Chiral Fermions, Anomalies and Chern-Simons Currents on the Lattice

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Abstract

I discuss the zeromode spectrum of lattice chiral fermions in the domain wall model suggested recently. In particular I give the critical momenta where the fermions cease to be chiral and show that the spectrum is directly related to the behaviour of the Chern-Simons current on the lattice. First results for the domain wall model on the finite lattice indicate that the relevant features of the model in the infinite system survive for the finite lattice.

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1 Introduction

In this talk I want to report some first results for the domain wall model suggested by Kaplan [1]. The basic idea of this model is to start with an odd dimensional –and therefore vectorlike– theory and to add a mass term for the fermions which depends on the extra dimension and has the form of a soliton or kink, generating in this way a domain wall. It is well known that in a situation like this one finds zeromodes bound to the domain wall [2]. Kaplan was able to show that these zeromodes represent chiral fermions on the lattice and that the unwanted doubler modes can be removed by introducing the usual Wilson-term. These results [1, 3], which hold for the free theory and for the infinite lattice, could be demonstrated to survive even on a finite lattice [4]. In addition the cancellation of the zeromode anomaly by the divergence of the Goldstone-Wilczek current as described in [1] could also been seen on the finite lattice [4].

In this talk I will discuss the peculiar behaviour of the zeromode spectrum. I will give the critical momenta of these zeromodes where they lack to be chiral and relate the zeromode spectrum to the behaviour of the Chern-Simons current on the lattice.

2 The infinite lattice

To be specific I will discuss the spectrum and also the anomaly for the case of a 3-dimensional model. The results generalized to arbitrary dimensions can be found in [3, 4]. I start with the Dirac-Wilson operator on an infinite lattice with lattice spacing $a = 1$

$$K_{3D} = \sum_{\mu=1}^{3} \sigma_\mu \partial_\mu + m\epsilon(s) + \frac{r}{2} \sum_{\mu=1}^{3} \Delta_\mu$$

(1)

where $\partial$ denotes the lattice derivative $\partial_\mu = \frac{1}{2}[\delta_{z,z+\mu} - \delta_{z,z-\mu}]$, $\Delta$ the lattice Laplacian $\Delta_\mu = [\delta_{z,z+\mu} + \delta_{z,z-\mu} - 2\delta_{z,z}]$, the $\sigma_\mu$ are the usual Pauli matrices and $r$ the Wilson coupling. I will denote by $s$ the extra dimension along which the mass defect appears, while $x, t$ are the 2-dimensional coordinates. The domain wall is taken to be a step function $\epsilon$,

$$\epsilon(s) = \begin{cases} 
-1 & s < 0 \\
0 & s = 0 \\
+1 & s > 0 
\end{cases}$$

(2)

where the height of the domain wall is given by the mass parameter $m$ which I will choose to be positive throughout this paper.

We are looking for solutions which are plane waves in the $(x, t)$-plane

$$\Psi^\pm = e^{i(p_xt + p_x x)}\Phi(s)u^\pm$$

(3)
where $u^\pm$ are the eigenspinors of $\sigma_3$, $\sigma_3 u^\pm = \pm u^\pm$. With this ansatz the Dirac operator becomes

$$K_{3D} = \sum_{i=1}^{2} i\sigma_i \sin(p_i) + \sigma_3 \partial_s + m\epsilon(s) + r \sum_{i=1}^{2} (\cos(p_i) - 1) + \frac{r}{2} \Delta_s .$$

(4)

The final goal is to diagonalize the 3 dimensional Dirac operator in such a way that it reduces to the 2 dimensional Dirac operator for free massless fermions,$K_{3D} \Psi = K_{2D} \Psi$ where $K_{2D}$ acting on $\Psi$ is given by

$$K_{2D} = \sum_{\mu=1}^{2} \sigma_\mu \partial_\mu = i(\sigma_1 \sin(p_1) + \sigma_2 \sin(p_2)) .$$

(5)

Hence the equation to solve is

$$\left[ \sigma_3 \partial_s + m\epsilon(s) - rF + \frac{r}{2} \Delta_s \right] \Phi u^\pm = 0$$

(6)

where $F = \sum_{i=t,x} (1 - \cos(p_i))$. Following [1] I choose an exponential ansatz for $\Phi$ away from the domain wall $\Phi(s+1) = z\Phi(s)$. Inserting this into (6) one finds four solutions

$$z = \frac{r - m_{\text{eff}} \pm \sqrt{m_{\text{eff}}(m_{\text{eff}} - 2r) + 1}}{r \pm 1}$$

(7)

where $m_{\text{eff}} = m\epsilon(s) - rF$, the $\pm$ in the nominator stand for the two roots and the $\pm$ in the denominator stand for the chirality. Note that in the limit $r = 1$ eq.(6) can be reduced to the corresponding expressions in [1].

