Hausdorff Continuous Solutions of Arbitrary Continuous Nonlinear PDEs through the Order Completion Method

Elemér E Rosinger

Department of Mathematics
and Applied Mathematics
University of Pretoria
Pretoria
0002 South Africa
eerosinger@hotmail.com

Abstract

In 1994 we showed that very large classes of systems of nonlinear PDEs have solutions which can be assimilated with usual measurable functions on the Euclidean domains of definition of the respective equations. Recently, the regularity of such solutions has significantly been improved by showing that they can in fact be assimilated with Hausdorff continuous functions. The method of solution of PDEs is based on the Dedekind order completion of spaces of smooth functions which are defined on the domains of the given equations. In this way, the method does not use functional analytic approaches, or any of the customary distributions, hyperfunctions, or other generalized functions.

Type independent existence and regularity results for large classes of systems of nonlinear PDEs

Ten years ago, in [4], the following significant threefold breakthrough was obtained with respect to solving large classes of nonlinear PDEs, see MR 95k:35002. Namely:
a) arbitrary nonlinear PDEs of the form

\[
F(x, U(x), \ldots, D^p U(x), \ldots) = f(x), \quad x \in \Omega
\]

with \( F \) jointly continuous in all its arguments, \( f \) in a class of measurable functions, \( \Omega \subseteq \mathbb{R}^n \) arbitrary open, \( p \in \mathbb{N}^n \), with \( |p| \leq m \), for \( m \in \mathbb{N} \) arbitrary given, and the unknown function \( U : \Omega \rightarrow \mathbb{R} \), were proven to have

b) solutions \( U \) which can be assimilated with usual measurable functions on \( \Omega \), and

c) the solution method was based on the Dedekind order completion of suitable spaces of smooth functions on \( \Omega \).

In fact, the conditions at a) can further be relaxed by assuming that \( F \) may admit certain discontinuities, namely, that it is continuous only on \((\Omega \setminus \Sigma) \times \mathbb{R}^{m^*}\), where \( \Sigma \) is a closed, nowhere dense subset of \( \Omega \), while \( m^* \) is the number of arguments in \( F \) minus \( n \). This relaxation on the continuity of \( F \) may be significant since such subsets of discontinuity \( \Sigma \) can have arbitrary large positive Lebesgue measure.

The method of order completion and the results on the existence and regularity of solutions can easily be extended to systems of nonlinear PDEs of the above form (1). Furthermore, initial and/or boundary value problems can be dealt with easily by this order completion method.

In this way, the solutions of the unprecedented large class of nonlinear PDEs in (1) can be obtained without the use of any sort of distribution, hyperfunctions, generalized functions, or of methods of functional analysis. Moreover, one obtains a general, blanket regularity, given by the fact that the solutions constructed can be assimilated with usual measurable functions on the corresponding domains \( \Omega \) in Euclidean spaces.

Recently, in collaboration with R. Anguelov, see [1], a further significant improvement of the above mentioned 1994 results was obtained.
Namely, this time we can further improve the regularity properties of the solutions by proving that they always belong to the significantly smaller class of Hausdorff continuous functions on the open domains $\Omega$, see Appendix for a short account on Hausdorff continuous functions.

It should be noted that the results in [4] on existence of solutions do for the first time in the literature manage fully to overcome the celebrated 1957 Hans Lewy impossibility, see [5], and in fact do so with a large nonlinear margin.

Also, the existence results in [4], and thus their mentioned recent improvement with respect to the regularity of solutions, when solving large classes of nonlinear PDEs, supersede to a good extent the earlier similar ones obtained through the algebraic nonlinear theory of generalized functions introduced by the author in the 1960s, and developed since then alone or in collaborations, see 46F30 in the AMS Subject Classification at www.ams.org/index/msc/46Fxx.html, as well as [6], [7], [3, p. 7] and the literature cited there, or MR 89g:35001, MR 92d:46098, Zbl.Math.717 35001, Bull.AMS, Jan. 1989, Vol. 20, No. 1, 96-101.

