On induced modules of inertial-invariant support $\tau$-tilting modules over blocks of finite groups$^*$

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December 6, 2022

Abstract

In this article, we prove that induced modules of support $\tau$-tilting modules over blocks of finite groups satisfying inertial-invariant condition are also support $\tau$-tilting modules.

1 Introduction and notation

Support $\tau$-tilting modules introduced in [2] form an important class of modules. These correspond bijectively to various representation theoretical objects, such as two-term silting complexes, functorially finite torsion classes, left finite semibricks, two-term simple-minded collections and more (see [2, 4, 5, 7]). Let $k$ be an algebraically closed field of characteristic $p > 0$, $G$ a finite group, $\tilde{G}$ a normal subgroup of $G$, $B$ a block of $kG$ and $\tilde{B}$ a block of $k\tilde{G}$ covering $B$, that is, the block of $k\tilde{G}$ satisfying that $1_B 1_{\tilde{B}} \neq 0$, where $1_B$ and $1_{\tilde{B}}$ mean the respective unit elements of $B$ and $\tilde{B}$. We denote the inertial group of the block $B$ in $\tilde{G}$ by $I_{\tilde{G}}(B)$ and the second group cohomology of the factor group $I_{\tilde{G}}(B)/G$ with coefficients in the unit group $k^\times$ of the field $k$ with trivial action by $H^2(I_{\tilde{G}}(B)/G, k^\times)$. In [8], construction-methods of support $\tau$-tilting modules over $\tilde{B}$ from the ones over $B$ using the induction functor $\text{Ind}_{\tilde{G}}^G$ were presented under the following conditions:

1. Any left finite brick $U$ in the category of $B$-module is $I_{\tilde{G}}(B)$-invariant, that is, $xU \cong U$ as $B$-modules for any $x \in I_{\tilde{G}}(B)$.

2. $H^2(I_{\tilde{G}}(B)/G, k^\times) = 1$.

3. The group algebra $k[I_{\tilde{G}}(B)/G]$ is basic as a $k$-algebra.

This paper presents the following results, which relaxes the assumptions above. First, we state a construction-method of support $\tau$-tilting modules over $k\tilde{G}$ from the ones over $kG$.

Main Theorem 1.1 (see Theorems 3.2 and 3.3). Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$, $B$ a block of $kG$, $\tilde{B}$ a block of $k\tilde{G}$ covering $B$ and $M$ a support $\tau$-tilting $B$-modules satisfying $xM \cong M$ as $B$-modules for any $x \in I_{\tilde{G}}(B)$. Then the induced module Ind$_{\tilde{G}}^G M$ is a support $\tau$-tilting $k\tilde{G}$-module. In particular, the module $B\text{Ind}_{\tilde{G}}^G M$ is a support $\tau$-tilting $\tilde{G}$-module.

We will demonstrate that there is an relation between $I_{\tilde{G}}(B)$-invariant support $\tau$-tilting $B$-modules and support $\tau$-tilting $k\tilde{G}$-modules. Now we recall that the set of support $\tau$-tilting sr-tilt $\Lambda$ has a partially ordered set structure for any finite dimensional algebra (see Definition-Proposition 2.2).

Main Theorem 1.2 (see Theorem 3.6). Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$, $B$ a block of $kG$, $\tilde{B}$ a block of $k\tilde{G}$ covering $B$ and $M$ a $B$-modules satisfying $xM \cong M$ as $B$-modules for any $x \in I_{\tilde{G}}(B)$. Then $M$ is a support $\tau$-tilting $B$-module if and only if Ind$_{\tilde{G}}^G M$ is a support $\tau$-tilting $k\tilde{G}$-module. Moreover, for any two $I_{\tilde{G}}(B)$-invariant support $\tau$-tilting $B$-modules $M$ and $M'$, $M \geq M'$ in sr-tilt $B$ if and only if Ind$_{\tilde{G}}^G M \geq \text{Ind}_{\tilde{G}}^G M'$ in sr-tilt $k\tilde{G}$.

$^*$Mathematics Subject Classification (2020). 20C20, 16G10.

$^1$Keywords. Support $\tau$-tilting modules, Induced modules, Blocks of finite groups.
Throughout this paper, we use the following notation and terminologies. Let $\Lambda$ be a finite dimensional $k$-algebra over a field $k$. Modules mean finitely generated left modules. We denote by $\Lambda$-mod the module category of $\Lambda$. For a $\Lambda$-module $U$, we denote by $P(U)$ the projective cover of $U$, by $\Omega(U)$ the syzygy of $U$, by $\tau U$ the Auslander–Reiten translate of $U$ and by $\text{add} U$ the full subcategory of $\Lambda$-mod whose objects are isomorphic to direct summands of finite direct sums of $U$.

This paper is organized as follows. In section 2, we introduce basic terminologies and some known results for $\tau$-tilting theory and modular representation theory of finite groups. In section 3, we give some lemmas and the main result, and present applications and examples.

