The $S - D$ mixing and di-electron widths of higher charmonium $1^{--}$ states

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Abstract

The di-electron widths of $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$, and their ratios are shown to be in good agreement with experiment, if in all cases the $S - D$ mixing with a large mixing angle $\theta \approx 34^\circ$ is taken. Arguments are presented why continuum states give small contributions to the wave functions at the origin. We find that the $Y(4360)$ resonance, considered as a pure $3^3S_1$ state, would have very small di-electron width, $\Gamma_{ee}(Y(4360)) = 0.060$ keV. On the contrary, for large mixing between the $4^3S_1$ and $3^3D_1$ states with the mixing angle $\theta = 34.8^\circ$, $\Gamma_{ee}(\psi(4415)) = 0.57$ keV coincides with the experimental number, while a second physical resonance, probably $Y(4360)$, has also a rather large $\Gamma_{ee}(Y(\sim 4400)) = 0.61$ keV. For the higher resonance $Y(4660)$, considered as a pure $5^3S_1$ state, we predict the di-electron width $\Gamma_{ee}(Y(4660)) = 0.70$ keV, but it becomes significantly smaller, namely 0.31 keV, if the mixing angle between the $5^3S_1$ and $4^3D_1$ states $\theta = 34^\circ$. The mass and di-electron width of the $6^3S_1$ charmonium state are calculated.

1 Introduction

Knowledge of the di-electron widths of higher charmonium states is important for many reasons. First of all, it can help to identify the nature of the newly discovered resonances with $J^{PC} = 1^{--}$ and distinguish between conventional $c\bar{c}$ mesons and, for example, tetraquarks which have much smaller di-electron

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As shown in [1], di-electron widths of compact four-quark systems, like $c\bar{c}q\bar{q}$, with $J^{PC} = 1^{--}$ appear to be two orders smaller than those of conventional $c\bar{c}$ mesons.

For higher charmonium states, which lie above the $D\bar{D}^*$ (or $D^*\bar{D}^*$) threshold, their wave functions (w.f.) can be strongly affected by the coupled-channel and threshold effects, being in general very complicated functions. Via open channels the w.f. of a resonance contains admixtures of other states with the same quantum numbers and also a contribution from continuum state(s). To define such a w.f. one needs to formulate the relativistic many-channel Hamiltonian in QCD, even if in some approximation [2]. In the simplest case, the continuum part consists of two open-charm mesons and to some extent this continuum part can be considered as a particular case of a four-quark system, $c\bar{c}q\bar{q}$. Since the contribution of any four-quark state to the w.f. at the origin is much smaller than that of a meson, it can be neglected. This effect occurs because the probability to collect four (and even three) particles at the origin is much smaller than for two particles. This fact does not exclude that at larger distances the continuum part can give an essential contribution to the w.f. and even dominates asymptotically. Just such a coupling to the continuum provides a shift down of the mass of the $P$-wave heavy-light mesons [2]. Therefore the w.f. at the origin of a higher vector resonance, which we are interested in here, can be calculated taking into account only the mixing between those vector states which masses, defined in single-channel approximation, have close values. Study of the charmonium spectrum shows that the $(n+1)^3S_1$ and $n^3D_1$ states ($n \geq 3$) have small mass differences, $\leq 60$ MeV, which decrease for higher radial excitations. On the other hand the mass differences between neighbouring $^3S_1$ states is of the order of several hundreds of MeV, so in first approximation mixing between these states can be neglected. Such a representation of the w.f. at the origin can be tested via concrete predictions for the di-electron widths of different vector states in heavy quarkonia.

In this picture the $S-D$ mixing between higher resonances can be considered in the same way as it has been done for $\psi'(3686)$ and $\psi''(3770)$ [3], [4], where the mixing angle $\theta = (12 \pm 2)^\circ$ is extracted from the ratio of their di-electron widths. Here we show that for higher vector states the $S-D$ mixing is significantly larger and the mixing angle $\theta \sim 34^\circ$. Just for such an angle the di-electron widths of $\psi(4040)$ and $\psi(4160)$ turn out to be almost equal, as in experiment [5]-[9].

There are also other arguments in favor of a large $S-D$ mixing. It is
known that the experimental di-electron widths of $\psi(4040)$, $\psi(4160)$, and $\psi(4415)$ are significantly smaller than the potential model predictions in single-channel approximation [10]-[13]. Moreover, even with the use of a many-channel w.f. at the origin (calculated in [12] for the Cornell coupled-channel model [14]) the di-electron widths of higher charmonium states appeared to be considerably larger than in experiment. It is well-known that the QCD-motivated gluon-exchange (GE) potentials (with the strong coupling, possessing the asymptotic freedom property) have much smaller w.f. at the origin than those with $\alpha_s = const$ [13]; nevertheless, even such potentials give di-electron widths of $\psi(4040)$ and $\psi(4415)$ which are still 50 – 70% larger than the experimental numbers [10]-[12]. In the recent paper [15] the $Y(4660)$ resonance [16], considered as a $5^3S_1$ state, has also large $\Gamma_{ee}(5^3S_1) = 1.34$ keV, which is even significantly larger than $\Gamma_{ee}(\psi(4415)) = 0.58(7)$ keV.

