On uniform K-stability of pairs

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Abstract: In this paper, we discuss stable pairs, which were first studied by S. Paul, and give a proof for a result I learned from him. As a consequence, we will show that the K-stability implies the CM-stability.

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1 Introduction

Let \( \mathbf{G} \) be the linear group \( \text{SL}(N + 1, \mathbb{C}) \) and \( \mathbf{V}, \mathbf{W} \) be its two rational representations. By the rationality, say for \( \mathbf{V} \), we mean that for all \( \alpha \in \mathbf{V}^\vee \) (dual space) and \( v \in \mathbf{V} \setminus \{0\} \), the matrix coefficient \( \varphi_{\alpha,v} \) is a regular function on \( \mathbf{G} \), that is, \( \varphi_{\alpha,v} \in \mathbb{C}[\mathbf{G}] \), where

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†Our arguments here also work for any classical subgroups of \( \text{GL}(N + 1, \mathbb{C}) \) after some modifications, moreover, all representations in this paper are finite dimensional and complex.
\[ \varphi_{\alpha,v} : G \mapsto \mathbb{C}, \quad \varphi_{\alpha,v}(\sigma) = \alpha(\rho(\sigma)v). \quad (1.1) \]

Recall that by an one-parameter subgroup of \( G \), we mean a homomorphism \( \lambda : \mathbb{C}^* \rightarrow G \), where \( \mathbb{C}^* = \mathbb{C}\{0\} \) is the multiplicative group consisting of all non-zero complex numbers. For any such a \( \lambda \) and \( v \in V \), we can associate a weight as the unique integer \( w_\lambda(v) \) such that there is a non-zero limit \( v_0 \) in \( V \):

\[ \lim_{t \to 0} t^{-w_\lambda(v)}\lambda(t)(v) = v_0 \neq 0. \quad (1.2) \]

Let \( T \) be a maximal algebraic torus of \( G \) and write \( \text{gl}(N+1,\mathbb{C}) \) as \( \text{gl} \) for simplicity. Given such a \( T \), its character lattice is defined by

\[ M_\mathbb{Z} = \text{Hom}_\mathbb{Z}(T, \mathbb{C}^*). \quad (1.3) \]

Its dual lattice, denoted by \( N_\mathbb{Z} \), consists of all one parameter subgroups contained in \( T \). More explicitly, for each \( \ell \in N_\mathbb{Z} \), the corresponding one-parameter subgroup \( \lambda_\ell \) is given by

\[ m(\lambda_\ell(t)) = t^{(\ell,m)}, \quad \forall m \in M_\mathbb{Z}, \quad t \in T, \]

where \((\cdot,\cdot)\) is the standard pairing: \( N_\mathbb{Z} \times M_\mathbb{Z} \mapsto \mathbb{Z} \).

We have an associated vector space \( M_\mathbb{R} = M_\mathbb{Z} \otimes \mathbb{R} \cong \mathbb{R}^N \).

Since \( V \) is rational, it decomposes under the action of \( T \) into weight spaces

\[ V = \bigoplus_{a \in A} V_a, \quad \text{where} \quad V_a = \{ v \in V \mid t \cdot v = a(t)v, \ t \in T \}. \quad (1.4) \]

Here \( A = \{ a \in M_\mathbb{Z} \mid V_a \neq 0 \} \).

Given any \( v \in V \setminus \{0\} \), we denote by \( A(v) \) the set of all \( a \in A \) with \( v_a \neq 0 \), where \( v_a \) is the projection of \( v \) into \( V_a \).

The weight polytope \( \mathcal{N}(v) \) of \( v \) is defined to be the convex hull of \( A(v) \) in \( M_\mathbb{R} \). Since \( V \) is a rational representation, \( \mathcal{N}(v) \) is a rational polytope.

There is a natural representation \( \text{gl}(N+1,\mathbb{C}) \), which consists of all \((N+1) \times (N+1)\) matrices, by left multiplication:

\[ G \times \text{gl}(N+1,\mathbb{C}) \mapsto \text{gl}(N+1,\mathbb{C}) : (\sigma,B) \mapsto \sigma B. \]

The weight polytope \( \mathcal{N}(I) \) of the identity matrix \( I \) in \( \text{gl} \) is the standard \( N \)-simplex which contains the origin in \( M_\mathbb{R} \). Then we define degree \( \text{deg}(V) \) of \( V \) by

\[ \text{deg}(V) = \min \{ k \in \mathbb{Z} \mid k > 0 \ \text{and} \ \mathcal{N}(v) \subset k\mathcal{N}(I) \ \text{for all} \ 0 \neq v \in V \}. \quad (1.5) \]
Clearly, this definition implies

$$qw_\lambda(I) \leq w_\lambda(v).$$

where $q = \deg(V)$.

Now we can introduce the notion of K-stability due to S. Paul [Pa13].

**Definition 1.1.** Let $v \in V \setminus \{0\}$ and $w \in W \setminus \{0\}$.

- We call $(v, w)$ K-semistable if for any one-parameter subgroup $\lambda$ of $G$, we have $w_\lambda(w) \leq w_\lambda(v)$.
- We call $(v, w)$ K-stable if it is K-semistable and $w_\lambda(w) < w_\lambda(v)$ whenever the one-parameter subgroup $\lambda$ satisfying: $\deg(V) w_\lambda(I) < w_\lambda(v)$.

Here is our main theorem.

**Theorem 1.2.** If $(v, w)$ is K-stable, then there is an integer $m > 0$ such that for any one-parameter subgroup $\lambda$ of $G$, we have

$$m (w_\lambda(v) - w_\lambda(w)) \geq w_\lambda(v) - \deg(V) w_\lambda(I).$$

(1.6)

**Remark 1.3.** We may also say that $(v, w)$ is uniformly K-stable if (1.6) is satisfied for any one-parameter subgroups.

We denote by $\| \cdot \|$ a Hermitian norm on both $V$ and $W$ and define

$$p_{v,w}(\sigma) = \log \| \sigma(w) \|^2 - \log \| \sigma(v) \|^2.$$  

(1.7)

Then we have

**Theorem 1.4.** If $(v, w)$ is K-stable, then there is an integer $m > 0$ and a uniform constant $C$ such that

$$mp_{v,w}(\sigma) \geq \deg(V) \log \| \sigma \|^2 - \log \| \sigma(v) \|^2 - C.$$  

(1.8)

The organization of this paper is as follows: In next section, we recall a theorem on K-semistability first given by S. Paul and a proof of this theorem by S. Boucksom, T. Hisamoto and M. Jonsson. In Section 3, we show a connection between K-stability and positivity of certain line bundle. This line bundle is actually the CM-bundle in the case of constant scalar curvature Kähler metrics. In Section 4, we prove Theorem 1.2 and its consequence, Theorem 1.4. In Section 5, we give an application of our main theorem. We show that the K-stability implies the CM-stability. In Appendix A, we give a proof of a variant of Richardson’s lemma due to S. Paul.

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In this section, we discuss a theorem which was first given by S. Paul in [Pa12a].

**Theorem 2.1.** Let \((v, w)\) be a pair in \((V \setminus \{0\}) \times (W \setminus \{0\})\), then \(p_{v,w}\) is bounded from below on \(G\) if and only if \((v, w)\) is K-semistable.

This theorem has an equivalent form. Consider orbits \(G[v, w] \subset \mathbb{P}(V \oplus W)\) and \(G[v, 0] \subset \mathbb{P}(V \oplus \{0\})\), where \([v, w]\) (resp., \([v, 0]\)) denotes the corresponding point in the projective space \(\mathbb{P}(V \oplus W)\) (resp. \(\mathbb{P}(V \oplus \{0\})\)). We denote their closures by \(\overline{G[v, w]}\) and \(\overline{G[v, 0]}\).

