Necessary conditions for partial and super-integrability of Hamiltonian systems with homogeneous potential

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Abstract. We consider a natural Hamiltonian system of \(n\) degrees of freedom with a homogeneous potential. Such system is called partially integrable if it admits \(1 < l < n\) independent and commuting first integrals, and it is called super-integrable if it admits \(n + l\), \(0 < l < n\) independent first integrals such that \(n\) of them commute. We formulate two theorems which give easily computable and effective necessary conditions for partial and super-integrability. These conditions are derived in the frame of the Morales-Ramis theory, i.e., from an analysis of the differential Galois group of variational equations along a particular solution of the system. To illustrate an application of the formulated theorems, we investigate three and four body problems on a line and the motion in a radial potential.

Key words: integrability, nonintegrability criteria, monodromy group, differential Galois group, hypergeometric equation, Hamiltonian equations

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1 Introduction

The fundamental problem in Hamiltonian mechanics is to decide whether a given system is integrable. Integrability in this context usually means the integrability in the Liouville sense \([1]\), but it is also important to consider the non-commutative integrability as it was defined in \([14]\). Moreover, there exist examples of systems which are super-integrable, i.e., systems with \(n\) degrees of freedom admitting \(m > n\) independent first integrals such that \(n\) of them commute. Such super-integrable systems attract much attention, see e.g. \([8, 22, 10, 7]\). On the other hand, even if a considered system is not integrable, it is anyway important to know if it admits one or more first integrals. These additional first integrals can be used, e.g. to reduce the dimension of the system.
In this paper, we consider Hamiltonian systems with \( n \) degrees of freedom given by a natural Hamiltonian function

\[
H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q),
\]

where \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \) are canonical coordinates, and \( V \) is a homogeneous function of degree \( k \in \mathbb{Z} \). Our aim is to find necessary conditions for:

1. the existence of a meromorphic first integral functionally independent with \( H \); later we call such integral an additional first integral;
2. the existence of commuting functionally independent meromorphic first integrals \( F_1 = H, F_2, \ldots, F_m \), for \( 2 < m \leq n \);
3. the existence of functionally independent meromorphic first integrals \( F_1 = H, F_2, \ldots, F_n + m \), for \( 0 < m < n \), such that \( F_1, \ldots, F_n \) commute.

In other words, our goal is to find necessary conditions for the partial commutative integrability and super-integrability.

To formulate the main results of this paper we have to recall a theorem of J.J. Morales-Ruiz and J.-P. Ramis which gives the strongest known necessary conditions for the integrability in the Liouville sense of Hamiltonian systems given by a Hamiltonian function of the form (1.1).

The basic assumption of the above mentioned theorem is that Hamilton’s equations generated by (1.1), i.e.,

\[
\frac{dq}{dt} = p, \quad \frac{dp}{dt} = -\frac{\partial V(q)}{\partial q},
\]

admit a straight line solution of the form

\[
q(t) = \varphi(t)d, \quad p(t) = \dot{\varphi}(t)d,
\]

where \( d \) is a non-zero vector in \( \mathbb{C}^n \), and \( \varphi(t) \) is a scalar function. Such solution exists iff at \( d \) gradient \( V'(d) \) is parallel to \( d \), i.e., \( V'(d) = \gamma d \) for a non-zero \( \gamma \). Such \( d \) is called the Darboux point of potential \( V \). The length of \( d \) can be fixed arbitrarily and traditionally \( d \) is normalised in such a way that it satisfies the following non-linear equation

\[
V'(d) = d. \tag{1.4}
\]

**Remark 1.1** If \( V'(d) = \gamma d \), then \( \tilde{d} := ad \) satisfies \( V'(d) = \gamma a^{k-2}\tilde{d} \). Thus, for \( k \neq 2 \) we can find such \( a \) that \( V'(\tilde{d}) = \tilde{d} \). For \( k = 2 \) we cannot generally assume that for a straight line solution (1.3) \( d \) satisfies (1.4).

Accepting the above convention, it is easy to see that (1.3) is a solution of (1.2) iff \( \varphi(t) \) satisfies \( \ddot{\varphi} = -\varphi^{k-1} \). For further considerations we choose a phase curve \( \Gamma \) corresponding to a non-zero energy level

\[
e = \frac{1}{2} \varphi^2 + \frac{1}{k} \varphi^k, \quad e \neq 0. \tag{1.5}
\]

The necessary conditions of the Morales-Ramis theorem were obtained by an analysis of the variational equations along the considered phase curve. These equations are of the form

\[
\ddot{x} = -\varphi(t)^{k-2} V''(d)x, \tag{1.6}
\]
where $V''(d)$ is the Hessian of $V$ calculated at $d$. Let us assume that $V''(d)$ is diagonalizable. Then, in an appropriate base, equations (1.6) split into a direct product of second order equations

$$
\ddot{y}_i = -\lambda_i \varphi(t)^{k-2} y_i, \quad 1 \leq i \leq n,
$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $V''(d)$. One of these eigenvalues, let us say $\lambda_n$ is $k - 1$. We call this eigenvalue trivial.

In [16] J. J. Morales-Ruiz and J. P. Ramis proved the following theorem.

**Theorem 1.1** (Morales-Ramis). *If the Hamiltonian system defined by Hamiltonian (1.1) with a homogeneous potential of degree $k \in \mathbb{Z}^*$ is integrable in the Liouville sense, then each pair $(k, \lambda_i)$ belongs to an item of the following list

| case | $k$ | $\lambda$ |
|------|-----|----------|
| 1.   | $\pm 2$ | $\lambda$ |
| 2.   | $k$ | $p + \frac{k}{2} p(p - 1)$ |
| 3.   | $k$ | $\frac{1}{2} \left( \frac{k - 1}{k} + p(p + 1)k \right)$ |
| 4.   | 3 | $-\frac{1}{24} + \frac{1}{6} (1 + 3p)^2$, $-\frac{1}{24} + \frac{3}{32} (1 + 4p)^2$ |
| 5.   | 4 | $-\frac{1}{8} + \frac{2}{9} (1 + 3p)^2$ |
| 6.   | 5 | $-\frac{9}{40} + \frac{5}{18} (1 + 3p)^2$, $-\frac{9}{40} + \frac{1}{10} (2 + 5p)^2$ |
| 7.   | $-3$ | $\frac{25}{24} - \frac{1}{6} (1 + 3p)^2$, $\frac{25}{24} - \frac{3}{32} (1 + 4p)^2$ |
| 8.   | $-4$ | $\frac{9}{8} - \frac{2}{9} (1 + 3p)^2$ |
| 9.   | $-5$ | $\frac{49}{40} - \frac{5}{18} (1 + 3p)^2$, $\frac{49}{40} - \frac{1}{10} (2 + 5p)^2$ |

(1.8) where $p$ is an integer and $\lambda$ is an arbitrary complex number.*

We formulate our main results in two theorems. The first gives necessary conditions for the partial integrability.

**Theorem 1.2.** *If a Hamiltonian system defined by Hamiltonian (1.1) with a homogeneous potential of degree $k \in \mathbb{Z}^*$ admits $1 \leq l \leq n$ functionally independent and commuting meromorphic first integrals $F_1 = H, F_2, \ldots, F_l$, then at least $l$ pairs of $(k, \lambda_i)$ belong to the list (1.8) from Theorem 1.1.*

Notice that the Morales-Ramis Theorem [16] is a corollary to the above theorem.

A super-integrable system is integrable in the Liouville sense. Thus necessary conditions for the super-integrability have to restrict the list (1.8) from Theorem 1.1. Our second theorem gives such restrictions. Notice that these restrictions are imposed on non-trivial eigenvalues.
Theorem 1.3. If a Hamiltonian system defined by Hamiltonian \((1.1)\) with a homogeneous potential of degree \(k \in \mathbb{Z}^*\) admits \(n+1\), \(1 < l < n\) functionally independent meromorphic first integrals \(F_1 = H, F_2, \ldots, F_{n+l}\), such that \(F_1 = H, F_2, \ldots, F_n\) commute, then each \((k, \lambda_i)\) belongs to the list \((1.8)\) from Theorem 1.1 and moreover

- if \(|k| \leq 2\), then at least \(l\) pairs \((k, \lambda_i)\), where \(\lambda_i\) is a non-trivial eigenvalue, belong to the following list

| case | \(k\) | \(\lambda\) |
|------|------|------|
| I.   | \(-2\) | \(1 - r^2\) |
| II.  | \(-1\) | 1    |
| III. | 1     | 0    |
| IV.  | 2     | \(r^2\) |

where \(r \in \mathbb{Q}^*\);

- if \(|k| > 2\), then at least \(l\) pairs \((k, \lambda_i)\) where \(\lambda_i\) is a non-trivial eigenvalue, belong to items 3–9 of table \((1.8)\).

Remark 1.2 Assume that \(d\) satisfies \(V'(d) = \gamma d\) with \(\gamma \neq 1\) then, working with straight line solution \((1.3)\), we arrive to the same results. However, to apply the above three theorems, we have to make the following modification. If \(\hat{\lambda}_1, \ldots, \hat{\lambda}_n\) are eigenvalues of \(V''(d)\), then we put \(\lambda_i = \hat{\lambda}_i / \gamma\), for \(i = 1, \ldots, n\), see \((1.3)\) for details. This remark is important only for \(k = 2\), as for \(k \neq 2\) we can always assume that for a straight line solution \((1.3)\) Darboux point \(d\) satisfies \(V'(d) = d\).

The rest of this paper, except for the last section, is devoted to present proofs of the above theorems. To this end, in the next section we recall basic facts from the Ziglin theory \([27, 28]\) and its differential Galois extension developed by A. Baider, R. C. Churchill, J. J. Morales, J.-P. Ramis, D. L. Rod, C. Simó and M. F. Singer, see \([4, 15, 19, 17]\) and references therein, which is called the Morales-Ramis theory. Section 3 contains a derivation of variational equations and their reduction to an algebraic form. It appears that these equations are a direct sum of a certain type of hypergeometric equations. In section 4 we give a detailed analysis of the differential Galois group of this type of hypergeometric equation. Obstructions for partial and super-integrability follow from the fact that the differential Galois group of variational equations must have an appropriate number of invariants. This problem, reformulated into the language of Lie algebra of the differential Galois group, is analysed in Section 5. Short proofs of Theorem 1.2 and 1.3 are given in Section 6. In the last section we present an application of our theorems. We analyse three and four body problems on a line and a radial potential. To make the paper self-contained, we collect in Appendix several known facts concerning the differential Galois group of a general second order equation with rational coefficients and the Riemann \(P\) equation.

2 Basic facts from the general theory

In this section we recall several basic facts from the Ziglin and Morales-Ramis theory in the setting needed in this paper. For detailed expositions, see e.g. \([2, 3, 17, 18, 15, 4]\).

