UNIONS OF CUBES IN $\mathbb{R}^n$, COMBINATORICS IN $\mathbb{Z}^n$ AND THE JOHN-NIRENBERG AND JOHN-STRÖMBERG INEQUALITIES

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Abstract. Suppose that the $d$-dimensional unit cube $Q$ is the union of three disjoint “simple” sets $E$, $F$ and $G$ and that the volumes of $E$ and $F$ are both greater than half the volume of $G$. Does this imply that, for some cube $W$ contained in $Q$, the volumes of $E \cap W$ and $F \cap W$ both exceed $s$ times the volume of $W$ for some absolute positive constant $s$?

Here, by “simple” we mean a set which is a union of finitely many dyadic cubes.

We prove that an affirmative answer to this question would have deep consequences for the important space $BMO$ of functions of bounded mean oscillation introduced by John and Nirenberg.

We recall and use the notion of a John-Strömberg pair which is closely related to the above question. The above mentioned result is obtained as a consequence of a general result about these pairs. We also present a number of additional results about these pairs.

1. Introduction

We are going to pose and discuss a rather simply formulated geometrical and almost combinatorical question. We shall refer to it as Question A(1/2). Although it may seem to have no connection with functional analysis, this question in fact has its origins in the study of a very important space of functions. The space $BMO$, or $BMO(D)$, which consists of all functions of bounded mean oscillation on a suitable subset $D$ of $\mathbb{R}^d$, plays a surprisingly wide range of different important roles in several branches of analysis. (For some details about those roles see, e.g., [3, p. 132] or [1, p. 4].)

One of our aims here is to warmly invite readers, including those with interests quite unrelated to spaces of functions or even to analysis in general, to study our Question A(1/2), and hopefully even resolve it. In particular (as our title seeks to indicate, and as confirmed by [1]) those readers with expertise in geometry in $\mathbb{R}^n$ or combinatorial problems in $\mathbb{Z}^n$ may well have some valuable insights. Those who choose to respond to this invitation will be able to effectively do so, without needing any familiarity whatsoever with the space $BMO$, nor with the proofs in this paper, nor with the contents of the earlier papers which gave rise to it.

Before formulating Question A(1/2) we shall mention another question which motivates it and then fix some necessary (but rather simple) terminology and notation.

This paper is a sequel to [3] and its preliminary more detailed version [1]. The main motivation for those papers, and also for this one, is the wish for a better understanding of the celebrated John-Nirenberg inequality [5]. This inequality is satisfied by every function in the above mentioned space $BMO(D)$.

In particular, it is hoped to ultimately find an answer to:
**Question J-N.** Can the constants in the John-Nirenberg inequality for BMO functions of $d$ variables be chosen to be independent of $d$?

We refer to [6], [7], [8], [9], [10] and [11], and also to references in these papers. For some rather recent results concerning best constants in certain versions of the John-Nirenberg inequality, these deal mainly with the case where $d = 1$. The results of [10] apply to all values of $d$. Some remarks on pp. 7-8 of [7] recall some reasons for being interested in the sizes of these constants, also for $d > 1$. We are grateful to Andrei Lerner, Pavel Shvartsman and Leonid Slavin for information about these and related results.

As will be seen in a moment, our Question A(1/2) is quite easy to formulate, and is expressed in terms readily accessible to a general mathematical audience. Its importance lies in the fact that an affirmative answer to it would imply an affirmative answer to Question J-N. It would seem to be at least slightly easier to answer Question A(1/2) than to answer certain very similar questions which have an analogous role and which are posed in [11] and [3], and are also discussed in the brief survey article [2]. In the formulation of Question A(1/2), and indeed throughout this paper, we shall use the following notation and terminology (much of which is standard, and most of which was also used in [1] and [3]).

**Definition 1.1.** We shall understand that

(i) $d$ is a positive integer, and that

(ii) the word **cube** means a closed cube in $\mathbb{R}^d$ with sides parallel to the axes, i.e., the cartesian product of $d$ bounded closed intervals, all of the same length. Special roles will be played by the $d$-dimensional **unit cube** $[0, 1]^d$ and its **dyadic subcubes** (i.e., the cubes of the form $\prod_{j=1}^d [(n_j - 1)2^{-k}, n_j 2^{-k}]$, where $k \in \mathbb{N}$ and the integers $n_j$ all satisfy $1 \leq n_j \leq 2^k$). It will sometimes be convenient to use the standard notation $Q(x, r)$ for the cube of side length $2r$ centred at $x \in \mathbb{R}^d$, i.e., $Q(x, r) = \left\{ y \in \mathbb{R}^d : \|y - x\|_{\ell_2} \leq r \right\}$.

Furthermore,

(iii) we shall let $Q(\mathbb{R}^d)$ denote the collection of all cubes in $\mathbb{R}^d$, and

(iv) the word **$d$-multi-cube** will mean a non-empty subset of $[0, 1]^d$ which is the union of finitely many dyadic subcubes of $[0, 1]^d$, and

(v) we will denote the $d$-dimensional Lebesgue measure of any Lebesgue measurable subset $E$ of $\mathbb{R}^d$ by $\lambda(E)$. (But, as explained below, for Question A(1/2) you do not need to know anything at all about Lebesgue measure.)

(The values of $d$ in (ii) and in (v) will be clear from the contexts in which they appear.)

In the formulation of Question A(1/2) we will not need any subtle properties of Lebesgue measure. In fact we will only be dealing with the simpler notion of $d$-dimensional volume. This is because the only Lebesgue measurable subsets $E$ of $\mathbb{R}^d$ that we will encounter in Question A(1/2) are cubes or $d$-multi-cubes and their intersections with other cubes. All of these are finite unions $E = \bigcup_{m=1}^M E_m$ of non-overlapping sets $E_m$, where each $E_m$ is the cartesian product of $d$ closed intervals. So we simply have that each $\lambda(E_m)$ is the product of the lengths of those $d$ intervals, and that $\lambda(E) = \sum_{m=1}^M \lambda(E_m)$.

With some small adjustments, we can replace consideration of $\lambda(E)$ for all the above mentioned sets $E$ by consideration of the cardinality of $E \cap 2^{-k}\mathbb{Z}^d$ for a suitably large choice of $k \in \mathbb{N}$. (This possibility was already very briefly hinted at in [2].) This is one reason for our suggestion above, in the opening paragraph of this paper, that Question A(1/2) might perhaps also be approachable via combinatorial considerations.

After these preparations, here at last is the promised question:
**Question A(1/2).** Does there exist an absolute constant \( s > 0 \) which has the following property?

**For every positive integer** \( d \), **whenever** \( E_+ \) **and** \( E_- \) **are two disjoint** \( d \)-**multicubes** **which satisfy**

\[
\min \{ \lambda(E_+), \lambda(E_-) \} > \frac{1}{2} (1 - \lambda(E_+) - \lambda(E_-)),
\]

**then there exists a cube** \( W \) **which is contained in** \([0, 1]^d\) **and for which**

\[
\min \{ \lambda(W \cap E_+), \lambda(W \cap E_-) \} \geq s \lambda(W).
\]

Most of the work which is required to show that an affirmative answer to Question A(1/2) implies an affirmative answer to Question J-N has already been done in [1] and [3]. Because of that we will not have any need at all here to deal with any details concerning any of the versions of the function space \( BMO \), nor to even recall their definitions. It was shown in [1] and [3] that an affirmative answer to Question J-N would be a consequence of an affirmative answer to the following question which was formulated at the beginnings of both of those papers, and is clearly quite similar to Question A(1/2).

**Question A.** Do there exist two absolute constants \( \tau \in (0, 1/2) \) and \( s > 0 \) which have the following property?

**For every positive integer** \( d \) **and for every cube** \( Q \) **in** \( \mathbb{R}^d \), **whenever** \( E_+ \) **and** \( E_- \) **are two disjoint measurable subsets of** \( Q \) **whose** \( d \)-**dimensional Lebesgue measures satisfy**

\[
\min \{ \lambda(E_+), \lambda(E_-) \} > \tau \lambda(Q \setminus E_+ \setminus E_-),
\]

**then there exists a cube** \( W \) **which is contained in** \( Q \) **and for which**

\[
\min \{ \lambda(W \cap E_+), \lambda(W \cap E_-) \} \geq s \lambda(W).
\]

It is obvious that an affirmative answer to Question A would imply an affirmative answer to Question A(1/2). Our task here is to show that the reverse implication also holds, namely that affirmatively answering the apparently at least slightly less demanding Question A(1/2) would suffice to also affirmatively answer Question A and therefore also Question J-N.

In the next section we shall discuss some known results related to Question A(1/2) including some limitations which can be anticipated in any attempts to solve it. Then, after recalling some more notions and providing some preliminary results in Section 3, we will obtain the above mentioned reverse implication in Section 4 as an immediate consequence (see Corollary 4.2 below) of the main result of this paper, Theorem 4.1, which is formulated in a slightly more general context than Questions A and A(1/2).

We note that Theorem 4.1 could also conceivably be used to deduce an affirmative answer for Question J-N from certain variants of Question A, which might perhaps be easier to answer than Question A(1/2). Some explorations of the possibility of such options, including some relevant numerical experiments, will perhaps be discussed in a future sequel to this paper. The reader may perhaps care to take note of some issues raised in Section 10 of [1] which might turn out to be relevant for resolving Question A(1/2) or its variants.

The above mentioned more general context in which Theorem 4.1 is formulated revolves around the notion of John-Strömberg pairs, whose definition we will recall in Section 3.
In Section 5 we shall present a number of additional results about these pairs, beyond those which will be needed for our main result.

Finally, the extremely brief Section 6 will make some minor comments about the papers [1] and [3].

2. Some constraints and some known results related to Question A(1/2)

As soon as we plunge into any attempt to answer Question A(1/2) we can almost immediately identify our real “enemies”. We can of course always assume that \( \lambda(E_-) \leq \lambda(E_+) \). If we restrict our attention to the case where \( r_0 \lambda(E_+) \leq \lambda(E_-) \leq \lambda(E_+) \) for some fixed \( r_0 \in (0, 1] \) then an obvious calculation shows that, by simply choosing \( W = [0, 1]^d \), we can obtain (1.2) for \( s = r_0/(3r_0 + 1) \). This shows that our above mentioned “enemies” are the cases where the ratio \( \lambda(E_-)/\lambda(E_+) \) is arbitrarily small.

It is known that if Question A(1/2) has an affirmative answer, then the positive number \( s \) for which that answer holds must satisfy
\[
(2.1) \quad s \leq \sqrt{5} - 2.
\]

This follows from the third of three results which have been obtained by Ron Holzman [4] in connection with Question A(1/2). It is a pleasure to describe these results. The first of them is an affirmative answer to a non-trivial special case of Question A(1/2): Holzman has shown, for every \( d \in \mathbb{N} \), that whenever \( E_+ \) and \( E_- \) are \( d \)-multi-cubes which are each finite unions of dyadic cubes, all of side length 1/2, and they satisfy (1.1), then there exists a cube \( W \) in \([0, 1]^d\) which satisfies (1.2) for \( s = 1/4 \). In fact, as will be explained below in Remark 5.8 our Lemma 5.7 shows that this value of \( s \) cannot be improved in the following sense: If the general form of Question A(1/2) has an affirmative answer, then the positive number \( s \) which appears in that answer must satisfy \( s \leq 1/4 \). So this already comes quite close to establishing (2.1).

