Elliptic algebra, Frenkel–Kac construction and root of unity limit

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Abstract

We argue that the level-1 elliptic algebra $U_{q,p}(\hat{\mathfrak{g}})$ is a dynamical symmetry realized as a part of 2d/5d correspondence where the Drinfeld currents are the screening currents to the $q$-Virasoro/W block in the 2d side. For the case of $U_{q,p}(\hat{\mathfrak{sl}}(2))$, the level-1 module has a realization by an elliptic version of the Frenkel–Kac construction. The module admits the action of the deformed Virasoro algebra. In a $r$th root of unity limit of $p$ with $q^2 \to 1$, the $\mathbb{Z}_r$-parafermions and a free boson appear and the value of the central charge that we obtain agrees with that of the 2d coset CFT with para-Virasoro symmetry, which corresponds to the 4d $\mathcal{N} = 2$ $SU(2)$ gauge theory on $\mathbb{R}^4/\mathbb{Z}_r$.

Keywords: elliptic algebra, deformed Virasoro symmetry, root of unity limit

1. Introduction

Since the proposal of AGT(W) relation [1, 2], 2d/4d correspondence and its generalizations have been intensively studied. Originally, it was the correspondence between the Nekrasov partition function of 4d $\mathcal{N} = 2$ supersymmetric $SU(N)$ gauge theory and the conformal block of 2d CFT with the $W_N$ symmetry. One of the generalizations is to consider the 4d $SU(N)$ gauge theory on $\mathbb{R}^4/\mathbb{Z}_r$ [3, 4]. For works in this direction, see for example [4–15]. The corresponding CFT is described by a coset

$$\frac{\hat{\mathfrak{sl}}(N)_r \oplus \hat{\mathfrak{sl}}(N)_\kappa}{\hat{\mathfrak{sl}}(N)_{r+\kappa}}.$$ (1.1)

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possessing the \( r \)th ‘para-\( W_N \) symmetry’ [6, 16]. Here \( \kappa \) is a parameter related to the \( \Omega \)-background parameters \( \epsilon_{1,2} \).

Another generalization is a 5d lift (or K-theoretic lift), i.e. to consider the 5d \( \mathcal{N} = 1 \) \( SU(N) \) gauge theories on \( \mathbb{R}^4 \times S^1 \) [17–23]. (For more general 2d/6d correspondence, see for example [24–28]). The corresponding 2d theories are no longer conformally invariant but are invariant under the deformed Virasoro [29–31] \( (N = 2) \) or deformed \( W_N \) symmetry \( (N \geq 3) \) [32, 33]. The deformed vertex operators [34–37] play important roles in constructing \( q \)-deformed conformal/W block. The \( q \)-Virasoro/W algebras themselves are, however, not sufficient to determine the \( q \)-vertex operators. This comes from their lack of the coalgebra structure—in particular, the coproduct. To construct the deformed vertex operators, one can take an approach that utilizes algebras having the coproduct and that is closely connected with the \( q \)-Viraroso/W algebras. There are at least two such algebras: the Ding–Iohara–Miki (DIM) algebras [38, 39] and the elliptic algebra \( U_{q,p} (\hat{\mathfrak{g}}) \) [40–47]. Here \( \hat{\mathfrak{g}} \) is an untwisted affine Lie algebra. They are different kinds of extension of the quantum group \( U_q (\mathfrak{g}) \). For research based on the DIM algebras, see, for example, [26, 48–55].

In this paper, we take the second approach to make exploit the elliptic algebra \( U_{q,p} (\hat{\mathfrak{g}}) \). One important property of this elliptic algebra with regard to the 2d/5d correspondence is that the Drinfeld currents act as the screening currents on the \( q \)-Virasoro/W block in the 2d side. The elliptic algebras are constructed based on the elliptic solutions to the Yang–Baxter equations. There are two class of elliptic solutions. One is related to the eight-vertex model (XYZ model) [56, 57]. The other is related to the face-type integrable lattice models (ABF [58] or RSOS models). The corresponding elliptic algebras are called vertex-type and face-type respectively. The Sklyanin algebra [59] and \( A_{q,p} (\mathfrak{sl}(N)) \) [60, 61] are vertex-type elliptic algebras.

The face-type elliptic solutions obey the dynamical Yang–Baxter equation (or the Gervais–Neveu–Felder equation) [62, 63]. The dynamical Yang–Baxter equation

\[ R_{12}(\lambda + h_2) R_{13}(\lambda) R_{23}(\lambda + h_1) = R_{23}(\lambda) R_{13}(\lambda + h_2) R_{12}(\lambda) \tag{1.2} \]

is equivalent to the star-triangle equation in solvable lattice models [64]. Here \( \lambda \) is a dynamical parameter and \( h \) is an element of the Cartan subalgebra. This equation first appeared in the study of the monodromy properties of the conformal blocks of the Liouville field theory [62]. (See also [65–67]).