We assume here that the relatively small values of the di-electron widths of the $n^3S_1$ states and the rather large widths of the $n^3D_1$ states, initially considered as pure states, occur mostly due to $S-D$ mixing. Here we do not use the same assumption as made in [11], where to reach agreement with experimental widths, “total” screening of the GE interaction at large distances has been supposed: such an assumption cannot explain why $\Gamma_{ee}(\psi(4160))$ is large, and also has no deep theoretical grounds.

Three experimental facts point to a possibly large mixing between the $(n+1)^3S_1$ and $n^3D_1$ vector charmonium states:

1. The measured di-electron width of $\psi(4160)$, which is usually considered as the $2^3D_1$ state, is large [8, 9]:

   $\Gamma_{ee}(\psi(4160)) = 0.83 \pm 0.06$ keV. \hspace{1cm} (1)

   Namely, it is only 5 – 10% smaller than the di-electron width of $\psi(4040)$ and $\sim$14 times larger than the width calculated here for a pure $2^3D_1$ state: $\Gamma_{ee}(2^3D_1) = 0.061$ keV (see Section 3). It is also about three times larger than $\Gamma_{ee}(\psi''(3770)) = 0.248(6)$ keV.

2. On the contrary, the experimental width of $\psi(4040)$, considered usually as the $3^3S_1$ state,

   $\Gamma_{ee}(\psi(4040)) = 0.86 \pm 0.07$ keV, \hspace{1cm} (2)

appears to be almost two times smaller than for a pure $3^3S_1$ state [12].
This situation can be resolved if the S-D mixing between these two states is taken into account. For levels above the $D^*\bar{D}^*$ threshold such a mixing can occur owing to short-range tensor forces (it gives a rather small effect) and the influence of open channel(s). Since at present there is no dynamical calculation of $S-D$ mixing, the influence of open channels can be taken into account in a phenomenological way through the introduction of a mixing angle, as for $\psi'(3686)$ and $\psi''(3770)$ [3, 4].

3. The third fact refers to the di-electron width of $\psi(4415)$. If this resonance is considered as the $4^3S_1$ state, then potential models give di-electron widths in the range $1.1 - 1.5$ keV [10, 12, 13], which are almost two times larger than in experiment [8, 9]:

$$\Gamma_{ee}(\psi(4415)) = 0.58 \pm 0.07 \text{ keV}.$$  \hspace{1cm} (3)

Such a decrease of the di-electron width could occur via mixing with a still unidentified $3^3D_1$ state, which in single-channel calculations has mass $M(3D) = 4.470(10)$ MeV, while $M(4S) = 4420(10)$ MeV, i.e., these two masses are rather close to the masses of the physical resonances $\psi(4415)$ and $Y(4360)$. One may expect that these $4^3S_1$ and $3^3D_1$ states could be strongly coupled to the $S$-wave decay channels, like $D_1(2420)D^*(2010)$, $D_0^*(2400)D^*(2010)$, and $D_{s0}^*(2317)D_s^*(2112)$, and due to this coupling the $4^3S_1$ and $3^3D_1$ levels are mixed and acquire hadronic downward mass shifts, which are typically $\sim 40 - 60$ MeV [14].

As a result, one of the shifted physical states goes over into the conventional $\psi(4415)$ charmonium, while the other one can possibly be identified with the $Y(4360)$ resonance, recently discovered by the Belle Collaboration [16]. (In our analysis here, the Belle resonance $Y(4360)$ with $\Gamma = 48(15)$ MeV [16] and the wide resonance $Y(4324)$ with $\Gamma = 172$ MeV, observed by the BaBar collaboration [17], are considered to be the same). Then the di-electron width of $\psi(4415)$ is calculated here, taking into account large $S-D$ mixing, while for the analysis it is inessential from which state, $4^3S_1$ or $3^3D_1$, the resonance $\psi(4415)$ originates. We show that for $\theta = 34^\circ$ the di-electron widths of both physical resonances have close values: $\Gamma_{ee}(Y(4360)) \sim \Gamma_{ee}(\psi(4415)) = 0.58$ keV.

In our picture it is convenient to define the mixing angle between higher vector states from the ratio of the di-electron widths, as in [3, 4]: in this
case the QCD factor $\beta_V$, occurring due to radiative corrections (see Sect. 4), is cancelled in the ratio. From such an analysis a large mixing angle is extracted and the absolute values of $\Gamma_{ee}(\psi(4040))$ and $\Gamma_{ee}(\psi(4160))$ are obtained in good agreement with experiment if the same QCD factor $\beta_V = 0.63$ is taken for all higher states.

Notice that the mass of the $5^3S_1$ state, $M(5^3S_1) = 4640(10)$ MeV, has been predicted in [12], before the Belle resonance $Y(4660)$ was discovered [16]. This resonance and its radiative transitions were studied in detail in [15] giving $\Gamma_{ee}(Y(4660)) = 1.34$ keV. In our calculations $\Gamma_{ee}(Y(4660))$ strongly depends on a possible admixture of the $4^3D_1$ state and is considerably smaller than in [15]: $\Gamma(Y(4660)) = 0.70$ keV, if this resonance is a pure $5^3S_1$ state, and about two times smaller, $\Gamma_{ee}(Y(4660)) = 0.31$ keV, if the mixing angle between $5^3S_1$ and the unobserved $4^3D_1$ state (with mass $\sim 4700$ MeV) is 34°, the same as for $\psi(4415)$.

