\text{or}_{(i, j)}(D) = \text{sgn}(\det(D_{11})^i \det(D_{22})^j)$$ 

where $i, j \in \{0, 1\}$.

It is straightforward to show that $\text{or}_{(i, j)}$ is well defined on $GL(M)$ i.e., does not depend on the chosen basis.

**Definition 2.4.3.** For $i, j \in \{0, 1\}$, we define $\text{or}_{(i, j)}(M)$ to be the set of equivalence classes of bases of $M$ where two bases $\beta, \beta'$ are equivalent if $\text{or}_{(i, j)}$ of the matrix corresponding to the change of basis from $\beta$ to $\beta'$ is 1.
2.5. Bilinear Forms. Let $A$ be a commutative super algebra over $\mathbb{R}$ with a sign homomorphism and $M$ a free $A$-module. A bilinear form on $M$ is a morphism $B : M \otimes M \to A$. A bilinear form $B$ induces a map

$$\hat{B} : M \to M^*$$

by

$$\hat{B}(m_1)m_2 = B(m_1, m_2).$$

If $B$ is a non-degenerate even bilinear form, then $\hat{B}$ is an even isomorphism and thus induces a map

$$\text{Ber}(\hat{B}) : \text{Ber}(M) \xrightarrow{\sim} \text{Ber}(M^*) \cong \text{Ber}(M)^*.$$

**Definition 2.5.1.** For $B$ a non-degenerate even bilinear form, we define $\text{Ber}(B) \in \text{Ber}(M^*)^\otimes 2$ to be the image of 1 under the map

$$A \cong \text{Ber}(M) \otimes \text{Ber}(M)^* \xrightarrow{(\text{Ber}(\hat{B}) \otimes \text{id})} \text{Ber}(M^*)^\otimes 2.$$

We also define the element $\text{Ber}^{-1}(B) \in \text{Ber}(M)^{\otimes 2}$, which is the unique element satisfying

$$\text{Ber}(B) \left( \text{Ber}^{-1}(B) \right) = 1$$

where we used the fact that for any module $N$,

$$N^* \otimes N^* \cong (N \otimes N)^*$$

where the isomorphism is given by

$$(n_1^* \otimes n_2^*)(n_1 \otimes n_2) = (-1)^{p(n_1)p(n_2)} n_1^*(n_1)n_2^*(n_2)$$

for $n_i \in N$ and $n_i^* \in N^*$.

In the case where both $A$ and $M$ are purely even, we will denote these elements by $\text{det}(B)$ and $\text{det}^{-1}(B)$ respectively.

**Definition 2.5.2.** For $B$ a non-degenerate even symmetric form, we define an element $\text{or}_{(0,1)}(B) \in \text{or}_{(0,1)}(M)$ by the following: a basis $\beta = \{e_i\} \in \text{or}_{(0,1)}(B)$ if the matrix $B_\beta = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ satisfies

$$\text{sgn}(\text{pfaff}(B_{22})) = 1$$

where $(B_\beta)_{ij} = B(e_i, e_j)$ and pfaff denotes the Pfaffian of a skew-symmetric matrix.

The main idea of this construction is that a symmetric bilinear form is skew-symmetric on the odd part of the module in the ungraded sense, and there defines an orientation the same way a symplectic form does classically.

2.6. Berezinian of an Odd Isomorphism. Let $A$ be a commutative super algebra and $M$ be a free $p|q$ module over $A$. Let $E$ be an odd automorphism of $M$. We will construct an element of $\text{Ber}(M)^{\otimes 2}$ from this automorphism. We have that $E$ defines an isomorphism

$$\hat{E} : \Pi M \to M$$

and therefore also

$$\text{Ber}(\hat{E}) : \text{Ber}(\Pi M) \to \text{Ber}(M).$$

Via the identification of $\text{Ber}(\Pi M)$ and $\text{Ber}(M^*)$ the above map becomes

$$\text{Ber}(\hat{E}) : \text{Ber}(M^*) \to \text{Ber}(M).$$

**Definition 2.6.1.** We denote by $\text{Ber}(E)$ the image of 1 in $\text{Ber}(M)^{\otimes 2}$ of the map

$$A \cong \text{Ber}(M^*) \otimes \text{Ber}(M) \xrightarrow{\text{Ber}(\hat{E}) \otimes \text{id}} \text{Ber}(M)^{\otimes 2}.$$
If we were to perform an analogous construction for an even automorphism, we would get an element of $\text{Ber}(M^*) \otimes \text{Ber}(M)$ which is naturally isomorphic to $A$.

In coordinates, $\text{Ber}(E)$ is given by the following expression: for $\beta$ a basis of $M$, $b_\beta \otimes b_\beta$ is a basis element of $\text{Ber}(M)^{\otimes 2}$ and we have

$$\text{Ber}(E) = \text{Ber}(E_\beta^\beta I') b_\beta \otimes b_\beta$$

where

$$I' = \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix}$$

and $E_\beta^\beta$ is the matrix representation of $E$ in the basis $\beta$.

### 2.7. Square Root of One-Dimensional Modules.

Let $A$ be a commutative super algebra over $\mathbb{R}$ with a sign homomorphism. Suppose we are given an automorphism of the positive elements of $A$

$$\sqrt{\cdot} : A^\times_0 \rightarrow A^\times_0$$

such that $(\sqrt{a})^2 = a$ for all $a \in A^\times_0$. For example, we have such map for the algebra of functions on a supermanifold.

**Definition 2.7.1.** For any (even or odd) one-dimensional module $M$, we define the map

$$\sqrt{\cdot} : (M^{\otimes 2})_\times \rightarrow M \otimes \text{or}_{(1,1)}(M)$$

where $M_\times$ denotes the set of elements of $M$ that constitute a basis. If $v$ is a basis element of $M$, then $v \otimes v$ is a basis element of $M^{\otimes 2}$ and we define

$$\sqrt{fv \otimes v} := \sqrt{|f|} v \otimes \text{or}_{(1,1)}(v)$$

where $f \in A^\times_0$ and $|f| = \text{sgn}(f) \cdot f$ is positive.

### 3. Calculus on Supermanifolds

**3.1. Supermanifolds.** Just like many other geometric objects (manifolds, schemes, analytic spaces), a supermanifold is a locally ringed space with a particular local model. The local model in this case is the topological space $\mathbb{R}^m$ with the structure sheaf $C^\infty(\mathbb{R}^m)[\theta^1, \ldots, \theta^n]$.

**Definition 3.1.1.** A supermanifold of dimension $m|n$ is a pair $M = (\mathcal{M}, \mathcal{O}_M)$ where $\mathcal{M}$ is a topological space and $\mathcal{O}_M$ is a sheaf of super algebras such that every point $m \in \mathcal{M}$ has a neighborhood $U \subset \mathcal{M}$ such that $(U, \mathcal{O}_M|_U) \cong (\mathbb{R}^m, C^\infty(\mathbb{R}^m)[\theta^1, \ldots, \theta^n]) =: \mathbb{R}^{m|n}$.

**Example 3.1.2.** Let $\overline{M}$ be an ordinary $m$-manifold and $E \rightarrow \overline{M}$ a rank $n$ vector bundle. Then the pair $(M, \Gamma(\bigwedge^* E))$ is a supermanifold of dimension $m|n$ which we denote by $\Pi E^*$.

A morphism of supermanifolds $f : M \rightarrow N$ is a continuous map

$$|f| : \overline{M} \rightarrow \overline{N}$$

and an even morphism of sheaves of rings over $\overline{M}$

$$f^* : |f|^{-1} \mathcal{O}_N \rightarrow \mathcal{O}_M$$

which on the level of stalks is a morphism of local rings.

