ON SOME NEW HERMITE-HADAMARD TYPE INEQUALITIES
FOR s-GEOMETRICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, some new integral inequalities of Hermite-Hadamard type related to the s-geometrically convex functions are established and some applications to special means of positive real numbers are also given.

1. INTRODUCTION

In this section, we firstly list several definitions and some known results.

Definition 1. Let $I$ be an interval in $\mathbb{R}$. Then $f : I \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 2. $^5$Let $s \in (0, 1]$. A function $f : I \subset [0, \infty) \rightarrow [0, \infty)$ is said to be $s$-convex in the second sense if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda^s f(x) + (1 - \lambda)^s f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 3 $^7$. A function $f : I \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+$ is said to be a geometrically convex function if

$$f(x^\lambda y^{1-\lambda}) \leq f(x)^\lambda f(y)^{1-\lambda}$$

for $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 4 $^7$. A function $f : I \subset \mathbb{R}_+ = (0, \infty) \rightarrow \mathbb{R}_+$ is said to be a $s$-geometrically convex function if

$$f(x^\lambda y^{1-\lambda}) \leq f(x)^\lambda f(y)^{(1-\lambda)s}$$

for some $s \in (0, 1]$, where $x, y \in I$ and $\lambda \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$(1.1) \quad f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$ 

Recently, several integral inequalities connected with the inequalities $^{11}$ for the $s$-convex functions have been established by many authors for example see $^1$ $^2$ $^3$ $^4$ $^5$. In $^7$, The authors has established some integral inequalities connected with
the inequalities (1.1) for the $s$-geometrically convex and monotonically decreasing functions. In [6], Tunc has established inequalities for $s$-geometrically and geometrically convex functions which are connected with the famous Hermite Hadamard inequality holding for convex functions. In [6], Tunc also has given the following result for geometrically convex and monotonically decreasing functions:

**Corollary 1.** Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ be geometrically convex and monotonically decreasing on $[a, b]$, then one has

\[(1.2) \quad f^2 \left( \sqrt{ab} \right) \leq \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx \leq f(a)f(b).\]

Note that, the inequalities (1.2) are also true without the condition monotonically decreasing and the inequalities (1.2) are sharp.

In this paper, the author give new identities for differentiable functions. A consequence of the identities is that the author establish some new inequalities connected with the inequalities (1.2) for the $s$-geometrically convex functions.

2. Main Results

In order to prove our results, we need the following lemma:

**Lemma 1.** Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ be differentiable on $I^o$, and $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then

\[
\begin{align*}
&f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx \\
= &\int_0^1 \frac{b}{2} \ln \left( \frac{a}{b} \right) \left( t - 1 \right) \left( \frac{a}{b} \right)^{\frac{t}{2}} f \left( a^{1-t} (ab)^{\frac{t}{2}} \right) f' \left( b^{1-t} (ab)^{\frac{t}{2}} \right) dt \\
&+ \frac{a}{2} \ln \left( \frac{b}{a} \right) (t - 1) \left( \frac{b}{a} \right)^{\frac{t}{2}} f' \left( a^{1-t} (ab)^{\frac{t}{2}} \right) f \left( b^{1-t} (ab)^{\frac{t}{2}} \right) dt,
\end{align*}
\]

(2.1)

\[
\begin{align*}
&f^2 \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx \\
= &\int_0^1 \frac{b}{2} \ln \left( \frac{a}{b} \right) t \left( \frac{a}{b} \right)^{\frac{t}{2}} f \left( a^{1-t} (ab)^{\frac{t}{2}} \right) f' \left( b^{1-t} (ab)^{\frac{t}{2}} \right) dt \\
&+ \frac{a}{2} \ln \left( \frac{b}{a} \right) t \left( \frac{b}{a} \right)^{\frac{t}{2}} f' \left( a^{1-t} (ab)^{\frac{t}{2}} \right) f \left( b^{1-t} (ab)^{\frac{t}{2}} \right) dt.
\end{align*}
\]

(2.2)
Proof. Integrating by part and changing variables of integration yields

\[
\int_0^1 \frac{b}{2} \ln \left( \frac{a}{b} \right) (t - 1) \left( \frac{a}{b} \right)^{\frac{t}{2}} f \left( a^{1-t} (ab)^\frac{t}{2} \right) f' \left( b^{1-t} (ab)^\frac{t}{2} \right)
+ \frac{a}{2} \ln \left( \frac{b}{a} \right) (t - 1) \left( \frac{b}{a} \right)^{\frac{t}{2}} f \left( a^{1-t} (ab)^\frac{t}{2} \right) f' \left( b^{1-t} (ab)^\frac{t}{2} \right) dt
= \int_0^1 (t - 1) \left[ f \left( a^{1-t} (ab)^\frac{t}{2} \right) f \left( b^{1-t} (ab)^\frac{t}{2} \right) \right]' dt
= (t - 1) f \left( a^{1-t} (ab)^\frac{t}{2} \right) f \left( b^{1-t} (ab)^\frac{t}{2} \right) \bigg|_0^1 - \int_0^1 f \left( a^{1-t} (ab)^\frac{t}{2} \right) f' \left( b^{1-t} (ab)^\frac{t}{2} \right) dt
= f(a)f(b) - \frac{2}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx.
\]

