STRUCTURAL THEOREMS FOR QUASIASYMPTOTICS OF ULTRADISTRIBUTIONS

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ABSTRACT. We provide complete structural theorems for the so-called quasiasymptotic behavior of ultradistributions. As an application of these results, we obtain descriptions of quasiasymptotic properties of regularizations at the origin of ultradistributions and discuss connections with Gelfand-Shilov spaces.

1. Introduction

Several asymptotic notions play a fundamental role in the theory of generalized functions. The subject has been studied by several authors and applications have been elaborated in areas such as mathematical physics, Tauberian theorems for integral transforms, number theory, and differential equations. See the monographs [7, 16, 17, 24] for an overview of results and the articles [6, 18, 27] for recent contributions.

The purpose of this article is to present a detailed structural study of the so-called quasiasymptotics of ultradistributions. The concept of quasiasymptotic behavior for Schwartz distributions was introduced by Zavyalov in [25] and further developed by him, Drozhzhinov, and Vladimirov in connection with their powerful multidimensional Tauberian theory for Laplace transforms [24]. A key aspect in the understanding of this concept is its description via so-called structural theorems and complete results in that direction were achieved in [21, 23] (cf. [11, 17]). In [14] Pilipović and Stanković naturally extended the definition of quasiasymptotic behavior to the context of (non-quasianalytic) one-dimensional ultradistributions and studied its basic properties. We shall obtain here complete structural theorems for quasiasymptotics of ultradistributions that generalize their distributional counterparts. Our main goal is thus to characterize those ultradistributions having quasiasymptotic behavior as infinite sums of derivatives of functions satisfying classical pointwise asymptotic relations.

The paper is organized as follows. In Section 2 we explain some notions and tools that will play a role in our arguments. Section 3 studies the quasiasymptotic behavior at infinity. A key idea we apply here will be to connect the quasiasymptotic behavior with the so-called $S$-asymptotic behavior, for which structural theorems are available, via an exponential change of variables. The nature of the problem under consideration requires to split our treatment in two cases, depending on whether the degree of the quasiasymptotic behavior is a negative integer or not. We obtain in Section 4 structural theorems quasiasymptotic behavior at the origin, our technique there is based on a

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reduction to the results from Section 3 by means of a change of variables and then regularization. Our method also yields asymptotic properties of regularizations at the origin of ultradistributions having prescribed asymptotic properties, generalizing results for distributions from [22]. It is also worth mentioning that our approach here differs from the one employed in the literature to deal with Schwartz distributions, and in fact can be used to produce new proofs for the classical structural theorems for the quasiasymptotic behavior of distributions. We conclude the article by studying extensions of quasiasymptotics to Gelfand-Shilov spaces in Section 5.

2. Preliminaries

In this section we fix the notation and collect some background material on ultradistributions and their asymptotic behavior. In preparation for the next sections, we also recall below the Faà di Bruno formula for higher order derivatives of compositions of functions.

2.1. Weight sequences and ultradistributions. Let \( \{M_p\}_{p \in \mathbb{N}} \) be a weight sequence of positive numbers and set \( m_p := M_p/M_{p-1}, \quad p \geq 1 \). We shall make use of various of the following conditions for weight sequences:

\[
(M.1): \quad M_p^2 \leq M_{p-1}M_{p+1}, \quad p \geq 1; \\
(M.2): \quad M_{p+q} \leq AH^{p+q}M_pM_q, \quad p, q \in \mathbb{N}, \text{ for certain constants } A \text{ and } H \geq 1; \\
(M.2)': \quad M_{p+1} \leq AH^pM_p, \quad p \in \mathbb{N}, \text{ for certain constants } A \text{ and } H \geq 1; \\
(M.3): \quad \sum_{p=1}^{\infty} 1/m_p \leq Cq/m_q, \quad q \geq 1, \text{ for a certain constant } C; \\
(M.3)': \quad \sum_{p=1}^{\infty} 1/m_p < \infty.
\]

Note that (M.1) says that \( m_p \) is increasing. The meaning of all these conditions is very well explained in [12].

Throughout the rest of the article we shall fix a weight sequence \( M_p \) and always assume it satisfies (M.1), (M.2), and (M.3). Let \( \Omega \subseteq \mathbb{R} \) be open and let \( K \subseteq \Omega \) be a compact subset. The space \( \mathcal{E}^{(M_p),h}(K) \) stands for the space of all \( \phi \in C^\infty(\Omega) \) that satisfy \( \|\phi\|_{K,h} = \sup_{k \in \mathbb{N},x \in K} |\phi^{(k)}(x)|/h^kM_k < \infty \), and \( \mathcal{D}^{(M_p)} \) stands for the subspace consisting of elements supported by \( K \). Next, we naturally define [12] the following locally convex spaces of ultradifferentiable functions

\[
\mathcal{E}^{(M_p)}(\Omega) = \lim_{K \in \Omega} \lim_{h \to 0} \mathcal{E}^{(M_p),h}(K) \\
\mathcal{D}^{(M_p)}(\Omega) = \lim_{K \in \Omega} \lim_{h \to 0} \mathcal{D}^{(M_p),h}(K) \\
\mathcal{D}^{(M_p),h}(K) = \mathcal{E}^{(M_p),h}(K), \\
\mathcal{D}^{(M_p)}(\Omega) = \lim_{K \in \Omega} \mathcal{D}^{(M_p),h}(K).
\]

As customary [12], the common notation for \( (M_p) \) and \( \{M_p\} \) will be \( * \), and in statements needing a separate treatment we will always talk first about the Beurling case, followed by the assertion for the Roumieu case in parenthesis. When \( \Omega = \mathbb{R} \), we simply write \( \mathcal{D}^* = \mathcal{D}^*(\mathbb{R}) \) and \( \mathcal{E}^* = \mathcal{E}^*(\mathbb{R}) \). The strong duals \( \mathcal{D}''(\Omega) \) and \( \mathcal{E}''(\Omega) \) are the spaces of ultradistributions and compactly supported ultradistributions, respectively, on \( \Omega \).
A \(*\)-ultradifferential operator is an infinite order differential operator

\[ P(D) = \sum_{m=0}^{\infty} a_m D^m, \quad a_m \in \mathbb{C}, \]

where the coefficients satisfy \(|a_m| \leq C\mu^m/M_m\) for some \(\mu > 0\) and \(C > 0\) (for every \(\mu > 0\) there is a \(C = C_\mu > 0\)). It is ensured by condition \((M.2)\) that \(P(D)\) acts continuously on \(\mathcal{D}^*(\Omega)\) and \(\mathcal{E}^*(\Omega)\) and hence, by duality, on the ultradistribution spaces \(\mathcal{D}^{*\prime}(\Omega)\) and \(\mathcal{E}^{*\prime}(\Omega)\).

### 2.2. Asymptotic behavior of ultradistributions.

The main subject of study of this article is the quasiasymptotic behavior of ultradistributions, which is defined via asymptotic comparison with regularly varying functions.

A real-valued measurable function \(L\) defined in some interval of the form \([x_0, \infty), x_0 > 0\), is called \(slowly\ varying\ at\ infinity\) if \(L\) is positive for large arguments and

\[ \lim_{x \to \infty} \frac{L(ax)}{L(x)} = 1, \]

for any \(a > 0\). Standard references for these types of functions are [1, 20]. We are only interested in the behavior of \(L\) for large arguments, so because of [20, Lemma 1.2] we may always assume \(L\) to be defined, positive, and locally bounded (or even continuous) on \([0, \infty)\). Applying [1, Theorem 1.5.4], we obtain the following useful property, known as Potter’s estimates: for any \(\gamma > 0\) there exists a \(C_\gamma > 0\) such that

\[ L(\lambda x) \leq C_\gamma \max\{x^{-\gamma}, x^{\gamma}\} \]

holds for any \(\lambda, x > 0\). Finally, we say that a function \(L\) on \((0, \infty)\) is \(slowly\ varying\ at\ the\ origin\) if \(\tilde{L}(x) := L(x^{-1})\) is slowly varying at infinity. Regularly varying functions of index \(\alpha\) are those of the form \(x^\alpha L(x)\), where \(L\) is a slowly varying function.

In accordance to [14, 17], we define the quasiasymptotic behavior of an ultradistribution at infinity or at the origin as follows.

**Definition 2.1.** Let \(L\) be a slowly varying function at infinity (at the origin, resp.). We say that \(f \in \mathcal{D}^{*\prime}\) has **quasiasymptotic behavior at infinity (at the origin)** in \(\mathcal{D}^{*\prime}\) **with respect to** \(\lambda^\alpha L(\lambda)\), \(\alpha \in \mathbb{R}\), if for some \(g \in \mathcal{D}^{*\prime}\) and every \(\phi \in \mathcal{D}^*\),

\[ \lim_{\lambda \to \infty} \left\langle \frac{f(\lambda x)}{\lambda^\alpha L(\lambda)}, \phi(x) \right\rangle = \left\langle g(x), \phi(x) \right\rangle \quad (\text{resp.,} \lim_{\lambda \to 0^+}) \]

(2.2)

If (2.2) holds, we also say that \(f\) has **quasiasymptotics of order \(\alpha\) at infinity (at the origin)** with respect to \(L\) and write in short: \(f(\lambda x) \sim \lambda^\alpha L(\lambda)g(x)\) in \(\mathcal{D}^{*\prime}\) as \(\lambda \to \infty\) (resp., \(\lambda \to 0^+\)).

