Topological Magnetic Excitations

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Topological properties play an increasingly important role in future research and technology. This also applies to the field of topological magnetic excitations which has recently become a very active and broad field. In this perspective article, we give an insight into the current theoretical and experimental investigations and try an outlook on future lines of research.

I. INTRODUCTION

Quantum magnetism continues to fascinate researchers in condensed matter physics. Insulating quantum magnets provide clean solid state systems which are characterized by only a few coupling constants. The magnetic degrees of freedom do not interact strongly with other degrees of freedom which has enabled a multitude of fundamental and applied achievements. Applications range from the compass needle in the distant past to deflection magnets in beamlines and magnetic data storage indispensable in today’s information technologies [1]. The next major revolution in this respect could be not only to store the information magnetically, but also to transmit and to process it by magnetic excitations.

Usually, signal transmission and procession are accompanied by dissipative losses. Topological systems have the outstanding feature to be robust against many perturbations, e.g., by suppressing backscattering and thereby reducing dissipation. This can enhance the efficiency of devices decisively. Since the discovery of the quantum Hall effect [2, 3], topological effects have appeared in condensed matter systems in various forms. Topological magnetic excitations are one important route of research in this area with promising potential for magnetic devices. Using neutral magnetic excitations instead of charge excitations avoids long-range interactions with stray electric fields which act as noise source. Thus, less dissipative devices based on topological magnets can be hoped for.

In the light of the above, this perspective article gives a survey over the current research on magnetic excitations with topological properties focussing on localized spins in insulators or magnetic semiconductors. It is organized in three parts. First, the theoretical point of view is presented. Second, the experimental state-of-the-art is discussed. Finally, we conclude with an outlook. Of course, a complete account over all activities in this booming research field is not possible.

II. THEORETICAL FOUNDATIONS

The symmetries of a system determine the possibility of non-trivial topology. The lower the symmetry, the more coupling terms are possible which may give the eigenstates a twist such that topological characteristics arise. For magnetic insulators the generic twisting couplings are the Dzyaloshinskii–Moriya (DM) interactions

$$\mathcal{H}_{ij,\text{DM}} = D_{ij} \cdot (S_i \times S_j)$$

(1)

resulting from spin-orbit coupling on the electronic level [2, 3]. The above mentioned symmetry aspects are given by Moriya’s rules [1] which constraints the direction a DM vector $D_{ij}$ can point. The selection rules and how spin-orbit coupling leads to DM terms are shown in the Supplemental Sects. V and VI.

An alternative coupling which can also lead to nontrivial topology is the scalar spin chirality [7]

$$\mathcal{H}_{ijk,\text{ssc}} = \chi S_i \cdot (S_j \times S_k),$$

(2)

involving three different spins and favoring their noncoplanar orientation. The cross product in both terms (1) and (2) implies the required magnetic anisotropy. The scalar spin chirality does not necessarily require spin-orbit coupling [8, 9]. Expressing the spin operators in terms of bosons allows one to address the elementary excitations, the quasi-particles, as bosons [10, 11].

On the level of a single particle, bosons and fermions behave in the same way so that features of fermionic models carry over to bosonic and magnonic ones, see also Supplemental Sect. VII. Common representations for ordering quantum magnets are the ones of Holstein and Primakoff [10], of Dyson and Maleev [8, 9] and of Schwinger [15] which describe quantized spin waves: magnons [16]. Disordered magnets can be valence bond solids with triplon excitations [17–19] which can be captured by bond operators [20, 21]. In one dimension (1D) and for strong frustration, fractionalization of the magnetic excitations to spinons [22] can arise. In 1D they can be understood as domain walls. They are intricate objects and do not allow for straightforward bosonic descriptions, see for instance Ref. [23]. Additionally, long-range ordered magnets can host stable solitonic static spin waves such as skyrmions. These four types of magnetic excitations form the body of this survey and are schematically shown in Fig. 1. The magnon excitation is the most common magnetic quasi-particle and therefore takes the largest part of the remainder. Note that the ground states are mostly topologically trivial, independent of the nature of the excitations.

The Berry phase is the basic concept in defining topological properties. In the general context of differential geometry, it is the change of a vector in a fibre bundle.
upon parallel transport along a closed path on the manifold. In quantum mechanics, it appears as the phase that an eigenstate of a parameter dependent Hamiltonian acquires when the parameters are changed along a closed path [12], see Supplemental Sect. VIII. Depending on the relevant parameter space several invariants can be defined by Berry phases. The closed path around the Brillouine zone in two dimensions (2D) defines the Chern number

$$C_n := \frac{1}{2\pi} \int \int_{\text{BZ}} F_{n,xy}(k) dk_x dk_y,$$

where $F_{n,xy}(k)$ is the Berry curvature of the band $n$ expressed by

$$F_{n,xy}(k) := i \left( \langle \partial_{k_x} n | \partial_{k_y} n \rangle - \langle \partial_{k_y} n | \partial_{k_x} n \rangle \right).$$

