A local true Hamiltonian for the CGHS model in new variables

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Following our previous work, a complete classical solution of the CGHS model in Hamiltonian formulation in new variables is given. We preform a series of analyses and transformations to get to the CGHS Hamiltonian in new variables from a generic class of two dimensional dilatonic gravitational systems coupled to matter. This gives us a second class system, a total Hamiltonian consisting of a Hamiltonian constraint, a diffeomorphism constraint and two second class constraints. We calculate the Dirac brackets, bring them to a standard form similar to the Poisson brackets by introducing a new variable. Then by rescaling lapse and shift, the Hamiltonian constraint is transformed into a form where it has an strong Abelian algebra with itself. This property holds both in vacuum case and in case with matter coupling. Then for each of the vacuum and the coupled-to-matter cases, we preform two gauge fixings, one set for each case, and solve the classical system completely in both cases. The gauge fixing of the case coupled to matter is done by implementing a method based on canonical transformation to a new set of variables and leads to a local true Hamiltonian.

We also show that our formalism is consistent with the original CGHS paper by showing that the equations of motion are the same in both cases. Finally we derive the relevant surface term of the model.
The two dimensional gravitational systems with black hole solutions, specially the CGHS model [1], have proven to be a very good test bench to try out various ideas about quantum gravity. There is an extensive study of these systems in the literature (for a short list of references see [2–8] and the references within them) and there have been numerous attempts to understand some quantum phenomena, such as black hole evaporation, information loss and the asymptotic fate of spacetime, using their black hole solutions. There is also the important question of whether quantum gravity, specially loop quantum gravity, eliminates the singularity? Loop quantum gravity has been progressing on the issue of addressing how singularities are affected by quantum gravity. Examples are replacing the Big Bang singularity by a Big Bounce in homogenous models [9] and some form of singularity resolution for the Schwarzschild black holes in spherically symmetric mid-superspaces [10]. But with all these attempts, the questions surrounding these issues has not been answered in a satisfactory way.

The purpose of this paper is twofold. On one hand, we address the formulation of the CGHS model in a manner that is suitable for applying loop quantum gravity techniques. We start by deriving the Ashtekar-like variables for the model and writing its Hamiltonian in terms of those variables. One of the differences of our method with many other works that can be found in the literature is that this formulation is pursued without a conformal transformation. This is important in the sense that one is working with variables that have direct geometrical meaning so there is no need to turn everything back to their original directly-geometric form at the end. It also makes it easier to read the physical implications off of the theory. Another advantage of our formulation is that it gives us a Hamiltonian constraint that commutes with itself and hence the algebra of the constraints becomes a Lie algebra, since now we have structure constants instead of structure functions. Thus this formulation of the system appears to be suitable for applying loop quantum gravity techniques.

The second purpose of the paper is to solve the classical system coupled to matter completely classically by some choice of gauge fixing such that it results in a local true Hamiltonian for the system. Local here means that the lapse will not be an integral of canonical variables and hence the Hamiltonian will not be an integral of an integral. This way the equations of motion will also be local. The locality is important because, among other things, it is obviously much harder to try to quantize a non-local theory. There have been some previous studies [11–13] which did not lead to a Hamiltonian that was the spatial integral of a local density, thus leading to non-local equations of motion. This in turn leads to difficulties upon trying to quantize the system. However, there have been some recent works [14, 15] leading to local true Hamiltonian for 3+1 spherically symmetric case which we follow in order to derive such a Hamiltonian for the CGHS model in new variables.

The structure of this paper is as follows: in section II, we rewrite a generic action for two dimensional gravitational systems coupled to a dilaton field and a scalar matter field (which includes the CGHS model as a special case) in a suitable way to include cases with or without kinetic term for the dilaton field in one action. In section III, the previous Lagrangian will be written in tetrad variables and is transformed into a Hamiltonian by a Legendre transformation. In section IV, we focus just on the CGHS model, derive the new Ashtekar-like variables for it similar to the 3+1 case and write its Hamiltonian in those variables. Sections V and VI are dedicated to the implementation of the Dirac procedure for a second class system in our Hamiltonian formalism. In section VII, the Hamiltonian constraint is written in a special form such that it has an strong Abelian algebra with itself. Section VIII is where equations of motions are derived for the Hamiltonian system. In sections IX and X, both vacuum and coupled-to-matter cases are gauge fixed and a local true Hamiltonian is derived for the case coupled to matter. In section XI, the boundary term for the formalism is derived and is compared to the standard boundary term of the CGHS model.

II. GENERIC ACTION IN METRIC VARIABLES

It has been shown [2, 16, 17] that the most general diffeomorphism invariant action yielding second order differential equations for the metric $g$ and a scalar (dilaton) field $\Phi$ in two dimensions is

$$S_{g,\text{dil}} = \int d^2 x \sqrt{-g} \left( D(\Phi) R(g) + \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + V(\Phi) \right). \quad (1)$$

The authors in [6] argue that the most general form is in fact $S = \int d^2 x \sqrt{-g} \left( D(\Phi) R(g) + V \left( (\nabla \Phi)^2, \Phi \right) \right)$.
In order to keep the generality, we can combine (9) and (12) in to the following form

\[ S = \int d^4x \sqrt{-\hat{g}} \left( D(\Phi) R(\hat{g}) + \frac{1}{2} \hat{g}^{ab} \partial_a \hat{\Phi} \partial_b \hat{\Phi} + V(\hat{\Phi}) \right) - \int d^2x \sqrt{-|g|} \hat{W}(\hat{\Phi}) \hat{g}^{ab} \partial_a \hat{\Phi} \partial_b \hat{\Phi}. \]  

which by using spherically symmetric ansatz

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu + \Phi^2 (d\theta^2 + \sin^2(\theta) d\phi^2), \]

and integrating over \( \theta \) and \( \phi \), becomes

\[ S = \int d^2x \sqrt{-|g|} \left( \frac{1}{4} \Phi^2 R(g) + \frac{1}{2} \hat{g}^{ab} \partial_a \hat{\Phi} \partial_b \hat{\Phi} + \frac{1}{2} \hat{\Phi}^2 \lambda^2 \right) - \frac{1}{2} \int d^2x \sqrt{-|g|} \hat{g}^{ab} \partial_a f \partial_b f. \]

In (4), \( x^0, x^1, \theta \) and \( \phi \) are some coordinates adapted to the spherical symmetry and \( g_{\mu\nu} \) is the metric on the \( x^0, x^1 \) plane. If \( \frac{\partial^2 \hat{\Phi}}{\partial \hat{\phi}^2} \neq 0 \), it is possible to rescale \( \hat{\Phi} \) to be identical to \( x^1 \). It is easily seen that (the gravitational part of) (4) is an example of (11) and for this case, \( \Phi \) mimics the dilaton field.

The second case is the CGHS model (1) whose action minimally coupled to matter is

\[ S_{\text{CGHS}} = \int d^2x \sqrt{-|g|} e^{-2\varphi} (R + 4g^{ab} \partial_a \varphi \partial_b \varphi + 4\lambda^2) - \frac{1}{2} \int d^2x \sqrt{-|g|} g^{ab} \partial_a f \partial_b f, \]

in which \( \varphi \) (not to be confused with the coordinate \( \phi \) in spherically symmetric model) corresponds to the dilaton field and \( \lambda^2 \) is the cosmological constant. By a redefinition of the dilatation field

\[ \hat{\Phi} = 2 \sqrt{2} e^{-\varphi}, \]

one gets

\[ S_{\text{CGHS}} = \int d^2x \sqrt{-|g|} \left( \frac{1}{8} \Phi^2 R(g) + \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{2} \Phi^2 \lambda^2 \right) - \frac{1}{2} \int d^2x \sqrt{-|g|} g^{ab} \partial_a f \partial_b f. \]

whose gravitation part can be seen to be an example of (11). So from these two cases we can infer that the minimally coupled matter part corresponding to (11), can be written as

\[ S_m = - \int d^2x \sqrt{-|g|} \hat{W}(\hat{\Phi}) \hat{g}^{ab} \partial_a f \partial_b f. \]

Thus the full general action with minimal coupling to matter will become

\[ S = \int d^2x \sqrt{-|g|} \left( D(\Phi) R(g) + \frac{1}{2} g^{ab} \partial_a \Phi \partial_b \Phi + \hat{W}(\Phi) \right) - \int d^2x \sqrt{-|g|} \hat{W}(\hat{\Phi}) \hat{g}^{ab} \partial_a f \partial_b f. \]