To further facilitate the understanding of the above mentioned results, it may be useful to point to the following. In his latest 2004 edition of his Springer Universitext book "Lectures on PDEs", see [2], V I Arnold starts on page 1 with the statement:

"In contrast to ordinary differential equations, there is no unified theory of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry ..." (italics added)

However, as the above mentioned results show, since [4], there is an existence and regularity of solutions theory for the large class of systems of nonlinear PDEs of the form in (1). Moreover, recently, the regularity result has been significantly improved by proving the existence of Hausdorff continuous solutions for such general nonlinear systems of PDEs.
Appendix. Hausdorff continuous functions

We shall deal with functions whose values can be usual or extended real numbers, that is, elements in

\[ \mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\} \]

Moreover, we shall allow the values of the functions to be not only numbers in \( \mathbb{R} \), but also closed intervals of such numbers, namely

\[ [a, b] \subseteq \mathbb{R}, \quad a, b \in \mathbb{R}, \quad a \leq b \]

It turns out to be quite surprising how much more appropriate such a framework is when one deals with large classes of nonsmooth functions in what is usually called Real Analysis. Indeed, by considering such interval valued functions one obtains a systematic and effective way to study and deal with a large variety of nonsmooth functions. Furthermore, one can gain important insights into the properties of such nonsmooth functions, properties which in fact are not available in the usual approach.

It appears therefore that with the emergence in the second half of the 19-th century of a rigorous approach to Analysis, and specifically, with the Dirichlet definition of a function as having values given only and only by one single number, a certain undesired limitation was imposed in an unintended manner, especially what the study of nonsmooth functions is concerned.

Somewhat later, towards the end of the 19-th century, Baire brought in the concepts of lower and upper semi-continuous functions, when dealing with nonsmooth real valued functions. And in effect, he associated with each real valued function \( f \), two other real, or extended real valued functions \( I(f) \) and \( S(f) \), with \( I(f) \leq f \leq S(f) \), which proved to be particularly helpful. However, following the prevailing mentality, each of these three functions were considered as being single valued.
As it turns out, however, by considering *interval valued* functions, such as for instance \( F(f) = [I(f), S(f)] \), one can significantly improve on the understanding and handling of nonsmooth functions.

The study of interval valued functions can, among others, show that the particular case of functions which have values given by one single number is appropriate for continuous functions only. On the other hand, nonsmooth functions are much better described by suitably associated interval valued functions.

Indeed, in the case of functions \( f \) which are *not* continuous, a much better description can be obtained by considering them given by a *pair* of usual point valued functions, namely \( f = [\underline{f}, \overline{f}] \), thus leading to interval valued functions. And then, a natural class which replaces, and also extends, the usual point valued continuous functions is that of *Hausdorff-continuous* interval valued functions. The distinctive and *essential* feature of these Hausdorff-continuous functions \( f = [\underline{f}, \overline{f}] \) is a condition of *minimality* with respect to the *gap* between \( \underline{f} \) and \( \overline{f} \), with the further requirement that \( \underline{f} \) be lower semi-continuous, and \( \overline{f} \) be upper semi-continuous.

In retrospect, it is surprising to see how near Baire came to such a treatment of nonsmooth functions, what deep results he obtained, and its correspondent for lower semi-continuous functions, and yet followed the prevailing trend which considered functions as having to have point, and not interval values.

A good measure of the *naturalness* of interval valued functions can be seen in the results related to the Dedekind order completion of various spaces of continuous functions. And it is precisely such recently obtained results which allow for the mentioned significantly increased regularity properties of solutions of PDEs.

These Dedekind order completions prove to be subspaces of Hausdorff-continuous, thus interval valued functions. By the way, the space of Hausdorff-continuous functions itself is order complete.

Such results extend easily to functions defined on large classes of topological spaces.

A further indication of the natural role interval values play in the study
of nonsmooth functions can be found in the Differential and Integral Calculus being presently developed for functions with such values.

It will be useful to start by introducing a few notations. Let

\[(A.1) \quad \mathbb{IR} = \{ [a, \pi] \mid a, \pi \in \mathbb{R} = \mathbb{R} \cup \{-\infty, +\infty\}, \ a \leq \pi \}\]

be the set of all finite or infinite closed intervals. The functions which we consider can be defined on arbitrary topological spaces \(\Omega\). For the purposes of the nonlinear PDEs studied in this book, however, it will be sufficient to assume that \(\Omega \subseteq \mathbb{R}^n\) are arbitrary open subsets.

