Strong Identification Codes for Graphs

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Abstract

For any graph $G$, a set of vertices $V$ is said to be dominating if every vertex of $G$ contains at least one node of $G$ and separating if each vertex $v$ contains a unique neighbour $u_v \in V$ that is adjacent to no other vertex of $G$. If $V$ is both dominating and separating, then $V$ is defined to be an identification code. In this paper, we study strong identification codes with an index $r$, by imposing the constraint that each vertex of $G$ contains at least $r$ unique neighbours in $V$. We use the probabilistic method to study both the minimum size of strong identification codes and the existence of graphs that allow an identification code with a given index.

Key words: Strong identification codes, strong neighbourhood graphs, probabilistic method.

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1 Introduction

An identification code of a graph $G$ is a set of vertices of $G$ that form both a dominating and a separating set (Karpovsky, Chakrabarty and Levitin [7]). Bounds for $\gamma(G)$, the minimum size of an identification code for $G$, have been evaluated for different classes of graphs: Gravier and Moncel [6] describe graphs whose minimum size identification code contains all vertices

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apart from exactly one vertex. Foucaud, Klasing, Kosowski and Raspaud [4] studied constructions of identification codes for triangle-free graphs, with size at most a constant fraction of the number of vertices. Later Foucaud and Perarnau [5] obtained bounds for $\gamma(G)$ in terms of the minimum vertex degree $\delta$ for graphs $G$ whose girth is at least five and also study the size of the identification codes for random regular graphs. Recently, [3] have investigated the use of graph spectra as a tool to characterize the identification codes.

In this paper, we study strong identification codes of a graph $G$ with an index $r \geq 1$, satisfying the additional constraint that each vertex contains at least $r$ unique neighbours in the code. (For formal definitions, we refer to Section 2). We use the probabilistic method to estimate the minimum size of an identification code and also to determine existence of graphs that allow strong identification codes with a given index.

The paper is organized as follows. In Section 2, we state and prove our main result, Theorem 1, regarding the minimum size of a strong identification code with index $r$. Next in Section 3, we state and prove Theorem 2 regarding existence of graphs that allow strong identification codes with a given index.

2 Strong Identification Codes

Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set $E$. For a vertex $v \in G$, we define $N(v)$ to be the set of all vertices adjacent to $v$, not including $v$ and let $N[v] = N(v) \cup \{v\}$. We also denote the elements of $N(v)$ as neighbours of $v$. Letting $\#A$ denote the number of elements in the set $A$, we have the following definition.

Definition 1. A set of vertices $C$ is said to be an identification code for $G$ with index $r$ if for any two distinct vertices $v, u \in G$ we have

$$\# \left( (N[v] \setminus N[u]) \cap C \right) \geq r.$$(2.1)

For $r = 1$, the above definition reduces to the concept of identification codes as studied in Gravier and Moncel [6] and Foucaud and Perarnau [5].

Clearly, one necessary condition for $G$ to have an identification code with index $r$ is that the neighbourhood of any vertex contains at least $r$ distinct vertices not present in any other vertex, i.e., for any two distinct ver-
indices \( u, v \in G \), we must have that
\[
\#(N[v] \setminus N[u]) \geq r. \tag{2.2}
\]
If (2.2) holds, we say that \( G \) has the \( r \)--strong neighbourhood property. In the next Section \ref{sec:next_section}, we use the probabilistic method to establish the existence of graphs satisfying (2.2).

If the \( r \)--strong neighbourhood property holds, then the whole vertex set of \( G \) is itself an identification code with index \( r \).

For graphs with stronger neighbourhoods, we have the following result regarding the minimum size of an identification code with a given index.

**Theorem 1.** Let \( G \) be a graph on \( n \) vertices satisfying the \((r + d + 1)\)--strong neighbourhood property for some integer \( d \geq 1 \). If \( \theta_n = \theta_n(G, r, d) \) is the minimum size of an identification code of \( G \) with index \( r \), then
\[
\frac{1}{\Delta + 1} \leq \frac{\theta_n}{n} \leq 1 - \frac{c(d, r)}{(\Delta + 1)(r+2)/d}, \tag{2.3}
\]
where \( \Delta \geq 2 \) is the maximum degree of a vertex in \( G \) and \( c(d, r) := d(d+1)^{-1} - 1 - (2r)^{-1}/d \).

Thus if \( G \) has \((r + d + 1)\)--strong neighbourhood property for some integer \( d \geq 1 \), then there exists a *nontrivial* fraction of vertices of \( G \) that act as an identification code with index \( r \).