2 Preliminaries

In this section, $\Lambda$ means a finite dimensional $k$-algebra.

2.1 Support $\tau$-tilting modules

We recall the definitions and basic properties of support $\tau$-tilting modules. For a $\Lambda$-module $M$, we denote by $|M|$ the number of isomorphism classes of indecomposable direct summands of $M$. In particular, $|\Lambda| := |\Lambda/\Lambda e\Lambda|$ means the number of isomorphism classes of simple $\Lambda$-modules.

Definition 2.1 ([2, Definition 0.1]). Let $\Lambda$ be a finite dimensional $k$-algebra and $M$ a $\Lambda$-module.

1. We say that $M$ is $\tau$-rigid if $\text{Hom}_\Lambda(M, \tau M) = 0$.

2. We say that $M$ is $\tau$-tilting if $M$ is a $\tau$-rigid module and $|M| = |\Lambda|$.

3. We say that $M$ is support $\tau$-tilting if there exists an idempotent $e$ of $\Lambda$ such that $M$ is a $\tau$-tilting $\Lambda/\Lambda e\Lambda$-module.

For support $\tau$-tilting $\Lambda$-modules $M$ and $M'$, we write $M \sim_{\text{add}} M'$ if $\text{add} M = \text{add} M'$. Then the relation $\sim_{\text{add}}$ is an equivalence relation. We denote $s_{\tau\text{-tilt}} \Lambda$ the set of equivalence classes of all support $\tau$-tilting $\Lambda$-modules under the equivalence relation $\sim_{\text{add}}$.

Definition-Proposition 2.2 ([2, Theorem 2.7]). For $M, M' \in s_{\tau\text{-tilt}} \Lambda$, we write $M \geq M'$ if there exist a positive integer $r$ and an epimorphism $M \oplus rM' \rightarrow M'$. Then we get a partial order on $s_{\tau\text{-tilt}} \Lambda$.

We denote by $H(s_{\tau\text{-tilt}} \Lambda)$ the Hasse diagram for the partially ordered set $s_{\tau\text{-tilt}} \Lambda$.

Remark 2.3 ([1, Proposition 2.3 (a), (b)]). Since $e = 0$ is an idempotent of $\Lambda$ and $\Lambda/\Lambda e\Lambda = \Lambda$, any $\tau$-tilting module is a support $\tau$-tilting module. Moreover, for any $\tau$-rigid $\Lambda$-module $M$, the following conditions are equivalent:

1. $M$ is a support $\tau$-tilting module.

2. There exist a projective $\Lambda$-module $P$ satisfying $\text{Hom}_\Lambda(P, M) = 0$ and $|M| + |P| = |\Lambda|$.

Proposition 2.4 ([2, Corollary 2.13]). Let $M$ be a $\tau$-rigid $\Lambda$-module and $P$ a projective $\Lambda$-module satisfying that $\text{Hom}_\Lambda(P, M) = 0$. Then the following conditions are equivalent:

1. $|M| + |P| = |\Lambda|$, that is, $M$ is a support $\tau$-tilting $\Lambda$-module (see Remark 2.3).

2. If $\text{Hom}_\Lambda(M, \tau X) = 0$, $\text{Hom}_\Lambda(X, \tau M) = 0$ and $\text{Hom}_\Lambda(P, X) = 0$, then $X \in \text{add} M$ for any $\Lambda$-module $X$.

The following proposition plays an important role in the proof of our main result.

Proposition 2.5 ([6, Proposition 2.14]). Let $\Lambda$ be a finite dimensional $k$-algebra and $M$ a $\tau$-rigid $\Lambda$-module. Then $M$ is a support $\tau$-tilting $\Lambda$-module if and only if there exists an exact sequence

\[
\Lambda \xrightarrow{f} M' \xrightarrow{f'} M'' \rightarrow 0
\]

in $\Lambda$-mod with $M', M'' \in \text{add} M$ and $f$ a left $\text{add} M$-approximation of $\Lambda$. 

2
2.2 Modules over blocks of finite group

Let $G$ be a finite group and $H$ a subgroup of $G$. We denote by $\text{Res}_{H}^{G}$ the restriction functor from $kG$-mod to $kH$-mod and $\text{Ind}_{H}^{G} := kG \otimes_{kH} \bullet$ the induction functor from $kH$-mod to $kG$-mod. The field $k$ can always be regarded as a $kG$-module by defining $gx = x$ for any $g \in G$ and $x \in k$. This module is called the trivial module and is denoted by $k_{G}$.

