Subnormal $n$-roots of quasinormal operators are quasinormal

Pawel Pietrzycki

Abstract. In the recent paper [7], R. E. Curto, S. H. Lee, J. Yoon, asked the following question: Let $A$ be a subnormal operator, and assume that $A^2$ is quasinormal. Does it follow that $A$ is quasinormal? In this paper, we give an affirmative answer to this question. In fact, we prove more general result that subnormal $n$-roots of quasinormal operators are quasinormal.

1. Introduction

The class of bounded quasinormal operators was introduced by A. Brown in [2]. A bounded operator $A$ on a (complex) Hilbert space $\mathcal{H}$ is said to be quasinormal if $A(A^*A) = (A^*A)A$. Two different definitions of unbounded quasinormal operators appeared independently in [14] and in [23]. As shown in [11, Theorem 3.1], these two definitions are equivalent. Following [23, 154 pp.], we say that a closed densely defined operator $A$ in $\mathcal{H}$ is quasinormal if $A$ commutes with the spectral measure $E$ of $|A|$, i.e. $E(\sigma)A \subset AE(\sigma)$ for all Borel subsets $\sigma$ of the nonnegative part of the real line. By [23, Proposition 1], a closed densely defined operator $A$ in $\mathcal{H}$ is quasinormal if and only if $U|A| \subset |A|U$, where $A = U|A|$ is the polar decomposition of $A$ (see [26, Theorem 7.20]). For more information on quasinormal operators we refer the reader to [2, 4, 24] for the bounded case, and to [14, 23, 16, 11, 3, 24] for the unbounded one.

In the recent paper [7], R. E. Curto, S. H. Lee and J. Yoon motivated by previous papers [5, 6] asked the following question:

Problem 1.1. (see [7, Problem 1.1]) Let $A$ be a subnormal operator, and assume that $A^2$ is quasinormal. Does it follow that $A$ is quasinormal?

With the additional assumption of left invertibility they showed that a left invertible subnormal operator $A$ whose square $A^2$ is quasinormal must be quasinormal (see [7, Theorem 2.3]). It remains an open question whether this is true in general without any assumption about left invertibility.

In the literature, similar properties to Problem 1.1 for other classes of operators are known. Namely, hyponormal $n$-roots of normal operators are normal (see [19, Theorem 5]). In turn, if $A$ is hyponormal operator and $A^n$ is subnormal then $A$ doesn’t have to be subnormal (see [20]).

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In this paper, we give an affirmative answer to question posed by R. E. Curto, S. H. Lee and J. Yoon. In fact, we prove more general result:

**Theorem 1.2.** If $A$ is a subnormal operator on $\mathcal{H}$ and there exist $n \in \mathbb{N}$ such that $A^n$ is quasinormal, then $A$ is quasinormal.

In the Section 3, we provide proof of this theorem. In this proof we will use the following Embry’s characterisation of quasinormal operators.

**Theorem 1.3.** (see [8, page 63]) Let $A$ be a bounded operator on $\mathcal{H}$. Then the following conditions are equivalent:

(i) $A$ is quasinormal,

(ii) $A^*A^n = (A^*A)^n$, $n \in \mathbb{N}$,

(iii) there exists a (unique) spectral Borel measure $E$ on $\mathbb{R}_+$ such that

$$A^*A^n = \int_{\mathbb{R}_+} x^n E(dx), \quad n \in \mathbb{Z}_+.$$ 

(cf. [11, Theorem 3.6]). The assertion (ii) of this theorem leads to the following question: is it necessary to assume that the equality in (1.1) holds for all $n \in \mathbb{N}$?