The wave vectors $k_x$ and $k_y$ parametrize the manifold, here the torus of the 2D Brillouin zone. The key feature of the Chern number is that it can only take integer values so that it cannot change its value continuously. This implies that at boundaries between phases of different Chern numbers the system changes non-perturbatively and thus energy gaps must close. This leads to the famous bulk-boundary correspondence implying the existence of in-gap states [25] which are often localized boundary states if protected by indirect energy gaps in the bulk [19]. Another integer topological invariant derived from the Berry phase is the Zak phase in 1D systems with inversion symmetry [27]. In addition, winding numbers characterize topological objects beyond Berry phases. They are based on covering mappings of closed manifolds onto spheres of various dimensions [28–30].

### A. Magnon excitations

A fermionic lattice model with a non-trivial Chern number was first proposed by Haldane [7] implying a quantized transversal conductance. Exploiting the analogy, Haldane and Arovas [32] transferred the concept also to a quantum spin system proposing transversal spin current due to a finite Chern number in a 2D chiral disordered gapped quantum magnet. Then it took 15 years before conventional ordered magnets were considered for topological transport phenomena [33]. The prediction of a thermal Hall effect [34] induced by topological magnon or spinon bands has triggered studies on topological quantum magnets since the transversal heat conductivity is accessible in experiments. Topological magnons were first proposed on the kagome lattice induced by scalar spin chirality [34, 35]. Magnon bands in an insulating kagome ferromagnet (FM) with DM interactions were calculated subsequently [36–39]. An external magnetic field can be added to fully polarize the ground state, to break time-reversal symmetry or to open energy gaps between bands.

Further FM lattices with DM interactions have been analyzed to show topological magnons with non-trivial Chern numbers such as the pyrochlore [40], the Shastry–Sutherland [19], the honeycomb [5, 6, 42], the bilayer honeycomb [44] and the Lieb lattice [45]. Studies of periodic arrays of FM islands have shown that magnetic dipole-dipole interactions can also lead to topological magnons on square and honeycomb lattices [46–48]. The elementary excitations of antiferromagnetically (AFM) ordered models are magnons as well, but magnons in AFMs differ from their FM counterparts in that they have a linear dispersion at low wave number and that their number is not conserved. In second quantization, this also requires to pass from the standard scalar product to an adapted symplectic product, also called ‘parannuity’, see e.g. Ref. [11] and references therein. Non-trivial Chern numbers of AFM magnon bands have been investigated in the honeycomb [6, 50], the pyrochlore [51], the kagome [52, 53], the Shastry–Sutherland [54] and the triangular [55] lattice.

The Kitaev model [56] is strongly frustrated due to the bond-directional couplings which favor exotic quantum states such as quantum spin liquids. The required anisotropy for non-trivial Chern numbers is intrinsically present if Kitaev interactions are combined with FM or AFM couplings [57, 58].

The bulk-boundary correspondence [25, 37] predicts chiral magnonic edge modes for non-trivial Chern numbers [47]. Here, the term ‘chiral’ means that the excitations propagates only in one direction along one edge [59, 60] like, e.g., skipping orbits in the quantum Hall effect. This non-reciprocity is caused by the Berry curvature which acts on the excitations like a magnetic field on charges. A temperature gradient unbalances the occupation of magnons at two opposite edges implying a thermally driven transversal spin currents, the so-called

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**FIG. 1:** Sketch of a magnon (a), a triplon (b), a spinon (c) and a skyrmion excitation (d). The blue and red bonds in (b) correspond to a singlet and a triplet state, respectively. Panel (d) shows a Néel (left) and a Bloch (right) skyrmion.
spin Nernst effect [35, 36, 44, 61–65].

The magnonic analogue of a fermionic Chern insulator is not the only kind of a magnetic topological phase. Magnons can also characterize a topological semi-metal phases where the term ‘metal’ is to be understood only as analogy to fermionic systems. Topological semi-metals are divided in three categories: Weyl, Dirac and nodal line semi-metals. These systems are characterized by finite quantized Berry phases around point degeneracies (monopoles) or line degeneracies (nodal lines) in momentum space. Their boundary states are Fermi arcs for monopoles and drumhead surface states for nodal lines. Magnonic Weyl systems have been proposed for 3D pyrochlore FMs and AFMs [66–69], a stacked honeycomb FM [70, 71], and a stacked kagome AFM [72, 73]. If time-reversal symmetry and inversion symmetry are preserved each energy on the Dirac cone is doubly degenerate. By analogy, the magnetic counterpart of a Dirac semimetal has been predicted [74–77]. Topological nodal lines indicate the degeneracy of two bands along this line in momentum space induced by topology, i.e., the Berry phase around such line does not vanish, but takes integer values. Magnonic analogues have also been found [77–80]. Lastly, a 3D topological insulator with magnons has also been suggested [81].

