The de Rham-Witt and $\mathbb{Z}_p$-cohomologies of an algebraic variety

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To Mike Artin on the occasion of his 70th birthday.

Abstract

We prove that, for a smooth complete variety $X$ over a perfect field,

$$H^i(X, \mathbb{Z}_p(r)) \cong \text{Hom}_{\mathbf{D}(\mathbb{R})} (1, R\Gamma(W\Omega^*_X(r))[i])$$

where $H^i(X, \mathbb{Z}_p(r)) = \lim_{\longleftarrow} H^{i-r}(X_{et}, \nu_n(r))$ (Milne 1986, p309), $W\Omega^*_X$ is the de Rham-Witt complex on $X$ (Illusie 1979b), and $\mathbf{D}(\mathbb{R})$ is the triangulated category of coherent complexes over the Raynaud ring (Illusie and Raynaud 1983, 1.3.10.1, p120).

Introduction

According to the standard philosophy (cf. Deligne 1994, 3.1), a cohomology theory $X \mapsto H^i(X, r)$ on the algebraic varieties over a fixed field $k$ should arise from a functor $R\Gamma$ taking values in a triangulated category $\mathbf{D}$ equipped with a $t$-structure and a Tate twist $\mathbf{D} \mapsto \mathbf{D}(r)$ (a self-equivalence). The heart $\mathbf{D}^\bigcirc$ of $\mathbf{D}$ should be stable under the Tate twist and have a tensor structure; in particular, there should be an essentially unique identity object $1$ in $\mathbf{D}^\bigcirc$ such that $1 \otimes D \cong D \cong D \otimes 1$ for all objects in $\mathbf{D}^\bigcirc$. The cohomology theory should satisfy

$$H^i(X, r) \cong \text{Hom}_{\mathbf{D}} (1, R\Gamma(X)(r)[i]).$$

(1)

For example, motivic cohomology $H^i_{\text{mot}}(X, \mathbb{Q}(r))$ should arise in this way from a functor to a category $\mathbf{D}$ whose heart is the category of mixed motives $k$. Absolute $\ell$-adic étale cohomology $H^i_{et}(X, \mathbb{Z}_\ell(r))$, $\ell \neq \text{char}(k)$, arises in this way from a functor to a category $\mathbf{D}$ whose heart is the category of continuous representations of $\text{Gal}(\overline{k}/k)$ on finitely generated $\mathbb{Z}_\ell$-modules (Ekedahl 1990). When $k$ is algebraically closed, $H^i_{et}(X, \mathbb{Z}_\ell(r))$ becomes the familiar group $\lim_{\longleftarrow} H^i_{et}(X, \mu_{\ell^n}^*)$ and lies in $\mathbf{D}^\bigcirc$; moreover, in this case, (1) simplifies to

$$H^i(X, r) \cong H^i(R\Gamma(X)(r)).$$

(2)

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Now let $k$ be a perfect field of characteristic $p \neq 0$, and let $W$ be the ring of Witt vectors over $k$. For a smooth complete variety $X$ over $k$, let $W\Omega_X^\bullet$ denote the de Rham-Witt complex of Bloch-Deligne-Illusie (see Illusie 1979b). Regard $\Gamma = \Gamma(X, -)$ as a functor from sheaves of $W$-modules on $X$ to $W$-modules. Then

$$H^i_{\text{crys}}(X/W) \cong H^i(\Gamma(W\Omega_X^\bullet))$$

(Illusie 1979a, 3.4.3), where $H^i_{\text{crys}}(X/W)$ is the crystalline cohomology of $X$ (Berthelot 1974). In other words, $X \mapsto H^i_{\text{crys}}(X/W)$ arises as in (2) from the functor $X \mapsto \Gamma(W\Omega_X^\bullet)$ with values in $D^+(W)$.

