Factorization of Twist-Four Gluon Operator Contributions

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We consider diagrams with up to four $t$-channel gluons in order to specify gluonic twist-four contributions to deep inelastic structure functions. This enables us to extend the method developed by R.K. Ellis, W. Furmanski, and R. Petronzio (EFP) to the gluonic case. The method is based on low-order Feynman diagrams in combination with a dimensional analysis. It results in explicitly gauge invariant expressions for the factorization of twist-four gluon-operator matrix elements and the corresponding coefficient functions.

Supported by the TMR Network “QCD and Deep Structure of Elementary Particles”. One of us (C.B.) is supported by Graduiertenkolleg Theoretische Elementarteilchenphysik.

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1 Introduction

The currently available experimental data on deep inelastic structure functions covers a large kinematical regime with high precision measurements. This provides an interesting challenge for theoretical physics when it comes to describing this data in the low-$Q^2$ domain. At sufficiently high $Q^2$, a perturbative approach is justified and within the classical DGLAP approach next-to-leading-order (NLO) fits are still the state-of-the-art. Although some authors [2] have been using NLO DGLAP down to rather low values of $Q^2$, it becomes more and more obvious [3] that such an approach is not sufficient for a precise description of deep inelastic structure function data. Missing are not primarily the sub-leading terms in powers of $\alpha_s$, but corrections which are proportional to the reciprocal value of the photon virtuality $Q^2$, viz. higher-twist terms. This statement gets some support from the experimental observation that there is a rather large number of events in which a longitudinally polarized photon scatters off a proton diffractively and results in a vector meson. Since this process has been shown to be calculable perturbatively and to belong to higher-twist [4–7], one should expect a sizeable amount of higher-twist contributions to deep inelastic structure functions in the same kinematical regime. However, it has been shown recently that the situation is more complicated [8]: one gets contributions with different signs which might lead to strong cancellations and an estimate of the amount of inclusive higher-twist contributions based exclusively on diffractive vector meson production is not sufficiently reliable. Therefore a systematic study of higher-twist is not only worthwhile but necessary. This is even more true, since at least some of the twist-four contributions, after removing the explicit $1/Q^2$-dependence, are known to evolve faster than the leading-twist contributions, and to increase faster with decreasing $x_B$.

Most computations based on twist-four operators for the inclusive process have been published in the early 1980s. S. Gottlieb [9] classified all twist-four operators and computed some of their anomalous dimensions. The anomalous dimensions of four-quark operators and of flavor octet, three-body operators were obtained by M. Okawa [10], and the Wilson coefficient functions of four-quark operators by S.P. Luttrell et.al. [11]. Calculations based on the complete set of twist-four operators introduced in [9] are difficult since this set is over-determined: different operators are related by the equations of motion. This deficiency was remedied by R.L. Jaffe and M. Soldate [12, 13] who introduced a special basis, which they called canonical. They showed that this basis is adequate for the necessary twist-four perturbative calculations and calculated the operators’ tree-level coefficient functions.

These approaches are all based on the operator product expansion (OPE) and make use of the fact that the hadronic tensor $W^{\alpha\beta}$ factorizes into two parts, viz. coefficient functions $C_i$ and parton correlation functions $f_i$ which depend on a set of light-cone momenta $\{x\}$:

$$W^{\alpha\beta} = \sum_{\tau = 2, \text{even}}^{\tau_{\text{even}}} \left( \frac{\Lambda^2}{Q^2} \right)^{\frac{-\tau}{2}} \sum_i \int d\{x\} C_i^{\alpha\beta}(\{x\}) f_i(\{x\}), \quad (1.1)$$

where $\tau$ corresponds to the twist of the quantities and we have dropped details like the dependence on logarithms of $Q^2$. On the other hand, R.K. Ellis, W. Furmanski and R. Petronzio (EFP) [1] followed a more intuitive approach and computed $1/Q^2$-corrections to the conventional leading-twist Born diagram starting from leading-order Feynman diagrams. Using special techniques to classify the spinor, color and momentum structure, the authors could perform the factorization of
quark-coefficient functions and operators and reduced the number of contributing terms with the help of equations of motion.

The computations in [1] can coarsely be divided into two steps. In the first step the authors considered the relevant diagrams with four $t$-channel partons at the Born level and performed the factorization of the coefficient functions and matrix elements of the relevant operators. In detail, these operators are (we are ignoring terms which involve a matrix $\gamma_5$)\footnote{Cf. Eqs. (2.45)-(2.50) and (5.23)-(5.24) in [1]. The calculations are performed in the axial gauge with the gauge vector $n_\mu$. Since we are interested in the general structure of the results, we postpone the definition of various quantities until later sections.}

\begin{align*}
A^\rho(\lambda) &= \frac{1}{4} \langle p | T \{ \bar{\psi}(0) \gamma^\rho \psi(\lambda n) \} | p \rangle, & (1.2) \\
B^{\rho\mu}(\eta, \lambda) &= \frac{1}{4} \langle p | T \{ \bar{\psi}(0) \gamma^\rho D^\mu(\eta n) \psi(\lambda n) \} | p \rangle, & (1.3) \\
C^{\rho\mu\nu}(\omega, \eta, \lambda) &= \frac{1}{4} \langle p | T \{ \bar{\psi}(0) \gamma^\rho D^\mu(\omega n) D^\nu(\eta n) \psi(\lambda n) \} | p \rangle & (1.4)
\end{align*}

and

\begin{align*}
Q(\omega, \eta, \lambda) &= \frac{1}{16} g^2 \langle p | T \{ [\bar{\psi}(0) \gamma^a \psi(\omega n)] [\bar{\psi}(\eta n) \gamma^a \psi(\lambda n)] \} | p \rangle. & (1.5)
\end{align*}

In the second step, EFP used equations of motion in order to express the contributions of the two-quark operators (1.2)-(1.4) in terms of linearly independent operators. For this they studied both the longitudinal basis (which corresponds to the canonical basis in [12]) and introduced a new one which they called the transverse basis. While within the longitudinal basis the expressions turned out to be rather complicated, in the transverse basis the twist-four contributions of the two-quark operator to the transverse and longitudinal structure functions $F_T$ and $F_L$ can be expressed in terms of

\begin{align*}
\Lambda^2 T_1(x) &= \frac{1}{4} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle p | T \{ \bar{\psi}(0) \gamma_\mu \gamma_\nu D^\mu(0) D^\nu(\lambda n) \psi(\lambda n) \} | p \rangle, & (1.6) \\
\Lambda^2 T_2(x_2, x_1) &= \frac{1}{4} \int \frac{d\lambda}{2\pi} \frac{d\eta}{2\pi} e^{i(\eta x_2 - \eta x_1)} e^{i\lambda x_1} \langle p | T \{ \bar{\psi}(0) \gamma_\mu \gamma_\nu D^\mu(\eta n) D^\nu(\eta n) \psi(\lambda n) \} | p \rangle, & (1.7)
\end{align*}

such that the twist-four contributions to $F_T$ and $F_L$ are (we are neglecting the proton mass)

\begin{align*}
F_T^{\tau=4} &= \frac{\Lambda^2}{Q^2} \left[ 4 T_1(x_B) - x_B \int dx_1 dx_2 \frac{\delta(x_B - x_2) - \delta(x_B - x_1)}{x_2 - x_1} T_2(x_2, x_1) \right], & (1.8) \\
F_L^{\tau=4} &= 4 \frac{\Lambda^2}{Q^2} T_1(x_B), & (1.9)
\end{align*}

plus terms involving the four-quark operator (1.5).

EFP’s results have been used as a starting point for many other studies. For example, introducing a quantity named special propagator, J. Qiu\footnote{Cf. Eqs. (2.45)-(2.50) and (5.23)-(5.24) in [1]. The calculations are performed in the axial gauge with the gauge vector $n_\mu$. Since we are interested in the general structure of the results, we postpone the definition of various quantities until later sections.} could reproduce the results obtained in [1] in a very
A.P. Bukhvostov et al. define a special class of operators which they call quasi-partonic, formulate the evolution equations for these operators and compute the integral kernels for all possible $2 \rightarrow 2$ transitions in the one-loop approximation.

All these computations of twist-four quantities are based on quark-operators. In the small-$x_B$ domain, on the other hand, we know that in the leading-twist case the gluon-operator is dominating since it leads to the maximum power in $1/x_B$. For higher-twist, one expects that in the small-$x_B$ domain it is, again, the gluon-operators which lead to the main contributions. A systematic analysis of twist-four gluon-operators is missing. Such an analysis could, again, be based on the OPE. For twist-four one has to take into account contributions of two-gluon operators,

$$\text{tr} \, F^{\mu_1\alpha} D^{\mu_2} \ldots D^{\mu_{n-1}} F^{\mu_n}_{\alpha} g_{\mu_i \mu_j} + \text{perm.},$$

as well as of four-gluon operators,

$$\text{tr} \, F^{\mu_1\alpha} \ldots F^{\mu_{i}\alpha} \ldots F^{\mu_{j}\beta} \ldots F^{\mu_n}_{\beta} ,$$

$$\text{tr} \, F^{\mu_1\alpha} \ldots F^{\mu_{i}\beta} \ldots F^{\mu_j}_{\alpha} \ldots F^{\mu_n}_{\beta} ,$$

$$\text{tr} \, F^{\mu_1\alpha} \ldots F^{\mu_{i}\beta} \ldots F^{\mu_{j}}_{\beta} \ldots F^{\mu_n}_{\alpha} .$$

Here the dots denote products of covariant derivatives. In a future analysis one will have to compute the corresponding coefficient functions and the anomalous dimensions. This is a demanding task since all these twist-four gluon-operators will mix under renormalization.

Alternatively, one can make use of the more intuitive approach of EFP in order to find expressions for the contributions of twist-four gluon-operators to inclusive DIS structure functions. This is what we want to discuss in this article. We base our computations on the diagrams in Fig. and consider amplitudes with two, three and four $t$-channel gluons. The method we are going to describe will lead to the factorization of gluonic coefficient functions and matrix elements of gluon-operators between proton states. For our analysis we are neglecting the target-mass corrections and we do not start from the classical way of defining twist as the difference of canonical dimension and spin; instead we will search for $1/Q^2$ corrections to the familiar leading-twist result. We will use a dimensional analysis to show that diagrams with more than four $t$-channel gluons lead to contributions of, at least, twist-six. We are working in the axial gauge, and we can show that, within this gauge, it is sufficient to restrict oneself to the diagrams in Fig. In this paper we do not compute explicit expressions for the upper quark-loop parts in the diagrams of Fig. Instead we derive some general expressions for the corresponding amplitudes and make use only of the color-, Lorentz- and momentum structure of these quark-loops.