Topological magnons in 1D can be characterized by the Zak phase, i.e., the Berry phase along the 1D Brilloin zone. So far, only a few magnetic chain systems are suggested with topologically non-trivial magnons [82–84]. The bulk-boundary correspondence predicts the existence of end states within the indirect gap between magnon branches of non-trivial Zak phase, cf. Ref. [11].

In addition to static topological models, constant in space, first ideas on Floquet engineering [85] have been advocated. The magnetic dipole moment of a magnon implies that it acquires a phase when propagating through an electric field. This constitutes the Aharonov–Casher effect [86] analogous to the commonly known Aharonov–Bohm effect for charges in a magnetic field [87]. Hence, magnonic Landau levels can appear [63, 88]. The electric field generated by a laser implies a time-periodic perturbation amenable to Floquet theory. This route to control the effective Hamiltonian has recently been studied [89–93].

The field of topological magnons has so far been strongly driven by transferring and adapting ideas from fermionic systems. But one has to keep in mind essential persisting differences. Fermionic bands can be completely filled or empty at zero temperature which is the basis of the quantization of the quantum Hall conductivity [13–16]. In bosonic systems, the notion of filled bands does not exist, hence no quantization of the transversal conductivities is to be expected. Furthermore, many electronic systems with topological properties preserve time-reversal symmetric systems so that the $\mathbb{Z}_2$ topological invariant applies. Magnetically ordered systems, however, break time-reversal symmetry so that the definition of a $\mathbb{Z}_2$ topological invariant is only possible in combination with additional symmetries. In this way, phenomena similar to the quantum spin Hall state can be established for magnetic excitations [65, 88, 98–100].

B. Triplon excitations

Besides conventional long-range ordered system there exist many quantum antiferromagnets which are valence bond solids. Certain bonds, dimer bonds, dominate over the other bonds. For instance, the rung bonds in a spin ladder dominate over the leg bonds [101–103]. Then the elementary excitations are triplons [18], i.e., dressed excitations of the dimer bonds from the singlet $S = 0$ to the triplet state with $S = 1$. In isotropic models, of course, the triplons are three-fold degenerate.

Strong frustration favors valence bond solids over long-range magnetic order as in the Shastry–Sutherland lattice [104], and hence favors triplons as elementary magnetic excitations. Their mobility as expressed by their dispersion is strongly suppressed by frustration, again impressively illustrated in the Shastry–Sutherland lattice [105, 106]. The reduced dispersion renders the degenerate triplon bands prone to topological twists induced by small perturbations such as DM terms inducing non-trivial topology [107–109]. The dimerized honeycomb lattice can display non-trivial triplon bands [110] in form of a bilayer of two honeycomb lattices which makes a triplon Hall effect possible [111].

Topological behavior of triplons in 1D systems such as spin ladders has been predicted by investigating the Zak phase [11, 112] or the winding number [113]. In BiCu$_2$PO$_6$ and Ba$_2$CuSi$_2$O$_6$Cl$_2$ it has been verified by inelastic neutron scattering data [11, 112]. Topological triplons in chains may imply localized end states, but only if they are protected by indirect band gaps [11].

By extension of spin dimers with spin-1 triplons, certain lattices may also trimerize with spin-1/2 doublets and spin-3/2 quartets excitations which become topologically non-trivial if a sufficiently strong DM term is present [114].

C. Spinon excitations

The term ‘spinon’ is used for fractional magnetic excitations of spin-1/2. Spinons appear also in correlated electronic systems where spin and charge separate in a spinon and a holon excitation. Hence, a number of studies addresses the question of topological properties of Mott insulators, i.e., electronic systems with sufficiently strong repulsive interactions so that they turn insulating [115–124]. These approaches are constructed to address systems without long-range magnetic order and without spontaneous or non-spontaneous dimerization. A widespread caveat consists in the use of mean-field approaches which start from the separation of spin and charge degrees of freedom. If one follows this line of reasoning
effective magnetic fields engender Lorentz forces, Landau levels, and concomitant transversal conductivities [34, 125]. Far from any isotropic spin models, for instance for variants of the Kitaev model without any long-range order, deconfined spinon descriptions with topological properties can be established [126, 127].

D. Skyrmion excitations

Talking about topological properties in magnets, skyrmions have to be mentioned. However, the so far considered topological excitations are itinerant and can be described by dispersive bands of which the topology is captured by Berry phases. In contrast, skyrmions are static, long-lived magnetic textures in the nanometer range in FMs or AFMs which are topologically protected. This means that their creation or annihilation requires to overcome a substantial energy barrier. They are not characterized by quantum Berry phases in momentum space, but by winding numbers in real space such as

$$ n = \frac{1}{4\pi} \iint dx dy \, m \cdot (\partial_x m \times \partial_y m), $$

where $m$ corresponds to the local magnetization. We do not go into detail because many excellent reviews of this field exist [128–134].

While skyrmions are \textit{eo ipso} static, they can be moved by various means, for instance by applying fields and/or currents. In this way, they can store information, transport it, and may allow for processing it. This triggers abundant studies on applications, see above cited reviews.

