UNIT-REGULAR ELEMENTS IN RESTRICTIVE SEMIGROUPS
OF TRANSFORMATIONS AND LINEAR OPERATORS

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Abstract. Let $T(X)$ be the full transformation semigroup on a set $X$ and let $L(V)$ be the semigroup under composition of all linear operators on a vector space $V$ over a field. For a nonempty subset $Y$ of $X$ and a subspace $W$ of $V$, we consider the restrictive semigroups $T(X,Y) = \{ f \in T(X) \mid Yf \subseteq Y \}$ and $L(V,W) = \{ f \in L(V) \mid Wf \subseteq W \}$ under composition. We characterize unit-regular elements in $T(X,Y)$ and $L(V,W)$. Utilizing these, we characterize unit-regularity of $T(X,Y)$ and $L(V,W)$. We prove that $f \in L(V)$ is unit-regular if and only if nullity($f$) = corank($f$). A transformation semigroup is called semi-balanced if all its elements are semi-balanced. We determine a necessary and sufficient condition for $T(X,Y)$ and $L(V,W)$ to be semi-balanced.

1. Introduction

An element $s$ of a semigroup $S$ with identity is called unit-regular if there exists a unit $u \in S$ such that $sus = s$. A unit-regular semigroup is a semigroup with identity in which every element is unit-regular. The notion of unit-regularity, which was introduced by Ehrlich [8] within the context of rings, has consecutively received wide attention from many semigroup theorists (see e.g., [1, 2, 3, 7, 9, 10, 18, 19, 27]). In 1980, Alarcao [7, Proposition 1] proved that a semigroup with identity is unit-regular if and only if it is factorizable.

Note that a unit-regular semigroup is a specific type of regular semigroup. Let $X$ be a nonempty set and $T(X)$ be the full transformation semigroup on $X$. It is well-known that $T(X)$ is regular [14, p. 63, Exercise 15]. Alarcao [7, Proposition 5] proved that $T(X)$ is unit-regular if and only if $X$ is finite. In 1966, for a fixed nonempty subset $Y$ of $X$, Magill [17] introduced an interesting semigroup $T(X,Y)$ under composition defined by $T(X,Y) = \{ f \in T(X) \mid Yf \subseteq Y \}$ and called it as the restrictive semigroup of transformations. If $Y = X$, then $T(X,Y) = T(X)$. To this extent, $T(X,Y)$ may regard as a generalization of $T(X)$. Several properties of $T(X,Y)$ have been investigated (see, e.g., [5, 6, 12, 20, 21, 23, 26]). For example, Nenthein et al. [20, Theorem 2.3] characterized regular elements in $T(X,Y)$.

Let $V$ be a vector space over a field and $L(V)$ be the semigroup under composition consisting of all linear operators on $V$. It is well-known that $L(V)$ is regular [14, p. 63, Exercise 19]. Ehrlich [8, Theorem 4] proved that $L(V)$ is not unit-regular when $V$ is infinite-dimensional. Jampachon et al. [15, Theorem 2] proved that...
L(V) is factorizable if and only if V is finite-dimensional. If V is finite-dimensional, Kemprasit [16] directly showed that L(V) is unit-regular. For a fixed subspace W of V, analogous to $\overline{T}(X, Y)$, Nenthein and Kemprasit [21] studied a semigroup $\overline{L}(V, W)$ under composition defined by

$$\overline{L}(V, W) = \{ f \in L(V) \mid Wf \subseteq W \}.$$  

If W is trivial, then $\overline{L}(V, W) = L(V)$. To this extent, $\overline{L}(V, W)$ may regard as a generalization of L(V). In [21] Proposition 3.1, the authors characterized regular elements in $\overline{L}(V, W)$ and subsequently proved that $\overline{L}(V, W)$ is regular if and only if W is trivial. Chaiya [23] Theorem 11] gave a necessary and sufficient condition for $\overline{L}(V, W)$ to be unit-regular. Several other interesting properties of $\overline{L}(V, W)$ have been studied (see, e.g., [2, 3, 13, 22]).

In 1998, Higgins et al. [11] p. 1356 introduced an interesting nonempty subset B of T(X) consisting of all elements $\alpha \in T(X)$ such that the collapse of $\alpha$ and the defect of $\alpha$ are same. The elements of B are called semi-balanced. We say that a transformation semigroup is semi-balanced if all its elements are semi-balanced. It is obvious that every subsemigroup of T(X) is semi-balanced if X is finite. We can point out from [24] Lemma 3.6 that T(X) is semi-balanced if and only if X is finite.

The rest of this paper is structured as follows. In Section 2, we introduce the notation and terminology which are needed for the sequel. In Section 3, we prove that $\overline{T}(X, Y)$ (resp. $L(V)$) is semi-balanced if and only if X is finite (resp. V is finite-dimensional). In Section 4, we characterize unit-regular elements in $\overline{T}(X, Y)$ and then determine a necessary and sufficient condition for $\overline{T}(X, Y)$ to be unit-regular. In Section 5, we characterize unit-regular elements in $\overline{L}(V, W)$. Moreover, using this, we determine a necessary and sufficient condition for $\overline{L}(V, W)$ to be unit-regular. We prove that $f \in L(V)$ is unit-regular if and only if nullity($f$) = corank($f$), and alternatively show that L(V) is unit-regular if and only if V is finite-dimensional.

