Vector breaking of replica symmetry in some low temperature disordered systems

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Abstract

We present a new method to study disordered systems in the low temperature limit. The method uses the replicated Hamiltonian. It studies the saddle points of this Hamiltonian and shows how the various saddle point contributions can be resummed in order to obtain the scaling behaviour at low temperatures. In a large class of strongly disordered systems, it is necessary to include saddle points of the Hamiltonian which break the replica symmetry in a vector sector, as opposed to the usual matrix sector breaking of spin glass mean field theory.

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1 Introduction

The use of the replica method has turned out to be very efficient in some disordered systems. It allows for a detailed characterization of the low temperature phase at least at the mean field level. In all the mean field spin glass like problems where one can expect the mean field theory to be exact, the Parisi scheme of replica symmetry breaking is successful, and at the moment there is no counterexample showing that it does not work. On the other hand, the low temperature phase of these systems is complicated enough, even at the mean field level. One might hope that the very low temperature limit could be easier to analyse, while its physical content should be basically the same. This very low temperature limit is also an extreme case where one might hope to get a better understanding of the finite dimensional problem. At first sight the low temperature
limit is indeed simpler since the partition function could be analysed at the level of a saddle point approximation. However it is easy to see that generically this limit does not commute with the limit of the number of replicas going to zero. There is a very basic origin to this non commutation, namely the fact that there still exist, even at zero temperature, sample to sample fluctuations. In this paper we try to develop a method of summation over all saddle point in replica space, in order to get the low temperature behaviour of glassy systems. The main aim of this paper is to propose this new method. We have tested it on some elementary problems which can be solved directly. As for its application to more difficult problems, we have also obtained some very good approximation to the zero temperature fluctuations a particle in a random medium, as well as some interesting scaling relations in the random field Ising model. Sect.2 presents the method and illustrates it on a variety of zero dimensional problems. In section 3 we discuss the case of directed polymers in random media with long range interactions, where we rederive the scaling exponents using this new method. In sect.4 we study the D dimensional random field Ising model. Perspectives are briefly summarized in sect.5.

2 Zero-dimensional systems

2.1 The Ising Model

To demonstrate in the simplest terms how the proposed procedure works, we consider first some trivial zero dimensional problems. The simplest example is one Ising spin $\sigma = \pm 1$ in a random field $h$. The Hamiltonian is:

$$H = \sigma h$$  \hspace{1cm} (2.1)\]

where the distribution for the random field is Gaussian:

$$P(h) = \frac{1}{\sqrt{2\pi h_0^2}} \exp\left(-\frac{h^2}{2h_0^2}\right)$$  \hspace{1cm} (2.2)\]

The free energy is:

$$-\beta F(h_0; \beta) = \ln \left[ \sum_{\sigma = \pm 1} \exp(-\beta \sigma h) \right] = \int_{-\infty}^{+\infty} Dx \ln[1 + \exp(2\beta h_0 x)]$$  \hspace{1cm} (2.3)\]

where $Dx$ is the centered gaussian measure of width one: $Dx = \frac{dx}{\sqrt{2\pi}} \exp(-\frac{1}{2} x^2)$. In particular, in the zero temperature limit one finds:

$$F(h_0; \beta \to \infty) = \frac{2h_0}{\sqrt{2\pi}}$$  \hspace{1cm} (2.4)\]

2
Let us consider now how this "problem" can be solved in terms of the replica approach:

\[ -\beta F(h_0; \beta) = \lim_{n \to 0} \frac{1}{n} (Z^n - 1) = \]

\[ \lim_{n \to 0} \frac{1}{n} \left[ \sum_{\sigma_a = \pm 1} \exp \left\{ \frac{1}{2} \beta^2 h_0^2 \left( \sum_{a=1}^n \sigma_a \right)^2 \right\} - 1 \right] = \tag{2.5} \]

In view of the application of the method to more complicated problems, we want to compute the behaviour at low temperature. This cannot be done naively from a saddle point evaluation of the sum at large \( \beta \), because of the non commutativity of the limits \( \beta \to \infty \) and \( n \to 0 \). Instead we proceed as follows. The term \( k = 0 \), which is the contribution from the 'replica symmetric (RS) configuration' \( \sigma_a = +1 \), is singled out; its contribution is equal to \( 1 + O(n^2) \), which cancels the \((-1)\) in eq. (2.5). The contributions of the rest of the terms (which could be interpreted as corresponding to the states with "replica symmetry breaking" (RSB) in the replica vector \( \{ \sigma_a \} \)) can be represented as follows:

\[ F(h_0; \beta) = -\lim_{n \to 0} \frac{1}{\beta n} \sum_{k=1}^{\infty} \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \exp \left\{ \frac{1}{2} \beta^2 h_0^2 (2k-n)^2 \right\} \] \tag{2.6}

Here the summation over \( k \) can be extended beyond \( k = n \) to \( \infty \) since the gamma function is equal to infinity at negative integers.

Now we perform the analytic continuation \( n \to 0 \), using the relation:

\[ \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \bigg|_{n \to 0} \simeq n \frac{(-1)^{k-1}}{k} \] \tag{2.7}

Thus, for the free energy (2.6) one obtains:

\[ -\beta F(h_0; \beta) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(\beta^2 h_0^2 k^2) = \int_{-\infty}^{+\infty} Dx \ln [1 + \exp(2\beta h_0 x)] \] \tag{2.8}

We see that this result coincides with the one (2.3) obtained by the direct calculation. This is of course no surprise since we have just done an exact replica computation. But it exemplifies some of the steps that we shall need below, in particular the proper definition and computation of the divergent series appearing in (2.3) through an integral representation.

### 2.2 The "Soft" Ising Model

Consider now the "soft" version of the Ising model described by the double-well Hamiltonian:
\[ H = -\frac{1}{2} \tau \phi^2 + \frac{1}{4} \phi^4 - h\phi \]  

(2.9)

where the random field is described by the Gaussian distribution (2.2). We concentrate again on the zero temperature limit. Besides, we assume that the typical value of the field \( h_0 \) is small (\( h_0 \ll \frac{\tau^3}{2} \)). In this case the field will not destroy the double-well structure of the Hamiltonian (2.9), and (at \( T \to 0 \)) the system must be equivalent to the discrete Ising model considered before. (The “opposite limit” of the random field Hamiltonian with only one ground state will be considered in Sec.2.3).

The direct calculation of the zero-temperature free energy is trivial. For a given value of the field \( h \ll \frac{\tau^3}{2} \) the ground states of the Hamiltonian (2.9) are:

\[ \phi_1 \approx +\sqrt{\tau} + h \tau, \quad \phi_2 \approx -\sqrt{\tau} + h \tau, \]

for \( h > 0 \); and \( \phi_1 \approx -\sqrt{\tau} + h \tau, \quad \phi_2 \approx +\sqrt{\tau} + h \tau \), for \( h < 0 \). In both cases the corresponding energy is

\[ E_g(h) \approx -\frac{1}{4} \tau^2 - |h|\sqrt{\tau}. \]

Thus, the zero-temperature averaged free energy is:

\[ F(h_0) \approx -\frac{1}{4} \tau^2 - 2\tau \int_0^{+\infty} \frac{dh}{\sqrt{2\pi h_0^2}} h \exp\left( -\frac{h^2}{2h_0} \right) = -\frac{1}{4} \tau^2 - \frac{2h_0\tau}{\sqrt{2\pi}} \]  

(2.10)

Consider now how this result can be obtained in terms of replicas. The replica Hamiltonian and the corresponding saddle-point equations are:

\[ H_n = -\frac{1}{2} \tau \sum_{a=1}^n \phi_a^2 + \frac{1}{4} \sum_{a=1}^n \phi_a^4 - \frac{1}{2} \beta h_0^2 (\sum_{a=1}^n \phi_a)^2 \]  

\[ -\tau \phi_a + \phi_a^3 = \beta h_0^2 (\sum_{a=1}^n \phi_a) \]  

(2.11)

(2.12)

The ”replica-symmetric” solution of these equations (in the limit \( n \to 0 \)) is: \( \phi_a = \phi_{rs} = \sqrt{\tau} \). The corresponding energy is \( E_{rs} = -\frac{1}{4} n \tau^2 \). This solution (in the limit \( n \to 0 \)) does not involve the contribution from the random field.

