New integrable systems
related to the relativistic Toda lattice

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Abstract. New integrable lattice systems are introduced, their different integrable discretization are obtained. Bäcklund transformations between these new systems and the relativistic Toda lattice (in the both continuous and discrete time formulations) are established.
1 Introduction

This paper is devoted to integrable equations of the classical mechanics. More precisely, we shall deal here with equations of motion in the Newtonian form.

We introduce here two new integrable continuous time lattice systems, and present several novel integrable discrete time systems.

The first continuous time system is:

\[ \ddot{x}_k = \dot{x}_{k+1} \exp(x_{k+1} - x_k) - \dot{x}_{k-1} \exp(x_k - x_{k-1}) + \exp(2(x_k - x_{k-1})). \quad (1.1) \]

The second one:

\[ \ddot{x}_k = \dot{x}_k^2 \left( \dot{x}_{k+1} \exp(x_{k+1} - x_k) - \dot{x}_{k-1} \exp(x_k - x_{k-1}) \right). \quad (1.2) \]

To the author’s knowledge, these systems have not appeared in the literature, despite their beauty and possible physical applications. However, they are in a close relation to another well known integrable lattice – a relativistic Toda lattice:

\[ \ddot{x}_k = \dot{x}_{k+1} \dot{x}_k \frac{g^2 \exp(x_{k+1} - x_k)}{1 + g^2 \exp(x_{k+1} - x_k)} - \dot{x}_{k-1} \dot{x}_k \frac{g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - x_{k-1})}. \quad (1.3) \]

More precisely, we shall show that (1.1) and (1.3) are connected by a sort of Bäcklund transformation, and the same is true for (1.2) and (1.3).

We now write down integrable discretizations we propose for the lattices (1.1), (1.2). In the difference equations below \( x_k = x_k(t) \) are supposed to be functions of the discrete time \( t \in h\mathbb{Z} \), and \( \bar{x}_k = x_k(t + h), \ x_k = x_k(t - h) \). Discretization of the first lattice (1.1):

\[ \exp(x_k - x) - \exp(x_k - x_k) = \]

\[ -\frac{1}{1 - h \exp(x_{k+1} - x_k)} + h \exp(x_{k+1} - x_k) + \frac{1}{1 - h \exp(x_k - \bar{x}_{k-1})} - h \exp(x_k - x_{k-1}). \quad (1.4) \]

Discretization of the second lattice (1.2)::

\[ \frac{h}{\exp(x_k - x) - 1} - \frac{h}{\exp(x_k - \bar{x}_k) - 1} = \]

\[ \exp(x_{k+1} - x_k) - \exp(x_{k+1} - x_k) - \exp(x_k - x_{k-1}) + \exp(x_k - \bar{x}_{k-1}). \quad (1.5) \]
The same Bäcklund transformation as for the continuous time systems relates these systems of difference equations to the discrete time relativistic Toda lattice:

\[
\frac{\exp(\bar{x}_k - x_k) - 1}{\exp(x_k - \bar{x}_k) - 1} = \frac{(1 + g^2 \exp(x_{k+1} - x_k))(1 + g^2 \exp(x_k - \bar{x}_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k))(1 + g^2 \exp(x_k - x_{k-1}))},
\]

(1.6)

A modification of the construction leading to the discrete time systems above allows to derive several further nice discretizations. For example, for the lattice (1.1):

\[
\exp(\bar{x}_k - 2x_k + x_{k+1}) = \frac{(1 + h \exp(x_{k+1} - x_k))(1 - h \exp(x_{k+1} - x_{k-1}))}{(1 + h \exp(x_k - x_{k-1}))(1 - h \exp(x_k - \bar{x}_{k-1}))},
\]

(1.7)

and for the relativistic Toda lattice (1.3):

\[
\frac{\exp(-\bar{x}_k + x_k) - 1}{\exp(-x_k + x_{k+1}) - 1} = \frac{(1 + g^2 \exp(x_{k+1} - x_k))(1 + g^2 \exp(x_k - \bar{x}_{k-1}))}{(1 + g^2 \exp(x_{k+1} - x_k))(1 + g^2 \exp(x_k - x_{k-1}))}.
\]

(1.8)

(The last system resembles very much the previous discretization of the relativistic Toda lattice (1.6), however the relation between them is far from trivial.)

All the systems above (continuous and discrete time ones) may be considered either on an infinite lattice \((k \in \mathbb{Z})\), or on a finite one \((1 \leq k \leq N)\). In the last case one of the two types of boundary conditions may be imposed: open–end \((x_0 = \infty, x_{N+1} = -\infty)\) or periodic \((x_0 \equiv x_N, x_{N+1} \equiv x_1)\). We shall be concerned only with the finite lattices here, consideration of the infinite ones being to a large extent similar.

The last remark: the “list” of references at the end of this paper might look strange; in fact most of the references listed in [1] are relevant, and we do not reproduce them here only in order to save place. An interested reader is advised to consult these references.

2 Simplest flows of the relativistic Toda hierarchy and their bi–Hamiltonian structure

We consider in this section two simplest flows of the relativistic Toda hierarchy. The first of them is:

\[
\dot{d}_k = d_k(c_k - c_{k-1}), \quad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}). \tag{2.1}
\]

The second flow of the relativistic Toda hierarchy is:

\[
\dot{d}_k = d_k\left(\frac{c_k}{d_{k+1}d_{k+1}} - \frac{c_{k-1}}{d_{k-1}d_k}\right), \quad \dot{c}_k = c_k\left(\frac{1}{d_k} - \frac{1}{d_{k+1}}\right). \tag{2.2}
\]
They may be considered either under open–end boundary conditions \((d_{N+1} = c_0 = c_N = 0)\), or under periodic ones (all the subscripts are taken \((\text{mod } N)\), so that \(d_{N+1} \equiv d_1, c_0 \equiv c_N, c_{N+1} \equiv c_1\)).

The both lattices \(\{1.1\}, \{1.3\}\) arise from the flow \(\{2.1\}\) under two different parametrizations of the variables \((c,d)\) by the canonically conjugated variables \((x,p)\). Analogously, the both lattices \(\{1.2\}, \{1.3\}\) may be considered as arising on the same way from the flow \(\{2.2\}\).