In some cases, it is desirable to eliminate the kinetic term \( g^{ab} \partial_a \Phi \partial_b \Phi \). This can be achieved by a conformal transformation

\[ \hat{g}_{ab} = \Omega^2(\Phi) g_{ab}, \]

with

\[ \Omega(\Phi) = C \exp \left( \frac{1}{4} \int d\Phi \frac{1}{D(\Phi)} \right) \]

and \( C \) being a constant of integration. In this case the general action becomes

\[ S = \int d^2x \sqrt{-|g|} \left( D(\Phi) R(g) + \Omega^{-2}(\Phi) V(\Phi) \right) - \int d^2x \sqrt{-|g|} \hat{W}(\hat{\Phi}) \hat{g}^{ab} \partial_a f \partial_b f. \]

In order to keep the generality, we can combine (9) and (12) into the following form

\[ S_{1+1} = \int d^2x \sqrt{-|g|} \left\{ Y(\Phi) R + \frac{1}{2} Z g^{ab} \partial_a \hat{\Phi} \partial_b \hat{\Phi} + V(\Phi) \right\} - \int d^2x \sqrt{-|g|} \hat{W}(\hat{\Phi}) \hat{g}^{ab} \partial_a f \partial_b f. \]
where we have introduced the variable $Z = 0, 1$, which plays the role of a “switch” that turns the kinetic term on or off. The gravitational and matter Lagrangian densities are now

\[
L_g = \sqrt{-g} \left\{ Y(\Phi) R + \frac{1}{2} Z g^{ab} \partial_a \Phi \partial_b \Phi + V(\Phi) \right\},
\]

(14)

\[
L_m = -\sqrt{-g} |W(\Phi)| g^{ab} \partial_a f \partial_b f.
\]

(15)

It is worth noting that the action (13) is similar to the action of $f(R)$ gravity theories with $V(\Phi)$ being the potential of the dilaton field. In the CGHS case, this potential is related to the cosmological constant. So the cosmological constant in this case may be seen as dynamical and coming from properties of the dilaton field rather than just being there.

### III. TETRAD FORMULATION

In order to get to the new variables for these cases, one needs to first write the theory in tetrad formulation in which

\[
g_{ab} = \eta_{IJ} e^I_a e^J_b
\]

(16)

and $\eta_{IJ}$ is the Minkowski metric, $e^I_a$ are the tetrads and $I, J$ are the internal indices while $a, b$ are the abstract ones. The curvature can be written in terms of the curvature of the spin connection and ultimately in terms of the spin connection $\omega^a_{IJ}$ itself as

\[
R = R_{abIJ} e^a_I e^b_J = (2\partial_{[a} \omega^b_{|a]} + [\omega^b_{|a]} \omega^a_{IJ}]) e^a_I e^b_J,
\]

(17)

where $[,]$ stands for the Lie commutator in the Lorentz Lie algebra and the indices $I, J$ take value in this algebra. Since the spin connection is antisymmetric in $I, J$, we can write it as

\[
\omega^a_{IJ} = \omega_a^{\ IJ}.
\]

(18)

This way, the curvature (17) becomes

\[
R = (2\partial_{[a} \omega^b_{|a]} + \omega^b_{[a} \omega^a_{IJ}]) e^a_I e^b_J = 2 \partial_{[a} \omega^b_{|a]} e^a_I e^b_J,
\]

(19)

where we have used the following fact about the Lie commutator in this case

\[
[\omega^b_{[a}, \omega^a_{b]} = \omega^b_a \epsilon^I_K e^J_I e^b_J - \omega^b_a \epsilon^I_K e^J_I e^b_J
\]

(20)

and also the fact that $\eta^J_I e^a_I e^b_J$ is symmetric in $a, b$ while $\omega^b_{[a} \omega^a_{b]}$ is antisymmetric and thus

\[
\omega^b_{[a} \omega^a_{b]} \eta^J_I e^a_I e^b_J = 0.
\]

(21)

We also would like to add the torsion free condition (contracted by a Lagrange multiplier) to the action (13). This condition reads

\[
0 = de^J_I + e^J_J \wedge e^I = 2\partial_{[a} e^b_{|a]} + 2e^J_J \omega_{[a} e^b_{|a]}.
\]

(22)

Since this is a 2-form, we need to contract it with $e^{ab}$ to get an scalar to be able to add it to the Lagrangian density. Doing so, contracting it with a Lagrange multiplier $-X_I$ and substituting everything in (14) yields

\[
L_g = -2X_I e^{ab} (\partial_{[a} e^b_{|a]} + e^J_J \omega_{[a} e^b_{|a]}) + 2Y \partial_{[a} \omega^b_{|a]} e^J_J e^a_I e^b_J + \frac{1}{2} Z \eta^J_I e^a_I e^b_J \partial_a \Phi \partial_b \Phi + eV.
\]

(23)

We can write $e^{ab}$ in terms of $e^{IJ}, e^a_I$ and its determinant $e = \det(e^a_I)$ as

\[
e^{ab} = -ee^a_I e^b_J e^{IJ}.
\]

(24)
If we integrate by parts in the first term in Lagrangian density (23) to bring the partial derivative to act on $X_I$, and then use the above result for $e^{ab}$ in both first two terms of this Lagrangian, the pure gravitational Lagrangian density can be written as

$$L_g = e \left( -2\partial_a(X_I)e_K^a e^{KI} - 2X_I e^{Ia} \omega_a + 2Y \partial_a \omega_b e^{IJ} e_I e_J + \frac{1}{2} Z \eta^{IJ} e_I e_J \partial_a \Phi \partial_b \Phi + V \right). \quad (25)$$

From (15), the matter Lagrangian density can also simply be written as

$$L_m = -W \eta^{IJ} e_I e_J \partial_a f \partial_b f. \quad (26)$$

Next, we decompose the Lagrangian by ADM method and perform a Legendre transformation to get to the Hamiltonian. Most of the details needed for these steps have been already discussed in our previous work [18]. So we just mention the result here. The generic Hamiltonian then will become.

$$H = N \left[ \frac{P_\Phi}{|P|} (X^1)' + \frac{P_I}{|P|} (X^2)' - 2 \frac{P_\omega}{|P|} \omega_1 X^1 - 2 \frac{P_2}{|P|} \omega_1 X^2 - \frac{Z}{|P|} \Phi^2 - \frac{P_2}{Z|P|} + 2 \frac{W}{|P|} \left( \frac{P_2}{2|P|} - \frac{|P|^2}{2} \right) \right] + N^1 \left[ P_\Phi \Phi' + P_I f' + \frac{P_1}{|P|} (X^1)' + \frac{P_2}{|P|} (X^2)' - P_1 \omega_1 X^2 - P_2 \omega_1 X^1 \right] + \omega_0 \left[ P_1 X^2 + P_2 X^1 - (2Y)' \right]. \quad (27)$$

Here $N$ is lapse, $N^1$ is the (one dimensional) shift vector, the momenta are

$$P_I = \frac{\partial L}{\partial X_I} = 2 \sqrt{\eta} n_I, \quad (28)$$

$$P_\omega = \frac{\partial L}{\partial \omega_1} = 2Y, \quad (29)$$

$$P_\Phi = \frac{\partial L}{\partial \Phi} = \frac{Z}{N} \left( N^1 \Phi' - \phi \right), \quad (30)$$

$$P_f = \frac{\partial L}{\partial f} = -\frac{2W}{N} \left( N^1 f' - F \right). \quad (31)$$

$n_I$’s are normals to the spatial hypersurfaces, $X^1 = \epsilon^{IJ} X_J$, $q$ is the determinant of the induced metric on the spatial hypersurfaces, $|P| = \sqrt{-\eta^{IJ} P_I P_J}$ is the norm of $P_I$ (which is a timelike vector, hence the negative sign) and the prime represents partial derivative with respect to the spatial coordinate $x^1$. From the definitions of momenta (29) and (30), one can see that if $Z = 1$, i.e., cases with kinetic term present (without conformal transformation), then $\Phi$ is a canonical variable and $P_\Phi$ is its momentum. In this case, since $Y$ involves $\Phi$, equation (29) will be a new primary constraint. This would not happen in the cases without kinetic term (with conformal transformation) like standard spherically symmetric case in Ashtekar variables, since there, $\Phi$ is not a canonical variable although equation (29) is still valid. Thus in the CGHS case, we should add (29) to the general Hamiltonian (27) to get the total Hamiltonian.

