Odderon effects in the differential cross-sections at Tevatron and LHC energies

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Abstract In the present paper, we extend the Froissaron-Maximal Odderon (FMO) approach at \( t \) different from 0. Our extended FMO approach gives an excellent description of the 2148 experimental points considered in a wide range of energies and momentum transferred. We show that the very interesting TOTEM results for proton-proton differential cross-section in the range 2.76-13 TeV, together with the Tevatron data for antiproton-proton at 1.8 and 1.96 TeV give further experimental evidence for the existence of the Odderon. One spectacular theoretical result is the fact that the difference in the dip-bump region between \( \bar{p}p \) and \( pp \) differential cross-sections is diminishing with increasing energies and for very high energies (say 100 TeV), the difference between \( \bar{p}p \) and \( pp \) in the dip-bump region becomes bigger than \( \bar{p}p \) at \(|t| \) about 1 GeV\(^2\). This is a typical Odderon effect. Another important - phenomenological - result of our approach is that the slope in \( pp \) scattering has a different behavior in \( t \) than the slope in \( \bar{p}p \) scattering. This is also a clear Odderon effect.

1 Introduction

The Odderon is certainly one the most important problems in strong interaction physics. It was introduced [1] in 1973 on the basis of asymptotic theorems [2], [3] and was rediscovered later in QCD [4], [5], [6]. In spite of the fact that its theoretical status is very solid, its experimental evidence from half a century is still scarce. This situation is not astonishing. The clear evidence for Odderon has to come by comparing the data at the same energy in hadron-hadron and antihadron-hadron scatterings. But we have not such accelerators! We therefore have to limit our search for evidence for the Odderon only in an indirect way. The search for the Odderon is crucial in order to confirm the validity of QCD. It is very fortunate that the TOTEM datum \( \rho^{pp} = 0.1\pm0.01 \) at 13 TeV[7] is the first experimental discovery of the Odderon, namely in its maximal form [8]. Moreover, we checked recently that the Froissaron-Maximal Odderon (FMO) approach is the only model in agreement with the LHC data. We generalized the FMO approach by relaxing the \( \ln^2 s \) constraints both in the even- and odd-under-crossing amplitude and we show that, in spite of a considerable freedom of a large class of amplitudes, the best fits bring us back to the maximality of strong interaction [9].

In the present paper, we extend the FMO approach at \( t \) different from 0. We show that the very interesting TOTEM results for proton-proton differential cross-section in the range 2.76-13 TeV, together with the D0 data for antiproton-proton at 1.96 TeV give further experimental evidence for the existence of the Odderon.

2 Extension of the FMO approach at \( t \) different from zero - General definitions

In general amplitude of \( pp \) forward scattering is

\[
F_{pp}(s,t) = F_+(s,t) + F_-(s,t)
\]
and the amplitude of antiproton-proton scattering is
\[ F_{pp}(s, t) = F_+(s, t) - F_-(s, t). \]  
(2)

In this model we used the following normalization of the physical amplitudes.
\[ \sigma_t(s) = \frac{1}{\sqrt{8(s-4m^2)}} \text{Im} F(s, 0), \]
\[ d\sigma_{tot} = \frac{1}{64\pi ks(s-4m^2)} |F(s, t)|^2 \]  
(3)

where \( k = 0.3893797 \text{ mb} \cdot \text{GeV}^2 \). With this normalization the amplitudes have dimension \( \text{mb} \cdot \text{GeV}^2 \).

Strictly speaking crossing-even (CE), \( F_+(s, t) \), and crossing-odd (CO), \( F_-(s, t) \), parts of amplitudes are defined as functions of \( z_t = (t+2s-4m^2)/(4m^2-t) \), where \( m \) is proton mass, with the property
\[ F_{\pm}(-z_t, t) = \pm F_{\pm}(z_t, t). \]  
(4)

In the FMO model CE and CO terms of amplitudes are defined as sums of the asymptotic contributions \( F^H(s, t) \), \( F^{MO}(s, t) \) and Regge pole contributions which are important at the intermediate and relatively low energies
\[ F_+(z_t, t) = F^H(z_t, t) + F^{R+}(z_t, t), \]
\[ F_-(z_t, t) = F^{MO}(z_t, t) + F^{R-}(z_t, t) \]  
(5)

where \( F^H(z_t, t) \) denotes the Froissaron contribution and \( F^{MO}(z_t, t) \) denotes the Maximal Odderon contribution. Their specified form will be defined below.

3 Regge poles and their double rescatterings

In the FMO model in the terms \( F^{R\pm}(s, t) \) we consider not only single Regge pole contributions but also their double rescatterings or double cuts. Their contributions, \( F^{R+}_{pp}(z_t, t), F^{R+}_{pp}(z_t, t) \), are the following
\[ F^{R+}_{pp}(z_t, t) = F^+(z_t, t) + F^+(z_t, t), \]
\[ F^{R-}_{pp}(z_t, t) = F^-(z_t, t) - F^-(z_t, t) \]  
(6)

where \( z_t = -1 + 2s/(4m^2 - t) \approx 2s/(4m^2 - t) \). For a convenience in further work with parameterizations in FMO model at \( t = 0 \) and \( t \neq 0 \) contrary to standard definition of \( z_t \) we put opposite sign for it.

\[ F^+(z_t, t) = F^+(z_t, t) + F^{R+}(z_t, t) + F^{PP}(z_t, t), \]
\[ F^-(z_t, t) = F^-(z_t, t) + F^{R-}(z_t, t) + F^{PO}(z_t, t). \]  
(7)

