Stochastic Echo Phenomena in Nonequilibrium Systems

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(Dated: May 5, 2014)

A thermodynamic system is driven out of equilibrium by a time-dependent force or a nonconservative force represented with a protocol \( \lambda(t) \). Dynamics of such a system is irreversible so that the ensemble of trajectories under a time-reversed protocol \( \lambda(-t) \) is not equivalent to that of time-reversed trajectories under \( \lambda(t) \). We raise a question whether one can find a suitable protocol under which the system exhibits time-reversed motions of the original system. Such a phenomenon is referred to as a stochastic echo phenomenon. We derive a condition for the optimal protocol that leads to the stochastic echo phenomenon in Langevin systems. We find that any system driven by time-independent nonconservative forces has a dual system exhibiting the stochastic echo phenomenon perfectly. The stochastic echo phenomena are also demonstrated for harmonic oscillator systems driven by time-dependent forces. Our study provides a novel perspective on the time-irreversibility of nonequilibrium systems.

PACS numbers: 05.70.Ln, 02.50.-r, 05.40.-a

Keywords: nonequilibrium, irreversibility, echo phenomenon, entropy production

I. INTRODUCTION

Dynamical systems with time-reversal symmetry exhibit an echo phenomenon. A classical mechanical system follows a time-reversed trajectory when the velocity of all particles is reversed. So does a quantum mechanical system when the Hamiltonian \( H \) is switched to \(-H\). An echo phenomenon also occurs in a driven macroscopic system. One of the most popular examples is a drop of dye placed in a highly viscous fluid filled in the gap between two concentric cylinders [1]. When the inner cylinder is rotated, the drop spreads apparently uniformly in the fluid. Surprisingly, the drop reappears when the rotation is reversed. In this system the high viscosity is the reason for the echo phenomenon. It prevents thermalization within an experimental time scale [2–4]. It is an interesting question whether one can observe an echo phenomenon in fully stochastic systems.

An equilibrium system with Langevin dynamics obeys the detailed balance condition [5]

\[
p_{eq}(\mathbf{x}_i)P(\mathbf{x}_f, t_f|\mathbf{x}_i, t_i) = p_{eq}(\mathbf{x}_f)P(\mathbf{x}_i, t_i|\mathbf{x}_f, t_f) ,
\]

where \( p_{eq} \) denotes the equilibrium probability density function (PDF) and \( P(\mathbf{x}', t'|\mathbf{x}, t) \) denotes the transition probability from \( \mathbf{x} \) at time \( t \) to \( \mathbf{x}' \) at time \( t' \). The detailed balance implies that any path or trajectory \( \mathbf{x}(0 \leq t \leq \tau) \) and its time-reversed one \( \mathbf{x}^R(t) \equiv \mathbf{x}(\tau - t) \) are equally probable. Hence, an equilibrium system displays a stochastic echo phenomenon (SEP), an echo phenomenon in the probabilistic sense.

When a thermodynamic system is exerted by a time-dependent force or a nonconservative force, it is driven into a nonequilibrium state [6–8]. We will represent a nonequilibrium driving with a protocol \( \lambda(t) \), which may depend on time or not. A nonequilibrium system does not obey the detailed balance. So, a path \( \mathbf{x}(t) \) under a protocol \( \lambda(t) \) and its time-reversed path \( \mathbf{x}^R(t) \) under a time-reversed protocol \( \lambda^R(t) \equiv \lambda(\tau - t) \) are not equally probable. In fact, the difference in the statistical weights between them is directly related to the entropy production, which is the basis for the various nonequilibrium fluctuation theorems in stochastic thermodynamics [9–14].

Due to the absence of the detailed balance, one does not expect the SEP with a time-reversed protocol. It raises a question whether one can control a nonequilibrium system with a suitable protocol to exhibit the SEP. In other words, is it possible to find an optimal protocol under which the system evolves in time as the original system in the time-reversed direction? In this paper, we derive the condition for the optimal protocol for the SEP in nonequilibrium systems following the Langevin dynamics. We begin with a review on stochastic thermodynamics for nonequilibrium Langevin systems in Sec. II. Then, the optimal condition for the SEP will be derived in Sec. III. The SEP will be further investigated in systems driven by nonconservative forces in Sec. IV and by time-dependent forces in Sec. V. We conclude the paper with summary and discussions in Sec. VI.

