Topologies on the symmetric inverse semigroup

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Abstract
The symmetric inverse semigroup $I(X)$ on a set $X$ is the collection of all partial bijections between subsets of $X$ with composition as the algebraic operation. We study the minimal Hausdorff inverse semigroup topology on $I(X)$. We present some characterizations of it. When $X$ is countable such topology is Polish.

Keywords Inverse semigroup · Topological semigroup · Polish semigroup

1 Introduction

The symmetric inverse semigroup $I(X)$ on a set $X$ is the collection of all partial bijections between subsets of $X$ with composition as the algebraic operation. Among inverse semigroups, $I(X)$ plays a role analogous to that played by the symmetric group $S_\infty(X)$ for groups: every inverse semigroup $S$ is isomorphic to a subsemigroup of $I(S)$. In this paper we study some semigroup topologies on $I(X)$. In the case of a countable $X$, the topologies on $I(X)$ can be found Polish (i.e. completely metrizable and separable).

The symmetric group $S_\infty(X)$ is a subsemigroup of $X^X$ and also of $I(X)$. The usual topology of $S_\infty(X)$ is the one it inherits from the product topology on $X^X$ (where $X$ has the discrete topology). We define a topology $\tau_{pp}$ on $I(X)$ that we call the partial product topology. It is a generalization of the product topology on $X^X$ and it also induces in $S_\infty(X)$ its usual product topology. It turns out that $\tau_{pp}$ is the minimal
inverse semigroup Hausdorff topology on $I(X)$. We present a characterization of $\tau_{pp}$ analogous to that well known fact that the product topology is the smallest topology making all projections continuous. We show that there is an onto map $\pi : S_\infty(Y) \to I(X)$ such that $\tau_{pp}$ is the quotient topology given by $\pi$. Moreover, $\tau_{pp}$ is the unique inverse semigroup Hausdorff topology on $I(X)$ with respect to which $\pi$ is continuous. The topology $\tau_{pp}$ was independently studied in [3]. Some of the results we present are coming from [9].

We remark that $X$ is always assumed to be an infinite discrete space, or equivalently, $X$ is just an infinite set without a topology. For a non discrete topological space $X$, the natural extension of our results to the semigroup of partial homeomorphism will be treated elsewhere.

Our motivation came from some recent works about groups and semigroups which admits a unique Polish group (semigroup) topology. A theorem of Kallman [6] says that $S_\infty(\mathbb{N})$ has a unique Polish group topology, namely the product topology (see [3, Exercise 2.3.9]). The same happens with the group of homeomorphisms of the Cantor space as shown by Rosendal and Solecki [10]. Mesyan et al. [8] showed that the product topology is the unique Polish semigroup topology on $\mathbb{N}^\mathbb{N}$. More recently, Elliott et al. made a very extensive study of Polish semigroup topologies [3]. In particular, they showed that there is a unique Polish inverse semigroup topology on $I(\mathbb{N})$, namely $\tau_{pp}$. Their proof is based on a general criterion for getting automatic continuity for $I(\mathbb{N})$ using the fact that $S_\infty(\mathbb{N})$ admits a unique Polish group topology. Our approach is quite different but we obtained a weaker result (see Theorem 6.6). We refer the reader to [2] (and references therein) to get a more complete account of this topic.

2 Preliminaries

A *semigroup* is a non-empty set $S$ together with an associative binary operation $\circ$. To simplify the notation we sometimes write $st$ in placed of $s \circ t$. A semigroup $S$ is *regular* if for all $s \in S$, there is $t \in S$ such that $sts = s$ and $tst = t$. In this case, $t$ is called an inverse of $s$. If each element have a unique inverse, $S$ is an *inverse* semigroup and, in this case, $s^*$ denotes the inverse of $s$. An element $s$ of a semigroup is *idempotent* if $ss = s$. We denote by $E(S)$ the collection of idempotents of $S$.

Let $\tau$ be a topology on a semigroup $S$. If the multiplication $S \times S \to S$ is continuous, we call $S$ a topological semigroup. An inverse semigroup $S$ is called topological if it is a topological semigroup and the function $i : S \to S$, $s \to s^*$ is continuous. We refer the reader to [1] as a general reference for topological semigroups.

We are going to present some general facts about metrizable inverse semigroups that will be needed in the sequel. Given a metric $d$, we denote by $\tau_d$ the topology given by $d$.

**Proposition 2.1** Let $S$ be an inverse semigroup and $\tau$ be a topology on $S$ such that $(S, \tau)$ is a Hausdorff topological semigroup. Suppose $(s_n)_n$ is a sequence on $S$ such that $s_n \to s$ and $s_n^* \to t$. Then $t = s^*$. 
Suppose \( s_n = s_n s^*_n s_n \) and \( s^*_n = s^*_n s_n s^*_n \) for all \( n \in \mathbb{N} \). By the continuity of the semigroup operation, we have that \( s_n = s_n s^*_n s_n \to sts \) and \( s^*_n = s^*_n s_n s^*_n \to tst \). Thus \( s = sts \) and \( t = tst \), in other words, \( t = s^* \).

Let \( S \) be an inverse semigroup and \( \rho \) a metric on \( S \). Define another metric as follow

\[
\rho^*(s, t) = \rho(s^*, t^*).
\]

**Proposition 2.2** Let \( S \) be an inverse semigroup and \( \rho \) a metric on \( S \).

(i) The map \( i : (S, \tau_\rho) \to (S, \tau_{\rho^*}) \), \( s \mapsto s^* \), is an isometry. Moreover, if \( \rho \) is complete, so is \( \rho^* \).

(ii) Suppose \( (S, \tau_\rho) \) is a topological semigroup, then \( (S, \tau_{\rho^*}) \) is a topological semigroup.

**Proof** It is straightforward recalling that \( (st)^* = t^* s^* \).

The following theorem is motivated by a similar result for metrizable groups (see [3, Corollary 2.2.2]).

**Theorem 2.3** Let \( S \) be an inverse semigroup and \( \rho \) a complete metric on \( S \) such that \( (S, \tau_\rho) \) is a topological semigroup. Let \( d = \rho + \rho^* \), then \( d \) is a complete metric on \( S \) and \( (S, \tau_d) \) is a topological inverse semigroup.

**Proof** Let us see that \( d \) is complete. Let \( (s_n)_n \) be a \( d \)-Cauchy sequence and \( \epsilon > 0 \), then there is \( N \in \mathbb{N} \) such that if \( n, m > N \), \( d(s_n, s_m) < \epsilon \). Thus \( \rho(s_n, s_m) < \epsilon \) and \( \rho^*(s_n, s_m) < \epsilon \) for all \( n, m > N \). Since \( \rho \) and \( \rho^* \) are complete, there are \( s, t \in S \) such that \( s_n \xrightarrow{d} s \) and \( s_n \xrightarrow{\rho^*} t \). By Proposition 2.1, \( t = s \). It is easy to see that \( s_n \xrightarrow{d} s \).

Now we show that \( \tau_d \) is an inverse semigroup topology. First, we note that \( i^{-1}(B_d(s, r)) = B_d(s^*, r) \), thus \( i \) is continuous. To see that the operation \( (S, \tau_d) \times (S, \tau_d) \to (S, \tau_d) \) is continuous, let \( (x_n)_n \) and \( (y_n)_n \) be sequences such that \( x_n \xrightarrow{d} x \) and \( y_n \xrightarrow{d} y \), we show that \( x_n y_n \xrightarrow{d} xy \). Clearly we have \( x_n \xrightarrow{\rho} x \), \( y_n \xrightarrow{\rho} y \), \( x^*_n \xrightarrow{\rho} x^* \) and \( y_n^* \xrightarrow{\rho} y^* \). Since the operation is \( \tau_\rho \)-continuous, \( x_n y_n \xrightarrow{\rho} xy \) and \( (x_n y_n)^* \xrightarrow{\rho} (xy)^* \), therefore \( x_n y_n \xrightarrow{d} xy \).

