Non-singular model universe from a perfect fluid scalar-metric cosmology

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To seek for a singularity free model universe from a perfect fluid scalar-metric cosmology, we work in the “Emergent Cosmology” (EC) paradigm which is a non-singular alternative for cosmological inflation. By using two methods including Linear Stability Theory and Effective Potential Formalism, we perform a classical analysis on the possible static solutions (that are called usually as Einstein Static Universes (ESU) in literature) in order to study EC paradigm in a FRW background. Our model contains a kinetic term of the scalar field minimally coupled to the background geometry without a potential term. The matter content of the model consists of a perfect fluid plus a cosmological constant \(\Lambda\) as a separate source. In the framework of a local dynamical system analysis, we show that in the absence or presence of \(\Lambda\), depending on some adopted values for the free parameters of the underlying cosmological model with flat and non-flat spatial geometries, one gets some static solutions which are viable under classical linear perturbations. By extending our study to a global dynamical system analysis, we show that in the presence of \(\Lambda\) with non-flat spatial geometries there is a future global de Sitter attractor in this model. Following the second method, we derive a new static solution that represents a stable ESU but this time without dependence on the free parameters of the cosmological model at hand. As a whole, our analysis suggests the possibility of graceful realization of a non-singular EC paradigm (i.e. leaving the initial static phase and entering the inflation period as the universe is evolving) through either preserving or violation of the strong energy condition.

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I. INTRODUCTION

Cosmic inflation as a paradigm for the early universe very soon has become a dominant paradigm among cosmologists, specially by successfully addressing some important unresolved problems of the standard Big-Bang cosmology (such as the flatness, horizon and initial seeds of perturbations for large scale structure formation)\(^{[1-4]}\). The greatest achievement of this scenario, which so far has been approved by observational data released with a very remarkable accuracy\(^ {[3]}\), is that it could provides a mechanism for producing a nearly scale invariant spectrum of the primordial density fluctuations. In fact, through this mechanism it was able for the first time to explain the origin of inhomogeneities in the universe within the framework of the causal physics. However, a question then arises is that why despite all the predictions confirmed by the current observations, still we can not claim that the cosmological inflation has really happened in the early stage of the cosmic history. Note that other scenarios which, as inflation, yield a nearly scale invariant spectrum of the primordial fluctuations are capable to make predictions consistent with current observational data of the large scale structure as well as the CMB anisotropies. Therefore, only matching with observational data cannot be a strict criterion for the definite acceptance of inflation as a phenomenon occurred in the early universe. Despite all the amazing phenomenological achievements gained so far in the context of the inflationary cosmology, it suffers from some non negligible conceptual issues (for more detailed studies in these issues there are some seminal review papers such as\(^ {[6, 7]}\)). This was a motivation for making a number of alternative (or pre-inflation) theories such as “Pre-Big-Bang scenario”\(^ {[8]}\), “Matter Bounce cosmology”\(^ {[9, 10]}\), “Ekpyrotic Scenario”\(^ {[11]}\), “Emergent Cosmology”\(^ {[12]}\) and some others\(^ 1\). In the present paper, by concerning

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\(^1\)We emphasize that all these alternatives to the cosmological inflation must be able to produce a nearly scale invariant spectrum of the primordial cosmological fluctuations in one side, and also explain the large scale structure formation of the current universe from the other side.
on Big-Bang singularity issue, we centralize our attention on the framework of “Emergent Cosmology” (EC) paradigm.

EC is one of the favorite alternatives in recent years in which, by running the time evolution of the universe from $-\infty$ to $+\infty$ and also by circumventing the Penrose and Hawking singularity theorems [13], one manages to provide a non-singular cosmological framework in contrast to inflationary cosmology models. According to the alternative scenarios, by following the evolution of spacetime (assuming homogeneity and isotropy) towards the past, the growth of the Hubble parameter $H$ stops in a certain scale and after that $H \to 0$. In another words, as the universe moves to the past infinity, the scale factor $a$ also gets closer to a constant value, $a \to a_\infty$. To be more technical, due to infinity of radius appointed from $H$ (that is, $R_H \sim H^{-1}$) in this quasi-solid initial state (the so called Einstein static universe (ESU)), the physical wavelength of fluctuations stays almost constant. So, at the end by falling the $R_H$ into a microscopic value, conditions provide the situation for passing the emergent phase to the standard cosmology framework. It is important to note that Ellis et al. suggested this paradigm within the standard general relativity (GR) by concentrating on the positive space curvature condition for the universe. It is noteworthy that this requirement also is supported observationally since observational data as analyzed by WMAP7 [14] address the possibility of a closed universe with 68 percent confidence level. However, this scenario was defeated at the beginning of its birth. Since it has been shown that ESU elicited from the basic gravitational theory (i.e. GR), is not able to survive stable against the prevailing perturbations in very early moments of the universe and therefore is expected to decay rapidly [12]. Therefore, this paradigm was developed into the modified gravity based models (including dynamic equations supported by some correction terms) with the hope to improve the stability conditions of the ESU at the high energy regime. For instance in references such as [16]-[38] that are working within some specific gravity frameworks, the authors have followed this issue. It is important to note that the EC paradigm (as a non-singular alternative scenario which is realizable successfully within the framework of any modified gravity), should satisfy the following two conditions:

- The issue of the existence as well as the stability of ESU against initial perturbations.
- Joining to the standard cosmological history by finding a successful graceful exit mechanism.

In this regard, some of the above mentioned references just have focused on the first condition without addressing to the latter one. In this regard, in some of the modified gravity based cosmological models reported these years there is still no possibility for deriving a non-singular model universe even from the EC point of view.

Due to important role played by scalar fields (from both phenomenological and theoretical viewpoints) in cosmological models, we study the above mentioned EC paradigm within a FRW model with a matter part consisting a perfect fluid plus a cosmological constant $\Lambda$ (as a separate source) along with a dimensionless scalar field that is minimally coupled to the background geometry. To be more specific, in our toy model the dimensionless scalar field is coupled to the Lorentz invariant measure $\sqrt{-gd^4x}$ constructed from the metric in order to extract a scalar-metric model universe which results in a non-singular cosmology. We note that the perfect fluid of our toy model is treated in the Schutz’s formalism [40, 41]. In recent years this formalism has been used frequently within different contexts, see [42-47] for instance. The Schutz’s formalism provides a context in which by using some canonical methods [48], one gets a super Hamiltonian in order to derive classical equations of dynamics in the underlying model universe. The rest of the present paper is devoted to a dynamical analysis of ESU(s) from two viewpoints: the linear stability theory and effective potential formalism, in the absence or presence of cosmological constant $\Lambda$. In other words, in the relevant sections we will try to find a well realization of the EC paradigm within a perfect fluid scalar-metric model universe to get a non-singular cosmological model. As we will show, both in the presence and absence of the cosmological constant (and yet with a classical viewpoint of spacetime), there is the chance to get rid of the initial Big Bang singularity in a fascinating manner.

II. THE EQUATIONS OF MOTION

The model universe that we are going to study the issue of EC is a FRW cosmological model that includes a matter sector with perfect fluid and a cosmological constant $\Lambda$. In this setup a scalar field minimally interacts with gravity with coupling to the background metric. To start, by setting the units as $c = \hbar = 16\pi G = 1$, we introduce the