A supermanifold has an underlying ordinary manifold. Let $J \subset \mathcal{O}_M$ be the ideal generated by odd elements. The locally ringed space $(\mathcal{M}, \mathcal{O}_M/J)$ is an ordinary smooth manifold since locally, the quotient $\mathcal{O}_M/J$ is isomorphic to $C^\infty(\mathbb{R}^m)$. This smooth manifold is called the reduced manifold of $M$ and denoted $M_{\text{red}}$. We have an embedding

$$M_{\text{red}} \hookrightarrow M$$

---

3See the footnote on page 16.
in the category of supermanifolds. From now on we’ll abandon the notation \( \overline{M} \) for the underlying topological space and denote it by \( M_{\text{red}}, \) i.e., \( M = (M_{\text{red}}, \mathcal{O}_M). \)

A supermanifold can be given by charts and gluing data. The gluing isomorphisms are maps of locally ringed spaces

\[ U \rightarrow V \]

where \( U, V \subset \mathbb{R}^{m|n} \). For ordinary manifolds, a map of locally ringed spaces as above is specified by the pullbacks of the coordinate functions of the image manifold. The same holds for supermanifolds: a morphism

\[ \phi : U \rightarrow V \]

where \( U \subset \mathbb{R}^{m|n}, V \subset \mathbb{R}^{m'|n'} \) is uniquely specified by \( \phi^* (y^j) \) and \( \phi^* (\xi^i) \) where \( y^j, \xi^i \) are the coordinate functions on \( V \). Conversely, any collection of \( m' \) even functions \( \psi_i \) and \( n' \) odd functions \( \eta_j \) on \( U \) such that \((\psi_1, \ldots, \psi_{m'}) \in V_{\text{red}} \subset \mathbb{R}^{m'} \) when evaluated on \( U_{\text{red}} \) give rise to a morphism \( \phi : U \rightarrow V \) such that \( \phi^* (y^j) = \psi_i \) and \( \phi^* (\xi^i) = \eta_j \). A notation for this morphism resembling that of an ordinary manifold is

\[ \phi(x^1, \ldots, x^m, \theta^1, \ldots, \theta^n) = (\psi_1, \ldots, \psi_{m'}, \eta_1, \ldots, \eta_{n'}) \]

where \( x^i, \theta^j \) are the coordinate functions on \( U \).

A vector bundle \( E \) of rank \( p|q \) over a supermanifold \( M \) is a sheaf of \( \mathcal{O}_M \) modules over \( M_{\text{red}} \) which is locally free of rank \( p|q \). Given a morphism of supermanifolds \( f : M \rightarrow N \), the pullback vector bundle is defined by

\[ f^* E = f^{-1} E \otimes_{f^{-1} \mathcal{O}_N} \mathcal{O}_M. \]

3.2. Vector Fields. The tangent bundle \( TM \) of a supermanifold is the sheaf of derivations of the structure sheaf i.e. \( \mathbb{R} \)-linear maps \( D : \mathcal{O} \rightarrow \mathcal{O} \) such that

\[ D(ab) = D(a)b + (-1)^{p(a)p(D)} aD(b). \]

Sections of the tangent bundle are referred to as vector fields. The rank of \( TM \) for a \((m|n)\)-supermanifold \( M \) is \((m|n)\) and in a local chart with coordinates \( x^1, \ldots, x^m, \theta^1, \ldots, \theta^n \), a basis of \( TM \) is given by \( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \theta^j} \), where

\[
\begin{align*}
\frac{\partial}{\partial x^i}(x^j) &= \delta^j_i; & \frac{\partial}{\partial x^i}(\theta^j) &= 0, \\
\frac{\partial}{\partial \theta^j}(\theta^i) &= \delta^i_j; & \frac{\partial}{\partial \theta^j}(x^i) &= 0.
\end{align*}
\]

A lot of geometric concepts having to do with vector fields carry over from classical geometry once we are able to phrase them in terms of maps of spaces and sheafs of sections. An important construction is the flow of a vector field. In its most naive sense it is only defined for even vector fields: If \( V \) is an even vector field on a supermanifold \( M \), then there exists a flow map

\[ \phi_V : D \rightarrow M \]

where \( D \subset M \times \mathbb{R}^{1|0} \) is an open neighborhood of \( M \times \{0\} \), having the property that for any function \( f \in \mathcal{O}_M \),

\[ \phi^*_V (V(f)) = \frac{\partial}{\partial t} \phi^*_V (f). \]

The flow map allows us to define the Lie derivative. We will state the definition for the Lie derivative of a vector field, but analogous definitions work for other geometric structures that will be defined later. Let \( V, W \in TM \) be vector fields, and assume that \( V \) is even. Then the Lie derivative of \( W \) with respect to \( V \) is

\[ \mathcal{L}_V W(f) := \frac{d}{dt} \bigg|_{t=0} W(i^*_V(\phi^*_V(f))) \]
where \(i_t: M \to M \times \mathbb{R}^{1|0}\) is the embedding \(i_t(m) = (m, t)\).

To define the Lie derivative with respect to an odd vector field \(Q\) on \(M\), we consider the even vector field \(\eta Q\) on \(M \times \mathbb{R}^{0|1}\) where \(\eta\) is the odd coordinate on \(\mathbb{R}^{0|1}\) and define \(L_Q\) by requiring
\[
L_{\eta Q}(\ast) = \eta L_Q(\ast).
\]

Similarly to the case of ordinary manifolds, the Lie derivative of vector fields can be expressed as the Lie bracket where in the case of supermanifolds we have to account for the Koszul sign rule. For \(V, W\) homogeneous vector fields, we define a new vector field \([V, W]\) of parity \(p(V) + p(W)\) by
\[
[V, W](f) = V(W(f)) - (-1)^{p(V)p(W)} W(V(f))
\]
where \(f\) is any function. We have
\[
L_V W = [V, W].
\]
Note that for an odd vector field \(Q\), we have \([Q, Q] = 2Q^2\) which in general need not vanish.

A \((i, j)\)-orientation of a supermanifold is a section of the bundle \(or_{(i, j)}(TM)\). A supermanifold might be \((i, j)\)-orientable for some pairs \((i, j)\) but not others. For example, a \((1, 0)\)-orientation of \(M\) is the same as orientation of \(M_{red}\) while a \((0, 1)\)-orientation of \(\Pi E\) where \(E \to N\) is an ordinary vector bundle, is an orientation of \(E\).

### 3.3. Integration

One can define differential forms on supermanifolds as sections of exterior powers of the cotangent bundle just as for ordinary manifolds, but if a supermanifold has non-zero odd dimension, then there are differential forms of arbitrary high degree. In particular, there is not a top degree form that one can integrate over a supermanifold. In order to get an integration theory, we need to generalize the top degree form in a different manner. For an ordinary vector space \(E\), the top degree exterior power is isomorphic to the determinant line. The generalization of the determinant to supergeometry is the Berezinian.

**Definition 3.3.1.** A section of \(\text{Ber}(TM^*)\) is called an integral form. A section of \(\text{Ber}(TM^*) \otimes or_{(1,0)}(TM)\) is called a density.