### 3 The partial product topology

The symmetric inverse semigroup on a nonempty set \( X \) is defined as follows:

\[
I(X) = \{ f : A \to B \mid A, B \subseteq X \text{ and } f \text{ is bijective} \}.
\]

For \( f : A \to B \) in \( I(X) \) we denote \( A = \text{dom}(f) \) and \( B = \text{im}(f) \). Let \( 2^X \) denote the power set of \( X \) endowed with the product topology of \( \{0, 1\}^X \) by the usual identification of a subset of \( X \) with its characteristic function. The symmetric group, \( S_\infty(X) \), is the collection of all bijections from \( X \) to \( X \). It is a topological group with...
respect to the topology it gets as a subset of $X^X$ endowed with the product topology (with $X$ discrete). Our main interest is when $X$ is an infinite set and this will be assumed from now on.

For each $x \in X$, let

$$D_x = \{ f \in I(X) : x \in \text{dom}(f) \}.$$  

The following functions play an analogous role as the projection functions for the product topology on $X^X$.

$$\text{dom} : I(X) \to 2^X, f \mapsto \text{dom}(f),$$

$$\text{im} : I(X) \to 2^X, f \mapsto \text{im}(f),$$

$$\text{ev}_x : D_x \to X, f \mapsto f(x), \text{ with } x \in X.$$  

The operation on $I(X)$ is the usual composition, namely, given $f, g \in I(X)$, then $f \circ g$ is defined by letting $\text{dom}(f \circ g) = g^{-1}(\text{dom}(f) \cap \text{im}(g))$ and if $x \in \text{dom}(f \circ g)$ then $(f \circ g)(x) = f(g(x))$. The idempotents of $I(X)$ are the partial identities $1_A : A \to A, 1_A(x) = x$ for all $x \in A$ and $A \subseteq X$. Notice that $1_\emptyset$ is the empty function which also belongs to $I(X)$. We remark that $1_\emptyset$ needs to be included in $I(X)$ in order that the operation $\circ$ is well defined, for instance, $1_A \circ 1_B = 1_\emptyset$ for $A$ and $B$ disjoint subsets of $X$.

For $x, y \in X$, let

$$v(x, y) = \{ f \in I(X) \mid x \in \text{dom}(f) \text{ and } f(x) = y \},$$

$$w_1(x) = \{ f \in I(X) \mid x \notin \text{dom}(f) \},$$

$$w_2(y) = \{ f \in I(X) \mid y \notin \text{im}(f) \}.$$  

It is clear that the sets $v(x, y)$ are motivated by the usual subbase for the product topology on $X^X$. As we will see, these sets are tightly related to any $T_1$ semigroup topology on $I(X)$.

Now we introduce some semigroup topologies on $I(X)$. Let $\tau_0$ be the topology generated by $\{v(x, y) : x, y \in X\}$, $\tau_1$ be the topology generated by $\{v(x, y), w_1(x) : x, y \in X\}$ and $\tau_{pp}$ be the topology generated by $\{v(x, y), w_1(x), w_2(y) : x, y \in X\}$. Some basic properties of these topologies follow from the next lemma.

**Lemma 3.1** Let $c : I(X) \times I(X) \to I(X)$ given by $c(f, g) = f \circ g$ and $i : I(X) \to I(X)$ given by $i(f) = f^{-1}$. Then, for all $x, y \in X$, we have

(i) $c^{-1}(v(x, y)) = \bigcup_{z \in X} (v(z, y) \times v(x, z)).$

(ii) $c^{-1}(w_1(x)) = (I(X) \times w_1(x)) \cup \bigcup_{z \in X} (w_1(z) \times v(x, z)).$

(iii) $c^{-1}(w_2(y)) = (w_2(y) \times (I(X)) \cup \bigcup_{z \in X} (v(z, y) \times w_2(z)).$

(iv) $i^{-1}(v(x, y)) = v(y, x).$

(v) $i^{-1}(w_1(x)) = w_2(x).$

(vi) $i^{-1}(w_2(y)) = w_1(y).$
Proof It is straightforward. □

These topologies were independently defined in [2] where it was shown essentially the following.

Theorem 3.2 (i) \((I(X), \tau_0)\) is a \(T_0\) topological inverse semigroup but it is not \(T_1\).

(ii) \((I(X), \tau_1)\) is a Hausdorff topological semigroup.

(iii) \((I(X), \tau_{pp})\) is a Hausdorff topological inverse semigroup.

Proof Let \(\tau\) be any of the topologies mentioned in the hypothesis. From Lemma 3.1 it follows that \(\tau\) is a semigroup topology on \(I(X)\).

(i) Clearly, from Lemma 3.1, the inversion map \(i\) is continuous with respect to \(\tau_0\). It remains to show that \(\tau_0\) is \(T_0\). In fact, let \(f, g \in I(X)\) with \(f \neq g\). There are several cases to consider, we treat only one, the others are analogous. Assume there is \(x \in \text{dom}(f) \cap \text{dom}(g)\) such that \(f(x) \neq g(x)\). Then \(f \in v(x, f(x))\) and \(g \notin v(x, f(x))\). Notice that \(I(X)\) is the only \(\tau_0\)-open set containing \(1_\emptyset\), thus \((I(X), \tau_0)\) is not \(T_1\).

(ii) It remains to show that \(\tau_1\) is \(T_2\). This is easily done by analyzing the following cases. Let \(f \neq g\) in \(I(X)\). (a) There is \(x \in \text{dom}(f) \triangle \text{dom}(g)\). (b) There is \(x \in \text{dom}(f) \cap \text{dom}(g)\) with \(f(x) \neq g(x)\).

(iii) It follows from (ii) and Lemma 3.1. □

We call \(\tau_{pp}\) the partial product topology. Perhaps a more appropriate but longer name would be “partial product topology for a discrete space \(X\)”.

The main objective of this paper is to study \(\tau_{pp}\). We include \(\tau_0\) to stress that the most natural generalization of the product topology on \(X^X\) gives an inverse semigroup topology on \(I(X)\) but it is not Hausdorff. To make it Hausdorff we add the sets \(w_1(x)\) and to keep it an inverse semigroup topology we need to add also the sets \(w_2(x)\).

The following results show that \(\tau_{pp}\) is minimal among all Hausdorff inverse semigroup topologies on \(I(X)\) and also that declaring open all sets \(w_1(x)\) and \(w_2(x)\) is somewhat unavoidable. We introduce a notation for some natural subsemigroups of \(I(X)\). For each cardinal \(\kappa \leq |X|\), let

\[ I_\kappa(X) = \{ f \in I(X) : |\text{dom}(f)| \leq \kappa \}. \]

Theorem 3.3 Suppose \((I(X), \tau)\) is a \(T_1\) topological semigroup. Then

(i) Each \(v(x, y)\) is clopen and \(w_1(y)\) and \(w_2(y)\) are closed for all \(y\).

(ii) Suppose \(\tau\) is an inverse semigroup topology. The relation \(\subseteq\) is closed on \(I(X) \times I(X)\) iff every \(w_1(x)\) is open.

(iii) If \(\tau\) is an inverse semigroup Hausdorff topology, then \(\tau_{pp} \subseteq \tau\). Moreover, if \(S\) is an inverse subsemigroup of \(I(X)\) such that \(I_1(X) \subseteq S\) and \(\rho\) is an inverse semigroup topology on \(S\), then \(\tau_{pp}|S \subseteq \rho\).

Proof (i) For each \(x, y \in X\), let \(u_{x,y} \in I(X)\) be such that \(\text{dom}(u_{x,y}) = \{x\}\) and \(\text{im}(u_{x,y}) = \{y\}\). Let \(\varphi : I(X) \to I(X)\) given by \(\varphi(h) = u_{y,x} \circ h \circ u_{y,x}\). Then \(\varphi\) is continuous and

\[ I(X) \setminus v(x, y) = \varphi^{-1}(1_\emptyset). \]
Thus, $v(x, y)$ is open. We also have that $v(x, y) = \varphi^{-1}(u_{y,x})$. Hence, $v(x, y)$ is also closed. To see that $w_1(x)$ and $w_2(y)$ are closed observe that $I(X) \setminus w_1(x) = \bigcup_{z \in X} v(x, z)$ and $I(X) \setminus w_2(y) = \bigcup_{z \in X} v(z, y)$.

