On the Nonstationary Stokes System in a Cone ($L_p$ Theory)

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Abstract. The authors consider the Dirichlet problem for the nonstationary Stokes system in a three-dimensional cone. In a previous work they had proved the existence and uniqueness of solution in weighted $L^2$ Sobolev spaces, where the weights are powers of the distance from the vertex of the cone. Now they extend these results to weighted $L_{q,p}$ Sobolev spaces.

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Introduction

The present paper deals with the Dirichlet problem for the nonstationary Stokes system in a three-dimensional cone $K$, i.e., we consider the problem

$$\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \nabla p &= f, \quad -\nabla \cdot u = g \quad \text{in} \ K \times (0, \infty), \\
u(x, t) &= 0 \quad \text{for} \ x \in \partial K, \ t > 0, \\
u(x, 0) &= 0 \quad \text{for} \ x \in K.
\end{align*}$$

Elliptic boundary value problems (including the stationary Stokes system) in domains with conical points or edges are well-studied in many papers (see e.g. the bibliography in the monographs [9,10,24,26]). For the stationary Stokes system, we refer also to [3,11,21–23,29]. However, parabolic problems in domains with singularities on the boundary had not been researched as intensively until now. There are several papers dealing with the behavior of solutions of the heat equation near conical points and edges (see [2,8,14,19,25,30,31]). In [12,13,15], the solutions of second order parabolic equations with time-dependent coefficients in cones and wedges were studied. A theory for general parabolic equations (not systems) with time-independent coefficients in domains with conical points was developed in the papers [5–7]. This theory includes, in particular, existence, uniqueness and regularity theorems for solutions in weighted $L^2$ Sobolev spaces and a description of the behavior of the solutions near the conical point. However, the class of problems considered in [5–7] does not include the Stokes system, and an extension of the results to the Stokes system is not trivial. In our recent paper [16], we were able to prove an existence and uniqueness theorem for solutions of the nonstationary Stokes system in weighted $L^2$ Sobolev spaces. An analogous result for a two-dimensional angle was proved in [28]. The existence and uniqueness results in [16] and [28] show that there are essential differences between the Stokes system on one hand and the heat equation (and the class of parabolic problems considered in [5–7]) on the other hand. In the case of the heat equation, the bounds for the weight parameter $\beta$ in the existence and uniqueness theorems depend on the eigenvalues of only one operator pencil. If one deals with the Stokes system, one has to consider the eigenvalues of two different operator pencils. In [17,18], we studied the behavior of the solutions of the Stokes system at infinity and near the vertex of the cone, respectively. It turns out that the asymptotics of solutions of the Stokes at infinity differs considerably from the asymptotics of solutions of the heat equation and other parabolic problems.
The goal of the present paper is to prove the existence and uniqueness of solutions in weighted $L_{q,p}$ Sobolev spaces and so generalize the results in [16]. As in [16], we consider first the problem
\[
 sU - \Delta U + \nabla P = F, \quad -\nabla \cdot U = G \quad \text{in} \quad K, \quad U = 0 \quad \text{on} \partial K
\]
with the complex parameter $s$ which arises if one applies the Laplace transform to the problem (1), (2). A large part of the paper (Sect. 2) is devoted to this parameter-depending problem. In [16] we proved an existence and uniqueness theorem for solutions of the problem (3) in weighted $L_2$ Sobolev spaces, regularity assertions and estimates for the solution. In Sects. 2.2–2.5, we extend these results to weighted $L_p$ Sobolev spaces. Let $V^1_{p,\beta}(K)$ be the weighted Sobolev space with the norm
\[
\|U\|_{V^1_{p,\beta}(K)} = \left( \int_K \sum_{|\alpha| \leq l} r^{p(|\beta|+|\alpha|)} |\partial^\alpha U(x)|^p \, dx \right)^{1/p},
\]
where $r = r(x)$ is the distance of $x$ from the vertex of the cone. All other function spaces which are used in the present paper are introduced in Sect. 1. In particular, we obtain the following existence and uniqueness result (see Theorem 2.3 for the heat equation and other parabolic problems. But, as was shown in [16], the above condition on $\beta$ turns out to be insufficient to prove the existence and uniqueness of solutions of the time-dependent problem (1), (2) in weighted $L_2$ Sobolev spaces, regularity assertions and estimates for the solution. In Sects. 2.2–2.5, we extend these results to weighted $L_p$ Sobolev spaces.

Suppose that $\text{Re} \ s \geq 0$, $F \in V^0_{p,\beta}(K)$, $G \in V^1_{p,\beta}(K)$ and $sG \in (V^1_{p,-\beta}(K))^*$, where $1 < p < \infty$, $p' = p/(p-1)$ and $\beta$ satisfies the condition
\[
- \min(\mu_2, \lambda_1 + 1) < 2 - \beta - \frac{3}{p} < \lambda_1, \quad 2 - \beta - \frac{3}{p} \neq 0.
\]
In the case $s \neq 0$, $\beta + \frac{3}{p} > 2$, we assume that the integral of $G$ over $K$ is zero. Then the problem (3) has a uniquely determined solution $(U, P)$ in the space $V^2_{p,\beta}(K) \times V^1_{p,\beta}(K)$ which satisfies the estimate
\[
\|U\|_{V^2_{p,\beta}(K)} + |s| \|U\|_{V^0_{p,\beta}(K)} + \|P\|_{V^1_{p,\beta}(K)} \\
\leq c \left( \|F\|_{V^0_{p,\beta}(K)} + \|G\|_{V^1_{p,\beta}(K)} + |s| \|G\|_{(V^1_{p,-\beta}(K))^*} \right)
\]
with a constant $c$ independent of $U$, $P$ and $s$.

The uniqueness of the solution holds even for $-(\lambda_1 + 1) < 2 - \beta - \frac{3}{p} < \mu_2 + 1$ (see Theorem 2.2). In Sect. 2.6, we obtain a similar estimate for the derivative $(U'(s), P'(s))$ of the solution with respect to the parameter $s$. The numbers $\lambda_1$ and $\mu_2$ in (5) are the smallest positive eigenvalues of certain operator pencils which are introduced in Sect. 1.3.

In the case $\mu_2 < \lambda_1 + 1$, the condition (5) on $\beta$ is more restrictive than the analogous condition for the heat equation and other parabolic problems. But, as was shown in [16], the above condition on $\beta$ is sharp. Theorem 2.3 with the condition (5) on $\beta$ turns out to be insufficient to prove the existence and uniqueness of solutions of the time-dependent problem (1), (2) in weighted $L_p$ Sobolev spaces if $p < 2$. For this reason, we consider the parameter-depending problem (3) also for data $F \in V^0_{p,\beta}(K)$ and $G = 0$, where $\beta$ satisfies the weaker condition
\[
-1 - \lambda_1 < 2 - \beta - \frac{3}{p} < \lambda_1.
\]
As is shown in Sect. 2.7, there exist solutions $(U, P)$ in a larger space $V^2_{p,\beta,\gamma}(K) \times V^1_{p,\beta,\gamma}(K)$ for such data. In Sect. 2.8, where we restrict ourselves to the case $p = 2$, we establish the asymptotics of the function $U \in V^2_{2,\beta,\gamma}(K)$ at infinity. Here, the remainder is an element of the weighted space $V^{3/2}_{2,\beta-\frac{1}{2}}(K)$ which is a subspace of $V^0_{p,\beta'-2}(K)$ for arbitrary $p \geq 2$ and $\beta' + \frac{3}{p} = \beta + \frac{3}{2}$.

The second part of the paper (Sect. 3) deals with the time-dependent problem (1), (2). The goal of this part is to prove the existence and uniqueness of solutions in the space $W^2_{q,p,\beta}(Q) \times L_q(\mathbb{R}_+; V^1_{p,\beta}(K))$ if $\beta$ satisfies the condition (5). Here, $Q = K \times \mathbb{R}_+$ and $W^2_{q,p,\beta}(Q)$ is a subspace of $L_q(\mathbb{R}_+; V^2_{p,\beta}(K))$ (see Sect. 1.2). The main result of the present paper is the following.
Suppose that \( f \in L_q(\mathbb{R}_+; V^0_{p,\beta}(K)) \), \( g \in L_q(\mathbb{R}_+; V^1_{p,\beta}(K)) \), \( g(x,0) = 0 \) for \( x \in K \), and \( \partial_t g \in L_q(\mathbb{R}_+; (V^1_{p,-\beta}(K))^*) \), where \( p, q \in (1, \infty) \) and \( \beta \) satisfies the inequalities (5). In the case \( \beta + \frac{2}{p} > 2 \), we assume that \( g \) satisfies the condition

\[
\int_K g(x,t) \, dx = 0 \quad \text{for almost all } t. \tag{8}
\]

Then there exists a uniquely determined solution \((u,p)\) of the problem (1), (2) in the space \( W^{2,1}_{q,p;\beta}(Q) \times L_q(\mathbb{R}_+; V^1_{p,\beta}(K)) \).

In order to prove this result, we proceed as follows. First, we prove in Sect. 3.1 that for arbitrary \( g \in L_q(\mathbb{R}_+; V^1_{p,\beta}(K)) \), \( \partial_t g \in L_q(\mathbb{R}_+; (V^1_{p,-\beta}(K))^*) \), \( g(x,0) = 0 \) for \( x \in K \), \( \beta + \frac{2}{p} \not\in \{-1,1,2,4\} \), there exists a vector function \( u \in W^{2,1}_{q,p;\beta}(Q) \) such that

\[ \nabla \cdot u = g \quad \text{in } Q, \]

\[ u(x,t) = 0 \quad \text{for } x \in \partial K \quad \text{and } u(x,0) = 0 \quad \text{for } x \in K. \]

In the case \( 2 < \beta + \frac{2}{p} < 4 \), we must assume for this that \( g \) satisfies the condition (8). This result allows us to restrict ourselves to the system (1), (2) with \( g = 0 \).

In Sect. 3.2, we prove some a priori estimates for the solutions in weighted spaces, while Sects. 3.3–3.6 deal with the solvability of the problem (1), (2) in weighted \( L_{q,p} \) Sobolev spaces for different \( p \) and \( q \).

In Sect. 3.3 we start with the case \( p = q \geq 2 \). We prove that for arbitrary \( f \in L_p(\mathbb{R}_+; V^0_{p,\beta}(K)) \) and for \( g = 0 \), there exists a unique solution of the problem (1), (2) in the space \( W^{2,1}_{p,p;\beta}(Q) \times L_p(\mathbb{R}_+; V^1_{p,\beta}(K)) \) if \( \beta \) satisfies the condition (5) (see Theorem 3.1). The method is similar to the proof of weighted \( L_p \) estimates for elliptic problems in domains with edges in [20]. We use the estimates for solutions of the parameter-dependent problem together with an extension of Mikhlin’s Fourier multiplier theorem to operator-valued functions.

In order to obtain the same existence and uniqueness result as in Theorem 3.1 for the case \( p = q < 2 \), it is not sufficient to apply a duality argument if \( \mu_2 < \lambda_1 + 1 \). This is also a reason why we are interested in solutions of the problem (1), (2) for right-hand sides \( f \in L_p(\mathbb{R}_+; V^0_{p,\beta}(K)) \), \( g = 0 \), where \( p \geq 2 \) and \( \beta \) satisfies the weaker condition (7).

Under this condition on \( p \) and \( \beta \), we show in Sect. 3.4 that there exists a solution of this problem satisfying the estimate

\[ \|u\|_{L_p(\mathbb{R}_+; V^0_{p,\beta-2}(K))} \leq c \|f\|_{L_p(\mathbb{R}_+; V^0_{p,\beta}(K))} \]

with a constant \( c \) independent of \( f \) (see Lemma 3.11). For this, we use the results of Sects. 2.7 and 2.8 and the above mentioned Fourier multiplier theorem. Lemma 3.11 enables us to extend the result of Theorem 3.1 to the case \( p < 2 \) what is done in Sect. 3.5. In the closing Sect. 3.6 we prove the above given existence and uniqueness result in weighted \( L_{q,p} \) Sobolev spaces for the case \( p \neq q \).

1. Notation, Definitions

Throughout this paper, \( K = \{x \in \mathbb{R}^3 : \omega = x/|x| \in \Omega\} \) is a cone with vertex at the origin. Here, \( \Omega \) is a subdomain of the unit sphere \( S^2 \) with smooth (of class \( C^{2,\alpha} \)) boundary \( \partial \Omega \), where \( \alpha \) is an arbitrarily small number, \( 0 < \alpha < 1 \). Furthermore, let \( Q = K \times \mathbb{R}_+ = K \times (0,\infty) \).

1.1. Weighted Sobolev Spaces in a Cone

For nonnegative integer \( l \) and real \( p \in (1, \infty) \), \( \beta \in \mathbb{R} \), we define the weighted Sobolev space \( V^l_{p,\beta}(K) \) as the set of all functions (or vector functions) with finite norm (4). The space \( V^0_{p,\beta}(K) \) is also denoted by \( L_{p,\beta}(K) \). The trace space of \( V^l_{p,\beta}(K) \), \( l \geq 1 \), on the boundary \( \partial K \) is denoted by \( V^{l-1/p}_{p,\beta}(\partial K) \). For a noninteger number \( s = l + \sigma \), where \( l \) is an integer, \( l \geq 0 \), and \( 0 < \sigma < 1 \), the space \( V^s_{p,\beta}(K) \) is defined as
the weighted Sobolev-Slobodetskii space with the norm
\[
\|U\|_{V^s_{p,\beta}(K)} = \left( \|U\|_{V^s_{p,\beta-\sigma}(K)}^p + \sum_{|\alpha|=l} |\partial^\alpha U|^p_{s,p,\beta} \right)^{1/p},
\]
where
\[
|U|^p_{s,p,\beta} = \int_K \int_K \frac{|x|^p \left| U(x) - U(y) \right|^p}{|x-y|^{3+p\sigma}} \, dx \, dy.
\]

Note that the set \( C_0^\infty(K\setminus\{0\}) \) of all infinitely differentiable functions with compact support in \( K\setminus\{0\} \) is dense in \( V^1_{p,\beta}(K) \) and \( V^s_{p,\beta}(K) \). Consequently, these spaces can be also defined as the completion of the set \( C_0^\infty(K\setminus\{0\}) \) with respect to the above norms. By \( \overset{0}{V}^1_{p,\beta}(K) \), we denote the closure of the set \( C_0^\infty(K) \) in \( V^1_{p,\beta}(K) \) or, what is the same, the space of all functions \( U \in V^1_{p,\beta}(K) \) which are zero on \( \partial K\setminus\{0\} \). By Hardy’s inequality, there exists a constant \( c \) such that
\[
\|U\|_{L^{p-1}(K)} \leq c \|\nabla U\|_{L^{p,\beta}(K)} \quad \text{for all } U \in C_0^\infty(K\setminus\{0\}) \quad \text{if } \beta \neq 1 - \frac{3}{p}.
\]

For \( U \in C_0^\infty(K) \) this estimate is valid without the above restriction on \( \beta \). This means that the \( V^1_{p,\beta}(K) \)-norm of a function \( U \in \overset{0}{V}^1_{p,\beta}(K) \) is equivalent to the \( L^{p,\beta}(K) \)-norm of \( \nabla U \). Under the condition \( \beta \neq 1 - \frac{2}{p} \), this is also true for the space \( V^1_{p,\beta}(K) \).

Let \( l \) be a nonnegative integer, and let \( p \) and \( \beta \) be real numbers, \( 1 < p < \infty \). Then \( E^l_{p,\beta}(K) \) is the weighted Sobolev space with the norm
\[
\|U\|_{E^l_{p,\beta}(K)} = \left( \int_K \sum_{|\alpha| \leq l} \left( r^{p\beta} + r^{p(|\beta|-l+|\alpha|)} \right) |\partial^\alpha U(x)|^p \, dx \right)^{1/p}.
\]

As is known, \( E^l_{p,\beta}(K) = V^l_{p,\beta}(K) \cap V^0_{p,\beta}(K) \), and the \( E^l_{p,\beta}(K) \)-norm is equivalent to the norm
\[
\|U\| = \|U\|_{V^l_{p,\beta}(K)} + \|U\|_{V^0_{p,\beta}(K)}.
\]

The trace space for \( E^l_{p,\beta}(K) \), \( l \geq 1 \), on the boundary is denoted by \( E^{l-1/p}_{p,\beta}(\partial K) \). Furthermore, \( \overset{0}{E}^1_{p,\beta}(K) \) is defined as the space of all functions \( u \in E^1_{p,\beta}(K) \) which are zero on the boundary.

The function \( G \) in (3) is mostly considered as an element of the space
\[
X^1_{p,\beta}(K) = V^1_{p,\beta}(K) \cap (V^1_{p,\gamma}(K))^*,
\]
where \( p \in (1, \infty) \) and \( p' = p/(p-1) \). Note that there is also the equality \( X^1_{p,\beta}(K) = E^1_{p,\beta}(K) \cap (V^1_{p,\gamma}(K))^* \) (see [16] in the case \( p = 2 \)).

Finally, we introduce the spaces
\[
E^2_{p,\beta,\gamma}(K) = E^2_{p,\beta}(K) + E^2_{p,\gamma}(K), \quad \text{and} \quad V^1_{p,\beta,\gamma}(K) = V^1_{p,\beta}(K) + V^1_{p,\gamma}(K)
\]
with different weight parameters \( \beta \) and \( \gamma \). Let \( \zeta \) be a smooth (two times continuously differentiable) function on \( K \) with compact support, \( \zeta(x) = 1 \) for \( |x| < 1 \), and let \( \eta = 1 - \zeta \). Then
\[
E^2_{p,\beta,\gamma}(K) = \{ U : \zeta U \in E^2_{p,\beta}(K), \ \eta U \in E^2_{p,\gamma}(K) \} \quad \text{if } \beta \geq \gamma.
\]

This means that the bigger parameter \( \beta \) in the space \( E^2_{p,\beta,\gamma}(K) \) controls the behavior of the functions near the vertex while the smaller parameter \( \gamma \) controls the behavior at infinity. The same is true for the space \( V^1_{p,\beta,\gamma}(K) \).
1.2. Weighted Sobolev Spaces in $Q$

Let $\mathcal{X}$ be a space with the norm $\| \cdot \|_{\mathcal{X}}$, and let $q \in (1, \infty)$. Then $L_q(\mathbb{R}^+; \mathcal{X})$ is the Lebesgue space of functions (or vector functions) on the interval $(0, \infty)$ with values in $\mathcal{X}$ and the norm

$$
\| u \|_{L_q(\mathbb{R}^+; \mathcal{X})} = \left( \int_0^{\infty} \| u(t) \|_{\mathcal{X}}^q \, dt \right)^{1/q}.
$$

Analogously, $L_q(0,T; \mathcal{X})$ is the Lebesgue space of functions on the interval $(0,T)$ with values in $\mathcal{X}$.

For $p, q \in (1, \infty)$, we define $L_{q,p; \beta}(Q) = L_q(\mathbb{R}^+; L_{p,\beta}(K))$. Furthermore, $W^{2,1}_{q,p; \beta}(Q)$ denotes the space of all vector functions $u$ in $Q = K \times \mathbb{R}^+$ such that $u \in L_q(\mathbb{R}^+; V^2_{p,\beta}(K))$ and $\partial_t u \in L_q(\mathbb{R}^+; V^0_{p,\beta}(K))$. This space is equipped with the norm

$$
\| u \|_{W^{2,1}_{q,p; \beta}(Q)} = \| u \|_{L_q(\mathbb{R}^+; V^2_{p,\beta}(K))} + \| \partial_t u \|_{L_q(\mathbb{R}^+; V^0_{p,\beta}(K))}.
$$

In the case $q = p$, we will always write $L_{p; \beta}(Q)$ and $W^{2,1}_{p; \beta}(Q)$ instead of $L_{p,p; \beta}(Q)$ and $W^{2,1}_{p,p; \beta}(Q)$, respectively.

1.3. Operator Pencils, the Eigenvalues $\lambda_1$ and $\mu_2$

We introduce the following operator pencils $\mathcal{L}(\lambda)$ and $\mathcal{N}(\mu)$ generated by the Dirichlet problem for the stationary Stokes system and the Neumann problem for the Laplacian in the cone $K$, respectively. For every complex $\lambda$, we define the operator $\mathcal{L}(\lambda)$ as the mapping

$$
\dot{W}^1_2(\Omega) \times L_2(\Omega) \ni \begin{pmatrix} u \\ p \end{pmatrix} \rightarrow \begin{pmatrix} r^{2-\lambda}( - \Delta r^\lambda u(\omega) + \nabla r^{\lambda-1} p(\omega)) \\ -r^{1-\lambda} \nabla \cdot (r^\lambda u(\omega)) \end{pmatrix} \in \dot{W}^{-1}_2(\Omega) \times L_2(\Omega),
$$

where $r = |x|$ and $\omega = x/|x|$. Another description of the operator pencil $\mathcal{L}$ is given in Sect. 3.1. As is known, the strip $-1 \leq \text{Re} \lambda \leq 0$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$ (see [11] or [10, Theorem 5.5.6]) and the strip $-2 \leq \text{Re} \lambda \leq 1$ contains only real eigenvalues (see [10, Theorem 5.3.1]). We denote the smallest positive eigenvalue of this pencil by $\lambda_1$. Then $\lambda_{-1} = -1 - \lambda_1$ is the greatest negative eigenvalue. The numbers $\lambda = 1$ and $\lambda = -2$ are always eigenvalues. Thus, $0 < \lambda_1 \leq 1$.

The operator $\mathcal{N}(\mu)$ is defined for any complex $\mu$ as

$$
\mathcal{N}(\mu) p = \left( - \delta_\omega p - \mu(\mu + 1)p, \frac{\partial p}{\partial n} \bigg|_{\partial \Omega} \right) \quad \text{for} \; p \in \dot{W}^2_2(\Omega),
$$

where $\delta_\omega$ denotes the Beltrami operator on the sphere $S^2$ and $n$ is the exterior normal vector to $\partial \Omega$. The eigenvalues of the pencil $\mathcal{N}(\mu)$ are real. The spectrum contains, in particular, the simple eigenvalues $\mu_1 = 0$ and $\mu_{-1} = -1$ with the eigenfunction $\phi_1 = \text{const}$. By $\mu_2$, we denote the smallest positive eigenvalue of the pencil $\mathcal{N}(\mu)$.

2. The Parameter-Depending Problem

We consider the parameter-depending problem (3). In [16] we obtained an existence and uniqueness result for solutions in the space $E^2_{2,\beta}(K) \times V^1_{2,\beta}(K)$. One goal of this section is to extend this result to weighted $L_p$ Sobolev spaces.
2.1. Some Imbeddings and Estimates for Weighted Spaces in $K$

Let $k, l$ be integers, $0 \leq k \leq l$. Then it follows directly from the definitions of the norms in $V^l_{p,\beta}(K)$ and $E^l_{p,\beta}(K)$ that

$$V^l_{p,\beta}(K) \subset V^{k}_{p,\beta-l+k}(K) \quad \text{and} \quad E^l_{p,\beta}(K) \subset E^k_{p,\gamma}(K) \quad \text{if} \quad \beta - l + k \leq \gamma \leq \beta.$$ 

The operators of these imbeddings are continuous. Using the inequality

$$|U(x) - U(y)| = \left| \int_0^1 \frac{\partial}{\partial t} U(x + t(y-x)) \, dt \right| \leq |x-y| \left| \int_0^1 |(\nabla U)(x + t(y-x))| \, dt, \right.$$ 

one can easily show that

$$|U|_{\sigma, p, \beta-1+\sigma} \leq c \|\nabla U\|_{V^0_{p,\beta}(K)}$$

for all $U \in V^1_{p,\beta}(K)$ with a constant $c$ independent of $U$. Similarly, one can show that

$$|U|_{\sigma, p, \gamma} \leq c \|U\|_{E^1_{p,\beta}(K)} \quad \text{if} \quad \beta - 1 + \sigma \leq \gamma \leq \beta.$$ 

Hence, the imbeddings

$$V^l_{p,\beta}(K) \subset V^{l-1+\sigma}_{p,\beta-1+\sigma}(K) \subset V^{l-1}_{p,\beta-1}(K) \quad \text{and} \quad E^l_{p,\beta}(K) \subset V^{l-1+\sigma}_{p,\gamma}(K) \quad (9)$$

hold for arbitrary $\sigma, 0 \leq \sigma < 1$, and arbitrary $\beta, \gamma, \beta - 1 + \sigma \leq \gamma \leq \beta$.

Let $\zeta_\nu = \zeta_\nu(x)$ be two times continuously differentiable functions on $K$ depending only on $r = |x|$ such that

$$\text{supp} \zeta_\nu \subset \{x \in \overline{K} : 2^\nu - 1 < |x| < 2^{\nu+1}\}, \quad \sum_{\nu=-\infty}^{+\infty} \zeta_\nu = 1$$

and

$$|\partial_\nu^\alpha \zeta_\nu(x)| \leq c 2^{-\nu|\alpha|} \quad \text{for} \quad |\alpha| \leq 2.$$ 

One can easily show (see [24, Lemma 1.2.1]) that there exist positive constants $c_1, c_2$ such that

$$c_1 \|U\|_{V^l_{p,\beta}} \leq \sum_{\nu=-\infty}^{+\infty} \|\zeta_\nu U\|_{V^l_{p,\beta}} \leq c_2 \|U\|_{V^l_{p,\beta}} \quad (10)$$

for all $U \in V^l_{p,\beta}(K)$. An analogous result is true for the norms in $V^s_{p,\beta}(K), E^l_{p,\beta}(K)$, in the trace spaces $V^{l-1/p}_{p,\beta}(\partial K)$ and $E^{l-1/p}_{p,\beta}(\partial K)$ and in the dual space $(V^l_{p,\beta}(K))^*$. Using (10), we can prove the following inequality:

**Lemma 2.1.** Suppose that $0 \leq s < l$, $1 < p < \infty$ and $\beta \in \mathbb{R}$. If $\varepsilon$ is an arbitrary real number, then there exists a constant $c(\varepsilon)$ such that

$$\|U\|_{V^s_{p,\beta-l+s}(K)} \leq \varepsilon \|U\|_{V^l_{p,\beta}(K)} + C(\varepsilon) \|U\|_{V^0_{p,\beta-l}(K)} \quad (11)$$

for all $U \in V^l_{p,\beta}(K)$.

**Proof.** Let $U_\nu(x) = U(2^n x)$ and $\xi_\nu(x) = \zeta_\nu(2^n x)$. Obviously, $\xi_\nu$ is zero outside the set $1/2 < |x| < 2$. Since the weighted and nonweighted Sobolev norms are equivalent on the set of functions with support in the set $1/2 \leq |x| \leq 2$, it follows from Ehrling’s lemma (see, e.g. [33, Chapter 1, § 7]) that

$$\|\xi_\nu U_\nu\|_{V^s_{p,\beta-l+s}(K)} \leq \varepsilon \|\zeta_\nu U_\nu\|_{V^l_{p,\beta}(K)} + C(\varepsilon) \|\zeta_\nu U_\nu\|_{V^l_{p,\beta-l}(K)}$$

with a constant $C(\varepsilon)$ independent of $\nu$. Using the coordinate transformation $2^n x = y$, we get

$$\|\xi_\nu U\|_{V^s_{p,\beta-l+s}(K)} \leq \varepsilon \|\zeta_\nu U\|_{V^l_{p,\beta}(K)} + C(\varepsilon) \|\zeta_\nu U\|_{V^l_{p,\beta-l}(K)}$$

with the same constants $\varepsilon$ and $C(\varepsilon)$. Summing up over all integer $\nu$ and using (10), we obtain (11). □
The following lemma on the traces of $V_{p,\beta}^s(K)$- and $E_{p,\beta}^l(K)$-functions is essentially proved in [20].

