LANTERN SUBSTITUTION AND NEW 
SYMPLECTIC 4-MANIFOLDS WITH $b_2^+ = 3$

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Abstract. Motivated by the construction of H. Endo and Y. Gurtas, chang-
ing a positive relator in Dehn twist generators of the mapping class group 
by using lantern substitutions, we show that 4-manifold $K3\#2\mathbb{CP}^2$ equipped 
with the genus two Lefschetz fibration can be rationally blown down along six 
disjoint copies of the configuration $C_2$. We compute the Seiberg-Witten invari-
ants of the resulting symplectic 4-manifold, and show that it is symplectically 
minimal. Using our example, we also construct an infinite family of pair-
wise non-diffeomorphic irreducible symplectic and non-symplectic 4-manifolds 
homeomorphic to $M = 3\mathbb{CP}^2 \# (19 - k)\overline{\mathbb{CP}^2}$ for $1 \leq k \leq 4$.

1. Introduction

There is a beautiful interplay between the algebra and the topology when one 
studies a Lefschetz fibration structure on a smooth 4-manifold. To every Lefschetz 
fibration over $\mathbb{S}^2$, one can associate a factorization of the identity element as a 
product of positive Dehn twists in the mapping class group of the fiber, and con-
versely, for such a factorization in the mapping class group, there is a corresponding 
Lefschetz fibration over $\mathbb{S}^2$ [19]. By the remarkable work of Donaldson [11, 12], ev-
ery closed symplectic 4-manifold admits a structure of Lefschetz pencil, which can 
be blown up at its base points to yield a Lefschetz fibration. Conversely, Gompf 
[19] showed that if $g \geq 2$, then the total space of a genus $g$ Lefschetz fibration 
admits a symplectic structure. As one changes the identity word by conjugations 
and relations of the mapping class group, the corresponding Lefschetz fibration 
also changes topologically. There are many efforts in trying to understand what 
all the relations in the mapping class groups mean topologically. One of the well 
understood relations is the lantern relation, which corresponds to the symplectic 
operation of rational blowdown [15, 19].

In this article, we start with the genus two Lefschetz fibration on $K3\#2\mathbb{CP}^2$ 
over $\mathbb{S}^2$ with global monodromy given by the relation $\varrho = (t_{c_5}t_{c_4}t_{c_3}t_{c_2}t_{c_1})^6 = 1$ 
in the mapping class group $M_2$ of a closed genus two surface, where each $c_i$ is a 
simple closed curve as in Figure 5 and $t_{c_i}$ is a right handed Dehn twist along the 
curve $c_i$, $i = 1, \cdots, 5$. We factorize the monodromy of the given Lefschetz fibra-
tion by a series of conjugations and braid relations to get a word upon which we 
can perform six lantern relation substitutions. Applying these lantern substitutions 
change the total space of our Lefschetz fibration topologically as a six rational blow-
downs on $K3\#2\mathbb{CP}^2$. Furthermore, using the Seiberg-Witten invariants, we show

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rational blowdown.
that the resulting symplectic 4-manifold is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2\#15\overline{\mathbb{CP}^2}$ and symplectically minimal. We would like to point out that the goal of this paper is not to construct exotic smooth structures on very small 4-manifolds with $b_2^+ = 3$, but rather to use the lantern relation substitutions to study smooth structures on various 4-manifolds. The study of exotic 4-manifolds with small Euler characteristics has been carried out by the first author and B. D. Park in [1, 2, 3].

In the following three sections, we present some background material and recall some results which will be needed in this paper. Section 2 discusses some well-known relations in the mapping class group, which will be used in our computations and study. In Section 3, we give a brief background information on Lefschetz fibrations, provide some examples that will be used to illustrate discussions in the paper, and prove various results that will be needed in the sequel. In Sections 4 and 5, we recall the rational blowdown technique of Fintushel and Stern, prove the theorem proved by H. Endo and Y. Gurtas relating the lantern substitution to the rational blowdown operation, state results of R. Gompf on rational blowdown along smooth $-4$ sphere, and discuss the knot surgery operation of Fintushel-Stern, respectively. Finally, in section 6, we prove our main theorems.

2. Mapping Class Groups

Let $\Sigma_g$ denote a 2-dimensional, closed, oriented, and connected surface of genus $g > 0$ surface.

**Definition 1.** Let $\text{Diff}^+ (\Sigma_g)$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$, and $\text{Diff}_0^+ (\Sigma_g)$ be the subgroup of $\text{Diff}^+ (\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ that are isotopic to the identity. The mapping class group $M_g$ of $\Sigma_g$ is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g$, i.e.,

$$M_g = \text{Diff}^+ (\Sigma_g) / \text{Diff}_0^+ (\Sigma_g).$$

**Definition 2.** Let $\alpha$ be a simple closed curve on $\Sigma_g$. A right handed Dehn twist $t_\alpha$ about $\alpha$ is the isotopy class of a self-diffeomorphism of $\Sigma_g$ obtained by cutting the surface $\Sigma_g$ along $\alpha$ and gluing the ends back after rotating one of the ends $2\pi$ to the right.

Notice that the conjugate of a Dehn twist is again a Dehn twist. Indeed, if $f : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism, then it is easy to check that $f \circ t_\alpha \circ f^{-1} = t_{f(\alpha)}$. Next, we briefly mention some relations that hold in the mapping class group $M_g$. This elementary fact and relations will be used quite often in our computation in Section 6.

2.1. Commutativity and Braid Relation. Let $\alpha$ and $\beta$ be two simple closed curves on $\Sigma_g$.

**Lemma 3.** If $\alpha$ and $\beta$ are disjoint, then we have the following commutativity relation: $t_\alpha t_\beta = t_\beta t_\alpha$. If $\alpha$ and $\beta$ transversely intersect at a single point, then the corresponding Dehn twists satisfy the following braid relation: $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.

