Boundedness of non regular pseudo-differential operators on variable exponent Triebel-Lizorkin-Morrey spaces

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Abstract. In this paper, we study the boundedness of non regular pseudo-differential operators on variable exponent Besov-Morrey spaces $E^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ with symbols $a(x,\xi)$ belonging to $C^\ell_{s,m1,\delta}$. For these symbols $x$-regularity is measured in Hölder-Zygmund spaces.

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1. Introduction

Pseudo-differential calculus is a well-established tool for the analysis of partial differential equations, especially non-linear ones. Indeed, in [16] one can find many applications of the calculus of non regular pseudo-differential operators to non-linear differential equations. The boundedness of these operators has been extensively addressed in several works. For boundedness on Lebesgue spaces, Besov spaces, Triebel-Lizorkin spaces and Sobolev spaces, we refer to [2], [6], [12] and [13].

The boundedness of pseudo-differential operators in Triebel-Lizorkin-Morrey spaces with constant exponents denoted $E^{s}_{p,u,q}$ was studied by Yoshihiro Sawano in [15].

Our focus in this paper concerns the boundedness of pseudo-differential operators on Triebel-Lizorkin-Morrey spaces with variable exponents denoted $E^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}$ (see [4]) with symbols in the class $C^\ell_{s,m1,\delta}$. The results of this paper are certainly relevant because they generalize those of [15].

Our approach is as follows. To treat the boundedness of these operators with non-regular symbols belonging to $C^\ell_{s,m1,\delta}$ we use elementary symbols as it was done in [2], [12], [14]

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Indeed, the symbol reduction method, due to Coifman and Meyer\cite{6}, makes it possible to be limited to symbols $a(x, \xi) \in C^s_{\delta} S_{m, \delta}$ of the form $a(x, \xi) = \sum_{j\geq 0} \sigma_j(x) \psi_j(\xi)$ (see [14] and [2]). Then, we rewrite the symbol as a sum of three parts, a “low-high”, a ”high-high”, and a ”high-low” part. Thus, the operator $a(x, D)$ with symbol $a$ can be resolved into three operators $a_1(x, D)$, $a_2(x, D)$ and $a_3(x, D)$ with symbols $a_1$, $a_2$ and $a_3$. Now it remains to study the boundedness of each elementary operators.

We structure this paper in 4 sections as follows. In Section 2 we give the preliminaries, where we recall the definitions of Morrey spaces and Besov-Morrey spaces with variable exponents. In Section 3, we recall necessary tools for the proofs of the lemmas and the main result that we give in Section 4.

2. Preliminaries

We denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ stands for the set of all integer numbers. We write $B(x, r)$ for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. We use $c$ as a generic positive constant, i.e. a constant whose value may change with each appearance. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant $c$, and $f \approx g$ means $f \lesssim g \lesssim f$. Throughout the paper we denote by $\mathcal{M}(\mathbb{R}^n)$ the family of all complex or extended real-valued measurable functions on $\mathbb{R}^n$.

By $\text{supp} f$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $\chi_E$ denotes its characteristic function.

We denote by $\mathcal{S}(\mathbb{R}^n)$ the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution $f$ is denoted by $\mathcal{F}f$ or $\hat{f}$ while its inverse transform is denoted by $\mathcal{F}^{-1}f$ or $\check{f}$.

2.1. Variable exponents

For more information on the results of this paragraph, see [11] and [7].

- By $\mathcal{P}(\mathbb{R}^n)$ we denote the set of all measurable functions $p : \mathbb{R}^n \rightarrow (0, +\infty)$ (called variable exponents) which are essentially bounded away from zero. We denote $p_{\mathbb{R}^n}^+ := \text{ess sup}_{\mathbb{R}^n} p(x)$ and $p_{\mathbb{R}^n}^- := \text{ess inf}_{\mathbb{R}^n} p(x)$; we abbreviate $p^+ = p_{\mathbb{R}^n}^+$ and $p^- = p_{\mathbb{R}^n}^-$. 

- The function $\phi_p$ is defined as follows:

$$\phi_p(t) = \begin{cases} 
\frac{t^p(x)}{} & \text{if } p(x) \in (0, +\infty), \\
0 & \text{if } p(x) = +\infty \text{ and } t \in [0, 1], \\
+\infty & \text{if } p(x) = +\infty \text{ and } t \in (1, +\infty]. 
\end{cases}$$

The variable exponent modular associated to $p(\cdot)$ is defined by

$$\varrho_p(f) := \int_{\mathbb{R}^n} \phi_p(|f(x)|) dx.$$
The variable exponent Lebesgue space $L_{p(\cdot)} := L_{p(\cdot)}(\mathbb{R}^n)$ is the family of (equivalence classes of) functions $f \in M(\mathbb{R}^n)$ such that $g_{p(\cdot)}(f/\lambda)$ is finite for some $\lambda > 0$.

$L_{p(\cdot)}$ is a quasi-Banach space equipped with the quasinorm

$$\|f\|_{p(\cdot)} := \inf \left\{ \mu > 0 : g_{p(\cdot)} \left( \frac{1}{\mu} f \right) \leq 1 \right\}.$$  

• We say that a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, abbreviated $g \in C^{l_{\log}}(\mathbb{R}^n)$, if there exists $c_{l_{\log}}(g) \geq 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{l_{\log}}(g)}{\log(e + 1/|x - y|)} \quad \text{for all } x, y \in \mathbb{R}^n. \quad (1)$$

The function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be globally log-Hölder continuous, abbreviated $g \in C_{\log}(\mathbb{R}^n)$, if it is locally log-Hölder continuous and there exists $g_{\infty} \in \mathbb{R}$ and $c_{\infty}(g) \geq 0$ such that

$$|g(x) - g_{\infty}| \leq \frac{c_{\infty}(g)}{\log(e + |x|)} \quad \text{for all } x \in \mathbb{R}^n.$$  

We write $g \in \mathcal{P}_{\log}(\mathbb{R}^n)$ if $0 < g^{-} \leq g(x) \leq g^{+} \leq +\infty$ with $1/g \in C_{\log}(\mathbb{R}^n)$.