To be more precise we ask for which subset $S \subset \mathbb{N}$ the following system of operator equations:

$$A^*A^s = (A^*A)^s, \quad s \in S,$$

implies the quasinormality of $A$. This problem has been studied by several authors (see [11, 12, 17, 18, 24, 25]). Below we collect some results on this problem. If operator $A$ satisfies one of the following conditions:

- $A$ is compact (log-hyponormal, hyponormal) (in particular, if the Hilbert space is finite dimensional), and satisfies (1.2) with $S = \{n\}$, where $n \in \mathbb{N}$ is fixed (see [24, page 198], [18, Theorem 5.3] and [25]),
- $A$ is closed densely defined operators and satisfies (1.2) with $S = \{2, 3\}$ (see [11, Theorem 3.6],
- $A$ is bounded and satisfies (1.2) with $S = \{p, m, m+p, n, n+p\}$ for some $p, m, n \in \mathbb{N}$, $m < n$ (see [18, Theorem 3.11]),

then $A$ is quasinormal.

On the other hand for every integer $n \geq 2$, there exists an operator $A$ such that

$$(A^*A)^n = A^*A^n \quad \text{and} \quad (A^*A)^k \neq A^kA^k \quad \text{for all} \quad k \in \{2, 3, \ldots\} \setminus \{n\}.$$ 

Examples of such operators are known in the class weighted shifts on directed trees and composition operators on $L^2$-space (see [11, Example 5.5] and [17, Theorem 4.3]).

2. Preliminaries

All Hilbert spaces considered in this paper are assumed to be complex. Let $A$ be a linear operator in a complex Hilbert space $\mathcal{H}$. Denote by $A^*$ the adjoint of $A$. We write $B(\mathcal{H})$ and $B^+_+(\mathcal{H})$, for the $C^*$-algebra of all bounded operators and the cone of all positive operators in $\mathcal{H}$, respectively. We say that $A \in B(\mathcal{H})$ is

- **projection** if $A = A^*$ and $A = A^2$,
- **normal** if $A^*A = AA^*$,
**quasinormal** if \( A^* A = (A^* A) A \),

**subnormal** if it is (unitarily equivalent to) the restriction of a normal operator to its invariant subspace.

Let \( \mathcal{B}(\mathbb{C}) \) stand for the \( \sigma \)-algebra of all Borel subsets of the complex plane \( \mathbb{C} \). We say that a mapping \( F : \mathcal{B}(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \) is a **semispectral measure** if \( \langle F(\cdot)f, f \rangle \) is a positive Borel measure for every \( f \in \mathcal{H} \), and \( F(\mathbb{C}) = I_{\mathcal{H}} \). Since all the Borel measures \( \langle F(\cdot)f, f \rangle, f \in \mathcal{H} \), being finite are automatically regular (cf. \[21\], Theorem 2.18), the closed support \( \text{supp} F \) of \( F \) always exists. A semispectral measure \( E : \mathcal{B}(\mathbb{C}) \to \mathcal{B}(\mathcal{H}) \) is said to be a **spectral measure** if \( E(\Delta) \) is an orthogonal projection for every \( \Delta \in \mathcal{B}(\mathbb{C}) \).

The following fact follows from the spectral theorem \[22\], Theorem 12.12] and plays an important role in our further investigations.

**Theorem 2.1.** If \( n \in \mathbb{N} \), then the commutants of a positive operator and its \( n \)-th root coincide.

Let \( I \subseteq \mathbb{R} \) be an interval (which may be open, half-open, or closed; finite or infinite) and \( f : I \to \mathbb{R} \) be a bounded borel function. A function \( f \) is said to be **operator monotone** if \( f(A) \leq f(B) \) for any two selfadjoint operators \( A, B \in \mathcal{B} \) such that \( A \leq B \) and \( \sigma(A), \sigma(B) \subseteq I \). In 1934 K. Löwner [15] proved that a function defined on an open interval is operator monotone if and only if it allows an analytic continuation into the complex upper half-plane that is an analytic continuation to a Pick function. The class of operator monotone functions is an important class of real-valued functions and it has various applications in other branches of mathematics. The operator monotone functions have very important properties, namely, they admit integral representations with respect to suitable Borel measures. In particular, a continuous function \( f : [0, \infty) \to \mathbb{R} \) is operator monotone if and only if there is a finite Borel measure \( \mu \) on \([0, \infty)\) such that \( \int_0^\infty \frac{1}{1+\lambda^2}d\mu(\lambda) < \infty \) and

\[
(2.1) \quad f(t) = \alpha + \beta t + \int_0^\infty \left( \frac{1}{\lambda - t} - \frac{\lambda}{1 + \lambda^2} \right)d\mu(\lambda),
\]

where \( \alpha \in \mathbb{R} \) and \( \beta \geq 0 \). Below, we give an example of a function which is operator monotone.