2. Preliminaries and Notation

By |A|, we mean the cardinality of a set A. Given any sets A and B, we write $A \setminus B$ to denote the set of elements of A which are not in B. Let $f$ be a map. We use dom($f$), codom($f$), and $R(\alpha)$ respectively to denote the domain, codomain, and range of $f$. We write the image of an element $x$ under $f$ by $xf$ and denote the composition of maps simply by juxtaposition. If $A \subseteq \text{dom}(f)$ and $B \subseteq \text{codom}(f)$, we set $Af = \{ af \mid a \in A \}$ and $Bf^{-1} = \{ x \in \text{dom}(f) \mid xf \in B \}$. Moreover, if $B = \{ b \} \subseteq \text{codom}(f)$ then we write $Bf^{-1}$ as $bf^{-1}$. By a selfmap on a set X we mean a map from X into itself. Let $f$ be a selfmap on a set X. For a subset $A \subseteq \text{dom}(f)$, the restriction of f to A is the map $f|_A : A \to X$ given by $x(f|_A) = xf$ for all $x \in A$. If $B \subseteq \text{codom}(f)$ such that $Xf \subseteq B$, the corestriction of f to B is the map from X to B agreeing with f. Moreover, we use $f|_B$ to denote the corestriction of the map $f|_A$ to B when $Bf \subseteq B$.

Let X be a nonempty set and $f$ be a selfmap on X. A transversal of an equivalence relation $\rho$ on X is a subset of X which contains exactly one element from each $\rho$-class. We let $D(f) = X \setminus R(f)$ and ker($f$) = $\{(x, y) \in X \times X \mid xf = yf\}$. We shall write $T_f$ to denote any transversal of the equivalence relation ker($f$). Note
that \(|T_f| = |R(f)|\). The defect \(d(f)\) of \(f\) is the cardinality of \(D(f)\) and the collapse \(c(f)\) of \(f\) is the cardinality of \(X \setminus T_f\) (cf. [11] p. 1356). Note that \(c(f)\) is independent of the choice of transversal of \(\ker(f)\), and \(c(f) = 0\) if and only if \(f\) is injective. A selfmap \(f\) is said to be semi-balanced if \(c(f) = d(f)\) (cf. [11] p. 1356). By \(B(S)\) we mean the set of all semi-balanced elements in a subsemigroup \(S\) of \(T(X)\). Note that \(B(S) = S \cap B\). For any semigroup \(M\) with identity, the set of all unit elements in \(M\) is denoted by \(U(M)\) and the set of all unit-regular elements in \(M\) is denoted by \(\text{ureg}(M)\). A subset \(A\) of \(X\) is said to be invariant under a map \(f \in T(X)\) if \(Af \subseteq A\). Note that if \(f \in U(T(X,Y))\), then \(f_{1_Y} \in U(T(Y))\).

Throughout this paper, let \(V\) denote a vector space over an arbitrary field. We shall denote the zero vector of \(V\) by 0. The subspaces \(\{0\}\) and \(V\) of \(V\) are called trivial subspaces. By \(\langle T \rangle\) we mean the subspace spanned by a subset \(T\) of \(V\). Let \(U\) be a subspace of \(V\). We say that \(U\) is a proper subspace of \(V\) if \(U \neq V\). The dimensions of \(U\) and the quotient space \(V/U\) are denoted by \(\dim(U)\) and \(\text{codim}_V(U)\), respectively. We denote by \(V = U \oplus W\) the (internal) direct sum of subspaces \(U\) and \(W\) of \(V\). If \(V = U \oplus W\), then we say that \(W\) is a complement of \(U\) in \(V\).

Let \(f\) be a linear operator on \(V\). We let \(R(f) = \{vf \mid v \in V\}\) and \(N(f) = \{v \in V \mid vf = 0\}\). Note that \(N(f)\) is a subspace of \(\text{dom}(f)\) and \(R(f)\) is a subspace of \(\text{codom}(f)\). We shall write \(\text{nullity}(f), \text{rank}(f),\) and \(\text{corank}(f)\) for \(\dim(N(f)), \dim(R(f)),\) and \(\dim(V/R(f))\), respectively. We shall write \(V \approx U\) if vector spaces \(V\) and \(U\) are isomorphic. Let \(B,B'\) be bases for \(V\). We shall write \(f\) to denote the unique linear operator on \(V\) obtained by linear extension of a map \(f : B \to V\) or a map \(f : B \to B'\) to the entire space \(V\) (cf. [23] Theorem 2.2]). Note that if \(f : B \to B'\) is a bijective map, then \(\bar{f} \in U(L(V))\). If \(f \in L(V,W)\), then \(f_{1_W} \in L(W)\) (cf. [23] p. 73]).

For further standard terminology in semigroup theory and linear algebra, we refer the reader to \([14]\) and \([23]\), respectively. In the rest of the paper, \(Y\) is a nonempty subset of a set \(X\) and \(W\) is a subspace of a vector space \(V\) over a field.

### 3. Semi-Balanced Semigroup of Transformations

In this section, we give a necessary and sufficient condition for \(\overline{T}(X,Y)\) to be semi-balanced. We also give a necessary and sufficient condition for \(L(V)\) to be semi-balanced. We begin with the following proposition.

