SYMMETRIES OF QUADRATIC FORMS CLASSES AND OF QUADRATIC SURDS CONTINUED FRACTIONS. 
PART II: CLASSIFICATION OF THE PERIODS’ PALINDROMES

FRANCESCA AICARDI

Abstract. The continued fractions of quadratic surds are periodic, according to a theorem by Lagrange. Their periods may have differing types of symmetries. This work relates these types of symmetries to the symmetries of the classes of the corresponding indefinite quadratic forms. This allows to classify the periods of quadratic surds and at the same time to find, for an arbitrary indefinite quadratic form, the symmetry type of its class and the number of integer points, for that class, contained in each domain of the Poincaré model of the de Sitter world, introduced in Part I [1]. Moreover, we obtain the same information for every class of forms representing zero, by the finite continued fraction related to a special representative of that class. We will see finally the relation between the reduction procedure for indefinite quadratic forms, defined by the continued fractions, and the classical reduction theory, which acquires a geometrical description by the results of Part I.

1. Definition of the palindromes

Definition 1. A finite continued fraction \([\alpha_0, \alpha_1, \ldots, \alpha_N]\) is said palindromic \(^1\) iff
\[\alpha_i = \alpha_{N-i}, \quad i = 0, \ldots, N.\]

Definition 2. A period of length \(P\) is a finite sequence of \(P\) natural numbers that cannot be written as a sequence of identical sub-sequences. For example, \([1,2,3,1,2,3]\) is not a period.

Definition 3. An infinite continued fraction \([\alpha_0, \alpha_1, \ldots]\) is said periodic if for some non negative integer \(N\) and some natural \(P:\)
\[\alpha_{N+j} = \alpha_{N+j \mod P}, \quad \forall j > 0.\]

It is denoted by:
\[\llbracket \alpha_0, \alpha_1, \ldots, \alpha_{N-1}, \alpha_N, \alpha_{N+1}, \ldots, \alpha_{N+P-1} \rrbracket,\]
and \([a_1, a_2, \ldots, a_P] := [\alpha_N, \alpha_{N+1}, \ldots, \alpha_{N+P-1}]\) is its period, of length \(P.\)
Definition 4. The *inverse* of the period of a periodic continued fraction is obtained by writing the period backwards. Example: the period \([a, b, c, d]\) is the inverse of the period \([d, c, b, a]\).

Definition 5. The period of a periodic continued fraction is said to be *palindromic* if there exists a cyclic permutation of it such that the permuted period is equal to its inverse period. For example, the following periods are palindromic

\[ [a, a, b, b] , \quad [a, a, b, a, a] \quad \text{and} \quad [a, b, a, c, c]. \]

Note that any period of a sole element is palindromic.

Definition 6. The period of a periodic continued fraction is said to be *even* if its length is even, and *odd* otherwise.

Remark. If a palindromic period is even, then there exist at least two different cyclic permutations of it such that the permuted periods are equal to their inverse. Example: \([abccba]\) and \([cbaabc]\).

Definition 7. The period of a periodic continued fraction is said to be *bipalindromic* if there is a cyclic permutation of it such that the permuted period can be subdivided into two palindromic odd sequences. For example, the periods

\[ \Gamma_1 = [a, b, c, b, a, d] \quad \text{and} \quad \Gamma_2 = [a, a, a, b] \]

are bipalindromic, since \(\Gamma_1\) can be written as \([(b, c, b)(a, d, a)]\) (or \([(b, a, d, a, b)(a)]\), etc.) and \(\Gamma_2\) as \([(a)(a, b, a)]\) (or \([(b)(a, a, a)]\)).

Note that any period of two different elements is, according to the definition above, bipalindromic. Hence a non palindromic period contains at least three different elements.

Remark. The palindromicity of a period of \(P\) elements (natural numbers) can be seen as the symmetry, with respect to an axis, of a plane polygon, whose vertices are labeled by these natural numbers. If \(P\) is odd, a vertex must belong to the symmetry axis, whereas if \(P\) even, either no vertices belong to the symmetry axis (and hence every element has its symmetric element), or two vertices belong to the symmetry axis (and hence these two vertices have no symmetric element). This last case corresponds to bipalindromicity.

2. The symmetry types of the classes of quadratic forms

By \(f\) we denote the triple of integer coefficients of the binary quadratic form \(f = mx^2 + ny^2 + kxy\).
In Part I we called $\mathcal{J}$ the group, isomorphic to $\text{PSL}(2, \mathbb{Z})$, acting on the space of the form coefficients $(m, n, k)$, and whose action is induced by that of $\text{SL}(2, \mathbb{Z})$ on the plane $(x, y)$, where the binary forms are defined.

The class of the form $f = (m, n, k)$ under $\mathcal{J}$ is denoted by $C(f)$ or by $C(m, n, k)$.

We recall the classification of the symmetry types of classes of indefinite binary quadratic forms (i.e., with discriminant $\Delta = k^2 - 4mn < 0$), already introduced in Part I. We have considered three commuting involutions, acting in the space of forms and defining, with $f$, 8 forms (see Figure 1-III). In particular, given $f = (m, n, k)$:

1. the form $f_c = (n, m, -k)$ is the complementary of the form $f$;
2. the form $\overline{f} = (m, n, -k)$ is the conjugate of the form $f$;
3. the form $f^* = (-n, -m, k)$ is the adjoint of the form $f$;
4. the form $\overline{f}^* = (-n, -m, -k)$ is the antipodal of the form $f$, and is the adjoint of the conjugate (or the conjugate of the adjoint) of the form $f$;
5. the form $-f = (-m, -n, -k)$ is the opposite of the form $f$, and it is the complementary of the adjoint of the form $f$.

Remarks.

- Any form $(m, n, 0)$ is selfconjugate, i.e., $\overline{f} = f$.
- Any form $(m, -m, k)$ is selfadjoint, i.e., $f^* = f$.

The complementary of a form $f = (m, n, k)$ belongs to the class of $f$, $C(m, n, k)$, while the conjugate and/or the adjoint forms of $f$ may or may not belong to the class of $f$.

However (see Proposition 1.2 in Part I) if a class contains a pair of forms related by some involution, or a form which is invariant under some involution, then the entire class is invariant under that involution.

Hence, according to the different symmetries of forms that we have considered, there are exactly five types of symmetries of the classes.

**Definition 8.** A class of forms is said to be:

1. **asymmetric** if it contains only pairs of complementary form;
2. **$k$-symmetric** if it contains only pairs of complementary forms and conjugate forms or isolated selfconjugate forms;
3. **$(m+n)$-symmetric** if it contains only pairs of complementary forms and adjoint forms and isolated selfadjoint forms;
4. **antisymmetric** if it contains only pairs of complementary forms and antipodal forms;
(5) *supersymmetric* if it contains all pairs of complementary, conjugate, adjoint (and thus antipodal) forms.

**Remark.** In term of the symmetries in the plane \((x, y)\), given a form \(f(x, y) (= f(-x, -y))\) in the class \(C(f)\),

1. the complementary form \(f(-y, x) = f(y, -x)\) belongs to \(C(f)\);
2. the form \(f(y, x)\) and its complementary form \(f(x, -y) = f(-x, y)\) belong to \(C(\bar{f})\);
3. the form \(-f(x, y)\) and its complementary form \(-f(y, -x) = -f(y, -x)\) belong to \(C(f^*)\);
4. the form \(-f(y, x)\) and its complementary form \(-f(x, -y) = -f(-x, y)\) belong to \(C(\bar{f}^*)\).

### 3. Results

In this section we enunciate the theorems, which will be proved in the next sections, using mainly the results of Part I [1].

#### 3.1. Basic Theorems

Let \(f := (m, n, k)\) be a triple of integers such that \(k^2 - 4mn > 0\). The ordered pair \((\xi^+(f), \xi^-(f))\) denotes the roots (the first with sign plus, the second with sign minus) of the quadratic equation \(m\xi^2 + k\xi + n = 0:\)

\[
\xi^\pm(f) = \frac{-k \pm \sqrt{k^2 - 4mn}}{2m}.
\]

Suppose \(\xi^\pm(f)\) be irrational.

**Theorem 3.1.** The continued fractions of roots \(\xi^+(f)\) and \(\xi^-(f)\) are periodic and their periods, up to cyclic permutations, are one the inverse of the other.

**Theorem 3.2.** The ordered pair of periods of the continued fractions of \((\xi^+(f), \xi^-(f))\), both periods considered up to cyclic permutations, is an invariant of the class \(C(m, n, k)\).

Because of Theorem 3.1, the symmetry properties of the periods of \(\xi^+\) and \(\xi^-\) coincide.

**Notation.** In the sequel, \(\Gamma(m, n, k)\) will denote the period of the continued fraction of \(\xi^+(m, n, k)\).

**Theorem 3.3.** Any period \(s\) defines the class \(C(m, n, k)\) such that \(\Gamma(m, n, k) = s\) up to a multiplicative positive integer constant and up antipodal involution.

\(^2\)This theorem was probably already known to Lagrange, Galois, etc. A geometrical proof of the first part is given in [4], and of the second part in [5].
**Definition.** A form $f$ is said to be *primitive* if cannot be written as $af'$ for another integer form $f'$ and $a > 0$. All forms in the same class are either primitive or non-primitive. A class of primitive forms is said primitive.

The theorems of the next section imply, moreover, the following

**Corollary 3.4.** Any period $s$ of length $P$ defines uniquely the primitive class $C(m, n, k)$ such that $\Gamma(m, n, k) = s$ if and only if $P$ is odd.

### 3.2. Theorems on the symmetries of the periods.

**Theorems 3.5–9** There is an one-to-one correspondence between the 5 symmetry types (in Definition 8) of the classes $C(m, n, k)$ and the 5 symmetry types of their corresponding periods $\Gamma(m, n, k)$.

This one-to-one correspondence is stated separately for each type of symmetry in the following 5 theorems.

**Theorem 3.5.** The period $\Gamma(m, n, k)$ is palindromic and even iff the class $C(m, n, k)$ is $(m+n)$-symmetric.

**Example.** $m = 5$, $n = -7$, $k = 9$. $\Gamma = [1, 1, 2, 2]$. In the same class we have:

| $m$ | $n$ | $k$ | $\Gamma$ |
|-----|-----|-----|---------|
| 5   | -7  | 9   | [1, 1, 2, 2] |
| 7   | -7  | -5  | [1, 2, 2, 1] |
| 7   | -5  | 9   | [2, 2, 1, 1] |
| 5   | -5  | -11 | [2, 1, 1, 2] |

Note that the class contains a pair of adjoint forms and two self-adjoint forms.

**Theorem 3.6.** The period $\Gamma(m, n, k)$ is palindromic and odd iff the class $C(m, n, k)$ is supersymmetric.

**Example.** $m = 1$, $n = -2$, $k = -3$. $\Gamma = [3, 1, 1]$. In the same class:

| $m$ | $n$ | $k$ | $\Gamma$ |
|-----|-----|-----|---------|
| 1   | -2  | -3  | [3, 1, 1] |
| 1   | -2  | 3   | [1, 1, 3] |
| 2   | -2  | -1  | [1, 3, 1] |
| 2   | -1  | 3   | [3, 1, 1] |
| 2   | -1  | -3  | [1, 1, 3] |
| 2   | -2  | 1   | [1, 3, 1] |

Note that the orbit contains two selfadjoint forms, which are conjugate.

**Theorem 3.7.** The period $\Gamma(m, n, k)$ is bipalindromic iff the class $C(m, n, k)$ is $k$-symmetric.
Example. $m = 1, n = -2, k = -5$. $\Gamma = [5, 2, 1, 2]$. In the same class:

| $m$ | $n$ | $k$ | $\Gamma$ |
|-----|-----|-----|----------|
| 1   | -2  | -5  | $[5, 2, 1, 2]$ |
| 1   | -2  | 5   | $[2, 1, 2, 5]$ |
| 3   | -2  | -3  | $[1, 2, 5, 2]$ |
| 3   | -2  | 3   | $[2, 5, 2, 1]$ |

Note that the orbit contains two pairs of conjugate forms.

Remark. The square root of a rational number $\sqrt{p/q}$ has continued fraction with period either odd and palindromic or bipalindromic, since it is the root of the equation $qx^2 - p = 0$, corresponding to a class of forms either $k$-symmetric or supersymmetric. This answers a question that Arnold posed in [3].

Theorem 3.8. The period $\Gamma(m, n, k)$ is non palindromic and odd iff the class $C(m, n, k)$ is antisymmetric.

Example. $m = 5, n = -3, k = -13$. $\Gamma = [2, 1, 4]$. In the same class:

| $m$ | $n$ | $k$ | $\Gamma$ |
|-----|-----|-----|----------|
| 5   | -3  | -13 | $[2, 1, 4]$ |
| 5   | -9  | 7   | $[1, 4, 2]$ |
| 3   | -9  | -11 | $[4, 2, 1]$ |
| 3   | -5  | 13  | $[2, 1, 4]$ |
| 9   | -5  | -7  | $[1, 4, 2]$ |
| 9   | -3  | 11  | $[4, 2, 1]$ |

Note that the orbit contains three pairs of antipodal forms.