In comparison to quarkonic operators considered by EFP, the case of gluonic operators is slightly more complicated, and we will carry out only the very first steps of the corresponding analysis. Since the photon couples to the four-gluon state only through the quark loop, already the “Born” approximation will contain contributions proportional to $\log Q^2$ (i.e., UV-divergences). In this first part of the analysis, we will ignore logarithms and we will not address the computation of the quark loops. The emphasis of the present paper lies on the analysis of the color and Lorentz structure and on the factorization of the quark loop and the proton matrix elements. We will not yet consider the moments of the structure functions and also the definition of an operator basis will be postponed to a future paper.

For our discussion it is useful to clarify the notion of leading-order (LO) and next-to-leading order. The diagrams in Figs. and are suppressed by factors $g$ and $g^2$, resp., compared to the diagram.
in Fig. 1a. Within our derivations we will factor these constants out and absorb them within the expressions for the lower blobs. This way we can treat all three quark-loop expressions as if they were of the same order in $\alpha_s$. On the other hand, computing these quark-loop expressions one will get terms proportional to logarithms in $Q^2$ and terms which are just constants. Terms which are proportional to logarithms correspond to LO quark-coefficient functions convoluted by Altarelli-Parisi splitting functions, while we denote the constants as NLO gluon-coefficient functions. For this first analysis we restrict ourselves to quark-loop diagrams which are one-particle irreducible (1PI). The three- and four-gluon vertices in Fig. 2 have, therefore, to be considered as if they were absorbed by the lower blobs and are part of the evolution.

For our computations it is convenient to use the axial gauge, i.e. we are introducing a light-like vector $n^\mu$ such that

$$n_\mu A_\mu^a = 0. \quad (1.12)$$

We are using the same vector $n$ within the decomposition of momentum vectors into Sudakov variables:

$$k_i^\mu = x_i p^\mu + \alpha_i n^\mu + k_i^\mu, \quad p^2 = n^2 = p \cdot k_i,T = n \cdot k_i,T = 0, \quad p \cdot n = 1. \quad (1.13)$$

Here $p$ represents the momentum of the target hadron, which we assume to be massless throughout this paper. For the color factors we have chosen the following normalization of the Gell-Mann matrices $t^a$:

$$\text{tr} t^a t^b = \frac{1}{2} \delta^{ab}, \quad (1.14)$$

and we set the number of colors $N_c = 3$.

The outline of this paper is as follows. In section 2 we define our basic quantities. In particular we distinguish between the upper quark-loop parts in Fig. 1, $R$, and the lower blobs, $\Gamma$. The expressions for the quark-loops will then be expanded about the longitudinal components of the gluon-momenta they depend on. For the derivatives which occur in these expansions, we compute some Ward identities in section 3. In section 4 we show how, in the axial gauge, the $A_\mu^a$-operators which occur in the matrix elements for the lower blobs in Fig. 1 can be expressed in terms of $F_{\mu\nu}^a$-operators. Section 5 is devoted to the dimensional analysis of the quantities under consideration. This analysis leads to the justification that we can restrict ourselves to the diagrams in Fig. 1 for a consistent analysis of gluonic twist-four contributions. Since the diagrams in Fig. 1 contain also leading-twist contributions and contributions of twist larger than four, we will provide, also in section 5, a method for projecting out the twist-four parts. Using the results of sections 2 through 5 the twist-four contributions we are interested in can be written in a completely gauge invariant form, which is factorized into coefficient functions and matrix elements between proton states. We give these results in section 6, viz. Eqs. (6.5), (6.10) and (6.25). Subsection 6.3 contains a comparison between our results and those obtained by EFP. Finally, in the conclusions, we discuss the steps which have to be completed before our results can be used in a phenomenological analysis.

## 2 Factorization

As outlined in the introduction it is our aim to specify twist-four gluon operators, write them in covariant form and perform the factorization of coefficient functions and matrix elements of
these operators between proton states. We will argue that it is sufficient to consider the diagrams in Fig. 1. Here the upper parts represent the quark-loops with two, three and four $t$-channel gluons. They are amputated in the gluon lines, and we denote them by $R_{\mu \nu}^{ab}(k_1)$, $R_{\mu \sigma \nu}^{abc}(k_1, k_2)$ and $R_{\mu \sigma \lambda \nu}^{abcd}(k_1, k_2, k_3)$, resp. These quark-loops will contribute to both NLO twist-four gluon-coefficient functions and terms which are convolutions of LO quark-coefficient functions with Altarelli-Parisi splitting functions. It is our aim to derive general results for the twist-four contributions of the diagrams in Fig. 1 without using explicit expressions for these quark-loops. All we need are some general properties of the functions $R$ concerning the Lorentz-, color- and momentum-structure.

The lower parts in Fig. 1 are denoted by $\Gamma_{\mu \nu}^{ab}(k_1)$, $\Gamma_{\mu \sigma \nu}^{abc}(k_1, k_2)$ and $\Gamma_{\mu \sigma \lambda \nu}^{abcd}(k_1, k_2, k_3)$, resp., and with these conventions the contributions of the diagrams in Fig. 1 are given by

\begin{align}
\tilde{H}_1(x_B, Q^2) &= \int \frac{d^4 k_1}{(2\pi)^4} R_{\mu \nu}^{ab}(k_1) \Gamma_{\mu \nu}^{ab}(k_1), \\
\tilde{H}_2(x_B, Q^2) &= \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} R_{\mu \sigma \nu}^{abc}(k_1, k_2) \Gamma_{\mu \sigma \nu}^{abc}(k_1, k_2), \\
\tilde{H}_3(x_B, Q^2) &= \int \frac{d^4 k_1}{(2\pi)^4} \frac{d^4 k_2}{(2\pi)^4} \frac{d^4 k_3}{(2\pi)^4} R_{\mu \sigma \lambda \nu}^{abcd}(k_1, k_2, k_3) \Gamma_{\mu \sigma \lambda \nu}^{abcd}(k_1, k_2, k_3),
\end{align}

where we have dropped the photon indices. As we will show in section 5 expressions (2.1)-(2.3) contain leading-twist and twist-four contributions of gluonic operators as well as some parts with twist larger than four. The twist-four parts can be projected out if one performs a dimensional analysis. Here, the convenience of the axial gauge becomes most explicit.

The lower blobs in Fig. 1 can be written as matrix elements of gluon-operators between proton
FIG. 2: Sample diagrams which contain three- and four-gluon vertices.

states:

\[ \Gamma^{\mu\nu}_{ab}(k_1) = \int d^4z_1 e^{ik_1z_1} \langle p|T\{A^\nu_b(0)A^\mu_a(z_1)\}|p\rangle, \]  

(2.4)

\[ \Gamma^{\mu\sigma\nu}_{abc}(k_1, k_2) = \int d^4z_1 d^4z_2 e^{ik_1z_1} e^{ik_2z_2} \langle p|T\{A^\nu_c(0)A^\sigma_b(z_2)A^\mu_a(z_1)\}|p\rangle, \]  

(2.5)

\[ \Gamma^{\mu\sigma\lambda\nu}_{abcd}(k_1, k_2, k_3) = \int d^4z_1 d^4z_2 d^4z_3 e^{ik_1z_1} e^{ik_2z_2} e^{ik_3z_3} \langle p|T\{A^\lambda_d(0)A^\sigma_c(z_3)A^\mu_a(z_1)A^\nu_b(z_2)\}|p\rangle. \]  

(2.6)

Here \( T \) denotes the time ordered product, and expressions (2.4)-(2.6) contain the propagators of the gluon lines in Fig. 1.

In order to perform the factorization procedure with expressions (2.1)-(2.3) it is convenient to expand the expressions for the quark-loops about the longitudinal components of the gluon-momenta (cf. (1.13)):

\[ R^{ab}_{\mu\nu}(k_1) = R^{ab}_{\mu\nu}(x_1p) + \frac{\partial R^{ab}_{\mu\nu}(x_1p)}{\partial k_1^\sigma} (k_1 - x_1p)^\sigma + \frac{1}{2} \frac{\partial^2 R^{ab}_{\mu\nu}(x_1p)}{\partial k_1^\sigma \partial k_1^\lambda} (k_1 - x_1p)^\sigma (k_1 - x_1p)^\lambda + \cdots, \]  

(2.7)

\[ R^{abc}_{\mu\sigma\nu}(k_1, k_2) = R^{abc}_{\mu\sigma\nu}(x_1p, x_2p) + \sum_{i=1}^2 \frac{\partial R^{abc}_{\mu\sigma\nu}(x_1p, x_2p)}{\partial k_i^\lambda} (k_i - x_i p)^\lambda + \cdots, \]  

(2.8)

\[ R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_2, k_3) = R^{abcd}_{\mu\sigma\lambda\nu}(x_1p, x_2p, x_3p) + \cdots. \]  

(2.9)

We will justify in section 3 that we can neglect additional terms in these series if we are interested only in terms up to twist-four.

Before we give the results of the factorization procedure, we compute some Ward identities for the derivatives in Eqs. (2.7) and (2.8), show how the expressions for the lower blobs \( \Gamma \) can be expressed with the help of gluonic \( F^{\mu\nu}_{ab} \)-operators, and perform a dimensional analysis of the expressions (2.1)-(2.3) in the next three sections.

3 Ward Identities

In order to find relations for the derivatives of the quark-loops in Eqs. (2.7) and (2.8) we will make use of Ward identities. Although these identities can be derived diagrammatically at the
one-loop level, this is a rather tedious task. We, therefore, make a digression into the path-integral formulation of QCD and derive these Ward identities in the most general way. For this, we first derive a relation for the generating functional of perturbatively calculated vertex functions $V$ in the next subsection. This relation, viz. Eq. (3.9), will then be used to derive the desired equations in the following two subsections.

We should stress that the Ward identities which will be derived in this section are valid for 1PI vertex functions. They can, therefore, not be applied to expressions for quark-loops which involve three- and four-gluon vertices like those in Fig. 2.