Suppose first that \(\mu\) is a compactly supported integral form on \(\mathbb{R}^{m|n}\). Let \(x^1, \ldots, x^m, \theta^1, \ldots, \theta^n\) be the coordinates on \(\mathbb{R}^{m|n}\). A basis \(\beta\) of \(TM^*\) is given by \(\beta = \{dx^1, \ldots, d\theta^n\}\) which is dual to the basis \(\{\partial_{x^1}, \ldots, \partial_{\theta^n}\}\) of \(TM\). The basis of \(\text{Ber}(TM^*)\) that we denoted \(b_\beta\) in \(\S 2.3\) is denoted by \([dx^1, \ldots, d\theta^n]\) in this context. We can thus write the integral form \(\mu\) as
\[
\mu = [dx^1, \ldots, d\theta^n] \left( \sum_{I \subset \{1, \ldots, n\}} f_I \theta^I \right)
\]
where \(\theta^I = \prod_{i \in I} \theta^i\) and \(f_I \in C^\infty_c(\mathbb{R}^m)\). We define the integral of \(\mu\) over \(\mathbb{R}^{m|n}\) to be
\[
\int_{\mathbb{R}^{m|n}} \mu = \int_{\mathbb{R}^{m|n}} [dx^1, \ldots, d\theta^n] \left( \sum_{I \subset \{1, \ldots, n\}} f_I \theta^I \right) = \int_{\mathbb{R}^m} dx^1 \ldots dx^n f_{\{1,2,\ldots,n\}}
\]
where the right side is an ordinary integral. We refer to [Man88] for a careful proof that this integral does not depend on the chosen coordinates \(x^1, \ldots, \theta^n\), but only on the induced orientation on \(\mathbb{R}^m\) or in other words on \(or_{(1,0)}(\mathbb{R}^{m|n})\).

Now suppose \(M\) is a \((1, 0)\)-oriented supermanifold and \(\mu\) a compactly supported form on \(M\). To calculate the integral of \(\mu\), we find a locally finite cover of \(M\) by oriented affine charts \(\{U_i\}\) and a partition of unity \(\{\phi_i\}\) subordinate to \(\{U_i\}\). The integral is defined by
\[
\int_M \mu = \sum_i \int_{U_i} \phi_i \mu
\]
where the sum on the right is finite since \( \mu \) is compactly supported and \( \{U_i\} \) is locally finite. The arguments for why the partition of unity exists and why this definition does not depend on the choices made is analogous to the classical case.

Instead of assuming that \( M \) is \((1,0)\)-oriented, we can instead integrate a density. In particular, a density can be integrated over a supermanifold regardless of whether it is \((1,0)\)-orientable.

One defines the Lie derivative of an integral form analogously to how we defined the Lie derivative of a vector field. Of importance in future sections is the following fact: if \( \mu \) is a compactly supported density on a supermanifold and \( V \) is a vector field, then

\[
\int_M L_V \mu = 0.
\]

This statement follows from the fact that integration is invariant under diffeomorphisms.

3.4. Subsupermanifolds. Let \( M \) be a supermanifold. A closed subsupermanifold \( N \) is given by a morphism \( i : N \to M \) such that restricted to the reduced manifolds, \( i \) is a closed embedding and \( di : TN \to i^*TM \) is an inclusion of a direct summand. Equivalently, a closed subsupermanifold can be given by a sheaf of ideals \( I \) of \( \mathcal{O}_M \) such that for every point \( p \in M_{\text{red}} \), there exists a coordinate chart \( U \ni p \) with coordinates \( \{x^1, \ldots, x^m, \theta^1, \ldots, \theta^n\} \) such that \( I|_U = \langle x^1, \ldots, x^k, \theta^1, \ldots, \theta^l \rangle \) for some \( k, l \in \mathbb{N} \). One can then define \( N_{\text{red}} \subset M_{\text{red}} \) as the set of points where the stalks of \( I \) are proper ideals of \( \mathcal{O}_M \) and the structure sheaf \( \mathcal{O}_N \) as \( i^{-1}_{\text{red}}(\mathcal{O}_M/I) \). Conversely, given an embedding \( i : N \to M \) as above, the sheaf of ideals defined by

\[
I(U) = \{ \psi \in \mathcal{O}_M | i^* \phi = 0 \}
\]

satisfies the desired property.

3.5. Vanishing Subsupermanifolds of Sections of Vector Bundles. Let \( E \) be a vector bundle on \( M \) and \( s \) a section of \( E \). We define the vanishing space of \( s \). We work locally on \( M_{\text{red}} \). Given a point \( p \in M_{\text{red}} \) there exists an open set \( U \ni p \) such that \( E \) is trivial when restricted to \( U \). Pick \( e_1, \ldots, e_k \), sections of \( E|_U \) trivializing the bundle. A section \( s \) can then be written as

\[
s = \sum_{i=1}^k e_i s_i
\]

where \( s_i \in \mathcal{O}_U \). We would like to define the vanishing space of \( s \) by the ideal

\[
I_s := \langle s_1, \ldots, s_k \rangle \subset \mathcal{O}_U.
\]

The ideal \( I_s \) does not depend on the choice of trivialization of \( E \). In general though, \( I_s \) does not define a subsupermanifold i.e. it is not locally generated by coordinate functions.

**Definition 3.5.1.** For \( s \) a section of a vector bundle, we say that the vanishing space of \( s \) is a subsupermanifold if the ideal \( I_s \) defines a subsupermanifold \( N \subset M \). In that case we call \( N \) the vanishing subsupermanifold of \( s \).

There is a more direct way to define the ideal \( I_Q \) for a vector field \( Q \) on \( M \). In particular, \( I_Q \) is generated by \( Q(\mathcal{O}_M) \). Assume that the vanishing space of \( Q \) is a subsupermanifold \( N \). Then, just as classically, \( L_Q \) gives rise to a linear map

\[
L : TM|_N \to TM|_N
\]

where \( TM|_N \) is the pullback \( i^*TM \) where \( i : N \to M \) is the inclusion of the subsupermanifold \( N \). Moreover, \( L \) vanishes on \( TN \subset TM \) and therefore descends to a map

\[
TM/TN =: \nu_N \to TM|_N \to \nu_N.
\]
Definition 3.5.2. Let \( Q \) be a vector field on \( M \) and assume that the vanishing space of \( Q \) is a subsupermanifold \( N \). We call the vanishing space non-degenerate if the map \( L : \nu_N \to \nu_N \) induced by the Lie derivative \( L_Q \) is an automorphism.

The critical space of a function \( \psi \in \mathcal{O}_M \) is defined as the vanishing space of the 1-form \( d\psi \). Similarly to the case of vector fields, the ideal \( I_{d\psi} \) is generated by \( \{ Q\psi | Q \in TM \} \). Assume that the critical space of \( \psi \) is a subsupermanifold \( N \). The Hessian of \( \psi \) is a symmetric bilinear form on \( TM|_N \) defined by

\[
\text{Hess}(\psi)(X,Y) = X'Y'\psi|_N
\]
where \( X', Y' \) are any vector fields on \( M \) which restrict to \( X, Y \) on \( N \). The Hessian vanishes on \( TN \) and therefore descends to the normal bundle \( \nu_N \).

Definition 3.5.3. Let \( \psi \) be a function on a supermanifold \( M \) whose critical subspace is a subsupermanifold \( N \). We say that \( N \subset M \) is non-degenerate if \( \text{Hess}(\psi) \) restricted to \( \nu_N \) is a non-degenerate bilinear form.

4. Stationary Phase Approximation

4.1. Stationary Phase for Ordinary Manifolds. We first state the result in 1-dimension.

**Proposition 4.1.1.** Let \( f \) be a compactly supported function on \( \mathbb{R} \) and \( S \) a smooth function on \( \mathbb{R} \) having a single non-degenerate critical point at \( 0 \). We then have

\[
\int f e^{i\lambda S} dx = e^{i\lambda S(0)} \frac{1}{\sqrt{|S''(0)|}} e^{i\text{sgn}(S''(0))} \left( \frac{2\pi}{\lambda} \right)^{\frac{1}{2}} (f(0) + O(\lambda^{-1}))
\]
as \( \lambda \to \infty \).