**Proposition 3.1.** If \(M\) is a submonoid of \(T(X)\), then \(\text{ureg}(M) \subseteq B(M)\).

**Proof.** Let \(f \in \text{ureg}(M)\). Since \(M\) is a submonoid of \(T(X)\), we have \(f \in \text{ureg}(T(X))\) and so \(f \in B\) by [24] Theorem 3.4]. Hence \(f \in M \cap B = B(M)\), as required. \(\square\)

We now have the following obvious corollary of Proposition 3.1.

**Corollary 3.2.** Every unit-regular submonoid of \(T(X)\) is semi-balanced.

In general, the converse of Corollary 3.2 is not true as shown in the following example.

**Example 3.3.** Let \(X\) be a finite set such that \(|X| \geq 3\) and let \(Y\) be a proper subset of \(X\) such that \(|Y| \geq 2\). It is obvious that \(\overline{T}(X,Y)\) is semi-balanced. But, \(\overline{T}(X,Y)\) is not regular by [21] Proposition 2.1(ii)] and consequently \(\overline{T}(X,Y)\) is not unit-regular.
The next theorem gives a necessary and sufficient condition for $\mathcal{T}(X,Y)$ to be semi-balanced.

**Theorem 3.4.** The restrictive semigroup $\mathcal{T}(X,Y)$ is semi-balanced if and only if $X$ is finite.

**Proof.** Suppose that $\mathcal{T}(X,Y)$ is semi-balanced. On the contrary, let us assume that $X$ is infinite. We consider the following two possibilities separately.

**Case (X \ Y is finite).** Then $Y$ is infinite and so there exists a map $\alpha: Y \to Y$ which is injective but not surjective. Define a map $f: X \to X$ by

$$xf = \begin{cases} x\alpha & \text{if } x \in Y, \\ x & \text{if } x \in X \setminus Y. \end{cases}$$

Observe that $f \in \mathcal{T}(X,Y)$. Since $\alpha$ is injective but not surjective, it follows that $f$ is injective but not surjective.

**Case (X \ Y is infinite).** Then there exists a map $\beta: X \setminus Y \to X \setminus Y$ which is injective but not surjective. Define a map $f: X \to X$ by

$$xf = \begin{cases} x & \text{if } x \in Y, \\ x\beta & \text{if } x \in X \setminus Y. \end{cases}$$

Observe that $f \in \mathcal{T}(X,Y)$. Since $\beta$ is injective but not surjective, it follows that $f$ is injective but not surjective.

Thus, in either case, there exists a selfmap $f \in \mathcal{T}(X,Y)$ which is injective but not surjective. Therefore $c(f) = 0$ but $d(f) \geq 1$. It follows that $f$ is not semi-balanced, a contradiction. Hence our assumption is wrong. Hence we must have $X$ is finite.

Conversely, suppose that $X$ is finite. Then every selfmap on $X$ is semi-balanced and so the restrictive semigroup $\mathcal{T}(X,Y)$ is semi-balanced. $\square$

The following theorem determines a necessary and sufficient condition for $L(V)$ to be semi-balanced.

**Theorem 3.5.** The semigroup $L(V)$ is semi-balanced if and only if $V$ is finite-dimensional.

**Proof.** Suppose that $L(V)$ is semi-balanced. On the contrary, let us assume that $V$ is infinite-dimensional. Let $B$ be a basis for $V$. Note that $B$ is infinite. Therefore there exists a map $\alpha: B \to B$ which is injective but not surjective. Let $f \in L(V)$ such that $xf = x\alpha$ for all $x \in B$. Then $f$ is injective but not surjective. This implies $c(f) = 0$ and $d(f) > 0$. Therefore $f$ is not semi-balanced, a contradiction. Hence $V$ must be finite-dimensional.

Conversely, suppose that $V$ is finite-dimensional. Then $L(V)$ is unit-regular by combining [7, Proposition 1] and [15, Theorem 2]. Hence the semigroup $L(V)$ is semi-balanced by Corollary [5.2] $\square$

4. Unit-regular elements in $\mathcal{T}(X,Y)$

In this section, we characterize unit-regular elements in $\mathcal{T}(X,Y)$. Using this characterization, we give a necessary and sufficient condition for $\mathcal{T}(X,Y)$ to be unit-regular. We begin with the following simple lemma.
Lemma 4.1. Let \( f \in \overline{T}(X, Y) \). If \( f \in \text{ureg}(\overline{T}(X, Y)) \), then \( f_{1_Y} \in \text{ureg}(T(Y)) \).

Proof. If \( f \in \text{ureg}(\overline{T}(X, Y)) \), then there exists \( g \in U(\overline{T}(X, Y)) \) such that \( fgf = f \). We therefore see that \( g_{1_Y} \in U(T(Y)) \) and \( f_{1_Y}g_{1_Y}f_{1_Y} = f_{1_Y} \). Hence \( f_{1_Y} \in \text{ureg}(T(Y)) \). \( \square \)

Note that \( \overline{T}(X, Y) \) contains the identity map on \( X \). The next theorem characterizes unit-regular elements in \( \overline{T}(X, Y) \).

