One Form Deformation of Sprays

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Abstract. In this paper, we introduce the notion of one form deformation of sprays. The metrizability of the new spray, when the background spray is flat, is characterized. Therefore, we obtain new projectively flat metrics of constant flag curvature. Moreover, these new metrics are not, generally, isometric to the Klein metric via affine transformations. New solutions for Hilbert’s fourth problem are obtained and constructed. Various examples are discussed and studied.

1. Introduction

The notion of sprays was introduced by W. Ambrose et al. [1] in 1960. A system of second order ordinary differential equations (SODE) with positively 2-homogeneous coefficients functions can be shown as a second order vector field, which is called a spray. All sprays are associated with a SODE and conversely, a spray can be associated with a SODE. If such a system introduces the variational (Euler-Lagrange) equations of the energy of a Finsler metric, then it is said to be Finsler metrizable and in this case the spray is the geodesic spray of the Finsler metric. The Finsler metrizability problem for a spray \( S \) looking for a Finsler structure whose geodesics coincide with the geodesics of \( S \). The metrizability problem can be considered as a special case of the inverse problem of the calculus of variation. Several interesting results on the metrizability problem can be found in the literature, we refer, for example, to [5, 9, 12, 13, 16] and the references therein.

The geodesics of a Finsler structure \( F \) on an open subset \( U \subset \mathbb{R}^n \) are straight lines if and only if the spray coefficients of \( F \) are given in the form

\[
G = \frac{y}{x}y^k b_k(x).
\]

Straight lines in \( U \) are parametrized by \( \sigma(t) = f(t) a + b \), where \( a, b \in \mathbb{R}^n \) are constant vectors and \( f(t) > 0 \) is a positive function. The regular case of Hilbert’s Fourth Problem is to characterize all locally projectively flat Finsler metrics; that is, the metrics whose geodesics are straight lines on an open subset of \( \mathbb{R}^n \).

Beltrami’s theorem states that a Riemannian metric is locally projectively flat if and only if it has constant sectional curvature. In Finslerian case, this is not true. There are non projectively flat Finsler metrics of constant flag curvature. Flag curvature is an analogue of sectional curvature in Finsler geometry.

Bucataru and Muzany \([5, 7]\) characterized the sprays which are metrizable by Finsler metrics of constant flag curvature \( \kappa \).

In this paper, we introduce the one form deformation of sprays. For a given spray \( S \) on a manifold \( M \), we define the one form deformation as the projective deformation of \( S \) by a one form \( \beta \) on \( M \). In other words we get a new spray \( \tilde{S} = S - 2\beta \mathcal{C} \) which has the same geodesics of \( S \). We focus our attention to the one form deformation of a flat spray; namely, \( S = S_0 - 2\beta \mathcal{C} \) and \( \beta(x, y) = y^k b_k(x) \) is a one form on the manifold \( M \). We study the Finsler metrizability of the deformation spray \( \tilde{S} \). We characterize the metrizability of \( S \) by a Finsler metric of constant flag curvature. We obtain a new family of projectively flat metrics of constant flag curvature and hence new solutions for Hilbert’s fourth problem.

The metric on \( \mathbb{B}^n \subset \mathbb{R}^n \) given by

\[
F_\mu = \sqrt{\frac{(1 + \mu |x|^2) |y|^2 - \mu \langle x, y \rangle^2}{(1 + \mu |x|^2)^2}}, \quad y \in T_x \mathbb{B}^n \simeq \mathbb{R}^n
\]

is projectively flat Riemannian metrics of constant (flag) curvature \( \mu \) with the projective factor

\[
P = \frac{\mu \langle x, y \rangle}{1 + \mu |x|^2}.
\]

It is known that every locally projectively flat Riemannian metric is locally isometric.
to $F_\mu$ for some constant $\mu$. In this paper, we obtained new family of projectively flat metrics. By calculating the Jacobi endomorphism, we conclude that this it has constant flag curvature 1. This family is given by

$$F = \sqrt{\frac{4h(x)c_{ij}y^iy^j - 4(c_{ij}, x^i)2 - 4(c', y)c_{ij}x^iy^j - (c', y)^2}{(2(h(x)))^2}}$$

and its projective factor is

$$P(x, y) = -\frac{2c_{ij}x^iy^j + (c', y)}{2(h(x))},$$

where $h(x) := c_{ij}x^ix^j + (c', x) + c$, $c_{ij} = c_{ji}$, $c' = (c_1, c_2, ..., c_n)$ are constants. Starting by $F_\mu$ (when $\mu = 1$), the general transformation, $x \to Ax + B$ and $y \to Ay$ where $A$ is an $n \times n$ invertible matrix and $B$ is an arbitrary $n \times 1$ matrix, generates projectively flat Riemannian metrics. The obtained class of projectively flat metrics is not, generally, isometric to $F_\mu$ metric via affine transformations.

Since the deformation spray $S$ of a flat spray $S_0$ is always isotropic and in the case that the curvature of $S$ is non zero, then the metric freedom of $S$ is unique up to some constants. Hence, in our case the deformation of a flat spray by the specific one form is metrizable by unique metric. However, we construct new projectively flat Finsler metrics and hence Finsler solutions for Hilbert’s fourth problem.

