This paper investigates the asset liability management problem for an ordinary insurance system incorporating the standard concept of proportional reinsurance coverage in a stochastic interest rate and stochastic volatility framework. The goal of the insurer is to maximize the expectation of the constant relative risk aversion (CRRA) of the terminal value of the wealth, while the goal of the reinsurer is to maximize the expected exponential utility (CARA) of the terminal wealth held by the reinsurer. We assume that the financial market consists of risk-free assets and risky assets, and both the insurer and the reinsurer invest on one risk-free asset and one risky asset. By using the stochastic optimal control method, analytical expressions are derived for the optimal reinsurance control strategy and the optimal investment strategies for both the insurer and the reinsurer in terms of the solutions to the underlying Hamilton-Jacobi-Bellman equations and stochastic differential equations for the wealths. Subsequently, a semi-analytical method has been developed to solve the Hamilton-Jacobi-Bellman equation. Finally, we present numerical examples to illustrate the theoretical results obtained in this paper, followed by sensitivity tests to investigate the impact of reinsurance, risk aversion, and the key parameters on the optimal strategies.

1. **Introduction.** One of the most important extension of the portfolio selection problem is to take consideration of liability, which is regarded as the asset liability management optimization problem. Sharp and Tint (1990) investigated the asset liability management problem in a static setting for the first time. Kell and Müller (1995) studied efficient portfolio management in the asset liability context.
A financial security system’s asset liability management must be conducted with inputs from both assets and liabilities. Management of assets or liabilities separately will lead to serious problems. All in one word, asset liability management is very important to all financial market participants, both individuals and institutions, regardless their size and shape. Many researchers devote themselves to generalize original models and solve them by new methodologies with the help of latest scientific tools. Till now, research in asset liability management still presents prospects for further research.

The stochastic control approach was used for the first time by Merton (1969, 1971) on the combined problem of portfolio selection and consumption rules under continuous-time setting. Since then, many efforts have been made to further generalize the results with application in various practical areas, for example, insurance and pension fund management. Björnerle (2005) derived the optimal reinsurance strategy for the classical Cramer-Lundberg model with dynamic proportional reinsurance under both benchmark criterion and mean-variance criterion. Bai and Guo (2008) solved the optimal value function and optimal investment strategy for insurer with proportional reinsurance and no-shorting constraint in a generalized financial market setting. Chen et al. (2010) studied an investment-reinsurance problem with the objective of finding the optimal policy to minimize the probability of ruin. The market risk that concerns the insurer and regulators is measured based on Value-at-Risk, and for related literature, see Cao and Wan (2009) and Chiu and Wong (2013).

The above-mentioned research work basically use positive constants or deterministic functions to describe the interest rate and volatility rate of the risky asset, which is obviously contradicted by our observations of market fluctuations due to various sources of uncertainty. Henceforth, plenty of results on the effect of stochastic volatility or stochastic interest rate have been reported. The common models used to formulate stochastic interest rate and stochastic volatility include the Cox-Ingersoll-Ross (CIR) model, the constant elasticity of variance (CEV) model, and Heston’s stochastic volatility model. The term structure of interest rates based on an intertemporal general equilibrium asset pricing model was reported in Cox et al. (1985). Deelstra et al. (2000) studied an optimal investment problem with stochastic interest rate which follows the Cox-Ingersoll-Ross dynamics. Gao (2009) used a CEV model, which is usually regarded as an extension of geometric Brownian motion, to describe the stock price dynamics and obtained the optimal investment strategy for a defined contribution pension plan. Gu et al. (2010) applied the CEV model in the proportional reinsurance and investment optimization problem. Kraft (2005) solved the portfolio selection problem under the Heston’s stochastic volatility with the objective of maximizing utility from terminal wealth. Yi et al. (2013) considered a robust optimal reinsurance and investment problem under Heston’s stochastic volatility model for an ambiguity-averse insurer and derived the optimal strategy and the corresponding value function. Some other notable research work includes Chacko and Viceira (2005), Gao (2010), Gu et al. (2012), Li et al. (2012), Taksar and Zeng (2012), Chang and Rong (2013), and Zhao et al. (2014).

However, no work has been done on the asset liability management optimization problem for an insurance system with reinsurance under both stochastic interest rate and stochastic volatility. Hence, in this paper, we study this problem in the framework with the constant relative risk aversion (CRRA) utility function. We develop the dynamics of wealth process with consideration of both investment and
reinsurance control under the CIR stochastic interest rate and Heston stochastic volatility framework. We also study some special cases with constant interest rate or constant volatility, so as to compare the optimal strategies under different situations. The power utility function degenerates to a logarithmic utility function as \(\gamma \to 1\), and we provide the optimal strategies in the logarithmic utility case. Besides, another important extension of our model is to describe both the risky asset and the claim by a generalized jump-diffusion geometric Brownian motion (an exponential Lévy process) and a jump-diffusion Brownian motion (a Lévy process), such that our model become more realistic from the economic point of view with consideration of discontinuity and jumps. The goal of the insurer is to maximize the expectation of the CRRA utility of terminal wealth by optimally allocating its wealth in one risk-free asset and one risky asset dynamically within the fixed investment horizon. To obtain the explicit solutions, we apply the stochastic optimal control, power transform and variable change technique.

Another contribution of our paper is that we study the optimal investment problem for the reinsurer. The reinsurer is allowed to invest in different assets which include a risk-free asset and a risky asset that is governed by the classic Heston stochastic volatility model. Based on the optimal reinsurance control strategy decided by the insurer, the reinsurer aims to maximize the expected exponential utility (CARA) of the terminal wealth. We apply similar methods to solve the corresponding Hamilton-Jacobi-Bellman equation and obtain the explicit results.

The rest of the paper is organized as follows. In Section 2, we describe the financial market setting, the insurance system with proportional reinsurance, and establish the asset liability management optimization problem. The optimal investment strategy for the insurer and the optimal reinsurance control strategy are respectively derived for the optimization problem by using the stochastic optimal control, power transform and variable change technique in Section 3, followed by the analytical results for some special cases. In Section 4, with the optimal reinsurance control strategy decided by the insurer, we derive the optimal investment strategy for the reinsurer in the sense of maximizing the expected exponential utility of terminal wealth. In Section 5, we present numerical analysis so as to demonstrate our theoretical results together with some sensitivity tests of the impact of reinsurance and the key parameters on the optimal investment strategies for the insurer and the reinsurer and the optimal reinsurance control strategy. Some conclusions are given in Section 6.

2. Model formulation. Consider a filtered probability space \((\Omega, \mathcal{P}, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T})\) with \(\mathcal{F}_t = \sigma\{W(s); 0 \leq s \leq t\}\) being augmented by all the \(\mathcal{P}\)-null sets in \(\mathcal{F}\), where \(\mathcal{F} = \mathcal{F}_T\). Let \(W(t)\) be a one-dimensional standard Brownian motion defined on \((\Omega, \mathcal{P}, \mathcal{F})\) over \([0, T]\), and a positive constant \(T\) denote the investment time horizon. All random variables considered in this paper are continuously differentiable and bounded over \([0, T]\). All stochastic processes introduced in this paper are assumed to be well-defined and adapted processes in this space.

2.1. The financial market. We consider a financial market with the standard assumptions: continuous trading is allowed; no transaction cost or tax is involved in trading; and all assets are infinitely divisible. The financial market consists of two risk-free assets and two risky assets.
Assets available for the insurer. The price of the risk-free asset $B(t)$ is governed by the following ordinary differential equation
\[
\begin{aligned}
\begin{cases}
\frac{dB(t)}{dt} = r(t)B(t)dt, & 0 \leq t \leq T, \\
B(0) = b_0 > 0,
\end{cases}
\end{aligned}
\]
where $b_0$ is the initial price of the risk-free asset; $r(t)$ is the risk-free interest rate and is assumed to evolve according to the Cox-Ingersoll-Ross (CIR) model (1985) as follows
\[
\begin{aligned}
\begin{cases}
\frac{dr(t)}{dt} = K(\alpha - r(t))dt + \beta \sqrt{r(t)}dW^r(t), & 0 \leq t \leq T, \\
r(0) = r_0 > 0,
\end{cases}
\end{aligned}
\]
where $r_0$ is the initial interest rate; $K$, $\alpha$, and $\beta$ are positive constants satisfying $2K\alpha > \beta^2$ with $K$ being the degree of mean reversion and $\alpha$ being the long-run mean of the interest rate; $r(t)$ is non-negative almost surely. As analyzed in Cox et al. (1985), the CIR model is a continuous time first-order autoregressive process where the randomly moving volatility is elastically pulled toward a central location or long-term value.

The interest rate and volatility of the risky asset are both stochastic. The price of the risky asset $S(t)$ satisfies the following stochastic differential equation system
\[
\begin{aligned}
\begin{cases}
\frac{dS(t)}{S(t)} = \left[\mu(t) + \frac{\nu}{2}S(t)\right]dt + \sqrt{\nu S(t)}dW^S(t), & 0 \leq t \leq T, \\
\frac{d\sigma(t)}{\sigma(t)} = \kappa(\theta - \sigma(t))dt + \xi \sqrt{\sigma(t)}dW^\sigma(t), & 0 \leq t \leq T, \\
S(0) = s_0 > 0, & \sigma(0) = \sigma_0 > 0,
\end{cases}
\end{aligned}
\]
where $s_0$ and $\sigma_0$ are respectively the initial price and initial volatility of the risky asset; $\tau$, $\kappa$, $\theta$, and $\xi$ are also positive constants satisfying $2\kappa \theta > \xi^2$; $\kappa$ is the degree of mean reversion; $\theta$ is the long-run mean of the volatility rate; $\sigma(t)$ is also non-negative almost surely. The system above is the classic Heston stochastic volatility model. The dynamic process for the $\sigma(t)$ is also the famous CIR model with different parameters from (2).