It is shown by S. Paul in [Pa12a] that

\[
p_{v,w}(\sigma) = \log \tan^2 d(\sigma([v, w]), \sigma([v, 0])),
\]

where \(d(\cdot, \cdot)\) is the distance function of the Fubini-Study metric on \(\mathbb{P}(V \oplus W)\). It follows that \(p_{v,w}\) is bounded from below on \(G\) if and only if there is a constant \(c > 0\) such that

\[
d(\sigma([v, w]), \sigma([v, 0])) \geq c \quad \text{on} \quad G.
\]

It follows that \(p_{v,w}\) is bounded from below on \(G\) if and only if

\[
\overline{G[v, w]} \cap \overline{G[v, 0]} = \emptyset.
\]

Therefore, we have

**Theorem 2.2.** The pair \((v, w)\) is K-semistable if and only if (2.2) holds.

**Remark 2.3.** In [Pa12a], S. Paul called \((v, w)\) semistable if (2.2) holds. In the case that \(V = \mathbb{C}\) is a trivial representation, we can take \(v = 1\), then \((1, w)\) is semistable if and only if 0 is not in the closure of the affine orbit \(Gw\). In other words, \(w\) is semistable in the usual sense of Geometric Invariant Theory. This shows that the K-stability generalizes the notion of stability in classical Geometric Invariant Theory.

Let us give a proof of Theorem 2.2 following S. Boucksom, T. Hisamoto and M. Jonsson in [BHJ17]. This proof is based on the following theorem ([BHJ17, Theorem 5.6]).

**Theorem 2.4.** Let \(U\) be any rational representation of \(G\). If the (Zariski) closure of the \(G\)-orbit of a point \(x \in \mathbb{P}(U)\) meets a \(G\)-invariant Zariski closed subset \(Z \subset \mathbb{P}(U)\), then some \(z \in Z \cap \overline{Gx}\) can be reached by an one-parameter subgroup \(\lambda\) of \(G\), i.e. \(\lambda(t)(x)\) converge to \(z\) as \(t\) tends to 0.

Note that a given \(z \in Z \cap \overline{Gx}\) may not be reachable by any one-parameter subgroup unless the stabilizer of \(z\) in \(G\) is reductive.

Clearly, in order to prove Theorem 2.2 we only need to prove that if \((v, w)\) is not K-stable, then two closures of orbits \(\overline{G[v, w]}\) and \(\overline{G[v, 0]}\) do not intersect. If it is not true, then \(Z \cap \overline{G[v, w]} \neq \emptyset\), where \(Z = \mathbb{P}(V \oplus \{0\})\). Since \(Z\) is a closed
$G$-invariant subset in $\mathbb{P}(V \oplus W)$, by Theorem 2.4, we have an one-parameter subgroup $\lambda$ of $G$ such that

$$\lim_{t \to 0} \lambda(t)([v,w]) \in \mathbb{P}(V \oplus \{0\}).$$

This is equivalent to saying that $w_\lambda(w) > w_\lambda(v)$. This contradicts to the $K$-semistability condition. So Theorem 2.1 is proved.

3 Positivity vs stability

In this section, we will interpret the stability in terms of positivity of certain line bundle. This interpretation will be used in proving our main theorems. We assume that $(v,w) \in V \times W$ be as before and is $K$-semistable.

Let $\pi : \tilde{\mathbb{P}}(V,W) \to \mathbb{P}(V \oplus W)$ be the blow-up of $\mathbb{P}(V \oplus W)$ along subvarieties $\mathbb{P}(V \oplus \{0\})$ and $\mathbb{P}(\{0\} \oplus W)$. Then $\tilde{\mathbb{P}}(V,W)$ is a smooth projective variety with a natural $G$-action and $\pi$ is an isomorphism on the complement of the exceptional loci over $\mathbb{P}(V \oplus \{0\})$ and $\mathbb{P}(\{0\} \oplus W)$. Since neither $v$ nor $w$ is zero, the orbit $G[v,w]$ lies in the complement of $\mathbb{P}(V \oplus \{0\})$ and $\mathbb{P}(\{0\} \oplus W)$, so it lifts to an orbit $\pi^{-1}(G[v,w])$ in $\tilde{\mathbb{P}}(V,W)$. Let $\tilde{G}$ be a smooth variety compactifying $G$. The action of $G$ induces a holomorphic map

$$\phi : G \to \tilde{\mathbb{P}}(V,W), \quad \phi(\sigma) = \pi^{-1}(\sigma([v,w])).$$

Since our assumption that the representations of $G$ on $V$ and $W$ are rational, by (1.1), $\phi$ is made of polynomials on $G$, so there is a blow-up $\tilde{\tilde{G}}$ of $\tilde{G}$ along $\tilde{G} \setminus G$ such that it extends to a holomorphic map

$$\tilde{\phi} : \tilde{\tilde{G}} \to \tilde{\mathbb{P}}(V,W).$$

There are two natural projections:

$$\pi_V : \tilde{\mathbb{P}}(V,W) \to \mathbb{P}(V \oplus \{0\}) \quad \text{and} \quad \pi_W : \tilde{\mathbb{P}}(V,W) \to \mathbb{P}(\{0\} \oplus W).$$

For any vector space $U$, we will denote by $H_U^1$ the hyperplane bundle over $\mathbb{P}(U)$. Then its inverse $H_U^{-1}$ is the universal bundle over $\mathbb{P}(U)$, so we have

$$H_{U}^{-1} \setminus Z_{U} = U \setminus \{0\}, \quad (3.1)$$

where $Z_U$ denotes the zero section of $H_U^{-1}$. It follows that for any non-zero $u \in U$, we can regard $\sigma(u)$ as a point in the fiber of $H_U^{-1}$ over $\sigma([u]) \subset \mathbb{P}(U)$. Since $\sigma(u) \neq 0$, $\sigma(u)^{-1}$ can be naturally regarded as a point in the fiber of $H_U$ over $\sigma([u])$. Now we define a line bundle over $\tilde{\mathbb{P}}(V,W)$:

$$L = \pi_V^*H_V^{-1} \otimes \pi_W^*H_W.$$  \hspace{1cm} (3.2)

Put

$$\tilde{\pi}_V = \pi_V \cdot \tilde{\phi} : \tilde{G} \to \mathbb{P}(V \oplus \{0\}) \quad \text{and} \quad \tilde{\pi}_W = \pi_W \cdot \tilde{\phi} : \tilde{G} \to \mathbb{P}(\{0\} \oplus W).$$

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Since they maps $G$ onto the orbits $G[v]$ and $G[w]$ respectively, we have a natural section $S_{v,w}$ of $\tilde{\phi}^*L$ over $G$:

$$S_{v,w}(\sigma) = \tilde{\pi}_V^*\sigma(v) \otimes \tilde{\pi}_W^*\sigma(w)^{-1}, \quad \forall \sigma \in G.$$  

The Hermitian norms on $V$ and $W$ induce Hermitian metrics on line bundles $H_V$ and $H_W$, consequently, we have a natural Hermitian metric $\|\cdot\|_L$ on $L$ and consequently, $\phi^*L$. We observe

$$p_{v,w}(\sigma) = -\log \|S_{v,w}\|^2_L.$$  

Then on $X$, we have

$$p_{v,w} \geq -2c \text{ if and only if } \|S_{v,w}\|_L \leq e^c. \quad (3.3)$$

Hence, by well-known extension theorem for holomorphic functions, $S_{v,w}$ can be extended to be a holomorphic section over $\tilde{G}$. The converse is also true. Hence, we have shown

**Lemma 3.1.** The functional $p_{v,w}$ is bounded from below if and only if $S_{v,w}$ extends as a holomorphic section over $\tilde{G}$, equivalently, $(v, w)$ is K-semistable if and only if $S_{v,w}$ extends as a holomorphic section.

Next we discuss the case of K-stability. Recall that $G$ has a natural representation $\mathfrak{gl} = \mathfrak{gl}(N+1, \mathbb{C})$. Set $U = \mathfrak{g}l^\otimes q$ and $u = I^\otimes d$, where $q = \text{deg}(V)$.

