Helmholtz equation in unbounded domains: some convergence results for a constrained optimization problem

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Abstract

We consider a constrained optimization problem arising from the study of the Helmholtz equation in unbounded domains. The optimization problem provides an approximation of the solution in a bounded computational domain. In this paper we prove some estimates on the rate of convergence to the exact solution.

1 Introduction

In this paper, we consider a constrained optimization problem which arises from the computational study of wave propagation in unbounded domains. We are interested in a classical scattering problem, which can be stated as follows. Let \( D \subset \mathbb{R}^d, \ d \geq 2, \) be a bounded domain and let \( u \) be the solution of

\[
\begin{cases}
\Delta u + k^2 u = 0, & \text{in } \mathbb{R}^d \setminus D, \\
u = f, & \text{on } \partial D, \\
\lim_{r \to +\infty} r^{\frac{d-1}{2}} \left( \frac{\partial u}{\partial r} - iku \right) = 0.
\end{cases}
\]

(1)

It is well-known that the solution of problem (1) can be written explicitly in terms of layer potentials (see [1] for instance). A challenging problem in real applications is how to approximate the solution of (1) in a bounded computational domain \( \Omega \), with \( D \subset \Omega \). Usually, the goal is to prescribe transparent boundary conditions on \( \partial \Omega \) in such a way that the corresponding solution approximates the exact solution on a good fashion.

Many methods have been studied and the research on this topic is still very active (see for instance [4, 5, 8, 12, 13, 14, 15, 16, 17, 19, 21] and references therein).

In a recent paper [8], the authors studied a new approach to the problem of transparent boundary conditions which is based on the minimization of an integral functional arising from the radiation condition at infinity. The approach

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in [8] works under quite general assumptions on the index of refraction. Indeed, it applies to the study of the Helmholtz equation
\[ \Delta u + k^2 n(x)^2 u = 0 \quad \text{in} \ \mathbb{R}^d \setminus D, \]
where the index of refraction \( n \) may have some angular dependency at infinity, i.e. \( n(x) \to n_\infty(x/|x|) \) as \( |x| \to +\infty \), as well as some (unbounded) perturbations. The novelty of the method is that it is not based on the knowledge of the exact solution in some exterior domain, but it relies on a different formulation of the radiation condition at infinity (see [20]); indeed, under suitable assumptions on \( n \), we have that there exists a unique solution of (2) which satisfies the radiation condition
\[ \int_{\mathbb{R}^d \setminus D} \left| \nabla u - iknu \frac{x}{|x|} \right|^2 \frac{dx}{1 + |x|} < \infty. \]
When the computational domain \( \Omega \) is considered, we can approximate the solution of (2)-(3) by the minimizer \( u_\Omega \) of the following constrained optimization problem
\[ \min_{J_\Omega(v)} = \int_{D \setminus D} \left| \nabla u - iknu \frac{x}{|x|} \right|^2 \frac{dx}{1 + |x|}, \]
where \( \Delta v + k^2 n(x)^2 v = 0 \) in \( \mathbb{R}^d \setminus D \), \( v = f \) on \( \partial D \).

In [8] it was proven that, if \( \Omega = B_R \) (a ball of radius \( R \) centered at the origin and containing the scatterer), then the minimizer \( u_{B_R} \) of (4) converges in \( H^1_{loc} \) norm to the solution of (2)-[3] as \( R \to +\infty \).

As already mentioned this approach works under very general assumptions on \( n \), even where most of the methods available in literature fails (at least applied in a standard way). Other advantages of this method are: (i) it works for very general choices of \( n \) and \( \Omega \) [8]; (ii) it is of easy implementation since it consists in minimizing a quadratic functional subject to a linear constraint; (iii) it is suitable to be generalized to the waveguide’s case by using the results in [6]–[10], [18].

In the present paper, we shall study the rate of convergence of this approach in the simplest case possible: \( n \equiv 1, \ d = 2, \ D = B_{R_0}, \ \Omega = B_R \), with \( R_0 < R \). It results that the rate of the \( H^1_{loc} \) norm convergence to the exact solution is \( R^{-1} \) as \( R \to +\infty \). Compared to the existing methods in literature, for \( n \equiv 1 \) this approach gives a slower rate of convergence. However, we believe that the understanding of this simple case gives a hint on the rate of convergence for much more general indexes of refraction, for which the method is more suitable. We mention that the every result of this paper can be easily generalized to the three-dimensional case and for other choices of the boundary condition on \( \partial D \).