**Proposition 2.6** (see [3, Lemma 8.5, Lemma 8.6]). Let $G$ be a finite group, $K$ a subgroup of $G$, $H$ a subgroup of $K$. Then the following hold:

1. $\text{Res}_{H}^{K} \cong \text{Res}_{H}^{G} \text{Res}_{K}^{G}$.
2. $\text{Ind}_{H}^{K} \cong \text{Ind}_{H}^{G} \text{Ind}_{K}^{G}$.
3. The functors $\text{Res}_{H}^{G}$ and $\text{Ind}_{H}^{G}$ are left and right adjoint to each other.
4. The functors $\text{Res}_{H}^{G}$ and $\text{Ind}_{H}^{G}$ send projective modules to projective modules.

Let $G$ be a normal subgroup of a finite group $\tilde{G}$ and $U$ a $kG$-module. For $\tilde{g} \in \tilde{G}$, we define a $kG$-module $\tilde{g}U$ consisting of symbols $\tilde{g}u$ as a set, where $u \in U$ and its $kG$-module structure is given by $\tilde{g}u + \tilde{g}u' := \tilde{g}(u + u')$, $\lambda(\tilde{g}u) := \tilde{g}(\lambda u)$ and $\tilde{g}(\tilde{g}u) := \tilde{g}(\tilde{g}^{-1}g\tilde{g}u)$ for any $u, u' \in U$, $\lambda \in k$ and $g \in G$. Let $U$ be a $kG$-module. If $U$ is projective or indecomposable, then $xU$ is also projective or indecomposable, respectively.

**Theorem 2.7** (Mackey’s decomposition formula for normal subgroups). Let $G$ be a normal subgroup of a finite group $\tilde{G}$ and $U$ a $kG$-module. Then we have an isomorphism

$$\text{Res}_{\tilde{G}}^{G}\text{Ind}_{\tilde{G}}^{G}U \cong \bigoplus_{x \in [\tilde{G}/G]} xU,$$

of $kG$-modules, where $[\tilde{G}/G]$ is a set of representatives of the factor group $\tilde{G}/G$.

We recall the definition of blocks of group algebras. Let $G$ be a finite group. The group algebra $kG$ has a unique decomposition

$$kG = B_{0} \times \cdots \times B_{l}$$

into the direct product of indecomposable $k$-algebras $B_{i}$. We call each indecomposable direct product component $B_{i}$ a block of $kG$ and the decomposition above the block decomposition. We remark that any block $B_{i}$ is a two-sided ideal of $kG$.

For any indecomposable $kG$-module $U$, there exists a unique block $B_{i}$ of $kG$ such that $U = B_{i}U$ and $B_{j}U = 0$ for all $j \neq i$. Then we say that $U$ lies in the block $B_{i}$ or simply $U$ is a $B_{i}$-module. We denote by $B_{0}(kG)$ the principal block of $kG$, in which the trivial $kG$-module $k_{G}$ lies.

Let $G$ be a normal subgroup of a finite group $\tilde{G}$, $B$ a block of $kG$ and $\tilde{B}$ a block of $k\tilde{G}$. We say that $\tilde{B}$ covers $B$ (or that $B$ is covered by $\tilde{B}$) if $1_{\tilde{B}}1_{B} \neq 0$.

**Remark 2.8** (see [3, Theorem 15.1, Lemma 15.3]). With the notation above, the following are equivalent:

1. The block $\tilde{B}$ covers $B$.
2. There exists a non-zero $\tilde{B}$-module $U$ such that $\text{Res}_{\tilde{G}}^{\tilde{B}}U$ has a non-zero direct summand lying in $B$.
3. For any non-zero $\tilde{B}$-module $U$, there exists a non-zero direct summand of $\text{Res}_{\tilde{G}}^{\tilde{B}}U$ lying in $B$.

We denote by $I_{\tilde{G}}(B)$ the inertial group of $B$ in $\tilde{G}$, that is $I_{\tilde{G}}(B) := \left\{ x \in \tilde{G} \mid xBx^{-1} = B \right\}$.

**Remark 2.9.** The principal block $B_{0}(kG)$ of $kG$ is covered by the principal block $B_{0}(k\tilde{G})$ of $k\tilde{G}$ and $I_{\tilde{G}}(B_{0}(k\tilde{G})) = \tilde{G}$.

**Remark 2.10.** Let $G$ be a normal subgroup of a finite group $\tilde{G}$, $B$ a block of $kG$ and $M$ a $B$-module. Then $xM$ is a $B$-module for $x \in \tilde{G}$ if and only if $x \in I_{\tilde{G}}(B)$.  


Proposition 2.11 (see [9, Theorem 5.5.10, Theorem 5.5.12]). Let $G$ be a normal subgroup of a finite group $\tilde{G}$, $B$ a block of $k\tilde{G}$ and $\beta$ a block of $kI(\tilde{G})$ covering $B$. Then the following hold:

1. For any $B$-module $V$, the induced module $\text{Ind}^{I(\tilde{G})}_{\tilde{G}}(B)$ is a direct sum of $kI(\tilde{G})$-module lying blocks covering $B$.