II. LANGEVIN DYNAMICS

We consider a system of coordinate \( \mathbf{x} = (x_1, \ldots, x_N) \) which is in thermal contact with a heat reservoir at temperature \( T \) and exerted by a force \( \mathbf{f}_{\lambda(t)}(\mathbf{x}) \). The force includes a time-independent nonconservative force or a conservative force with a time-dependent external parameter. Such a non-equilibrium force is represented with a protocol \( \lambda(t) \). The Langevin equation reads [2]

\[
\dot{\mathbf{x}}(t) = \mathbf{f}_{\lambda(t)}(\mathbf{x}(t)) + \sqrt{2T \eta(t)} ,
\]

where \( \eta(t) \) is the \( \delta \)-correlated thermal noise with \( \langle \eta_i(t) \rangle = 0 \) and \( \langle \eta_i(t)\eta_j(t') \rangle = \delta_{ij}\delta(t - t') \). The damping
coefficient and the Boltzmann constant are set to 1.

Given the initial PDF \( p_i(x(0)) \) for \( x(0) \), the dynamics determines the PDF \( p_f(x(\tau)) \) for \( x(\tau) \) at time \( t = \tau \). The probability density functional for a path \( x(t) \) in the interval \( 0 \leq t \leq \tau \) is given by

\[
P_{\lambda,p_i}[x] = p_i(x(0)) e^{-\mathcal{L}[x;f_\lambda]} \tag{3}
\]
with the action

\[
\mathcal{L}[x;f_\lambda] = \int_0^\tau dt \left\{ \frac{\dot{x} - f_\lambda(x)}{4T} + \frac{1}{2} \nabla_x \cdot f_\lambda(x) \right\}. \tag{4}
\]

For convenience, we adopt the Stratonovich convention for the stochastic integral throughout the paper [3].

Suppose that the system follows a path \( x(t) \) under a protocol \( \lambda(t) \). The total entropy production along the path is related to the irreversibility as [8, 10, 11]

\[
\Delta S_{\text{tot}}[x] = \ln \left[ \frac{P_{\lambda,p_i}[x]}{P_{\lambda^R,p_i}^R[x]} \right], \tag{5}
\]
where \( P_{\lambda,p_i}[x] \equiv P_{\lambda,p_i}^R[x^R] \) denotes the probability density functional for a time-reversed path \( x^R(t) = x(\tau - t) \) under a time-reversed protocol \( \lambda^R(t) \equiv \lambda(\tau - t) \). It is decomposed as \( \Delta S_{\text{tot}}[x] = \Delta S_{\text{sys}}[x] + \Delta S_{\text{res}}[x] \) with

\[
\Delta S_{\text{sys}}[x] = -\ln \left[ \frac{p_f(x(\tau))}{p_i(x(0))} \right] \tag{6}
\]
and

\[
\Delta S_{\text{res}}[x] = \mathcal{L}[x^R;\lambda^R] - \mathcal{L}[x;\lambda]. \tag{7}
\]

Note that \( \Delta S_{\text{sys}} \) is the change in the Shannon entropy of the system and \( \Delta S_{\text{res}}[x] \) is the entropy change of the heat reservoir with the heat

\[
Q[x] = \int_0^\tau dt \cdot \dot{x}(t) \cdot f_\lambda(t)(x(t)). \tag{8}
\]

The quantity \( -Q[x] \) is equal to the work done by the damping force and the random force. So, \( Q[x] \) is the heat dissipated into the heat reservoir.

