ON MULTIPLICITY OF SEMI-CLASSICAL SOLUTIONS TO NONLINEAR DIRAC EQUATIONS OF SPACE-DIMENSION $n$

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Abstract. In this paper, we study multiplicity of semi-classical solutions to nonlinear Dirac equations of space-dimension $n$:

$$-i\hbar \sum_{k=1}^{n} \alpha_k \partial_k u + a\beta u + V(x)u = f(x,|u|)u, \text{ in } \mathbb{R}^n,$$

where $n \geq 2$, $\hbar > 0$ is a small parameter, $a > 0$ is a constant, and $f$ describes the self-interaction which is either subcritical: $W(x)|u|^{p-2}$, or critical: $W_1(x)|u|^{p-2} + W_2(x)|u|^{2^*-2}$, with $p \in (2, 2^*)$, $2^* = \frac{2n}{n-1}$. The number of solutions obtained depending on the ratio of $\min V$ and $\liminf_{|x| \to \infty} V(x)$, as well as $\max W$ and $\limsup_{|x| \to \infty} W(x)$ for the subcritical case and $\max W_j$ and $\limsup_{|x| \to \infty} W_j(x), j = 1, 2$, for the critical case.

1. Introduction and main results. In quantum theory in order to describe the translation from quantum to classical mechanics, existence of semi-classical solutions of stationary quantum systems possesses an important physical interest. There have been large amounts of works on existence, multiplicity and concentration phenomenon of semi-classical solutions of nonlinear Schrödinger equations for $n$-dimensional space arising from non-relativistic quantum mechanics. We refer the reader to, e.g., [2, 10, 13, 18, 12, 31] and the references therein. In comparison, only a few similar results are known for nonlinear Dirac equations of space-dimension $n$ arising from relativistic mechanics. Mathematically, the nonlinear Dirac equation is more difficult because, unlike the spectrum of the Laplacian which is bounded below, the spectrum of the Dirac operator is neither bounded below nor above.

In this paper, we are mainly interested in utilizing variational methods to obtain multiplicity results for the Dirac equation of space-dimension $n$, by introducing

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some conditions depending on the behaviors of the potentials near the infinity, which can be directly verified. There are two main ingredients. One is to give a representation of ground state of the associated linear autonomous problem (the so-called limit equation) which yields the comparison conditions and etc. The other is to construct subspaces on which the relative energy functional is bounded above, say by \( b \), and satisfies the Palais-Smale condition below the level \( b \), and thus we are able to apply an abstract critical point theorem. Moreover, we will deal with the case of critical nonlinearity.

We now recall the problems and state our results. Consider the nonlinear Dirac equation of space-dimension \( n \), given by

\[-i\hbar \partial_t \psi = i\hbar \sum_{k=1}^{n} \alpha_k \partial_k \psi - mc^2 \beta \psi - P(x) \psi + G_\psi(x, \psi)\]  

(1)

for the function \( \psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}^N \) which stands for the wave. Here \( c \) is the speed of light, \( m > 0 \) is the mass of the electron, \( \hbar \) denotes Planck’s constant, and \( \{ \alpha_k \}_{k=1}^{n}, \beta \) is an \( (n+1) \)-tuple of Dirac matrices:

1. \( \beta^* = \beta \) and \( \alpha_k^* = \alpha_k \) for \( k = 1, \cdots, n \), i.e. \( \beta \) and \( \alpha_k \) are self-adjoint.
2. \( \beta^2 = 1, \alpha_k \beta + \beta \alpha_k = 0 \) and \( \alpha_i \alpha_j + \alpha_j \alpha_i = 2 \delta_{ij} I_{N \times N} \) for \( i, j = 1, \cdots, n \).

Proposition 1.1. [33] There is an \( n+1 \)-tuple of Dirac matrices in \( M_N(\mathbb{C}) \) when \( N = 2^{\lceil \frac{n+1}{2} \rceil} \), where \( \lceil r \rceil \) is the integral part of a nonnegative real number \( r \). Moreover, we have \( \{ \alpha_k \}_{k=1}^{n}, \beta \) has the form

\[ \beta = \left( \begin{array}{cc} I_{\frac{n}{2}} & 0 \\ 0 & -I_{\frac{n}{2}} \end{array} \right), \alpha_k = \left( \begin{array}{cc} 0 & \alpha_k \\ \alpha_k^* & 0 \end{array} \right), \quad k = 1, \cdots, n, \]

where the \( \alpha_k \) are \( \frac{n}{2} \times \frac{n}{2} \) matrices (which are Hermitian if \( n \) is odd).

For the nonlinear Dirac equation of space-dimension \( n \), \( P(x) \) denotes the potential, and \( G_\psi(x, \psi) \) describes the self-interaction in Quantum electrodynamics. It is considered to be possible basic models for unified field theories (see [28, 26] etc. and references therein). For more physical background about the nonlinear Dirac equation of space-dimension \( n \), we refer the reader to [16, 33] and the references therein. Here, we assuming that \( G(x, e^{i\theta} \psi) = G(x, \psi) \) for all \( \theta \in [0, 2\pi] \), a standing wave solution of (1) is a solution of the form \( \psi(t, x) = e^{i\omega t} w(x) \). It is clear that \( \psi(t, x) \) solves (1) if and only if \( w(x) \) solves the equation

\[-i\hbar \sum_{k=1}^{n} \alpha_k \partial_k w + a \beta w + V(x) w = F_w(x, w)\]  

(2)

where \( a = mc, V(x) = \frac{P(x)}{c} + \mu I_4 \) and \( F(x, w) = \frac{G(x, w)}{c} \).

In the two space-dimensional case, dealing with the Cauchy problem of (1), N.Bournaveas in [7] proved a local well-posedness result for the Yukawa interaction model in which the nonlinear term has the form \( G_\psi(x, \psi) \psi = \phi \beta \psi \) with \( \phi \) being Klein-Gordon field determined by \( \psi \). This result is later improved by P. d’Ancona, D. Foschi and S. Selberg [11]. Their proof relied on the null structure of the nonlinear system. And in [27] A. Grünrock and H. Pecher obtained the first global well-posedness result for large data in two space dimensions of Yukawa model is established by using Bourgain type function spaces.

In the literature, there are many results concerning existence of solutions of (2) with \( n = 3 \) under various hypotheses on the potential and the nonlinearity (see
{23} for a survey). When $V(x) \equiv \omega$, F. Merle in {29} studied the following model nonlinearity:

$$F(w) = \frac{1}{2} |\omega \hat{w}| + b |\alpha \omega \hat{w}|,$$

where $w \omega \hat{w} = (\beta w, \alpha w)_{C^4}$, $\alpha = \alpha_1 \alpha_2 \alpha_3$

with $b > 0$. In {3, 4, 8, 25} the so-called Soler model

$$F(w) = \frac{1}{2} H(w \hat{w}), \ H \in C^2(\mathbb{R}, \mathbb{R}), \ H(0) = 0$$

was investigated by using shooting methods. Such nonlinearities were later studied by using for the first time a variational method in {24}, where more general super-

results for (2) with $V(x)$ and $F(x, w)$ depending periodically on $x$ were obtained in

by using a critical point theory. For non-periodic potentials (the Coulomb-type potential is a typical example), existence and multiplicity of solutions were studied in {19} for asymptotically quadratic nonlinearities and in {21} for super-quadratic subcritical nonlinearities, where $V(x)$ and $F(x, w)$ were assumed to have limits as $|x| \to \infty$.

For small $\hbar$, the standing waves are referred to as semi-classical states. To describe the translation from quantum to classical mechanics, existence of solutions $w_\hbar$, $\hbar$ small, is of great physical importance. Existence and concentration phenomena of semi-classical ground states of the Dirac equation (2) with $n = 3, N = 4$ where the nonlinearity $F_w(x, w) = W(x) h(w)$ have been studied, in {14} for $V(x) \equiv 0$ and $h(w)$ super-linear and subcritical, in {17} for $V: \mathbb{R}^3 \to \mathbb{R}$ and $h(w)$ super-linear and subcritical and in {20} for $V: \mathbb{R}^3 \to \mathbb{R}$ and $h(w) = g(|w|) + |w|, g(|w|) w$ critical. In {16}, Y. Ding, Q. Guo and T. Xu obtained existence and concentration phenomena of semi-classical ground states of the Dirac equation (2) with $n \geq 2$ for subcritical or critical nonlinearities. For the multiplicity of semi-classical solutions of nonlinear Dirac equations, in {34} Z. Wang and X. Zhang obtained an interesting result for autonomous nonlinear Dirac equations with $n = 3, N = 4$ where the nonlinear term is subcritical. They constructed an infinite sequence of bound state solutions for small values of $\varepsilon$, particularly, these solutions are of higher topological type. Very recently, Y. Chen, Y. Ding and T. Xu in {9} treated subcritical and critical nonlinearities for autonomous nonlinear Dirac equations with $n = 3, N = 4$ and assumed that the nonlinearities $f$ satisfied $\lim_{x \to \infty} f(x) = \kappa$. They got the number of solutions with the topology of the set where the potential attains its minimum for small $\kappa$. The main objective of this paper is to study non-autonomous nonlinear Dirac equations of space-dimension $n \geq 2$ with subcritical or critical nonlinearities. The number of solutions obtained is described by the maximum and behavior at infinity of linear and nonlinear potentials.

Firstly, we deal with the subcritical case, writing $\varepsilon \equiv h$ and $\alpha \cdot \nabla = \sum_{k=1}^{n} \alpha_k \partial_k$,

$$-i\varepsilon \alpha \cdot \nabla w + \alpha \beta w + V(x) w = W(x)|w|^{p-2} w,$$  

where $p \in (2, 2^*)$ with $2^* = \frac{2n}{n-2}$.

