A STOCHASTIC REPRESENTATION FOR BACKWARD INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT. By reversing the time variable we derive a stochastic representation for backward incompressible Navier-Stokes equations in terms of stochastic Lagrangian paths, which is similar to Constantin and Iyer’s forward formulations in [6]. Using this representation, a self-contained proof of local existence of solutions in Sobolev spaces are provided for incompressible Navier-Stokes equations in the whole space. In two dimensions or large viscosity, an alternative proof to the global existence is also given. Moreover, a large deviation estimate for stochastic particle trajectories is presented when the viscosity tends to zero.

1. Introduction

The classical Navier-Stokes equations describe the evolution of velocity fields of an incompressible fluid, and takes the following form with the external force zero:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p &= 0, \quad t \geq 0, \\
\nabla \cdot u &= 0, \\
\int_0^t u(0) &= u_0,
\end{align*}
\]

where column vector field \( u = (u^1, u^2, u^3)^t \) denotes the velocity field, \( p \) is the pressure and \( \nu \) is the kinematic viscosity. When the viscosity \( \nu \) vanishes, the above equation becomes the classical Euler equation:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= 0, \quad t \geq 0, \\
\nabla \cdot u &= 0, \\
\int_0^t u(0) &= u_0,
\end{align*}
\]

which describes the motion of an ideal incompressible fluid. The mathematical theory about Navier-Stokes equations and Euler equations has been extensively studied and the existence of regularity solutions is still a big open problem in modern PDEs.

Recently, Constantin and Iyer [6] presented an elegant stochastic representation for incompressible Navier-Stokes equations based on stochastic particle paths, which is realized by an implicit stochastic differential equation: the drift term is computed as the expected value of an expression involving the stochastic flow defined by itself. More precisely, let \( (u, X) \) solve the following stochastic system:

\[
\begin{align*}
X_t(x) &= x + \int_0^t u_s(X_s(x))ds + \sqrt{2\nu}B_t, \quad t \geq 0, \\
A_t &= X_t^{-1}, \\
\int_0^t u_t &= \mathbb{E}P[\nabla^t A_t(u_0 \circ A_t)],
\end{align*}
\]

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where $B_t$ is a 3-dimensional Brownian motion, $P$ is the Leray-Hodge projection onto divergence free vector fields, and $\nabla^t A_t$ denotes the transpose of Jacobi matrix $\nabla A_t$. Then $u$ satisfies equation (1) with initial data $u_0$. One of the proofs given in [6] is based on a stochastic partial differential equation satisfied by the inverse flow $A$. By using this representation, a self-contained proof of the existence of local smooth solutions is provided in [12, 13].

Let $(u, p)$ solve (1). Notice that if we make the time change:

$$\tilde{u}(t, x) := -u(-t, x), \quad \tilde{p}(t, x) = p(-t, x) \text{ for } t \leq 0,$$

then $\tilde{u}$ satisfies the following equation (called backward Navier-Stokes equation here):

$$\begin{cases}
\partial_t \tilde{u} + (\tilde{u} \cdot \nabla) \tilde{u} + \nu \Delta \tilde{u} + \nabla \tilde{p} = 0, & t \leq 0, \\
\nabla \cdot \tilde{u} = 0, & \tilde{u}(0) = u_0.
\end{cases}$$

(4)

The purpose of the present paper is to give a slightly different representation for $\tilde{u}$ by using backward particle paths. More precisely, let $(\tilde{u}, X)$ solve the following stochastic system

$$\begin{cases}
X_{t,s}(x) = x + \int_t^s \tilde{u}_r(X_{t,r}(x))dr + \sqrt{2\nu}(W_s - W_t), & t \leq s \leq 0, \\
\tilde{u}_t = \mathbb{E}[P[(\nabla^t X_{t,0})(u_0 \circ X_{t,0})]], & t \leq 0.
\end{cases}$$

(5)

Then $\tilde{u}$ satisfies backward incompressible Navier-Stokes equation (1) with final value $u_0$. Intuitively, by reversing the time, the starting point is changed as the end point. Hence, representation (5) is essentially the same as (3). But, equation (5) is easier to be dealt with mathematically since the unpleasant term $A = X^{-1}$ which usually incurs extra mathematical calculations does not appear in (5). Such a representation will be proved in Section 2. We emphasize that representations (3) and (5) are useful in numerical computations (cf. [19, 15]). Direct calculations shows that the second equation in (5) is equivalent to

$$\begin{align*}
\tilde{\omega}_t &= \nabla \times \tilde{u}_t = \mathbb{E}[(\nabla^{-1} X_{t,0})(\omega_0 \circ X_{t,0})], \\
\tilde{u}_t &= -\Delta^{-1} \nabla \times \tilde{\omega}_t,
\end{align*}$$

(6) (7)

where $\tilde{\omega}_t$ is the vorticity, $\nabla^{-1} X_{t,0}$ is the inverse of Jacobi matrix $\nabla X_{t,0}$ and (7) is exactly the Biot-Savart law.

We also mention other stochastic formulations for incompressible Navier-Stokes equations. In [9], a representation formula for the vorticity of three dimensional Navier-Stokes equations was given by using stochastic Lagrangian paths, however, there is no a self-contained proof of the existence given there. In [18], Le Jan and Sznitman used a backward-in-time branching process in Fourier space to express the velocity field of a three-dimensional viscous fluid as the average of a stochastic process, which then leads to a new existence theorem. In [3], basing on Girsanov’s transformation and Bismut-Elworthy-Li’s formula, Busnello introduced a purely probabilistic treatment to the existence of a unique global solution for two dimensional Navier-Stokes equations, where the stretching term disappears, and the non-linear equation obeyed by the vorticity has the form of Fokker-Planck equation. Later on, Busnello, Flandoli and Romito in [4] carefully analyzed an implicit probabilistic representation for the vorticity of three dimensional Navier-Stokes equations, and a local existence was given. In that paper, much attentions were also paid on a probabilistic representation formula for a general system of linear parabolic equations. Moreover, in [3, 4], an interesting probabilistic representation for the Biot-Savart law was also given and analyzed so that they can recover the velocity from the vorticity.
by probabilistic approach. Recently, Cipriano and Cruzeiro in [5] described a stochastic variational principle for two dimensional incompressible Navier-Stokes equations by using the Brownian motions on the group of homeomorphisms on the torus. More recently, Cruzeiro and Shamara\u0161ov\u0161 [7] established a connection between equation (4) and a system of infinite dimensional forward-backward stochastic differential equations on the group of volume-preserving diffeomorphisms of a flat torus.

This paper is organized as follows: In Section 3, we shall give a self-contained proof of local existence in Sobolev spaces. The proof is based on successive approximation or fixed point method as in [12]. Therein, Iyer considered the spatially periodic case and worked in H\"oder continuous function spaces. When Sobolev spaces are considered, we have to overcome the difficulty due to the non-closedness of Sobolev spaces under pointwise multiplications and compositions. Thus, it seems to be hard to exhibit the same proof in Section 3 for representation (3) due to the presence of $A = X^{-1}$.

A key point in the proof lies in that the flow map $x \mapsto X_{t,s}(x)$ preserves the Lebesgue measure, i.e., $\det(\nabla X_{t,s}) = 1$.