Proceeding along the lines of the Section 2.1, we have to look also for the solution of the eqs. (2.12) which would involve the ”replica symmetry breaking” in the replica vector \( \{\phi_a\} \):

\[ \phi_a = \begin{cases} \phi_1 & \text{for } a = 1, \ldots, k \\ \phi_2 & \text{for } a = k + 1, \ldots, n \end{cases} \]  

(2.13)

In terms of this Ansatz in the limit \( n \to 0 \) the replica summations can be performed according to the following simple rule: \( \sum_{a=1}^n \phi_a = k\phi_1 + (n - k)\phi_2 \to k(\phi_1 - \phi_2) \). The saddle-point eqs. (2.12) then turn into two equations for \( \phi_1 \) and \( \phi_2 \):

\[ -\tau \phi_{1,2} + \phi_{1,2}^3 = \beta k h_0^2 (\phi_1 - \phi_2) \]  

(2.14)
Assuming that $\beta k h_0^2 \ll \tau$ (the explanation of this strange assumption - considering that we are interested in the $\beta \to \infty$ limit! - will be given below), in the leading order one gets:

$$\phi_1 \simeq +\sqrt{\tau}; \quad \phi_2 \simeq -\sqrt{\tau}$$ \hspace{1cm} (2.15)

From the eq. (2.11) one obtains the corresponding energy of the above "RSB" saddle-point solution:

$$E_k = -\frac{1}{2} k \tau (\phi_1^2 - \phi_2^2) + \frac{1}{4} k (\phi_1^4 - \phi_2^4) - \frac{1}{2} \beta h_0^2 k^2 (\phi_1 - \phi_2)^2 \simeq$$

$$\simeq -2 \beta k^2 h_0^2 \tau + O(h_0^4)$$ \hspace{1cm} (2.16)

Now, similarly to the calculations of Sec.2.1 for the zero-temperature free energy one obtains, summing the contributions from all these saddle points:

$$F(h_0) = -\lim_{n \to 0} \frac{1}{\beta n} (Z_n - 1) \simeq$$

$$= - \lim_{n \to 0} \frac{1}{\beta n} (Z_{rs} - 1) - \lim_{n \to 0} \frac{1}{\beta n} Z_{rsb} =$$

$$= - \lim_{n \to 0} \frac{1}{\beta n} [\exp(\frac{1}{4} \beta n \tau^2) - 1] - \frac{1}{\beta} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(2 \beta^2 k^2 h_0^2 \tau) =$$

$$= -\frac{1}{4} \tau^2 - \frac{1}{\beta} \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi}} \exp(-\frac{1}{2} x^2) \ln[1 + \exp(2 \beta h_0 x \sqrt{\tau})]$$

Taking the limit $\beta \to \infty$ one finally gets the result:

$$F(h_0) \simeq -\frac{1}{4} \tau^2 - \frac{1}{\beta} 2 \beta h_0 \sqrt{\tau} \int_{0}^{+\infty} \frac{dx}{\sqrt{2\pi}} x \exp(-\frac{1}{2} x^2) = -\frac{1}{4} \tau^2 - \frac{2 h_0 \sqrt{\tau}}{\sqrt{2\pi}}$$ \hspace{1cm} (2.17)

which coincides with eq. (2.10).

It is worth to note that the summation of the series in eq. (2.17) can also be performed in the other way:

$$F_{rsb} = -\frac{1}{\beta} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(2 \beta^2 k^2 h_0^2 \tau) = \frac{1}{2i \beta} \int_{C} \frac{dz}{z \sin(\pi z)} \exp(2 \beta^2 z^2 h_0^2 \tau)$$ \hspace{1cm} (2.18)

where the integration goes over the contour in the complex plane shown in Fig.1a. Then we can move the contour to the position shown in Fig.1b, and after the change of integration variable:

$$z \to [2 \beta^2 h_0^2 \tau]^{-1/2} i x$$ \hspace{1cm} (2.19)
in the limit $\beta \to \infty$ we have:

$$\sin(\pi z) \simeq \frac{1}{\beta} i\pi [2h_0^2 \tau]^{-1/2} x$$  \hspace{1cm} (2.21)

Then, taking into account also the contribution from the pole at $x = 0$ for the integral in eq.(2.19) we get:

$$F_{rsb} = \frac{h_0 \sqrt{2\tau}}{2\pi} \int_{-\infty}^{+\infty} \frac{dx}{x^2} \left\{ \exp(-x^2) - 1 \right\} = -\frac{2h_0 \sqrt{\tau}}{\sqrt{2\pi}}$$  \hspace{1cm} (2.22)

which again coincides with the results (2.10) and (2.18).

This little exercise with the integral representation of the divergent series in eq.(2.19) shows in particular that the “effective” value of the parameter $\beta k \to \beta z$ which enter into the saddle-point equations (2.14) scales (according to (2.20)) as $(h_0 \sqrt{\tau})^{-1}$. That is why in the zero temperature limit the ”effective” value of the factor $\beta k h_0^2 \sim h_0 / \sqrt{\tau}$ in the eq.(2.14) can be assumed to be small compared to $\tau$ (for small fields $h_0 \ll \tau^{3/2}$).

**Replica fluctuations**

Because of the non commutativity of the limits $n \to 0$ and $\beta \to \infty$, one cannot get the exact result by keeping only the saddle-point states of the replica Hamiltonian. Actually, averaging over quenched disorder involves the effects of sample to sample fluctuations which in terms of the replica formalism manifest themselves as the contribution from the replica fluctuations. In other words, to get exact result in terms of replicas the contribution from the saddle-points is not enough, and one has to integrate over replica fluctuations even in the zero-temperature limit.

This phenomenon can be easily demonstrated for the above example of the ”soft” Ising model. Let us take into account the contribution from the Gaussian replica fluctuations near the ”replica-symmetric” saddle-point $\phi_a = \phi_{rs} = \sqrt{\tau}$:

$$\phi_a = \phi_{rs} + \varphi_a$$  \hspace{1cm} (2.23)

From the eq.(2.14) for the ”replica-symmetric” part of the partition function we get:

$$Z_{rs} = \exp\left(\frac{1}{4} \beta n \tau^2 \right) \int d\varphi_a \exp\left\{ -\beta \sum_{a,b} (\tau \delta_{ab} - \frac{1}{2} \beta h_0^2) \varphi_a \varphi_b \right\} \simeq \exp\left\{ \frac{1}{4} \beta n \tau^2 + \frac{hn^2}{4\tau} - \frac{1}{2} n \ln(\beta \tau) \right\}$$  \hspace{1cm} (2.24)

Therefore, in the zero-temperature limit one obtains the following contribution to the free energy:

$$F_{rs} = -\frac{1}{4} \tau^2 - \frac{h_0^2}{4\tau}$$  \hspace{1cm} (2.25)
We see that at $T = 0$ there exists a finite contribution $\sim h_0^2/\tau$ due to the replica fluctuations. In the particular example considered the value of $h_0$ was assumed to be small, and this contribution can be treated as a small correction. However, we should keep in mind that the contribution from the replica fluctuations in general could appear to be of the same order as that from the saddle points. Therefore, the calculations we are going to perform in next sections for less trivial examples taking into account only saddle-point states can not pretend to give exact results, giving only the scaling dependence from the parameters of a model.

**Saddle points**

In the above calculations of the free energy for the "soft" Ising system we have taken into account only the contribution from the two minima of the double-well potential. The existence of the third saddle-point, which is the maximum at $\phi = 0$, has been ignored. In this particular example such an algorithm looks natural. However, in less trivial systems very often it is not easy to distinguish the types of the saddle points involved. Moreover, it could be very hard to impose a simple and robust "discrimination" rule with respect to different types of saddle-points, which would not block the calculations at the very start.