For a proof see [22].
2.2. Lantern Relation. Let $\Sigma_{0,4}$ be a sphere with 4 boundary components.

**Lemma 4.** If $\delta_1, \delta_2, \delta_3, \delta_4$ are the boundary curves of $\Sigma_{0,4}$ and $\alpha, \beta, \gamma$ are the simple closed curves as shown in Figure 1, then we have

$$t_\gamma t_\beta t_\alpha = t_{\delta_1} t_{\delta_2} t_{\delta_3} t_{\delta_4},$$

where $t_{\delta_i}, 1 \leq i \leq 4,$ denote the Dehn twists about $\delta_i$.

For a proof see [22].

![Figure 1. Lantern Relation](image)

This relation was known to Dehn. Later on it was rediscovered by D. Johnson and named as lantern relation by him [22]. For more results on lantern relation see [23].

3. LEFSCHETZ FIBRATION

Let us first recall the definition of Lefschetz fibration.

**Definition 5.** Let $X$ be a compact, connected, oriented, smooth 4-manifold. A Lefschetz fibration on $X$ is a smooth map $f : X \to \Sigma_h$, where $\Sigma_h$ is a compact, oriented, smooth 2-manifold of genus $h$, such that $f$ is surjective and each critical point of $f$ has an orientation preserving chart on which $f : \mathbb{C}^2 \to \mathbb{C}$ is given by $f(z_1,z_2) = z_1^2 + z_2^2$.

It is a corollary of the Sard’s theorem that $f$ is a smooth fiber bundle away from finitely many critical points, say $p_1, \cdots, p_k$. The genus of the regular fiber of $f$ is defined to be the genus of the Lefschetz fibration. If a fiber passes through critical point set $p_1, \cdots, p_k$, then it is called a singular fiber which is an immersed surface with a single transverse self-intersection. A singular fiber can be described by its monodromy, an element of the mapping class group $M_g$, where $g$ is the genus of the Lefschetz fibration. This element is a right handed Dehn twist along a simple closed curve on $\Sigma$, called the vanishing cycle. If this curve is a nonseparating curve, then the singular fiber is called nonseparating, otherwise it is called separating. For a genus $g$ Lefschetz fibration over $S^2$, the product of right handed Dehn twists $t_{\alpha_i}$ along the vanishing cycles $\alpha_i$, for $i = 1, \cdots, k$, gives us the global monodromy of the Lefschetz fibration, the relation $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_k} = 1$ in $M_g$. Conversely, such
a relation defines a genus $g$ Lefschetz fibration over $S^2$ with the vanishing cycles $\alpha_1, \ldots, \alpha_k$.

Let $c_1$, $c_2$, $c_3$, $c_4$, and $c_5$ be the simple closed curves as in Figure 5. For convenience we shall denote the right handed Dehn twists $t_{c_i}$ along the curve $c_i$ by $c_i$. It is well-known that the following relations hold in the mapping class group $M_2$:

\begin{align*}
(c_1)(c_2c_3c_4c_5c_2c_1)^2 &= 1, \\
(c_1)(c_2c_3c_4c_5)^6 &= 1, \\
(c_1)(c_2c_3c_4)^{10} &= 1.
\end{align*}

For each relation above, we have a corresponding genus two Lefschetz fibrations over $S^2$ with total spaces $\mathbb{CP}^2 \# 13\mathbb{CP}^2$, $K3\# 2\mathbb{CP}^2$, and the Horikawa surface $H$, respectively. In this paper, we will consider genus two Lefschetz fibration on $K3\# 2\mathbb{CP}^2$ with global monodromy $(c_1c_2c_3c_4c_5)^6 = 1$.

The following result is well-known. For the convenience of the reader, we sketch the proof.

**Lemma 6.** The genus two Lefschetz fibration on $K3\# 2\mathbb{CP}^2$ over $S^2$ with the monodromy factorization $(c_1c_2c_3c_4c_5)^6 = 1$ can be obtained as the double branched covering of $\mathbb{CP}^2 \# \mathbb{CP}^2$ branched along a smooth algebraic curve $B$ in the linear system $|6L|$, where $L$ is the proper transform of line $L$ in $\mathbb{CP}^2$ avoiding the blown-up point.

**Proof.** We closely follow the proof of Lemma 3.1 in [4], see also the discussion in [5]. Let $D$ denote a degree $d$ algebraic curve in $\mathbb{CP}^2$. Fix a generic projection $\pi : \mathbb{CP}^2 \setminus pt \to \mathbb{CP}^1$ whose pole does not belong to $D$. According to the work of B. Moishezon and M. Teicher, the braid monodromy describing a degree $d$ branch curve $D$ in $\mathbb{CP}^2$ is given by a braid factorization. In fact, it is shown that the braid monodromy around the point at infinity in $\mathbb{CP}^1$, which is given by the central element $\Delta^2$ in $B_d$, can be written as the product of the monodromies about the critical points of the projection map $\pi$. More precisely, the following factorization $\Delta^2 = (\sigma_1 \cdots \sigma_{d-1})^d$ holds in the braid group $B_d$, where $\sigma_i$ is a positive half-twist exchanging two points, and fixing the remaining $d - 2$ points.

We first degenerate the smooth algebraic curve $B$ in $\mathbb{CP}^2 \# \mathbb{CP}^2$ into a union of 6 lines in a general position (see Figure 2). The braid group factorization corresponding to the configuration $B$ is given by $\Delta^2 = (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5)^6$. By lifting this braid factorization to the mapping class group of the genus two surface, we obtain that the monodromy factorization $(c_1c_2c_3c_4c_5)^6 = 1$ for the corresponding double branched covering.