Theorem 4.2. Let \( f \in \overline{T}(X, Y) \). Then \( f \in \text{ureg}(\overline{T}(X, Y)) \) if and only if

(i) \( f_{1_Y} \in \text{ureg}(T(Y)) \);
(ii) \( R(f_{1_Y}) = Y \cap R(f) \);
(iii) \( |C(f) \setminus C(f_{1_Y})| = |D(f) \setminus D(f_{1_Y})| \) where \( C(f) = X \setminus T_f \) and \( C(f_{1_Y}) = Y \setminus T_{f_{1_Y}} \) for some transversals \( T_f \) and \( T_{f_{1_Y}} \) of \( \ker(f) \) and \( \ker(f_{1_Y}) \), respectively, such that \( T_{f_{1_Y}} = Y \cap T_f \).

Proof. Suppose first that \( f \in \text{ureg}(\overline{T}(X, Y)) \). Then there exists \( g \in U(\overline{T}(X, Y)) \) such that \( fgf = f \).

(i) From Lemma 4.1, we have \( f_{1_Y} \in \text{ureg}(T(Y)) \).
(ii) Since \( Y \) is invariant under \( f \), it is clear that \( R(f_{1_Y}) \subseteq Y \cap R(f) \). For the reverse inclusion, let \( y \in Y \cap R(f) \). Then there exists \( x \in X \) such that \( xf = y \). Since \( y \in Y \) and \( Y \) is invariant under \( g \), we have
\[
y = xf = (xf)gf = (yg)f \in R(f_{1_Y})\]
and so \( Y \cap R(f) \subseteq R(f_{1_Y}) \). Thus \( R(f_{1_Y}) = Y \cap R(f) \).
(iii) Observe that \( f_{1_Y}g_{1_Y}f_{1_Y} = f_{1_Y} \). Since \( fgf = f \) and \( f_{1_Y}g_{1_Y}f_{1_Y} = f_{1_Y} \), it follows from [24, Lemma 3.1] that \( T_f = R(fg) \) and \( T_{f_{1_Y}} = R(f_{1_Y}g_{1_Y}) \) are transversals of \( \ker(f) \) and \( \ker(f_{1_Y}) \), respectively. Clearly \( T_{f_{1_Y}} = Y \cap T_f \).

Let \( C(f) = X \setminus T_f \) and \( C(f_{1_Y}) = Y \setminus T_{f_{1_Y}} \). Since \( g \) and \( g_{1_Y} \) are bijective selfmaps on \( X \) and \( Y \), respectively, we have
\[
D(f)g = (X \setminus R(f))g = Xg \setminus R(f) = X \setminus T_f = C(f)
\]
and
\[
D(f_{1_Y})g_{1_Y} = (Y \setminus R(f_{1_Y}))g_{1_Y} = Yg_{1_Y} \setminus R(f_{1_Y}g_{1_Y}) = Y \setminus T_{f_{1_Y}} = C(f_{1_Y}).
\]
Therefore
\[
(D(f) \setminus D(f_{1_Y}))g = D(f)g \setminus D(f_{1_Y})g = D(f)g \setminus D(f_{1_Y})g_{1_Y} = C(f) \setminus C(f_{1_Y})
\]
and hence \( |C(f) \setminus C(f_{1_Y})| = |D(f) \setminus D(f_{1_Y})| \).

Conversely, let \( f \) satisfies all the three given conditions. From (iii), note that \( T_f \subseteq X \) and \( T_{f_{1_Y}} \subseteq Y \) are transversals of \( \ker(f) \) and \( \ker(f_{1_Y}) \), respectively. Therefore the corestrictions of the maps \( f_{1_{Y_f}} \) and \( (f_{1_Y})_{|T_{f_{1_Y}}} \) to \( R(f) \) and \( R(f_{1_Y}) \), respectively, are bijective. Let \( g_0 \) be the inverse of the corestriction map of \( f_{1_{Y_f}} \) to \( R(f) \), and let \( h_0 \) be the inverse of the corestriction map of \( (f_{1_Y})_{|T_{f_{1_Y}}} \) to \( R(f_{1_Y}) \).

Since \( T_{f_{1_Y}} = Y \cap T_f \) by (iii), it is clear that \( xg_0 = xh_0 \) for all \( x \in R(f_{1_Y}) \) and so \( R(f_{1_Y})g_0 = T_{f_{1_Y}} \). Write \( C(f) = X \setminus T_f \) and \( C(f_{1_Y}) = Y \setminus T_{f_{1_Y}} \). From (i) and Proposition 4.1, we see that \( f_{1_Y} \) is semi-balanced. Therefore there exists a bijection \( g_1 : D(f_{1_Y}) \to C(f_{1_Y}) \). Also, there exists a bijection \( g_2 : D(f) \setminus D(f_{1_Y}) \to C(f) \) and \( gh_0 = f_{1_Y} \). Therefore, we have \( f_{1_Y} = f_{1_{Y_f}}(x)g_0 = (f_{1_Y})_{|T_{f_{1_Y}}}h_0 \). Hence \( f_{1_Y} \) is semi-balanced.