Because of this, we would like to propose a somewhat modified scheme of calculations which takes into account all saddle points. In the above example of the "soft" Ising model the third saddle-point (the maximum) is at $\phi = 0$. Then, instead of the Ansatz (2.13) let us represent the replica vector $\phi_a$ as follows:

$$
\phi_a = \begin{cases} 
+\sqrt{\tau} & \text{for } a = 1, \ldots, k \\
-\sqrt{\tau} & \text{for } a = k + 1, \ldots, k + l \\
0 & \text{for } a = k + l + 1, \ldots, n 
\end{cases}
$$

For the corresponding "energy" (in the limit $n \to 0$) from the replica Hamiltonian (2.11) one easily finds:

$$
H_{kl} = -\frac{1}{4} \tau^2 (k + l) - \frac{1}{2} h_0^2 \tau (k - l)^2 + O(h_0^4)
$$

(2.27)

Note that in terms of the Ansatz (2.26) the "replica symmetric" state $(k = l = 0)$, $\phi_a = \phi_0 = 0$ has zero energy, so that it gives no contribution to the free energy.

The combinatoric factor in the $n \to 0$ limit is now:

$$
\frac{n!}{k! l! (n - k - l)!} \to n \frac{(-1)^{k+l-1} (k + l)!}{k + l} \frac{1}{k! l!}
$$

(2.28)

Thus, for the free energy (for $\beta \to \infty$) we obtain:
\[ F(h_0) = -\frac{1}{\beta} \sum_{k+l=1}^{\infty} \frac{(-1)^{k+l-1}}{k+l} \frac{(k+l)!}{k!l!} \exp\left\{ \frac{1}{4} \beta \tau^2 (k + l) + \frac{1}{2} \beta^2 h_0^2 \tau (k - l)^2 \right\} \] (2.29)

This series can be summed up in a similar way as the ones in eqs. (2.8) and (2.17):

\[ F(h_0) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} Dx \ln \left[ 1 + \exp\left\{ \frac{\beta}{4} \tau^2 + \beta h_0 \sqrt{\tau} x \right\} + \exp\left\{ \frac{\beta}{4} \tau^2 - \beta h_0 \sqrt{\tau} x \right\} \right] \] (2.30)

In the limit \( \beta \to \infty \) one finds:

\[ F(h_0) = -\frac{1}{\beta} \int_{-\infty}^{+\infty} Dx \left[ \frac{\beta}{4} \tau^2 + \beta h_0 \sqrt{\tau} |x| \right] = - \frac{1}{4} \tau^2 - \frac{2h_0 \sqrt{\tau}}{\sqrt{2\pi}} \] (2.31)

Again, we get the correct result. While it might seem at first sight somewhat "magic", at least some aspects of this computation can be understood. In the example considered (as well as in the further examples to be studied below) the relevant states, which contribute to the free energy, have negative energy \(-E(h)\). Then, in the low temperature limit the partition function of a given sample is \( Z \simeq \exp(\beta E(h)) \). Therefore, in the limit \( \beta \to \infty \) the free energy can be represented with exponential accuracy as follows:

\[ F(h_0) = -\frac{1}{\beta} \ln Z \simeq - \frac{1}{\beta} \ln (1 + Z) \] (2.32)

One can easily check that after averaging \( Z^m \equiv Z_m \) and taking into account the contributions of the two minima of the corresponding replica Hamiltonian \( H_m \) one recovers the series in eq. (2.29).

The only "magic" rule which should be followed in the direct replica calculations is that the "background" state, \( \phi_0 = 0 \) (the one with zero energy) in the Ansatz for the replica vector \( \phi_a \) of the type \( (2.26) \) should be placed in the last group of replicas. Using this rule, the series obtained for the free energy will correspond to the above interpretation (2.32).

### 2.3 One-Well Potential

Consider now how the method works in the case where the Hamiltonian has only one minimum:

\[ H = \frac{1}{\alpha} \phi^\alpha - h_0 \phi \] (2.33)
where $\phi \geq 0$ and $\alpha \geq 2$, and the random field $h$ is again described by the Gaussian distribution (2.2). For $\alpha = 4$ this system can be interpreted as the variant of the Hamiltonian (2.9) in the limit of strong magnetic fields.

In the zero-temperature limit the free energy is defined by the ground state $\phi(h) = (h^2)_{\alpha = 1}^{\gamma}$ for $h > 0$, and $\phi = 0$ for $h \leq 0$. Its energy is $E(h) = -\frac{\alpha}{\alpha - 1}h^{\alpha/(\alpha - 1)}$ for $h > 0$, and $E = 0$ for $h \leq 0$. Therefore, for the averaged zero-temperature free energy we find:

$$F(h_0) = -\frac{\alpha}{\alpha - 1} \int_{0}^{+\infty} \frac{dh}{\sqrt{2\pi h_0}}h^{\frac{\alpha}{\alpha - 1}} + \frac{\beta h_0^2}{\alpha - 1}$$

In terms of replicas, the corresponding replicated Hamiltonian is:

$$H_n = \frac{1}{\alpha} \sum_{a=1}^{n} \phi_a^\alpha - \frac{1}{2} \beta h_0^2 \sum_{a,b=1}^{n} \phi_a \phi_b$$

(2.35)

This Hamiltonian has a trivial ”background” extremum at $\phi = 0$ with zero energy. Therefore, following the scheme proposed in the previous subsection, we look for non-trivial saddle-point solutions in terms of the following Ansatz:

$$\phi_a = \begin{cases} 
\phi & \text{for } a = 1, \ldots, k \\
0 & \text{for } a = k + 1, \ldots, n
\end{cases}$$

(2.36)

For the corresponding Hamiltonian and the saddle-point equation (in the limit $n \to 0$) one gets:

$$H_k = \frac{1}{\alpha} k\phi^\alpha - \frac{1}{2} \beta h_0^2 k^2 \phi^2$$

(2.37)

$$\phi^{\alpha-1} - \beta h_0^2 k \phi = 0$$

(2.38)

The solution on this equation and the corresponding energy are:

$$\phi = (\beta k h_0^2)^{\frac{1}{\alpha - 2}}$$

(2.39)

$$H_k = -\frac{1}{\beta} \frac{\alpha - 2}{2\alpha} (\beta k)^{\frac{2(\alpha - 1)}{\alpha - 2} h_0^{\frac{2\alpha}{\alpha - 2}}}$$

(2.40)

(Note, that although one can try with more ”RSB” steps in the replica vector $\phi_a$ it can be easily proved that there exists only one type of the non-trivial solution given by the Ansatz (2.36)). Then, in terms of the procedure described above for the free energy we have:

$$F(h_0) = -\frac{1}{\beta} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp\left\{\frac{\alpha - 2}{2\alpha} (\beta k)^{\frac{2(\alpha - 1)}{\alpha - 2} h_0^{\frac{2\alpha}{\alpha - 2}}} \right\}$$

(2.41)
The summation of this series can be performed in terms of the integral representation eq.(2.19):

\[
F(h_0) = \frac{1}{2i\beta} \int_C \frac{dz}{z \sin(\pi z)} \exp\left\{ \frac{\alpha - 2}{2\alpha} (\beta z)^{\frac{2(\alpha-1)}{\alpha-2}} h_0^{\frac{2\alpha}{\alpha-2}} \right\} \tag{2.42}
\]

where the integration goes over the contour in the complex plane shown in Fig.1a. Then, again, we move the contour to the position shown in Fig.1b and redefine the integration variable:

\[
z \rightarrow \frac{1}{\beta} h_0^{-\frac{\alpha}{\alpha-1}} ix \tag{2.43}
\]

In the limit \(\beta \rightarrow \infty\) we have:

\[
\sin(\pi z) \simeq \frac{1}{\beta} i\pi h_0^{-\frac{\alpha}{\alpha-1}} x \tag{2.44}
\]

and

\[
F(h_0) = -h_0^{\frac{\alpha}{\alpha-1}} \frac{1}{2\pi} \int_{C_1} \frac{dx}{x^2} \exp\left\{ \frac{\alpha - 2}{2\alpha} (ix)^{\frac{2(\alpha-1)}{\alpha-2}} \right\} \equiv B(\alpha) \times h_0^{\frac{\alpha}{\alpha-1}} \tag{2.45}
\]

Thus, we have obtained the correct scaling of the free energy as a function of \(h_0\). Note however, that although it is also possible to calculate the value of the coefficient \(B(\alpha)\) in the integral (2.45), such a calculation would not make much sense because to obtain the correct coefficient (which is given by the integral in (2.34)) one would need to take into account replica fluctuations which we have neglected here.