We define $\frac{1}{g_{\infty}} := \lim_{|x| \to +\infty} \frac{1}{g(x)}$ and we use the convention $\frac{1}{\infty} = 0$.

### 2.2. Variable exponent Triebel-Lizorkin-Morrey spaces

We refer to the papers [4], [18], [3], [5], [17] and [9], for further results on Triebel-Lizorkin-Morrey spaces and variable exponent Triebel-Lizorkin-Morrey spaces.

• Morrey spaces

**Definition 1.** For $p, u \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^{-} \leq p(x) \leq u(x) \leq +\infty$, the variable exponent Morrey space $M_{p(\cdot), u(\cdot)} := M_{p(\cdot), u(\cdot)}(\mathbb{R}^n)$ consists of all functions $f \in M(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p(\cdot), u(\cdot)}} := \sup_{x \in \mathbb{R}^n, r > 0} \| r_u^{u(x)} f \|_{L_{p(\cdot)}(B(x,r))}.$$

**Definition 2.** Let $p, q, u \in \mathcal{P}(\mathbb{R}^n)$ with $p(x) \leq u(x)$. The mixed space $M_{p(\cdot), u(\cdot)}(\ell_{q(\cdot)})$ consists of all sequences $(f_{\nu})_{\nu} \subset M(\mathbb{R}^n)$ such that,

$$\| (f_{\nu})_{\nu} \|_{M_{p(\cdot), u(\cdot)}(\ell_{q(\cdot)})} := \left( \sum_{\nu=0}^{+\infty} |f_{\nu}(\cdot)|^{q(\cdot)} \right)^{1/q(\cdot)} \quad < +\infty.$$  

(3)
Remark 1. [4] Note that \( \|f\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \) defined a quasinorm on \( M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)}) \). It is a norm when \( \min(p^-, q^-) \geq 1 \).

Proposition 1. Let \( f \) and \( g \) be two measurable functions with \( 0 \leq f(x) \leq g(x) \) for a.e. \( x \in \mathbb{R}^n \). Then it holds
\[
\|f\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \leq \|g\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.
\]

Proposition 2. Let \( p, q, u \in \mathcal{P}(\mathbb{R}^n) \) with \( p(x) \leq u(x) \) and \( 0 < t < +\infty \).
Let \( (f_{\nu})_{\nu} \subset \mathcal{M}(\mathbb{R}^n) \)
\[
\| \langle |f_{\nu}|^t \rangle_{\nu} \|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} = \| (f_{\nu})_{\nu} \|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}
\]
with the usual modification every time \( q(x) = +\infty \).

**Triebel-Lizorkin-Morrey spaces.**
We first recall a Littlewood-Paley partition of unity \( \{\psi_{\nu}\}, \nu \geq 0 \).
The functions \( \psi_{\nu} \) are defined as follows. Let \( \psi_0 \in C_0^\infty(\mathbb{R}^n) \) such that \( \psi_0 \equiv 1 \) on \( B(0; 1) \) and \( \text{supp} \psi_0 \subset B(0; 2) \).
Set
\[
\psi_{\nu}(x) = \psi_0(2^{-\nu}x) - \psi_0(2^{-\nu+1}x) \quad \text{for all } \nu \in \mathbb{N}.
\]
Then \( \psi_{\nu} \) is supported on the dyadic shell
\[
D_{\nu} = \{ \xi \in \mathbb{R}^n : 2^{\nu-1} \leq |\xi| \leq 2^{\nu+1} \}.
\]
If \( f \in \mathcal{S}' \), then
\[
f = \sum_{\nu \geq 0} \psi_{\nu} f.
\]
The Fourier multiplier \( \psi_{j}(D) \) with symbol \( \psi_{j} \) is defined as
\[
\psi_{\nu}(D)f(x) = \mathcal{F}^{-1}(\psi_{\nu} \cdot \hat{f})(x) = \int_{\mathbb{R}^n} \psi_{\nu}(\xi)\hat{f}(\xi)e^{ix\cdot\xi}d\xi.
\]

Definition 3. Let \( \{\psi_{\nu}\} \) be the usual Littlewood-Paley partition of unity. Let \( s : \mathbb{R}^n \to \mathbb{R} \), \( p, q \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and \( u \in \mathcal{P}(\mathbb{R}^n) \) such that \( 0 < p^- \leq p(x) \leq u(x) \leq \sup u < +\infty \) and \( q^-, q^+ \in (0, +\infty) \). The Triebel-Lizorkin-Morrey spaces \( \mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)} \) consists of all distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that
\[
\|f\|_{\mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)}} := \| \psi_{0}(D)f\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} + \| \left(2^{\nu s(\cdot)}\psi_{\nu}(D)f_{\nu} \right)_{\nu \geq 1}\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} < +\infty. \tag{4}
\]

Remark 2. [4](remark4.4) Note that \( \|f\|_{\mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)}} \) defined a quasinorm on \( \mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)}^{s(\cdot)} \). It is a norm when \( \min(p^-, q^-) \geq 1 \).
3. Basic tools

In this section we present some useful results for the last section. At First, we recall the $\eta$-functions defined by

$$\eta_{\nu,m}(x) = 2^{n\nu} (1 + 2^n|x|)^{-m}, \quad \nu \in \mathbb{N}_0, \ m > 0.$$ 

Note that $\eta_{\nu,m} \in L^1$ for $m > n$ and the corresponding $L^1$-norm does not depend on $\nu$.

