Polarons as stable solitary wave solutions to the Dirac–Coulomb system

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Abstract

We consider solitary wave solutions to the Dirac–Coulomb system both from physical and mathematical points of view. Fermions interacting with gravity in the Newtonian limit are described by the model of Dirac fermions with the Coulomb attraction. This model also appears in certain condensed matter systems with emergent Dirac fermions interacting via optical phonons. In this model, the classical soliton solutions of equations of motion describe the physical objects that may be called polarons, in analogy to the solutions of the Choquard equation. We develop analytical methods for the Dirac–Coulomb system, showing that the no-node gap solitons for sufficiently small values of charge are linearly (spectrally) stable.

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1. Introduction

The Dirac–Maxwell system and other models of fermion fields with self-interaction (such as the massive Thirring model [1] and the Soler model [2]) have been attracting the interest of both physicists and mathematicians for many years. These models, just like the nonlinear Schrödinger equation, have localized solitary wave solutions of the form \( \phi(x) e^{-i\omega t} \), where \( \phi(x) \) is exponentially localized in space. For the Dirac–Maxwell system the localized solutions with \( \omega \in (-m, m) \) have been shown to exist (first numerically and then analytically) in [3–6]. These solutions may encode certain properties of the theory which cannot be obtained via perturbative analysis. The role of these classical solutions in high energy physics has long been the topic of intense discussion (see, for example, [7–12]). It seems that there is no physical meaning of such solitary waves in quantum electrodynamics. These classical localized states are formed due to the attraction of the spinor field to itself, which takes place at the energies \( \omega \gtrsim -m \). On the other hand, the associated quantum field theory admits the appearance of antiparticles. To describe antiparticles, it is necessary to change the order of the
fermion creation–annihilation operators. This results in the additional change of sign at the scalar potential. Thus, in the quantum theory for $\omega \gtrsim -m$ instead of the self-attraction (which could lead to the creation of a localized mode) one again ends up with the self-repulsion in the antiparticle sector. This anticommuting nature of fermion variables is ignored in the classical Dirac–Maxwell system.

In the present paper we make an attempt to determine how the solitary waves of the classical equations of motion could play a role in the quantum field theory. We show that the Dirac equation with the Coulomb attraction emerges in the semiclassical description of fermions interacting with optical phonons or with the gravitational field.

Let us give more specifics. It is well known that relativistic Dirac fermions may emerge in condensed matter systems. This occurs, for example, on the boundary of 3D topological insulators and in some two-dimensional (2D) structures like graphene [13, 14]. Moreover, massless Dirac fermions appear in 3D materials (see, for example, [15] and references therein) at the phase transition between a topological and a normal insulator. Recently, the existence of massless fermions at the phase transition between the insulator states with different values of topological invariants has been proven for a wide class of relativistic models [16]. When the interaction with other fields is taken into account, these massless fermions gain mass. Fermion excitations in various materials interact with phonons [17]. As a result, attractive interaction between the fermions appears. Such an interaction gives rise to the formation of Cooper pairs in microscopic theories of superconductivity [17, 18]. In the case of the exchange by optical phonons [19] the interaction has the form of the Coulomb attraction. This leads to the formation of the polaron, as in the nonrelativistic Landau–Pekar approach [20, 21].

While in the above model the Lorentz symmetry is broken due to the Coulomb forces, the relativistically-invariant version of a similar system is given by the Dirac fermions interacting with the gravitational field. This problem may be thought of as a true relativistic polaron problem. We show that in the Newtonian limit we again arrive at the system of Dirac fermions interacting via Coulomb-like forces. Such polarons may emerge in the unified theories.

It is worth mentioning that positronium (the bound state of electron–positron pair) has nothing to do with the solitary waves discussed in the present paper. Positronium can be described by the Dirac equation in the external Coulomb field. This is in contrast to our solitary waves, which are described by the Dirac equation in the potential created by the spinor field itself.

In the second part of the paper we analyze the stability of solitary wave solutions in the Dirac–Coulomb system. We show that certain solitary waves are linearly stable, i.e. the spectrum of the equation linearized at a particular solitary wave has no eigenvalues with positive real part. Our approach is based on the fact that the nonrelativistic limit of the Dirac–Coulomb system is the Choquard equation. In particular, the solitary wave solutions to the Dirac–Coulomb system are obtained as a bifurcation from the solitary waves of the Choquard equation. (It is worth mentioning that the Choquard equation appears in the conventional polaron problem [20, 21].) It also follows that the eigenvalue families of the Dirac–Coulomb system linearized at a solitary wave are deformations of eigenvalue families corresponding to the Choquard equation (this has been rigorously proved in [22] in the context of nonlinear Dirac equations). The latter could be analyzed via the Vakhitov–Kolokolov stability criterion [23]. The delicate part is the absence of bifurcations of eigenvalues from the continuous spectrum.

We explain that this follows from the limiting absorption principle for the free Dirac operator. We emphasize that we present one of the first results for the linear stability of spatially localized fermion modes. Prior attempts at their stability properties included the analysis of stability of Dirac solitary waves with respect to particular families of perturbations (such as dilations), see e.g. [24–27] and related numerical results [28–31]. Yet neither the linear stability...
nor orbital nor asymptotic stability were understood. Our latest results on linear stability and
instability for the nonlinear Dirac equation are in [32, 33]. There are also recent results on
asymptotic stability of solitary wave solutions to the nonlinear Dirac equation in 1D and in
3D [34, 35] (proved under the assumption that a particular solitary wave is linearly stable).
Interestingly, the orbital stability has been proven for small amplitude solitary waves in the
completely integrable massive Thirring model [36]. It should be mentioned that in contrast
to those of the nonlinear Dirac equation, the questions of linear, orbital, and even asymptotic
stability of solitary waves in the nonlinear Schrödinger equation are essentially settled (see
e.g. [23, 37–39]).

We would also like to mention that Einstein–Dirac equations were shown to have
particle-like solutions which are linearly stable with respect to spherically symmetric
perturbations [40].

The paper is organized as follows. In section 2, we briefly describe the model under
investigation. In section 3 we discuss the appearance of the considered solitary waves in a
system of Dirac fermions interacting with optical phonons. In section 4 we consider the relation
of solitary waves to the gravitational polaron. In section 5.2 we sketch the proof of existence
of solitary waves and then address the question of their linear stability. Our conclusions are
given in section 6. In the appendix we give the details of the Vakhitov–Kolokolov stability
criterion [23] in its application to the Choquard equation.

2. Dirac–Coulomb system

Below, we choose the units so that \( \hbar = c = 1 \). We consider the model with the action

\[
S_E = \int d^3x \, dt \, \bar{\zeta} (i D \phi - m) \zeta - \int d^3x \, dt \, \frac{(\nabla \phi)^2}{2}, \tag{2.1}
\]

where the Dirac fermion field interacts with itself via an instantaneous Coulomb interaction,
\( x \in \mathbb{R}^3, \, t \in \mathbb{R}, \, e^2 \) is the coupling constant, \( \zeta(x, t) \in \mathbb{C}^4 \) is a four-component Dirac field,
\( \phi(x, t) \in \mathbb{R} \) is the Coulomb field, and

\[
D \phi = \gamma^0 (\partial_0 + i e \phi) + \sum_{j=1}^3 \gamma^j \partial_j,
\]

where \( \partial_0 = \frac{\partial}{\partial t}, \, \partial_j = \frac{\partial}{\partial x_j}, \, 1 \leq j \leq 3 \). The Dirac matrices \( \gamma^\mu, 0 \leq \mu \leq 3 \), satisfy
the Euclidean–Clifford algebra \( [\gamma^\mu, \gamma^\nu] = 2g^{\mu\nu} \gamma^0 \), with \( g^{\mu\nu} \) the inverse of the metric tensor
\( g_{\mu\nu} = \text{diag}[1, -1, -1, -1] \). Above, \( e \) is the charge of the spinor field, \( \phi(x, t) \) is the (real-valued)
external scalar field (such as the potential of the electric field in \( \mathbb{R}^3 \)). We denote
\( (\nabla \phi)^2 = \sum_{j=1}^3 (\partial_j \phi)^2 \). It is worth mentioning that in this model the Lorentz symmetry is
broken due to the Coulomb forces.

The dynamical equations corresponding to (2.1) are given by the following Dirac–
Coulomb system:

\[
\begin{align*}
    i \partial_t \zeta &= -i \alpha \cdot \nabla \zeta + m \beta \zeta + e \phi \zeta, \\
    \Delta \phi &= e \zeta^* \zeta, \tag{2.2}
\end{align*}
\]

where \( \zeta(x, t) \in \mathbb{C}^4, \phi(x, t) \in \mathbb{R}, x \in \mathbb{R}^3 \), and \( \Delta = \sum_{j=1}^3 \partial_j^2 \). Above, \( \alpha = (\alpha^1, \alpha^2, \alpha^3) \), where
the self-adjoint Dirac matrices \( \alpha^j \) and \( \beta \) are related to \( \gamma^\mu \) by

\[
\gamma^j = \gamma^0 \alpha^j, \quad 1 \leq j \leq 3; \quad \gamma^0 = \beta.
\]
They satisfy $(\alpha')^2 = \beta^2 = I_4, \alpha'\alpha' + \alpha\alpha' = 2I_2 \delta_{jk}, \alpha'\beta + \beta\alpha' = 0; 1 \leq j, k \leq 3$. Above, $\zeta = (\beta\xi)^* = \xi^*\beta$, with $\xi^*$ the Hermitian conjugate of $\xi$. A particular choice of the Dirac matrices does not matter; we take the Dirac matrices in the common form

$$
\alpha' = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.
$$

(2.3)

where $I_2$ is the $2 \times 2$ unit matrix and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

**Remark 2.1.** Note that according to the second equation in (2.2), if $\varphi \to 0$ at infinity, then $e\varphi$ is strictly negative and behaves like an attractive Coulomb potential in the first equation in (2.2) (for energies near $m$), leading to the existence of bound states for $\omega \lesssim m$.