2 The masses of the $J^{PC} = 1^{--}$ charmonium states

The hyperfine (HF) and fine-structure splittings of higher radial excitations are small ($\leq 20$ MeV) [11], [18], therefore their masses practically coincide with the centroid masses, $M_{cog}(nL)$, which we need to determine with good accuracy. To calculate them we use here the relativistic string (RS) Hamiltonian with universal (for all mesons) interaction [19], [20]. For charmonium one contribution to the mass formula, namely the small string correction ($\leq 5$ MeV) for the states with $L \neq 0$ can be neglected, while the self-energy correction, $\sim -20$ MeV, is taken into account here.

In heavy quarkonia the mass $M_{cog}(nL)$ is just given by the eigenvalue (e.v.) of the spinless Salpeter equation (SSE) [20]:

$$\left\{ 2\sqrt{p^2 + m^2_c + V_B(r)} \right\} \psi_{nL}(r) = M_{cog}(nL)\psi_{nL}(r).$$

Here we also use the RS Hamiltonian written in the Einbein approximation (EA) [21]. In this case the spin-averaged mass can be presented as:

$$M_{cog}(nL) = \omega_{nL} + \frac{m^2_c}{\omega_{nL}} + E_{nL}(\omega_c) + \Delta_{SE},$$

5
where the e.v. \( E_{nL} \) are the solutions of the so-called einbein equation \[21, 22\]:

\[
\left( \frac{p^2}{\omega_{nL}} + V_0(r) \right) \varphi_{nL}(r) = E_{nL} \varphi_{nL},
\]

which together with the mass \( \omega_{nL} \) should be defined in a selfconsistent way:

\[
\omega_{nL}^2 = m_c^2 - \frac{\partial E_{nL}}{\partial \omega_{nL}}.
\]

For the \( n^3D_1 \) state the small HF contribution to the mass will be neglected here, i.e., its mass \( M(n^3D_1) = M_{\text{cog}}(nD) \), while for the \( n^3S_1 \) states we still keep the small HF correction: \( M(n^3S_1) = M_{\text{cog}}(nS) + \frac{1}{4} \delta_{\text{HF}}(nS) \) with \( \delta_{\text{HF}}(nS) = M(n^3S_1) - M((n^1S_0)). \) The values of \( \delta_{\text{HF}}(nS) = 48(48), 16(20), 12(16), 6(10) \text{ MeV} \) \( \text{MeV} \), calculated in \[11\] and \[18\] (in parentheses), are used here.

Our calculations are performed with the universal potential \( V_B(r) \) from \[20, 22\]:

\[
V_B(r) = \sigma(r) \cdot r - \frac{4}{3} \frac{\alpha_B(r)}{r},
\]

where the vector coupling \( \alpha_B(r) \) is taken in two-loop approximation: it has the asymptotic freedom behavior at small \( r \), freezes (saturates) at large \( r \), and depends on the number of flavors \( n_f \). For charmonium we use \( n_f = 4 \) and the QCD constant \( \Lambda_{\text{MS}}^{(4)} \) = 254 MeV, which gives the vector QCD constant \( \Lambda_V(n_f = 4) = 1.4238 \cdot \Lambda_{\text{MS}}^{(4)}(n_f = 4) = 360 \text{ MeV} \) \[22\]. The freezing (critical) value of \( \alpha_B(r) \) is expressed through \( \Lambda_V \) and the so-called background mass \( M_B = 1.0 \text{ GeV} \):

\[
\alpha_B(r \rightarrow \infty) = \alpha_B(q = 0) = \frac{4\pi}{\beta_0 t_0} \left( 1 - \frac{\beta_1}{\beta_0^2} \frac{\ln t_0}{t_0} \right) = 0.546,
\]

where \( t_0 = \ln \left( \frac{M_B}{\Lambda_V} \right) \), \( \beta_0 = 11 - \frac{2}{3} n_f \), and \( \beta_1 = 102 - \frac{38}{3} n_f \).

For low-lying states which have relatively small sizes (with r.m.s. radius \( R(nL) \leq 0.8 \text{ fm} \), a linear confining potential with constant string tension, \( \sigma = \sigma_0 \approx 0.18 \text{ GeV}^2 \), can be used \[23\]. However, for higher states, which lie above open thresholds and have large radii \( R(nL) \geq 1.0 \text{ fm} \), it is important to take into account the creation of virtual light-quark pairs \( (q\bar{q}) \), even in single-channel approximation. Due to virtual loops the surface inside the Wilson
Table 1: The charmonium masses $M(n^3S_1)$ (in MeV) for the potential (8).

| state | SSE$^a)$ $m_c = 1.425$ GeV | EA$^a)$ $m_c = 1.410$ GeV | BGS[18] $m_c = 1.4794$ GeV | exp.[9] |
|-------|------------------|------------------|------------------|--------|
| 1S    | 3105             | 3095             | 3090             | 3097   |
| 2S    | 3678             | 3682             | 3672             | 3686   |
| 3S    | 4078             | 4096             | 4072             | 4039(1)|
| 4S    | 4398             | 4426             | 4406             | 4421(4)| 4361(18)$^b)$ |
| 5S    | 4642             | 4672             | 4664(16)$^b)$    |        |
| 6S    | 4804             | 4828             |                  |        |

$^a)$ The self-energy corrections to the masses, $\Delta_{SE}(nS) \approx -20$ MeV, are taken into account both in the relativistic case (SSE) and in the einbein approximation.