### 2.4 The Toy Model

Let us consider now a slightly less trivial example of a zero-dimensional system which cannot be solved by elementary algebra. This system, generally called the "toy model", consists of a single degree of freedom \(\phi\) evolving in an energy landscape which is the sum of a quadratic well and a Brownian potential. The Hamiltonian is:

\[
H = \frac{1}{2} \mu \phi^2 + V(\phi) \tag{2.46}
\]

where \(V(\phi)\) is the random potential described by the Gaussian distribution:

\[
P[V(\phi)] \sim \exp\left\{ -\frac{1}{4g} \int d\phi (\frac{dV}{d\phi})^2 \right\} \tag{2.47}
\]
The $V$ distribution is characterized by its first two moments:

$$
(V(\phi) - V(\phi'))^2 = 2g|\phi - \phi'|;
$$

$$
V(\phi) = 0;
$$

$$
V(\phi)V(\phi') = C - g|\phi - \phi'|
$$

where $C$ is an irrelevant constant This problem was introduced originally as a toy, zero dimensional version of the interface in the random field Ising model [5]. It has the virtue of showing explicitly how the most standard field theoretic methods like perturbation theory, iteration methods or supersymmetry get fooled in this problem, in the low $\mu$, low temperature limit, by the existence of many metastable states [5, 6, 7, 8]. The main point is that at low enough temperatures the usual perturbation theory does not work and a qualitatively reasonable theory must involve the effects of the replica symmetry breaking. This has been demonstrated within the replica gaussian variational approximation [6, 7].

One quantity which one would like to calculate in such a system is the value of $\langle \phi^2 \rangle$ in the limit of the zero temperature. Using simple energy arguments one can easily estimate what must be the scaling dependence of this quantity on the parameters $\mu$ and $g$. For a given value of $\phi$ the loss of energy due to the attractive potential in the Hamiltonian (2.46) is $\sim \mu \phi^2$. Possible gain of energy due to the random potential according to statistics (2.47)-(2.48) can be estimated as $\sim \sqrt{g}\sqrt{\phi}$. Optimizing the total energy $E \sim \mu \phi^2 - \sqrt{g}\sqrt{\phi}$ with respect to $\phi$ one finds that

$$
\langle \phi^2 \rangle = C_2 \frac{g^{2/3}}{\mu^{4/3}}
$$

This result tells that the typical energy minimum of the Hamiltonian (2.46) lies at a finite distance from the origin. The scaling (2.49), which is obviously right, is not so easy to derive from some field theoretic methods which could be also used in higher dimension, and there is no known exact result for the constant $C_2$ at the moment.

Let us try to calculate the value of $\langle \phi^2 \rangle$ in the zero temperature limit using the method considered above. The replicated Hamiltonian of the system (2.46) is:

$$
H_n = \frac{1}{2} \mu \sum_{a=1}^{n} \phi_a^2 + \frac{1}{2} \beta g \sum_{a,b=1}^{n} |\phi_a - \phi_b|
$$

The corresponding saddle-point equations are:

$$
\mu \phi_a + \beta g \sum_{b=1}^{n} \text{Sign}(\phi_a - \phi_b) = 0
$$
(Note that in this formula, whenever there is some ambiguity, one should always assume that there is at some intermediate step a short scale regularization. Therefore, one must interpret for instance \( \text{Sign}(0) = 0 \).) Let us first look for non-trivial solutions of the eqs. (2.51). It can be easily proven that within the "one-step" RSB Ansatz (2.13) there exist no non-trivial solutions. Let us consider the "two-steps" Ansatz for the replica vector \( \phi_a \):

\[
\phi_a = \begin{cases} 
\phi_1 & \text{for } a = 1, \ldots, k \\
\phi_2 & \text{for } a = k + 1, \ldots, k + l \\
\phi_3 & \text{for } a = k + l + 1, \ldots, n 
\end{cases} \quad (2.52)
\]

From eqs. (2.51) one finds the following equations for \( \phi_{1,2,3} \) (in the limit \( n \to 0 \)):

\[
\begin{align*}
\mu \phi_1 + \beta g l \text{Sign}(\phi_1 - \phi_2) - \beta g (k + l) \text{Sign}(\phi_1 - \phi_3) &= 0 \\
\mu \phi_1 + \beta g k \text{Sign}(\phi_2 - \phi_1) - \beta g (k + l) \text{Sign}(\phi_2 - \phi_3) &= 0 \\
\mu \phi_3 + \beta g k \text{Sign}(\phi_3 - \phi_1) + \beta g l \text{Sign}(\phi_3 - \phi_2) &= 0
\end{align*} \quad (2.53)
\]

The solution of these equations is:

\[
\phi_1 = -\frac{g}{\mu} \beta k; \quad \phi_2 = \frac{g}{\mu} \beta l; \quad \phi_3 = \frac{g}{\mu} \beta (l - k) \quad (2.54)
\]

and the corresponding energy is (in the limit \( n \to 0 \)):

\[
E_{kl} = -\frac{\beta^2 g^2}{2\mu} kl (k + l) \quad (2.55)
\]

It can be proven that there exist no other solutions of the saddle-point equations (2.51) with a number of RSB steps larger than two.

Therefore, (after taking the limit \( n \to 0 \)) for the "RSB" part of the free energy we get the following series (see eq. (2.28)):

\[
F_{rsb} = -\frac{1}{\beta n} \sum_{k+l=1}^{n} \frac{n!}{k!(n-k-l)!} \exp(-\beta E_{kl}) \rightarrow
\]

\[
-\frac{1}{\beta} \sum_{k+l=1}^{\infty} \frac{(-1)^{k+l-1}}{k+l} \frac{1}{k!l!} \exp\{\lambda kl (k + l)\}
\]

where

\[
\lambda = \frac{\beta^2 g^2}{2\mu} \rightarrow \infty \quad (2.56)
\]

We again carry the summation of the asymptotic series (2.56) with the integral method mentioned in section 2.2:
\[ F_{\text{rsb}} = \frac{1}{\beta(2i)^2} \int \int_C dz_1 dz_2 \frac{\Gamma(z_1 + z_2 + 1)}{(z_1 + z_2) \sin(\pi z_1) \sin(\pi z_2) \Gamma(z_1 + 1) \Gamma(z_2 + 1)} \exp\{\lambda z_1 z_2 (z_1 + z_2)\} \]  

(2.58)

where the integrations over \( z_1, z_2 \) both go around the contour in the complex plane shown in Fig.1a.