Therefore, \( f \in \text{ureg}(\overline{T}(X, Y)) \).
That means (iii). Using these three bijections \(g_0, g_1,\) and \(g_2,\) we define a map \(g: X \to X\) by

\[
xg = \begin{cases} 
  xg_0 & \text{if } x \in R(f), \\
  xg_1 & \text{if } x \in D(f_{1'}) \\
  xg_2 & \text{if } x \in D(f) \setminus D(f_{1'}). 
\end{cases}
\]

Clearly \(g\) is bijective. Moreover,

\[
Yg = (R(f_{1'}) \cup D(f_{1'}))g = R(f_{1'})g_0 \cup D(f_{1'})g_1 = T(f_{1'}) \cap C(f_{1'}) \subseteq Y.
\]

Therefore \(g \in U(T(X, Y))\). We can also verify in a routine manner that \(fgf = f\) and hence \(f \in \text{ureg}(T(X, Y))\). \(\square\)

We need the following lemma to prove Theorem 4.4 that characterizes unit-regularity of \(T(X, Y)\).

**Lemma 4.3.** \(R(f_{1'}) = Y \cap R(f)\) for all \(f \in T(X, Y)\) if and only if \(|Y| = 1\) or \(Y = X\).

**Proof.** Suppose that \(R(f_{1'}) = Y \cap R(f)\) for all \(f \in T(X, Y)\). If \(|X| \leq 2\), then it is obvious. So, we let \(|X| \geq 3\).

On the contrary, let us assume that \(Y\) is a proper subset of \(X\) such that \(|Y| \geq 2\). Let \(a, b \in Y\) be distinct elements. Define a map \(f: X \to X\) by

\[
xf = \begin{cases} 
  a & \text{if } x \in Y, \\
  b & \text{otherwise.} 
\end{cases}
\]

Observe that \(Y\) is invariant under \(f\) and so \(f \in T(X, Y)\). Also, we see that \(R(f_{1'}) = Yf = \{a\}\) while \(Y \cap R(f) = \{a, b\}\). This is clearly a contradiction, and therefore our assumption is wrong. Hence we must have \(|Y| = 1\) or \(Y = X\).

The converse part is immediate. \(\square\)

**Theorem 4.4.** The restrictive semigroup \(T(X, Y)\) is unit-regular if and only if

(i) \(X\) is finite;
(ii) \(|Y| = 1\) or \(Y = X\).

**Proof.** Suppose that \(T(X, Y)\) is unit-regular.

(i) Then \(T(X, Y)\) is semi-balanced by Corollary 3.2 and hence \(X\) is finite by Theorem 3.4.

(ii) Note that each map of \(T(X, Y)\) is unit-regular. It follows from Theorem 4.2(ii) that \(R(f_{1'}) = Y \cap R(f)\) for all \(f \in T(X, Y)\) and hence \(|Y| = 1\) or \(Y = X\) by Lemma 4.3.

Conversely, suppose that the given two conditions hold and let \(f \in T(X, Y)\). Note that \(Y\) is finite by (i). It follows from [7, Proposition 5] that \(f_{1'} \in \text{ureg}(T(Y))\). That means \(f\) satisfies Theorem 4.2(i).

Since \(|Y| = 1\) or \(Y = X\) by (ii), we have \(R(f_{1'}) = Y \cap R(f)\) by Lemma 4.3. That means \(f\) satisfies Theorem 4.2(ii).

Since \(X\) and \(Y\) are finite by (i), it is clear that \(c(f) = d(f)\) and \(c(f_{1'}) = d(f_{1'})\). It is also immediate that there exist transversals \(T_f\) and \(T(f_{1'})\) of \(\text{ker}(f)\) and \(\text{ker}(f_{1'})\), respectively, such that \(T(f_{1'}) = Y \cap T_f\). Write \(C(f) = X \setminus T_f\) and
and then alternatively show that the restrictive semigroup characterization, we prove that and.

Proof. That means is a monomorphism, it is clear that is injective, the corestriction of satisfies Theorem 4.2. Thus is unit-regular, there exists a unique such that . For the reverse inclusion, let . We now claim that . Let . Then . This gives . Now, define a map : whenever . To see this, let . Then and so . This gives .

Now, we define a map : by for all . Since is a monomorphism, it is clear that is also a monomorphism. To show is surjective, let . Since , we simply have . Since is injective, the corestriction of to is bijective and so there exists a unique such that . It remains to show that . These two conditions can easily verify in a routine manner. Hence is surjective and thus .

If and are isomorphic subspaces of isomorphic vector spaces and , respectively, it is not true in general that (cf. [23, p. 93, line 7]). However, we have the following.

Lemma 5.2. Let and be subspaces of vector spaces and , respectively. If there exists an isomorphism such that , then .

Proof. Define a map by . By using linearity of , it is clear that is linear. To see is surjective, let . Since and is bijective, there exists a unique such that for . Then and so is surjective. Thus by the first isomorphism theorem.

We now claim that . Let . Then . This gives and so . Since and is bijective, we have and therefore . For the reverse inclusion, let . Since , we simply have and so and so . This gives and so . Therefore . Hence and thus .

Lemma 5.3. Let . Then there exists a subspace of such that and are transversals of and , respectively.