### 3. Polaron due to interaction of fermion field with optical phonons

#### 3.1. Field-theoretical description of the generalized Frohlich model

It was mentioned in the introduction that Dirac fermions may emerge in various 3D systems at the phase transition between the insulating states with different values of momentum space topological invariants. Similarly to the ordinary electrons in crystals, these Dirac fermions may interact with optical phonons, thus giving rise to the polaron problem. The model Hamiltonian for a Dirac particle interacting with optical phonons can be obtained via the generalization of the conventional Frohlich Hamiltonian [19–21, 41, 42]:

$$
\mathcal{H} = \sum_{j=1}^{3} \sum_{p} c_p^{\dagger} \left[ g \gamma^j \gamma^i \hat{p}_j + m \gamma^0 \right] c_p + \sum_{k,p} \left[ i \tilde{\alpha} c_{p+k}^{\dagger} \frac{1}{|k|} \hat{a}_k c_p + \text{h.c.} \right] + \Omega \sum_{k} \hat{a}_k^{\dagger} \hat{a}_k.
$$

(3.1)

Here $\tilde{\alpha}$ and $\Omega$ are coupling constants, $c_p^{\dagger}$ are the electron creation operators, and $\hat{a}_k^{\dagger}$ are the phonon creation operators. We introduce the phonon field $\varphi(x) = i\sqrt{2\Omega} \sum_k \frac{1}{|k|} \hat{a}_k e^{i \Omega t} + \text{h.c.}$ and the electron field $\psi(x) = \sum_{p} c_p e^{i \varphi t}$.

We express $\text{Tr} e^{-i\mathcal{H}T}$, where $T$ is time, as a functional integral. This is done as follows. First, we subdivide the time interval $[0, T]$ into smaller intervals $\Delta T$ and represent $e^{-i\mathcal{H}T} = e^{-i\mathcal{H}_1 \Delta T} \ldots e^{-i\mathcal{H}_3 \Delta T}$. Next, we substitute

$$
1 = \frac{1}{2\pi i} \int d\eta d\bar{\eta} d\zeta d\bar{\zeta} e^{-\bar{\zeta} \eta \zeta} (\eta | \zeta).
$$

where the coherent states are $| \eta \zeta \rangle = e^{-\eta \zeta} | 0 \rangle$, $\zeta \in \mathbb{C}$, and $\eta$ is the Grassmann variable. The functional integral over $\zeta, \eta$ appears. Then, the integration variables $\varphi$ and $\psi$ are introduced as a result of the action of the corresponding operator fields on the coherent states. Therefore, the Frohlich Hamiltonian gives rise to the quantum field theory of the interacting fermion and phonon fields, with the partition function given by

$$
Z = \int d\psi d\bar{\psi} \int d\varphi d\bar{\varphi} \exp \left( i \int d^3 x \bar{\psi} \left[ i \frac{\partial}{\partial \varphi} - m - e\varphi \right] \psi + i \int d^3 x \right) \frac{\left( \nabla \psi \right)^2}{2 \Omega^2} \frac{1}{2} \frac{\left( \nabla \varphi \right)^2}{2}.
$$

(3.2)

where $\varphi = \gamma^j \partial_j$, with the summation over $\mu = 0, \ldots, 3$; $(\nabla \psi)^2 = \sum_{j=1}^{3} \left( \frac{\partial \psi}{\partial x_j} \right)^2$. Above, we denote $e = \tilde{\alpha} \sqrt{\frac{\Omega}{2\pi}}$. We formalize this as follows:

**Lemma 3.1.** The quantum-mechanical system with the Frohlich Hamiltonian (3.1) is equivalent to the field theory with the partition function (3.2).
It is worth mentioning that usually the nontrivial vierbein appears when Dirac fermions emerge in condensed matter models [43]. Moreover, this vierbein fluctuates. Under certain circumstances, such fluctuations may give rise to the emergent gravity [43]. We, therefore, assume that the emergent vierbein has small fluctuations and can be transferred to the unity action given by equation (2).

In the low energy approximation $E \ll \Omega$, we arrive at the partition function with the action given by equation (2.1):

\[
Z = \int d\bar{\psi} d\psi \exp \left( i \int d^3x dt \bar{\psi} \left( i\gamma \partial_t - m - e\gamma^0 \phi \right) \psi - i \int d^3x dt \left( \nabla \phi \right)^2 / 2 \right).
\]

After the Wick rotation ($t \to -it$, $\phi \to i\phi$) to Euclidean space-time we arrive at

\[
Z = \int d\bar{\psi} d\psi \exp \left( - \int d^3x dt \bar{\psi} \left[ \Gamma^0 \left( \partial_0 + ie\phi \right) + \Gamma^j / \partial_j + m \right] \psi + \int d^3x dt \left( \nabla \phi \right)^2 / 2 \right).
\]

Here the Euclidean gamma-matrices $\Gamma^\mu$ satisfy $[\Gamma^\mu, \Gamma^\nu] = 2\delta^{\mu\nu}$, $0 \leq \mu, \nu \leq 3$.

**Remark 3.2.** In fact, the integration in (3.4) is not convergent due to the positive sign at the kinetic term for $\phi$. This is exactly the same problem as for the Euclidean functional integral for the gravitational theory with the Einstein–Hilbert action (see below). This shows that the theory defined by the partition function (3.3) can be considered only as an effective low energy model. At some (large) energies, the action has to be redefined in order to make the Euclidean functional integral convergent. As well as for the quantum gravity, this can be done if the term with higher powers of $\partial \phi$ is added to the action. This, in turn, regularizes the Coulomb interaction at small distances. In physical applications, such an additional term in the action appears at scales at which the attraction due to optical phonons no longer dominates, and some other interactions come into play (say, the Coulomb repulsion due to photons).

### 3.2. Application of semiclassical methods to the model

In this subsection we follow the approach of [44] on the semiclassical methods for fermion systems and obtain similar results. Here it is important that we consider the phonon field constant in time. We come to the model with the following partition function:

\[
Z = \int d\bar{\psi} d\psi d\phi \exp \left( i \sum_\eta T \int d^3x \bar{\psi}_\eta \left[ \eta - \mathcal{H}_\phi \right] \psi_\eta - i \int d^3x dt \left( \nabla \phi \right)^2 / 2 \right),
\]

where

\[
\mathcal{H}_\phi = \phi^0 \left[ -i \sum_{j=1}^3 \gamma^j / \partial_j + m + e\gamma^0 \phi \right].
\]

Here the system is considered with the anti-periodic in time boundary conditions: $\psi(t + T, x) = -\psi(t, x)$. We use the decomposition

\[
\psi(t, x) = \sum_{\eta = \frac{\pi}{T} (2k+1), k \in \mathbb{Z}} e^{-i\eta t} \psi_\eta(x).
\]

We represent $\psi$ as $\psi_\eta(x) = \sum_n c_{\eta,n} \Psi_n^\eta(x)$, where $\Psi_n^\eta$ is the eigenfunction of $\mathcal{H}_\phi$ corresponding to the eigenvalue $E_n^\eta$ and normalized to unity ($\int d^3x \Psi_n^\eta \bar{\Psi}_n^\eta = 1$):

\[
Z = \int d\bar{\psi} d\psi d\phi \exp \left( i \sum_{\eta,n} T \bar{\psi}_{\eta,n} \left[ \eta - E_n^\eta \right] \psi_{\eta,n} - i \int d^3x dt \left( \nabla \phi \right)^2 / 2 \right).
\]
Integrating out the Grassmann variables \( c_n \) we come to:

\[
Z = \int d\psi \exp \left( -i \int d^3x \, d\tau \frac{(\nabla \psi)^2}{2} \right) \prod_{\eta} \prod_{n} \left( (\eta - E_n^\psi)T \right)
\]

\[
= C \int d\psi \exp \left( -i \int d^3x \, d\tau \frac{(\nabla \psi)^2}{2} \right) \prod_{n} \cos \frac{T E_n^\psi}{2},
\]

(3.6)

where \( C \) depends on the details of the regularization but depends neither on \( T \) nor on the spectrum in the continuum limit. The values \( E_n^\psi \) depend on the parameters of the Hamiltonian, with the index \( n \) enumerating these values.

Equation (3.6) is derived as follows. Recall that in (3.5) the summation is over \( \eta = \frac{\pi}{2}(2k + 1) \). The product over \( k \) can be calculated as in [44]:

\[
\prod_{k \in \mathbb{Z}} \left( 1 + \frac{E_k^\psi T}{\pi (2k + 1)} \right) = \cos \frac{E_k^\psi T}{2},
\]

(3.7)

where \( T = 2Na \). Here we imply that the lattice regularization is introduced, and \( a \) is the lattice spacing while \( 2N \) is the lattice size in the imaginary time direction. In the limit \( a \to 0 \) we come to \( N \to \infty \). Thus,

\[
\text{Det}(i\partial_\tau - \mathcal{H}_\psi) = C \prod_{n} \cos \frac{E_n^\psi T}{2}.
\]

(3.8)

We get (see also [44, 45]):

\[
Z = C \sum_{\{K\}_{\eta}=0,1} \int d\psi \exp \left( -i T \int d^3x \frac{(\nabla \psi)^2}{2} + \frac{iT}{2} \sum_{n} E_n^\psi - iT \sum_{n} K_n E_n^\psi \right)
\]

\[
= C \sum_{\{K\}_{\eta}=0,1} \int d\psi \, e^{i\mathcal{Q}_{K}(\psi)}.
\]

(3.9)

Following [44], we interpret equation (3.9) as follows. \( K_n \) represents the number of occupied states with the energy \( E_n^\psi \). These numbers may be 0 or 1. The term \( \sum_n E_n^\psi \) vanishes if \( \psi = 0 \), since in this case the values \( E_n \) come in pairs with opposite signs.