Assets available for the reinsurer. The price of the risk-free asset $B^{rc}(t)$ is modelled by
\[
\begin{aligned}
\begin{cases}
\frac{dB^{rc}(t)}{B^{rc}(t)} = \mu dt, & 0 \leq t \leq T, \\
B^{rc}(0) = b_0^{rc} > 0
\end{cases}
\end{aligned}
\]
where $b_0^{rc}$ is the initial price of the risk-free asset; $\mu$ is a positive constant risk-free interest rate. The price of the risky assets $S^{rc}(t)$ follows the Heston model as below
\[
\begin{aligned}
\begin{cases}
\frac{dS^{rc}(t)}{S^{rc}(t)} = \left[\mu + a\sigma(t)\right]dt + b\sqrt{\sigma(t)}dW^{S^{rc}}(t), & 0 \leq t \leq T, \\
S^{rc}(0) = s_0^{rc} > 0, & \sigma(0) = \sigma_0 > 0
\end{cases}
\end{aligned}
\]
where $s_0^{rc}$ is the initial price; $a$ and $b$ are positive constants; $\sigma(t)$ follows the same CIR dynamic process as in the risky asset’s price system available for the insurer (3).

2.2. The insurance system with reinsurance.

The wealth process for the insurer. Based on the basic structure of an ordinary insurance system incorporating the standard proportional reinsurance coverage (see Zimbidis, 2008), we develop the equation for the wealth increasement during the
time interval \([t, t + \Delta t]\) held by the insurer as follows

\[
[\text{Wealth Increase}] = [\text{Investment Return}] + [\text{Premium Received}] \\
- [\text{Claims Paid}] - [\text{Reinsurance Premiums}] \\
+ [\text{Claims Recovered by the Reinsurer}].
\] (6)

Assume that the insurer with an initial wealth \(x_0\) at time 0 and invests its wealth in the market dynamically in the horizon \([0, T]\). Denote by \(x(t)\) the wealth of the insurer at time \(t\), and \(\pi(t)\) be the amount invested in the risky asset at time \(t\). Then the capital amount invested in the risk-free asset at time \(t\) is \(x(t) - \pi(t)\). Taking the investment return into consideration, the insurer’s wealth (6) can be modelled by the following stochastic differential equation

\[
dx(t) = (x(t) - \pi(t)) \frac{dB(t)}{B(t)} + \pi(t) \frac{dS(t)}{S(t)}
\]

\[
+ (1 + \eta(t))m(t)dt - dL(t)
\]

\[
- (1 + \zeta(t))m(t)(1 - \psi(t))dt + (1 - \psi(t))dL(t),
\] (7)

where \(\eta(t)\) is the safety loading of the insurer at time \(t\), \(\zeta(t)\) is the safety loading of the reinsurer, at time \(t\), \(1 - \psi(t)\) is the proportion of the reinsurance coverage at time \(t\), \(\psi(t) \in [0, 1]\) is the proportion retained by the insurer at time \(t\). So, \(\psi(t)\) should satisfy \(0 \leq \psi(t) \leq \frac{n(t)}{\sigma(t)} \leq 1\). The dynamics of the claims payment \(L(t)\) evolves according to the following stochastic differential equation

\[
\begin{cases}
    dL(t) = m(t)dt + \phi(t)dW^L(t), & 0 \leq t \leq T, \\
    L(0) = l_0 \geq 0,
\end{cases}
\] (8)

where \(l_0\) is the initial value of the liability; \(m(t)\) and \(\phi(t)\) are respectively the appreciation and the volatility of the aggregate claim \(L(t)\) which are assumed to be deterministic functions of time \(t\).

**The wealth process for the reinsurer.** From the reinsurer’s perspective of view, its wealth should satisfy the following equation

\[
[\text{Wealth Increase}] = [\text{Investment Return}] + [\text{Premium Received}] \\
- [\text{Claims Paid}].
\] (9)

Assume that the reinsurer has an initial wealth \(y_0\) at time \(t = 0\) and invests its wealth in the market dynamically in the horizon \([0, T]\). Denote by \(y(t)\) the wealth of the insurer at time \(t\), and \(u(t)\) be the amount invested in the risky asset at time \(t\). Then the capital amount invested in the risk-free asset at time \(t\) is \(y(t) - u(t)\). Taking the investment return into consideration, the reinsurer’s wealth (9) could be expressed in the following stochastic differential equation

\[
dy(t) = (y(t) - u(t)) \frac{dB^{re}(t)}{B^{re}(t)} + u(t) \frac{dS^{re}(t)}{S^{re}(t)}
\]

\[
+ (1 + \zeta(t))m(t)(1 - \psi(t))dt - (1 - \psi(t))dL(t).
\] (10)

**Remark 1.** It should be noted that the risky asset price, the liability (claims) dynamics, the stochastic interest rate and the stochastic volatility process use different stochastic factor to describe the dynamics. \(W^r(t), W^S(t), W^\sigma(t)\), and \(W^L(t)\) are all standard Brownian motions with some conditions: (i) \(W^S(t)\) and \(W^\sigma(t)\) are correlated with \(\text{Cov}(W^S(t), W^\sigma(t)) = \rho\) and \(-1 < \rho < 1\); (ii) \(W^r(t)\) is independent of \(W^S(t), W^\sigma(t)\), and \(W^L(t)\); (iii) \(W^L(t)\) is independent of \(W^S(t), W^\sigma(t)\), and \(W^r(t)\).
Remark 2. To avoid risk-free profit for the insurer, the inequality holds for any time $t$
\[\eta(t) \leq \zeta(t), \quad 0 \leq t \leq T.\]
Otherwise, the insurer can earn arbitrary profit and transfer all the insurance risks to the reinsurer by paying a reinsurance premium smaller than the insurance premium.

Substituting (1), (2), (3) and (8) into (7), we get the following wealth system for the insurer
\[
\begin{aligned}
\begin{cases}
    dx(t) &= \left( r(t) + \tau \sigma(t) \pi(t) + (x(t) - \pi(t)) r(t) + m(t) \eta(t) - (1 - \psi(t)) \zeta(t) \right) dt \\
    &+ \pi(t) \nu \sqrt{\sigma(t)} dW^S(t) - \psi(t) \phi(t) dW^L(t), \quad 0 \leq t \leq T, \\
    x(0) &= x_0 > 0.
\end{cases}
\end{aligned}
\tag{11}
\]

Inspired by Zimbidis (2008), we define a new variable $\omega(t) = \zeta(t) - \eta(t) \geq 0$ to represent the difference between the insurer’s loading and the reinsurer’s loading. Throughout this paper we only consider the case that $\omega(t) = 0$. Under this situation, the condition for $\psi(t)$ changes to $0 \leq \psi(t) \leq 1$. Then, the dynamics of $x(t)$ can be simplified as follows
\[
\begin{aligned}
\begin{cases}
    dx(t) &= \left( \tau \sigma(t) \pi(t) + x(t) r(t) + m(t) \psi(t) \zeta(t) \right) dt \\
    &+ \pi(t) \nu \sqrt{\sigma(t)} dW^S(t) - \psi(t) \phi(t) dW^L(t), \quad 0 \leq t \leq T, \\
    x(0) &= x_0 > 0.
\end{cases}
\end{aligned}
\tag{12}
\]

Substituting (4), (5) and (8) into (10), we get the following system for the reinsurer
\[
\begin{aligned}
\begin{cases}
    dy(t) &= \left( (\mu + a \sigma(t)) u(t) + (y(t) - u(t)) \mu + m(t) (1 - \psi(t)) \zeta(t) \right) dt \\
    &+ u(t) b \sqrt{\sigma(t)} dW^S(t) - (1 - \psi(t)) \phi(t) dW^L(t), \quad 0 \leq t \leq T, \\
    y(0) &= y_0 > 0.
\end{cases}
\end{aligned}
\tag{13}
\]

2.3. Asset liability management optimization problem.

Definition 2.1 (Admissible Strategy). Denote $\mathbf{L}^2_F(t,T;\mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued and measurable stochastic processes $f(s)$ adapted to $\{\mathcal{F}_s\}_{s\geq t}$ on $[0, T]$ such that
\[
E \left[ \int_t^T |f(s)|^2 ds \right] < +\infty.
\]

For the insurer, a strategy pair $[\pi(\cdot), \psi(\cdot)] = \{[\pi(t), \psi(t)]; t \in [0, T]\}$ is called admissible if $[\pi(\cdot), \psi(\cdot)] \in \mathbf{L}^2_F(t,T;\mathbb{R}^n)$, and the triple $[x(\cdot), \pi(\cdot), \psi(\cdot)]$ is a unique solution to the stochastic differential equation (12). In addition, we denote $\Pi(0, T)$ the set of all such admissible solutions over $[0, T]$.

For the reinsurer, a strategy $u(\cdot) = \{u(t); t \in [0, T]\}$ is called admissible if $u(\cdot) \in \mathbf{L}^2_F(t,T;\mathbb{R}^n)$, and $[y(\cdot), u(\cdot)]$ is a unique solution to the stochastic differential equation (13). We denote $\mathcal{U}(0, T)$ the set of all such admissible strategies over $[0, T]$.

The asset liability management problem for an ordinary insurance system with reinsurance in a stochastic interest rate and stochastic volatility framework refers to the problem of finding the optimal admissible strategy and optimal reinsurance control strategy such that the utility functions for the insurer and the reinsurer is maximized. We express the optimization problems as follows.
Problem P.

\[
\max_{[\pi(t), \psi(t)]} \mathbb{E}[U(x(T))]
\]
subject to (12),

where \( U(x) \) is the utility function which is strictly concave with positive first-order derivative and negative second-order derivative. We use the power (the so-called constant relative risk aversion CRRA) utility function for the insurer in our investigation, which is given by

\[
U(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0 \quad \text{and} \quad \gamma \neq 1,
\]

where \( \gamma \) is the insurer’s relative risk aversion coefficient.

Problem Q.

\[
\max_u \mathbb{E}[U_{re}(y(T))]
\]
subject to (13) and \( \psi^*(t) \),

Where \( U_{re}(y) \) is the utility function which is also strictly concave with positive first-order derivative and negative second-order derivative. We use the exponential (the so-called constant absolute risk aversion CARA) utility function for the reinsurer in our investigation, which is given by

\[
U_{re}(y) = -\frac{1}{q}e^{-qy}, \quad q > 0,
\]

where \( q \) is the reinsurer’s absolute risk aversion coefficient.