Let $R$ be the Raynaud ring, let $D(X, R)$ be the derived category of the category of sheaves of graded $R$-modules on $X$, and let $D(R)$ be the derived category of the category of graded $R$-modules (Illusie 1983, 2.1). Then $\Gamma$ derives to a functor

$$\Gamma : D(X, R) \to D(R).$$

When we regard $W\Omega_X^\bullet$ as a sheaf of graded $R$-modules on $X$, $\Gamma(W\Omega_X^\bullet)$ lies in the full subcategory $D^b_c(R)$ of $D(R)$ consisting of coherent complexes (Illusie and Raynaud 1983, II 2.2), which Ekedahl has shown to be a triangulated subcategory with $t$-structure (Illusie 1983, 2.4.8). In this note, we define a Tate twist $(r)$ on $D^b_c(R)$ and prove that

$$H^i(X, \mathbb{Z}_p(r)) \cong \text{Hom}_{D^b_c(R)}(1, \Gamma(W\Omega_X^\bullet)(r)[i]).$$

Here $H^i(X, \mathbb{Z}_p(r)) = \lim_{\leftarrow n} H^{i-r}_\text{et}(X, \mathbb{Z}_p(n))$ with $\mathbb{Z}_p(n)$ the additive subsheaf of $W_n\Omega_X^\bullet$ locally generated for the étale topology by the logarithmic differentials (Milne 1986, §1), and $1$ is the identity object for the tensor structure on graded $R$-modules defined by Ekedahl (Illusie 1983, 2.6.1). In other words, $X \mapsto H^i(X, \mathbb{Z}_p(r))$ arises as in (1) from the functor $X \mapsto \Gamma(W\Omega_X^\bullet)$ with values in $D^b_c(R)$.

This result is used in the construction of the triangulated category of integral motives in Milne and Ramachandran 2005.

It is a pleasure for us to be able to contribute to this volume: the $\mathbb{Z}_p$-cohomology was introduced (in primitive form) by the first author in an article whose main purpose was to prove a conjecture of Artin, and, for the second author, Artin’s famous 18.701-2 course was his first introduction to real mathematics.

The Tate twist

According to the standard philosophy, the Tate twist on motives should be $N \mapsto N(r) = N \otimes \mathbb{T}^\otimes r$ with $\mathbb{T}$ dual to $\mathbb{L}$ and $\mathbb{L}$ defined by $Rh(\mathbb{P}^1) = 1 \oplus \mathbb{L}[-2]$.

The Raynaud ring is the graded $W$-algebra $R = R^0 \oplus R^1$ generated by $F$ and $V$ in degree 0 and $d$ in degree 1, subject to the relations $FV = p = VF, F\sigma = \sigma F, aV = V \cdot \sigma a, ad = da (a \in W), d^2 = 0$, and $FdV = d; \text{ in particular, } R^0$ is the Dieudonné ring $W_a[F, V]$ (Illusie 1983, 2.1). A graded $R$-module is nothing more than a complex

$$M^\bullet = (\cdots \to M^i \xrightarrow{d} M^{i+1} \to \cdots)$$
of $W$-modules whose components $M^i$ are modules over $R^0$ and whose differentials $d$ satisfy $FdV = d$. We define $T$ to be the functor of graded $R$-modules such that $(TM)^i = M^{i+1}$ and $T(d) = -d$. It is exact and defines a self-equivalence $T : D^b_c(R) \to D^b_c(R)$.

The identity object for Ekedahl’s tensor structure on the graded $R$-modules is the graded $R$-module $\mathbb{1} = (W, F = \sigma, V = p\sigma^{-1})$ concentrated in degree zero (Illusie 1983, 2.6.1.3). It is equal to the module $E_{0/1} = df R^0/(F - 1)$ of Ekedahl 1985, p. 66.

There is a canonical homomorphism $\mathbb{1} \oplus T^{-1}(\mathbb{1})[-1] \to R\Gamma(W\Omega^*_X)$ (in $D^b_c(R)$), which is an isomorphism because it is on $W_1\Omega^{*}_{p^1} = \Omega^{*}_{p^1}$ and we can apply Ekedahl’s “Nakayama lemma” (Illusie 1983, 2.3.7). See Gros 1985, I 4.1.11, p21, for a more general statement. This suggests our definition of the Tate twist $r$ (for $r \geq 0$), namely, we set $M(r) = T^r(M)[-r]$ for $M$ in $D^b_c(R)$.

Ekedahl has defined a nonstandard $t$-structure on $D^b_c(R)$ the objects of whose heart $\Delta$ are called diagonal complexes (Illusie 1983, 6.4). It will be important for our future work to note that $T = T(\mathbb{1})[-1]$ is a diagonal complex: the sum of its module degree $(-1)$ and complex degree $(+1)$ is zero. The Tate twist is an exact functor which defines a self-equivalence of $D^b_c(R)$ preserving $\Delta$.