Shifting the contour of integration to the position shown in Fig1.b, and redefining the integration variables: \( z_1, z_2 \rightarrow \lambda^{-1/3} i x_1, x_2 \) in the limit \( \beta \rightarrow \infty (\lambda^{-1/3} \rightarrow 0) \) one gets:

\[ F_{\text{rsb}} = \frac{1}{\beta} \frac{\lambda^{1/3}}{2\pi^2} \int_0^{+\infty} dx_1 dx_2 \left\{ \frac{\sin(x_1 x_2 (x_1 + x_2))}{x_1 x_2 (x_1 + x_2)} + \frac{\sin(x_1 x_2 (x_1 - x_2))}{x_1 x_2 (x_1 - x_2)} \right\} \]  

(2.59)

Substituting the value of \( \lambda = \beta^3 g^2 / 2\mu \) we finally get the result for the zero-temperature free energy:

\[ F_{\text{rsb}} = \frac{g^{2/3}}{\mu^{1/3}} \frac{\sqrt{3} \Gamma(1/6)}{4\pi^{3/2}} \]  

(2.60)

To this piece we must now add the replica symmetric contribution. The saddle point equations have the trivial solution: \( \phi_a = 0 \) with the corresponding energy \( E_0 \equiv H_n[\phi_a = 0] = 0 \). As we want to get a quantitative result for the constant \( C_2 \), we must also include the contribution from the replica fluctuations around this saddle point. This cannot be done just at the level of integrating the quadratic fluctuations. We shall rather make the following (strong) assumption, namely that this whole ‘RS’ part of the free energy, including the replica fluctuations, is given by the Gaussian replica variational method \[4, 9, 10\]. We do not have a very convincing argument to support this hypothesis; we just point out that this gaussian variational method involves the Gaussian integration over replica fields which in a sense is ”symmetric” with respect to the point \( \phi_a = 0 \). In the end the hypothesis is best supported by the good result one gets for \( C_2 \). We denote the gaussian variational contribution by \( F_{\text{rv}} \), and our conjecture is that \( F = F_{\text{rv}} + F_{\text{rsb}} \).

According to eq.(2.46):

\[ \langle \phi^2 \rangle = 2 \frac{\partial F}{\partial \mu} = \langle \phi^2 \rangle_{\text{rv}} - \frac{g^{2/3}}{\mu^{4/3}} \frac{\Gamma(1/6)}{2\sqrt{3} \pi^{3/2}} \]  

(2.61)

Using the result of \[9\] for the value of \( \langle \phi^2 \rangle_{\text{rv}} \) we finally get:

\[ \langle \phi^2 \rangle = \frac{g^{2/3}}{\mu^{4/3}} \left( \frac{3}{(4\pi)^{1/3}} - \frac{\Gamma(1/6)}{2\sqrt{3} \pi^{3/2}} \right) \simeq 1.00181 \frac{g^{2/3}}{\mu^{4/3}} \]  

(2.62)

We have compared this result with some numerical simulations of the problem. The scaling in \( \mu \) and \( g \) is obviously correct, the only point to check is the prefactor \( C_2 \). Choosing for instance the values of the parameters \( \mu = 1 \) and \( g = 2\sqrt{\pi} \) (when the replica
variational method gives $\langle \phi^2 \rangle_{rv} = 3$ we obtain from (2.62) the analytical prediction: $\langle \phi^2 \rangle \simeq 2.3291$. The numerical simulation was done at zero temperature, with these same values of $\mu$ and $g$. The $\phi$ interval $[-8, 8]$ is discretized in $2N$ points, on which one generates a random potential as in (2.46). The exhaustive scan gives the minimum, from which one computes $\langle \phi^2 \rangle$. We average over 100000 samples. The number of points $2N$ ranged from $2^8$ to $2^{16}$, in this regime there is no systematic $N$ dependance. There is no systematic error due to the finite width of the interval since we have checked that, within our statistics, there is no sample for which the minimum is found with $|\phi| \geq 7$. The result of the simulation is $\langle \phi^2 \rangle \simeq 2.45 \pm .02$. The value predicted by our replica saddle point summation is rather close, although there is a clear small discrepancy.

The result for $\langle \phi^4 \rangle$

To be sure that this relatively good agreement of our prediction for $\langle \phi^2 \rangle$ with the numerical result is not just a coincidence we have performed similar calculations for the next order correlator $\langle \phi^4 \rangle$. The computations, which are similar to the ones we have just presented but more cumbersome, are given in the appendix. The result is:

$$\langle \phi^4 \rangle = \langle \phi^4 \rangle_{rv} + \langle \phi^4 \rangle_{rsb} = \frac{g^{4/3}}{\mu^{8/3}} \left( \frac{27}{(4\pi)^{2/3}} - \frac{17\sqrt{3} [\sin(\pi/12) + \cos(\pi/12)]}{3\sqrt{\pi} \Gamma(1/6) \sin(\pi/6)} \right)$$  (2.63)

For the values of the parameters $\mu = 1$ and $g = 2\sqrt{\pi}$ (when the replica variational method gives $\langle \phi^4 \rangle_{rv} = 27$) we obtain: $\langle \phi^4 \rangle \simeq 16.25$. The numerical result is obtained with the same procedure as above and gives $\langle \phi^4 \rangle \simeq 17.05 \pm .2$. Again these number are close but there is a significative difference.

Clearly, the vector type of rsb that we have been using on all these zero dimensional problems is somewhat strange, and we cannot assert that we control all of its aspects (in particular the fact that the replica fluctuations around the rs saddle point are summed by the gaussian variational method is still unclear). However in all these cases, and in particular in the non-trivial case of the toy model, we have obtained good results using this simple receipe. Therefore we now turn to its application to more elaborate problems, starting with systems in one dimension.

3 Directed Polymers in Random Media

The problem of a directed polymer in a random medium is an important problem which has been much studied recently [11]. Although the situation in 1+1 dimension, with a delta correlated potential, is relatively well understood, there are still a lot of uncertainties about more complicated cases.
We shall consider a one dimensional case with long range correlations of the potential. It is described by a one-dimensional scalar field system with the following Hamiltonian:

\[ H = \int_0^L dx \left[ \frac{1}{2} \left( \frac{d\phi(x)}{dx} \right)^2 + V(x, \phi) \right] \]  

(3.1)

where the random potentials \( V(x, \phi) \) are described by the Gaussian distribution with non-local correlations with respect to the fields \( \phi \):

\[ V(x, \phi) V(x', \phi') = \delta(x - x') \left[ \text{const} - g(\phi - \phi')^{2\alpha} \right] \]  

(3.2)

where \( 0 < \alpha < 2 \).

This problem naturally arises, with \( \alpha = 1/2 \), when one considers an interface in the two dimensional random field Ising model at low temperatures: then the field \( \phi \) just describes the lateral fluctuations in the interface, in a solid on solid approximation.

One first basic question that we would like to answer concerns the scaling behaviour of the lateral fluctuations. Let the value of the field \( \phi(x) \) be stucked to zero at the origine: \( \phi(x = 0) \equiv 0 \). Then one would like to know how the average value of the field at \( x = L \), \( \langle \phi(L) \rangle \), scales with \( L \):

\[ \langle \phi(L)^2 \rangle \equiv \left( Z^{-1} \int d\phi_0 \phi_0^2 \int_{\phi(0)=0}^{\phi(L)=\phi_0} D\phi(x) \exp(-\beta H[\phi(x), V]) \right) \sim L^{2\zeta} \]  

(3.3)

where the partition function \( Z \) (for a given realization of the random potential) is given by the integration over all the trajectories \( \phi(x) \) with only one boundary condition \( \phi(x = 0) = 0 \). The ‘wandering exponent’ \( \zeta \) has been computed in the case of local correlations of the random potential, it is then equal to \( 2/3 \) \[12\]. In the case of non local correlations such as (3.2), it is believed that this exponent should be equal to \( 3/2(2 - \alpha) \) at small enough \( \alpha \). This is the result that is obtained from the gaussian variational Ansatz \[4\], and it can also be derived from a mapping to the Burgers (or the KPZ) equation and a study of this equation through a dynamical renormalization group procedure \[13\].

