Asymptotic behavior of a three-dimensional haptotactic cross-diffusion system modeling oncolytic virotherapy

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Abstract

This paper deals with an initial-boundary value problem for a doubly haptotactic cross-diffusion system arising from the oncolytic virotherapy

\[
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla v) + \mu u(1 - u) - uz, \\
    v_t &= -(u + w)v, \\
    w_t &= \Delta w - \nabla \cdot (w \nabla v) - w + uz, \\
    z_t &= D_z \Delta z - z - uz + \beta w,
\end{align*}
\]

in a smoothly bounded domain \(\Omega \subset \mathbb{R}^3\) with \(\beta > 0\), \(\mu > 0\) and \(D_z > 0\). Based on a self-map argument, it is shown that under the assumption \(\beta \max\{1, \|u_0\|_{L^\infty(\Omega)}\} < 1 + (1 + \frac{1}{\min_{x \in \Omega} u_0(x)})^{-1}\), this problem possesses a uniquely determined global classical solution \((u, v, w, z)\) for certain type of small data \((u_0, v_0, w_0, z_0)\). Moreover, \((u, v, w, z)\) is globally bounded and exponentially stabilizes towards its spatially homogeneous equilibrium \((1, 0, 0, 0)\) as \(t \to \infty\).

1 Introduction

Oncolytic virus (OV) is either a natural or genetic-engineered virus which can specifically infect cancer cells, enlarge the quantities through replication inside the cancer cells and eventually lyse them, while ideally leave normal cells unharmed\([9, 11]\). Accordingly, comparing
with the traditional treatments towards cancer disease, oncolytic virus therapy has recently been recognized as a promising alternatives and nowadays has been used in current clinical trials\[4, 6, 18, 23\]. However, such partially success in some clinical trials still reveals the limitation in implementation. In fact, the efficacy of the so-called virotherapy will be limited by a plenty of factors, like the deposits of extracellular matrix (ECM), immune system, or the circulating antibodies\[10, 19, 33, 38\]. Therefore, to better understand the underlying mechanisms that limit the efficacy of clinical treatment, Alzahrani et al.\[1\] introduced a mathematical model to simulate the interaction between the uninfected cancer cells \(u\), infected cancer cells \(w\), extracellular matrix \(v\) and oncolytic virus \(z\), which is formulated by

\[
\begin{aligned}
    u_t &= D_u \Delta u - \xi_u \nabla \cdot (u \nabla v) + \mu_u u (1 - u) - \rho_u u z, & x \in \Omega, \ t > 0 \\
    v_t &= - (\alpha_u u + \alpha_w w)v + \mu_v v (1 - v), & x \in \Omega, \ t > 0 \\
    w_t &= D_w \Delta w - \xi_w \nabla \cdot (w \nabla v) - \delta_w w + \rho_w w z, & x \in \Omega, \ t > 0 \\
    z_t &= D_z \Delta z - \delta_z z - \rho_z u z + \beta w, & x \in \Omega, \ t > 0
\end{aligned}
\]  

in a smoothly bounded domain \(\Omega \subset \mathbb{R}^N\). The parameters \(D_u, D_w, D_z, \xi_u, \xi_w, \alpha_u, \alpha_w, \mu_u, \delta_w, \delta_z, \beta\) are positive while \(\mu_v, \rho_u, \rho_w, \rho_z\) are nonnegative. The underlying modeling hypothesize are that apart from its random diffusion, both type of cancer cells will be attracted simultaneously due to some macromolecules bounded in the extracellular matrix, and that uninfected tumor cells, except perhaps proliferating logically, are converted into a state of infection in contact with virus particles, and infected cells die due to cytolysis. In addition, the virus particles will adhere on the surface of cancer cells and then the newly infectious virus will be released from its inside of tumor with rate \(\beta > 0\); beyond this, both of uninfected and infected cancer cells can degrade the static ECM due to the matrix degrading enzymes and the ECM is possibly re-established according to a logistic law.

Due to the relevance with several biological contexts, inter alia the cancer invasion or angiogenesis\[2, 3\], the haptotactic mechanism has received considerable attention in current literatures\[7, 12, 13, 14, 16, 17, 20, 21, 15, 24, 26, 34, 39, 40\]. It is noted that as the striking feature thereof, two simultaneous haptotactic cross-diffusion terms distinguishes (1.1) from the most of the studied chemotaxis-haptotaxis system\[12, 13, 14, 20, 26, 39\] and haptotaxis system\[7, 16, 17, 21, 34, 40\]; apart from that, the respective mechanism in (1.1)
is potentially enhanced through the zero-order nonlinear production term $+ \rho_w u z$ even in
the two-dimensional case, which brings significant challenges to rigorous analysis even at the
level of elementary solution theory. This reflects the situation that the comprehensive result
on (1.1) seems to be rather thin, particularly in the physically relevant three-dimensional
setting, although the qualitative information has been successfully achieved for some simple
variants of (1.1) [5, 22, 15, 30, 32, 36]. For example, the issue of the global solvability of
system (1.1) was addressed in [25] in the spatially two-dimensional framework, while it is
still unaddressed in the corresponding three-dimensional situation. Recently, based on the
previous result of global solvability, the $L^\infty$-exponential convergence property towards its
equilibrium has been achieved in [35]. On the other hand, the existing literatures in series
gradually indicate that the virus production rate $\beta$ relative to the death rate of infected
cancer cells is critical for the large-time behavior of the corresponding solution at least
in some simple variant of (1.1) inter alia through neglecting the haptotactic movement of
infected cancer cells and the renewed effect of ECM [27, 28, 29, 30]. Specially, for the cross-
diffusion system with single haptotaxis:

$$
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla v) + \mu u (1 - u) - \rho u z, \quad x \in \Omega, \ t > 0, \\
    v_t &= -(u + w) v, \quad x \in \Omega, \ t > 0, \\
    w_t &= \Delta w - w + u z, \quad x \in \Omega, \ t > 0, \\
    z_t &= D z \Delta - z - u z + \beta w, \quad x \in \Omega, \ t > 0,
\end{align*}
$$

(1.2)
in a smooth bounded domain $\Omega \subset \mathbb{R}^2$, it is proved in [29] that if $\beta > 1$ and the initial
data $u_0 > \frac{1}{\beta - 1}$, the corresponding solution to (1.2) with $\mu = \rho = 0$ will be blow-up in
infinite time, while the solution will be bounded if $u_0 < \frac{1}{(\beta - 1)_+}$ and $v_0 \equiv 0$. However,
the solution component $u$ will possess a positive lower bound when $0 < \beta < 1, \rho > 0$ [30].
Futhermore, based on the outcome of [30], the asymptotic behavior of (1.2) is established
in [27] if $0 < \beta < 1$. It should be mentioned that in the case of $\rho > 0$ and $\mu > 0$, the
convergence property is also discussed in [5]. Particularly, it is shown in [28] that for any
prescribed level $\gamma \in (0, \frac{1}{(\beta - 1)_+})$, the corresponding solution of (1.2) with $\mu = 0, \rho \geq 0$
will tend to the constant equilibrium $(u_\infty, 0, 0, 0)$ with some $u_\infty > 0$ whenever the initial
deviation from $(\gamma, 0, 0, 0)$ is suitably small, which provided a complement to the result of
It is observed that for two-dimensional haptotactic cross-diffusion model (1.2), the result on the asymptotic behavior seems to be rather comprehensive when $0 < \beta < 1$, while it becomes rudimentary in the case of $\beta > 1$ \cite{29, 28}, let alone that for the doubly haptotactic cross-diffusion system in the three-dimensional scenario. Accordingly, the purpose of this work is to investigate the dynamical features involving the doubly haptotactic processes in the three dimensional setting without the constraint of $0 < \beta < 1$. More precisely, we shall consider a simplified version of (1.1) by neglecting the renewal of extracellular matrix, which is given as follows

$$
\begin{align*}
  \frac{\partial u}{\partial t} &= \Delta u - \nabla \cdot (u\nabla v) + \mu u(1 - u) - uz, & x \in \Omega, \ t > 0, \\
  \frac{\partial v}{\partial t} &= -(u + w)v, & x \in \Omega, \ t > 0, \\
  \frac{\partial w}{\partial t} &= \Delta w - \nabla \cdot (w\nabla v) - w + uz, & x \in \Omega, \ t > 0, \\
  \frac{\partial z}{\partial t} &= D_z \Delta z - z - uz + \beta w, & x \in \Omega, \ t > 0, \\
  (\nabla u - u\nabla v) \cdot \nu &= (\nabla w - w\nabla v) \cdot \nu = \nabla z \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) &= u_0, \ v(x, 0) = v_0, \ w(x, 0) = w_0, \ z(x, 0) = z_0, & x \in \Omega,
\end{align*}
$$