Proof. Suppose that is a basis for . Then for all . Therefore, for each , we can choose a fixed vector such that and . Now,
Lemma 5.5. Let $v f^{-1} \neq \emptyset$ for all $v \in B_2$. Therefore, for each $v \in B_2$, we can choose a fixed vector $\bar{v}$ of $v f^{-1}$. Let $U = \langle C \rangle$ where $C = \{u' \in u f^{-1} \cap W \mid u \in B_1\} \cup \{\bar{v} \in v f^{-1} \mid v \in B_2\}$. We now claim that $U$ and $U \cap W$ are transversals of $\ker(f)$ and $\ker(f_{1w})$, respectively.

First, we show that $U$ is a transversal of $\ker(f)$. Let $w \in R(f)$ if $B_1 \cup B_2$ is a basis for $R(f)$, we have $w = c_1 u_1 + \cdots + c_m u_m + d_1 v_1 + \cdots + d_n v_n$ for some $u_1, \ldots, u_m \in B_1$ and $v_1, \ldots, v_n \in B_2$ where $m, n \geq 0$. Consider the vector $w' = c_1 u_1' + \cdots + c_m u_m' + d_1 v_1' + \cdots + d_n v_n' \in U$ where $u_1', \ldots, u_m', v_1', \ldots, v_n' \in C$. Observe that $w' f = w$ and so $w' \in w f^{-1} \cap U$. Recall that $C$ is a basis for $U$. Therefore, by construction of $C$, we see that $w f^{-1} \cap U = \{w'\}$ and so $|w f^{-1} \cap U| = 1$. Since $w \in R(f)$ is arbitrary, the subspace $U$ of $V$ is a transversal of $\ker(f)$.

Similarly, we can show that the subspace $U \cap W$ of $W$ is a transversal of $\ker(f_{1w})$.

Lemma 5.4. Let $f \in L(V)$ and let $T_f$ be a transversal of $\ker(f)$. If $B_0$ is a basis for $N(f)$ and $B$ is a basis for $R(f)$, then $B_0 \cup (T_f \cap B f^{-1})$ is a basis for $V$.

Proof. Note that $B_0 \cap (T_f \cap B f^{-1}) = \emptyset$. Observe that $T_f \cap B f^{-1}$ is linearly independent and therefore $B_0 \cup (T_f \cap B f^{-1})$ is linearly independent. We further need to show that $\langle B_0 \cup (T_f \cap B f^{-1}) \rangle = V$, and so we consider a complement $U$ of $N(f)$ in $V$. Then $B_0 \cup B(f_{1w})^{-1}$ is a basis for $V$, where the isomorphism $f_{1w}$ also denotes the corestriction of $f_{1w}$ to $R(f)$. Now, let $u \in B(f_{1w})^{-1}$. Then $u \in U$ and so $v = u + w' \in T_f \cap B f^{-1}$ for some $w' \in N(f)$. Therefore $u = v - w' \in \langle B_0 \cup (T_f \cap B f^{-1}) \rangle$. Since $u$ is arbitrary, we have $B(f_{1w})^{-1} \subseteq \langle B_0 \cup (T_f \cap B f^{-1}) \rangle$ and so $B_0 \cup B(f_{1w})^{-1} \subseteq \langle B_0 \cup (T_f \cap B f^{-1}) \rangle$. Thus $\langle B_0 \cup (T_f \cap B f^{-1}) \rangle = V$ and hence $B_0 \cup (T_f \cap B f^{-1})$ is a basis for $V$.

Lemma 5.5. Let $f \in L(V, W)$. If $f \in \text{ureg}(L(V, W))$, then $f_{1w} \in \text{ureg}(L(W))$.

Proof. If $f \in \text{ureg}(L(V, W))$, then there exists $g \in U(L(V, W))$ such that $g f = f$. We therefore see that $g_{1w} \in U(L(W))$ and $f_{1w} g_{1w} f_{1w} = f_{1w}$. Hence $f_{1w} \in \text{ureg}(L(W))$.

Note that $\overline{L}(V, U)$ contains the identity linear operator on $V$. We next characterize the unit-regular elements in $\overline{L}(V, W)$ as follows.

Theorem 5.6. Let $f \in \overline{L}(V, W)$. Then $f \in \text{ureg}(\overline{L}(V, W))$ if and only if

1. $R(f_{1w}) = W \cap R(f)$;
2. $\text{nullity}(f_{1w}) = \text{corank}(f_{1w})$;
3. $\text{codim}_W (W + T_f) = \text{codim}_W (W + R(f))$ for some subspace $T_f$ of $V$ such that $T_f$ and $W \cap T_f$ are transversals of $\ker(f)$ and $\ker(f_{1w})$, respectively.

Proof. Suppose first that $f \in \text{ureg}(\overline{L}(V, W))$. Then there exists $g \in U(\overline{L}(V, W))$ such that $g f = f$.

1. Since $R(f_{1w}) \subseteq W$, we see that $R(f_{1w}) \subseteq W \cap R(f)$. For the reverse inclusion, let $w \in W \cap R(f)$. Then there exists $v \in V$ such that $v f = w$. Since $w \in W$ and $W$ is invariant under $g$, we have

$$w = v f = (w f g) = (w g) f \in R(f_{1w})$$

and thus $W \cap R(f) \subseteq R(f_{1w})$. Hence $R(f_{1w}) = W \cap R(f)$. 


