Complete complementarity relations and their Lorentz invariance

Marcos L. W. Basso and Jonas Maziero

Departamento de Física, Centro de Ciências Naturais e Exatas, Universidade Federal de Santa Maria, Avenida Roraima, 1000, Santa Maria, Rio Grande do Sul 97105-900, Brazil

It is well known that entanglement under Lorentz boosts is highly dependent on the boost scenario in question. For single-particle states, a spin-momentum product state can be transformed into an entangled state. However, entanglement is just one of the aspects that completely characterizes a quantum system. The other two are known as the wave-particle duality. Although the entanglement entropy does not remain invariant under Lorentz boosts, and neither do the measures of predictability and coherence, we show here that these three measures taken together, in a complete complementarity relation (CCR), are Lorentz invariant. Peres et al. (Peres et al. 2002 Phys. Rev. Lett. 88, 230402. (doi:10.1103/PhysRevLett.88.230402)) realized that even though it is possible to formally define spin in any Lorentz frame, there is no relationship between the observable expectation values in different Lorentz frames. Analogously, one can, in principle, define complementary relations in any Lorentz frame, but there is no obvious transformation law relating complementary relations in different frames. However, our result shows that the CCRs have the same value in any Lorentz frame, i.e. there is a transformation law connecting the CCRs. In addition, we explore relativistic scenarios for single and two-particle states, which helps in understanding the exchange of different aspects of a quantum system under Lorentz boosts.

1. Introduction

Entanglement is one of the most intriguing characteristics that turns apart the quantum world from the classical world. Its fundamental importance in quantum
foundations [1,2], together with its application in several areas, such as quantum information and
quantum computation [3–5], has made the entanglement theory achieve great progress in recent
decades. Moreover, there has been more and more interest in how entanglement behaves under
relativistic settings [6]. For instance, in [7] the author considered the relativistic version of the
famous Einstein–Podolsky–Rosen experiment with massive spin-1/2 particles. Czachor argued
that the degree of violation of the Bell inequality is dependent on the velocity of the particles,
leading to implications for quantum cryptography. A few years later, the authors in [8,9] showed
that the entanglement of Bell states depends on the velocity of an observer. On the other hand,
Alsing & Milburn [10] argued that the entanglement fidelity of a Bell state remains invariant for
a Lorentz-boosted observer. However, in the same year, it was demonstrated by Peres et al. [11]
that the entropy of a single massive spin-1/2 particle does not remain invariant under Lorentz
boosts. Thereafter, the behaviour of entanglement under Lorentz boosts has been receiving a lot
of attention by researchers [12–20].

As pointed out by Palge & Dunningham [21], the main aspect to be noticed here is that many
of these apparently conflicting results involve systems containing different particle states and
boost geometries. Therefore, entanglement under Lorentz boosts is highly dependent on the
boost scenario in question [22]. For single-particle states, a spin-momentum product state can be
transformed into an entangled state. Beyond that, Lorentz boosts can be regarded as controlled
quantum operations where momentum plays the role of the control system, whereas the spin can
be taken as the target qubit, as argued in [17]. This implies that Lorentz boosts perform global
transformations on single-particle systems. As in [16,18,21], by using discrete momentum states,
in this article we discuss the fact that for a spin-momentum product state to be transformed into
an entangled state it needs coherence between the momentum states. Otherwise, if the momentum
state is completely predictable, the spin-momentum state remains separable and the Lorentz
boost will at most generate superposition between the spin states. In addition, we discuss similar
results for the two-particle states under Lorentz boosts. As already noticed in [8,21], the state
and entanglement changes of the different degrees of freedom depend considerably on the initial
states involved, as well as on the geometry of the boost scenario. Whereas some states and
geometries leave the overall entanglement invariant, others create entanglement.

Besides, it is known that entanglement is just one of the aspects that completely characterizes
a quanton. (According to Lévy-Leblond [23], the term ‘quanton’ was given by M. Bunge. The
usefulness of this term is that one can refer to a generic quantum system without using
words like particle or wave.) The other two, which also are intriguing characteristics that
turn apart the quantum world from the classical world, are known as the wave-particle
duality. This distinguished aspect is generally captured, in a qualitative way, by Bohr’s
complementarity principle [24]. For instance, in the Mach–Zehnder interferometer or in the
double-slit interferometer, the wave aspect is characterized by interference fringes visibility,
meanwhile the particle nature is given by the which-way information of the path along the
interferometer. A quantitative version of the wave-particle duality was first investigated by
Wootters & Zurek [25], and later captured by a complementarity inequality in [26,27]:

$$P^2 + V^2 \leq 1,$$

where $P$ is an a priori predictability, for which the particle aspect is inferred once the quanton
is more likely to follow one path than the other and it is directly related to the probability
distribution given by the diagonal elements of the density operator, as we will discuss further
later. Besides, $V$ is the visibility of the interference pattern. Recently, several steps have been taken
towards the quantification of the wave-particle duality, with the establishment of minimal and
reasonable conditions that visibility and predictability measures should satisfy [28,29]. As well,
with the development of the field of quantum information, it was suggested that the quantum
coherence [30] would be a good generalization of the visibility measure [31–34]. Until now, many
approaches were taken for quantifying the wave-particle properties of a quantum system [35–
39]. As pointed out by Qian et al. [40], complementarity relations like equation (1.1) do not really
predict a balanced exchange between $P$ and $V$ simply because the inequality permits a decrease of $P$ and $V$ together, or an increase by both. It even allows the extreme case $P = V = 0$ to occur (neither wave or particle) while, in an experimental set-up, we still have a quanton on hand. Such a quanton cannot be nothing. Thus, one can see that something must be missing from equation (1.1). As noticed by Jakob & Bergou [41], this lack of knowledge about the system is due to entanglement, or, more generally, to quantum correlations [42]. This means that the information is being shared with another system and this kind of quantum correlation can be seen as responsible for the loss of purity of each subsystem such that, for pure maximally entangled states, it is not possible to obtain information about the local properties of the subsystems. Hence, to completely quantify a quanton, one has also to regard its correlations with other systems, such that the entire system is pure.

In this paper, we study how these different aspects of a quanton behave under Lorentz boosts. Even though entanglement entropy does not remain invariant under Lorentz boosts, and neither do measures of predictability and coherence, we show that these three measures together, in what is known as a complete complementarity relation (CCR), are Lorentz invariant. In [11], the authors showed that, even though it is possible to formally define spin in any Lorentz frame, there is no relationship between the observable expectation values in different Lorentz frames. Here the situation is different. First, one can define complementary relations in any Lorentz frame, but there is no obvious transformation law relating complementary relations in different frames. However, since the purity of a state is preserved under transformations between inertial frames, the complementary relations have the same value in any Lorentz frame, i.e. there is a transformation law connecting the CCRs. In addition, we explore several relativistic scenarios for single and two-particle states, which helps in understanding the exchange of these different aspects of a quanton under Lorentz boosts.

The organization of this article is as follows. In §2, we discuss the representations of the Poincaré group in the Hilbert space, as well as the Wigner’s little group, by focusing on spin-1/2 massive particles. In §3, we obtain CCRs for multipartite pure quantum systems, and show that CCRs are Lorentz invariant. Thereafter, in §4, we turn to the study of the behaviour of CCRs in relativistic scenarios for several single and two-particle states. Lastly, in §5, we give our conclusion.

2. Representations of the Poincaré group in the Hilbert space

One of the fundamental questions when studying the relativistic formulation of the quantum theory is how quantum states behave under Lorentz boosts. In the language of group theory, we are seeking to represent an element of the Lorentz group by a unitary operator on the Hilbert space that the quantum states belongs to. More specifically, single-particle quantum states are classified by their transformation under the inhomogeneous Lorentz group, or Poincaré group, which consists of homogeneous Lorentz transformations $\Lambda$ and translations $a$ [43]. For our discussion, we adopt the following notation: Greek indices run over the 4-spacetime coordinate labels $\{0, 1, 2, 3\}$; Latin indices run over the three spacial coordinates labels $\{1, 2, 3\}$; the Minkowski metric $\eta_{\mu \nu}$ is diagonal with elements $\{-1, 1, 1, 1\}$; 4-vectors are in un-boldfaced type while spacial vectors are represented by an arrow. For instance, the 4-momentum for a particle with mass $m$ is given by $p = (p^0, p^1, p^2, p^3) = (p^0, p)$, with norm $p^2 := p_\mu p^\mu = \eta_{\mu \nu} p^\nu p^{\mu} = -(p^0)^2 + p^2 = -m^2$, where we use natural units, i.e. $c = \hbar = 1$.