Theorem 3.9. The period $\Gamma(m, n, k)$ is non palindromic and even iff the class $C(m, n, k)$ is asymmetric.

Example. $m = 5, n = -15, k = 18$. $\Gamma = [1, 2, 3, 4]$. In the same class:

| $m$ | $n$ | $k$ | $\Gamma$ |
|-----|-----|-----|----------|
| 5   | -15 | 18  | $[1, 2, 3, 4]$ |
| 8   | -15 | -12 | $[2, 3, 4, 1]$ |
| 8   | -7  | 20  | $[3, 4, 1, 2]$ |
| 5   | -7  | -22 | $[4, 1, 2, 3]$ |

In [2], Arnold posed the question whether the roots of all quadratic equations of type $x^2 + kx + n = 0$ are palindromic. The answer is given by the following corollary.

Corollary 3.10. The continued fractions of the quadratic surds corresponding to a form whose class represents 1 have period either odd and palindromic or even and bipalindromic.
3.3. **Theorems on the numbers of representatives.** The theorems below complete the results of Part I and refer to particular domains of the space of forms, which are defined there.

We will see, proving the theorems above, that the period \( \Gamma(m, n, k) \) is related to the cycle (of to half of it) composed by the representatives of the class \( C(m, n, k) \) satisfying \( m > 0 \) and \( n < 0 \) (thus belonging to \( H^0 \)).

Every class \( C(m, n, k) \) has the same symmetry of the cycle of its representatives in \( H^0 \). However, a cycle with some of the symmetries of the classes that we have considered, could possess a priori some higher symmetry, namely that of an \( n \)-gon. The following theorem excludes this possibility.

**Theorem 3.11.** The cycle in \( H^0 \) cannot have symmetries other than those of its class.

Let \( \Gamma(m, n, k) = [a_1, \ldots, a_P] \).

**Definition.** If \( P \) is odd, define

\[
\Pi(m, n, k) := \Gamma^2 = [a_1, a_2, \ldots, a_P, a_{P+1}, \ldots, a_P]
\]

where \( p = 2P \) and \( a_{P+i} = a_i \), for \( i = 1, \ldots, P \); otherwise, \( \Pi(m, n, k) = \Gamma(m, n, k) \) and \( p = P \).

**Theorem 3.12.** Let \((m, n, k)\) be any triple of integers such that \( k^2 - 4mn > 0 \) is different from a square number, and \( \Pi(m, n, k) = [a_1, a_2, \ldots, a_p] \). Define

\[
t_{\text{odd}} := \sum_{i \text{ odd}}^p a_i, \quad t_{\text{even}} := \sum_{i \text{ even}}^p a_i, \quad t := \sum_{i}^p a_i.
\]

Class \( C(m, n, k) \) has \( t \) points in \( H^0 \) and in \( H^0_B \); has \( t_{\text{odd}} \) points in every domain of \( G_A \) and \( G_A \) and has \( t_{\text{even}} \) points in every domain of \( G_B \) and \( G_B \) (or vice versa).

Moreover \( t_{\text{odd}} = t_{\text{even}} = t/2 \) if \( \Gamma \) is either odd or even and palindromic, i.e., if the corresponding form is supersymmetric, antisymmetric, or \((m + n)\)-symmetric.

In Section 4 of Part I we have seen that if \( \Delta \) is equal to a square number, each class has representatives on the boundaries of the domains of \( C_H \). In particular, Theorem 4.14 says that there are \( k \) distinct classes with discriminant equal to \( k^2 \). These \( k \) classes have a fixed number of representatives inside each domain. The following theorems deduce the number of points inside the domains of \( C_H \) of every class and its symmetry type from the finite continued fraction of a rational number, related to a representative of that class.

**Remark.** The last element of a finite continued fraction is greater than 1.
**Definition.** We call odd continued fraction of a rational number \( r > 1 \) the finite continue fraction \([a_1, \ldots, a_N]\) of \( r \), if \( N \) is odd, otherwise the continue faction \([a_1, a_2, \ldots, a_{N-1}, 1]\).

Similarly, we call even continued fraction of a rational number \( r > 1 \) the finite continue fraction \([a_1, \ldots, a_N]\) of \( r \), if \( N \) is even, otherwise the continue faction \([a_1, a_2, \ldots, a_{N-1}, 1]\).

Note that the odd (even) continued fraction of \( r = [a_1, \ldots, a_N] \), when \( N \) is even (resp., odd), still represents \( r \), indeed \([a_1, \ldots, (a_N-1), 1] = a_1 + \frac{1}{\cdots + \frac{1}{(a_N-1)+\ddots}} = a_1 + \frac{1}{\cdots + \frac{1}{a_N}} = [a_1, \ldots, a_N]\).

**Theorem 3.13.** Let \( k > m > 0 \) and \([a_1, \ldots, a_L]\) be the even continued fraction of the rational number \( k/m \). Define

\[
(t)_{\text{odd}} := \sum_{i \text{ odd}}^{L} a_i - 1, \quad (t)_{\text{even}} := \sum_{i \text{ even}}^{L} a_i - 1, \quad t := \sum_{i}^{L} a_i - 1.
\]

The following statements hold:

i) Class \( C(m, 0, k) \) in \( H^0 \) and \( H^0_B \) has \( t \) points, in every domain of \( G_A \) and \( G_A^{-} \) has \( t_{\text{odd}} \) points and in every domain of \( G_B \) and \( G_B^{-} \) it has \( t_{\text{even}} \) points.

ii) Moreover, \( t_{\text{odd}} = t_{\text{even}} = (t - 1)/2 \) if \( C(m, 0, k) \) is \((m + n)\)-symmetric.

**Theorem 3.14.** Let \( \Delta = k^2 \) and \( 0 \leq m < |k| \).

i) Class \( C(m, 0, k) \) is supersymmetric iff \( m = 0 \) or if \( k \) is even and \( m = k/2 \) (the discriminant is in this case divisible by 4).

ii) \( C(m, 0, k) \) is \((m + n)\)-symmetric iff the even continued fraction of \( k/m \) is palindromic.

iii) \( C(m, 0, k) \) is \(k\)-symmetric iff the odd continued fraction of \( k/m \) is palindromic.

iv) Class \( C(m, 0, k) \) cannot be antisymmetric.

v) \( C(m, 0, k) \) is asymmetric iff neither the odd nor the even continued fraction of \( k/m \) are palindromic.

The Appendix contains examples of the theorems above.

The last section 5 is devoted to the reduction theory for indefinite forms, from the geometrical view point of our model. We will prove, moreover, the following theorem on the periodic modular fractions (see eqq. (12) and (13)):

**Theorem 3.15.** Let \((c_1, c_2, \ldots, c_t)\) be the period of the modular fraction of a quadratic surd. Then

\[
\sum_{i=1}^{t} c_i = 3t
\]

if the corresponding class is supersymmetric, antisymmetric or \((m + n)\)-symmetric.
4. Proofs

4.1. Fundamental lemmas. In Part I we defined, for any $\Delta > 0$ such that $\Delta \mod 4 = 0, 1$:

$$H_{\Delta} = \{(m, n, k) \in \mathbb{R}^3 : k^2 - 4mn = \Delta\}.$$ 

This is the space of quadratic forms

$$f = mx^2 + ny^2 + kxy$$

with real coefficients and fixed discriminant (see Figure I-I).

Moreover, we have defined the projection $Q$ of the hyperboloid $H_{\Delta}$ to the open cylinder $C_H$ (Figure I-III).

Lemma 4.1. There exists an one-to-one correspondence between the cylinder $C_H$ and the space

$$\Xi = \{(\xi^+(f), \xi^-(f)) \in \mathbb{R}P^1 \times \mathbb{R}P^1 \setminus \{(\xi, \xi) | \xi \in \mathbb{R}P^1\}, \ f \in H_{\Delta}\}.$$ 

where $\xi^+(f)$ and $\xi^-(f)$ are the roots of the equation $f = 0$ for the variable $\xi = \frac{x}{y} \in \mathbb{R}P^1$.

Proof.

Map $Q : H_{\Delta} \to C_H$ (see eq. (15) in Part I) is a homeomorphism. Also eq. (1) defines a homeomorphism between $H_{\Delta}$ and $\Xi$. We want, however, to see explicitly the one-to-one correspondence between $C_H$ and $\Xi$.

In Figure I-II the cylinder $C_H$ is depicted replacing the curved segments bounding some of its domains by straight line segments.

Note that cylinder I-I is obtained by the torus $\mathbb{R}P^1 \times \mathbb{R}P^1$ minus its diagonal (Figure I-IV). The circles $c_1$ and $c_2$, frontier of $C_H$, represent the limit points at infinite of the hyperboloid, which coincide with those of the cone $\Delta = 0$. For these limit values of the coefficients, the roots of the corresponding quadratic equations tend to a same value. Hence the two circles correspond to the diagonal $\xi^+ = \xi^-$. The root $\xi^+$ vanishes when $n = 0$ and $k > 0$, and $\xi^-$ vanishes when $n = 0$ and $k < 0$. The values of $\xi^+$ and $\xi^- = \pm \infty$ are reached when $m = 0$, and they, too, changes sign when $m$ changes sign. Note that lines $m = 0$ and $n = 0$ are the boundaries of domains $H_0^0$ and $H_0^R$. The rhomboidal regions $H_0^0$ and $H_0^R$ in $C_H$ (Figure I-III and Figures 8 and 10 of Part I) are represented by true rhombi in Figure I-II. These regions are thus represented in $\Xi$ by the square regions $\xi^+ \cdot \xi^- < 0$, denoted by $H_0^0$ and $H_0^R$ in Figure I-IV as well. Outside $H_0^0$ and $H_0^R$ coefficients $m$ and $n$ have the same sign and there are four important domains: $H_A$ and $H_A$, where $m$ and $m$ are positive, $m + n < k$, $k > 0$ ($H_A$) and $m + n < -k$, $k < 0$ ($H_A$); $H_B$ and $H_B$,
where $m$ and $n$ are negative, $m + n > k$, $k < 0$ ($H_B$) and and $m + n > -k$, $k < 0$ ($H_B$).

They are mapped, respectively, to the domains:

$$
H_A = \{(\xi^+, \xi^-) : -1 < \xi^+ < 0 , \quad \xi^- < -1\};
$$
$$
H_A^- = \{(\xi^+, \xi^-) : \quad \xi^+ > 1 , \quad 0 < \xi^- < 1\};
$$
$$
H_B = \{(\xi^+, \xi^-) : \quad \xi^+ < -1 , \quad -1 < \xi^- < 0\};
$$
$$
H_B^- = \{(\xi^+, \xi^-) : \quad 0 < \xi^+ < 1 , \quad \xi^- > 1\}.
$$
These considerations are sufficient to determine the complete correspondence. To obtain the ‘square’ of Figure 1-IV we have to cut the ‘rectangle’ of Figure 1-II along the lines $m = 0$, so obtaining two triangles: one containing $H^0$ (with circle $c_1$ as base) and the other containing $H^0_R$ (with circle $c_2$ as base, see Figure 2). Then, we put the triangle containing $H^0$ above the other, as shown in figure. Finally, turn the figure so obtained by $\pi/4$ counterclockwise. Note that this procedure preserves the continuity. \hfill $\square$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Correspondence between $C_H$ and $\Xi$.}
\end{figure}

Remark. The complementary form $f_c$ of the form $f$ in $C_H$ is represented by a point with the same ordinate as $f$ and shifted by $\pi$ in the horizontal direction, whereas the conjugate, the adjoint and the antipodal forms are symmetrical with respect to $f$ as shown in Figure 1-III.

The following relations hold among the pairs $(\xi^+, \xi^-)$ of the triples obtained from the triple $f = (m, n, k)$ by all the considered involutions.

| $f$   | $(m, n, k)$ | $\xi^+$ | $\xi^-$ | $f_c = -f^*$ | $(n, m, -k)$ | $-1/\xi^+$ | $-1/\xi^-$ |
|-------|-------------|---------|---------|-------------|-------------|-----------|-----------|
| $f^*$ | $(m, n, -k)$ | $-\xi^-$ | $-\xi^+$ | $f_c = -f$  | $(n, m, k)$ | $1/\xi^-$  | $1/\xi^+$  |
| $f^*$ | $(n, -m, k)$ | $-1/\xi^-$ | $-1/\xi^+$ | $f_c = -f$  | $(-m, -n, -k)$ | $\xi^-$  | $\xi^+$  |
| $f^*$ | $(n, -m, -k)$ | $1/\xi^-$ | $1/\xi^-$ | $f_c = -f$  | $(-m, -n, k)$ | $-\xi^+$  | $-\xi^-$  |

Table 1

Therefore, in $\Xi$ the complementary forms of the forms in $H^0$ are obtained moving $H^0$ by a translation over $H^0_R$, and vice versa, and those outside $H^0$ and outside $H^0_R$ by moving the upper-right quarter of $\Xi$ over the lower-left, and vice versa. The antipodal symmetry, which is a reflection with respect to the centre of $H_0$ and $H^0_R$, becomes the reflection with respect to point $(1, -1)$ or $(-1, 1)$, etc.