**IV. THE CGHS HAMILTONIAN IN NEW VARIABLES**

From now on we focus only on the CGHS case and hence we substitute the explicit forms of $Y(\Phi), V(\Phi)$ and $W(\Phi)$ in the previous generic Hamiltonian and also add (29) as a new primary constraint to it to get

$$H = N \left( \frac{2P_2}{|P|} \partial_1 X^1 + \frac{2P_1}{|P|} \partial_1 X^2 - 2 \frac{P_1}{|P|} \omega_1 X^1 - 2 \frac{P_2}{|P|} \omega_1 X^2 - \frac{|P|^2}{4} \lambda^2 \Phi^2 \right).$$

In fact this can be seen more clearly in the Lagrangian which leads to this Hamiltonian. There, all the terms containing $\Phi$ are multiplied by $Z$. 
- \frac{\Phi'^2}{|P|} - \frac{P^2_\Phi}{|P|} + \frac{(f')^2}{|P|} + \frac{P^2_f}{|P|} \\
+ N^1 (P_1 \partial_1^* X^1 + P_2 \partial_2^* X^2 - P_3^* X^1 \omega_1 - P_1^* X^2 \omega_1 + \Phi' P_\Phi + f' P_f) \\
+ \omega_0 \left( P_1^* X^2 + P_2^* X^1 - \left( \frac{1}{4} \Phi^2 \right)^{1/2} \right) \\
+ M \left( P_\omega - \frac{1}{4} \Phi^2 \right), \quad (32)

where M is a Lagrange multiplier. In order to transform to the Ashtekar variables and following a similar pattern as for Ashtekar variables in the 3+1 model [18], we introduce the following new momenta with a canonical transformation:

\begin{align*}
P_\omega &= E^x, \\
|P| &= 2E^\varphi, \\
P_1 &= 2 \cosh(\eta) E^\varphi, \\
P_2 &= 2 \sinh(\eta) E^\varphi,
\end{align*}

where \( E^x, E^\varphi \) and \( \eta \) are the new momenta. This gives us the generating function

\[ F(q, P) = 2^* X^1 \cosh(\eta) E^\varphi + 2^* X^2 \sinh(\eta) E^\varphi + \omega_1 E^x + \Phi P_\Phi + f P_f. \quad (37) \]

Using \( F(q, P) \), we can find the new canonical variables as

\begin{align*}
Q_\eta &= \frac{\partial F}{\partial P_\eta} = 2^* X^1 \sinh(\eta) E^\varphi + 2^* X^2 \cosh(\eta) E^\varphi, \\
K_\varphi &= \frac{\partial F}{\partial E^\varphi} = 2^* X^1 \cosh(\eta) + 2^* X^2 \sinh(\eta), \\
A_x &= \frac{\partial F}{\partial E^x} = \omega_1,
\end{align*}

where \( Q_\eta, K_\varphi \) and \( A_x \) correspond to \( \eta, E^\varphi \) and \( E^x \) respectively. From the above equations, we can find \(^* X^1, ^* X^2\) and \( \omega_1 \) as

\begin{align*}
^* X^1 &= \frac{1}{2} \left( K_\varphi \cosh(\eta) - \frac{Q_\eta \sinh(\eta)}{E^\varphi} \right), \\
^* X^2 &= -\frac{1}{2} \left( K_\varphi \sinh(\eta) - \frac{Q_\eta \cosh(\eta)}{E^\varphi} \right), \\
\omega_1 &= A_x.
\end{align*}

In order to write the Hamiltonian density \( \mathcal{H} \) in these new variables, we substitute \( \{Q_\eta, K_\varphi, A_x\} \) and \( \{X^1, X^2, \omega_1\} \) in the total Hamiltonian \( \mathcal{H} \), and then make a field redefinition

\[ A_x = K_x - \eta' \quad (44) \]

to get rid of \( \eta \) in the Hamiltonian and get

\begin{align*}
H &= N \left( \frac{Q_\eta}{E^\varphi} - \frac{Q_\eta E^\varphi'}{E^\varphi} - \frac{1}{2} E^\varphi \Lambda^2 \Phi^2 - K_\varphi K_x - \frac{\Phi'^2}{2E^\varphi} - \frac{P^2_\Phi}{2E^\varphi} + \frac{(f')^2}{2E^\varphi} + \frac{P^2_f}{2E^\varphi} \right) \\
&\quad + N^1 (E^\varphi K_\varphi' - Q_\eta K_x + \Phi' P_\Phi + f' P_f) \\
&\quad + \omega_0 \left( Q_\eta - \frac{1}{4} \Phi^2 \right)^{1/2} + M \left( E^x - \frac{1}{4} \Phi^2 \right). \quad (45)
\end{align*}

We can see from here that the total Hamiltonian is just the sum of four constraints as is expected for a totally constrained system. The first constraint multiplied by the lapse function \( N \) is the Hamiltonian constraint. The other that is multiplied by the shift vector \( N^1 \) is the diffeomorphism constraint. The constraint that is multiplied by \( \omega_0 \) is
the Gauss constraint and the last one is the one we got from the definition of the momentum $P_\eta$. Solving the Gauss constraint in the above Hamiltonian and substituting the resultant $Q_\eta$ from it back into the Hamiltonian yields

$$H = N \left( \frac{\Phi \Phi'}{2E^\varphi} - \frac{\Phi \Phi' E^{\varphi'}}{2E^{\varphi'}} - \frac{1}{2} E^\varphi \lambda^2 \Phi^2 - K_\varphi K_x - \frac{P_\Phi^2}{2E^\varphi} + \frac{(f')^2}{2E^\varphi} + \frac{P_f^2}{2E^\varphi} \right)$$

$$+ N^1 \left( \frac{1}{2} \Phi \Phi' K_x + \Phi' P_\Phi + f' P_f + \Phi' P_\varphi + \Phi' P_f + \Phi' P_\varphi + \Phi' P_f + \frac{1}{4} \Phi^2 K_x' \right).$$

(46)

Since we now know our canonical variables and momenta, we can write their Poisson brackets as

$$\{K_x(x), E^\varphi(y)\} = \delta(x - y),$$

(47)

$$\{K_\varphi(x), E^\varphi(y)\} = \delta(x - y),$$

(48)

$$\{\Phi(x), P_\Phi(y)\} = \delta(x - y),$$

(49)

$$\{f(x), P_f(y)\} = \delta(x - y),$$

(50)

and we have not written the Poisson bracket of $(Q_\eta, \eta)$ pair because they no longer appear in the Hamiltonian. The rest of the Poisson brackets are strongly zero.

V. THE CONSISTENCY CONDITIONS ON CONSTRAINTS

Following the Dirac procedure, we should check the preservation of the constraints to see if there are any new secondary constraints and/or to find the value of the Lagrange multipliers in terms of canonical variables. This means that the constraints $C$, being the constants of motion, should remain weakly vanishing during the evolution

$$\dot{C} = \{C, H\} \approx 0,$$

(51)

where $\approx$ represents weak inequality. The Poisson brackets of the Hamiltonian and diffeomorphism constraints with $H$ vanishes weakly. Let’s check the consistency of the constraint

$$\mu = E^\varphi - \frac{1}{4} \Phi^2.$$  

(52)

For this, we need (47) and (49). The preservation condition of $\mu$ constraint leads to a new, and by definition secondary, constraint which we call $\alpha$:

$$\dot{\alpha} = \{\mu, H\} \approx 0 \Rightarrow \alpha = K_\varphi + \frac{1}{2} P_\Phi \Phi \approx 0.$$  

(53)

We also need to check the preservation of the new $\alpha$ constraint. This leads to a relation between the Lagrange multipliers $N$, $N^1$ and $M$ (and canonical variables). Finding $M$ from this relation and substituting it into the total Hamiltonian [46] yields

$$H = N \left( - K_\varphi K_x - \frac{2E^\varphi \Phi'}{2E^{\varphi'}} + \frac{2E^\varphi \Phi'}{2E^{\varphi'}} - \frac{P_\Phi^2}{2E^\varphi} - \frac{2E^\varphi \Phi'}{2E^{\varphi'}} - \frac{1}{2} E^\varphi \lambda^2 \Phi^2 - \frac{1}{2} \frac{\Phi P_\Phi K_x}{E^\varphi} + \frac{2P_\Phi K_x E^\varphi}{\Phi E^{\varphi'}} \right)$$

$$+ \frac{2E^\varphi P_f^2}{\Phi^2 E^{\varphi'}} - \frac{2E^\varphi E^\varphi}{\Phi^2 E^{\varphi'}} + \frac{2E^\varphi}{2E^\varphi + \frac{2E^\varphi}{\Phi^2 E^{\varphi'}}}$$

$$+ N^1 \left( - \frac{1}{2} \Phi \Phi' K_x + \Phi' P_\Phi + f' P_f + E^\varphi K_x' + E^\varphi K_\varphi' - \frac{1}{4} \Phi^2 K_x' \right).$$