One has to impose the condition that the solutions are normalizable to obtain sensible wavefunctions. This means that $|z| > 1$ for $s < 0$ and $|z| < 1$ for $s > 0$. The boundaries of the regions where chiral solutions exist are obtained by setting $|z| = 1$. Explicit matching of the normalizable solutions for positive and negative $s$ at $s = 0$ enables one to determine the regions with chiral fermions. One finds that existence and chirality of the solutions is independent of the sign of $r$ and that a negative $m$ leads to opposite chiralities. Depending on the values of $m/r$ one gets $m = rF$ and $m = r(F+2)$ as the boundaries for the critical momenta, where $F$ is defined as above.

The results can be summarized as follows. Starting with $m/r = 0$ one finds no chiral fermions. For increasing $0 < m/r < 2$ the region in momentum space around $\vec{p} = (0,0)$ where chiral modes exist grows. This region is bounded by $m = rF$ which gives the upper critical momenta. Increasing $m/r$ above $m/r = 2$ opens up the two “doubler” modes at $\vec{p} = (0,\pi)$ and $\vec{p} = (\pi,0)$ which have flipped chirality, while the original mode at $\vec{p} = (0,0)$ disappears. Here the boundaries of the regions in momentum space are given by $m = rF$ for the lower and $m = r(F+2)$ for the upper critical momenta. For $m/r > 4$ the two “doublers” disappear and one gets a zero mode at $\vec{p} = (\pi,\pi)$ with the same chirality as the mode at $\vec{p} = (0,0)$. The boundary for the lower critical momenta is given by $m = r(F+2)$. This mode is finally also lost as $m/r$ is increased to $m/r \geq 6$.

It should be remarked that this spectrum stems from $\Psi^+$ solutions only, and that there are no $\Psi^-$ solutions for positive $m$. The change of the chirality is the usual reinterpretation of the chirality at different corners of the Brillouin zone.
The zero mode spectrum as found here is directly related to the coefficient of the lattice Chern-Simons current induced by heavy fermions. This current is responsible for the anomaly cancellation. As there is a chiral zeromode bound to the domain wall, there exists the corresponding anomaly. However, we started with a vectorlike theory and consequently the theory should be anomaly free. As is explained in [1] this contradiction is resolved by the fact that the domain wall induces a Goldstone-Wilczek current [6] far off the domain wall the divergence of which exactly cancels the zeromode anomaly on the domain wall [7].

Recently, this current was also calculated on the lattice. The radiatively induced Chern-Simons action in d=3 dimensions is given by

$$\Gamma_{CS} = \epsilon_{\mu\nu\rho} \int d^3 x A_\mu \partial_\nu A_\rho .$$

The effective action is then given by $c \Gamma_{CS}$. The coefficient $c$ can be calculated in perturbation theory [9, 5] and is given by

$$c = i \epsilon_{\mu\nu\rho} \frac{\partial}{\partial \langle q \rangle_\nu} I \bigg|_{q=0}$$

with $I$ the following integral

$$I = \int \frac{d^3 p}{(2\pi)^3} Tr \left[ S(p)\Lambda_\mu(p, p-q) S(p-q)\Lambda_\rho(p+q, p) \right]$$

where $S$ is the fermion propagator and $\Lambda_\mu$ denotes the photon vertex. Imposing the Ward identity

$$\Lambda_\mu(p, p) = -i \partial/\partial p_\mu S^{-1}(p)$$

the relevant integral becomes

$$\int \frac{d^3 p}{(2\pi)^3} Tr \left[ S\partial_\mu S^{-1} \right] \left[ S\partial_\nu S^{-1} \right] \left[ S\partial_\rho S^{-1} \right].$$

The fermion propagator can be written in a generic form, $S^{-1}(p) = a(p) + i\vec{b}(p)\vec{\sigma}$ which has the structure of $S^{-1}(p) = N(p)V(p)$, where $V(p)$ is a $SU(2)$ matrix and $N(p)$ a normalization factor. Provided that $S^{-1}$ does not vanish for any $p$ the integral eq.(12) does not depend on $N(p)$ and has a simple topological interpretation: It is the winding number of the map $V$ from the torus $T^3$ to $SU(2)$ or the sphere $S^3$. (In general this map is a map from the torus $T^d$ to the sphere $S^d$.)

