MAXIMAL INEQUALITIES AND WEIGHTED BMO PROCESSES

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ABSTRACT. For a general adapted integrable right-continuous with left limits (RCLL) process $(X_t)_{t \geq 0}$ taking values in a metric space $(E, d)$, we show (among other things) that for every $m \in (1, \infty)$ and every $\tau > 0$

$$\kappa_1(m) \| \sup_{t \in [0, \tau]} \mathbb{E}(d(X_{t-}, X_t) | \mathcal{F}_t) \|_m \leq \| \sup_{t \in [0, \tau]} d(X_0, X_t) \|_m \leq \kappa_2(m) \| \sup_{t \in [0, \tau]} \mathbb{E}(d(X_{t-}, X_t) | \mathcal{F}_t) \|_m$$

with some universal constants $\kappa_1(m), \kappa_2(m)$ independent from $\tau$ such that $\kappa_1(m) = O(1)$ and $\kappa_2(m) = O(m)$ as $m \to \infty$. This is a probabilistic version of Fefferman–Stein estimate for the sharp maximal functions. While the former inequality is derived easily from Doob’s martingale inequality, the later inequality is a consequence of John–Nirenberg inequalities for weighted BMO processes, which are obtained in this note. We explain how John–Nirenberg inequalities can be utilized to obtain inequalities for martingales, both old and new alike in a unified way.

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1. INTRODUCTION AND MAIN RESULTS

Doob’s maximal martingale inequality ([Doo53]) and Burkholder–Davis–Gundy inequality ([BDG72]) have ultimately become indispensable and in the development of probability theory. They are however only available within martingales and their derivatives. In most situations, it is possible to introduce some auxiliary martingales such that these inequalities can be applied. A recent example is the stochastic sewing lemma from [Lê20] which has led to a wide range of new applications from regularization by noise [ABLM20], stochastic numerics [DGL22, LL21, BDG21] to rough stochastic differential equations [FHL21]. However, when applying to certain problems, one encounters many technical issues and some unexplanatory conditions. This raises a question whether martingale method is the correct toolbox for these problems, which we do not attempt to answer herein. Nevertheless, as an initiative step forward, we derive in this note maximal inequalities which are valid for general adapted integrable stochastic processes having RCLL (right-continuous with left limits) sample paths. To state the results, we first fix some notation. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions. For each stopping time $S$, $\mathbb{E}_S$ denotes the conditional expectation with respect to $\mathcal{F}_S$ and for each $G \in \mathcal{G}$, $\mathbb{P}_S(G) := \mathbb{E}_S(1_G)$. For an integrable random variable $\Xi$, $\mathbb{E}_\cdot \Xi$ always denotes a RCLL version of $t \mapsto \mathbb{E}_t \Xi$. For each
\[ m \in (0, \infty), \text{ we denote } \| \cdot \|_m = \left( \mathbb{E} | \cdot |^m \right)^{1/m} \text{ and } \| \cdot \|_\infty = \text{ess sup}_\omega | \cdot |. \] We define the constants
\[
(c_m)^m = m \left( 1 + \frac{1}{m} \right)^{(m+1)^2} \quad \text{and} \quad M = 1 + \sum_{m=1}^{\infty} \frac{(c_m)^m}{m!} e^{-3m}.
\]
Note that \( \sup_{m \geq 1} c_m < \infty \) and \( M \) is finite by the ratio test.

**Theorem 1.1.** Let \((\mathcal{E}, d)\) be a metric space and \(X : [0, \infty) \times \Omega \to (\mathcal{E}, d)\) be a RCLL adapted integrable process. We define for each \( t \leq \tau, \)
\[
X^\#_{t, \tau} = \sup_{s \in [t, \tau]} \mathbb{E}_s d(X_{s-}, X_t) \quad \text{and} \quad X^\#_\tau = X^\#_{0, \tau}. \tag{1.2}
\]
Then for every \( \tau > 0 \) and \( m \in (0, \infty), \)
\[
\| \sup_{t \in [0, \tau]} d(X_0, X_t) \|_m \leq (2cm)m \| \sup_{t \in [0, \tau]} \mathbb{E}_t X^\#_{t, \tau} \|_m, \tag{1.3}
\]
\[
\sup_{t > 0} \left\| \sup_{t \in [0, \tau]} \frac{d(X_0, X_t)}{\epsilon + \sup_{s \in [0, t]} \mathbb{E}_s X^\#_{s, \tau}} \right\|_m \leq 2cm, \tag{1.4}
\]
\[
\sup_{t > 0} \mathbb{E} \left[ \lambda \sup_{t \in [0, \tau]} \frac{d(X_0, X_t)}{\epsilon + \sup_{s \in [0, t]} \mathbb{E}_s X^\#_{s, \tau}} \right] \leq M \quad \text{for every } \lambda \leq (2e^3)^{-1}. \tag{1.5}
\]
In addition,
\[
\frac{m - 1}{2m - 1} \| X^\#_{t, \tau} \|_m \leq \| \sup_{t \in [0, \tau]} d(X_0, X_t) \|_m \leq \frac{2cm}{m - 1} \| X^\#_{t, \tau} \|_m \quad \text{for every } m \in (1, \infty), \tag{1.6}
\]
and
\[
\| \sup_{t \in [0, \tau]} d(X_0, X_t) \|_m \leq \frac{2cm}{(1 - m)^{1/m}} \| \mathbb{E}_0 X^\#_{t, \tau} \|_m \quad \text{for every } m \in (0, 1). \tag{1.7}
\]

That \( \sup_{t \in [0, \tau]} \mathbb{E}_t X^\#_{t, \tau} \) is a well-defined random variable needs some explanation. Observe that \( t \mapsto \mathbb{E}_t X^\#_{t, \tau} \) is a supermartingale. In addition, \( t \mapsto \mathbb{E}[\mathbb{E}_t X^\#_{t, \tau}] = \mathbb{E} \sup_{s \in [t, \tau]} \mathbb{E}_s d(X_{s-}, X_t) \) is right-continuous by the Lebesgue monotone convergence theorem. Hence, be [RY99, Chapter II (2.9)], the process \( t \mapsto \mathbb{E}_t X^\#_{t, \tau} \) has a RCLL version for which we always use in **Theorem 1.1** and hereafter. This shows that \( \sup_{t \in [0, \tau]} \mathbb{E}_t X^\#_{t, \tau} \) is measurable.