**Notations:** In order to describe our results some notations are in order:

$$\tau := \min_{x \in \mathbb{R}^n} V(x), \ \mathcal{V} := \{ x \in \mathbb{R}^n : V(x) = \tau \}, \ \tau_\infty := \liminf_{|x| \to \infty} V(x),$$

$$\kappa := \max_{x \in \mathbb{R}^n} W(x), \ \mathcal{W} := \{ x \in \mathbb{R}^n : W(x) = \kappa \}, \ \kappa_\infty := \limsup_{|x| \to \infty} W(x),$$

$$\lambda := \min_{x \in \mathbb{R}^n} \frac{1}{W(x)}, \ \mathcal{L} := \{ x \in \mathbb{R}^n : \frac{1}{W(x)} = \lambda \}, \ \lambda_\infty := \liminf_{|x| \to \infty} \frac{1}{W(x),}$$

$$\lambda := \max_{x \in \mathbb{R}^n} W(x), \ \mathcal{W} := \{ x \in \mathbb{R}^n : W(x) = \lambda \}, \ \lambda_\infty := \limsup_{|x| \to \infty} \frac{1}{W(x).}$$
Assume that

\( \text{Theorem 1.2.} \)

\[ x_e \in \mathcal{V} \text{ such that } \kappa_e := W(x_e) = \max_{x \in \mathcal{V}} W(x), \]

\[ x_w \in \mathcal{W} \text{ such that } \tau_w := V(x_w) = \max_{x \in \mathcal{W}} V(x). \]

Assuming that the potentials satisfies:

\( (P_0) \) \( \mathcal{V}, \mathcal{W} \in C^1 \cap L^\infty(\mathbb{R}^n, \mathbb{R}), |V|_\infty < a, \inf_{x \in \mathbb{R}^n} W(x) > 0; \)

\( (P_1) \) \( \tau < \tau_\infty \) and there exists \( R_\nu > 0 \) such that \( \kappa_\nu \geq W(x) \) for all \( |x| \geq R_\nu; \)

\( (P_2) \) \( \kappa > \kappa_\infty \) and there exists \( R_w > 0 \) such that \( \tau_w \leq V(x) \) for all \( |x| \geq R_w. \)

Then all the conclusions of Theorem 1.1 remain true.

If \( (P_1) \) holds, we can set

\[ \mathcal{H}_1 = \{ x \in \mathcal{V} : W(x) = \kappa_e \} \cup \{ x \notin \mathcal{V} : W(x) > \kappa_e \}. \]

If \( (P_2) \) holds, we can set

\[ \mathcal{H}_2 = \{ x \in \mathcal{W} : V(x) = \tau_w \} \cup \{ x \notin \mathcal{W} : V(x) < \tau_w \}. \]

Moreover, if \( \mathcal{V} \cap \mathcal{W} \neq \emptyset \), then \( \kappa_e = \kappa, \tau_w = \tau \), consequently \( \mathcal{H}_1 = \mathcal{V} \cap \mathcal{W} \) or \( \mathcal{H}_2 = \mathcal{V} \cap \mathcal{W}. \)

Now we state our main results as follows.

**Theorem 1.1.** Assume that \( (P_0) \) and \( (P_1) \) hold, let

\[ m = \left[ \left( \frac{a + \tau_\infty}{a + \tau} \right)^{\frac{p}{p-2} - n} \left( \frac{\kappa}{\kappa_\infty} \right)^{\frac{2}{p-2}} \right]. \]

Then there is \( \varepsilon_m > 0 \) such that equation (3) possesses at least \( m \) pairs of solutions \( w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}(\mathbb{R}^n, \mathbb{C}^N) \) provided \( \varepsilon \leq \varepsilon_m. \)

**Theorem 1.2.** Assume that \( (P_0) \) and \( (P_2) \) hold. Let

\[ m = \left[ \left( \frac{a + \tau_\infty}{a + \tau_w} \right)^{\frac{p}{p-2} - n} \left( \frac{\kappa}{\kappa_\infty} \right)^{\frac{2}{p-2}} \right]. \]

Then all the conclusions of Theorem 1.1 remain true.

**Remark 1.1.** If additionally \( \nabla V \) and \( \nabla W \) are bounded, then among the solutions, the ground state (the least energy solution) denoted by \( w_\varepsilon \) satisfies (see [16]):

(i) There exists a maximum point \( x_e \) of \( |w_\varepsilon| \) such that, up to a subsequence, \( x_e \to x_0(\varepsilon \to 0), \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{H}_1) = 0 \) (we replace \( (\mathcal{H}_1) \) by \( (\mathcal{H}_2) \) for Theorem 1.2), and \( v_\varepsilon(x) := \omega_\varepsilon(\varepsilon x + x_\varepsilon) \) converges in \( H^1(\mathbb{R}^n, \mathbb{C}^N) \) to a ground state solution of

\[ -i\alpha \cdot \nabla u + a\beta u + V(x_0)u = W(x_0)|u|^{p-2}u, \text{ in } \mathbb{R}^n. \]

In particular if \( \mathcal{V} \cap \mathcal{W} \neq \emptyset \), then \( \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, \mathcal{V} \cap \mathcal{W}) = 0 \), and up to a subsequence, \( v_\varepsilon \) converges in \( H^1(\mathbb{R}^n, \mathbb{C}^N) \) to a ground state solution of

\[ -i\alpha \cdot \nabla u + a\beta u + \tau u = \kappa|u|^{p-2}u, \text{ in } \mathbb{R}^n. \]

(ii) There are positive constants \( C_1, C_2 \) independent of \( \varepsilon \) such that

\[ |\omega_\varepsilon(x)| \leq C_1 e^{-C_2|x-x_\varepsilon|}, \forall x \in \mathbb{R}^n. \]

Next we consider the equation with critical nonlinearity

\[ -i\varepsilon\alpha \cdot \nabla w + a\beta w + V(x)w = W_1(x)|w|^{p-2}w + W_2(x)|w|^{2^* - 2}w. \]

In the following let \( S \) denote the best constant of the Sobolev inequality

\[ S|u|_{L^2}^2 \leq ||u||^2 \text{ for all } u \in H^1(\mathbb{R}^n, \mathbb{C}^N). \]
We use the notation $\hat{H}_{\mathbb{S}}^2(\mathbb{R}^n, \mathbb{C}^N)$ to express the completion space of $C_{c}^{\infty}(\mathbb{R}^n, \mathbb{C}^N)$ with the norm

$$\|u\|_{H_{\mathbb{S}}^2}^2 := \int_{\mathbb{R}^n} |\zeta| \cdot |\hat{u}(\zeta)|^2 d\zeta,$$

where $u \to \hat{u}$ is the Fourier transform of $u \in L^2$. Let $\gamma_p$ denote the least energy of the equation

$$- i \alpha \cdot \nabla u + a \beta u = |u|^{p-2} u.$$

Set

$$\mathcal{R}_p := \left( \frac{S^n}{2n \cdot \gamma_p} \right)^{p-2}.$$ 

We also need the following notations: for $j = 1, 2$

$$\kappa_j := \max_{x \in \mathbb{R}^n} W_j(x), \quad \mathcal{W}_j := \{x \in \mathbb{R}^n : W_j(x) = \kappa_j\}, \quad \kappa_{j_\infty} := \limsup_{|x| \to \infty} W_j(x),$$

$$x_{j\nu} \in \mathcal{V} \text{ such that } \kappa_{j\nu} := W_j(x_{j\nu}) = \max_{x \in \mathcal{V}} W_j(x),$$

$$x_w \in \mathcal{W} \text{ such that } \tau_w := V(x_w) = \max_{x \in \mathcal{W}} V(x), \text{ where } \mathcal{W} := \mathcal{W}_1 \cap \mathcal{W}_2.$$ 

For $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ in $\mathbb{R}^2$, we use $\vec{x} \geq \vec{y}$ to denote $x_1 \geq y_1$ and $x_2 \geq y_2$, and $\vec{x} > \vec{y}$ if $\vec{x} \geq \vec{y}$ with $\min \{x_1 - x_2, y_1 - y_2\} > 0$. In what follows, denote, for $\mu \in (-a, \tau_\infty)$ and $\vec{\nu} = (\nu_1, \nu_2) \in \mathbb{R}^2$ with $\vec{\nu} \geq \vec{0}$,

$$m(\mu, \vec{\nu}) = \min \left\{ \left( \frac{a + \tau_\infty}{a + \mu} \right)^{\frac{1}{a^+}} \left( \frac{\nu_1}{\kappa_{1\infty}} \right)^{\frac{1}{\kappa_{1\infty}}}, \left( \frac{\nu_2}{\kappa_{2\infty}} \right)^{\frac{1}{\kappa_{2\infty}}} \right\}$$

and let $\vec{\kappa} = (\kappa_1, \kappa_2), \vec{\kappa}_{\infty} = (\kappa_1, \kappa_{2\infty}), \vec{\kappa}_\nu = (\kappa_{1\nu}, \kappa_{2\nu}).$

Assuming that the potentials satisfies: for $j = 1, 2$

(Q0) $\mathcal{V}, \mathcal{W}_j \in C^1 \cap L^\infty(\mathbb{R}^n, \mathbb{R}), -a < V(x) \leq 0, \inf_{x \in \mathbb{R}^n} W_j(x) > 0$;

(Q1) $\tau < \tau_\infty$ and there exists $R_\nu > 0$ such that $\kappa_{j\nu} \geq W_j(x)$ for all $|x| \geq R_\nu$;

(Q2) $\vec{\kappa} > \vec{\kappa}_\nu$ and there exists $R_\nu > 0$ such that $\tau_w \leq V(x)$ for all $|x| \geq R_w$;

(Q3) $\left( \frac{a}{a + \tau_\infty} \right)^{(2n-1)(p-2)-2} \frac{\kappa_{2\nu}}{\kappa_{1\infty}} < \mathcal{R}_p.$

If (Q1) holds, we can set

$$\mathcal{H}_\nu = \{x \in \mathcal{V} : W_j(x) = \kappa_{j\nu}, j = 1, 2\} \cup \{x \notin \mathcal{V} : W_j(x) > \kappa_{j\nu}, j = 1, 2\}.$$

If (Q2) holds, we can set

$$\mathcal{H}_\nu = \{x \in \mathcal{W} : V(x) = \tau_w\} \cup \{x \notin \mathcal{W} : V(x) < \tau_w\}.$$ 

Moreover, if $\mathcal{V} \cap \mathcal{W} \neq \emptyset$, then $\kappa_{j\nu} = \kappa_j, \tau_w = \tau$, consequently $\mathcal{H}_\nu = \mathcal{V} \cap \mathcal{W}$ or $\mathcal{H}_\nu = \mathcal{V} \cap \mathcal{W}.$

Now we state our main results as follows.

