ISOMONODROMIC DEFORMATIONS AND TWISTED YANGIANS ARISING IN TEICHMÜLLER THEORY

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Abstract. In this paper we build a link between the Teichmüller theory of hyperbolic Riemann surfaces and isomonodromic deformations of linear systems whose monodromy group is the Fuchsian group associated to the given hyperbolic Riemann surface by the Poincaré uniformization. In the case of a one-sheeted hyperboloid with \( n \) orbifold points we show that the Poisson algebra \( \mathcal{D}_n \) of geodesic length functions is the semiclassical limit of the twisted \( q \)-Yangian \( Y_q'(\mathfrak{o}_n) \) for the orthogonal Lie algebra \( \mathfrak{o}_n \) defined by Molev, Ragoucy and Sorba. We give a representation of the braid group action on \( \mathcal{D}_n \) in terms of an adjoint matrix action. We characterize two types of finite-dimensional Poissonian reductions and give an explicit expression for the generating function of their central elements. Finally, we interpret the algebra \( \mathcal{D}_n \) as the Poisson algebra of monodromy data of a Frobenius manifold in the vicinity of a non-semisimple point.

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1. INTRODUCTION

In recent years Teichmüller theory has attracted interest from the mathematical physics community due to the manifestation of the Teichmüller space as the Hilbert space for three-dimensional quantum gravity [38]. The Teichmüller space possesses its canonical (Weil–Petersson) Poisson structure, whose symmetry group is the mapping class group of orientation-preserving homeomorphisms modulo isotopy. The algebra of observables is the collection of length functions of geodesic representatives of homotopy classes of essential closed curves together with its natural mapping class group action.

Algebras of geodesic length functions appearing in studies of Teichmüller spaces of hyperbolic Riemann surfaces are closely related to those appearing in isomonodromy problems. For example, the Nelson–Regge algebra [33], [34] appearing as the algebra of geodesic–length–functions on a genus \( g \) Riemann surface with 1 or 2 holes [9], [11], and its isomorphic algebras \( A_n \) of geodesic functions on a disk with \( n \) orbifold points [7], coincide with the Poisson algebras of monodromies in Fuchsian systems arising in Frobenius manifold theory [37] and algebras of groupoid of upper triangular matrices [4].

This coincidence between algebras of geodesic length functions appearing in Teichmüller theory and algebras of monodromy data of isomonodromic systems remained a mystery so far. In this paper we characterize a natural isomonodromic connection on the punctured \( \mathbb{P}^1 \) (the Chern–Simons connection) whose monodromy group is given by the Fuchsian group of a disk with \( n \) orbifold points. This shows that the \( A_n \) algebras coincide with the Poisson algebras of the monodromy data of a \( 2 \times 2 \) Fuchsian system with \( n + 1 \) poles.

We then generalize this correspondence and introduce a new type of Poisson algebras, whose geometrical origin are algebras of geodesic functions on a one-sheeted hyperboloid (or topologically an annulus) with \( n \) orbifold points. On the analytical side, we can obtain the corresponding Fuchsian system starting from an \( A_{n+m} \)-system and clashing \( m \) regular singularities to produce a new hole. The corresponding Poisson algebra of monodromy data is now a quadratic algebra independent on the number \( m \) of the clashed poles that can be interpreted as an abstract algebra for infinitely many generators \( G^{(k)}_{i,j} \), \( i, j = 1, \ldots, n, k \in \mathbb{Z}_{\geq 0} \) (see theorem [6,1]). We call this algebra the \( \mathfrak{D}_n \) algebra. At level 0, i.e. for the generators \( G^{(0)}_{i,j} \), \( i, j = 1, \ldots, n \), this algebra restricts to the Nelson Regge algebra.

We show that the \( \mathfrak{D}_n \) algebra is the semiclassical limit of the twisted \( q \)-Yangian \( Y_q'(o_n) \) for the orthogonal Lie algebra \( o_n \) [32], or, in other words, the defining relations of \( \mathfrak{D}_n \) algebra are the semiclassical limit of the well-known reflection equation.

Beside the Poisson structure, another common property of the algebras of geodesic–length–functions on a Teichmüller space are the braid-group relations, which generate the mapping class group. The braid group invariants are simultaneously the
central elements of the Poisson algebra, therefore constructing a convenient representation of the braid group is always helpful in finding the central elements of the Poisson algebra. Such a representation in terms of the adjoint matrix action for the $A_n$ algebras was constructed in [13] and was used in [4] for constructing the Poisson invariants of the corresponding algebra. On the analytic side, the action of the braid group corresponds to the analytic continuation of the solutions to the isomonodromic problem [14].

In this paper we give a representation of the braid group action on $D_n$ in terms of an adjoint matrix action (see proposition 6.6). Due to topological considerations, this action is represented by $n$ generators, the generators $\beta_{i,i+1}$, $i = 1, \ldots, n - 1$, interchanging the $i$-th orbifold point with the $(i+1)$-th one, and a new generator $\beta_{n,1}$ interchanging the first and the last orbifold points from the other side of the new hole. This new generator acts in a non-trivial way mixing different levels.

We characterize two types of finite-dimensional Poissonian reductions, the so-called level $p$ reductions and the $D_n$ reduction, and give an explicit expression for the generating function of their central elements.\footnote{We draw the attention of the reader to the fact that throughout this paper we deal with two distinct objects: the $D_n$ algebra and the $D_n$ algebra.} Let us briefly describe these two reductions from a geometric point of view. The level $p$ reduction corresponds to collapsing the newly created hole to an orbifold point of order $p$. The $D_n$ reduction is the reduction to a finitely generated cubic algebra produced in [5], where the corresponding braid-group action was also constructed. However, a procedure for finding central elements (or the braid-group invariants of this algebra) was lacking in [5]. We fill this gap in this paper.

The quantum braid-group action representation for the $D_n$ algebra was found in [5], [7]. Since the reduction of the $D_n$ algebra to the $D_n$ algebra can be presented in the matrix form, it is clear that the very same representation of the quantum braid group must be simultaneously a representation for the quantum braid group (or quantum mapping class group) for the $D_n$ algebra as well as for all its $p$-level (quantum) reductions. Using this insight we show that the subgroup generated by $\beta_{i,i+1}$ for $i = 1, \ldots, n - 1$ is quantized to the one acting on the twisted quantized enveloping algebra $U_q^p(\mathfrak{a}_n)$ studied in [31], while the action of $\beta_{n,1}$ (quantized or not) is new.

The fact that the Nelson–Regge algebra coincides with the Poisson algebras of monodromies of Fuchsian systems arising in Frobenius manifold theory poses the natural question of characterizing the special class of Frobenius manifolds coming from Teichmüller theory. This is a highly non-trivial problem that we postpone to subsequent work [10]. In this paper, we show that in two special limiting cases of the $A_n$ algebra, that we call $A_n^*$ and $A_n^*$ respectively, the Teichmüller space carries the same Frobenius manifold structure as the respective quantum cohomology rings $H^*(\mathbb{CP}^2)$ and $H^*(\mathbb{CP}^3)$.

Finally we interpret our $D_n$ algebra as the Poisson algebra of the Stokes data of a Frobenius manifold in the vicinity of a non semi-simple point.

The paper is organized as follows. In Sec. 2 we briefly recall the combinatorial description of Teichmüller spaces of Riemann surfaces with holes and with orbifold points and describe the Goldman bracket [18] of geodesic functions. The special case of the Nelson–Regge algebras are considered in Subsec. 2.1 and the one of the $D_n$ algebras in Subsec. 2.2. In Sec. 3 we consider the isomonodromic deformations of a
Fuchsian system with \( n + 1 \) poles and introduce its monodromy data. The Poisson brackets on the set of these monodromy data are the Korotkin–Santtleben brackets \([25]\) described in Sec. 4. In Sec. 5 we introduce the procedure of pole clashing and use it to generate a new hole. The new (infinite-dimensional) Poisson algebras \( \mathfrak{D}_n \) of the monodromy data for the one-sheeted hyperboloid with \( n \) orbifold points are introduced in Sec. 6 where we also prove that \( \mathfrak{D}_n \) algebra is the semiclassical limit of the twisted \( q \)-Yangian \( \mathfrak{Y}_q^{(o_n)} \) for the orthogonal Lie algebra \( o_n \) (Subsec. 6.3). In Subsec. 6.4 we construct the braid-group representation in matrix form and in Subsec. 6.5 we quantize it. We study various reductions of these algebras in Sec. 7 where we introduce the \( p \)-level reductions, the algebras \( \mathfrak{D}_n^{(p)} \), which enjoy the same braid-group representation in the adjoint matrix form as the general algebra, enabling us to evaluate their central elements. We then turn to the case of the \( D_n \)-algebra and show that it can be obtained by a special reduction (based on the skein relations) from the “ambient” \( \mathfrak{D}_n \) algebra and with the same braid-group representation as above. This fact enables us to construct \( n \) central elements of the \( D_n \) algebra. We prove the algebraic independence of these central elements and that the algebra \( D_n \) admits in general no more than \( n \) algebraically independent central elements.

The link with the Frobenius manifold theory and the quantum cohomology of projective spaces is carried out in Sec. 8.

Finally, in Appendix A, we provide the description of monodromy data for a general \( n \times n \) Fuchsian system. In Appendix B, we present the proof of the Jacobi identities for the brackets of the \( \mathfrak{D}_n \) algebra brackets, and in Appendix C, we present the proof of the algebraic independence for the \([np/2]\) central elements of the algebra \( \mathfrak{D}_n^{(p)} \).

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2. Orbifold Riemann surfaces

The graph description of the Teichmüller theory of surfaces with orbifold points was proposed in \([5]\), \([6]\)\(^2\). This theory is formulated in terms of hyperbolic geometry by introducing new parameters (the number of orbifold points on a Riemann surface with holes). Let us denote by \( \Sigma_{g,s,n} \) a Riemann surface of genus \( g \) with \( s \) holes and \( n \) orbifold points of order two. By the Poincaré uniformization theorem

\[
\Sigma_{g,s,n} \sim \mathbb{H}/\Delta_{g,s,n},
\]

\(^2\)In \([5]\), it was developed for the bordered Riemann surfaces, the interpretation in terms of the orbifold Riemann surfaces was given in \([6]\), but all the algebraic formulas in \([5]\) are identical for the both geometrical interpretations.
where
\[ \Delta_{g,s,n} = \langle \gamma_1 \cdots \gamma_{2g+s+n-1} \rangle, \quad \gamma_1 \cdots \gamma_{2g+s+n-1} \in \text{PSL}(2, \mathbb{R}) \]
is a Fuchsian group, the fundamental group of the surface \( \Sigma_{g,s,n} \). In particular for orbifold Riemann surfaces, the Fuchsian group \( \Delta_{g,s,n} \) is almost hyperbolic, i.e. all its elements are either hyperbolic (when all the holes have nonzero perimeters; parabolic elements are allowed when a hole degenerates into a puncture) or have trace equal to zero.

Let us remind the Thurston shear-coordinate description of the Teichmüller spaces of Riemann surfaces with holes and, possibly, orbifold points (see [6]). The main idea [15] is to decompose each hyperbolic matrix \( \gamma \in \Delta_{g,s,n} \) as a product of the form

\[ \gamma = (-1)^K R^{k_1} X_{z_1} \cdots R^{k_p} X_{z_p}, \quad i_j \in I, \quad k_{i_j} = 1, 2, \quad K := \sum_{j=1}^{p} k_{i_j} \]

where \( I \) is a set of integer indices and the matrices \( R, L \) and \( X_{z_i} \) are defined as follows:

\[
\begin{align*}
R &:= \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \\
L &:= -R^2 := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \\
X_{z_i} &:= \begin{pmatrix} 0 & -\exp \left( \frac{z_i}{2} \right) \\ \exp \left( -\frac{z_i}{2} \right) & 0 \end{pmatrix},
\end{align*}
\]

and to decompose each traceless element as

\[ \gamma_0 = \gamma^{-1} F \gamma, \]

where \( \gamma \) is decomposed as in (2.1) and

\[ F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \]

The decomposition of each element in the Fuchsian group \( \Delta_{g,s,n} \) is obtained by looking at the closed geodesic corresponding to it in the fat–graph associated to \( \Sigma_{g,s,n} \).

Let us briefly recall how to associate a fat-graph to a Riemann surface with holes but without orbifold points [15] [16]. In this case, one considers a spine \( \Gamma_{g,s} \) corresponding to the Riemann surface \( \Sigma_{g,s} \) with \( g \) handles and \( s \) boundary components (holes). The spine, or fat–graph \( \Gamma_{g,s} \) is a connected graph that can be drawn without self-intersections on \( \Sigma_{g,s} \), it has all vertices of valence three, it has a prescribed cyclic ordering of labeled edges entering each vertex, and it is a maximal graph in the sense that its complement on the Riemann surface is a set of disjoint polygons (faces), each polygon containing exactly one hole (and becoming simply connected after gluing this hole). Since a graph must have at least one face, only Riemann surfaces with at least one hole, \( s > 0 \), can be described in this way. The hyperbolicity condition also implies \( 2g - 2 + s > 0 \).

In the case where no orbifold points are present, the Fuchsian group \( \Delta_{g,s} \) is strictly hyperbolic if all the holes have nonzero perimeters, and only the elements that correspond to holes degenerated into punctures are parabolic ones. The decomposition (2.1) can be obtained by establishing a one-to-one correspondence between
elements of the Fuchsian group and closed paths in the spine starting and terminating at the same directed edge. Each time the path \( A \) corresponding to the element \( \gamma_A \) (or, equivalently, to its invariant closed geodesic) passes through the \( \alpha \)th edge, an edge–matrix \( X_{\alpha} \) with the real coordinate \( Z_{\alpha} \) appears in the decomposition of \( \gamma \). At the end of the edge, the path can either turn right or left, and a matrix \( R \) or \( L \) respectively appears in the decomposition [15].

The introduction of orbifold points is achieved by considering \textit{new types of graphs} with pending vertices [5]. Then, if a geodesic line comes to a pending vertex, it undergoes an \textit{inversion}, which corresponds to inserting the inversion matrix \( F \), into the corresponding string of \( 2 \times 2 \)-matrices. The edge terminating at a pending vertex is called \textit{pending edge}.

All possible paths in the spine (graph) that are closed and may experience an arbitrary number of inversions at pending vertices of the graph must be taken into account.

The algebras of geodesic length functions were constructed in [5] by postulating the Poisson relations on the level of the shear coordinates \( X_\alpha \) of the Teichmüller space:

\[
\{ f(X), g(X) \} = \sum_{\text{3-valent vertices } \alpha = 1}^{4g + 2s + n - 4} \sum_{i=1}^{3 \mod 3} \left( \frac{\partial f}{\partial X_\alpha} \frac{\partial g}{\partial X_{\alpha_i}} - \frac{\partial g}{\partial X_\alpha} \frac{\partial f}{\partial X_{\alpha_i+1}} \right),
\]

where the sum ranges all the three-valent vertices of a graph and \( \alpha_i \) are the labels of the cyclically (counterclockwise) ordered \((\alpha_4 \equiv \alpha_1)\) edges incident to the vertex with the label \( \alpha \). This bracket gives rise to the \textit{Goldman bracket} on the space of geodesic length functions [18].

We recall an important relation valid in \( \mathbb{P}SL(2) \):

\[
\text{Tr}\gamma_A \text{Tr}\gamma_B = \text{Tr}(\gamma_A \gamma_B) + \text{Tr}(\gamma_A \gamma_B^{-1}).
\]

This relation corresponds to resolving the crossing between the two corresponding geodesics \( A \) and \( B \) as in Fig. 1 and it referred to as \textit{skein relation}.