Skyrmion lattices, i.e., periodic arrays of skyrmions, can be seen to generate local magnetic field twisting the local quantization axes. This in turn generates Berry phases for the magnons, the itinerant bosonic excitations on top of the skyrmion lattices, for FM [135, 136] and AFM systems [137, 138]. They show phenomena like the topological Hall effect [139, 140]. This findings were triggered by the observation that a magnon can acquire a Berry phase by propagating along a closed contour in non-trivial magnetization profiles [141].

III. EXPERIMENTAL TECHNIQUES AND RESULTS

While there are abundant theoretical studies on a large variety of quantum magnets with topological properties, experimental observations are still scarce. So far, there are essentially two kinds of studies, namely inelastic neutron scattering and transport measurements. The dispersion of the itinerant magnetic excitations is measured by inelastic neutron scattering (INS). This only provides the eigenenergies, but no direct information on the eigenstates. However, it is the momentum dependence of eigenstates which defines the band topology, cf. Eq. (4). Hence, only the comparison of the INS data to the results of theoretical models allows one to establish non-trivial topology. In this way, topological magnons are confirmed in magnonic insulators [38, 142] and in magnonic semimetal [143–147]. Topological triplons are established in the Chern insulator SrCu$_2$(BO$_3$)$_2$ [109] and in quasi-1D spin ladders by their Zak phase [11, 112].

Detecting transversal conductivities is the second experimental pillar to find non-trivial topology. Since magnetic excitations are neutral, one aims at the thermal Hall effect. Separating phononic from magnonic effects is an issue in this respect. If the temperature is low enough, the magnetic contribution dominates over the phononic one. So far, the thermal magnon Hall effect has been observed only for few material [148–152]. In a quantum spin ice magnet a thermal Hall effect was found and assigned to neutral excitations, but they could not unambiguously identified as magnons or phonons [153]. For SrCu$_2$(BO$_3$)$_2$, no transversal conductivities could be measured [154] in spite of the INS results [109].

IV. OUTLOOK AND CONCLUSION

This outlook represents a subjective view on ongoing or imminent developments as well as important issues that need to be addressed in the authors’ opinion. Glancing at the above survey, it suggests itself that further experimental progress is more urgent than theoretical one. Nevertheless, there are also important theoretical challenges.

A. Experimental Challenges

It is desirable to establish a set of well-characterized model compounds which allow for detailed experimental studies. With future applications in mind, the robustness and transport properties of the edge states are important. Efficient spin channels should display low dissipation, stability against disorder and be subjected to low Joule heating [155]. Up to now, more FM systems than AFM systems are known because the generation and in particular the detection of magnetic excitations are easier for the former. Yet, it is certainly indicated to enforce the search for more AFM compounds. Their linear dispersion at low energies implies much faster propagation of magnetic excitations than in their FM counterparts. In addition, the linear dispersion implies a significantly reduced phase space for magnon decay into two or three magnons. In FMs, the quadratic dispersion implies a continuum of scattering states of two magnons into which a single magnon can decay.

Having information processing in mind, magnetic transport is the key feature. Thus the energetically low-lying excitations matter more than the high-lying ones unless high-lying excitations can be specifically generated. So the specific generation and detection of mag-
magnetic excitations is crucial. So far, bulk properties were the focus of the investigations. But the most interesting physics goes on at the boundaries of topologically non-trivial phases. Hence, one needs experiments which can directly address the edge modes so that the signal does not need to be separated from a bulk signal. Clearly, this requires spatial resolution so that the boundaries can be excited separately. Furthermore, temporal resolution is required to measure velocities of signal transmission along the edges which constitutes a key quantity of edge modes [156–158]. Then, control and manipulation of topological magnonic systems can be envisaged which will trigger a new boost to the field. A conceivable way to achieve spatial and temporal control is optical pumping by inverse Faraday effect [159, 160] or by parametric pumping [161].

Finally, one wants to have interfaces between topological magnonics and conventional electronics. To this end, spin-to-charge conversion is a necessity for which first experimental results are available [162, 163] and further theoretical ideas are proposed [164].

Between generation and detection, the control of the magnonic information is essential. This implies switching signals, amplifying and delaying them, having them interfere with one another. These issues are still fully at their infancy experimentally. Yet, there exist ideas as to how spin-wave diodes, spin-wave beam splitters and spin-wave interferometers can be realized and controlled by manipulation of domain walls [165].

Exemplarily, we outline a way to tune the velocity of signal transmission; for magnetic systems this is the spin wave velocity. Tailored boundaries of magnetic samples, see Fig. 2, can reduce this velocity by hybridizing propagating edge modes with local, non-dispersive modes in the attached bays. For electronic systems this has been studied thoroughly for topological systems with chiral edge modes [156–158]. Adjusting the single-ion anisotropy [166] in order to change the dispersion is a promising route.