In the weak coupling approximation when \( \frac{\pi}{2T} \ll 1 \) the energy levels can be represented as \( E_n^\psi \approx E_n^\psi 0 + \sum_n E_n^\psi \). Then the integral over \( \psi \) is Gaussian; it is equal to \( \sim \exp(i\mathcal{Q}_{K}(\psi_{\text{class}})T) \), where \( \psi_{\text{class}} \) satisfies the variational problem \( \delta \mathcal{Q}_{K}(\psi) = 0 \). In this limit, the dominant contributions of the \( \psi \)-configurations satisfy the following variational problem:

\[
0 = \delta \left[ -\int d^3x \frac{(\nabla \psi)^2}{2} + \sum_n \left( E_n^\psi + K_n E_n^\psi \right) \right]
\]

\[
= \delta \int d^3x \left[ -\frac{\nabla \psi)^2}{2} - \sum_n \left( K_n \xi_n^+ \mathcal{H}_\psi \xi_n - \xi_n^+ \mathcal{H}_\psi \xi_n \right) \right].
\]

(3.10)

In the right-hand side, the variation over \( \xi_0 \) with the constraint \( \int d^3x \, \xi_n^+ \xi_0 = 1 \) gives, in addition, the one-fermion wave functions \( \xi_n \). The variational problem can be written as

\[
0 = \delta \int d^3x \left[ \sum_n \left( \frac{1}{2} - K_n \right) \xi_n^+ \mathcal{H}_\psi - \lambda_n |\xi_n = -\frac{(\nabla \psi)^2}{2} \right]
\]

\[
= \delta \int d^3x \left[ \sum_n \left( K_n - \frac{1}{2} \right) \xi_n^+ \mathcal{H}_\psi - \lambda_n |\xi_n - m - e\gamma^0 \psi |\xi_n \right] = \delta \int d^3x \left[ \sum_n \left( K_n - \frac{1}{2} \right) \xi_n^+ \mathcal{H}_\psi - \lambda_n |\xi_n - m - e\gamma^0 \psi |\xi_n \right] = \left[ \sum_n \left( K_n - \frac{1}{2} \right) \xi_n^+ \mathcal{H}_\psi - \lambda_n |\xi_n - m - e\gamma^0 \psi |\xi_n \right].
\]

(3.11)
Here $\lambda_n$ are the Lagrange multipliers. We introduced the time dependence into $\zeta$: the variation is performed with respect to the functions of the form $\zeta_n = e^{-i\omega t} \xi_n(x)$ and with respect to the time-independent phonon cloud $\varphi$. In this form, the functional to be used in the variational problem almost coincides with the action from (3.2). The difference is that we assume the special form of $\zeta \sim e^{-i\omega t}$ and also that $\varphi$ does not depend on time. Also, instead of the Grassmann variables, we substitute ordinary wave functions and take into account filling factors for the fermion states.

The additional constraint is that the wave functions $\zeta_n$ are different, so that there are no states that are occupied more than once. The variation is performed with the numbers $K_n$ being fixed. After the variational problem is solved one says that the state with the wave function $\zeta_n$ that has been found is occupied if $K_n \neq 0$ for the corresponding value of $n$. In equation (3.9) we need to sum up all such configurations with different arrays $K_n$. The calculated values of $\varphi$ are to be substituted into the exponent in equation (3.9) while the values of $E_n$ are given by $E_n = \zeta_n^+ H \zeta_n$. This approach is similar to the conventional Hartree–Fock approximation.

We come to the following result:

**Theorem 3.3.** The partition function for the system of Dirac electrons interacting with optical phonons at low energies $E \ll \Omega$ is given by equation (3.9). In the weak coupling limit, the integral over $\varphi$ in (3.9) is evaluated in the stationary phase approximation, resulting in the variational problem (3.11).

**Remark 3.4.** CP-invariance implies that for any state $n$ with the energy $E_n^+ \neq 0$ there exists a state $\bar{n}$ with the energy $E_{\bar{n}}^+ = -E_n^-$. In particular, for $\varphi = 0$ we obtain $\sum_n E_n = 0$. States with positive values $E_n$ are interpreted as electrons, while states with negative $E_n$ correspond to holes. The vacuum state is this case the state with all negative levels $E_n$ occupied. The situation is changed when $\varphi \neq 0$. However, if $\max \epsilon \varphi \leq m$, the states with $E_n > 0$ are also interpreted as electrons while the states with $E_n < 0$ are interpreted as holes. If $\max \epsilon \varphi > 2m$, then there could be states which cannot be considered as either electrons or holes. Instead, these states correspond to the Schwinger pair creation process. The appearance of such states, however, may be avoided in the weak coupling limit, when $\epsilon$ is small and, therefore, one almost always has $\epsilon \varphi < m$. That is why in the weak coupling the vacuum can again be considered as the state with all negative levels of $E_n$ occupied and all positive levels of $E_n$ empty. At large enough values of $\alpha = \frac{e \varphi}{2m}$ this pattern may be changed due to the creation of pairs that may lead to the change of vacuum. In this case the fermion condensate may appear; the description of the theory in terms of the collection of one-fermion states is no longer relevant.

### 3.3. One-polaron problem

Now let us consider the usual polaron problem, i.e. the problem of one electron interacting with the phonon cloud. Recall that $Z = \text{Tr} e^{-iHt}$. Therefore, the sum in equation (3.9) corresponds to the sum over many fermion states. The state with $K_n^{\text{vac}} = \frac{1}{2}(1 + \text{sign} E_n)$ corresponds to the vacuum. Then the state with $K_n(q) = K_n^{\text{vac}} + \delta_{nq}$ for some $E_q > 0$ corresponds to the state that consists of the vacuum (the Dirac sea of negative levels) and the bound state of one electron and the phonon cloud surrounding it. This is a polaron. The vacuum is translation-invariant and CP-invariant. That is why $\varphi_{\text{vac}} = 0$. The vacuum energy is defined as $E_{\text{vac}} = \sum_n K_n^{\text{vac}} E_n^0$. The polaron energy is equal to

$$E_q = \int d^3x \frac{(\nabla \varphi_q)^2}{2} + \sum_n \left(K_n(q) - \frac{1}{2}\right) E_n^{\varphi_q},$$

(3.12)
where \( \varphi_q \) is defined by solving the variational problem (3.11). Infinite vacuum energy has to be subtracted from \( E_q \); the quantity \( E_q = E_q - E_{\text{vac}} \) is finite and is considered as a renormalized polaron state energy in physical applications. In the general case, the renormalized polaron energy contains the contribution from the virtual electron–hole pairs which are born when large enough \( \varphi_q \) appears. The interaction with the Dirac sea also contributes to \( E_q \). However, under certain circumstances, these contributions can be neglected. This is the so-called quenched approximation when the interactions between different fermions are neglected while the interaction between the phonon cloud and the electron is taken into account. This means that the probability that the electron–hole pair is created from vacuum is small, while the total electric charge \( Q \) for the given problem is implied equal to unity. For the case of the conventional polaron these conditions are assumed [19–21, 41, 42]. In our case the setup for the one-polaron problem should include \( Q = 1 \) and \( |E_q - m| \ll m \). The latter condition provides that the probability for electron–hole pairs to be created is small. In this case, the polaron energy is calculated as

\[
E_q = \int d^3x \frac{(\nabla \varphi_q)^2}{2} + E_{\varphi_q}^0, \tag{3.13}
\]

where \( \varphi_q \) is calculated via the variational problem

\[
0 = \delta \int d^3x \left\{ \bar{\zeta} \left[ \frac{i}{\partial} - m - e \gamma^0 \varphi_q \right] \zeta - \frac{(\nabla \varphi_q)^2}{2} \right\}. \tag{3.14}
\]

This problem, in turn, leads to the following equations:

\[
i\partial_t \zeta = -i\alpha \cdot \nabla \zeta + m\beta \zeta + e\varphi_q(x, t) \zeta(x, t), \tag{3.15}
\]

\[
\Delta \varphi_q(x, t) = e|\zeta(x, t)|^2. \tag{3.16}
\]

where \( x \in \mathbb{R}^3, \zeta(x, t) \in \mathbb{C}^4 \). That is, the potential \( \varphi_q(x, t) \in \mathbb{R} \) is generated by the spinor field itself. That is why we come to equation (2.2) with the important constraint on the wave function \( \zeta \):

\[
\int d^3x \zeta^+ \zeta = 1. \tag{3.17}
\]

We arrive at the following statement:

**Lemma 3.5.** Let us consider the problem of bound states of one electron surrounded by the phonon cloud (polaron) in the field theory with partition function (3.2). In the low energy approximation \( E \ll \Omega \) the system of equations (3.15), (3.16), (3.17), as well as the variational problem (3.14), solve this problem in the first order approximation of the weak coupling expansion.

**Remark 3.6.** It is important that only those solutions of system (2.2) which are normalized according to equation (3.17) have a physical meaning.

It is worth mentioning that the given variational problem may be relevant for the solution of polaron problem not only in the weak coupling regime (see, for example, [20, 21, 42]). In the nonrelativistic case, this variational problem has appeared in the approach due to Landau and Pekar [20, 21]. However, in these papers the trial functions were used for the minimization, while we consider the given variational problem exactly.