We shall often employ the parameter \(\varepsilon \to 0^+\) instead of \(\lambda \to 0^+\) when treating separately the quasiasymptotic behavior at the origin in order to make a notational distinction.
Remark 2.2. Although we have defined quasiasymptotic behavior over $\mathcal{D}^*$, it is not necessary to restrict ourselves to this specific space of test functions. Thus, as in [17], given a barreled topological vector space $\mathcal{F}$ of test functions with continuous action of dilations, we will say that $f \in \mathcal{F}'$ has quasiasymptotic behavior at infinity (at the origin) in $\mathcal{F}'$ if (2.2) holds for any $\phi \in \mathcal{F}$. Some restrictions do impose themselves however, as follows from the following observation.

If $f(\lambda x) \sim \lambda^\alpha L(\lambda)g(x)$ in $\mathcal{D}''$ as $\lambda \to \infty$ (as $\lambda \to 0+$), it can easily be shown [7, 17] that this forces $g$ to be a homogeneous ultradistribution of degree $\alpha$. An adaptation of the proof of [7, Theorem 2.6.1] shows that all homogeneous ultradistributions are exactly the homogeneous distributions.

Proposition 2.3. Let $g \in \mathcal{D}''$ be a homogeneous ultradistribution of degree $\alpha$. If $\alpha \neq -1, -2, -3, \ldots$, there exist constants $c_+$ and $c_-$ such that

\[ g(x) = c_+ x_+^\alpha + c_- x_-^\alpha. \]

If $\alpha = -n$, with $n \in \mathbb{Z}_+$, then there are constants $c_1$ and $c_2$ such that

\[ g(x) = c_1 x^{-n} + c_2 \delta^{(n-1)}(x). \]

Proof. The proof of this proposition only makes use of the conditions $(M.1)$, $(M.2)'$, and $(M.3)'$ for the weight sequence. Suppose that $g(\lambda x) = \lambda^\alpha g(x)$ for all $\lambda > 0$, then one verifies that

\[ xg'(x) = \alpha g(x). \]

This differential equation can be solved locally on $\mathbb{R} \setminus \{0\}$, so that $g$ takes the form

\[ g(x) = c_+ x_+^\alpha + c_- x_-^\alpha + f(x) \]

if $\alpha \notin \mathbb{Z}_-$, or

\[ g(x) = c_1 x^\alpha + f(x) \]

if $\alpha \in \mathbb{Z}_-$, where $f \in \mathcal{D}''$ is homogeneous of degree $\alpha$ with support in $\{0\}$. Then, the Fourier-Laplace transform $\hat{f}$ is an entire function of exponential type 0, homogeneous of degree $-\alpha - 1$. Since homogeneous entire functions are polynomials, it follows that

\[ \hat{f} = 0 \text{ if } \alpha \notin \mathbb{Z}_- \text{ or } \hat{f}(\xi) = (-i)^{-\alpha-1} c_2 \xi^{-\alpha-1} \text{ for some constant } c_2 \text{ if } \alpha \in \mathbb{Z}_-. \]

As $\{e^{ix}: x \in \mathbb{R}\}$ is a dense subspace of $\mathcal{E}^*$ (cf. [12, Theorem 7.3, p. 75]), it follows that $f = 0$ if $\alpha \notin \mathbb{Z}_-$ or $f = c_2 \delta^{(-\alpha-1)}$ if $\alpha \in \mathbb{Z}_-$.

We mention that we will employ the notation $H(x) = x_0^+$ for the Heaviside function, i.e., the characteristic function of $(0, \infty)$. In addition, we shall make use of the special (non homogeneous!) distributions $\text{Pf}(H(\pm) x^{-k})$, $k \in \mathbb{Z}_+$, where Pf stands for Hadamard finite part regularization [4].

Closely related to the quasiasymptotics is the S-asymptotic behavior, which takes into account translation rather than dilation. It is defined in the following way.

Definition 2.4. Let $c$ be a positive real-valued function defined on $[0, \infty)$. An ultradistribution $f \in \mathcal{D}''$ has $S$-asymptotic behavior with respect to $c$ if there exists a $g \in \mathcal{D}''$ such that for any $\phi \in \mathcal{D}^*$,

\[ \lim_{h \to \infty} \left\langle \frac{f(x+h)}{c(h)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle. \]
We write in short: \( f(x + h) \sim c(h)g(x) \) in \( \mathcal{D}^\ast \) as \( h \to \infty \).

The S-asymptotic behavior of generalized functions is an extensively studied and well understood concept, for which many results in ultradistributional spaces are at hand. For a thorough overview, we refer the reader to [17]. Concerning our purposes in this article, we will make use of the following result, originally shown in [15].

**Theorem 2.5** ([17, Theorem 1.10]). Suppose \( M_p \) satisfies (M.1), (M.2) and (M.3). Let \( f \in \mathcal{D}^\ast \). Then \( f \) has S-asymptotics related to \( c \) on \([0, \infty)\) if and only if for a given \( a > 0 \) there exist an ultradifferential operator \( P(D) \) of class * and continuous functions \( f_1 \) and \( f_2 \) on \([-a, \infty)\) such that

\[
\lim_{h \to \infty} \frac{f_i(x + h)}{c(h)}, \quad i = 1, 2,
\]

exist uniformly in \( x \in (-a, a) \), \( i = 1, 2 \), and \( f = P(D)f_1 + f_2 \) on \((-a, \infty)\).

2.3. Faà di Bruno formula. As we shall often be working with substitutions, some of our manipulations will involve derivatives of compositions and an explicit formula will be needed. Therefore, we will employ the well-known Faà di Bruno formula [10], which makes use of the Bell polynomials. For any \( n, k \in \mathbb{N} \), \( 0 \leq k \leq n \), these are defined as

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! \cdots j_{n-k+1}!} \prod_{i=1}^{n-k+1} \left( \frac{t_i}{i!} \right)^{j_i}, \tag{2.4}
\]

where the sum is taken over all finite sequences \( j_1, \ldots, j_{n-k+1} \) of non-negative integers that satisfy

\[
j_1 + \cdots + j_{n-k+1} = k, \quad 1 \cdot j_1 + \cdots (n-k+1) \cdot j_{n-k+1} = n.
\]

For any two \( f, g \in C^\infty \) and any \( n \in \mathbb{N} \), the Faà di Bruno formula is then,

\[
\frac{d^n}{dx^n} (f \circ g)(x) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k} \left( g^{(1)}(x), \ldots, g^{(n-k+1)}(x) \right). \tag{2.5}
\]

For any sequence \( \{x_j\}_{j \in \mathbb{N}} \) and \( k \in \mathbb{N} \), the generating function of the Bell polynomial \( B_{n,k} \) is

\[
\frac{1}{k!} \left( \sum_{j=1}^{\infty} x_j t^j \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, \ldots, x_{n-k+1}) \frac{t^n}{n!}. \tag{2.6}
\]

An important example is that of the **Stirling numbers of the second kind** (see e.g. [8]), given by

\[
S(n, k) := B_{n,k}(1, \ldots, 1), \tag{2.7}
\]

and note that \( S(0, 0) = 1. \)
3. The structure of quasiasymptotics at infinity

This section is devoted to studying the quasiasymptotic behavior at infinity. Our main results are Theorem 3.6 and Theorem 3.7 where we provide a full description of the structure of quasiasymptotics at infinity. Some auxiliary lemmas used in their proofs are shown in Subsection 3.1. Throughout this section $L$ stands for a slowly varying function at infinity.

3.1. Some lemmas. We start with the ensuing useful estimates for the weight sequence $M_p$, which we shall often exploit throughout the article.

Lemma 3.1. For any $\ell > 0$ there is $C_\ell > 0$ (independent of $p$) such that

\[ \sum_{k=p}^{\infty} \frac{k! \ell^k}{M_k} \leq C_\ell \frac{p! L^p}{M_p} \tag{3.1} \]

and

\[ \sum_{k=p}^{\infty} S(k+1, p+1) \frac{\ell^k}{M_k} \leq C_\ell \frac{(2\ell)^p}{M_p}. \tag{3.2} \]

Proof. The proof of this lemma only requires the weight sequence to satisfy (M.1) and

\[ \lim_{p \to \infty} \frac{m_p}{p} = \infty. \tag{3.3} \]

(Notice that (3.3) certainly holds if (M.1) and (M.3)' are satisfied [12, Lemma 4.1, p. 55].)

Clearly, it is enough to show (3.1) just for sufficiently large $p$. Using (3.3), there is $p_0$ such that for any $p \geq p_0$ we have $p/m_p \leq (2\ell)^{-1}$. Hence, it follows that for $p$ in this range

\[
\sum_{k=p}^{\infty} \frac{k! \ell^k}{M_k} = \frac{p!}{M_p} \left( \ell^p + \sum_{k=p+1}^{\infty} \frac{(p+1) \cdot \ldots \cdot k \cdot \ell^k}{m_{p+1} \cdot \ldots \cdot m_k} \right) \\
\leq \frac{p!}{M_p} \left( \ell^p + \sum_{k=p+1}^{\infty} \frac{\ell^k}{(2\ell)^{k-p}} \right) \leq 2 \frac{p! \ell^p}{M_p}.
\]

For (3.2), we start by bounding the Stirling numbers. For such numbers [19, Theorem 3] we have the estimates

\[ S(k, p) \leq \binom{k}{p} p^{k-p}, \quad 1 \leq p \leq k. \]

Using the trivial upper bound $2^k$ for $\binom{k}{p}$, we get

\[ S(k+1, p+1) \leq 2^{k+1} (p+1)^{k-p} \leq 2^{k+1} \frac{k!}{p!} \]
for $k \geq p$. The rest follows by application of (3.1),

$$\sum_{k=p}^{\infty} S(k+1, p+1) \frac{f^k}{M_k} \leq \frac{1}{p!} \sum_{k=p}^{\infty} k!(2f)^k \leq 2C_{2f} \frac{(2f)^p}{M_p}.$$ 

□

In [21], the structure of distributional quasiasymptotics at infinity was found by noting that certain primitives preserve the asymptotic behavior, being of a higher degree, and using the fact that eventually the primitives are continuous functions. As the latter part does not hold in general for ultradistributions, a more careful analysis is needed, although we may carry over some of the distributional results.