(54)

Next step is to check if the constraints are first class or second class. Calculating the Poisson brackets of constraints among themselves shows that $\mu$ and $\alpha$ are second class and do not commute with each other. In other words, their Poisson bracket with each other does not vanish weakly. Since there are second class constraints in the theory, now we should abandon the Poisson bracket and move on to the Dirac bracket and also put the second class constraint strongly equal to zero and eliminate some of the variables in term of others. By doing this, we can get rid of the $(\Phi, P_\Phi)$ pair in the Hamiltonian as can be seen below. Equating both the $\mu$ and $\alpha$ constraints strongly to zero yields

$$\mu = 0 \Rightarrow \Phi = 2\sqrt{E^\varphi},$$

(55)

$$\alpha = 0 \Rightarrow P_\Phi = -\frac{K_x E^\varphi}{\sqrt{E^\varphi}}.$$  

(56)
Using this and (58), the general form of the Dirac bracket for our theory becomes

\[ H = N \left(-K_ϕ K_x - \frac{E^{x'} E^{x''}}{E^ϕ} + \frac{1}{2} E^{x''} + \frac{E^{x''}}{E^ϕ} - \frac{1}{2} K_ϕ^2 \right) \frac{E^ϕ}{E^x} - 2E^ϕ E^x χ^2 + \frac{1}{2} \frac{P_ϕ}{E^ϕ} + \frac{1}{2} f^2 \]

+ \( N^1 \left(-K_ϕ E^{x'} + f' P_ϕ - \frac{K_ϕ E^{x'} E^{x''}}{E^ϕ} + E^ϕ K'_ϕ \right) \).

(57)

where now we are only left with a Hamiltonian and a diffeomorphism constraint.

VI. DIRAC BRACKET AND THE ALGEBRA OF CANONICAL VARIABLES

In order to switch to the Dirac bracket, we need to find the general form of the Dirac bracket for our theory. For a field theory (where the variables have continuous indices), the Dirac bracket is

\[ \{A(x), B(y)\}_D = \{A(x), B(y)\} - \int dw \int dz \{A(x), χ_ρ(w)\} C^{ρσ}(w, z) \{χ_σ(z), B(y)\} \],

(58)

where the \( \{, \}_D \) refers to the Dirac bracket, \( χ \)'s are the second class constraints and \( C^{ρσ}(w, z) \) are the elements of the inverse of the matrix of the Poisson brackets between the \( ρ \)'th and \( σ \)'th second class constraints

\[ C^{ρσ}(w, z) = \{χ_ρ(w), χ_σ(z)\}. \]

(59)

In our model, there are only two second class constraints, \( μ \) and \( α \). Thus the matrix of the Poisson brackets of the second class constraints will be

\[ C = C^{ρσ}(x, y) = \begin{pmatrix} \{μ(x), μ(y)\} & \{μ(x), α(y)\} \\ \{α(x), μ(y)\} & \{α(x), α(y)\} \end{pmatrix} = \begin{pmatrix} 0 & \{μ(x), α(y)\} \\ \{α(x), μ(y)\} & 0 \end{pmatrix}. \]

(60)

To compute the elements of this matrix we use (52) and (53) along with (49) to get

\[ \{μ(x), α(y)\} = \left\{ E^{x'}(x) - \frac{1}{4} Φ(x)^2, K_ϕ(y) + \frac{1}{2} \frac{P_ϕ(y) Φ(y)}{E^ϕ(y)} \right\} \]

\[ = - \frac{1}{8 E^ϕ(y)} \{Φ(x)^2, P_ϕ(y)\} \]

\[ = - \frac{1}{4 E^ϕ(y)} Φ^2 δ(x - y). \]

(61)

The same method of computations gives

\[ \{μ(x), α(y)\} = \frac{1}{4} \frac{Φ^2(x)}{E^ϕ(x)} δ(x - y). \]

(62)

To calculate the elements of \( C^{-1} \), we use the property \( CC^{-1} = 1 \), or in terms of their elements

\[ \int C^{ρσ}(x, z) C^{σβ}(z, y) dz = δ_ρ^β δ(x - y), \]

(63)

which yields

\[ C^{-1} = C^{ρσ}(x, y) = \begin{pmatrix} 0 & \frac{4E^ϕ(x)}{Φ^2(x)} \\ -\frac{4E^ϕ(x)}{Φ^2(x)} & 0 \end{pmatrix} δ(x - y). \]

(64)

Using this and (53), the general form of the Dirac bracket for our theory becomes

\[ \{A(x), B(y)\}_D = \{A(x), B(y)\} + \int dw \int dz \left( \{A(x), μ(w)\} \frac{4E^ϕ(w)}{Φ^2(w)} δ(w - z) \{μ(z), B(y)\} \right) - \int dw \int dz \left( \{A(x), α(w)\} \frac{4E^ϕ(w)}{Φ^2(w)} δ(w - z) \{α(z), B(y)\} \right). \]

(65)
If we use this formula and the Poisson brackets (67)-(69), we can find the Dirac brackets of the canonical variables between each other as

\[
\{K_x(x), E^\varphi(y)\}_D = \{K_\varphi(x), E^\varphi(y)\}_D = \{f(x), P_f(y)\}_D = \delta(x - y),
\]

\[
\{K_x(x), K_\varphi(y)\}_D = \frac{K_\varphi}{E^\varphi} \delta(x - y),
\]

\[
\{K_x, E^\varphi\}_D = -\frac{E^\varphi}{E^x} \delta(x - y),
\]

\[
\{E^x, K_\varphi\}_D = \{E^\varphi, E^\varphi\}_D = \{f, \bullet\}_D = \{P_f, \bullet\}_D = 0,
\]

where \(\bullet\) means everything except \(P_f\) and the \(\bullet\) means everything except \(f\).

Looking at the above Dirac brackets, we can make an important observation: by introducing a new variable

\[
U_x = K_x + \frac{E^\varphi K_\varphi}{E^x},
\]

the Dirac brackets (66)-(69) can be brought to the standard from

\[
\{U_x(x), E^\varphi(y)\}_D = \{K_\varphi(x), E^\varphi(y)\}_D = \{f(x), P_f(y)\}_D = \delta(x - y),
\]

with other brackets being zero. This is an important step since it makes Dirac brackets look like the standard form of the Poisson brackets of canonical pairs. Using (70), the total Hamiltonian (57) becomes

\[
H = N \left( -K_\varphi U_x - \frac{E^\varphi E^\varphi'}{E^x} - \frac{1}{2} K_\varphi^2 E^\varphi E^x + \frac{E^\varphi E^\varphi'}{E^x} + \frac{1}{2} K_\varphi^2 E^\varphi E^x - \frac{1}{2} E^\varphi E^x \lambda^2 + \frac{1}{2} P_f^2 + \frac{1}{2} f^2 \right)
\]

\[
+ N^1 \left( -U_x E^\varphi + f' P_f + E^\varphi K_\varphi' \right).
\]

**VII. TRANSFORMING THE VACUUM HAMILTONIAN CONSTRAINT INTO A TOTAL DERIVATIVE**

We can omit \(U_x\) in the Hamiltonian constraint in (72) by a redefinition of the shift

\[
\overline{N} = N^1 + \frac{NK_\varphi}{E^x}.
\]

Substituting this into (72) gives

\[
H = N \left( -\frac{E^\varphi E^\varphi'}{E^x} - \frac{1}{2} \frac{E^\varphi E^\varphi'}{E^x} + \frac{E^\varphi E^\varphi'}{E^x} + \frac{1}{2} K_\varphi^2 E^\varphi E^x - \frac{1}{2} E^\varphi E^x \lambda^2 + \frac{1}{2} P_f^2 + \frac{1}{2} f^2 \right)
\]

\[
+ \overline{N}^1 \left( -U_x E^\varphi + f' P_f + E^\varphi K_\varphi' \right).
\]