An inertial reference frame $O$ is related to another inertial frame $O'$ via a Poincaré transformation

$$x'^\mu := T(\Lambda, a)x^\nu = \Lambda^\mu_\nu x^\nu + a^\mu,$$

with $x = (x^0, x)$ being the coordinates of $O$, and similarly for $O'$. Then $T(\Lambda, a)$ induces a unitary transformation on quantum states characterized by

$$|\Psi\rangle \rightarrow U(\Lambda, a)|\Psi\rangle,$$
which satisfies the same composition rule of $T(\Lambda, a)$:

$$U(\Lambda_1, a_1) U(\Lambda_2, a_1) = U(\Lambda_1 \Lambda_2, \Lambda_1 a_2 + a_1). \quad (2.3)$$

Single-particle quantum states can be denoted by $|p\rangle \otimes |\sigma\rangle := |p, \sigma\rangle$, where $p$ labels the 4-momenta and $\sigma$ labels the spin for massive particles. The quantum states $|p, \sigma\rangle$ are eigenvectors of the momentum operator $P^\mu$ with eigenvalues $p^\mu$, i.e. $P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle$. This corresponds to a basis of plane waves and, thus, transforms under translations as $U(l, a) |p, \sigma\rangle := U(a) |p, \sigma\rangle = e^{-ipa} |p, \sigma\rangle$, where $pa := p_\mu a^\mu = p^\mu a_\mu$. Meanwhile, a general Lorentz boost $\Lambda$ takes the eigenvalue $p^\mu \rightarrow \Lambda^\mu_\nu p^\nu$, and therefore $U(\Lambda, 0) |p, \sigma\rangle := U(\Lambda) |p, \sigma\rangle$ must be a linear combination of all states with momentum $\Lambda p$, i.e.

$$U(\Lambda) |p, \sigma\rangle = \sum_{\lambda} D_{\lambda, \sigma}(\Lambda, p) |\Lambda p, \lambda\rangle. \quad (2.4)$$

As $U(\Lambda)$ is a representation, it preserves the group structure, and imposes conditions on the values of $D_{\lambda, \sigma}$. To see this, let us recall that $U(\Lambda)$ leaves $p^2 := p_\mu p^\mu = p \cdot p - E^2 = -m^2$ and the sign of $p^0 = E$ unchanged for a particle with mass $m$. Hence, we can use these two invariant quantities to classify states into specific classes. For each value of $p^2$ and for each sign ($p^0$), it is possible to choose a ‘standard’ $4$-momentum $k$ that identifies a specific class of quantum states [44]. For massive particles, we can fix the standard momentum $k$ to be the particle’s momentum in the rest frame, i.e. $k = (m, 0, 0, 0)$. Then, any momenta $p$ can be expressed in terms of the standard momentum, i.e. $p^\mu = (L(p)k)^\mu = L(p)^\mu_\nu k^\nu$, where $L(p)$ is a Lorentz transformation that depends on $p$ and takes $k \rightarrow p$. Therefore, quantum states $|p, \sigma\rangle$ can be defined in terms of the standard momentum state $|k, \sigma\rangle$:

$$|p, \sigma\rangle = U(L(p)) |k, \sigma\rangle. \quad (2.5)$$

Now, if we apply a Lorentz boost $\Lambda$ on $|p, \sigma\rangle$, then

$$U(\Lambda) |p, \sigma\rangle = U(\Lambda) U(L(p)) |k, \sigma\rangle \quad (2.6)$$

$$= U(L) U(AL(p)) |k, \sigma\rangle \quad (2.7)$$

$$= U(L(\Lambda p) L^{-1}(\Lambda p)) U(AL(p)) |k, \sigma\rangle \quad (2.8)$$

$$= U(L(\Lambda p)) U(L^{-1}(\Lambda p) AL(p)) |k, \sigma\rangle \quad (2.9)$$

$$= U(L(\Lambda p)) U(W(\Lambda, p)) |k, \sigma\rangle, \quad (2.10)$$

where $W(\Lambda, p) = L^{-1}(\Lambda p) AL(p)$ is called Wigner rotation, which leaves the standard momentum $k$ invariant, and only acts on the internal degrees of freedom of $|k, \sigma\rangle$: $k \xrightarrow{L} p \xrightarrow{A} \Lambda p \xrightarrow{L^{-1}} k$. Hence, the final momentum in the rest frame is different from the original one by a Wigner rotation, i.e. $U(W(\Lambda, p)) |k, \sigma\rangle = \sum_{\lambda} D_{\lambda, \sigma}(W(\Lambda, p)) |k, \lambda\rangle$. On the other hand, $U(L(\Lambda p))$ takes $k \rightarrow \Lambda p$ without affecting the spin, by definition. Therefore,

$$U(\Lambda) |p, \sigma\rangle = U(L(\Lambda p)) U(W(\Lambda, p)) |k, \sigma\rangle \quad (2.11)$$

$$= U(L(\Lambda p)) \sum_{\lambda} D_{\lambda, \sigma} W(\Lambda, p) |k, \lambda\rangle \quad (2.12)$$

$$= \sum_{\lambda} D_{\lambda, \sigma}(W(\Lambda, p)) |\Lambda p, \lambda\rangle. \quad (2.13)$$

It is worth mentioning that the subscripts of $D_{\lambda, \sigma}(W(\Lambda, p))$ can be suppressed, and we can write $U(\Lambda) |p, \sigma\rangle = |\Lambda p\rangle \otimes D(W(\Lambda, p)) |\sigma\rangle$. The set of Wigner rotations forms a group known as the little group, which is a subgroup of the Poincaré group [45]. In other words, under a Lorentz transformation $\Lambda$, the momenta $p$ goes to $\Lambda p$, and the spin transforms under the representation $D(W(\Lambda, p))$ of the little group $W$. For massive particles, the little group is the well-known group of rotations in three dimensions, $SO(3)$. However, it is also known that $SO(3)$ is homomorphic to $SU(2)$, and the irreducible unitary representations of $SU(2)$ span a Hilbert space of $2j + 1$ dimensions, with $j = n/2$, where $n$ is an integer [46,47]. The value of $j$ is what we usually refer
to as the spin of the massive particle. In this article, we will be interested in spin-1/2 particles, hence the representation of the Wigner rotation is given by [48,49]

$$D(W(A,p)) = \frac{(p^0 + m) \cosh(\omega/2) I_{2 \times 2} + (p \cdot \hat{e}) \sinh(\omega/2) - i \sinh(\omega/2) \sigma \cdot (p \times \hat{e})}{\sqrt{(p^0 + m)((p^0 + m)\Lambda)^0 + m}}$$

(2.14)

$$= \cos \frac{\phi}{2} I_{2 \times 2} + i \sin \frac{\phi}{2} (\sigma \cdot \hat{n}),$$

(2.15)

with $I_{2 \times 2}$ being the identity matrix, meanwhile $\sigma$ are the Pauli matrices, and

$$\cos \frac{\phi}{2} = \frac{\cosh(\omega/2) \cos(\alpha/2) + \sinh(\omega/2) \sin(\alpha/2)(\hat{e} \cdot \hat{p})}{\sqrt{\frac{1}{2}(1 + \cosh \omega \cos \alpha + \sinh \omega \sin \alpha(\hat{e} \cdot \hat{p}))}},$$

(2.16)

and

$$\sin \frac{\phi}{2} \hat{n} = \frac{\sinh(\omega/2) \sin(\alpha/2)(\hat{e} \times \hat{p})}{\sqrt{\frac{1}{2}(1 + \cosh \omega \cos \alpha + \sinh \omega \sin \alpha(\hat{e} \cdot \hat{p}))}},$$

(2.17)

where $\cosh \alpha = p^0/m$, $\omega = \tanh^{-1} v$ is the rapidity of the boost [50], $\hat{e}$ is the unit vector pointing in the direction of the boost, $p$ is the 4-momenta of the particle in $\mathcal{O}$, and $\Lambda p$ is the 4-momenta of the particle in $\mathcal{O}'$. As an example, if the momentum is in the $x$-direction of the reference frame $\mathcal{O}$ and the boost is given in the $z$-axis, then

$$D(W(A,p)) = \cos \frac{\phi}{2} I_{2 \times 2} - i \sin \frac{\phi}{2} \sigma_y = \begin{pmatrix} \cos \frac{\phi}{2} & -\sin \frac{\phi}{2} \\ \sin \frac{\phi}{2} & \cos \frac{\phi}{2} \end{pmatrix},$$