Lemma 3.2. Let $f \in D^\ast$. Suppose $f$ has quasiasymptotics with respect to $\lambda^\alpha L(\lambda)$, $\alpha \in \mathbb{R}$.

(i) If $\alpha \notin \mathbb{Z}_-$: for any $n \in \mathbb{N}$ and any $n$-primitive $F_n$ of $f$ there exists a polynomial $P$ of degree at most $n - 1$ such that $F_n + P$ has quasiasymptotics with respect to $\lambda^{\alpha+n} L(\lambda)$ in $D^\ast$.

(ii) If $\alpha = -k$, $k \in \mathbb{Z}_+$: there is some $(k-1)$-primitive $F$ of $f$ such that $F$ has quasiasymptotics with respect to $\lambda^{-1} L(\lambda)$ in $D^\ast$.

Proof. One may retread the proofs of [21, Section 2] (see also [17, Section 2.10]), keeping in mind Proposition 2.3 to conclude that the results also hold for ultradistributions. (The conditions needed here on $M_p$ are (M.1), (M.2)', and (M.3)'). □

The previous lemma roughly speaking shows that in order to find the structure of quasiasymptotics for arbitrary degree, it suffices to discover the structure for degrees $\geq -1$, where extra care is needed for the case $-1$. It should also be noticed that the converse statements for (i) and (ii) from Lemma 3.2 trivially hold true.

As ultradistributions with compact support may be evaluated at any Taylor approximation of the test functions, it follows that they are bounded by some negative integral degree at infinity. In fact, they satisfy the moment asymptotic expansion [7]. It is worth mentioning that the constants $\mu_n$ occurring in the next lemma are the moments of the ultradistribution, that is, $\mu_n = \langle f(x), x^n \rangle$.

Lemma 3.3. For any $f \in E^\ast$ and any $N \in \mathbb{N}$, there exist constants $\mu_0, \ldots, \mu_{N-1}$ such that

$$\langle f(\lambda x), \phi(x) \rangle = \sum_{n=0}^{N-1} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n! \lambda^{n+1}} + O(\lambda^{-(N+1)})$$

holds in $E^\ast$ as $\lambda \to \infty$.

Proof. Analogously as in [7, Theorem 3.3.1]. (The assumptions need here on the weight sequence are (M.1), (M.2)', and (M.3)'). □

Consequently, the quasiasymptotic behavior of degree $> -1$ is a local property at infinity, which in some arguments enables us to remove the origin from the support of the ultradistribution in our analysis.
Corollary 3.4. Suppose that $f_1, f_2 \in \mathcal{D}^*_{\alpha}$ and that for some $a > 0$, $f_1$ and $f_2$ coincide on $\mathbb{R} \setminus [-a, a]$. Suppose that $f_1(\lambda x) \sim \lambda^\alpha L(\lambda)g(x)$ in $\mathcal{D}^*_{\alpha}$ as $\lambda \to \infty$, where $\alpha > -1$. Then, also $f_2(\lambda x) \sim \lambda^\alpha L(\lambda)g(x)$ in $\mathcal{D}^*_{\alpha}$.

3.2. Structural theorem for $\alpha \notin \mathbb{Z}_-$. We study in this subsection quasiasymptotics of degree $\alpha \notin \mathbb{Z}_-$. Part of our analysis reduces the general case to that when $\alpha > -1$, i.e., the case when the quasiasymptotic behavior is local. Consequently, we may restrict our discussion to those ultradistributions whose support lie in the complement of some zero neighborhood. As both the negative and positive half-line can be treated symmetrically, it is natural to start the analysis with ultradistributions that are supported on the positive half-line. In the next crucial lemma we further normalize the situation by assuming that our ultradistribution is supported in $(e, \infty)$.

Lemma 3.5. Let $\alpha \in \mathbb{R}$ and let $f \in \mathcal{D}^*$ be such that $\text{supp } f \subset (e, \infty)$ and $f$ has quasiasymptotic behavior at infinity with respect to $\lambda^\alpha L(\lambda)$ in $\mathcal{D}^*(0, \infty)$. Then, there are continuous functions $f_m$ such that $\text{supp } f_m \subset (e, \infty)$, $f = \sum_{m=0}^{\infty} f_m$, the limits

$$\lim_{x \to \infty} \frac{f_m(x)}{x^{\alpha+m}L(x)}$$

exist, and furthermore, for some $\ell > 0$ (any $\ell > 0$) there is a $C = C_\ell > 0$ such that,

$$|f_m(x)| \leq C\frac{x^\ell}{M_m} x^{\alpha+m}L(x), \quad m \in \mathbb{N}, \quad x > 0.$$

Proof. Suppose $f(\lambda x) \sim \lambda^\alpha L(\lambda)g(x)$ in $\mathcal{D}^*(0, \infty)$ as $\lambda \to \infty$. Since composition with a real analytic function induces continuous mappings between spaces of ultradifferentiable functions (see e.g. [9, Prop. 8.4.1, p. 281]), we obtain that the composition $f(e^x)$ is an element of $\mathcal{D}^*$. Also, $\psi \in \mathcal{D}^*$ if and only if $\psi(x) = \varphi(e^x)$ with $\varphi \in \mathcal{D}^*(0, \infty)$.

These key observations allow us to make a change of variables in order to apply Theorem 2.5. In fact, we set $u(x) := f(e^x)$, $w(x) := g(e^x)$ and $c(h) := e^{\varphi(h)}$. (Notice that $w$ has actually the form $w(x) = Be^\alpha x$ for some $B > 0$.) A quick computation shows that

$$u(x + h) \sim c(h)w(x) \quad \text{in } \mathcal{D}^* \text{ as } h \to \infty.$$

Taking into account the well known uniform convergence theorem for regularly varying functions [10, 20], we then obtain continuous functions $u_1$ and $u_2$ such that

$$\lim_{h \to +\infty} \frac{u_i(x + h)}{c(h)}, \quad i = 1, 2,$$

exist uniformly for $x$ in compacts of $(0, \infty)$ and for an ultradifferential operator $P(D)$ of class $*$,

$$u = P(D)u_1 + u_2,$$

on $(0, \infty)$. Inspection in the proof of Theorem 2.5 given in [17] shows that we may let the supports of the $u_i$ lie in $(1, \infty)$ since $\text{supp } u \subset (1, \infty)$. 

Take any $\varphi \in D^\ast(0, \infty)$ and put $\psi(x) = e^x \varphi(e^x)$, then the substitution $y = e^x$ yields

$$\langle f(y), \varphi(y) \rangle = \left\langle f(y), \frac{\varphi(y)}{y} \right\rangle = \langle u(x), \psi(x) \rangle = \langle P(D)u_1(x), \psi(x) \rangle + \langle u_2(x), \psi(x) \rangle.$$

Let us consider both terms of the sum individually. The latter is simply

$$\langle u_2(x), \psi(x) \rangle = \int_1^\infty u_2(x)\psi(x)dx = \int_\infty^\infty u_2(\log y)\varphi(y)dy.$$

So, by setting $f_2(y) := u_2(\log y)$, we get

$$\langle u_2(x), \psi(x) \rangle = \langle f_2(y), \varphi(y) \rangle,$$

and the existence of

$$\lim_{y \to \infty} \frac{f_2(y)}{y^\alpha L(y)}.$$

For the first term, we will need to explicitly calculate the derivatives of $\psi$. Using the Faà di Bruno formula (2.5) and (2.7), one readily verifies that

$$(3.4) \quad \frac{d^k}{dx^k}(\varphi(e^x)) = \sum_{m=0}^{k} \varphi^{(m)}(e^x) e^{mx} S(k, m).$$

for any $k \in \mathbb{N}$. Consequently, (3.4) yields for each $n \in \mathbb{N}$,

$$\psi^{(n)}(x) = e^x \sum_{k=0}^{n} \binom{n}{k} \frac{d^k}{dx^k}(\varphi(e^x)) = e^x \sum_{m=0}^{n} \varphi^{(m)}(e^x) e^{mx} \sum_{k=m}^{n} \binom{n}{k} S(k, m)$$

$$= e^x \sum_{m=0}^{n} S(n+1, m+1) e^{mx} \varphi^{(m)}(e^x),$$

where we have applied [3, Theorem 5.3.B],

$$S(n+1, m+1) = \sum_{k=m}^{n} \binom{n}{k} S(k, m).$$

Then,

$$\int_1^\infty u_1(x)\psi^{(n)}(x)dx = \sum_{m=0}^{n} S(n+1, m+1) \int_\infty^\infty u_1(\log y)\varphi^{(m)}(y)y^m dy.$$

If $P(D) = \sum_{n=0}^{\infty} a_n D^n$, then by (3.2) from Lemma 3.1, we may consider the following constants,

$$c_m = (-1)^m \sum_{n=m}^{\infty} (-1)^n a_n S(n+1, m+1),$$

and it follows that $c_m \leq C\mu^m/M_m$ for some $\mu > 0$ (for any $\mu > 0$) and some $C_\mu = C > 0$. Collecting everything together, we obtain

$$\langle P(D)u_1(x), \psi(x) \rangle = \sum_{m=0}^{\infty} (-1)^m c_m \int_\infty^\infty u_1(\log y)\varphi^{(m)}(y)y^m dy.$$
So if we define $f_{1,m}(y) := u_1(\log y)y^m$, $m \in \mathbb{N}$, we get

\begin{equation}
\langle P(D)u_1(x), \psi(x) \rangle = \sum_{m=0}^{\infty} c_m \langle f_{1,m}^{(m)}(y), \varphi(y) \rangle,
\end{equation}

and the limits

\begin{equation}
\lim_{y \to \infty} \frac{f_{1,m}(y)}{y^{\alpha+m}L(y)}
\end{equation}
exist. This completes the proof of the lemma. 

\[\Box\]

We are ready to discuss the general case.