B. Theoretical Challenges

The everlasting task of theory is to propose models and mechanisms which explain experimental observations. In addition, theory can propose setups and systems to get better handles on the relevant experimental signatures. Furthermore, there are also intrinsic theoretical issues.

Conceptually, damping or decay of the excitations which carry the information is a crucial issue. The signal damping is an experimental fact so that any measurement needs to be faster than the decay of the signal. But there resides also a deeper conceptual issue: how is a Berry phase properly defined if there are no well-defined elementary particles because they hybridize with the continua of scattering states? The first steps are undertaken [58, 75, 167, 168], but deeper insight is still missing. Strong magnon-magnon or triplon-triplon interactions can give rise to bound states. Their interplay with the elementary magnons or triplons also needs to be taken into account [82, 83].

In addition, the interaction between magnetic and phononic degrees of freedom can also play an important role either by leading to topological magnons [169] or to topological phonons [170] or to topological polarons [171].

C. Conclusion

On the one hand, topological magnonics, i.e., using dispersive magnetic excitations for information storage, transmission and processing, is a very promising and fundamentally exciting research area. On the other hand, the experimental and theoretical issues to be tackled and solved are challenging. This should trigger our curiosity and ingenuity to come up with seminal solutions!

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Supplemental Material for "Topological Magnetic Excitations"

This supplement consists of four sections. (I) We provide the symmetry analysis of spin lattices according to the five selection rules of Moriya. (II) We illustrate the derivation of the Dzyaloshinskii-Moriya interaction from spin-orbit coupling. (III) A ferromagnetic Heisenberg model on a honeycomb lattice illustrates the similarity between electronic and magnonic systems. By applying a bosonic spin representation we deduce a magnon model analogous to the Haldane model. (IV) We introduce the concept of the Berry phase and illustrate it by the example of the Su-Schrieffer–Heeger model. The latter model also enables us to illustrate boundary states as implied by the bulk-boundary correspondence.

V. DZYALOSHINSKII-MORIYA INTERACTION

The Dzyaloshinskii-Moriya (DM) interaction can be seen as the main driving mechanism to introduce a topological twist to eigenstates of magnetic systems. The point group symmetries of the model restrict the directions of the DM vectors $D$ which in turn determine the DM interactions according to the five selection rules formulated by Moriya [S1]. Here, we briefly present these five selection rules considering two coupled ions with spin.

One spin is at site $A$ interacting with the spin at site $B$. The center point of the connecting line $AB$ is labeled by $C$.

1. If $C$ is a center of inversion, $D = 0$ is implied.
2. In case a mirror plane perpendicular to $AB$ exists and comprises $C$, $D \perp AB$ holds so that $D$ lies in this mirror plane.
3. If a mirror plane comprises sites $A$ and $B$, the vector $D$ must be perpendicular to it.
4. If a twofold rotation axis exists perpendicular to $AB$ and passing through $C$, the vector $D$ is parallel to this rotation axis.
5. In case of an $n$-fold rotation axis ($n \geq 2$) along $AB$ exists, $D \parallel AB$ holds.

As stated the selection rules are based on inversion, mirror and rotation symmetry. The degree of restriction of the DM vector increases for increasing symmetry. Thus, lower symmetry is favorable for topological effects.

VI. DERIVATION OF THE DZYALOSHINSKII-MORIYA INTERACTION

The aim is to derive the term

$$\mathcal{H}_{DM} = \sum_{(i,j)} D_{ij} \cdot (S_i \times S_j) \quad (S1)$$

for two sites $i \neq j$ where we restrict ourselves to nearest-neighbour sites $\langle i,j \rangle$ for clarity. The fermionic Hamiltonian belongs to a Hubbard model and is split into a local on-site Hamiltonian $\mathcal{H}_0$ and a hopping Hamiltonian $\mathcal{H}_1$

$$\mathcal{H}_0 = E \sum_{i,\sigma} a_{i,\sigma}^\dagger a_{i,\sigma} + U \sum_{i} a_{i,\uparrow}^\dagger a_{i,\uparrow} a_{i,\downarrow}^\dagger a_{i,\downarrow} \quad (S2a)$$

$$\mathcal{H}_1 = t \sum_{\langle i,j \rangle, \sigma} a_{i,\sigma}^\dagger a_{j,\sigma} + \sum_{\langle i,j \rangle, \alpha\beta} C_{ij} \cdot \sigma_{\alpha\beta} a_{i,\alpha}^\dagger a_{j,\beta} \quad (S2b)$$

where $E$ is the local single-particle energy, $t$ the isotropic hopping element, $U$ the local Hubbard repulsion, $\sigma$ is the usual vector of Pauli matrices, $\alpha, \beta \in \{\uparrow, \downarrow\}$ and the vector $C_{ij}$ determines the spin dependent hopping resulting from spin orbit coupling.