(ii) Suppose $\subseteq$ is closed. Then $f \notin w_1(x)$ iff $u_{x,y} \subseteq f^{-1} \circ f$. Thus the complement of $w_1(x)$ is closed. Conversely, $f \notin g$ iff for some $x, y \in X$ we have that $f \notin v(x, y)$ and $g \in ((I(X) \setminus v(x, y)) \cup w_1(x)$. From this it follows that the complement of $\subseteq$ is open.

(iii) Suppose that $\tau$ is an inverse semigroup Hausdorff topology on $I(X)$. From (i) we have that each $v(x, y)$ is $\tau$-open. To see that each $w_1(x)$ is $\tau$-open, by (ii), it suffices to show that $\subseteq$ is $\tau \times \tau$-closed. In fact, notice that $f \subseteq g$ iff $f = f \circ g$. Since $\circ$ is $\tau$-continuous, then $\subseteq$ is closed. Finally, since $\tau$ is an inverse semigroup topology and each $w_1(x)$ is $\tau$-open, from Lemma 3.1 we conclude that each $w_2(y)$ is also $\tau$-open. Thus $\tau_{pp} \subseteq \tau$. The proof of the last statement about inverse subsemigroups of $I(X)$ is a straightforward relativization of the previous argument. \qed

**Corollary 3.4** $(I(X), \tau_{pp})$ is a regular space.

**Proposition 3.5** Let $\tau$ be a semigroup topology on $I(X)$. If $w_1(x)$ is open for some $x$, then $w_1(y)$ is open for every $y$. Moreover, if $\tau$ is an inverse semigroup topology on $I(X)$ and $w_1(x)$ is open for some $x$, then $w_2(y)$ is open for every $y$.

**Proof** Let $g \in S_\infty(X)$ be such that $g(x) = y$. Let $\varphi : I(X) \to I(X)$ be given by $\varphi(f) = f \circ g$. Since $\tau$ is a semigroup topology, clearly $\varphi$ is a homeomorphism and $w_1(y) = \varphi^{-1}(w_1(x))$. The second claim follows from the first and Lemma 3.1. \qed

Next result shows that $\tau_1$ and $\tau_{pp}$ are different.

**Proposition 3.6** $w_2(y)$ is $\tau_1$-nowhere dense for all $y \in X$.

**Proof** By Theorem 3.2, $\tau_1$ is a $T_1$ semigroup topology and by Proposition 3.3 each $w_2(y)$ is $\tau_1$-closed. We show that each $w_2(y)$ has empty $\tau_1$-interior. Let $V$ be the basic $\tau_1$-open set

$$V = \bigcap_{i=1}^n v(x_i, y_i) \cap \bigcap_{i=1}^m w_1(z_i).$$

If $y = y_i$ for some $i$, then $V \cap w_2(y) = \emptyset$. Otherwise, as $X$ is infinite, pick $x \notin \{x_1, \ldots, x_n, z_1, \ldots, z_m\}$. Then $V \cap v(x, y) \neq \emptyset$ and $V \cap v(x, y) \cap w_2(y) = \emptyset$. \qed

Now we present a generalization of the fact that the product topology is the smallest topology with respect to which all projections are continuous.

**Theorem 3.7** (i) $\tau_{pp}$ is the smallest topology such that $\text{dom} : I(X) \to 2^X$, $\text{im} : I(X) \to 2^X$ and $\text{ev}_x : D_x \to X (x \in X)$ are continuous, where $2^X$ is endowed with the product topology and $X$ with the discrete topology.

(ii) $\text{dom}$ and $\text{im}$ are open maps when $I(X)$ is endowed with the topology $\tau_{pp}$.
(iii) Let $Y$ be a topological space and $\varphi : Y \to (I(X), \tau_{pp})$ a map. Then $\varphi$ is continuous if, and only if, $\text{dom} \circ \varphi$, $\text{im} \circ \varphi$ are continuous and $\text{ev}_x \circ \varphi$ is continuous on $\varphi^{-1}(D_x)$ for all $x \in X$.

**Proof** (i) For each $x \in X$, let $v_x = \{A \subseteq X : x \in A\}$. Each $v_x$ is clopen in $2^X$ and they form a subbasis for the product topology on $2^X$. The result follows immediately from the next identities:

$$w_1(x) = (\text{dom})^{-1}(2^X \setminus v_x), \quad w_2(x) = (\text{im})^{-1}(2^X \setminus v_x), \quad v(x, y) = \text{ev}_x^{-1}(\{y\}).$$

(ii) Let $\{x_i : 1 \leq i \leq n\}, \{y_i : 1 \leq i \leq n\}, \{u_j : 1 \leq j \leq m\}$ and $\{z_k : 1 \leq k \leq l\}$ be finite subsets of $\mathbb{N}$ and consider the basic open set in $I(\mathbb{N})$:

$$V = \bigcap_{i=1}^n v(x_i, y_i) \cap \bigcap_{j=1}^m w_1(u_j) \cap \bigcap_{k=1}^l w_2(z_k).$$

Then

$$\{\text{dom}(f) : f \in V\} = \{A \in 2^X : \forall i \leq n(x_i \in A) \text{ and } \forall j \leq m(u_j \notin A)\}.$$

is clearly open. Analogously

$$\{\text{im}(f) : f \in V\} = \{B \in 2^X : \forall i \leq n(y_i \in B) \text{ and } \forall k \leq l(z_k \notin B)\}.$$

(iii) Let $\varphi : Y \to I(X)$ satisfy the hypothesis. We will show that $\varphi$ is continuous. Let $V \subseteq I(X)$ be a basic open set as in (ii). Let $y \in \varphi^{-1}(V)$, then $x_i \in \text{dom}(\varphi(y))$ for $1 \leq i \leq n$. By (ii), $\text{dom}(V)$ and $\text{im}(V)$ are open in $2^X$. We claim that

$$y \in (\text{dom} \circ \varphi)^{-1}(\text{dom}(V)) \cap (\text{im} \circ \varphi)^{-1}(\text{im}(V)) \cap \bigcap_{i=1}^n (\text{ev}_{x_i} \circ \varphi)^{-1}(\{y_i\}) \subseteq \varphi^{-1}(V).$$

In fact, since $\text{dom}(\varphi(y)) \in \text{dom}(V), u_i \notin \text{dom}(\varphi(y))$ for all $j \leq m$. Analogously, $z_k \notin \text{im}(\varphi(y))$ for all $k \leq l$. Finally, as $y \in (\text{ev}_{x_i} \circ \varphi)^{-1}(\{y_i\})$, then $\varphi(y)(x_i) = y_i$. Thus $\varphi(y) \in V$. We have shown that $\varphi^{-1}(V)$ is open and hence $\varphi$ is continuous. □

Next we present a characterization of when the collection of idempotents of $I(X)$ is compact. Recall that $2^X$ is endowed with the product topology.

**Theorem 3.8** Let $\tau$ be an inverse semigroup Hausdorff topology on $I(X)$. The map $A \mapsto 1_A$ from $2^X$ to $I(X)$ is continuous if the collection of idempotents is compact. In particular, the collection of idempotents of $I(X)$ is $\tau_{pp}$-compact.