Notice that a regular fiber of the given fibration is a two fold cover of a sphere with homology class $f = h - e_1$ branched over 6 points, where $h$ denotes the hyperplane class in $\mathbb{CP}^2$. Thus a regular fiber is a surface of genus two. The exceptional sphere $e_1$ in $\mathbb{CP}^2 \# \mathbb{CP}^2$, which intersects $f = h - e_1$ positively at one point, gives rise to two disjoint $-1$ sphere sections to the given genus two Lefschetz fibration on $K3\# 2\mathbb{CP}^2$.

Below we prove key Proposition, which plays an important role in the proof of our main Theorems. The following proposition also gives an alternative and
Proposition 7. The genus two Lefschetz pencil on $K3$ with two base points given above can be constructed by symplectic fiber summing two copies of $E(1) = \mathbb{CP}^2 \# 9\mathbb{CP}^2$ along a regular torus fiber.

Proof. Since the fiber of the elliptic fibration on $E(1) = \mathbb{CP}^2 \# 9\mathbb{CP}^2$ is a blow up of a generic cubic curve in $\mathbb{CP}^2$, its homology class is equal to $f = 3h - e_1 - e_2 - \cdots - e_9$, where $h$ denotes the hyperplane class in $\mathbb{CP}^2$ and $e_i$ is the homology class of the exceptional sphere of the $i$th blow-up. We first consider the pencil of lines in $\mathbb{CP}^2$ all passing through the fixed point $p_1$ away from the cubic curve $C$ in $\mathbb{CP}^2$. Observe that each generic line $L$, which is a sphere of self-intersection 1, in this pencil intersects the cubic curve $C$ at three distinct points by Bezout’s Theorem.

Furthermore, by applying the Riemann-Hurwitz formula to the restriction of the map $\pi : \mathbb{CP}^2 \setminus p_1 \to \mathbb{CP}^1$ to $C$, we compute the degree of the ramification divisor $R$: $\text{deg}(R) = 2g(C) - 2 - 3(2g(\mathbb{CP}^1) - 2) = 6$. Consequently, for a generic smooth cubic $C$, there are exactly six lines in the pencil above that are tangent to $C$.

Next, we choose the regular torus fiber $F$ along which the fiber sum of two copies of $E(1) = \mathbb{CP}^2 \# 9\mathbb{CP}^2$ will be performed. Since the generic elliptic fiber of $E(1) = \mathbb{CP}^2 \# 9\mathbb{CP}^2$ intersects the lines of the corresponding pencil at three points (see Figure 4), the generic line of the pencil in each $E(1)$ intersects the boundary of $E(1) \setminus \nu(F)$ in three disjoint circles. We choose a gluing diffeomorphism $\psi = id_F \times (\text{complex conjugation})$, that identities these circles as in Figure 3 to obtain a pencil of genus two curves in $E(2) = K3$ surface. Since the pencils in each copy of $E(1)$ are holomorphic and the gluing map on the boundary 3-torus is identity on the elliptic fiber, the resulting genus two pencil is holomorphic as well. By holomorphically blowing up this genus two pencils at base points of the pencil $p_1$ and $p_2$ in $K3$ surface, we obtain the genus two holomorphic fibration on $K3 \# 2\mathbb{CP}^2$. 

Figure 2. Branch Locus for $K3 \# 2\mathbb{CP}^2$
This holomorphic fibration has six singular fibers resulting from the gluing of six tangent lines mentioned above in each copy of $E(1)$. These singular fibers are not a Lefschetz type, and each singularity is topologically a sphere. Furthermore, using the analysis of the singular fibers, we see that each singular fiber can be perturbed into five Lefschetz type singularities with non-separating vanishing cycles (see the braid monodromy discussion in [4], page 5). Finally, using the result of B. Siebert and G. Tian, Theorem A in [25], on holomorphicity of genus two Lefschetz fibration, we see that the fibration is isomorphic to the one given in Lemma 6.

Remark 8. This description of the genus two pencil allows us to see rim tori and Gompf nuclei in elliptic surface $K3$ (See Example 9 below and [18] for an explanation).

Figure 3. A Genus Two Surface via Fiber Sums of $E(1)$

Example 9. In this example, we study $K3$ surfaces $E(2) = E(1)\#^2\overline{E}(1)$ in some detail. As a consequence of our discussion, we will also derive some useful facts about the genus two Lefschetz pencil on $K3$ with two base points given above. Let us think of $K3$ surface as the fiber sum of two copies of $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ along a torus fiber as in Proposition 7. We choose the following basis for the intersection form of $E(1): \langle f \rangle = 3h - e_1 - \ldots - e_9, e_9, e_1 - e_2, e_2 - e_3, \ldots, e_7 - e_8, -h + e_6 + e_7 + e_8 \rangle$. Note that the last 8 classes can be represented by spheres of self-intersection $-2$ and generate the intersection matrix $-E_8$, where $E_8$ is the matrix corresponding to the Dynkin diagram of the exceptional Lie algebra $E_8$. The class $f$ is fiber of an elliptic fibration on $E(1) = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ and $e_9$ is a
Thus, the genus two fiber Σ is disjoint from the rim tori that descend to \( \{K\} \). The reader can notice from the intersection form of the sphere section of self-intersection \(-1\). When we perform the fiber sum along torus to obtain \( E(2) \), it is not hard to see the surfaces that generate the intersection form \( 2(-E_8) \oplus 3H \) for \( E(2) \), where \( H \) is a hyperbolic pair. The two copies of the Milnor fiber \( \Phi(1) \in E(1) \) are in \( E(2) \), providing 16 spheres of self-intersection \(-2\) (corresponding to the classes \( \{e_1 - e_2, e_2 - e_3, \ldots, e_7 - e_8, -h + e_6 + e_7 + e_8\} \) and \( \{e'_1 - e'_2, e'_2 - e'_3, \ldots, e'_7 - e'_8, -h' + e'_6 + e'_7 + e'_8\} \) mentioned above), realize two copies of \(-E_8\). One copy of hyperbolic pair \( H \) comes from an identification of the torus fibers \( f \) and \( f' \), and a sphere section \( \sigma \) of self-intersection \(-2\) obtained by sewing the sphere sections \( e_9 \) and \( e'_9 \), i.e. from the Gompf’s nucleus \( N(2) \) in \( E(2) \).