**Theorem 1.3.** Assume that (Q0), (Q1) and (Q3) hold, let $m = [m(\tau, \vec{\kappa}_\nu)]$. Then there is $\varepsilon_m > 0$ such that equation (8) possesses at least $m$ pairs of solutions $w_\varepsilon \in \bigcap_{s \geq 2} W^{1,s}(\mathbb{R}^n, \mathbb{C}^N)$ provided $\varepsilon \leq \varepsilon_m$.

**Theorem 1.4.** Assume that (Q0), (Q2) and (Q3) hold, let $m = [m(\tau_w, \vec{\kappa}_\nu)]$, then all the conclusions of Theorem 1.3 remain true.
Remark 1.2. Assuming either $p < 3$ or \( \left( \frac{n}{a + \tau} \right)^{(2n-1)(p-2)-2} \frac{\kappa_0}{\kappa_1^{n-1}(p-2)} < \mathcal{R}_p \). If additionally $V$ and $\nabla W_j, j = 1, 2$, are bounded, then among the solutions, the ground state (the least energy solution), denoted by $w_\varepsilon$ satisfies (see [16]):

(i) There exists a maximum point $x_\varepsilon$ of $|w_\varepsilon|$ such that, up to a subsequence, $x_\varepsilon \to x_0(\varepsilon \to 0), \lim_{\varepsilon \to 0} dist(x_\varepsilon, \mathcal{H}_\varepsilon) = 0$ (we replace $\mathcal{H}_\varepsilon$ by $\mathcal{H}_w$ for Theorem 1.4), and $v_\varepsilon(x) := \omega_\varepsilon(\varepsilon x + x_\varepsilon)$ converges in $H^1(\mathbb{R}^n, \mathbb{C}^N)$ to a ground state solution of

\[-i\alpha \cdot \nabla w + a\beta w + V(x_\varepsilon)w = W_1(x_0)|w|^{p-2}w + W_2(x_0)|w|^{2^*-2}w, \text{ in } \mathbb{R}^n.
\]

In particular if $V \cap \mathcal{V} \neq \emptyset$, then $\lim_{\varepsilon \to 0} dist(x_\varepsilon, V \cap \mathcal{V}) = 0$, and up to a subsequence, $v_\varepsilon$ converges in $H^1(\mathbb{R}^n)$ to a ground state solution of

\[-i\alpha \cdot \nabla w + a\beta w + \tau w = \kappa_1|w|^{p-2}w + \kappa_2|w|^{2^*-2}w, \text{ in } \mathbb{R}^n.
\]

(ii) There are positive constants $C_1, C_2$ independent of $\varepsilon$ such that

\[|\omega_\varepsilon(x)| \leq C_1 e^{-\frac{C_2}{2} |x-x_\varepsilon|}, \forall x \in \mathbb{R}^n.
\]

Remark 1.3. For the subcritical case, observe that in (4), $m$ can be sufficiently large if $\tau$ closes sufficiently to $-a$. One may make similar comments on (5). For the critical case, we can obtain in (11) that if $\kappa_1, \kappa_2$ are small, $m$ can be sufficiently large. In this condition, $\kappa_1, \kappa_2$ can be sufficiently large, which is an open problem in [9].

Notation. Throughout this paper, we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^n$, $B_R(x)$ denotes the ball of radius $R$ centered at $x$;
- $\cdot \cdot_q$ denotes the usual norm of the space $L^q(\mathbb{R}^n, \mathbb{C}^N), 1 \leq q \leq \infty$;
- $\alpha_\varepsilon(1)$ denotes $\alpha_\varepsilon(1) \to 0$ as $\varepsilon \to 0$;
- $u \cdot v$ denotes the scalar product in $\mathbb{C}^N$ of $u$ and $v$, i.e., $u \cdot v = \sum_{i=1}^N u_i \bar{v}_i$;
- $C$ or $C_i (i = 1, 2, \cdots)$ are some positive constants may change from line to line.

2. The functional-analytic setting. For convenience, let

\[H_0 := -i\alpha \cdot \nabla + a\beta
\]

denotes the Dirac operator. It is well known that $H_0$ is a selfadjoint operator on $L^2(\mathbb{R}^n, \mathbb{C}^N)$ with domain $D(H_0) = H^1(\mathbb{R}^n, \mathbb{C}^N)$, then by [16, Lemma 2.1], we know that

\[\sigma(H_0) = \sigma_c(H_0) = \mathbb{R} \setminus (-a, a)
\]

where $\sigma(H_0)$ and $\sigma_c(H_0)$ denote the spectrum and the continuous spectrum of $H_0$, respectively. Thus the space $L^2(\mathbb{R}^n, \mathbb{C}^N)$ possesses the orthogonal decomposition:

\[L^2 = L^- \oplus L^+, \ u = u^- + u^+
\]

so that $H_0$ is negative definite on $L^-$ and positive definite on $L^+$. Let $|H_0|$ denote the absolute value of $H_0$ and $|H_0|^\frac{1}{2}$ denote its square root. Define $E := D(|H_0|^\frac{1}{2}) = H^2(\mathbb{R}^n, \mathbb{C}^N)$ be the Hilbert space with the inner product

\[(u, v) = \Re(|H_0|^\frac{1}{2} u, |H_0|^\frac{1}{2} v)_{L^2}.
\]
and the induced norm \( \|u\| = \langle u, u \rangle^{\frac{1}{2}} \), where \( \mathbb{R} \) stands for the real part of a complex number. Since \( \sigma(H_0) = \mathbb{R} \setminus (-a, a) \), one has
\[
a|u|^2 \leq \|u\|^2, \quad \text{for all } u \in E. \tag{12}
\]
Note that this norm is equivalent to the usual \( H^1 \)-norm, hence \( E \) embeds continuously into \( L^q \) for all \( q \in [2, 2^*) \) and compactly into \( L^q_{\text{loc}} \) for all \( q \in [1, 2^*) \). That is, there exists constant \( d_q > 0 \) such that
\[
|u| \leq d_q \|u\|, \quad \text{for all } u \in E. \tag{13}
\]
Moreover, it is clear that \( E \) possesses the following decomposition
\[E = E^- \oplus E^+, \quad \text{where } E^\pm = E \cap L^\pm\]
which is decomposition orthogonal with respect to inner products \( \langle \cdot, \cdot \rangle_{L^2} \) and \( \langle \cdot, \cdot \rangle \).

Furthermore, from [22, Proposition 2.1], this decomposition induces of \( E \) also a natural decomposition of \( L^q \), hence there is \( c_q > 0 \) such that
\[
c_q |u|^q_\pm \leq \|u\|_q^q \quad \text{for all } u \in E. \tag{14}
\]

In the following, let
\[
V_\varepsilon(x) = V(\varepsilon x), \quad W_\varepsilon(x) = W(\varepsilon x), \quad W_j(\varepsilon x) = W_j(\varepsilon x), \quad j = 1, 2;
\]
\[
f(x, |u|) = \begin{cases} W(x)|u|^{p-2}, & \text{in subcritical}, \\ W_1(x)|u|^{p-2} + W_2(x)|u|^{2^*-2}, & \text{in critical}; \end{cases}
\]
\[
F_\varepsilon(x, |u|) = \int_0^{|u|} f(\varepsilon x, t)dt, \quad \Psi_\varepsilon(u) = \int_{\mathbb{R}^n} F_\varepsilon(x, |u|)dx.
\]
Making the change of variable \( x \mapsto \varepsilon x \), we can rewrite the equation (3) or (8) as the following equivalent equation
\[\quad -i\alpha \cdot \nabla u + a\beta u + V(\varepsilon x)u = f(\varepsilon x, |u|)u, \quad \text{in } \mathbb{R}^n. \tag{15}\]
On \( E \), we define the energy functional \( \Phi_\varepsilon \) corresponding to equation (15) by
\[
\Phi_\varepsilon(u) = \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 + \frac{1}{2} \int_{\mathbb{R}^n} V_\varepsilon(x)|u|^2dx - \Psi_\varepsilon(u),
\]
for \( u = u^+ + u^- \in E \). It follows by standard arguments that \( \Phi_\varepsilon \in C^1(E, \mathbb{R}) \). Also, for any \( u, v \in E \), one has
\[
\Phi'_\varepsilon(u)v = (u^+ - u^-, v) + \Re \int_{\mathbb{R}^n} V(\varepsilon x)u \cdot vdx - \Psi'_\varepsilon(u)v.
\]
Moreover, in [17, Lemma 2.1] and [20, Lemma 3.3] it is proved that critical points of \( \Phi_\varepsilon \) are weak solutions of equation (15).