2. There exists a unique block $\tilde{B}$ of $k\tilde{G}$ covering $B$ such that the induction functor

\[ \text{Ind}^{I(\tilde{G})}_{\tilde{G}}(B) : kI(\tilde{G})\text{-mod} \rightarrow k\tilde{G}\text{-mod} \]

restricts to a Morita equivalence

\[ \text{Ind}^{I(\tilde{G})}_{\tilde{G}}(B) : \beta\text{-mod} \rightarrow \tilde{B}\text{-mod} . \]

and the mapping $\beta$ to $\tilde{B}$ is a bijection between the set of blocks of $kI(\tilde{G})$ covering $B$ and the one of $k\tilde{G}$ covering $B$.

3 The main results and their applications

In this section, we give some lemmas and our main theorem. After that, we give some applications and examples of the main results.

3.1 Main theorems and their proof

The next lemma plays a key role.

Lemma 3.1. Let $G$ be a normal subgroup of a finite group $\tilde{G}$ and $M$ a $kG$-module satisfying $xM \cong M$ as $kG$-modules for any $x \in \tilde{G}$. Then the following hold:

1. $xP(M) \cong P(M)$ for any $x \in \tilde{G}$.

2. $x\Omega(M) \cong \Omega(M)$ for any $x \in \tilde{G}$.

3. $\text{Ind}^{\tilde{G}}_{G}(\Omega(M)) \cong \Omega(\text{Ind}^{\tilde{G}}_{G}(M))$.

4. $\tau(\text{Ind}^{\tilde{G}}_{G}(M)) \cong \text{Ind}^{\tilde{G}}_{G}(\tau M)$.

Proof. For any $x \in \tilde{G}$, we have an isomorphism $\phi : xM \rightarrow M$ by the assumption. We consider the following commutative diagram in $kG$-mod with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & x\Omega(M) & \rightarrow & xP(M) & \xrightarrow{\pi_M} & xM & \rightarrow & 0 \\
\downarrow{\phi''} & & \downarrow{\phi'} & & \downarrow{\phi} & & & & \\
0 & \rightarrow & \Omega(M) & \rightarrow & P(M) & \xrightarrow{\pi_M} & M & \rightarrow & 0.
\end{array}
\]

Since $\pi_M$ is an essential epimorphism and $\phi$ is an isomorphism, the vertical morphisms $\phi'$ and $\phi''$ are isomorphisms and so (1) and (2) holds.

By Proposition 2.6 (4), we have the following commutative diagram in $k\tilde{G}$-mod with exact rows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \text{Ind}^{\tilde{G}}_{G}(\Omega(M)) & \rightarrow & \text{Ind}^{\tilde{G}}_{G}(P(M)) & \xrightarrow{\text{Ind}^{\tilde{G}}_{G}(\pi_M)} & \text{Ind}^{\tilde{G}}_{G}(M) & \rightarrow & 0 \\
\downarrow{\phi'} & & \downarrow{\phi} & & \downarrow{\text{Id}_{\text{Ind}^{\tilde{G}}_{G}(M)}} & & & & \\
0 & \rightarrow & \Omega(\text{Ind}^{\tilde{G}}_{G}(M)) & \rightarrow & P(\text{Ind}^{\tilde{G}}_{G}(M)) & \xrightarrow{\pi_{\text{Ind}^{\tilde{G}}_{G}(M)}} & \text{Ind}^{\tilde{G}}_{G}(M) & \rightarrow & 0.
\end{array}
\]
Since \( \pi_{\text{Ind}}^{G}M \) is an essential epimorphism, we have that the vertical morphisms \( \varphi \) and \( \varphi' \) are split epimorphisms, \( \text{Ker} \varphi \cong \text{Ker} \varphi' \) are projective \( kG \)-modules and that \( \Omega(\text{Ind}_{G}^{G}M) \oplus \text{Ker} \varphi \cong Ind_{G}^{G}\Omega(M) \). By Theorem 2.7 and (2), we have

\[
\Omega(M)^{\oplus [G:G]} \cong \bigoplus_{x \in [G:G]} x\Omega(M) \\
\cong \text{Res}_{G}^{\tilde{G}}\text{Ind}_{G}^{G}\Omega(M) \\
\cong \text{Res}_{G}^{\tilde{G}}\Omega(\text{Ind}_{G}^{G}M) \oplus \text{Res}_{G}^{\tilde{G}}\text{Ker} \varphi'.
\]

Since \( \text{Res}_{G}^{\tilde{G}}\text{Ker} \varphi' \) is projective by Proposition 2.6 (4) and \( \Omega(M) \) has no non-zero projective summands by the self-injectivity of the group algebra \( kG \), we have that \( \text{Ker} \varphi \cong \text{Ker} \varphi' = 0 \). This finishes the proof of (3).