3. Solution scheme for problem P. In this section, we first derive the stochastic differential equation of wealth process by considering the dynamics of the risk-free asset, the risky asset, and the liability. Then, we obtain the Hamilton-Jacobi-Bellman equation associated with the Optimization Problem P. At last, we find the explicit solutions for the CRRA utility function via power transformation and variable change method, followed by the analytical solutions for some special cases.

3.1. Derivation of the main results. We define the value function as follows

\[
H(t, x, r, \sigma) = \sup_{[\pi(t), \psi(t)]} \mathbb{E}[U(x(T))|x(t) = x, r(t) = r, \sigma(t) = \sigma], \quad 0 \leq t \leq T,
\]

with boundary condition \( H(T, x, r, \sigma) = U(x(T)) \). The Hamilton-Jacobi-Bellman equation associated to the Optimization Problem P is

\[
H_t + \sup_{[\pi(t), \psi(t)]} \left\{ (\tau \sigma \pi(t) + xr + m(t)\psi(t)\zeta(t))H_x + K(\alpha - r)H_r \\
+ \kappa(\theta - \sigma)H_\sigma + \frac{1}{2}(\pi^2(t)\nu^2\sigma + \psi^2(t)\phi^2(t))H_{xx} \\
+ \frac{1}{2}\beta^2 rH_{rr} + \frac{1}{2}\xi^2 \sigma H_{\sigma\sigma} + \pi(t)\nu \rho \xi \sigma H_{\sigma} \right\} = 0,
\]

where \( H_t, H_x, H_r, H_\sigma, H_{xx}, H_{rr}, H_{\sigma\sigma}, \) and \( H_{\sigma} \) denote partial derivatives of first-order and second-order with respect to time \( t \), wealth \( x(t) \), interest rate \( r(t) \), and volatility \( \sigma(t) \). We apply the same symbol to denote partial derivatives in the rest.
of this paper. By the first-order necessary and sufficient condition for maximization with respect to $\psi(t)$ and $\pi(t)$ we have

$$\psi^*(t) = -\frac{m(t)\zeta(t)}{\phi^2(t)} \cdot \frac{H_x}{H_{xx}}.$$ (18)

$$\pi^*(t) = -\frac{\tau\sigma H_x + \nu\rho\xi\sigma H_{xx}}{\nu^2\sigma H_{xx}}.$$ (19)

Substituting (18) and (19) into (17), we obtain the partial differential equation for the value function $H(t, x, r, \sigma)$

$$H_t + x r H_x + K(\alpha - r) H_x + \kappa (\theta - \sigma) H_x + \frac{1}{2} \beta^2 r H_{xx}$$

$$+ \frac{1}{2} \xi^2 \sigma H_{xx} - \frac{m^2(t)\zeta^2(t)H_x^2}{2\phi^2(t)H_{xx}} - \frac{\left(\frac{\tau\sigma H_x + \nu\rho\xi\sigma H_{xx}}{\nu^2\sigma H_{xx}}\right)^2}{2} = 0.$$ (20)

Then we use power transformation and variable change method to solve the partial differential equation (20). The results we obtained can be summarized by the following theorem.

**Assumption 1.** To avoid discussion of some subcases in the optimal reinsurance control strategy, $\psi^*(t)$, we assume that $0 < \frac{m(t)\zeta(t)}{\phi^2(t)} \cdot \frac{x(t)}{\gamma} < 1$.

**Theorem 3.1.** Under the assumption, $\gamma > \max \left\{ \frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho\sigma\nu + \xi^2\sigma^2}{\xi\rho\sigma\nu + \xi^2\sigma^2}, 0 \right\}$, $\gamma \neq 1$, and $0 < \frac{m(t)\zeta(t)}{\phi^2(t)} \cdot \frac{x(t)}{\gamma} < 1$, the optimal utility, the optimal reinsurance control strategy, and the optimal investment strategy for the insurer are as follows

$$H(t, x, r, \sigma) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( P(t) + Q(t) r + R(t) \sigma \right),$$ (21)

$$\psi^*(t) = \frac{m(t)\zeta(t)}{\phi^2(t)} \cdot \frac{x(t)}{\gamma};$$ (22)

$$\pi^*(t) = \left( \frac{\tau}{\nu^2} + \frac{\rho\xi}{\nu} R(t) \right) \frac{x(t)}{\gamma};$$ (23)

where

$$P(t) = \int_t^T \left( K\alpha Q(s) + \kappa \theta R(s) + \frac{m^2(s)\zeta^2(s)}{\phi^2(s)} \frac{1-\gamma}{2\gamma} \right) ds,$$

$$Q(t) = \frac{\lambda_1\lambda_2\exp \left( -\sqrt{\Delta Q(T-t)} \right) - \lambda_1\lambda_2}{\lambda_1 \exp \left( -\sqrt{\Delta Q(T-t)} \right) - \lambda_2},$$

$$R(t) = \frac{n_1n_2\exp \left( -\sqrt{\Delta R(T-t)} \right) - n_1n_2}{n_1 \exp \left( -\sqrt{\Delta R(T-t)} \right) - n_2}.$$ (24)
\[
\begin{align*}
\Delta_Q &= K^2 - 2\beta^2(1 - \gamma), \\
\lambda_{1,2} &= \frac{K + \sqrt{\Delta_Q}}{\beta^2}, \\
\Delta_R &= \kappa^2 - \frac{2\xi\rho\kappa\tau(1 - \gamma)}{\nu^2\gamma} - \frac{\xi^2\tau^2(1 - \gamma)}{\nu^2\gamma}, \\
n_{1,2} &= \frac{\gamma\kappa - (1 - \gamma)\xi\rho\kappa\tau\gamma\sqrt{\Delta_R}}{\gamma\xi^2\tau(1 - \gamma)\xi^2\tau^2}.
\end{align*}
\] (25)

Proof. See Appendix A. \qed

Then we continue with discussing about the properties of \(P(t), Q(t)\) and \(R(t)\).

**Proposition 1.** Under the assumption, \(1 > \gamma > \max\left\{\frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho\kappa\tau(1 - \gamma)}{\nu^2\gamma} + \frac{\xi^2\tau^2}{\nu^2\gamma}, 0\right\}\), \(Q(t)\) is a decreasing function with respect to time \(t\) and satisfies
\[
0 \leq Q(t) \leq \frac{\lambda_1\lambda_2\exp\left(-\sqrt{\Delta_Q}T\right) - \lambda_1\lambda_2}{\lambda_1\exp\left(-\sqrt{\Delta_Q}T\right) - \lambda_2},
\] (26)

under the assumption, \(\gamma > 1\), \(Q(t)\) is an increasing function with respect to time \(t\) and satisfies
\[
0 \geq Q(t) \geq \frac{\lambda_1\lambda_2\exp\left(-\sqrt{\Delta_Q}T\right) - \lambda_1\lambda_2}{\lambda_1\exp\left(-\sqrt{\Delta_Q}T\right) - \lambda_2},
\] (27)

where \(\lambda_1\) and \(\lambda_2\) are given in Theorem 3.1.

Proof. Through simple calculation, we have
\[
\frac{dQ(t)}{dt} = \frac{\lambda_1\lambda_2(\lambda_1 - \lambda_2)\sqrt{\Delta_Q}\exp\left(-\sqrt{\Delta_Q}(T-t)\right)}{\left(\lambda_1\exp\left(-\sqrt{\Delta_Q}(T-t)\right) - \lambda_2\right)^2}.
\] (28)

and
\[
\lambda_1 = \frac{K - \sqrt{\Delta_Q}}{\beta^2} = \begin{cases}
> 0, & 1 > \gamma > \max\left\{\frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho\kappa\tau(1 - \gamma)}{\nu^2\gamma} + \frac{\xi^2\tau^2}{\nu^2\gamma}, 0\right\}, \\
< 0, & \gamma > 1,
\end{cases}
\]

\[
\lambda_2 = \frac{K + \sqrt{\Delta_Q}}{\beta^2} > 0,
\]

\[
\lambda_1 < \lambda_2.
\]

Hence,
\[
\frac{dQ(t)}{dt} = \begin{cases}
> 0, & \gamma > 1, \\
< 0, & 1 > \gamma > \max\left\{\frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho\kappa\tau(1 - \gamma)}{\nu^2\gamma} + \frac{\xi^2\tau^2}{\nu^2\gamma}, 0\right\}.
\end{cases}
\]

Substituting \(t\) by 0 and \(T\) in the expression of \(Q(t)\), we get the (26) and (27). \qed

**Proposition 2.** Under the assumption, \(1 > \gamma > \max\left\{\frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho\kappa\tau(1 - \gamma)}{\nu^2\gamma} + \frac{\xi^2\tau^2}{\nu^2\gamma}, 0\right\}\), \(R(t)\) is a decreasing function with respect to time \(t\) and satisfies
\[
0 \leq R(t) \leq \frac{n_1n_2\exp\left(-\sqrt{\Delta_R}T\right) - n_1n_2}{n_1\exp\left(-\sqrt{\Delta_R}T\right) - n_2};
\] (29)
under the assumption, $\gamma > 1$, $R(t)$ is an increasing function with respect to time $t$ and satisfies
\[ 0 \geq R(t) \geq \frac{n_1 n_2 \exp\left(-\sqrt{\Delta R} t\right) - n_1 n_2}{n_1 \exp\left(-\sqrt{\Delta R} t\right) - n_2}, \tag{30} \]
where $n_1$ and $n_2$ are given in Theorem 3.1.