The paper is organized as follows. In Section 2 we state the problem, recall the main results in [8] and prove some preliminary result. In Section 3 we find an explicit representation of the solution by means of Fourier series. As a consequence, we obtain the convergence estimates.

## 2 Preliminaries

In this section we introduce some notation and recall some results from [8] which will be useful in the rest of the paper. Some preliminary results will also be proven.
Let $R_0 > 0$ be fixed and let $\psi$ be the solution of
\[
\begin{aligned}
\begin{cases}
\Delta \psi + k^2 \psi = 0 & \text{in } \mathbb{R}^2 \setminus \bar{B}_{R_0}, \\
u = f & \text{on } \partial B_{R_0}, \\
\lim_{r \to +\infty} r^\frac{1}{2} \left( \frac{\partial \psi}{\partial r} - ik\psi \right) = 0,
\end{cases}
\end{aligned}
\]
where $r := |x|$, $f \in L^2(\partial B_{R_0})$. We consider the polar coordinates $x = r(\cos \omega, \sin \omega)$, where $r = |x|$ and $\omega \in [0, 2\pi)$, so that
\[
\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \omega^2},
\]
and
\[
\nabla u = \frac{\partial u}{\partial r} e_r + \frac{1}{r} \frac{\partial u}{\partial \omega} e_\omega,
\]
where $e_r = x/|x|$ and $e_\omega = (-x_2, x_1)/|x|$.

By separating the variables, a solution $u$ of $\Delta u + k^2 u = 0$ can be written as
\[
u(r, \omega) = \sum_{n \in \mathbb{Z}} [a_n J_n(kr) + b_n Y_n(kr)] e^{in\omega};
\]
here $J_n(r)$ and $Y_n(r)$ are, respectively, the Bessel and Neumann functions of order $n$ and they satisfy (see [2])
\[
\lim_{r \to \infty} r^\frac{1}{2} \left( J_n'(r) - iY_n(r) \right) = 0,
\]
which implies that the outgoing solution of (20) can be written in terms of $H_n^{(1)}$, $n \in \mathbb{N}$.

In [8] the authors proposed a method for approximating $\psi$ on
\[
\mathcal{A}_R := B_R \setminus \bar{B}_{R_0},
\]
which is based on the following minimization problem:

\[
\text{Minimize } J_R(u) := \int_{A_R} |\nabla u - iku \frac{x}{|x|}|^2 dx,
\]

where \(\Delta u + k^2 u = 0\) in \(A_R\), \(u = f\) on \(\partial B_{R_0}\). \(12\)

We will denote by \(u_{A_R}\) the minimizer of \(12\) (see [8] for the existence and uniqueness of the minimizer). As already mentioned in the introduction, the problem considered in [8] is much more general than problem \(5\) both for the choice of the domain and for the coefficient \(n\), which here is fixed to be \(n \equiv 1\) while in [8] may have angular dependence as well as perturbations.

The reader will notice that the functional in \(12\) differs from the one mentioned in the Introduction in the absence of the weight \((1 + |x|)^{-1}\). However, the two integral formulations of the radiation condition are equivalent when \(n \equiv 1\), as follows from an asymptotic expansion at infinity of the solution (see also Section 3 in [8]). The choice of the functional without the weight is just to simplify the computations. In the present paper we will deal only with a constant index of refraction, since in this case we know the explicit solution and accurate convergence results can be obtained analytically.

The main results in [8] were: (i) the existence and uniqueness of the minimizer \(u_{A_R}\) for \(12\); (ii) \(u_{A_R} \to \psi\) in \(H^1_{loc}\) norm as \(R \to +\infty\). We summarize these results in the following theorem (the results are stated for the particular case studied in this paper).

