Distributed computation of equilibria in misspecified convex stochastic Nash games

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Abstract

The distributed computation of Nash equilibria is assuming growing relevance in engineering where such problems emerge in the context of distributed control. Accordingly, we consider static stochastic convex games complicated by a parametric misspecification, a natural concern when competing in any large-scale networked engineered system. We present two sets of schemes in which firms learn the equilibrium strategy and correct the misspecification by leveraging noise-corrupted observations.

(1) Monotone stochastic Nash games: We present a scheme that combines a distributed stochastic approximation scheme with a learning update that leverages a least-squares estimator derived from a collection of observations. We proceed to show that the resulting sequence of estimators converges to the true equilibrium strategy and the true parameter in mean-square and in probability, respectively; (2) Stochastic Nash-Cournot games with unobservable aggregate output: We refine (1) to a Cournot setting where the tuple of strategies is not observable and assume that payoff functions and strategy sets are public knowledge (a common knowledge assumption). When noise-corrupted prices are observable, iterative fixed-point schemes are developed, allowing for simultaneously learning the equilibrium strategies and the parameter in an almost-sure sense; Preliminary numerics support the theoretical statements and demonstrate the superiority over sequential counterparts in which learning precedes computation.

I. INTRODUCTION

In networked engineered systems, a common challenge lies in designing distributed control architectures that ensure the satisfaction of a system-wide criterion in environments complicated by nonlinearity, uncertainty, and dynamics. Such control-theoretic problems take on a variety...
of forms and arise in a variety of networked settings, including networks of unmanned aerial vehicles (UAVs), traffic networks, wireline and wireless communication networks, and energy systems, amongst others. Such systems are defined by the absence of a designated central entity that either has system-wide control or has access to global information. As a consequence, control will be effected through distributed decision-making and local interactions that rely on limited information. Game-theoretic approaches represent one avenue for designing such protocols. Game theory has seen wide applicability in the social, economic, and engineered sciences in a largely descriptive role while there has been an immense recent interest in a prescriptive role [3] that designs a game whose equilibria represent the solutions to the control problem of interest [4], [5]; consequently, the distributed learning of Nash equilibria assumes immediate relevance.

Learning in Nash games has seen much study in the last several decades [6]–[8]. In continuous strategy regimes, convex static games are of significant importance in engineered systems such as communication networks [9]–[12], signal processing [13]–[15], and power markets [16], [17]. Dynamic fictitious and gradient play are employed for learning Nash equilibria [18] while utility-based dynamics are examined in [19].

An oft-used assumption in game-theoretic models is one which requires that player payoffs are public knowledge, allowing every player to correctly forecast the choices of his adversaries. As noted by Kirman [20], a firm’s view of the game may be corrupted or misspecified in at least two distinct ways in a Cournot setting where firms decide production levels given a price function: (i) a firm might have a correct description of the price function but an incorrect estimate of its parameters; and (ii) it may have an incorrect structure of the price function and incorrectly conclude that prediction errors are a consequence of misspecified parameters. Kirman [20] considered such a learning process, and showed that by observing true demand, the suggested learning process guarantees that the firm strategies converge to the noncooperative Nash equilibrium. Further inspiration may be drawn from studies by Bischi [21], [22], Szidarovsky [23], [24], amongst others [25], where firms competing in a deterministic Nash-Cournot game learn a parameter of the demand function while playing the game repeatedly. Note that an inherent assumption of a low discount rate is imposed that discounts the future effect of any player’s strategies. Recently, analogous questions of optimization and estimation have also been studied in the context of revenue management where models may be misspecified. Cooper et al. [26] consider a joint process of forecasting and optimization in a regime where
the underlying model may be erroneous and show that the resulting revenues can systematically reduce over time, a phenomenon referred to as the *spiral-down* effect.

When designing protocols for practical engineered systems, particularly in the absence of a centralized controller, it is often possible that the associated parameters of the utility functions are misspecified. For instance, in power market models that enlist Nash-Cournot models [16], [17], the precise nature of the price function is assumed to be given. Similarly, the expected capacity or availability of renewable generation assets is rarely known a priori. Similarly, when developing distributed protocols for networked UAVs, the prescribed utility functions may rely on agent-specific information that can only be learnt via through observations. Faced by such challenges, our goal lies in the development and analysis of general purpose algorithms that combine computation of Nash equilibria with a learning phase to correct misspecification.

Relatively is available via general-purpose schemes for learning equilibria in the face of imperfect information. Accordingly, we present two distributed learning schemes in which firms learn their Nash strategy while correcting the misspecification in their payoffs:

(1) **Stochastic gradient schemes for stochastic Nash games**: Our first class of algorithms addresses a broad class of convex static stochastic Nash games with monotone maps (referred to hereafter as Nash games). Such problems emerge from stochastic generalizations of problems arising in communication networks [10]–[12], [27], signal processing [13]–[15], and power markets [16]. We present a distributed stochastic approximation framework that updates each player’s strategy. These updates rely on a least-squares estimator of the misspecified parameter based on a collection of observations derived from the sequence of updates. The resulting sequence of estimators of the Nash equilibrium and the misspecified vector are shown to converge to their true counterparts in a mean-square and in probability, respectively and error bounds are provided for constant steplength variants.

(2) **Iterative fixed-point schemes for stochastic Nash-Cournot games**: We refine these statements to a Cournot regime where aggregate output is unobservable and one parameter of the demand function is misspecified. Under a common-knowledge assumption, agents develop an estimate of aggregate output and the misspecified price function parameter by observing noisy prices. These estimates allow for an iterative fixed-point schemes that produces iterates that are shown to converge to the Nash-Cournot equilibrium in an almost-sure sense. Additionally, firms learn the true parameter in an almost-sure sense. The result can be extended to nonlinear price
functions and finite termination statements for noise-free regimes are provided.

**Remark:** Before proceeding, we comment on a sequential two stage framework:

\[
\begin{align*}
\text{Step 1. Learn } \theta^* \quad \text{Step 2. Compute } x^*(\theta^*),
\end{align*}
\]

where \( \theta^* \) is to be learnt and \( x^*(\theta^*) \) is the (stochastic) Nash equilibrium, given \( \theta^* \). Unfortunately, such an approach is complicated by several challenges. First, Step 1. needs to be completed in a finite number of iterations, practically impossible for stochastic learning problems. Second, premature termination of Step 1. leads to an erroneous estimate \( \hat{\theta} \) leading to an incorrect Nash equilibrium \( \hat{x} \). In fact, in stochastic regimes, one often cannot prescribe the amount of learning effort required in a priori sense. Preliminary numerics reveal that sequential schemes may perform orders of magnitude worse when compared with iterative fixed-point schemes (see Table IV). Third, offline or a priori observations may be unavailable as required by Step 1.

The rest of the paper is organized as follows. In Section II, we define a misspecified stochastic Nash game and present a gradient-based schemes that leverage least-squares estimates for learning. In Section III, we develop iterative fixed-point schemes in Cournot settings where aggregate output is unobservable. Proofs for supporting results are provided in in Section IV. Finally, in Section V discusses the performance of the presented schemes on a networked Nash-Cournot game and the paper concludes with a set of remarks in Section VI.