The remaining two copies of \( H \) come from 2 rim tori and their dual \(-2\) spheres (see discussion in [19], page 73)). These 22 classes (19 spheres and 3 tori) generate \( H_2 \) of \( E(2) \). Since \( c_1(E(n)) = (2 - n)f, E(2) \) has a trivial canonical class. As we can see from the Figure 4, the class of the genus two surface of square 2 of the genus two Lefschetz pencil on \( K3 \) is \( h + h' \), we simply add the homology classes of the surfaces. As a consequence, the class of the genus two fiber in \( K3\#2\mathbb{CP}^2 \) is given by \( h + h' - E_1 - E_2 \), where \( E_1 \) and \( E_2 \) are the homology classes of the exceptional spheres of the blow-ups at the points \( p_1 \) and \( p_2 \). We can also verify the symplectic surface \( \Sigma \), given by the class \( h + h' - E_1 - E_2 \), has genus two by applying the adjunction formula to \( (K3\#2\mathbb{CP}^2, \Sigma) \): \( g(\Sigma) = 1 + 1/2(K3\#2\mathbb{CP}^2 \cdot [\Sigma] + [\Sigma]^2) = 1 + ((E_1 + E_2) \cdot (h + h' - E_1 - E_2) + (h + h' - E_1 - E_2)^2)/2 = 1 + (2 + 0)/2 = 2. \) The reader can notice from the intersection form of \( K3 \) that both rim tori has no intersections with the genus surface in the pencil given by the homology class \( h + h' \).

Thus, the genus two fiber \( \Sigma \) is disjoint from the rim tori that descend to \( K3\#2\mathbb{CP}^2 \).

4. Rational Blowdown and Lantern Relations

The basic idea of the rational blowdown surgery is that if a smooth 4-manifold \( X \) contains a particular configuration \( C_p \) of transversally intersecting 2-spheres whose boundary is the lens space \( L(p^*, 1 - p) \), then one can replace \( C_p \) with rational ball \( B_p \) to construct a new manifold \( X_p \). If one knows the Seiberg-Witten invariants of the original manifold \( X \), then one can determine the same invariants of \( X_p \). Below
we briefly discuss the rational blow-down. We refer the reader to [15, 24] for full details.

Let \( p \geq 2 \) and \( C_p \) be the smooth 4-manifold obtained by plumbing disk bundles over the 2-sphere according to the following linear diagram

\[
\begin{array}{cccccc}
& -(p+2) & -2 & \cdots & -2 & \\
u_{p-1} & u_{p-2} & & & u_1 \\
\end{array}
\]

where each vertex \( u_i \) of the linear diagram represents a disk bundle over 2-sphere with the given Euler number.

According to Casson and Harer [6], the boundary of \( C_p \) is the lens space \( L(p^2,1-p) \) which also bounds a rational ball \( B_p \) with \( \pi_1(B_p) = \mathbb{Z}_p \) and \( \pi_1(\partial B_p) \to \pi_1(B_p) \) surjective. If \( C_p \) is embedded in a 4-manifold \( X \) then the rational blowdown manifold \( X_p \) is obtained by replacing \( C_p \) with \( B_p \), i.e., \( X_p = (X \setminus C_p) \cup B_p \). If \( X \) and \( X \setminus C_p \) are simply connected, then so is \( X_p \).

**Lemma 10.** \( b_2^+(X_p) = b_2^-(X) \), \( \sigma(X_p) = \sigma(X) + (p-1) \), \( c_1^2(X_p) = c_1^2(X) + (p-1) \), and \( \chi_h(X_p) = \chi_h(X) \).

**Proof.** Notice that the manifold \( C_p \) is negative definite, we have \( b_2^+(X_p) = b_2^+(X) \) and \( b_2^-(X_p) = b_2^-(X) - (p-1) \). Thus, \( \sigma(X_p) = \sigma(X) + (p-1) \). Using the formulas \( c_1^2 = 3\sigma + 2e \) and \( \chi_h = (\sigma + e)/4 \), we have \( c_1^2(X_p) = 3\sigma(X) + 2e(X_p) = 3(\sigma(X) + (p-1)) + 2(e(X) - (p-1)) = c_1^2(X) + (p-1) \) and \( \chi_h(X_p) = (\sigma(X) + (p-1) + e(X) - (p-1))/4 = \chi_h(X) \).

\( \square \)

**Theorem 11.** [15, 24]. Suppose \( X \) is a smooth 4-manifold with \( b_2^+(X) > 1 \) which contains a configuration \( C_p \). If \( L \) is a SW basic class of \( X \) satisfying \( L \cdot u_i = 0 \) for any \( i \) with \( 1 \leq i \leq p-2 \) and \( L \cdot u_{p-1} = \pm p \), then \( L \) induces a SW basic class \( L \) of \( X_p \) such that \( SW_{X_p}(L) = SW_X(L) \).

**Theorem 12.** [15, 24] If a simply connected smooth 4-manifold \( X \) contains a configuration \( C_p \), then the SW-invariants of \( X_p \) are completely determined by those of \( X \). That is, for any characteristic line bundle \( L \) on \( X_p \), with \( SW_{X_p}(L) \neq 0 \), there exists a characteristic line bundle \( L \) on \( X \) such that \( SW_X(L) = SW_{X_p}(L) \).

In this paper we only use the rational blowdown surgery along configuration \( C_2 \), i.e. the rational blowdowns along the \(-4\) spheres.