We have said that $(v, u)$ is K-semistable, moreover, for some uniform $c > 0$, we have

$$p_{v,u}(\sigma) = \text{deg}(V) \log \|\sigma\| - \log \|\sigma(v)\| \geq -c, \quad \forall \sigma \in G.$$  

Here $\|\sigma\|$ is actually the Hilbert-Schmidt norm of $\sigma \in \mathfrak{gl}$.

As above, we have a blow-up variety $\tilde{P}(V, U)$ of $P(V \oplus U)$ along subvarieties $P(V \oplus \{0\})$ and $P(\{0\} \oplus U)$, together with a holomorphic map

$$\phi' : G \mapsto \tilde{P}(V, U), \quad \phi'(\sigma) = [(v, u)].$$  

Moreover, we may choose the blow-up $\tilde{G}$ of $G$ above such that $\psi$ extends to a holomorphic map

$$\tilde{\psi} : \tilde{G} \mapsto \tilde{P}(V, U).$$

As above, we have a line bundle on $\tilde{P}(V, U)$

$$L' = \pi_V^*H_V^{-1} \otimes \pi_U^*H_U. \quad (3.4)$$

The orbit $G([v, w], [v, u])$ can be also lifted to a $G$-orbit in $\tilde{P}(V, W) \times \tilde{P}(V, U)$ and induces a holomorphic map:

$$\tilde{\psi} = (\tilde{\phi}, \tilde{\psi}) : \tilde{G} \mapsto \tilde{P}(V, W) \times \tilde{P}(V, U).$$

We also have two holomorphic sections $S_{v,w}$ and $S_{v,u}$ of line bundles $\phi^*L$ and $(\phi')^*L'$, which are both equipped with natural Hermitian metrics $\|\cdot\|_L$ and $\|\cdot\|_{L'}$, over $G$. Theorem 1.4 can be put in an equivalent form:
**Theorem 3.2.** If \((v, w)\) is K-stable, then there is an integer \(m > 0\) such that for some constant \(C > 0\),

\[
\| S_{v, w} \|^m_L \leq C \| S_{v, u} \|_{L'}.
\]  

(3.5)

Because the vanishing order of \(S_{v, u}\) along each irreducible divisor \(D \subset \tilde{G} \setminus G\) is always finite and there are only finitely many of irreducible divisors \(\tilde{G} \setminus G\), (3.5) is the same as saying: For any irreducible divisor \(D \subset \tilde{G} \setminus G\), \(S_{v, w}\) vanishes along \(D\) so long as \(S_{v, u}\) does. This last statement is plausible, but a direct proof is not available. Instead, we will prove Theorem 1.4 or 3.2 through consideration of maximal tori in \(G\).

4 Proof of main theorem

In this section, we prove Theorem 1.2 and Theorem 1.4. We will adopt the notations in last section. Let \(T\) be a maximal torus of \(G\), then any other maximal torus \(T'\) is of the form \(T' = \tau \cdot T \cdot \tau^{-1}\). We will fix a maximal torus \(T\). It admits a compactification \(\bar{T}\) which is simply a product of \(\mathbb{CP}^1\) in the case of \(G = \text{SL}(N + 1, \mathbb{C})\). Put

\[
P = \tilde{P}(V, W) \times \tilde{P}(V, U).
\]

Consider the holomorphic map induced by the \(G\)-action:

\[
f : T \times P \mapsto P, \quad f(\sigma, x) = \sigma(x).
\]

By (1.1), this map can be made of polynomials on \(G = \text{SL}(N + 1, \mathbb{C})\), so there is a blow-up variety \(\pi : \tilde{Z} \mapsto Z\), where \(Z = T \times P\), such that it extends to a holomorphic map

\[
\tilde{f} : \tilde{Z} \mapsto P.
\]

For each \(x \in P\), its preimage in \(T \times P\) is given by

\[
f^{-1}(x) = \{ (\sigma^{-1}, \sigma(x)) \mid \sigma \in T \} \subset T \times P.
\]

This is isomorphic to \(T\). We denote by \(T_x\) the closure of \(f^{-1}(x)\) in \(\tilde{Z}\), clearly, we have \(T_x \subset \tilde{f}^{-1}(x)\) which is closed in \(\tilde{Z}\).

**Lemma 4.1.** Let \(\omega\) be a fixed Kähler metric on \(\tilde{Z}\), then there is a constant \(C = C(\omega, T)\), which is independent of \(x \in P\), satisfying:

\[
\int_{T_x} \omega^N \leq C,
\]

where \(N\) is the dimension of \(T\).

**Proof.** There is a subvariety \(B \subset P\) of complex codimension at least 2 such that for any \(x\) outside \(B\), \(\tilde{f}^{-1}(x)\) is a subvariety of dimension \(N\) and consequently, it
coincides with $\tilde{T}_x$. Clearly, all $\tilde{T}_x$ for $x \in P \setminus B$ are homologous to each other, so we have
\[ \int_{\tilde{T}_x} \omega^N = C(\omega, T), \quad \forall x \in P \setminus B, \tag{4.1} \]
i.e., these integrals stay as a constant. For any $x \in B$, we can choose a sequence of $x_i \in P \setminus B$ converging to $x$ such that $\lim x_i = x$. By (4.1), $\tilde{T}_{x_i}$ form a bounded family in $P$, so by taking a subsequence if necessary, we may assume that $\tilde{T}_{x_i}$ converge to a subvariety (possibly non-reduced) $D_\infty$ in $\tilde{Z}$. Clearly, $\tilde{T}_x$ is contained in $D_\infty$ as one component. Hence, by using (4.1) for $x_i$, we get
\[ \int_{\tilde{T}_x} \omega^N \leq C. \tag{4.2} \]

The lemma is proved.

Next we recall $L$ over $\tilde{P}(V, W)$ and $L'$ over $\tilde{P}(V, U)$ introduced in last section. Through projections, they pull back to two line bundles $\tilde{L}$ and $\tilde{L}'$ over $\tilde{Z}$. For any $\tau \in G$, we put $x = (\tau([v, w]), \tau([v, u]))$, then restrictions of $L$ and $L'$ to each $\tilde{T}_x \subset \tilde{Z}$ have two holomorphic sections $S_{v, w}$ and $S_{v, u}$ constructed in a similar way as we did in last section. Actually, if we identify $\tilde{T}_x$ with a closure of the orbit $T \cdot \tau \subset G$ in $\tilde{G}$, then these are simply those $S_{v, w}$ and $S_{v, u}$ over $\tilde{G}$ from last section restricted to $\tilde{T}_x$. Note that both $\tilde{L}$ and $\tilde{L}'$ have naturally induced Hermitian metrics $||\cdot||_{\tilde{L}}$ and $||\cdot||_{\tilde{L}'}$.

**Proposition 4.2.** There is an uniform integer $m > 0$ such that for $\tau \in G$, there is a constant $C_{x, \omega}$ such that $||S_{v, w}||_{\tilde{L}^m} \leq C_{x, \omega} ||S_{v, u}||_{\tilde{L}'}$ on $\tilde{T}_x$. \[
\tag{4.3}
\]

**Proof.** We will prove (4.3) in two steps. Without loss of generality, we may take $\tau = Id$. First we claim that for any irreducible divisor $D \subset \tilde{T}_x \setminus T(x)$, $S_{v, w}$ vanishes along $D$ whenever $S_{v, u} = 0$ on $D$. This can be proved by applying a variant of Richardson’s lemma which already appeared in [Pa13].

**Lemma 4.3.** For any $z \in T_x(T(x))$, there is a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that $\lambda(t)(x) \to \varsigma(z)$ for some $\varsigma \in T$ as $t \to 0$.

Now we choose $z$ to be a generic point in $D$, by the above lemma, we have a one-parameter subgroup $\lambda$ satisfying:
\[ \lim_{t \to 0} \lambda(t)(x) = \varsigma(z). \]

Since $D$ is $T$-invariant, $\varsigma(z)$ is a generic point in $D$. By the K-stability condition, $S_{v, w}$ restricted to the closure of $\lambda(\mathbb{C}^*)$ must vanish at $\varsigma(z)$, so $S_{v, w}$ vanishes along $D$.