The second of Holzman’s recent results is that this largest possible value 1/4 for \( s \) is indeed attained in another special case of Question A(1/2), where \( d \) is restricted to take only one value, namely \( d = 1 \), but where there is no restriction on the form of the disjoint measurable subsets \( E_+ \) and \( E_- \) of \([0, 1]\). (See also Remark 5.11.) Holzman’s third result (see Remark 5.8 for details) uses a particular example when \( d = 2 \) to show that the analogue of his second result, when the single chosen value for \( d \) is greater than 1, does not hold. This example shows that for each \( d \geq 2 \), as already stated above, the relevant value of \( s \) cannot be greater than \( \sqrt{5} - 2 \).

3. Some Further Definitions and Preliminary Results

The following definition (which is effectively the same as Definition 7.10 of [1] p. 29 and Definition 7.9 of [3] pp. 153–154) recalls a notion which plays a central role in [1] and [3], and which is of course closely related to Question A.

**Definition 3.1.** Let \( d \) be a positive integer and let \( \mathcal{E} \) be a non-empty collection of Lebesgue measurable subsets of \( \mathbb{R}^d \). Suppose that each \( E \in \mathcal{E} \) satisfies \( 0 < \lambda(E) < \infty \). Let \( \tau \) and \( s \) be positive numbers with the following property:

(*) Let \( Q \) be an arbitrary set in \( \mathcal{E} \) and let \( E_+ \) and \( E_- \) be arbitrary disjoint measurable subsets of \( Q \). Suppose that
\[
(3.1) \quad \min \{ \lambda(E_+), \lambda(E_-) \} > \tau \lambda(Q \setminus E_+ \setminus E_-).
\]

Then there exists a set \( W \subset Q \) which is also in \( \mathcal{E} \) and for which
\[
(3.2) \quad \min \{ \lambda(E_+ \cap W), \lambda(E_- \cap W) \} \geq s \lambda(W).
\]
Then we will say that \((\tau, s)\) is a **John-Strömberg pair** for \(\mathcal{E}\).

**Remark 3.2.** In the context of the preceding definition, if \(E_+\) and \(E_-\) and \(W\) are measurable subsets of \(Q\) which satisfy \(E_+ \cap E_- = \emptyset\) and \(\lambda(W) > 0\) and (3.2) for some \(s > 0\), then

\[
\lambda(W) \geq \lambda(E_+ \cap W) + \lambda(E_- \cap W) \geq 2 \min \{\lambda(E_+ \cap W), \lambda(E_- \cap W)\} \geq 2s\lambda(W) > 0
\]

and therefore \(s \leq 1/2\). Consequently, for any choice of \(\mathcal{E}\), any pair \((\tau, s)\) which is a John-Strömberg pair for \(\mathcal{E}\) must satisfy \(0 < s \leq 1/2\). (Cf. Remark 7.16 of [1, p. 30].)

Several “natural” choices of the collection \(\mathcal{E}\) are mentioned on pp. 4–5 of [1] and pp. 132–133 of [3]. These are relevant for studying a number of variants of the function space \(Q\) on line 11 of page 47.

It will sometimes be convenient, for each \(d \in \mathbb{N}\), to let \(\text{JS}(d)\) denote the set of all ordered pairs \((\tau, s)\) of positive real numbers which are John-Strömberg pairs for \(Q(\mathbb{R}^d)\).

Obviously, an equivalent reformulation of Question A is:

**Question A \((\tau, s)\).** Do there exist two absolute constants \(\tau \in (0, 1/2)\) and \(s > 0\) for which \((\tau, s) \in \text{JS}(d)\) for every \(d \in \mathbb{N}\)?

This makes it very relevant to use the following known result. (Note that below in Subsection 5.1 we will present a result containing some new slight variants of it.)

**Theorem 3.4.** Let \(d\) be a positive integer. The ordered pair of positive numbers \((\tau, s)\) is a John-Strömberg pair for \(Q(\mathbb{R}^d)\) if and only if it has the following property:

\(\star\) Whenever \(F_+\) and \(F_-\) are disjoint \(d\)-multi-cubes which satisfy

\[
\min \{\lambda(F_+), \lambda(F_-)\} > \tau \lambda([0, 1]^d \setminus F_+ \setminus F_-),
\]

then there exists a cube \(W\) contained in \([0, 1]^d\) for which

\[
\min \{\lambda(W \cap F_+), \lambda(W \cap F_-)\} \geq s\lambda(W).
\]

**Proof.** Obviously every John-Strömberg pair \((\tau, s)\) for \(Q(\mathbb{R}^d)\) must have the property \(\star\). The fact that property \(\star\) implies that \((\tau, s)\) is a John-Strömberg pair for \(Q(\mathbb{R}^d)\) is exactly the content of Theorem 10.2 of [1, p. 43], and is proved on pp. 44–47 of [1]. However, some other small issues remain to be clarified: It should be mentioned that there are a few minor misprints in the proof of Theorem 10.2 of [1]. But they do not effect its validity. On page 46, in the third and fourth lines above the inequality (10.16) the notation \(\tilde{F}_s\) should of course be changed to \(F_\#\) (in two places) and \(\Omega_s\) should be changed to \(\Omega_\#\). Then on line 11 of page 47 \(r_\#^d\) should be \((2r_\#)^d\) and \(r^d_\#\) should be \((2r_\#)^d\). Similarly, four factors
of $2^d$ have been (harmlessly) omitted on line 5 of page 63 of [1] in the proof of Lemma 10.1, which is an ingredient in the proof of Theorem 10.2. That line should be

$$= (2R_n)^d - (2r_\tau)^d + (2R_n)^d - (2r_\tau)^d.$$ 

It should also be mentioned that, in the formulation of Theorem 10.2 of [1], it is stated that the number $\tau$ must satisfy $0 < \tau < 1/2$. But the proof is valid for all $\tau > 0$. (The requirement that $\tau < 1/2$ was imposed only because this is relevant in Theorem 9.1 of [1, p. 41] and [2, p. 164].)

The clarification of these small issues completes the proof of Theorem 3.4. □

4. The main result

We can now present our main result, and its obvious corollary for dealing with Question A(1/2).

**Theorem 4.1.** Let $d$ be a positive integer. Suppose that $(\tau, s)$ is a John-Strömberg pair for $Q(\mathbb{R}^d)$. Then

$$0 < s \leq 1/2$$ (4.1)

and, for each $\theta \in (0, s/(1-s))$, the pair $((1-\theta)\tau, s - \theta(1-s))$ is also a John-Strömberg pair for $Q(\mathbb{R}^d)$.

**Corollary 4.2.** Suppose that the answer to Question A(1/2) is affirmative, i.e., suppose that $(1/2, s)$ is a John-Strömberg pair for $Q(\mathbb{R}^d)$ for some positive number $s$ which does not depend on $d$. Then, if we choose $\theta = s/(2-2s)$ in Theorem 3.4, we obtain that $\left(\frac{1}{2} \cdot \frac{2-3s}{2-2s}, \frac{s}{2}\right)$ is a John-Strömberg pair for $Q(\mathbb{R}^d)$ for all $d \in \mathbb{N}$, which gives us an affirmative answer to Question A($\tau, s$) and therefore also to Question A and Question J-N.

**Proof of Theorem 4.1.** Let us explicitly choose particular (necessarily positive) numbers $\tau$, $s$ and $\theta$ which satisfy the hypotheses of the theorem.

The inequalities (4.1) as observed in Remark 3.2 follow from the fact that $(\tau, s)$ is a John-Strömberg pair for $Q(\mathbb{R}^d)$. Note that (4.1) and the conditions imposed on $\theta$ ensure that $0 < \theta < 1$ and also that $0 < \theta(1-s) < s$. Consequently, the numbers $(1-\theta)\tau$ and $s - \theta(1-s)$ are both strictly positive (as indeed they must be if they are to form a John-Strömberg pair).

Throughout this proof we will let $Q$ denote the $d$-dimensional unit cube, $Q = [0,1]^d$. Let $F_+$ and $F_-$ be two arbitrary disjoint subsets of $Q$ which are both $d$-multi-cubes and satisfy

$$\min \{\lambda(F_+), \lambda(F_-)\} > (1-\theta)\tau \lambda(Q \setminus F_+ \setminus F_-)$$ (4.2)

In view of Theorem 3.4 it will suffice to show that there exists a subcube $W$ of $[0,1]^d$ such that

$$\min \{\lambda(W \cap F_+), \lambda(W \cap F_-)\} \geq (s - \theta(1-s)) \lambda(W).$$ (4.3)

For some sufficiently large integer $N$, both of the sets $F_+$ and $F_-$ are finite unions of dyadic cubes all of the same side length $2^{-N}$, as of course is the whole unit cube $Q$. Since the distance between $F_+$ and $F_-$ must be positive, there must be some dyadic subcubes of $Q$ of side length $2^{-N}$ which are not contained in $F_+ \cup F_-$, and thus their interiors are all contained in $Q \setminus F_+ \setminus F_-. Therefore the set $Q \setminus F_+ \setminus F_-$ is also, at least to within some set of measure zero, a finite union of dyadic cubes of side length $2^{-N}$.

Let $F_\pm$ denote the collection of all dyadic cubes of side length $2^{-N}$ which are contained in $F_\pm$. For each $\delta \in (0,2^{-N-1})$ and for each cube $W = \prod_{j=1}^n [a_j, a_j + 2^{-N}]$ in the collection...
\(F_+\), let \(H(W, \delta)\) denote the cube concentric with \(W\) contained in the interior of \(W\) whose side length is \(2^{-N} - 2\delta\), i.e., \(H(W, \delta) = \prod_{j=1}^{n}[a_j + \delta, a_j + 2^{-N} - \delta]\). We will later use the following obvious fact: If \(V\) is a cube which intersects with \(H(W, \delta)\) and has side length less than \(\delta\), then
\[
(4.4) \quad V \subset W.
\]

We now introduce a special subset \(H^+_\delta\) of \(F_+\) which is “slightly smaller” than \(F_+\). It is defined by
\[
H^+_\delta := \bigcup_{W \in F_+} H(W, \delta).
\]

With the help of this set, we will now choose a particular suitable value for \(\delta\) which must remain unchanged for the rest of this proof. Obviously \(\lambda(F_+) - \lambda(H^+_\delta) = \lambda(F_+ \setminus H^+_\delta)\) for each \(\delta \in (0, 2^{-N-1})\) and \(\lim_{\delta \searrow 0} \lambda(F_+ \setminus H^+_\delta) = 0\). Therefore, since the inequality in (4.2) is strict, we can and will choose our fixed value of \(\delta\) to be sufficiently small to ensure that
\[
(4.5) \quad \min \{\lambda(H^+_\delta), \lambda(F_-)\} > (1 - \theta) \tau \lambda(Q \setminus F_+ \setminus F_-) + \tau \lambda(F_+ \setminus H^+_\delta).
\]

Let \(G\) denote the collection of all dyadic cubes of side length \(2^{-N}\) whose interiors are contained in \(Q \setminus F_+ \setminus F_-\). Given an arbitrary positive integer \(k\), we divide each cube in the collection \(G\) into \(2^d k\) dyadic subcubes of side length \(2^{-N-k}\). Let \(G_k\) denote the collection of all dyadic cubes obtained in this way. In other words, \(G_k\) is simply the collection of all dyadic cubes of side length \(2^{-N-k}\) whose interiors are contained in \(Q \setminus F_+ \setminus F_-\). For each \(W \in G_k\) we let \(U(W, k)\) be the (closed) cube concentric with \(W\) whose volume satisfies
\[
(4.6) \quad \lambda(U(W, k)) = \theta \lambda(W).
\]

We now introduce two more sets which will play particularly useful roles for us. First we define
\[
V_k := \bigcup_{W \in G_k} U(W, k)
\]
and then use \(V_k\) to define the disjoint union
\[
H^-_k := F_- \cup V_k
\]
which contains and will be “controllably” larger than \(F_-\).