(1.3)

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary and $\nu$ is a outward unit normal vector of $\Omega$. Moreover, we henceforth assume that the initial data satisfies

$$
\begin{align*}
  u_0, w_0, z_0 \text{ and } v_0 \text{ are nonnegative functions from } C^{2+\vartheta}(\bar{\Omega}) \text{ for some } \vartheta \in (0, 1), \\
  \text{with } u_0 \neq 0, \ w_0 \neq 0, \ z_0 \neq 0, \ v_0 \neq 0 \text{ on } \partial \Omega.
\end{align*}
$$

(1.4)

With regard to the qualitative information of doubly haptotactic cross-diffusion systems in the three-dimensional setting, our subsequence analysis reveals that even in the case of $\beta > 1$, the uninfected cancer cells $u$ in (1.3) is uniformly persistent for certain type of small data $(u_0, v_0, w_0, z_0)$; in particular, whenever $u_0(x) \leq 1$, the virus production rate making possible for the decay of uninfected cancer cells is increasing with respect to $\min_{x \in \Omega} u_0(x)$, which can be stated as follows

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, $D_z > 0, \mu > 0$ and $\beta \max\{1, \|u_0\|_{L^\infty(\Omega)}\} < 1 + (1 + \frac{1}{\min_{x \in \Omega} u_0(x)})^{-1}$. Then there exists $\varepsilon = \varepsilon(\beta, \mu, u_0) > 0$ with the property such that whenever the initial data $(u_0, v_0, w_0, z_0)$ fulfills (1.4),

$$
\|v_0\|_{L^\infty(\Omega)} < \varepsilon,
$$

(1.5)
and
\[ \|w_0\|_{L^\infty(\Omega)} < \varepsilon \] (1.6)
as well as
\[ \|z_0\|_{L^\infty(\Omega)} < \varepsilon, \] (1.7)
the problem (1.3) has a unique non-negative global classical solution \((u, v, w, z)\) which is bounded in the sense that
\[ \sup_{t>0} \{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,5}(\Omega)} \} < \infty, \]
and moreover
\[ \|u(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty \] (1.8)
and
\[ \|v(\cdot, t)\|_{W^{1,4}(\Omega)} \to 0 \quad \text{as } t \to \infty \] (1.9)
and
\[ \|w(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty \] (1.10)
as well as
\[ \|z(\cdot, t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty. \] (1.11)

Our approach is based on a self-map argument, which presupposes a certain decay property on \(z\) within an appropriate time interval, and then proves that it is actually valid in the whole interval \([0, T_{\text{max}}]\) with the help of some necessary a priori estimates, inter alia the lower pointwise bound of \(a = u e^{-v}\), which results from a parabolic comparison argument on the basis of an absorptive parabolic inequality fulfilled by \(a\) (see (3.8)). Indeed, due to the lower pointwise bound of \(a\), one can arrive at the exponential decay of \(v\) (see Lemma 3.2). As a consequence thereof, one can achieved a suitable upper bounds for \(b = w e^{-v}\) as well as \(a\) by adequately exploiting the corresponding cooperative parabolic system for \((a, b)\) (see Lemma 3.3), which is somewhat different from that in [28]. Note that the assumption in Theorem 1.1 makes it possible to construct the desired super-solutions thereof. Thereafter we shall be able to prove that \(z\) actually decays exponentially (see Lemma 3.4) and thereby complete
the self-map type reasoning. Furthermore, based on the $L^\infty(\Omega)$ of $(a, v, b, z)$ achieved in previous section, we prove that $(a, v, b, z)$ is actually global, that is $T_{\max} = \infty$. To this end, we need to estimate $\nabla v$ with respect to the norm in $L^5(\Omega)$ according to extensibility criteria (2.2). With respect to this, we turn to estimate the coupled quantity

$$\int_\Omega |\nabla a|^2 + \int_\Omega |\nabla b|^2 + \eta \int_\Omega |\nabla v|^4$$

with some $\eta > 0$ through the testing processes (Lemma 4.1). In Section 5, (1.8) is shown by means of the bootstrap method. Indeed, as the first step toward this we show the convergence of integral $\int_0^\infty \int_\Omega |\nabla v|^2$ (Lemma 5.2). As the consequence thereof, we will then successively obtain the exponential decay property for $\|\nabla v(\cdot, t)\|_{L^2(\Omega)}$, $\|u(\cdot, t) - 1\|_{L^p(\Omega)}$ for any $p < 6$, $\|\nabla v(\cdot, t)\|_{L^4(\Omega)}$ and $\|u(\cdot, t) - 1\|_{L^\infty(\Omega)}$, thanks to the explicit expression of $\nabla v$ and the Gagliardo–Nirenberg type inequality.

2 Preliminaries

For the convenience of subsequent analysis, we introduce the variable transformation

$$a = ue^{-v}, \quad b = we^{-v},$$

upon which system (1.3) is converted to the following system

$$\begin{cases}
    a_t = e^{-v} \nabla \cdot (e^v \nabla a) + f(a, b, v, c), & x \in \Omega, t > 0, \\
    b_t = e^{-v} \nabla \cdot (e^v \nabla b) + g(a, b, v, c), & x \in \Omega, t > 0, \\
    v_t = -(ae^v + be^v)v, & x \in \Omega, t > 0, \\
    z_t = Dz \Delta z - z - uz + \beta w, & x \in \Omega, t > 0, \\
    \frac{\partial a}{\partial \nu} = \frac{\partial b}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \\
    a(x, 0) = u_0(x)e^{-v_0(x)}, \quad b(x, 0) = w_0(x)e^{-v_0(x)}, & x \in \Omega, \\
    v_0(x, 0) = v_0(x), \quad z(x, 0) = z_0(x), & x \in \Omega
\end{cases}$$

(2.1)

with

$$f(a, b, v, c) := -az + a(a + b)e^v v + \mu a(1 - e^v a)$$

with
as well as

\[ g(a, b, v, c) := -b + az + b(a + b)e^v v. \]

Note that (1.3) and (2.1) are equivalent in the framework of the classical solution. In this framework the following statements on local-in-time existence and a convenient extensibility criteria of the classical solution to (2.1) can be proved by slightly adapting the arguments detailed in [20].

Lemma 2.1. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary; \( D_z \) and \( \beta \) are positive, and initial data \((u_0, v_0, w_0, z_0)\) satisfies (1.4). Then there exist \( T_{\text{max}} \in (0, \infty] \) and a unique quadruple \((a, v, b, z)\) \( \in C^2(\Omega \times (0, T_{\text{max}})) \cap C^0(\Omega \times [0, T_{\text{max}})) \) which solves (2.1) classically in \( \Omega \times (0, T_{\text{max}}) \), and are such that if \( T_{\text{max}} < \infty \), then

\[
\limsup_{t \nearrow T_{\text{max}}} \left\| a(\cdot, t) \right\|_{L^\infty(\Omega)} + \left\| b(\cdot, t) \right\|_{L^\infty(\Omega)} + \left\| z(\cdot, t) \right\|_{L^\infty(\Omega)} + \left\| \nabla v(\cdot, t) \right\|_{L^5(\Omega)} \to \infty.
\]

(2.2)

3 A priori estimates for solutions

In order to set the frame for the self-map type argument under the assumption of \( \beta \) in Theorem 1.1, it seems necessary for us to formulate the following elementary observation.