It is instructive to consider how equation (3.14) appears from the consideration of the two-point Green function

\[
G(t_2 - t_1) = \frac{1}{Z} \int d\psi \, d\bar{\psi} \, e^{i \int d^3x \, dt \left\{ \bar{\psi} \left[ \frac{i}{\partial} - m - e \gamma^0 \psi \right] \psi - \frac{\Delta \varphi_q^2}{2} \right\} \psi^+(t_1, x) \psi(t_2, x) d^3x}. \tag{3.18}
\]
Let us mention that the consideration of arbitrary values of $t_1, t_2$ requires a more complicated technique, when $\varphi$ is constant as a function of time except at the points $t_1, t_2$. This is because the insertion of $\tilde{\psi}$ and $\tilde{\psi}$ disturbs vacuum in such a way that the value of $\varphi$ is changed at $t_1$ and $t_2$. This technique uses the so-called Floquet indices and will be applied in the next section to the consideration of gravitational polarons, which are the relativistic generalizations of the objects considered in this section. Here we restrict ourselves to the case $t_1 = 0, t_2 = T$. Then

$$G(T) = \frac{\text{const}}{17Z} \sum_{\{K_q\}, q = 0, 1} \int d\varphi \ e^{-\text{i}T \int d^4x (\bar{\psi} D^\mu \psi + \mu \text{det} E_{\mu}^a \bar{\psi} \gamma^a \psi)} \sum_{\bar{n} = \frac{T}{2(2k + 1)}, q} \frac{e^{-\text{i}nT}}{\eta - E^q}. \quad (3.19)$$

Using the Poisson summation formula [46, 47], we get

$$g(t, E^q) = \sum_{\bar{n} = \frac{T}{2(2k + 1)}} \frac{e^{-\text{i}nT}}{\eta - E^q} = \frac{-\text{i}T e^{-\text{i}E^q t}}{1 + e^{-\text{i}E^q T}}.$$

Here it is implied that the energy levels have small imaginary parts (as usual in the quantum field theory). After the Wick rotation ($T \rightarrow -1/T$, $T$ being the temperature) the given expression would become the usual finite temperature Matsubara Green function. Therefore, we arrive at

$$G(T) = \sum_{q} \sum_{\{K_q\}, q = 0, 1} \int d\varphi \ e^{\text{i}T \int d^4x (\bar{\psi} D^\mu \psi + \mu \text{det} E_{\mu}^a \bar{\psi} \gamma^a \psi)} \sum_{\bar{n} = \frac{T}{2(2k + 1)}, q} \frac{e^{-\text{i}nT}}{\eta - E^q}. \quad (3.20)$$

Here in each term of the summation over $K_q$ the field $\varphi$ is to be determined in the stationary phase approximation via solving the variational problem (3.11). Again, in the quenched approximation we come to

$$G(T) = \sum_{q} \int d\varphi \ e^{-\text{i}T \int d^4x (\bar{\psi} D^\mu \psi + \mu \text{det} E_{\mu}^a \bar{\psi} \gamma^a \psi)} = \sum_{q} Z_q e^{-\text{i}E^q T}.$$

Here $E_q$ is the energy of the polaron in the $q$th state calculated according to lemma 3.5. The factors $Z_q$ are the pre-exponential factors of the stationary phase approximation.

4. Gravitational polarons

4.1. Semiclassical description of fermions coupled to the gravitational field

In the previous section, we considered the quantum system of Dirac electrons interacting via the attractive Coulomb potential. This system may appear in certain nonrelativistic models at the quantum phase transition between the phases of the fermionic systems with different values of topological invariants [16]. However, the Coulomb interaction breaks the emergent Lorentz symmetry. It is interesting, therefore, to consider the relativistic extension of the model defined by the partition function (3.3); one such extension is discussed in this section.

We consider the relativistic Dirac fermion interacting with the gravitational field. The action of a Dirac spinor in Riemann space has the form [48–52]

$$S_f = \int (\text{i} \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi) \left| E \right| d^4x. \quad (4.1)$$

Here $\left| E \right| = \text{det} E_{\mu}^a$, where $E_{\mu}^a$ is the inverse vierbein, $\gamma^\mu = E_{\mu}^a \gamma^a$, and $\bar{\psi} = \psi^+ \gamma^0$. The covariant derivative is

$$D_\mu = \partial_\mu + \frac{1}{4} \omega_{\mu}^{ab} \gamma_a \gamma_b. \quad (4.2)$$
with \( \gamma_{a} \gamma_{b} = \frac{1}{2} (\gamma_{a} \gamma_{b} - \gamma_{b} \gamma_{a}) \). The torsion-free spin connection is denoted by \( \omega_{\mu}^{a} \). It is related to \( E_{\mu}^{a} \) and the affine connection \( \Gamma_{\mu \nu}^{\rho} \) as follows:

\[
\nabla_{\nu} E_{\mu}^{a} = \partial_{\nu} E_{\mu}^{a} - \Gamma_{\mu \nu}^{\rho} E_{\rho}^{a} + \omega_{\mu}^{a} E_{\nu}^{\rho} = 0,
\]

\[
D_{\nu} E_{\mu}^{a} = \partial_{\nu} E_{\mu}^{a} + \omega_{\mu \nu}^{a} E_{\nu}^{\rho} = 0.
\]

(4.3)

This results in:

\[
\Gamma_{\mu \nu}^{\rho} = \left[ \Gamma_{\mu \nu}^{\rho} \right] = \frac{1}{2} g^{\rho \lambda} \left( \partial_{\nu} g_{\lambda \gamma} + \partial_{\gamma} g_{\lambda \nu} - \partial_{\lambda} g_{\nu \gamma} \right),
\]

\[
\omega_{\mu \nu}^{a} = \frac{1}{2} (c_{abc} - c_{cab} + c_{cba}) E_{\mu}^{a}.
\]

(4.4)

Here \( c_{abc} = \eta_{ad} E_{b}^{a} E_{c}^{d} \partial_{\nu} E_{\mu}^{d} \), \( g_{\mu \nu} = E_{\mu}^{a} E_{\nu}^{a} \delta_{\mu \nu} \), and \( \Gamma_{\mu \nu}^{\rho} = \Gamma_{\nu \mu}^{\rho} = 0 \); indices are lowered and raised with the aid of \( g \) and \( E \).

The partition function of the model is given by

\[
Z = \int d\bar{\psi} \, d\psi \, dE \, e^{i \int d^{4}x (E) \bar{\psi} [\partial \cdot m] \psi - \frac{1}{2 \pi} R(E)}
\]

where \( \partial \) is \( \gamma^{\mu} D_{\mu} \), with \( D_{\mu} \) given by equation (4.2).

**Remark 4.1.** The integral over the Grassmann variables \( \psi \) in continuum field theory requires additional discussion. There are several ways to define the functional integral: via lattice discretization, via re-expressing it as a functional determinant, etc. In all these cases, the presence of a nontrivial metric leads to additional difficulties. Below we assume that the functional integral is defined in such a way that

\[
\int d\bar{\psi} \, d\psi \, e^{i \int d^{4}x (E) \bar{\psi} \phi \psi} = \text{Det} \hat{Q} = \prod_{n} \lambda_{n},
\]

(4.5)

where \( \lambda_{n} \) are eigenvalues of \( \hat{Q} \). (The spectrum of \( \hat{Q} \) is discrete if we consider the system in the finite 4-volume \( V_{4} = \int d^{4}x \left| E \right| \).) We assume the toroidal topology, and also that in a synchronous reference frame the boundary conditions are antisymmetric in time and symmetric in the spatial coordinates. Equation (4.5) can be rewritten as

\[
\text{Det} \hat{Q} = \int d\bar{\psi} \, d\psi \, e^{i \sum_{n} \lambda_{n} \int d^{4}x \left| E \right| \bar{\Psi}_{n}^{\dagger} (x) \Psi_{n} (x) \gamma_{0} c_{n}} = \int d\bar{c} \, dc \text{Det} \left( \frac{\partial (\bar{c}, c)}{\partial (\bar{\psi}, \psi)} \right) e^{i \sum \lambda_{n} \bar{c}_{n} c_{n}}
\]

(4.6)

where we used the decompositions

\[
\psi = \sum_{n} \Psi_{n} (x) c_{n}, \quad \psi^{\dagger} = \sum_{n} \Psi_{n}^{\dagger} (x) \bar{c}_{n}.
\]

(4.7)

Here \( \Psi_{n} (x) \) are eigenfunctions of \( \hat{Q} \) corresponding to eigenvalues \( \lambda_{n} \), while \( c_{n}, \bar{c}_{n} \) are new Grassmann variables. The operator \( \hat{Q} \) is assumed to be Hermitian with respect to the inner product \( (\psi, \phi) = \int d^{4}x \left| E \right| \bar{\psi} \phi \). Therefore, the eigenfunctions satisfy \( \int d^{4}x \left| E \right| \bar{\Psi}_{n}^{\dagger} (x) \Psi_{n} (x) = \delta_{nm} \). The key assumption about the integration measure over \( \psi \) is that with this normalization one has \( \text{Det} \left( \frac{\partial (\bar{c}, c)}{\partial (\bar{\psi}, \psi)} \right) = 1 \). Equation (4.6) also allows us to calculate various correlation functions \( \langle \psi^{\dagger} (x_{1}) \cdots \psi (x_{N}) \rangle \). Namely, we first represent \( \psi \) as the series (4.7), and then the integral over \( \bar{c}, c \) is evaluated as usually.

Again, the integral in the Euclidean space is not convergent, and should be redefined at high energies; see the discussion above (cf remark 3.2). In order to bring the theory into a form suitable for the considerations similar to that of [44], let us consider the system in the gauge corresponding to a synchronous reference frame. In this gauge, \( E_{\mu}^{0} E_{\nu}^{0} \delta^{\mu \nu} = \delta^{00} \); we also set \( E_{\mu}^{0} = \delta_{\mu}^{0} \) via the rotation of the reference frame in its internal space and the corresponding \( SO(3, 1) \) transformation of spinors. That is why the gauge is fixed both with respect to the
Here const depends neither on the gravitational field nor on general coordinate transformations and with respect to the inner $SO(3, 1)$ rotations of the reference frame. We denote
\[ \gamma^{\mu} \left[ i E_{\nu}^E \gamma^{\nu} \left( \partial_{\mu} + \frac{1}{2} \omega_{\mu}^{\alphah} \eta_{\alpha \gamma} \right) - m \right] = i \delta_0 - \mathcal{H}, \]
\[ - \mathcal{H} = i E_{\nu}^E \gamma^{\nu} \partial_{\mu} + i E_{\nu}^E \gamma^{\nu} \frac{1}{2} \omega_{\mu}^{\alphah} \eta_{\alpha \gamma}(\partial)_{\nu} - \gamma^{0} m, \]
where the summation in $j$ is over $j = 1, 2, 3$.