Thus let us consider a complex holomorphic system of differential equations

\[
\frac{d}{dt} x = v(x), \quad x \in U \subset \mathbb{C}^n, \quad t \in \mathbb{C},
\]

\(2.1\)
where $U$ is an open and connected subset of $\mathbb{C}^n$. The Ziglin and Morales-Ramis theory are based on the linearization of the original system around a particular non-equilibrium solution. Hence, let $\varphi(t)$ be a non-equilibrium solution of \((2.1)\). Usually it is not a single-valued function of the complex time $t$. Thus, we associate with $\varphi$ a Riemann surface $\Gamma$ with $t$ as a local coordinate. The variational equations along $\Gamma$ have the form

$$\dot{\xi} = A(t)\xi, \quad A(t) = \frac{\partial \varphi}{\partial x}(\varphi(t)), \quad \xi \in T_tU.$$ \hspace{1cm} (2.2)

With these equations we can associate two groups. The first one, called the monodromy group is defined as follows. Let us fix a point $p \in \Gamma$, and let $\Xi(t)$ be the local fundamental system of solutions of \((2.2)\) satisfying $\Xi(t_0) = E$, where $E$ is the identity matrix, and $\varphi(t_0) = p$. Matrix $\Xi(t)$ is holomorphic for a sufficiently small disc $D_\varepsilon(t_0) := \{t \in \mathbb{C} \mid |t - t_0| < \varepsilon\}$. Then an analytical continuation of $\Xi(t)$ along a closed curve $\gamma$ with a base point $p$ gives rise to a new fundamental system $\tilde{\Xi}(t)$ for $t \in D_\varepsilon(t_0)$. Solutions of a linear system \((2.2)\) form a $\mathbb{C}$-linear space, hence each column in $\Xi(t)$ is a linear combination of columns of $\Xi(t)$. Thus we have $\tilde{\Xi}(t) = \Xi(t)M_{\gamma}$, for a certain $M_{\gamma} \in \text{GL}(n, \mathbb{C})$. It can be shown that matrix $M_{\gamma}$, called the monodromy matrix, does not depend on a specific choice of $\gamma$ only on its homotopy class. If $\gamma$ is a product of two loops $\gamma = \gamma_1 \cdot \gamma_2$ (first go along loop $\gamma_1$, and then go along $\gamma_2$), then

$$M_{\gamma} = M_{\gamma_1 \cdot \gamma_2} = M_{\gamma_2}M_{\gamma_1},$$

in other words, an analytic continuation of solutions of system \((2.2)\) along closed paths with a fixed point $p$, gives an anti-representation of the first fundamental group $\pi_1(p, \Gamma)$ of $\Gamma$. The image $\mathcal{M}$ of this anti-representation is called the monodromy group of equation \((2.2)\). In the above definition a point $p \in \Gamma$ appeared, so we should write $\mathcal{M}_p$. However, if we choose $q \in \Gamma$, $q \neq p$, then the obtained monodromy group $\mathcal{M}_q$ is isomorphic with $\mathcal{M}_p$. More precisely, there exists a matrix $C \in \text{GL}(n, \mathbb{C})$ such that every element $A$ of $\mathcal{M}_q$ is uniquely given by $C^{-1}BC$, where $B \in \mathcal{M}_p$. Thus, we do not specify the dependence on the base point later.

To define the differential Galois group of equation \((2.2)\) we have to switch to the algebraic language. We can consider the entries of matrix $A(t)$ in equation \((2.2)\) as elements of field $K := \mathcal{M}(\Gamma)$ of functions meromorphic on $\Gamma$. This field with the differentiation with respect to $t$ as a derivation is a differential field. Only constant functions from $K$ have a vanishing derivative, so the subfield of constants of $K$ is $\mathbb{C}$. It is obvious that solutions of \((2.2)\) are not necessarily elements of $K^n$. The fundamental theorem of the differential Galois theory guarantees that there exists a differential field $L \supset K$ such that $n$ linearly independent (over $\mathbb{C}$) solutions of \((2.2)\) are contained in $L^n$. The smallest differential extension $L \supset K$ with this property is called the Picard-Vessiot extension of $K$. A group $S$ of differential automorphisms of $L$ which do not change $K$ is called the differential Galois group of equation \((2.2)\). It can be shown that $S$ is a linear algebraic group. Thus, it is a union of a finite number of disjoint connected components. One of them, containing the identity, is called the identity component and is denoted by $S^0$.

Let $\xi = (\xi_1, \ldots, \xi_n)^T \in L^n$ be a solution of equation \((2.2)\), and $g$ an element of its differential Galois group. Then, $g(\xi) := (g(\xi_1), \ldots, g(\xi_n))^T$ is also its solution. In fact, by definition $g$ commutes with the time differentiation, so we have

$$\frac{d}{dt}g(\xi) = g(\dot{\xi}) = g(A(t)\xi) = A(t)g(\xi),$$

as $g$ does not change elements of $K$. Thus, if $\Xi \in \mathcal{M}(n, L)$ is a fundamental matrix of \((2.2)\), i.e., its columns are linearly independent solutions of \((2.2)\), then $g(\Xi) = \Xi M_g$. 

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where $M_g \in \text{GL}(n, \mathbb{C})$. In other words, we can look at the differential Galois group as a matrix group.

It is known that the monodromy group is contained in the differential Galois group. Moreover, in a case when equations (2.1) are Hamiltonian, then both these groups are subgroups of $\text{Sp}(n, \mathbb{C})$.

Now we explain why the monodromy and differential Galois groups of variational equations are important in a study of integrability. At first, we introduce a few definitions. Let us consider a holomorphic function $F$ defined in a certain connected neighbourhood of solution $\varphi(t)$. In this neighbourhood we have the expansion

$$F(\varphi(t) + \xi) = F_m(\xi) + O(||\xi||^{m+1}), \quad F_m \neq 0.$$  

(2.3)

Then the leading term $f$ of $F$ is the lowest order term of the above expansion i.e., $f(\xi) := F_m(\xi)$. Note that $f(\xi)$ is a homogeneous polynomial of variables $\xi = (\xi_1, \ldots, \xi_n)$ of degree $m$; its coefficients are polynomials in $\varphi(t)$. If $F$ is a meromorphic function, then it can be written as $F = P/Q$ for certain holomorphic functions $P$ and $Q$. Then the leading term $f$ of $F$ is defined as $f = p/q$, where $p$ and $q$ are leading terms of $P$ and $Q$, respectively. In this case $f(\xi)$ is a homogeneous rational function of $\xi$.

One can prove that if $F$ is a meromorphic (holomorphic) first integral of equation (2.1), then its leading term $f$ is a rational (polynomial) first integral of variational equations (2.2). If system (2.1) has $m \geq 2$ functionally independent meromorphic first integrals $F_1, \ldots, F_m$, then their leading terms can be functionally dependent. However, by the Ziglin Lemma [27, 2, 4], we can find $m$ polynomials $G_1, \ldots, G_m \in \mathbb{C}[z_1, \ldots, z_m]$ such that leading terms of $G_i(F_1, \ldots, F_m)$, for $1 \leq i \leq m$ are functionally independent.

Additionally, if $\mathcal{G} \subset \text{GL}(n, \mathbb{C})$ is the differential Galois group of (2.2), and $f$ is its rational first integral, then $f(g(\xi)) = f(\xi)$ for every $g \in \mathcal{G}$, see [2, 15]. This means that $f$ is a rational invariant of group $\mathcal{G}$. Thus we have a correspondence between the first integrals of the system (2.1) and invariants of $\mathcal{G}$.

**Lemma 2.1.** If equation (2.1) has $k$ functionally independent first integrals which are meromorphic in a connected neighbourhood a non-equilibrium solution $\varphi(t)$, then the differential Galois group $\mathcal{G}$ of the variational equations along $\varphi(t)$, as well as their monodromy group, have $k$ functionally independent rational invariants.

As mentioned above, a differential Galois group is a linear algebraic group, thus, in particular, it is a Lie group, and one can consider its a Lie algebra. This Lie algebra reflects only the properties of the identity component of the group. It is easy to show that if a Lie group has an invariant, then also its Lie algebra has an integral. Let us explain what the last expression means. Let $g \subset \text{GL}(n, \mathbb{C})$ denote the Lie algebra of $\mathcal{G}$. Then an element $Y \in g$ can be considered as a linear vector field: $x \mapsto Y(x) := Yx$, for $x \in \mathbb{C}^n$. We say that $f \in \mathbb{C}(x)$ is an integral of $g$, iff $Y(f)(x) = df(x) \cdot Y(x) = 0$, for all $Y \in g$.

**Proposition 2.1.** If $f_1, \ldots, f_k \in \mathbb{C}(x)$ are algebraically independent invariants of an algebraic group $\mathcal{G} \subset \text{GL}(n, \mathbb{C})$, then they are algebraically independent first integrals of the Lie algebra $g$ of $\mathcal{G}$.

The above facts are the starting points for applications of differential Galois methods to a study of integrability.

If the considered system is Hamiltonian, then we have additional constrains. First of all, the differential Galois group of variational equations is a subgroup of the symplectic group. Secondly, commutation of first integrals imposed by the Liouville integrability implies commutation of variational first integrals. The following lemma plays the crucial role and this is why it was called The Key Lemma see Lemma III.3.7 on page 72 in [2].
Lemma 2.2. Assume that Lie algebra \( g \subset \text{sp}(2k,\mathbb{C}) \) admits \( k \) functionally independent and commuting first integrals. Then \( g \) is Abelian.

Hence, if \( g \) in the above lemma is the Lie algebra of a Lie group \( \mathcal{G} \), then the identity component \( \mathcal{G}^0 \) of \( \mathcal{G} \) is Abelian.

Using all these facts Morales and Ramis proved the following theorem \([15, 17] \).

Theorem 2.1 (Morales-Ramis). Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve \( \Gamma \), and that variational equations along \( \Gamma \) are Fuchsian. Then the identity component of the differential Galois group of the variational equations is Abelian.

Generally, it is difficult to determine the differential Galois group of a given system of variational equations when its dimension is greater than two. This is a reason why, instead of the variational equations, it is convenient to work with their reduced form called the normal variational equations. For a general definition of this notion see e.g. \([15] \). Here we define the normal variational equations for a case when the considered Hamiltonian system is defined on \( \mathbb{C}^{2n} \) with \( z = (q_1, p_1, \ldots , q_n, p_n) \) as canonical coordinates. Let us assume that the system admits a two dimensional symplectic invariant plane

\[ \Pi := \{ z \in \mathbb{C}^{2n} \mid q_i = p_i = 0 \quad \text{for} \quad i = 1, \ldots , n - 1 \}. \]  

(2.4)

Thus, if \( H \) is the Hamiltonian of the system, then

\[ \frac{\partial H}{\partial q_i}(0, \ldots , 0, q_n, p_n) = \frac{\partial H}{\partial p_i}(0, \ldots , 0, q_n, p_n) = 0 \quad \text{for} \quad i = 1, \ldots , n - 1. \]  

(2.5)

Now, for a particular solution \( \psi(t) = (0, \ldots , q_n(t), p_n(t)) \), the matrix of the variational equations has a block diagonal form

\[ A(t) = \begin{bmatrix} N(t) & 0 \\ B(t) & T(t) \end{bmatrix}, \]  

(2.6)

where \( N(t) \), \( B(t) \) and \( T(t) \) are \( 2(n-1) \times 2(n-1), 2 \times 2(n-1) \) and \( 2 \times 2 \) matrices, respectively. Hence, the variational equations are a product of two systems

\[ \frac{d}{dt} \xi = N(t) \xi, \quad \xi \in \mathbb{C}^{2(n-1)} \quad \text{and} \quad \frac{d}{dt} \eta = B(t) \xi + T(t) \eta, \quad \eta \in \mathbb{C}^2. \]  

(2.7)

The first of them is called the normal variational equations.