Here \( F^+(z_t, t), F^-(z_t, t) \) are simple j-pole Pomeron and Odderon contributions and \( F^{R+}(z_t, t), F^{R-}(z_t, t) \) are effective f and \( \omega \) simple j-pole contributions, where \( j \) is an angular momenta of these reggeons. \( F^{PP}(z_t, t), F^{FO}(z_t, t), F^{PO}(z_t, t) \), are double PP, OO, PO cuts, correspondingly. We consider the model at \( t \neq 0 \) and at energy \( \sqrt{s} > 19 \text{ GeV} \), so we neglect the rescatterings of secondary reggeons with \( P \) and \( O \). In the considered kinematical region they are small. Besides, because \( f \) and \( \omega \) are effective, they can take into account small effects from the cuts. The standard Regge pole contributions have the form
\[ F^{R\pm}(z_t, t) = \left( \frac{1}{i} \right) c_2 m^2 C_{R\pm} e^{\mp i z_t}, \]  
(8)

where \( R_\pm = P, O, R+, R- \) and \( c_2(0) = 1 + 2P, O, R+ = 1 \). The factor \( 2m^2 = z_t/t (s \gg m^2) \) is inserted in amplitudes \( F^{R\pm}(z_t, t) \) in order to have the normalization for amplitudes and dimension of coupling constants (in mb) coinciding with those in [3]. The same is made for all other amplitudes, including Froissaron and Maximal Odderon (see below). For the coupling function \( C_{R\pm}(t) \) we have considered two possibilities. The first one is a simple exponential form
\[ C_{R\pm}(t) = C_{R\pm} e^{b R z t}, \quad C_{R\pm}(0) = C_{R\pm}. \]  
(9)

The second case is a linear combination of exponents for Standard Pomeron and Odderon terms which allow to take into account some possible effects of non-exponential behavior of coupling function. Secondary reggeons still are parameterized in the simplest exponential form, because we did not consider low energies where terms \( R_\pm(s, t) \) are more important.

\[ C^{P,O}(t) = C^{P,O} \left[ \Psi^{P,O}(i) \right]^2, \]
\[ \Psi^{P,O}(i) = c^{P,O} e^{b^{P,O} t} + (1 - c^{P,O}) e^{b^{O,O} t}. \]  
(10)

The double cuts are written in a simplified form as compared with the exact form of a cut. They can be considered also as effective PP, OO, PO cuts. Namely,
\[ F^{PP}(z_t, t) = -\frac{2m^2 C^{PP}}{\ln(-i z_t)} (\alpha_{PP} + i \alpha'_{PP}) e^{2\phi e^{PP}}, \]
\[ \alpha_{PP}(t) = 1 + \alpha'_{PP}, \quad \alpha'_{PP} = \frac{\alpha'_{PP}}{2}, \]  
(11)

\[ F^{OO}(z_t, t) = -\frac{2m^2 C^{OO}}{\ln(-i z_t)} (\alpha_{OO} + i \alpha'_{OO}) e^{2\phi e^{OO}}, \]
\[ \alpha_{OO}(t) = 1 + \alpha'_{OO}, \quad \alpha'_{OO} = \frac{\alpha'_{OO}}{2}, \]  
(12)

\[ F^{PO}(z_t, t) = i \frac{2m^2 C^{PO}}{\ln(-i z_t)} (\alpha_{PO} + i \alpha'_{PO}) e^{2\phi e^{PO}}, \]
\[ \alpha_{PO}(0) = 1 + \alpha'_{PO}, \quad \alpha'_{PO} = \frac{\alpha'_{PO}}{2}. \]  
(13)
4 Froissaron and Maximal Odderon at $t \neq 0$

4.1 Partial amplitudes for Froissaron and Odderon

Let us start from the Froissaron amplitude in $(s,t)$-representation at high $s$. The amplitude can be expanded in the series of partial amplitudes $\phi(\omega,t)$. In accordance with the standard definition of partial amplitude

$$F(z_t,t) = 16\pi \sum_{j=0}^{\infty} (2j+1) P_j(-z_t) \phi(j,t).$$

(14)

With such definition partial amplitude satisfies the unitarity equation in the form

$$\text{Im} \phi(j,t) = \rho(t) |\phi(j,t)|^2,$$  

$$\rho(t) = \sqrt{1 - 4m^2/t}$$  

(15)

We use of the Sommerfeld-Watson transform amplitude (here and in what follows $\omega = j - 1$ and $j$ is complex angular momentum) which can be written as follows

$$F^\zeta(z_t,t) = 16\pi \sum_{\xi = -1,1} \frac{d\omega}{2\pi t} (2\omega + 3) \frac{1 - \xi e^{-i\pi\omega}}{-\sin(\pi\omega)} \times \phi^\zeta(\omega, t) P_{1+\omega}(z_t)$$

$$= 16\pi \sum_{\xi = -1,1} \frac{d\omega}{2\pi t} (2\omega + 3) \times e^{-i\pi\omega/2} \frac{-\xi - \xi e^{-i\pi\omega/2}}{-\sin(\pi\omega)} \phi^\zeta(\omega, t)$$

$$= z_t \sum_{\xi = -1,1} \frac{d\omega}{2\pi t} \phi^\zeta(\omega, t).$$

(16)

where $\xi$ is the signature of the term, contour $C$ is a straight line parallel to imaginary axis and at the right of all singularities of $\phi^\zeta(\omega, t)$, $\zeta = \ln(z_t) - i\pi/2 \equiv \ln(-i\omega z_t)$ and

$$\phi^\zeta(\omega, t) = 16\pi(2\omega + 3) e^{\xi\pi\omega/2} - \xi e^{-i\pi\omega/2} \pi^{-1/2} 2^{1+w+1} \times \frac{\Gamma(\omega + 3/2)}{\Gamma(\omega + 2)} \phi^\zeta(\omega, t)$$

(17)

Thus for crossing even amplitude ($\xi=+1$) we have

$$\varphi^+(\omega, t) = i32\sqrt{\pi}(2\omega + 3) \frac{\Gamma(\omega + 3/2)}{\Gamma(\omega + 2)} 2^{\omega} \varphi^+(\omega, t)$$

(18)

and for crossing odd amplitude ($\xi=-1$)

$$\varphi^-(\omega, t) = -32\sqrt{\pi}(2\omega + 3) \frac{\Gamma(\omega + 3/2)}{\Gamma(\omega + 2)} 2^{\omega} \varphi^-(\omega, t)$$

(19)

Inverse transformation is

$$\varphi^\pm(\omega, t) = \int_0^\infty d\zeta e^{-\omega\zeta} F^\pm(z_t,t), \quad z_t = e^\zeta.$$

(20)

One can show that in order to have maximal growth of total cross section $\sigma_{tot}(s) \propto \xi^2$ at $s \to \infty$, to have a growing elastic cross section bounded by

$$\sigma_{el}(s)/\sigma_{tot}(s) \to \text{const} \quad \text{at} \quad s \to \infty$$

and to provide the correct analytical properties of amplitude at $t \approx 0$ necessary to write the partial amplitude $\phi(\omega,t)$ in the following form (more details are given in the Appendices, Section A)

$$\varphi^\pm(\omega, t) = \left( \begin{array}{c} i \beta^\pm(\omega, t) \\ [\omega^2 + R^2 q^2]^3/2 \end{array} \right).$$

