Creating quantum discord through local generalized amplitude damping

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Abstract. We show that two qubits initially in completely classical state can create quantum discord through a local generalized amplitude damping channel, but high temperature will impede the creating of quantum discord.

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1. Introduction

Quantum entanglement is one of the most striking features of quantum physics. It remarkably reveals nonlocality and may lead to powerful applications in quantum information and quantum computation \[1,2\]. But entanglement is incredibly fragile under environmental noises, and any local operations can not increase it. These are serious frustrations for using entanglement.

Quantum discord \[3,4\] is another quantum correlation other than entanglement, and also can be used for quantum computation \[5,6,7\]. It has been shown that quantum discord is more robust than entanglement \[8\]. Furthermore, recent studies \[9,10,11\] indicate that local environmental noises can create quantum discord. These are exciting results compared to the case of entanglement.

In \[10\], F. Ciccarello and V. Giovannetti showed that a zero discord state of two qubits

\[
\tau_{\text{ini}} = \frac{1}{2}(|+\rangle \langle +| \otimes |0\rangle \langle 0| + |-\rangle \langle -| \otimes |1\rangle \langle 1|) \tag{1}
\]

can create quantum discord under an amplitude damping channel, where

\[
|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \tag{2}
\]

In this paper we will investigate the more general case that the initial state is

\[
\rho_{\text{ini}} = \frac{1}{2}(|\psi_0(\lambda)\rangle \langle \psi_0(\lambda)| \otimes |0\rangle \langle 0| + |\psi_1(\lambda)\rangle \langle \psi_1(\lambda)| \otimes |1\rangle \langle 1|) \tag{3}
\]

where

\[
|\psi_0(\lambda)\rangle = \sqrt{\lambda}|0\rangle + \sqrt{1-\lambda}|1\rangle, \quad |\psi_1(\lambda)\rangle = \sqrt{1-\lambda}|0\rangle - \sqrt{\lambda}|1\rangle, \quad \lambda \in [0,1]. \tag{4}
\]
and the noise is the generalized amplitude damping channel. We know generalized amplitude damping describes temperature effects, so we are especially interested in the temperature effect on the quantum discord. This paper is organized as follows. In Sec. 2, we briefly recall some basics about quantum discord and generalized amplitude damping as preparations. In Sec. 3, we investigate the quantum discord of two qubits whose initial state as in Eq.(3) and undergoes a local generalized amplitude damping. Sec. 4 is a brief summary.

2. Quantum discord and generalized amplitude damping

Suppose two quantum systems A and B are described by the Hilbert spaces $H^A$ and $H^B$, respectively. The composite system AB is then described by the Hilbert space $H^A \otimes H^B$. For a state $\rho$ on $H^A \otimes H^B$, the quantum discord (with respect to A) of $\rho$ is defined as

$$D_A(\rho) = S(\rho^A) - S(\rho) + \inf_{\{\Pi_i\}} \sum_i p_i S(\rho_i^B),$$

where $\rho^A = \text{tr}_B \rho$, $\Pi_i = |i\rangle\langle i|$, $\inf$ takes all projective measurements $\{\Pi_i\}_i$ on system A, $p_i = \text{tr}_B \langle i|\rho|i\rangle$, $\rho_i^B = \langle i|\rho|i\rangle/p_i$, $S(\cdot)$ is the Von Neumann entropy. It can be proven that

$$D_A(\rho) = 0 \iff \rho = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^B,$$

where $\{|i\rangle\}_i$ is an orthonormal basis for $H^A$, $\{|\rho_i^B\}_i$ are density operators on $H^B$, $p_i \geq 0$, $\sum_i p_i = 1$.