Lemma 3.1. Let \( \beta \max \{1, \left\| u_0 \right\|_{L^\infty(\Omega)} \} < 1 + (1 + \frac{1}{\min_{x \in \Omega} u_0(x)})^{-1} \). Then there exist \( K = K(\beta, u_0) > 0 \), \( \delta \in (0, 1) \) and \( \varepsilon_0 \in (0, 1) \) such that for all \( 0 < \varepsilon < \varepsilon_0 \), we have

\[
\frac{1}{1 - \delta} \left( \max \left\{ \frac{\mu + \varepsilon e}{\mu - \varepsilon}, \left\| u_0 \right\|_{L^\infty(\Omega)} \right\} + \sqrt{\varepsilon} e \max \{3, \left\| u_0 \right\|_{L^\infty(\Omega)} + 1 \} \right)
< K < \frac{1}{\beta e^\varepsilon} \left( 1 + \left( \frac{\mu e^\varepsilon}{\mu - \sqrt{\varepsilon} + \frac{\varepsilon}{\min_{x \in \Omega} u_0(x)}} \right)^{-1} - \delta \right) (1 - \frac{3}{2} \sqrt{\varepsilon}).
\]

(3.1)

Proof. By the hypothesis \( \beta \max \{\left\| u_0 \right\|_{L^\infty(\Omega)}, 1\} < 1 + (1 + \frac{1}{\min_{x \in \Omega} u_0(x)})^{-1} \), one can see that

\[
K = \frac{1}{2} \left( \max \{1, \left\| u_0 \right\|_{L^\infty(\Omega)} \} + \frac{1 + (1 + \frac{1}{\min_{x \in \Omega} u_0(x)})^{-1}}{\beta} \right)
\] satisfies

\[
\max \{1, \left\| u_0 \right\|_{L^\infty(\Omega)} \} < K < \frac{1 + (1 + \frac{1}{\min_{x \in \Omega} u_0(x)})^{-1}}{\beta}.
\]

(3.2)

Therefore, the claim follows by means of an argument based on continuous dependence. \( \square \)
Now we further take $\varepsilon \in (0, 1)$ small enough such that
\[
\varepsilon < \varepsilon_1 := \min\left\{\frac{\beta^2}{4}, \frac{\mu}{3e}\right\} \quad (3.3)
\]
and henceforth suppose that the initial data $(u_0, v_0, w_0, z_0)$ fulfills (1.4)-(1.7), and then consider the set
\[
S = \left\{ T_1 > 0 \mid \|z(\cdot, t)\|_{L^\infty(\Omega)} < \sqrt{\varepsilon}e^{-\delta t} \quad \text{for all } t \in (0, T_1 \cap T_{\max}) \right\} \quad (3.4)
\]
which is not empty due to (1.7). In particular, $T = \sup S \in (0, T_{\max}]$, the maximal interval on which $\|z(\cdot, t)\|_{L^\infty(\Omega)} < \sqrt{\varepsilon}e^{-\delta t}$ holds, is well-defined.

As the first step of the proof of Theorem 1.1, we make sure that if $T_{\max} < \infty$, then
\[
T = T_{\max}. \quad (3.5)
\]
To this end, we suppose that $T < T_{\max}$ and then derive the pointwise lower-bounded estimate of $a$ on interval $(0, T)$ from the hypothesis included in (3.4) by a simple comparison argument.

**Lemma 3.2.** Let the assumptions in Theorem 1.1 hold and the initial data $(u_0, v_0, w_0, z_0)$ fulfills (1.4)-(1.7) with $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$. Then we have
\[
a(x, t) \geq \left\{ \frac{e^{\sqrt{\varepsilon} + \varepsilon}}{\min_{x \in \Omega} u_0(x)} + \frac{\mu e^\varepsilon}{\mu - \sqrt{\varepsilon}} \right\}^{-1} \quad \text{for all } x \in \Omega, t \in (0, T) \quad (3.6)
\]
as well as
\[
v(x, t) \leq \varepsilon \exp\left\{ -\left( \frac{e^{\sqrt{\varepsilon} + \varepsilon}}{\min_{x \in \Omega} u_0(x)} + \frac{\mu e^\varepsilon}{\mu - \sqrt{\varepsilon}} \right) t \right\} \quad \text{for all } x \in \Omega, t \in (0, T). \quad (3.7)
\]

**Proof.** From the definition of $T$ and $v(x, t) \leq v_0(x) \leq \varepsilon$, one can see that for all $(x, t) \in \Omega \times (0, T)$,
\[
a_t \geq e^{-v} \nabla \cdot (e^v \nabla a) + (\mu - \sqrt{\varepsilon}e^{-\delta t})a - \mu e^\varepsilon a^2
\]
\[
= \Delta v - \nabla v \cdot \nabla a + (\mu - \sqrt{\varepsilon}e^{-\delta t})a - \mu e^\varepsilon a^2. \quad (3.8)
\]
Thus, if we let $a(t) \in C^1((0, T))$ denote the solution of
\[
\left\{ \begin{array}{l}
a_t = (\mu - \sqrt{\varepsilon}e^{-\delta t})a - \mu e^\varepsilon a^2, \quad t \in (0, T), \\
a(0) = a_0 := \min_{x \in \Omega} a_0(x),
\end{array} \right. \quad (3.9)
\]
On the other hand, by the nonnegativity of functions with (3.10) yields (3.7) immediately.

Proof. Let and thereby $(a, b)$ is a sub-solution of the cooperative parabolic system.

In order to construct a appropriate super-solution $(\hat{a}, \hat{b})$ to problem (3.12), we let

$$\hat{a}(x, t) = \varphi(t), \quad \hat{b}(x, t) = K\varepsilon e^{-\delta t} e^{-\delta t} \text{ for } x \in \Omega, t \in (0, T),$$

where $K > 0$, $\varepsilon > 0$ and $\delta > 0$ are given in Lemma 3.1.

Based on above estimates, one can obtain suitable upper bounds for $b$ as well as for $a$ by adequately exploiting a cooperative parabolic system.

**Lemma 3.3.** Suppose that the assumptions in Lemma 3.2 hold, then

$$a(x, t) \leq \max\{\|u_0\|_{L^\infty(\Omega)}, 2\}, \quad b(x, t) \leq K\varepsilon e^{-\delta t}$$

where $K > 0$, $\varepsilon > 0$ and $\delta > 0$ are given in Lemma 3.1.

**Proof.** Let $a_* = \left\{ \frac{e^{-\varepsilon t}}{\min_{x \in \Omega} u_0(x)} + \frac{\mu e^\varepsilon}{\mu - \sqrt{\varepsilon}} \right\}^{-1}$. Then due to the positivity of $a$, $b$ and $z$, it follows from (3.4), (3.7) that

$$\begin{cases} 
    a_t \leq e^{-\varepsilon} \nabla \cdot (a \nabla a) + \varepsilon e^\varepsilon \nabla a \cdot (a + b) e^{-a_* t} + \mu a - \mu a^2, & x \in \Omega, t \in (0, T), \\
    b_t \leq e^{-\varepsilon} \nabla \cdot (b \nabla b) - b + \varepsilon e^\varepsilon b \cdot (a + b) e^{-a_* t} + \sqrt{\varepsilon} a e^{-\delta t}, & x \in \Omega, t \in (0, T),
\end{cases}$$

and thereby $(a, b)$ is a sub-solution of the cooperative parabolic system

$$\begin{cases} 
    \tilde{a}_t = e^{-\varepsilon} \nabla \cdot (\tilde{a} \nabla \tilde{a}) + \epsilon e^\varepsilon \nabla \tilde{a} \cdot (\tilde{a} + \tilde{b}) e^{-a_* t} + \mu \tilde{a} - \mu \tilde{a}^2, & x \in \Omega, t \in (0, T), \\
    \tilde{b}_t = e^{-\varepsilon} \nabla \cdot (\tilde{b} \nabla \tilde{b}) - \tilde{b} + \epsilon e^\varepsilon \nabla \tilde{b} \cdot (\tilde{a} + \tilde{b}) e^{-a_* t} + \sqrt{\varepsilon} \tilde{a} e^{-\delta t}, & x \in \Omega, t \in (0, T), \\
    \tilde{a}(x, 0) = u_0(x) e^{-\varphi_0(x)}, \quad \tilde{b}(x, 0) = w_0(x) e^{-\varphi_0(x)}.
\end{cases}$$

In order to construct a appropriate super-solution $(\hat{a}, \hat{b})$ to problem (3.12), we let

$$\hat{a}(x, t) = \varphi(t), \quad \hat{b}(x, t) = K\varepsilon e^{-\delta t} e^{-\delta t} \text{ for } x \in \Omega, t \in (0, T),$$

where $K$ is given in Lemma 3.1 and $\varphi \in C^1([0, T])$ is the solution of the Bernoulli-type IBV

$$\begin{cases} 
    \varphi'(t) = (\varepsilon e + \mu) \varphi(t) - (\mu - \varepsilon) \varphi^2(t), \\
    \varphi(0) = \|u_0\|_{L^\infty(\Omega)}
\end{cases}$$
which satisfies
\[ \varphi(t) \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \frac{\varepsilon e + \mu}{\mu - \varepsilon e}\} \] (3.15)
according to the explicit solution thereof.