By the following equality, we obtain the inequality (2.1)

\[
\int_a^{\sqrt{ab}} \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx = \int_a^{\sqrt{ab}} \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx.
\]

\[
\int_0^1 \frac{b}{2} \ln \left( \frac{a}{b} \right) t \left( \frac{a}{b} \right)^{\frac{t}{2}} f \left( a^{1-t} (ab)^\frac{t}{2} \right) f' \left( b^{1-t} (ab)^\frac{t}{2} \right)
+ \frac{a}{2} \ln \left( \frac{b}{a} \right) t \left( \frac{b}{a} \right)^{\frac{t}{2}} f \left( a^{1-t} (ab)^\frac{t}{2} \right) f' \left( b^{1-t} (ab)^\frac{t}{2} \right) dt
= \int_0^1 t \left[ f \left( a^{1-t} (ab)^\frac{t}{2} \right) f \left( b^{1-t} (ab)^\frac{t}{2} \right) \right]' dt
= tf \left( a^{1-t} (ab)^\frac{t}{2} \right) f \left( b^{1-t} (ab)^\frac{t}{2} \right) \bigg|_0^1 - \int_0^1 f \left( a^{1-t} (ab)^\frac{t}{2} \right) f' \left( b^{1-t} (ab)^\frac{t}{2} \right) dt
= f^2 \left( \sqrt{ab} \right) - \frac{2}{\ln b - \ln a} \int_a^{\sqrt{ab}} \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx
= f^2 \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx.
\]

This completes the proof of Lemma \( \Box \)

**Theorem 1.** Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( s \)-geometrically convex on \( [a, b] \) for \( q \geq 1 \) and \( s \in (0, 1] \),
then

\[(2.3)\]

\[
\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b f(x) f\left(\frac{ab}{x}\right) dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2 - \frac{1}{s}} H_{1}(s, q; h_{1}(\theta), h_{1}(\vartheta)),
\]

\[(2.4)\]

\[
\left| f^2\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b f(x) f\left(\frac{ab}{x}\right) dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2 - \frac{1}{q}} H_{2}(s, q; h_{2}(\theta), h_{2}(\vartheta)),
\]

where \(M_{1} = \max_{x \in [a, \sqrt{ab}]} |f(x)|\), \(M_{2} = \max_{x \in [\sqrt{ab}, b]} |f(x)|\).

\[(2.5)\]

\[
h_{1}(u) = \begin{cases} \frac{1}{\ln u - u - 1}, & u = 1 \\ \frac{1}{\ln u - u + 1}, & u \neq 1 \\ \end{cases},
\]

\[
h_{2}(u) = \begin{cases} \frac{1}{\ln u - u - 1}, & u = 1 \\ \frac{1}{\ln u - u + 1}, & u \neq 1 \\ \end{cases},
\]

\[
\theta = \left( \frac{a |f'(a)|^s}{b |f'(b)|^s} \right)^{\frac{q}{2}}, \quad \vartheta = \left( \frac{b |f'(b)|^s}{a |f'(a)|^s} \right)^{\frac{q}{2}}.
\]

\[(2.6)\]

\[
H_{i}(s, q; h_{i}(\theta), h_{i}(\vartheta)) = \begin{cases} b |f'(b)|^s M_{1} h_{i}^{1/q}(\theta) + a |f'(a)|^s M_{2} h_{i}^{1/q}(\vartheta), \\ |f'(a)|, \ |f'(b)| \leq 1, \\ b |f'(b)| |f'(a)|^{1-s} M_{1} h_{i}^{1/q}(\theta) + a |f'(a)| |f'(b)|^{1-s} M_{2} h_{i}^{1/q}(\vartheta), \\ |f'(a)|, \ |f'(b)| \geq 1, \\ b |f'(b)| M_{1} h_{i}^{1/q}(\theta) + a |f'(a)| M_{2} h_{i}^{1/q}(\vartheta), \\ |f'(a)| \leq 1 \leq |f'(b)|, \\ b |f'(b)|^{1-s} M_{1} h_{i}^{1/q}(\theta) + a |f'(a)| M_{2} h_{i}^{1/q}(\vartheta), \\ |f'(b)| \leq 1 \leq |f'(a)|. \\ \end{cases}, \quad i = 1, 2
\]

