Note on the Diameter of Path-Pairable Graphs

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Abstract
A graph on $2k$ vertices is path-pairable if for any pairing of the vertices the pairs can be joined by edge-disjoint paths. The so far known families of path-pairable graphs have diameter of length at most 3. In this paper we present an infinite family of path-pairable graphs with diameter $d(G) = O(\sqrt{n})$ where $n$ denotes the number of vertices of the graph. We prove that our example is extremal up to a constant factor.

Introduction
Given a fixed integer $k$, a graph $G$ on at least $2k$ vertices is $k$-path-pairable if for any pair of disjoint sets of vertices $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$ of $G$ there exist $k$ edge-disjoint paths $P_i$ such that $P_i$ is a path from $x_i$ to $y_i$, $1 \leq i \leq k$. The path-parability number of a graph $G$ is the largest positive integer $k$ for which $G$ is $k$-path-pairable. The motivation of setting edge-disjoint paths between certain pairs of nodes naturally arose in the study of communication networks. There are various reasons to measure the capability of the network by its path-pairability number, that is, the maximum number of pairs of users for which the network can provide separated communication channels without data collision (see [1] for additional details). The nodes corresponding to the users are often called terminal nodes or terminals. A graph $G$ on $n = 2m$ vertices is path-pairable if it is $m$-path-pairable, that is, given a pairing of the vertex set $V(G) = \{x_1, y_1, \ldots, x_m, y_m\}$ there exist edge disjoint paths $P_1, \ldots, P_m$ joining $x_1$ to $y_1$, $x_m$ to $y_m$, respectively. By definition, path-pairable graphs are $k$-path-pairable for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

The three dimensional cube $Q_3$ and the Petersen graph $P$ are both known to be path-pairable. The graph shown in Figure 1 is the only path-pairable graph with maximal degree 3 on 12 or more vertices. Apart from such small and rather sporadic examples we only know a few path-pairable families. Certainly, the complete graph $K_{2k}$ on $n = 2k$ vertices is path-pairable. It can be proved easily that the $n$-partite graph $K_{a_1, \ldots, a_n}$ is path-pairable as long as $\sum_{i=1}^{n} a_i$ is even. Particular species of the latter family, the star graphs $K_{2a+1,1}$ show that path-pairability is achievable even in the presence of vertices of small degrees. They also illustrate that vertices of large degrees are easily accessible transfer stations to carry out linking in the graphs without much effort. That motivates the study of $k$-path-pairable and path-pairable graphs with small maximum degree. Faudree, Lehel and Gyárfás [3] gave examples of $k$-pairable graphs for every $k \in \mathbb{N}$ with maximum degree 3. Note that their construction has exponential size in terms of $k$ and is not path-pairable. Unlike in case of $k$-path-pairability, the

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maximum degree $\Delta(G)$ must increase together with the size of a path-pairable graph $G$. Faudree, Gyárfás and Lehel [4] proved that if $G$ is a path-pairable graph on $n$ vertices with maximum degree $\Delta$ then $n \leq 2\Delta^\Delta$. The theorem gives an approximate lower bound of $\log(n) / \log\log(n)$ on $\Delta(G)$. By contrast, the graphs of the above presented families have maximum degree $\frac{n}{2}$ or more. Kubicka, Kubicki and Lehel [5] investigated path-pairability of complete grid graphs and proved that the two-dimensional complete grid $K_a \times K_b$ of size $n = ab$ is path-pairable. For $a = b$ that gives examples of path-pairable graphs with maximum degree $\Delta = 2a - 2 < 2\sqrt{m}$. In the same paper they raised the question about similar properties of three-dimensional complete grids. Note that if path-pairable, the grid $K_m \times K_m \times K_m$ yields an even better example of size $n = m^3$ and maximum degree $\Delta = 3(m - 1) = O(\sqrt[3]{n})$.

We mention that one of the most interesting and promising path-pairable candidate is the $n$-dimensional hypercube $Q_n$. $Q_1 = K_2$ is path-pairable while $Q_2 = C_4$ is not as pairing of the nonadjacent vertices of the cycle cannot be linked. In fact, it has been proved that $Q_n$ is not path-pairable for even $n$ [2]. One can prove by lengthy case-by-case analysis that the already mentioned $Q_3$ is path-pairable. The question for higher odd-dimensions has yet to be answered.

Conjecture 1 ([1]). The $(2k + 1)$-dimensional hypercube $Q_{2k+1}$ is path-pairable for all $k \in \mathbb{N}$.