Remember that we denoted by $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the generators of $\text{SL}(2, \mathbb{Z})$ acting on the $(x, y)$-plane and by $A$, $B$, $R$ the corresponding generators of $\mathcal{F}.$
Definition. The operators \( \alpha, \beta, \) and \( \sigma, \) acting on \( \mathbb{R}P^1, \) and corresponding to the operators \( A, B, R \) of \( \mathcal{T}, \) are defined by

\[
\alpha(\xi^{\pm}(f)) = \xi^{\pm}(A(f)), \quad \beta(\xi^{\pm}(f)) = \xi^{\pm}(B(f)), \quad \sigma(\xi^{\pm}(f)) = \xi^{\pm}(R(f)).
\]

Lemma 4.2. The actions of the operators \( \alpha, \beta \) and \( \sigma \) on the roots \( \xi^{\pm} \) coincide with those of the inverse of the homographic operators \( A, B \) and \( R \) defined by the generators \( A, B \) and \( R \) of \( \text{SL}(2, \mathbb{Z}). \)

Proof. The actions of the inverse homographic operators \( A^{-1} \) and \( B^{-1} \) (see equations (9) and (12) of Part I), are:

\[
A^{-1} : \xi \rightarrow \frac{\xi - 1}{1}; \quad B^{-1} : \xi \rightarrow \frac{1}{-1 + 1/\xi}; \quad R^{-1} = R : \xi \rightarrow -\frac{1}{\xi}.
\]

On the other hand, if \( \xi^{\pm} \) are the two roots of \( f(x, y)|_{y=1} = 0, \) then \( \alpha(\xi^{\pm}) \) are by definition the corresponding roots of \( f(x + y, y)|_{y=1} = f(x + 1, 1) = 0, \) and hence they are equal to \( \xi^{\pm} - 1. \)

By definition, \( \beta(\xi^{+}) \) is the first root of the equation \( f(x, x + y)|_{y=1} = 0. \) Note that \( 1/\xi^{+} = \frac{1}{2n} \sqrt{\Delta} \) is the second root, \( w^{-} \), of the equation \( f(1, y) = 0. \) Hence \( \beta(w^{-}) \) is, by the above definition, the second root of \( f(x, y + x)|_{x=1} = f(1, y + 1) = 0. \) We thus have \( \beta(1/\xi^{+}) = 1/\xi^{+} - 1. \) The first root of the equation \( f(x, x + y)|_{y=1} = 0 \) is thus equal to \( 1/\beta(1/\xi^{+}) = \frac{1}{-1 + 1/\xi^{+}}. \) For \( \beta(\xi^{-}) \) the proof is analogous (exchanging \( + \) with \( - \), and first with second).

Finally, note that \( -1/\xi^{\pm} = \frac{k + \sqrt{\Delta}}{2n} \) are exactly the first and second roots of \( f(-y, x)|_{y=1}, \) i.e., \( -1/\xi^{\pm} = \xi^{\pm}(R(f)) \) and hence they are equal, by definition, to \( \sigma(\xi^{\pm}(f)). \)

Denote by \( \bar{\alpha} \) and \( \bar{\beta} \) the elements \( \alpha^{-1} \) and \( \beta^{-1}. \)

Remark. The above lemma defines an isomorphism between the group \( \mathcal{T}, \) acting on the space of forms, and the group generated by \( \alpha, \beta, \) and \( \sigma, \) acting on the space of ordered pairs of quadratic equation roots.

The actions of the operators \( \bar{\alpha} \) and \( \bar{\beta} \) are

\[
\bar{\alpha}(\xi) = \xi + 1; \quad \bar{\beta}(\xi) = \frac{1}{1 + 1/\xi}.
\]

The following lemma holds for all real numbers, with continued fraction non necessarily periodic and infinite.

Here \( \alpha^n \) and \( \beta^n \) denote the n-th iterations of \( \alpha \) and \( \beta. \)

Lemma 4.3. If \( \xi > 1, \) with continued fraction \( \xi = [a, b, c, \ldots] \), then

\[
\alpha^n(\xi) = [0, b, c, \ldots].
\]
If $0 < \xi < 1$, with continued fraction $\xi = [0, d, e, g, \ldots]$, then
\[ \beta^d(\xi) = [e, g, \ldots]. \]

**Proof.** By Lemma 4.2

\[ \alpha(\xi) = \xi - 1, \quad \text{and} \quad \alpha^a(\xi) = \xi - a. \]

Hence, if

\[ \xi = a + \frac{1}{b + \frac{1}{c + \frac{1}{d} + \ldots}}, \]

then

\[ \alpha^a(\xi) = \frac{1}{b + \frac{1}{c + \frac{1}{d} + \ldots}} = [0, b, c, \ldots]. \]

Moreover,

\[ \beta(\xi) = \frac{1}{-1 + \xi}, \quad \text{and} \quad \beta^d(\xi) = \frac{1}{-d + \frac{1}{\xi}}. \]

Hence, if

\[ \xi = \frac{1}{d + \frac{1}{e + \frac{1}{g + \ldots}}}, \]

then

\[ \beta^d(\xi) = \frac{1}{-d + \frac{1}{\frac{1}{d + \frac{1}{e + \frac{1}{g + \ldots}}}}} = \frac{1}{-d + \frac{1}{e + \frac{1}{g + \ldots}}} = e + \frac{1}{g + \ldots} = [e, g, \ldots]. \]

\[ \square \]

**Lemma 4.4.** If $\xi < -1$, with continued fraction $\xi = [-a, b, c, \ldots]$, then

\[ \bar{\alpha}^a(\xi) = -[0, b, c, \ldots]. \]

If $-1 < \xi < 0$, with continued fraction $\xi = [-0, d, e, g, \ldots]$, then

\[ \bar{\beta}^d(\xi) = -[e, g, \ldots]. \]

**Proof.** It is analogous to the proof of the preceding lemma: indeed, by Lemma 4.2 and the Remark following it,

\[ \bar{\alpha}^a = \alpha^{-a}(\xi) = \xi + a; \]

so, if $\xi = [-a, b, c, \ldots]$, i.e.,

\[ \xi = -a - \frac{1}{b + \frac{1}{c + \frac{1}{d} + \ldots}}, \]

then

\[ \xi + a = -\frac{1}{b + \frac{1}{c + \frac{1}{d} + \ldots}}. \]
Moreover,
\[ \beta^d(\xi) = \beta^{-d}(\xi) = \frac{1}{d + 1/\xi}; \]
so, if \( \xi = [-0, d, e, g, \ldots] \), i.e.,
\[ \xi = -\frac{1}{d + 1/\xi} + \frac{1}{e + 1/\xi} + \cdots, \]
then
\[ \frac{1}{d + 1/\xi} = -\frac{1}{e + 1/\xi}. \]

We group in the following Lemma some observations.

Let a black arrow from the point \( f \) to the point \( g \) in \( C_H \) indicate that \( Af = g \), and a white arrow that \( Bf = g \).

**Lemma 4.5.** If the points \( f \) and \( g \) in \( C_H \) are joined by an arrow from \( f \) to \( g \), then
1) the points \( f^* \) and \( g^* \) in \( C_H \), symmetric with respect to the horizontal line of points \( f \) and \( g \), are joined by an arrow from \( g^* \) to \( f^* \) of the opposite color (see Figure 3-I,II);
2) the points \( \mathbf{f} \) and \( \mathbf{g} \) in \( C_H \), symmetric with respect to the the vertical line \( k = 0 \) of points \( f \) and \( g \), are related by an arrow from \( \mathbf{g} \) to \( \mathbf{f} \) of the same color (see Figure 3-II,III);
3) the points \( \mathbf{f}^* \) and \( \mathbf{g}^* \) in \( C_H \), symmetric with respect to the centre of \( H^0 \) of points \( f \) and \( g \), are related by an arrow from \( \mathbf{f}^* \) to \( \mathbf{g}^* \) of the opposite color (see Figure 3-II-IV).

**Proof.** The proof of the corresponding identities:

\[
\begin{align*}
1) \quad & (Af)^* = Bf^*; \quad (Bf)^* = Af^*; \\
2) \quad & Af = A \mathbf{f}; \quad Bf = B \mathbf{f}; \\
3) \quad & (Af)^* = Bf^*; \quad (Bf)^* = Af^*.
\end{align*}
\]

is given in Part I, Lemma 1.3.

4.2. **Classes non representing zero.** Because of the correspondence stated by Lemma 4.1, the same symbols will denote the regions in \( C_H \) and in \( \Xi \), like in Figure 1.

**Proof of Theorems 3.1 and 3.2.** We prove initially that in \( H^0 \) the continued fractions of \( \xi^+ \) and \( \xi^- \) are periodic and their periods are one the inverse of the other.

**Definition.** We call principal point a point \( f \) belonging to a cycle in \( H^0 \) such that its consecutive point is reached by \( A \) iff \( f \) is reached by \( B \) from its preceding point, or vice versa.

Let \( h \) be a principal point in \( H^0 \). It belongs, by Theorem 4.4 and Corollary 4.12 of Part I, to a cycle \( \gamma_h(T_1, \ldots T_t) \), being \( T_i \) equal either to \( A \) or to \( B \). Let \( T_1 = A \).
Hence we can write \( Th = h \), where \( T = B^{a_p}A^{a_{p-1}} \ldots B^{a_2}A^{a_1} \), grouping together in \( p \) groups, subsequent operators \( T_i \) of type \( A \) or of type \( B \), so that \( \sum_{i=1}^{p} a_i = t \). The cycle of \( h \) will be thus \( \gamma_h(A^{a_1}, B^{a_2}, \ldots, A^{a_{p-1}}, B^{a_p}) \), indicating that \( h_{a_1} = A^{a_1}h, \ h_{a_1+a_2} = B^{a_2}A^{a_1}h, \) etc, till \( h_t = Th = h \). Let \( (\xi^+, \xi^-) \) be the pair of roots associated to \( h \). By Lemmas 4.1 and 4.2, there is an analogous operator to \( T \), say \( \tau \), obtained from \( T \) by translating \( A \) into \( \alpha \) and \( B \) into \( \beta \), satisfying:

\[
(9) \quad \tau(\xi^+) = \xi^+, \quad \tau(\xi^-) = \xi^-.
\]

We observe now that, since \( h \) is in \( H^0 \), \( \xi^+ \) is positive and \( \xi^- \) is negative. Write \( \xi^+ = [b_1, b_2, \ldots] \). Applying \( A^{a_1+1} \) to \( h \), we exit from \( H^0 \), by Lemma 4.6 of Part I; so, we obtain \( \alpha^{a_1} \xi^+ = [b_1 - a_1, b_2, \ldots] = [0, b_2, \ldots] \) and hence \( b_1 = a_1 \). Analogously, applying \( B^{a_2+1} \) to \( A^{a_1}h \), we exit from \( H^0 \); so \( \beta^{a_2} \circ \alpha^{a_1}(\xi^+) = [b_3, b_4, \ldots] \), i.e., \( b_2 = a_2 \). In this way we obtain \( b_i = a_i \) for \( i = 1, \ldots, p \). But, at the end of the cycle, we get, by eq. (9)

\[
\tau(\xi^+) = [b_{p+1}, b_{p+2}, \ldots] = [a_1, a_2, \ldots, a_p, b_{p+1}, b_{p+2}, \ldots].
\]

Similarly, applying by recurrence to \( \xi^+ \), for every natural \( j \), the \( j \)-th iterate \( \tau^j \) of \( \tau \):

\[
\tau^j(\xi^+) = [b_{jp+1}, b_{jp+2}, \ldots] = [a_1, a_2, \ldots, a_p, a_1, a_2, \ldots],
\]

we obtain that \([a_1, a_2, \ldots, a_{p-1}, a_p]\) are the first \( p \) elements of the continued fraction obtained from \( \xi^+ \) canceling out the first \( jp \) elements, so concluding

\[
\xi^+ = [a_1, a_2, a_3, \ldots, a_p].
\]

Note that \( p \) is even, since \( T \) begins with a power of \( B \) and ends with a power of \( A \).