\[ H^2 = \frac{8\pi G}{3} \rho - \Lambda, \quad R^2 = \frac{3}{2} \frac{8\pi G}{3} \rho - \Lambda, \quad a^2 = \frac{3}{2} \frac{8\pi G}{3} \rho - \Lambda, \quad R_H = \frac{3}{2} \frac{8\pi G}{3} \rho - \Lambda, \quad \rho = \frac{8\pi G}{3} \rho - \Lambda. \]
following total action \(^3\)
\[
S = \int_M d^4x \sqrt{-g} \left[ R - g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right] + 2 \int_{\partial M} d^3x \sqrt{|h|} h_{ab} K^{ab} + \int_M d^4x \sqrt{-g} (p + \Lambda) ,
\]
for the model universe. In this action \(R, K^{ab}\) and \(h_{ab}\) represent the Ricci scalar, the extrinsic curvature and induced metric on the three-dimensional spatial hypersurfaces, respectively. This action corresponds to a simple form of the scalar-tensor theory introduced by Saez and Ballester (SB) \(^4\) in which by minimal coupling of the metric with a dimensionless scalar field in a simple manner, provides a scalar-metric framework of gravitation. Concerning phenomenological worthy of the SB scalar field theory in classical level, it solves the problem of missing matter in cosmologies with non-flat spatial geometry and also suggests some quite reliable descriptions related to the weak fields. We note that all matter content of the theory is not addressed by the third part in the above action (which contains the pressure of the fluid \(p\) plus a cosmological constant \(\Lambda\)), but the scalar field component of SB theory also contributes. Here, unlike the standard inflationary model, the role of the scalar field in matter budget of the model universe arises just from the standard kinetic term so that the self-interaction potential term has no contribution. The second term in the above action may seems a bit vague. The origin of this boundary term back to this point that Ricci scalar in the gravitational section of the action \(^{11}\) contains second derivatives of the metric tensor, which is an unconventional trait of field theories. Of course, with a not so complicated calculation, one can show that this boundary term will be removed through variation of \(\int_M d^4x \sqrt{-g} R\). We are interested in taking the standard equation of state (EoS) \(p = \omega \rho\) as a perfect fluid with energy density \(\rho\). To be more clarified, we give a little more explanation about the matter term including the perfect fluid in action \(^{11}\). One, by referring to \(^{13}\), will meet a action as \(S_{P,F} = - \int d^4x \sqrt{-g} (1 + \nu)\) (here \(\zeta\) and \(\nu\) are called the density of fluid’s and internal energy, respectively) relevant to the perfect fluid which is known as Hawking-Ellis action. There it is shown that by definition of \(\rho = \zeta (1 + \nu)\) and \(p = \zeta^2 \frac{d\nu}{d\zeta}\), for the fluid’s energy density and pressure, via variation of the action \(S_{P,F}\) in terms of metric, one gets the standard energy-momentum tensor. Astonishingly, it has been stated that this result, just with the same method, can be repeated through taking the perfect fluid within the Lagrangian, see \(^{11} 11\) for more details. Generally, in the FRW models based Schutz’s formalism, there exists no torsion. Namely, the fluid’s four-velocity can be expressed as \(^{10} 11\)
\[
U_\nu = \frac{1}{h} (\epsilon_\nu + \theta S_\nu) ,
\]
in which thermodynamical potentials \(h\) and \(S\) denote the specific enthalpy and the specific entropy, respectively. It is also noteworthy that in Schutz’s formalism there is no explicit physical interpretation for the variables \(\epsilon\) and \(\theta\). The above mentioned fluid’s four velocity satisfies the normalization condition \(U_\mu U^\mu = -1\). By taking the isotropic and homogeneous assumptions in the model universe at hand, the geometry of spacetime can be characterized by the FRW metric as
\[
ds^2 = N^2(t) dt^2 - a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \right) ,
\]
where \(N\) and \(a(t)\) show the lapse function and scale factor respectively. Also, for the curvature constant \(k\) there are three normalized values \(+1, 0, 1\) by concerning a closed, flat and open universe respectively. The fluid sector of the action \(^{11}\) can be reexpress via some thermodynamic equations as follows
\[
\rho = \rho_0 (1 + \Pi) ,
\]
\[
h = 1 + \Pi + \frac{P}{\rho_0} ,
\]
\[
\tau dS = d\Pi + Pd \left( \frac{1}{\rho_0} \right) .
\]
The quantities \(\tau, \rho, \rho_0\) and \(\Pi\) in the above equations are temperature, total mass energy density, rest mass density and specific internal energy energy respectively. Using these thermodynamic relations, it is easy to demonstrate that
\[
P = \frac{\omega}{(1 + \omega)^{\frac{1}{1 - \omega}}} h^{-\frac{2}{1 - \omega}} e^{-\frac{\phi}{h}} .
\]
\(^3\) Note that although the first part of this action may looks like as the action for an inflationary toy model, this is not actually the case since there is no self-interaction potential term, \(V(\phi)\), that is necessary in standard inflation paradigm.
\(^4\) We note that there are some different extensions for SB scalar theory. Its simplest extension includes a non-standard kinetic term via a generic function \(F(\phi)\phi_{,\mu} \phi_{,\nu}\) without a scalar potential term \(^{50}\). However, in this extension it can be seen easily that by a re-parametrization of the scalar field as \(\phi \rightarrow \phi\), one can always make \(F(\phi) = 1\) which means that the generic function is just a gauge quantity. There are other generalizations of the SB scalar field theory as a toy model of cosmological inflation in which a potential term is included in the action, see for instance \(^{51} 52\).
In a comoving system including a perfect fluid four velocity $U_{\mu} = (N, 0, 0, 0)$ supported by a cosmological constant $\Lambda$, we get the following relation

$$
\mathcal{L}_\Lambda = a^3N^{-\frac{1}{2}} \frac{\omega(\dot{\epsilon} + \theta \dot{S})^{\frac{\omega+1}{\omega}}}{(1 + \omega)^{\frac{\omega+1}{\omega}}} e^{-\frac{\dot{\theta}}{\omega}} + Na^3\Lambda ,
$$

(6)

for the Lagrangian which $h > 0$ and $(\dot{\epsilon} + \theta \dot{S}) > 0$. Then, the Hamiltonian of the system is obtained as follows

$$
H_{M-\Lambda} = \dot{\epsilon} P_{\epsilon} + \dot{\theta} P_{\theta} - L_{M-\Lambda} = N \frac{P_T}{a^{3\omega}} - Na^3\Lambda .
$$

(7)

This Hamiltonian is a linear function of $P_T$. Also, conjugate momentums, depending on parameters $\epsilon$ and $S$, are defined as $P_\epsilon = \frac{\partial}{\partial \epsilon} \mathcal{L}_\Lambda$ and $P_S = \frac{\partial}{\partial S} \mathcal{L}_\Lambda$ respectively. One also can obtain the following point-like expression for the Lagrangian of the gravitational part of the model at hand

$$
\mathcal{L}_G = \frac{6a^2\dot{a}}{N} + 6kNa - \frac{1}{N} F(\phi) \dot{\phi}^2 a^3 .
$$

(8)

Now, by having the Lagrangian relevant to the matter and gravitational sectors of the theory, we are able to construct the Hamiltonian for our model in terms of the conjugate momenta by using the following relation

$$
H = \dot{a} P_a + \dot{\phi} P_\phi + \dot{\epsilon} P_\epsilon + \dot{\theta} P_\theta - L ,
$$

(9)

so that $H = H_G + H_{M-\Lambda}$, $L = \mathcal{L}_G + \mathcal{L}_{M-\Lambda}$ and also $P_q = \frac{\partial \mathcal{L}}{\partial q^q}$ where $q = a, \phi$. Therefore, the super Hamiltonian relevant to the minisuperspace of the model universe at hand takes the following form

$$
H = H_G + H_{M-\Lambda} = N \left( -\frac{1}{24} \frac{P_a^2}{a} + \frac{1}{4a^3} P_\phi^2 - 6ka + \frac{P_T}{a^{3\omega}} - \Lambda a^3 \right) .
$$

(10)

The last two terms including $P_T$ and $\Lambda$ in this Hamiltonian clearly address the matter part of the model. In Eq. (10), $N$ can be labeled as the Lagrange multiplier holding the constraint equation $H = 0$. By regarding $\hat{T} = \{ T, H \} = \frac{N}{2a^{3\omega}}$, in the case of selecting a gauge as

$$
N = a^{3\omega} ,
$$

(11)

the variable $T$ in Eq. (10) takes the role of a time coordinate (i.e. $T = t$) in the cosmological model at hand. At this point, for a classical description of the dynamics in this model universe, we present the following Hamiltonian equations

$$
\begin{cases}
\dot{a} = -\frac{a^{3\omega-1}}{12} P_a , \\
\dot{P}_a = -\frac{a^{3\omega-2}}{24} P_a^2 + \frac{3a^{3+3\omega}}{24F(\phi)a^{3\omega}} P_\phi^2 + 6ka^{3\omega} + 3\omega a^{-1} P_0 + 2\Lambda a^{3\omega+2} , \\
\dot{\phi} = \frac{a^{3\omega-3}}{2F(\phi)} P_\phi , \\
\dot{P}_\phi = 0 , \quad \Rightarrow P_\phi = \text{Constant} , \\
\dot{P}_T = 0 , \quad \Rightarrow P_T = \text{Constant} ,
\end{cases}
$$

(12)

which are actually the same as the classical equations of motion. By taking time derivit of the third equation in the above system, we get

$$
\frac{\ddot{\phi}}{\phi} + 3(1 - \omega) \left( \frac{\dot{a}}{a} \right) = 0 ,
$$

(13)

5 To achieve this form of the Hamiltonian, which is a function of $P_T$, we have applied the following canonical transformations

$$
T = -PS e^{-\phi} P_T^{-\omega+1} \quad \text{and} \quad P_T = P_T e^{\phi} P_T^{-\omega+1} e^\phi .
$$