We discuss now a Hamiltonian structure of the both flows \(\{2.1\}, \{2.2\}\). It is easy to see that they are Hamiltonian with respect to two different compatible Poisson brackets. The first bracket is linear:

\[
\{c_k, d_{k+1}\}_1 = -c_k, \quad \{c_k, d_k\}_1 = c_k, \quad \{d_k, d_{k+1}\}_1 = c_k, \tag{2.3}
\]

(only the non–vanishing brackets are written down), and Hamiltonian functions generating the flows \(\{2.1\}, \{2.2\}\) in this bracket are equal to

\[
H_+^{(1)} = \frac{1}{2} \sum_{k=1}^{N} (d_k + c_{k-1})^2 + \sum_{k=1}^{N} (d_k + c_{k-1})c_k, \quad H_-^{(1)} = - \sum_{k=1}^{N} \log(d_k). \tag{2.4}
\]

The second Poisson bracket is quadratic:

\[
\{c_k, c_{k+1}\}_2 = -c_k c_{k+1}, \quad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1}, \quad \{c_k, d_k\}_2 = c_k d_k \tag{2.5}
\]

the corresponding Hamiltonian functions being

\[
H_+^{(2)} = \sum_{k=1}^{N} (d_k + c_k), \quad H_-^{(2)} = \sum_{k=1}^{N} \frac{d_k + c_k}{d_k d_{k+1}}. \tag{2.6}
\]

We turn now to the integrable discretizations of the flows \(\{2.1\}, \{2.2\}\), derived in \([1]\). An integrable discretization of the flow \(\{2.1\}\) is given by the difference equations

\[
\bar{d}_k = d_k \frac{a_{k+1} - h d_{k+1}}{a_k - h d_k}, \quad \bar{c}_k = c_k \frac{a_{k+1} + h c_{k+1}}{a_k + h c_k}, \tag{2.7}
\]

where \(a_k = a_k(c,d)\) is defined as a unique set of functions satisfying the recurrent relation

\[
a_k = 1 + h d_k + \frac{h c_{k-1}}{a_{k-1}} \tag{2.8}
\]

together with an asymptotic relation

\[
a_k = 1 + h (d_k + c_{k-1}) + O(h^2). \tag{2.9}
\]
In the open–end case, due to $c_0 = 0$, we obtain from (2.8) the following finite continued fractions expressions for $a_k$:

$$a_1 = 1 + hd_1; \quad a_2 = 1 + hd_2 + \frac{hc_1}{1 + hd_1}; \quad \ldots ;$$

$$a_N = 1 + hd_N + \frac{hc_{N-1}}{1 + hd_{N-1} + \frac{hc_{N-2}}{1 + hd_{N-2} + \ldots + \frac{hc_1}{1 + hd_1}}}.$$  

In the periodic case (2.8), (2.9) uniquely define $a_k$'s as $N$-periodic infinite continued fractions. It can be proved that for $h$ small enough these continued fractions converge and their values satisfy (2.9).

An integrable discretization of the flow (2.2) is given by the difference equations

\begin{equation}
\tilde{d}_k = d_{k+1} - h\tilde{d}_{k-1}, \quad \tilde{c}_k = c_{k+1} + h\tilde{d}_k,
\end{equation}

where $\tilde{d}_k = \tilde{d}_k(c, d)$ is defined as a unique set of functions satisfying the recurrent relation

\begin{equation}
\frac{c_k}{\tilde{d}_k} = d_k - h - h\tilde{d}_{k-1},
\end{equation}

together with an asymptotic relation

\begin{equation}
\tilde{d}_k = \frac{c_k}{d_k} + O(h).
\end{equation}

In the open–end case we obtain from (2.11) the following finite continued fractions expressions for $\tilde{d}_k$:

$$\tilde{d}_1 = \frac{c_1}{d_1 - h}; \quad \tilde{d}_2 = \frac{c_2}{d_2 - h - \frac{hc_1}{d_1 - h}}; \quad \ldots ;$$

$$\tilde{d}_{N-1} = \frac{c_{N-1}}{d_{N-1} - h - \frac{hc_{N-2}}{d_{N-2} - h - \ldots - \frac{hc_1}{d_1 - h}}}.$$  

In the periodic case (2.11), (2.12) uniquely define $\tilde{d}_k$'s as $N$-periodic infinite continued fractions. It can be proved that for $h$ small enough these continued fractions converge and their values satisfy (2.12).

It can be proved [1] that the maps (2.7), (2.10) are Poisson with respect to the both brackets (2.3), (2.5).
3 Lax representations

Recall [1] that both continuous time systems (2.1), (2.2) and discrete time ones (2.7), (2.10) admit Lax representations, the Lax matrices being the same for the both cases.

Namely, the following statement holds. Introduce two $N$ by $N$ matrices depending on the phase space coordinates $c_k, d_k$ and (in the periodic case) on the additional parameter $\lambda$:

$$L(c, d, \lambda) = \sum_{k=1}^{N} d_k E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (3.1)$$

$$U(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} - \lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}. \quad (3.2)$$

Here $E_{jk}$ stands for the matrix whose only nonzero entry on the intersection of the $j$th row and the $k$th column is equal to 1. In the periodic case we have $E_{N+1,N} = E_{1,N}, E_{N,N+1} = E_{N,1}$; in the open–end case we set $\lambda = 1$, and $E_{N+1,N} = E_{N,N+1} = 0$. Consider also following two matrices:

$$T_+(c, d, \lambda) = L(c, d, \lambda)U^{-1}(c, d, \lambda), \quad T_-(c, d, \lambda) = U^{-1}(c, d, \lambda)L(c, d, \lambda). \quad (3.3)$$

**Proposition 1.** The flow (2.1) is equivalent to the following matrix differential equations:

$$\dot{L} = LB - AL, \quad \dot{U} = UB - AU, \quad (3.4)$$

which imply also

$$\dot{T}_+ = [T_+, A], \quad \dot{T}_- = [T_-, B], \quad (3.5)$$

where

$$A(c, d, \lambda) = \sum_{k=1}^{N} (d_k + c_{k-1})E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}, \quad (3.6)$$

$$B(c, d, \lambda) = \sum_{k=1}^{N} (d_k + c_k)E_{kk} + \lambda \sum_{k=1}^{N} E_{k+1,k}. \quad (3.7)$$

**Proposition 2.** The map (2.7) is equivalent to the following matrix difference equations:

$$\bar{L} = A^{-1}LB, \quad \bar{U} = A^{-1}UB, \quad (3.8)$$

which imply also

$$\bar{T}_+ = A^{-1}T_+A, \quad \bar{T}_- = B^{-1}T_-B, \quad (3.9)$$
where

\[ A(c, d, \lambda) = \sum_{k=1}^{N} a_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \]  

\[ B(c, d, \lambda) = \sum_{k=1}^{N} b_k E_{kk} + h\lambda \sum_{k=1}^{N} E_{k+1,k}, \]  

and the quantities \( b_k \) are defined by

\[ b_k = a_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k} = a_{k-1} \frac{a_k + hc_k}{a_{k-1} + hc_{k-1}}. \]  

Note that the compatibility of the two expressions for \( b_k \) in (3.12) is an immediate consequence of (2.8), and that it follows from (3.12), (2.9) that

\[ b_k = 1 + h(d_k + c_k) + O(h^2). \]  