Eq.(5) is hard to optimize, even for two qubits case till now only few states are found possessing analytical expressions [12, 13]. B. Dakic, V. Vedral and C. Brukner proposed a geometric measure of quantum discord as

$$D_G^A(\rho) = \inf_{\sigma} \{\text{tr}[(\rho - \sigma)^2] : D_A(\sigma) = 0\}.$$ (7)

Evidently,

$$D_A(\rho) = 0 \iff D_G^A(\rho) = 0.$$ (8)

As an elegant result, $D_G^A(\rho)$ allows analytical expressions for all two qubits states [14]. More specifically, for a two qubits state

$$\rho = \frac{1}{4}(I \otimes I + \sum_{i=1}^3 x_i \sigma_i \otimes I + \sum_{j=1}^3 y_j I \otimes \sigma_j + \sum_{i,j=1}^3 T_{ij} \sigma_i \otimes \sigma_j),$$ (9)

where, $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$ are Pauli matrices, $\{x_i\}_{i=1}^3$, $\{y_j\}_{j=1}^3$, $\{T_{ij}\}_{i,j=1}^3$ are all real numbers with

$$x_i = \text{tr}[\rho \sigma_i \otimes I], y_j = \text{tr}[\rho I \otimes \sigma_j], T_{ij} = \text{tr}[\rho \sigma_i \otimes \sigma_j],$$ (10)

then [14]

$$D_G^A(\rho) = \frac{1}{4}(\sum_{i=1}^3 x_i^2 + \sum_{i,j=1}^3 T_{ij}^2 - \lambda_{\text{max}}).$$ (11)
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Where $\lambda_{\text{max}}$ is the largest eigenvalue of the matrix $xx^t + TT^t$, $x = (x_1, x_2, x_3)^t$, $T$ is the matrix $(T_{ij})$.

Generalized amplitude damping describes the effect of dissipation to an environment at finite temperature ([1], 8.3.5). Suppose two qubits systems A and B, given the bases $\{|0\rangle, |1\rangle\}$, $\{|0\rangle, |1\rangle\}$ for $H^A$ and $H^B$, the operation elements for generalized amplitude damping are

$$E_0 = \sqrt{p} \begin{pmatrix} 1 & 0 & \sqrt{1-\gamma} \\ 0 & \sqrt{1-\gamma} & 0 \end{pmatrix}, \quad E_1 = \sqrt{p} \begin{pmatrix} 0 & \sqrt{\gamma} & 0 \\ \sqrt{\gamma} & 0 & 0 \end{pmatrix},$$
$$E_2 = \sqrt{1-p} \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_3 = \sqrt{1-p} \begin{pmatrix} 0 & \sqrt{\gamma} & 0 \\ \sqrt{\gamma} & 0 & 0 \end{pmatrix}. \quad (12)$$

Where $p \in [0,1]$, $\gamma \in [0,1]$. $p = p(T)$ and $\gamma = \gamma(T)$ are functions of temperature $T$. $\gamma(T)$ describes the transition probability between $|0\rangle$ and $|1\rangle$ at temperature $T$. $p(T)$ describes temperature effects, $p(0) = 1$ is the case of amplitude damping. We can assume that (see for example [15]) $p(T)$ is a monotonically decreasing function when $T$ varies from 0 to $\infty$, and

$$\lim_{T \to 0} p(T) = 1, \lim_{T \to \infty} p(T) = 0.5. \quad (13)$$

From Eq.(12) we can see that $p$ varies from 0 to 0.5 is symmetric to that $p$ varies from 1 to 0.5 in the sense exchanging states $|0\rangle$ and $|1\rangle$, so we only need to consider the case $p$ varies from 1 to 0.5.

A state $\rho$ after this generalized amplitude damping will be a state

$$E(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger + E_2\rho E_2^\dagger + E_3\rho E_3^\dagger. \quad (14)$$

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Now suppose two qubits initially is in the state as in Eq.(3), through the generalized amplitude damping channel in Eq.(12), the output state $\rho = E(\rho_{\text{ini}})$ can be calculated directly by Eqs.(3,12,14), will be

$$\rho = \frac{1}{4}(I \otimes I + x_3\sigma_3 \otimes I + T_{13}\sigma_1 \otimes \sigma_3 + T_{33}\sigma_3 \otimes \sigma_3), \quad (15)$$

where

$$x_3 = (2p - 1)\gamma, \quad T_{13} = 2\sqrt{\lambda(1-\lambda)(1-\gamma)}, \quad T_{33} = (1-\gamma)(2\lambda - 1). \quad (16)$$

We first ask, for what initial states and what generalized amplitude damping channel, the quantum discord of the output states, $D_A(\rho)$, are vanishing? This is the Proposition 1 below.