Finally, we prove the assertion (4). Since \( \tilde{G} \) and \( G \) are symmetric \( k \)-algebras, it holds that \( \tau M \cong \Omega \Omega(M) \) and \( \tau(\text{Ind}_{G}^{G}M) \cong \Omega \Omega(\text{Ind}_{G}^{G}M) \) for any \( kG \)-module \( M \). Therefore, (4) immediately follows from (3).

**Theorem 3.2.** Let \( G \) be a normal subgroup of a finite group \( \tilde{G} \) and \( M \) a support \( \tau \)-tilting \( kG \)-module satisfying \( xM \cong M \) as \( kG \)-modules for any \( x \in \tilde{G} \). Then the induced module \( \text{Ind}_{G}^{G}M \) of \( M \) is a support \( \tau \)-tilting \( k\tilde{G} \)-module.

**Proof.** The similar proof of [10, Theorem 4.2] works in this setting. By Lemma 3.1 (3), Proposition 2.6 (3), Theorem 2.7, the \( I_{G}(B) \)-invariance of and the \( \tau \)-rigidity of \( M \), we have the following:

\[
\text{Hom}_{kG}(\text{Ind}_{G}^{G}M, \tau \text{Ind}_{G}^{G}M) \cong \text{Hom}_{kG}(\text{Ind}_{G}^{G}M, \text{Ind}_{G}^{G}\tau M) \\
\cong \text{Hom}_{kG}(\text{Res}_{G}^{\tilde{G}}\text{Ind}_{G}^{G}M, \tau M) \\
\cong \text{Hom}_{kG}(\bigoplus_{x \in [G:G]} xM, \tau M) \\
\cong \bigoplus_{x \in [G:G]} \text{Hom}_{kG}(M, \tau M) \\
= 0.
\]

Therefore, we have that \( \text{Ind}_{G}^{G}M \) is \( \tau \)-rigid. By Proposition 2.5, there exists an exact sequence

\[
kG \xrightarrow{f} M' \xrightarrow{f'} M'' \longrightarrow 0 \tag{3.1.1}
\]

with \( M', M'' \in \text{add } M \) and \( f \) a left add \( M \)-approximation of \( kG \). Applying the functor \( \text{Ind}_{G}^{G} \) to the exact sequence (3.1.1), we get the exact sequence

\[
k\tilde{G} \cong \text{Ind}_{G}^{G}kG \xrightarrow{\text{Ind}_{G}^{G}f} \text{Ind}_{G}^{G}M' \xrightarrow{\text{Ind}_{G}^{G}f'} \text{Ind}_{G}^{G}M'' \longrightarrow 0
\]

satisfying that \( \text{Ind}_{G}^{G}M', \text{Ind}_{G}^{G}M'' \in \text{add } \text{Ind}_{G}^{G}M \). Then by Proposition 2.5, we only have to prove that \( \text{Ind}_{G}^{G}f \) is a left add \( \text{Ind}_{G}^{G}M \)-approximation of \( k\tilde{G} \), that is, the map

\[
\text{Hom}_{k\tilde{G}}(\text{Ind}_{G}^{G}M', X) \xrightarrow{\bullet \text{Ind}_{G}^{G}f} \text{Hom}_{k\tilde{G}}(k\tilde{G}, X) \tag{3.1.2}
\]

is surjective for any \( X \in \text{add } \text{Ind}_{G}^{G}M \). First we prove that the map

\[
\text{Hom}_{k\tilde{G}}(\text{Ind}_{G}^{G}M', \text{Ind}_{G}^{G}M) \xrightarrow{\bullet \text{Ind}_{G}^{G}f} \text{Hom}_{k\tilde{G}}(k\tilde{G}, \text{Ind}_{G}^{G}M) \tag{3.1.3}
\]
is surjective. By Proposition 2.6 (3), Theorem 2.7 and the assumption, we get the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_{k\tilde{G}}(\text{Ind}_{G}^{\tilde{G}}M', \text{Ind}_{G}^{\tilde{G}}M) & \xrightarrow{\bullet \text{Ind}_{G}^{\tilde{G}}f} & \text{Hom}_{k\tilde{G}}(k\tilde{G}, \text{Ind}_{G}^{\tilde{G}}M) \\
\downarrow & & \downarrow \\
\text{Hom}_{kG}(M', \text{Res}_{G}^{\tilde{G}}\text{Ind}_{G}^{\tilde{G}}M) & \xrightarrow{\bullet \phi} & \text{Hom}_{kG}(kG, \text{Res}_{G}^{\tilde{G}}\text{Ind}_{G}^{\tilde{G}}M) \\
\downarrow & & \downarrow \\
\text{Hom}_{kG}(M', \bigoplus_{x \in \tilde{G}/G} xM) & \xrightarrow{\bullet \phi} & \text{Hom}_{kG}(kG, \bigoplus_{x \in \tilde{G}/G} x \text{Ind}_{G}^{\tilde{G}}(B)) \\
\downarrow & & \downarrow \\
\text{Hom}_{kG}(M', \tilde{G}^{\oplus \tilde{G}/G}) & \xrightarrow{\bullet \phi} & \text{Hom}_{kG}(kG, M^{\oplus \tilde{G}/G}).
\end{array}
\]