We will study the multiplicity of critical points of \( \Phi_\varepsilon \) via linking arguments. The following results can be found in [16].

**Lemma 2.1.** One has

(i) \( \Phi_\varepsilon \) is weakly sequentially lower semi-continuous and \( \Phi_\varepsilon \) is weakly sequentially continuous.

(ii) \( \Phi_\varepsilon \) possesses the linking structure:

1° There exist \( r > 0 \) and \( \rho > 0 \) independent of \( \varepsilon \) such that \( \Phi_\varepsilon|_{B_r^+} \geq 0 \) and \( \Phi_\varepsilon|_{S_r^+} \geq \rho \), where \( B_r^+ = \{u \in E^+: \|u\| \leq r\} \) and \( S_r^+ = \{u \in E^+: \|u\| = r\} \);
2° For any finite dimensional linear subspace $H \subset E^+$, there exist $R = R_H > 0$ and $C = C_H > 0$ such that $\Phi_\varepsilon(u) < 0$ for all $u \in \hat{E}_H \setminus B_R$ and $\max \Phi_\varepsilon(\hat{E}_H) \leq C$, where $\hat{E}_H = E^- \oplus H$.

Let $c_\varepsilon$ denote the minimax level of $\Phi_\varepsilon$ deduced by the linking structure

$$
c_\varepsilon := \inf_{c \in \mathcal{K}_\varepsilon} \max_{u \in E_c} \Phi_\varepsilon(u) = \inf_{c \in \mathcal{K}_\varepsilon} \max_{u \in E_c} \Phi_\varepsilon(u). \tag{16}
$$

Denote $\mathcal{K}_\varepsilon := \{u \in E : \Phi_\varepsilon(u) = 0\}$ the critical set of $\Phi_\varepsilon$. Note that if $u \in \mathcal{K}_\varepsilon$ then

$$
\Phi_\varepsilon(u) = \Phi_\varepsilon(u) - \frac{1}{2} \Phi'_\varepsilon(u)u = \int_{\mathbb{R}^n} \frac{1}{2} f(\varepsilon x, |u|)u^2 - F(\varepsilon x, |u|)dx \geq 0.
$$

Using the iterative argument of [24, Proposition 3.2] one checks easily the following lemma (see [16]).

**Lemma 2.2.** If $u \in \mathcal{K}_\varepsilon$ with $\Phi_\varepsilon(u) \leq C_1$, $|u|_2 \leq C_2$, $q \in [2, \infty)$, $u \in W^{1,q}(\mathbb{R}^n, \mathbb{C}^N)$ with $|u|_{W^{1,q}} \leq \Lambda_q$ where $\Lambda_q$ depends only on $C_1, C_2$ and $q$.

To describe furthermore $c_\varepsilon$ we recall the Mountain-Pass type (see [1] also [21, 30, 32]). Consider, for a fixed $u \in E^+$, the map defined by $\phi_\varepsilon(v) = \Phi_\varepsilon(u + v)$. Observe that, for any $v, w \in E$

$$
\phi_\varepsilon''(v, w) = -|w|^2 + \int_{\mathbb{R}^n} V_\varepsilon(x)|w|^2dx - \Psi_\varepsilon(u + v)[w, w].
$$

Since $|V|_\infty < a$ and $\Psi_\varepsilon$ is strictly convex, there is a unique $h_\varepsilon$ that

$$
\phi_\varepsilon(h_\varepsilon(u)) = \max_{v \in E^-} \phi_\varepsilon(v). \tag{17}
$$

It is clear that $v \neq h_\varepsilon(u)$ if and only if $\Phi_\varepsilon(u + v) < \Phi_\varepsilon(u + h_\varepsilon(u))$. Define $I_\varepsilon : E^+ \to \mathbb{R}$ by $I_\varepsilon(u) = \Phi_\varepsilon(u + h_\varepsilon(u))$, that is

$$
I_\varepsilon(u) = \frac{1}{2} \left( |u|^2 - |h_\varepsilon(u)|^2 \right) + \frac{1}{2} \int_{\mathbb{R}^n} V_\varepsilon(x)|u + h_\varepsilon(u)|^2dx - \Psi_\varepsilon(u + h_\varepsilon(u)).
$$

Set

$$
\mathcal{K}_\varepsilon := \{u \in E^+ \setminus \{0\} : I_\varepsilon(u)u = 0\}.
$$

In the following we will call $\{h_\varepsilon(\cdot), I_\varepsilon(\cdot), \mathcal{K}_\varepsilon\}$ the “Mountain-pass” reduction for the equation:

$$
-ia \cdot \nabla u + a \beta u + V_\varepsilon(x)u = f_\varepsilon(x, |u|)u.
$$

Plainly,

$$
c_\varepsilon = \inf_{u \in \mathcal{K}_\varepsilon} I_\varepsilon(u)
$$

(see [1, 15, 21]). This, jointly with (16), implies

**Lemma 2.3.** There is a sequence $\{e_n\} \subset E^+ \setminus \{0\}$ such that, denoting $u_n = e_n + h_\varepsilon(e_n)$

$$
\Phi_\varepsilon(u_n) \to c_\varepsilon \quad \text{and} \quad \Phi'_\varepsilon(u_n) \to 0, \quad n \to \infty.
$$

In addition, we have

**Lemma 2.4.** Let $\{u_n = u_n^+ + u_n^-\}$ be a $(PS)_c$ sequence for $\Phi_\varepsilon$ and set $v_n = u_n^+ + h_\varepsilon(u_n^+), z_n = u_n^- - h_\varepsilon(u_n^+)$. Then $\|z_n\| \to 0$ and $\{v_n\}$ is also a $(PS)_c$ sequence for $\Phi_\varepsilon$, that is, $u_n^+$ is a $(PS)_c$ sequence for $I_\varepsilon$. Consequently, either $c = 0$ or $c \geq c_\varepsilon$. 
Proof. It suffices to show that \( \|z_n\| \to 0 \). Observe that
\[
0 = \Phi'_\varepsilon (v_n) z_n = -(h_\varepsilon (u_n^+), z_n) + \Re \int_{\mathbb{R}^n} V_\varepsilon v_n z_n \, dx - \Psi'_\varepsilon (v_n) z_n
\]
and since \( \{u_n\} \) is a \( (PS)_c \) sequence,
\[
o(1) = \Phi'_\varepsilon (u_n) z_n = -(u_n^-, z_n) + \Re \int_{\mathbb{R}^n} V_\varepsilon u_n z_n \, dx - \Psi'_\varepsilon (u_n) z_n.
\]
Thus,
\[
o(1) = \|z_n\|^2 - \int_{\mathbb{R}^n} V_\varepsilon |z_n|^2 \, dx + (\Psi'_\varepsilon (v_n + z_n) - \Psi'_\varepsilon (v_n)) z_n.
\]
Since \( F_\varepsilon (x,|u|) \) is strictly convex, \( (\Psi'_\varepsilon (v_n + z_n) - \Psi'_\varepsilon (v_n)) z_n \geq 0 \), which, together with the fact that \( |V|_\infty < a \), implies
\[
o(1) \geq \left( 1 - \frac{|V|_\infty}{a} \right) \|z_n\|^2.
\]
Thus, \( \|z_n\| \to 0 \). Finally, it follows from (16) that if \( c \neq 0 \) then \( c \geq c_\varepsilon \). \( \square \)

Below, for notational convenience, we denote by \( \Phi_0 \) the energy functional of the equation
\[
-i\alpha \cdot \nabla u + a\beta u + V(0)u = f(0,|u|)u. \tag{18}
\]
We define correspondingly \( c_0 \), the critical set \( \mathcal{N}_0 \), and the “Mountain-pass” reduction \( \{h_0, I_0, \mathcal{M}_0\} \) for (18). The proofs of the following conclusions can be found in [17] for the subcritical case and [20] for the critical case.

**Lemma 2.5.** We have

1. For any \( u \in E^+ \setminus \{0\} \), there is a unique \( t_\varepsilon = t_\varepsilon (u) > 0 \) such that \( t_\varepsilon u \in \mathcal{N}_\varepsilon \). Moreover, \( \lim_{\varepsilon \to 0} t_\varepsilon (u) = t_0(u) \), \( \|h_\varepsilon (t_\varepsilon u) - h_0 (t_0 u)\| \to 0 \) and \( \limsup_{\varepsilon \to 0} c_\varepsilon \leq I_0(t_0 u) \). In addition, if \( u \in \mathcal{N}_0 \) then \( t_0 = 1 \).

2. \( \lim_{\varepsilon \to 0} c_\varepsilon = c_0 \).

In order to establish our multiplicity results, we recall an abstract critical point theorem, see [5, 14]. Let \( X, Y \) be Banach spaces with \( X \) being separable and reflexive, and set \( E = X \oplus Y \). Let \( S \subset X^* \) be a countable dense subset. Let \( \mathcal{P} \) be the family of semi-norms on \( E \):
\[
p_s : E = X \oplus Y \to \mathbb{R}, \quad p_s(x + y) = |s(x)| + ||y||, \quad s \in S.
\]
Denote by \( \mathcal{T}_\mathcal{P} \) the topology on \( E \) induced by \( \mathcal{P} \). Let \( \mathcal{T}_{w^*} \) be the weak* topology of \( E^* \).

For a functional \( \Phi : E \to \mathbb{R} \) and numbers \( a, b \in \mathbb{R} \) we write \( \Phi^a := \{ u \in E : \Phi(u) \leq a \} \), \( \Phi_a := \{ u \in E : \Phi(u) \geq a \} \), and \( \Phi^0_a := \Phi_a \cap \Phi^a \). Assume

(\( \Phi_1 \)) \( \Phi \in C^1(E, \mathbb{R}) \); \( \Phi : (E, \mathcal{T}_\mathcal{P}) \to \mathbb{R} \) is upper semi-continuous, and \( \Phi' : (\Phi_a, \mathcal{T}_\mathcal{P}) \to (E^*, \mathcal{T}_{w^*}) \) is continuous for every \( a \in \mathbb{R} \).