As an advantage, these canonical transformations let us to follow a dynamical system with more parameters [14, 47].
for the field \( \phi \). Integrating the second order equation we arrive at the following expression

\[
\dot{\phi}^2 = Ca^{6\omega - 6},
\]  

(14)

with a numerical constant as \( C > 0 \). At the end, in the case of canceling the momenta from the system (12), the differential form of the Hamiltonian constraint \( \mathcal{H} = 0 \), becomes

\[
-6a^{1-3\omega} \dot{a}^2 + \dot{\phi}^2 a^{3-3\omega} - 6ka^{1+3\omega} + P_T - \Lambda^{3\omega + 3} = 0,
\]

(15)

where putting (14) into this relation the first equation of motion acquires the following form

\[
(\frac{\dot{a}}{a})^2 = \frac{C}{6}a^{6\omega - 6} - \frac{P_T}{6}a^{6\omega - 2} + \frac{\Lambda}{6}a^{6\omega}.
\]

(16)

By taking differential of equation (16) with respect to time, the second equation of motion takes the following form

\[
\ddot{a} = \frac{C}{6}a^{6\omega - 6} - \frac{3k\omega a^{6\omega - 2}}{12} + \frac{P_T}{12}(3\omega - 1)a^{6\omega} - \frac{2\Lambda}{3}a^{6\omega}.
\]

(17)

To obtain above two equations of motion we have used the continuity equation \( \dot{\rho} + \frac{4}{3} \dot{a}(1 + \omega) = 0 \) as well as the result of its integration i.e. \( \rho = \rho_0 a^{-3(1+\omega)} \) (we have set \( \rho_0 = 1 \)) due to perfect fluid energy density. As the final words in this section, there are two points for more attention: firstly, the terms including numerical constant \( C \) in the rhs of the Friedmann and Raychaudhuri equations, address the matter contribution of the SB scalar field. Secondly, these generalized dynamical equations are not the same things as we know from the standard cosmology. More precisely, the absence of a scalar potential term \( V(\phi) \) in the action (1) distinguishes equations (16) and (17) from their standard counterparts which are derived from original (standard) inflationary action including a scalar field minimally coupled to Einstein gravity with self-interaction potential term, see Refs. [12, 15].

III. FIRST ORDER DYNAMIC SYSTEM ANALYSIS OF ESU WITHOUT COSMOLOGICAL CONSTANT, \( \Lambda = 0 \)

Now we perform a first order dynamical system analysis on ESU(s) emerging from the cosmological model at hand. We consider the case with \( \Lambda = 0 \). It is expected that at the end of the analysis there will be a clear answer to the question “is it possible a well realization of the EC paradigm to circumvent the Big-Bang singularity within the mentioned toy model universe? If yes, then under what conditions?”. In the first step, using the background equations (16) and (17) we extract the relevant static solutions (ESU(s)) and check the validity of physical conditions for each of these solutions. As the next step, we investigate the stability of derived static solutions by employing a first order dynamical system analysis. At the final step, we discuss on the issue of “natural transition from ESU to the standard cosmology evolution”.

From equations (16) and (17), one can show that the static solution (with \( a = a_s \) and \( \dot{a} = 0 = \ddot{a} \)) fulfills the following equations for the case with \( \Lambda = 0 \)

\[
\frac{C}{6}a_s^{6\omega - 6} - ka_s^{6\omega - 2} + \frac{P_T}{6}a_s^{6\omega} \rho_s = 0,
\]

(18)

and

\[
\frac{C}{6}a_s^{6\omega - 6} - 3k\omega a_s^{6\omega - 2} + \frac{P_T}{12}(3\omega - 1)a_s^{6\omega} \rho_s = 0,
\]

(19)

respectively. We begin our dynamical system analysis for the case of flat model, \( k = 0 \). Solving equation (18) we arrive at the following relation

\[
\rho_s = \frac{C}{P_T a_s^6},
\]

(20)

between \( a_s \) and \( \rho_s \). Here, to save the positivity of the energy density (\( \rho_s > 0 \)), we have to choose only the negative values for \( P_T \). There is no fixed value for the critical point \( a_s \) and it can take any positive value. Having the relation
In this situation, we have

\[ x_1 = a, \quad x_2 = \dot{a}. \]  

(21)

In this situation, we have

\[ \dot{x}_1 = x_2 = Y_1(x_1, x_2) \quad \text{and} \quad \dot{x}_2 = \frac{6}{(3\omega - 2)}x_1^{6\omega - 5} - 3k\omega x_1^{6\omega - 1} + \frac{P_T}{12}(3\omega - 1)x_1^{6\omega + 1} \rho = Y_2(x_1, x_2). \]  

(22)

To treat the stability of ESU(s), one just needs to determine eigenvalues \( \lambda^2 \) of the following Jacobian matrix

\[ J(Y_1(x_1, x_2), Y_2(x_1, x_2)) = \left( \begin{array}{cc} \frac{\partial Y_1}{\partial x_1} & \frac{\partial Y_1}{\partial x_2} \\ \frac{\partial Y_2}{\partial x_1} & \frac{\partial Y_2}{\partial x_2} \end{array} \right) \implies \lambda^2 = \frac{\partial Y_2}{\partial x_1}(a_s, \rho_s). \]  

(23)

where only in the case with \( \lambda^2 < 0 \) one can assign a stable ESU to the relevant physical critical point\(^7\). In a spatially flat universe case, \( k = 0 \), we have

\[ \lambda^2 = \frac{C}{6}(3\omega - 2)(6\omega - 5)a_s^{6\omega - 6} + \frac{P_T}{12}(3\omega - 1)(6\omega + 1)a_s^{6\omega}\rho_s, \]  

(24)

where after application of the relation \( \rho = \frac{6}{1 + 3\omega} \) as well as the condition \( \omega = 1 \) into \( \rho_0 \), we find \( \lambda^2 < 0 \). This means that if the ESU is perturbed from its original condition by a small perturbation, then the universe experiences an infinite series of oscillations around the the initial state, since by fixing \( \omega = 1 \) in \( \lambda \), we have

\[ \ddot{a} - \left( \frac{C + P_T}{6} \right)a = 0. \]  

(25)

Since \( P_T < 0 \), then if \( |P_T| > C \), this equation results in an oscillatory solution as \( e^{\sqrt{\frac{|C + P_T|}{6}}t} \). Although this result seems to be interesting, it suffers from lack of transition to the inflation epoch. As has been mentioned previously, the ESU only for the EoS parameter \( \omega = 1 \) is physically meaningful. Therefore, one expects it never get out of the stable situation to entree into the inflation epoch as the universe evolves. In another words, despite the realization of a stable ESU in this model with a spatially flat geometry, the initial singularity yet remains.

We continue our analysis by focusing on the contribution of the non-flat spatial geometries \((k \neq 0)\). From the background equations \( 18 \) and \( 19 \) in the presence of the curvature constant we obtain the following critical points

\[ a_s^4 = \frac{C}{2k} \left( \frac{1 - \omega}{1 + 3\omega} \right) \quad \text{and} \quad \rho_s = \frac{1}{P_T} \left( \frac{6k}{a_s^2} - \frac{C}{a_s^2} \right). \]  

(26)

With a straightforward calculation one can find that in particular for the case with \( k = +1 \), both of the above critical points are physically meaningful (i.e. \( a_s > 0 \) and \( \rho_s > 0 \)) provided that

\[ P_T > 0, \quad \frac{1}{3} < \omega < 1, \quad \frac{1}{3} \leq \omega < 1. \]  

(27)

Now, the relevant eigenvalues \( \lambda^2 \) can be written as

\[ \lambda^2 = \frac{C}{6}(3\omega - 2)(6\omega - 5) \left( \frac{C(1 - \omega)}{2 + 6\omega} \right)^{\frac{1}{2}} - 3\omega(6\omega - 1) \left( \frac{C(1 - \omega)}{2 + 6\omega} \right)^{\frac{3}{2}} + \frac{C^{\frac{1}{2}}}{12}(3\omega - 1)(6\omega + 1) \left( 6 \left( \frac{1 - \omega}{2 + 6\omega} \right)^{\frac{1}{2}} - \left( \frac{1 - \omega}{2 + 6\omega} \right)^{\frac{3}{2}} \right) \]  

(28)

\(^6\) EoS as \( \rho = p \) points out an exotic component known as stiff fluid which for the first time was studied by Zeldovich.\(^53\). There are a large number of studies on cosmological models with stiff fluid. For instance, in some inhomogeneous cosmological models with EoS parameter \( \omega = 1 \) some exact solutions without singularity have been extracted.\(^54\)\(^55\). In Ref.\(^58\) some other important cosmological implications of the stiff fluid are presented.