(2.18)

where the Wigner angle $\phi$ is given by

$$\cos \frac{\phi}{2} = \frac{\cosh(\omega/2) \cos(\alpha/2)}{\sqrt{\frac{1}{2}(1 + \cosh \omega \cos \alpha)}},$$

(2.19)

$$\sin \frac{\phi}{2} \hat{n} = \frac{\sinh(\omega/2) \sin(\alpha/2)\hat{y}}{\sqrt{\frac{1}{2}(1 + \cosh \omega \cos \alpha)}},$$

(2.20)

and

$$\tan \phi = \frac{\sinh \omega \sin \alpha}{\cosh \omega + \cosh \alpha},$$

(2.21)

which implies that $\phi \in [0, \pi/2]$. Hence, the transformation law for the spin-1/2 particle with momentum $p$ along the $x$-axis of $\mathcal{O}$ is given by

$$U(A)|p, 0\rangle = |\Lambda p\rangle \otimes \left( \cos \frac{\phi}{2} |0\rangle + \sin \frac{\phi}{2} |1\rangle \right)$$

(2.22)

and

$$U(A)|p, 1\rangle = |\Lambda p\rangle \otimes \left( -\sin \frac{\phi}{2} |0\rangle + \cos \frac{\phi}{2} |1\rangle \right),$$

(2.23)

where $|0\rangle$ means spin ‘up’ and $|1\rangle$ stands for spin ‘down’ along the $z$-axis. Therefore, as one can see, for separable and completely predictable states, a Lorentz boost will only generate superposition between the possible states of the spin of the particle, as already noticed in [44].

### 3. The Lorentz invariance of CCRs

In [42], we developed a general framework to obtain CCRs for a subsystem that belongs to an arbitrary multipartite pure quantum system, just by exploring the purity of the multipartite quantum system. To make our investigation easier, we begin by assuming that momenta can be treated as discrete variables [16,18,21]. This can be justified once we can consider narrow distributions centred around different momentum values such that it is possible to represent them by orthogonal state vectors, i.e. $(p_i^j | p_j^i) = \delta_{ij}$. Although narrow momenta are an idealization, it is a system worth studying since it helps to understand more realistic systems, and, also, it is possible
to approximate continuous momenta as a finite (but large) number of discrete momenta. Also, throughout this article, we will consider only massive particles of spin 1/2. By doing this, we are considering a particular representation of the Wigner little group. However, the result obtained in this section does not depend on the particular choice of representation, once the representation is unitary.

So, let us consider \( n \) massive quantons with spin 1/2 in a pure state described by \( |\psi\rangle_{A_1,\ldots,A_2n} \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2n} \) with dimension \( d = d_{A_1} d_{A_2} \cdots d_{A_{2n}} \), in the reference frame \( \mathcal{O} \). For instance, \( A_1, A_2 \) are referred to as the momentum and spin of the first quanton, and so on. By defining a local orthonormal basis for each degree of freedom (DOF) \( A_m, \{ |m\rangle_{A_m} \}_{m=1,\ldots,2n} \), the state of the multipartite quantum system can be written as [51]

\[
\rho_{A_1,\ldots,A_2n} = |\psi\rangle_{A_1,\ldots,A_2n} \langle \psi| = \sum_{i_1,\ldots,i_{2n},j_1,\ldots,j_{2n}} \rho_{i_1\ldots i_{2n},j_1\ldots j_{2n}} |i_1,\ldots,j_{2n}\rangle_{A_1,\ldots,A_{2n}} \langle j_1,\ldots,j_{2n}|_{A_1,\ldots,A_{2n}}.
\]  

(3.1)

Without loss of generality, let us consider the state of the DOF \( A_1 \), which is obtained by tracing over the other subsystems,

\[
\rho_{A_1} = \sum_{i_1,j_1} \rho_{i_1,j_1}^A |i_1\rangle_{A_1} \langle j_1|_{A_1} = \sum_{i_1,j_1} \rho_{i_1z\ldots i_{2n},j_1z\ldots j_{2n}} |i_1\rangle_{A_1} \langle j_1|_{A_1},
\]

(3.2)

for which the Hilbert–Schmidt quantum coherence and the corresponding predictability measure in terms of the density matrix elements are given by

\[
C_{hs}(\rho_{A_1}) = \sum_{i_1 \neq j_1} |\rho_{i_1,j_1}^A|^2 = \sum_{i_1 \neq j_1} \sum_{i_2,\ldots,i_{2n}} |\rho_{i_1z\ldots i_{2n},j_1z\ldots j_{2n}}|^2.
\]

(3.3)

and

\[
P_{I}(\rho_{A_1}) = \left( \sum_{i_1} |\rho_{i_1,j_1}^A|^2 - \frac{1}{d_{A_1}} \right) \left( \sum_{i_2,\ldots,i_{2n}} |\rho_{i_1z\ldots i_{2n},i_2z\ldots i_{2n}}|^2 - \frac{1}{d_{A_1}} \right).
\]

(3.4)

Besides, such a predictability measure can be first defined as \( P_{I}(\rho_{A_1}) := S_{I}^\text{max} - S_{I}(\rho_{A_1}) \) [39], where \( \rho_{A_1}\text{diag} \) corresponds to the diagonal elements of \( \rho_{A_1} \) and \( S_I(\rho) := 1 - \text{Tr}\rho^2 \) is the linear entropy. To get more intuition about this predictability measure, let us consider the projector onto the state index \( i_1 : \Pi_{i_1} := |i_1\rangle_{A_1} \langle i_1|_{A_1} \), which can be one of the paths of a Mach–Zehnder interferometer. Now, the uncertainty of the state \( i_1 \) is given by its variance \( \mathcal{V}(\rho_{A_1}, \Pi_{i_1}) = \langle \Pi_{i_1}^2 \rangle - \langle \Pi_{i_1} \rangle^2 = \rho_{i_1}^A - (\rho_{i_1})^2 \) such that sum of the uncertainties of all the possible states (or paths) is given by \( \sum_{i_1} \mathcal{V}(\rho_{A_1}, \Pi_{i_1}) = 1 - \sum_{i_1} (\rho_{i_1})^2 \), which represents the total uncertainty of all states. One can easily see that \( \sum_{i_1} \mathcal{V}(\rho_{A_1}, \Pi_{i_1}) \) is exactly the linear entropy of \( \rho_{A_1}\text{diag} : S_I(\rho_{A_1}) = 1 - \text{Tr}\rho_{A_1}^2 \). Thus, after repeating the same experiment several times, we obtain a probability distribution given by \( \rho_{00}^A, \ldots, \rho_{d_{A_1}-1,d_{A_1}-1}^A \), which represents a probability of the quanton being measured in the state \( |0\rangle, \ldots, |d_{A_1}-1\rangle \). From this, it is possible to calculate the uncertainty about the paths through \( \rho_{A_1}\text{diag} \) such that \( P_{I}(\rho_{A_1}) := S_{I}^\text{max} - S_{I}(\rho_{A_1}) \) offers a measure of the capability to predict what outcome will be obtained in the next run of the experiment. For instance, if we obtain a uniform probability distribution, i.e., \( \rho_{i_1}^A = 1/d_{A_1} \), then \( P_{I}(\rho_{A_1}) = 0 \). Therefore, it is possible to see that equation (3.4) is a way of quantifying how much the probability distribution expressed by \( \rho_{A_1}\text{diag} \) differs from the uniform probability distribution. While the Hilbert–Schmidt quantum coherence can be defined as \( C_{hs}(\rho_{A_1}) := \min_{t \in \mathcal{T}} ||\rho_{A_1} - t||_{hs}^2 \), where \( \mathcal{T} \) is the set of all incoherent states (diagonal density operators), and the Hilbert–Schmidt’s norm of a matrix \( M \in \mathbb{C}^{d_{A_1} \times d_{A_1}} \) is defined as \( ||M||_{hs} := \sqrt{\sum_{j,k} |M_{jk}|^2} \). The minimization procedure yields equation (3.3). Therefore, the Hilbert–Schmidt quantum coherence is measuring how distant the density operator \( \rho_{A_1} \) is in comparison with its closest incoherent state, which in this case is \( \rho_{A_1}\text{diag} \), given the Hilbert–Schmidt’s norm. Loosely speaking, the quantum coherence is measuring ‘how much’ orthogonal
superposition is encoded in a given density operator $\rho_A$ and it is directly related to the off-diagonal elements of $\rho_A$. Besides, we showed in [39] that these are bona-fide measures of visibility and predictability, respectively. From these equations, an incomplete complementarity relation, $P_h(\rho_A) + C_h(\rho_A) \leq (d_A - 1)/d_A$, is obtained by exploring the mixedness of $\rho_A$, i.e. $1 - \text{Tr}\rho_A^2 \geq 0$.