**Proof.** (1) Let \(M_{1} = \max_{x \in [a, \sqrt{ab}]} |f(x)|\), \(M_{2} = \max_{x \in [\sqrt{ab}, b]} |f(x)|\). Since \(|f'|^q\) is \(s\)-geometrically convex on \([a, b]\), from lemma \(\Box\) and power mean inequality, we have

\[
\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b f(x) f\left(\frac{ab}{x}\right) dx \right|
\]

\[
\leq \int_0^1 \frac{b}{2} \ln \left( \frac{a}{b} \right) |t - 1| \left( \frac{a}{b} \right)^{\frac{1}{q}} \left| f\left(a^{1-t}(ab)^{\frac{1}{q}}\right) \right| \left| f'\left(b^{1-t}(ab)^{\frac{1}{q}}\right) \right| dt
\]

\[
+ \frac{a}{2} \ln \left( \frac{b}{a} \right) |t - 1| \left( \frac{b}{a} \right)^{\frac{1}{q}} \left| f'\left(a^{1-t}(ab)^{\frac{1}{q}}\right) \right| \left| f\left(b^{1-t}(ab)^{\frac{1}{q}}\right) \right| dt
\]
\[ \frac{b}{2} \ln \left( \frac{a}{b} \right) M_1 \int_0^1 (1-t) \left( \frac{a}{b} \right)^{\frac{a}{b}} \left| f' \left( b^{1-t} (ab) \right) \right| dt \]

\[ + \frac{a}{2} \ln \left( \frac{b}{a} \right) M_2 \int_0^1 (1-t) \left( \frac{b}{a} \right)^{\frac{b}{a}} \left| f' \left( a^{1-t} (ab) \right) \right| dt \]

\[ \leq \frac{b}{2} \ln \left( \frac{a}{b} \right) M_1 \left( \int_0^1 (1-t) \left( \frac{a}{b} \right)^{\frac{a}{b}} \left| f' \left( b^{1-t} (ab) \right) \right| dt \right)^{\frac{1}{2}} \]

\[ + \frac{a}{2} \ln \left( \frac{b}{a} \right) M_2 \left( \int_0^1 (1-t) \left( \frac{b}{a} \right)^{\frac{b}{a}} \left| f' \left( a^{1-t} (ab) \right) \right| dt \right)^{\frac{1}{2}} \]

\[ \leq \frac{b \ln \left( \frac{b}{a} \right) M_1}{2} \left( \int_0^1 (1-t) \left( \frac{a}{b} \right)^{\frac{a}{b}} \left| f' \left( b \right) \right|^{q(1/2)^2} \left| f' \left( b \right) \right|^{q((2-t)/2)^2} dt \right)^{\frac{1}{2}} \]

\[ + \frac{a \ln \left( \frac{b}{a} \right) M_2}{2} \left( \int_0^1 (1-t) \left( \frac{b}{a} \right)^{\frac{b}{a}} \left| f' \left( b \right) \right|^{q(1/2)^2} \left| f' \left( a \right) \right|^{q((2-t)/2)^2} dt \right)^{\frac{1}{2}}. \]

If \( 0 < \mu \leq 1 \leq \eta, \ 0 < t, s \leq 1 \), then

\[ (2.9) \quad \mu^{t^s} \leq \mu^{t^s}, \quad \eta^{t^s} \leq \eta^{t^s+1-s}. \]

(i) If \( 1 \geq |f'(a)|, |f'(b)| \), by (2.9) we obtain that

\[ \int_0^1 (1-t) \left( \frac{a}{b} \right)^{\frac{a}{b}} \left| f' \left( a \right) \right|^{q((t/2)^2)} \left| f' \left( b \right) \right|^{q((2-t)/2)^2} dt \]

\[ \leq \int_0^1 (1-t) \left( \frac{a}{b} \right)^{\frac{a}{b}} \left| f' \left( a \right) \right|^{q((t/2)^2)} \left| f' \left( b \right) \right|^{q((2-t)/2)^2} \left| f' \left( a \right) \right|^{\theta} dt = |f' \left( b \right)|^{q \theta} h_1 (\theta), \]

\[ (2.10) \quad \int_0^1 (1-t) \left( \frac{b}{a} \right)^{\frac{b}{a}} \left| f' \left( b \right) \right|^{q((t/2)^2)} \left| f' \left( a \right) \right|^{q((2-t)/2)^2} dt \]

\[ \leq \int_0^1 (1-t) \left( \frac{b}{a} \right)^{\frac{b}{a}} \left| f' \left( b \right) \right|^{q((t/2)^2)} \left| f' \left( a \right) \right|^{q((2-t)/2)^2} \left| f' \left( b \right) \right|^{\theta} dt = |f' \left( a \right)|^{q \theta} h_1 (\theta). \]
(ii) If $1 \leq |f'(a)|$, $|f'(b)|$, by (2.9) we obtain that

$$
\int_{0}^{1} (1-t) \left( \frac{a}{b} \right)^{\frac{1}{q}} |f'(a)|^{q(t/2)^{r}} |f'(b)|^{q(2-t)/2^{r}} \, dt \\
\leq \int_{0}^{1} (1-t) \left( \frac{a}{b} \right)^{\frac{1}{q}} |f'(a)|^{q(\frac{b}{a} + 1-s)} |f'(b)|^{q(1-\frac{b}{a})} \, dt \\
= \left( |f'(a)|^{1-s} |f'(b)| \right)^{q} h_{1}(\theta),
$$

