Research Article

Chaos Synchronization of a Finance Chaotic System with an Integral Sliding Mode Controller

Jingjing Wang and Chunzhi Yang

Department of Mathematics, Huainan Normal University, Huainan 232038, China

Correspondence should be addressed to Chunzhi Yang; yangcz2002@126.com

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1. Introduction

In the past few decades, chaotic synchronization, due to its unique characteristics, has been widely used in engineering, finance, chemistry, confidential communications, and so on [1–5]. In recent decades, due to the complexity and inherent randomness of economic factors, chaotic nonlinear financial system has attracted extensive attention [6–12]. For example, Chao and Xingyuan [6] studied the existence of Hopf bifurcation and topological horseshoe of a class of finance chaotic system. Yu et al. [7] proposed a feedback control method to stabilize the hyperchaotic finance system to its equilibrium point. Huang and Cao [8] employed active control strategy to realize synchronization and antisynchronization of a class of fractional chaotic financial systems. In [11], a linear feedback control scheme was proposed to realize chaos synchronization of two identical financial chaotic systems. In addition to the feedback control method and active control method, there are other control methods that can realize synchronization of financial chaotic systems, such as adaptive control [13–15], impulsive control [16, 17], sliding mode control [15, 18–24], and so on.

Among above methods, more attention is paid to the sliding mode control method because of its simple design and good robustness. Sliding mode control is generally divided into two steps: the first step is to design the sliding mode surface; the second step is to design the controller so that the system state can enter the sliding mode surface. For example, Chiu [21] proposed two integral sliding mode control methods to make the tracking error reach the sliding surface in finite time. Using same two integral sliding modes, Zhao et al. [24] proposed two effective control methods, which inhibited the influence of external disturbances and realized the effective state estimation of the continuous stirred tank reactor. These two types of integral sliding mode are written as (1) $s = e + \lambda \int_{0}^{t} \text{sign}(e) \, dr$; (2) $s = e + \lambda \int_{0}^{t} \left| e \right| \text{sign}(e) \, dr$, $\gamma \in (0, 1)$, $\lambda > 0$. Since these integral sliding mode need to have \text{sign}(\cdot) function, the controllers designed by them will lead to chattering phenomenon. Therefore, to modify the above integral sliding mode has become a research direction.

Inspired by the above works, this paper investigates the synchronization of two finance chaotic systems. The main contributions of this paper are summarized as follows.

(1) Compared with the traditional integral sliding mode, the integrated sliding mode proposed in this paper can estimate the ultimate synchronization error bound by its own dynamics.

(2) Compared with the traditional integral sliding mode control method, the proposed method can eliminate the chatter phenomenon in error states and controller.

The rest of this work is arranged as follows. Some assumptions and lemmas and the problem statement are presented in Section 2. Section 3 gives sliding mode control design and stability analysis. In Section 4, some comparison results are presented to show the validity of the proposed method. At last, the conclusion is included in Section 5.
2. Preliminaries

In general, the finance chaotic system is described as

\[ \begin{align*}
\dot{x}_1 &= x_3 + (x_2 - a_1)x_1, \\
\dot{x}_2 &= 1 - \beta_1 x_2 - x_1^2, \\
\dot{x}_3 &= -x_1 - \sigma_1 x_3,
\end{align*} \tag{1} \]

where \( x = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \) is the state vector, \( x_1 \) denotes the interest rate, \( x_2 \) denotes the investment demand, \( x_3 \) denotes the price exponent, parameter \( a_1 \) denotes the saving amount, parameter \( \beta_1 \) denotes the investment cost, and parameter \( \sigma_1 \) denotes the elasticity of the demands of commercials. We regard system (1) as the master system, and the slave system is described as

\[ \begin{align*}
\dot{y}_1 &= y_3 + (y_2 - a_2)y_1 + d_1(t) + \text{sat}(u_1), \\
\dot{y}_2 &= 1 - \beta_2 y_2 - y_1^2 + d_2(t) + \text{sat}(u_2), \\
\dot{y}_3 &= -y_1 - \sigma_2 y_3 + d_3(t) + \text{sat}(u_3),
\end{align*} \tag{2} \]