Now, since $\rho_{A_1,\ldots,A_{2n}}$ is a pure quantum system, then $1 - \text{Tr}\rho_{A_1,\ldots,A_{2n}}^2 = 0$, or equivalently,

$$1 - \left( \sum_{(i_1,\ldots,i_{2n})=(j_1,\ldots,j_{2n})}^{(i_1,\ldots,i_{2n})\neq(j_1,\ldots,j_{2n})} + \sum_{(i_1,\ldots,i_{2n})\neq(j_1,\ldots,j_{2n})} \right) |\rho_{i_1i_2\ldots i_{2n}j_1\ldots j_{2n}}|^2 = 0,$$

(3.5)

where

$$\sum_{(i_1,\ldots,i_{2n})\neq(j_1,\ldots,j_{2n})} = \sum_{i_1 \neq j_1} + \sum_{i_1 = j_1} + \sum_{i_2 \neq j_2} + \sum_{i_2 = j_2} + \cdots + \sum_{i_{2n} \neq j_{2n}} + \sum_{i_{2n} = j_{2n}} .$$

(3.6)

The purity condition (3.5) can be rewritten as a CCR:

$$P_l(\rho_A) + C_h(\rho_A) + S_l(\rho_A) = \frac{d_A - 1}{d_A_A},$$

(3.7)

with $S_l(\rho_A)$ being the linear entropy of the subsystem $A_1$ given by

$$S_l(\rho_A) := \sum_{i_1 \neq j_1} \sum_{(i_2,\ldots,i_{2n})\neq(j_2,\ldots,j_{2n})} \left( |\rho_{i_1i_2\ldots i_{2n}j_1\ldots j_{2n}}|^2 - \rho_{i_1i_2\ldots i_{2n}j_1\ldots j_{2n}}^* \rho_{i_1i_2\ldots i_{2n}j_1\ldots j_{2n}} \right).$$

(3.8)

It is worthwhile mentioning that the CCR in equation (3.7) is a natural generalization of the complementarity relation obtained by Jakob & Bergou [52,53] for bipartite pure quantum systems. More generally, $E = \sqrt{2S_l(\rho_A)}$, where $E$ is the generalized concurrence obtained in [54] for multiparticle pure states. Now, for the boosted observer $O^\prime$ of $\S 2$, the same $n$ massive quantons system is described by $|\Psi_A^*\rangle_{A_1,\ldots,A_{2n}} = U(\Lambda)|\Psi_A\rangle_{A_1,\ldots,A_{2n}}$, and the density matrix of the multiparticle pure quantum system can be written as [55,56]

$$\rho_{A_1,\ldots,A_{2n}} = |\Psi_A\rangle_{A_1,\ldots,A_{2n}} \langle \Psi_A | = U(\Lambda)\rho_{A_1,\ldots,A_{2n}} U^\dagger(\Lambda),$$

(3.9)

which implies that $\text{Tr}(\rho_{A_1,\ldots,A_{2n}}^2) = \text{Tr}(\rho_{A_1,\ldots,A_{2n}}^2)$, and the whole system remains pure under the Lorentz boost. As we used the purity of the density matrix to obtain the CCR, then, from $1 - \text{Tr}(\rho_{A_1,\ldots,A_{2n}}^2) = 0$, we can obtain

$$P_l(\rho_{A_1}^A) + C_h(\rho_{A_1}^A) + S_l(\rho_{A_1}^A) = \frac{d_A - 1}{d_A_A},$$

(3.10)

This proves our claim that CCRs are invariant under Lorentz transformations. For continuous momenta, the result shown here remains valid if applied to the discrete degrees of freedom with the continuous momenta traced out, since we defined $P_l$, $C_h$, and $S_l$ only for the discrete degrees of freedom. Therefore, one can see that there is a transformation law connecting the CCRs for different Lorentz frames.

4. Relativistic settings

(a) Single-particle system scenarios

We begin by considering three different single-particle states where the particle is moving in two opposing directions along the $y$-axis and the spins are aligned with the $z$-axis irrespective of the
direction of the boost in the reference frame $O$:

\[
|\Psi\rangle = \frac{1}{\sqrt{2}}(|p\rangle + |-p\rangle) \otimes |0\rangle, \quad (4.1)
\]

\[
|\Sigma\rangle = \frac{1}{\sqrt{2}}(|p,0\rangle + |-p,1\rangle) \quad (4.2)
\]

and

\[
|\Phi\rangle = \frac{1}{2}(|p\rangle + |-p\rangle) \otimes (|0\rangle + |1\rangle), \quad (4.3)
\]

where $|0\rangle$ means spin ‘up’, and $|1\rangle$ spin ‘down’. In addition, $|-p\rangle$ is describing the state whose spatial momentum has opposite direction in comparison with $|p\rangle$. It is worthwhile mentioning that the states $|\Psi\rangle$, $|\Phi\rangle$ are separable states, while $|\Sigma\rangle$ is a maximal entangled state. Moreover, the state $|\Psi\rangle$ has maximal coherence in the momentum DOF, and maximal predictability in the spin DOF. Meanwhile $|\Phi\rangle$ is maximally coherent in both degrees of freedom, and $|\Sigma\rangle$ has no local properties. Now, let us consider an observer $O'$ boosted with velocity $v$ in a direction orthogonal to the momentum of the particle in the frame $O$, i.e. in the $x-z$ plane, making an angle $\theta \in [0, \pi/2]$ with the $x$-axis. Hence, the direction of boost is given by $\hat{\mathbf{v}} = \cos \theta \hat{x} + \sin \theta \hat{z}$, and the Wigner rotation follows directly:

\[
D(W(\Lambda, \pm p)) = \cos \frac{\phi}{2} I_{2 \times 2} + i \sin \frac{\phi}{2} (\mp \sin \theta \sigma_x \pm \cos \theta \sigma_z)
\]

\[
= \begin{pmatrix}
\cos \frac{\phi}{2} \pm i \sin \frac{\phi}{2} \cos \theta & \mp i \sin \frac{\phi}{2} \sin \theta \\
\mp i \sin \frac{\phi}{2} \sin \theta & \cos \frac{\phi}{2} \mp i \sin \frac{\phi}{2} \cos \theta
\end{pmatrix},
\]

since $\mp \hat{\mathbf{v}} = \mp \hat{\mathbf{y}}$. Therefore, the observer in $O'$ assigns in general a different state to the same system. For instance, the state given by equation (4.1) in $O'$ is described by

\[
|\Psi_{\Lambda}\rangle = U(\Lambda) |\Psi\rangle = \frac{1}{\sqrt{2}} ((|Ap\rangle \otimes D(W(\Lambda, p)) |0\rangle + |-Ap\rangle \otimes D(W(\Lambda, -p)) |0\rangle)
\]

\[
= \frac{1}{\sqrt{2}} \left[ |Ap\rangle \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right) |0\rangle - i \sin \frac{\phi}{2} \sin \theta |1\rangle \right] \\
+ |-Ap\rangle \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \cos \theta \right] |0\rangle + i \sin \frac{\phi}{2} \sin \theta |1\rangle),
\]

which in general is an entangled state. The reduced density matrix of the spin (momentum) is obtained by tracing out the momentum (spin) states:

\[
\rho_{\Lambda s} = \text{Tr}_{Ap} |\Psi_{\Lambda}\rangle \langle \Psi_{\Lambda}| = \begin{pmatrix}
\cos \frac{\phi}{2} + \sin \frac{\phi}{2} \cos^2 \theta & - \sin \frac{\phi}{2} \sin \theta \cos \theta \\
- \sin \frac{\phi}{2} \sin \theta \cos \theta & \sin \frac{\phi}{2} \sin^2 \theta
\end{pmatrix}, \quad (4.8)
\]

and

\[
\rho_{\Lambda p} = \text{Tr}_{As} |\Psi_{\Lambda}\rangle \langle \Psi_{\Lambda}| = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} (\cos \phi + i \sin \phi \cos \theta) \\
\frac{1}{2} (\cos \phi - i \sin \phi \cos \theta) & \frac{1}{2}
\end{pmatrix}, \quad (4.9)
\]

In figure 1, we plotted the different aspects of the degrees of freedom of the quanton for different values of $\theta$. For instance, if there is no boost, i.e. $\phi = 0$, the state remains unchanged, regardless of the direction of the boost. Also, if the boost is along the $x$-axis, $\theta = 0$, the state remains the same. Now, if the boost is along the $z$-axis, the entanglement between the moment and the spin of the particle increases with the increase of the Wigner angle. In exchange, the coherence of the momentum and the predictability of the spin decrease with $\phi$. Beyond that, for any $\theta, \phi \in [0, \pi/2]$, the CCR $P_{lz} + C_{lz} + S_{l} = 1/2$ is always satisfied.
Figure 1. The different aspects of the degrees of freedom in the state $|\Psi_A\rangle$ for different values of $\theta$. (a) $S(\rho_{AA}) = S(\rho_{AB})$ as a function of the Wigner angle, (b) $P(\rho_{AA})$ as a function of the Wigner angle, (c) $Ch_\phi(\rho_{AA})$ as a function of the Wigner angle, (d) $Ch_\phi(\rho_{AB})$ as a function of the Wigner angle. (Online version in colour.)