A common attribute of the known path-pairable graphs is their small diameter. For each pair $(x, y)$ of terminals in the above examples the length of the shortest $x, y$ paths is at most 3. While terminal pairs of an actual pairing may not always be joined by shortest paths, small diameter gives the advantage of quick accessibility of the vertices and makes designation of edge-disjoint paths easier. The question concerning the existence of an infinite family of path-pairable graphs with unbounded diameter naturally arises. We use the notation $d(G)$ for the diameter of the graph $G$ and $d_{\text{max}}(G, n)$ for the maximal diameter of a path-pairable graph on $n$ vertices. We mention that if true, Conjecture 1 proves the lower bound $\log_2 n \leq d_{\text{max}}(G, n)$ for $n = 2^{2k+1}, k \in \mathbb{N}$.

The main goal of this note is to study the largest possible diameter of path-pairable graphs. We present a family of path-pairable graphs $\{G_n\}$ such that $G_n$ has $n$ vertices and diameter $O(\sqrt{n})$ for infinitely many values of $n$. We show that our construction is optimal up to a constant factor by proving the following theorem.

Theorem 1. If $G$ is a path-pairable graph on $n$ vertices with diameter $d$ then $d \leq 6\sqrt{2} \cdot \sqrt{n}$.

Before the proofs we fix further notation. For a subgraph $H$ of a graph $G$, $|H|$ denotes the number of vertices in $H$. The degree of a vertex $x$ and the distance of vertices $x$ and $y$ are denoted by $d(x)$ and $d(x, y)$, respectively. For additional details on path-pairable graphs we refer the reader to [1], [2] and [4].
Construction

We construct our example by the graph operation called "blowing-up". In a blown-up graph $H$ of a graph $G$ we substitute each vertex of $G$ by a class of independent vertices and join two vertices of different classes by an edge if the corresponding vertices of $G$ are joined. As an example, any complete $n$-partite graph $K_a_1,...,a_n$ can be derived from the complete graph $K_n$ on vertex set $\{x_1,...,x_n\}$ by blowing up the vertex $x_i$ to an independent set of size $a_i$ for $i = 1,2,...,n$. In the current construction we blow up a cycle of even length. Let $n = (2m) \cdot (4m + 3)$ and define $G$ as an equally blown up graph of the cycle $C_{2m}$ of size $n$, that is, $V(G) = \{x_{i,j} : 0 \leq i \leq 2m, 0 \leq j \leq 4m + 3\}$ and $x_{i,j}$ and $x_{i',j'}$ are connected if $i - i' = 1$ or $i - i' = -1$ modulo $2m$. We use the notation $S_i = \{x_{i,j} \in V(G) : 0 \leq j \leq 4m + 3\}$ and refer to the set as the $i$th class of $G$. Easy to see that $G$ has diameter $m > \frac{1}{4}\sqrt{n}$. We mention that $G$ also has maximum degree $O(\sqrt{n})$, the same order of magnitude as in [5] which is the best known result for path-pairable graphs with small degree.

Set an arbitrary pairing of the vertices of $G$. We accomplish the linking of the pairs in two phases. During the first phase we ship exactly one terminal of each pair to the class of its partner. If two terminals belong to the same class then the pair simply skips the first phase of the linking. In order to choose the traveling terminal of each pair we direct our cycle $C_{2m}$ and the blown-up graph $G$ counterclockwise and label each pair $x,y$ such that there exist a directed $x \rightarrow y$ path of length at most $m$. Between classes $S_i$ and $S_{i+1}$ we fix $m$ matchings $M_1^i,...,M_m^i$ of size $4m + 3$ and color them by $1,...,m$.

We divide shipping in phase one into $m$ steps and set the following rule: a terminal $x$ of the $i$th class traveling to the class of its pair at distance $1 \leq d \leq m$ chooses the edge of $M_j^i$ adjacent to $x$ to move to the next $S_{i+1}$ (modulo $m$) class. In step $2 \leq j \leq d$ the terminal takes the corresponding edge of $M_{j+1}^{i+j-1}$ to proceed. Having reached the requested class at step $d$ it stops and skips the remaining $m - d$ steps.

Observe that traveling terminals $x$ and $y$ starting at the same class never arrive to the same vertex of any class. Now assume that edge $e = (x_{i,j},x_{i+1,k})$ has been utilized by two paths $P_1$ and $P_2$ of two traveling terminals. By the shipping procedure it means that $e \in M_j^i$ for some $1 \leq t \leq m$ and that $P_1$ and $P_2$ must have started in the same class. However, in order to share an edge they also have to share a vertex which contradicts our previous observation. It proves that phase one terminates without edge-collision and summons pairs.

