Girvin-MacDonald-Platzman Collective Mode at General Filling Factors: Magneto-Roton Minimum at Half-Filled Landau Level

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The single mode approximation has proved useful for the excitation spectrum at \( \nu = 1/3 \). We apply it to general fractions and find that it predicts \( n \) magneto-roton minima in the dispersion of the Girvin-MacDonald-Platzman collective mode for the fractional quantum Hall states at \( \nu = n/(2n+1) \), and one magneto-roton minimum for both the composite Fermi sea and the paired composite fermion state. Experimental relevance of the results will be considered.

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The single mode approximation (SMA) was employed by Girvin, MacDonald, and Platzman (GMP) to gain insight into the excitation spectrum of the fractional quantum Hall effect (FQHE) at filling factor \( \nu = 1/3 \), where a good wave function for the ground state was known due to Laughlin. In analogy to Feynman’s theory of superfluid \( ^4 \)He, GMP considered a density wave excitation, referred to below as the GMP collective mode, and predicted a “roton” minimum in the energy dispersion, later confirmed in exact diagonalization studies. A sharp peak observed by Pinczuk et al. in inelastic Raman scattering experiments has been interpreted as the zero wave vector limit of the GMP mode, and another one at lower energy as the roton minimum. Now that accurate wave functions are known for all FQHE ground states, based on the physics of composite fermion (CF), we have investigated in this work the GMP mode for general filling fractions. One of our motivations is to explore what insight the SMA provides for the CF Fermi sea at \( \nu = 1/2 \), and for the fully spin polarized paired CF state, a promising candidate for the FQHE at \( \nu = 5/2 \).

The CF wave function for the ground state at \( \nu = n/(2n+1) \) is given by

\[
\Phi_n^{CF} = P \Phi_n^2 \Phi_n^1 .
\]

\( \Phi_n \) is the wave function for \( n \) filled Landau levels of electrons and \( \Phi_n^2 \) attaches two vortices to each electron to convert it into a composite fermion, hence the interpretation of \( \Phi_n^{CF} \) as \( n \) filled Landau levels of composite fermions. \( P \) is the lowest Landau level (LL) projection operator. (We will be restricting the Hilbert space to the lowest LL of electrons throughout this work, as appropriate in the limit of very strong magnetic fields. Also, a strictly two-dimensional system will be considered; finite thickness corrections, which have been found to lower the energy gaps by as much as 50%, must be incorporated in the theory when comparing quantitatively to experiment.) \( \Phi_n^{CF} \) is an extremely good representation of the actual FQHE state at \( \nu = n/(2n+1) \) and can be taken as exact for all practical purposes. Just as \( \Phi_n \) develops into the electron Fermi sea in the limit of \( n \to \infty \), \( \Phi_n^{CF} \) evolves into the CF Fermi sea as \( \nu = 1/2 \) is approached.

Following GMP, consider the following ansatz for the excited state:

\[
P \rho_k \Phi_n^{CF} \equiv \overline{\rho}_k \Phi_n^{CF}
\]

where \( \rho_k = \sum_j (-i k \mathbf{r}_j) \) is the usual density operator, and \( \overline{\rho}_k \) is the projected density operator. The energy of this state is given by

\[
\Delta E_k = \frac{< \Phi_n^{CF} | \overline{\rho}_k | V - E_0 | \overline{\rho}_k | \Phi_n^{CF}>}{< \Phi_n^{CF} | \overline{\rho}_k \Phi_n^{CF}>} = \frac{J_k}{\overline{\rho}_k}
\]

\( V = \frac{1}{2} \sum_{j \neq k} \frac{e^2}{|r_j - r_k|} \) is the Coulomb energy and \( E_0 \) is the energy of the ground state. The projected static structure factor \( (\overline{s}_k) \) can be obtained from the ordinary static structure factor by using the relation \( \overline{s}_k = s_k - (1 - e^{-k^2/2}) \), and the projected oscillator strength \( (\overline{J}_k) \) is given by the expression (with magnetic length \( l_0 = 1 \))

\[
\overline{J}_k = \overline{s}_k + \overline{\rho}_k
\]

\[
\overline{s}_k = \exp\left[-\frac{k^2}{2}\right] \int_0^\infty dq V(q) s_k[q J_0(qk) - 1]
\]

\[
\overline{\rho}_k = 2 \exp\left[-\frac{k^2}{2}\right] \int \frac{d^2q}{(2\pi)^2} V(|q - k|) s_q e^{k \cdot q} \sin^2\left(\frac{k \times q}{2}\right)
\]

where \( V(q) = 2\pi e^2/q \) is the two-dimensional Fourier transform of the Coulomb interaction. Thus, \( \Delta E_k \) can be calculated from the knowledge of the structure factor \( s_k \), which we obtain by a Fourier transformation of the pair distribution function, \( g(r) \).

