PP-Wave Strings from Membrane and from String in the Spacetime with Two Time Directions

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Abstract

In this paper we obtain strings that propagate in the quantized pp-wave backgrounds. We can obtain these strings from the solutions of membrane. The other way is the propagation of a massless string in a spacetime with two time dimensions. This string sweeps a worldvolume, which enables us to obtain other strings in the quantized pp-wave backgrounds in the spacetime with one time direction. The associated algebras and Hamiltonians of these massive strings will be studied.

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1 Introduction

The plane wave metric supported by a Ramond-Ramond 5-form background [1], provides examples of exactly solvable string models [2, 3]. Note that, string theory in this background tests AdS/CFT correspondence [4]. This background by Penrose limit [5], also is related to the $AdS_5 \times S^5$ background [6].

Combining membrane theory [7, 8] with string theory in the pp-wave background, we can find a structure for the membrane [9]. In fact, an enlarged framework for studying strings is the compactified membrane theory [10]. In this paper, from the solutions of the compacted membrane, we shall obtain massive strings which propagate in the pp-wave backgrounds. Some properties of these strings such as algebra and Hamiltonian will be studied.

Similar to the membrane theory, we shall also find strings in the pp-wave backgrounds from a massless string that propagates in the spacetime with two time directions. The worldvolume of this massless string has one space and two time directions [11, 12]. One of these two time coordinates (both of the spacetime and of the string worldvolume) is compacted on a circle. The resulted massive strings will be studied in detail.

This paper is organized as follows. In section 2, relations between the massive strings with the membrane are given. In section 3, from the solution of the membrane, we shall obtain the solutions of the massive closed strings. In section 4, the same will be done for the massive open strings. In section 5, by compactifying the extra time coordinates of the spacetime and of the string worldvolume, we shall obtain another solutions for the massive closed and open strings.

2 Membrane in terms of massive strings

By choosing appropriate gauges [8], the membrane action can be written as

$$S = -\frac{T_2}{2} \int d^2\sigma d\rho (\eta^{AB} \partial_A Z^I \partial_B Z^I),$$

(1)

where $I \in \{1, 2, ..., 8\}$. The metric of the membrane worldvolume is $\eta_{AB} = \text{diag}(-1, 1, 1)$ and $T_2$ is its tension. Therefore, the equation of motion of the membrane is

$$(-\partial^2_\tau + \partial^2_\sigma + \partial^2_\rho) Z^I(\tau, \sigma, \rho) = 0.$$  

(2)

It is possible to expand $Z^I(\tau, \sigma, \rho)$ as in the following

$$Z^I(\tau, \sigma, \rho) = \sum_{n=-\infty}^{\infty} X_n^I(\tau, \sigma) q_n(\rho),$$

(3)
where \( q_n(\rho) = \exp(\frac{in\rho}{R}) \). The ranges of \( \sigma \) and \( \rho \) are \( 0 \leq \sigma \leq 2\pi \alpha'p^+ \) and \( 0 \leq \rho \leq 2\pi R \). In fact, it is assumed that one of the spacetime and one of the membrane directions are identified and wrapped on a circle with radius \( R \). The radius of compactification is \( R = \frac{1}{4\pi\alpha' T_2} \). Since the membrane coordinate \( Z^I(\tau, \sigma, \rho) \) is real, we should have

\[
X^I_n(\tau, \sigma) = X^I_{-n}(\tau, \sigma). \tag{4}
\]

The equation of motion of a string with mass number \( n \) is

\[
(-\partial_\tau^2 + \partial_\sigma^2 - \mu_n^2) X^I_n(\tau, \sigma) = 0, \tag{5}
\]

where \( \mu_n = \frac{n}{R} \) is the mass of the string. This equation also can be obtained from the equations (2) and (3).

The Hamiltonian of this system of strings is

\[
H = \frac{1}{4\pi\alpha'} \sum_{n \in \mathbb{Z}} \int_0^{2\pi \alpha' p^+} d\sigma \left( \partial_\tau X^I_n \partial_\tau X^I_{-n} + \partial_\sigma X^I_n \partial_\sigma X^I_{-n} + \mu_n^2 X^I_n X^I_{-n} \right). \tag{6}
\]

This Hamiltonian describes infinite number of massive strings that interact with each other. Strings with symmetric mass numbers (i.e., \( n \) and \( -n \)) couple to each other. There is no coupling between strings with the same mass signs. One interpretation is that, the membrane is a distribution of infinite number of massive strings with quantized masses.