Based on the face-type elliptic solutions, various elliptic quantum groups have been introduced. Some of them are \( E_{\gamma,\eta} (\mathfrak{sl}(2)) \) [63, 64], \( B_{q,\lambda} (\mathfrak{g}) \) [61] and \( U_{q,p} (\hat{\mathfrak{g}}) \). The latter two, \( B_{q,\lambda} (\mathfrak{g}) \) and \( U_{q,p} (\hat{\mathfrak{g}}) \), are face-type algebras closely related each other. They are quite similar, but have different Hopf algebra-like structures. \( B_{q,\lambda} (\mathfrak{g}) \) is a quasi-Hopf algebra [68] whose coproduct is not coassociative, while \( U_{q,p} (\hat{\mathfrak{g}}) \) is a \( H \)-Hopf algebra [69] whose coproduct is coassociative. Due to this coassociativity, \( U_{q,p} (\hat{\mathfrak{g}}) \) has a simpler coalgebra structure than \( B_{q,\lambda} (\mathfrak{g}) \) does.

Let us consider the face-type elliptic algebra \( U_{q,p} (\hat{\mathfrak{g}}) \) with level \( k \). It is an elliptic deformation of the algebra of screening charges of the coset CFT [40, 41]

\[ \hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_{-k-2}. \tag{1.3} \]

This CFT is closely related to the \( k \)-fusion RSOS model of type \( \mathfrak{g} \). For the level \( k = 1 \), \( U_{q,p} (\hat{\mathfrak{g}}) \) is closely related to the deformed \( W \) algebra of type \( \mathfrak{g} \) [32, 33]. In the CFT limit (\( q \to 1 \) limit), the coset model \( \hat{\mathfrak{g}}_1 \oplus \hat{\mathfrak{g}}_{-3} \) shows the ordinary \( W(\mathfrak{g}) \)-symmetry.

In this paper, we would like to argue that the level-1 \( U_{q,p} (\hat{\mathfrak{sl}}(N)) \) algebra is a dynamical symmetry in its connection with the 5d \( \mathcal{N} = 1 \) \( SU(N) \) gauge theory on \( \mathbb{R}^4 \times S^1 \). Let us denote
the radius of $S^1$ by $R$. We propose the following dictionary between the deformation parameters and gauge theory parameters:

\[ q = e^{(1/2)R(n + n^2)}, \quad p = e^{Re}, \quad p^* = pq^{-2} = e^{-Re}. \]  

(1.4)

Taking $R \to 0$ limit is equivalent to a CFT limit of $U_{q,p}(\hat{sl}(N))$ with $q \to 1$ ($p,p^* \to 1$). Hence we obtain ordinary 2d/4d correspondence: 4d $SU(N)$ gauge theory on $\mathbb{R}^4$ and 2d CFT with $W_N$ symmetry.

We also propose another CFT limit: a root of unity limit of the parameters

\[ p \to \omega^{\ell}, \quad p^* \to \omega^{\ell}, \quad q^2 \to 1, \]  

(1.5)

where $\omega$ is the primitive $r$th root of unity and $\ell$ is an integer such that $\omega^{\ell} \neq 1$. If the level 1 elliptic algebra $U_{q,p}(\hat{sl}(N))$ is the 2d side symmetry of 2d/5d correspondence and deformed blocks are given by the correlation functions of vertex operators of this algebra, this root of unity limit automatically leads to the correspondence between the $W$ block of the coset CFT $\hat{sl}(N) \oplus \hat{sl}(N)$ and the Nekrasov instanton partition function on $\mathbb{R}^4/\mathbb{Z}_r$.

For simplicity, we consider the $N = 2$ case: $U_{q,p}(\hat{sl}(2))$ [40]. Generalization to general $N$ is straightforward (though it may be tedious).

For the level $k = 1$ elliptic algebra $U_{q,p}(\hat{sl}(2))$, the closely related RSOS model is known as the Andrews–Baxter–Forrester (ABF) face model [58]. The $q$-deformed Virasoro algebra plays the role of dynamical symmetry of the ABF model [29, 70–72]. It has a general level $k$ realization by a deformed $Z$-algebra or parafermions. As we here only deal with its connection with the deformed Virasoro algebra, we consider a simple level-1 realization, i.e. an elliptic deformation of the Frenkel–Kac construction. In the $p \to 0$ limit, the elliptic algebra $U_{q,p}(\hat{sl}(2))$ essentially goes to the quantum group $U_q(\hat{sl}(2))$, and if we take further limit $q \to 1$, it goes to the affine Lie algebra $\hat{sl}(2)_k$.

This paper is organized as follows: in the next section, we review the elliptic algebra $U_{q,p}(\hat{sl}(2))$. In section 3, level 1-modules of $U_{q,p}(\hat{sl}(2))$ algebra based on an elliptic version of the Frenkel–Kac construction is explained. In section 4, we discuss the root of unity limit of the level-1 $U_{q,p}(\hat{sl}(2))$ algebra. In appendix, we briefly recall the Frenkel–Kac construction of the affine Lie algebra $\hat{sl}(2)_k$.

### 2. Elliptic algebra $U_{q,p}(\hat{sl}(2))$

In this section, we review the elliptic algebra $U_{q,p}(\hat{sl}(2))$ [40]. The face-type elliptic algebra $U_{q,p}(\hat{sl}(2))$ is an elliptic deformation of the affine Lie algebra $\hat{sl}(2)$. If we take the deformation parameters $p \to 0$ and $q \to 1$, then $U_{q,p}(\hat{sl}(2))$ goes to the $\hat{sl}(2)$ current algebra (and a Heisenberg algebra). We essentially follow the convention of [73].