**Proof.** Through simple calculation, we have
\[ \frac{dR(t)}{dt} = \frac{n_1 n_2(n_1 - n_2)\sqrt{\Delta R} \exp\left(-\sqrt{\Delta R}(T - t)\right)}{(n_1 \exp\left(-\sqrt{\Delta R}(T - t)\right) - n_2)^2}, \tag{31} \]
and
\[
\begin{align*}
n_1 &= \frac{\gamma \kappa - (1 - \gamma)\xi \rho \xi}{\gamma \xi^2 + (1 - \gamma)\xi^2 \rho^2}, \\
n_2 &= \frac{\gamma \kappa - (1 - \gamma)\xi \rho \xi + \gamma \sqrt{\Delta R}}{\gamma \xi^2 + (1 - \gamma)\xi^2 \rho^2}, \\
n_1 < n_2, \\
n_1 n_2 &= \frac{\xi^2 \tau^2 (1 - \gamma)\left(\rho + (1 - \rho)\gamma\right)}{\nu^2 \left(\gamma \xi^2 + (1 - \gamma)\xi^2 \rho^2\right)^2}.
\end{align*}
\]
Since $0 \leq \rho \leq 1$ and $\gamma > 0$, we have
\[ n_1 n_2 = \begin{cases} 
< 0, & \gamma > 1, \\
> 0, & 1 > \gamma > \max\left\{\frac{2\gamma^2 - K^2}{2\beta^2}, \frac{2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}{\kappa^2 + 2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}, 0\right\}.
\end{cases} \]
Hence,
\[ \frac{dR(t)}{dt} = \begin{cases} 
> 0, & \gamma > 1, \\
< 0, & 1 > \gamma > \max\left\{\frac{2\gamma^2 - K^2}{2\beta^2}, \frac{2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}{\kappa^2 + 2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}, 0\right\}.
\end{cases} \]
Substituting $t$ by $0$ and $T$ in the expression of $R(t)$, we get the (29) and (30). \qed

According to the properties of $Q(t)$ and $R(t)$, we can easily derive the properties of $P(t)$, which is summarized in the following Proposition.

**Proposition 3.** Under the assumption, $1 > \gamma > \max\left\{\frac{2\gamma^2 - K^2}{2\beta^2}, \frac{2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}{\kappa^2 + 2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}, 0\right\}$, $P(t)$ is a decreasing function with respect to time $t$ and satisfies
\[ 0 \leq P(t) \leq \int_0^T \left(\kappa \alpha Q(s) + \kappa \theta R(s) + \frac{m^2(s)\zeta^2(s)}{\phi^2(s)} \frac{1 - \gamma}{2\gamma}\right) ds; \tag{32} \]
under the assumption, $\gamma > 1$, $P(t)$ is an increasing function with respect to time $t$ and satisfies
\[ 0 \geq P(t) \geq \int_0^T \left(\kappa \alpha Q(s) + \kappa \theta R(s) + \frac{m^2(s)\zeta^2(s)}{\phi^2(s)} \frac{1 - \gamma}{2\gamma}\right) ds, \tag{33} \]
Proof. Calculate the first-order derivative with respect to $t$, we get
\[
\frac{dP(t)}{dt} = - \left( K\alpha Q(t) + \kappa \theta R(t) + \frac{m^2(t)\zeta^2(t)}{\phi^2(t)} \frac{1 - \gamma}{2\gamma} \right).
\] (34)

Since
\[
\frac{m^2(t)\zeta^2(t)}{2\gamma \phi^2(t)} > 0,
\]
with the results in Proposition 1 and Proposition 2, we have
\[
\frac{dP(t)}{dt} = \left\{ \begin{array}{ll}
> 0, & \gamma > 1, \\
< 0, & 1 > \gamma > \max \left\{ \frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho \kappa \tau \nu + \xi^2 \tau^2}{\kappa \tau^2 + 2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}, 0 \right\}.
\end{array} \right.
\]

Substituting $t$ by 0 and $T$ in the expression of $P(t)$, we get the (32) and (33).

Remark 3. The optimal reinsurance control strategy, $\psi^*(t)$, has no direct relationship with the parameters of the risk-free asset system and the risky asset system, but an indirect relationship through $x(t)$. We also conclude that the stochastic interest rate has indirect influence on the optimal reinsurance control strategy and the optimal investment strategy for the insurer through $x(t)$.

Remark 4. For the CRRA utility function, the risk tolerance is defined as
\[
- \frac{U_x}{U_{xx}} = \frac{x}{\gamma},
\] (35)
which increases with wealth. Also, from (22) and (23), both the optimal reinsurance control strategy $\psi^*(t)$ and the optimal investment strategy $\pi^*(t)$ are proportional to the current wealth $x(t)$ and inverse proportional to the risk aversion $\gamma$, which means that the claim retained by the insurer, and the investment amount in the risky asset increase as the risk tolerance increase, which is consistent with our common intuition.

Remark 5. The financial market in this paper is not complete due to the correlation between the risky asset’s price $S(t)$ and the stochastic volatility $\sigma(t)$. The optimal investment strategy consists of two parts, the first part, $\frac{\rho \kappa \tau \nu x(t)}{\beta \gamma} R(t)$, is the optimal investment strategy in the complete market; the second part, $\frac{\rho \kappa \tau \nu x(t)}{\beta \gamma} R(t)$, is introduced from the stochastic volatility.

According to Proposition 2, we know that in the case $\rho > 0$, which means the uncertainties of the risky asset’s price process and the stochastic volatility process change in the same sense, the insurer should initially invest a greater proportion of wealth in the risky asset and then reduce the proportion gradually if $1 > \gamma > \max \left\{ \frac{2\beta^2 - K^2}{2\beta^2}, \frac{2\xi\rho \kappa \tau \nu + \xi^2 \tau^2}{\kappa \tau^2 + 2\xi \rho \kappa \tau \nu + \xi^2 \tau^2}, 0 \right\}$; and the insurer should initially invest a lower proportion of wealth in the risky asset and then increase the proportion as time passes if $\gamma > 1$. In the case $\rho < 0$, the situation is just the opposite to the above analysis and explanation.

We also observe that in the case $W^\sigma(t)$ is independent of $W^S(t)$, $\rho = 0$ and $\pi^*(t)$ reduces to the strategy in a complete market. In addition, $\pi^*(t)$ does not depend on $Q(t)$ because we assume that $W^T(t)$ is independent of $W^S(t)$. If we relax these assumptions in Remark 1, the problem becomes more complex and we need to explore another new technique to find the solutions.
Remark 6. The utility function in (14) will degenerate to a logarithmic utility function \( U(x) = \ln(x) \) as the risk aversion \( \gamma \) approaches to 1. We give the corresponding optimal reinsurance control strategy and the optimal investment strategy for a logarithmic utility function in the next section as a special case to our original model.

Remark 7. In this paper, we assume that the safety loading of the insurer \( \eta(t) \) is the same as the safety loading of the reinsurer \( \zeta(t) \), which is an essential condition for the closed-form solutions of the optimal strategy. If we relax the assumption, it is necessary to explore another new approach to tackle the problem.

3.2. Special cases. Special case I: Constant interest rate and stochastic volatility. If \( K = \beta = 0 \), the interest rate \( r(t) \) becomes a constant. The Problem \( P \) degenerates to an optimization problem with constant interest rate and stochastic volatility with the following results under the assumption \( \gamma > \frac{2\xi \rho \tau \nu + \xi^2 \tau^2}{\kappa^2 \nu^2 + 2\xi \rho \nu \tau + \xi^2 \tau^2}, \quad \gamma \neq 1 \).

\[
H(t, x, r, \sigma) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( P(t) + Q(t) r + R(t) \sigma \right),
\]

\[
\psi^*(t) = \frac{m(t) \zeta(t)}{\phi^2(t)} \cdot \frac{x(t)}{\gamma},
\]

\[
\pi^*(t) = \left( \frac{\tau}{\nu^2} + \frac{\rho \xi}{\nu} R(t) \right) \frac{x(t)}{\gamma},
\]

where

\[
P(t) = \int_t^T \left( \kappa \theta R(s) + \frac{m^2(s) \zeta^2(s) 1-\gamma}{2\gamma} \right) ds,
\]

\[
Q(t) = (1-\gamma)(T-t),
\]

\[
R(t) = \frac{n_{12} \exp \left( -\sqrt{\Delta R(T-t)} \right) - n_{12}}{n_{12} \exp \left( -\sqrt{\Delta R(T-t)} \right) - n_{12}},
\]

\[
\Delta R = \kappa^2 - \frac{2\xi \rho \tau \nu (1-\gamma)}{\kappa^2 \nu^2 + 2\xi \rho \nu \tau + \xi^2 \tau^2},
\]

\[
n_{12} = -\frac{\gamma \kappa - (1-\gamma) \rho \xi \Delta R + \gamma \sqrt{\Delta R}}{\gamma \kappa + (1-\gamma) \rho \xi \Delta R + \gamma \sqrt{\Delta R}}.
\]

Special case II: Stochastic interest rate and constant volatility. If \( \kappa = \xi = 0 \), the volatility \( \sigma(t) \) becomes a constant. The Problem \( P \) degenerates to an optimization problem with stochastic interest rate and constant volatility with the following results under the assumption \( \gamma > \frac{2\beta^2 - \kappa^2}{2\beta^2}, \quad \gamma \neq 1 \).

\[
H(t, x, r, \sigma) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( P(t) + Q(t) r + R(t) \sigma \right),
\]

\[
\psi^*(t) = \frac{m(t) \zeta(t)}{\phi^2(t)} \cdot \frac{x(t)}{\gamma},
\]

\[
\pi^*(t) = \frac{\tau}{\nu^2} \frac{x(t)}{\gamma},
\]
where
\[ P(t) = \int_t^T \left( K\alpha Q(s) + \frac{m^2(s)\zeta^2(s)}{\phi^2(s)} \frac{1-\gamma}{2\gamma} \right) ds, \]
\[ Q(t) = \frac{\lambda_1 \lambda_2 \exp \left( -\sqrt{\Delta_Q(T-t)} \right) \lambda_1 \lambda_2}{\lambda_1 \exp \left( -\sqrt{\Delta_Q(T-t)} \right) \lambda_2}, \]
\[ R(t) = \frac{\tau (1-\gamma)}{2\nu^2 \gamma} (T-t), \]
\[ \Delta_Q = K^2 - 2\beta^2 (1-\gamma), \]
\[ \lambda_{1,2} = \frac{K \mp \sqrt{\Delta_Q} \beta^2}{\beta^2}. \]

**Special case III: Constant interest rate and constant volatility.** If \( \kappa = \xi = 0 \) and \( K = \beta = 0 \), the interest rate and volatility term both reduce to constants. The Problem \( P \) degenerates to an optimization problem with constant interest rate and constant volatility with the following results under the condition \( \gamma > 0 \) and \( \gamma \neq 1 \).

