Wave propagation in channels and cracks with elastic walls

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Abstract. In this paper wave propagation in the uniform and nonuniform elastic channels filled with fluid or gas is investigated. The weak discontinuities theory approach is proposed to determine the criteria of gradient catastrophe phenomena to occur for the cases of uniform and nonuniform channels. The results of the paper can be applied to studying of the nonlinear wave propagation phenomena in elastic pores, cracks and channels filled with fluid or gas.

1. Short review
The first fundamental research of hydroelastic wave propagation in pipeline systems was provided by N. E. Zhukovsky in the end of the nineteenth century. The results of his experimental and theoretical research are published in [1]. Historically, hydroelastic waves in channels with elastic walls were called hydraulic shock or hydro hummer. This terms come from above mentioned fundamental work "On the hydraulic shock in water pipes" [1]. The number of publications in this field is constantly growing. The theory of wave propagation in such pipeline systems can be found in [2]. A description of application of hydroelastic pipe waves to geophysical survey is provided in [3]. The main goal of the current research is to carry out a qualitative investigation of the nonlinear phenomena that occur in the hydroelastic wave propagation process in nonuniform channels and cracks. Nonlinear effects are usually connected to wave front steeping and result in strong discontinuity of the solution. In the present work the influence of the nonlinearity of the base equations and formation of the gradient catastrophe in the nonuniform elastic channels is studied using the theory of weak discontinuity (the discontinuity of the solution derivatives) propagation [4]. To the best of our knowledge such studies have not been conducted so far. The results of our investigation are protected by a Russian patent [5].

2. Problem definition and reference model
2.1. Hypotheses and assumptions
Let the uniform or nonuniform channel with elastic walls be filled with medium (fluid or gas). Then \( \rho = \rho(p) \), \( p = p(x,t) \) and \( u = u(x,t) \) are the density, pressure and velocity of the medium, where \( x \) is the axis coordinate of the channel and \( t \) is time. Let the section area of the channel in the point \( x \) be equal to \( A(p,x) \) (see Figure 1). We assume that density of the medium \( \rho = \rho(p) \) and channel section area \( A(p,x) \) are the known functions of pressure \( p \) and coordinate \( x \). Let's also assume that \( A(x,p) = a(x)b(p) \).
If the wavelength is considerably greater than the channel cross-section then the displacement transverse to the channel axis can be neglected. Waves satisfying these conditions are called longitudinal or hydraulic waves.

Mass and momentum conservation laws for the longitudinal waves are as follows

\[
\begin{align*}
(A \rho) + (A \rho u) &= 0, \\
u_t + uu_u + p_x / \rho &= 0,
\end{align*}
\]

here and after \( p = p(x,t) \) and \( u = u(x,t) \) are unknown functions.

### 2.2. Mathematical model

The disturbance propagation in channels with elastic walls is described by the system of partial derivative equations (PDE) (1) which can be directly obtained from mass and momentum conservation laws

\[
\left( \frac{p}{u} \right)_t + \left( \frac{u}{1/\rho} \frac{\rho x^2}{u} \right)_x = 0.
\]

One can easily check that PDE system (1) is a quasi-linear hyperbolic PDE system, and the characteristic directions of the system are \( u + c \) and \( u - c \), where \( c = \left( \frac{A}{(A \rho)_{\rho}} \right)^{1/2} \) is the local velocity of the hydroelastic wave in the channel. By direct evaluation one can obtain that the sound velocity in fluid \( c_f \) is greater than the velocity of the hydroelastic waves \( c = c(p,x) < c_f = \left( \frac{\partial p}{\partial \rho} \right)^{1/2} \), assuming that the entropy is constant in the fluid and in the channel walls.

PDE system (1) describes the wave propagation in the pipes and channels with elastic walls including waves in the well fracturing channels which are widely used to increase the effective radius of the oil wells and to degas coal beds. Multiple applications of the hydraulic fracturing technology show the present paper is timely. Also, the PDE system (1) can be applied to model the wave propagation process in elastic media with open pores or cracks.