**Theorem 2.1** Let \(\psi\) be given by \(10\) and let \(R > \rho > R_0\). We have the following results

(i) There exists a unique minimizer \(u_{A_R}\) of Problem \(12\);

(ii) \(u_{A_R}\) is a solution of

\[
\Delta u + k^2 u = 0 \quad \text{in} \quad A_R, \quad u = f \, \text{on} \, \partial B_{R_0}.
\]

(iii) the minimizer of \(12\) converges to \(\psi\) as \(R \to +\infty\) in the \(H^1_{loc}\) norm, that is:

\[
\lim_{R \to +\infty} \|u_{A_R} - \psi\|_{H^1(A_R)} = 0.
\]

For any \(u, v \in H^1(A_R)\), it will be useful to define the following semidefinite positive hermitian product:

\[
\langle u, v \rangle_R = \text{Re} \int_{A_R} \nabla u \cdot \nabla \bar{v} dx,
\]

and the associated seminorm

\[
[u]_R = \langle u, v \rangle_R^{\frac{1}{2}}.
\]

We have the following Lemma.

**Lemma 2.2** Let \(u, v \in H^1(A_R)\) and let

\[
u_n(r) = \sum_{n \in \mathbb{Z}} u_n(r) e^{in\omega}, \quad v_n(r) = \sum_{n \in \mathbb{Z}} v_n(r) e^{in\omega}.
\]
Then we have that
\[ (u, v)_R = \sum_{n \in \mathbb{Z}} \int_{R_0}^R [pu_n'(\rho)\bar{v}'(\rho) + \frac{n^2}{\rho}u_n(\rho)\bar{v}_n(\rho)]d\rho. \] (16)

Proof. Let \( u, v \in C^1(\mathcal{A}_R) \). From (6), we obtain that
\[ \nabla u(r, \omega) = \sum_{n \in \mathbb{Z}} [u_n'(r)e_r + \frac{in}{r}v_n(r)e_\omega]e^{in\omega}, \]
and an analogous formula holds for \( v \). Fubini-Tonelli’s Theorem and Parseval’s identity yield (16). If \( u, v \in H^1(\mathcal{A}_R) \), then the conclusion follows from a standard approximation argument.

3 Convergence estimates

In this section we prove our main result on the convergence of the approximating solution. Our strategy is to write a minimization problem for solutions of the homogeneous Helmholtz equation which is equivalent to Problem (12) and then we use Fourier representation to obtain an explicit expression of the minimizer.

For any function \( u \in H^1(\mathcal{A}_R) \), we define \( U \in H^1(\mathcal{A}_R) \) as follows:
\[ U(x) = e^{-ik|x|}u(x). \] (17)

By using this notation, the functional \( J_R \) in (12) can be written as
\[ J_R(u) = (U, U)_R = [U]_{2,R}^2. \]

In the following lemma we write a minimization problem for solutions of the homogeneous Helmholtz equation which is equivalent to Problem (12).

Lemma 3.1 For a fixed \( R > R_0 \), let \( u_{\mathcal{A}_R} \) be the minimizer of Problem (12) and set
\[ v_{\mathcal{A}_R} = u_{\mathcal{A}_R} - \psi, \] (18)
with \( \psi \) given by (10). Then, \( v_{\mathcal{A}_R} \) is the unique minimizer of the following problem
\[ \text{Minimize } I_R(v) := (V + 2\Psi, V)_R, \] (19)
where \( v \) is a solution of
\[ \begin{cases} 
\Delta v + k^2v = 0, & \text{in } \mathcal{A}_R; \\
v = 0, & \text{on } \partial B_{R_0}; 
\end{cases} \] (20)
here \( V \) and \( \Psi \) are the functions associated to \( v \) and \( \psi \) by (17), respectively.

Proof. For any \( u \in H^1(\mathcal{A}_R) \), we define \( v \in H^1(\mathcal{A}_R) \) by \( v = u - \psi \). Hence the functional \( J_R \) in (12) is given by
\[ J_R(v + \psi) = (\Psi, \Psi)_R + (V + 2\Psi, V)_R, \]
where \( v \) is a solution of (20). Since \( (\Psi, \Psi)_R \) is fixed, we conclude.