The map in the last row is surjective since \( f \) is left \( M \)-approximation of \( kG \), which implies that the map in the first row, which is the map (3.1.3), is surjective. Hence, we get that

\[
\text{Hom}_{k\tilde{G}}(\text{Ind}_{G}^{\tilde{G}}M', \text{Ind}_{G}^{\tilde{G}}M^{\oplus m}) \xrightarrow{\bullet \text{Ind}_{G}^{\tilde{G}}f} \text{Hom}_{k\tilde{G}}(k\tilde{G}, \text{Ind}_{G}^{\tilde{G}}M^{\oplus m})
\]

is surjective for any \( m \in \mathbb{N} \). Now take \( X \in \text{add Ind}_{G}^{\tilde{G}}M \) and \( h \in \text{Hom}_{k\tilde{G}}(k\tilde{G}, X) \) arbitrarily. Then there exists \( m \in \mathbb{N} \) and a split exact sequence

\[
0 \longrightarrow X \longrightarrow \text{Ind}_{G}^{\tilde{G}}M^{\oplus m} \longrightarrow Y \longrightarrow 0
\]

in \( k\tilde{G} \)-mod. Let \( \gamma : \text{Ind}_{G}^{\tilde{G}}M^{\oplus m} \to X \) be a retraction of \( \alpha \), that is, a \( k\tilde{G} \)-homomorphism satisfying \( \gamma \circ \alpha = \text{Id}_X \). Since the map (3.1.4) is surjective and \( \alpha \circ h \in \text{Hom}_{k\tilde{G}}(k\tilde{G}, \text{Ind}_{G}^{\tilde{G}}M^{\oplus m}) \), there exists \( h' \in \text{Hom}_{k\tilde{G}}(\text{Ind}_{G}^{\tilde{G}}M', \text{Ind}_{G}^{\tilde{G}}M^{\oplus m}) \) such that \( h' \circ \text{Ind}_{G}^{\tilde{G}}f = \alpha \circ h \). Hence, we have that

\[
h = \text{Id}_X \circ h = \gamma \circ \alpha \circ h = \gamma \circ h' \circ \text{Ind}_{G}^{\tilde{G}}f.
\]

Therefore, the map (3.1.2) is surjective. \( \square \)

The following result makes the assumption in Theorem 3.2 weaker in case where that the module \( M \) lies in \( \tilde{G} \) not only \( k\tilde{G} \)-module.

**Theorem 3.3.** Let \( G \) be a normal subgroup of a finite group \( \tilde{G} \), \( B \) a block of \( kG \), \( \tilde{B} \) a block of \( k\tilde{G} \) covering \( B \) and \( M \) a support \( \tau \)-tilting \( B \)-module satisfying \( xM \cong M \) as \( B \)-modules for any \( x \in I_G(B) \). Then \( \text{Ind}_{G}^{\tilde{G}}M \) is a support \( \tau \)-tilting \( k\tilde{G} \)-module. In particular, \( \tilde{B}\text{Ind}_{G}^{\tilde{G}}M \) is a support \( \tau \)-tilting \( \tilde{B} \)-module.

**Proof.** Let \( \tilde{B}_1 = \tilde{B}, \ldots, \tilde{B}_e \) be the all blocks of \( k\tilde{G} \) covering \( B \). By Proposition 2.11 (2), we can take \( \beta_1, \ldots, \beta_e \) the blocks of \( kI_{\tilde{G}}(B) \) satisfying the induction functor \( \text{Ind}_{I_{\tilde{G}}(B)}^{\tilde{G}} \) restricts to a Morita equivalence

\[
\text{Ind}_{I_{\tilde{G}}(B)}: \beta_i\text{-mod} \longrightarrow \tilde{B}_i\text{-mod}
\]

for any \( i = 1, \ldots, e \). By Theorem 3.2, the induced module \( \text{Ind}_{G}^{I_{\tilde{G}}(B)}M \) is a support \( \tau \)-tilting \( kI_{\tilde{G}}(B) \)-module and hence \( \beta_i\text{Ind}_{G}^{I_{\tilde{G}}(B)}M \) is a support \( \tau \)-tilting \( \beta_i \)-module for any \( i = 1, \ldots, e \). Therefore, we have that \( \text{Ind}_{I_{\tilde{G}}(B)}\beta_i\text{Ind}_{G}^{I_{\tilde{G}}(B)}M \) is a support \( \tau \)-tilting \( \tilde{B}_i \)-module. By Proposition 2.11 (1) and Proposition 2.6 (2), we have

\[
\bigoplus_{i=1}^e \text{Ind}_{I_{\tilde{G}}(B)}\beta_i\text{Ind}_{G}^{I_{\tilde{G}}(B)}M \cong \text{Ind}_{G}^{I_{\tilde{G}}(B)}M
\]

\[
\cong \text{Ind}_{I_{\tilde{G}}(B)}M \cong \text{Ind}_{G}^{\tilde{G}}M.
\]