(\( \Phi_2 \)) there exists \( r > 0 \) with \( \rho := \inf \Phi (S_r Y) > \Phi(0) = 0 \) where \( S_r Y := \{ y \in Y : ||y|| = r \} \).

(\( \Phi_3 \)) there exist a finite-dimensional subspace \( Y_0 \subset Y \) and \( R > r \) such that we have for \( E_0 := X \times Y_0 \) and \( B_0 := \{ u \in E_0 : ||u|| \leq R \} : b := \sup \Phi (E_0) < \infty \) and \( \sup \Phi (E_0 \setminus B_0) < \inf \Phi (B_0 Y) \).

We consider the set \( \mathcal{M} (\Phi^c) \) of maps \( g : \Phi^c \to E \) with the properties

(i) \( g \) is \( \mathcal{P} \)-continuous and odd;
(ii) \( g(\Phi^a) \subset \Phi^a \) for all \( a \in [\rho, b] \);  
(iii) each \( u \in \Phi^c \) has a \( \mathcal{P} \)-open neighbourhood \( O \subset E \) such that the set \( (id - g)(O \cap \Phi^c) \) is contained in a finite-dimensional linear subspace.

The pseudo-index of \( \Phi^c \) is defined by  
\[
\psi(c) := \min \{ \text{gen}(g(\Phi^c) \cap S_Y) : g \in \mathcal{M}(\Phi^c) \} \in \mathbb{N}_0 \cup \{\infty\},
\]
where \( \text{gen}(\cdot) \) denotes the usual symmetric index. Additionally, set for \( d > 0 \) fixed \( \mathcal{M}_0(\Phi^d) := \{ g \in \mathcal{M}(\Phi^d) : g \text{ is a homeomorphism from } \Phi^d \text{ to } g(\Phi^d) \} \).

Then we define for \( c \in [0, d] \) \( \psi_d(c) := \min \{ \text{gen}(g(\Phi^c) \cap S_Y) : g \in \mathcal{M}_0(\Phi^d) \} \).

Note that, by definition, \( \psi(c) \leq \psi_d(c) \) for all \( c \in [\rho, b] \).

**Theorem 2.1.** (See [5, 14]) Let \( (\Phi_1) - (\Phi_3) \) be satisfied, and assume that \( \Phi \) is even and satisfies the \((PS)_c\) condition for \( c \in [\rho, b] \). Then \( \Phi \) has at least \( n := \dim Y_0 \) pairs of critical points with critical values given by  
\[
c_i := \inf \{ c \geq 0 : \psi(c) \geq i \} \in [\rho, b], \quad i = 1, \ldots, n.
\]

If \( \Phi \) has only finitely many critical points in \( \Phi^b \), then \( \rho < c_1 < c_2 < \cdots < c_n \leq b \).

**Remark 2.1.** Setting \( X = E^- \) and \( Y = E^+ \), it follows from the definition and Lemma 2.1 that the functional \( \Phi = \Phi_c \) is even and satisfies \((\Phi_1)\) and \((\Phi_2)\).

3. Preliminary results. Firstly, we recall a result on the representation of the energy to certain constant coefficient systems.

**Lemma 3.1.** Let \( u \) be a solution of  
\[
-\text{i}\alpha \cdot \nabla u + a\beta u + \mu u = \nabla F(u), \quad u \in H^1(\mathbb{R}^n, \mathbb{C}^N),
\]
where \( F(u) = \frac{\nu}{p}|u|^p \) or \( F(u) = \frac{\nu_1}{p}|u|^p + \frac{\nu_2}{2^*}|u|^{2^*} \). Then the energy  
\[
\Phi(u) = \frac{n - 2}{2n} \int_{\mathbb{R}^n} \langle -\text{i}\alpha \cdot \nabla u, u \rangle \, dx.
\]

**Proof.** By the Pohozaev’s identity ([24])  
\[
\int_{\mathbb{R}^n} \langle -\text{i}\alpha \cdot \nabla u, u \rangle \, dx = \frac{n}{2} \int_{\mathbb{R}^n} \left( -\langle a\beta u, u \rangle - \mu |u|^2 + 2F(u) \right) \, dx.
\]

On the other hand,  
\[
\int_{\mathbb{R}^n} \langle -\text{i}\alpha \cdot \nabla u, u \rangle \, dx = \int_{\mathbb{R}^n} \left( -\langle a\beta u, u \rangle - \mu |u|^2 + \nabla F(u)\bar{u} \right) \, dx.
\]

Thus,  
\[
\frac{n - 2}{2} \int_{\mathbb{R}^n} \langle a\beta u, u \rangle + \mu |u|^2 \, dx = \int_{\mathbb{R}^n} (nF(u) - \nabla F(u)\bar{u}) \, dx.
\]
so the energy functional
\[ \Phi(u) = \Phi(u) - \frac{1}{n} \Phi'(u)u \]
\[ = \frac{n-2}{2n} \int_{\mathbb{R}^n} \langle -i \alpha \cdot \nabla u, u \rangle \, dx + \frac{n-2}{2n} \int_{\mathbb{R}^n} \langle a \beta u, u \rangle + \mu u^2 \, dx \]
\[ - \int_{\mathbb{R}^n} \left( F(u) - \frac{1}{n} \nabla F(u) \tilde{u} \right) \, dx \]
\[ = \frac{n-2}{2n} \int_{\mathbb{R}^n} \langle -i \alpha \cdot \nabla u, u \rangle \, dx. \]

This completes the proof. \( \square \)

### 3.2. The limit equation: subcritical case.

Consider, for any \( \tau \leq \mu \leq \tau_\infty \) and \( \kappa_\infty \leq \nu \leq \kappa \),

\[-i \alpha \cdot \nabla u + a \beta u + \mu u = \nu |u|^{p-2} u. \tag{19}\]

Its solutions are critical points of the functional
\[ \Gamma_{\mu \nu}(u) := \frac{1}{2} \left( \|u^+\|^2 - \|u^-\|^2 \right) + \frac{\mu}{2} \int_{\mathbb{R}^n} |u|^2 \, dx - \frac{\nu}{p} \int_{\mathbb{R}^n} |u|^p \, dx \]
defined for \( u = u^+ + u^- \in E \). Denote the critical set, the least energy, and the set of least energy solutions of \( \Gamma_{\mu \nu} \) as follows

\[ \mathcal{L}_{\mu \nu} := \{ u \in E : \Gamma'_{\mu \nu}(u) = 0 \}, \]
\[ \gamma_{\mu \nu} := \inf \{ \Gamma_{\mu \nu}(u) : u \in \mathcal{L}_{\mu \nu} \setminus \{0\} \}, \]
\[ \mathcal{R}_{\mu \nu} := \{ u \in \mathcal{L}_{\mu \nu} : \Gamma_{\mu \nu}(u) = \gamma_{\mu \nu}, |u(0)| = |u|_\infty \}. \]

The following conclusions are from [17, 20]:

(i) \( \mathcal{L}_{\mu \nu} \neq \emptyset, \gamma_{\mu \nu} > 0 \), and \( \mathcal{L}_{\mu \nu} \subset \bigcap_{q \geq 2} W^{1,q} \),

(ii) \( \gamma_{\mu \nu} \) is attained, and \( \mathcal{R}_{\mu \nu} \) is compact in \( H^1(\mathbb{R}^n, \mathbb{C}^N) \),

(iii) there exist \( C, c > 0 \) such that \( |u(x)| \leq C \exp(-c|x|) \) for all \( x \in \mathbb{R}^n \) and \( u \in \mathcal{R}_{\mu \nu} \).

Using \( \gamma_p \) we have the following representation (see [16, Lemma 3.4]).

**Lemma 3.2.** The corresponding least energy of (19) denoted by \( \gamma_{\mu \nu} \), then

\[ \gamma_{\mu \nu} \leq \left( \frac{a}{a + \mu} \right)^{\frac{2}{n+2} + \nu \frac{2}{n+2}} \gamma_p. \]

**Lemma 3.3.** Let \( -a < \mu_j < a \) and \( \nu_j > 0, j = 1, 2 \), with \( \min \{ \mu_2 - \mu_1, \nu_1 - \nu_2 \} > 0 \). Then \( \gamma_{\mu_1 \nu_1} < \gamma_{\mu_2 \nu_2} \). In particular, \( \gamma_{\mu_1 \nu} < \gamma_{\mu_2 \nu} \) if \( \mu_1 < \mu_2 \), and \( \gamma_{\mu_1 \nu} > \gamma_{\mu_2 \nu} \) if \( \nu_1 < \nu_2 \).

The following Lemma describes a comparison between the ground state energy values for different parameters \( \mu \in (-a, a) \) and \( \nu > 0 \), which will play an important role in proving the existence result in Section 4. This conclusion follows directly from the representation of \( \gamma_{\mu \nu} \).