Thanks to Lemma 3.1, we can find an explicit formula for \( v_{\mathcal{A}_R} \). In particular, we have the following theorem.
Theorem 3.2 Let \( v_{AR} \) be the minimizer of Problem (19). Then,

\[
v_{AR}(r, \omega) = \sum_{n \in \mathbb{N}} v_n^R \eta_n(kr)e^{in\omega},
\]

where

\[
\eta_n(\rho) = Y_n(kR_0)J_n(\rho) - J_n(kR_0)Y_n(\rho), \quad \rho > 0,
\]

and

\[
v_n^R = - \frac{f_n^R}{c_n^R},
\]

with

\[
c_n^R = \int_{R_0}^{R} \left[ \rho k^2 \eta'_n(\rho)^2 + \left( \rho k^2 + \frac{n^2}{\rho} \right) \eta_n(\rho)^2 \right] d\rho,
\]

and

\[
\gamma_n^R = \frac{2}{\pi} ki(R - R_0) + \frac{1}{H_n^{(1)}(kR_0)} \int_{R_0}^{R} [k^2 \rho H_n^{(1)}(\rho)\eta'_n(\rho) + (k^2 \rho + \frac{n^2}{\rho})H_n^{(1)}(\rho)\eta_n(\rho)] d\rho.
\]

**Proof.** Since \( u_{AR} \) solves (13), then \( v_{AR} = u_{AR} - \psi \) solves (20). By separation of variables and from the homogeneous boundary condition on \( \partial B_{R_0} \), we write a solution \( v \) of (20) as

\[
v(r, \omega) = \sum_{n \in \mathbb{N}} v_n \eta_n(kr)e^{in\omega},
\]

where \( \eta_n \) is given by (22). Since \( V(r, \omega) = e^{-ikr}v(r, \omega) \), then

\[
V(r, \omega) = \sum_{n \in \mathbb{N}} v_n \tilde{\eta}_n(kr)e^{in\omega},
\]

where we set

\[
\tilde{\eta}_n(r) = e^{-ikr} \eta_n(kr).
\]

By letting \( \Psi(r, \omega) = e^{-ikr}\psi(r, \omega) \), we have that

\[
\Psi(r, \omega) = \sum_{n \in \mathbb{N}} \frac{f_n^R}{H_n^{(1)}(kR_0)} \tilde{\eta}_n(kr)e^{in\omega},
\]

where

\[
\tilde{\eta}_n(r) = e^{-ikr}H_n^{(1)}(kr).
\]

We notice that

\[
\tilde{\eta}'_n(\rho) = ke^{-ik\rho}(\eta_n(\rho) - i\eta_n(\rho));
\]

from Lemma 2.2 and since \( \eta_n \) is real-valued, we have that

\[
(V, V)_R = \sum_{n \in \mathbb{N}} |v_n|^2 c_n^R.
\]
where \( c_n^R \) is given by \( \text{(23)} \). Analogously, from
\[
(\Psi, V)_R = \text{Re} \sum_{n \in \mathbb{N}} f_n \bar{\gamma}^R_n \nu_n \int_{R_0}^R \rho \bar{H}_n^r(\rho) \eta_n(\rho) d\rho,
\]
we obtain that
\[
\rho \bar{H}_n^r(\rho) \eta_n(\rho) + \frac{n^2}{\rho} \bar{h}_n(\rho) \eta_n(\rho) =
\]
\[
= \rho k^2 (H_n^{(1)}(k\rho) - iH_n^{(1)}(k\rho)) \eta_n(k\rho) + \rho k \eta_n(k\rho) + \frac{n^2}{\rho} H_n^{(1)}(k\rho) \eta_n(k\rho). \tag{27}
\]
Some computations yield
\[
H_n^{(1)}(k\rho) \eta_n(k\rho) - H_n^{(1)}(k\rho) \eta'_n(k\rho) = H_n^{(1)}(kR_0) [J_n(k\rho) Y_n'(k\rho) - J'_n(k\rho) Y_n(k\rho)],
\]
and, from
\[
J_n(r) Y_n'(r) - J'_n(r) Y_n(r) = \frac{2}{\pi r}
\]
(see formula 9.1.16 in \( \text{(24)} \)), we obtain that
\[
H_n^{(1)}(k\rho) \eta_n(k\rho) - H_n^{(1)}(k\rho) \eta'_n(k\rho) = \frac{2}{\pi k \rho} H_n^{(1)}(k\rho). \tag{28}
\]
From \( \text{(27)} \) and \( \text{(28)} \) we have that
\[
(\Psi, V)_R = \text{Re} \sum_{n \in \mathbb{N}} f_n \gamma^R_n \nu_n,
\]
with \( \gamma^R_n \) given by \( \text{(25)} \) and hence
\[
(\Psi, V)_R = \sum_{n \in \mathbb{N}} c_n^R |\nu_n|^2 + 2 \text{Re} f_n \gamma^R_n \nu_n.
\]
By minimizing each term of the sum we obtain \( \text{(23)} \).