![Figure 1. The classical skein relation.](image)

\[2.1. \text{ } A_n \text{ algebra.} \text{ The simplest case of orbifold Riemann surface is a Poincaré disk with } n \geq 3 \text{ orbifold points in the interior; we denote it by } \Sigma_{0,1,n}. \text{ In this case, the fat-graph } \Gamma_{0,1,n} \text{ is a tree-like graph depicted in Fig. 2 for } n = 3, 4, \ldots. \text{ We enumerate the } n \text{ dot-vertices counterclockwise, } i, j = 1, \ldots, n, \text{ and consider the algebra of all geodesic functions.}^3\]

\[3\text{Note that in the cluster algebra terminology (see [17]) these algebras were denoted by } A_{n-2}.\]
2.1.1. Poisson relations for $A_n$ algebra. Let $G_{i,j}$ with $i < j$ denote the geodesic function corresponding to the geodesic line that encircles exactly two pending vertices with the indices $i$ and $j$. Examples for $n = 3$ and $n = 4$ are in the figure. It turns out that these geodesic functions suffice for closing the Poisson algebra:

$$
\{G_{i,k}, G_{j,l}\} = 0, \quad \text{for } i < k < j < l, \quad \text{and for } i < j < l < k,
$$

$$
\{G_{i,k}, G_{j,l}\} = 2 (G_{i,j} G_{k,l} - G_{i,l} G_{k,j}), \quad \text{for } i < j < k < l,
$$

$$
\{G_{i,k}, G_{i,l}\} = - (G_{i,k} G_{i,l} - 2G_{k,l}), \quad \text{for } i < j < k < l.
$$

(2.5)

Note that the left-hand side is doubled in this case as compared to Nelson–Regge algebras recalled in \[11\].

$$
\text{Figure 2. Generating graphs for } A_n \text{ algebras for } n = 3, 4, \ldots. \text{ We indicate character geodesics whose geodesic functions } G_{i,j} \text{ enter bases of the corresponding algebras.}
$$

In this paper we consider a basis $\gamma_1, \ldots, \gamma_n$ in the Fuchsian group $\Delta_{0,1,n}$ such that

$$
-\text{Tr}(\gamma_i \gamma_j) = G_{i,j}.
$$

(The sign convention is such that when we interpret $G_{i,j}$ as being the geodesic functions related to lengths $\ell_{i,j}$ of closed geodesics, we have $G_{i,j} = 2 \cosh(\ell_{i,j}/2) \geq 2$.) In this case, for convenience we let $Z_i$ denote the coordinates of pending edges and $Y_j$ all other coordinates. In the case where we do not distinguish between pending and internal edges, we preserve the notation $X_\alpha$ for all the coordinates. This basis is given by the following (we write it in $SL(2, \mathbb{R})$):

$$
\gamma_1 = F,
$$

$$
\gamma_2 = -X_1 L X_2 F X_2 R X_1,
$$

$$
\gamma_3 = -X_1 R X_2 L X_3 F X_3 R X_2 L X_1,
$$

$$
\gamma_i = -X_1 R X_Y L X_2 \ldots R X_{Y_{i-2}} L X_{Z_1} F X_{Z_1} R X_{Y_{i-2}} L \ldots X_{Y_1} L X_{Z_1},
$$

$$
\gamma_{i+1} = -X_1 R X_Y L X_2 \ldots R X_{Y_{i-1}} L X_{Z_{i-1}} F X_{Z_{i-1}} R X_{Y_{i-1}} L X_{Z_1},
$$

$$
\ldots
$$

(2.6)

$$
\gamma_n = -X_1 R X_Y L X_2 \ldots R X_{Y_{n-2}} L X_{Z_{n-2}} F X_{Z_{n-2}} R X_{Y_{n-2}} L \ldots X_{Y_1} L X_{Z_1},
$$

Observe that $\text{Tr} \gamma_i = 0$, $i = 1, \ldots, n$. It is not hard to check that the matrix

$$
\gamma_\infty := (\gamma_1 \gamma_2 \ldots \gamma_n)^{-1}
$$
has eigenvalues \((-1)^{n-1}e^{\pm P/2}\), where \(P\) is the length of the perimeter around the hole:

\[
P = 2 \sum_{i=1}^{n} Z_i + 2 \sum_{j=1}^{n-3} Y_j.
\]  

(2.7)

2.1.2. Mapping-class group action on \(A_n\). Observe that there is a degree of arbitrariness in the choice of the fat graph \(\Gamma_{g,s,n}\) associated to a Riemann surface \(\Sigma_{g,s,n}\). This arbitrariness is described by the Whitehead moves [35] and their generalization to the case of pending edges [5]. Using these moves, or flip morphisms, one can establish a morphism between any two algebras corresponding to surfaces of the same genus, same number of boundary components, and same number of orbifold points. If, after a series of morphisms, a graph of the same combinatorial type as the initial one (disregarding marking of edges) is obtained, then this morphism is associated to a mapping class group operation, therefore passing from the groupoid of morphisms to the mapping class group.

For the \(A_n\) algebra, the action of the mapping class group corresponds to the following action of the braid group [5]: let us construct the upper-triangular matrix

\[
\beta_{i,i+1} A = \tilde{A},
\]

where

\[
\begin{cases}
G_{i,i+1} = G_{i,i}, & j > i + 1, \\
G_{i,i+1} = G_{j,i}, & j < i, \\
G_{i,i+1} = G_{i,i+1} - G_{i+1,j}, & j > i + 1, \\
G_{j,i+1} = G_{j,i} - G_{j,i+1}, & j < i, \\
G_{i,i+1} = G_{i,i+1}.
\end{cases}
\]  

(2.9)

A very convenient way to present this transformation is by introducing the special matrices \(B_{i,i+1}\) of the block-diagonal form (see [13])

\[
B_{i,i+1} = \begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & G_{1,i+1} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\]

(2.10)

Then, the action of the braid group generator \(\beta_{i,i+1}\) on \(A\) acquires merely a matrix product form:

\[
\beta_{i,i+1} A = B_{i,i+1} A B_{i,i+1}^T
\]  

(2.11)
where $B^T_{i,i+1}$ denotes the matrix transposed to $B_{i,i+1}$. In this setting it is easy to prove the braid group relations:

$$\beta_{i-1,i} \beta_{i,i+1} \beta_{i-1,i} = \beta_{i,i+1} \beta_{i-1,i} \beta_{i,i+1}, \quad 2 \leq i \leq n-1,$$

and also the extra relation [13]:

$$(\beta_{n-1,n} \beta_{n-2,n} \cdots \beta_{2,3} \beta_{1,2})^n = \text{Id}.$$ (2.12)

**Remark 2.1.** Observe that upper triangular matrices of the form (2.8) can be interpreted as Stokes matrices of certain linear system of ordinary differential equations appearing in the theory of Frobenius Manifolds [13]. A study of the special class of Frobenius manifolds arising in Teichmüller theory is in progress [10]. We show some preliminary results in Sec. 8.

### 2.2. $D_n$ algebra.

The simplest case of orbifold Riemann surface with two holes is an annulus with $n \geq 2$ marked points, $\Sigma_{0,2,n}$. The fat graph $\Gamma_{0,2,n}$ with $n = 4$ is shown in Fig. 3.

2.2.1. *Poisson relations for the $D_n$ algebra.* Again, the algebras of geodesic functions were constructed in [5] by postulating the Poisson relations on the level of the shear coordinates of the Teichmüller space. To close this Poisson algebra more geodesic functions are needed than in the case of $A_n$, they are: $\hat{G}_{i,i}$, the geodesic containing the $i$-th pending vertex and the hole, and for each $i, j = 1, \ldots, n$ two geodesics containing the pending vertices $i$ and $j$: $\hat{G}_{i,j}$ and $\hat{G}_{j,i}$. Here, the order of subscripts indicates the direction of encompassing the hole (the second boundary component of the annulus), see Fig. 3. Obviously, $\hat{G}_{i,j}$ with $1 \leq i < j \leq n$ constitute one (among $n$ possible) $A_n$-subalgebras of the $D_n$ algebra. The total number of generators of the $D_n$ algebra is therefore $n^2$. In this paper, we indicate the geodesic functions from this set by the hat symbol to distinguish them from the level $k$ geodesic functions which will be introduced in Section 6.

The relevant Poisson brackets are cubic, and can be found in [5], [7]. One of the aims of this paper is to describe the $D_n$-algebras as reductions of the $D_n$ algebras which will be constructed in section 6 below. We shall realize this reduction first in the geometric case, i.e., using the skein relations, then in the analytical case and finally in the abstract algebraic case.

Let us briefly describe the mapping class group action in terms of the braid group. Note that both the Poisson brackets and the action of the braid group do not depend on the perimeter of the hole, so these $D_n$ algebras can be considered as abstract Poisson algebras, i.e. as algebras for $n^2$ formal objects $\hat{G}_{i,j}$, $i, j = 1, \ldots, n$. 
2.2.2. **Braid group relations for** $D_n$-**algebras.** The action of the braid group on the generators $\tilde{G}_{i,j}$, $i, j = 1, \ldots, n$ of the $D_n$ algebra can be presented in the explicit form as follows:

\[
\begin{align*}
\tilde{G}_{i+1,k} &= \tilde{G}_{i,k} & k &\neq i, i+1, \\
\tilde{G}_{i,k} &= \tilde{G}_{i,k} \tilde{G}_{i,i+1} - \tilde{G}_{i+1,k} & k &\neq i, i+1, \\
\tilde{G}_{k,i+1} &= \tilde{G}_{k,i} & k &\neq i, i+1, \\
\tilde{G}_{k,i} &= \tilde{G}_{k,i} \tilde{G}_{i,i+1} - \tilde{G}_{k,i+1} & k &\neq i, i+1, \\
\tilde{G}_{i,i+1} &= \tilde{G}_{i,i} \\
\tilde{G}_{i+1,i+1} &= \tilde{G}_{i,i} \\
\tilde{G}_{i} &= \tilde{G}_{i,i} \tilde{G}_{i,i+1} - \tilde{G}_{i,i+1,i+1} \\
\tilde{G}_{i+1,i} &= \tilde{G}_{i,i} + \tilde{G}_{i,i+1} \tilde{G}_{i,i} - 2 \tilde{G}_{i,i} \tilde{G}_{i,i+1,i+1}
\end{align*}
\]

for $1 \leq i \leq n - 1$ and

\[
\begin{align*}
\tilde{G}_{1,k} &= \tilde{G}_{n,k} & k &\neq n, 1, \\
\tilde{G}_{n,k} &= \tilde{G}_{n,k} \tilde{G}_{n,1} - \tilde{G}_{1,k} & k &\neq n, 1, \\
\tilde{G}_{k,1} &= \tilde{G}_{k,n} & k &\neq n, 1, \\
\tilde{G}_{k,n} &= \tilde{G}_{k,n} \tilde{G}_{n,1} - \tilde{G}_{k,1} & k &\neq n, 1, \\
\tilde{G}_{n,1} &= \tilde{G}_{n,1} \\
\tilde{G}_{1,1} &= \tilde{G}_{n,n} \\
\tilde{G}_{n,n} &= \tilde{G}_{n,n} \tilde{G}_{1,1} - \tilde{G}_{1,1} \\
\tilde{G}_{1,n} &= \tilde{G}_{1,n} + \tilde{G}_{1,n} \tilde{G}_{n,1} - 2 \tilde{G}_{1,n} \tilde{G}_{1,1}
\end{align*}
\]

**Lemma 2.2.** For any $n \geq 2$, we have the braid group relation for the transformations (2.13), (2.14):

\[
\beta_{i+1,i} \tilde{G}_{k,l} = \tilde{G}_{k,l}:
\]

\[
\begin{align*}
\beta_{i+1,i} \beta_{i,i} \beta_{i-1,i} &= \beta_{i,i+1} \beta_{i-1,i} \beta_{i,i+1}, i = 1, \ldots, n \mod n,
\end{align*}
\]

where for $i = n$ the element $\beta_{n,n+1}$ stands for $\beta_{n,1}$.

Note that the second braid-group relation (2.12) is lost in the case of $D_n$-algebras. This is due to the topological restriction imposed by the extra hole.

Presenting the braid-group action in the matrix-action (covariant) form (2.11) is a nontrivial problem. In fact special combinations of $\tilde{G}_{i,j}$ admit similar transformation laws under the subgroup $\langle \beta_{1,2}, \ldots, \beta_{n-1,n} \rangle$ of braid-group transformations generated by relations (2.13) alone. In fact the following result was proved in [5]:

**Lemma 2.3.** Consider the $n \times n$ skewsymmetric matrix $\overline{R}$ of entries:

\[
(\overline{R})_{i,j} := \begin{cases} 
-\tilde{G}_{j,i} - \tilde{G}_{i,j} & j < i \\
\tilde{G}_{i,j} + \tilde{G}_{j,i} & j > i \\
0 & j = i
\end{cases}
\]

the symmetric matrix $\overline{S}$ of entries:

\[
(\overline{S})_{i,j} := \tilde{G}_{i,i} \tilde{G}_{j,j} \quad \text{for all} \quad 1 \leq i, j \leq n;
\]
and the upper triangular matrix $\hat{A}$ of entries

$$\hat{A}_{i,j} = \begin{cases} \hat{G}_{i,j} & i < j \\ 0 & i > j \\ 1 & i = j \end{cases}$$

Then any linear combination $w_1 \hat{A} + w_2 \hat{A}^T + \rho \hat{R} + \sigma \hat{S}$ with complex $w_1$, $w_2$, $\rho$, and $\sigma$ transforms by formula (2.11) under the subgroup $\langle \beta_1, \ldots, \beta_{n-1,n} \rangle$ of braid-group transformations generated by relations (2.13) alone.

Below we construct the matrix representation of the total braid group action and find the central elements of the $D_n$ algebra (see sub-section 7.3.2).

3. Monodromy preserving deformations

In this section we interpret the matrices $\gamma_1, \ldots, \gamma_n$ as monodromy matrices of a Fuchsian system with $2 \times 2$ residue matrices $A_j$ independent on $\lambda$:

$$\frac{d}{d\lambda} \Phi = \left( \sum_{k=1}^{n} \frac{A_k}{\lambda - u_k} \right) \Phi,$$

where $u = (u_1, \ldots, u_n)$ are the pairwise distinct pending vertices in the fat graph. The residue matrices $A_j$ satisfy the following conditions:

$$\text{eigen} (A_j) = \pm \frac{1}{4} \quad \text{and} \quad -\sum_{k=1}^{n} A_k = A_{\infty},$$

where, given

$$\mu := \begin{cases} \frac{\nu}{2\pi i} + \frac{1}{2}, & \text{for } n \text{ odd} \\ \frac{\nu}{2\pi i}, & \text{for } n \text{ even} \end{cases},$$

$$A_{\infty} := \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad \text{for } \mu \neq 0 \quad \text{and} \quad A_{\infty} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{for } \mu = 0.$$

The description of the monodromy data of the system (3.19) is recalled in Appendix A. It is convenient to fix the base point of the fundamental group at $\infty$ so that one actually considers the monodromy matrices

$$M_i = C_{\infty}^{-1} \gamma_i C_{\infty}, \quad M_{\infty} = C_{\infty}^{-1} \gamma_{\infty} C_{\infty},$$

where $C_{\infty}$ is the matrix of the eigenvalues of $\gamma_{\infty}$ so that

$$M_{\infty} = \begin{pmatrix} (-1)^n e^{P/2} & 0 \\ 0 & (-1)^n e^{-P/2} \end{pmatrix}, \quad \text{for } P \neq 0,$$

$$M_{\infty} := \begin{pmatrix} (-1)^n & 1 \\ 0 & (-1)^n \end{pmatrix}, \quad \text{for } P = 0.$$

Given a point in the Teichmüller space, specified by $\gamma_1, \ldots, \gamma_n$, or equivalently $M_1, \ldots, M_n$, there exists a Fuchsian system having monodromy matrices $M_1, \ldots, M_n$. More precisely the following general theorems hold true:
Theorem 3.1. [12] Given \( n \) arbitrary \( 2 \times 2 \) matrices \( M_1, \ldots, M_n \) and an arbitrary number \( \mu \) such that

\[
M_\infty := (M_1 M_2 \ldots M_{n-1} M_n)^{-1}
\]
is given by

\[
M_\infty = \begin{cases} 
\begin{pmatrix} e^{2i\pi\mu} & 0 \\
0 & e^{-2i\pi\mu} \end{pmatrix}, & \text{for } \mu \not\in \mathbb{Z}, \frac{1}{2} + \mathbb{Z}, \\
\begin{pmatrix} 1 & 1 \\
0 & 1 \end{pmatrix}, & \text{for } \mu \in \mathbb{Z}, \\
\begin{pmatrix} -1 & 1 \\
0 & -1 \end{pmatrix}, & \text{for } \mu \in \frac{1}{2} + \mathbb{Z},
\end{cases}
\]

and fixed a point \( u^0 = (u^0_1, \ldots, u^0_n) \in X_n, X_n := \mathbb{C}^n \setminus \{\text{diagonals}\} \), for any neighbourhood \( U \subset X_n \) of \( u^0 \) there exists \( u \in U \) and a Fuchsian system

\[
\frac{d}{d\lambda}\Phi = \left(\sum_{k=1}^{n} A_k \frac{1}{\lambda - u_k}\right) \Phi,
\]

with the given monodromy matrices \( M_1, \ldots, M_n \) and with \( A_\infty \) given by (3.20).

Indeed there is a whole family of Fuchsian systems with the same monodromy matrices, they are given by the solutions of the Schlesinger equations (3.25). In fact the following theorem is true in any dimension:

Theorem 3.2. [28, 29] Let \( M_1, \ldots, M_n \) be the monodromy matrices of the Fuchsian system

\[
\frac{d}{d\lambda}\Phi^0 = \left(\sum_{k=1}^{n} A_k^0 \frac{1}{\lambda - u^0_k}\right) \Phi^0,
\]

with \( u^0 = (u^0_1, \ldots, u^0_n) \in X_n \). Then there exists a neighbourhood \( U \subset X_n \) of \( u^0 \) such that for any \( u \in U \) there exists a unique \( n \)-uple \( A_1(u), \ldots, A_n(u) \) of analytic valued matrix functions such that

\[
A_i(u^0) = A_i^0, \quad i = 1, \ldots, n,
\]

and the monodromy matrices of the system

\[
\frac{d}{d\lambda}\Phi = \left(\sum_{k=1}^{n} A_k(u) \frac{1}{\lambda - u_k}\right) \Phi,
\]

with respect to the same basis of loops, coincide with \( M_1, \ldots, M_n \). The matrices \( A_1(u), \ldots, A_n(u) \) are solutions of the Schlesinger equations:

\[
\frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}.
\]

The solution \( \Phi^0(\lambda) \) of (3.24) can be uniquely continued, for \( \lambda \not= u_i \), to an analytic function

\[
\Phi(\lambda, u), \quad u \in U,
\]
such that $\Phi(\lambda, u^0) = \Phi^0(\lambda)$. This continuation is the local solution of the Cauchy problem with the initial data $\Phi^0$ for the following system:

$$\frac{\partial}{\partial u_i} \Phi = -\frac{A_i}{\lambda - u_i} \Phi.$$  

Moreover the functions $A_1(u), \ldots, A_n(u)$ and $\Phi(\lambda, u)$ can be continued analytically to global meromorphic functions on the universal coverings of $X_n$ and $\mathbb{P}^1 \setminus \{u_1, \ldots, u_n\} \otimes X_n$ respectively.