The main step is to apply perturbation theory to the half-filled states with precisely one electron at each site in second order of $\mathcal{H}_1$. The resulting effective Hamiltonian is given by

$$\mathcal{H}_{\text{eff}} = P_0 \mathcal{H}_1 P_1 (E_0 - \mathcal{H}_0)^{-1} P_1 \mathcal{H}_1 P_0 \quad (S3)$$

where $P_n$ is the projection operator onto the Hilbert subspace with precisely $n$ double occupancies and $E_0$ the ground state energy. The hopping term automatically creates one double occupancy so that $(E - \mathcal{H}_0)^{-1} = (-U)^{-1}$ holds. The second application of $\mathcal{H}_1$ annihilates the double occupancy so that the system reaches again the states with all sites singly occupied. States with additionally created double occupancies are projected out by $P_0$. If both hoppings are isotropic, the known antiferromagnetic superexchange $J = 4t^2/U$ results

$$H_{\text{isotropic}} = J \sum_{\langle i,j \rangle} S_i \cdot S_j \quad , \quad (S4)$$
where we use
\begin{align}
a_{i,\downarrow}^{\dagger}a_{i,\uparrow} - a_{i,\uparrow}^{\dagger}a_{i,\downarrow} & = 2S_i^z \quad (S5a) \\
a_{i,\downarrow}^{\dagger}a_{i,\downarrow} & = S_i^z \quad (S5b) \\
a_{i,\uparrow}^{\dagger}a_{i,\uparrow} & = S_i^{-} \quad (S5c)
\end{align}

But in presence of spin-dependent hopping one hopping can be isotropic (t) and the other spin-dependent (C) leading to the antisymmetric DM coupling in Eq. \((S1)\) with
\[ D_{ij} = \frac{4i}{U} [tC_{ij} - tC_{ji}] \quad (S6) \]

There are also symmetric couplings arising in second order in \(C\), but we do not discuss them here and refer the interested reader to specialized references \([S2, S3]\).

\section{VII. Similarity Between Electronic and Magnonic Models}

In order to clearly see the similarity between electronic and magnonic systems it is helpful to express the spin operators in terms of elementary excitations of bosonic nature. For illustration, we present the example of topological magnons in a ferromagnetic honeycomb lattice \([S5, S6]\). The aim is to show that single magnons in this model essentially behave like single fermions in the Haldane model. We choose the Haldane model \([S7]\) because it is one of the first and paradigmatic fermionic models with non-trivial topology and often used for proof-of-principle studies.

The spin Hamiltonian consists of three parts
\begin{align}
\mathcal{H} & = \mathcal{H}_{NNN} + \mathcal{H}_{NNN} + \mathcal{H}_{DM} \quad (S7a) \\
\mathcal{H}_{NN} & = -J \sum_{\langle ij \rangle} S_i \cdot S_j \quad (S7b) \\
\mathcal{H}_{NNN} & = -J' \sum_{\langle\langle ij \rangle\rangle} S_i \cdot S_j \quad (S7c) \\
\mathcal{H}_{DM} & = \sum_{\langle\langle ij \rangle\rangle} D_{ij} \cdot (S_i \times S_j) \quad (S7d)
\end{align}

with two isotropic Heisenberg couplings \(J\) and \(J'\) and one DM coupling \(D_{ij}\). The isotropic couplings are ferromagnetic \(J, J' > 0\) while pairs of nearest neighbors (NN) and next-nearest neighbors (NNN) are denoted by \(\langle ij \rangle\) and by \(\langle\langle ij \rangle\rangle\), respectively. For clarity, the DM couplings are restricted to a direction perpendicular to the plane of the lattice \(D_{ij} = \nu_{ij}D_\phi\hat{e}_z\) with \(\nu_{ij} = \pm 1\). The anti-clockwise hopping from site \(i\) to site \(j\) around a honeycomb is counted positive while the clockwise hopping is counted negative. We assume a fully polarized ground state in \(z\) direction even in presence of DM interaction.

There are several bosonic representations of the spin operators. In leading order in \(1/(2S)\), both the Dyson–Maleev representation \([S8, S9]\) and the Holstein-Primakoff representation \([S10]\) correspond to
\begin{align}
S_i^+ & = \sqrt{2S}(b_i + O((2S)^{-2})) \quad (S8a) \\
S_i^- & = \sqrt{2S}(b_i^\dagger + O((2S)^{-2})) \quad (S8b) \\
S^z & = b_i^\dagger b_i - S \quad (S8c)
\end{align}

In order to stay in a one-particle framework and to focus on low-lying excitations we use the harmonic approximation which means that we keep only bilinear terms. Thus, the first Heisenberg term is expressed as
\[ \mathcal{H}_{NN} = -JS \sum_{\langle ij \rangle} \left[ (b_i^\dagger b_j + b_j^\dagger b_i) - (b_i^\dagger b_i + b_j^\dagger b_j) + S \right] \quad (S9b) \]