**II. Gradient-based schemes for convex Nash games**

**A. Problem description**

We consider an \( N \)-person stochastic Nash game in which the \( i \)th player solves \( \text{Opt}(x_{-i}) \):

\[
\min_{x_i \in K_i} \mathbb{E}[f_i(x; \theta^*, \xi(\omega))] \quad (\text{Opt}(x_{-i}))
\]

where \( K_i \subseteq \mathbb{R}^{n_i} \), \( \theta^* \in \mathbb{R}^m \), \( \xi : \Omega \to \mathbb{R}^d \), \( n = \sum_{i=1}^N n_i \), and \( f_i : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R} \) is a real-valued function in \( x_i \), and \( \omega \in \Omega \). The associated Nash equilibrium is given by a tuple \( x^* = (x_i^*)_{i=1}^N \) where \( x_i^* \in \text{SOL}(\text{Opt}(x_{-i}^*)) \) for \( i = 1, \ldots, N \), \( \text{SOL}(\text{Opt}(x_{-i}^*)) \) denotes the solution of \( \text{Opt}(x_{-i}) \). Under the convexity assumptions (introduced in (A1)) on \( \text{Opt}(x_{-i}) \), the equilibrium conditions of the game can be compactly stated as a stochastic variational inequality problem \( \text{VI}(K, F(\bullet; \theta^*)) \) where \( K \) and \( F(x; \theta) \) are defined as

\[
K \triangleq \prod_{i=1}^N K_i \quad \text{and} \quad F(x; \theta) \triangleq \left( \mathbb{E}[\nabla_{x_i} f_i(x; \theta, \xi)] \right)_{i=1}^N,
\]

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variables. The least squares estimator \( \theta \) and observations, \( y \) scheme where the independent noise and this analysis is left as future research. When an agent has collected \( k \) observations, \( y^j := p(x^j; \theta^*) = \sum_{\ell=1}^m \theta^*_\ell g_\ell(x^j) + \zeta^j \), for \( j = 1, \cdots, k \), where \( \zeta^j \) are i.i.d. random variables. The least squares estimator \( \theta^k \) is given by the solution to the following problem:

\[
\min_{\theta^k \in \Theta} \frac{1}{k} \sum_{j=1}^k \left( y^j - \sum_{\ell=1}^m \theta^*_\ell g_\ell(x^j) \right)^2.
\]

Note that the observations are not independent and consistency of the associated estimators may be claimed from the following set of conditions [28].

**Definition 1 (Grenander’s conditions):** Consider the regression model \( y = X\beta + \epsilon \), where \( y = (y_1; \cdots; y_k) \), \( X = (x_1; \cdots; x_m) \) with \( x_j = (x_{1j}; \cdots; x_{kj}) \) for \( j = 1, \cdots, m \), \( \beta = (\beta_1; \cdots; \beta_m) \), and \( \epsilon = (\epsilon_1, \cdots, \epsilon_k)^T \) with \( \epsilon_1, \cdots, \epsilon_k \) are i.i.d. random variables. Suppose

(a) For each \( j \), \( \lim_{k \to \infty} d_{kj} = +\infty \), where \( d_{kj} = x_j^T x_j \).

(b) For each \( j \), \( \lim_{k \to \infty} x_{lj}^2 / d_{kj} = 0 \), for all \( l = 1, \cdots, k \).

(c) Let \( R_k \) be the sample correlation matrix of the columns of \( X \), excluding the constant term if there is one. Then, \( \lim_{k \to \infty} R_k = C \), a positive definite matrix.

**Lemma 1:** [28, Pg. 65] Suppose \( \hat{\beta}_k \) denotes the least squares estimator of \( \beta \) and suppose \( X \) and \( y \) satisfy Grenander’s conditions as prescribed by Def. 1. Then, \( \hat{\beta}_k \xrightarrow{P} \beta \) as \( k \to \infty \).

**b) Distributed computational scheme:** We consider a distributed stochastic approximation scheme where the \( i \)th agent employs its belief regarding \( \theta^* \) to take a (stochastic) gradient step:

\[
x_{i}^{k+1} := \Pi_{K_i} \left( x_{i}^{k} - \gamma_i^k (\nabla x_i f_i(x^k; \theta^k) + w_i^k) \right), \quad k \geq 0, \quad i = 1, \cdots, N.
\]
We may specify our joint scheme for learning and computation as follows:

| Algorithm I: Gradient response and learning. |
|------------------------------------------------|
| Let \( \theta^0 \in \Theta, x_0 \in K, \{ \gamma^k_i \} > 0 \), for \( i = 1, \ldots, N \), and \( k = 0 \). |
| Step 1: \[
  x^{k+1}_i := \Pi_{K_i} \left( x^k_i - \gamma^k_i \left( \nabla_x f_i(x^k; \theta^k) + w^k_i \right) \right), \quad k \geq 0, \quad i = 1, \ldots, N \]  (Computation) |
| \( \theta^{k+1} := \arg\min_{\theta \in \Theta} \frac{1}{k} \sum_{\ell=1}^k \left( y^\ell - \sum_{i=1}^m \theta^\ell_i g_i(x^\ell_i) \right)^2 \). \] (Learning) |
| Step 2: if \( k > K \), stop; else \( k := k + 1 \) and go to Step 1. |

**B. Assumptions and background**

We now present the main assumptions employed in deriving convergence properties of Algorithm I. Foremost amongst these is a set of convexity assumptions on agent problems.

**Assumption 1 (A1):** For \( i = 1, \ldots, N \), suppose the function \( f_i(x; \theta) \) is convex and continuously differentiable function in \( x_i \) for every \( x_{-i} \in \prod_{j \neq i} K_j \) and every \( \theta \in \Theta \). Furthermore, suppose \( \Theta \) is a closed, convex, and bounded set and for \( i = 1, \ldots, N \), \( K_i \subseteq \mathbb{R}^{n_i} \) is a nonempty, closed, convex and bounded set. Furthermore, suppose the following hold:

(a) For every \( \theta \), \( F(x; \theta) \) is both strongly monotone and Lipschitz continuous in \( x \) with constants \( \mu_x \) and \( L_x \); for every \( \theta \), \( (F(x; \theta) - F(y; \theta))^T (x - y) \geq \mu_x \| x - y \|^2 \), and \( \| F(x; \theta) - F(y; \theta) \| \leq L_x \| x - y \| \).

(b) \( F(x^*; \theta) \) is Lipschitz continuous in \( \theta \) with constant \( L_{\theta} \).

Note that monotone Nash games include stable Nash games, a class of games for which it has been shown that a range of evolutionary dynamics allow for convergence to Nash equilibria [29]. In fact, in recent work [30], a notion of passivity has been developed. Adhering to the spirit of distributed control, we impose the following informational assumption.

**Assumption 2 (A2):** For \( i = 1, \ldots, N \), the \( i \)th agent knows only his objective \( f_i \) and and strategy set \( K_i \). Furthermore, the vector \( x \) is assumed to be observable.

Given the presence of noise, we define two forms of convergence that will assume relevance.

**Definition 2:** Let \( \{X_k\} \) be a sequence of random variables on the probability space \((\Omega, \mathcal{F}_x, \mathbb{P}_x)\), and \( X \) a random variables on the same probability space. Then, (a) \( X_k \xrightarrow{L^2} X \) if \( \lim_{k \to \infty} \mathbb{E}(\|X_k - X\|^2) = 0 \); (b) \( X_k \xrightarrow{P} X \) if for all \( \epsilon > 0 \), \( \lim_{k \to \infty} \mathbb{P}(\|X_k - X\| \geq \epsilon) = 0 \).

The proof of the distributed gradient schemes will leverage the following lemma from [31].