In order to obtain estimates on the convergence, it will be useful to write the coefficients \( c_n^R \) and \( \gamma^R_n \) more explicitly. We will need the following two lemmas.

**Lemma 3.3** Let \( C_n \) and \( D_n \) be two cylindric functions, with \( n \in \mathbb{Z} \). Then
\[
\begin{align*}
C_{-n}(r) &= (-1)^n C_n(r), \tag{29} \\
\int_{r}^r \rho C_n(\rho) D_n(\rho) d\rho &= \frac{r^2}{2} [C_n(r) D_n(r) - C_{n+1}(r) D_{n-1}(r)], \tag{30} \\
- \int_{r}^r \frac{2n}{\rho} C_n(\rho) D_n(\rho) d\rho &= C_0(r) D_0(r) + C_n(r) D_n(r) + 2 \sum_{m=1}^{n-1} C_m(r) D_m(r), \tag{31} \\
& \quad n \neq 0, \\
\int_{r}^r \rho C_0'(\rho) D_0(\rho) d\rho &= rC_0'(r) D_0(r) + \frac{r^2}{2} [C_0(r) D_0(r) + C_1(r) D_1(r)], \tag{32}
\end{align*}
\]
and
\[
\int^r \rho C_n'(\rho)D'_n(\rho)d\rho = rC_n'(r)D_n(r) + \frac{1}{2n}[C_0(r)D_0(r) + C_n(r)D_n(r) + 2 \sum_{m=1}^{n-1} C_m(r)D_m(r)] + \frac{\rho^2}{2}[C_n(r)D_n(r) - C_{n+1}(r)D_{n-1}(r)], \quad n \neq 0. \quad (33)
\]

Proof. Formulas (29), (30) and (31) are formulas 9.1.5, 11.3.31 and 11.3.36 in [2], respectively. Since \( C_n \) is a solution of the Bessel equation
\[
(rC_n'(r))' + \left( r - \frac{n^2}{\rho} \right) C_n(r) = 0,
\]
we have that
\[
rC_n'(r) = - \int^r \left( \rho - \frac{n^2}{\rho} \right) C_n(\rho)d\rho. \quad (34)
\]

By multiplying (34) by \( D_n(r) \), integrating in \( dr \) and using an integration by parts, we obtain
\[
\int^r \rho C_n'(\rho)D'_n(\rho)d\rho = -D_n(r) \int^r \left( \rho - \frac{n^2}{\rho} \right) C_n(\rho)d\rho + \int^r \left( \rho - \frac{n^2}{\rho} \right) C_n(\rho)D_n(\rho)d\rho.
\]

From (34), formula 11.3.36 in [2] (by letting \( \mu = \nu = 0 \)) and by using (29) in the case \( n = 0 \), we conclude. \( \Box \)

Lemma 3.4 Let \( n \in \mathbb{N} \) be fixed and let \( c_n^R \) and \( \gamma_n^R \) be given by (24) and (25). Then
\[
c_0^R = \left\{ r\eta_{0,0}(r)\eta_{0,0}(r) + \frac{r^2}{2}(\eta_{0,0}(r)^2 + \eta_{0,1}(r)\eta_{0,1}(r)) \right\}^k_{R_o}, \quad (35a)
\]
\[
c_n^R = \left\{ r\eta_{n,n}(r)\eta_{n,n}(r) + \frac{1}{2}(n - \frac{1}{n})[\eta_{n,0}(r)^2 + \eta_{n,n}(r)^2 + 2 \sum_{m=1}^{n-1} \eta_{m,m}(r)^2] + \frac{r^2}{2}(\eta_{n,n}(r)^2 - \eta_{n,n+1}(r)\eta_{n,n+1}(r)) \right\}^k_{R_o}, \quad n \neq 0, \quad (35b)
\]
and
\[
\gamma_0^R = \frac{2}{\pi} ki(R - R_o) + r^2[H_0^{(1)}(r)\eta_{0,0}(r) + H_1^{(1)}(r)\eta_{0,1}(r)]^k_{R_o}, \quad (36a)
\]
\[
\gamma_n^R = \frac{2}{\pi} ki(R - R_o) + \frac{1}{H_n^{(1)}(k R_o)} \left\{ rH_n^{(1)}(r)\eta_{n,n}(r) + \frac{1}{2}(n - \frac{1}{n})[H_0^{(1)}(r)\eta_{n,0}(r) + H_1^{(1)}(r)\eta_{n,n}(r) + 2 \sum_{m=1}^{n-1} H_m^{(1)}(r)\eta_{m,m}(r)] + r^2[H_n^{(1)}(r)\eta_{n,n}(r) - H_{n+1}^{(1)}(r)\eta_{n,n+1}(r)] \right\}^k_{R_o}, \quad n \neq 0, \quad (36b)
\]
where
\[ \eta_{n,m}(r) = Y_n(kR_0)J_m(r) - J_n(kR_0)Y_m(r), \quad r > 0, \] (37)
for every \( m \in \mathbb{N} \).