Next we show that there is an uniform upper bound on vanishing order of $S_{v, u}$ along any irreducible components of $\tilde{T}_x \setminus T(x)$. To see this, we take the
Kähler metric $\omega$ on $\tilde{Z}$ as in Lemma 4.1 and denote by $R_{L'}$ the curvature form of the Hermitian metric $\omega$. Then we have

$$\sum_i m_i \langle D_i \rangle - R_{L'} = \sqrt{-1} \partial \bar{\partial} \log \| S_{\nu, u} \|^2_{L'},$$

(4.4)

where $\{ D_i \}$ is the collection of irreducible components of $\tilde{T}_x \setminus T(x)$ and each $m_i$ is the vanishing order of $S_{\nu, u}$ along $D_i$. Integrating $\omega^{N-1}$ on both sides of (4.4), we get

$$\sum_i m_i \int_{D_i} \omega^{N-1} = \int_{T_x} R_{L'} \wedge \omega^{N-1}.$$  

(4.5)

Since the curvature $R_{L'}$ is bounded by a multiple of $\omega$, by Lemma 4.1, the right side of (4.5) is bounded from above by an uniform constant. Hence, if we let $m$ be a positive integer bigger than the right side of (4.5), we have $m_i \leq m$ for each $i$. We have seen above that $S_{\nu, w}$ vanishes along $D_i$ so long as $m_i \geq 1$, so $S_{\nu, w}^m \cdot S_{\nu, w}^{-1}$ extends to a holomorphic section of $L \otimes (L')^{-1}$ over $T_x$. Thus (4.3) holds and our proposition is proved.

Now let $m$ be given in Proposition 4.2, put $x = ([v, w], [v, u])$ and consider a function $F_x$ on $G$:

$$F_x(\sigma) = m (\log \| \sigma(u) \|^2 - \log \| \sigma(v) \|^2) - (\log \| \sigma(u) \|^2 - \log \| \sigma(v) \|^2).$$  

(4.6)

It is the same as

$$F_x(\sigma) = - \log \left( \frac{\| S_{\nu, u} \|^2_{L'}}{\| S_{\nu, u} \|^2_{L'}} \right).$$  

(4.7)

For each one-parameter subgroup $\lambda \subset G$, we have

$$F_x(\lambda(t)) = [m(w_\lambda(w) - w_\lambda(v)) - (q w_\lambda(I) - w_\lambda(v))] \log |t|^2 + O(1),$$

(4.8)

where $O(1)$ denotes a bounded quantity.

Each one-parameter subgroup $\lambda$ is contained in a maximal torus $\tau \cdot T \cdot \tau^{-1}$ of $G$. Then $\lambda = \tau^{-1} \cdot \lambda \cdot \tau$ is an one-parameter subgroup in $T$. Note that

$$F_x(\lambda(t)) = F_x(\tau \cdot \bar{\lambda} \cdot \tau^{-1}).$$

Using this and (4.6), we can deduce the following expansion as we did in (4.8),

$$F_x(\lambda(t)) = [m(w_\lambda(\bar{w}) - w_\lambda(\bar{v})) - (q w_\lambda(\tau) - w_\lambda(\bar{v}))] \log |t|^2 + O(1),$$  

(4.9)

where $\bar{w} = \tau^{-1}(w)$ and $\bar{v} = \tau^{-1}(v)$. This, together with (4.8), implies that the coefficients in front of $\log |t|^2$ in (4.8) and (4.9) are equal. In fact, by similar arguments using expansions, we have

$$w_\lambda(w) - w_\lambda(v) = w_\lambda(\bar{w}) - w_\lambda(\bar{v})$$

and

$$q w_\lambda(I) - w_\lambda(v) = q w_\lambda(\bar{v}) - w_\lambda(\bar{v}).$$


where \( \tau = \tau^{-1} \). We also have
\[
F_{t(x)}(\tilde{\lambda}(t)) = [m(w_\chi(\bar{\omega}) - w_\chi(\bar{v})) - (q w_\chi(I) - w_\chi(v))] \log |t|^2 + O(1). \tag{4.10}
\]

By (4.7) and Proposition 4.2, \( F_{t(x)}(\tilde{\lambda}(t)) = F_{t(x)}(\tilde{\lambda}(t) \cdot \varsigma) \) is bounded from below on \( T \). It follows from this and (4.10) that
\[
m(w_\chi(\bar{\omega}) - w_\chi(\bar{v})) - (q w_\chi(I) - w_\chi(v)) \leq 0.
\]
Consequently, we have
\[
m(w_\lambda(\bar{\omega}) - w_\lambda(\bar{v})) - (q w_\lambda(I) - w_\lambda(v)) \leq 0. \tag{4.11}
\]

This concludes the proof of Theorem 1.2.

Next, by using (4.11) and Theorem 2.1 (also [BHJ17], Theorem 5.4 (ii)), we conclude the proof of Theorem 1.4.

5 K-stability vs CM-stability

In this section, as an application of Theorem 1.2, we give a proof of the main theorem in [Ti14]. We will follow closely discussions in [Ti14].

Let \( M \) be a projective manifold polarized by an ample line bundle \( L \). By the Kodaira embedding theorem, for \( \ell \) sufficiently large, a basis of \( H^0(M, L^\ell) \) gives an embedding \( \phi_\ell : M \hookrightarrow \mathbb{C}P^N \), where \( N = \dim_\mathbb{C} H^0(M, L^\ell) - 1 \). Any other basis gives an embedding of the form \( \sigma \cdot \phi_\ell \), where \( \sigma \in G = SL(N + 1, \mathbb{C}) \). We fix such an embedding.

The CM-stability introduced in [Ti97] is defined in terms of Mabuchi’s K-energy:
\[
M_{\omega_0}(\varphi) = -\frac{1}{V} \int_0^1 \int_M \varphi (\text{Ric}(\omega_{t \varphi}) - \mu \omega_{t \varphi}) \wedge \omega_{t \varphi}^{-1} \wedge dt, \tag{5.1}
\]
where \( \omega_0 \) is a Kähler metric with Kähler class \( 2\pi c_1(L) \) and
\[
\omega_{t \varphi} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi \quad \text{and} \quad \mu = \frac{c_1(M) \cdot c_1(L)^{n-1}}{c_1(L)^n}. \tag{5.2}
\]
Given an embedding \( M \subset \mathbb{C}P^N \) by \( K_M^\ell \), we have an induced function on \( G = SL(N + 1, \mathbb{C}) \) which acts on \( \mathbb{C}P^N \):
\[
F(\sigma) = M_{\omega_0}(\psi_\sigma), \tag{5.3}
\]
where \( \psi_\sigma \) is defined by
\[
\frac{1}{\ell} \sigma^* \omega_{FS} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi_\sigma. \tag{5.4}
\]

2The proof in [Ti14] needs additional arguments for the reduction from \( G \) to its maximal tori. It was thought that one can complete this reduction by applying a result in [Pa12a]. It turns out that additional arguments are needed and provided here.
Note that $F(\sigma)$ is well-defined since $\psi_\sigma$ is unique modulo addition of constants. Similarly, we can define $J$ on $G$ by

$$J(\sigma) = J_{\omega_0}(\psi_\sigma),$$  \hspace{1cm} (5.5)

where

$$J_{\omega_0}(\varphi) = \sum_{i=0}^{n-1} \frac{i + 1}{n + 1} \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega_i^\alpha \wedge \omega_{n-i}^\alpha. \hspace{1cm} (5.6)$$

**Definition 5.1.** We call $M$ CM-semistable with respect to $L$ if $F$ is bounded from below and CM-stable with respect to $L$ if $F$ bounded from below and is proper modulo $J$, i.e., for any sequence $\sigma_i \in G$,

$$F(\sigma_i) \to \infty \text{ whenever } J(\sigma_i) \to \infty. \hspace{1cm} (5.7)$$

We say $(M, L)$ CM-stable (resp. CM-semistable) if $M$ is CM-stable (resp. CM-semistable) with respect to $L$ for all sufficiently large $\ell$.