**Lemma 2:** Let \( u_k \) be a sequence of nonnegative numbers and such that \( v_{k+1} \leq (1 - u_k) v_k + \beta_k \) for all \( k \geq 0 \), where \( 0 \leq u_k \leq 1, \beta_k \geq 0 \), \( \sum_{k=0}^{\infty} u_k = \infty \), \( \lim_{k \to \infty} \frac{\beta_k}{u_k} = 0 \). Then, \( v_k \to 0 \) a.s.
We define a new probability space \((Z, \mathcal{F}, P)\), where \(Z \triangleq \Omega \times \Lambda\), \(\mathcal{F} \triangleq \mathcal{F}_x \times \mathcal{F}_\theta\) and \(P \triangleq \mathbb{P}_x \times \mathbb{P}_\theta\). We use \(\mathcal{F}_k\) to denote the sigma-field generated by the initial points \((x^0, \theta^0)\) and errors \((w^l, \zeta^l)\) for \(l = 0, 1, \ldots, k - 1\), i.e., \(\mathcal{F}_0 = \{(x^0, \theta^0)\}\) and \(\mathcal{F}_k = \{(x^0, \theta^0), ((w^l, \zeta^l), l = 0, 1, \ldots, k - 1)\}\) for \(k \geq 1\). We make the following assumptions on the filtration and errors.

**Assumption 3 (A3):** Let the following hold:

(i) \(\mathbb{E}[w^k | \mathcal{F}_k] = 0\) and \(\mathbb{E}[\zeta^k | \mathcal{F}_k] = 0\) a.s. for all \(k\).

(ii) \(\mathbb{E}[\|w^k\|^2 | \mathcal{F}_k] \leq \nu^2\) a.s. for all \(k\).

In an effort to construct truly distributed schemes that require no coordination in terms of setting parameters, we allow each agent to independently set steplengths and as long as a suitable relationship between these steplengths holds, convergence follows. Specifically, the \(i\)th agent employs a diminishing steplength sequence given by \(\gamma_k^i\). Furthermore, we define \(\gamma_{\text{min}}^k \triangleq \min_{1 \leq i \leq N} \{\gamma_k^i\}\) and \(\gamma_{\text{max}}^k \triangleq \max_{1 \leq i \leq N} \{\gamma_k^i\}\) for all \(k\) and make the following assumptions on the steplengths of the algorithm.

**Assumption 4 (A4):** Let \(\{\gamma_k^i\}\) be chosen such that:

(a) \(\sum_{k=1}^{\infty} \gamma_{\text{min}}^k = \infty\), \(\sum_{k=1}^{\infty} (\gamma_{\text{max}}^k)^2 < \infty\).

(b) \(\lim_{k \to \infty} \frac{\gamma_{\text{max}}^k - \gamma_{\text{min}}^k}{\gamma_{\text{max}}^k} = 0\).

Throughout the paper, we use \(\|x\|\) to denote the Euclidean norm of a vector \(x\), i.e., \(\|x\| = \sqrt{x^T x}\). We use \(\Pi_K\) to denote the Euclidean projection operator onto a set \(K\), i.e., \(\Pi_K(x) \triangleq \text{argmin}_{y \in K} \|x - y\|\). A square matrix \(H\) is said to be a \(P\)-matrix if every principal minor of \(H\) is positive. Similarly, \(H\) is a \(P_0\)-matrix if every principal minor of \(H\) is nonnegative.

**C. Analysis**

We provide a contraction result employed in developing our convergence statements.

**Lemma 3:** Suppose \(H : K \to \mathbb{R}^n\) is strongly monotone and Lipschitz continuous over \(K\) with constant \(\mu\) and constant \(L\), respectively. If \(q \triangleq \sqrt{1 - 2\mu \gamma + \gamma^2 L^2}\), then for any \(\gamma > 0\), we have

\[
\|\Pi_K(x - \gamma H(x)) - \Pi_K(y - \gamma H(y))\| \leq q\|x - y\|. \tag{5}
\]

If (A1) holds, implying that \(\Theta\) and \(K\) are bounded, then we can assume that \(\|\theta^k - \theta^*\| \leq m_\theta\) and \(\|x^k - x^*\| \leq m_x\) for some positive constants \(m_\theta\) and \(m_x\). Our first result provides a convergence statement for the sequence of iterates produced by Algorithm I and can be stated as follows.
Lemma 4: Suppose (A1), (A2), (A3) and (A4) hold. Let \( \{x^k, \theta^k\} \) be computed via Algorithm I. Then the following holds for any \( k \geq 0 \):

\[
\mathbb{E} \left[ \|x^{k+1} - x^*\|^2 \right] \leq \alpha_k \mathbb{E}[\|x^k - x^*\|^2] + \beta_k,
\]

where \( \alpha_k = 1 - 2\gamma_k \mu_x + 2(\gamma_k - \gamma_k \gamma_k \max L_x + 2(\gamma_k \gamma_k \max L_x^2 + 2\gamma_k \gamma_k \max L_\theta \max m_x \mathbb{E}[\|\theta^k - \theta^*\|]) + (\gamma_k \gamma_k \max \theta^k \max^2 v^2.
\]

Proof: On one hand, by the nonexpansivity of the Euclidean projector, we have

\[
\|x^{k+1} - x^*\|^2 = \sum_{i=1}^N \|\Pi_{K_i}(x_i^k - \gamma_i^k (F_i(x^k; \theta^k) + w_i^k)) - \Pi_{K_i}(x_i^\gamma - \gamma_i^k F_i(x^*; \theta^*))\|^2
\]

\[
\leq \sum_{i=1}^N \|x_i^k - x_i^* - \gamma_i^k (F_i(x^k; \theta^k) - F_i(x^*; \theta^*)) - \gamma_i^k w_i^k\|^2.
\]

A little algebra reveals that the right hand side can be expanded as follows:

\[
\text{RHS of (7)} = \sum_{i=1}^N \|x_i^k - x_i^*\|^2 + (\gamma_i^k)^2 \|F_i(x^k; \theta^k) - F_i(x^*; \theta^*)\|^2 + (\gamma_i^k)^2 \|w_i^k\|^2
\]

\[
- 2 \sum_{i=1}^N \gamma_i^k (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^*))
\]

\[
- 2 \sum_{i=1}^N \gamma_i^k (x_i^k - x_i^*)^T w_i^k + 2 \sum_{i=1}^N (\gamma_i^k)^2 (F_i(x^k; \theta^k) - F_i(x^*; \theta^*))^T w_i^k.
\]

RHS of (8) \leq \|x^k - x^*\|^2 + (\gamma_i^k)^2 \|F(x^k; \theta^k) - F(x^*; \theta^*)\|^2 + (\gamma_i^k)^2 \|w^k\|^2

\[
- 2 \sum_{i=1}^N \gamma_i^k (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^*))
\]

\[
- 2 \sum_{i=1}^N \gamma_i^k (x_i^k - x_i^*)^T w_i^k + 2 \sum_{i=1}^N (\gamma_i^k)^2 (F_i(x^k; \theta^k) - F_i(x^*; \theta^*))^T w_i^k.
\]

By (A1), term 1 in (9) may be bounded by leveraging the Lipschitz continuity of \( F(x; \theta) \), as shown next:

\[
\|x^k - x^*\|^2 + 2(\gamma_i^k)^2 \|F(x^k; \theta^k) - F(x^*; \theta^*)\|^2 + 2(\gamma_i^k)^2 \|F(x^*; \theta^k) - F(x^*; \theta^*)\|^2
\]

\[
\leq \|x^k - x^*\|^2 + 2(\gamma_i^k)^2 L_\theta^2 \|x^k - x^*\|^2 + 2(\gamma_i^k)^2 \|F(x^*; \theta^k) - F(x^*; \theta^*)\|^2
\]

\[
\leq (1 + 2(\gamma_i^k)^2 L_\theta^2) \|x^k - x^*\|^2 + 2(\gamma_i^k)^2 L_\theta^2 \|\theta^k - \theta^*\|^2.
\]

