LIOUVille THEOREMS FOR STABLE WEAK SOLUTIONS OF ELLIPTIC PROBLEMS INVOLVING GRUSHIN OPERATOR

PHUONG LE
Division of Computational Mathematics and Engineering
Institute for Computational Science, Ton Duc Thang University, Ho Chi Minh City, Vietnam
Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Abstract. We consider the boundary value problem
\begin{align*}
\begin{cases}
-\text{div}_G(w_1 \nabla_G u) = w_2 f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align*}
where $\Omega$ is a bounded or unbounded $C^1$ domain of $\mathbb{R}^N$, $w_1, w_2 \in L^1_{\text{loc}}(\Omega) \setminus \{0\}$ are nonnegative functions, $f$ is an increasing function, $\nabla_G$ and $\text{div}_G$ are Grushin gradient and Grushin divergence, respectively. We prove some Liouville theorems for stable weak solutions of the problem under suitable assumptions on $\Omega$, $w_1$, $w_2$ and $f$. We also show the sharpness of our results when $\Omega = \mathbb{R}^N$ and $f$ has power or exponential growth.

1. Introduction and main results. Throughout this article we always assume that $\Omega$ is a bounded or unbounded $C^1$ domain of $\mathbb{R}^N$, $w_1, w_2 \in L^1_{\text{loc}}(\Omega) \setminus \{0\}$ are nonnegative functions and $f \in C^1((a, b)) \cap C^2((a, b) \setminus Z_f)$ is an increasing function, where $-\infty \leq a < b \leq +\infty$ and $Z_f$ is the set of zeros of $f$. Clearly, $Z_f$ contains at most one element due to the monotonicity of $f$. We study the nonexistence of stable weak solutions of the problem
\begin{align}
\begin{cases}
-\text{div}_G(w_1 \nabla_G u) = w_2 f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align}
where the Grushin gradient $\nabla_G$ and Grushin divergence $\text{div}_G$ are defined by
\[
\nabla_G u = (\nabla_x u, |x|^\alpha \nabla_y u),
\]
\[
\text{div}_G v = \text{div}_x v + |x|^\alpha \text{div}_y v,
\]
for $\mathbf{x} = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} = \mathbb{R}^N$ and $\alpha \geq 0$. We also denote $\Delta_G u := \text{div}_G(\nabla_G u) = \Delta_x u + |x|^2 \Delta_y u$ and call $\Delta_G$ the Grushin operator, which reduces to the well-known Laplace one when $\alpha = 0$. For $\mathbf{x} \in \mathbb{R}^N$, we define the Grushin distance from 0 to $\mathbf{x}$ as
\[
|x|_G = \left( |x|^2(\alpha + 1) + (\alpha + 1)^2 |y|^2 \right)^{\frac{1}{2(\alpha + 1)}}.
\]

2000 Mathematics Subject Classification. Primary: 35J25, 35H20; Secondary: 35B53, 35B35.
Key words and phrases. Elliptic problems, Grushin operator, stable solutions, nonexistence, Liouville theorems.
This research is funded by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2017.307.
We may then define Grushin ball $B_G(x_0, R) = \{x \in \mathbb{R}^N : |x - x_0|_G < R\}$ for $x_0 \in \mathbb{R}^N$ and $R > 0$. Finally, we denote $N_\alpha = N_1 + (\alpha + 1)N_2$, which is called the homogeneous dimension associated to the Grushin operator $\Delta_G$.

Let us introduce a functional setting for problem (1). Denote

$$H^{1,G}(\Omega, w_1) = \{u \in L^2(\Omega) : \sqrt{w_1} |\nabla_G u| \in L^2(\Omega)\},$$

then $H^{1,G}(\Omega, w_1)$ is a Banach space endowed with the norm

$$\|u\| = \left(\int_\Omega u^2 \, dx + \int_\Omega w_1 |\nabla_G u|^2 \, dx\right)^{\frac{1}{2}}.$$

We also denote by $H^{1,G}_0(\Omega, w_1)$ the closure of $C_c^\infty(\Omega)$ in $H^{1,G}(\Omega, w_1)$. Since $w_1$ and $w_2$ are not necessarily locally bounded, so are solutions of (1). Therefore, it is natural to study solutions of (1) in the following weak sense.

**Definition 1.1.** We say that $u \in H^{1,G}_0(\Omega, w_1)$ is a weak solution of problem (1) if $w_2f(u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega [w_1 \nabla_G u \cdot \nabla_G \varphi - w_2 f(u)\varphi] \, dx = 0, \quad \text{for all } \varphi \in H^{1,G}_0(\Omega, w_1). \tag{2}$$

A weak solution $u$ of problem (1) is said to be stable if

$$\int_\Omega [w_2 f'(u)\varphi^2 - w_1 |\nabla_G \varphi|^2] \, dx \leq 0, \quad \text{for all } \varphi \in H^{1,G}_0(\Omega, w_1). \tag{3}$$

The definition of stability is motivated by a phenomenon in physical sciences. A system is called in a stable state if it can recover from perturbations. A small change will not prevent the system from returning to equilibrium. From that intuition, stable solutions are those that make the energy of the system attain a local minimum. In other words, a solution $u$ is stable if the second variation at $u$ of the energy functional is nonnegative. Physical backgrounds and recent developments on stable solutions of semilinear elliptic equations can be found in the interesting monograph [11] by Dupaigne.

Liouville theorem for stable solutions, which concern about nonexistence of this particular type of solutions, have drawn much attention in the last decade. In the pioneering work [12], Farina established a sharp Liouville theorem for stable classical solutions of equation

$$-\Delta u = |u|^{q-1}u \quad \text{in } \mathbb{R}^N$$

where $q > 1$. Indeed, he proved that the equation has no nontrivial stable classical solution if and only if $q < q_c$, where

$$q_c = \begin{cases} +\infty & \text{if } N \leq 10, \\ \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & \text{if } N \geq 11 \end{cases}$$

is call the Joseph-Lundgren exponent (see [15]). It should be notice that this exponent is larger than the usual Sobolev one $q_S = \frac{N + 2}{N - 2}$ and condition $q < q_c$ is equivalent to $N < 2 + \frac{4(q + \sqrt{q(q-1)})}{q-1}$. Later, some of Farina’s results were extended to the weighted case in [5, 8, 23]. In [8], the authors proved the nonexistence of nontrivial stable weak solutions of $-\Delta u = |x|^{\alpha} |u|^{q-1}u$ under the restriction that the solutions are locally bounded. This restriction was removed in [23]. Although the work [5] only deals with stable classical solutions, it consider more general
A typical result in [5] states that if $q > 1$, $w_1 = (|x|^2 + 1)^{q_1/2}$, $w_2 = (|x|^2 + 1)^{q_2/2}$ and
\[
N < \frac{2(q + \sqrt{q(q - 1)})(q_2 - q_1 + 2)}{q - 1} + 2 - q_1,
\]
then there is no positive stable classical solution of $-\text{div}(w_1 \nabla u) = w_2 u^q$ in $\mathbb{R}^N$. Since then, these results have been generalized to the weighted quasilinear equation $-\text{div}(w_1 |\nabla u|^{q-2} \nabla u) = w_2 f(u)$ in $\mathbb{R}^N$ with $f(u) = e^u$, $|u|^q - 1 u$ where $q > 1$ or $-u^{-q}$ where $q > 0$. Such generalizations may be found in [6, 17, 19, 20] and references therein. Some attempts to extend Farina’s results to elliptic problems with nonlinearity $f$ belonging to a wider class of positive and convex functions were also made in [4, 10, 16, 18].