In accordance with usual very standard notation, we will denote the interior and the boundary of any given cube \(W\) by \(W^o\) and \(\partial W\) respectively. We of course have
\[
(4.7) \quad Q \setminus F_+ \setminus F_- = \left( \bigcup_{W \in G_k} W^o \right) \cup Z
\]
for some set \(Z \subset \bigcup_{W \in G_k} \partial W\). Since \(\lambda(Z) = 0\), this implies that
\[
(4.8) \quad \lambda(Q \setminus F_+ \setminus F_-) = \sum_{W \in G_k} \lambda(W).
\]

In the following calculation we shall use (4.7) in the second line, and then, in the third line, the fact that, for every cube \(W \in G_k\), the set \(Z\) of zero measure introduced in (4.7) is disjoint from the cube \(W^o\) and therefore also from \(U(W, k)\). The final line of the calculation will use the facts that, for each pair of distinct cubes \(W\) and \(W'\) in \(G_k\), we have \(U(W', k) \subset (W')^o\) and also \(W^o \cap (W')^o = \emptyset\) and therefore \(W^o \cap U(W', k) = \emptyset\).

Thus we obtain that
\[
Q \setminus F_+ \setminus H_k^- = Q \setminus F_+ \setminus F_- \setminus \left( \bigcup_{W \in \mathcal{G}_h} U(W, k) \right) = \left( Z \cup \left( \bigcup_{W \in \mathcal{G}_h} W^\circ \right) \right) \setminus \left( \bigcup_{W \in \mathcal{G}_h} U(W, k) \right) = Z \cup \left( \left( \bigcup_{W \in \mathcal{G}_h} W^\circ \right) \setminus \left( \bigcup_{W \in \mathcal{G}_h} U(W, k) \right) \right) = Z \cup \left( \bigcup_{W \in \mathcal{G}_h} (W^\circ \setminus U(W, k)) \right).
\]

Since \( \lambda(Z) = 0 \), the preceding equalities, together with (4.6) and (4.8), imply that
\[
\lambda(Q \setminus F_+ \setminus H_k^-) = \sum_{W \in \mathcal{G}_h} \lambda(W^\circ \setminus U(W, k)) = (1 - \tau) \sum_{W \in \mathcal{G}_h} \lambda(W) = (1 - \theta)\lambda(Q \setminus F_+ \setminus F_-).
\]

Any point in the set \( Q \setminus H^+_\delta \setminus H_k^- \) which is not in \( F_+ \) must be in \( Q \setminus F_+ \setminus H_k^- \). This shows that
\[
Q \setminus H^+_\delta \setminus H_k^- \subset (Q \setminus F_+ \setminus H_k^-) \cup (F_+ \setminus H^+_\delta).
\]

Since the two sets in parentheses on the right side of (4.10) are disjoint, it follows that
\[
\lambda(Q \setminus H^+_\delta \setminus H_k^-) \leq \lambda(Q \setminus F_+ \setminus H_k^-) + \lambda(F_+ \setminus H^+_\delta).
\]

Now we can first use the fact that \( F_- \subset H_k^- \) and then invoke (4.5) followed by (4.9) and then (4.11), to obtain that
\[
\begin{align*}
\min \{ \lambda(H^+_\delta), \lambda(H^-_k) \} &\geq \min \{ \lambda(H^+_\delta), \lambda(F_-) \} \\
&> (1 - \theta)\tau \lambda(Q \setminus F_+ \setminus F_-) + \tau \lambda(F_+ \setminus H^+_\delta) \\
&= \tau \lambda(Q \setminus F_+ \setminus H_k^-) + \tau \lambda(F_+ \setminus H^+_\delta) \\
&\geq \tau \lambda(Q \setminus H^+_\delta \setminus H_k^-).
\end{align*}
\]

Since \( V_k \subset Q \setminus F_+ \setminus F_- \) and \( F_+ \cap F_- = \emptyset \), we have
\[
H_k^- \subset F_- \cup (Q \setminus F_+ \setminus F_-) = Q \setminus F_+.
\]

Furthermore \( H^+_\delta \subset F_+ \) and so \( H^+_\delta \) and \( H^-_k \) are disjoint measurable subsets of \( Q \). According to our hypotheses, \((\tau, s)\) is a John-Strömberg pair for \( \mathbb{Q}(\mathbb{R}^d) \). Therefore, for each \( k \in \mathbb{N} \), since the quantity in the first line of (4.12) is strictly larger than the quantity in its last line, we deduce that there must exist some subcube \( W_k \) of \( Q \) which satisfies
\[
\min \{ \lambda(W_k \cap H^+_\delta), \lambda(W_k \cap H^-_k) \} \geq s\lambda(W_k).
\]

We now claim that the side length of \( W_k \), which we can conveniently write as \( (\lambda(W_k))^{1/d} \), must satisfy
\[
(\lambda(W_k))^{1/d} \geq \delta.
\]

To show this we first observe that, since the cube \( W_k \) intersects with \( H^+_\delta \), it must intersect with the cube \( H(W, \delta) \) for at least one cube \( W \) in the collection \( \mathcal{F}_+ \). If (4.15) does not
hold, i.e., if $W_k$ has side length less than $\delta$, then (cf. the discussion immediately preceding (4.4)) $W_k$ must be completely contained in that particular cube $W$ and therefore also in $F_\pm$. Consequently $W_k$ cannot intersect with the set $H_k^-$. (Here we have used (4.13) once more.) This contradicts (4.14) and shows that (4.15) does hold.

Let $\tilde{W}_k$ be a new cube containing $W_k$ and concentric with $W_k$. More precisely, if $W_k = \prod_{j=1}^d [a_j, b_j]$, then we choose $\tilde{W}_k$ to be $\prod_{j=1}^d [a_j - 2^{-N-k}, b_j + 2^{-N-k}]$. Clearly $\tilde{W}_k$ contains all dyadic cubes of side length $2^{-N-k}$ which intersect with $W_k$, and the side length $(\frac{\lambda(\tilde{W}_k)}{\lambda(W_k)})^{1/d}$ of $\tilde{W}_k$ satisfies

$$(\frac{\lambda(\tilde{W}_k)}{\lambda(W_k)})^{1/d} = (\frac{\lambda(W_k)}{\lambda(W_k)})^{1/d} + 2^{1-N-k}.$$  

This, together with (4.15), gives us that

$$(\frac{\lambda(\tilde{W}_k)}{\lambda(W_k)})^{1/d} = (\lambda(W_k))^{1/d} + 2^{1-N-k}.$$  

It follows that

$$\lambda(\tilde{W}_k) - \lambda(W_k) = \lambda(W_k) \left( \frac{\lambda(\tilde{W}_k)}{\lambda(W_k)} - 1 \right) \leq \lambda(W_k) \left( 1 + \frac{2^{1-N-k}}{\delta} \right)^d - 1.$$  

(4.16)

Let $H_k$ be the collection of all cubes in $G_k$ which intersect with $W_k$. Clearly

$$(4.17) \bigcup_{W \in H_k} W^\circ \subset (Q \setminus F_+ \setminus F_-) \cap \tilde{W}_k.$$  

Our definitions of $V_k$ and of $H_k$ also immediately give us that

$$W_k \cap V_k = W_k \cap \bigcup_{W \in G_k} U(W, k)$$  

$$= W_k \cap \bigcup_{W \in H_k} U(W, k)$$  

$$\subset \bigcup_{W \in H_k} U(W, k).$$

This and then (4.6) will enable us to obtain the first line in the following calculation. Its second line will use the fact that the interiors of the cubes in $G_k$ and therefore also in $H_k$, are pairwise disjoint. Its third line will use (4.17). Its fourth line will use the fact that $H_k^+ \subset F_+$. Its sixth line will follow from the obvious inclusion $W_k \subset \tilde{W}_k$. The justifications of all other steps should be evident. (We wonder, casually, whether it might somehow be possible to replace the simple-minded transition from the sixth to the
seventh line by a sharper estimate, which might then lead to a (slightly) stronger version of Theorem 4.1)

\[ \lambda(W_k \cap V_k) \leq \sum_{W \in \mathcal{H}_k} \lambda(U(W, k)) = \theta \sum_{W \in \mathcal{H}_k} \lambda(W) \]

for all sufficiently large \( \lambda \) and \( \theta \). So, in view of (4.19), for all sufficiently large \( \lambda \) and \( \theta \), we can set

\[ \lim_{\lambda \to \infty} \epsilon_k = 0. \]

Combining the result of this calculation with (4.16), we deduce that, for each \( k \in \mathbb{N} \),

\[ \lambda(W_k \cap V_k) \leq \theta \lambda(W_k \cap H^\varepsilon_k) + \epsilon_k \lambda(W_k) \]

where \( \epsilon_k := \theta \left( \left( 1 + \frac{2^{1-N-k}}{\delta} \right)^d - 1 \right) \) and we will later use the obvious fact that

\[ \lim_{k \to \infty} \epsilon_k = 0. \]

In view of (4.14), we have that

\[ \lambda(W_k \setminus H^\varepsilon_k) = \lambda(W_k) - \lambda(W_k \cap H^\varepsilon_k) \leq (1 - s)\lambda(W_k). \]

We now have the ingredients needed to estimate \( \lambda(W_k \cap F_-) \) from below. Since \( \lambda(W_k \cap H^-_k) = \lambda(W_k \cap F_-) + \lambda(W_k \cap V_k) \) we can use (4.14) and (4.18) and then (4.20) to obtain that

\[ \lambda(W_k \cap F_-) \geq s\lambda(W_k) - \theta \lambda(W_k \setminus H^\varepsilon_k) - \epsilon_k \lambda(W_k) \]

where \( \epsilon_k := \theta \left( \left( 1 - s \right) \right) \lambda(W_k) \)

Furthermore, again with the help of (4.14), we also obviously have that

\[ \lambda(W_k \cap F_+) \geq \lambda(W_k \cap H^\varepsilon_k) \geq s\lambda(W_k). \]

As already observed at the beginning of this proof, \( s - \theta(1 - s) \) and \( \theta(1 - s) \) are both strictly positive. So, in view of (4.19), for all sufficiently large \( k \), we will also have \( s > (s - \theta(1 - s) - \epsilon_k) \) and \( (s - \theta(1 - s) - \epsilon_k) > 0 \). Therefore, we can now deduce from (4.21) and (4.22) and (4.22) that \( ((1 - \theta) \tau, s - \theta(1 - s) - \epsilon_k) \) is a John-Strömberg pair for \( Q(\mathbb{R}^d) \) for all sufficiently large \( k \). This seems to be very close to our required result. Indeed it will only require a little more effort to obtain that result.