### 2.3. Characteristics.

The differential equation for the characteristics of system (1) is \( \frac{dx}{dt} = u \pm c \). As \( A = a(x) b(p) \) the velocity \( c \) of the hydroelastic waves depends only on the pressure in the fluid, \( c = c(p) \), the characteristics of the state of rest are straight lines in the \((x,t)\) space. The characteristics of the state of rest are satisfying the equation \( x = c_o t + \xi \), where \( c_o = c(p_o) \) is the velocity of the hydroelastic waves propagating in the state of rest, \( p_o \) is the pressure in the fluid at rest, \( \xi \) is the initial position of the

![Figure 1. Elastic channel filled with fluid.](image)
characteristic. It is convenient to apply the propagation theory of the weak discontinuities to predict the gradient catastrophe phenomena in the uniform and nonuniform elastic channels [3, 4].

2.4. Propagation of the weak discontinuity along state of rest characteristic
Let's assume that weak discontinuity of the solution is propagating along the state of rest characteristic 
$$x = c_{t}t + \xi.$$ Then the intensity of the weak discontinuity $$\sigma = \sigma(t)$$ is the solution of the Cauchy problem for the Riccati differential equation
$$\frac{d}{dr} \sigma + \sigma \frac{c_{0}A_{x}}{2A} + \sigma^{2}\left(1 + \frac{1}{2}\left(\rho \chi^{2}\right) + \frac{1}{2}\rho_{0}^{2}c_{0}^{2}\left(\frac{1}{\rho}\right)\right) = 0, \quad (2)$$
$$\sigma(0) = \sigma_{0},$$
where \(\begin{bmatrix} p_{r} \\ p_{t} \end{bmatrix} = \sigma\begin{bmatrix} \rho_{0}c_{0} \\ 1 \end{bmatrix};\) \([f]\) is the jump of function \(f\) on the characteristic \(x = c_{t}t + \xi\), and \(\sigma(0) = \sigma_{0}\) is some initial condition for the intensity of the weak discontinuity. Generally, equation (2) can be analyzed numerically, but it is important to study some particular cases of the Cauchy problem (2).

Elastic properties of the system channel-fluid are defined by fluid compressibility and channel elasticity. Geometric properties of the channel can be defined by \(\frac{A_{x}}{A}\) ratio: a uniform channel has \(\frac{A_{x}}{A} = 0\), a V-shaped channel has \(\frac{A_{x}}{A} = \frac{1}{x}\), a conical channel has \(\frac{A_{x}}{A} = \frac{2}{x}\), an exponential channel \(\frac{A_{x}}{A} = const\) (Fig. 2). In the present paper we study the basic cases of uniform, V-shaped and conical channels filled with an incompressible fluid.

![Figure 2. Different types of channels: (a) - uniform channel, (b) - V-shaped channel, (c) - conical channel, (d) - exponential channel.](image)

2.4.1. Gradient catastrophe in the uniform channel filled with an incompressible fluid. In this case equation (2) transforms to
$$\frac{d}{dr} \sigma + \sigma^{2}q = 0, \quad (3)$$
where \(q = \left[1 + \frac{1}{2}\left(\rho \chi^{2}\right)\right].\) The solution of Cauchy problem (2) is given by the formula \(\sigma = \frac{\sigma_{0}}{qt + \sigma_{0} + 1}.\)

According to the physical formulation of the problem \(q > 0\), gradient catastrophe occurs if the initial
intensity $\sigma_0 < 0$. The moment of gradient catastrophe is $t^* = \frac{-1}{q \sigma_0}$, while the coordinate is $x^* = c_0 t^* + \xi = \frac{-1}{q \sigma_0} c_0 + \xi$.