In phase two we finish the linking. Observe that each vertex of the graph hosts at most $m + 1$ terminals after phase 1. For terminals $x$ and $y$ being stationed in $S_i$ consider the yet unused edges of the bipartite subgraph $H_i$ spanned by $S_i$ and $S_{i+1}$. As $d_H(x),d_H(y) \geq 3m + 3$ there exists $z \in S_{i+1}$ such that $(x,z),(y,z) \in E(G)$ hence $x$ and $y$ can be joined by that path of length $2$. Together with the path generated during phase one for $x$ linking can be completed. One can easily verify that no edge of $H_i$ has been used multiple times during phase two. That completes the proof.

Proof of Theorem 1

Assume $G$ is a path pairable graph on $n$ vertices with diameter $d \geq 8$. Let $x,y \in V(G)$ such that $d(x,y) = d$. Define $S_i = \{z \in V(G) : d(x,z) = i\}$ for $i = 0,1,2,...,d$ and $U_i = \bigcup_{j=0}^{i} S_j$. Set the notation $s_i = |S_i|$ and $u_i = |U_i|$. Note that there is no edge between any $S_i$ and $S_j$ ($i < j$) classes unless they are consecutive, that is, $j = i + 1$. We divide our sets along the path into three parts creating left, middle and right segments $A,B$ and $C$ as follows:
Lemma 3. A stronger but less practical result can be proved about \( s \) before, placing \( \left\lfloor \frac{2k+1}{4} \right\rfloor \) terminals in \( U_{2k+2} \) and their pairs in \( V(G) - U_{2k+2} \) there must be space for at least \( u_{2k+1} \) edge-disjoint paths passing from \( S_{2k+2} \) to \( S_{2k+3} \). By induction hypothesis \( u_{2k+1} \geq k(2k+1) \) while the number of edges between the two classes is at most \( s_{2k+2}s_{2k+3} \leq \left( \frac{s_{2k+2}+s_{2k+3}}{2} \right)^2 \). It yields \( k(2k+1) \leq \left( \frac{s_{2k+2}+s_{2k+3}}{2} \right)^2 \), that is, \( s_{2k+2} + s_{2k+3} \geq \sqrt{2k(2k+1)} \geq k+1 \) if \( k \geq 1 \).

Lemma 2. \(|A|, |C| \geq \min \left( \frac{n}{2}, \frac{d^2}{100} \right) \).

Proof. Assume \(|V(A)| < \frac{n}{2} \). Using Lemma 1 we know that \(|A| = u_{i} = s_{0} + \cdots + s_{i} \geq 0 + \cdots + \left\lfloor \frac{2k+1}{4} \right\rfloor \geq \frac{d^2}{100} \). The result concerning \(|C|\) can be proved similarly.

If \( \left\lfloor \frac{2k+1}{4} \right\rfloor < t < \left\lfloor \frac{3}{4} d \right\rfloor \), the number of edges between \( S_{t} \) and \( S_{t+1} \) is at least \( \min \left( \frac{n}{2}, \frac{d^2}{100} \right) \). As seen before, placing \( \min \left( \frac{n}{2}, \frac{d^2}{100} \right) \) terminals in \( A \) and their pairs in \( C \) the set \( S_{t} \) has to be able to bridge \( \min \left( \frac{n}{2}, \frac{d^2}{100} \right) \) disjoint paths to \( S_{t+1} \). The number of crossing edges between these two sets is at most \( s_{t}, s_{t+1} \). It means \( s_{t} + s_{t+1} \geq \sqrt{s_{t} s_{t+1}} \geq \min \left( \frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10} \right) \). That gives us the requested lower bound on \(|B|\):

\[
|B| \geq \sum_{t=\left\lfloor \frac{2k+1}{4} \right\rfloor +1}^{\left\lfloor \frac{3}{4} d \right\rfloor} s_{i} + s_{i+1} \geq \frac{d}{3} \min \left( \frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10} \right) \geq \frac{d}{6} \min \left( \frac{\sqrt{n}}{\sqrt{2}}, \frac{d}{10} \right)
\]

As \(|B| \leq n\), our equation proves that \( d \leq 6\sqrt{2\sqrt{n}} \).

Summary

We proved that the maximal diameter \( d_{max}(G) \) of a path pairable graph on \( n \) vertices has order of magnitude \( O(\sqrt{n}) \), that is, \( \frac{1}{4} \sqrt{n} < d_{max}(G, n) < 6\sqrt{2\sqrt{n}} \). We mention that in Lemma 1 an even stronger but less practical result can be proved about \( s_{2k} + s_{2k+1} \).

Lemma 3. Given \( \varepsilon > 0 \), \( s_{2k} + s_{2k+1} \geq (2 - \varepsilon) \cdot k \) holds for sufficiently large \( k \), as long as \( u_{2k+1} \leq \frac{n}{2} \).

This result with more careful calculation may help reducing the gap between the presented upper and lower bounds of the constant.

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