(21)

where $r_\pm$ are some constants, $q^2_\pm = -t$ and $\beta(\omega, t)$ has not singularity at $\omega^2 + R^2 q^2 = 0$. In fact a choice of the sign in $\phi^-(\omega, t)$ does not matter because the crossing odd terms contribute to $pp$ and $pp$ amplitude with the opposite signs. In order to have agreement with parametrization and parameters which we used in the papers devoted to analysis of the data at $t = 0$, we should replace -1 for for +1 in front of $\phi^-(\omega, t)$

At $\omega = 0$, function $\varphi^-(\omega, t)$ has singularity in $t$ if $\beta^-(0, t) \neq 0$, namely, $\varphi^-(0, t) \propto (-t)^{3/2}$. One of arguments against the Maximal Odderon is that this singularity in partial amplitude means the existing of massless particle in the model. However as we seen above $\varphi^-(\omega, t)$ is not the real physical partial amplitude which is

$$\varphi^-(\omega, t) = \left[ 32\sqrt{\pi}(2\omega + 3) \frac{\Gamma(\omega + 3/2)}{\Gamma(\omega + 2)} 2^{\omega} \times \sin(\pi\omega/2) \varphi^-(\omega, t) \right]^{-1}$$

(22)

and it equals to 0 at $\omega = 0$ because of $\sin(\pi\omega/2)$ coming from signature factor.

Now let us suppose that in accordance with the structure of the singularity of $\varphi_{\pm}(\omega, t)$ at $\omega^2 + \omega^2_0 = 0$ ($\omega^2_0 = R^2 q^2$) the functions $\beta_{\pm}(\omega, t)$, depending on $\omega$ through the variable $\kappa_{\pm} = (\omega^2 + \omega^2_0)^{1/2}$, can be expanded in powers of $\kappa_{\pm}$

$$\varphi^\pm(\omega, t) = \left( \begin{array}{c} i \beta^\pm(\omega, t) + \kappa_{\pm} \beta^\pm_3(\omega, t) + \kappa_{\pm}^2 \beta^\pm_5(\omega, t) \\ \kappa_{\pm}^3 \end{array} \right).$$

(23)

Then making use the table integrals (see the Section A) we obtain the expressions for $F^\pm(z_t, t)$ which are written in the next Section.

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4.2 Froissaron and Maximal Odderon in \((s,t)\)-representation

At \(t = 0\) Froissaron and Maximal Odderon have the universal form independently of any extension to \(t \neq 0\):

\[
F^H(z_t,t = 0) = iz[H_1 \ln^2(-iz_t) + H_2 \ln(-iz_t) + H_3],
\]

\[
F^{MO}(z_t,t = 0) = z[O_1 \ln^2(-iz_t) + O_2 \ln(-iz_t) + O_3]
\]

where \(z = 2m^2z_t\). At \(t = 0\) we have \(z_t = (s-2m^2)/(2m^2)\).

The Froissaron and the Maximal Odderon defined at \(t = 0\) by above Eqs. (24,25) allow various extensions to analytical \(t\)-dependencies. Probably it is impossible a priori to choose the best of them. In the present work we consider an extension of Eqs. (24,25). Comparing the various cases we have found that the best description of the data is achieved if third terms do not contain Bessel functions of \(r_+\tau_\xi\), while they have a more complicated than simple exponent functions of \(t\).

\[
-\frac{1}{iz}F^H(z_t,t) = H_1\zeta^2\frac{2J_1(r_+\tau_\xi)}{r_+\tau_\xi} \Phi_{H,1}(t) + H_2\zeta \frac{\sin(r_+\tau_\xi)}{r_+\tau_\xi} \Phi_{H,2}(t) + (H_3 - C^P)\Phi_{H,3}(t),
\]

\[
\Phi_{H,i}(t) = \exp(b^H_iz_+), \quad i = 1,2.
\]

\[
\Phi_{H,3}(t) = h \exp(b^H_3z_+) + (1-h) \exp(b^H_tz_+),
\]

\(z_+ = 2m_\pi - \sqrt{4m^2_\pi - t}\).

\[
-\frac{1}{z}F^{MO}(z_t,t) = O_1\zeta^2\frac{2J_1(r_-\tau_\xi)}{r_-\tau_\xi} \Phi_{MO,1}(t) + O_2\zeta \frac{\sin(r_-\tau_\xi)}{r_-\tau_\xi} \Phi_{MO,2}(t) + (O_3 + C^O)\Phi_{MO,3}(t),
\]

\[
\Phi_{MO,i}(t) = \exp(b^O_iz_-), \quad i = 1,2.
\]

\[
\Phi_{MO,3}(t) = o \exp(b^O_3z_-) + (1-o) \exp(b^O_tz_-),
\]

\(z_- = 3m_\pi - \sqrt{9m^2_\pi - t}\).

In above equations \(\zeta = \ln(-iz_t)\). Due to the factor \(z\) (instead of \(z_t\)) the amplitudes \(F^H(z_t,t)\) and \(F^{MO}(z_t,t)\) have the required normalization with additional factor \(2m^2\).

In the Froissaron and Maximal Odderon contributions the couplings \(H_3\) and \(O_3\) are redefined. The standard Pomeron and Odderon were included to constant terms \(H_3\) and \(O_3\), correspondingly, in the fit at \(t = 0\).

At \(t \neq 0\) contribution of \(P\) and \(O\) are important and they have to be added to amplitudes. In order to save parameter \(H_2(2O_3)\) as sum of Froissaron and standard Pomeron (Maximal Odderon and standard Odderon) at \(t = 0\) but distinguish them at \(t \neq 0\) we made the above mentioned replacement. Thus the real couplings of Froissaron and Maximal Odderon are \(H_3 - C^P\) and \(O_3 + C^O\) correspondingly.

