Constraint on the fifth force through perihelion precession of planets

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The equivalence principle is important in fundamental physics. The fifth force, as a describing formalism of the equivalence principle, may indicate the property of an unknown theory. Dark matter is one of the most mysterious objects in the current natural science. It is interesting to constrain the fifth force of dark matter. We propose a new method to use perihelion precession of planets to constrain the long-range fifth force of dark matter. Due to the high accuracy of perihelion precession observation, and the large difference of matter composition between the Sun and planets, we get one of the strongest constraints on the fifth force of dark matter. In the near future, the BepiColombo mission will be capable to improve the test by another factor of ten.

I. INTRODUCTION

The equivalence principle (EP) is fundamental to both Newtonian theory and Einsteinian theory [1]. Consequently experimental examination of the EP is very important to fundamental physics [2–4]. We have the Eötvös parameter to describe the violation of EP,

$$\eta_{(A,B)} = \frac{2(a_A - a_B)}{a_A + a_B},$$

where $a_i$ ($i = A, B$) denotes the acceleration of two test particles $A$ and $B$ relative to the central attractor. In general the attracting force arising from the central attractor depends on the composition of two test particles and the central object when EP is violated, so does the above Eötvös parameter. We need to distinguish between different Eötvös parameters arising from different objects. For example, (i) the MICROSCOPE satellite [4] considered a Ti-Pt pair with respect to the Earth, and obtained $\eta_{(\text{Ti},\text{Pt})} \lesssim 10^{-14}$; (ii) the Eöt-Wash experiments [2] considered Be-Al and Be-Ti pairs with respect to the Sun, and obtained $\eta_{(\text{Be},\text{Al})}$, $\eta_{(\text{Be},\text{Ti})} \lesssim 10^{-13}$; and (iii) Lunar laser ranging [3] considered the Earth-Moon pair with respect to the Sun, and obtained $\eta_{(\text{Earth},\text{Moon})} \lesssim 10^{-13}$. Using the idea proposed by Stubbs [5] and the above results [2, 3], we have $\eta_{\text{DM}}$, $\eta_{\text{DM}}$, $\eta_{\text{DM}} \lesssim 10^{-8}$ when considering the Galactic dark matter (DM) as the attractor (see Refs. [2, 6] for the framework of effective field theory).

Equivalently we can use the concept of fifth force to describe the violation of EP [7]. Compared to the Eötvös parameter, the concept of fifth force is more straightforward to be related to a fundamental theory. Based on the fifth force, we can take the advantage of large difference in the matter composition between two test bodies to better constrain the fifth force of an object in question. This idea has been successfully applied in Ref. [8] where binary pulsar PSR J1713+0747 constrained a neutron star (NS)-white dwarf (WD) pair with respect to the DM, $\eta_{\text{DM}} \lesssim 0.004$ [9].

Dark matter is one of the most mysterious objects in current natural science. It is interesting to study the fifth force behavior of dark matter which will help people to advance knowledge about dark matter. Due to the great efforts in searching for dark matter particles, we nowadays have stringent constraints on the interaction cross-section between ordinary matter and dark matter. Those studies mostly focus on the possible short-range interactions between dark matters and the nucleons. We here investigate another possibility with the long-range fifth-force formalism originally proposed by the authors of [10]. Notice that the long-range fifth force we are studying is extremely weak from the point of view of particle physics. We will see that it is even weaker than the gravity interaction, thus it does not contradict any constraints from dark matter searches. This is a largely unexplored territory, thus it is interesting to see whether dark matter could have a sizeable long-range interaction with ordinary matter. The strongest constraint for the long-range fifth force of dark matter comes from the lunar laser ranging (LLR) experiment. The NS-WD binary PSR J1713+0747 also gives an interesting constraint using the large difference of matter composition between NS and WD [8, 9, 11]. Here we propose a new method to use the perihelion precession of planets to constrain the fifth force of dark matter. Due to both the large difference of matter composition between the Sun and the planets, and the high observational accuracy of perihelion precession, we can get a very good constraint of the fifth force of the dark matter in the Galaxy.