$^b)$ Belle data [16]

loop decreases, making the string tension dependent on the $Q\bar{Q}$ separation $r$ [24]:

$$\sigma(r) = \sigma_0(1 - \gamma f(r)).$$

(10)

Such a flattening of the confining potential is common to all mesons of large sizes, and therefore for charmonium the form and parameters of such modified string tension can be taken from the analysis of the radial Regge trajectories for light mesons [24]:

$$\gamma = 0.40; \quad f(r \to 0) = 0, \quad f(r \to \infty) = 1.0.$$  

(11)

As shown in [24], due to flattening the masses of all higher levels are shifted down and these mass shifts increase with $n$. For example, the shift of the $5^3S_1$ state reaches $\sim 100$ MeV [23]. In Tables 1 and 2 we give the masses of pure $n^3S_1$ and $n^3D_1$ states for the potential (8), which are calculated using the SSE and the EA (6), and also the masses calculated in [18], where in the Cornell potential a constant coupling is used, equal to our freezing value (9). From Table 1 one can see that in our calculations the mass of the $5^3S_1$ state agrees with that of the $Y(4660)$ resonance [16]. For the $6^3S_1$ level the predicted mass is $M(6^3S_1) = 4815(15)$ MeV.
Table 2: The charmonium masses $M(n^3D_1)$ (in MeV) for the potential $\delta$.

| state $n$ | SSE$^a$ $m_c = 1.425$ GeV | EA$^a$ $m_c = 1.410$ GeV | BGS$^{[18]}$ $m_c = 1.4794$ GeV | Exp. $^9$ |
|-----------|--------------------------|--------------------------|-------------------------------|-------|
| 1$D$      | 3800                     | 3779                     | 3806                          | 3770(3) |
| 2$D$      | 4156                     | 4165                     | 4167                          | 4159(3) |
| 3$D$      | 4464                     | 4477                     |                               | 4421 |
|           |                          |                          |                               | 4361$^b$, 4324$^c$ |
| 4$D$      | 4690                     | 4707                     |                               |       |
| 5$D$      | 4840                     | 4855                     |                               |       |

$^a$ See the footnote $^a$ to Table 1
$^b$ See the footnote $^b$ to Table 1
$^c$ BaBar data $^{[17]}$

As seen in Tables 1 and 2, the masses $M((n+1)S)$ and $M(nD)$ are close to each other, even in single-channel approximation. The difference between them decreases for larger radial excitations, so that for $n = 5$ it is only $\sim 30$ MeV. It is worthwhile to notice that besides the “correlated” mass shifts of higher levels—due to virtual pair creation—some levels, which lie near thresholds, can have additional downward shifts due to strong coupling to a continuum channel. We denote these mass shifts as decay-channel (DC) shifts; they can be calculated only within a multi-channel approach. The masses of the $2S$, $4S$, and $2D$ levels calculated here (see Tables 1 and 2) agree with experiment within 20 MeV, i.e., they have essentially no DC shifts. However, the mass of the $3^3S_1$ level is $\sim 40$ MeV larger than the experimental one, because this level can be affected by the $D^*\bar{D}^*$ channel and we estimate its DC shift as $\sim 40$ MeV. Therefore, the masses predicted in our paper, have different accuracies, which is better than 20 MeV for the levels without DC shifts, and than 40 MeV for the levels strongly coupled to nearby continuum decay channels.

Thus from our analysis we conclude that the mass difference,

$$\Delta_n M = M(n^3D_1) - M((n+1)^3S_1),$$

(12)

decreases from the value $\Delta_2 M(\text{exp}) = 120$ MeV for $n = 2$ to $\sim 30$ MeV for $n = 4$. Therefore higher levels are almost degenerate and the $S-D$ mixing
for them, as well as the mixing angle, become larger than for \(\psi'\) and \(\psi''\). Also in single-channel approximation \(M_{\text{cog}}((n + 1)S)\) is always smaller than \(M_{\text{cog}}(nD)\).

### 3 Vector Decay Constants

The decay constants of vector (V) mesons are calculated here using the analytical expressions derived in [25] in the framework of the Field Correlator Method, where \(f_V(nS)\) and \(f_V(nD)\) are given by

\[
f^3_V(nS) = 12 \frac{|\psi_{nS}(0)|^2}{M_V(nS)} \xi_V = \frac{3}{\pi} \frac{|R_{nS}(0)|^2}{M_V(nS)} \xi_V,
\]

where the relativistic factor \(\xi_V\) is defined as

\[
\xi_V(nL) = \frac{m^2 + \omega^2_{nL} + \frac{1}{3} \langle \mathbf{p}^2 \rangle}{2\omega^2_{nL}}.
\]

The same expression \(13\) can be applied to the decay constants \(f_V(n^3D_1)\), if in \(13\) the w.f. \(R_{nD}(0)\) is defined as in \(15\) below.

All matrix elements (m.e.) which are needed to calculate the decay constants for the \(nS\) and \(nD\) states, are given in Tables 3 and 4. An interesting fact is that for the \((n + 1)S\) and \(nD\) states the m.e. like \(\omega_{nL}, \langle \mathbf{p}^2 \rangle_{nL}\), and \(\xi_V(nL)\) coincide with an accuracy better than 1% and therefore the difference between \(f_V((n + 1)S)\) and \(f_V(nD)\) comes only from their w.f. at the origin and the small differences in their masses \(M_V(nL)\).