**Proposition 3.** The flow (2.2) is equivalent to the following matrix differential equations:

\[ \dot{L} = LD - CL, \quad \dot{U} = UD - CU, \]  

which imply also

\[ \dot{T}_+ = [T_+, C], \quad \dot{T}_- = [T_-, D], \]  

where

\[ C(c, d) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_{k+1}} E_{k,k+1}, \]  

\[ D(c, d) = -\lambda^{-1} \sum_{k=1}^{N} \frac{c_k}{d_k} E_{k,k+1}. \]  

**Proposition 4.** The map (2.10) is equivalent to the following matrix difference equations:

\[ \bar{L} = CLD^{-1}, \quad \bar{U} = CUD^{-1} \]  

which imply also

\[ \bar{T}_+ = C T_+ C^{-1}, \quad \bar{T}_- = D T_- D^{-1}, \]  

where

\[ C(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1} \sum_{k=1}^{N} c_k E_{k,k+1}, \]  

\[ D(c, d, \lambda) = \sum_{k=1}^{N} E_{kk} + h\lambda^{-1} \sum_{k=1}^{N} d_k E_{k,k+1}. \]
and the quantities \( c_k \) are defined by

\[
c_k = \frac{d_k - h\delta_{k-1}}{d_{k+1} - h\delta_k} = \frac{c_k + h\delta_k}{c_{k+1} + h\delta_{k+1}}. \tag{3.22}
\]

The compatibility of the two expressions for \( c_k \) in (3.22) is an immediate consequence of (2.11), and it follows from (3.22), (2.12) that

\[
c_k = \frac{c_k}{d_{k+1}} + O(h). \tag{3.23}
\]

The spectral invariants of the matrices \( T_\pm(c, d, \lambda) \) serve as integrals of motion for the flows (2.1), (2.2), as well as for the maps (2.7), (2.10). In particular, it is easy to see that the Hamiltonian functions (2.4), (2.6) are spectral invariants of the Lax matrices:

\[
H_+^{(1)} = \frac{1}{2} \text{tr}(T_+^2), \quad H_-^{(1)} = -\text{tr} \log(T_\pm),
\]

\[
H_+^{(2)} = \text{tr}(T_+), \quad H_-^{(2)} = \text{tr}(T_-^{-1}).
\]

Moreover, it can be proved [1] that the maps (2.7), (2.10) admit in both Poisson brackets (2.3), (2.5) interpolation by Hamiltonian flows with the Hamiltonian functions being certain spectral invariants of the matrices \( T_\pm \).

4 Parametrization of the linear bracket by canonically conjugated variables

In what follows we shall consider different Poisson maps from the standard symplectic space \( \mathbb{R}^{2N}(x, p) \) into the Poisson space \( \mathbb{R}^{2N}(c, d) \), the latter being equipped with different Poisson brackets, while the former always with the canonical one:

\[
\{x_k, x_j\} = \{p_k, p_j\} = 0, \quad \{p_k, x_j\} = \delta_{kj}. \tag{4.1}
\]

We shall call such maps parametrizations of the corresponding Poisson bracket on \( \mathbb{R}^{2N}(c, d) \) through canonically conjugated variables \((x, p)\).

For example, the linear Poisson bracket (2.3) may be parametrized by the canonically conjugated variables \((x, p)\) according to the formulas:

\[
d_k = p_k - \exp(x_k - x_{k-1}), \quad c_k = \exp(x_{k+1} - x_k). \tag{4.2}
\]

Let us see how do the equations of motion look in this parametrization. We start with (2.1), (2.7).
Obviously, the function $H_+^{(1)}$ takes the form
\[
H_+^{(1)} = \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N} p_k \exp(x_{k+1} - x_k). \tag{4.3}
\]

Correspondingly, the flow (2.1) takes the form of canonical equations of motion:
\[
\dot{x}_k = \frac{\partial H_+^{(1)}}{\partial p_k} = p_k + \exp(x_{k+1} - x_k),
\]
\[
\dot{p}_k = -\frac{\partial H_+^{(1)}}{\partial x_k} = p_k \exp(x_{k+1} - x_k) - p_{k-1} \exp(x_k - x_{k-1}).
\]
As an immediate consequence of these equations one gets the Newtonian equations of motion (1.1). A standard procedure allows to find a Lagrangian formulation of these equations. Indeed, one has to express
\[
\mathcal{L} = \sum_{k=1}^{N} \dot{x}_k p_k - H \tag{4.4}
\]
in terms of $(x_k, \dot{x}_k)$, which in the present case leads to
\[
\mathcal{L}_+^{(1)}(x, \dot{x}) = \frac{1}{2} \sum_{k=1}^{N} \left( \dot{x}_k - \exp(x_{k+1} - x_k) \right)^2. \tag{4.5}
\]

Note that the results of the previous section provide us with a Lax representation of our new lattice (1.1): one needs only to set in the formulas of the Proposition 1
\[
c_k = \exp(x_{k+1} - x_k), \quad d_k = \dot{x}_k - \exp(x_k - x_{k-1}) - \exp(x_{k+1} - x_k).
\]

We turn now to a less straightforward case of discrete equations of motion.

**Theorem 1.** In the parametrization (4.2) the equations of motion (2.7) may be presented in the form of the following two equations:
\[
hp_k = \exp(\bar{x}_k - x_k) - \frac{1}{1 - h \exp(x_k - \bar{x}_{k-1})} + h \exp(x_k - x_{k-1}) - h \exp(x_{k+1} - x_k), \tag{4.6}
\]
\[
h\bar{p}_k = \exp(\bar{x}_k - x_k) - \frac{1}{1 - h \exp(x_{k+1} - \bar{x}_k)}, \tag{4.7}
\]
which imply also the Newtonian equations of motion (1.4).

**Proof.** The second equation of motion in (2.7) together with the parametrization $c_k = \exp(x_{k+1} - x_k)$ implies that the following quantity is constant, i.e. does not depend on $k$:
\[
\exp(-\bar{x}_k + x_k)(a_k + hc_k) = \text{const}.
\]
Choosing this constant to be equal to 1, we get:

\[ a_k + h c_k = \exp(\bar{x}_k - x_k), \]  

(4.8)

hence

\[ a_k = \exp(\bar{x}_k - x_k) - h \exp(x_{k+1} - x_k) = \exp(\bar{x}_k - x_k) \left( 1 - h \exp(x_{k+1} - \bar{x}_k) \right). \]  

(4.9)

Substituting the last two formulas into (2.8), we get

\[ a_k - h d_k = 1 + \frac{h c_{k-1}}{a_{k-1}} = \frac{1}{1 - h \exp(x_k - \bar{x}_{k-1})}, \]  

(4.10)

or

\[ h d_k = \exp(\bar{x}_k - x_k) \left( 1 - h \exp(x_{k+1} - \bar{x}_k) \right) - \frac{1}{1 - h \exp(x_k - \bar{x}_{k-1})}. \]  

(4.11)

Now the first equation of motion in (2.7) may be rewritten with the help of (4.10) as

\[ \bar{d}_k = d_k \frac{1 - h \exp(x_k - \bar{x}_{k-1})}{1 - h \exp(x_{k+1} - \bar{x}_k)}, \]

which together with (1.22) implies:

\[ h \bar{d}_k = \exp(\bar{x}_k - x_k) \left( 1 - h \exp(x_k - \bar{x}_{k-1}) \right) - \frac{1}{1 - h \exp(x_{k+1} - \bar{x}_k)}. \]  

(4.12)

Under the parametrization \( d_k = p_k - \exp(x_k - x_{k-1}) \) the equations (4.11), (4.12) are equivalent to (1.6), (1.7).