### 3.1 Generating Functional

The generating functional for Green’s functions in the axial gauge can be written as \[ W[J^a_\mu, K^a] = N \int \mathcal{D}A^a_\mu \mathcal{D}\lambda^a \exp \left\{ i \int d^4x \left[ \mathcal{L}_{\text{QCD}} - \lambda^a n \cdot A^a + A^a_\mu J^a_\mu + \lambda^a K^a \right] \right\} . \] (3.1)

Here $N$ is some normalization factor, $J^a_\mu$ and $K^a$ are sources, the term $\lambda^a n \cdot A^a$ has been introduced in order to define the axial gauge. The generating functional must be invariant under gauge transformations, which for infinitesimal $\omega(x)$ can be written as

\[ A^a_\mu \rightarrow A^a_\mu + D^{ab}_\mu (A) \omega^b = A^a_\mu + \partial_\mu \omega^a - gf^{abc} A^c_\mu \omega^b . \] (3.2)

Inserting (3.2) into (3.1), substituting fields that occur as factors within the integral by functional derivatives and making use of the arbitrariness of $\omega(x)$, one gets

\[ 0 = D^{ab}_\mu (J) \left( n^\mu \frac{\delta}{\delta K^b(x)} - J^a_\mu (x) \right) W[J^a_\mu, K^a] , \] (3.3)

where the covariant derivative is given by

\[ D^{ab}_\mu (J) = \delta^{ab} \partial_\mu - gf^{abc} \partial_\mu \frac{\delta}{\delta J^c_\mu} . \] (3.4)

For our analysis we need the generating functional $V$ for vertex functions. This can be obtained with the help of the following procedure. First, define the functional $\tilde{V}$ via Legendre transformation of the generating functional of the connected Green’s functions, $Z$,

\[ \tilde{V}[A^a_\mu, \Lambda^a] = Z[J^a_\mu, K^a] - \int d^4x \left[ A^a_\mu J^a_\mu + K^a \Lambda^a \right] , \] (3.5)

where $Z$ is defined as

\[ Z[J^a_\mu, K^a] = -i \ln W[J^a_\mu, K^a] . \] (3.6)

With these definitions we get from (3.3)

\[ D^{ab}_\mu (A) \left( n^\mu \Lambda_b + \frac{\delta \tilde{V}}{\delta A^a_\mu} \right) = 0 . \] (3.7)
Now, $V$ can be obtained via subtraction \cite{16, 17}:

$$V[A_\mu^a, \Lambda^a] = \tilde{V}[A_\mu^a, \Lambda^a] + \int d^4x \Lambda_a n \cdot A^a$$

(3.8)

and the Ward identity (3.7) finally becomes

$$D^a_\mu(A) \frac{\delta V}{\delta A^a_\mu} = 0.$$  

(3.9)

### 3.2 Relations Between Two- and Three-Gluon Quark-Loops

We are now deriving an equation which relates the derivative of the two-gluon quark-loop $R^{ab}_{\mu\nu}$ with the three-gluon quark-loop $R^{abc}_{\mu\nu\rho}$. The relation between the perturbatively computed quark-loops and the generating functional $V$ is as follows (see Fig. 3 for our conventions on the gluon momenta: the $n$th momentum 4-vector $k_n$, i.e. the $n$th argument of the $R$-functions, is the sum of momenta of the first $n$ gluons):

$$\int d^4x_1 d^4x_2 e^{ik_1 \cdot x_1} e^{ik_2 \cdot x_2} \frac{\delta^2 V[0]}{\delta A_{\mu}^a(x_1) \delta A_{\nu}^b(x_2)} = (2\pi)^4 \delta(k_1 + k) R^{ab}_{\mu\nu}(k_1),$$

(3.10)

$$\int d^4x_1 d^4x_2 e^{ik_1 \cdot x_1} e^{ik_2-k_1 \cdot x} e^{ik_2 \cdot x_2} \frac{\delta^3 V[0]}{\delta A_{\mu}^a(x_1) \delta A_{\nu}^b(x) \delta A_{\rho}^c(x_2)} = (2\pi)^4 \delta(k_2 + k) R^{abc}_{\mu\nu\rho}(k_1, k_2),$$

(3.11)

$$\int d^4x_1 d^4x_2 d^4x_3 e^{ik_1 \cdot x_1} e^{ik_2-k_1 \cdot x} e^{ik_3-k_2 \cdot x} e^{ik_3 \cdot x_3} \frac{\delta^4 V[0]}{\delta A_{\mu}^a(x_1) \delta A_{\nu}^b(x) \delta A_{\rho}^c(x_2) \delta A_{\sigma}^d(x_3)} = (2\pi)^4 \delta(k_3 + k) R^{abcd}_{\mu\nu\rho\sigma}(k_1, k_2, k_3).$$

(3.12)

Relabelling Lorentz- and color indices in Eq. (3.9) and differentiating the equation by $A_{\mu}^a(x_1)$ and $A_{\nu}^c(x_2)$ one gets

$$\frac{\partial^\sigma}{\partial A_{\mu}^a(x_1) \delta A_{\nu}^a(x) \delta A_{\sigma}^c(x_2)} = g \left\{ f^{abc} \delta(x - x_1) \frac{\partial^2 V[0]}{\delta A_{\rho}^b(x) \delta A_{\sigma}^c(x_2)} - f^{bcd} \delta(x - x_2) \frac{\partial^2 V[0]}{\delta A_{\mu}^a(x_1) \delta A_{\sigma}^d(x)} \right\}.$$  

(3.13)

Now, multiplying Eq. (3.13) by appropriate exponentials, integrating over $x, x_1$ and $x_2$ and keeping Eqs. (3.10) and (3.11) in mind one ends up with

$$-i k^\sigma R^{abc}_{\mu\lambda\nu}(k_1, k + k_1) = g \left\{ f^{abc} R^{\mu\nu\rho}_{\lambda\rho}(k + k_1) - f^{bcl} R^{\mu\rho\lambda}_{\nu\rho}(k_1) \right\}.$$  

(3.14)

The two gluons at the lower end of the quark loop must be in the color singlet. We can, therefore, define

$$R^{ab}_{\mu\nu}(k) \equiv \delta^{ab} R_{\mu\nu}(k).$$

(3.15)

With this definition we can take the limit $k \to 0$ in Eq. (3.14) and finally get the desired result:

$$i g f^{abc} \frac{\partial R_{\mu\nu}(k_1)}{\partial k_1^\sigma} = R^{abc}_{\mu\nu}(k_1, k_1).$$

(3.16)
3.3 Relations Involving Four-Gluon Quark-Loops

Relations between derivatives of the three-gluon quark-loop $R^{abc}_{\mu\sigma\lambda\nu}$ and the four-gluon quark-loop $R^{abcd}_{\mu\sigma\lambda\nu}$ can be obtained in a completely analogous way. If we differentiate (3.9) three times, multiply the resulting equation by suitable exponentials and perform the $x_i$-integrations, we get, similar to (3.14):

$$-ik^\sigma R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_1 + k, k_2 + k) = g\left\{ f^{ab'}b'^{cd}R^{b'd}_{\mu\lambda\nu}(k_1 + k, k_2 + k) - f^{bc'd}R^{ac'd}_{\mu\lambda\nu}(k_1, k_2 + k) - f^{b'db'}R^{ac'd}_{\mu\lambda\nu}(k_1, k_2) \right\}.$$  (3.17)

Before we can take the limit $k \to 0$ we have to explore the color structure of $R^{abc}_{\mu\sigma\lambda\nu}$. First, consider the one-loop case. Since $R^{abc}_{\mu\sigma\lambda\nu}$ is the sum of all diagrams with three external gluons, we can, in the one-loop case, always group two diagrams together. For each diagram there is another diagram which gives the same momentum dependence (up to the sign) but different color structure (see Fig. 3 for illustration). Since the gluon-system must have even C-parity this color structure is always of the form

$$\text{tr}(t^at^bt^c) - \text{tr}(t^ct^b) = \frac{1}{2}f^{abc}.$$  (3.18)

For each pair of diagrams. We, therefore, define

$$R^{abc}_{\mu\lambda\nu}(k_1, k_2) \equiv gf^{abc}R^{abc}_{\mu\lambda\nu}(k_1, k_2),$$  (3.19)

where the factor $g$ has been factored out for convenience. We expect that Eq. (3.19) reflects the general ansatz for the color structure of a three-gluon system in the case of even C-parity and is valid beyond the one-loop level. If we, now, insert (3.19) into (3.17) and make use of the Jacobi identity,

$$f^{abc'}f^{c'cd} - f^{acc'}f^{c'bd} + f^{ade'}f^{e'bc} = 0,$$  (3.20)

we can take the limit $k \to 0$ and obtain

$$ig^2 \left\{ f^{abc'}f^{c'cd}\frac{\partial R^{abc}_{\mu\lambda\nu}(k_1, k_2)}{\partial k_1} + f^{acc'}f^{c'bd}\frac{\partial R^{abc}_{\mu\lambda\nu}(k_1, k_2)}{\partial k_2} \right\} = R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_1, k_2).$$  (3.21)

For our computations we need relations for the derivatives of $R^{abc}_{\mu\lambda\nu}$ with respect to $k_1$ or $k_2$. Since Eq. (3.21) contains these derivatives only as a sum, we have to project with suitable color tensors. Now, we realize that there is more than one color tensor which projects out the desired quantity in Eq. (3.21). For example both

$$P_1^{abcd} \equiv \delta^{ac}\delta^{bd}$$  (3.22)

and

$$P_2^{abcd} \equiv d^{ace'}d^{c'bd}$$  (3.23)

\[\text{In the case of odd C-parity the coupling of the gluon to quarks and anti-quarks involves a sign change. In this case, one therefore gets a factor } \text{tr}(t^at^bt^c) + \text{tr}(t^ct^b) = \frac{1}{2}d^{abc}.\]
FIG. 3: Two sample diagrams which enter into $R_{\mu \nu \sigma \lambda}$ at leading order in $\alpha_s$. With an adequate choice of the loop integration variable the traces over Dirac-matrices give the same (up to the sign) momentum dependence for both diagrams. The first diagram contributes with a color factor $\text{tr} t^a t^b t^c t^d$ while the second one contributes with a color factor $\text{tr} t^c t^b t^a$.

project out $\partial R_{\mu \nu \sigma \lambda} / \partial k_1^\sigma$. This fact will lead to some important restrictions for the four-gluon quark-loop $R_{\mu \sigma \lambda \nu}^{abcd}$ (Eq. (3.28) below).