We consider now \( \xi^- \). In this case it is convenient to write the right equation in (9) as

\[
\xi^- = \tau^{-1}(\xi^-),
\]

where \( \tau^{-1} = \bar{\alpha}^{a_1} \circ \bar{\beta}^{a_2} \ldots \bar{\alpha}^{a_{p-1}} \circ \bar{\beta}^{a_p} \). Again by Lemma 4.6 of Part I, we know that, applying \( \bar{A} \) to \( h \), we exit from \( H^0 \), i.e, \( \bar{\alpha}(\xi^-) = \xi^- + 1 > 0 \), and hence \( 0 > \xi^- > -1 \), i.e., \( \xi^- = [-0, c_1, c_2, \ldots] \). For analogous arguments, we will find, using again Lemma 4.4 that \( \bar{\beta}^{a_p}(\xi^-) = [-c_2, c_3, \ldots] \) and hence \( c_1 = a_p \). Analogously, applying \( \bar{A}^{a_p-1} \) to \( \bar{B}^{a_1}h \), we obtain \( \bar{\alpha}^{a_{p-1}} \circ \bar{\beta}^{a_p}(\xi^-) = [-c_3, c_4, \ldots] \), i.e., \( c_2 = a_{p-1} \). In this way, we obtain \( c_i = a_{p+1-i} \), for \( i = 1, \ldots, p \). So, at the end of the cycle, we get, by eq. (9)

\[
\tau^{-1}(\xi^-) = [-c_{p+1}, c_{p+2}, \ldots] = [-a_p, a_{p-1}, \ldots, a_2, a_1, c_{p+1}, c_{p+2}, \ldots],
\]

and hence also that \([a_p, a_{p-1}, \ldots, a_2, a_1]\) are the first \( p \) elements of the continued fraction of any iterate of \( \tau^{-1} \) on \( \xi^- \):

\[
\tau^{-j}(\xi^-) = [-c_{jp+1}, c_{jp+2}, \ldots] = [-a_p, a_{p-1}, \ldots, a_1, a_p, a_{p-1}, \ldots],
\]
so concluding
\[ \xi^- = -[0, [a_p, a_{p-1}, \ldots, a_2, a_1]]. \]

We have proved that the roots \( \xi^\pm(h) \) are periodic and their periods are one the inverse of the other when \( h \in H^0 \) is a principal point, i.e., the operator \( T \) of \( T^+ \) satisfying \( Th = h \), starts by \( A \) and ends by \( B \) (or vice versa).

It is now clear that a non principal point in \( H^0 \), in the cycle of \( h \) between \( h \) and \( h_{a_1} \) (whenever \( a_1 > 1 \)), is defined by
\[ h_j = A^j h, \]
for some \( j < a_1 \), and satisfies
\[ \xi^+(h_j) = [a_1 - j, a_2, \ldots] = [a_1 - j, [a_2, a_3, \ldots, a_p, a_1]], \]
and it has the same period as \( \xi^+(h) \), being the period defined up to cyclic permutations. Moreover
\[ \xi^-(h_j) = -[j, [a_p, a_{p-1}, \ldots] = -[j, [a_p, a_{p-1}, \ldots, a_1]]. \]

Since we may repeat the reasoning for any point of the cycle between two principal points, the periods of the continued fractions of \( \xi^\pm(h) \) are one the inverse of the other for every \( h \in H^0 \).

To complete the proof of Theorem 3.1 we have to consider points outside \( H^0 \). Every point belongs to an orbit and every orbit possess, by the results of Part I, a representative inside \( H^0 \). Any point of the orbit can be thus written as \( p = Th \), with \( h \in H^0 \) and \( T \in T \). Now, every element \( T \) of the group can be written as a finite product of the generators \( A, B \) and \( R \). By Lemma 4.2, we translate \( T \) into \( \tau \), composed by the corresponding generators \( \alpha, \beta \) and \( \sigma \). The action of each of these generators on a continued fraction affects evidently only its initial elements, and hence \( \tau \) affects only a finite initial part of \( \xi^\pm(h) \), so that the periods of \( \xi^\pm(\tau(h)) \) remain unchanged, up to cyclic permutations.

For instance, the periods of the continued fractions of \( \xi^\pm(h_c) \) are the same as those of \( \xi^\pm(h) \), because of the relations shown in Table 1, since \( h_c = Rh \). We have completed the proof of Theorem 3.1 and at the same time that of Theorem 3.2.

\[ \square \]

Remark. The above proof shows, for every class \( C(m, n, k) \) of indefinite forms with discriminant different from a square number, a correspondence between the cycle in \( H^0 \) (Theorem 4.13 of Part I) and the periods of continued fractions of \( \xi^\pm(m, n, k) \). As we remarked, the number \( p \) of groups of operators of type \( A \) and \( B \) which alternate in the cycle is necessarily even. The length \( P \) of the period of the continued fraction can be, however, odd. In this case the cycle corresponds indeed to the double of the period, and \( p = 2P \), as we will see in detail in Theorems 3.6 and 3.8.
Proof of Theorem 3.3. By the proof of Theorem 3.1, a root $\xi^+(f)$ or $\xi^-(f)$ is immediately periodic (i.e., there are no elements different from 0 before the period) only if $f$ is a principal point of $H^0$ or of $H^0_R$, and every class of forms, with discriminant different from a square number, has a principal representative in $H^0$ and in $H^0_R$.

Note that $\xi^+$ is positive and $\xi^-$ is negative in $H^0$, and vice versa in $H^0_R$. Remember also that inverting the order of the pair of roots (and hence of periods) corresponds to inverting the signs of the coefficients of the equation (i.e., of $f$).

Given a sequence $s = [a_1, \ldots, a_P]$, we firstly suppose that $[[a_1, \ldots, a_P]]$ is the first root of a quadratic equation with integer coefficients. To know such an equation, we write:

$$\xi = a_1 + \frac{1}{a_2 + \frac{1}{\ldots + \frac{1}{a_P + \frac{1}{\xi}}}}.$$

Call $(m, n, k)$ its coefficients and $f = (m, n, k)$.

If we suppose that the first root is less than 1, i.e., it is equal to $[0, [a_1, a_2, \ldots, a_P]]$, we obtain another equation:

$$\xi = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots + \frac{1}{a_P + \xi}}}}.$$

which will correspond to $f^*$ (see Table I).

Alternatively, supposing the first root be negative and greater than $-1$, i.e., equal to $[0, [a_1, a_2, \ldots, a_P]]$, we obtain another equation which correspond to $-f^*$.

Finally, supposing the first root be negative and less than $-1$, i.e., equal to $-[a_1, a_2, \ldots, a_P]]$, we obtain another equation which correspond to $-f$.

Since $-f^* = f^*$, and $-f = f^*$, the four triples of coefficients that we have obtained belong to only two classes, related by the antipodal symmetry. □

We give here the proofs of the theorems on the symmetries of the periods. We illustrate theorems 3.5-3.9 in Figure 3. The cycles in $H^0$ correspond indeed to the periods related to the forms considered in the examples. In these figures the small circles indicates the forms. Black circles correspond to principal points. The black arrow indicate the operator $A$ and the white arrow the operator $B$.

Lemma 4.6. If a cycle contains two points related by some symmetry, the cycle possesses that symmetry.

Proof. By Lemma 4.6 of Part I, a point of a cycle determines uniquely both the successive points and the preceding one, and, hence, all points of the cycle. Because of relations stated by Lemma 4.5, two symmetric points have symmetric arrows entering and exiting
Figure 3. The elements of the periods are equal to the number of arrows between two principal points (black dots).

from them, and hence two symmetric points determine the symmetries of their neighboring in the cycle, and so the symmetry of the entire cycle. \[\square\]

Note that the symmetries of a cycle in \(H^0\) concerns only its points and the segments joining them: the directions of the arrows and their colors are necessarily related by the rules given by Lemma 4.5.

Lemma 4.7. There are no vertical arrows in a cycle which contains more than two points.

Proof. Suppose a vertical white arrow relates two symmetric points \(f\) and \(f^*\). By the lemma above, a black arrow must relate \(f^*\) to \((f^*)^* = f\). So either these points form a cycle, or they cannot be joined by an arrow. \[\square\]

Lemma 4.8. A cycle \((m+n)\)-symmetric contains exactly two selfadjoint points. The selfadjoint points are principal.
Proof. Let \( f \) and \( f^* \) be two different points of the cycle (so they lie in the same vertical line). Let \( g \) be the successive of \( f \) (i.e., either \( Af = g \) or \( Bf = g \)). The symmetric \( g^* \) is in the cycle and is related to \( f^* \) by \( Bf^* \) or \( \bar{A}f^* \). Continuing with the successive points according to the arrows after \( g \), and their symmetric, according to the inverse arrows after \( g^* \), we have to close that part of the cycle, from \( f \) to \( f^* \). But since there are no arrows between adjoint points by Lemma 4.7, we must have, for some pair of adjoint points \( h \) and \( h^* \), that \( Ah = j \) and \( \bar{B}h^* = j \), with \( j \) selfadjoint (belonging to the line \((m + n) = 0\)). Point \( j \) is thus a principal point. In the remaining part of the cycle, from \( f^* \) to \( f \), there is a selfadjoint point for the same argument. We have now to prove that there are no other selfadjoint points. By the above argument, starting by any pair of adjoint points an reaching the successive pairs following the arrows in both directions, when we arrive the second selfadjoint point, we close the cycle. Since a point of the cycle cannot be visited twice by results of Part I, no other selfadjoint points are possible. \( \square \)

Lemma 4.9. If an orbit contains a selfadjoint point, is symmetric with respect to the axis \((m + n) = 0\).

Proof. Let \( h = h^* \) be the selfadjoint point. For the arguments of the preceding Lemma, if the successive of \( h \) is \( Ah \), then the preceding of \( h \) is \( \bar{B}h \). \( Ah \) and \( \bar{B}h \) are a pair of adjoint points. By Lemma 4.6, the orbit is at least \((m + n)\)-symmetric. \( \square \)

Proof of Theorem 3.5 (See Figure 3-I).

Suppose the class be \((m + n)\)-symmetric. This means that the cycle is invariant under reflection with respect to the horizontal axis \((m + n = 0)\). Choose one of the two selfadjoint points, \( h \), of the cycle (it exists by Lemma 4.8). Consider all points of the cycle between \( h \) and the second selfadjoint point, \( h' \), following the arrows. This procedure defines an operator \( M \in \mathcal{T}^+ \) such that

\[
Mh = h'.
\]

The points between \( h' \) and \( h \), forming the other part of the cycle, are the adjoint of the points from \( h \) to \( h' \). By Lemma 4.5

\[
\check{M}h = h',
\]

where \( \check{M} \) is obtained by \( M \) exchanging \( A \) with \( B \), and \( B \) with \( \bar{A} \). Equivalently, we may write

\[
\check{M}^{-1}h' = h,
\]

where \( \check{M}^{-1} \) is obtained by \( M \) inverting the order of the factors and exchanging \( A \) with \( B \). But thus \( \check{M}^{-1} = M' \), i.e., the transpose of \( M \), since the transpose of \( A \) and \( B \) are \( B \)
and $A$, respectively. Hence
\[ MM^\vee h = h. \]

So, if $M$ is the product of $q$ groups of generators of type $A$ and $B$: $M = A^{a_1}B^{a_2} \cdots A^{p-1}B^q$, the operator $T = MM^\vee$, defining the cycle, is he product $A^{a_1}B^{a_2} \cdots A^{q-1}B^q A^q B^{q-1} \cdots B^{a_2}A^{a_1}$.

The roots $\xi^\pm(h)$, satisfying: $\tau(\xi^\pm) = \xi^\pm$ are thus
\[ \xi^+ = [[a_1, a_2, \ldots, a_q, a_q, \ldots a_2, a_1]], \quad \xi^- = [0, [a_q, a_q-1, \ldots, a_1, a_1, a_2, \ldots, a_q]]. \]

Their periods have length $P = 2q$ and are palindromic.

On the other hand, given a period $\Gamma$ of length $P$, we associate to it a cycle, as we explained in the proof of Theorem 1, using Lemma 4.1. Now, if $\Gamma$ is even and palindromic, also the sequence of groups of operators $A$ and $B$ is even and palindromic in $T$, satisfying $Th = h$, for some principal point $h$. We thus can write $T = MM^\vee$, and
\[ h = MM^\vee h = (M^\vee)^{-1}M^{-1}h. \]

Since $M^\vee = \hat{M}$, and $\hat{M}^{-1} = M^\vee$, we have, by the second equality above
\[ h = \hat{M}M^{-1}h, \quad \text{and} \quad h^* = MM^\vee h^*. \]

Hence $h = h^*$ is selfadjoint. By Lemma 4.9, the cycle is $(m + n)$-symmetric. \(\Box\)

Remark. For the forms of such a class, $\Pi = \Gamma$ and $p = P = 2q$.

Proof of Theorem 3.6. (See Figure 3-II).