Let $q$ and $p$ be two parameters. The elliptic algebra $U_{q,p}(\hat{sl}(2))$ is a unital associative algebra generated by the following elements

\[ P, h, e_m, f_m, \alpha_n, K^\pm, q^{\pm(1/2)k}, d, \quad (m \in \mathbb{Z}, n \in \mathbb{Z} \setminus \{0\}). \]  

(2.1)

$K^\pm$ are invertible, $k$ is a central element and $d$ is a grading operator

\[ [d, e_m] = m e_m, \quad [d, f_m] = m f_m, \quad [d, \alpha_n] = n \alpha_n, \quad [d, P] = [d, h] = [d, K^\pm] = 0. \]  

(2.2)

The eigenvalue of $k$ on a $U_{q,p}(\hat{sl}(2))$-module is called the level of the module.
It is convenient to introduce the elliptic currents:

\[ e(z) = \sum_{m \in \mathbb{Z}} e_m z^{-m-1}, \quad f(z) = \sum_{m \in \mathbb{Z}} f_m z^{-m-1}, \]

(2.3)

\[ \psi^+(q^{-1/2}z) = K^+ \exp \left( -\left( q - q^{-1} \right) \sum_{m > 0} \frac{\alpha_m}{1 - p^m} z^m \right) \exp \left( \left( q - q^{-1} \right) \sum_{m > 0} \frac{p^m \alpha_m}{1 - p^m} z^{-m} \right), \]

(2.4)

\[ \psi^-(q^{1/2}z) = K^- \exp \left( -\left( q - q^{-1} \right) \sum_{m > 0} \frac{p^m \alpha_m}{1 - p^m} z^m \right) \exp \left( \left( q - q^{-1} \right) \sum_{m > 0} \frac{\alpha_m}{1 - p^m} z^{-m} \right). \]

(2.5)

Then the remaining defining relations are given by

\[ [P, h] = 0, \quad [P, e(z)] = -2 e(z), \quad [h, e(z)] = 2 e(z), \]

(2.6)

\[ [P, f(z)] = 0, \quad [h, f(z)] = -2 f(z), \quad [P, \alpha_n] = 0, \quad [h, \alpha_n] = 0, \]

(2.7)

\[ [P, K^\pm] = -2K^\pm, \quad [h, K^\pm] = 0, \]

(2.8)

\[ K^\pm e(z) = q^{\mp 2} e(z) K^\pm, \quad K^\pm f(z) = q^{\mp 2} f(z) K^\pm, \]

(2.9)

\[ [\alpha_m, \alpha_n] = \delta_{m+n,0} \frac{[2m]_q}{m} \frac{1 - p^{|m|}}{1 - p^{|n|}} q^{-k|m|}, \]

(2.10)

\[ [\alpha_m, e(z)] = \frac{[2m]_q}{m} \frac{1 - p^{|m|}}{1 - p^{|n|}} q^{-k|m|} \zeta^m e(z), \quad [\alpha_m, f(z)] = - \frac{[2m]_q}{m} \zeta^m f(z), \]

(2.11)

\[ z_1 \frac{(q^{-2} z_1 / z_2 ; p^*)^L}{(p^* q^{-2} z_1 / z_2 ; p^*)^L} e(z_1) e(z_2) = - z_2 \frac{(q^2 z_1 / z_2 ; p^*)^L}{(p^* q^2 z_1 / z_2 ; p^*)^L} e(z_2) e(z_1), \]

(2.12)

\[ z_1 \frac{(q^{-2} z_1 / z_2 ; p^*)^L}{(p^* q^{-2} z_1 / z_2 ; p^*)^L} f(z_1) f(z_2) = - z_2 \frac{(q^2 z_1 / z_2 ; p^*)^L}{(p^* q^2 z_1 / z_2 ; p^*)^L} f(z_2) f(z_1). \]

(2.13)

\[ [e(z_1), f(z_2)] = \frac{1}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-2} z_1 / z_2) \psi^-(q^{1/2} z_2) - \delta(q^2 z_1 / z_2) \psi^+(q^{-1/2} z_2) \right). \]

(2.14)

Here \( p^* = pq^{-2k} \) and

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad (x; \xi)_\infty = \prod_{n = 0}^{\infty} (1 - x \xi^n), \quad \delta(z) = \sum_{n \in \mathbb{Z}} z^n. \]

(2.15)

2.1. \( p \to 0 \) limit: \( U_q(\widehat{sl}(2)) \)

In the \( p \to 0 \) limit, the elliptic algebra \( U_{q,p}(\widehat{sl}(2)) \) becomes the direct product of quantum group \( U_q(\widehat{sl}(2)) \) and the algebra \( \mathcal{H} \) generated by \( \{ P, e^{\pm 2k} \} \) where \( [P, Q] = 1 \).
\[ U_{q,p}(\mathfrak{sl}(2)) \rightarrow U_{q}(\mathfrak{sl}(2)) \otimes \mathcal{H}, \quad [U_{q}(\mathfrak{sl}(2)), \mathcal{H}] = 0, \quad (2.16) \]

\[ P \rightarrow P, \quad h \rightarrow h, \quad q^{\pm (1/2)k} \rightarrow q^{\pm (1/2)k}, \quad d \rightarrow d, \quad (2.17) \]

\[ e(z) \rightarrow x^+(z) e^{-2Q}, \quad f(z) \rightarrow x^-(z), \quad \alpha_n \rightarrow \tilde{\alpha}_n, \quad K^\pm \rightarrow q^{\pm h} e^{-2Q}, \quad (2.18) \]

\[ \psi^+(q^{-k/2}z) \rightarrow \varphi(q^{-k/2}z) e^{-2Q}, \quad \psi^-(q^{k/2}z) \rightarrow \psi(q^{k/2}z) e^{-2Q}, \quad (2.19) \]

where

\[ \varphi(q^{-k/2}z) = q^{-h} \exp \left( - (q - q^{-1}) \sum_{m > 0} \tilde{\alpha}_m z_m^m \right), \quad (2.20) \]

\[ \psi(q^{k/2}z) = q^h \exp \left( (q - q^{-1}) \sum_{m > 0} \tilde{\alpha}_m z_m^{-m} \right). \quad (2.21) \]