For a proof, see for example [BO99, Bot97]. In [Bot97], the higher order terms are also computed. To extend this result to higher dimensional manifolds, we need the Morse-Bott Lemma. See [BH04] for a proof.

**Proposition 4.1.2** (Morse-Bott Lemma). Let \( S \) be a function on a manifold of dimension \( n \) such that its critical space is a non-degenerate submanifold \( N \). Then around each point \( p \in N \), there exists a coordinate system \( \{ x^i \}_{i=1}^n \) on a neighborhood \( U \ni p \) in which \( S \) is expressed as

\[
S = S(p) + (x^1)^2 + \cdots + (x^\alpha)^2 - (x^{\alpha+1})^2 - \cdots - (x^k)^2
\]
where \( k \) is the codimension of \( N \).

Under the assumptions of the proposition, the Hessian \( H \) of \( S \) defines a non-degenerate bilinear form on the normal bundle \( \nu_N \) of \( N \) and therefore an element

\[
\sqrt{\text{det}^{-1}(H)} \in \text{det}(\nu_N) \otimes (\nu_N)
\]
as explained in §2.5 and §2.7.

Since we have the exact sequence

\[
0 \to TN \to TM \to \nu_N \to 0,
\]
we have that

\[
\text{det} \; TM^* \otimes \text{or}(TM) \cong (\text{det} \; TN^* \otimes \text{or}(TN)) \otimes (\text{det} \; \nu_N^* \otimes \text{or}(\nu_N)).
\]

In particular, contracting with the section \( \sqrt{\text{det}^{-1}(H)} \), gives a map

\[
\langle \sqrt{\text{det}^{-1}(H)}, \bullet \rangle : \text{det} \; TM^* \otimes \text{or}(TM) \to \text{det} \; TN^* \otimes \text{or}(TN)
\]
from densities on \( M \) to densities on \( N \).
Proposition 4.1.3. Let $M$ be a compact manifold and $\mu$ a density. Let $S$ be a smooth function such that its critical space is a non-degenerate connected submanifold $N$ of codimension $k$ and denote $H = \text{Hess}(S)$. We have

$$\int_M e^{i\lambda S} = e^{\text{sgn}(H)\frac{\pi i}{4}} \left( \int_N e^{i\lambda S} \langle \sqrt{\det^{-1} H}, \mu \rangle + O(\lambda^{-1}) \right)$$

as $\lambda \to \infty$ where $\text{sgn}(H)$ is the signature of $H$ (the number of positive eigenvalues minus the number of negative eigenvalues).

Proof. If $S$ has no critical points in the support of $\mu$ then $\int_M e^{i\lambda S} = O(\lambda^{-\infty})$. To see it, note that there exists a vector field $V$ such that $V(S)$ is invertible on the support of $\mu$. By invariance of integration under diffeomorphisms we have

$$0 = \int L_V \left( \frac{1}{i\lambda V(S)} e^{i\lambda S} \right) = \frac{1}{i\lambda} \int L_V \left( \frac{\mu}{V(S)} \right) e^{i\lambda S} + \int e^{i\lambda S}.$$

We have $|\int L_V \left( \frac{1}{V(S)} \right) e^{i\lambda S}| \leq C$ for some constant $C$ since $|e^{i\lambda S}| = 1$ for all $\lambda$ and therefore $\int e^{i\lambda S} = O(\lambda^{-1})$. Applying the same argument $l$ times we get $\int e^{i\lambda S} = O(\lambda^{-l})$ for any $l \in \mathbb{N}$.

This shows that the asymptotic behavior of $Z(\lambda)$ only depends on a neighborhood of $N$. The actual expression is gotten by a simple application of partition of unity, the Morse-Bott Lemma (Proposition 4.1.2) and the statement in the case of one dimension ( Proposition 4.1.1).

As an example, assume that $N$ is an isolated point and we are working in a fixed coordinate system $\{x^1, \ldots, x^n\}$. Then $\langle \sqrt{\det^{-1} H}, dx^1 \wedge \cdots \wedge dx^n \rangle = \frac{1}{\sqrt{\det H}}$ and the first degree approximation is easily computed.

4.2. Stationary Phase for Supermanifolds. Let $M$ be a supermanifold, $\mu$ a compactly supported density and $S$ an even function. We again study the asymptotic behavior of

$$Z(\lambda) = \int_M e^{i\lambda S}.$$

If $dS$ is nowhere vanishing, then $Z(\lambda) = O(\lambda^{-\infty})$ for the same reason as in the proof of Proposition 4.1.3. We proceed in the same way as we did in the classical case. We first show that near a non-degenerate critical subsupermanifold, an even function $S$ has the standard form in appropriate coordinates. We will then write a formula for stationary phase approximation that is easy to verify in standard coordinates.

Theorem 4.2.1. Let $S$ be an even function on a supermanifold of dimension $n \mid m$ such that its critical space is a non-degenerate subsupermanifold $N$ of codimension $k \mid l'$. Then $l' = 2l$ is even and for each point $p \in N_{\text{red}}$, there exists a coordinate system $\{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\}$ on a neighborhood $U \ni p$ in which $S$ is expressed as

$$S = S(p) + (x_1)^2 + \cdots + (x^\alpha)^2 - (x^{\alpha+1})^2 - \cdots - (x^{k})^2 + \theta_1^2 + \theta_2^2 + \cdots + \theta^{2l-1} \theta^{2l}.$$

Proof. To see that $l'$ is even, consider the vector bundle $j^*(\nu_N)$ where $j : N_{\text{red}} \hookrightarrow N$ and $\nu_N$ is the normal bundle to $N$. This bundle has a canonical decomposition into even and odd submodules since $O_{N_{\text{red}}}$ is purely even. In particular, the restriction of $H := \text{Hess}(S)$ to $j^*(\nu_N)_1$ is a non-degenerate skew-symmetric bilinear form since $H$ is non-degenerate. This forces $j^*(\nu_N)_1$ to be even dimensional which implies that $l'$ is even.

We prove the theorem by induction where we induct on $a \in \mathbb{N}$ in the statement that there exists a coordinate system $\{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\}$ on a neighborhood $U \ni p$ such that

$$S = S(p) + (x_1)^2 + \cdots + (x^\alpha)^2 - (x^{\alpha+1})^2 - \cdots - (x^{k})^2 + \theta_1^2 + \theta_2^2 + \cdots + \theta^{2l-1} \theta^{2l} \mod J^{2a}.$$
where $J$ is the ideal generated by odd functions. The theorem is proved by noting that when $a > \frac{m}{2}$, the ideal $J^{2a}$ is the zero ideal. We first consider the base case where $a = 2$.

Pick a coordinate system $\{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\}$ on a neighborhood $U$ of $p \in N$ in which $N$ is given by vanishing of the ideal $I = \langle x^1, \ldots, x^k, \theta^1, \ldots, \theta^{2l} \rangle$. By the Morse-Bott Lemma (Proposition 4.1.2), by changing the even coordinates and subtracting the constant $S(p)$, we may assume that

\[ S = (x^1)^2 + \cdots + (x^a)^2 - (x^{a+1})^2 - \cdots - (x^k)^2 + \sum_{i,j=1}^{m} S_{ij} \theta^i \theta^j \mod J^4, \]

where $(S_{ij})$ is a skew-symmetric matrix of functions on $U_{\text{red}}$. Non-degeneracy of $N$ implies that $(S_{ij})_{i,j=1}^{2l}$ is non-degenerate on $N_{\text{red}} \cap U_{\text{red}}$. Since non-degeneracy is an open condition we may take $U$ small enough so that $(S_{ij})_{i,j=1}^{2l}$ is non-degenerate on $U_{\text{red}}$. By a method analogous to the Gram-Schmidt process, every symplectic vector space possesses a symplectic basis. By a parametrized version of the same method, a symplectic vector bundle locally possesses a symplectic basis of sections. Applying this to the trivial $\mathbb{R}^{2l}$ bundle over $U_{\text{red}}$ with the symplectic form $(S_{ij})_{i,j=1}^{2l}$, there exists a non-degenerate matrix $(T_{ij})_{i,j=1}^{2l}$ of functions on $U_{\text{red}}$ such that

\[ T^T S = \frac{1}{2} \begin{pmatrix} 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \]