Hence, \( \text{Ind}_{G}^{\tilde{G}}M \) is a support \( \tau \)-tilting \( k\tilde{G} \)-module. Therefore, we get that \( \tilde{B}\text{Ind}_{G}^{\tilde{G}}M \) be a support \( \tau \)-tilting \( \tilde{B} \)-module. \( \square \)
Corollary 3.4. Let $G$ be a normal subgroup of a finite group $\tilde{G}$, $B$ a block of $kG$ and $\tilde{B}$ a block of $k\tilde{G}$ covering $B$. If $M \geq M'$ in s$t$-tilt $B$ for $I_G(B)$-invariant support $\tau$-tilting $B$-modules $M$ and $M'$, then $\mathcal{B} \text{Ind}_G^B M \geq \mathcal{B} \text{Ind}_\tilde{G}^{\tilde{B}} M'$ in $s\tau$-tilt $\tilde{B}$.

Proof. By the exactness of the induction functor $\text{Ind}_G^B$ and Theorem 3.3, the statement is obvious. □

We will demonstrate that there is an interrelation between the orders of $I_G(B)$-invariant support $\tau$-tilting $B$-modules and support $\tau$-tilting $k\tilde{G}$-modules.

Proposition 3.5. Let $M$ be an $I_G(B)$-invariant $B$-module. If the induced module $\text{Ind}_G^B M$ is a support $\tau$-tilting $B$-module, then $M$ is a support $\tau$-tilting $B$-module.

Proof. By Remark 2.3, we can take a projective $k\tilde{G}$-module $\tilde{P}$ satisfying that $\text{Hom}_{k\tilde{G}}(\tilde{P}, \text{Ind}_G^B M) = 0$ and $|\tilde{P}| + |\text{Ind}_G^B M| = |\tilde{B}|$. By Proposition 2.4 and Proposition 2.6 (4), we enough to show the following:

1. $M$ is a $\tau$-rigid $B$-module.
2. $\text{Hom}_B(B \text{Res}_G^G \tilde{P}, M) = 0$.
3. If $\text{Hom}_B(M, \tau X) = 0$, $\text{Hom}_B(X, \tau M) = 0$ and $\text{Hom}_B(B \text{Res}_G^G \tilde{P}, X) = 0$, then $X \in \text{add } M$ for any $B$-module $X$.

By Remark 2.10, the $I_G(B)$-invariance of $M$, Theorem 2.7, Proposition 2.6 (3), Lemma 3.1 and the $\tau$-rigidity of $\text{Ind}_G^B M$, we have that

$$\text{Hom}_B(M, \tau M)_{[I_G(B)/G]} \cong \text{Hom}_{kG}(\bigoplus_{x \in [I_G(B)/G]} xM, \tau M) \oplus \text{Hom}_{kG}(\bigoplus_{x \notin I_G(B)} xM, \tau M)$$

$$\cong \text{Hom}_{kG}(\bigoplus_{x \in [G/G]} xM, \tau M)$$

$$\cong \text{Hom}_{kG}(\text{Res}_G^G \text{Ind}_G^B M, \tau M)$$

$$\cong \text{Hom}_{kG}(\text{Ind}_G^B M, \text{Ind}_G^B \tau M)$$

$$\cong \text{Hom}_{kG}(\text{Ind}_G^B M, \tau \text{Ind}_G^B M) = 0.$$
Similarly, we have that $\text{Hom}_{k\tilde{G}}(\text{Ind}_{\tilde{G}}^G X, \tau \text{Ind}_{\tilde{G}}^G M) = 0$. Also, we have that

$$
\text{Hom}_{k\tilde{G}}(\tilde{P}, \text{Ind}_{\tilde{G}}^G X) \cong \text{Hom}_{kG}(\text{Res}_{\tilde{G}}^G \tilde{P}, X) \\
\cong \text{Hom}_B(B \text{Res}_{\tilde{G}}^G \tilde{P}, X) = 0.
$$

Hence, we have that $\text{Ind}_{\tilde{G}}^G X \in \text{add} \text{Ind}_{\tilde{G}}^G M$ by Proposition 2.4. Therefore, we have that $\text{Res}_{\tilde{G}}^G \text{Ind}_{\tilde{G}}^G X \in \text{add} \text{Res}_{\tilde{G}}^G \text{Ind}_{\tilde{G}}^G M$. In particular, we have that $X \in \text{add} M$ since $\text{Res}_{\tilde{G}}^G \text{Ind}_{\tilde{G}}^G X \cong \bigoplus_{x \in [\tilde{G}/G]} X$ and $\text{Res}_{\tilde{G}}^G \text{Ind}_{\tilde{G}}^G M \cong \bigoplus_{x \in [\tilde{G}/G]} M$, which implies that $M$ is a support $\tau$-tilting $B$-module by Proposition 2.4.