Suppose the class be supersymmetric. This means that the cycle is invariant under reflection with respect to the horizontal axis $(m + n = 0)$ as well as with respect to the vertical axis $(k = 0)$. Since a supersymmetric orbit has, in particular, the symmetry of a $(m + n)$-symmetric orbit, it has two selfadjoint points, by Lemma 4.6. Since these points lie on the horizontal line $(m + n) = 0$, and the cycle is supersymmetric, these points must be symmetric with respect to the vertical axis $k = 0$, i.e., they are conjugate. Call them $h$ and $\overline{h}$. Since $h^* = h$ and $\overline{h}^* = \overline{h}$, being these points selfadjoint, they must be principal points. Let $Ah$ and $\overline{B}h$ be the successive and preceding point of $h$. The successive and preceding of $\overline{h}$ are, by Lemma 4.5, $B\overline{h}$ and $A\overline{h}$, and
\[ A\overline{h} = \overline{A}h, \quad B\overline{h} = B\overline{h}. \]

If we consider the successive points of the cycle, after $h$ and $Ah$, we must meet the point $\overline{h}$, and necessarily, before it, $A\overline{h}$. To every point $f = Th$, successive to $h$, there correspond, in the same arc of cycle from $h$ to $\overline{h}$, a symmetric point, $\overline{f} = T\overline{h}$, where $T$ is obtained by $T$ substituting each $A$ by $\hat{A}$, and each $B$ by $\hat{B}$. Therefore we can write
\[ \overline{h} = Mh, \quad h = M\overline{h}, \]
obtaining \( M = \bar{M}^{-1} \). I.e., \( M \) consists in a palindromic sequence

\[
M = A^{a_1}B^{a_2} \cdots B^{a_q}A^{a_1}.
\]

Note that the sequence have to end by \( A \) (or by \( B \)) if it starts by \( A \) (respectively, by \( B \)). Hence the number of groups of \( A \) and \( B \) in \( M \) is odd.

Using Theorem 3.5, to close the cycle from \( \vec{h} \) to \( h \), we multiply \( M \) by \( M^\vee \), which is composed by the same sequence of \( q \) groups of generators, in inverse order and exchanging \( B \) with \( A \). But since \( M \) is itself palindromic, it is invariant under the inversion of the order of factors, and hence \( M^\vee \) is obtained by \( M \) simply exchanging \( A \) with \( B \).

Using Lemmas 4.1 and 4.3 as in proof of Theorem 3.1 we thus obtain that the continued fractions of \( \xi^\pm(h) \) are odd, 2 periods of them corresponding in fact to a cycle in \( H^0 \).

Conversely, if we have a continued fraction with odd palindromic period

\[
[a_1, a_2, \ldots, a_{q-1}, a_q, a_q-1, \ldots a_2, a_1]
\]

we associate to it an operator \( M = A^{a_1}B^{a_2} \cdots B^{a_2}A^{a_1} \), and solving \( MM^\vee h = h \), we find a selfadjoint form, and, by the symmetry of \( M \), the cycle of \( h \) results to be supersymmetric. \( \square \)

**Remark.** For the forms of such a class we thus have \( \Pi = \Gamma^2 \) and \( p = 2P = 2(2q + 1) \).

**Proof of Theorem 3.7.** (See Figure 3-III).

Suppose the class be \( k \)-symmetric. This means that the cycle is invariant under reflection with respect to the vertical axis (\( k = 0 \)). Choose any pair in the cycle of conjugate points, and call them \( h \) and \( \bar{h} \). Let \( A\bar{h} \) and \( \bar{B}h \) be the successive and preceding point of \( h \). The successive and preceding of \( \bar{h} \) are, by Lemma 4.5, \( B\bar{h} \) and \( A\bar{h} \), and

\[
\bar{A}\bar{h} = \bar{A}h, \quad \bar{B}h = B\bar{h}.
\]

If we consider the successive points of the cycle, after \( h \) and \( A\bar{h} \), we must meet the point \( \bar{h} \), and necessarily, before it, \( \bar{A}\bar{h} \). To every point \( f = Th \), successive to \( h \), there correspond, in the same arc of cycle from \( h \) to \( \bar{h} \), a symmetric point, \( \bar{f} = T\bar{h} \), where \( T \) is obtained by \( T \) replacing each \( A \) by \( \bar{A} \), and each \( B \) by \( \bar{B} \). We write finally

\[
\bar{h} = Mh, \quad h = \bar{M}\bar{h},
\]

obtaining \( M = \bar{M}^{-1} \). I.e., \( M \) consists in a palindromic sequence

\[
M = A^{a_1}B^{a_2}A^{a_3} \cdots B^{a_q}A^{a_{q-1}} \cdots A^{a_3}B^{a_2}A^{a_1}.
\]

Note that the sequence have to end by \( A \) (or by \( B \)) if it starts by \( A \) (respectively, by \( B \)). The central group is \( A^{a_q} \) if \( p \) is odd, \( B^{a_q} \) is \( q \) is even. The number of groups of \( A \) and \( B \)
in $M$ is $2q + 1$, and hence is odd. Applying the same argument to the arc of circle from $h$ to $h^*$, we obtain $h = NH$, where $N$ is (if $M$ starts with $A$),

$$N = B^{b_1} A^{b_2} B^{b_3} \cdots B^{b_{r-1}} A^{b_r} B^{b_{r-1}} \cdots B^{b_3} A^{b_2} B^{b_1},$$

and, as before, the central group is $A^{b_r}$ is $r$ is even, $B^{b_r}$ is $r$ is odd.

We thus obtain

$$h = NMh,$$

where $T = NM$, defining the cycle, is composed by two sequences palindromic and odd:

$$[(a_1, \ldots, a_q, \ldots a_1)(b_1, \ldots, b_r, \ldots, b_1)].$$

The resulting sequence is, by Theorem 3.1, the period of the root of the quadratic equation associated to $h$, and this period is evidently even and bipalindromic.

Conversely, having a period bipalindromic, we subdivide it into two palindromic odd periods, of length $2q + 1$ and $2r + 1$, $[a_1, \ldots, a_q, \ldots a_1]$ and $[b_1, \ldots, b_r, \ldots, b_1]$ and build, for instance, the operators $M = A^{a_1} B^{a_2} \cdots A^{a_1}$ and $N = B^{b_1} A^{b_2} \cdots B^{b_1}$. By Theorem 3.1 this is the period of the root $\xi^+(f)$, where $f$ satisfy:

$$f = NMf$$

Using Lemma 4.5 we immediately see that points $f$ and $Mf$, which are in the cycle, are conjugate, and all points $Af, A^2f, \ldots, A^qf, A^{a_1}Bf$, etc, belonging to that part of the cycle, defined by $M$, are conjugate of the corresponding points of the same part of the cycle $A^rMf, A^2Mf, \ldots, A^rMf, A^{a_1}B^rM$, etc. Note that, if $a_q$ is even, there is a central point, on the axis $k = 0$, which is self-conjugate. Similarly the pairs of points obtained by the second part of the cycle, from $Mf$ to $NMf = f$, are conjugate. So we obtain that the cycle, being symmetric with respect to the vertical axis $(k = 0)$, is $k$-symmetric. □

**Remark.** In this case $\Pi = \Gamma$ and $p = P = 2(r + q) + 2.$

**Proof of Theorem 3.8.** (See Figure 3-IV)

Suppose the class be antisymmetric. This means that the cycle is invariant under reflection with respect to the centre of $H^0$. Choose any pair in the cycle of antipodal points, and call them $h$ and $h^*$. Let $Ah$ and $Bh$ be the successive and preceding point of $h$. The successive and preceding of $h^*$ are, by Lemma 4.5, $Bh^*$ and $Ah^*$, and

$$\overline{(Ah)^*} = Bh^*, \quad \overline{Bh^*} = Ah^*.$$  \hspace{1cm} (10)

If we consider the successive points of the cycle, after $h$ and $Ah$, we must meet the point $h^*$. To every point $f = Th$, successive to $h$, there correspond, in the antipodal part of
the cycle from \( h \) to \( \overline{h} \), a symmetric point, \( \overline{f} = \bar{T}\overline{h} \), where \( \bar{T} \) is obtained by \( T \) replacing each \( A \) by \( B \), and each \( B \) by \( A \). So, we write finally
\[
\overline{h}^* = Mh, \quad h = \bar{M}\overline{h}^*.
\]
Note that the last operator of \( M \) must be the same as the initial (in this case \( A \)), because of eq. (10).

So we obtain
\[
h = \bar{M}Mh,
\]
where \( M \), because of the preceding remark, has an odd number \( q \) of groups of operators \( A \) and \( B \). The operator \( T = \bar{M}M \) defining the cycle is thus
\[
T = B^{a_1}A^{a_2} \cdots B^{a_q}A^{a_1}B^{a_2} \cdots A^{a_q}.
\]

Hence the roots \( \xi^\pm(h) \) have odd periods, non palindromic,
\[
[a_1, a_2, \ldots, a_q], \quad [a_q, a_{q-1}, \ldots, a_1].
\]

\[\square\]

Remark. In this case \( \Pi = \Gamma^2 \) and \( p = 2P = 2q \).

Proof of Theorem 3.4. (See Figure 3-V).

The asymmetric case is the simplest. The number \( p \) of groups of operators \( A \) and \( B \), factors of the operator \( T \in \mathfrak{T}^+ \) satisfying \( Th = h \) (\( h \) being a principal point of the cycle), is necessarily even. Since there are no symmetries, the periods of the roots \( \xi^\pm(h) \) are non symmetric and contain thus \( p \) (even) elements.

\[\square\]

Remark. In this case \( \Pi = \Gamma \) and \( p = P \).

Proof of Corollary 3.4. By Theorem 3.3, the sequence \( s \) defines uniquely a primitive class iff this class is invariant under antipodal symmetry. Such a class is either supersymmetric or antisymmetric. By Theorems 3.5, 3.9 these cases are the only cases in which the periods are odd.

\[\square\]

Proof of Theorem 3.11. Suppose that the cycle in \( H^0 \) have the symmetry of a regular \( n \)-gon, with \( n > 2 \). This means that there exists an operator \( M \) of \( \mathfrak{T}^+ \) and a point \( h \in H^0 \) such that \( h = M^nh \). By the results of Part I and Theorem 3.11 all points \( h_i \) of the cycle satisfy \( \bar{M}^n_ih_i = h_i \), for some \( \bar{M}^n_i \), obtained from \( M^n \) by a cyclic permutation of its factors. Among them, there is a principal point \( f \) such that \( \xi^+(f) \) is immediately periodic
\[
\xi^+(f) = [(a_1, \ldots, a_p)_1, (a_1 \ldots a_p)_2, \ldots, (a_1, \ldots, a_p)_n],
\]
being for it $\tilde{M}^n = (B^{a_1}A^{a_2} \ldots A^{a_p})^n$, with $p$ even. But $\xi^+(f)$ satisfies
\[
\mu(\xi^+(f)) = \xi^+(f),
\]
$\mu$ being obtained from $(B^{a_1}A^{a_2} \ldots A^{a_p})$ by translating $A$ into $\alpha$ and $B$ into $\beta$. By Lemmas 4.1 and 4.2 also $f$ satisfies
\[
f = (B^{a_1}A^{a_2} \ldots A^{a_p})f,
\]
and thus belongs to a cycle of $p$ elements. Since a point of the cycle cannot be visited twice, no other points can belong to the same cycle, so that the cycle has no the symmetry of a regular $n$-gon.

**Proof of Theorem 3.12.** By Theorem 4.13 of Part I, the number of points inside $G_A$ and $G_\alpha$ of class $C(m,n,k)$ is equal to the number $t_B$ of factors of type $B$ in the operator $T \in \mathcal{T}^+$, which defines the cycle in $H^0$. Theorem 3.1 establishes a correspondence between $\Pi = [a_1,a_2,\ldots,a_p]$ and the sequence of factors $A$ and $B$ of $T$. Hence $t_B$ is equal either to $t_{odd}$ or to $t_{even}$, depending whether the generators to the powers $a_j$, with $j$ odd, are $B$ or $A$.

If the period $\Gamma$ is even and palindromic, $P$ is even and $a_i$ is equal to $a_{P+1-i}$. Hence, when $i$ is even, $(P + 1 - i)$ is odd, and vice versa. Hence values $t_{odd}$ and $t_{even}$, given by eq. (2), coincide, and being their sum equal to $t$, are equal to $t/2$.

If the period is even and non palindromic, $p = P$, and values $t_{odd}$ and $t_{even}$, given by eq. (2), do not coincide necessarily.

If the period $\Gamma$ is odd, $P$ is odd. In $\Pi$, by definition, $a_{i+P} = a_i$, for all $i = 1,\ldots,P$ and when $i$ is even, $(i + P)$ is odd and vice versa. Also in this case $t_{odd} = t_{even}$.

On the other hand, the equality $t_{odd} = t_{even}$ do not imply, evidently, the symmetry of the period.