where \( y = [y_1, y_2, y_3]^T \in \mathbb{R}^3 \) is the state vector and \( d(t) = [d_1(t), d_2(t), d_3(t)]^T \) is the external disturbance vector. Parameters \( a_2, \beta_2, \) and \( \sigma_2 \) are the same as \( a_1, \beta_1, \) and \( \sigma_1 \) in system (1). \( \text{sat}(u) = [\text{sat}(u_1), \text{sat}(u_2), \text{sat}(u_3)]^T \) is control input, which is subject to saturation nonlinearity. And \( \text{sat}(u_i) \) is described as

\[ \text{sat}(u_i) = \begin{cases} 
\frac{u_i^b}{\eta_1} \text{sign}(u_i), & \text{if } |u_i| \geq u_i^b, \\
0, & \text{if } |u_i| < u_i^b,
\end{cases} \tag{3} \]

where \( u_i^b \) is the unknown bound of \( u_i, i = 1, 2, 3 \). Define the synchronization error vector \( e = y - x = [y_1 - x_1, y_2 - x_2, y_3 - x_3]^T = [e_1, e_2, e_3]^T \). According to (1) and (2), we obtain the following error dynamic system:

\[ \begin{align*}
\dot{e}_1 &= g_1(y) - f_1(x) + d_1(t) + \text{sat}(u_1), \\
\dot{e}_2 &= g_2(y) - f_2(x) + d_2(t) + \text{sat}(u_2), \\
\dot{e}_3 &= g_3(y) - f_3(x) + d_3(t) + \text{sat}(u_3),
\end{align*} \tag{4} \]

where \( f_1(x) = x_3 + (x_2 - a_1)x_1, g_1(y) = y_3 + (y_2 - a_2)y_1, f_2(x) = 1 - \beta_1 x_2 - x_1^2, g_2(y) = 1 - \beta_2 y_2 - y_1^2, f_3(x) = -x_1 - \sigma_1 x_3, g_3(y) = -y_1 - \sigma_2 y_3. \)

In order to achieve effective state synchronization between the master chaotic system and the slave chaotic system, we make the following assumptions about \( f_i(x), g_i(y), \) and \( d_i(t), i = 1, 2, 3. \)

**Assumption 1.** Nonlinear functions \( f_i(x) \) and \( g_i(y) \) are unknown but bounded, and states \( x \) and \( y \) are measurable.

**Assumption 2.** The external disturbance \( d_i(t) \) is bounded, i.e., \( |d_i(t)| \leq d_i^* \), where \( d_i^* \) is unknown.

**Remark 1.** In [11], the linear feedback control method is used to synchronize two identical finance chaotic systems, but the unknown function and saturation nonlinearity input of the finance chaotic system are not considered. At the same time, in order to make the controller have good robustness, an integral sliding mode control method is used to discuss the synchronization of master system (1) and slave system (2).

**Remark 2.** Fuzzy logic systems are used in this paper to estimate unknown functions \( f_i(x) \) and \( g_i(y) \), and the related fuzzy estimation theory can be found in the literature [25]. In addition, \( d_i^* \) is only used in stability analysis and not involved in the design of \( u_i \).

By using FLSs, \( f_i(x) \) and \( g_i(y) \) can be approximated as

\[ f_i(x) = \theta_i^T \varphi_i(x) + \epsilon_{f_i}, \quad |\epsilon_{f_i}| \leq \epsilon_{f_i}^*, \tag{5} \]

\[ g_i(y) = \theta_i^T \varphi_i(y) + \epsilon_{g_i}, \quad |\epsilon_{g_i}| \leq \epsilon_{g_i}^*, \tag{6} \]

where \( \theta_i^* \) are ideal weight vectors, \( \epsilon_{f_i} \) and \( \epsilon_{g_i} \) are approximation errors, and \( \epsilon_{f_i}^* \) and \( \epsilon_{g_i}^* \) are upper bounds of \( \epsilon_{f_i} \) and \( \epsilon_{g_i} \), respectively. \( \varphi_i(x) \) and \( \varphi_i(y) \) are basis functions.