We point out that the operator $i \delta_0 - \mathcal{H}$ is Hermitian (while $\mathcal{H}$ is not). We have
\[ Z = \int dE \text{Det}(i \delta_0 - \mathcal{H}) e^{-\frac{i}{\hbar} \int d^4x R[E]}. \]

In order to calculate the determinant $\text{Det}(i \delta_0 - \mathcal{H})$, we use anti-periodic in time boundary conditions. Suppose that we find the solution $\zeta$ of the equation $(i \delta_0 - \mathcal{H}) \zeta = 0$ such that $\zeta(t + T) = e^{-i \Omega T} \zeta$. (Here $\Omega T$ is the Floquet index [44]). Then $\Psi_{k, \Omega} = e^{i \frac{\chi (2k + 1)}{\Omega T}} \zeta$ is the eigenfunction of the operator $(i \delta_0 - \mathcal{H})$:
\[ (i \delta_0 - \mathcal{H}) \Psi_{k, \Omega} = - \left( \frac{\pi}{T} (2k + 1) + \Omega \right) \Psi_{k, \Omega}. \] (4.8)

With the derivation similar to that of equations (3.6) and (3.7) we come to
\[ \text{Det}(i \delta_0 - \mathcal{H}) = \text{const} \prod_n \cos \frac{\Omega_n^E T}{2}. \] (4.9)

Here const depends neither on the gravitational field nor on $T$, while the product is over the different values $\Omega_n^E T$ of the Floquet index [44]. These values depend on the vierbein field $E^\mu$. The index $n$ enumerates them. The partition function takes the following form (cf equation (3.9)):
\[ Z \sim \text{const} \sum_{|K_n| = 0, 1} \int dE \exp \left( - i m_p^2 \int d^4x E[R + \frac{1}{2} \sum_n \Omega_n^E - i T \sum_n K_n \Omega_n^E] \right). \] (4.10)

Here $m_p$ is the Planck mass. Following [44], we interpret equation (4.10) as follows. The numbers $K_n$ represent the number of occupied states with the Floquet index $\Omega_n^E T$. These numbers may be 0, 1. The vacuum here corresponds to the negative ‘energies’ (Floquet indices) occupied and positive ‘energies’ empty.

Here the physical meaning of the numbers $K_n$ is the same as in the previous section. The only difference is that now the Floquet indices appear in place of the energy levels and that the gravitational field depends on time. The semiclassical approximation for the gravitational field now leads to the variational problem
\[ 0 = \delta \left\{ - \sum_n \left[ K_n - \frac{1}{2} \right] \Omega_n^E T - m_p^2 \int d^4x R[E] \right\} \]
\[ = \delta \int d^4x \sum_n \left[ K_n - \frac{1}{2} \right] \Psi_{k, \Omega_n^E}^\dagger \left( i \delta_0 - \mathcal{H} + \frac{\pi}{T} (2k + 1) \right) \Psi_{k, \Omega_n^E} - m_p^2 R \] (4.11)

Here $E$ is varied, $k$ is arbitrary, and the normalization is $\int d^4x |E| \Psi_{k, \Omega_n^E}^\dagger \Psi_{k, \Omega_n^E} = T$. Let us also introduce the wave function
\[ \zeta_{n, \lambda, E} = e^{-i \frac{\pi (2k + 1) + \lambda_n^R}} \Psi_{k, \Omega_n^E}. \] (4.12)
The values $\lambda_n$ play the role of Lagrange multipliers. At $\Omega_n = \lambda_n$, the functions $\zeta_{n, \lambda, E}$ satisfy the following conditions:
\[ \zeta_{n, \lambda, E}(t + T) = e^{-i T \zeta_{n, \lambda, E}(t)}, \]
\[ \int d^4x |E| \zeta_{n, \lambda, E} E^A_{\alpha} \gamma^A \zeta_{n, \lambda, E} = T, \]
\[ [i \partial - m] \zeta_{n, \lambda, E} = 0. \] (4.13)
Equation (4.12) can be rewritten as

$$0 = \int d^4x \left( \sum_n (K_n - 1/2) \bar{\zeta}_{n,\gamma,E} [\delta [i\partial - m] |E|] \zeta_{n,\gamma,E} - m_0^2 [\delta R |E|] \right).$$  \hspace{1cm} (4.14)

One can see that the variation of $\zeta_{n,\gamma,E}$ does not enter this expression that defines the field $E$ for any given $\zeta_{n,\gamma,E}$. At the same time one can see that the variation of $\zeta_{n,\gamma,E}$ would give the Dirac equation (4.13). That is why for the determination of both $\zeta_{n,\gamma,E}$ and $E$ we may use the variational problem

$$0 = \delta \int d^4x \left\{ \sum_n \left( K_n - 1/2 \right) \bar{\zeta}_{n,\gamma,E} [\delta [i\partial - m] \zeta_{n,\gamma,E} - m_0^2 R] |E| \right\},$$  \hspace{1cm} (4.15)

where the gravitational field and the wave functions $\zeta_n$ are varied. The wave functions are normalized so that $\int d^4x |E| \zeta^\dagger \zeta = T$. The additional constraint is that the wave functions $\zeta_n$ are different, so that there are no states that are occupied more than once. The variation is performed with the fixed values of $K_n$. The final form of the variational problem is gauge invariant, although it was derived in a synchronous reference frame. As a result, the gravitational field is defined by the Einstein equations with the energy–momentum tensor defined by the set of one-fermion states that represent the sea of occupied energy levels and the fermion–antifermion excitations given by the set $K_n$. Fermion wave functions are defined by the Dirac equation in the given external gravitational field.

After the variational problem is solved one says that the state with the wave function $\zeta_n$ that has been found is occupied if $K_n \neq 0$ for the corresponding value of $n$. In equation (4.10) we need to sum up all such configurations with different arrays $K_n$. The calculated values of $E$ are to be substituted into the exponent in equation (4.10) while the values of $\Omega_n^E$ are given by $\Omega_n^E = \zeta_n^+ H \zeta_n$. This is the generalization of the Hartree–Fock approximation.

We came to the following result:

**Theorem 4.2.** The partition function for the system of Dirac fermions interacting with the gravitational field is given by equation (4.10). In the semiclassical approximation (for the energies $E \ll m_0$) the stationary phase approximation leads to the variational problem (4.15).

**Remark 4.3.** The variational problem (4.15) is gauge invariant, while the normalization of the wave function is not. We need

$$\int d^4x |E| \zeta^\dagger = T,$$  \hspace{1cm} (4.16)

where the spinor $\zeta$ and the time extent $T$ are defined in a synchronous reference frame. Here $T$ is a global characteristic of the space-time, $|E|d^4x$ is the invariant 4-volume, while $\zeta^\dagger = \zeta E_0^\gamma \gamma^\alpha \zeta$ is the time-component of the 4-vector.

4.2. Gravitational one-polaron problem

In order to investigate one-polaron states, we consider the two-point Green function

$$G(t_2 - t_1) = \frac{1}{Z} \int \bar{\psi} \psi \ dE \ e^{i \int d^4x [E(\bar{\psi}[i\partial - m] \psi - \frac{1}{m_0^2} \int d^4x R(|E|)] \bar{\psi}^+ (t_1, x) \psi (t_2, x) d^3x |E (t_2, x)|}. \hspace{1cm} (4.17)$$

Here again the system is considered in a synchronous reference frame. It is implied that the corresponding terms are present in the integration measure over $E$. With all the notations introduced above, we can represent $G$ as follows:

$$G(t_2 - t_1) = \frac{1}{Z} \sum_q \sum_{\{K_n\} = 0.1} \int dE \ e^{-i m_0^2 \int d^4x |R(-iT \sum_q |K_n - 1/2 |\Omega_n^E - \Omega_n^E(-i(t-t_1))\zeta_n^+ F^E (t_1, t_2)}.$$
FE approximation. When varying the terms in the exponent, it is necessary to take into account large values of $m_P$ allow us to calculate integrals over $E$ in the stationary phase approximation. When varying the terms in the exponent, it is necessary to take into account $F^E(t_1, t_2)$ which disturbs the effective action at $t = t_1, t_2$. However, if $t_2 - t_1 = T$, then, as in the previous section, we come to the following simplification:

$$
G(T) = \frac{1}{Z} \sum_q \sum_{(K_0) = 0, 1} \int dE \exp \left( -i m_P^2 \int d^4x |E - i T \sum_{n \neq q} [K_n - 1/2] G_{E_n}^F - i T G_{q,E}^F \right).$$

(4.18)

The lemma follows:

**Lemma 4.4.** In the quenched approximation, at energies much less than $m_P$, the one-polaron problem is reduced to the variational problem

$$
0 = \delta \int d^4x \left\{ [i \partial - m] \xi - m_P^2 R \right\} |E|.

(4.19)

Here $\xi$ is the fermion wave function normalized according to remark 4.3.

4.3. Newtonian limit

In the nonrelativistic limit, the energy–momentum tensor for the Dirac field is given by $T^\mu\nu = \bar{\psi} \gamma^\mu \partial^\nu \psi \sim m \zeta^+ \partial^\mu \delta^\nu_0$. The gravitational field is considered in the linear approximation $g_{\mu\nu} = \eta_{\mu\nu} + f_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. In the gauge $\partial^\nu h_{\mu\nu} = 0$ (where $h_{\mu\nu} = f_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} f^E$) equation (4.19) takes the form

$$
0 = \delta \int d^4x \left\{ \bar{\psi} \left[ i \partial - m - \frac{1}{2} f_{\mu\nu} \gamma_\mu \partial_\nu \right] \psi - m_P^2 \left[ (\partial_\mu f^\nu_\gamma)^2 - \frac{1}{2} (\partial_\mu f^\gamma_\nu)^2 \right] \right\}.