**4.3. Classes representing 1.**

**Proof of Corollary 3.10.** We prove this corollary by showing that the class of forms representing 1 is either $k$-symmetric or supersymmetric. The statements will follow from

---

3Alternative proof: The class of quadratic form $C(1,n,k)$ is the sole class, with discriminant $k^2 - 4n$, representing 1. This class is the group identity of the class group. Since the inversion in the class group corresponds to the conjugation, the identity class is self-conjugate, and hence it is invariant under reflection with respect to the axis $k = 0$. The sole classes having this symmetry are the supersymmetric classes (which have period odd and palindromic) and the $k$-symmetric classes (which have period even and bipalindromic).
Theorems 3.7 and 3.6. Indeed, in such a class there is a representant \( f \) with coefficients \((1, n, k)\), i.e., a form

\[
f = x^2 + kyx + ny^2.
\]

Let \( T = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \). If \( v = (x, y) \), \( g(v) = f(Tv) = x^2 - kxy + ny^2 \). We observe that \( g = (1, n, -k) = T \) and hence \( T \in C(1, n, k) \). Therefore the class \( C(1, n, k) \) is either \( k \)-symmetric or supersymmetric.

4.4. Classes representing 0. Classes of forms representing 0 have discriminant equal to a square number. The roots of the corresponding quadratic equations are rational, and have thus a finite continued fraction.

Proof of Theorem 3.13. The rational number \( k/m \) is equal to \( \xi^+(h) \), \( h = (m, 0, -k) \). The quadratic form \( h \), for which \( \xi^+(h) > 0 \) and \( \xi^-(h) = 0 \), belongs to set \( F_A \) (see Section 4.4 in Part I). By Theorem 4.14 of Part I, \( h \) is one of the \( k \) forms with discriminant \( \Delta = k^2 \) and it is the sole form in \( F_A \) of its class.

Class \( C(m, 0, k) \) is the conjugate of class \( C(m, 0, -k) \), being \( \overline{h} = (m, 0, k) \). Conjugate classes, being symmetric with respect to the plane \( k = 0 \), have the same number of points in each domain.

The form \( f = Ah \) is inside \( H^0 \), and satisfy \( \overline{A}f \in F_A \). By Theorem 4.19 of Part I, it is the starting point of a chain, containing all \( t \) points in \( H^0 \) of its class. The final point of that chain represents form \( g \) satisfying: \( g = Tf \) for some operator \( T \in \mathcal{J}^+ \), product of \( t - 1 \) generators. Moreover, \( p = Ag \in F_A \), and \( \xi^+(p) = 0 \).

By hypothesis \([a_1, a_2, \ldots, a_L] \) is the even continued fraction of \( \xi^+(h) \). Using Lemmas 4.2 and 4.3, we thus obtain:

\[
\xi^+(f) = [a_1 - 1, a_2, \ldots, a_L], \quad \text{and} \quad \xi^+(g) = [1].
\]

Define \( \tau \) by \( \xi^+(g) = \tau(\xi^+(f)) \). Since \( L \) is even

\[
\tau = \beta^a_{L-1} \circ \alpha^a_{L-1} \cdots \beta^a_2 \circ \alpha^a_1.
\]

Hence operator \( T \) sending \( f \) to \( g \) is equal to

\[
T = B^a_{L-1} A^a_{L-1} \cdots B^a_2 A^a_1.
\]

According to Theorem 4.19 of Part I, the number of points of \( C(m, n, k) \) inside every domain in \( G_A \) and in \( \overline{G}_A \) is equal to the number of factors of type \( B \) in \( T \), and it is therefore equal to \( t_{\text{even}} \), given by eq. 3, whereas the number of points inside every
domain in $G_B$ and in $G_{\bar{B}}$ is equal to the number of factors of type $A$, and it is thus equal to $t_{\text{odd}}$.

When the class is supersymmetric or $(m+n)$-symmetric, it is symmetric with respect to the horizontal axis $(m+n) = 0$. The number of points inside the domains in $G_A$ and $G_{\bar{A}}$ coincides with that inside the domains in $G_B$ and $G_{\bar{B}}$.

Since $t_{\text{odd}} + t_{\text{even}} = t - 1$, we have

$$t_{\text{odd}} = t_{\text{even}} = (t - 1)/2. \quad (11)$$

The number of points inside $H^0$ is thus odd, when $t_A = t_B$. The theorem is proved \( \square \)

**Remark.** Eq. (11) says that, when the class is supersymmetric or $(m+n)$-symmetric, the number of points inside $H^0$ is odd.

**Proof of Theorem 3.14.**

For $m = 1, \ldots, k - 1$, the rational number $k/m$ is the non zero root of the equation $m\xi^2 - k\xi = 0$, i.e., $\xi^+(h) = k/m$ and $h = (m, 0, -k)$ is the representative of the class $C(m, 0, -k)$.

Form $(m, 0, k)$ is the conjugate of $h$, and its class has the same type of symmetry as the class of $h$.

i) Lemma 4.17 of Part I proves that $C(0, 0, k)$ is supersymmetric. If $m = -k/2$, let $h = (m, 0, -2m)$. Then form $f := Ah = (m, -m, 0)$ is the central point of $H^0$. By Lemma $4.5$, $Bf = h^*$, $Af = \overline{h}$, $\overline{Bf} = \overline{h}^*$, i.e., the four forms obtained applying the generators $A$ and $B$ and their inverse to $f$ are symmetric and belong to the boundary of $H^0$. Hence $f$ is the only point of the orbit inside $H^0$ and the orbit is supersymmetric.

The fact that if an orbit $C(m, 0, k)$ is supersymmetric then $m = 0$ or $m = k/2$ is proved by the following reasoning. The initial point $i$ of the chain in $H^0$ is joined by two inverse arrows to $\bar{A}i$ and $\bar{Bi}$, and the final point $f$ by two arrows to $Af$ and $Bf$. Since the orbit is supersymmetric, $\bar{A}i$ and $\bar{Bi}$ are antipodal as well as the arrows from them, and $Af$ and $Bf$ are antipodal, as well as the arrow to them. Therefore, the only possibility in order the chain, with its four points on the boundary of $H^0$, have the supersymmetry, is that the initial point and the final point either both coincide with the central point of $H^0$, or do not exist. The first case is that of $C(k/2, 0, k)$, whereas the second case is that of $C(0, 0, k)$.

ii) Let $h = (m, 0, -k)$, $m > 0$ and $m \neq k/2$. Let, moreover, $m/k = [a_1, \ldots, a_L]$, with $L$ even. Suppose $C(m, 0, -k)$ be $(m+n)$-symmetric. By the same arguments proving Theorem 3.13, we obtain

$$\beta^{a_L} \circ \alpha^{a_{L-1}} \cdots \beta^{a_2} \circ \alpha^{a_1}(\xi^+(h)) = 0,$$
whereas, by the same arguments proving Theorem 3.5

\[ B^L A^{aL-1} \cdots B^{a2} A^{a1} h = h^*. \]

The final point \( g \) of the chain starting at \( f = Ah \) must be sent by \( \bar{B} \) to \( h^* \), because of the symmetry of the orbit. Every point of the chain obtained as \( Th, T \in \mathcal{T}^+ \), must have its symmetric, as well as every arrow, according to Lemma 4.5. In this way we obtain a palindromic sequence of groups of operators \( A \) and \( B \), which must be even, since it starts by \( A \) and ends by \( B \). Like in the proof of Theorem 3.5 we conclude that every \((m+n)\)-symmetric chain contains one (only one in this case) selfadjoint point \((m+n = 0)\) in \( H^0 \). On the other hand, if we have \( \xi^+(m/k) \) even and palindromic, using Lemma 4.3 we reach zero, i.e., a point of the boundary of \( H^0 \), and by the symmetry of the corresponding operator of \( \mathcal{T}^+ \), we conclude that this point is \( h^* \), and hence the orbit is at least \((m+n)\)-symmetric. Such orbit could be, in fact, supersymmetric, but this is excluded by point (i).

iii) Let \( h = (m, 0, -k) \), \( m > 0 \) and \( m \neq k/2 \). Let, moreover, \( m/k = [a_1, \ldots, a_L] \), with \( L \) odd. Suppose \( C(m, 0, -k) \) be \( k \)-symmetric. By the same arguments proving Theorem 3.13 we obtain

\[ \alpha^{a_L} \circ \beta^{a_{L-1}} \cdots \beta^{a_2} \circ \alpha^{a_1} \xi^+ = 0, \]

whereas, by the same arguments proving Theorem 3.6

\[ A^L B^{a_{L-1}} \cdots B^{a_2} A^{a_1} h = \bar{h}. \]

Indeed, the final point \( g \) of the chain starting at \( f = Ah \) must be sent by \( A \) to \( \bar{h} \), because of the symmetry of the orbit. Every point of the chain obtained as \( Th, T \in \mathcal{T}^+ \), must have its symmetric, as well as every arrow, according to Lemma 4.5. In this way we obtain a palindromic sequence of groups of operators \( A \) and \( B \), which must be odd, since it starts by \( A \) and ends by \( A \). On the other hand, if \( \xi^+(m/k) \) is odd and palindromic, using Lemma 4.3 we reach zero, i.e., a point of the boundary of \( H^0 \), and by the symmetry of the corresponding operator of \( \mathcal{T}^+ \), we conclude that this point is \( \bar{h} \), and hence the orbit is at least \( k \)-symmetric. Such orbit could be, in fact, supersymmetric, but we must exclude this case because of point (i).

iv) For an antisymmetric orbit we should have \( t_A = t_B \), because of the complementarity between \( G_A \) and \( G_B \), as well as between \( G_{\bar{A}} \) and \( G_{\bar{B}} \). By Theorem 3.13 the number of points in \( H^0 \) should thus be odd. An antisymmetric chain in \( H^0 \) containing an odd number of points must contain the centre of \( H^0 \), but in this case the orbit is supersymmetric, by point (i).
v) The remaining possibility is that the chain as well as the orbit $C(m, 0, k)$ is asymmetric. Hence this happens iff neither the odd nor the even continued fraction of $k/m$ are palindromic. □

5. Reduction theory

Theorem 3.1 has as natural consequence a reduction procedure (probably already known), described by the following corollary. We will see moreover in this section how this reduction method is related to another method, to which we refer as ‘classical reduction theory’ ([6], [7], [8]).

Given any indefinite form $f = mx^2 + ny^2 + kxy$, for which $mn \geq 0$, the procedure to transform it in a form of the same class $m'x^2 + n'y^2 + k'xy$ such that $m'n' \leq 0$ is given by the following corollary.

Corollary 5.1. Let $f = (m, n, k)$, with $mn \geq 0$ and $\tilde{f} := (\tilde{m}, \tilde{n}, \tilde{k})$ be obtained applying to $f$ one of the involutions, such that $\xi^+(\tilde{f}) > 0$. Let $[\alpha_0, \alpha_1, \ldots, \alpha_N, [a_1, \ldots, a_p]]$ represent the continued fraction of the root $\xi^+(\tilde{f})$ (the possible absence of the period indicates that the root is rational). Then the form

$$f' := (m', n', k') = \tilde{L}(\tilde{f})$$

satisfies $m'n' \leq 0$, where $L = C^{a_N} \cdots A^{a_2} B^{a_1} A^{a_0}$, being $C = A$ if $N$ is even and $C = B$ if $N$ is odd.

Proof. Note that the involutions do not change the product $mn$; to find the required involution, look at Table 1, and choose it depending on the value of $\xi^+(f)$. Form $\tilde{f}$ may not belong to the class of $f$. For this reason at the end of the procedure we apply the same involution to the form obtained as $\tilde{L}\tilde{f}$. Indeed, the orbit of two forms related by an involution, either coincide (if the class is invariant under that involution), or have no common elements, each element of one class being related by that involution to an element of the other class.

The expression of $L$ follows from Lemma 4.3. We obtain in this way, if $N$ is even, that

$$\xi^+(\tilde{L}\tilde{f}) = [[a_1, \ldots, a_p]],$$

otherwise

$$\xi^+(\tilde{L}\tilde{f}) = [0, [a_1, \ldots, a_p]].$$

The condition $\xi^+(\tilde{f}) > 0$ means that $\tilde{f}$ is either in $G_A$ or in $G_B$ (where $m > 0$ and $n > 0$). The operator $L$ belongs indeed to $\mathcal{T}^+$, and its sequence of factors of type $A$ and $B$ can
be read as a path from \( \tilde{f} \) towards \( H^0 \), lying entirely in \( G_A \) or in \( G_B \). This path is unique because of Theorem 4.2 of Part I.

Being \( \xi^+(L\tilde{f}) \) immediately periodic, we conclude that \( L\tilde{f} \in H^0 \) (\( \xi^+(L\tilde{f}) = 0 \) if the period is absent) and hence that the form \( L\tilde{f} \) satisfies \( \tilde{m}'\tilde{n}' < 0 \) (or \( \tilde{m}'\tilde{n}' = 0 \)). Again, the involution will not change the product \( mn \), and we obtain that form \( f' = L\tilde{f} \) belongs to the same class of \( f \).

\( \square \)

**Remark.** The operator \( L' \) of \( PSL(2, \mathbb{Z}) \) such that \( f' = L'(f) \) is deduced from \( L \) using Lemma 4.5. It is still an operator of \( T^+ \) or of \( T^- \), whose factors indicate the unique path from \( f \) towards \( H^0 \) which lies entirely either in \( G_A \) or in \( G_B \) or in \( \bar{G}_A \) or in \( \bar{G}_B \).