It is known that, [3], one of the conditions for a spray $S$ with non-vanishing Ricci curvature to be metrizable by a Finsler function of non-zero constant flag curvature is rank $dd_J(\text{Tr } \Phi) = 2n$. As an application of the deformation of a flat spray by a one form, we answer the following question:

**Does any spray of non-vanishing Ricci curvature satisfy the condition rank $dd_J(\text{Tr } \Phi) = 2n$?**

By an example, we show that for a spray $S$, if $S$ has non vanishing Ricci curvature, then the rank of the form $dd_J(\text{Tr } \Phi)$ is not necessarily maximal; that is, the condition rank $dd_J(\text{Tr } \Phi) = 2n$ is sharp for the metrizability of $S$.

2. **Preliminaries**

Let $M$ be an $n$-dimensional manifold and $(TM, \pi_M, M)$ be its tangent bundle and $(T^*M, \pi, M)$ the subbundle of nonzero tangent vectors. We denote by $(x^i)$ local coordinates on the base manifold $M$ and by $(x^i, y^j)$ the induced coordinates on $TM$. The vector 1-form $J$ on $TM$ defined, locally, by $J = \frac{\partial}{\partial y^j} \otimes dx^i$ is called the natural almost-tangent structure of $TM$. The vertical vector field $\mathcal{C} = y^j \frac{\partial}{\partial y^j}$ on $TM$ is called the canonical or the Liouville vector field.

A vector field $S \in X(TM)$ is called a spray if $JS = \mathcal{C}$ and $[\mathcal{C}, S] = S$. Locally, a spray can be expressed as follows

$$S = y^j \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^j},$$

where the spray coefficients $G^i = G^i(x, y)$ are 2-homogeneous functions in the $y = (y^1, ..., y^n)$ variable. A curve $\sigma : I \to M$ is called regular if $\sigma' : I \to TM$, where $\sigma'$ is the tangent lift of $\sigma$.

A regular curve $\sigma$ on $M$ is called geodesic of a spray $S$ if $S \circ \sigma' = \sigma''$. Locally, $\sigma(t) = (x^i(t))$ is a geodesic of $S$ if and only if it satisfies the equation

$$\frac{d^2x^i}{dt^2} + 2G^i \left( x, \frac{dx}{dt} \right) = 0. \tag{2.2}$$

An orientation preserving reparameterization $t \to \tilde{t}(t)$ of the system (2.2) leads to a new spray $\tilde{S} = S - 2PC$. The scalar function $P \in C^\infty(TM)$ is 1-homogeneous and it is related to the new parameter by

$$\frac{d^2\tilde{t}}{dt^2} = P \left( x^i(t), \frac{dx^i}{dt} \frac{dt}{d\tilde{t}} \right) \frac{dt}{d\tilde{t}} > 0. \tag{2.3}$$

**Definition 2.1.** Two sprays $S$ and $\tilde{S}$ are projectively related if their geodesics coincide up to an orientation preserving reparameterization. $\tilde{S}$ is called the projective deformation of spray $S$. 

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A nonlinear connection is defined by an n-dimensional distribution \( H : u \in T M \rightarrow H_u \in T_u(T M) \) that is supplementary to the vertical distribution, which means that for all \( u \in T M \), we have \( T_u(T M) = H_u(T M) \oplus V_u(T M) \).

Every spray \( S \) induces a canonical nonlinear connection through the corresponding horizontal and vertical projectors,

\[
h = \frac{1}{2} (Id + [J, S]), \quad v = \frac{1}{2} (Id - [J, S])
\]

Equivalently, the canonical nonlinear connection induced by a spray can be expressed in terms of an almost product structure \( \Gamma = [J, S] = h - v \). With respect to the induced nonlinear connection, a spray \( S \) is horizontal, which means that \( S = hS \).

Locally, the two projectors \( h \) and \( v \) can be expressed as follows

\[
h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes \delta y^i,
\]

The Jacobi endomorphism is defined by

\[
\Phi = v \circ [S, h] = R^j_i \frac{\partial}{\partial y^i} \otimes dx^j = \left( 2 \frac{\partial G^i_j}{\partial x^i} - S(N^i_j) - N^k_i N^i_k \right) \frac{\partial}{\partial y^j} \otimes dx^i.
\]

The two curvature tensors are related by

\[
3R = [J, \Phi], \quad \Phi = i_S R.
\]

The Ricci curvature, \( \text{Ric} \), and the Ricci scalar, \( \rho \in C^\infty(T M) \) [2] and [15], are given by

\[
\text{Ric} = (n - 1) \rho = R^k_i = \text{Tr}(\Phi).
\]

**Definition 2.2.** A spray \( S \) is called isotropic if the Jacobi endomorphism has the form

\[
\Phi = \rho J - \alpha \otimes C,
\]

where \( \alpha \) is a semi-basic 1-form \( \alpha \in \Lambda^1(T M) \).

Due to the homogeneity condition, for isotropic sprays, the Ricci scalar is given by \( \rho = i_S \alpha \).

**Definition 2.3.** A Finsler manifold of dimension \( n \) is a pair \( (M, F) \), where \( M \) is a differentiable manifold of dimension \( n \) and \( F \) is a map

\[
F : T M \rightarrow \mathbb{R},
\]

such that:

(a): \( F \) is smooth and strictly positive on \( T M \) and \( F(x, y) = 0 \) if and only if \( y = 0 \),

(b): \( F \) is positively homogenous of degree 1 in the directional argument \( y : \mathcal{L}_C F = F \),

(c): The metric tensor \( g_{ij} = \frac{\delta^2 E}{\delta y^i \delta y^j} \) has rank \( n \) on \( T M \), where \( E := \frac{1}{2} F^2 \) is the energy function.