The most well known system of celestial mechanics is the Sun-Mercury system. The famous “43 arcseconds” problem hastened the birth of general relativities. Later the observation about the Mercury perihelion precession became more and more accurate [12]. The current most accurate detection is done by the MESSENGER mission [13]. In the near future much more accurate detection will be achieved by the BepiColombo mission [12] which was launched last year. These missions directly detect the relative distance and velocity between Mercury and

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the Earth. Combining these data with other related information, people can construct accurate ephemerides for Mercury. Based on the accurate ephemerides it is straightforward to get the perihelion advance of Mercury. Representative ephemerides for Mercury include the JPL DE series [14], the EPM series [15] and the INPOP series [16]. Take the EPM2004 ephemerides as an example; the estimated accuracy for the perihelion advance of Mercury is about 10⁻³ as/cy [17]. EPM2004 is rather old; more recent ephemerides give much more accurate perihelion advance of Mercury.

Although our knowledge is limited about the perihelion precession observation for planets other than Mercury, we find that the Jupiter may result in a better constraint than Mercury. This is because when one converts the constraint of the extra perihelion precession to the Eötvös parameter, the planet orbit information plays an important role. We will calculate the detailed conversion relation and explain such a dependence in the next section. There an analysis of the Eötvös parameter based on the perihelion precession observation will also be presented. After that we relate the Eötvös parameter constraint to the fifth force in Sec. III. At last, some related discussion and the summary are given in Sec. IV. To complement the main text, some necessary calculation detail is included in the Appendix.

II. EFFECTS OF THE FIFTH FORCE ON PERIHELION ADVANCE OF THE MERCURY

The fifth force results in a relative acceleration of the Mercury with respect to the Sun [8],

$$\vec{a}_{\eta_{DM}} = \eta_{DM} a_{DM}, \quad (2)$$

where $\vec{a}_{DM}$ is the gravitational acceleration acted on the Mercury-Sun binary system by the DM in the Galaxy.

The acceleration in Eq. (2) generates an additional perihelion precession. This kind of physical picture has been investigated before by other authors including Damour and Schäfer [18] and Freire et al. [19]. These authors were concerned about the orbital eccentricity variation. Differently our concern is the precession of the perihelion. The variation of the longitude of the perihelion can be expressed as (see the detailed calculation in Appendix A),

$$\dot{\varpi}_5 = -\frac{3 \eta_{DM} a_{DM}}{GM_p} \pi a^2 \left( \frac{\sqrt{1 - e^2}}{e} F_1 + \frac{e}{\sqrt{1 - e^2}} F_2 \right), \quad (3)$$

where

$$F_1 = \cos(\Phi - \Omega) \cos \omega \sin \Theta + \cos \Theta \sin \omega + \sin \omega \cos \epsilon \sin \Omega \sin(\Phi - \Omega) \quad (4)$$

$$F_2 = \tan \frac{\omega}{2} \sin \epsilon \cos \Theta - \sin \epsilon \sin \Theta \sin(\Phi - \Omega) \quad (5)$$

Here $G$ is the gravitational constant, $M$ is the total mass of the Sun and the planet and $P$ is the orbital period. We have adopted traditional notation above for the planet orbit, where $\Omega$ is the angle between the $x$ direction and the ascending node, $\omega$ is the angle between the perihelion and the ascending node of the ecliptic plane, $a$ is the semi-major axis, $e$ is the orbit eccentricity, and $\epsilon$ is the inclination of the orbit. In addition, $\Theta$ is the angle between the Galactic center and the spin axis of the Sun, $\Phi$ is the angle between the $x$ direction and the projected direction of the Galactic center to the $x-y$ plane. The orbit layout is illustrated in Fig. 1. Specifically for the Galaxy center we have $\Theta = 117.1^\circ$ and $\Phi = 192.9^\circ$.