The Lax representations for the system (1.4) is given by the Proposition 2, where the expressions for the coefficients \( c_k, d_k, a_k, b_k \) in terms of the variables \( x_k \) and their discrete time updates \( \bar{x}_k \) are given by \( c_k = \exp(x_{k+1} - x_k) \), (4.11), (4.12), (4.9), and

\[ b_k = \exp(\bar{x}_k - x_k) \left( 1 - h \exp(x_k - \bar{x}_{k-1}) \right). \]

The last formula following from (3.12), (1.9), and (4.11).

Note also that the equations (1.6), (4.7) not only immediately imply (1.4) from the introduction, but, moreover, allow to find a Lagrangian interpretation of this equation. Indeed, the general theory says that if the equations of motion are represented in the Lagrange form

\[ \partial \left( \Lambda(\bar{x}, x) + \Lambda(x, \bar{x}) \right) / \partial x_k = 0, \]  

(4.13)

then the momenta \( p_k \) canonically conjugated to \( x_k \) are given by

\[ p_k = -\partial \Lambda(\bar{x}, x) / \partial x_k, \]  

(4.14)
so that
\[ \tilde{p}_k = \partial \Lambda(\tilde{x}, x)/\partial \tilde{x}_k. \] (4.15)

Identifying (4.6), (4.7) with (4.14), (4.15), respectively, we see that the Lagrange function for the equation (1.4) can be chosen in the form
\[ \Lambda^{(1)}_+(\tilde{x}, x) = \sum_{k=1}^{N} \varphi(\tilde{x}_k - x_k) - h^{-1} \sum_{k=1}^{N} \log \left(1 - h \exp(x_{k+1} - \tilde{x}_k)\right) - \sum_{k=1}^{N} \exp(x_{k+1} - x_k), \] (4.16)

where
\[ \varphi(\xi) = h^{-1} \left( \exp(\xi) - 1 - \xi \right). \]

Obviously, this function serves a finite difference approximation to (4.5).

We turn now to the equations of motion (2.2), (2.10), and find out how they look in the parametrization (4.2).

The function \( H^{(1)} \) takes the form
\[ H^{(1)} = - \sum_{k=1}^{N} \log \left( p_k - \exp(x_k - x_{k-1}) \right). \] (4.17)

Correspondingly, the canonical equations of motion for the flow (2.2) takes the form:
\[ \dot{x}_k = \partial H^{(1)}/\partial p_k = - \frac{1}{\dot{p}_k - \exp(x_k - x_{k-1})}, \]
\[ \dot{p}_k = - \partial H^{(1)}/\partial x_k = \frac{\exp(x_{k+1} - x_k)}{p_{k+1} - \exp(x_{k+1} - x_k)} - \frac{\exp(x_k - x_{k-1})}{p_k - \exp(x_k - x_{k-1})}. \]

As a consequence of these equations, one gets:
\[ p_k = \exp(x_k - x_{k-1}) - \frac{1}{\dot{x}_k}, \quad \dot{p}_k = -\dot{x}_{k+1} \exp(x_{k+1} - x_k) + \dot{x}_k \exp(x_k - x_{k-1}), \]
and the Newtonian equations of motion (1.2) follow. A standard procedure (4.4) leads to a Lagrangian formulation of these equations. One has:
\[ L^{(1)}(x, \dot{x}) = - \sum_{k=1}^{N} \log(\dot{x}_k) + \sum_{k=1}^{N} \dot{x}_k \exp(x_k - x_{k-1}). \] (4.18)

In order to get a Lax representation of the lattice (1.2) one needs only to set in the formulas of the Proposition 3:
\[ c_k = \exp(x_{k+1} - x_k), \quad d_k = -\frac{1}{\dot{x}_k}. \]
Turn to the discrete equations of motion (2.10), we get:

**Theorem 2.** In the parametrization (4.2), the equations of motion (2.10) may be presented in the form of the following two equations:

\[ p_k = -\frac{h}{\exp(\bar{x}_k - x_k) - 1} + \exp(x_k - \bar{x}_{k-1}), \]
\[ (4.19) \]

\[ \bar{p}_k = -\frac{h}{\exp(\bar{x}_k - x_k) - 1} + \exp(x_{k+1} - \bar{x}_k) - \exp(\bar{x}_{k+1} - \bar{x}_k) + \exp(\bar{x}_k - \bar{x}_{k-1}), \]
\[ (4.20) \]

which imply also the Newtonian equations of motion (1.4).

**Proof.** The second equation of motion in (2.10), rewritten as

\[ \tilde{c}_k = \frac{c_k 1 + h\delta_k / c_k}{1 + h\delta_k / c_k}, \]

together with the parametrization \( c_k = \exp(x_{k+1} - x_k) \) implies that the following quantity is constant, i.e. does not depend on \( k \):

\[ \exp(\bar{x}_k - x_k) \left( 1 + \frac{h\delta_k}{c_k} \right) = \text{const}. \]

Choosing this constant to be equal to 1, we get:

\[ \frac{c_k}{\delta_k} + h = \frac{h}{\exp(\bar{x}_k - x_k) - 1}, \]
\[ (4.21) \]

hence

\[ h\delta_k = \exp(x_{k+1} - \bar{x}_k) - \exp(x_{k+1} - x_k) = -\exp(x_{k+1} - \bar{x}_k) \left( \exp(\bar{x}_k - x_k) - 1 \right). \]
\[ (4.22) \]

The recurrent relation (2.11) implies:

\[ d_k - h\delta_{k-1} = \frac{c_k}{\delta_k} + h, \]
\[ (4.23) \]

which together with (4.22), (4.21) implies:

\[ d_k = -\frac{h}{\exp(\bar{x}_k - x_k) - 1} - \exp(x_k - \bar{x}_{k-1}) \left( \exp(\bar{x}_{k-1} - x_{k-1}) - 1 \right). \]
\[ (4.24) \]

Now we can rewrite the first equation of motion in (2.10), taking into account (4.23), (4.24):

\[ \tilde{d}_k = d_{k+1} \frac{\exp(\bar{x}_{k+1} - x_{k+1}) - 1}{\exp(\bar{x}_k - x_k) - 1}, \]
The last equation together with (4.24) implies:
\[
\ddot{d}_k = -\frac{h}{\exp(\bar{x}_k - x_k) - 1} - \exp(x_{k+1} - \bar{x}_k) \left( \exp(\bar{x}_{k+1} - x_{k+1}) - 1 \right).
\] (4.25)

Under the parametrization \(d_k = p_k - \exp(x_k - x_{k-1})\) the equations (4.24), (4.25) are equivalent to (4.19), (4.20).