Since the 2-form \( dd_J E \) is non-degenerate, the Euler-Lagrange equation

\[
\omega_E := i_S ddd_J E - d(E - \mathcal{L}_C E) = 0
\]

uniquely determines a spray \( S \) on \( T M \). This spray is called the geodesic spray of the Finsler function. The \( \omega_E \) is called the Euler-Lagrange form associated to \( S \) and \( E \).

**Definition 2.4.** A spray \( S \) on a manifold \( M \) is called Finsler metrizable if there exists a Finsler function \( F \) such that the geodesic spray of the Finsler manifold \( (M, F) \) is \( S \).

**Definition 2.5.** The function \( F \) is said to be of scalar flag curvature if there exists a function \( \kappa \in C^\infty(T M) \) such that

\[
\Phi = \kappa (F^2 J - Fd_J F \otimes C).
\]

It follows that for a Finsler function \( F \), of scalar flag curvature \( \kappa \), its geodesic spray \( S \) is isotropic, with Ricci scalar \( \rho = \kappa F^2 \) and the semi-basic 1-form \( \alpha = \kappa Fd_J F \).

**Definition 2.6.** A Finsler metric \( F = F(x, y) \) on an open subset \( U \subset \mathbb{R}^n \) is said to be projectively flat if all geodesics are straight lines in \( U \). A Finsler metric \( F \) on a manifold \( M \) is said to be locally projectively flat if at any point, there is a local coordinate system \( (x^i) \) in which \( F \) is projectively flat.
From now on, we use the notations $\partial_i$ for the partial differentiation with respect to $x^i$ and $\dot{\partial}_i$ for the partial differentiation with respect to $y^i$.

By [11], a Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies the following system of equations,

$$y^i \dot{\partial}_i \partial_j F - \partial_i F = 0,$$

In this case, $G^i = P y^i$ where $P = P(x, y)$, the projective factor of $F$, given by $P = \frac{\partial_i F y^i}{2F}$.

3. One form deformation

For a given spray $S$ on a manifold $M$, we mean by the one form deformation of $S$ the projective deformation of $S$ defined by

$$\tilde{S} = S - 2\beta C,$$

where $\beta$ is a one form on the manifold $M$. In this paper, we will focus our attention to the one form deformation of flat sprays. Let $S = S_0 - 2\beta C$, where $S_0$ is a flat spray; that is, $S_0 = y^i \partial_i$ and $\beta = b_i(x)y^i$ is one form on $M$, then by [3], one has the following

Lemma 3.1. For the deformation spray $S = S_0 - 2\beta C$ of a flat spray $S_0$. The corresponding horizontal projectors and Jacobi endomorphisms of the two sprays are related as follows:

(a): $h = h_0 - \beta J - dJ \beta \otimes C$,

(b): $\Phi = (\beta^2 - S_0 \beta) J - (\beta dJ \beta + dJ(S_0 \beta) - 3dh_0 \beta) \otimes C$,

In [7], Bucataru and Muzsnay characterized the metrizability of projective deformation of flat sprays as follows.

Theorem 3.2. [7] The spray $S = S_0 - 2PC$ is metrizable by a Finsler function of non zero constant flag curvature if and only if

(i): $dJ \alpha = 0$,

(ii): $dh_0 = 0$,

(iii): $\text{rank}(ddJ \rho) = 2n$,

where $\alpha$ is a semi-basic 1-form given by $\alpha = PdJ P + dJ(S_0 P) - 3dh_0 P$ and $\rho$ is the Ricci scalar given by $\rho = P^2 - S_0 P$.

Proposition 3.3. Let $S = S_0 - 2\beta C$ be a one form deformation of a flat spray $S_0$ with non-vanishing Ricci curvature. Then, necessary conditions for the properties $dJ \alpha = 0$ and rank $ddJ (Tr \Phi) = 2n$ to be hold are

(a): $\dot{\partial}_i b_j - \dot{\partial}_j b_i = 0$, i.e $b_i$ is gradient ($\beta$ is closed on $M$).

(b): $\det(\dot{\partial}_i b_j + b_i b_j) \neq 0$.

Proof. Let $dJ \alpha = 0$, since $dJ \alpha = -3dh_0 dJ \beta$, then we have

$$dh_0 dJ \beta(\dot{\partial}_i, \dot{\partial}_j) = \dot{\partial}_i \dot{\partial}_j \beta - \dot{\partial}_j \dot{\partial}_i \beta = 0.$$

Using the fact $\dot{\partial}_i b_j = b_j$, we get $\dot{\partial}_i b_j - \dot{\partial}_j b_i = 0$, i.e $b_i$ is gradient.

Since $\text{Tr} \Phi = (n - 1)(\beta^2 - S_0 \beta)$, then by a direct calculations and using that $b_i$ is gradient, we obtain

$$ddJ (Tr \Phi) = 2(n - 1)((\dot{\partial}_i b_j + b_i b_j) dx^i \wedge dy^j + (b_i \dot{\partial}_j \beta - b_j \dot{\partial}_i \beta) dx^i \wedge dx^j).$$

Consequently, rank $ddJ (Tr \Phi) = 2n$ if $\det(\dot{\partial}_i b_j + b_i b_j) \neq 0$. \hfill $\square$

The above proposition together with [13] show the following

Corollary 3.4. Let $S = S_0 - 2\beta C$ be a one form deformation of a flat spray $S_0$ with non-vanishing Ricci curvature. Then, necessary conditions for $S$ to be metrizable are

(a): $\dot{\partial}_i b_j - \dot{\partial}_j b_i = 0$, i.e $b_i$ is gradient,

(b): $\det(\dot{\partial}_i b_j + b_i b_j) \neq 0$.