Thanks to the latter, \((\hat{a}, \hat{b})\) is actually a super-solution of (3.13). Indeed, due to (3.3),
\[ \frac{\varepsilon e + \mu}{\mu - \varepsilon e} < 2 \text{ and } K\sqrt{\varepsilon} < 1. \]
Hence from (3.15) and (3.1), it follows that
\[
K(\varphi(t) + K \varepsilon e^{-\delta t})e^{\varepsilon} + \varphi(t) \\
\leq K(\max\{\|u_0\|_{L^\infty(\Omega)}, \frac{\varepsilon e + \mu}{\mu - \varepsilon e}\} + K \varepsilon e) + \max\{\|u_0\|_{L^\infty(\Omega)} e^{\varepsilon e} + \max\{\|u_0\|_{L^\infty(\Omega)}, \frac{\varepsilon e + \mu}{\mu - \varepsilon e}\}\} \\
\leq K\varepsilon e \max\{\|u_0\|_{L^\infty(\Omega)} + 1, 3\} + \max\{\|u_0\|_{L^\infty(\Omega)} e^{\varepsilon e} + \max\{\|u_0\|_{L^\infty(\Omega)}, \frac{\varepsilon e + \mu}{\mu - \varepsilon e}\}\} \\
\leq K(1 - \delta),
\]
which implies that
\[
\hat{b}_t \geq e^{-\varepsilon} \nabla \cdot (e^{\varepsilon} \nabla \hat{b}) - \hat{b} + \varepsilon e \hat{b}(\hat{a} + \hat{b}) e^{-a* t} + \sqrt{\varepsilon} a e^{-\delta t}, x \in \Omega, t \in (0, T). \]

In addition, due to (1.6) and (3.2),
\[
\hat{b}(x, 0) \geq \|w_0\|_{L^\infty(\Omega)} \geq w_0(x) e^{-v_0(x)}.
\]

Apart from that, in light of (3.14), we have
\[
\hat{a}_t \geq e^{-\varepsilon} \nabla \cdot (e^{\varepsilon} \nabla \hat{a}) + \varepsilon e \hat{a}(\hat{a} + \hat{b}) e^{-a* t} + \mu \hat{a} - \mu \hat{a}^2, \quad \hat{a}(x, 0) \geq u_0(x) e^{-v_0(x)}.
\]

Therefore \((\hat{a}, \hat{b})\) is readily verified to be a super-solution of problem (3.12), and thereby by comparison principle
\[
a(x, t) \leq \hat{a}(x, t) \leq \max\{\|u_0\|_{L^\infty(\Omega)} + 2\}, \quad b(x, t) \leq \hat{b}(x, t) = K\sqrt{\varepsilon} e^{-\delta t}.
\]

Based on the outcomes in Lemma 3.2 and Lemma 3.3, we can verify that actually \(T = T_{\text{max}}\) provided \(T_{\text{max}} < \infty\), which is stated as follows.
Lemma 3.4. Let the assumptions in Theorem 1.1 hold and the initial data \((u_0, v_0, w_0, z_0)\) fulfills (1.4)–(1.7) with \(\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}\). Then

\[
z(x, t) \leq (\sqrt{\varepsilon} - \varepsilon/2) e^{-\delta t} \quad \text{for all} \quad x \in \Omega, t \in (0, T), \tag{3.17}
\]

as well as \(T = T_{\text{max}}\) provided \(T_{\text{max}} < \infty\).

Proof. According to Lemma 3.2 and Lemma 3.3, we have

\[
u(x, t) \geq a(x, t) \geq a^*: = \left\{ e^{\sqrt{\varepsilon} \eta + \varepsilon} + \frac{\mu^\varepsilon}{\mu - \sqrt{\varepsilon}} \right\}^{-1}
\]

and

\[
w(x, t) \leq b(x, t) e^{\varepsilon} \leq K\sqrt{\varepsilon} e^{\varepsilon - \delta t}.
\]

Hence in light of the fourth equation of (2.1), \(z\) satisfies

\[
z_t \leq D_2 \Delta z - a^* z - z + \beta K\sqrt{\varepsilon} e^{\varepsilon - \delta t}, \quad x \in \Omega, 0 < t < T. \tag{3.18}
\]

Now we may compare \(z\) to spatially homogeneous functions having a supersolution property with regard to the parabolic operator in (3.18). Indeed, if \(\hat{z}(t)\) denotes the solution of the initial-value problem

\[
\left\{ \begin{array}{l}
y_t = -(a^* + 1) y + \beta K\sqrt{\varepsilon} e^{\varepsilon - \delta t}, \quad 0 < t < T, \\
y(0) = \|z_0\|_{L^\infty(\Omega)},
\end{array} \right.
\]

then from the comparison principle we infer that

\[
z(x, t) \leq \hat{z}(t) = \|z_0\|_{L^\infty(\Omega)} e^{-(a^* + 1)t} + \beta K\sqrt{\varepsilon} e^{\varepsilon} \int_0^t e^{-(a^* + 1)(t-s)} e^{-\delta s} ds \leq \varepsilon e^{-(a^* + 1)t} + \frac{\beta K\sqrt{\varepsilon} e^\varepsilon}{1 + a^* - \delta} e^{-\delta t} \leq (\sqrt{\varepsilon} - \varepsilon/2) e^{-\delta t}, \tag{3.19}
\]

where inequality (3.1) is used in the last inequality.

At this position, supposed that \(T < T_{\text{max}}\), then according to the definition of \(T\) and continuity of \(z\), we have \(\|z(\cdot, T)\|_{L^\infty(\Omega)} = \sqrt{\varepsilon} e^{-\delta T}\), which contradicts with (3.19) and thereby shows that actually \(T = T_{\text{max}}\).
4 Global solvability

The purpose of this section is to show that under the assumptions in Theorem 1.1 \((a, v, b, z)\) is actually a global solution of (2.1). To this end, in light of (2.2) and the outcomes of Lemma 3.3 and Lemma 3.4, we need to establish a bound for \(\nabla v\) with respect to the norm in \(L^5(\Omega)\), rather than \(L^4(\Omega)\) in the two-dimensional setting. With respect to this, we first derive the spatial-temporal estimates of \(\Delta a\) through the testing processes, thanks to the \(L^\infty(\Omega)\)-estimates of \(a, b\) and \(z\) just asserted.

**Lemma 4.1.** Let \((a, v, b, z)\) be the classical solution of (2.1) on \((0, T_{\text{max}})\) obtained in previous section. Then if \(T_{\text{max}} < \infty\), one can find \(C > 0\) fulfilling

\[
\int_0^t \int_\Omega (|\Delta a|^2 + |\Delta b|^2) \leq C \quad \text{for all } t \in (0, T_{\text{max}})
\]

as well as

\[
\int_\Omega |\nabla v(\cdot, t)|^4 \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]

**Proof.** According to the outcomes of Lemma 3.3 and Lemma 3.4 we have

\[
a(x, t) \leq c_1 := \max\{|u_0|_{L^\infty(\Omega)}, 2\}, \quad v(x, t) \leq 1, \quad b(x, t) \leq 1 \quad \text{and} \quad z(x, t) \leq 1,
\]

which definitely entail that

\[
|f(a, b, v, z)| \leq c_2 := c_1 + c_1(c_1 + 1)e + \mu c_1^2 e.
\]

Multiplying the equation

\[
a_t = \Delta a + \nabla v \cdot \nabla a + f(a, b, v, z)
\]

by \(-\Delta a\) and integrating by parts, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla a|^2 + \int_\Omega |\Delta a|^2
\]

\[
= - \int_\Omega \Delta a \nabla v \cdot \nabla a - \int_\Omega f \Delta a
\]

\[
\leq \frac{1}{2} \int_\Omega |\Delta a|^2 + \int_\Omega |\nabla v \cdot \nabla a|^2 + c_2^2 |\Omega|.
\]
Here combining the Gagliardo–Nirenberg inequality with the standard elliptic regularity, one can find $c_3 > 0$ such that

$$\|\nabla \varphi\|_{L^2(\Omega)}^2 \leq c_3 \|\Delta \varphi\|_{L^2(\Omega)} \|\varphi\|_{L^\infty(\Omega)}$$

for all $\varphi \in C^2(\Omega)$ with $\frac{\partial \varphi}{\partial \nu} = 0$

which together with (4.3) shows that

$$\int_\Omega |\Delta a|^2 \geq \frac{1}{c_2 c_3} \int_\Omega |\nabla a|^4$$

for all $0 < t < T_{\text{max}}$. (4.4)

Therefore applying (4.3) and the Young inequality, we have

$$\frac{d}{dt} \int_\Omega |\nabla b|^2 + c_6 \int_\Omega |\nabla b|^4 + c_6 \int_\Omega |\Delta b|^2 \leq c_7 \int_\Omega |\nabla v|^4 + c_7.$$ (4.6)

To compensate the right summand of (4.5) and (4.6), we test the $v$-equation in (2.1) against $|\nabla v|^2 \nabla v$ to obtain that for all $0 < t < T_{\text{max}},$

$$\frac{1}{4} \frac{d}{dt} \int_\Omega |\nabla v|^4 = -\int_\Omega ve^v|\nabla v|^2 \nabla v \cdot (\nabla a + \nabla b) - \int_\Omega e^v(v + 1)(a + b)|\nabla v|^4$$

$$\leq -\int_\Omega ve^v|\nabla v|^2 \nabla v \cdot (\nabla a + \nabla b) - \int_\Omega ae^v|\nabla v|^4.$$ (4.7)

Recalling the lower bound for $a$ in (3.6), one can find $c_8 > 0$ such that

$$\int_\Omega ae^v|\nabla v|^4 \geq c_8 \int_\Omega |\nabla v|^4.$$ 

So we combine (4.7) and (4.3) with the Young inequality to see that

$$\frac{d}{dt} \int_\Omega |\nabla v|^4 + c_9 \int_\Omega |\nabla v|^4 \leq c_{10} \left( \int_\Omega |\nabla a|^4 + \int_\Omega |\nabla b|^4 \right)$$ (4.8)

for some $c_9 > 0, c_{10} > 0$.