For each \( k \in \mathbb{N} \), let \( x_k \) be the centre of the cube \( W_k \) and let \( r_k \) be half its side length. I.e., we can set \( W_k = Q(x_k, r_k) \) in the standard notation recalled above in Definition 1.1(ii).
Of course \( x_k \in Q \) and, by (4.15) and the fact that \( W_k \subset Q \), we also have \( \delta/2 \leq r_k \leq 1/2 \). Therefore there exists a strictly increasing sequence \( \{n_k\}_{k \in \mathbb{N}} \) of positive integers such that the sequences \( \{x_{nk}\}_{k \in \mathbb{N}} \) and \( \{r_{nk}\}_{k \in \mathbb{N}} \) converge, respectively, to a point \( x \in Q \) and a number \( r \in [\delta/2, 1/2] \). Let \( W \) be the cube \( W = Q(x, r) \). Then, by Lemma 10.1 of [12, p. 43] and (4.21) and (4.19), it follows that

\[
\lambda(W \cap F_{\tau}) = \lim_{k \to \infty} \lambda(W_{nk} \cap F_{\tau}) \\
\geq \lim_{k \to \infty} (s - \theta(1 - s) - \varepsilon_{nk}) \lambda(W_{nk}) \\
= (s - \theta(1 - s)) \lambda(W) .
\]

Similarly, by replacing \( k \) by \( n_k \) in (4.22) and then passing to the limit as \( k \) tends to \( \infty \), we obtain that

\[
\lambda(W \cap F_{\tau}) \geq s \lambda(W) \geq (s - \theta(1 - s)) \lambda(W) .
\]

All this shows that the subcube \( W \) of \( Q \) satisfies (4.3) and therefore completes the proof of the theorem. \( \square \)

**Remark 4.3.** It seems quite possible that a more elaborate version of the preceding proof might show that the hypotheses of Theorem 4.1 can yield a stronger result, namely that \( (\tau', s') \) is a John-Strömberg pair for some positive number \( \tau' \) smaller than \( (1 - \theta) \tau \) and/or for some number \( s' \) greater than \( s - \theta(1 - s) \). Perhaps a strategy for proving this might involve separately considering the two cases where \( \lambda(F_+) > c \lambda(F_-) \) and where \( \lambda(F_-) \leq \lambda(F_+) \leq c \lambda(F_-) \) for some suitably chosen constant \( c > 1 \).

5. Some further results about John-Strömberg pairs

5.1. Other characterizations of John-Strömberg pairs. It will be convenient to begin by introducing some “technical” terminology.

**Definition 5.1.** Let \( F_+ \) and \( F_- \) be two disjoint measurable subsets of \([0, 1]^d\) which both have positive measure and which also satisfy one of the following two conditions:

(i) \( \lambda\left([0, 1]^d \setminus F_+ \setminus F_-\right) = 0 \).

(ii) There exists a non-empty open subset \( \Omega \) of \([0, 1]^d\) for which

\[
\max \{\lambda(F_+ \cap \Omega), \lambda(F_- \cap \Omega)\} < \lambda(\Omega) \quad \text{and} \quad \min \{\lambda(F_+ \cap \Omega), \lambda(F_- \cap \Omega)\} = 0 .
\]

Then we shall say that \( (F_+, F_-) \) is a **tame couple** in \([0, 1]^d\).

**Remark 5.2.** Obviously the two conditions (i) and (ii) are mutually exclusive. Furthermore, the following simple argument shows that the condition (ii) is equivalent to

\[
(\text{ii}') \quad \text{There exists a dyadic cube } V \text{ contained in } [0, 1]^d \text{ for which}
\]

\[
\max \{\lambda(F_+ \cap V), \lambda(F_- \cap V)\} < \lambda(V) \quad \text{and} \quad \min \{\lambda(F_+ \cap V), \lambda(F_- \cap V)\} = 0 .
\]

The Lebesgue measures of each of the sets appearing in (5.2) remain unchanged if the dyadic cube \( V \) in them is replaced by its interior \( V^\circ \). Therefore condition (ii)' implies condition (ii). Conversely, suppose that some open set \( \Omega \) satisfies (5.1). We can suppose that \( 0 = \lambda(F_- \cap \Omega) \leq \lambda(F_+ \cap \Omega) < \lambda(\Omega) \). (Otherwise simply reverse the roles of \( F_+ \) and \( F_- \) in what is to follow.) By Theorem 1.11 of [12, p. 8], \( \Omega \) can be expressed as the union of a sequence of non-overlapping dyadic cubes \( \Omega = \bigcup_{n \in \mathbb{N}} V_n \). So \( \sum_{n=1}^{\infty} \lambda(F_+ \cap V_n) < \sum_{n=1}^{\infty} \lambda(V_n) \) and consequently \( \lambda(F_+ \cap V_n) < \lambda(V_n) \) for at least one \( n \) which of course also satisfies \( \lambda(F_- \cap V_n) = 0 \). So, for that choice of \( n \), the dyadic cube \( V_n \) satisfies (5.2). Therefore condition (ii) implies condition (ii)' and so (ii) and (ii)' are indeed equivalent.
We can now present the new variant of Theorem 3.4 to which we referred in the preamble to that theorem. It will specify two more properties of a pair $(\tau, s)$, which we shall label as $(\ast \ast)$ and $(\ast \ast \ast)$, and which are each equivalent to the property that $(\tau, s) \in JS(d)$. Note that the only difference between the statement of property $(\ast \ast)$ in Theorem 5.3 and the statement of property $(\ast)$ in Theorem 3.4 is that the strict equality “$>$” which appears in (3.3) in the statement of $(\ast)$ has been replaced by “$\geq$” in (5.3) in the statement of $(\ast \ast)$. Property $(\ast \ast \ast)$ of Theorem 5.3 is more elaborate, and requires the terminology of Definition 5.1.

**Theorem 5.3.** Let $d$ be a positive integer and let $(\tau, s)$ be an ordered pair of positive numbers. Then each of the following two properties is equivalent to the property that $(\tau, s)$ is a John-Strömberg pair for $Q(R^d)$.

$(\ast \ast)$ Whenever $F_+$ and $F_-$ are disjoint $d$-multi-cubes which satisfy

$$\min \{\lambda(F_+), \lambda(F_-)\} \geq \tau \lambda([0,1]^d \setminus F_+ \setminus F_-),$$

then there exists a cube $W$ contained in $[0,1]^d$ for which

$$\min \{\lambda(W \cap F_+), \lambda(W \cap F_-)\} \geq s \lambda(W).$$

$(\ast \ast \ast)$ Whenever $(F_+, F_-)$ is a tame couple of measurable subsets of $[0,1]^d$ which satisfies (5.3), then there exists a cube $W$ contained in $[0,1]^d$ for which (5.4) holds.

**Proof.** Let us first show that property $(\ast \ast \ast)$ implies property $(\ast \ast)$. Suppose that $F_+$ and $F_-$ are arbitrary disjoint $d$-multi-cubes. By the same simple reasoning as was given in the paragraph immediately following (4.3) in the proof of Theorem 4.1 we see that the set $[0,1]^d \setminus F_+ \setminus F_-$ must contain the interior $V^\circ$ of at least one dyadic subcube $V$ of $[0,1]^d$. The set $\Omega = V^\circ$ of course satisfies the condition (ii) in Definition 5.1. Consequently, $(F_+, F_-)$ is a tame couple in $[0,1]^d$. It immediately follows that property $(\ast \ast \ast)$ indeed does imply property $(\ast \ast)$.

Obviously, if the pair $(\tau, s)$ of positive numbers has property $(\ast \ast)$, then it also has property $(\ast)$ of Theorem 3.4 and therefore $(\tau, s)$ is a John-Strömberg pair for $Q(R^d)$.

Now suppose that $(\tau, s)$ is a John-Strömberg pair for $Q(R^d)$. Then, by letting $(\tau_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ be the special constant sequences $\tau_n = \tau$ and $s_n = s$, we see that $(\tau, s)$ satisfies the hypotheses of Lemma 5.5 which we will formulate and prove below. In view of part (b) of that lemma, $(\tau, s)$ has property $(\ast \ast \ast)$ . This completes the proof of the theorem. $\Box$

**Remark 5.4.** It is very natural to wonder whether property $(\ast \ast \ast)$ is equivalent to a stronger and more simply expressed variant of that property in which the same implication is required to hold for all pairs of disjoint measurable subsets $F_+, F_-$ of $[0,1]^d$ which both have positive measure, thus omitting the requirement that $F_+$ and $F_-$ should form a tame couple. (In other words, as in Remark 7.17 of [11, p. 30], we are essentially wondering whether in Definition 3.1 at least in the case where $\mathcal{E} = Q(R^d)$, it would be equivalent to replace “$>$” by “$\geq$” in (3.1), of course then with the necessary proviso that $\lambda(E_+)$ and $\lambda(E_-)$ are both positive.) We are unable to answer this question, but Lemma 5.5 shows that if its answer is negative, then the sets $F_+$ and $F_-$ which provide a counterexample, must both have very intricate structure (and it would not be inappropriate to refer to them as forming a “wild” couple).

**Lemma 5.5.** Let $d$ be a positive integer. Let $(\tau_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ be two sequences of positive numbers which converge to positive limits, $\tau$ and $s$ respectively. Suppose that $(\tau_n, s_n)$ is a John-Strömberg pair for $Q(R^d)$ for every $n \in \mathbb{N}$. Then
Proof. In this proof it will be convenient to use a trivial generalization of the standard notation for cubes recalled in Definition 5.3(ii) and to introduce some additional obvious terminology as follows:

**Definition 5.6.** We permit ourselves to extend the notation $Q(x, r)$ for a cube of side length $r$ centred at $x$, for each $x \in \mathbb{R}^d$ and $r > 0$, also to the case where $r = 0$, by letting $Q(x, 0)$ denote the singleton $\{x\}$. Let $\{x_k\}_{k \in \mathbb{N}}$ be a convergent sequence in $\mathbb{R}^d$ and let $\{r_k\}_{k \in \mathbb{N}}$ be a convergent sequence of positive numbers. For each $k \in \mathbb{N}$ let $W_k$ be the cube $Q(x_k, r_k)$. Then we refer to $\{W_k\}_{k \in \mathbb{N}}$ as a convergent sequence of cubes, and define its limit $\lim_{k \to \infty} W_k$ to be the cube or singleton $Q(\lim_{k \to \infty} x_k, \lim_{k \to \infty} r_k)$.