The above theorems establish the Riemann–Hilbert correspondence:

$$\mathcal{M} / \mathcal{D} \leftrightarrow A / \mathcal{D}$$

where $\mathcal{D} = \{ D \in GL(2, \mathbb{C}) \text{ diagonal matrix} \}$ and

$$\mathcal{M} := \{(M_1, \ldots, M_n) \in SL(2, \mathbb{C}) : \text{Tr} M_i = 0, (M_1 M_2 \ldots M_n)^{-1} = M_\infty \text{ given in (3.22)}\}$$

and

$$A := \{(A_1, \ldots, A_n) \in \mathfrak{sl}(2, \mathbb{C}) : \text{eigen}(A_i) = \pm \frac{1}{4}, \sum_{k=1}^n A_k = -A_\infty, A_\infty \text{ as in (3.20)}\}.$$  

The choice of the generators $\gamma_1, \ldots, \gamma_n$ in the Fuchsian group $\Delta_n$ allows us to extend the Riemann–Hilbert correspondence to the "suitably" complexified Teichmüller space, where "suitably" means the complexification of the Teichmüller space that corresponds to the handle–body case. We postpone the study of the extension of the Riemann–Hilbert correspondence to the handle–body Teichmüller space to subsequent publications.

3.1. Analytic continuation of the Schlesinger equations solutions and braid group action. The procedure of the analytic continuation of the solutions to the Schlesinger equations in terms of the action of the braid group $\mathcal{B}_n = \langle \beta_1, \ldots, \beta_{n-1} \rangle$ on the monodromy matrices $M_1, \ldots, M_n$ was obtained in [14]. Let us recall here the main ideas of this derivation.

According to Theorem 3.2 any solution of the Schlesinger equations can be continued analytically from a point $u_0$ to any other point $u \in X_n$ provided that the end-points are not the poles of the solution. The result of the analytic continuation depends only on the homotopy class of the path in $X_n$, i.e. one obtains a natural action of the pure braid group $\mathcal{P}_n$

$$\mathcal{P}_n = \pi_1(X_n, u_0)$$

on the space of solutions of the Schlesinger equations. By using the fact that thanks to Theorem 3.1 the solutions of the Schlesinger equations are locally uniquely determined by the monodromy matrices $M_1, \ldots, M_n$, one can describe the procedure of analytic continuation by an action of the pure braid group on the monodromy matrices. For technical simplicity, we deal with the action of the full braid group:

$$\mathcal{B}_n = \pi_1(X_n \setminus S_n, u_0),$$

where $S_n$ is the symmetric group. This action is given by

$$\beta_{i,i+1}(M_j) = M_j, \quad \text{for } j = 1, \ldots, i - 1, i + 2, \ldots, n$$

$$\beta_{i,i+1}(M_i) = M_i M_{i+1} M_i^{-1}, \quad \beta_{i,i+1}(M_{i+1}) = M_i.$$

(3.26)

By using the skein relation, it is a straightforward computation to show that on $G_{i,j} := -\text{Tr}(M_i M_j)$ the braid group action coincides with the action (2.9), so
that the action of the mapping class group on the Teichmüller space of a disk with \( n \) marked points corresponds to the procedure of analytic continuation of the corresponding solution to the Schlesinger equations.

4. Korotkin–Samtleben bracket

In this section we remind the Hamiltonian formulation of the Schlesinger equations, the definition of the Korotkin–Samtleben bracket and we show how to obtain the \( A_n \) Poisson algebra from it.

4.1. Hamiltonian formulation of the Schlesinger equations. The Hamiltonian description of the Schlesinger equations in any dimension \( m \) was derived [20] from the general construction of a Poisson bracket on the space of flat connections in a principal \( G \)-bundle over a surface with boundary using Atiyah–Bott symplectic structure (see [1]). Explicitly this approach yields the following well known formalism representing the Schlesinger equations in Hamiltonian form with \( n \) time variables \( u_1, \ldots, u_n \) and \( n \) commuting time–dependent Hamiltonian flows on the dual space to the direct sum of \( n \) copies of the Lie algebra \( \mathfrak{sl}(m) \)

\[
\mathfrak{g} := \oplus_n \mathfrak{sl}(m) \ni (A_1, A_2, \ldots, A_n).
\]

**Theorem 4.1.** [22] The dependence of the solutions \( A_k, k = 1, \ldots, n \), of the Schlesinger equations upon the variables \( u_1, \ldots, u_n \) is determined by Hamiltonian systems on (4.27) with time-dependent quadratic Hamiltonians

\[
H_k = \sum_{l \neq k} \frac{\text{Tr}(A_k A_l)}{u_k - u_l},
\]

(4.28)

\[
\frac{\partial}{\partial u_k} A_l = \{ A_l, H_k \}.
\]

(4.29)

Because of isomonodromicity the Hamiltonian equations (4.29) can be restricted onto the symplectic leaves

\[
\mathcal{O}_1 \times \cdots \times \mathcal{O}_n \in \mathfrak{g}
\]

obtained by fixation of the conjugacy classes \( \mathcal{O}_1, \ldots, \mathcal{O}_n \) of the matrices \( A_1, \ldots, A_n \). The matrix \( A_\infty \) is a common integral of the Schlesinger equations. Applying the procedure of symplectic reduction [27] one obtains the reduced symplectic space

\[
\{ A_1 \in \mathcal{O}_1, \ldots, A_n \in \mathcal{O}_n, A_\infty = \text{given diagonal matrix} \}
\]

(4.30)

modulo simultaneous diagonal conjugations.

The dimension of this reduced symplectic leaf in the generic situation is equal to \( 2g \) where

\[
g = \frac{m(m-1)(n-1)}{2} - (m-1).
\]

In the \( 2 \times 2 \) case, i.e. for \( m = 2 \) the dimension of the symplectic leaves is \( 2(n-2) \), which coincides with the dimension of the Teichmüller space.
4.2. Korotkin–Samtleben bracket. The standard Lie–Poisson bracket on $\mathfrak{g}^*$ can be represented in r-matrix formalism:

$$\left\{ A(\lambda_1) \otimes A(\lambda_2) \right\} = \left[ \frac{1}{2} A(\lambda_1) + \frac{1}{2} \lambda_1 \right],$$

where $r(z) = \Omega \frac{\Omega}{z}$ is a classical r-matrix, i.e. a solution of the classical Yang–Baxter equation. In the case of $\mathfrak{g} := \oplus_n \mathfrak{sl}(m)$, $\Omega$ is the exchange matrix $\Omega = \sum_{i,j} E_{ij} \otimes E_{ji}$ (we identify $\mathfrak{sl}(m)$ with its dual by using the Killing form $(A,B) = \text{Tr} AB$, $A,B \in \mathfrak{sl}(m)$).

The standard Lie–Poisson bracket on $\mathfrak{sl}(m,\mathbb{C})$ is mapped by the Riemann–Hilbert correspondence to the Korotkin–Samtleben bracket:

$$\left\{ M_i \otimes M_i \right\} = \frac{1}{2} \left( \frac{1}{2} M_i \Omega M_i - M_i \Omega M_i \right),$$

(4.31)

$$\left\{ M_i \otimes M_j \right\} = \frac{1}{2} \left( \frac{1}{2} M_i \Omega M_j + \frac{1}{2} M_j \Omega M_i - \frac{1}{2} \Omega M_i M_j - \frac{1}{2} M_j M_i \Omega \right), \quad \text{for } i < j.$$

This bracket does not satisfy the Jacobi identity - however it restricts to a Poisson bracket on the adjoint invariant objects.

Lemma 4.2. The $A_n$ Poisson algebra $\mathfrak{g}(2,2)$ is the Korotkin–Samtleben bracket restricted to the adjoint invariant objects $\mathfrak{g}_{i,j} := -\text{Tr}(\gamma_i \gamma_j) = -\text{Tr}(M_i M_j)$.

Proof. We show how to prove relation:

$$(4.32) \quad \{ G_{i,k}, G_{j,l} \} = 2 \{ G_{i,j} G_{k,l} - G_{i,l} G_{k,j} \}, \quad \text{for } i < j < k < l.$$

By definition of $G_{i,j}$ we have:

$$\{ G_{i,k}, G_{j,l} \} = \{ \text{Tr}(M_i M_k), \text{Tr}(M_j M_l) \} = \frac{1}{2} \text{Tr} \left[ \left\{ \frac{1}{2} M_i \otimes M_j \right\} \frac{1}{2} M_k M_l + \right.$$

$$+ \frac{1}{2} M_j \left\{ \frac{1}{2} M_i \otimes M_l \right\} M_k + \frac{1}{2} M_i \left\{ \frac{1}{2} M_k \otimes M_j \right\} M_l + \left.$$  

$$+ \frac{1}{2} M_i M_j \left\{ \frac{1}{2} M_k \otimes M_l \right\} \right].$$

Applying the Korotkin–Samtleben bracket (4.31), one gets:

$$\{ G_{i,k}, G_{j,l} \} = \frac{1}{2} \text{Tr} \left[ \left( M_i \Omega M_j + M_j \Omega M_i - M_i M_j M_i \right) \frac{1}{2} M_k M_l + \right.$$

(4.33)

$$+ \frac{1}{2} M_j \left( M_i \Omega M_l + M_l \Omega M_i - M_i M_l M_i \right) \frac{1}{2} M_k + \left.$$  

$$- \frac{1}{2} M_i \left( M_k \Omega M_j + M_j \Omega M_k - M_k M_j M_k \right) \frac{1}{2} M_l + \left.$$  

$$+ \frac{1}{2} M_i M_j \left( M_k \Omega M_l + M_l \Omega M_k - M_k M_l M_k \right) \right].$$

The Poisson algebra $\mathfrak{g}(2,2)$ was obtained in [27] as the restriction of the Korotkin–Samtleben bracket to the traces of products of $n \times n$ monodromy matrices.
This is a rather long computation. To simplify it we introduce a graphic representation (this will be useful also in the proof of Theorem 6.1) for the restriction of the Korotkin–Samtleben bracket (4.31) to traces of products of matrices. We represent the term of type

$$M_1^{12}M_2^{12}M_3^{12}M_4^{12}$$

as in Fig. 4.

![Figure 4](Graphic representation of the Korotkin–Samtleben bracket. The horizontal dashed line separates the spaces one (on top) and two (bottom). The vertical dashed line divides left from right: matrices at the left (resp. right) of $\Omega$ are on the left (resp. right) half plane. The two diagonal lines represent the freedom of transferring matrices through the exchange matrix.)

The trace is obtained by mapping all matrices to the right (or to the left) through the diagonal lines and taking the trace of the product of the contribution in space one with the contribution in space two (see examples in Fig. 5 and 6).

![Figure 5](Graphic representation of the relation $\text{Tr}_{12}(M_1^{12}M_2^{12}M_3^{12}M_4^{12}) = \text{Tr}(M_1M_2M_3M_4)$.)

Using this graphic representation we immediately obtain that the first two lines on the right hand side of (4.33) cancel each other and:

$$\{G_{i,k}, G_{j,l}\} = \text{Tr}(M_1M_2M_3M_4) + M_2M_3M_4M_1 - M_1M_2M_3M_4 - M_1M_2M_3M_1.$$

Since $M_i^{-1} = -M_i$ and $\text{Tr} M_i = 0$ for all $i = 1, \ldots, n$ by the skein relation (2.4) we obtain the final result. In fact

$$\begin{align*}
\text{Tr}(M_1M_2M_3M_4) &= \text{Tr}(M_1M_2)\text{Tr}(M_3M_4) - \text{Tr}(M_1M_2M_3M_4) \\
\text{Tr}(M_1M_2M_3M_4) &= \text{Tr}(M_1M_2)\text{Tr}(M_3M_4) - \text{Tr}(M_1M_2M_3M_4) \\
\text{Tr}(M_1M_2M_3M_4) &= \text{Tr}(M_1M_2)\text{Tr}(M_3M_4) - \text{Tr}(M_1M_2M_3M_4) \\
\text{Tr}(M_1M_2M_3M_4) &= \text{Tr}(M_1M_2)\text{Tr}(M_3M_4) - \text{Tr}(M_1M_2M_3M_4).
\end{align*}$$
\begin{align*}
\frac{d}{d\lambda} \tilde{\Phi} &= \left( \sum_{k=1}^{\tilde{n}} \frac{\tilde{A}_k}{\lambda - \tilde{u}_k} \right) \tilde{\Phi},
\end{align*}

with \( \tilde{n} = n - m + 1 \) for some positive integer \( m < n \), monodromy matrices \( \tilde{M}_1, \ldots, \tilde{M}_{\tilde{n}} \), where
\[
\tilde{M}_i = M_i, \quad \text{for } i = 1, \ldots, \tilde{n} - 1, \quad \text{and } \tilde{M}_{\tilde{n}} = M_{n-m+1} \cdots M_{n-1} M_n.
\]

The main idea is that system \((5.34)\) is obtained from system \((3.19)\) by clashing \( m \) poles \([24]\). To this aim we set
\[
\tilde{u} := (u_1, \ldots, u_{\tilde{n}-1}),
\]

and
\[
A_i(\tilde{u}, t) := A_i(\tilde{u}, tv_{\tilde{n}}, \ldots, tv_n), \quad \text{for } i = 1, \ldots, \tilde{n} - 1,
\]
\[
B_j(\tilde{u}, t) := A_{\tilde{n}-1+j}(\tilde{u}, tv_{\tilde{n}}, \ldots, tv_n), \quad \text{for } j = 1, \ldots, m.
\]

The Schlesinger equations in the variable \( t \) become:
\begin{align*}
\frac{\partial A_i}{\partial t} = & \sum_{j=1}^{m} \frac{v_j}{tv_j - u_i} [B_j, A_i], \quad & \frac{\partial B_j}{\partial t} = & \frac{1}{t} \sum_{k \neq j} [B_k, B_j] - \sum_{i=1}^{\tilde{n}-1} \frac{v_j}{tv_j - u_i} [B_j, A_i].
\end{align*}

The following theorem gives the conditions under which system \((5.34)\) can be obtained as the limit for \( t \to 0 \) of system \((3.19)\). We state it in full generality, namely for any dimension \( m \) of the Fuchsian systems involved.

**Theorem 5.1.** \([24]\) Let \( A_1^0, \ldots, A_{\tilde{n}-1}^0, B_1^0, \ldots, B_m^0 \) be some constant in \( t \) matrices such that
\[
\Lambda := \sum_{j=1}^{m} B_j^0
\]
The traces of the corresponding elements in the Fuchsian group are:

\[ \text{intersections} \] around the new hole perimeter will contribute to the Poisson algebra.

Note that

\[ \text{the monodromy matrix around the hole as} \]

\[ \tilde{\varphi}(\lambda, t) := \lim_{t \to \infty} \Phi(\lambda, t) \]

then the following three statements are true:

For any \( \varphi \), there exists an \( \varepsilon > 0 \) such that the Schlesinger equations (5.36) admits a unique solution in the sector \( \{ t \in \mathbb{C}, |t| < \varepsilon, \arg(t) < \varphi \} \) such that the following estimates on the asymptotic behavior hold true:

\[ |A_i(t) - A_i^0| \leq K|t|^{1-\sigma} \quad |t^{-\Lambda}B_j(t)t^\Lambda - B_j^0| \leq K|t|^{1-\sigma}, \]

for some \( \sigma \) such that \( 1 > \sigma > \vartheta \).

Let \( \Phi(\lambda, t) \) be the corresponding solution of the system (3.19) normalized at infinity. Then the limit \( \tilde{\Phi}(\lambda) := \lim_{t \to 0} \Phi(\lambda, t) \) exists and it satisfies the system (5.34) with

\[ \tilde{A}_i = A_i^0 \text{ for } i = 1, \ldots, \tilde{n} - 1, \quad \text{and } \tilde{A}_{\tilde{n}} = \Lambda. \]

The corresponding monodromy matrices of the system (5.34) under the conditions (5.40) are \( \tilde{M}_1, \ldots, \tilde{M}_{\tilde{n}} \), where

\[ \tilde{M}_i = M_i, \quad \text{for } i = 1, \ldots, \tilde{n} - 1, \quad \text{and } \tilde{M}_{\tilde{n}} = M_{n-\tilde{n}+1} \ldots M_{n-1}M_n. \]

In our case, i.e. when system (3.19) comes from the \( A_n \) algebra, i.e. the monodromy matrices are given by (3.21) and (2.9), we are always able to clash odd numbers of poles. In fact for an odd number \( m = n - \tilde{n} + 1 \),

\[ \text{Tr}(M_{n-m+1} \ldots M_{n-1}M_n) = \text{Tr}((\gamma_{n-m+1} \ldots \gamma_{n-1} \ldots \gamma_n) = 2 \cos(\pi \vartheta) \]

is always a real number bigger than 2. This implies that \( \vartheta \) is purely imaginary, hence \( \Re(\vartheta) = 0 \) and the hypotheses of theorem 5.1 are satisfied.