It can be shown that if the Hamiltonian system possesses a first integral \( F \), then the normal variational equations also have a first integral which is an invariant of their differential Galois group \( \mathcal{G}_N \subset \text{Sp}(2(n-1),\mathbb{C}) \). Moreover, if \( F_1 \) and \( F_2 \) are commuting first integrals functionally independent together with \( H \), then we can assume that the corresponding first integrals \( f_1 \) and \( f_2 \) of the normal variational equations are independent and commuting, see \([4, 15] \). These facts imply that the statement of Theorem \(2.1 \) remains valid if in its formulation the normal variational equations are used instead of the variational equations.

3 Necessary conditions for Liouville integrability

Let \( G(k,\lambda) \) denote the differential Galois group of equation

\[ \ddot{y} = -\lambda \dot{y}^k - 2y. \]  

(3.1)
It is a subgroup of $\text{SL}(2, \mathbb{C}) \simeq \text{Sp}(2, \mathbb{C})$.

It is clear that the differential Galois group $G$, of equations (1.7) is a direct product

$$G = G(k, \lambda_1) \times \cdots \times G(k, \lambda_n) \subset \text{Sp}(2n, \mathbb{C}).$$  \hspace{1cm} (3.2)

Hence, $G^\circ$ is Abelian if and only if groups $G(k, \lambda_i)^\circ$ are Abelian, for $i = 1, \ldots, n$. It follows that we know for which values of $k$ and $\lambda$ the identity component $G(k, \lambda)^\circ$ of the differential Galois group $G(k, \lambda)$ of equation (3.1) is Abelian.

To solve this problem we introduce a new independent variable in equation (3.1), as it was proposed in [26], namely, assuming that $k \neq 0$ and $e \neq 0$ we put

$$t \to z := \frac{1}{ek} \varphi(t)^k.$$  \hspace{1cm} (3.3)

Then, equation (3.1) is transformed to the following one

$$z(1-z)y'' + \left(\frac{k-1}{k} - \frac{3k-2}{2k} z\right)y' + \frac{\lambda}{2k} y = 0,$$  \hspace{1cm} (3.4)

where prime denotes the differentiation with respect to $z$. It is the Gauss hypergeometric equation

$$z(1-z)y'' + [c - (a + b + 1)z]y' - aby = 0,$$  \hspace{1cm} (3.5)

with parameters

$$a + b = \frac{k-2}{2k}, \quad ab = -\frac{\lambda}{2k}, \quad c = 1 - \frac{1}{k}.$$  \hspace{1cm} (3.6)

The differences of exponents at $z = 0, 1, \infty$ for equation (3.5) are

$$\rho = 1 - c = \frac{1}{k}, \quad \sigma = c - a - b = \frac{1}{2}, \quad \tau = a - b = \frac{1}{2k} \sqrt{(k-2)^2 + 8k\lambda},$$  \hspace{1cm} (3.7)

respectively.

Let $\hat{G}(k, \lambda)$ denote the differential Galois group of equation (3.4). Notice that $\hat{G}(k, \lambda)$ is a subgroup of $\text{GL}(2, \mathbb{C})$, and is different from $G(k, \lambda)$. However, it can be shown, see [16, 15], that $G(k, \lambda)^\circ$ and $\hat{G}(k, \lambda)^\circ$ are isomorphic.

Now, the change of the independent variable (3.3), transforms the variational equations (1.7) into a direct product of hypergeometric equations

$$z(1-z)y''_i + \left(\frac{k-1}{k} - \frac{3k-2}{2k} z\right)y'_i + \frac{\lambda_i}{2k} y_i = 0, \quad 1 \leq i \leq n,$$  \hspace{1cm} (3.8)

whose differential Galois group $\hat{G}$ is a direct product

$$\hat{G} = \hat{G}(k, \lambda_1) \times \cdots \times \hat{G}(k, \lambda_n).$$

A necessary condition for the integrability is now following: all groups $\hat{G}(k, \lambda_i)^\circ$ have to be Abelian, and thus solvable. Exactly this reasoning was used in the proof of Theorem 1.1 given in [15, 16].

4 Group $G(k, \lambda)^\circ$

From the previous section it follows that it is important to know precisely the identity component of the differential Galois group of hypergeometric equation (3.5) with parameters $a, b$ and $c$ given by (3.6). This is the aim of this section.
As we have already mentioned the differential Galois group of (3.5) is not a subgroup of SL(2, C). It causes some technical problems. To avoid them, we transform equation (3.5) to the normal form putting

\[ w = y \exp \int p \, dz, \quad p := \frac{c - (a + b + 1)z}{z(1 - z)}. \]  (4.1)

Then we obtain

\[ w'' = \frac{\rho^2 - 1 + z(1 - \rho^2 - \tau^2 + \sigma^2) + z^2(\tau^2 - 1)}{4z^2(z - 1)^2} w. \]  (4.2)

For this equation exponents at 0, 1 and at the infinity are

\[ \left\{ \frac{1}{2}(1 - \rho), \frac{1}{2}(1 + \rho) \right\}, \left\{ \frac{1}{2}(1 - \sigma), \frac{1}{2}(1 + \sigma) \right\}, \left\{ -\frac{1}{2}(1 - \tau), -\frac{1}{2}(1 + \tau) \right\}, \]  (4.3)

respectively. Its monodromy and differential Galois groups are now subgroups of SL(2, C). It is important to remark here that the identity components of the differential Galois groups of (3.5) and (4.2) are the same. Notice also that the differences of exponents at singular points were unchanged.

Assuming that \( \rho, \sigma \) and \( \tau \) are defined by (3.7), we denote by \( \mathcal{G}(k, \lambda) \) the differential Galois group of equation (4.2). In what follows we describe properties of \( \mathcal{G}(k, \lambda)^\circ \), but, as we explained, groups \( \mathcal{G}(k, \lambda)^\circ, \hat{G}(k, \lambda)^\circ \) and \( G(k, \lambda)^\circ \) are isomorphic, so, as a result, we obtain a characterisation of \( \hat{G}(k, \lambda)^\circ \).

At first, we recall the following fact which explains the origin of table (1.8) given in Theorem 1.1.

**Proposition 4.1.** Group \( \mathcal{G}(k, \lambda)^\circ \) is solvable if and only if \((k, \lambda)\) belongs to an item in table (1.8).

**Proof.** Equation (4.2) is the Riemann P equation. The Kimura theorem A.1 gives necessary and sufficient conditions for the solvability of the identity component of its differential Galois group. Table (1.8) is just a specification of these conditions for \( \rho, \sigma \) and \( \tau \) given by (3.7).

A necessary condition for the integrability is that \( \mathcal{G}(k, \lambda)^\circ \) is Abelian. As not all solvable groups are Abelian, one can think that conditions of Theorem 1.1 can be sharpened. We show that it is not like that, i.e., we prove that if \( \mathcal{G}(k, \lambda)^\circ \) is solvable, then it is Abelian. Suppose that \( \mathcal{G}(k, \lambda)^\circ \) is solvable but not Abelian. Then, as it is explained in Appendix, there is only one possibility: \( \mathcal{G}(k, \lambda) = \mathcal{G}(k, \lambda)^\circ = \mathcal{T} \), where \( \mathcal{T} \) is the triangular subgroup of SL(2, C). So, such case can appear only if the considered equation is reducible. Let us recall, see Appendix, that equation (4.2) is reducible iff it has a solution \( w = \exp[\int \omega] \) where \( \omega \in \mathcal{C}(z) \).

**Proposition 4.2.** Equation (4.2) is reducible if and only if \( \lambda = p + kp(p - 1)/2 \) for some \( p \in \mathbb{Z} \).

**Proof.** To proof this lemma it is enough to check directly one of equivalent conditions given in Lemma A.3.

If equation (4.2) is reducible, then respective exponents at singular points 0, 1 and infinity are following

\[ \left\{ \frac{1}{2} - \frac{1}{2k}, \frac{1}{2} + \frac{1}{2k} \right\}, \left\{ \frac{1}{4}, \frac{3}{4} \right\}, \left\{ -\frac{2 + k(l + 2)}{4k}, \frac{2 + k(l - 2)}{4k} \right\}. \]  (4.4)
where \( l \) is an odd integer.

Now, we can show that if equation (4.2) is reducible, then the identity component of its differential Galois group \( \mathcal{G}(k, \lambda)^\circ \) is a proper subgroup of the triangular group \( \mathcal{T} \), and thus it is Abelian.

**Lemma 4.1.** Assume that equation (4.2) is reducible. Then its differential Galois group \( \mathcal{G}(k, \lambda) \) is a proper subgroup of the triangular group.

**Proof.** The difference of exponents for singular point \( z = 1 \) is \( 1/2 \). Thus, from Lemma A.4, it follows that if equation (4.2) is reducible, then it possesses a solution of the form:

\[
    w = z^r(1 - z)^s h(z),
\]

where \( h(z) \) is a polynomial, and \( r \) is an exponent at \( z = 0 \), and \( s \) in an exponent at \( z = 1 \).

As \( r \) and \( s \) are rational, there exists \( j \in \mathbb{N} \) such that \( w^j \in \mathbb{C}(z) \). Now, by Lemma A.2, \( \mathcal{G}(k, \lambda) \) is either a proper subgroup of the diagonal group, or a proper subgroup of the triangular group.

For our further analysis it is important to know the dimension of \( \mathcal{G}(k, \lambda)^\circ \) in a case when \( \mathcal{G}(k, \lambda) \) is reducible. By the Lemma A.2, either \( \mathcal{G}(k, \lambda) \) is a finite cyclic group, and then \( \mathcal{G}(k, \lambda)^\circ = \{E\} \), or \( \mathcal{G}(k, \lambda) \) is a proper subgroup of the triangular group, and then

\[
    \mathcal{G}(k, \lambda)^\circ = \mathcal{T}_1 := \left\{ \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \mid c \in \mathbb{C} \right\}.
\]

**Proposition 4.3.** Assume that \( \mathcal{G}(k, \lambda) \) is diagonal. Then \( k \in \{\pm 1, \pm 2\} \).

**Proof.** If \( \mathcal{G}(k, \lambda) \) is diagonal, then the monodromy group of equation (4.2) is diagonal. This last group is generated by two elements \( M_0, M_1 \in \text{SL}(2, \mathbb{C}) \) which can be assumed diagonal. Then from Lemma A.4 it follows that at least one of matrices \( M_0, M_1 \) or \( M_0M_1 \) is \( \pm E \). The eigenvalues of \( M_0 \) are \( \exp(2\pi i r_{1,2}) \), where \( r_{1,2} \) are exponents at \( z = 0 \) for equation (4.2), i.e.

\[
    r_{1,2} = \frac{1}{2} \left( 1 \pm \frac{1}{k} \right).
\]

Hence, if \( M_0 = E \), then \( k = \pm 1 \), and it is impossible that \( M_0 = -E \). Similar arguments show that \( M_1 \neq \pm E \), and if \( M_0M_1 = \pm E \), then \( k = \pm 2 \).