**Lemma 3.4.** Let \( \tau \leq \mu \leq \tau_\infty \) and \( \kappa_\infty \leq \nu \leq \kappa \),

\[ \gamma_{\mu \nu} \leq \left( \frac{\kappa_\infty}{\nu} \right)^{\frac{2}{n+1}} \left( \frac{a + \mu}{a + \tau_\infty} \right)^{\frac{p}{n} - 1} \gamma_\infty, \]

where \( \gamma_\infty = \gamma_{\tau_\infty \kappa_\infty} \).
Proof. Let \( u \) be a least energy solution of (19) where \( \mu = \tau_\infty, \nu = \kappa_\infty \) and set
\[
v(x) = bu(\xi x), \quad b = \left( \frac{\kappa_\infty (a + \mu)}{\nu (a + \tau_\infty)} \right)^{\frac{1}{p-2}}, \quad \xi = \frac{a + \mu}{a + \tau_\infty}.
\]
Writing \( u = (u_1, u_2) \in \mathbb{C}^2 \times \mathbb{C}^2 \), observe that (19) is equivalent to
\[
\begin{cases}
- i \sigma \cdot \nabla u_2 + (a + \mu) u_1 = \nu |u|^{p-2} u_1, \\
- i \sigma \cdot \nabla u_1 - (a - \mu) u_2 = \nu |u|^{p-2} u_2,
\end{cases}
\]
with the energy functional
\[
\Gamma_{\mu\nu}(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left( - i \alpha \cdot \nabla v, v \right) + (a + \mu) |v_1|^2 - (a - \mu) |v_2|^2 \, dx - \frac{\nu}{p} \int_{\mathbb{R}^n} |v|^p \, dx.
\]
Setting
\[
\eta = \frac{(a + \mu) (a - \tau_\infty)}{(a + \tau_\infty) (a - \mu)},
\]
the function \( v \) is a least energy solution of
\[
\begin{cases}
- i \sigma \cdot \nabla v_2 + (a + \mu) v_1 = \nu |v|^{p-2} v_1 \\
- i \sigma \cdot \nabla v_1 - \eta (a - \mu) v_2 = \nu |v|^{p-2} v_2
\end{cases}
\]
with energy
\[
I(v) := \frac{1}{2} \int_{\mathbb{R}^n} \left( - i \alpha \cdot \nabla v, v \right) + (a + \mu) |v_1|^2 - \eta (a - \mu) |v_2|^2 \, dx - \frac{\nu}{p} \int_{\mathbb{R}^n} |v|^p \, dx.
\]
Since
\[
\eta - 1 = \frac{2a (\mu - \tau_\infty)}{(a + \tau_\infty) (a - \mu)} \leq 0,
\]
i.e., \( \eta \leq 1 \), one has \( \Gamma_{\mu\nu}(v) \leq I(v) \) which, together with Lemma 3.1 implies
\[
\gamma_{\mu\nu} \leq \left( \frac{\kappa_\infty}{\nu} \right)^{\frac{2}{p-2}} \left( \frac{a + \mu}{a + \tau_\infty} \right)^{\frac{p}{p-2} - n} \gamma_\infty.
\]
\[\square\]

As a consequence we have

**Lemma 3.5.** There holds \( m \gamma_{\mu\nu} \leq \gamma_\infty \) where
\[
m = \left[ \left( \frac{\kappa_\infty}{\nu} \right)^{\frac{2}{p-2}} \left( \frac{a + \mu}{a + \tau_\infty} \right)^{\frac{p}{p-2} - n} \right].
\]

### 3.2. The limit equation: critical case.

Next, we consider, for any \( \tau \leq \mu \leq \tau_\infty \) and \( \kappa_\infty \leq \kappa_1, \kappa_2 \leq \kappa \),
\[
- i \alpha \cdot \nabla u + a \beta u + \mu u = \nu_1 |u|^{p-2} u + \nu_2 |u|^{2^*-2} u.
\]
Its solutions are critical points of the functional
\[
\Gamma_{\mu\nu}(u) := \Gamma_{\mu\nu_1}(u) - \frac{\nu_2}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} \, dx = \Gamma_{\mu\nu_2}(u) - \frac{\nu_1}{p} \int_{\mathbb{R}^n} |u|^p \, dx
\]
on \( u \in E \), where
\[
\Gamma_{\mu\nu_1}(u) = \frac{1}{2} \left( \|u^+\|^2 - \|u^-\|^2 \right) + \frac{1}{2} \int_{\mathbb{R}^n} \mu |u|^2 \, dx - \frac{\nu_1}{p} \int_{\mathbb{R}^n} |u|^p \, dx,
\]
\[
\Gamma_{\mu\nu_2}(u) = \frac{1}{2} \left( \|u^+\|^2 - \|u^-\|^2 \right) + \frac{1}{2} \int_{\mathbb{R}^n} \mu |u|^2 \, dx - \frac{\nu_2}{2^*} \int_{\mathbb{R}^n} |u|^{2^*} \, dx.
\]
Let $\gamma_{\mu\nu}, \gamma_{\mu\nu_1}, \gamma_{\mu\nu_2}$ denote the linking levels of $\Gamma_{\mu\nu}, \Gamma_{\mu\nu_1}, \Gamma_{\mu\nu_2}$, respectively. One has
\[ \gamma_{\mu\nu} < \gamma_{\mu\nu_1}, \quad \gamma_{\mu\nu} < \gamma_{\mu\nu_2}. \] (23)

Define $h_0 : E^+ \to E^-$ by $\Gamma_{\mu\nu}(u + h_0(u)) = \max_{\nu \in E^-} \Gamma_{\mu\nu}(u + v)$ for $u \in E^+$, and $I_{\mu\nu}(u) = \Gamma_{\mu\nu}(u + h_0(u))$. Set $\mathcal{M}_{\mu\nu} = \{ u \in E^+ : I'_{\mu\nu}(u) = 0 \}$.

The following result is from [16, Lemma 3.6].

**Lemma 3.6.** $\gamma_{\mu\nu}$ is attained provided the following:
\[ \left( \frac{a}{a + \mu} \right)^{(2n-1)(p-2)} \frac{\nu_2^{(n-1)(p-2)}}{\nu_1^2} < \mathcal{R}_p. \] (24)

In the sequel, by $\tilde{v}_1 \leq \tilde{v}_2$ we mean that $\min\{\nu_1^2 - \nu_1^1, \nu_2^2 - \nu_2^1\} \geq 0$ for every vectors $\tilde{v}_j = (\nu_1^j, \nu_2^j)$. The following conclusion is clear.

**Lemma 3.7.** Let $\mu_j \in (-a, a)$ and $\tilde{v}_j > 0, j = 1, 2$, with $\mu_1 \leq \mu_2$ and $\tilde{v}_1 \geq \tilde{v}_2$, then $\gamma_{\mu_1, \tilde{v}_1} \leq \gamma_{\mu_2, \tilde{v}_2}$. If $\min\{\mu_2 - \mu_1, \tilde{v}_1 - \tilde{v}_2\} > 0$ then $\gamma_{\mu_1, \tilde{v}_1} < \gamma_{\mu_2, \tilde{v}_2}$.

Below, let $u$ be a least energy solution of
\[ -i\alpha \cdot \nabla u + \beta u + \tau_\infty u = \kappa_\infty |u|^{p-2} u + \kappa_\infty |u|^{2^*-2} u \] (25)
with the energy denoted by $\gamma_\infty$ which is attained. For $\tau \leq \mu \leq \tau_\infty$ and $\kappa_\infty \leq \nu_j \leq \kappa_j$, set
\[ v(x) = bu(\xi x), \quad \xi = \frac{a + \mu}{a + \tau_\infty}, \quad b = \max\{b_1, b_2\} \]
where
\[ b_1 = \left( \frac{\xi \kappa_\infty}{\nu_1} \right)^{\frac{1}{p-2}} \quad \text{and} \quad b_2 = \left( \frac{\xi \kappa_\infty}{\nu_2} \right)^{\frac{1}{p-2}}. \]
Then, $v$ solves
\[ -i\alpha \cdot \nabla v + a\beta v + M_{\mu}v = \frac{\kappa_\infty (a + \mu)}{b_2} \nu_1 |v|^{p-2} v + \frac{\kappa_\infty (a + \mu)}{b_2} \nu_2 |v|^{2^*-2} v \]
with energy denoted by $I^*(v)$, where $M_{\mu} = \begin{pmatrix} \mu_1 I_2 & 0 \\ 0 & \mu_2 I_2 \end{pmatrix}$ with
\[ \mu_1 = \xi (a + \tau_\infty) - a = \mu, \quad \mu_2 = a - \xi (a - \tau_\infty) > \mu. \]
By definition, $\gamma_{\mu\nu} \leq I^*(v)$. Then, by Lemma 3.1 it is clear that
\[ \gamma_{\mu\nu} \leq I^*(v) = b^2 \left( \frac{a + \tau_\infty}{a + \mu} \right)^{n-1} \gamma_\infty. \]

Set
\[ m(\mu, \tilde{v}) = \begin{cases} \left( \frac{a + \tau_\infty}{a + \mu} \right)^{p-2-n} \left( \frac{\nu_1}{\kappa_\infty} \right)^{\frac{2}{p-2}}, & \text{if } b_1 \geq b_2, \\ \left( \frac{\nu_1}{\kappa_\infty} \right)^{\frac{2}{p-2}}, & \text{otherwise}, \end{cases} \] (26)
we have
\[ m(\mu, \tilde{v}) I^*(v) = \gamma_\infty. \]
Thus it follows from the above definitions the following

**Lemma 3.8.** For $\tau \leq \mu \leq \tau_\infty$ and $\kappa_\infty \leq \nu_j \leq \kappa_j$, there holds
\[ m(\mu, \tilde{v}) \gamma_{\mu\nu} < \gamma_\infty. \]
3.3. The cut-off functions. Finally, we consider
\[-i\alpha \cdot \nabla u + a \beta u + V'' u = W'_\varepsilon |u|^{p-2} u,\] (27)
where \(V' = \max\{\mu, V(x)\}, V'' = V'(\varepsilon x)\); \(W'' = \min\{\nu, W(x)\}, W'\varepsilon = W'(\varepsilon x)\), and
\[-i\alpha \cdot \nabla u + a \beta u + V'' u = W'_\varepsilon |u|^{p-2} u + W''_\varepsilon |u|^{2^*-2} u,\] (28)
where \(W''_j = \min\{\nu_j, W_j(x)\}, W'\varepsilon = W'\varepsilon (\varepsilon x)\).