Let us fix an arbitrary $d \in \mathbb{N}$, and arbitrary sequences $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{s_n\}_{n \in \mathbb{N}}$ of positive numbers which tend respectively to the positive numbers $\tau$ and $s$, and have the property that $(\tau_n, s_n) \in JS(d)$ for each $n \in \mathbb{N}$.

In order to deduce that $(\tau, s)$ has property (**) and/or property (***) we begin by fixing two arbitrary disjoint measurable sets $F_+$ and $F_-$ which are contained in of $[0, 1]^d$, which form a tame couple in $[0, 1]^d$, and which also satisfy (5.3). To obtain part (a) of the lemma we have the task of proving, for these choices of $F_+$, $F_-$, $\{\tau_n\}_{n \in \mathbb{N}}$, $\{s_n\}_{n \in \mathbb{N}}$, $\tau$ and $s$, that there exists a subcube $W$ of $[0, 1]^d$ which satisfies (5.4). If needed at any stage of our proof of that fact, we may make the additional assumption that $F_+$ and $F_-$ are both $d$-multi-cubes.

An analogous task is required to obtain part (b) of the lemma, with the difference that in our proof this time, instead of being able to assume that $F_+$ and $F_-$ are $d$-multi-cubes, we may make the additional assumption, if needed, that $s_n \geq s$ for all $n$.

There is quite a lot of overlap in the ingredients which will be used for performing these two tasks, and it may help avoid some confusion if we give some general description, in advance, of each of the four steps which we shall use to accomplish both of them almost simultaneously. We stress that neither of the two above mentioned additional assumptions will be needed in the first two of these four steps.

In Step 1 of the proof, we shall use the given measurable sets $F_+$ and $F_-$ and the given sequences $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{s_n\}_{n \in \mathbb{N}}$ to construct a special sequence $\{W(n_k)\}_{k \in \mathbb{N}}$ of subcubes of $[0, 1]^d$ which converges either to a cube or to a singleton set, in the sense of Definition 5.6. We shall perform this construction in two different (and unexplained and sometimes complicated) ways, depending on which of the two conditions of Definition 5.1 is satisfied by $(F_+, F_-)$. It is only in later steps of the proof that we will be able to properly see the usefulness of the particular features of these constructions.

In Step 2, we shall see that if the sequence $\{W(n_k)\}_{k \in \mathbb{N}}$ obtained in the preceding step converges to a cube $W$, then that cube is contained in $[0, 1]^d$ and satisfies (5.4).

In Step 3, we shall see that whenever the disjoint measurable sets $F_+$ and $F_-$ are both required to also be $d$-multi-cubes, then the sequence $\{W(n_k)\}_{k \in \mathbb{N}}$ can be always be constructed so that it converges to a subcube rather than a singleton. In view of Step 2, this will complete the proof of part (a) of the lemma.

Finally, in Step 4, in order to complete the proof of part (b), it will remain (again in view of Step 2) only to deal with the case where the sequence $\{W(n_k)\}_{k \in \mathbb{N}}$ converges to a singleton. We will do this by showing that, in this case, when we also impose the...
requirement that the sequence \( \{s_n\}_{n \in \mathbb{N}} \) satisfies \( s_n \geq s \) for each \( n \), then there exists an integer \( k_0 \) for which the subcube \( W = W(n_{k_0}) \) is contained in \([0, 1]^d\) and satisfies (5.4).

**STEP 1: Construction of the special convergent sequence** \( \{W(n_k)\}_{k \in \mathbb{N}} \).

Since \( (F_+, F_-) \) is a tame couple, it must satisfy either condition (i) or condition (ii) of Remark 5.2.

Our construction of \( \{W(n_k)\}_{k \in \mathbb{N}} \) will be quite simple in the case where condition (i) holds, i.e., when

\[
\lambda \left( [0, 1]^d \setminus F_+ \setminus F_- \right) = 0. \tag{5.5}
\]

The strict inequalities \( 0 < \lambda(F_+) < \lambda\left([0, 1]^d\right) \) enable us to use Lemma 7.1 of [1, p. 25] or of [3, p. 150] (whose simple proof is a special case of the proof of Lemma 7.5 on pages 26-27 of [1]) to ensure the existence of a cube \( W = Q(x, r) \subset [0, 1]^d \) (of course with \( r > 0 \)) for which

\[
\lambda(W \cap F_+) = \lambda(W \setminus F_+) = \frac{1}{2} \lambda(W). \tag{5.6}
\]

The condition (5.5) is of course equivalent to \( \lambda(F_+ \cup F_-) = 1 \) and to the fact that the sets \( F_- \) and \([0, 1]^d \setminus F_+\) coincide to with sets of measure zero. Therefore we also have

\[
\lambda(W \cap F_-) = \lambda(W \setminus F_+). \tag{5.7}
\]

In this case we will construct our required special convergent sequence \( \{W(n_k)\}_{k \in \mathbb{N}} \) by simply setting \( x_k = x, \ r_k = r, \ W(k) = W \) and \( n_k = k \) for each \( k \in \mathbb{N} \). Then the sequence \( \{W(n_k)\}_{k \in \mathbb{N}} \) of course converges, not to a singleton, but to the cube \( W \) which is its constant value.

It remains to consider the more complicated case when \( F_+ \) and \( F_- \) satisfy condition (ii) or equivalently condition (ii)' of Remark 5.2.

In that case, since the roles of \( F_+ \) and \( F_- \) are interchangeable in properties (**) and (** *) and in Definition 5.1 and in (5.4), we can assume, without loss of generality, that, by Remark 5.2 there exists a dyadic cube \( V \) contained in \([0, 1]^d\) for which

\[
0 = \lambda(F_- \cap V) \leq \lambda(F_+ \cap V) < \lambda(V). \tag{5.8}
\]

Since the interior \( V^\circ \) of \( V \) satisfies

\[
\lambda(V^\circ \setminus F_+) = \lambda(V \setminus F_+) = \lambda(V) - \lambda(V \cap F_+) > 0,
\]

the Lebesgue differentiation theorem guarantees the existence of a point \( z \in V^\circ \setminus F_+ \) for which \( \lim_{r \to 0} \frac{\lambda(Q(z, r) \setminus F_+)}{\lambda(Q(z, r))} = 1 \). Since \( Q(z, r) \subset V^\circ \) for all sufficiently small \( r \), we also have \( \lim_{r \to 0} \frac{\lambda(Q(z, r) \cap F_+)}{\lambda(Q(z, r))} = 1 \) and, consequently, \( \lim_{r \to 0} \frac{\lambda(Q(z, r) \cap F_-)}{\lambda(Q(z, r))} = 0 \). These properties of \( V^\circ \) and \( z \) enable us to assert the existence of a sequence \( \{\rho_k\}_{k \in \mathbb{N}} \) of positive numbers tending to monotonically to zero, such that for each \( k \) we have

\[
\lambda(Q(z, \rho_k)) < \lambda(V^\circ \setminus F_+) \tag{5.9}
\]

and

\[
Q(z, \rho_k) \subset V^\circ \tag{5.10}
\]

and therefore also
For each $k \in \mathbb{N}$,

\[
\lambda (Q(z, \rho_k) \setminus F_+) > 0.
\]

Let us now define a sequence $\{F_+(k)\}_{k \in \mathbb{N}}$ of measurable sets by setting

\[
F_+(k) = F_+ \cup Q(z, \rho_k) \quad \text{for each } k \in \mathbb{N}.
\]

Note that, in view of (5.11),

\[
\lambda (F_+(k)) = \lambda(F_+) + \lambda (Q(z, \rho_k) \setminus F_+) > \lambda(F_+).
\]

We also introduce the set

\[
G := F_- \setminus V^o.
\]

Note that

\[
F_+(k) \cap G = \emptyset \quad \text{for each } k \in \mathbb{N}
\]

since in fact

\[
F_+ \cap G = (F_+ \cup Q(z, \rho_k)) \cap G = (F_+ \cap G) \cup (Q(z, \rho_k) \setminus G)
\]

\[
\subseteq (F_+ \cap F_-) \cup (Q(z, \rho_k) \setminus V^o) = \emptyset \cup \emptyset.
\]

We also claim that

\[
\lambda \left([0,1]^d \setminus F_+(k) \setminus G\right) > 0 \quad \text{for all } k \in \mathbb{N}.
\]

To show this we first observe that, obviously, $V^o \setminus F_+(k) = V^o \setminus F_+(k) \setminus G$. Consequently,

\[
\lambda \left([0,1]^d \setminus F_+(k) \setminus G\right) \geq \lambda (V^o \setminus F_+(k) \setminus G) = \lambda (V^o \setminus F_+(k))
\]

\[
= \lambda (V^o \setminus F_+ \setminus Q(z, \rho_k)) = \lambda (V^o \setminus F_+) - \lambda ((V^o \setminus F_+) \cap Q(z, \rho_k))
\]

\[
\geq \lambda (V^o \setminus F_+) - \lambda (Q(z, \rho_k)).
\]

By (5.16) this last expression is strictly positive for each $k$, which completes our proof of (5.16).

We next observe that, by the first part of (5.8) (and of course also (5.14)),

\[
\lambda(F_-) = \lambda(F_- \cap V^o) + \lambda(F_- \setminus V^o) = \lambda(F_- \setminus V^o) = \lambda(G).
\]

We use (5.16) then (5.15) then (5.13) and (5.17), and finally the disjointness of $F_+$ and $F_-$ to obtain that

\[
0 < \lambda \left([0,1]^d \setminus F_+(k) \setminus G\right) = 1 - \lambda(F_+(k)) - \lambda(G)
\]

\[
< 1 - \lambda(F_+) - \lambda(F_-) = \lambda \left([0,1]^d \setminus F_+ \setminus F_-\right).
\]

Since $\tau > 0$, this implies that

\[
0 < \tau \lambda \left([0,1]^d \setminus F_+(k) \setminus G\right) < \tau \lambda \left([0,1]^d \setminus F_+ \setminus F_-\right).
\]

Then, since $\min \{\lambda(F_+(k)), \lambda(G)\} \geq \min \{\lambda(F_+), \lambda(F_-)\}$, it follows from (5.18) and the fact that $F_+$ and $F_-$ satisfy (5.3) that

\[
\min \{\lambda(F_+(k)), \lambda(G)\} > \tau \lambda([0,1]^d \setminus F_+(k) \setminus G) > 0 \quad \text{for each } k \in \mathbb{N}.
\]

We use these strict inequalities to construct a sequence $\{j(k)\}_{k \in \mathbb{N}}$ of positive integers such that $j(k) \geq k$ and $j(k)$ is sufficiently large to ensure that $\tau_{j(k)}$ is sufficiently close to $\tau$ to imply that

\[
\min \{\lambda(F_+(k)), \lambda(G)\} > \tau_{j(k)} \lambda([0,1]^d \setminus F_+(k) \setminus G) \quad \text{for each } k \in \mathbb{N}.
\]
Since \((\tau_j(k), s_j(k)) \in JS(d)\) and since \(F_+(k)\) and \(G\) are disjoint, we see, in accordance with Definition 3.1 (for \(E = Q(\mathbb{R}^d)\)), that there exists a subcube \(W(k)\) of \([0, 1]^d\) which satisfies
\[
\min \{\lambda(W(k) \cap F_+(k)), \lambda(W(k) \cap G)\} \geq s_j(k)\lambda(W(k)).
\]
The fact that \(j(k) \geq k\) for each \(k\) ensures that
\[
\lim_{k \to \infty} s_j(k) = s.
\]

Since
\[
W(k) \cap F_+ \subset W(k) \cap F_+(k) = (W(k) \cap F_+) \cup (W(k) \cap Q(z, r_k)) \subset (W(k) \cap F_+) \cup Q(z, r_k)
\]
and \(\lambda(Q(z, r_k)) = (2r_k)^d\), we obtain that
\[
\lambda(W(k) \cap F_+) \leq \lambda(W(k) \cap F_+(k)) \leq \lambda(W(k) \cap F_+) + (2\rho_k)^d
\]
for all \(k \in \mathbb{N}\).