By changing the odd coordinates $\theta^1, \ldots, \theta^{2l}$ in $U$ by

\[ \theta^i_{\text{old}} = \sum_{i=1}^{2l} T_{ij} \theta^i_{\text{new}} \]

we have

\[ S = (x^1)^2 + \cdots + (x^a)^2 - (x^{a+1})^2 - \cdots - (x^k)^2 + \theta^1 \theta^2 + \cdots + \theta^{2l-1} \theta^{2l} + \sum_{i=1}^{m} S'_{ij} \theta^i \theta^j \mod J^4 \]

for some functions $S'_{ij}$ such that $S'_{ij} = -S'_{ji}$ if $i,j > 2l$.

To simplify notation, define a function $S_{st}$ of $k$ even variables and $2l$ odd variables by

\[ S_{st}(x^1, \ldots, \theta^{2l}) = (x^1)^2 + \cdots + (x^a)^2 - (x^{a+1})^2 - \cdots - (x^k)^2 + \theta^1 \theta^2 + \cdots + \theta^{2l-1} \theta^{2l} \]

so that we have

\[ (4.1) \quad S = S_{st}(x^1, \ldots, \theta^{2l}) + \sum_{i=1}^{m} S'_{ij} \theta^i \theta^j \mod J^4. \]

Assume without loss of generality that $N$ is connected. The function $S$ vanishes on $N_{\text{red}}$ and since it is constant on $N$, it vanishes on $N$. Since both $S$ and $dS$ vanish on $N$, it follows that $S \in I^2$ where $I = \langle x^1, \ldots, x^k, \theta^1, \ldots, \theta^{2l} \rangle$ is the ideal of functions vanishing on $N$. The choice of coordinates $\{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\}$ introduces a grading by $(\mathbb{Z}/2\mathbb{Z})^m$ on $O_M$ in which functions of degree $\{d_i\}_{i=1}^{m} \in (\mathbb{Z}/2\mathbb{Z})^m$ are those of the form $f(x^1, \ldots, x^n) \prod_{i=1}^{m} (\theta^i)^{d_i}$. The ideal $I^2$ is a graded ideal with respect to this grading and therefore by decomposing $S$ into its homogeneous components, each term $S'_{ij} \theta^i \theta^j$ in equation (4.1) belongs to $I^2$. We thus have $S'_{ij}$ vanishes on $N_{\text{red}} \cap U_{\text{red}}$ at least to second order if $i > 2l$ and at least to first order if $i \leq 2l$. Thus
\[ S = S_{st}(x^1, \ldots, \theta^{2l}) + \sum_{\beta, \gamma=1}^{k} x^\beta x^\gamma F_{\beta\gamma} + \sum_{i=1}^{2l} G_i \theta^i \mod J^4 \]

for some functions \( F_{\beta\gamma} \in J^2 \) and \( G_i \in J \cap \langle x^1, \ldots, x^k \rangle \) where

\[
\sum_{\beta, \gamma=1}^{k} x^\beta x^\gamma F_{\beta\gamma} = \sum_{i=2l+1}^{m} S'_{ij} \theta^i \theta^j,
\]

\[
2l \sum_{i=1}^{2l} G_i \theta^i = \sum_{i=1}^{2l} \sum_{j=2l+1}^{m} S'_{ij} \theta^i \theta^j.
\]

Consider the following change of coordinates:

\[
\tilde{\theta}^i = \begin{cases} 
\theta^i + (-1)^i G_{i+(-1)^{i+1}} & i \leq 2l \\
\theta^i & i > 2l 
\end{cases}
\]

Since \( G_i \in \langle x^1, \ldots, x^k \rangle \), up to taking a smaller neighborhood \( U \ni p \), the above is an invertible transformation on \( U \) and

\[ S = S_{st}(x^1, \ldots, \tilde{\theta}^{2l}) \mod J^4 \]

for some functions \( F'_{\beta\gamma} \in J^2 \) satisfying \( F'_{\beta\gamma} = F_{\gamma\beta} \). Consider the following change of coordinates:

\[
\tilde{x}^\beta = \begin{cases} 
x^\beta + \frac{1}{2} \sum_{\gamma=1}^{k} x^\gamma F'_{\beta\gamma} & \beta \leq \alpha \\
x^\beta - \frac{1}{2} \sum_{\gamma=1}^{k} x^\gamma F'_{\beta\gamma} & \alpha < \beta \leq k \\
x^\beta & k < \beta
\end{cases}
\]

This too is an invertible transformation on a neighborhood of \( p \) and

\[ S = S_{st}(\tilde{x}^1, \ldots, \tilde{\theta}^{2l}) \mod J^4. \]

This establishes the base case for the induction.

We now consider the inductive step. Assume there exists a coordinate system \( \{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\} \)

on a neighborhood \( U \ni p \) such that

\[ S = S_{st}(x^1, \ldots, \theta^{2l}) \mod J^{2a} \]

with \( a \geq 2 \). As above, since \( S \in J^2 \), we have

\[ S = S_{st}(x^1, \ldots, \theta^{2l}) + \sum_{\beta, \gamma=1}^{k} x^\beta x^\gamma F_{\beta\gamma} + \sum_{i=1}^{2l} G_i \theta^i \mod J^{2(a+1)} \]

for some functions \( F_{\beta\gamma} \in J^{2a} \) and \( G_i \in J^{2a-1} \). Changing the coordinates by

\[
\tilde{x}^\beta = \begin{cases} 
x^\beta + \frac{1}{2} \sum_{\gamma=1}^{k} x^\gamma F_{\beta\gamma} & \beta \leq \alpha \\
x^\beta - \frac{1}{2} \sum_{\gamma=1}^{k} x^\gamma F_{\beta\gamma} & k < \beta \leq \alpha \\
x^\beta & \beta > k
\end{cases}
\]

\[
\tilde{\theta}^i = \begin{cases} 
\theta^i + (-1)^i G_{i+(-1)^{i+1}} & i \leq 2l \\
\theta^i & i > 2l
\end{cases}
\]

we have

\[ S = S_{st}(\tilde{x}^1, \ldots, \tilde{\theta}^{2l}) \mod J^{2(a+1)}. \]
We have used that $G_i G_j \in J^{da-2} \subset J^{2(a+1)}$ which is only true when $a \geq 2$. This is why we had to be more careful when discussing the base case.

\begin{align*}
\text{Theorem 4.2.2.} & \quad \text{Let } N \subset M \text{ be a non-degenerate connected submanifold of codimension } k |2l|. \text{ Let } \mu \in \text{Ber}(TM^{*}) \otimes \text{or}_{(1,0)} \text{ be a compactly supported density and define } Z(\lambda) := \int_{M} \mu e^{i\lambda S}. \text{ Denote by } H \text{ the Hessian of } S \text{ and by } H_{\text{red}} \text{ the Hessian of } S|_{M_{\text{red}}}. \text{ We have }
\int_{M} \langle \nabla \text{Ber}^{-1} H , \mu \rangle \cdot (1 + O(\lambda^{-1}))
\end{align*}

where $\text{sgn}(H_{\text{red}})$ is the signature of $(H_{\text{red}})$.