**Proposition 1.** The quantum discord $D_A(\rho)$ of the state $\rho$ in Eq.(15) is zero if one of the conditions below is satisfied

(i). $\gamma = 0$;

(ii). $\gamma = 1$;

(iii). $\lambda = 0$;

(iv). $\lambda = 1$;
(v). \( p = 0.5 \).

**Proof.** From Eq.(8), we only need to prove \( D_A^G(\rho) = 0 \). Using Eq.(11) to Eq.(15), after some calculations, we get

\[
D_A^G(\rho) = \frac{1}{4} \{ (x_3^2 + T_{13}^2 + T_{33}^2) - \sqrt{[(x_3 + T_{13})^2 + T_{33}^2][(x_3 - T_{13})^2 + T_{33}^2]} \}. \tag{17}
\]

From Eq.(16), it is easy to see that

(vi). if \( \gamma = 0 \), then \( x_3 = 0 \);

(vii). if \( \gamma = 1 \), then \( T_{13} = T_{33} = 0 \);

(viii). if \( \lambda = 0 \), then \( T_{13} = 0 \);

(ix). if \( \lambda = 1 \), then \( T_{13} = 0 \);

(x). if \( p = 0.5 \), then \( x_3 = 0 \).

Each of the cases above can result in \( D_A^G(\rho) = 0 \). We then complete this proof.

We next investigate the quantum discord \( D_A(\rho) \) for the state \( \rho \) in Eq.(15) for more general cases, that is \( \gamma \in (0, 1), \lambda \in (0, 1), p \in (0.5, 1) \).

Any orthonormal basis of \( H^A \) can be expressed as an unitary matrix \( U \) multiplied by the orthonormal basis \( \{ |0\>, |1\> \} \). Since \( U \) can be written as

\[
U = \begin{pmatrix} t + ic & -b + ia \\ b + ia & t - ic \end{pmatrix},
\]

where \( t, a, b, c \) are all real numbers and satisfy

\[
t^2 + a^2 + b^2 + c^2 = 1,
\]

then any orthonormal basis of \( H^A \) can be expressed as

\[
|\varphi_0\rangle = (t + ic)|0\rangle + (b + ia)|1\rangle,
\]

\[
|\varphi_1\rangle = (-b + ia)|0\rangle + (t - ic)|1\rangle.
\]

Notice that

\[
I|0\rangle = |0\rangle, \quad \sigma_x|0\rangle = |1\rangle, \quad \sigma_y|0\rangle = i|1\rangle, \quad \sigma_z|0\rangle = |0\rangle,
\]

\[
I|1\rangle = |1\rangle, \quad \sigma_x|1\rangle = |0\rangle, \quad \sigma_y|1\rangle = -i|0\rangle, \quad \sigma_z|1\rangle = -|1\rangle.
\]

Denote \( p_0\rho_0 = \langle \varphi_0 | \rho | \varphi_0 \rangle \) and \( p_1\rho_1 = \langle \varphi_1 | \rho | \varphi_1 \rangle \), then from Eqs.(15,20,21), we have

\[
p_0\rho_0 = \frac{I}{4} \{1 + (t^2 + c^2 - a^2 - b^2)x_3\} + \frac{\sigma_z}{4} [2(tb + ac)T_{13} + (t^2 + c^2 - a^2 - b^2)T_{33}]
\]

\[
p_0 = tr \langle \varphi_0 | \rho | \varphi_0 \rangle = \frac{1}{2} (1 + x_3) - (a^2 + b^2)x_3,
\]

\[
p_1\rho_1 = \frac{I}{4} \{1 + (a^2 + b^2 - t^2 - c^2)x_3\} + \frac{\sigma_z}{4} [-2(tb + ac)T_{13} + (a^2 + b^2 - t^2 - c^2)T_{33}]
\]

\[
p_1 = tr \langle \varphi_1 | \rho | \varphi_1 \rangle = \frac{1}{2} (1 - x_3) + (a^2 + b^2)x_3.
\]