6. \( \mathcal{D}_n \) Algebras

In this section, we interpret system (5.34) as a system on an annulus with \( n \) marked points, the hole at the center of the annulus being the result of clashing \( m \) poles. For convenience we change our notation: we start with a system with \( n + m \) poles, clash \( m \) of them and we call the final number of orbifold points \( n \). We denote the monodromy matrix around the hole as

\[ M_h := M_{n+1} \ldots M_{n+m-1}M_{n+m}. \]

Now the paths that join the points \( u_i \) and \( u_j \) winding \( k \) times (possibly with self-intersections) around the new hole perimeter will contribute to the Poisson algebra. The traces of the corresponding elements in the Fuchsian group are:

\[ G_{i,j}^{(k)} := -\text{Tr}(M_iM_h^kM_jM_h^{-k}). \]

Note that

\[ G_{i,j}^{(k)} = G_{j,i}^{(-k)}, \]
so that in particular the level 0 elements $G_{ij}^{(0)}$ are symmetric and

$$G_{i,i}^{(0)} = 2.$$  

The Poisson algebra for the elements $G_{ij}^{(k)}$ is described in the following:

**Theorem 6.1.** The geodesic length functions $G_{ij}^{(k)}$ satisfy the following Poisson relations, for $0 \leq k$

$$\{G_{j,i}^{(0)}, G_{p,l}^{(k)}\} = (\epsilon(j - l) - \epsilon(i - l))(G_{i,j}^{(0)}G_{p,j}^{(k)} - G_{i,j}^{(0)}G_{p,l}^{(k)}) +$$

$$+(\epsilon(j - p) - \epsilon(i - p))(G_{i,p}^{(0)}G_{j,l}^{(k)} - G_{i,j}^{(0)}G_{p,l}^{(k)}).$$

and for $0 < m < k$:

$$\left\{G_{i,j}^{(m)}, G_{p,l}^{(k)}\right\} =$$

$$\epsilon(i - l)(G_{p,i}^{(m)}G_{j,l}^{(k)} - G_{i,j}^{(0)}G_{p,j}^{(m)}) + \epsilon(i - p)(G_{j,p}^{(m)}G_{i,l}^{(k)} - G_{i,j}^{(0)}G_{p,l}^{(m)}) +$$

$$+\epsilon(j - l)(G_{p,j}^{(m)}G_{i,l}^{(k)} - G_{i,j}^{(0)}G_{p,j}^{(m)}) + \epsilon(j - p)(G_{p,i}^{(m)}G_{j,l}^{(k)} - G_{i,j}^{(0)}G_{p,l}^{(m)}).$$

where $\epsilon$ denotes the sign function ($\epsilon(x) = \{-1, x < 0; 0, x = 0; 1, x > 0\}$). We introduce the generating function

$$G_{i,j}(\lambda) := A_{i,j}^{(0)} + \sum_{k=1}^{\infty} G_{i,j}^{(k)} \lambda^{-k},$$

where $A^{(0)}$ is an upper-triangular matrix with the entries

$$A_{i,j}^{(0)} = \left\{ \begin{array}{ll} G_{i,j}^{(0)} & \text{for } i < j \\ 1 & \text{for } i = j \\ 0 & \text{for } i > j. \end{array} \right.$$  

The Poisson bracket then becomes

$$\{G_{j,i}(\lambda), G_{p,l}(\mu)\} =$$

$$\left(\epsilon(j - p) - \frac{\lambda + \mu}{\lambda - \mu}\right) G_{p,i}(\lambda)G_{j,l}(\mu) +$$

$$+ \left(\epsilon(i - l) + \frac{\lambda + \mu}{\lambda - \mu}\right) G_{j,i}(\lambda)G_{i,l}(\mu) +$$

$$+ \left(\epsilon(i - p) - \frac{1 + \lambda\mu}{1 - \lambda\mu}\right) G_{j,p}(\lambda)G_{i,l}(\mu) +$$

$$+ \left(\epsilon(j - l) + \frac{1 + \lambda\mu}{1 - \lambda\mu}\right) G_{i,j}(\lambda)G_{p,l}(\mu).$$  

This is an abstract infinite-dimensional Poisson algebra.

Note that at level zero the relation (6.42) produces the Nelson–Regge algebra (2.3) for $G_{ij}^{(0)}$. 
Before proving this theorem it is important to stress that the number $m$ of clashed poles does not appear in the formulae. Indeed, if we consider the elements $G_{i,j}^{(k)}$ as infinitely many independent elements, the brackets \((6.44)\) and \((6.45)\) define a Poisson algebra satisfying the Jacobi identity.

**Definition 6.2.** We call the Poisson relations \((6.44)\) and \((6.45)\) the $D_n$-algebra of the elements $G_{i,j}^{(k)}$.

In Subsec. \(6.3\) we prove that $D_n$ is the semiclassical limit of the twisted $q$–Yangian $Y'_q(\mathfrak{o}_n)$ for the orthogonal Lie algebra $\mathfrak{o}_n$ introduced in \([32]\). This gives an alternative proof to the Jacobi identity.

Geometrically, we interpret the $D_n$ algebra as the algebra of geodesic length functions on an annulus with $n$ marked points. To understand this it is enough to look at the geometric construction of a hole by clashing two poles.

### 6.1. Example: Clashing two poles

Observe first that the product of two traceless elements is a hyperbolic element. This is consistent with the fact that by clashing two points we get a hole. We begin with the algebra $A_{n+2}$ and interpret the element $M_h = M_{n+1}M_{n+2}$ as an element corresponding to going around a new hole.

We consider the Fuchsian group $\Delta_{0,2,n}$ generated by $M_h$ and $M_i$, $i = 1, \ldots, n$. Obviously, the group thus constructed is a subgroup of $\Delta_{0,1,n+2}$, the Fuchsian group in the $A_{n+2}$ case.

When clashing two poles $u_{n+1}$ and $u_{n+2}$ we create a hole thus obtaining a new hole perimeter, the loop containing $u_{n+1}$ and $u_{n+2}$. As a consequence, we can consider the subgraph in the right-hand side of Fig. 7 instead of the tree-like subgraph in the left-hand side.

![Figure 7](image-url)

The product of matrices corresponding to paths that go around $u_{n+1}$ and $u_{n+2}$ in the left-hand side is then preserved: we have the matrix equality

\[
X_Y RX_{Z_{n+2}} FX_{Z_{n+1}} RX_{Z_{n+1}} RX_Y = X_{Y_h} RX_{Z_h} RX_{Y_h}.
\]

This equality holds provided the new coordinates $Z_h$ and $Y_h$ are defined as follows:

\[
Y + Z_{n+1} + Z_{n+2} = Y_h + \frac{Z_h}{2},
\]

\[
G_{n+1,n+2} = e^{Z_{n+1}+Z_{n+2}} + e^{-Z_{n+1}+Z_{n+2}} + e^{Z_{n+1}-Z_{n+2}} + e^{-Z_{n+1}-Z_{n+2}} = e^{Z_h/2} + e^{-Z_h/2}.
\]

Obviously, $G_{n+1,n+2}$ commutes with all elements of the (sub)group $\Delta_{0,2,n}$.

\(^5\)From a geometric view point we have no constraint on the number of clashed poles. The constraint that $m$ needs to be odd arises when we want to carry out the clashing in the analytic framework.
To show that the Fuchsian group $\Delta_{0,2,n}$ generated by $M_h$ and $M_i$, $i = 1, \ldots, n$ can be seen as a subgroup of $\Delta_{0,1,n+2}$, the Fuchsian group in the $A_{n+2}$ case, we show that the elements $G_{ij}^{(k)}$ can be expressed in terms of elements in $\Delta_{0,1,n+2}$. For example:

$$
G_{ij}^{(1)} = G_{i,n+2} G_{n+1,n+2} G_{j,n+1} - G_{i,n+1} G_{j,n+1} - G_{i,n+2} G_{j,n+2} + G_{i,j},
$$

where (6.51) follows from the skein relation (2.4), see figure 8.

6.2. Proof of Theorem 6.1

The proof of this theorem is organized as follows: first we construct the Poisson algebra of the elements $G_{ij}$ in a subgroup of $\Delta_{0,1,n+2}$ (see Appendix B). Then we prove that the abstract algebra $\mathfrak{O}_n$ is actually an infinite dimensional Poisson algebra (see Appendix B).

We begin by extending the Korotkin–Samtleben bracket to the monodromy matrices of the system (5.34):

**Proposition 6.3.** If $M_i$, $i = 1, \ldots, n + m$, satisfy the brackets (4.31), then the set of new matrices $\tilde{M}_i$, $i = 1, \ldots, n + 1$ such that $\tilde{M}_i = M_i$ for $i = 1, \ldots, n$ and $\tilde{M}_{n+1} = M_h$ satisfy the same brackets (4.31). We also have the following brackets for the powers of $M_h$:

$$
\left\{ M_i \otimes M_h^k \right\} = \frac{1}{2} \left( \frac{1}{M_i} \Omega M_h^k + \frac{2}{M_h} \Omega M_i - \frac{1}{M_i} M_h^k \frac{1}{M_h} \right), \forall k \in \mathbb{Z},
$$

(6.52)

$$
\left\{ M_h \otimes M_h^k \right\} = \frac{1}{2} \left( \frac{2}{M_h} \Omega M_h + \frac{1}{M_h} M_h^k \Omega - \frac{1}{M_h} M_h^k \frac{1}{M_h} \right), \forall k \in \mathbb{Z}.
$$

(6.53)

**Proof.** First we prove the case when $m = 2$, i.e. $M_h = M_{n+1} M_{n+2}$:

$$
\left\{ M_i \otimes M_h \right\} = \frac{1}{2} \left( \frac{1}{M_i} \Omega M_{n+1} + \frac{2}{M_{n+1}} \Omega M_i - \frac{1}{M_i} M_{n+1} + \frac{2}{M_{n+1}} M_i \Omega \right) M_{n+2} -
$$

$$
+ \frac{1}{2} M_{n+1} \left( \frac{1}{M_i} \Omega M_{n+2} + \frac{2}{M_{n+2}} \Omega M_i - \frac{1}{M_i} M_{n+2} + \frac{2}{M_{n+2}} M_i \Omega \right) =
$$

$$
= \frac{1}{2} \left( \frac{1}{M_i} \Omega M_{n+1} + \frac{2}{M_{n+1}} \Omega M_i - \frac{1}{M_i} M_{n+1} + \frac{2}{M_{n+1}} M_i \Omega \right).
$$
The proof for any \( m \) is a straightforward induction on \( m \) and we omit it. The proof of (6.52) is a simple consequence of the fact that \( M_1, \ldots, M_n, M_h \) satisfy the brackets (4.31). The proof of (6.53) is again the same sort of computations, using the freedom of transferring \( M_h \Omega = \Omega M_h \) and \( M_h^2 \Omega = \Omega M_h^2 \) through the exchange matrix \( \Omega \).

\[ \square \]

Proof: To conclude the proof of Theorem 6.1 we first outline how to obtain (6.44). We compute

\[
\{ G_{j,i}^{(m)}, G_{p,i}^{(k)} \} = \{ \text{Tr} \left( M_j M_h^m M_i M_h^{-m} \right), \text{Tr} \left( M_p M_h^k M_i M_h^{-k} \right) \},
\]

by applying the Leibnitz rule and by using the extended Korotkin–Samtleben bracket (6.52) and (6.53). By a rather long but straightforward computation using the graphic representation explained in the proof of Lemma 4.2 we arrive to a sum of traces that we break up by using the skein relation.

The equivalence of (6.44) and (6.48) is a standard exercise; the most important part is to verify the Jacobi relations for these brackets. For this, the representation (6.46) seems to be the most convenient one. The computation is purely technical and we present it in Appendix B.

\[ \square \]

6.3. \( \mathcal{D}_n \) as semi–classical limit of \( Y_q(\mathfrak{o}_n) \). Here, we prove that \( \mathcal{D}_n \) is the semi-classical limit of the twisted \( q \)-Yangian \( Y_q(\mathfrak{o}_n) \) for the orthogonal Lie algebra \( \mathfrak{o}_n \) introduced in [32]. The latter is the algebra generated by the matrix elements \( G_{i,j}^{(k)} \), \( i, j = 1, \ldots, n, k \in \mathbb{Z}_{\geq 0} \) subject to the defining relations:

\[
R(\lambda, \mu) G(\lambda) R(\lambda^{-1}, \mu)^{T_1} G(\mu) = G(\mu) R(\lambda^{-1}, \mu)^{T_1} G(\lambda) R(\lambda, \mu)
\]

where the \( R \)-matrix is given by

\[
R(\lambda, \mu) = (\lambda - \mu) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1} - q) \sum_i E_{ii} \otimes E_{ii} + (q^{-1} - q) \lambda \sum_{i < j} E_{ij} \otimes E_{ji} + (q^{-1} - q) \mu \sum_{i > j} E_{ij} \otimes E_{ji}
\]

and it is a solution of the Yang–Baxter equation and the apex \( T_1 \) indicates the transposition in space one.

The semiclassical limit is obtained by putting \( q = -\exp(i\pi \hbar) \) and taking the terms of order \( \hbar \) in the Laurent expansion as \( \hbar \) tends to zero. The \( R \) matrix is expanded as

\[
R(\lambda, \mu) = (\lambda - \mu) \mathbb{1} \otimes \mathbb{1} + i\pi \hbar \, r(\lambda, \mu),
\]

where \( r \) is a classical \( r \)-matrix:

\[
r(\lambda, \mu) = (\lambda + \mu) \sum_{ij} E_{ii} \otimes E_{jj} + 2\lambda \sum_{i < j} E_{ij} \otimes E_{ji} + 2\mu \sum_{i > j} E_{ij} \otimes E_{ji},
\]

while the matrix \( G(\lambda) \) remains the same. The reflection equation (6.54) in the semiclassical limit becomes:

\[
\{ G(\lambda) \otimes G(\mu) \} = \left[ \frac{r(\lambda, \mu)}{\lambda - \mu}, G(\lambda) \right] G(\mu) + G(\lambda) \frac{r(\lambda^{-1}, \mu)^{T_1}}{\lambda^{-1} - \mu} G(\mu) - G(\mu) \frac{r(\lambda^{-1}, \mu)^{T_1}}{\lambda^{-1} - \mu} G(\lambda).
\]
It is a straightforward computation to show that formula (6.57) coincides with our formula (6.48).  

6.4. Braid-group relations for \( G_{ij}^{(k)} \). The braid group (and the mapping class group) in the case of \( \mathcal{D}_n \) algebra is generated by \( n \) generators. Besides the standard generators \( \beta_{i,i+1} \), \( i = 1, \ldots, n-1 \), each of which interchanges the \( i \)th and the \( i+1 \)th orbifold points, we have one additional generator \( \beta_{1,n} \) interchanging the first and the last, \( n \)th orbifold points. The action of this element is not trivial due to the presence of the new hole:

**Theorem 6.4.** Let \( \mathcal{G}^{(k)} \) be a matrix of entries \( G_{ij}^{(k)} \). The action of the braid-group elements \( \beta_{i,i+1} \) for \( i = 1, \ldots, n-1 \) is given by

\[
\beta_{i,i+1} G^{(k)} = \tilde{G}^{(k)} : \begin{cases} 
\tilde{G}^{(k)}_{i,i+1,j} = G^{(k)}_{i,i+1,j}, & j > i + 1, \\
\tilde{G}^{(k)}_{i+1,i,j} = G^{(k)}_{i+1,i,j}, & j < i, \\
\tilde{G}^{(k)}_{i,i+1,j} = G^{(k)}_{i,i+1,j} - G^{(k)}_{i+1,i,j}, & j > i + 1, \\
\tilde{G}^{(k)}_{i,i,j} = G^{(k)}_{i,i,j} - G^{(k)}_{i,i+1,j}, & j < i, \\
\tilde{G}^{(k)}_{i,i,j} = G^{(k)}_{i,i,j} - G^{(k)}_{i,i+1,j} - G^{(k)}_{i+1,i+1,j} + G^{(k)}_{i+1,i+1,j}, \\
\tilde{G}^{(k)}_{i+1,i+1,j} = G^{(k)}_{i+1,i+1,j}.
\end{cases}
\]

The action of the last generator, \( \beta_{n,1} \) is

\[
\beta_{n,1} G^{(k)} = \tilde{G}^{(k)} : \begin{cases} 
\tilde{G}^{(k)}_{n,j} = G^{(k+1)}_{n,j}, & n > j, \\
\tilde{G}^{(k)}_{1,n} = G^{(k-1)}_{1,n}, & 1 < j < n, \\
\tilde{G}^{(k)}_{n,j} = G^{(k)}_{n,j} - G^{(k-1)}_{n,j}, & n > j, \\
\tilde{G}^{(k)}_{1,n} = G^{(k)}_{1,n} - G^{(k+1)}_{1,n}, & n > j, \\
\tilde{G}^{(k)}_{n,j} = G^{(k)}_{n,j} \left( G^{(1)}_{n,j} \right)^2 - G^{(k+1)}_{n,j} - G^{(k-1)}_{n,j} G^{(1)}_{n,j} + G^{(k)}_{n,j}, \\
\tilde{G}^{(k)}_{1,n} = G^{(k)}_{1,n} - G^{(k+1)}_{1,n} - G^{(k-1)}_{1,n}, \\
\tilde{G}^{(k)}_{j,n} = G^{(k)}_{j,n} - G^{(k+1)}_{j,n}, \\
\tilde{G}^{(k)}_{1,1} = G^{(k)}_{1,1},
\end{cases}
\]

where we imply the relations (6.42) and (6.43), in particular, we have

\[
\tilde{G}^{(1)}_{n,1} = G^{(0)}_{n,1} - G^{(1)}_{1,n} = G^{(1)}_{n,1}.
\]

**Proof.** We deduce the braid group relations for \( G_{ij}^{(k)} \) defined by (6.41) by the braid group action (3.26) on the monodromy matrices \( M_1, \ldots, M_n, M_h \). Setting:

\[
\tilde{G}^{(k)}_{ij} = \text{Tr} \left( \beta(M_i) \beta(M_j^{(k)}) \beta(M_j) \beta(M_j^{-k}) \beta(M_j^{(k)}) \right),
\]

one immediately obtains (6.58). In the same way, defining

\[
\beta_{n,1} := \beta_{n,h} \beta_{h,1} \beta_{n,h}^{-1},
\]

we get (6.59).\[\square\]

\footnote{Observe that in [32] a different normalization for \( \mathcal{G} \) was used: the level 0 was taken lower triangular. As a consequence the \( R \)-matrix was globally transposed.}
Remark 6.5. The braid group action (6.58), (6.59) does not depend on the number $m$ of clashed poles, and it is therefore well defined in the abstract case as well. Observe that the action of the braid-group elements $\beta_{i,i+1}$ for $i = 1, \ldots, n - 1$ is defined separately for each of $n \times n$ matrices $G^{(k)}$ (i.e. all relations involve one and the same level $k$) and it has precisely the same form for all of them, while the action of the of the last generator $\beta_{n,1}$ mixes different levels (labeled by the index $(k)$). The following proposition is straightforward to prove:

**Proposition 6.6.** The braid group transformations for $D_n$ algebra have the following matrix representation in terms of the matrix $G(\lambda)$ (6.46):

\begin{equation}
\beta_{i,i+1} G(\lambda) = B_{i,i+1} G(\lambda) (B_{i,i+1}^{-1})^T, \quad i = 1, \ldots, n - 1
\end{equation}

where the matrices $B_{i,i+1}$ depend only on $G^{(0)}_{ij}$ and have the form (2.10) (with $G_{ij}$ replaced by $G^{(0)}_{ij}$). The action of $\beta_{n,1}$ is

\begin{equation}
\beta_{n,1} G(\lambda) = B_{n,1} G(\lambda) (B_{n,1}^{-1})^T,
\end{equation}

where

$$B_{n,1}(\lambda) = \begin{pmatrix}
0 & 0 & \ldots & 0 & \lambda \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & 1 & 0 \\
-\lambda^{-1} & 0 & \ldots & 0 & G^{(1)}_{n,1}
\end{pmatrix}.$$  

6.5. Quantum braid group relations. In this subsection, we assume $G^{(k)h}_{ij}$ to be Hermitian operators, subject to quantum exchange relations (6.54). The action of the braid group then follows from the one for the quantum $D_n$ algebra in [7].