In Section 4, we shall give an alternative proof to the global existence when the spatial dimension is two or the viscosity is large enough in any dimensions. Such results are well known. For two dimensional Navier-Stokes and Euler equations in the whole space, the global existence of smooth solutions are referred to [19]. The global existence of regularity solutions for large viscosity is referred to [16]. Recently, Iyer [14] presented an alternative proof to the global existence for small Reynolds. His proof is based on the decay of heat flows and stochastic representation (3). Following [14], we will give a different proof based on Bismut’s formula and representation (5).

Let $(u^{\nu}, X^{\nu})$ denote the solution of equation (5). In Section 5, as $\nu$ goes to zero, an asymptotic probability estimate of $X^{\nu}$ in diffeomorphism group is presented by the well known large deviation estimate for stochastic diffeomorphism flows.

2. STOCHASTIC REPRESENTATION OF BACKWARD INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

We begin with some notational conventions. Fix $d \geq 2$ and put

$\mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $\mathbb{R}_- := (-\infty, 0]$, $\mathbb{I} := (d \times d)$-unit matrix

and

$I^2 := \{\sigma = (\sigma_k)_{k \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} : \|\sigma\|_{I^2}^2 := \sum \sigma_k^2 < +\infty\}.$

For a differentiable transformation $X$ of $\mathbb{R}^d$, the Jacobi matrix of $X$ is given by

$$\nabla X := \begin{pmatrix}
\partial_1 X^1, & \partial_2 X^1, & \cdots, & \partial_d X^1 \\
\partial_1 X^2, & \partial_2 X^2, & \cdots, & \partial_d X^2 \\
\vdots & \vdots & \ddots & \vdots \\
\partial_1 X^d, & \partial_2 X^d, & \cdots, & \partial_d X^d
\end{pmatrix},$$

where $\partial_i = \frac{\partial}{\partial x_i}$. We use $\nabla^t X$ to denote the transpose of $\nabla X$. For $k \in \mathbb{N}_0$, let $C^k_b(\mathbb{R}^d; \mathbb{R}^d)$ denote the space of $k$-order continuous differentiable vector fields on $\mathbb{R}^d$ with the norm:

$$\|u\|_{C^k_b} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^d} |D^\alpha u(x)| < +\infty,$$

where $D^\alpha$ denotes the derivative with respect to the multi index $\alpha$. 
Let $u \in C(\mathbb{R}_-; C^3_b(\mathbb{R}^d, \mathbb{R}^d))$ and $t \mapsto \sigma_t = \sigma(t) \in L^2_{loc}(\mathbb{R}_-; \mathbb{R}^d \times I^2)$ satisfy

$$\sum_{k=1}^{\infty} \sigma_k(t) \sigma_k(t) = I. \tag{8}$$

Let $\{X_{t,s}(x), t \leq s \leq 0\}$ solve the following SDE

$$X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + \sqrt{2v} \int_t^s \sigma_r dB_r, \tag{9}$$

where $B_t := \{B_t^k, t \leq 0, k \in \mathbb{N}\}$ is a sequence of independent standard Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Thanks to \(\mathbb{L}\), the diffusion operator associated to equation (9) is given by

$$L_t g \equiv \nu \Delta g + (u_t \cdot \nabla) g.$$

We first prove the following result.

**Theorem 2.1.** Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be a $C^2$-vector field satisfying

$$|D^\alpha \phi(x)| \leq C(1 + |x|^\beta), \quad |\alpha| \leq 2, \quad \beta > 0,$$

and $f \in C(\mathbb{R}_-; C^2_b(\mathbb{R}^d, \mathbb{R}^d))$. Define

$$\Lambda_\phi(t, x) := (\nabla^t X_{t,0}(x)) \phi(X_{t,0}(x)), \quad \Lambda_f(t, x) := \int_t^0 (\nabla^t X_{t,r}(x)) f_r(X_{t,r}(x)) dr$$

and

$$w_t(x) := \mathbb{E} \Lambda_\phi(t, x) - \mathbb{E} \Lambda_f(t, x).$$

Then $w \in C^{1,2}((-\infty, 0) \times \mathbb{R}^d)$ satisfies the following backward Kolmogorov’s equation:

$$\partial_t w_t + L_t w_t + (\nabla^t u_t)^t w_t = f_t, \quad \lim_{t \to 0} w_t(x) = \phi(x). \tag{10}$$

**Proof.** Let $g$ be a twice continuously differentiable function satisfying

$$|D^\alpha g(x)| \leq C(1 + |x|^\beta), \quad |\alpha| \leq 2, \quad \beta > 0.$$

For $h > 0$, by Itô’s formula we have

$$\mathbb{E} g(X_{t-h,t}(x)) = g(x) + \mathbb{E} \left[ \int_{t-h}^t (L_r g)(X_{t-h,r}(x)) dr \right].$$

From this, it is easy to see that

$$\frac{1}{h} \left[ \mathbb{E} g(X_{t-h,t}(x)) - g(x) \right] = \frac{1}{h} \mathbb{E} \left[ \int_{t-h}^t (L_r g)(X_{t-h,r}(x)) dr \right] \to (L_t g)(x) \quad \text{as} \quad h \to 0. \tag{11}$$

Noticing that

$$X_{t-h,0}(x) = X_{t,0} \circ X_{t-h,t}(x),$$

we have

$$\nabla X_{t-h,0}(x) = (\nabla X_{t,0}) \circ X_{t-h,t}(x) \cdot \nabla X_{t-h,t}(x).$$

Thus, by the independence of $X_{t-h,t}(x)$ with $X_{t,0}(x)$, we have

$$\mathbb{E} \Lambda_\phi(t - h, x) = \mathbb{E} [(\nabla^t X_{t-h,t}(x)) \Lambda_\phi(t, X_{t-h,t}(x))]$$

$$= \mathbb{E} \left[ (\nabla^t X_{t-h,t}(x))(\mathbb{E} \Lambda_\phi(t) \circ X_{t-h,t}(x)) \right]$$

and

$$\mathbb{E} \Lambda_f(t - h, x) = \mathbb{E} \left[ (\nabla^t X_{t-h,t}(x)) \Lambda_f(t, X_{t-h,t}(x)) \right].$$
Hence, we may write

\[ w_{t-h}(x) = E \left[ (\nabla^t X_{t-h,t}(x)) w_t(X_{t-h,t}(x)) \right] \]

and

\[ \frac{1}{h}(w_t(x) - w_{t-h}(x)) = -\frac{1}{h} E \left[ (\nabla^t X_{t-h,t}(x) - \mathbb{1}) w_t(X_{t-h,t}(x)) \right] \]

\[ -\frac{1}{h} \left[ E (w_t(X_{t-h,t}(x))) - w_t(x) \right] + \frac{1}{h} \left[ \int_{t-h}^{t} (\nabla^t X_{t-h,r}(x)) f_r(X_{t-h,r}(x)) dr \right] \]

\[ =: I_1^h(t, x) + I_2^h(t, x) + I_3^h(t, x). \]

Observing that

\[ \nabla X_{t-h,t}(x) - \mathbb{1} = \int_{t-h}^{t} (\nabla u_s) \circ X_{t-h,s}(x) \cdot \nabla X_{t-h,s}(x) ds. \]

we deduce

\[ \lim_{h \to 0} I_1^h(t, x) = -[(\nabla^t u_t) w_t](x). \]

By (11) we have

\[ \lim_{h \to 0} I_2^h(t, x) = -(L_t w_t)(x). \]

Moreover, a simple limit procedure also gives

\[ \lim_{h \to 0} I_3^h(t, x) = f_t(x). \]

Combining the above calculations, we conclude that

\[ \lim_{h \to 0} \frac{1}{h}(w_t(x) - w_{t-h}(x)) = -(L_t w_t)(x) - [(\nabla^t u_t) w_t](x) + f_t(x). \]

Equation (10) now follows (see [11, p.124] for more details).