\[ H(t,x,r,\sigma) = \frac{x^{1-\gamma}}{1-\gamma} \exp \left( P(t) + Q(t)r + R(t)\sigma \right), \quad (42) \]
\[ \psi^*(t) = \frac{m(t)\zeta(t)}{\phi^2(t)} x(t), \quad (43) \]
\[ \pi^*(t) = \frac{\tau}{\nu^2 \gamma} x(t), \quad (44) \]

where
\[ P(t) = \int_t^T \left( \frac{m^2(s)\zeta^2(s)}{\phi^2(s)} \frac{1-\gamma}{2\gamma} \right) ds, \]
\[ Q(t) = (1-\gamma)(T-t), \]
\[ R(t) = \frac{\tau (1-\gamma)}{2\nu^2 \gamma} (T-t). \]

**Special case IV: Logarithmic utility.** If \( \gamma = 1 \), the power utility function degenerates into a logarithmic utility. After some calculation, we get
\[ P(t) = 0, \quad (45) \]
\[ Q(t) = 0, \quad (46) \]
\[ R(t) = 0, \quad (47) \]
\[ \pi^*(t) = \frac{\tau}{\nu^2} x(t), \quad (48) \]
\[ \psi^*(t) = \frac{m(t)\zeta(t)}{\phi^2(t)} x(t), \quad (49) \]
\[ H(t,x,r,\sigma) = \ln(x(t)). \quad (50) \]

**Remark 8.** The parameters in the stochastic interest rate model and the stochastic volatility model have no explicit influence on \( \pi^*(t) \) or \( \psi^*(t) \), but through the current wealth \( x(t) \). \( \pi^*(t) \) and \( \psi^*(t) \) are both proportional to the current wealth \( x(t) \). To gain the influence of the parameters on the optimal strategies, one would need to carry out a numerical investigation which will be done in Section 5. Besides, \( \pi^*(t) \) is also proportional to \( \tau \) and inverse proportional to \( \nu^2 \). \( \psi^*(t) \) also depends on the parameter of the claim process and the safety loadings of the insurer/reinsure (we assume that \( \eta(t) = \zeta(t) \)).
4. Solution scheme for problem $Q$. The insurer plays the leading role in business and chooses the optimal reinsurance control strategy, $\psi^*(t)$, based on the CRRA utility function. The reinsurer can also invest its wealth (surplus) in financial market to pursue maximum profit. Inspired by Li et al. (2015), in this section, we investigate the optimal investment strategy for the reinsurer with the chosen $\psi^*(t)$ under the CARA utility function of terminal wealth.

4.1. Derivation of the main results. Following similar procedures, we define the value function as follows

$$V(t, y, \sigma) = \sup_{u(t)} E[U^{re}(y(T)) | y(t) = y, \sigma(t) = \sigma], \quad 0 \leq t \leq T, \quad (51)$$

with boundary condition $V(T, y, \sigma) = U^{re}(y(T))$. The Hamilton-Jacobi-Bellman equation associated to the Optimization Problem $Q$ is

$$V_t + \sup_{u(t)} \left\{ (a\sigma u(t) + y\mu + m(t)(1 - \psi^*(t))\zeta(t))V_y + \kappa(\theta - \sigma)V_\sigma 
+ \frac{1}{2} \left(u^2(t)b^2\sigma + (1 - \psi^*(t))^2\phi^2(t)\right)V_{yy} + \frac{1}{2}\xi^2\sigma V_{\sigma\sigma} + u(t)b\rho\xi\sigma V_{y\sigma}\right\} = 0, \quad (52)$$

where $V_t$, $V_y$, $V_\sigma$, $V_{yy}$, $V_{\sigma\sigma}$, and $V_{y\sigma}$ denote partial derivatives of first-order and second-order with respect to time $t$, wealth $y(t)$, and volatility $\sigma(t)$. The first-order necessary and sufficient condition for the optimal investment strategy of the reinsurer is

$$u^*(t) = -\frac{a\sigma V_y + b\rho\xi\sigma V_{y\sigma}}{b^2\sigma V_{yy}}. \quad (53)$$

Remark 9. By comparing (19) with (53), we find that (i) the optimal investment strategies for the insurer and the reinsurer have similar format, which is because the assets available in the financial market for the insurer and the reinsurer are similar, including one risk-free asset and one risky asset which is described by the Heston model; and (ii) both of the two optimal investment strategies have no relationship with $\psi^*(t)$, which is because we assume that the claim process and the risky assets’ price processes are independent, i.e., no correlation.

Substituting (53) into (52), we get the partial differential equation for $V(t, y, \sigma)$ as follows

$$V_t + \left(y\mu + m(t)(1 - \psi^*(t))\zeta(t)\right)V_y + \kappa(\theta - \sigma)V_\sigma + \frac{1}{2}\xi^2\sigma V_{\sigma\sigma} 
+ \frac{1}{2}(1 - \psi^*(t))^2\phi^2(t)V_{yy} - \frac{(a\sigma V_y + b\rho\xi\sigma V_{y\sigma})^2}{2b^2\sigma V_{yy}} = 0. \quad (54)$$

We then use the variable change method to solve the partial differential equation (54). The results we obtained can be summarized by the following theorem.

**Theorem 4.1.** Based on the optimal reinsurance control strategy decided by the insurer, the optimal utility and the optimal investment strategy for the reinsurer in the sense of maximizing the expected CARA utility function of terminal wealth are
as follows
\[
V(t, y, \sigma) = -\frac{1}{q} \exp \left\{ -q [ye^\mu(T-t) + f(t) + \hat{P}(t)\sigma + \hat{Q}(t)] \right\},
\]

(55)

and satisfies
\[
u^*(t) = \frac{a - b\rho\xi q \hat{P}(t)}{b^2 q e^\mu(T-t)},
\]

(56)

where
\[
f(t) = \left( \frac{m(t)(1-\psi^*(t))\zeta(t)}{\mu} \right) \left( e^\mu(T-t) - 1 \right) - \frac{q(1-\psi^*(t))^2\theta^2(t)}{4\mu} \left( e^{2\mu(T-t)} - 1 \right),
\]

\[
\hat{P}(t) = \left\{
\begin{array}{ll}
\hat{b}_1 \varphi_1 \varphi_2 \exp \left\{ -\sqrt{\Delta_{\hat{P}}(T-t)} - \varphi_1 \varphi_2 \right\}, & \rho^2 q^2 \neq 1, \\
\hat{b}_1 \varphi_1 \exp \left\{ -\sqrt{\Delta_{\hat{P}}(T-t)} - \varphi_2 \right\}, & \rho^2 q^2 = 1,
\end{array}
\right.
\]

\[
\hat{Q}(t) = \int_t^T \hat{P}(s)\kappa\theta ds,
\]

(57)

and
\[
\Delta_{\hat{P}} = (\kappa + \frac{a\rho\xi q^2}{b})^2 + \xi q^2(1-\rho^2 q^2)^2\sigma^2,
\]

(58)

Proof. See Appendix B.

Then we discuss about the properties of \( \hat{P}(t) \).

**Proposition 4.** In the case \( \rho^2 q^2 = 1 \), \( \hat{P}(t) \) is a decreasing function with respect to time \( t \) and satisfies
\[
0 \leq \hat{P}(t) \leq \frac{qa^2}{2b^2\kappa + 2ab\rho\xi q^2} \left\{ 1 - \exp\left\{ -(\kappa + \frac{a\rho\xi q^2}{b})T \right\} \right\}.
\]

(59)

In the case \( \rho^2 q^2 > 1 \), \( \hat{P}(t) \) is an increasing function with respect to time \( t \) and satisfies
\[
0 \geq \hat{P}(t) \geq \frac{\varphi_1 \varphi_2 \exp\left\{ -\sqrt{\Delta_{\hat{P}}T} - \varphi_1 \varphi_2 \right\}}{\varphi_1 \exp\left\{ -\sqrt{\Delta_{\hat{P}}T} - \varphi_2 \right\}};
\]

(60)

in the case \( \rho^2 q^2 < 1 \), \( \hat{P}(t) \) is a decreasing function with respect to time \( t \) and satisfies
\[
0 \leq \hat{P}(t) \leq \frac{\varphi_1 \varphi_2 \exp\left\{ -\sqrt{\Delta_{\hat{P}}T} - \varphi_1 \varphi_2 \right\}}{\varphi_1 \exp\left\{ -\sqrt{\Delta_{\hat{P}}T} - \varphi_2 \right\}},
\]

(61)

where \( \varphi_1 \) and \( \varphi_2 \) are given in Theorem 4.1.

Proof. In the case \( \rho^2 q^2 = 1 \),
\[
\frac{d\hat{P}(t)}{dt} = -(\kappa + \frac{a\rho\xi q^2}{b}) \frac{qa^2}{2b^2\kappa + 2ab\rho\xi q^2} \exp\left\{ -(\kappa + \frac{a\rho\xi q^2}{b})T - t \right\} < 0.
\]

(62)

Thus \( \hat{P}(t) \) decreases with respect to \( t \). Substituting \( t \) by 0 and \( T \) in the expression of \( \hat{P}(t) \), we get (59).
In the case $\rho^2 q^2 \neq 1$,
\[
\frac{d\hat{P}(t)}{dt} = \frac{\varrho_1 \varrho_2 (\varrho_1 - \varrho_2) \sqrt{\Delta \hat{P}} \exp \left( -\sqrt{\Delta \hat{P}}(T-t) \right)}{\left( \varrho_1 \exp \left( -\sqrt{\Delta \hat{P}}(T-t) \right) - \varrho_2 \right)^2},
\]
(63)
and
\[
\begin{align*}
\varrho_1 &= \frac{-(\kappa b + a\rho \xi q^2) + b\sqrt{\Delta \hat{P}}}{\xi^2 q(1 - \rho^2 q^2)b}, \\
\varrho_2 &= \frac{-(\kappa b + a\rho \xi q^2) - b\sqrt{\Delta \hat{P}}}{\xi^2 q(1 - \rho^2 q^2)b}, \\
\varrho_1 \varrho_2 &= \frac{\xi^2 q^2 a^2 (1 - \rho^2 q^2)}{\left( \xi^2 q(1 - \rho^2 q^2)b \right)^2}, \\
\varrho_2 &= \frac{-\xi^2 q^2 a^2 (1 - \rho^2 q^2)}{\left( \xi^2 q(1 - \rho^2 q^2)b \right)^2}.
\end{align*}
\]
So we have
\[
\frac{d\hat{P}(t)}{dt} = \begin{cases} 
> 0, & \rho^2 q^2 > 1, \\
< 0, & \rho^2 q^2 < 1.
\end{cases}
\]
Substituting $t$ by 0 and $T$ in the expression of $\hat{P}(t)$, we get (61) and (60).