If a local solution near a singular points contains a logarithm, then the monodromy group contains the following element

\[
    M = \begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix},
\]

and, moreover it can be shown that such element belongs to the identity component of the differential Galois group of considered equation. Hence we can use this fact for checking whether \( \mathcal{G}(k, \lambda)^\circ = \mathcal{T}_1 \).

**Proposition 4.4.** Assume that \( k = 1 \) and that equation (4.2) is reducible. Then singular point \( z = 0 \) is logarithmic except for the case \( \lambda = 0 \).

**Proof.** We apply Lemma A.6 from Appendix, and we use notation introduced just before it. For \( k = 1 \) exponents at \( z = 0 \) are \( \rho_1 = 1 \) and \( \rho_2 = 0 \), so \( m := \rho_1 - \rho_2 = 1 \), and \( \langle m \rangle = \{1\} \). Thus, by Lemma A.6 singularity \( z = 0 \) is logarithmic if an only if for
arbitrary exponents $s$ and $t$ at $z = 1$ and $z = \infty$, respectively, we have $1 + s + t \neq 1$. Using (4.4) we obtain
\[
1 \pm \frac{p}{2} \neq 1 \quad \text{and} \quad \frac{3}{2} + \frac{p}{2} \neq 1, \quad \text{for} \quad p \in \mathbb{Z}.
\] (4.6)
This system of inequalities is satisfied for $p \in \mathbb{Z} \setminus \{-1, 0\}$. For $p = 0$ and $p = -1$ we have $\lambda = 0$. This finishes the proof.

In a similar way one can show the following.

**Proposition 4.5.** Assume that $k = -1$, and that equation (4.2) is reducible. Then singular point $z = 0$ is logarithmic except for the case $\lambda = 1$.

For $k = \pm 2$, the singular point at infinity can be logarithmic.

**Proposition 4.6.** Assume that $k = 2$ and that equation (4.2) is reducible. Then singular point $z = \infty$ is logarithmic if and only if $\lambda = 0$.

**Proof.** Under the given assumptions, $\lambda = p^2$, exponents at infinity are $\tau_1 = (p - 1)/2$ and $\tau_2 = -(p + 1)/2$, where $p \in \mathbb{Z}$. Thus, $m := \tau_1 - \tau_2 = p$. If $p = 0$, then the singularity is logarithmic. Assume that $p > 0$. By Lemma A.6, singularity $z = \infty$ is logarithmic if an only if for arbitrary exponents $r$ and $s$ at $z = 0$ and $z = 1$, respectively, we have
\[
\frac{1}{2}(p - 1) + r + s \notin \langle m \rangle.
\]
Using (4.4) we obtain the following condition: none of the three numbers
\[
\frac{1}{2}p, \quad \frac{1}{2}(p + 1), \quad \frac{1}{2}(p + 2),
\]
belongs to $\langle m \rangle = \{1, \ldots, p\}$. This is not true, as either the first or the second belongs to $\langle m \rangle$. Similar arguments work for $p < 0$, and this finishes the proof.

**Proposition 4.7.** Assume that $k = -2$ and that equation (4.2) is reducible. Then singular point $z = \infty$ is logarithmic if and only if $\lambda = 1$.

Let us summarise our analysis.

**Corollary 4.1.** Assume that equation (4.2) is reducible. Then $\mathcal{G}(k, \lambda)^\circ = \mathcal{T}_1$ except for the following cases:

1. $k = -2$ and $\lambda = 1 - p^2$, $p \in \mathbb{Z}^*$,
2. $k = -1$ and $\lambda = 1$,
3. $k = 1$ and $\lambda = 0$,
4. $k = 2$ and $\lambda = p^2$, $p \in \mathbb{Z}^*$,

when $\mathcal{G}(k, \lambda)^\circ = \{E\}$.

Let us assume now that $\mathcal{G}(k, \lambda)^\circ$ is not reducible but solvable. Then we have two possibilities. Either $\mathcal{G}(k, \lambda)$ is primitive and finite, and then $\mathcal{G}(k, \lambda)^\circ = \{E\}$, or $\mathcal{G}(k, \lambda)$ is a subgroup of $\mathcal{D}\mathcal{P}$ group, see Appendix.

**Proposition 4.8.** Assume that equation (4.2) is not reducible. Then $\mathcal{G}(k, \lambda)$ is a subgroup of $\mathcal{D}\mathcal{P}$ group only if and only if either:
1. $k = -2$; in this case $G(k, \lambda)^\circ = \{E\}$ if and only if $\lambda = 1 - r^2$ for some $r \in \mathbb{Q} \setminus \mathbb{Z}$, and $G(k, \lambda)^\circ = \mathbb{D}$ otherwise, or

2. $k = 2$; in this case $G(k, \lambda)^\circ = \{E\}$ if and only if $\lambda = r^2$ for some $r \in \mathbb{Q} \setminus \mathbb{Z}$, and $G(k, \lambda)^\circ = \mathbb{D}$ otherwise, or

3. $|k| > 2$ and

$$\lambda = \frac{1}{2} \left( \frac{k - 1}{k} + p(p + 1)k \right), \quad p \in \mathbb{Z},$$

and in this case $G(k, \lambda)$ is finite, so $G(k, \lambda)^\circ = \{E\}$.

Proof. We apply Lemma A.7 from Appendix. A necessary condition for $G(k, \lambda)$ to be a subgroup of $\mathbb{D}\mathbb{P}$ group is following: at least two differences of exponents are half integers. As the difference of exponents at $z = 1$ is $\sigma = 1/2$, we have two possibilities: either $k = \pm 2$ and then $\rho = \pm 1/2$, or

$$\lambda = \frac{1}{2} \left( \frac{k - 1}{k} + p(p + 1)k \right), \quad (4.7)$$

for some $p \in \mathbb{Z}$. Moreover, if $G(k, \lambda)$ is a subgroup of $\mathbb{D}\mathbb{P}$ group, then it is a finite group if and only if, at two singular points, the differences of exponents are half integers and exponents at the remaining point are rational, otherwise $G(k, \lambda)^\circ = \mathbb{D}\mathbb{P}$. Hence, under the assumption of our lemma, for $|k| > 2$, group $G(k, \lambda)$ is a subgroup of $\mathbb{D}\mathbb{P}$ group iff $\lambda$ is given by (4.7). But, for these values of $\lambda$ all exponents are rational, and this implies that the group is finite. This proves case 3.

For $k = \pm 1$ and $\lambda$ given (4.7) equation (4.2) is reducible, but we assumed that it is not reducible, so this case is excluded.

Let $k = -2$. Then exponents at infinity are rational if $\tau$ is rational, see (4.3). From (3.7) we have

$$\tau = -\sqrt{1 - \lambda}.$$ 

Thus $G(-2, \lambda)$ is a finite subgroup of $\mathbb{D}\mathbb{P}$ group iff $\lambda = 1 - r^2$ for a rational $r$. However, if $r \in \mathbb{Z}$, then equation (4.2) is reducible. This proves case 1.

For $k = 2$ we have $\tau = \sqrt{\lambda}$, so $G(2, \lambda)$ is a finite subgroup of $\mathbb{D}\mathbb{P}$ group iff $\lambda = r^2$ for a rational $r$ but if $r$ is an integer, then equation (4.2) is reducible, thus we have to exclude these values.

We summarise our analysis in the following corollary.

**Corollary 4.2.** Assume that the identity component $G(k, \lambda)^\circ$ of the differential Galois group of equation (4.2) is solvable. Then $G(k, \lambda)^\circ$ is Abelian. Moreover, $G(k, \lambda)^\circ = \{E\}$ if and only if either

1. $|k| > 2$ and $(k, \lambda)$ belongs to an item 3–9 of table (1.8), or

2. $|k| \leq 2$ and $(k, \lambda)$ belongs to an item of the following table

| case | $k$ | $\lambda$ |
|------|-----|----------|
| I,   | -2, | $\lambda = 1 - r^2$ |
| II.  | -1, | 1        |
| III. | 1   | 0        |
| IV.  | 2   | $\lambda = r^2$ |

where $r \in \mathbb{Q}^\ast$. 

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5 Certain Poisson algebra

As was mentioned for a Hamiltonian system, the differential Galois group $\mathcal{G}$ of variational equations along a particular solution is a subgroup of the symplectic group $\text{Sp}(2n, \mathbb{C})$, thus the Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\text{sp}(2n, \mathbb{C})$. The necessary conditions for the integrability in the Liouville sense from Theorem 2.1 are expressed in terms of the identity component of $\mathcal{G}$. The properties of this component are encoded in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. To find the necessary conditions for partial and super-integrability, we have to characterise Lie algebras $\mathfrak{g}$ which admit a certain number of first integrals. And this is the main goal of this section. Here we follow the ideas and methods introduced in [20].

An element $Y$ of Lie algebra $\text{sp}(2n, \mathbb{C})$, considered as a linear vector field, is a Hamiltonian vector field given by a global Hamiltonian function $H : \mathbb{C}^{2n} \to \mathbb{C}$, which is a degree 2 homogeneous polynomial of $2n$ variables $(x, y) := (x_1, \ldots, x_n, y_1, \ldots, y_n)$, i.e. $H \in \mathbb{C}[x, y]$. In this way we identify $\text{sp}(2n, \mathbb{C})$ with a $\mathbb{C}$-linear vector space $\mathbb{C}[x, y]$ with the canonical Poisson bracket as the Lie bracket. Thus, for a Lie algebra $\mathfrak{g} \subset \text{sp}(2n, \mathbb{C}) \simeq \mathbb{C}[x, y]$, a rational function $f \in \mathbb{C}[x, y]$ is a first integral of $\mathfrak{g}$, iff $\{H, f\} = 0$, for all $H \in \mathfrak{g}$. A field of rational first integrals of $\mathfrak{g}$ we denote by $\mathbb{C}(x, y)^0$.

Now, we consider the case when $\mathfrak{g}$ is a Lie subalgebra of $\text{sp}(2, \mathbb{C})$. It is easy to show that Lie algebra $\text{sp}(2, \mathbb{C})$ does not admit any non-constant first integral.

**Proposition 5.1.** A rational function $f \in \mathbb{C}(x, y)$ is a first integral of $\text{sp}(2, \mathbb{C})$, iff $f \in \mathbb{C}$.

**Proof.** Let $f \in \mathbb{C}(x, y)$ be a first integral of $\text{sp}(2, \mathbb{C}) \simeq \mathbb{C}[x, y]$. Thus, $\{f, H\} = 0$, for each $H \in \mathbb{C}[x, y]$. Let us take $H = x^2$. Then,

$$\{f, H\} = -2x \frac{\partial f}{\partial y} = 0,$$

and this shows that $f$ does not depend on $y$, i.e., $f \in \mathbb{C}(x)$. Taking $H = y^2$, we show that $f$ does not depend on $x$. Hence $f \in \mathbb{C}$. \hfill \Box

The above proposition shows that only proper subalgebras of $\text{sp}(2, \mathbb{C})$ can have non-constant first integrals.

**Proposition 5.2.** If $\mathfrak{g}$ is a Lie subalgebra of $\text{sp}(2, \mathbb{C})$ and $\dim \mathbb{C} \mathfrak{g} > 0$, then the number of algebraically independent rational first integrals of $\mathfrak{g}$ is not greater than one.