2.4.2. $V$-shaped channel filled with incompressible fluid. In this case equation (2) transforms to

$$\frac{d}{dt} \sigma + \sigma \left( \frac{1}{2(t + \xi/c_0)} + \sigma^2 q \right) = 0. \tag{4}$$

The solution of the Cauchy problem is given by formula (5)

$$\sigma(t) = \frac{\sigma_0}{
\exp \left( \frac{c_0}{q \xi \sigma_0} \ln \left( \frac{c_0 \xi}{q \xi \sigma_0} \right) - 1 \right) - \frac{c_0}{q \xi \sigma_0} - \xi)
+ q \xi^2 \sigma_0 \ln \left( \frac{c_0 \xi}{q \xi \sigma_0} \right) + \frac{c_0 \xi}{q \xi \sigma_0} - \frac{c_0 \xi}{q \xi \sigma_0} \ln \left( \frac{c_0 \xi}{q \xi \sigma_0} \right) - \frac{q \xi^2 \sigma_0 \ln \left( \frac{c_0 \xi}{q \xi \sigma_0} \right)}{q \xi^2 \sigma_0 \ln \left( \frac{c_0 \xi}{q \xi \sigma_0} \right)}.
\tag{7}$$

Gradient catastrophe occurs in $t^* = \frac{c_0 - 4q \xi \sigma_0}{4q^2 \xi^2 \sigma_0}$ at coordinate $x^* = c_0 t^* + \xi$ if the initial intensity $\sigma_0 < 0$.

2.4.3. Conical channel filled with an incompressible fluid.

In that case Riccati equation (2) changes to equation

$$\frac{d}{dt} \sigma + \sigma \left( \frac{1}{t + \xi/c_0} \right) = 0 \tag{6}$$

and the solution of the Cauchy problem is given by formula

$$\sigma(t) = \frac{\sigma_0}{
\exp \left( \frac{c_0}{q \xi \sigma_0} \ln \left( \frac{c_0}{q \xi \sigma_0} \right) - 1 \right) - \frac{c_0}{q \xi \sigma_0} - \xi)
+ q \xi^2 \sigma_0 \ln \left( \frac{c_0}{q \xi \sigma_0} \right) + \frac{c_0 \xi}{q \xi \sigma_0} - \frac{c_0 \xi}{q \xi \sigma_0} \ln \left( \frac{c_0}{q \xi \sigma_0} \right) - \frac{q \xi^2 \sigma_0 \ln \left( \frac{c_0}{q \xi \sigma_0} \right)}{q \xi^2 \sigma_0 \ln \left( \frac{c_0}{q \xi \sigma_0} \right)}.
\tag{7}$$

Gradient catastrophe occurs in $t^* = \frac{\xi}{c_0} \left( \exp \left( \frac{-c_0}{q \xi \sigma_0} \right) - 1 \right)$ on the characteristic $x^* = c_0 t^* + \xi$ if the initial intensity $\sigma_0 < 0$.

2.5. Nonlinear effects and wave direction relative to $V$-shaped or conical channel expansion or reduction.

In the formulae (4-7) we considered the case that the channel cross-section expands. It means that gradient catastrophes, described by formulae (5, 7) occur for the waves propagating in the expanding channels. In case of channel cross-section reduces the sign of the coefficient related to $\sigma$ changes to negative. From the mechanical point of view the influence of nonlinear effects increases for the waves that propagate in tapered channels, but we do not consider such effects in the present paper. In Cauchy problem (2), nonlinear effects of the system (1) are taken into account by the coefficient $\sigma^2$.

3. Conclusion

In the present paper criteria for the formation of gradient catastrophes in expanding channels filled with an incompressible fluid are determined. It should be noted that taking into account the
compressibility of the fluid in the channel makes the Riccati equation more complex, but does not significantly change it. Methods for predicting gradient catastrophes in channels with elastic walls based on the theory of propagation of weak discontinuities have shown high efficiency.

In most of the cases the fluid flow rates in the pipes and channels are considerably smaller than the sound velocity in fluid and the velocity of the hydroelastic waves. The proposed approach can be used to describe nonlinear effects of wave propagation in channels and pipes with a fluid flow. Since the typical velocities of the fluid flow through pores and cracks are smaller than the velocity of hydroelastic wave propagation, all the above methods and approaches can be used to study nonlinear hydroelastic waves in the hydraulic fracturing channels.

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