Here \( \tilde{\alpha}_n \) satisfies the following commutation relations:

\[ [\tilde{\alpha}_m, \tilde{\alpha}_n] = \delta_{m+n,0} \frac{[2m]_q [km]_q}{m} q^{-k|m|}. \quad (2.22) \]

The mode expansion of the Drinfeld currents \( x^\pm(z) \) is given by

\[ x^\pm(z) = \sum_{m \in \mathbb{Z}} x_m^m z^{m-1}. \quad (2.23) \]

Then, the quantum group \( U_{q}(\mathfrak{sl}(2)) \) is generated by

\[ x_m^\pm, \tilde{\alpha}_n, h, q^{\pm (1/2)k}, \tilde{d}, \quad (m \in \mathbb{Z}; n \in \mathbb{Z} \setminus \{0\}). \quad (2.24) \]

The defining relations are given as follows: \( k \) is a central element,

\[ [\tilde{d}, x_m^\pm] = mx_m^\pm, \quad [\tilde{d}, \tilde{\alpha}_n] = n \tilde{\alpha}_n, \quad [\tilde{d}, h] = 0, \quad (2.25) \]

\[ [h, \tilde{\alpha}_n] = 0, \quad [h, x_m^\pm] = \pm 2x_m^\pm, \quad (2.26) \]

\[ [\tilde{\alpha}_m, \tilde{\alpha}_n] = \delta_{m+n,0} \frac{[2m]_q [km]_q}{m} q^{-k|m|}, \quad (2.27) \]

\[ [\tilde{\alpha}_m, x^+(z)] = \frac{[2m]_q}{m} q^{-k|m|} z^{m} x^+(z), \quad [\tilde{\alpha}_m, x^-(z)] = -\frac{[2m]_q}{m} z^{m} x^-(z), \quad (2.28) \]

\[ z_1 (1 - q^{k/2} z_2 / z_1) x^+(z_1) x^+(z_2) = -z_2 (1 - q^{k/2} z_1 / z_2) x^+(z_2) x^+(z_1). \quad (2.29) \]

\[ [x^+(z_1), x^-(z_2)] = \frac{1}{(q - q^{-1})z_1 z_2} \left( \delta(q^{-k} z_1 / z_2) \psi(q^{k/2} z_2) - \delta(q^k z_1 / z_2) \varphi(q^{-k/2} z_2) \right). \quad (2.30) \]

### 2.1.1 \( q \rightarrow 1 \): \( \widehat{\mathfrak{sl}(2)_k} \) current algebra

Furthermore, if we also take the \( q \rightarrow 1 \) limit, \( U_{q}(\mathfrak{sl}(2)) \) goes to the \( \widehat{\mathfrak{sl}(2)_k} \) current algebra with level \( k \):

\[ U_{q}(\mathfrak{sl}(2)) \rightarrow \widehat{\mathfrak{sl}(2)_k}, \quad (2.31) \]
\[ x^\pm(z) \rightarrow J^\pm(z), \quad \tilde{\alpha}_n \rightarrow a_n, \quad \frac{\psi(q^{1/2}z) - \varphi(q^{-1/2}z)}{(q - q^{-1})z} \rightarrow 2J^3(z), \]  
\tag{3.22}

where
\[ 2J^3(z) = h z^{-1} + \sum_{m \neq 0} a_m z^{-m-1}. \]  
\tag{3.23}

Here
\[ [a_m, a_n] = 2k m \delta_{m+n,0}. \]  
\tag{3.24}

The commutation relations for the \( \hat{\mathfrak{sl}}(2) \) currents are given by
\[ [J^3(z_1), J^\pm(z_2)] = \pm \delta(z_2/z_1) \frac{J^\pm(z_2)}{z_2}, \quad [J^3(z_1), J^3(z_2)] = \frac{(k/2)}{z_1 z_2} \delta'(z_2/z_1), \]  
\tag{3.25}

\[ [J^+(z_1), J^-(z_2)] = \frac{k}{z_1 z_2} \delta'(z_2/z_1) + \delta(z_2/z_1) \frac{2J^3(z_2)}{z_2}. \]  
\tag{3.26}

Here \( \delta'(x) = \sum_{n \in \mathbb{Z}} nx^n \). These commutation relations are equivalent to the following OPE:
\[ J^3(z_1) J^\pm(z_2) \sim \frac{\pm J^\pm(z_2)}{z_1 - z_2}, \quad J^3(z_1) J^3(z_2) \sim \frac{k/2}{(z_1 - z_2)^2}, \]  
\tag{3.27}

\[ J^+(z_1) J^-(z_2) \sim \frac{k}{(z_1 - z_2)^2} + \frac{2J^3(z_2)}{z_1 - z_2}. \]  
\tag{3.28}

3. Level 1 modules of \( U_{q,p}(\hat{\mathfrak{sl}}(2)) \)

It is well-known that the level \( k = 1 \) modules of the \( \hat{\mathfrak{sl}}(2) \) current algebra can be obtained by a free massless chiral boson compactified on a circle with the self-dual radius. The Fock space of the compactified boson is decomposed into the two irreducible \( \hat{\mathfrak{sl}}(2)_1 \) modules with highest weights \( \Delta_0 \) and \( \Lambda_1 \). It is the so-called the Frenkel–Kac construction [74] (see appendix for a brief review).