When \( X \) is a real valued discrete martingale, (1.6) can be traced back at least to Garsia [Gar73, Theorem III.5.2] and Stroock [Str73]. For general stochastic processes, (1.6) seems to be new and has no counterpart in literatures, as far as the author’s knowledge.

We note that the estimate \( \| \sup_{t \in [0, \tau]} d(X_0, X_t) \|_m \leq c(m) \| \sup_{t \in [0, \tau]} \mathbb{E}_t d(X_t, X_t) \|_m \) fails for \( m > 2 \) even for discrete martingale. A counter example given by Osekowski in [Os15] is the martingale \( g = (0, g_1, g_1, g_1, \ldots) \) with \( \mathbb{P}(g_1 = -\epsilon) = 1 - \mathbb{P}(g_1 = \epsilon^{-1}) = (1 + \epsilon^2)^{-1}. \)

We will derive **Theorem 1.1** from John–Nirenberg inequalities for stochastic processes of bounded weighted mean oscillation (weighted BMO).
Definition 1.2. Let $\tau > 0$ be a fixed number, $(\phi_t)_{t \in [0, \tau]}$ be a positive RCLL adapted process and $(V_t)_{t \in [0, \tau]}$ be a real valued RCLL adapted process. $V$ belongs to BMO$_\phi$ if

$$[V]_{\text{BMO}_\phi} := \sup_{0 \leq S \leq T \leq \tau} \| \phi_{S,T}^{-1} \mathbb{E}_S |V_T - V_S| \|_{\infty} < \infty,$$

where the supremum is taken over all stopping times $S, T$. For a process $V$ in BMO$_\phi$, its modulus of oscillation $\rho^\phi(V)$ is defined by

$$\rho^\phi_{S,T}(V) = \sup_{s \leq S \leq T \leq t} \| \mathbb{E}_S (\phi_{S,T}^*)^{-1} |V_T - V_S| \|_{\infty}, \quad 0 \leq s \leq t \leq \tau,$$

where $\phi_{S,T}^* = \sup_{r \in [s,t]} \phi_r$ and $S,T$ denote generic stopping times. We also define

$$\kappa^\phi_{S,T}(V) = \lim_{h \downarrow 0} \sup_{0 \leq u \leq S_T} \rho^\phi_{u,h}(V).$$

We note that $\rho^\phi$ is well-defined and

$$\rho^\phi_{S,T} \leq \sup_{0 \leq S \leq T \leq \tau} \| \phi_{S,T}^{-1} \mathbb{E}_S |V_T - V_S| \|_{\infty} \leq [V]_{\text{BMO}_\phi}. $$

The relation between Theorem 1.1 and weighted BMO processes is that any RCLL adapted integrable stochastic process is BMO with a suitable weight. Indeed, let $X$ be as in Theorem 1.1 and fix $\tau > 0$. For stopping times $S \leq T \leq \tau$, we have

$$\mathbb{E}_S d(X_T, X_T) = \mathbb{E}_S 1_{\{T < \tau\}} \lim_{\varepsilon \downarrow 0} \mathbb{E}_{T+\varepsilon} d(X_{T+\varepsilon}, X_T) \leq \mathbb{E}_S \sup_{S \leq t \leq T} \mathbb{E}_t d(X_t, X_T) = \mathbb{E}_S \mathbb{E}_S X_{S,t}^\#$$

and hence

$$\mathbb{E}_S [d(X_0, X_T) - d(X_0, X_{S,-})] \leq \mathbb{E}_S d(X_T, X_T) + \mathbb{E}_S d(X_{S,-}, X_T) \leq 2 \mathbb{E}_S X_{S,t}^\#.$$

This shows that for every $\varepsilon > 0$, $d(X_0, X_T) \in \text{BMO}_{2\mathbb{E}_S X_{S,t}^\#, \varepsilon}$ with $[X]_{\text{BMO}_{2\mathbb{E}_S X_{S,t}^\#, \varepsilon}} \leq 1$.

John–Nirenberg inequalities for weighted BMO processes are described in the following result.

Theorem 1.3. Let $(V_t)_{t \in [0, \tau]}$ be a process in BMO$_\phi$. Then for every $r \in [0, \tau]$ and every $m \in (0, \infty)$,

$$\mathbb{E}_r \sup_{r \leq t \leq \tau} |V_t - V_r|^m \leq \left( c_m m \rho^\phi_{r,\tau}(V) \right)^m \cdot \mathbb{E}_r |\phi_{r,\tau}^*|^m \quad \text{a.s.}$$

and

$$\mathbb{E}_r \left( \sup_{r \leq t \leq \tau} \frac{|V_t - V_r|}{\phi_{r,t}^*} \right)^m \leq \left( c_m m \rho^\phi_{r,\tau}(V) \right)^m \quad \text{a.s.}$$