The Lax representations for the system (1.5) is given by the Proposition 4, where the expressions for the coefficients \(c_k, d_k, \tilde{d}_k, c_k\) in terms of the variables \(x_k\) and their discrete time updates \(\bar{x}_k\) are given by \(c_k = \exp(x_{k+1} - x_k), (4.24), (4.23), (4.22)\), and
\[
h\tilde{c}_k = -\exp(x_{k+1} - \bar{x}_k) \left( \exp(\bar{x}_{k+1} - x_{k+1}) - 1 \right)
\]
(the last formula following from (4.22), (4.22), and (4.21)).

Identifying (4.19), (4.20) with (4.14), (4.15), respectively, we get the Lagrange function for the equation (1.5) in the form
\[
\Lambda^{(1)}(\bar{x}, x) = -h \sum_{k=1}^{N} \psi(\bar{x}_k - x_k) + \sum_{k=1}^{N} \left[ \exp(\bar{x}_k - \bar{x}_{k-1}) - \exp(x_k - \bar{x}_{k-1}) \right],
\]
where
\[
\psi(\xi) = \int_{0}^{\xi} \frac{d\eta}{\exp(\eta) - 1} = \log(\exp(\xi) - 1) - \xi.
\]
This function clearly is a finite difference approximation of (4.18).

### 5 Parametrization of the quadratic bracket by canonically conjugated variables

We give for completeness the results corresponding to another parametrization of the variables \(c_k, d_k\) by means of canonically conjugated variables \(x_k, p_k\), namely the parametrization leading to the quadratic bracket (2.5). The relativistic Toda lattice arises on this way. The corresponding formulas were given in [1], but in an ad hoc manner, without derivation. We take an opportunity to fill in this gap here.

The parametrization leading to the quadratic bracket (2.5) reads:
\[
d_k = \exp(p_k), \quad c_k = g^2 \exp(x_{k+1} - x_k + p_k),
\] (5.1)
\((g^2 \in \mathbb{R} \text{ is a coupling constant})\).
In terms of these variables
\[ H^{(2)}_+ = \sum_{k=1}^{N} \exp(p_k) \left( 1 + g^2 \exp(x_{k+1} - x_k) \right), \]
\[ H^{(2)}_- = \sum_{k=1}^{N} \exp(-p_k) \left( 1 + g^2 \exp(x_k - x_{k-1}) \right). \]  
(5.2)

Hence the equation of motion corresponding to (2.1) take the canonical form
\[ \dot{x}_k = \frac{\partial H^{(2)}}{\partial p_k} = \exp(p_k) \left( 1 + g^2 \exp(x_{k+1} - x_k) \right), \]
\[ \dot{p}_k = -\frac{\partial H^{(2)}}{\partial x_k} = g^2 \exp(x_{k+1} - x_k + p_k) - g^2 \exp(x_k - x_{k-1} + p_{k-1}). \]  

This can be put into a Newtonian form (1.3).

A standard procedure allows also to find a Lagrangian formulation of these equations, with a Lagrange function
\[ L^{(2)}_+ (x, \dot{x}) = \sum_{k=1}^{N} [\dot{x}_k \log(\dot{x}_k) - \dot{x}_k] - \sum_{k=1}^{N} \dot{x}_k \log \left( 1 + g^2 \exp(x_{k+1} - x_k) \right). \]  
(5.3)

The Lax representation for these equations are given by the Proposition 1 with
\[ d_k = \dot{x}_k / \left( 1 + g^2 \exp(x_{k+1} - x_k) \right), \quad c_k = g^2 \exp(x_{k+1} - x_k)d_k. \]

Analogously, the canonical equations of motion corresponding to (2.2) are:
\[ \dot{x}_k = \frac{\partial H^{(2)}}{\partial p_k} = -\exp(-p_k) \left( 1 + g^2 \exp(x_k - x_{k-1}) \right), \]
\[ \dot{p}_k = -\frac{\partial H^{(2)}}{\partial x_k} = g^2 \exp(x_{k+1} - x_k - p_{k+1}) - g^2 \exp(x_k - x_{k-1} - p_k). \]

The Newtonian equations following from these ones are just the same (1.3) as before. they correspond, however, to a different form of a Lagrange function:
\[ L^{(2)}_- (x, \dot{x}) = -\sum_{k=1}^{N} [\dot{x}_k \log(-\dot{x}_k) - \dot{x}_k] + \sum_{k=1}^{N} \dot{x}_k \log \left( 1 + g^2 \exp(x_k - x_{k-1}) \right). \]  
(5.4)

Respectively, the Lax representation for these equations is given by the Proposition 3 with the identifications
\[ d_k = \frac{1 + g^2 \exp(x_k - x_{k-1})}{\dot{x}_k}, \quad c_k = g^2 \exp(x_{k+1} - x_k)d_k. \]

We turn now to the discrete time systems (2.7), (2.10).
Theorem 3. In the parametrization (5.1) the map (2.7) takes the form of the following two equations:

\[ h \exp(p_k) = \frac{\left( \exp(x_k - x_{k-1}) - 1 \right)}{\left( 1 + g^2 \exp(x_{k+1} - x_k) \right)} \left( 1 + g^2 \exp(x_k - x_{k-1}) \right), \] (5.5)

\[ h \exp(\tilde{p}_k) = \frac{\left( \exp(x_k - x_{k-1}) - 1 \right)}{\left( 1 + g^2 \exp(x_{k+1} - x_k) \right)}. \] (5.6)

This implies also a Newtonian form (1.6) of equations of motion.