Taking the usual Wilson fermion propagator the winding number can be calculated by using the fact that the integral has to be computed only in infinitesimal regions near the Brillouin corners. One obtains (see [3] for more details)

$$c = \frac{-i}{32\pi^3} \sum_{k=0}^{3} (-1)^k \frac{m - 2rk}{|m - 2rk|} .$$

This expression should be compared with the (unregulated) continuum result [7]

$$c_{cont} = \frac{-i}{32\pi^3} \frac{m}{|m|} .$$
Therefore one finds that for $m/r < 0$, $c = 0$. For $m/r < 2$ the lattice result is twice the continuum value. For $2 < m/r < 4$ it is $-4c_{\text{cont}}$ and for $4 < m/r < 6$ it is again twice the continuum value.

For the case of the domain wall model in which we are interested here the result has the following implication: For the calculation of the Goldstone-Wilczek current one goes far off the domain wall and assumes there the mass to be constant. Then one can calculate the current with the method described above. For $m/r < 2$ one finds therefore that the current only flows on one side of the domain wall and has twice the value of the continuum result. Of course, the divergencies will come out the same. The behaviour of the lattice Chern-Simons coefficient finds its exact correspondence in the zero mode spectrum as discussed above where we found 1 lefthanded, 2 righthanded and 1 lefthanded chiral fermion for $0 < m/r < 2$, $2 < m/r < 4$ and $4 < m/r < 6$, respectively.

### 3 And the Finite Lattice

Any numerical work on this system will necessarily involve finite lattices, and so I now compare the results obtained on the infinite system with the ones of a finite lattice. On the finite lattice one has to choose some boundary conditions. Taking periodic boundary conditions generates a second anti-domain wall. The mass term is therefore modified to be $m\epsilon_L(s)$ with

$$
\epsilon_L(s) = \begin{cases} 
-1 & 2 \leq s \leq \frac{L_s}{2} \\
+1 & \frac{L_s}{2} + 2 \leq s \leq L_s \\
0 & s = 1, \frac{L_s}{2} + 1
\end{cases}.
$$

The zero modes on the finite lattice can be searched for by solving the Hamiltonian problem numerically. If one again assumes plane waves in the $x$-direction the Hamiltonian is given by

$$
H = -\sigma_1 \left[ i\sigma_2 \sin(p_x) + \sigma_3 \partial_s + m\epsilon_L(s) + r(\cos(p_x) - 1) + \frac{r}{2} \Delta_s \right] \tag{16}
$$

The eigenvalues and eigenfunctions of the Hamiltonian eq. (16) were calculated numerically. To find the critical momenta the ratio

$$
R = \frac{\bar{\Psi}\Psi}{\bar{\Psi}\sigma_1\Psi} \tag{17}
$$

was studied, which is a normalized measure for whether the fermions are chiral or not. It is zero if the fermions are chiral and $R > 0$ for non-chiral modes (see fig.2b in [4]). To determine whether one still has chiral fermions a threshold value for $R$ was defined. If $R < 0.01$ the fermions were regarded to be chiral.

Comparing the results from the infinite system with the finite lattice calculations with $L = 100$ [3], one finds that the two are practically indistinguishable. For $L = 20$, a lattice
size realistic for simulations, a small shift occurs. Fixing $m$ and $r$ we find for $m/r < 2$ a smaller value and for $2 < m/r < 4$ a larger value of the critical momentum.

It is also possible to extract the Chern-Simons current on the finite lattice in the presence of a smooth external gauge field configuration. The current is most easily computed by the inverse fermion matrix of the model which can be obtained by standard methods like conjugate gradient. Note that this is an exact solution of the numerical problem and that no simulation is involved [4].

I show in fig. 1 the Chern-Simons current on a $16^3$ lattice with $m = 0.81$ and $r = 0.9$. Note that these values of $m, r$ are quite large. The picture nicely demonstrates that the advocated flow of the current as obtained above is reproduced on the finite lattice: It flows only on one side of the domain wall and is zero on the other.

It is a simple task to get the divergence of the current and one can demonstrate [4] that it obeys the continuum anomaly equation to a very good accuracy. It is also possible to find non-trivial anomaly cancellation like in the 3-4-5 model where the individual fermion currents are anomalous but the sum of them cancel.

In summary the domain wall model shows a lot of promising features on the finite lattice like the chiral zeromode spectrum and the correct anomaly behaviour. Although these properties were only found for the free theory or with weak external gauge fields they certainly point into the right direction and are encouraging. The next step is to include dynamical gauge fields and see whether one can find the desired behaviour.

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Figure Caption

Fig.1 I show the Chern-Simons current in arbitrary units as obtained on a $16^3$ lattice. It shows the expected peculiar behaviour that it flows only on one side of the domain wall and is zero on the other. The locations of the domain walls is at $s=1$ and at $s=9$. 