**Remark 10.** For the CARA utility function, the risk tolerance is defined as
\[
-\frac{U_y}{U_{yy}} = \frac{1}{q},
\]
(64)
which is a constant and independent of the current wealth held by the reinsurer $y(t)$. So the optimal investment strategy $u^*(t)$ is independent of $y(t)$, which is quite different from the CRRA utility function.

**Remark 11.** The optimal investment strategy for the reinsurer expressed by (56) can be decomposed into two parts. The first part, $\frac{a}{b q} e^{-\mu(T-t)}$, is closely related to the risky asset’s price process and the risk-free asset’s price process. The second part, $-\frac{b}{q} \hat{P}(t) e^{-\mu(T-t)}$, is the so-called ‘modification/correction factor’, which represents a supplementary term introduced from the Heston’s stochastic volatility model and reflects how the reinsurer hedges the volatility risk. Since $e^{-\mu(T-t)}$ is a monotone positive increasing function with respect to time $t$. The modification factor mainly depends on the properties of $\rho$ and $\hat{P}(t)$.

According to Proposition 4, we know that in the case $\rho > 0$, that is the uncertainties of the risky asset’s price process and the stochastic volatility process change in the same sense, the reinsurer should initially invest a larger proportion of wealth in the risky asset and then reduce the proportion gradually if $\rho^2 q^2 > 1$; otherwise, the reinsurer should initially invest a smaller proportion of wealth in the risky asset and then increase the proportion steadily. In the case $\rho < 0$, the situation is just the opposite to the above analysis and explanation.

We also observe that in the case where $W^S(t)$ is independent of $W^S(t)$, $\rho = 0$ and the second part of $u^*(t)$ reduces to zero consequently.
4.2. Special case. In the Case with Constant Interest Rate and Constant Volatility, \( \kappa = \xi = 0 \), and the volatility \( \sigma(t) \) reduces to a constant. The Problem \( Q \) degenerates to a optimization problem with constant interest rate and constant volatility with the following results

\[
V(t,y,\sigma) = -\frac{1}{q} \exp \left\{ -q(ye^{\mu(T-t)} + f(t) + \hat{P}(t)\sigma) \right\},
\]

(65)

\[
u^*(t) = \frac{a}{b^2} e^{-\mu(T-t)},
\]

(66)

where

\[
f(t) = \frac{m(t)(1 - \psi^*(t))\zeta(t)}{\mu} \left( e^{\mu(T-t)} - 1 \right) - \frac{q(1 - \psi^*(t))^2 \varphi^2(t)}{4\mu} \left( e^{2\mu(T-t)} - 1 \right),
\]

(67)

\[
\hat{P}(t) = \frac{qa}{2b^2} (T - t).
\]

(68)

Remark 12. In this case, the optimal investment strategy for the reinsurer is a deterministic function and it is independent of the current wealth \( x(t) \) due to the constant risk tolerance defined according to the exponential (CARA) utility function. Moreover, it is a monotone increasing function with respect to time \( t \), which means that the reinsurer can forecast the investment amount in the risky asset from the beginning and increase it from the initial amount \( \frac{a}{b^2} e^{-\mu T} \) to \( \frac{a}{b^2} \) steadily.

5. Numerical analysis. In this section, we provide some numerical examples to illustrate the evolution of the risky assets for the insurer and for the reinsurer, the dynamic behaviour of the wealth processes held by the insurer and the reinsurer, and the dynamic behavior of the optimal strategies in the first subsection. Then in the subsequent subsection, we examine the impact of the key parameters on the optimal reinsurance control strategy, and the optimal investment strategy respectively for the insurer and the reinsurer.

5.1. Simulation of the dynamic behaviour. Throughout the numerical analysis, unless otherwise stated, the basic parameters are given in Table 1, including the CIR stochastic interest rate system, the risky asset \( S(t) \) system, the risk-free asset available for the reinsurer, the risky asset \( S^{re}(t) \) system, and the insurance system with reinsurance, as defined in Section 2.

As shown in Figure 1, we observe the market fluctuations described by the CIR stochastic interest rate model (Equation (2)) and the Heston stochastic volatility model (Equation (3)). Accordingly, the prices of the risky assets held by the insurer (Equation (3)) and the reinsurer (Equation (5)) are also fluctuate with respect to time \( t \) as shown in Figure 2. The difference is that \( S(t) \) is with stochastic interest rate and stochastic volatility, whiles \( S^{re}(t) \) is only under the stochastic volatility framework. Hence, \( S(t) \) is slightly more unstable than \( S^{re}(t) \). It can also be noted that, the evolutions of \( S(t) \) and \( S^{re}(t) \) present some similarities and this is because we use the same Heston stochastic volatility model. The wealth processes for the insurer (Equation (7)) and the reinsurer (Equation (10)) illustrate similar properties as shown in Figure 3.

Based on the results obtained in Theorem 3.1 and Theorem 4.1, Figure 4 depicts the dynamic behaviour of the optimal reinsurance control strategy \( \psi^*(t) \) (Equation
Table 1. Parameter values for the original model

| Symbol | Value | Symbol | Value | Symbol | Value |
|--------|-------|--------|-------|--------|-------|
| $T$    | 5     | $\nu$  | 1     | $r_0$  | 0.05  |
| $\phi(t)$ | 1.2   | $\theta$ | 0.06  | $b_0$  | 1     |
| $m(t)$ | 0.6   | $\kappa$ | 2     | $b_0^e$ | 1     |
| $\zeta(t)$ | 0.8   | $\xi$   | 0.1   | $\sigma_0$ | 0.04 |
| $\alpha$ | 0.1   | $a$     | 1     | $s_0$  | 1     |
| $\beta$ | 0.1   | $b$     | 1     | $s_0^e$ | 1     |
| $K$    | 0.15  | $\mu$  | 0.1   | $l_0$  | 2     |
| $\tau$ | 1.5   | $\rho$ | 0.5   | $x_0$  | 5     |
| $\gamma$ | 4     | $q$     | 0.5   | $y_0$  | 5     |

(22)), the optimal investment strategy $\pi^*(t)$ (Equation (23)) for the insurer and the optimal investment strategy $u^*(t)$ (Equation (56)) for the reinsurer over time. As demonstrated in Propositions 2 and 4, $x(t)$ is an increasing function with respect to $t$, $R(t)$ is also an increasing time-dependent function under the case $\gamma > 1$, and $\hat{P}$ is a decreasing function of $t$ under the case $\rho^2 q^2 < 1$, we hence conclude that $\psi^*(t)$, $\pi^*(t)$ and $u^*(t)$ are all increasing functions with respect to $t$. This is consistent with the illustration in Figure 4. Compared with $\psi^*(t)$ and $\pi^*(t)$, $u^*(t)$ varies with time more smoothly due to the independence of the wealth process involved as demonstrated in Remarks 4 and 10.

Figure 1. Evolutions of the CIR stochastic interest rate $r(t)$ and Heston stochastic volatility $\sigma(t)$ within the investment horizon $[0, T]$. 
5.2. Sensitivity analysis. Sensitivity analysis I: The impact on the optimal insurance/reinsurance control strategy. Figure 5 and 6 show the effects of $\zeta(t)$ and $\gamma$ on $\psi^*(t)$. We observe that a greater $\zeta(t)$ leads to a higher $\psi^*(t)$, and this is because we assume $\eta(t) = \zeta(t)$, the insurer tends to purchase less reinsurance as the cost of reinsurance increases. We also observe that $\psi^*(t)$ decreases with respect to $\gamma$, and this is because the more risk averse the insurer is, the more reinsurance it tends to purchase so as to improve financial stability.
The optimal reinsurance control strategy

The optimal investment strategy for the insurer

The optimal investment strategy for the reinsurer

Figure 4. The dynamic behaviour of (a) the optimal reinsurance control strategy $\psi^*(t)$, (b) the optimal investment strategy for the insurer $\pi^*(t)$ and (c) the optimal investment strategy for the reinsurer $u^*(t)$.

The effect of safety loading on the optimal reinsurance control strategy

Figure 5. Sensitivities of $\psi^*(t)$ with respect to $\zeta(t)$.

Sensitivity analysis II: The impact on the optimal investment strategy for the insurer. Figure 7 and 8 illustrate the impact of $\gamma$ and $\nu$ on $\pi^*(t)$. We see that $\pi^*(t)$ decreases as $\gamma$ increases, and this is because the insurer will reduce the investment in the risky asset as it becomes less risk tolerant. We also find that $\pi^*(t)$ decreases as the volatility of the risky asset increases, and this is because in the case $\rho > 0$, the insurer will invest less when the uncertainties of the risky
Sensitivity analysis III: The impact on the optimal investment strategy for the reinsurer. From Figure 9, we notice that the optimal investment strategy for the reinsurer $u^*(t)$ decreases with respect to risk-free interest rate $\mu$, which means no matter the correlation coefficient $\rho$ is positive or negative, the reinsurer will reduce investment in the risky asset as $\mu$ increases. Figure 10 shows that the asset’s price process and the stochastic volatility process move in the same sense, as demonstrated in Proposition 2.
The effect of \( \nu \) on the optimal investment strategy for the insurer

Figure 8. Sensitivities of \( \pi^*(t) \) with respect to the parameter \( \nu \).

reinsurer will invest less as the absolute risk aversion coefficient \( q \) increases, which is true under both cases (i) \( \rho > 0 \) and (ii) \( \rho < 0 \). This illustrates the intuitive observation that the reinsurer will apply more conservative investment strategy if the risk tolerance decreases as proposed in Remark 10. Figure 11 and 12 depict the influences of the risk aversion coefficient on the optimal investment strategy under two types of cases (i) \( \rho > 0 \) and (ii) \( \rho < 0 \). It should be addressed that investment amount in the risky asset decreases as \( \rho \) increases if \( \rho > 0 \); while it increases as \(|\rho|\) increases if \( \rho < 0 \), which is consistent with the demonstration in Remark 11.