Therefore, through an appropriate combination of (4.5), (4.6) and (4.8), one can find $\eta > 0, c_{11} > 0$ and $c_{12} > 0$ such that

$$\frac{d}{dt} \left( \int_\Omega |\nabla a|^2 + \int_\Omega |\nabla b|^2 + \eta \int_\Omega |\nabla v|^4 \right) + c_{11} \int_\Omega (|\Delta a|^2 + |\Delta b|^2)$$

$$\leq c_{12} \left( \int_\Omega |\nabla a|^2 + \int_\Omega |\nabla b|^2 + \eta \int_\Omega |\nabla v|^4 \right) + c_{12}.$$ (4.9)
At this position, writing \( y(t) := \int_{\Omega} |\nabla a(\cdot, t)|^2 + \int_{\Omega} |\nabla b(\cdot, t)|^2 + \eta \int_{\Omega} |\nabla v(\cdot, t)|^4 + 1 \), we then have
\[
y'(t) + c_{11} \int_{\Omega} (|\Delta a|^2 + |\Delta b|^2) \leq c_{12} y(t)
\]
for all \( 0 < t < T_{\text{max}} \), which leads to
\[
y(t) \leq y_0 e^{c_{12} T_{\text{max}}}
\]
as well as
\[
\int_0^t \int_{\Omega} (|\Delta a|^2 + |\Delta b|^2) \leq C(T_{\text{max}})
\]
for all \( 0 < t < T_{\text{max}} \), and thereby completes the proof.

Now making efficient use of the spatial-temporal estimates of \( \Delta a \) provided by Lemma 4.1, one can derive the further regularity property of \( \nabla v \) in the three-dimensional setting beyond that in (4.2).

**Lemma 4.2.** Assume that \( T_{\text{max}} < \infty \), one can find \( C > 0 \) fulfilling
\[
\int_{\Omega} |\nabla v(\cdot, t)|^5 \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]  
(4.10)

**Proof.** Multiplying the third equation in (2.1) by \( |\nabla v|^3 \nabla v \), integrating by parts and applying Young’s inequality, we obtain that for all \( 0 < t < T_{\text{max}} \),
\[
\frac{1}{5} \frac{d}{dt} \int_{\Omega} |\nabla v|^5 = - \int_{\Omega} v e^v |\nabla v|^3 \nabla v \cdot (\nabla a + \nabla b) - \int_{\Omega} e^v (v + 1)(a + b) |\nabla v|^4
\leq - \int_{\Omega} v e^v |\nabla v|^3 \nabla v \cdot (\nabla a + \nabla b)
\leq e \left\{ \int_{\Omega} |\nabla v|^5 \right\} ^{\frac{4}{5}} \left\{ \left\{ \int_{\Omega} |\nabla a|^5 \right\} ^{\frac{1}{5}} + \left\{ \int_{\Omega} |\nabla b|^5 \right\} ^{\frac{1}{5}} \right\}.
\]  
(4.11)

Now we see that \( y(t) := \int_{\Omega} |\nabla v(\cdot, t)|^5 \), \( t \in (0, T_{\text{max}}) \) satisfies
\[
y'(t) \leq 5 e \left( \|\nabla a(\cdot, t)\|_{L^5(\Omega)} + \|\nabla b(\cdot, t)\|_{L^5(\Omega)} \right)^4 y^\frac{4}{5}(t)
\]
and hence
\[
\|\nabla v(\cdot, t)\|_{L^5(\Omega)} \leq \|\nabla v_0\|_{L^5(\Omega)} + e \int_0^t \left( \|\nabla a(\cdot, s)\|_{L^5(\Omega)} + \|\nabla b(\cdot, s)\|_{L^5(\Omega)} \right) ds.
\]  
(4.12)
Furthermore, as
\[ \| \nabla \varphi \|_{L^5(\Omega)} \leq c_1 \| \Delta \varphi \|_{L^4(\Omega)}^{\frac{1}{2}} \| \varphi \|_{L^\infty(\Omega)}^{\frac{1}{2}} \quad \text{for all} \ \varphi \in C^2(\Omega) \text{ with} \ \frac{\partial \varphi}{\partial \nu} = 0 \]
with constant \( c_1 > 0 \), we get
\[ \| \nabla v(\cdot, t) \|_{L^5(\Omega)} \leq \| \nabla v_0 \|_{L^5(\Omega)} + c_2 \int_0^t (\| \Delta a(\cdot, s) \|_{L^2(\Omega)}^2 + \| \Delta b(\cdot, s) \|_{L^2(\Omega)}^2 + 1) ds \]
for \( c_2 > 0 \), which along with (4.1) yields (4.10) immediately. \qed

**Corollary 4.1.** Under the assumptions in Theorem 1.1 problem (1.3) admits a unique global solution \((u, v, w, z)\).

**Proof.** According to the equivalence of (1.3) and (2.1) in the considered framework of classical solutions, the global existence readily results from Lemma 3.2, Lemma 3.3, Lemma 4.2 and the extensibility criteria (2.2) in Lemma 2.1. \qed

## 5 Exponential decay of \( u - 1 \)

Throughout this section, \((a, v, b, z)\) and \((u, v, w, z)\) are the global classical solution of (2.1) and (1.3), respectively. Combining the outcomes of Lemma 3.2, Lemma 3.3 and Lemma 3.4 with Corollary 4.1, we can see that (1.9)–(1.11) in Theorem 1.1 have been proved already, and thereby only need to show that (1.8) is valid.

The decay property (1.8) will be shown by means of the bootstrap method adapted from [35], the latter is concerned with the two-dimensional version of (1.2) with \( \beta < 1 \). As the start point of the derivation of (1.8), we establish the following basic stabilization feature of \( a(= e^{-\nu} u) \) upon the decay property of \( v \) with respect to \( L^\infty(\Omega) \).

**Lemma 5.1.** Let the assumptions in Theorem 1.1 hold. Then
\[ \int_0^\infty \int_{\Omega} \frac{|\nabla a|^2}{a^2} < \infty \quad (5.1) \]
and
\[ \int_0^\infty \int_{\Omega} (u - 1)^2 < \infty. \quad (5.2) \]
Proof. In view of $s - 1 - \log s > 0$ for all $s > 0$ and $v_t < 0$, we can conclude that

$$\frac{d}{dt} \int_\Omega e^v (a - 1 - \log a)
= \int_\Omega e^v (a - 1 - \log a) v_t + \int_\Omega \frac{a - 1}{a} e^v a_t$$
$$\leq - \int_\Omega e^v \frac{\|\nabla a\|^2}{a^2} + \mu \int_\Omega e^v (a - 1)(1 - u)
+ \int_\Omega e^v (a - 1)(u + w)v - \int_\Omega e^v z(a - 1).$$

(5.3)

Here by Young’s inequality,

$$(1 - a)(1 - u) = (1 - u)^2 + (u - a)(1 - u)$$
$$\geq \frac{1}{2}(1 - u)^2 - a^2(e^v - 1)^2.$$

(5.4)