It is noteworthy that the average value of \( \Delta S_{\text{tot}} \) is the relative entropy (or Kullback-Leibler divergence) of \( \mathcal{P}_{\lambda,p_i}[x] \) with respect to \( \mathcal{P}_{\lambda^R,p_i}^R[x] \) [12]:

\[
\langle \Delta S_{\text{tot}} \rangle_{\lambda,p_i} = D \left( \mathcal{P}_{\lambda,p_i} || \mathcal{P}_{\lambda^R,p_i}^R \right). \tag{9}
\]

The relative entropy of a PDF \( p(x) \) with respect to another PDF \( q(x) \) is defined as

\[
D(p||q) = \langle \ln \frac{p(x)}{q(x)} \rangle_p = \int dx \ p(x) \ln \frac{p(x)}{q(x)},
\]
It is nonnegative for any \( p \) and \( q \), and \( D(p||q) = 0 \) if and only if \( p(x) = q(x) \) almost everywhere.

So the entropy production vanishes only for equilibrium systems satisfying the detailed balance. In nonequilibrium systems, the path probabilities are not equal and the entropy production is always positive.

III. STOCHASTIC ECHO PHENOMENON

In order to find the optimal protocol for the SEP, we introduce an virtual system subject to a force \( f_\kappa(t)(x) \) characterized by a protocol \( \kappa(t) \). The PDF for \( x(0) \) of this system is taken to be \( p_f \) that corresponds to the PDF of the original system with \( f_\lambda \) at time \( \tau \). Then, the probability density functional for a path \( x(t) \) is given by \( \mathcal{P}_{\kappa(t),p_f}[x(t)] = p_f(x(0)) e^{-\mathcal{L}[x;f_\lambda]} \).

The entropy production in Eq. (9) suggests that the dissimilarity between the paths of the original and the virtual systems under the time reversal can be measured by the relative entropy

\[
S[\kappa;\lambda] = D \left( \mathcal{P}_{\lambda,p_i} || \mathcal{P}_{\kappa,p_i}^R \right). \tag{10}
\]

The optimal protocol is found by minimizing the relative entropy with respect to \( \kappa \). The relative entropy is useful since it is nonnegative and zero only for the perfect SEP with \( \mathcal{P}_{\lambda,p_i}[x] = \mathcal{P}_{\kappa,p_i}^R[x] \). It reduces to the total entropy production for the special choice \( \kappa = \lambda^R \). Consequently, it provides a lower bound for the total entropy production, \( \langle \Delta S_{\text{tot}} \rangle = S[\lambda^R;\lambda] \geq \min_{\kappa} S[\kappa;\lambda] \geq 0 \).

Before proceeding, we remark that the optimal protocol has also been considered in a different context. Suppose that a system is driven by a time-dependent external parameter \( \lambda(t) \) under the constraint that \( \lambda(0) = \lambda_i \) and \( \lambda(\tau) = \lambda_f \). One may seek for the optimal control of \( \lambda(t) \) that yields the minimum entropy production [18, 19].

Note that this optimal protocol for the minimum entropy production is different from the optimal protocol for the SEP.

For a notational simplicity, we introduce \( \mu(t) \equiv \lambda^R(t) = \kappa(\tau - t) \). Then, the relative entropy is rewritten as

\[
S[\kappa;\lambda] = \langle \Delta S_{\text{sys}} \rangle_{\lambda,p_i} + \frac{1}{T} \langle Q[x] \rangle_{\lambda,p_i}, \tag{11}
\]
where

\[
Q[x] = -\mathcal{L}[x;f_\lambda] + \mathcal{L}[x^R;f_\lambda] = \int_0^\tau dt \left[ \dot{x} \cdot (f_\lambda + f_\mu)/2 - (f_\lambda^2 - f_\mu^2)/4 \right]
\]
\[\quad - T \nabla_x \cdot (f_\lambda - f_\mu)/2 \tag{12}\]
with \( f_\lambda = f_\lambda(t)(x(t)) \) and \( f_\mu = f_\mu(t)(x(t)) \). Deriving the second equality in Eq. (12), we made an expression of a variable \( t \to \tau - t \) for \( \mathcal{L}[x^R;f_\lambda] \). The first term \( \langle \Delta S_{\text{sys}} \rangle_{\lambda,p_i} \) is independent of \( \mu(t) \). So it suffices to minimize the average of \( Q \). The functional \( Q \) will be called a pseudo-heat since it reduces to the physical heat \( Q \) for a particular choice \( \mu(t) = \lambda(t) \) (see Eq. (9)).