**Proof** Let $J = \{1_A : A \subseteq X\}$ be the collection of idempotents of $I(X)$. Thus, if the map $A \mapsto 1_A$ is continuous, then $J$ is compact. Conversely, suppose $J$ is compact. By Theorem 3.7, the function $\text{dom} : (I(X), \tau_{pp}) \to 2^X$ is continuous and $\tau_{pp} \subseteq \tau$ (by Theorem 3.3), thus $\text{dom}$ is also continuous with respect to $\tau$. Thus,
For all $x \in \text{dom}J : (J, \tau) \rightarrow 2^\mathbb{N}$ is a continuous bijection and hence a homeomorphism as $J$ is compact Hausdorff. Its inverse is the map $A \mapsto 1_A$.

For the last claim, let $\varphi : 2^X \rightarrow (I(X), \tau_{pp})$ be given by $\varphi(A) = 1_A$. We need to show that $\varphi$ is continuous. Notice that $(\text{dom} \circ \varphi)(A) = (\text{im} \circ \varphi)(A) = A$ for all $A$, and $(\text{ev}_x \circ \varphi)(A) = x$ for all $A$ with $x \in A$. Thus, by Theorem 3.7 (iii), $\varphi$ is continuous.

**Remark 3.9** A semigroup $S$ is called $H$-closed if $S$ is a closed subsemigroup of any topological Hausdorff semigroup which contains $S$ as a subsemigroup. $S$ is called absolutely $H$-closed if any continuous homomorphic image of $S$ into a topological Hausdorff semigroup is $H$-closed. More information about these notions can be found in [4,5]. They showed that if $(I_n(X), \tau)$ is a topological semigroup and $E(I_n(X))$ is $\tau$-compact, then $(I_n(X), \tau)$ is an absolutely $H$-closed topological semigroup. By an abuse of notation, we will also denote by $\tau_{pp}$ the subspace topology $\tau_{pp}|_S$ (see [4, Theorem 3.7]). Using that result, we claim that $(I_n(X), \tau_{pp})$ is an absolutely $H$-closed topological semigroup. In fact, it is easy to see that $I_n(X)$ is $\tau_{pp}$-closed and thus $E(I_n(X))$ is a $\tau_{pp}$-closed subset of $E(I(X))$ and therefore compact, by Theorem 3.8.

They also showed that there is only one compact Hausdorff semigroup topology on $E(I_1(X))$. Below we extend their result to $I_1(X)$.

**Theorem 3.10** $\tau_{pp}$ is the unique Hausdorff inverse semigroup topology on $I_1(X)$ such that $E(I_1(X))$ is compact.

**Proof** Let $\tau$ be an inverse semigroup Hausdorff topology on $I_1(X)$ such that $E(I_1(X))$ is compact. By the minimality of $\tau_{pp}$ (see Theorem 3.3) we have that $\tau_{pp} \subseteq \tau$. Since $E(I(X))$ is compact with respect to both topologies and they are Hausdorff, $\tau$ and $\tau_{pp}$ are $\tau_{pp}$-isolated in $E(I(X))$. We claim that every $f \in I_1(X)$ which is not idempotent is $\tau_{pp}$-isolated in $I_1(X)$. Let $x \in X$ be such that $y = f(x) \neq x$. Then $f \in v(x, y)$ and $v(x, y) \cap I_1(X) = \{f\}$.

**Remark 3.11** From the previous result, we conclude that $\tau_{pp}$ restricted to $I_1(X)$ is equal to the topology $\tau^c$ defined in Proposition 3.6 of [4].

Next we show a characterization of the convergence of nets with respect to $\tau_{pp}$.

**Lemma 3.12** Let $(f_\lambda)_{\lambda \in \Lambda}$ be a net in $I(X)$ and $f \in I(X)$. Then, $f_\lambda \xrightarrow{\tau_1} f$ if and only if the following conditions hold.

(i) For all $x \in \text{dom}(f)$ there is $\lambda_0 \in \Lambda$ such that $x \in \text{dom}(f_\lambda)$ and $f_\lambda(x) = f(x)$ for all $\lambda \geq \lambda_0$.

(ii) For all $x \notin \text{dom}(f)$ there is $\lambda_0 \in \Lambda$ such that $x \notin \text{dom}(f_\lambda)$ for all $\lambda \geq \lambda_0$.

**Proof** It is straightforward from the definition of $\tau_1$.

**Theorem 3.13** Let $(f_\lambda)_{\lambda \in \Lambda}$ be a net in $I(X)$ and $f \in I(X)$. Then, $f_\lambda \xrightarrow{\tau_{pp}} f$ if, and only if, $f_\lambda \xrightarrow{\tau_1} f$ and $f_\lambda^{-1} \xrightarrow{\tau_1} f^{-1}$.

**Proof** It follows immediately from the fact that $f \in w_1(y)$ iff $f^{-1} \in w_2(y)$.
We end this section making some comments about our assumption that \( X \) is discrete. Suppose \( X \) is a topological space. A natural and important inverse subsemigroup of \( I(X) \) is the collection \( S \) of all homeomorphisms between open subsets of \( X \). Clearly, if \( X \) is discrete, then \( S = I(X) \). When \( S \) is viewed as a topological inverse subsemigroup of \( (I(X), \tau_{pp}) \), we may loose some interesting properties when \( X \) is non discrete. First, the collection of idempotents of \( S \) might not be compact and second, the domain of the evaluation function \( \{(f, x) \in S \times X : x \in \text{dom}(f)\} \) might not be open in \( S \times X \). As we said in the introduction, the natural extension of our results to this more general setting will be treated elsewhere.

4 The case \( X \) countable

In this section we study \( \tau_{pp} \) when \( X \) is a countable set, which can be assumed, without loss of generality, to be equal to \( \mathbb{N} \). In this case, \( (I(\mathbb{N}), \tau_{pp}) \) is a Hausdorff, regular and second-countable space, thus metrizable by the Urysohn theorem. Moreover, we show that \( (I(\mathbb{N}), \tau_{pp}) \) is Polish. In order to define a metric compatible with \( \tau_{pp} \) we introduce some auxiliary functions. Let \( f, g \in I(\mathbb{N}) \) and consider the functions \( a_{(f,g)}, b_{(f,g)} \in 2^\mathbb{N} \) defined by

\[
a_{(f,g)}(n) = \begin{cases} 
0 & \text{if } n \in (\text{dom}(f) \cap \text{dom}(g)) \cup ((\text{dom}(f))^c \cap (\text{dom}(g))^c), \\
1 & \text{otherwise.}
\end{cases}
\]

\[
b_{(f,g)}(n) = \begin{cases} 
0 & \text{if } n \notin \text{dom}(f) \cap \text{dom}(g), \\
\min\{1, |f(n) - g(n)|\} & \text{if } n \in \text{dom}(f) \cap \text{dom}(g).
\end{cases}
\]

Now consider the following metric

\[
\rho(f, g) = \sum_{n \in \mathbb{N}} \frac{a_{(f,g)}(n) + b_{(f,g)}(n)}{2^n}.
\]

We show next that \( (I(\mathbb{N}), \tau_1) \) is a Polish semigroup (but not a topological inverse semigroup).

**Proposition 4.1** \( (I(\mathbb{N}), \tau_1) \) is metrizable and \( \rho \) is a compatible metric for \( \tau_1 \).

**Proof** It is easy to verify that \( \rho \) is indeed a metric. Since \( (I(\mathbb{N}), \tau_1) \) is metrizable, to show that \( \rho \) is compatible with \( \tau_1 \) it suffices to show that they induce the same convergent sequences.

