Geometric Formulation of Nonlinear Quantum Mechanics for Density Matrices

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Abstract

Proposals for nonlinear extensions of quantum mechanics are discussed. Two different concepts of “mixed state” for any nonlinear version of quantum theory are introduced: (i) genuine mixture corresponds to operational “mixing” of different ensembles, and (ii) a mixture described by single density matrix without having a canonical operational possibility to pick out its specific convex decomposition is called here an elementary mixture. Time evolution of a class of nonlinear extensions of quantum mechanics is introduced. Evolution of an elementary mixture cannot be generally given by evolutions of components of its arbitrary convex decompositions. The theory is formulated in a “geometric form”: It can be considered as a version of Hamiltonian mechanics on infinite dimensional space of density matrices. A quantum interpretation of the theory is sketched.

1 Introduction

In popular and well written book [1] on conceptual foundations of quantum mechanics (QM) there is a subsection [Chap. 9-4, p.278] on nonlinear Schrödinger equation inserted into the section entitled Some impossible processes, and containing a proof of inconsistency of any nonlinear evolution in QM “if we retain the other postulates of quantum theory without any change”. These “other postulates” include statistical interpretation and time evolution of density matrices corresponding to some of our genuine mixtures, as well as the definition of entropy by von Neumann which is appropriate just to our elementary mixtures.

*The talk presented at the conference: “New Insights in Quantum Mechanics”, Goslar, Aug. 31st – Sept. 4th, 1998.
(cf. Subsection 2.1). The nonlinear extensions of QM appearing usually in literature [3, 4, 5] are formulated for time evolution of vector states (i.e. wave functions); trials to extend such dynamics to evolutions of density matrices led to inconsistencies, cf. e.g. criticism [5] of [3].

We argue that (at least some of) the inconsistencies mentioned above are consequences of improper interpretation of “mixtures” and of definition of their time–evolution. We shall introduce two concepts of “mixed states” in this contribution, as well as a class of nonlinear extensions of dynamics avoiding the mentioned inconsistencies. Our formulation of nonlinear quantum theory [6] was inspired by symplectic reformulation of QM [7], and was encouraged by publication of the Weinberg’s proposal [3].

2 Two concepts of “mixture” and nonlinearity in QM

In standard elementary formulation of QM (linear, without superselection rules, describing systems with “finite number of degrees of freedom”) the set of bounded “observables” is described by the set $\mathcal{L}(\mathcal{H})_s$ of all bounded selfadjoint operators in a complex Hilbert space $\mathcal{H}$, and the set of “states” is described by the set of density matrices $\mathcal{S}_* := T(\mathcal{H})_{+1}$ consisting of positive trace–class operators of unit trace. Any density matrix $\varrho \in \mathcal{S}_*$ is either of the form

\[ \varrho = P_{\varphi} = P_{\varphi}^2 = P_{\varphi}^* \in \mathcal{L}(\mathcal{H})_s, \quad \varphi \in \mathcal{H}, \]  

(1)

corresponding to a “pure state”, or it is a nontrivial convex combination of one dimensional projections $P_{\varphi}$. Each density matrix corresponding to a nontrivial mixture, i.e. $\varrho^2 \neq \varrho$, can be written in infinitely many mutually different ways in the form of its convex decompositions:

\[ \varrho = \sum_j \lambda_j \varrho_j = \left( \sum_k \kappa_k \nu_k \right), \quad \varrho_j, \nu_k \in T(\mathcal{H})_{+1}, \quad \lambda_j, \kappa_k \in [0, 1) \subset \mathbb{R}_+. \]  

(2)

The density matrices of the form (2) represent “mixed states”, resp. “mixtures”.

Each such a state is uniquely determined by determination of expectations of “sufficiently many” observables $A \in \mathcal{L}(\mathcal{H})_s$, i.e. by numbers

\[ \langle A \rangle_\varrho := Tr(\varrho \cdot A), \quad \forall A \in \mathcal{L}(\mathcal{H})_s. \]  

(3)

Such a set of state–determining expectations corresponding to a nontrivial mixture (2) can arise in QM in two different experimental situations corresponding to two possible interpretations of density matrices, cf. Subsection 2.1. These two possibilities are not empirically distinguishable in the standard framework of QM, but their distinction becomes crucial, as we shall show in this paper, in any nonlinear generalization of the theory.
2.1 Two concepts of mixed states

Let us distinguish here two kinds of preparation procedures for states described in QM by a density matrix $\rho$, and let us also introduce the corresponding two concepts of “mixed states”.