Another possible generalization corresponds to elliptic problems involving Grushin operator, i.e., problem (1). It is well-known that the operator $\Delta_G$ belongs to the wide class of subelliptic operators studied by Franchi et al. in [14] (see also [2, 3]). Via Kelvin transform and the method of moving planes, the optimal Liouville type theorem for nonnegative solutions of the problem $-\Delta_G u = u^p$ in $\mathbb{R}^N$ has been established in [21, 24] for the case $1 < p < \frac{N + 2}{N - 2}$. More recently, some Liouville type theorems for stable solutions of elliptic problems involving Grushin operator were established in [1, 9, 22]. In [9], the authors proved the nonexistence of nontrivial stable weak solutions of the equation
\[
-\Delta_G u + \nabla_G w \cdot \nabla_G u = |x|^\alpha |u|^{q-1} u \quad \text{in } \mathbb{R}^N
\]
in the case $N_\alpha < N^\#$, where $N^\#$ is explicitly computed. It should be notice that this equation is a special case of (1) where $\Omega = \mathbb{R}^N$, $w_1 = e^{-w}$, $w_2 = |x|^\alpha e^{-w}$ and $f(u) = |u|^{q-1} u$. The work [22] concerns about nonexistence of stable classical solutions of similar equation with no convection (i.e, $w = 0$) but with more general subelliptic operator. Moreover, the nonexistence of classical stable solutions of equation
\[
-\Delta_G u = e^u \quad \text{in } \mathbb{R}^N
\]
was derived in [1] for the case $2 < N_\alpha < 10$. However, in all of the above works, questions on the sharpness of Liouville theorems are left open, except for the case $\alpha = 0$. It is due to the difficulty of building explicit stable solutions of such equations when $\alpha > 0$.

As far as we know, Liouville type theorems for stable weak solutions of (1) has not been fully studied in the literature except for some special cases in [1, 9] mentioned above. The purpose of this paper is to establish Liouville results for stable weak solutions of (1) in general domain $\Omega$, with general nonlinearity $f$ and general nonnegative weights $w_1, w_2$.

To state our main result, let us denote by $z_f$ the unique zero of $f$ if $Z_f \neq \emptyset$. We also denote by $B(x_0, R)$ the usual ball in $\mathbb{R}^{N_1}$ centered at $x_0$ with radius $R$. Similar notation is used for the ball $B(y_0, R)$ in $\mathbb{R}^{N_2}$. For a measurable set $U \subset \mathbb{R}^N$, we denote by $|U|$ its Lebesgue measure. We also use the convention that $\int_U \frac{1}{x} \, dx = +\infty$ if $v$ is a nonnegative function which is equal to zero in a subset of $U$ having positive Lebesgue measure. Finally, for $c \in \mathbb{R}$ we define $\text{sign}(c) = 1$ if $c \geq 0$ and $\text{sign}(c) = -1$ otherwise.

The main result in this article is the following Liouville theorem, which extends some previous results in [1, 5, 8, 9, 12, 13, 23].
Theorem 1.2. Assume that there exist $\Gamma, \gamma, C > 0$ such that
\[
C |f(t)|^{2T} \leq \gamma f'(t)^2 \leq f(t)f''(t) \quad \text{for all } t \in (a, b) \setminus Z_f
\]  
and there exists $\beta \in (1, 1 + 2\sqrt{\gamma})$ such that
\[
\lim_{R \to +\infty} R^{-2(\Gamma + \beta + 1)} \int_{\Omega \cap B_G(0,4R) \setminus B_G(0,R)} \left( \frac{w_1^{\Gamma+\beta+1}}{w_2^{\beta+1}} \right)^{\frac{1}{2}} \, dx = 0.
\]  
Assume in addition that $f(0) = 0$ or $\Omega = \mathbb{R}^N$. Then $w_2 f(u) \equiv 0$ if $u$ is a stable weak solution of (1). Consequently,

(i) If $Z_f = \emptyset$, then (1) has no stable weak solution.

(ii) If $Z_f \neq \emptyset$ and $w_2 > 0$ in $\Omega$, then (1) has a unique stable weak solution $u \equiv z_f$.

Although this paper is motivated by the idea of Farina [12] and related works, it should be mentioned that the use of available techniques in our case is not straightforward and many nontrivial additional ideas are introduced to overcome the difficulties caused by the generality of our statements.

- We consider a large class of weights $w_1, w_2$ and nonlinearity $f$ and we neither assume positivity nor convexity of $f$. We also consider the equation in a proper domain $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition. This is different from other works which deal with general nonlinearity (see [4, 10, 16]) and therefore our results require very delicate analysis.

- The fact that weak solutions are not locally bounded also leads to another difficulty. The usual cut-off functions as used in [9, 23] do not work with general nonlinearity $f$. To overcome this difficulty, we use new cut-off functions inspired by [17, 19, 20] and show that our key estimates are still valid for these functions.

- We also construct some examples to show the sharpness of our Liouville theorems. To prove the stability of constructed solutions, we use a version of Hardy inequalities related to Grushin type operators established in [7].

As a consequence of Theorem 1.2, we have the following Liouville result which is valid for problems in a bounded domain in any dimension.

Proposition 1. If $\Omega$ is bounded, $w_2 > 0$ in $\Omega$, $f$ satisfies (4) and $f(0) = 0$, then (1) has a unique stable weak solution $u \equiv 0$.

For the next result, we say that $f$ is positive (negative) if $f(t) > 0$ for all $t \in (a, b)$ (resp., $f(t) < 0$ for all $t \in (a, b)$).