**Remark 2.2.** A more general Feynman-Kac formula for a deterministic system of parabolic equations was given in [4]. However, the proof is simpler in our case. In representation (5), if we define \( w_t := E[(\nabla^t A_t)(u_0 \circ A_t)] \), then \( w_t \) also satisfies (11) with \( f = 0 \) (see [6, p.343, (4.5)]).

Basing on this theorem, we can give a stochastic representation for backward Navier-Stokes equation (4) as in [6].
Theorem 2.3. Let $\nu \geq 0$ and $u_0 \in C^2_b(\mathbb{R}^d; \mathbb{R})$ a deterministic divergence-free vector field, and $f \in C(\mathbb{R}_-; C^2_b(\mathbb{R}^d; \mathbb{R}))$. Suppose that $\sigma$ satisfies (5), and $(u, X)$ solves the stochastic system:

\begin{align}
X_{t,s}(x) &= x + \int_t^s u_r(X_{t,r}(x))dr + \sqrt{2\nu} \int_t^s \sigma_r dB_r, \quad t \leq s \leq 0, \quad (12) \\
u_t &= \mathbf{P}\Lambda^u_0(t) - \mathbf{P}\Lambda^w_f(t), \quad t \leq 0, \quad (13)
\end{align}

where $\mathbf{P}$ is the Leray-Hodge projection onto divergence free vector fields, and $\Lambda^u_0$ and $\Lambda^w_f$ are given by

\begin{align}
\Lambda^u_0(t, x) &= (\nabla^t X_{t,0}(x))u_0(X_{t,0}(x)), \\
\Lambda^w_f(t, x) &= \int_0^t (\nabla^t X_{t,r}(x))f_r(X_{t,r}(x))dr.
\end{align}

Then $u$ satisfies the backward incompressible Navier-Stokes equation:

\begin{align}
\begin{cases}
\partial_t u + (u \cdot \nabla)u + \nu \Delta u + \nabla p = f, \quad t \leq 0, \\
\nabla \cdot u = 0, \quad u(0, x) = u_0(x).
\end{cases} \quad (14)
\end{align}

Conversely, if $u$ solves backward Navier-Stokes equation (14), then $u$ is given by (13).

Proof. First of all, let

\begin{align}
w_t(x) := \mathbb{E}\Lambda^u_0(t, x) - \mathbb{E}\Lambda^w_f(t, x). \quad (15)
\end{align}

By Theorem 2.1, $w(t, x) = w_t(x)$ solves the following backward Kolmogorov’s equation:

\begin{align}
\partial_t w + (u \cdot \nabla)w + (\nabla^t u)w + \nu \Delta w = f, \quad w(0, x) = u_0(x). \quad (16)
\end{align}

In view of $u = \mathbf{P}w$, we may write

\begin{align}
w = u + \nabla q.
\end{align}

Substituting it into equation (16), one finds that

\begin{align}
\partial_t u + (u \cdot \nabla)u + \nu \Delta u + \nabla p = f, \quad u(0, x) = u_0(x),
\end{align}

where

\begin{align}
p = \partial_t q + (u \cdot \nabla)q + \nu \Delta q + \frac{1}{2}|u|^2.
\end{align}

Conversely, let $(u, p)$ solve (14). As above, if we define $w$ by (15), then $w$ satisfies equation (16). We need to show that $u = \mathbf{P}w$, or equivalently, for some scalar valued function $q$

\begin{align}
v := w - u = \nabla q.
\end{align}

By (16) and (14), $v$ solves the following equation:

\begin{align}
\partial_t v + (u \cdot \nabla)v + (\nabla^t u)v + \nu \Delta v = \nabla p - \frac{1}{2}\nabla|u|^2, \quad v(0, x) = 0. \quad (17)
\end{align}

Let

\begin{align}
q(t, x) := \mathbb{E} \left( \int_0^t \left[ \frac{1}{2}|u(r, X_{t,r}(x))|^2 - p(r, X_{t,r}(x)) \right] dr \right).
\end{align}

Then $q$ solves the following equation (cf. (11)):

\begin{align}
\partial_t q + (u \cdot \nabla)q + \nu \Delta q = p - \frac{1}{2}|u|^2, \quad q(0, x) = 0.
\end{align}

Taking gradients for both sides of the above equation yields

\begin{align}
\partial_t \nabla q + (u \cdot \nabla)\nabla q + (\nabla^t u)(\nabla q) + \nu \Delta \nabla q = \nabla p - \frac{1}{2}\nabla|u|^2.
\end{align}
By the uniqueness of solutions to linear equation \([17]\), \(v = \nabla q\).

**Remark 2.4.** In the above proof, we have assumed that the solutions are regular enough so that all the calculations are valid. The existence of regular solutions will be proven in the next section.

3. A Proof of Local Existence in Sobolev Spaces

With a little abuse of notations, in this and next sections we use \(p\) to denote the integrability index since the pressure will not appear below. For \(k \in \mathbb{N}_0\) and \(p > 1\), let \(W^{k,p}(\mathbb{R}^d; \mathbb{R}^d)\) be the usual \(\mathbb{R}^d\)-valued Sobolev space on \(\mathbb{R}^d\), i.e, the completion of \(\mathcal{C}_0^\infty(\mathbb{R}^d; \mathbb{R}^d)\) with respect to the norm:

\[
\|u\|_{k,p} := \|u\|_p + \sum_{j=1}^k \|\nabla^j u\|_p,
\]

where \(\| \cdot \|_p\) is the usual \(L^p\)-norm, and \(\nabla^j\) is the \(j\)-order gradient operator. Note that \(W^{0,p}(\mathbb{R}^d; \mathbb{R}^d) = L^p(\mathbb{R}^d; \mathbb{R}^d)\) and the following Sobolev’s embedding holds: for \(p > d\) (cf. [10])

\[
W^{1,p}(\mathbb{R}^d; \mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d; \mathbb{R}^d), \text{ i.e. } \| \cdot \|_{\infty} \leq c \| \cdot \|_{1,p},
\]

where \(c = c(p, d)\). Below, we shall use \(c\) to denote a constant which may change in different occasions, and whose dependence on parameters can be traced carefully from the context. Let \(W^{k,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)\) be the local Sobolev space on \(\mathbb{R}^d\). We introduce the following Banach space of transformations of \(\mathbb{R}^d\):

\[
\mathbb{X}^{k+2,p} := \left\{ X \in W^{k+2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d) : |X(0)| + \|\nabla X\|_\infty + \|\nabla^2 X\|_{k,p} < +\infty \right\}.
\]

**Definition 3.1.** The Weber operator \(W : L^p(\mathbb{R}^d; \mathbb{R}^d) \times \mathbb{X}^{2,p} \to L^p(\mathbb{R}^d; \mathbb{R}^d)\) is defined by

\[
W(v, \ell) := P[(\nabla^t \ell)v],
\]

where \(P\) is the Leray-Hodge projection onto divergence free vector fields.