A slight variant of the simple reasoning in (5.17), again using the first part of (5.8) and (5.20), gives us that
\[
\lambda(W(k) \cap F_-) = \lambda(W(k) \cap F_- \cap V^o) + \lambda(W(k) \cap F_- \setminus V^o)
\]
\[
= \lambda(W(k) \cap F_- \setminus V^o) = \lambda(W(k) \cap G)
\]
for all \(k \in \mathbb{N}\).

For each \(k \in \mathbb{N}\) we let the point \(x_k\) in \([0, 1]^d\) and the number \(r_k \in (0, 1/2]\) be the centre and half-side length respectively of the subcube \(W(k)\). I.e., we have \(W(k) = Q(x_k, r_k)\).

There exists a strictly increasing sequence \(\{n_k\}_{k \in \mathbb{N}}\) of positive integers such that the sequences \(\{x_{nk}\}_{k \in \mathbb{N}}\) and \(\{r_{nk}\}_{k \in \mathbb{N}}\) converge, respectively, to a point \(x \in [0, 1]^d\) and a number \(r \in [0, 1/2]\). We can now declare the sequence \(\{W(n_k)\}_{k \in \mathbb{N}} = \{Q(x_{nk}, r_{nk})\}_{k \in \mathbb{N}}\) to be the special convergent sequence which we set out to construct in this step of the proof in the case where \((F_+, F_-)\) satisfies condition (ii) of Definition 5.1.

Since we have already specified our construction of \(\{W(n_k)\}_{k \in \mathbb{N}}\) when \((F_+, F_-)\) satisfies condition (i) of Definition 5.1 this completes Step 1 of our proof.

**STEP 2: A proof that whenever the limit of \(\{W(n_k)\}_{k \in \mathbb{N}}\) is a cube, then that cube has the two properties required to immediately complete the proof of the theorem.**

Suppose that the limit of the sequence \(\{W(n_k)\}_{k \in \mathbb{N}} = \{Q(x_{nk}, r_{nk})\}_{k \in \mathbb{N}}\) which was constructed in the previous step, is indeed a cube \(W = Q(x, r)\), i.e., that
\[
r := \lim_{k \to \infty} r_{nk} > 0.
\]

Let us now prove that this implies that \(W\) is contained in \([0, 1]^d\) and that it satisfies (5.4). (We defer treatment of the case where \(\lim_{k \to \infty} r_{nk} = 0\) to Step 4.)

In the case which was dealt with at the beginning of the previous step, where \(F_+\) and \(F_-\) satisfy condition (i) of Definition 5.1 and where indeed we always have \(r > 0\), it is already known that the cube \(W\) is contained in \([0, 1]^d\). We note that, by the reasoning in Remark 3.2 we have \(s_n \leq 1/2\) for each \(n \in \mathbb{N}\) and therefore also \(s \leq 1/2\). So (5.4) follows immediately from (5.6) and (5.7).

We turn to the remaining case, where \(F_+\) and \(F_-\) satisfy condition (ii) of Definition 5.1 and therefore the sequence \(\{W(n_k)\}_{k \in \mathbb{N}}\) is constructed in the more elaborate way described in the second and much longer part of the previous step. Here the positivity of the limit \(r\) permits us to apply Lemma 10.1 of [1, p. 43] (whose proof was briefly discussed above near the end of the first paragraph of the proof of Theorem 3.4) to the sequence
(5.24) \[ \lambda(W \cap F_+) = \lim_{k \to \infty} \lambda(W(n_k) \cap F_+) \]
and (here also using (5.22)) that
(5.25) \[ \lambda(W \cap F_-) = \lim_{k \to \infty} \lambda(W(n_k) \cap F_-) = \lim_{k \to \infty} \lambda(W(n_k) \cap G) = \lambda(W \cap G), \]
and also that
(5.26) \[ \lambda(W) = \lim_{k \to \infty} \lambda(W(n_k)). \]
From (5.24) and (5.21) we see that
(5.27) \[ \lambda(W \cap F_+) = \lim_{k \to \infty} \lambda(W(n_k) \cap F_+(n_k)), \]
and from (5.26) and (5.20) we see that
(5.28) \[ s\lambda(W) = \lim_{k \to \infty} s_j(n_k) \lambda(W(n_k)). \]
In view of (5.19) we have that
\[ \lambda(W(n_k) \cap F_+(n_k)) \geq s_j(n_k) \lambda(W(n_k)) \quad \text{and} \quad \lambda(W(n_k) \cap G) \geq s_j(n_k) \lambda(W(n_k)) \]
for each \( k \in \mathbb{N} \). If we take the limit as \( k \) tends to \( \infty \) in each of these two inequalities, and apply (5.27), (5.26) and (5.28), then the two resulting inequalities can be rewritten as the required single inequality (5.24).

Thus we have shown that if (5.23) holds, then in both cases, i.e., whether it is condition (i) or condition (ii) of Definition 5.1 which applies to \( F_+ \) and \( F_- \), it follows that the limiting cube \( W \) indeed has both the properties required to complete the proof of part (a) and also part (b) of the lemma.

It remains to explain, as we shall do in the remaining two steps of the proof, how we can sometimes guarantee that (5.23) does hold, and how we can proceed when it does not hold.

**STEP 3: Completion of the proof of part (a) of the lemma.**

Our preceding treatment of the case where (5.23) holds, will now enable us to complete the proof of part (a) of the lemma in full generality. Part (a) refers to the property (**), so in our proof of it we can and must assume that the disjoint sets \( F_+ \) and \( F_- \) are both \( d \)-multi-cubes.

We once again refer (as we did at the beginning of the proof of Theorem 5.3) to the simple reasoning after (4.3) in the proof of Theorem 1.1 which shows that there exists a dyadic cube \( V \) whose interior is contained in \([0,1]^d \setminus F_+ \setminus F_- \) and which therefore must satisfy (5.3). So we can construct the sequence of cubes \( \{W(n_k)\}_{k \in \mathbb{N}} \) for this particular choice of \( V \) in exactly the way that was done in the second part of Step 1 of this proof. This choice of \( V \) implies that the set \( G \) introduced in (5.12) must satisfy \( G = F_- \setminus V^\circ = F_- \).

Since they are \( d \)-multi-cubes, \( F_+ \) and \( F_- = G \) are compact and so of course is \( Q(z, \rho_k) \). Consequently, the set \( F_+(k) \) (defined by (5.12)) is also compact. In view of (5.10), it is disjoint from \( G \). So the distance, which we denote by \( \text{dist}_\infty(F_+(k), G) \), between \( F_+(k) \) and \( G \) with respect to the \( \ell^\infty \) metric on \( \mathbb{R}^d \), must be positive. In fact, since \( F_+(k) \subset F_+(1) \) (because \( \rho_k \leq \rho_1 \)), we have that
(5.29) \[ \text{dist}_\infty(F_+(k), G) \geq \text{dist}_\infty(F_+(1), G) > 0 \text{ for all } k \in \mathbb{N}. \]

Now we proceed more or less similarly to the last steps of the proof of Theorem 4.1. By (5.19) the cube \( W(k) = Q(x_k, r_k) \) intersects both of the sets \( F_+(k) \) and \( G \). So its side
length $2r_k$ cannot be smaller than $\text{dist}_\infty (F_+(k), G)$. Consequently, also using (5.29), we see that
\[
\frac{1}{2} \geq r_k \geq \delta_0 := \frac{1}{2} \text{dist}_\infty (F_+(1), G) > 0 \text{ for all } k \in \mathbb{N}.
\]
Consequently $r \geq \delta_0$ and so (5.23) holds, permitting us to use the reasoning of Step 2 to ensure the existence of cube $W$ which has the properties required to complete the proof of part (a).

**STEP 4: Completion of the proof of part (b) of the lemma.**

In view of Step 2, if the sequence of cubes $\{W(n_k)\}_{k \in \mathbb{N}}$ constructed in Step 1 converges to the cube $W$, then that cube is contained in $[0, 1]^d$ and satisfies (5.4) and no further reasoning is required to complete the proof of part (b). Thus it remains only to deal with the case where $r = \lim_{k \to \infty} r_{n_k} = 0$. It is clear from the first part of Step 1 that this cannot happen if $(F_+, F_-)$ satisfies condition (i) of Definition 5.1. So the sequence $\{W(n_k)\}_{k \in \mathbb{N}}$ has necessarily been constructed as in the second part of Step 1, via the sets $F(k)$ and other sets introduced there.

Let us first see that, in this case, the point $x = \lim_{k \to \infty} x_{n_k}$ cannot coincide with the point $z$ which appears in the definition (5.12) of the sets $F_+(k)$. If $x = z$ and is therefore in the interior $V^\circ$ of the dyadic cube $V$ used in the construction, then there exists some $k$ which is sufficiently large to ensure that $r_{n_k} + \|x_{n_k} - x\|_\infty < \text{dist}_\infty (x, \partial V)$. This means that every point $y$ in the cube $W(n_k) = Q(x_{n_k}, r_{n_k})$ satisfies $\|y - x\|_\infty \leq \|y - x_{n_k}\|_\infty + \|x_{n_k} - x\|_\infty < \text{dist}_\infty (x, \partial V)$. Consequently $W(n_k) \subset V^\circ$ and therefore $W(n_k) \cap G = \emptyset$. (Recall that $G$ is defined by (5.14).) Since this contradicts (5.19), we have indeed shown that $x \neq z$.