In this section, the elliptic analog of the Frenkel–Kac construction is explained. Let us introduce elliptic bosons by
\[ \Phi(z) = 2Q_h + h \log z - \sum_{m \neq 0} \frac{\alpha_m}{|m|_q} z^{-m}, \]  
\tag{3.29}

\[ \Phi^\vee(z) = 2Q_h + h \log z - \sum_{m \neq 0} \frac{\alpha_m}{|m|_q} \frac{(1 - p^{|m|})}{(1 - p^{|m|})} q^{|m|} z^{-m}, \]  
\tag{3.30}

where the modes obey the following commutation relations
\[ [\alpha_m, \alpha_n] = \delta_{m+n,0} \frac{2|m|_q |m|_q}{m} \frac{(1 - p^{|m|})}{(1 - p^{|m|})} q^{|m|}, \quad [h, Q_h] = 1, \quad (m, n \neq 0). \]  
\tag{3.31}
Note that the commutation relations among the non-zero modes \( \alpha_n \) are the \( k = 1 \) case of (2.10). We also introduce an additional algebra generated by \( \{ P, e^{\pm 2Q} \} \) with \( [P, Q] = 1 \).

We assume that the elliptic bosons are ‘compactified on a circle of self-dual radius’. This means that the eigenvalues of \( h \) on the elliptic boson Fock space are integers\(^4\). Hence \( Q_h \) can appear only in the form \( e^{\pm q} \) (\( n \in \mathbb{Z} \)).

Let \( |0\rangle \) be the Fock vacuum characterized by
\[
h |0\rangle = 0, \quad P |0\rangle = 0, \quad \alpha_n |0\rangle = 0, \quad (n > 0).
\]
The Fock space \( \mathcal{F} \) of the elliptic bosons and the additional algebra is spanned by the following vectors
\[
\alpha_{n_1} \alpha_{n_2} \cdots \alpha_{n_k} e^{2m_0Q + m_2Q_h} |0\rangle, \quad (0 < n_1 \leq n_2 \leq \cdots \leq n_k; m_1, m_2 \in \mathbb{Z}). \tag{3.5}
\]

The action of the level 1 \( U_q(\widehat{\mathfrak{sl}}(2)) \) on the Fock space \( \mathcal{F} \) is realized by
\[
e(z) = e^{-2Q} e^{\Phi(z)} : \quad f(z) = e^{-\Phi(z)} :. \tag{3.6}
\]
\[
K^\pm = e^{-2Q} q^{\mp h} \tag{3.7}
\]
Here \( p^* = pq^{-2} \). The normal ordering is defined by moving \( Q \) and \( Q_h \) to the left of \( P \) and \( h \), the creation operators \( \alpha_{-n} (n > 0) \) to the left of the annihilation operators \( \alpha_n (n > 0) \). Hence,
\[
e(z) = e^{-2Q + 2\alpha_0 z^h} \exp \left( \sum_{m>0} \frac{\alpha_m z^m}{m!q} \right) \exp \left( -\sum_{m>0} \frac{\alpha_m z^{-m}}{m!q} \right). \tag{3.8}
\]
\[
f(z) = e^{-2Q} q^{-h} \alpha_m \exp \left( -\sum_{m>0} \frac{\alpha_m (1 - p^{*m})}{m!q} q^m z^m \right) \exp \left( \sum_{m>0} \frac{\alpha_m (1 - p^{m})}{m!q} q^m z^{-m} \right). \tag{3.9}
\]

The currents \( \varphi^\pm (q^{\mp_{1/2}} z) \) are given by (2.4) and (2.5) for level 1 \( \alpha_m \) and with \( K^\pm \) substituted by (3.7). The grading operator is realized as
\[
d = -\frac{1}{4} h^2 - \sum_{m>0} \frac{m^2}{2m!q} \frac{(1 - p^{*m})}{(1 - p^m)} q^m \alpha_m \alpha_m. \tag{3.10}
\]

As in the case of the affine Lie algebra, the Fock space \( \mathcal{F} \) decomposes into two irreducible level 1 \( U_q(\widehat{\mathfrak{sl}}(2)) \)-modules according to the eigenvalues of \( (-1)^h \):
\[
\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_- \tag{3.11}
\]
where \( \mathcal{F}_\pm = \{ v \in \mathcal{F} | (-1)^h v = \pm v \} \).