**Proof.** As $\dim \mathbb{C} \mathfrak{g} > 0$, there exists a non-zero $H \in \text{sp}(2, \mathbb{C}) \simeq \mathbb{C}[x, y]$. The number of rational algebraically independent first integrals of a non-zero linear Hamiltonian vector field $X_H$ in $\mathbb{C}^2$ is at most one. \hfill \Box

**Proposition 5.3.** If $\mathfrak{g}$ is a Lie subalgebra of $\text{sp}(2, \mathbb{C})$ and $\dim \mathbb{C} \mathfrak{g} = 2$, then $\mathbb{C}(x, y)^0 = \mathbb{C}$.

**Proof.** All two dimensional Lie algebras are solvable so $\mathfrak{g}$ is solvable. Thus a connected Lie group $G \subset \text{sp}(2, \mathbb{C})$ with Lie algebra $\mathfrak{g}$ is solvable. By the Lie-Kolchin theorem $G$ is conjugate to the triangular group

$$\mathcal{T} := \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{C}^*, \ b \in \mathbb{C} \right\}.$$  \hfill (5.1)

The Lie algebra $\mathfrak{t}$ of $\mathcal{T}$ is isomorphic to $\mathfrak{g}$, and is generated by two elements

$$h_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$  \hfill (5.2)
Let $H_1$ and $H_2$ be Hamiltonian functions from $\mathbb{C}^2[x,y]$ such that linear vector fields $X_{H_1}$ and $X_{H_2}$ have matrices $h_1$ and $h_2$, respectively. It is easy to check that

$$H_1 = xy, \quad H_2 = \frac{1}{2}y^2.$$  \hfill (5.3)

We show that $\mathbb{C}(x,y)^1 = \mathbb{C}$. Assume that there exists $f \in \mathbb{C}(x,y)^1 \setminus \mathbb{C}$. Hence $\{f, H_i\} = 0$ for $i = 1, 2$. But

$$\{f, H_2\} = y\frac{\partial f}{\partial x} = 0,$$

so, $f \in \mathbb{C}(y)$. However, for $f \in \mathbb{C}(y)$, we have

$$\{f, H_1\} = -y\frac{\partial f}{\partial y} = 0,$$

and this implies that $f \in \mathbb{C}$. A contradiction with assumption that $f$ is not a constant shows that $\mathbb{C}(x,y)^1 = \mathbb{C}$. Moreover, as Lie algebras $t$ and $\mathfrak{g}$ are isomorphic, we have also $\mathbb{C}(x,y)^0 = \mathbb{C}$. \hfill \(\square\)

Now, we consider a case adopted for a variational equation of the form \((1.7)\). For such equations the differential Galois group $\mathfrak{G}$ is a direct product

$$\mathfrak{G} = \mathfrak{G}_1 \times \cdots \times \mathfrak{G}_n,$$  \hfill (5.4)

where $\mathfrak{G}_i$ is an algebraic subgroup of $\text{Sp}(2, \mathbb{C})$, for $i = 1, \ldots, n$. Hence, the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$ is also a direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n,$$  \hfill (5.5)

where $\mathfrak{g}_i$ is a Lie subalgebra of $\text{sp}(2, \mathbb{C})$, for $i = 1, \ldots, n$. Let us denote by $\mathfrak{s}_n$ the Lie algebra which is the direct sum of $n$ copies of $\text{sp}(2, \mathbb{C})$

$$\mathfrak{s}_n := \underbrace{\text{sp}(2, \mathbb{C}) \oplus \cdots \oplus \text{sp}(2, \mathbb{C})}_{n\text{-times}},$$  \hfill (5.6)

and by $\pi_i : \mathfrak{s}_n \to \text{sp}(2, \mathbb{C})$, the projection onto the $i$-th component of $\mathfrak{s}_n$, for $i = 1, \ldots, n$. If we make identification $\text{sp}(2n, \mathbb{C}) \cong \mathbb{C}^2[x,y]$, then $\mathfrak{s}_n$ is viewed as

$$\mathfrak{s}_n = \bigoplus_{i=1}^n \mathbb{C}_2[x_i,y_i].$$  \hfill (5.7)

**Lemma 5.1.** Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{s}_n$, and $\mathfrak{g}_i = \pi_i(\mathfrak{g})$ for $i = 1, \ldots, n$. Assume that $f \in \mathbb{C}(x,y)$ is a non-constant rational first integral of $\mathfrak{g}$, and that there exists $1 \leq j \leq n$, such that $\mathfrak{g}_j$ is not Abelian, Then $f$ does not depend on $x_j$ and $y_j$.

**Proof.** As $\mathfrak{g}_j$ is not Abelian its dimension is greater than one. Let us consider the case $\dim_{\mathbb{C}} \mathfrak{g}_j = 3$. Then $\mathfrak{g}_j = \text{sp}(2, \mathbb{C})$, and we proceed as in the proof of Proposition 5.1. First integral $f$ of $\mathfrak{g}$ is, in particular, a first integral of $\mathfrak{g}_j$. Hence, for each $H \in \mathfrak{g}_j \cong \mathbb{C}_2[x_j,y_j]$ we have $\{H, f\} = 0$. For $H = x_j^2$, we obtain

$$0 = \{H, f\} = -2x_j\frac{\partial f}{\partial y_j},$$

so $f$ does not depend on $y_j$. Taking $H = y_j^2$ we show that $f$ does not depend on $x_j$.

If $\dim_{\mathbb{C}} \mathfrak{g}_j = 2$, then we proceed in a similar way using arguments from the proof of Proposition 5.3. \hfill \(\square\)
From the above lemma we have the following consequences.

**Corollary 5.1.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \mathfrak{s}_n \), and \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). If \( \mathfrak{g}_i \) is not Abelian for \( i = 1, \ldots, n \), then \( \mathbb{C}(x, y)^\mathfrak{g} = \mathbb{C} \).

**Corollary 5.2.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \mathfrak{s}_n \), and \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). If \( f \in \mathbb{C}(x, y) \setminus \mathbb{C} \) is a first integral of \( \mathfrak{g} \), then there exists \( 1 \leq j \leq n \) such that \( \mathfrak{g}_j \) is Abelian.

Now, we consider a case when \( \mathfrak{g} \) admits more than one independent first integral.

**Lemma 5.2.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \mathfrak{s}_n \), and \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). If \( \mathfrak{g} \) admits two algebraically independent and commuting first integrals \( f, g \in \mathbb{C}(x, y) \), then there exist \( 1 \leq i < j \leq n \), such that \( \mathfrak{g}_i \) and \( \mathfrak{g}_j \) are Abelian.

**Proof.** Neither \( f \) nor \( g \) is a constant. Thus, by Corollary 5.2 there exists \( 1 \leq i \leq n \) such that \( \mathfrak{g}_i \) is Abelian. Without loss of generality we can assume that \( \mathfrak{g}_1 \) is Abelian. Suppose that \( \mathfrak{g}_j \) for \( 2 \leq j \leq n \) are not Abelian. Then, from Lemma 5.1 it follows that \( f \) and \( g \) do not depend on \( (x_j, y_j) \) for \( 2 \leq j \leq n \), and hence \( f, g \in \mathbb{C}(x_1, y_1) \). But \( f \) and \( g \) commute, thus

\[
0 = \{f, g\} = \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial y_1} - \frac{\partial f}{\partial y_1} \frac{\partial g}{\partial x_1} = \frac{\partial (f, g)}{\partial (x_1, y_1)}.
\]

so \( f \) and \( g \) are functionally, and thus algebraically dependent. A contradiction finishes the proof.

To prove the next lemma we need the following well known fact, see e.g. Proposition 3.7 on page 63 in [15].

**Proposition 5.4.** Consider \( \mathbb{C}^{2m} \) as a linear symplectic space with the canonical coordinates \( (q, p) = (q_1, \ldots, q_m, p_1, \ldots, p_m) \). If \( f_1, \ldots, f_l \in \mathbb{C}(q, p) \) are algebraically independent and commuting, then \( l \leq m \).

**Lemma 5.3.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \mathfrak{s}_n \), and \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). If \( \mathfrak{g} \) admits first integrals \( f_1, \ldots, f_p \in \mathbb{C}(x, y) \), where \( 1 < p \leq n \), which are algebraically independent and commuting, then there exist \( 1 \leq i_1 < \ldots < i_p \leq n \), such that \( \mathfrak{g}_{i_1}, \ldots, \mathfrak{g}_{i_p} \) are Abelian.

**Proof.** We prove this lemma by induction with respect to \( p \). We have already proved this lemma for \( p = 2 \). Assume that this lemma is valid for \( p = j \), where \( 2 < j < n \). We show that it is valid for \( p = j + 1 \).

We have \( j + 1 \) commuting and independent first integrals \( f_1, \ldots, f_{j+1} \). From the inductive assumption it follows that among all \( \mathfrak{g}_i \) at least \( j \) are Abelian. We can assume that \( \mathfrak{g}_1, \ldots, \mathfrak{g}_j \) are Abelian. We have to show that there exits \( j < l \leq n \) such that \( \mathfrak{g}_l \) is Abelian. We prove it by contraction. Thus assume that \( \mathfrak{g}_{j+1}, \ldots, \mathfrak{g}_n \) are not Abelian. Then, from Lemma 5.1 it follows that integrals \( f_1, \ldots, f_{j+1} \) do not depend on \( (x_i, y_i) \) for \( i = j + 1, \ldots, n \). Thus, \( f_1, \ldots, f_{j+1} \in \mathbb{C}(x_1, \ldots, x_j, y_1, \ldots, y_j) \), and we have a contradiction with Proposition 5.4. This finishes the proof.

**Corollary 5.3.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( \mathfrak{s}_n \), and \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). If \( \mathfrak{g} \) admits algebraically independent and commuting first integrals \( f_1, \ldots, f_n \in \mathbb{C}(x, y) \), then \( \mathfrak{g}_i \) is Abelian for \( i = 1, \ldots, n \).

**Proposition 5.5.** Let \( \mathfrak{h} \) be a one dimensional Lie subalgebra of \( \mathfrak{sp}(2, \mathbb{C}) \cong \mathbb{C}_2[x_1, y_1] \). If \( f \in \mathbb{C}(x_1, \ldots, x_n, y_1, \ldots, y_n) \setminus \mathbb{C} \) is a rational first integral of \( \mathfrak{h} \), then there exists an element \( h \in \mathbb{C}[x_1, y_1] \setminus \mathbb{C} \) such that \( f \in \mathbb{C}(h, x_2, \ldots, x_n, y_2, \ldots, y_n) \).
Proof. Let us assume that \( h \) is nilpotent, i.e., \( h \) is generated by \( H = y^2 \). Then we have

\[ 0 = \{H, f\} = -2y_1 \frac{\partial f}{\partial x_1} . \]

Thus, \( f \) does not depend on \( x_1 \), so for this case we choose \( h = y_1 \).