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Combining (7) with (9), (10), (12) and (13), we obtain

\[
\gamma_i^k \leq \gamma_{\max}^k \text{ for all } i:
\]

\[
- 2 \sum_{i=1}^{N} \gamma_i^k (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^k)) = - 2 \sum_{i=1}^{N} \gamma_{\max}^k (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^k))
\]

\[
- 2 \sum_{i=1}^{N} (\gamma_i^k - \gamma_{\max}^k) (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^k))
\]

The right hand side of (11) can be bounded as follows:

\[
\text{RHS} \leq -2\gamma_{\max}^k \sum_{i=1}^{N} (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^k)) + 2(\gamma_{\max}^k - \gamma_{\min}^k) \sum_{i=1}^{N} \|x_i^k - x_i^*\| \|F_i(x^k; \theta^k) - F_i(x^*; \theta^k)\|
\]

Proceeding further, we may leverage Hölder’s inequality, the Lipschitz continuity of \(F(x; \theta)\) and (A1), to obtain the following sequence of inequalities:

\[
- 2\gamma_{\max}^k \sum_{i=1}^{N} (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^k))
\]

\[
+ 2(\gamma_{\max}^k - \gamma_{\min}^k) \sum_{i=1}^{N} \|x_i^k - x_i^*\| \|F_i(x^k; \theta^k) - F_i(x^*; \theta^k)\|
\]

\[
\leq -2\gamma_{\max}^k (x^k - x^*)^T (F(x^k; \theta^k) - F(x^*; \theta^k)) + 2(\gamma_{\max}^k - \gamma_{\min}^k) \|x^k - x^*\| \|F(x^k; \theta^k) - F(x^*; \theta^k)\|
\]

\[
\leq -2\gamma_{\max}^k \mu_x \|x^k - x^*\|^2 + 2(\gamma_{\max}^k - \gamma_{\min}^k) L_x \|x^k - x^*\|^2.
\]

Again by (A1), term 3 in (9) can also be similarly be bounded from above, again by employing the Cauchy-Schwartz inequality, Hölder’s inequality and the Lipschitz continuity of \(F(x; \theta)\), as we show in the next sequence of inequalities:

\[
-2 \sum_{i=1}^{N} \gamma_i^k (x_i^k - x_i^*)^T (F_i(x^k; \theta^k) - F_i(x^*; \theta^k)) \leq 2\gamma_{\max}^k \sum_{i=1}^{N} \|x_i^k - x_i^*\| \|F_i(x^k; \theta^k) - F_i(x^*; \theta^k)\|
\]

\[
\leq 2\gamma_{\max}^k \|x^k - x^*\| \|F(x^k; \theta^k) - F(x^*; \theta^k)\| \leq 2\gamma_{\max}^k L_\theta \|x^k - x^*\| \|\theta^k - \theta^*\|.
\]

Combining (7) with (9), (10), (12) and (13), we obtain

\[
\|x^{k+1} - x^*\|^2 \leq (1 - 2\gamma_{\max}^k \mu_x) \|x^k - x^*\|^2 + 2(\gamma_{\max}^k - \gamma_{\min}^k) L_x \|x^k - x^*\|^2 + 2(\gamma_{\max}^k)^2 L_x^2 \|x^k - x^*\|^2
\]

\[
+ 2(\gamma_{\max}^k)^2 L_\theta \|\theta^k - \theta^*\|^2 + 2\gamma_{\max}^k L_\theta \|x^k - x^*\| \|\theta^k - \theta^*\| + (\gamma_{\max}^k)^2 \|w^k\|^2
\]

\[
- 2 \sum_{i=1}^{N} \gamma_i^k (x_i^k - x_i^*)^T w_i^k + 2 \sum_{i=1}^{N} (\gamma_i^k)^2 (F_i(x^k; \theta^k) - F_i(x^*; \theta^*))^T w_i^k.
\]
The right hand side can be expressed as follows:

\[
\text{RHS} = (1 - 2\gamma_{\max}^k \mu_x + (2(\gamma_{\max}^k - \gamma_{\min}^k) L_x + 2(\gamma_{\max}^k)^2 L_x^2) \|x^k - x^*\|^2 \\
+ 2(\gamma_{\max}^k)^2 L_x^2 \|\theta^k - \theta^*\|^2 + (\gamma_{\max}^k)^2 \|w^k\|^2 + 2\gamma_{\max}^k \theta^k \|x^k - x^*\| \|\theta^k - \theta^*\| \\
- 2\sum_{i=1}^N \gamma_i^k (x^k_i - x^*_i)^T w_i^k + 2\sum_{i=1}^N (\gamma_i^k)^2 (F_i(x^k; \theta^k) - F_i(x^*; \theta^*))^T w_i^k.
\]

Recall that \(\Theta\) and \(K\) are bounded, implying that \(\|\theta^k - \theta^*\|^2 \leq m_{\theta}^2\), \(\|x^k - x^*\| \leq m_x\) and

\[
\|x^{k+1} - x^*\|^2 \leq (1 - 2\gamma_{\max}^k \mu_x + (2(\gamma_{\max}^k - \gamma_{\min}^k) L_x + 2(\gamma_{\max}^k)^2 L_x^2) \|x^k - x^*\|^2 \\
+ 2(\gamma_{\max}^k)^2 L_x^2 m_{\theta}^2 + (\gamma_{\max}^k)^2 \|w^k\|^2 + 2\gamma_{\max}^k \theta^k \|x^k - x^*\| + (\gamma_{\max}^k)^2 \nu^2.
\]

By taking conditional expectations and by recalling that \(\mathbb{E}[w^k \mid \mathcal{F}_k] = 0\) and \(\mathbb{E}[\|w^k\|^2 \mid \mathcal{F}_k] \leq \nu^2\), we obtain the following bound:

\[
\mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] \leq (1 - 2\gamma_{\max}^k \mu_x + (2(\gamma_{\max}^k - \gamma_{\min}^k) L_x + 2(\gamma_{\max}^k)^2 L_x^2) \mathbb{E}[\|x^k - x^*\|^2] \\
+ 2(\gamma_{\max}^k)^2 L_x^2 m_{\theta}^2 + 2\gamma_{\max}^k \theta^k \mathbb{E}[\|x^k - x^*\|] + (\gamma_{\max}^k)^2 \nu^2.
\]

Taking expectations once again, we obtain the following inequality:

\[
\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq (1 - 2\gamma_{\max}^k \mu_x + (2(\gamma_{\max}^k - \gamma_{\min}^k) L_x + 2(\gamma_{\max}^k)^2 L_x^2) \mathbb{E}[\|x^k - x^*\|^2] \\
+ 2(\gamma_{\max}^k)^2 L_x^2 m_{\theta}^2 + 2\gamma_{\max}^k \theta^k \mathbb{E}[\|x^k - x^*\|] + (\gamma_{\max}^k)^2 \nu^2.
\]

**Theorem 1:** Suppose (A1), (A2), (A3) and (A4) hold. Let \(\{x^k, \theta^k\}\) be computed via Algorithm I. Then, \(x^k \overset{L^2}{\longrightarrow} x^*\) and \(\theta^k \overset{P}{\longrightarrow} \theta^*\) as \(k \to \infty\).