Table 3: The matrix elements \(\omega_{nS}\) (GeV), \(\langle \mathbf{p}^2 \rangle\) (GeV\(^2\)), and the w.f. at the origin \(R_{nS}(0)\) (GeV\(^3/2\)) (no mixing) for the potential [8]

| State | \(\omega_{nS}\) (GeV) | \(R_{nS}(0)\) | \(\langle \mathbf{p}^2 \rangle\) | \(\xi_{nS}\) |
|-------|----------------|-------------|----------------|----------|
| 1\(S\) | 1.59 | 0.905 | 0.541 | 0.929 |
| 2\(S\) | 1.65 | 0.767 | 0.722 | 0.910 |
| 3\(S\) | 1.69 | 0.714 | 0.882 | 0.899 |
| 4\(S\) | 1.71 | 0.655 | 0.947 | 0.894 |
| 5\(S\) | 1.66 | 0.531 | 0.775 | 0.908 |
| 6\(S\) | 1.63 | 0.445 | 0.665 | 0.916 |
Table 4: The matrix elements $\omega_{nD}$ (GeV), $\langle p^2 \rangle$ ((GeV/c)^2), the w.f. at the origin $R_{nD}(0)$ (GeV^3/2), and the second derivative $R''_{nD}(0)$ (GeV^7/2) for the potential (8).

| State | $\omega_{nD}$ (GeV) | $R''_{nD}(0)$ | $R_{nD}(0)$ | $\langle p^2 \rangle$ | $\xi_{nD}$ |
|-------|----------------------|---------------|-------------|----------------|------------|
| 1D    | 1.65                 | 0.145         | 0.095       | 0.721           | 0.909      |
| 2D    | 1.69                 | 0.213         | 0.132       | 0.881           | 0.899      |
| 3D    | 1.71                 | 0.248         | 0.150       | 0.939           | 0.893      |
| 4D    | 1.65                 | 0.221         | 0.144       | 0.745           | 0.911      |
| 5D    | 1.64                 | 0.206         | 0.135       | 0.682           | 0.912      |

The w.f. at the origin $R_{nD}(0)$ is defined here as in [26], being expressed via the second derivative $R''_{nD}(0)$:

$$R_{nD}(0) = \frac{5}{2\sqrt{2}\omega_{nD}^2}R''_{nD}(0).$$ \hspace{1cm} (15)

It is interesting that $R_{nS}(0)$ and $R_{nD}(0)$ have different behavior for growing $n$: while $R_{nS}(0)$ decreases for higher radial excitations, the second derivative $R''_{nD}(0)$ and $R_{nD}(0)$ grow with increasing $n$, if a linear confining potential is used. For the flattening potential used here, $R_{nS}(0)$ decreases even faster, while $R_{nD}(0)$ increases for the 2S and 3S states and then practically saturates for higher levels. This growth of $R_{nD}(0)$ is possibly one of the reasons why higher radial excitations have large S − D mixing.

In Table 5 the decay constants of the charmonium states with $J^{PC} = 1^{--}$ are given in two cases: without and with $S − D$ mixing. The mixing angle between the $2^3S_1$ and $1^3D_1$ levels has already been calculated in [3] and [4], and also in [23], where the mixing angle $\theta = 11^\circ$ has been extracted from the ratio of experimental di-electron widths, $\Gamma_{ee}(\psi'(3686))$ and $\Gamma_{ee}(\psi''(3770))$. Other mixing angles are calculated below. The matrix elements and other numbers in Tables 3 and 4 are calculated here with the use of the EA equation (10), which provides regular behavior of the w.f. at the origin, in contrast to the SSE for which the S−wave w.f. diverges at the origin and needs to be regularized.

As seen in Table 5 the decay constants $f_V(nD)$ are very small, $\sim 50$ MeV ($\theta = 0$) for pure D-wave states. However, if $S − D$ mixing is large ($\theta \sim 34^\circ$) the decay constants $f_V(\theta)$ ($n \geq 2$) of physical, “mixed” states appear to be practically equal for $\psi(4040)$ and $\psi(4160)$, and also for $\psi(4415)$ and the still
unidentified second charmonium state, which originates from the 2D level and it is denoted below as \( \tilde{\psi}(4415) \) (although its mass is \( \sim 4470(10) \) MeV in single-channel approximation). For large \( S - D \) mixing all decay constants lie in the range \( 220 \pm 20 \) MeV \( (n = 2, 3) \). Precisely this fact provides close values of the di-electron widths of the \( (n+1)S \) and \( nD \) states.

4 Di-electron widths

In [23] the di-electron widths: \( \Gamma_{ee}(J/\psi), \Gamma_{ee}(\psi(3686)), \) and \( \Gamma_{ee}(\psi(3770)) \) have been calculated with high precision, \( \leq 5\% \), using the theoretical formula where the di-electron width is expressed via the decay constant \( \xi_V \), containing the relativistic correction \( \xi_V \), and includes QCD radiative corrections (this expression is the relativistic generalization of the van Royen-Weisskopf formula [27] in the framework of Field Correlator Method). The QCD correction, known in one-loop approximation, enters as the multiplicative factor denoted here as \( \beta_V = 1 - \frac{16}{3\pi}\alpha_s(M_V) \). Then

\[
\Gamma_{ee}(n^3S_1) = \frac{4\pi e^2\alpha^2}{3M_{nS}} f_{nS}^2 \beta_V = \frac{4\pi e^2\alpha^2}{M_{nS}^2} |R_{nS}(0)|^2 \xi_{nS} \beta_V, \tag{16}
\]

\[
\Gamma_{ee}(n^3D_1) = \frac{4\pi e^2\alpha^2}{3M_{nD}} f_{nD}^2 \beta_V = \frac{4\pi e^2\alpha^2}{M_{nD}^2} |R_{nD}(0)|^2 \xi_{nD} \beta_V. \tag{17}
\]