**Theorem and corollaries**

Regard $W\Omega^*_X$ as a sheaf of graded $R$-modules on $X$, and write $R\Gamma$ for the functor $D(X, R) \to D(R)$ defined by $\Gamma(X, -)$. As we noted above, $R\Gamma(W\Omega^*_X)$ lies in $D^b_c(R)$.

**Theorem.** For any smooth complete variety $X$ over a perfect field $k$ of characteristic $p \neq 0$, there is a canonical isomorphism

$$H^i(X, \mathbb{Z}_p(r)) \cong \text{Hom}_{D^b_c(R)}(\mathbb{1}, R\Gamma(W\Omega^*_X)(r)[i]).$$

**Proof.** For a graded $R$-module $M^*$,

$$\text{Hom}(\mathbb{1}, M^*) = \ker(1 - F : M^0 \to M^0).$$

To obtain a similar expression in $D^b_c(R)$ we argue as in Ekedahl 1985, p90. Let $\hat{R}$ denote the completion $\lim \leftarrow R/(V^n R + dV^n R)$ of $R$ (ibid. p60). Then right multiplication by $1 - F$ is injective, and $\mathbb{1} \cong \hat{R}^0/\hat{R}^0(1 - F)$. As $F$ is topologically nilpotent on $\hat{R}^1$, this shows that the sequence

$$0 \to \hat{R} \xrightarrow{(1-F)} \hat{R} \to \mathbb{1} \to 0,$$

(3)
is exact. Thus, for a complex of graded $R$-modules $M$ in $D^b(R)$,

$$\text{Hom}_{D(R)}(1, M) \cong H^0(R \text{Hom}(1, M)) \cong H^0(R \text{Hom}(\hat{R} \to \hat{R}, M)).$$

If $M$ is complete in the sense of Illusie 1983, 2.4, then $R \text{Hom}(\hat{R}, M) \cong R \text{Hom}(R, M)$ (Ekedahl 1985, I 5.9.3ii, p78), and so

$$\text{Hom}_{D(R)}(1, M) \cong H^0(\text{Hom}(R \to \hat{R}, R), M)) \cong H^0(\text{Hom}(R, M) \to \text{Hom}(R, M)).$$

Following Illusie 1983, 2.1, we shall view a complex of graded $R$-modules as a bicomplex $M^{••}$ in which the first index corresponds to the $R$-grading: thus the $j$th row $M^{•j}$ of the bicomplex is the $R$-module $(\cdots \to M^{i-j} \to M^{i+1-j} \to \cdots)$, and the $i$th column $M^{i•}$ is a complex of (ungraded) $R^0$-modules. The $j$th-cohomology $H^j(M^{••})$ of $M^{••}$ is the graded $R$-module

$$(\cdots \to H^j(M^{i•}) \to H^j(M^{i+1•}) \to \cdots).$$

Now, $\text{Hom}(R, M^{••}) = M^{0•}$, and so

$$H^0(\text{Hom}(R, M^{••}(r)[i])) = H^{i-r}(M^{••}).$$

The complex of graded $R$-modules $R\Gamma(W\Omega^•_X)$ is complete (Illusie 1983, 2.4, Example (b), p33), and so (4) gives an isomorphism

$$\text{Hom}_{D(R)}(1, R\Gamma(W\Omega^•_X)(r)[i]) \cong H^0(\text{Hom}(R, R\Gamma(W\Omega^•_X)(r)[i])) \to \text{Hom}(R, R\Gamma(W\Omega^•_X)(r)[i])).$$

The $j$th-cohomology of $R\Gamma(W\Omega^•_X)$ is obviously

$$H^j(R\Gamma(W\Omega^•_X)) = (\cdots \to H^j(X, W\Omega^i_X) \to H^j(X, W\Omega^{i+1}_X) \to \cdots)$$

(Illusie 1983, 2.2.1), and so (5) allows us to rewrite (6) as

$$\text{Hom}_{D(R)}(1, R\Gamma(W\Omega^•_X)(r)[i]) \cong H^{i-r}(R\Gamma(W\Omega^•_X(1-F) \to R\Gamma(W\Omega^•_X)).$$