Proposition 2. Assume that $f(0) = 0$ or $\Omega = \mathbb{R}^N$, and one of the following two conditions occurs

(F1) there exist $\Gamma \geq \gamma > 0$ such that
\[
\gamma f'(t)^2 \leq f(t)f''(t) \leq \Gamma f'(t)^2 \quad \text{for all } t \in (a, b) \setminus Z_f,
\]
\[
\liminf_{t \to a^+} |f(t)|^{-\frac{1}{\Gamma}} f'(t) > 0 \quad \text{if } f \text{ is not positive},
\]
\[
\liminf_{t \to b^-} |f(t)|^{-\frac{1}{\Gamma}} f'(t) > 0 \quad \text{if } f \text{ is not negative},
\]

(F2) there exist $\gamma \geq \Gamma > 0$ such that
\[
\gamma f'(t)^2 \leq f(t)f''(t) \quad \text{for all } t \in (a, b) \setminus Z_f,
\]
\[
\liminf_{t \to a^+} |f(t)|^{-\frac{1}{\gamma}} f'(t) > 0 \quad \text{if } f \text{ is positive},
\]
\[
\liminf_{t \to b^-} |f(t)|^{-\frac{1}{\gamma}} f'(t) > 0 \quad \text{if } f \text{ is negative},
\]
\[
\liminf_{t \to z_f} |f(t)|^{-\Gamma} f'(t) > 0 \text{ if } Z_f \neq \emptyset.
\]

We also assume in addition that

(W) there exist \( R_0, \delta > 0, q_1 \in \mathbb{R} \) and \( q_2 > q_1 - 2 \) such that \( w_2(x) > 0 \) for all \( |x|_G < R_0 \), moreover

\[
\frac{w_1(x)^{1+\beta+1}}{w_2(x)^{\beta+1}} \leq C_\beta |x|_G^{q_1(1+\beta+1)-q_2(\beta+1)}
\]

for all \( |x|_G > R_0 \), all \( \beta \in (1+2\sqrt{\gamma} - \delta, 1+2\sqrt{\gamma}) \) and some \( C_\beta \) independent of \( x \).

If

\[
\lim_{R \to +\infty} R^{-N^\#+\varepsilon} |\Omega \cap B_G(0, 4R) \setminus B_G(0, R)| = 0
\]

for some \( \varepsilon > 0 \), where

\[
N^\# := \frac{2(1 + \sqrt{\gamma})(q_2 - q_1 + 2)}{\Gamma} + 2 - q_1,
\]

and problem (1) has a stable weak solution \( u \), then \( Z_f \neq \emptyset \) and \( u \equiv z_f \).

A sufficient condition for (W) is that

(W') there exist \( C_1, C_2, R_0 > 0, q_1 \in \mathbb{R} \) and \( q_2 > q_1 - 2 \) such that

\[
\frac{w_1(x)}{|x|_G^{q_1}} \leq C_1 \quad \text{and} \quad \frac{w_1(x)}{|x|_G^{q_1}} \leq C_2 \frac{w_2(x)}{|x|_G^{q_2}}
\]

for all \( |x|_G > R_0 \),

\[
w_2(x) > 0 \text{ for all } |x|_G < R_0.
\]

Moreover, since

\[
B_G(0, 4R) \subset B_x(0, 4R) \times B_y \left( 0, \left( \frac{4R}{\alpha + 1} \right)^{\alpha+1} \right),
\]

we have \( |\Omega \cap B_G(0, 4R) \setminus B_G(0, R)| \leq |B_G(0, 4R)| \leq CR^{N_\alpha} \). Therefore, (6) is satisfied if \( N_\alpha < N^\# \) (we may choose \( \varepsilon < N^\# - N_\alpha \)). If \( f(0) = 0 \), then (6) is also satisfied in the case that \( \Omega \) has finite Lebesgue measure and \( N^\# > 0 \).

If we focus our attention to the class of bounded stable weak solutions, then we are able to establish a Liouville theorem in \( \mathbb{R}^N \) without any restriction on the homogeneous dimension \( N_\alpha \).

**Theorem 1.3.** Assume that \( f \) is positive and \( a = -\infty \), or \( f \) is negative and \( b = +\infty \). Assume in addition that there exists \( \gamma > 0 \) such that \( \gamma f'(t)^2 \leq f(t) f''(t) \) for all \( t \in (a, b) \) and \( w_1, w_2 \) satisfy assumption (W'). Then the problem

\[
-\text{div}_G(w_1 \nabla_G u) = w_2 f(u) \text{ in } \mathbb{R}^N
\]

has no bounded stable weak solution.

As an application, we may derive the following Liouville theorems for problem (1) with Lane-Emden, negative exponent, exponential or singular nonlinearity.

**Proposition 3.** Assume that \( q > 1 \) and \( w_1, w_2 \) satisfy (W) with \( \Gamma = \gamma = \frac{2-1}{q} \). Then zero is the unique stable weak solution of problem

\[
\begin{align*}
-\text{div}_G(w_1 \nabla_G u) &= w_2 |u|^{q-1}u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

provided that

\[
\lim_{R \to +\infty} R^{-N^\#+\varepsilon} |\Omega \cap B_G(0, 4R) \setminus B_G(0, R)| = 0
\]
for some \( \varepsilon > 0 \), where

\[
N^\# := \frac{2(q + \sqrt{q(q - 1)})(q_2 - q_1 + 2)}{q - 1} + 2 - q_1.
\]

In particular, the conclusion holds in one of the following cases

(i) \( \Omega \) is bounded,
(ii) \( \Omega \) has finite Lebesgue measure and \( N^\# > 0 \),
(iii) \( N_\alpha < N^\# \).

**Proposition 4.** Assume that \( q < 0 \) and \( w_1, w_2 \) satisfy (W) with \( \Gamma = \gamma = \frac{2-1}{q} \). Then equation

\[
\text{div}_G(w_1 \nabla_G u) = w_2 u^q \quad \text{in } \mathbb{R}^N
\]

has no positive stable weak solution in homogeneous dimension

\[
N_\alpha < N^\# := \frac{2(q + \sqrt{q(q - 1)})(q_2 - q_1 + 2)}{q - 1} + 2 - q_1.
\]

Moreover, the equation has no bounded positive stable weak solution in any dimension if \( w_1, w_2 \) satisfy (W').

**Proposition 5.** Assume that \( w_1, w_2 \) satisfy (W) with \( \Gamma = \gamma = 1 \). Then equation

\[
-\text{div}_G(w_1 \nabla_G u) = w_2 e^u \quad \text{in } \mathbb{R}^N
\]

has no stable weak solution in homogeneous dimension \( N_\alpha < N^\# := 10 - 5q_1 + 4q_2 \). Moreover, the equation has no bounded stable weak solution in any dimension if \( w_1, w_2 \) satisfy (W').

**Proposition 6.** Assume that \( w_1, w_2 \) satisfy (W') and \( f : (0, +\infty) \to \mathbb{R} \) has one of the following forms

(i) \( f(t) = -e^{r-t-s} \),
(ii) \( f(t) = -t^{-r} - t^{-s} \),
(iii) \( f(t) = -e^{-rt} - t^{-s} \),

where \( r, s > 0 \). Then equation

\[
-\text{div}_G(w_1 \nabla_G u) = w_2 f(u) \quad \text{in } \mathbb{R}^N
\]

has no bounded positive stable weak solution.