The Hamiltonian (6) can be written as \( H = \sum_{n \in \mathbb{Z}} H_n \). The corresponding action also has the form \( S = \sum_{n \in \mathbb{Z}} S_n \). Each action \( S_n \) describes a massive string with the mass \( \mu_n \). In fact, after the gauge fixing [2], each \( S_n \) is related to a plane wave metric.

According to the equation (3) we have

\[
X^I_n(\tau, \sigma) = \frac{1}{2\pi R} \int_0^{2\pi R} d\rho \left( q_{-n}(\rho) Z^I(\tau, \sigma, \rho) \right). \tag{7}
\]

Therefore, integration of the membrane coordinates by the weight \( q_{-n}(\rho) \) over the worldvolume coordinate \( \rho \), produces massive string coordinates.

### 3 Massive closed strings

A solution of the equation of motion of the membrane, i.e., equation (2), can be written as

\[
Z^I(\tau, \sigma, \rho) = z^I(\tau, \rho) + \frac{i\sqrt{2\alpha'}}{2\alpha'^2} \sum_{l,k \neq (0,0)} \left[ \frac{1}{\Omega_{lk}} \exp[-i(\Omega_{lk}\tau - \mu_k \rho)] \left( A^I_{lk} e^{il\sigma} + \tilde{A}^I_{lk} e^{-il\sigma} \right) \right], \tag{8}
\]
where the zero modes are given by the function \( z^I(\tau, \rho) \) as in the following

\[
z^I(\tau, \rho) = \sum_{k \in \mathbb{Z}} \left[ q_k(\rho) \left( x^I_k \cos(\mu_k \tau) + \frac{p^I_k}{p^+} \sin(\mu_k \tau) \right) \right].
\]

Also \( \Omega_{lk} \) and \( \bar{l} \) are

\[
\Omega_{lk} = \text{sgn}(l) \sqrt{\left( \frac{l}{\alpha' p^+} \right)^2 + \left( \frac{k}{R} \right)^2}, \quad \bar{l} = \frac{l}{\alpha' p^+}.
\]

The worldvolume frequencies \( \{ \Omega_{lk} \} \) imply that the indices “\( l \)” and “\( k \)”, in the oscillating part of the solution (8), simultaneously can not be zero. The Fourier coefficients have been chosen so that \( Z^I \) is real coordinate. Therefore, we should have

\[
x^I_k = x^I_{-k}, \quad p^I_k = p^I_{-k}, \quad A^I_{lk} = A^I_{-l,-k}, \quad \tilde{A}^I_{lk} = \tilde{A}^I_{-l,-k}.
\]

Furthermore, the algebra of the membrane modes are

\[
[A^I_{lk}, A^{J'}_{l'k'}] = [\tilde{A}^I_{lk}, \tilde{A}^{J'}_{l'k'}] = \alpha' p^+ \Omega_{lk} \delta^{IJ} \delta_{l+l',0} \delta_{k+k',0}.
\]

\[
[x^I_k, p^J_{k'}] = i \delta^{IJ} \delta_{k+k',0}.
\]

Combining the solution (8) with the equation (7), we obtain the closed string coordinates with the mass number \( n \),

\[
X^I_n(\tau, \sigma) = x^I_n \cos(\mu_n \tau) + \frac{p^I_n \sin(\mu_n \tau)}{\mu_n} + \frac{i \sqrt{2\alpha'}}{2\alpha' p^+} \sum_{(l,n) \neq (0,0)} \left[ \frac{1}{\Omega_l} e^{-i\Omega_l \sigma} (A^I_{ln} e^{i\sigma} + \tilde{A}^I_{ln} e^{-i\sigma}) \right].
\]

This string coordinate satisfies the condition (4). Note that \( A^I_{l,-n} \) for positive indices \( l \) and \( n \) is creation operator, while \( A^I_{ln} \) is annihilation operator. Also \( A^I_{-l,n} \) with respect to the index \( l \) is creation operator and with respect to the index \( n \) is annihilation operator. Similar interpretation also holds for \( A^I_{l,-n} \).