**Proof:** From Lemma 4, we have that (6) holds.

\[
\mathbb{E}[\|x^{k+1} - x^*\|^2] \leq (1 - 2\gamma_{\max}^k \mu_x + (2(\gamma_{\max}^k - \gamma_{\min}^k) L_x + 2(\gamma_{\max}^k)^2 L_x^2) \mathbb{E}[\|x^k - x^*\|^2] \\
+ 2(\gamma_{\max}^k)^2 L_x^2 m_{\theta}^2 + 2\gamma_{\max}^k \theta^k \mathbb{E}[\|x^k - x^*\|] + (\gamma_{\max}^k)^2 \nu^2.
\]

First, we note that by assumption \(\theta_k \to \theta^*\) in probability (by Lemma 1) and \(\theta_k\) is a uniformly integrable sequence of random variables. It follows that \(\theta_k \to \theta^*\) in an \(L_1\) sense as \(k \to \infty\). To
guarantee the convergence of $\mathbb{E}[\|x^k - x^*\|^2]$ to zero, by Lemma 2, it suffices to prove that (a) $\alpha_k < 1$ for $k \geq K$, (b) $\sum_{k=1}^{\infty} (1 - \alpha_k) = \infty$, and (c) $\lim_{k \to \infty} \frac{\beta_k}{1 - \alpha_k} = 0$. We begin by noting that

$$\alpha_k = \left(1 - 2\gamma_k^k \mu_x + 2(\gamma_{\max}^k - \gamma_{\min}^k)L_x + 2(\gamma_{\max}^k)^2 L_x^2\right)$$

$$= \left(1 - 2\gamma_k^k \left(\mu_x - \frac{\gamma_{\max}^k - \gamma_{\min}^k}{\gamma_{\max}^k} L_x\right) + 2(\gamma_{\max}^k)^2 L_x^2\right).$$

By assumption, $\lim_{k \to \infty} \frac{(\gamma_k^k - \gamma_{\min}^k)}{\gamma_{\max}^k} = 0$, implying that for $k \geq K_1$, $\mu_x - \frac{(\gamma_k^k - \gamma_{\min}^k)}{\gamma_{\max}^k} L_x > \bar{\mu}_x$. As a consequence, for $k > K_1$, $\alpha_k < (1 - 2\bar{\mu}_x \gamma_{\max}^k + 2(\gamma_{\max}^k)^2 L_x^2) = (1 - 2\gamma_{\max}^k (\bar{\mu}_x - \gamma_{\max}^k L_x^2)).$

Furthermore, for $k > K_2$, we have that $\alpha_k < 1$. We now proceed to show that $\sum_{k=1}^{\infty} (1 - \alpha_k) = \infty$.

$$\sum_{k=1}^{\infty} (1 - \alpha_k) = \sum_{k=1}^{\infty} \left(2\gamma_{\max}^k \mu_x - 2(\gamma_{\max}^k - \gamma_{\min}^k)L_x - 2(\gamma_{\max}^k)^2 L_x^2\right)$$

$$= \sum_{k=1}^{\infty} \left(2\gamma_{\max}^k \left(\mu_x - \frac{\gamma_{\max}^k - \gamma_{\min}^k}{\gamma_{\max}^k} L_x\right) - 2(\gamma_{\max}^k)^2 L_x^2\right)$$

$$= \sum_{k=1}^{K_1-1} \left(\gamma_{\max}^k \left(2\mu_x - 2\left(\frac{\gamma_{\max}^k - \gamma_{\min}^k}{\gamma_{\max}^k}\right) L_x\right) - 2(\gamma_{\max}^k)^2 L_x^2\right)$$

$$+ \sum_{k=K_1}^{\infty} \left(\gamma_{\max}^k \left(2\mu_x - 2\left(\frac{\gamma_{\max}^k - \gamma_{\min}^k}{\gamma_{\max}^k}\right) L_x\right) - 2(\gamma_{\max}^k)^2 L_x^2\right)$$

$$= \sum_{k=1}^{K_1-1} \left(\gamma_{\max}^k \left(2\mu_x - 2\left(\frac{\gamma_{\max}^k - \gamma_{\min}^k}{\gamma_{\max}^k}\right) L_x\right) - 2(\gamma_{\max}^k)^2 L_x^2\right)$$

$$+ \sum_{k=K_1}^{\infty} \left(2\gamma_{\max}^k \bar{\mu}_x - 2(\gamma_{\max}^k)^2 L_x^2\right) = \infty,$$

where the final equality is a consequence of noting that $\sum_{k=K_1}^{\infty} \gamma_{\max}^k = \infty$. It remains to show (c):

$$\lim_{k \to \infty} \frac{\beta_k}{1 - \alpha_k} = \lim_{k \to \infty} \frac{2(\gamma_{\max}^k)^2 L_{\theta}^2 m_{\theta}^2 + 2\gamma_{\max}^k L_{\theta} m_{\theta} x \mathbb{E}[\|\theta^k - \theta^*\|]}{(2\gamma_{\max}^k \mu_x - 2(\gamma_{\max}^k - \gamma_{\min}^k)L_x - 2(\gamma_{\max}^k)^2 L_x^2)}$$

$$= \lim_{k \to \infty} \frac{2\gamma_{\max}^k L_{\theta}^2 m_{\theta}^2 + 2L_{\theta} m_{\theta} x \mathbb{E}[\|\theta^k - \theta^*\|]}{(2\mu_x - 2(\gamma_{\max}^k - \gamma_{\min}^k)L_x - 2(\gamma_{\max}^k)^2 L_x^2)}$$

$$= \lim_{k \to \infty} \left(2\gamma_{\max}^k L_{\theta}^2 m_{\theta}^2 + 2L_{\theta} m_{\theta} x \mathbb{E}[\|\theta^k - \theta^*\|]\right) = 0.$$

where the third equality follows from $\lim_{k \to \infty} \frac{\gamma_{\max}^k - \gamma_{\min}^k}{\gamma_{\max}^k} \to 0$ and $\lim_{k \to \infty} \gamma_{\max}^k = \infty$ and the last equality is a consequence of noting that $\mathbb{E}[\|\theta^k - \theta^*\|] \to 0$ as $k \to \infty$. By Lemma 2 and (14),
we may conclude that $E[\|x^k - x^*\|^2] \to 0$ or $x^k \overset{L^2}{\to} x^*$ as $k \to \infty$. Note that $E[\|\theta^k - \theta^*\|] \to 0$ as $k \to \infty$.

A natural concern is whether the rule that relates the steplengths can be implemented in a distributed fashion without coordination. We propose a rule, first suggested by [32], in which every agent chooses a positive integer and the required coordination statement holds. We view this as a protocol that may be employed for developing distributed schemes. The next result ensures that for such a choice, the required assumptions hold [32].

Lemma 5 (Choice of steplength sequences): Suppose for $i = 1, \ldots, N$, agent $i$ employs a steplength sequence, denoted by $\gamma_i^k$, which is defined as $\gamma_i^k = \frac{1}{(k+N_i)\alpha}$, where $N_i$ is a positive integer and $\frac{1}{2} < \alpha < 1$. Then, $\sum_{k=1}^{\infty} \gamma_i^k = \infty$, $\sum_{k=1}^{\infty} (\gamma_i^k)^2 < \infty$ and $\lim_{k \to \infty} \frac{\gamma_i^k - \gamma_{\min}^k}{\gamma_{\max}^k} = 0$.