**Remark 1.** The assumption on \( N_\alpha < N^\# \) in Proposition 3 can be rewritten as

\[
\begin{cases}
1 < q < q_c(N_\alpha, q_1, q_2), & \text{if } N_\alpha > 10 - 5q_1 + 4q_2, \\
1 < q < \infty, & \text{if } N_\alpha \leq 10 - 5q_1 + 4q_2,
\end{cases}
\]

where

\[
q_c(N_\alpha, q_1, q_2) = \frac{2(q_2 - q_1 + 2)\sqrt{(q_2 - q_1 + 2)(2N_\alpha + q_1 + q_2 - 2)}}{(N_\alpha + q_1 - 2)(N_\alpha - 10 + 5q_1 - 4q_2)} + \frac{N_\alpha^2 + 2(q_1^2 + q_2 - q_1 - 2)N_\alpha + (q_1^2 - 2q_2^2 + 2q_1q_2 - 4q_1 - 4q_2 + 4)}{(N_\alpha + q_1 - 2)(N_\alpha - 10 + 5q_1 - 4q_2)}.
\]

Note that \( q_c(N, 0, 0) \) is the Joseph-Lundgren exponent as mentioned in the introduction of this paper. Proposition 3 recovers some known results in [5, 8, 12, 23] when \( \alpha = 0 \). A result similar to Proposition 3 has been established in [9] for \( \Omega = \mathbb{R}^N \) under more restrictions on \( w_1 \) and \( w_2 \). The main novelty of Proposition 3 is therefore the conclusion in the case \( \Omega \neq \mathbb{R}^N \).
We also point out that the first part of Proposition 5 (i.e., nonexistence of stable weak solutions) has been obtained in [1, Corollary 1.3] under additional assumptions that \( w_1 = w_2 = 1 \) and \( u \in C^2(\mathbb{R}^N) \).

**Remark 2.** Proposition 4 and 6 are completely new to the best of our knowledge. To prove Proposition 6, it suffice to show that \( \frac{f(t)f''(t)}{f'(t)^2} > 1 \), then to apply Theorem 1.3. For instance, if \( f(t) = -e^{-rt} - t^{-s} \) we have

\[
\frac{f(t)f''(t)}{f'(t)^2} = \frac{(r^2 e^{-rt} + s(s + 1)t^{-s-2}) (e^{-rt} + t^{-s})}{(re^{-rt} + st^{-s-1})^2} = 1 + \frac{st^{-s-2} (e^{-rt} + t^{-s}) + (r - st^{-1})^2 t^{-s} e^{-rt}}{(re^{-rt} + st^{-s-1})^2} > 1.
\]

Let us emphasize that Proposition 2 is sharp in the sense that problem (1) may have nontrivial stable weak solutions if (6) is not satisfied for any \( \varepsilon > 0 \), which is the case if \( \Omega = \mathbb{R}^N \) and \( N_\alpha \geq N^# \). Therefore, we may call \( N^# \) a critical homogeneous dimension, which divides the range of instability of problem (1). Indeed, if \( \Omega = \mathbb{R}^N \) and \( f \) has power or exponential growth, we are able to give some examples of such stable weak solutions.

**Proposition 7.** Let \( q \in (-\infty, 0) \cup (1, +\infty) \), \( q_2 > q_1 - 2 > -N_\alpha + \max \left\{ \frac{q_2 - q_1 + 2}{q - 1}, 0 \right\} \) and \( s > \max \left\{ 2\alpha - N_1, \frac{2\alpha(3\alpha - 1 + 2\sqrt{q(q-1)})}{q-1} \right\} \). We define

\[
\begin{align*}
    f(u) &= \text{sign}(q) u^q, \\
    w_1(x) &= |x|^{q_1+2\alpha-s} |x|^{s-2\alpha}, \\
    w_2(x) &= |x|^{q_2-s} |x|^s.
\end{align*}
\]

Then, \( f, w_1 \) and \( w_2 \) satisfy (F1) and (W) with \( \Gamma = \gamma = \frac{q-1}{q} \). Furthermore, if

\[
N_\alpha \geq N^# := \frac{2(q + \sqrt{q(q-1)})(q_2 - q_1 + 2)}{q-1} + 2 - q_1, \tag{7}
\]

then \( U(x) = |M|^\frac{1}{1-q} |x|^{\frac{q_2-q_1+2}{1-q}} |x|^{\frac{q_2-q_1+2}{1-q}} \) is a stable weak solution of

\[-\text{div}_{\mathcal{G}}(w_1 \nabla_G u) = w_2 f(u) \quad \text{in} \ \mathbb{R}^N,\]

where

\[
M = \frac{q_2 - q_1 + 2}{1-q} \left( N_\alpha + q_1 - 2 + \frac{q_2 - q_1 + 2}{1-q} \right).
\]

**Proposition 8.** Let \( q_2 > q_1 - 2 > -N_\alpha \) and \( s > \max \{ 2\alpha - N_1, 10\alpha \} \). We define

\[
\begin{align*}
    f(u) &= e^u, \\
    w_1(x) &= |x|^{q_1+2\alpha-s} |x|^{s-2\alpha}, \\
    w_2(x) &= |x|^{q_2-s} |x|^s.
\end{align*}
\]

Then, \( f, w_1 \) and \( w_2 \) satisfy (F1) and (W) with \( \Gamma = \gamma = 1 \). Furthermore, if

\[
N_\alpha \geq N^# := 10 - 5q_1 + 4q_2,
\]

then \( U(x) = \ln(N_\alpha + q_1 - 2) + \ln(q_2 - q_1 + 2) + (q_1 - q_2 - 2) \ln |x|_\mathcal{G} \) is a stable weak solution of

\[-\text{div}_{\mathcal{G}}(w_1 \nabla_G u) = w_2 f(u) \quad \text{in} \ \mathbb{R}^N.\]
In what follows, we denote by $C$ a generic constant whose concrete values may change from line to line or even in the same line. If this constant depends on an arbitrary small number $\varepsilon$, then we may denote it by $C_\varepsilon$. For the sake of simplicity, we also denote by $\int v$ the integral $\int_\Omega v\,dx$.

We prove our main results, namely Theorem 1.2, 1.3 and Proposition 2, in the next section. Section 3 is devoted to the sharpness of our Liouville theorems where we prove Proposition 7 and 8.

2. Liouville type theorems for stable weak solutions. The key estimate in the proof of our main results is (17). In order to obtain it, we firstly derive an integral estimate which is valid for all $f \in C^1((a,b))$.