**Lemma 2.2.** There exists a constant $c$ such that
\[
\|r^{\beta-s+1/p}U\|_{L_p(\partial K)} \leq c\|U\|_{V_{p,\beta}^s(K)} \tag{12}
\]
for all $U \in V_{p,\beta}^s(K)$, $s > \frac{1}{p}$, and
\[
\|(r^{\beta} + r^{\beta-l+1/p})U\|_{L_p(\partial K)} \leq c\|U\|_{E_{p,\beta}^l(K)} \tag{13}
\]
for all $U \in E_{p,\beta}^l(K)$, $l \geq 1$.

**Proof.** For the estimate (13), we refer to [20, Lemma 1.6]. In the case of integer $s \geq 1$, the estimate (12) is given in [20, Lemma 1.4]. If $s = l + \sigma \geq 1$, then the inequality (12) follows from the imbedding $V_{p,\beta}^{l+\sigma}(K) \subset V_{p,\beta-l-\sigma+1}(K)$ and from [20, Lemma 1.4]. It remains to prove (12) for $\frac{1}{p} < s < 1$. Let $U_\nu$ and $\zeta_\nu$ be defined as in the proof of Lemma 2.1. Since the $W_{p,\beta}^q$ and $V_{p,\beta}^s$ norms are equivalent on the subset of functions with the support in the set $1/2 \leq |x| \leq 2$, we have
\[
\|r^{\beta-s+1/p}\zeta_\nu U_\nu\|_{L_p(\partial K)}^p \leq c\|\zeta_\nu U_\nu\|_{V_{p,\beta}^s(K)}^p.
\]
Using the coordinate transformation $2^\nu x = y$, we get this inequality (with the same constant $c$) for the function $\zeta_\nu U$. Therefore, summing up this inequality over all integer $\nu$, we obtain (12). □

In the next lemma, we give some weighted $L_q$ estimates for functions $U \in E_{p,\beta}^l(K)$, $p \geq q$.

**Lemma 2.3.** (1) If $U \in E_{p,\beta}^l(K)$, $q \leq p$ and $\beta + \frac{2}{p} < \gamma + \frac{3}{q} < \beta + 1 + \frac{3}{p}$, then
\[
\|U\|_{V_{q,\gamma-1}^{\beta+1}(K)} \leq c\|U\|_{E_{p,\beta}^l(K)}.
\]

In the case $q = p$ this is even true if $\beta \leq \frac{\gamma}{q} < \beta + 1$.

(2) If $U \in E_{p,\beta}^l(K)$, $q \leq p$ and $\beta + \frac{2}{p} < \gamma + \frac{3}{q} < \beta + 1 + \frac{2}{p}$, then
\[
\|r^{\gamma-1+1/q}U\|_{L_q(\partial K)} \leq c\|U\|_{E_{p,\beta}^l(K)}.
\]

In the case $q = p$, this is even true $\beta \leq \frac{\gamma}{q} = \beta + 1 - \frac{1}{p}$.

**Proof.** For the case $q = p$, we refer to (9) and (13). Suppose that $q < p$.

(1) Let $a(r) = r$ for $r < 1$ and $a(r) = r^{-1}$ for $r \geq 1$. Then Hölder’s inequality implies
\[
\|r^{\gamma-1}U\|_{L_q(K)} \leq c\left(\int_K r^{\gamma-1+3/q} a(r)^{-\varepsilon/q} |U|^p \, dx\right)^{1/p}
\]
where $c = \left(\int_K r^{-3} a(r)^{\varepsilon/(p-q)} \, d\sigma\right)^{(p-q)/(pq)}$ is finite for positive $\varepsilon$. We can choose $\varepsilon$ such that $\beta + \frac{2}{p} \leq \gamma + \frac{3}{q} - \frac{\varepsilon}{pq}$ and $\gamma + \frac{3}{q} + \frac{\varepsilon}{pq} \leq \beta + 1 + \frac{3}{p}$. Then we get
\[
\|r^{\gamma-1}U\|_{L_q(K)} \leq c\left(\int_K r^{\beta + \frac{\beta-1}{p}} |U|^p \, d\sigma\right)^{1/p} \leq c\|U\|_{E_{p,\beta}^l(K)}.
\]

(2) By Hölder’s inequality,
\[
\|r^{\gamma-1+1/q}U\|_{L_q(\partial K)} \leq c\left(\int_{\partial K} r^{\gamma-1+3/q} a(r)^{-\varepsilon/q} |U|^p \, d\sigma\right)^{1/p},
\]
where $c = \left(\int_{\partial K} r^{-2} a(r)^{\varepsilon/(p-q)} \, d\sigma\right)^{(p-q)/(pq)}$ is finite for positive $\varepsilon$. Let $\varepsilon$ be such that $\beta + \frac{3}{p} \leq \gamma + \frac{3}{q} - \frac{\varepsilon}{pq}$ and $\gamma + \frac{3}{q} + \frac{\varepsilon}{pq} \leq \beta + 1 + \frac{2}{p}$. Then, by means of Lemma 2.2, we obtain
\[
\|r^{\gamma-1+1/q}U\|_{L_q(\partial K)} \leq c\|(r^{\beta} + r^{\beta-1+1/p}) U\|_{L_p(\partial K)} \leq c'\|U\|_{E_{p,\beta}^l(K)}.
\]

This proves the second assertion.
The case $p \leq q$ is considered in the next lemma.

**Lemma 2.4.** (1) If $1 < p \leq q < \infty$, $s - \frac{3}{p} > l - \frac{3}{q}$, $l \geq 0$ and $\beta + \frac{3}{p} = \gamma + \frac{3}{q}$, then $V_{p,\beta}^s(K) \subset V_{q,\gamma-s+l}^l(K)$.
(2) If $1 < q < \infty$, $\frac{2}{p} - \frac{2}{q} < 1$ and $\beta + \frac{2}{p} = \gamma + \frac{2}{q}$, then
\[ \|r^{\gamma-1+1/q}U\|_{L_q(\partial K)} \leq c \|U\|_{V_{p,\beta}^{1-1/r}(\partial K)}. \]

**Proof.** (1) Let $U \in V_{p,\beta}^s(K)$, and let $U_\nu$ and $\tilde{\zeta}_\nu$ be the same functions as in the proof Lemma 2.1. Then it follows from the continuity of the imbedding $W_p^s \subset W_q^l$ for bounded domains that
\[ \|\tilde{\zeta}_\nu U_\nu\|_{V_{q,\gamma-s+l}^l(K)} \leq c \|\tilde{\zeta}_\nu U_\nu\|_{V_{p,\beta}^s(K)} \]
with a constant $c$ independent of $U$ and $\nu$. Using the coordinate transformation $2^\nu x = y$, we get the same estimate (with the same constant $c$) for the function $\tilde{\zeta}_\nu U$. Since $p \leq q$ this implies
\[ \|U\|_{V_{q,\gamma-s+l}^l(K)} \leq c_1 \sum_{\nu=-\infty}^{+\infty} \|\zeta_\nu U\|_{V_{q,\gamma-s+l}^l(K)} \leq c_1 c^{q(1)} \sum_{\nu=-\infty}^{+\infty} \|\zeta_\nu U\|_{V_{p,\beta}^s(K)}. \]
\[ \leq c_1 c^{q(1)} \left( \sum_{\nu=-\infty}^{+\infty} \|\zeta_\nu U\|_{V_{p,\beta}^s(K)}^{p/q} \right)^{q/p} \leq c_2 \|U\|_{V_{p,\beta}^s(K)}. \]
(2) Obviously, the estimate is valid for all functions with support in the set $\{x \in K : \frac{1}{2} < |x| < 2\}$. In particular, the functions $\tilde{\zeta}_\nu U_\nu$ satisfy this estimate. Thus, by means of the coordinate transformation $2^\nu x = y$, we get the estimate
\[ \|r^{\gamma-1+1/q} \tilde{\zeta}_\nu U\|_{L_q(\partial K)} \leq c \|\zeta_\nu U\|_{V_{p,\beta}^{1-1/r}(\partial K)} \]
with a constant $c$ independent of $\nu$. Hence,
\[ \|r^{\gamma-1+1/q}U\|_{L_q(\partial K)} = c_1 \sum_{\nu=-\infty}^{+\infty} \|r^{\gamma-1+1/q} \tilde{\zeta}_\nu U\|_{L_q(\partial K)} \leq c_1 c^{q(1)} \sum_{\nu=-\infty}^{+\infty} \|\zeta_\nu U\|_{V_{p,\beta}^s(K)}. \]
\[ \leq c_1 c^{q(1)} \left( \sum_{\nu=-\infty}^{+\infty} \|\zeta_\nu U\|_{V_{p,\beta}^s(K)}^{p/q} \right)^{q/p} \leq c_2 \|U\|_{V_{p,\beta}^s(K)}. \]
This proves the lemma.

Finally, the following lemma can be easily proved by means of Hölder’s inequality.

**Lemma 2.5.** Suppose that $p \leq q$, $p \neq q$ and $s(q-p) \delta = p(q-s) \beta + q(s-p) \gamma$. If $F \in L_{p,\beta}(K) \cap L_{q,\gamma}(K)$, then $F \in L_{s,\delta}(K)$ and
\[ \|r^\delta F\|_{L_{s}(K)} \leq \|r^\beta F\|_{L_{p}(K)} \|r^\gamma F\|_{L_{q}(K)}. \]

**Proof.** We set $\tau_1 = p(q-s)(q-p)^{-1}$ and $\tau_2 = q(s-p)(q-p)^{-1}$. Since $\tau_1 + \tau_2 = s$, $p^{-1} \tau_1 + q^{-1} \tau_2 = 1$ and $s \delta = \tau_1 \beta + \tau_2 \gamma$, we obtain
\[ \int_K |r^\delta F|^s \, dx = \int_K |r^\beta F|^\tau_1 \ |r^\gamma F|^\tau_2 \, dx \leq \left( \int_K |r^\beta F|^p \, dx \right)^{\tau_1/p} \left( \int_K |r^\gamma F|^q \, dx \right)^{\tau_2/q} \]
bys means of Hölder’s inequality. This proves the lemma.

It follows from the last lemma that $V_{p,\beta}^l(K) \cap V_{q,\gamma,s+l}^l(K) \subset V_{s,\delta}^l(K)$ for arbitrary $l \geq 0$ under the conditions of Lemma 2.5 on $p, q, s$ and $\beta, \gamma, \delta$. 
2.2. Estimates for Solutions of the Parameter-Depending Problem

In the sequel, \( p \) is always a real number in the interval \( 1 < p < \infty \) and \( p' = p/(p-1) \). The following lemma can be easily deduced from [32, Theorem 3.1] (for the case \( p = 2 \) see [16, Lemma 2.2]).

Lemma 2.6. Let \( \mathcal{D} \) be a bounded domain with smooth (of class \( C^2 \)) boundary \( \partial \mathcal{D} \). Suppose that \( U \in W^{2}_p(\mathcal{D}), \nabla P \in L_p(\mathcal{D}), \)

\[
sU - \Delta U + \nabla P = F, \quad -\nabla \cdot U = G \quad \text{in} \quad \mathcal{D}, \quad U = 0 \quad \text{on} \quad \partial \mathcal{D}.
\]

Then

\[
\sum_{|\alpha| = 2} \| \partial_\alpha^p U \|_{L_p(\mathcal{D})} + |s| \| U \|_{L_p(\mathcal{D})} + \| \nabla P \|_{L_p(\mathcal{D})} \leq c \left( \| F \|_{L_p(\mathcal{D})} + \| G \|_{W^{1,2}_p(\mathcal{D})} + |s| \| G \|_{(W^{1,2}_p(\mathcal{D}))^*} + \| U \|_{L_p(\mathcal{D})} \right),
\]

where \( c \) is a constant independent of \( U, P \) and \( s \).

Let \( \zeta_\nu \) be the same smooth functions on \( K \) as in Sect. 2.1, and let \( \eta_\nu = \zeta_{\nu - 1} + \zeta_\nu + \zeta_{\nu + 1} \). Obviously, \( \zeta_\nu \eta_\nu = \zeta_\nu \) for every integer \( \nu \). As in [16], we can prove the following assertion by means of Lemma 2.6.

Lemma 2.7. Suppose that \( U \in W^{2}_p(\mathcal{D}), P \in W^{1}_p(\mathcal{D}) \) for every bounded subdomain \( \mathcal{D} \subset K, \overline{\mathcal{D}} \subset \overline{K} \{0\} \), and that \( (U, P) \) is a solution of the problem (3).

Then

\[
\| \zeta_\nu U \|_{V^{2,p}_p(K)} + |s| \| \zeta_\nu U \|_{V^{1,p}_p(K)} + \| \zeta_\nu P \|_{V^{1,p}_p(K)} \leq c \left( \| \zeta_\nu F \|_{V^{1,p}_p(K)} + \| \zeta_\nu G \|_{V^{1,p}_p(K)} \right),
\]

where \( c \) is independent of \( \nu \) and \( s \).

Proof. Obviously,

\[
s \zeta_\nu U - \Delta (\zeta_\nu U) + \nabla (\zeta_\nu P) = F', \quad -\nabla \cdot (\zeta_\nu u) = G' \quad \text{in} \quad K,
\]

where \( F'_i = \zeta_\nu F_i - 2\nabla \zeta_\nu \cdot \nabla U_i + U_i \Delta \zeta_\nu + P \partial_{z_i} \zeta_\nu \) for \( i = 1, 2, 3 \) and \( G' = \zeta_\nu G - U \cdot \nabla \zeta_\nu \). We define

\[ V(x) = \zeta_\nu (2^\nu r) U(2^\nu x), \quad Q(x) = 2^\nu \zeta_\nu (2^\nu r) P(2^\nu x). \]

By (15), we have

\[ 2^{\nu} s V - \Delta V + \nabla Q = \Phi, \quad -\nabla \cdot V = \Psi \quad \text{in} \quad K, \]

where \( \Phi(x) = 2^{2\nu} F'(2^\nu x) \) and \( \Psi(x) = 2^{2\nu} G'(2^\nu x) \). Since \( V \) and \( Q \) vanish outside the set \( K_0 = \{ x \in K : 1/2 < |x| < 2 \} \), it follows from Lemma 2.6 that

\[
\| D^2 V \|_{L_p(K)} + 2^{\nu} |s| \| V \|_{L_p(K)} + \| \nabla Q \|_{L_p(K)} \leq c \left( \| \Phi \|_{L_p(K_0)} + \| \Psi \|_{W^{1,p}_p(K_0)} + 2^{2\nu} |s| \| \Psi \|_{(W^{1,p}_p(K_0))^*} + \| V \|_{L_p(K_0)} \right),
\]

where the constant \( c \) is independent of \( \nu \) and \( s \). Here, \( D^2 V \) denotes the vector of all second order derivatives of \( V \). One can easily check that

\[
\| \zeta_\nu U \|_{V^{0,\beta}_p(K)} \leq c 2^{\nu(\beta + 3\nu/p)} \| V \|_{L_p(K)}, \quad \| D^2 (\zeta_\nu U) \|_{V^{0,\beta}_p(K)} \leq c 2^{\nu(\beta - 2) + 3\nu/p} \| D^2 V \|_{L_p(K)}
\]

and

\[
\| \nabla (\zeta_\nu P) \|_{V^{0,\beta}_p(K)} \leq c 2^{\nu(\beta - 2) + 3\nu/p} \| \nabla Q \|_{L_p(K)}.
\]

Furthermore, we obtain the estimates

\[
\| \Phi \|_{L_p(K)} + \| \Psi \|_{W^{1,p}_p(K)} \leq c 2^{\nu(2\nu - 3\nu/p)} \left( \| \zeta_\nu F \|_{V^{0,\beta}_p(K)} + \| \zeta_\nu G \|_{V^{1,\beta}_p(K)} \right) + |s| \| \zeta_\nu P \|_{V^{1,\beta}_p(K)}.
\]
Moreover,
\[ 2^{\nu + 3\nu/p} \| \Psi \|_{(W^1_p(K_0))^*} \leq c \| G' \|_{(V^1_{p',1-\beta}(K))^*} \leq c \left( \| \zeta_r G \|_{V^1_{p',1-\beta}(K)} + \| U \cdot \nabla \zeta_r \|_{V^1_{p',1-\beta}(K)} \right). \]

Thus, we get (14).

Since \( \zeta_r(x) \) depends only on \( r = |x| \), we have \( \nabla \zeta_r = \zeta'_r(r) r^{-1} x \), where \( \zeta'_r(r) \leq c r^{-1} \). Consequently,
\[ \| U \cdot \nabla \zeta_r \|_{(V^1_{p',1-\beta}(K))^*} \leq c \| \eta_r x \cdot U \|_{(V^1_{p',1-\beta}(K))^*}. \]

The last norm can be estimated by means of the following lemma.

**Lemma 2.8.** Suppose that \( (U, P) \in E^2_{p,\beta}(K) \times V^1_{p,\beta}(K) \) is a solution of (3). Then
\[ \| s \|_{L_p(\Omega)} \leq \| F \|_{(V^1_{p',1-\beta}(K))^*}, \]
where \( \sigma \) is an arbitrary real number greater than \( p^{-1} \) and \( c \) is independent of \( U, P \) and \( s \).

**Proof.** By (3), \( s x \cdot U = x \cdot (F + \Delta U - \nabla P) \). Here
\[ \| x \cdot F \|_{(V^1_{p',1-\beta}(K))^*} \leq \| F \|_{(V^1_{p',1-\beta}(K))^*} \leq \| F \|_{(V^1_{p,\beta}(K))}. \]

Since \( x \cdot n = 0 \) on \( \partial K \) for the normal vector \( n \), we get
\[ \left| \int_K (x \cdot \nabla P) Q \, dx \right| = \left| \int_K P \nabla (xQ) \, dx \right| \leq \| P \|_{V^1_{p,\beta-1}(K)} \| Q \|_{V^1_{p',1-\beta}(K)} \]
for \( Q \in V^1_{p',1-\beta}(K) \). Furthermore,
\[ \int_K (x \cdot \Delta U) Q \, dx = \sum_{j=1}^3 \left( -\int_K \nabla U_j \cdot \nabla (x_j Q) \, dx + \int_{\partial K} x_j \frac{\partial U_j}{\partial n} Q \, d\sigma \right) \]
for \( Q \in V^1_{p',1-\beta}(K) \), where
\[ \left| \int_K \nabla U_j \cdot \nabla (x_j Q) \, dx \right| \leq \| U \|_{V^1_{p,\beta-1}(K)} \| Q \|_{V^1_{p',1-\beta}(K)} \]
and
\[ \left| \int_{\partial K} x \frac{\partial U}{\partial n} Q \, d\sigma \right| \leq \| r^{\beta - 1 + \beta p} \|_{L_p(\partial K)} \| Q \|_{V^1_{p',1-\beta}(K)}. \]

for \( \sigma > p^{-1} \) (see Lemma 2.2). This proves the lemma.

Now it is easy to deduce the following estimate from Lemma 2.7.

**Lemma 2.9.** Suppose that \( U \in W^2_p(\Omega) \), \( P \in V^1_{p',\beta}(\Omega) \) for every bounded subdomain \( \Omega \subset K \), \( \overline{\Omega} \subset \overline{K} \{0\} \), and that \( U \in V^1_{p,\beta-1}(K) \), \( s x \cdot U \in (V^1_{p',2-\beta}(K))^* \), \( P \in V^0_{p,\beta-1}(K) \). If \( (U, P) \) satisfies (3) with data \( F \in V^0_{p,\beta}(K), G \in (V^1_{p',1-\beta}(K))^* \), then \( U \in V^2_p(K), sU \in V^0_{p,\beta}(K), P \in V^1_{p,\beta}(K) \) and
\[
\begin{align*}
&\| U \|_{V^2_p(K)} + \| s \|_{L_p(\Omega)} \| U \|_{V^1_{p,\beta}(K)} + \| P \|_{V^1_{p,\beta}(K)} \leq c \left( \| F \|_{V^0_{p,\beta}(K)} + \| G \|_{(V^1_{p',1-\beta}(K))^*} + \| U \|_{V^1_{p,\beta-2}(K)} + \| P \|_{V^0_{p,\beta-1}(K)} \right),
\end{align*}
\]
where \( c \) is independent of \( U, p \) and \( s \).
Proof. Summing up the inequality (14) over all integer \( \nu \) and using Lemma 2.8, we obtain (16) with the additional term

\[ c \|U\|_{V_{p,\beta-1+\sigma}^0(K)}^p \]
on the right-hand side, where \( \sigma \) is an arbitrary real number, \( p^{-1} < \sigma < 1 \). By (11), we have

\[ \|U\|_{V_{p,\beta-1+\sigma}^0(K)} \leq \varepsilon \|U\|_{V_{p,\beta}^2(K)} + C(\varepsilon) \|U\|_{V_{p,\beta}^0(K)} \].

Here, \( \varepsilon \) can be chosen arbitrarily small. This proves (16). \( \square \)

2.3. Normal Solvability of the Parameter-Depending Problem

Our goal is to show that (under certain conditions on \( p \) and \( \beta \)) the operator

\[ (E_{p,\beta}^2(K) \cap E_{p,\beta}^0(K)) \times V_{p,\beta}^1(K) \ni (U, P) \rightarrow (F, G) = (sU - \Delta U + \nabla P, -\nabla \cdot U) \in E_{p,\beta}^0(K) \times X_{p,\beta}^1(K) \] (17)
of the problem (3) has closed range if \( s \neq 0 \). For this, it is sufficient to show that

\[ \|U\|_{E_{p,\beta}^1(K)} + \|P\|_{V_{p,\beta}^1(K)} \leq c \left( \|F\|_{V_{p,\beta}^0(K)} + \|G\|_{X_{p,\beta}^1(K)} + \|U\|_{W_{p,\beta}^1(S)} + \|P\|_{L_p(S)} \right), \] (18)

where \( S \) is a bounded subdomain of \( K \) with positive distance from the vertex of \( K \).

Suppose that \((U, P)\) is a solution of the problem (3), \( U \in V_{p,\beta}^2(K), P \in V_{p,\beta}^1(K), sU \in V_{p,\beta}^0(K) \). Then

\[ \int_K \nabla P \cdot \nabla Q \, dx = \langle \Phi, Q \rangle \quad \text{for all} \quad Q \in V_{p,\beta}^1(K), \] (19)

where

\[ \langle \Phi, Q \rangle = \int_K \left( (F + \Delta U) \cdot \nabla Q - sGQ \right) \, dx. \]

We define

\[ \langle \Phi_1, Q \rangle = \int_K \left( (F - \nabla G) \cdot \nabla Q - sGQ \right) \, dx, \quad \langle \Phi_2, Q \rangle = \int_K \left( \Delta U + \nabla G \right) \cdot \nabla Q \, dx. \] (20)

Then

\[ \langle \Phi, Q \rangle = \langle \Phi_1, Q \rangle + \langle \Phi_2, Q \rangle. \]

Obviously, \( \Phi_1 \) and \( \Phi_2 \) are functionals on \( V_{p,\beta}^1(K) \) if \( U \in V_{p,\beta}^2(K), sU \in V_{p,\beta}^0(K) \) and \( P \in V_{p,\beta}^1(K) \),

\[ \|\Phi_1\|_{(V_{p,\beta}^1(K))^*} \leq \|F\|_{V_{p,\beta}^0(K)} + \|G\|_{X_{p,\beta}^1(K)} + |s| \|G\|_{(V_{p,\beta}^1(K))^*}, \]

\[ \|\Phi_2\|_{(V_{p,\beta}^1(K))^*} \leq c \|U\|_{V_{p,\beta}^2(K)} \cdot \] (21)

Furthermore, the following assertion is true for the functional \( \Phi_2 \).

Lemma 2.10. Suppose that \( U \in E_{p,\beta}^2(K), s \neq 0 \) and \( \beta \leq \gamma \leq \beta + \frac{1}{p} \). Then the mapping

\[ Q \rightarrow \langle \Phi_2, Q \rangle = \int_K \left( \Delta U - \nabla \nabla \cdot U \right) \cdot \nabla Q \, dx \]

defines a linear and continuous functional on \( V_{p,\beta-1-\gamma}^2(K) \), and the estimate

\[ \|\Phi_2\|_{(V_{p,\beta-1-\gamma}^2(K))^*} \leq c |s|^{(\beta-\gamma)/2} \left( \|U\|_{V_{p,\beta}^2(K)} + |s| \|U\|_{V_{p,\beta}^0(K)} \right) \]
is valid with a constant \( c \) independent of \( U \) and \( s \).
Proof. Suppose that $|s| = 1$. One can easily show (cf. [16, Lemma 2.5]) that

$$\langle \Phi_2, Q \rangle = \int_{\partial K} \sum_{i,j=1}^3 \partial U_i \frac{\partial Q}{\partial x_j} \left( n_j \frac{\partial Q}{\partial x_i} - n_i \frac{\partial Q}{\partial x_j} \right) d\sigma$$

and, consequently,

$$|\langle \Phi_2, Q \rangle| \leq \|r^{-1+1/p}\nabla U\|_{L_p(\partial K)} \|r^{-1+1/p}\nabla Q\|_{L_p(\partial K)}.$$ 

By Lemma 2.2,

$$\|r^{-1+1/p}\nabla U\|_{L_p(\partial K)} \leq \left( \int_{\partial K} \left( r^{p(\beta-1)+1} + r^{p\beta} \right) \left| \nabla U \right|^p dx \right)^{1/p} \leq c \|U\|_{E_{p,\beta}(K)}^2$$

and

$$\|r^{-1+1/p}\nabla Q\|_{L_p(\partial K)} \leq c \|\nabla Q\|_{V_{p,1-\gamma}(K)}.$$ 

This proves the estimate for the $(V_{p,1-\gamma}(K))^*$-norm of $\Phi_2$ in the case $|s| = 1$. In the case $|s| \neq 1$, we consider the functions $V(x) = U(|s|^{-1/2}x)$ and $R(x) = Q(|s|^{-1/2}x)$. With the substitution $|s|^{-1/2}x = y$, we obtain

$$\langle \Phi_2, Q \rangle = \int_K (\Delta V - \nabla \nabla \cdot V) \cdot \nabla R \, dx =: \langle \Psi_2, R \rangle$$

As is shown above, there is the estimate

$$|\langle \Phi_2, Q \rangle| = |\langle \Psi_2, R \rangle| \leq c \left( \|V\|_{V_{p,\beta}^2(K)} + \|V\|_{V_{p,\beta}^0(K)} \right) \|R\|_{V_{p,1-\gamma}^2(K)}.$$

Since

$$\|V\|_{V_{p,\beta}^2(K)} + \|V\|_{V_{p,\beta}^0(K)} = |s|^{(p\beta-2p+3)/2} \left( \|U\|_{V_{p,\beta}^2(K)} + |s|^p \|U\|_{V_{p,\beta}^0(K)} \right)$$

and

$$\|R\|_{V_{p,1-\gamma}^2(K)} = |s|^{(-p'\gamma-p'+3)/2} \|Q\|_{V_{p,1-\gamma}^2(K)}$$

we obtain the desired estimate for the $(V_{p,1-\gamma}(K))^*$-norm of $\Phi_2$ in the case $|s| \neq 1$. 

The following lemma was proved in [16] for the case $p = 2$. Similarly, we can prove this lemma for arbitrary $p$.

**Lemma 2.11.** (1) Suppose that $\Phi \in (V_{p,1-\gamma}(K))^*$ and that $1 - \beta - \frac{3}{p}$ is not an eigenvalue of the pencil $\mathcal{N}(\mu)$. Then there exists a unique solution $P \in V_{p,1-\gamma}^1(K)$ of the problem (19).