(4.20)

Here the variation is performed with respect to the wave functions $\zeta$ and with respect to the graviton cloud $f$. As a result, the graviton cloud is formed in accordance with the (linearized) Einstein equations with $T^{\mu\nu} = \bar{\psi} \gamma^\mu \partial^\nu \psi \zeta$.

In the nonrelativistic limit (we neglect gravitational waves that are not caused by the given spinor field, see [53, section 99] on the Newtonian limit of general relativity and the definition of $\phi$):

$$
\begin{align*}
& f^0_0 = 2 \phi, \quad f^0_b = -2 \phi \delta^b_0, \quad a, b = 1, 2, 3; \\
& i f^{\mu\nu} \gamma_\mu \partial_\nu \psi \sim 2 \phi m \gamma^0 \psi.
\end{align*}
$$

And $f^a_0 = 0$, $f^a_b = 0$ for $a = 1, 2, 3$. We arrive at the following system of equations:

$$
\begin{align*}
\Box \phi + \frac{m}{4m_P^2} \bar{\zeta} \gamma^0 \zeta &= 0, \\
[i \partial - m - m \gamma^0] \phi &= 0.
\end{align*}

(4.21)

Next, we require that $\phi$ does not depend on time and denote $m \phi = e \psi, \ e = \frac{m}{m_P}$. As a result we arrive at

$$
\begin{align*}
[i \partial - \mathbf{a} \cdot \nabla \zeta + m \beta \zeta + e \psi(x,t) \zeta(x,t)] \\
\Delta \psi(x,t) = e \zeta^*(x,t) \zeta(x,t)
\end{align*}

(4.22)
with the normalization
\[ \int d^4x \bar{\zeta} \gamma^0 \zeta \sqrt{-g} = T. \] (4.23)

Here \( \zeta \) is spinor field in a synchronous reference frame, \( \gamma^0 = E_a^0 \gamma^a \) is the time-component of covariant gamma-matrices (also in a synchronous reference frame).

The normalization of the spinor field is the subject of careful investigation. In the reference frame defined by the harmonic gauge, both the spinor field and the gravitational field \( \phi \) are independent of time. However, in a general situation, this is not the case in a synchronous reference frame. We may represent \( \bar{\zeta} \gamma^0 \zeta \) in the original reference frame. Let us mention that the solitary wave for the Einstein–Dirac system and the Einstein–Dirac–Maxwell system; for the Dirac–Maxwell system with \( J^\mu = \bar{\zeta} \gamma^\mu \zeta \) is defined in the original reference frame. In this frame, in the weak coupling, we have \( \sqrt{-g} \sim 1 - 2\phi \) and \( J^\mu \frac{\partial \nu \zeta}{\partial x^\nu} \sim \zeta^+ \zeta \). (Recall that \( g^{\mu\nu} \approx \eta^{\mu\nu} - \eta^{\nu\mu} \), \( J^\mu \) indices are lowered and raised by \( \eta_{\mu\nu} \) and \( \eta^{\mu\nu} \). Then \( f^{00} = 2\phi, f^{ab} = +2\phi \delta^{ab} \).)

The latter follows from the Hamilton–Jacobi equation that defines a synchronous reference frame \( g^{\mu\nu} \frac{\partial [\zeta, x^\rho]}{\partial x^\nu} = 1 \). In the Newtonian approximation \( \phi \ll 1 \), and \( \frac{\partial x^\rho}{\partial x^\nu} \), \( \nu = 1, 2, 3 \) are of the same order as \( \phi \). Therefore, up to the terms linear in \( \phi \), we get \( g^{\mu\nu} [\partial [\zeta, x^\rho]]^2 = [E^\mu_0 \partial [\zeta, x^\rho]]^2 = 1 \).

Also \( J^\mu, \mu, \nu = 1, 2, 3 \) are of the same order as \( \phi \). Therefore, up to the terms linear in \( \phi \), we have \( J^\mu \frac{\partial \nu \zeta}{\partial x^\nu} = \partial ^\rho \frac{\partial [\zeta, x^\rho]}{\partial x^\nu} = \bar{\zeta} \gamma^0 \zeta E^\mu_0 \partial [\zeta, x^\rho] \sim \zeta^+ \zeta \).

That is why we come to the following normalization (valid in the original reference frame in the weak coupling):
\[ \int d^3x \zeta^+ \zeta \sqrt{-g} \approx \int d^3x \zeta^+ \zeta \left(1 - \frac{1}{m_p} \varphi \right) = 1. \] (4.24)

**Lemma 4.5.** In the Newtonian limit in the harmonic gauge, the gravitational polaron problem is reduced to the system of equations (3.15), (3.16), (4.24) with \( e = \frac{m}{2m_p} \).

5. Stability of solitary waves in the Dirac–Coulomb system

5.1. Existence of solitary waves

In this section, we substitute \( \zeta \) by \( \frac{1}{2} \zeta \) and \( \varphi \) by \( \frac{1}{2} \varphi \), so that \( e \) disappears from the system (2.2). As a result, instead of the normalization condition (3.17) (polarons in condensed matter systems) we have
\[ \int d^3x \zeta^+ \zeta = e^2. \] (5.1)

For gravitational polarons, we have the constraint
\[ \int d^3x \zeta^+ \zeta \left(1 - \frac{1}{em_p} \varphi \right) = e^2. \] (5.2)

We express \( \varphi = \Delta^{-1} |\zeta|^2 \), where \( \Delta^{-1} \) in \( \mathbb{R}^3 \) is the operator of convolution with \( \frac{1}{4\pi|x|} \), and write the Dirac–Coulomb system (2.2) as the following Dirac–Choquard equation:
\[ i\partial \xi = -i\alpha \cdot \nabla \xi + m \beta \xi + \xi \Delta^{-1} |\zeta|^2, \] (5.3)
where \( \zeta(x, t) \in \mathbb{C}^4, x \in \mathbb{R}^3 \). The solitary wave solutions \( \phi_\omega e^{-i\omega t} \) with \( \omega \lesssim m \) can be constructed by rescaling from the solutions to the nonrelativistic limit of the model. Such a method was employed in [54, 55] for the nonlinear Dirac equation and in [56–58] for the Einstein–Dirac system and the Einstein–Dirac–Maxwell system; for the Dirac–Maxwell system, this approach has been implemented in [59]. Let us mention that the solitary wave solutions to (5.3) with \( \omega \lesssim m \) correspond to the solitary wave solutions of the Dirac–Maxwell system with \( \omega \gtrsim -m \) when the magnetic field is neglected; such solitary waves
such a solution exists due to \[60–62\]. Pick a unit vector self-interaction: in the Dirac–Maxwell system, the self-interaction is repulsive for \(\omega \lesssim m\) and attractive for \(\omega \gtrsim -m\); in the Dirac–Choquard equation (5.3), it is the opposite. The profile of the solitary wave \(\xi(x, t) = \phi_\omega(x) e^{-i\omega t}\) satisfies
\[
\omega \phi_\omega = -i \sigma \cdot \nabla \phi_\omega + m \beta \phi_\omega + \phi_\omega \Delta^{-1} |\phi_\omega|^2.
\] (5.4)

Let \(\phi_\omega(x) = \frac{\phi_\omega(x, \omega)}{\phi_\omega(x, \omega)}\), with \(\phi_\omega, \phi_\omega \in \mathbb{C}^2\) the ‘electron’ and ‘positron’ components. In terms of \(\phi_\omega\) and \(\phi_\omega\), (5.4) is written as
\[
\omega \phi_\omega = -i \sigma \cdot \nabla \phi_\omega + m \phi_\omega + \phi_\omega \Delta^{-1} (|\phi_\omega|^2 + |\phi_\omega|^2),
\]
\[
\omega \phi_\omega = -i \sigma \cdot \nabla \phi_\omega - m \phi_\omega + \phi_\omega \Delta^{-1} (|\phi_\omega|^2 + |\phi_\omega|^2),
\] (5.5)

where \(\sigma \cdot \nabla = \sum_{j=1}^3 \sigma_j \partial_j\), with \(\sigma_j\) the Pauli matrices. Let \(\epsilon > 0\) be such that \(\epsilon^2 = m^2 - \omega^2\).

We introduce functions \(\Phi_\epsilon(y, \epsilon), \Phi_\epsilon(y, \epsilon) \in \mathbb{C}^2\) by the relations
\[
\phi_\sigma(x, \omega) = \epsilon^2 \Phi_\epsilon(\epsilon x, \epsilon), \quad \phi_\omega(x, \omega) = \epsilon^3 \Phi_\epsilon(\epsilon x, \epsilon).
\]

Let \(\nabla_y, \Delta_y\) be the gradient and the Laplacian with respect to the coordinates \(y = \epsilon x\), so that \(\nabla_x = \epsilon \nabla_y, \Delta_x = \epsilon^2 \Delta_y\). Then equations (5.5) take the form
\[
\frac{- \Phi_\epsilon}{m + \omega} = -i \sigma \cdot \nabla_y \Phi_\epsilon + \Phi_\epsilon \Delta_y^{-1} (|\Phi_\epsilon|^2 + \epsilon^2 |\Phi_\epsilon|^2),
\] (5.6)
\[
(m + \omega) \Phi_\epsilon = -i \sigma \cdot \nabla_y \Phi_\epsilon + \epsilon^2 \Phi_\epsilon \Delta_y^{-1} (|\Phi_\epsilon|^2 + \epsilon^2 |\Phi_\epsilon|^2).
\] (5.7)

Let \(u \in H^\infty(\mathbb{R}^3, \mathbb{R})\) be a spherically symmetric strictly positive smooth solution to the Choquard equation,
\[
-\frac{1}{2m} u = -\frac{1}{2m} \Delta u - u \Delta^{-1} u^2;
\] (5.8)
such a solution exists due to [60–62]. Pick a unit vector \(n \in \mathbb{C}^2\). Then
\[
\hat{\Phi}_\epsilon = n u \in H^\infty(\mathbb{R}^3, \mathbb{C}^4),
\]
\[
\hat{\Phi}_\epsilon = -\frac{1}{2m} \sigma \cdot \nabla y \hat{\Phi}_\epsilon \in H^\infty(\mathbb{R}^3, \mathbb{C}^4)
\]
is a solution to (5.6), (5.7) corresponding to \(\epsilon = 0\). By [59], the perturbation theory allows to construct solutions to (5.4) with \(\omega \in (\omega_0, m)\), with some \(\omega_0 < m\), such that
\[
\phi_\omega(x) = \begin{bmatrix} \phi_\omega(x, \omega) \\ \phi_\omega(x, \omega) \end{bmatrix} = \begin{bmatrix} \epsilon^2 \hat{\Phi}_\epsilon(\epsilon x) + o(\epsilon^2) \\ \epsilon^3 \hat{\Phi}_\epsilon(\epsilon x) + o(\epsilon^3) \end{bmatrix},
\] (5.9)
where \(\omega\) and \(\epsilon\) are related by \(\omega = \sqrt{m^2 - \epsilon^2}\).