Now, we are in a position to announce and prove the first result of this work.
Theorem 3.5. The one form deformation $S = S_0 - 2\beta C$, $\beta(x, y) = y^k b_k(x)$, of a flat spray $S_0$, is Finsler metrizable if and only if

$$b_k(x) = \frac{2c_{ik} x^i + c_k}{2(c_{ij} x^i x^j + \langle c', x \rangle + c)},$$

where $c_{ij} = c_{ji}$, $c$, $c' = (c_1, c_2, ..., c_n)$ are constants.

Proof. We are going to prove that the conditions (i)-(iii) in Theorem 3.2 are satisfied if and only if $b_k(x)$ given by (3.1). Since $d_J \alpha = -3d_h d_J \beta$, then we have

$$d_h d_J (\partial_1, \partial_2) = \partial_1 \partial_2 \beta - \partial_i \partial_i \beta = \partial_i |b_i - \partial_i b_i|.$$

Using the property that $c_{ij}$ is symmetric, $b_i$ is gradient and therefore $d_J \alpha = 0$.

To calculate $\rho$, let’s compute $S_0(\beta) = y^k \partial_2 \beta$,

$$S_0(\beta) = -\frac{2h(x)c_{ij} y^i y^j - (2c_{ij} x^i y^j + \langle c', y \rangle)^2}{2(h(x))^2},$$

for simplicity, we use $h(x) := c_{ij} x^i x^j + \langle c', x \rangle + c$. Using the formula of $\beta$ together with (3.2), we have

$$\rho = \frac{4h(x)c_{ij} y^i y^j - 4(c_{ij} x^i y^j)^2 - 4\langle c', y \rangle c_{ij} x^i y^j - \langle c', y \rangle^2}{(2h(x))^2}.$$

Differentiating (3.2) with respect to $\partial_k$ and $\partial_i$, we obtain:

$$\partial_k \rho = \frac{4c_{ij} y^i y^j (2c_{ik} x^i + c_k) - 6c_{ij} x^i y^j c_{ik} y^j - 4\langle c', y \rangle c_{ik} y^j}{(2h(x))^2},$$

$$\partial_i \rho = \frac{8h(x)c_{ik} y^i - 8c_{ij} x^i y^j c_{ik} x^j - 4c_{ij} x^i y^j - 4\langle c', y \rangle c_{ik} x^j - 2\langle c', y \rangle c_k}{(2h(x))^2}.$$

By using Lemma 3.1(a), we have $d_h \rho = d_h \rho - \beta d_J \rho - 2p d_J \beta$ which given, locally, by

$$d_h \rho (\partial_1) = \partial_i \rho - \beta \partial_i \rho - 2p b_i.$$

Now, substituting from (3.3), (3.4) and (3.5) into the above equation, we get $d_h \rho (\partial_1) = 0$.

Putting $\rho_i := \partial_i \rho$, then we have

$$\rho_i = \frac{4c_{ij} h(x) - 4(c_{ij} x^i) (c_{ij} x^j) - 2c_{ij} c_{jk} x^k - 2c_{ij} c_{lk} x^k - c_i c_j}{(2h(x))^2}.$$

The condition (iii) (regularity condition) is satisfied if $\det(\rho_i) \neq 0$. Consequently, for appropriate constants $c_{ij}$, $c_i$ and $c$ such that $\det(\rho_i) \neq 0$, the spray $S$ is metrizable.

Conversely, let $S$ be metrizable. Since the condition (i) is satisfied if and only if there exists a locally defined, 0-homogeneous, smooth function $g$ on $\Omega \times \mathbb{R}^n \setminus \{0\}, \Omega$ is open subset of $\mathbb{R}^n$, such that

$$d_J \beta = d_h g.$$  

Then, we have

$$d_J (\partial_i) = d_h g (\partial_i) \Rightarrow \partial_i (b_j y^j) = \partial_i g \Rightarrow b_i = \partial_i g.$$

Since $b_i$ is a function of $x$, then $g(x, y) = g_1(x) + g_2(y)$, $g_2(y)$ is 0-homogenous function. Then, we can write $\beta = y^i b_i (x) = S_0 (g)$ and $b_h(x) = \partial_i g$. The condition (ii) is satisfied if and only if

$$d_h \rho - S_0 (g) d_J \rho - 2p d_h g = 0.$$

Applying the above equation on $\partial_i$ and using that $S_0(h) = \beta$ and $\rho = \beta^2 - S_0 (\beta)$, we have

$$\partial_i \rho - \beta \partial_i \rho - 2p \partial_i g = 0.$$

Making use of $\beta = y^i \partial_i g$ and $\rho = \beta^2 - S_0 \beta$, the solution of (3.6) is given by

$$g(x, y) = \frac{1}{2} \ln \left( c_{ij} x^i x^j + \langle c', x \rangle + c \right) + g_2(y),$$
Differentiating (3.7) with respect to $\partial_k$ we have
\begin{equation}
(3.8) \quad b_k(x) = \partial_k g = -\frac{2c_{ik}x^i + c_k}{2(c_{ij}x^j + (c', x) + c)}.
\end{equation}
This completes the proof. \hfill $\Box$

Making use of the above theorem we have the following

**Theorem 3.6.** With appropriate constants $c_{ij}$, $c$, $c' = (c_1, c_2, ..., c_n)$, the family
\begin{equation}
(3.9) \quad F = \sqrt{4h(x)c_{ij}y^i y^j - 4(c_{ij}x^i y^j)^2 - 4(c', y)c_{ij}x^i y^j - (c', y)^2}/(2h(x))^2,
\end{equation}
h(x) = c_{ij}x^i x^j + (c', x) + c, is a family of projectively flat metrics.