(2) Suppose that $\Phi \in (V_{q,1-\gamma}(K))^*$ and that $1 - \gamma - \frac{3}{q}$ is not an eigenvalue of the pencil $\mathcal{N}(\mu)$. Then there exists a unique solution $(P, \Psi) \in V_{q,1-\gamma}^0(K) \times (V_{q,1-\gamma}(K))^*$ of the problem

$$- \int_K P \Delta Q \, dx + \int_{\partial K} \Psi \frac{\partial Q}{\partial n} \, d\sigma = \langle \Phi, Q \rangle \text{ for all } Q \in V_{q,1-\gamma}^1(K)$$

(3) Suppose that $P \in V_{p,1-\gamma}^1(K)$ is a solution of the problem (19), where $\Phi \in (V_{p,1-\gamma}(K))^* \cap (V_{q,1-\gamma}(K))^*$, $\beta + \frac{3}{p} \leq \gamma + \frac{3}{q}$. We assume that the numbers $1 - \beta - \frac{3}{p}$ and $1 - \gamma - \frac{3}{q}$ are not eigenvalues of the pencil $\mathcal{N}(\mu)$. Then

$$P = \sum c_j r^{\mu_j} \phi_j(\omega) + P',$$

where $\mu_j$ are the eigenvalues of the pencil $\mathcal{N}(\mu)$ in the interval $1 - \gamma - \frac{3}{q} < \lambda < 1 - \beta - \frac{3}{p}$, $\phi_j$ are corresponding eigenfunctions, $P' \in V_{q,1-\gamma}^0(K)$ and

$$\|P'\|_{V_{q,1-\gamma}^0(K)} \leq c \|\Phi\|_{(V_{q,1-\gamma}(K))^*}$$

with a constant $c$ independent of $F$. 

Proof. (1) The first assertion can be found e.g. in [24, Theorem 7.7.3].
(2) If \(1 - \gamma - \frac{2}{q}\) is not an eigenvalue of the pencil \(N(\mu)\), then the operator

\[
V_{q',1-\gamma}^2 \ni U \to \left( -\Delta U, \frac{\partial U}{\partial n} \right) \in V_{q',1-\gamma}(K) \times V_{q',1-\gamma}^{1-1/q}\prime (\partial K)
\]

is an isomorphism. Consequently, the adjoint operator is also an isomorphism.
(3) The proof of the third assertion proceeds analogously to [16, Lemma 2.6].

As a simple consequence of the last lemma, we obtain the following regularity assertion for solutions of the problem (19).

**Corollary 2.1.** Let \(P \in V_{p,\beta}^1(K)\) be a solution of the problem (19), where \(\Phi \in (V_{p',-\beta}(K))^* \cap (V_{q',1-\gamma}(K))^*\), \(\beta + \frac{2}{p} < q + \frac{2}{q}\). If the interval \([1 - \gamma - \frac{3}{q}, 1 - \beta - \frac{3}{p}]\) is free of eigenvalues of the pencil \(N(\mu)\), then \(P \in V_{q,\gamma-1}^0(K)\). The same is true if \(1 - \gamma - \frac{3}{q} < 1 - \beta - \frac{3}{p}\), \(\mu = 1\) is the only eigenvalue of the pencil \(N(\mu)\) in the interval \([1 - \gamma - \frac{3}{q}, 1 - \beta - \frac{3}{p}]\), and \(\langle \Phi, 1 \rangle = 0\).

Proof. The first assertion is obvious. Suppose that \(1 - \gamma - \frac{3}{q} < 1 - \beta - \frac{3}{p}\) and \(\mu = 1\) is the only eigenvalue of the pencil \(N(\mu)\) in the interval \([1 - \gamma - \frac{3}{q}, 1 - \beta - \frac{3}{p}]\). Then it follows from Lemma 2.11 that \(P = c_{-1} r^{-1} + P'\), where \(P' \in V_{q,\gamma-1}^0(K)\). By the well-known formula for the coefficients \(c_j\) in (22) which can be found, e.g., in [26, Chapter 3, Theorem 5.8], the coefficient \(c_{-1}\) is a multiple of \(\langle \Phi, 1 \rangle\). Thus, \(P = P'\) if \(\langle \Phi, 1 \rangle = 0\).

Note that the constant function \(Q = 1\) belongs to the space \(V_{p',-\beta}^1(K) + V_{q',1-\gamma}(K)\) if \(1 - \gamma - \frac{3}{q} < 1 - \beta - \frac{3}{p}\). For the above introduced functional \(\Phi = \Phi_1 + \Phi_2\) (see (20)), the condition \(\langle \Phi, 1 \rangle = 0\) in Corollary 2.1 means that

\[
s \int_K G(x) \, dx = 0.
\]

**Theorem 2.1.** Suppose that \(s \neq 0\), that the line \(\text{Re} \lambda = 2 - \beta - \frac{3}{p}\) does not contain eigenvalues of the pencil \(L(\lambda)\), and that \(1 - \beta - \frac{2}{p}\) is not an eigenvalue of the pencil \(N(\mu)\). Then the range of the operator (17) is closed and the kernel has finite dimension.

Proof. We prove the estimate (18) for solutions of the problem (3) in the space \(E_{p,\beta}^2(K) \times V_{p,\beta}^1(K)\). If the support of \((U, P)\) is contained in the ball \(|x| < \varepsilon\), then

\[
\|U\|_{V_{p,\beta}^0(K)} \leq \varepsilon^2 \|U\|_{V_{p,\beta}^2(K)}
\]

and it follows from well-known results for the stationary Stokes system that

\[
\|U\|_{V_{p,\beta}^0(K)} + \|P\|_{V_{p,\beta}^1(K)} \leq c \left( \|F - sU\|_{V_{p,\beta}^0(K)} + \|G\|_{V_{p,\beta}^1(K)} \right).
\]

If \(\varepsilon\) is sufficiently small, we get

\[
\|U\|_{E_{p,\beta}^2(K)} + \|P\|_{V_{p,\beta}^1(K)} \leq c \left( \|F\|_{V_{p,\beta}^0(K)} + \|G\|_{V_{p,\beta}^1(K)} \right).
\]

Assume now that \((U(x), P(x))\) are zero for \(|x| < N\) and \(N\) is sufficiently large. Then

\[
\|U\|_{V_{p,\beta-1}^1(K)} \leq c N^{-1} \|U\|_{E_{p,\beta}^2(K)}.
\]

We estimate the norm of \(P\) in \(V_{p,\beta-1}^0(K)\). If \(1 - \beta - \frac{3}{p}\) is not an eigenvalue of the pencil \(N(\mu)\), then the problem (19) has a uniquely determined solution \(P = P_1 + P_2 \in V_{p,\beta}^1(K)\), where \(P_j, j = 1, 2\), are the (uniquely determined) solutions of the problems

\[
\int_K \nabla P_j \cdot \nabla Q \, dx = \langle \Phi_j, Q \rangle \quad \text{for all } Q \in V_{p',-\beta}^1(K)
\]
and $\Phi_j$ are the functionals (20). Here,
\[
\|P_1\|_{V_{p,\beta}^0(K)} \leq \|\Phi_1\|_{(V_{p,\gamma}^1(K))^*} \leq c \left( \|F\|_{V_{p,\beta}^0(K)} + \|G\|_{X_{p,\beta}^1(K)} \right)
\]
(see (21)). Suppose that $\beta < \gamma \leq \beta + \frac{1}{p}$ and that there are no eigenvalues of the pencil $\mathcal{N}(\mu)$ in the interval $1 - \gamma - \frac{2}{p} \leq \lambda \leq 1 - \beta - \frac{3}{p}$. Then it follows from Lemmas 2.10 and 2.11 that
\[
\|P_2\|_{V_{p,\gamma-1}(K)} \leq c \|U\|_{E_{p,\beta}^2(K)},
\]
where $c$ is independent of $N$. Since $P_2(x) = -P_1(x)$ for $|x| < N$, we obtain
\[
\|P_2\|_{V_{p,\beta}^0(K)}^p \leq \|P_1\|_{V_{p,\beta}^1(K)}^p + N^p(\beta - \gamma) \int_{|x| > N} r^{p(\gamma-1)} |P_2|^p \, dx \leq c \left( \|F\|_{V_{p,\beta}^0(K)}^p + \|G\|_{X_{p,\beta}^1(K)}^p + N^p(\beta - \gamma) \|U\|_{E_{p,\beta}^2(K)}^p \right).
\]
Thus, Lemma 2.9 yields
\[
\|U\|_{E_{p,\beta}^2(K)} + \|P\|_{V_{p,\beta}^1(K)} \leq c \left( \|F\|_{V_{p,\beta}^0(K)} + \|G\|_{X_{p,\beta}^1(K)} \right)
\]
if $N$ is sufficiently large.

Let $(U, P) \in E_{p,\beta}^2(K) \times V_{p,\beta}^1(K)$ be a solution of (3) with arbitrary support in $\overline{K}$, and let $S$ denote the set $\{x \in K : \frac{1}{2} < |x| < 2N\}$. Multiplying $(U, P)$ by suitable cut-off functions, we get the decomposition $(U, P) = (U_1, P_1) + (U_2, P_2) + (U_3, P_3)$, where $U_k \in E_{p,\beta}^2(K)$, $P_k \in V_{p,\beta}^1(K)$, $U_k = 0$ on $\partial K \setminus \{0\}$ for $k = 1, 2, 3$, supp $(U_1, P_1) \subset S$, supp $(U_2, P_2)$ is contained in the ball $|x| \leq \varepsilon$ and supp $(U_3, P_3)$ is contained in the set $|x| \geq N$. Furthermore,
\[
\|U_1\|_{W_{p,\beta}^1(S)} + \|P_1\|_{L_p(S)} \leq c \left( \|U\|_{W_{p,\beta}^1(S)} + \|P\|_{L_p(S)} \right),
\]
and the functions $F_k = (s - \Delta)U_k + \nabla P_k$ and $G_k = -\nabla \cdot U_k$ satisfy the estimate
\[
\|F_k\|_{V_{p,\beta}^0(K)} + \|G_k\|_{X_{p,\beta}^1(K)} \leq c \left( \|F\|_{V_{p,\beta}^0(K)} + \|G\|_{X_{p,\beta}^1(K)} + \|U\|_{W_{p,\beta}^1(S)} + \|P\|_{L_p(S)} \right)
\]
for $k = 1, 2, 3$. From Lemma 2.9 it follows that
\[
\|U_1\|_{E_{p,\beta}^2(K)} + \|P_1\|_{V_{p,\beta}^1(K)} \leq c \left( \|F_1\|_{V_{p,\beta}^0(K)} + \|G_1\|_{X_{p,\beta}^1(K)} + \|U\|_{W_{p,\beta}^1(S)} + \|P\|_{L_p(S)} \right).
\]
As was shown above, the vector functions $(U_2, P_2)$ and $(U_3, P_3)$ satisfy the estimate
\[
\|U_k\|_{E_{p,\beta}^2(K)} + \|P_k\|_{V_{p,\beta}^1(K)} \leq c \left( \|F_k\|_{V_{p,\beta}^0(K)} + \|G_k\|_{X_{p,\beta}^1(K)} \right)
\]
for $k = 2$ and $k = 3$ if $\varepsilon$ is sufficiently small and $N$ is sufficiently large. Combining the last inequalities, we obtain (18). This proves the theorem.

2.4. A Regularity Assertion for the Solution of (3)

The following lemma was proved for $p = q = 2$ in [16, Lemma 2.9]. We prove this for arbitrary $p, q \in (1, \infty)$.

**Lemma 2.12.** Suppose that $s \neq 0$ and that $(U, P) \in E_{p,\beta}^2(K) \times V_{p,\beta}^1(K)$ is a solution of the problem (3), where
\[
F \in V_{p,\beta}^0(K) \cap V_{q,\gamma}^0(K), \quad G \in X_{p,\beta}^1(K) \cap X_{q,\gamma}^1(K).
\]
We assume that one of the following three conditions is satisfied:
(i) $\beta + \frac{3}{p} < \gamma + \frac{3}{q}$ and the interval $1 - \gamma - \frac{3}{q} \leq \lambda \leq 1 - \beta - \frac{2}{p}$ does not contain eigenvalues of the pencil $\mathcal{N}(\mu)$,
(ii) $\beta + \frac{3}{p} > \gamma + \frac{3}{q}$ and the strip $2 - \beta - \frac{3}{p} \leq \text{Re}\lambda \leq 2 - \gamma - \frac{3}{q}$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$. 

\[\]
(iii) $\beta + \frac{3}{p} = \gamma + \frac{3}{q}$, the number $1 - \beta - \frac{3}{p}$ is not an eigenvalue of the pencil $\mathcal{N}(\mu)$, and the line $\text{Re}\lambda = 2 - \beta - \frac{3}{p}$ is free of eigenvalues of the pencil $\mathcal{L}(\lambda)$.

Then $U \in E^2_{q,\gamma}(K)$, $P \in V^1_{p,\gamma}(K)$.

**Proof.** (1) Suppose that $p = q$ and the condition (i) is satisfied. First, let $\gamma < \beta + 1 - \frac{1}{p}$. Then $P$ is a solution of the problem (19) with a functional $\Phi \in (V^1_{p,\beta-1}(K))^* \cap (V^2_{p,\gamma-1}(K))^*$ (see Lemma 2.10), and Lemma 2.11 implies $P \in V^0_{p,\gamma-1}(K)$. Furthermore, $U \in E^2_{p,\beta}(K) \subset V^1_{p,\beta-1+\sigma}(K)$ for $\beta \leq \gamma \leq \beta + 1 - \sigma$ (cf. (9)). From Lemma 2.8 it follows that $x \cdot U \in (V^1_{p,\gamma-1}(K))^*$. Using Lemma 2.9, we conclude that $U \in E^2_{p,\beta}(K)$ and $P \in V^1_{p,\gamma}(K)$.

Now, let $\beta + 1 - \frac{1}{p} \leq \gamma < \beta + 2 - \frac{2}{p}$. Since $F \in V^0_{p,\gamma}(K)$ and $G \in X^1_{p,\gamma}(K)$ for arbitrary $\gamma' \in [\beta, \gamma]$ it follows that $U \in V^2_{p,\gamma}(K)$ and $P \in V^1_{p,\gamma}(K)$ for arbitrary $\gamma'$, $\beta \leq \gamma' < \beta + 1 - \frac{1}{p}$. Consequently, $U \in E^2_{p,\gamma}(K)$ and $P \in V^1_{p,\gamma}(K)$ if $\gamma < \gamma' < \gamma + 1 - \frac{1}{p}$, i.e., $\gamma < \beta + 2 - \frac{2}{p}$. Analogously, we conclude by induction that $U \in E^2_{p,\gamma}(K)$ and $P \in V^1_{p,\gamma}(K)$ if $\gamma < \beta + k - \frac{k}{p}$ and $k \geq 3$.

(2) Suppose that $p = q$ and the condition (ii) is satisfied. Then we consider first the case that $\gamma \geq \beta - 2$. Let $\zeta$ be a smooth (two times continuously differentiable) function on $\mathbb{R}_+$ with support in $[0, 1]$ which is equal to one on the interval $(0, \frac{1}{2})$, and let $\eta = 1 - \zeta$. The functions $\zeta, \eta$ can be considered as smooth functions on $K$ if one defines $\zeta(x) = \zeta(|x|)$ and $\eta(x) = \eta(|x|)$. Obviously $\eta(U, P) \in E^2_{p,\gamma}(K) \times V^1_{p,\gamma}(K)$ (since $\beta > \gamma$), and $\zeta(U, P)$ is a solution of the problem

$$-\Delta(\zeta U) + \nabla(\zeta P) = F' + s\zeta U, \quad -\nabla \cdot (\zeta U) = G' \quad \text{in} \quad K,$$

where $F' = \zeta F - [\Delta, \zeta] U + P \nabla \zeta$ and $\zeta$ and $U$ belong to $V^0_{p,\gamma}(K)$ and $G' = \zeta G - U \cdot \nabla \zeta \in V^1_{p,\gamma}(K)$. Since the strip $2 - \beta - \frac{3}{p} \leq \text{Re}\lambda \leq 2 - \gamma - \frac{3}{p}$ does not contain eigenvalues of the pencil $\mathcal{L}(\lambda)$, it follows that $\zeta(U, P) \in V^2_{p,\gamma}(K) \times V^1_{p,\gamma}(K)$ and, consequently, $(U, P) \in E^2_{p,\gamma}(K) \times V^1_{p,\gamma}(K)$.

If $\beta - 4 \leq \gamma < \beta - 2$, then we obtain $(U, P) \in E^2_{p,\gamma}(K) \times V^1_{p,\gamma}(K)$ for each $\gamma' \in [\beta, 2, \beta)$ since $F \in V^0_{p,\gamma}(K)$ and $G \in V^1_{p,\gamma'}(K)$. We can choose $\gamma'$ such that $\gamma \geq \gamma' - 2$. Then we can conclude that $(U, P) \in E^2_{p,\gamma}(K) \times V^1_{p,\gamma'}(K)$. Analogously, we can show by induction that $U \in E^2_{p,\gamma}(K)$ and $P \in V^1_{p,\gamma}(K)$ if $\gamma \geq 2 - 2k$ for $k \geq 3$. Thus, the lemma is proved for the case $p = q$.

(3) Suppose that $p > q$. Then it can be easily shown by means of Hölder’s inequality that $\zeta(U, P) \in E^2_{2,\gamma'+\epsilon}(K) \times V^1_{2,\gamma'-\epsilon}(K)$ and $\eta(U, P) \in E^2_{q,\gamma'-\epsilon}(K) \times V^1_{q,\gamma'-\epsilon}(K)$, where $\gamma' = \beta + \frac{3}{p} - \frac{3}{q}$ and $\epsilon$ is an arbitrary positive number. Since

$$(s - \Delta)(\zeta U) + \nabla(\zeta P) = F' + s\zeta U \in V^0_{p,\gamma}(K), \quad -\nabla \cdot (\zeta U) = G' \in X^1_{q,\gamma}(K)$$

($F', G'$ are the same functions as in part 2) of the proof), it follows from parts 1) and 2) of the proof that $\zeta(U, P) \in E^2_{q,\gamma}(K)$ and $\zeta \in V^1_{q,\gamma}(K)$. Analogously, we obtain $\eta(U, P) \in E^2_{q,\gamma}(K)$ and $\eta \in V^1_{q,\gamma}(K)$.

(4) We consider the case $p < q$. If $\frac{3}{p} - \frac{3}{q} < 1$, then Lemma 2.4 yields $U \in V^1_{q,\gamma'-1}(K)$, $P \in V^0_{q,\gamma'-1}(K)$ and $x \cdot U \in V^0_{p,\beta-1}(K) \subset (V^1_{q,2-\gamma}(K))^*$, where $\gamma' = \beta + \frac{3}{p} - \frac{3}{q}$. If in addition $\gamma \leq \gamma'$ (i.e., $\gamma < \gamma' + \frac{3}{p} - \frac{3}{q}$), then $\zeta F \in V^0_{q,\gamma}(K)$, $\zeta G \in X^1_{q,\gamma'}(K)$, and Lemma 2.9 implies $U \in E^2_{q,\gamma}(K)$ and $P \in V^1_{q,\gamma}(K)$. Thus, it follows from part 2) of the proof of $\zeta U \in E^2_{q,\gamma}(K)$ and $\zeta \in V^1_{q,\gamma}(K)$. Furthermore, $\eta \in V^1_{q,\gamma-1}(K)$, $\eta P \in V^0_{q,\gamma-1}(K)$ and $\eta x \cdot U \in (V^1_{q,2-\gamma}(K))^*$ if $\gamma \leq \gamma'$. Applying again Lemma 2.9, we conclude that $\zeta(U, P) \in E^2_{q,\gamma}(K)$ and $\eta \in V^1_{q,\gamma}(K)$. Analogously, it can be shown in the case $\gamma > \gamma'$ that both $\zeta(U, P)$ and $\eta(U, P)$ belong to the space $E^2_{q,\gamma}(K) \times V^1_{q,\gamma}(K)$ if $p < q$ and $\frac{3}{p} - \frac{3}{q} < 1$.

In the case $\frac{3}{p} - \frac{3}{q} \geq 1$ it follows from Lemma 2.5 that $F \in V^1_{q,\delta}(K)$ and $G \in X^1_{q,\gamma}$ (K) with arbitrary $q_1 \in (p, q)$ and $q_1 \gamma = \tau_1 \beta + \tau_2 \gamma$, where $\tau_1 = \tau_1(q - q_1)(q - p) - 1$ and $\tau_2 = q(q_1 - q)(q - p) - 1$. Obviously, $\tau_1 + \frac{3}{q_1} = q_1 \tau_1(\beta + \frac{3}{p}) + q_1 \tau_2(\gamma + \frac{3}{q})$ and $q_1 \tau_1 + \tau_2 = 2$. This means that $\gamma + \frac{3}{q_1}$ lies between $\beta + \frac{3}{p}$ and $\gamma + \frac{3}{q}$. If $\frac{3}{p} - \frac{3}{q} < 1$, we can conclude that $(U, P) \in E^2_{q,\gamma}(K) \times X^1_{q,\gamma}(K)$ and $(U, P) \in E^2_{q,\gamma}(K) \times V^1_{q,\gamma}(K)$ if $q_1 \in (p, q)$ and $q_2 \in (q_1, q)$ are chosen such that $\frac{3}{p} - \frac{3}{q} < 1$, $\frac{3}{p} - \frac{3}{q_2} < 1$ and $\frac{3}{q_1} - \frac{3}{q} < 1$. Here, $\gamma + \frac{3}{q}$
lies between $\beta + \frac{3}{p}$ and $\gamma + \frac{3}{q}$. Thus it follows from parts 1) and 2) of the proof that $U \in E_{q,\gamma}^2(K)$ and $P \in V_{q,\gamma}^1(K)$. The proof is complete.

**Remark 2.1.** The assertion of the last lemma is also true if $\beta + \frac{3}{p} < 2 < \gamma + \frac{3}{q}$ and the interval $1 - \gamma - \frac{3}{p} \leq \lambda \leq 1 - \beta - \frac{3}{p}$ contains only the eigenvalue $\lambda = -1$ of the pencil $\mathcal{N}(\mu)$. However, then the function $G$ must satisfy the condition (23). In order to show this, one has to apply Corollary 2.1 instead of Lemma 2.11.

### 2.5. Bijectivity of the Operator (17)

First, we prove the following generalization of [16, Lemma 2.10].

**Theorem 2.2.** Suppose that $\text{Re} s \geq 0$ and $1 - \mu_2 < \beta + \frac{3}{p} < 3 + \lambda_1$. Then the operator (17) is injective.

**Proof.** In the case $s = 0$, the operator $(U, P) \to (F, G) = (-\Delta U + \nabla P, -\nabla \cdot U)$ is even injective in the larger space $(V_{p,\beta}^2(K) \cap \nabla^1_p,\beta_1(K)) \times V_{p,\beta}^1(K)$ as is known from the elliptic theory. Let $(U, P)$ be a solution of the problem

$$sU - \Delta U + \nabla P = 0, \quad -\nabla \cdot U = 0 \quad \text{in } K, \quad U = 0 \quad \text{on } \partial K,$$

$$U \in E_{p,\beta}^2(K), \quad P \in V_{p,\beta}^1(K), \quad \text{Re } s \geq 0, \quad s \neq 0.$$

We have to show that $U = 0$ and $P = 0$ under the conditions of the lemma. We consider the following cases.

1. $p = 2, \quad 0 \leq \beta \leq 1$. Then the solution coincides with the variational solution in the space $E_{2,\gamma}^2(K) \times (L^2(K) + V_{2,\gamma}^0(K))$ (see [16, Lemma 2.1]). Since this solution is unique by [16, Theorem 1.1], it follows that $U = 0$ and $P = 0$.

2. $1 < \beta + \frac{3}{p} < 1 + 3$. Then there exists a number $\gamma \in [0, \frac{1}{2})$ such that either

$$\lambda_{-1} < 2 - \beta - \frac{3}{p} < 2 - \gamma - \frac{3}{2} < \lambda_1 \quad \text{or} \quad \mu_{-1} = -1 < 1 - \gamma - \frac{3}{2} < 1 - \beta - \frac{3}{p} < \mu_1 = 0.$$

In both cases, Lemma 2.12 implies $(U, P) \in E_{p,\gamma}^2(K) \times V_{p,\gamma}^1(K)$, and from the first part of the proof it follows that $U = 0$ and $P = 0$.

3. $\max(1 - \mu_2, \frac{1}{p}) < \beta + \frac{3}{p} \leq 1$. In this case, we choose a number $\gamma$ such that $1 < \gamma + \frac{3}{p} < \beta + 1 + \frac{2}{p}$. Let $\zeta$ and $\eta$ be the same functions as in the proof of Lemma 2.12. Then $\zeta(U, P) \in E_{p,\gamma}^2(K) \times V_{p,\gamma}^1(K)$ and $\eta(U, P) \in E_{p,\beta - \varepsilon}^2(K) \times V_{p,\beta - \varepsilon}^1(K)$ for arbitrary $\varepsilon \geq 0$. Here, $\eta P$ is a solution of the problem

$$\int_K \nabla(\eta P) \cdot \nabla Q \, dx = \langle \Phi, Q \rangle \quad \text{for all } \Phi \in V_{p',\varepsilon-\beta}^1(K),$$

where

$$\langle \Phi, Q \rangle = \int_K \left( \Delta U \cdot \nabla(\eta Q) + \nabla \eta \cdot (P \nabla Q - Q \nabla P) \right) \, dx.$$

By Lemma 2.10, the functional $\Phi$ is continuous on $V_{p',\varepsilon-\beta}^1(K)$ and on $V_{p',1-\gamma}^2(K)$ since the interval $1 - \gamma - \frac{3}{p} \leq \mu \leq 1 + \varepsilon - \beta - \frac{3}{p}$ contains only the eigenvalue $\mu_1 = 0$ of the pencil $\mathcal{N}(\mu)$ for small $\varepsilon > 0$ and the eigenfunctions corresponding to this eigenvalues are constant, it follows from Lemma 2.11 that $\eta P = c + P_0$, where $P_0 \in V_{p,\gamma-1}^0(K)$. Consequently, $P = c + P_1$, where $P_1 = P_0 + \zeta P \in V_{p,\gamma-1}^0(K)$. Obviously, $(U, P_1)$ is also a solution of the problem (25). Furthermore, $U \in E_{p,\gamma}^2(K) \subset V_{p,\gamma+1-\sigma}^1(K)$ for $\sigma \leq \beta - \gamma + 1$ (see (9)). Since $\beta - \gamma + 1 > \frac{1}{p}$, we can assume that $\sigma > \frac{1}{p}$. Then Lemma 2.8 yields $x \cdot U \in (V_{p',2-\gamma}^1(K))^*$. Applying Lemma 2.9, we conclude that $U \in E_{p,\gamma}^2(K)$ and $P_1 \in V_{p,\gamma}^1(K)$. By what has been shown in the second part, it follows from this that $U = 0$ and $P_1 = 0$. But then $P$ must be constant. However, $P = 0$ is the only constant element of the space $V_{p,\beta}^1(K)$. 


(4) $1 - \mu_2 < \beta + \frac{3}{p} \leq \frac{1}{p}$. Then it follows from Lemma 2.12 that $U \in E^2_{p,\gamma}(K)$ and $P \in V^1_{p,\gamma}(K)$ with a certain $\gamma \in (-\frac{2}{p}, 1 - \frac{2}{3})$. Thus $U = 0$ and $P = 0$ by the third part of the proof.

Next, we prove the bijectivity of the operator (17). In the case $2 - \frac{3}{p} < \beta < 4 - \frac{3}{p}$, the following assertion has to be taken into account.