Consequently,

$$\sup_{r \in [0, \tau]} \left\| \mathbb{E}_r \exp \left( \lambda \sup_{r \leq t \leq \tau} \frac{|V_t - V_r|}{\phi_{r,t}^*} \right) \right\|_\infty < \infty \quad \text{for every } \lambda \leq (e^3 \kappa^\phi_{r,\tau}(V))^{-1}.$$

Additionally, (1.8)–(1.10) hold with $V_{r-}$ in place of $V_r$. 
Although not stated, Theorems 1.1 and 1.3 have natural discrete analogues. Neither one of (1.8) and (1.9) implies the other. However, (1.9) has an advantage of being meaningful even when \( \phi_{s,t} \) is not \( L^m \)-integrable. While (1.8) may be comparable to [Gei05, Theorem 1], [Gar73, Theorem III.5.2] and [SV06, Lemma A.1.2], the estimates (1.9) and (1.10) are genuine and inspired from similar phenomenon in the theory of singular integrals [Kar02, OCPR13]. Normalized estimates such as (1.10) play important roles in the theory of self-normalized processes [dlPnLS09] and self-normalized large deviations [Sha97, JSW03]. The proofs of Theorems 1.1 and 1.3 are presented in Section 2.

To put Theorem 1.3 into a perspective, we apply it to martingales for which many classical inequalities are readily available. We rediscover in a systematic and natural way the close relations between a martingale with its square functions, which is the heart of the works of Burkholder, Davis and Gundy [BDG72, Bur73, Dav70]. Although we are not able to recover all of the known results solely on this method, Theorem 1.3 turns out to be very robust and we discover new inequalities for martingales. These results are presented in Section 3. Theorem 1.1 could also be used here, however, we have learned that applying Theorem 1.3 directly to martingales gives better results.

A discussion on weighted BMO processes would be incomplete without mentioning processes with vanishing weighted mean oscillation. Results for these processes are of independent interests and are presented in Section 4. For further applications of BMO processes, we refer to the companion paper [Lê22] and the references therein. Applications of (1.6) will be discussed elsewhere for conciseness.

Relation to previous works. The later estimate in (1.6) is closely related to Fefferman–Stein’s estimate for the sharp maximal function in [FS72]. Indeed, let \( f : \mathbb{R}^d \to \mathbb{R} \) be a locally integrable function and define the maximal functions

\[
    f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy, \quad Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,
\]

where \( Q \) denotes a cube in \( \mathbb{R}^d \), \( |Q| \) is the Lebesgue measure of \( Q \) and \( f_Q = \frac{1}{|Q|} \int_Q f(y)dy \). Then [FS72, Theorem 5] shows that

\[
    \|Mf\|_{L^p(\mathbb{R}^d)} \leq A_p \|f^\#\|_{L^p(\mathbb{R}^d)}, \quad p \in (1, \infty),
\]

which is comparable to the second estimate in (1.6). For further connections with the theory of singular integrals, we refer to [FS72, Str73], appendix A.1 of [SV06], remark III.5.1 of [Gar73] and chapter 3 of [HvNVW16].

Weighted BMO sequences and discrete martingales were considered earlier by Garsia [Gar73], Stroock [Str73], Stroock and Varadhan [SV06], and weighted BMO processes were considered by Geiss [Gei05]. Inequality (1.8) is known (at least implicitly) to Geiss. However, John–Nirenberg inequalities are formulated as equivalence of norms in [Gei05] and require an additional condition on the weight process \( \phi \). In contrast, we have tried our best to frame the results in Theorem 1.3 under simplest assumptions. This is important and beneficial because Theorem 1.1 and martingale inequalities (Section 3) are consequences of Theorem 1.3 but not earlier results. The relation between the exponential constant and modulus of mean oscillation

\[
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has been observed in the author’s previous work [Lê22] in the absence of weight. The proof of Theorem 1.3 is different from [Lê22] but still follows similar arguments which appeared in earlier works. However our presentation simplifies previous proofs. In literatures on probability theory, John–Nirenberg inequalities are often derived for BMO martingales and their validity for other processes is rarely discussed. Some connections between weighted BMO martingales and BDG inequalities can be found in [Gar73] through a different point of view from the current article.

2. Proofs of main results

Lemma 2.1. If $X$ and $Y$ are nonnegative random variables satisfying

\[ \mathbb{P}(Y > \alpha + \beta) \leq \theta \mathbb{P}(Y > \alpha) + \mathbb{P}(X > \theta \beta) \]

for every $\alpha > 0$, $\beta > 0$ and $\theta \in (0, 1)$; then for for every $m \in (0, \infty)$,

\[ \|Y\|_m \leq c_m \|X\|_m \]

where the constant $c_m$ is defined in (1.1).

Proof. We choose $\beta = h \alpha$ for some $h > 0$ and integrate the inequality with respect to $m \alpha^{m-1} d\alpha$ over $(0, k/(1+h))$ to get that

\[ (1+h)^{-m} \int_0^k m \alpha^{m-1} \mathbb{P}(Y > \alpha) d\alpha \leq \theta \int_0^k m \alpha^{m-1} \mathbb{P}(Y > \alpha) d\alpha + \int_0^\infty m \alpha^{m-1} \mathbb{P}(X > \theta h \alpha) d\alpha. \]

Sending $k \to \infty$ and using the layer cake representation $\mathbb{E}X^m = \int_0^\infty m \alpha^{m-1} \mathbb{P}(X > \alpha) d\alpha$, we obtain that

\[ [(1+h)^{-m} - \theta] \mathbb{E}Y^m \leq (\theta h)^{-m} \mathbb{E}X^m. \]

We now choose $h = \frac{1}{m}$ and $\theta = \left( \frac{m}{m+1} \right)^{m+1}$ to obtain the result. \qed

We note that the condition in Lemma 2.1 is satisfied if $\mathbb{P}(Y > \alpha + \beta, X \leq \theta \beta) \leq \theta \mathbb{P}(Y > \alpha)$ which is closely related to the condition for Burkholder’s good $\lambda$-inequality [Bur73, Lemma 7.1]. The formulation in Lemma 2.1 is more convenient for our purpose.