The perihelion precession of planets is detectable. Until now the detected perihelion precession of planets can be explained without the fifth force contribution (3). So we can constrain $\eta_{DM}$ through Eq. (3) based on the observational precision. Note that Eq. (3) can be viewed as,

$$\dot{\varpi}_5 = \frac{\eta_{DM}}{\Xi} \quad (6)$$

where the factor $\Xi$ is determined completely by the planet’s orbit information and the dark matter distribution. Based on the observational precision $\epsilon_\varpi$, we can roughly constrain the Eötvös parameter to,

$$\eta_{DM} < \Xi \cdot \epsilon_\varpi. \quad (7)$$

In order to determine the factor $\Xi$, we need to investigate the dark matter distribution in our Galaxy. We assume for the dark matter halo of our Galaxy,

$$\rho = \begin{cases} \rho_{G\text{FW}}(r) \rho_0(r), & r_0 \leq r < R_{sp} \\ \rho_0(r), & r \geq R_{sp} \end{cases} \quad (8)$$

$$\rho_{G\text{FW}}(r) = \frac{\rho_0}{(r/R_s)^\gamma (1 + r/R_s)^{3-\gamma}} \quad (9)$$

The inner part of the halo is called the “spike”. $\rho_{sp} = \alpha r^{-\gamma_{sp}}$ is the distribution of the spike and $R_{sp}$ is its radius; $\gamma_{sp} = \frac{9-2\gamma}{3-\gamma}$ [20]. $\rho_0 = \beta r^{-\gamma_0}$ has taken into account the DM particles’ annihilation cross section. $\gamma_0$
depends on the annihilation mechanism: for $s$-wave annihilation $\gamma_{in} \simeq 0.5$, and for $p$-wave annihilation $\gamma_{in} \simeq 0.34$ [21]. The outer part is the generalized Navarro-Frenk-White (GNFW) profile. Specifically for our galaxy $\gamma$ ranging from 1.0 to 1.4 [8] and $R_S = 20$ kpc. Moreover, the parameter $\rho_0 = (2/7)\gamma \times 3.2 \times 10^{64} \text{GeV}/\text{kpc}^3$ which is determined through the condition that $\rho_{\text{GNFW}}(r \simeq 8\text{kpc}) = 1.2 \times 10^{64} \text{GeV}/\text{kpc}^3$.

Integrating the density from $r_0 = 10^{-5}$ kpc [21] to $R = 8$ kpc where our Solar system locates, we obtain the total DM mass,

$$m_\gamma = \int_{r_0}^{R} 4\pi r^2 \rho(r) dr$$

$$\in [8.4 \times 10^{10} \text{kg}, 1.0 \times 10^{11} \text{kg}],$$

for $\gamma \in [1.0, 1.4]$. We find that the spike part contributes little to the total DM mass. So the parameter $R_{\text{sp}}$ is negligible in the current discussion [8]. Consequently, the gravitational acceleration of DM can be estimated as,

$$a_{\text{DM}} = G\frac{m_\gamma}{R^2} \in [9.2 \times 10^{-11} \text{m/s}^2, 1.1 \times 10^{-10} \text{m/s}^2].$$

In the following analysis we take $a_{\text{DM}} \approx 10^{-10} \text{m/s}^2$.

Based on the above dark matter distribution information and planet orbit information, we can determine the precession factor $\Xi$. Besides Mercury we have also investigated several other planets, including Mars, Jupiter and Saturn. We list these corresponding $\Xi$’s in Table I.

| Planet   | Mercury | Mars  | Jupiter | Saturn |
|----------|---------|-------|---------|--------|
| $\Xi$    | 0.274   | 0.038 | 0.0081  | 0.0031 |