We define the quantum $G^h(\lambda)$ to be

\begin{equation}
G^h_{i,j}(\lambda) := A^h_{i,j} + \sum_{k=1}^{\infty} G^{(k)h}_{i,j} \lambda^{-k},
\end{equation}

where $A^h$ is an upper-triangular matrix with the entries $A^h_{i,j} = \{G^{(0)h}_{i,j}, i < j; q^{-1}, i = j; 0, i > j\}$. Recall that $q^1 = q^{-1}$.

**Proposition 6.7.** The braid group transformations for the quantum $D_n$ algebra have the following matrix representation in terms of the matrix $G^h(\lambda)$ (6.63):

\begin{equation}
\beta^h_{i,i+1} G^h(\lambda) = B^h_{i,i+1} G^h(\lambda) (B^h_{i,i+1})^T, \quad i = 1, \ldots, n - 1
\end{equation}

where

\begin{equation}
B^h_{i,i+1} = \begin{pmatrix}
1 & & & & \\
\vdots & \ddots & 1 & & \\
0 & \ddots & 0 & q G^{(0)h}_{i,i+1} & -q^2 \\
\vdots & & 0 & 1 & \\
& & & & 1
\end{pmatrix}.
\end{equation}
The action of $\beta_{n,1}^{\hbar}$ is

$$\beta_{n,1}^{\hbar} G^{\hbar}(\lambda) = B_{n,1}^{\hbar}(\lambda) G^{\hbar}(\lambda) (B_{n,1}^{\hbar}(\lambda^{-1}))^\dagger,$$

where

$$B_{n,1}^{\hbar}(\lambda) = \begin{pmatrix} 0 & 0 & \ldots & 0 & \lambda \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \vdots & \ddots & 1 & 0 \\ -q^2\lambda^{-1} & 0 & \ldots & 0 & qG^{(1)}_{n,1}^{\hbar} \end{pmatrix}.$$

7. Central elements of the $\mathfrak{D}_n$ algebras and of its Poissonian reductions

Thanks to the matrix form of the full braid group action given in Proposition 6.6 above, Molev and Ragoucy’s result that the central elements of the $\mathfrak{D}_n$ algebra are given by the coefficients of $\lambda$ in the polynomial

$$\det (G(\lambda))$$

still holds true for the full braid group action.

We are now going to study two types of finite-dimensional reductions, the level-$p$ reductions and the reductions to the $D_n$ case (see Sec. 2.2), the corresponding braid group actions and their central elements. Before describing these two types of reductions let us recall how to produce the central elements in the $A_n$ case.

7.1. Central elements of $A_n$. The general central elements of the Poisson algebra are simultaneously the braid-group invariants (which translates into mapping-class-group invariant terms in the geometrical setting). This can be easily seen from that the relation (2.11) holds for the transposed matrix $A$ as well,

$$\beta_{i,i+1} A^T = B_{i,i+1} A^T B_{i,i+1}^T,$$

and so for any linear combination $\lambda A + \lambda^{-1} A^T$, whose determinant is therefore braid-group invariant object [4]. Because $\det A = 1$, the generating function for Poisson central elements is

$$\det(A^{-T} A - \varphi) = \det(\lambda A + \lambda^{-1} A^T)\lambda^{-n}, \quad \varphi = -\lambda^2,$$

which coincides with the generating function for the invariants of the braid group. Therefore when considering the Nelson–Regge algebra (2.5) as an abstract algebra for $\binom{n(n-1)}{2}$ elements, the total number of possibly independent central elements is $\binom{n}{2}$.

In the geometric case, the Poisson commuting elements (the Casimir elements) are traces of monodromies. Whereas $\text{Tr} M^k_i = \text{Tr}(F^k)$ is a constant, the monodromy at infinity is nontrivial and its trace is equal to

$$\text{Tr} M_{\infty} = (-1)^{n-1} \cosh(P),$$

where $P$ is the perimeter of the hole defined in (2.7). We can prove that in this case the only non trivial braid group invariants generated by (7.70) is precisely $P$ [10].
7.2. **Level** $p$-*reductions.* We obtain the level $p$ reduction if we set

$$M_h^p = 1$$

for some integer $p$. In the $2 \times 2$ monodromy case, $\text{Tr} \ M_h = 2 \cos(2\pi k/p) = e^{\pi/h^2} + e^{-\pi/h^2}$, where $k$ is an integer and $P_h = 4\pi i k/p$ is the complex-valued perimeter of the second hole. The condition (7.71) is Poissonian because substituting it into relations (6.52) and (6.53) for $k = p$ we obtain identities: the left-hand sides of these relations vanish.

In the geometric setting, this condition means that, instead of a new hole, we introduce a new orbifold point of order $p$. On the level of elements of the $D_n$ algebra, it means that $G^{(k+p)}_{i,j} = G^{(k)}_{i,j}$ or, since $G^{(-k)}_{j,i} = G^{(k)}_{i,j}$, we obtain

$$G^{(k)}_{i,j} = G^{(p-k)}_{j,i} \text{ for } k = 0, \ldots, p - 1.$$ 

Due to this reduction, the generating function simplifies to $G(\lambda) = \frac{1}{\lambda^{p-1}} G_p(\lambda)$ where

$$G_p(\lambda) := A^{(0)} + \frac{G^{(1)}}{\lambda} + \cdots + \frac{G^{(p-1)}}{\lambda^{p-1}} + \frac{A^{(0)\text{T}}}{\lambda^p},$$

so the algebra becomes finite and we reserve for it the notation $D_n^{(p)}$.

**Proposition 7.1.** The braid group relations (6.60) and (6.61) imposed retain their matrix forms provided we replace $G(\lambda)$ by $G_p(\lambda)$ given in (7.73). There are exactly $\left[\frac{np}{2}\right]$ algebraically independent central elements in the algebra $D_n^{(p)}$. They are generated by $\det G_p(\lambda)$.

The proof of this proposition is very technical and we set it in Appendix C.

**Remark 7.2.** Observe that for $p = 1$, $\det(G_1(\lambda))$ becomes the generating function of the braid group invariants for the $A_n$ algebra given in (7.70).

We can now compute the dimension of the Poisson leaves corresponding to the algebra $D_n^{(p)}$. From condition (7.72) we have that the number of generators $G^{(k)}_{i,j}$ of the algebra $D_n^{(p)}$ is $n^2 p/2$ for even $p$ and $n(np-1)/2$ for odd $p$. Having $\left[\frac{np}{2}\right]$ generally algebraically independent central elements, we find that the highest dimensional symplectic leaves of the algebra $D_n^{(p)}$ have dimension always even:

$$\text{Poisson leaf dim} = \begin{cases} 
\frac{n^2 p}{2} - np = \frac{n(n-1)p}{2}, & \text{for } p \text{ even, any } n, \\
\frac{n(n-1)p}{2} - \frac{n}{2}, & \text{for } p \text{ odd, } n \text{ even,} \\
\frac{n(n-1)p}{2} - \frac{n-1}{2}, & \text{for } p \text{ odd, } n \text{ odd.}
\end{cases}$$

As a consequence, the geometric case corresponds to highly degenerated symplectic leaves, as their dimension is $2(n - 2)$. It is an interesting problem to provide the complete classification of the dimensions of symplectic leaves of $D_n^{(p)}$ algebras in the spirit of such the classification for the $A_n$ algebras constructed by Bondal [4]; we leave it for future studies.

7.3. **Reduction of $D_n$ to the $D_n$ algebra.**
7.3.1. Basic relations of the reduction $\mathcal{D}_n \to D_n$. We begin with that we naturally identify those elements of the algebra $\mathcal{D}_n$ that correspond to geodesics without self-intersections with the corresponding elements of the $D_n$ algebra (see Figure 3):

$$G_{i,j}^{(0)} \to \hat{G}_{i,j}, \quad 1 \leq i < j \leq n$$

and

$$G_{i,j}^{(1)} \to \hat{G}_{i,j}, \quad 1 \leq j < i \leq n.$$  

We now use the skein relations to present elements $G_{i,j}^{(1)}$ with $i \leq j$:

$$G_{i,i}^{(1)} \to 2\hat{G}_{i,i}\hat{G}_{j,j} - \hat{G}_{j,i} + (\Pi^2 - 2)\hat{G}_{i,j}, \quad 1 \leq i < j \leq n,$$

(7.75)

or, in the graphical form,

$$= 2 \quad \begin{array}{c} \includegraphics{diagram1.png} \end{array} - \begin{array}{c} \includegraphics{diagram2.png} \end{array} + \begin{array}{c} \includegraphics{diagram3.png} \end{array} -2 \cdot \begin{array}{c} \includegraphics{diagram4.png} \end{array}.$$  

(To obtain these relations we also used that when resolving the skein relations, the empty loop is equal $-2$.) Note, first, the appearance of the parameter

$$\Pi := e^{P_h/2} + e^{-P_h/2},$$

which is the geodesic function for the second hole with the perimeter $P_h$, and, second, that we can rewrite the relation (7.76) as

$$G_{i,j}^{(1)} \to 2\hat{G}_{i,i}\hat{G}_{j,j} - \hat{G}_{j,i} + (\Pi^2 - 2)\hat{G}_{i,j}, \quad 1 \leq i < j \leq n,$$

(7.76)

or, recalling that $G^{(k)} = (G^{(-k)})^T$,

$$G_{i,j}^{(1)} \to 2\hat{G}_{i,i}\hat{G}_{j,j} - \hat{G}_{i,j}^{(-1)} + (\Pi^2 - 2)\hat{G}_{i,j}^{(0)}, \quad 1 \leq i < j \leq n, \quad \Pi \neq 0.$$  

It is especially useful to express this reduction in terms of the matrices $\hat{A}$, $\hat{R}$, and $\hat{S}$ defined in (2.18), (2.16) and (2.17) respectively:

$$G^{(1)} \to \hat{R} + \hat{S} + (\Pi^2 - 1)\hat{A} - \hat{A}^T.$$  

\footnote{Note that all the elements $\hat{G}_{i,j}$ with $i \neq j$ can be expressed as polynomial expressions of the elements $G_{a,b}$ of the $A_{n+m}$-algebra; this property is however lacking for the diagonal elements $\hat{G}_{i,i}$, which cannot be presented as polynomial functions of the generators of the $A_{n+m}$-algebra.}
We now continue expressing higher $G^{(k)}$ fixing the parameter $i$ and moving $j$ counterclockwise as shown below (in the l.h.s. moving $j$ counterclockwise corresponds to constructing the geodesic which winds around the hole twice):

$$
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure1.png}
\end{array}
\end{align*}
$$

After completing this counterclockwise rotation of $j$, we obtain in the left-hand side the element $G^{(2)}_{i,j}$ whereas in the right-hand side the product $\hat{G}_{i,i} \hat{G}_{j,j}$ is cyclically symmetric, the term $-G^{(-1)}_{i,j}$ becomes $-G^{(0)}_{i,j}$, and the last term $(\Pi^2 - 2)G^{(0)}_{i,j}$ becomes $(\Pi^2 - 2)G^{(1)}_{i,j}$, and we again apply the reduction (7.77) and obtain that the matrix $G^{(2)}$ can be in turn presented as a linear combination of the matrices $\hat{R}$, $\hat{S}$, $\hat{A}$, and $\hat{A}^T$. It is not difficult to solve the obtained recurrent relations and obtain the following reduction law for $G^{(k)}$:

$$
G^{(k)} \rightarrow e^{kP_h} - e^{-kP_h} \frac{e^{kP_h} - 2 + e^{-kP_h}}{(e^{P_h} - 1)(1 - e^{P_h})} \hat{S} + \frac{e^{kP_h} - e^{-kP_h}}{1 - e^{P_h}} \left[ \frac{e^{(k-1)P_h}}{e^{P_h} - 1} - \frac{e^{-(k-1)P_h}}{e^{P_h} - 1} \right] \hat{A}^T, \ k \geq 1.
$$

(7.77)

The corresponding law for $G^{(-k)}$ can be obtained by transposing these relations. Now, the main statement follows.

**Theorem 7.3.** The expressions (6.60) and (6.61) are faithful representations for the braid-group action (2.13), (2.14) on elements of the $D_n$ algebra for any non-zero $\Pi$ provided the matrices $G^{(k)}$ are expressed through the elements $\hat{G}_{i,j}$ and the parameter $\Pi \neq 0$ using formulas (7.77).

**Proof.** We can verify directly that if we substitute the reduction formula (7.77) for every block $G^{(k)}$ in the matrix representation (6.40) and perform the braid-group transformations in the matrix form (6.60) and (6.61), then, in each matrix entry, the transformed quantities $\hat{G}_{i,j}$ will satisfy relations (2.13) and (2.14). □

Note that one of the most important features of the braid group relations for both the $D_n$ algebra and the $D_n$ algebra is that neither of them depends on the hole perimeters $P$ and $P_h$. So, since these relations both describe the same mapping class group transformations in the geometrical case, it is natural to expect that they will also coincide if we apply the reduction procedure based on the skein relation. In fact this is the case as shown by the following:

**Corollary 7.4.** A $p$-level reduction is consistent with the $D_n$-reduction provided $(e^{P_h})^p = 1$. In particular, the braid-group representation for $G_p(\lambda)$ (7.77) generates then the braid-group relations (2.13) and (2.14) of the algebra $D_n$. 

Remark 7.5. The case Π = 0 corresponding to the reduction of level 2 would correspond to the $D_n$ reduction $G_{i,j}^{(1)} = \hat{G}_{j,i} = G_{j,i}^{(1)}$. In this case, however, the algebraic elements $\hat{G}_{i,j}$ become dependent ($\hat{G}_{i,j} \hat{G}_{j,j} = \hat{G}_{i,j} + \hat{G}_{j,i}$) and we are lacking the complete $D_n$ algebra.

7.3.2. Central elements of the $D_n$ algebra. We now substitute reduction formulas (7.77) into the representation (6.46) and perform the explicit summation over powers of $\lambda^{-1}$. The results reads

$$G(\lambda) \to \frac{\lambda}{(\lambda-1)(e^{-\rho_b u} - 1)(e^{\rho_b u} - 1)} \times$$

$$\times [(\lambda - 1) \hat{K} + (\lambda + 1) \hat{S} + (\lambda^2 - 1) \hat{A} - (\lambda - \lambda^{-1}) \hat{A}^T],$$

(7.78)

so we come to the following proposition.

Proposition 7.6. The $D_n$ algebra admits exactly $n$ algebraically independent central elements $c_1, \ldots, c_n$. They are generated by

$$\det[(\lambda - 1) \hat{K} + (\lambda + 1) \hat{S} + (\lambda^2 - 1) \hat{A} - (\lambda - \lambda^{-1}) \hat{A}^T] =$$

$$= (\lambda - 1)^n \left[\lambda^{n+1} + \sum_{i=1}^n \lambda^i c_i + (-1)^{n+1} \sum_{i=1}^n \lambda^{1-i} c_i + (-1)^{n+1} \lambda^{-n}\right].$$

Proof. The fact that the central elements are generated by $\det(G(\lambda))$ follows from Theorem 7.3 and the formula for the central elements of $D_n$. The fact that this determinant takes the form given by the second row of (7.80) follows from the substitution (7.77) and the fact that the matrix $\hat{S}$ has rank one, so no more than one element of this matrix can enter the products when expanding the determinant over products of entries, and all other entries are proportional to $(\lambda - 1)$. To prove the algebraic independence of $c_1, \ldots, c_n$, let us consider the particular case where $\hat{G}_{i,j} = 0$ for $i \neq j$ and $\hat{G}_{i,i} \neq 0$ and $\hat{G}_{i,i}^2 \neq \hat{G}_{j,j}^2$ for $i \neq j$. In this case, $G(\lambda)$ becomes

$$(\lambda + \lambda^{-1})(\lambda - 1)E + \text{diag}\hat{G}_{i,i}$$

$$= \begin{pmatrix} 1 + \lambda & 2\lambda & \ldots & 2\lambda \\ 2 & 1 + \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2\lambda \\ 2 & \ldots & 2 & 1 + \lambda \end{pmatrix} \text{diag}\hat{G}_{i,i}$$

(here $\text{diag}\hat{G}_{i,i}$ is the diagonal matrix with the entries $\hat{G}_{i,i}$), and evaluating the determinant by the minors of the second matrix, we obtain that

$$\det G(\lambda) = \sum_{k=0}^n (\lambda - 1)^n (\lambda + \lambda^{-1})^{n-k} \left[\frac{\lambda + 1}{\lambda - 1}\right]^{1 - (-1)^n} \times$$

$$\text{SYM}_k(\hat{G}_{i,1}^2, \ldots, \hat{G}_{n,n}^2),$$

where SYM$_k$ is the symmetrical function of order $k$ of $n$ pairwise distinct variables $\hat{G}_{i,i}$, and these functions are obviously algebraically independent for $k = 1, \ldots, n$.