The w.f. at the origin \( R_{nD}(0) \) in (17) has been defined in (15) and the average kinetic energy \( \omega_{nL} = \langle \sqrt{p^2 + \mu^2} \rangle_{nL} \), calculated from equation (17), plays the
role of a constituent quark mass being different for different $nL$ states. The $\omega_{nL}$ have the following characteristic feature: For a linear confining potential with $\sigma = \sigma_0 = const.$ it grows for higher radial excitations, while for the flattening potential first it grows for $n = 2, 3, 4$ and then saturates around the value $\omega_{nL} \sim 1.65$ GeV for $n \geq 5$ (the values $\omega_{nS}, \omega_{nD}$ are given in Tables 3 and 4).

The expressions (16) and (17) contain the $c$-quark charge $e_c = 2/3$, $\alpha = 1/137$, and the mass $M_{nS}(M_{nD})$ of the $n^3S_1(n^3D_1)$ vector mesons. The w.f. at the origin $R_{nS}(0)$, $R_{nD}(0)$, and $R'_{nD}(0)$ are given in Tables 3 and 4.

As we discussed in the Introduction and Section 2, one may expect that the physical $\psi$-mesons represent a mixing of the $(n + 1)S$ and $nD$ states with close mass values, and our goal here is to determine the mixing angle between higher radial excitations. To this end we introduce the w.f. of the physical $\psi$-mesons, denoted here by $\varphi_{nS}(0)$ and $\varphi_{nD}(0)$, where the symbols $nS$ and $nD$ simply remind about the origin of those states:

$$\varphi_{nS}(0) = \cos \theta_n R_{nS}(0) - \sin \theta_n R_{(n-1)D}(0),$$

$$\varphi_{nD}(0) = \cos \theta_n R_{(n+1)S}(0) - \sin \theta_n R_{nD}(0).$$  \hspace{1cm} (18)

The di-electron widths of the $\psi$-mesons are expressed via the physical w.f. at the origin (18) in the same way as in (16) and (17). For a given mixing angle $\theta$ the w.f. at the origin (18) are easily calculated through the w.f. $R_{nS}(0)$ and $R_{nD}(0)$ for pure $S$- and $D$-wave states (they are given in Tables 3 and 4).

Table 6: The wave functions at the origin $\varphi_{(n+1)S}(0)$ and $\varphi_{nD}(0)$ in GeV$^{3/2}$ of the physical states for $n = 1, 2, 3, 4$.

| $n$ | 1  | 2  | 3  | 4  |
|-----|----|----|----|----|
| $\theta$ | 11° | 34.8° | 34° | 34° |
| $\varphi_{(n+1)S}(0)$ | 0.735 | 0.511 | 0.459 | 0.360 |
| $\varphi_{nD}(0)$ | 0.240 | 0.516 | 0.491 | 0.416 |

$^a$ The uncertainty in the mass value used gives rise to a theoretical error less than 1%.
5 $3^3S_1 - 2^3D_1$ mixing

To determine the mixing angle between the $3^3S_1$ and $2^3D_1$ states we use here the ratio of the di-electron widths of the $\psi(4040)$ and $\psi(4160)$ mesons as in [4] and [22]. This ratio does not depend on the QCD factor $\beta_V$ and the experimental di-electron widths are given in (1) and (2) with their ratio close to unity. Such a large ratio turns out to be possible only if the mixing angle between $3^3S_1$ and $2^3D_1$ states is large.

Taking in the ratio the w.f. at the origin (18), which are expressed via the numbers $R_{nS}(0)$ and $R_{nD}(0)$ from Tables 3 and 4, one can extract the mixing angle $\theta_2 = 34.8^\circ$ and determine the physical w.f. at the origin, as well as the decay constants of $\psi(4040)$ and $\psi(4160)$. Then for both charmonium states the di-electron widths appear to be in precise agreement with experiment (see (21)), if the QCD factor $\beta_V = 0.63$ is taken. This value of $\beta_V$ is smaller (i.e. the radiative corrections are larger) than for $J/\psi$, $\psi(3686)$, and $\psi(3770)$, where in all cases the larger value $\beta_V = 0.72$ gives precise agreement with experiment [23].