Currently the observational accuracy of perihelion precession of Mercury is $10^{-3}$ as/century [13, 17, 22]. In the near future, the European-Japanese BepiColombo mission will improve it for Mercury to about $10^{-4}$ as/century [12, 23]. Currently the observational accuracy of perihelion precession of Saturn is $10^{-3}$ as/century [24]. If more dedicated analysis is paid to construct an ephemerides, a more accurate perihelion precession will be obtained [25]. In the current paper we concern only the precession directly deduced from experiments. Based on the current observational accuracy, the Eötvös parameter can be constrained to $\eta_{\text{DM}}^{(\text{Sun}, \text{Mercury})} < 2.71 \times 10^{-4}$. In the near future the BepiColombo mission will improve such a constraint to $\eta_{\text{DM}}^{(\text{Sun}, \text{Mercury})} \lesssim 3 \times 10^{-5}$. A comparable accuracy as Mercury for Mars and Saturn will make the constraint one order of magnitude better. For Jupiter it will make the constraint two orders of magnitude better. But we are not sure about the observational accuracy of perihelion precession for Mars, Jupiter and Saturn. In the following analysis for the fifth force constraint we only discuss the result from the Mercury observation.

### III. THE FIFTH FORCE OF THE GALACTIC DM

We assume that the fifth force between DM and ordinary matter can be described by a Yukawa potential [2, 10],

$$V(r) = \frac{g^2 q DM}{4\pi r} e^{-r/\xi} \label{eq:13}$$

where $g$ is the coupling constant, $q_{\text{DM}}$ and $q$ are the dimensionless charges of DM and ordinary matter respectively, and $\xi$ is the range of effective interaction which is related to the mass of the intermediate particle through $\xi = h/(mc)$. The $\mp$ sign corresponds to scalar $(-)$ and vector $(\mp)$ interactions respectively. For an electrically neutral body consisting of atoms, the charge $q$ is parametrized as [2],

$$q = Z \cos \psi + N \sin \psi \label{eq:14}$$

where $Z$ is the proton number, $N$ is the neutron number, $\tan \psi \equiv q_n/(q_\mu + q_e)$, and $q_\mu$, $q_n$, $q_e$ are the fifth force charge carried by a proton, a neutron and an electron respectively. For a specific mixing, the value of $\psi$ can be derived; for example, in the $B - L$ scenario, $\psi = 90^\circ$.

The Yukawa potential (13) gives rise to a relative acceleration of two test bodies,

$$\Delta a = \mp \frac{g^2}{4\pi} q_{\text{DM}} \left[ \frac{q_A}{m_A} - \frac{q_B}{m_B} \right] \left[ \frac{1}{r^2} + \frac{1}{r \xi} \right] e^{-r/\xi} \label{eq:15}$$

where $q_{A,B}$ are the fifth force charges of these two bodies. Notice that if we had replaced in the above equation the fifth-force charge $q_{\text{DM}}$ with, say, $q_{\odot}$, the abnormal acceleration was tightly constrained by equivalence-principle experiments (e.g. by the Eötvös group and lunar laser ranging). Therefore, the fifth-force charges for ordinary matter are already constrained to be extremely small. In contrast, the fifth-force charges for dark matter are not so well bounded. In principle, dark matter can possess large fifth-force charges yet be un-noticed. This is exactly why it is interesting, and our study in this work is motivated. Due to this relative acceleration, the EP will be violated with an Eötvös parameter [2]

$$\eta_{\text{DM}}^{(A,B)} \approx \frac{\Delta a}{a_{\text{DM}}} \label{eq:16}$$

$$= \mp \frac{g^2}{4\pi G m^2} \left[ \frac{q_A}{\mu} - \frac{q_B}{\mu} \right] \frac{1}{1 + \frac{r}{\xi}} e^{-r/\xi},$$

where $a_{\text{DM}}$ is the acceleration resulted by the gravitational force of the DM,

$$a_{\text{DM}} = \frac{Gm_{\text{DM}}}{r^2}. \label{eq:17}$$
and we have introduced atomic mass unit \( u \) to express mass \( m = \mu u \). Based on the long-range interaction approximation to the fifth force \( \xi \to \infty \) [8], the Eötvös parameter becomes,

\[
\eta_{\text{DM}}^{(A,B)} \simeq \frac{g^2}{4\pi \mu u^2} \left( \frac{\mu}{\mu} \right)_{\text{DM}} \left[ \left( \frac{\mu}{\mu} \right)_A - \left( \frac{\mu}{\mu} \right)_B \right].
\]  

(18)

Using this expression, replacing the source of DM by some ordinary objects such as the Earth, the Sun or various man-made objects, the EP violation test can be formulated [2].