**Lemma 2.13.** If $U \in E^2_{p,\beta}(K) \cap V^1_{p,\beta}(K)$ and $2 < \beta + \frac{3}{p} < 4$, then the function $G = -\nabla \cdot U$ satisfies the condition (23).

The proof of this lemma proceeds analogously to the case $p = 2$ in [16, Lemma 2.12].

**Theorem 2.3.** Suppose that $\text{Res} \geq 0$, $s \neq 0$, $F \in V^0_{p,\beta}(K)$ and $G \in X^1_{p,\beta}(K)$, where $\beta$ satisfies the condition (5). If $\beta + \frac{3}{p} > 2$, we assume in addition that $G$ satisfies the condition (23). Then there exists a unique solution of the problem (3) in the space $E^2_{p,\beta}(K) \times V^1_{p,\beta}(K)$ satisfying the estimate (6).

**Proof.** By Theorem 2.1, the operator (17) has closed range. Furthermore, it follows from Theorem 2.2 that the kernel of this operator is trivial. Let $F \in C^\infty_0(\overline{K} \setminus \{0\})$ and $G \in C^\infty_0(\overline{K} \setminus \{0\})$. By [16, Theorem 1.1], there exists a unique variational solution $(U, P) \in \hat{E}^1_{2,0}(K) \times (L_2(K) + V^1_{2,0}(K))$ of the problem (3). We denote by $\zeta$ and $\eta$ the same smooth functions as in the proof of Lemma 2.12. Obviously,

$$\eta U \in V^1_{2,\beta-1}(K), \quad \eta P \in V^0_{2,\beta-1}(K) \quad \text{and} \quad \eta x \cdot U \in V^0_{2,\beta-1}(K) \subset (V^1_{2,\beta-1}(K))^*$$

for arbitrary $\delta \leq 0$. Since

$$(s - \Delta)(\eta U) + \nabla(\eta P) \in V^0_{2,\gamma}(K) \quad \text{and} \quad \nabla \cdot (\eta U) \in X^1_{2,\gamma}(K) \quad \text{for arbitrary } \gamma,$$

we conclude from Lemma 2.9 that $\eta U \in E^2_{2,\beta}(K)$ and $\eta P \in V^1_{2,\beta}(K)$ for arbitrary $\delta \leq 0$. Using Lemma 2.12, we conclude that $\eta U \in E^2_{2,\gamma}(K)$ and $\eta P \in V^1_{2,\gamma}(K)$ for arbitrary $\gamma < \frac{1}{2}$.

Furthermore, $\zeta(U, P)$ is a vector function in the space $V^1_{2,0}(K) \times L_2(K)$ satisfying the equations (24), where $F' - s\zeta U \in V^0_{2,\gamma}(K)$ and $G' \in X^1_{2,\gamma}(K)$ for arbitrary $\gamma \geq -1$. If $\frac{1}{2} - \lambda_1 < \gamma < 1$, then the strip $-\frac{1}{2} \leq \text{Re} \lambda \leq \frac{1}{2} - \gamma$ does not contain eigenvalues of the pencil $L(\lambda)$. Then it follows from well-known regularity results for the stationary Stokes system that $\zeta U \in E^2_{2,\gamma}(K)$ and $\zeta P \in V^1_{2,\gamma}(K)$. This proves that $(U, P) \in E^2_{2,\gamma}(K) \times V^1_{2,\gamma}(K)$ for arbitrary $\gamma$, $\frac{1}{2} - \lambda_1 < \gamma < \frac{1}{2}$. Applying again Lemma 2.12 (see also Remark 2.1), we obtain $U \in E^2_{p,\beta}(K)$ and $P \in V^1_{p,\beta}(K)$ for arbitrary $p$ and $\beta$ satisfying the condition (5). In the case $\beta + \frac{3}{p} > 2$, we must assume that $G$ satisfies the condition (23). Thus the solvability in the space $E^2_{p,\beta}(K) \times V^1_{p,\beta}(K)$ is shown for smooth data $F$ and $G$. Since the range of the operator (17) is closed, the problem (3) is solvable in this space for arbitrary data $F \in V^0_{p,\beta}(K)$, $G \in X^1_{p,\beta}(K)$.

In order to prove (6), it suffices to consider the case $|s| = 1$, $\text{Re} s \geq 0$. For such $s$, one can show (see [16, Theorem 1.1]) that the variational solution satisfies the estimate

$$\|U\|_{E^1_{2,0}(K)} + \|P\|_{V^1_{2,0}(K) + L_2(K)} \leq c \left( \|F\|_{(E^1_{2,0}(K))^*} + \|G\|_{L_2(K)} + \|G\|_{(V^1_{2,0}(K))^*} \right),$$

where $c$ does not depend on $\text{arg } s$. Thus, the constant $c$ in (16) is independent of $\text{arg } s$ for $|s| = 1$. If $|s| \neq 1$, then we obtain (23) by means of the coordinate transformation $x = |s|^{-1/2} y$.

2.6. An Estimate for the Derivatives of the Solution

Suppose that $\text{Res} \geq 0$, $s \neq 0$, and that $\beta$ satisfies the condition (5). Then for arbitrary $F \in V^0_{p,\beta}(K)$, there exists a uniquely determined solution $(U(s), P(s)) \in E^2_{p,\beta}(K) \times V^1_{p,\beta}(K)$ of the problem

$$su(s) - \Delta U(s) + \nabla P(s) = F, \quad \nabla \cdot U(s) = 0 \quad \text{in } K, \quad U(s) = 0 \quad \text{on } \partial K.$$
Obviously, the derivative $(U'(s), P'(s))$ is a solution of the problem
\[ sU'(s) - \Delta U'(s) + \nabla P'(s) = -U(s), \quad \nabla \cdot U'(s) = 0 \quad \text{in } K, \quad U'(s) = 0 \quad \text{on } \partial K. \]

Applying Theorem 2.3, we obtain the following estimate for the solution $(U, P)$ and its derivative $(U', P')$.

**Lemma 2.14.** Suppose that $\beta$ satisfies the condition (5). Then
\[
\|U^{(k)}(s)\|_{V^2_{p,\beta}(K)} + |s| \|U^{(k)}(s)\|_{V^0_{p,\beta}(K)} + \|P^{(k)}(s)\|_{V^1_{p,\beta}(K)} \leq c |s|^{-k} \|F\|_{V^0_{p,\beta}(K)}
\]
for $F \in V^0_{p,\beta}(K)$, $Res \geq 0$, $s \neq 0$, $k = 0$ and $k = 1$. The constant $c$ is independent of $F$ and $s$.

**Proof.** For $k = 0$, we refer to Theorem 2.3. Obviously, the functions $U_h(s) = h^{-1}(U(s+h) - U(s))$ and $P_h(s) = h^{-1}(P(s+h) - P(s))$ are elements of the spaces $E^2_{p,\beta}(K)$ and $V^1_{p,\beta}(K)$ for $h \neq 0$. Since
\[
(s - \Delta) U_h(s) + \nabla P_h(s) = -U(s+h) \quad \text{and} \quad \nabla \cdot U_h(s) = 0 \quad \text{in } K,
\]
we obtain the estimate
\[
\|U_h(s)\|_{V^2_{p,\beta}(K)} + |s| \|U_h(s)\|_{V^0_{p,\beta}(K)} + \|P_h(s)\|_{V^1_{p,\beta}(K)} \\
\leq c \|U(s+h)\|_{V^0_{p,\beta}(K)} \leq c |s+h|^{-1} \|F\|_{V^0_{p,\beta}(K)}
\]
for small $|h|$. This proves (27) for $k = 1$. \qed

In the following lemma, we denote by $\zeta_\nu$ the same functions on $K$ as in Sect. 2.1.

**Lemma 2.15.** Suppose $Res \geq 0$, $s \neq 0$, and that $\beta$ satisfies the condition (5). If $\zeta_\nu F \in V^0_{p,\beta}(K)$ and $(U_\nu(s), P_\nu(s))$ is the solution of the problem
\[
sU(s) - \Delta U(s) + \nabla P(s) = \zeta_\nu F, \quad \nabla \cdot U(s) = 0 \quad \text{in } K, \quad U(s) = 0 \quad \text{on } \partial K,
\]
in the space $E^2_{p,\beta}(K) \times V^1_{p,\beta}(K)$, then the estimate
\[
\|\zeta_\nu U^{(k)}_\nu(s)\|_{V^2_{p,\beta}(K)} + |s| \|\zeta_\nu U^{(k)}_\nu(s)\|_{V^0_{p,\beta}(K)} + \|\zeta_\nu P^{(k)}_\nu\|_{V^1_{p,\beta}(K)} \leq c 2^{-\varepsilon}|\mu - \nu|^{-k} \|\zeta_\nu F\|_{V^0_{p,\beta}(K)}
\]
with a certain positive $\varepsilon$ is satisfied for $k = 0$ and $k = 1$. The constant $c$ in this estimate is independent of $\mu$, $\nu$, and $F$.

**Proof.** Obviously, $\zeta_\nu F \in V^0_{p,\beta \pm \varepsilon}(K)$ for arbitrary $\varepsilon$. Consequently, it follows from Lemma 2.12 that $U^{(k)}_\nu(s) \in E^2_{p,\beta \pm \varepsilon}(K)$ and $P^{(k)}_\nu(s) \in V^1_{p,\beta \pm \varepsilon}(K)$ if $\varepsilon$ is sufficiently small. Furthermore, by Lemma 2.14,
\[
\|U^{(k)}_\nu(s)\|_{V^2_{p,\beta \pm \varepsilon}(K)} + \|P^{(k)}_\nu(s)\|_{V^1_{p,\beta \pm \varepsilon}(K)} \leq c |s|^{-k} \|\zeta_\nu F\|_{V^0_{p,\beta \pm \varepsilon}(K)}.
\]
Using the estimates
\[
\|\zeta_\nu F\|_{V^0_{p,\beta \pm \varepsilon}(K)} \leq c 2^{\varepsilon|\mu - \nu|} \|\zeta_\nu F\|_{V^0_{p,\beta}(K)} \quad \text{and} \quad \|\zeta_\nu U^{(k)}_\nu\|_{V^2_{p,\beta \pm \varepsilon}(K)} \leq c 2^{\varepsilon|\mu - \nu|} \|U^{(k)}_\nu\|_{V^2_{p,\beta \pm \varepsilon}(K)},
\]
and the analogous estimate for the $V^1_{p,\beta}(K)$-norm of $\zeta_\nu P^{(k)}_\nu$, we obtain the desired result. \qed

**2.7. Existence of Solutions in the Case $2 - \lambda_1 < \beta + \frac{3}{p} < \lambda_1 + 3$**

Let $F \in V^0_{p,\beta}(K)$, where $2 - \lambda_1 < \beta + \frac{3}{p} < \lambda_1 + 3$. If in addition $2 \neq \beta + \frac{3}{p} < \mu_2 + 2$, then there exists a solution $(U, P)$ of the problem (26) in the space $E^2_{p,\beta}(K) \times V^1_{p,\beta}(K)$. This is not true, in general, if $\mu_2 + 2 \leq \beta + \frac{3}{p} < \lambda_1 + 3$ or $\beta + \frac{3}{p} = 2$. However, in these cases, we can construct a solution in the wider space $E^2_{p,\beta \gamma}(K) \times V^1_{p,\beta \gamma}(K)$ which was introduced in Sect. 1. Let again $\zeta$ be a smooth (two times continuously differentiable) function on $\mathbb{R}_+$ with support in $[0, 1]$ which is equal to one on the interval $(0, \frac{1}{4})$. With the definition $\zeta(x) = \zeta(|x|)$, one can consider $\zeta$ also as a function on $K$. Furthermore, let the function $\chi_s$ be defined as
\[
\chi_s(x) = \zeta(|s|^{1/2}|x|)
\]
for arbitrary $s \neq 0$. If $\beta \geq \gamma$, then $U \in E_{p,\beta,\gamma}^2(K)$ if and only if $\chi s U \in E_{p,\beta}^2(K)$ and $(1 - \chi s) U \in E_{p,\beta,\gamma}^2(K)$. An analogous assertion is true for the space $V_{p,\beta,\gamma}^1(K)$.

**Lemma 2.16.** Suppose that $Re s \geq 0$, $s \neq 0$ and

$$2 - \lambda_1 < \gamma + \frac{3}{p} \leq \beta + \frac{3}{p} < \lambda_1 + 3, \quad 2 \neq \gamma + \frac{3}{p} < \mu_2 + 2. \tag{30}$$

Then for arbitrary $F \in V_{p,\beta}^0(K)$, there exists a unique solution $(U, P) \in E_{p,\beta,\gamma}^2(K) \times V_{p,\beta,\gamma}^1(K)$ of the problem (26). This solution is independent of the choice of $\gamma$ in the given interval. Furthermore, the estimate

$$\|\chi s U\|_{V_{p,\beta}^2(K)} + \|\chi s P\|_{V_{p,\beta}^1(K)} + |s|^{(\gamma - \beta)/2} \left(\|(1 - \chi s) U\|_{V_{p,\beta,\gamma}^2(K)} + |s| \|(1 - \chi s) U\|_{V_{p,\beta,\gamma}^1(K)}\right) + \|(1 - \chi s) P\|_{V_{p,\beta,\gamma}^1(K)} \leq c \|F\|_{V_{p,\beta}^0(K)} \tag{31}$$

holds with a constant $c$ independent of $F$ and $s$. In particular, $U \in V_{p,\beta,-2}^0(K)$ and

$$\|U\|_{V_{p,\beta,-2}^0(K)} \leq c \|F\|_{V_{p,\beta}^0(K)}, \tag{32}$$

where $c$ is independent of $F$ and $s$.

**Proof.** We prove the uniqueness. Suppose that $(U, P)$ is a solution of the problem (25) in the space $E_{p,\beta,\gamma}^2(K) \times V_{p,\beta,\gamma}^1(K)$. Then $\zeta(U, P) \in E_{p,\beta,\gamma}^2(K) \times V_{p,\beta}^1(K)$ and $\eta(U, P) \in E_{p,\beta}^2(K) \times V_{p,\beta}^1(K)$, where $\eta = 1 - \zeta$. Furthermore,

$$(s - \Delta)(\zeta U) + \nabla(\zeta P) = -(s - \Delta)(\eta U) - \nabla(\eta P) \in V_{p,\beta}^0(K)$$

and $\nabla \cdot (\zeta U) = -\nabla \cdot (\eta U) \in X_{p,\gamma}^1(K)$. Under the conditions on $\beta$ and $\gamma$, the strip $2 - \beta - \frac{3}{p} \leq \text{Re} \lambda \leq 2 - \gamma - \frac{3}{p}$ is free of eigenvalues of the pencil $\mathcal{P}(\lambda)$. Thus, it follows from Lemma 2.12 that $\zeta U \in E_{p,\beta}^2(K)$ and $\zeta P \in V_{p,\beta}^1(K)$. Consequently, $U \in E_{p,\beta}^2(K)$ and $P \in V_{p,\beta}^1(K)$. Applying Theorem 2.3, we conclude that $U = 0$ and $P = 0$.

We prove the existence of solutions. Suppose first that $|s| = 1$. Let $B_\rho$ denote the ball $|x| < \rho$, and let $G_1$ be a domain such that $K \cap B_1 \subset G_1 \subset K \cap B_2$ and the boundary $\partial G_1$ is smooth outside the conical point $x = 0$. Furthermore, let $G_\rho = \rho G_1 = \{y = \rho x : x \in G_1\}$ for arbitrary $\rho > 0$. Since $2 - \lambda_1 < \gamma + \frac{3}{p} < \lambda_1 + 3$, the problem

$$-\Delta U_0 + \nabla P_0 = F, \quad \nabla \cdot U_0 = 0 \text{ in } G_\rho, \quad U_0|_{\partial G_\rho \setminus \{0\}} = 0, \quad \int_{G_\rho} P_0 \, dx = 0$$

has a uniquely determined solution $(U_0, P_0) \in V_{p,\beta}^2(G_\rho) \times V_{p,\beta}^1(G_\rho)$ satisfying the estimate

$$\|U_0\|_{V_{p,\beta}^2(G_\rho)} + \|P_0\|_{V_{p,\beta}^1(G_\rho)} \leq c \|F\|_{V_{p,\beta}^0(G_\rho)} \tag{33}$$

(see [21,22, Theorem 6.1]). One can easily show (using the coordinate transformation $y = \rho x$) that the constant $c$ in (33) is independent of $\rho$. Since

$$\|sU_0\|_{V_{p,\beta}^0(G_\rho)} \leq 4\rho^2 \|U_0\|_{V_{p,\beta}^2(G_\rho)},$$

it follows that the problem

$$(s - \Delta) U_0 + \nabla P_0 = F, \quad \nabla \cdot U_0 = 0 \text{ in } G_\rho, \quad U_0|_{\partial G_\rho \setminus \{0\}} = 0, \quad \int_{G_\rho} P_0 \, dx = 0$$

has a uniquely determined solution $(U_0, P_0) \in V_{p,\beta}^2(G_\rho) \times V_{p,\beta}^1(G_\rho)$ satisfying the estimate (33) if $\rho$ is sufficiently small. Let $(U_0, P_0)$ be this solution. Then $\chi_\rho U_0 \in E_{p,\beta}^2(K)$ and $\chi_\rho P_0 \in V_{p,\beta}^1(K)$ (if we extend these functions by zero outside $B_\rho$) and

$$\|\chi_\rho U_0\|_{V_{p,\beta}^2(K)} + \|\chi_\rho P_0\|_{V_{p,\beta}^1(K)} \leq c \|F\|_{V_{p,\beta}^0(K)}.$$
Furthermore, the functions
\[ \Phi = F - (s - \Delta)(\chi_\rho U_0) - \nabla(\chi_\rho P_0) = (1 - \chi_\rho)F + [\Delta, \chi_\rho]U_0 - P_0 \nabla \chi_\rho \]
and \( G = -\nabla \cdot (\chi_\rho U_0) = -U_0 \cdot \nabla \chi_\rho \) are elements of the spaces \( V^0_{p,\gamma}(K) \) and \( X^1_{p,\gamma}(K) \), respectively. Since \( \Phi(x) \) and \( G(x) \) are zero for \( |x| < \rho/2 \), we obtain the estimate
\[ ||\Phi||_{V^0_{p,\gamma}(K)} + ||G||_{X^1_{p,\gamma}(K)} \leq c ||F||_{V^0_{p,\gamma}(K)} \]
with a constant \( c \) depending on \( \rho \). By Theorem 2.3, there exists a unique solution \( (U_1, P_1) \in E^2_{p,\gamma}(K) \times V^1_{p,\gamma}(K) \) of the problem
\[ (s - \Delta) U_1 + \nabla P_1 = \Phi, \quad \nabla \cdot U_1 = G \text{ in } K, \quad U_1|_{\partial K \setminus 0} = 0 \]
satisfying the estimate
\[ ||U_1||_{E^2_{p,\gamma}(K)} + ||P_1||_{V^1_{p,\gamma}(K)} \leq c ||F||_{V^0_{p,\gamma}(K)}. \]
Obviously, \( (U, P) = \chi_\rho(U_0, P_0) + (U_1, P_1) \) is a solution of the problem (25). From the above estimates for \( (U_0, P_0) \) and \( (U_1, P_1) \), we obtain (31) if \( |s| = 1 \).

Let \( s \) be an arbitrary number in the half-plane \( \Re s \geq 0, s \neq 0 \), and let \( H(x) = |s|^{-1}F(|s|^{-1/2}x). \) Then the problem
\[ \left( \frac{s}{|s|} - \Delta \right) V + \nabla Q = H, \quad \nabla \cdot V = 0 \text{ in } K, \quad V = 0 \text{ on } \partial K \setminus \{0\} \]
has a solution \( (V, Q) \) satisfying the estimate (31) with \( |s| = 1 \). Then the functions \( U(x) = V(|s|^{1/2}x) \) and \( P(x) = |s|^{1/2}Q(|s|^{1/2}x) \) satisfy (26). Furthermore, the estimate (31) holds. Since \( \beta - 2 + \frac{3}{p} < \lambda_1 + 1 < \mu_2 + 2 \), we can choose \( \gamma \) such that \( \beta \geq 2 \). Then
\[ ||U||_{V^0_{p,\beta-2}(K)} \leq ||\chi_\rho U||_{V^2_{p,\gamma}(K)} + |s|^{(\gamma - \beta + 2)/2} ||(1 - \chi_\rho)U||_{V^0_{p,\gamma}(K)} , \]
and (31) yields (32).

The independence of \( (U, P) \) on the choice of \( \gamma \) follows from the imbeddings \( E^2_{p,\beta,\gamma'}(K) \subseteq E^2_{p,\beta,\gamma}(K) \) and \( V^1_{p,\beta,\gamma}(K) \subseteq V^1_{p,\beta,\gamma}(K) \) for \( \gamma < \gamma' \leq \beta \) and from the uniqueness of the solution in the space \( E^2_{p,\beta,\gamma}(K) \times V^1_{p,\beta,\gamma}(K) \). The proof is complete. \( \square \)

Furthermore, we can prove the following regularity assertion for the solution \( (U, P) \) introduced in the last lemma.

**Lemma 2.17.** Suppose that \( (U, P) \in E^2_{p,\beta,\gamma}(K) \times V^1_{p,\beta,\gamma}(K) \) is a solution of the problem (26), where \( \Re s \geq 0, s \neq 0, \) and \( \beta, \gamma \) satisfy the inequalities (30). If \( F \in V^0_{p,\beta}(K) \cap V^0_{p,\beta'}(K) \) and \( \beta \geq \gamma \), then \( U \in E^2_{p,\beta',\gamma}(K) \) and \( P \in V^1_{p,\beta,\gamma}(K) \) if \( F \in V^0_{p,\beta}(K) \cap V^0_{p,\beta'}(K) \).

**Proof.** In the case \( \beta' \geq \beta \geq \gamma \), the assertion follows from the imbeddings \( E^2_{p,\beta,\gamma}(K) \subseteq E^2_{p,\beta',\gamma}(K) \) and \( V^1_{p,\beta,\gamma}(K) \subseteq V^1_{p,\beta',\gamma}(K) \). Suppose that \( \gamma \leq \beta' < \beta \). Then \( \zeta(U, P) \in V^2_{p,\beta}(K) \times V^1_{p,\beta}(K), \nabla \cdot (\zeta U) = U \cdot \nabla \zeta \subseteq X^1_{p,\beta}(K) \) and
\[ (s - \Delta)(\zeta U) + \nabla (\zeta P) = \zeta F - [\Delta, \zeta]U + \zeta \nabla P \in V^0_{p,\beta}(K), \]
where \( [\Delta, \zeta]U = \Delta(\zeta U) - \zeta \Delta U \). Thus, Lemma 2.12 implies \( \zeta U \in E^2_{p,\beta}(K) \) and \( \zeta P \in V^1_{p,\beta'(K)} \). This proves the lemma. \( \square \)

Next, we prove an estimate for the derivatives \( U'(s) \) and \( P'(s) \).

**Lemma 2.18.** Let \( F \in V^0_{p,\beta}(K) \), and let \( (U(s), P(s)) \in E^2_{p,\beta,\gamma}(K) \times V^1_{p,\beta,\gamma}(K) \) be a solution of the problem (26), where \( \Re s \geq 0, s \neq 0, \) and \( \beta \) and \( \gamma \) satisfy the conditions (30) and \( \gamma \geq \beta - 2 \). Then \( U'(s) \in E^2_{p,\gamma}(K), \) \( P'(s) \in V^1_{p,\gamma}(K) \) and
\[ ||U'(s)||_{V^2_{p,\gamma}(K)} + |s| ||U'(s)||_{V^0_{p,\gamma}(K)} + ||P'(s)||_{V^1_{p,\gamma}(K)} \leq c |s|^{(\beta - \gamma - 2)/2} ||F||_{V^0_{p,\gamma}(K)} \]
(34)
with a constant $c$ independent of $F$ and $s$. In particular,
\[ \|U'(s)\|_{V^0_{p,\beta-2}(K)} \leq c|s|^{-1} \|F\|_{V^0_{p,\beta}(K)}, \]
where $c$ is independent of $F$ and $s$.

**Proof.** Since $\beta - 2 \leq \gamma \leq \beta$, every of the spaces $E^2_{p,\beta}(K)$, $E^2_{p,\gamma}(K)$ and $E^2_{p,\beta,\gamma}(K)$ is a subspace of $V^0_{p,\gamma}(K)$. Furthermore, it follows from (31) that
\[ \|U\|_{V^0_{p,\gamma}(K)} \leq \|\chi_s U\|_{V^0_{p,\gamma}(K)} + \|(1 - \chi_s) U\|_{V^0_{p,\beta-2}(K)} \leq |s|^{(\beta-\gamma-2)/2} \|\chi_s U\|_{V^0_{p,\gamma}(K)} + \|(1 - \chi_s) U\|_{V^0_{p,\beta-2}(K)} \leq c|s|^{(\beta-\gamma-2)/2} \|F\|_{V^0_{p,\beta}(K)}. \tag{35} \]
Let $U_h(s) = h^{-1}(U(s + h) - U(s))$ and $P_h(s) = h^{-1}(P(s + h) - P(s))$ for $h \neq 0$. Since $(U_h(s), P_h(s))$ is a solution of the problem (28) with the right-hand side $U(s + h) \in V^0_{p,\gamma}(K)$, Lemma 2.17 implies $U_h \in E^2_{p,\gamma}(K)$ and $P_h \in V^1_{p,\gamma}(K)$. By Theorem 2.3, the estimate
\[ \|U_h(s)\|_{V^2_{p,\gamma}(K)} + \|s\| \|U_h(s)\|_{V^0_{p,\gamma}(K)} + \|P_h\|_{V^1_{p,\gamma}(K)} \leq c \|U(s + h)\|_{V^0_{p,\gamma}(K)} \]
with a constant $c$ independent of $U$, $s$ and $h$ holds. Consequently, $U'(s) \in E^2_{p,\gamma}(K)$ and
\[ \|U'(s)\|_{V^0_{p,\gamma}(K)} + \|s\| \|U'(s)\|_{V^0_{p,\gamma}(K)} + \|P'(s)\|_{V^1_{p,\gamma}(K)} \leq c \|U(s)\|_{V^0_{p,\gamma}(K)}. \]
This together with (35) implies (34). The estimate for the $V^0_{p,\beta-2}(K)$-norm of $U'(s)$ follows from (34) and from the inequality $(|s|^2)^{\beta-2} \leq (|s|^2)^{\gamma-2} + (|s|^2)^{\gamma}$. \qed

The function $\chi_s$ introduced above is not differentiable with respect to the complex variable $s$. For this reason, we replace it by a holomorphic function $\Psi_s$ which is defined as follows. Let $\psi$ be a $C^\infty$-function on $(-\infty, +\infty)$ with support in the interval $[0, 1]$ satisfying the conditions
\[ \int_0^1 \psi(t) dt = 1, \quad \int_0^1 t^j \psi(t) dt = 0 \] for $j = 1, \ldots, N$,
where $N$ is a sufficiently large integer. By $\Psi$, we denote the Laplace transform of $\psi$. The function $\Psi$ is analytic in $\mathbb{C}$ and satisfies the conditions $\Psi(0) = 1$, $\Psi^{(j)}(0) = 0$ for $j = 1, \ldots, N$. Hence,
\[ \left| \frac{d\Psi(t)}{ds} \right| \leq c_j |s|^{-N+1-j} \]
for $\Re s \geq 0$. Since $s^n \Psi^{(j)}(s)$ is the Laplace transform of the function $(-1)^j \frac{d^n}{dt^n} (t^j \psi(t))$, it follows that
\[ |\Psi^{(j)}(s)| \leq c_{j,n} |s|^{-n} \]
for every $j \geq 0$ and $n \geq 0$, where $c_{j,n}$ is independent of $s$, $\Re s \geq 0$. We set $\Psi_s(x) = \Psi(xs^2)$. Obviously, $(\Psi_s - \chi_s) U \in E^2_{p,\gamma}(K) \cap E^2_{p,\gamma}(K)$ for arbitrary $U \in E^2_{p,\beta,\gamma}(K)$. Hence, the cut-off function $\chi_s$ can be replaced by $\Psi_s$ in Lemma 2.16.