It remains to prove that in $D_n$ there are no more than $n$ central elements. For this let us consider the $D_n$ Poisson structure for $\hat{G}_{i,j}$ treating the elements $\hat{G}_{i,j}$ with $i, j = 1, \ldots, n$ as coordinates of a linear space $\mathbb{C}^n$. The Poisson brackets then define locally a structure of a bi-vector field, and if this structure has degeneracy
at most \( n \) in a vicinity of just one point, then there are no more than \( n \) central elements in the (global) Poisson algebra of \( D_n \).

A convenient choice of such a point is again \( \hat{G}_{i,j} = 0 \) for \( i \neq j \) and \( \hat{G}_{i,i} \neq 0 \) and \( \hat{G}_{i,i}^2 \neq \hat{G}_{j,j}^2 \) for \( i \neq j \). Then, the Poisson brackets from [5, 7] take the following form in the vicinity of this point in the configuration space \( \mathbb{C}^n \):

\[
\begin{align*}
\{\hat{G}_{i,j}, \hat{G}_{i,i}\} &= 2\hat{G}_{j,j} + O(\hat{G}_{\alpha,\beta}), \quad \alpha \neq \beta, \\
\{\hat{G}_{i,j}, \hat{G}_{j,j}\} &= -2\hat{G}_{i,i} + O(\hat{G}_{\alpha,\beta}), \quad \alpha \neq \beta, \\
\{\hat{G}_{i,j}, \hat{G}_{j,i}\} &= 2\hat{G}_{j,j}^2 - 2\hat{G}_{i,i}^2 
\end{align*}
\]

and all other brackets are either zero or of order \( O(\hat{G}_{\alpha,\beta}) \) with \( \alpha \neq \beta \) i.e., they are at least of the linear order in the variables that are small in the vicinity of the given point. The brackets (7.80) for the variables \( \hat{G}_{i,j} \) with \( i \neq j \) are obviously non-degenerate (because these variables just come in \( n(n+1)/2 \) pairs \( \hat{G}_{i,j}, \hat{G}_{j,i} \) and commute in the given approximation order \( O(1) \) with such the variables from all other pairs). So, in fact, the Poisson leaf has the dimension at least \( n(n-1) = n^2 - n \) in the vicinity of the given point, and we therefore have no more than \( n \) central elements of the \( D_n \) algebra.

\[\square\]

**Remark 7.7.** In fact, if we disregard all terms of order \( O(\hat{G}_{\alpha,\beta}) \) with \( \alpha \neq \beta \) in the brackets (7.80), then the Poisson dimension of the obtained system is exactly \( n(n-1) \); this is a simple but nice exercise in linear algebra which we leave to the reader. So, the highest Poisson leaf dimension of the \( D_n \) algebra is \( n^2 - n = n(n-1) \).

### 7.3.3. Central elements for \( D_2 \) and \( D_3 \)

In the \( D_2 \) algebra, we have the following two central elements:

\[
\begin{align*}
C_1^{(2)} &= \hat{G}_{1,1}\hat{G}_{2,2} - \hat{G}_{1,2} - \hat{G}_{2,1}, \\
C_2^{(2)} &= \hat{G}_{1,2}\hat{G}_{2,1} - \hat{G}_{2,2}^2 - \hat{G}_{1,1}^2.
\end{align*}
\]

In the \( D_3 \) algebra case there are three central elements:

\[
\begin{align*}
C_1^{(3)} &= \hat{G}_{1,1}\hat{G}_{2,2}\hat{G}_{3,3} - \hat{G}_{1,1}(\hat{G}_{3,2} + \hat{G}_{2,3}) - \hat{G}_{2,2}(\hat{G}_{1,3} + \hat{G}_{3,1}) - \hat{G}_{3,3}(\hat{G}_{2,1} + \hat{G}_{1,2}), \\
C_2^{(3)} &= \hat{G}_{1,2}\hat{G}_{2,3}\hat{G}_{3,1} - \hat{G}_{1,2}\hat{G}_{2,1} - \hat{G}_{2,3}\hat{G}_{3,2} - \hat{G}_{3,1}\hat{G}_{1,3} + \hat{G}_{2,1}^2 + \hat{G}_{2,2}^2 + \hat{G}_{3,3}^2, \\
C_3^{(3)} &= \hat{G}_{1,3}\hat{G}_{2,1}\hat{G}_{3,2} - \hat{G}_{1,2}\hat{G}_{2,1}\hat{G}_{3,3} - \hat{G}_{2,3}\hat{G}_{3,2}\hat{G}_{1,1} - \hat{G}_{3,3}\hat{G}_{1,3}\hat{G}_{2,2} + 2\hat{G}_{1,1}\hat{G}_{2,2}(\hat{G}_{2,3}\hat{G}_{3,1} - \hat{G}_{2,1} - \hat{G}_{1,2}) + 2\hat{G}_{2,2}\hat{G}_{3,3}(\hat{G}_{3,1}\hat{G}_{1,2} - \hat{G}_{3,2} - \hat{G}_{2,3}) + 2\hat{G}_{3,3}\hat{G}_{1,1}(\hat{G}_{3,2}\hat{G}_{1,2} - \hat{G}_{3,2} - \hat{G}_{2,3}) + \hat{G}_{2,1}^2 + \hat{G}_{2,2}^2 + \hat{G}_{1,3}^2 \\
&\quad - \hat{G}_{1,2}\hat{G}_{2,3}\hat{G}_{3,1} - \hat{G}_{2,3}\hat{G}_{3,1}\hat{G}_{2,1} - \hat{G}_{3,1}\hat{G}_{1,2}\hat{G}_{3,2} + \hat{G}_{1,2}^2 + \hat{G}_{2,3}^2 + \hat{G}_{3,1}^2 \\
&\quad + (\hat{G}_{1,1}^2 + 1)(\hat{G}_{2,2}^2 + 1) + (\hat{G}_{2,2}^2 + 1)(\hat{G}_{3,3}^2 + 1) + (\hat{G}_{3,3}^2 + 1)(\hat{G}_{2,1}^2 + 1)
\end{align*}
\]

8. Frobenius manifolds in the vicinity of a non-semi-simple point

In this section we interpret our \( \mathcal{D}_n \) algebra as the Poisson algebra of the Stokes data of a Frobenius manifold in the vicinity of a non-semi-simple point.

Frobenius manifolds where introduced by Dubrovin [13] as coordinate-free formulation of the famous Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations.
Loosely speaking, Frobenius manifolds are $n$-dimensional complex manifolds together with a smooth structure of Frobenius algebra on the tangent space. This is a commutative associative algebra with unity and with an invariant non–degenerate bilinear form (some other conditions must be satisfied, such as the existence of a grading vector field; a precise definition may be found in [13]).

Semi–simple Frobenius manifolds of dimension $n$ can be realized as the space of parameters $u = (u_1, \ldots, u_n)$ together with an $n \times n$ skew-symmetric matrix function $V(u)$ such that the linear differential operator

$$\Lambda(z) := \frac{d}{dz} - U - \frac{V(u)}{z},$$

$U$ being a diagonal matrix of entries $u_1, \ldots, u_n$, has constant monodromy data [13].

Generically, the monodromy data of $\Lambda(z)$ are encoded in the so–called Stokes matrix $S$, an upper triangular matrix with 1 on the diagonal (for a general definition of the monodromy data see [22, 23, 21]). Equivalently, semi–simple Frobenius manifolds are identified with the space of monodromy preserving deformations of a $n$–dimensional Fuchsian system:

$$\frac{d\Psi}{dz} = \sum_{k=1}^{n} A_k(u) \Psi - u_k \Psi,$$

where the matrices $A_1(u), \ldots, A_n(u)$ are solutions of the Schlesinger equations (3.25) and have the form:

$$A_k(u) = -E_k \left(V(u) + \frac{1}{2} \mathbb{I}\right), \quad \text{where } (E_k)_{ij} = \delta_{ik}\delta_{kj}, \quad k = 1, \ldots, n.$$

Let $S$ be the Stokes matrix associated to the differential operator $\Lambda(z) := \frac{d}{dz} - U - \frac{V}{z}$.

If rank $(G) = n$, where $G = S + S^T$, then the monodromy matrices of this system (8.81) have the form:

$$M_k = \mathbb{I} - E_k \left(S + S^T\right) = \mathbb{I} - E_k G, \quad k = 1, \ldots, n.$$

By using the Korotkin–Samtleben bracket and the relation

$$\text{Tr} (M_i M_j) = n - 4 + S_{ij}^2,$$

Ugaglia constructed the Poisson bracket among the entries $S_{ij}$ of the Stokes matrix [37]. She obtained the same formula as (2.5) with $S_{ij}$ in place of $G_{i,j}$ (up to a factor $-\frac{1}{2}$).

Our interpretation of the $\mathcal{D}_n$ algebra as the Poisson algebra of the Stokes data of a Frobenius manifold in the vicinity of a non semi–simple point is based on the observation that non semi–simple points correspond to the critical points of the Schlesinger equations, i.e. to the clashing of two or more poles in the Fuchsian system (8.81).

We use the same notation as in Sec. 5. In particular we fix a number $\tilde{n}$, we set

$$\tilde{u} := (u_1, \ldots, u_{\tilde{n}-1}), \quad \text{and } u_j := tv_j, \quad j = \tilde{n}, \ldots, \tilde{n} + m - 1 = n$$

and we define $A_i(\tilde{u}, t)$ for $i = 1, \ldots, \tilde{n} - 1$ and and $B_j(\tilde{u}, t)$ for $j = 1, \ldots, m = n - \tilde{n} + 1$ as in (5.32), so that the Schlesinger equations in the variable $t$ assume the form (5.36).
Theorem 8.1. Assume that $V$ is non resonant and that its eigenvalues $\mu_1, \ldots, \mu_n$ have real part $\Re \mu_i \in (-\frac{1}{2}, \frac{1}{2})$. If rank $(S + ST) = n$, then there exist some matrix functions $A_1^0(\tilde{u}), \ldots, A_{n-1}^0(\tilde{u})$ and $B_1^0(\tilde{u}), \ldots, B_{n-\tilde{n}+1}^0(\tilde{u})$ such that the Fuchsian system

$$
\frac{d\Phi}{dz} = \sum_{k=1}^{\tilde{n}} \frac{\tilde{A}_k}{\lambda - u_k} \Phi,
$$

with

$$
u_0 = 0, \quad \tilde{A}_i = A_i^0, \quad \text{for } i = 1, \ldots, \tilde{n} - 1, \quad \text{and } \tilde{A}_{\tilde{n}} = \sum_{j=1}^{n-\tilde{n}+1} B_j^0,
$$

has monodromy matrices $\mathcal{M}_1, \ldots, \mathcal{M}_{\tilde{n}-1}, \mathcal{M}_h$ where

$$
\mathcal{M}_h = \mathcal{M}_{\tilde{n}} \mathcal{M}_{\tilde{n}+1} \cdots \mathcal{M}_n.
$$

Moreover, the estimates (5.38), (5.39) for the solutions to the Schlesinger equations (5.30) hold true for $\sigma$ such that $\vartheta < \sigma < 1$, where $\vartheta$ is given by

$$
\vartheta = \max_{i,j = \tilde{n}, \ldots, n} |\Re (\mu_i - \mu_j)|.
$$

Remark 8.2. Observe that the hypothesis that the eigenvalues $\mu_1, \ldots, \mu_n$ have real part $\Re \mu_i \in (-\frac{1}{2}, \frac{1}{2})$ is not restrictive, in fact in the non–resonant case one can always perform a sequence of elementary Schlesinger transformations to shift the eigenvalues by integer and reduce to this case [23].

Proof. First let us prove that if the symmetric matrix $G := S + ST$ has rank $n$ then the group

$$
\tilde{\mathcal{M}} := \langle \mathcal{M}_1, \ldots, \mathcal{M}_{\tilde{n}-1}, \mathcal{M}_h \rangle
$$

is irreducible. This is a simple consequence of (8.83) and the definition of $\mathcal{M}_h = \mathcal{M}_{\tilde{n}} \mathcal{M}_{\tilde{n}+1} \cdots \mathcal{M}_n$. In fact assume by contradiction that the group $\tilde{\mathcal{M}}$ admits an invariant subspace and pick a vector $\mathbf{v} = (v_1, \ldots, v_n)^T$ in it. Then $\mathbf{v}$ is invariant w.r.t. the full monodromy group $\mathcal{M} := \langle \mathcal{M}_1, \ldots, \mathcal{M}_{\tilde{n}-1}, \mathcal{M}_{\tilde{n}}, \ldots, \mathcal{M}_n \rangle$. In fact by definition, $\mathcal{M}_h$ is given by

$$
\mathcal{M}_h = \begin{vmatrix}
1 - \sum_{i=1}^{n} E_i G + \sum_{i=\tilde{n}, j=i+1}^{n} E_i G E_j G - \sum_{i=\tilde{n}, j=i+1}^{n-1} \sum_{l=j+1}^{n} E_i G E_j G E_l G + \\
\cdots + (-1)^m E_{\tilde{n}} \cdots E_{n} G,
\end{vmatrix}
$$

which is a matrix whose $j$-th row for $j = \tilde{n}, \ldots, n$ coincides with the $j$-th row of:

$$
E_j - E_j G + E_j G \sum_{i=j+1}^{n} E_i G - E_j G \sum_{l=j+1}^{n} E_l G + \cdots E_j G E_{j+1} G \cdots E_n G.
$$

Then $\mathcal{M}_h \mathbf{v} = \mathbf{v}$ iff $\mathcal{M}_j \mathbf{v} = \mathbf{v}$ for $j = \tilde{n}, \ldots, n$. This proves that $\mathbf{v}$ is invariant w.r.t. the full monodromy group $\mathcal{M}$. But:

$$
\mathcal{M}_j \mathbf{v} = \mathbf{v} \forall j = 1, \ldots, n \iff E_j G \mathbf{v} = 0 \forall j = 1, \ldots, n \Rightarrow G \mathbf{v} = 0,
$$

which for rank$(G) = n$ gives a contradiction as we wanted.

Existence of the matrix functions $A_1^0(\tilde{u}), \ldots, A_{n-1}^0(\tilde{u})$, $B_1^0(\tilde{u}), \ldots, B_{n-\tilde{n}+1}^0(\tilde{u})$ follows from the fact that the monodromy group is irreducible [3]. To prove that the new system (8.83) with conditions (8.85) has monodromy matrices $\mathcal{M}_1, \ldots, \mathcal{M}_{\tilde{n}-1}, \mathcal{M}_h$ and to obtain the estimates (5.38), (5.39) we apply the clashing theorem [5.4]. The
only assumption we have to verify is that the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\tilde{A}_n$ satisfy the technical assumption (5.37). Thanks to relation (8.82), we have that
\[
eigenvalues \left( \sum_{j=\tilde{n}}^{n} B_j^0 \right) = -\frac{1}{2}, \ldots, -\frac{1}{2}, -\mu_{\tilde{n}} - \frac{1}{2}, \ldots, -\mu_n - \frac{1}{2},
\]
so that the technical condition is always satisfied when $V$ is non-resonant. $\square$

This theorem states that in the vicinity of a non semi-simple point the Frobenius manifold is identified with the space of monodromy preserving deformations of the $n$–dimensional Fuchsian system (8.84) with the conditions (8.85).

**Theorem 8.3.** The Poisson algebra of the monodromy data of the system (8.84) is given by $D_n$.

**Proof.** The Poisson algebra of the monodromy data of the system (8.84) is given by the Korotkin–Samtleben bracket restricted to the adjoint invariant functions such as
\[
\text{Tr}(M_i M_j), \quad \text{and} \quad \text{Tr}(M_i M_{\tilde{n}+1}^k M_{\tilde{n}+1} M_{\tilde{n}}^{-k}),
\]
where now the monodromy matrices are $n \times n$. Due to (8.83), it is easy to prove that
\[
\text{Tr}(M_i M_{\tilde{n}+1}^k M_{\tilde{n}+1} M_{\tilde{n}}^{-k}) = \text{Tr} \left( M_j - E_i G + E_i G M_{\tilde{n}+1}^k E_j G M_{\tilde{n}}^{-k} \right) = n - 4 + (G M_{\tilde{n}}^{-1})^2_{ij}
\]
where the last step is due to the identity $G M_{\tilde{n}} = M_{\tilde{n}}^{-1} G$, which is a straightforward consequence of (8.83) and (8.84). Defining
\[
G^{(k)}_{i,j} = (G M_{\tilde{n}}^k)_{ij}
\]
we get that the Poisson brackets among the elements $G^{(k)}_{i,j}$ coincide with (6.44) up to a factor $-\frac{1}{2}$. $\square$

### 8.1. Level-$p$ reduction in the case of Frobenius manifolds.

Here we study which restrictions on the elements $G_{i,j}$, $i, j = 1, \ldots, n$, we must impose to ensure the satisfaction of the level-$p$ reduction condition $M_{\tilde{n}}^p = I$ (7.71).

**Proposition 8.4.** The product of monodromy matrices $M_{\tilde{n}} := M_{\tilde{n}}, M_{\tilde{n}+1} \ldots M_n$ has the following block-matrix structure
\[
(8.88) \quad \mathcal{M}_{\tilde{n}} = \left[ \begin{array}{c|c} I_{(\tilde{n}-1) \times (\tilde{n}-1)} & \emptyset \\ \hline B & -\tilde{S}^{-1} \tilde{S}^T \end{array} \right],
\]
where $B$ is an $(n-\tilde{n}+1) \times (n-\tilde{n})$ matrix whose entries are polynomials in $G_{i,j}$ with $i = 1, \ldots, n$ and $j = \tilde{n}, \ldots, n$ and $\tilde{S}$ is the $(n-\tilde{n}+1) \times (n-\tilde{n}+1)$ upper-triangular matrix with unities on the diagonal and with its $(i,j)$ entry above the diagonal equal to $G_{\tilde{n}+i-1,\tilde{n}+j-1}$.