The only other possibility is that \( h \) is diagonal, i.e., it is generated by \( H = x_1y_1 \). First, let us assume that \( f \) is a polynomial in \( (x_1, y_1) \). We can consider \( f \) as an element of ring \( R[x_1, y_1] \), where \( R = \mathbb{C}(x_2, \ldots, x_n, y_2, \ldots, y_n) \). We can write \( f \) uniquely as a sum of homogeneous components. Here ‘homogeneity’ means the homogeneity with respect to \( (x_1, y_1) \). It is clear that if \( f \) is a first integral of \( H \), then each homogeneous component of \( f \) is also a first integral of \( H \). Thus, let us assume that \( f \) is homogeneous of degree \( s \) and let us represent it in the form

\[ f = \sum_{i=0}^{s} f_i x_1^i y_1^{s-i}, \quad f_i \in R \quad \text{for} \quad i = 1, \ldots, s. \]  

(5.8)

Then we obtain

\[ 0 = \{H, f\} = \sum_{i=0}^{s} f_i (s - i)x_1^i y_1^{s-i} - \sum_{i=0}^{s} if_i x_1^i y_1^{s-i} = \sum_{i=0}^{s} f_i (s - 2i)x_1^i y_1^{s-i}. \]

Hence, \( f_i = 0 \) for \( 2i \neq s \), and if for even \( s = 2r \), \( f_r \neq 0 \), then \( f = f_r(x_1y_1)^r \). This implies that every homogeneous, and thus arbitrary, polynomial first integral \( f \in R[x_1, y_1] \) of \( H \) is an element of \( R[h] \), where \( h = x_1y_1 \).

Now, assume that \( f \) is a rational first integral of \( H \). Then we can write \( f = P/Q \) where \( P \) and \( Q \) are relatively prime polynomials in \( R[x_1, y_1] \). Hence we have

\[ 0 = \{H, P/Q\} = \frac{1}{Q^2} (Q\{H, P\} - P\{H, Q\}) , \]

so \( Q\{H, P\} = P\{H, Q\} \). As \( P \) and \( Q \) are relatively prime this implies that

\[ \{H, P\} = \gamma P \quad \text{and} \quad \{H, Q\} = \gamma Q, \]  

(5.9)

for a certain \( \gamma \in R[x_1, y_1] \). Comparing the degrees of both sides in the above equalities, we deduce that \( \gamma \in \mathbb{C} \). If \( \gamma = 0 \), then \( P \) and \( Q \) are polynomial first integrals of \( H \), so in this case we have that \( f \in R(h) = \mathbb{C}(h, x_2, \ldots, x_n, y_2, \ldots, y_n) \).

We show that case \( \gamma \neq 0 \) is impossible. Let us assume that \( \gamma \neq 0 \). It is easy to see that if \( P \in R[x_1, y_1] \) satisfies equation

\[ \{H, P\} = \gamma P, \]  

(5.10)

then its every homogeneous component also satisfies this equation. Thus let us assume that \( P \) is homogeneous of degree \( s \). If we write

\[ P = \sum_{i=0}^{s} P_i x_1^i y_1^{s-i}, \quad P_i \in R \quad \text{for} \quad i = 1, \ldots, s, \]  

(5.11)

then, equation (5.10) leads to the following equality

\[ \sum_{i=0}^{s} P_i (s - 2i - \gamma)x_1^i y_1^{s-i} = 0. \]  

(5.12)
This implies that if coefficient \( P_i \neq 0 \), then \( \gamma = s - 2i \) and \( P = P_i x_1^i y_1^{s-i} \). Thus, every homogeneous solution of (5.10) is a monomial of the form \( P_i x_1^i y_1^{s-i} \gamma \), where \( \gamma \) is a non-zero integer and \( i \) is a non-negative integer such that \( i + \gamma \geq 0 \). Thus a non-homogeneous solution of (5.10) is a finite sum

\[
P = \sum_{i+\gamma>0} p_i x_1^i y_1^{s+i+\gamma}.
\]

But \( Q \) satisfies the same equation (5.10), so we have also

\[
Q = \sum_{j+\gamma>0} q_j x_1^j y_1^{s+j+\gamma}.
\]

If \( \gamma > 0 \), then \( P \) and \( Q \) are not relatively prime because they have a common factor \( y_1^\gamma \). On the other hand, if \( \gamma < 0 \), then they are not relatively prime either because they have a common factor \( x_1 \). We have a contradiction and this finishes the proof. \( \square \)

**Lemma 5.4.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( s_n \), \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). Assume that \( f \in \mathbb{C}(x, y) \setminus \mathbb{C} \) is a first integral of \( \mathfrak{g} \). If \( \dim_{\mathbb{C}} \mathfrak{g}_i = 1 \) for \( i = 1, \ldots, n \), then there exist \( h_i \in \mathbb{C}[x_i, y_i] \setminus \mathbb{C} \) for \( i = 1, \ldots, n \) such that \( f \in \mathbb{C}(h_1, \ldots, h_n) \).

**Proof.** It is enough to apply \( n \)-times Proposition 5.5 taking \( \mathfrak{g}_i \) as \( \mathfrak{h} \) for \( i = 1, \ldots, n \). \( \square \)

**Lemma 5.5.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( s_n \), \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). Assume that \( f_1, \ldots, f_{n+1} \in \mathbb{C}(x, y) \) are algebraically independent first integrals of \( \mathfrak{g} \) and, moreover, \( f_1, \ldots, f_n \) commute. Then there exists \( 1 \leq i \leq n \) such that \( \dim_{\mathbb{C}} \mathfrak{g}_i = 0 \).

**Proof.** As \( \mathfrak{g} \) admits \( n \) commuting and independent first integrals, by Corollary 5.3 \( \mathfrak{g}_i \) is Abelian, and thus \( \dim_{\mathbb{C}} \mathfrak{g}_i \leq 1 \) for \( i = 1, \ldots, n \).

We prove the statement of the lemma by contradiction. Thus let us assume that \( \dim_{\mathbb{C}} \mathfrak{g}_i = 1 \) for \( i = 1, \ldots, n \). Then, by Lemma 5.4 \( f_i \in \mathbb{C}(h_1, \ldots, h_n) \) for \( i = 1, \ldots, n+1 \). By assumption \( f_1, \ldots, f_{n+1} \) are algebraically independent. But in \( \mathbb{C}(h_1, \ldots, h_n) \) any set of \( s > n \) elements is algebraically dependent. A contradiction finishes the proof. \( \square \)

The above lemma can be generalised in the following way.

**Lemma 5.6.** Let \( \mathfrak{g} \) be a Lie subalgebra of \( s_n \), \( \mathfrak{g}_i = \pi_i(\mathfrak{g}) \) for \( i = 1, \ldots, n \). Assume that \( f_1, \ldots, f_{n+s} \in \mathbb{C}(x, y) \), \( 1 \leq s < n \) are algebraically independent first integrals of \( \mathfrak{g} \) and, moreover, \( f_1, \ldots, f_n \) commute. Then there exist \( 1 \leq i_1 < \cdots < i_s \leq n \) such that \( \dim_{\mathbb{C}} \mathfrak{g}_{i_j} = 0 \), for \( j = 1, \ldots, s \).

This lemma can be easily proved by induction. We leave a proof to the reader.

### 6 Proofs of Theorem 1.2 and 1.3

Having the results collected in the two previous sections proofs of theorems 1.2 and 1.3 are very simple.

**Proof of Theorem 1.2** The differential Galois group of variational equations (1.7) has the form of product (3.2), hence its Lie algebra \( \mathfrak{g} \) is a Lie subalgebra of \( s_n \). If the considered system admits \( l \) functionally independent and commuting first integrals, then by Proposition 2.1 \( \mathfrak{g} \) has \( l \) algebraically independent and commuting first integrals. By Lemma 5.3 there exist \( 1 \leq i_1 < \cdots < i_l \leq n \) such that algebras \( \mathfrak{g}_{i_1} = \pi_{i_1}(\mathfrak{g}) \) are Abelian for \( s = 1, \ldots, l \). As the identity components of the differential Galois groups of equation (3.1) and (3.4) are the same, we have that \( \mathfrak{g}(s, \lambda_i) \) are Abelian for \( s = 1, \ldots, l \). The statement of the theorem follows directly from Proposition 4.1. \( \square \)
Proof of Theorem 1.3. The normal variational equations for the considered solution are a direct product of first \( n - 1 \) of equations (1.7). Hence, the Lie algebra \( g \) of their differential Galois group is a Lie subalgebra \( s_{n-1} \). By assumption, Lie algebra \( g \) admits first integrals \( f_2, \ldots, f_{n+l} \), such that \((n - 1)\) of them \( f_2, \ldots, f_n \) commute. By Lemma (5.6), \( l \) among Lie algebras \( g_i = \pi_i(s_{n-1}) \), for \( i = 1, \ldots, n - 1 \), are zero dimensional. Without loss of the generality we can assume that \( \dim \mathbb{C} g_i = 0 \) for \( i = 1, \ldots, l \). Then \( \hat{G}(k, \lambda_i)^\circ \simeq G(k, \lambda_i)^\circ = \{E\} \) for \( i = 1, \ldots, l \). Assume that \( |k| > 2 \). Then, by point 1. of Corollary 4.2, \( (k, \lambda_i) \) belongs to an item 3–9 in table (1.8). For \( |k| \leq 2 \), by point 2. of Corollary 4.2, \( (k, \lambda_i) \) belongs to an item of table (4.8), for \( i = 1, \ldots, l \). \( \square \)

7 Examples

As the first example, we consider the following potential

\[
V = \frac{1}{k} \left[ (q_1 - q_2)^k + (q_2 - q_3)^k + (q_3 - q_1)^k \right],
\]

(7.1)

with an integer \( k \). We exclude the uninteresting cases, \( k = 0, 1 \), from the beginning. This system describes a motion of three particles with equal masses on a line, with coordinates \( q_1, q_2, q_3 \), respectively. For an arbitrary \( k \) this system is partially integrable because it has a first integral

\[
F_2 = p_1 + p_2 + p_3,
\]

which is the total momentum of the system. Furthermore, this system is integrable in the Liouville sense for \( k \in \{4, 2, -2\} \), and even super-integrable for \( k = \pm 2 \). Indeed, the additional first integrals for each case are the following:

- for \( k = 4 \):
  \[
  F_3 = p_1(q_2 - q_3) + p_2(q_3 - q_1) + p_3(q_1 - q_2),
  \]

- for \( k = 2 \):
  \[
  F_3 = p_1(q_2 - q_3) + p_2(q_3 - q_1) + p_3(q_1 - q_2),
  \]

  \[
  F_4 = (p_2 - p_3)^2 + 3(q_2 - q_3)^2,
  \]

- for \( k = -2 \):

\[
F_3 = \frac{2}{3}(p_1^3 + p_2^3 + p_3^3) - \frac{p_1 + p_2}{(q_1 - q_2)^2} - \frac{p_2 + p_3}{(q_2 - q_3)^2} - \frac{p_3 + p_1}{(q_3 - q_1)^2},
\]

\[
F_4 = (q_1 + q_2 + q_3) \left[ \frac{q_1 + q_2}{(q_1 - q_2)^2} - \frac{q_2 + q_3}{(q_2 - q_3)^2} - \frac{q_3 + q_1}{(q_3 - q_1)^2} \right]
\]

\[
- (p_1 + p_2 + p_3)(p_1 q_1 + p_2 q_2 + p_3 q_3),
\]

\[
F_5 = (p_1 + p_2 + p_3) \left[ 2(p_1^2 q_1 + p_2^2 q_2 + p_3^2 q_3) - \frac{q_1 + q_2}{(q_1 - q_2)^2} - \frac{q_2 + q_3}{(q_2 - q_3)^2} - \frac{q_3 + q_1}{(q_3 - q_1)^2} \right]
\]

\[
- 3(q_1 + q_2 + q_3) \left[ 2(p_1^3 + p_2^3 + p_3^3) - \frac{p_1 + p_2}{(q_1 - q_2)^2} - \frac{p_2 + p_3}{(q_2 - q_3)^2} - \frac{p_3 + p_1}{(q_3 - q_1)^2} \right].
\]

Case \( k = 2 \) is just a three particle harmonic oscillator, while case \( k = -2 \) is a special case of the Calogero-Moser system. The Calogero-Moser system has a Lax pair representation and this fact allows to prove its super-integrability, see [24]. On the other hand, it seems that the case \( k = 4 \) was first realized to be integrable in [25]. Until now
no further integrals and no further values of \( k \) for which the system is integrable or super-integrable have been found.