The number of forms of that class satisfying \( mn < 0 \) is given by Theorem 3.12.

### 5.1. Classical reduction theory seen in the de Sitter world.

**Definition.** The sequence (finite or infinite) of integers \( b_i \geq 2, (b_0, b_1, b_2, \ldots) \), denotes the **modular fraction** of the number \( \xi \):

\[
\xi = b_0 - \frac{1}{b_1 - \frac{1}{b_2 - \cdots}}.
\]

An irrational number has infinite modular fraction, and periodic modular fractions (with period of length \( t \)) are defined exactly as the periodic continued fractions (see Definition 3), and are denoted by

\[
(b_0, b_1, \ldots, b_{N-1}, (b_N, b_{N+1}, \ldots, b_{N+t-1})).
\]

The following theorems represent a synthesis of the classical reduction theory. The proofs that we give here show a geometrical description of this theory in the Sitter world.

**Definition.** An indefinite form \( h = (m, n, k) \) such that \( C(m, n, k) \) does not represent zero is said to be **reduced** iff satisfies \( m > 0, n > 0, k < 0, m + n < |k| \).

**Theorem 5.2.** Let \( h = (m, n, k) \) be a reduced form. Then the modular fraction of \( \xi^+(h) \) is immediately periodic, and the number of its elements is equal to the number of reduced forms of the class \( C(m, n, k) \). The first roots \( \xi^+ \) associated to the other reduced forms are given by the cyclic permutations of the elements of \( \xi^+(f) \).

**Theorem 5.3.** Let \( f = (m, n, k) \) satisfy \( m > 0, n > 0, k < 0 \). Then the modular fraction of \( \xi^+(f) \) is periodic:

\[
\xi^+(f) = (b_0, b_1, \ldots, b_M, (c_1, c_2, \ldots, c_t))
\]

and the operator

\[
L = RA^{b_M} \cdots RA^{b_2} RA^{b_1} RA^{b_0}
\]
satisfies: \( Lf = h \), where \( h \) is reduced.

**Proof of Theorem 5.2.** The proof follows from the results of this work using the following

**Lemma 5.4.** Let \( f = (m, n, k) \) and \( \xi^+(f) = (a, b, c, d, \ldots) \). Then

\[
\xi^+(RA^a f) = (b, c, d, \ldots);
\]
i.e., the cancellation of the first element in the modular fraction of \( \xi^+(f) \) corresponds to the action on \( f \) of the operator \( A \), iterated a number of times equal to that element, followed by the action of \( R \).

**Proof.** By Lemma 4.2,

\[
\xi^+(A^a f) = \alpha^a(\xi^+(f)) = -\frac{1}{b - c + \frac{1}{d - 1}},
\]

\[
\xi^+(RA^a f) = \sigma \circ \alpha^a(\xi^+(f)) = b - \frac{1}{c - d + \frac{1}{a}}.
\]

\( \square \)

A reduced form \( h \) of a class \( C(m, n, k) \) non representing zero belongs to \( H_A \). By Lemma 4.1, it is represented in \( \Xi \) by a point with \( \xi^+ > 1 \) and \( 0 < \xi^- < 1 \).

We know that point \( Ah \) is in \( H^0 \) (see Figure 4). We consider now all the successive points \( A^2h, A^3h, \ldots \) that belongs to \( H^0 \). At some iteration of order \( c_1 \) point \( A^{c_1}h \) exits from \( H^0 \) and lies necessarily in \( H_A \). At this point we apply operator \( R \), going back to \( H_A \). The point \( h' = RA^{c_1}h \) coincides with \( h \) only if \( f \) contains a sole point. Indeed, if \( RA^{c_1}h = h \), consider \( f = Ah \), which is in \( H^0 \). It satisfies \( ARA^{c_1}f = f \). Using the relation \( R = \bar{A}BA \), we write

\[
BA^{c_1-1}f = f,
\]

so concluding that the cycle in \( H^0 \) contains only two principal points and, by Theorem 3.12, that in \( H_A \) the orbit has only one element. If \( h' \) is different from \( h \), we repeat the procedure till

\[
RA^{c_1}RA^{c_1-1} \cdots RA^{c_1}h = h.
\]

By the same reason, this occurs when all points in \( H_A \), as well as in \( H^0 \), are visited (see figure). By Lemma 5.4, using the same arguments of the proof of Theorem 3.1, we obtain that the modular fraction of \( \xi^+(h) \) is periodic, namely

\[
\xi^+(h) = ((c_1, c_2, \ldots, c_t)).
\]
We have thus proved that the modular fraction of the first root of an equation corresponding to a form in $H_\bar{A}$ is immediately periodic and the length $t$ of its period is equal to the number of points of the class in $H_\bar{A}$, i.e., the number of reduced forms. □

Proof of Theorem 5.3. We observe now that a form $f$ satisfying the conditions of the theorem lies in $G_\bar{A}$. Hence there is a form $h$ in $H_\bar{A}$ such that $f = T^{-1}h$, where $T \in T^-$. We rewrite $T$ as product of operators $A$ and $B$ by the following procedure: we replace each operator $B$ in $T^{-1}$ by $A \bar{A}BA \bar{A}$. Hence we replace $\bar{A}BA$ by $R$. We obtain a sequence containing the same factors of type $A$ as $T^{-1}$ and where every factor of type $B^i$ is replaced by $(ARA)(ARA)\cdots(ARA)$. Since the modular fraction of $\xi^+(h)$ is immediately periodic (say, $((c_1, c_2, \ldots, c_t))$, by Theorem 5.2 using Lemma 5.4 we conclude that

$$T^{-1} = RA^{b_M}RA^{b_{M-1}}, \ldots, RA^{b_1}RA^{b_0}$$

iff

$$\xi^+(f) = (b_0, b_1, \ldots, b_M, (c_1, c_2, \ldots, c_t)).$$

□

We obtain the following corollary:

Corollary 5.5. If the period $\Pi$ of the continued fraction of $\xi^+(h)$, $h \in H_\bar{A}$ is $\Pi = [a_1, a_2, \ldots, a_p]$, then the period of the modular fraction of $\xi^+(h)$ is obtained in the following way:

1. replace every element $a_{2i+1}$ of the continued fraction having odd index by $a_{2i+1} + 2$;
2. replace every element $a_{2i}$ of the continued fraction having even index by a sequence containing a number of "2" equal to $a_{2i} - 1$. 

Figure 4. Relation between a cycle in $H_\bar{A}$, represented by $[1,1,3,1,1,3]$ and a cycle in $H^0$, represented by $(3,5,3,2,2)$. The black dot in $H_\bar{A}$ corresponds to the form $h$ in the proof of Theorem 5.2 and in the Example.
Remark. The length of period of the modular fraction gives the number of reduced forms in $H_A$, whereas the sum of all elements in the period of the continued fraction gives the number of elements in $H^0$.

Example. The supersymmetric class represented in Figure 4 is that of the Example of Theorem 3.6. Let $f = (2, -1, -3)$. We have $\xi^+(f) = 3/4 + \sqrt{17}/4 = [1, 1, 3]$. Hence $\Pi = [1, 1, 3, 1, 1, 3]$. There are indeed 10 points inside $H^0$. Point $h = \bar{A}f = (2, 4, -7)$ is in $H_A$ and $\xi^+ (h) = 7/4 + \sqrt{17}/4$ has modular fraction $\xi^+ (h) = ((3, 5, 3, 2, 2))$.

There are indeed 5 points in $H_A$.

We are now able to give the proof of Theorem 3.15.

Proof of Theorem 3.15. By the corollary above, given the period $(c_1, c_2, \ldots, c_t)$ of a modular fraction, we associate to it the period $[a_1, a_2, \ldots, a_p]$ of the corresponding continued fraction, replacing every element $c_i$ greater than 2 by an element $a_j$ equal to $c_i - 2$, and substituting the group of $r \geq 0$ successive elements $c_{i+1}, c_{i+2}, \ldots, c_{i+r}$ equal to 2, by an element $a_{j+1}$ equal to $r + 1$ (hence $a_{j+1} = 1$ if $c_{i+1} > 2$). We obtain a period of $p$ elements with $p$ even, which is in fact the the period $\Pi$ of the continued fraction. By Theorem 3.12 the sum of the elements of index odd of the continued fraction is equal to the sum of elements of index even if the class has the mentioned symmetries, and is also equal to the number of points in $H_A$ and in $H_A$ of the orbit. By Theorem 5.2 the number of points in $H_A$ of the orbit equals the number $t$ of elements of the period of the modular fraction considered. Hence by the equation

$$\sum_{i=1}^{t} (c_i - 2) = t$$

we obtain the statement of the theorem. □

Remark. For a non symmetric class we have to consider the modular fractions of the two inverse periods, that are different.

Acknowledgements

I am grateful to Vladlen Timorin for having told me the classical reduction theory and for his interest in this work. I apologize number theorists for my own terminology on symmetries and continued fractions: I hope the risk of confusion be anyway low, the notions I use being all elementary.
Appendix: Tables of classes of indefinite forms with small discriminant

The following tables contain a representative of every class with $\Delta \leq 100$; moreover, if $\Delta$ is not a square integer, we give the period $\Gamma$, its length $P$, the numbers $t^\uparrow$ of points inside each domain in $G_A$ and $G_{\overline{A}}$, the number $t^\downarrow$ of points inside each domain in $G_B$ and $G_{\overline{B}}$, and the type of symmetry. A star indicates that the class is non primitive, i.e., is obtained by a primitive class by multiplication by an integer greater than 1.

If $\Delta$ is equal to a square number, then instead of $\Gamma$ and $P$ we give the continued fraction of $k/m$, its length $L$, the number $t$ of points in $H^0$ and in $H^0_R$.

Remark. The classes of forms non representing zero are either supersymmetric or $k$-symmetric if $\Delta \leq 100$.

Indeed, the first class (i.e., with minimal discriminant) non representing zero having the $(m+n)$-symmetry is that in the Example of Theorem 3.5. It has period $[1, 2, 2, 1]$ and discriminant 221. The first antisymmetric class has period $[1, 2, 3]$ and discriminant 148, and that asymmetric has period $[1, 1, 2, 3]$ and discriminant 396.

The tables show that there are classes of forms representing zero with $\Delta \leq 100$ that have all the possible types of symmetry.

Moreover, we plot in the following figure the fractions of the total number of classes with a given discriminant, versus the discriminant $< 10^4$, corresponding to the different types of symmetries (indicated by different symbols).

Figure 5. rhombi: $k$-symmetric; circles: supersymmetric; crosses: $(m+n)$-symmetric; windows: asymmetric; squares: antisymmetric
Tables of classes non representing zero with $\Delta < 100$