In Section 6, we shall use the following theorem of H. Endo and Y. Gurtas.

**Theorem 13.** Let \( g, g' \) be positive relators of \( M_g \) and \( M_{g'}, M_{g'}' \) the corresponding Lefschetz fibrations over \( S^2 \), respectively. If \( g' \) is obtained by applying a lantern substitution to \( g \), then the 4-manifold \( M_{g'} \) is a rational blowdown of \( M_g \) along a configuration \( C_2 \subset M_g \).

We will also need the following lemmas, which are due to R. Gompf, to analyze the symplectic 4-manifolds constructed in Section 6. For the proof we refer the reader to [18] and [8]. See also the work of J. Dorfmeister [7, 8] who gives a related criteria on symplectic minimality and how the symplectic Kodaira dimension changes under the rational blowdown along a symplectic \(-4\) sphere (see also related work in [9]).
Lemma 14. Let \((X, V_X)\) be a relatively minimal smooth pair with \(V_X\) an embedded \(-4\) sphere. If \(X\) contains a smoothly embedded exceptional sphere transversely intersecting the hypersurface \(V_X\) in a single positive point, then the manifold obtained under \(-4\) blow-down of \(V_X\) is diffeomorphic to the blow-down of \(X\) along this sphere.

Lemma 15. Let \((X, V_X)\) be a relatively minimal smooth pair with \(V_X\) an embedded \(-4\) sphere. If \(X\) contains two disjoint smoothly embedded exceptional spheres each transversely intersecting the hypersurface \(V_X\) in a single positive point, then the manifold obtained under \(-4\) blow-down of \(V_X\) is diffeomorphic to the blow-down of \(X\) along one of these spheres.

5. Knot Surgery

Let \(X\) be a 4-manifold with \(b_2^+(X) > 1\) and contain a homologically essential torus \(T\) of self-intersection 0. Let \(N(K)\) be a tubular neighborhood of \(K\) in \(S^3\), and let \(T \times D^2\) be a tubular neighborhood of \(T\) in \(X\). The knot surgery manifold \(X_K\) is defined by \(X_K = (X \setminus (T \times D^2)) \cup (S^1 \times (S^3 \setminus N(K)))\) where two pieces are glued in a way that the homology class of \([pt \times \partial D^2]\) is identified with \([pt \times \lambda]\) where \(\lambda\) is the class of the longitude of knot \(K\). Fintushel and Stern proved the theorem that shows Seiberg-Witten invariants of \(X_K\) can be completely determined by the Seiberg-Witten invariant of \(X\) and the Alexander polynomial of \(K\) [16]. Furthermore, if \(X\) and \(X \setminus T\) are simply connected, then so is \(X_K\).

Theorem 16. Suppose that \(\pi_1(X) = \pi_1(X \setminus T) = 1\) and \(T\) lies in a cusp neighborhood in \(X\). Then \(X_K\) is homeomorphic to \(X\) and Seiberg-Witten invariants of \(X_K\) is \(\text{SW}_{X_K} = \text{SW}_X \cdot \Delta_K(t^2)\), where \(t = t_T\) and \(\Delta_K\) is the symmetrized Alexander polynomial of \(K\). If the Alexander polynomial \(\Delta_K(t)\) of knot \(K\) is not monic then \(X_K\) admits no symplectic structure. Moreover, if \(X\) is symplectic and \(K\) is a fibered knot, then \(X_K\) admits a symplectic structure.

We refer the reader to [16] for the details.

6. Construction of exotic 4-manifolds via lantern substitution

In this section, we first construct a simply connected, minimal symplectic 4-manifold \(X\) homeomorphic but not diffeomorphic to \(3\mathbb{CP}^2 \# 15\mathbb{CP}^2\) starting from \(K3\# 2\mathbb{CP}^2\) and applying the sequence of six rational blowdowns via lantern substitutions. Next, by performing knot surgery on a homologically essential torus of self-intersection 0, we obtain an infinite family of simply connected, symplectic and non-symplectic 4-manifolds all homeomorphic but not diffeomorphic to \(X\). Using Seiberg-Witten invariants, we distinguish their smooth structures.

Theorem 17. There exists an irreducible symplectic 4-manifold \(X\) homeomorphic but not diffeomorphic to \(3\mathbb{CP}^2 \# 15\mathbb{CP}^2\) that can be obtained using the genus two Lefschetz fibration on \(K3\# 2\mathbb{CP}^2\) over \(S^2\) with global monodromy given by the relation \(q = (c_5c_4c_3c_2c_1)^6 = 1\) in the mapping class group \(M_2\) by applying six lantern substitutions.

In order to prove this theorem, we first prove the following two lemmas.

Lemma 18. The word \(c_5c_4c_3c_2c_1c_5c_4c_3c_2c_1\) in the mapping class group \(M_2\) can be conjugated to contain the lantern relation in three different ways.
Figure 5. The Curves $c_1$, $c_2$, $c_3$, $c_4$, and $c_5$

Figure 6. Special Curves $k$, $h$

Figure 7. Special Curves $\bar{k}$, $\bar{h}$

Proof. Below we denote the lantern relation substitution, the braid relation substitution, the conjugation, and the arrangement using the commutativity by $\xrightarrow{L}$, $\xrightarrow{B}$, $\xrightarrow{C}$, $\xrightarrow{\sim}$ respectively. For the convenience of the reader, we have highlighted the changes that occur in each step.