The optimal protocol \( \mu_{\text{opt}} \) is obtained from the condition \( \delta(Q)/\delta\mu \big|_{\mu_{\text{opt}}} = 0 \), which yields

\[
\left. \Delta \left( \dot{x} \cdot \frac{\partial f_\mu}{\partial \mu} + f_\mu \cdot \frac{\partial f_\mu}{\partial \mu} + T \nabla_x \cdot \frac{\partial f_\mu}{\partial \mu} \right) \right|_{\lambda,p_i} = 0. \tag{13}
\]
To proceed further, we consider the decomposition of the form

\[ f_\lambda(x) = g(x) + \lambda(t)h(x) \]  

(14)

with auxiliary force fields \( g \) and \( h \). The physical meaning of the decomposition will be explained later. In this case, the SEP is achieved with the force \( f_{\mu op}(t) = g(x) + \mu_{op}(t-t)h(x) \) with the optimal protocol

\[ \mu_{op}(t) = -\frac{\langle (\dot{x}(t) + g(x(t)) + T\nabla_x h(x(t))) \rangle_{P_{\lambda,p}}}{\langle h(x(t)) \rangle_{P_{\lambda,p}}} . \]  

(15)

In the Stratonovich calculus, \( \dot{x}(t) \cdot F(x(t)) \) should be interpreted as \( \lim_{\delta t \to 0} \frac{1}{\delta t}(x(t + \delta t) - x(t)) \cdot \frac{1}{2} \hat{F}(x(t + \delta t)) + F(x(t)) \) for any vector field \( F(x) \). Using the Langevin equation, one finds that

\[ \langle \dot{x} \cdot F(x) \rangle_{P_{\lambda,p}} = \langle (f_\lambda + T\nabla_x V) \cdot F(x) \rangle_{P_{\lambda,p}} . \]  

It further simplifies the optimal condition to the form

\[ \mu_{op} = \lambda - 2\frac{\langle \dot{x} \cdot h \rangle_{P_{\lambda,p}}}{\langle h^2 \rangle_{P_{\lambda,p}}} = -\lambda - 2\frac{\langle g + T\nabla_x h \rangle_{P_{\lambda,p}}}{\langle h^2 \rangle_{P_{\lambda,p}}} , \]  

(17)

where function arguments are omitted for simplicity. Hereafter, the average \( \langle \cdot \rangle \) is to be taken with respect to \( P_{\lambda,p} \), unless stated otherwise. Note that \( \delta^2 \langle \hat{Q} \rangle / \delta \mu^2 \mid_{\mu_{op}} = \langle h^2 \rangle \geq 0 \). Hence, \( \delta \langle \hat{Q} \rangle / \delta \mu \mid_{\mu_{op}} = 0 \) indeed provides the minimum of the pseudo-heat.

Deviation from the perfect SEP can be measured by the relative entropy with the optimal protocol, \( S_{op} \equiv S[\kappa = \mu_{op}; \lambda] = \langle S_{sys} \rangle + \langle Q_{op} \rangle / T \) with \( Q_{op} = \hat{Q} \mid_{\mu_{op}} \). We define the pseudo-heat dissipation rate \( \dot{Q}_{op} \) as the integrand of Eq. (12) with \( \mu = \mu_{op} \). Inserting in Eq. (14) into Eq. (12) and using Eq. (17), one can show that

\[ \langle \dot{Q}_{op} \rangle = \langle \hat{Q} \rangle - \frac{1}{4} (\lambda - \mu_{op})^2 \langle h^2 \rangle , \]  

(18)

where the average heat dissipation rate is given by

\[ \langle \hat{Q} \rangle = \langle \dot{x} \cdot f_\lambda \rangle = \langle (f_\lambda + T\nabla_x V) \cdot f_\lambda \rangle . \]  

(19)

It is obvious that \( \langle \hat{Q} \rangle \geq \langle \dot{Q}_{op} \rangle \).