Define \( \tilde{\theta}_i = \theta_i^* - \hat{\theta}_i \), \( \tilde{\varphi}_i = \theta_i^* - \hat{\varphi}_i \) where \( \hat{\theta}_i \) and \( \hat{\varphi}_i \) are the estimations of \( \theta_i^* \) and \( \theta_i^* \), respectively. So, we obtain

\[ f_i(x) = \tilde{\theta}_i^T \tilde{\varphi}_i(x) + \tilde{\theta}_i \varphi_i(x) + \epsilon_{f_i}(x), \tag{7} \]

\[ g_i(y) = \tilde{\theta}_i^T \tilde{\varphi}_i(y) + \tilde{\theta}_i \varphi_i(y) + \epsilon_{g_i}(y). \]

Substituting (7) into (4), error system (4) can be rewritten as

\[ \begin{align*}
\dot{e}_1 &= \tilde{\theta}_1^T \tilde{\varphi}_1(y) + \tilde{\theta}_2^T \tilde{\varphi}_2(y) - \tilde{\theta}_1 \varphi_1(y) - \tilde{\theta}_2 \varphi_2(y) + \tilde{\varphi}_1 \varphi_1(y) + \tilde{\varphi}_2 \varphi_2(y) + \tilde{d}_1 + u_1, \\
\dot{e}_2 &= \tilde{\theta}_2^T \tilde{\varphi}_2(y) + \tilde{\theta}_3^T \tilde{\varphi}_3(y) - \tilde{\theta}_2 \varphi_2(y) - \tilde{\theta}_3 \varphi_3(y) + \tilde{\varphi}_2 \varphi_2(y) + \tilde{\varphi}_3 \varphi_3(y) + \tilde{d}_2 + u_2, \\
\dot{e}_3 &= \tilde{\theta}_3^T \tilde{\varphi}_3(y) + \tilde{\theta}_1^T \tilde{\varphi}_1(y) - \tilde{\theta}_3 \varphi_3(y) - \tilde{\theta}_1 \varphi_1(y) + \tilde{\varphi}_3 \varphi_3(y) + \tilde{\varphi}_1 \varphi_1(y) + \tilde{d}_3 + u_3,
\end{align*} \tag{8} \]

where \( \tilde{d}_i = \epsilon_{g_i}(y) - \epsilon_{g_i}(x) + d_i(t) + \Delta u_i, \Delta u_i = \text{sat}(u_i) - u_i \).

Obviously, there exists \( \tilde{d}_i \) such that \( |\tilde{d}_i| \leq |\tilde{d}_i| \).

The following lemma is introduced for the subsequent discussions.

**Lemma 1** (see [14]). For any \( z \in \mathbb{R}, 1 > \gamma \geq 0, \) and \( \epsilon > 0, \) the following inequality holds:

\[ |z|^\gamma - z|z|^\gamma \tanh \left( \frac{z}{\gamma} \right) \leq \eta_1 |\epsilon|^{\gamma+1}, \tag{9} \]

where \( \eta_1 = \varphi^*(2\varphi - \gamma - 1) > 0 \) and \( \varphi \) is the unique solution of \( \varphi + \tanh(\varphi) = \gamma + 1 \).

**Remark 3.** In this paper, inequality (9) of Lemma 1 is used to discuss the ultimate boundary of tracking error \( e \).

The aim of this paper is to design a novel integral sliding mode controller to make the synchronization error \( e \) stable, an integral sliding

3. Sliding Mode Control Design and Stability Analysis

In order to design a sliding mode robust controller to make the synchronization error \( e \) stable, an integral sliding mode control method is used to discuss the synchronization of master system (1) and slave system (2).
variable needs to be designed. In this paper, we design the sliding mode as follows:

\[
\begin{align*}
    s_1 &= e_1 + \int_0^t \left( \mu_1 \Tanh_{\epsilon_1}^c (e_1) + \mu_2 \Tanh_{\epsilon_1}^0 (e_1) \right) \, dt, \\
    s_2 &= e_2 + \int_0^t \left( \mu_1 \Tanh_{\epsilon_1}^c (e_2) + \mu_2 \Tanh_{\epsilon_1}^0 (e_2) \right) \, dt, \\
    s_3 &= e_3 + \int_0^t \left( \mu_1 \Tanh_{\epsilon_1}^c (e_3) + \mu_2 \Tanh_{\epsilon_1}^0 (e_3) \right) \, dt,
\end{align*}
\]