Let us consider the following cases
• Making the lantern substitution \( c_5 c_2 c_4 \) for \( c_3 \delta x \).
\[
\begin{align*}
& c_5 c_4 c_2 c_1 c_5 c_3 c_2 c_1 \\
\sim & c_5 (c_4) \cdot c_3 c_2 c_5 c_1 c_5 c_4 c_3 \cdot c_{_{1^{-1}}}(c_2) \\
\sim & c_5 (c_4) \cdot c_3 c_2 \cdot c_5^2 c_1^2 \cdot c_4 c_3 \cdot c_{_{1^{-1}}}(c_2) \\
\xrightarrow{L} & c_5 (c_4) \cdot c_3 c_2 \cdot c_3 \delta x \cdot c_4 c_3 \cdot c_{_{1^{-1}}}(c_2)
\end{align*}
\]

• Making the lantern substitution \( c_1 c_3 c_1 c_3 \) for \( \bar{k}h c_5 \).
\[
\begin{align*}
& c_5 c_4 c_2 c_1 c_5 c_4 c_3 c_2 c_1 \\
\sim & c_5 c_4 c_3 c_2 c_4 c_1 c_3 c_2 c_1 \\
\sim & c_5 c_4 c_5 \cdot c_3 (c_2 c_4) \cdot c_3^2 c_3^3 \cdot c_{_{1^{-1}}}(c_2) \\
\xrightarrow{L} & c_5 c_4 c_5 \cdot c_3 (c_2 c_4) \cdot \bar{k}h c_5 \cdot c_{_{1^{-1}}}(c_2)
\end{align*}
\]

• Making the lantern substitution \( c_3 c_2 c_4 \) for \( c_1 k h \).
\[
\begin{align*}
& c_5 c_4 c_2 c_1 c_5 c_4 c_3 c_2 c_1 \\
\sim & c_5 (c_4) \cdot c_5 c_3 c_2 c_3 (c_4) c_3 \cdot c_1 c_2 c_1 \\
\sim & c_5 (c_4) \cdot c_3 (c_2) \cdot c_3^2 c_3^3 \cdot c_{_{1^{-1}}}(c_4) \cdot c_1 c_2 c_1 \\
\xrightarrow{L} & c_5 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_{_{3^{-1}}}(c_4) \cdot c_1 c_2 c_1
\end{align*}
\]

Applying the above lemma several times, we next show how the word \( g = (c_5 c_4 c_3 c_2 c_1)^6 = 1 \) can be conjugated to contain six lantern relation.

**Lemma 19.** The global monodromy of genus two Lefschetz fibration on \( K3#2\mathbb{CP}^2 \) over \( S^2 \) given by the relation \( g = (c_5 c_4 c_3 c_2 c_1)^6 = 1 \) can be conjugated and braided to contain six lantern relations.

**Proof.** We start with the identity word: \( (c_5 c_4 c_3 c_2 c_1)^6 = 1 \)

By applying three identities of Lemma 18, we can convert the word \( (c_5 c_4 c_3 c_2 c_1)^6 = ((c_5 c_4 c_3 c_2 c_1)^2)^3 \) into the following word:

\[
\begin{align*}
& \sim c_5 (c_4) \cdot c_3 c_2 \cdot c_3 \delta x \cdot c_4 c_3 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 c_4 c_5 \cdot c_3 (c_2 c_4) \cdot \bar{k}h c_5 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_{_{3^{-1}}}(c_4) \cdot c_1 c_2 c_1 \\
\sim & c_5 (c_4) \cdot c_3 c_2 \cdot c_3 (\delta x) \cdot c_3 (c_4) \cdot c_3^2 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 (c_2 c_4) \cdot \bar{k}h c_5 \cdot c_{_{1^{-1}}}(c_2) \cdot c_3 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_{_{3^{-1}}}(c_4) \cdot c_1 c_2 c_1 \\
\xrightarrow{B} & c_5 (c_4) \cdot c_3 c_2 \cdot c_3 (\delta x) \cdot c_3 (c_4) \cdot c_3^2 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 (c_2 c_4) \cdot \bar{k}h c_5 \cdot c_{_{1^{-1}}}(c_2) \cdot c_3 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_{_{3^{-1}}}(c_4) \cdot c_1 c_2 c_1 \\
\sim & c_5 (c_4) \cdot c_3 c_2 \cdot c_3 (\delta x) \cdot c_3 (c_4) \cdot c_3^2 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 (c_2 c_4) \cdot \bar{k}h c_5 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_{_{3^{-1}}}(c_4) \cdot c_2 c_1 c_2 \\
\xrightarrow{L} & c_5 (c_4) \cdot c_3 c_2 \cdot c_3 (\delta x) \cdot c_3 (c_4) \cdot c_3 k h \cdot c_{_{3^{-1}}}(c_2) \cdot c_5 (c_2 c_4) \cdot \bar{k}h c_5 \cdot c_{_{1^{-1}}}(c_2) \cdot c_5 (c_4) \cdot c_3 (c_2) \cdot c_1 k h \cdot c_{_{3^{-1}}}(c_4) \cdot c_2 c_1 c_2
\end{align*}
\]
\[ \sim c_5(c_4) \cdot c_3 c_2 c_1 \cdot c_6(\delta x) \cdot c_5(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_3(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_1^{-1}(c_4) \cdot c_2 c_1 c_2 \]

\[ c_1 c_2 \cdot c_5(c_4) \cdot c_3 c_2 c_1 \cdot c_3(\delta x) \cdot c_4(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_3(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \]

\[ \sim c_5(c_4) \cdot c_3 c_2 c_1 \cdot c_3(\delta x) \cdot c_3(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_1^{-1}(c_4) \cdot c_2 \]

\[ \sim c_5(c_4) \cdot c_3 c_2 c_1 \cdot c_3(\delta x) \cdot c_3(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_1^{-1}(c_4) \cdot c_2 \]

\[ B \sim c_5(c_4) \cdot c_1 c_3 c_2 c_1 \cdot c_3(\delta x) \cdot c_3(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_1^{-1}(c_4) \cdot c_2 \]

\[ c_1 c_2 \cdot \tilde{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_3(\delta x) \cdot c_3(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_1^{-1}(c_4) \cdot c_2 \]