In the following sections, we will apply the formalism to systems driven by a nonconservative force and a time-dependent force, separately.

### IV. NONCONSERVATIVE FORCE CASE

Consider a thermodynamic system exerted by a force \( f(x) \) which is a sum of a conservative force \( f_c(x) \) and a nonconservative force \( f_{nc}(x) \). A conservative force is given by a gradient of a potential function \( V(x) \) as \( f_c(x) = -\nabla_x V(x) \), while a nonconservative force does not have a potential function. A nonconservative force drives a system out of equilibrium \([6, 8]\). Along the line of Eq. (14), the force is decomposed as

\[ f(x) = f_c(x) + \lambda f_{nc}(x) \]  

(20)

with \( g = f_c \) and \( h = f_{nc} \). Here, it is useful to keep \( \lambda = 1 \) to represent the presence of a nonequilibrium driving.

Suppose that the system is in the nonequilibrium steady state (NESS) with the PDF denoted by \( p_{ss}(x) = e^{-\phi_{ss}(x)} \). It is the steady state solution of the Fokker-Planck equation \([5]\)

\[ \frac{\partial p}{\partial t} = \nabla_x \cdot ( -f + T\nabla_x p ) . \]  

(21)

Inserting \( p_{ss}(x) = e^{-\phi_{ss}(x)} \) into Eq. (21), one finds that \( \phi_{ss}(x) \) should satisfy

\[ -\nabla_x \phi_{ss} + \nabla_x \cdot [f + T\nabla_x \phi_{ss}] = 0 . \]  

(22)

In the NESS, the system entropy does not change while the heat is dissipated at a constant rate \( \langle \hat{Q} \rangle_{ss} = \langle \dot{x} \cdot f \rangle_{ss} = \lambda \langle \dot{x} \cdot f_{nc} \rangle_{ss} \), where \( \langle \cdot \rangle_{ss} \) denotes the average over the NESS \([5]\). Note that the average power of the conservative force is zero in the NESS since \( \langle \dot{x} \cdot f_c \rangle_{ss} = -\frac{d}{dt} \langle V(x(t)) \rangle_{ss} = 0 \).

The optimal protocol in Eq. (17) is given by

\[ \mu_{op} = -\lambda - 2\frac{\langle \dot{x} \cdot f + T\nabla_x f_{nc} \rangle_{ss}}{\langle f_{nc} \rangle_{ss}^2} . \]  

(23)

It is constant in time since the averages are taken over the NESS. From Eqs. (18) and (19), the average pseudo-heat dissipation rate is given by

\[ \langle \dot{Q}_{op} \rangle_{ss} = \langle \dot{x} \cdot f_{nc} \rangle_{ss} (\lambda \langle f_{nc} \rangle_{ss}^2 - \langle \dot{x} \cdot f_{nc} \rangle_{ss}) . \]  

(24)

It does not vanish in general. Therefore the relative entropy \( S_{op} \) is nonzero and the SEP is imperfect.

Note that the decomposition of a given force \( f(x) \) in Eq. (20) is not unique. One can add and subtract a gradient of any scalar function to the conservative force and from the nonconservative force, respectively \([5]\). By using this degree of freedom, one can always choose

\[ f_c(x) = -T\nabla_x \phi_{ss}(x) \]  

(25)

using the steady-state PDF \( p_{ss}(x) = e^{-\phi_{ss}(x)} \). This particular choice turns out to be extremely useful. Due to the steady state condition in Eq. (22), \( f_c \) and \( f_{nc} \) satisfy

\[ (f_c + T\nabla_x) \cdot f_{nc} = 0 . \]  

(26)

Therefore, the optimal protocol in Eq. (23) is given by

\[ \mu_{op} = -\lambda , \]  