(2.11)

(iii) If $|f'(a)| \leq 1 \leq |f'(b)|$, by (2.9) we obtain that

$$
\int_{0}^{1} (1-t) \left( \frac{a}{b} \right)^{\frac{1}{q}} |f'(a)|^{q(t/2)^{r}} |f'(b)|^{q(2-t)/2^{r}} \, dt \\
\leq \int_{0}^{1} (1-t) \left( \frac{a}{b} \right)^{\frac{1}{q}} |f'(a)|^{q(\frac{b}{a} + 1-s)} |f'(b)|^{q(1-\frac{b}{a})} \, dt \\
= \left( |f'(b)|^{1-s} |f'(a)| \right)^{q} h_{1}(\theta),
$$

(2.12)

(iv) If $|f'(b)| \leq 1 \leq |f'(a)|$, by (2.9) we obtain that

$$
\int_{0}^{1} (1-t) \left( \frac{a}{b} \right)^{\frac{1}{q}} |f'(b)|^{q(t/2)^{r}} |f'(a)|^{q(2-t)/2^{r}} \, dt \\
\leq \int_{0}^{1} (1-t) \left( \frac{a}{b} \right)^{\frac{1}{q}} |f'(b)|^{q(\frac{b}{a} + 1-s)} |f'(a)|^{q(\frac{b}{a})} \, dt \\
= \left( |f'(a)|^{1-s} |f'(b)|^{s} \right)^{q} h_{1}(\theta),
$$
\begin{align}
(2.13) \quad &\int_0^1 \left(1 - t \right) \left( \frac{b}{a} \right)^{\frac{q}{q(t/2)^r}} \left| f' (b) \right|^{q(t/2)^r} \left| f' (a) \right|^{q((2-t)/2)^r} dt \\
&\leq \int_0^1 \left(1 - t \right) \left( \frac{b}{a} \right)^{\frac{q}{q(t/2)^r}} \left| f' (b) \right|^{q(1-\frac{q}{q(t/2)^r})} \left| f' (a) \right|^{q((2-t)/2)^r} dt = \left| f' (a) \right|^q h_1 (\theta).
\end{align}

From \textbf{(2.8)} to \textbf{(2.13)}, \textbf{(2.3)} holds.

(2) Let $M_1 = \max_{x \in [a, \sqrt{ab}]} |f(x)|$, $M_2 = \max_{x \in [\sqrt{ab}, b]} |f(x)|$. Since $|f'|^q$ is $s$-geometrically convex on $[a, b]$, from \textbf{lemma 1} and Hölder inequality, we have

\begin{align}
&\left| f^2 \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f (x)}{x} f \left( \frac{ab}{x} \right) dx \right| \\
&\leq \int_0^1 \frac{b}{2} \ln \left( \frac{b}{a} \right) \left( \int_0^1 t \left( \frac{a}{b} \right)^{\frac{q}{q(t/2)^r}} \left| f' (a) \right|^{q(1-t)/2} \right) \left( \int_0^1 t \left( \frac{b}{a} \right)^{\frac{q}{q(t/2)^r}} \left| f' (b) \right|^{q(1-t)/2} \right) dt \\
&= \frac{b}{2} \ln \left( \frac{b}{a} \right) M_1 \left( \int_0^1 t dt \right)^{1/4} \left( \int_0^1 t \left( \frac{a}{b} \right)^{\frac{q}{q(t/2)^r}} \left| f' (a) \right|^{q(1-t)/2} \right)^{\frac{1}{4}} \\
&= \frac{b}{2} \ln \left( \frac{b}{a} \right) M_2 \left( \int_0^1 t dt \right)^{1/4} \left( \int_0^1 t \left( \frac{b}{a} \right)^{\frac{q}{q(t/2)^r}} \left| f' (b) \right|^{q(1-t)/2} \right)^{\frac{1}{4}} \\
&\leq \frac{b \ln \left( \frac{b}{a} \right)}{2} M_1 \left( \int_0^1 t dt \right)^{1/4} \left( \int_0^1 t \left( \frac{a}{b} \right)^{\frac{q}{q(t/2)^r}} \left| f' (a) \right|^{q(1-t)/2} \right)^{\frac{1}{4}} \\
&= \frac{a \ln \left( \frac{b}{a} \right)}{2} M_2 \left( \int_0^1 t dt \right)^{1/4} \left( \int_0^1 t \left( \frac{b}{a} \right)^{\frac{q}{q(t/2)^r}} \left| f' (b) \right|^{q(1-t)/2} \right)^{\frac{1}{4}}.
\end{align}

(i) If $1 \geq |f'(a)|$, $|f'(b)|$, by \textbf{LEM} we obtain that

\begin{align}
\int_0^1 t \left( \frac{a}{b} \right)^{\frac{q}{q(t/2)^r}} \left| f' (a) \right|^{q(1-t)/2} \left| f' (b) \right|^{q(1-t)/2} dt \leq \left| f' (b) \right|^{qa} h_2 (\theta),
\end{align}
Let Corollary 2.