For the remainder of this section we will restrict ourselves to the one-loop approximation of the quark loop. In this case we can use the same arguments that led to the definition (3.19). For four gluons we define the quantity (cf. (3.18))

$$d_{abcd} \equiv \text{tr}(t^a t^b t^c t^d) + \text{tr}(t^d t^c t^b t^a) = -\frac{1}{4} f^{abc} f^{bcd} + \frac{1}{4} d^{abc} d^{bcd} + \frac{1}{6} \delta^{abc} \delta^{bcd}. \quad (3.24)$$

We realize that there are three linearly independent tensors $d$, which can be obtained by permutation of color indices in Eq. (3.24) and, therefore, write the color structure of $R_{\mu \sigma \lambda \nu}^{abcd}$ as

$$R_{\mu \sigma \lambda \nu}^{abcd}(k_1, k_2, k_3) \equiv g^2 \left\{ d_{abcd} R_{\mu \sigma \lambda \nu}^1(k_1, k_2, k_3) + d^{abc} R_{\mu \sigma \lambda \nu}^2(k_1, k_2, k_3) + d^{abcd} R_{\mu \sigma \lambda \nu}^3(k_1, k_2, k_3) \right\}. \quad (3.25)$$

Each diagram that contributes to $R_{\mu \sigma \lambda \nu}^{abcd}$ at the one-loop level enters into one of the functions $R_{\mu \sigma \lambda \nu}^i$. Inserting (3.25) into (3.21) and projecting with the tensors $P_{abcd}^1$ and $P_{abcd}^2$ we end up with two equations, viz.

$$24i \frac{\partial R_{\mu \sigma \lambda \nu}^1(k_1, k_2)}{\partial k_1^\sigma} = -\frac{4}{3} R_{\mu \sigma \lambda \nu}^1(k_1, k_1, k_2) + \frac{32}{3} R_{\mu \sigma \lambda \nu}^2(k_1, k_1, k_2) + \frac{32}{3} R_{\mu \sigma \lambda \nu}^3(k_1, k_1, k_2), \quad (3.26)$$

$$20i \frac{\partial R_{\mu \sigma \lambda \nu}^2(k_1, k_2)}{\partial k_1^\sigma} = -\frac{40}{9} R_{\mu \sigma \lambda \nu}^1(k_1, k_1, k_2) + \frac{50}{9} R_{\mu \sigma \lambda \nu}^2(k_1, k_1, k_2) + \frac{50}{9} R_{\mu \sigma \lambda \nu}^3(k_1, k_1, k_2). \quad (3.27)$$

These two equations can simultaneously be fulfilled only if the following equation holds:

$$R_{\mu \sigma \lambda \nu}^1(k_1, k_1, k_2) + R_{\mu \sigma \lambda \nu}^2(k_1, k_1, k_2) + R_{\mu \sigma \lambda \nu}^3(k_1, k_1, k_2) = 0. \quad (3.28)$$

Note that this equation holds only true if one of the gluon-momenta vanishes (see Fig. 4 for our conventions on the gluon-momenta). Inserting the latter equation into Eq. (3.26) we end up with the desired results:

$$i \frac{\partial R_{\mu \sigma \lambda \nu}^{abc}(k_1, k_2)}{\partial k_1^\sigma} = -\frac{1}{2} g f^{abc} R_{\mu \sigma \lambda \nu}^1(k_1, k_1, k_2) = -\frac{1}{2} g f^{abc} R_{\mu \sigma \lambda \nu}^3(k_1, k_2, k_2) \quad (3.29)$$

There are more tensors than those in Eqs. (3.22) and (3.23) which project out $\partial R_{\mu \lambda \nu} / \partial k_1^\sigma$ in (3.21). Using these gives a consistent picture but no new results.
and
\[ i \frac{\partial R_{\mu \nu}^{abc}(k_1, k_2)}{\partial k_2^\lambda} = -\frac{1}{2} g f^{abc} R_{\mu \sigma \lambda \nu}^{1}(k_1, k_2, k_2) = -\frac{1}{2} g f^{abc} R_{\mu \lambda \sigma \nu}^{3}(k_1, k_1, k_2). \] (3.30)

Here we have also projected out the derivative of \( R_{\mu \sigma \lambda \nu}^{abc} \) with respect to \( k_2 \) in Eq. (3.21) and we have used the fact that we have some freedom in choosing Lorentz- and color-indices and momentum variables in defining the four-gluon quark-loop in Eq. (3.12). \( R_{\mu \sigma \lambda \nu}^{abc} \) must be invariant under the simultaneous exchange of momentum variables, Lorentz- and color indices. This leads to the possibility to express the derivatives of \( R_{\mu \sigma \lambda \nu}^{abc} \) in Eqs. (3.29) and (3.30) either by \( R_{\mu \sigma \lambda \nu}^{1} \) or by \( R_{\mu \sigma \lambda \nu}^{3} \).

Finally, differentiating Eq. (3.16) by \( k_1^\lambda \) and using (3.29) and (3.30) we get
\[ i \frac{\partial R_{\mu \nu}^{ad}(k_1)}{\partial k_1^\lambda} = \frac{1}{2} \delta^{ad} \left\{ R_{\mu \sigma \lambda \nu}^{1}(k_1, k_1, k_1) + R_{\mu \sigma \lambda \nu}^{3}(k_1, k_1, k_1) \right\} = -\frac{1}{2} \delta^{ad} R_{\mu \sigma \lambda \nu}^{2}(k_1, k_1, k_1), \] (3.31)

where we have also made use of (3.28).

### 4 Gluonic Operators

As outlined in section 2, the lower blobs in Fig. 1 can be written as matrix-elements of gluonic operators (Eqs. (2.4)-(2.6)). Since it is our aim to find expressions for the gluonic twist-four contributions which are explicitly gauge invariant, we have to find relations in which all operators consist of gluonic operators \( F_{\mu \nu}^{cd} \) and covariant derivatives \( D_{\mu}^{ab} \). In this section we show how, in the axial gauge, \( A_{\mu}^{a} \)-operators within matrix elements can be substituted by \( F_{\mu \nu}^{cd} \)-operators. For this we make use of the relation
\[ n_{\mu} F_{\mu \nu}^{cd} = (n \cdot \partial) A_{\nu}^{d}. \] (4.1)

Let us, first, define\(^6\)
\[ \int d^4 z \ d^4 z' \ e^{ik \cdot z} e^{-ik' \cdot z'} \langle p | T \left\{ A_{\mu}^{d}(z') A_{\nu}^{a}(z) \right\} | p \rangle \equiv (2\pi)^4 \delta(k - k') G_{\mu \nu}^{ab}(k), \] (4.2)
and examine the expression
\[ \int d^4 z \ d^4 z' \ e^{ik \cdot z} e^{-ik' \cdot z'} \langle p | T \left\{ (n \cdot \partial') A_{\nu}^{d}(z') (n \cdot \partial) A_{\mu}^{a}(z) \right\} | p \rangle = -i n \cdot k i n \cdot k' \int d^4 z \ d^4 z' \ e^{ik \cdot z} e^{-ik' \cdot z'} \langle p | T \left\{ A_{\mu}^{d}(z') A_{\nu}^{a}(z) \right\} | p \rangle = n \cdot k n \cdot k' (2\pi)^4 \delta(k - k') G_{\mu \nu}^{ab}(k), \] (4.3)
\(^6 We are not writing the \( p \)-dependence of \( G_{\mu \nu}^{ab} \) explicitly.
where we have used partial integration. If we perform the integration over \( k' \) in Eq. (4.3), keeping (4.1) in mind and inserting \( x = n \cdot k \), we end up with

\[
\int d^4 z e^{ik \cdot z} \langle p | T \{ A_\beta^\nu(0) A_\alpha^\mu(z) \} | p \rangle = \frac{1}{x^2} \int d^4 z e^{ik \cdot z} n_\alpha n_\beta \langle p | T \{ F_\beta^\nu(0) F_\alpha^\mu(z) \} | p \rangle,
\]

which is the desired result. In complete analogy, one can find relations for matrix-elements with more than two gluon-operators.

5 Dimensional Analysis

Before we present the results of the factorization procedure we, first, perform the dimensional analysis in this section, in order to justify that we have restricted ourselves to diagrams with up to four \( t \)-channel gluons (Fig. 1) and to those terms in the expansions of the quark-loop expressions \( R \) which are given in Eqs. (2.7)-(2.9). By this, we also derive the projectors which are needed to obtain contributions of definite twist from the expressions under consideration.

To understand the reasoning of the following discussion, it will be sufficient to anticipate that after factorization the upper and lower parts of the diagrams in Fig. 1 are connected by integrals over momentum fractions \( x_i \), which we can write as

\[
H_1 = \int dx_1 R_{\mu\nu}(x_1 p) 2G^{\mu\nu}(x_1), \\
H_2 = \int dx_1 dx_2 R_{\mu\sigma\nu}(x_1 p, x_2 p) 3G^{\mu\sigma\nu}(x_1, x_2), \\
H_3 = \int dx_1 dx_2 dx_3 R_{\mu\sigma\lambda\nu}(x_1 p, x_2 p, x_3 p) 4G^{\mu\sigma\lambda\nu}(x_1, x_2, x_3).
\]

It will also be sufficient to know that the functions \( G \) are matrix elements of gluon operators which may contain derivatives. The number of Lorentz-indices corresponds to the number of gluon-lines plus the number of derivatives. For convenience we are using the above relations already in this section. The precise definition of the functions \( G \) will be given below in Eqs. (6.6), (6.11) and (6.26) and the functions \( H_i \), which are not the same as the functions \( \tilde{H}_i \) in (2.1)-(2.3), will be discussed in (5.1).

We expect an infrared cut-off to enter into the expressions for the quark-loops only within logarithms since infrared and mass singularities are logarithmic at most. Moreover, renormalizability guarantees that one-loop diagrams with two external photons and any number of gluons do not contain UV divergences. Therefore, after factorization, the only dimensional quantity the quark-loops \( R \) can depend on is \( Q^2 \). On the other hand, the only dimensional quantity the matrix-elements \( G \) can depend on is \( \Lambda \). If we, therefore, determine the dimension of a particular function \( R \) with Lorentz indices projected out, we can relate this dimension to the powers of \( 1/Q^2 \) this function must be proportional to.

A simple analysis shows that the dimension of the quark-loops is given by the expression

\[
\dim R = 2 - B,
\]

(5.4)
where $B$ is the number of gluon-lines plus the number of derivatives:

$$B = n_G + n_D. \tag{5.5}$$

With the help of (5.1)-(5.3) we conclude that

$$\text{dim}_4 G = \text{dim}_3 G + 1 = \text{dim}_2 G + 2. \tag{5.6}$$

5.1 Two Gluons

In order to split up $H_1$, (5.1), into leading- and next-to-leading twist expressions, we, now, explore the tensor structure of $2G^\mu\nu$ (which is given by Eq. (6.6), below):

$$2G^\mu\nu(x_1) = d^\mu\nu A_1(x_1) + n^\mu n^\nu A_2(x_1). \tag{5.7}$$

Since we are working in the axial gauge, i.e. $n \cdot A = 0$, there can be no terms proportional to $p^\mu p^\nu$ in (5.7) (cf. (2.4)). The tensor

$$d^\mu\nu \equiv g^\mu\nu - p^\mu n^\nu - n^\mu p^\nu \tag{5.8}$$

has been introduced since it serves as a projector, such that

$$A_1 = \frac{1}{2} d^\mu\nu 2G^\mu\nu, \tag{5.9}$$

$$A_2 = p_\mu p_\nu 2G^\mu\nu. \tag{5.10}$$

Now, since $\text{dim}_p = 1$, $\text{dim} d^\mu\nu = 0$ and $\text{dim}_n = -1$, we conclude with the help of (5.6) that $\text{dim}_A_2 = \text{dim}_A_1 + 2$. It follows that in Eq. (5.7) the term proportional to $A_1$ gives the leading-twist contribution, while the term proportional $A_2$ contributes to twist-four, i.e. in (5.1) the corresponding expression is proportional to $\Lambda^2/Q^2$.