The following lemma is from [8](Lemma19) and [10](Lemma6.1)

**Lemma 1.** Let $\alpha \in C^0_{\text{loc}}(\mathbb{R}^n)$ and let $m \geq 0$, $R \geq c_{\text{log}} \log(\alpha)$, where $c_{\text{log}}$ is the constant from (1) for $\alpha$. Then

$$2^{m\nu} \eta_{\nu,m}(x - y) \leq c 2^{m\nu} (\eta_{\nu,m}(x))^1/t, \quad x, y \in \mathbb{R}^n \text{ and } \nu \in \mathbb{N}_0.$$ 

The following lemma is from [10](lemma A.6).

**Lemma 2.** Let $t > 0$, $\nu \in \mathbb{N}_0$ and $m > n$. Then there exists $c = c(t, m, n)$ such that for all $g \in S'((\mathbb{R}^n)$ with $\text{supp} Fg \subset \xi \in \mathbb{R}^n: |\xi| \leq 2^{\nu+1}$, We have

$$|g(x)| \leq c (\eta_{\nu,m} * |g|^t(x))^{1/t}, \quad x \in \mathbb{R}^n.$$ 

The following lemma is from [4](theorem3.3).

**Lemma 3.** Let $p, q \in \mathcal{P}^0(\mathbb{R}^n)$ and $u \in \mathcal{P}(\mathbb{R}^n)$ such that $1 < p^- \leq p(x) \leq u(x) \leq \sup u < +\infty$ and $q^-, q^+ \in (1, +\infty)$. If

$$m > n + n \max \left\{ 0, \sup_{x \in \mathbb{R}^n} \left( \frac{1}{p(x)} - \frac{1}{u(x)} \right) - \frac{1}{p^\infty} \right\},$$

then there exists $c > 0$ such that for all sequences $(f_\nu)_\nu \subset M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})$.

$$\| (\eta_{\nu,m} * f_\nu)_\nu \|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})} \leq c \| (f_\nu)_\nu \|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.$$ 

The following lemma is from [1](Corollary 4.8.)

**Lemma 4.** Let $p \in \mathcal{P}^0(\mathbb{R}^n)$ and $u \in \mathcal{P}$ with $1 < p^- \leq p(x) \leq u(x) \leq \sup u < +\infty$. If

$$m > n + n \max \left\{ 0, \sup_{x \in \mathbb{R}^n} \left( \frac{1}{p(x)} - \frac{1}{u(x)} \right) - \frac{1}{p^\infty} \right\},$$

Then there exists $c > 0$ such that

$$\| \eta_{\nu,m} * f \|_{M_{p(\cdot),u(\cdot)}} \leq c \| f \|_{M_{p(\cdot),u(\cdot)}}.$$ 

The following lemma is from [4](Lemma 3.7).
Lemma 5. Let \( p, u, q \in \mathcal{P}(\mathbb{R}^n) \) with \( p(x) \leq u(x) \). Let \( \delta > 0 \). For any sequence \( (g_j)_{j \in \mathbb{N}_0} \) of non negative measurable functions on \( \mathbb{R}^n \), we denote
\[
G_\nu(x) := \sum_{j=0}^{+\infty} 2^{-|\nu-j|\delta} g_j(x), \quad x \in \mathbb{R}^n, \ \nu \in \mathbb{N}_0.
\]
Then it holds
\[
\|G_\nu\|_{L^q(\mathbb{R}^n, u(x)\,dx)} \leq c(\delta, q) \|g_j\|_{L^q(\mathbb{R}^n, u(x)\,dx)}
\]
where
\[
c(\delta, q) = \max \left( \sum_{\nu \in \mathbb{Z}} 2^{-|\nu\delta|}, \left[ \sum_{\nu \in \mathbb{Z}} 2^{-|\nu\delta q|} \right]^{1/q} \right).
\]

4. Boundedness of pseudo-differential operators

We will use symbols for which \( x \)-regularity is measured in Hölder-Zygmund spaces.

Definition 4. [14] The function \( a(x, \xi) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) belongs to the symbol class \( C^\ell_{s,1,\delta} \), \( \delta \in [0, 1] \), \( \ell > 0 \) if it is smooth in \( \xi \) and satisfies the following estimates:
\[
\begin{align*}
\|\partial^\alpha a(\cdot, \xi)\|_{C^\ell_{s,1,\delta}} &\leq c_\alpha (\xi)^{m-|\alpha|+\ell\delta} \quad \text{and} \\
|\partial^\alpha a(x, \xi)| &\leq c_\alpha (\xi)^{m-|\alpha|}.
\end{align*}
\]

In (5), \( \langle \xi \rangle \) stand for \( (1 + |\xi|^2)^{1/2} \).