To clarify statement of the theorem above, recall from §2.5 and §2.7 that $\sqrt{\text{Ber}^{-1} H} \in \text{Ber} (\nu_{N}) \otimes \text{or}_{(1,1)} (\nu_{N})$ and therefore

\begin{align*}
\sqrt{\text{Ber}^{-1} H} \otimes \text{or}_{(1,0)}(H) \in \text{Ber}(\nu_{N}) \otimes \text{or}_{(1,0)}(\nu_{N}).
\end{align*}

The exact sequence

\begin{align*}
0 \to TN \to TM \to \nu_{N} \to 0
\end{align*}

gives an isomorphism

\begin{align*}
\text{Ber}(TM^{*}) \otimes \text{or}_{(1,0)}(TM) \cong (\text{Ber}(\nu_{N}^{*}) \otimes \text{or}_{(1,0)}(\nu_{N})) \otimes (\text{Ber}(TN^{*}) \otimes \text{or}_{(1,0)}(TN)).
\end{align*}

The contraction $\langle \sqrt{\text{Ber}^{-1} H} \otimes \text{or}_{(1,0)}(H), \mu \rangle \in \text{Ber}(TN^{*}) \otimes \text{or}_{(1,0)}(TN)$ is then a density on $N$.

We note that a similar statement with the integrand being real, i.e. $\mu e^{-\lambda S}$, is discussed in §3.10.

\begin{proof}
It suffices to check this proposition locally on $N$ with the general statement following by using a partition of unity. By the Morse-Bott Lemma generalized to supermanifolds (Theorem 4.2.1), there exists a coordinate system in which

\begin{align*}
S = C + (x^{1})^{2} + \cdots + (x^{\alpha})^{2} - (x^{\alpha+1})^{2} - \cdots - (x^{k})^{2} + \theta^{1}\theta^{2} + \cdots + \theta^{2l-1}\theta^{2l}
\end{align*}

where $C = S|_{M_{\text{red}}}$. To simplify notation, let

\begin{align*}
S_{0} & \quad = C + (x^{1})^{2} + \cdots + (x^{\alpha})^{2} - (x^{\alpha+1})^{2} - \cdots - (x^{k})^{2}
S_{1} & \quad = \theta^{1}\theta^{2} + \cdots + \theta^{2l-1}\theta^{2l}
S & \quad = S_{0} + S_{1}.
\end{align*}

Assume that $\mu$ is supported in this coordinate chart and thus can be expressed as $\mu = [dxd\theta] f$. We have

\begin{align*}
Z(\lambda) = \int_{R^{n+l}} [dxd\theta] f \cdot (1 + i\lambda S_{1} + \cdots + \frac{(i\lambda S_{1})^{l}}{l!}) e^{i\lambda S_{0}}.
\end{align*}

\footnote{Recall that to define $\sqrt{\text{Ber}^{-1} H}$ we need a homomorphism $\text{sgn} : O_{M}^{*} \to \{\pm 1\}$ and a square root operator $\sqrt{\cdot} : (O_{M}^{*})_{>0} \to (O_{M}^{*})_{>0}$. The sign is given by restricting to $M_{\text{red}}$. The square root is given by the following local formula: picking a system of coordinates, a positive function is given by $f = f_{0} + \psi$ where $f_{0}$ is a positive function on $M_{\text{red}}$ and $\psi \in J$ where $J$ is the ideal generated by odd functions. We then have $\sqrt{\mathcal{J}} = \sqrt{f_{0}} \left(1 + \frac{\psi}{\sqrt{f_{0}}} \right) = \sqrt{f_{0}} \sum_{n=0}^{\infty} (-1)^{n}(2n)! \frac{(\psi)}{\sqrt{f_{0}}}^{n}$.}

The above sum is finite since $\psi$ is nilpotent. One can check that $\sqrt{\mathcal{J}}$ is the unique positive function satisfying $(\sqrt{\mathcal{J}})^{2} = f$ and hence does not depend on the chosen coordinates.
The highest degree term in $\lambda$ is $\int [i\alpha S]^l e^{i\lambda S_0}$. Since $\frac{S_l}{H_l} = \theta^1 \ldots \theta^{2l}$, the integration of this term picks up the coefficient of $\theta^{2l+1} \theta^{2l+2} \ldots \theta^q$ in $f$. We have

$$Z(\lambda) = (i\lambda)^l \int_{\mathbb{R}^p} dx^1 \ldots dx^p \left( \frac{\partial}{\partial \theta^q} \ldots \frac{\partial}{\partial \theta^{2l+1}} f \right)_{\mathbb{R}^p} e^{i\lambda S_0} (1 + O(\lambda^{-1}))$$

and therefore applying the ordinary stationary phase approximation (Proposition 5.1.2) we get

$$Z(\lambda) = e^{i\lambda C} e^{\frac{2\pi}{\lambda} i(a-(k-\alpha))} \left( \frac{2\pi}{\lambda} \right)^{\frac{q}{2}} (i\lambda)^l \int_{\mathbb{R}^{p-k}} dx^{k+1} \ldots dx^p \left( \frac{\partial}{\partial \theta^q} \ldots \frac{\partial}{\partial \theta^{2l+1}} f \right)_{\mathbb{R}^{p-k}} \cdot (1 + O(\lambda^{-1}))$$

The last equality is simply a calculation of $\langle \sqrt{\text{Ber}^{-1} H} \otimes \text{or}(0,1)(H), [dx d\theta] f \rangle$ in local coordinates. The factor of $(-1)^l$ in the above equation comes from the fact that the matrix of $H$ in the basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^p}, \frac{\partial}{\partial \theta^1}, \ldots, \frac{\partial}{\partial \theta^q}$ of $\nu_N$ is

$$\begin{pmatrix}
1 & \ldots & 0 & -1 \\
0 & 1 & 0 & \ldots \\
& \ddots & \ddots & \ddots \\
& & 0 & -1 \\
& & 1 & 0
\end{pmatrix}.$$ 

Since

$$\text{pf} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -1,$$

the orientation $\text{or}(0,1)(H)$ differs from the one given by $\frac{\partial}{\partial \theta^1}, \ldots, \frac{\partial}{\partial \theta^q}$ by $(-1)^l$. \qed

5. Localization Theorem

5.1. General Localization Statement. The constructions and proofs in this subsection are taken from [SZ97]. We include the proofs for completeness.

Let $M$ be a supermanifold. We begin with a series of definitions.

**Definition 5.1.1.** We call an even vector field $V$ on $M$ compact if there exists an action by a compact torus $T$ on $M$ and an element $X \in \mathfrak{t}$ in the Lie algebra of $T$ which induces the vector field $V$.

**Definition 5.1.2.** We denote the normal bundle to $M_{\text{red}}$ by $\alpha_M$. It is a vector bundle on the ordinary manifold $M_{\text{red}}$.

An odd vector field $Q$ on $M$ naturally defines a section $\alpha(Q)$ of $\alpha_M$. Similarly, an odd function $\sigma$ defines a section $\alpha^*(\sigma)$ of the conormal bundle $\alpha^*_M$ induced by the 1-form $d\sigma$. If $\{x^1, \ldots, x^n, \theta^1, \ldots, \theta^m\}$ are coordinates in some neighborhood such that an odd vector field $Q$ is given by

$$Q = \sum_{\alpha=1}^m (b^\alpha(x) + \ldots) \frac{\partial}{\partial \theta^\alpha} + \sum_{i=1}^n (\ldots) \frac{\partial}{\partial x^i}$$

where the terms in $(\ldots)$ belong to the ideal $J$ generated by odd functions, then $\alpha(Q) = \sum_{\alpha=1}^m b^\alpha \frac{\partial}{\partial \theta^\alpha}$. The section $\alpha^*(\sigma)$ for $\sigma$ an odd function is defined similarly from an expression of $d\sigma$. 