5.2 Three Gluons

The twist analysis of (5.2) can be performed in complete analogy. Exploring the tensor structure of $3G^{\mu\sigma\nu}$ (defined in Eq. (6.11), below) one gets:

$$3G^{\mu\sigma\nu} = d^{\mu\nu} n^\sigma B_1^1 + d^{\mu\sigma} n^\nu B_1^2 + d^{\nu\sigma} n^\mu B_1^3 + n^\mu n^\nu n^\sigma B_2. \tag{5.11}$$

Here, we have made use of the fact that

$$n_\alpha, 3G^{\alpha_1\alpha_2\alpha_3} = 0. \tag{5.12}$$

As mentioned before we postpone the derivation of some of the results we are using here to the next section. Eq. (5.12) follows since the indices of $3G^{\mu\sigma\nu}$ belong either to terms like $A^\alpha$ or like $\omega^\alpha, \partial^\alpha$, where the tensor $\omega^\alpha$ is introduced in (5.7). It subtracts the longitudinal component of the

\[ 7 \text{Additionally, terms proportional to } n_\alpha p_\beta \epsilon^{\mu\nu\sigma\alpha\beta} \text{ (in Eq. (5.7)) or } n_\alpha \delta^{\mu\sigma\nu\alpha} \text{ (in Eq. (5.11)) would be present in the case of polarized scattering or if a virtual Z boson scatters off the proton.} \]
momentum vector it is applied to and \( n_\alpha \omega^\alpha_{\alpha'} = 0 \). The various terms in (5.11) can be projected out in the following way

\[
B_1^1 = \frac{1}{2} d_{\mu\nu} p_\sigma 3 G^{\mu\sigma\nu} \text{ etc.,} \\
B_2 = p_\mu p_\nu p_\sigma 3 G^{\mu\sigma\nu},
\]

and a dimensional analysis shows that the terms proportional \( B_1^i \) contribute to twist-four, while the term proportional \( B_2 \) contributes to twist-six.

5.3 Four Gluons

The tensor structure of \( 4 G^{\mu\sigma\lambda\nu} \) (see Eqs. (6.26)-(6.28) below) is given by

\[
4 G^{\mu\sigma\lambda\nu} = d^{\mu\nu} d^{\sigma\lambda} C_1^1 + \text{perm.} + d^{\mu\nu} n^\sigma n^\lambda C_2^1 + \text{perm.} + n^\mu n^\nu n^\sigma n^\lambda C_3,
\]

with

\[
C_1^1 = d_{\mu\sigma\lambda} 4 G^{\mu\sigma\lambda\nu} \text{ etc.,} \\
C_2^1 = \frac{1}{2} d_{\mu\nu} p_\sigma p_\lambda 4 G^{\mu\sigma\lambda\nu} \text{ etc.,} \\
C_3 = p_\mu p_\nu p_\sigma p_\lambda 4 G^{\mu\sigma\lambda\nu},
\]

where

\[
d_{\mu\sigma\lambda} = \frac{1}{8} (3 d_{\mu\nu} d_{\sigma\lambda} - d_{\mu\sigma} d_{\nu\lambda} - d_{\mu\lambda} d_{\nu\sigma}).
\]

In this case, the functions \( C_1^i \) give the twist-four and the functions \( C_2^i \) the twist-six contributions, while \( C_3 \) belongs to twist-eight.

Terms with more than four \( t \)-channel gluons and/or derivatives can be analysed in the same way. The term with lowest dimension of the corresponding five-gluon contribution \( 5 G^{\mu\sigma\lambda\nu} \) would be proportional to \( d^{\mu\nu} d^{\sigma\lambda} n^\rho \) (perm.) and, therefore, belongs to twist-six. Increasing the number of gluon-lines or derivatives also increases the twist. We can, therefore, justify that we have restricted ourselves to the diagrams in Fig. 1 and to the terms in Eqs. (2.7)-(2.9).

6 Results of the Factorization Procedure

Before we present the results of the factorization procedure, let us summarize the preceding results. Our computations are based on the diagrams in Fig. 1, where the expressions for the quark-loops \( R \) are expanded about the longitudinal components of the gluon-momenta (2.7)-(2.8). For the derivatives in (2.7) and (2.8) we have computed some Ward identities in section 3, viz. (3.16), (3.29), (3.30) and (3.31). For the derivation of these relations we had to explore the color structure of the quark-loop expressions, (3.15), (3.19) and (3.25), which also led to an important relation for the four-gluon quark-loop, (3.28), if one of the gluon-momenta vanishes. In section 4 we have developed a method which is useful in order to replace, within matrix elements, \( A \)-operators by
As outlined before, it is our aim to write the gluonic twist-four contributions to the inclusive deep-inelastic structure functions in an explicitly gauge invariant form. In order to get these expressions it is, now, necessary to reorder the terms which enter into the sum of the three diagrams in Fig. 1 after expanding the quark loops about the longitudinal components of the gluon-momenta as in Eqs. (2.7)-(2.9). In particular, we write the sum of these contributions as

$$\tilde{H}_1 + \tilde{H}_2 + \tilde{H}_3 = H_1 + H_2 + H_3 + O \left( \frac{1}{Q^4} \right). \quad (6.1)$$

Here \( H_1 \) consists only of the zeroth-order term of the Taylor expansion of the two-gluon quark-loop in Eq. (2.7) inserted in (2.1), while \( H_2 \) is the sum of the first-order two-gluon quark-loop (from Eq. (2.7)) and the zeroth-order three-gluon quark-loop (from Eq. (2.8)) inserted in Eqs. (2.1) and (2.2). \( H_3 \) consists of the sum of expressions (2.1) through (2.3) where only the zeroth-order term of the four-gluon quark-loop (2.9), the first-order three-gluon quark-loop (2.8) and the second-order two-gluon quark-loop (2.7) are taken into account. With these conventions the steps needed to write \( H_1 \) and \( H_2 \) in an explicitly gauge invariant form are straightforward, while the situation is more peculiar in the case of \( H_3 \). We review the results in the next subsections.

### 6.1 Two and Three Gluons

Inserting the zeroth-order term in Eq. (2.7) into (2.1), we realize that the expression for the quark-loop depends only on the longitudinal component of the four-momentum \( k_1 \), and we can simplify the integrations over \( k_1 \) and \( z \) if we introduce the expression (cf. (1.13))

$$\int \frac{d^4k_1}{(2\pi)^4} e^{ik_1 \cdot z} \delta(x_1 - k_1 \cdot n) = \int \frac{d\lambda}{2\pi} e^{i\lambda x_1} \delta^{(4)}(z - \lambda n). \quad (6.3)$$

The result is

$$H_1 = \int dx_1 R_{\mu\nu}^a(x_1p) \int \frac{d\lambda}{2\pi} e^{i\lambda x_1} \langle p | T \left\{ A^\mu_b(0) A^\nu_a(\lambda n) \right\} | p \rangle. \quad (6.4)$$

With the help of the results of section 4 we can substitute the \( A \)-operators by \( F \)-operators, and with the help of Eqs. (5.7) and (5.10) we can project out the twist-four contribution:

$$H_1|_{\tau=4} = \int dx_1 n^\mu n^\nu R_{\mu\nu}(x_1p) p_\mu p_\nu 2G^{\mu\nu}(x_1), \quad (6.5)$$

where

$$2G^{\mu\nu}(x_1) = \frac{n_\alpha n_\beta}{x_1} \int \frac{d\lambda}{2\pi} e^{i\lambda x_1} \langle p | T \left\{ \delta_{ab} F_{\mu^\beta}^a(0) F_{\nu}^{\alpha}(\lambda n) \right\} | p \rangle. \quad (6.6)$$
In the case of $H_2$ we get the additional factor $(k_1 - x_1 p)^\sigma$ from the first-order term in Eq. (2.7). This factor can be written as a derivative applied to the exponential in (2.4) if we introduce the tensor
\[
\omega^\mu_\mu \equiv g^\mu_\nu - p^\mu n_\nu, \quad \omega^\mu_\mu k_1^\mu = (k_1 - x_1 p)^\mu, \quad (6.7)
\]
which subtracts the longitudinal component of the momentum vector it is applied to. Now, using partial integration in order to apply this derivative to the gluon operator and making use of Eq. (3.16) we get
\[
H_2 = \int dx_1 dx_2 R_{\mu\sigma\nu}(x_1 p, x_2 p) \int \frac{d\lambda d\eta}{2\pi} e^{i\lambda x_1} e^{i\eta(x_2 - x_1)} \omega^\sigma_\sigma (p \{ A^\nu_\epsilon (0) D^\sigma_\alpha (\eta n) A_\alpha^\mu (\lambda n) \} | p \rangle, \quad (6.8)
\]
where the partial derivative within the covariant derivative is due to the first-order term in the Taylor expansion (2.7), and the gluon operator within $D^\sigma_\alpha$ is due to the zeroth-order term in (2.8). For the derivation of (6.8) we have also made use of the fact that, within the axial gauge, the $\omega$-tensor applied to a gluon-operator yields the same operator:
\[
\omega^\mu_\mu A_\alpha^\mu = A_\alpha^\mu. \quad (6.9)
\]
Projecting out the twist-four contribution with the help of Eqs. (5.11) and (5.13) and applying the method developed in section 4 in order to substitute the $A$-operators by $F$-operators we end up with
\[
H_2 |_{\tau=4} = \int dx_1 dx_2 \left\{ d^{\mu\nu} n^\sigma R_{\mu\sigma\nu}(x_1 p, x_2 p) \frac{1}{2} d_{\mu'\nu'} p_{\sigma'} 3G^{\mu'\sigma'\nu'}(x_1, x_2) + \text{perm.} \right\}, \quad (6.10)
\]
where
\[
3G^{\mu\sigma\nu}(x_1, x_2) = \frac{n_\alpha n_\beta}{x_1 x_2} \int \frac{d\lambda d\eta}{2\pi} e^{i\lambda x_1} e^{i\eta(x_2 - x_1)} \omega^\sigma_\sigma (p \{ F^\nu_\epsilon (0) D^\sigma_\alpha (\eta n) F_\alpha^\mu (\lambda n) \} | p \rangle. \quad (6.11)
\]
For the permutation in (6.10) one has to keep Eqs. (5.11) and (5.13) in mind, i.e. indices of the projector which is applied to $R_{\mu\sigma\nu}$ and of the projector which is applied to $3G^{\mu'\sigma'\nu'}$ have to be permuted simultaneously. Similar to $H_1$, (6.5), $H_2$ contains only contributions of the two-gluon operator.