Let us now consider the set of interval valued functions

\[(A2) \quad \mathbb{A}(\Omega) = \{ f : \Omega \rightarrow \mathbb{IR} \}\]

By identifying the point \(a \in \mathbb{R}\) with the degenerate interval \([a, a] \in \mathbb{IR}\), we consider \(\mathbb{R}\) as a subset of \(\mathbb{IR}\). In this way \(\mathbb{A}(\Omega)\) will contain the set of functions with extended real values, namely

\[(A3) \quad \mathcal{A}(\Omega) = \{ f : \Omega \rightarrow \mathbb{R} \} \subseteq \mathbb{A}(\Omega)\]

We define a partial order \(\leq\) on \(\mathbb{IR}\) by

\[(A4) \quad [a, \pi] \leq [b, \nu] \iff a \leq b, \ a \leq \nu \leq \pi \leq \nu\]

Now on \(\mathbb{A}(\Omega)\) we define the partial order induced by (A2.1.4) in the usual point-wise way, namely, for \(f, g \in \mathbb{A}(\Omega)\), we have

\[(A5) \quad f \leq g \iff f(x) \leq g(x), \ x \in \Omega\]

Clearly, when restricted to \(\mathcal{A}(\Omega)\), the above partial order on \(\mathbb{A}(\Omega)\) reduces to the usual one among point valued functions.

Given an interval \(a = [a, \pi] \in \mathbb{IR}\), we denote
\[ w(a) = \begin{cases} \overline{a} - a & \text{if } a, \overline{a} \text{ finite} \\ \infty & \text{if } \overline{a} = \infty \text{ and } a \text{ finite}, \text{ or } a = -\infty \text{ and } \overline{a} \text{ finite}, \text{ or } a = -\infty \text{ and } \overline{a} = \infty \\ 0 & \text{if } a = \overline{a} = \pm \infty \end{cases} \]

which is called the width of the interval \( a \). Also, we denote by

\[ |a| = \max\{ |a|, |\overline{a}| \} \]

the modulus of the interval \( a = [a, \overline{a}] \in \mathbb{IR} \).

In this way

\[ \mathcal{A}(\Omega) = \{ f \in \mathcal{A}(\Omega) \mid w(f(x)) = 0, \ x \in \Omega \} \subseteq \mathcal{A}(\Omega) \]

Let \( f \in \mathcal{A}(\Omega) \). For every \( x \in \Omega \), the value of \( f \) is an interval, namely

\[ f(x) = [\underline{f}(x), \overline{f}(x)], \text{ with } \underline{f}(x), \overline{f}(x) \in \mathbb{R}, \underline{f}(x) \leq \overline{f}(x) \]

Hence, every function \( f \in \mathcal{A}(\Omega) \) can be written in the form

\[ f = [\underline{f}, \overline{f}], \text{ with } \underline{f}, \overline{f} \in \mathcal{A}(\Omega), \underline{f} \leq f \leq \overline{f} \]

and

\[ f \in \mathcal{A}(\Omega) \iff \underline{f} = f = \overline{f} \]

In the particular case of functions in \( \mathcal{A}(\Omega) \), that is, with extended real, but point, and not nondegenerate interval values, a number of results in the sequel were obtained by Baire.

Most of the more general results concerning functions in \( \mathcal{A}(\Omega) \), that is, with values finite or infinite closed intervals, have recently been developed by Anguelov.

For \( x \in \Omega \), we denote by \( \mathcal{V}_x \) the set of all neighbourhoods \( V \subseteq \Omega \) of \( x \). Let us consider the pair of mappings \( I, S : \mathcal{A}(\Omega) \rightarrow \mathcal{A}(\Omega) \), called
lower and upper Baire operators, respectively, where for every function $f \in \mathbf{A}(\Omega)$, we define

\[(A11) \quad I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf \{ z \in f(y) \mid y \in V \}\]

\[(A12) \quad S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup \{ z \in f(y) \mid y \in V \}\]

In Baire, these two operators were considered and studied in the particular case of functions $f \in \mathbf{A}(\Omega)$.

