REGULARITY OF LERAY-HOPF SOLUTIONS TO NAVIER-STOKES EQUATIONS (I)-CRITICAL REGULARITY IN WEAK SPACES

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ABSTRACT. We consider the regularity of Leray-Hopf solutions to impressible Navier-Stokes equations on critical case \( u \in L^2_w(0,T; L^\infty(\mathbb{R}^3)) \). By a new embedding inequality in Lorentz space we prove that if \( \|u\|_{L^2_w(0,T; L^\infty(\mathbb{R}^3))} \) is small then as a Leray-Hopf solution \( u \) is regular. Particularly, an open problem proposed in [8] is solved.

1. Introduction

We consider the regularity of weak solutions to impressible Navier-Stokes equations

\[
\begin{cases}
\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, & \text{in } \mathbb{R}^3 \times (0, T) \\
\text{div} u = 0, & \text{in } \mathbb{R}^3 \times (0, T)
\end{cases}
\]

where \( u \) and \( p \) denote the unknown velocity and pressure of incompressible fluid respectively. \( u : (x, t) \in \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3 \) is called a weak solution of (1.1) if it is a Leray-Hopf solution. Precisely, it satisfies

\[
\begin{align*}
(1) & \quad u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)), \\
(2) & \quad \text{div} u = 0 \quad \text{in } \mathbb{R}^3 \times (0, T), \\
(3) & \quad \int_0^T \int_{\mathbb{R}^3} \{-u \cdot \partial_t \phi + \nabla u \cdot \nabla \phi + (u \cdot \nabla u) \cdot \phi\} dxdt = 0
\end{align*}
\]

for all \( \phi \in C_0^\infty(\mathbb{R}^3 \times (0, T)) \) with \( \text{div} \phi = 0 \) in \( \mathbb{R}^3 \times (0, T) \).

In this paper, we prove the following critical regularity of the Leray-Hopf solutions to the Navier-Stokes equations in weak spaces, which was an open problem proposed in [8].

**Theorem 1.1.** There is a constant \( \epsilon > 0 \) such that if \( u \) is a weak solution of the Navier-Stokes equations (1.1) in \( \mathbb{R}^3 \times (0, T) \) and if

\[ \|u\|_{L^2_w(0,T; L^\infty(\mathbb{R}^3))} \leq \epsilon \]

then \( u \) is regular in \( \mathbb{R}^3 \times (0, T) \).

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Here \( L^p_w(0, T; L^q(\Omega)) \) \((1 < p < \infty, 1 \leq q \leq \infty)\) denote the spaces of functions \( v : (x, t) \in \Omega \times (0, T) \to \mathbb{R}^3 \) with
\[
\|v\|_{L^p_w(0, T; L^q(\Omega))} := \sup_{\sigma} \sigma^{1/p} \left\{ t \in (0, T) : \|v(\cdot, t)\|_{L^q(\Omega)} > \sigma \right\}^{1/p} < \infty.
\]
It is known that the weak spaces \( L^p_w \) are special cases of the more general Lorentz spaces \( L^{p,r} \) and \( L^p_w = L^{p,\infty} \) (see [1]).

As a corollary of Theorem 1.1, we have that there is a constant \( C > 0 \) such that if \( u \) is a weak solution of (1.1) and
\[
|u(x, t)| \leq \frac{C}{(T-t)^{1/2}}, \quad \forall (x, t) \in \mathbb{R}^3 \times (T-R, T)
\]
then \( u \) is bounded in \( \mathbb{R}^3 \times (T-R, T) \).

Combining our Theorem 1.1 with the former results of Sohr [14], Kim and Kozono [8], we have

**Corollary 1.2.** For all \( r \in [3, \infty] \), there is a constant \( \epsilon > 0 \) depending only on \( r \), such that if \( u \) is a weak solution of the Navier-Stokes equations (1.1) and if
\[
\|u\|_{L^s_w(0, T; L^r_w(\mathbb{R}^3))} \leq \epsilon, \quad \text{with} \quad \frac{2}{s} + \frac{3}{r} = 1,
\]
then \( u \) is regular in \( \mathbb{R}^3 \times (0, T] \).