**Remark 3.2.** \(P = I - \nabla(-\Delta)^{-1}\text{div}\) is a singular integral operator (SIO) which is bounded in \(L^p\)-space for \(p \in (1, \infty)\) (cf. [22]).

We now prepare several lemmas for later use.

**Lemma 3.3.** (i) For any \(k \in \mathbb{N}_0\) and \(p > d\), there exists a constant \(c = c(k, p, d) > 0\) such that for all \(v \in W^{k+2,p}(\mathbb{R}^d; \mathbb{R}^d)\) and \(\ell \in \mathbb{X}^{k+2,p}\),

\[
\| \nabla W(v, \ell) \|_{k+1,p} \leq c(\|\nabla \ell\|_{\infty} + \|\nabla^2 \ell\|_{k,p}) \cdot \|\nabla v\|_{k+1,p} \tag{19}
\]

(ii) For \(p > d\), there exists a constant \(c = c(p, d) > 0\) such that for all \(v_1, v_2 \in W^{2,p}(\mathbb{R}^d; \mathbb{R}^d)\) and \(\ell_1, \ell_2 \in \mathbb{X}^{2,p}\) with \(\ell_1 - \ell_2 \in L^p(\mathbb{R}^d; \mathbb{R}^d)\)

\[
\|W(v_1, \ell_1) - W(v_2, \ell_2)\|_p \leq c(\|v_1\|_{2,p} \|\ell_1 - \ell_2\|_p + \|\nabla \ell_2\|_{\infty} \|v_1 - v_2\|_p). \tag{20}
\]

**Proof.** (i) Noting that

\[
P((\nabla^t \ell)v) + P((\nabla^t v)\ell) = P(\nabla(\ell \cdot v)) = 0,
\]

we have

\[
\partial_i W(v, \ell) = P[(\nabla^t \partial_i v) + (\nabla^t \ell)\partial_i v] = P[-(\nabla^t v)^t \partial_i \ell + (\nabla^t \ell) \partial_i v]
\]

and

\[
\partial_j \partial_i W(v, \ell) = P[-(\nabla^t v)^t \partial_j \partial_i \ell - (\nabla^t \partial_j v) \partial_i \ell + (\nabla^t \ell) \partial_j \partial_i v + (\nabla^t \partial_j \ell) \partial_i v].
\]
Hence, by (18) we have
\[ \| \partial_t \mathbf{W}(v, \ell) \|_p \leq c(\| \nabla^t \mathbf{W}(v, \ell) \|_p + \| \nabla^t \partial_t \mathbf{W}(v, \ell) \|_p) \leq c(\| \nabla v \|_p \| \partial_t \ell \|_\infty + \| \nabla \ell \|_\infty \| \partial_t v \|_p) \leq c\| \nabla v \|_p \| \nabla \ell \|_\infty \] and
\[ \| \partial_j \partial_i \mathbf{W}(v, \ell) \|_p \leq c\| \nabla^2 \ell \|_p \| \nabla v \|_p + \| \nabla \ell \|_\infty \| \nabla v \|_p \leq c(\| \nabla \ell \|_\infty + \| \nabla^2 \ell \|_p) \cdot \| \nabla v \|_{1,p} \]
which produces
\[ \| \nabla \mathbf{W}(v, \ell) \|_{1,p} \leq c(\| \nabla \ell \|_\infty + \| \nabla^2 \ell \|_p) \cdot \| \nabla v \|_{1,p}. \]
The higher derivatives can be estimated similarly.

(ii) By (21), we have
\[ \mathbf{W}(v_1, \ell_1) - \mathbf{W}(v_2, \ell_2) = \mathbf{P}(\nabla^t(\ell_1 - \ell_2))v_1 + \mathbf{P}(\nabla^t v_2)(v_1 - v_2) \]
\[ = -\mathbf{P}(\nabla^t v_1)(\ell_1 - \ell_2) + \mathbf{P}(\nabla^t v_2)(v_1 - v_2). \]
So,
\[ \| \mathbf{W}(v_1, \ell_1) - \mathbf{W}(v_2, \ell_2) \|_p \leq c\| \nabla v_1 \|_p \| \ell_1 - \ell_2 \|_p + c\| \nabla v_2 \|_p \| \ell_1 - \ell_2 \|_p \]
which yields (20) by (18). \qed 

**Lemma 3.4.** (i) For $k \in \mathbb{N}_0$ and $p > d$, there exist constants $c = c(k, p, d) > 0$ and $\alpha_k \in \mathbb{N}_0$ such that for all $u \in W^{k+2,p}(\mathbb{R}^d; \mathbb{R}^d)$ and all $X \in \mathbb{X}^{k+2,p}$ preserving the volume,
\[ \| \nabla(u \cdot X) \|_{k+1,p} \leq c\| \nabla u \|_{k+1,p} (1 + \| \nabla X \|_{k+2}^{\alpha_k} + \| \nabla^2 X \|_{k,p}^{\alpha_k}). \] (22)
(ii) For $p > d$, there exists a constant $c = c(p, d) > 0$ such that for all $u \in W^{2,p}(\mathbb{R}^d; \mathbb{R}^d)$ and $X, \tilde{X} \in \mathbb{X}^{k,p}$ with $X - \tilde{X} \in \mathcal{L}^p(\mathbb{R}^d; \mathbb{R}^d)$,
\[ \| u \cdot X - u \cdot \tilde{X} \|_p \leq c\| \nabla u \|_{1,p} \cdot \| X - \tilde{X} \|_p. \] (23)

**Proof.** (i) Since $X$ preserves the volume, we have
\[ \| u \cdot X \|_p = \| u \|_p. \] (24)
Observe that for $m \geq 2$
\[ \nabla^k(u \cdot X) = (\nabla^m u) \cdot X \cdot (\nabla X)^m + \cdots + (\nabla u) \cdot X \cdot \nabla^m X. \] (25)
(22) follows by (18) and (24).
(ii) It follows from
\[ u \cdot X - u \cdot \tilde{X} = \int_0^1 (\nabla u) \cdot (sX + (1 - s)\tilde{X}) \cdot (X - \tilde{X}) ds \] (26)
and (18). \qed 