(i). A state of a given system in QM described by a density matrix $\rho \neq \rho^2$ is often interpreted as representation of the statistical ensemble $\{\rho\}$ of copies of the considered physical system consisting of subensembles numbered by an index $j \in J (\equiv \text{an index set})$, each of which is prepared by its own macroscopically distinguishable preparation procedure preparing the system in a state $\rho_j \in \mathcal{S}$, occurring in $\{\rho\}$ with probability $\lambda_j$. Hence the term “mixture” corresponds here to intuitive notion of “mixing” of a set of copies of the system occurring in states $\rho_j$ with relative frequencies $\lambda_j$. There are infinitely many of such mutually different (possibly quantummechanically mutually incompatible) preparation procedures leading to the same density matrix $\rho$ which are in the framework of QM indistinguishable [1] by measurements of the quantities (3).

Each of the “components” $\rho_j$ such that $\rho_j \neq \rho_j^2$ could be decomposed further into “less mixed” components, $\rho_j = \sum_k \lambda_{jk} \rho_{jk}$, etc. Let the components $\rho_j$ of the decomposition corresponding to the described preparation procedure of the ensemble $\{\rho\}$ be empirically indecomposable, i.e. they are elementary mixtures in the sense of the next point (ii). Then the state corresponding to the ensemble $\{\rho\}$ will be called a genuine mixture. It is characterized by a specific convex combination of elementary mixtures: $\{\rho\} \equiv \{\rho\}_\lambda := \{\lambda_j; \rho_j : j \in J\}$. Other such decompositions of the same density matrix will correspond to different genuine mixtures (empirically distinguishable in some generalizations of QM, cf. later in this paper).

(ii). Let a density matrix $\rho \in \mathcal{S}$ associated with the considered system $S$ be given. It is always possible (at least mathematically) to find another QM–system $S'$ and a vector state $\Phi \in \mathcal{H} \otimes \mathcal{H}'$ of the combined system $S + S'$ such that its restriction to the subsystem $S$ is $\rho$:

$$Tr(P_\Phi \cdot A \otimes I_{\mathcal{H}'}) \equiv Tr(\rho \cdot A), \quad \forall A \in \mathcal{L}(\mathcal{H})_s,$$

where $I_{\mathcal{H}'} \in \mathcal{L}(\mathcal{H}')$ is the unit observable of the attached system $S'$ (called sometimes an “ancilla”).

Let us assume that the state $\rho$ of $S$ is obtained by such a restriction from an empirically prepared state $P_\Phi$ of a system $S+S'$, the observables of which contain all elements of $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}')_s$ (hence, no superselection rules). The systems $S$ and $S'$ are dynamically independent. The state $\rho$ of $S$ (and the corresponding statistical ensemble of copies of $S$) prepared in this way will be called an elementary mixture. These states are a special case of genuine mixtures corresponding to the trivial decomposition of $\rho$ into a single element.

This second method of preparation of a state $\rho$ do not specify any decomposition of $\rho$ into “simpler”, or “purer” states. We need not specify in the following a way of preparation of an elementary mixture; in the forthcoming
considerations, any elementary mixture \( \rho \) will be a “full fledged” description of a quantum state, in an equal degree for nontrivial mixtures \( \rho^2 \neq \rho \), as well as for pure states. From the point of view of physical intuition, the elementary mixtures are distinguished by impossibility to specify their decomposition to subensembles determined by their physical preparation.

**Remark 1** We shall see how nonlinear quantum dynamics can distinguish different genuine mixtures with the same “barycentre” \( \rho \) by measuring the quantities in (3) only. Detailed specification of conditions under which we are dealing either with “genuine”, or with “elementary” mixture will be actual only after observing some nonlinear quantum evolution: It is also a question for experimentalists. Here we stress just theoretical possibilities and their connections.

These different possible interpretations of density matrices are usually ignored because of their experimental indistinguishability in (linear) QM.\(^1\) They could be distinguished by considering also states of preparation apparatuses, i.e. by considering correlations of states of \( S \) with some “macroscopic parameters” (we shall not go into details of this point here, cf., however, Remark 2).