**Theorem 3.6.** Suppose $\alpha \notin \mathbb{Z}_-$ and let $k \in \mathbb{N}$ be the smallest non-negative integer such that $-(k+1) < \alpha$. Then, an ultradistribution $f \in \mathcal{D}^\ast$ has quasiasymptotic behavior

\begin{equation}
f(\lambda x) \sim \lambda^\alpha L(\lambda)(c_-x_\alpha^- + c_+x_\alpha^+) \quad \text{in } \mathcal{D}^\ast \quad \text{as } \lambda \to \infty
\end{equation}

if and only if there exist continuous functions $f_m$ on $\mathbb{R}$, $m \geq k$, such that

\begin{equation}f = \sum_{m=k}^{\infty} f_m^{(m)},\end{equation}

the limits

\begin{equation}\lim_{x \to \pm \infty} \frac{f_m(x)}{x^m|x|^\alpha L(|x|)} = c_m^\pm, \quad m \geq k,
\end{equation}
exist, and for some $\ell > 0$ (any $\ell > 0$) there is a $C = C_\ell > 0$ such that

\begin{equation}|f_m(x)| \leq \begin{cases}
C \ell^m \frac{M_m}{M_m}, & |x| < 1, \\
C \ell^m |x|^\alpha M_m |x|^{\alpha+m} L(|x|), & |x| \geq 1,
\end{cases}
\end{equation}

for all $m \geq k$. Furthermore, in this case we have

\begin{equation}c_\pm = \sum_{m=k}^{\infty} c_m^\pm \frac{\Gamma(\alpha + m + 1)}{\Gamma(\alpha + 1)}.
\end{equation}

**Proof.** In view of Lemma 3.2(i), we may assume that $\alpha > -1$ so that $k = 0$.

Suppose then first that $f$ has quasiasymptotic behavior (3.7). We write $f = f_- + f_c + f_+$, where $f_c \in \mathcal{E}^\ast$ coincides with $f$ on an open interval containing $[-e, e]$ and supp $f_- \subset (-\infty, -e)$ and supp $f_+ \subset (e, \infty)$. Then, by Corollary 3.4 each $f_\pm$ has quasiasymptotic behavior with respect to $\lambda^\alpha L(\lambda)$ in $\mathcal{D}^\ast(-\infty, 0)$ and $\mathcal{D}^\ast(0, \infty)$, respectively. Using Lemma 3.5 we find continuous functions $f_{1,m}^{\pm}$, $m \in \mathbb{N}$ with supports in $(-\infty, -e)$ and $(e, \infty)$, respectively, such that the identity

\begin{equation}f_\pm = \sum_{m=0}^{\infty} \frac{d^m}{dx^m} f_{1,m}^{\pm},\end{equation}
holds, the limits
\[ c_m^\pm = (-1)^m \lim_{x \to \pm \infty} \frac{f_{1,m}^\pm(x)}{x^{\alpha + mL(x)}} \]
extist, and the bounds \(|f_{1,m}^\pm(x)| \leq C' \ell^m |x|^{\alpha + mL(|x|)} / M_m\) are satisfied for some \(\ell > 0\) (any \(\ell > 0\)) and some \(C' = C'_\ell > 0\). Applying Komatsu’s first structural theorem for ultradistributions [12] Theorem 8.1 and Theorem 8.7] (which hold under\(^1\) (M.1), (M.2)', and (M.3)') one can also find continuous functions \(g_m\), whose supports lie in some (arbitrarily chosen) neighborhood of \(\text{supp } f\), \(R > 0\) and suppose that for some \(\alpha > 0\) has for any \(m\) such that for all \(\phi \in D\) and \(\alpha \in \mathbb{R}\) given by (3.11). As we indicated before, we may assume that the inequality (2.1) holds holds true in the quasianalytic case under milder conditions, \(\alpha > 0\). Actually, the first structural theorem holds true in the quasianalytic case under milder conditions, 

\[ \lim_{\lambda \to \infty} \left< \frac{f(\lambda x)}{x^{\alpha L(\lambda)}}, \phi(x) \right> = \lim_{\lambda \to \infty} \sum_{m=0}^{\infty} (-1)^m \int_{-\infty}^{\infty} \frac{f_m(\lambda x)}{x^{\alpha + mL(\lambda)}} \phi^{(m)}(x) dx \]

\[ = \sum_{m=0}^{\infty} (-1)^m \int_{-\infty}^{\infty} (c_m x_+^\alpha + c_m^+ x_+^\alpha) x^m \phi^{(m)}(x) dx \]

\[ = c_- \int_{-\infty}^{0} x^\alpha \phi(x) dx + c_+ \int_{0}^{\infty} x^\alpha \phi(x) dx, \]

with \(c_-\) and \(c_+\) given by (3.11). □

\(^1\)Actually, the first structural theorem holds true in the quasianalytic case under milder conditions, see [4] Proposition 4.1 and Proposition 4.7].
3.3. Structural Theorem for negative integral degree. We now address the case of quasiasymptotics of degree $\alpha \in \mathbb{Z}_-$. The next structural theorem is the second main result of this section.

**Theorem 3.7.** Let $k \in \mathbb{Z}_+$ and $f \in \mathcal{D}^\ast$. Then, $f$ has the quasiasymptotic behavior

$$f(\lambda x) \sim \frac{L(\lambda)}{\lambda^k} (\gamma \delta^{(k-1)}(x) + \beta x^{-k})$$

in $\mathcal{D}^\ast$ as $\lambda \to \infty$ if and only if there exist continuous functions $f_m$ on $\mathbb{R}$, $m \geq k - 1$, such that

$$f = \sum_{m=k-1}^{\infty} f_m(m),$$

the limits

$$\lim_{x \to \pm \infty} \frac{f_m(x)}{x^{m-k}L(|x|)} = c_m^\pm, \quad m \geq k - 1,$$

and

$$\lim_{x \to \infty} \frac{1}{L(x)} \int_{-x}^{x} f_{k-1}(t)dt = c_{k-1}^k$$

exist, and for some $\ell > 0$ (any $\ell > 0$) there is $C = C_\ell > 0$ such that

$$|f_m(x)| \leq \begin{cases} C \frac{\ell^m}{M_m}, & |x| < 1, \\ C \frac{\ell^m}{M_m} |x|^\alpha L(|x|), & |x| \geq 1, \end{cases}$$

for all $m \geq k - 1$. Furthermore, we must have

$$\gamma = c_{k-1}^k + \sum_{m=k}^{\infty} (c_m^+ - c_m^-) \quad \text{and} \quad \beta = (-1)^{k-1}(k-1)!c_{k-1}^k = (-1)^{k-1}(k-1)!c_{k-1}^-.$$

**Proof.** In view of Lemma 3.2(ii) we may assume that $k = 1$.

**Necessity.** We start showing the necessity of the conditions if $f$ has the quasiasymptotic behavior (3.12). Our strategy consists of modifying the quasiasymptotics to one of order 0 by multiplying $f$ by $x$, applying Lemma 3.5, and then studying the structure it imposes on $f$. Take a compactly supported ultradistribution $f_e$ that coincides with $f$ on $[-e,e]$ and consider $\tilde{f} = f - f_e$, so that $\text{supp}(f - f_e) \cap [-e,e] = \emptyset$. We set $g(x) = x(f(x) - f_e(x))$, which, in view of Lemma 3.3, has quasiasymptotic behavior

$$g(\lambda x) \sim \beta L(\lambda)$$

in $\mathcal{D}^\ast$ as $\lambda \to \infty$.