**Proof.** From (2.7) it follows:

\[ \frac{\tilde{c}_k}{d_k} = \frac{c_k}{d_k} \frac{a_{k+1} + hc_{k+1}}{a_k - hd_k}, \]

\[ \frac{a_k}{a_k - hd_k} = \exp(x_k - x_{k-1}). \]

Since \( c_k/d_k = g^2 \exp(x_{k+1} - x_k) \), this implies that the following quantity is constant, i.e. does not depend on \( k \):

\[ \exp(x_k - x_{k-1}) \frac{a_k - hd_k}{a_k + hc_k} = \text{const}. \]

Setting this constant equal to 1, we get:

\[ \frac{a_k + hc_k}{a_k - hd_k} = \exp(x_k - x_{k-1}). \]

This implies:

\[ a_k = h \frac{d_k \exp(x_k - x_k) + c_k}{\exp(x_k - x_k) - 1} \]

\[ = hd_k \frac{\exp(x_k - x_k)(1 + g^2 \exp(x_{k+1} - x_k))}{\exp(x_k - x_k) - 1}. \] (5.7)

As a consequence, we get:

\[ a_k + hc_k = h \exp(x_k - x_k) \frac{d_k + c_k}{\exp(x_k - x_k) - 1} \]

\[ = hd_k \frac{\exp(x_k - x_k)(1 + g^2 \exp(x_{k+1} - x_k))}{\exp(x_k - x_k) - 1}. \] (5.8)

Substituting (5.7), (5.8) in the recurrent relation (2.8), we get:

\[ a_k - hd_k = 1 + \frac{hc_{k-1}}{a_{k-1}} = \frac{1 + g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - x_{k-1})}. \] (5.9)
Substituting in the left–hand side of this formula the expression (5.7) for $a_k$, we arrive at:

$$hd_k = \frac{(\exp(\bar{x}_k - x_k) - 1)}{(1 + g^2 \exp(x_{k+1} - x_k))} \left(\frac{1 + g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - \bar{x}_{k-1})}\right),$$

(5.10)

Further, from the first equation of motion in (2.7) and (5.9) it follows:

$$h\bar{d}_k = \frac{\exp(\bar{x}_k - x_k) - 1}{1 + g^2 \exp(x_{k+1} - \bar{x}_k)}.$$  

(5.11)

Now (5.5), (5.6) follow from (5.10), (5.11) under the parametrization $d_k = \exp(p_k)$.

The Lax representation for the system (5.5), (5.6) is given by the Proposition 2 with the following expressions through $x_k$, $\bar{x}_k$: (5.10), (5.11) for $d_k$, $\bar{d}_k$; $c_k = g^2 \exp(x_{k+1} - x_k)d_k$;

$$a_k = \exp(\bar{x}_k - x_k) \left(\frac{1 + g^2 \exp(x_{k+1} - \bar{x}_k)}{1 + g^2 \exp(x_{k+1} - x_k)}\right) \left(\frac{1 + g^2 \exp(x_k - x_{k-1})}{1 + g^2 \exp(x_k - \bar{x}_{k-1})}\right),$$

(5.12)

$$b_k = \exp(\bar{x}_k - x_k).$$

(5.13)

Indeed, (5.12) follows from (5.7), (5.10), and (5.13) follows from (3.12), (5.12), and (5.9).

Identifying (5.5) and (5.6) with (4.14), (4.15), we get a Lagrange function

$$\Lambda_+^{(2)}(\bar{x}, x) = \sum_{k=1}^{N} \Phi(\bar{x}_k - x_k) + \sum_{k=1}^{N} [\Psi(x_{k+1} - \bar{x}_k) - \Psi(x_{k+1} - x_k)],$$

(5.14)

where the two functions $\Phi(\xi), \Psi(\xi)$ are defined by

$$\Phi(\xi) = \int_{0}^{\xi} \log \left| \frac{\exp(\eta) - 1}{h} \right| d\eta, \quad \Psi(\xi) = \int_{0}^{\xi} \log \left( 1 + g^2 \exp(\eta) \right) d\eta.$$  

(5.15)

It is easy to see that this Lagrangian function serves as a finite difference approximation to (5.3).

**Theorem 4.** In the parametrization (5.1) the map (2.10) takes the form of the following two equations:

$$\exp(p_k) = \frac{h(1 + g^2 \exp(x_k - \bar{x}_{k-1}))}{(1 - \exp(\bar{x}_k - x_k))},$$

(5.16)

$$\exp(\tilde{p}_k) = \frac{h(1 + g^2 \exp(\bar{x}_k - \bar{x}_{k-1}))}{(1 - \exp(\bar{x}_k - x_k))} \left(\frac{1 + g^2 \exp(x_{k+1} - \bar{x}_k)}{1 + g^2 \exp(x_{k+1} - \bar{x}_k)}\right).$$

(5.17)
This implies the same Newtonian equations of motion (1.10) as for the map (2.7).

Proof. From (2.10), (2.11) we deduce:

\[
\frac{c_k}{d_k} = \frac{c_{k+1}}{d_{k+1}} - h \frac{d_{k+1} - h \vartheta_k}{c_{k+1} + h \vartheta_{k+1}} \frac{d_k}{d_{k+1}} \frac{\vartheta_k}{\vartheta_{k+1}}.
\]

Because of \( c_k/d_k = g^2 \exp(x_{k+1} - x_k) \), this implies that the following quantity does not depend on \( k \):

\[
\vartheta_k \exp(\bar{x}_k - x_{k+1}) = \text{const}.
\]

Setting this constant equal to \( g^2 \), we get:

\[
\vartheta_k = g^2 \exp(x_{k+1} - \bar{x}_k).
\] (5.18)

Substituting this formula into the recurrence (2.11), we get:

\[
d_k = \frac{h \left(1 + g^2 \exp(x_k - \bar{x}_{k-1})\right)}{1 - \exp(\bar{x}_k - x_k)}.
\] (5.19)

Formulas (5.16), (5.19) imply also

\[
d_k - h \vartheta_{k-1} = \frac{h \left(1 + g^2 \exp(\bar{x}_k - \bar{x}_{k-1})\right)}{1 - \exp(\bar{x}_k - x_k)}.
\] (5.20)

This last formula together with the first equation in (2.10) implies:

\[
\delta_k = \frac{h \left(1 + g^2 \exp(\bar{x}_k - \bar{x}_{k-1})\right)}{1 - \exp(\bar{x}_k - x_k)} \frac{1 + g^2 \exp(x_{k+1} - \bar{x}_k)}{1 + g^2 \exp(\bar{x}_{k+1} - \bar{x}_k)}.
\] (5.21)

Finally, (5.16), (5.17) is equivalent to (5.19), (5.21), due to the parametrization \( d_k = \exp(p_k) \).

The Lax representation for the system (5.16), (5.17) is given by the Proposition 4 with the following expressions through \( x_k, \bar{x}_k \): (5.19), (5.21) for \( d_k, \delta_k \); \( c_k = g^2 \exp(x_{k+1} - \bar{x}_k) d_k \); (5.18) for \( \vartheta_k \); and

\[
c_k = g^2 \exp(x_{k+1} - \bar{x}_k) \frac{1 - \exp(\bar{x}_{k+1} - x_k)}{1 - \exp(\bar{x}_k - x_k)} (1 + g^2 \exp(\bar{x}_k - \bar{x}_{k-1})) (1 + g^2 \exp(\bar{x}_{k+1} - \bar{x}_k))
\]

The last formula follows from (3.22), (5.18), and (5.20).