A pseudo-differential operator on \( \mathcal{E}^{s(\cdot)}_{\mathcal{P}(\cdot), u(\cdot), q(\cdot)} \) with symbol \( a \in C^\ell_{s,1,\delta} \) is defined by
\[
a(x, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x, \xi) \mathcal{F} f(\xi) \, d\xi, \quad f \in \mathcal{E}^{s(\cdot)}_{\mathcal{P}(\cdot), u(\cdot), q(\cdot)}.
\]

Definition 5. We call elementary symbol in the class \( C^\ell_{s,1,\delta} \), \( \delta \in [0, 1] \), \( \ell > 0 \) an expression of the form
\[
a(x, \xi) = \sum_{j \geq 0} a_j(x) \psi_j(\xi)
\]
where \( \psi_0 \) is smooth supported on the ball \( B(0, 2) \), \( \psi_j(\xi) = \psi(2^{-j} \xi) \) and \( \psi \in C^\infty_0 \) is supported on the dyadic shell \( D_0 = \{ \xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2 \} \), while \( a_j \) is uniformly bounded sequence such that
\[
\|a_j\|_{C^\ell_{s,1,\delta}} \leq c_2^j(m+\ell\delta).
\]

Since \( a(x, D) \) and \( \psi_j(D) \) do not commute, to study boundedness of \( a(x, D) \), the symbol reduction method due to Coifman and Meyer[6] makes it possible to be limited to elementary symbols. Therefore, the operator \( a(x, D) \) with symbol \( a \) can be resolved into ”elementary operators” \( a_k(x, D) \) with symbols \( a_k \). This idea has been exploited to establish continuity of pseudo-differential operators with non-regular symbols in inhomogeneous Sobolev spaces \( H^{s,p} \) and Hölder-Zygmund spaces \( C^\ell_s \) (see [12] and [2]).
Lemma 6. [14] Let \( f = \sum_{j \geq 0} f_j \) in \( S' \), with \( \text{supp} f_j \subset B(0, A 2^j) \) for some \( A > 0 \). Then, for \( \ell > 0 \),
\[
\|f\|_{C^\ell} \leq c(A) \sup_{j \geq 0} \left\{ 2^{j\ell} \|f_j\|_{L_\infty} \right\}.
\]
(6)

The following lemmas plays a fundamental role in the proof of the boundedness of pseudo-differential operators on \( \mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)} \).

Lemma 7. Let \( c_1, c_2 > 0 \), \( s \in C^\log_{loc} \), \( p, q \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and \( u \in \mathcal{P}(\mathbb{R}^n) \) such that \( 0 < p^- \leq p(x) \leq u(x) \leq \sup u < \infty \) and \( q^-, q^+ \in (0, +\infty) \). Let \( \{f_k\}_{k \in \mathbb{N}_0} \) be a sequence of tempered distributions such that
\[
\text{supp} \mathcal{F} f_0 \subset B(0, 2c_2)
\]
and
\[
\text{supp} \mathcal{F} f_k \subset \left\{ \xi \in \mathbb{R}^n : c_1 2^{k-1} < |\xi| < c_2 2^{k+1} \right\} \quad \text{for } k > 0
\]
Then
\[
\left\| \sum_{k=0}^{+\infty} f_k \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} \lesssim \left\| \left( 2^{ks(\cdot)} f_k \right)_k \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.
\]

Proof. Let \( \{\psi_j\} \) be the Littlewood-Paley partition of unity defined above. By hypothesis, \( \psi_j, j \geq 1 \) are supported on the dyadic shell \( D_j \), while \( \psi_0 \) is supported on the ball \( B(0; 2) \). Hence, there is \( N_1, N_2 \in \mathbb{N}_0 \) such that
\[
\psi_0(D) \left( \sum_{k=0}^{+\infty} f_k \right) = \psi_0(D) \left( \sum_{k=0}^{N_1} f_k \right)
\]
and
\[
\psi_j(D) \left( \sum_{k=0}^{+\infty} f_k \right) = \psi_j(D) \left( \sum_{k=j-N_1}^{j+N_2} f_k \right)
\]
Then
\[
\left\| \sum_{k=0}^{+\infty} f_k \right\|_{\mathcal{E}^{s(\cdot)}_{p(\cdot),u(\cdot),q(\cdot)}} = \left\| \sum_{k=0}^{N_1} \tilde{\psi}_0 * f_k \right\|_{M_{p(\cdot),u(\cdot)}} + \left\{ \left( 2^{js(\cdot)} \sum_{k=j-N_1}^{j+N_2} \tilde{\psi}_j * f_k \right) \right\}_{j \geq N_1} \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}.
\]
(7)

• Let us first estimate \( \left\{ \left( 2^{js(\cdot)} \sum_{k=j-N_1}^{j+N_2} \tilde{\psi}_j * f_k \right) \right\}_{j \geq N_1} \right\|_{M_{p(\cdot),u(\cdot)}(\ell_{q(\cdot)})}
\]
Since \( \tilde{\psi}_j * f_k \in S' \) and \( \text{supp} \mathcal{F} (\tilde{\psi}_j * f_k) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j+1} \} \), then, by lemma 2,
\[
|\tilde{\psi}_j * f_k| \lesssim (n_{j,m} * |f_k|^\ell)^{1/\ell}, \quad k = j - N_1, \ldots, j + N_2.
\]
for any \( m > n + c\log(s) + n \max \left\{ 0, \sup_{x \in \mathbb{R}^n} \left( \frac{1}{p(x)} - \frac{1}{u(x)} \right) - \frac{1}{p'} \right\} \) and any \( t > 0 \).

