Caching with Time Windows and Delays

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Abstract

We consider two generalizations of the classical weighted paging problem that incorporate the notion of delayed service of page requests. The first is the (weighted) Paging with Time Windows (PageTW) problem, which is like the classical weighted paging problem except that each page request only needs to be served before a given deadline. This problem arises in many practical applications of online caching, such as the “deadline” I/O scheduler in the Linux kernel and video-on-demand streaming. The second, and more general, problem is the (weighted) Paging with Delay (PageD) problem, where the delay in serving a page request results in a penalty being assessed to the objective. This problem generalizes the caching problem to allow delayed service, a line of work that has recently gained traction in online algorithms (e.g., Emek et al. STOC ’16, Azar et al. STOC ’17, Azar and Touitou FOCS ’19).

We give $O(\log k \log n)$-competitive algorithms for both the PageTW and PageD problems on $n$ pages with a cache of size $k$. This significantly improves on the previous best bounds of $O(k)$ for both problems (Azar et al. STOC ’17). We also consider the offline PageTW and PageD problems, for which we give an $O(1)$ approximation algorithms and prove APX-hardness. These are the first results for the offline problems; even NP-hardness was not known before our work. At the heart of our algorithms is a novel “hitting-set” LP relaxation of the PageTW problem that overcomes the $\Omega(k)$ integrality gap of the natural LP for the problem. To the best of our knowledge, this is the first example of an LP-based algorithm for an online algorithm with delays/deadlines.

1 Introduction

In the caching/paging problem, page requests from a universe of $n$ pages arrive over time. They have to be served by swapping pages in and out of a cache that can hold only $k < n$ pages at a time. In weighted paging, each page $p$ has a weight $w_p$, and the goal is to minimize the sum of weights of evicted pages. In this paper we consider situations where page requests do not need to be served immediately, but can be delayed for some time. For instance, in mixed-workload environments such as those arising in cloud computing or operating systems, requests from time-sensitive applications (such as interactive ones) have short deadlines, but batch processes can tolerate longer wait times. (Indeed, the “deadline” I/O scheduler in the Linux kernel is precisely for this purpose, although the way it currently handles deadlines is not very sophisticated [Lin].) A different application arises in network streaming, e.g., in video-on-demand, where a server needs to cache segments appearing in multiple video streams (see, e.g., [CBD+15, DDH+12]). Depending on when these segments are required, various streams set different deadlines for each of these segments. In all these applications, the key feature is that individual page requests can be delayed, but only until a given deadline. Specifically, the request $r_t = (p, d)$ at time $t$ for a page $p$ includes a deadline $d$, and the algorithm must ensure that the page is in the cache at some time in the interval $[t, d]$. We call this the (weighted) Paging with Time Windows (PageTW) problem; if the deadline is the same as the time of the request, we get back the weighted paging problem.

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A more general setting is one where the page requests do not have specific deadlines, but the algorithm incurs a cost that is monotonically non-decreasing with the delay in serving individual requests. This is related to the recent line of work in online algorithms with delay, where problems such as online matching [EKW16, ACK17, AAC+17, AF20] and online network design [AT19, Aza] have been considered. In particular, our work relates to the “online service with delays” problem [AGGP17, BKS18, AT19], and can be interpreted as a generalization of this problem to \( k \) servers but for the special case of a star metric. In our problem, each request is specified by a triple \((p, t, F)\), where \( p \) is the requested page, \( t \) is the time at which this request is made, and \( F: \{t, t + 1, \ldots, \} \to \mathbb{R}_{\geq 0} \) denotes the non-decreasing loss function associated with it. The objective is to minimize the sum of two quantities: the sum of weights of pages evicted from the cache (the usual objective in weighted paging) and the total delay losses incurred over all the individual page requests. We call this the \textit{(weighted) Paging with Delay (PageD)} problem. Note that PageTW is a special case of this problem where the delay loss is 0 till the deadline, and \( \infty \) thereafter.

\textbf{Theorem 1.1} (Main Results: Online Algorithm). There is an \( O(\log k \log n) \)-competitive randomized algorithm for the PageD problem in the online setting, where \( n \) is the number of pages and \( k \) is the size of the cache. As a consequence, there is also an \( O(\log k \log n) \)-competitive randomized algorithm for the special case of the PageTW problem in the online setting.

Previously, an \( O(k) \)-competitive deterministic algorithm was given by Azar \textit{et al.} [AGGP17] for both problems. PageD and PageTW inherit an \( \Omega(\log k) \)-competitiveness lower bound from the classical paging problem; closing the gap between our upper bound and this lower bound remains open.

While we stated the above theorem for the more general PageD problem, and derived the bound for PageTW as a corollary, we will actually prove the theorem for the special case of the PageTW problem first, and then show that we can reduce the PageD problem to the PageTW problem. More precisely, we extend our PageTW algorithm to a generalization that we call the PageTWPenalties problem, where every page request has a non-negative penalty that the algorithm can choose to incur instead of satisfying the request. Then, we give a reduction from the PageD problem to the PageTWPenalties problem in Section 6 without changing the objective; moreover, this reduction can be performed online. So, the rest of this section, and much of the subsequent sections, focus on the PageTW and PageTWPenalties problems.

At the heart of our algorithm is a novel “hitting-set” LP relaxation of the PageTW problem that overcomes the \( \Omega(k) \) integrality gap of the natural LP relaxation for this problem (see Appendix B.2). From a theoretical perspective, the PageTW problem is in the category of online optimization problems with delays/deadlines that has attracted significant interest recently (e.g., [EKW16, AGGP17, ACK17, BFNT17, AT19, BBB+16]).

To the best of our knowledge, our work is the first example of an LP-based algorithm in this line of research. Given the great success of LP-based techniques in online algorithms in general, we hope that our work spurs further progress in this area.

We also study the offline versions of the PageTW and PageD problems, where the request sequence is given up-front. Here, the first question is tractability: since weighted paging is solvable in polynomial time offline, it is conceivable that so are PageTW and PageD. We show that the PageTW problem (and therefore, by generalization, the PageD problem) is APX-hard. We complement this lower bound with an \( O(1) \)-approximation for the offline PageTW and PageTWPenalties problems, which again by our reduction from the PageD problem to the PageTWPenalties problem, implies an \( O(1) \)-approximation for the offline PageD problem.

\textbf{Theorem 1.2} (Main Results: Offline Algorithm). The PageTW problem is NP-hard (and APX-hard) even when the cache size \( k = 1 \), and the pages have unit weight. As a consequence, the PageD problem is also NP-hard (and APX-hard) under these restrictions. Moreover, there are \( O(1) \)-approximation deterministic algorithms for the PageTW and PageD problems, based on rounding a linear program to show a constant integrality gap.
1.1 Our Techniques

The weighted paging problem has an “interval covering” IP formulation [BBN12, You91]:

$$\min \left\{ \sum_{p,j} w_p x_{p,j} : \sum_{p \neq p_t} x_{p,j(p_t)} \geq n - k \ \forall t, x_{p,j} \in \{0, 1\} \ \forall p, j \right\}.$$

For every page $p$, define an interval starting at each request for it, and ending just before the next request. Because of the request, this page $p$ must be present in the cache at the start of each such interval, but may be evicted at some subsequent point: the IP variable $x_{p,j} \in \{0, 1\}$ indicates if a page is evicted before its next request. While this IP does not explicitly indicate when a page is evicted, any online algorithm solving it must raise a variable $x_{p,j}$ from 0 to 1 at a specific time between the $j$th and $(j+1)$st request for page $p$. We visualize this using a 2-dimensional picture indexed by the pages and time, recording the eviction of page $p$ at time $t$ by putting a star at location $(p,t)$ (see Fig. 1). In classical paging, the intervals for any page partition its row into disjoint, tightly-fitting segments. The capacity constraint of the cache forces the following property: of the $n$ intervals (for different pages) containing time $t$ (these are indexed $j(p,t)$ for page $p$), at least $n - k$ contain a star at some time $\leq t$. In other words, at least $n - k$ pages must have been evicted from the cache since their last request.

The situation is more complex in PageTW. Previously, it sufficed to record page evictions, because page insertions were entirely dictated by the requests: whenever a page is requested, it must be inserted in the cache if it were evicted after its previous request. So, for an insertion to be feasible, it suffices to just ensure that sufficiently many pages are evicted since their previous request. In PageTW, however, page requests can be fulfilled at a later time, so evictions alone do not completely describe the state of the cache. One option is to explicitly encode page insertions via IP variables, but then we need packing constraints on these variables to enforce the size of the cache. Handling such packing constraints in online IPs seems beyond the scope of current techniques in online algorithms. Another idea is to reduce the ambiguity of when pages are inserted in the cache, e.g., by enforcing that all page insertions are done the end of their request intervals (if the page is not in the cache at the beginning of the interval). This would be a useful property, because the state of the cache could then be completely described by variables for page evictions. This property, however, is false: forcing a page request to be satisfied at the start/end of its request interval can be much costlier than doing it somewhere in the middle. E.g., if a heavy page is being evicted, we should serve some outstanding light requests while there is an empty slot in the cache (see Appendix B.1).

The Hitting Set IP Relaxation. To overcome these challenges, we first re-interpret the interval covering IP for classical paging. We again use $x_{p,t}$ variables (saying page $p$ is evicted at time $t$). The cache-size
constraint at any time \( t' \) insists that at least \( n - k \) pages are evicted at times \( \leq t' \) since their last request. To implement this, we define an interval for each page \( p \) starting at the last request for \( p \) and ending at \( t' \), and write a covering constraint saying at least \( n - k \) of these intervals have a star within them (i.e., \( x_{p,t} = 1 \) for times \( t \) within these intervals). Note: there is nothing special about the last request for a page before \( t' \)—we could have written these constraints for every choice of request of every page before \( t' \). The additional constraints would be redundant given the one containing the last requests, and would unnecessarily lead to an exponential-sized IP.

In the PageTW problem, however, the request intervals for a page might overlap, or may even be nested, so it is easier to write constraints for every request, rather than to identify some (non-canonical) last request before time \( t' \). Extending the previous intuition, we define the following intervals for time \( t' \): corresponding to a request interval \( I = (s(I), e(I)) \) for a page with \( s(I) < t' \), there is a constraint interval \( (s(I), \max(e(I), t')) \). Note that if the request interval extends beyond \( t' \), i.e., \( e(I) > t' \), then we must extend the constraint interval rightwards to \( e(I) \), since the page might be served after \( t' \). Now, we enforce the same constraint as earlier: for any choice of such constraint intervals, one for each distinct page, at least \( n - k \) must have a star in them. We call these constraint intervals the right extensions of their respective request intervals at time \( t' \) (see Fig. 2 for an example).

In classical paging, these constraints are valid even if we exclude the page currently requested at time \( t' \). In other words, of the remaining \( n - 1 \) pages, the constraint ensures that at least \( n - k \) have been evicted ensuring a cache slot for the currently requested page. All feasible solutions satisfy this constraint since the requested page must be in the cache at time \( t' \). This stronger constraint, however, does not hold for PageTW.

If we write the above constraints for \( n - 1 \) pages, then it would reserve a cache slot for the remaining page at the current time, thereby excluding feasible solutions that do not satisfy this property. Conversely, the (weaker) constraints summing over all \( n \) (and not \( n - 1 \)) pages are not sufficient: they do not reserve a cache slot for a requested page at any time during the request interval.

So we need a new set of constraints. These reserve a cache slot for a page \( p \) within each request interval \( I_t = (s_p, t_p) \) for it. Let us exclude this page \( p \) and choose a request \( I = (s(I), e(I)) \), where \( e(I) \leq t_p \), for each of the remaining \( n - 1 \) pages. For each such request (say for a page \( p' \)), consider a different extended constraint interval \( (\min(s_p, s(I)), t_p) \). Now we are guaranteed that in any feasible solution, one of two things happens: either page \( p' \) resides in the cache for the entire extended interval \( (\min(s_p, s(I)), t_p) \) and therefore also for the sub-interval \( (s_p, t_p) \), or it is “hit” (inserted or evicted) during the constraint interval. Since page \( p \) must be served in its request interval \( (s_p, t_p) \), at most \( k - 1 \) pages can be resident in the cache during \( (s_p, t_p) \), i.e., at least \( n - k \) of the \( n - 1 \) pages are hit during these extended constraint intervals. We call these extended intervals double extensions of their request intervals for time \( t_p \) (again, see Fig. 2). Our “hitting set”

\[ \text{Figure 2: The left figure illustrates right extensions Rext(I,t) for request intervals I shown by solid lines. The dotted lines show how these intervals are extended. The figure on the right shows double extensions Dext(I,t), with the critical interval I_t in red.} \]
IP comprises these two sets of requests, for right extensions and double extensions. We give details of this formulation in §2.

**Solving the Hitting Set IP Online.** Loosely, we extend ideas from Bansal et al. [BBN12] for solving the weighted paging IP online to our hitting set IP. There are some challenges, however. Firstly, the hitting set IP is of exponential size, since we wrote covering constraints for every choice of request interval for every page. Secondly, unlike in weighted paging, there are two sets of constraints, one on \( n - 1 \) pages and the other on \( n \) pages; the weighted paging IP only has the first set. Thirdly, the decision variables are for \((p, t)\) pairs, and do not uniquely correspond to constraint intervals. Nevertheless, we show that, as long as the request intervals for the pages are “non-nested”, the techniques of [BBN12] can be adapted to our hitting set IP. When the request intervals are nested, we solve the problem on two carefully selected subsets of the input where the request intervals are non-nested; then we show, somewhat surprisingly, that the combined solution satisfies the general instance. The competitive ratio of this algorithm is \( O(\log k) \), asymptotically the same as weighted paging. This algorithm appears in §5.

**Converting IP Solution to Cache Schedule Online.** As described earlier, the IP solution only gives us a set of stars, indicating “hits” for each page where each hit might either represent insertion or eviction of the page from the cache. Moreover, the IP solution does not necessarily give all the insertions and evictions. E.g., in the case of instantaneous request intervals representing classical weighted paging, our IP is identical to the standard interval covering IP and only gives page evictions. Indeed, the bulk of our technical work is in converting a feasible IP solution to an actual cache schedule that satisfies all requests. This is further complicated by the fact that this translation has to be done online.

The main difficulty is the following: when a request interval \( I \) for a page \( p \) arrives, we don’t know how long to wait before serving it. For instance, suppose the IP has a “hit” for a heavy page. The example in Appendix B.1 shows that we must use this opportunity to serve requests for light pages that are currently waiting. But, which pages should we serve? Suppose we serve a page \( p \) at some time \( t \in I \) by loading \( p \) in the cache, and evict it soon after to serve other light pages. Now if another interval \( I' \) for \( p \) arrives after \( t \) and \( I' \) overlaps with (or is even nested in) \( I \), it is clear that we should have waited to load \( p \) till \( I' \) arrives. In the offline case, we can use a reverse-delete step where we undo such mistakes. But, in the online setting, we must find a careful balance between waiting “long enough” and servicing outstanding requests. Specifically, we build a tree structure over the request intervals (which may not be laminar in general), and use the structural properties to argue that our algorithm can find a balance between these two competing goals. The online conversion algorithm appears in §3.

While we cannot show that our algorithm achieves the ultimate goal of being \( O(\log k) \)-competitive, we do not know any worse gaps for our approach. Indeed, the fact that the integrality gap of the hitting set formulation is constant, as evidenced by our offline solution, gives us hope that the ideas here will lead to further improvements.

### 1.2 Related Work

Azar et al. [AGGP17] study the online service problem with delays, where a single server services requests in a metric space. Each request has an associated monotone delay function that gives the cost of serving requests at each time after its arrival. The server pays for the total movement plus delay costs. They give an \( O(h^3) \)-competitive algorithm for HSTs of height \( h \). They extend the result to \( k \) servers at a loss of a factor of \( k \), which gives an \( O(k) \)-competitiveness for PageTW. (Progress on related problems appears in [AT19].) A related problem is online multilevel aggregation [BBB+16] where a single server sits at the root of a tree, requests arrive at the leaves, and the server occasionally goes to service some subset of requests and returns to the root. The cost is again the sum of movement and delay costs. Buchbinder et al. gave an \( O(h) \)-competitive algorithm for \( h \)-level HSTs [BFNT17], improving on [BBB+16]; the model itself combines elements of TCP
acknowledgment [KKR03] and online joint replenishment [BKL+13]. Online problems with delays were first proposed by Emek et al. [EKW16] for online matching; see [ACK17, AAC+17] for other work.