The constructions and proofs in this subsection are taken from [SZ97].
Definition 5.1.3. For \( Q \) an odd vector field, let \( R_Q \subset M_{\text{red}} \) be the vanishing set of \( \alpha(Q) \).

Note that \( R_Q \) need not have a nice form for an arbitrary \( Q \).

Proposition 5.1.4 ([SZ97 Theorem 1]). Let \( M \) be a supermanifold, \( Q \) an odd vector field and \( \mu \in \text{Ber}(TM^*) \otimes \omega_{(1,0)}(TM) \) a compactly supported density. If \( \mu \) is \( Q \)-invariant \((\mathcal{L}_Q \mu = 0)\), and \( Q^2 \) is compact then for any neighborhood \( U \supset R_Q \) in \( M_{\text{red}} \) there exists an even \( Q \)-invariant function \( g \) which takes the value 1 on some neighborhood \( O \supset R_Q \) and vanishes outside \( U \). Moreover, for every such \( g \) we have

\[
\int_{M} \mu = \int_{M} \mu \cdot g.
\]

Lemma 5.1.5 ([SZ97 Lemma 1]). Under the assumptions of the above proposition, there exists an odd function \( \sigma \) such that the zero set of \( Q\sigma|_{M_{\text{red}}} \) coincides with \( R_Q \) and \( Q^2 \sigma = 0 \).

Before proving this lemma, we note that this function will be central to most of this section so the importance of this lemma should not be overlooked.

Proof. Since \( Q^2 \) comes from an action of some compact torus \( T \), and induces an action on \( \alpha_M \), there exists a \( Q^2 \) invariant inner product \( g \) on \( \alpha_M \). Let us denote the induced isomorphism between \( \alpha_M \) and \( \alpha^*_M \) also by \( g \). There exists an odd function \( \sigma' \) on \( M \) such that \( \alpha^*(\sigma') = g(\alpha(Q)) \). There are numerous ways of seeing it, and perhaps the most direct way is to note that the sheaf of sections of \( \alpha^*_M \) is naturally isomorphic to \( J/J^2 \) where \( J \) is the sheaf of ideals generated by odd functions. We can now average out by the action of \( T \) to get a \( T \) invariant odd function \( \sigma \). In particular \( Q^2 \sigma = 0 \).

Since \( Q \) and \( g \) are \( T \) invariant, we conclude that so is \( g(\alpha(Q)) \) and thus in particular \( \alpha^*(\sigma) = \alpha^*(\sigma') \). We now take a look at how \( \sigma \) looks in local coordinates. If \( \{x_1, \ldots, x_n, \theta_1, \ldots, \theta_m\} \) are coordinates in some neighborhood such that

\[
Q = (b^\alpha(x) + \ldots) \frac{\partial}{\partial \theta^\alpha} + (\ldots) \frac{\partial}{\partial x^i},
\]

then

\[
\sigma = g_{\alpha \beta} b^\alpha \theta^\beta + \ldots
\]

and

\[
Q \sigma = g_{\alpha \beta} b^\alpha \theta^\beta + \ldots.
\]

In particular, \( Q\sigma|_{M_{\text{red}}} = ||\alpha(Q)||_g \) which shows that the vanishing set of \( Q\sigma|_{M_{\text{red}}} \) coincides with \( R_Q \).

Proof of Proposition 5.1.4 Define a function

\[
\beta = \frac{\sigma}{Q \sigma}
\]

where \( \sigma \) is the function from the lemma. The function \( \beta \) is defined on the complement of \( R_Q \) and satisfies \( Q\beta = 1 \).

We first construct a convenient partition of unity of \( M \). Considering the action of \( Q^2 \) on \( M_{\text{red}} \) we can find a \( Q^2 \) invariant locally finite open cover \( \{U_i\} \) and an open neighborhood \( O \supset R_Q \) such that \( O \subset U_0 \subset U \) and \( U_i \cap O = \emptyset \) for \( i \neq 0 \). Taking any partition of unity \( \{f_i\} \) subordinate to \( \{U_i\} \) and averaging out by the action of the torus \( T \) inducing \( Q^2 \), we can assume that \( \{f_i\} \) are \( Q^2 \) invariant. Consider now the functions \( g_i \) given by

\[
g_i = \begin{cases} Q(\beta f_i) & \text{for } i > 0 \\ 1 - \sum_{i \neq 0} g_i & \text{for } i = 0. \end{cases}
\]
It is now not difficult to see that $g_0$ satisfies the requirements of the proposition. Moreover, note that for $i \neq 0$, we have
\[ \int_M \mu g_i = \int_M \mathcal{L}_Q(\mu \beta f_i) = 0 \]
where the last equality follows from invariance of integration under diffeomorphisms. We now have
\[ \int_M \mu = \sum_i \int_M \mu g_i = \int_M \mu g_0. \]
It is left to the reader to show that the above localization statement holds for any $g$ as in the statement of the proposition.

5.2. Localization Formula. The previous section tells us when an integral of a density $\mu$ localizes to the vanishing set of an odd vector field $Q$. In this section we will calculate this integral in terms of the local data of $\mu$ and $Q$ near this vanishing space in the case the vanishing space of $Q$ is a non-degenerate subsupermanifold.

Definition 5.2.1. Let $V$ be a vector space over $\mathbb{R}$ and $E$ an automorphism of $V$ such that the closure of the group generated by $e^{tE}$ in $GL(V)$ is compact. Let $g$ be any inner product such that $e^{tE} \subset O(g)$. Define
\[ \text{or}(E) \in \text{or}(V) \]
to be the orientation determined by the symplectic form $(x, y) \mapsto \langle x, Ey \rangle_g$.

Note that the above construction is well defined since the space of inner products $g$ satisfying $e^{tE} \subset O(g)$ is a convex set. This can be seen from the fact that the condition $e^{tE} \subset O(g)$ is equivalent to $E$ being skew-adjoint with respect to $g$ which is a linear condition on $g$.

We recall that a symplectic form $w \in \bigwedge^2 E^*$ on a $2n$ dimensional vector space defines an orientation via the volume form $w^n \in \bigwedge^{2n} E^*$. Equivalently, a basis $\beta$ belongs to the orientation defined by $\omega$ if $\text{sgn}(\text{pfaff}(\omega_\beta)) = 1$ where $\omega_\beta$ is the matrix of $\omega$ in the basis $\beta$. Combining this with the definition of $\text{or}(E)$, we see that a basis $\beta$ belongs to $\text{or}(E)$ if and only if for any positive definite symmetric matrix $g$ such that $gE_\beta$ is skew-symmetric,
\[ \text{sgn}(\text{pfaff}(gE_\beta)) = 1 \]
where $E_\beta$ is the matrix of $E$ in the basis $\beta$.

Theorem 5.2.2. Let $M$ be a supermanifold, $Q$ and odd vector fields such that $Q^2$ is compact and $\mu$ a $Q$-invariant compactly supported density. Suppose that the vanishing space of $Q$ is a non-degenerate subsupermanifold $N$. Let $\nu$ be the normal bundle of $N$ and $L := \mathcal{L}_Q|_\nu$, which is an odd automorphism. Then $\text{codim}(N) = 2l|2l|$ for some $l \in \mathbb{N}$, $L^2$ restricted to $\nu_1|_{N_{\text{red}}}$ defines an element $o \in \text{or}(0, 1)(\nu)$ as in definition 5.2.1 and the integral of $\mu$ over $M$ is given by
\[ \int_M \mu = (-2\pi)^l \int_N \langle \sqrt{\text{Ber}(L) \otimes o}, \mu \rangle \]
where $\sqrt{\text{Ber}(L)} \in (\text{Ber}(\nu) \otimes \text{or}(1, 1)(\nu))$ is constructed in \[2.6, 2.7\].