Figures 13 - 18 plot the effects of key parameters of the price system of risky asset that is available for the reinsurer on \( u^*(t) \), including (i) the rate at which \( \sigma(t) \) revert to \( \theta \), i.e., \( \kappa \); (ii) the volatility of the volatility of the Heston model, i.e., \( \xi \); and (iii) the influence coefficients \( a \) and \( b \). We find that

- The reinsurer will invest more as \( \kappa \) increases if \( \rho > 0 \), while the reinsurer will invest less as \( \kappa \) increases if \( \rho < 0 \). Since a larger \( \kappa \) indicates a more stable volatility of \( S^{re}(t) \), and \( \rho > 0 \) means the uncertainties of \( S^{re}(t) \) and \( \sigma(t) \) move in the same sense, hence the reinsurer will apply more aggressive investment strategy. In the case \( \rho < 0 \), the situation is just the opposite to the above analysis and explanation.

- The reinsurer will invest less as \( \xi \) increases if \( \rho > 0 \), while the reinsurer will invest more as \( \xi \) increases if \( \rho < 0 \). As \( \rho > 0 \) shows the same meaning as above, and a greater \( \xi \) leads to more fluctuating volatility of \( S^{re}(t) \), the reinsurer will apply more conservative investment strategy. However, under the case \( \rho < 0 \), more severe fluctuating volatility give a higher \( S^{re}(t) \), so the reinsurer will increase investment in the risky asset.

- The reinsurer will invest more as \( a \) increases no matter whether \( \rho > 0 \) or \( \rho < 0 \). The reason is that the appreciation rate of \( S^{re}(t) \) rises with respect to \( a \).

- The reinsurer will invest less as \( b \) increases no matter whether \( \rho > 0 \) or \( \rho < 0 \). The reason is that the volatility of \( S^{re}(t) \) increases with respect to \( b \).
Figure 9. Sensitivities of the optimal investment strategy $u^*(t)$ for the reinsurer with respect to the interest rate $\mu$.

Figure 10. Sensitivities of the optimal investment strategy $u^*(t)$ for the reinsurer with respect to the risk aversion coefficient $q$.

Figure 11. Sensitivities of the optimal investment strategy $u^*(t)$ for the reinsurer with respect to the positive correlation coefficient.

Figure 12. Sensitivities of the optimal investment strategy $u^*(t)$ for the reinsurer with respect to the negative correlation coefficient.

6. Conclusions. In this paper, we consider (i) the optimal investment strategy for the insurer and the reinsurer and (ii) the problem of insurance and reinsurance control for the asset liability management problem for an ordinary insurance system with reinsurance under a framework of CIR stochastic interest rate and Heston stochastic volatility. Hence, it is worth to further extend our research into another framework, which is CIR stochastic interest rate and CEV stochastic volatility. The insurer can allocate the wealth in one risk-free asset and one risky asset with the objective to maximize the expected utility of the terminal wealth in the fixed investment horizon. Similarly, the reinsurer is also allowed to invest in one risk-free
Figure 13. Sensitivities of the optimal investment strategy \( u^*(t) \) for the reinsurer with respect to the mean reversion speed \( \kappa \) when \( q = 4 \) and \( \rho > 0 \).

Figure 14. Sensitivities of the optimal investment strategy \( u^*(t) \) for the reinsurer with respect to the mean reversion speed \( \kappa \) when \( q = 2 \) and \( \rho < 0 \).

Figure 15. Sensitivities of the optimal investment strategy \( u^*(t) \) for the reinsurer with respect to “volatility of volatility” \( \xi \) when \( q = 2 \) and \( \rho < 0 \).

Figure 16. Sensitivities of the optimal investment strategy \( u^*(t) \) for the reinsurer with respect to “volatility of volatility” \( \xi \) when \( q = 2 \) and \( \rho > 0 \).

asset and one risky asset which are described under the Heston stochastic volatility framework.

The explicit expressions of the optimal investment strategies and the optimal insurance control strategy are obtained by applying the stochastic optimal control, power transform and variable change technique. In addition, we also study some special cases for the optimization problem for the insurer, including (i) the case with
constant interest rate and stochastic volatility; (ii) the case with stochastic interest rate and constant volatility; (iii) the case with constant interest rate and constant volatility and (iv) the case with logarithmic utility. Besides, we discuss our results and explore the reasons in the context of reinsurance business.

At last, some numerical examples are provided to illustrate the theoretical results and the effect of reinsurance, risk aversion, stochastic interest rate and stochastic volatility on the optimal investment strategies.

However, our optimization problems have some limitations, for example, we ignore the correlation between the dynamics of the risky asset, the stochastic interest rate and the stochastic volatility. It would be interesting to extend our model in several ways in future research, including (i) considering different type of reinsurance, i.e., the excess-of-loss reinsurance and stop loss cover; (ii) using different criteria, for example, mean-capital-at-risk (mean-CaR), mean-Conditional-value-at-risk (mean-CVaR), or the mean-variance criterion that minimize the variance of terminal wealth for a given expectation of terminal wealth; (iii) taking use of Pontryagin’s maximum principle (MP) approach to tackle the problem; and (iv) taking into account the uncertain investment horizon, the minimum guarantee or other constraints; (v) describing both the risky asset and the claims by a generalized jump-diffusion geometric Brownian motion (an exponential Lévy process) and a jump-diffusion Brownian motion (a Lévy process) in order to make our model more realistic from the economic point of view with consideration of discontinuity and jumps; (vi) study the problem with constraints, including uncertain investment horizon, regime switching, minimum benefit guarantee, and various market risks, for example, the inflation risk.
Appendix A. The proof of Theorem 3.1. According to the utility function in (14), we conjecture a solution to (20) with the following form:

\[ H(t, x, r, \sigma) = \frac{x^{1-\gamma}}{1-\gamma} g(t, r, \sigma), \quad g(T, r, \sigma) = 1. \] (69)

For simplicity, let \( g = g(t, r, \sigma) \) and \( H = H(t, x, r, \sigma) \), which has positive first-order derivative with respect to \( x \) and negative second-order derivative with respect to \( x \). Differentiating \( H \) with \( f_t \) denoting \( \frac{df}{dt} \), we have

\[
\begin{align*}
H_t &= \frac{H}{g}g_t, \\
H_x &= \frac{1-\gamma}{x} H, \\
H_r &= \frac{H}{g}g_r, \\
H_\sigma &= \frac{H}{g}g_\sigma, \\
H_{xx} &= -\gamma(1-\gamma)x^{-2}H, \\
H_{rr} &= \frac{H}{g}g_{rr}, \\
H_{\sigma\sigma} &= \frac{H}{g}g_{\sigma\sigma}, \\
H_{x\sigma} &= \frac{H}{gx}(1-\gamma)g_\sigma.
\end{align*}
\] (70)

Substituting (70) into (20) and eliminating the dependence on \( x \), we get the partial differential equation for \( g \) as follows

\[
\begin{align*}
\frac{g_r}{g} + r(1-\gamma) + K(\alpha - r)\frac{g_r}{g} + \kappa(\theta - \sigma)\frac{g_\sigma}{g} + \frac{1}{2}\beta^2 r\frac{g_{rr}}{g} + \frac{1}{2}\xi^2 \sigma^2 g_{\sigma\sigma} \\
+ \frac{m^2(t)\zeta^2(t)}{\phi^2(t)} \frac{1-\gamma}{2}\frac{\left((1-\gamma)\tau\sigma + \nu \rho \xi \sigma (1-\gamma)\frac{g_\sigma}{g}\right)^2}{2\nu^2 \sigma(1-\gamma)} = 0.
\end{align*}
\] (71)

Inspired by Chang and Rong (2013) and Guan and Liang (2014), we guess that \( g(t, r, \sigma) \) has the following expression and we verify this latter,

\[ g(t, r, \sigma) = \exp\left(P(t) + Q(t)r + R(t)\sigma\right). \] (72)

For simplicity, let \( P = P(t), \ Q = Q(t), \) and \( R = R(t) \). Calculating the first-order and second-order partial derivatives of \( g \) with respect to \( t, r, \) and \( \sigma \), we have

\[
\begin{align*}
g_t &= g\left(P_t + Q_tr + R_t\sigma\right), \\
g_r &= gQ, \\
g_\sigma &= gR, \\
g_{rr} &= gQ^2, \\
g_{\sigma\sigma} &= gR^2.
\end{align*}
\] (73)
Substituting (72) and (73) into (71) and arranging the coefficient of $r$, $\sigma$, and $r^0\sigma^0$, we get the following ordinary differential equations for $P(t)$, $Q(t)$, and $R(t)$ with boundary conditions

$$P_t + K\alpha Q + \kappa\theta R + \frac{m^2(t)\zeta^2(t)}{\sigma^2(t)} \frac{1 - \gamma}{2\gamma} = 0, \quad P(T) = 0; \quad (74)$$

$$Q_t - KQ + \frac{1}{2}\beta^2 Q^2 + 1 - \gamma = 0, \quad Q(T) = 0; \quad (75)$$

$$R_t - \kappa R + \frac{1}{2}\xi^2 R^2 + \frac{(1 - \gamma)(\tau + \nu\rho\xi R)}{2\nu^2\gamma} = 0, \quad R(T) = 0. \quad (76)$$