In the classical paging/caching problem with instantaneous requests, each interval is of length zero and must be satisfied immediately. Belady’s offline algorithm (Farthest in Future) is optimal for the number of evictions [Bel66]; in contrast, the offline PageTW problem is APX-hard. We know deterministic \(k\)-competitive and randomized \(O(\ln k)\)-competitive algorithms; both are optimal [ST85, FKL+91]. Weighted paging is equivalent to the \(k\)-server problem on a weighted star, so deterministic \(k\)-competitiveness follows from the algorithm \(k\)-server on trees [CKPV91]. Bansal et al. [BBN12] gave a randomized \(O(\ln k)\)-competitive algorithm for weighted paging, illustrating the power of the primal-dual technique for these problems. They used an interval covering IP given by [BBF+01, CK99], which we extend in our work.

**Paper Outline.** In Section 2, we describe the new IP formulation for PageTW. In Section 3, we show how a solution to this IP can be used to generate a caching schedule online. We give the corresponding offline algorithm in Section 4. We show how to (approximately) solve the IP, both offline and online, in Section 5. The reduction from the PageD problem to the PageTW problem that changes the objective by at most an \(O(1)\)-factor appears in Section 6. We prove APX-hardness of PageTW in Appendix A, and give some illustrative examples in Appendix B.

## 2 The IP Relaxations for PageTW and PageTWPenalties

There is a universe of \(n\) pages, and the cache can hold \(k\) pages at any time. Each page \(p\) incurs a cost when we evict it from the cache, which is denoted by its weight \(w(p) \geq 0\). In the (weighted) Paging with Time Windows (PageTW) problem, each request specifies a page \(p\) and an interval \(I = [s(I), t(I)]\): the page \(p\) must be in the cache at some time during this interval \(I\). Since the only times of interest in the problem are the start and end times of intervals, we assume without loss of generality (wlog) that \(s(I), e(I) \in \mathbb{Z}\), so the interval \(I := [s(I), \ldots, e(I)]\). Note that in the traditional paging problem, each interval \(I\) contains a single timestep, i.e., \(I = \{t\}\) for some \(t\). In the online setting, a request comprising the identity of the page and the end time of the interval \(e(I)\) (i.e., the deadline) is revealed at its start time \(s(I)\). This is known as the clairvoyant setting in the literature; strong lower bounds are known for the non-clairvoyant setting where the deadline is only revealed at time \(t(I)\) [AGGP17].

We write a “hitting set” integer programming relaxation for this problem: this IP does not capture the PageTW problem exactly, but we show that (a) it contains only valid constraints, and hence provides a lower bound on the optimal cost, (b) it can be solved approximately in polynomial time, and (c) the “relaxation” gap is small, i.e., a solution to this IP can be used to obtain a feasible solution to the original PageTW problem.

The IP has Boolean variables \(x_{pt}\) for each page-time pair \((p, t)\), with this variable being set if the page \(p\) is “hit” at time \(t\): it is either brought into or evicted from the cache at time \(t\). We assume that for each time \(t \in \mathbb{Z}\), there is exactly one request interval \(I\) having \(e(I) = t\): this incurs no loss of generality, since we can remove timesteps with no deadlines, and split times with multiple intervals ending at it. Hence each request interval \(I\) corresponds to a unique page \(\text{page}(I) \in [n]\). For time \(t\), let \(I_t\) and \(p_t\) be the unique interval ending at time \(t\) and its corresponding page; we call these the critical interval and page for time \(t\).

As described in the introduction, we use two sets of extensions for request intervals to define these constraints:

\[
\text{if } s(I) \leq t \quad \Rightarrow \quad \text{right extension of } I \quad \text{Rext}(I, t) := [s(I), \ldots, \max(t, e(I))].
\]

\[
\text{if } e(I) \leq t \quad \Rightarrow \quad \text{double extension of } I \quad \text{Dext}(I, t) := [\min(s(I), s(I)), \ldots, t].
\]
(See Figure 2.) The “Hitting Set IP” below has variables $x_{p,t} \in \{0, 1\}$:

$$
\min \sum_{p,t} w(p) x_{p,t} \\
\sum_{I \in \mathcal{C}} \sum_{t' \in \text{Rext}(I,t)} x_{\text{page}(I), t'} \geq 1 \quad \forall t \forall \mathcal{C} \text{ with } k + 1 \text{ requests for distinct pages starting before } t
$$

(R1)

$$
\sum_{I \in \mathcal{C}} \sum_{t' \in \text{Dext}(I,t)} x_{\text{page}(I), t'} \geq 1 \quad \forall t \forall \mathcal{C} \text{ with } k \text{ requests for distinct pages (excluding } p_t \text{) ending before } t
$$

(D1)

We now show that these sets of constraints are valid:

**Claim 2.1.** The constraints (R1) and (D1) are valid for any solution to PageTW.

**Proof.** Fix a solution for PageTW. Set $x_{p,t}$ to 1 if this page $p$ is evicted at time $t$ or loaded into the cache at time $t$ in this solution. Consider the constraint (R1) for a collection $\mathcal{C}$ and time $t$. At time $t$, one of the $k + 1$ pages corresponding to $\mathcal{C}$ is not in the cache—let the corresponding request interval be $I \in \mathcal{C}$ for page $p = \text{page}(I)$. Two cases arise: in the solution the page $p$ is in the cache either (a) at some time during $[s(I), t)$, or (ii) at some time during $[t, e(I)]$. (The latter case arises only if $t \leq e(I)$.) In the first case, $p$ must have been evicted during $[s(I), t]$, whereas in the second case it must have been brought into the cache during $[t, e(I)]$. In either case the $x_{p,t'}$ variables sum to at least 1 over the right-extended interval for $I$ with respect to $t$.

Now consider the constraint (D1) for a collection $\mathcal{C}$ and time $t$, where the critical request at time $t$ is $I_t$ with $p_t := \text{page}(I_t)$. In the solution, let page $p_t$ be in the cache at some time $\tau \in I_t$. At this time, at least one of the $k$ pages corresponding to the intervals in $\mathcal{C}$ is not in the cache; say this interval is $I$ for page $p := \text{page}(I)$. Again two cases arise: in the optimal solution this page is in the cache either (i) at some time during $[s(I), \tau]$ (where this case arises only if $s(I) \leq \tau$), or (ii) at some time $t$ during $[\tau, e(I)]$ (again, this case arises only if $\tau \leq e(I)$—note that at least one of these two cases must happen). If the former case happens, then $p$ must have been evicted during $[s(I), \tau]$, whereas if the second case happens, then $p$ must have been brought in the cache during $[\tau, e(I)]$. Since $\tau \in I_t \in [s(I_t), e(I_t) = t]$, in both cases the $x_{p,t'}$ variables sum to at least 1 for the doubly-extended interval for $I$ with respect to $t$. $\square$

In the (weighted) Paging with Time Windows and Penalties (PageTWPenalties) problem, each request interval $I$ has an associated penalty value $\ell(I) \geq 0$, which is the penalty for not satisfying the request associated with interval $I$. Not all requests must be satisfied, but if some request is not satisfied we must pay the penalty for it. The IP for the PageTWPenalties problem is very similar, where $y_I$ is the indicator for whether we choose to take the penalty.

$$
\min \sum_{p,t} w(p) x_{p,t} + \sum_I \ell(I) y_I \\
\sum_{I \in \mathcal{C}} \left( y_I + \sum_{t' \in \text{Rext}(I,t)} x_{\text{page}(I), t'} \right) \geq 1 \quad \forall t \forall \mathcal{C} \text{ with } k + 1 \text{ requests for distinct pages starting before } t
$$

(R1)

$$
\sum_{I \in \mathcal{C}} \left( y_I + \sum_{t' \in \text{Dext}(I,t)} x_{\text{page}(I), t'} \right) \geq 1 - y_I, \quad \forall t \forall \mathcal{C} \text{ with } k \text{ requests for distinct pages (excluding } p_t \text{) ending before } t
$$

(D1)

This IP is a strict generalization of (IP), since we can set $\ell(I) = \infty$ to force the $y_I = 0$. The proof of validity of (IPp) for the PageTWPenalties problem is identical to Claim 2.1 above, and is omitted.
3 Solving PageTW and PageTWPenalties Online using Online Solutions to (IP) and (IPp)

Now that we have the IPs, we need to solve them online, and also show how to convert a solution into one for the PageTW or PageTWPenalties problem. Indeed, (IP) and (IPp) do not have any explicit capacity constraints, so we need to extract a “schedule” from the IP solution in an online manner. In this section we show the latter step; we discuss solving the IPs in Section 5.

Theorem 3.1. There is an online algorithm that converts an $\alpha$-competitive integral solution to (IPp) into a valid $O(\alpha \log n)$-competitive solution for the PageTWPenalties instance. As a special case, there is an online algorithm that converts an $\alpha$-competitive integral solution to (IP) into a valid $O(\alpha \log n)$-competitive solution for the PageTW instance.

Although we stated this theorem in terms of the more general PageTWPenalties problem, we will actually prove it for the simpler PageTW problem. This is without loss of generality since an integer solution to (IPp) for an instance of the PageTWPenalties problem already specifies the page requests that are being satisfied by the solution, and the ones where the solution incurs the penalty. Given an instance of the PageTWPenalties problem, we simply remove the requests in the latter set to create an equivalent instance of the PageTW problem. On this instance, we apply the above theorem for the PageTW problem to recover the theorem for the PageTWPenalties instance.

In the rest of the paper, we move between solutions $x$ to (IP) and their characteristic set $A^* := \{(p, t) \mid x_{p,t} = 1\}$. Visually, thinking of time as the $x$-axis and the $n$ pages as the $y$-axis, the solution $A^*$ corresponds to a set of “stars” in the 2-dimensional plane. Let us list some properties of the online solution $A^*_t$ to (IP) (which should satisfy all the constraints corresponding to times $t$ and earlier) that are maintained by the algorithm in Section 5.

(A1) **Monotonicity:** $A^*_t \subseteq A^*_{t+1}$ for all $t$.

(A2) **Past-Preservation:** At time $t$ the algorithm only adds stars corresponding to times $t$ or later. Ideally, at time $t$, it should only add stars at time $t$, with the following exception.

(A3) **Sparsity:** for every page $p$, $A^*_t$ contains at most one star $(p, t')$ with $t' > t$. Furthermore, if $A^*_t$ has such a star, then this star hits all the request intervals for $p$ which contain time $t$. In fact, our online algorithm for PageTW does not need to know the exact location of the stars after time $t$—it just needs to know the set of pages $p$ for which the solution $A^*_t$ contains such a star.

The main idea of the algorithm is that if the cache is full and we need to evict a heavy page $p$, we should spend about $w(p)$ amount of weight in serving other outstanding requests at time $t$. The requests that need to be serviced need to be carefully chosen, because there are conflicting goals: (i) we want to service the cheaper requests, because this way we can service many of these, (ii) we want to go by EDF (Earliest Deadline First) order because the ones ending soon are more critical, and finally (iii) we prefer to service the requests which are hit by the solution $A^*_t$ because we can directly pay for these service costs. Interestingly, we show that we can simultaneously take care of all of these three requirements. Moreover, we can identify a weight $w$ such that we can take care of all outstanding requests which are cheaper than $w$ and are not hit by $A^*_t$.

3.1 The Online Algorithm for Non-Overlapping Requests

In this section, we assume that no two request intervals for the same page overlap—that is, for any pair of requests $I, I'$ for the same page, $I \cap I' = \emptyset$. This gives a simpler algorithm than for the general case, which follows the same approach but has to deal with the case that multiple request intervals for the same page may try to charge to the same star in $A^*_t$. (See §3.2 for the online algorithm for the general case, and Section 4 for the offline algorithm).
Algorithm 1 (see Figure 3) shows how to convert an online solution $A^*_t$ to (IP) into a feasible solution to the underlying PageTW instance. At each time $t$, we begin with some pages $C(t)$ in the cache. If the unique request $I_t$ ending at time $t$ is not already satisfied, and the cache is full, we evict the cheapest page $p_{\min}$ in the cache. We then potentially serve some other pending requests by bringing in and then evicting them, and also potentially remove some other pages from the current cache. (These services and removals help pay for evicting $p_{\min}$.)

Specifically, for every page $p$ in $C(t)$, define $I^p_t$ to be the most recent request for $p$ which ends before $t$ — this is well-defined because requests don’t overlap. Define $Z^*$ to be the pages in $C(t)$ for which the interval $\text{Dext}(I^p_t,t)$ is hit by $A^*_t$. (Since $\text{Dext}(I^p_t,t)$ ends at time $t$, this only requires the knowledge of stars in $A^*_t$ at or before time $t$). We can directly pay for evicting these pages from the cache. But the situation is tricky—some $\text{Dext}(I^p_t,t')$ for a future time $t'$ may also be hit by the same star in $A^*_t$. So we evict a subset of $Z^*$—ones for which we are sure that the corresponding stars of $A^*_t$ won’t be charged again in the future.

To do this, the first simple observation is that we need to do this charging only when the critical page is not much heavier than the cheapest page in the cache, else we can charge the eviction to the much heavier page in the cache. We define $U$ to be the set of outstanding requests at time $t$ and $U^o$ to be the subset of $U$ which are not hit by $A^*_t$ (lines 10–11)—by the sparsity property, these are the pages $p$ for which $A^*_t$ does not currently have a star beyond time $t$. We service all the requests in $U \setminus U^o$ immediately (we service a request by loading the corresponding page in the cache, and “evict a request” by evicting the corresponding page from the cache)—these request intervals are hit by $A^*_t$ and can be directly paid for (because of the non-overlapping intervals). It is trickier to decide which requests in $U^o$ to service. In Lemma 3.4 we show there is a page $p^*$ in $Z^*$ such that $w(U^o_{\leq w(p^*)}) \leq 2 \cdot w(Z^*_{\leq w(p^*)})$, where the notation $X_{\leq a}$ denotes all the stars in $X$ of weight at most $a$. We service all the requests in $U^o_{\leq 2w(p^*)}$ and evict the pages $Z^*_{\leq w(p^*)}$. This ensures that all the remaining unsatisfied requests are much heavier than the current pages remaining in the cache. By the observation at the start of this paragraph, the stars in $A^*_t$ which are being charged for the
We now show that the algorithm is well-defined. We first need some supporting claims to show that the algorithm is well-defined, and then bound the cost.

Lemma 3.4. There exists a page \( p^* \in Z^* \) such that

\[
w(U_{\leq 2w(p^*)}) \leq 2w(Z_{\leq 2w(p^*)}^*).
\]

Proof. We first claim that \( |U^*| \leq |Z^*| \). Indeed, define a set of \( k+1 \) request intervals as follows. For each page \( p \in C(t) \setminus Z^* \), consider the request interval \( I_p \) for page \( p \) as defined in line 8. Since \( p \not\in Z^* \), we must have \( (p,t') \not\in A_t^* \) for all times \( t' \in \text{Dext}(I_p^*, t) \). But since the interval ends before \( t \), we get \( \text{Rext}(I_p^*, t) \subseteq \text{Dext}(I_p^*, t) \)

\( (p,t) \), some of which corresponds to a star \((p,t)\) in the natural manner. The set \( Z^* = \{p_1,p_2,p_3\} \) and assuming \( I_6,I_7,I_8 \) are unsatisfied at time \( t \), the set \( U^* = \{I_6,I_7\} \).

eviction of \( Z_{\leq w(p^*)}^* \) are not going to be charged again.

Finally, we serve \( I_1 \) by bringing \( p_1 = \text{page}(I_1) \) into the cache if still needed. Observe that the cache \( C(t+1) \) at the start of time \( t+1 \) is contained within \( C(t) \cup \{p_1\} \), since all other pages we satisfy at time \( t \) are also evicted. Moreover, if \( C(t) \) was full, the cheapest page in \( C(t) \) is evicted, and other pages from \( C(t) \) may be evicted too.

3.1.1 The Analysis

We first need some supporting claims to show that the algorithm is well-defined, and then bound the cost.

Claim 3.2. Suppose a page \( p \) is evicted from the cache at time \( t_1 \) but is in the cache at the end of time \( t_2 > t_1 \). Then there must exist a request interval \( I \) for page \( p \) with \( t_1 < s(I) \leq e(I) \leq t_2 \).