Since Leray(1934) [10] and Hopf(1951) [6] proved the global existence of weak solutions, it has been a fundamental open problem to prove the uniqueness and regularity of weak solutions to the Navier-Stokes equations. For \( 3 < r < \infty \), Corollary 1.2 in \( L^s_w(0, T; L^r_w(\mathbb{R}^3)) \) were proved by Sohr [14]. Corollary 1.2 in \( r = 3 \) was proved by Kim and Kozono [8]. On the other hand, similar results in Lebesgue spaces on \( \Omega \times (0, T) \) have been proved by Serrin [13], Struwe [15] and Takahashi [16], and similar results in Lebesgue spaces on \( \mathbb{R}^3 \times (0, T) \) have been proved by Giga [5], E.B. Fabes, B.F. Jones, N.M. Rivere [4], Kozono, Taniuchi [9] and Iskauriaza, Serëgin, Shverak [7] (see also W. von Wahl [17]).

Notice that the global case of the open problem proposed by Kim and Kozono in [KK pp.87 line 12-14] is solved by using Theorem 1.1. The local case of the open problem was claimed in [T]. But as pointed by Kim and Kozono in [KK pp.99 line 9-11], the critical local case can not be treated by the method given in [T] and developed in [KK].

To prove Theorem 1.1, a key step is to prove a priori estimate for vorticity equation (see Proposition 3.1), where we estimate the nonlinear terms by \( \|v\|_Q \) and the norm of \( u \) in Lorentz space. To this aim, we first prove a new embedding inequality in Lorentz space in section 2.
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2. Embedding inequality in Lorentz space

Let $\chi \in C_0^\infty(B_{1/3}(0))$ and $\varphi \in C_0^\infty(B_{8/3}(0) \setminus B_{3/4}(0))$ be the Littlewood-Paley dyadic decomposition that satisfy (see [12]):

$$
(2.1) \quad \chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad 1/3 \leq \chi^2(\xi) + \sum_{q \geq 0} \varphi^2(2^{-q}\xi) \leq 1, \quad \forall \xi \in \mathbb{R}^3.
$$

Denote

$$
\Delta_{-1}v = F^{-1}[\chi(\xi)F[v](\xi)], \quad \Delta_q v = F^{-1}[\varphi(2^{-q}\xi)F[v](\xi)], \quad \forall q \geq 0
$$

and define

$$
V(Q[0,T]) := \{v \in L^2(0,T; H^1(\mathbb{R}^3)) : ||v||_{Q[0,T]} < \infty\}
$$

and

$$
(2.2) \quad ||v||_{Q[0,T]}^2 = \sum_{q \geq -1} \sup_{0 \leq t < T} \frac{1}{2} \int_{\mathbb{R}^3} |\Delta_q v(x,t)|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\nabla v(x,t)|^2 dxdt.
$$

We shall use the notation $||v||_Q$ and $V(Q)$ to denote $||v||_{Q[0,1]}$ and $V(Q[0,1])$ respectively.

**Lemma 2.1.** There is a constant $C > 0$, such that for all $f \in L^{2,\infty}(0,1)$ and $v \in V(Q)$,

$$
(2.3) \quad \sum_{q \geq -1} \sum_{-2 \leq j \leq q + 4} \int_0^1 |f(t)| \int_{\mathbb{R}^3} |\Delta_j v(x,t)| |\nabla \Delta_q v(x,t)| dxdt \leq C ||f||_{L^{2,\infty}} ||v||^2_Q.
$$