Thus

\[
\left\| \left\{ 2^{js} \sum_{k=j-N_1}^{j+N_2} \tilde{u}_j * f_k \right\} \right\|_{L^2} \lesssim \left\| \left\{ 2^{js} \left( \eta_{j,m} * |f_k|^t \right)^{1/t} \right\} \right\|_{L^2},
\]

By lemma 1, we can move \( 2^{js} \) inside the convolution

\[
2^{js} \left( \eta_{j,m} * |f_k|^t \right)^{1/t} \lesssim \left( \eta_{j,m-c\log(s)} * 2^{js} |f_k|^t \right)^{1/t}.
\]

Then

\[
\left\| \left\{ 2^{js} \sum_{k=j-N_1}^{j+N_2} \tilde{u}_j * f_k \right\} \right\|_{L^2} \lesssim \left\| \left\{ \sum_{k=j-N_1}^{j+N_2} \left( \eta_{j,m-c\log(s)} * 2^{js} |f_k|^t \right)^{1/t} \right\} \right\|_{L^2}.
\]

With \( t \in (0, \min \{ 1, p^-, q^- \}) \), lemma 4 yields

\[
\left\| \left\{ 2^{js} \sum_{k=j-N_1}^{j+N_2} \tilde{u}_j * f_k \right\} \right\|_{L^2} \lesssim \left\| \left\{ \sum_{k=0}^{N_1+N_2} \left( 2^{js} |f_{j+k-N_1}|^t \right)^{1/t} \right\} \right\|_{L^2}.
\]

\( \bullet \) Now we estimate the first term.

Since \( \text{supp} \mathcal{F} (\tilde{u}_0 * f_k) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \} \), then by lemma 2, \( |\tilde{u}_0 * f_k| \lesssim |f_k| \).

Thus

\[
\left\| \sum_{k=0}^{N_1} \tilde{u}_0 * f_k \right\|_{L^2} \lesssim \sum_{k=0}^{N_1} |f_k|_{L^2} = \sum_{k=0}^{N_1} \left\| (0, \ldots, f_k, 0, \ldots) \right\|_{L^2}.
\]
\[ \lesssim \left\| \left( 2^{k s(t)} f_k \right)_k \right\|_{\ell_q(t_{q(\mathbb{P}(\mathbb{R}^n)})}. \]

The proof is completed. \qed

**Lemma 8.** Let \( c > 0, s \in C^{\log}_{\text{loc}}, p, q \in \mathcal{P}^{\log}(\mathbb{R}^n) \) and \( u \in \mathcal{P}(\mathbb{R}^n) \) such that \( 0 < p^- \leq p(x) \leq u(x) \leq \sup u < +\infty, s^+ > 0 \) and \( q^-, q^+ \in (0, +\infty) \). Let \( \{f_k\}_{k \in \mathbb{N}_0} \) be a sequence of tempered distributions such that

\[ \text{supp} \mathcal{F} f_k \subset B(0, c2^{k+1}) \]

Then

\[ \left\| \sum_{k=0}^{+\infty} f_k \right\|_{\ell_q(t_{q(\mathbb{P}(\mathbb{R}^n))})} \lesssim \left\| \left( 2^{k s(t)} f_k \right)_k \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} \]

**Proof.** In view of the hypothesis on \( \text{Supp} \psi_j \), there is \( N \in \mathbb{N}_0 \) such that

\[ \left\| \sum_{k=0}^{+\infty} f_k \right\|_{\ell_q(t_{q(\mathbb{P}(\mathbb{R}^n))})} = \left\| \psi_0(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} + \left\{ 2^{j s(t)} \psi_j(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\}_{j \geq N, M_{p^+}(u^*)_{\ell_q(t)}}. \]

(i) At first we estimate

\[ \left\| \left\{ 2^{j s(t)} \psi_j(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\}_{j \geq N, M_{p^+}(u^*)_{\ell_q(t)}} \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} \]

We have

\[ \left\| \left\{ 2^{j s(t)} \psi_j(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\}_{j \geq N, M_{p^+}(u^*)_{\ell_q(t)}} \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} = \left\| \left\{ \sum_{k=0}^{+\infty} 2^{j s(t)} \left( \psi_j * f_k \right) \right\}_{j \geq N, M_{p^+}(u^*)_{\ell_q(t)}} \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} \]

Since

\[ \left\{ \text{supp} \mathcal{F} \left( \psi_j * f_k \right) \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j+1} \} \right\}, \]

by lemma 2 ,

\[ 2^{j s(t)} \left( \psi_j * f_k \right) \lesssim 2^{j s(t)} \left( \eta_{j,m} * |f_k| \right)^{1/t}, \]

for \( m > n + c_{\log}(1/q) + c_{\log}(s) + n \max \left\{ 0, \sup_{x \in \mathbb{R}^n} \left( \frac{1}{p(x)} - \frac{1}{u(x)} \right) - \frac{1}{p} \right\} \) and \( t > 0 \).

Therefore

\[ \left\| \left\{ 2^{j s(t)} \psi_j(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\}_{j \geq N, M_{p^+}(u^*)_{\ell_q(t)}} \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} \lesssim \left\| \left\{ \sum_{k=0}^{j} 2^{j s(t)} \left( \eta_{j,m} * |f_k| \right)^{1/t} \right\}_{j \geq N, M_{p^+}(u^*)_{\ell_q(t)}} \right\|_{M_{p^+}(u^*)_{\ell_q(t)}} \]
Let us estimate each one of the two terms on the right-hand side. Using lemmas 1 we can move $2^{\nu s(\cdot)}$ inside the convolution $2^{\nu s(\cdot)} (\eta_{\nu,m} |f_k|^t)^{1/t}$. And we have

$$2^{\nu s(\cdot)} (\eta_{\nu,m} |f_k|^t)^{1/t} \lesssim (\eta_{\nu,m_0} 2^{\nu s(\cdot)} |f_k|^t)^{1/t}, \nu = j \text{ or } k \text{ where } m_0 = m - c \log(s).$$

Thus

$$\| \left\{ \sum_{k=j-N}^{j} 2^{js(\cdot)} (\eta_{j,m} |f_k|^t)^{1/t} \right\}_{j \in M_p(u)(\ell_q(\cdot))} \| \lesssim \sum_{k=-N}^{0} \left\| \left\{ \eta_{j,m_0} 2^{js(\cdot)} |f_{k+j}|^t \right\}_{j \in M_p(u)(\ell_q(\cdot))} \right\|. \quad (2022)$$