In view of the main interest here in this book in interval valued functions $f \in \mathbf{A}(\Omega)$, it is useful to consider as well the following third mapping, namely, $F : \mathbf{A}(\Omega) \to \mathbf{A}(\Omega)$, defined by

\[(A13) \quad F(f)(x) = [ I(f)(x), S(f)(x) ], \quad f \in \mathbf{A}(\Omega), \ x \in \Omega, \]

and called the graph completion operator.

The lower and upper Baire operators $I$ and $S$, and consequently, the graph completion operator $F$, applied to any interval valued function $f = [ \underline{f}, \overline{f} ] \in \mathbf{A}(\Omega)$ can now be conveniently represented in terms of the functions $\underline{f}$ and $\overline{f}$. Indeed, from (A2.1.11), (A2.1.12) it is easy to see that

\[(A14) \quad I(f) = I(\underline{f}), \quad S(f) = S(\overline{f})\]

Hence $F(f)$ can be written in the form

\[(A15) \quad F(f) = [ I(f), S(f) ] = [ I(\underline{f}), S(\overline{f}) ]\]

Let us note that for every function $f \in \mathbf{A}(\Omega)$ we have the relations, see (A5), (A9)

\[(A16) \quad I(f) = I(\underline{f}) \leq \underline{f} \leq f \leq \overline{f} \leq S(\overline{f}) = S(f)\]

and, thus, the inclusions

\[(A17) \quad f(x) \subseteq F(f)(x), \ x \in \Omega\]
Furthermore, the lower Baire operator \( I : f \rightarrow I(f) \), the upper Baire operator \( S : f \rightarrow S(f) \) and the graph completion operator \( F : f \rightarrow F(f) = [ I(f), S(f) ] \) are all monotone with respect to the order \( \leq \) in (A5) on \( \mathbf{A}(\Omega) \), which means that for every two functions \( f, g \in \mathbf{A}(\Omega) \) we have

\[
(A18) \quad f \leq g \implies I(f) \leq I(g), \quad S(f) \leq S(g), \quad F(f) \leq F(g)
\]

The operator \( F \) is also monotone with respect to inclusion, namely

\[
(A19) \quad f(x) \subseteq g(x), \quad x \in \Omega \implies F(f)(x) \subseteq F(g)(x), \quad x \in \Omega
\]

With an immediate extension of Baire, one can also show that all three operators are idempotent, that is, for every \( f \in \mathbf{A}(\Omega) \), we have

\[
(A20) \quad I(I(f)) = I(f), \quad S(S(f)) = S(f), \quad F(F(f)) = F(f)
\]

**Definition A1**

A function \( f \in \mathbf{A}(\Omega) \) is called *segment-continuous*, or in short, *s-continuous*, if and only if

\[
(A21) \quad F(f) = f
\]

In view of (A17), it is obvious that condition (A21) is equivalent with

\[
(A21^*) \quad F(f)(x) \subseteq f(x), \quad x \in \Omega
\]

Furthermore, (A20), (A21) give

\[
(A22) \quad F(f) \text{ is s-continuous for } f \in \mathbf{A}(\Omega)
\]

**Example A1**
Let us illustrate the concept of s-continuity in the simplest case of functions with one variable and one single discontinuity. Thus, with \( \Omega = \mathbb{R} \), we take \( f : \Omega \rightarrow \mathbb{R} \), or in other words, \( f \in \mathcal{A}(\Omega) \), defined by

\[
    f(x) = \begin{cases} 
        a & \text{if } x < 0 \\
        b & \text{if } x = 0 \\
        c & \text{if } x > 0 
    \end{cases}
\]

where \( a, b, c \in \mathbb{R}, a \neq c \). Then \( f \) is not s-continuous.

Let us now take \( f : \Omega \rightarrow \mathbb{R} \), that is, \( f \in \mathcal{A}(\Omega) \), defined by

\[
    f(x) = \begin{cases} 
        a & \text{if } x < 0 \\
        [b, c] & \text{if } x = 0 \\
        d & \text{if } x > 0 
    \end{cases}
\]

where \( a, b, c, d \in \mathbb{R}, a \leq d \) and \( b \leq c \). Then \( f \) is s-continuous, if and only if \( b \leq a \) and \( d \leq c \). Similarly, if \( a \geq d \) and \( b \leq c \), then \( f \) is s-continuous, if and only if \( b \leq d \) and \( a \leq c \).