In view of this fact, there exists an integer $k_0$ which is large enough to ensure that
\[
\|x - x_{n_{k_0}}\|_\infty + \rho_{n_{k_0}} + r_{n_{k_0}} < \|x - z\|_\infty, \tag{5.30}
\]
where $\rho_{n_{k_0}}$ is a element of the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ with limit 0 which is used in the definition (5.12) of the sets $F_+(k)$. If $y$ is a point in the intersection of the two cubes $W(n_{k_0}) = Q(x_{n_{k_0}}, r_{n_{k_0}})$ and $Q(z, \rho_{n_{k_0}})$, then
\[
\|x - z\|_\infty \leq \|x - x_{n_{k_0}}\|_\infty + \|x_{n_{k_0}} - y\|_\infty + \|y - z\|_\infty \\
\leq \|x - x_{n_{k_0}}\|_\infty + \rho_{n_{k_0}} + r_{n_{k_0}}.
\]
But this contradicts (5.30) and enables us to conclude that $W(n_{k_0})$ and $Q(z, \rho_{n_{k_0}})$ must be disjoint and therefore that
\[
W(n_{k_0}) \cap F_+(n_{k_0}) = W(n_{k_0}) \cap F_+.
\]
We apply this, together with (5.22) for $k = n_{k_0}$ and then (5.19) for $k = n_{k_0}$, to obtain that the cube $W := W(n_{k_0})$ satisfies
\[
\min \{\lambda (W \cap F_+), \lambda (W \cap F_-)\} = \min \{\lambda (W \cap F_+(n_{k_0})), \lambda (W \cap G)\} \\
\geq s_{j(n_{k_0})} \lambda (W). \tag{5.31}
\]

The cube $W$, like all other cubes in the sequence $\{W(n_k)\}_{k \in \mathbb{N}}$ is contained in $[0, 1]^d$. Finally we have to recall that in the statement of part (b) of the lemma, the sequence $\{s_n\}_{n \in \mathbb{N}}$ is required to satisfy $s_n \geq s$ for each $n$. So (5.31) shows that, also in this last remaining case, we have obtained a subcube $W$ of $[0, 1]^d$ which satisfies (5.4) for the given sets $F_+$ and $F_-$. This completes the proof of the lemma. \square
5.2. Some pairs which are not John-Strömberg pairs. The following result is a more elaborate variant of Remark 3.2.

Lemma 5.7. For each $d \in \mathbb{N}$ and each $\tau > 0$, and for every $s > 1/(2 + 1/\tau)$, the pair $(\tau, s)$ is not in $JS(d)$.

Proof. If $s > 1/2$ then the result follows from Remark 3.2. So we can assume that $1/(2 + 1/\tau) < s \leq 1/2$. Let us choose some number $a \in (1/(2 + 1/\tau), s)$ and then let $E_- = \{(x, t) : x \in [0, 1]^{d-1}, 0 \leq t \leq a\}$ and $E_+ = \{(x, t) : x \in [0, 1]^{d-1}, 1 - a \leq t \leq 1\}$. (Obviously when $d = 1$ we have to interpret the previous definition to mean that $E_- = [0, a]$ and $E_+ = [1 - a, 1]$.) Since $a < 1/2$, these two sets are disjoint. Furthermore

$$\lambda([0, 1]^d \setminus E_+ \setminus E_-) = 1 - 2a = \frac{1 - 2a}{\lambda} \min \{\lambda(E_+), \lambda(E_-)\}$$

or, equivalently,

$$\min \{\lambda(E_+), \lambda(E_-)\} = \frac{1}{1 - 2a} \lambda([0, 1]^d \setminus E_+ \setminus E_-).$$

Since $a > 1/(2 + 1/\tau)$ it follows that $\frac{1}{1 - 2a} > \tau$ and so

$$\min \{\lambda(E_+), \lambda(E_-)\} > \tau \lambda([0, 1]^d \setminus E_+ \setminus E_-).$$

We will complete the proof of this lemma by showing that, although the two disjoint measurable subsets $E_+$ and $E_-$ of $[0, 1]^d$ satisfy (5.32), there does not exist any subcube $W$ of $[0, 1]^d$ which satisfies $\min \{\lambda(E_+ \cap W), \lambda(E_- \cap W)\} \geq s \lambda(W)$. This will follow from the inequality

$$\frac{\min \{\lambda(E_+ \cap W), \lambda(E_- \cap W)\}}{\lambda(W)} \leq a$$

which will be seen to hold for every subcube $W$ of $[0, 1]^d$. This inequality seems intuitively quite obvious, but let us nevertheless give a detailed (and perhaps not optimally elegant) proof.

Let $W$ be an arbitrary subcube of $[0, 1]^d$ and let $\theta$ be its side length. Of course $\theta \in (0, 1]$ and $W$ must be of the form $W = \{(x, t) : x \in W_0, t \in [\beta, \beta + \theta]\}$ where $W_0$ is some subcube of $[0, 1]^{d-1}$ of side length $\theta$ and $[\beta, \beta + \theta] \subset [0, 1]$. (If $d = 1$ then $W$ is simply the interval $[\beta, \beta + \theta]$.) We only need to consider the case where $[\beta, \beta + \theta]$ has a non-empty intersection with each one of the intervals $[0, a]$ and $[1 - a, 1]$, since otherwise at least one of the two sets $E_+ \cap W$ and $E_- \cap W$ is empty and (5.33) is a triviality. The non-emptiness of the above mentioned two intersections implies that

$$\beta \leq a \text{ and } \beta + \theta \geq 1 - a.$$

When $\theta$ and $\beta$ satisfy all the above mentioned conditions we have that $\lambda(E_- \cap W) = \theta^{d-1}(a - \beta)$ and $\lambda(E_+ \cap W) = \theta^{d-1}(\beta + \theta - (1 - a))$. Therefore

$$\min \{\lambda(E_+ \cap W), \lambda(E_- \cap W)\} \leq \theta^{d-1} \sup_{\beta \in \mathbb{R}} \min \{a - \beta, \beta + \theta - (1 - a)\}. $$

For each fixed choice of $\theta$, the expression in the square brackets on the right side of (5.34) is the minimum of a strictly decreasing function of $\beta$ and a strictly increasing function of $\beta$. Therefore its supremum and thus its maximum is attained when $\beta$ takes the unique
value for which these two functions are equal, namely when \( \beta = (1 - \theta)/2 \). This shows that

\[
(5.35) \quad \frac{\min \{ \lambda(E_+ \cap W), \lambda(E_- \cap W) \}}{\lambda(W)} \leq \frac{\theta^{d-1}}{\theta^d} \left( a - \frac{1 - \theta}{2} \right) = \frac{1}{2} - \frac{1}{\theta} \left( \frac{1}{2} - a \right).
\]

Finally, the facts that \( \theta \in (0, 1) \) and \( \frac{1}{2} - a > 0 \) imply that the right side of (5.35) is bounded above by

\[
\frac{1}{2} - \left( \frac{1}{2} - a \right) = a
\]

which establishes (5.33) and so completes the proof of the lemma. \( \square \)

Remark 5.8. In particular, when \( \tau = 1/2 \), Lemma 5.7 shows that

\[
(5.36) \quad \left( \frac{1}{2}, \frac{1}{4} + \varepsilon \right) \notin JS(d) \text{ for every } d \in \mathbb{N} \text{ and every } \varepsilon > 0.
\]

Thus the second of the three results obtained recently by Ron Holzman (see Section 2), namely that \( \left( \frac{1}{2}, \frac{1}{4} \right) \in JS(1) \), shows that \( s = \frac{1}{4} \) is the largest possible value of \( s \) for which \( \left( \frac{1}{2}, s \right) \in JS(1) \). In the light of (5.36), the first of his three results can also be considered, in some sense, to be best possible. At least in the non-trivial special case that he considered there he obtained that there exists a cube \( W \) in \([0, 1]^d\) which satisfies (1.2) for \( s = 1/4 \). (Apparently one cannot exclude the possibility that, at least in that special case, one might be able to also obtain a cube \( W \) in \([0, 1]^d\) which satisfies (1.2) for some \( s > 1/4 \).)

These results tempt one to wonder whether perhaps the property \( \left( \frac{1}{2}, \frac{1}{4} \right) \in JS(d) \) might hold for all \( d \in \mathbb{N} \), if so, that would of course also answer Question A(1/2) affirmatively, and in an optimally strong, even “dramatically strong” way. However, in the third of his results relating to this question, Holtzman has analysed the following example suggested by the author and has shown that this is an “impossible dream”. This property fails to hold already for \( d = 2 \). Therefore (cf. Theorem 5.14 below) it also does not hold for any other \( d > 1 \).

Let \( F_+ \) be the rectangle \( F_+ = [0, 1] \times [2/3, 1] \) and let \( F_- \) be the union of the two squares \([0, 1/3] \times [0, 1/3]\) and \([2/3, 1] \times [0, 1/3]\). Then \( (F_+, F_-) \) is a tame couple in \([0, 1]^2\). The areas of the three disjoint sets \( F_+, F_- \) and \([0, 1]^2 \setminus F_+ \setminus F_- \) are respectively \( 1/3, 2/9 \) and \( 4/9 \), and this ensures that \( F_+ \) and \( F_- \) satisfy (5.3) for \( \tau = 1/2 \) and \( d = 2 \). (In this example \( \lambda \) will of course always denote two-dimensional Lebesgue measure.) Let \( \mathcal{W} \) denote the collection of all closed squares with sides parallel to the axes which are contained in \([0, 1]^2\). For each \( W \in \mathcal{W} \) let

\[
f(W) = \min \{ \lambda(W \cap F_+), \lambda(W \cap F_-) \} / \lambda(W).
\]

The third of Ron Holzman’s results is that, for these choices of \( F_+ \) and \( F_- \),

\[
\sup \{ f(W) : W \in \mathcal{W} \} = \max \{ f(W) : W \in \mathcal{W} \} = \sqrt{5} - 2.
\]

In view of part (*** of Theorem 5.3, this shows that, indeed, \( (1/2, 1/4) \notin JS(2) \) and, furthermore (again recalling Theorem 5.14), that \( (1/2, s) \notin JS(d) \) for every \( s > \sqrt{5} - 2 \) and \( d \geq 2 \).

It is tempting to wonder whether a sequence of appropriate variants of this example for subsets of \([0, 1]^d\), where \( d \) tends to \( \infty \), might lead to a negative answer to Question A(1/2). Initial attempts to find such a sequence have not yielded anything decisive.
5.3. Some further properties of the set $JS(d)$.

**Fact 5.9.** For each $d \in \mathbb{N}$ and each $\tau > 0$ there exists some $s > 0$ such that $(\tau, s) \in JS(d)$.

*Proof.* This is a consequence of Theorem 7.8 of [1] pp. 28–29 a.k.a. Theorem 7.7 of [3] pp. 152–153] which, for each $\tau > 0$, provides a positive number $s$ depending on $\tau$ and $d$ such that $(\tau, s) \in JS(d)$. (The explicit formula for $s$ will be recalled and used in Theorem 5.12 below. □

The preceding result ensures that the supremum in the following definition is taken over a non-empty set.

**Definition 5.10.** For each $d \in \mathbb{N}$ and $\tau > 0$ let

$$
\sigma(\tau, d) := \sup \{ s > 0 : (\tau, s) \in JS(d) \}.
$$

**Remark 5.11.** Thus the second and third results of Ron Holzman mentioned in Section 2 (cf. also Remark 5.8) can be written as

$$
\sigma \left( \frac{1}{2}, 1 \right) = \frac{1}{4} \quad \text{and} \quad \sigma \left( \frac{1}{2}, 2 \right) \leq \sqrt{5} - 2.
$$

We can now readily establish several properties of the set $JS(d)$ and the function $\sigma(\tau, d)$.

**Theorem 5.12.** For each fixed $d \in \mathbb{N}$,

(i) $(\tau, \sigma(\tau, d)) \in JS(d)$ for each $\tau > 0$.