The pair distribution function is computed numerically by performing Monte Carlo on systems of 50 to 60 composite fermions. The technique for dealing with the projection operator in the CF wave function has been described in the literature. The distance between electrons is measured along the arc (as opposed to chord), which is believed to minimize the curvature effects. For Fourier transformation, it is useful to have...
an analytic function for \( g(r) \), which is obtained by expanding \( e^{-r^2/4}(g(r) - 1) + e^{-r^2/4} \) in the power series \( \sum_m C_m r^{4m+2} \) and then adjusting the coefficients \( C_m \) to obtain the best fit. It is necessary to keep a large number of terms in the expansion to ensure that all of the oscillations in \( g(r) \) are captured properly, which in turn is crucial for obtaining the oscillations in the energy dispersion of the GMP mode. At \( \nu = 5/11 \), we keep terms up to \( r^{162} \), with a total of 41 fitting parameters; the number of required terms increases very rapidly as the filling factor approaches 1/2. The fitting is done in the standard manner, by minimizing the chi-square function; the condition that the derivative of chi-square with respect to all the fitting parameters vanish reduces to solving a set of linear equations for the fitting parameters. Since we are dealing with a huge number of parameters and rather subtle details of raw data, we use the technique of “singular value decomposition” for solving these equations, which fixes the roundoff error sensitivity of the usual normal equation solution through Gauss-Jordan elimination.

For Laughlin’s wave function, the pair distribution function in the quantum mechanical ground state is identical to the thermal pair distribution function of a classical two-dimensional one-component plasma (2DOCP) with logarithmic interactions. It must satisfy certain constraints, which are tantamount to requiring that \( \Sigma_k \rightarrow \frac{1}{2} \nu k^2 \) as \( k \to 0 \); in other words, they fix the coefficients of the \( k^0 \), \( k^2 \) and the \( k^4 \) term in the expansion of \( \Sigma_k \). In terms of the 2DOCP, the absence of the \( k^2 \) term is a consequence of charge neutrality, the absence of the \( k^2 \) term of perfect screening, and the coefficient of the \( k^4 \) term is fixed from the compressibility sum rule. The first two are quite generally expected for incompressible FQHE states; given that \( g_k \rightarrow k^4 \) as \( k \to 0 \), a finite gap at small \( k \) is possible only if \( \Sigma_k \sim k^4 \) at small \( k \). But the coefficient of \( k^4 \) given above appears to be special to Laughlin’s wave function, since the other FQHE states do not enjoy a mapping into a 2DOCP. However, Lopez and Fradkin have argued that for any general incompressible state, the small \( k \) properties are correctly described by a wave function whose modulus is given by \( |\prod_{j<k}(z_j - z_k)|^{1/\nu}\exp[- \sum_j |z_j|^2/4] \); a plasma analogy on this wave function will again produce the above coefficient of \( \frac{1}{2\nu} \) for the \( k^4 \) term. Therefore, we fit the numerical \( g(r) \) to the above power series subject to all three constraints. The fits are excellent as seen in Fig. 4 for \( \nu = 5/11 \). The goodness of the fit may be taken as a corroboration of the assertion made by Lopez and Fradkin with regard to universal long distance properties of general incompressible fractional Hall states. It is stressed that the fixing of the coefficient of \( k^4 \) in this manner is not crucial for the results below; if we fit \( g(r) \) subject to only the first two constraints, the dispersions of the GMP mode are affected only slightly at small \( k \).

The Fourier transformation is readily performed with the help of the analytic form of \( g(r) \). The projected structure factors are shown in Fig. 5 for the principal \( \nu = n/(2n + 1) \) FQHE states at 1/3, 2/5, 3/7, 4/9, and 5/11, and the GMP-mode dispersions obtained from them in Fig. 4. Contrary to what one might have expected, the energy in the \( k \to 0 \) limit increases with \( n \). The small \( k \) region is sensitive to the finite system size, through the curvature of the spherical geometry. We note that our \( k \to 0 \) limit of the energy for \( \nu = 1/3 \) GMP mode is in complete agreement with GMP who had employed much bigger systems in their calculations, which gives us reasonable confidence that our results are reliable even at small \( kl_0 \), even though we have not investigated systematically the particle-number \( (N) \) dependence of our results. There are \( n \) inflection points in \( \Sigma_k \) for the state with \( n \) filled CF-LLs, producing \( n \) minima in the dispersion curve. The two exterior minima are the strongest, with the interior minima becoming progressively weaker with increasing \( n \). In particular, the minimum at \( kl \approx 2 \) is quite robust to variations in \( n \), and appears to survive all the way to \( \nu = 1/2 \) to produce a roton minimum at \( k \approx 2k_F \). In the wave vector range \( kl_0 > 0.5 \), the dispersion is quite insensitive to \( n \) (especially if we ignore \( n = 1 \) and 2), and a smoothed version, shown in Fig. 6, presumably gives a reasonable approximation to the dispersion at \( \nu = 1/2 \). No conclusions can be made for the \( 1/2 \) dispersion at smaller \( k \), due to a substantial \( n \) dependence of the curves, and also because the wave function approach is anyway not expected to provide a reliable account of the long distance properties of the CF Fermi sea. The significance of the roton minimum is that it is observable in light scattering experiments due to a divergence in the density of states.