Proof. The only step in the algorithm when a page is brought into the cache (and not evicted immediately afterwards) is line 17. Hence, there must be an unsatisfied request interval \( I \) ending at some time \( e(I) \leq t_2 \) which brought in (and kept) \( p \) in the cache after it had been evicted at time \( t_1 \). If \( s(I) \leq t_1 \), then \( I \) would have been satisfied at time \( t_1 \), which is not true.

We now show that the algorithm is well-defined.

Claim 3.3. The set \( Z^* \) defined in lines 6–9 is non-empty.

Proof. For each page \( p \in C(t) \), let \( I_p^* \) be the request interval defined in line 8—such a request interval exists because of Claim 3.2 (we assume that the cache is empty initially). Applying the IP constraint (D1) to time \( t \) and these \( k \) request intervals implies that at least one of their doubly-extended intervals is hit by \( A_t^* \), and hence the corresponding page belongs to \( Z^* \).
and so $A_t^*$ does not hit the right-extended interval for $I_t^p$ either. To this collection of $k - |Z^*|$ intervals, add the request intervals corresponding to $U^\circ$—all these request intervals contain $t$, so the right-extension operation does not extend them. Moreover, we have at most one interval per page, and they are all unsatisfied, so the collection now has $|U^\circ| + k - |Z^*|$ many intervals for distinct pages. And none of their right-extensions are hit by $A_t^*$, so by constraint (R1) this collection has size at most $k$. This proves that $|U^\circ| \leq |Z^*|$. 

Let $A$ be the set of pages in $Z^*$ and $B$ the set of pages in $U^\circ$. We set up a bipartite graph on $(A, B)$ with an edge between $p \in A$ and $p' \in B$ if $w(p') \leq 2w_p$. If this graph has a perfect matching, then $w(B) \leq 2w(A)$. We choose $p^*$ to be the highest-weight page in $Z^*$. 

Else such a perfect matching does not exist. Let $A' \subseteq A$ be a minimal Hall set, and $B'$ be the neighborhood of $A'$. Let $a$ be any page in $A'$. The pages in $A' \setminus \{a\}$ can be matched with $B'$. Therefore, $w(B') \leq 2w(A' \setminus \{a\}) \leq 2w(A')$. Now choose $p^*$ to be the highest weight page in $A'$, to get $w(U^\circ \leq 2w(p^*)) \leq 2w(Z^*_\leq w(p^*))$.

Therefore when we reach line 14, a page $p^*$ of the desired form exists, and the algorithm is well-defined. Finally the next claim shows that the request interval $I_t$ gets served.

**Claim 3.5.** Suppose $I_t$ is unsatisfied at time $t$. If $w(p_t) \leq 2w(p_{\min})$, then the page $p_t$ belongs to either $U \setminus U^\circ$ in line 12 or to $U^\circ \leq 2w(p^*)$ in line 16, and is served and evicted. Else $p_t$ is served by line 17, and remains in the cache.

**Proof.** The page $p_t$ is unsatisfied at the beginning of time $t$ and so $I_t$ belongs to $U$. First assume $w(p_t) \leq 2w(p_{\min})$. If $I_t$ is not in $U^\circ$, it is served in line 12. Else we reach line 16. In this case, the page $p^*$ chosen in line 14 has $2w(p^*) \geq 2w(p_{\min}) \geq w(p_t)$, so $p_t$ beings to $U^\circ \leq 2w(p^*)$ and is served in line 12. The other case (where $w(p_t) > 2w(p_{\min})$) is clear by construction. 

### 3.1.2 The Cost Guarantee

We want to bound the total cost incurred till time $T$. The high-level cost analysis goes as follows. If the cache has room we can just satisfy $I_t$, so suppose the cache is full and we need to pay to evict $p_{\min}$. If the page $p_t$ is twice as heavy as $p_{\min}$, we can charge $p_{\min}$ to $p_t$ and pay when $p_t$ is subsequently evicted. Else, if the unsatisfied intervals crossing time $t$ which are hit by $A_t^*$ have large weight, i.e., if $w(U \setminus U^\circ) \geq w(p_{\min})$, we can serve and evict them and then charge to them—this can pay for $p_{\min}$. Finally, we evict some pages from the current cache that are hit by $A_t^*$; they pay for both evicting $p_{\min}$ and for serving some more of the outstanding requests. These pages are evicted from the cache to ensure they are not charged again.

We now show how to pay for the possible evictions in lines 4, 12, and 15-16 (since bringing in pages is for free). We maintain the invariant that each page $p$ in the cache has at most $w(p)$ “load” on it; pages outside the cache have zero load. The load measures the evictions which have not been paid for till now. If we bring in $p_t$ and if $w(p_t) \geq 2w(p_{\min})$, its load becomes the load of $p_{\min}$ plus the cost of evicting $p_{\min}$; thus the total load on $p_t$ is at most its weight, and $p_t$ remains in the cache, maintaining the invariant. At the end of the algorithm, the total load over all pages is at most the weight of the pages in the cache, which is at most the optimum cost. This adds one to the competitive ratio. Else if $w(p_t) < 2w(p_{\min})$, we evict at least one page in $Z^*$ (by Claim 3.3) and can charge evicting $p_{\min}$ to the eviction of that page—which we show below how to charge to $A_t^*$.

Next: we charge evicting $U \setminus U^\circ$ to $w(A_t^*)$ as follows. Each interval in $U \setminus U^\circ$, say for page $p$, contains some star at $(p, t')$ in $A_t^*$ (and hence in $A_t^*$, and we can charge to star). Since the request intervals for a page are disjoint (by our simplifying assumption), $(p, t)$ cannot lie in any other request interval for $p$, and will not be charged by line 12 again.

Finally, we charge the eviction cost for lines 15-16. This cost is $O(w(Z^*_\leq w(p^*)))$ by our choice of $p^*$ in line 14. Observe that for each page $p I$ in $Z^*_\leq w(p^*)$, the doubly-extended interval $\text{Dext}(I^p_t, t)$ is hit by an star at
The algorithm from Section 3.1 assumes the requests for a page are non-overlapping. We now extend it to
This proves Theorem 3.1 (without losing the extra log $n$ factor) in the case of non-overlapping requests for any
page $p$. The general case gets trickier. Indeed, consider the example with a page $p$ having request intervals
$[t_1, t], [t_2, t], \ldots, [t_k, t]$, where $t_1 < t_2 < \ldots < t_k < t$. Suppose we have a star $(p, t) \in A^*_t$. Consider a time
$t' \in [t_1, t]$ when the algorithm reaches line 10. If any of these intervals is not satisfied at $t'$, then they will get
counted in $U \setminus U^*$, and so we will charge the star at $(p, t)$ for servicing $p$ at time $t'$. But this can happen for
multiple values of $t'$, and we would have only one star in $A^*$ to charge to. Moreover, we cannot say that we will
take care of all these requests at the ending time $t$—since all the pages in the cache may be very expensive at
that time. In the off-line case (which appears in Section 4), one can add a reverse delete step, where we look at
all these times when we service some of these requests, and realize that a subset of them would suffice.
However, we discuss the more involved online case in the next section.

### 3.2 Online Algorithm for the General Setting

The algorithm from Section 3.1 assumes the requests for a page are non-overlapping. We now extend it to handle
overlapping requests in an online fashion. Algorithm 2 (see Figure 5) gives the online algorithm—the lines changed from Algorithm 1 are highlighted. We call a request interval $I$ non-dominating if it does not contain another request interval for page$(I)$—we know whether $I$ is non-dominating only at time $e(I)$. Notice that the definition of $I^*_t$ in line 8 looks only at non-dominating intervals.

Since the request intervals for a particular page are no longer disjoint, we do not serve and evict all the
intervals in $U \setminus U^*$ when we create space at time $t$ (as Algorithm 1 would do in line 1). Instead we only serve the requests hit by $A^*_t$ before time $t$, and some small set of requests that are hit by $A^*_t$ after time $t$. As shown in the example at the end of Section 3.1, serving all such requests may lead to unbounded number of chargings to a star in $A^*_t$. These requests are considered in the earliest deadline order and their total weight is a constant times the weight of the cheapest page in $Z^*$ (denoted by $p^1$). It is also worth noting that we perform these steps only if $I_t$ is hit by $A^*_t$ (line 12).

It is also worth noting that line 16 is the only place in the algorithm where we need to know the right end-point of an existing request for a page.

By Lemma 3.4 the page $p^*$ in line 2 exists, and hence the algorithm is well-defined. For the correctness we need to show that each page is served. Indeed, for an unsatisfied request $I_t$ at its deadline $t$, either $p_t$ has twice the weight of $p_{\min}$ and is handled in line 2. Else, either the request interval $I_t$ is hit by $A^*_t$ and so it
We charge the weight of evicting $Z^*$.

**Proof.**

Claim 3.7. $\mathcal{U}_{t \in \mathcal{T}} w(U^*_t) \leq w(A^*_T)$.  

**Proof.** We charge the weight of evicting $p = \text{page}(I)$ for some request interval $I$ in $U^*_t$ to the element $p_t$. 

Algorithm 2: ConvertOnline((IP) solution $A^*_T$ appearing online)

1. **foreach** $t = 0, 1, \ldots$ **do**
2. \hspace{1em} let $I_t$ be the interval with deadline $t$, and let $p_t = \text{page}(I_t)$
3. \hspace{1em} if cache $C(t)$ full and $I_t$ not satisfied **then**
4. \hspace{2em} evict the least-weight page $p_{\text{min}}$ in $C(t)$
5. \hspace{1em} if $w(p_t) \leq 2w(p_{\text{min}})$ **then**
6. \hspace{2em} $Z^* \leftarrow \emptyset$.
7. \hspace{2em} for every page $p$ in $C(t)$ **do**
8. \hspace{3em} $I^p_t \leftarrow$ non-dominating request interval $I$ with $\text{page}(I) = p$ and largest ending time $e(I) < t$.
9. \hspace{3em} if $\text{Dext}(I^p_t, t)$ is hit by $A^*_t$ then add $I^p_t$ to $Z^*$.
10. \hspace{2em} $U \leftarrow$ unsatisfied request intervals active at time $t$ (one per page, if there are multiple choose one with earliest deadline).
11. \hspace{2em} $U^2 \leftarrow \{I \in U \mid \exists t' \in I \text{ with } (\text{page}(I), t') \in A^*_t\}$ be intervals in $U$ not hit by $A^*_t$.
12. \hspace{2em} if $I_t \not\in U^2$ **then**
13. \hspace{3em} $U^*_t \leftarrow$ request intervals $I$ in $(U \setminus U^*)$ which are hit by $A^*_t$ at some time $\leq t$.
14. \hspace{3em} serve and evict all requests in $U^*_t$.
15. \hspace{3em} evict the cheapest page $p^*$ in $Z^*$ (this may be the same as $p_{\text{min}}$)
16. \hspace{3em} sort intervals in $U \setminus (U^* \cup U^*_t)$ with weights $\leq 2w(p^*)$ in ascending order of end-times.
17. \hspace{3em} serve and evict a maximal prefix of these intervals with total weight at most $4w(p^*)$.
18. \hspace{2em} let $U^*_{\leq w}$ and $Z^*_{\leq w}$ denote pages in $U^*$ and $Z^*$ with weight at most $w$.
19. \hspace{2em} let $p^*$ be a page in $Z^*$ such that $w(U^*_{\leq 2w(p^*)}) \leq 2 \cdot w(Z^*_{\leq w(p^*)})$.
20. \hspace{2em} evict all pages in $Z^*_{\leq w(p^*)}$.
21. \hspace{2em} serve and evict all requests in $U^*_{\leq 2w(p^*)}$.
22. **if** $I_t$ not satisfied **then** bring page $p_t$ into cache.

Figure 5: Online Algorithm to Service Request Intervals

belongs to $U^*_t$ and is served/evicted in line 2, or it is not hit by $A^*_t$ and so belongs to $U^*_{\leq 2w(p^*)}$ (by Claim 3.5) and is served/evicted in line 2. It remains to estimate the total eviction cost of this algorithm.

### 3.2.1 The Cost Analysis

Consider the run of the algorithm until time $T$; we bound the total cost incurred until this time, in terms of $w(A^*_T)$. Observe that evictions can only happen on lines 4, 14–17, and 20–21. The first and the last of these can be dealt with as in Section 3.1. Indeed, paying for $p_{\text{min}}$ and its load is done by either putting a load on $p_t$ if $w(p_t) \geq 2w(p_{\text{min}})$ or else at least one other page from $C(t)$ is evicted and charged for, and we can handle $p_{\text{min}}$ by charging a constant factor more. The total cost incurred during lines 20–21 is at most $2w(A^*_T)$, since the proof for Lemma 3.6 remains unchanged. Indeed, at each time $t$ when we perform those evictions, we charge the stars in $A^*_T$ which hit the intervals in $Z^*_{\leq w(p^*)}$, and these stars are never charged again due to Lemma 3.6.

It remains to bound the cost incurred during lines 14–17. Let $\mathcal{T} \subseteq [T]$ contain the times when we reach those lines. Let $Z^*_t, U_t, U^*_t,$ and $U^*_t$ denote the corresponding sets at time $t$, and $p^*_t, p^*_t$ denote the pages chosen in line 2 and line 2. The evictions in line 2 are easy to pay for:

**Claim 3.7.** $\mathcal{U}_{t \in \mathcal{T}} w(U^*_t) \leq w(A^*_T)$.  

**Proof.** We charge the weight of evicting $p = \text{page}(I)$ for some request interval $I$ in $U^*_t$ to the element

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(p, t') ∈ A∗ where t' ∈ [0, t] ∩ I. We claim that no element in A∗ will get charged twice this way. Indeed, if we charge to (p, t') at time t ≥ t', we have satisfied all existing request intervals for page p containing the time t'—and no future requests can arrive that contain it.

Some more notation: let U_i w be the prefix of request intervals serviced in line 2. For a time t ∈ T, define the effective cost at time t to be w(p_i w) + w(U_i w)—this is the remaining cost incurred at time t in lines 2 and 2. For an interval [a, b], let A∗ [a, b] denote the set of (p, t') in A∗ where t' ∈ [a, b]. Let P∗ [a, b] be the set of pages corresponding to which there is at least one star in A∗ [a, b]. Note that w(P∗ [a, b]) ≤ w(A∗ [a, b]) for any time t and interval [a, b].

Claim 3.8. Suppose times t_1, t_2 ∈ T are such that t_1 < t_2 and I_{t_2} contains time t_1. Then the effective cost at time t_1 is at most α w(P_{t_1}^*[t_1, t_2]).

Proof. By design, w(U_{t_1}^w) ≤ 4w(p_{t_1} w), so the effective cost at time t_1 is at most 5w(p_{t_1} w). Thus it suffices to bound w(p_{t_1} w). Since t_2 ∈ T, the interval I_{t_2} was not served at time t_1. The possible reasons are:

1. The interval I_{t_2} ∈ U_{t_1} and w(p_{t_2}) > 2w(p_{t_1} w). Since w(p_{t_1} w) ≤ w(p_{t_1} w) by the choice of p_{t_1} w, so w(p_{t_1} w) < \frac{1}{2}w(p_{t_2}). Since I_{t_2} is hit by A∗ (because t_2 ∈ T), the past preserving property of the online solution A∗ implies that there must be a star at (p_{t_2}, t') in A∗ for some t' ∈ [t_1, t_2]. Therefore, w(p_{t_2}) ≤ w(P_{t_1}^*[t_1, t_2]).

2. I_{t_2} ∉ U_{t_1} w but w(p_{t_2}) > 2w(p_{t_1} w), so it was not considered in the sorted ordering (in line 16). However, I_{t_2} was not in U_{t_1}, so it was hit by A∗ at some time after t_1—this means w(p_{t_2}) is counted in w(P_{t_1}^*[t_1, t_2]).

3. I_{t_2} ∉ U_{t_1} w and w(p_{t_2}) ≤ 2w(p_{t_1} w) but we did not add I_{t_2} to U_{t_1} w at time t_1: So w(U_{t_1} w) must have been more than 4w(p_{t_1} w) - w(p_{t_2}) ≥ 2w(p_{t_1} w). We added intervals to U_{t_1} w in the earliest-deadline-first order, so all the weight added to w(U_{t_1} w) before considering I_{t_2} belongs to w(A_{t_1}^*[t_1, t_2]). Chaining these inequalities, w(p_{t_1} w) ≤ \frac{1}{2}w(U_{t_1} w) ≤ \frac{1}{2}w(P_{t_1}^*[t_1, t_2]).