If we separate the acceleration of the ordinary material results from DM into the gravitational part \( a_{\text{DM}} \) and the fifth force part \( a_{\text{DM}} \), as \( a_{\text{tot}} = a_{\text{DM}} + a_{\text{DM}} \), the Eötvös parameter is related to the acceleration ratio through [26],

\[
a_{\text{DM}} = a_{\text{tot}} \frac{\eta_{\text{DM}}^{(A,B)}}{\sin \psi [\Delta(N/\mu)] + \cos \psi [\Delta(Z/\mu)]}. 
\]  

(19)

where \( \Delta(\cdot) \) means the difference between bodies A and B. The acceleration ratio is determined up to an unknown \( \psi \) once the magnitude of \( \eta_{\text{DM}} \) and the compositions of matter in the experiment are given.

TABLE II: The ratios of proton number and neutron number to the (dimensionless) mass for related objects in the current paper [6, 8]. Here the solid planets mean planets like the Earth, Mercury, Mars and others.

|          | \( Z/\mu \) | \( N/\mu \) |
|----------|-------------|-------------|
| NS       | 0           | 1.19        |
| WD       | 0.5         | 0.5         |
| solid planets | 0.49      | 0.51        |
| Sun      | 0.86        | 0.14        |
| Moon     | 0.502       | 0.498       |

We list the involved matter composition for kinds of objects in Table II. Based on the current constraint \( \eta_{\text{DM}}^{(\text{Sun,Mercury})} < 2.71 \times 10^{-4} \) and a future expected constraint \( \eta_{\text{DM}}^{(\text{Sun,Mercury})} \lesssim 3 \times 10^{-5} \) we can determine the constraint of fifth force as shown in Fig. 2 for neutral hydrogen. In the figure, the regions above curves represent the excluded parameter space. For comparison we reproduce the result for the NS-WD binary PSR J1713+0747 and that for LLR [8]. Although the observational constraint of PSR J1713+0747 on the Eötvös parameter is not as tight \( \eta_{\text{DM}}^{(\text{NS-WD})} \lesssim 0.004 \), the large difference of matter composition makes the resulted fifth force constraint comparable to other experiments. Regarding the LLR, although the observation accuracy makes the constraint to the Eötvös parameter very tight \( \eta_{\text{DM}}^{(\xi)} \lesssim 10^{-5} \), the similar matter composition for the Moon and the Earth makes the constraint to the fifth force not quite outstanding among the experiments. Interestingly our new perihelion precession method takes both the advantage of high observation accuracy and a mediate matter composition difference. As shown in Fig. 2, the constraint resulted from current observation is already similar to the result of LLR. In the near future one more order of magnitude improvement will be achieved.

FIG. 2: The fifth force constraint for neutral hydrogen from current perihelion precession measurement and an expected near-future measurement based on the BepiColombo mission. For comparison the constraints from the variation of orbital eccentricity of the NS-WD binary PSR J1713+0747 [8] and that from the LLR measurement [5, 27] are also included.

IV. DISCUSSION AND SUMMARY

The equivalence principle (EP) is important to gravitational theory and high energy physics theory. Alternatively the equivalence principle can be expressed as a fifth force. Since the equivalence principle, or the fifth force, depends on the difference in the matter composition, it is useful to investigate different kinds of matter. Due to the mysteries of dark matter and the possibility that dark matter might possess un-noticed large fifth-force charges, it is quite interesting to investigate the equivalence principle related to the dark matter from experiments and observations.

Regarding the dark matter in our Galaxy, the most stringent constraint comes from the LLR detection. The authors of Ref. [8] took the advantage of the large matter composition difference between a NS and a WD, and obtained a compelling constraint.