### 2.8. Asymptotics of the Solution at Infinity

In this subsection, we study the behavior of the solutions of the problem (26) at infinity. Here, we restrict ourselves to the case $p = 2$. Let $F \in V^0_{p,\beta}(K)$, where $\frac{1}{2} - \lambda_1 < \beta < \lambda_1 + \frac{3}{2}$. By Lemma 2.16, there exists a solution $(U, P)$ of the problem (26) in the space $E^2_{p,\beta,\gamma}(K) \times V^1_{p,\beta,\gamma}(K)$, where $\frac{1}{2} - \lambda < \gamma \leq \beta$, $\gamma \neq \frac{1}{2}$ and $\gamma < \mu_2 + \frac{1}{2}$. If $\beta \neq \frac{1}{2}$ and $\beta < \mu_2 + \frac{1}{2}$, then it follows from Lemma 2.17 that $U \in E^2_{p,\beta}(K)$ and $P \in V^1_{p,\beta}(K)$. In particular, we have $U \in V_{p,\beta-1/2}^3(K)$ for $\beta \neq \frac{1}{2}$, $\beta < \mu_2 + \frac{1}{2}$. This is not true if $\mu_2 + \frac{1}{2} \leq \beta < \lambda_1 + \frac{3}{2}$. The goal of this subsection is to find a decomposition of the function $U$ with some “singular” terms and a remainder $V \in V_{p,\beta-1/2}^3(K)$. This decomposition together with the imbedding $V_{p,\beta-1/2}^3(K) \subset V_{p,\beta'}^{0}(K)$ for $p \geq 2$, $\beta' = \beta + \frac{3}{2} - \frac{3}{p}$ (cf. Lemma 2.4) will be applied in Sect. 3.4 for the proof of the existence of solutions in weighted $L_p$ spaces.
We use the following notation. Let \( \mu_j \) be an eigenvalue of the pencil \( \mathcal{N}(\mu) \), and let \( \phi_{j,k}, k = 1, \ldots, \sigma_j \), be orthonormalized eigenfunctions corresponding to \( \mu_j \). Then \( \mu_{-j} = -1 - \mu_j \) is also an eigenvalue of this pencil with the same eigenfunctions \( \phi_{j,k} \). The functions
\[
p_j^{\mu_j}(x) = \mu_j \phi_{j,k}(x)
\]
satisfy the equation \( \Delta p_0^{\mu_j} = 0 \) in \( K \) and the Neumann condition \( \partial p_0^{\mu_j} / \partial n = 0 \) on \( \partial K \setminus \{0\} \). Furthermore, let \( u^{\mu_j} = -\nabla p_0^{\mu_j} \) and let \( \nu(x) \) denote the distance of the point \( x \in K \) from the boundary \( \partial K \). The function \( \nu = \nu(x) \) is positively homogeneous of degree 1 and two times continuously differentiable in the set \( \{x \in K : \nu(x) < \delta |x|\} \), where \( \delta \) is sufficiently small. We define \( v^{\mu_j}(x) = u^{\mu_j} - (u^{\mu_j} \cdot \nabla) \nabla u^{\mu_j} \) in this neighborhood and set
\[
u_0^{\mu_j}(x,s) = s^{-1} \left( u^{\mu_j}(x) - \chi \left( \frac{\nu}{s} \right) e^{-\nu \sqrt{s} v^{\mu_j}(x)} \right),
\]
where \( \chi \) is a two times continuously differentiable function on \((0, \infty)\) with support in \([0, \delta]\) which is equal to 1 in \([0, \delta/2]\), and \( \sqrt{s} \) denotes the square root of \( s \) with positive real part. The vector function \( u_0^{\mu_j}(x) \) is zero on \( \partial K \setminus \{0\} \). In the set \( \{x \in K : \nu(x) > \delta |x|\} \), it coincides with \( s^{-1} u^{\mu_j}(x) \).

Suppose that \( 0 \leq \mu_j < \lambda_1 \). Then, by [18, Corollary 2.1], there exists a solution \((V^{\mu_j}, Q^{\mu_j})\) of the problem
\[
(s - \Delta) V + \nabla Q = 0, \quad \nabla \cdot V = 0 \text{ in } K, \quad V = 0 \text{ on } \partial K \setminus \{0\}
\]
which has the form
\[
V^{\mu_j} = (1 - \Psi_j) u_0^{\mu_j} + u^{\mu_j}, \quad Q^{\mu_j} = (1 - \Psi_j) p_0^{\mu_j} + q^{\mu_j},
\]
where \( w^{\mu_j} \in E_{2,\delta}^2(K), q^{\mu_j} \in V_{1,\delta}^1(K) \) with arbitrary \( \delta \in (\frac{1}{2}, \frac{1}{2} - \mu_j) \) and \( w^{\mu_j} = 0 \) on \( \partial K \setminus \{0\} \).

Lemma 2.19. Suppose that \( Res \geq 0, s \neq 0 \) and \( F \in V_{0,\beta}^0(K) \), where \( \frac{1}{2} - \lambda_1 < \beta < \lambda_1 + \frac{3}{2} \). If \((U, P)\) is the solution of the problem \((25)\) in the space \( E_{2,\beta,\gamma}^2(K) \times V_{2,\beta,\gamma}^1(K) \), where \( \frac{1}{2} - \lambda_1 < \gamma \leq \beta, \gamma < \mu_2 + \frac{1}{2} \) and \( \gamma \neq \frac{1}{2} \), then \( U \) admits the decomposition
\[
U(s) = (1 - \Psi_j) \sum_{j \in I_\beta} \sum_{k = 1}^{\sigma_j} c_{j,k}(s) u_0^{(-j,k)}(\cdot, s) + V(s),
\]
where \( V(s) \in V_{2,\beta-1/2}^{3/2}(K) \), \( V'(s) \in V_{2,\beta-1/2}^{3/2}(K) \), \( I_\beta \) is the set of all \( j \) such that \( 0 < \mu_j \leq \beta - \frac{3}{2} \) (\( I_\beta = \emptyset \) if \( \beta < \mu_2 + \frac{3}{2} \)) and
\[
c_{j,k}(s) = -\frac{s}{1 + 2\mu_j} \int_K F(x) \cdot V^{(j,k)}(x, s) \, dx.
\]
Furthermore,
\[
\|V(s)\|_{V_{2,\beta-1/2}^{3/2}} + \|s\| \|V'(s)\|_{V_{2,\beta-1/2}^{3/2}} \leq c \|F\|_{V_{0,\beta}^0(K)}
\]
and
\[
|c_{j,k}(s)|^2 \leq c |s|^{\beta - \mu_j - 1/2} \|F\|_{V_{2,\beta}^0(K)}^2
\]
with a constant \( c \) independent of \( F \) and \( s \).

Proof. By Lemma 2.16, the solution \((U, P)\) does not depend on the choice of \( \gamma \). If \( \beta < \mu_2 + \frac{1}{2} \) and \( \beta \neq \frac{1}{2} \), then \( U^{(k)}(s) \in E_{2,\beta}^2(K) \subset V_{2,\beta-1/2}^{3/2}(K) \) and
\[
|s|^k \|U^{(k)}(s)\|_{V_{2,\beta-1/2}^{3/2}} \leq c |s|^k \|U^{(k)}(s)\|_{V_{2,\beta}^0(K)} \leq c \|F\|_{V_{0,\beta}^0(K)}
\]
for \( k = 0 \) and \( k = 1 \) (cf. Theorem 2.3 and Lemma 2.14). In the case \( \beta = \frac{1}{2} \), we may assume that \( U^{(k)}(s) \) is an element of the space \( E_{2,\beta,\gamma}^2(K) \), with some \( \gamma \) in the interval \( (\beta - \frac{1}{2}, \beta) \). But this is also a subspace of \( V_{2,\beta-1/2}^{3/2}(K) \) (see (9)).

Suppose that \( \mu_2 + \frac{1}{2} \leq \beta < \lambda_1 + \frac{3}{2} \). Then the vector function \((1 - \Psi_s)(U, P)\) lies in the space \( E_{2,\gamma}^2(K) \times V_{2,\gamma}^1(K) \) with arbitrary \( \gamma \in (\lambda_1 + \frac{1}{2}, \mu_2 + \frac{1}{2}) \) and satisfies the equations

\[
(s - \Delta)(U - \Psi_s U) + \nabla(P - \Psi_s P) = \Phi,\quad -\nabla \cdot (U - \Psi_s U) = G,
\]

where

\[
\Phi = (1 - \Psi_s) F + [\Delta, \Psi_s] U - P \nabla \Psi_s \quad \text{and} \quad G = U \cdot \nabla \Psi_s.
\]

Here \([\Delta, \Psi_s] U = \Delta(\Psi_s U) - \Psi_s \Delta U\). Obviously, \( \Phi \in V_{2,\beta+\varepsilon-1/2}^0(K) \) and \( G \in X_{2,\beta+\varepsilon-1/2}^1(K) \) for arbitrary \( \varepsilon \leq \frac{1}{2} \). If \( \beta - 1 \) is not an eigenvalue of the pencil \( \mathcal{N}(\mu) \), then we set \( \varepsilon = 0 \). Otherwise, let \( \varepsilon \) be a positive number less than \( \frac{1}{2} \) such that the interval \( \beta - 1 < \lambda \leq \beta - 1 + \varepsilon \) does not contain eigenvalues of the pencil \( \mathcal{N}(\mu) \). Then by [18, Lemma 2.5], the functions \((1 - \Psi_s) U\) and \((1 - \Psi_s) P\) admit the decompositions

\[
(1 - \Psi_s) U(s) = (1 - \Psi_s) \Sigma_1(s) + W(s), \quad (1 - \Psi_s) P(s) = (1 - \Psi_s) \Sigma_2(s) + Q(s),
\]

where \( W(s) \in E_{2,\beta+\varepsilon-1/2}^2(K), Q(s) \in V_{2,\beta+\varepsilon-1/2}^2(K) \)

\[
\Sigma_1(s) = \sum_{0< \mu_j \leq \beta-1} \sum_{k=1}^{\Sigma_1} c_{j,k}(s) U(-j,k)(x,s), \quad \Sigma_2(s) = \sum_{0< \mu_j \leq \beta-1} \sum_{k=1}^{\Sigma_1} c_{j,k}(s) P(-j,k)(x,s).
\]

Here, \( U(-j,k) = u_0^{(-j,k)} \) and \( P(-j,k) = p_0^{(-j,k)} \) if \( \mu_j > \beta - 2 \). In the case \( \mu_j \leq \beta - 2 \),

\[
U^{(-j,k)} = u_0^{(-j,k)} + u_1^{(-j,k)}, \quad P^{(-j,k)} = p_0^{(-j,k)} + p_1^{(-j,k)}
\]

with functions \( u_1^{(-j,k)} \) and \( p_1^{(-j,k)} \) of the form (cf. [18, Lemma 2.3])

\[
u(-j,k) = s^{-3/2} r^{-\mu_j/2} \left( \frac{1}{\mu_j} \sum_{n=0}^{\infty} a_n(x) \log^n r - \chi \left( \frac{\mu_j}{r} \right) e^{-\sqrt{\mu_j}} \sum_{m,n=0}^{\infty} b_{m,n} \omega_n \left( \mu_j \sqrt{s} \right)^m \log^n r \right)
\]

and

\[
u(-j,k) = s^{-1} r^{-\mu_j/2} \left( \frac{1}{\mu_j} \sum_{n=0}^{\infty} a_n(x) \log^n r - \chi \left( \frac{\mu_j}{r} \right) e^{-\sqrt{\mu_j}} \sum_{m,n=0}^{\infty} b_{m,n} \omega_n \left( \mu_j \sqrt{s} \right)^m \log^n r \right).
\]

Furthermore, by [18, Theorem 2.1] and (31), the estimate

\[
\|W\|_{E_{2,\beta+\varepsilon-1/2}^2(K)} + \|Q\|_{V_{2,\beta+\varepsilon-1/2}^2(K)} + \sum_{j,k} |c_{j,k}(s)| \leq c \|F\|_{V_{2,\beta}^0(K)}
\]

is satisfied with a constant \( c \) independent of \( F \) and \( s \) if \( |s| = 1 \).

We show that \((1 - \Psi_s) u_0^{(-j,k)} \in V_{2,\beta-1/2}^2(K) \) for \( \mu_j > \beta - 3/2 \). Since \( v^{(-j,k)} = -\nabla p_0^{(-j,k)} \) has the form \( r^{-\mu_j/2} \phi(\omega) \), it follows that \( v^{(-j,k)} \in V_{2,\beta}^2(K) \subset V_{2,\beta-1/2}^2(K) \). Using the inequality

\[
\int_{K \setminus B_1} r^{2\beta-1} r^{-\mu_j/2} \left( \frac{\nu(x)}{r} \right) e^{-\sqrt{\nu(x)}} dx \leq c \int_1^\infty r^{2\beta-2\mu_j/2} \nu(\omega) e^{-\nu(\omega)\Re \sqrt{\nu}} d\omega \leq c \int_1^\infty r^{2\beta-2\mu_j/2} d\omega,
\]

one can show that \((1 - \Psi_s) v^{(-j,k)} \in E_{2,\beta-1/2}^2(K) \subset V_{2,\beta-1/2}^2(K) \) if \( \mu_j > \beta - 3/2 \). Thus,

\[
(1 - \Psi_s) u_0^{(-j,k)} \in V_{2,\beta-1/2}^2(K) \text{ for } \mu_j > \beta - 3/2.
\]

Analogously, one can prove that \((1 - \Psi_s) u_1^{(-j,k)} \in \ldots \)
$V_{2, \beta-1/2}^{3/2}(K)$ if $\mu_j > \beta - \frac{5}{2}$. Since $\beta < \lambda_1 + \frac{3}{2} \leq \frac{5}{2}$, we get $(1 - \Psi_s) u_1^{(-j,k)} \in V_{2, \beta-1/2}^{3/2}(K)$ for all $\mu_j > 0$. Thus, (41) implies

$$U(s) = (1 - \Psi_s) \sum_{0 < \mu_j \leq \beta - 3/2} \sum_{k=1}^{2} \sigma_j c_j,k(s) u_0^{(-j,k)}(s, s) + V(s),$$

where

$$V = W + \Psi_s U + (1 - \Psi_s) \left( \sum_{j \in J_{\beta}^*} \sum_{k=1}^{2} \sigma_j c_j,k u_0^{(-j,k)} + \sum_{j \in J_{\beta}^*} \sum_{k=1}^{2} \sigma_j c_j,k u_1^{(-j,k)} \right)$$

(42)

with the notation $J_{\beta}^* = \{ j : \max(0, \beta - 3/2) < \mu_j \leq \beta - 1 \}$ and $J_{\beta}^* = \{ j : 0 < \mu_j \leq \beta - 2 \}$. Since $W \in E_{2, \beta+\epsilon-1/2}^{2}(K) \subset V_{2, \beta-1/2}^{3/2}(K)$ and $\Psi_s U \in V_{2, \beta}^{3/2}(K) \subset V_{2, \beta-1/2}^{3/2}(K)$, it follows that $V \in V_{2, \beta-1/2}^{3/2}(K)$. For $|s| = 1$, we obtain the estimate

$$||V||_{V_{2, \beta-1/2}^{3/2}(K)} \leq c ||F||_{V_{2, \beta}^{2}(K)}$$

with a constant $c$ independent of $s$.

We prove the formula (38) for the coefficients $c_{j,k}(s)$. Since $\mu_j < \lambda_1 \leq 1$ for $j \in I_{\beta}$, the coefficients $c_{j,k}(s)$ are given by the formula

$$c_{j,k}(s) = \frac{s}{1 + 2 \mu_j} \int_K \left( \Phi(x) \cdot V^{(j,k)}(x, s) + G(x) Q^{(j,k)}(x, s) \right) dx$$

(43)

(cf. [18, Theorem 2.1]), where $(V^{(j,k)}, Q^{(j,k)})$ is the solution of the problem (36) introduced above, i. e., $V^{(j,k)} = (1 - \Psi_s) u_0^{(j,k)} + w^{(j,k)}$ and $Q^{(j,k)} = (1 - \Psi_s) p_0^{(j,k)} + q^{(j,k)}$, where $w^{(j,k)} \in E_{2, \delta}^{2}(K)$, $q^{(j,k)} \in V_{2, \delta}^{1}(K)$ with arbitrary $\delta \in (\frac{1}{2} - \lambda_1, \frac{1}{2} - \mu_j)$. We can choose $\delta$ such that $2 - \delta \geq \beta$. Then $\Psi_s U \in V_{2, \delta}^{2}(K)$, $\Psi_s P \in V_{2, \delta-1}(K)$, and integration by parts yields

$$\int_K ((s - \Delta) (\Psi_s U) + \nabla (\Psi_s P)) \cdot w^{(j,k)} dx - \int_K q^{(j,k)} \nabla \cdot (\Psi_s U) dx$$

$$= \int_K \Psi_s U \cdot ((s - \Delta) w^{(j,k)} + \nabla q^{(j,k)}) dx - \int_K \Psi_s P \nabla \cdot v^{(j,k)} dx.$$ 

Obviously, this equality is also valid if we replace $w^{(j,k)}$ and $q^{(j,k)}$ by $(1 - \Psi_s) u_0^{(j,k)}$ and $(1 - \Psi_s) p_0^{(j,k)}$, respectively. Consequently, we get

$$\int_K ((s - \Delta) (\Psi_s U) + \nabla (\Psi_s P)) \cdot V^{(j,k)} dx - \int_K Q^{(j,k)} \nabla \cdot (\Psi_s U) dx = 0$$

and (43) implies (38).

We prove that $V'(s) \in V_{2, \beta-1/2}^{3/2}(K)$. Differentiating the equation

$$(s - \Delta)W + \nabla Q = (s - \Delta) (1 - \Psi_s)(U - \Sigma_1) + \nabla (1 - \Psi_s)(P - \Sigma_2),$$

we get

$$(s - \Delta)W' + \nabla Q'(s) = -W(s) + \frac{d}{ds} \left( (s - \Delta) (1 - \Psi_s)(U - \Sigma_1) + \nabla (1 - \Psi_s)(P - \Sigma_2) \right)$$

$$= -W(s) - \frac{d\Psi_s}{ds} \left( F - (s - \Delta) \Sigma_1 - \nabla \Sigma_2 \right) + \frac{d}{ds} \left( \Delta, \Psi_s \right) (U - \Sigma_1) - (P - \Sigma_2) \nabla \Psi_s$$

$$- (1 - \Psi_s) \frac{d}{ds} \left( (s - \Delta) \Sigma_1 + \nabla \Sigma_2 \right).$$

The terms $\frac{d\Psi_s}{ds} \left( F - (s - \Delta) \Sigma_1 - \nabla \Sigma_2 \right)$ and $\frac{d}{ds} \left( \Delta, \Psi_s \right) \left( U - \Sigma_1 \right) - \left( P - \Sigma_2 \right) \nabla \Psi_s$ belong to $V_{2, \beta+\epsilon-1/2}^{0}(K)$ since the derivatives of the function $\Psi_s(x) = \Psi(r^2)$ are rapidly decreasing for $r \to 0$ and $r \to \infty$. Using
[18, Lemma 2.3], one can easily show that the same is true for the term \((1 - \Psi_s) \frac{d}{ds} (s - \Delta) \Sigma_1 + \nabla \Sigma_2\). Hence, \((s - \Delta)W'(s) + \nabla Q'(s) \in V_0^{1, \beta + \varepsilon - 1/2}(K)\). Analogously,

\[
\nabla \cdot W'(s) = \frac{d}{ds} \nabla \cdot (1 - \Psi_s) (U - \Sigma_1) = -\frac{d}{ds} (U - \Sigma_1) \nabla \Psi_s \in X_2^{1, \beta + \varepsilon - 1/2}(K).
\]

The norms of \((s - \Delta)W'(s) + \nabla Q'(s)\) and \(\nabla \cdot W'(s)\) can be estimated by means of Lemmas 2.18 and 2.19. Let \(W_h(s) = h^{-1}(W(s + h) - W(h))\). Then by means of the estimate in Theorem 2.3, we get

\[
\|W'(s)\|_{E_2^{1, \beta + \varepsilon - 1/2}(K)} = \lim_{h \to 0} \|W_h(s)\|_{E_2^{1, \beta + \varepsilon - 1/2}(K)}
\]

\[
\leq C(s) \left( \| (s - \Delta)W(s) + \nabla Q(s) \|_{V_0^{1, \beta + \varepsilon - 1/2}(K)} + \| \nabla \cdot W(s) \|_{X_2^{1, \beta + \varepsilon - 1/2}(K)} \right).
\]

For \(|s| = 1\), we obtain the estimate

\[
\|W'(s)\|_{E_2^{1, \beta + \varepsilon - 1/2}(K)} \leq c \|F\|_{V_0^{1, \beta}(K)}
\]

with a constant \(c\) independent of \(s\). The same estimate can be easily proved for the derivative with respect to \(s\) of the functions \((1 - \Psi_s) c_{j,k}(s) u_{j,k}^{-1}(x, s)\) and, in the case \(\mu_j > \beta - 3/2\), for the derivatives of the functions \((1 - \Psi_s) c_{j,k}(s) u_{0,k}^{-1}(x, s)\). Furthermore, it follows from Lemmas 2.16 and 2.18 that

\[
\left\| \frac{d}{ds} \Psi_s U(s) \right\|_{V_2^{1, \beta}(K)} \leq c \|F\|_{V_0^{1, \beta}(K)}
\]

for \(|s| = 1\). Since \(V(s)\) is given by (42) and the spaces \(E_2^{2, \beta + \varepsilon - 1/2}(K)\) and \(V_2^{2, \beta}(K)\) are subspaces of \(V_2^{3/2, \beta - 1/2}(K)\), it follows that \(V'(s) \in V_2^{3/2, \beta - 1/2}(K)\). In the case \(|s| = 1\), we obtain the estimate

\[
\|V'(s)\|_{V_2^{3/2, \beta - 1/2}(K)} \leq c |s|^{-1} \|F\|_{V_0^{1, \beta}(K)}
\]

with a constant \(c\) independent of \(F\) and \(s\).

It remains to prove the estimates (39) and (40) for arbitrary \(s \neq 0\), \(\Re s \geq 0\). Let \((U, P)\) be the solution of the problem (25). For an arbitrary positive real number \(a\), we define

\[
U_a(x, s) = U(a^{-1/2}x, as), \quad P_a(x, s) = a^{-1/2} P(a^{-1/2}x, as), \quad F_a(x, s) = a^{-1} F(a^{-1/2}x, as).
\]

Then

\[
(a^{-1} s - \Delta) U_a(x, a^{-1} s) + \nabla P(x, a^{-1} s) = F(x, a^{-1} s), \quad \nabla \cdot U_a(x, a^{-1} s) \text{ in } K
\]

and \(U_a = 0\) on \(\partial K\). As was shown above, the function \(U_a\) admits the decomposition

\[
U_a(x, a^{-1} s) = (1 - \Psi_s/a) \sum_{j \in I_\beta} \sum_{k=1}^{\sigma_j} d_{j,k}(s, a) u_{0,k}^{-1}(x, a^{-1} s) + V_a(x, a^{-1} s),
\]

where \(V_a(\cdot, a^{-1} s) \in V_2^{3/2, \beta - 1/2}(K)\) and

\[
d_{j,k}(s, a) = -\frac{s}{(1 + 2\mu_j)a} \int_{K} F_a(x) \cdot V_{j,k}(x, a^{-1} s) \, dx.
\]

For \(|s| = a\), the functions \(V_a(\cdot, a^{-1} s)\) and \(d_{j,k}(s, a)\) satisfy the estimates

\[
\|V_a(\cdot, a^{-1} s)\|_{V_2^{3/2, \beta - 1/2}(K)} + \|V_a'(\cdot, a^{-1} s)\|_{V_2^{3/2, \beta - 1/2}(K)} + \sum |d_{j,k}(s, a)|
\]

\[
\leq c \|F_a\|_{V_2^{1, \beta}(K)},
\]

(44)

where \(V'_a(x, s) = \partial_s V_a(x, s)\). It can be easily seen that

\[
u_{0,k}^{-1}(x, a^{-1} s) = a^{-\mu_j/2} u_{0,k}^{-1}(a^{-1/2}x, s)\quad \text{and} \quad V_{j,k}(x, a^{-1} s) = a^{(\mu_j+1)/2} V_{j,k}(a^{-1/2}x, s).
\]
Consequently, with the substitution \( a^{-1/2} x = y \), we obtain

\[
d_{j,k}(s,a) = -\frac{a^{\mu_j/2}s}{2\mu_j + 1} \int_K F(y) \cdot V^{(j,k)}(y,s) \, dy = a^{\mu_j/2} c_{j,k}(s)
\]

and

\[
U(y,s) = (1 - \Psi_s(y)) \sum_j \sum_k c_{j,k}(s) u_0^{(-j,k)}(y,s) + V_a(a^{1/2}y,a^{-1}s).
\]

The last equality together with (37) imply \( V(y,s) = V_a(a^{1/2}y,a^{-1}s) \). Using (44) and the equalities

\[
\| F_a \|^2_{V_{2,\beta}^0(K)} = a^{\beta - 1/2} \| F \|^2_{V_{2,\beta}^0(K)}, \quad \| V_a(\cdot,a^{-1}s) \|^2_{V_{2,\beta-1/2}^{3/2}(K)} = a^{\beta - 1/2} \| V(\cdot,s) \|^2_{V_{2,\beta-1/2}^{3/2}(K)}
\]

and

\[
\| V_a(\cdot,a^{-1}s) \|^2_{V_{2,\beta-1/2}^{3/2}(K)} = a^{\beta + 3/2} \| V_a(\cdot,s) \|^2_{V_{2,\beta-1/2}^{3/2}(K)}
\]

we obtain (39) and (40). The proof is complete. \( \square \)

Let \( \zeta_\nu \) be the same smooth functions on \( K \) as in Sect. 2.1. Analogously to Lemma 2.15, we obtain the following assertion.

**Lemma 2.20.** Suppose \( \frac{1}{2} - \lambda_1 < \gamma \leq \beta < \lambda_1 + \frac{3}{2} \), \( \gamma < \mu_2 + \frac{1}{2} \), \( \gamma \neq \frac{3}{2} \), \( Re s \geq 0 \) and \( s \neq 0 \). If \( F \in V_{2,\beta}^0(K) \) and \( (U_\nu(s),P_\nu(s)) \) is the solution of the problem (29) in the space \( E_{2,\beta,\gamma}^2(K) \times V_{2,\beta,\gamma}^1(K) \), then

\[
U_\nu(s) = (1 - \Psi_s) \sum_j \sum_{k=1}^{\sigma_j} c_{j,k,\nu}(s) u_0^{(-j,k)}(\cdot,s) + V_\nu(s),
\]

where \( V_\nu^{(k)}(s) \in V_{2,\beta-1/2}^{3/2}(K) \) for \( k = 1 \) and \( k = 2 \) and

\[
c_{j,k,\nu}(s) = -\frac{s}{1 + 2\mu_j} \int_K \zeta_\nu(x) F(x) \cdot V^{(j,k)}(x,s) \, dx.
\]

Furthermore,

\[
\| \zeta_\nu V_\nu^{(k)}(s) \|^2_{V_{2,\beta-1/2}^{3/2}(K)} + |s| \| \zeta_\nu V_\nu'(s) \|^2_{V_{2,\beta-1/2}^{3/2}(K)} \leq c 2^{-\varepsilon|\mu - \nu|} \| \zeta_\nu F \|^2_{V_{2,\beta}^0(K)}
\]

with a certain positive \( \varepsilon \), where \( c \) is independent of \( F,s,\mu,\nu \).