**Proof.** By differentiating (3.9) with respect to $x^k$ and $y^k$ respectively, we have
\begin{equation}
(3.10) \quad \partial_k F = \frac{1}{2F} \left[ 4(2c_{ik}x^i + c_k)(c_{ij}y^i y^j - 8c_{ij}y^i x^j y^j - 4(c', y)c_{ij}y^j - 2(c', y)c_k) - 2c_{ik}x^i + c_k \right],
\end{equation}
\begin{equation}
(3.11) \quad \dot{\partial}_k F = \frac{1}{2F} \left[ 8h(x)c_{ik}y^i - 8(c_{ik}x^i y^j + 4c_{ik}x^i x^j y^j - 4(c', y)c_{ik}x^i y^j - 2(c', y)c_k) - 4c_{ik}x^i + c_k \right].
\end{equation}
Again, differentiating (3.11) with respect to $x^j$ gives
\begin{equation}
(3.12) \quad \partial_j \dot{\partial}_k F = \frac{1}{2F} \left[ 8h(x)c_{ij}y^i - 8(c_{ij}x^i y^j + 4c_{ij}x^i x^j y^j - 4(c', y)c_{ij}y^j - 4(c', y)c_k) - 4c_{ij}x^i + c_k \right],
\end{equation}
\begin{equation}
(3.13) \quad \partial_j y^k \partial_k F = -\frac{2c_{ik}x^i + (c', y)}{2F},
\end{equation}
\begin{equation}
(3.14) \quad y^j \partial_j \partial_k F = -\frac{2}{F} \left( \frac{2c_{ik}x^i + (c', y) c_{ij}y^j}{(2h(x))^2} - \frac{2}{F} \left( \frac{2c_{ik}x^i + c_k}{(2h(x))^2} \right) \right).
\end{equation}
By (3.10) and (3.14), we find that metric (3.9) satisfies the system
\begin{equation}
y^j \partial_j \partial_k F = \partial_k F
\end{equation}
which assures that $F$ is projectively flat metric. \hfill $\Box$

**Proposition 3.7.** The geodesic spray $G^i$ and the Jacobi endomorphism $R^i_j$ of the metric (3.9) are given by
\begin{equation}
(3.15) \quad G^i = \frac{4h(x)c_{ij}y^j - (2c_{ij}x^i y^j + (c', y))^2}{(2h(x))^2},
\end{equation}
\begin{equation}
(3.16) \quad R^i_j = \frac{4h(x)c_{ij}y^j - (2c_{ij}x^i y^j + (c', y))^2}{(2h(x))^2}.
\end{equation}
Moreover, the metric (3.9) has constant flag curvature 1.

**Proof.** By making use of Theorem 3.6 the metric (3.9) is projectively flat and hence its geodesic spray is given by
\begin{equation}
G^i = P(x, y)y^i, \quad P(x, y) = \frac{y^k \partial_k F}{2F},
\end{equation}
Now, using (3.13) we get the required formula for the geodesic spray $G^i$.

The Jacobi endomorphism $R^i_j$ has the form
\begin{equation}
R^i_j = 2\partial_j G^i - y^k \partial_k N^i_j + 2G^k G^i_j - N^i_j N^k_j
\end{equation}
where $N^i_j = \partial_j G^i$ and $G^i_j = \partial_k N^i_j$. By using the formula of $G^i$, we have the following
\begin{equation}
\partial_j G^i = \frac{(2h(x))2c_{js}y^s - (2c_{js}x^s y^s + (c', y))(2(2c_{rs}x^r + c_j))}{(2h(x))^2} y^i,
\end{equation}
\begin{equation}
\partial_k N^i_j = \frac{(2h(x))2c_{ks}y^s - (2c_{ks}x^s y^s + (c', y))(2(2c_{rs}x^r + c_j))}{(2h(x))^2} y^i.
\end{equation}
Proof. The proof of
As an application of the deformation of a flat spray by a one form, we answer the following question:

Corollary 3.8. Taking
If rank \(a\):
By plugging the above quantities into the formula of \(R^i\), the result follows.

Hence, the Ricci scalar is given by \(\rho = F^2\) and so \(F\) has constant curvature 1.

As a special case, we have the following

Corollary 3.9. Let \(S = S_0 - 2\beta C\) be a one form deformation of a flat spray \(S_0\),

A necessary condition for \(S\) to be metrizable is \(c_{ij} \neq 0\).