Since $e^s \leq 1 + 2s$ for all $s \in [0, \log 2]$, (3.7) allows us to fix a $t_1 > 1$ suitable large such that for all $t \geq t_1$, $(e^v - 1)^2 \leq 4v^2$, which along with (5.4) implies that

$$(1 - a)(1 - u) \geq \frac{1}{2}(1 - u)^2 - 4a^2v^2$$

for $t \geq t_1$. Hence thanks to the outcomes of Lemma 3.3 and Lemma 3.4, one can obtain from (5.3) that for all $t \geq t_1$

$$\frac{d}{dt} \int_\Omega e^v (a - 1 - \log a)
+ \int_\Omega \frac{\|\nabla a\|^2}{a^2} + \frac{\mu}{2} \int_\Omega (u - 1)^2$$
$$\leq c_1 \int_\Omega z + c_1 \int_\Omega v$$

with $c_1 > 0$. Upon a time integration over $(t_1, t)$, the latter arrives at

$$\int_{t_1}^t \int_\Omega \frac{\|\nabla a\|^2}{a^2} + \frac{\mu}{2} \int_{t_1}^t (u - 1)^2$$
$$\leq c_1 \int_{t_1}^t \int_\Omega z + c_1 \int_{t_1}^t \int_\Omega v + \int_\Omega e^{v(t_1)} (a(\cdot, t_1) - 1 - \log a(\cdot, t_1)),$$

which together with (3.6) and (3.7) makes sure that both (5.1) and (5.2) hold. □

In order to make sure that (5.2) implies the time decay property of $u - 1$, it seems desirable to consider the exponential decay properties of $\int_\Omega |\nabla v(\cdot, t)|^2$, rather than the integrability of $a_t$ in $L^2((0, \infty); L^2(\Omega))$. As the first step toward this, we first show the convergence of integral $\int_0^\infty \int_\Omega |\nabla v|^2$, which is stated below.
Lemma 5.2. Suppose that the assumptions in Theorem 1.1 hold. Then we have
\begin{equation}
\int_0^\infty \int_\Omega |\nabla v|^2 < \infty. \tag{5.5}
\end{equation}

Proof. Testing the second equation in (2.1) against \(e^v b\) and integrating by parts, we get
\begin{align*}
\frac{d}{dt} \int_\Omega e^v b^2 + 2 \int_\Omega e^v |\nabla b|^2 + 2 \int_\Omega e^v b^2 \\
= 2 \int_\Omega abze^v + 2 \int_\Omega e^v b^2 (ae^v + be^v)v,
\end{align*}
which together with the global-in-time boundedness property of \(a\) and \(z\), implies that
\begin{equation}
\frac{d}{dt} \int_\Omega e^v b^2 + \int_\Omega |\nabla b|^2 + 2 \int_\Omega e^v b^2 \leq c_1 \int_\Omega b
\end{equation}
for some \(c_1 > 0\). Hence according to Lemma 3.3, we can get
\begin{equation}
\int_0^\infty \int_\Omega |\nabla b|^2 < \infty. \tag{5.6}
\end{equation}

Now since \(\nabla v_t = -(\nabla u + \nabla w)v - (u + w)\nabla v\), a direct computation shows that
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla v|^2 + \int_\Omega (u + w)|\nabla v|^2 \\
= - \int_\Omega ve^v b |\nabla v|^2 - \int_\Omega ve^v \nabla v \cdot \nabla b - \int_\Omega v \nabla v \cdot \nabla u \\
\leq - \int_\Omega ve^v \nabla v \cdot \nabla b - \int_\Omega ve^v \nabla v \cdot \nabla a.
\end{align*}
Therefore, thanks to the point-wise lower bound in (5.6), and the boundedness of \(a\) and \(v\), we can find find \(c_2 > 0\) such that
\begin{equation}
\frac{d}{dt} \int_\Omega |\nabla v|^2 + a_* \int_\Omega |\nabla v|^2 \leq c_2 (\int_\Omega |\nabla b|^2 + \int_\Omega \frac{|\nabla a|^2}{a^2}). \tag{5.7}
\end{equation}
Let \(y(t) := \int_\Omega |\nabla v|^2\) and \(h(t) := c_2 (\int_\Omega |\nabla b|^2 + \int_\Omega \frac{|\nabla a|^2}{a^2})\). Then we infer from (5.7) that for \(t > 0\),
\begin{equation}
y'(t) + a_* y(t) \leq h(t), \tag{5.8}
\end{equation}
where (5.6) and (5.1) warrant the existence of \(c_3 > 0\) such that
\begin{equation}
\int_0^\infty h(s) ds \leq c_3. \tag{5.9}
\end{equation}
Therefore thanks to (5.9), an integration of (5.8) yields (5.5). \(\Box\)
Upon estimates (5.5), (5.6) and (5.1), we make use of the explicit expression of $\nabla v$ to verify the exponential decay property of $\int_{\Omega} |\nabla v|^2$.

**Lemma 5.3.** There exists $C > 0$ such that

$$\int_{\Omega} |\nabla v|^2 \leq C(t + 1)e^{-2a_* t} \text{ for all } t > 0,$$

(5.10)

where $a_* = \left\{ \frac{\frac{\sqrt{\delta}}{\min_{x \in \Omega} u_0(x)}}{\mu - \sqrt{\varepsilon}} + \frac{ue^{\varepsilon}}{\mu - \sqrt{\varepsilon}} \right\}^{-1}$.

**Proof.** According to the second equation in (1.3), we have

$$\nabla v(\cdot, t) = \nabla v(\cdot, 0)e^{-\int_0^t (u+w)(\cdot, s)ds} - v(\cdot, 0)e^{-\int_0^t (u+w)(\cdot, s)ds} \int_0^t (\nabla u(\cdot, s) + \nabla w(\cdot, s))ds,$$

which, together with the fact that $w \geq 0$, $u = ae^v \geq a_*$ due to (3.6) and thereby $u(x, t) + w(x, t) \geq a_*$ for $x \in \Omega, t > 0$, implies that

$$\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq 2e^{-2a_* t}\|\nabla v_0\|_{L^2(\Omega)}^2 + 4te^{-2a_* t} \left( \int_0^t \int_{\Omega} |\nabla u|^2 ds + \int_0^t \int_{\Omega} |\nabla w|^2 ds \right).$$

(5.11)

Further, since

$$|\nabla w| = |e^v \nabla b + e^v b \nabla v| \leq e(|\nabla v|b + |\nabla b|)$$

as well as

$$|\nabla u| = |e^v \nabla a + e^v a \nabla a| \leq e(|\nabla v|a + |\nabla a|),$$

we infer from (5.11) that there exists $c_1 > 0$ such that for $t > 0$,

$$\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq c_1 e^{-2a_* t} + c_1 t e^{-2a_* t} \int_0^t \int_{\Omega} (|\nabla b|^2 + |\nabla a|^2 + |\nabla v|^2) ds \leq c_1 e^{-2a_* t} + c_1 t e^{-2a_* t} \int_0^\infty \int_{\Omega} (|\nabla b|^2 + |\nabla a|^2 + |\nabla v|^2) ds$$

(5.12)

and thereby thanks to estimates (5.5), (5.6) and (5.1), we get

$$\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq c_2 (t + 1)e^{-2a_* t}$$

with some $c_2 > 0$. \qed
On the basis of smoothing estimates for the Neumann heat semigroup on $\Omega$, we merely turn the decay information provided by Lemma 5.3 into the exponential decay of $u - 1$ with respect to the norm in $L^p(\Omega)$ for arbitrary $p < 6$.

**Lemma 5.4.** There exists $\gamma > 0$ such that for every $p < 6$,

$$\| u(\cdot, t) - 1 \|_{L^p(\Omega)} \leq C(p)e^{-\gamma t} \tag{5.13}$$

with some $C(p) > 0$ for all $t > 0$.

**Proof.** Testing the first equation in (1.3) by $u - 1$ and integrating by parts, we have

$$\frac{d}{dt} \int_{\Omega} (u - 1)^2 + 2 \int_{\Omega} |\nabla u|^2 + 2\mu \int_{\Omega} u(u - 1)^2 = 2 \int_{\Omega} u \nabla v \cdot \nabla u - 2 \int_{\Omega} (u - 1)uz.$$

We thereupon make use of (3.6) and Lemma 3.3 along with the Young inequality to get

$$\frac{d}{dt} \int_{\Omega} (u - 1)^2 + \int_{\Omega} |\nabla u|^2 + 2\mu a_\ast \int_{\Omega} (u - 1)^2 \leq \int_{\Omega} u^2 |\nabla v|^2 + 2 \int_{\Omega} uz \leq c_1 \int_{\Omega} |\nabla v|^2 + c_1 \int_{\Omega} z \tag{5.14}$$

with some $c_1 > 0$.