(10)

where \( \Tanh_{\epsilon_i}^c (e_i) = |e_i|^{\gamma_i} \tanh (e_i/\epsilon_i), \) \( \Tanh_{\epsilon_i}^0 (e_i) = \tanh (e_i/\epsilon_i), \) \( \mu_1, \mu_2 \) are designed positive parameters, \( \gamma_i \in (0, 1), \) and \( \epsilon_i \) is a small positive constant, \( i = 1, 2, 3. \)

**Remark 4.** Different from traditional sliding mode control method that the occurrence of chatter phenomenon is mainly due to the hysteresis of control switching, the integral sliding variable (10) used in this paper can avoid chatter phenomenon and estimate the ultimate boundary of the tracking error \( e. \)

According to (8) and (10), the time derivative of \( s_i \) is obtained as follows:

\[
\dot{s}_i = \tilde{\theta}_{g_i}^T \Phi_{g_i} (y) + \tilde{\theta}_{f_i}^T \Phi_{g_i} (y) - \tilde{\theta}_{f_i}^T \Phi_{f_i} (x) - \tilde{\theta}_{f_i}^T \Phi_{f_i} (x) + \tilde{d}_i (t) + u_i + \mu_1 \Tanh_{\epsilon_1}^c (e_i) + \mu_2 \Tanh_{\epsilon_1}^0 (e_i).
\]

(11)

Based on the above analysis, we give the main result as follows.

**Theorem 1.** Consider the synchronization error system (8) with the external disturbance; if the integral sliding variable is constructed as (10) and the controller \( u_i \) is designed as

\[
u_i = -\mu_3 s_i - \mu_4 \Tanh_{\epsilon_1}^c (s_i) - \tilde{\theta}_{g_i}^T \Phi_{g_i} (y) + \tilde{\theta}_{f_i}^T \Phi_{f_i} (x)
\]

\[-\mu_1 \Tanh_{\epsilon_1}^c (e_i) - \mu_2 \Tanh_{\epsilon_1}^0 (e_i),
\]

and parameter adaptive laws are proposed as

\[
\dot{\tilde{\theta}}_{g_i} = \eta_{\theta_i} \left[ \delta_i \Phi_{g_i} (y) - \delta_i \tilde{\theta}_{g_i} \right],
\]

\[
\dot{\tilde{\theta}}_{f_i} = \eta_{\theta_i} \left[ -\tilde{\theta}_{f_i} \Phi_{f_i} (x) - \delta_2 \tilde{\theta}_{f_i} \right],
\]

(13)

where \( \Tanh_{\epsilon_1}^c (s_i) = |s_i|^{\gamma_i} \tanh (s_i/\epsilon_i), \) \( \delta_1, \delta_2, \mu_3, \mu_4 \) are design positive parameters, \( \gamma_i \) is a small positive constant, then all the closed-loop system signals in (14) are uniformly ultimately bounded.

**Proof.** Consider the Lyapunov function as

\[
V = V_1 + V_2 + V_3,
\]

(14)

where

\[
V_1 = \sum_{i=1}^{3} \frac{1}{\eta_{\theta_i}} |\tilde{\theta}_{g_i} - \theta_i|^2,
\]

\[
V_2 = \sum_{i=1}^{3} \frac{1}{\eta_{\theta_i}} |\tilde{\theta}_{f_i} - \theta_i|^2,
\]

\[
V_3 = \sum_{i=1}^{3} \frac{1}{\eta_{\theta_i}} |\tilde{\theta}_{g_i} - \theta_i|^2.
\]

(15)