**Lemma 3.5.** For $k \in \mathbb{N}_0$, $U > 0$ and $T := \frac{1}{U}$, there exist constants $c_1 = c_1(p, d) > 0$ and $c_2 = c_2(k, p, d) > 0$ such that for any divergence free vector field $u \in C([-T, 0]; W^{k+2,p})$ satisfying $\sup_{t \in [-T, 0]} \| \nabla u_t \|_{k+1,p} \leq U$, the solution $X_{t,s}$ to (12) belongs to $\mathbb{X}^{k+2,p}$ a.s., and for all $t \in [-T, 0]$
\[ \| \nabla X_{t,0} \|_\infty \leq c_1, \quad \| \nabla^2 X_{t,0} \|_{k,p} \leq c_2. \] (27)
Proof. Noting that
\[ \nabla X_{t,s} = I + \int_t^s (\nabla u_r) \circ X_{t,r} \cdot \nabla X_{t,r} \, dr, \]
we have
\[ \| \nabla X_{t,s} \|_\infty \leq 1 + \int_t^s \| \nabla X_{t,r} \|_\infty \cdot \| \nabla u_r \|_\infty \, dr. \]
By Gronwall’s inequality, we obtain by (18)
\[ \text{By (29) and (18)} \]
x\[ \text{So,} \]
\[ \text{By Gronwall’s inequality again we get} \]
we have
\[ \text{Proof.} \]
\[ \text{Lemma 3.6.} \]
\[ \text{Higher derivatives can be estimated similarly step by step.} \]
\[ \text{Lemma 3.6. For } p > d \text{ and } T > 0, \text{ let } u, \tilde{u} \in C([-T,0];W^{2,p}(\mathbb{R}^d;\mathbb{R}^d)), \text{ and } X, \tilde{X} \text{ solve SDE (2) with drifts } u \text{ and } \tilde{u} \text{ respectively. Then for some } c = c(p,d) > 0 \text{ and any } t \in [-T,0], \]
\[ \|X_{t,0} - \tilde{X}_{t,0}\|_p \leq \exp \left[ \frac{c}{t} \|\nabla u_t\|_{1,p} \right] \cdot \int_0^t \|u_r - \tilde{u}_r\|_p \, dr. \]
\[ \text{Proof. We have} \]
\[ X_{t,s}(x) - \tilde{X}_{t,s}(x) = \int_t^s (u_r(X_{t,r}(x)) - \tilde{u}_r(\tilde{X}_{t,r}(x))) \, dr \]
\[ = \int_t^s (u_r(X_{t,r}(x)) - u_r(\tilde{X}_{t,r}(x))) \, dr \]
\[ + \int_t^s (u_r(\tilde{X}_{t,r}(x)) - \tilde{u}_r(\tilde{X}_{t,r}(x))) \, dr. \]
For } R > 0, \text{ let } B_R := \{ x \in \mathbb{R}^d : \| x \| \leq R \}. \text{ By virtue of } x \mapsto \tilde{X}_{t,r}(x) \text{ preserving the volume and formula (26), we have}
\[ \|X_{t,s} - \tilde{X}_{t,s}\|_{L^p(B_R)} \leq \int_t^s \|u_r \circ X_{t,r} - u_r \circ \tilde{X}_{t,r}\|_{L^p(B_R)} \, dr + \int_t^s \|u_r - \tilde{u}_r\|_p \, dr \]
\[ \leq \sup_{r \in [-T,0]} \| \nabla u_r \|_\infty \int_t^s \|X_{t,r} - \tilde{X}_{t,r}\|_{L^p(B_R)} \, dr \]
By Gronwall’s inequality and (13), we get
\[ \|X_{t,s} - \tilde{X}_{t,s}\|_{L^p(BR)} \leq \exp \left[ c \sup_{t \in [-T,0]} \|\nabla u_t\|_{1,p} \right] \cdot \int_{t}^{0} \|u_r - \tilde{u}_r\|_p dr. \] (31)

Letting \( R \) go to infinity gives (30). \( \Box \)

We are now in a position to prove the following local existence result.

**Theorem 3.7.** For \( \nu \geq 0, k \in \mathbb{N}_0 \) and \( p > d \), there exists a constant \( c_0 = c_0(k, p, d) > 0 \) independent of \( \nu \) such that for any \( u_0 \in W^{k+2,p}(\mathbb{R}^d; \mathbb{R}^d) \) divergence free and \( T := (c_0\|\nabla u_0\|_{k+1,p})^{-1} \), there is a unique pair \((u, X)\) with \( u \in C([-T,0]; W^{k+2,p}) \) satisfying

\[
\begin{align*}
X_{t,s}(x) &= x + \int_{t}^{s} u_r(X_{t,r}(x))dr + \sqrt{2\nu} \int_{t}^{s} \sigma_r dB_r, \quad t \leq s \leq 0, \\
u_t &= \mathbb{P}E[(\nabla^t X_{t,0})(u_0 \circ X^n_{t,0})], \quad t \leq 0. 
\end{align*}
\] (32)

Moreover, for any \( t \in [-T,0] \)
\[ \|\nabla u_t\|_{k+1,p} \leq c_0 \|\nabla u_0\|_{k+1,p}. \] (33)

**Proof.** Set \( u^n_t(x) := u_0(x) \). Consider the following Picard’s iteration sequence

\[
\begin{align*}
X^n_{t,s}(x) &= x + \int_{t}^{s} u^n_r(X^n_{t,r}(x))dr + \sqrt{2\nu} \int_{t}^{s} \sigma_r dB_r, \quad t \leq s \leq 0, \\
u^n_t &= \mathbb{P}E[(\nabla^t X^n_{t,0})(u_0 \circ X^n_{t,0})], \quad t \leq 0.
\end{align*}
\] (34)

Noting that
\[ \mathbb{P}E[(\nabla^t X^n_{t,0})(u_0 \circ X^n_{t,0})] = \mathbb{E}W(u_0 \circ X^n_{t,0}, X^n_{t,0}), \]
we have by (19) and (22)
\[ \|\nabla u_t^{n+1}\|_{k+1,p} \leq \mathbb{E}\|\nabla W(u_0 \circ X^n_{t,0}, X^n_{t,0})\|_{k+1,p} \leq c_0 \mathbb{E}\left[ (\|\nabla X^n_{t,0}\|_\infty + \|\nabla^2 X^n_{t,0}\|_{k,p}) \cdot \|\nabla (u_0 \circ X^n_{t,0})\|_{k+1,p} \right] \leq c_3 \mathbb{E}\left[ (1 + \|\nabla X^n_{t,0}\|_{\infty}^{k+3} + \|\nabla^2 X^n_{t,0}\|_{k.p}) \cdot \|\nabla u_0\|_{k+1,p} \right], \]

where \( \beta_k \in \mathbb{N} \) only depends on \( k \) and \( c_0 = c_3(k, p, d) > 1 \).

Set
\[ c_0 := c_3 (1 + c_1^{k+3} + c_2^{\beta_k}) \geq 1, \]
where \( c_1 \) and \( c_2 \) are from Lemma 3.5. Choosing \( U := c_0\|\nabla u_0\|_{k+1,p} \) and \( T := 1/U \) in Lemma 3.5 we have by induction and Lemma 3.5
\[ \sup_{t \in [-T,0]} \|\nabla u^n_t\|_{k+1,p} \leq U, \quad \forall n \in \mathbb{N}. \] (35)

On the other hand, we also have by (27)
\[ \|u_t^{n+1}\|_p \leq c_0 \mathbb{E}\|\nabla^t X^n_{t,0})(u_0 \circ X^n_{t,0})\|_p \leq c_0 \mathbb{E}\left[ \|\nabla^t X^n_{t,0}\|_\infty \cdot \|u_0 \circ X^n_{t,0}\|_p \right] \leq c \|u_0\|_p, \]
which together with (35) gives the following uniform estimate:
\[ \sup_{n \in \mathbb{N}} \sup_{t \in [-T,0]} \|u^n_t\|_{k+2,p} < +\infty. \] (36)
Now by (20) and (23) (30), we have
\[
\|u_{t}^{n+1} - u_{t}^{m+1}\|_{p} \leq cE \left[ \|u_{0} \circ X_{t,0}^{n}\|_{2, p} \cdot \|X_{t,0}^{n} - X_{t,0}^{m}\|_{p} \right. \\
+ \|\nabla X_{t,0}^{m}\|_{\infty} \cdot \|u_{0} \circ X_{t,0}^{n} - u_{0} \circ X_{t,0}^{m}\|_{p}\] \\
\leq c \int_{t}^{0} \|u_{r}^{n} - u_{r}^{m}\|_{p} dr,
\]
where \(c = c(p, d, U)\) is independent of \(n, m\). From this we derive that
\[
\limsup_{n, m \to \infty} \sup_{t \in [-T, 0]} \|u_{t}^{n} - u_{t}^{m}\|_{p} = 0.
\]
By (36) and interpolation inequality, we further have
\[
\limsup_{n, m \to \infty} \sup_{t \in [-T, 0]} \|u_{t}^{n} - u_{t}^{m}\|_{k+1, p} = 0.
\]
Therefore, there is a \(u \in C([-T, 0]; W^{k+1, p}(\mathbb{R}^{d}; \mathbb{R}^{d}))\) such that
\[
\lim_{n \to \infty} \sup_{t \in [-T, 0]} \|u_{t}^{n} - u_{t}\|_{k+1, p} = 0.
\]
Taking limits for (34), one finds that \(u\) is a solution of (32). Estimate (33) follows from (35).