Now redefining the lapse in the above total Hamiltonian as

\[
\overline{N} = N \frac{E^\varphi E^x}{E^\varphi'}
\]

will yield

\[
H_T = \overline{N} \left( \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{E^\varphi E^\varphi'}{E^x} - 2 E^\varphi \lambda^2 - \frac{1}{2} K_\varphi^2 \right) - \frac{f' P_f K_\varphi}{E^\varphi E^x} + \frac{1}{2} \frac{P_f^2 E^\varphi E^x}{2 E^x E^\varphi} + \frac{1}{2} \frac{f^2 E^\varphi}{2 E^x E^\varphi} \right)
\]

\[
+ \overline{N}^1 \left( -U_x E^\varphi + f' P_f + E^\varphi K_\varphi' \right).
\]

The Hamiltonian constraint above

\[
\mathcal{H} = \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{E^\varphi E^\varphi'}{E^x} - 2 E^\varphi \lambda^2 - \frac{1}{2} K_\varphi^2 \right) - \frac{f' P_f K_\varphi}{E^\varphi E^x} + \frac{1}{2} \frac{P_f^2 E^\varphi E^x}{2 E^x E^\varphi} + \frac{1}{2} \frac{f^2 E^\varphi}{2 E^x E^\varphi},
\]

will yield
has a remarkable property: its complete form or its vacuum form \((f = 0 = P_f)\) both have a strong Abelian algebra with itself, namely

\[
\{ \mathcal{H}(x), \mathcal{H}(y) \}_D = 0, \tag{78}
\]

\[
\left\{ \mathcal{H}(x) \bigg|_{f=0, P_f=0}, \mathcal{H}(y) \bigg|_{f=0, P_f=0} \right\}_D = 0. \tag{79}
\]

This way the algebra of the constraints becomes a Lie algebra since now we have structure constants instead of structure functions.

\section{VIII. Equations of Motion}

We are going to write the equations of motion in two equivalent cases, before and after rescaling lapse and shift and introducing \(U_x\). The reason is that to compare the original CGHS equations of motion with our formulation, it is much easier to use the Hamiltonian equations of motion before rescaling lapse and shift. This is because in order to be able to compare equations of motion between our formulation and the original Lagrangian formulation, we need to find the second order equations of motion from the Hamiltonian equations of motion. This is achieved much easier using explicit form of the Hamiltonian before rescaling.

\subsection{A. Before introducing \(U_x\) and rescaling \(N\) and \(N^1\)}

Here we can use the Hamiltonian \(57\) and the Dirac brackets \(66, 69\) to find the equations of motion, \(\dot{F} = \{ F(x), \int dy H(y) \}_D\), for any function of the phase space \(F\). Using these, one gets for the canonical pairs

\[
\dot{K}_x = N \left( -\frac{K_x K_x}{E^x} + \frac{1}{2} \frac{f'^2}{E^x E^x} + \frac{1}{2} \frac{P^2_p}{E^x E^x} + \frac{E'^{x''}}{E^x E^x} - \frac{E^{x'} E^0}{E^x E^2} - \frac{E^{x''} E^0}{E^2 E^x} \right) \\
+ (N^1 K_x)' + \frac{N'^{x''}}{E^x E^x}, \tag{80}
\]

\[
\dot{E}_x = N K_x + N^1 E^{x'}, \tag{81}
\]

\[
\dot{K}_x = N \left( -\frac{1}{2} \frac{f'^2}{E^x E^x} - \frac{1}{2} \frac{P^2_p}{E^x E^x} + \frac{1}{2} \frac{E^{x'}}{E^x E^x} + \frac{1}{2} \frac{K_x^2}{E^x E^x} + \frac{1}{2} \frac{E^{x''}}{E^x E^x} \right) \\
+ \frac{N' E^{x'}}{E^x E^x} + N K_x', \tag{82}
\]

\[
\dot{E}_x = N K_x + N^1 E^{x'} + N^1 E^{x''}, \tag{83}
\]

\[
\dot{f} = N P_f + N^1 f', \tag{84}
\]

\[
\dot{P}_f = \left( N \frac{f'}{E^x} + N^1 P_f \right)' . \tag{85}
\]

\subsection{B. After introducing \(U_x\) and rescaling \(N\) and \(N^1\)}

In this case, we can use the total Hamiltonian \(76\) and the brackets \(71\) to calculate the equations of motion, \(\dot{F} = \{ F(x), \int dy H_T(y) \}_D\). For the canonical pairs we get:

\[
\dot{U}_x = \bar{N} \left( -\frac{K_x P_f f'}{E^x E^x E^x} - \frac{f'^2}{E^x E^x E^x} - \frac{P^2_p}{E^x E^x E^x} + \frac{E'^{x'}(P^2_p + f'^2)}{E^x E^x E^x} \right) \\
+ \bar{N}' \left( -\frac{1}{2} \frac{P^2_p + f'^2}{E^x E^x E^x} - \frac{2}{E^x E^x E^x} - \frac{1}{2} \frac{K_x^2}{E^x E^x E^x} + \frac{E^{x''}}{E^x E^x E^x} + \frac{1}{2} \frac{E^{x''}}{E^x E^x E^x} + 2 \lambda^2 \right),
\]
\[ N'' \left( \frac{E'^t}{E^x E^\varphi^2} \right) + (\bar{N}^1 U_x)', \quad (86) \]

\[ \dot{E}_x = \bar{N}^1 E'^x, \quad (87) \]

\[ K_\varphi = \bar{N} \left( - \frac{E'^t \left( P_j^2 + f'^2 \right)}{E^x E^\varphi^3} + \frac{f' P_j K_\varphi^2}{E^x E^\varphi^2} \right) + \bar{N}^1 \frac{E'^x}{E^x E^\varphi^3} + \bar{N}^1 K_\varphi', \quad (88) \]

\[ \dot{E}_\varphi = \bar{N} \frac{P_j f'}{E^x E^\varphi^2} - \bar{N}^1 K_\varphi + (\bar{N}^1 E^x)', \quad (89) \]

\[ \dot{f} = \bar{N} \left( \frac{P_j E'^x}{E^x E^\varphi^2} - \frac{K_\varphi f'}{E^x E^\varphi^2} \right) + \bar{N}^1 f', \quad (90) \]

\[ \dot{P}_j = \left[ \bar{N} \left( \frac{E'^t f'}{E^x E^\varphi^2} - \frac{K_\varphi P_j}{E^x E^\varphi^2} \right) + \bar{N}^1 P_j \right]'. \quad (91) \]

**IX. GAUGE FIXING THE VACUUM THEORY**

Using (76), the vacuum Hamiltonian (with \( f = P_j = 0 \)) after an integration by parts can be written as

\[ H_0 = \bar{N}' H_d + \bar{N}^1 D \]

\[ = \bar{N} \left( \frac{1}{2} \frac{E'^x}{E^x E^\varphi^3} - 2E^x \chi^2 - \frac{1}{2} \frac{K_\varphi^2}{E^x} \right) + \bar{N}^1 \left( -U_x E'^t + E^x K_\varphi' \right), \quad (92) \]

where the prime sign on \( \bar{N}' \) can be taken to mean that we have a new lapse after integration by parts or to show that this lapse is the derivative of the previous lapse \( \bar{N} \) with respect to \( x \). To completely solve the vacuum classical theory, one can use two gauge fixings. We choose the first gauge fixing with an explicit coordinate dependence as

\[ \chi_1 = E^x(x) - e^{-2\chi_x} \approx 0. \quad (93) \]

Since this gauge fixing does not contain any explicit time dependence, its preservation condition in time gives

\[ \dot{\chi}_1(x) \approx 0 \]

\[ \int dy \{ \chi_1(y), H_0(y) \}_D \approx 0 \]

\[ \bar{N}'(x) E'(x) \approx 0, \quad (96) \]

which implies that the shift (not the original shift but the rescaled one) vanishes,

\[ \bar{N}' = 0. \quad (97) \]

The motivation for the gauge fixing (93) is the following: the coordinate transformations

\[ \bar{x}^+ = e^{\lambda(t+x)} \]

\[ \bar{x}^- = -e^{-\lambda(t-x)} \]

will lead us to the formulation of the theory in the conformal gauge in the original CGHS paper [1], where \( (\bar{x}^-, \bar{x}^+) \) are the null coordinates in that gauge. Now if we use these transformations in (93), the form of \( E^x \) will becomes exactly the same as the form it gets in the vacuum case in the conformal gauge in [1].