For this system, let us see how Theorems 1.2 and 1.3 work. For this purpose we need solutions of the algebraic equation \( V'(d) = d \). We do not know how to find all of them for an arbitrary \( k \), nevertheless, it is sufficient to know some of them. Here we use two solutions. The first one is

\[
d_1 = (c, 0, -c), \quad c^{k-2} = \frac{1}{1 + (-1)^k 2^{k-1}},
\]

and it exists for all \( k \neq 2 \). The second one, which exists only when \( k > 2 \) is an even integer, is

\[
d_2 = (c, -2c, c), \quad c^{k-2} = \frac{1}{3^{k-1}}.
\]

The eigenvalues of the Hessian matrix \( V''(d_1) \) are

\[
(\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}) = \left( \frac{3(k-1)}{1 + (-1)^k 2^{k-1}}, 0, k-1 \right),
\]

and eigenvalues of \( V''(d_2) \) are following

\[
(\lambda_{2,1}, \lambda_{2,2}, \lambda_{2,3}) = \left( \frac{(k-1)}{3}, 0, k-1 \right).
\]

First, the considered system is partially integrable because of the existence of two commuting first integrals, \( F_1 = H \) and \( F_2 \). Then Theorem 1.2 requires that for each \( i = 1, 2 \) at least two pairs of \((k, \lambda_{i,j})\) belong to the list (1.8). Indeed this is the case, as \( \lambda_{i,2} = 0 \) and \( \lambda_{i,3} = k-1 \) for \( i = 1, 2 \) are always in item 2 of the list (1.8).

We show that for values of \( k \) different from given above there is no integrable cases.

**Lemma 7.1.** Assume that \( k \in \mathbb{Z} \setminus \{-2, 0, 1, 2, 4\} \). Then the Hamiltonian system with potential (7.1) is not integrable in the Liouville sense.

**Proof.** We prove the statement of the lemma by a contradiction. Thus let \( k \in \mathbb{Z} \setminus \{-2, 0, 1, 2, 4\} \) and the system is integrable in the Liouville sense. Then, from Theorem 1.1 or 1.2 it follows that \((k, \lambda_{i,1})\) are in the list (1.8). We show that for each \( k \in \mathbb{Z} \setminus \{-2, 0, 1, 2, 4\} \) either \( \lambda_{i,1} \) or \( \lambda_{2,1} \) does not belong to the list.

Assume that \( k \geq 3 \) is an odd integer. Then \( \lambda_{1,1} = 3(k-1)/(1 - 2^{k-1}) < 0 \) but for positive \( k \) the allowed values in the list are non-negative.

Assume \( k \geq 6 \) is an even integer. We show that \( \lambda_{2,1} = (k-1)/3 \) does not belong to an item of table (1.8). We have only two possibilities: either \( \lambda_{2,1} \) belongs to item 2 or to item 3. On the other hand, item 2 and item 3 with integer \( p \) gives a strictly increasing sequence of numbers

\[
\left( 0, \frac{k-1}{2k}, 1, k-1, k + \frac{k-1}{2k}, k + 2, 3k - 2, 3k + \frac{k-1}{2k}, \ldots \right),
\]

while \( 1 < (k-1)/3 < k-1 \) when \( k \geq 6 \). Thus \( \lambda_{2,1} \) does not belong to an item of table (1.8).

When \( k = -1 \), \( \lambda_{1,1} = -8 \) does not belong to the list.

Assume \( k \leq -3 \) is a negative integer. We show that \( \lambda_{1,1} = 3(k-1)/(1 + (-1)^k 2^{k-1}) \) does not belong to an item of table (1.8). We first show that \( \lambda_{1,1} \) does not belong to item...
2 nor to item 3. Indeed, item 2 and item 3 with integer $p$ gives a strictly decreasing sequence of numbers
\[
\left(1, \frac{k-1}{2k}, 0, k+2, k + \frac{k-1}{2k}, k-1, 3k+3, 3k + \frac{k-1}{2k}, 3k-2, 6k+4, \ldots \right).
\]

On the other hand one can easily verify the inequality
\[
3k-2 > \frac{3(k-1)}{1+(-1)^{2k-1}} > 6k+4,
\]
when $k \leq -3$. Thus $\lambda_{1,1}$ does not belong to an item 2 and 3 of table (1.8).

When $k \in \{-5, -4, -3\}$, we have to check also that $\lambda_{1,1}$ does not belong to items 7-9 in table (1.8). This task is reduced to checking if a certain quadratic polynomial has an integer root. Let us consider for example case $k = -3$. Then $\lambda_{1,1} = -64/3$. For $k = -3$ items 2, 3 and 7 are allowed. Assume that $\lambda_{1,1}$ is given by the first expression in item 7. Then equation
\[
\frac{25}{24} - \frac{1}{6} (1+3p)^2 = -\frac{64}{3},
\]
must have an integer solution for $p$. But as it easy to show that it has not such a root. 

Finally, let us apply Theorem 1.3 to potential (7.1) and check how the conditions for the super-integrability are verified. Among three integrable values of $k$, the case $k = 4$ cannot be super-integrable since the set of eigenvalues is
\[
(\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}) = (\lambda_{2,1}, \lambda_{2,2}, \lambda_{2,3}) = (1, 0, 3)
\]
(7.2)
and none of $\lambda_{i,j}$ belongs to items 3-9 of the list. On the other hand, case $k = -2$ gives $(\lambda_{1,1}, \lambda_{1,2}, \lambda_{1,3}) = (-8, 0, -3)$. The non-trivial eigenvalues are $-8$ and 0. These values are compatible with the maximal super-integrability, as they are given by the first item in table (1.9).

For $k = 2$, equation $V'(d) = \gamma d$ has a non-zero solution only for $\gamma = 3$. Eigenvalues $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)$ of $V''(d)$ do not depend on $d$, and we have $(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = (3, 0, 3)$, so $(\lambda_1, \lambda_2, \lambda_3) = (1, 0, 1)$, see Remark 1.2. The nontrivial eigenvalues are 0 and 1. Only one of them, namely 1, belongs to the forth item in table (1.9), so the system is super-integrable but it cannot be maximally super-integrable. The fact that the system cannot be maximally super-integrable also follows from the following. The Hamiltonian of the system is a quadratic function of canonical variables. Thus we can transform it into the normal form. It is easy to show that this normal form is following
\[
K = \frac{1}{2} \sqrt{3}(x_1^2 + y_1^2) + \frac{1}{2} \sqrt{3}(x_2^2 + y_2^2) + \frac{1}{2} y_3^2.
\]

Hence the phase curves of the systems lie on two dimensional cylinders. But for a maximally super-integrable system the maximal dimension of an invariant set is one.

Potential (7.1) has the following higher dimensional generalisation
\[
V = \frac{1}{k} \sum_{i=1}^{n} (q_i - q_{i+1})^k, \quad q_{n+1} \equiv q_1.
\]

(7.3)

For $k = -2$, as it was shown in [24], this potential is maximally super-integrable. The question appears whether it is integrable for $k = 4$ and $n > 3$. We show that for $n = k = 4$ this potential is only partially integrable with no more than one additional first integral.
Lemma 7.2. Assume that \( n = k = 4 \). Then the Hamiltonian system with potential (7.3) admits no more than two functionally independent meromorphic and commuting first integrals, namely the Hamiltonian and \( F_2 = p_1 + p_2 + p_3 + p_4 \).

Proof. For \( n = k = 4 \) algebraic equation \( V'(d) = d \) has a solution

\[
d = \frac{1}{4}(-1, 1, 1, -1).
\]

For this Darboux point eigenvalues of \( V''(d) \) are following

\[
(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left( \frac{3}{2}, \frac{3}{2}, 0, 3 \right).
\]

It is easy to verify that \( \lambda_3 \) and \( \lambda_4 \) are given by the second item in table (1.8) but there is no item giving \( \lambda_1 = \lambda_2 = 3/2 \). Hence, by Theorem (1.2), the system admits no more than two independent and commuting first integrals.

The classical Bertrand theorem [5], states that the only radial potentials \( V(r) = \alpha r^k \), for which all bounded orbits are periodic, are those with \( k = -1 \) and \( k = 2 \). The condition that all bounded orbits of a Hamiltonian system are periodic means that the system is degenerated, i.e., all invariant tori are one dimensional. Such degeneration appears if the system is maximally super-integrable. Having this in mind, let us apply Theorem 1.3 to a radial potential of the form

\[
V = \alpha r^k, \quad \alpha \neq 0, \quad r = \sqrt{q_1^2 + q_2^2}, \quad (7.4)
\]

with an integer \( k \). The Hamiltonian system with this potential is integrable as it admits the angular momentum integral, \( F_2 = q_1 p_2 - q_2 p_1 \). We show the following.

Lemma 7.3. The Hamiltonian system with potential (7.4) is super-integrable if and only if \( k = -1 \) and \( k = 2 \).

Proof. Potential (7.4) admits infinitely many Darboux points, but, at each of them, the eigenvalues of the Hessian \( V'' \) are \((1, k - 1)\), and the non-trivial one is \( \lambda = 1 \). Let us apply Theorem 1.3 to check whether this system can be super-integrable. Assume that \(|k| > 2\). Then, by Theorem 1.3, the non-trivial eigenvalue should be given by an item 3–9 in table 1.8, but all these items give non-integer values of \( \lambda \). If \(|k| \leq 2\), then \((k, \lambda) = (1, 1)\) do not belong to this table while \((k, \lambda) = (2, 1)\) and \((k, \lambda) = (-1, 1)\) do. Both cases \( k = -1 \) and \( k = 2 \) are indeed super-integrable, because of the existence of the third integral,

\[
F_3 = p_2(q_1 p_2 - q_2 p_1) + \alpha \frac{q_1}{r},
\]

for \( k = -1 \), and

\[
F_3 = p_1 p_2 + 2\alpha q_1 q_2,
\]

for \( k = 2 \).

Thus, at least as far as the above example is concerned, Theorem 1.2 and 1.3 together have the full predicting power. They predict all integrable, partially integrable and super-integrable cases without any exceptions.
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A Appendix

A.1 Second order differential equations with rational coefficients

Let us consider a second order differential equation of the following form

\[ y'' = ry, \quad r \in \mathbb{C}(z), \quad \equiv \frac{d}{dz}. \tag{A.1} \]

For this equation its differential Galois group \( \mathcal{G} \) is a linear algebraic subgroup of \( SL(2, \mathbb{C}) \).