**Theorem 5.2.** Let $(M, L)$ be a polarized projective manifold which is K-stable. Then $M$ is CM-stable with respect to any $L^\ell$ which is very ample. In particular, $(M, L)$ is CM-stable.

We refer the readers to [Ti13] for the definition of the K-stability. Clearly, Theorem 5.2 follows from the following theorem.

**Theorem 5.3.** Let $(M, L)$ be a polarized projective manifold which is K-stable with respect to $L^\ell$. Then there are positive constants $\delta$ and $C$, which may depend on $\ell$, such that

$$F(\sigma) \geq \delta J(\sigma) - C \text{ on } G. \hspace{1cm} (5.8)$$

The rest of this section is devoted to the proof of Theorem 5.3.

First we recall Theorem 2.4 in [Ti14] which relates the K-stability to the asymptotic behavior of the K-energy.

**Theorem 5.4.** If $(M, L)$ is K-stable with respect to $L^\ell$, then $F$ is proper along any one-parameter subgroup $\lambda$ of $G$.

Here by properness along $\lambda$, we mean that $F$ is bounded from below along $\lambda$ and for any sequence $t_i \to 0$, $F(\lambda(t_i))$ diverge to $\infty$ whenever $J(\lambda(t_i)) \to \infty$.

Next we recall the Chow coordinate and Hyperdiscriminant of $M$ ([Pa08]): Let $G(N - n - 1, N)$ the Grassmannian of all $(N - n - 1)$-dimensional subspaces in $\mathbb{C}P^N$. We define

$$Z_M = \{ P \in G(N - n - 1, N) | P \cap M \neq \emptyset \}. \hspace{1cm} (5.9)$$

Then $Z_M$ is an irreducible divisor of $G(N - n - 1, N)$ and determines a non-zero homogeneous polynomial $R_M \in \mathbb{C}[M_{(n+1) \times (N+1)}]$, unique modulo scaling, of degree $(n + 1)d$, where $M_{k \times l}$ denotes the space of all $k \times l$ matrices. We call $R_M$ the Chow coordinate or the $M$-resultant of $M$. 
Next consider the Segre embedding:

\[ M \times \mathbb{C}P^{n-1} \subset \mathbb{C}P^n \times \mathbb{C}P^{n-1} \mapsto \mathbb{P}(M_{n \times (N+1)}^r), \]

where \( M_{k \times l}^r \) denotes its dual space of \( M_{k \times l} \). Then we define

\[
Y_M = \{ H \subset \mathbb{P}(M_{n \times (N+1)}^r) \mid T_p(M \times \mathbb{C}P^{n-1}) \subset H \text{ for some } p \}. \tag{5.10}
\]

Then \( Y_M \) is a divisor in \( \mathbb{P}(M_{n \times (N+1)}^r) \) of degree \( \tilde{d} = (n(n+1) - \mu) d \). This determines a homogeneous polynomial \( \Delta_M \) in \( \mathbb{C}[M_{n \times (N+1)}^r] \), unique modulo scaling, of degree \( \tilde{d} \). We call \( \Delta_M \) the hyperdiscriminant of \( M \).

Set

\[
r = (n + 1) \tilde{d} d, \quad V = C_r[M_{(n+1) \times (N+1)}], \quad W = C_r[M_{n \times (N+1)}],
\]

where \( C_r[\mathbb{C}^k] \) denotes the space of homogeneous polynomials of degree \( r \) on \( \mathbb{C}^k \).

Following [Pa12b], we associate \( M \) with the pair \( (R(M), \Delta(M)) \) in \( V \times W \), where \( R(M) = R_M^d \) and \( \Delta(M) = \Delta_M^{(n+1)d} \). It follows from [Pa08] that

\[
|F(\sigma) - a_n p_{R(M), \Delta(M)}(\sigma)| \leq C, \quad \forall \sigma \in G \tag{5.11}
\]

where \( a_n > 0 \) and \( C \) are uniform constants.

For each one-parameter subgroup \( \lambda \in \mathbb{N}_\mathbb{Z} \), we have

\[
p_{R(M), \Delta(M)}(\lambda(t)) = (w_\lambda(\Delta(M)) - w_\lambda(R(M))) \log |t|^2 + \mathcal{O}(1). \tag{5.12}
\]

Since \( F \) is bounded from below on \( G \), we deduce from (5.11) and (5.12) that

\[
w_\lambda(R(M)) - w_\lambda(\Delta(M)) \geq 0. \tag{5.13}
\]

Hence, \( (R(M), \Delta(M)) \) is K-semistable as a pair. Moreover, we see

\[
\lim_{t \to 0} F(\lambda(t)) = \infty \iff w_\lambda(R(M)) - w_\lambda(\Delta(M)) > 0. \tag{5.14}
\]

On the other hand, by Lemma 3.2 in [Ti14], we have

\[
|J(\sigma) - p_{R(M), I^r}(\sigma)| \leq C, \quad \forall \sigma \in G. \tag{5.15}
\]

Here, as in previous sections, \( I \) denotes the identity in \( \mathfrak{gl} \) and \( I^r \in U = \mathfrak{gl}^{\otimes r} \).

For any one-parameter subgroup \( \lambda \), we also have

\[
p_{R(M), I^r}(\lambda(t)) = (r w_\lambda(I) - w_\lambda(R(M))) \log |t|^2 + \mathcal{O}(1). \tag{5.16}
\]

Combining this with (5.15), we have

\[
\lim_{t \to 0} J(\lambda(t)) = \infty \iff w_\lambda(R(M)) - r w_\lambda(I) > 0. \tag{5.17}
\]

Using (5.14), (5.17) and Theorem 5.4, we show that \( (R(M), \Delta(M)) \) is K-stable as a pair.

Hence, Theorem 5.3 follows from Theorem 1.4
6 A final remark

In this section, we discuss another approach to proving Theorem 1.2. I strongly believe that this approach can be worked out. If so, the resulting proof would be simpler than the one we gave in Section 4. We will adopt notations from Section 1 and 2. For simplicity, we first recall an interpretation of K-semistability of S. Paul in terms of polytopes (cf. [Pa08]).

Proposition 6.1. Let \( (v, w) \), \( V, W, G \) and \( T \) as in Section 1. Then \( (v, w) \) is K-semistable if and only if for any \( \tau \in G \), we have

\[
\mathcal{N}(\tau(v)) \subset \mathcal{N}(\tau(w)).
\]

Let us explain why this proposition holds. We observe that for any one-parameter subgroup \( \lambda \subseteq T \) (\( \ell \in \mathbb{N} \)),

\[
w_{\lambda}(\tau(w)) = \inf\{ (\ell, m) | m \in A(\tau(w)) \}, \quad (6.2)
\]

\[
w_{\lambda}(\tau(v)) = \inf\{ (\ell, m) | m \in A(\tau(v)) \}. \quad (6.3)
\]

If (6.1) is false, then we can find \( m' \in A(\tau(v)) \) which is not contained in \( \mathcal{N}(\tau(w)) \). This means that there is a linear function \( \ell : M \mathbb{R} \rightarrow \mathbb{R} \) such that \( \ell_{|A(\tau(w))} \geq 0 \) and \( \ell(m') < 0 \). Since both \( V \) and \( W \) are rational representations, we may take \( \ell \) to be integral, i.e., \( \ell \in \mathbb{N} \). Thus,

\[
w_{\lambda}(\tau(v)) \leq (\ell, m') < 0 \quad \text{while} \quad w_{\lambda}(\tau(w)) \geq 0.
\]

This implies that \( (v, w) \) is not K-semistable. So the K-semistability implies (6.1). The other direction is clear.

Similarly, we can express the K-stability of \( (v, w) \) in terms of polytopes \( \mathcal{N}(\tau(v)), \mathcal{N}(\tau(w)) \) and \( \mathcal{N}(\tau) \) corresponding to representations \( V, W \) and \( gl \).