### 3. The Time-Dependent Problem

We pass to the problem (1), (2).

#### 3.1. Existence of Solutions of the Equation \( \nabla \cdot u = g \)

Suppose that \( g \in L_q(\mathbb{R}_+;V_{p,\beta}^1(K)) \), \( \partial_t g \in L_q(\mathbb{R}_+;(V_{p,-\beta}^1(K))^*) \) and \( g(x,0) = 0 \) for \( x \in K \). Our goal is to show that there exists a function \( u \in W_{q,p,\beta}^{2,1}(Q) \) satisfying the equation \( \nabla \cdot u = g \) in \( Q \), the initial condition \( u(x,0) = 0 \) for \( x \in K \) and the boundary condition \( u(x,t) = 0 \) for \( x \in \partial K \), \( t > 0 \).

We show this by means of solvability and regularity results for the stationary Stokes system in a thin cone. But for this, it is necessary to show that the eigenvalues of the pencil \( \mathcal{L} \), except \( \lambda = 1 \) and \( \lambda = -2 \) lie outside a wide strip \( |Re \lambda + \frac{1}{2}| < m \) of the complex plane if the domain \( \Omega \) on the sphere \( S^2 \) is small.

Let \( r, \varphi, \theta \) denote the spherical coordinates of the point \( x \), and let \( u_r, u_\varphi, u_\theta \) be the spherical components of the vector function \( u = (u_1, u_2, u_3) \). By [10, Lemma 3.2.1], the vector function \( u \) belongs to
\[ \mathcal{W}^1_2(\Omega) \] if and only if \( u_r \in \mathcal{W}^1_2(\Omega) \) and \( u_\omega = (u_\varphi, u_\theta) \in \mathcal{H}^1_2(\Omega) \), where \( \mathcal{H}^1_2(\Omega) \) is the completion of the set \( C^\infty_0(\Omega) \) with respect to the norm
\[
\| u_\omega \|_{\mathcal{H}^1_2(\Omega)} = \left( Q(u_\omega, u_\omega) + \| u_\omega \|^2_{L^2(\Omega)} \right)^{1/2}.
\]

Here, \( Q \) denotes the sesquilinear form
\[
Q(u_\omega, v_\omega) = \int_\Omega \left( \partial_\theta u_\theta \cdot \partial_\theta v_\theta + \partial_\varphi u_\varphi \cdot \partial_\varphi v_\varphi + \frac{1}{\sin^2 \theta} \left( \partial_\varphi u_\varphi - \cos \theta u_\varphi \right) \cdot \left( \partial_\varphi v_\varphi - \cos \theta v_\varphi \right) + \frac{1}{\sin^2 \theta} \left( \partial_\varphi u_\varphi + \cos \theta u_\varphi \right) \cdot \left( \partial_\varphi v_\varphi + \cos \theta v_\varphi \right) \right) \, d\omega,
\]
where \( d\omega = \sin \theta \, d\theta \, d\varphi \). We denote by
\[
\nabla_\omega \cdot u_r = \frac{1}{\sin \theta} \left( \partial_\theta (\sin \theta u_\theta) + \partial_\varphi u_\varphi \right) \quad \text{and} \quad \nabla_\omega u_r = \left( \sin \theta \right)^{-1} \partial_\varphi u_r
\]
the spherical divergence of \( u_\omega \) and spherical gradient of \( u_r \), respectively. As was shown in [11] (see also [10, Subsection 5.2.1]), the eigenvectors \((u_r, u_\omega, p)\) of the pencil \( L \) corresponding to the eigenvalue \( \lambda \) satisfy the integral identity
\[
Q(u_\omega, v_\omega) + \int_\Omega (\nabla_\omega u_r) \cdot \nabla_\omega v_r \, d\omega - 2 \int_\Omega \left( (\nabla_\omega u_r) \cdot v_\omega - (\nabla_\omega \cdot u_\omega) \right) \, d\omega
\]
\[
- \int_\Omega \left( (\lambda^2 + \lambda - 2) u_r v_r + (\lambda^2 + \lambda - 1) u_\omega \cdot v_\omega \right) \, d\omega
\]
\[
- \int_\Omega \left( p (\nabla_\omega \cdot v_\omega + (1 - \lambda) v_r) + (\nabla_\omega \cdot u_\omega + (\lambda + 2) u_r) \right) \, d\omega = 0 \quad (45)
\]
for all \( u_r \in \mathcal{W}^1_2(\Omega) \), \( u_\omega \in \mathcal{H}^1_2(\Omega) \) and \( q \in L^2(\Omega) \).

We may assume without loss of generality that \( \Omega = K \cap S^2 \) does not contain the south pole \((0, 0, -1)\). Then we define the domain \( \Omega_\delta \) on the unit sphere \( S^2 \) as follows. Let \( K \) be the set of all \((\varphi, \theta) \in [0, 2\pi) \times [0, \pi)\) such that \((\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta) \in \Omega\). For arbitrary positive \( \delta < 1 \), we define \( \Omega_\delta \) as the set of all \( \omega \in S^2 \) which have the form \( \omega = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta) \), where \((\varphi, \delta^{-1} \theta) \in K \) (see Fig. 1).

The above introduced operator pencil for the domain \( \Omega_\delta \) is denoted by \( L_\delta \). Using the representation (45), we can prove the following lemma on the eigenvalues of the pencil \( L_\delta \).

**Lemma 3.1.** Suppose that \( \Omega \subset S^2 \setminus \{(0, 0, -1)\} \) and \( m > \frac{3}{2} \). Then there exists a positive number \( \delta_0 \) depending on \( \Omega \) and \( m \) such that the strip \( \text{Re} \lambda + \frac{1}{2} \leq m \) contains only the simple eigenvalues \( \lambda = 1 \) and \( \lambda = -2 \) of the pencil \( L_\delta \) if \( \delta < \delta_0 \).

**Proof.** First note that for arbitrary \( g \in L^2(\Omega_\delta) \), \( \int_{\Omega_\delta} g \, d\omega = 0 \), there exists a function \( u_\omega \in \mathcal{H}^1_2(\Omega_\delta) \) such that
\[
\nabla_\omega \cdot u_\omega = g \quad \text{in} \quad \Omega_\delta \quad \text{and} \quad \| u_\omega \|_{\mathcal{H}^1_2(\Omega_\delta)} \leq c \| g \|_{L^2(\Omega_\delta)}.
\]
Here, the constant \( c \) can be chosen independent of \( \delta \). Indeed, we may assume that \( \theta < \pi - \varepsilon \) with a certain positive \( \varepsilon \) for \((\varphi, \theta) \in G\). Then the function \( h(\varphi, \theta) = g(\varphi, \delta \theta) \left( \sin \theta \right)^{-1} \sin(\delta \theta) \) belongs to \( L^2(\Omega) \) for arbitrary \( g \in L^2(\Omega_\delta) \) and
\[
\| h \|_{L^2(\Omega)} \leq c \| g \|_{L^2(\Omega_\delta)}
\]
with a constant \( c \) independent of \( \delta \). Furthermore, the integral of \( h \) over \( \Omega \) is zero. If \( u_\omega = (u_\theta, u_\varphi) \) is a solution of the equation
\[
\nabla_\omega \cdot u_\omega = \frac{1}{\sin \theta} \left( \partial_\theta (\sin \theta u_\theta) + \partial_\varphi v_\varphi \right) = h \quad \text{in} \quad \Omega,
\]
then \( u_{\theta}(\varphi, \theta) = \delta v_{\theta}(\varphi, \delta^{-1}\theta) (\sin \theta)^{-1} \sin(\delta^{-1}\theta) \) and \( u_{\varphi}(\varphi, \theta) = v_{\varphi}(\varphi, \delta^{-1}\theta) \) satisfy the equation \( \nabla_\omega \cdot u_\omega = g \) in \( \Omega_\delta \). Furthermore, one can easily verify that

\[
\|u_\omega\|_{H^1_0(\Omega_\delta)} \leq c \|v_\omega\|_{H^1_0(\Omega)}
\]

with a constant \( c \) independent of \( \delta \).

By [10, Theorems 5.2.2 and 5.3.1], it suffices to prove the nonexistence of real eigenvalues except \( \lambda = 1 \) in the interval \(-\frac{1}{2} < \lambda < m - \frac{1}{2} \) for small \( \delta \). Suppose that \( u_r \) and \( u_\omega \) satisfy (45) for arbitrary \( v_r \in \tilde{W}^{1,2}(\Omega) \) and \( v_\omega \in H^1_0(\Omega) \times L^2(\Omega) \). If we set \( v_r = 0 \) and \( v_\omega = 0 \), we obtain the equation

\[
\nabla_\omega \cdot u_\omega + (\lambda + 2) u_r = 0 \quad \text{in} \quad \Omega_\delta.
\]

In particular, it follows that

\[
\int_{\Omega_\delta} u_r \, d\omega = 0.
\]

Thus, for \( v_r = u_r \) and \( v_\omega = 0 \), the integral identity (45) implies

\[
\int_\Omega |\nabla_\omega u_r|^2 \, d\omega - (\lambda^2 + 3\lambda + 2) \int_\Omega |u_r|^2 \, d\omega = (1 - \lambda) \int_{\Omega_\delta} (p - \hat{p}) \, d\omega,
\]

where \( \hat{p} \) is the mean value of \( p \). This implies

\[
|\lambda - 1| \int_{\Omega_\delta} |(p - \hat{p}) \, u_r| \, d\omega \geq (\Lambda_\delta - \lambda^2 - 3\lambda - 2) \int_{\Omega_\delta} |u_r|^2 \, d\omega,
\]

where

\[
\Lambda_\delta = \inf_{0 \neq u_r \in \tilde{W}^{1,2}(\Omega_\delta)} \frac{\int_{\Omega_\delta} |\nabla_\omega u_r|^2 \, d\omega}{\int_{\Omega_\delta} |u_r|^2 \, d\omega}
\]

is large if \( \delta \) is small.
We estimate $u_\omega$ and $p - \hat{p}$. For $v_r = 0$, the identity (45) takes the form

$$Q(u_\omega, v_\omega) - (\lambda^2 + \lambda - 1) \int_{\Omega_\delta} u_\omega \cdot \vec{v}_\omega \, d\omega - \int_{\Omega_\delta} (p - \hat{p} - 2u_r) \nabla \cdot \vec{v}_\omega \, d\omega = 0. \quad (47)$$

Let $w_\omega$ be a function $\hat{h}^{1/2}(\Omega_\delta)$ satisfying the equation

$$\nabla \cdot w_\omega = -(\lambda + 2) u_r \quad \text{in} \quad \Omega_\delta$$

and the estimate (see the statement at the beginning of the proof)

$$\|w_\omega\|_{h^{1/2}(\Omega_\delta)} \leq C_1 |\lambda + 2| \|u_r\|_{L^2(\Omega_\delta)}$$

with a constant $C_1$ independent of $\delta$. Then $\nabla \cdot (u_\omega - w_\omega) = 0$. If we insert $v_\omega = u_\omega - w_\omega$ into (47), we obtain

$$Q(u_\omega, u_\omega) - Q(u_\omega, w_\omega) - (\lambda^2 + \lambda - 1) \int_{\Omega_\delta} u_\omega \cdot (\vec{u}_\omega - \vec{w}_\omega) \, d\omega = 0.$$

For small $\delta$, the $L^2(\Omega_\delta)$-norm of $u_\omega$ is small compared to $\sqrt{Q(u_\omega, u_\omega)}$. Consequently,

$$\|u_\omega\|_{h^{1/2}(\Omega_\delta)} \leq c \|w_\omega\|_{h^{1/2}(\Omega_\delta)} \leq C_2 |\lambda + 2| \|u_r\|_{L^2(\Omega_\delta)}$$

for small $\delta$, where $C_2$ is independent of $\delta$. Finally, let $v_\omega$ be a function in $\hat{h}^{1/2}(\Omega_\delta)$ satisfying the equation

$$\nabla \cdot v_\omega = p - \hat{p} - 2u_r \quad \text{in} \quad \Omega_\delta$$

and the estimate

$$\|v_\omega\|_{h^{1/2}(\Omega_\delta)} \leq C_1 |p - \hat{p} - 2u_r|_{L^2(\Omega_\delta)}.$$

Then (47) implies

$$\|p - \hat{p} + 2u_r\|^2_{L^2(\Omega_\delta)} = Q(u_\omega, v_\omega) - (\lambda^2 + \lambda - 1) \int_{\Omega_\delta} u_\omega \cdot \vec{v}_\omega \, d\omega \leq C_3 \|u_\omega\|_{h^{1/2}(\Omega_\delta)} \|p - \hat{p} - 2u_r\|_{L^2(\Omega_\delta)}$$

with a constant $C_3$ independent of $\delta$. This together with the above estimate for the norm of $u_\omega$ yields

$$\|p - \hat{p}\|_{L^2(\Omega_\delta)} \leq (2 + C_1 C_3 |\lambda + 2|) \|u_r\|_{L^2(\Omega_\delta)}.$$

For $u_r \neq 0$, this contradicts (46) if $-1/2 < \lambda < m - 1/2$ and $\Lambda_\delta$ is large, i.e., $\delta$ is small. If $u_r = 0$, then it follows that $p - \hat{p} = 0$ and $u_\omega = 0$. In this case, (45) holds for arbitrary $(v_r, v_\omega) \in \hat{h}^{1/2}(\Omega_\delta) \times \hat{h}^{1/2}(\Omega_\delta)$ only if $\lambda = 1$. This proves the lemma.

The last lemma together with regularity result for solutions of the stationary Stokes system allow us to prove the following assertion.

**Lemma 3.2.** Suppose that $g \in L_\eta(\mathbb{R}_+; V^1_{p,\beta}(K))$, $\partial_1 g \in L_\eta(\mathbb{R}_+; (V^1_{p',-\beta}(K))^*)$, $g(x, 0) = 0$ for $x \in K$, and that $\beta + \frac{2}{p}$ is not equal to one of the numbers $-1, 1, 2, 4$. In the case $2 < \beta + \frac{2}{p} < 4$, we assume that $g$ satisfies the condition (8). Then there exists a vector function $u \in W^{2,1}_{q,p,\beta}(Q)$ such that

$$\nabla \cdot u = g \quad \text{in} \quad Q = K \times \mathbb{R}_+, \quad u(x, t) = 0 \quad \text{for} \quad x \in \partial K, \quad u(x, 0) = 0 \quad \text{for} \quad x \in K$$

and

$$\|u\|_{W^{2,1}_{q,p,\beta}(Q)} \leq c \left(\|g\|_{L_\eta(\mathbb{R}_+; V^1_{p,\beta}(K))} + \|\partial_1 g\|_{L_\eta(\mathbb{R}_+; (V^1_{p',-\beta}(K))^*)}\right) \quad (48)$$

with a constant $c$ independent of $g$.
Proof. First, we prove the assertion for the cone \( K_\delta = \{ x : \omega = x/|x| \in \Omega_\delta \} \), where \( \Omega_\delta \) is the above defined subset of the unit sphere and \( \delta \) is a small positive number. By Lemma 3.1, the strip \(-\beta - \frac{3}{p} \leq \Re \lambda \leq 2 - \beta - \frac{3}{p}\) contains at most the simple eigenvalues \( \lambda = 1 \) and \( \lambda = -2 \) of the pencil \( \mathcal{L}_\delta(\lambda) \) if \( \delta \) is sufficiently small. By the assumptions of the lemma, the lines \( \Re \lambda = -\beta - \frac{3}{p} \) and \( \Re \lambda = 2 - \beta - \frac{3}{p} \) (and, consequently, also the line \( \Re \lambda = 2 + \beta - \frac{3}{p} \)) are free of eigenvalues of this pencil. Thus, the operator

\[
(V^2_{p',-\beta}(K_\delta) \cap \tilde{V}^1_{p',-\beta-1}(K_\delta)) \times V^1_{p,-\beta}(K_\delta) \ni (V, Q) \\
\rightarrow (-\Delta V + \nabla Q, -\nabla \cdot V) \in V^0_{p',-\beta}(K_\delta) \times V^1_{p',-\beta}(K_\delta)
\]

of the stationary Stokes system is an isomorphism (see, e.g., [26, Chapter 3, Theorem 5.2]). This means in particular that for every \( t > 0 \), there exists a uniquely determined weak solution \((\mathbf{v}(\cdot,t), \mathbf{q}(\cdot,t))\) of the problem

\[
-\Delta \mathbf{v}(\cdot,t) + \nabla \mathbf{q}(\cdot,t) = 0, \quad \nabla \cdot \mathbf{v}(\cdot,t) = \partial_t \mathbf{g}(\cdot,t) \quad \text{in} \quad K_\delta, \quad \mathbf{v}(\cdot,t) = 0 \quad \text{on} \quad \partial K_\delta
\]

in the space \( V^0_{p,\beta}(K_\delta) \times (V^1_{p',-\beta}(K_\delta))^* \), i.e., a vector function \((\mathbf{v}(\cdot,t), \mathbf{q}(\cdot,t))\) satisfying the equation

\[
\int_{K_\delta} \left( \mathbf{v} \cdot (-\Delta \mathbf{V} + \nabla \mathbf{Q}) - \mathbf{q} \nabla \cdot \mathbf{V} \right) dx = \int_{K_\delta} \partial_t \mathbf{g}(x,t) Q(x) dx
\]

for all \( V \in V^2_{p',-\beta}(K_\delta) \), \( Q \in V^1_{p',-\beta}(K_\delta) \), \( V = 0 \) on \( \partial K_\delta \{0\} \). This solution satisfies the estimate

\[
\|\mathbf{v}(\cdot,t)\|_{V^0_{p,\beta}(K_\delta)} + \|\mathbf{q}(\cdot,t)\|_{(V^1_{p',-\beta}(K_\delta))^*} \leq c \|\partial_t \mathbf{g}(\cdot,t)\|_{(V^1_{p',-\beta}(K_\delta))^*} \tag{49}
\]

for all \( t \), where \( c \) is independent of \( t \). Obviously, the functions

\[
\mathbf{u}(x,t) = \int_0^t \mathbf{v}(x,\tau) d\tau, \quad \mathbf{p}(x,t) = \int_0^t \mathbf{q}(x,\tau) d\tau
\]

satisfy the equations

\[
-\Delta \mathbf{u}(x,t) + \nabla \mathbf{p}(x,t) = 0, \quad \nabla \cdot \mathbf{u}(x,t) = \mathbf{g}(x,t) \quad \text{for} \quad x \in K_\delta, \quad \mathbf{u}(x,t) = 0 \quad \text{for} \quad x \in \partial K_\delta.
\]

Furthermore, \( \mathbf{u}(x,0) = 0 \) for \( x \in K_\delta \). Clearly, \( \mathbf{u}(\cdot,t) \in V^0_{p,\beta}(K_\delta) \) and \( \mathbf{p}(\cdot,t) \in (V^1_{p',-\beta}(K_\delta))^* \) for all \( t \). Since \( \mathbf{g}(\cdot,t) \in V^1_{p,\beta}(K_\delta) \), it follows that \( \mathbf{u}(\cdot,t) \in V^0_{p,\beta}(K_\delta) \) and \( \mathbf{p}(\cdot,t) \in V^1_{p,\beta}(K_\delta) \) for all \( t \) if the strip \(-\beta - \frac{3}{p} < \Re \lambda < 2 - \beta - \frac{3}{p}\) is free of eigenvalues of the pencil \( \mathcal{L}_\delta \). If the strip \(-\beta - \frac{3}{p} < \Re \lambda < 2 - \beta - \frac{3}{p}\) contains the eigenvalue \( \lambda = 1 \), then we conclude that \( \mathbf{u}(\cdot,t) \in V^2_{p,\beta}(K_\delta) \) and \( \mathbf{p}(\cdot,t) = c(t) \in V^1_{p,\beta}(K_\delta) \) with a function \( c \) of the variable \( t \). This follows from the fact that the eigenvectors of the pencil \( \mathcal{L}_\delta \) corresponding to the eigenvalue \( \lambda = 1 \) are the pairs \((0,c)\) with constant \( c \). Suppose that the strip \(-\beta - \frac{3}{p} \leq \Re \lambda \leq 2 - \beta - \frac{3}{p}\) contains the eigenvalue \( \lambda = -2 \). Then the function \( u \) admits the decomposition

\[
\mathbf{u}(x,t) = c(t) r^{-2} \phi(\omega) + \mathbf{w}(x,t), \quad \text{where} \quad \mathbf{w}(\cdot,t) \in V^2_{p,\beta}(K_\delta),
\]

\((\phi, \psi)\) is an eigenfunction corresponding to eigenvalue \( \lambda = -2 \) and

\[
c(t) = C \int_{K_\delta} \mathbf{g}(x,t) dt
\]

with a certain constant \( C \). Here, we used the formula for the coefficients in the asymptotics of solutions of elliptic problems (see, e.g., [26, Chapter 3]) and the fact that the eigenvectors corresponding to the eigenvalue \( \lambda = 1 \) are multiples of the constant vector \((0,1)\). Since \( 2 < \beta + \frac{3}{p} < 4 \) it follows from the assumptions of the lemma that \( c(t) = 0 \). Thus in all three cases, we conclude that \( \mathbf{u}(\cdot,t) \in V^2_{p,\beta}(K_\delta) \) for all \( t \). Furthermore, the estimate

\[
\|\mathbf{u}(\cdot,t)\|_{V^2_{p,\beta}(K_\delta)} \leq c \|\mathbf{g}(\cdot,t)\|_{V^1_{p,\beta}(K_\delta)} \tag{50}
\]

holds with a constant \( c \) independent of \( \mathbf{g} \) and \( t \). Integrating (49) and (50) with respect to the variable \( t \), we obtain (48). Thus the lemma is proved for the cone \( K_\delta \) if \( \delta \) is sufficiently small.
Now, let \( g \) be a function on \( K \times \mathbb{R}_+ \), and let \( \delta \) be sufficiently small. We define the function \( h \) on \( K_\delta \times \mathbb{R}_+ \) in spherical coordinates \( r, \varphi, \theta \) as

\[
h(r, \varphi, \theta, t) = g(r, \varphi, \delta^{-1} \theta, t) (\sin \theta)^{-1} \sin(\delta^{-1} \theta).
\]

Then

\[
\|h\|_{L_q(K_\delta)} + \|\partial_r h\|_{L_q(K_\delta)} \leq c \left( \|g\|_{L_q(K)} + \|\partial_r g\|_{L_q(K)} \right)
\]

with a constant \( c \) depending on \( \delta \). As was shown above, there exists a vector function \( v \) on \( K_\delta \) satisfying the equation \( \nabla \cdot v = h \) and the estimate

\[
\|v\|_{W^{2,1}_{q,p,\beta}(Q)} \leq c \left( \|h\|_{L_q(K)} + \|\partial_r h\|_{L_q(K)} \right).
\]

This means that the spherical components \( v_r, v_\varphi \) and \( v_\theta \) of \( v \) satisfy the equation

\[
\frac{1}{r^2} \frac{\partial (r^2 v_r \sin \theta)}{\partial r} + \frac{1}{r} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} = h(r, \varphi, \theta, t) \sin \theta.
\]

Then the functions

\[
u_r(r, \varphi, \theta, t) = v_r(r, \varphi, \delta \theta, t) (\sin \theta)^{-1} \sin(\delta \theta), \quad u_\theta(r, \varphi, \theta, t) = \delta^{-1} v_\theta(r, \varphi, \delta \theta, t) (\sin \theta)^{-1} \sin(\delta \theta)
\]

and

\[
u_\varphi(r, \varphi, \theta, t) = v_\varphi(r, \varphi, \delta \theta, t)
\]

satisfy the equation

\[
\frac{1}{r^2} \frac{\partial (r^2 u_r \sin \theta)}{\partial r} + \frac{1}{r} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} + \frac{1}{r} \frac{\partial u_\varphi}{\partial \varphi} = g(r, \varphi, \theta, t) \sin \theta.
\]

This means that the vector function \( u \) with the spherical components \( u_r, u_\varphi \) and \( u_\theta \) is a solution of the equation \( \text{div} u = g \). Furthermore, there exists a constant \( c \) depending on \( \delta \) such that

\[
\|u\|_{W^{2,1}_{q,p,\beta}(Q)} \leq c \|v\|_{W^{2,1}_{q,p,\beta}(Q)}.
\]

Thus, \( u \) satisfies (48).

**Remark 3.1.** Obviously, the operator \( g \to u \) in the last lemma is linear. Furthermore, using regularity results for the solutions of the stationary Stokes system, one can easily prove that \( u \in W^{2,1}_{q,p,\beta}(Q) \cap W^{2,1}_{2,\gamma}(Q) \) if \( g \in L_q(K) \cap L_2(K) \) and

\[
\partial_t g \in L_q(K) \cap L_2(K), \quad \partial_\varphi g \in L_q(K) \cap L_2(K).
\]

**3.2. A Local Estimate**

The following lemma can be easily derived from solvability and uniqueness assertions for the Stokes system in bounded domains with smooth boundary which can be found in [32].

**Lemma 3.3.** Let \( a \) be an arbitrary positive real number, and let \( (u, p) \in W^{2,1}_{q,p,\beta}(Q) \times L_2(K) \) be a solution of the problem (1), (2) such that \( u(x, t), p(x, t) = 0 \) for \( |x| < a/2 \) and \( |x| > 2a \). Furthermore, let \( f \in L_{q,p,\beta}(Q), \ g \in L_q(K), \) and \( \partial_\varphi g \in L_q(K) \). Then \( u \in W^{2,1}_{q,p,\beta}(Q), \ p \in L_q(K) \) and

\[
\|u\|_{W^{2,1}_{q,p,\beta}(Q)} + \|\nabla p\|_{L_{q,p,\beta}(Q)} \leq c \left( \|f\|_{L_{q,p,\beta}(Q)} + \|g\|_{L_q(K)} + \|\partial_\varphi g\|_{L_q(K)} + \|u\|_{L_{q,p,\beta-2}(Q)} \right).
\]

Here, the constant \( c \) is independent of \( u, p \) and \( a \).
Proof. First, let $\alpha = 1$. Then the functions $u, p, f, g$ are zero outside the area $\frac{1}{2} < |x| < 2$. Using [32, Theorem 3.1], we obtain

$$
\|u\|_{L_q(0,2;W^2_p(K))} + \|\nabla p\|_{L_q(0,2;L_p(K))} \\
\leq c \left( \|f\|_{L_q(0,2;L_p(K))} + \|g\|_{L_q(0,2;W^2_p(K))} + \|\partial_t g\|_{L_q(0,2;W^{1,1}_p(K)^*)} \right).
$$

(52)

Let $\chi_k$ be smooth (of class $C^2$) functions on $\mathbb{R}_+$ such that $\chi_k(t) = 0$ for $t < k$ and $\chi_k(t) = 1$ for $t > k + 1$, $k = 1, 2, \ldots$. Obviously

$$(\partial_t - \Delta) (\chi_k u) + \nabla (\chi_k p) = \chi_k f + \chi_k u, \quad \nabla \cdot (\chi_k u) = \chi_k g \quad \text{in } Q.$$  

Moreover, $(\chi_k u)(x,t) = 0$ for $x \in \partial K \setminus \{0\}$ and $(\chi_k u)(x,k) = 0$ for $x \in K$, $k = 1, 2, \ldots$. Thus, it follows from [32, Theorem 3.1] that

$$
\|\chi_k u\|_{L_q(k,k+2;W^2_p(K))} + \|\chi_k \nabla p\|_{L_q(k,k+2;L_p(K))} \\
\leq c \left( \|\chi_k f + \chi_k u\|_{L_q(k,k+2;L_p(K))} + \|\chi_k g\|_{L_q(k,k+2;W^1_p(K))} + \|\chi_k \partial_t g\|_{L_q(k,k+2;W^{1,1}_p(K)^*)} \right),
$$

where $c$ is independent of $u, p$ and $k$. Consequently,

$$
\|u\|_{L_q(k+1,k+2;W^2_p(K))} + \|\nabla p\|_{L_q(k+1,k+2;L_p(K))} \leq c \left( \|f\|_{L_q(k+1,k+2;L_p(K))} + \|u\|_{L_q(k+1,k+2;L_p(K))} \\
+ \|g\|_{L_q(k+1,k+2;W^1_p(K))} + \|\partial_t g\|_{L_q(k+1,k+2;W^{1,1}_p(K)^*)} \right).
$$

(53)

Summing up the inequalities (52) and (53) for $k = 1, 2, \ldots$, we obtain

$$
\|u\|_{L_q(\mathbb{R}_+;W^2_p(K))} + \|\nabla p\|_{L_q(\mathbb{R}_+;L_p(K))} \leq c \left( \|f\|_{L_q(\mathbb{R}_+;L_p(K))} + \|u\|_{L_q(\mathbb{R}_+;L_p(K))} \\
+ \|g\|_{L_q(\mathbb{R}_+;W^1_p(K))} + \|\partial_t g\|_{L_q(\mathbb{R}_+;W^{1,1}_p(K)^*)} \right).
$$

Since the $W^1_p$ and $V^{1,1}_{p,\beta}$ norms are equivalent on the set of the functions with support in the set $\frac{1}{2} \leq |x| \leq 2$, the estimate (51) is proved for the case $\alpha = 1$.