| $\Delta$ | $m$ | $n$ | $k$ | $\Gamma$ | $P$ | $t^1 - t^1_1$ | symm. | n.p. |
|---------|-----|-----|-----|---------|-----|---------------|-------|------|
| 5       | 1   | -1  | 1   | $[1]$   | 1   | 1-1           | super |      |
| 8       | 1   | -1  | 2   | $[2]$   | 1   | 2-2           | super |      |
| 12      | 2   | -1  | 2   | $[2, 1]$| 2   | 2-1           |       | k    |
|         | 1   | -2  | 2   | $[1, 2]$| 2   | 1-2           |       | k    |
| 13      | 1   | -1  | 3   | $[3]$   | 1   | 3-3           | super |      |
| 17      | 2   | -2  | 1   | $[1, 3, 1]$| 3 | 5-5     | super |      |
| 20      | 1   | -1  | 4   | $[4]$   | 1   | 4-4           | super |      |
|         | 2   | -2  | 2   | $[1]$   | 1   | 1-1           | super | *    |
| 21      | 3   | -1  | 3   | $[3, 1]$| 2   | 3-1           |       | k    |
|         | 1   | -3  | 3   | $[1, 3]$| 2   | 1-3           |       | k    |
| 24      | 2   | -1  | 4   | $[4, 2]$| 2   | 4-2           |       | k    |
|         | 1   | -2  | 4   | $[2, 4]$| 2   | 2-4           |       | k    |
| 28      | 3   | -2  | 2   | $[1, 1, 4, 1]$| 4 | 5-2     |       | k    |
|         | 2   | -3  | 2   | $[1, 4, 1, 1]$| 4 | 2-5     |       | k    |
| 29      | 1   | -1  | 5   | $[5]$   | 1   | 5-5           | super |      |
| 32      | 2   | -2  | 4   | $[2]$   | 1   | 2-2           | super | *    |
|         | 1   | -4  | 4   | $[1, 4]$| 2   | 1-4           |       | k    |
|         | 4   | -1  | 4   | $[4, 1]$| 1   | 4-1           |       | k    |
| 33      | 2   | -1  | 5   | $[5, 2, 1, 2]$| 4 | 6-4     |       | k    |
|         | 1   | -2  | 5   | $[2, 1, 2, 5]$| 4 | 4-6     |       | k    |
| 37      | 3   | -3  | 1   | $[1, 5, 1]$| 3   | 7-7           | super |      |
| 40      | 3   | -3  | 2   | $[1, 2, 1]$| 3   | 4-4           | super |      |
|         | 1   | -1  | 6   | $[6]$   | 1   | 6-6           | super |      |
| 41      | 2   | -2  | 5   | $[2, 1, 5, 1, 2]$| 5 | 11-11    |      | super |
| 44      | 2   | -1  | 6   | $[6, 3]$| 2   | 6-3           |       | k    |
|         | 1   | -2  | 6   | $[3, 6]$| 2   | 3-6           |       | k    |
| 45      | 3   | -3  | 3   | $[1]$   | 1   | 1-1           | super | *    |
|         | 1   | -5  | 5   | $[1, 5]$| 2   | 1-5           |       | k    |
|         | 5   | -1  | 5   | $[5, 1]$| 2   | 5-1           |       | k    |
| 48      | 1   | -3  | 6   | $[2, 6]$| 2   | 2-6           |       | k    |
|         | 3   | -1  | 6   | $[6, 2]$| 2   | 6-2           |       | k    |
|         | 4   | -2  | 4   | $[2, 1]$| 2   | 1-2           |       | k    |
|         | 2   | -4  | 4   | $[1, 2]$| 2   | 1-2           |       | *    |
| 52      | 3   | -3  | 4   | $[1, 1, 6, 1, 1]$| 5 | 10-10   | super |      |
|         | 2   | -2  | 6   | $[3]$   | 1   | 3-3           | super | *    |
| 53      | 1   | -1  | 7   | $[7]$   | 1   | 7-7           | super |      |
| 56      | 5   | -2  | 4   | $[2, 1, 6, 1]$| 4   | 8-2           |       | k    |
|         | 2   | -5  | 4   | $[1, 6, 1, 2]$| 4   | 2-8           |       | k    |
| 57      | 4   | -3  | 3   | $[1, 1, 3, 7, 3, 1]$| 6 | 7-9     |       | k    |
|         | 3   | -4  | 3   | $[1, 3, 7, 3, 1, 1]$| 6 | 9-7     |       | k    |
| $\Delta$ | $m$ | $n$ | $k$ | $\Gamma$ | $P$ | $t^\uparrow - t_\downarrow$ | symm. | n.p. |
|---|---|---|---|---|---|---|---|---|
| 60 | 2 | -3 | 6 | [2, 3] | 2 | 2-3 | $k$ |  |
|   | 3 | -2 | 6 | [3, 2] | 2 | 3-2 | $k$ |  |
|   | 1 | -6 | 6 | [1, 6] | 2 | 1-6 | $k$ |  |
|   | 6 | -1 | 6 | [6, 1] | 2 | 6-1 | $k$ |  |
| 61 | 3 | -3 | 5 | [2, 7, 2] | 3 | 11-11 | super |  |
| 65 | 4 | -4 | 1 | [1, 7, 1] | 3 | 9-9 | super |  |
|   | 2 | -2 | 7 | [3, 1, 3] | 3 | 7-7 | super |  |
| 68 | 1 | -1 | 8 | [8] | 1 | 8-8 | super | * |
|   | 4 | -4 | 2 | [1, 3, 1] | 3 | 5-5 | super |  |
| 69 | 5 | -3 | 3 | [1, 1, 7, 1] | 4 | 8-2 | $k$ |  |
|   | 3 | -5 | 3 | [1, 7, 1, 1] | 4 | 2-8 | $k$ |  |
| 72 | 3 | -3 | 6 | [2] | 1 | 2-2 | super | * |
|   | 1 | -2 | 8 | [4, 8] | 2 | 8-4 | $k$ |  |
|   | 2 | -1 | 8 | [8, 4] | 2 | 8-4 | $k$ |  |
| 73 | 4 | -4 | 3 | [1, 2, 3, 1, 7, 1, 3, 2, 1] | 9 | 21-21 | super |  |
| 76 | 3 | -1 | 8 | [8, 2, 1, 3, 1, 2] | 6 | 10-7 | $k$ |  |
|   | 1 | -3 | 8 | [2, 1, 3, 1, 2, 8] | 6 | 7-10 | $k$ |  |
| 77 | 1 | -7 | 7 | [1, 7] | 2 | 1-7 | $k$ |  |
|   | 7 | -1 | 7 | [7, 1] | 2 | 7-1 | $k$ |  |
| 80 | 4 | -4 | 4 | [1] | 1 | 1-1 | super | * |
|   | 2 | -2 | 8 | [4] | 1 | 4-4 | super | * |
|   | 1 | -4 | 8 | [2, 8] | 2 | 2-8 | $k$ |  |
|   | 4 | -1 | 8 | [8, 2] | 2 | 8-2 | $k$ |  |
| 84 | 6 | -2 | 6 | [3, 1] | 2 | 3-1 | $k$ | * |
|   | 2 | -6 | 6 | [1, 3] | 2 | 1-3 | $k$ | * |
|   | 4 | -3 | 6 | [2, 1, 1, 8, 1, 1] | 6 | 4-10 | $k$ |  |
|   | 3 | -4 | 6 | [1, 1, 8, 1, 1, 2] | 6 | 10-4 | $k$ |  |
| 85 | 3 | -3 | 7 | [2, 1, 2] | 3 | 5-5 | super |  |
|   | 1 | -1 | 9 | [9] | 1 | 9-9 | super |  |
| 88 | 2 | -3 | 8 | [2, 1, 8, 1, 2, 4] | 6 | 12-6 | $k$ |  |
|   | 4 | -2 | 8 | [4, 2, 1, 8, 1, 2] | 6 | 6-12 | $k$ |  |
| 89 | 4 | -4 | 5 | [1, 1, 4, 9, 4, 1, 1] | 7 | 21-21 | super |  |
| 92 | 1 | -7 | 8 | [1, 3, 1, 8] | 4 | 2-11 | $k$ |  |
|   | 7 | -1 | 8 | [8, 1, 3, 1] | 4 | 11-2 | $k$ |  |
| 93 | 1 | -3 | 9 | [3, 9] | 2 | 3-9 | $k$ |  |
|   | 3 | -1 | 9 | [9, 3] | 2 | 9-3 | $k$ |  |
| 96 | 5 | -3 | 6 | [2, 1, 1, 1] | 4 | 3-2 | $k$ |  |
|   | 3 | -5 | 6 | [1, 1, 1, 2] | 4 | 2-3 | $k$ |  |
|   | 2 | -4 | 8 | [2, 4] | 2 | 2-4 | $k$ | * |
|   | 4 | -2 | 8 | [4, 2] | 2 | 4-2 | $k$ | * |
|   | 1 | -8 | 8 | [1, 8] | 2 | 1-8 | $k$ |  |
|   | 8 | -1 | 8 | [8, 1] | 2 | 8-1 | $k$ |  |
| 97 | 2 | -2 | 9 | [4, 1, 2, 2, 9, 2, 2, 1, 4] | 9 | 27-27 | super |  |
Tables of classes representing zero with $\Delta \leq 100$

| $\Delta$ | $m$ | $n$ | $k$ | $k/m$ | $L$ | $t^1 - t_i$ | symm. | n.p. |
|---------|-----|-----|-----|-------|-----|-------------|-------|------|
| 1       | 0   | 0   | 1   | 0     | 0   | 0           | 0-0   | super |
| 4       | 0   | 0   | 2   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 2   | [2]   | 1   | 1           | 0-0   | super |
| 9       | 0   | 0   | 3   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 3   | [3]   | 1   | 2           | 1-0   | $k$   |
|         | 2   | 0   | 3   | [1, 1, 1] | 3   | 2           | 0-1   | $k$   |
| 16      | 0   | 0   | 4   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 4   | [4]   | 1   | 3           | 2-0   | $k$   |
|         | 2   | 0   | 4   | [2]   | 1   | 1           | 0-0   | super |
|         | 3   | 0   | 4   | [1, 2, 1] | 3   | 3           | 0-2   | $k$   |
| 25      | 0   | 0   | 5   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 5   | [5]   | 1   | 4           | 3-0   | $k$   |
|         | 2   | 0   | 5   | [2, 2] | 2   | 3           | 1-1   | $m+n$ |
|         | 3   | 0   | 5   | [1, 1, 1, 1] | 4   | 3           | 1-1   | $m+n$ |
|         | 4   | 0   | 5   | [1, 3, 1] | 3   | 4           | 0-3   | $k$   |
| 36      | 0   | 0   | 6   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 6   | [6]   | 1   | 5           | 4-0   | $k$   |
|         | 2   | 0   | 6   | [3]   | 1   | 2           | 1-0   | $k$   |
|         | 3   | 0   | 6   | [2]   | 1   | 1           | 0-0   | super |
|         | 4   | 0   | 6   | [1, 1, 1] | 3   | 2           | 0-1   | $k$   |
|         | 5   | 0   | 6   | [1, 4, 1] | 3   | 5           | 0-4   | $k$   |
| 49      | 0   | 0   | 7   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 7   | [7]   | 1   | 6           | 5-0   | $k$   |
|         | 2   | 0   | 7   | [3, 1, 1] | 3   | 4           | 2-1   | asymm |
|         | 3   | 0   | 7   | [2, 2, 1] | 3   | 4           | 1-2   | asymm |
|         | 4   | 0   | 7   | [1, 1, 2, 1] | 4   | 4           | 2-1   | asymm |
|         | 5   | 0   | 7   | [1, 2, 1, 1] | 4   | 4           | 1-2   | asymm |
|         | 6   | 0   | 7   | [1, 5, 1] | 3   | 6           | 0-5   | $k$   |
| 64      | 0   | 0   | 8   | 0     | 0   | 0           | 0-0   | super |
|         | 1   | 0   | 8   | [8]   | 1   | 7           | 6-0   | $k$   |
|         | 2   | 0   | 8   | [4]   | 1   | 3           | 2-0   | $k$   |
|         | 3   | 0   | 8   | [2, 1, 2] | 3   | 4           | 2-1   | $k$   |
|         | 4   | 0   | 8   | [2]   | 1   | 1           | 0-0   | super |
|         | 5   | 0   | 8   | [1, 1, 1, 1, 1] | 5   | 4           | 1-2   | $k$   |
|         | 6   | 0   | 8   | [1, 2, 1] | 3   | 3           | 0-2   | $k$   |
|         | 7   | 0   | 8   | [1, 6, 1] | 3   | 7           | 0-6   | $k$   |
| $\Delta$ | $m$ | $n$ | $k$ | $k/m$ | $L$ | $t$ | $t_l - t$ | symm. | n.p. |
|------|-----|-----|-----|-------|-----|-----|----------|-------|-----|
| 81   | 0   | 0   | 9   | 0     | 0   | 0   | 0-0      | super | *   |
|      | 1   | 0   | 9   | [9]   | 1   | 8   | 7-0      | $k$   |     |
|      | 2   | 0   | 9   | [4, 1, 1] | 3   | 5   | 3-1      | asymm |     |
|      | 3   | 0   | 9   | [3]   | 1   | 2   | 1-0      | $k$   | *   |
|      | 4   | 0   | 9   | [2, 3, 1] | 3   | 5   | 1-3      | asymm |     |
|      | 5   | 0   | 9   | [1, 1, 3, 1] | 4   | 5   | 3-1      | asymm |     |
|      | 6   | 0   | 9   | [1, 1, 1] | 3   | 2   | 0-1      | $k$   | *   |
|      | 7   | 0   | 9   | [1, 3, 1, 1] | 4   | 5   | 1-3      | asymm |     |
|      | 8   | 0   | 9   | [1, 7, 1] | 3   | 8   | 0-7      | $k$   |     |
| 100  | 0   | 0   | 10  | 0     | 0   | 0   | 0-0      | super | *   |
|      | 1   | 0   | 10  | [10]  | 1   | 9   | 8-0      | $k$   |     |
|      | 2   | 0   | 10  | [5]   | 1   | 4   | 3-0      | $k$   | *   |
|      | 3   | 0   | 10  | [3, 3] | 2   | 5   | 2-2      | $m+n$ |     |
|      | 4   | 0   | 10  | [2, 2] | 2   | 3   | 1-1      | $m+n$ | *   |
|      | 5   | 0   | 10  | [2]   | 1   | 1   | 0-0      | super | *   |
|      | 6   | 0   | 10  | [1, 1, 1, 1] | 4   | 3   | 1-1      | $m+n$ | *   |
|      | 7   | 0   | 10  | [1, 2, 2, 1] | 4   | 5   | 2-2      | $m+n$ |     |
|      | 8   | 0   | 10  | [1, 3, 1] | 3   | 4   | 0-3      | $k$   | *   |
|      | 9   | 0   | 10  | [1, 8, 1] | 3   | 9   | 0-8      | $k$   |     |

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(F. Aicardi) SISTIANA 56 PR (Trieste), Italy

*E-mail address: aicardi@sissa.it*