The following lemma describes all possible types of \( \mathcal{G} \) and relates these types to the forms of solutions of (A.1), see [12, 15].

**Lemma A.1.** Let \( \mathcal{G} \) be the differential Galois group of equation (A.1). Then one of four cases can occur.

1. \( \mathcal{G} \) is reducible (it is conjugate to a subgroup of triangular group); in this case equation (A.1) has an exponential solution of the form \( y = \exp \int \omega \), where \( \omega \in \mathbb{C}(z) \),

2. \( \mathcal{G} \) is conjugate with a subgroup of

\[
\mathcal{D}P = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \mid c \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} \mid c \in \mathbb{C}^* \right\} ,
\]

in this case equation (A.1) has a solution of the form \( y = \exp \int \omega \), where \( \omega \) is algebraic over \( \mathbb{C}(z) \) of degree 2,

3. \( \mathcal{G} \) is primitive and finite; in this case all solutions of equation (A.1) are algebraic,

4. \( \mathcal{G} = SL(2, \mathbb{C}) \) and equation (A.1) has no Liouvillian solution.

We need a more precise characterisation of case 1 in the above lemma. It is given by the following lemma, see Lemma 4.2 in [21].

**Lemma A.2.** Let \( \mathcal{G} \) be the differential Galois group of equation (A.1) and assume that \( \mathcal{G} \) is reducible. Then either

1. equation (A.1) has a unique solution \( y \) such that \( y' / y \in \mathbb{C}(z) \), and \( \mathcal{G} \) is conjugate to a subgroup of the triangular group

\[
\mathcal{T} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a, b \in \mathbb{C}, a \neq 0 \right\} .
\]

Moreover, \( \mathcal{G} \) is a proper subgroup of \( \mathcal{T} \) if and only if there exists \( m \in \mathbb{N} \) such that \( y^m \in \mathbb{C}(z) \). In this case \( \mathcal{G} \) is conjugate to

\[
\mathcal{T}_m = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \mid a, b \in \mathbb{C}, a^m = 1 \right\} ,
\]

where \( m \) is the smallest positive integer such that \( y^m \in \mathbb{C}(z) \), or
2. equation (A.1) has two linearly independent solutions \( y_1 \) and \( y_2 \) such that \( y_i'/y_i \in \mathbb{C}(z) \), then \( \mathcal{G} \) is conjugate to a subgroup of the diagonal group

\[
\mathcal{D} = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \mid a \in \mathbb{C}, a \neq 0 \right\}.
\]

In this case, \( y_1 y_2 \in \mathbb{C}(z) \). Furthermore, \( \mathcal{G} \) is conjugate to a proper subgroup of \( \mathcal{D} \) if and only if \( y_m^i \in \mathbb{C}(z) \) for some \( m \in \mathbb{N} \). In this case \( \mathcal{G} \) is a cyclic group of order \( m \) where \( m \) is the smallest positive integer such that \( y_m^i \in \mathbb{C}(z) \).

### A.2 Riemann \( P \) equation

The Riemann \( P \) equation \([23]\) is the most general second order differential equation with three regular singularities. If we place, using homography, these singularities at \( z = 0 \), \( z = 1 \) and \( z = \infty \), then it has the form

\[
\frac{d^2w}{dz^2} + \left( \frac{1 - \rho_1 - \rho_2}{z} + \frac{1 - \sigma_1 - \sigma_2}{z - 1} \right) \frac{dw}{dz} + \left( \frac{\rho_1 \rho_2}{z^2} + \frac{\sigma_1 \sigma_2}{(z - 1)^2} + \frac{\tau_1 \tau_2 - \rho_1 \rho_2 - \sigma_1 \sigma_2}{z(z - 1)} \right) w = 0,
\]

(A.2)

where \((\rho_1, \rho_2), (\sigma_1, \sigma_2)\) and \((\tau_1, \tau_2)\) are the exponents at the respective singular points. These exponents satisfy the Fuchs relation

\[
\sum_{i=1}^{2} (\rho_i + \sigma_i + \tau_i) = 1.
\]

We denote the differences of exponents by

\[
\rho = \rho_1 - \rho_2, \quad \sigma = \sigma_1 - \sigma_2, \quad \tau = \tau_1 - \tau_2.
\]

The following lemma gives the necessary and sufficient condition for (A.2) to be reducible. It is a classical, well known fact, see \([3]\).

**Lemma A.3.** Equation (A.2) is reducible if and only there exist \( i, j, k \in \{1, 2\} \), such that

\[
\rho_i + \sigma_j + \tau_k \in \mathbb{Z}.
\]

(A.3)

Equivalently, equation (A.2) is reducible if and only if at least one number among

\[
\rho + \sigma + \tau, \quad -\rho + \sigma + \tau, \quad \rho - \sigma + \tau, \quad \rho + \sigma - \tau,
\]

(A.4)

is an odd integer.

From the above lemma it follows that if equation (A.2) is reducible, then we can always renumber exponents in such a way that

\[
\rho_1 + \sigma_1 + \tau_1 \in -\mathbb{N}_0,
\]

where \( \mathbb{N}_0 \) denotes the set of nonnegative integers. But then, from the Fuchs relation, we also have

\[
\rho_2 + \sigma_2 + \tau_2 \in \mathbb{N}.
\]

Hence, if (A.2) is reducible, we assume from now on that the exponents are numbered in this way.

For a more precise characterisation of the monodromy and differential Galois groups we need the following two lemmas. The first describes one solution of (A.2) in a case when it is reducible, see Lemma 4.3.6, p. 90 in \([9]\).
Lemma A.4. Assume that equation (A.2) is reducible, and moreover, at least one of the exponents’ differences $\rho$, $\sigma$, $\tau$ is not an integer. Then equation (A.2) has a solution of the form

$$w(z) = z^{\rho_1} (1 - z)^{\sigma_1} h(z),$$

where $h(z)$ is a polynomial, and $\deg h(z) \leq n := -\rho_1 - \sigma_1 - \tau_1$.

We also need one fact concerning the monodromy group of equation (A.2). This group is generated by two matrices $M_0, M_1 \in \text{GL}(2, \mathbb{C})$. These matrices correspond to homotopy classes $[\gamma_0]$ and $[\gamma_1]$ of loops with one common point encircling once in the positive sense singularities $z = 0$ and $z = 1$, respectively. Then we have the following lemma, see Lemma 4.3.5 on p. 90 in [9].

Lemma A.5. Assume that $M_0$ and $M_1$ are simultaneously diagonalisable. Then at least one of matrices $M_0$, $M_1$ or $M_0 M_1$ is a scalar matrix.

If the difference of exponents at a singular point is an integer, then it can happen that a local solution around this singularity contains a logarithm. Such a singularity is called logarithmic. In the case of equation (A.2), it is enough to know the exponents to decide which singularity is logarithmic. To formulate the next lemma which gives the necessary and sufficient conditions for a singularity of (A.2) to be logarithmic we introduce the following notation. For a non-negative integer $m \in \mathbb{N}_0$ we define

$$\langle m \rangle := \begin{cases} \emptyset & \text{if } m = 0, \\ \{1, \ldots, m\} & \text{otherwise.} \end{cases}$$

For $s \in \{0, 1, \infty\}$ let $e_{s,1}$ and $e_{s,2}$ denote exponents of equation (A.2), ordered in such a way that $\text{Re} e_{s,1} \geq \text{Re} e_{s,2}$. With the above notation we have the following.

Lemma A.6. Let $r \in \{0, 1, \infty\}$. Then $r$ is a logarithmic singularity of equation (A.2) if and only if $m := e_{r,1} - e_{r,2} \in \mathbb{N}_0$, and

$$e_{r,1} + e_{s,i} + e_{t,j} \notin \langle m \rangle, \quad \text{for } i, j \in \{1, 2\}, \quad (A.5)$$

where $r, s, t$ are pairwise different elements of $\{0, 1, \infty\}$.

For the proof, see Lemma 4.7 and its proof on pp. 91–93 in [9].

For equation (A.2) the necessary and sufficient conditions for solvability of the identity component of its differential Galois group are given by the following theorem due to Kimura [11], see also [15].

Theorem A.1 (Kimura). The identity component of the differential Galois group of equation (A.2) is solvable if and only if

A: at least one of four numbers $\rho + \sigma + \tau$, $-\rho + \sigma + \tau$, $\rho - \sigma + \tau$, $\rho + \sigma - \tau$, is an odd integer, or

B: the numbers $\rho$ or $-\rho$ and $\sigma$ or $-\sigma$ and $\tau$ or $-\tau$ belong (in an arbitrary order) to some of the following fifteen families
$\begin{array}{|c|c|c|c|}
\hline
1 & 1/2 + l & 1/2 + s & \text{arbitrary complex number} \\
\hline
2 & 1/2 + l & 1/3 + s & 1/3 + q \\
\hline
3 & 2/3 + l & 1/3 + s & 1/3 + q \quad l + s + q \text{ even} \\
\hline
4 & 1/2 + l & 1/3 + s & 1/4 + q \\
\hline
5 & 2/3 + l & 1/4 + s & 1/4 + q \quad l + s + q \text{ even} \\
\hline
6 & 1/2 + l & 1/3 + s & 1/5 + q \\
\hline
7 & 2/5 + l & 1/3 + s & 1/3 + q \quad l + s + q \text{ even} \\
\hline
8 & 2/3 + l & 1/5 + s & 1/5 + q \quad l + s + q \text{ even} \\
\hline
9 & 1/2 + l & 2/5 + s & 1/5 + q \quad l + s + q \text{ even} \\
\hline
10 & 3/5 + l & 1/3 + s & 1/5 + q \quad l + s + q \text{ even} \\
\hline
11 & 2/5 + l & 2/5 + s & 2/5 + q \quad l + s + q \text{ even} \\
\hline
12 & 2/3 + l & 1/3 + s & 1/5 + q \quad l + s + q \text{ even} \\
\hline
13 & 4/5 + l & 1/5 + s & 1/5 + q \quad l + s + q \text{ even} \\
\hline
14 & 1/2 + l & 2/5 + s & 1/3 + q \quad l + s + q \text{ even} \\
\hline
15 & 3/5 + l & 2/5 + s & 1/3 + q \quad l + s + q \text{ even} \\
\hline
\end{array}
$

where $l, s, q \in \mathbb{Z}$.

If the identity component $G^0$ of the differential Galois group $G$ of equation (A.2) is solvable, but the equation is not reducible, i.e., if case A in the Kimura theorem does not occur, then the differential Galois group is either an imprimitive finite group (families 2–15), or it is a subgroup of $D^\Phi$ group. In the last case $G$ can be finite or whole $D^\Phi$ group. The following lemma gives a criterion for distinction of these two cases.

**Lemma A.7.** Suppose equation (A.2) is not reducible. Then its differential Galois group $G$ is a subgroup of $D^\Phi$ group if and only if at two singular points the differences of exponents are half integers. Moreover, $G$ is a finite group if and only if the exponents at the remaining singular point are rational.

The above lemma is just case (b) of Theorem 2.9 from [6].

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