The solutions of (27) are critical points of
\[\Phi^{\mu\nu}_\varepsilon(u) = \frac{1}{2} \left(\|u^+\|^2 - \|u^+\|^2\right) + \frac{1}{2} \int_{\mathbb{R}^n} V'_\varepsilon |u|^2 dx - \frac{1}{p} \int_{\mathbb{R}^n} W''_\varepsilon(x)|u|^p dx\]
on \(u \in E = E^+ \oplus E^-\). Let \(c^{\mu\nu}_\varepsilon\) denote the Minimax level of \(\Phi^{\mu\nu}_\varepsilon\) deduced by the linking structure (see (16)). Write \(h^{\mu\nu}_\varepsilon, I^{\mu\nu}_\varepsilon, M^{\mu\nu}_\varepsilon\), and so on, for the notations associated to the Mountain-Pass reduction. Recall that, for any \(u \in E^+ \setminus \{0\}\), there is a unique \(t = t(u) > 0\) such that \(t(u) u \in \mathcal{N}^{\mu\nu}_\varepsilon\). It is easy to check that \(c^{\mu\nu}_\varepsilon = \inf \{I^{\mu\nu}_\varepsilon(u) : u \in \mathcal{N}^{\mu\nu}_\varepsilon\}\). In the sequel we denote
\[\Phi^{\infty}_\varepsilon = \Phi^{\infty\infty}_\varepsilon, \quad c^{\infty}_\varepsilon = c^{\infty\infty}_\varepsilon, \quad \mathcal{N}^{\infty\varepsilon}_\varepsilon = \mathcal{N}^{\infty\infty}_\varepsilon \quad \text{and} \quad \Gamma^{\infty}_\varepsilon = \Gamma^{\infty\infty}_\varepsilon, \gamma^{\infty}_\varepsilon = \gamma^{\infty\infty}_\varepsilon.\]
As a consequence of Lemma 2.5 we have

**Lemma 3.9.** \(c^{\infty}_\varepsilon \to \gamma^{\infty}_\varepsilon\) as \(\varepsilon \to 0\).

**Remark 3.1.** Similarly, one obtains easily that \(\lim_{\varepsilon \to 0} c^{\mu\nu}_\varepsilon = \gamma^{\mu\nu}_\varepsilon\).

As a consequence one has

**Lemma 3.10.** \(\Phi^{\mu\nu}_\varepsilon\) satisfies the \(PS)\_\varepsilon\) condition for \(c < \gamma^{\infty}_\varepsilon\) if \(\varepsilon \) small.

**Proof.** Writing \(I(u) = \Phi^{\mu\nu}_\varepsilon(u)\), let \(I(u_n) \to c\) and \(I'(u_n) \to 0\). Then \(u_n\) is bounded and we can assume that \(u_n \to u\). Clearly \(I'(u) = 0\). Set \(z_n = u_n - u\), note that \(z_n \to 0\) in \(E, z_n \to 0\) in \(L^q_{\text{loc}}\) for \(q \in [1, 2^*)\), and \(z_n(x) \to 0\) a.e. in \(x\). Using the Brezis-Lieb lemma[35], it is easy to check that \(\Phi^{\infty\varepsilon}_\varepsilon(z_n) \to c - \Phi^{\mu\nu}_\varepsilon(u)\) and \((\Phi^{\infty\varepsilon}_\varepsilon)'(z_n) \to 0\). If \(c = \gamma^{\mu\nu}_\varepsilon(u)\) then \(z_n \to 0\) and we are done. If \(c - \gamma^{\mu\nu}_\varepsilon(u) \geq c^{\infty\varepsilon}_\varepsilon\) then \(c \geq c^{\mu\nu}_\varepsilon + c^{\infty}_\varepsilon\), a contradiction. \(\Box\)

Solutions of the equation (28) are critical points of
\[\Phi^{\mu\nu\bar{\nu}}_\varepsilon(u) = \Phi^{\mu\nu\bar{\nu}}_\varepsilon(u) - \frac{1}{2^*} \int_{\mathbb{R}^n} W''_\varepsilon(x)|u|^{2^*} dx\]
with \(\bar{\nu} = (\nu_1, \nu_2)\). Let \(c^{\mu\nu\bar{\nu}}_\varepsilon\) be the linking level (see (16)). Writing \(\Phi^{\infty\varepsilon}_\varepsilon\) and \(c^{\infty}_\varepsilon\) for \(\mu = \tau^{\infty}\) and \(\bar{\nu} = (\kappa^{\infty}_1, \kappa^{\infty}_2)\) we have, as Lemma 3.9, the following

**Lemma 3.11.** \(c^{\infty\varepsilon}_\varepsilon \to \gamma^{\infty}_\varepsilon\) as \(\varepsilon \to 0\).

Also as Lemma 3.10, there holds the following

**Lemma 3.12.** \(\Phi^{\mu\bar{\nu}}_\varepsilon\) satisfies the \((PS)\_\varepsilon\) condition for all \(c < \gamma^{\infty}_\varepsilon\).

**Proof.** Denote \(I(u) = \Phi^{\mu\bar{\nu}}_\varepsilon\) and let \(I(u_n) \to c, I'(u_n) \to 0\). One can assume \(u_n \to u\) and set \(z_n = u_n - u\). Then \(z_n\) is a \((PS)\_\varepsilon\) sequence for \(\Phi^{\infty\varepsilon}_\varepsilon\) where \(\bar{\varepsilon} = c - I(u)\). By Lemma 3.11, if \(I(u) \neq c\) then \(c - I(u) \geq \gamma^{\infty}_\varepsilon\), a contradiction. \(\Box\)
4. Proofs of main results: the subcritical case. Setting \( u(x) = w(\varepsilon x) \), the equation (3) is equivalent to the following

\[
- i \alpha \cdot \nabla u + a \beta u + V_\varepsilon(x)u = W_\varepsilon(x)|u|^{p-2}u.
\]  
(29)

Proof of Theorem 1.1. Without loss of generality, we may assume that \( 0 \in \mathcal{Y} \) and \( x_v = 0 \). Observe that \( \tau = V(0) \) and \( \kappa_v = W(0) \). Solutions of (29) are critical points of the functional \( \Phi_\varepsilon(u) := \Phi_\varepsilon^\infty(u) \). For notational convenience we denote \( \Phi_0(u) = \Gamma_{\tau\kappa_v} \). We will utilize Theorem 2.1. Obversely, \( \Phi_\varepsilon \) is even, and in virtue of Remark 2.1 the conditions (\( \Phi_1 \)) and (\( \Phi_2 \)) are satisfied. It remains to verify (\( \Phi_3 \)).

Let \( u \in \mathcal{H}_{\tau\kappa_v} \) and let \( \chi_\tau \in C_0^\infty(\mathbb{R}^+) \) be such that \( \chi_\tau(s) = 1 \) if \( s \leq r \) and \( \chi_\tau(s) = 0 \) if \( s \geq r + 1 \). Set \( u_\tau(x) = \chi_\tau(|x|)u(x) \). Recall that \( |u(x)| \leq Ce^{-c|x|} \) for some \( C, c > 0 \) and all \( x \in \mathbb{R}^n \), hence \( \|u_r - u\| \to 0 \) as \( r \to \infty \). Then \( \|u_\tau + u\| - \|u_\tau - u\| \to 0, \) \( \Phi_0(u_\tau) \to \gamma_{\tau\kappa_v} \) and \( \Phi_0(u_r) \to 0 \) as \( r \to \infty \). Let \( h_0 : E^+ \to E^- \) be defined so that \( \Phi_0(u + h_0(u)) = \max \Phi_0(u + v)(\text{see (17)}). \) Plainly, \( \|u_r - h_0(u_r^+)\| \to 0 \) and

\[
\max_{v \in E^-} \Phi_0(u_r^+ + v) = \Phi_0(\hat{u}_r) = \Phi_0(u_r) + o(1) = \gamma_{\tau\kappa_v} + o(1).
\]

Observe that since \( V(\varepsilon x) \to \tau \) and \( W(\varepsilon x) \to \kappa_v \) as \( \varepsilon \to 0 \) uniformly in \( |x| \leq r + 1 \), we have that, for any \( \delta > 0 \), there are \( \varepsilon_\delta > 0 \) and \( \varepsilon \delta > 0 \) such that

\[
\max_{w \in E^- \oplus \mathcal{H}_{\tau\kappa_v}} \Phi_\varepsilon(w) < \gamma_{\tau\kappa_v} + \delta
\]

for all \( r \geq r_\delta \) and \( \varepsilon \leq \varepsilon_\delta \). Let \( y^j = (2j(r + 1), 0, \ldots, 0) \), define

\[
u_j(x) = u(x - y^j) = u(x_1 - 2j(r + 1), x_2, \ldots, x_n), \quad u_{rj}(x) = u_r(x - y^j),
\]

for \( j = 0, 1, \ldots, m - 1 \). Setting \( r_m = (2m - 1)(r + 1) \), it is clear that \( \text{supp} \ u_{rj} \subset B_{r_m}(0) \). Obviously \( \{u_{rj}^+\}_{j=0}^{m-1} \) are linearly independent. Indeed, if \( w^+ = \sum_{j=0}^{m-1} c_j u_{rj}^+ = 0 \), denoting \( w = \sum_{j=0}^{m-1} c_j u_{rj} \) one has \( w = w^- + w^+ \) and

\[- \|w^-\|^2 = a_r(w) = \sum_{j} c_j^2 a_r(u_{rj}^+) = a_r(u_r) \sum_{j} c_j^2,
\]

which implies \( c_j = 0, j = 0, 1, \ldots, m - 1 \). Now set

\[E_m = E^- \oplus \text{span} \{u_{rj} : j = 0, \ldots, m - 1\} = E^- \oplus \text{span} \{u_{rj}^+ : j = 0, \ldots, m - 1\}.
\]