The mixing angle between the $(n+1)^3S_1$ and $n^3D_1$ states, denoted here as $\theta_n$, can be calculated if at least one of the di-electron widths is known from experiment. For the $3S$ and $2D$ states both di-electron widths are known and $\theta_2$ is easily determined. It is important to notice that for a pure $2^3D_1$ state the di-electron width is very small: $\Gamma_{ee}(2^3D_1) = 0.059$ keV, i.e., $\sim 14$ times smaller than the experimental number (1), and one can expect large mixing between the $3^3S_1$ and $2^3D_1$ states. Such a large mixing can occur via the nearby open $D^* \bar{D}^*$ channel and partly through short-ranged tensor forces which, however, do not provide a large mixing angle, $\theta(\text{tensor}) \lesssim 7^\circ$. On the contrary, from the ratio:

$$ \eta = \frac{\Gamma_{ee}(\psi(4040))}{\Gamma_{ee}(\psi(4160))} = 1.04 \pm 0.17, \quad (19) $$

one obtains two solutions with a large magnitude of $\theta_2$: a positive and a negative one:

$$ \theta_2 = 34.8^\circ \quad \text{or} \quad \theta_2 = -55.7^\circ. \quad (20) $$

For these angles and using (18) the physical w.f. $\varphi(\psi(4040), r = 0) = 0.511$ GeV$^{3/2}$ and $\varphi(\psi(4160), r = 0) = 0.516$ GeV$^{3/2}$, appear to be almost equal. Then from (16) and (17) with $\beta_V = 0.63$ one calculates the following di-
electron widths:
\[ \Gamma_{ee}(\psi(4040)) = 0.87 \text{ keV}, \quad \Gamma_{ee}(\psi(4160)) = 0.83 \text{ keV}, \quad (21) \]
which just coincide with the central values of the experimental values, \(^{11}\) and \(^{2}\). The QCD factor \( \beta = 0.63 \) extracted simultaneously, corresponds to the strong coupling \( \alpha_s(M_V) = 0.217 \). Later this value of \( \beta_V = 0.63 \) is used to determine \( \theta_n \) for higher excitations \((n = 3, 4)\). Notice that the same mixing angle \( \theta_2 = 35^\circ \) (or \( \theta_2 = -55^\circ \)) has been obtained in the analysis of \( \psi(4040) \) and \( \psi(4160) \) \(^{28}\).

6 Large mixing between \( 4^3S_1 \) and \( 3^3D_1 \) states

In constituent quark models (in single-channel approximation) two vector states, \( 4^3S_1 \) and \( 3^3D_1 \), are expected in the mass region around 4.4 GeV (see Tables \(^{1}\) and \(^{2}\)). Our calculations give the masses \( M(4^3S_1) \sim 4.42 \) GeV and \( M(3^3D_1) \sim 4.47(1) \) GeV with their mass difference \( \sim 50 \text{ MeV} \). However, one cannot exclude that due to strong coupling to the \( D^*D_1(2420) \) and \( D^*D_2(2460) \) channels the \( 4^3S_1 \) and \( 3^3D_1 \) states are mixed and the mass of one or probably both states is shifted down. Then one of these mixed (physical) states can be identified with \( \psi(4415) \) and the other one with the newly discovered resonance \( Y(4360) \) \(^{16}\). (Note that in charmonium the values of the DC shifts are typically \( \sim 40 \text{ MeV} \) \(^{14}\)).

From experiment only the di-electron width \( \Gamma_{ee}(\psi(4415)) \) is presently known, \( \Gamma_{ee}(\psi(4415)) = 0.58 \pm 0.07 \text{ keV} \), while for the \( Y(4360) \) resonance two possible numbers have been measured for the product \(^{16}\).

\[ B(Y(4360) \rightarrow \psi(2S)\pi^+\pi^-) \times \Gamma_{ee}(Y(4360)) = \begin{cases} 10.4 \pm 3.2 \text{ eV} \\ 11.8 \pm 3.2 \text{ eV} \end{cases} \quad (22) \]

Still, this restricted information allows one to draw an important conclusion. First, for pure \( 4S \) and \( 3D \) states \((\theta_3 = 0)\) with the w.f. at the origin \( R_{4S}(0) = 0.65 \text{ GeV}^{3/2} \) and \( R_{3D}(0) = 0.150 \text{ GeV}^{3/2} \) (from Tables \(^{2}\) and \(^{4}\)), their di-electron widths are the following:

\[ \Gamma_{ee}(4^3S_1) = 1.19 \text{ keV}, \quad \Gamma_{ee}(3^3D_1) = 0.06 \text{ keV}, \quad (\theta = 0). \quad (23) \]
i.e., \( \Gamma_{ee}(4S) \) is two times larger than the experimental number \(^{3}\) while \( \Gamma_{ee}(3D) \) is small. To reach agreement with experiment for \( \psi(4415) \) we need
to take a large mixing angle, namely $\theta_3 = 34^\circ$, as for the $3S - 2D$ mixing, for which
\[ \Gamma_{ee}(\psi(4415))|_{\text{theory}} = 0.57 \text{ keV} \]  
(24)
is completely in agreement with the central experimental value (3).

Then for the same angle the di-electron width of the second physical state, which can be denoted as $\bar{\psi}(4470)$ (in many-channel approximation its mass may be smaller), appears to be ten times larger than for a pure $3^3D_1$ state:
\[ \Gamma_{ee}(\bar{\psi}(4470)) = 0.63 \text{ keV}. \]  
(25)
Moreover this width is even slightly larger than that of $\psi(4415)$ (here we take $\beta_V = 0.63$ as for the $3S$ and $2D$ states). Since in this case the di-electron widths coincide within 10% accuracy, in the framework of the single-channel approximation it is difficult to decide which of these states should be identified with $\psi(4415)$ or with $Y(4360)$. From the experimental value (22) and the di-electron width (28) one obtains an estimate of the branching ratio
\[ B(Y(4360) \to \psi(2S)\pi^+\pi^-), \]  
(26)
which is rather large. Thus for large mixing angle ($\theta_3 \approx 34^\circ$) one cannot a priori conclude which resonance, $\psi(4415)$ or $Y(4360)$, originates from the $4^3S_1$. A decisive test for their identification could come from the study of their radiative transitions to the $\chi_{cJ}$ states, because radiative transitions are very sensitive to the mixing angle, as has been shown for $\psi'(3686)$ and $\psi''(3770)$ in [3, 29].