We choose our next gauge fixing as

\[ \chi_2 = E^\varphi(x) - 1 \approx 0. \quad (100) \]

The reason behind this gauge fixing is the observation that in the conformal gauge, \( E^\varphi \) is the only independent metric component and upon using the coordinate transformations (93) and (98), the vacuum metric becomes flat (i.e.
$E^\varphi = 1$). Preserving the $\chi_2$ constraint gives
\begin{align}
\dot{\chi}_2(x) & \approx 0 \quad (101) \\
\int dy \{ \chi_2(x), H_1(y) \}_D & \approx 0 \quad (102) \\
\overline{N}(x)K_\varphi(x)e^{2\lambda x} & \approx 0 \quad (103)
\end{align}
which yields
\begin{equation}
\overline{N}(x) = 0. \quad (104)
\end{equation}
This is not an issue since this is the derivative of the transformed original lapse. Up to now, we have $\overline{N}$, $\overline{N}'$, $E^x$ and $E^\varphi$ explicitly. Now we can solve for $K_\varphi$ from the weakly vanishing of the Hamiltonian constraint in (92). This, together with (93) and (100) yields
\begin{equation}
K_\varphi = 0. \quad (105)
\end{equation}
Finally $U_x$ can be found from the weakly vanishing the diffeomorphism constraint as
\begin{equation}
U_x = \frac{E^\varphi K'}{E^{ex}}, \quad (106)
\end{equation}
which upon substituting the relevant values of the variables from above gives
\begin{equation}
U_x = 0. \quad (107)
\end{equation}
This way, the vacuum case is completely solved classically.

**A. The original lapse and shift**

Using the results we just obtained for the vacuum case, we can express the original lapse and shift in (72) in terms of the ones we obtained using the gauge fixings. Using (74), (104), (93) and (100) we get
\begin{align}
\overline{N}' = 0 \quad (108) \\
\left( N \frac{E^\varphi E^x}{E^{ex}} \right)' = 0 \quad (109) \\
N' = 0, \quad (110)
\end{align}
which implies
\begin{equation}
N = g(t) + C, \quad (111)
\end{equation}
for which we can simply choose
\begin{equation}
N = 1. \quad (112)
\end{equation}
Using (73), (97), (98) and (100) we find
\begin{align}
\overline{N}' = 0 \quad (113) \\
N'(x) + \frac{N(x)K_\varphi}{E^{ex}} = 0 \quad (114) \\
N'(x) = 0. \quad (115)
\end{align}
X. GAUGE FIXING THE CASE COUPLED TO MATTER

The above gauge fixings which we used for the vacuum case will not work for the case with matter field. Among other reasons, one relatively obvious reason is that the two gauge fixings (93) and (100), are related to the form that $E^x$ and $E^\phi$ take in the vacuum case. Also we can not turn the whole Hamiltonian constraint into a total derivative and therefore if we get a vanishing lapse, it would be the original lapse itself that vanishes not its derivative. By inspecting the total Hamiltonian (76), one can see that perhaps a good choice for the first gauge fixing is

$$\zeta_1 = E^x - h(x) \approx 0.$$  \hspace{1cm} (116)

with $h(x)$ an arbitrary function of $x$ coordinate. Since the only nonvanishing Dirac bracket between $\zeta_1$ and $H_T$ in (76) comes from the term in the diffeomorphism constraint containing $U_x$, and since $U_x$ appears without derivatives there, the preservation of $\zeta_1$ will give us an algebraic equation for $\bar{N}^1$. In more precise way we have

$$\dot{\zeta}_1 = \{\zeta_1(x), \int dy H_T(y)\} \approx 0$$  \hspace{1cm} (117)

$$- \int dy \bar{N}^1(y) \{E^\phi(x), U_x(y)\} E^\phi(y) \approx 0$$  \hspace{1cm} (118)

$$\int dy \bar{N}^1(y) E^\phi(y) \delta(x - y) \approx 0$$  \hspace{1cm} (119)

$$\bar{N}^1(x) E^\phi(x) \approx 0,$$  \hspace{1cm} (120)

which means that the shift vanishes,

$$\bar{N}^1 = 0.$$  \hspace{1cm} (121)

Now, from (76), (116) and (121), the partially gauge fixed total Hamiltonian will become

$$H_{TF} = \bar{N} \left( \frac{\partial}{\partial x} \left( \frac{1}{2} \frac{h'^2}{h E^\phi} - 2h^2 \frac{K^2}{2h} \right) - \frac{f'}{h E^\phi} K^2 h^2 + \frac{1}{2} \frac{P^2 h'}{2h E^\phi} + \frac{1}{2} \frac{h'^2}{h E^\phi} \right).$$  \hspace{1cm} (122)

For the second gauge fixing, we follow the procedure suggested in [14, 15]. The basic idea is to make a canonical transformation and define a true Hamiltonian that gives the correct equations of motion. To start, we identify the terms inside the total derivative of the Hamiltonian constraint in (122) as our new canonical variable $X$,

$$X = \frac{1}{2} \frac{h'^2}{h E^\phi} - 2h^2 \frac{K^2}{2h},$$  \hspace{1cm} (123)

This suggests that we can find a generating function of the third kind as

$$F_3(p, Q) = F_3(K^\phi, X),$$  \hspace{1cm} (124)

by means of the equation

$$E^\phi(X, K^\phi) = \frac{\partial F_3(K^\phi, X)}{\partial K^\phi}.$$  \hspace{1cm} (125)

One can find $E^\phi$ from (123), substitute it into the above and integrate with respect to $K^\phi$ to get

$$F_3(K^\phi, X) = h' \ln (K^\phi + \Omega),$$  \hspace{1cm} (126)

with

$$\Omega = \sqrt{K^\phi + 2hX + 4\lambda^2 h^2}.$$  \hspace{1cm} (127)

Now $P_X$, the momentum conjugate to $X$, can be found as

$$P_X = \frac{\partial F_3(K^\phi, X)}{\partial X} = -\frac{h h'}{\Omega(K^\phi + \Omega)}.$$  \hspace{1cm} (128)
The above equation will be the Hamiltonian constraint after writing $K\phi$ in terms of $X, f$ and $P_f$. To do this, we find $E^{\phi}$ from (123), substitute it in (122), find $K\phi$ from vanishing of it and substitute the resulting $K\phi$ into (128). This way, our total Hamiltonian will be

$$H_{tot} = \tilde{N} P_X + H_{true} = \tilde{N} \left( P_X + \frac{hh'}{\Omega(X,f,P_f)(K\phi(X,f,P_f) + \Omega(X,f,P_f))} \right).$$

(129)

The next step is to introduce the second gauge condition as

$$\zeta_2 = X - b(x,t) \approx 0,$$

(130)

with $b(x,t)$ a function of the coordinates. Since in (129) $P_X$ only appears in the first term in the parenthesis, the preservation of the above constraint

$$\dot{\zeta}_2 = \{\zeta_2(x), \int dy H_{tot}(y)\} + \frac{\partial b(x,t)}{\partial t} \approx 0$$

(131)

gives

$$\tilde{N} = \dot{b}.$$

(132)

Because $f$ and $P_f$ commute with $P_X$, the evolution equations will be

$$\dot{f} = \left\{ f(x), \int dy H_{tot}(y) \right\} = \left\{ f(x), \int dy H_{true}(y) \right\},$$

(133)

$$\dot{P}_f = \left\{ P_f(x), \int dy H_{tot}(y) \right\} = \left\{ P_f(x), \int dy H_{true}(y) \right\},$$

(134)

with the true Hamiltonian being

$$H_{true} = \dot{b} \left( \frac{hh'}{\Omega(f,P_f)(K\phi(f,P_f) + \Omega(f,P_f))} \right).$$

(135)

The true Hamiltonian (135) is a local Hamiltonian density in the sense that $\tilde{N}$ has not been given in terms of an integral of canonical variables. This happens thanks to the present method of gauge fixing which gives an algebraic equation for lapse instead of a differential equation. If $\tilde{N}$ was given in terms of an integral of canonical variables, then the Hamiltonian would have been given in terms of an integral of an integral and would have been nonlocal in that sense.

This local true Hamiltonian gives the correct equations of motion for our system as it can be checked by comparing the equations (133) and (134) with the equations of motion derived before gauge fixing in section (VIII B), and then substituting in them the gauge fixing conditions.