Now, the state $|\Xi\rangle$ given by equation (4.2) is described in $O'$ as

$$|\Xi_A\rangle = \frac{1}{\sqrt{2}} (|\Lambda p\rangle \left[ \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right)|0\rangle - i \sin \frac{\phi}{2} \sin \theta |1\rangle \right]$$
$$+ |\Lambda p\rangle \left[ i \sin \frac{\phi}{2} \sin \theta |0\rangle + \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \cos \theta \right)|1\rangle \right]),$$

while the reduced density matrices are given by

$$\rho_{AS} = \rho_{AB}^\dagger = \begin{pmatrix} \frac{1}{2} & i \cos \frac{\phi}{2} \sin \frac{\phi}{2} \sin \theta \\ -i \cos \frac{\phi}{2} \sin \frac{\phi}{2} \sin \theta & \frac{1}{2} \end{pmatrix}.$$ (4.11)

In this example, by inspecting figure 2, if there is no boost, i.e. $\phi = 0$, the state remains unchanged, regardless of the direction of the boost. Also, if the boost is along the $x$-axis, $\theta = 0$, the state remains the same. However, for $\theta \in (0, \pi/2]$ and $\phi \neq 0$, there is an increase of the coherence of both degrees of freedom, in exchange of the consumption of the entanglement between the momenta and spin of the particle. In the extreme case where $\theta = \pi/2$ and $\phi \to \pi/2$, both degrees of freedom have maximal coherence and the state $|\Xi_A\rangle$ becomes separable

$$|\Xi_A\rangle_{\phi=\pi/2; \phi=\pi/2} = \frac{1}{2} (|\Lambda p\rangle + i |\Lambda p\rangle) \otimes (|0\rangle - i |1\rangle).$$ (4.12)

Lastly, in the boosted frame $O'$, the state $|\Phi\rangle$ given by equation (4.3) is described by

$$|\Phi_A\rangle = \frac{1}{2} \left( |\Lambda p\rangle \left[ \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} (\cos \theta - \sin \theta) \right)|0\rangle + \left( \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} (\cos \theta + \sin \theta) \right)|1\rangle \right]$$
$$+ |\Lambda p\rangle \left[ \left( \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} (\cos \theta - \sin \theta) \right)|0\rangle + \left( \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} (\cos \theta + \sin \theta) \right)|1\rangle \right]),$$ (4.13)
with the reduced density matrices being

\[
\rho_{As} = \begin{pmatrix}
\frac{1}{2}(\cos^2 \phi - \sin^2 \phi \cos 2\theta) & \frac{1}{2}(\cos^2 \phi - \sin^2 \phi \cos 2\theta) \\
\frac{1}{2}(\cos^2 \phi - \sin^2 \phi \cos 2\theta) & \frac{1}{2}(\cos^2 \phi - \sin^2 \phi \cos 2\theta)
\end{pmatrix}
\] (4.15)

and

\[
\rho_{Ap} = \begin{pmatrix}
\frac{1}{2}(\cos \phi + i \sin \phi \sin \theta) & \frac{1}{2}(\cos \phi - i \sin \phi \sin \theta) \\
\frac{1}{2}(\cos \phi - i \sin \phi \sin \theta) & \frac{1}{2}(\cos \phi + i \sin \phi \sin \theta)
\end{pmatrix}
\]. (4.16)

In contrast to the second example, here the entanglement between momentum and spin increases with the Wigner angle, in exchange of the consumption of the coherence of both degrees of freedom. However, for \(\theta = \pi/2\) the state is separable:

\[
|\Phi_A\rangle_{\phi=\pi/2} = \frac{1}{2}(|Ap\rangle + A^*|\Lambda p\rangle) \otimes (|0\rangle + |1\rangle),
\] (4.17)

where \(\Lambda = \cos(\phi/2) - i \sin(\phi/2)\) and \(A^*\) is the complex conjugate of \(A\). The coherence and entropy of the momentum have the same qualitative behaviour as for the spin, which is plotted in figure 3.

From these examples, we can see that if the state of the system is separable, and has no superposition between the momentum states in the reference frame \(O\), then a Lorentz boost cannot generate entanglement between the momentum and spin degrees of freedom. This result helps to explain the reported generation of entanglement between momenta and spin in one of the first studies of single-particle systems carried out by Peres et al. [11], where they considered a particle with mass \(m\) whose momentum wave function in the rest frame is given by \(\psi(p) = (2\pi)^{-3/4}w^{3/2}e^{-p^2/2w^2}\), with \(w\) being the width of the wave packet. This is a Gaussian state of

\[
Figure 2. The different aspects of the degrees of freedom of the quanton in the state \(|\Sigma_\Lambda\rangle\) for different values of \(\theta\). (a) \(S_i(\rho_{\Lambda s})\), \(j = s, p\), as a function of the Wigner angle, (b) \(C_{hs}(\rho_{\Lambda s})\), as a function of the Wigner angle. (Online version in colour.)
\]

\[
Figure 3. The different aspects of the spin of the quanton in the state \(|\Phi_\Lambda\rangle\) for different values of \(\theta\). (a) \(S_i(\rho_{\Lambda s})\) as a function of the Wigner angle, (b) \(P(\rho_{\Lambda s})\) as a function of the Wigner angle, (c) \(C_{hs}(\rho_{\Lambda s})\) as a function of the Wigner angle. (Online version in colour.)
\]
minimum uncertainty, but still represents a continuous superposition. Therefore, in this case it is possible to generate entanglement via a Lorentz transformation.

(b) Two-particle system scenarios

As shown in §4a, if the momentum states have no coherence, then it is not possible to generate entanglement between the momenta and spin degrees of freedom. In this section, we begin by discussing this issue for the two-particle case and end this section giving two examples. Now, let us consider a two-particle state described in $\mathcal{O}$ as

$$|\Psi\rangle_{A,B} = \sum_{\sigma,\lambda} \psi_{\sigma} \psi_{\lambda} |p, q\rangle_{A,B} \otimes |\sigma, \lambda\rangle_{A,B},$$  

(4.18)

where $\sum_{j} |\psi_{j}|^2 = 1$ for $j = \sigma, \lambda$. In addition, $|p, q\rangle_{A,B} = |p\rangle_{A} \otimes |q\rangle_{B}$ denotes the momentum state of particles A and B, respectively, meanwhile $|\sigma, \lambda\rangle_{A,B} = |\sigma\rangle_{A} \otimes |\lambda\rangle_{B}$ represents the state of the spins of the particles A and B. The state $|\Psi\rangle_{A,B}$ is separable and has no coherence between momentum states in the reference frame $\mathcal{O}$. Now, in the boosted frame $\mathcal{O}'$, we have $|\Psi\rangle_{A,B} = U(A)|\Psi\rangle_{A,B}$, i.e.

$$|\Psi\rangle_{A,B} = \sum_{\sigma,\lambda} \psi_{\sigma} \psi_{\lambda} |Ap, Aq\rangle_{A,B} \otimes D(W(A,p))|\sigma\rangle_{A} \otimes D(W(A,q))|\lambda\rangle_{B}$$

(4.19)

$$= |Ap, Aq\rangle_{A,B} \otimes \sum_{\sigma} \psi_{\sigma} D(W(A,p))|\sigma\rangle_{A} \otimes \sum_{\lambda} \psi_{\lambda} D(W(A,q))|\lambda\rangle_{B},$$