**Remark 5.1.** Let us mention that among the solitary waves considered above, only those with the discrete values \(\omega_{n,\epsilon} \in (\omega_0, m)\) satisfy the constraint (5.1) (or the constraint (5.2)). These values are parametrized by the number of nodes \(n\) of the corresponding solution to the Choquard equation (5.8), the quantum number \(\kappa = \pm 1\), and the value of charge \(\epsilon\) that enters the constraint (5.1) (or (5.2)). In the context of the Dirac–Maxwell system, this pattern is described in detail in [4].
5.2. Linear stability of solitary waves

We assume that \( \omega_0 < m \) is such that for \( \omega \in (\omega_0, m) \) there are solitary wave solutions \( \phi_\omega(x) e^{-i\omega t} \) to (5.3). Taking the ansatz \( \xi(x, t) = (\phi_\omega(x) + \rho(x, t)) e^{-i\omega t} \), we derive the linearization at the solitary wave \( \phi_\omega(x) e^{-i\omega t} \):

\[
i\dot{\rho} = (D_m - \omega + \Delta^{-1}|\phi_\omega|^2)\rho + \Delta^{-1}(\rho^*\phi_\omega + \phi_\omega^*\rho)\phi_\omega,
\]

where \( D_m = -i\alpha \cdot \nabla + m\beta \). We are looking for the eigenvalues of the linearization operator in the right-hand side. That is, we substitute \( \rho(x, t) = \xi(x) e^{it} \), with \( \xi \in L^2(\mathbb{R}^3, \mathbb{C}^4) \), \( \xi \neq 0 \), getting

\[
i\lambda \xi = (D_m - \omega + \Delta^{-1}|\phi_\omega|^2)\xi + \phi_\omega\Delta^{-1}(\xi^*\phi_\omega + \phi_\omega^*\xi),
\]

and we would like to know possible values of \( \lambda \). If there is \( \text{Re} \lambda > 0 \) corresponding to \( \xi \neq 0 \), then the linearization at a solitary wave is linearly unstable, and we would expect that the solitary wave is (‘dynamically’) unstable under perturbations of the initial data.

**Theorem 5.2.** There exists \( \omega_1 \in (\omega_0, m) \) such that the ‘no-node’ solitary waves with \( \omega \in (\omega_1, m) \) are linearly stable, so that there are no solutions \( \lambda \in \mathbb{C}, \xi \in L^2(\mathbb{R}^3, \mathbb{C}^4) \) to (5.10) with \( \text{Re} \lambda \neq 0 \) and \( \xi(x) \neq 0 \).

**Remark 5.3.** By the ‘no-node’ solitary waves we mean the solutions (5.9) constructed from the strictly positive solution to the Choquard equation (5.8).

Let us mention that such ‘no-node’ solutions that satisfy the constraint (5.1) (or the constraint (5.2)) only exist if the value of \( e^2 \) is sufficiently small. That is why the above results on the existence and stability may be reformulated as follows:

**Theorem 5.4.** There is charge \( \frac{1}{2} > 0 \) such that the Dirac–Coulomb system (2.2) with \( e^2 \in (0, \text{charge}^2) \) has ‘no-node’ solitary wave solutions \( \xi(x, t) = \phi(x) e^{-i\omega t} \) which satisfy the constraint (5.1) (or (5.2)) and are linearly stable.

**Remark 5.5.** According to the scaling in the ansatz (5.9), one has \( \int |\phi_\omega|^2 \, d^3x \sim \epsilon \sim (m - \omega)^{1/2} \), hence \( e^2 \) and \( \omega \) in theorem 5.4 are related by

\[
e^2 \sim (m - \omega)^{1/2}, \quad \omega \lesssim m.
\]

We point out that (5.10) is \( \mathbb{R} \)-linear but not \( \mathbb{C} \)-linear, because of the presence of \( \xi^* \). Let us rewrite (5.10) in the C-linear form. For this, we introduce the following notations:

\[
\xi = \begin{bmatrix} \text{Re} \xi \\
\text{Im} \xi \end{bmatrix}, \quad \Phi_\omega = \begin{bmatrix} \text{Re} \phi_\omega \\
\text{Im} \phi_\omega \end{bmatrix}, \quad J = \begin{bmatrix} 0 & L_1 \\
-L_4 & 0 \end{bmatrix},
\]

\[
\alpha = \begin{bmatrix} \text{Re} \alpha \\
\text{Im} \alpha \end{bmatrix}, \quad \beta = \begin{bmatrix} \text{Re} \beta & -\text{Im} \beta \\
\text{Im} \beta & \text{Re} \beta \end{bmatrix}.
\]

Then (5.10) can be written as

\[
\lambda \xi = J L(\omega) \xi,
\]

where

\[
L(\omega) = (D_m - \omega + \Delta^{-1}|\Phi_\omega|^2) + 2 \Phi_\omega \Delta^{-1}(\Phi_\omega^* \xi),
\]

\[
D_m = \sum_{j=1}^3 J \alpha_j \partial_j + \beta \cdot m.
\]

The operators \( D_m \) and \( L(\omega) \) considered on the domain \( H^1(\mathbb{R}^3, \mathbb{C}^4) \) are self-adjoint.

Theorem 5.2 is the immediate consequence of the following lemma.
Lemma 5.6. Let \( \omega_k \in (0, m) \), \( k \in \mathbb{N} \); \( \omega_k \to m \) as \( k \to \infty \). Then there is no sequence \( \lambda_k \in \sigma_p(JL(\omega_k)) \) with \( \text{Re} \lambda_k \neq 0 \).

Proof. The lemma is proved in several steps, which we now sketch; more details will appear in [22]. First, one shows that if there were a sequence of eigenvalues \( \lambda_k \in \sigma_p(JL(\omega_k)) \) such that \( \lim_{k \to \infty} \lambda_k \) existed, then we would have

\[
\lim_{k \to \infty} \lambda_k \subset [0, \pm 2mi].
\]

The proof of this statement follows from the fact that in the limit \( \omega \to m \), as \( \|\phi_\omega\|_{L^\infty} \to 0 \)
(cf (5.9)), the operator \( JL(\omega) \) turns into \( J(D_m - m) \). According to [63], there is the limiting absorption principle for the free Dirac operator \( D_m \); its resolvent, \( (D_m - z)^{-1} \), is uniformly bounded from \( L^2_s(\mathbb{R}^3, \mathbb{C}^4) \) to \( L^2_{s,k}(\mathbb{R}^3, \mathbb{C}^4) \), for any \( s > 1/2 \) and uniformly for \( |\text{Re}z| > m + \delta \) (for any fixed \( \delta > 0 \)) and \( \Im z \neq 0 \). (Recall that \( L^2_s(\mathbb{R}^n) = \{u \in L^2_{loc}(\mathbb{R}^n); \|u\|_{L^2_s}^2 := \int_{\mathbb{R}^n} (1 + x^2)^s |u(x)|^2 \, dx < \infty \}. \) This implies that the resolvent of \( JL(\omega) \) is bounded in these weighted spaces outside the union of \( i\mathbb{R} \) with open neighborhoods of ‘thresholds’ \( \lambda = 0 \) and \( \lambda = \pm 2mi \), as long as \( \omega \) is sufficiently close to \( m \). In turn, this implies that as \( \omega_k \to m \), the eigenvalues \( \lambda_k \) cannot accumulate but to these three threshold points.

Further, the eigenvalues \( \lambda_k \) with \( \text{Re} \lambda_k \neq 0 \) cannot accumulate at \( \pm 2mi \). This follows from the fact that if \( \lambda_k \to \lambda_0 \in i\mathbb{R}\backslash 0 \) as \( \omega_k \to \omega_0 \in i\mathbb{R}\backslash 0 \), then \( \lambda_0 \) itself has to belong to the point spectrum of \( JL(\omega_0) \) (corresponds to the \( L^2 \) eigenfunction); this result is again based on the limiting absorption principle. At the same time, there can be no \( L^2 \) eigenfunctions of a constant coefficient operator \( J(D_m - m) \).

Finally, one has to study the most involved case \( \lambda_k \to 0 \). One first proves that if \( \lambda_k \to 0 \) and \( \text{Re} \lambda_k \neq 0 \) as \( \omega_k \to m \), then necessarily \( \lambda_k = O(m - \omega_k) \). Then one studies the rescaled equation. The conclusion is that the families of eigenvalues for the linearization of the Dirac–Choquard equation, \( \lambda_k \in \sigma_p(JL(\omega_k)) \), are deformations of families of eigenvalues for the linearization of the Choquard equation, which is a nonrelativistic limit of the Dirac–Choquard equation; in the context of the nonlinear Dirac equation, this has been rigorously done in [22]. The presence of eigenvalues with nonzero real part in the linearization of Choquard equation is controlled by the Vakhitov–Kolokolov stability criterion [23]; for the linearization at no-node solutions, this criterion prohibits existence of such eigenvalues. This finishes the proof of the lemma. \( \square \)

We reproduce the Vakhitov–Kolokolov stability criterion [23] in application to the Choquard equation in the appendix (see lemma A.1 below).