The equation (6) gives the Hamiltonian of the closed strings as in the following

\[
H = \frac{1}{2p^+} \sum_{n \in \mathbb{Z}} \left( p^I_{-n} p^I_n + m_n^2 x^I_{-n} x^I_n \right) + \frac{1}{2\alpha' p^+} \sum_{n \in \mathbb{Z}} \sum_{(l,n) \neq (0,0)} \left( A^I_{l,-n} A^I_{ln} + \tilde{A}^I_{l,-n} \tilde{A}^I_{ln} \right).
\]
Now define the new oscillators $a'_{ln}$ and $\tilde{a}'_{ln}$ as

$$a'_{ln} = \frac{1}{\sqrt{\alpha' p^+ |\Omega_{ln}|}} A'_{ln},$$
$$\tilde{a}'_{ln} = \frac{1}{\sqrt{\alpha' p^+ |\Omega_{ln}|}} \tilde{A}'_{ln}. \quad (16)$$

These oscillators have the algebras

$$[a'_{ln}, a'_{l'n'}] = [\tilde{a}'_{ln}, \tilde{a}'_{l'n'}] = \text{sgn}(l) \delta^{ll'} \delta_{n+n',0}. \quad (17)$$

Therefore, the oscillating part of the Hamiltonian (15) takes the form

$$H_{\text{osc}} = \sum_{n=1}^{\infty} \left[ |\Omega_{0n}| (a'_{0,-n} a'_{0n} + \tilde{a}'_{0,-n} \tilde{a}'_{0n}) \right] + \sum_{n \in \mathbb{Z}} \sum_{l=1}^{\infty} \left[ |\Omega_{ln}| (a'_{l,-n} a'_{ln} + \tilde{a}'_{l,-n} \tilde{a}'_{ln}) \right] + C + \tilde{C}. \quad (18)$$

The constants $C$ and $\tilde{C}$ come from the normal ordering of the Hamiltonian. After regularization, they are

$$C = \tilde{C} = -\frac{1}{3} \left( \frac{1}{R} + \frac{1}{\alpha' p^+} \right) + 8 \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \Omega_{ln}. \quad (19)$$

In $H_{\text{osc}}$, the first summation is normal ordered. Also the second summation with respect to the index $l$ is normal ordered.

## 4 Massive open strings

The procedure of closed string also can be applied for open string. That is, for the equation (2), we can consider the solution

$$Z^I(\tau, \sigma, \rho) = z^I(\tau, \rho) + \frac{i \sqrt{2\alpha'}}{2\alpha' p^+} \sum_{(l,k) \neq (0,0)} \left[ \frac{1}{\Omega'_{lk}} A^I_{lk} \exp[-i(\Omega'_{lk} \tau - \mu_k \rho)] \cos \left( \frac{l \sigma}{2\alpha' p^+} \right) \right], \quad (20)$$

where $z^I(\tau, \rho)$ has been given by the equation (9), and $\Omega'_{lk}$ is

$$\Omega'_{lk} = \text{sgn}(l) \sqrt{\left( \frac{l}{2\alpha' p^+} \right)^2 + \left( \frac{k}{R} \right)^2}. \quad (21)$$

Since the coordinate $Z^I(\tau, \sigma, \rho)$ is real the Fourier modes should satisfy the conditions

$$x'^{I+}_k = x'^{I-}_k, \quad p'^{I+}_k = p'^{I-}_k, \quad A'^{I+}_{lk} = A'^{I-}_{l,-k}. \quad (22)$$
The open string coordinates resulted from the membrane coordinates (20), satisfy the Neumann boundary condition. For satisfying the Dirichlet boundary condition, it is sufficient to drop \( z^I(\tau, \rho) \) and change \( \cos(\frac{l\sigma}{2\alpha' p^+}) \) to \( -i \sin(\frac{l\sigma}{2\alpha' p^+}) \).