3.1. \( p \to 0 \) limit

In the \( p \to 0 \) limit, we obtain the free boson representation of level 1 \( U_q(\widehat{\mathfrak{sl}}(2)) \) [75]:
\[
e(z) \to e^{-2Q} x^+ (z), \quad f(z) \to x^- (z), \quad \alpha_n \to \tilde{\alpha}_n, \tag{3.12}
\]
\(^4\)See a remark on the last paragraph of appendix.
where
\[ x^+(z) = e^{2Q_0 z^h} : \exp \left( - \sum_{m \neq 0} \tilde{\alpha}_m \frac{z^{-m}}{|m|_q} \right) : , \]  
(3.13)

\[ x^-(z) = e^{-2Q_0 z^{-h}} : \exp \left( \sum_{m \neq 0} \frac{q^{|m|} \tilde{\alpha}_m}{|m|_q} z^{-m} \right) : , \]  
(3.14)

with

\[ [\tilde{\alpha}_m, \tilde{\alpha}_n] = \delta_{m+n,0} \sum_{m \neq 0} \frac{m}{|m|_q} q^{-|m|}. \]  
(3.15)

Note that

\[ \tilde{d} = -\frac{1}{4} h^2 - \sum_{m > 0} \frac{m^2}{|m|_q} q^m \tilde{\alpha}_m \tilde{\alpha}_m. \]  
(3.16)

3.1.1. \( q \to 1 \) limit. Furthermore, if we take \( q \to 1 \) limit, we obtain the Frenkel–Kac construction of the level 1 \( \hat{sl}(2) \) current algebra:

\[ x^\pm (z) \to J^\pm (z), \quad \tilde{\alpha}_n \to a_n, \]  
(3.17)

where

\[ J^\pm (z) = : e^{\pm \Phi_0(z)} : = e^{\pm 2Q_0 z^\pm h} : \exp \left( \pm \sum_{m \neq 0} \frac{a_m}{m} z^{-m} \right) :, \]  
(3.18)

\[ \Phi_0(z) = 2Q_0 + h \log z - \sum_{m \neq 0} \frac{a_m}{m} z^{-m}, \quad [a_m, a_n] = 2m \delta_{m+n,0}. \]  
(3.19)

Also, we find

\[ J^3(z) = \frac{1}{2} \partial \Phi_0(z). \]  
(3.20)

Note that the normalization of the free boson \( \Phi(z) \) is chosen as \( \langle \Phi_0(z_1) \Phi_0(z_2) \rangle = 2 \log(z_1 - z_2) \).

In the \( q \to 1 \) limit, the grading operator goes to

\[ \tilde{d} \to -L_0, \]  
(3.21)

where

\[ L_0 = \frac{1}{4} h^2 + \frac{1}{2} \sum_{m > 0} a_{-m} a_m. \]  
(3.22)

This operator \( L_0 \) is one of the Virasoro generators with the central charge \( c = 1 \):

\[ T(z) = \frac{1}{4} : (\partial \Phi_0(z))^2 : = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}. \]  
(3.23)
3.2. \textit{q}-Virasoro algebra

It is known that on the level-1 $U_{q,p}(\mathfrak{sl}(2))$-modules, the action of the deformed Virasoro algebra can be defined \cite{40}.

The deformed Virasoro algebra is introduced in \cite{29–31}. It contains two parameters, usually denoted by $q$ and $t$ (and $p = q/t$). But in order to avoid confusion with those of $U_{q,p}(\hat{\mathfrak{sl}}(2))$, we denote them by $\tilde{q}$ and $\tilde{t}$ (and $\tilde{p} = \tilde{q}/\tilde{t}$) in this paper.

The generators of $q$-Virasoro algebra are combined into the generating operator as

$$T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}. \quad (3.24)$$

The defining relations of the $q$-Virasoro algebra are given by

$$f(z_2/z_1)T(z_1)T(z_2) - f(z_1/z_2)T(z_2)T(z_1) = \frac{(1 - \tilde{q})(1 - \tilde{t}^{-1})}{(1 - \tilde{p})} [\delta(\tilde{p}z_1/z_2) - \delta(\tilde{p}^{-1}z_1/z_2)], \quad (3.25)$$

where

$$f(z) = \exp \left( \sum_{n > 0} 1 - n \frac{(1 - \tilde{q}^n)(1 - \tilde{t}^{-n})}{(1 - \tilde{p}^n)} z^n \right). \quad (3.26)$$

Let

$$T(z) = \tilde{\Lambda}_1(z) + \tilde{\Lambda}_2(z), \quad (3.27)$$

with

$$\tilde{\Lambda}_1(z) = q^{p_{+1}}(p^*)^{-(1/2)h} : \exp \left( \sum_{m \neq 0} \frac{1 - p^*m}{[2m]_q} \alpha_m z^{-m} \right) :, \quad (3.28)$$

$$\tilde{\Lambda}_2(z) = q^{-p_{-1}}(p^*)^{(1/2)h} : \exp \left( - \sum_{m \neq 0} \frac{1 - p^*m}{[2m]_q} \alpha_m (q^2 z)^{-m} \right) :, \quad (3.29)$$

This operator defines the action of the $q$-Virasoro algebra on the level 1 module of $U_{q,p}(\hat{\mathfrak{sl}}(2))$. The parameters of $q$-Virasoro algebra $\tilde{q}$, $\tilde{t}$ are respectively identified with those of the elliptic algebra as follows:

$$\tilde{q} = p, \quad \tilde{t} = p^* = pq^{-2}. \quad (3.30)$$

Note that $\tilde{p} = \tilde{q}/\tilde{t} = q^2$. The parameter $\beta$ is defined by $\tilde{t} = \tilde{q}^\beta$. Then we have $p^* = p^\beta$. If we write

$$p = q^{2M}, \quad p^* = q^{2(M-1)}, \quad (3.31)$$

then we can read off the value of the parameter $\beta$:

$$\beta = \frac{M - 1}{M}. \quad (3.32)$$

The deformation parameters $\tilde{q}$ and $\tilde{t}$ is related to the gauge theory parameters as $\tilde{q} = e^{R \epsilon z}$, $\tilde{t} = e^{-R \epsilon z}$. Therefore, (3.30) leads to (1.4).
Let us introduce the following currents:

\[
\tilde{e}(z) = e(z)z^{-(M-1)^{-1}p}, \quad \tilde{f}(z) = f(z)z^{M-1(L_2+\beta H)}.
\] (3.33)