Proposition 9. Assume that $u$ is a stable weak solution of (1). Let $\phi, \psi \in C^1((a,b))$ such that $\phi', \psi' \in L^\infty((a,b))$, $\psi'(t) \geq \phi'^2(t)$ and $\phi'(t) > 0$ for a.e. $t \in (a,b)$. Then for all $\eta \in C^1_c(\Omega)$ and $\varepsilon \in (0,1)$ we have

$$\int w_2 (f'(u)\phi(u) - (1 + \varepsilon)f(u)(\psi(u)))\eta^2 \leq C_\varepsilon \int w_1 \left( \phi(u)^2 + \frac{\psi(u)^2}{\psi'(u)} \right) |\nabla G\eta|^2,$$

where we use a convention that $\psi(t_0) = \limsup_{t \to t_0} \psi(t)$ if $\psi'(t_0) = 0$.

The same conclusion also holds for all $\eta \in C^1_c(\mathbb{R}^N)$ if $\phi(0) = \psi(0) = 0$.

Proof of Proposition 9. Since $\psi'$ is bounded and $\eta \in C^1_c(\Omega)$ or $\psi(0) = 0$, we may use $\varphi = \psi(u)\eta^2$ as a test function in (2) to obtain

$$\int w_1 \psi'(u)\eta^2 |\nabla G u|^2 + 2 \int w_1 \psi(u)\eta \nabla G u \cdot \nabla G \eta = \int w_2 f(u)\psi(u)\eta^2.$$

Hence,

$$\int w_1 \psi'(u)\eta^2 |\nabla G u|^2 - \int w_2 f(u)\psi(u)\eta^2 \leq 2 \int w_1 |\psi(u)|\eta |\nabla G u||\nabla G \eta|$$

$$\leq \varepsilon \int w_1 \psi'(u)\eta^2 |\nabla G u|^2 + \frac{4}{\varepsilon} \int w_1 \psi(u)^2 |\nabla G \eta|^2,$$

which implies

$$\int w_1 \psi'(u)\eta^2 |\nabla G u|^2 \leq \frac{1}{1 - \varepsilon} \int w_2 f(u)\psi(u)\eta^2 + \frac{4}{\varepsilon (1 - \varepsilon)} \int w_1 \psi(u)^2 |\nabla G \eta|^2. \quad (8)$$

Since $\phi'$ is bounded and $\eta \in C^1_c(\Omega)$ or $\phi(0) = 0$, we may apply (3) with $\varphi = \phi(u)\eta$ to get

$$\int w_2 f'(u)\phi(u)\eta^2 \leq \int w_1 |\phi'(u)\eta \nabla G u + \phi(u) \nabla G \eta|^2$$

$$\leq \int w_1 \phi'(u)^2 \eta^2 |\nabla G u|^2 + \int w_1 \phi(u)^2 |\nabla G \eta|^2 + 2 \int w_1 |\phi(u)|\phi'(u)\eta |\nabla G u||\nabla G \eta|.$$
From (8), (9) and the fact that $\psi' \geq \phi^2$ we obtain
\[ \int w_2 f'(u) \phi(u)^2 \eta^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon} \int w_2 f(u) \psi(u) \eta^2 + C \varepsilon \int w_1 \left( \phi(u)^2 + \frac{\psi(u)}{\psi'(u)} \right) |\nabla \eta|^2. \]

Finally, we may replace $\varepsilon$ with $\frac{\varepsilon}{\varepsilon^2}$ to get the conclusion. \qed

**Proof of Theorem 1.2.** Let $\phi(t) = |f(t)|^{2+\gamma} f(t)$ and $\psi(t) = \int_c^t \phi'(s)^2 \, ds$, where
\[ c = \begin{cases} a, & \text{if } f \text{ is positive,} \\ b, & \text{if } f \text{ is negative,} \\ z_f, & \text{otherwise.} \end{cases} \]

We prove that for all $t \in (a,b) \setminus \{c\}$,
\[ |\psi(t)| \leq \frac{(\beta + 1)^2}{4(\beta + \gamma)} \frac{f'(t)}{|f(t)|} \phi(t)^2. \tag{10} \]

Indeed, if $c < t < b$, then $f(t) > 0$, $\psi(t) > 0$ and for all $d \in (c,t)$,
\begin{align*}
\frac{4\beta}{(\beta + 1)^2} \int_d^t \phi'(s)^2 \, ds & = \beta \int_d^t f(s)^{\beta-1} f'(s)^2 \, ds \\
& = f(t)^{\beta} f'(t) - f(d)^{\beta} f'(d) - \int_d^t f(s)^{\beta} f''(s) \, ds \\
& \leq f(t)^{\beta} f'(t) - \gamma \int_d^t f(s)^{\beta-1} f'(s)^2 \, ds \\
& \leq \frac{f'(t)}{f(t)} \phi(t)^2 - \frac{4\gamma}{(\beta + 1)^2} \int_d^t \phi'(s)^2 \, ds.
\end{align*}

Therefore, $\int_d^t \phi'(s)^2 \, ds \leq \frac{(\beta + 1)^2}{4(\beta + \gamma)} \frac{f'(t)}{|f(t)|} \phi(t)^2$. Now let $d \to c^+$, we conclude that $\psi(t)$ is finite and we obtain (10). By a similar argument, (10) also holds for $a < t < c$. Thus, (10) holds for all $t \in (a,b) \setminus \{c\}$.

As a consequence of (10), we have
\[ \frac{\psi(t)^2}{\psi'(t)} \leq C \frac{f'(t)^2 \phi(t)^4}{f(t)^2 \phi'(t)^2} = C \phi(t)^2. \tag{11} \]

Clearly, $\phi$ and $\psi$ are increasing in $(a,b)$ and belongs to $C^1((a,b))$. However, since we do not know if $\phi', \psi' \in L^\infty((a,b))$, we have to truncate $\phi$ and $\psi$ before applying Proposition 8. For large enough $k \in \mathbb{N}$, we denote by $x_k$ the unique solution of equation $\phi(t) = -k$ if the solution exists and $x_k = a$ if the equation has no solution. Similarly, let $y_k$ be the unique solution of equation $\phi(t) = k$ if the solution exists and $y_k = b$ if the equation has no solution. We now define
\[ \phi_k(t) = \begin{cases} \phi(t), & \text{if } x_k < t < y_k, \\ (t-x_k) \phi'(x_k) + \phi(x_k), & \text{if } a < t \leq x_k, \\ (t-y_k) \phi'(y_k) + \phi(y_k), & \text{if } y_k \leq t < b \end{cases} \]