**Proof.** Step 1. Note that the weak space $L_w^2(0,1)$ is equivalent to the Lorentz space $L^{2,\infty}(0,1)$, and the norm on $L^{2,\infty}(0,1)$ can be defined equivalently by

$$
||f||_{L^{2,\infty}(0,1)} = \sup\{||E||^{-1/2} \int_E |f(t)| dt ; \quad E \in \mathcal{L}\}
$$

where $\mathcal{L}$ is the collection of all Lebesgue measurable sets with a positive measure (see [12]). Instead of the Lebesgue measurable sets, the original version in [12](18.5) used the collection of all Borel sets with a positive measure. Since for all Lebesgue measurable sets $E$ and $1 < p < \infty$

$$
\int_E |f(t)| dt = \int_{E \cap \{|f| \geq \sigma\}} |f(t)| dt + \int_{E \setminus \{|f| \geq \sigma\}} |f(t)| dt
$$

$$
\leq \int_0^\infty \lambda^p |\{t \in E : |f(t)| > \lambda\}| \frac{d\lambda}{\lambda^p} + \sigma |E|
$$

$$
\leq C ||f||_{L_w^p(E)} |E|^{1/p'}
$$

by taking $\sigma = ||f||_{L_w^p(E)} |E|^{1/(1-p')}$, nothing is lost when we use Lebesgue measurable sets to replace Borel measurable sets.

It is known that $L^{2,\infty}(0,1)$ is the dual space of $L^{2,1}(0,1)$, where for $g \in L^{2,1}(0,1)$ the norm is defined by the infimum of $\sum_{j \geq 0} |c_j|$, the sums of the coefficients of
the atom decomposition

\[ g(t) = \sum_{j \geq 0} c_j a_j(t) \]

over all possible expansions of \( g \).

Step 2. Note that for \( q \geq -1 \)

\[
\|\nabla \Delta_q v\|_{L^2(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} \sum_{1 \leq j \leq 3} |\nabla_{x_j} \Delta_q v|^2 dx \right)^{1/2}
\]

\[
= \left( \int_{\mathbb{R}^3} \sum_{1 \leq j \leq 3} |i^{j} \xi_j \mathcal{F}[\Delta_q v](\xi)|^2 d\xi \right)^{1/2}
\]

\[
\leq \left( \frac{8}{3} \right)^{2^q} \|\Delta_q v\|_{L^2(\mathbb{R}^3)},
\]

and for \( q \geq 0 \)

\[
\|\nabla \Delta_q v\|_{L^2(\mathbb{R}^3)} \geq \left( \frac{3}{4} \right)^{2^q} \|\Delta_q v\|_{L^2(\mathbb{R}^3)}.
\]

Denote

\[ M(v) = \sup_{0 \leq t < 1} (\|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)}) \]

and for \( k = 1, 2, 3, \ldots \), define

\[ E_k = \{ t \in (0, 1) : 2^{-k} \leq M(v)^{-1}(\|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)}) \leq 2^{-k+1} \}. \]

Since \( \|v\|_Q \) is bounded, \( M(v) \) is bounded and \( E_k \) are Lebesgue measurable.

Note that for \( t \in E_k \)

\[
\left( \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \right) \leq 2^{-(k-1)} M(v)
\]

\[
< \frac{2}{|E_k|} \int_{E_k} \left( \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \right) dt.
\]

Step 3. Denote

\[ h(t) = \left( \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \right) \]