These are the screening currents of the \(q\)-Virasoro algebra as we have stressed in the introduction:

\[
[T(z_1), \tilde{e}(z_2)] = (1-p)(1-p^*) \frac{d\tilde{e}}{dz_2} \left[ \delta(qz_1/z_2)z_2O_1(z_2) \right].
\] (3.34)

\[
[T(z_2), \tilde{f}(z_2)] = (1-p)(1-p^*) \frac{d\tilde{f}}{dz_2} \left[ \delta(q^2z_1/z_2)z_2O_2(z_2) \right].
\] (3.35)

Here

\[
\frac{d\tilde{e}}{dz}f(z) = \frac{f(z) - f(\xi z)}{(1-\xi)z}, \quad (\xi = p, p^*).
\] (3.36)

The operators \(O_1(z)\) and \(O_2(z)\) are given by

\[
O_1(z) = p^{-1}e^{-2\Omega_2q_0}q^{\rho_2(1/2)h}z^{-(M-1)^{-1}p+h}
\times \exp \left( - \sum_{m \neq 0} \frac{(1+p^m)}{[2m]_q} \alpha_m (qz)^{-m} \right) \quad :.
\] (3.37)

\[
O_2(z) = (p^*)^{-1}e^{-2\Omega_2q_0}q^{-(p+1)(1/2)h}z^{M-1(p+h)-h}
\times \exp \left( \sum_{m \neq 0} \frac{(1-p^m)(1+p^m)}{[2m]_q(1-p^m)} \alpha_m (q^{-2}z)^{-m} \right) \quad :.
\] (3.38)

### 4. Root of unity limit

We have obtained the dictionary between the parameters of the deformed Virasoro algebra and those of \(U_{q,p}(\hat{\mathfrak{sl}}(2))\). We can take the same root of unity limit of the parameters as was done in \[13, 14\].

Let us consider the following \(r\)th root of unity limit of the level-1 representations of \(U_{q,p}(\hat{\mathfrak{sl}}(2))\).

\[
p \to \omega^\ell, \quad p^* \to \omega^\ell, \quad q^2 \to 1,
\] (4.1)

where \(\omega\) is the primitive \(r\)th root of unity \(\omega = \exp(2\pi i/r)\), and \(\ell\) is an integer such that \(\omega^\ell \neq 1\).

A branch of \(\log p\) and \(\log p^*\) are chosen as

\[
\log p = 2\pi i \left( k_1 + \frac{\ell}{r} \right) - \frac{1}{\sqrt{3}}R, \quad (k_1 \in \mathbb{Z}),
\] (4.2)

\[
\log p^* = 2\pi i \left( k_2 + \frac{\ell}{r} \right) - \sqrt{3}R, \quad (k_2 \in \mathbb{Z}),
\] (4.3)
and the root of unity limit is meant by the limit of $R \to 0$. For simplicity, we assume $k_1 \neq k_2$.

The parameter $\beta$ is restricted to the value

$$\beta = \frac{k_2}{k_1} + \frac{\ell}{r} = \frac{r k_2 + \ell}{r k_1 + \ell},$$

(4.4)

from the consistency of the relation

$$\beta = \frac{\log p^*}{\log p}.$$  

(4.5)

For later convenience, let

$$m := \frac{r k_1 + \ell}{k_1 - k_2}.$$ 

(4.6)

Then

$$m - r = \frac{r k_2 + \ell}{k_1 - k_2}, \quad \beta = \frac{m - r}{m}.$$  

(4.7)

By repeating the analysis of [14], we see that the level 1 $U_{q,p}(\hat{sl}(2))$ goes to the tensor product of an algebra generated by $\{P, e^{\pm 2q}\}$, the $\mathbb{Z}_r$-parafermions and a free boson. The parafermions and the boson are fields on the $w$-plane where $w = z'$. The $\mathbb{Z}_r$-parafermions and the boson in a backgound charge $Q_E/\sqrt{r}$ are described by a conformal field theory (CFT) with the central charge

$$c = c_{\text{parafermion}} + c_{\text{boson}},$$

(4.8)

with

$$c_{\text{parafermion}} = \frac{2(r - 1)}{r + 2}, \quad c_{\text{boson}} = 1 - 6 \left(\frac{Q_E}{\sqrt{r}}\right)^2, \quad Q_E = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}.$$  

(4.9)

Then

$$c = \frac{2(r - 1)}{r + 2} + 1 - \frac{6r}{m(m - r)}$$

$$= \frac{3r}{r + 2} + \frac{3(m - 2 - r)}{m - r} - \frac{3(m - 2)}{m},$$

(4.10)

which is the central charge of the coset CFT:

$$\hat{sl}(2)_r \oplus \hat{sl}(2)_{m - 2 - r} \oplus \hat{sl}(2)_{m - 1}.$$  

(4.11)

We remark that by setting $b = i\sqrt{\beta}$, this central charge (4.10) can be expressed as

$$c = \frac{3r}{r + 2} + \frac{6}{r} (b + 1/b)^2.$$  

(4.12)

Hence the coset (4.11) may be described by the $r$th para-Liouville theory [16, 76].
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Appendix. Frenkel–Kac construction for $\hat{sl}(2)_1$

We briefly review the Frenkel–Kac construction [74] for the case of the level 1 $\hat{sl}(2)$ current algebra.