Note that For any \( \tau \in G \), \( \mathcal{N}(\tau) \) is always the standard N-simplex \( \mathcal{N}(I) \). In view of (6.2) and (6.3), we see that \( (v, w) \) is K-stable if and only if it is K-semistable and for any \( \tau \in G \) and \( \ell \in \mathbb{N} \),

\[
q \inf\{ (\ell, m) | m \in \mathcal{N}(I) \} < \inf\{ (\ell, m) | m \in A(\tau(v)) \}.
\]

\[
\Rightarrow \quad \inf\{ (\ell, m) | m \in \mathcal{N}(\tau(w)) \} < \inf\{ (\ell, m) | m \in A(\tau(v)) \} \quad (6.4)
\]

Theorem 1.2 is equivalent to that for any there is an integer \( k \) such that \( \tau \in G \) and \( \ell \in \mathbb{N} \),

\[
k \inf\{ (\ell, m) | m \in A(\tau(v)) \} - \inf\{ (\ell, m) | m \in A(\tau(w)) \} \quad (6.5)
\]

\[
\geq \quad \inf\{ (\ell, m) | m \in A(\tau(v)) \} - q \inf\{ (\ell, m) | m \in \mathcal{N}(I) \}.
\]

By the semi-continuity, there should be only finitely many possibly different \( \mathcal{N}(\tau(v)) \) and \( \mathcal{N}(\tau(w)) \), so we believe that there will be a direct proof of (6.5).

This above approach to proving (6.5) may be applied to proving uniform K-stability for a K-stable polarized manifold \( (M, L) \) as in Section 5. For each sufficiently large \( \ell \), we have a pair \( (R_{\ell}(M), \Delta_{\ell}(M)) \) associated to the embedding.
given by $H^0(M, L^\ell)$. As we have shown in Section 5, such a pair $(R_\ell(M), \Delta_\ell(M))$ is uniformly K-stable. Since the ring $R(M, L)$ is finitely generated, where

$$R(M, L) = \bigoplus_{\ell=1}^{\infty} H^0(M, L^\ell),$$

the related polytopes $N(R_\ell(M))$ and $N(\Delta_\ell(M))$ for all $\ell$ should be determined by finitely many such polytopes, so we may have an uniform estimate for $k$ appeared in (6.5) for all $\ell$. This would lead to a proof of the uniform K-stability of $(M, L)$. However, because the relations among those polytopes $N(R_\ell(M))$ and $N(\Delta_\ell(M))$ are implicit, the proof may be tricky.

7 Appendix A: Proof of Lemma 4.3

In this appendix, following [Pa13], we give a proof of Lemma 4.3 which we restate as follows:

**Lemma 7.1.** For any $z \in \tilde{T}\setminus T_\mathfrak{x}$, where $x = ([v, w], [v, u])$, there is a one-parameter subgroup $\lambda : \mathbb{C}^* \to T$ such that $\lambda(t)x \to \tau(z)$ for some $\tau \in T$ as $t \to 0$.

**Proof.** We may assume that $\tilde{T}$ is a smooth subvariety in a projective space $\mathbb{P}(E)$ for a $G$-representation $E$. We can have the following decomposition:

$$E = \sum_{a \in A} E_a, \quad \text{where } E_a = \{ u \in E \mid t \cdot u = a(t) u , t \in T \}.$$

Accordingly, we can write

$$z = \sum_{a \in A} z_a, \quad z_a \in E_a.$$

Pick up a $z_{a_0} \neq 0$, say the first one $z_{a_0}$, then we have $E = \mathbb{C} \oplus F$ and can regard $F$ as a subspace in $\mathbb{P}(E)$, moreover, $F$ admits $T$-action and a decomposition:

$$F = \sum_{a \in A'} F_a, \quad \text{where } F_a = \{ u \in F \mid t \cdot u = a(t) u , t \in T \}, \quad (7.1)$$

where $a' = A \setminus \{a_0\}$. We fix a basis $\{e_i\}_{1 \leq i \leq d'}$ of $F$ such that $t \cdot e_i = a_i(t)e_i$ for some $a_i \in A'$. Since $z \in F \subset \mathbb{P}(E)$, $T_\mathfrak{x} \subset F$. Clearly, $Z = \tilde{T} \cap F\setminus T_\mathfrak{x}$ is $T$-invariant and closed, furthermore, there are $t_\ell \in T$ such that $t_\ell \to 0$ and $t_\ell \cdot x$ converge to $z \in Z$ as $\ell$ goes to $\infty$.

By rearranging the indices, we may write

$$x = \sum_{i=1}^{d'} x_i e_i, \quad z = \sum_{j=k}^{d'} z_j e_j.$$

3For $i \neq j$, we may still have $a_i = a_j$.
where \( 1 \leq k \leq d', x_i \neq 0 \) and \( z_j \neq 0 \). Our assumption implies that \( a_i(t_\ell)x_i \) converge to 0 for \( i < k \) and converge to \( z_i \) for \( i \geq k \). Since \( x_i \neq 0 \) and \( z_j \neq 0 \), we get

\[
\lim_{\ell \to \infty} a_i(t_\ell) = 0 \quad \forall i < k \quad \text{and} \quad \lim_{\ell \to \infty} a_j(t_\ell) = \frac{z_j}{x_j} = 1 \quad \forall j \geq k.
\]

Consider the quotient

\[
\pi : M_R \mapsto W = M_R/M_z, \quad \text{where } M_z = \bigoplus_{j=k}^{d'} R \cdot a_j.
\]

Denote by \( \Delta \) the convex hull (in \( W \)) of \( a_i \) for \( i = 1, \ldots, k-1 \).

We claim that \( 0 \not\in \Delta \). This can be shown as follows: If the claim is false, then there are real constants \( r_1, \ldots, r_{k-1} \geq 0 \) such that

\[
\text{some } r_i > 0, \quad \sum_{i=1}^{k-1} r_i a_i = 0 \mod M_z,
\]

hence, there are \( c_k, \ldots, c_{d'} \) such that

\[
\sum_{i=1}^{k-1} r_i a_i = \sum_{j=k}^{d'} c_j a_j.
\]

Hence, for all \( t \in T \), we have

\[
\prod_{i=1}^{k-1} |a_i(t)|^{r_i} = \prod_{j=k}^{d'} |a_j(t)|^{c_j}. \tag{7.2}
\]

By plugging in the sequence \( \{t_\ell\} \), we get a contradiction since the left side of (7.2) tends to zero while the right side does not. This proves our claim.

Using this claim and the Hyperplane Separation Theorem, one can get a linear functional \( f : W \mapsto \mathbb{R} \) such that \( f(\pi(a_i)) > 0 \) for \( i < k \). Furthermore, one can choose this to be rational. Next we lift \( f \) to

\[
F = f \cdot \pi : M_R \mapsto \mathbb{R}.
\]

Then \( F \) is a rational linear functional on \( M_R \). Multiplying it by an integer, we may even assume that \( F \) is integer-valued. Therefore, it induces an one-parameter subgroup \( \lambda : \mathbb{C}^* \mapsto T \) satisfying:

\[
\lim_{t \to 0} \lambda(t)x = \sum_{j=k}^{d'} x_j e_j = x'. \tag{7.3}
\]

Since \( a(t_\ell)x_j \) converge to \( z_j \) for any \( j \geq k \), we have \( Tz \subset Tx' \). On the other hand, both orbits \( Tz \) and \( Tx' \) have the same dimension, so \( Tz = Tx' \). It follows that there is a \( \tau \in T \) such that

\[
\lim_{t \to 0} \lambda(t)x = \tau(z). \tag{7.4}
\]
The lemma is proved.

8 Appendix B: A variant proof of Theorem 1.2
by Li Yan and Xiaohua Zhu

In this appendix, we give an element proof of Theorem 1.2 by an approach in Section 6.