Proof of Theorem 1.3. We fix $r \in [0, \tau)$ and assume without loss of generality that $\rho^{\phi}_{r, \tau}(V) = 1$. We put $V^* = \sup_{t \in [r, \tau]} |V_t - V_r|$ and $\phi^* = \sup_{t \in [r, \tau]} \phi_t$. Let $\alpha, \beta$ be two positive numbers and define

\[ S = \tau \wedge \inf\{t \in [r, \tau] : |V_t - V_r| > \alpha\}, \quad T = \tau \wedge \inf\{t \in [r, \tau] : |V_t - V_r| > \alpha + \beta\}, \]

with the standard convention that $\inf(\emptyset) = \infty$. Clearly $S$ and $T$ are stopping times and $r \leq S \leq T \leq \tau$. On the event $\{V^* > \alpha + \beta\}$, we have $|V_r - V_{t^*}| \geq \alpha + \beta$, $|V_{S^*} - V_r| \geq \alpha$ and $|V_{S^*} - V_t| \leq \alpha$. The last inequality needs some justification. By right-continuity, there is $\epsilon > 0$ such that
$|V_s - V_r| < \alpha$ for every $s \in [r, r + \epsilon]$, which implies that $S > r$. Then one has $|V_{S-} - V_r| \leq \alpha$ by definition of $S$. (The case when $V_r$ is replaced by $V_{r-}$, the inequality $|V_{S-} - V_{r-}| \leq \alpha$ is trivial if $S = r$.)

Using the triangle inequality $|V_T - V_r| \leq |V_T - V_{S-}| + |V_{S-} - V_r|$, this implies that

$$\{ V^* > \alpha + \beta \} \subset \{ |V_T - V_{S-}| \geq \beta, V^* > \alpha \}.$$

It follows that for every $G \in \mathcal{F}_r$ and every $\theta \in (0, 1)$,

$$\mathbb{P}(V^* > \alpha + \beta, G) \leq \mathbb{P}(|V_T - V_{S-}| \geq \beta, V^* > \alpha, G) \leq \mathbb{P}(|V_T - V_{S-}| \geq \theta^{-1} \phi_{S,T}^*, V^* > \alpha, G) + \mathbb{P}(\phi^* > \theta \beta, V^* > \alpha, G).$$

By conditioning, noting that $\{ V^* > \alpha \}$ is $\mathcal{F}_S$-measurable, and using definition of BMO$_\phi$, we have

$$\mathbb{P}(|V_T - V_{S-}| \geq \theta^{-1} \phi_{S,T}^*, V^* > \alpha, G) \leq \theta \mathbb{P}(V^* > \alpha, G).$$

Hence, we obtain from the above that

$$\mathbb{P}(V^* > \alpha + \beta, G) \leq \theta \mathbb{P}(V^* > \alpha, G) + \mathbb{P}(\phi^* > \theta \beta, G).$$

Applying Lemma 2.1,

$$||V^* 1_G||_m \leq c_m m ||\phi^* 1_G||_m.$$ 

Since $G$ is arbitrary in $\mathcal{F}_r$, we obtain (1.8).

To show (1.9), we follow a similar argument. This time, for $\alpha, \beta > 0$, we define

$$S = \tau \wedge \inf \{ t \in [r, \tau] : (\phi_{r,t}^*)^{-1} |V_r - V_t| > \alpha \},$$

$$T = \tau \wedge \inf \{ t \in [r, \tau] : (\phi_{r,t}^*)^{-1} |V_r - V_t| > \alpha + \beta \}.$$

We have by triangle inequality and monotonicity,

$$\frac{|V_T - V_r|}{\phi_{r,T}^*} \leq \frac{|V_T - V_{S-}|}{\phi_{S,T}^*} + \frac{|V_{S-} - V_r|}{\phi_{r,S-}^*},$$

so that putting $(V/\phi)^* = \sup_{r \leq t \leq \tau} |V_r - V_t|/\phi_{r,t}^*$, we have

$$\left\{ \left( \frac{V}{\phi} \right)^* > \alpha + \beta \right\} \subset \left\{ \frac{|V_T - V_{S-}|}{\phi_{S,T}^*} \geq \beta, \left( \frac{V}{\phi} \right)^* > \alpha \right\}.$$

From here, we obtain (1.9) through Lemma 2.1 using similar arguments as previously.

Inequalities (1.8) and (1.9) with $V_{r-}$ in place of $V_r$ are obtain analogously.

For $\lambda \kappa_{r,T}(V) < e^{-3}$, there is an $h_0 > 0$ such that $\lambda \rho_{r,t}(V) \leq e^{-3}$ whenever $t - s \leq h_0$ and $s, t \in [r, \tau]$. For such $s, t$, we have by Taylor’s expansion and (1.9) that

$$\left\| \mathbb{E}_s \exp \left( \lambda \sup_{u \in [s,t]} \frac{|V_u - V_s|}{\phi_{s,u}^*} \right) \right\|_{\infty} \leq \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \mathbb{E}_s \left( \sup_{u \in [s,t]} \frac{|V_u - V_s|}{\phi_{s,u}^*} \right)^m.$$
\[ \leq 1 + \sum_{m=1}^{\infty} \frac{\lambda^{m}}{m!} (c_{m}m)^{m} (\rho_{s,t}(V))^{m} \leq M, \]