5 Comparison of the FMO model with the data

We give here the results of the fit to the data in the following region of \(s\) and \(|t|\):

for \(\sigma_{tot}(s), \rho(s)\) at \(5\ \text{GeV} \leq \sqrt{s} \leq 13\ \text{TeV},\)

for \(d\sigma/dt\) at \(9\ \text{GeV} \leq \sqrt{s} \leq 13\ \text{TeV}\) and \(0.05\ \text{GeV}^2 \leq |t| \leq 14.2\ \text{GeV}^2\).

The \(t\)-region is chosen in such a way that we can ignore the contribution of the Coulomb part of amplitudes which in given region are one order of magnitude or less than 1% of the nuclear amplitude.

For 13 TeV TOTEM data we used the data at \(t = 0\) for \(\sigma_{tot}\) and \(\rho\) and data for \(d\sigma/dt\) at \(t \neq 0\) presented at the 4th Elba Workshop on Forward Physics @ LHC Energy by F. Nemes [11] and at the 134th open LHCC meeting by F. Rava [12].

5.1 A new method of minimization for global fits

By the term "global fit" we mean a simultaneous fit at \(t = 0\) (experimental observables \(\sigma_{tot}, \sigma_{PP}, \sigma_{P\bar{P}}\) and \(\rho_{PP}, \rho_{P\bar{P}}\)) and at \(t \neq 0\) (\(d\sigma^{PP}/dt, d\sigma^{P\bar{P}}/dt\)). In the standard method in which the sum of all \(\chi^2\) for each experimental point is minimized, we face with the following problem.

The number of experimental points at \(t = 0\) in the chosen kinematic region is about 250, whereas number of points at \(t \neq 0\) is about 2000. It is clear that in such a case an influence of 2000 points for differential cross sections is much stronger than those for cross sections and ratios of real to imaginary part of amplitude at \(t = 0\). As result we have all parameters mainly fixed by differential cross sections. Thus we see some destruction of the good fit at \(t = 0\) which can be performed separately, the best parameters controlling behavior of the forward observables are changed and a worse value of \(\chi^2\) for data at \(t = 0\).

There are two ways to fix the problem. In the first method we fix all parameters which are critical for \(t = 0\) and then perform the fit at \(t \neq 0\). It works for simple models where free parameters can be split for two relatively independent groups: parameters responsible for \(t = 0\) and parameters responsible for \(t\) different from 0. However this method does not work, for example, in the models which take into account multiple rescatterings. In this case parameters such as slopes of Regge trajectories, slopes \(B\)-s of residual functions (in the simplest
case they are proportional to $\exp(Bt)$ and so on) are important for value of amplitude at $t \neq 0$. At the same time, the expressions for the total cross sections and ratios $\rho$ also contain these parameters. For such models we can’t do separate fits at $t = 0$ and at $t \neq 0$. Thus we come back to the above mentioned problem in a global fit.

To solve the problem we propose a new method of minimization. In this method we minimize not the total sum of the individual $\chi^2$-s but another combination of these $\chi^2$-s. Namely the most adapted quantity for minimization is the weighted $\chi^2$. Let we have $M$ kinds of observables and for each them $i = 1, 2, ..., M$ we have $N_i$ experimental points. We define the weighted $\chi^2$ as

$$\chi^2_w = \frac{1}{M} \sum_{i=1}^{N_i} \chi^2(k),$$

$$\chi^2(k) = \left[ \frac{f(k) - f_{exp}(k)}{\delta f_{exp}(k)} \right]^2. \quad (28)$$

In fact we sum $\chi^2(i)/N_i(M)$-s over kinds of observables, where $\chi^2(i) = \sum \chi^2(k)$ is sum of individual $\chi^2(k)$ for the given kind of observable, their number being $N_i$. In the ideal case after minimization all $\chi^2(i)/N_i$ would be $\sim 1$ and $\chi^2_w \sim 1$. In our case we have $M = 6, N_1 = 110, N_2 = 59, N_3 = 67, N_4 = 12, N_5 = 1511, N_6 = 389$.

In the next Sections we present and shortly comment the results of the standard minimization and of weighted one performed in accordance with Eq. (28).

We would like to notice that it is very easy to calculate the usual total $\chi^2$ and $\chi^2$/dof or to present the results of the fit in the set of $\chi^2(i)/N_i$ along with $\chi^2_w$.

### 5.2 Total $pp$ and $\bar{p}p$ cross sections and parameters $\rho_{pp}$ and $\rho_{\bar{p}p}$

In this Section we give the results of the standard minimization at $t = 0$. The obtained in FMO model values of $\chi^2$ in the fit at $t = 0$ with double rescattering of the standard Pomeron and Odderon (Eqs. [14][15]) are given in the Table 1. One can see from this Table that agreement of the model with data at $t = 0$ is excellent.

| Process | Observable | $N_i$ number of data | $\chi^2/N$ FMO-model |
|---------|------------|----------------------|-----------------------|
| $pp$    | $\sigma_{tot}$ | 110                 | 0.813814E+00          |
| $\bar{p}p$ | $\sigma_{tot}$ | 59                  | 0.789203E+00          |
| $pp$    | $\rho$      | 67                   | 0.155244E+01          |
| $\bar{p}p$ | $\rho$      | 12                  | 0.450656E+00          |

$\chi^2_{tot} = 245.525, \chi^2/NDF = 1.045$

**Table 1 Number of experimental points and the quality of their description when the standard minimization in FMO model is applied**

The values of parameters obtained from the fit at $t = 0$ are given in the Table 2. Experimental data and theoretical curves are presented in the Fig. 1.

| Name (dimension) | Value | Error |
|------------------|-------|-------|
| $H_1$ (mb)       | 0.298 | 0.001 |
| $H_2$ (mb)       | -1.951| 0.004 |
| $H_3$ (mb)       | 46.439| 0.021 |
| $O_1$ (mb)       | -0.0747| 0.001 |
| $O_2$ (mb)       | 1.783 | 0.010 |
| $O_3$ (mb)       | -12.426| 0.086 |
| $\alpha_+(0)$    | 0.533 | 0.001 |
| $C^+$ (mb)       | 73.427| 0.20  |
| $\alpha_-(0)$    | 0.264 | 0.002 |
| $C^-$ (mb)       | 59.940| 0.436 |
| $C^{O+}$ (mb)    | 25.928| 0.250 |
| $C^{O-}$ (mb)    | 21.096| 0.107 |
| $C^{P+}$ (mb)    | 23.945| 0.233 |

**Table 2 Parameters and their errors obtained in the standard fit of FMO model at $t = 0$**

5.3 $pp$ and $\bar{p}p$ differential cross sections $d\sigma/dt$

Here we present results for both methods of minimization, the standard method and the weighted one.