Finally, we conclude this section with a constant steplength error bound.

Proposition 1 (Constant steplength error bound): Suppose (A1), (A2), (A3) and (A4) hold. Let $\gamma_i^k = \gamma$ for $i = 1, \ldots, N$ for all $k$ where $\gamma < \mu_x/L_x^2$. Let $\{x^k, \theta^k\}$ be computed via Algorithm I. Then, the following holds:

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq \frac{2\gamma L_x^2 m_\theta^2 + \gamma \nu^2}{2\mu_x - 2\gamma L_x^2}.$$

Proof: From Lemma 4, we have that (6) holds.

$$E[\|x^{k+1} - x^*\|^2] \leq (1 - 2\gamma_{\max}^k \mu_x + 2(\gamma_{\max}^k - \gamma_{\min}^k) L_x + 2(\gamma_{\max}^k)^2 L_x^2) E[\|x^k - x^*\|^2]$$

$$+ 2(\gamma_{\max}^k)^2 L_\theta^2 m_\theta^2 + 2\gamma_{\max}^k L_\theta m_x E[\|\theta^k - \theta^*\|] + (\gamma_{\max}^k)^2 \nu^2.$$

Let $\gamma_i^k = \gamma$ for $i = 1, \ldots, N$ for all $k$. Then, $\gamma_{\max}^k = \gamma_{\min}^k = \gamma$ for all $k$, which implies

$$E[\|x^{k+1} - x^*\|^2] \leq (1 - 2\gamma \mu_x + 2\gamma^2 L_x^2) E[\|x^k - x^*\|^2]$$

$$+ 2\gamma^2 L_\theta^2 m_\theta^2 + 2\gamma_L_\theta m_x E[\|\theta^k - \theta^*\|] + \gamma^2 \nu^2.$$

By taking the supremum limit on both sides, we get

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq (1 - 2\gamma \mu_x + 2\gamma^2 L_x^2) \limsup_{k \to \infty} E[\|x^k - x^*\|^2]$$

$$+ 2\gamma^2 L_\theta^2 m_\theta^2 + 2\gamma_L_\theta m_x \limsup_{k \to \infty} E[\|\theta^k - \theta^*\|] + \gamma^2 \nu^2,$$

or

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq \frac{2\gamma L_\theta^2 m_\theta^2 + 2\gamma_L_\theta m_x \limsup_{k \to \infty} E[\|\theta^k - \theta^*\|] + \gamma \nu^2}{2\mu_x - 2\gamma L_x^2}.$$

Note by proof of Theorem 1 that $E[\|\theta^k - \theta^*\|] \to 0$ as $k \to \infty$. Thus, we have

$$\limsup_{k \to \infty} E[\|x^k - x^*\|^2] \leq \frac{2\gamma L_\theta^2 m_\theta^2 + \gamma \nu^2}{2\mu_x - 2\gamma L_x^2}.$$
III. ITERATIVE FIXED-POINT SCHEMES FOR MISSPECIFIED NASH-COURNOT GAMES

In Section III-C, we show that the proposed iterative fixed-point scheme generates a sequence of iterates guaranteed to converge to the Nash-Cournot equilibrium in an a.s. sense. Additionally, the beliefs associated with the unknown parameter converge to their true counterparts in an a.s. sense. This result allows for noise-corrupted observations in price. Notably, when price observations are noise-free, this scheme learns the correct parameter in a single step, as shown in Section III-D. Finally, in Section III-E, we relax the affine assumption on prices.

A. Problem description

In this part of the paper, we consider a Nash-Cournot game wherein 

\[ f_i(x) \triangleq c_i(x_i) - p(X; \theta)x_i, \]

where \( x_{-i} \triangleq (x_j)_{i \neq j=1}^N, X \triangleq \sum_{i=1}^N x_i, x_i \) denotes the output of firm \( i \), \( c_i(\cdot) \) denotes firm \( i \)'s cost function, \( \theta^* \) denotes the true value of the parameters of the price function and \( K_i \) denotes the strategy set of firm \( i \). The price of the commodity, denoted by \( p(X; \theta^*) \), is defined as

\[ p(X; \theta^*) \triangleq \left( \theta_0^* - \sum_{i=1}^m \theta_i^* X^\alpha_i \right). \] (15)

In this section, we consider a special case with \( \theta^* \triangleq (a^*, b^*) \), which means that \( p(X; \theta^*) \triangleq a^* - b^* X \). We assume that \( \theta^* \) is not known to the firms, and consider a scheme in which firms play this game repeatedly, as in [22], [33], with the intent of learning the parameters by examining the discrepancies between estimated and actual prices. More specifically, we consider a regime where noisy prices may be observed. In particular, we consider two different settings:

**Case 1 (Learning \( a^* \)):** Here, firms have access to \( b^* \) but are unaware of \( a^* \). Therefore, if the \( i \)th firm harbors a belief on \( a^* \) denoted by \( a_i \) and estimates the aggregate output \( X \) by \( X_i \), then the \( i \)th firm’s price estimate is given by \( a_i - b^* X_i \). If \( \xi : \Lambda \rightarrow \mathbb{R} \) and \( (\Lambda, \mathcal{F}_\theta, \mathbb{P}_\theta) \) denotes the probability space, the true prices may be corrupted by additive noise and are given by

\[ p(X; \xi) \triangleq (a^* + \xi) - b^* X. \] (16)

**Case 2 (Learning \( b^* \)):** Here, firms have access to \( a^* \) but are unaware of \( b^* \). Analogously, if the \( i \)th firm harbors a belief on \( b^* \) denoted by \( b_i \) and estimates the aggregate output \( X \) by \( X_i \), then the \( i \)th firm’s price estimate is given by \( a^* - b_i X_i \). Furthermore, the true price may be corrupted by noise scaled by the aggregate output and is given by

\[ p(X; \xi) \triangleq a^* - (b^* + \xi)X. \] (17)
The next assumption formalizes these two cases.

**Assumption 5 (A5):** One of the following holds:

(A5a) Firms know $b^*$ but do not know $a^*$ and the price is defined by (16).

(A5b) Firms know $a^*$ but do not know $b^*$ and the price is defined by (17).

Furthermore, the random variable $\xi$ is defined by $\xi : \Lambda \rightarrow \mathbb{R}$, $(\Lambda, \mathcal{F}_\theta, \mathbb{P}_\theta)$ is the associated probability space and $\xi^1, \ldots, \xi^k$ are i.i.d. random variables for all $k$.