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and notice that

\[
\begin{align*}
\int_0^1 |f(t)| \int_{\mathbb{R}^3} |\Delta_j v(x, t)| |\nabla \Delta_q v(x, t)| dxdt \\
\leq \int_0^1 |f(t)| \left( \int_{\mathbb{R}^3} |\Delta_j v(x, t)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla \Delta_q v(x, t)|^2 dx \right)^{1/2} dt \\
\leq \left( \frac{8}{3} \right)^2 2^q \int_0^1 |f(t)| h(t) dt \quad (\text{by } (2.4)) \\
\leq \left( \frac{8}{3} \right)^2 2^{q+1} \sum_{k \geq 1} |E_k|^{-1} \int_{E_k} |f(t)| dt \int_{E_k} h(t) dt \quad (\text{by } (2.6)) \\
\leq \left( \frac{8}{3} \right)^2 2^{q+1} \|f\|_{L^2, \infty} \sum_{k \geq 1} \frac{1}{|E_k|^{1/2}} \int_{E_k} h(t) dt \quad (\text{by step 1}) \\
\leq \left( \frac{8}{3} \right)^2 2^{q+1} \|f\|_{L^2, \infty} \left( \sum_{k \geq 1} (\sup_{E_k} h)^{1/2} \right) \left( \int_{E_k} h(t) dt \right)^{1/2} \\
\leq \left( \frac{8}{3} \right)^2 2^{q+1} \|f\|_{L^2, \infty} (\sum_{k \geq 1} \sup_{E_k} h)^{1/2} \left( \sum_{k \geq 1} \int_{E_k} h(t) dt \right)^{1/2} \\
(2.7) \quad \leq \left( \frac{8}{3} \right)^2 2^{q+1} \|f\|_{L^2, \infty} \sqrt{2} M(v)^{1/2} \left( \int_0^1 h(t) dt \right)^{1/2} \quad (\text{by } (2.6)) \\
\end{align*}
\]

For \(j, q \geq 0\), by (2.5) we have

the right of (2.7)

\[
\leq C\|f\|_{L^2, \infty} M(v)^{1/2} \left( \int_0^1 \left( \int_{\mathbb{R}^3} |\nabla \Delta_j v(x, t)|^2 dx \right)^{1/2} \left( \int_{\mathbb{R}^3} |\nabla \Delta_q v(x, t)|^2 dx \right)^{1/2} dt \right)^{1/2},
\]

and for \(j = -1, q \geq 0\)

the right of (2.7)

\[
\leq C\|f\|_{L^2, \infty} M(v)^{1/2} \left( \int_0^1 \left( \int_{\mathbb{R}^3} |\Delta_{-1} v(x, t)|^2 dxdt \right)^{1/4} \left( \int_0^1 \left( \int_{\mathbb{R}^3} |\nabla \Delta_q v(x, t)|^2 dxdt \right)^{1/4} \right)^{1/4}.
\]
So, by (2.1) we have
\[
\sum_{q \geq -1} \sum_{q-2 \leq j \leq q+4} \int_0^1 |f(t)| dt \int_{\mathbb{R}^3} |\Delta_j v(x, t)||\nabla \Delta_q v(x, t)| dx
\]
\[
\leq C\|f\|_{L^2, \infty} \sum_{q \geq 0} (\sup_{0 \leq t < 1} \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)}) (\int_0^1 \int_{\mathbb{R}^3} |\nabla \Delta_q v(x, t)|^2 dx dt)^{1/2}
\]
\[
+C\|f\|_{L^2, \infty} (\sup_{0 \leq t < 1} \int_{\mathbb{R}^3} |\Delta_{-1} v(x, t)|^2 dx)^{1/2} (\int_0^1 \int_{\mathbb{R}^3} |\Delta_{-1} v(x, t)|^2 dx dt)^{1/2}
\]
\[
\leq C\|f\|_{L^2, \infty} \left\{ \sum_{q \geq -1} \sup_{0 \leq t < 1} \int_{\mathbb{R}^3} |\Delta_q v(x, t)|^2 dx \right\}^{1/2} \sup_{0 \leq t < 1} \int_{\mathbb{R}^3} |\Delta_q v(x, t)|^2 dx
\]
\[
\leq C\|f\|_{L^2, \infty} \|v\|_{Q}^2. \quad \square
\]

3. Proof of theorem 1.1

Without loss generality, we assume \( T = 1 \). We consider the Cauchy problem for the vorticity equation which follows the Navier-Stokes equations (1.1)
\[
(3.1)
\begin{cases}
\partial_t v - \Delta v + \text{div}(Bv) = 0, & \forall (x, t) \in \mathbb{R}^3 \times (0, 1) \\
v(x, 0) = v_0(x), & \forall x \in \mathbb{R}^3,
\end{cases}
\]
where \( Bv = v \otimes u - u \otimes v \), and \( v = \text{curl } u \). The following a priori estimate for (3.1) will be proved in section 4.