From (2.14) to (2.18), (2.4) holds. This completes the required proof.

\[(2.15) \quad \int_0^1 t \left( \frac{b}{a} \right)^\frac{q}{s} |f'(b)|^{q(1/2)^r} |f'(a)|^{q((2-t)/2)^r} dt \leq |f'(a)|^q h_2(\theta).\]

(ii) If \(1 \leq |f'(a)|, |f'(b)|\), by (2.14) we obtain that

\[(2.16) \quad \int_0^1 t \left( \frac{b}{a} \right)^\frac{q}{s} |f'(a)|^{q(1/2)^r} |f'(b)|^{q((2-t)/2)^r} dt \leq \left( |f'(b)|^{1-s} |f'(a)| \right)^q h_2(\theta).\]

(iii) If \(|f'(a)| \leq 1 \leq |f'(b)|\), by (2.9) we obtain that

\[(2.17) \quad \int_0^1 t \left( \frac{b}{a} \right)^\frac{q}{s} |f'(a)|^{q(1/2)^r} |f'(b)|^{q((2-t)/2)^r} dt \leq |f'(b)|^q h_2(\theta),\]

(iv) If \(|f'(b)| \leq 1 \leq |f'(a)|\), by (2.9) we obtain that

\[(2.18) \quad \int_0^1 t \left( \frac{b}{a} \right)^\frac{q}{s} |f'(a)|^{q(1/2)^r} |f'(b)|^{q((2-t)/2)^r} dt \leq |f'(a)|^q h_2(\theta).\]

From (2.14) to (2.18), (2.4) holds. This completes the required proof. □

If taking \(s = 1\) in Theorem \[\[\]\[\]\] we can derive the following corollary.

**Corollary 2.** Let \(f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+\) be differentiable on \(I^o\), and \(a, b \in I^o\) with \(a < b\) and \(f' \in L[a, b]\). If \(|f'|^q\) is geometrically convex on \([a, b]\) for \(q \geq 1\), then

\[
\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2-\frac{q}{2}} H_1(1, q; h_1(\theta), h_1(\theta)),
\]

\[
\left| f^2 \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right)^{2-\frac{q}{2}} H_2(1, q; h_2(\theta), h_2(\theta)),
\]

where \(\theta, \vartheta, H_1, H_2, h_1\) and \(h_2\) are the same as in Theorem \[\[\]\[\].

If taking \(q = 1\) in Theorem \[\[\]\[\] we can derive the following corollary.
Corollary 3. Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'| \) is \( s \)-geometrically convex on \([a, b]\) for \( s \in (0, 1] \), then

\[
\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right) H_1(s, 1; h_1(\theta), h_1(\vartheta)),
\]

\[
\left| f^2(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right) H_2(s, 1; h_2(\theta), h_2(\vartheta)),
\]

where \( \theta, \vartheta, H_1, H_2, h_1, h_2 \) are the same as in Theorem 2.

Theorem 2. Let \( f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+ \) be differentiable on \( I^o \), and \( a, b \in I^o \) with \( a < b \) and \( f' \in L[a, b] \). If \( |f'|^q \) is \( s \)-geometrically convex on \([a, b]\) for \( q > 1 \) and \( s \in (0, 1] \), then

\[
(2.19)
\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right) \left( \frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} H_3(s, q; h_3(\theta), h_3(\vartheta)),
\]

\[
(2.20)
\left| f^2(\sqrt{ab}) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) \, dx \right| \leq \ln \left( \frac{b}{a} \right) \left( \frac{1}{2} \right) \left( \frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} H_3(s, q; h_3(\theta), h_3(\vartheta)),
\]

where \( M_1 = \max_{x \in [a, \sqrt{ab}]} |f(x)|, \quad M_2 = \max_{x \in [\sqrt{ab}, b]} |f(x)|, \)

\[
h_3(u) = \begin{cases} 
1, & u = 1 \\
\frac{u}{\ln u}, & u \neq 1, \quad u > 0
\end{cases}
\]