where \( M \) is defined in (1.1). Putting \( Z_{t} = \sup_{u \in [r,t]} \frac{|V_{u} - V_{r}|}{\phi_{s,u}} \), since

\[ Z_{t} - Z_{s} \leq \sup_{u \in [s,t]} \frac{|V_{u} - V_{s}|}{\phi_{s,u}}, \]

the previous estimate also implies that \( \| \mathbb{E}_{s} e^{\lambda(Z_{t} - Z_{s})} \|_{\infty} \leq M \) whenever \( t - s \leq h_{0} \). Now partition \([0, \tau]\) by points \( 0 = t_{0} < t_{1} < \ldots < t_{n} = \tau \) so that \( \max_{1 \leq k \leq n} (t_{k} - t_{k-1}) \leq h_{0} \). Then

\[ \mathbb{E}_{r} e^{\lambda(Z_{t} - Z_{r})} = \mathbb{E}_{r} e^{\lambda(Z_{t_{n-1}} - Z_{r})} e^{\lambda(Z_{t_{n-1}} - Z_{t_{n-1}})} \leq \mathbb{E}_{r} e^{\lambda(Z_{t_{n-1}} - Z_{r})} \| \mathbb{E}_{t_{n}} e^{\lambda(Z_{t_{n-1}} - Z_{t_{n-1}})} \|_{\infty}. \]

Iterating the previous inequality yields

\[ \left\| \mathbb{E}_{r} e^{\lambda(Z_{r} - Z_{t})} \right\|_{\infty} \leq \left\| \mathbb{E}_{r} e^{\lambda(Z_{t_{j-1}} - Z_{t})} \right\|_{\infty} \prod_{k=j+1}^{n} \left\| \mathbb{E}_{t_{k-1}} e^{\lambda(Z_{t_{k-1}} - Z_{t_{k-1}})} \right\|_{\infty}, \]

where \( j \) is such that \( t_{j-1} \leq r < t_{j} \). This implies that \( \| \mathbb{E}_{r} e^{\lambda(Z_{t} - Z_{r})} \|_{\infty} \leq M^{n} \) which shows (1.10). To show (1.10) with \( V_{r} \), we use \( \hat{Z}_{t} = \sup_{u \in [r,t]} \frac{|V_{u} - V_{r}|}{\phi_{s,u}} \) instead of \( Z_{t} \) and follow the same arguments.

**Proof of Theorem 1.1.** Let \( \phi \) be a RCLL version of the process \( t \mapsto 2\mathbb{E}_{t} X_{t,t}^{\#} \). We have seen earlier in Section 1 that for every \( \varepsilon > 0 \), \( V := d(X_{0}, X_{\cdot}) \) belongs to \( \text{BMO}_{\phi+\varepsilon} \) with \( [V]_{\text{BMO}_{\phi+\varepsilon}} \leq 1 \).

Applying (1.8) and sending \( \varepsilon \downarrow 0 \), we have for every \( m \in (0, \infty) \) that

\[ \| \sup_{t \in [0,\tau]} |V_{t}| \|_{m} \leq (2c_{m}m) \| \sup_{t \in [0,\tau]} \mathbb{E}_{t} X_{t,t}^{\#} \|_{m}, \]

which shows (1.3). (1.4) is a direct consequence of (1.9) and (1.5) follows from (1.4). When \( m > 1 \), combining with the Doob’s maximal inequality

\[ \| \sup_{t \leq \tau} \mathbb{E}_{t} X_{t,t}^{\#} \|_{m} \leq \| \sup_{t \leq \tau} \mathbb{E}_{t} X_{t,t}^{\#} \|_{m} \leq \frac{m}{m-1} \| X_{t,t}^{\#} \|_{m}, \]

we obtain the later inequality in (1.6). We observe that by triangle inequality,

\[ X_{t,t}^{\#} \leq \sup_{t \in [0,\tau]} (\mathbb{E}_{t} d(X_{0}, X_{t}) + d(X_{0}, X_{\tau})). \]

Hence, by the Doob’s maximal inequality, we have \( \| X_{t,t}^{\#} \|_{m} \leq (m/(m-1)+1) \| \sup_{t \in [0,\tau]} d(X_{0}, X_{t}) \|_{m} \), showing the first inequality in (1.6).

To show (1.7), we note that \( (\mathbb{E}_{t} X_{t,t}^{\#}) \) is a non-negative supermartingale. Hence by [RY99, Chapter II (1.15)] \( \mathbb{P}(\sup_{t \leq \tau} \mathbb{E}_{t} X_{t,t}^{\#} > \alpha) \leq \frac{1}{\alpha} \mathbb{E}_{t} X_{t,t}^{\#} \). Applying [KS22, Lemma 2.22] we have

\[ \| \sup_{t \leq \tau} \mathbb{E}_{t} X_{t,t}^{\#} \|_{m} \leq (1 - m)^{-1/m} \| \mathbb{E}_{0} X_{t,t}^{\#} \|_{m}, \]

which, together with (1.6), shows (1.7). \( \Box \)
3. Inequalities for martingales

In order to reveal the role of weighted BMO processes, we apply Theorem 1.3 to martingales and positive sequences. We discover new ranges for classical inequalities from the works of Burkholder, Davis and Gundy. We start with the following result, which is the dual of Doob’s martingale inequality.