XI. COMPARE WITH THE ORIGINAL LAGRANGIAN THEORY

In order to make a connection with the original Lagrangian formulation of the CGHS and also to check the consistency of our formulation, we transform our equations of motion into the ones in the null coordinates in the conformal gauge and compare them to the original Lagrangian ones.

A. Metric and other variables in null coordinates

The first step to make a connection between the two formalisms is finding the relations between the form of the metric and other canonical variables in both formulations. The coordinates used in the original CGHS formulation are the null coordinates

$$x^+ = x^0 + x^1,$$

(136)

$$x^- = x^0 - x^1,$$

(137)
where $x^0 = t$ and $x^1 = x$ are the coordinates used here up to now. We can find the relation between the components of the null and non-null metrics using the general transformation

$$ g_{ab} = \frac{\partial x^a'}{\partial x^a} \frac{\partial x^b'}{\partial x^b} \bar{g}_{ab}' . $$

(138)

In the original CGHS model, the metric components in the conformal gauge are

$$ \bar{g}_{++} = - \frac{1}{2} e^{2\rho} , $$

(139)

$$ \bar{g}_{--} = \bar{g}_{+-} = 0 . $$

(140)

Thus the relations between the components in two coordinate systems are

$$ g_{00} = 2\bar{g}_{+-} = - e^{2\rho} , $$

(141)

$$ g_{11} = - 2\bar{g}_{+} = e^{2\rho} , $$

(142)

$$ g_{01} = g_{10} = g_{++} - g_{--} = 0 . $$

(143)

The relations between the partial derivatives in the two coordinates thus become

$$ \partial_t = \partial_+ + \partial_-, $$

(144)

$$ \partial_x = \partial_+ - \partial_-, $$

(145)

$$ \partial_t \partial_t = \partial_+ \partial_+ + \partial_- \partial_- + 2 \partial_+ \partial_-, $$

(146)

$$ \partial_x \partial_x = \partial_+ \partial_+ + \partial_- \partial_- - 2 \partial_+ \partial_-, $$

(147)

$$ \partial_t \partial_x = \partial_+ \partial_+ - \partial_- \partial_- . $$

(148)

We can also write $E^x$ in terms of the dilaton field. Using the relation between $\Phi$ and $\phi$ (the dilaton field in original CGHS paper)

$$ \Phi = 2 \sqrt{2} e^{-\phi} . $$

(149)

and substituting it into the $\mu$ constraint (52) and equating this second class constraint strongly to zero, we get

$$ E^x = \frac{1}{4} \Phi^2 = 2 e^{-2\phi} . $$

(150)

The variable $E^\varphi$ can also be written as

$$ E^\varphi = \frac{|P|}{2} = \sqrt{q} = \sqrt{q_{11}} = \sqrt{\bar{g}_{11}} = \sqrt{2\bar{g}_{+-}} = \sqrt{e^{2\rho}} = e^\rho , $$

(151)

where we have used the canonical transformation (34) and the fact that $q_{ab}$ has only one independent component so that $q = q_{11}$. The details of these calculations can be found in [18]. Using the ADM formalism, the form of the metric components in 2D can be expressed as

$$ g_{00} = - N^2 + q_{11} (N^1)^2 , $$

(152)

$$ g_{11} = q_{11} , $$

(153)

$$ g_{01} = - q_{11} N^1 . $$

(154)

From this, one can find $N$ and $N^1$ in terms of metric components as

$$ N^1 = - \frac{g_{01}}{q_{11}} = - \frac{g_{01}}{g_{11}} , $$

(155)

$$ N = \sqrt{q_{11} (N^1)^2 - g_{00}} = \sqrt{\frac{g_{01}^2}{q_{11}} - g_{00}} = \sqrt{\frac{g_{01}^2}{g_{11}} - g_{00}} . $$

(156)

Substituting (141)-(143) in the above two equations and using (151) yields

$$ N^1 = 0 , $$

(157)

$$ N = \sqrt{-g_{00}} = e^\rho = E^\varphi . $$

(158)

Now we are ready to compare the equations of motion.
B. The equations of motion in null coordinates

In order to compare our equations of motion with the original ones in the CGHS paper, we need to transform ours into the second order form and then bring them into the null form. Starting from the equations for the matter field and its conjugate, if we find \( P_f \) from (54), substitute it into (55) and then use (144)-(148), (151), (157) and (158) we get

\[
\partial_+ \partial_- f = 0. \tag{159}
\]

Finding \( K_\phi \) from (51) and substituting it in (52) and using (144)-(148), (150), (151), (157) and (158) yields

\[
V_1 = e^{-\rho}e^{-2\phi} (e^{2\phi} \rho + (\partial_- f)^2 + (\partial_- f)^2) + 4e^{2\rho} \lambda^2 - 4\partial_+^2 \phi - 4\partial_-^2 \phi - 8\partial_+ \partial_- \phi + 8\partial_+ \rho \partial_+ \phi + 8\partial_- \rho \partial_- \phi = 0. \tag{160}
\]

For the next second order equation, we find \( K_\phi \) from (53) and substitute it in (50). Then upon using (144)-(148), (150), (151), (157), (158) and the values of \( P_f \) and \( K_\phi \) from (54) and (51) respectively, we get

\[
V_2 = \frac{1}{2} e^{2\phi} \rho + (\partial_- f)^2 + (\partial_- f)^2 + 2\partial_+^2 \phi + 2\partial_-^2 \phi - 4\partial_+ \partial_- \phi + 4\partial_+ \rho \partial_+ \phi - 4\partial_- \rho \partial_- \phi = 0. \tag{161}
\]

We can follow the same procedure and find the Hamiltonian and diffeomorphism constraints in (51) in the null coordinate as

\[
\mathcal{H} = e^{-\rho}e^{-2\phi} (e^{2\phi} \rho + (\partial_- f)^2 + (\partial_- f)^2) - 4e^{2\rho} \lambda^2 - 4\partial_+^2 \phi - 4\partial_-^2 \phi + 8\partial_+ \partial_- \phi + 8\partial_+ \rho \partial_+ \phi + 8\partial_- \rho \partial_- \phi = 0, \tag{162}
\]

\[
\mathcal{D} = e^{-2\phi} (8\partial_+ \partial_- \phi - 8\partial_- \rho \partial_- \phi - 4\partial_+^2 \phi + 4\partial_-^2 \phi - (\partial_- f)^2) = 0. \tag{163}
\]

C. Identifying our equations of motion with those of the CGHS paper

If we compare the above equations with the equations of motion of the original CGHS model, we can note the following:

The matter field equation is identically the same in both methods and is given by (160). The original energy momentum equations, \( T_{++} = 0 \) and \( T_{--} = 0 \) in [1] are combined in the diffeomorphism constraint (162) as

\[
T_{++} - T_{--} = \frac{1}{2} \mathcal{D} = \left[ e^{-2\phi} (4\partial_+ \rho \partial_+ \phi - 2\partial_+^2 \phi) + \frac{1}{2} (\partial_+ f)^2 \right] - \left[ e^{-2\phi} (4\partial_- \rho \partial_- \phi - 2\partial_-^2 \phi) + \frac{1}{2} (\partial_- f)^2 \right] = 0. \tag{164}
\]

The original \( T_{+-} = 0 \) equation can be obtained by combining the Hamiltonian constraint equation (162) and equation (160) as following:

\[
T_{+-} = -\frac{e^\rho}{8} (V_1 - \mathcal{H}) = e^{-2\phi} (2\partial_+ \rho \partial_- \phi - 4\partial_+ \phi \partial_- \phi - \lambda^2 e^{2\rho}) = 0, \tag{165}
\]

and finally the dilaton field equation of motion in CGHS paper is obtained by combining \( V_1 \) and \( V_2 \) i.e. equations (160) and (161) as

\[
\frac{e^\rho e^{2\phi}}{4} V_1 + \frac{1}{2} V_2 = -4\partial_+ \rho \partial_+ \phi + 4\partial_+ \phi \partial_- \phi + 2\partial_+ \partial_- \phi + \lambda^2 e^{2\rho} = 0. \tag{166}
\]
XII. BOUNDARY TERMS

It is important to take care of the boundary conditions in our theory \[19\]. One important reason is that energy in general relativity is related to the surface integral or boundary term at infinity \[3\].