**Proof.** From the explicit form of $\mathcal{M}_i$ (8.83) it follows that only the last $n-\tilde{n}+1$ lines of $M_{\tilde{n}}$ differ from the unit matrix; only $G_{i,j}$ with both $i$ and $j$ greater or equal $\tilde{n}$ contribute to the expression in the lower right square block and we let $\mathcal{M}_r$ with $r = \tilde{n}, \ldots n$ denote the lower-right $(n-\tilde{n}+1) \times (n-\tilde{n}+1)$-matrix blocks of the
corresponding monodromy matrices $\mathcal{M}_r$. The proposition assertion then follows from the Dubrovin’s identity $\tilde{\mathcal{M}}_{\bar{n}}\tilde{\mathcal{M}}_{\bar{n}+1}\cdots\tilde{\mathcal{M}}_{\bar{n}} = -\tilde{S}^{-1}\tilde{S}^T$.

We now introduce the notation $\tilde{\mathcal{M}}_h := -\tilde{S}^{-1}\tilde{S}^T$. We then have the following lemma.

**Lemma 8.5.** The condition (7.71) is satisfied if $(\tilde{\mathcal{M}}_h)^p = \mathbb{I}$ and the symmetric form $\tilde{S} + \tilde{S}^T$ is nondegenerate.

**Proof.** From the explicit form of $\mathcal{M}_h$ we have that the condition (7.71) is equivalent to the simultaneous satisfaction of the two conditions:

(i) $(\mathbb{I} + \tilde{\mathcal{M}}_h + (\tilde{\mathcal{M}}_h)^2 + \cdots + (\tilde{\mathcal{M}}_h)^{p-1}) B = 0$ and

(ii) $(\mathcal{M}_h)^p = \mathbb{I}$.

Multiplying the first condition by $(\tilde{\mathcal{M}}_h - \mathbb{I})$ and using the second condition we obtain the identity. If the matrix $(\tilde{\mathcal{M}}_h - \mathbb{I})$ is nondegenerate this implies the satisfaction of the first condition. But this nondegeneracy condition is exactly the condition of the nondegeneracy of the symmetric form $\tilde{S} + \tilde{S}^T$ (upon the multiplication by $\tilde{S}$ from the right).

**Example 8.6.** Let us consider the case of arbitrary $m = n - \bar{n} + 1 \geq 2$ and $p = m + 1$. Then, a convenient choice is $G_{i,j} \equiv 1$ for $\bar{n} \leq i < j \leq n$. Indeed, we then have that the characteristic equation $\det(\tilde{\mathcal{M}}_h - \eta \mathbb{I}) = 0$ is equivalent to

$$
\det\begin{bmatrix}
1 + \eta & \eta & \cdots & \eta \\
1 & 1 + \eta & \cdots & \eta \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 + \eta
\end{bmatrix}
= 1 + \eta + \cdots + \eta^m = 0,
$$

and we have exactly $m = p - 1$ single roots that are $e^{2\pi ik/p}$, $k = 1, \ldots, p - 1$; each root corresponds to a nondegenerate eigenvalue with the corresponding eigenvector of $\tilde{\mathcal{M}}_h$, and we therefore have that all the conditions of Lemma 8.5 are satisfied.

### 8.2. Quantum cohomology of $\mathbb{CP}^2$ and $\mathbb{CP}^3$.**

The fact that the Poisson algebra (2.5) coincides with the Ugaglia bracket on the space of Stokes matrices of a semi–simple Frobenius manifold poses the natural question of characterizing the special class of semi–simple Frobenius manifolds coming from Teichmüller theory. This is a highly non trivial problem that we postpone to subsequent work [10]. In this section we concentrate on two particular cases: $A_3$ in the limit $Z_1 = Z_2 = Z_3 = 0$ and $A_4$ in the limit $Z_1 = -Z_2 = Z_3 = -Z_4 = \frac{\log(2)}{2}$, $Y = 0$, let us dub them $A^*_3$ and $A^*_4$ respectively. Through the identification of the matrix $A$ defined in (2.8) with the Stokes matrix $S$ associated to the Frobenius manifold structure, we build a link between the $A^*_3$ and $A^*_4$ and the quantum cohomology rings $H^*(\mathbb{CP}^2)$ and $H^*(\mathbb{CP}^3)$ respectively. In fact:

---

Recall that this formula follows from the chain of matrix equalities: $SE_1 = E_1, E_iS^T E_j = 0$ for $i < j$, $E_iSE_j = \delta_{i,j}E_j$ for $i \geq j$; then $SM_1 = S - E_1(S + S^T) = (E_2 + \cdots + E_n)S - E_1S^T$. Multiplying this expression by $\mathcal{M}_2$ from the right and using the above formulas, we obtain $(E_3 + \cdots + E_n)S - (E_1 + E_2)S^T$ as the result; we then multiply it by $\mathcal{M}_3$ from the right and continue until we obtain that $SM_1\mathcal{M}_2\cdots\mathcal{M}_n = -(E_1 + E_2 + \cdots + E_n)S^T = -S^T$. 

Theorem 8.7. The matrices $A$ in the cases $A^3_3$ and $A^4_4$ have the form

$$
\begin{pmatrix}
1 & 3 & 3 \\
0 & 1 & 3 \\
0 & 0 & 1 
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
1 & 4 & 6 & 4 \\
0 & 1 & 4 & 6 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 1 
\end{pmatrix}
$$

which coincide with the Stokes matrix of the respective quantum cohomology rings $H^*(\mathbb{CP}^2)$ [13] and $H^*(\mathbb{CP}^3)$ [19].

Proof. The proof of this theorem is straightforward, it is simply based on plugging in $G_{i,j} = \text{Tr}(\gamma_i \gamma_j)$ the appropriate values of the shear coordinates. □

This link with the quantum cohomology of the projective space is only valid in low dimension, i.e. for $n = 3, 4$. However it gives some insight on the nature of the solutions of the Schlesinger equations related to Teichmüller theory: we expect them to be transcendental. In fact, on the one side we observe that generically the monodromy group $\langle \gamma_1, \ldots, \gamma_n \rangle$ is irreducible and none of the matrices $\gamma_i$ is a multiple of the identity, therefore there is no evidence that these solutions should be special functions [28]. On the other side, the Schlesinger equations associated to $H^*(\mathbb{CP}^2)$ are solved in terms of Painlevé VI transcendents. For the moment we can only prove the following:

Theorem 8.8. The solutions to the $2 \times 2$ Schlesinger equations (3.25) with monodromy matrices $\gamma_1, \ldots, \gamma_n$ given by (2.6) are not algebraic in $u_1, \ldots, u_n$.

Proof. This is a simple consequence of the fact that the analytic continuation of the solutions to the Schlesinger equations is given by the action of the braid group on the monodromy matrices (3.26), which in terms of $G_{i,j} = -\text{Tr}(\gamma_i \gamma_j)$ is given by formulae (2.9). In the geometric case, i.e. when the monodromy matrices $\gamma_1, \ldots, \gamma_n$ are given by (2.6), it is easy to verify that $|G_{i,j}| > 2$ and the braid group orbits are therefore infinite as proved in [14]. Therefore the corresponding solution to the Schlesinger equations cannot be algebraic. □

Appendix A Monodromy data

A general description of monodromy data of linear systems of ODE can be found in [22, 23, 21]. Here we remind the notations and definitions for $m \times m$ Fuchsian systems. We work in the basis where $A_\infty$ is diagonal.

Fix a real number $\varphi \in [0, 2\pi]$ and consider the open subset $U \in X_n$ such that the rays $L_1, \ldots, L_n$ defined by

$$
L_j := \{ u_j + ipe^{-i\varphi} \mid 0 \leq \rho < \infty \}
$$

(A.1)

do not intersect. We assume that the points $(u_1, \ldots, u_n) \in U$ are ordered in such a way that the rays $L_1, \ldots, L_n$ exit from infinity in clockwise order.

Let us fix a fundamental matrix solution near all singular points $u_1, \ldots, u_n, \infty$. To this end we fix branch cuts on the complex plane along the rays $L_1, \ldots, L_n$ and choose the branches of logarithms $\log(\lambda - u_1), \ldots, \log(\lambda - u_n), \log \lambda^{-1}$. Assume $A_1, \ldots, A_n$ to be diagonalizable

$$
A_i = \Gamma_i^{-1} \Theta_i \Gamma_i,
$$

where $\Theta_i$ is a diagonal matrix.
We can fix fundamental matrices analytic on
\[(A.2) \quad \lambda \in \mathbb{C} \setminus \bigcup_{k=1}^{n} L_k,\]
as follows:
\[(A.3) \quad \Phi_k(\lambda) = (\Gamma_k + \mathcal{O}(\lambda - u_k)) (\lambda - u_k)^{\Theta_k}, \quad \lambda \to u_k, \quad k = 1, \ldots, n,\]
and
\[(A.4) \quad \Phi(\lambda) := \Phi_\infty(\lambda) = \left( \mathbb{I} + \mathcal{O}\left(\frac{1}{\lambda}\right) \right) \lambda^{-\Theta_\infty} \lambda^{-R(\infty)}, \quad \text{as} \quad \lambda \to \infty,\]
where the two linear operators \(\Theta_\infty, R(\infty)\) are an admissible pair i.e. the operator \(\Theta_\infty\) is semisimple and the operator \(R(\infty)\) is nilpotent and they satisfy the relation
\[(A.5) \quad e^{2\pi i \Theta_\infty} R(\infty) = R(\infty) e^{2\pi i \Theta_\infty}.\]

Define the connection matrices by
\[(A.6) \quad \Phi_\infty(\lambda) = \Phi_k(\lambda) C_k,\]
where \(\Phi_\infty(\lambda)\) is to be analytically continued in a vicinity of the pole \(u_k\) along the positive side of the branch cut \(L_k\).

The monodromy matrices \(M_k, k = 1, \ldots, n, \infty\) are defined with respect the basis \(l_1, \ldots, l_n\) of loops in the fundamental group
\[\pi_1(\mathbb{C} \setminus \{u_1, \ldots, u_n\}, \infty),\]
chosen by imposing that the small loops \(l_1, \ldots, l_n\) encircle counter–clockwise the points \(u_1, \ldots, u_n\). Denote \(l_j^* \Phi_\infty(\lambda)\) the result of analytic continuation of the fundamental matrix \(\Phi_\infty(\lambda)\) along the loop \(l_j\). The monodromy matrix \(M_j\) is defined by
\[(A.7) \quad l_j^* \Phi_\infty(\lambda) = \Phi_\infty(\lambda) M_j, \quad j = 1, \ldots, n.\]
The monodromy matrices satisfy
\[(A.8) \quad M_\infty M_1 \cdots M_n = \mathbb{I}, \quad M_\infty = C_\infty^{-1} \exp\left(2\pi i A_\infty\right) \exp\left(2\pi i R(\infty)\right) C_\infty,\]
for some constant matrix \(C_\infty\), because of the choice of the ordering of the branch cuts \(L_1, \ldots, L_n\). Clearly one has
\[(A.9) \quad M_k = C_k^{-1} \exp\left(2\pi i \Theta_k\right) C_k, \quad k = 1, \ldots, n.\]

The collection of the local monodromy data \(\Theta_1, \ldots, \Theta_n, \Theta_\infty, R(\infty)\) together with the central connection matrices \(C_1, \ldots, C_n, C_\infty\) uniquely fix the Fuchsian system with given poles. They are defined up to an equivalence that we now describe. The eigenvalues \(\Theta_\infty\) of the matrix \(A_\infty\) are defined up to permutations. Fixing the order of the eigenvalues, we define the class of equivalence of the nilpotent part \(R(\infty)\) and of the connection matrices \(C_1, \ldots, C_n, C_\infty\) by factoring out the transformations of the form
\[(A.10) \quad C_k \mapsto G_k^{-1} C_k G_\infty, \quad k = 1, \ldots, n, \quad C_\infty \mapsto G_\infty^{-1} C_k G_\infty\]
where \(G_k \in \text{GL}(n, \mathbb{C})\) is such that
\[\quad [G_k, \Theta_k] = 0\]
and \(G_\infty \in \text{GL}(n, \mathbb{C})\) is such that
\[(A.11) \quad \lambda^{-\Theta_\infty} G \lambda^{\Theta_\infty} = G_0 + \frac{G_1}{\lambda} + \frac{G_2}{\lambda^2} + \ldots,\]
for some constant matrices $G_0, G_1, G_2, \ldots$.

Observe that the monodromy matrices (A.9) will transform by a simultaneous conjugation

$$M_k \mapsto G_\infty^{-1} M_k G_\infty, \quad k = 1, 2, \ldots, n, \infty.$$  

**Definition A.9.** The class of equivalence (A.10) of the collection

$$(A.12) \quad \Theta_1, \ldots, \Theta_n, \Theta_\infty, R(\infty), C_1, \ldots, C_n, C_\infty$$

is called monodromy data of the Fuchsian system with respect to a fixed ordering of the eigenvalues of the matrix $A_\infty$ and a given choice of the branch cuts.

**Lemma A.10.** Two Fuchsian systems of the form (3.19) with the same poles $u_1, \ldots, u_n$ and $\infty$ and the same matrix $A_\infty$ coincide, modulo conjugations by matrices $G$ such that $[A_\infty, G] = 0$, if and only if they have the same monodromy data with respect to the same system of branch cuts $L_1, \ldots, L_n$.

**Appendix B**  
PROOF OF JACOBI IDENTITIES FOR THE $\mathfrak{D}_n$ BRACKET

In this appendix, we prove the Jacobi identity:

$$(B.1) \quad \{ \{ G_{j,i}(\lambda), G_{p,i}(\mu) \}, G_{q,r}(\nu) \} + \text{cyclic permutations} = 0$$

for the bracket (6.48). We proceed in three steps.

**1.** Without restricting the generality, we segregate all the terms containing the terms $G_{j,r}$ with all possible arguments. There are three cases.

**1a.** Terms in $G_{j,r} G_{q,i} G_{p,i}$ with any choice of the arguments. In this case, only the first two lines of formula (6.48) contribute. Then, with accounting for the cyclic...
permutations, we have the sum of twelve terms
\[
\begin{align*}
&\left(\epsilon(j-p) + \frac{\lambda + \mu}{\lambda - \mu}\right)\left(\epsilon(j-q) + \frac{\mu + \nu}{\mu - \nu}\right) G_{p,i}(\lambda) G_{q,i}(\mu) G_{j,r}(\nu) \\
&+ \left(\epsilon(j-p) + \frac{\lambda + \mu}{\lambda - \mu}\right)\left(\epsilon(l-r) - \frac{\mu + \nu}{\mu - \nu}\right) G_{p,i}(\lambda) G_{q,i}(\nu) G_{j,r}(\mu) \\
&+ \left(\epsilon(i-l) - \frac{\lambda + \mu}{\lambda - \mu}\right)\left(\epsilon(j-q) + \frac{\lambda + \nu}{\lambda - \nu}\right) G_{p,i}(\mu) G_{q,i}(\lambda) G_{j,r}(\nu) \\
&+ \left(\epsilon(i-l) - \frac{\lambda + \mu}{\lambda - \mu}\right)\left(\epsilon(l-r) - \frac{\lambda + \nu}{\lambda - \nu}\right) G_{p,i}(\mu) G_{q,i}(\nu) G_{j,r}(\lambda) \\
&+ \left(\epsilon(p-q) + \frac{\mu + \nu}{\mu - \nu}\right)\left(\epsilon(p-j) + \frac{\lambda + \nu}{\lambda - \lambda}\right) G_{p,i}(\lambda) G_{q,i}(\mu) G_{j,r}(\nu) \\
&+ \left(\epsilon(p-q) + \frac{\mu + \nu}{\mu - \nu}\right)\left(\epsilon(r-i) - \frac{\nu + \lambda}{\nu - \lambda}\right) G_{p,i}(\nu) G_{q,i}(\mu) G_{j,r}(\lambda) \\
&+ \left(\epsilon(l-r) - \frac{\mu + \nu}{\mu - \nu}\right)\left(\epsilon(p-j) + \frac{\lambda + \nu}{\lambda - \lambda}\right) G_{p,i}(\lambda) G_{q,i}(\nu) G_{j,r}(\mu) \\
&+ \left(\epsilon(l-r) - \frac{\mu + \nu}{\mu - \nu}\right)\left(\epsilon(r-i) - \frac{\nu + \lambda}{\nu - \lambda}\right) G_{p,i}(\nu) G_{q,i}(\mu) G_{j,r}(\lambda) \\
&+ \left(\epsilon(q-j) + \frac{\nu + \lambda}{\nu - \lambda}\right)\left(\epsilon(q-p) + \frac{\lambda + \mu}{\lambda - \lambda}\right) G_{p,i}(\lambda) G_{q,i}(\mu) G_{j,r}(\nu) \\
&+ \left(\epsilon(q-j) + \frac{\nu + \lambda}{\nu - \lambda}\right)\left(\epsilon(i-l) - \frac{\lambda + \mu}{\lambda - \lambda}\right) G_{p,i}(\mu) G_{q,i}(\nu) G_{j,r}(\lambda) \\
&+ \left(\epsilon(r-i) - \frac{\nu + \lambda}{\nu - \lambda}\right)\left(\epsilon(q-p) + \frac{\lambda + \mu}{\lambda - \lambda}\right) G_{p,i}(\nu) G_{q,i}(\mu) G_{j,r}(\lambda) \\
&+ \left(\epsilon(r-i) - \frac{\nu + \lambda}{\nu - \lambda}\right)\left(\epsilon(i-l) - \frac{\nu + \mu}{\nu - \mu}\right) G_{p,i}(\nu) G_{q,i}(\nu) G_{j,r}(\lambda)
\end{align*}
\]

Note the cancelations between the lines with the numbers 2 and 7, 3 and 10, and 6 and 11. The lines with the numbers 1, 5, and 9 are proportional to the term \(G_{p,i}(\lambda) G_{q,i}(\mu) G_{j,r}(\nu)\) with the proportionality coefficient
\[
\begin{align*}
&(\epsilon(j-p) + \frac{\lambda + \mu}{\lambda - \mu}) (\epsilon(j-q) + \frac{\mu + \nu}{\mu - \nu}) \\
&+ (\epsilon(p-q) + \frac{\mu + \nu}{\mu - \nu}) (\epsilon(p-j) + \frac{\nu + \lambda}{\nu - \lambda}) \\
&+ (\epsilon(q-j) + \frac{\nu + \lambda}{\nu - \lambda}) (\epsilon(q-p) + \frac{\lambda + \mu}{\lambda - \lambda}) \\
&= (\epsilon(j-p)\epsilon(j-q) + \epsilon(p-q)\epsilon(p-j) + \epsilon(q-j)\epsilon(q-p)) \\
&+ \left(\frac{\lambda + \mu}{\lambda - \mu} \cdot \frac{\mu + \nu}{\mu - \nu} \cdot \frac{\nu + \lambda}{\nu - \lambda} \cdot \frac{\lambda + \mu}{\lambda - \mu}\right).
\end{align*}
\]

The last line of this expression sums up to \(-1\) whereas the combination of the \(\epsilon\)-factors in the next to the last line is 1 unless \(j = p = q\) in which case it vanishes. Therefore this coefficient is
\[-\delta_{jp}\delta_{pq}.
\]
Analogously, the lines with the numbers 4, 8, and 12 are proportional to $G_{p,l}(\mu)G_{q,r}(\nu)G_{j,r}(\lambda)$ with $\delta_{il}\delta_{lr}$.  