\[\square\]

**Remark 3.8.** The constant \(c_{0}\) in (35) is usually strictly greater than 1. If \(c_{0}\) equals 1, then we can invoke the standard bootstrap method to obtain the global existence. This will be studied in the next section when the periodic boundary is considered and the viscosity is large enough.

Since the existence time interval in Theorem 3.7 is independent of the viscosity \(\nu\), we also obtain the local existence of solutions to Euler equation (2). Moreover, as \(\nu \to 0\), the solution of Navier-Stokes equation converges to the solution of Euler equation as given below.

**Proposition 3.9.** Keep the same assumptions as in Theorem 3.7. For \(\nu \geq 0\) and \(u_{0} \in W^{k+2, p} \cap L^{2}\), let \((u_{t}^{\nu}, X_{t}^{\nu})\) be the solution of (32) corresponding to viscosity \(\nu\) and initial value \(u_{0}\). Then for any \(j = 0, \cdots, k+1\), there exists \(c = c(k, j, p, d, \|u_{0}\|_{k+2, p}, \|u_{0}\|_{2}) > 0\) such that for all \(\nu \geq 0\) and \(t \in [-T, 0]\)
\[
\|u_{t}^{\nu} - u_{t}^{0}\|_{C_{t}^{j}} \leq c(\nu|t|)^{(k+2+j - \frac{1}{p})/(1 + \frac{k+2}{d} - \frac{1}{p})}.
\]

**Proof.** Note that \(u_{0} \in W^{k+2, p} \cap L^{2}\) guarantees \(u_{t}^{\nu} \in W^{k+2, p} \cap L^{2}\). By
\[
\partial_{t}(u_{t}^{\nu} - u_{t}^{0}) + \nu u_{t}^{\nu} + P[(u_{t}^{\nu} \cdot \nabla)u_{t}^{\nu} - (u_{t}^{0} \cdot \nabla)u_{t}^{0}] = 0,
\]
we have
\[
-\partial_{t}\|u_{t}^{\nu} - u_{t}^{0}\|_{2}^{2} = \nu\langle \Delta u_{t}^{\nu}, u_{t}^{\nu} - u_{t}^{0}\rangle_{2} + \langle(u_{t}^{\nu} \cdot \nabla)u_{t}^{\nu} - (u_{t}^{0} \cdot \nabla)u_{t}^{0}\rangle_{2} + \langle((u_{t}^{\nu} - u_{t}^{0}) \cdot \nabla)u_{t}^{\nu} - (u_{t}^{\nu} - u_{t}^{0})\rangle_{2}
\]
\[
\leq \nu\|\Delta u_{t}^{\nu}\|_{2} \cdot \|u_{t}^{\nu} - u_{t}^{0}\|_{2} + \|\nabla u_{t}^{\nu}\|_{\infty} \|u_{t}^{\nu} - u_{t}^{0}\|_{2},
\]
i.e.,
\[
-\partial_{t}\|u_{t}^{\nu} - u_{t}^{0}\|_{2} \leq \nu\|\Delta u_{t}^{\nu}\|_{2} + \|\nabla u_{t}^{\nu}\|_{\infty} \|u_{t}^{\nu} - u_{t}^{0}\|_{2}.
\]
By Gronwall’s inequality and (33) we obtain
\[
\|u_{t}^{\nu} - u_{t}^{0}\|_{2} \leq \nu \int_{t}^{0} \|\Delta u_{s}^{\nu}\|_{2} ds \cdot \exp \left[ \int_{t}^{0} \|\nabla u_{s}^{\nu}\|_{\infty} ds \right] \leq c\nu|t|.
\]
The desired estimate now follows by the Sobolev embedding (cf. [10]): for \( u \in W^{k+2,p} \cap L^2 \)
\[
\|\nabla^j u\|_{\infty} \leq c_{k,j,p,d} \|u\|_{k+2,p}^{\alpha} \|u\|_2^{1-\alpha},
\]
where \( \alpha = (\frac{j}{d} + \frac{1}{2})/(\frac{1}{2} + \frac{k+2}{d} - \frac{1}{p}) \).

\[ \Box \]

**Remark 3.10.** We cannot prove a convergence rate \( O(\sqrt{\nu t}) \) as in [12] starting from (32) because \( x \mapsto (X_{t,0}(x) - X_{t,0}(x)) \) does not belong to any \( L^p \)-spaces.

### 4. Existence of Global Solutions

#### 4.1. Global Existence in Two Dimensions

First of all, we recall the following Beale-Kato-Majda’s estimate about SIOs, which can be proved as in [19, p.117, Proposition 3.8], we omit the details.

**Lemma 4.1.** For \( p > d \), let \( u \in W^2_p(\mathbb{R}^d; \mathbb{R}^d) \) be a divergence free vector field and \( \omega := \text{curl} u \). Then, for some \( c = c(p,d) \)
\[
\|\nabla u\|_{\infty} \leq c(1 + \log^+ \|\omega\|_{1,p})(1 + \|\omega\|_{\infty}),
\]
where \( \log^+ x := \max\{\log x, 0\} \) for \( x > 0 \).

In two dimensional case, taking the curl for the second equation in (32), one finds that \( \omega_t := \text{curl} u_t := \partial_1 u^2_t - \partial_2 u^1_t = E[\omega_0 \circ X_t, 0] \).

From this, we clearly have
\[
\|\omega_t\|_{p} \leq \|\omega_0\|_{p}, \quad 1 \leq p \leq \infty.
\]

Basing (37) and representation (38), we may prove the following global existence for 2D Navier-Stokes and Euler equations.

**Theorem 4.2.** In two dimensions, for \( \nu \geq 0 \), \( k \in \mathbb{N}_0 \), \( p > 2 \) and \( u_0 \in W^{k+2,p}(\mathbb{R}^2; \mathbb{R}^2) \) divergence free, there exists a unique global solution \((u, X)\) to equation (32).

**Proof.** We only need to prove the following a priori estimate: for all \( t \in \mathbb{R}_- \)
\[
\|u_t\|_{k+2,p} \leq c(\|u_0\|_{k+2,p}, k, p, t) < +\infty,
\]
where \( c(\|u_0\|_{k+2,p}, k, p, t) \) continuously depends on its parameters.