We require that both Hamiltonian and diffeomorphism constraints

\[
H_c = \int dx N \mathcal{H},
\]

\[
D_c = \int dx N^1 \mathcal{D},
\]

be functionally differentiable. This means that if the variations of \(H_c\) and \(D_c\) leads to

\[
\delta H_c = \int_{-\infty}^{\infty} dx N \delta \mathcal{H} = \int_{-\infty}^{\infty} dx \left( \frac{\delta H_c}{\delta q^i} \delta q^i + \frac{\delta H_c}{\delta p^i} \delta p^i \right) + \int_{-\infty}^{\infty} dx \partial_x (\delta S_H), \tag{169}
\]

\[
\delta D_c = \int_{-\infty}^{\infty} dx N^1 \delta \mathcal{D} = \int_{-\infty}^{\infty} dx \left( \frac{\delta D_c}{\delta q^i} \delta q^i + \frac{\delta D_c}{\delta p^i} \delta p^i \right) + \int_{-\infty}^{\infty} dx \partial_x (\delta S_D), \tag{170}
\]

then for \(H_c\) and \(D_c\) to be functionally differentiable, we need to add \(-\delta S_H\) and \(-\delta S_D\) to them respectively. This means that the overall variation of surface term that should be added to the variation of the action is

\[
\delta S_{\text{surface}} = -\delta \int dt (S_H + S_D), \tag{171}
\]

and clearly the surface term to be added to the action will be

\[
S_{\text{surface}} = -\int dt (S_H + S_D). \tag{172}
\]

In the language of formulation of the CGHS that has been presented so far, i.e. using the total Hamiltonian \[76\], the terms that obstruct functional differentiability for the diffeomorphism constraint turn out to be

\[
\partial_x (\delta S_D) = \partial_x \left[ \tilde{N}^1 (E^\varphi \delta K_\varphi + P_f \delta f - U_x \delta E^x) \right] \tag{173}
\]

and the corresponding terms for Hamiltonian constraint are

\[
\partial_x (\delta S_H) = \partial_x \left[ \tilde{N} \left( \frac{K^2}{2 E^x} \delta E^x + \frac{E^{\varphi\varphi} E^x}{E^\varphi} \delta E^x + \frac{1}{2} \frac{E^{\varphi f} E^x}{E^\varphi} \delta E^x + \frac{E^{\varphi f} \delta f}{E^\varphi} \right) - \frac{2 \lambda^2 \delta E^x}{E^\varphi} - \frac{E^{\varphi\varphi} E^x}{E^\varphi} \delta E^x - \frac{1}{2} \frac{E^{\varphi f} \delta E^x}{E^\varphi} - \frac{K_\varphi \delta K_\varphi}{E^\varphi} + \frac{3}{2} \frac{E^{\varphi f} \delta E^x}{E^\varphi} \right] \right]
\]

\[
+ \partial_x \left[ \frac{\tilde{N}}{E^\varphi} \frac{E^{\varphi f} \delta E^x}{E^\varphi} \right]. \tag{174}
\]

---

3 More precisely it is identified as the conserved quantity associated to the invariance of the action under time translations at infinity, i.e. under transformation generated by a timelike killing vector field at infinity.
So the total variation of the surface term is

\[
\delta S_{\text{surface}} = - \int dt \left( \int_{-\infty}^{\infty} dx \partial_x (\delta S_H) + \int_{-\infty}^{\infty} dx \partial_x (\delta S_H) \right)
= - \int dt \left[ \delta S_H + \delta S_D \right]_{-\infty}^{\infty}
= - \int dt \left\{ \tilde{N}^1 (E^x \delta K_\varphi + P_f \delta f - U_\varphi \delta E_x) \right.
\]
\[
\left. + \tilde{N} \left[ \left( \frac{1}{2} \frac{K_\varphi^2}{E^{x2}} + \frac{E^{\varphi''}}{E^{\varphi2} E_x^2} + \frac{1}{2} \frac{f'^2}{E^{\varphi2} E_x^2} - 2\lambda^2 \right)
\right. \right.
\]
\[
\left. - 2 \frac{E^x \delta E_x}{E^\varphi E_x} + 3 \frac{E^{x2}}{2 E^{\varphi2} E_x^2} + \frac{1}{2} \frac{P_f^2}{E^{\varphi2} E_x^2} \right) \delta E_x
\]
\[
\left. + \left( \frac{E^x f'}{E^{\varphi2} E_x} - \frac{K_\varphi P_f}{E^\varphi E_x} \delta f - \frac{E^{x2} \delta E_x}{E^{\varphi2} E_x} - \frac{K_\varphi \delta K_\varphi}{E^\varphi} \right) \right.
\]
\[
\left. \left. + \partial_x \left( \frac{\tilde{N} E_x f'}{E^{\varphi2} E_x} \delta E_x \right) \right) \right\}_{-\infty}^{\infty}. \tag{175}
\]

If we use the prescription at infinity for the matter field

\[
\delta f|_{x=\pm\infty} = 0, \tag{176}
\]
then the variation of the surface term will be

\[
\delta S_{\text{surface}} = - \int dt \left\{ \tilde{N}^1 (E^x \delta K_\varphi - U_\varphi \delta E_x) \right.
\]
\[
\left. + \tilde{N} \left[ \left( \frac{1}{2} \frac{K_\varphi^2}{E^{x2}} + \frac{E^{\varphi''}}{E^{\varphi2} E_x^2} + \frac{1}{2} \frac{f'^2}{E^{\varphi2} E_x^2} - 2\lambda^2 \right)
\right. \right.
\]
\[
\left. - 2 \frac{E^x \delta E_x}{E^\varphi E_x} + 3 \frac{E^{x2}}{2 E^{\varphi2} E_x^2} + \frac{1}{2} \frac{P_f^2}{E^{\varphi2} E_x^2} \right) \delta E_x
\]
\[
\left. - \frac{E^{x2} \delta E_x}{E^{\varphi2} E_x} - \frac{K_\varphi \delta K_\varphi}{E^\varphi} \right) \left. \left. + \partial_x \left( \frac{\tilde{N} E_x f'}{E^{\varphi2} E_x} \delta E_x \right) \right) \right\}_{-\infty}^{\infty}. \tag{177}
\]

For the present gauge fixing, one can arrive at a surface term by substituting the gauge fixings and the values of lapse and shift into the general form of the variation of the total surface term \[177\], and also putting \( \delta h(x) = 0 = \delta b(x,t) \) (since they are just functions of the coordinates):

\[
S_{\text{surface}} = \int dt \left[ \frac{1}{2} \frac{b \left( K_\varphi^2 - \frac{h'^2}{E^{\varphi2}} \right)}{h} \right]_{-\infty}^{\infty}. \tag{178}
\]

Comparing the above term to the standard form of the boundary term of the CGHS \[20\],

\[
S_{\text{surface}} = - \int dt \left[ NM \right]_{-\infty}^{\infty},
\]

where \( M \) is the ADM mass, and since \( \tilde{N} \approx \dot{b} \) by our gauge fixing, one gets

\[
M = \frac{1}{2h} \left( \frac{h'^2}{E^{\varphi2}} - K_\varphi^2 \right).
\]

By using \[123\], the above formula can also be written as

\[
M = X + 2h \lambda^2
= b + 2h \lambda^2
\]

where we used the second gauge fixing in the second line.
XIII. CONCLUSION

We have analyzed the CGHS model without any conformal transformation in terms of new variables similar to the standard Ashtekar variables of the 3+1 spherically symmetric model. This means that not only the variables have direct geometric meaning and there is no need to turn everything back to their directly-geometric form at the end, but also it might be much easier to read off the physics out of the this formulation. Then, by means of rescaling lapse and shift, the Hamiltonian constraint of the system was cast into a form such that it commutes with itself both in vacuum and coupled-to-matter cases. This makes the system suitable for analysis using loop quantum gravity techniques because, among other things, the system is expressed in Ashtekar-like variables and the algebra of constraints is now a Lie algebra. In the next step, we solved both vacuum and coupled-to-matter cases completely classically by introducing two gauge fixings for each case. In the case where there is matter coupling, we arrive at a local true Hamiltonian hence leading to local equations of motion.

As one possible future direction, in not-gauge-fixed or partially gauge fixed case, it is desirable to polymerize the relevant variables that captures the semiclassical behavior of loop quantum gravity. In the totally gauge fixed case, one can follow different quantization schemes suitable for a local true Hamiltonian theory. These can be starting points for addressing the issue of singularity resolution in further works. Also the possibility of completing the Dirac procedure is in principle there, since the constraint algebra is now a Lie algebra. In addition, it is also worth noting that the algebra of Hamiltonian constraint with itself is very simple and this might be a good aid in a possible process of quantization.

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