**Remark 2** The above physically intuitive determination of difference between the two concepts of “mixed states” can be made mathematically clear after accepting an extended set of “observables” for the considered system \( S \). Let the \( C^* \)-algebra of observables be the set \( C_*(S_*, L(H)) \) of continuous (in some conveniently specified topologies) functions \( \alpha : \nu \mapsto \alpha(\nu) \in L(H) \) on “conveniently” compactified \( S_* \), \( S_* \ni \nu \), with bounded operator values. The centre of this \( C^* \)-algebra contains classical quantities consisting of the commutative \( C^* \)-algebra of scalar–valued continuous functions \( \alpha \in C(S_*) \). The elementary mixtures are then just the pure states of this commutative subalgebra. Intuitively, this extension of the set of observables corresponds to the extension of the physical system \( S \) by macroscopic parameters of (preparation procedures of) its (microscopic) states.

### 2.2 Nonlinear transformations distinguish the two kinds of mixed states

The above mentioned indistinguishability of different decompositions (3) of any given mixture described by a density matrix \( \rho \) is conserved by linear time evolutions (or any unitary transformations)

\[
\varphi_t(\rho) := u_t \rho u_t^*, \quad u_t \equiv \exp(-itH).
\]

We have then for all \( t \in \mathbb{R} \):

\[
\varphi_t(\rho) = \sum_j \lambda_j \varphi_t(\rho_j) = \sum_k \kappa_k \varphi_t(\nu_k), \quad \varphi_{t=0}(\rho) \equiv \rho.
\]

\(^1\)The only situations where the present author noticed a discussion of these different interpretations were works on quantum measurement problem.\(^2\) The genuine mixtures are called there also Gemenge, and the interpretation: ignorance interpretation.
Conversely, if a continuous group of invertible transformations \( \varphi_t \) of \( S_\ast \) satisfies (6) for all decompositions (2) of all density matrices \( \varrho \), then it is described by some unitary operators \( u_t \equiv \exp(-itH) \) [Theorem 3.2.8, and Example 3.2.14]. In these cases, determination of time evolution for pure states is sufficient to determine evolution of all states uniquely.

Let us assume that \( \varphi_t \) is a nonlinear family of transformations of \( S_\ast \), i.e. (6) is no more identically valid. Hence there is \( \varrho \) and its decompositions (2) such that for a “nonlinear \( t \)”:

\[
(\varphi_t(\varrho) \neq \lambda_j \varphi_t(\varrho_j) \neq \sum_k \kappa_k \varphi_t(\nu_k)).
\]

(7)

In other words, the mappings \( \varphi_t : S_\ast \mapsto S_\ast \) are not all affine now, and the evolution of components \( \varrho_j \) of a decomposition (2) of a density matrix \( \varrho \) need not determine a “corresponding” evolution of \( \varrho \). Evolutions of elementary mixtures have to be specified on the whole space of quantum states \( S_\ast \). Moreover, eq. (7) shows that nonlinear evolution can distinguish between different genuine mixtures corresponding to the same initial density matrix.

3 Nonlinear Extensions of Quantum Dynamics

Let us specify here a class of evolutions generalizing the linear ones (5). We shall ignore here problems with unboundedness of generators. After defining an evolution of all elementary mixtures\(^2\), we shall stress its distinction from the corresponding evolutions for genuine mixtures. We shall mention briefly the statistical interpretation of the generalized quantum theory. More details on the theory can be found in [6].

3.1 Dynamics on the quantum phase space

Let us consider the space \( S_\ast \) of all density matrices as the quantum phase space. The dynamics on \( S_\ast \) is just Hamiltonian (better: Poisson) classical dynamics on the infinite-dimensional submanifold (with boundary) \( S_\ast \) of the linear space \( \mathcal{T}_s(\mathcal{H}) \) (\( \equiv \) symmetric trace class) endowed with the trace norm and with the Poisson brackets

\[
\{f, h\}(\varrho) \equiv iTr(\varrho[D_\varrho f, D_\varrho h]), \quad f, h \in C^\infty(\mathcal{T}_s(\mathcal{H}), \mathbb{R}).
\]

(8)

Here \( D_\varrho f \) is the differential of \( f \in C^\infty(\mathcal{T}_s(\mathcal{H}), \mathbb{R}) \) considered (in a canonical way\(^3\), cf. Remark [3]) as a bounded linear operator on \( \mathcal{H} \), and \( [\cdot , \cdot] \) is the commutator in \( \mathcal{L}(\mathcal{H}) \).