**Proposition 3.1.** Let \( z_1, z_2, \ldots \) be positive random variables. Then for every \( m \in (0, \infty) \), we have

\[
\| \sum_{i=1}^{\infty} \mathbb{E}_i z_i \|_m \leq 2c_m m \| \sum_{i=k}^{\infty} z_i \|_m. \tag{3.1}
\]

If \( m \in (1, \infty) \),

\[
\| \sum_{i=1}^{\infty} \mathbb{E}_i z_i \|_m \leq 2c_m m' \| \sum_{i=k}^{n} z_i \|_m. \tag{3.2}
\]

**Proof.** Define \( V_k = \sum_{i=1}^{k} \mathbb{E}_i z_i \). From \( \mathbb{E}_k(V_n - V_{k-1}) = \mathbb{E}_k \sum_{i=k}^{n} z_i \), we see that \( (V_k)_{k \leq n} \) is BMO with the weight \( \langle \phi \rangle = \mathbb{E}_k \sum_{i=k}^{n} z_i \). Applying Theorem 1.3, we have for every \( m \in (0, \infty) \)

\[
\| \sum_{i=1}^{n} \mathbb{E}_i z_i \|_m \leq 2c_m m \| \sum_{i=k}^{n} z_i \|_m.
\]

Sending \( n \to \infty \) yields the first claim. The second claim follows from the first and Doob’s maximal inequality. \( \square \)

In [BDG72], Burkholder–Davis–Gundy show (3.2) with a different constant for all \( m \in [1, \infty) \) (note that the case \( m = 1 \) is trivial). The estimate (3.1) appears to be new. We take the chance to mention that it is shown in [Bur73, Theorem 20.1] that

\[
\| \sum_{i=1}^{\infty} \mathbb{E}_i z_i \|_m \geq 2^{-1/m} \| \sum_{i=1}^{\infty} z_i \|_m, \quad m \in (0, 1).
\]

Let \( f = (f_k)_{k \geq 0} \) be a square integrable discrete martingale, \( f_{-1} = 0 \). Define for each \( k \leq n \)

\[
df_{k} = f_{k} - f_{k-1}, \quad f_{k,n} = \sup_{k \leq j \leq n} |f_{j} - f_{k}|, \quad f_{n} = f_{0,n},
\]

\[
[f]_{k,n} = \sum_{j=k}^{n} |df_j|^2, \quad [f]_{n} = [f]_{0,n},
\]

\[
\langle f \rangle_{k,n} = \sum_{j=k+1}^{n+1} \mathbb{E}_j |df_j|^2, \quad \langle f \rangle_{n} = \langle f \rangle_{0,n}.
\]

From the martingale property, we have the following relations

\[
\mathbb{E}_i |f_k - f_{i-1}|^2 = \mathbb{E}_i [f]_{i,k} = \mathbb{E}_i ([f]_{k} - [f]_{i-1}), \tag{3.3}
\]
and
\[ E_i(\langle f \rangle_k - \langle f \rangle_{i-1}) = E_i(\langle f \rangle_{i,k} = E_i[f_{k+1} - f_i]^2 = E_i[f]_{i+1,k+1}. \]  

(3.4)

**Theorem 3.2.** For \( m \in (0, \infty) \)
\[
\sup_{k \leq n} \| f_{k,n}^{*} \|_{2m} \leq 4c_{2m} m \| \sup_{k \leq n} \| E_k[f]_{k,n} \|_{m}^{1/2} \leq 4c_{2m} m \| \sup_{k \leq n} (E_k(\langle f \rangle_{k,n-1} + |d f_k|^2))^{1/2} \|_{m}, \]  

(3.5)

\[
\| \langle f \rangle_{n} \|_{m} \leq 2c_{m} m \| \sup_{k \leq n} E_k[f_{k,n+1}^{*}] \|_{m} \]  

(3.6)

(3.7)

If \( m \in (1, \infty), m' = m/(m - 1), \) we have
\[
\sup_{k \leq n} \| f_{k,n}^{*} \|_{m} \leq 4c_{2m mm'} \| \langle f \rangle_{n} \|_{m}^{1/2}, \]  

(3.8)

\[
\sup_{k \leq n} \| f_{k,n}^{*} \|_{2m} \leq 4c_{2m} m \left( (m')^{1/2} \| \langle f \rangle_{n-1} \|_{m}^{1/2} + \| \sup_{k \leq n} |d f_k| \|_{2m} \right), \]  

(3.9)

\[
\| \langle f \rangle_{n} \|_{m} \leq 8c_{mm'} \| f_{n}^{*} \|_{2m}, \]  

(3.10)

\[
\| \langle f \rangle_{n} \|_{m} \leq 8c_{mm'} \min \left\{ \| \langle f \rangle_{n+1} \|_{m}, \| f_{n+1}^{*} \|_{2m} \right\}. \]  

(3.11)

In addition, there is a universal constant \( \lambda > 0 \) such that for every \( n \geq k \geq 0 \) and each pair \((V_k, \Phi_{k,n})\) in
\[
\left\{ \left( f_k, \sup_{i \leq k} (E_i[f]_{i,n})^{1/2} \right), \left( f_k, \sup_{i \leq k} E_i[f_{i,n}^{*}] \right), \left( \langle f \rangle_k, \sup_{i \leq k} \min \{ E_i[f]_{i+1,n+1}, E_i[f_{i,n+1}^{*}] \} \right) \right\}, \]  

we have \( \sup_{n \geq 0} E e^{\lambda \sup_{k \leq n} \| \Phi_{k,n} \|_{m}} \leq M. \)