1b. Terms in $G_{j,r}G_{q,p}G_{l,i}$ with any choice of the arguments.  
In this case, all four lines of (6.48) contribute and we have the sum of five terms  
\[
\begin{align*}
&\left( \epsilon(i-p) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) \left( \epsilon(j-q) + \frac{\lambda + \nu}{\lambda - \nu} \right) G_{i,l}(\mu)G_{q,p}(\nu)G_{j,r}(\lambda) \\
+ &\left( \epsilon(i-p) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) \left( \epsilon(p-r) - \frac{\lambda + \nu}{\lambda - \nu} \right) G_{i,l}(\mu)G_{q,p}(\nu)G_{j,r}(\lambda) \\
+ &\left( \epsilon(p-r) - \frac{1 + \mu \nu}{1 - \mu \nu} \right) \left( \epsilon(r-i) - \frac{1 + \mu \lambda}{1 - \mu \lambda} \right) G_{i,l}(\mu)G_{q,p}(\nu)G_{j,r}(\lambda) \\
+ &\left( \epsilon(q-j) + \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(i-p) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) G_{i,l}(\mu)G_{q,p}(\nu)G_{j,r}(\lambda) \\
+ &\left( \epsilon(r-i) - \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(i-p) + \frac{1 + \mu \nu}{1 - \mu \nu} \right) G_{i,l}(\mu)G_{q,p}(\nu)G_{j,r}(\lambda)
\end{align*}
\]
Here, again, the first line is canceled with the fourth line and the remaining lines are all proportional to the term $G_{i,l}(\mu)G_{q,p}(\nu)G_{j,r}(\lambda)$ with the proportionality coefficient  
\[
\begin{align*}
&\left( \epsilon(i-p) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) \left( \epsilon(p-r) - \frac{\lambda + \nu}{\lambda - \nu} \right) \\
+ &\left( \epsilon(p-r) - \frac{1 + \mu \nu}{1 - \mu \nu} \right) \left( \epsilon(r-i) - \frac{1 + \mu \lambda}{1 - \mu \lambda} \right) \\
+ &\left( \epsilon(r-i) - \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(i-p) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) \\
+ &\left( \epsilon(q-j) + \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(i-p) + \frac{1 + \mu \nu}{1 - \mu \nu} \right) \\
= &\left( \epsilon(i-p)\epsilon(p-r) + \epsilon(p-r)\epsilon(r-i) + \epsilon(r-i)\epsilon(i-p) \right) \\
+ &\left( -\frac{1 + \lambda \mu}{1 - \lambda \mu} \frac{\lambda + \nu}{\lambda - \nu} + \frac{1 + \mu \nu}{1 - \mu \nu} \frac{1 + \mu \lambda}{1 - \mu \lambda} \frac{\nu + \lambda}{\nu - \lambda} \frac{1 + \mu \nu}{1 - \mu \nu} \right) \\
= &\delta_{il}\delta_{lr}.
\end{align*}
\]

1c. Terms in $G_{j,r}G_{p,q}G_{l,i}$ with any choice of the arguments.  
In this case we again have the sum of five terms  
\[
\begin{align*}
&\left( \epsilon(l-j) - \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) \left( \epsilon(q-j) + \frac{1 + \mu \nu}{1 - \mu \nu} \right) G_{i,l}(\lambda)G_{p,q}(\mu)G_{j,r}(\nu) \\
+ &\left( \epsilon(q-l) + \frac{1 + \mu \nu}{1 - \mu \nu} \right) \left( \epsilon(j-l) + \frac{\nu + \lambda}{\nu - \lambda} \right) G_{i,l}(\lambda)G_{p,q}(\mu)G_{j,r}(\nu) \\
+ &\left( \epsilon(q-l) + \frac{1 + \mu \nu}{1 - \mu \nu} \right) \left( \epsilon(i-r) - \frac{\nu + \lambda}{\nu - \lambda} \right) G_{i,l}(\nu)G_{p,q}(\mu)G_{j,r}(\lambda) \\
+ &\left( \epsilon(j-q) + \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(l-q) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) G_{i,l}(\lambda)G_{p,q}(\mu)G_{j,r}(\nu) \\
+ &\left( \epsilon(i-r) - \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(l-q) + \frac{1 + \mu \nu}{1 - \mu \nu} \right) G_{i,l}(\nu)G_{p,q}(\mu)G_{j,r}(\lambda).
\end{align*}
\]
Here the third line cancels with the fifth line, and the remaining lines are all proportional to the term $G_{i,i}(\lambda) G_{p,q}(\mu) G_{j,r}(\nu)$ with the proportionality coefficient
\[
\left( \epsilon(l-j) - \frac{1+\lambda \mu}{1-\lambda \mu} \right) \left( \epsilon(q-j) + \frac{1+\mu \nu}{1-\mu \nu} \right) +
\left( \epsilon(q-l) + \frac{1+\mu \nu}{1-\mu \nu} \right) \left( \epsilon(j-l) + \frac{\nu + \lambda}{\nu - \lambda} \right) +
\left( \epsilon(j-q) + \frac{\nu + \lambda}{\nu - \lambda} \right) \left( \epsilon(l-q) - \frac{1+\lambda \mu}{1-\lambda \mu} \right) =
\delta_{ij} \delta_{ql}.
\]

2. From the above cases 1a–1c, we see that the Jacobi identity is satisfied separately for every distribution of indices unless at least three among the indices $j$, $i$, $p$, $l$, $q$, and $r$ coincide. So, let us consider the bracket with three coinciding indices, $\{G_{s,i}(\lambda), G_{s,i}(\mu), G_{s,r}(\nu)\} +$ cyclic permutations. Let us follow the term $G_{s,i}(\nu) G_{s,i}(\lambda) G_{s,r}(\mu)$. The coefficient by this term is \((\epsilon(i-l) - \frac{\lambda + \mu}{\lambda - \mu}) (\epsilon(i-r) - \frac{\mu + \nu}{\mu - \nu}) +\)
cyclic permutations and it is again easy to see that this term vanishes unless $i = l = r$. So finally we are left with the case:

\[
\{G_{s,i}(\lambda) , G_{s,i}(\mu) , G_{s,i}(\nu)\} + \{G_{s,i}(\lambda) , G_{s,i}(\mu) , G_{s,s}(\nu)\} + \{G_{s,i}(\mu) , G_{s,i}(\nu) , G_{s,s}(\lambda)\}.
\]

In this case, we have

\[
\{G_{s,s}(\lambda) , G_{s,i}(\mu) \} = \left( \epsilon(i-s) + \frac{1+\lambda \mu}{1-\lambda \mu} \right) (G_{s,s}(\lambda) G_{i,i}(\mu) - G_{s,s}(\mu) G_{i,i}(\lambda))
\]

and the result of the double bracket before applying the cyclic symmetry reads

\[
\{G_{s,i}(\lambda) , G_{s,i}(\mu) , G_{s,i}(\nu)\} = \left( \epsilon(i-s) + \frac{1+\lambda \mu}{1-\lambda \mu} \right) \times
\times \left\{ \frac{\lambda + \nu}{\lambda - \nu} (G_{s,i}(\nu) [G_{i,i}(\lambda) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\lambda)] -
-G_{s,s}(\lambda) [G_{i,i}(\nu) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\nu)] +
+ \frac{\mu + \nu}{\mu - \nu} (G_{s,i}(\nu) [G_{i,i}(\lambda) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\lambda)] -
-G_{s,s}(\nu) [G_{i,i}(\lambda) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\nu)] +
+ \frac{1+\lambda \nu}{1-\lambda \mu} (G_{s,i}(\nu) [G_{i,i}(\lambda) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\lambda)] -
-G_{i,s}(\lambda) [G_{i,i}(\nu) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\nu)] +
+ \frac{1+\mu \nu}{1-\mu \nu} (G_{s,i}(\nu) [G_{i,i}(\lambda) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\lambda)] -
-G_{i,s}(\nu) [G_{i,i}(\lambda) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\nu)] +
+ \epsilon(s-i) (\{G_{s,i}(\lambda) , G_{i,i}(\lambda)\} [G_{i,i}(\nu) G_{s,s}(\mu) + G_{i,i}(\mu) G_{s,s}(\nu)] -
- \epsilon(s-i) (\{G_{s,i}(\mu) , G_{i,i}(\nu)\} [G_{i,i}(\lambda) G_{s,s}(\lambda) + G_{i,i}(\lambda) G_{s,s}(\nu)] -
\right\}.
\]
In this expression, the term proportional to the product of two \( \epsilon \)-functions gives zero under the cyclic permutation, all the terms proportional to a single \( \epsilon \)-function are mutually canceled as well as do all the terms proportional to the products of \( \lambda \)-factors, so the result is zero, as expected.

3. When five or six indices coincide, the Jacobi identity is satisfied identically because

\[ \{ \mathcal{G}_{s,s}(\lambda), \mathcal{G}_{s,s}(\mu) \} = 0. \]

We have therefore proved the satisfaction of the Jacobi identities for all cases of indices distribution in the formula (6.48).

**Appendix C  Proof of Proposition 7.1**

The proof of the first statement of the proposition is an obvious consequence of the fact that \( \mathcal{G}(\lambda) = \frac{1}{n!} \mathcal{G}_p(\lambda) \). The fact that the coefficients of \( \det (\mathcal{G}_p(\lambda)) \) are central elements is an obvious consequence of the braid group action. To prove that there are exactly \( \left\lfloor \frac{np}{2} \right\rfloor \) algebraically independent central elements, let us write \( \mathcal{G}_p(\lambda) \) in terms of the variable \( u \) where \( \lambda = u^2 \). Then, up to irrelevant multiplier \( u^p \),

\[ \mathcal{G}_p(u) := A^{(0)}u^p + \sum_{k=1}^{p-1} G^{(k)} u^{p-2k} + A^{(0)}T u^{-p}, \]

and the symmetry \( \mathcal{G}_p(u) = (\mathcal{G}_p(u^{-1}))^T \) becomes obvious, so \( \det \mathcal{G}_p(u) = \det \mathcal{G}_p(u^{-1}) \).

We then must prove that elements with nonnegative powers of \( u \) in the expansion of this determinant (except the highest term that is just \( 1 \cdot u^{pn} \)) are algebraically independent. For this, let us consider the form

\[ d \det \mathcal{G}_p(u) = \det \mathcal{G}_p(u) \operatorname{Tr} (\mathcal{G}_p^{-1}(u)d\mathcal{G}_p(u)) \]

in the vector space of the differentials \( dG_{i,j}^{(k)} \) (with the constraint (7.72) imposed) at the special point at which all \( G_{i,j}^{(k)} \equiv 1 \).

The matrix \( \mathbf{G}_p(u) \) for all \( G_{i,j}^{(k)} \) equal to unity has the entries \( u^p + u^{p-2} + \cdots + u^2 - p + u^p \) on the diagonal, the entries \( u^p + u^{p-2} + \cdots + u^2 - p \) above the diagonal, and the entries \( u^p + u^{p-2} + \cdots + u^2 - p + u^p \) below the diagonal. In fact, it is not difficult to find \( \det \mathbf{G}_p(u) \cdot \mathbf{G}_p^{-1}(u) \). This is the matrix with all the diagonal terms equal to \( u^{p(n-1)} + u^{p(n-1)-2} + \cdots + u^{-p(n-1)+2} + u^{-p(n-1)} \), with the \( \{i,j\} \) entry above the diagonal equal to \( -u^{p(n-2)(j-i)}(u^p + u^{p-2} + \cdots + u^2 - p) \), and with the \( \{i,j\} \) entry below the diagonal equal to \( -u^{-p(n-2)(j-i)}(u^p + \cdots + u^2 - p + u^p) \). We introduce the standard scalar multiplication on the linear space of differentials (taking into account the symmetry (7.72)):

\[ \left\langle dG_{i,j}^{(k)} \left| dG_{i,m}^{(s)} \right. \right\rangle = \delta_{i,j} \delta_{j,m} \delta_{k,s} + \delta_{i,m} \delta_{j,k} \delta_{k,p-s}, \text{ for } i \neq j \text{ or } s \neq p/2. \]

We then segregate the coefficients standing by nonnegative powers of \( u \) and \( v \) in the bilinear form:

\[ \left\langle \det \mathbf{G}_p(u) \operatorname{Tr} (\mathbf{G}_p^{-1}(u)d\mathcal{G}_p(u)) \right| \det \mathbf{G}_p(v) \operatorname{Tr} (\mathbf{G}_p^{-1}(v)d\mathcal{G}_p(v)) \right\rangle. \]

(We perform the calculations for even \( n \) and odd \( p \), other cases can be treated analogously.) The bilinear form of variations of central elements (its \( \{i,j\} \) entries are the coefficients of \( n \cdot u^{pn-2i} v^{pn-2j} \), \( i, j = 1, \ldots, pn/2 \) in the above expression) is
the sum of the following four \((np/2) \times (np/2)\) matrices (the first two of them come from the brackets between differentials of nondiagonal entries and the last two arise from the brackets between diagonal term differentials)

\[
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
1 & 2 & 2 & \ldots & 2 & 1 & 0 & \ddots & \\
1 & 2 & 3 & \ldots & 3 & 2 & 1 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \\
1 & 2 & 3 & \ldots & p & 3 & 2 & \ddots & 1 \\
0 & 1 & 2 & 3 & \ldots & p & \vdots & \ddots & 2 \\
0 & 0 & 1 & 2 & 3 & \ldots & p & \vdots & 3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 0 & 1 & 2 & 3 & \ldots & p \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 1 & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & 1 & 2 & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & \ldots & 1 & 1 & \ldots & 1 \\
1 & 2 & \ldots & 2 & 2 & \ldots & 2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & p-1 & p-1 & \ldots & p-1 \\
1 & 2 & \ldots & p-1 & p-1 & \ldots & p-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & p-1 & p-1 & \ldots & p-1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \ldots & 0 & 1 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 2 & \ldots & p-1 & p-1 & \ldots & p-1 \\
1 & 2 & \ldots & p-1 & p-1 & \ldots & p-1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 2 & \ldots & p-1 & p-1 & \ldots & p-1 \\
\end{pmatrix}
\]

We now perform the following row and column operations: first, we subtract the first row from \(p - 1\) subsequent rows, then subtract the second row from \(p - 1\) subsequent rows, etc. Then, we perform the same operation with columns: subtract the first column from \(p - 1\) subsequent columns, etc. The first matrix then becomes just the \((np/2) \times (np/2)\) unit matrix, the second matrix will contain just one nonzero \(p \times p\) block in the lower right corner: in this block we obtain the matrix with unities
on the antidiagonal and zeros elsewhere; the third and fourth matrices become \((np/2) \times (np/2)\)-matrices composed from \(n^2/4\) equal \(p \times p\) blocks: these blocks are

\[
\begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix}, \quad \text{and} \quad
\begin{pmatrix}
0 & \cdots & 0 & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}
\]

for the respective third and fourth matrices. We then subtract, first, the upper \(p \times (np/2)\) row-block from all the lower blocks and then subtract the last \((np/2) \times p\) column-block from all the preceding ones. In the last two matrices only the upper-right \(p \times p\) block remains nonzero whereas in the sum of the first two matrices we have nonzero \(p \times p\) blocks on the diagonal and on the bottom and left sides. In the obtained matrix, we perform the following recurrent procedure: expand over \((p-1)\)st row (only \(E_{p-1,p-1}\) entry is nonzero), then subtract the very last row from the \(p\)th row from the bottom and expand over the obtained \(p\)th line from the bottom (only the very first entry, \(E_{1,pn/p-1,1}\), is nonzero). Then subtract the first line from the \((p-1)\)th row and expand, first, over the very last column (only the entry \(E_{np/p,2np/2}\) is nonzero) and then over \(pn/2-p+1\)th column (only the first entry, \(E_{pn/2-p+1,1,1}\), is nonzero). Eventually, expand over the first and the last rows in all the intermediate blocks (only entries on the diagonal are nonzero). After this chain of operations, we come to the block-diagonal matrix, which has the same form as the initial one, but has all blocks of size \((p-2) \times (p-2)\) instead of \(p \times p\). Continuing this procedure (in the case of odd \(p\), we come on the last stage to the block matrix with blocks of size \(1 \times 1\). The upper right block of such a matrix is zero, the lower-right block is \(2\), all the diagonal blocks are \(1\), and the matrix is lower-triangular, therefore its determinant is \(2\). The proposition is proved.

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