8.1 Stability and polytopes of weights

Let $V$ be a linear space with a $r$-dimensional torus $T$-action. Then we can decompose it into a direct sum of $T$-invariant subspaces such that

$$V = \bigoplus_{\mu \in I_{V}} m_{\mu} \hat{V}_{\mu},$$

where $I_{V}$ is a finite set of characters of $T$, $\hat{V}_{\mu}$’s are one-dimensional irreducible representations of $T$ with multiplicity $m_{\mu}$. Thus for any $v \in V$, there are some $\alpha \in I_{V}$ and non zero constants $c_{\alpha}^{v} \in \mathbb{C}$ such that

$$v = \sum_{\alpha} c_{\alpha}^{v} e_{\alpha},$$

where $e_{\alpha} \in m_{\alpha} \hat{V}_{\alpha}$. It follows that for any one parameter sub-group $\lambda(t)$ of $T$ with its Lie algebra $\lambda \in \mathbb{Z}^{r}$,

$$\lambda(t)(v) = \sum_{\alpha} c_{\alpha}^{v} t^{(\lambda, \Lambda_{\alpha})} e_{\alpha},$$

where $\Lambda_{\alpha}$ are weights as characters of $T$ associated to $v$.

Set a convex hull of weights $\{\Lambda_{\alpha}\}$ by $\mathcal{N}(v)$ and Let $v_{\mathcal{N}(v)}(\cdot)$ be a support function of $\mathcal{N}(v)$. Then by (1.2), it is easy to see that the weight $w_{\lambda}(v)$ of $v$ associated to $\lambda(t)$ is equal to

$$w_{\lambda}(v) = \min_{\alpha} (\lambda, \Lambda_{\alpha}) = -v_{\mathcal{N}(v)}(-\lambda).$$

Now we assume that $V$ is a linear presentation space of a reductive Lie group $G$ with a maximal torus $T$. Then for any one parameter subgroup $\lambda(t)'$, there is a $\sigma \in G$ and one parameter sub-group $\lambda(t)$ of $T$ such that

$$\lambda(t)' = \sigma \cdot \lambda(t) \cdot \sigma^{-1} \subseteq T' = \sigma \cdot T \cdot \sigma^{-1}.$$  

Thus $\sigma(\hat{V}_{\mu})$ are one-dimensional irreducible representations of torus $T'$ with property

$$V = \bigoplus_{\mu \in I_{V}} m_{\mu} \sigma(\hat{V}_{\mu}).$$
In fact, for the vector \( v' = \sigma^{-1}v \), there are non zero constants \( c_{\alpha'} \in \mathbb{C} \) such that
\[
v' = \sum_{\alpha'} c_{\alpha'} e_{\alpha'}
\]
and
\[
\lambda(t)v' = \sum_{\alpha'} c_{\alpha'} t^{\langle \lambda, \Lambda_{\alpha'} \rangle} e_{\alpha'},
\]
where \( e_{\alpha'} \in m_{\alpha'} \sigma(\tilde{V}_{\alpha'}) \) and \( \Lambda_{\alpha'} \) are weights as characters of \( T \) associated to \( v' \). Denote by the convex hull of weights \( \{\Lambda_{\alpha'}\} \) by \( N_{\sigma}(v) \). It is clear that such convex hulls \( N_{\sigma}(v) \) are finitely many since \( V \) is a finitely dimensional linear space. Moreover, the weight \( w_{\lambda}(v) \) of \( \lambda(t)v' \) is given by
\[
w_{\lambda}(v) = \min_{\alpha'} \langle \lambda, \Lambda_{\alpha'} \rangle = -N_{\sigma}(v)(-\lambda).
\] (8.3)

Hence, by (8.3), we have the following proposition.

**Proposition 8.1.** (1) Pair \((v,w)\) is K-semistable with respect to \( G \) iff
\[
N_{\sigma}(v) \subseteq N_{\sigma}(w), \forall \sigma \in G.
\] (8.4)

(2) Pair \((v,w)\) is K-stable with respect to \( G \) iff (8.4) is satisfied and
\[
\{x \mid v_{\sigma}(x) = v_{\sigma}(w)(x)\} \subseteq \{x \mid v_{\sigma}(x) = v_{\deg(V)N_{\sigma}(I)}(x)\}, \forall \sigma \in G. \quad (8.5)
\]

(3) Pair \((v,w)\) is uniform K-stable with respect to \( G \) iff there is an \( m \in \mathbb{N}_{+} \) such that
\[
\left(1 - \frac{1}{m}\right)N_{\sigma}(v) + \frac{1}{m} \deg(V)N_{\sigma}(I) \subseteq N_{\sigma}(w).
\] (8.6)

**8.2 \( K \)-stability implies \( K \)-uniform stability**

As we discussed Section 8.1, the numbers of convex hulls \( N_{\sigma}(v), N_{\sigma}(w) \) and \( N_{\sigma}(I) \) are all finite. Thus by Proposition 8.1(3), Theorem 1.2 is reduced to prove

**Theorem 8.2.** Let \( V, W \) be two linear space with a torus \( T \)-action. Suppose that pair \((v,w) \in (V \setminus \{0\}) \times (W \setminus \{0\}) \) is K-stable with respect to \( T \). Then there is an \( m \in \mathbb{N}_{+} \) such that
\[
\left(1 - \frac{1}{m}\right)N(v) + \frac{1}{m} \deg(V)N(I) \subseteq N(w).
\] (8.7)

To prove Theorem 8.2, we first state a result for the support function of convex polytope. Let
\[
P = \cap_{A \in A} \{l_A(y) = a_A - u_A(y) \geq 0\}
\] (8.8)
be a convex polytope and \( P_{\sigma}^P \subseteq \{l_A(y) = 0\} \) be its (codim 1) facets. For \( I \subseteq A \), we denote
\[
P_{\sigma}^P = \cap_{i \in I} F_{\sigma}^P.
\]
Lemma 8.3. The set
\[ \Omega^P_I = \{ x \mid v_P(x) = \langle x, y \rangle, \forall y \in F^P_I \} = \text{Span}_{\mathbb{R}_{\geq 0}} \{ u_i \mid i \in I \}. \]

Proof. By definition
\[ \Omega^P_I = \{ x \mid \langle x, y - y' \rangle \geq 0, \forall y \in F^P_I \} \]
By the convexity of \( P \), this is equivalent to
\[ \Omega^P_I = \left( \text{Span}_{\mathbb{R}_{\geq 0}} \{ y - y' \mid y \in F^P_I \} \right)^\vee. \]
Thus the lemma is true. \( \square \)

Proof of Theorem 8.2. Since \((v,w)\) is \( K \)-stable, by Proposition 8.1 (1), we have (8.4). In particular, if
\[ \text{dist}(\partial \mathcal{N}(v), \partial \mathcal{N}(w)) \geq \epsilon_0 > 0, \]
there is a \( \delta_0 > 0 \) such that for any \( \delta \in (0, \delta_0) \),
\[ (1 - \delta) \mathcal{N}(v) + \delta \deg(V) \mathcal{N}(I) \subseteq \mathcal{N}(w), \]
since \( \mathcal{N}(I) \) is compact. The theorem then follows from Proposition 8.1 (3).