**Proof of Theorem 1**

The lower bound in (2.3) is obtained as follows. If \( C \) is an identification code with index \( r \), then \( C \) is also a dominating set in the sense that \( \bigcup_{v \in C} N[v] = \{1, 2, \ldots, n\} \). Since \( \#N[v] \leq \Delta + 1 \), it follows that \( \#C \geq \frac{n}{\Delta + 1} \).

To prove the upper bound in (2.3), we use the probabilistic method similar to Theorem 1.2.2, pp. 4, Alon and Spencer 1, that concerns the minimum size of a dominating set. We select each vertex of \( G \) independently with probability \( q \) to be determined later and let \( Z \) be the random set of vertices selected. Say that a vertex \( v \in G \) is *bad* if one of the following two conditions hold: either (a) \( \#(N[v] \cap Z) \leq r - 1 \) or (b) \( \#(N[v] \cap Z) \geq r \) but there exists a vertex \( w \) satisfying
\[
\#((N[v] \setminus N[w]) \cap Z) \leq r - 1. \tag{2.4}
\]
A vertex of $G$ which is not bad is said to be \textit{good}.

If $Y_b$ is the set of all bad vertices and $Z_b = \bigcup_{v \in Y_b} N[v]$, then the set $Y := Z \cup Z_b$ is an identification code for $G$ with index $r$ : Indeed, if $v$ is a good vertex, then for any vertex $u \neq v$ we have

$$\# \left( (N[v] \setminus N[u]) \cap Y \right) \geq \# \left( (N[v] \setminus N[u]) \cap Z \right) \geq r.$$ 

On the other hand if $v$ is a bad vertex, then for any vertex $u \neq v$, we have by the $(r + d + 1)$—strong neighbourhood property that

$$\# \left( (N[v] \setminus N[u]) \cap Y \right) \geq \# \left( (N[v] \setminus N[u]) \cap Z_b \right) \geq r + d + 1.$$ 

We now estimate the size of $Z_b$ by bounding the probability that a vertex $v \in G$ is bad. If $\delta(v) := \# N[v]$, then the probability that case (a) occurs is

$$f_1(q, v) := \sum_{l=0}^{r-1} \binom{\delta(v)}{l} q^l (1-q)^{\delta(v)-l}. \quad (2.5)$$

Now if possibility (b) occurs and some vertex $w$ satisfies (2.4), then either $w$ is a neighbour of $v$ or $w$ is at distance two from $v$ and shares a common neighbour with $v$. Letting $\delta(v, w) := \# (N[v] \setminus N[w])$, the probability that case (b) occurs is then bounded above by

$$f_2(q, v) := \max_w \sum_{l=0}^{r-1} \binom{\delta(v, w)}{l} q^l (1-q)^{\delta(v, w)-l}, \quad (2.6)$$

where the maximum is taken over all vertices $w$ at a distance of at most two from $v$. Since the maximum degree of any vertex is at most $\Delta$, there are at most $\Delta + \Delta(\Delta - 1) = \Delta^2$ neighbours at a distance of at most two from $v$. Consequently, the probability that vertex $v$ is bad is upper bounded by $f_1(q, v) + \Delta^2 f_2(q, v)$ and so the expected number of vertices in $Y_b$ is at most $n (f_1(q, v) + \Delta^2 f_2(q, v))$. This in turn implies that the expected number of vertices in $Z_b = \bigcup_{v \in Y_b} N[v]$ is at most $n (\Delta f_1(q, v) + \Delta^3 f_2(q, v))$.

Summarizing, we get from the discussion in the previous paragraph that the expected number of vertices in $Z \cup Z_b$ is at most

$$n \left( q + \Delta \max_v f_1(q, v) + \Delta^3 \max_v f_2(q, v) \right). \quad (2.7)$$
To find suitable bounds for \( f_1(q, v) \) and \( f_2(q, v) \), we first let
\[
\delta := \min_v \#N(v) \geq r + d
\]
be the minimum degree of a vertex in \( G \). From (2.5) we have that
\[
f_1(q, v) \leq (1 - q)^{\delta - r + 1} \sum_{l=0}^{r-1} \binom{\delta(v)}{l} q^l \leq (1 - q)^d \sum_{l=0}^{r-1} \binom{\delta(v)}{l} q^l
\]
and using \( \binom{\delta(v)}{l} \leq \delta^l \leq (\Delta + 1)^l \) and \( q \leq 1 \), we get that
\[
f_1(q, v) \leq (1 - q)^d \sum_{l=0}^{r-1} (\Delta + 1)^l \leq r(\Delta + 1)^{r-1}(1 - q)^d + 1. \tag{2.8}
\]