Splitting $g$ as the sum of two distributions supported on $(-\infty, -e)$ and $(e, \infty)$ respectively, we can apply Lemma 3.5 to obtain its structure as

$$g = \sum_{m=0}^{\infty} g_m^{(m)},$$
where each of the functions has support in \((-\infty,-e) \cup (e,\infty)\), satisfies the corresponding bounds implied by the lemma, and is such that the limits \(\lim_{x \to \pm \infty} x^{-m} g_m(x) / L(|x|)\) exist. Define, for any \(j \in \mathbb{N}\), the following continuous functions

\[
\tilde{f}_j(x) = \begin{cases} 
0, & x = 0, \\
\frac{x^{j-1}}{j!} \sum_{m=j}^{\infty} m! g_m(x) x^{-m}, & x \neq 0.
\end{cases}
\]

Let us verify they satisfy the requirements that the \(f_j\) should satisfy. First of all, for some \(\ell > 0\) (any \(\ell > 0\)) and \(C = C_\ell > 0\),

\[
\left| \tilde{f}_j(x) \right| \leq C \frac{|x|^{j-1}}{j!} L(|x|) \sum_{m=j}^{\infty} \frac{m! \ell^m}{M_m} \leq C' |x|^{j-1} \frac{\ell^j}{M_j} L(|x|),
\]

by (3.1) from Lemma 3.1. This not only shows that each \(\tilde{f}_j\) is well-defined and continuous on \(\mathbb{R}\), but also provides the bounds (3.16) for them. From dominated convergence we infer the existence of

\[
\lim_{x \to \pm \infty} \tilde{f}_j(x) = \lim_{x \to \pm \infty} \frac{1}{j!} \sum_{m=j}^{\infty} m! g_m(x) x^{-m} L(|x|) = \frac{1}{j!} \sum_{m=j}^{\infty} \lim_{x \to \pm \infty} m! g_m(x).
\]

Take an arbitrary \(\phi \in D^*\) and let \(\varphi \in D^*\) be another corresponding test function that coincides with \(\tilde{\phi}\) on \(\mathbb{R} \setminus (-e,e)\), while its support not containing the origin. We then have

\[
\left\langle \tilde{f}(x), \phi(x) \right\rangle = \left\langle g(x), \frac{\varphi(x)}{x} \right\rangle = \sum_{m=0}^{\infty} \sum_{j=0}^{m} (-1)^m \binom{m}{j} \left( \left\langle g_m(x), (-1)^{m-j}(m-j)! \frac{\varphi^{(j)}(x)}{x^{m-j+1}} \right\rangle \right)
\]

\[
= \sum_{j=0}^{\infty} \frac{(-1)^j j!}{j!} \sum_{m=j}^{\infty} \left\langle m! x^{j-1} g_m(x), \varphi^{(j)}(x) \right\rangle
\]

\[
= \sum_{j=0}^{\infty} \left\langle \tilde{f}_j^{(j)}, \varphi(x) \right\rangle.
\]

Applying the first structural theorem to \(f_c\) as in the proof of Theorem 3.6, we obtain compactly supported continuous functions \(g_m\) such that \(f_m = \tilde{f}_m + g_m\) satisfy (3.13), (3.14), and (3.16). The necessity of (3.15) follows from (3.18) below. That (3.17) must necessarily hold will also be shown below in the proof of the converse.

**Sufficiency.** Conversely, assume that (3.13) holds with \(f_m\) fulfilling (3.14), (3.15) and (3.16) (recall we work with the reduction \(k = 1\)). We assume without loss of generality that \(L(x)\) is everywhere continuous and vanishes for \(x \leq 1\). We consider

\[
g = \sum_{m=1}^{\infty} f_m^{(m-1)}.
\]
It follows from Theorem 3.6 that \( g \) has quasiasymptotic behavior of degree 0 with respect to \( L(\lambda) \), and differentiation then yields 
\[
f(\lambda x) - f_0(\lambda x) = g'(\lambda x) \sim (\gamma - c_0^+ L(\lambda)) \delta(x) \quad \text{in } \mathcal{D}'' \quad \text{as } \lambda \to \infty,
\]
with \( \gamma \) precisely given as in (3.17). It thus remains to determine the quasiasymptotic properties of \( f_0 \). Write \( F(x) = \int_0^x f_0(t) dt \). Since \( f_0(\pm x) \sim \pm c_0^+ L(x)/x, \ x \to \infty, \) one readily shows that 
\[
F(\lambda x)H(\pm x) = F(\pm \lambda)H(\pm x) + c_0^\pm \int_\lambda^{\pm \lambda x} L(t) t dt + o(L(\lambda))
\]
\[
= F(\pm \lambda)H(\pm x) + c_0^\pm L(\lambda)H(\pm x) \log |x| + o(L(\lambda)), \quad \lambda \to \infty,
\]
uniformly for \( x \) on compact intervals, and in particular the relation holds in \( \mathcal{D}'' \). Differentiating 
\[
F(\lambda x) = F(-\lambda)H(-x) + F(\lambda)H(x) + L(\lambda) \left( c_0^- H(-x) + c_0^+ H(x) \right) \log |x| + o(L(\lambda)),
\]
we conclude that 
\[
(3.18) \quad f_0(\lambda x) = \frac{F(\lambda) - F(-\lambda)}{\lambda} \delta(x) + \frac{L(\lambda)}{\lambda} \left( c_0^- \text{Pf} \left( \frac{H(-x)}{x} \right) + c_0^+ \text{Pf} \left( \frac{H(x)}{x} \right) \right) + o \left( \frac{L(\lambda)}{\lambda} \right),
\]
whence the result follows. \( \square \)

3.4. **Extension from \( \mathbb{R} \setminus \{0\} \) to \( \mathbb{R} \).** The methods employed in the previous two subsections also allow us to study the following question. Suppose that the restriction of \( f \in \mathcal{D}'' \) to \( \mathbb{R} \setminus \{0\} \) is known to have quasiasymptotic behavior in \( \mathcal{D}''(\mathbb{R} \setminus \{0\}) \), what can we say about the quasiasymptotic properties of \( f \)? In view of symmetry considerations, it is clear that it suffices to restrict our attention to ultradistributions supported on \([0, \infty)\).

**Theorem 3.8.** Suppose that \( f \in \mathcal{D}'' \) is supported in \([0, \infty)\) and has quasiasymptotic behavior 
\[
f(\lambda x) \sim c \lambda^\alpha L(\lambda) x^\alpha \quad \text{in } \mathcal{D}''(0, \infty) \quad \text{as } \lambda \to \infty.
\]

(i) If \( \alpha > -1 \), then \( f \) has the quasiasymptotics \( f(\lambda x) \sim c \lambda^\alpha L(\lambda) x^\alpha_+ \) in \( \mathcal{D}'' \) as \( \lambda \to \infty \).

(ii) If \( \alpha < -1 \) and \( N \in \mathbb{N} \) is such that \(-N + 1 < \alpha < -N\), then there exist constants \( a_0, \ldots, a_{N-1} \) such that 
\[
f(\lambda x) - \sum_{n=0}^{N-1} a_n \delta^{(n)}(x) \sim c \lambda^\alpha L(\lambda) x^\alpha_+ \quad \text{in } \mathcal{D}'' \quad \text{as } \lambda \to \infty,
\]

(iii) If \( \alpha = -k \in \mathbb{Z}_- \), then there is a function \( b \) satisfying\(^2\) for each \( a > 0 \)
\[
(3.19) \quad b(ax) = b(x) + \int_1^a \frac{(-1)^{k-1} L(x) \log a + o(L(x))}{(k-1)!} \, dx, \quad x \to \infty,
\]
\(^2\)Such functions are called associate homogeneous of degree 0 with respect to \( L \) in [17, 21]. They coincide with functions of the so-called De Haan class [1].
and constants \(a_0, \ldots, a_{k-1}\) such that

\[
(3.20) \quad f(\lambda x) = c \frac{L(\lambda)}{\lambda^k} \text{Pf} \left( \frac{H(x)}{x^k} \right) + b(\lambda) \delta^{(k-1)}(x) + \sum_{j=0}^{k-1} a_j \frac{\delta^{(j)}(x)}{\lambda^{j+1}} + o \left( \frac{L(\lambda)}{\lambda^k} \right),
\]

in \(D^{**}\) as \(\lambda \to \infty\).

**Proof.** Lemma 3.3 says that we may assume that, say, \(\text{supp } f \subset (e, \infty)\) by removing a neighborhood of the origin. So, we can apply exactly the same argument as in the proof of Theorem 3.6 (via Lemma 3.5 and Lemma 3.2(i)) to show parts (i) and (ii). For (iii), we assume without loss of generality that \(k = 1\) (Lemma 3.2(ii)) and apply the same argument as in the proof of Theorem 3.7 to conclude that

\[
f(\lambda x) = f_0(\lambda x) + \gamma \frac{L(\lambda)}{\lambda} \delta(x) + o \left( \frac{L(\lambda)}{\lambda} \right)
\]
in \(D^{**}\).

where the continuous function \(f_0\) has also support in \((e, \infty)\) and satisfies \(f_0(x) \sim cL(x)/x, x \to \infty\), in the ordinary sense. At this point the result can be derived from [21, Theorem 4.3] (see also [17, Theorem 2.38, p. 155]), but we might argue directly as follows. In fact, we proceed in the same way we arrived at (3.18). Set \(b(x) = \int_1^x f_0(t)dt\), then, uniformly for \(x\) in compact subsets of \((0, \infty)\),

\[
b(\lambda x) = b(\lambda)H(x) + c \int_\lambda^{\lambda x} \frac{L(t)}{t} dt + o(L(\lambda)) = b(\lambda)H(x) + cL(\lambda)H(x) \log x + o(L(\lambda)),
\]

so that differentiation finally shows

\[
f_0(\lambda x) = \frac{b(\lambda)}{\lambda} \delta(x) + c \frac{L(\lambda)}{\lambda} \text{Pf} \left( \frac{H(x)}{x} \right) + o \left( \frac{L(\lambda)}{\lambda} \right),
\]

actually in \(D'\). \(\square\)

4. **The structure of quasiasymptotics at the origin**

We now focus our attention on quasiasymptotic behavior at the origin. The reader should notice that Lemma 3.2 holds for quasiasymptotics at the origin as well. Furthermore, it is a simple consequence of the definition that quasiasymptotics at the origin is a local property, in the sense that two ultradistributions that coincide in a neighborhood of the origin must have precisely the same quasiasymptotic properties. Throughout this section \(L\) stands for a slowly varying function at the origin and we set \(\tilde{L}(x) = L(1/x)\). From now on, by convention the parameters \(\varepsilon \to 0^+\) and \(\lambda \to \infty\).