\[
H_3(s, q; h_3(\theta), h_3(\vartheta)) = \begin{cases} 
\frac{b}{4} |f''(b)|^s M_1 h_3^{1/q}(\theta) + a |f'(a)|^s M_2 h_3^{1/q}(\vartheta), \\
|f''(a)|, \quad |f''(b)| \leq 1, \\
\frac{b}{2} |f''(b)| |f'(a)|^{1-s} M_1 h_3^{1/q}(\theta) + a |f'(a)||f''(b)|^{1-s} M_2 h_3^{1/q}(\vartheta), \\
|f''(a)|, \quad |f''(b)| \geq 1, \\
\frac{b}{4} |f''(b)| M_1 h_3^{1/q}(\theta) + a |f'(a)||f''(b)|^{1-s} M_2 h_3^{1/q}(\vartheta), \\
|f''(a)| \leq 1 \leq |f''(b)|, \\
\frac{b}{2} |f''(b)|^s M_1 h_3^{1/q}(\theta) + a |f'(a)| M_2 h_3^{1/q}(\vartheta), \\
|f''(a)| \leq 1 \leq |f''(a)|.
\end{cases}
\]

and \( \theta, \vartheta \) are the same as in (2.17).
Proof. (1) Let $M_1 = \max_{x \in [a, \sqrt{ab}]} |f(x)|$, $M_2 = \max_{x \in [\sqrt{ab}, b]} |f(x)|$. Since $|f'|^q$ is $s$-geometrically convex on $[a, b]$, from lemma [1] and Hölder inequality, we have

\[
\left| f(a) f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left( \frac{ab}{x} \right) \, dx \right| \leq \frac{b}{2} \left| \ln \left( \frac{a}{b} \right) \right| \left| t - 1 \right| \left( \frac{a}{b} \right)^{\frac{1}{2}} \left| f' \left( a^{1-t} (ab) \frac{t}{2} \right) \right| \left| f' \left( b^{1-t} (ab) \frac{t}{2} \right) \right| \frac{M_1}{M_2} \int_0^1 \left( (1-t) \left( \frac{a}{b} \right)^{\frac{1}{2}} \left| f' \left( a^{1-t} (ab) \frac{t}{2} \right) \right| \right) \frac{1}{q} \, dt + \frac{a}{2} \ln \left( \frac{b}{a} \right) M_2 \left( \int_0^1 \left( (1-t) \left( \frac{b}{a} \right)^{\frac{1}{2}} \right) \frac{1}{q} \right) \left| f' \left( a^{1-t} (ab) \frac{t}{2} \right) \right| \frac{1}{q} \, dt
\]

(2.21) $\leq \frac{b \ln \left( \frac{b}{a} \right) M_1}{2} \left( \frac{a - 1}{2q - 1} \right)^{1-\frac{1}{q}} \left( \int_0^1 \left( \frac{a}{b} \right)^{\frac{q}{2}} \left| f' \left( a^{q(t/2)^r} \right) \right| \frac{1}{q} \right) \left| f' \left( b^{q((2-t)/2)^r} \right) \right|^q \, dt + \frac{a \ln \left( \frac{b}{a} \right) M_2}{2} \left( \frac{q - 1}{2q - 1} \right)^{1-\frac{1}{q}} \left( \int_0^1 \left( \frac{b}{a} \right)^{\frac{q}{2}} \left| f' \left( b^{q(t/2)^r} \right) \right| \frac{1}{q} \right) \left| f' \left( a^{q((2-t)/2)^r} \right) \right|^q \, dt$.

(i) If $1 \geq |f'(a)|$, $|f'(b)|$, by (2.9) we have

\[
\int_0^1 \left( \frac{a}{b} \right)^{\frac{q}{2}} |f'(a)|^{q(t/2)^r} |f'(b)|^{q((2-t)/2)^r} \, dt = |f'(b)|^{q^*} h_3 (\theta),
\]

(2.22) \[
\int_0^1 \left( \frac{b}{a} \right)^{\frac{q}{2}} |f'(b)|^{q(t/2)^r} |f'(a)|^{q((2-t)/2)^r} \, dt = |f'(a)|^{q^*} h_3 (\theta).
\]

(ii) If $1 \leq |f'(a)|$, $|f'(b)|$, by (2.7) we have

\[
\int_0^1 \left( \frac{a}{b} \right)^{\frac{q}{2}} |f'(a)|^{q(t/2)^r} |f'(b)|^{q((2-t)/2)^r} \, dt \leq \left( |f'(b)| |f'(a)|^{1-s} \right)^q h_3 (\theta),
\]
From (2.26) and (2.22) to (2.25), (2.20) holds.

\[(2.23)\]
\[
\int_0^1 \left(\frac{b}{a}\right)^{2q} |f'(b)|^{q(t/2)^r} |f'(a)|^{q((2-t)/2)^r} \, dt \leq \left(\frac{|f'(a)| |f'(b)|^{1-s}}{|f'(b)|^{1-s}}\right)^q h_3(\vartheta).
\]