Let \( (f_n)_{n \in \mathbb{N}} \) be a sequence and \( f \in I(\mathbb{N}) \) be such that \( f_n \xrightarrow{\rho} f \). We use Theorem 3.12 to show that \( f_n \xrightarrow{\tau_1} f \). Let \( m \in \mathbb{N} \), there is \( N \in \mathbb{N} \) such that if \( n \geq N \), \( \rho(f_n, f) < \frac{1}{2^m} \), therefore \( a_{(f,f_n)}(m) = 0 \) and \( b_{(f,f_n)}(m) = 0 \), for all \( n \geq N \). We consider two cases: (a) Suppose \( m \in \text{dom}(f) \). Since \( a_{(f,f_n)}(m) = 0 \) and \( b_{(f,f_n)}(m) = 0 \), for all \( n \geq N \), we have that \( m \in \text{dom}(f_n) \) and \( f_n(m) = f(m) \), for all \( n \geq N \). (b) Suppose \( m \notin \text{dom}(f) \). Since \( a_{(f,f_n)}(m) = 0 \), for all \( n \geq N \), we have that \( m \notin \text{dom}(f_n) \), for all \( n \geq N \). Then, by Theorem 3.12, \( f_n \xrightarrow{\tau_1} f \).\[ \square \]
For the other direction, suppose \( g_n \xrightarrow{\tau_1} g \). Let \( k \in \mathbb{N} \). Consider the following sets

\[
A = \{ m \in \mathbb{N} \mid m \leq k \text{ and } m \in \text{dom}(g) \}
\]

and

\[
B = \{ m \in \mathbb{N} \mid m \leq k \text{ and } m \notin \text{dom}(g) \}.
\]

Since \( g_n \xrightarrow{\tau_1} g \), there is \( N \in \mathbb{N} \) such that if \( n \geq N \), \( g_n \in v(m, g(m)) \), for all \( m \in A \) and \( g_n, g \in w_1(m) \), for all \( m \in B \). Thus, \( a_{(g, g_n)}(m) = 0 \) and \( b_{(g, g_n)}(m) = 0 \) for all \( m \leq k \) and \( n \geq N \). Finally, we have that for all \( n \geq N \)

\[
\rho(g, g_n) = \sum_{t=1}^{\infty} \frac{a_{(g, g_n)}(t) + b_{(g, g_n)}(t)}{2^t} \leq \sum_{t=k+1}^{\infty} \frac{1}{2^t} = \frac{1}{2^{k+1}} < \frac{1}{2^k}.
\]

Therefore \( g_n \xrightarrow{\rho} g \).

The following fact shows that \( 2^\mathbb{N} \) is naturally embedded into \( I(\mathbb{N}) \). Its easy proof is left to the reader.

**Proposition 4.2** Let \( \eta \) be given by \( \eta(A, B) = \rho(1_A, 1_B) \), for \( A, B \in 2^\mathbb{N} \). Then \( \eta \) is a compatible metric for \( 2^\mathbb{N} \).

**Proposition 4.3** \((I(\mathbb{N}), \tau_1)\) is a Polish semigroup and \( \rho \) is a complete compatible metric.

**Proof** By Proposition 4.1, we only need to show that \( \rho \) is complete. Let \((f_n)_n\) be a \( \rho \)-Cauchy sequence. Then \((\text{dom}(f_n))_n\) is Cauchy in \( 2^\mathbb{N} \) (by Proposition 4.2), so it is convergent. Let \( A = \lim_{n \to \infty} \text{dom}(f_n) \). It is easy to verify that, for each \( p \in A \), \((\text{ev}_p(f_n))_n\) is Cauchy in \( \mathbb{N} \) (with the discrete metric) and therefore eventually constant to a value \( f(p) \). Thus we have defined a function \( f \) such that \( \text{dom}(f) = A \) and \( f_n \xrightarrow{\rho} f \). It is easy to see that \( f \) is injective. Finally, from Theorem 3.12, \( f_n \xrightarrow{\tau_1} f \). \( \Box \)

Recall that we also have a metric \( \rho^* \) on \( I(\mathbb{N}) \) given by

\[
\rho^*(f, g) = \rho(f^{-1}, g^{-1}).
\]

From the previous result and Proposition 3.6 we conclude that neither \( \rho \) nor \( \rho^* \) is compatible with \( \tau_{pp} \). However, as in Theorem 2.3, we define another metric as follows:

\[
d(f, g) = \rho(f, g) + \rho^*(f, g).
\]

**Theorem 4.4** \((I(\mathbb{N}), \tau_{pp})\) is a Polish inverse semigroup and \( d \) is a complete compatible metric.

**Proof** The proof of the compatibility of \( d = \rho + \rho^* \) with \( \tau_{pp} \) is similar to the proof of Theorem 4.1. The completeness of \( d \) follows from Theorem 2.3 and Proposition 4.3. \( \Box \)
5 Open operation

For \( f \in I(X) \), let \( r_f : I(X) \to R_f \) be given by \( r_f(g) = g \circ f \) and

\[
R_f = \{ g \circ f \mid g \in I(X) \}.
\]

The corresponding \( l_f \) and \( L_f \), for the left operation, are analogously defined.

In contrast to what happens with topological groups, for a topological semigroup it is not true that \( Vx \) is open when \( V \) is open. Nevertheless, we show that \( r_f \) is an open map. We need this fact in the next section for proving a uniqueness result for \( \tau_{pp} \).

Lemma 5.1 Let \( x, y, w, z \in X \). Then

1. \( v(x, y) \circ v(z, x) = v(z, y) \).
2. \( v(x, y) \circ v(z, w) = I(X) \setminus v(z, y) \), for \( w \neq x \).
3. \( w_1(x) \circ w_1(y) = w_1(y) \).
4. \( w_2(x) \circ w_2(y) = w_2(x) \).
5. \( w_2(x) \circ w_1(y) = w_2(x) \cap w_1(y) \).
6. \( w_1(x) \circ w_2(y) = I(X) \).
7. \( w_1(y) \circ v(x, y) = w_1(x) \).
8. \( w_1(z) \circ v(x, y) = I(X) \) if \( z \neq y \).
9. \( v(x, y) \circ w_1(z) = w_1(z) \).
10. \( w_2(z) \circ v(x, y) = [v(y, x) \circ w_1(z)]^{-1} = [w_1(z)]^{-1} = w_2(z) \).
11. \( v(x, y) \circ w_2(z) = [w_1(z) \circ v(y, x)]^{-1} \).

Proof All items are proved in an analogous way.

1. It is clear that \( v(x, y) \circ v(z, x) \subseteq v(z, y) \). For the other inclusion, let \( f \in v(z, y) \) and pick \( A \subseteq X \) such that \( x \in A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection such that \( h(z) = x \) and let \( g = f \circ h^{-1} \). Notice that \( g : A \to \text{im}(f) \) and \( g(x) = y \), therefore \( h \in v(z, x) \), \( g \in v(x, y) \) and \( f = g \circ h \).

2. Suppose \( w \neq x \). Let \( f \in v(x, y) \circ v(z, w) \), then \( f = g \circ h \), with \( h \in v(z, w) \) and \( g \in v(x, y) \). We consider two cases:
   (a) Suppose \( w \in \text{dom}(g) \). Since \( w \neq x \), \( g(w) \neq y \) Notice that \( f(z) = g(h(z)) = g(w) \neq y \), therefore \( f \in I(X) \setminus v(z, y) \).
   (b) Suppose \( w \notin \text{dom}(g) \). We have that \( h(z) = w \notin \text{dom}(g) \), therefore \( z \notin \text{dom}(g \circ h) = \text{dom}(f) \). Thus \( f \in I(X) \setminus v(z, y) \).

Now let us see that \( I(X) \setminus v(z, y) \subseteq v(x, y) \circ v(z, w) \). Let \( f \in I(X) \setminus v(z, y) \). We consider four cases.

(a) Suppose \( z \notin \text{dom}(f) \) and \( y \notin \text{im}(f) \). Let \( A \subseteq X \) be such that \( w, x \in A \) and \( |A| = |\text{dom}(f) \cup \{z\}| \). Let \( h : \text{dom}(f) \cup \{z\} \to A \) be any bijection such that \( h(z) = w \), \( h(f^{-1}(y)) = x \) and let \( g = f \circ h^{-1} \). Since \( h(f^{-1}(y)) = x \) we have that \( g(x) = f(h^{-1}(x)) = y \), therefore \( h \in v(z, x) \), \( g \in v(x, y) \) and \( f = g \circ h \).