We will reduce the analysis of the structure of quasiasymptotics at the origin to that of the quasiasymptotics at infinity via a substitution. Our starting key observation is the following lemma:

**Lemma 4.1.** If \(f \in D''(\mathbb{R} \setminus \{0\})\) has quasiasymptotic behavior with respect to \(\varepsilon^\alpha L(\varepsilon), \alpha \in \mathbb{R}\), then \(f(x) := f(1/x)\) has quasiasymptotics in \(D''(\mathbb{R} \setminus \{0\})\) with respect to \(\lambda^{-\alpha} \tilde{L}(\lambda)\).
Proof. Take any $\phi \in D^*(\mathbb{R} \setminus \{0\})$ and set $\tilde{\phi}(x) := \phi(1/x)$. Suppose that $f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon)g(x)$ in $D^*(\mathbb{R} \setminus \{0\})$. If we set $\lambda := \varepsilon^{-1}$, then we get
\[
\lim_{\lambda \to \infty} \left\langle \frac{\tilde{f}(\lambda x)}{\lambda^{-\alpha} L(\lambda)}, \phi(x) \right\rangle = \lim_{\lambda \to \infty} \left\langle \frac{f(x)}{\lambda^{-\alpha+1} L(\lambda)}, \tilde{\phi}(\lambda x)x^{-2} \right\rangle
\]
\[
= \lim_{\varepsilon \to 0^+} \left\langle \frac{f(x)}{\varepsilon^{\alpha+1} L(\varepsilon)}, \phi(\frac{x}{\varepsilon}) (\frac{x}{\varepsilon})^{-2} \right\rangle
\]
\[
= \left\langle g(x), \tilde{\phi}(x)x^{-2} \right\rangle = \left\langle g(1/x), \phi(x) \right\rangle.
\]
\[\square\]

We would now like to proceed applying the structure theorem to $\tilde{f}$ and transform back via the change of variables $x \leftrightarrow 1/x$. We therefore need to see how this substitution acts on derivatives, which can be done via Faà di Bruno’s formula.

Lemma 4.2. Let $\phi \in C^\infty(\mathbb{R} \setminus \{0\})$ and set $\psi(x) := x^{-2}\phi(1/x)$. Then for any $m \in \mathbb{N}$, there exist constants $c_{m,0}, \ldots, c_{m,m}$ such that
\[
(4.1) \quad \frac{d^m}{dx^m}(\psi(x)) = \sum_{j=0}^{m} c_{m,j} \frac{\phi^{(j)}(1/x)}{x^{m+j+2}},
\]
where we have the bounds
\[
(4.2) \quad |c_{m,j}| \leq \frac{m!}{j!} 4^m, \quad 0 \leq j \leq m.
\]

Proof. By (2.5), it follows that
\[
\frac{d^k}{dx^k}(\phi(1/x)) = \sum_{j=1}^{k} (-1)^{k-j} x^{-(k+j)} \phi^{(j)}(1/x) B_{k,j}(1!, 2!, \ldots, (k-j+1)!).
\]

From (2.6) we get
\[
\frac{1}{j!} \left( \frac{t}{1-t} \right)^j = \sum_{k=j}^{\infty} B_{k,j}(1!, \ldots, (k-j+1)!) \frac{t^k}{k!},
\]
whence we infer that
\[
B_{k,j}(1!, \ldots, (k-j+1)!) = \frac{d^k}{dt^k} \left( \frac{1}{j!} \left( \frac{t}{1-t} \right)^j \right) \bigg|_{t=0} = \frac{k!(k-1)!}{j!(j-1)!(k-j)!}.
\]

Therefore, we obtain that (4.1) holds with
\[
c_{m,0} = (-1)^m (m+1)! \quad \text{and} \quad c_{m,j} = (-1)^m \frac{m!}{j!} \sum_{k=j}^{m} (m-k+1)(k-1)\binom{k-1}{j-1}
\]
when $0 < j \leq m$. In the latter case,
\[
|c_{m,j}| \leq \frac{m!}{j!} \frac{(m-j+1)(m-j+2)}{2} \frac{(m-1)}{j-1} \leq \frac{m!}{j!} m(m+1)2^{m-2},
\]
Theorem 4.3. Let $\alpha \notin \mathbb{Z}_-$ and let $k \in \mathbb{N}$ be the smallest integers such that $-(k+1) < \alpha$. Then, $f \in \mathcal{D}'$ has quasiasymptotic behavior

$$f(\varepsilon x) \sim \varepsilon^\alpha L(\varepsilon)(c_- x^\alpha_ - + c_+ x^\alpha_ +) \quad \text{in } \mathcal{D}' \text{ as } \varepsilon \to 0^+$$

if and only if there exist functions $f_m \in L^1(-1,1)$, $m \geq k$, that are continuous on $[-1,1] \setminus \{0\}$ such that

$$f = \sum_{m=k}^{\infty} f_m^{(m)}, \quad \text{on } (-1,1),$$

$$c_m^\pm = \lim_{x \to 0^\pm} \frac{f_m(x)}{x^m |x|^\alpha L(|x|)}, \quad m \geq k,$$

exist, and furthermore, for some $\ell > 0$ (for any $\ell > 0$) there is a $C > 0$ such that

$$|f_m(x)| \leq C \frac{f_m}{M_m} x^{\alpha + m} L(|x|), \quad 0 < |x| \leq 1,$$

for all $m \geq k$. Moreover, the relation (3.11) must hold.

Proof. The proof of sufficiency can be done analogously as in Theorem 3.6. Hence we are only left with necessity. If we can show the theorem for degree larger than $-1$, then the full structure theorem will follow from Lemma 3.2(i), hence we assume that $\alpha > -1$ (hence $k = 0$). If $f$ has quasiasymptotic behavior with respect to $\varepsilon^\alpha L(\varepsilon)$, then $\tilde{f}(x) := f(1/x)$ has quasiasymptotic behavior in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ with respect to $\lambda^{-\alpha} \tilde{L}(\lambda)$, where $\tilde{L}(x) := L(1/x)$. Then by Theorem 3.6 or Theorem 3.7 if $\alpha \in \mathbb{Z}_+$ and keeping in mind our observations from Section 3.4, there exist continuous $\tilde{f}_m$ in $\mathbb{R} \setminus \{0\}$, $m \geq 0$, that satisfy (3.8), (3.9) and (3.10). Consider now for any $m \geq 0$,

$$f_m(x) := \sum_{k=m}^{\infty} (-1)^{k+m} c_{k,m} \tilde{f}_k(1/x) x^{m+k},$$

where the $c_{k,m}$ are as in Lemma 4.2. By (3.10) and (4.2) it follows that for some $\ell > 0$ (for any $\ell > 0$) and any $0 < |x| \leq 1$,

$$|f_m(x)| = \left| \sum_{k=m}^{\infty} (-1)^{k+m} c_{k,m} \tilde{f}_k(1/x) x^{m+k} \right|$$

$$\leq \sum_{k=m}^{\infty} \frac{k!}{m!} 4^k : C \frac{\ell^k}{M_k} |x|^\alpha L(|x|)|x|^{m+k}$$

$$= C |x|^{\alpha + m} L(|x|) \frac{1}{m!} \sum_{k=m}^{\infty} k! (4\ell)^k \frac{C}{M_k} \leq CC_4 \frac{(4\ell)^m}{M_m} |x|^\alpha L(|x|),$$
by (3.1) from Lemma 3.1. This not only shows existence and continuity in \([-1,1] \setminus \{0\}\), but also shows that the \(f_m\) satisfy (4.6). By (3.9) and dominated convergence, it also follows that for these functions the limits (4.5) exist. Now take any \(\phi \in \mathcal{D}^* (\mathbb{R} \setminus \{0\})\) with \(\text{supp} \phi \subseteq (-1,1)\) and set \(\psi(x) := \phi(1/x)x^{-2}\). Then,

\[
\langle f(x), \phi(x) \rangle = \langle \tilde{f}(x), \psi(x) \rangle = \sum_{k=0}^{\infty} \langle \tilde{f}_k(x), (-1)^k \psi^{(k)}(x) \rangle.
\]

Since for any \(k \in \mathbb{N}\), by Lemma 4.2,

\[
\int_{-\infty}^{\infty} \tilde{f}_k(x)\psi^{(k)}(x)dx = \sum_{m=0}^{k} c_{k,m} \int_{-\infty}^{\infty} \tilde{f}_k(x)\phi^{(m)}(1/x)x^{m+k+2}dx
\]

\[
= \sum_{m=0}^{k} c_{k,m} \int_{-\infty}^{\infty} \tilde{f}_k(1/x)\phi^{(m)}(x)x^{m+k}dx,
\]

it follows by switching the order of summation that

\[
f = \sum_{m=0}^{\infty} f^{(m)}_m,
\]

in \(\mathcal{D}^*((-1,1) \setminus \{0\})\). Now as \(\alpha > -1\), the latter sum is an element of \(\mathcal{D}^*\), so that there is some \(g \in \mathcal{D}^*\) with \(\text{supp} g \subseteq \{0\}\) for which

\[
f = \sum_{m=0}^{\infty} f^{(m)}_m + g,
\]

in \(\mathcal{D}^*[-1,1]\). Since we have already shown sufficiency, the sum has quasiasymptotics with respect to \(\varepsilon^n L(\varepsilon)\), implying that the same holds for \(g\). If \(g \neq 0\), we can find an ultradifferentiable operator \(P(D) = \sum_{n \geq n_0} a_n D^n\) of type * such that \(g = P(D)\delta\) and \(a_{n_0} \neq 0\). Then, for any \(\phi \in \mathcal{D}^*\),

\[
\langle \frac{g(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \phi(x) \rangle = \sum_{n=n_0}^{\infty} (-1)^n a_n \frac{\varepsilon^{-n-\alpha-1}}{L(\varepsilon)} \phi^{(n)}(0).
\]

But if \(\phi(x) = x^{n_0}\) in a neighborhood of 0, we conclude that

\[
\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^{n_0+\alpha+1}L(\varepsilon)} = \frac{(-1)^{n_0}}{a_{n_0}n_0!} \lim_{\varepsilon \to 0^+} \left\langle \frac{g(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)}, \phi(x) \right\rangle.
\]

On the contrary, by (2.1), it follows that for any \(n \in \mathbb{N}\) and \(\alpha > -1, \varepsilon^{-n-\alpha-1}/L(\varepsilon) \to \infty\) as \(\varepsilon \to 0^+\), leading to a contradiction. Therefore, \(g\) must be identically 0 and this completes the proof of the theorem.