In the following, we assume that
\[ \text{dist}(\partial \mathcal{N}(v), \partial \mathcal{N}(w)) = 0. \]
Then
\[ \{ x \mid v_{\mathcal{N}(w)}(x) = v_{\mathcal{N}(w)}(x) \} \]
\[ = \{ x \mid v_{\mathcal{N}(w)}(x) = \langle x, y \rangle \text{ and } v_{\mathcal{N}(w)}(x) = \langle x, y \rangle, \text{ for some } y \in \partial \mathcal{N}(v) \cap \partial \mathcal{N}(w) \}. \quad (8.10) \]
By Proposition 8.1 (2), we see that for any \( x \) as above there is \( y \in \partial(\deg(V) \mathcal{N}(I)) \cap \mathcal{N}(v) \cap \partial \mathcal{N}(w) \) such that
\[ v_{\mathcal{N}(v)}(x) = v_{\deg(V) \mathcal{N}(I)}(x) = \langle x, y \rangle. \]
In particular,
\[ (\partial \mathcal{N}(v) \cap \partial \deg(V) \mathcal{N}(I) \cap \partial \mathcal{N}(w)) \neq \emptyset. \quad (8.11) \]
Recall by the definition of \( \deg(V) \) that
\[ \mathcal{N}(v) \subseteq \deg(V) \mathcal{N}(I). \]
Then, by (8.11), there are some facets \( F^N_{\mathcal{I}}(v), F^\deg(V) \mathcal{N}(I) \) and \( F^N_{\mathcal{K}}(w) \) of \( \mathcal{N}(v), \deg(V) \mathcal{N}(I) \) and \( \mathcal{N}(w) \), respectively, such that
\[ F^N_{\mathcal{I}}(v) \subseteq \left( F^\deg(V) \mathcal{N}(I) \cap F^N_{\mathcal{K}}(w) \right). \]
where each $F_n^\diamondsuit$ above contains a point $y \in (\partial N(v) \cap \partial \deg(V)N(I) \cap \partial N(w))$ in its relative interior. Thus by Proposition 8.1 (2) together with Lemma 8.3 and (8.10), we get (see Figure 1-2 for 4 possible stable cases in dimension 2, according to different codimensions of $F_n^\diamondsuit$),

$$\Omega_{K_n}^{N(w)} \subseteq \Omega_{\deg(V)N(I)}^{\deg} \subseteq \Omega_{\deg(V)N(I)}^{N(v)}.$$ 

Hence, by the duality property (8.9), there are compact sets $\Omega_1 \subseteq \Omega_2$, both contain $F_n^{\diamondsuit}$ such that (for simplicity, see Figure 3 for the case (d) in Figure 2),

$$(\Omega_i \cap N(v)) \subseteq (\Omega_i \cap \deg(V)N(I)) \subseteq (\Omega_i \cap N(w)), \ i = 1, 2. \quad (8.12)$$
Without loss of generality, we may assume there is only one facet $F_\mathcal{N}(v)$ of $\mathcal{N}(v)$ such that

$$F_\mathcal{N}(v) \subseteq (\partial \mathcal{N}(v) \cap \partial \text{deg}(V) \mathcal{N}(\text{Id}) \cap \partial \mathcal{N}(w)). \quad (8.13)$$

We consider the following two cases.

(i) Outside $\Omega_1$. Then we have

$$\text{dist}(\mathcal{N}(v) \setminus \Omega_1, \partial \mathcal{N}(w)) \geq \epsilon_0 > 0.$$

Thus there is a $\delta'_0 > 0$ such that for any $\delta' \in (0, \delta'_0)$,

$$(1 - \delta') (\mathcal{N}(v) \setminus \Omega_1) + \delta' \text{deg}(V) \mathcal{N}(\text{Id}) \subseteq \mathcal{N}(w).$$

(ii) Inside $\Omega_1$. Then by (8.12), (8.13) and convexity of each $\mathcal{N}(\cdot)$, there is a $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$,

$$(1 - \delta) (\Omega_1 \cap \mathcal{N}(v)) + \delta \text{deg}(V) \mathcal{N}(\text{Id}) \subseteq (\Omega_2 \cap \mathcal{N}(w)).$$

Therefore, by choosing

$$m = \left\lceil \frac{1}{\min\{\delta_0, \delta'_0\}} \right\rceil + 1,$$

we get (8.7) immediately.

\[ \square \]

### 8.3 CM-stability

In this subsection, we first give a direct proof of Theorem 2.1, then we use Theorem 1.2 to derive Theorem 1.4.

Recall the functional on $G$ associated to the pair $(v, w) \in V \setminus \{0\} \times W \setminus \{0\}$ in Section 2,

$$p_{v,w}(\sigma) = \log ||\sigma(w)||^2 - \log ||\sigma(v)||^2, \sigma \in G.$$
Without loss of generality, we may assume that $\| \cdot \|$ is an euclidean norm of vector in $V$ or $W$ and it is invariant under the maximal compact group $K$ which complexifies $G$. We note that there is a uniform constant $\delta_0$ such that for any indices $i, l$ it holds
\[
\inf\{|(kv)_i|, (kw)_i \neq 0, k \in K\} \geq \delta_0 \quad \text{and} \quad \inf\{|(kw)_i|, (kv)_i \neq 0, k \in K\} \geq \delta_0.
\]
Thus for any one parameter subgroup $\lambda(s)$, and $k, k'. \in K$, we get
\[
p_{v:w}(k' \lambda(s)k) = p_{v:w}(k^{-1} \lambda(s)k) = p_{v:w}(\lambda(s)k)
= -[\max_{\alpha\in\Lambda(kv)}(a, t_\lambda) - \max_{\alpha'\in\Lambda(kw)}(a', t_\lambda)]\log\frac{1}{s} + O(1)
= (w_k^{-1} \lambda_k(v) - w_k^{-1} \lambda_k(w))\log\frac{1}{s} + O(1), \quad \forall |s| \ll 1,
\]
where $O(1)$ means a uniform bounded constant and $t_\lambda$ denote the Lie algebra of $\lambda(s)$.

By the semi-stability of $(v, w)$, the function
\[
f_{kv,kw}(t) = [\max_{\alpha'\in\Lambda(kw)}(a', t) - \max_{\alpha\in\Lambda(kv)}(a, t)] \geq 0
\]
for all $t \in \mathbb{Z}^n$ and so for all $t \in \mathbb{Q}^n$. Thus, $f_{kv,kw}(t)$ can be extended to a non-negative continuous function in $\mathbb{R}^n$.

Fix a small $s > 0$, for example, $s = \frac{1}{100}$. Then, for any $t = (t_1, ..., t_n) \in T$, there is $t = (t_1, ..., t_n) \in \mathbb{R}^n$ such that
\[
(|t_1|, ..., |t_n|) = (s^{t_1}, ..., s^{t_n}).
\]
It follows that
\[
p_{v:w}(k'tk) = p_{v:w}(tk) = f_{kv,kw}(t)\log\frac{1}{s} + O(1)
\geq -C. \tag{8.14}
\]
Hence, we prove

**Theorem 8.4.** (1) Pair $(v, w)$ is $K$-semistable with respect to $G$. Then
\[
p_{v:w}(\sigma) \geq -C, \quad \forall \sigma \in G. \tag{8.15}
\]
(2) Pair $(v, w)$ is $K$-stable with respect to $G$. Then there is an $m \in \mathbb{N}_+$ such that
\[
mp_{v:w}(\sigma) \geq p_{v:1}(\sigma) - C
\geq \deg(V)\log||\sigma||^2 - \log||\sigma(v)||^2 - C, \quad \forall \sigma \in G. \tag{8.16}
\]

**Proof.** By the $K \times K$ decomposition of $G$, for any $\sigma \in G$, there are $k, k' \in K$ and $t \in T$ such that $\sigma = k'tk$. Thus (8.15) follows from (8.14). Next, we can consider the following energy $\tilde{p}_{v:w,1}$,
\[
\tilde{p}_{v:w,1}(\sigma) = mp_{v:w}(\sigma) - p_{v:1}(\sigma), \quad \forall \sigma \in G.
\]

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Then as in the proof of (8.14), we have

\[ \tilde{p}_{v,w;k}(k'tk) = \tilde{f}_{kv,kw,kt}(t) \log \frac{1}{s} + \mathcal{O}(1), \]

where

\[
\tilde{f}_{kv,kw,kt}(t) = m[\max_{a' \in \Lambda(kv)} \langle a', t \rangle - \max_{a \in \Lambda(kv)} \langle a, t \rangle] - [\max_{a'' \in \Lambda(kt)} \langle a'', t \rangle - \max_{a \in \Lambda(kv)} \langle a, t \rangle]
\]

is a non-negative continuous function in \( \mathbb{R}^n \) by Theorem 1.2. Thus we get

\[ \tilde{p}_{v,w;1}(\sigma) \geq -C, \quad \forall \sigma \in G, \]

which implies (8.16).

\[ \square \]

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