\(^7\) Physically, \( \lambda^2 < 0 \) means that the critical point \((a_s, 0)\) is a stable concentric center so that if one applies a small perturbation, that point does not collapse. However, \( \lambda^2 > 0 \) can address an unstable saddle point. Note however that \( \lambda^2 > 0 \) does not necessarily means that it is an unstable saddle point always. For instance if in Jacobian matrix \( 29 \), \( \frac{\partial Y_1}{\partial x_1} \neq 0 \) and \( \frac{\partial Y_2}{\partial x_2} = 0 = \frac{\partial Y_2}{\partial x_2} \), then \( \lambda^2 > 0 \) addresses a stable node.
FIG. 1: $\lambda^2$ versus $\omega$ for the case with $k = +1$ and by satisfying the seconded constraint in (27). We have set the numerical value as $C = 1$ and $C = 25$ for the left and right panels respectively. Note that the general behavior of $\lambda^2 - \omega$ in the case of fixing the numerical values $0 < C < 25$ is similar to the left panel while if $C \geq 25$ it is similar to the right panel.

By regarding the first constraint mentioned in (27), one infers that here there is no possibility of extracting stable ESU since for any desired value $C > 0$, we have $\lambda^2 > 0$. However, the second constraint in (27) addresses a different situation, see Fig. 1. We see that by choosing some numerical values for $C$, ESU may experience both stable and unstable phases with time evolution of the universe. More precisely, for any values limited to the interval $0 < C < 25$, there are the possibility of having both positive and negative values for $\lambda^2$ in terms of the allowed values of $\omega$. However, for $C \geq 25$, the eigenvalue is utterly negative. As an example, if we fix the value of $C$ as $C = 1$ (the left panel of Fig. 1) in the toy model at hand, then ESU is stable if $0.4 < \omega < 1$. While with growth of the cosmic time and subsequently reducing the EoS parameter to $\omega = 1/3$, the state changes towards instability. This changing phase of the critical point (from stability to instability) with cosmic time can be thought as a graceful transition of the ESU to the standard cosmological evolution as is expected and required. On the other hand, if we fix the value of $C$ as $C = 25$ (the right panel of Fig. 1) and even larger values, the universe remains in a stable static phase forever without beginning the standard thermal history. This situation clearly is shown in Fig. 2 for three cases $\omega = 1/3$, $2/3$, $4/5$. Now, if the scale factor $a_s$ be affected by a slight perturbation, there will be no decay and the universe oscillates around an equilibrium point. Note that here the Big Bang singularity problem still endures.

From equation (26), for case with negative spatial curvature $k = -1$, the critical points $\rho_s$ and $a_s$ are physical only in case with

$$P_T < 0, \quad \omega < -\frac{1}{3}. \quad (29)$$

As a result, the eigenvalues $\lambda^2$ take the following form

$$\lambda^2 = \frac{C}{6} (3\omega - 2)(6\omega - 5) \left( \frac{C(\omega - 1)}{2 + 6\omega} \right) \frac{\omega - \frac{1}{2}}{2} + 3\omega(6\omega - 1) \left( \frac{C(\omega - 1)}{2 + 6\omega} \right) \frac{\omega - \frac{1}{2}}{2} - \frac{C \frac{\omega}{2} - \frac{3}{2}}{12} (3\omega - 1)(6\omega + 1) \left\{ 6 \left( \frac{\omega - 1}{2 + 6\omega} \right) \frac{\omega - \frac{1}{2}}{2} + \left( \frac{\omega - 1}{2 + 6\omega} \right) \frac{\omega - \frac{1}{2}}{2} \right\}. \quad (30)$$

One can simply show that for any values of $C > 0$ and $P_T < 0$, by fixing the values of the EoS parameter as $\omega < -\frac{1}{3}$, the eigenvalue $\lambda^2$ is positive. Therefore, from the cosmological model at hand with $k < 0$, in the absence of cosmological constant $\Lambda$, a sustainable ESU cannot be extracted. In Table 1 we have listed all the possibilities in this cosmological setup with $\Lambda = 0$ which results in a steady ESU.

IV. FIRST ORDER DYNAMICAL SYSTEM ANALYSIS OF THE ESU WITH A COSMOLOGICAL CONSTANT

Regarding the contribution of the cosmological constant $\Lambda$ in the matter part of the action (1), by the same way as the previous section here we focus further on the dynamics of ESUs. Static solutions in the presence of a cosmological
FIG. 2: The scale factor, $a$, versus the cosmic time $t$ for the case with $k = +1$ corresponding to three EoS parameter $\omega = 1/3$ (blue curve), $\omega = 2/3$ (red curve) and $\omega = 4/5$ (green curve) with numerical values $C = 25$ and $P_T = -1$.

TABLE I: Summary of the stable physical critical points (ESU) for $\Lambda = 0$. The case with $k = +1$ is the case that leads to fulfilling the conditions of the EC in order to eliminate the Big-Bang singularity.

| $k$  | Physical Critical Points | Stability Conditions |
|------|--------------------------|----------------------|
| $k = 0$ | $\rho_s = -\frac{C}{P_Ta_s^6}$, $a_s$ can be any positive desired value | $P_T < 0, \, \omega = 1$ |
| $k = +1$ | $a_s^6 = \frac{C}{2} \left( \frac{1}{1 + 3\omega} \right)$ and $\rho_s = \frac{P_T}{a_s^6} \left( \frac{1}{2} - \frac{C}{a_s^6} \right)$ | $P_T < 0, \, \frac{1}{3} \leq \omega < 1$ |

constant, $\Lambda$, should satisfy the following equations

$$\frac{C}{6} a_s^{6\omega - 6} - ka_s^{6\omega - 2} + \frac{P_T}{6} a_s^{6\omega} \rho_s - \frac{\Lambda}{6} a_s^{6\omega} = 0 \; ,$$

(31)

and

$$\frac{C}{6} a_s^{6\omega - 6} - 3k\omega a_s^{6\omega - 2} + \frac{P_T}{12} (3\omega - 1) a_s^{6\omega} \rho_s - \frac{2\Lambda}{3} a_s^{6\omega} = 0 \; .$$

(32)

As before, let us begin our analysis from the case of $k = 0$ which leads to the critical points as

$$a_s^{-6} = \frac{\Lambda}{C} \frac{9 - 3\omega}{1 - 3\omega} \quad \text{and} \quad \rho_s = -\frac{\Lambda}{P_T} \left( \frac{8}{1 - 3\omega} \right) .$$

(33)

These critical points have physical meaning if

$$\Lambda > 0, \quad P_T < 0, \quad \omega < \frac{1}{3} , \quad$$

$$\Lambda < 0, \quad P_T < 0, \quad \frac{1}{3} \leq \omega < 3 , \quad$$

$$\Lambda > 0, \quad P_T > 0, \quad \omega > 3 .$$

(34)

Adding the contribution of $\Lambda$ to the first order differential equation (23), the eigenvalue $\lambda^2$ generally can be read as

$$\lambda^2 = \frac{C}{6} (3\omega - 2)(6\omega - 5)x_1^{6\omega - 6} - 3k\omega(6\omega - 1)x_1^{6\omega - 2} + \frac{P_T}{12} (3\omega - 1)(6\omega + 1)x_1^{6\omega} \rho - \frac{2\Lambda}{3} (6\omega + 1)x_1^{6\omega} .$$

(35)

Putting the critical points (33) into (35), for the case of $k = 0$, the final form of the eigenvalue $\lambda^2$ is obtained as

$$\lambda^2 = \frac{C}{6} (3\omega - 2)(6\omega - 5) \left( \frac{\Lambda(9 - 3\omega)}{C(1 - 3\omega)} \right)^{1-\omega} .$$

(36)

Note that with three constraints listed in (34), the contribution of $\Lambda$ can not lead to the derivation of a sustained static solution (ESU) since $\lambda^2 > 0$. 
In what follows, we consider non-flat cases with \( k \neq 0 \). The critical point corresponding to energy density \( \rho_s \) reads as

\[
\rho_s = \frac{1}{P_T} (\Lambda + \frac{6k}{a_s^2} - \frac{C}{a_s^2}) .
\]  

(37)

However, in deriving \( a_s^{-2} \) we dealing with a cubic equation as follows

\[
\frac{C}{4}(1 - \omega)z^3 - \frac{3k}{2}(\omega + \frac{1}{3})z + \frac{\Lambda}{4}(\omega - 3) = 0 ,
\]

(38)
in which \( z \equiv a_s^{-2} \). Generally, the solutions of the above equation can be given via the following expression

\[
\Delta = -4\alpha \beta^3 - 27\alpha^2 \gamma^2 ,
\]