\[ c_1 c_2 \cdot \tilde{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_3(\delta x) \cdot c_3(c_4) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5 \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_1^{-1}(c_4) \cdot c_2 \]

\[ \sim c_5(c_4) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_5(k h) \cdot c_5\overline{c}_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_3^{-1}(c_4) \cdot c_2 \]

\[ \sim c_4 \cdot c_1^{-1}(c_5(c_4(c_2(c_4)) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_6\overline{c}_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_3^{-1}(c_4) \cdot c_2 \]

\[ \sim c_5(c_4)(c_2(c_4) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_5(k h) \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_3^{-1}(c_4) \cdot c_2 c_4 \]

\[ \sim c_5(c_4)(c_2(c_4) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_5(k h) \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_3^{-1}(c_4) \cdot c_2 c_4 \]

\[ B \sim c_2(c_4) \cdot c_5(c_4(c_2(c_4)) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_5(k h) \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_3^{-1}(c_4) \cdot c_4 \]

\[ B \sim c_2(c_4) \cdot c_5(c_4(c_2(c_4)) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_5(k h) \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_3^{-1}(c_4) \cdot c_4 \]

\[ \sim c_2(c_4) \cdot c_5(c_4(c_2(c_4)) \cdot \tilde{k} h \cdot c_3(c_3(c_4(c_2(c_4)) \cdot k h \cdot c_1^{-1}(c_2) \cdot c_5(c_2 c_4) \cdot \bar{k} h c_5) \cdot c_5(k h) \cdot c_1^{-1}(c_2) \cdot c_5(c_4) \cdot c_5(c_2) \cdot c_1 k h \cdot c_4 c_3 \]
The Loops $c_1$, $c_2$, $c_3$, $c_4$, and $c_5$

Let us denote a regular fiber of the genus two Lefschetz fibration on $X$ as $\Sigma_2$ and the generators of fundamental group of $\Sigma_2$ as $c_1$, $c_2$, $c_3$, $c_4$, and $c_5$. We continue to
use the notation of the Figure [5] by choosing a base point $p$ as in $\Sigma_2$ as in Figure [8]. From the long exact homotopy sequence (see, for example [10], page 290), we deduce that

$$\pi_1(\Sigma_2) \to \pi_1(X) \to \pi_1(S^2) = 1$$

Moreover, $\pi_1(X)$ is finitely presented group and generated by the images of the standard generators, which we again denote by $c_1, c_2, c_3, c_4,$ and $c_5,$ under the surjection map. Furthermore, the loops $\beta_1, \beta_2, \beta_3, \ldots, \beta_{23}, \beta_{24}$ on the regular fiber are all nullhomotopic in $\pi_1(X)$, where $\beta_1, \beta_2, \beta_3, \ldots, \beta_{23}, \beta_{24}$ are the curves corresponding to $24$ Dehn twists $D_i$ of the genus two Lefschetz fibration on $X$ given below

$$D_1 = c_4^{-1}c_5(c_4), \quad D_2 = c_2c_4^{-1}(\bar{k}), \quad D_3 = c_2c_4^{-1}(\bar{h}), \quad D_4 = c_2c_4^{-1}c_1^{-1}c_3^{-1}(c_2), \quad D_5 = c_2c_4^{-1}c_3(\delta), \quad D_6 = c_2c_4^{-1}c_3(x), \quad D_7 = c_2c_4^{-1}c_5(c_4), \quad D_8 = c_2c_4^{-1}(k), \quad D_9 = c_2c_4^{-1}(h), \quad D_{10} = c_2c_4^{-1}(c_2),$$

$$D_{11} = c_2c_5(c_2), \quad D_{12} = c_2c_5(c_4), \quad D_{13} = c_2c_5(\bar{k}), \quad D_{14} = c_2c_5(\bar{h}), \quad D_{15} = c_2c_4^{-1}(c_2),$$

$$D_{16} = c_4^2(c_4), \quad D_{17} = c_1, \quad D_{18} = k, \quad D_{19} = h, \quad D_{20} = c_4^{-1}(c_2),$$

$$D_{21} = c_1, \quad D_{22} = k, \quad D_{23} = h, \quad D_{24} = c_4^{-1}(c_4).$$

We observe that $\beta_{21} = 1$ in $\pi_1(X)$, implies that $c_1$ is trivial element in $\pi_1(X)$. Using $\beta_{10} = \beta_{20} = 1$, we obtain $c_2$ and $c_3$ are trivial in $\pi_1(X)$. Furthermore, the relations $\beta_5 = \beta_{22} = \beta_{23} = \beta_{24} = 1$ imply that the elements $c_4$ and $c_5$ are null-homotopic and thus $X$ is simply connected.

Using the blow up formula for the Seiberg-Witten function [16], we have

$$SW_{K3\#2\mathbb{CP}^2} = SW_{K3} \cdot \prod_{j=1}^{2} (e^{E_1} + e^{-E_1}) = (e^{E_1} + e^{-E_1})(e^{E_2} + e^{-E_2}),$$

where $E_i$ is an exceptional class coming from the $i^{th}$ blow up. Consequently, it follows from this formula, the set of basic classes of $K3\#2\mathbb{CP}^2$ are given by $\pm E_1 \pm E_2$, and the Seiberg-Witten invariants on these classes are $\pm 1$. Moreover, after performing two rational blow-downs along a copy of the configuration $C_2$, the resulting manifold is diffeomorphic to $K3$ by Lemma [14]. Thus, the only basic class is the zero class, which descends from the top classes $\pm (E_1 + E_2)$ in $K3\#2\mathbb{CP}^2$. Next, using the Corollary 8.6 in [15], we observe that $X$ has Seiberg-Witten simple type. Furthermore, by applying Theorem [11] and Theorem [12] we completely determine the Seiberg-Witten invariants of $X$ using the basic classes and invariants of $K3$: Up to sign the symplectic manifold $X$ has only one basic class which descends from the canonical class of $K3$. By Theorem [12] (or by Taubes theorem [24]), the value of the Seiberg-Witten function on these classes, $\pm K_X$, are $\pm 1$.