(27)

which means that the SEP is achieved by applying the nonconservative force in the opposite direction, \( f_c = -T\nabla_x \phi_{ss}(x) \).
The pseudo-heat dissipation rate can be evaluated from Eq. (12). Using $\mu_{op} = -\lambda$ and Eq. (20), one obtains that $Q_{op} \propto x \cdot f = -T \frac{d}{dt} \phi_{ss}(x(t))$. It is the total time derivative, so its average value vanishes in the NESS. Therefore, we conclude that a nonequilibrium system driven by a nonconservative force can exhibit a perfect SEP. Precisely speaking, a system being exerted by a force $f$ and characterized by the steady-state PDF $p_{ss} = e^{-\phi_{ss}}$ is equivalent to a dual system defined by a dual force $f_{dual} = -T \nabla_x \phi_{ss} - (f + T \nabla_x \phi_{ss})$ under the time reversal. Both systems share the same steady-state PDF $p_{ss}(x)$.

We add a remark on the force decomposition. Suppose that the total force is decomposed as in Eq. (20) with $\lambda = 1$. It has been shown that the special choice of $f_c = -T \nabla_x \phi_{ss}$ guarantees Eq. (20). One may ask whether the converse is also true. Suppose that the force decomposed as in Eq. (20) and that $f_c$ and $f_{nc}$ satisfy Eq. (20). Then one can show that the conservative part is indeed equal to $f_c = T \nabla_x \ln p_{ss}$ with the steady-state PDF $p_{ss}$ when the steady state is unique. The steady state condition along this line was discussed in Ref. [22].

V. TIME-DEPENDENT FORCE CASE

In this case, we demonstrate the SEP with two solvable model systems. First, consider a simple harmonic oscillator in one dimension whose stable position is dragged according to a given protocol $\lambda(t)$. The force is given by

$$f_{\lambda}(t)(x) = -k(x - \lambda(t))$$  \hspace{1cm} (28)

with a force constant $k$. Without loss of generality, we will set $\lambda(0) = 0$. This system describes a bead trapped by an optical tweezer or an electric charge in an electric circuit [20, 21].

We assume that the particle follows the Boltzmann distribution $p_t(x(0)) \propto e^{-Kx(t)/2T}$ at $t = 0$. The PDF for $x(t)$ at all $t$ is known exactly (see e.g., Ref. [23]). It follows the Gaussian distribution with the mean $(x(t)) = \Lambda(t)$ and the variance $(\langle (x(t) - \Lambda(t))^2 \rangle = T/k$ with $\Lambda(t) = k \int_0^t ds \langle \lambda(s) \rangle e^{-k(t-s)}$.

The system corresponds to the case in Eq. (14) with $g(x) = -kx$ and $h(x) = k$. Hence, the optimal protocol from Eq. (17) is given by

$$\mu_{op}(t) = -\lambda(t) + 2 \langle x(t) \rangle = -\lambda(t) + 2 \Lambda(t)$$  \hspace{1cm} (29)

The PDF for $x(t)$ is given by the Gaussian distribution with a constant variance at all $t$. So the system entropy does not change. The pseudo-heat, evaluated from Eqs. (18) and (19), vanishes while $\langle Q \rangle = k^2(\lambda - \Lambda)^2$. Therefore the relative entropy $S_{op}$ vanishes and the SEP is perfect.

Figure [1](a) illustrates the SEP with a protocol $\lambda(t) = vt$. The initial position $x(0)$ is drawn from the Boltzmann distribution $p_t(x(0)) \propto e^{-Kx(0)^2/(2T)}$. In the interval $0 \leq t \leq \tau$, the particle is driven under the protocol $\lambda(t)$.

Subsequently, in the interval $\tau \leq t \leq 2\tau$, the particle is driven under the time-reversed protocol $\lambda(2\tau - t)$ or the optimal protocol $\mu_{op}(2\tau - t)$ where $\mu_{op}(t) = -\lambda(t) + 2\Lambda(t) = vt - 2v(1 - e^{-Kt/k})/k$. Figure [1](b) shows that paths tend to lag behind the driving. Hence, the reversibility is broken under the time reversed protocol. On the other hand, the optimal protocol generates paths which are symmetric under the time reversal.