**Proposition 3.1.** There exists \( \epsilon > 0 \) such that if
\[
(3.2)
\|u\|_{L^2, \infty(0, 1; L^\infty(\mathbb{R}^3))} \leq \epsilon
\]
then for all \( t_1 \in (0, 1] \), for all solutions \( v \) of (3.1) in \( V(Q[0, t_1]) \), we have
\[
(3.3)
\|v\|_{Q[0, t_1]}^2 \leq C \|v_0\|^2_{L^2(\mathbb{R}^3)}
\]
where the constant \( C \) is independent of \( v \) and \( t_1 \).

**Proof of Theorem 1.1** Note that the weak space \( L^2_w(0, 1) \) is equivalent to the Lorentz space \( L^{2, \infty}(0, 1) \). Proposition 3.1 implies a priori estimate for the solutions of (3.1) provided that
\[
\|u\|_{L^2_w(0, 1; L^\infty(\mathbb{R}^3))} \leq \epsilon.
\]
If \( u \) is a Leray-Hopf solution to (1.1), then \( u \in L^2(0, 1; H^1(\mathbb{R}^3)) \). So for any \( \delta_0 > 0 \) there is \( \delta \in (0, \delta_0) \) such that \( \|\nabla u(\delta)\|_{L^2(\mathbb{R}^3)} < \infty \). Take
\[
v_0(x) = \text{curl } u(x, \delta)
\]
and consider the Cauchy problem
\[
(3.4)
\begin{cases}
\partial_t v - \Delta v + \text{div}(Bv) = 0, & \text{in } \mathbb{R}^3 \times (\delta, 1) \\
v(x, \delta) = v_0(x), & \text{in } \mathbb{R}^3.
\end{cases}
\]
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The solution $v$ of (3.4) is regular at least in a short time interval $(\delta, t_1)$ with $t_1 \leq 1$. So for any small $\delta_1 > 0$, $v \in V(\delta, t_1 - \delta_1)$. We can use the a priori estimate in Proposition 3.1 to get

$$\|v\|_{Q[\delta, t_1]}^2 \leq \limsup_{\delta_1 \to 0^+} \|v\|_{Q[\delta, t_1 - \delta_1]}^2 \leq C \|v_0\|_{L^2(\mathbb{R}^3)}^2$$

because the constant $C$ is independent of $\delta_1$. Then for all $t \in [\delta, t_1)$, $\|v(t)\|_{L^2(\mathbb{R}^3)}$ is uniformly bounded. So $v$ is regular at $t = t_1$. Similarly by the initial data $v(x, t_1)$ and so on we can prove that the solution $v$ of (3.4) is regular in $(\delta, 1]$ provided that $\|u\|_{L^2_{w}(0, 1; L^\infty(\mathbb{R}^3))} \leq \epsilon$.

Let $\delta_0 \to 0^+$. So $v(x, t)$ and $u(x, t)$ are regular for $t \in (0, 1]$ provided that $\|u\|_{L^2_{w}(0, 1; L^\infty(\mathbb{R}^3))} \leq \epsilon$.