Let $\phi(z)$ be the free massless chiral boson on a circle of radius $R$:

$$\phi(z) = \hat{q} - i\hat{p} \log z + i \sum_{n \neq 0} \frac{1}{n} \hat{a}_n z^{-n},$$  \hspace{1cm} (A.1)

where

$$[\hat{q}, \hat{p}] = i, \quad [\hat{a}_m, \hat{a}_n] = m\delta_{m+n,0}. \hspace{1cm} (A.2)$$

Note that $\langle \phi(z_1) \phi(z_2) \rangle = - \log(z_1 - z_2)$. Let $|0\rangle$ be the Fock vacuum defined by

$$\hat{p} |0\rangle = 0, \quad \hat{a}_n |0\rangle = 0, \quad (n > 0). \hspace{1cm} (A.3)$$

Since the boson is compactified on the circle with radius $R$, the eigenvalues of the momentum operator $\hat{p}$ must be $n/R \ (n \in \mathbb{Z})$. Let us denote the corresponding momentum eigenstates by

$$|n; R\rangle = e^{i(n/R)\hat{q}} |0\rangle, \quad \hat{p} |n; R\rangle = \frac{n}{R} |n; R\rangle. \hspace{1cm} (A.4)$$

The compactified boson Fock space $\mathcal{F}_R$ is obtained from the Fock vacuum $|0\rangle$ by acting the creation operators $\hat{a}_m \ (m > 0)$ and $e^{i(n/R)\hat{q}} \ (n \in \mathbb{Z})$. On this Fock space the action of the position operator $\hat{q}$ is allowed only through the form of $e^{i(n/R)\hat{q}} \ (n \in \mathbb{Z})$. Hence the vertex operators

$$V_u(z) = : e^{iu\phi(z)} : = e^{iu\hat{q}z\partial_z} \exp \left( u \sum_{n > 0} \frac{1}{n} \hat{a}_{-n} z^n \right) \exp \left( -u \sum_{n > 0} \frac{1}{n} \hat{a}_n z^{-n} \right),$$  \hspace{1cm} (A.5)

with $Ru \in \mathbb{Z}$ can act on the Fock space. Let

$$J^\pm(z) = V_{\pm \sqrt{2}}(z) = : e^{\pm i \sqrt{2} \phi(z)} :, \quad J^3(z) = \frac{i}{\sqrt{2}} \partial \phi(z), \hspace{1cm} (A.6)$$

with mode expansion

$$J^a_m = \sum_{m \in \mathbb{Z}} J^a_m z^{-m-1}, \quad a = \pm, 3. \hspace{1cm} (A.7)$$

It is well-known that $J^a_m$ realize the affine Lie algebra $\hat{sl}(2)$ with level 1. In order to make the Fock space $\mathcal{F}_R$ be a $\hat{sl}(2)_1$-module, the compactification radius $R$ must be an integer multiple of $1/\sqrt{2}$.
In particular, at $R = \sqrt{2}$ (the self-dual radius), $J_{n}^{\pm}$ shifts the momentum $\hat{p} = n/\sqrt{2}$ to $\hat{p} = (n \pm 2)/\sqrt{2}$. Therefore, the boson Fock space $F_{R=\sqrt{2}}$ decomposes into two irreducible $\mathfrak{sl}(2)$-modules according to the values of $(-1)^{\hat{p}}$:

$$F_{\sqrt{2}} = L(\Lambda_0) \oplus L(\Lambda_1),$$

where

$L(\Lambda_0) = \{ v \in F_{\sqrt{2}} \mid (-1)^{\hat{p}} v = (+1)v \}, \quad L(\Lambda_1) = \{ v \in F_{\sqrt{2}} \mid (-1)^{\hat{p}} v = (-1)v \}.$

(A.8)

Here $\Lambda_0$ and $\Lambda_1$ are the fundamental weights of the affine Lie algebra $\mathfrak{sl}(2)$ and $L(\Lambda_i)$ are the integrable highest-weight module with highest weight $\Lambda_i$. The highest-weight vector of the basic module $L(\Lambda_0)$ and that of the defining module $L(\Lambda_1)$ are respectively given by

$|\Lambda_0 \rangle = |0; \sqrt{2} \rangle = |0 \rangle \in L(\Lambda_0), \quad |\Lambda_1 \rangle = |1; \sqrt{2} \rangle = e^{(i/\sqrt{2})\hat{q}} |0 \rangle \in L(\Lambda_1).$

(A.9)

**Remark.** The boson $\Phi_0(z)$ (3.19) is related to the canonically normalized boson $\phi(z)$ (A.1) as $\Phi_0(z) = i\sqrt{2} \phi(z)$. In particular, $h$ is identified with $\sqrt{2} \hat{p}$. Hence at the self-dual radius, the eigenvalues of $h$ must be integers.

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