For $t \in (0, \min \{p^-, q^-\})$, lemma 3 yields

$$\| \left\{ \sum_{k=j-N}^{j} 2^{js(\cdot)} (\eta_{j,m} |f_k|^t)^{1/t} \right\}_{j \in M_p(u)(\ell_q(\cdot))} \| \lesssim \sum_{k=-N}^{0} \left\| \left\{ 2^{js(\cdot)} |f_{k+j}|^t \right\}_{j \in M_p(u)(\ell_q(\cdot))} \right\|. \quad (2022)$$

Then

$$\| \left\{ \sum_{k=j-N}^{j} 2^{js(\cdot)} (\eta_{j,m} |f_k|^t)^{1/t} \right\}_{j \in M_p(u)(\ell_q(\cdot))} \| \lesssim \left\| \left\{ 2^{js(\cdot)} f_j \right\}_{j \in M_p(u)(\ell_q(\cdot))} \right\|. \quad (2022)$$

And

$$\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-(k-j)s(\cdot)} 2^{ks(\cdot)} (\eta_{k,m} |f_k|^t)^{1/t} \right\}_{j \in M_p(u)(\ell_q(\cdot))} \| \lesssim \left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-|j-k|s(\cdot)} (\eta_{k,m_0} 2^{ks(\cdot)} |f_k|^t) \right\}_{j \in M_p(u)(\ell_q(\cdot))} \right\| \lesssim \left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-|j-k|s^-} (\eta_{k,m_0} 2^{ks(\cdot)} |f_k|^t) \right\}_{j \in M_p(u)(\ell_q(\cdot))} \right\|.$$
for \( m > n \)

Then by lemma 1

\[
\left\| \sum_{k=0}^{+\infty} 2^{-|j-k|s^-} \left( \eta_{k,m_0} \ast 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})}
\]

By lemma 5,

\[
\left\| \sum_{k=0}^{+\infty} 2^{-|j-k|s^-} \left( \eta_{k,m_0} \ast 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} \lesssim \left\| \left( \eta_{k,m_0} \ast 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})}
\]

For \( t \in (0, \min p^-, q^-) \), lemma 3 yields

\[
\left\| \left( \eta_{k,m_0} \ast 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} \lesssim \left\| \left( 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})}
\]

Then

\[
\left\| \left\{ \sum_{k=j+1}^{+\infty} 2^{-(k-j)s(-)2^{ks(-)} \left( \eta_{k,m_0} \ast |f_k|^t \right)^{1/t} } \right\} \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} \lesssim \left\| \left( 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})}
\]

(ii) Now we estimate \( \left\| \psi_0(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} \).

Since

\[ \text{supp} F (\psi_0 \ast f) \subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{k+1} \right\} , \]

Then

\[
\left\| \psi_0(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} = \left\| \sum_{k=0}^{N} \psi_0 \ast f_k \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} \lesssim \left\| \sum_{k=0}^{\infty} \left( \eta_{k,m} \ast |f_k|^t \right)^{1/t} \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} ,
\]

for \( m > n + c\log (1/q) + c\log (s) + n \max \left\{ 0, \sup_{x \in \mathbb{R}^n} \left( \frac{1}{p(x)} - \frac{1}{u(\frac{x}{2})} \right) - \frac{1}{p_*} \right\} \) by lemma 2.

Then by lemma 1

\[
\left\| \psi_0(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})} \lesssim \left\| \sum_{k=0}^{+\infty} 2^{-ks^-} \left( \eta_{k,m-c\log(s)} \ast 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})}
\]

\[
= \left\| \sum_{k=0}^{+\infty} 2^{-ks^-} \left( \eta_{k,m-c\log(s)} \ast 2^{ks(-)|f_k|^t} \right) \right\|_{M_{\frac{(1)}{p_*}}(\ell_{\frac{(s^0)}{p_*}})}
\]

Thus
\[ \left\| \psi_0(D) \left( \sum_{k=0}^{+\infty} f_k \right) \right\|_{M_p(u,\cdot)} \lesssim \left\| \left( \eta_{k,m-c_{\log}(s)} \ast 2^{ks(\cdot)} f_k \right)_k \right\|_{M_p(u,\cdot)(\ell^q)} \]
\[ \lesssim \left\| \left( 2^{ks(\cdot)} f_k \right)_k \right\|_{M_p(u,\cdot)(\ell^q)} \]
by lemma 5 and lemma 2.

The proof is completed. □

Theorem 1.
Let \( a(x, \xi) \in C^\ell_0 S^{m=\delta}_{1,\delta} \) where \( m \in \mathbb{R}, \delta \in [0,1] \) and \( \ell > 0 \). Let \( 1 \leq p^- \leq p(x) \leq u(x) \leq \sup u < +\infty \) and \( q^-, q^+ \in [1, +\infty) \). Let \( s \in C^\log_{\text{loc}} \) such that \( 0 < s^- \leq s^+ < \ell \). Then
\[ a(x, D) : \mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)} \rightarrow \mathcal{E}_{p(\cdot),u(\cdot),q(\cdot)} \]
is bounded.