Thus we proved Theorem 1.1. □

4. PROOF OF PROPOSITION 3.1

Without loss generality, we assume $t_1 = 1$. We introduce Bony’s paraproduct from the Littlewood-Paley analysis. We denote

$$S_j u = \sum_{-1 \leq k \leq j-1} \Delta_k u, \quad \Delta_j u = S_{j+1}(u) - S_j(u).$$

For the product $uv$ of $u$ and $v$, we shall decompose it as the sum

$$uv = T_u v + T_v u + R(u, v)$$

of paraproducts

$$T_u v := \sum_{j \geq 1} S_{j-1} u \Delta_j v, \quad T_v u := \sum_{j \geq 1} \Delta_j u S_{j-1} v,$$

and remainder

$$R(u, v) := \sum_{j \geq -1} \sum_{j-1 \leq k \leq j+1} \Delta_k u \Delta_j v,$$

where

$$S_j v = \sum_{-1 \leq k \leq j-1} \Delta_k v, \quad S_0 v = \Delta_{-1} v, \quad S_{-1} v = 0.$$

Note that for $q \geq -1$,

$$\Delta_q(T_u v) = \Delta_q(\sum_{q-2 \leq j \leq q+4} \Delta_j u S_{j-1} v)$$

and

$$\Delta_q(R(u, v)) = \Delta_q(\sum_{j \geq q-3} \sum_{k=j-1}^{j+1} \Delta_k u \Delta_j v),$$

because (see [12] Lemma 16), for example,

$$\text{spt}(\Delta_j u S_{j-1} v) \subset \{ (\frac{3}{4} - \frac{2}{3})j^2 \leq |\xi| \leq (\frac{8}{3} + \frac{2}{3})j^2 \}, \forall j \geq 2,$$
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\[ \text{spt}(\Delta_q) \subset \left\{ \left( \frac{3}{4} \right)^{2^q} \leq |\xi| \leq \left( \frac{8}{3} \right)^{2^q} \right\}, \quad \forall q \geq 0, \]

the necessary condition of \( \text{spt}(\Delta_j u S_{j-1} v) \cap \text{spt}(\Delta_q) \neq \emptyset \) is \( q - 2 \leq j \leq q + 4 \).

Step 1. Applying the operator \( \Delta_q \) to (3.1) we get

\[
\tag{4.1}
\begin{align*}
\partial_t \Delta_q v - \Delta \Delta_q v - \text{div} \Delta_q (u \otimes v - v \otimes u) &= 0, \quad \forall (x, t) \in \mathbb{R}^3 \times (0, 1) \\
\Delta_q v(x, 0) &= \Delta_q v_0(x), \quad \forall x \in \mathbb{R}^3.
\end{align*}
\]

Taking inner products with \( \Delta_q v \) in the two sides of the equations (4.1), we have

\[
\tag{4.2}
\frac{1}{2} (\partial_t - \Delta) |\Delta_q v(x, t)|^2 + |\nabla \Delta_q v(x, t)|^2
\]

\[
= -[\text{div}(\Delta_q R(B, v))] \cdot \Delta_q v(x, t) - [\text{div}(\Delta_q T_v B)] \cdot \Delta_q v(x, t) \\
- [\text{div}(\Delta_q T_B v)] \cdot \Delta_q v(x, t), \quad \forall (x, t) \in \mathbb{R}^3 \times (0, 1),
\]

\[
|\Delta_q v(x, 0)|^2 = |\Delta_q v_0(x)|^2, \quad \forall x \in \mathbb{R}^3.
\]

Here the notations \( R(B, v) \), \( T_v B \), \( T_B v \) may be understood as \( R(u, v) \), \( T_v u \), \( T_u v \).

Integrating (4.2) over \( \mathbb{R}^3 \times [0, 1] \) we have

\[
\sup_{0 \leq t < 1} \frac{1}{2} \int_{\mathbb{R}^3} |\Delta_q v(x, t)|^2 dx - \int_{\mathbb{R}^3} |\Delta_q v_0(x)|^2 dx + \int_0^1 dt \int_{\mathbb{R}^3} |\nabla \Delta_q v(x, t)|^2 dx
\]

\[
\leq 2 \int_0^1 dt \int_{\mathbb{R}^3} \{ \Delta_q R(B, v) + \Delta_q T_v B + \Delta_q T_B v \} \cdot \nabla \Delta_q v(x, t) dx
\]

\[
=: J_1 + J_2 + J_3
\]

where \( J_1, J_2, J_3 \) denote the integrations corresponding to \( \Delta_q R(B, v) \), \( \Delta_q T_v B \), \( \Delta_q T_B v \).