The equations (7) and (20) give the coordinates of the open string with the mass number \( n \),

\[
X^i_n(\tau, \sigma) = x^i_n \cos(\mu_n \tau) + \frac{p^i_n}{p^+} \sin(\mu_n \tau) + i\sqrt{2\alpha'} \frac{1}{2\alpha' p^+} \sum_{(l,n)\neq(0,0)} \left[ \frac{1}{\Omega'_ln} A^i_{ln} e^{-i\alpha'_{ln} \tau} \cos \left( \frac{l\sigma}{2\alpha' p^+} \right) \right],
\]

for the Neumann directions, and

\[
X^a_n(\tau, \sigma) = \sqrt{2\alpha'} \frac{1}{2\alpha' p^+} \sum_{(l,n)\neq(0,0)} \left[ \frac{1}{\Omega'_ln} A^a_{ln} e^{-i\alpha'_{ln} \tau} \sin \left( \frac{l\sigma}{2\alpha' p^+} \right) \right],
\]

for the Dirichlet directions. The coordinates (23) and (24) satisfy the condition (4). For the massless string i.e., \( n = 0 \), these are the usual open string coordinates.

The Hamiltonian of this system of open strings is sum of the Dirichlet and Neumann Hamiltonians. Therefore, we have

\[
H = \frac{1}{2p^+} \sum_{n \in Z} \left( p^i_{-n} p^i_n + m^2_n x^i_{-n} x^i_n \right) + \frac{1}{4\alpha' p^+} \sum_{n \in Z} \sum_{(l,n)\neq(0,0)} (A^I_{-l,-n} A^I_{ln}).
\]

The algebras of the open string modes are equation (13) and

\[
[A^I_{ln}, A'^J_{ln'}] = 2\alpha' p^+ \Omega'_ln \delta^I_J \delta_{l+l',0} \delta_{n+n',0}, \quad I, J \in \{i, a\}.
\]

Define the oscillators \( \{a^I_{ln}\} \) as in the following

\[
a^I_{ln} = \frac{1}{\sqrt{2\alpha' p^+ |\Omega'_ln|}} A^I_{ln}.
\]

Then the algebra of these oscillators becomes

\[
[a^I_{ln}, a'^J_{ln'}] = \text{sgn}(l) \delta^I_J \delta_{l+l',0} \delta_{n+n',0}, \quad I, J \in \{i, a\}.
\]

In terms of these oscillators and after normal ordering, the oscillating part of the Hamiltonian (25) takes the form

\[
H_{osc} = \sum_{n=1}^{\infty} \left( \Omega'_ln |a^I_{0,-n} a^I_{ln} \right) + \sum_{n \in Z} \sum_{l=1}^{\infty} \left( |\Omega'_ln| a^I_{-l,-n} a^I_{ln} \right) + C',
\]

where the normal ordering constant \( C' \) is

\[
C' = -\frac{1}{3} \left( \frac{1}{R} + \frac{1}{2\alpha' p^+} \right) + 8 \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} \Omega'_ln.
\]
5 Strings in the pp-wave backgrounds from string in the spacetime with two time directions

Some efforts have been devoted to imagining a world with more than one time dimension. There are many possibilities for the numbers of space dimension, time dimension, and also for the space and time dimensions of the extended objects, that propagate in the spacetime [11, 12]. For example, it is possible to have a membrane that its worldvolume has two space and one time dimensions. This membrane propagates in a spacetime with ten space and one time dimensions. There is another interesting possibility, i.e., a string that its worldvolume contains one space and two times. This string lives in a spacetime with nine spaces and two times. So, both the (10, 1; 2, 1) and the (9, 2; 1, 2) cases yield $SO(3, 2) \times SO(8)$ as the bosonic group [11].

In addition to $Z^0$, let the direction $Z^{10}$ also be a time coordinate of the spacetime. Therefore, the action of a massless string in this spacetime is

$$\bar{S} = -\frac{T}{2} \int d\sigma d^2\tau (\eta_{\mu\nu} \eta^{AB} \partial_A Z^\mu \partial_B Z^n - 1),$$  \hspace{1cm} (31)

where $\mu, \nu \in \{0, 1, ..., 10\}$. The parameters $\tau$ and $\tau'$ are time coordinates and $\sigma$ is space coordinate of the string worldvolume. The metrics of the spacetime and the worldvolume are $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, ..., 1, -1)$ and $\eta_{AB} = \text{diag}(-1, -1, 1)$. The constant $T$ is tension of the string in this spacetime. It is not equal to $\frac{1}{2\pi \alpha'}$ which is string tension in the usual spacetime (i.e., spacetime with one time direction). By the identification $Z^{10} = \tau'$ and choosing the light-cone gauge, the action (31) takes the form

$$\bar{S} = -\frac{T}{2} \int d^2\tau \int_0^{2\pi \alpha' p^+} d\sigma (\eta^{AB} \partial_A Z^I \partial_B Z^I).$$  \hspace{1cm} (32)