(b) Suppose \( z \in \text{dom}(f) \) and \( y \notin \text{im}(f) \). Let \( A \subseteq X \) be such that \( w \in A, x \notin A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection such that \( h(z) = w \), and let \( g : A \cup \{x\} \to \text{im}(f) \cup \{y\} \) be such that \( g(x) = y \) and \( g = f \circ h^{-1} \) in \( A \). Notice that \( h \in v(z, x) \), \( g \in v(x, y) \) and \( f = g \circ h \).
Suppose \( z \in \text{dom}(f) \) and \( y \in \text{im}(f) \). Let \( A \subseteq X \) be such that \( w, x \in A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection such that \( h(z) = w, h(f^{-1}(y)) = x \) and let \( g = f \circ h^{-1} \). Notice that \( g : \text{im}(f) \to A \) and \( g(x) = y \), therefore \( h \in v(z, x) \), \( g \in v(x, y) \) and \( f = g \circ h \).

(d) Suppose \( z \notin \text{dom}(f) \) and \( y \notin \text{im}(f) \). Let \( A \subseteq X \) be such that \( w, x \in A \) and \( |A| = |\text{dom}(f) \cup \{z\}| \). Let \( h : \text{dom}(f) \cup \{z\} \to A \) be any bijection such that \( h(z) = w \), and let \( g : A \cup \{x\} \to \text{im}(f) \cup \{y\} \) be such that \( g(x) = y \) and \( g = f \circ h^{-1} \) in \( A \). Notice that \( h \in v(z, x) \), \( g \in v(x, y) \) and \( f = g \circ h \).

3. It is obvious that \( w_1(x) \circ w_1(y) \subseteq w_1(y) \). For the other inclusion, let \( f \in w_1(y) \) and pick \( A \subseteq X \) such that \( x \notin A \) and \( A = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection and let \( g = f \circ h^{-1} \). Notice that \( g : \text{im}(f) \to A \), and therefore \( h \in w_1(y) \), \( g \in w_1(x) \) and \( f = g \circ h \).

4. It follows from part 3., as \([w_1(y) \circ w_1(x)]^{-1} = w_2(x) \circ w_2(y)\).

5. It is easy to see that \( w_2(x) \circ w_1(y) \subseteq w_2(x) \cap w_1(y) \). Let \( f \in w_2(x) \cap w_1(y) \). Pick \( A \subseteq X \) such that \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection and let \( g = f \circ h^{-1} \). Then \( h \in w_1(y) \), \( g \in w_2(x) \) and \( f = g \circ h \).

6. Let \( f \in I(X) \). Pick \( A \subseteq X \) such that \( x, y \notin A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection and let \( g = f \circ h^{-1} \). Then \( h \in w_1(y) \), \( g \in w_2(x) \) and \( f = g \circ h \).

7. It is easy to see that \( w_1(y) \circ v(x, y) \subseteq w_1(x) \). Let \( f \in w_1(x) \). Pick \( A \subseteq X \) such that \( y \notin A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection such that \( h(x) = y \), and let \( g = f \circ h^{-1} \). Then \( h \in v(x, y) \), \( g \in w_1(y) \) and \( f = g \circ h \).

8. Let \( f \in I(X) \). We have two cases.

(a) Suppose that \( x \in \text{dom}(f) \). Pick \( A \subseteq X \) such that \( y \in A, z \notin A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \to A \) be any bijection such that \( h(x) = y \), and let \( g = f \circ h^{-1} \). Then \( h \in v(x, y) \), \( g \in w_1(z) \) and \( f = g \circ h \).

(b) Suppose that \( x \notin \text{dom}(f) \). Pick \( A \subseteq X \) such that \( y, z \notin A \) and \( |A| = |\text{dom}(f)| \). Let \( h : \text{dom}(f) \cup \{x\} \to A \) be any bijection such that \( h(x) = y \), and let \( g = f \circ h^{-1} \). Then \( h \in v(x, y) \), \( g \in w_1(z) \) and \( f = h \circ g \).

9. Let us see that \( w_1(z) \subseteq v(x, y) \circ w_1(z) \). Let \( f \in w_1(z) \) and suppose that \( y \in \text{im}(f) \). Let \( A \subseteq X \) be such that \( x \in A \) and \( |A| = |\text{dom}(f)| \). Let \( g : \text{dom}(f) \to A \) be any bijection such that \( g(f^{-1}(y)) = x \) and let \( h = f \circ g^{-1} \). Then \( f = h \circ g, h \in v(x, y) \).

Now, suppose that \( f \in w_1(z) \) and \( y \notin \text{im}(f) \). Let \( A \subseteq X \) be such that \( x \notin A \) and \( |A| = |\text{dom}(f)| \). Let \( g : \text{dom}(f) \to A \) be any bijective function. Let \( h : A \cup \{x\} \to \text{im}(f) \cup \{y\} \) be such that \( h(x) = y \) and \( h \circ g = f \circ v \) for \( v \in \text{dom}(f) \). We have that \( h \in v(x, y) \), \( g \in w_1(z) \) and \( f = h \circ g \), that is, \( f \in v(x, y) \circ w_1(z) \).

Finally, 10. and 11. are evident. □

Lemma 5.2 Let \( A, B, C \subseteq I(X) \), then \( A \circ (B \cap C) \subseteq (A \circ B) \cap (A \circ C) \).

Proof It is straightforward. □

Now we can show that \( r_f \) is an open map.

Theorem 5.3 Let \( f \in I(X) \). Then, \( r_f : I(X) \to R_f \) is an open map where \( R_f \) is endowed with the relative topology as a subspace of \( (I(X), \tau_{pp}) \).

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Proof Let \( U \) be a non empty basic \( \tau_{pp} \)-open set of the form:

\[
U = \bigcap_{i=1}^{n} v(x_i, y_i) \cap \bigcap_{i=1}^{m} w_1(z_i) \cap \bigcap_{i=1}^{p} w_2(w_i).
\]

We will show that \( U \circ f \) is open. Consider the sets:

\[
\begin{align*}
\hat{M} &= \{x_i : 1 \leq i \leq n\} & M &= \hat{M} \cap \text{im}(f), \\
\hat{N} &= \{z_i : 1 \leq i \leq m\} & N &= \hat{N} \cap \text{im}(f), \\
\hat{O} &= \{w_i : 1 \leq i \leq p\} & O &= \hat{O} \cap \text{im}(f),
\end{align*}
\]

and

\[
Q = \bigcap_{x_i \in M} v(f^{-1}(x_i), y_i) \cap \bigcap_{z_i \in N} w_1(f^{-1}(z_i)) \cap \bigcap_{w_i \in O} w_2(w_i).
\]

We claim that \( U \circ f = Q \cap R_f \). First, we show that \( U \circ f \subseteq Q \). Let

\[
R = \bigcap_{x_i \in M} v(f^{-1}(x_i), x_i) \cap \bigcap_{z_i \in N} v(f^{-1}(z_i), z_i) \cap \bigcap_{w_i \in O} v(f^{-1}(w_i), w_i),
\]

\[
S = \bigcap_{x_i \in \hat{M} \setminus M} w_2(x_i) \cap \bigcap_{x_i \in \hat{N} \setminus N} w_2(z_i) \cap \bigcap_{x_i \in \hat{O} \setminus O} w_2(w_i).
\]

Then \( f \in R \cap S \). By Lemma 5.1, we have the following:

- \( v(x_i, y_i) \circ v(f^{-1}(x_i), x_i) = v(f^{-1}(x_i), y_i) \), for all \( x_i \in M \).
- \( w_1(z_i) \circ v(f^{-1}(z_i), z_i) = w_1(f^{-1}(z_i)) \), for all \( z_i \in N \).
- \( w_2(w_i) \circ v(f^{-1}(w_i), w_i) = w_2(w_i) \), for all \( w_i \in O \).
- \( v(x_i, y_i) \circ w_2(x_i) = w_2(y_i) \), for all \( x_i \in \hat{M} \setminus M \).
- \( w_1(z_i) \circ w_2(z_i) = I(X) \), for all \( z_i \in \hat{N} \setminus N \).
- \( w_1(w_i) \circ w_2(w_i) = w_2(w_i) \), for all \( w_i \in \hat{O} \setminus O \).