(4.20)

which is also separable. In this case, the Wigner rotation will only change the coherences of the spin states of the particles A and B. Now, let us consider a state in $\mathcal{O}$ with superposition in the momentum states of particle A

$$|\Phi\rangle_{A,B} = \sum_{p} \psi_{p} |p, q\rangle_{A,B} \otimes |\sigma, \lambda\rangle_{A,B},$$

(4.21)

with $\sum_{p} |\psi_{p}|^2 = 1$. Then, a Lorentz boost can generate entanglement between the momentum and spin of particle A

$$|\Phi\rangle_{A,B} = \sum_{p} \psi_{p} |Ap, Aq\rangle_{A} \otimes D(W(A,p))|\sigma\rangle_{A} \otimes |Aq\rangle_{B} \otimes D(W(A,q))|\lambda\rangle_{B}.$$  

(4.22)

However, there is no entanglement between particles A and B. Similarly, if we consider that the state in $\mathcal{O}$ has coherence in the momentum states of A and B, there will be no entanglement between particles A and B. To obtain an entangled state of the whole system in $\mathcal{O}'$, we have to consider a state in $\mathcal{O}$ already entangled in the momentum degrees of freedom, i.e.

$$|\Xi\rangle_{A,B} = \sum_{p,q} \psi_{p,q} |p, q\rangle_{A,B} \otimes |\sigma, \lambda\rangle_{A,B},$$

(4.23)

with $\sum_{p,q} |\psi_{p,q}|^2 = 1$. Hence, in boosted frame $\mathcal{O}'$ we have

$$|\Xi\rangle_{A,B} = \sum_{p,q} \psi_{p,q} |Ap, Aq\rangle_{A,B} \otimes D(W(A,p))|\sigma\rangle_{A} \otimes D(W(A,q))|\lambda\rangle_{B}.$$  

(4.24)

For instance, if we consider the particles moving in two opposing directions along the $y$-axis and the spins are aligned with the $z$-axis irrespective of the direction of the boost in the reference frame $\mathcal{O}$,

$$|\Xi\rangle_{A,B} = \frac{1}{\sqrt{2}}(|p, -p\rangle + |-p, p\rangle) \otimes \langle 0, 0|,$$

(4.25)

as before, the observer $\mathcal{O}'$ is boosted with velocity $v$ in a direction orthogonal to the momentum of particle in the frame $\mathcal{O}$, i.e. in the $x-z$ plane, making an angle $\theta \in [0, \pi/2]$ with the
The states of particles A and B are given by \( \rho \), which represents the behaviour of the different aspects of the spin of particle A. The aspects of the spin of particle B display similar behaviour. It is worth emphasizing that it is not the momentum–momentum density matrix of the particles A and B that increases, but the entanglement of the spins of the particles is the bipartite coherence of the reduced momentum–momentum density matrix of the particles A and B. In [8], the authors discussed the generation of spin–spin entanglement between two particles under Lorentz boosts. Meanwhile, the momentum states of particles A and B are given by \( \rho^A_{\Lambda p} = \rho^B_{\Lambda p} = \frac{1}{2}(|\Lambda p, -\Lambda p\rangle\langle \Lambda p, -\Lambda p| + |-\Lambda p, \Lambda p\rangle\langle -\Lambda p, \Lambda p|) \). In addition, figure 4a,b and c represent the behaviour of the different aspects of the spin of particle A. The aspects of the spin of particle B display similar behaviour. It is worth emphasizing that it is not the entanglement between spin–spin that increases, but the entanglement of the spin of one of the particles with all the other degrees of freedom. In [8], the authors discussed the generation of spin–spin entanglement for two particles under Lorentz boosts. Meanwhile, the momentum–momentum density matrix of the particles A and B is described by

\[
|\Sigma_{AB}\rangle = \frac{1}{\sqrt{2}}(|\Lambda p, -\Lambda p\rangle + |-\Lambda p, \Lambda p\rangle) \otimes \left[ \left( \cos^2 \frac{\phi}{2} + \sin^2 \frac{\phi}{2} \cos^2 \theta \right) |0, 0\rangle + \sin^2 \frac{\phi}{2} \sin^2 \theta |1, 1\rangle - \sin^2 \frac{\phi}{2} \sin \theta \cos \theta (|0, 1\rangle + |1, 0\rangle) \right] + \frac{i \sin(\phi/2) \sin(\phi/2) \sin \theta}{\sqrt{2}} (|\Lambda p, -\Lambda p\rangle - |-\Lambda p, \Lambda p\rangle) \otimes (|0, 1\rangle - |1, 0\rangle),
\]

which is an entangled state between all degrees of freedom. In this case, the resource consumed to generate entanglement of the spins of the particles is the bipartite coherence of the reduced momentum–momentum density matrix of the particles A and B.

\[
\rho_{\Lambda p, \Lambda p} = \frac{1}{2}(|\Lambda p, -\Lambda p\rangle\langle \Lambda p, -\Lambda p| + |-\Lambda p, \Lambda p\rangle\langle -\Lambda p, \Lambda p| + t.c.),
\]

where t.c. stands for transpose conjugated. In figure 4d, we plotted the coherence and the linear entropy of \( \rho_{\Lambda p, \Lambda p} \) as a function of \( \phi \), where \( S_l(\rho_{\Lambda p, \Lambda p}) \) is measuring the entanglement of \( \rho_{\Lambda p, \Lambda p} \) as a whole with the rest of the degrees of freedom. Meanwhile, the concurrence measure [57] \( E \) of \( \rho_{\Lambda p, \Lambda p} \) decreases monotonically with the Wigner angle, which means the momentum–momentum entanglement decreases with \( \phi \), once \( E(\rho_{\Lambda p, \Lambda p}) = 2C_{ls}(\rho_{\Lambda p, \Lambda p}) \). In addition, figure 4a,b and c represent the behaviour of the different aspects of the spin of particle A. The aspects of the spin of particle B display similar behaviour. It is worth emphasizing that it is not the entanglement between spin–spin that increases, but the entanglement of the spin of one of the particles with all the other degrees of freedom. In [8], the authors discussed the generation of spin–spin entanglement for two particles under Lorentz boosts. Meanwhile, the momentum states of particles A and B are given by \( \rho^A_{\Lambda p} = \rho^B_{\Lambda p} = \frac{1}{2}(|\Lambda p, \Lambda p\rangle\langle \Lambda p| + |-\Lambda p, \Lambda p\rangle\langle -\Lambda p|) \), which implies that \( S_l(\rho^j_{\Lambda p}) = 1/2, j = A, B \), and the overall entanglement between the momentum of particle A (B) with the rest of the system does not change under the Lorentz boost, even though the

- Figure 4. The different aspects of \( |\Sigma_{A}\rangle \) for different values of \( \theta \). (a) \( S_l(\rho_{\Lambda A}) \) as a function of the Wigner angle, (b) \( P(\rho_{\Lambda A}) \) as a function of the Wigner angle, (c) \( C_{ls}(\rho_{\Lambda A}) \) as a function of the Wigner angle, (d) \( C_{ls}(\rho_{\Lambda A, \Lambda p}) \) as a function of the Wigner angle. (Online version in colour.)
The reduced spin density matrices of each particle are given by

$$\rho \text{ and } \rho_A, \rho_B$$

with the momentum of the particles along the

$$\rho \text{ and } \rho_A, \rho_B$$

the spin of particle A (B) increases, as shown in figure 5

A (B) with the rest of the system decreases with the Wigner angle. In exchange, the coherence of

and B, once

coherence increases, since it is related to the momentum–momentum entanglement of particles A

as a whole with the spins, therefore the entanglement of momentum–momentum

has to decrease.

By contrast, we also can redistribute entanglement to generate coherence in the spin states of

particles A and B. For instance, let us consider the following two-particle states in

O

has to decrease.

The different aspects of $|\gamma_A\rangle$. (a) $C_{as}(\rho_A), S(I)\rho_A$ as a function of the Wigner angle, (b) $C_{as}(\rho_{AP,AP}), S(I)\rho_A$ as a function of the Wigner angle. (Online version in colour.)

entanglement of momentum–momentum decreases. Hence, in this case, the entanglement of

the momentum of particle A (B) is redistributed among the others degrees of freedom. For $\phi = 0$ (no boost), just the momentum of the particles are entangled. In the limit $\phi = \pi/2$, the momentum of

the particles are entangled with the spins, therefore the entanglement of momentum–momentum

entanglement of momentum–momentum decreases. Hence, in this case, the entanglement of

the momentum of particle A (B) is redistributed among the others degrees of freedom. For $\phi = 0$ (no boost), just the momentum of the particles are entangled. In the limit $\phi = \pi/2$, the momentum of

the particles are entangled with the spins, therefore the entanglement of momentum–momentum

has to decrease.