Proof. We recall that the symbol reduction method, due to Coifman and Meyer[6], makes it possible to be limited to symbols \( a(x, \xi) \in C^\ell_0 S^{m=\delta}_{1,\delta} \) of the form (see [14] and [2])
\[ a(x, \xi) = \sum_{j \geq 0} \sigma_j(x) \psi_j(\xi) \]
where \( \sigma_j \) satisfies
\[ \| \sigma_j \|_{C^\ell_0} \leq c 2^{j(m+\ell\delta)} \]
and
\[ \| \sigma_j \|_{L_\infty} \leq c \]
with \( c \) depending on \( \delta \) and \( \ell \) but not on \( j \). And \( \psi_j \) is exactly a Littlewood-Paley function. We have
\[ \sigma_j(x) = \sum_{k=0}^{+\infty} \psi_j(D) \sigma_j(x). \]
Then
\[ \sigma_j(x) \psi_j(\xi) = \left( \sum_{k=0}^{+\infty} \psi_j(D) \sigma_j(x) \psi_j(\xi) \right). \]
Therefore
\[ a(x, \xi) = \sum_{j=0}^{+\infty} \left( \sum_{k=0}^{+\infty} \psi_k(D) \sigma_j(x) \right) \psi_j(\xi). \]

Set
\[ a_{kj} = \psi_k(D) \sigma_j. \]
Then
\[ a(x, \xi) = \sum_{j=0}^{+\infty} \left( \sum_{k=0}^{+\infty} a_{kj} \right) \psi_j(\xi). \]
(i) At first, it’s necessary to estimate \( \|a_{kj}\|_{L^\infty} \).
We recall the quasinorm of \( C^\ell_\sigma \): 
\[
\|\psi_k(D)\sigma_j\|_{C^\ell_\sigma} = \sup_k 2^{k\ell} \|\psi_k(D)\sigma_j\|_{L^\infty}.
\]
Since 
\[
\|\psi_k(D)\sigma_j\|_{C^\ell_\sigma} \leq c \|\sigma_j\|_{C^\ell_\sigma}.
\]
Then 
\[
\sup_k 2^{k\ell} \|\psi_k(D)\sigma_j\|_{L^\infty} \leq c \|\sigma_j\|_{C^\ell_\sigma}.
\]
Using (9), we obtain 
\[
\|a_{kj}\|_{L^\infty} \leq c 2^{j(m+\delta)} 2^{-k\ell}.
\]
Note that \((1 - \Delta)^m\), \( m \in \mathbb{R} \) is an isomorphism that composes well with pseudo-differential operators (see [14] and [15]). Therefore, it is enough to examine the case \( m = 0 \).
If \( m = 0 \) then 
\[
\|a_{kj}\|_{L^\infty} \leq c 2^{j\ell\delta} 2^{-k\ell}.
\]
(ii) Now we rewrite the symbol as a sum of three parts
\[
a(x, \xi) = \sum_{j \geq 0} \left( \sum_{k=0}^{j-4} a_{kj}(x) + \sum_{k=j-3}^{j+3} a_{kj}(x) + \sum_{k=j+4}^{\infty} a_{kj}(x) \right) \psi_j(\xi)
\]
where
\[
a_1(x, D)f = \sum_{j=0}^{+\infty} \left( \sum_{k=0}^{j-4} a_{kj}\psi_j(D)f \right),
\]
\[
a_2(x, D)f = \sum_{j=0}^{+\infty} \left( \sum_{k=j-3}^{j+3} a_{kj}\psi_j(D)f \right),
\]
\[
a_3(x, D)f = \sum_{j=0}^{+\infty} \left( \sum_{k=j+4}^{\infty} a_{kj}\psi_j(D)f \right).
\]
We have
\[
\mathcal{F} \left( \sum_{k=0}^{j-4} a_{kj} f_j \right) = \sum_{k=0}^{j-4} \mathcal{F}(\psi_k(D)\sigma_j) * \mathcal{F}(\psi_j(D)f)
\]
\[
= \sum_{k=0}^{j-4} (\psi_k \mathcal{F}\sigma_j) * (\psi_j \mathcal{F}f).
\]
Using the fact that \( \text{supp}(f * g) \subset \text{supp}f + \text{suppg} \) for all compactly supported distributions \( f, g \in \mathcal{S}' \), we have \( \text{supp} \mathcal{F} \left( \sum_{k=0}^{j-4} a_{kj} f_j \right) \subset \{ \xi \in \mathbb{R}^n : c_1 2^{j-1} \leq |\xi| \leq c_2 2^{j+1} \} \) with \( c_1, c_2 > 0 \). Then lemma 7 yields
\[ \|a_1(x, D)f\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}} = \left\| \sum_{j=0}^{+\infty} \left( \sum_{k=0}^{j-4} a_{kj}\psi_j(D)f \right) \right\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}} \]

\[ \lesssim \left\| \left( 2^{js(\cdot)} \sum_{k=0}^{j-4} a_{kj}\psi_j(D)f \right) \right\|_{M_{p(\cdot), u(\cdot)}(\ell_q(\cdot))} \]

\[ \lesssim \left\| \left( \sum_{k=0}^{j-4} \|\sigma_j\|_{L_\infty} 2^{js(\cdot)}\psi_j(D)f \right) \right\|_{M_{p(\cdot), u(\cdot)}(\ell_q(\cdot))} \]

\[ \lesssim \left\| \left( 2^{js(\cdot)}\psi_j(D)f \right) \right\|_{M_{p(\cdot), u(\cdot)}(\ell_q(\cdot))} \]

Then

\[ \|a_1(x, D)f\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}} \lesssim \|f\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}}. \]

• For the second part \( \|a_2(x, D)f\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}} = \left\| \sum_{j=0}^{+\infty} \left( \sum_{k=j-3}^{j+3} a_{kj}f_j \right) \right\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}} \), we observe that

\[ \mathcal{F} \left( \sum_{k=j-3}^{j+3} a_{kj}f_j \right) = \sum_{k=j-3}^{j+3} \mathcal{F}(\psi_k(D)\sigma_j) \ast \mathcal{F}(\psi_j(D)f) \]

\[ = \sum_{k=j-3}^{j+3} (\psi_k \mathcal{F}\sigma_j) \ast (\psi_j \mathcal{F}f). \]

Then \( \mathcal{F} \left( \sum_{k=j-3}^{j+3} a_{kj}f_j \right) \) is supported on the ball \( B(0, 2^{j+4}) \).