In the case $\alpha \neq 1$, we introduce the new coordinates $x' = a^{-1}x$, $t' = a^{-2}t$. Obviously, the functions $u(x',t') = u(ax',a^2t')$ and $q(x',t') = a p(ax',a^2t')$ are zero outside the area $\frac{1}{2} < |x| < 2$ and satisfy the equations

$$
\frac{\partial u}{\partial t'} - \Delta u + \nabla x' q = a^2 f(ax',a^2t'), \quad -\nabla x' \cdot v = a g(ax',a^2t')
$$

and the initial and boundary conditions (2). Thus, $u, q$ satisfy the estimate (51), where $f(x',t')$, $g(x',t')$ are replaced by the functions $a^2 f(ax',a^2t')$ and $a g(ax',a^2t')$, respectively. Using the equality

$$
\|u\|^q_{L_q(\mathbb{R}_+;V^{1,1}_{p,\beta}(K))} = a^{2+q(\beta-2)+3q/p} \|u\|^q_{L_q(\mathbb{R}_+;V^{1,1}_{p,\beta}(K))}
$$

and analogous relations for the norm of $p, f$ and $g$, we obtain (51) with a constant $c$ independent of $a$. □

In the sequel, we restrict ourselves to the case $g = 0$, i. e., we consider the problem

$$
\frac{\partial u}{\partial t} - \Delta u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in } K \times (0,\infty),
$$

(54)

$$
u(x,t) = 0 \quad \text{for } x \in \partial K \setminus \{0\}, \quad t > 0, \quad u(x,0) = 0 \quad \text{for } x \in K.
$$

(55)

Let $\zeta_\nu$ be the same cut-off functions as in Sect. 2.1, and let $\eta_\nu = \zeta_{\nu-1} + \zeta_\nu + \zeta_{\nu+1}$ for arbitrary integer $\nu$. Then we can easily deduce the following assertion from Lemma 3.3.
Lemma 3.4. Let \((u, p) \in W_{q,p;\beta}^{2,1}(Q) \times L_q(\mathbb{R}^+; V_{p,\beta}^1(K))\) be a solution of the problem (54), (55), and let 
\[ \sigma > \frac{1}{p} \] 
Then
\[
\|\zeta u\|_{W_{q,p;\beta}^{2,1}(Q)} + \|\zeta p\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} \leq c \left( \|\zeta u\|_{L_q(\mathbb{R}^+; V_{p,\beta}^{1+\sigma}(K))} + \|\eta u\|_{L_q(\mathbb{R}^+; V_{p,\beta-1+\sigma}^1(K))} + \|\eta p\|_{L_q(\mathbb{R}^+; V_{p,\beta-1}^1(K))} \right),
\]
where \(c\) is independent of \(u, p\) and \(\nu\).

Proof. Obviously,
\[
(\partial_t - \Delta)(\zeta u) + \nabla(\zeta u) = f_\nu, \quad \nabla \cdot (\zeta u) = g_\nu \quad \text{in} \quad Q,
\]
where \(f_\nu = \zeta \bar{f} - [\Delta, \zeta \nu] u + p \nabla \zeta u\) and \(g_\nu = u \cdot \nabla \zeta u\). Here, \([\Delta, \zeta \nu]\) denotes the commutator of \(\Delta\) and \(\zeta \nu\), i.e., \([\Delta, \zeta \nu] u = \Delta(\zeta u) - \zeta \nu \Delta u\). By Lemma 3.3,
\[
\|\zeta u\|_{W_{q,p;\beta}^{2,1}(Q)} + \|\zeta p\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} \leq c \left( \|f_\nu\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} + \|g_\nu\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} + \|\partial_t g_\nu\|_{L_q(\mathbb{R}^+; V_{p,\beta-2}^1(K))} \right) + \|\zeta u\|_{L_q(\mathbb{R}^+; V_{p,\beta-1}^1(K))} + \|\zeta p\|_{L_q(\mathbb{R}^+; V_{p,\beta-1}^1(K))}.
\]

Obviously,
\[
\|f_\nu\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} \leq c \|\zeta f\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} + \|\eta u\|_{L_q(\mathbb{R}^+; V_{p,\beta-1}^1(K))} + \|\eta p\|_{L_q(\mathbb{R}^+; V_{p,\beta-1}^1(K))}
\]
and
\[
\|g_\nu\|_{L_q(\mathbb{R}^+; V_{p,\beta}^1(K))} \leq c \|\eta u\|_{L_q(\mathbb{R}^+; V_{p,\beta-1}^1(K))}.
\]
Since \(\partial_t g_\nu = \partial_t u \cdot \nabla \zeta u = r^{-1} \zeta_\nu x \cdot \partial_t u + \partial_t u = f + \Delta u - \nabla p\), we obtain the estimate
\[
\|\partial_t g_\nu(\cdot, t)\|_{V_{p,2-\beta}^1(K)^*} \leq c \|\eta_\nu x \cdot \partial_t u(\cdot, t)\|_{V_{p,2-\beta}^1(K)^*} + c \|\eta_\nu f(\cdot, t)\|_{V_{p,\beta}^1(K)} + \|\eta_\nu p(\cdot, t)\|_{V_{p,\beta-1}^1(K)} + \|\eta_\nu u(\cdot, t)\|_{V_{p,\beta-1+\sigma}^1(K)}
\]
analogously to Lemma 2.8, where \(c\) is independent of \(t\). This proves the lemma. \(\square\)

By (10), there exist positive constants \(c_1\) and \(c_2\) such that
\[
\|u\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))^p} \leq \sum_{\nu = -\infty}^{+\infty} \|\zeta u\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))^p} \leq c_1 \|u\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))^p} \leq c_2 \|u\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))^p}
\]
for arbitrary \(u \in L_p(\mathbb{R}^+; V_{p,\beta}^1(K))\). Thus, we can deduce the following result from Lemma 3.4.

Corollary 3.1. Let \((u, p) \in W_{p,\beta}^{2,1}(Q) \times L_p(\mathbb{R}^+; V_{p,\beta}^1(K))\) be a solution of the problem (54), (55). Then
\[
\|u\|_{W_{p,\beta}^{2,1}(Q)} + \|p\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))} \leq c \left( \|f\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))} + \|u\|_{L_p(\mathbb{R}^+; V_{p,\beta-2}^1(K))} + \|p\|_{L_p(\mathbb{R}^+; V_{p,\beta-1}^1(K))} \right)
\]
with a constant \(c\) independent of \(u\) and \(p\).

Proof. By Lemma 3.4 and (56), we have
\[
\|u\|_{W_{p,\beta}^{2,1}(Q)} + \|p\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))} \leq c \left( \|f\|_{L_p(\mathbb{R}^+; V_{p,\beta}^1(K))} + \|u\|_{L_p(\mathbb{R}^+; V_{p,\beta-1+\sigma}^1(K))} + \|p\|_{L_p(\mathbb{R}^+; V_{p,\beta-1}^1(K))} \right).
\]
Using the inequality
\[
\|u\|_{L_p(\mathbb{R}^+; V_{p,\beta-1+\sigma}^1(K))} \leq \varepsilon \|u\|_{L_p(\mathbb{R}^+; V_{p,\beta}^2(K))} + C(\varepsilon) \|u\|_{L_p(\mathbb{R}^+; V_{p,\beta-2}^1(K))}
\]
(cf. (11)), we obtain the desired result. \(\square\)
3.3. Solvability in Weighted $L_p$ Sobolev Spaces, the Case $p \geq 2$

In this subsection, we assume always that $\beta$ satisfies (5) and $\gamma = \beta + \frac{3}{p} - \frac{3}{2}$, i. e.,

$$2 - \lambda_1 < \beta + \frac{3}{p} < 2 + \min(\mu_2, \lambda_1 + 1), \quad 2 - \frac{3}{p} \neq 2.$$  \hfill (57)

Using Lemma 2.14 and an extension of Mikhlin’s Fourier multiplier theorem to operator-valued functions (see [1, Theorem 6.1.6]), we can prove the following lemma.

**Lemma 3.5.** Suppose that $f \in L_{2;\gamma}(Q)$, where $\gamma$ satisfies the condition (57). Then there exists a unique solution $(u, p) \in W_{2;\gamma}^{2,1}(Q) \times L_2(\mathbb{R}^+; V_{2;\gamma}^1(K))$ of the problem (54), (55) satisfying the estimate

$$\|u\|_{W_{2;\gamma}^{2,1}(Q)} + \|p\|_{L_2(\mathbb{R}^+; V_{2;\gamma}^1(K))} \leq c \|f\|_{L_{2;\gamma}(Q)}.$$  \hfill (58)

If in addition $f \in L_{p,2;\gamma}(Q)$ for some $p$, $1 < p < \infty$, then $u \in W_{p,2;\gamma}^{2,1}(Q)$, $p \in L_2(\mathbb{R}^+; V_{2;\gamma}^1(K))$ and

$$\|u\|_{W_{p,2;\gamma}^{2,1}(Q)} + \|p\|_{L_2(\mathbb{R}^+; V_{2;\gamma}^1(K))} \leq c \|f\|_{L_{p,2;\gamma}(Q)}.$$  \hfill (59)

In both estimates, the constant $c$ is independent of $f$.

**Proof.** For the first assertion, we refer to [16, Theorem 3.1]. Let $U(x, s)$, $P(x, s)$ and $F(x, s)$ be the Laplace transforms of $u(x, t)$, $p(x, t)$ and $f(x, t)$, respectively. Then

$$u(x, t) = \frac{1}{2\pi i} \int_{-\infty}^{+i\infty} e^{st} U(x, s) ds, \quad p(x, t) = \frac{1}{2\pi i} \int_{-\infty}^{+i\infty} e^{st} P(x, s) ds.$$  \hfill (50)

We denote the operator $V^0_{2,\gamma}(K) \ni F \to U \ni V^2_{2,\gamma}(K)$ of the problem (26) by $\Phi(s)$. If we extend $f$ by zero to $K \times (-\infty, 0)$, we have the representation

$$u = \mathcal{F}^{-1}_{\tau \to t} \Phi(i\tau) \mathcal{F}_{t \to \tau} f,$$

where $\mathcal{F}_{t \to \tau}$ denotes the Fourier transform with respect to $t$. By Lemma 2.14, there is the estimate

$$\|\Phi(k)(s)\|_{V^0_{2,\gamma}(K) \to V^2_{2,\gamma}(K)} \leq c |s|^{-k} \quad \text{for Re } s \geq 0, \quad s \neq 0$$

if $k = 0$ or $k = 1$. Using the above mentioned extension of Mikhlin’s multiplier theorem, we conclude that $\mathcal{F}^{-1}_{\tau \to t} \Phi(i\tau) \mathcal{F}_{t \to \tau}$ is a continuous operator from $L_p(\mathbb{R}; V^0_{2,\gamma}(K))$ into $L_p(\mathbb{R}; V^2_{2,\gamma}(K))$ and

$$\|u\|_{L_p(\mathbb{R}; V^2_{2,\gamma}(K))} \leq c \|f\|_{L_p(\mathbb{R}; V^0_{2,\gamma}(K))}$$

with a constant $c$ independent of $f$. Analogously, the estimate for the $L_p(\mathbb{R}^+; V^1_{2,\gamma}(K))$-norm of $p$ holds. Finally, the desired estimate for the $L_{p,2;\gamma}(Q)$-norm of $\partial_t u$ follows from the equation $\partial_t u = f + \Delta u - \nabla p$. \hfill $\square$

Let $\zeta_\nu$ be the same smooth functions on $K$ as in Sects. 2.1 and 2.2. We consider the solution $(u_\nu, p_\nu)$ of the problem

$$\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \nabla p &= \zeta_\nu f, & \nabla \cdot u &= 0 \quad \text{in } K \times (0, \infty), \\
u(x, t) &= 0 \quad \text{for } x \in \partial K, \quad t > 0, \quad u(x, 0) = 0 \quad \text{for } x \in K.
\end{align*}$$  \hfill (58)

(59)

Analogously to Lemma 3.5, we obtain the following assertion.

**Lemma 3.6.** Let $\zeta_\nu f \in L_{2;\gamma}(Q) \cap L_{p,2;\gamma}(Q)$, where $1 < p < \infty$ and $\gamma$ satisfies the condition (57). Furthermore, let $(u_\nu, p_\nu)$ be the solution of the problem (58), (59) in the space $W_{2;\gamma}^{2,1}(K) \times L_2(\mathbb{R}^+; V_{2;\gamma}^1(K))$. Then

$$\|\zeta_\mu u_\nu\|_{W_{2;\gamma}^{2,1}(Q)} + \|\zeta_\nu p_\nu\|_{L_p(\mathbb{R}^+; V_{2;\gamma}^1(K))} \leq c 2^{-\varepsilon|\mu-\nu|} \|\zeta_\nu f\|_{L_{p,2;\gamma}(Q)}$$

with a constant $c$ independent of $\mu, \nu$ and $f$. 

Lemma 3.7. Using the same notation as in the proof of Lemma 3.5, we have

\[ \zeta_\mu u_\nu = F_{T^{-1}_r}^T \zeta_\mu \Phi(tT) \zeta_\mu F_{-r} \]

where

\[ \| \zeta_\mu \Phi(k) s \zeta_\nu \|_{V^2_{\epsilon_2}(K) \rightarrow V^2_{\epsilon_2}(K)} \leq c |s|^{-k} 2^{-\epsilon |\mu - \nu|} \quad \text{for Re} \ s \geq 0, \ s \neq 0 \]

(see Lemma 2.15). Applying the same multiplier theorem as in the proof of Lemma 3.5, we get

\[ \| \zeta_\mu u_\nu \|_{L_p(\mathbb{R}^+: V^1_{2;\gamma}(K))} \leq c 2^{-\epsilon |\mu - \nu|} \| \zeta_\nu f \|_{L_p,2;\gamma}(Q) . \]

The analogous estimate holds for the \( L_p(\mathbb{R}^+: V^1_{2;\gamma}(K)) \)-norm of \( \zeta_\nu p_\nu \). Since \( \partial_t u_\nu = \zeta_\nu f + \Delta u_\nu - \nabla p_\nu \), it follows that

\[ \| \zeta_\mu \partial_t u_\nu \|_{L_p,2;\gamma}(Q) \leq c 2^{-\epsilon |\mu - \nu|} \| \zeta_\nu f \|_{L_p,2;\gamma}(Q) . \]

This proves the lemma.

Suppose now that \( \zeta_\nu f \in L_{p;\beta}(Q) \). Then \( \zeta_\nu f \in L_{p;\beta}(Q) \) for \( 1 < q \leq p, \beta + \frac{3}{p} = \beta' + \frac{3}{q}, \) and

\[ \| \zeta_\nu f \|_{L_{p,\beta}(Q)} \leq c \| \zeta_\nu f \|_{L_{p,\beta}(Q)} \]

with a constant \( c \) independent of \( f \) and \( \nu \). Indeed, by Hölder’s inequality, we have

\[ \| \zeta_\nu f (\cdot, t) \|_{V^0_{p;\beta}(K)} \leq \left( \int_K r^{p\beta} \| \zeta_\nu f \|_p \, dx \right)^{q/p} \left( \int_{2^{\nu-1} \leq |x| < 2^{\nu+1}} r^{-3} \, dx \right)^{(p-q)/p} \]

where \( c \) is independent of \( f, \nu \) and \( t \). In particular,

\[ \| \zeta_\nu f \|_{L_{p,2;\gamma}(Q)} \leq c \| \zeta_\nu f \|_{L_{p,\beta}(Q)} \quad (60) \]

if \( p \geq 2 \) and \( \beta + \frac{3}{p} = \gamma + \frac{3}{2} \).

Lemma 3.7. Suppose that \( \zeta_\nu f \in L_{p;\beta}(Q) \cap L_{2;\gamma}(Q) \), where \( p \geq 2 \) and \( \beta, \gamma \) satisfy the condition (57). Furthermore, let \( (u_\nu, p_\nu) \) be the solution of the problem (58), (59) in the space \( W^{2,1}_{2;\gamma}(Q) \times L_2(\mathbb{R}^+: V^1_{2;\gamma}(K)) \). Then

\[ \| \zeta_\mu u_\nu \|_{L_p(\mathbb{R}^+: V^2_{\epsilon_2}(K))} + \| \zeta_\mu p_\nu \|_{L_p(\mathbb{R}^+: V^1_{\epsilon_2}(K))} \leq c 2^{-\epsilon |\mu - \nu|} \| \zeta_\nu f \|_{L_{p,\beta}(Q)} \quad (61) \]

with a constant \( c \) independent of \( \mu, \nu \) and \( f \).

Proof. Applying Lemma 3.6 and (60), we obtain the estimate

\[ \| \eta_\mu u_\nu \|_{L_p(\mathbb{R}^+: V^2_{\epsilon_2}(K))} + \| \eta_\mu p_\nu \|_{L_p(\mathbb{R}^+: V^1_{\epsilon_2}(K))} \leq c 2^{-\epsilon |\mu - \nu|} \| \zeta_\nu f \|_{L_{p,\beta}(Q)} . \]

Suppose that \( 2 \leq q \leq p, q < 4 \) and \( \frac{1}{q} < \sigma < \frac{3}{q} - \frac{1}{2} \). Then there are the continuous imbeddings

\[ V_{2,\gamma}(K) \subset V^1_{q;\beta' - 1 + \sigma}(K), \quad V_{2;\gamma}(K) \subset V^0_{q,\beta' - 1}(K), \]

where \( \beta' = \gamma + \frac{3}{2} - \frac{3}{q} \) (see Lemma 2.4). Thus, we get

\[ \| \eta_\mu u_\nu \|_{L_p(\mathbb{R}^+: V^1_{q;\beta' - 1 + \sigma}(K))} + \| \eta_\mu p_\nu \|_{L_p(\mathbb{R}^+: V^0_{q,\beta' - 1}(K))} \leq c 2^{-\epsilon |\mu - \nu|} \| \zeta_\nu f \|_{L_{p,\beta}(Q)} , \]

where \( c \) is independent of \( \mu, \nu \) and \( f \). Lemma 3.4 implies

\[ \| \zeta_\mu u_\nu \|_{L_p(\mathbb{R}^+: V^2_{\epsilon_2}(K))} + \| \zeta_\mu p_\nu \|_{L_p(\mathbb{R}^+: V^1_{\epsilon_2}(K))} \leq c 2^{-\epsilon |\mu - \nu|} \| \zeta_\nu f \|_{L_{p,\beta}(Q)} . \]

In the case \( p < 4 \), we can choose \( q = p \), and the lemma is proved. In the case \( p \geq 4 \), we choose \( q \) and \( \sigma \) such that \( 3 < q < 4 \) and \( \frac{1}{p} < \sigma < 1 + \frac{3}{p} - \frac{3}{q} \). From the first part of the proof and from the imbeddings

\[ V_{2,\gamma}(K) \subset V^1_{p,\beta - 1 + \sigma}(K), \quad V_{2;\gamma}(K) \subset V^0_{p,\beta - 1}(K) \]
it follows that
\[ \|\eta \mu u\|_{L^p(\mathbb{R}^+_+; V^{1+s}_{p,\beta-1}(K))} + \|\eta \mu p\|_{L^p(\mathbb{R}^+_+; V^0_{p,\beta-1}(K))} \leq c 2^{-|\nu|} \|\zeta_\nu f\|_{L^{p,\beta}(Q)} . \]
Applying again Lemma 3.4, we obtain (61).

Using the inequalities (56), we can deduce the following result from Lemma 3.7.

**Lemma 3.8.** Suppose that \( f \in L^{p,\beta}(Q) \cap L^{2,\gamma}(Q) \), where \( p \geq 2 \) and \( \beta, \gamma \) satisfy the condition (57). If \((u, p)\) is the solution of the problem (54), (55) in the space \( W^{2,1}_{2,\gamma}(Q) \times L^2(\mathbb{R}^+_+; V^{1,\gamma}_{2,\gamma}(K)) \), then \( u \in W^{2,1}_{p,\beta}(Q) \), \( p \in L^p(\mathbb{R}^+_+; V^{1,\gamma}_{p,\beta}(K)) \) and
\[ \|u\|_{W^{2,1}_{p,\beta}(Q)} + \|p\|_{L^p(\mathbb{R}^+_+; V^{1,\gamma}_{p,\beta}(K))} \leq c \|f\|_{L^{p,\beta}(Q)} \]
with a constant \( c \) independent of \( f \).

**Proof.** Obviously,
\[ u = \sum_{\nu = \infty}^{+\infty} u\nu \text{ and } p = \sum_{\nu = \infty}^{+\infty} p\nu, \]
where \((u\nu, p\nu)\) are the (uniquely determined) solutions of the problem (58), (59) in the space \( W^{2,1}_{2,\gamma}(Q) \times L^2(\mathbb{R}^+_+; V^{1,\gamma}_{2,\gamma}(K)) \). Using (56) and (61), we obtain
\[ \|u\|_{L^p(\mathbb{R}^+_+; V^{2,\beta}_{p,\beta}(K))} \leq c \sum_{\nu = \infty}^{+\infty} \|\zeta \mu u\|_{L^p(\mathbb{R}^+_+; V^{2,\beta}_{p,\beta}(K))} = c \sum_{\nu = \infty}^{+\infty} \|\zeta \nu f\|_{L^{p,\beta}(Q)} \]
\[ \leq c \sum_{\nu = \infty}^{+\infty} \left( \sum_{\nu = \infty}^{+\infty} \|\zeta \nu u\|_{L^p(\mathbb{R}^+_+; V^{2,\beta}_{p,\beta}(K))} \right)^{\frac{1}{\nu}} \leq c \sum_{\nu = \infty}^{+\infty} \left( \sum_{\nu = \infty}^{+\infty} 2^{-|\nu|} \|\zeta f\|_{L^{p,\beta}(Q)} \right)^{\frac{1}{\nu}} . \]
Since the operator of discrete convolution with the kernel \( \{2^{-|k|}\}_{k = \infty}^{+\infty} \) is continuous in \( l^p \), we obtain
\[ \|u\|_{L^p(\mathbb{R}^+_+; V^{2,\beta}_{p,\beta}(K))} \leq c \sum_{\nu = \infty}^{+\infty} \|\zeta f\|_{L^{p,\beta}(Q)} \leq C \|f\|_{L^{p,\beta}(Q)} . \]
Analogously,
\[ \|p\|_{L^p(\mathbb{R}^+_+; V^{1,\beta}_{p,\beta}(K))} \leq c \|f\|_{L^{p,\beta}(Q)} . \]
The analogous estimate for the norm of \( \partial_t u \) in \( L^{p,\beta}(Q) \) follows from the equation \( \partial_t u = f + \Delta u - \nabla p \). This proves the lemma.

Now it is easy to prove the following theorem.

**Theorem 3.1.** Suppose that \( f \in L^{p,\beta}(Q) \), where \( p \geq 2 \) and \( \beta \) satisfies the inequalities (5). Then there exists a unique solution \((u, p)\) of the problem (54), (55) in the space \( W^{2,1}_{p,\beta}(Q) \times L^p(\mathbb{R}^+_+; V^{1,\gamma}_{p,\beta}(K)) \) satisfying the estimate
\[ \|u\|_{W^{2,1}_{p,\beta}(Q)} + \|p\|_{L^p(\mathbb{R}^+_+; V^{1,\gamma}_{p,\beta}(K))} \leq c \|f\|_{L^{p,\beta}(Q)} \]
with a constant \( c \) independent of \( f \).

**Proof.** Existence of solutions. Let \( \gamma = \beta + \frac{3}{p} - \frac{3}{2} \) and let \( \{f_k\} \) be a sequence of functions \( f_k \in L^{p,\beta}(Q) \cap L^{2,\gamma}(Q) \) converging to \( f \) in \( L^{p,\beta}(Q) \). Furthermore, let \((u_k, p_k)\) be the solutions of the problem (54), (55) with the right-hand side \( f_k \) in the space \( W^{2,1}_{2,\gamma}(Q) \times L^2(\mathbb{R}^+_+; V^{1,\gamma}_{2,\gamma}(K)) \). By Lemma 3.8, the sequence \((u_k, p_k)\) converges in \( W^{2,1}_{p,\beta}(K) \times L^p(\mathbb{R}^+_+; V^{1,\gamma}_{p,\beta}(K)) \). The limit \((u, p)\) is a solution of the problem (54), (55) and satisfies the estimate (62).
Uniqueness of the solution. Let \( f = 0 \) and let \((u, p)\) be a solution of the problem (54), (55) in the space \( W^{2,1}_{p,\beta}(Q) \times L_p(\mathbb{R}_+; V^{1,\gamma}_{p,\beta}(K)) \). There exist sequences \( \{u_k\} \) and \( \{p_k\} \) in \( W^{2,1}_{p,\beta}(Q) \cap W^{2,\gamma}_{p,\gamma}(Q) \) and \( L_p(\mathbb{R}_+; V^{1,\gamma}_{p,\beta}(K)) \cap L_2(\mathbb{R}_+; V^{1,\gamma}_{2,\gamma}(K)) \) which converge to \( u \) and \( p \), respectively, satisfy the boundary condition \( u_k(x,t) = 0 \) for \( x \in \partial K \setminus \{0\} \) and the initial condition \( u_k(x,0) = 0 \). Furthermore, let \( f_k = (\partial_t - \Delta) u_k + \nabla p_k \) and \( g_k = \nabla \cdot u_k \). Then the sequences \( \{f_k\}, \{g_k\}, \{\partial_t g_k\} \) converge to zero in \( L_p(\beta)(Q), L_p(\mathbb{R}_+; V^{1,\gamma}_{p,\beta}(K)) \) and \( L_p(\mathbb{R}_+; (V^{1,\gamma}_{p,\beta}(K))^*) \), respectively. Suppose that \( \beta + \frac{3}{p} = \gamma + \frac{3}{2} \). Then by Lemma 3.2 (see also Remark 3.1), there exist functions \( v_k \in W^{2,1}_{p,\beta}(Q) \cap W^{2,\gamma}_{p,\beta}(Q) \) such that \( \nabla \cdot v_k = g_k \) and \( v_k \to 0 \) in \( W^{2,1}_{p,\beta}(Q) \) as \( k \to \infty \). Applying Lemma 3.8, we conclude that the sequences \( \{u_k - v_k\} \) and \( \{p_k\} \) converge to zero in \( W^{2,1}_{p,\beta}(Q) \) and \( L_p(\mathbb{R}_+; V^{1,\gamma}_{p,\beta}(K)) \). Hence, \( u = 0 \) and \( p = 0 \).