Therefore, using Lemma 5.2 and the claims above, we easily have that

\[
U \circ f \subseteq U \circ (R \cap S)
\]

\[
\leq \bigcap_{x_i \in M} v(f^{-1}(x_i), y_i) \cap \bigcap_{z_i \in N} w_1(f^{-1}(z_i)) \cap \bigcap_{w_i \in O} w_2(w_i) \cap \bigcap_{x_i \in \hat{M} \setminus M} w_2(w_i) \cap \bigcap_{w_i \in \hat{O} \setminus O} w_2(w_i)
\]

\[
= \bigcap_{x_i \in M} v(f^{-1}(x_i), y_i) \cap \bigcap_{z_i \in N} w_1(f^{-1}(z_i)) \cap \bigcap_{w_i \in O} w_2(w_i) \cap \bigcap_{x_i \in \hat{M} \setminus M} w_2(w_i) \cap \bigcap_{w_i \in \hat{O} \setminus O} w_2(w_i)
\]

\[
= Q.
\]

Now, we are going to show that \( Q \cap R_f \subseteq U \circ f \). Let \( h \in Q \cap R_f \) and define \( g = h \circ f^{-1} \). Notice that \( g(x_i) = y_i \), for all \( x_i \in M \), since \( h \in v(f^{-1}(x_i), y_i) \) for
Let $x_i \in M$. Notice also that that if $x_i \in \hat{M} \setminus M$, then $x_i \notin \text{dom}(g)$, and $y_i \notin \text{im}(g)$ since $h \in w_2(y_i)$ for $x_i \in \hat{M} \setminus M$. Therefore we can extend $g$ to a function $\hat{g}$ such that $\text{dom}(\hat{g}) = \text{dom}(g) \cup (\hat{M} \setminus M)$ and $g(x_i) = y_i$, for all $x_i \in \hat{M} \setminus M$.

To finish the proof it suffices to show that $h = \hat{g} \circ f$ and $\hat{g} \in U$. First, we show that $\hat{g} \in U$.

- $\hat{g}(x_i) = y_i$, for all $i$, by construction.
- $z_i \notin \text{dom}(\hat{g})$, for all $z_i$. In fact, if $z_i \in \hat{N} \setminus N$, then $z_i \notin \text{im}(f)$, therefore $z_i \notin \text{dom}(g)$ and thus $z_i \notin \text{dom}(\hat{g})$. On the other hand, if $z_i \in N$, then $h \in w_1(f^{-1}(z_i))$ and $z_i \notin \text{dom}(\hat{g})$.
- As $h \in w_2(w_i)$ for all $w_i$, we have that $w_i \notin \text{im}(\hat{g})$, for all $w_i$.

Thus $\hat{g} \in U$. Finally, since $g = h \circ f^{-1}$ and $\text{dom}(\hat{g}) \cap \text{im}(f) = \text{dom}(g)$, $h = \hat{g} \circ f$.

**Theorem 5.4** Let $f \in I(X)$ and consider $L_f = \{ f \circ g \mid g \in I(X) \}$. Then, $\circ_f : I(X) \to L_f$, given by $\circ_f(g) = f \circ g$, is an open function, where $L_f$ is endowed with the relative topology.

**Proof** Similar to the proof of Theorem 5.3.

We conjecture that $\circ$ is an open map. To show it, it would require analyzing many cases in order to extend Lemma 5.1. We did not pursue it further, since we do not need this result for this paper.

### 6 Uniqueness of $\tau_{pp}$

In this section we present some results showing that $\tau_{pp}$ is unique in some sense. To that end we first show that $(I(X), \tau_{pp})$ is a quotient of a symmetric group $S_\infty(Y)$ (for some $Y$) with the usual product topology.

Let $X \subseteq Y$. For each $f \in S_\infty(Y)$ we define $\hat{f} \in I(X)$ as follows:

$$\hat{f} = \{(x, f(x)) : x \in X \text{ and } f(x) \in X\}.$$ 

Thus $\hat{f}$ is the restriction of $f$ to $f^{-1}(X) \cap X$. Let $\pi : S_\infty(Y) \to I(X)$ be given by $\pi(f) = \hat{f}$.

**Proposition 6.1** Let $X$ be an infinite set and $X \subseteq Y$. Then $\pi$ is onto if, and only if, $|X| \leq |Y \setminus X|$.

**Proof** Suppose $|X| \leq |Y \setminus X|$ and let $g \in I(X)$. Let $A = \text{dom}(g)$ and $B = \text{im}(g)$. Let $C \subseteq Y \setminus X$ and $D \subseteq Y \setminus X$ be such that $|C| = |X \setminus A|$, $|D| = |X \setminus B|$, $|Y \setminus (X \cup C)| = |Y \setminus (X \cup D)| = |Y|$. All these conditions can be fulfilled as $|X| \leq |Y \setminus X|$. Now take any extension of $g$ to a bijection $f : Y \to Y$ such that $f[X \setminus A] = C$, $f[D] = X \setminus B$. Then $\hat{f} = g$.

Conversely, suppose $|X| > |Y \setminus X|$. Let $X = A \cup B$ be a partition of $X$ into sets of equal cardinality. Then there is no $f \in S_\infty(Y)$ such that $\hat{f} = 1_A$. \qed
Let us observe that the map $\pi$ is not a semigroup homomorphism. In fact, let $y \in Y \setminus X$, $x \in X$, and $f \in S_\infty(Y)$ be such that $f(x) = y$, $f(y) = x$ and $f(z) = z$ for all $z \notin \{x, y\}$. Then $f \circ f = 1_y$, $\pi(1_y) = 1_X$ but $\pi(f) \circ \pi(f) = 1_{X \setminus \{x\}}$. In general, $\pi(f) \circ \pi(g) \subseteq \pi(f \circ g).

**Proposition 6.2** Let $X \subseteq Y$ with $|X| \leq |Y \setminus X|$. The map $\pi : S_\infty(Y) \to (I(X), \tau_{pp})$ is continuous, onto and open.

**Proof** First, we will show that $\pi$ is continuous. For $x, y \in Y$, let $u(x, y)$ denote the subbasic open set of $S_\infty(Y)$ given by $\{f \in S_\infty(Y) : f(x) = y\}$. The continuity of $\pi$ follows from the following identities.

(i) $\pi^{-1}(u(x, y)) = u(x, y)$ for all $x, y \in X$.

(ii) $\pi^{-1}(u_1(x)) = \bigcup_{y \in Y \setminus X} u(x, y)$ for $x \in X$.

(iii) $\pi^{-1}(u_2(y)) = \bigcup_{x \in Y \setminus X} u(x, y)$ for $y \in X$.

To see that $\pi$ is open, let $x_i, y_i$ in $Y$ for $1 \leq i \leq n$. Let $1 \leq k_1 \leq k_2 \leq k_3 \leq n$ be such that the following four conditions are satisfied: (1) $x_i, y_i \in X$ for $1 \leq i \leq k_1$, (2) $x_j \in X$ and $y_j \notin X$ for $k_1 < j \leq k_2$, (3) $x_j \notin X$ and $y_j \in X$ for $k_2 < j \leq k_3$ and (4) $x_j \notin X$ and $y_j \notin X$ for $k_3 < j \leq n$. Then

\[
\pi\left(\bigcap_{i=1}^{k_1} u(x_i, y_i)\right) = \bigcup_{i=1}^{k_1} u(x_i, y_i) \cap \bigcup_{j=k_1+1}^{k_2} w_1(x_j) \cap \bigcup_{j=k_2+1}^{k_3} w_2(y_j).
\]

The direction $\subseteq$ is straightforward. For $\supseteq$ we use an analogous construction as in the proof that $\pi$ is onto (Proposition 6.1). $\square$

**Theorem 6.3** Let $X \subseteq Y$ with $|Y \setminus X| \geq |X|$. Then $\tau_{pp}$ is the only inverse semigroup Hausdorff topology on $I(X)$ with respect to which $\pi : S_\infty(Y) \to I(X)$ is continuous.