The structure for negative integral degree can be described as follows.

**Theorem 4.4.** Let \(f \in \mathcal{D}^*\) and \(k \in \mathbb{Z}_+\). Then, \(f\) has quasiasymptotic behavior

\[
f(\varepsilon x) \sim \frac{L(\varepsilon)}{\varepsilon^k} (\gamma \delta^{(k-1)}(x) + \beta x^{-k}) \quad \text{in } \mathcal{D}^* \text{ as } \varepsilon \to 0^+
\]
if and only if there are continuous functions $F$ and $f_m$ on $[-1, 1] \setminus \{0\}$, $m \geq k$, such that

\begin{equation}
\tag{4.8}
f = F^{(k)} + \sum_{m=k}^{\infty} f_m^{(m)} \quad \text{on } (-1, 1),
\end{equation}

the limits

\begin{equation}
\tag{4.9}
c_m^\pm = \lim_{x \to 0^\pm} \frac{f_m(x)}{x^{m-k} L(|x|)}, \quad m \geq k,
\end{equation}

exist, for some $\ell > 0$ (for any $\ell > 0$) there exists $C = C_\ell > 0$ such that

\begin{equation}
\tag{4.10}
|f_m(x)| \leq C \frac{f_m}{M_m} |x|^{\alpha+m} L(|x|), \quad 0 < |x| \leq 1,
\end{equation}

for all $m \geq k$, and for any $a > 0$ the limit

\begin{equation}
\tag{4.11}
\lim_{x \to 0^+} \frac{F(ax) - F(-x)}{L(x)} = c_1^* + c_2^* \log a
\end{equation}

exists. In this case,

\begin{equation}
\tag{4.12}
\gamma = c_1^* + \sum_{m=k}^{\infty} (c_m^+ - c_m^-) \quad \text{and} \quad \beta = (-1)^{k-1}(k-1)!c_2^*.
\end{equation}

**Proof.** For the sufficiency, applying Theorem 4.3 to the series $\sum_{m=k}^{\infty} f_m^{(m-1)}$, one deduces

\[ f(x) = \sum_{m=0}^{\infty} f_m^{(m)} \quad \text{on } (-1, 1) \setminus \{0\}, \]

with continuous functions $f_0, f_1, \ldots$ on $[-1, 1] \setminus \{0\}$ such that the limits (4.9) exist including the case $m = 0$. Applying again Theorem 4.3 to the series $\sum_{m=1}^{\infty} f_m^{(m-1)}$, we deduce that $f_0$ has an extension $g_0$ to $\mathbb{R}$ with quasiasymptotic behavior of order $-1$ with respect to $L(\varepsilon)$. Let $F$ be a first order primitive of $g_0$. Due to the fact that $F' = f_0$ off the origin and the quasiasymptotic behavior of $F'$, it is clear that $F$ is integrable at the origin and that it must have the form

\[ F(x) = \begin{cases} 
- \int_{-1}^{x} f_0(t) dt + C_+ & \text{if } x > 0, \\
\int_{-x}^{0} f_0(t) dt + C_- & \text{if } x < 0.
\end{cases} \]
Similarly as in the proof of Theorem 3.7, we conclude that
\[ c_0 = \lim_{x \to 0^+} \frac{F(x) - F(-x)}{L(x)} \]
must exist by comparing with the quasiasymptotics of \( g_0 \). Hence, for each \( a > 0 \)
\[ \lim_{x \to 0^+} \frac{F(ax) - F(-x)}{L(x)} = c_0 + \lim_{x \to 0^+} \frac{1}{L(x)} \int_x^{ax} f_0(t) \, dt = C_0 + c_0^+ \log a. \]

□

Our method also yields:

**Theorem 4.5.** Suppose that \( f_0 \in \mathcal{D}^\ast'(0, \infty) \) has quasiasymptotic behavior
\[ f_0(\varepsilon x) \sim c\varepsilon^\alpha L(\varepsilon)x^\alpha \quad \text{in} \quad \mathcal{D}^\ast'(0, \infty) \quad \text{as} \quad \varepsilon \to 0^+. \]
Then \( f_0 \) admits extensions to \( \mathbb{R} \). Let \( f \in \mathcal{D}^\ast' \) be any of such extensions with support in \([0, \infty)\). Then:

(I) If \( \alpha \notin \mathbb{Z}_- \), then there is \( g \in \mathcal{D}^\ast' \) with \( \operatorname{supp} g \subseteq \{0\} \) such that
\[ f(\varepsilon x) - g(\varepsilon x) \sim c\varepsilon^\alpha L(\varepsilon)x^\alpha \quad \text{in} \quad \mathcal{D}^\ast' \quad \text{as} \quad \varepsilon \to 0^+. \]

(II) If \( \alpha = -k \in \mathbb{Z}_- \), then there are a function \( b \) satisfying for each \( a > 0 \)
\[ b(ax) = b(x) + c \frac{(-1)^{k-1}}{(k-1)!} L(x) \log a + o(L(x)), \quad x \to 0^+, \]
and an ultradistribution \( g \in \mathcal{D}^\ast' \) with \( \operatorname{supp} g \subseteq \{0\} \) such that
\[ f(\varepsilon x) = c \frac{L(\varepsilon)}{\varepsilon^k} \operatorname{Pf} \left( \frac{H(x)}{\varepsilon^k} \right) + b(\varepsilon) \delta^{(k-1)}(x) + g(\varepsilon x) + o \left( \frac{L(\varepsilon)}{\varepsilon^k} \right), \quad x \to 0^+. \]

5. Quasiasymptotic behavior in \( S_1^\ast' \)

As an application of our structural theorems, we now discuss some other extension results for quasiasymptotics of ultradistributions. For distributions, the connection between tempered distributions and the quasiasymptotic behavior has been extensively studied [17, 18, 21, 23, 26]. The following properties are well known:

1. If \( f \in \mathcal{D}' \) has quasiasymptotic behavior at infinity, then \( f \in \mathcal{S}' \) and it has the same quasiasymptotic behavior in \( \mathcal{S}' \).

2. If \( f \in \mathcal{S}' \) has quasiasymptotic behavior at the origin in \( \mathcal{D}' \), then it has the same quasiasymptotic behavior in \( \mathcal{S}' \).

Our goal is to obtain ultradistributional analogs of these results. In this context, the natural counterparts of \( \mathcal{S}' \) are the Gelfand-Shilov type spaces [2, 13], defined as follows. We consider two sequences \( \{A_p\}_{p \in \mathbb{N}} \) and \( \{B_p\}_{p \in \mathbb{N}} \) of positive numbers; here we
shall assume that both satisfy (M.1) and $B_0 = A_0 = 1$. For any $h > 0$, $\mathcal{S}^{A_p, h}_{B_p, h}$ denotes the Banach space of all $\varphi \in C^\infty$ for which the norm

$$
\|\varphi\|_{A_p, B_p, h} := \sup_{x \in \mathbb{R}, m, n \in \mathbb{N}} \frac{(1 + |x|)^n |\varphi^{(m)}(x)|}{h^{m+n} A_m B_n}
$$

is finite. Then we consider the following locally convex spaces

$$
\mathcal{S}^{(A_p)}_{(B_p)} = \lim_{h \to 0^+} \mathcal{S}^{A_p, h}_{B_p, h}, \quad \mathcal{S}^{(A_p)}_{(B_p)} = \lim_{h \to \infty} \mathcal{S}^{A_p, h}_{B_p, h},
$$

corresponding to the Beurling and Roumieu case, and where we will use $\mathcal{S}^{*}_{(\gamma)}$ as a common notation for these two cases.

Let us now study the quasiasymptotic at infinity. As usual the relation $N_p \subset M_p$ between two weight sequences means that there are $C, \mu > 0$ for which $N_p \leq C \mu^p M_p$, $p \in \mathbb{N}$.

**Theorem 5.1.** Suppose that $A_p B_p \subset M_p$ and $A_p$ satisfies (M.2)' If $f \in \mathcal{D}'$ has quasiasymptotic behavior with respect to $\lambda^\alpha L(\lambda)$, with $L$ slowly varying at infinity and $\alpha \in \mathbb{R}$, then $f \in \mathcal{S}^{*}_{(\gamma)}$ and it has the same quasiasymptotic behavior in $\mathcal{S}^{*}_{(\gamma)}$.