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\(^2\)Similar dynamics for density matrices was formulated by Czachor et al. [4] in terms of Lie–Nambu brackets.
Remark 3

The correspondence between $D_\varrho f$ and operators in $L(H)_s$ is given by the duality between the space of selfadjoint trace class operators $\mathcal{T}_s(H)$ ($\ni A, B$) representing (real-)linear functionals $\nu \mapsto \langle B; \nu \rangle \in \mathbb{R}$ on $\mathcal{T}_s(H)$ so that $\langle B; \nu \rangle := \text{Tr}(B \cdot \nu)$. Hence the differential $D_\varrho f$ calculated on the vector $\nu$ according to its definition as a linear functional on $\mathcal{T}_s(H)$, can be represented by the operator $\tilde{D}_\varrho f \in L(H)_s$ determined by the relation

$$D_\varrho f(\nu) \equiv \langle D_\varrho f ; \nu \rangle := \frac{d}{dt} \bigg|_{t=0} f(\varrho + tv) \equiv \text{Tr}(\nu \cdot \tilde{D}_\varrho f).$$  \hspace{1cm} (9)

Being this valid for all $\nu \in \mathcal{T}_s(H)$, the operator $\tilde{D}_\varrho f$ is determined unambiguously by (9). We shall use $\tilde{D} \equiv D$ in the following.

Let $Q \in C^\infty(\mathcal{T}_s(H), \mathbb{R})$ be a Hamiltonian for our dynamics (for “more realistic” cases this function $Q$ is not everywhere defined and it is unbounded) \cite{6}. Then the Hamiltonian flow $\varphi^Q_t$ can be described \cite{6} by unitary cocycle $u_Q$ consisting of the set of unitary operators $\{u_Q(t, \varrho) : t \in \mathbb{R}, \varrho \in S_1\}$ on $H$ satisfying a version of the nonlinear Schrödinger equation:

$$i \frac{d}{dt} u_Q(t, \varrho) = D_{\varphi^Q_t} Q \cdot u_Q(t, \varrho), \quad u_Q(0, \varrho) \equiv 0;$$  \hspace{1cm} (10)

and fulfilling the cocycle identity:

$$u_Q(t + s, \varrho) = u_Q(s, \varphi^Q_t(\varrho))u_Q(t, \varrho), \quad \forall s, t, \in \mathbb{R}.$$  \hspace{1cm} (11)

Here enters the nonlinearity via

$$\varrho(t) \equiv \varphi^Q_t(\varrho) := u_Q(t, \varrho)\varrho u_Q(t, \varrho)^*, \quad \varrho(0) := \varrho.$$  \hspace{1cm} (12)

The equation (10) leads to the evolution equation written directly for $\varrho(t)$:

$$i \frac{d}{dt} \varrho(t) = [D_{\varphi^Q_t} Q, \varrho(t)].$$  \hspace{1cm} (13)

Let us rewrite eq. (10) into a common form of nonlinear Schrödinger equation for wave functions $\psi(t) := u_Q(t, P_\psi) \psi \in H$ (we set $D_\psi \equiv D_{P_\psi}$):

$$i \frac{d}{dt} \psi(t) = D_{\psi(t)} Q \cdot \psi(t).$$  \hspace{1cm} (14)

It is seen from (12), and from unitarity of $u_Q(t, \varrho)$ that the evolution $\varphi^Q_t$ leaves spectral characteristics of all density matrices invariant. Hence it conserves also purity of states. If the evolution were known for vector states only, evolution for density matrices would remain undefined. In specific cases, it is possible \cite{10} to find a “natural extension” of the Hamiltonian $Q$ from the subset $P(H) \subset S_*$ of vector states to other parts of $S_*$; mathematical guides for such an extension might be, e.g. continuity, symmetry, or some aesthetical, resp. formal considerations.
Example 1 Let us take, e.g. $H=\mathcal{L}^2(\mathbb{R}^n)$ with $\langle \psi|\varphi \rangle := \int \overline{\psi}(x)\varphi(x)\,d^n x$. Let us write density matrices $\rho$ “in the $x$–representation” with a help of their operator kernels $\rho(x,y)$:

$$[\rho \psi](x) \equiv \int \rho(x,y)\psi(y)\,d^n y, \quad \psi \in \mathcal{H}. \quad \quad (15)$$