**Proof.** The proof follows from Lemma 3.3 after tedious but straightforward computations. \( \square \)

The rest of the paper is devoted to find estimates of the rate of convergence of the solution of the approximating problem to the exact solution. It will be enough to estimate the rate of convergence of the solution of \( \varepsilon_{A_R} \) to zero, which clearly gives the desired \( H^1 \) estimate of the difference between the exact and the approximating solutions, as follows from (13).

We shall make use of the following asymptotic formulas (see formulas 9.2.1–9.2.10 in [2]):

\[ J_n(r) = \sqrt{\frac{2}{\pi r}} \left[ \cos(r - \alpha_n) - \frac{\mu_n}{r} \sin(r - \alpha_n) + \mathcal{O}(r^{-2}) \right], \] (38)
\[ Y_n(r) = \sqrt{\frac{2}{\pi r}} \left[ \sin(r - \alpha_n) + \frac{\mu_n}{r} \cos(r - \alpha_n) + \mathcal{O}(r^{-2}) \right], \] (39)
\[ H_n^{(1)}(r) = \sqrt{\frac{2}{\pi r^2}} e^{i(r-\alpha_n)} \left[ 1 + \frac{i\mu_n}{r} + \mathcal{O}(r^{-2}) \right], \] (40)
as \( r \to +\infty \) and where
\[ \alpha_n = \frac{1}{2} n\pi, \quad \text{and} \quad \mu_n = \frac{4n^2 - 1}{8}. \] (41)

From (38) and (39), we readily obtain that
\[ \eta_{n,m}(R) = \sqrt{\frac{2}{\pi kR}} \left[ Y_n(kR_0) \left( \cos(kR - \alpha_m) - \frac{\mu_n}{kR} \sin(kR - \alpha_m) \right) + J_n(kR_0) \left( \sin(kR - \alpha_m) + \frac{\mu_n}{kR} \cos(kR - \alpha_m) \right) + \mathcal{O}(R^{-2}) \right]. \]

We have the following result.

**Theorem 3.5** Let \( n \in \mathbb{N} \) be fixed and let \( c_n^R \) and \( \gamma_n^R \) be given by (24) and (25). Then
\[ \frac{\gamma_n^R}{c_n^R} = \mathcal{O}\left( \frac{1}{R} \right), \] (42)
as \( R \to +\infty \).

**Proof.** From (39) and by using the first order expansions in (38)–(40), we obtain that
\[ c_n^R = \frac{Rk}{\pi} \left[ J_n(kR_0)^2 + Y_n(kR_0)^2 + \mathcal{O}(R^{-1}) \right], \] (43)
as \( R \to +\infty \). Analogously, from (38), (38)–(40) and very tedious calculations we obtain that
\[ k^2 R^2 [H_n^{(1)}(kR)\eta_{n,n}(kR) - H_{n+1}^{(1)}(kR)\eta_{n,n-1}(kR)] = \]
\[ = -\frac{2}{\pi} ikRH_n^{(1)}(kR_0) + 2\pi \left[ nH_n^{(1)}(kR_0) + \frac{1}{2} H_n^{(2)}(kR_0)e^{-2i(kR-\alpha_n)} \right] + \mathcal{O}(R^{-2}), \]
which implies that
\[ \gamma_R^n = O(1), \]
as \( R \to +\infty \), and we conclude. \[ \square \]

**Corollary 3.6** Let \( \psi \) and \( u_{A_R} \) be the solutions of (5) and (12), respectively, and assume that
\[ f(\omega) = \sum_{n=-N}^{N} f_n e^{i n \omega}, \]
with \( \omega \in [0, 2\pi] \) and \( N \in \mathbb{N} \). Let \( R_0 > R_0 \) be fixed. Then
\[ \| \psi - u_{A_R} \|_{H^1(A_{R_0})} = O(R^{-1}), \]
as \( R \to +\infty \).