6. Conclusions

In the present paper we considered solitary waves in the system of Dirac fermions interacting via the Coulomb attraction from both physics and mathematics viewpoints. On the physical side the solitary waves describe polarons that may appear in two situations. The first one corresponds to certain condensed matter systems in which massive Dirac fermions interact with optical phonons. The polarons in the system of true relativistic Dirac fermions interacting with gravity in the Newtonian limit are also described by the above-mentioned solitary waves.

A possible application of our construction for the gravitational case is related to the situation when the gravitational interaction between elementary particles is strong enough. The corresponding problem may only appear in the models that rely on quantum gravity. It is worth mentioning here that the role of the gravitational interaction may be played by the
emergent gravity [43, 64] with the scale much lower than the Planck mass. Such models may be relevant for the description of TeV-scale physics [65].

On the mathematical side we develop analytical methods for the investigation of solitary waves. These methods are based on the observation that these localized solutions are obtained as a bifurcation from the solitary waves of the Choquard equation. Based on this approach, we demonstrate that the no-node gap solitons for sufficiently small values of \( e \) are linearly stable.

It is worth mentioning that the solitary waves similar to those considered in the present paper may also exist in 2D systems like the boundary of topological insulators or graphene. This may occur if the interaction between the electrons of the 2D system with the bulk phonons (with the substrate phonons in the case of graphene) is strong enough. We postpone the consideration of the corresponding 2D solitary waves to future publications.

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Appendix. Vakhitov–Kolokolov criterion for the Choquard equation

We consider the Choquard equation,

\[
i\partial_t \zeta = -\frac{1}{2m} \Delta \zeta + m\xi + \zeta \Delta^{-1} |\zeta|^2,
\]

where \( \zeta(x, t) \in \mathbb{C}^3 \) and \( x \in \mathbb{R}^3 \). We are interested in the solitary wave solutions \( \zeta(x, t) = u_\omega(x) e^{-i\omega t}, \omega \in \mathbb{R}; u_\omega \) satisfies

\[
- (m - \omega) u_\omega = -\frac{1}{2m} \Delta u_\omega + u_\omega \Delta^{-1} u_\omega^2.
\]

Given a solution to (5.8),

\[
- \frac{1}{2m} u = -\frac{1}{2m} \Delta u + u \Delta^{-1} u^2,
\]

then, for any \( \omega < m \), the profiles

\[
u_\omega(x) = 2m(m - \omega) v(x\sqrt{m(m - \omega)})
\]

(A.3)
correspond to a family of solitary wave solutions to (A.2). Note that this scaling is the same as that of \( \phi_\epsilon \) in (5.9).

**Lemma A.1.** For \( \omega < m \), the no-node solitary wave solutions \( \zeta_\omega(x, t) = u_\omega(x) e^{-i\omega t} \) to the Choquard system are linearly stable.

The linear stability of no-node solitary waves of the Choquard equation follows from the Vakhitov–Kolokolov stability criterion [23] which is applicable to systems of the Schrödinger type. It also follows from [62] (where the orbital stability of these solitary waves is proved). Let us sketch the argument. First, we notice that, by (A.3), the charge of the solitary wave \( u_\omega(x) e^{-i\omega t} \) is given by

\[
Q(\phi_\omega) = \int_{\mathbb{R}^3} |\phi_\omega(x)|^2 \, d^3x \sim (m - \omega)^{1/2}, \quad \omega \lesssim m.
\]
therefore,
\[ \frac{dQ(u_\omega)}{d\omega} < 0, \quad \omega < m. \tag{A.4} \]

We consider the solution to the Choquard equation in the form of a perturbed solitary wave,
\[ \zeta(x, t) = (u_\omega(x) + R(x, t) + iS(x, t)) e^{-i\omega t}, \]
with \( R, S \) real-valued. The linearized equation on \( R, S \) is given by
\[ \frac{\partial [R]}{S} = j(\omega) \begin{bmatrix} R \\ S \end{bmatrix}, \tag{A.5} \]
where
\[ j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad I(\omega) = \begin{bmatrix} L_1(\omega) & 0 \\ 0 & L_0(\omega) \end{bmatrix}. \tag{A.6} \]
and
\[ \begin{align*}
L_0(\omega) &= -\frac{1}{2m} \Delta + m - \omega + \Delta^{-1} \mu_0^2, \\
L_1(\omega) &= L_0(\omega) + 2\Delta^{-1}(\mu_0)u_\omega.
\end{align*} \]
Both operators \( L_0(\omega) \) and \( L_1(\omega) \) are self-adjoint, with \( \sigma_{\text{ess}}(L_0(\omega)) = \sigma_{\text{ess}}(L_1(\omega)) = [m - \omega, +\infty) \). Clearly, \( L_0(\omega)u_\omega = 0 \), with \( 0 \in \sigma_L(\omega) \) an eigenvalue corresponding to a positive eigenfunction \( u_\omega \); it follows that 0 is a simple eigenvalue of \( L_0 \), with the rest of the spectrum separated from zero. Taking the derivatives of the equality \( L_0(\omega)u_\omega = 0 \) with respect to \( \omega \), we get:
\[ L_1(\omega)\partial_\omega u_\omega = 0, \quad L_1(\omega)\partial_\omega u_\omega = u_\omega. \tag{A.7} \]

The first relation shows that \( \lambda_1 = 0 \) is the point eigenvalue of \( L_1(\omega) \), and since \( \partial_\omega u_\omega \) vanishes on a hyperplane \( x_j = 0 \), there is one negative eigenvalue \( \lambda_0 < 0 \) of \( L_1(\omega) \).

Now we may determine the spectrum of \( j(\omega) \begin{bmatrix} R \\ S \end{bmatrix} \). We closely follow [23].
If \( \begin{bmatrix} R \\ S \end{bmatrix} \) is an eigenfunction corresponding to the eigenvalue \( \lambda \in \mathbb{C} \), then \( -\lambda^2 R = L_0 L_1 R \). If \( \lambda \neq 0 \), then one concludes that \( R \) is orthogonal to \( \ker L_0 = \text{Span}(u_\omega) \), hence we can apply \( L_0^{-1} \); taking then the inner product with \( u_\omega \), we have:
\[ -\lambda^2 \begin{bmatrix} R \\ L_0^{-1} R \end{bmatrix} = \begin{bmatrix} R \\ L_1 R \end{bmatrix}. \tag{A.8} \]

With \( L_0, L_1 \) being self-adjoint, this relation implies that \( \lambda^2 \in \mathbb{R} \). Since \( L_0(\omega) \) is non-negative and \( R \perp \ker L_0 \), one has \( \langle u_\omega, L_0^{-1} u_\omega \rangle > 0 \). The solution to
\[ \mu := \inf \{ \langle R, L_1(\omega) R \rangle; \| R \| = 1, \langle u_\omega, R \rangle = 0 \} \]
satisfies \( L_1(\omega)R = \mu R + \nu u_\omega \), where \( \mu, \nu \in \mathbb{R} \) play the role of the Lagrange multipliers. Due to the condition \( \langle u_\omega, R \rangle = 0 \), \( \mu \) delivers the zero value to the function
\[ f(z) = \langle u_\omega, (L_1(\omega) - z)^{-1} u_\omega \rangle, \quad z \in \rho(L_1(\omega)), \]
with \( \rho(L_1) \) denoting the resolvent set of \( L_1 \). Since \( \ker L_1 \) is spanned by \( \partial_\omega u_\omega \) and therefore is orthogonal to \( u_\omega \), we can extend \( f(z) \) to \( z \in (\lambda_0, \lambda_2) \), where \( \lambda_0 = \inf \sigma(L_1(\omega)) < 0 \) and \( \lambda_2 \) is the smallest positive eigenvalue of \( L_1 \) in the interval \( (0, m - \omega) \) (or the edge of the essential spectrum, \( \lambda = m - \omega \)). We need to know whether \( \mu \) is positive or negative. Since \( f'(z) > 0 \), the sign of \( \mu \) is opposite to
\[ f(0) = \langle u_\omega, L_1(\omega)^{-1} u_\omega \rangle = \langle u_\omega, \partial_\omega u_\omega \rangle = \frac{\partial_\omega Q(u_\omega)}{2}. \]
In the second equality, we used the second relation from (A.7). From (A.4), we conclude that $f(0) < 0$; thus, $\mu > 0$. By (A.8), $\lambda^2 \leq 0$, leading to $\sigma(\mathbf{J}) \subset \mathbb{R}$.

This shows that there are no families of eigenvalues of $\mathbf{J}(\omega)$ with nonzero real part bifurcating from $\lambda = 0$ at $\omega = m$. Since bifurcations of eigenvalues from $\lambda = 0$ for the linearizations of the Choquard equation and the Dirac–Choquard equation (5.3) (which is equivalent to the Dirac–Coulomb system) have the same asymptotics as $\omega \to m$, we conclude that neither are there families of eigenvalues of $\mathbf{J L}(\omega)$ with nonzero real part.

**Remark A.2.** The rigorous proof of linear stability of solitary wave solutions to the Dirac–Choquard equation requires a more detailed analysis of the spectrum of $\mathbf{J L}(\omega)$. Namely, one needs to know whether there are resonances or embedded eigenvalues. Theoretically, resonances or embedded eigenvalues of higher algebraic multiplicity could bifurcate off the imaginary axis into the complex domain, yielding a family of eigenvalues $\lambda_k \in \sigma_p(\mathbf{J L}(\omega_k))$ with $\omega_k \to m$, $\lambda_k = O(m - \omega_k)$, $\text{Re} \lambda_k \neq 0$ (and resulting in the instability of Dirac–Choquard solitary waves), although we expect that generically this does not happen.

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