An understanding of the low-energy excitations is intimately related to an understanding of the physics of the ground state, and was clarified by the CF theory: given that the actual FQHE ground state is described as the state with \( n \) filled CF-LLs, it is natural to consider excited states in which one composite fermion is promoted from the \( n \)th CF-LL to the \((n+1)\)st, producing an exciton of composite fermions. This provides an excellent quantitative description of the low-energy excitation branch at general FQHE. In particular, the CF exciton has lower energy than the GMP mode; for example, at \( \nu = 2/5 \), the minimum energy of the GMP mode is approximately 40% higher than that of the CF exciton.

Even though the GMP mode is not the smallest energy excitation, it has a precise and important physical significance: It provides information about excited states that are connected to the ground state by the density operator, which are also the excitations that are probed by perturbations that couple to the density, as for example, in light scattering experiments. While the SMA is exact when the density operator couples the ground state only to a single mode, or to states in a narrow range of energy, it continues to provide the average energy (in fact, the exact first moment of the energy) of the density-coupled states quite generally. Is the GMP mode observable? In the case of the ordinary electron Fermi liquid at zero
magnetic field, the analogous mode is the plasmon which is sharply defined outside the single particle excitation (SPE) continuum, but is Landau damped inside. However, it does not disappear immediately upon entering the SPE continuum; its line-width broadens only slowly as it extends deeper into the SPE region. We expect that the GMP mode behaves similarly, and predict that it will appear (say, in Raman experiments) as a broad peak centered at the SMA energy, possibly in addition to a sharper peak at lower energies coming from the CF-exciton. Of course, the CF exciton ceases to exist for the CF sea, but the GMP collective mode may still be well defined and observable.

Another interesting possibility at the half-filled Landau level is pairing of composite fermions. A variational wave function for the paired CF state is given by

$$Pf[(z_j - z_k)^{-1}] \prod_{j<k}(z_j - z_k)^2 \exp(-\frac{1}{4} \sum_j |z_j|^2) \quad (5)$$

where $z_j = x_j + i y_j$, and, apart from an overall normalization factor, the Pfaffian has the form of the real space BCS wave function: $Pf[(z_j - z_k)^{-1}] = A \prod_{j<k}(z_{j+1} - z_{2j+1})^{-1}$, $A$ being the antisymmetrization operator. This state has an energy slightly higher than the CF sea in the lowest LL, but slightly lower in the second LL (i.e. at $\nu = 5/2$), consistent with the experimental observation that a compressible state is observed at 1/2 but FQHE at 5/2. After correcting for particle-hole symmetry, the Pfaffian state has also been shown to have a high degree of overlap with the exact ground state at $\nu = 5/2$ in finite system studies. All this taken together supports the view that the physics of the 5/2 FQHE lies in pairing of composite fermions. We apply the SMA to this state in order to learn about its collective excitations. The pair distribution function of this wave function has a "shoulder" at smaller $k$ relative to the pair distribution function of the CF sea, indicative of a real space pairing of composite fermions in this wave function, and results in a structure factor, shown in Fig. 2, peaked at a smaller wave vector than the $s_{\nu}$ of the CF sea. We have computed the dispersion of the GMP mode for both the lowest and the first excited Landau levels, appropriate for $\nu = 1/2$ and $\nu = 5/2$, respectively; the latter is obtained by using an effective interaction in the lowest Landau level that is equivalent to the Coulomb interaction in second Landau level, following the method outlined in Park et al. The resulting dispersion is shown in Fig. 3; it again contains a roton minimum, although much broader than for the CF sea.

In summary, application of the single mode approximation to composite fermion states has resulted in new experimentally verifiable predictions. This work was supported in part by the National Science Foundation under grant no. DMR-9615005, and by NCSA Origin 2000 under grant no. DMR970015N.

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FIG. 1. The pair distribution function $g(r)$ for 5/11. The points are from Monte Carlo calculations, and the solid line is the analytical fit. $l_0$ denotes the magnetic length.

FIG. 2. Projected static structure factors $\overline{s}_k$ for the FQHE states at 1/3, 2/5, 3/7, 4/9, and 5/11.

FIG. 3. Dispersions of the GMP mode at 1/3, 2/5, 3/7, 4/9, and 5/11. The energies are in given in units of $e^2/\epsilon l_0$, where $\epsilon$ is the dielectric constant of the background material.

FIG. 4. The estimated dispersion of the GMP mode for the fully polarized composite Fermi sea at the half-filled Landau level. The Fermi wave vector is given by $k_F = \sqrt{4\pi \rho}$, $\rho$ being the electron density.

FIG. 5. Projected static structure factors $\overline{s}_k$ for the paired composite fermion state for $N = 50$. The full structure factor $s_k$ is shown in the inset. The structure factor of the 5/11 state is also shown for comparison.
FIG. 6. Dispersion for the GMP collective mode for the paired composite fermion state, both for the lowest \((n = 0)\) and the first excited \((n = 1)\) Landau levels, corresponding to \(\nu = 1/2\) and \(\nu = 5/2\), respectively.