Since A_{t_1}^* ⊆ A_{t_2}^*, we get the desired result.

Claim 3.9. Suppose t_1, t_2 ∈ T are such that t_1 < t_2, and I_{t_2} does not contain t_1. Suppose p_{t_1} w = p_{t_2} w—call this page p l—then there is a star for (p l, t') in A_{t_2}^* for some t' ∈ (t_1, t_2).

Proof. Page p l is evicted at time t_1, and so it must have been brought in by an unsatisfied request R; this request must start after t_1 (else it would be satisfied at t_1) and end before t_2 (since p l is in the cache at time t_2). We claim that I_{t_2}^p is also contained in (t_1, t_2). Suppose not. So s(I_{t_2}^p) ≤ t_1.

The interval I is either itself non-dominating, or contains a non-dominating request for p l. In either case, there is a non-dominating request interval for p l which is contained in (t_1, t_2)—call this I' (it could be same as I). Now, I' is not designated as I_{t_2}^p. It must be the case that t(I_{t_2}^p) ≥ t(I'). But then I_{t_2}^p contains I', which contradicts the fact that it is non-dominating.

Since I_{t_2} is also contained in (t_1, t_2), we see that Dext(I_{t_2}^p, t_2) is also contained in (t_1, t_2). Since p l ∈ Z^* at time t_2, Dext(I_{t_2}^p, t_2) is hit by A_{t_2}^*. This proves the claim.

The Charging Forest. Motivated by Claims 3.8 and 3.9, we define a directed forest F = (T, E) as follows. For time t ∈ T, if time t' is the smallest time such that t' > t and the critical interval I_{t'} for t' contains t, we define t' to be the parent of t, i.e., we add an arc (t, t'). If no such time t' > t exists, then t has no parent (i.e., zero out-degree). The following lemma gives some natural properties of the forest F.

Lemma 3.10. Suppose t, t' ∈ T and t < t'.

(a) If I_{t'} contains t, then t' is an ancestor of t in F.
The parents of their right end-points. The corresponding forest $F$ is shown on the right.

(b) If $t'$ is not an ancestor of $t$ in $F$, then then any node $t''$ in the subtree rooted at $t'$ satisfies $t'' > t$.

Proof. The first property follows by a simple induction, which we omit. For the second property, suppose for a contradiction that $t'' < t$, and let $t'' = t_0,t_1,\ldots,t_r = t'$ be the path from $t''$ to $t'$ in the forest $F$. Since $t' > t$, there is an $i$ for which $t \in [t_i,t_{i+1}]$. Since the interval $I_{t_{i+1}}$ contains $t_i$, it also contains $t$. But then, $t_{i+1}$ should be an ancestor of $t$ by property (a), which gives a contradiction.

We now divide pages and times into classes. For each class $c$, we will consider a sub-forest $F_c$ of $F$. We say that a page $p$ is of class $c$ if $w(p)$ lies in the range $[2^c,2^{c+1})$. We say that a node $t$ in the charging forest $F$ is of class $c$ if the corresponding page $p_t^l$ is of class $c$. Let $V^c$ be the vertices of class $c$ in $F$. Let $F^c$ be the minimal sub-graph of $F$ which preserves the connectivity between $V^c$ (as in $F$). So the leaves of $F^c$ belong to $V^c$, but there could be internal vertices belonging to other classes. We now show how to account for the cost incurred for the vertices in $V^c$. Let $A_T^*(c)$ be the total weight of the stars in $A_T^*$ corresponding to pages of class $c$. We say that a node in $F^c$ is a lone-child if it is the only child of its parent.

Claim 3.11. The total effective cost incurred during the leaf nodes in $F^c$ and the internal nodes of class $c$ in $F^c$ which are not lone-children is $O(A_T^*(c))$.

Proof. The effective cost incurred during each time of class $c$ is a constant times $2^c$. Since the number of internal nodes which are not lone children is bounded above by the number of leaf nodes, it is enough to bound the effective cost incurred at the leaf nodes. For a page $p$ of class $c$, let $F^c(p)$ be the leaf nodes $t$ in $F^c$ for which $p_t^l = p$. Let the times in $F^c(p)$ in increasing order be $t_1,t_2,\ldots,t_k$. Note that $I_{t_k}$ does not contain $t_{i-1}$ for $i = 2,\ldots,k$—otherwise $t_i$ will be an ancestor of $t_{i-1}$ (Lemma 3.10).

Claim 3.9 now implies that $A_T^*$ contains a star for page $p$ during $(t_{i-1},t_i]$. Thus, the total effective cost incurred during $F^c(p)$ can be charged to the stars in $A_T^*$ corresponding to page $p$. Since all the leaf nodes in $F^c$ belong to class $c$, he result follows.

It remains to account for the times in $V^c$ which have only one child in $F^c$.

Claim 3.12. Let $t_1$ and $t_2$ be two distinct times of class $c$ which are lone-child nodes in $F^c$. Let $t_1'$ and $t_2'$ be the parents of $t_1$ and $t_2$ respectively. Then the intervals $[t_1,t_1']$ and $[t_2,t_2']$ are internally disjoint.

Proof. Suppose not. Say $t_1 < t_2 \leq t_1'$. First assume $t_2' > t_1'$. Then $I_{t_2'}$ contains $t_1'$ and so $t_2'$ must be an ancestor of $t_1'$. If $t_1'$ is same as $t_2$, then the result follows easily, otherwise $t_1'$ is a descendant of $t_2$ (since $t_2'$ has only one child). But then $t_1' < t_2$, a contradiction.

The other case happens when $t_2' < t_1'$. In this case $I_{t_1'}$ contains $t_2'$ (since it contains $t_1$ and $t_1 < t_2'$). If $t_1 = t_2'$, the result again follows trivially. Otherwise $t_2'$ is a descendant of $t_1$, a contradiction.

The above Claim along with Claim 3.8 and Claim 3.11 show that the total cost incurred by times of class $c$ can be charged to $w(A_T^*)$. Thus, if there are $K$ different classes, we get $O(K)$ approximation. To convert this into $O(\log n)$ approximation, we observe the following refinement of Claim 3.8. For a class $c$, times $t_1 < t_2$, let $A_T^*(c,[t_1,t_2])$ be the stars of $A_T^*[t_1,t_2]$ which are of class $c$. 

![Diagram](https://via.placeholder.com/150)
Claim 3.13. Suppose times $t_1, t_2 \in \mathbb{T}$ are such that $t_1 < t_2$ and $I_{t_2}$ contains time $t_1$. Let $p_{t_1}^\dagger$ be of class $c$. Then the effective cost at time $t_1$ is at most most
\[
10 \left( \sum_{c' = c - \log n - 3}^c w(A_T^r(c', [t_1, t_2])) + \sum_{c' > c} \frac{w(A_T^r(c', [t_1, t_2]))}{2^{c' - c}} \right).
\]

Proof. Claim 3.8 shows that $w(P_T^*[t_1, t_2])$ is at least $\frac{w(p_{t_1}^\dagger)}{2}$. Let $P'$ be the pages of weight at most $w(p_{t_1}^\dagger)/4n$ in $P_T^*[t_1, t_2]$—since the total weight of these pages is at most $w(p_{t_1}^\dagger)/4$. Thus the pages of class $c - \log n - 2$ and higher contribute at least half of $w(P_T^*[t_1, t_2])$. Further, if $P_T^*[t_1, t_2]$ contains a page $p$ of weight higher than $w(p)$, then we can just charge it $w(p_{t_1}^\dagger)$ (and may not even charge to other pages in $P_T^*[t_1, t_2]$). The desired result now follows from Claim 3.8.

Claim 3.11 and Claim 3.12 along with Claim 3.13 imply that the total cost incurred during times of class $c$ is a constant times
\[
\sum_{c' = c - \log n - 3}^c w(A_T^r(c')) + \sum_{c' > c} \frac{w(A_T^r(c'))}{2^{c' - c}}.
\]

Summing over all classes yields Theorem 3.1.

4 Offline Algorithm for the PageTW and PageTWPenalties Problems

In Section 3, we gave an online algorithm for PageTW and PageTWPenalties using online (integer) solutions to (IP) and (IPp). We shall now prove the following offline version of Theorem 3.1.

Theorem 4.1. There is a polynomial time algorithm that converts an $\alpha$-approximate integral solution to (IPp) into a solution for the PageTWPenalties instance and has approximation ratio of $O(\alpha)$. As a consequence, there is a polynomial time algorithm that converts an $\alpha$-approximate integral solution to (IP) into a solution for the PageTW instance and has approximation ratio of $O(\alpha)$.

As in Theorem 3.1, we will actually prove the above theorem for the more restricted PageTW problem. This is sufficient for the more general PageTWPenalties problem as well, by the same reduction as the one we used in Theorem 3.1. Namely, the requests that are satisfied by the integer solution to an PageTWPenalties instance are used to create an (equivalent) instance of the PageTW problem, and then the above theorem for the PageTW problem is applied to this instance to derive a valid solution for the original PageTWPenalties instance.

In the offline setting, we can assume that a request interval for a page $p$ does not contain another interval for the same page—otherwise we can always remove the outer interval. Let $A^\star$ be an integral solution to (IP), and we want to convert it to a feasible solution to the underlying PageTW instance. As discussed in Section 3, this will be done by adding a reverse delete step to Algorithm 1 (which considered the special case when all the request intervals for a particular page were mutually disjoint).

The algorithm is shown in Algorithm 3. The first part of the algorithm until line 3 is same as in Algorithm 1. However, we cannot pay for all the evictions in line 12. Therefore, we remove some of these evictions in lines 18–23. We describe the details of this process now. We use $Sat$ to denote the set of requests serviced during line 3. Let $Sat_p$ be the requests in $Sat$ that correspond to $p$ (and $T_p$ be the time at which they are served), and $Sat_p'$ be a maximal collection of disjoint intervals in $Sat_p$. Since each of the intervals in $Sat_p'$ is hit by a distinct element of $A^\star$, we can pay for the service of $Sat_p'$. We define $T_p'$ to be the time instances in $T_p$ which are closest on each side to the end-points of the intervals in $Sat_p'$, and hence $|T_p'| \leq 4|Sat_p'|$. It is not
Claim 4.2. Suppose a page $p$ is evicted from the cache at time $t_1$ but is in the cache at the end of time $t_2 > t_1$. Then there must exist a request interval $I$ for page $p$ with $t_1 < s(I) \leq e(I) \leq t_2$.

Claim 4.3. The set $Z^*$ defined in lines 6–9 is non-empty.

Lemma 4.4. There exists a page $p^* \in Z^*$ such that

$$w(U_{\leq 2w(p^*)}^o) \leq 2w(Z_{\leq 2w(p^*)}^*)$$

Lemma 4.4 shows the existence of page $p^*$ in line 3. The proof of the following claim is same as that of Claim 3.5.

Claim 4.5. Let $I_t$ be unsatisfied at time $t$. If $w(p_t) \leq 2w(p_{\min})$, then the page $p_t$ belongs to either $U \setminus U^o$ in
line 12 or to \( U_{\geq 2w(p')} \) in line 16, and is served and evicted. Else \( p_t \) is served by line 17, and remains in the cache.

We now show that even after removing some of the services for a page \( p \) in lines 20–23, the algorithm services all the requests in \( \text{Sat}_p \). In the claim below, we use the notation in lines 20–23.

**Claim 4.6.** Every request interval in \( \text{Sat}_p \) has non-empty intersection with \( T'_p \).

**Proof.** Let \( I \) be a request interval in \( \text{Sat}_p \). First assume that it lies in \( \text{Sat}'_p \), and let \( t \in T_p \) be the time at which it is services in line 12. Since \( t \) lies between \( s(I) \) and \( e(I) \), the closest time in \( T_p \) to the right of \( s(I) \), call it \( t' \), must lie between \( s(I) \) and \( t \). Since \( t' \in T'_p \), the result follows.

Now assume \( I \notin \text{Sat}'_p \). So there must be an interval \( I' \in \text{Sat}'_p \) which overlaps with \( I \). Since any two requests for the same page are non-nested, \( I' \) contains either \( s(I) \) or \( t(I) \). Suppose it contains \( s(I) \) (the other case is similar). Then \( s(I') < s(I) < e(I') < e(I) \). Let \( t \) be the time at which \( I \) is serviced in line 12. Say \( t \) lies to the right of \( e(I') \). Then the time in \( T_p \) which is closest to \( e(I') \) on the right side, call it \( t' \), lies in \( T'_p \). But \( t' \in [e(I'), t] \) and so it lies in \( I \). The case when \( t \) is to the left of \( e(I') \) is similar—there will be a time in \( T_p \) which lies in the interval \([s(I), e(I)]\) and so belongs to \( I \) as well.

The above claim proves that the algorithm services all the request intervals.

We now analyze the cost incurred by the algorithm. The analysis is again very similar to that in Section 3.1.2.

To pay for evictions in line 4, we maintain the invariant that each page \( p \) in the cache has at most \( w(p) \) “load” on it; pages outside the cache have zero load. If we bring in \( p_t \) and if \( w(p_t) \geq 2w(p_{\text{min}}) \), its load becomes the load of \( p_{\text{min}} \) plus the cost of evicting \( p_{\text{min}} \); thus the total load on \( p_t \) is at most its weight, and \( p_t \) remains in the cache, maintaining the invariant. At the end of the algorithm, the total load over all pages is at most the optimum cost. This adds one to the competitive ratio.

Else if \( w(p_t) < 2w(p_{\text{min}}) \), we evict at least one page in \( Z^* \) (by Claim 4.3) and can charge evicting \( p_{\text{min}} \) to the eviction of that page—which we show below how to charge to \( A^* \).

Now we consider the cost incurred during line 12. Because of lines 18–23, we do not pay for all of these request intervals. Instead, for a particular page \( p \), we serve at most \( 4|\text{Sat}_p| \) requests for \( p \) in this line. Since the requests in \( \text{Sat}_p \) are disjoint and each of them is hit by \( A^* \), we can charge the service cost to the elements of \( A^* \).

Finally, we charge the eviction cost for lines 15-16. This cost is \( O(w(Z^*_{\geq w(p')})) \) by our choice of \( p' \) in line 14. Observe that for each page \( p \) in \( Z^*_{\leq w(p')} \), the doubly-extended interval \( \text{Dext}(I^p, t) \) is hit by an element \((p, t') \in A^* \), so we want to charge to this element of \( A^* \). Moreover, each page in \( Z^*_{\leq w(p')} \) is at least as heavy as \( p_{\text{min}} \), so any of these elements of \( A^* \) can pay to evict \( p_{\text{min}} \) (and its load). We finally show that no element of \( A^* \) can be charged twice in this manner. The proof is identical to that of Claim 3.6.

**Lemma 4.7.** No element in \( A^* \) can be charged twice because of evictions in lines 15-16

This completes the proof of Theorem 4.1.

## 5 Solving the Integer Program (\( IP_p \)) for PageTWPenalties

We now give algorithms to solve the integer program (\( IP_p \)), both in the offline and online settings. The main challenge in the offline case is that the LP relaxation has an unbounded integrality gap, so just relaxing the integrality constraints and then rounding will not suffice. Instead, we write a compact IP that has a smaller gap, and also has fewer constraints. Let us consider an example problem, that of picking \( n - k \) out of \( n \) items. All items have unit weight, so any feasible solution has cost at least \( n - k \). If variable \( x_i \in [0, 1] \) indicates that we should pick item \( i \), we can write an integer linear constraint for every choice \( C \) of \( k + 1 \) items saying
that $\sum_{i \in C} x_i \geq 1$. But the LP relaxation of this IP admits the fractional solution where $x_i := \frac{1}{k+1}$ for all $i$, and hence total cost $\frac{1}{k+1} \ll n - k$, showing a large integrality gap. However, replacing these $\binom{n}{k+1}$ linear constraints by the compact form $\{ \sum_i x_i \geq n - k, x \in [0,1]^n \}$ gives a formulation having no integrality gap; we use analogous ideas to address both the challenges above. In the online setting, we need to solve and round the resulting LPs online, which will require us to refine the primal-dual algorithms of Bansal et al. [BBN10].