Number of experimental points in $pp$ and $\bar{p}p$ differential cross sections used in the standard fit (all parameters listed in the Table 1 are fixed in this case) and quality of fit are shown in the Table 3.

The values of $\chi^2$ for the both methods of fit at $t \neq 0$ are presented in the Table 4. One can see that the both methods of minimization lead to comparable description of the data at $t = 0$ but the weighted minimization gives lower (by about 3%) total $\chi^2$.

| Process | Observable | $N_i$ number of data | $\chi^2/N$ FMO-model |
|---------|------------|----------------------|-----------------------|
| $pp$    | $d\sigma/dt$ | 1511            | 1.491                 |
| $\bar{p}p$ | $d\sigma/dt$ | 389              | 1.370                 |

$\chi^2_{tot} = 2785.445, \chi^2/NDF = 1.489$

**Table 3 Number of experimental points for differential cross sections and the quality of their description in FMO model by the standard method of minimization (parameters determining the amplitudes at $t = 0$ are fixed in this fit)**
Fig. 1 Total cross sections and ratios $\rho$ in FMO model with the $PP, PO, OO$ terms added

| Observable | Number of points, $N_p$ | Standard fit | Weighted fit |
|------------|-------------------------|--------------|--------------|
| $\sigma_{pp}^{tot}$ | 110 | $x^2/N_p$ | $x^2/N_p$ |
| $\sigma_{\bar{p}p}^{tot}$ | 59 | 0.789 | 0.805 |
| $\rho_{pp}$ | 67 | 1.552 | 1.554 |
| $\rho_{\bar{p}p}$ | 12 | 0.451 | 0.414 |
| $d\sigma_{pp}/dt$ | 1511 | 1.491 | 1.457 |
| $d\sigma_{\bar{p}p}/dt$ | 389 | 1.370 | 1.118 |

Table 4 Comparison of the fit results with two methods of minimization

The values of parameters and their errors obtained in two fits, standard and weighted, within the FMO model are given in the Table 5 (parameters of the Froissarson and Maximal Odderon terms) and in the Table 6 (parameters of the standard Pomeron and Odderon, of their double rescatterings and of secondary reggeons).

In Figs. 2 and 3 we show the differential cross-sections at energies bigger than 19 GeV. In Fig. 4 we show the differential cross-sections at the LHC energies 7, 8 and 13 TeV and in Fig. 5 we show our predictions at 2.76 TeV. In Fig. 6 we show in a magnified way the differential cross-sections at 53 GeV. In Fig. 7 we show the $\bar{p}p$ differential cross section at 1.18-1.96 TeV and $pp$ differential cross section at 7 TeV.

As one can see from these figures our description of the data in a wide range of energies is very good. In

Fig. 8 we show the evolution of the dip-bump structure in $pp$ and $\bar{p}p$ differential cross sections with increasing energy. In Fig. 9 we show in a magnified way the dip-bump region at different energies and in Fig. 10 we show the evolution of the ratio $R_\sigma = (d\sigma(\bar{p}p)/dt)/(d\sigma(pp)/dt)$ with increasing energy. A remarkable prediction can be seen from these last three figures: the difference in the dip-bump region between $pp$ and $\bar{p}p$ differential cross sections is diminishing with increasing energies and, for very high energies (say 100 TeV, see Fig. 9), the ratio in the dip-bump region goes to 1. At ISR energies until $\sim 60$ GeV the ratio $R_\sigma > 1$ and then it becomes less than 1 but increases to maximum at some $t_0$. After maximum the value of $R_\sigma$ decreasing and equals to 1 at $t_0$ which is going to lower $t$ with increasing energy. At higher $t$ however $R_\sigma$ is oscillating around of 1 when $t$ increase. This is a spectacular Odderon effect.
Odderon effects in the differential cross-sections at Tevatron and LHC energies

Table 5 Parameters of Froissaron and Maximal Odderon and their errors in FMO model determined from the standard and weighted fits to the data on $d\sigma/dt$ and their double rescatterings, of secondary Reggeons and their $\rho$ and $\omega$ and ratios $\rho/\omega$ were also included in fitting the data.

| Name (dimension) | Standard minimization | Weighted minimization |
|------------------|-----------------------|-----------------------|
| $H_1$ (mb)       | 0.208 fixed           | 0.287 fixed           |
| $H_2$ (mb)       | -1.95 fixed           | -1.579 fixed          |
| $H_3$ (mb)       | 46.439 fixed          | 43.221 fixed          |
| $b_1^1$ (GeV$^{-1}$) | 2.402 0.001           | 2.595 0.056           |
| $b_1^2$ (GeV$^{-1}$) | 6.505 0.033           | 7.573 2.227           |
| $b_1^3$ (GeV$^{-1}$) | 1.584 0.020           | 1.837 0.421           |
| $h$              | 0.0201 0.0003         | 0.099 0.038           |
| $b_1^4$ (GeV$^{-1}$) | 3.638 0.001           | 3.654 0.121           |
| $r_+=r_-$ (GeV$^{-1}$) | 0.2602 0.0001        | 0.296 0.004           |
| $O_1$ (mb)       | -0.0747 fixed         | -0.0566 0.0088        |
| $O_2$ (mb)       | 1.783 fixed           | 1.294 0.152           |
| $O_3$ (mb)       | -12.426 fixed         | -8.606 0.842          |

Table 6 Parameters of standard Pomeron and Odderon, of their double rescatterings, of secondary Reggeons and their errors in FMO model determined from the standard and weighted fits to the data on $d\sigma/dt$. Total cross sections $\sigma_{tot}$ and ratios $\rho$ were also included in the weighted fit.