### 3.4. Existence of Solutions in \( L_p;\beta-2(Q) \)

Suppose that \( f \in L_{p,\beta}(Q) \), where \( p \geq 2 \) and

\[
2 - \lambda_1 < \beta + \frac{3}{p} = \gamma + \frac{3}{2} < \lambda_1 + 3.
\]

(63)

Our goal is to prove that there exists a vector function \((u, p)\) of the problem (54), (55), where \( u \in L_{p,\beta-2}(Q) \). In the case \( p \geq 2, 2 - \lambda_1 < \beta + \frac{3}{p} < \min(\lambda_1 + 3, \mu_2 + 2) \), \( \beta + \frac{3}{p} \neq 2 \), this follows directly from Theorem 3.1. But we allow now that \( \beta + \frac{3}{p} \geq \mu_2 + 2 \). First, we obtain the following result by means of Lemmas 2.16 and 2.19.

**Lemma 3.9.** Suppose that \( f \in L_{2,\gamma}(Q) \), where \( \gamma \) satisfies the inequalities (63). Then there exists a solution \((u, p)\) of the problem (54), (55), where \( u \) has the form

\[
u = K \frac{f}{\varphi} + v \]

(64)

with a function \( v \in L_2(\mathbb{R}_+; V^{3/2}_{2,\gamma-1}(K)) \) and an integral operator

\[
(K \varphi)(x, t) = \int_0^t \int_K \sum_j \sum_k K_{j,k}(x, y, t - \tau) \varphi(y, \tau) \, dy \, d\tau.
\]

Here, the functions \( K_{j,k} \) satisfy the estimate

\[
|K_{j,k}(x, y, t)| \leq c t^{-3/2} \left(1 + \frac{|x|}{\sqrt{t}} \right)^{-2-\mu_j} \left(1 + \frac{|y|}{\sqrt{t}} \right)^{\mu_j-1} \left(\frac{|y|}{|y| + \sqrt{t}} \right)^{\lambda_1 - \epsilon}
\]

(65)

with arbitrarily small \( \epsilon > 0 \). Furthermore,

\[
\|u\|_{L_{2,\gamma-2}(Q)} + \|v\|_{L_2(\mathbb{R}_+; V^{3/2}_{2,\gamma-1}(K))} \leq c \|f\|_{L_{2,\gamma}(Q)}
\]

with a constant \( c \) independent of \( f \). If moreover \( f \in L_p(\mathbb{R}_+; V^{0,\gamma}_2(K)) \), \( 1 < p < \infty \), then \( v \in L_p(\mathbb{R}_+; V^{3/2}_{2,\gamma-1}(K)) \) and

\[
\|v\|_{L_p(\mathbb{R}_+; V^{3/2}_{2,\gamma-1}(K))} \leq c \|f\|_{L_p(\mathbb{R}_+; V^{0,\gamma}_2(K))},
\]

(66)

where \( c \) is independent of \( f \).

**Proof.** Let \( F(x, s) \) be the Laplace transform of \( f(x, t) \), and let \((U(s), P(s))\) be the (uniquely determined) solution of the parameter-depending problem (26) in the space \( E_{2,\gamma'}( \gamma(K) ) \times V^{1,\gamma}_{2,\gamma}(K) \), \( \frac{1}{2} - \lambda_1 < \gamma' \leq \gamma \), \( \gamma' < \mu_2 + \frac{1}{2} \) (see Lemma 2.16). Then the vector function

\[
(u, p)(x, t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} e^{st} (U, P)(x, s) \, ds
\]
is a solution of the problem (54), (55). By Lemma 2.19, the function $U$ has the decomposition (37) with a remainder $V \in V^{3/2}_{2, \gamma - 1/2}(K)$. Since $V$ satisfies the estimate (39) with $\gamma$ instead of $\beta$, it is the Laplace transform of a function $v \in L_2(\mathbb{R}^+; V^{3/2}_{2, \gamma - 1/2}(K))$. The sum

$$(1 - \Psi_s) \sum_{0 < \mu_j \leq \gamma - 3/2} \sum_{k=1}^{\sigma_j} c_{j,k}(s) v_0(-j,k)(x, s)$$

in the decomposition (37) is the Laplace transform of $(Kf)(x, t)$ (cf. [18, Theorem 3.4]). The estimate (65) is given in [18, Lemma 3.1]. Using Lemma 2.19 and the same multiplier theorem as in the proof of Lemma 3.6, we obtain (66).

We consider the integral operator $K$ in the decomposition (64) of the function $u$.

**Lemma 3.10.** Suppose that $0 < \mu_j \leq \gamma - \frac{3}{2} = \beta - 3 + \frac{3}{p} < \lambda_1$ and that the kernel $K_{j,k}$ satisfies the estimate (65). Then the integral operator

$$(K_{j,k} f)(x, t) = \int_0^t \int_K K_{j,k}(x, y, t - \tau) f(y, \tau) dy d\tau$$

realizes a linear and continuous mapping from $L_{p,\beta}(Q)$ into $L_{p,\beta-2}(Q)$.

**Proof.** Since $3 < \beta + \frac{3}{p} < \lambda_1 + 3 \leq 4$, we can choose a real number $\alpha$ such that

$$\max(2 - 3p, -1 - p\beta) < p\alpha < \min(2 - 2p, -1 - p\beta + p\mu_j + p).$$

Furthermore, there exist real numbers $\alpha_1, \alpha_2$ such that

$$\mu_j - 1 - \alpha - \beta < \alpha_1 < \frac{3}{p} - 3 - \alpha < \alpha_2 < \lambda_1 - \alpha - \beta - \varepsilon$$

if $\varepsilon$ is sufficiently small. We define $\gamma_1 = \mu_j - 1 - \alpha_1$ and $\gamma_2 = \lambda_1 - \alpha_2 - \varepsilon$. By Hölder’s inequality,

$$|(K_{j,k} f)(x, t)| \leq A^{1/p'} B^{1/p},$$

where $p' = p/(p - 1),

A = \int_0^t \int_K (t - \tau)^{-3/2} \left(1 + \frac{|x|}{\sqrt{t - \tau}}\right)^{-p'} \left(1 + \frac{|y|}{\sqrt{t - \tau}}\right)^{p'\alpha_1} \left(\frac{|y|}{|y| + \sqrt{t - \tau}}\right)^{p'\alpha_2} |y|^{p\alpha} dy d\tau$$

and

$$B = \int_0^t \int_K (t - \tau)^{-3/2} \left(1 + \frac{|x|}{\sqrt{t - \tau}}\right)^{-p(1+\mu_j)} \left(1 + \frac{|y|}{\sqrt{t - \tau}}\right)^{p\gamma_1} \times \left(\frac{|y|}{|y| + \sqrt{t - \tau}}\right)^{p\gamma_2} |y|^{-p\alpha} |f(y, \tau)|^p dy d\tau.$$
\[
\leq c \int_{0}^{\infty} \int_{K} |y|^{-p\alpha} |f(y, \tau)|^p \int_{\tau}^{\infty} (t - \tau)^{-3/2} \left( 1 + \frac{|y|}{\sqrt{t - \tau}} \right)^{p\gamma_1} \left( \frac{|y|}{|y| + \sqrt{t - \tau}} \right)^{p\gamma_2} D(t - \tau) \, dt \, dy \, d\tau,
\]
where
\[
D(t - \tau) = \int_{K} |x|^{p(\alpha + \beta) - 2} \left( 1 + \frac{|x|}{\sqrt{t - \tau}} \right)^{-p(1 + \mu_j)} \, dx
\]
\[
= (t - \tau)^{(p\alpha + p\beta + 1)/2} \int_{K} |z|^{p(\alpha + \beta) - 2} (1 + |z|)^{-p(1 + \mu_j)} \, dz = c (t - \tau)^{(p\alpha + p\beta + 1)/2}
\]
since \(p(\alpha + \beta) > -1\) and \(p(\alpha + \beta - 1 - \mu_j) < -1\). This means that
\[
\|K_{j,k}\|_{L_{p,\beta-2}(Q)}^p \leq c \int_{0}^{\infty} \int_{K} |y|^{-p\alpha} |f(y, \tau)|^p E(y, \tau) \, dy \, d\tau,
\]
where
\[
E(y, \tau) = \int_{\tau}^{\infty} (t - \tau)^{(p\alpha + p\beta - 2)/2} \left( 1 + \frac{|y|}{\sqrt{t - \tau}} \right)^{p\gamma_1} \left( \frac{|y|}{|y| + \sqrt{t - \tau}} \right)^{p\gamma_2} \, dt.
\]
With the substitution \(|x|/\sqrt{t - \tau} = s\), we obtain
\[
E(y, \tau) = |y|^{p(\alpha + \beta)} \int_{0}^{\infty} s^{-p(\alpha + \beta) - 1} (1 + s)^{p\gamma_1} \left( \frac{s}{1 + s} \right)^{p\gamma_2} \, ds = c |y|^{p(\alpha + \beta)}
\]
since \(\gamma_1 - \alpha - \beta < 0 < \gamma_2 - \alpha - \beta\). Thus,
\[
\|K_{j,k}\|_{L_{p,\beta-2}(Q)} \leq c \|f\|_{L_{p,\beta}(Q)}.
\]
This proves the lemma. \(\square\)

Let \((u, p)\) be the solution of the problem (54), (55) given in Lemma 3.9. By Lemma 3.9, there exist also solutions \((u_\nu, p_\nu)\) of the problem (58), (59), where \(u_\nu\) has the representation
\[
u_k = \mathcal{K}(\zeta_k) + v_\nu
\]
with a function \(v_\nu \in L_p(\mathbb{R}^+; V^{3/2}_{\alpha, \gamma-1/2}(K))\). Obviously,
\[
u = \sum_{\nu = -\infty}^{+\infty} u_\nu \quad \text{and} \quad v = \sum_{\nu = -\infty}^{+\infty} v_\nu.
\]
By the same arguments as in the proof of 3.9, we can deduce the estimate
\[
\|\zeta_{\nu} v_\nu\|_{L_p(\mathbb{R}^+; V^{3/2}_{\alpha, \gamma-1/2}(K))} \leq c 2^{-\varepsilon|\mu - \nu|} \|\zeta_{\nu} f\|_{L_p(\mathbb{R}^+; V^0_{\alpha, \gamma}(K))}
\]
from Lemma 2.20. Here, the constant \(c\) is independent of \(\nu, \nu\) and \(f\).

**Lemma 3.11.** Let \(f \in L_{p,\beta}(Q)\), where \(p \geq 2\) and \(\beta\) satisfies (63). Then there exists a solution \((u, p)\) of the problem (54), (55) such that \(u \in L_{p,\beta-2}(Q)\) and
\[
\|u\|_{L_{p,\beta-2}(Q)} \leq c \|f\|_{L_{p,\beta}(Q)}
\]
with a constant \(c\) independent of \(f\).

**Proof.** Suppose first that \(f \in L_{p,\beta}(Q) \cap L_{p,\beta}(Q), \gamma + 3/2 = \beta + 3/p\). Then by Lemma 3.9, there exists a solution \((u, p), \) where \(u\) has the representation (64) with a remainder \(v \in L_p(\mathbb{R}^+; V^{3/2}_{\alpha, \gamma-1/2}(K))\). Using the imbedding \(L_p(\mathbb{R}^+; V^{3/2}_{\alpha, \gamma-1/2}(K)) \subset L_p(\mathbb{R}^+; V^0_{p,\beta-2}(K))\) (cf. Lemma 2.4), we obtain
\[
\|v\|^p_{L_{p,\beta-2}(Q)} \leq c \sum_{\mu = -\infty}^{+\infty} \|\zeta_{\mu} v\|^p_{L_{p,\beta-2}(Q)} \leq c \sum_{\mu = -\infty}^{+\infty} \left( \sum_{\nu = -\infty}^{+\infty} \|\zeta_{\mu} v_\nu\|^p_{L_{p,\beta-2}(Q)} \right)^p
\]
\[
\leq c \sum_{\mu = -\infty}^{+\infty} \left( \sum_{\nu = -\infty}^{+\infty} \|\zeta_{\mu} v_\nu\|^p_{L_p(\mathbb{R}^+; V^{3/2}_{\alpha, \gamma-1/2}(K))} \right)^p
\]

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Lemma 3.12. Suppose that \((u, p) \in W_{p; \beta}^{2,1}(Q) \times L_p(\mathbb{R}_+; V_{p; \beta}^{1}(K))\) is a solution of (54), (55), where \(p < 2\), \(2 - \lambda_1 < \beta + \frac{3}{p} < \lambda_1 + 3\). Then

\[
\|u\|_{L_{p; \beta-2}(Q)} \leq c \|f\|_{L_{p; \beta}(Q)}.
\]

Proof. It suffices to prove the estimate (68) for smooth vector functions \((u, p)\) with compact support in \((\overline{K}\setminus\{0\}) \times \mathbb{R}_+.\) Let also \(h\) be a continuous vector function with compact support in \((\overline{K}\setminus\{0\}) \times \mathbb{R}_+.\) From the condition on \(p\) and \(\beta\), it follows that \(2 - \lambda_1 < 2 - \beta + \frac{3}{p} < \lambda_1 + 3\). Thus by Lemma 3.11, there exists a solution \((\nu, q)\) of the system

\[
\partial_t \nu - \Delta \nu + \nabla q = h, \quad \nabla \cdot \nu = 0 \quad \text{in} \quad K \times \mathbb{R}_+
\]

with the initial and boundary conditions (55) satisfying the estimate

\[
\|\nu\|_{L_{p'; 1-\beta}(Q)} \leq c \|h\|_{L_{p'; 2-\beta}(Q)}.
\]

By means of Lemmas 2.12 and 2.17, it can be easily shown that \(\nu \in W_{2; \beta'}^{2,1}(Q)\) and \(q \in L_2(\mathbb{R}_+; V_{2, \beta}^{1}(K))\) with some \(\beta' < \min(\mu_2, \lambda_1 + 1) + \frac{1}{2}\). Thus, integrating by parts, we obtain

\[
\int_0^T \int_K u(x, t) \cdot h(x, T - t) \, dx \, dt = \int_0^T \int_K u(x, T - t) \cdot (\partial_t \nu - \Delta \nu + \nabla q)(x, t) \, dx \, dt
\]

\[
= \int_0^T \int_K (\partial_t u - \Delta u + \nabla p)(x, t) \cdot (\nu(x, T - t) \, dx \, dt = \int_0^T \int_K f(x, t) \cdot \nu(x, T - t) \, dx \, dt.
\]

Then (69) yields

\[
\left| \int_0^T \int_K u(x, t) \cdot h(x, T - t) \, dx \, dt \right| \leq c \|f\|_{L_{p; \beta}(Q)} \|h\|_{L_{p'; 2-\beta}(Q)},
\]

where \(c\) is independent of \(T\). This proves the lemma.

Next, we prove the following estimate for the pressure \(p\).

Lemma 3.13. Suppose that \((u, p) \in W_{p; \beta}^{2,1}(Q) \times L_p(\mathbb{R}_+; V_{p; \beta}^{1}(K))\) is a solution of (54), (55), where \(\beta + \frac{3}{p} \neq 2 + \mu_j\) for all \(j\) \((i.e. \ 1 - \beta - \frac{3}{p} \) is not an eigenvalue of the pencil \(\mathcal{N}(\mu))\). Then

\[
\|p\|_{L_{p; \beta-1}(Q)} \leq c \left( \|f\|_{L_{p; \beta}(Q)} + \sum_{j=1}^3 \|\partial_{x_j} u\|_{L_p(\mathbb{R}_+; L_{p; \beta-1+1/p}(\partial K))} \right),
\]

where \(c\) is independent of \(u\) and \(p\).
Proof. (54) and (55) yield
\[ \int_K \nabla p \cdot \nabla q \, dx = \langle F, q \rangle \] for all \( q \in V_{p',-\beta}^1(K) \), where
\[ \langle F, q \rangle = \int_K (f + \Delta u) \cdot \nabla q \, dx = \int_K f \cdot \nabla q \, dx + \int_{\partial K} \sum_{i,j=1}^3 \frac{\partial u_i}{\partial x_j} \left( n_j \frac{\partial q}{\partial x_j} - n_j \frac{\partial q}{\partial x_i} \right) \, ds \]
(cf. [16, Lemma 2.5]). Obviously,
\[
\| \langle F, q \rangle \| \leq \| f \|_{L_{p,\beta}(K)} \| \nabla q \|_{L_{p',-\beta}(K)} + \sum_{j=1}^3 \| \partial x_j u \|_{L_{p,\beta-1+1/p}(\partial K)} \| \nabla q \|_{L_{p',-\beta+1/p}(\partial K)}
\]
\[
\leq \left( \| f \|_{L_{p,\beta}(K)} + \sum_{j=1}^3 \| \partial x_j u \|_{L_{p,\beta-1+1/p}(\partial K)} \right) \| q \|_{V_{p',1-\beta}^2(K)}
\]
for all \( q \in V_{p',1-\beta}^2(K) \) (cf. Lemma 2.2). Using Lemma 2.11, we obtain
\[
\| p(\cdot, t) \|_{L_{p,\beta-1}(K)}^p \leq c \left( \| f(\cdot, t) \|_{L_{p,\beta}(K)}^p + \sum_{j=1}^3 \| \partial x_j u \|_{L_{p,\beta-1+1/p}(\partial K)}^p \right),
\]
where \( c \) is independent of \( t \). Integrating the last inequality with respect to \( t \), we obtain (70). \( \square \)

The last two lemmas imply the following result.

Lemma 3.14. Suppose that \((u, p) \in W_{p;\beta}^{1,1}(Q) \times L_p(\mathbb{R}^+; V_{p;\beta}^1(K))\) is a solution of (54), (55), where \( p < 2, 2 - \lambda_1 < \beta + \frac{3}{p} < \lambda_1 + 3 \) and \( \beta + \frac{3}{p} \neq 2 + \mu_j \) for all \( j \). Then
\[
\| u \|_{W_{p;\beta}^{2,1}(Q)} + \| p \|_{L_p(\mathbb{R}^+; V_{p;\beta}^1(K))} \leq c \| f \|_{L_{p,\beta}(Q)},
\]
where \( c \) is independent of \( u \) and \( p \).

Proof. By Lemmas 2.1 and 2.2, we have
\[
\| \partial x_j u \|_{L_{p,\beta-1+1/p}(\partial K)} \leq \varepsilon \| u \|_{V_{p;\beta}^1(K)} + c(\varepsilon) \| u \|_{L_{p,\beta-2}(K)},
\]
where \( \varepsilon \) can be chosen arbitrarily small. This inequality together with Lemma 3.13 yields
\[
\| p \|_{L_{p,\beta-1}(Q)} \leq c \left( \| f \|_{L_{p,\beta}(Q)} + \varepsilon \| u \|_{L_p(\mathbb{R}^+; V_{p;\beta}^1(K))} + c(\varepsilon) \| u \|_{L_{p,\beta-2}(Q)} \right).
\]
Estimating the norm of \( u \) in \( L_{p,\beta-2}(Q) \) by means of Lemma 3.12 and using Corollary 3.1, we obtain (71). \( \square \)

Theorem 3.2. Suppose that \( f \in L_{p;\beta}(Q), g \in L_p(\mathbb{R}^+; V_{p;\beta}^1(K)), g(x, 0) = 0 \) for \( x \in K \), and \( \partial g \in L_p(\mathbb{R}^+; (V_{p;\beta}^1(K))^*) \), where \( 1 < p < \infty \) and \( \beta \) satisfies the inequalities (5). In the case \( \beta + \frac{3}{p} > 2 \), we assume that \( g \) satisfies the condition (8). Then there exists a uniquely determined solution \((u, p) \in W_{p;\beta}^{2,1}(Q) \times L_p(\mathbb{R}^+; V_{p;\beta}^1(K))\) of the problem (1), (2) satisfying the estimate
\[
\| u \|_{W_{p;\beta}^{2,1}(Q)} + \| p \|_{L_p(\mathbb{R}^+; V_{p;\beta}^1(K))} \leq c \left( \| f \|_{L_{p,\beta}(Q)} + \| g \|_{L_p(\mathbb{R}^+; V_{p;\beta}^1(K))} + \| \partial g \|_{L_p(\mathbb{R}^+; (V_{p;\beta}^1(K))^*)} \right)
\]
with a constant \( c \) independent of \( f \) and \( g \).

Proof. Because of Lemma 3.2, we can restrict ourselves in the proof to the case \( g = 0 \). In this case, we assume that \( p < 2 \). Then by Lemma 3.14, every solution \((u, p) \in W_{p;\beta}^{2,1}(Q) \times L_p(\mathbb{R}^+; V_{p;\beta}^1(K))\) of the problem (54), (55) satisfies the estimate (71). Consequently,
the solution is unique and the range of the operator of this problem is closed in $L_{p;\beta}(Q)$. We prove the existence of solutions in the case $p < 2$. Suppose that
\[ f \in L_p(\mathbb{R}^+; V_{\gamma}^0(K) \cap V_{\gamma+\epsilon}^0(K)) \cap L_2(\mathbb{R}^+; V_{\gamma+\epsilon}^0(K) \cap V_{\gamma-\epsilon}^0(K)) \tag{72} \]
where $\gamma + \frac{3}{2} = \beta + \frac{3}{p}$ and $\epsilon$ is a sufficiently small positive number. Using Hölder’s inequality, one can easily show that $V_{\gamma+\epsilon}^0(K) \cap V_{\gamma-\epsilon}^0(K) \subset V_{p;\beta}^0(K)$. By [16, Theorems 3.1 and 3.2], there exists a solution $(u, p)$ of the problem (54), (55) in the space
\[ (W_{\gamma+\epsilon}^0(K) \cap V_{\gamma-\epsilon}^0(K)) \times L_2(\mathbb{R}^+; V_{\gamma+\epsilon}^0(K) \cap V_{\gamma-\epsilon}^0(K)). \]
Using Lemma 3.5, we conclude that $(u, p)$ is an element of the space
\[ (W_{\gamma+\epsilon}^0(K) \cap V_{\gamma-\epsilon}^0(K)) \times (L_p(\mathbb{R}^+; V_{\gamma+\epsilon}^0(K)) \cap L_p(\mathbb{R}^+; V_{\gamma-\epsilon}^0(K))). \]
But this is a subspace of $W_{p;\beta}^0(Q) \times L_p(\mathbb{R}^+; V_{p;\beta}^0(K))$. Since the space (72) is dense in $L_{p;\beta}(Q)$, it follows that the problem (54), (55) is solvable in $W_{p;\beta}^0(Q) \times L_p(\mathbb{R}^+; V_{p;\beta}^0(K))$ for arbitrary $f \in L_{p;\beta}(Q)$. The proof of the theorem is complete.\]

3.6. Solvability in Weighted $L_{q,p}$ Sobolev Spaces

In order to extend the result of the last theorem to weighted $L_{q,p}$ Sobolev spaces, we use a generalization of Mikhlin’s Fourier multiplier theorem which follows directly from [4, Theorem 1.1].

**Lemma 3.15.** Let $X$ and $Y$ be Banach spaces with Fourier type $\geq 1$, and let $A(\tau)$ be a linear and continuous operator from $X$ into $Y$ satisfying the estimate
\[ \|A^{(k)}(\tau)\|_{X \rightarrow Y} \leq c |\tau|^{-k} \quad \text{for } k = 0 \text{ and } k = 1 \text{ and for all real } \tau \neq 0, \]
where $c$ is independent of $\tau$. If the operator $F_{-\tau^{-1}} A(\tau) F_{-\tau}$ is bounded from $L_p(\mathbb{R}; X)$ into $L_p(\mathbb{R}; Y)$ for some $p \in (1, \infty)$, then it is bounded from $L_q(\mathbb{R}; X)$ into $L_q(\mathbb{R}; Y)$ for all $q \in (1, \infty)$.

Note that (weighted or nonweighted) $L_p$ spaces have Fourier type $\geq 1$ (Fourier type $p$ if $p \leq 2$ and Fourier type $p'$ if $p \geq 2$, see [27]). Thus, Lemma 3.15 enables us to extend the assertion of Theorem 3.2 to weighted $L_{q,p}$ spaces.

**Theorem 3.3.** Suppose that $f \in L_{q,p;\beta}(Q)$, $g \in L_p(\mathbb{R}^+; V_{p;\beta}^1(K))$, $g(x,0) = 0$ for $x \in K$, and $\partial_t g \in L_q(\mathbb{R}^+; V_{p;\beta}^1(K)')$, where $p, q \in (1, \infty)$ and $\beta$ satisfies the inequalities (5). In the case $\beta + \frac{3}{p} > 2$, we assume that $g$ satisfies the condition (8). Then there exists a uniquely determined solution $(u, p)$ of the problem (1), (2) satisfying the estimate
\[ \|u\|_{W_{q,p;\beta}^2(Q)} + \|p\|_{L_q(\mathbb{R}^+; V_{p;\beta}^1(K))} \leq c \left( \|f\|_{L_{q,p;\beta}(Q)} + \|g\|_{L_q(\mathbb{R}^+; V_{p;\beta}^1(K))} + \|\partial_t g\|_{L_q(\mathbb{R}^+; V_{p;\beta}^1(K)')} \right) \]
with a constant $c$ independent of $f$ and $g$.

**Proof.** By Lemma 3.2, we may restrict ourselves to the case $g = 0$. Let $\Phi_\alpha(s)$ denote the operator $V_{p;\beta}^0(K) \ni F \mapsto \partial_x^\alpha U \in V_{p;\beta-2|\alpha|}^0(K)$, where $(U, P)$ is the uniquely determined solution of the problem (26) in the space $V_{p;\beta}^0(K) \times V_{p;\beta}^1(K)$. As was noted in the proof of Lemma 3.5, the derivatives of the velocity component $u$ of the solution $(u, p)$ of the problem (1), (2) have the representation
\[ \partial_x^\alpha u = \mathcal{F}_{-\tau}^{-1} \Phi_\alpha(i\tau) \mathcal{F}_{-\tau} f \]
for $|\alpha| \leq 2$. Let $X$ and $Y$ denote the spaces $V_{p;\beta}^0(K)$ and $V_{p;\beta-2|\alpha|}^0(K)$, respectively. By Lemma 2.14, there is the inequality
\[ \|\Phi_\alpha^{(k)}(s)\|_{X \rightarrow Y} \leq c |s|^{-k} \]
for $\text{Re } s \geq 0, s \neq 0, |\alpha| \leq 2, k = 0$ and $k = 1$. By Theorem 3.2, the operator $\mathcal{F}_{\tau-t}^{-1} \Phi_{\alpha}(i\tau) \mathcal{F}_{t-\tau}$ is bounded from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$ for $|\alpha| \leq 2$. Consequently, it is also bounded from $L_q(\mathbb{R}, X)$ into $L_q(\mathbb{R}, Y)$, i.e., the estimate

$$\|u\|_{L_q(\mathbb{R}^d; V_{p, \beta}^2(K))} \leq c \|f\|_{L_q(\mathbb{R}, V_{p, \beta}^1(K))}$$

holds. Analogously, we can estimate the $L_q(\mathbb{R}^d; V_{p, \beta}^1(K))$-norm of the pressure $p$. This proves the theorem. □

Compliance with ethical standards
Conflicts of interest The authors declares that they have no competing interests.

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