We expect that Eqs. (6.5) and (6.10) are of more general validity, since the Ward identities we have used are valid to all orders and our ansätze for the color structure of $R_{\mu\nu}^{ab}$ and $R_{\mu\lambda\nu}^{abc}$, (3.15) and (3.19), are of the most general nature.

### 6.2 Four Gluons
6.2.1 Basics

The situation is more complicated in the case of \( H_3 \). Putting together all terms which enter into \( H_3 \) and making use of Eqs. (3.29), (3.30) and (3.31) we get

\[
H_3 = \int d x_1 d x_2 d x_3 \left\{ R_{\mu \sigma \lambda \nu}^{abcd}(x_1 p, x_2 p, x_3 p) 4 G_{\mu \sigma \lambda \nu}^{abcd}(x_1, x_2, x_3) + \right.
\]
\[
\left. + \frac{i}{2} g f^{acd} R^1_{\mu \sigma \lambda \nu}(x_1 p, x_2 p, x_3 p) 3 G_{\mu \sigma \lambda \nu}^{abcd,1}(x_1, x_3) \delta(x_1 - x_2) + \right.
\]
\[
\left. + \frac{i}{2} g f^{abd} R^1_{\mu \sigma \lambda \nu}(x_1 p, x_2 p, x_3 p) 3 G_{\mu \sigma \lambda \nu}^{abcd,2}(x_1, x_2) \delta(x_2 - x_3) + \right.
\]
\[
\left. + \frac{1}{4} \delta^{\mu \delta} \left[ R^3_{\mu \sigma \lambda \nu}(x_1 p, x_2 p, x_3 p) + R^3_{\mu \sigma \lambda \nu}(x_1 p, x_2 p, x_3 p) \right] 2 G_{\mu \sigma \lambda \nu}^{abcd}(x_1) \delta(x_1 - x_2) \delta(x_2 - x_3) \right\},
\]

where

\[
4 G_{\mu \sigma \lambda \nu}^{abcd}(x_1, x_2, x_3) = \int \frac{d \lambda}{2 \pi} \frac{d \eta}{2 \pi} \frac{d \tau}{2 \pi} e^{i \lambda x_1} e^{i \eta(x_2 - x_1)} e^{i \tau(x_3 - x_2)} (p) |T \left\{ A_{\mu}^\nu(0) A_{\sigma}^\lambda(\tau n) A_{\lambda}^\sigma(\eta n) A_{\nu}^\mu(\lambda n) \right\} |p\rangle,
\]

\[
3 G_{\mu \sigma \lambda \nu}^{abcd,1}(x_1, x_3) = \omega^\sigma_{\mu \nu} \int \frac{d \lambda}{2 \pi} \frac{d \eta}{2 \pi} e^{i \lambda x_1} e^{i \eta(x_2 - x_1)} (p) |T \left\{ A_{\mu}^\nu(0) A_{\sigma}^\lambda(\tau n) i \delta^{\nu \sigma} A_{\lambda}^\sigma(\eta n) \right\} |p\rangle,
\]

\[
3 G_{\mu \sigma \lambda \nu}^{abcd,2}(x_1, x_2) = \omega^\lambda_{\mu \nu} \int \frac{d \lambda}{2 \pi} \frac{d \eta}{2 \pi} e^{i \lambda x_1} e^{i \eta(x_2 - x_1)} (p) |T \left\{ A_{\mu}^\nu(0) i \delta^{\nu \lambda} A_{\sigma}^\lambda(\eta n) A_{\lambda}^\sigma(\eta n) \right\} |p\rangle,
\]

\[
2 G_{\mu \sigma \lambda \nu}^{abcd}(x_1) = \omega^\sigma_{\mu \nu} \omega^\lambda_{\lambda \nu} \int \frac{d \lambda}{2 \pi} e^{i \lambda x_1} (p) |T \left\{ A_{\mu}^\nu(0) i \delta^{\nu \lambda} i \delta^{\lambda \sigma} A_{\lambda}^\sigma(\eta n) \right\} |p\rangle.
\]

The derivatives in Eqs. (6.14)-(6.16) have to be understood as \( \partial^\nu A(\lambda n) \equiv \partial A(z)/\partial z^\nu |_{z=\lambda n} \), etc.

Due to our definition (3.25), we can split up Eq. (6.12) into three parts

\[
H_3 = H_3^1 + H_3^2 + H_3^3,
\]

where \( H_3^i \) corresponds to those terms in (6.12) which contain the function \( R_{\mu \sigma \lambda \nu}^{i} \). Since we intend to write the expression for \( H_3 \) in a form which is explicitly gauge invariant (similar to (6.5) and (6.10)) we have to modify it in such a way that partial derivatives occur only within covariant derivatives. This can most easily be done with the help of the following relation:

\[
f^{abcd} A_{\lambda}^{\sigma}(\tau n) \partial^\sigma = f^{a'c'd} A_{\lambda}^{\sigma}(\tau n) D^\sigma_{a' \sigma}(\eta n) + g f^{ab'c} f^{a'c'd} A_{\lambda}^{\sigma}(\tau n) A_{\sigma}^{\sigma}(\eta n).
\]

Inserting (5.18) into (6.12) we realize that the results for the functions \( H_3^i \) consist of terms with two-, three- and four-gluon operators. In the spin-averaged case the three-gluon operator does not contribute. The decomposition (6.17) is not unique, since the Ward identities could be expressed with the help of different combinations of the functions \( R_{\mu \sigma \lambda \nu}^{i} \) in Eqs. (3.29), (3.30) and (3.31). We could, therefore, also write Eq. (6.12) with different combinations of these functions. By exploring symmetry relations for \( R_{\mu \sigma \lambda \nu}^{abcd}(k_1, k_2, k_3) \) we will show how to derive expressions for \( H_3 \) which involve only one of the functions \( R^{i} \) and contain no three-gluon operator.
| $R^1_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$ | $R^2_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$ | $R^3_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$ |
|---------------------------------|---------------------------------|---------------------------------|
| $R^2_{\mu\sigma\lambda}(k_2 - k_1, k_2, k_3)$ | $R^1_{\sigma\mu\lambda\nu}(k_2 - k_1, k_2, k_3)$ | $R^3_{\mu\sigma\lambda\nu}(k_2 - k_1, k_2, k_3)$ |
| $R^2_{\mu\sigma\lambda\nu}(k_1, k_2, k_2 - k_3)$ | $R^1_{\sigma\mu\lambda\nu}(k_1, k_2, k_2 - k_3)$ | $R^3_{\mu\sigma\lambda\nu}(k_1, k_2, k_2 - k_3)$ |
| $R^3_{\mu\lambda\sigma\nu}(k_1, k_1 - k_2 + k_3, k_3)$ | $R^2_{\mu\lambda\sigma\nu}(k_1, k_1 - k_2 + k_3, k_3)$ | $R^1_{\mu\lambda\sigma\nu}(k_1, k_1 - k_2 + k_3, k_3)$ |
| $R^2_{\sigma\lambda\mu\nu}(k_2 - k_1, k_3 - k_1, k_3)$ | $R^3_{\sigma\lambda\mu\nu}(k_2 - k_1, k_3 - k_1, k_3)$ | $R^1_{\sigma\lambda\mu\nu}(k_2 - k_1, k_3 - k_1, k_3)$ |
| $R^3_{\sigma\lambda\mu\nu}(k_3 - k_2, k_1 - k_2 + k_3, k_3)$ | $R^1_{\sigma\lambda\mu\nu}(k_3 - k_2, k_1 - k_2 + k_3, k_3)$ | $R^2_{\sigma\lambda\mu\nu}(k_3 - k_2, k_1 - k_2 + k_3, k_3)$ |
| $R^1_{\lambda\sigma\mu\nu}(k_3 - k_2, k_3 - k_1, k_3)$ | $R^3_{\lambda\sigma\mu\nu}(k_3 - k_2, k_3 - k_1, k_3)$ | $R^2_{\lambda\sigma\mu\nu}(k_3 - k_2, k_3 - k_1, k_3)$ |
| $R^3_{\sigma\mu\lambda\sigma}(k_1, k_1 - k_3, k_1 - k_2)$ | $R^3_{\mu\sigma\lambda\nu}(k_1, k_1 - k_3, k_1 - k_2)$ | $R^2_{\mu\sigma\lambda\nu}(k_1, k_1 - k_3, k_1 - k_2)$ |

TABLE 1: Making use of symmetry relations for $R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$ one finds relations between the three functions $R^i_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$.

6.2.2 Symmetry Relations of the Four-Gluon Quark-Loop Expression

The expression for the four-gluon quark-loop, $R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$, has to be completely symmetrical under the simultaneous exchange of gluon-momenta, Lorentz- and color indices. This can be seen in two ways. First, within the path-integral formulation we have defined $R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$ in Eq. (6.12). Since the gluon is a boson the derivatives in (3.12) commute and the proclaimed symmetry follows immediately. Secondly, it can be seen diagrammatically. Clearly, $R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_2, k_3)$ corresponds to the expression which is obtained if one takes into account all diagrams up to a specified order, i.e. in leading order one has to consider all permutations for the coupling of four t-channel gluons to quarks and anti-quarks. Consider, for example, the diagram in Fig. 3a. If we set $k_1 = k'_2 - k'_1$, $k_2 = k'_2$, $k_3 = k'_3$ and exchange $\mu \leftrightarrow \sigma$ and $a \leftrightarrow b$ we get an analytic expression which is the same as the one we get from the diagram in Fig. 3b. Since such a relation can be found for each diagram that enters into $R^{abcd}_{\mu\sigma\lambda\nu}$, we have in general

$$R^{abcd}_{\mu\sigma\lambda\nu}(k_1, k_2, k_3) = R^{bada}_{\mu\sigma\lambda\nu}(k_2, k_1, k_3).$$

(6.19)

Next, with the help of the color structure (3.23) of $R^{abcd}_{\mu\sigma\lambda\nu}$ we can find relations between the three functions $R^i_{\mu\sigma\lambda\nu}$ if we realize that for a $\leftrightarrow$ b: $d^{abcd} \rightarrow d^{dabc}$, $d^{abcd} \rightarrow d^{abc}$, $d^{abcd} \rightarrow d^{dabc}$, as follows from the definition of $d^{abcd}$ (3.24). With these relations we get

$$R^1_{\mu\sigma\lambda\nu}(k_1, k_2, k_3) = R^2_{\mu\sigma\lambda\nu}(k_2, k_1, k_3),$$

(6.20)

$$R^2_{\mu\sigma\lambda\nu}(k_1, k_2, k_3) = R^3_{\mu\sigma\lambda\nu}(k_2, k_1, k_3),$$

(6.21)

$$R^3_{\mu\sigma\lambda\nu}(k_1, k_2, k_3) = R^1_{\mu\sigma\lambda\nu}(k_2, k_1, k_3).$$

(6.22)

Exchanging other gluon lines one gets a large number of relations. Those which are useful in the subsequent derivations and a few additional ones are given in Table 1.