| Name (dimension) | Standard minimization | Weighted minimization |
|------------------|-----------------------|-----------------------|
| $\alpha_p$ (GeV$^{-2}$) | 0.2664 0.0002         | 0.289 0.028           |
| $C_1^p$ (mb)     | 118.440 0.105         | 73.056 3.570          |
| $c_1^p$          | 0.231 0.002           | 0.187 0.020           |
| $b_1^1$ (GeV$^{-2}$) | 14.542 0.175          | 17.096 5.900          |
| $b_1^2$ (GeV$^{-2}$) | 2.085 0.002           | 1.873 0.101           |
| $\alpha_1$ (GeV$^{-2}$) | 0.1927 0.0003         | 0.191 0.008           |
| $C_1^\omega$ (mb) | 27.927 0.059          | 25.146 2.006          |
| $c_1^\omega$     | 0.605 0.001           | 0.690 0.023           |
| $b_1^1$ (GeV$^{-2}$) | 2.016 0.004           | 1.917 0.123           |
| $b_1^2$ (GeV$^{-2}$) | 0.114 0.003           | 0.060 0.052           |
| $\alpha_+$ (GeV$^{-2}$) | 0.533 fixed          | 0.561 0.028           |
| $\alpha_- (GeV^{-2})$ | 0.800 0.277         | 0.800 0.207           |
| $C_+^p$ (mb)     | 73.427 fixed          | 68.597 8.308          |
| $\alpha_+$ (Ge$^{-2}$) | 0.264 fixed          | 0.297 0.100           |
| $C_- (mb)$       | 59.940 fixed          | 25.146 2.006          |
| $\alpha_-$ (GeV$^{-2}$) | 0.800 0.035         | 1.198 0.391           |
| $b_-$ (GeV$^{-2}$) | 0.059 0.074          | 0.0 fixed             |
| $C_+^{OO}$ (mb)  | 25.328 0.250          | 44.723 6.565          |
| $C_+^{PO}$ (mb)  | -21.096 0.107         | 3.759 6.565           |
| $b_+^{PP}$ (GeV$^{-2}$) | 23.954 0.233         | 14.134 2.373          |
| $b_+^{OO}$ (GeV$^{-2}$) | 3.160 0.353          | 2.427 1.101           |
| $b_+^{PO}$ (GeV$^{-2}$) | 3.209 0.267          | 3.504 13.101          |
| $b_+^{OO}$ (GeV$^{-2}$) | 0.295 0.003          | 2.479 0.042           |

Fig. 2 $pp$ differential cross sections at $\sqrt{s}$ > 19 GeV

Fig. 3 $pp$ differential cross sections at $\sqrt{s}$ from 19 GeV up to 1.96 TeV
Fig. 4 \( pp \) differential cross sections at \( \sqrt{s} = 7, 8, 13 \) TeV

Fig. 5 Prediction of FMO model for \( pp \) differential cross sections at \( \sqrt{s} = 2.76 \) TeV

Fig. 6 \( pp \) and \( \bar{p}p \) differential cross sections at \( \sqrt{s} = 53 \) GeV

Fig. 7 \( \bar{p}p \) differential cross section at 1.18-1.96 TeV and \( pp \) differential cross section at 7 TeV
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6 Slope $B(s, t)$

The slope $B(s, t)$ is a very interesting quantity in the search for Odderon effects. It is defined by

$$B(s, t) = \frac{d}{dt} \ln \frac{d\sigma}{dt}.$$  

If we consider the dependence of slope on energy and compare this dependence with available experimental data we have to take into account that slopes in any realistic model depend on $t$. Dependence of slope on $t$ at various energies in the FMO model is illustrated in Fig. 11 (left panel). Therefore we must to calculate the slope $< B(s) >$ averaged in some interval of $t$. We did that in the interval $|t| \in (0.05, 0.2)\text{GeV}^2$ which approximately is in agreement to the intervals from which the experimental data on $B$ is determined.

$$< B(s) > = \frac{1}{\Delta t} \int_{t_{\text{min}}}^{t_{\text{max}}} dt \frac{d}{dt} (\ln(d\sigma/dt)) = \frac{1}{\Delta t} \left[ \ln \left( \frac{d\sigma(t_{\text{max}})/dt}{d\sigma(t_{\text{min}})/dt} \right) \right]$$

where $\Delta t = t_{\text{max}} - t_{\text{min}}$.

We show in Table 7 our predictions for the averaged slopes in the TeV region of energy as compared with experiments at Tevatron and LHC.

In Fig. 11 (right panel) we show the increasing of the averaged slopes at $t=0$ with increasing energy. One can see that the slopes are approaching the $\ln^2 s$ increase at high energies.
| Energy (TeV) | Experiment | $<B_{pp}(s)>$ (GeV$^{-2}$) | $<B_{pp}(s)>$ (GeV$^{-2}$) |
|-------------|------------|----------------|----------------|
|             | Experimental data | FMO model | Experimental data | FMO model |
| 1.8         | E710       | -      | 16.81 | 16.3±0.5 |
| 1.8         | CDF        | -      | 16.81 | 16.98±0.25 |
| 1.96        | D0         | -      | 16.94 | 16.86±0.25 |
| 2.76        | TOTEM      | 17.1±0.26 | 17.49 | - |
| 7           | TOTEM      | 19.9±0.3 | 19.14 | - |
| 7           | ATLAS      | 19.73±0.39 | 19.14 | - |
| 8           | TOTEM      | 19.9±0.3 | 19.40 | - |
| 8           | ATLAS      | 19.74±0.31 | 19.40 | - |
| 13          | TOTEM      | 20.36±0.19 | 20.40 | - |

Table 7: Experimental values of slopes of $pp$ and $\bar{p}p$ differential cross sections at TeV energies and the averaged slopes calculated in FMO model.

Fig. 11: Slopes $B_{pp}(t)$ and $B_{\bar{p}p}(t)$ at increasing energy (left panel) and the $s$-dependence of the averaged slopes $<B_{pp}(s)>$, $<B_{\bar{p}p}(s)>$ together with experimental data (right panel).

Fig. 12: Slope $B(s, t)$ for $pp$ (left panel) and $\bar{p}p$ (right panel) at selected energies.
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In Fig. 12 we plot the slopes as function of $t$ in $pp$ and $\bar{p}p$ scatterings. We discover from the $t$-dependence of the slopes an extremely interesting phenomenon. The slope in $pp$ scattering has a different behaviour in $t$ than the slope in $\bar{p}p$ scattering. In the left panel of Fig. 12 we see that in $pp$ scattering the slopes are first nearly constant and after that they fall sharply, they cut a first time the $B(t) = 0$ line and finally they reach an approximately constant value for higher $t$. The two crossing points of the $B(t) = 0$ line move towards smaller $t$ when energy increases. In the right panel of Fig. 12 we see a very different behaviour in $pp$ scattering. In this case, at energies higher than ISR ones, $B(t)$ marginally crosses zero, but no so deeply and sharply as in $pp$ scattering. For completeness, we show in Fig. 13 the slope parameter for $pp$ scattering at 7 and 13 TeV as compared with the slope parameter in $\bar{p}p$ scattering at 1.96 TeV, where we can see the same phenomenon.