Step 2. We have

\[
\tag{4.3}
\sum_q |J_q| \leq C \| u \|_{L^2(\mathbb{R}^3)} \| v \|_{Q}^2.
\]

As in the proof of Lemma 2.1, we denote

\[
h(t) = \| \Delta_q v(t) \|_{L^2(\mathbb{R}^3)} \| \Delta_j v(t) \|_{L^2(\mathbb{R}^3)}
\]

\[
M = \sup_{0 \leq t < 1} h(t)
\]

and define

\[
E_k = \{ t \in (0, 1) : 2^{-k} < M^{-1} h(t) \leq 2^{-k-1} \}
\]

where \( M \) is bounded and \( E_k \) are Lebesgue measurable because \( \| v \|_{Q} \) is bounded.
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As in (2.7) we have

\[
\int_0^1 \|u(t)\|_{L^\infty(\mathbb{R}^3)} h(t) dt \\
\leq C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \sum_{k \geq -1} \frac{1}{|E_k|^{1/2}} \int_{E_k} h(t) dt
\]

(4.5)

\[
\leq C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \sum_{k \geq -1} \left( \sup_{E_k} h(t) \right)^{1/2} \left( \int_{E_k} h(t) \right)^{1/2}
\]

\[
\leq C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} M^{1/2} \left( \int_0^1 h(t) dt \right)^{1/2}.
\]

So

\[
|J_1| \leq 2 \int_0^1 dt \int_{\mathbb{R}^3} \Delta_q \left( \sum_{j \geq -3} \sum_{k = j-1}^{j+1} \Delta_k u \Delta_j v \right) \cdot \nabla \Delta_q v(x, t) dx
\]

\[
\leq C^2 q \sum_{j \geq -3} \int_0^1 \|u(t)\|_{L^\infty(\mathbb{R}^3)} \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} dt
\]

\[
\leq C^2 q \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \left\{ \sup_{0 \leq t < 1} \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \left( \int_0^1 \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} dt \right)^{1/2} \right\}^{1/2}
\]

\[
\times \sum_{j \geq -3} \left\{ \sup_{0 \leq t < 1} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \left( \int_0^1 \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} dt \right)^{1/2} \right\}^{1/2}
\]

by using (4.5) and Hölder inequality, and

\[
\sum_{q \geq -1} |J_1|
\]

\[
\leq C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \left( \sum_{q \geq -1} 2^q \sup_{0 \leq t < 1} \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \left( \int_0^1 \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} dt \right)^{1/2} \right)^{1/2}
\]

\[
\times \left( \sum_{q \geq -1} 2^{q-3} \left( \sum_{j \geq q-3} \left\{ \sup_{0 \leq t < 1} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \left( \int_0^1 \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} dt \right)^{1/2} \right\}^2 \right)^{1/2} \right)^{1/2}
\]

\[
\leq C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \left( \sum_{q \geq -1} \sup_{0 \leq t < 1} \|\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} \left( \int_0^1 \|\nabla\Delta_q v(t)\|_{L^2(\mathbb{R}^3)} dt \right)^{1/2} \right)^{1/2}
\]

\[
\times \left( \sum_{q \geq -1} \left( \sum_{j \geq q-3} 2^{q-3-j} \left\{ \sup_{0 \leq t < 1} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \left( \int_0^1 \|\nabla\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} dt \right)^{1/2} \right\}^2 \right)^{1/2} \right)^{1/2}
\]

\[
+ C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \sup_{0 \leq t < 1} \|\Delta_{-1} v(t)\|_{L^2(\mathbb{R}^3)}^2
\]

\[
\leq C \|u\|_{L^p(0,1;L^\infty(\mathbb{R}^3))} \|v\|_Q^2,
\]

9
where the Hardy-Young inequality

\[(\sum_{q \geq -1} (\sum_{j \geq q-2} 2^{q-2j} a_j)^2)^{1/2} \leq C(\sum_{q \geq -1} a_q^2)^{1/2}, \quad (a_j \geq 0),\]

is used in the last step.