By lemma 8,

\[ \|a_2(x, D)f\|_{E^{s(\cdot), p(\cdot), u(\cdot), q(\cdot)}} \lesssim \left\| \left( 2^{js(\cdot)} \sum_{k=j-3}^{j+3} a_{kj}\psi_j(D)f \right) \right\|_{M_{p(\cdot), u(\cdot)}(\ell_q(\cdot))} \]

\[ \lesssim 2^{-m} \left\| \left( \sum_{k=j-3}^{j+3} \|\sigma_j\|_{L_\infty} 2^{js(\cdot)}\psi_j(D)f \right) \right\|_{M_{p(\cdot), u(\cdot)}(\ell_q(\cdot))}. \]
One have
\[
\sum_{k=j-3}^{j+3} \|a_{kj}\|_{L_{\infty}} \lesssim \sum_{k=-3}^{3} 2^{-k\ell} < +\infty \quad \text{(with } \delta = 1). \]

Then
\[
\|a_2(x, D)f\|_{\mathcal{E}_{p(\cdot), u(\cdot), q(\cdot)}^s} \lesssim \left\| \left(2^{js(\cdot)}\psi_j(D)f\right)\right\|_{M_{p(\cdot), u(\cdot), q(\cdot)}(\ell_{q(\cdot)})} \lesssim \|f\|_{\mathcal{E}_{p(\cdot), u(\cdot), q(\cdot)}^s}.
\]

**Now let us estimate last part. Since** \(F\left(\sum_{k=j+4}^{+\infty} a_{kj}f_j\right)\) **is not supported on any ball or shell, we cannot directly use neither lemma 7 nor lemma 8. However, in \(S'\) we can write**
\[
\sum_{j=0}^{+\infty} \sum_{k=j+4}^{+\infty} a_{kj}f_j = \sum_{k=4}^{+\infty} \sum_{j=0}^{k-4} a_{kj}f_j.
\]

We have
\[
F\left(\sum_{j=0}^{k-4} a_{kj}f_j\right) = \sum_{j=0}^{k-4} (\psi_k\mathcal{F}a_j) \ast (\psi_j\mathcal{F}f).
\]

We have \(\text{supp}\mathcal{F}\left(\sum_{k=0}^{j-4} a_{kj}f_j\right) \subset \{\xi \in \mathbb{R}^n | c_1 2^{j-1} \leq |\xi| \leq c_2 2^{j+1}\}\) with \(c_1, c_2 > 0\).

Thus we can use lemma 7.
\[
\|a_3(x, D)f\|_{\mathcal{E}_{p(\cdot), u(\cdot), q(\cdot)}^s} = \left\| \sum_{k=4}^{+\infty} \left(\sum_{j=0}^{k-4} a_{kj}f_j\right)\right\|_{\mathcal{E}_{p(\cdot), u(\cdot), q(\cdot)}^s} \lesssim \left\| \left(2^{ks(\cdot)}\sum_{j=0}^{k-4} a_{kj}f_j\right)\right\|_{M_{p(\cdot), u(\cdot), q(\cdot)}(\ell_{q(\cdot)})} \lesssim \left\| \sum_{j=0}^{k-4} \|a_{kj}\|_{L_{\infty}} 2^{ks(\cdot)}\psi_j(D)f\right\|_{M_{p(\cdot), u(\cdot), q(\cdot)}(\ell_{q(\cdot)})}.
\]

If we use (13) with \(\delta = 1\), we have
\[
\|a_3(x, D)f\|_{\mathcal{E}_{p(\cdot), u(\cdot), q(\cdot)}^s} \lesssim \left\| \sum_{j=0}^{k-4} 2^{j\ell} 2^{-k\ell} 2^{ks(\cdot)}\psi_j(D)f\right\|_{M_{p(\cdot), u(\cdot), q(\cdot)}(\ell_{q(\cdot)})}.
\]
\[
= \left\| \sum_{j=0}^{k-4} 2^{(k-j)(s(-) - \ell)} 2^{j s(-)} \psi_j(D)f \right\|_{k M_{p(-),u(-)}(\ell_{q(-)})} \\
\leq \left\| \sum_{j=0}^{k-4} 2^{-k-j\|s-\ell\|} 2^{j s(-)} \psi_j(D)f \right\|_{k M_{p(-),u(-)}(\ell_{q(-)})} \\
\leq \left\| \sum_{j=0}^{\infty} 2^{-k-j\|s-\ell\|} 2^{j s(-)} \psi_j(D)f \right\|_{k M_{p(-),u(-)}(\ell_{q(-)})} .
\]

By hypothesis \(|s - \ell| > 0\). Therefore, by lemma 5
\[
\left\| \left( \sum_{j=0}^{k-4} 2^{-k-j\|s-\ell\|} 2^{j s(-)} \psi_j(D)f \right) \right\|_{k M_{p(-),u(-)}(\ell_{q(-)})} \lesssim \left\| \left( 2^{j s(-)} \psi_j(D)f \right) \right\|_{k M_{p(-),u(-)}(\ell_{q(-)})} .
\]

Then
\[
\|a_3(x, D)f\|_{E^{s(-)}_{p(-),u(-),q(-)}} \lesssim \|f\|_{E^{s(-)}_{p(-),u(-),q(-)}} .
\]

The proof is completed. \(\square\)

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