We handle the constraints for the right and double extensions separately, in §5.1 and §5.2 respectively; this at most doubles the cost of the solution. In both cases, we reduce to the following interval covering problem:

**Definition 5.1** (Tiled Interval Cover). In the tiled interval cover problem (TiledIC), for each page $p \in [n]$, we are given a collection $I_p$ of disjoint intervals that partition the entire timeline. All intervals in $I_p$ have the same weight $w$. The goal is to select a minimum-weight subset of intervals from $I := \cup_p I_p$ such that for every time $t$, at least $n - k$ of these selected intervals contain $t$.

The offline algorithms to solve TiledIC will rely on total-unimodularity, and the online ones will reduce to primal-dual algorithms for the classical paging problem. The details of these solutions to TiledIC appear in Appendix D.

### 5.1 IP Solution for Right Extension Constraints

In this section, the focus is only on the right-extension constraints, i.e., the following IP:

$$\min_{x,y \text{ Boolean}} \sum_{p,t} w(p) x_{p,t} + \sum_{I} \ell(I) y_{I}$$

$$(\text{IP-Rp})$$

$$\sum_{I \in C} \min \left( 1, y_{I} + \sum_{t' \in \text{Rext}(I,t')} x_{\text{page}(I),t'} \right) \geq 1 \quad \forall t \forall C,$$

$$(R1p)$$

again $C$ consisting of $k + 1$ requests, each for a distinct page, and each starting before time $t$. The discussion about getting a compact IP above can be used to show that constraint (R1p) is equivalent (for integral solutions) to the following constraint:

$$\sum_{I \in C} \min \left( 1, y_{I} + \sum_{t' \in \text{Rext}(I,t')} x_{\text{page}(I),t'} \right) \geq n - k \quad \forall t \forall C,$$

$$(R2p)$$

where $C$ now consists of $n$ requests, one for each of the pages, and each starting before time $t$.

We now show how to approximately solve (IP-Rp) using an algorithm for TiledIC. Consider an instance $I$ of (IP-Rp). We assume that at time 0, there is a request interval $[0,0]$ with infinite penalty for every page; this only changes the optimum value by $\sum_p w(p)$. We now create an instance $I'$ of TiledIC by creating a collection of intervals $K_p$ for each page $p$ using the procedure in Figure 9; each of these intervals will have weight $w(p)$. Essentially each such interval is obtained by a minimal collection of original request intervals corresponding to $p$ in $I$ such that their total penalty exceeds $w(p)$. $\ll$Anupam 5.1: Figure fix. $\gg$ 

**Lemma 5.2** (Forward Direction). Consider an integral solution $(x,y)$ to $I$. Then there is a solution $S$ to $I'$ of cost at most $2(\sum_{p,t} w(p)x_{p,t} + \sum_{I} \ell(I)y_{I})$.

**Proof.** For an interval $I'$ in $K_p$, let $rt(I')$ to be the interval in $K_p$ which lies immediately to the right of $I'$. For every page $p$ and interval $I' \in K_p$, we add both $I'$ and $rt(I')$ to the solution $S$ if either of these conditions is satisfied: (i) $x_{p,t} = 1$ for some time $t \in I'$, or (ii) $y_I = 1$ for every request interval $I \in I$ for page $p$ such that the interval $I$ is contained within the interval $I'$. The cost guarantee is easy to see. We charge the cost of $I'$ and $rt(I')$ to $w(p)x_{p,t}$ in case (i), and to the total penalty of request intervals for page $p$ contained within $I'$ in case (ii).
Proof. In the first case we must have set $I_t = (i)$, corresponding to a set $C$ of $(1, 2, \ldots, k)$ and $x,y \in (x_1, x_2, \ldots, x_n)$, and hence $|C| \geq n - k$. Moreover, the collection $P$ contains the intervals $I_t$ for every request interval. Let $P_2 := [n] \setminus P_1$ be the remaining set of pages, i.e., pages such that there is at least one request interval for it (call this request interval $I_t$) contained within $I_t$ such that $y_{I_t} = 0$.

Let $C$ be the collection of the pages $I_t$ defined above, one for each page $p$. Applying constraint $(R2p)$ to $C$, we get

$$|P_1| + \sum_{p \in P_2} \sum_{t' \in \text{Rest}(I_t(p),t)} \min(1, x_{p,t'}) \geq n - k.$$ 

Hence, there is a set $P_2' \subseteq P_2$ of cardinality $n - k - |P_1|$ such that for any page $p \in P_2'$, there is a time $t' \in I_t \subseteq I_t(p)$ with $x_{p,t'} = 1$. It follows that we will pick intervals $I_t(p)$ and $I_t(I_t(p))$ into the solution $S$. Moreover, the collection $S$ contains the intervals $I_t(p)$ and $I_t(I_t(p))$ for each page $p \in P_1$. Since all these intervals $I_t(I_t(p))$ contain $t$, we have chosen $|P_1 \cup P_2'| \geq n - k$ intervals containing $t$, and hence $S$ is a feasible solution to $I_t$.

Lemma 5.3 (Reverse Direction). Let $S$ be a integral solution to the instance $I_t$. Then there is a solution $(x, y)$ to $I_t$ of cost at most $3w(S)$.

Proof. The solution $(x, y)$ is as follows: for each page $p$ and interval $I' = [t_1', t_2'] \in K_p \cap S$, (i) set $x_{p,t_1'} = x_{p,t_2'} = 1$, and also (ii) for every request interval $I = [t_1, t_2]$ for page $p$ having $t_1' \leq t_1 \leq t_2 < t_2'$ (i.e., intervals contained within such a chosen interval $I'$ and ending strictly earlier) set $y_{I_t} = 1$. Since $t_2$ is strictly smaller than $t_2'$, the construction of $K_p$ ensures that the total penalty cost of such intervals is at most $w(p)$. The cost guarantee for $(x, y)$ follows immediately. It remains to show feasibility.

Fix a time $t$. For each page $p$, let $I_t(p) \in K_p$ be the interval containing $t$. By the feasibility of $S$, there is a set $P_1 \subseteq S$ of $n - k$ pages such that their corresponding intervals $I_t(p)$ contain $t$. Consider some constraint $(R1p)$ corresponding to a set $C$ of requests for instance $I_t$. For each page $p$, let $I_t(p) \in C$ be the request interval for $p$. We know that $I_t(p)$ starts before time $t$, but it may end after time $t$. For a page $p \in P_1$, two cases arise: (i) $I_t(p)$ is strictly contained in $I_t'(p)$, or (ii) $\text{Rest}(I_t(p),t)$ contains one of the two end-points of $I_t'(p)$. In the first case we must have set $y_{I_t(p)}$ to 1, whereas in the second case there is a time $t'$ (which is one
of the two end-points of \( I'(p) \) in \( \text{Rext}(I(p), t) \) such that \( x_{p,t'} = 1 \). Since \( |P_1| = n - k \), we infer that \((x, y)\) satisfies Equation (R1p).

Combining these with Lemma D.1, we see that there is a 6-approximation offline algorithm for \( \text{SolveRextP} \).

### 5.1.1 Implementing the Solution Online

For the **online setting**, we can construct the set \( K_p \) online, and also approximately solve the \( \text{TiledIC} \) problem in an online fashion using the algorithm from Appendix D. However, the translation given in Lemma 5.3 cannot be implemented online. Specifically, it sets the \( x \) variables for start and end times of the chosen intervals \( I(p) \) and also \( y \) variables for all intervals strictly contained within \( I(p) \), but (i) setting \( x_{p,t} \) variables for times \( t_1 < t \) and \( y_I \) variables for past intervals violates the past-preserving property required in \( \S 3 \), and (ii) the online construction of \( K_p \) means the end time \( t_2 \) of the interval \( I(p) \in K_p \) is not known at time \( t \).

The first issue is easy to fix: if we change the proof of Lemma 5.3 so that when interval \( I\) is added to \( S \) at some time \( t \), we set \( x_{p,t} = 1 \) instead of \( x_{p,t'} \), and also set \( y_I = 1 \) for interval \( I \) that are contained with \( I'(p) \) only from now on. This change makes it past-preserving, and maintains correctness. As for the second issue, that the online algorithm may not know the right end-point \( t'_2 \) of \( I'(p) \in K_p \) at time \( t \), and hence cannot add the star in the future, there is at most one such “to-be-set” variable for each page \( p \); moreover, all intervals for page \( p \) containing the current time \( t \) are already hit variables corresponding to past times, or by this single to-be-set variable. This satisfies the sparsity property of \( \S 3 \).

Hence, combining the above reductions with the algorithmic results on solving \( \text{TiledIC} \) in Appendix D, and implementing these changes in the online setting, we get:

**Lemma 5.4 (Right-Extension Algorithms).** There is an online \( O(\log k) \)-competitive algorithm to solve the right-extension constraints with penalties (IP-Rp). The solution satisfies the monotonicity, past-preserving, and sparsity properties required in \( \S 3 \). Finally, there is an offline 6-approximation algorithm for this problem.

### 5.2 IP Solution for Double Extension Constraints

We now want to solve the double-extension constraints (D1):

\[
\min_{x,y} \text{ Boolean } \sum_{p,t} w(p) x_{p,t} + \sum_{I} \ell(I) y_I \tag{IP-D1p}
\]

\[
\sum_{I \in C} \min \left(1, y_I + \sum_{t' \in \text{Dest}(I,t)} x_{\text{page}(I),t'} \right) \geq 1 - y_I, \quad \forall t \forall C, \tag{D1p}
\]

where \( C \) consists of \( k \) requests, each for a distinct page (not equal to page \( p_i \)), and each request interval ending before time \( t \). As in the previous section, we can replace (D1p) by the following constraints and get exactly the same integer solutions.

\[
\sum_{I \in C} \min \left(1, y_I + \sum_{t' \in \text{Dest}(I,t)} x_{\text{page}(I),t'} \right) \geq (n - k)(1 - y_I), \quad \forall t \forall C. \tag{D2p}
\]

Here the set \( C \) consists of \( n - 1 \) requests, one for each distinct page different from page \( p_i \), where each of these request intervals ends before time \( t \). The variable \( y_I \) has two roles—the variable \( y_{I'} \) on the right denotes whether we need to consider the constraints (D3p) corresponding to time \( t \), whereas the ones on the left denote whether those requests were satisfied, or if their penalty is paid instead. It turns out that we can drop the occurrences of the \( y_I \) variables on the left, and get a polynomial-sized covering IP instead.

For each page \( p \), let \( K_p \) be as defined in Section 5.1, and let \( T_p \) denote the right end-points of the intervals in \( K_p \). For time \( t \) and page \( p \), define time \( \tau^p_t \) as follows: let \( I_t = [t_1, t] \) be the request interval which ends at
time \( t \), and \( p_t \) denote page(\( I_t \)). If \( I'(p) \) is the last interval in \( \mathcal{K}_p \) which ends before \( t_1 \), then \( \tau^p_t \) is the right end-point of this interval \( I'(p) \). (Of course, the time \( \tau^p_t \) lies in the set \( \mathcal{T}_p \).) Define the interval \( D^p_t := [\tau^p_t, t] \).

Consider the following compact IP, which is equivalent to (IP-D1p) up to constant factors, as we show next.

\[
\min_{x,y \text{ Boolean}} \sum_{p,t} w(p) x_{p,t} + \sum_I \ell(I) y_I 
\]  

(IP-D3p)

\[
\sum_{p \neq p_t} \min \left( 1, \sum_{t' \in D^p_t} x_{p,t'} \right) \geq (n - k)(1 - y_t) \quad \forall t. 
\]  

(D3p)

**Lemma 5.5.** Given an integer solution \((x, y)\) to (IP-D1p), there is a solution \((x', y')\) to (IP-D3p) with cost at most three times as much. Conversely, if \((x', y')\) is an integer solution to (IP-D3p), there is a corresponding solution to (IP-D1p) of at cost at most twice that of \((x', y')\).

**Proof.** For the first part, define \( y'_I = y_{I_t} \) for all \( I \), and \( x'_{p,t} = x_{p,t} \) for all \( p \) and \( t \). For an interval \( I \in \mathcal{K}_p \), define \( rt(I) \) to be the interval immediately to its right in \( \mathcal{K}_p \). For each page \( p \) and interval \( I \in \mathcal{K}_p \), we perform the following steps: let \( t_1 \) and \( t_2 \) be the right end-points of \( I \) and \( rt(I) \) respectively. We set \( x'_{p,t_1} = x'_{p,t_2} = 1 \) if either of these conditions hold: (i) there is a time \( t \in I \) such that \( x_{p,t} = 1 \), or (ii) \( y_{I_t} = 1 \) for every request interval \( I' \) for \( p \) which is contained within \( I \). The cost guarantee for \((x', y')\) follows easily.

It remains to show that \((x', y')\) is feasible, consider a time \( t \) and assume \( y'_{I_t} = y_{I_t} = 0 \) (otherwise (D3p) follows immediately). Let \( I_t = [t_1, t] \), and for page \( p \neq p_t \), let \( I'(p) \) be the last interval in \( \mathcal{K}_p \) ending before \( t_1 \), so that \( \tau^p_t \) is the right end-point of \( I'(p) \). If there is a request interval \( I \) for page \( p \) which is contained in \( I'(p) \) and \( y_I = 0 \), we define \( I(p) \) to be such an interval \( I \), otherwise \( I(p) \) is any request interval for \( p \) contained within \( I'(p) \). Let \( P_1 \) denote the pages for which the first case holds and \( P_2 \) be the second set of pages. Feasibility of (D1p) for these set of intervals implies that

\[
|P_2| + \sum_{p \in P_1} \min \left( 1, \sum_{t' \in \operatorname{Dext}(I(p), t)} x_{p,t'} \right) \geq n - k.
\]

For pages \( p \in P_2 \), we would have set \( x'_{p,\tau^p_{p,t}} = 1 \) due to condition (ii). Furthermore, the interval \( \operatorname{Dext}(I(p), t) \) contains \( D^p_t \) and is contained in \( I'(p) \). Now for page \( p \in P_1 \), suppose \( t' \in \operatorname{Dext}(I(p), t) \) is such that \( x_{p,t'} = 1 \). Then either \( t' \in D^p_t \) in which case \( x'_{p,t'} = x_{p,t'} \), or else \( t' \in I'(p) \) and we would have set \( x'_{p,\tau^p_{p,t}} = 1 \) by condition (i). In either case, \( \sum_{t' \in D^p_t} x_{p,t'} \geq 1 \). This shows that \((x', y')\) is a feasible solution to (IP-D1p).

We now show the converse. Let \((x', y')\) be a feasible solution to (IP-D3p). We construct a feasible solution \((x, y)\) to (IP-D1p). As above, we first set \( x = x', y = y' \). Furthermore, for every \( (p, t) \) such that \( x'_{p,t} = 1 \), let \( I = [t_1, t_2] \) be the interval in \( \mathcal{K}_p \) containing \( t \), and set \( x_{p,t_2} = 1 \). The cost guarantee for \( x \) follows easily. To show feasibility, fix a time \( t \) for which \( y_{I_1} = 0 \), and let \( P \) be the set of pages for which the LHS equals 1 in constraint (D3p). For a page \( p \), recall that \( I'(p) \) is the interval in \( \mathcal{K}_p \) which ended before \( I_1 \) started, but \( rt(I'(p)) \) intersects \( I_1 \). Now consider the constraint (D1p) for time \( t \) and set of pages \( P \). We claim that the LHS term for every page \( p \in P \) equals 1. If \( \operatorname{Dext}(I, t) \supseteq D^p_t \), then this claim follows from the fact that \( p \in P \). So assume that \( \operatorname{Dext}(I, t) \supsetneq D^p_t \), and since both intervals share the same right endpoint, \( \operatorname{Dext}(I, t) \subseteq D^p_t \). But
then, by the greedy construction process, \( r_t(I'(p)) \) is contained in \( D^p_t \). Since \( p \in P \) we know that \( x^p_{p,t'} = 1 \) for some \( t' \in D^p_t \) and hence in \( r_t(I'(p)) \cup I_t \). If \( t' \in I_t \), then \( t' \in \text{Dext}(I,t) \) where \( I \in C \) is the request interval for \( p \). Since \( x^p_{p,t'} = 1 \) as well, we see that the LHS term for \( p \) in (D1p) equals 1. On the other hand, if \( t' \in r_t(I'(p)) \), we will set \( x^p_{p,t''} \) to 1, where \( t'' \) is the right end-point of \( r_t(I'(p)) \). Since \( t'' \in I_t \), we have that \( t'' \in \text{Dext}(I,t) \), and so the same conclusion holds.