In this paper we propose a new method to use the perihelion precession observation of planets to constrain the fifth force acted by the dark matter in our Galaxy. We interestingly find that the Eötvös parameter is proportional to an extra perihelion precession rate. The proportional coefficient depends on, and only on, the gravitational acceleration at the Solar system acted by dark matter and the orbit information of the specific planet. Such dependence is shown in Eq. (3). Based on this relation,
the Eötvös parameter can be constrained by the observational accuracy of the perihelion precession [see Eq. (7)]. For a given observation accuracy of the perihelion precession, different planets result in different constraints on the Eötvös parameter. Due to the much smaller factor $\Xi$, Jupiter gives two orders stronger constraint than Mercury if comparable accurate precession can be detected for Jupiter.

Thanks to the high observation accuracy of the perihelion precession, the current constraint on the Eötvös parameter has achieved $\eta_{\text{DM}}^{(\text{Sun,Mercury})} \approx 2.71 \times 10^{-4}$. Thanks to the big difference of the matter composition between planets and the Sun, the constraint on the Eötvös parameter will result in a good constraint on the fifth force. Besides our results, currently the strongest constraint on the fifth force of the dark matter comes from the LLR. Current observation accuracy of the perihelion precession for the Mercury results in a similar constraint. After the BepiColombo mission, one more order of magnitude improvement in the constraint is expected in the near future. It is worth mentioning that it is amazing to see that an extremely weak fifth force from the viewpoint of particle physics, even weaker than the gravity, can be constrained with celestial dynamics. This kind of study complements the searches for dark matter particles from, say, the Large Hadron Collider and underground laboratories.

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Appendix A: The effect of the Fifth force on the planet perihelion precession

We consider the fifth force as a perturbation to the binary orbit elements. Then the perturbative equations of the binary orbit elements are (see page 158 in Ref. [28])

$$\frac{d\omega}{dt} = \sqrt{\frac{\mu}{e(1 + e \cos f)}} \left( \frac{1}{2} (\cos 2f - 2 e \cos f - 3) F_e - \sin f (e + 3 \cos f + 2 \cos^2 f) F_m \right)$$  \hspace{1cm} (A4)

$$\frac{dM}{dt} = n - \frac{1 - e^2}{\mu e (1 + e \cos f)} \left[ (2 e \sin f + e \cos^2 f) - e \sin f \cos^2 f + 2 \cos f - \sin f \cos f \right] F_e$$

$$+ \left( - e \cos^3 f + 2 \cos f - e \sin f \cos f \right) F_m$$

$$- 2 \sin f \cos f \cos^2 f) F_m$$  \hspace{1cm} (A5)

$$\frac{df}{dt} = \sqrt{\frac{\mu}{p^3}} (1 + e \cos f)^2 \left[ (2 \sin f + \cos f) + e \cos^3 f + e \sin f \cos f \right] F_e + \left( \cos f \sin f + e \cos^2 f \sin f - 2 \cos f - e \cos^2 f \cos f \right) F_m$$

$$+ (\cos f + e \cos f) F_m$$  \hspace{1cm} (A6)

where $\mu = GM_{\text{tot}}$, $p = a(1 - e^2)$, $n = \sqrt{\mu/a^3}$, $F_e = \vec{F} \cdot \hat{c}$, $F_m = \vec{F} \cdot \hat{m}$, $F_l = \vec{F} \cdot \hat{l}$ with $M_{\text{tot}}$ the total mass of the binary, $\hat{c}$ the unit vector from the center of mass towards the perihelion, $\hat{l}$ the unit vector pointing along the orbital angular momentum and $\hat{m} = \hat{l} \times \hat{c}$. $\vec{F}$ is the perturbed force. $\left( a, \iota, \Omega, e, \omega, M, f \right)$ are the binary orbit elements, respectively semimajor axis, inclination angle, ascending node, periastron angle, mean anomaly and true anomaly.