Projection operators $P_\psi$ have the kernels $P_\psi(x,y) \equiv \|\psi\|^2\overline{\psi}(x)\psi(y)$. Let the Hamiltonian function $Q$ will be taken as the (unbounded) functional

$$Q(P_\psi) := \text{Tr}(P_\psi \cdot H_0) + \frac{\varepsilon}{\alpha + 1} \int P_\psi(x,x)^{\alpha+1} d^n x, \quad \quad (16)$$

with $H_0$ some selfadjoint (linear) operator on $\mathcal{L}^2(\mathbb{R}^n)$, and $\alpha > 0$. Let $t \mapsto P_\psi(t), \psi(0) := \psi$ be any differentable curve through $P_\psi \in P(\mathcal{H})$, and let $\dot{P}_\psi \in T_{P_\psi} P(\mathcal{H})$ be its tangent vector expressed by an operator according to Remark 3. Then the (unbounded, nonlinear) Hamiltonian $D_\psi Q$ can be expressed by:

$$\text{Tr}(D_\psi Q \cdot \dot{P}_\psi) := \frac{d}{dt} \bigg|_{t=0} Q(P_\psi(t)), \quad \quad (17)$$

what leads to the corresponding form of “nonlinear Schrödinger wave–equation” for $\psi_t := \psi(t)$:

$$i \left[ \frac{d}{dt} \psi_t \right](x) = [H_0 \psi_t](x) + \varepsilon |\psi_t(x)|^{2\alpha} \psi_t(x), \quad \|\psi_t\| = 1. \quad \quad (18)$$

One possible extension of this nonlinear dynamics to the whole space $\mathcal{S}_*$ is obtained by “the substitution $\rho \mapsto P_\psi$”, i.e. by the choice of the Hamiltonian

$$Q(\rho) := \text{Tr}(\rho \cdot H_0) + \frac{\varepsilon}{\alpha + 1} \int \rho(x,x)^{\alpha+1} d^n x, \quad \quad (19)$$

and the corresponding dynamics is then described by (13) with

$$D_\rho Q(\nu) \equiv \text{Tr}(\nu \cdot H_0) + \varepsilon \int \rho(x,x)^\alpha \nu(x,x)\,d^n x. \quad \quad (20)$$

We shall compare in the next subsection the evolutions of the mixed states described by the same initial density matrix (2) for the two distinguished interpretations.

3.2 Genuine mixtures, their dynamics and interpretation

Since genuine mixtures $\{\rho\} := \{\rho\}_\lambda := \{\lambda_j; \rho_j \} : j \in J$ are considered as probability distributions over the space of “elementary events” $\mathcal{S}_*$ (in the sense of

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3We shall proceed, in the presented “Examples”, in a heuristic way, by “plausible” formal manipulations; the necessary mathematical comments are omitted here.
the classical Kolmogorov probability theory) with probabilities $\lambda_j$ concentrated in the points $\varrho_j \in S_*$, adequate description of genuine mixtures are probability measures $\mu_{\{\varrho\}}$ on $S_*$. Let us denote $\delta_\varrho$ the Dirac probability measures: $\delta_\varrho(\Lambda) = 0 \iff \varrho \notin \Lambda \subset S_*$. The measure $\mu_{\{\varrho\}}$ describing the state–decomposition $\{\varrho\}$ describes the integral of functions $f \in L^1(S_*,\mu_{\{\varrho\}})$:

$$\mu_{\{\varrho\}}(f) = \sum_j \lambda_j \delta_{\varrho_j}(f) = \sum_j \lambda_j f(\varrho_j).$$

(21)

In this language, the elementary mixtures might be considered as the special case of genuine ones, in which the decomposition $\{\varrho\}$ of $\varrho$ according to (2) is trivial: $\lambda_1 := 1$, $\varrho_1 := \varrho$, $\lambda_j \equiv 0$ ($\forall j \neq 1$).

If there is given a Hamiltonian time evolution $\varphi^Q_t$ on $S_*$, the evolution of the state described by $\mu_{\{\varrho\}}$ can be described by the corresponding evolution of the distinguished decomposition:

$$\{\varrho\} := \{(\lambda_j; \varrho_j) : j \in J\} \rightarrow \{\varrho\}_t := \{(\lambda_j; \varphi^Q_t(\varrho_j)) : j \in J\}.$$ (22)

This evolution is described equivalently by the evolution of measures $\mu_t$ on $S_*$:

$$(t; \mu) \mapsto \mu_t \equiv \mu \circ \varphi^Q_{-t}. (23)$$

Let us illustrate, by explicit calculation, the difference between time evolutions of the same initial density matrix considered in its two different interpretations.