Step 3. We have

\[
\sum_q |J_2| = \sum_q \left| \int_0^1 \int_{\mathbb{R}^3} \Delta_q (\sum_{q-2 \leq j \leq q+4} \Delta_j u S_{j-1} v) \cdot \nabla \Delta_q v \, dx \, dt \right|
\leq C \sum_{q \geq -1} \sum_{q-2 \leq j \leq q+4} \int_0^1 \|\Delta_j u(t)\|_{L^\infty(\mathbb{R}^3)} \|S_{j-1} v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_q \nabla v(t)\|_{L^2(\mathbb{R}^3)} \, dt
\]

\[+ C \sum_{q \geq -1} \sum_{q-2 \leq j \leq q+4} \int_0^1 \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \|S_{j-1} u(t)\|_{L^\infty(\mathbb{R}^3)} \|\Delta_q \nabla v(t)\|_{L^2(\mathbb{R}^3)} \, dt
\]

\[\leq C \sum_{q \geq -1} \sum_{q-2 \leq j \leq q+4} \int_0^1 \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \|S_{j-1} u(t)\|_{L^\infty(\mathbb{R}^3)} \|\Delta_q \nabla v(t)\|_{L^2(\mathbb{R}^3)} \, dt
\]

because (see [2])

\[\|S_{j-1} v(t)\|_{L^\infty(\mathbb{R}^3)} \leq C \|S_{j-1} \nabla u(t)\|_{L^\infty(\mathbb{R}^3)} \leq C 2^j \|S_{j-1} u(t)\|_{L^\infty(\mathbb{R}^3)},
\]

\[\|\Delta_j u(t)\|_{L^2(\mathbb{R}^3)} \leq C 2^{-j} \|\Delta_j \nabla u(t)\|_{L^2(\mathbb{R}^3)} \leq C 2^{-j} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)}.
\]

So by using Lemma 2.1, we have

\[(4.6) \quad \sum_q |J_2| \leq C \|u\|_{L^2(\infty(0,1;L^\infty(\mathbb{R}^3)))} \|v\|_Q^2.
\]

Step 4. Notice that

\[
\sum_q |J_3| = 2 \sum_q \left| \int_0^1 \int_{\mathbb{R}^3} \Delta_q (\sum_{q-2 \leq j \leq q+4} \Delta_j v S_{j-1} u) \cdot \nabla \Delta_q v \, dx \, dt \right|
\leq C \sum_{q \geq -1} \sum_{q-2 \leq j \leq q+4} \int_0^1 \|S_{j-1} u(t)\|_{L^\infty(\mathbb{R}^3)} \|\Delta_j v(t)\|_{L^2(\mathbb{R}^3)} \|\Delta_q \nabla v(t)\|_{L^2(\mathbb{R}^3)} \, dt
\]

\[\leq C \|u\|_{L^2(\infty(0,1;L^\infty(\mathbb{R}^3)))} \|v\|_Q^2
\]

by using Lemma 2.1 again.

So from (4.3), (4.4), (4.6) and (4.7) we have

\[\|v\|_Q^2 \leq C \|u\|_{L^2(\infty(0,1;L^\infty(\mathbb{R}^3)))} \|v\|_Q^2 + \|v_0\|_{L^2(\mathbb{R}^3)}^2.
\]

Take \(\epsilon = 1/(2C)\). If \(\|u\|_{L^2(\infty(0,1;L^\infty(\mathbb{R}^3)))} < \epsilon\), we have \(\|v\|_Q^2 \leq 2\|v_0\|_{L^2(\mathbb{R}^3)}^2\).
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