The compact LP can clearly be solved offline. The following theorem, whose proof is deferred the appendix, shows that the fractional relaxation of (IP-D3p) can also be solved online losing only a logarithmic factor.

**Theorem 5.6 (Solving the LP Online).** There is an \( O(\log k) \)-competitive online algorithm which maintains a solution to the fractional relaxation of (IP-D3p). At time \( t \), the fractional solution only changes (and in fact, increases) the variables \( x_{p,t} \) for all pages \( p \), and \( y_{I,t} \).

**Corollary 5.7 (Integral Penalty Variables).** Given an online fractional solution \((\bar{x}, \bar{y})\) to (IP-D3p), we can maintain another online fractional solution \((\bar{x}, \bar{y})\) whose cost is at most twice that of \((\bar{x}, \bar{y})\), such that \( \bar{y}_I \) is integral for every \( I \). This solution also has the same property that at time \( t \), it only increases the variables \( x_{p,t} \) and \( y_{I,t} \), corresponding to time \( t \).

**Proof.** Fix a time \( t \) and the corresponding solution \((\bar{x}, \bar{y})\). We set \( \bar{y}_I = 1 \) if \( \bar{y}_I > 1/2 \), and to 0 otherwise; we also set \( \bar{x}_{p,t} = \min(1, 2\bar{x}_{p,t}) \). This at most doubles the cost of the solution, and maintains feasibility. Furthermore, at time \( t \), \((\bar{x}, \bar{y})\) only increases the variables \( \bar{x}_{p,t} \) for all pages \( p \) and \( \bar{y}_I \).

Let the problem defined by (IP-D3p) be called \text{SolveDextP}, and fix an instance \( I \) of this problem. For rest of the discussion, we maintain an online fractional solution \((\bar{x}, \bar{y})\) to \( I \) with the properties mentioned in Corollary 5.7. In order to maintain an online integral solution to \( I \), we first solve the problem for non-nested instances, and then prove an “extension” theorem to translate from non-nested instances to all instances.

### 5.2.1 Solving Globally Non-Nested Cases of \text{SolveDextP}

We say that interval \( I = [t_1, t_2] \) is strictly nested within \( I' = [t'_1, t'_2] \) if \( t'_1 \leq t_1 \leq t_2 \leq t'_2 \), and either the first or the last inequality is strict. (We drop the use of strict, and simply say “nested” henceforth.) Two intervals are nested if one of them is nested within another. Let \( T' \) be some subset of the timeline \( T \) such that for every \( t_1 \neq t_2 \in T' \), their critical intervals \( I_{t_1}, I_{t_2} \) are not nested. We define the \text{NonNestDextP} problem, which solves the problem (IP-D3p) where the constraints correspond to times \( t \in T' \) and we are also given that \( \bar{y}_{I,t} = 0 \) for all \( t \in T' \) (i.e., we are not paying the penalty at these times). First, we show that the intervals \( D^p_t = [t^p_t, t] \) for any fixed page are also non-nested.

**Claim 5.8.** For times \( t, t' \in T' \) and a page \( p \) such that \( p \notin \{p_1, p_2\} \), the intervals \( D^p_t, D^p_{t'} \) are non-nested.

**Proof.** Assume wlog that \( t < t' \). Let \( I_t = [t_1, t], I_{t'} = [t'_1, t'] \). By the non-nested property, \( t_1 \leq t'_1 \). Therefore, by construction, \( t^p_t \leq t'^p_{t'} \).

We reduce the instance \text{NonNestDext} to a tiled interval cover problem with exclusions (\text{TiledICEx}) instance. \text{TiledICEx} is like \text{TiledIC}, where additionally for each time \( t \) we are specified a page \( p_t \), and we cannot use the intervals in \( T_p \) for the coverage requirement at time \( t \). In Appendix D we give a constant-factor approximation algorithm and \( O(\log k) \)-competitive algorithm for \text{TiledICEx}.

The reduction of the instance \((I, T')\) of \text{NonNestDextP} to an instance \( \mathcal{T}' \) of \text{TiledICEx} proceeds as follows. For each page \( p \), we build a disjoint collection of intervals \( D^p_t \) as shown in Figure 11, by greedily picking a set of non-overlapping intervals in \( \{D^p_t: t \geq 0\} \) and extending them to partition the timeline. In the instance \( \mathcal{T}' \), the set \( \mathcal{T}'(p) \) of intervals for page \( p \) is given by the intervals in \( D^p_t \), and the excluded page \( p_t \) is the page requested at time \( t \). Furthermore, the cost of each interval in \( \mathcal{T}'(p) \) is given by \( w(p) \), and the
covering requirement $R$ at each time $t$ is $n - k$. The natural LP relaxation for $T'$ has a variable $z_I$ for each interval $I$ such that for every time $t$:

$$
\sum_{I : I \in \mathcal{I}'(p), p \neq p, t \in I} z_I \geq n - k. \tag{3}
$$

1. Initialize $\mathcal{D}_p \leftarrow \emptyset$, $t^* \leftarrow 0$.
2. for $t = 1, 2, \ldots$ do
   3. if $D_p^t$ does not contain $t^*$ in the interior then
      4. Add $[t^*, t]$ to $\mathcal{D}_p$
      5. Update $t^* \leftarrow t$.


![Figure 11: The online procedure to construct the partition $\mathcal{D}_p$ for page $p$.](image)

**Lemma 5.9.** Given the instance $(\mathcal{I}, T')$ as above, let $T'$ be the corresponding **TiledICEx** instance.

(i) Let $(\bar{x}, \bar{y})$ be a fractional solution to (IP-D3p) with $\bar{y}_{t_1} = 0$ for all $t \in T'$. Then there is a fractional solution to the above LP relaxation for $T'$ of cost at most twice that of $(\bar{x}, \bar{y})$.

(ii) Let $S$ be an integral solution to $T'$. Then there is an integral solution to the **NonNestDextP** instance $(\mathcal{I}, T')$ of cost at most twice that of $S$.

Moreover, both the above constructions can be done efficiently.

**Proof.** Let $(\bar{x}, \bar{y})$ be a fractional solution to (IP-D3p). We construct a fractional solution $z$ for the instance $T'$ as follows. For every variable $x_{p,t_1}$, let $I$ be the interval in $D_p$ containing $t$, and let $1t(I)$ be the interval in $D_p$ immediately to the left of $I$. We raise both $z_I$ and $z_{1t(I)}$ by $x_{p,t}$. Finally, if any $z_I$ variable exceeds one, we cap it at one. The cost of $z$ is at most twice that of $\bar{x}$. To show feasibility, consider a time $t$. Let $I_1$ be the interval in $D_p$ containing $t$ and $I_2 = rt(I_1)$ be the interval immediately to the right of $I_1$ in $D_p$. Both $I_1$ and $I_2$ contain intervals $D_p^{t_1}$ and $D_p^{t_2}$ for some times $t_1$ and $t_2$ respectively. Therefore, Claim 5.8 implies that $D_p^t$ is contained in $I_1 \cup I_2$. It follows that we will raise $z_{I_1}$ by at least $\min(1, \sum_{t' \in D_p^t} x_{p,t'})$. From (D3p) it follows that $z$ is a feasible solution.

We now prove the second part of the lemma. Let $S$ be a feasible solution to $T'$. We will build an integral solution $(x, y)$, where $y_{t_1} = 0$ for all $t \in T'$, for $(\mathcal{I}, T')$. For every interval $I = [t_1, t_2] \in S \subseteq D_p$, we set $x_{p,t_1} = x_{p,t_2} = 1$. The cost guarantee follows easily. To verify feasibility, consider a time $t$. For each page $p \neq p_t$, let $I(p)$ be the interval in $T'(p)$ which contains $t$. Let $P(t)$ be the set of pages for which $S$ contains $I(p)$. By feasibility of the $z$-solution, $|P(t)| \geq (n - k)$. The interval $D_p^t$ and $I(p)$ overlap (both of them contain time $t$). Also, $D_p^t$ cannot be contained in $I(p)$, by the way the set $D_p$ is constructed. Therefore, $D_p^t$ must contain one of the two end-points of $I(p) := [t_1, t_2]$. Since we set both $x_{p,t_1}, x_{p,t_2}$ to 1, it follows that the LHS term corresponding to page $p$ in (D3p) (for time $t$) is also 1. This shows that $(x, y)$ is feasible.

Combining Lemma 5.9 with the 2-approximation for **TiledICEx** from Lemma D.1 gives an 8-approximation algorithm for **NonNestDextP**. As in Section 5.1.1, the reduction from the proof of Lemma 5.9(ii) can be carried out in an online manner. If the online algorithm for **TiledICEx** selects an interval $I := [t_1, t_2]$ at a time $t$ (note that $t \leq t_2$), we need to add the stars $(p, t_1)$ and $(p, t_2)$ in our solution for $\mathcal{I}$. It turns out that the proof of Lemma 5.9 holds if we add the stars $(p, t)$ and $(p, t_2)$ instead. Further, the star $(p, t_2)$ can be added at time $t_2$. Since the set of constraints (D3p) corresponding to time $t$ involve variables at $t$ and earlier only, the online algorithm need not remember at time $t$ the stars which will appear in future – it can keep track of all the stars which have been added at time $t$, and any such star which corresponds to time $t' > t$ will only appear at time $t'$ in the algorithm. Thus, the algorithm satisfies the property that at any time $t$, it will only
This algorithm can be implemented in online manner as well. More formally, assume there is a present preserving online algorithm which generates a set \( \bar{A} \) of stars, where \( \bar{A} := \{ (p, t) \mid x_{p, t} = 1 \} \). For a time \( t \) and a set of elements \( A^* \), let \( P(A^*, t) \) denote the set of pages for which the corresponding intervals intervals \( D_p^t \) are hit by \( A^* \). I.e., we can rephrase constraint (D3p) as wanting to find a set \( A^* \) such that \( P(A^*, t) \setminus \{ p_i \} \) has at least \( n - k \) pages. The main technical ingredient is the following extension result:

**Theorem 5.12 (Extension Theorem).** There is an algorithm that takes a set \( T' \) of times, a non-nested net \( N' \subseteq T' \), the associated monotone map \( \varphi \), and a set \( A^* \), and outputs another set \( B^* \supseteq A^* \) such that

\[
\begin{align*}
(i) \quad P(B^*, t) &\supseteq P(A^*, \varphi(t)) \quad \text{for all } t \in T' \setminus N', \text{ and} \\
(ii) \quad w(B^*) &\leq 3 w(A^*).
\end{align*}
\]

This algorithm can be implemented in online manner as well. More formally, assume there is a present preserving online algorithm which generates the set \( A^*_t \) at time \( t \in T \). Then there is a present preserving online algorithm which generates \( B^*_t \) at time \( t \in T \) and satisfies conditions (i) and (ii) above (with \( A^* \) and \( B^* \) replaced by \( A^*_t \) and \( B^*_t \) respectively).

We defer the proof to Appendix C, and instead explain how to use the result in the off-line setting first. We invoke the extension theorem twice. For the first invocation, we use Theorem 5.12 with the entire set of times \( T \), a net \( N \) and the associated monotone map \( \varphi \), and with \( A^* \) being a solution of weight at most \( 8 \text{opt}(T) \) given by Lemma 5.10 on the sub-instance \( N \). This outputs a set \( B^* \) with \( w(B^*) \leq 24 \text{opt}(T) \). Moreover, since \( A^* \) is feasible for \( N' \), it follows from the first property of Theorem 5.12 that \( |P(B^*, t) \setminus \{ p_i \}| \geq |P(A^*, \varphi(t))| - 1 \geq n - k - 1 \) for every time \( t \in T \setminus T' \).
For the second invocation, let $T_1 \subseteq T \setminus \mathcal{N}$ be the subset of times $t$ such that $|P(B^*, t) \setminus \{p_i\}| = n - k - 1$, i.e., those with unsatisfied demand. We use Theorem 5.12 again, this time with $T_1$, a net $\mathcal{N}_1$ and the associated monotone map $\varphi_1$, and a solution $A^*_1$ obtained by using Lemma 5.10 on the sub-instance $\mathcal{N}_1$. This gives us $B_1^*$ with weight at most $24 \cdot \text{opt}(T)$. We output $B^* \cup B_1^*$ as our solution. Somewhat surprisingly, this set $B_1^*$ gives us the extra coverage we want, as we show next.

**Lemma 5.13** (Feasibility). For any time $t$, $|P(B^* \cup B_1^*, t) \setminus \{p_i\}| \geq n - k$.

**Proof.** We only need to worry about times $T_1 \setminus \mathcal{N}_1$. Consider such a time $t_1$. Let $t_2 := \varphi_1(t_1)$ and $t_3 := \varphi(t_2)$. For sake of brevity, let $p_i$ denote $p_{ti}$, and $I_t$ denote $I_{ti}$. Note that $I_3 \subseteq I_2 \subseteq I_1$, and since there are no nested intervals of the same page, also $p_1 \neq p_2 \neq p_3$. Recall: we want to show $|P(B^* \cup B_1^*, t_1) \setminus \{p_i\}| \geq n - k$.

Note that $P(B^*, t_2)$ contains $P(A^*, t_3)$, and the latter has size at least $n - k$ by construction. Hence, for $t_2$ to appear in $T_1$, we must have $|P(B^*, t_2)| = n - k$ and $p_2 \in P(A^*, t_3)$. Since $I_1$ contains $I_3$, $D_{t_1}^{p_2}$ contains $D_{t_3}^{p_2}$, and so, $B^*$ hits $D_{t_3}^{p_2}$, i.e., $p_2 \in P(B^*, t_1)$.

Since $P(A^*, t_2) \setminus \{p_2\}$ has size $n - k$ (by construction of $A^*_1$), and $P(B_1^*, t_1)$ contains $P(A^*_1, t_2)$, it follows that $P(B_1^*, t_1) \setminus \{p_2\}$ also has size at least $n - k$. Therefore, $P(B^* \cup B_1^*, t_1)$ has size at least $n - k + 1$ because $P(B^*, t_1)$ contains $p_2$. This proves the lemma. \qed

Since $w(B^* \cup B_1^*) \leq 48 \cdot \text{opt}(T)$, we get a 48-approximation algorithm for $\text{SolveDextP}$. It is easy to check that these arguments carry over to the online case as well; we briefly describe the main steps. The set $\mathcal{N}$ can be generated in an online manner using a greedy algorithm (as mentioned in the beginning of this section). We invoke the online algorithm in Lemma 5.10 to get a present restricted solution $A^*_1$ for all $t \in \mathcal{N}$. Theorem 5.12 implies that the present restricted solution $B^*_1$ can be constructed at time $t$. Given $B^*_1$, we can tell whether a particular time $t$ qualifies for being in $T_1$. The same argument can now be repeated to show that we can maintain $(B^*_1)_t$ for all $t \in T_1$. Combining this with Lemma 5.10, we get

**Lemma 5.14.** There is an online $O(\log k)$-competitive present restricted algorithm to $\text{SolveDextP}$. Moreover, there is an offline algorithm 48-approximation algorithm for $\text{SolveDextP}$.  

**Corollary 5.15.** There is a constant factor (offline) approximation algorithm for (IPp). Further, there is an $O(\log k)$-competitive online algorithm that satisfies the past-preserving and sparsity property as in §3.