A simple derivation of this scaling can be obtained by an energy balance argument a la Imry Ma \[14\]. Let the value of the field be equal to \( \phi_0 \) at \( x = L \). Then the loss of the energy due to the gradient term in the Hamiltonian \( (3.1) \) can be estimated as \( E_g \sim \phi_0^2 / L \). The gain of energy due to the random potential term, according to eq.\( (3.2) \), can be estimated as \( E_V \sim -\sqrt{L} \sqrt{g} \phi_0^{\alpha} \). Optimizing \( E_g \) and \( E_V \) with respect to \( \phi_0 \) one finds:

\[ \phi_0 \sim L^{\frac{3}{2(2 - \alpha)}} g^{\frac{1}{2(2 - \alpha)}} \]  

(3.4)

In this section we will demonstrate how this result can be obtained in the zero-temperature limit in terms of the proposed replica saddle point method. The replicated Hamiltonian is:
\[ H_n = \int_0^L dx \left[ \frac{1}{2} \sum_{a=1}^{n} \left( \frac{d\phi_a(x)}{dx} \right)^2 + \frac{1}{2} \beta g \sum_{a,b=1}^{n} (\phi_a(x) - \phi_b(x))^{2\alpha} \right] \quad (3.5) \]

Strictly speaking, the systematic way of solving this problem following our general method is the following: one must find \( n \) saddle point trajectories \( \phi_a(x) \) for fixed \( n \) boundary conditions \( \phi_a(L) \), then one has to derive the corresponding energy \( \tilde{H}_n[\phi_a(L)] \), and finally one has to find the saddle point solutions with respect to the values of \( \phi_a(L) \).

Here we shall follow a much simpler strategy. Since it is obvious that there always exists the trivial solution \( \phi(x) \equiv 0 \), we will suppose that the correct scaling can be obtained simply by taking into account one non-trivial saddle-point trajectory. In other words, from the very beginning we are going to look for the saddle point solutions within the following Ansatz:

\[
\phi_a(x) = \begin{cases} 
\phi(x) & \text{for } a = 1, \ldots, k \\
0 & \text{for } a = k + 1, \ldots, n
\end{cases} \quad (3.6)
\]

Comparing this Ansatz to the zero dimensional exercises of the previous section, we see that it should amount to assuming that the lowest lying configuration dominates. This is certainly true since one knows [15, 16] that the metastable states have an excitation energy which scales as \( L^\omega \), with \( \omega = 2\zeta - 1 \). Substituting this Ansatz into the replica Hamiltonian (3.5) in the limit \( n \to 0 \) one gets:

\[
H_k = k \int_0^L dx \left[ \frac{1}{2} \left( \frac{d\phi(x)}{dx} \right)^2 - \beta g \phi^{2\alpha}(x) \right] \quad (3.7)
\]

As usual (see the previous section) the free energy is defined by the series:

\[
F(L) \sim -\frac{1}{\beta} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(-\beta H_k) \quad (3.8)
\]

where the value \( H_k \) is defined by the corresponding saddle-point solution for \( \phi(x) \).

The saddle point trajectory is defined by the following differential equation:

\[
\frac{d^2 \phi}{dx^2} = -2\alpha \beta kg \phi^{2\alpha - 1} \quad (3.9)
\]

with the boundary conditions: \( \phi(0) = 0 \) and \( \phi(L) = \phi_0 \). This equation can be easily integrated:

\[
\int_0^{\phi(x)} \frac{d\phi}{\sqrt{\lambda - \phi^{2\alpha}}} = x \sqrt{2\beta kg} \quad (3.10)
\]

where the integration constant \( \lambda \) is defined by the boundary condition:
\[
\int_0^{\phi_0} \frac{d\phi}{\sqrt{\lambda - \phi^{2\alpha}}} = L\sqrt{2\beta k g}
\] (3.11)

Substituting this solution into the Hamiltonian (3.7), we obtain after some simple algebra:

\[
H_k = k \left[ -\beta k g \lambda L + \sqrt{2\beta k g} \int_0^{\phi_0} d\phi \sqrt{\lambda - \phi} \right]
\] (3.12)

Taking derivative of \(H_k\) with respect to \(\phi_0\) (and taking into account the constrain (3.11)) one finds the following saddle-point solution:

\[
\phi_0 \sim L^{\frac{1}{1-x}} (\beta k g)^{\frac{1}{x(x-1)}}
\] (3.13)

and \(\lambda = \phi_0^{2\alpha}\). Its energy (3.12) is:

\[
H_k = -\frac{(const)}{\beta} (\beta k)^{\frac{1}{x(x-1)}} L^{\frac{1}{1-x}} g^{\frac{1}{x-1}}
\] (3.14)

Now we proceed as before, introducing an integral representation of the series (3.8) and a rescaling of the integration variable by a factor \(\frac{1}{\beta} L^{-\frac{1}{x}+\alpha} g^{-\frac{1}{x-1}}\). Then we get the scaling of the free energy:

\[
F(L) \sim L^{\frac{1+\alpha}{x(x-1)}} g^{\frac{1}{x(x-1)}}
\] (3.15)

from which we obtain the scaling of \(\phi_0\) as a function of \(L\):

\[
\phi_0(L) \sim L^{\frac{3}{2(x-1)}} g^{\frac{1}{2(x-1)}}
\] (3.16)

which coincides with the result (3.4) given by the naive energy arguments, as well as by more elaborate calculations.

Although the example demonstrated in this section provides no new results we hope that the proposed method could turn out to be also useful when applied for directed polymers with smaller \(\alpha\), or in larger dimension.

4 Random Field Ising Model in \(D\) dimensions

Since the topic of the random field Ising model covers an enormous amount of litterature (see e.g [17]), it would be rather difficult to give any brief introductory review. Here, however, we are mainly concerned with how the method we have proposed before works in various situations. Therefore, we will concentrate only onto one particular aspect of the problem.

17
It is well known that the main problem in the studies of the low temperature phase in the random field Ising model is that one has to perform the summation over numerous local minima states, which seems to be impossible to do within the framework of the usual perturbation theory [17]. It has been proposed recently that, because of these local minima states a special "intermediate" (separating paramagnetic and ferromagnetic phase) spin-glass-like thermodynamic state could set in around the critical point, and moreover, this state is characterized by a replica symmetry breaking in the corresponding correlation functions [18]. At low temperature, and when the width of the distribution of the random field is not too small, the same phenomenon must be present. It was proposed long ago [2], and elaborated later on in [3], that the metastable states in this regime should be characterized by some "instanton in replica space". Our method provides one more step in the elaboration of this idea.

We consider the random field Ising model in terms of the usual Ginzburg-Landau Hamiltonian in $D$ dimensions:

$$H = \int d^D x \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \tau \phi^2 + \frac{1}{4} g \phi^4 - h(x) \phi \right]$$

(4.1)

where the random fields $h(x)$ are described by the $\delta$-correlated Gaussian distribution:

$$P[h(x)] = \prod_x \frac{1}{\sqrt{2\pi h_0^2}} \exp \left( -\frac{h^2(x)}{2h_0^2} \right)$$

(4.2)

The corresponding replica Hamiltonian is:

$$H_n = \int d^D x \left[ \sum_{a=1}^n (\nabla \phi_a)^2 + \frac{1}{2} \tau \sum_{a=1}^n \phi_a^2 + \frac{1}{4} g \sum_{a=1}^n \phi_a^4 - \frac{1}{2} h_0^2 \sum_{a,b=1}^n \phi_a \phi_b \right]$$

(4.3)

According to the procedure developed in previous sections we are going to look for the most simple non-trivial saddle-point solutions at the background of the trivial one, $\phi_a(x) \equiv 0$. In terms of the Ansatz:

$$\phi_a(x) = \begin{cases} \phi(x) & \text{for } a = 1, \ldots, k \\ 0 & \text{for } a = k+1, \ldots, n \end{cases}$$

(4.4)

The replica Hamiltonian (4.3) reads in the $n \to 0$ limit:

$$H_k = k \int d^D x \left[ \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} (h_0^2 k - \tau) \phi^2 + \frac{1}{4} g \phi^4 \right]$$

(4.5)

Consider for simplicity the situation at $\tau = 0$ (Notice that here we work close to the critical temperature. The use of our saddle point technique allows to study the system at the tree level, which is supposed to give the leading singularities close to $T_c$ [19]). The corresponding saddle-point equation is:
\[ -\Delta \phi - \lambda \phi + g\phi^3 = 0 \]  
(4.6)

where \( \lambda = h_0^2 k \). As usual, the free energy is given by the series:

\[ F(h_0) \sim -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \exp(-H_k) \]  
(4.7)

where the value of \( H_k \) is defined by the corresponding saddle-point solution of the eq. (4.6).