Consequently, returning to the first example above, it follows that \( f \) is s-continuous, if and only if \( a = b = c \), that is, if and only if \( f \) is continuous, see (A2.1.32) below, for the general case of functions \( f \in \mathcal{A}(\Omega) \).

\[\square\]

The fundamental concept is presented now in

**Definition A2**

A function \( f \in \mathcal{A}(\Omega) \) is called **Hausdorff-continuous**, or in short, **H-continuous**, if and only if \( f \) is s-continuous, and in addition, for every s-continuous function \( g \in \mathcal{A}(\Omega) \), we have satisfied the *minimality con-
dition on \( f \):

\[(A23) \quad g(x) \subseteq f(x), \ x \in \Omega \quad \implies \quad g = f\]

We shall denote by \( H(\Omega) \) the set of all Hausdorff-continuous interval valued functions on \( \Omega \).

**Example A2**

Let us again consider the second function in Example A1 above. Then \( f \) is H-continuous, if and only if \( a = b \) and \( c = d \)

Let us give three further examples.

First, let us define \( \alpha : \mathbb{R} \rightarrow \overline{\mathbb{R}} \) by

\[
\alpha(x) = \begin{cases} 
-1 & \text{if } x < 0 \\
[ -1, 1 ] & \text{if } x = 0 \\
1 & \text{if } x > 0
\end{cases}
\]

and then we can define \( \beta : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}} \) by

\[
\beta(x, y) = \begin{cases} 
\alpha(\sin(1/(x^2 + y^2))) & \text{if } (x, y) \neq (0, 0) \\
[ -1, 1 ] & \text{if } (x, y) = (0, 0)
\end{cases}
\]

It is easy to see that both \( \alpha \) and \( \beta \) are H-continuous.

The third example is a typical shock wave solution of the well known nonlinear PDE in Fluid Dynamics

\[U_t + UU_x = 0, \quad t \geq 0, \ x \in \mathbb{R}\]

which corresponds to the initial value problem
\( U(0, x) = \begin{cases} 
1 & \text{if } x \leq -1 \\
-x & \text{if } -1 \leq x \leq 0 \\
0 & \text{if } x \geq 0
\end{cases} \)

Namely, with \( \Omega = [0, \infty) \times \mathbb{R} \), we have the solution \( U : \Omega \rightarrow \mathbb{R} \) given by

\[
U(t, x) = \begin{cases} 
1 & \text{if } 0 \leq t < 1, \ x < t - 1 \\
x/(t - 1) & \text{if } 0 \leq t < 1, \ t - 1 \leq x \leq 0 \\
0 & \text{if } 0 \leq t < 1, \ x > 0 \\
1 & \text{if } t \geq 1, \ x < (t - 1)/2 \\
[ -1, 1 ] & \text{if } t \geq 1, \ x = (t - 1)/2 \\
0 & \text{if } t \geq 1, \ x > (t - 1)/2
\end{cases}
\]

Then \( U \) is H-continuous.

**Remark A1**

The *minimality* condition (A23) in the above definition of H-continuous functions proves to play a fundamental role.

\[\boxdot\]

As for the significance of the *regularity* property of being Hausdorff continuous, here we an important *similarity* between usual continuous, and on the other hand, Hausdorff-continuous functions, on the other. Namely, both of them are determined *uniquely* if they are known on a *dense* subset of their domains of definition.

This property comes in spite of the fact that Hausdorff-continuous functions can have discontinuities on sets of first Baire category, and such sets can have arbitrary large positive Lebesgue measure.
Indeed, we have

**Theorem A1**

Let \( f = [f, \overline{f}], \ g = [g, \overline{g}] \in A(\Omega) \) be two \( H \)-continuous functions, and suppose given any dense subset \( D \subseteq \Omega \). Then with the partial order in (A5), we have

a) \( f(x) \leq g(x), \ x \in D \implies f \leq g \) on \( \Omega \)

b) \( f(x) \leq g(x), \ x \in D \implies f \leq g \) on \( \Omega \)

c) \( f(x) \leq g(x), \ x \in D \implies f \leq g \) on \( \Omega \)

Also

d) \( f(x) = g(x), \ x \in D \implies f = g \) on \( \Omega \)

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