This gives an exact sequence

$$\cdots \to \text{Hom}(1, R\Gamma(W\Omega^•_X)(r)[i]) \to H^{i-r}(X, W\Omega^•_X(1-F) \to H^{i-r}(X, W\Omega^•_X) \to \cdots \quad (7)$$

On the other hand, there is an exact sequence (Illusie 1979b, I 5.7.2)

$$0 \to \nu_•(r) \to W•\Omega^•_X \to W•\Omega^•_X \to 0$$
of prosheaves on $X_{et}$, which gives rise to an exact sequence
\[ \cdots \to H^i(X, \mathbb{Z}_p(r)) \to H^{i-r}(X, W\bullet \Omega_X^r) \xrightarrow{1-F} H^{i-r}(X, W\bullet \Omega_X^r) \to \cdots \]  
(Milne 1986, 1.10). Here $\nu_n(r)$ denotes the projective system $(\nu_n(r))_{n \geq 0}$, and $H^i(X, W\bullet \Omega_X^r) = \lim_{\leftarrow n} H^i(X, W_n \Omega_X^r)$ (étale or Zariski cohomology — they are the same).

Since $H^r(X, W\Omega_X^r) \cong H^r(X, W\bullet \Omega_X^r)$ (Illusie 1979a, 3.4.2, p101), the sequences (7) and (8) will imply the theorem once we check that there is a suitable map from one sequence to the other, but the right hand square in
\[
\begin{array}{ccc}
W\Omega_X^r & \xrightarrow{1-F} & W\Omega_X^r \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
R\Gamma W\Omega_X^r & \xrightarrow{1-F} & R\Gamma W\Omega_X^r \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
W\bullet \Omega_X^r & \xrightarrow{1-F} & W\bullet \Omega_X^r \\
\downarrow & & \downarrow
\end{array}
\begin{array}{ccc}
R\Gamma W\bullet \Omega_X^r & \xrightarrow{1-F} & R\Gamma W\bullet \Omega_X^r \\
\downarrow & & \downarrow
\end{array}
\]
gives rise to such a map. \qed

As in Milne 1986, p309, we let $H^i(X, (\mathbb{Z}/p^n\mathbb{Z})(r)) = H^{i-r}_{et}(X, \nu_n(r))$.

**Corollary 1.** There is a canonical isomorphism
\[ H^i(X, (\mathbb{Z}/p^n\mathbb{Z})(r)) \cong \text{Hom}_{D^b(\mathcal{R})}(\mathbf{1}, R\Gamma W_n \Omega_X^* (r)[i]). \]

**Proof.** The canonical map $\nu_*(r)/p^n\nu_*(r) \to \nu_n(r)$ is an isomorphism (Illusie 1979b, I 5.7.5, p. 598), and the canonical map $W\Omega_X^*/p^nW\Omega_X^* \to W_n \Omega_X^*$ is a quasi-isomorphism (ibid. I 3.17.3, p577). The corollary now follows from the theorem by an obvious five-lemma argument. \qed

Lichtenbaum (1984) conjectures the existence of a complex $\mathbb{Z}(r)$ on $X_{et}$ satisfying certain axioms and sets $H^i_{mot}(X, r) = H^i_{et}(X, \mathbb{Z}(r))$. Milne (1988, p68) adds the “Kummer $p$-sequence” axiom that there be an exact triangle
\[ \mathbb{Z}(r) \xrightarrow{p^n} \mathbb{Z}(r) \to \nu_n(r)[-r] \to \mathbb{Z}(r)[1]. \]

Geisser and Levine (2000, Theorem 8.5) show that the higher cycle complex of Bloch (on $X_{et}$) satisfies this last axiom, and so we have the following result.

**Corollary 2.** Let $\mathbb{Z}(r)$ be the higher cycle complex of Bloch on $X_{et}$. Then there is a canonical isomorphism
\[ H^i_{et}(X, \mathbb{Z}(r) \xrightarrow{p^n} \mathbb{Z}(r)) \cong \text{Hom}_{D^b(\mathcal{R})}(\mathbf{1}, R\Gamma W_n \Omega_X^* (r)[i]). \]

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