**Example 2.2.** The function \( f : [0, \infty) \ni x \to x^p \in \mathbb{R} \) for \( p \in (0, 1) \) is operator monotone and has an integral representation

\[
x^p = \frac{\sin p\pi}{\pi} \int_0^\infty \frac{x\lambda^{p-1}}{x+\lambda}d\mu(\lambda), \quad x \in [0, \infty).
\]

The fact that function from Example 2.2 is operator monotone is known as the Löwner-Heinz inequality.

**Theorem 2.3 (Löwner-Heinz inequality [10, 15]).** If \( A, B \in \mathcal{B}_+(\mathcal{H}) \) are such that \( B \leq A \) and \( p \in [0, 1] \), then \( B^p \leq A^p \).

The other inequality related to operator monotone functions needed in this paper is the Hansen inequality, which was established in [9] by F. Hansen. In [24, Lemma 2.2] M. Uchiyama gave a necessary and sufficient condition for the equality in the Hansen inequality. The key ingredient of its proof is the integral representation of operator monotone functions given in (2.1).
Theorem 2.4 (Hansen inequality [9, 24]). Let $\mathcal{H}$ be a separable Hilbert space, $A \in B_+(\mathcal{H})$, $P$ be a non-trivial projection and $f : [0, \infty) \to \mathbb{R}$ be an operator monotone function with $f(0) \geq 0$. Then we have

$$Pf(A)P \preceq f(PAP).$$

Moreover the equality holds, only in the case of $PA = AP$ and $f(0) = 0$, if $f$ is not an affine function.

3. Proofs of the Main Theorem

In this section, we will give proof of Theorem 1.2. In this proof we use the properties of operator monotone functions and condition of equality in Hansen's inequality.

First proof of Theorem 1.2. Let $N \in B(\mathcal{K})$ be a normal extension of $A$ and $P$ be the projection from $\mathcal{K}$ onto $\mathcal{H}$:

$$P = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Note that

$$(3.1) \quad P(N^*N)^kP = PN^*kN^kP = \begin{bmatrix} A^kA^k & 0 \\ 0 & 0 \end{bmatrix}$$

for every $k \in \mathbb{N}$. Since $A^n$ is quasinormal then by assertion (ii) of Theorem 1.3

$$(3.2) \quad (A^n)^*(A^n)^k = [(A^n)^*(A^n)]^k, \quad k \in \mathbb{N}.$$ 

Combining (3.1) with (3.2), we have

$$(3.3) \quad P(N^*N)^n = \begin{bmatrix} A^nA^n & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (A^{2n}A^{2n})^2 & 0 \\ 0 & 0 \end{bmatrix} = (P(N^*N)^{2n})^\frac{1}{2}.$$ 

Let $f : (0, \infty) \to \mathbb{R}$ be a function given by $f(x) = \sqrt{x}$. It follows from Example 2.2 and Theorem 2.3 that $f$ is operator monotone function. Therefore (3.3) implies that

$$Pf((N^*N)^{2n})P = f(P(N^*N)^{2n}P).$$

We conclude from Theorem 2.4 that $(N^*N)^{2n}$ commutes with $P$. By Theorem 2.1 $N^*N$ commutes with $P$. This in turn implies that

$$\begin{bmatrix} A^kA^k & 0 \\ 0 & 0 \end{bmatrix} = P(N^*N)^kP = (P(N^*N)P)^k = \begin{bmatrix} (A^*A)^k & 0 \\ 0 & 0 \end{bmatrix}$$

for all $k \in \mathbb{N}$. Hence $A^kA^k = (A^*A)^k$. This and Theorem 1.3 implies that $A$ is quasinormal. 

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Wydział Matematyki i Informatyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków
E-mail address: pawel.pietrzycki@im.uj.edu.pl