**Theorem 3.6.** Let $M$ and $M'$ be $I_G(B)$-invariant $B$-modules. Then the following hold:

1. $M$ is a support $\tau$-tilting $B$-module if and only if $\text{Ind}_{\tilde{G}}^G M$ is a support $\tau$-tilting $k\tilde{G}$-module.

2. Assume that $M$ and $M'$ are support $\tau$-tilting $B$-modules. Then $M \geq M'$ in $sr$-tilt $B$ if and only if $\text{Ind}_{\tilde{G}}^G M \geq \text{Ind}_{\tilde{G}}^G M'$ in $sr$-tilt $k\tilde{G}$.

**Proof.** (1) is clear by Theorem 3.3 and Proposition 3.5. In order to prove (2), we only show that if $\text{Ind}_{\tilde{G}}^G M \geq \text{Ind}_{\tilde{G}}^G M'$ in $sr$-tilt $k\tilde{G}$ then $M \geq M'$ in $sr$-tilt $B$ by Corollary 3.4, but it follows from the fact the restriction functor $\text{Res}_{\tilde{G}}^G$ is an exact functor, the $I_G(B)$-invariance of $M$ and Theorem 2.7.

### 3.2 Examples

Finally, we illustrate our main results with the following examples.

**Example 3.7.** Let $\tilde{G}$ be a finite group, $G$ a normal subgroup of $\tilde{G}$ with cyclic Sylow $p$-subgroup such that the quotient group $\tilde{G}/G$ is a $p$-group, $B$ a block of $kG$ and $\tilde{B}$ a block of $k\tilde{G}$ covering $B$. Then any support $\tau$-tilting $B$-module is $I_G(B)$-invariant (see [8, Lemma 4.2 (2)]). Therefore, we can always apply Theorem 3.3.

**Example 3.8.** Let $G_1$ and $G_2$ be arbitrary finite groups and $M$ a support $\tau$-tilting $kG_1$-module. Then the group $G_1$ is a normal subgroup of the direct product group $G_1 \times G_2$, and it is clear that $M \cong xM$ for any $x \in G_1 \times G_2$. Therefore, the induced module $\text{Ind}_{G_1 \times G_2}^{G_1} M \cong kG_2 \otimes_k M$ is support $\tau$-tilting $k[G_1 \times G_2]$-module by Theorem 3.2.

**Example 3.9.** Let $k$ be an algebraically closed field of characteristic $p = 2$, $G$ the alternating group $A_4$ of degree 4 and $\tilde{G}$ the symmetric group $S_4$ of degree 4. The principal blocks of $kA_4$ and $kS_4$ are themselves, respectively. Moreover, the block $kA_4$ is covered by $kS_4$. The algebras $kA_4$ and $kS_4$ are Brauer graph algebras associated to the Brauer graphs in Figure 1(a) and Figure 1(b), respectively:

![Brauer graphs](image)

(a) The Brauer graph of $kA_4$

(b) The Brauer graph of $kS_4$

**Figure 1:** Brauer graphs

Now we draw the Hasse diagram $\mathcal{H}(sr$-tilt $kA_4)$ of the partially ordered set $sr$-tilt $kA_4$ as follows:
Figure 2: The Hasse diagram of $\mathcal{H}(\text{st-tilt } kA_4)$.
The colored support $\tau$-tilting modules in Figure 2 are all of the invariant support $\tau$-tilting modules under the action of $S_4$. Next we draw the Hasse diagram $\mathcal{H}(s\tau$-tilt $kS_4)$ of partially ordered set $s\tau$-tilt $kS_4$ as follows:

$$\mathcal{H}(s\tau$-tilt($kS_4$)) :

```
\begin{array}{c}
P_1 \oplus P_2 \\
\downarrow \\
2' \\
\downarrow \\
1' \oplus 1' \\
\downarrow \\
P_1 \oplus P_2 \\
\end{array}
```

Figure 3: The Hasse diagram of $s\tau$-tilt $kS_4$

The functor $\text{Ind}_{A_4}^{S_4}$ takes each colored $S_4$-invariant support $\tau$-tilting $kA_4$-module in Figure 2 to that in Figure 3 with the same color. We remark that even if support $\tau$-tilting $kA_4$-module $M$ is basic, its induction $\text{Ind}_{A_4}^{S_4} M$ is not necessarily basic. For example, the induced module $\text{Ind}_{A_4}^{S_4}(1 \oplus 1' \oplus 1' \oplus 1')$ is not basic.

Acknowledgements

The author would like to thank Yuta Kozakai for useful advice, discussions and comments.

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