Under a common-knowledge assumption (Def. A6), every firm can construct estimates about its competitors’ strategies based on their current belief regarding the unknown parameter. Specifically, suppose $x^k_{ij}$ denotes the estimate of the $j$th firm’s strategy by the $i$th player at the $k$th step. Then, $x^k_{jj}$ denotes the $j$th firm’s true strategy and consequently, the true aggregate $X^k$ is given by $X^k \triangleq \sum_{j=1}^N x^k_{jj}$ for $k \geq 0$ and $i = 1, \ldots, N$. Then, the price observed at the $k$th step is given by

$$
P(X^k; \theta, \xi^k) = \begin{cases} 
(a^* + \xi^k) - b^*X^k, & \text{under (A5a),} \\
(a^* - (b^* + \xi^k)X^k, & \text{under (A5b).}
\end{cases}
$$

In an abuse of notation, we denote the parameter to be learnt by $\theta$; therefore, if $a$ is being learnt, then $\theta = a$. Furthermore, agent $i$ may use his estimate of the aggregate $X^k_i$ to compute $\vartheta^k_i$:

$$
\vartheta^k_i \triangleq \begin{cases} 
p(X^k; \theta, \xi^k) + b^*X^k_i, & \text{if } \theta = a, \\
(a^* - p(X^k; \theta, \xi^k))/X^k_i, & \text{if } \theta = b.
\end{cases}
$$

Note that $X^k_i$ is maintained as strictly positive by assuming that at least one of the strategy sets requires strictly positive output. Consequently, the average of $\vartheta^k_i$ after $k$ steps is given by

$$
\bar{\vartheta}_i^k = \frac{(k - 1)\bar{\vartheta}_i^{k-1} + \vartheta^k_i}{k}.
$$

Later in this section, we show that $\bar{\vartheta}_i^k$ is precisely the sample average of the set of random variables $\theta^* + \xi^1, \ldots, \theta^* + \xi^k$ after $k$ steps. Next, we describe our iterative fixed-point scheme for the simultaneous distributed learning of $\theta^*$ and $x^*$. Throughout this section, let $x_i = (x_{i1}, \cdots, x_{iN})$ for $i = 1, \cdots, N$.

Note that (Opt($z_{-i}$)) can be stated as a fixed point problem. More specifically, suppose firm $i$’s problem is compactly stated as follows:

$$
\min_{x_i \in K_i} f_i(x_i, x_{-i}^k),
$$

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where \( x^k_{-i} \) denotes the tuple associated with decisions of firm \( i \)’s rivals. Under convexity assumptions on \( K_i \) and \( f_i \) in \( x_i \) given \( x_{-i} \), \( x^*_i \) is a solution to the fixed-point problem:

\[
x_i = \Pi_{K_i} \left( x_i - \gamma \nabla_x f_i(x_i; x_{-i}) \right),
\]

(20)

where \( \gamma > 0 \). Note that in Algorithm II, the \( i \)th agent solves a fixed-point problem given by

\[
x_{ij}^{k+1} = \Pi_{K_j} \left( x_{ij}^{k+1} - \gamma_x \left( \nabla_{x_{ij}} f_j(x_i^{k+1}; \hat{\theta}_i^{k+1}) + \epsilon^{k+1}_{x_{ij}} \right) \right), \quad j = 1, \ldots, N,
\]

(21)

which provides its belief of the (regularized) Nash equilibrium, given its beliefs \( X^k_i \) and \( \hat{\theta}_i^{k+1} \).

Algorithm II: Iterative fixed-point and learning

Given a sequence \( \{\epsilon^k\} \downarrow 0 \), and \( \gamma_x, \gamma_\theta, k = 0; \sum_{j=1}^N x^0_{ij} = X^0; p(X^0; \theta, \xi^0) := a^* - b^* X^0, \epsilon^0 > 0; \bar{\theta}^0_i = 0 \) for \( i = 1, \ldots, N \).

Step 1. For \( i = 1, \ldots, N \)

\[
x_{ij}^{k+1} = \Pi_{K_j} \left( x_{ij}^{k+1} - \gamma_x \left( \nabla_{x_{ij}} f_j(x_i^{k+1}; \hat{\theta}_i^{k+1}) + \epsilon^{k+1}_{x_{ij}} \right) \right), \quad j = 1, \ldots, N,
\]

(Fixed point)

\[
\theta_i^{k+1} = \Pi_{\Theta} \left( \theta_i^k - \gamma_\theta \left( \hat{p}_i^k + \epsilon^k \theta_i^{k+1} \right) \right),
\]

(Learning)

where \( \bar{\theta}^k_i \) is given by (18), \( \hat{\theta}_i^{k+1} := \frac{1}{k+1} \hat{\theta}_i^k + \frac{k}{k+1} \bar{\theta}^k_i \), and

\[
\hat{p}_i^k := \begin{cases} 
p(X_k^{k+1}; \hat{\theta}_i^{k+1}, 0) - p(X_k^{k+1}; \theta, \xi^k), & \text{if } \theta = a, \\
p(X_k^k; \theta, \xi^k) - p(X_k^{k+1}; \hat{\theta}_i^{k+1}, 0), & \text{if } \theta = b.
\end{cases}
\]

Step 2. If \( k > K \), stop; else \( k := k + 1 \) and go to Step 1.

B. Assumptions and background

An assumption that is often employed in games is that of common knowledge, whereby firms are aware of the costs functions and strategy sets of their competitors (see [6]). Formally, this assumption is given by the following:

Assumption 6 (A6): The common knowledge assumption holds with regard to the cost functions \( c_i(x_i) \) and strategy sets \( K_i \) for \( i = 1, \ldots, N \). Furthermore, aggregate output is assumed to be unobservable.

The common knowledge assumption requires clarification before proceeding. As forwarded by [34], the notion of “common knowledge” extends beyond agents having access to information (often referred to as “mutual knowledge”); specifically, two agents are assumed to have common knowledge of an event, if both agents know the event, agent 1 knows that agent 2 knows it, agent 2 knows that agent 1 knows it, agent 1 knows that agent 2 knows that agent 1 knows it and so on. However, in many settings, the common knowledge assumption, partly encapsulated by (A6), cannot be expected to hold. In this section, we consider a special case where the price
\( p(X; \theta^*) \) is defined as
\[
p(X; a^*, b^*) \triangleq a^* - b^* X, \tag{22}
\]
where \( \theta^* \triangleq (a^*, b^*) \). We also assume that firms cannot observe aggregate output and firms employ a belief of aggregate output, relying on the knowledge of the cost functions and strategy sets of their competitors. Such a knowledge is assured through a common knowledge assumption. Collectively, these two assumptions are captured by (A6). Several motivating examples exist in the literature detailing common knowledge; these include instances provided by [35] (the barbecue problem) and [36] (the department store problem), amongst others. In a Cournot setting, a common knowledge assumption may be imposed when modeling bidding by generation firms in power markets. Cournot models have proved extremely popular in such regimes, as seen in the work by [16]. In such markets, firms are often mandated to provide details regarding their available capacities and heat-rates (cost functions).

Recall that our goal lies in developing schemes for learning equilibria and misspecified parameters. This is effected by having each firm maintain a belief of the aggregate which is updated after every step. Using this belief, a firm may determine its output and the estimated output of its competitors, contingent on its belief of the aggregate. Simultaneously, every firm updates its belief regarding the misspecified parameter by employing the discrepancy between observed and estimated prices, the latter being computable by using the belief of the aggregate. This model, while aligned, with that suggested by [22], [33] enjoys distinctions at several levels; specifically, we allow for constrained problems with nonlinear cost functions with noisy price observations arising from possibly nonlinear price functions.

We now make the following convexity and continuity assumptions on the cost functions, which can be seen as a special case of (A1).

**Assumption 7 (A7):** Suppose the cost function \( c_i(x_i) \) is a convex, continuously differentiable function in \( x_i \). Suppose \( c_i'(x_i) \) is Lipschitz continuous on \( K_i \) with constant \( M_i \). Furthermore, suppose \( \Theta \) is a closed, convex, and bounded set and for \( i = 1, \ldots, N \), \( K_i \) is a nonempty, closed, convex and bounded set.

The strong monotonicity and Lipschitz continuity of the mapping \( F(x) \) can be easily shown under (A7).

**Lemma 6:** Consider the mapping \( F(x) \) defined by (1) and suppose (A7) holds. Then \( F(x) \) is
a strongly monotone Lipschitz continuous mapping.