Following the proof of Lemma 3.5, we have
\[
\|\nabla X_{t,0}\|_{\infty} \leq \exp \left[ \int_t^0 \|\nabla u_r\|_{\infty} dr \right].
\]

Noting that
\[
\nabla \omega_t = E(\nabla \omega_0 \circ X_{t,0} \cdot \nabla X_{t,0}),
\]
we have
\[
\|\nabla \omega_t\|_{p} \leq \|\nabla \omega_0\|_{p} \cdot \|\nabla X_{t,0}\|_{\infty}
\]
and by (39) and (40)
\[
\|\omega_t\|_{1,p} \leq \|\omega_0\|_{1,p} \cdot \left( 1 + \exp \left[ \int_t^0 \|\nabla u_r\|_{\infty} dr \right] \right).
\]

Hence, by (37) (39) and (41)
\[
\|\nabla u_t\|_{\infty} \leq c(1 + \log^+ \|\omega_t\|_{1,p})(1 + \|\omega_t\|_{\infty})
\]
\[
\leq c + c \int_t^0 \|\nabla u_r\|_{\infty} dr,
\]
where \( c = c(\|\omega_0\|_{1,p}, p) \). By Gronwall’s inequality we obtain
\[
\|\nabla u_t\|_\infty \leq c e^{c|t|}.
\]
Substituting this into (40) and (41) gives
\[
\|\nabla X_{t,0}\|_\infty \leq c e^{c|t| e^{c|t|}},
\]
and by Calderon-Zygmund’s inequality about SIOs (cf. [22])
\[
\|\nabla u_t\|_{1,p} \leq \|\omega_t\|_{1,p} \leq \|\omega_0\|_{1,p} \cdot \left(1 + e^{c|t| e^{c|t|}}\right).
\]
Moreover,
\[
\|u_t\|_p \leq c E(\|\nabla X_{t,0}\|_\infty \cdot \|u_0 \circ X_{t,0}\|_p) \leq c \|u_0\|_p \cdot e^{c|t| e^{c|t|}}.
\]
Thus,
\[
\|u_t\|_{2,p} \leq c(\|u_0\|_{2,p}, p, t) < +\infty.
\]
Starting from (38) and as in Lemma 3.5 higher derivatives can be estimated similarly. \( \Box \)

4.2. Global Existence for Large Viscosity. In this section, we study the existence of global solutions for large viscosity and work on the \( d \)-dimensional torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \). Let \( W^{k,p}(\mathbb{T}^d, \mathbb{R}) \) be the \( \mathbb{R}^d \)-valued Sobolev spaces on \( \mathbb{T}^d \) with vanishing mean. Instead of (12), we consider
\[
X_{t,s}(x) = x + \int_t^s u_r(X_{t,r}(x))dr + \sqrt{2\nu}(B_s - B_t),
\]
where \( B \) is the standard Wiener process on \( \Omega := C(\mathbb{R}_+; \mathbb{R}^d) \), i.e., for \( \omega \in \Omega \), \( B(\omega) = \omega(\cdot) \).

We recall the following Bismut’s formula (cf. [2, 8]). For the reader’s convenience, a short proof is provided here.

**Theorem 4.3.** For any \( t < 0 \) and \( f \in C_0^1(\mathbb{T}^d; \mathbb{R}) \), it holds that
\[
(\nabla E f(X_{t,0}))(x) = \frac{1}{t \sqrt{2\nu}} E \left[ f(X_{t,0}(x)) \int_t^0 \left( s(\nabla u_s) \circ X_{t,s} - \mathbb{1} \right)dB_s \right]. \tag{43}
\]
In particular, for any \( p > d \) and some \( c = c(p, d) \)
\[
\|\nabla E f(X_{t,0})\|_p \leq \frac{c}{\sqrt{\nu |t|}} \|f\|_p \|t| \cdot \sup_{s \in [t,0]} \|\nabla u_s\|_{1,p} + 1 \]. \tag{44}
\]

**Proof.** Fix \( t < 0 \) and \( y \in \mathbb{R}^d \) below and define
\[
h(s) := \frac{1}{t \sqrt{2\nu}} \left[ (t-s)y + \int_t^s [(\nabla u_r) \circ X_{t,r}(x)] \cdot (ry)dr \right], \quad s \in [t,0].
\]
Consider the Malliavin derivative of \( X_{t,s} \) with respect to the sample path along the direction \( h \), i.e.,
\[
D_h X_{t,s}(x, \omega) = \lim_{\varepsilon \to 0} \frac{X_{t,s}(x, \varepsilon h + \omega) - X_{t,s}(x, \omega)}{\varepsilon}, \quad \omega \in \Omega.
\]
From (12) one sees that
\[
D_h X_{t,s}(x) = \int_t^s [(\nabla u_r) \circ X_{t,r}(x)] \cdot D_h X_{t,r}(x)dr + \sqrt{2\nu} h(s)
\]
\[
= \frac{(t-s)y}{t} + \int_t^s [(\nabla u_r) \circ X_{t,r}(x)] \cdot \left[ D_h X_{t,r}(x) + \frac{ry}{t} \right]dr.
\]
On the other hand, we have
\[ \nabla X_{t,s}(x) \cdot y = y + \int_t^s [(\nabla u_r) \circ X_{t,r}(x)] \cdot \nabla X_{t,r}(x) \cdot y \, dr. \]

By the uniqueness of solutions, we get
\[ \nabla X_{t,s}(x) \cdot y = D_h X_{t,s}(x) + \frac{s y}{t}. \]

In particular,
\[ \nabla X_{t,0}(x) \cdot y = D_h X_{t,0}(x). \]

Now
\[ \nabla \mathbb{E} f(X_{t,0}) \cdot y = \mathbb{E} \left[ (\nabla f) \circ X_{t,0} \cdot \nabla X_{t,0} \cdot y \right] \]
\[ = \mathbb{E} [(\nabla f) \circ X_{t,0} \cdot D_h X_{t,0}] \]
\[ = \mathbb{E} [D_h (f \circ X_{t,0})] \]
\[ = \mathbb{E} \left[ (f \circ X_{t,0}) \int_t^0 \dot{h}(s) dB_s \right], \]

where the last step is due to the integration by parts formula in the Malliavin calculus (cf. [20]). Formula (43) now follows.

For estimation (44), by Hölder’s inequality and Itō’s isometry, the square of the right hand side of (43) is controlled by
\[ \frac{1}{2\nu t^2} \mathbb{E} |f(X_{t,0}(x))|^2 \mathbb{E} \left[ \int_t^0 |s(\nabla u_s) \circ X_{t,s}(x) - \mathbb{I}|^2 ds \right] \]
\[ \leq \frac{c}{\nu t} \mathbb{E} |f(X_{t,0}(x))|^2 \left[ |t|^3 \sup_{s \in [t,0]} \|\nabla u_s\|_{\infty} + |t| \right]. \]

Hence, by (18)
\[ \|\nabla \mathbb{E} f(X_{t,0})\|_p \leq \frac{c}{\sqrt{\nu t}} \|f\|_p \left[ |t| \cdot \sup_{s \in [t,0]} \|\nabla u_s\|_{\infty} + 1 \right] \]
\[ \leq \frac{c}{\sqrt{\nu t}} \|f\|_p \left[ |t| \cdot \sup_{s \in [t,0]} \|\nabla u_s\|_{1,p} + 1 \right]. \]

The proof is complete. \( \square \)

We now prove the following global existence result (see also [16, 14]).