Substituting (12) into (11), the derivative of \( V_1 \) can be obtained as

\[
V_1 = \sum_{i=1}^{3} \frac{3}{\eta_{\theta_i}} |s_i|^2 - \sum_{i=1}^{3} \frac{\mu_i}{\eta_{\theta_i}} |s_i|^{\gamma_i} \tanh \left( \frac{s_i}{\epsilon_i} \right) - \sum_{i=1}^{3} |s_i| \tilde{d}_i (t) + \sum_{i=1}^{3} |s_i| \tilde{\theta}_{g_i} (y) + \sum_{i=1}^{3} |s_i| \tilde{\theta}_{f_i} (x).
\]

(16)

From (13), the time derivative of \( V_2 \) and \( V_3 \) yields

\[
\dot{V}_2 = \sum_{i=1}^{3} \frac{1}{\eta_{\theta_i}} \tilde{\theta}_{g_i} (y) + \delta_1 \tilde{\theta}_{g_i} \tilde{\theta}_{g_i},
\]

(17)

\[
\dot{V}_3 = \sum_{i=1}^{3} \frac{1}{\eta_{\theta_i}} \tilde{\theta}_{f_i} (x) + \delta_2 \tilde{\theta}_{f_i} \tilde{\theta}_{f_i}.
\]

(18)

Using (16), (17), and (18), one gets
\[
\dot{V} = -\mu_3 \sum_{i=1}^{3} \frac{s_i^3}{2} - \sum_{i=1}^{3} \mu_i s_i |s_i|^2 \tanh \left( \frac{s_i}{\epsilon_2} \right) + \sum_{i=1}^{3} s_i \hat{d}_i(t) \\
+ \sum_{i=1}^{3} \delta_i \tilde{\theta}^{T}_i \hat{\phi}_i + \sum_{i=1}^{3} \delta_i \tilde{\theta}^{T}_j \hat{\phi}_j; \\
\]  
(19)

Since the following inequalities hold:
\[
\begin{align*}
\delta_i \tilde{\phi}^{T}_i \hat{\phi}_i &\leq -\delta_i \frac{2}{3} \| \hat{\phi}_i \|^2, \\
\delta_i \tilde{\theta}^{T}_i \hat{\phi}_i &\leq -\delta_i \frac{2}{3} \| \hat{\phi}_i \|^2, \\
\delta_i \tilde{\phi}^{T}_j \hat{\phi}_j &\leq -\delta_i \frac{2}{3} \| \hat{\phi}_j \|^2,
\end{align*}
\]  
(20)

by using Lemma 1, one has
\[
-\sum_{i=1}^{3} \mu_i s_i |s_i|^2 \tanh \left( \frac{s_i}{\epsilon_2} \right) \leq -\sum_{i=1}^{3} \mu_i |s_i|^{\gamma_i+1} + 3\mu_3 \eta_{\gamma_i} \epsilon_2^{\gamma_i+1}. \\
(21)
\]

Substituting (20) and (21) into (19) results in
\[
\begin{align*}
\dot{V} &\leq -3 \left( \mu_3 - \frac{1}{2} s_i^2 - \sum_{i=1}^{3} \mu_i |s_i|^{\gamma_i+1} \right) \\
&\quad - \delta_i \frac{3}{2} \sum_{i=1}^{3} \| \hat{\phi}_i \|^2 - \delta_i \frac{3}{2} \sum_{i=1}^{3} \| \hat{\phi}_j \|^2 + r_0,
\end{align*}
\]  
(22)

where \( r_0 = \sum_{i=1}^{3} (d_i^2/2 + 3\mu_3 \eta_{\gamma_i} \epsilon_2^{\gamma_i+1}) + (\delta_i/2) \sum_{i=1}^{3} \| \hat{\phi}_i \|^2 + (\delta_i/2) \sum_{i=1}^{3} \| \hat{\phi}_j \|^2. \) And let \( r_{\min} = \min \{2\mu_3 - 1, \eta_\delta \delta_1, \eta_\delta \delta_2 \}. \) It can be obtained as
\[
\dot{V} \leq -r_{\min} V + r_0. \\
(23)
\]

From (23), we know that
\[
\begin{align*}
V &\leq \frac{r_0}{r_{\min}} + \left( V(0) - \frac{r_0}{r_{\min}} \right) e^{-r_{\min}t} \leq 2 \left( V(0) + \frac{r_0}{r_{\min}} \right).
\end{align*}
\]  
(24)

Obviously, it can be concluded that all signals in (14) are ultimately uniformly bounded. The proof is completed.