The equation of motion of string, extracted from this action, is

$$(-\partial_\tau^2 - \partial_{\tau'}^2 + \partial_\sigma^2) Z^I(\tau, \tau'; \sigma) = 0.$$  \hspace{1cm} (33)

In addition, for the closed string there is the condition

$$Z^I(\tau, \tau'; \sigma + 2\pi \alpha' p^+) = Z^I(\tau, \tau'; \sigma).$$  \hspace{1cm} (34)

The open string also should satisfy the conditions

$$(\partial_\sigma Z^I)_{\sigma_0} = 0,$$ \hspace{1cm} (35)
(36) for the Neumann directions, and
\[(\partial_{\tau} Z^I)_{\sigma_0} = 0,\]
\[(\partial_{\sigma} Z^I)_{\sigma_0} = 0,\] for the Dirichlet directions. The constant \(\sigma_0 = 0, 2\pi \alpha' p^+\) shows the ends of the open string.

Assume that the extra time direction \(Z^{10}\) to be compact on a circle with the radius \(R\). Since the functions \(\{q_n(\tau') = \exp\left(\frac{in\pi' \tau'}{R}\right)\}\) are periodic we can write
\[Z^I(\tau, \tau'; \sigma) = \sum_{n=-\infty}^{\infty} X^I_n(\tau, \sigma) q_n(\tau').\] (37)

Again the reality of \(Z^I\) leads to the condition (4) for the coordinate \(X^I_n(\tau, \sigma)\). According to the equation (37) we have
\[X^I_n(\tau, \sigma) = \frac{1}{2\pi R} \int_{0}^{2\pi R} d\tau' \left(q_{-n}(\tau') Z^I(\tau, \tau'; \sigma)\right).\] (38)

The equation (33) gives the equation of motion of the string coordinate \(X^I_n(\tau, \sigma)\) as
\[(-\partial^2_{\tau} + \partial^2_{\sigma} + \mu^2_n)X^I_n(\tau, \sigma) = 0.\] (39)

Also the string tension \(T\) depends on the radius of compactification, i.e.,
\[T = \frac{1}{4\pi^2 \alpha' R}.\] (40)

According to the equation (37), the action (32) becomes
\[\bar{S} = -\frac{1}{4\pi \alpha'} \sum_{n \in \mathbb{Z}} \int d\tau \int_{0}^{2\pi \alpha' p^+} d\sigma \left(\eta^{ab} \partial_a X^I_n \partial_b X^I_{-n} - \mu^2_n X^I_n X^I_{-n}\right).\] (41)

In fact, the equation of motion extracted from this action, is the equation (39). This action leads to the following Hamiltonian
\[\bar{H} = \frac{1}{4\pi \alpha'} \sum_{n \in \mathbb{Z}} \int_{0}^{2\pi \alpha' p^+} d\sigma \left(\partial_\tau X^I_n \partial_\tau X^I_{-n} + \partial_\sigma X^I_n \partial_\sigma X^I_{-n} - \mu^2_n X^I_n X^I_{-n}\right).\] (42)

The action (41) and the associated Hamiltonian (42) can be written as \(\bar{S} = \sum_{n \in \mathbb{Z}} \bar{S}_n\) and \(\bar{H} = \sum_{n \in \mathbb{Z}} \bar{H}_n\). Each light-cone action \(\bar{S}_n\) corresponds to the plane wave metric
\[d\bar{s}_n^2 = 2dX^+dX^- + \mu^2_n \sum_{I=1}^{8} X^I_n X^I_{-n}(dX^+)^2 + \sum_{i=1}^{8} dX^I_i dX^I_{-i}.\] (43)

For achieving to this metric, the role of the coordinates \(X^0\) and \(X^9\) should be changed. In other words, in the usual plane wave metric, we should apply the exchanges \(X^0 \rightarrow \pm iX^9\).
and $X^9 \rightarrow \pm iX^9$. Therefore, the light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^9)$ transform as
$X^+ \rightarrow \pm iX^+$ and $X^- \rightarrow \mp iX^-$. 