(39)
where \( \alpha = \frac{C}{4}(1 - \omega) \), \( \beta = -\frac{3k}{2}(\omega + \frac{1}{3}) \) and \( \gamma = \frac{3}{4}(\omega - 3) \). In particular, by setting \( k = \pm 1 \), the above expression gives \( \Delta < 0 \), if

\[
\begin{align*}
\Lambda > 0 \text{ or } \Lambda < 0, & \quad \omega < -1, \quad -1 \leq \omega < -\frac{1}{3}, \quad \omega > 1, \\
\Lambda > \sqrt{-\frac{32 - 288\omega - 864(\omega^2 + \omega^3)}{-243C + 405C\omega - 189C\omega^2 + 27C\omega^3}} & \text{ or } \Lambda < -\sqrt{-\frac{32 - 288\omega + 864(\omega^2 + \omega^3)}{-243C + 405C\omega - 189C\omega^2 + 27C\omega^3}}, \quad -\frac{1}{3} < \omega < 1 ;
\end{align*}
\]

(40)
and

\[
\begin{align*}
\Lambda > 0 \text{ or } \Lambda < 0, & \quad -1 < \omega < 1, \\
\Lambda > \sqrt{-\frac{32 + 288\omega + 864(\omega^2 + \omega^3)}{-243C + 405C\omega - 189C\omega^2 + 27C\omega^3}} & \text{ or } \Lambda < -\sqrt{-\frac{32 + 288\omega - 864(\omega^2 + \omega^3)}{-243C + 405C\omega - 189C\omega^2 + 27C\omega^3}}, \quad \omega < -\frac{1}{3}, \quad 1 < \omega < 3 ,
\end{align*}
\]

(41)
corresponding to the closed and open spatial geometries, respectively. The negative sign of \( \Delta \) means that the cubic equation \( (38) \) has only a real solution (two other solutions are imaginary and unacceptable) which is given as

\[
\frac{1}{a_s^2} = \frac{1}{3\alpha} (D + \frac{\Delta \pm}{D}) ,
\]

(42)
so that

\[
\Delta_{0\pm} = \pm \frac{9C}{8}(1 - \omega)(\omega + \frac{1}{3}), \quad \Delta_1 = \frac{27}{64}C^2\Lambda(\omega - 3)(1 - \omega)^2, \quad \text{and} \quad D = \left( \frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_{0\pm}^3}}{2} \right) .
\]

(43)
Notice that the sign \pm in these relations corresponds to the positive and negative spatial curvature respectively. Now let us to see whether these critical points satisfy the physical conditions \( \frac{1}{a_s^2} > 0 \) and \( \rho_s > 0 \) or not. We then infer that the above mentioned critical points represent a factual ESU with respect to the following conditions

\[
\begin{align*}
\Lambda > 0, \quad P_T > 0, & \quad \omega < -1, \quad -1 \leq \omega < -\frac{1}{3}, \\
\Lambda < 0, \quad P_T < 0, & \quad \omega > 1 ,
\end{align*}
\]

(44)
and

\[
\Lambda > 0, \quad P_T < 0, \quad -\frac{1}{3} < \omega < 1 ,
\]

(45)
corresponding to the cases \( k = \pm 1 \) respectively. As is seen from these conditions, the second case listed in \( (40) \) and \( (41) \) can not result in a factual ESU. As a result inserting the above critical points into \( (35) \), we get

\[
\begin{align*}
\lambda_\pm^2 = & \quad \frac{C}{6}(3\omega - 2)(6\omega^2 - 5) \bigg\{ -\frac{1}{3\alpha} (D + \frac{\Delta \pm}{D}) \bigg\}^{3 - 3\omega} \mp 3\omega(6\omega - 1) \bigg\{ -\frac{1}{3\alpha} (D + \frac{\Delta \pm}{D}) \bigg\}^{-3 - 3\omega} - \\
& \quad \left( -\frac{1}{3\alpha} (D + \frac{\Delta \pm}{D}) \right)^{3 - 3\omega} \left\{ \frac{-2\Lambda}{3} (6\omega + 1) + \frac{(3\omega - 1)(6\omega + 1)}{12} \left( \Lambda - \frac{2}{\alpha} (D + \frac{\Delta \pm}{D}) \right) + \frac{C}{27\alpha^3} (D + \frac{\Delta \pm}{D})^3 \right\} .
\end{align*}
\]

(46)
Interestingly, we find that for \( k = 1 \), the sign of \( \lambda^2 \) in the case where the conditions listed in \( (43) \) are satisfied, depends on the values of \( C \) and \( \Lambda \). In other words, while by attributing some values for \( C \) and \( \Lambda \), we get a stable ESU with \( \lambda_+^2 < 0 \), setting some other values can lead to an unstable ESU, i.e. \( \lambda_-^2 > 0 \). Moreover, we note that under the conditions \( (43) \), no transition occurs from stability to instability (as the EoS parameter \( \omega \) reduces, the sign of
FIG. 3: The scale factor versus the cosmic time in the presence of $\Lambda$ and with the EoS parameter $\omega = 1$ corresponding to three physical conditions [48] from the blue to green curve respectively. We have set the numerical values $C = 8$, $P_T = 1$, $\Lambda = -1$ (blue curve), $C = 8$, $P_T = 1$, $\Lambda = 5$ (red curve) and $C = 8$, $P_T = -1$, $\Lambda = 3/2$ (green curve).

$\lambda^2$ does not change to positive one) to enter the standard thermal history of the universe. However, for the case of $k = -1$, if the conditions (45) are satisfied, we are dealing with an unstable ESU ($\lambda^2 > 0$), so that this result is independent of fixing what amount to the $C$ and $\Lambda$.

The relation (49) with $\omega = 1$ gives the conditions for fulfillment of $\Delta = 0$. In this case, we are dealing with the critical points as

$$\frac{1}{a_3^2} = -\frac{\Lambda}{4k} \quad \text{and} \quad \rho_s = \frac{1}{P_T} \left(-\frac{\Lambda}{2} \frac{CA^3}{64k^4}\right).$$

(47)

It is not difficult to demonstrate that the above mentioned critical points are physical provided that

$$k = 1, \quad \Lambda < 0, \quad P_T > 0,$$

$$k = -1, \quad \Lambda > \sqrt{\frac{32}{C}}, \quad P_T > 0,$$

$$k = -1, \quad 0 < \Lambda < \sqrt{\frac{32}{C}}, \quad P_T < 0.$$  \hspace{1cm} (48)

Now by having the critical points (17) and by setting $\omega = 1$ in (55) we arrive at the following expression

$$\lambda^2 = \frac{13C}{6} + \frac{784k^3}{3\Lambda^3} - 480\sqrt{\frac{k^7}{\Lambda^5}},$$

(49)

for the eigenvalue $\lambda^2$. Surprisingly, for all three physical conditions listed in (48) we face with a sustainable ESU i.e. $\lambda^2 < 0$. So, as is displayed in Fig. 3, in response to external small perturbations, ESU does not destroy since it fluctuates steadily. This means that in this situation also despite the existence of a stable static solution, there is no possibility to exit from ESU and consequently the early singularity problem stays unresolved. As the final case, provided that

$$k = 1, \quad -\frac{1}{3} < \omega < 1, \quad \Lambda > 0 \quad \text{or} \quad \Lambda < 0,$$

$$k = -1, \quad \omega < -1, \quad -1 \leq \omega < -\frac{3}{5}, \quad \Lambda > 0 \quad \text{or} \quad \Lambda < 0,$$

we have $\Delta > 0$ which addresses the existence of three separate real solutions for the cubic equation (48). These three real solutions include

$$(Z_+)_1 = A_+ \cos\left(\frac{\phi_+}{3}\right), \quad (Z_+)_2 = A_+ \cos\left(\frac{\phi_+}{3} + \frac{2\pi}{3}\right), \quad (Z_+)_3 = A_+ \cos\left(\frac{\phi_+}{3} + \frac{4\pi}{3}\right),$$

(51)

and

$$(Z_-)_1 = A_- \cos\left(\frac{\phi_-}{3}\right), \quad (Z_-)_2 = A_- \cos\left(\frac{\phi_-}{3} + \frac{2\pi}{3}\right), \quad (Z_-)_3 = A_- \cos\left(\frac{\phi_-}{3} + \frac{4\pi}{3}\right),$$

(52)

for the closed and open universes respectively where

$$A_\pm = \sqrt{\pm \frac{4}{3C} \frac{6\omega + 2}{1 - \omega}}, \quad \phi_\pm = \cos^{-1}\left(\frac{3\Lambda(3-\omega)}{C(1-\omega)}\sqrt{\pm \frac{4}{3C} \frac{6\omega + 2}{1 - \omega}}\right).$$