(iii) If \(|f'(a)| \leq 1 \leq |f'(b)|\), by (2.20) we obtain that
\[
\int_0^1 \left(\frac{a}{b}\right)^{2q} |f'(a)|^{q(t/2)^r} |f'(b)|^{q((2-t)/2)^r} \, dt \leq |f'(b)|^q h_3(\vartheta),
\]

\[(2.24)\]
\[
\int_0^1 \left(\frac{b}{a}\right)^{2q} |f'(b)|^{q(t/2)^r} |f'(a)|^{q((2-t)/2)^r} \, dt \leq \left(\frac{|f'(a)| |f'(b)|^{1-s}}{|f'(b)|^{1-s}}\right)^q h_3(\vartheta).
\]

(iv) If \(|f'(b)| \leq 1 \leq |f'(a)|\), by (2.20) we obtain that
\[
\int_0^1 \left(\frac{a}{b}\right)^{2q} |f'(a)|^{q(t/2)^r} |f'(b)|^{q((2-t)/2)^r} \, dt \leq |f'(a)|^q h_3(\vartheta),
\]

\[(2.25)\]
\[
\int_0^1 \left(\frac{b}{a}\right)^{2q} |f'(b)|^{q(t/2)^r} |f'(a)|^{q((2-t)/2)^r} \, dt \leq |f'(a)|^q h_3(\vartheta).
\]

From (2.21) to (2.25), (2.19) holds.

(2) Let \(M_1 = \max_{x \in [a, \sqrt{ab}]} |f(x)|, \ M_2 = \max_{x \in [\sqrt{ab}, b]} |f(x)|\) is \(s\)-geometrically convex on \([a, b]\), from lemma [4] and Hölder inequality, we have

\[
\left| f^2 \left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b f(x) \ f \left(\frac{ab}{x}\right) \ dx \right| \leq \frac{b}{2} \ln \left(\frac{b}{a}\right) \ M_1 \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{a}{b}\right)^{2q} |f'(a)|^{q(t/2)^r} |f'(b)|^{q((2-t)/2)^r} \, dt\right)^{\frac{1}{q}}
\]

\[
(2.26) \frac{a}{2} \ln \left(\frac{b}{a}\right) \ M_2 \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} \left(\int_0^1 \left(\frac{b}{a}\right)^{2q} |f'(b)|^{q(t/2)^r} |f'(a)|^{q((2-t)/2)^r} \, dt\right)^{\frac{1}{q}}.
\]

From (2.26) and (2.22) to (2.25), (2.20) holds. \(\square\)

If taking \(s = 1\) in Theorem 2 we can derive the following corollary.

**Corollary 4.** Let \(f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+\) be differentiable on \(I^o\), and \(a, b \in I^o\) with \(a < b\) and \(f' \in \mathcal{L}[a, b]\). If \(|f'|^q\) is \(s\)-geometrically convex on \([a, b]\) for \(q > 1\), then

\[
\left| f(a)f(b) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left(\frac{ab}{x}\right) \ dx \right| \leq \frac{1}{2} \ln \left(\frac{b}{a}\right) \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} H_3(1, q; h_3(\vartheta), h_3(\vartheta)),
\]

\[
\left| f^2 \left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} f \left(\frac{ab}{x}\right) \ dx \right| \leq \frac{1}{2} \ln \left(\frac{b}{a}\right) \left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}} H_3(1, q; h_3(\vartheta), h_3(\vartheta)),
\]

for any \(q > 1\).
where $\theta$, $\vartheta$, $H_3$ and $h_3$ are the same as in Theorem 2.

3. APPLICATION TO SPECIAL MEANS

Let us recall the following special means of two nonnegative number $a, b$ with $b > a$ :

1. The arithmetic mean

$$A = A(a, b) := \frac{a + b}{2}.$$ 

2. The geometric mean

$$G = G(a, b) := \frac{a + b}{2}.$$ 

3. The logarithmic mean

$$L = L(a, b) := \frac{b - a}{\ln b - \ln a}.$$ 

4. The $p$-logarithmic mean

$$L_p = L_p(a, b) := \left(\frac{b^{p+1} - a^{p+1}}{(p + 1)(b - a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$ 

Let $f(x) = \left(\frac{x^s}{s} + 1\right)$, $x \in (0, 1)$, $0 < s < 1$, $q \geq 1$, and then the function $|f'(x)|^q = x^{(s-1)q}$ is $s$-geometrically convex on $(0, 1)$ for $0 < s < 1$ (see [7]).