**Closed strings in pp-waves**

For a closed string the solution of the equation of motion (33) is

$$Z^I(\tau, \tau'; \sigma) = z^I(\tau, \tau') + \frac{i\sqrt{2\alpha'}}{2\alpha'p^+} \sum_{k \in \mathbb{Z}} \sum_{|l| \geq |l_k|+1} \left[ \frac{1}{\omega_{lk}} \exp[-i(\omega_{lk} \tau - \mu_k \tau')] \left( A^I_{lk} e^{i\sigma} + \tilde{A}^I_{lk} e^{-i\sigma} \right) \right].$$

(44)

where the zero modes are given by the function

$$z^I(\tau, \tau') = \sum_{k \in \mathbb{Z}} \left[ q_k(\tau') \left( x^I_k \cosh(\mu_k \tau) + \frac{p^I_k}{p^+} \frac{\sinh(\mu_k \tau)}{\mu_k} \right) \right].$$

(45)

Also $\omega_{lk}$, $\tilde{l}$ and $l_k$ are

$$\omega_{lk} = \text{sgn}(l) \sqrt{\left( \frac{l}{\alpha'p^+} \right)^2 - \left( \frac{k}{R} \right)^2}, \quad \tilde{l} = \frac{l}{\alpha'p^+}, \quad l_k = \left[ \frac{\alpha'p^+}{R} |k| \right].$$

(46)

In fact, the number $l_k$ is integer part of $\frac{\alpha'p^+}{R} |k|$. Reality of the coordinate (44) implies that the Fourier coefficients should obey the conditions (11).

The coordinate of the closed string with the mass number $n$, in the usual spacetime is

$$X^I_n(\tau, \sigma) = x^I_n \cosh(\mu_n \tau) + \frac{p^I_n}{p^+} \frac{\sinh(\mu_n \tau)}{\mu_n} + \frac{i\sqrt{2\alpha'}}{2\alpha'p^+} \sum_{|l| \geq |l_n|+1} \left[ \frac{1}{\omega_{ln}} \exp[-i\omega_n \tau] \left( A^I_{ln} e^{i\sigma} + \tilde{A}^I_{ln} e^{-i\sigma} \right) \right].$$

(47)

This coordinate satisfies the equation of motion (39) and the condition (4). The algebra of the modes of this coordinate, has been given by the equations (12) and (13).

The equations (42) and (47) give the Hamiltonian of the closed strings as

$$H = \frac{1}{2p^+} \sum_{n \in \mathbb{Z}} \left( p^I_{-n} p^I_n - m^2_n x^I_{-n} x^I_n \right) + \frac{1}{2\alpha'p^+} \sum_{n \in \mathbb{Z}} \sum_{|l| \geq |l_n|+1} \left( A^I_{l-n} A^I_{ln} + \tilde{A}^I_{l-n} \tilde{A}^I_{ln} \right).$$

(48)

Using the definition (16), with $\omega_{ln}$ instead of $\Omega_{ln}$, the oscillating part of this Hamiltonian becomes

$$H_{osc} = \sum_{n \in \mathbb{Z}} \sum_{l = l_n + 1}^{\infty} \left[ \omega_{ln} \left( a^I_{l-n} a^I_{ln} + \tilde{a}^I_{l-n} \tilde{a}^I_{ln} \right) \right] + A + \tilde{A}.$$

(49)

The normal ordering constants $A$ and $\tilde{A}$ come from the algebra (17) and they are

$$A = \tilde{A} = -\frac{1}{3\alpha'p^+} + 8 \sum_{n=1}^{\infty} \sum_{l = l_n + 1}^{\infty} \omega_{ln}.$$  

(50)

**Open strings in pp-waves**
The equation of motion (33) gives the following solutions for the open strings

\[ Z^i(\tau, \tau'; \sigma) = z^i(\tau, \tau') + \frac{i\sqrt{2\alpha'}}{2\alpha'p^+} \sum_{k \in \mathbb{Z}} \sum_{|l| \geq l_k+1} \left[ \frac{1}{\omega_{l_k}^i} A_{l_k}^i \exp[-i(\omega_{l_k}^i \tau - \mu_k \tau')] \cos \left( \frac{l \sigma}{2\alpha'p^+} \right) \right], \tag{51} \]

for the Neumann boundary condition, where \( z^i(\tau, \tau') \) has been given by the equation (45) and \( \omega_{l_k}^i \) and \( l_k \) are

\[ \omega_{l_k}^i = \text{sgn}(l) \sqrt{\left( \frac{l}{2\alpha'p^+} \right)^2 - \left( \frac{k}{R} \right)^2}, \quad l_k = \left\lfloor \frac{2\alpha'p^+}{R}|k| \right\rfloor. \tag{52} \]