and $\psi_k(t) = \int_c^t \phi_k'(s)^2 \, ds$. Then $\phi_k, \psi_k \in C^1((a,b))$ and $\phi_k', \psi_k' \in L^\infty((a,b))$. We show that, for all $t \in (a,b) \setminus \{c\}$,
\[ |\psi_k(t)| \leq \frac{(\beta + 1)^2}{4(\beta + \gamma)} \frac{f'(t)}{|f(t)|} \phi_k(t)^2. \tag{12} \]
Indeed, (12) holds for $t \in (x_k, y_k) \setminus \{c\}$ thanks to (10). In order to prove (12) for $y_k \leq t < b$, it suffices to show that

$$\int_{y_k}^{t} \phi_k(s)^2 \, ds \leq \frac{(\beta + 1)^2}{4(\beta + \gamma)} \left( \frac{f'(t)}{f(t)} \phi_k(t)^2 - \frac{f'(y_k)}{f(y_k)} \phi_k(y_k)^2 \right),$$

or

$$(t - y_k)\phi'(y_k)^2 \leq \frac{\beta + 1}{2(\beta + \gamma)} \left( \phi'(t)/(\phi(t)) \left((t - y_k)\phi'(y_k) + \phi(y_k))^2 - \phi'(y_k)\phi(y_k) \right) \right),$$

which is equivalent to

$$h_k(t) := g(t) - \frac{d(t - y_k + g(y_k))}{t - y_k + dg(y_k)} \leq 0, \quad (13)$$

where $d = \frac{\beta + 1}{2(\beta + \gamma)}$ and $g(t) = \frac{\phi(t)}{\phi(t)} = \frac{2}{\beta + 1} f(t)$. We have

$$h_k(t) = \frac{2}{\beta + 1} \frac{f'(t)^2 - f(t) f''(t)}{f'(t)^2} - \frac{d(t - y_k + g(y_k))(t - y_k + 2dg(y_k) - g(y_k))}{(t - y_k + dg(y_k))^2}$$

$$\leq \frac{2}{\beta + 1} (1 - \gamma) - l_k(t - y_k),$$

where

$$l_k(t) = \frac{d(t + g(y_k))(t + 2dg(y_k) - g(y_k))}{(t + dg(y_k))^2}.$$

Since

$$l_k'(t) = \frac{2d(d - 1)g(y_k)^2}{(t + dg(y_k))^3} \geq 0 \quad \text{for } t \geq 0,$$

we obtain that for $t \geq y_k$,

$$h_k'(t) \leq \frac{2}{\beta + 1} (1 - \gamma) - l_k(0) = 0.$$

Hence $h_k(t) \leq h_k(y_k) = 0$ and (13) is verified. Therefore, (12) is proved for all $y_k \leq t < b$. A similar argument can be applied to the case $a < t \leq x_k$. Hence, (12) holds for all $t \in (a, b) \setminus \{c\}$. As a consequence,

$$f'(t)\phi_k(t)^2 - (1 + \varepsilon) f(t) \psi_k(t) \geq \left( 1 - \frac{(1 + \varepsilon)(\beta + 1)^2}{4(\beta + \gamma)} \right) f'(t)\phi_k(t)^2.$$

Note that $1 - \frac{(\beta + 1)^2}{4(\beta + \gamma)} > 0$ by the assumption $\beta < 1 + 2\sqrt{\gamma}$. We may choose and fix some $\varepsilon > 0$ such that $1 - \frac{(1 + \varepsilon)(\beta + 1)^2}{4(\beta + \gamma)} > 0$. Hence, together with assumption (4), we obtain

$$f'(t)\phi_k(t)^2 - (1 + \varepsilon) f(t) \psi_k(t) \geq C f'(t)\phi_k(t)^2 \geq C|f(t)|^2 \phi_k(t)^2. \quad (14)$$

We now prove a truncated version of (11). More precisely, we prove that for all $t \in (a, b) \setminus \{c\}$,

$$\frac{\psi_k(t)^2}{\psi_k'(t)} \leq C\phi(t)^2. \quad (15)$$

Indeed, (15) holds for $t \in (x_k, y_k) \setminus \{c\}$ thanks to (11). Note that (15) is equivalent to

$$\int_{t}^{\varepsilon} \phi_k(s)^2 \, ds \leq C|\phi(t)|^2 \phi_k(t).$$
In order to prove this for \( y_k \leq t < b \), it suffices to show that
\[
\int_{y_k}^{t} \phi'_k(s)^2 \, ds \leq \phi(t)\phi'_k(t) - \phi(y_k)\phi'_k(y_k)
\]
or
\[
\phi(t) \geq \phi(y_k) + (t - y_k)\phi'(y_k) = \phi_k(t).
\]

The last inequality is verified by the fact that
\[
\phi''(t) = \frac{\beta + 1}{2} f(t) \frac{\beta - 3 + 1}{2} \left( \frac{\beta - 1}{2} f'(t)^2 + f(t)f''(t) \right) \geq 0 \quad \text{for} \ t > c.
\]

Hence, (15) holds for \( y_k \leq t < b \). A similar argument may be carried out for the remaining case \( a < t \leq x_k \). Therefore, (15) holds for all \( t \in (a,b) \setminus \{c\} \). The above proof also implies \(|\phi_k(t)| \leq |\phi(t)|\). As a consequence,
\[
\phi_k(t)^2 + \frac{\psi_k(t)^2}{\phi_k(t)} \leq C\phi(t)^2 = C|f(t)|^{\beta + 1}.
\]

Collecting (14), (16) and with the aid of Proposition 9, we have
\[
\int w_2|f(u)|^\Gamma \phi_k(u)^2 \eta^2 \leq C \int w_1|f(u)|^{\beta + 1}|\nabla_G \eta|^2
\]
for all \( \eta \in C^1_c(\mathbb{R}^N) \).

Fatou's Lemma yields
\[
\int w_2|f(u)|^\Gamma + \beta + 1 \eta^2 \leq C \int w_1|f(u)|^{\beta + 1}|\nabla_G \eta|^2.
\]

Applying this inequality for \( \eta = \xi^m \), where \( m = \frac{\Gamma + \beta + 1}{\beta + 1} \) and \( \xi \in C^1_c(\mathbb{R}^N) \),
\[
\int w_2|f(u)|^{\Gamma + \beta + 1} \xi^{2m} \leq C \int w_1|f(u)|^{\beta + 1} \xi^{2(m-1)}|\nabla_G \xi|^2.
\]