By contrast, we also can redistribute entanglement to generate coherence in the spin states of

particles A and B. For instance, let us consider the following two-particle states in $O$

$|\gamma_A\rangle_{AB} = \frac{1}{\sqrt{2}} (|p, 0\rangle_A \otimes |-p, 1\rangle_B + |p, 1\rangle_A \otimes |p, 0\rangle_B)$

$$= \frac{1}{\sqrt{2}} (|p, -p\rangle_{AB} \otimes |0, 1\rangle_{AB} + |p, p\rangle_{AB} \otimes |1, 0\rangle_{AB}), \quad (4.28)$$

with the momentum of the particles along the y-axis. Now, for a boosted frame $O'$ along the z-axis, the Wigner rotation is given by equation (4.4) imposing $\theta = \pi/2$. Hence,

$|\gamma_A\rangle_{AB} = \frac{1}{\sqrt{2}} [i \cos \frac{\phi}{2} \sin \frac{\phi}{2} (|Ap, -Ap\rangle + |-Ap, Ap\rangle) \otimes (|0, 0\rangle - |1, 1\rangle)]$

$$+ \frac{1}{\sqrt{2}} |Ap, -Ap\rangle \otimes \left( \cos^2 \frac{\phi}{2} |0, 1\rangle + \sin^2 \frac{\phi}{2} |1, 0\rangle \right)$$

$$+ \frac{1}{\sqrt{2}} |-Ap, Ap\rangle \otimes \left( \sin^2 \frac{\phi}{2} |0, 1\rangle + \cos^2 \frac{\phi}{2} |1, 0\rangle \right). \quad (4.29)$$

The reduced spin density matrices of each particle are given by

$$\rho^A_{as} = \rho^B_{as} = \begin{pmatrix} \frac{1}{2} & i \cos \frac{\phi}{2} \sin \frac{\phi}{2} \\ -i \cos \frac{\phi}{2} \sin \frac{\phi}{2} & \frac{1}{2} \end{pmatrix}, \quad (4.30)$$

and $\rho^A_{AP} = \rho^B_{AP} = \frac{1}{2} I_{2 \times 2}$, where $I_{2 \times 2}$ is the identity matrix. The entanglement of the spin of particle A (B) with the rest of the system decreases with the Wigner angle. In exchange, the coherence of

the spin of particle A (B) increases, as shown in figure 5a. In addition, the entanglement of $\rho_{AP, AP}$ as a whole with the spins of the particles also decreases with $\phi$. From figure 5b, the bipartite coherence increases, since it is related to the momentum–momentum entanglement of particles A and B, once $E(\rho_{AP, AP}) = \sqrt{2}C_{as}(\rho_{AP, AP})$, as we also can see from

$$\rho_{AP, AP} = \frac{1}{2} \left( |Ap, -Ap\rangle \langle Ap, -Ap| + |-Ap, Ap\rangle \langle -Ap, Ap| \right)$$

$$+ \left( 2 \cos^2 \frac{\phi}{2} \sin^2 \frac{\phi}{2} |Ap, -Ap\rangle \langle -Ap, Ap| + t.c. \right). \quad (4.31)$$
Hence, the entanglement of the momentum of particle A (B) with the rest of the degrees of the system remains the same under the Lorentz boost, although it is shuffled around among the degrees of freedom. For instance, when $\phi = 0$ (no boost), the momentum of particle A is entangled with all other degrees of freedom. However, in the limit $\phi = \pi/2$, the momentum of particle A is entangled just with the momentum of particle B, since

$$|\mathcal{Y}_{\phi=\pi/2}\rangle_{A,B} = \frac{1}{\sqrt{2}} \left( (|A\rangle - |\Lambda\rangle)_{A,B} + (|\Lambda\rangle - |A\rangle)_{A,B} \right) \otimes \frac{1}{\sqrt{2}} \left( (|0\rangle_A + |1\rangle_A) \otimes (|0\rangle_B - i|1\rangle_B) \right), \quad (4.32)$$

and $S_{\lambda}(\rho_{AP\Lambda P}) = 0$ for $\phi = \pi/2$.

5. Conclusion

Although the entanglement entropy does not remain invariant under Lorentz boosts, and neither do the measures of predictability and coherence, we showed in this work that these three measures taken together, in a CCR, are Lorentz invariant. Even though it is possible to formally define spin in any Lorentz frame, there is no relationship between the observable expectation values in different Lorentz frames, according to Peres et al. [11]. Here the situation is quite different. First, it is possible to formally define complementarity in any Lorentz frame and, in principle, there is no relationship between the complementarity relations in different Lorentz frames. However, our results showed that it is possible indeed to connect CCRs in different Lorentz frames. Therefore, we showed how the connection between the CCRs defined in different Lorentzian frames is possible, and disclosed interesting aspects of the redistribution of quantum features for the relativistic dynamics of one- and two-particle states and how the role of CCRs helps to keep track of how this redistribution of quantum features is done.

Data accessibility. This article has no additional data.

Authors’ contributions. M.L.W.B. and J.M. conceived and developed the project, discussed the results and contributed to the manuscript. M.L.W.B. made the calculations, proofs and plots and wrote the first version of the manuscript. J.M. critically revised the manuscript. Finally, both authors gave final approval for publication and agree to be held accountable for the work performed therein.

Competing interests. We declare we have no competing interests.

Funding. This work was supported by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), process 88882.427924/2019-01, and by the Instituto Nacional de Ciência e Tecnologia de Informação Quântica (INCT-IQ), process 465469/2014-0.

References

1. Schrödinger E. 1935 Discussion of probability relations between separated systems. Math. Proc. Camb. Phil. Soc. 31, 555–563. (doi:10.1017/S0305004100013554)
2. Horodecki R, Horodecki P, Horodecki M, Horodecki K. 2009 Quantum entanglement. Rev. Mod. Phys. 81, 865. (doi:10.1103/RevModPhys.81.865)
3. Popescu S. 1994 Bell’s inequalities versus teleportation: What is nonlocality?. Phys. Rev. Lett. 72, 797. (doi:10.1103/PhysRevLett.72.797)
4. Preskill J. 2000 Quantum information and physics: some future directions. J. Mod. Opt. 47, 127. (doi:10.1080/09500340008244031)
5. Cavalcanti PSD, Šupić I. 2017 All entangled states can demonstrate nonclassical teleportation. Phys. Rev. Lett. 119, 110501. (doi:10.1103/PhysRevLett.119.110501)
6. Terno DR. 2006 Introduction to relativistic quantum information. (http://arxiv.org/abs/quant-ph/0508049)
7. Czachor M. 1997 Einstein-Podolsky-Rosen-Bohm experiment with relativistic massive particles. Phys. Rev. A 55, 72. (doi:10.1103/PhysRevA.55.72)
8. Gingrich RM, Adami C. 2002 Quantum entanglement of moving bodies. Phys. Rev. Lett. 89, 270402. (doi:10.1103/PhysRevLett.89.270402)
9. Terashima H, Ueda M. 2003 Relativistic Einstein-Podolsky-Rosen correlation and Bell’s inequality. Int. J. Quantum Inform. 1, 93. (doi:10.1142/S0219749903000061)
10. Alsing PM, Milburn GJ. 2002 Lorentz invariance of entanglement. Quantum Inf. Comput. 2, 487.
11. Peres A, Scudo PF, Terno DR. 2002 Quantum entropy and special relativity. *Phys. Rev. Lett.* **88**, 230402. (doi:10.1103/PhysRevLett.88.230402)

12. Gingrich R, Bergou A, Adami C. 2003 Entangled light in moving frames. *Phys. Rev. A* **68**, 042102. (doi:10.1103/PhysRevA.68.042102)

13. Li H, Du J. 2003 Relativistic invariant quantum entanglement between the spins of moving bodies. *Phys. Rev. A* **68**, 022108. (doi:10.1103/PhysRevA.68.022108)

14. Moon YH, Hwang SW, Ahn D. 2004 Relativistic entanglements of Spin 1/2 particles with general momentum. *Prog. Theor. Phys.* **112**, 219–240. (doi:10.1143/PTP.112.219)