**Proof.** Since \( v_{A_R} = \psi - u_{A_R} \), the proof follows straightforwardly from Theorem 3.2 and Corollary 3.6. \[ \square \]

**References**

[1] H. Ammari, H. Kang. Reconstruction of small inhomogeneities from boundary measurements. Lecture Notes in Mathematics, 1846. Springer-Verlag, Berlin, 2004.

[2] M. Abramowitz and I. A. Stegun. Handbook of mathematical functions. Dover, New York (1965).

[3] O. Alexandrov and G. Ciraolo. Wave propagation in a 3-D optical waveguide. Math. Models Methods Appl. Sci. (M3AS), 14 (2004), no.6, 819-852.

[4] P. Bettess. Infinite Elements, Penshaw Press, Sunderland, UK, 1992.

[5] J.-P. Bérenger. A perfectly matched layer for the absorption of electromagnetic waves, J. Comput. Phys. 114 (1994), 185–200.

[6] G. Ciraolo, A method of variation of boundaries for waveguide grating couplers, Applicable Analysis, 87 (2008), 1019–1040.

[7] G. Ciraolo, A radiation condition for the 2-D Helmholtz equation in stratified media, Comm. Part. Diff. Eq., 34 (2009), 1592–1606.

[8] G. Ciraolo, F. Gargano, V. Sciacca. A computational method for the Helmholtz equation in unbounded domains based on the minimization of an integral functional. Journal of Computational Physics, 246 (2013), 78–95.

[9] G. Ciraolo and R. Magnanini. Analytical results for 2-D non-rectilinear waveguides based on the Green’s function. Math. Methods Appl. Sci., 31 (2008), no.13, 1587–1606.
[10] G. Ciraolo and R. Magnanini, *A radiation condition for uniqueness in a wave propagation problem for 2-D open waveguides*, Math. Methods Appl. Sci., 32 (2009), 1183–1206.

[11] D. Colton and R. Kress. *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag Berlin, Heidelberg, New York, 1992.

[12] B. Engquist, A. Majda. *Radiation boundary conditions for acoustic and elastic wave calculations*, Comm. Pure Appl. Math., 32 (1979), 314–358.

[13] D. Givoli. *Numerical Methods for Problems in Infinite Domains. Studies in Applied Mechanics*, vol. 33, Elsevier Scientific Publishing Co., Amsterdam, 1992.

[14] M.J. Grote, J.B. Keller. *On nonreflecting boundary conditions*, J. Comput. Phys., 122 (1995), 231–243.

[15] I. Harari. *A survey of finite element methods for time-harmonic acoustics*, Comput. Methods Appl. Mech. Engrg., 195 (2006), 1594–1607.

[16] F. Ihlenburg. *Finite Element Analysis of Acoustic Scattering*, Springer-Verlag, New York, 1998.

[17] J.B. Keller, D. Givoli. *Exact nonreflecting boundary conditions*, J. Comput. Phys., 82 (1989), 172–192.

[18] R. Magnanini, F. Santosa. *Wave propagation in a 2-D optical waveguide*, SIAM J. Appl. Math., 61 (2001), 1237–1252.

[19] M. Medvinsky, E. Turkel, U. Hetmaniuk. *Local absorbing boundary conditions for elliptical shaped boundaries*. J. Comput. Phys. 227 (2008), no. 18, 8254–8267.

[20] B. Perthame, L. Vega. *Energy concentration and Sommerfeld condition for Helmholtz equation with variable index at infinity*, GAFA Geom. Funct. Anal., 17 (2008), 1685–1707.

[21] S. A. Sauter, C. Schwab. *Boundary element methods*. Springer Series in Computational Mathematics, 39. Springer-Verlag, Berlin, 2011.