At this stage we see that the situation is getting rather different from the ones studied in the previous sections. If we would choose the obvious space-independent solution \( \phi = (\lambda/g)^{1/2} \), we would find that the value of \( H_k \) is proportional to the volume \( V \) of the system: \( H_k = \frac{1}{4} k(\lambda^2/g)V = -\frac{1}{4g} k^3 h_0^4 V \). Then, the summation of the series (4.7) would immediately yield a free energy proportional to \( V^{1/3} \) and not to \( V \). Therefore this solution, as well as any other solution with an energy \( H_k \) proportional to the volume of the system, is irrelevant for the bulk properties.

Thus, we have to look for localized solutions: the ones which are local in space (breaking translation invariance) and which have finite energy. Let us assume to start with that such an ”instanton”-type solution exists (see below), and that for a given \( k \) it is characterized by the spatial size \( R(k) \). Then, if we take into account only one-instanton contribution (or in other words we consider a gas of non-interacting instantons), due to the trivial entropy factor \( V/R^D \) (this is the number of positions of the object of the size \( R \) in the volume \( V \)) we get a free energy proportional to the volume:

\[ F(h_0) \sim -\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{V}{R^D} \exp(-H_k) \]  
(4.8)

where \( H_k \) must be finite (volume independent).

It is easy to understand that the equation (4.6) indeed has localized solutions. Let us assume that the value of the field \( \phi(x) \) is such that \( \lambda\phi^2 \gg g\phi^4 \). Then in a first approximation the saddle-point equation (4.6) is linear:

\[ \Delta \phi + \lambda \phi = 0 \]  
(4.9)

The simplest possible spherically-symmetric solutions of this equation in \( D \) dimensions are the well known Bessel-type functions. In particular there exist oscillating solutions which have a finite value \( \phi(r = 0) \equiv \phi_0 \) at the origin and which decay to zero at \( r \to \infty \) (like \( \sim r^{-(D-1)/2} \sin r \)). For example, in dimension \( D = 3 \) this solution is simply:

\[ \phi(r) = \phi_0 \frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}} \]  
(4.10)

In dimensions \( D \) these solutions have a finite spatial scale:
\[ R(k) = \lambda^{-1/2} = (h_0^2 k)^{-1/2} \] (4.11)

and finite energy:

\[ H_k = -(const) k \phi_0^2 \lambda^{-D/2} \] (4.12)

At the level of the equation (4.9) itself, the value of \( \phi_0 \) remains arbitrary (the equation is linear). On the other hand, from the point of view of the energy this is not an extremum since the energy explicitly depends on the value of \( \phi_0 \) (this is the saddle-point solution for the fixed boundary condition \( \phi(r = 0) = \phi_0 \)). If we would let the value of \( \phi_0 \) be free in the absence of the non-linear term \( g\phi^4 \) it would, of course, fall down to infinity. However, if we take into account the term \( g\phi^4 \) in the ”exact” Hamiltonian (4.5) it is natural to expect that \( \phi_0 \) will stabilize around the saddle-point value

\[ \phi_0^2 = \frac{\lambda}{g} \] (4.13)

The above qualitative arguments can be easily verified for the model double-well potential:

\[ \tilde{U}(\phi) = -\frac{1}{2} \phi^2 \text{ for } |\phi| \leq \sqrt{\lambda/g} \text{ and } \tilde{U}(\phi) = +\infty \text{ for } |\phi| > \sqrt{\lambda/g}, \text{ taken instead of the ”real” one: } U(\phi) = -\frac{1}{2} \phi^2 + \frac{1}{4} g\phi^4. \]

In this case for any \( |\phi_0| \leq \sqrt{\lambda/g} \) there exists the exact Bessel-like saddle-point solution with finite energy (4.12), and real extremum of the Hamiltonian would be achieved at \( \phi_0 = \pm \sqrt{\lambda/g} \).

Let us calculate the contribution of such solutions to the free energy. Substituting into the series (4.8) the energy of the solution (4.12), the estimate for the value of \( \phi_0 \) (4.13) and the characteristic size of the solution (4.11), together with \( \lambda = h_0^2 k \) we get:

\[ F(h_0) \sim -V \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (h_0^2 k)^{D/2} \exp \left[ \frac{(const)}{gh_0^2} (h_0^2 k)^{6-D} \right] \] (4.14)

We see that the series is getting strongly divergent only at dimensions \( D < 6 \). This is the only regime where the considered saddle-point solutions provide a relevant contribution.

Now, following the scheme developed in the previous sections, we turn to the integral representation and rescale the integration variable by a factor \( (gh_0^2)^{\frac{4-D}{2}} h_0^{-2} \), which gives a free energy with the following scaling in the limit \( gh_0^2 \ll 1 \):

\[ \frac{F(h_0)}{V} \sim \frac{1}{g} (gh_0^2)^{\frac{4-D}{2}} \] (4.15)

Besides, using the same scaling \( k \sim (gh_0^2)^{\frac{2-D}{2}} h_0^{-2} \) for the characteristic spatial scale of the saddle-point solutions (1.11), which could be interpreted as a kind of disorder induced \textit{finite} correlation length (near \( T = T_c \), as we shall see in more details below), we obtain:
\[ R_c(h_0) \sim (gh_0^2)^{-\frac{1}{\nu}} \] (4.16)

In the same way one gets the estimate for the value of the "disorder parameter" \( \phi^2 \sim (gh_0^2) \):

\[
\phi_0^2 \sim \frac{1}{g} (gh_0^2)^{\frac{2}{\nu}}
\] (4.17)

Finally, one can easily obtain the estimate for the value of the temperature interval \( \tau_c \) around \( T_c \) where all the above qualitative calculations make sense. Formally the derivation of the saddle-point solutions has been done for \( \tau = 0 \). Actually, according to the replica Hamiltonian (4.5) the calculations should remain correct until \( |\tau| \ll h_0^2k \).

Using the above scale estimate for \( k \) one finds the upper bound for the value of \( \tau \):

\[
|\tau| \ll \tau_c \sim (gh_0^2)^{\frac{2}{\nu}}
\] (4.18)

This value of \( \tau_c \) can be interpreted as the estimate for the temperature interval around \( T_c \) where the supposed disorder dominated (spin-glass type) phase sets in.

Of course, the procedure proposed in this section is still incomplete. In a selfconsistent approach one should study the effects produced by the interactions between these instanton solutions, not talking about the effects of the critical fluctuations. At the present stage we are not able to say anything about the ferromagnetic phase transition itself and in particular about the behaviour of the corresponding ferromagnetic order parameter.