Rewrite (75) as below

$$Q_t = -\frac{1}{2}\beta^2 Q^2 + KQ - (1 - \gamma). \quad (77)$$

Let $\Delta_Q$ denote the discriminant of the following quadratic equation

$$-\frac{1}{2}\beta^2 Q^2 + KQ - (1 - \gamma) = 0. \quad (78)$$

By simple calculation, we have

$$\Delta_Q = K^2 - 2\beta^2(1 - \gamma). \quad (79)$$

Under the condition $\Delta_Q > 0$, we calculate the two roots of (78) as follows

$$\lambda_{1,2} = \frac{K \pm \sqrt{\Delta_Q}}{\beta^2}, \quad \gamma > \frac{2\beta^2 - K^2}{2\beta^2}. \quad (80)$$

In this case, we integrate (74) on both sides with respect to $t$ and get

$$\frac{1}{\lambda_1 - \lambda_2} \int_t^T \left( \frac{1}{Q(s) - \lambda_1} - \frac{1}{Q(s) - \lambda_2} \right) dQ(s) = -\frac{1}{2}\beta^2(T - t). \quad (81)$$

Solving (81) with boundary condition $Q(T) = 0$, we obtain

$$Q(t) = \frac{\lambda_1\lambda_2 e^{\frac{\lambda_1\lambda_2}{\lambda_1 e^{\frac{-\sqrt{\Delta_Q}Q(T - t)}}}} - \lambda_1\lambda_2}{\lambda_1 e^{\frac{-\sqrt{\Delta_Q}Q(T - t)}} - \lambda_2}. \quad (82)$$

The ordinary differential equation (76) is similar to (75), so we apply the same method to solve for $R(t)$. Firstly, we derive the discriminant of the corresponding quadratic equation, $\Delta_R$; then, we derive the condition of $\gamma$ based on $\Delta_R > 0$ and the two roots, $n_1$ and $n_2$; finally, we integrate (76) with respect to $t$ and obtain the
result of $R(t)$ as follows

$$R(t) = \frac{n_1n_2e^{\left(-\sqrt{\Delta R(T - t)}\right)} - n_1n_2}{n_1e^{\left(-\sqrt{\Delta R(T - t)}\right)} - n_2}.$$

$$\Delta R = \kappa^2 - \frac{2\xi\rho\kappa T(1 - \gamma)}{\nu} - \frac{\xi^2\tau^2(1 - \gamma)}{\nu^2\gamma},$$

$$n_{1,2} = \frac{\gamma\kappa - (1 - \gamma)\xi\rho T + \gamma\sqrt{\Delta R}}{\gamma\xi^2 + (1 - \gamma)\xi^2\rho^2},$$

$$\gamma > \frac{2\xi\rho\kappa T\nu + \xi^2\tau^2}{\kappa^2\nu^2 + 2\xi\rho\kappa T\nu + \xi^2\tau^2}.$$ (86)

Rewriting (74) and taking integration on both sides, we get

$$P(t) = \int_t^T \left(K\alpha Q(s) + \kappa\theta R(s) + \frac{m^2(s)g^2(s) - 1 - \gamma}{\phi^2(s) 2\gamma} \right) ds.$$

(87)

With the results of $P(t)$, $Q(t)$, and $R(t)$, we obtain the solution for $H(t, x, r, \sigma)$. Substituting $H(t, x, r, \sigma)$ into (18) and (19), the optimal reinsurance control strategy and the optimal investment strategy can be obtained as shown in the theorem.

**Appendix B. The proof of Theorem 4.1.** According to the CARA utility function described in (15), we conjecture a solution which has the following expression

$$V(t, y, \sigma) = -\frac{1}{q} e^{\left(-q\mu_{e^t(T-t)} + f(t) + g(t, \sigma)\right)},$$

(88)

with $f(T) = 0$, $g(T, \sigma) = 0$. For simplicity, we let $V = V(t, y, \sigma)$ and $g = g(t, \sigma)$. $V$ has positive first-order derivative with respect to $y$ and negative second-order derivative with respect to $\sigma$. Then we calculate the first-order and second-order derivatives of $V$ with respect to $t$, $y$, and $\sigma$, and get

$$V_t = qV(\mu ye^t(T-t) - \frac{df}{dt} - g_t),$$

$$V_y = -qV e^t(T-t),$$

$$V_\sigma = -qV g_\sigma,$$

$$V_{yy} = q^2V e^{2t(T-t)},$$

$$V_{\sigma\sigma} = q^2V e^t(T-t) g_\sigma,$$

$$V_{y\sigma} = q^2V e^t(T-t) g_\sigma.$$

(89)

Substituting (89) into (54) yields

$$qV \left(\mu ye^t(T-t) - \frac{df}{dt} - g_t\right) - qV e^{t(T-t)} \left(y\mu + m(t)(1 - \psi^*(t))\xi(t)\right) - qV g_\sigma \kappa(\theta - \sigma)$$

$$+ \left(q^2V g_\sigma^2 - qV g_\sigma \sigma\right) \frac{1}{2}\xi^2 + q^2V e^{2t(T-t)} \frac{1}{2}(1 - \psi^*(t))^2\phi^2(t)$$

$$- \left(-qV e^{t(T-t)} \alpha_\sigma + b_p\xi\sigma q^2V e^{t(T-t)} g_\sigma\right)^2$$

$$2b^2\sigma q^2V e^{2t(T-t)} = 0.$$ (90)

After simplification, we decompose the above equation into two equations

$$-\frac{df}{dt} - e^{t(T-t)} \left(m(t)(1 - \psi^*(t))\xi(t)\right) + \frac{1}{2}(1 - \psi^*(t))^2\phi^2(t)q e^{2t(T-t)} = 0,$$

(91)
and

\[-g_t - g_\sigma \kappa (\theta - \sigma) - \frac{q \left( -a\sigma + b\rho \xi \sigma q\gamma \sigma \right)^2}{2b^2\sigma} + \frac{1}{2} \xi^2 \sigma \left( q\gamma^2 - g_{\sigma\sigma} \right) = 0. \tag{92}\]

Solve (91) under the boundary condition \( f(T) = 0 \), we get

\[ f(t) = \frac{m(t)(1 - \psi^*(t))\zeta(t)}{\mu} \left( e^{\kappa(T-t)} - 1 \right) - \frac{q(1 - \psi^*(t))^2 \phi^2(t)}{4\mu} \left( e^{2\mu(T-t)} - 1 \right). \tag{93}\]

To solve (92), we conjecture a solution as follows

\[ g(t, \sigma) = \hat{P}(t)\sigma + \hat{Q}(t), \tag{94}\]

with \( \hat{P}(T) = \hat{Q}(T) = 0 \). Substituting the above expression into (92), we get

\[ \sigma \left( \frac{d\hat{P}}{dt} - \hat{P}_\kappa + q \left( -a + b\rho \xi q\hat{P} \right)^2 \frac{1}{2b^2} - \frac{1}{2} \xi^2 q\hat{P}^2 \right) + \frac{d\hat{Q}}{dt} + \hat{P}_\kappa \theta = 0. \tag{95}\]

To solve the above differential equation, we decompose it into two equations by eliminating dependence on \( \sigma \)

\[ \frac{d\hat{P}}{dt} - \hat{P}_\kappa + q \left( -a + b\rho \xi q\hat{P} \right)^2 \frac{1}{2b^2} - \frac{1}{2} \xi^2 q\hat{P}^2 = 0, \tag{96}\]

and

\[ \frac{d\hat{Q}}{dt} + \hat{P}_\kappa \theta = 0. \tag{97}\]

Rewrite (96) as follows

\[ \frac{d\hat{P}}{dt} = \frac{1}{2} \xi^2 q(1 - \rho^2 q^2)\hat{P}^2 + (\kappa + \frac{a\rho \xi q^2}{b})\hat{P} - \frac{q\sigma^2}{2b^2}. \tag{98}\]

In the case that \( \rho^2 q^2 \neq 1 \), let \( \Delta_{\hat{P}} \) denote the discriminant of the quadratic equation

\[ \frac{1}{2} \xi^2 q(1 - \rho^2 q^2)\hat{P}^2 + (\kappa + \frac{a\rho \xi q^2}{b})\hat{P} - \frac{q\sigma^2}{2b^2} = 0. \tag{99}\]

Then we have

\[ \Delta_{\hat{P}} = (\kappa + \frac{a\rho \xi q^2}{b})^2 + \frac{\xi^2 q^2(1 - \rho^2 q^2)a^2}{b^2} > 0. \tag{100}\]

So the two roots of (99) are as follows

\[ \varrho_{1,2} = \frac{-\left( \kappa + \frac{a\rho \xi q^2}{b} \right) \pm \sqrt{\Delta_{\hat{P}}} \xi^2 q(1 - \rho^2 q^2)b}{\xi^2 q(1 - \rho^2 q^2)b}. \tag{101}\]

The differential equation (96) can be solved by

\[ \frac{1}{\varrho_1 - \varrho_2} \int_t^T \left( \frac{1}{\hat{P}(s) - \varrho_1} - \frac{1}{\hat{P}(s) - \varrho_2} \right) d\hat{P}(s) = \frac{1}{2} \xi^2 q(1 - \rho^2 q^2)(T - t). \tag{102}\]

With the boundary condition, we derive the solution for \( \hat{P}(t) \)

\[ \hat{P}(t) = \frac{\varrho_1 \varrho_2 \exp \left( -\sqrt{\Delta_{\hat{P}}}(T - t) \right) - \varrho_1 \varrho_2}{\varrho_1 \exp \left( -\sqrt{\Delta_{\hat{P}}}(T - t) \right) - \varrho_2}. \tag{103}\]
In the case that $\rho^2 q^2 = 1$, the differential equation (96) reduces to
\[
\frac{d\hat{P}}{dt} = \left(\kappa + \frac{a\rho \xi q^2}{b}\right)\hat{P} - \frac{qa^2}{2b^2}, \quad \hat{P}(T) = 0. \tag{104}
\]
Take integration of both sides and we get
\[
\hat{P}(t) = \frac{qa^2}{2b^2\kappa + 2ab\rho \xi q^2} \left(1 - \exp\left\{-\left(\kappa + \frac{a\rho \xi q^2}{b}\right)(T - t)\right\}\right). \tag{105}
\]
After we obtain the solution of (96), it is easy to derive the solution of (97). The proof is completed.

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ASSET LIABILITY MANAGEMENT WITH STOCHASTIC DIFFERENTIAL EQUATIONS

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