Identifying (5.16) and (5.17) with (4.14), (4.15), we get a Lagrange function

\[
\Lambda_2(\bar{x}, x) = - \sum_{k=1}^N \Phi(\bar{x}_k - x_k) + \sum_{k=1}^N \left[ \Psi(\bar{x}_k - \bar{x}_{k-1}) - \Psi(x_k - \bar{x}_{k-1}) \right],
\] (5.22)

with the same functions \( \Phi(\xi), \Psi(\xi) \) (5.15) as before. It is easy to see that this Lagrangian function is a finite difference approximation to (5.4).
6 Parametrization of the mixed brackets by canonically conjugated variables

It turns out that there exist still another parametrizations of the variables \((c, d)\) by canonically conjugated variables \((x, p)\) leading to interesting discretizations. As we shall see, these parametrizations lead to Poisson brackets which are linear combinations of the two homogeneous ones \((2.3)\) and \((2.3)\). In some sense (which will be clear from the proofs of the Theorems below) these parametrizations are specially designed to obtain nice Newtonian equations from the maps \((2.7)\), \((2.10)\).

We start from the parametrization leading the linear combination
\[
\{\cdot, \cdot\}_1 + h\{\cdot, \cdot\}_2,
\]
which turns out to admit nice discrete Newtonian formulation when applied to \((2.7)\). Clearly, we get an alternative discretization of the lattice \((1.1)\) on this way. Consider following parametrization:
\[
h d_k = \exp(hp_k) - 1 - h \exp(x_k - x_{k-1}), \quad c_k = \exp(x_{k+1} - x_k + hp_k).
\] (6.1)

(Obviously, in the limit \(h \to 0\) we recover the parametrization of the linear bracket \((1.2)\)). Simple calculations show that the Poisson brackets between \((c, d)\) induced by \((6.1)\) read:
\[
\begin{align*}
\{c_{k+1}, c_k\} &= hc_{k+1}c_k, \quad \{d_{k+1}, d_k\} = -c_k, \\
\{d_{k+1}, c_k\} &= c_k + hd_{k+1}c_k, \quad \{d_k, c_k\} = -c_k - hd_kc_k,
\end{align*}
\]
which is exactly \(\{\cdot, \cdot\}_1 + h\{\cdot, \cdot\}_2\). Let us look at the equations of motion generated by these \(h\)-dependent Poisson bracket.

**Theorem 5.** In the parametrization \((6.1)\) the map \((2.7)\) takes the form of the following two equations:
\[
\begin{align*}
\exp(hp_k) &= \exp(\bar{x}_k - x_k) \frac{(1 + h \exp(x_k - x_{k-1}))(1 - h \exp(x_k - \bar{x}_{k-1}))}{(1 + h \exp(x_{k+1} - x_k))}, \quad (6.2) \\
\exp(h\bar{p}_k) &= \exp(\bar{x}_k - x_k)(1 - h \exp(x_{k+1} - \bar{x}_k)). \quad (6.3)
\end{align*}
\]
This implies the Newtonian equations of motion \((1.7)\).

**Proof.** The crucial observation consists in the following: we can extract from the relations \((6.1)\) the following consequence:
\[
\exp(hp_k) = 1 + hd_k + \frac{hc_{k-1}}{\exp(hp_{k-1})}.
\]
Comparing this with (2.8), we see that \( a_k \) and \( \exp(hp_k) \) satisfy one and the same recurrent relation. Due to the uniqueness of its solution, they must coincide, so that we obtain:

\[
a_k = \exp(hp_k).
\] (6.4)

As a consequence, we get immediately:

\[
a_k - hd_k = 1 + h \exp(x_k - x_{k-1}), \quad a_k + hc_k = \exp(hp_k) \left( 1 + h \exp(x_{k+1} - x_k) \right).
\] (6.5)

These expressions together with (6.1), being substituted into (2.7), allow to rewrite the latter in the form:

\[
\exp(h\tilde{p}_k) - h \exp(\tilde{x}_k - \tilde{x}_{k-1}) = \exp(hp_k) \frac{1 + h \exp(x_{k+1} - x_k)}{1 + h \exp(x_k - x_{k-1})} - h \exp(x_{k+1} - x_k);
\]

\[
\exp(\tilde{x}_{k+1} - \tilde{x}_k + h\tilde{p}_k) = \exp(x_{k+1} - x_k + hp_{k+1}) \frac{1 + h \exp(x_{k+2} - x_{k+1})}{1 + h \exp(x_{k+1} - x_k)}.
\] (6.6)

The formula arising when excluding \( \exp(\tilde{p}_k) \) from these two equations, can be we written after some manipulations as:

\[
\exp(x_{k+1} - \tilde{x}_{k+1} + hp_{k+1}) \frac{1 + h \exp(x_{k+2} - x_{k+1})}{1 + h \exp(x_{k+1} - x_k)} + h \exp(x_{k+1} - \tilde{x}_k) =
\]

\[
\exp(x_k - \tilde{x}_k + p_k) \frac{1 + h \exp(x_{k+1} - x_k)}{1 + h \exp(x_k - x_{k-1})} + h \exp(x_k - \tilde{x}_{k-1}).
\]

So, the expression on the right–hand side is constant, i.e. does not depend on \( k \). Setting this constant equal to \( 1 \), we arrive at (6.2). Substituting (6.2) into (6.6), we get (6.3).

It is not difficult to extract from this proof the expressions for coefficients of the matrices forming the Lax representation of the system (1.7) following from the Proposition 2. Also a Lagrangian formulation of this system can be obtained in a standard way: the Lagrange function corresponding to (6.2), (6.3) is

\[
\Lambda^{(\text{mixed})}(\bar{x}, x) = \sum_{k=1}^{N} \frac{(\bar{x}_k - x_k)^2}{2h} - h^{-1} \sum_{k=1}^{N} \left[ \phi_1(x_{k+1} - \bar{x}_k) + \phi_2(x_{k+1} - x_k) \right],
\] (6.7)

where

\[
\phi_1(\xi) = \int_{0}^{\xi} \log \left( 1 - h \exp(\eta) \right) d\eta, \quad \phi_2(\xi) = \int_{0}^{\xi} \log \left( 1 + h \exp(\eta) \right) d\eta.
\]

This is a finite difference approximation to (4.7) different from (1.16).
We turn now to another parametrization leading to a mixed Poisson bracket, namely to the bracket \( \{ \cdot, \cdot \}_2 - h \{ \cdot, \cdot \}_1 \). The corresponding formulas are:

\[
d_k = \exp(p_k) + h \left( 1 + g^2 \exp(x_k - x_{k-1}) \right), \quad c_k = g^2 \exp(x_{k+1} - x_k + p_k).
\]

As is easy to calculate, the resulting Poisson brackets between the variables \((c,d)\) are:

\[
\{c_{k+1},c_k\} = c_{k+1} c_k, \quad \{d_{k+1},d_k\} = -hc_k,
\]

\[
\{d_{k+1},c_k\} = d_{k+1} c_k - hc_k, \quad \{d_k,c_k\} = -d_k c_k + hc_k.
\]

i.e. the linear combination \( \{ \cdot, \cdot \}_2 - h \{ \cdot, \cdot \}_1 \). The equations arising from (2.10) under this parametrization, naturally approximate the relativistic Toda lattice (1.3).