It is known that, \[6\], one of the conditions for a spray \(S\) with non-vanishing Ricci curvature to be metrizable by a Finsler function of non-zero constant flag curvature is rank \(dd_J(Tr \Phi) = 2n\). As an application of the deformation of a flat spray by a one form, we answer the following question:

Does any spray of non-vanishing Ricci curvature satisfy the condition rank \(dd_J(Tr \Phi) = 2n\)?
where \( e' = (c_1, c_2, \ldots, c_n) \), \( c_i \) and \( c \) are arbitrary constants. Since \( \text{Tr } \Phi = \text{Ric} = (n - 1)(\beta^2 - S_0\beta) \), \( \text{Ric} \) is the Ricci curvature. Then, we get
\[
\text{Ric} = -(n - 1)\frac{\langle e', y \rangle^2}{4(\langle e', x \rangle + c)^2}.
\]
Since \( n \neq 1 \), we have \( \text{Ric} \neq 0 \). Straightforward calculations lead to
\[
dd J(\text{Tr } \Phi) = 2(n - 1)(\alpha_{ij} dx^i \wedge dy^j + \beta_{ij} dx^i \wedge dx^j),
\]
where \( \alpha_{ij} = \frac{\langle e_i, e_j \rangle}{4(\langle e', x \rangle + c)^2} \). Then, \( \dd J(\text{Tr } \Phi) \) has maximal rank if \( \det(\alpha_{ij}) \neq 0 \), but \( \det(\alpha_{ij}) = 0 \) and moreover rank(\( \alpha_{ij} \)) = 1, then the result follows. \( \square \)

4. Affine transformations of Klein metric

The metric on \( B^n \subset \mathbb{R}^n \) given by
\[
F_\mu = \sqrt{\frac{\langle 1 + \mu|x|^2 \rangle y^2 - \mu(x, y)^2}{(1 + \mu|x|^2)^2}}, \quad P = -\frac{\mu(x, y)}{1 + \mu|x|^2}, \quad y \in T_x B^n \simeq \mathbb{R}^n
\]
is projectively flat Riemannian metrics of constant (flag) curvature \( \mu \). It is known that every locally projectively flat Riemannian metric is locally isometric to \( F_\mu \) for some constant \( \mu \). Starting by the Klein metric, the affine transformation \( x \to Ax + B \) and \( y \to Ay \) where \( A \) is an \( n \times n \) invertible matrix and \( B \) is an arbitrary \( n \times 1 \) matrix, produces projectively flat Riemannian metrics.

In this section, taking \( \mu = 1 \), we show that the family \( \{F_\mu \} \) is not isometric to the Klein metric via affine transformations.

**Theorem 4.1.** The family \( \{F_\mu \} \) is not, generally, isometric to the Klein metric (\( \mu = 1 \)) via affine transformations.

**Proof.** Consider the affine transformation \( \pi = Ax + B \) and \( \overline{y} = Ay \), where \( A \) is an \( n \times n \) invertible matrix and \( B \) is an arbitrary \( n \times 1 \) matrix,
\[
A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.
\]
Then we have
\[
|\pi|^2 = \sum_{k=1}^n a_{ki} a_{kj} x^i x^j + 2 \sum_{k=1}^n a_{ki} b_{kj} x^i + |b|^2,
\]
\[
|\overline{y}|^2 = \sum_{k=1}^n a_{ki} a_{kj} y^i y^j + \sum_{k=1}^n a_{ki} b_{kj} x^i,
\]
where \( |b|^2 = b_1^2 + b_2^2 + \ldots + b_n^2 \). Therefore, the Klein metric transforms to
\[
F = \sqrt{\frac{H(x)(\sum_{k=1}^n a_{ki} a_{kj}) y^i y^j - (\sum_{k=1}^n a_{ki} a_{kj}) x^i y^j)^2 - 2(b', y)(\sum_{k=1}^n a_{ki} a_{kj}) x^i y^j - (B', y)^2}{(H(x))^2}},
\]
where \( H(x) := 1 + ((\sum_{k=1}^n a_{ki} a_{kj}) x^i x^j + 2(\sum_{k=1}^n a_{ki} b_{kj} x^i + |b|^2) \), \( B' = (B_1, \ldots, B_n) \), \( B_i = \sum_{k=1}^n a_{ki} b_{kj} \). Now, comparing the equations \( \{F_\mu \} \) and \( \{\text{Klein} \} \), we get
\[
2c_{ij} = \sum_{k=1}^n a_{ki} a_{kj}, \quad c_i = \sum_{k=1}^n a_{ki} b_{kj}, \quad 2c = 1 + |b|^2.
\]
Thus we get, formally, the class \( \{F_\mu \} \) provided that the above system is consistent. But, generally, the above system is inconsistent. So once you have the transformation then you get the \( c's \), but if you have the \( e's \) then the transformation not necessarily exist.
For example, let
\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \]

Then the constants \( c \)'s are given by
\[
\begin{align*}
c_{11} &= \frac{1}{2}(a_{11}^2 + a_{21}^2), \\
c_{22} &= \frac{1}{2}(a_{12}^2 + a_{22}^2), \\
c_{12} &= c_{21} = a_{11}a_{12} + a_{21}a_{22}, \\
c_1 &= a_{11}b_1 + a_{21}b_2, \\
c_2 &= a_{12}b_1 + a_{22}b_2, \\
c &= \frac{1}{2}(1 + (b_1^2 + b_2^2)).
\end{align*}
\]