As a second example, we consider a particle in one dimension subject to a force

$$f_{\lambda}(t)(x) = -\lambda(t)x$$  \hspace{1cm} (30)

with a time dependent force constant $\lambda(t)$ [22]. The solution of the Langevin equation is given by $x(t) = x(0)e^{-K(t,0)/2T} + \sqrt{2T} \int_0^t dt' e^{-K(t,t')/2T} \eta(t')$, where $K(t,t') = \int_0^t ds \langle \lambda(s) \rangle$. The initial position $x(0)$ is also taken to follow the Boltzmann distribution $p_t(x(0)) \propto e^{-\lambda(0)x(0)^2/(2T)}$. Then, $x(t)$, being the sum of independent Gaussian random variables, follows the Gaussian distribution with the mean $(x(t)) = 0$ and the $t$-dependent variance

$$\langle (x(t))^2 \rangle = \frac{T}{\lambda(0)} e^{-2K(t,0)} + 2T \int_0^t dt' e^{-2K(t,t')}$$  \hspace{1cm} (31)

This system corresponds to the case in Eq. (14) with $g(x) = 0$ and $h(x) = -x$. From Eq. (17), we find that the optimal protocol is given by

$$\mu_{op}(t) = \frac{2T}{\langle (x(t))^2 \rangle} - \lambda(t)$$  \hspace{1cm} (32)

In this case, the PDF for $x(t)$ changes its shape. So the system entropy varies with time as (see Eq. (4))

$$\langle \Delta S_{sys} \rangle = \langle (x(t))^2 \rangle - \langle (x(0))^2 \rangle = \frac{1}{2} \ln \frac{\langle (x(t))^2 \rangle}{\langle (x(0))^2 \rangle}$$  \hspace{1cm} (33)

Differentiating it with respect to $t$ and using Eq. (31), one finds that the system entropy increase rate is given...
by

\[ \langle \hat{S}_{\text{sys}} \rangle = -\frac{\lambda(t)}{\langle x(t)^2 \rangle} \left( \langle x(t)^2 \rangle - \frac{T}{\lambda(t)} \right). \]  

(34)

The pseudo-heat production rate calculated from Eqs. \(19\) and \(18\) is given by

\[ \langle Q_{\text{op}} \rangle = \frac{\lambda(t)T}{\langle x(t)^2 \rangle} \left( \langle x(t)^2 \rangle - \frac{T}{\lambda(t)} \right). \]  

(35)

Both \( \langle \hat{S}_{\text{sys}} \rangle \) and \( \langle \hat{Q}_{\text{op}} \rangle \) are nonzero, while \( \langle \hat{S}_{\text{sys}} \rangle + \langle \hat{Q}_{\text{op}} \rangle / T = 0 \). Therefore the optimal protocol also leads to the perfect SEP.

VI. SUMMARY AND DISCUSSION

We have investigated the SEP in nonequilibrium systems driven by a nonconservative force or a time-dependent force. By minimizing the relative entropy in Eq. \(11\), we have derived the optimal condition for the SEP in Eq. \(13\). Our study shows that stochastic motions of nonequilibrium system can be reversed under the precise control using the optimal protocol.

The harmonic oscillator system, whose stable position or force constant is varying with time, is shown to display the perfect SEP. It is uncertain whether the harmonic oscillator is an exceptional case or not. Further studies are necessary to investigate whether the perfect SEP is possible in more complicated systems. We have shown that any nonequilibrium system driven by a constant nonconservative force can exhibit the perfect SEP. Any driven system is shown to have a dual system that is equivalent under the time reversal. The duality may shed some light on the general property of the NESS. The pseudo-heat plays an important role in the characterization of time-reversibility. The physical meaning of the pseudo-heat is an open question. The expression in Eq. \(24\) suggests that it might be related to the violation of the fluctuation dissipation relation in the NESS \(26\)\(28\). It should be scrutinized further in future.

Acknowledgments

This work was supported by the Basic Science Research Program through the NRF Grant No. 2013R1A2A2A05006776. We thank Prof. Hyunggyu Park and Prof. Juyeon Yi for helpful discussions.

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