Young’s inequality with \( q = \frac{\Gamma + \beta + 1}{\beta + 1} \), \( q' = \frac{\Gamma + \beta + 1}{\beta + 1} \) and \( \varepsilon = \frac{1}{2} \) leads now to
\[
\int w_2|f(u)|^{\Gamma + \beta + 1} \xi^{2m} \leq \frac{1}{2} \left( w_2^1 |f(u)|^{\beta + 1} \xi^{2(m-1)} \right)^{\frac{1}{q'}} + C \left( w_1 w_2^{-\frac{\beta + 1}{2}} |\nabla_G \xi|^2 \right)^{\frac{1}{q'}}
\]
\[
= \frac{1}{2} \int w_2|f(u)|^{\Gamma + \beta + 1} \xi^{2m} + C \int w_1^{\frac{1}{q'}} w_2^{-\frac{\beta + 1}{2q'}} |\nabla_G \xi|^2,
\]
which implies
\[
\int w_2|f(u)|^{\Gamma + \beta + 1} \xi^{\frac{2(\Gamma + \beta + 1)}{\beta + 1}} \leq C \int \left( \frac{w_1^{\Gamma + \beta + 1}}{w_2^{\beta + 1}} \right)^{\frac{1}{q'}} |\nabla_G \xi|^{\frac{2(\Gamma + \beta + 1)}{\beta + 1}}. \tag{17}
\]

Let \( \xi_0 \in C^1_c(\mathbb{R}, [0, 1]) \) be a cut-off function satisfying \( \xi_0(t) = 1 \) for \( |t| \leq 1 \) and \( \xi_0(t) = 0 \) for \( |t| \geq 2 \). For \( R \) large enough, we define \( \xi_R(x,y) = \xi_0(\frac{|x|}{R})\xi_0(\frac{\alpha+1}{R+1}|y|) \), then \( \xi_R \in C^1_c(\mathbb{R}^N, [0, 1]) \).

It is clear that \( \xi_R = 1 \) in \( B_x(0, R) \times B_y(0, \frac{R^{\alpha+1}}{\alpha+1}) \) and \( \xi_R = 0 \) outside of \( B_x(0, 2R) \times B_y(0, \frac{2R^{\alpha+1}}{\alpha+1}) \). Therefore, \( \nabla_G \xi_R = 0 \) outside of \( B_x(0, 2R) \times B_y(0, \frac{2R^{\alpha+1}}{\alpha+1}) \). Moreover, it is straightforward to check that \( |\nabla_G \xi_R| = \sqrt{(\nabla_x \xi_R)^2 + (\nabla_y \xi_R)^2} / R \leq CR^{-1} \) for some \( C \) depending only on \( \xi_0 \). We also have
\[
B_G(0, R) \subset B_x(0, R) \times B_y \left( 0, \frac{R^{\alpha+1}}{\alpha+1} \right).
\]
and
\[ B_x(0, 2R) \times B_y \left( 0, \frac{2R^{\alpha+1}}{\alpha+1} \right) \subset B_G \left( 0, (2^{2\alpha+2} + 4)^{\frac{1}{2\alpha+\gamma}} R \right) \subset B_G(0, 4R). \]

With this test function, inequality (17) yields
\[
\int_{\Omega \cap B_G(0,R)} \beta f(\varepsilon w_2) \Gamma^{\beta+1} \, dx \leq CR^{-2\beta+1} \int_{\Omega \cap B_G(0,4R) \setminus B_G(0,R)} \frac{w_2^{\Gamma+\beta+1}}{w_2^{\beta+1}} \, dx.
\]

Letting \( R \to +\infty \) and using assumption (5), we get \( \int \beta f(\varepsilon w_2) \Gamma^{\beta+1} = 0 \). That is, \( w_2 \beta f(\varepsilon) \equiv 0 \).

**Proof of Proposition 2.** We have
\[
\left( \frac{f'(t)}{|f(t)|^\Gamma} \right)' = \frac{(f(t)f''(t) - \Gamma f'(t)^2) f(t)}{|f(t)|^{\Gamma+2}}.
\]

Therefore, if \( f \) satisfies (F1), then \( f'/|f|^\Gamma \) is non-decreasing in \((a,c)\) and non-increasing in \((c,b)\), where \( c \) is defined as in proof of Theorem 1.2. Hence, \( f'(t) \geq C|f(t)|^\Gamma \) and (4) is satisfied. Otherwise, if \( f \) satisfies (F2), then \( f'/|f|^\Gamma \) is non-increasing in \((a,c)\) and non-decreasing in \((c,b)\). Hence also in this case, \( f'(t) \geq C|f(t)|^\Gamma \) and (4) is satisfied.

On the other hand, choosing \( \beta \in (\max\{1, 1 + 2\sqrt{\gamma} - \delta\}, 1 + 2\sqrt{\gamma}) \). Then for \( R > R_0 \) we have
\[
R^{-2\beta+1} \int_{\Omega \cap B_G(0,4R) \setminus B_G(0,R)} \frac{w_2^{\Gamma+\beta+1}}{w_2^{\beta+1}} \, dx \leq R^{-N\beta} |\Omega \cap B_G(0,4R) \setminus B_G(0,R)|,
\]
where
\[
N\beta = \frac{(2 - q_1)(\Gamma + \beta + 1) + q_2(\beta + 1)}{\Gamma}.
\]

Since \( \lim_{\beta \to (1 + 2\sqrt{\gamma})^-} N\beta = N\# \), we may choose \( \beta \) sufficiently close to \( 1 + 2\sqrt{\gamma} \) if necessary such that \( N\beta > N\# - \varepsilon \) and then let \( R \to +\infty \) in (18) to obtain (5). Now the conclusion follows immediately from Theorem 1.2.

**Proof of Theorem 1.3.** We point out that \( f'(t) > 0 \) for all \( t \in (a, b) \). Indeed, assume that \( f'(t_0) = 0 \) for some \( t_0 \in (a, b) \). If \( f \) is positive, then \( f''(t) \geq 0 \) for all \( t \in (-\infty, b) \). Therefore, \( f'(t) \leq f'(t_0) = 0 \) for all \( t \in (-\infty, t_0) \). But this contradicts to the fact that \( f \) is increasing in \( t \in (-\infty, t_0) \). Similar argument can be carried out in the case that \( f \) is negative.

By contradiction, assume that (1) has a bounded stable weak solution \( u \). Then we may restrict \( f \) into interval \((-\|u\|_{L^\infty(\mathbb{R}^N)} - 1, \|u\|_{L^\infty(\mathbb{R}^N)} + 1) \cap (a, b)\) and find out that \( f \) satisfies (F2) in its new domain for any \( \Gamma \in (0, \gamma) \). If we choose \( \Gamma \) sufficiently close to 0 such that
\[
\frac{2(1 + \sqrt{\gamma})(q_2 - q_1 + 2)}{\Gamma} + 2 - q_1 > N\alpha,
\]
we reach a contradiction by applying Proposition 2.
3. The sharpness of the critical homogeneous dimension. To show the sharpness of the critical homogeneous dimension $N^\#$ in Proposition 2, we utilize the following Hardy type inequality involving Grushin gradient.