Now, take \( c_{11} = c_{22} = 0, c_{12} = c_{21} = 1, c_1 = 1, c_2 = 1, c = 1 \), we get a projectively flat metric and at the same time by substitution in the above system one obtains inconsistent system. For this choice of the \( c \)'s, we have
\[
F = \sqrt{\frac{(8x_1x_2y_1y_2 - 4x_1^2y_1^2 - 4x_2^2y_2^2 + 4x_1y_1y_2 - 4x_1^2y_2^2 - 4x_2y_1y_2 + 4x_2y_1y_2 - y_1^2 + 6y_1y_2 - y_2^2)}{4(2x_1x_2 + x_1 + x_2 + 1)^2}},
\]
and the projective factor is given by
\[
\beta = \frac{1}{2} \frac{2x_1y_2 + 2x_2y_1 + y_1 + y_2}{2x_1x_2 + x_1 + x_2 + 1}.
\]

So one can say that the affine transformation of Klein metric is contained in (3.9) but not any metric in (3.9) can be obtained by an affine transformation. \( \square \)

Now, the question is

**What is the isometry (transformation) between the klein metric and the family (3.9)?**

### 5. Finsler solutions for Hilbert fourth problem and examples

Since the deformation spray \( S \) of a flat spray \( S_0 \) is always isotropic and in the case which the curvature of \( S \) is non zero, then the metric freedom \([10]\) of \( S \) is unique up to some constants. Therefore, in our case the deformation of a flat spray by the specific one form \( \beta = b_i(x)y^i \) given by (3.1) is metrizable by unique Riemannian metric given in (3.9). However, in this section, we introduce some new projectively flat Finsler metrics and hence new Finsler solutions for Hilbert’s fourth problem. Although, Lots of new projectively flat Finsler metrics can be constructed, we will mention only two examples.

For simplicity we consider the following special case.

**Corollary 5.1.** Putting \( c_{ij} = \lambda \delta_{ij} \), we have
\[
(5.1) \quad F = \sqrt{\frac{4\lambda(x|x|^2 + \langle c', x \rangle + c)|y|^2 - 4\lambda^2\langle x, y \rangle^2 - 4\lambda\langle c', y \rangle\langle x, y \rangle - \langle c', y \rangle^2}{4\lambda|x|^2 + \langle c', x \rangle + c}},
\]
is a family of projectively flat metrics with the projective factor
\[
\beta = -\frac{2\lambda(x, y) + \langle c', y \rangle}{2\lambda|x|^2 + \langle c', x \rangle + c}.
\]

By making use of the above corollary, since \( \beta \) is closed one form on \( M \) and \( F \) is projectively flat Riemannian metric, then we have the following example of projectively flat Finsler metric.

**Example 1.** The family of metrics
\[
F = \sqrt{\frac{4\lambda(x|x|^2 + \langle c', x \rangle + c)|y|^2 - 4\lambda^2\langle x, y \rangle^2 - 4\lambda\langle c', y \rangle\langle x, y \rangle - \langle c', y \rangle^2 + (2\lambda(x, y) + \langle c', y \rangle)}{2\lambda|x|^2 + \langle c', x \rangle + c}}
\]
is new family of projectively flat Finsler metrics. Where

\[ G^i = P(x, y)g^i, \quad P(x, y) = -\frac{2\lambda(x, y) + \langle c', y \rangle}{2\lambda|x|^2 + \langle c', x \rangle + c} + \left(F - \frac{\lambda|y|^2}{2F(\lambda|x|^2 + \langle c', x \rangle + c)}\right). \]

Consequently, we have new Finsler solutions for Hilbert’s fourth problem.

By the help of [8] (Example 8.2.2, Page 156), we have another Finsler solution for Hilbert’s fourth problem as follows.

**Example 2.** The metric

\[
\Theta(x, y) = \frac{\langle c', y \rangle(c', x) - 4\lambda(x, y)}{4c\lambda|x|^2 - c^2} + \frac{\sqrt{16\lambda^2c^2(x, y)^2 - |x|^2|y|^2 + \langle c', y \rangle^2(c', x)^2 + 8\lambda\langle c', y \rangle(c', x)(x, y) + 4\lambda^2|y|^2}}{4c\lambda|x|^2 - c^2}
\]

is Funk metric and, moreover, it is projectively flat with the projective factor \( P = \frac{2\lambda(x, y)}{2}. \) Thus \( \Theta(x, y) \) is projectively flat with constant flag curvature \(-\frac{1}{2}\).

**Proof.** Using (5.1), we have

\[ F(0, y) = \phi(y) = \sqrt{\frac{4\lambda|y|^2 - \langle c', y \rangle^2}{4c^2}}. \]

Define

\[ \Theta(x, y) = \phi(y + \Theta(x, y)x) = \sqrt{\frac{4\lambda|y + \Theta(x, y)x|^2 - \langle c', y + \Theta(x, y)x \rangle^2}{4c^2}}. \]

Squaring both sides of the above equation and solving it for \( \Theta \), we get the required formula. Since \( \phi(y) \) is a Minkowski norm, then \( \Theta(x, y) \) is Funk metric and it is projectively flat metric with the projective factor \( P = \frac{2\lambda(x, y)}{2}. \)

The following example shows a one form deformation of a non flat spray which is not metrizable.