The first round bracket in Eq. \((S9b)\) describes NN hopping of magnons on the honeycomb lattice while the second round bracket shifts the dispersion. The NNN Heisenberg couplings yield the analogous result. The same procedure is applied to the DM interaction
\begin{align}
\mathcal{H}_{DM} & = \sum_{\langle\langle ij \rangle\rangle} D_{ij} S_i \times S_j \quad (S10a) \\
& = -\frac{iD_z}{2} \sum_{\langle\langle ij \rangle\rangle} \nu_{ij}(S_i^+ S_j^- - S_i^- S_j^+) \quad (S10b) \\
& = -iD_z S \sum_{\langle\langle ij \rangle\rangle} \nu_{ij}(b_i^\dagger b_j - b_j^\dagger b_i) \quad (S10c)
\end{align}

Combining both hopping terms to the NNN yields the complex hopping known from the Haldane model
\[ [\mathcal{H}_{NNN} + \mathcal{H}_{DM}]_{\text{hop}} = -S \sum_{\langle\langle ij \rangle\rangle} (J' + \nu_{ij}D_z) b_i^\dagger b_j \quad (S11a) \]
\[ = -S \sqrt{J'^2 + D^2_z} \sum_{\langle\langle ij \rangle\rangle} e^{i\nu_{ij}\phi} b_i^\dagger b_j \quad (S11b) \]

with \(\phi = \arctan(J'/D_z)\). This shows how a finite DM interaction induces a complex hopping for magnons on the honeycomb lattice which is responsible for the topological twist. Eventually, we obtain the Haldane model in terms of bilinear bosonic operators representing the magnons of the original spin Hamiltonian. The constant on-site terms do not affect the topology in the bulk because they do not influence the eigenstates. Similarly, combinations of isotropic Heisenberg and anisotropic DM interactions can be used on any lattice model to emulate electronic models by a spin model on the level of single elementary excitations. In analogy, the crossproduct in the scalar spin chirality can act like the DM interaction.
In case of antiferromagnetic couplings on bipartite lattices, similar calculations can be done once one has mapped the Néel state to the fully polarized state by rotating the spins of one sublattice by 180°. But even on the bilinear level, terms creating or annihilating pairs of bosons appear which break the conservation of particle number. Still, a suitable Bogoliubov transformation diagonalizes the model yielding well-defined dispersions and eigenstates. The necessary Bogoliubov transformations are no longer represented by unitary matrices so that the subsequent calculation of Berry phases and Chern numbers requires to use a generalization of the standard scalar product to a so-called ‘symplectic’ or ‘paraunitary’ product for the involved operators, see for instance the Appendix of Ref. [S11] for a comprehensive presentation of this subtlety.

Of course, the harmonic, bilinear approximation used above neglects effects such as two-particle interactions and spontaneous quasi-particle decay. Furthermore, one still has to keep in mind that the bosonic quasi-particles obey another statistics than fermions. Nevertheless, one expects that particular topological properties of the magnetic excitations are retained.

Finally, we point out that besides the representation of elementary excitations in ordered spin systems by ferromagnetic or antiferromagnetic magnons the elementary excitations in disordered spin systems can often be described by triplon operators, see main text. For them, DM interactions lead similarly to complex hopping processes yielding topological twists.

**VIII. BERRY PHASE AND BOUNDARY STATES**

The Berry phase [S12] is one of the fundamental concepts to describe topological characteristics. For instance the quantum Hall effect is classified by topological invariants [S13–S16] based on the Berry phase. We will introduce the notion of the Berry phase and briefly present the example of the Su-Schrieffer–Heeger (SSH) model [S17, S18] to gain deeper insight.

We start by considering an adiabatic evolution of a general Hamiltonian depending on a parameter set \( P \) along a closed path \( \Gamma \). According to the adiabatic theorem the non-degenerate eigenstate remains in its instantaneous eigenstate if the variation of continuous parameters is slow enough and the considered eigenstate is separated from other states by an energy gap. The instantaneous eigenstates of the Hamiltonian can be calculated by diagonalization of

\[
H(P) |n, P\rangle = E_n(P) |n, P\rangle ,
\]

at each point \( P \) in parameter space where \( n \) is the label of the energy eigenstates. Note that the instantaneous eigenstates are not completely determined by this equation since nothing fixes their phase. In order to remove this arbitrariness we assume that a smooth gauge of the phase can be found such that \( f(P, P') := \langle n, P|n, P'\rangle \) is a differentiable function of both arguments. The Berry phase \( \gamma_n \) depends only on the contour \( \Gamma \) in parameter space along which the system is changed and on the state \( n \) considered. It also depends on the phase gauge unless the contour \( \Gamma \) is closed.