Thanks to the outcomes of Lemma 5.3 and Lemma 3.4, (5.14) readily leads to

$$\int_{\Omega} (u - 1)^2 \leq c_2 e^{-2\eta t} \tag{5.15}$$

with $\eta := \min\{\mu a_\ast, a_\ast, \frac{\delta}{2}\}$ and $c_2 > 0$ for all $t > 0$.

According to known smoothing estimates for the Neumann heat semigroup on $\Omega \subset \mathbb{R}^3$ [37], there exist $c_3 = c_3(p, q) > 0$, $c_4 = c_4(p, q) > 0$ fulfilling

$$\| e^{\sigma \Delta} \varphi \|_{L^p(\Omega)} \leq c_3 \sigma^{-\frac{3}{2}(1-q)} \| \varphi \|_{L^q(\Omega)} \tag{5.16}$$

for each $\varphi \in C^0(\Omega)$, and for all $\varphi \in (L^q(\Omega))^3$,

$$\| e^{\sigma \Delta \nabla} \varphi \|_{L^p(\Omega)} \leq c_4 (1 + \sigma^{-\frac{1}{2} - \frac{3}{2}(1-q)} e^{-\lambda_1 \sigma}) \| \varphi \|_{L^q(\Omega)} \tag{5.17}$$

with $\lambda_1 > 0$ the first nonzero eigenvalue of $-\Delta$ in $\Omega$ under the Neumann boundary condition.
Applying a variation-of-constants representation of \( u \) related to the first equation in (1.3) and utilizing (5.16) and (5.17), we infer that

\[
\| (u - 1)(\cdot, t) \|_{L^p(\Omega)} \\
\leq \| e^{t(\Delta - \eta)}(u_0 - 1) \|_{L^p(\Omega)} + \int_0^t \| e^{(t-s)(\Delta - \eta)\nabla \cdot (u \nabla v)} \|_{L^p(\Omega)} ds \\
+ \int_0^t \| e^{(t-s)(\Delta - \eta)}((\mu u - \eta)(1 - u) - u \bar{z}) \|_{L^p(\Omega)} ds \\
\leq c_3(p)e^{-\eta t}\| u_0 - 1 \|_{L^p(\Omega)} + c_4(p)\int_0^t (1 + (t-s)^{-\frac{3}{2} + \frac{3}{2p}}) e^{-(\eta + \lambda_1)(t-s)}\| \nabla v(\cdot, s) \|_{L^2(\Omega)} ds \\
+ c_5(p)\int_0^t (1 + (t-s)^{-\frac{3}{2} + \frac{3}{2p}}) e^{-\eta(t-s)}\| (u - 1)(\cdot, s) \|_{L^2(\Omega)} ds \\
+ c_5(p)\int_0^t (1 + (t-s)^{-\frac{3}{2} + \frac{3}{2p}}) e^{-\eta(t-s)}\| z(\cdot, s) \|_{L^2(\Omega)} ds \\
\leq c_3(p)e^{-\eta t}\| u_0 - 1 \|_{L^p(\Omega)} + c_4(p)\int_0^t (1 + (t-s)^{-\frac{3}{2} + \frac{3}{2p}}) e^{-(\eta + \lambda_1)(t-s)}\| \nabla v(\cdot, s) \|_{L^2(\Omega)} ds \\
+ c_5(p)\int_0^t (1 + (t-s)^{-\frac{3}{2} + \frac{3}{2p}}) e^{-\eta(t-s)}\| (u - 1)(\cdot, s) \|_{L^2(\Omega)} ds \\
+ c_5(p)\int_0^t (1 + (t-s)^{-\frac{3}{2} + \frac{3}{2p}}) e^{-\eta(t-s)}\| z(\cdot, s) \|_{L^2(\Omega)} ds
\]

(5.18)

for some \( c_5(p) > 0 \). Therefore by (5.15), (5.10), (3.17) and the fact that for \( \alpha \in (0, 1) \) and \( \delta_1 \) positive constants with \( \gamma_1 \neq \delta_1 \), there exists \( c_6 > 0 \) such that

\[
\int_0^t (1 + (t-s)^{-\alpha})e^{-\gamma_1 s}e^{-\delta_1(t-s)} ds \leq c_6 e^{-\min\{\gamma_1, \delta_1\}t},
\]

(5.18) readily yields (5.13) with \( \gamma = \eta \) and some \( C(p) > 0 \).

It is noted that due to the fact that the integrability exponent in (5.10) does not exceed the considered spatial dimension \( N = 3 \), the uniform Hölder bounds for \( u \) seems to be unavailable so far, though \( \nabla v \in L^\infty_{loc}((0, \infty), L^5(\Omega)) \) achieved in Lemma 4.2. On the other hand, according to the extensibility criteria of the classical solution to (2.1), we need to establish the global boundedness of \( \| \nabla v(\cdot, t) \|_{L^5(\Omega)} \). To this end, we first turn to make sure that \( \int_\Omega |\nabla v|^4 \) decays exponentially and inter alia \( a, b \) enjoy some higher regularity, which results from a series of testing procedures.

**Lemma 5.5.** Let the hypothesis in Theorem 1.1 hold. Then there exist \( \alpha > 0 \) and \( C > 0 \) such that for all \( t > 0 \),

\[
\int_\Omega (|\nabla v(\cdot, t)|^4 + |\nabla a(\cdot, t)|^2 + |\nabla b(\cdot, t)|^2) \leq Ce^{-\alpha t}
\]

(5.19)

as well as

\[
\int_0^\infty \int_\Omega (|\Delta a|^2 + |\Delta b|^2) < \infty.
\]

(5.20)
Proof. Testing the identity

\[ a_t = \Delta a + \nabla v \cdot \nabla a + f(x, t), \quad x \in \Omega, \quad t > 0 \]

with \( f(x, t) = \mu(a(1 - u) - az + a(u + w)v) \) by \(-\Delta a\), and using Young’s inequality, we get

\[
\frac{d}{dt} \int_{\Omega} |\nabla a|^2 + 2 \int_{\Omega} |\Delta a|^2 = -2 \int_{\Omega} (\nabla a \cdot \nabla v) \Delta a - 2 \int_{\Omega} f \Delta a \\
\leq \int_{\Omega} |\Delta a|^2 + 2 \int_{\Omega} |\nabla a|^2 |\nabla v|^2 + 2 \int_{\Omega} |f|^2.
\]

Proceeding as in the proof of (4.5), we can find \( c_1 > 0, c_2 > 0 \) such that

\[
\frac{d}{dt} \left\| \nabla a \right\|_{L^2(\Omega)}^2 + c_1 \left\| \nabla a \right\|_{L^2(\Omega)}^2 + c_1 \left\| \Delta a \right\|_{L^2(\Omega)}^2 \\
\leq c_2 \left\| \nabla v \right\|_{L^4(\Omega)}^4 + c_2 \left\| f \right\|_{L^2(\Omega)}^2.
\]

Likely, we also have

\[
\frac{d}{dt} \left\| \nabla b \right\|_{L^2(\Omega)}^2 + c_3 \left\| \nabla b \right\|_{L^2(\Omega)}^2 + c_3 \left\| \Delta b \right\|_{L^2(\Omega)}^2 \\
\leq c_4 \left\| \nabla v \right\|_{L^4(\Omega)}^4 + c_4 \left\| g \right\|_{L^2(\Omega)}^2
\]

for some \( c_3 > 0, c_4 > 0 \), where \( g(x, t) = -b + ue^v z + b(u + w)v \).

To appropriately compensate the first summand on right-hand side of (5.22) and (5.23), we use the third equation in (2.1) to see that

\[
\frac{1}{4} \frac{d}{dt} \int_{\Omega} |\nabla v|^4 = -\int_{\Omega} |\nabla v|^2 \nabla v \cdot \nabla v_t \\
= -\int_{\Omega} a(v + 1)e^v|\nabla v|^4 - \int_{\Omega} ve^v|\nabla v|^2 \nabla v \cdot \nabla a \\
- \int_{\Omega} b(v + 1)e^v|\nabla v|^4 - \int_{\Omega} ve^v|\nabla v|^2 \nabla v \cdot \nabla b.
\]

Here recalling the uniform positivity of \( a \) stated in Lemma 3.2, we can pick \( c_5 > 0 \) fulfilling

\[
\int_{\Omega} a(v + 1)e^v|\nabla v|^4 \geq c_5 \int_{\Omega} |\nabla v|^4
\]

and thus infer by the Young inequality and Lemma 3.4 that for all \( t > 0 \)

\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^4 + c_5 \int_{\Omega} |\nabla v|^4 \\
\leq 2 e^{-\delta t} \int_{\Omega} (|\nabla a|^4 + |\nabla b|^4) \\
\leq c_6 e^{-\delta t} \int_{\Omega} (|\Delta a|^2 + |\Delta b|^2)
\]
with constant $c_6 > 0$, where we have used the Gagliardo–Nirenberg type inequality (4.4) in the last inequality.