**Proposition 10** (Hardy type inequality [7]). Let $r, s \in \mathbb{R}$ be such that $N_1 + 2 > r - s$ and $N_1 > 2 \alpha - s$. Then for every $\varphi \in C^1_c(\mathbb{R}^N)$, we have

$$
\left( \frac{N_1 + s - r}{2} \right)^2 \int_{\mathbb{R}^N} |x|^{-r} |x|^s \varphi^2 \, dx \leq \int_{\mathbb{R}^N} |x|^{2(\alpha + 1) - r} |x|^{s - 2\alpha} |\nabla_G \varphi|^2 \, dx.
$$

The above inequality is actually a special case of [7, Theorem 3.1], but it is sufficient for our purpose. We also need some basic formulas whose proofs are provided for readers’ convenience.

**Proposition 11.** If $f \in C^1(\mathbb{R})$, then

$$
\nabla_G f(|x|) = f'(|x|) |x|^{-2\alpha - 1} |x|^{\alpha} ((|x|) x, (\alpha + 1) y).
$$

Moreover, if $f \in C^2(\mathbb{R})$, we have

$$
\text{div}_G(|x|^{-r} |x|^s \nabla_G f(|x|)) = |x|^{-r + s - 2\alpha} \left( f''(|x|) + \frac{N_1 + r + s - 1}{|x|} f'(|x|) \right).
$$

**Proof of Proposition 11.** We have

$$
\nabla_x f(|x|) = f'(|x|) |x|^{-2\alpha - 1} |x|^{\alpha} x,
$$

$$
\nabla_y f(|x|) = (\alpha + 1) f'(|x|) |x|^{-2\alpha - 1} y.
$$

Therefore,

$$
\nabla_G f(|x|) = f'(|x|) |x|^{-2\alpha - 1} |x|^{\alpha} ((|x|) x, (\alpha + 1) y).
$$

Consequently,

$$
\text{div}_G(|x|^{-r} |x|^s \nabla_G f(|x|))
= \text{div}_x(|x|^{-r} |x|^s \nabla_G f(|x|)) + |x|^{\alpha} \text{div}_y (|x|^{-r} |x|^s \nabla_G f(|x|))
= \text{div}_x(f'(|x|) |x|^{-2\alpha - 1} |x|^{s + 2\alpha} x) + (\alpha + 1) |x|^{\alpha} \text{div}_y (f'(|x|) |x|^{-2\alpha - 1} |x|^{s + \alpha})
= |x|^{-r + s - 2\alpha} \left( f''(|x|) + \frac{N_1 + r + s - 1}{|x|} f'(|x|) \right).
$$

We are now in a position to prove our main results in this section.

**Proof of Proposition 7.** Clearly, $f$ satisfies (F1) with $\Gamma = \gamma = \frac{q - 1}{q}$. For $\beta \in \left(1, 1 + 2 \sqrt{\frac{q - 1}{q}}\right)$,

$$
\frac{u_1^{\Gamma + \beta + 1}}{u_2^{\beta + 1}} = |x|_{G_{\Gamma + \beta + 1}} (q_1 + 2\alpha - s) (\beta + 1)(q_2 - s) |x|^{s - 2\alpha} (\Gamma + \beta + 1).
$$

We have $s\Gamma - 2\alpha (\Gamma + \beta + 1) > 0$ thanks to $s > \frac{2\alpha (3q - 1 + 2\sqrt{q(q - 1)})}{q - 1}$ and $\beta < 1 + 2 \sqrt{\frac{q - 1}{q}}$. Since $|x| \leq |x|_G$,

$$
\frac{u_1^{\Gamma + \beta + 1}}{u_2^{\beta + 1}} \leq |x|_{G_{\Gamma + \beta + 1}} (q_1 (\Gamma + \beta + 1) - q_2 (\beta + 1)).
$$

Therefore, $w_1$ and $w_2$ satisfy (W).
Note that $\text{sign}(M) = \text{sign} \left( \frac{2s-2q+2}{1-q} \right) = -\text{sign}(q)$. We may use Proposition 11 to check that $U$ is a weak solution of (1). It is remain to prove that $U$ is stable. By density arguments, it suffice to show (3) for $\varphi \in C^1_c(\mathbb{R}^N)$. By Hardy inequality (see Proposition 10),
\[
\int [w_1|\nabla_G \varphi|^2 - w_2 f'(u) \varphi^2] = \int \left( |x|^{q_1+2\alpha-s} |x|^{s-2\alpha} |\nabla_G \varphi|^2 - |qM||x|^{\frac{q_1-s-2}{s-2}} |x|^s \varphi^2 \right) 
\geq \left( \frac{N_\alpha + q_1 - 2}{2} \right)^2 + qM \int |x|^{\frac{q_1-s-2}{s-2}} |x|^s \varphi^2.
\]

On the other hand, direct calculation yields that (7) is equivalent to
\[
\left( \frac{N_\alpha + q_1 - 2}{2} \right)^2 + qM \geq 0.
\]
Therefore, $U$ is stable.

**Proof of Proposition 8.** Clearly, $f$ satisfies (F1) with $\Gamma = \gamma = 1$. For $\beta \in (1, 3)$,
\[
\frac{u_1^{\Gamma+\beta+1}}{u_2^{\beta+1}} = |x|^{(\Gamma+\beta+1)(q_1+2\alpha-s)-(\beta+1)(q_2-s)} |x|^{s-2\alpha(\Gamma+\beta+1)}.
\]
We have $s\Gamma - 2\alpha(\Gamma + \beta + 1) > 0$ thanks to $s > 10\alpha$ and $\beta < 3$. From $|x| \leq |x|_G$, we have
\[
\frac{u_1^{\Gamma+\beta+1}}{u_2^{\beta+1}} \leq |x|^{q_1(\Gamma+\beta+1) - q_2(\beta+1)}.
\]
Therefore, $w_1$ and $w_2$ satisfy (W).

We may use Proposition 11 to check that $U$ is a weak solution of (1). It is remain to prove that $U$ is stable. By density arguments, it suffice to show (3) for $\varphi \in C^1_c(\mathbb{R}^N)$. We have
\[
\int [w_1|\nabla_G \varphi|^2 - w_2 f'(u) \varphi^2] 
\geq \left( \frac{N_\alpha + q_1 - 2}{2} \right)^2 - (N_\alpha + q_1 - 2)(q_2 - q_1 + 2) \int |x|^{\frac{q_1-s-2}{s-2}} |x|^s \varphi^2 
\geq \frac{1}{4}(N_\alpha + q_1 - 2)(N_\alpha + 5q_1 - 4q_2 - 10) \int |x|^{\frac{q_1-s-2}{s-2}} |x|^s \varphi^2 
\geq 0.
\]
Therefore, $U$ is stable.

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Received January 2019; revised May 2019.

E-mail address: lephuong@tdtu.edu.vn