**Proof.** Let $I$ be an instance of $\text{PageTwPenalties}$. Let $A_i^*$ and $A_j^*$ be (offline) solutions for (IP-Rp) and (IP-D3p) as guaranteed by Lemma 5.4 and Lemma 5.14 respectively. Lemma 5.5 shows that $A_2^*$ can be mapped to a solution $A_3^*$ that satisfies (IP-D1p), and $\text{cost}(A_3^*) \leq 2 \cdot \text{cost}(A_2^*)$. Further $A^* := A_1^* \cup A_2^*$ is a feasible solution to (IPp). Since (IP-Rp) and (IP-D1p) are special cases of (IPp), Lemmas 5.4, 5.5 and 5.14 imply that

\[
\text{cost}(A_1^*) + \text{cost}(A_3^*) \leq 6 \cdot \text{opt}(I) + 2 \cdot \text{cost}(A_2^*) \leq 6 \cdot \text{opt}(I) + 96 \cdot \text{opt}(\text{IP-D1p}) \leq O(1) \cdot \text{opt}(I).
\]

The online version follows analogously. Note that the conversion from $A_2^*$ to $A_3^*$ in Lemma 5.5 can be carried out in an online manner, and if $A_3^*$ is present restricted, then so is $A_1^*$. Since $A_2^*$ and $A_3^*$ are past preserving, so is $A^*$. Also the sparsity of $A_1^*$ and the fact that $A_3^*$ does not add any start in the future implies that $A^*$ also satisfies the sparsity property. \qed

Corollary 5.15, along with Theorem 3.1 and Theorem 4.1, implies Theorem 1.1 and Theorem 1.2 respectively. The integrality gap of (IPp) is constant for the following reason – the integrality gap of the LP relaxations for $\text{SolveDext}$ and $\text{NonNestDext}$ are $O(1)$, and the reductions in Lemmas 5.2, 5.3 and 5.9 also hold between the fractional solutions to the corresponding problems.
6 Extension to Paging with Delay

In this section, we show a simple reduction from the (weighted) Paging with Delays (PageD) problem to the PageTWPenalties problem. which allows us to translate the results of the previous sections giving an $O(1)$-approximate offline algorithm and an $O(\log k \log n)$-competitive online algorithm for the PageTWPenalties problem to get the same asymptotic performance for the PageD problem.

We transform an instance $I$ of PageD to an instance $I'$ of PageTWPenalties as follows. Recall that each request in $I$ is specified by a triple $(p,t,F)$, where $p$ is the requested page, $t$ is the time at which this request is made, and $F : \{t,t+1,\ldots\} \rightarrow \mathbb{R}_{\geq 0}$ denotes the non-decreasing loss function associated with it.

We may assume without loss of generality that $F(t) = 0$, since otherwise we can work with the function $F'(t) := F(t') - F(t)$, and the competitive ratio is no worse. To model this request, we create an ensemble of intervals $[t,t']$ for each $t' \geq t$ in the PageTWPenalties instance $I'$, where the penalty for the interval $I := [t,t']$ is $F(t' + 1) - F(t')$, for each $t' \geq t$.

To see the equivalence, suppose this request $(p,t,F)$ is served at time $t'$—i.e., the page $p$ enters the cache after time $t$ only at time $t'$. Then all intervals in its ensemble ending at later times are also satisfied. Moreover, intervals ending at earlier times $t,t+1,\ldots,t'-1$ are not satisfied, and their penalty adds up to $F(t') - F(t) = F(t')$, as desired. Given this equivalence and the algorithmic results for the PageTWPenalties problem, we get:

**Theorem 6.1.** There is an $O(\log k \log n)$-competitive online algorithm and an $O(1)$-approximate offline algorithm for the PageD problem.

This completes the proof of Theorems 1.1 and 1.2.

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A NP-Hardness of PageTW

We now show that the PageTW is APX-hard, even when the cache size $k = 1$ and we have unit weights. The reduction is the same as that of Nonner and Souza [NS09] for the joint-replenishment problem, and we give it here for completeness. The reduction is from the (unweighted) Vertex Cover problem on bounded-degree graphs.

Consider an instance $I$, consisting of a graph $G = (V, E)$, of the Vertex Cover problem. We reduce it to an instance $I'$ of the PageTW problem. In the instance $I'$, we have one page $p_e$ for every edge $e \in E$. We also have a special page $p^\star$. All pages have unit weight and the cache size $k$ is 1. We now specify the request intervals for each page. The timeline $T$ is the line $[0, |V| + 1]$. For the page $p^\star$ we have request intervals $[t, t]$ for every integer $t \in T$, i.e., this page must be in the cache (or brought into the cache) at each integer time $t$. Now consider the page $p_e$ for the edge $(u, v) \in E$. Assume wlog that $u < v$. We have three request intervals for this page $e$: $I^1_e = [0, u], I^2_e = [u, v], I^3_e = [v, |V| + 1]$, where $n$ denotes $|V|$. Note that these are closed intervals. This completes the description of the reduction. We first prove the easier direction (see Figure 12 for an example).

![Figure 12: Illustration of the reduction from Vertex Cover to PageTW. The page $p^\star$ is requested at each time. All the other pages have three request intervals, shown by solid lines with end-point delimiters.](image)

Claim A.1. Suppose there is a vertex cover of $G$ of size at most $r$ (in the instance $I$). Then there is a solution to $I'$ of cost at most $r + 2|E| + 1$.

Proof. Let $U$ be a vertex cover of size $r$. The caching schedule is as follows: we will ensure that at the end of each time $t$, the page $p^\star$ is in the cache. This will ensure that all requests for $p^\star$ are satisfied. For every $u \in U$, we do the following: let $N(u)$ be the edges incident to $u$ in $G$. We bring each of the pages $p_e \in N(u)$ in the cache and the evict it. At the end of this process (at time $u$) we bring $p^\star$ back in the cache.

For every edge $e \in E$, we have ensured that we bring $e$ in the cache either at time $u$ or $v$ (or both). If we bring in $e$ at both the times, we have satisfied all the requests for $p_e$. Otherwise, we would have satisfied two out of the three request for $p_e$, and the unsatisfied request would be either $I^1_e$ or $I^3_e$. Let $E_1$ be the set of edges $e$ for which the request $I^1_e$ is unsatisfied, and let $I^2_e$ be the set of edges $e$ for which $I^3_e$ is unsatisfied. At
time 0, we bring in and then evict all pages in $I_e^1$. Then we bring in page $p^*$. At time $n + 1$, we evict $p^*$ and bring in and then evict all the pages in $E_3$. This yields a feasible caching schedule. The total number of times $p^*$ is evicted is at most $r + 1$ (at each of the times in $U$, and maybe at time $n + 1$). Every page $p_e$ is evicted exactly twice. This proves the claim. 

Claim A.2. Suppose there is a solution to $I'$ of cost at most $r + 2|E| + 1$. Then there is a vertex cover of $G$ of size at most $r + 1$.

Proof. Let $S$ be a solution to the caching problem. For an edge $e$, let $T_e$ be the timesteps when $e$ is brought into the cache. Since $p^*$ must be present at the end of each integer time, $T_e$ must have non-empty intersection with each of the three request intervals for $e$. We now modify $S$ to a solution $S'$ which has the following property: (i) The total cost of $S'$ is at most that of $S$, (ii) for every edge $e = (u,v)$, the corresponding set $T_e'$ in $S'$ has non-empty intersection with $\{u,v\}$.

Initialize $S'$ and $T_e'$ to $S$ and $T_e$ respectively. While there is an edge $e = (u,v)$ such that $T_e'$ does not contain $u$ or $v$, we do the following: $T_e'$ must contain a distinct time in each of the intervals $I_e^1, I_e^2, I_e^3$—let $t_1, t_2, t_3$ denote these three times. Note that $t_2$ must be in the interior of $I_e^2$. Assume wlog that $u < v$. Instead of bringing in $p_e$ at times $t_1$ and $t_2$ (and evicting them at these times), we will bring in $p_e$ at time $u$. This will save us a cost of 1 in the total eviction cost of $p_e$. However, it may happen that earlier $p^*$ was not getting evicted at time $u$, and now we will need to evict it (and then bring it back into cache) at time $u$. Still, this will not increase the cost of the solution.

Thus, we see that $\cup_e T_e'$ must contain a vertex cover $U$ of $G$. Since $p^*$ must be getting evicted at each of these times, the total cost of $S'$ (and hence, that of $S$) is at least $2|E| + |U|$. This implies the claim. 

Using the above two claims, we show that the PageTW problem is APX-hard.

Lemma A.3. Let $G$ be a graph of maximum degree 4. Suppose there is an $(1 + \varepsilon)$-approximation for PageTW problem. Then there is a $(1 + 9\varepsilon)$-approximation for Vertex Cover on $G$.

Proof. Let $A$ be the $\alpha$-approximation algorithm for PageTW. The algorithm for Vertex Cover on $G$ is as follows: use the reduction described above to get an instance $I'$ of PageTW. Run $A$, and then use the proof of Claim A.2 to get a vertex cover for $G$.

Suppose $G$ has a vertex cover of size $r$. Since the maximum degree of $G$ is 4, we know that $|E| \leq 4r$. Now Claim A.1 implies that $I'$ has a solution of cost at most $2|E| + r + 1$, and so $A$ outputs a solution of cost at most $(1 + \varepsilon)(2|E| + r + 1) = 2|E| + 2\varepsilon|E| + (1 + \varepsilon)(r + 1) \leq 2|E| + (1 + 9\varepsilon)r + O(1)$. Claim A.2 now shows that there is a vertex cover of size at most $(1 + 9\varepsilon)r + O(1)$ in $G$. This proves the lemma; the additive 1 can be ignored because we can take multiple copies of $G$ and make $r$ as large as we want.

Finally, the fact that vertex cover is hard to approximate to within $\approx 1.02$ on 4-regular graphs [CC06] implies that PageTW is $\approx 1.002$-hard, and completes the proof.

B Some Illustrative Examples

B.1 Evictions at Endpoints are Insufficient

It is easy to check that we cannot hope to service every interval $I$ at either $s(I)$ or $t(I)$, which we can do for the unweighted case. Indeed, consider the following input: suppose $k = 1$ and there is a very heavy page which is requested at each time, and so we need to have it in the cache at every time. Now there are $n$ unit weight pages, but there request intervals are $[0, n], [1, n + 1], [2, n + 2], \ldots$.

The optimal solution is to service all these requests at time $n$, because then we will evict the heavy page only once. Thus, our algorithm needs to use these windows of opportunity to service as many cheap requests as possible.
B.2 An Integrality Gap for the Interval Hitting LP

We now consider a natural LP relaxation for PageTW which extends that for weighed caching, and show that it has large integrality gap. We have variables $x_{p,J}$ for pages $p$ and intervals $J \subseteq [T]$, indicating that $J$ is maximal interval during which the page $p$ is in the cache for the entire interval $J$. Recall that we are allowed to service many requests at each timestep, so each timestep may have up to $n$ loads and $n$ evictions. To handle this situation, we “expand” the timeline so that all such “instantaneous” services can be thought of as loading each page in the cache for a tiny amount of time, and then evicting it. This will ensure that we can write a packing constraint in the LP relaxation which says that no more than $k$ pages are in the cache at any particular time.

Let $N$ be a large enough integer ($N \geq n$, where $n$ is the number of distinct pages will suffice). We assume that all $s(I), t(I)$ values for any request interval $I$ are multiples of $N$ (this can be easily achieved by rescaling). Let $E$ denote the set of end-points of the request intervals (so each element in $E$ is a multiple of $N$). As above, we have variables $x_{p,J}$, where the end-points of $J$ are integers (which need not be multiples of $N$).

The idea is that between two consecutive intervals of $E$, we can pack $N$ distinct unit size intervals, each of which may correspond to loading and then evicting a distinct page. We can now write the LP relaxation:

$$\text{min} \sum_{p,J} w(p) \cdot x_{p,J}$$

subject to:

$$\sum_{J : J \cap I \neq \emptyset} x_{p,J} \geq 1 \quad \forall \text{ request intervals } I \text{ with } p = \text{page}(I)$$

$$\sum_{p} \sum_{J : t \in J} x_{p,J} \leq k \quad \forall \text{ integer times } t \quad (5)$$

$$x_{p,J} \geq 0$$

Theorem B.1. The above LP has an integrality gap of $\Omega(k)$.

Proof. Suppose we have $k$ “heavy” pages with weight $k$ each, and 1 “light” page with weight 1. The request intervals for each of the pages are $E_0, E_1, \ldots, E_T$, where $E_i = [ikN, (i+1)kN]$, and $N$ and $T$ are suitable large parameters ($T, N > k^3$ will suffice).

We first argue that any integral solution must have $\Omega(T)$ cost. To see this, consider the request intervals $E_0, E_4, E_8, \ldots$ for the light page $p$. The page $p$ must be brought at least once during each of these intervals — say at timeslots $t_0, t_4, t_8, \ldots$, where $t_{4i} \in E_{4i}$ for all $i$. Notice that $E_{4i+2}$ lies strictly between $t_{4i}$ and $t_{4i+4}$, and hence each of the heavy pages must be present at least once during $E_{4i+2}$. Since there can be at most $k-1$ heavy pages in the cache at time $t_{4i}$, it follows that at least one heavy page must be brought into the cache during $[t_{4i}, t_{4i+4}]$. This argument shows that the cost of any integral solution must be $\Omega(Tk)$.

Now we argue that there is a fractional solution to the LP of total cost $O(T)$. For each heavy page $q$, we define $x_{q,J} = 1 - 1/k^2$, where $J = [0, (T+1)kN]$, i.e., $J$ is the entire timeline. Notice that each interval $E_i$ is of length $Nk$. We can therefore find $(k+1)k^2$ disjoint intervals of length 1 each in it, and “assign” $k^2$ of these intervals to each of the $k+1$ pages. Let $S_{i,p}$ be the set of $k^2$ unit length intervals assigned to page
p (which could be the light page or one of the heavy pages). For each heavy page q and each unit length interval H assigned to it, we set \( x_{q,H} \) to \( 1/k^4 \). For the light page \( p \) and each unit length interval \( H \) assigned to it, we set \( x_{p,H} \) to \( 1/k^2 \).

Now we check feasibility of this solution. Consider a heavy page \( q \) and the request interval \( E_i \) for it. The LHS of constraint (4) for this request is \( (1 - 1/k^2) + k^2/k^4 = 1 \), where the first term corresponds to \( x_{p,J} \) and the second term comes from the \( k^2 \) unit length intervals in \( S_{i,q} \). For the light page \( p \) and the request interval \( E_i \) for it, the LHS of this constraint is 1, because each of the \( k^2 \) unit length intervals \( H \) in \( S_{i,p} \) has \( x_{p,H} \) equal to \( 1/k^2 \). The constraint (5) is easy to check – for any time \( t \), the LHS is at most \( k(1 - 1/k^2) + 1/k^2 \leq k \), because the first term comes from \( x_{q,J} \) for each heavy page \( q \), and the second term comes from the fact that all the unit length intervals are disjoint.

Let us now compute the cost of this solution. For a heavy page \( q \), the total cost is \( k(1 - 1/k^2) + T/k \), where the first term comes because of the long interval \( J \) and the second term is because of the unit length intervals. This is \( O(T/k) \). Summing over all heavy pages, this cost is \( O(T) \). For the light page, we have \( Tk^2 \) unit length intervals, each to a fractional extent of \( 1/k^2 \). Therefore the total cost here is \( T \) as well. This proves the integrality gap of \( \Omega(k) \).

The essential problem with this LP is that the heavy pages are being almost completely fractionally assigned, leaving a tiny \( \varepsilon \) amount of space. Since all the requests are long, they can be slowly satisfied over \( 1/\varepsilon \) time periods, which is much less cost than the cost of actually evicting a heavy page.

C Proof of The Extension Theorem

**Theorem 5.12** (Extension Theorem). There is an algorithm that takes a set \( \mathcal{T}' \) of times, a non-nested net \( \mathcal{N}' \subseteq \mathcal{T}' \), the associated monotone map \( \varphi \), and a set \( A^* \), and outputs another set \( B^* \supseteq A^* \) such that

1. \( P(B^*, t) \supseteq P(A^*, \varphi(t)) \) for all \( t \in \mathcal{T}' \setminus \mathcal{N}' \), and
2. \( w(B^*) \leq 3w(A^*) \).

This algorithm can be implemented in online manner as well. More formally, assume there is a present preserving online algorithm which generates the set \( A^*_t \) at time \( t \in \mathcal{T} \). Then there is a present preserving online algorithm which generates \( B^*_t \) at time \( t \in \mathcal{T} \) and satisfies conditions (i) and (ii) above (with \( A^* \) and \( B^* \) replaced by \( A^*_t \) and \( B^*_t \) respectively).