Proof. Because \( E_i[f]_{i,k} \leq E_i[f]_{i,n} \) whenever \( i \leq k \leq n \), we deduce from (3.3) that \((f_k)_{k \leq n}\) is BMO with weight \( (E_k[f]_{k,n})^{1/2} \) \( k \leq n \). Identity (3.3) can also be read from right to left. In particular, because \( E_i[f_k - f_{i-1}]^2 \leq E_k[f_{k-1,n}]^2 \), it implies that \((E_k[f]_{k,n})_{k \leq n}\) is BMO with weight \((E_k[f_{k-1,n}]^2)_{k \leq n}\). Similarly, (3.4) implies that \((\langle f \rangle_k)_{k \leq n}\) is BMO with either weight \((E_k[f]_{k+1,n+1})_{k \leq n}\) or weight \((E_k[f_{k,n+1}]^2)_{k \leq n}\). Applying Theorem 1.3, we obtain (3.5)-(3.7). The second estimate in (3.5) is due to \( E_k[f]_{k,n} = E_k(\langle f \rangle_{k,n-1} + |d f_k|^2) \). We note that \( |d f_k|^2 \leq \| f \|_{n} \) and \( f_{k,n}^* \leq 2d f_k^* \). Hence, applying Doob’s martingale inequality, we obtain (3.8)-(3.11) from (3.5)-(3.7). The exponential integrability of \( V/\Phi \) is a direct consequence of (1.10). \( \square \)

We observe that necessary conditions for Theorem 3.2 are the identities (3.3) and (3.4), which are valid as long as \( E_i d f_j d f_k = 0 \) whenever \( i \leq j < k \). In a more general case, one would need to control \( E_i \sum_{j < k} d f_j d f_k \) by a suitable weight, however we have not explored this idea. Results for continuous square integrable martingales are stated without proof.
Theorem 3.3. Let \((X_t)_{t \geq 0}\) be a square integrable continuous martingale with quadratic variation \((\langle X \rangle_t)_{t \geq 0}\). Define \(X^*_{s,t} = \sup_{s \leq r \leq t} |X_r - X_s|\). There is a universal constant \(\lambda > 0\) such that for every \(\tau > 0\),

\[
\sup_{\varepsilon > 0} \mathbb{E} \exp \left\{ \lambda \sup_{t \leq \tau} \varepsilon + \sup_{s \in [0,t]} (\mathbb{E}_s \langle X \rangle_{s,t})^{1/2} \right\} \leq M \tag{3.12}
\]

and

\[
\sup_{\varepsilon > 0} \mathbb{E} \exp \left\{ \lambda \sup_{t \leq \tau} \varepsilon + \sup_{s \in [0,t]} \mathbb{E}_s |X^*_{s,t}|^2 \right\} \leq M. \tag{3.13}
\]

Estimates (3.5)-(3.7) for the range \(m \in (0, 1)\) and the exponential integrability appear to be new, as far as the author’s knowledge. Estimates with the conditional square functions can be found in [Gar73, Bur73, Bur73, Wan91]. Inequality (3.9) is also known as Burkholder–Rosenthal inequality which is valid for \(m \in [1, \infty)\) ([Bur73]). Although we did not recover the case \(m = 1\) from John–Nirenberg inequality, we have obtained its weak form, namely (3.5), which is valid for all \(m \in (0, \infty)\). The exponential estimates in Theorems 1.1, 3.2 and 3.3 are complementary to known estimates for self-normalized processes in [dlPnLS09, Chapter 12].

4. Weighted VMO processes

Definition 4.1. Let \((V_t)_{t \in [0, \tau]}\) be a process in \(\text{BMO}_\phi, \ p \in [1, \infty)\) and \(\alpha \in (0, 1]\).

(i) \(V\) is \(\text{VMO}_\phi\) if \(\kappa_{0,\tau}^\phi(V) = 0\).

(ii) \(V\) is \(\text{VMO}_\phi^{p-\text{var}}\) if \(V\) is \(\text{VMO}_\phi\) and \(\rho^\phi(V)\) has finite \(p\)-variation.

(iii) \(V\) is \(\text{VMO}_\phi^\alpha\) if \(V\) is \(\text{VMO}_\phi\) and \(\rho^\phi(V)\) is \(\alpha\)-Hölder continuous on the diagonal.

We define

\[
[V]_{\text{VMO}_\phi^{p-\text{var}}; [0, \tau]} := \left( \sup_{\pi \in \mathcal{P}([0, \tau])} \left( \sum_{[s, t] \in \pi} \rho^\phi_{s,t}(V) |p| \right)^{1/p} \right) \tag{4.1}
\]

and

\[
[V]_{\text{VMO}_\phi^\alpha; [0, \tau]} := \sup_{0 \leq s < t \leq \tau} \left( \frac{\rho^\phi_{s,t}(V)}{\Gamma(1 - \alpha)} (t - s)^\alpha \right) \tag{4.2}
\]

where \(\mathcal{P}([0, \tau])\) is the set of all partitions on \([0, \tau]\).

It is evident that \(\text{VMO}_\phi^\epsilon \subset \text{VMO}_\phi^{1/\alpha-\text{var}}\) and \([V]_{\text{VMO}_\phi^{1/\alpha-\text{var}}; [0, \tau]} \leq \tau^\alpha [V]_{\text{VMO}_\phi^\alpha; [0, \tau]}\). Reasoning as in [Lê22], processes in \(\text{VMO}_\phi\) are necessarily continuous.

Proposition 4.2. Let \((V_t)_{t \in [0, \tau]}\) be a process in \(\text{VMO}_\phi^{p-\text{var}}\) and define the function

\[
(s, t) \mapsto w^\phi_{s,t}(V) := ([V]_{\text{VMO}_\phi^{p-\text{var}}; [s, t]})^p.
\]

Then \(w(V) : \{(s, t) \in [0, \tau]^2 : s \leq t\} \to [0, \infty)\) is a control.
Proof. For every $s \leq u \leq t$, we have
\[
\|E_s(\phi_{s,t}^*)^{-1}|V_t - V_s||_{\infty} \leq \|E_s(\phi_{s,u}^*)^{-1}|V_u - V_s||_{\infty} + \|E_u(\phi_{u,t}^*)^{-1}|V_t - V_u||_{\infty}
\]
so that $\rho_{s,t}(V) \leq \rho_{s,u}(V) + \rho_{u,t}(V)$. The rest of the proof proceeds in the same way as in [Lê22, Section 3].