By virtue of Lemma 2.5, let \( t_{rj} > 0 \) be such that \( t_{rj} u_{rj}^+ \in \mathcal{N} \). Observe that

\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} t_{rj} = \lim_{\varepsilon \to 0} t_{rj} = 1,
\]

(31)

\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} h_\varepsilon(t_{rj} u_{rj}^+) = \lim_{\varepsilon \to 0} h_\varepsilon(t_{rj} u_{rj}^+) = h_0(u^+) = u^-,
\]

(32)

\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} \|h_\varepsilon(t_{rj} u_{rj}^+) - t_{rj} u_{rj}^-\| = \lim_{\varepsilon \to 0} \|h_\varepsilon(t_{rj} u^+) - t_{rj} u^-\| = 0.
\]

(33)
It is not difficult to check the following

\[
\max_{w \in E_m} \Phi_\varepsilon(w) = \Phi_\varepsilon \left( \sum_{j=0}^{m-1} t_{ej} u_{r_j}^+ + h_e \left( t_{ej} u_{r_j}^- \right) \right)
\]

\[
= \Phi_\varepsilon \left( \sum_{j=0}^{m-1} t_{ej} u_{r_j}^+ + t_{ej} u_{r_j}^- \right) + o(1_r)
\]

\[
= \Phi_\varepsilon \left( \sum_{j=0}^{m-1} t_{ej} u_{r_j}^- \right) + o(1_r)
\]

\[
= \sum_{j=0}^{m-1} \Phi_\varepsilon (t_{ej} u_{r_j}) + o(1_r)
\]

\[
= \sum_{j=0}^{m-1} \Phi_\varepsilon (t_{ej} u_{r_j}^+ + t_{ej} u_{r_j}^-) + o(1_r)
\]

\[
= \sum_{j=0}^{m-1} \Phi_\varepsilon (t_{ej} u_{r_j}^+ + h_e \left( t_{ej} u_{r_j}^+ \right)) + o(1_r)
\]

\[
= \sum_{j=0}^{m-1} \Phi_0 (t_{0j} u_{r_j}^- + h_0 \left( t_{0j} u_{r_j}^- \right)) + o(1_{r_e})
\]

\[
= \sum_{j=0}^{m-1} \Phi_0 (u) + o(1_{r_e})
\]

\[
= m \gamma_{r_\varepsilon} + o(1_{r_e})
\]

where \( o(1_r) \) means arbitrary small as \( r \to \infty \), and \( o(1_{r_e}) \) means arbitrary small as \( r \) sufficiently large and \( \varepsilon \) sufficiently small.

Now, by assumptions and Lemma 3.5 for any \( 0 < \delta < \gamma_\infty - m \gamma_{r_\varepsilon} \), one may choose \( r > 0 \) large and then \( \varepsilon_m > 0 \) small such that, for all \( \varepsilon \leq \varepsilon_m \),

\[
\max_{w \in E_m} \Phi_\varepsilon(w) \leq \gamma_\infty - \delta.
\]

By Lemma 2.2 we see that the solutions are in \( \bigcap_{s \geq 2} W^{1,s} \).

Now, as (30) one can choose \( r > 0 \) and \( \varepsilon_m > 0 \) such that, if \( \varepsilon \leq \varepsilon_m \)

\[
\Phi_\varepsilon(w) < \gamma_\infty \quad \text{for all } w \in E_m.
\]

It follows from Lemma 3.10 that \( \Phi_\varepsilon \) satisfies the \((PS)_c\) condition for all \( c < \gamma_\infty \), that is, the general condition \((\Phi_3)\) is satisfied. Now by applying Theorem 2.1 one sees that either \( \Phi_\varepsilon \) has infinitely many critical points, or has at least \( m \) pairs of critical points with different critical values \( 0 < c_\varepsilon^0 < \cdots < c_\varepsilon^{m-1} \leq \sup_{w \in H_m} \Phi_\varepsilon(w) < \gamma_\infty \).

The proof is hereby complete. \( \square \)

Proof of Theorem 1.2. We are sketchy. Assume \( x_w = 0 \) and consider \( \mu = \tau_w = V(0), \nu = \kappa = W(0) \) and \( \Phi_\varepsilon = \Phi_{\tau_0 \kappa}, \Phi_0 = \Gamma_{r_\kappa} \). Let \( u \in H_{\tau_0 \kappa}, \Phi_0(u) = \gamma_{r_0 \kappa} \). As before, define \( u_\varepsilon \) and \( u_{r_j}, j = 0, \cdots, m - 1 \), and set the \( m \)-dimensional subspace \( E_m \). Then one checks that

\[
\max_{w \in E_m} \Phi_\varepsilon(w) = m \gamma_{r_\varepsilon} + o(1_{r_\varepsilon}).
\]
Now, by assumptions and Lemma 3.5 for any \( 0 < \delta < \gamma_\infty - m \gamma_{\tau_u} \), one may choose \( r > 0 \) large and then \( \varepsilon_m > 0 \) small such that, for all \( \varepsilon \leq \varepsilon_m \), \( \max_{w \in E_m} \Phi_\varepsilon(w) \leq \gamma_\infty - \delta \).

Define, for \( p \in [2, 2^*] \),
\[
\ell(w) := \frac{a(w)}{|w|_p^2}, \quad \text{where} \quad a(w) = \int_{\mathbb{R}^n} \langle H_0 w, w \rangle dx + \int_{\mathbb{R}^n} \tau \langle |w|^2 \rangle dx, \quad w \in E_m.
\]

For \( w_0 \in E_m \) with \( \ell(w_0) = \max \ell(E_m) \), by \( (P_2) \), \( (11) \) and Lemma 3.4, we have
\[
\Phi_0(w_0) = \frac{p-2}{2p_k \tau_{\infty}} \ell(w_0)^{\frac{p}{p-2}} \leq \frac{p-2}{2p_k \tau_{\infty}} (m \ell(w_0))^{\frac{p}{p-2}} + o(1) \leq \left( \frac{a + \tau_{\infty}}{a + \tau_u} \right)^{\frac{p}{p-2} - n} \left( \frac{\kappa}{\kappa_\infty} \right)^{\frac{2}{p-2}} \gamma_\infty m^{\frac{p}{p-2}} < \gamma_\infty.
\]

Now by Theorem 2.1 one gets the multiplicity conclusion. \( \square \)

5. **Proofs of main results: the critical case.** Setting \( u(x) = \varphi(x) \), the equation (8) is equivalent to the following
\[
- i \alpha \cdot \nabla u + \beta u + V_u(x) u = W_{1\varepsilon}(x) |u|^{p-2} u + W_{2\varepsilon}(x) |u|^{2^* - 2} u. \tag{34}
\]

**Proof of Theorem 1.2.** We may assume that \( 0 \in \mathcal{Y}, x_v = 0 \) and \( \tau = V(0) \kappa_{jv} = W_j(0) \). Solutions of (34) are critical points of the functional \( \Phi_u(\mu) := \Phi_{\mu}^{\bar{\kappa}_v}(\mu) \) with \( \bar{\kappa}_v = (\kappa_{1v}, \kappa_{2v}) \). Denote \( \Phi_0(\mu) = \gamma \kappa_{\infty} \). We will adopt an argument different from that of Theorem 1.1. Let \( u \in \mathcal{A}_{r, \kappa_v} \) be a solution of (22) with \( \mu = \tau \) and \( \bar{\nu} = \bar{\kappa}_v \).

Define \( u_r, u_{rj}, j = 0, \ldots, m - 1 \), and set \( E_m \) as before.

By virtue of Lemma 2.5, let \( t_{\varepsilon rj} > 0 \) be such that \( t_{\varepsilon rj} u_{rj}^+ \in \mathcal{N}_{t_r} \). Observe that
\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} t_{\varepsilon rj} = \lim_{\varepsilon \to 0} t_{\varepsilon rj} = 1,
\]
\[
\lim_{\varepsilon \to 0} h_\varepsilon (t_{\varepsilon rj} u_{rj}^+) = \lim_{\varepsilon \to 0} h_\varepsilon (t_{\varepsilon rj} u_{rj}^-) = h_0 (u^+) = u^-,
\]
\[
\lim_{\varepsilon \to 0} \lim_{r \to \infty} \| h_\varepsilon (t_{\varepsilon rj} u_{rj}^+) - t_{\varepsilon rj} u_{rj}^- \| = \lim_{\varepsilon \to 0} \| h_\varepsilon (t_{\varepsilon rj} u^+) - t_{\varepsilon rj} u^- \| = 0.
\]

One checks easily the following
\[
\max_{w \in E_m} \Phi_\varepsilon^*(w) = \Phi_\varepsilon^* \left( \sum_{j=0}^{m} t_{\varepsilon j} u_{rj}^+ + h_\varepsilon (t_{\varepsilon rj} u_{rj}^+) \right) + o(1_r)
\]
\[
\leq \Phi_\varepsilon^* \left( \sum_{j=0}^{m-1} t_{\varepsilon j} u_{rj}^+ + t_{\varepsilon j} u_{rj}^- \right) + o(1_r)
\]
\[
= \sum_{j=0}^{m-1} \Phi_\varepsilon^* (t_{\varepsilon j} u_{rj}) + o(1_r)
\]
large and then by Theorem 2.1 we obtain the multiplicity conclusion.

sufficiently large and where by Theorem 2.1 one gets the multiplicity conclusion.

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