15. Lee D, Chang-Young E. 2004 Quantum entanglement under Lorentz boost. *New J. Phys.* **6**, 67. (doi:10.1088/1367-2630/6/1/067)

16. Jordan TF, Shaji A, Sudarshan ECG. 2007 Lorentz transformations that entangle spins and entangle momenta. *Phys. Rev. A* **75**, 022101. (doi:10.1103/PhysRevA.75.022101)

17. Dunningham J, Palge V, Vedral V. 2009 Entanglement and nonlocality of a single relativistic particle. *Phys. Rev. A* **80**, 044302. (doi:10.1103/PhysRevA.80.044302)

18. Friis N, Bertlmann RA, Huber M, Hiesmayr BC. 2010 Relativistic entanglement of Spin 1/2 particles with general momentum. *Phys. Rev. A* **81**, 042114. (doi:10.1103/PhysRevA.81.042114)

19. Esfahani BN, Aghae M. 2011 Relativistic entanglement for spins and momenta of a massive three-particle system. *Int. J. Quantum Inf.* **9**, 1255–1265. (doi:10.1142/S0219749911007897)

20. V. Bittencourt VAS, Bernardini AE, Blasone M. 2018 Effects of Lorentz boosts on Dirac bispinor entanglement. *J. Phys.: Conf. Ser.* **1071**, 012001. (doi:10.1088/1742-6596/1071/1/012001)

21. Palge V, Dunningham J. 2015 Entanglement of two relativistic particles with discrete momenta. *Ann. Phys.* **363**, 275. (doi:10.1016/j.aop.2015.09.028)

22. Palge V, Dunningham J. 2012 Generation of maximally entangled states with sub-luminal Lorentz boost. *Phys. Rev. A* **85**, 042322. (doi:10.1103/PhysRevA.85.042322)

23. Lévy-Leblond J-M. 2003 On the nature of quantons. *Sci. Educ.* **12**, 495. (doi:10.1023/A:1025382113814)

24. Bohr N. 1928 The quantum postulate and the recent development of atomic theory. *Nature* **121**, 580–590. (doi:10.1038/121580a0)

25. Wootters WK, Zurek WH. 1979 Complementarity in the double-slit experiment: quantum nonseparability and a quantitative statement of Bohr’s principle. *Phys. Rev. D* **19**, 473. (doi:10.1103/PhysRevD.19.473)

26. Englert B-G. 1996 Fringe visibility and which-way information: an inequality. *Phys. Rev. Lett.* **77**, 2154. (doi:10.1103/PhysRevLett.77.2154)

27. Greenberger DM, Yasin A. 1988 Simultaneous wave and particle knowledge in a neutron interferometer. *Phys. Lett. A* **128**, 391–394. (doi:10.1016/0375-9601(88)90114-4)

28. Dür S. 2001 Quantitative wave-particle duality in multibeam interferometers. *Phys. Rev. A* **64**, 042113. (doi:10.1103/PhysRevA.64.042113)

29. Englert B-G, Kaszlikowski D, Kwek LC, Chee WH. 2008 Wave-particle duality in multi-path interferometers: general concepts and three-path interferometers. *Int. J. Quantum Inf.* **6**, 129–157. (doi:10.1142/S0219749908003220)

30. Baumgratz T, Cramer M, Plenio MB. 2014 Quantifying coherence. *Phys. Rev. Lett.* **113**, 140401. (doi:10.1103/PhysRevLett.113.140401)

31. Bera MN, Qureshi T, Siddiqui MA, Pati AK. 2015 Duality of quantum coherence and path distinguishability. *Phys. Rev. A* **92**, 012118. (doi:10.1103/PhysRevA.92.012118)

32. Bagan E, Bergou JA, Cottrell SS, Hillery M. 2016 Relations between coherence and path information. *Phys. Rev. Lett.* **116**, 160406. (doi:10.1103/PhysRevLett.116.160406)

33. Qureshi T. 2019 Coherence, interference and visibility. *Quanta* **8**, 24. (doi:10.12743/quantav8i1.87)

34. Mishra S, Venugopalan A, Qureshi T. 2019 Decoherence and visibility enhancement in multi-path interference. *Phys. Rev. A* **100**, 042122. (doi:10.1103/PhysRevA.100.042122)

35. Angelo RM, Ribeiro AD. 2015 Wave-particle duality: an information-based approach. *Found. Phys.* **45**, 1407–1420. (doi:10.1007/s10701-015-9913-6)

36. Coles PJ. 2016 Entropic framework for wave-particle duality in multipath interferometers. *Phys. Rev. A* **93**, 062111. (doi:10.1103/PhysRevA.93.062111)

37. Bagan E, Calsamiglia J, Bergou JA, Hillery M. 2018 Duality games and operational duality relations. *Phys. Rev. Lett.* **120**, 050402. (doi:10.1103/PhysRevLett.120.050402)

38. Roy P, Qureshi T. 2019 Path predictability and quantum coherence in multi-slit interference. *Phys. Scr.* **94**, 095004. (doi:10.1088/1402-4896/ab1cd4)
39. Basso MLW, Chrysosthemos DSS, Maziero J. 2020 Quantitative wave-particle duality relations from the density matrix properties. Quantum Inf. Process. 19, 254. (doi:10.1007/s11128-020-02753-y)

40. Qian XF, Konthasinghe K, Manikandan SK, Spiecker D, Vamivakas AN, Eberly JH. 2020 Turning off quantum duality. Phys. Rev. Res. 2, 012016. (doi:10.1103/PhysRevResearch.2.012016)

41. Jakob M, Bergou JA. 2010 Quantitative complementarity relations in bipartite systems: entanglement as a physical reality. Opt. Commun. 283, 827–830. (doi:10.1016/j.optcom.2009.10.044)

42. Basso MLW, Maziero J. 2020 Complete complementarity relations for multipartite pure states. J. Phys. A: Math. Theor. 53, 465301. (doi:10.1088/1751-8121/abc361)

43. Weinberg S. 1995 The quantum theory of fields I. Cambridge, UK: Cambridge University Press.

44. Lanzagorta M. 2014 Quantum information in gravitational fields. Williston, VT: Morgan & Claypool Publishers.

45. Ohnuki Y. 1988 Unitary Representations of the Poincaré group and Relativistic Wave Equations. Singapore: World Scientific.

46. Sexl RU, Urbantke HK. 2001 Relativity, groups, particles: special relativity and relativistic symmetry in field and particle physics. New York, NY: Springer.

47. Tung W-K. 1985 Group theory in physics. Philadelphia, PA: World Scientific.

48. Ahn D, Lee H, Moon YH, Hwang SW. 2003 Relativistic entanglement and Bell’s inequality. Phys. Rev. A 67, 012103. (doi:10.1103/PhysRevA.67.012103)

49. Halpern FR. 1968 Special relativity and quantum mechanics. Hoboken, NJ: Prentice-Hall.

50. Rhodes JA, Semon MD. 2004 Relativistic velocity space. Wigner rotation and Thomas precession. Am. J. Phys. 72, 943–960. (doi:10.1119/1.1652040)

51. Bergou JA, Hillery M. 2013 Introduction to the theory of quantum information processing. New York, NY: Springer.

52. Jakob M, Bergou JA. 2007 Complementarity and entanglement in bipartite Qudit systems. Phys. Rev. A 76, 052107. (doi:10.1103/PhysRevA.76.052107)

53. Jakob M, Bergou JA. 2006 Generalized complementarity relations in composite quantum systems of arbitrary dimensions. Int. J. Mod. Phys. B 20, 1371–1381. (doi:10.1142/S0217979206033851)

54. Bhaskara VS, Panigrahi PK. 2017 Generalized concurrence measure for faithful quantification of multiparticle pure state entanglement using Lagrange’s identity and wedge product. Quantum Inf. Process. 16, 118. (doi:10.1007/s11128-017-1568-0)

55. Caban P, Rembielinski J. 2005 Lorentz-covariant reduced spin density matrix and EPR-Bohm correlations. Phys. Rev. A 72, 012103. (doi:10.1103/PhysRevA.72.012103)

56. Debarba T, Vianna RO. 2012 Quantum state of a free spin-1/2 particle and the inextricable dependence of spin and momentum under Lorentz transformations. Int. J. Quantum Info. 10, 1230003. (doi:10.1142/S0219749912300033)

57. Wootters WK. 1998 Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245–2248. (doi:10.1103/PhysRevLett.80.2245)