(53)
Among the three solutions presented in (51) and (52), only the first and third ones satisfy the physical condition \( \frac{1}{a^2} > 0 \) with the following conditions:

\[
\begin{align*}
(Z_+)_1 & > 0, \quad \text{if } \Lambda > 0, \quad \text{or } \Lambda < 0 \quad \text{and } |A| \ll 1 \\
(Z_+)_3 & > 0, \quad \text{if } \Lambda < 0 \quad \text{and } |A| \ll 1 \\
(Z_-)_1 & > 0, \quad \text{if } \Lambda > 0, \quad \text{or } \Lambda < 0 \quad \text{and } |A| \ll 1 \\
(Z_-)_3 & > 0, \quad \text{if } \Lambda > 0 \quad \text{and } |A| \ll 1. 
\end{align*}
\]  

(54)

Now by substituting the mentioned physical critical points into (55), one gets

\[
\begin{align*}
(r_{+1})_1 &= \frac{1}{PF} \left( \Lambda + 6A_+ \cos(\phi_+^3) - CA_+^3 \cos^3(\phi_+^3) \right), \\
(r_{+1})_3 &= \frac{1}{PF} \left( \Lambda + 6A_+ \cos(\phi_+^3 + \frac{4\pi}{3}) - CA_+^3 \cos^3(\phi_+^3 + \frac{4\pi}{3}) \right), \\
(r_{-1})_1 &= \frac{1}{PF} \left( \Lambda - 6A_- \cos(\phi_+^3) - CA_+^3 \cos^3(\phi_+^3) \right), \\
(r_{-1})_3 &= \frac{1}{PF} \left( \Lambda - 6A_- \cos(\phi_+^3 + \frac{4\pi}{3}) - CA_+^3 \cos^3(\phi_+^3 + \frac{4\pi}{3}) \right). 
\end{align*}
\]  

(55)

By focusing on the first two cases relevant to \( k = 1 \), we find that for \((r_{+1})_1\) and \((r_{+1})_3\), the positive energy condition holds if in addition to fulfillment of the relevant constraints (54), we set \( P_F < 0 \) and \( P_F > 0 \) in the cosmological model at hand, respectively. Eventually, the relevant expressions of the eigenvalues \( \lambda^2 \) take the following forms

\[
(\lambda^2)_1 = \frac{C}{6}(3\omega - 2)(6\omega - 5) \left( A_+ \cos(\phi_+^3) \right)^{3-3\omega} - 3\omega(6\omega - 1) \left( A_+ \cos(\phi_+^3) \right)^{\frac{1}{2}-3\omega} - \\
\left( A_+ \cos(\phi_+^3) \right)^{3-3\omega} - \left\{ -\frac{2\Lambda}{3}(6\omega + 1) + \frac{(3\omega - 1)(6\omega + 1)}{12} \left( \Lambda + 6A_+ \cos(\phi_+^3) - C(A_+ \cos(\phi_+^3)^3) \right) \right\},
\]  

(56)

and

\[
(\lambda^2)_3 = \frac{C}{6}(3\omega - 2)(6\omega - 5) \left( A_+ \cos(\phi_+^3 + \frac{4\pi}{3}) \right)^{3-3\omega} - 3\omega(6\omega - 1) \left( A_+ \cos(\phi_+^3 + \frac{4\pi}{3}) \right)^{\frac{1}{2}-3\omega} - \\
\left( A_+ \cos(\phi_+^3 + \frac{4\pi}{3}) \right)^{3-3\omega} - \left\{ -\frac{2\Lambda}{3}(6\omega + 1) + \frac{(3\omega - 1)(6\omega + 1)}{12} \left( \Lambda + 6A_+ \cos(\phi_+^3 + \frac{4\pi}{3}) \right) \right\},
\]  

(57)

\( \) respectively. Unlike the case of positive spatial curvature, here both physical critical points \((Z_-)_1\) and \((Z_-)_3\) lead
Stability conditions. The plot of $\lambda^2$ versus $\omega$ in a closed universe model ($k = +1$) corresponding to two physical static solutions $(Z_+)_1$ (left panel) and $(Z_+)_3$ (right panel) by setting the numerical values $C = 50$, $\Lambda = 0.0001$.

**TABLE II:** Summary of the stable physical critical points (ESU) in the presence of constant $\Lambda$. The case marked with * is the case that leads to fulfilling the conditions of the EC in order to eliminate the Big-Bang singularity.

| $k$ | Physical critical points | Stability conditions |
|-----|--------------------------|---------------------|
| $k = +1$ | $a_s^2 = -\frac{1}{\omega} \left( D + \frac{\Lambda}{a^2} \right)$, $\rho_s = \frac{1}{P_T} \left( \Lambda + \frac{6 \omega}{\gamma^2} - \frac{C}{\gamma^2} \right)$ | $\lambda > 0$, $P_T > 0$, $\omega < -1$, $-1 \leq \omega < -\frac{1}{4}$ $\lambda < 0$, $P_T < 0$, $\omega > 1$ |
| $k = +1$ | $a_s^2 = -\frac{\Lambda}{\omega}$, $\rho_s = \frac{1}{P_T} \left( -\frac{2}{3} - \frac{C\Lambda}{6\omega} \right)$ | $\omega = 1$, $\Lambda < 0$, $P_T > 0$ |
| $k = -1$ | $a_s^2 = \frac{\Lambda}{\omega}$, $\rho_s = \frac{1}{P_T} \left( -\frac{2}{3} - \frac{C\Lambda}{6\omega} \right)$ | $\omega = 1$, $\Lambda > \sqrt{\frac{2\rho}{C}}$, $P_T > 0$ |
| * | $(Z_+)_1$, $(\rho_{+1})_1$ | $-\frac{1}{2} < \omega < 1$, $\Lambda > 0$ or $\Lambda < 0$, $|\Lambda| < 1$, $P_T < 0$ |
| * | $(Z_+)_3$, $(\rho_{+3})_3$ | $-\frac{1}{2} < \omega < 1$, $\Lambda < 0$, $|\Lambda| < 1$, $P_T > 0$ |
| $k = -1$ | $a_s^2 = \frac{\Lambda}{\omega}$, $\rho_s = \frac{1}{P_T} \left( -\frac{2}{3} - \frac{C\Lambda}{6\omega} \right)$ | $\omega = 1$, $0 < \Lambda < \sqrt{\frac{2\rho}{C}}$, $P_T < 0$ |

In Table 2, we have listed all the possible cases leading to a stable ESU in the presence of a cosmological constant $\Lambda$.

In comparison between Tables 1 and 2 we see that adding the cosmological constant term to the matter sector of the action make more and severe requirements for stability of ESU since now we are encountering a wider parameter space of the model. In each of these two Tables only one case has the potential for well realization of EC paradigm within underlying model universe. In this case the Big-Bang singularity is completely removed. This is the main achievement of this detailed study.

**V. GLOBAL DYNAMICAL SYSTEM ANALYSIS OF THE MODEL**

So far we have performed a local dynamical system analysis of the ES solutions. We saw that in the framework of a local dynamical system analysis there is the possibility of removing the early universe singularity through a full realization of the EC scenario. However, for the EC paradigm to be viable one definitely needs a supplementary analysis which covers the whole phase space i.e. a global dynamical system analysis. Hence, to complete the study of the dynamical system analysis of the model we focus on the ES solutions at infinite phase space. It is done through the following coordinate transformations

$$(x_1, x_2) \Rightarrow (X_1, X_2) : X_1 = \frac{1}{x_1} = \frac{1}{a}, \quad X_2 = \frac{x_2}{x_1} = \frac{\dot{a}}{a}; \quad 0 < X_1 < +\infty, \quad -\infty < X_2 < +\infty$$  \hspace{1cm} (60)

in the phase plane to cover a circle at infinity. Now the background equations (18) and (19) are re-expressed as follows

$$\frac{C}{6} X_1^{6-6\omega} - k X_1^{2-6\omega} + \frac{P_T \rho_s}{6} X_1^{-6\omega} = 0 ,$$  \hspace{1cm} (61)
FIG. 5: Existence regions for dS-attractor of critical points relevant to a spatially closed (left panel) and spatially open (right panel) universes in the \((p_T - \Lambda)\) parameter space. Here we have set positive values for \(C\) and \(\rho_s\).

and

\[
C \frac{6}{6} X_{1s}^{6-6\omega} - 3k\omega X_1^{2-6\omega} + \frac{p_T \rho_s}{12} (3\omega - 1) X_{1s}^{-6\omega} = 0 ,
\]

(62)

respectively. As a result, the autonomous system transforms to

\[
\dot{X}_1 = -X_1 X_2, \quad \dot{X}_2 = \frac{C}{6}(3\omega - 1)X_1^{6-6\omega} - 3k\omega X_1^{2-6\omega} + \frac{p_T \rho_s}{12}(3\omega - 1)X_{1s}^{-6\omega} - X_2^2 .
\]

(63)

The conditions for realization of the ES solution \((a = a_s, \rho = \rho_s, \dot{a} = 0)\) require \((X_{1s}, X_{2s}) = 0\) as the critical points on a circle at infinity for the autonomous system (63). In this global analysis with the phase space with infinite domain, the existence of a stable global de Sitter (dS) attractor is essential for the EC paradigm to be successful. Therefore, in the following analysis we fix \(\omega = -1\).