**Example 2** Let us take the system with its “extended” dynamics from Example 4, and let us fix a nontrivial mixture $\varrho$ of several vector states $P_{\psi_j}$: $\varrho = \sum_j \lambda_j P_{\psi_j}$. Let us calculate the difference between the derivatives with respect to the time in $t = 0$ of the two evolutions: (i) of the genuine mixture evolution $\sum_j \lambda_j \varphi^Q_t(P_{\psi_j})$, and (ii) of the elementary mixture evolution $\varphi^Q_t(\varrho)$. We shall calculate the right side of (4) for the two cases and take their difference. Let us write the kernel “in $x$–representation” of $\varrho$ as the convex combination of the vector–state kernels:

$$\varrho(x, y) = \sum_j \lambda_j \|\psi_j\|^2 \psi_j(x) \bar{\psi}_j(y).$$ (24)

The (symbolic) “kernel” of the Hamiltonian $D_\varrho Q$ can be written:

$$D_\varrho Q(x, y) = H_0(x, y) + \varepsilon \delta(x - y) \varrho(x, x)^n.$$ Here, $\delta(\cdot)$ is the Dirac distribution on $R^n$. We have to express the difference $\Delta_t^{\{\varrho\}}(x, y)$ between the kernels (in $x$–representation) of the operators

$$\sum_j \lambda_j [D_{\psi_j(t)}Q, P_{\psi_j}], \text{ and } [D_{\varrho(t)}Q, \varrho(t)],$$

(25)
what expresses the difference between time derivatives of “the same density matrix” \( \rho = \sum \lambda_j P_{\psi_j} \) in the two interpretations. The linear operator \( H_0 \) does not contribute into this difference. The kernels of commutators entering into the calculation are (for all \( \nu \in \mathcal{S}_\ast \)) of the form

\[
[D, Q, \nu](x, y) = [H_0, \nu](x, y) + \varepsilon \nu(x, y)(\nu(x, x)^\alpha - \nu(y, y)^\alpha).
\]

We can (and we shall) take all \( \|\psi_j\| \equiv 1 \). Let us denote

\[
\chi_{\{\rho\}}^j(x) := |\psi_j(x)|^{2\alpha} - \left( \sum_k \lambda_k |\psi_k(x)|^2 \right)^\alpha.
\]

Then the wanted difference at \( t = 0 \) is

\[
\Delta_{\{\rho\}}(x, y) := \Delta_{\{\rho\}}^0(x, y) = \varepsilon \sum_j \lambda_j \psi_j(x) \overline{\psi}_j(y)(\chi_{\{\rho\}}^j(x) - \chi_{\{\rho\}}^j(y)). \tag{25}
\]

By proving that the operator \( \Delta_{\{\rho\}} \) is not identical zero for all \( \{\rho\} \), we can prove nontrivial difference of the two time evolutions explicitly. This can be easily proved for \( \lambda_1 := 1 - \lambda_2 \), and \( \psi_1, \psi_2 \) specific two–valued functions concentrated on disjoint subsets of \( \mathbb{R}^n \).

**Remark 4** Our proposal for abstract interpretation scheme of the theory looks as follows:

The set of observables is \( \mathcal{C}_s(\mathcal{S}_\ast, \mathcal{L}(\mathcal{H})) \) (cf. Remark 3). Their expectations in the states \( \mu_{\{\rho\}} \) are

\[
\langle \hat{a} \rangle_{\{\rho\}} \equiv \sum_j \lambda_j \text{Tr}(\rho_j a(\rho_j)). \tag{26}
\]

The higher momenta are \( \langle \hat{a}^k \rangle_{\{\rho\}} \), \( k = 2, 3, \ldots \), what allows us to calculate probability distributions of (also microscopic) observables. The “classical” distribution of elementary mixtures composing a genuine one can be given by an arbitrary (not only discrete) measure \( \mu \) on \( \mathcal{S}_\ast \) (endowed by a Borel structure \( \mathcal{B} \)). Then we have

\[
\langle \hat{a} \rangle_\mu = \int_{\mathcal{S}_\ast} \text{Tr}(\rho a(\rho)) \mu(d\rho). \tag{27}
\]

The system is here described as a composition of a QM–system on \( \mathcal{H} \), and a classical system with phase space \( \mathcal{S}_\ast \). The above considered nonlinear dynamics of the quantum system can be described as linear dynamics of this composed system \( \mathcal{H} \).

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