This phenomenon is a clear Odderon effect. The odd-under crossing amplitude makes the difference between $pp$ and $\bar{p}p$ scatterings and this amplitude is dominated at high energy by the Maximal Odderon.

7 Comparison with other approaches

To our knowledge, the present model is the only model which fits forward and non forward data in a wide range of energies (including TeV region), without explicit theoretical defects (like the violation of the unitarity).

However, it is important to note that our results concerning the slopes are in complete agreement with those obtained recently by Csörgő et al. [13], who performed a very useful mirroring between the discontinuous experimental data (points) and continuous analytic functions (scattering amplitudes) by using an expansion in terms of Lévy polynomials. In such a way they get a very clear Odderon effect concerning the slopes. Their analysis have no dynamical content: it is a parametrization of experimental data in terms of big number of parameters.

This agreement is very important from two points of view. On one side, the Odderon existence is reinforced by two quite different analysis, one model-independent and the other one having a dynamical content.

On another side, the fact that the Maximal Odderon is in agreement with a model-independent analysis re-inforce the status of the Maximal Odderon.

8 Conclusion

In our paper we present an extension of the Froissargent-Maximal Odderon (FMO) approach for $t$ different from zero, which satisfies rigorous theoretical constraints. Our extended FMO approach gives an excellent description of a general QCD light front formalism [15].

The two quite different analysis have no dynamical content: it is a parametrization of experimental data (points) and continuous analytic functions (scattering amplitudes) by using an expansion in terms of Lévy polynomials. In such a way they get a very clear Odderon effect concerning the slopes. Their analysis have no dynamical content: it is a parametrization of experimental data in terms of big number of parameters.

This agreement is very important from two points of view. On one side, the Odderon existence is reinforced by two quite different analysis, one model-independent and the other one having a dynamical content.

On another side, the fact that the Maximal Odderon is in agreement with a model-independent analysis re-inforce the status of the Maximal Odderon.

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Fig. 13 Dependence on $t$ of the slopes $B(s,t)$ for $pp$ scattering at 7 and 13 TeV and for $\bar{p}p$ scattering at 1.96 TeV
A Appendix

A.1 General constraints

Let us reiterate here that the model with \( \sigma_t(s) \propto \ln^2 s \) is not compatible with a linear pomeron trajectory having the intercept 1. Indeed, let us assume that

\[
\alpha_P(t) = 1 + \alpha_P't \tag{31}
\]

and the partial wave amplitude has the form

\[
\varphi(j, t) = \eta(j) \frac{\beta(j, t)}{[j - 1 - \alpha_P't]!^n} \approx \frac{i\beta(1, t)}{[j - 1 - \alpha_P't]!^n}, \tag{32}
\]

\[
\eta(j) = \frac{1 + \xi e^{-\pi j}}{-\sin \pi j}. \tag{33}
\]

For Pomeron (simple or double pole) and Froissaron signature, \( \xi = +1 \).

In \((s, t)\)-representation amplitude \( \varphi(j, t) \) is transformed to

\[
a(s, t) = \frac{1}{2\pi i} \int dj \varphi(j, t) e^{j\xi j}, \quad \xi = \ln(s/s_0). \tag{34}
\]

Then, we have pomeron contribution at large \( s \) as

\[
a(s, t) \approx -\tilde{\beta}(t) \ln(-is/s_0)^{n-1}(is/s_0)^{1+\alpha_P't} \tag{35}
\]

where

\[
\tilde{\beta}(t) = \beta(t)/\sin(\pi \alpha_P(t)/2). \tag{36}
\]

If as usually \( \tilde{\beta}(t) = \tilde{\beta} \exp(bt) \) then we obtain

\[
\sigma_t(s) \propto \ln^{n-1} s, \quad \sigma_{el}(s) \propto \frac{1}{s^2} \int dt |a(s, t)|^2 \propto \ln^{2n-3} s. \tag{37}
\]

According to the obvious inequality,

\[
\sigma_{el}(s) \leq \sigma_t(s) \tag{38}
\]

we have

\[
2n - 3 \leq n - 1 \quad \Rightarrow \quad n \leq 2. \tag{39}
\]

Thus we come to the conclusion that the model with \( \sigma_t(s) \propto \ln^2 s \) \((n=3)\) is incompatible with a linear pomeron trajectory. In other words the partial amplitude Eq. (32) with \( n = 3 \) is incorrect.

If \( n = 1 \) we have a simple \( j \)-pole leading to constant total cross section and vanishing at \( s \to \infty \) elastic cross section. However such a behaviour of the cross sections is not supported by experimental data.

If \( n = 2 \) we have the model of dipole pomeron \( \sigma_t(s) \propto \ln(s) \) and would like to emphasize that double \( j \)-pole is the maximal singularity of partial amplitude settled by unitarity bound [38] if its trajectory is linear at \( t \approx 0 \).

We would like to notice here that TOTEM data for the pp total cross section exclude the dipole pomeron model which is unable to describe with a reasonable \( \chi^2 \) the high values of \( \sigma_{el}(s) \) at LHC energies.

Thus, constructing the model leading to cross section which increases faster than \( \ln(s) \), we need to consider a more complicated case (we consider at the moment a region of small \( t \) and \( j \approx 1 \)):

\[
\varphi_{\pm}(j, t) = \frac{\beta(j, t)}{[j - 1 + \tau(-t)^{1/\mu}]^n} \approx \frac{i\beta(1, t)}{[j - 1 + \tau(-t)^{1/\mu}]^n}. \tag{40}
\]

Making use of the same arguments as above, we obtain

\[
\sigma_t(s) \propto \ln^{n-1} s, \quad \sigma_{el}(s) \propto \ln^{2n-3-\mu} s \quad \text{and} \quad \mu \geq n - 1. \tag{41}
\]

However in this case amplitude \( a(s, t) \) has a branch point at \( t = 0 \) which is forbidden by analyticity of amplitude \( a(s, t) \).