6.2.3 Gauge Invariant Expressions

With the help of the relations in Table 1 we can, now, write Eq. (6.12) as an explicitly gauge invariant expression. For example, using the relation in row one, column two of Table 1 we can, in
the first line of (6.12), replace the function $R^2_{\mu\sigma\lambda\nu}$ by $R^1_{\sigma\mu\lambda\nu}$ and after an appropriate transformation of integration variables ($x_1 \to x_2 - x_1$, $\lambda \leftrightarrow \eta$) and an exchange of Lorentz- and color indices $R^1_{\mu\sigma\lambda\nu}$ can be factored out. Similarly the terms involving $R^3_{\mu\sigma\lambda\nu}$ in (6.12) can be brought into the desired form with the help of the expression in row three, column three of Table 1. We end up with

$$H_3 = \int dx_1 dx_2 dx_3 \int \frac{d\lambda}{2\pi} \frac{d\eta}{2\pi} e^{i\lambda x_1} e^{i\eta(x_2-x_1)} e^{i\tau(x_3-x_2)} R^1_{\mu\sigma\lambda\nu}(x_1 p, x_2 p, x_3 p) \omega^\sigma_\nu \omega^\lambda_\nu \times$$

$$\times \langle p| T\{ [g_1^a b c d] A^\lambda_\nu(\tau n) A^\nu_\lambda(\eta n) - \frac{1}{2} D^\lambda_\sigma(\tau n) D^\nu_\sigma(\eta n)] A^\mu_\alpha(\lambda n) \} | p \rangle, \quad (6.23)$$

where

$$l^{abcd}_1 = f^{aced} f^c_{bd} + \frac{1}{4} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) =$$

$$= \frac{1}{2} f^{aced} f^c_{bd} - \frac{3}{2} d^{aced} d^c_{bd} + \frac{1}{2} \left( \delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) + \frac{1}{4} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) =$$

$$= \frac{1}{2} f^{aced} f^c_{bd} + 3 d^{abcd}. \quad (6.24)$$

From Eq. (6.12) we realize that there is a term proportional $R^3_{\mu\sigma\lambda\nu}$ in the first line. Due to (3.25) this yields terms with color factors $d^{abcd}$ and corresponding permutations of color indices. For the derivation of (6.23) one has to add and subtract terms which involve four $A$-operators in order to get covariant derivatives. Combining the subtracted expressions with those in the first line of Eq. (6.12) the resulting color factor is no longer proportional to $d^{abcd}$. Instead one gets the term in Eq. (6.24).

Again, with the help of (5.13) and (5.16) we can project out the twist-four contributions and with the help of the results in section 4 we can substitute the $A$-operators by $F$-operators:

$$H_3|_{\tau=4} = \int dx_1 dx_2 dx_3 \left\{ d^{\mu\nu} d^\lambda \Gamma^1_{\mu\sigma\lambda\nu}(x_1 p, x_2 p, x_3 p) d_{\mu\nu} d^\lambda \Gamma^4_{\mu\sigma\lambda\nu}(x_1, x_2, x_3) + \text{perm.} \right\}, \quad (6.25)$$
where
\[ 4G_1^{\mu \sigma \lambda \nu}(x_1, x_2, x_3) = \int \frac{d\lambda}{2\pi} \frac{d\eta}{2\pi} e^{i\lambda x_1} e^{i\eta(x_2-x_1)} e^{i\eta(x_3-x_2)} \times \] (6.26)
\[ \times \left[ \frac{-1}{x_1 x_3(x_2-x_1)(x_3-x_2)} n_\alpha n_\beta n_\gamma n_\delta \langle p| T \left\{ g^2 l_{abcd}^{\mu \nu} F_d^{\mu \nu}(0) F_\epsilon^{\lambda \gamma}(\tau n) F_b^{\alpha \beta}(\eta n) F_a^{\mu \alpha}(\lambda n) \right\} | p \rangle \right] \]
\[ - \frac{1}{2 x_1 x_3} n_\alpha n_\beta \langle p| T \left\{ F_\delta^{\mu \nu}(0) \omega_\lambda^\tau D_\delta^\lambda(\tau n) \omega_\sigma^\tau D_\sigma^\mu(\eta n) F_a^{\mu \alpha}(\lambda n) \right\} | p \rangle \right], \]
and the permutations in (5.25) have to be understood in the same sense as in (5.10) (cf. (5.11), (5.11)). For \( H_3 \) we get contributions of the two-gluon operator as well as of the four-gluon operator.

The representation of \( H_3 \) in a gauge invariant form as in Eq. (5.25) is not unique. For example, we can derive expressions where the quark-loop is represented by \( R^2 \) or \( R^3 \) instead of \( R^3 \). These expressions can be obtained in two ways. One can either start with Eq. (6.12), make use of the relations in Table I and apply the same methods which led to (5.25). Or one starts with (6.25) and rewrite \( R^3 \) in terms of \( R^2 \) or \( R^3 \) with the help of the expressions in Table I. The results can be summarized as follows. If we exchange \( R_{1 \sigma \lambda}^{\mu \nu}(x_1 p, x_2 p, x_3 p) \) by \( R_{2 \sigma \lambda}^{\mu \nu}(x_1 p, x_2 p, x_3 p) \) or \( R_{3 \sigma \lambda}^{\mu \nu}(x_1 p, x_2 p, x_3 p) \) in (5.25) we also have to exchange \( 4G_1^{\mu \sigma \lambda \nu}(x_1, x_2, x_3) \) by \( 4G_2^{\mu \sigma \lambda \nu}(x_1, x_2, x_3) \) or \( 4G_3^{\mu \sigma \lambda \nu}(x_1, x_2, x_3) \), resp. The latter functions can be written as
\[ 4G_2^{\mu \sigma \lambda \nu}(x_1, x_2, x_3) = \int \frac{d\lambda}{2\pi} \frac{d\eta}{2\pi} e^{i\lambda x_1} e^{i\eta(x_2-x_1)} e^{i\eta(x_3-x_2)} \times \] (6.27)
\[ \times \left[ \frac{-1}{x_1 x_3(x_2-x_1)(x_3-x_2)} n_\alpha n_\beta n_\gamma n_\delta \langle p| T \left\{ g^2 l_{abcd}^{\mu \nu} F_d^{\mu \nu}(0) F_\epsilon^{\lambda \gamma}(\tau n) F_b^{\alpha \beta}(\eta n) F_a^{\mu \alpha}(\lambda n) \right\} | p \rangle \right] \]
\[ - \frac{1}{2 x_1 x_3} n_\alpha n_\beta \langle p| T \left\{ F_\delta^{\mu \nu}(0) \omega_\lambda^\tau D_\delta^\lambda(\tau n) \omega_\sigma^\tau D_\sigma^\mu(\eta n) F_a^{\mu \alpha}(\lambda n) \right\} | p \rangle \right], \]
and
\[ 4G_3^{\mu \sigma \lambda \nu}(x_1, x_2, x_3) = \int \frac{d\lambda}{2\pi} \frac{d\eta}{2\pi} e^{i\lambda x_1} e^{i\eta(x_2-x_1)} e^{i\eta(x_3-x_2)} \times \] (6.28)
\[ \times \left[ \frac{-1}{x_1 x_3(x_2-x_1)(x_3-x_2)} n_\alpha n_\beta n_\gamma n_\delta \langle p| T \left\{ g^2 l_{abcd}^{\mu \nu} F_d^{\mu \nu}(0) F_\epsilon^{\lambda \gamma}(\tau n) F_b^{\alpha \beta}(\eta n) F_a^{\mu \alpha}(\lambda n) \right\} | p \rangle \right] \]
\[ - \frac{1}{2 x_1 x_3} n_\alpha n_\beta \langle p| T \left\{ F_\delta^{\mu \nu}(0) \omega_\lambda^\tau D_\delta^\lambda(\tau n) \omega_\sigma^\tau D_\sigma^\mu(\eta n) F_a^{\mu \alpha}(\lambda n) \right\} | p \rangle \right], \]
where
\[ l_{abcd}^{\mu \nu} = f^{ace'} f^{c'bd} + \frac{1}{4} \left( \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) = \frac{1}{2} f^{ace'} f^{c'be} - \frac{3}{2} d^{ace'} d^{c'be} + \frac{1}{2} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc} \right) + \frac{1}{4} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) \] (6.29)
and
\[ l_{abcd}^{\mu \nu} = - f^{ace'} f^{c'bd} + \frac{1}{4} \left( \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) = \] (6.30)
\[ \frac{1}{2} f^{ace'} f^{c'cd} - \frac{3}{2} d^{ace'} d^{c'cd} + \frac{1}{2} \left( -\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right) + \frac{1}{4} \left( \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc} \right). \]
The color factors $l_{i}^{abcd}$ are related by permutations of color indices:

$$l_{1}^{abcd} = l_{2}^{abdc} = l_{3}^{acbd}. \quad (6.31)$$

Note the different Lorentz- and color structure and the different dependence on integration variables of the two-gluon operators in (6.26), (6.27) and (6.28).