Nevertheless, we shall now show that these simple replica instanton estimates are quite reasonable and can in fact be recovered in terms of (completely independent) simple scaling arguments. Indeed, let us come back to the original random field Hamiltonian (4.1). Configurations of the field \( \phi(x) \) which correspond to local minima satisfy the saddle-point equation:

\[ -\Delta \phi(x) + \tau \phi(x) + g\phi^3(x) = h(x) \] (4.19)

Let us estimate at which spatial and temperature scales the random fields give a dominant contribution. We consider a large region \( \Omega_L \) of linear size \( L \gg 1 \). The spatially averaged value of the random field in this region is:

\[
h(\Omega_L) \equiv \frac{1}{L^D} \int_{x \in \Omega_L} d^D x h(x)
\] (4.20)

Correspondingly, the typical average value of the random field in this region of size \( L \) is:

\[
h_L \equiv \left[ h^2(\Omega_L) \right]^{1/2} = h_0 L^{-D/2}
\] (4.21)
Then the estimate for the typical value of the order parameter field $\phi_L$ in this region can be obtained from the saddle-point equation:

$$\tau\phi_L + g\phi_L^3 = h_L$$  \hspace{1cm} (4.22)

Then, as long as:

$$\tau\phi_L \ll g\phi_L^3$$  \hspace{1cm} (4.23)

the typical value of $\phi_L$ inside such clusters is dominated by the random field:

$$\phi_L \sim (h_L/g)^{1/3} \sim \left(\frac{h_0}{g}\right)^{1/3}L^{-D/6}$$  \hspace{1cm} (4.24)

Now let us estimate up to which characteristic size of the cluster the external fields can dominate. According to (4.23) and (4.24) one gets:

$$L \ll \frac{(gh_0^3)^{1/D}}{\tau^{3/D}}$$  \hspace{1cm} (4.25)

On the other hand, the estimation of the order parameter in terms of the equilibrium equation (4.22) can be correct only on length scales much larger than the size of the fluctuation region which is equal to the correlation length (of the pure system) $R_c \sim \tau^{-\nu}$. Thus, one has the lower bound for $L$:

$$L >> \tau^{-\nu}$$  \hspace{1cm} (4.26)

Therefore, the region of parameters where the external fields dominate is:

$$\tau^{-\nu} << \frac{(gh_0^3)^{1/D}}{\tau^{3/D}}$$  \hspace{1cm} (4.27)

or

$$\tau^{3-\nu D} << gh_0^2$$  \hspace{1cm} (4.28)

Such a region of temperatures near $T_c$ exists only if:

$$\nu D < 3$$  \hspace{1cm} (4.29)

In this case the temperature interval near $T_c$ in which the order parameter configurations are mainly defined by the random fields is:

$$\tau_c(h_0) \sim (gh_0^2)^{1/1+\nu D}$$  \hspace{1cm} (4.30)
In the mean field theory (which correctly describes the phase transition in the pure system for \( D > 4 \) \( \nu = 1/2 \). Thus, according to the condition (4.29) the above non-trivial temperature interval \( \tau_c \) exists only at dimensions \( D < 6 \). Substituting \( \nu = 1/2 \) into (4.30) we get:

\[
\tau_c(h_0) \sim (gh_0^2)^{\frac{2}{6-D}}
\]

(4.31)

Then, the random field defined spatial scale can be estimated from (4.25):

\[
L_c(h_0) \sim (gh_0^2)^{-\nu/D}
\]

(4.32)

Correspondingly, the typical value of the order parameter field at scales \( L_c(h_0) \) is obtained from the eq.(4.24):

\[
\phi_{L_c}^2 \sim \frac{1}{g} (gh_0^2)^{\frac{2}{6-D}}
\]

(4.33)

The energy density is estimated as \( \frac{E}{V} \sim \phi_{L_c} h_{L_c} \). Taking into account (4.21) and (4.33) we find:

\[
\frac{E}{V} \sim \frac{1}{g} (gh_0^2)^{\frac{4}{6-D}}
\]

(4.34)

We see that we get through these simple arguments a region around \( T_c \) where the disorder induces a finite correlation length. Furthermore the estimates for \( \frac{E}{V}, L_c, \phi_{L_c} \) and \( \tau_c \) perfectly coincide with the results obtained in terms of our previous replica saddle-point method, eqs.(4.15)-(4.18). Both approaches clearly hold only in a regime where critical fluctuations can be neglected.

5 Conclusions

We have proposed a method to analyse random systems by summing up various saddle point contributions in the replicated Hamiltonian. We think that it may open a new route in this type of study. In particular, the application to finite dimensional systems, which we started here with the directed polymer on one hand, and with the random field Ising model on the other hand, looks quite interesting. Indeed we have seen on this last case how this method allows to take into account instanton contributions which are usually out of reach of most analytic methods in these systems. Such instanton contributions have been argued to be important for a long time ([2, 3]). We think we can get them under control with the present approach.

Clearly our method is still not totally understood in all details. We have pointed out that it involves one single basic rule, stating the way one has to order the various saddle points in replica space. Within this hypothesis it gives reasonable results in all
the cases we have checked so far, but of course more studies are needed to justify this hypothesis.

6 Appendix: Computation of the fourth moment in the toy model

Using the saddle point solution (2.54) we have:

\[
\langle \phi^4 \rangle_{rsb} = \sum_{k+l=1}^{n} \frac{n!}{k!l!(n-k-l)!} [k\phi_1^4 + l\phi_2^4 + (n-k-l)\phi_0^4] \exp\{-\beta E_{kl}\} \rightarrow \tag{6.1}
\]

\[
(\frac{2\mu}{\beta})^4 \sum_{k+l=1}^{n} \frac{(-1)^{k+l-1}(k+l)!}{k!l!} kl(k+l)(3k^2 + 3l^2 - 5kl) \exp\{\lambda kl(k+l)\}
\]

Proceeding similarly to the calculations of the free energy \( F_{rsb} \) (2.58)-(2.59) we get:

\[
\langle \phi^4 \rangle_{rsb} = (\frac{2\mu}{\beta})^4 \frac{\partial}{\partial \lambda} \left\{-\frac{1}{2\pi^2} \int F_C \frac{dz_1 dz_2}{(z_1+z_2)\sin(\pi z_1)\sin(\pi z_2)} \frac{\Gamma(z_1+z_2+1)}{\Gamma(z_1+1)\Gamma(z_2+1)} (3z_1^2 + 3z_2^2 - 5z_1z_2) \times \right.
\]

\[
\left. \exp[\lambda z_1z_2(z_1 + z_2)]\right\} \tag{6.2}
\]

Shifting contour to the position in Fig.1b and redefining \( z_{1,2} \to \lambda^{-1/3}i x_{1,2} \) in the limit \( \beta \to \infty \) \((\lambda^{-1/3} \to 0)\) we find:

\[
\langle \phi^4 \rangle_{rsb} = (\frac{2\mu}{\beta})^4 \frac{\partial}{\partial \lambda} \left\{-\frac{\lambda^{-1/3}}{2\pi^2} \int F_C \frac{dx_1 dx_2}{(x_1+x_2)\sin(\pi x_1)\sin(\pi x_2)} \frac{\Gamma(x_1+x_2+1)}{\Gamma(x_1+1)\Gamma(x_2+1)} (3x_1^2 + 3x_2^2 - 5x_1x_2) \times \right.
\]

\[
\left. \exp[-i\lambda x_1x_2(x_1 + x_2)]\right\} \tag{6.3}
\]

Taking into account the contribution from the pole at \( x_{1,2} = 0 \) after somewhat painful algebra we finally obtain the following result:

\[
\langle \phi^4 \rangle_{rsb} = -\frac{g^{4/3}}{\mu^{8/3}} \frac{17\sqrt{3}[\sin(\pi/12) + \cos(\pi/12)]}{3\sqrt{\pi \Gamma(1/6)\sin(\pi/6)}} \tag{6.4}
\]

Taking into account the contribution from the replica fluctuations \[9\]:

\[
\langle \phi^4 \rangle_{rv} = \frac{g^{4/3}}{\mu^{8/3}} \frac{27}{(4\pi)^{2/3}} \tag{6.5}
\]

for the fourth order correlator we get the final result:

\[
\langle \phi^4 \rangle = \langle \phi^4 \rangle_{rv} + \langle \phi^4 \rangle_{rsb} = \frac{g^{4/3}}{\mu^{8/3}} \frac{27}{(4\pi)^{2/3}} - \frac{g^{4/3}}{\mu^{8/3}} \frac{17\sqrt{3}[\sin(\pi/12) + \cos(\pi/12)]}{3\sqrt{\pi \Gamma(1/6)\sin(\pi/6)}} \tag{6.6}
\]
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Figure captions

Fig. 1 The contours of integration in the complex plane used for summing the series.
a) The original contour. b) The deformed contour.