In order to investigate the secular change of the orbital elements, we transform the derivatives with respect to $t$ to the ones with respect to the true anomaly $f$,

$$\frac{da}{df} = \frac{2 p^3}{\mu} \left( 1 - e^2 \right)^2 \sin f \left( \frac{e + \cos f}{(1 + e \cos f)^2} F_e + \frac{(e + \cos f)}{(1 + e \cos f)^2} F_m \right)$$

$$\times \frac{1}{\mu} (1 + e \cos f)^3 F_i$$  \hspace{1cm} (A8)

$$\frac{d\iota}{df} = \frac{p^2 \sin (\omega + f)}{\mu (1 + e \cos f)^2} F_i$$  \hspace{1cm} (A9)

$$\frac{df}{df} = \frac{p^2}{\mu (1 + e \cos f)^2} \left( (1 + \cos^2 f + 2 e \cos f) \right) F_m$$

$$- \sin f \left( e + 3 \cos f + 2 \cos^2 f \right) F_m$$

$$- \sin f \left( e + 3 \cos f + 2 \cos^2 f \right) F_e$$

$$\frac{d\omega}{df} = \frac{p^2}{\mu e (1 + e \cos f)^2} \left[ \frac{1}{2} (\cos 2f - 2 e \cos f - 3) F_e$$

$$+ \sin f \cos f \cos f + e \cos f \right] F_i$$  \hspace{1cm} (A10)

$$\frac{dM}{df} = - \frac{p^2}{\mu e (1 + e \cos f)^2} \left[ (2 e \sin f + e \cos^2 f$$

$$- \sin f \left( e + 3 \cos f + 2 \cos^2 f \right) F_m$$

$$- \sin f \left( e + 3 \cos f + 2 \cos^2 f \right) F_e$$

$$+ \left( - e \cos^3 f + 2 \cos f - e \sin f \cos f \right) F_m$$

$$- 2 \sin f \cos f \cos^2 f) F_m$$

$$- 2 \sin f \cos f \cos^2 f) F_m$$
For the Sun-planet binary systems with respect to the dark matter in the Galaxy, the fifth force can be approximated as a constant force. Using this fact, and integrating the above equations with respect to $t$ over the range $(0, 2\pi)$, we obtain the change for a period of the binary orbit. Combining this perturbation from the fifth force and the general relativistic effect up to the first post-Newtonian order, the secular change of the binary elements can be expressed as

$$\left( \frac{da}{dt} \right)_{\text{sec}} = 0$$  
$$\left( \frac{d\Omega}{dt} \right)_{\text{sec}} = -\frac{3\mu}{2a\sqrt{1-e^2}} \sin \omega F_i$$  
$$\left( \frac{d\omega}{dt} \right)_{\text{sec}} = -\frac{3\mu}{c^2a(1-e^2)} - \frac{3}{2a}\times\left( \frac{1}{e} - \frac{1}{\sqrt{1-e^2}} \right)\frac{F}{e} - \frac{e}{\sqrt{1-e^2}} \cot \omega F_i$$  
$$\left( \frac{dM}{dt} \right)_{\text{sec}} = -\frac{n^3a^2}{c^2e^2\sqrt{1-e^2}} \times \left[ 5e^2 + (6 - 7\eta) \left( 1 - \sqrt{1-e^2} \right) \right] + \frac{1}{2na} \times \left\{ 2\left( -1 + 3e^2 + e^4 + (1 - e^2)^{5/2} \right) F_e + (5 - 2e^2)F_m \right\}$$  

(A13)  

(A14)  

(A15)  

(A16)  

(A17)  

(A18)  

(A19)  

(A20)

where $\varpi = \omega + \Omega$. Equivalently we can use vector notation $\vec{i} \equiv \sqrt{1 - e^2} \, \vec{e}$, $\vec{e} \equiv e \, \vec{e}$ to denote the above relations.

These two equations correspond to the second and the third expressions of Eq. (3) in [18].

Converting to the variables involved in the main text, Eq. (A20) reduces to Eq. (3).

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