To see how the Berry phase appears we follow the Gedankenexperiment that the system is varied in time in parameter space along the closed path \( P(t) \) starting at \( t = 0 \) and finishing at \( t = t_f \) with \( P(t_f) = P(t = 0) \). According to the adiabatic theorem one can describe the time evolution of the state by the ansatz

\[
|\psi(t)\rangle = \exp(i\theta(t))|n, P(t)\rangle
\]

with a smooth phase \( \theta(t) \). Inserting the ansatz into the the Schrödinger equation (\( \hbar = 1 \)) and multiplying with \( \langle n, P(t) \rangle \) from the left yields

\[
-E_n(P(t)) + i\langle n, P(t) \rangle \frac{d}{dt}|n, P(t)\rangle = \frac{d}{dt}\theta(t) \tag{S14}
\]

which is solved by

\[
\theta(t) = -\int_0^t E_n(P(t')) dt' \tag{S15a}
\]

\[
+ i\int_0^t \langle n, P(t') \rangle \frac{d}{dt'}|n, P(t')\rangle dt' . \tag{S15b}
\]

The first term (S15a) is the dynamic phase growing linearly in time if the process is extended to longer and longer times, i.e., performed slower and slower. The second term (S15b) describes the geometric phase; this is the Berry phase. Note that it does not matter how fast the path \( P(t) \) is followed. The oscillent time dependence is removed if the explicit dependence on \( P \) is used instead

\[
\frac{d}{dt}|n, P(t')\rangle dt' = \nabla_P|n, P\rangle \tag{S16}
\]

yielding eventually

\[
\gamma_n = i \int_\Gamma \langle n, P|\nabla_P|n, P\rangle \cdot dP \tag{S17a}
\]

\[
= \int_\Gamma A_n(P) \cdot dP \tag{S17b}
\]

\[
A_n(P) := i\langle n, P|\nabla_P|n, P\rangle \tag{S17c}
\]

where the quantity \( A_n(P) \) is called Berry connection.

If one considers the closed contour around the Brillouin zone, i.e., \( P = k \), one can define a topological invariant from the Berry phase. For instance, we consider the one-dimensional Su-Schrieffer-Heeger (SSH) model which is described by the tight-binding model

\[
\mathcal{H}_{\text{SSH}} = \sum_i \left( wc_{i,A}^t c_{i,A} + wc_{i+1,A}^t c_{i+1,B} \right) + \text{h.c.} . \tag{S18}
\]
The dispersion of the two bands is easily found resulting from Fourier transformation of (S18). The bulk Hamiltonian at given wave vector \( k \) is given by shown by the yellow shaded box and the lattice constant by red and green bonds, respectively. The unit cell is highlighted by yellow shading. The two limiting cases are shown in (b) and (c).

The bipartite chain is shown in Fig. S1 where the intracellular hoppings \( v \) and intercell hoppings \( w \) are displayed by red and green bonds, respectively. The unit cell is shown by the yellow shaded box and the lattice constant is given by \( a \). The \( 2 \times 2 \) matrix representation \( H_k \) of the Hamiltonian at given wave vector \( k \) can be denoted by

\[
H_k = d \cdot \sigma ,
\]

where \( \sigma \) is the vector of Pauli matrices and

\[
d(k) := (v + w \cos(k), w \sin(k), 0)
\]

resulting from Fourier transformation of (S18). The bulk dispersion of the two bands is easily found

\[
E_{\pm}(k) = \pm d = \pm \sqrt{d_x^2 + d_y^2 + d_z^2}.
\]

The two bands are separated for \( v \neq w \) while \( v = w \) defines the transition line between the two topological phases of the SSH model where the energy gap between both bands closes. The two corresponding eigenstates can be expressed by

\[
|\pm\rangle = \frac{1}{\sqrt{2d(d \pm d_z)}} \left( d_z \pm d \right) .
\]

Consequently, the Berry connection is given by

\[
A_{\pm} = i(\pm |\partial_k|) = \frac{-1}{2d(d \pm d_z)}(d_y \partial_k d_x - d_x \partial_k d_y) .
\]

The Berry phase is calculated by integration of \( A_{\pm} \) over \( k \) from 0 to \( 2\pi \) as defined in Eq. (S17b). For \( |v| > |w| \) the Berry phase is 0 while for \( |v| < |w| \) it takes the finite value \( \pm \pi \) which corresponds to a nontrivial topological character of this quantum phase. Note that swapping \( v \leftrightarrow w \) does not change the bulk dispersion, but does change the eigenvectors. This exemplifies that the topological twist is encoded in the eigenstates of the system and not in its eigenenergies.

The SSH model also allows us to illustrate the bulk-boundary correspondence by considering finite chains in the two limiting cases shown in Fig. S1(b) and (c). The non-trivial topological case for \( v = 0 \) displays localized edge states at the ends of the finite piece of chain while the topologically trivial case \( w = 0 \) leads only to isolated dimers. This illustrates that generically the quantum phase with non-trivial Berry phase \( \gamma_n \neq 0 \) displays localized boundary states while the trivial phase with Berry phase \( \gamma_n = 0 \) does not. But we emphasize that the topological edge states may delocalize in spite of non-trivial topological invariants if they are not protected additionally by an indirect energy gap, for details see Ref. [S11, S19].

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