In the case of \(k = 0\), by setting \(X_{1s} = \left(\frac{-p_T \rho_s}{C}\right)^{1/6}\) with \(p_T < 0\) and \(\rho_s > 0\) into the relevant Jacobian matrix, the squared eigenvalue is obtained to be a positive value which addresses an unstable global dS solution. As a result, the relevant fixed point cannot play the role of a dS-attractor.

For the cases \(k = \pm 1\), we get the following real critical points

\[
X_{1s\pm} = \left(\pm 2^{4/3} \left(\frac{C^2 p_T \rho_s (1 + \frac{1}{C p_T^2 \rho_s^2})}{2C} \right)^{-1/3} + \left(\frac{p_T \rho_s}{2C} \left(1 + \frac{1}{2C p_T^2 \rho_s^2}\right)\right)^{1/3}\right)
\]

(64)

on a circle at infinity, respectively. We found that for these two fixed points to represent global dS-attractors (i.e. \(\lambda^2 < 0\)), the integration constant \(C\) should be negative, which is in contrary to what the equation (14) imposes on the model. Therefore, in these cases existence of a global dS-attractor to have a successful realization of EC scenario is not possible. To proceed further, we continue our global stability analysis now with a new ingredient: the presence of a cosmological constant \(\Lambda\). With the same coordinate transformations, this time the autonomous system transforms to

\[
\dot{X}_1 = -X_1 X_2, \quad \dot{X}_2 = \frac{2}{3} C X_1^{12} + 3k X_1^{8} - \left(\frac{p_T}{3} \rho + \frac{2\Lambda}{3}\right) X_1^6 - X_2^2 .
\]

(65)

\(^8\) Note that for this autonomous system in a finite domain of the phase space there are also other ES solutions which previously have been taken into account.
The squared eigenvalue of the above autonomous system reads as
\[ \lambda^2 = 8CX_{1s}^{12} - 24kX_{1s}^8 + (2p_T\rho_s - 4\Lambda)X_{1s}^6. \] (66)

By setting the fixed points \( X_{1s} = \left( \frac{4\Lambda}{C} \right)^{1/6} \), \( \rho_s = \frac{2\Lambda}{p_T} \) for the case of flat spatial geometry, we find \( \lambda^2 = \frac{26\Lambda^2}{C} > 0 \). So, we obtain the real critical points
\[ X_{1s\pm} = \pm \frac{2^{4/3}}{3}(9C^2(\Lambda - p_T\rho_s))^{1/3} + \left( \frac{\Lambda - p_T\rho_s}{2C}(1 + \sqrt{1 \pm \frac{864}{C(\Lambda - p_T\rho_s)^2}}) \right)^{1/3} \] (67)
on a circle at infinity for the non-flat cases \( k = \pm 1 \), respectively. As can be seen from Fig. 5 this time, unlike the previous cases, by tuning some values of \( p_T \) and \( \Lambda \), fixed points (67) could represent global dS-attractors. Therefore, to have a successful EC scenario in a perfect fluid scalar-metric scenario, the existence of a cosmological constant \( \Lambda \) is necessary.

VI. EC SCENARIO FROM EFFECTIVE POTENTIAL APPROACH

Following our classical stability analysis, here, inspired by [20] we intend to see the ESU stability issue and subsequently a full realization of the EC scenario from another approach known as "effective potential formalism". Indeed, by viewing ESU as a classical particle which is affected by a one-dimensional potential as \( U \), we shall derive the required information by concerning on the stability status of the ESU.

For this purpose, the first Friedmann equation can be rewritten in terms of the effective potential \( U(a) \) as follows
\[ \frac{1}{2} \dot{a}^2 + U(a) = E \equiv 0, \quad \text{where} \quad U(a) = \frac{C}{12}a^{6\omega-4} + \frac{k}{2}a^{6\omega} - \frac{p_T}{12}a^{3\omega-1} + \frac{\Lambda}{12}a^{6\omega+2} \] (68)
Translating the conditions required for realization of the ESU (i.e. \( \dot{a} = 0 = \ddot{a} \)) in the context of effective potential formalism leads us to the following conditions
\[ U(a_s) = 0 \quad \text{and} \quad U'(a_s) = 0 \] (69)
which can be read as follows
\[ U(a_s) = -\frac{C}{12}a_s^{6\omega-4} + \frac{k}{2}a_s^{6\omega} - \frac{p_T}{12}a_s^{3\omega-1} + \frac{\Lambda}{12}a_s^{6\omega+2} = 0, \] (70)
and
\[ U'(a_s) = -\frac{C}{6}(3\omega - 2)a_s^{6\omega-5} + 3k\omega a_s^{6\omega-1} - \frac{p_T}{12}(3\omega - 1)a_s^{3\omega-2} + \frac{\Lambda}{6}(3\omega + 1)a_s^{6\omega+1} = 0 \] (71)
respectively. At the first sight, one can see that under any condition \( a_s = 0 \) can be a trivial extremum point (EP) for the above equations. It is not so hard to prove that the aforementioned EP cannot be a physical point. However, equations (67) and (71) under some constraints on the free parameters \( C \) and \( p_T \) suggest another EP as \( a_s = 1 \). In what follows, once in the absence and then in the presence of cosmological constant \( \Lambda \), for the underlying cosmological toy model we investigate possible realization of a non-singular EC paradigm by concerning on the stability/instability status of the local EP that represents an ESU.

A. \( \Lambda = 0 \)

By neglecting the contribution of the cosmological constant \( \Lambda \) in our toy model, for the case of flat spatial geometry \( k = 0 \), there is an EP as \( a_s = 1 \) moreover an unacceptable case \( a_s = 0 \) if \( C = -p_T \) and \( w = 1 \). However, it is clear that with the mentioned constraints we have \( U''(a_s = 1) = 0 \) which means that \( a_s = 1 \) is not a local EP, rather is a saddle point. As a result, here the stability status of the saddle point \( a_s = 1 \) is complicated and unclear.
However, for the non-flat cases \((k = \pm 1)\), the EP \(a_s = 1\) can be saved provided that

\[
P_T = \pm \frac{8}{1 - \omega} \quad \text{and} \quad C = \pm \frac{2 + 6\omega}{\omega - 1}. \tag{72}
\]

These conditions are very valuable in the sense that they make our analysis no longer dependent on the model free parameters \((C\) and \(P_T\)) unlike the phase analysis presented in previous sections. By applying aforementioned conditions, we come to the following equations

\[
U''(a_s) = -\frac{2a_s^{-3+3\omega}(-2 + 3\omega)(-1 + 3\omega)}{3(1 - \omega)} \pm \frac{a_s^{-6+6\omega}(-5 + 6\omega)(-4 + 6\omega)(2 + 6\omega)}{12(-1 + \omega)} \pm 3a_s^{-2+6\omega}(1 + 6\omega) \tag{73}
\]

which are corresponding to the closed (+ sign) and open (- sign) spatial geometries, respectively. The stability/instability status of the ESU \((a_s = 1)\) is recognizable through \(U''(a_s = 1) > 0\) and \(U''(a_s = 1) < 0\), respectively. Therefore, we arrive at the following constraints on parameter \(\omega\)

\[
w > \frac{1}{3} \quad \Rightarrow \quad U''(a_s = 1) > 0 \tag{74}
\]

\[
w < \frac{1}{3} \quad \Rightarrow \quad U''(a_s = 1) < 0
\]

and

\[
w < \frac{1}{3} \quad \Rightarrow \quad U''(a_s = 1) > 0 \tag{75}
\]

\[
w > \frac{1}{3} \quad \Rightarrow \quad U''(a_s = 1) < 0
\]

for the cases of \(k = \pm 1\), respectively. The above constraints suggest that only for the case with \(k = +1\) it is possible to exit the initial static mode in order to enter into the inflation period. As is shown qualitatively in Fig. 6, by decreasing the EoS parameter \(\omega\) (with this idea in mind that in infinitely past time i.e. \(t \to -\infty\) it was a constant value) from \(\omega > -1/3\) towards \(\omega < -1/3\) as universe moves forward, the stable EP \(a_s = 1\) converts into an unstable counterpart so that through an explicit violation of SEC it prepares to enter the accelerating non-singular EC. We note that within the context of effective potential formalism, the unstable state suggests two different paths for ESU: because of small inhomogeneities ESU moves towards standard expanding thermal history i.e. EC or contracts to an initial singularity, as we can see qualitatively in the right panel of Fig. 6.