Let

\[
a = \sqrt{r} \cos \theta_1, \quad b = \sqrt{r} \sin \theta_1, \quad c = \sqrt{1 - \sqrt{r} \cos \theta_2}, \quad t = \sqrt{1 - r} \cos \theta_2,
\]

then

\[
a^2 + b^2 = r, \quad tb + ac = \sqrt{r(1 - r)} \sin \theta_3,
\]
where
\[ r \in [0, 1], \quad \theta_1 \in [0, 2\pi), \quad \theta_2 \in [0, 2\pi), \quad \theta_3 = \theta_1 + \theta_2. \] (28)

Eqs.(22-25) then read
\[
p_0 \rho_0 = \frac{I}{4} [1 + (1 - 2r)x_3] + \frac{\sigma_z}{4} [2\sqrt{r(1 - r)} \sin \theta_3 T_{13} + (1 - 2r)T_{33}],
\] (29)
\[
p_0 = \frac{1}{2} [1 + (1 - 2r)x_3],
\] (30)
\[
p_1 \rho_1 = \frac{I}{4} [1 - (1 - 2r)x_3] + \frac{\sigma_z}{4} [-2\sqrt{r(1 - r)} \sin \theta_3 T_{13} - (1 - 2r)T_{33}],
\] (31)
\[
p_1 = \frac{1}{2} [1 - (1 - 2r)x_3].
\] (32)

Let
\[
\alpha = 1 - 2r, \quad \beta = 2\sqrt{r(1 - r)} \sin \theta_3,
\] (33)
then Eqs.(29-32) become
\[
p_0 \rho_0 = \frac{I}{4} [1 + \alpha x_3] + \frac{\sigma_z}{4} [\beta T_{13} + \alpha T_{33}],
\] (34)
\[
p_0 = \frac{1}{2} [1 + \alpha x_3],
\] (35)
\[
p_1 \rho_1 = \frac{I}{4} [1 - \alpha x_3] + \frac{\sigma_z}{4} [-\beta T_{13} - \alpha T_{33}],
\] (36)
\[
p_1 = \frac{1}{2} [1 - \alpha x_3],
\] (37)

with
\[
\alpha^2 + \beta^2 \leq 1.
\] (38)

From Eqs.(34-37), it is easy to find that \( \rho_0 \) has two eigenvalues
\[
\frac{1 + \alpha(x_3 + T_{33}) + \beta T_{13}}{2(1 + \alpha x_3)}, \quad 1 - \frac{1 + \alpha(x_3 + T_{33}) + \beta T_{13}}{2(1 + \alpha x_3)},
\] (39)
and \( \rho_1 \) has two eigenvalues
\[
\frac{1 - \alpha(x_3 + T_{33}) - \beta T_{13}}{2(1 - \alpha x_3)}, \quad 1 - \frac{1 - \alpha(x_3 + T_{33}) - \beta T_{13}}{2(1 - \alpha x_3)}.
\] (40)

Thus
\[
p_0 S(\rho_0) + p_1 S(\rho_1) = F(\alpha, \beta)
= \frac{1}{2} (1 + \alpha x_3) h\left(\frac{1 + \alpha(x_3 + T_{33}) + \beta T_{13}}{2(1 + \alpha x_3)}\right) + \frac{1}{2} (1 - \alpha x_3) h\left(\frac{1 - \alpha(x_3 + T_{33}) - \beta T_{13}}{2(1 - \alpha x_3)}\right),
\] (41)
where \( h(\cdot) \) is the binary entropy
\[
h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x),
\] (42)
h(\( x \)) is defined on \( x \in [0, 1] \), \( h(x) \) is a concave function, and \( h(x) = h(1 - x) \).

We say \( F(\alpha, \beta) \) is a concave function in \( (\alpha, \beta) \), this can be seen from the concavity of \( h(x) \) and the facts below:
(xi). If \( f(x) \) and \( g(x) \) are convex functions, then so is \( f(x) + g(x) \) (\cite{16}, 3.2.1); (xii). Convexity is invariant under affine maps: that is, if \( f(x) \) is convex with \( x \in R^m \), then so is \( g(y) = f(Ay + x_0) \) with \( y \in R^m, x_0 \in R^n \), \( A \) an \( n \times m \) real matrix (\cite{16}, 3.2.2); (xiii). If \( f(x) \) is convex, then so is \( g(x, t) = tf(x/t) \) with \( t > 0 \) (\cite{16}, 3.2.6).