**Theorem 6.** In the parametrization (6.8) the map (2.10) takes the form of the following two equations:

\[
\exp(p_k) = \frac{h \left( 1 + g^2 \exp(x_k - \bar{x}_{k-1}) \right)}{\exp(-\bar{x}_k + x_k) - 1},
\]

\[
\exp(\bar{p}_k) = \frac{h \left( 1 + g^2 \exp(\bar{x}_k - \bar{x}_{k-1}) \right)}{\exp(-\bar{x}_k + x_k) - 1} \frac{\left( 1 + g^2 \exp(x_{k+1} - \bar{x}_k) \right)}{\left( 1 + g^2 \exp(\bar{x}_{k+1} - \bar{x}_k) \right)}.
\]

This implies the Newtonian equations of motion (1.8).

**Proof.** This time the crucial observation consists in the following: as a consequence of the relations (6.8) we have:

\[
\frac{c_k}{g^2 \exp(x_{k+1} - x_k)} = d_k - h - hg^2 \exp(x_k - x_{k-1}).
\]

Comparing this with (2.11), we see that \( \vartheta_k \) and \( g^2 \exp(x_{k+1} - x_k) \) satisfy one and the same recurrent relation. Due to the uniqueness of its solution, they must coincide, so that we obtain:

\[
\vartheta_k = g^2 \exp(x_{k+1} - x_k).
\]

As a consequence, we get immediately:

\[
d_k - h \vartheta_{k-1} = \exp(p_k) + h, \quad c_k + h \vartheta_k = g^2 \exp(x_{k+1} - x_k) \left( \exp(p_k) + h \right).
\]

Substituting these expressions together with (6.8) into (2.10), we bring the latter to the form:

\[
\exp(\bar{p}_k) + hg^2 \exp(\bar{x}_k - \bar{x}_{k-1}) = \exp(p_k) + hg^2 \exp(x_{k+1} - x_k) \frac{\exp(p_k) + h}{\exp(p_{k+1}) + h},
\]

\[
\exp(\bar{x}_{k+1} - \bar{x}_k + \bar{p}_k) = \exp(x_{k+1} - x_k + p_{k+1}) \frac{\exp(p_k) + h}{\exp(p_{k+1}) + h}.
\]
Excluding $\exp(\tilde{p}_k)$ from these two equations, we get the formula which after some manipulations can be written as:

$$\exp\left(\frac{x_{k+1} - \tilde{x}_{k+1} + p_{k+1} - hg^2 \exp(x_{k+1} - \tilde{x}_k)}{\exp(p_{k+1}) + h}\right) = \exp\left(\frac{x_k - \tilde{x}_k + p_k - hg^2 \exp(x_k - \tilde{x}_{k+1})}{\exp(p_k) + h}\right).$$

So, the expression on the right–hand side is constant, i.e. does not depend on $k$. Setting this constant equal to 1, we arrive at (6.9), which is equivalent also to

$$\exp(p_k) + h = \frac{h \exp(-\tilde{x}_k + x_k) \left(1 + g^2 \exp(\tilde{x}_k - \tilde{x}_{k-1})\right)}{\exp(-\tilde{x}_k + x_k) - 1}. \quad (6.14)$$

Finally, substituting (6.14) into (6.13), we get

$$\exp(\tilde{p}_k) = \exp(p_{k+1}) \frac{\exp(-\tilde{x}_{k+1} + x_{k+1}) - 1}{\exp(-\tilde{x}_k + x_k) - 1} \frac{1 + g^2 \exp(\tilde{x}_k - \tilde{x}_{k-1})}{1 + g^2 \exp(\tilde{x}_{k+1} - \tilde{x}_k)}$$

which together with (6.9) implies (6.10).

The Lax representation for (1.8) is given by the Proposition 4; it is not difficult to extract from the proof above the expressions for the entries of the matrices forming the Lax representation. Also the Lagrangian formulation of (1.8) easily follows from (6.9), (6.10). The corresponding Lagrange function is

$$\Lambda^{(\text{mixed})}(\tilde{x}, x) = \sum_{k=1}^{N} \Phi(-\tilde{x}_k + x_k) + \sum_{k=1}^{N} \left[\Psi(\tilde{x}_k - \tilde{x}_{k-1}) - \Psi(x_k - \tilde{x}_{k-1})\right], \quad (6.15)$$

with the functions $\Phi(\xi), \Psi(\xi)$ given in (5.13).

Let us note that, though the equations (1.6), (1.8) are very similar, there exist no obvious changes of variables bringing one of them into another. The only way to do this, i.e. to connect two corresponding sets of $(x_k, \tilde{x}_k)$, is to identify the corresponding $(c_k, d_k)$, given for (1.6) by (5.1), (5.16), (5.17), and for (1.8) by (6.8), (6.9), (6.10). The resulting change of variables is a rather nontrivial Bäcklund transformation.

7 Conclusion

The main message of the present note is following: the field of integrable systems of classical mechanics, even in its best studied parts, is far from being exhausted. Namely, the well known flows of the relativistic Toda hierarchy (2.1), (2.2) have a much more rich dynamical content as usually assumed. Even more is this true for the recently derived discretizations (2.7), (2.10) of these flows. Namely, different parametrizations of
the variables \((c,d)\) by canonically conjugated variables \((x,p)\) (corresponding to the bi–Hamiltonian structure of the relativistic Toda hierarchy) allowed us to derive two new integrable continuous time lattices and four new integrable discretizations, in addition to the previously known ones. All the obtained systems are connected by means of highly nontrivial Bäcklund transformations, which can be obtained by identifying the variables \((c_k,d_k)\) for different models corresponding to one and the same (continuous or discrete time) flow (2.1), (2.2), (2.7), or (2.10).

The methods of this paper can be used also in a more simple situation of the usual Toda lattice, where they also lead to interesting findings. They will be reported in a separate paper.

**References**

[1] Yu.B.Suris. A discrete–time relativistic Toda lattice. – *J.Physics A: Math. & Gen.*, 1996, v.29, p.451–465.