### B. \(\Lambda \neq 0\)

By adding the cosmological constant \(\Lambda\) into our cosmological model, the EP \(a_s = 1\) can be represents as ESU for the case \(k = 0\) if

\[
C = \Lambda \left(\frac{\omega + 1}{\omega - 1}\right), \quad P_T = \frac{2\Lambda}{1 - \omega} \tag{76}
\]

![FIG. 6: The scheme of the effective potential for a closed universe raised by the toy model at hand with \(\Lambda = 0\) consisting of two types matter fields with \(w = \frac{1}{3}\) (left panel) and \(w = -\frac{2}{3}\) (right panel). The minimum of the effective potential in the left panel represents a stable ESU while its counterpart’s maximum in the right panel corresponds to an unstable ESU since against an small perturbations it moves either into an accelerating emergent universe (EU) or into a contracting universe (CU).](image-url)
By putting this expressions into the effective potential we get

$$U''(a_s) = \frac{a_s^{-6+6\omega}(1+\omega)(-5+6\omega)(-4+6\omega)\Lambda + a_s^{-3+3\omega}(-2+3\omega)(-1+3\omega)(8+2\Lambda)}{12(\omega - 1)}$$

(77)

In the $\omega - \Lambda$ plane the EP $a_s = 1$ shows a stable or unstable ESU (see the two figures in upper panel of Fig. 6 with relevant descriptions).

As is shown qualitatively in the upper couple of the figures in Fig. 7, in the presence of a positive or negative cosmological constant for a flat geometry universe, a graceful realization of the EC is possible. Focusing on the
same argument as already been raised, we see in both plots that by reduction of $\omega$ (as the universe evolves) the EP $a_s = 1$ as a representative of the ESU, transits from stability region to the instability region to joint to the standard cosmology (here we mean inflation period). Note that for the case with $\Lambda > 0$, it is possible to have stability/instability switching even without violating SEC. If $\Lambda < 0$, only for small values of $\Lambda$ this possibility may be present so that for larger values, SEC explicitly violates.

Using the conditions (70) and (71) for the cases of $k = \pm 1$, we can rewrite the free parameters $C$ and $P_T$ in terms of $\omega$ and $\Lambda$ as follows

$$C = \frac{\Lambda(\omega + 1) + 6\omega + 2}{\omega - 1}, \quad P_T = \frac{2\Lambda + 8}{1 - \omega}, \quad (78)$$

and

$$C = \frac{\Lambda(\omega + 1) - 6\omega - 2}{\omega - 1}, \quad P_T = \frac{8 - 2\Lambda}{\omega - 1}, \quad (79)$$

respectively. As a result, putting the above expressions into the effective potential, we arrive at

$$U''(a_s) = \frac{a_s^{6\omega - 6}(6\omega - 5)(6\omega - 4)(\Lambda(1 + \omega) \pm (6\omega + 2)) \pm a_s^{3\omega - 3}(3\omega - 2)(3\omega - 1)(8 \pm 2\Lambda)}{12(\omega - 1)} + 3a_s^{6\omega - 2}\omega(6\omega - 1), \quad (80)$$

corresponding to the cases $k = \pm 1$ respectively. The relevant stability/instability regions within $(\omega - \Lambda)$ parameter space are depicted in the middle and lower couple of plots in Fig. 7, respectively. So, we can argue that for both the closed and open universes, whether $\Lambda$ is positive or negative, a full realization of the non-singular EC scenario could be achievable. The important thing is that independent of the sign of $\Lambda$, EC occurs in spatially closed universes by respecting the SEC, but in its open counterpart it occurs only by violation of SEC.

VII. CONCLUSIONS

"Emergent Cosmology" paradigm is going to be a successful non-singular alternative for cosmological inflation. This scenario consists of existence and stability of static solutions called Einstein static universe (ESU) along with the issue of graceful transition to the expected thermal history of the standard cosmology. In an attempt for realization of a non-singular emergent cosmology, here we have presented a total action $\mathcal{I}$ from a perfect fluid scalar-metric model universe with a perfect fluid plus a cosmological constant $\Lambda$ as a distinct sources. We have proceeded by obtaining the classical equations of motion using a super Hamiltonian approach within the Schutz representation. The central idea in study of EC paradigm is "beginning with no beginning". Namely it contains the idea that our current universe may has been appeared from a primordial rigid state past eternally known as Einstein static universe (ESU) instead of initial Big-Bang singularity. In this paper, we have studied in details this pre-inflation scenario within the context of the above mentioned model universe by using the "linear stability theory" as well as "effective potential formalism" by concerning on the analysis of the static solutions. This dynamical system analysis within the mentioned methods consists of two cases: $\Lambda = 0$ and $\Lambda \neq 0$.

By focusing on the first order dynamical system, We have shown that in the absence of the cosmological constant, $\Lambda = 0$, in the case of fixing some values for the free parameters $C$ and $p_T$ of the underlying cosmological model and also by imposing some constraints on the equation of state (EoS) parameter $\omega$, a permanent ESU is realizable for both flat and closed universes. As listed in Table 1 for the cases of flat and closed universes, the EoS parameter $\omega$ satisfies the conditions $\omega = 1$ and $\frac{1}{2} \leq \omega < 1$, respectively. However, we found that just for the latter case, the conditions provide a graceful exit from the frozen state towards the cosmological inflation and then beginning of the standard thermal history. To be more concrete, Fig. 1 (the left panel) obviously shows that the sign of the eigenvalue $\lambda^2$ switches from negative to the positive one as the EoS parameter $\omega$ reduces slowly with the cosmic time. This means that the universe eternally shall not be caught in the initial static state so that finally joins the standard cosmology and in this way the Big-Bang singularity issue could be resolved.

By following a first order dynamical system analysis, regarding the contribution of $\Lambda$ term in the matter sector of the action $\mathcal{I}$ in addition to the perfect fluid, we have obtained further and more severe stability conditions for ESU in comparison with $\Lambda = 0$ case, see Table 2. Here the stable ESU can be realized in closed and open underlying model universes by setting some admitted values for the free parameters $C$ and $P_T$ as well as $\omega$. Our analysis has revealed that in the presence of $\Lambda$, to reach a sustainable ESU we only need a tiny proportion of the cosmological
constant, that is, $|\Lambda| \ll 1$. Nonetheless, we found that all cases listed in Table 2 cannot result in a graceful realization of the EC paradigm. Rather, only for the third case of a closed spatial geometry an emergent cosmology happens gracefully. Although this result, in terms of the type of spatial geometry, is similar to our former case (with $\Lambda = 0$), here the admitted range of EoS parameter $\omega$ is wider, that is, $-\frac{1}{3} < \omega < 1$.

For an EC paradigm to be a successful scenario, existence of a future de Sitter (dS) global attractor is essential. For this purpose, we have extended our dynamical system analysis into a phase space with infinite extension. Our analysis shows that in the presence of $\Lambda$ under certain constraints on $(p_T - \Lambda)$ parameter space with non-flat spatial geometries, there are dS global attractors.

Concerning on the effective potential formalism we have applied this formalism as another approach to investigate the dynamical properties of the EC scenario within the cosmological toy model at hand. Indeed here ESU is realized by translation of the required conditions within the effective potential formalism through derivation of the relevant extremum points. By doing this analysis in the absence an also presence of $\Lambda$, we have derived a new ESU which if certain constraints on the $\omega$ are met, then it can result in a full realization of EC scenario. Of course, some of these constraints may cause an explicit violation of SEC. However, the strength of this analysis within the framework of the underlying cosmological model is that relevant outcomes are independent of fixing some values for free parameters $C$ and $p_T$ of the model, unlike the first method. Figs 6 and 7 qualitatively reflect the main results of this dynamical system analysis.

In summary and as our main conclusion, in this study we have shown that in the absence or presence of the cosmological constant $\Lambda$, there is the possibility of a graceful realization of Emergent Cosmology (and therefore a resolution of the Big-Bang singularity problem) within a non-flat underlying model universe with a perfect fluid supported by a variety of matter sources which some of them are committed to SEC and some other violate it.

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