It follows that the minimum of \( F(\alpha, \beta) \) in Eq.(41) over the domain \( \alpha^2 + \beta^2 \leq 1 \) can be achieved on the unit circle \( \alpha^2 + \beta^2 = 1 \). Further, from Eq.(41), it is evident that

\[
F(\alpha, \beta) = F(-\alpha, -\beta),
\]

hence,

\[
\min[p_0S(\rho_0) + p_1S(\rho_1) : \text{all } \{\Pi_i\}_{i=1}^2] = \min\{F(\theta) : 0 \leq \theta \leq \pi\},
\]

where

\[
F(\theta) = \frac{1}{2}(1 + x_3 \cos \theta)h\left(\frac{1 + (x_3 + T_{33}) \cos \theta + T_{13} \sin \theta}{2(1 + x_3 \cos \theta)}\right) + \frac{1}{2}(1 - x_3 \cos \theta)h\left(\frac{1 - (x_3 + T_{33}) \cos \theta - T_{13} \sin \theta}{2(1 - x_3 \cos \theta)}\right).
\]

Finally, we get the expression of \( D(\rho) \) as

\[
D(\rho) = S(\rho^A) - S(\rho) + \min\{F(\theta) : 0 \leq \theta \leq \pi\},
\]

where \( S(\rho^A) \) and \( S(\rho) \) can be calculated directly by Eq.(15) as

\[
S(\rho^A) = h\left(\frac{1 + x_3}{2}\right),
\]

\[
S(\rho) = 1 - \frac{1}{2}h\left(\frac{1 + \sqrt{(x_3 + T_{33})^2 + T_{13}^2}}{2}\right) - \frac{1}{2}h\left(\frac{1 + \sqrt{(x_3 - T_{33})^2 + T_{13}^2}}{2}\right).
\]

From these expressions, we have Proposition 2 below.

**Proposition 2.** Quantum discord \( D_A(\rho) \) in Eq.(46) for the state \( \rho \) in Eq.(15) has the same value for \((p, \lambda, \gamma)\) and \((p, 1 - \lambda, \gamma)\), that is, \( D_A(\rho) \) is symmetric about \( \lambda = 0.5 \).

**Proof.** From Eq.(16), if \((p, \lambda, \gamma)\) generates \((x_3, T_{13}, T_{33})\), then \((p, 1 - \lambda, \gamma)\) generates \((x_3, T_{13}, -T_{33})\). \((x_3, T_{13}, T_{33})\) and \((x_3, T_{13}, -T_{33})\) generate the same value for \( S(\rho^A) - S(\rho) \), see Eqs.(47-48). Also, in \( F(\theta) \) of Eq.(45), by use of \( h(x) = h(1 - x) \), we can get

\[
F(\theta, x_3, T_{13}, T_{33}) = F(\pi - \theta, x_3, T_{13}, -T_{33}).
\]

Thus, we can readily attain Proposition 2 and end this proof.
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Figure 1. Discord versus $\lambda$ and $\gamma$ when $p = 1$, $p = 0.8$, $p = 0.6$, $p = 0.55$.

Fig.1 depicts $D_A(\rho)$ as a function of $(\lambda, \gamma)$ at $p = 1$, $p = 0.8$, $p = 0.6$, $p = 0.55$, respectively. From Fig.1 we see that, when $\lambda$ is close to $\lambda = 0.5$ and $p$ close to 1, $D_A(\rho)$ is relatively large. That is to say, more superposition of the states $|0\rangle$ and $|1\rangle$ in Eq.(3) can boost the creating of quantum discord, while high temperature, conversely, can impede the creating. Fig.2 shows $D_A(\rho)$ as a function of $(p, \gamma)$ at $\lambda = 0.5$.

Figure 2. Discord versus $p$ and $\gamma$ when $\lambda = 0.5$.

4. Summary

We investigated the quantum discord of two qubits which initially in completely classical state then experienced a local generalized amplitude damping channel. We showed that a completely classical state can create quantum discord through a local generalized amplitude damping, while high temperature will impede the creating of quantum discord.

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