**Proof.** The procedure to obtain \( B^* \) from \( A^* \) is a simple greedy procedure, and appears in Figure 13: it goes over the times in \( \mathcal{T}' \setminus \mathcal{N}' \), and fixes any violations to the containment condition of the theorem by adding a new element to \( B^* \). For brevity, define \( \mathcal{T}'' := \mathcal{T}' \setminus \mathcal{N}' \). It immediately follows that for any \( t' \in \mathcal{T}'' \), the set

```
1 Initialize \( B^* \leftarrow A^* \).
2 for \( t \in \mathcal{T}' \setminus \mathcal{N}' \) in increasing order do
3     for every page \( p \in P(A^*, \varphi(t)) \setminus P(B^*, t) \) do
4         Add \((p, t)\) to \( B^* \).
5 return \( B^* \).
```

Figure 13: The extension procedure to prove Theorem 5.12

\( P(A^*, \varphi(t)) \) is a subset of \( P(B^*, t) \). We just need to bound the cost of \( B^* \).

Let the times in \( \mathcal{T}'' \) be \( t_1 < t_2 < \ldots \) and **Claim 5.11** shows that \( \varphi(t_1) \leq \varphi(t_2) \leq \cdots \). The desired result now easily follows from the following claim.
Claim C.1. Suppose we add \((p,t_i),(p,t_j),(p,t_k)\) to the set \(B^*\) for some page \(p\) and times \(t_i < t_j < t_k\). Then there is a time \(t \in [t_i,t_k]\) such that \((p,t) \in A^*\).

Proof. Fix a page \(p\) with \(t_i, t_j, t_k\) as in the statement above, and let \(t \leq t_k\) be the largest time such that \((p,t) \in A^*\). We now make a sequence of observations:

(i) We claim that \(\varphi(t_k) \geq t_j\) and \(I_{\varphi(t_k)} \subseteq [t_j,\varphi(t_k)]\). If either is false then \(I_{t_k}\) contains \(t_j\), in which case there is no need to add \((p,t_k)\) to \(B^*\), because \(B^*\) already contains \((p,t_j)\).

(ii) Moreover, the interval \(D^p_{t_j} \subseteq [t_i,t_j]\). Clearly \(D^p_{t_j}\) ends at \(t_j\) (by definition). If it starts before \(t_i\), then there is no need to add \((p,t_j)\) to \(B^*\).

(iii) Next, \(D^p_{\varphi(t_k)} \subseteq [t_i,\varphi(t_k)]\): Since \(\varphi(t_k) \geq t_j\), \(D^p_{\varphi(t_k)}\) starts after (or at the same time as) \(D^p_{t_j}\) starts, and so this follows by (ii) above.

Now since \(p\) lies in \(P(A^*,\varphi(t_k))\), statement (iii) implies that we have \((p,t) \in A^*\) for some \(t \in [t_i,\varphi(t_k)]\). Claim C.1 means that we can charge the three elements added to \(B^*\) to this element \((p,t) \in A^*\) that lies in between \([t_i,t_j]\). This proves the cost bound, and hence Theorem 5.12.

In the online setting, \(B^*_1\) can be easily constructed from \(A^*_1\) using the procedure in Figure 13; and it is easy to check it is also present restricted.

D The Tiled Interval Cover Problem

In the tiled interval cover problem (TiledIC), the input is the following. For each page \(p \in [n]\), we have a collection \(I_p\) of disjoint intervals that cover the entire timeline, with each such interval having weight \(w_p\). We also have a requirement \(n - k\). The goal is to pick some set of intervals from \(I = \cup_p I_p\) that minimize their total weight, such that every time \(t\) is covered by \(n - k\) different intervals. In the version with exclusions (TiledICEx), the interval \(E_t\) ending at time \(t\) does not count towards the requirement of \(n - k\). (As always we assume that a unique interval ends at each time.)

D.1 The Offline Case

Lemma D.1. The linear relaxation for the TiledIC problem is integral, whereas the that for the TiledICEx problem has an integrality gap of at most 2.

Proof. We can even show this for the case where the weights and requirements are non-uniform, i.e., each time \(t\) has a potentially different requirement \(R_t\), and each interval has a different weight \(w_I\). Indeed, for the TiledIC problem, the constraint matrix

\[
\begin{align*}
\min_{z \in [0,1]^{|\mathcal{S}|}} & \sum_I w_I z_I \\
\sum_{I \subseteq \mathcal{S}, t \in I} z_I & \geq R_t \quad \forall t
\end{align*}
\]

has the consecutive-ones property and forms a totally-unimodular system, so the linear relaxation has integer extreme points and an optimal integer solution can be found in polynomial time.

Now let \(z\) be a solution to the LP relaxation for (note the exclusion of \(E_t\) from the sum)

\[
\begin{align*}
\min_{z \in [0,1]^{|\mathcal{S}|}} & \sum_I w_I z_I \\
\end{align*}
\]
Recall that $E_t$ is the interval ending at $t$ (though we will not need this for our solution). To construct an integer solution $S'$, first add to $S'$ all the intervals $I$ with $z_I \geq 1/2$. Now for each time $t$, let $R'_t$ be the residual coverage needed; i.e., define $R'_t := R_t - \#\{I \in S' \mid t \in I, I \not\in E_t\}$. Moreover, define $\hat{z}_I := 2z_I$ for $I \not\in S'$, and $z_I = 0$ for $I \in S'$. Clearly, $\sum_{I \in S' \mid t \not\in I, I \not\in S'} \hat{z}_I \geq 2R'_t$. Treat this as a solution to a TiledIC instance on the subcollection $S \setminus S'$ (crucially, ignoring the exclusions) with these adjusted requirements $2R'_t$, and let $S''$ be an optimal integer solution. For each time $t$, there are now $(R_t - R'_t)$ non-excluded sets in $S'$, and at least $(2R'_t - 1)^+ \geq \max(R'_t, 0)$ non-excluded sets from $S''$ covering it, which gives the desired coverage level of $R_t$. Due to the rounding up by a factor of 2, the cost of the solution is at most $2w^* \hat{z}$.

\section{The Online Case}

The online model for TiledIC and TiledICEx is that intervals are revealed online: specifically, the endpoint of an interval is revealed only when it ends (and since we are dealing with tiled instances, the next interval for that page starts immediately thereafter).

In the online case, the TiledICEx happens to be essentially identical to formulation used in online primal-dual algorithms for weighted paging, e.g., by [BBN12]. There are $n$ pages and a cache of size $k$, so these constraints say that at time $t$, there must have been $n - k$ pages apart from $p$ that are evicted since they were last requested. Hence, the intervals for a page start just after each request for the page, and end at the time of the next request. This means we can simulate the end of intervals in $I_p$ by requesting page $p$. The integer program is the following, where the page $I_t$ corresponds to the page $p_t$ requested at time $t$.

$$\min \left\{ \sum_p \sum_{I \in I_p} w_p x_I \mid \sum_{I \in I'} x_I \geq n - k \ \forall t, \ x_I \in \{0, 1\} \ \forall I \right\}.$$  

Using this connection and the result of [BBN12] immediately gives us an $O(\log k)$ randomized online algorithm for TiledICEx. To make the online model closer to the rest of the paper, let us reformulate the above IP as follows:

$$\min \left\{ \sum_{p,t} w_p x_{p,t} \mid \sum_{I \in I_p \setminus \{I_t\}} \min_{t' \in I'} x_{p,t'} \geq n - k \ \forall t, \ x_{p,t} \in \{0, 1\} \ \forall p,t \right\}.$$  

It is easy to between these two formulations, using the correspondence that at some time $t$, the variable $x_I$ has value equal to $\min(1, \sum_{t' \in I' \mid t' \leq t} x_{p_t})$. The algorithm from [BBN12] gives us an algorithm that only changes the variables at the current time $t$; hence this is clearly a past preserving algorithm.

To get an algorithm for TiledIC, we change the instance slightly: we add in a new page $p_0$ (so there are $n + 1$ pages) and make the cache of size $k + 1$. This new page has weight zero, so it can be brought in and evicted at will. Now we request page $p_0$ immediately after a request for any other page. (Denote the original request times by integers, and the requests for $p_0$ by half-integers.) Observe that at times $t - \frac{1}{2}$ when $p_0$ is requested, the constraints force $(n + 1) - (k + 1) = n - k$ “real” intervals covering time $t$ to have been chosen, which is precisely what we wanted. Now, suppose time $t$ corresponds to the interval in $I_{p_0}$ ending, and causing us to request the page $p_0$. The paging constraint then asks for $n - k$ pages except page $p_0$ to be chosen. But since page $p_0$ has zero weight, we can choose it, so we need to only choose $n - k - 1$ intervals from the rest of the pages except $\{p_0, p_t\}$. Since $p_0$ will always be chosen, this constraint is implied by the constraint at time $t - \frac{1}{2}$. So this reduction to paging exactly models the TiledIC problem, and we get an $O(\log k)$-competitive algorithm from [BBN12] again. We summarize the discussion of this section in the following lemma:
Lemma D.2. There are randomized online algorithms for the \textit{TiledIC} and \textit{TiledICEx} problems that are $O(\log k)$-competitive against oblivious adversaries.

E Proof of Theorem 5.6

Recall the Integer Program (IP-D3p):
\[
\min \sum_{p,t} w_p x_{p,t} + \sum_t L(I)y_I
\quad \text{(IP-Dp)}
\]
\[
\sum_{p: p \neq p_t} \min(1, \sum_{t' = T(p,t)} x_{p,t'}) \geq (n-k)(1-y_t) \quad \forall t
\quad \text{(D2p)}
\]

In this section, we prove the following result:

\textbf{Theorem 5.6} (Solving the LP Online). There is an $O(\log k)$-competitive online algorithm which maintains a solution to the fractional relaxation of (IP-D3p). At time $t$, the fractional solution only changes (and in fact, increases) the variables $x_{p,t}$ for all pages $p$, and $y_t$.

For sake of brevity, we rename $y_t$ as $y_t$, $L_I$ as $L_t$, $(n-k)$ as $R$, and the interval $[T(p,t), t]$ as $I(p,t)$: the only fact we use about this interval is that $I(p,t)$ always moves to the right (Claim 5.8). We can now rewrite the linear relaxation of the above IP as
\[
\min \sum_{p,t} w_p x_{p,t} + \sum_t L_t y_t
\quad \text{(LP-p)}
\]
\[
\sum_{p: p \neq p_t} \min(1, \sum_{t' = I(p,t)} x_{p,t'}) + R y_t \geq R \quad \forall t
\quad \text{(8)}
\]
\[
x_{p,t}, y_t \geq 0 \quad \forall p, t \quad \text{(9)}
\]

Observe that the cost of all $x_{p,t}$ variables corresponding to the same page $p$ is the same. If the penalty costs $L_t = \infty$ we get the hard covering problem. The following algorithm is a simple extension of a result of Bansal et al. [BBN10]; we give it here for sake of completeness.

All the variables are initialized to 0. For an interval $I$ and page $p$, let $x_{p,I}$ denote $\sum_{t \in I} x_{p,t}$. Let $\delta = \frac{1}{k+1}$. The algorithm is simple: at each time $t$, if the corresponding constraint for time $t$ is violated, then we raise some variables. Imagine this happening via a continuous process, with a clock starting at $\tau = 0$ and continuously increasing until the constraint is satisfied. Let $P_\tau = \{p \mid p \neq p_t, x_{p,I(p,t)} < 1\}$ be the pages in this constraint that are “active” at clock value $\tau$, i.e., the variables $x_{p,I(p,t)}$ are not already at their maximum value. We must have $|P_\tau| \geq k+1$, else the LHS of the constraint would have $R = n-k$ of the $x_{p,I(p,t)}$ values already at 1, and the constraint would be satisfied. Now raise the variables $x_{p,t}$ for every page $p$ in $P_\tau$ at the following rate:
\[
\frac{dx_{p,t}}{d\tau} = \frac{x_{p,I(p,t)} + \delta}{w_p}.
\]

Also,
\[
\frac{dy_t}{d\tau} = \frac{y_t R + \delta(|P_\tau| - k)}{L_t}.
\]

Note that we raise only the last variable $x_{p,t}$ for each interval $I(p, t)$, but it is raised proportional to the value of the entire interval. As these values rise, more pages fall out of the set $P_\tau$ until the constraint is satisfied.
To show the competitiveness, let \( x^* \) denote the optimal integer solution to \((\text{IP-D3}p)\) after satisfying the constraint for time \( t \). Also, the interval \( I_{p,t} \) is not defined for \( p = p_t \), we define it for the sake of analysis to be same as \( I(p,t') \) where \( t' < t \) is the most recent time such that \( p \neq p_{t'} \). Now let the potential be

\[
\Phi_t := 3 \sum_{p:x_{p,t}^* \geq 1} w_p \log \left( \frac{1 + \delta}{\min(1, x_{p,t}^* + \delta)} \right) + 3 \sum_{s \leq t: y_t^* = 1} L_s \log \left( \frac{1 + \delta}{y_s + \delta} \right).
\]

Note that each term in the potential is non-negative. We show that the amortized cost of the algorithm with respect to this potential can be paid for by the optimal cost times \( O(\log(1 + 1/\delta)) \).

First, suppose the constraint at time \( t \) is revealed. This causes the current intervals \( I(p,t) \) to possibly change, and hence some terms from the potential may disappear (since current intervals only move to the right). But dropping terms can only decrease the potential function. Next, let \( \text{OPT} \) augment its solution. Suppose \( \text{OPT} \) decides to set \( y_t^* = 1 \), then \( \text{OPT}'s \) cost is \( L_t \), whereas the potential goes up by \( 3L_t \log(\frac{1+\delta}{\delta}) \). Moreover, for each variable \( x_{p,t}^* \) that is set to 1 (there is no reason to raise any other variable), the cost to \( \text{OPT} \) is \( w_p \), whereas the potential increase is at most \( 3w_p \log(\frac{1+\delta}{\delta}) \). Hence we have

\[\Delta \Phi \leq \Delta \text{OPT} \cdot 3 \log(1 + 1/\delta).\]

Observe that since the current intervals \( I(p,t) \) only move rightwards, we charge each optimal variable only once.

Finally, the algorithm moves via the continuous process above. The instantaneous cost incurred by the algorithm is

\[
\frac{d\text{ALG}}{d\tau} = \sum_{p \in P_\tau} w_i \frac{dx_{p,t}}{d\tau} + L_t \frac{dy_t}{d\tau} \tag{10}
\]

\[
= \sum_{i \in P_\tau} (x_{p,t} + \delta) + R y_t + (|P_\tau| - k) \delta \tag{11}
\]

\[
= \sum_{p \in P_\tau} x_{p,t} + R y_t + \delta (|P_\tau| + |P_\tau| - k) \tag{12}
\]

\[
< R - (R - |P_\tau|) + \delta (|P_\tau| + |P_\tau| - k) \tag{13}
\]

\[
< (|P_\tau| - k) + 2|P_\tau| \delta. \tag{14}
\]

But since \( |P_\tau| \geq k + 1 \) and \( \delta = \frac{1}{k+1} \), we get \(|P_\tau| \delta \leq |P_\tau| - k\); this is the first time we use the value of \( \delta \). Hence

\[
\frac{d\text{ALG}}{d\tau} \leq 3(|P_\tau| - k).
\]

Finally, using the chain rule and the definition of the continuous process, the decrease in potential is:

\[
-\frac{d\Phi}{d\tau} = 3 \sum_{p \in P_\tau: x_{p,t}^* \geq 1} w_p \frac{x_{p,t}^* + \delta}{w_p} + \frac{L_t}{y_t + \delta} + 3 \frac{R y_t + (|P_\tau| - k) \delta}{L_t} \cdot 1_{y_t^* = 1}
\]

\[
\geq 3 \cdot \#\{p \in P_\tau: x_{p,t}^* \geq 1\} + 3(|P_\tau| - k) \cdot 1_{y_t^* = 1}
\]

The first equality above uses the fact that for pages in \( P_\tau, x_{p,t} \) is strictly less than 1, and so the truncation by 1 does not have any effect. Now either \( y_t^* = 1 \), in which case the second term gives us \( 3(|P_\tau| - k) \). Or else \( y_t^* = 0 \) and the second term is not present, but then \( x^* \) is a feasible solution to the covering constraint (9) at time \( t \). Therefore, at most \( n - R = k \) intervals \( I(p,t) \) are not hit by \( x^* \). So the contribution of the first term
in this case is at least $3(|P_\tau| - k)$. Putting these together, we get that $-\frac{d\Phi}{d\tau} \geq 3(|P_\tau| - k)$, and hence

$$\frac{d}{d\tau}(ALG + \Phi) \leq 0.$$ 

This shows log$(1 + 1/\delta)$-competitiveness. Using the setting of $\delta = \frac{1}{k+1}$ completes the proof of Theorem 5.6.