**Proof** Let $\tau$ be an inverse semigroup Hausdorff topology on $I(X)$ such that $\pi$ is continuous. Then by Theorem 3.3 we have that $\tau_{pp} \subseteq \tau$. Since $\pi$ is continuous, we conclude, by Proposition 6.2, that $\tau \subseteq \tau_{pp}$. $\square$

We now present a proof that $\tau_{pp}$ is the unique inverse semigroup Polish topology on $I(\mathbb{N})$ (satisfying some additional conditions). Our approach is different than the one used in [2]. We need two auxiliary results.

**Lemma 6.4** Let $R_A = \{f \circ 1_A : f \in I(X)\}$ and $r_A : I(X) \to R_A, r_A(f) = f \circ 1_A$. Let $\tau$ be an inverse semigroup Hausdorff topology on $I(X)$. Suppose $r_A$ is a $\tau$-open map for every $A \subseteq X$ cofinite. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence on $I(X)$ such that

(i) $f_k \xrightarrow{\tau_{pp}} f$.

(ii) There is a cofinite set $A \subseteq X$ such that $f_k \circ 1_A \xrightarrow{\tau} f \circ 1_A$.

Then $f_k \xrightarrow{\tau} f$. $\square$
**Proof** Let \( A \) be as in (ii). Let \( V \in \tau \) with \( f \in V \). Since \( \tau_{pp} \subseteq \tau \) (by Theorem 3.3), we can assume that \( V \subseteq v(x, f(x)) \cap w_1(z) \) for all \( x \in \text{dom}(f) \cap (X \setminus A) \) and all \( z \in (X \setminus A) \setminus \text{dom}(f) \). By hypothesis, \( V \circ 1_A \) is open in \( R_A \). By (iii), there is \( k_0 \) such that \( f_k \circ 1_A \in V \circ 1_A \) for all \( k \geq k_0 \). Thus, there is \( g_k \in V \) such that \( f_k \circ 1_A = g_k \circ 1_A \) for all \( k \geq k_0 \). By (i) and Lemma 3.12, there is \( k_1 \geq k_0 \) such that \( \text{dom}(f_k) \cap (X \setminus A) = \text{dom}(f) \cap (X \setminus A) \) and \( f_k(x) = f(x) \) for all \( x \in \text{dom}(f) \cap (X \setminus A) \) and all \( k \geq k_1 \). Since \( g_k \in V, \text{dom}(g_k) \cap (X \setminus A) = \text{dom}(f) \cap (X \setminus A) \) and \( g_k(x) = f(x) \) for all \( x \in \text{dom}(f) \cap (X \setminus A) \). As \( f_k \circ 1_A = g_k \circ 1_A \), then \( f_k = g_k \) and hence \( f_k \in V \) for all \( k \geq k_1 \). \( \square \)

**Lemma 6.5** Let \( \tau \) be a Polish inverse semigroup topology on \( I(\mathbb{N}) \) such that \( A \mapsto 1_A \) from \( 2^\mathbb{N} \) to \( (I(\mathbb{N}), \tau) \) is continuous. Let \( \mathbb{N} \subseteq Y \) with \( Y \setminus \mathbb{N} \) infinite and \( f_k \in S_\infty(Y) \), \( k \in \mathbb{N} \), be such that \( f_k \to f \). There is a dense \( G_\delta \) set \( G \subseteq S_\infty(Y) \) such that

\[
\hat{f}_k \circ \hat{g} \xrightarrow{\tau} \hat{f} \circ \hat{g} \quad \text{for all } g \in G.
\]

**Proof** By Theorem 3.3 we have that \( \tau_{pp} \subseteq \tau \). Since both topologies are Polish, by a well known classical result, they have the same Borel sets (see [7, Exercise 15.4]). As \( \pi \) is continuous with respect to \( \tau_{pp} \), we have that \( \pi : S_\infty(Y) \to (I(\mathbb{N}), \tau) \) is Borel measurable. Thus, there is a dense \( G_\delta \) set \( H \subseteq S_\infty(Y) \) such that \( \pi|_H : H \to (I(\mathbb{N}), \tau) \) is continuous (see [7, Theorem 8.38]). Let

\[
L_{k+1} = \{ g \in S_\infty(Y) : f_k \circ g \in H \}, \quad L_0 = \{ g \in S_\infty(Y) : f \circ g \in H \}.
\]

Let \( G = \bigcap_k L_k \). Since each \( L_k \) is dense \( G_\delta \), so is \( G \). As \( f_k \circ g \to f \circ g \) (in \( S_\infty(Y) \)) for all \( g \in G \) and \( \pi \) is continuous in \( H, \pi(f_k \circ g) \xrightarrow{\tau} \pi(f \circ g) \) for all \( g \in G \).

As we said before, \( \pi \) is not a homomorphism, however, we have the following

\[
\pi(f_k) \circ \pi(g) = \pi(f_k \circ g) \circ 1_{\text{dom}(\pi(f_k) \circ \pi(g))}.
\]

Notice that \( \pi(f_k) \xrightarrow{\tau_{pp}} \pi(f) \) as \( \pi \) is continuous with respect to \( \tau_{pp} \) (by Proposition 6.2). As the function \( \text{dom} \) is continuous (by Theorem 3.7), we have

\[
\text{dom}(\pi(f_k) \circ \pi(g)) \to \text{dom}(\pi(f) \circ \pi(g))
\]

where the convergence is in the Cantor space \( 2^\mathbb{N} \). Finally, by hypothesis, the map \( A \mapsto 1_A \) is continuous with respect to \( \tau \). Thus we conclude

\[
\pi(f_k) \circ \pi(g) \xrightarrow{\tau} \pi(f) \circ \pi(g)
\]

for all \( g \in G \). \( \square \)

**Theorem 6.6** \( \tau_{pp} \) is the unique inverse semigroup Polish topology on \( I(\mathbb{N}) \) such that the collection of idempotents is compact and \( r_A \) is a \( \tau \)-open map for every \( A \subseteq \mathbb{N} \).
Proof By Theorem 5.3, $r_A$ is a $\tau_{pp}$-open map and, by Theorem 3.8, the collection of idempotents is $\tau_{pp}$-compact. Conversely, let $\tau$ be a topology on $I(\mathbb{N})$ as in the hypothesis. Let $\mathbb{N} \subseteq Y$ be such that $Y \setminus \mathbb{N}$ is countable. To have that $\tau = \tau_{pp}$, it suffices to show, by Lemma 6.5, that $\pi : S_\infty(Y) \to (I(\mathbb{N}), \tau)$ is continuous.

Let $f_k \in S_\infty(Y)$ be a sequence converging to $f$. Let $G \subseteq S_\infty(Y)$ be a dense $G_\delta$ as in Proposition 6.5. Since $\{h \in S_\infty(Y) : \text{im}(h) \text{ is cofinite}\}$ is a $G_\delta$ dense set, let $g \in G$ be such that $\text{im}(g)$ is cofinite. Now notice that $\hat{g} \circ \hat{g}^{-1} = 1_{\text{im}(g)}$. Let $A = \text{im}(g)$. Since $\hat{f_k} \circ \hat{g} \overset{\tau}{\longrightarrow} \hat{f} \circ \hat{g}$, by the continuity of $\circ$, we have $\hat{f_k} \circ 1_A \overset{\tau}{\longrightarrow} \hat{f} \circ 1_A$. Thus, by Lemma 6.4, we conclude that $\hat{f_k} \overset{\tau}{\longrightarrow} \hat{f}$.

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