Now by the appropriate linear combination of (5.22), (5.23) and (5.25), we can see that there exists $t_1 > 1$ suitably large such that for all $t > t_1$,

$$
\frac{d}{dt} \left( \| \nabla a \|^2_{L^2(\Omega)} + \| \nabla b \|^2_{L^2(\Omega)} + c_7 \| \nabla v \|^4_{L^4(\Omega)} \right) + c_8 (\| \triangle a \|^2_{L^2(\Omega)} + \| \triangle b \|^2_{L^2(\Omega)})
$$
$$
+ \ c_8 (\| \nabla a \|^2_{L^2(\Omega)} + \| \nabla b \|^2_{L^2(\Omega)} + c_7 \| \nabla v \|^4_{L^4(\Omega)})
\leq \frac{1}{c_8} (\| f \|^2_{L^2(\Omega)} + \| g \|^2_{L^2(\Omega)}).
$$

Due to the global boundedness of $a, b, z, v$ achieved in the previous Lemmas, we have

$$
|f(x, t)|^2 + |g(x, t)|^2 \leq c_9 (|u(x, t) - 1|^2 + |b(x, t)|^2 + |z(x, t)|^2 + |v(x, t)|^2)
$$

with $c_9 > 0$, and thereby there exist $\eta_1 > 0$ and $c_{10} > 0$ such that

$$
\int_\Omega |f(\cdot, t)|^2 + |g(\cdot, t)|^2 \leq c_{10} e^{-\eta_1 t} \quad \text{for all } t > t_1,
$$

thanks to Lemma 5.4. Lemma 3.2, Lemma 3.3 and Lemma 3.4. Therefore from (5.26) and (5.27), it follows that function $y(t) := \| \nabla a \|^2_{L^2(\Omega)} + \| \nabla b \|^2_{L^2(\Omega)} + c_7 \| \nabla v \|^4_{L^4(\Omega)}$ satisfies

$$
y'(t) + c_8 y(t) + c_8 (\| \triangle a \|^2_{L^2(\Omega)} + \| \triangle b \|^2_{L^2(\Omega)}) \leq \frac{c_{10}}{c_8} e^{-\eta_1 t},
$$

and thereby (5.19) is readily valid with $\alpha = \min\{c_8, \eta_1\}$. Thereafter (5.20) results from an integration of (5.28).

Now we can turn the information contained in (5.19) and (5.20) into the global boundedness of $\| \nabla v(\cdot, t) \|_{L^5(\Omega)}$ in the three-dimensional framework beyond that in (4.2).

**Lemma 5.6.** Let the hypothesis in Theorem 1.1 hold. Then there exists $C > 0$ such that for all $t > 0$,

$$
\int_\Omega |\nabla v(\cdot, t)|^5 \leq C.
$$

**Proof.** Proceeding as in the proof of (4.12), we have

$$
\| \nabla v(\cdot, t) \|_{L^5(\Omega)} \leq \| \nabla v_0 \|_{L^5(\Omega)} + e \int_0^t (\| \nabla a(\cdot, s) \|_{L^5(\Omega)} + \| \nabla b(\cdot, s) \|_{L^5(\Omega)}) ds.
$$
Note that there exist $c_1 > 0, c_2 > 0$ such that for all $\varphi \in C^2(\Omega), \frac{\partial \varphi}{\partial \nu} = 0, \| \varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi \|_{W^{2,2}(\Omega)} \leq c_1 \| \Delta \varphi \|_{L^2(\Omega)}$ and $\| \varphi - \frac{1}{|\Omega|} \int_{\Omega} \varphi \|_{L^2(\Omega)} \leq c_2 \| \nabla \varphi \|_{L^2(\Omega)}$, so we have

$$\| \nabla a(\cdot, s) \|_{L^2(\Omega)} \leq c_3 \| \Delta a(\cdot, s) \|_{L^2(\Omega)} \| \nabla a(\cdot, s) \|_{L^2(\Omega)}$$

as well as

$$\| \nabla b(\cdot, s) \|_{L^2(\Omega)} \leq c_3 \| \Delta b(\cdot, s) \|_{L^2(\Omega)} \| \nabla b(\cdot, s) \|_{L^2(\Omega)}$$

for $c_3 > 0$, and thereby

$$\int_0^t \| \nabla a(\cdot, s) \|_{L^2(\Omega)} ds \leq c_3 \int_0^t \| \Delta a(\cdot, s) \|_{L^2(\Omega)} \| \nabla a(\cdot, s) \|_{L^2(\Omega)} ds$$

$$\leq c_3 \left\{ \int_0^t \| \Delta a(\cdot, s) \|_{L^2(\Omega)}^2 ds \right\} \left\{ \int_0^t \| \nabla a(\cdot, s) \|_{L^2(\Omega)}^2 ds \right\}^{\frac{1}{2}}$$

$$\leq c_4 \left\{ \int_0^t \| \Delta a(\cdot, s) \|_{L^2(\Omega)}^2 ds \right\} \left\{ \int_0^t e^{-\frac{\sigma}{4} s} ds \right\}^{\frac{1}{2}}$$

$$< c_5$$

as well as

$$\int_0^t \| \nabla b(\cdot, s) \|_{L^2(\Omega)} ds < c_6$$

for some $c_5 > 0, c_6 > 0$, where we have used (5.20) and (5.19). Hence (5.29) results readily from (5.30)–(5.32).

At this position, thanks to the known smoothing estimates for the Neumann heat semigroup again, we can readily turn the information contained in Lemma 5.5 into the exponential decay property of $u - 1$ with respect to $L^\infty(\Omega)$-norm.

**Lemma 5.7.** Let the assumptions in Theorem 1.1 hold. Then there exist $\vartheta > 0$ and $C > 0$ fulfilling

$$\| u(\cdot, t) - 1 \|_{L^\infty(\Omega)} \leq C e^{-\vartheta t}. \quad (5.33)$$

**Proof.** Since the proof is similar to that of Lemma 5.4 we only give a short proof of (5.33).

In view to known smoothing estimates for the Neumann heat semigroup on $\Omega \subset \mathbb{R}^2$ (37), there exist $c_1 > 0, c_2 > 0$ fulfilling

$$\| e^{\sigma \Delta} \varphi \|_{L^\infty(\Omega)} \leq c_1 \sigma^{-\frac{3}{4}} \| \varphi \|_{L^2(\Omega)} \quad (5.34)$$
for each \( \varphi \in C^0(\Omega) \), and for all \( \varphi \in (L^4(\Omega))^3 \),

\[
\| e^{\sigma \Delta} \nabla \cdot \varphi \|_{L^\infty(\Omega)} \leq c_2 (1 + \sigma^{-\frac{7}{8}}) e^{-\lambda_1 \sigma} \| \varphi \|_{L^4(\Omega)}
\]

(5.35)

with \( \lambda_1 > 0 \) the first nonzero eigenvalue of \(-\Delta\) in \( \Omega \) under the Neumann boundary condition.

According to the variation-of-constants representation of \( u \) related to the first equation in (1.3), we utilize (5.34) and (5.35) to infer that

\[
\| (u - 1)(\cdot, t) \|_{L^\infty(\Omega)} \leq e^{-t} \| u_0 - 1 \|_{L^\infty(\Omega)} + c_3 \int_0^t (1 + (t-s)^{-\frac{7}{8}}) e^{-(1+\lambda_1)(t-s)} \| \nabla v(\cdot, s) \|_{L^4(\Omega)} ds
\]

\[
+ c_3 \int_0^t (1 + (t-s)^{-\frac{7}{8}}) e^{-(t-s)} (\| (u - 1)(\cdot, s) \|_{L^2(\Omega)} + \| z(\cdot, s) \|_{L^2(\Omega)}) ds
\]

(5.36)

with some \( c_3 > 0 \). This readily establishes (5.33) with appropriate \( \vartheta > 0 \) in view of (5.19), (5.13) and (3.17).

Thereby our main result has essentially been proved already.

**Proof of Theorem 1.1.** The statement on global boundedness of classical solutions has been asserted by Lemma 3.2–Lemma 3.4 and Lemma 5.6. The convergence properties in (1.8)–(1.11) are precisely established by Lemma 3.2–Lemma 3.4 and Lemma 5.7, respectively.

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