(ii) From Lemma 5.3, there exists \( g_{|W} \in U(L(W)) \) such that \( f_{|W} g_{|W} f_{|W} = f_{|W} \). Note that \( R(g_{|W} f_{|W}) = R(f_{|W}) \) and \( g_{|W} f_{|W} \) is an idempotent of \( L(W) \). Combining this with [23] Theorem 2.22, we obtain \( N(g_{|W} f_{|W}) \cap R(f_{|W}) = W \) and so \( N(g_{|W} f_{|W}) \approx W/R(f_{|W}) \). Since \( g_{|W} \) is bijective, it follows from Lemma 5.3 that \( N(f_{|W}) \approx N(g_{|W} f_{|W}) \) and subsequently \( N(f_{|W}) \approx W/R(f_{|W}) \). Hence nullity\( (f_{|W}) = \text{corank}(f_{|W}) \).

(iii) Clearly \( R(fg) \) and \( R(f_{|W} g_{|W}) \) are subspaces of \( V \) and \( W \), respectively. Denote by \( T_f \) and \( T_{(f_{|W})} \) the subspaces \( R(fg) \) and \( R(f_{|W} g_{|W}) \), respectively. Since \( fgf = f \) and \( f_{|W} g_{|W} f_{|W} = f_{|W} \), it follows from [23] Lemma 3.1 that \( T_f \) and \( T_{(f_{|W})} \) are transversals of \( \ker(f) \) and \( \ker(f_{|W}) \), respectively. Since \( g \) is bijective, we have \( W \cap T_f = T_{(f_{|W})} \) and so \( (W + R(f))g = W + T_f \). Thus \( V/(W + R(f)) \cong V/(W + T_f) \) by Lemma 5.3 and hence \( \text{codim}_V(W + T_f) = \text{codim}_V(W + R(f)) \).

Conversely, suppose that the given three conditions hold for \( f \in \overline{T}(V, W) \) and let \( B_1 \) be a basis for \( R(f) \cap W \). Extend \( B_1 \) to bases \( B_1 \cup B_2 \) and \( B_1 \cup B_3 \) for \( R(f) \) and \( W \), respectively, where \( B_2 \subseteq R(f) \setminus \langle B_1 \rangle \) and \( B_3 \subseteq W \setminus \langle B_1 \rangle \). Clearly \( B_1 \cup B_2 \cup B_3 \) is a basis for \( W + R(f) \). Now, extend \( B_1 \cup B_2 \cup B_3 \) to a basis \( B = B_1 \cup B_2 \cup B_3 \cup B_4 \) for \( V \) where \( B_4 \subseteq V \setminus (W + R(f)) \).

By (iii), the subspace \( T_f \) of \( V \) is a transversal of \( \ker(f) \). Therefore the corestriction of \( f_{|T_f} \) to \( R(f) \) is an isomorphism. Denote by \( g_0 \) the inverse of this corestriction map. Note that \( g_0 : R(f) \to T_f \) is an isomorphism. Write \( B_1 g_0 = C_1 \) and \( B_2 g_0 = C_2 \). Since \( B_1 \cup B_2 \) is a basis for \( R(f) \), it is immediate that \( C_1 \cup C_2 \) is a basis for \( T_f \).

By (iii), the subspace \( T_{(f_{|W})} \) of \( W \) is a transversal of \( \ker(f_{|W}) \). Therefore the corestriction of \( (f_{|W})_{|T_{(f_{|W})}} \) to \( R(f_{|W}) \) is an isomorphism. Notice that the inverse of this isomorphism agrees with \( g_0 \) on \( R(f_{|W}) \). Recall that \( B_1 \) is a basis for \( W \cap R(f) \). Since \( R(f_{|W}) = W \cap R(f) \) by (i), we see that \( C_1 \) is a basis for \( T_{(f_{|W})} \).

Let \( C_3 \) be a basis for \( N(f_{|W}) \). Then \( C_1 \cup C_3 \) is a basis for \( W \) by Lemma 5.3, and subsequently \( C_1 \cup C_2 \cup C_3 \) is a basis for \( W + T_f \). Extend \( C_1 \cup C_2 \cup C_3 \) to a basis \( C = C_1 \cup C_2 \cup C_3 \cup C_4 \) for \( V \) where \( C_4 \subseteq V \setminus (W + T_f) \).

Recall that \( B_1 \) and \( B_1 \cup B_3 \) are bases for \( W \cap R(f) \) and \( W \), respectively. Since \( R(f_{|W}) = W \cap R(f) \) by (i), it follows that \( \text{corank}(f_{|W}) = |B_3| \) and so \( |C_3| = \text{nullity}(f_{|W}) = |B_3| \) by (ii). Thus there exists a bijection \( \alpha : B_3 \to C_3 \).

Recall that \( B \) and \( C \) are bases for \( V \) containing \( B_1 \cup B_2 \cup B_3 \) and \( C_1 \cup C_2 \cup C_3 \), respectively. Since \( B_1 \cup B_2 \cup B_3 \) and \( C_1 \cup C_2 \cup C_3 \) are bases for \( W + R(f) \) and \( W + T_f \), respectively, we then have \( \text{codim}_V(W + R(f)) = |B_4| \) and \( \text{codim}_V(W + T_f) = |C_4| \) and so \( |B_4| = |C_4| \) by (iii). Therefore there exists a bijection \( \beta : B_4 \to C_4 \).