Theorem 1 shows that the designed sliding mode controller (12) guarantees that all signals in (14) are bounded. Now, we further study the convergence of the synchronization error \( \xi. \) According to (10), (11), and (12), we have
\[
\begin{align*}
\dot{\xi}_i &= \dot{s}_i - \mu_1 \text{Tanh}^{\gamma_i}(e_i) - \mu_2 \text{Tanh}^{\gamma_i}(e_i) \\
&= \psi_i(t) - \mu_1 \text{Tanh}^{\gamma_i}(e_i) - \mu_2 \text{Tanh}^{\gamma_i}(e_i), \\
\end{align*}
\]  
(25)

where \( \psi_i(t) = \tilde{\theta}_i \tilde{\phi}_i(y) - \tilde{\theta}_j \tilde{\phi}_j(x) + \tilde{d}_i(t) - \mu_3 s_i - \mu_4 \text{Tanh}^{\gamma_i}(s_i). \) Through the conclusion of Theorem 1, there exists a positive constant \( \bar{\omega}_i \) such that
\[
|\psi_i(t)| \leq \bar{\omega}_i. \\
(26)
\]

Consider the following Lyapunov function:
\[
V = \frac{1}{2} \sum_{i=1}^{3} e_i^2. \\
(27)
\]

The derivative of \( V \) is calculated as
\[
\dot{V} = \sum_{i=1}^{3} \left( e_i \psi_i(t) - \mu_1 e_i \text{Tanh}^{\gamma_i}(e_i) - \mu_2 e_i \text{Tanh}^{\gamma_i}(e_i) \right) \\
\leq \sqrt{e_1^2 + e_2^2 + e_3^2} \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} - \sum_{i=1}^{3} \mu_i e_i e_i^{\gamma_i+1} \text{Tanh}^{\gamma_i}(e_i) \\
- \sum_{i=1}^{3} \mu_2 e_i \text{Tanh}^{\gamma_i}(e_i). \\
(28)
\]

By using Lemma 1, the following inequalities hold:
\[
-3 \sum_{i=1}^{3} \mu_1 |e_i| e_i^{\gamma_i+1} \text{Tanh}^{\gamma_i}(e_i) \leq - \sum_{i=1}^{3} \mu_i |e_i|^{\gamma_i+1} + 3\mu_3 \eta_{\gamma_i} e_i^{\gamma_i+1} \\
\leq - \mu_1 (e_1^2 + e_2^2 + e_3^2)^{\gamma_i+1/2} + 3\mu_3 \eta_{\gamma_i} e_i^{\gamma_i+1}, \\
(29)
\]

\[
- \sum_{i=1}^{3} \mu_2 e_i \text{Tanh}^{\gamma_i}(e_i) \leq - \sum_{i=1}^{3} \mu_2 |e_i| + 3\mu_3 \eta_{\gamma_i} e_i \\
\leq - \mu_2 (e_1^2 + e_2^2 + e_3^2)^{\gamma_i+1/2} + 3\mu_3 \eta_{\gamma_i} e_i. \\
(30)
\]

Substituting (29) and (30) into (28), we have
\[
\dot{V} \leq -(\mu_2 - \bar{\omega}) (e_1^2 + e_2^2 + e_3^2 - \mu_1 (e_1^2 + e_2^2 + e_3^2)^{\gamma_i+1/2} + \epsilon_0, \\
(31)
\]

where \( \bar{\omega} = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}, \epsilon_0 = 3\mu_3 \eta_{\gamma_i} e_i^{\gamma_i+1} + 3\mu_3 \eta_{\gamma_i} e_i. \) Selecting \( \mu_2 \) satisfies \( \mu_2 > \bar{\omega}, \) and define the compact set as
\[
\Omega_c = \left\{ e \in R^3 | \mu_1 (e_1^2 + e_2^2 + e_3^2)^{\gamma_i+1/2} \leq \epsilon_0 \right\}, \\
(32)
\]