For estimating (2.6), we argue similarly and use \( q \leq 1 \) and the fact that
\[
\delta(v, w) = \#(N[v] \setminus N[w]) \geq r + d + 1
\]
by the \((r + d + 1)\)-strong neighbourhood property, to get that
\[
f_2(q, v) \leq (1 - q)^{d+2} \sum_{l=0}^{r-1} \binom{\delta(v, w)}{l} \leq r(\Delta + 1)^{r-1}(1 - q)^{d+2}. \tag{2.9}
\]

Plugging the estimates (2.8) and (2.9) into (2.7), we get that the expected number of vertices in the identification code \( Z \cup Z_b \) is at most
\[
\mathbb{E}(\#(Z \cup Z_b)) \leq nq + r(\Delta + 1)^{r}(1 - q)^{d+1} + r(\Delta + 1)^{r+2}(1 - q)^{d+2}
\]
\[
\leq n(q + 2r(\Delta + 1)^{r+2}(1 - q)^{d+1})
\]
\[
=: n\Gamma(q). \tag{2.10}
\]

The solution \( q_0 \) to the equation \( \Gamma'(q) = 0 \) satisfies
\[
1 - 2r(\Delta + 1)^{r+2}(d + 1)(1 - q_0)^d = 0
\]
and so
\[
\Gamma(q_0) = q_0 + \frac{1 - q_0}{d + 1} = 1 - \frac{d}{d + 1} \left( \frac{1}{2r(\Delta + 1)^{r+2}(d + 1)} \right)^{\frac{1}{d}}.
\]

Consequently, there exists at least one identification code with index \( r \) and size at most \( n\Gamma(q_0) \). \qed
3 Strong Neighbourhood Graphs

In this section, we use the probabilistic method to obtain $r$–strong neighbourhood graphs for any integer $r \geq 1$.

The cycle $C_n$ on $n \geq 4$ vertices satisfies the 1–strong neighbourhood property. The following result establishes the existence of bounded degree graphs with $w$–strong neighbourhoods, for any integer $w \geq 2$.

**Theorem 2.** Let $w \geq 2$ be fixed. There exists an absolute constant $C > 0$ not depending on $w$ such that if $n \geq \max(C, 160(w + 1)^2 + 1) := M(w)$, then there exists a connected graph $G$ on $n$ vertices satisfying the $w$–strong neighbourhood property and having maximum vertex degree

$$\Delta(G) \leq \max(10 \log(2M(w)), 8(w + 1)) + 1.$$

Thus for any $w \geq 2$ and all $n$ large, there are bounded degree graphs on $n$ vertices satisfying the $w$–strong neighbourhood property.

We use the probabilistic method to prove Theorem 2, beginning with the following auxiliary Lemma.

**Lemma 3.** Let $y \geq 3$ be fixed. There exists an absolute constant $C > 0$ not depending on $y$ such that if $n \geq \max(C, 160y^2 + 1)$, then there exists a connected graph $G$ satisfying the $(y - 1)$–strong neighbourhood property and having maximum vertex degree of at most $\max(32 \log n, 8y)$.

To prove Lemma 3 we use the following standard deviation estimate. Let $\{X_j\}_{1 \leq j \leq L}$ be independent Bernoulli random variables with $\Pr(X_j = 1) = 1 - \Pr(X_j = 0) > 0$. If $T_L = \sum_{j=1}^L X_j, \theta_L = \mathbb{E}T_L$ and $0 < \epsilon \leq \frac{1}{2}$, then

$$\Pr(|T_L - \theta_L| \geq \theta_L \epsilon) \leq 2 \exp \left( -\frac{\epsilon^2}{4} \theta_L \right). \quad (3.1)$$

For a proof of (3.1), we refer to Corollary A.1.14, pp. 312 of Alon and Spencer (2008).

**Proof of Lemma 3** We use the probabilistic method. Let $H = G(n, p)$ be the random graph obtained by setting each edge of the complete graph $K_n$ on $n$ vertices to be independently present with probability $p$, where

$$p = \frac{\max(16 \log n, 4y)}{n - 1} \quad (3.2)$$
and absent otherwise.

Letting $N(1)$ denote the neighbours of the vertex 1, we have that
$\mathbb{E}\#N(1) = (n-1)p$ and so using (3.1) with $\epsilon = \frac{1}{2}$, we get
\[
\Pr \left( \frac{(n-1)p}{2} \leq \#N(1) \leq \frac{3(n-1)p}{2} \right) \geq 1 - \exp \left( -\frac{(n-1)p}{4} \right) \geq 1 - \frac{1}{n^4},
\] (3.3)
by (3.2). Letting
\[
A_{tot} := \bigcap_{1 \leq i \leq n} \left\{ \frac{(n-1)p}{2} \leq \#N(1) \leq \frac{3(n-1)p}{2} \right\},
\]
we get by the union bound and (3.3) that
\[
\Pr(A_{tot}) \geq 1 - \frac{1}{n^3}.
\] (3.4)
Thus, with high probability, i.e., with probability converging to one as $n \to \infty$, each vertex in the random graph $H$ contains at most $2(n-1)p = \max(32 \log n, 8y)$ neighbours.