(ii) $JS(d) = \{(x, y) : x > 0, 0 < y \leq \sigma(x, d)\}$

(iii) The function $x \mapsto \sigma(x, d)$ is non-decreasing and continuous and satisfies

$$
\varphi(x, d) \leq \sigma(x, d) \leq \frac{1}{2 + \frac{1}{2}} \quad \text{for all} \ x > 0,
$$

where

$$
\varphi(x, d) = \left\{ \begin{array}{ll}
2^{-d}(x - x^2)/(1 + x) & , \quad 0 < x \leq \sqrt{2} - 1 \\
2^{-d}(3 - 2\sqrt{2}) & , \quad x \geq \sqrt{2} - 1.
\end{array} \right.
$$

*Proof.* For part (i) let us fix an arbitrary $\tau > 0$ and first note the obvious fact that

$$
(\tau, s) \in JS(d) \Rightarrow (\tau, s') \in JS(d) \quad \text{for all} \ s' \in (0, s),
$$

which implies that $(\tau, s') \in JS(d)$ for every $s' \in (0, \sigma(\tau, d))$. Therefore the sequences $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{s_n\}_{n \in \mathbb{N}}$ which we define by $\tau_n := \tau$ and $s_n := (1 - 2^{-n})\sigma(\tau, d)$ must satisfy $(\tau_n, s_n) \in JS(d)$ for every $n \in \mathbb{N}$. For the proof of part (i) we now simply apply part (a) of Lemma 5.3 to the sequences $\{\tau_n\}_{n \in \mathbb{N}}$ and $\{s_n\}_{n \in \mathbb{N}}$ and then apply Theorem 5.12. Part (ii) then follows immediately from part (i) and (5.38). For part (iii) we first use another obvious fact, namely that

$$(\tau, s) \in JS(d) \Rightarrow (\tau', s) \in JS(d) \quad \text{for all} \ \tau' > \tau$$

to immediately deduce that the function $x \mapsto \sigma(x, d)$ is non-decreasing. This latter property means that, in order to show that this function is continuous, it will suffice to show that

$$
\lim_{n \to \infty} \sigma((1 + 1/n)\tau, d) \leq \sigma(\tau, d) \quad \text{and} \quad \lim_{n \to \infty} \sigma((1 - 1/n)\tau, d) \geq \sigma(\tau, d) \quad \text{for each} \ \tau > 0.
$$

We obtain the first of these inequalities by again using Theorem 5.3 together with part (a) of Lemma 5.3 to show that the limit of the sequence $\{(1 + 1/n)\tau, \sigma((1 + 1/n)\tau, d)\}_{n \in \mathbb{N}}$ of points in $JS(d)$ must also be a point in $JS(d)$. We next remark that, since $(\tau, \sigma(\tau, d)) \in$
$JS(d)$, it follows from Theorem 4.1 that \((1 - 1/n)\tau, \sigma(\tau, d) - (1 - \sigma(\tau, d))/n \in JS(d)\) for all sufficiently large \(n \in \mathbb{N}\). Therefore \(\sigma((1 - 1/n)\tau, d) \geq \sigma(\tau, d) - (1 - \sigma(\tau, d))/n\) for these same values of \(n\). This suffices to prove the second inequality in (5.40) and complete the proof of continuity.

The formula (5.38) for \(\varphi(x, d)\) in the estimate from below in (5.37) is obtained by once more appealing to Theorem 7.8 of [1] pp. 28–29] a.k.a. Theorem 7.7 of [3] pp. 152–153], and using the formulae appearing at the end of the statement of that theorem in the particular case where the collection of sets \(\mathcal{E}\) is chosen to be \(Q(\mathbb{R}^d)\) the collection of all cubes in \(\mathbb{R}^d\). We can replace \(M\) in those formulae by \(2^d\) using the fact that, in the terminology introduced just before the statement of that theorem, \(Q(\mathbb{R}^d)\) is \(2^d\)-decomposable. We can also replace \(\delta\) there by \(1/2\) using the fact (see Definition 7.4 of [1] p. 26] or [3] p. 151] and the remark immediately following it) that \(1/2\) is a bi-density constant for \(Q(\mathbb{R}^d)\). The estimate from above in (5.37) follows from Lemma 5.7.

This completes the proof of part (iii) and therefore of the whole theorem. \(\square\)

Remark 5.13. There is another different kind of lower bound for \(\sigma(\tau, d)\), in terms of values of \(\sigma(\tau', d)\) for appropriate numbers \(\tau'\) greater than \(\tau\), which can be obtained from Theorem 4.1. We have not bothered to explicitly state it here.

The following result seems intuitively completely obvious. But we shall provide an explicit proof.

Theorem 5.14. The inclusion \(JS(d + 1) \subset JS(d)\) and consequently also the inequality \(\sigma(\tau, d + 1) \leq \sigma(\tau, d)\) both hold for every \(d \in \mathbb{N}\) and \(\tau > 0\).

Proof. This is the one place in this paper where we need to use the more explicit notation \(\lambda_d\) instead of \(\lambda\) to denote \(d\)-dimensional Lebesgue measure. We will use Theorem 3.4.

Suppose that \((\tau, s) \in JS(d + 1)\). Let \(F_+\) and \(F_-\) be two arbitrarily chosen disjoint \(d\)-multi-cubes which satisfy (3.3), i.e., the inequality
\[
\min \{\lambda_d(F_+), \lambda_d(F_-)\} > \tau \lambda_d([0, 1]^d \setminus F_+ \setminus F_-),
\]
for this given value of \(\tau\). We define two subsets \(H_+\) and \(H_-\) of \([0, 1]^{d+1}\) as the cartesian products \(H_+ = F_+ \times [0, 1]\) and \(H_- = F_- \times [0, 1]\). They are disjoint, since \(F_+\) and \(F_-\) are disjoint. For each dyadic subcube \(E\) of \([0, 1]^d\), the cartesian product \(E \times [0, 1]\) is the union of \(2^k\) dyadic subcubes of \([0, 1]^{d+1}\), where \(k\) is such that the side length of \(E\) is \(2^{-k}\). It follows that \(H_+\) and \(H_-\) are both \((d + 1)\)-multi-cubes.

Now, and also again later, we shall use the very standard facts that
(i) for each bounded closed interval \([a, b]\), the set \(E \times [a, b]\) is a Lebesgue measurable subset of \(\mathbb{R}^{d+1}\) whenever \(E\) is a Lebesgue measurable subset of \(\mathbb{R}^d\), and that
(ii) \(\lambda_{d+1}(E \times [a, b]) = (b - a)\lambda_d(E)\) for each such \(E\).

These two facts together with the fact that \([0, 1]^{d+1} \setminus H_+ \setminus H_- = ([0, 1]^d \setminus F_+ \setminus F_-) \times [0, 1]\), and together with our assumption that \(F_+\) and \(F_-\) satisfy (5.41), imply that
\[
\min \{\lambda_{d+1}(H_+), \lambda_{d+1}(H_-)\} > \tau \lambda_{d+1}([0, 1]^{d+1} \setminus H_+ \setminus H_-).
\]
Consequently, by Theorem 3.4, our assumption that \((\tau, s) \in JS(d + 1)\) guarantees the existence of a subcube \(V\) of \([0, 1]^{d+1}\) for which
\[
\min \{\lambda_{d+1}(V \cap H_+), \lambda_{d+1}(V \cap H_-)\} \geq s \lambda_{d+1}(V).
\]
We can write \(V\) as the cartesian product \(V = W \times [a, b]\), where \(W\) is a subcube of \([0, 1]^d\) and \([a, b]\) is a subinterval of \([0, 1]\) whose length \(b - a\) of course equals the side length of \(V\) and of \(W\). Obviously \(V \cap H_+ = (W \cap F_+) \times [a, b]\) and \(V \cap H_- = (W \cap F_-) \times [a, b]\). So we
can again apply the standard facts (i) and (ii), which were recalled in an earlier step of this proof, to obtain the formulae $\lambda_{d+1}(E) = (b - a)\lambda_{d}(E)$ in the three cases where $E$ is $W \cap F_+$ or $W \cap F_-$ or $W$. When we substitute these formulae in (5.42) and divide both sides of the inequality by $b - a$, we obtain that the subcube $W$ of $[0,1]^d$ satisfies (3.4) of Theorem 3.4 i.e., that

$$\min\{\lambda_{d}(W \cap F_+), \lambda_{d}(W \cap F_-)\} \geq s\lambda_{d}(W).$$

Since $F_+$ and $F_-$ were chosen arbitrarily, we can once more apply Theorem 3.4 to deduce that $(\tau, s) \in JS(d)$, and so complete the proof of the present theorem. □

6. SOME COMMENTS AND MINOR CORRECTIONS FOR THE PAPERS [1, 3].

In the first paragraph of the proof of Theorem 7.8 [1, p. 30] a.k.a. Theorem 7.7 [3, pp. 154–155] it is shown that it suffices to consider the case where the two sets $E_+$ and $E_-$ are both compact. The justification of this is a little clearer if in the formula on the third line of the proof one replaces $G$ by $Q \setminus H_+ \setminus H_-$, which is obviously permissible.

We refer to the remarks made above in the course of the proof of Theorem 3.4 for some other small corrections and clarifications of some small issues in [1].

REFERENCES

[1] M. Cwikel, Y. Sagher and P. Shvartsman, A new look at the John-Nirenberg and John-Strömberg theorems for BMO. Lecture Notes. arXiv:1011.0766v1 [math.FA] (Posted on 2 Nov 2010).
[2] M. Cwikel, Y. Sagher and P. Shvartsman, A geometrical/combinatorial question with implications for the John-Nirenberg inequality for BMO functions, Banach Center Publ. 95 (2011), 45–53.
[3] M. Cwikel, Y. Sagher and P. Shvartsman, A new look at the John-Nirenberg and John-Strömberg theorems for BMO, J. Funct. Anal. 263 (2012), 129–166.
[4] R. Holzman, Private Communication.
[5] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961), 415–426.
[6] A. K. Lerner, The John-Nirenberg inequality with sharp constants, C. R. Math. Acad. Sci. Paris 351 (2013), 463–466.
[7] L. Slavin, The John–Nirenberg constant of BMO$^p$, $1 \leq p \leq 2$, arXiv:1506.04969v1 [math.CA] (Posted on 16 Jun 2015).
[8] L. Slavin and V. Vasyunin, Inequalities for BMO on α-Trees. IMRN (On line as of 1 Oct 2015).
[9] L. Slavin and V. Vasyunin, The John–Nirenberg constant of BMO$^p$, $p > 2$, arXiv:1601.03848 [math.CA] (Posted on 15 Jan 2016).
[10] L. Slavin and P. Zatitskii, Dimension-free estimates for semigroup BMO and $A_p$, arXiv:1908.02602 [math.CA] (Posted on 7 Aug 2019 and revised on 22 Aug 2019).
[11] D. Stolyarov and P. Zatitsky, Sharp transference principle for BMO and $A_p$, arXiv:1908.09497 [math.CA] (Posted on 26 Aug 2019).
[12] R. Wheeden and A. Zygmund, Measure and integral. An introduction to real analysis. Pure and Applied Mathematics, Vol. 43. Marcel Dekker, Inc., New York-Basel, 1977.

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