where \( \zeta \in (0, 1) \) is any number. If \( e \notin \Omega_c, \) then
\[
\dot{V} \leq - \mu_1 (e_1^2 + e_2^2 + e_3^2)^{\gamma_i+1/2} + \epsilon_0 < - \mu_1 (e_1^2 + e_2^2 + e_3^2)^{\gamma_i+1/2} = -2^{\gamma_i+1/2} \mu_1 \text{Tanh}^{\gamma_i+1/2} \\
= 2^{\gamma_i+1/2} \mu_1 \text{Tanh}^{\gamma_i+1/2}. \\
(33)
\]

Obviously, \( \dot{V} \) decreases monotonically outside the set \( \Omega_c \) until it enters the minimal level set of \( V \) containing \( \Omega_c. \) Therefore, we obtain the following theorem.

**Theorem 2.** Based on the conditions and results of Theorem 1 and selecting the appropriate parameters \( \mu_1 \) and \( \mu_2 \) in (10), the
tracking error vector $e$ is ultimately bounded and the ultimate bound is $\|e\| \leq (\varepsilon_0/\mu_1 (1 - \varsigma))^{1/c_1} + 1$.

Remark 5. The synchronization method of chaotic systems in this paper can be extended to generalized projective synchronization, lag synchronization, and so on.

4. Synchronization of Finance Chaotic System

In this section, the chaotic finance system [11] is introduced to demonstrate the effectiveness of the proposed control method (12). Parameters in the master system (1) and the slave system (2) are selected as $\alpha_1 = 0.8, \alpha_2 = 0.9, \beta_1 = 0.2, \beta_2 = 0.25, \sigma_1 = 1.9, \sigma_2 = 1.8$, and external disturbances $d_1(t) = 2.0 \sin(t), d_2(t) = 2.0 \cos(2t), d_3(t) = 2.0 \sin(3t)$. Here, initial values of $x$ and $y$ are chosen as $x(0) = [1, 1.5, 1.2]^T$ and $y(0) = [2, -1, -2.5]^T$, and other parameters are chosen as $\eta_\theta = \eta_f = 50, \mu_1 = \mu_2 = \mu_3 = \mu_4 = 5, \gamma_1 = 1/7, \gamma_2 = 1/5, \gamma_3 = 3, \mu_1 = \mu_2 = \mu_3 = 8, \delta_1 = \delta_2 = 0.01$. The fuzzy membership functions are selected as

$$
\theta_{f_1}(\zeta) = \exp\left[-\frac{1}{2}\left(\frac{\zeta + 4.5 - 1.5k_1}{0.7}\right)^2\right],
$$
$$
\theta_{f_2}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{\gamma_i + 4.5 - 1.5k_1}{0.7}\right)^2\right],
$$
$$
\theta_{f_3}(\zeta) = \exp\left[-\frac{1}{2}\left(\frac{\zeta + 5.5 - 1.5k_1}{0.7}\right)^2\right],
$$
$$
\theta_{g_1}(\zeta) = \exp\left[-\frac{1}{2}\left(\frac{\zeta + 4.5 - 1.5k_1}{0.7}\right)^2\right],
$$
$$
\theta_{g_2}(x_i) = \exp\left[-\frac{1}{2}\left(\frac{\gamma_i + 5.5 - 1.5k_1}{0.7}\right)^2\right],
$$
$$
\theta_{f_1}(\zeta) = \exp\left[-\frac{1}{2}\left(\frac{\zeta + 3.0 - 1.5k_2}{0.7}\right)^2\right],
$$
$$
\theta_{g_1}(\zeta) = \exp\left[-\frac{1}{2}\left(\frac{\zeta + 3.0 - 1.5k_2}{0.7}\right)^2\right].
$$

Figure 1: The trajectories of tracking errors $e_1, e_2, e_3$ by method (35) in time period (a) [0, 10 s] and (b) [5 s, 10 s].
Figure 2: The trajectories of (a) $\text{sat}(u_1), \text{sat}(u_2), \text{sat}(u_3)$ by method (35) in time period $[0, 10\text{s}]$ and (b) $u_1, u_2, u_3$ by method (35) in time period $[5\text{s}, 10\text{s}]$.