**Proof.** Let $k \in \mathbb{N}$ be the smallest natural number such that $-(k + 1) \leq \alpha$. Then by either Theorem 3.6 or Theorem 3.7 we find some $\ell > 0$ (for any $\ell > 0$) and a $C = C_\ell > 0$ such that (3.10) holds for all $m \geq k$. Wet set $n = \lceil \alpha + 1 \rceil$. Employing Potter’s estimate (2.1) (with $\gamma = \lambda = 1$) and (M.1) for $A_p$, we find that for any $\varphi \in \mathcal{S}^{*}_{(\gamma)}$ and any $m \geq k$ we have

$$
\left| \int_{-\infty}^{\infty} f_m(x) \varphi^{(m)}(x) \, dx \right| \leq C' \frac{\ell m}{M_m} \left( \int_{-1}^{1} |\varphi^{(m)}(x)| \, dx + \int_{|x| \geq 1} |x|^{m+n} |\varphi^{(m)}(x)| \, dx \right)
$$

$$
\leq 2C' \|\varphi\|_{A_p, B_p, h} \frac{\ell m}{M_m} \left( A_m h^m + A_m B_{m+n+2} h^{2m+n+2} \right)
$$

$$
\leq 2C' \|\varphi\|_{A_p, B_p, h} \frac{\ell m}{M_m} \left( A_m h^m + A_1^{-n-2} A_m B_{m+n+2} h^{2m+n+2} \right),
$$

and as $h \ell$ may be chosen freely, it follows from (M.2)' for $M_p$ that this is absolutely summable over $m \geq k$. Consequently, $f = \sum_{m=k}^{\infty} f^{(m)} \in \mathcal{S}^{*}_{(\gamma)}$.

For the quasiasymptotic behavior of $f$, the case where $\alpha$ is not a negative integer can be shown in a similar fashion as the sufficiency proof of Theorem 3.6. For $\alpha = -k \in \mathbb{Z}_-$, it is clear that we only need to treat the case $k = 1$, as the general case then automatically follows by differentiating (due to the fact that the weight sequence $A_p$ satisfies (M.2)''). By Theorem 3.7 there exist continuous functions $f_m$, $m \in \mathbb{N}$, satisfying (3.14), (3.15), and (3.16) such that

$$
f = f_0 + \sum_{m=1}^{\infty} f^{(m)}.
$$

The infinite sum in the previous identity clearly has a primitive with quasiasymptotic behavior with respect to $L(\lambda)$, so that its quasiasymptotic behavior may be extended to the whole of $\mathcal{S}^{*}_{(\gamma)}$, and in turn its derivative $\sum_{m=1}^{\infty} f^{(m)}$ has quasiasymptotic behavior.
with respect to $\lambda^{-1}L(\lambda)$ in $S_2'$. By (3.14), $f_0$ has quasiasymptotic behavior with respect $\lambda^{-1}L(\lambda)$ in $D'$, hence, by [21, Remark 3.1] (see also [17, Theorem 2.41, p. 158]), it has the same quasiasymptotic behavior in $S_2'$, hence certainly also in $S_1'$. Therefore, the same also holds for $f$.

Let us now turn our attention to the case at the origin. We start by proving an extension theorem for quasiasymptotics of compactly supported ultradistributions. We need to introduce further spaces. Given $k \in \mathbb{N}$ and $h > 0$, we define the Banach spaces $S_{0,k}^{M_p,h}$ of all $\varphi \in C^\infty$ such that

$$\sup_{x \in \mathbb{R}, m \in \mathbb{N}} \frac{(1 + |x|)^n |\varphi^{(m)}(x)|}{h^m M_m} < \infty,$$

and then consider the locally convex spaces

$$S_0^{(M_p)} = \lim_{n \to \infty} \lim_{h \to 0^+} S_{0,k}^{M_p,h}, \quad S_0^{(M_p)} = \lim_{n \to \infty} \lim_{h \to \infty} S_{0,k}^{M_p,h}.$$

**Theorem 5.2.** Let $L$ be a slowly varying function at the origin and $\alpha \in \mathbb{R}$. If $f \in E'$ has quasiasymptotics with respect to $\varepsilon^\alpha L(\varepsilon)$ in $D'$, then $f$ has the same quasiasymptotic behavior in $S_0'$.

**Proof.** We may normalize the situation and assume that $\text{supp} f \subset (-1, 1)$. We first treat the $\alpha > -1$. So assume $f$ satisfies (4.3). From Theorem 4.3 we find continuous functions $f_m$ on $[-1, 1] \setminus \{0\}$, vanishing outside $[-1, 1]$, satisfying (4.5) and (4.6), and such that

$$f = \sum_{m=0}^{\infty} f^{(m)}_m,$$

on the whole of $\mathbb{R}$. Take any $\psi \in S^*_0$ and decompose it as $\psi = \psi_- + \psi_0 + \psi_+$ where $\text{supp} \psi_- \subset (-\infty, -1]$, $\psi_0$ has compact support, and $\text{supp} \psi_+ \subset [1, \infty)$. By hypothesis

$$\lim_{\varepsilon \to 0^+} \left( \frac{f(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)} , \psi_0(x) \right) = C_+ \left( x^\alpha, \psi_+ \right) + C_- \left( x^\alpha, \psi_-(x) \right);$$

hence, it suffices to show that the same limit holds for $\psi_\pm$ placed instead of $\psi_0$. As the two cases are symmetrical, we only look at $\psi_+$. It follows from (2.1), (1.6) and the Lebesgue dominated convergence theorem that for any $m \in \mathbb{N},$

$$\lim_{\varepsilon \to 0^+} \left( \frac{f^{(m)}_m(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)} , \psi_+(x) \right) = (-1)^m \lim_{\varepsilon \to 0^+} \int_{0^+}^{1/\varepsilon} \left( \frac{L(\varepsilon x)}{\varepsilon^\alpha L(\varepsilon)} \right) x^{\alpha + m} \psi_+^{(m)}(x) dx$$

$$= (-1)^m \int_0^{\infty} x^{\alpha + m} \psi_+^{(m)}(x) dx$$

$$= \frac{\Gamma(m + \alpha + 1)}{\Gamma(\alpha + 1)} \int_0^{\infty} x^\alpha \psi_+(x) dx.$$
The general case $\alpha \notin \mathbb{Z}_-$ now follows from Lemma 3.5 and a differentiation argument. Finally, $\alpha \in \mathbb{Z}_-$ can established exactly as in the proof of [23, Theorem 6.1] (see also [17, Theorem 2.35, p. 151]).

The next proposition discusses ultradistributions vanishing in a neighborhood of the origin.

**Proposition 5.3.** Suppose that $M_p B_p \subset A_p, B_p$ satisfies (M.2), and (M.2)' holds for $A_p$. If $f \in S_p'$ vanishes in a neighborhood of the origin, then for each $\varphi \in S_1^*$ there is $r > 0$ (for each $r > 0$) such that

$$\langle f(\varepsilon x), \varphi(x) \rangle = O(r(\varepsilon^{-B(r/\varepsilon)}), \quad \text{as } \varepsilon \to 0^+. $$

where $B$ is the associated function of $B_p$, namely,

$$B(t) = \sup_{p \in \mathbb{N}} \log \frac{t^p}{B_p}.$$

**Proof.** We only give the proof in the Beurling case; the Roumieu case can be shown analogously by employing their well-known projective description (cf. [2, 5]). There are $0 < R < 1$ and $\ell, C > 0$ (for each $\ell$ there is $C$) such that

$$|\langle f(x), \phi(x) \rangle| \leq C \sup_{|x| \geq R, n, m \in \mathbb{N}} \left(1 + |x|^n \right) \phi^{(m)}(x), \quad \phi \in S_{(B_p)}^*(A_p).$$

Suppose $M_p B_p \leq C_1 \mu^p A_p$ and $B_{p+q} \leq C_2 H^{p+q} B_p B_q$, where $\mu, H \geq 1$. Taking $\phi(x) = \varepsilon^{-1} \varphi(x/\varepsilon)$ with $\varphi \in S_{(B_p)}^{(M_p)}$ and we keeping arbitrary $0 < \varepsilon < 1$ and $k \in \mathbb{Z}_+$, we have

$$|\langle f(\varepsilon x), \varphi(x) \rangle|$$

$$\leq C_3 \sup_{|x| \geq R, n, m \in \mathbb{N}} \left(\frac{\mu H}{\ell} \right)^{n+m} \frac{(1 + |x|)^{n+m+k} \varphi^{(m)}(x/\varepsilon)}{M_m B_{n+m}} \frac{1}{\varepsilon^{1+m(1 + |x|)^{m+k}}}$$

$$\leq C_4 (\ell \varepsilon)^{k-1} B_{k-1} \|\varphi\|_{M_p B_p, \ell R_p^{-1} H^{-2}};$$

whence the assertion follows. \qed

Combining Theorem 5.2 and Proposition 5.3 one obtains:

**Corollary 5.4.** Let $L$ be a slowly varying function at the origin and $\alpha \in \mathbb{R}$. Suppose that $M_p B_p \subset A_p, B_p$ satisfies (M.2), and (M.2)' holds for $A_p$. If $f \in S_1'$ has quasiasymptotics with respect to $\varepsilon^\alpha L(\varepsilon)$ in $D'$, then $f$ has the same quasiasymptotic behavior in $S_1'$. 

We end this article by mentioning that it would be interesting to determine optimal assumptions on the weight sequences so that the assertions of Theorem 5.1 and Proposition 5.3 remain valid.
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