For the Dirichlet boundary condition we have

\[ Z^a(\tau, \tau'; \sigma) = \frac{\sqrt{2\alpha'}}{2\alpha'p^+} \sum_{k \in \mathbb{Z}} \sum_{|l| \geq l_k+1} \left[ \frac{1}{\omega_{l_k}^a} A_{l_k}^a \exp[-i(\omega_{l_k}^a \tau - \mu_k \tau')] \sin \left( \frac{l \sigma}{2\alpha'p^+} \right) \right]. \tag{53} \]

The reality of the coordinates \( Z^i \) and \( Z^a \) leads to the conditions (22) for the open string modes.

The open string coordinates in the usual spacetime are

\[ X^i_n(\tau, \sigma) = x^i_n \cosh(\mu_n \tau) + \frac{p^i_n \sinh(\mu_n \tau)}{\mu_n} + \frac{i\sqrt{2\alpha'}}{2\alpha'p^+} \sum_{|l| \geq l_n+1} \frac{1}{\omega_{l_n}^i} A_{l_n}^i e^{-i\omega_{l_n}^i \tau} \cos \left( \frac{l \sigma}{2\alpha'p^+} \right), \tag{54} \]

for the Neumann directions, and

\[ X^a_n(\tau, \sigma) = \frac{\sqrt{2\alpha'}}{2\alpha'p^+} \sum_{|l| \geq l_n+1} \frac{1}{\omega_{l_n}^a} A_{l_n}^a e^{-i\omega_{l_n}^a \tau} \sin \left( \frac{l \sigma}{2\alpha'p^+} \right), \tag{55} \]

for the Dirichlet directions. These solutions also satisfy the condition (4).

The Hamiltonian of this system of open strings is

\[ H' = \frac{1}{2p^+} \sum_{n \in \mathbb{Z}} \left( p^i_n p^i_{-n} - m_n^2 x^i_n x^i_{-n} \right) + \frac{1}{4\alpha'p^+} \sum_{n \in \mathbb{Z}} \sum_{|l| \geq l_n+1} (A^i_{-l-n} A^i_{l_n}). \tag{56} \]

Apply the definition (27), with \( \omega_{l_n}^i \) instead of \( \Omega_{l_n}^i \), we obtain the oscillating part of \( H' \) as

\[ H'_{osc} = \sum_{n \in \mathbb{Z}} \sum_{l=l_n+1}^{\infty} \left( |\omega_{l_n}^i| a^i_{-l-n} a^i_{l_n} \right) + A'. \tag{57} \]

The algebra (28) gives the normal ordering constant \( A' \),

\[ A' = -\frac{1}{6\alpha'p^+} + 8 \sum_{n=1}^{\infty} \sum_{l=l_n+1}^{\infty} \omega_{l_n}^i. \tag{58} \]

Note that the equation (13) gives the algebra of \( x^i_n \) and \( p^i_n \) for both closed strings and the Neumann part of open strings.
6 Conclusions

In terms of the membrane solutions, we obtained the solutions of the massive closed and open strings, the algebras of the modes and the associated Hamiltonians. In fact, the algebra of the strings modes in the quantized pp-wave backgrounds is the same as the algebra of the membrane modes. The normal ordering constants of the Hamiltonians depend on both the string length \(2\pi\alpha'p^+\) and the radius of compactification of the membrane.

We also obtained strings in the quantized pp-wave backgrounds from a massless string that propagates in the spacetime with two time directions. One of these time coordinates is compacted on circle. The resulted massive strings are open or closed. The open string coordinates satisfy the Neumann or Dirichlet boundary conditions, as the usual case. Reality of the massless string coordinates projects out some of the oscillators from the solutions of the resulted massive strings and therefore, from the associated Hamiltonians.

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