Next, we estimate the number of common neighbours between the two vertices $i$ and $j$ as follows. Letting $T_{ij} := \#(N(i) \cap N(j))$ and using the Chernoff bound, we have for $s, t > 0$ that
\[
\Pr(T_{ij} > t) \leq \left( \mathbb{E}e^{sT_{ij}} \right)^{n-2} e^{-st}.
\] (3.5)
Each vertex $z \neq i, j$ is adjacent to both $i$ and $j$ with probability $p^2$, independent of the other vertices and so
\[
1 < \mathbb{E}e^{sT_{ij}} = 1 + (e^s - 1)p^2 \leq \exp \left( (e^s - 1)p^2 \right).
\] (3.6)
Substituting (3.6) into (3.5) and using $(\mathbb{E}e^{sT_{ij}})^{n-2} \leq (\mathbb{E}e^{sT_{ij}})^{n-1}$, we get
\[
\Pr(T_{ij} > t) \leq \exp \left( (e^s - 1)(n-1)p^2 \right) e^{-st}.
\] (3.7)
For $s = 2$, we have
\[
(e^s - 1)(n-1)p^2 = (e^2 - 1)\max(256(\log n)^2, 16y^2) \frac{1}{n-1} \leq \frac{1}{n-1} \max(2560(\log n)^2, 160y^2),
\]
which is at most one, provided that $n \geq 1 + \max(160y^2, n_0)$, where $n_0$ is the smallest integer $x$ such that $x > 2560(\log x)^2$. For all $n \geq 1 + \max(160y^2, n_0)$, we therefore have from (3.7) that
\[
P\left(T_{ij} > \left(\frac{n-1}{4}p\right)\right) \leq e^{\exp\left(\frac{-(n-1)}{2}p\right)} \leq \frac{e}{n^8},
\] since $(n-1)p \geq 16 \log n$ by (3.2). Thus,
\[
P\left(\#(N(i) \cap N(j)) \leq \left(\frac{n-1}{4}p\right)\right) \geq 1 - \frac{e}{n^8},
\] and, using the union bound, we have
\[
P\left(\bigcap_{i \neq j} \#(N(i) \cap N(j)) \leq \left(\frac{n-1}{4}p\right)\right) \geq 1 - \frac{e}{n^8}. \quad (3.9)
\]

From (3.8) and (3.9), we get that with high probability, the maximum degree of the random graph $H$ is at most $2(n-1)p = \max(32 \log n, 8y)$. In addition, for any two vertices $v \neq u$, the corresponding neighbourhood sets satisfy
\[
\#(N[v] \setminus N[u]) \geq \frac{(n-1)}{4}p - 1 \geq y - 1.
\]
Consequently, with high probability, the graph $H$ satisfies the $(y-1)$–strong neighbourhood property and because $p \geq \frac{16 \log n}{n}$, the graph $H$ is also connected with high probability (pp. 164–165, Bollobás [2]). Summarizing, there exists an absolute constant $C > 0$ not depending on $y$ such that if $n \geq \max(C, 160y^2 + 1)$, then with high probability $H$ is connected, satisfies the $(y-1)$–strong neighbourhood property and has a maximum vertex degree of at most $\max(32 \log n, 8y)$.

**Proof of Theorem 4** Let $w \geq 2$ and $M(w)$ be as in the statement of Theorem 2. For $n \geq M(w)$, split $\{1, 2, \ldots, n\}$ into $T$ sets $V_1, \ldots, V_T$ such that $V_i, 1 \leq i \leq T - 1$, has size exactly $M(w)$ and $V_T$ has size at least $M(w)$ and at most $2M(w)$. Using Lemma 3 with $w = y - 1$, we know that there exists a connected graph $G_i$ with vertex set $V_i$ satisfying the $w$–strong neighbourhood property and having a maximum vertex degree of at most $\Delta_0 = \max(32 \log(2M(w)), 8(w + 1))$. We pick one vertex $u_i \in V_i$ and connect $u_i$ to $u_{i+1}$ for $1 \leq i \leq T - 1$. The resulting graph $G$ is connected, satisfies
the \( w \)-strong neighbourhood property and has a maximum vertex degree of at most \( \Delta_0 + 1 \).

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