**Theorem 4.4.** Let \( k \in \mathbb{N}_0 \) and \( p > d \), \( u_0 \in W^{k,p}(T^d, \mathbb{R}^d) \) be divergence free and mean zero. Let \((u, X)\) be the local solution of (32) in Theorem 3.7. Then, there exist \( T_0 = T_0(k,p,d,\|\nabla u_0\|_{k+1,p}) \) and \( \delta = \delta(k,p) > 0 \) such that if \( \nu \geq \delta \|\nabla u_0\|_{k+1,p} \), then
\[ \|\nabla u_{T_0}\|_{k+1,p} \leq \|\nabla u_0\|_{k+1,p}, \]

and there is a global solution to equation (32).

**Proof.** Let \((u, X)\) be the local solution of (32) on \([-T,0]\) in Theorem 3.7 where \( T = (c_0\|\nabla u_0\|_{k+1,p})^{-1} \). Recalling the estimations in Lemma 3.5 and Theorem 3.7, we have for all \( t \in [-T,0] \)
\[ \|\nabla X_{t,0}\|_{\infty} \leq c_1, \quad \|\nabla^2 X_{t,0}\|_{k,p} \leq c_2 \] (45)
and
\[ \| \nabla u_t \|_{k+1,p} \leq c_0 \| \nabla u_0 \|_{k+1,p}. \] (46)

Write
\[ u_t = \mathbf{P} \mathbb{E}[(\nabla t X_{t,0} - I)(u_0 \circ X_{t,0})] + \mathbf{P} \mathbb{E}(u_0 \circ X_{t,0}). \] (47)

We separately deal with the first term and the second term. For the first term in (47), using (45), (46) and as in Lemma 3.5, one may prove that for some \( c = c(k, p, d) \) and all \( t \in [-T, 0] \)
\[ \| \nabla X_{t,0} - I \|_{k+2,p} \leq c \| \nabla u_0 \|_{k+1,p} \cdot |t|. \]
Using this estimate as well as (22), (45) and (46), one finds that
\[ \| \nabla \mathbf{P} \mathbb{E}[(\nabla t X_{t,0} - I)(u_0 \circ X_{t,0})] \|_{k+1,p} \leq c \| \nabla u_0 \|_{k+1,p}^2 \cdot |t|. \] (48)

For the second term in (47), by (44), (46) and Poincare’s inequality, we have
\[ \| \nabla \mathbf{P} \mathbb{E}(u_0 \circ X_{t,0}) \|_p \leq \frac{c}{\sqrt{p}|t|} \| u_0 \|_p \leq \frac{c}{\sqrt{p}|t|} \| \nabla u_0 \|_p. \] (49)
Note that
\[ \nabla^2 \mathbb{E}(u_0 \circ X_{t,0}) = \nabla \mathbb{E}((\nabla u_0) \circ X_{t,0}) + \nabla \mathbb{E}(((\nabla u_0) \circ X_{t,0})(\nabla X_{t,0} - I)). \]
As above, we have
\[ \| \nabla \mathbb{E}((\nabla u_0) \circ X_{t,0}) \|_p \leq \frac{c}{\sqrt{p}|t|} \| \nabla u_0 \|_p \]
and
\[ \| \nabla \mathbb{E}(((\nabla u_0) \circ X_{t,0})(\nabla X_{t,0} - I)) \|_{k,p} \leq c \| \nabla u_0 \|_{k+1,p}^2 \cdot |t|. \]
So,
\[ \| \nabla^2 \mathbb{E}(u_0 \circ X_{t,0}) \|_p \leq \frac{c}{\sqrt{p}|t|} \| \nabla u_0 \|_p + c \| \nabla u_0 \|_{k+1,p}^2 \cdot |t|. \] (50)
Continuing the above calculations we get
\[ \| \nabla^{k+2} \mathbb{E}(u_0 \circ X_{t,0}) \|_p \leq \frac{c}{\sqrt{p}|t|} \| \nabla^{k+1} u_0 \|_p + c \| \nabla u_0 \|_{k+1,p}^2 \cdot |t|. \] (51)
Combining (49), (50) and (51), we find
\[ \| \nabla \mathbb{E}(u_0 \circ X_{t,0}) \|_{k+1,p} \leq \frac{c}{\sqrt{p}|t|} \| \nabla u_0 \|_{k,p} + c \| \nabla u_0 \|_{k+1,p}^2 \cdot |t|. \] (52)
Summarizing (47), (48) and (52) yields
\[ \| \nabla u_t \|_{k+1,p} \leq \left[ \frac{c_3}{\sqrt{p}|t|} + c_4 \| \nabla u_0 \|_{k+1,p} \cdot |t| \right] \| \nabla u_0 \|_{k+1,p}, \]
where \( c_3 = c_3(k, p, d) \) and \( c_4 = c_4(k, p, d) \). Now, taking \( T_0 = \frac{1}{2c_4 \| \nabla u_0 \|_{k+1,p}} \) and \( \delta = 8c_3^2 c_4 \), we have for \( \nu \geq \delta \| \nabla u_0 \|_{k+1,p} \)
\[ \| \nabla u_{T_0} \|_{k+1,p} \leq \| \nabla u_0 \|_{k+1,p}. \]
The proof is thus finished. \( \square \)
5. A Large Deviation Estimate for Stochastic Particle Paths

Let $\mathbb{G}^k$ denote the $k$-order diffeomorphism group on $\mathbb{R}^d$, which is endowed with the locally uniform convergence topology together with its inverse for all derivatives up to $k$. Then $\mathbb{G}^k$ is a Polish space. Let $\mathbb{G}_0^k$ be the subspace of $\mathbb{G}^k$ in which each transformation preserves the Lebesgue measure, equivalently,

$$\mathbb{G}_0^k := \{X \in \mathbb{G}^k : \det(\nabla X) = 1\}.$$ 

Then $\mathbb{G}_0^k$ is a closed subspace of $\mathbb{G}^k$, therefore, a Polish space.

It is clear that $t \mapsto X_{t}^\nu(\cdot) \in \mathbb{G}_0^k$ is continuous by the theory of stochastic flow (cf. [17]). We now state a large deviation principle of Freidlin-Wentzell’s type, which follows from the results in [1, 21] by using Proposition 3.9.

**Theorem 5.1.** Keep all the things as in Proposition 3.9. For any Borel set $E \subset C([−T, 0]; \mathbb{G}_0^k)$, we have

$$- \inf_{Y \in \bar{E}} I(Y) \leq \liminf_{\nu \to 0} \nu \log \mathbb{P}(X^\nu \in E) \leq \limsup_{\nu \to 0} \nu \log \mathbb{P}(X^\nu \in E) \leq - \inf_{Y \in \bar{E}} I(Y),$$

where $E^\circ$ and $\bar{E}$ denotes the interior and the closure respectively in $C([−T, 0]; \mathbb{G}_0^k)$, and $I(Y)$ is the rate function defined by

$$I(Y) := \frac{1}{2} \inf_{\{h \in L^2([−T, 0]; \mathfrak{g}^k) : S(h) = Y\}} \int_{−T}^{0} \|h_s\|^2 l_2 ds, \quad Y \in C([−T, 0]; \mathbb{G}_0^k),$$

where $S(h) = Y$ solves the following ODE:

$$Y_s(x) = x + \int_{−T}^{s} u^\nu_r(Y_r(x))dr + \int_{−T}^{s} \langle \sigma_r, h_r \rangle l_2 dr, \quad s \in [−T, 0].$$

**Remark 5.2.** In two dimensions, the $T$ in the above theorem can be arbitrarily large by Theorem 4.2.

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