This extends the result of [SZ97] to the case where $N$ is not an isolated point.

Proof. The main tool for proving this theorem will be the function $\sigma$ constructed in Lemma 5.1.5. Recall that $\sigma$ is an odd function such that $Q^2 \sigma = 0$. Define
\[ Z(\lambda) := \int_M \mu e^{i\lambda Q}\sigma. \]
We then have
\[ \frac{d}{d\lambda} Z(\lambda) = i\lambda \int_M \mu Q\sigma e^{i\lambda Q}\sigma = i\lambda \int_M \mathcal{L}_Q(\mu \sigma e^{i\lambda Q}\sigma) = 0. \]
In particular $Z(\lambda)$ does not depend on $\lambda$. We can thus calculate $\int_M \mu = Z(0)$ via stationary phase approximation as $\lambda \to \infty$ from previous section once we show that the critical space of $Q\sigma$ is $N$ and calculate its Hessian.

We first show that the critical subsupermanifold of $Q\sigma$ contains $N$. For $V$ a vector field tangent to $N$, we have

$$VQ\sigma|_N = 0$$

since $Q\sigma$ is constant on $N$. For any vector field $V$ on $M$, we have

$$\mathcal{L}_Q(V)Q\sigma = \mathcal{L}_Q(VQ\sigma) - VQ^2\sigma = \mathcal{L}_Q(VQ\sigma)$$

and since $Q|_N = 0$ we have

$$\mathcal{L}_Q(V)Q\sigma|_N = 0.$$  

Since $N$ is non-degenerate, the map $\mathcal{L}_Q|_\nu$ is an automorphism and thus in particular any section of $TM|_N$ can be written as a sum of a restriction of a vector field tangent to $N$ and an image of $\mathcal{L}_Q$. This shows that $d(Q\sigma)$ vanishes on $N$.

The even codimension of $N$ equals the odd codimension of $N$ because $\mathcal{L}_Q|_\nu$ is an odd automorphism. The odd codimension of $N$ is even because $\mathcal{L}_Q^2|_\nu$ exponentiates to an action of a torus.

We now need to calculate $H := \text{Hess}(Q\sigma)$. For the time being, when referring to $H$ we will mean the restriction of $H$ to $N$ (So far, we only showed that the critical subsupermanifold of $Q\sigma$ contains $N$, not that it equals $N$). Consider any two vector fields $\xi_1, \xi_2$ on $M$. We have

$$\xi_1\xi_2Q^2\sigma = 0$$

and thus, applying the Leibniz rule to $\mathcal{L}_Q$, we get that

$$0 = (-1)^{p(\xi_1) + p(\xi_2)} \xi_1\xi_2Q^2\sigma = \mathcal{L}_Q(\xi_1\xi_2Q\sigma) - \mathcal{L}_Q(\xi_1)\xi_2Q\sigma - (-1)^{p(\xi_1)}\xi_1\mathcal{L}_Q(\xi_2)Q\sigma$$

and in particular, restricting to $N$ we get

$$\mathcal{L}_Q(\xi_1)\xi_2Q\sigma + (-1)^{p(\xi_1)}\xi_1\mathcal{L}_Q(\xi_2)Q\sigma = 0.$$  

In terms of the $H$ and $L$, this gives us

$$H(L\xi_1, \xi_2) + (-1)^{p(\xi_1)}H(\xi_1, L\xi_2) = 0.$$  

In terms of maps, the above equality states that

\begin{equation}
\hat{H} \circ L = -L^* \circ \hat{H}
\end{equation}

where

$$\hat{H} : \nu \to \nu^*$$

is the map induced by $H$. The maps in equation (5.1) are odd maps from $\nu$ to $\nu^*$. In terms of matrices with respect to some basis $\beta$ of $\nu$ the above equality is expressed by

\begin{equation}
\hat{H} L = -L^t \hat{H}.
\end{equation}

If in block form,

$$\hat{H} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$L = \begin{pmatrix} U & V \\ W & X \end{pmatrix},$$

then the above equality implies that modulo odd variables,

\begin{equation}
AV = W^t D.
\end{equation}
By construction of $\sigma$, one can see that $A$ is non-degenerate while $V$ and $W$ are non-degenerate by the fact that $L$ is an automorphism. Thus in particular $D$ is non-degenerate and $N$ is the critical subsupermanifold of $Q\sigma$.

Defining the matrix

$$I' = \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix}$$

as in the coordinate description of the Berezinian of an odd automorphism, it is not hard to see that

$$\text{Ber}(L^{st}I') = \text{Ber}(LI')^{-1}$$
$$\text{Ber}(I'\hat{H}I') = \text{Ber}(\hat{H})^{-1}.$$ 

In particular, it follows from equation (5.2) that

$$\text{Ber}(\hat{H}LI')^2 = 1.$$ 

We now show that $\text{Ber}(\hat{H}LI') = 1$ modulo odd variables. Let $J$ be the ideal of $\mathcal{O}_N$ generated by odd functions. We have

$$\hat{H}LI' \equiv \begin{pmatrix} AV & 0 \\ 0 & DW \end{pmatrix} \mod(J).$$

It thus follows from equation (5.3) and the fact that the dimension of $\nu_1$ is even that

$$\text{Ber}(\hat{H}LI') = 1 \mod(J).$$

We now show that in fact $\text{Ber}(\hat{H}LI') = 1$. Let $\alpha = \text{Ber}(\hat{H}LI') - 1$. We have that $\alpha \in J^2$ and satisfies

$$2\alpha + \alpha^2 = 0$$

since $(1 + \alpha)^2 = 1$. It follows from the equation above that if $\alpha \in J^k$ then also $\alpha \in J^{2^k}$. Since for sufficiently large $k$ we have that $J^k = 0$, we conclude that $\alpha = 0$.

Recalling the definition of $\text{Ber}(L)$ and $\text{Ber}(H)$, the equality

$$\text{Ber}(\hat{H}LI') = 1$$

implies that

$$\text{Ber}(L) = \text{Ber}^{-1}(H).$$

We now show that the $(0, 1)$ orientation defined by $H$ differs from the one defined by $L^2$ by $(-1)^l$. For the remaining of this paragraph, all of the quantities considered will be taken restricted to $N_{\text{red}}$ or equivalently modulo the ideal $J$ generated by odd functions. Denote by $H_\beta$ the matrix $H(e_i, e_j)$. We have $H_\beta = \hat{H}^{st}$ and therefore the $(0, 1)$ orientation defined by $H$ is given by the sign of $\text{pfaff}(D^t)$. The orientation defined by $L^2$ is given by the automorphism of $\nu_1$ which in the fixed basis is given by the matrix $WV$. The desired relationship between orientations will follow from equation (5.3)

$$AV = W^t D.$$ 

Define

$$g := W^{-t}AW^{-1}.$$ 

From the definition of $\sigma$, it follows that $A$ is positive definite, and therefore so is $g$. We then have

$$D = gWV$$

and therefore orientation defined by $WV$ is determined by

$$\text{pfaff}(gWV) = \text{pfaff}(D) = (-1)^l \text{pfaff}(D^t).$$

This shows that $\text{or}_{(0, 1)}(H) = (-1)^l o \in \text{or}_{(0, 1)}(\nu)$ where $o$ is the orientation of $\nu_1$ defined by $L^2|_{N_{\text{red}}}$. The theorem now follows by applying stationary phase approximation (Theorem 4.2.2) to $Z(\lambda)$, noting that $Q\sigma$ vanishes on $N$ and that $\text{sgn}(H_{\text{red}}) = 2l$. 
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