A proper form of amplitude leading to \( t_{eff} \) decreasing faster than \( \ln^{-1} s \) (it is necessary for \( \sigma_t \) rising faster than \( \ln s \) is the following

\[
\varphi_{\pm}(j, t) = \frac{\beta(j, t)}{[j - 1 + \tau(-t)^{1/\mu}]^n}. \tag{43}
\]

Now we have \( m \) branch points colliding at \( t = 0 \) in \( j \)-plane and creating the pole of order \( mn \) at \( j = 1 \) (but there is no branch point in \( t \) at \( t = 0 \)). At the same time \( t_{eff} \propto 1/\ln^m s \) and from \( \sigma_{el} \propto \ln^{2mn-2-\mu} s \leq \sigma_t \propto \ln^{n-1} s \leq \ln^2 s \) one obtains

\[
\begin{align*}
mn & \leq m + 1, \\
nm & \leq 3. \tag{44}
\end{align*}
\]

If \( \sigma_{el} \propto \sigma_t \) then \( n = 1 + 1/m \). Furthermore, if \( \sigma_t \propto \ln s \) then \( m = 1 \) and \( n = 2 \) which corresponds just to the dipole pomeron model. In the Froissaron (or tripole pomeron) model \( m = 2 \) and \( n = 3/2 \). It means that \( \sigma_t \propto \ln^2 s \).

A.2 Partial amplitudes

As it follows from Eq. (44) for the dominating at \( s \to \infty \) contribution in a Froissaron model with \( \sigma_t(s) \propto \ln^2(s) \), i.e. \( n = 2, m = 3/2 \), we have to take (here and in what follows we used a more convenient notations \( \omega = j - 1 \) and \( \omega_{\pm} = r_{\pm} \tau = r_{\pm} \sqrt{-1/\tau} \), \( \tau_0 = 1 GeV^2 \)). Then

\[
\varphi_{\pm}(\omega, t) = \eta_{\pm}(\omega) \frac{\beta_{\pm}(\omega, t)}{(\omega^2 + \omega_{\pm}^2)^{3/2}} \tag{45}
\]

where

\[
\eta_{\pm}(\omega) = \frac{1 + e^{-i\pi \omega}}{\sin \pi \omega}. \tag{46}
\]

For even signature

\[
\tilde{\beta}_{\pm}(\omega, t) = \beta_{\pm}(\omega, t)/\cos(\omega_{\pi}/2) \tag{47}
\]

\( \tau_{eff} \) can be defined by behaviour of elastic scattering amplitude at \( s \to \infty \). If \( a(s, t) \approx sf(s)F(t/t_{eff}(s)) \) then

\[
\sigma_{el}(s) \propto |f(s)|^2 \int_0^\infty dt |F(t/t_{eff})|^2 = t_{eff}(s)F(1)|^2. \tag{48}
\]
and for odd signature
\[ \beta_-(\omega, t) = \beta_-(\omega, t)/\sin(\omega\pi/2). \]  
(48)

Now let us suppose that in agreement with the structure of the singularity of \( \phi_+ \) at \( \omega^2 + \omega_0^2 = 0 \) the functions \( \beta_\pm(\omega, t) \) depend on \( \omega \) through the variable \( \kappa_\pm = (\omega^2 + \omega_0^2)^{1/2} \) and it can be expanded in powers of \( \kappa_\pm \)
\[ \phi_{\pm}(\omega, t) = \left(\frac{i}{2}\right)e^{-i\omega\pi/2} \beta_{1\pm}(t) + \frac{\kappa_\pm}{2} \beta_{2\pm}(t) + \frac{\kappa_\pm^2}{2} \beta_{3\pm}(t). \]  
(49)

There are different ways to add to partial amplitude \( \varphi(j, t) \) terms which at \( s \to \infty \) are small corrections (they can be named as subasymptotic terms).

Thus we can expand the “residue” \( \beta(\omega, t) \) in powers of \( \omega \) (if \( \beta(\omega, t) \) has not branch point in \( \omega = 0 \)) or in powers of \( (\omega^2 + \omega_0^2)^{1/2} \). Then, for the first case
\[ \tilde{\beta}(\omega, t) = \tilde{\beta}_1(t) + \omega \tilde{\beta}_2(t) + \omega^2 \tilde{\beta}_3(t), \]  
(50)

and in the second case we have (just this case is explored in the Section 4.2)
\[ \tilde{\beta}(\omega, t) = \tilde{\beta}_1(t) + (\omega^2 + \omega_0^2)^{1/2} \tilde{\beta}_2(t) + (\omega^2 + \omega_0^2)^3 \tilde{\beta}_3(t). \]  
(51)

Let us notice that the main terms in \( \varphi(j, t) \equiv \varphi(\omega, t) \) for both cases are coinciding having a pair of branch points colliding at \( \omega_0 = 0 \) (\( t = 0 \)) and generating a triple pole in partial amplitude.

Taking into account the table integrals
\[ \int_0^\infty dx x^{m-1} e^{-\omega x} J_\nu(\omega_0 x) = I_\nu^m(\omega, \omega_0) \]  
(52)

where
\[ I_{\nu+1} = \frac{(2\omega_0)^\nu}{\sqrt{\pi}} F(\nu + 1/2), \]  
\[ I_{\nu+3} = 2\omega \frac{(2\omega_0)^\nu}{\sqrt{\pi}} F(\nu + 3/2), \]  
(53)

one can find
\[ \frac{1}{(\omega^2 + \omega_0^2)^{3/2}} \int_C \frac{dw}{2\pi i} = \frac{1}{\omega_0} \int_0^\infty dx e^{-x\omega} I_1(\omega_0 x), \]  
(54)

\[ \frac{1}{(\omega^2 + \omega_0^2)^{3/2}} \int_C \frac{dw}{2\pi i} e^{x\omega} \frac{1}{\omega_0} = \int_0^\infty dx e^{-x\omega} \sin(x\omega_0), \]  
(55)

\[ \frac{1}{(\omega^2 + \omega_0^2)^{1/2}} \int_C \frac{dw}{2\pi i} = \int_0^\infty dx e^{-x\omega} J_0(\omega_0 x), \]  
(56)

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