Now, we define a map \( g : B \to C \) by setting

\[
vg = \begin{cases} 
  v g_0 & \text{if } v \in B_1 \cup B_2, \\
  v \alpha & \text{if } v \in B_3, \\
  v \beta & \text{if } v \in B_4.
\end{cases}
\]

Observe that the map \( g \) is bijective and so the unique linear map \( \bar{g} : V \to V \) is bijective. Also,

\[
W \bar{g} = \langle B_1 \cup B_3 \rangle \bar{g} = \langle B_1 g_0 \cup B_3 \alpha \rangle = \langle C_1 \cup C_3 \rangle \subseteq W.
\]
Therefore \( g \in U(T(V, W)) \). We can also easily verify in a routine manner that \( fg = f \) and hence \( f \in \text{ureg}(T(V, W)) \).

If \( W = V \), the conditions (i) and (iii) of Theorem 5.6 trivially hold and so we have the following straightforward corollary of Theorem 5.6 which characterizes unit-regular elements in \( L(V) \).

**Corollary 5.7.** Let \( f \in L(V) \). Then \( f \in \text{ureg}(L(V)) \) if and only if nullity\((f) = \text{corank}(f) \).

We need the following lemma to give an alternative proof of Theorem 5.9 which can also be obtained by combining both [7, Proposition 1] and [15, Theorem 2].

**Lemma 5.8.** nullity\((f) = \text{corank}(f) \) for all \( f \in L(V) \) if and only if \( V \) is finite-dimensional.

**Proof.** Suppose first that nullity\((f) = \text{corank}(f) \) for all \( f \in L(V) \). On the contrary, let us assume that \( V \) is infinite-dimensional. Let \( v \in V \) be a nonzero vector. Then \( \dim(\langle v \rangle) = 1 \). Note that \( V = \langle v \rangle \oplus U \) for some complement \( U \) of \( \langle v \rangle \) in \( V \). It follows from [23, Theorem 1.14] that \( \dim(V) = \dim(U) \) and there exists an isomorphism \( \phi: V \to U \). Then \( \phi: V \to V \) is a monomorphism and so nullity\((\phi) = 0 \). But, it is immediate that \( \text{corank}(\phi) = 1 \) which is a contradiction. Hence \( V \) is finite-dimensional.

Conversely, suppose that \( V \) is finite-dimensional and let \( f \in L(V) \). Then \( \text{corank}(f) = \dim(V) - \text{rank}(f) \). It follows from the rank plus nullity theorem that nullity\((f) = \dim(V) - \text{rank}(f) = \text{corank}(f) \). Since \( f \) is arbitrary, the proof is complete. \( \square \)

From Corollary 5.7 and Lemma 5.8, we thus have the following.

**Theorem 5.9.** The semigroup \( L(V) \) is unit-regular if and only if \( V \) is finite-dimensional.

We need the following lemma to prove Theorem 5.11.

**Lemma 5.10.** \( R(f|_W) = W \cap R(f) \) for all \( f \in T(V, W) \) if and only if \( W \) is trivial.

**Proof.** Suppose that \( R(f|_W) = W \cap R(f) \) for all \( f \in T(V, W) \), and we show that \( W \) is trivial. On the contrary, let us assume that \( W \) is nontrivial. Then there exists a nontrivial subspace \( U \) of \( V \) such that \( V = W \oplus U \). Let \( B \) and \( C \) be bases for \( W \) and \( U \), respectively. Then \( B \cup C \) is a basis for \( V \). Let \( w \) and \( u \) be fixed elements of \( B \) and \( C \), respectively. Define a map \( f: B \cup C \to V \) by setting

\[
vf = \begin{cases} 
    w & \text{if } v = u, \\
    0 & \text{if } v \in B \cup (C \setminus \{u\}).
\end{cases}
\]

Note that \( B \bar{f} = Bf = \{0\} \). Therefore \( W \bar{f} = \{0\} \subseteq W \) and so \( \bar{f} \in T(V, W) \). We also observe that \( W \cap R(f) = \langle w \rangle \) and so \( R(\bar{f}|_W) \neq W \cap R(\bar{f}) \), a contradiction. Hence \( W \) is trivial.

The converse is immediate. \( \square \)

We can now easily obtain a new proof of the following known theorem (see [2, Theorem 11]).

**Theorem 5.11.** The restrictive semigroup \( T(V, W) \) is unit-regular if and only if
(i) $W$ is trivial;
(ii) $V$ is finite-dimensional.

Proof. Suppose first that $L(V, W)$ is unit-regular.

(i) Then by Theorem 5.6(i), we have $R(f|_W) = W \cap R(f)$ for all $f \in L(V, W)$ and so $W$ is trivial by Lemma 5.10.

(ii) From (i), it is clear that $L(V, W) = L(V)$. Then, by assumption, $L(V)$ is unit-regular and so $V$ is finite-dimensional by Theorem 5.9.

Conversely, suppose that the given two conditions hold. By (i), we have $L(V, W) = L(V)$. Then, by (ii) and [15, Theorem 2], $L(V, W)$ is factorizable and so $L(V, W)$ is unit-regular by [7, Proposition 1]. □

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