7 \hspace{1cm} Y(4660), Y(4815)

The higher level $Y(4660)$ with $M = 4660 \pm 16$ MeV, recently discovered in [16], has a surprisingly small width, $\Gamma = 48 \pm 18$ MeV. This state lies close to the $S$-wave threshold $D_{s1}^*D_{s1}(2535)$ (with the threshold mass $M_{\text{th}} = 4647$ MeV) and to the $P$-wave threshold $D(2^3S_1)\bar{D}^*$ with $M_{\text{th}} = 4647$ MeV (our calculations give the mass $M(D(2^3S_1)) \approx 2640$ MeV). Our predictions for the masses of the $5^3S_1$ and $4^3D_1$ states in single-channel approximation (see Tables [1 and 2] are
\[ M(5^3S_1) = 4655(15) \text{ MeV}, \quad M(4^3D_1) = 4700(10) \text{ MeV}, \]  
(27)
where the theoretical uncertainty is taken into account. These masses differ only $\sim 50$ MeV and large mixing between the two states can be expected. Unfortunately, at present their di-electron widths remain unknown, and here we calculate them, assuming that the QCD factor $\beta_V = 0.63$ and $\theta_4 = 34^\circ$ as it takes place for $4S$ and $3D$ states, and also first consider pure $5S$ and $4D$ states ($\theta_4 = 0$), for which

$$\Gamma_{ee}(5^3S_1) = 0.73 \text{ keV}, \quad \Gamma_{ee}(4^3D_1) = 0.055 \text{ keV}, \quad (\theta_4 = 0)$$

(28)

i.e., $\Gamma_{ee}(5S)$ is rather large, being even larger than $\Gamma_{ee}(4S)$ (23). On the contrary $\Gamma_{ee}(4D)$ is very small.

For large $S - D$ mixing with $\theta_4 = 34^\circ$ (like $\theta_2$ and $\theta_3$) we obtain

$$\Gamma_{ee}(\tilde{\psi}(4660)) = 0.32 \text{ keV}, \quad (\theta_4 = 34^\circ)$$

(29)

which is two times smaller than (28). For the second state, denoted as $\tilde{\psi}(4690)$ we find,

$$\Gamma_{ee}(\tilde{\psi}(4690)) = 0.45 \text{ keV}, \quad (\theta_4 = 34^\circ)$$

(30)

the width appears to be eight times larger than for the pure $4D$ state in (28) and even larger than $\Gamma_{ee}(\psi(4660))$. Notice that equal widths are obtained for a bit smaller angle, $\theta_4 = 30^\circ$:

$$\Gamma_{ee}(\tilde{\psi}(4660)) = \Gamma_{ee}(\tilde{\psi}(4690)) = 0.39 \text{ keV}.$$ 

(31)

We predict also the $6^3S_1$ state although this state has very large r.m.s. radius, $R \cong 2.5$ fm, even in closed-channel approximation. Its mass is $M(6^3S_1) = 4815 \pm 15$ MeV and $\Gamma_{ee}(6^3S_1) = 0.20$ keV for $\theta_5 = 34^\circ$. The existence of so high a resonance would be important for the theory.

8 Conclusions

We have studied the di-electron widths of higher $n^3S_1$ and $n^3D_1$ radial excitations in charmonium and shown that

1. The almost equal values of $\Gamma_{ee}(4040)$ and $\Gamma_{ee}(4160)$, as well as the small value of $\Gamma_{ee}(4415)$, can be explained, if large S-D mixing between $(n + 1)^3S_1$ and $n^3D_1$ states takes place.
2. For $\psi(4040)$ and $\psi(4160)$ precise agreement with experiment is obtained taking the mixing angle $\theta_2 = 34.8^\circ$.

3. For $\psi(4415)$ the calculated di-electron width coincides with the central value of the experimental width for a mixing angle $\theta_3 = 34^\circ$.

4. In all cases the QCD radiative corrections appear to be important and the same strong coupling $\alpha_s(M_V) = 0.217$ is taken, giving $\sim 30\%$ effect.

5. In the single-channel approximation used here DC mass shifts (due to strong coupling to a nearby threshold) cannot be calculated. Therefore it remains unclear which physical resonance, $\psi(4415)$ or the Belle resonance $Y(4360)$, corresponds to the $3^3D_1(4^3S_1)$ state. For both states we predict close values of their di-electron widths: $\Gamma_{ee}(\psi(4415)) = 0.57$ keV and $\Gamma_{ee}(Y(4360)) = 0.63(7)$ keV (they coincide within the experimental error).

6. Assuming that the $5^3S_1$ and $4^3D_1$ states have also large $S - D$ mixing, with $\theta_4 = (32 \pm 2)^\circ$, we obtain: $\Gamma_{ee}(\tilde{\psi}(4660)) = 0.35(4)$ keV, $\Gamma_{ee}(\tilde{\psi}(4690)) = 0.40(5)$ keV.

7. One cannot exclude that a $6^3S_1$ state also exists, for which we predict the mass $M(6S) = 4815(20)$ MeV and di-electron width $\Gamma_{ee} = 0.20$ keV.

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