### 6.2.4 Color Structure

As we have seen, $H_3$ contains contributions of the two- and four-gluon operators. In this subsection we will explore the color factors $l_{i}^{abcd}$, which enter into the contribution of the four-gluon operator and are given by Eqs. (6.24), (6.29) and (6.31) depending on which representation of $H_3$ we are considering. If we have a look at Table [], we realize that there are also relations which transform one of the functions $R_i$ into the same function $R_i$ and only the dependence on the momenta and the Lorentz-indices changes. For example, if we start with the *four-gluon operator* part of (6.25), make use of the relation in line seven, row one of Table [], exchange the integration variables such that $x_3 \rightarrow x_1 - x_2$, $x_2 \rightarrow x_1 - x_3$, $\lambda \rightarrow \lambda - \eta$, $\eta \rightarrow -\eta$ and $\tau \rightarrow \tau - \eta$ and make use of Lorentz-invariance we get an expression which has exactly the same form for the *four-gluon operator* part as in Eq. (6.26) but the color factor has to be exchanged by $l_{1}^{adcb}$, i.e. indices $b$ and $d$ are exchanged. Now, the color factor $l_{1}^{abcd}$ in (6.24) has been written in three ways. There is the compact representation in the first line of (6.24), and in the second line it has been split up into symmetric and anti-symmetric parts with respect of exchange of color indices $a$ and $c$ or $b$ and $d$. The anti-symmetric part is given by the expression proportional $f_{acc}f_{c'db}$. Since the expression we got after use of the relation in Table [] must be the same as in (6.25), we conclude that the term proportional $f_{acc}f_{c'db}$ in (6.24) within (6.25) must give a vanishing contribution. In other words, the color factor that survives has the form

$$\tilde{l}_{1}^{abcd} = 3d^{abcd}. \quad (6.32)$$

We can repeat this argument with those expressions in which $H_3$ is expressed with the help of $R^2$ or $R^3$, (6.27), (6.28). Again, the surviving color tensors have the form

$$\tilde{l}_{2}^{abcd} = 3d^{abdc}, \quad (6.33)$$
$$\tilde{l}_{3}^{abcd} = 3d^{acbd}. \quad (6.34)$$

### 6.3 Discussion of the Results

In the previous sub-sections we have presented our main results on the gluonic twist-four contributions, viz. Eqs. (6.5), (6.10) and (6.25). While the color structure of the expressions for $H_1$ and $H_2$ is valid beyond one loop, we have some restrictions in the case of $H_3$. For a four-gluon system in the color singlet, there are five linearly independent states in the case of even C-parity. For example, one can expand the color structure with the help of the five states $1, 8_{A}, 8_{S}, 10 + \overline{10}$ and

---

*The Ward identities derived in section 3 are valid only for 1PI vertex functions. Therefore, contributions of diagrams which involve three- and four-gluon vertices like those in Fig. have to be incorporated separately into the expressions for $H_1$ through $H_3$.**
27. However, a closer inspection of our ansatz Eq. (3.25) for the color structure of the four-gluon quark-loop shows that the state \( 10 + \bar{10} \) does not occur and there are, in addition to the state \( 8A \), only two specific linear combinations of the three states \( 1, 8S, \) and \( 27 \). In an analysis valid to all orders one should expect that all color states contribute.

Let us, now, compare our results with those obtained by EFP [1]. As mentioned in the introduction, the computations in [1] can be divided into two steps. In the first step one considers the relevant diagrams and performs the factorization of the coefficient functions and matrix elements of the relevant operators. This leads to the operators in Eqs. (1.2)-(1.5). They can be compared with the two- and four-gluon operators in Eqs. (6.6), (6.11), and (6.26), which involve the same number of covariant derivatives as the corresponding quark operators.

In the second step, EFP used equations of motion in order to express the contributions of the two-quark operators (1.2)-(1.4) in terms of linearly independent quantities. Within the transverse basis, the resulting twist-four contributions to \( F_T \) and \( F_L \) are given in Eqs. (1.8) and (1.9) plus terms which involve the four-quark operator (1.5). In our case this second step has not yet been performed. It cannot be done without an explicit computation of the quark-loops \( R \).

On the other hand, one should try to find the relation between our results and the operators (1.10) and (1.11) occurring in the OPE. For this, one has, again, to compute the expressions for the quark-loops \( R \). As has been described in [1], the connection to the operators (1.10) and (1.11) can be established for moments of contributions to the structure functions with definite signature only, i.e. the expressions for \( 3G^{\mu\nu} \) and \( 4G^{\mu\nu\lambda\omega} \) in (6.11) and (6.26) have to be separated into symmetric and anti-symmetric parts. Note that the Lorentz-indices of the operators in (6.11) and (6.26) are not symmetrized. The permutations involved in Eqs. (6.10) and (6.25) imply the simultaneous exchange of Lorentz-indices of the quark-loop expressions and of the operators (cf. (5.11) and (5.15)). Taking moments of Eqs. (5.3), (5.10) and (5.25) will then result in an expansion of the operators in (5.9), (6.11) and (6.24) about the origin, so that finally the operators (1.10) and (1.11) will be recovered.

In summary, the approach used here is rather intuitive since one starts with certain diagrams which yield properties of the coefficient functions which, in turn, imply restrictions on the contributing operators. Within the OPE, on the other hand, one starts with a certain class of operators and has to find the corresponding coefficient functions.

As mentioned before, the computation of the quark-loop expressions \( R \) will yield terms which are proportional to logarithms in \( Q^2 \) and terms which are just constants. The terms which are proportional to logarithms have, actually, to be interpreted as part of the evolution of the contributions in Eqs. (1.6) and (1.7). They are due to configurations with collinear momenta of quarks and gluons and can be interpreted as contributions where one of the gluons splits up into a quark-antiquark pair. Non-logarithmic terms, on the other hand, correspond to the NLO twist-four gluon coefficient functions. For the computation of the quark-loop expressions \( R \) it will, therefore, be necessary to re-identify the diagrams considered by EFP.

7 Conclusions

For the description of deep inelastic structure functions in the low-\( Q^2 \) regime, the determination of the higher-twist contributions remains one of the outstanding theoretical challenges. At least in the small-\( x_B \) domain, some of these higher-twist contributions are known to increase faster with
decreasing $x_B$ than leading-twist terms. In addition, their $1/Q^2$-suppression is weakened due to evolution which is stronger than in the leading-twist case. In this paper we have presented results on gluonic twist-four operators, which are expected to give the main twist-four contributions at small-$x_B$. For the derivation of these results we have extended the method developed by EFP. Based on the diagrams in Fig. 1 and working in the axial gauge we could perform the factorization of the upper quark-loop parts and the lower blobs. The latter are expressed as matrix-elements of gluonic operators between proton states. This factorization procedure requires the use of Ward identities, which have been derived in section 3. The axial gauge proves to be very convenient for two reasons. First, gluonic $A$-operators occurring between proton states can easily be expressed in terms of $F$-operators which is necessary in order to obtain gauge invariant expressions (section 3). Secondly, diagrams with more than four $t$-channel gluons lead to contributions of, at least, twist-six. This statement is proved in section 5. Working in a covariant gauge, one would have to take into account contributions of diagrams with arbitrarily many $t$-channel gluons.

After the factorization had been performed we ended up with three contributions. The two- and three-gluon quark-loops are convoluted with matrix elements of the two-gluon operator, Eqs. (6.5) and (6.10), while the four-gluon quark-loop is convoluted with both, the two- and the four-gluon operator, Eq. (6.25). In order to derive the latter result, we had to make intensive use of symmetry relations for the expression of the four-gluon quark-loop. These symmetry relations also led to restrictions on the color structure of the four-gluon operator contribution. As the next step we have to define a minimal set of operators in terms of which we can express the operators in (6.5), (6.10) and (6.25). This requires the introduction of signature of the two- and four-gluon $t$-channel states, i.e. a further step is symmetrization of the color and Lorentz indices. It is only after this decomposition into contributions with definite signature that we can return to (1.10) and (1.11) and determine contributions to the structure functions.

It is worthwhile to compare the results presented here with some earlier work. The traditional approach to small-$x_B$ physics is the high-energy (small-$x_B$) approximation, which, in leading order, leads to the BFKL equation. The BFKL equation is usually described by a ladder diagram of two reggeized $t$-channel gluons. It sums all leading logarithms in $x_B$ and contains contributions of all twists. Contributions of diagrams with three and four reggeized $t$-channel gluons have been studied in the generalized leading logarithmic approximation in \[ 19, 20 \]. The results of these calculations have been used to compute the twist-four part in the double-logarithmic approximation (DLA), and it is interesting to compare them with the results of the present paper.

First, in \[ 20 \] it was shown that the three-gluon amplitude can be written as a sum of expressions for the two-gluon (BFKL) amplitude. Since this phenomenon generalizes the reggeization of the gluon, it has been termed reggeization, as well. Color structure and reggeization are independent of the twist. The fact that the three-gluon amplitude can be written as a sum of two-gluon amplitudes is therefore also true for the twist-four part. This result can be confronted with our new result that the three-gluon quark-loop couples only to the two-gluon operator (6.10). Next, the four-gluon amplitude studied in \[ 20 \] splits up into two terms: there is, again, a reggeizing part, which can be expressed with the help of expressions for the two-gluon amplitude, and a second part which is irreducible with respect to reggeization. We are, now, tempted to draw a relation with Eq. (6.25) which also contains contributions from two-gluon and four-gluon operators. In particular, one might associate the second matrix element in (6.25) with the reggeizing part of \[ 20 \], whereas the

\[ 9 \] Recently, C. Ewerz \[ 21 \] has performed an analysis of the five- and six-gluon amplitudes.
first term in (6.25) looks analogous to the irreducible four-gluon amplitude. Here, however, a word
of caution is in place. First, in [20] it is the reggeizing part, which contains color factors like $d^{abcd}$,
while the irreducible part is symmetric under simultaneous exchange of color and momentum. This
is in contrast to the color structure of two- and four-gluon operators in (6.25). Secondly, the
calculations use different approximations. Ref. [20] studies the leading-log($1/x_B$) approximation.
This has lead to the result that the lowest-order diagram with four $t$-channel gluons belongs to the
reggeizing part, which suggest that there is no direct coupling of the four-gluon operator to the
quark loop. In the present paper we have taken a different route: our analysis of $1/Q^2$ corrections
has not been restricted to the small-$x_B$ approximation, and we find contributions of the two-
and four-gluon operator already at leading-order. This can be considered as a hint that one will
encounter simplifications if one uses explicit expressions for the four-gluon quark loop in those
terms which are coupled to the four-gluon operator in Eq. (6.25) and then takes the limit $x_B \to 0$.
Agreement between the results of [20] and the present approach holds only in DLA; to go beyond
this approximation is one of the main motivations of the present paper.

In summary, we have performed the first step towards a consistent treatment of gluonic twist-
two contributions at small-$x_B$. What remains to be done is, first of all, the computation of the
expressions for the quark-loops. While this task might be straightforward, one has to keep in mind
that the expressions will contain contributions proportional to $\log Q^2$ and, therefore, contribute to
leading-order quark-coeffcient functions convoluted by Altarelli-Parisi splitting functions as well as
to next-to-leading order gluon coefficient functions. A major task will be the computation of
anomalous dimensions of the gluon operators. Since several gluonic operators contribute to twist-
two, one will have to handle also the problem of operator mixing. First results are contained in [3]
and [13], but a complete LO analysis does not exist. Moreover, although we have argued that at small-$x_B$
gluon-operators are expected to dominate, the rôle of quark-operators has not been clarified. For a fully consistent treatment of twist-two contributions one should also take
contributions of quark-operators into account. In such an analysis one will encounter the problem
of operator-mixing, as well. We feel that calculations along these lines are urgently needed: a more
complete understanding of deep inelastic $ep$ scattering at low $Q^2$ and small $x_B$ cannot be reached
unless we know the influence of higher twist.

Acknowledgements: We wish to thank L.N.Lipatov and R.K.Ellis for stimulating and very
helpful discussions.

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