**Theorem 4.3.** Let $(V_t)_{t \in [0,r]}$ be $\text{VMO}_\phi$. Then
\[
\sup_{r \in [0,r]} \left\| E_r \exp \left( \lambda \sup_{t \in [r,\tau]} \frac{|V_t - V_r|}{\phi_{r,t}^*} \right) \right\|_{\infty} < \infty \text{ for every } \lambda > 0.
\]
(4.3)

If $V$ is in $\text{VMO}^{p-\text{var}}_\phi$ for some $p \in (1, \infty)$ then there are constants $c_p, C_p$ such that
\[
\sup_{r \in [0,r]} \left\| E_r \exp \left( \lambda \sup_{t \in [r,\tau]} \frac{|V_t - V_0|}{\phi_{r,t}^*} \right) \right\|_{\infty} \leq M^{1+(e^3\lambda)p} w_0^\phi(V) \text{ for every } \lambda > 0,
\]
(4.4)

\[
E \exp \left( \lambda \sup_{r \in [0,r]} \frac{|V_r - V_0|^p}{\phi_{r,0}^*} \right) < \infty \text{ whenever } \lambda(w_0^\phi(V))^{1/p} < c_p.
\]
(4.5)

\[
\left\| E_s \sup_{t \in [r,\tau]} \frac{|V_t - V_r|^m}{|\phi_{r,t}^*|^m} \right\|_{\infty} \leq C_p \Gamma(m/p + 1)(w_{s,t}^\phi(V))^{m/p} \text{ for every } m \geq 1.
\]
(4.6)

In (4.5), we have set $p' = p/(p - 1)$. If $V$ is in $\text{VMO}^{1-\text{var}}_\phi$ then (4.4) still holds with $p = 1$ and
\[
\mathbb{P}(|V_t - V_r| \leq e^3 \phi_{s,t}^* w_{s,t}^\phi(V) \text{ for all } s \leq t \leq \tau) = 1.
\]
(4.7)

Proof. (4.3) is a direct consequence of (1.10). We only show (4.4) because the other estimates are derived as in [Lê22, Section 3]. We define $t_0 = 0$ and for each integer $k \geq 1$,
\[
t_k = \sup\{t \in [t_{k-1}, \tau] : \lambda |w_{t_{k-1},t_k}^\phi(V)|^{1/p} < e^{-3}\}.
\]

We have $\lambda x_{t_{k-1},t_k}^\phi(V) \leq \lambda \rho_{t_{k-1},t_k}^\phi(V) \leq \lambda |w_{t_{k-1},t_k}^\phi(V)|^{1/p} \leq e^{-3}$. The argument in the proof of Theorem 1.3 shows that the left-hand side of (4.4) is at most $M^n$ with $M$ defined in (1.1). By continuity of $w^\phi$, we have $\lambda \rho_{t_{k-1},t_k}^\phi = e^{-3}p$ for $k = 1, \ldots, n - 1$ and $\lambda \rho_{t_{n-1},t_n}^\phi \leq e^{-3}p$. By definition of controls, we have
\[
\frac{n - 1}{(e^3\lambda)^p} \leq \sum_{k=1}^n w_{t_{k-1},t_k}^\phi(V) \leq w_{0,\tau}^\phi(V),
\]
which implies that $n \leq 1 + (e^3\lambda)^p w_{0,\tau}^\phi(V)$. \hfill \Box

**Corollary 4.4.** Let $(V_t)_{t \in [0,r]}$ be a process in $\text{BMO}^{p-\text{var}}_\phi$ with $p \in (1, \infty)$. Assume that there is $q \in (1, \infty)$ such that
\[
E \exp \left( \lambda |\phi_{r,\tau}^{|1/p} \right) < \infty \text{ for every } \lambda > 0.
\]
Then
\[ \mathbb{E} \exp \left( \lambda \sup_{t \in [0,r]} |V_t - V_0|^{\frac{p'}{q'}} \right) < \infty \text{ for every } \lambda > 0. \]

Proof. If \( w^\phi_{0,r}(V) = 0 \), \( V \) is a constant and the claim is trivial. We thus assume that \( w^\phi_{0,r}(V) \neq 0 \).

Put \( X = \sup_{t \in [0,r]} |V_t - V_0| \), \( Y = \phi^*_t \). By Doob’s maximal inequality, we have
\[ \mathbb{E} e^{\lambda |\phi^*_t|^{1/(q-1)}} \leq C \mathbb{E} e^{\lambda^{1/(q-1)}} < \infty \text{ for every } \lambda > 0. \]

Let \( \lambda, \epsilon \) some positive numbers. Using Young inequality
\[ \lambda \frac{\epsilon'}{q} \leq \frac{\epsilon^q}{q} \left( \frac{X}{Y} \right)^{p'} + \frac{\epsilon^{-1} \lambda}{q'} Y^q \]
and Hölder inequality, we have
\[ \mathbb{E} e^{\lambda X^{p'/q}} \leq \left[ \mathbb{E} e^{\epsilon q (X)_{p'}} \right]^{1/q} \left[ \mathbb{E} e^{(\epsilon^{-1} \lambda) q' Y^{q'/q}} \right]^{1/q'}. \]

Observing that \( \frac{\epsilon'}{q} \leq \sup_{t \in [0,r]} \frac{|V_t - V_0|}{\phi^*_t} \) and \( q'/q = 1/(q-1) \), we can choose \( \epsilon \) sufficiently small and apply (4.5) to obtain the result. \( \square \)

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