**Proposition 1.** Let $0 < a < b \leq 1$, $0 < s < 1$, and $q \geq 1$. Then

$$\left|G^2\left(\frac{a^s}{s} + 1, \frac{b^s}{s} + 1\right) - \frac{2}{s^2}A\left(G^2\left(a^s, b^s\right), s^2\right) - \frac{2}{s}L_{s-1}^2(a, b)L(a, b)\right| \leq \frac{1}{G(s-1)^2(a, b)} \left(\frac{1}{(s^2 - s + 1)q}\right)^{\frac{1}{q}} \left(\frac{b - a}{4L(a, b)}\right)^{1 - \frac{1}{q}} \times \left[M_1G\left(a^{-(s-1)^2}, b^s\right)\left\{b\left(\frac{a^{(s-1)^2}}{2} - L\left(a\frac{(s-1)^q}{2}, b\frac{(s-1)^q}{2}\right)\right)\right\} \right]^\frac{1}{q} \times \left[M_2G\left(b^{-(s-1)^2}, a^s\right)\left\{L\left(a\frac{(s-1)^q}{2}, b\frac{(s-1)^q}{2}\right) - a\left(\frac{(s-1)^q}{2}\right)\right\} \right]^\frac{1}{q},$$

$$\left|\left(\frac{G^s(a, b)}{s} + 1\right)^2 - \frac{2}{s^2}A\left(G^2\left(a^s, b^s\right), s^2\right) - \frac{2}{s}L_{s-1}^2(a, b)L(a, b)\right| \leq \frac{1}{G(s-1)^2(a, b)} \left(\frac{1}{(s^2 - s + 1)q}\right)^{\frac{1}{q}} \left(\frac{b - a}{4L(a, b)}\right)^{1 - \frac{1}{q}} \times \left[M_1G\left(a^{-(s-1)^2}, b^s\right)\left\{L\left(a\frac{(s-1)^q}{2}, b\frac{(s-1)^q}{2}\right) - a\left(\frac{(s-1)^q}{2}\right)\right\} \right]^\frac{1}{q} \times \left[M_2G\left(b^{-(s-1)^2}, a^s\right)\left\{b\left(\frac{(s-1)^q}{2} - L\left(a\frac{(s-1)^q}{2}, b\frac{(s-1)^q}{2}\right)\right)\right\} \right]^\frac{1}{q}.$$
where $M_1 = \left( \sqrt{ab}/s \right) + 1$ and $M_2 = (b^s/s) + 1$.

**Proof.** Let $f(x) = (x^s/s) + 1$, $x \in (0, 1]$ , $0 < s < 1$. Then $|f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \geq 1$, $M_1 = \max_{x \in [a, \sqrt{ab}]} f(x) = \left( \sqrt{ab}/s \right) + 1$, $M_2 = \max_{x \in [\sqrt{ab}, b]} f(x) = (b^s/s) + 1$. Thus, by Theorem 1, Proposition 1 is proved.

**Proposition 2.** Let $0 < a < b \leq 1$, $0 < s < 1$, and $q > 1$. Then

$$
\left| G^2 \left( \frac{a^s}{s} + 1, \frac{b^s}{s} + 1 \right) - \frac{2}{s^2} A \left( G^2 \left( a^s, b^s \right), s^2 \right) - \frac{2}{s} L_{s-1}^s \left( a, b \right) L \left( a, b \right) \right|
\leq \frac{b-a}{2L(a,b)} \frac{q-1}{2q-1} 1 - \frac{1}{q} \frac{L^1 a \left( \frac{-1}{2}, \frac{-1}{2} \right)}{G \left( s-1 \right)^2 \left( a, b \right)} \times \left\{ M_1 G \left( a, -\left( s-1 \right)^2, b^s \right) + M_2 G \left( b, -\left( s-1 \right)^2, a^s \right) \right\}
$$

$$
\left| \left( \frac{G^s (a,b)}{s} + 1 \right) - \frac{2}{s^2} A \left( G^2 \left( a^s, b^s \right), s^2 \right) - \frac{2}{s} L_{s-1}^s \left( a, b \right) L \left( a, b \right) \right|
\leq \frac{b-a}{2L(a,b)} \frac{q-1}{2q-1} 1 - \frac{1}{q} \frac{L^1 a \left( \frac{-1}{2}, \frac{-1}{2} \right)}{G \left( s-1 \right)^2 \left( a, b \right)} \times \left\{ M_1 G \left( a, -\left( s-1 \right)^2, b^s \right) + M_2 G \left( b, -\left( s-1 \right)^2, a^s \right) \right\}
$$

where $M_1 = \left( \sqrt{ab}/s \right) + 1$ and $M_2 = (b^s/s) + 1$.

**Proof.** Let $f(x) = (x^s/s) + 1$, $x \in (0, 1]$ , $0 < s < 1$. Then $|f'(a)| = a^{s-1} > b^{s-1} = |f'(b)| \geq 1$, $M_1 = \max_{x \in [a, \sqrt{ab}]} f(x) = \left( \sqrt{ab}/s \right) + 1$, $M_2 = \max_{x \in [\sqrt{ab}, b]} f(x) = (b^s/s) + 1$. Thus, by Theorem 2, Proposition 2 is proved. \qed

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