Alternatively, we can determine the Seiberg-Witten invariants of $X$ directly by computing the algebraic intersection number of the classes $\pm E_1 \pm E_2$, with the classes of $-4$ spheres of six $C_2$ configurations. Observe that these $-4$ spheres are the components of the singular fibers of $K3\#2\mathbb{CP}^2$. Furthermore, by considering three regions on the genus two surface, where the rational blowdowns are performed (see Figures [6] and [7], and the location of the two points where we did blow up the genus two pencil (see Figures [3] and [4]), we compute the intersection numbers as follows: Let $S$ denote the homology class of $-4$ sphere of $C_2$. We have $S \cdot E_1 = S \cdot E_2 = 1$. Consequently, $S \cdot (E_1 + E_2) = \pm 2$ and $S \cdot (E_1 - E_2) = 0$. Since among the four classes $\pm E_1 \pm E_2$ only $E_1 + E_2$ and $-(E_1 + E_2)$ have intersection $\pm 2$ with
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−4 spheres of $C_2$, it follows from Theorem 11 that these are only two classes that descend to $X$.

Next, we apply the connected sum theorem for the Seiberg-Witten invariant and show that $SW$ function is trivial for $3\mathbb{C}P^2 \# 15\mathbb{C}P^2$. Since the Seiberg-Witten invariants are diffeomorphism invariants, we conclude that $X$ is not diffeomorphic to $3\mathbb{C}P^2 \# 15\mathbb{C}P^2$.

The minimality of $X$ follows from the fact that $X$ has no two basic classes $K$ and $K'$ such that $(K - K')^2 = -4$. Notice that $(K_X - (-K_X))^2 = 4(K_X^2) = 16$ in our case.

□

Theorem 20. There exist an infinite family of irreducible symplectic and an infinite family of irreducible non-symplectic pairwise non-diffeomorphic 4-manifolds all homeomorphic to $X$.

Proof. We will use both the branched cover and the fiber sum descriptions of $K3#2\mathbb{C}P^2$ given as in Lemma 10 and Proposition 7 to show that $X$ contains at least two disjoint tori that are disjoint from the singular fibers of genus two Lefschetz fibration on $K3#2\mathbb{C}P^2$ over $S^2$. An alternative proof, using the homology classes, is given in Example 9. These tori descend from Gompf nucleus of $E(2) = K3$ (See Example 9), and survive in $X$ after the rational blowdowns along $C_2$. Let us explain this more precisely using a geometric argument. First, note that according to the proof of Lemma 8 any genus two fiber, which arises from a two-fold cover of the sphere $h - e_1$, meets the torus $a \times b$ descending from an elliptic fiber of $K3$ at two points. Let $f_1, \ldots, f_6$ denote the complicated singular fibers of the genus two fibration on $K3#2\mathbb{C}P^2$. We perturb these singular fibers into Lefschetz type upon which we perform the rational blowdowns along $C_2$. Consider the tubular neighborhoods of these singular fibers in the manifold $K3#2\mathbb{C}P^2$. One should think of each neighborhood where we perturb one complicated singular fiber into five Lefschetz type singular fibers. The normal disks of these neighborhoods on the torus $a \times b$ are denoted in Figure 9 as $\delta_1, \ldots, \delta_6$. The dots in the disks indicate there are 5 singulars fibers over each disk $\delta_i$. We choose an open set $U$ on the torus $a \times b$ which contains the disks $\delta_1, \ldots, \delta_6$, and away from the loops $a$ and $b$ (See Figure 9). Thus, we can assume that the rational blowdown surgeries have no effect on the outside the tubular neighborhood of $U$. Next, we will choose a rim circle $\mu$ away from the tubular neighborhood of $U$. The rim tori that are not affected by six rational blowdowns are $\mu \times a$ and $\mu \times b$ (Figure 9). Note that each of these tori has a dual sphere of self-intersection $-2$, arising from the dual circles $a$ and $b$ (See Example 9). These tori are Lagrangian, but we can perturb the symplectic form so that one of them, say $T = \mu \times a$ becomes symplectic. Moreover, $\pi_1(X \setminus T) = 1$, which follows from the Van Kampen's Theorem using the facts that $\pi_1(X) = 1$ and the rim torus has a dual sphere (see Proposition 1.2 in [20], or Gompf [17], page 564). Hence, we have a symplectic torus $T$ in $X$ of self-intersection 0 such that $\pi_1(X \setminus T) = 1$. By performing a knot surgery on $T$, inside $X$, we acquire an irreducible 4-manifold $X_K$ that is homeomorphic to $X$. By varying our choice of the knot $K$, we can realize infinitely many pairwise non-diffeomorphic 4-manifolds, either symplectic or nonsymplectic.

□

Remark 21. Let $X(m)$ denote the symplectic 4-manifold obtained from $K3#2\mathbb{C}P^2$ by applying $m$ copies of the lantern substitutions, performed in the order given as
in Lemma 19 where $1 \leq m \leq 5$. Using Gompf’s result 14 and 15, we observe that $X(1)$ and $X(2)$ are diffeomorphic to $K3#\mathbb{CP}^2$ and $K3$ respectively. Similarly as in Theorem 16, we show that $X(m)$ is an exotic copy of $3\mathbb{CP}^2#(21 - m)\mathbb{CP}^2$ for $3 \leq m \leq 5$. Moreover, the knot surgery on torus yields the infinite family of symplectic and non-symplectic irreducible 4-manifolds all homeomorphic but not diffeomorphic to $3\mathbb{CP}^2#(21 - m)\mathbb{CP}^2$. We leave the details to the reader as an exercise.

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