**Example 3.**

Let \( M = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 2\} \) and \( S_0 \) be a spray given by the coefficients

\[ G^1_0 := \frac{(y^1)^2}{2x^2}, \quad G^2_0 := 0, \]

and take \( \beta = y^1 + y^2 \). Now consider the deformation \( S = S_0 - 2\beta C \). The new coefficients are given by

\[ G^1 := \frac{(y^1)^2}{2x^2} + y^1(y^1 + y^2), \quad G^2 := y^2(y^1 + y^2), \]

The spray \( S \) is isotropic and the coefficients of the nonlinear connection are given by

\[ N^1_1 = \frac{y^1}{x^2} + 2y^1 + y^2, \quad N^1_2 = y^1, \quad N^2_1 = y^2, \quad N^2_2 = y^1 + 2y^2. \]

The horizontal basis is \( \{h_1, h_2\} \) where

\[
h_1 = \frac{\partial}{\partial x^1} - \left(\frac{y^1}{x^2} + 2y^1 + y^2\right)\frac{\partial}{\partial y^1} - y^2\frac{\partial}{\partial y^2},
\]

\[
h_2 = \frac{\partial}{\partial x^2} - y^1\frac{\partial}{\partial y^1} - (y^1 + 2y^2)\frac{\partial}{\partial y^2}.
\]

We have

\[
v_1 := [h_1, h_2, h_1] = -\left(\frac{(x^2)^2y^1 + (x^2)^2y^2 + x^2y^1 - y^2x^2 + y^1}{(x^2)^2}\right)\frac{\partial}{\partial y^1} + \left(\frac{(x^2)^2y^1 + (x^2)^2y^2 + 2x^2y^1 + y^2y^2 + y^1}{(x^2)^2}\right)\frac{\partial}{\partial y^2},
\]

\[
v_2 := [h_1, h_2, h_2] = -\left(\frac{(x^2)^3y^1 + (x^2)^3y^2 + 2y^1}{(x^2)^3}\right)\frac{\partial}{\partial y^1} + \left(\frac{(x^2)^2y^1 + (x^2)^2y^2 - x^2y^1 + 2y^1}{(x^2)^2}\right)\frac{\partial}{\partial y^2}.
\]
Being \( v_1 \) and \( v_2 \) linearly independent we have \( \mathcal{H} = \text{Span}\{h_1, h_2, v_1, v_2\} = TTM \), where \( \mathcal{H} \) is the holonomy distribution generated by the horizontal vectors and their successive Lie brackets. Consequently, the Liouville vector field \( C \in \mathcal{H} \) hence the spray is not metrizable.

The following example introduces a one form deformation of a flat spray which is not metrizable.

**Example 4.**
Let \( M = \{(x^1, x^2) \in \mathbb{R}^2 : x^2 > 0\} \) and \( S_0 \) be a flat spray. So the coefficients are given by
\[
G_0^1 = G_0^2 = 0,
\]
and take \( \beta = y^1 + y^2 \). Now consider the deformation \( S = S_0 - 2\beta C \). The new coefficients are given by
\[
G^1 := y^1(y^1 + y^2), \quad G^2 := y^2(y^1 + y^2),
\]
The spray \( S \) is isotropic and the coefficients of the nonlinear connection are given by
\[
N^1_1 = 2y^1 + y^2, \quad N^1_2 = y^1, \quad N^2_1 = y^2, \quad N^2_2 = y^1 + 2y^2.
\]
The horizontal basis is \( \{h_1, h_2\} \) where
\[
h_1 = \frac{\partial}{\partial x^1} - (2y^1 + y^2) \frac{\partial}{\partial y_1} - y^2 \frac{\partial}{\partial y^2},
\]
\[
h_2 = \frac{\partial}{\partial x^2} - y^1 \frac{\partial}{\partial y_1} - (y^1 + 2y^2) \frac{\partial}{\partial y^2}.
\]
We have
\[
v_1 := [h_1, h_2] = -(y^1 + y^2) \frac{\partial}{\partial y^1} + (y^1 + y^2) \frac{\partial}{\partial y^2}.
\]
The successive Lie brackets of \( h_1 \) and \( h_2 \) produce no more linearly independent vectors and hence the holonomy distribution \( \mathcal{H} = \text{Span}\{h_1, h_2, v_1\} \). The metric freedom of \( S \) is unique. Now we can check if we have regular energy function metricizes \( S \) or not.

The spray \( S \) is Finsler metrizable if there exists a function \( E \) satisfying the following system of partial differential equations
\[
\mathcal{L}_C E = 2E, \quad d_h E = 0,
\]
which can be written in the form
\[
y_1 \hat{\partial}_1 E + y_2 \hat{\partial}_2 E - 2E = 0,
\]
\[
\frac{\partial E}{\partial x^1} - (2y^1 + y^2) \frac{\partial E}{\partial y^1} - y^2 \frac{\partial E}{\partial y^2} = 0,
\]
\[
\frac{\partial E}{\partial x^2} - y^1 \frac{\partial E}{\partial y^1} - (y^1 + 2y^2) \frac{\partial E}{\partial y^2} = 0,
\]
\[
-(y^1 + y^2) \frac{\partial E}{\partial y^1} + (y^1 + y^2) \frac{\partial E}{\partial y^2} = 0.
\]
The above system has the solution
\[
E = C_1 e^{4(x^1 + x^2)}(y^1 + y^2)^2.
\]
The matrix \( (g_{ij}) \) associated with \( E \) is singular and hence the spray is not metrizable. Here in this example \( \beta \) is closed but \( \det(\hat{\partial}_i b_j + b_i \hat{\partial}_j) = 0 \).

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