Figure 3: The trajectories of tracking errors $\epsilon_1, \epsilon_2, \epsilon_3$ by the proposed method (12) in time period (a) $[0, 10\text{s}]$ and (b) $[5\text{s}, 10\text{s}]$. 

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where \( \zeta = x_1, x_2, x_3; \xi = y_1, y_2, y_3; k_1 = 1, 2, 3, 4, 5; k_2 = 1, 2, 3. \)

According to Theorem 2, the ultimate bound of the tracking error \( e \) can be estimated in the interval \((-2.4 \times 10^{-3}, 2.4 \times 10^{-3})\) when \( \gamma_1 = 1/5. \) Now, we give the traditional integral sliding mode control method as follows:

\[
s_i = e_i + \int_0^t \mu_1 \text{Sign}(e_i) \, dt,
\]

\[
u_i = -\mu_2 s_i - \mu_4 \text{Sign}^{-1}_2(s_i) - \hat{\theta}_{g_i}^T \phi_{g_i}(y) + \hat{\theta}_{f_i}^T \phi_{f_i}(x)

- \mu_5 \text{Sign}(e_i), \tag{35}
\]

\[
\hat{\theta}_{g_i} = \eta \left[s_i \phi_{g_i}(y) - \delta \hat{\theta}_{g_i}\right],
\]

\[
\hat{\theta}_{f_i} = \eta \left[-s_i \phi_{f_i}(x) - \delta \hat{\theta}_{f_i}\right],
\]

where \( \text{Sign}(e_i) = \text{sign}(e_i), \text{Sign}^{-1}_2(s_i) = |s_i|^{1/2} \text{sign}(s_i), i = 1, 2, 3. \) Obviously, the approximation range of \( e \) cannot be obtained directly from (35).

Firstly, by using method (35), the corresponding simulation results are shown in Figures 1 and 2. From Figure 1, it can be found that all three synchronization errors \( e_1, e_2, e_3 \) can be controlled into an estimated range, however, \( \text{Se}_1, \text{Se}_2, \text{Se}_3 \) appear the chattering phenomenon. Meanwhile, Figure 2 shows that \( \text{sat}(u_i) \) and \( u_i \) also appear a large chattering phenomenon, which is mainly caused by the sign \((\cdot)\) function in method (35). In order to overcome the influence of the sign \((\cdot)\) function, this paper uses the tanh \((\cdot)\) function to design the controller. The simulation results are shown in Figures 3 and 4. From Figure 3, we can find that \( e_1, e_2, \) and \( e_3 \) are limited within a small range, which is smaller than the estimated range, and there is no chattering phenomenon for \( e_1, e_2, \) and \( e_3. \) Finally, comparing Figures 2 and 4, it can be seen that method (12) is significantly better than method (35) in terms of controller stability. Based on the above discussion, it can be seen that the proposed method (12) in this paper can effectively and quickly synchronize the master system and the slave system, can overcome the chattering phenomenon, and can estimate the predetermined range of synchronization error.

5. Conclusion

This paper considered the synchronization problem of a class of finance chaotic systems by using an integrated sliding mode control method. Firstly, the unknown functions \( f_i(x) \) and \( g_i(y) \) are estimated by using fuzzy logic systems. Then, a novel integral sliding variable \( s_i \) is proposed. The proposed control strategy \( u_i \) and parameter adaptive rules \( \tilde{\theta}_i \) can ensure that all signals of the closed-loop system are semiglobally uniformly stable, and the ultimate bound of
the tracking error can be estimated. Comparison results of simulation show that the proposed control method in this paper can ensure that the error fluctuation range is completely within the estimated range and the trajectories of errors and controllers have no chattering phenomenon. Therefore, compared with the traditional integrated sliding mode control method, the proposed control method in this paper has better control performance.

**Data Availability**

All datasets generated for this study are included in this paper.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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