This allows for claiming the existence and uniqueness of a Nash-Cournot equilibrium when the price function is affine.

**Proposition 2:** Consider a Nash-Cournot game in which the $i$th player solves $(\text{Opt}(x_{-i}))$ and the price is determined by (22). Furthermore, suppose (A7) holds. Then, the associated Nash-Cournot game admits a unique equilibrium.

**Proof:** From Lemma 6, it follows that the associated variational inequality $\text{VI}(K,F)$ has a strongly monotone mapping $F(x)$ over $K$. As a consequence, $\text{VI}(K,F)$ admits a unique solution [37].

If the strong monotonicity and Lipschitz continuity constants are denoted by $\eta$ and $L$ respectively, then this allows for developing a simple distributed learning scheme of the form (cf. [37]):

$$x^{k+1}_i = \Pi_K \left( x^k_i - \gamma \nabla_{x_i} f_i(x^k_i; \theta^*) \right), \quad i = 1, \ldots, N. \quad (23)$$

where $\gamma < 2\eta/L^2$ and $\Pi_K(y)$ denotes the projection of $y$ on $K$. Then the sequence $\{x^k\}$ produced by (23) converges to the unique solution of the $\text{VI}(K,F)$.

A select number of results will rely on boundedness of strategy sets, as specified by (A8).

**Assumption 8 (A8):** Suppose the estimator set $\Theta$ is a compact convex set in $\mathbb{R}_+$ given by $[\delta, \Delta]$ and $0 < \delta < \theta^* + \xi_k < \Delta$ for all $k$. Furthermore, suppose the sets $K_1, \ldots, K_N$ are bounded.

**C. Analysis of noise-corrupted iterative fixed-point schemes**

**Summary of scheme:** Before commencing with the analysis of Algorithm II, we briefly summarize the algorithm. At each iteration, firm $i$ computes its output and the estimated output of the competing firms (since it has access to all the cost information) using its belief of the parameter of the price function. Simultaneously, firm $i$ updates its belief regarding the price function parameter by using observational information. The terms with $\epsilon_k$ denote “regularization terms” that ensure the uniqueness of the associated fixed-point problems.

We begin the analysis of Algorithm II by noting that, given $p(X^k; \theta, \xi^k)$ and $\{\bar{\theta}^k_i\}_{i=1}^N$, the fixed-point problem (Fixed point) can be decomposed into $N$ subproblems. If $z_i$ is defined as $z_i = (x_{i1}, \ldots, x_{iN}, \theta_i)^T$, then the $i$th agent solves the following problem:

$$z^{k+1}_i = \Pi_K \left( z^k_i - \gamma \left( F^k(z^k_i; p(X^k; \theta, \xi^k), \bar{\theta}^k_i) + \epsilon^k z^{k+1}_i \right) \right), \quad (24)$$
where $\hat{K} \triangleq \prod_{j=1}^{N} K_j \times \Theta$, and $F^k(z_i; p, \bar{\theta}_i)$ is defined as
\[
F^k(z_i; p, \bar{\theta}_i) = \left( \nabla_{x_1} f_1(x_i; \bar{\theta}_i), \ldots, \nabla_{x_N} f_N(x_i; \bar{\theta}_i), g(X_i; \hat{\theta}_i, p) \right)^T,
\] (25)
where $\hat{\theta}_i = \frac{1}{k+1} \theta_i + \frac{k}{k+1} \bar{\theta}_i$ and $z_i = (x_i, \theta_i) = (x_{i1}, \ldots, x_{iN}, \theta_i)^T$. By analyzing the properties of $F^k$, the uniqueness of the solution to the fixed-point problem may be guaranteed.

**Proposition 3:** Suppose (A5), (A6) and (A7) hold. If $k \geq 0$ and $\epsilon^k > 0$, then the following hold:

(a) If $\theta = a$, then given $p(X^k; \theta, \xi^k)$ and $\{\bar{\xi}_i^k\}_{i=1}^{N}$, the solution to (Fixed point) is a singleton.
(b) If $\theta = b$, then given $p(X^k; \theta, \xi^k)$ and $\{\bar{\xi}_i^k\}_{i=1}^{N}$, the solution to (Fixed point) is a singleton.

Having shown that the fixed-point problem (or its regularization) leads to a unique solution, we proceed to show a Lipschitzian property on the solution set of (24) with respect to the parameter $\theta$. This proof is inspired by a related result presented by [38].

**Proposition 4:** Consider a VI$(K, F(.; \theta))$ where $F(x; \theta)$ is strongly monotone in $x$ over $K$ for all $\theta \in \Theta$, Lipschitz continuous in $x$ for all $\theta \in \Theta$ and Lipschitz continuous in $\theta$ for all $x \in K$. Then, the following hold:

(a) If $x(\theta)$ denotes the solution of VI$(K, F(.; \theta))$, then $x(\theta)$ is Lipschitz continuous in $\theta$ for all $\theta \in \Theta$.
(b) Given an $\epsilon > 0$, if $x(\theta, \epsilon)$ denotes the solution of VI$(K, F(.; \theta) + \epsilon I)$, then $x(\theta, \epsilon)$ is Lipschitz continuous in $\theta$ and $\epsilon$.

We are now prepared to show that the iterative fixed-point scheme produces a sequence of iterates that converge almost surely to the true Nash-Cournot equilibrium and allow for learning the true parameter.

**Theorem 2 (Global convergence of iterative fixed-point scheme):** Suppose (A5), (A6), (A7) and (A8) hold. Let $\{x_i^k, \hat{\theta}_i^k\}$ be computed via Algorithm II for $i = 1, \ldots, N$. Then $\hat{\theta}_i^k \to \theta^*$ almost surely for $i = 1, \ldots, N$ and $x_i^k \to x^*$ almost surely for $i = 1, \ldots, N$, where $x^*$ is a solution of the variational inequality (2).

**Proof:** Suppose $k \geq 0$. At the $k$th iteration, $\hat{p}_i^k$ is a function of $\hat{\theta}_i^{k+1}$, which is a function of $\hat{\theta}_i^k$. Consequently, the fixed-point problem (Fixed point) is a function of $\hat{\theta}_i^k$; at the outset, $\bar{\theta}_i^0$ is zero for $i = 1, \ldots, N$ and every agent is faced by (Fixed point) with the same parametrization. Since (Fixed point) has a unique solution (Prop. 3), it follows that $x_i^* = x_j^*$ for $i \neq j$ and
Therefore, Given $p(X^k; \theta, \xi^k)$ and $\{\hat{\vartheta}_i^k\}_{i=1}^N$, the solution $(x_i^{k+1}, \theta_i^{k+1})$ to (Fixed point) satisfies $x_i^{k+1} = x_j^{k+1}$ for all $i, j$. Thus, for all $k \geq 0$ and all $i$, we have that

$$p(X^k; \theta, \xi^k) = \begin{cases} (a^* + \xi^k) - b^* \sum_{j=1}^{N} x_j^k = (a^* + \xi^k) - b^* \sum_{j=1}^{N} x_{ij}, & \text{if } \theta = a, \\ a^* - (b^* + \xi^k) \sum_{j=1}^{N} x_j^k = a^* - (b^* + \xi^k) \sum_{j=1}^{N} x_{ij}, & \text{if } \theta = b. \end{cases}$$

Since for all $k \geq 0$ and all $i$,

$$\hat{\vartheta}_i^k = \begin{cases} p(X^k; \theta, \xi^k) + b^* X_i^k, & \text{if } \theta = a, \\ (a^* - p(X^k; \theta, \xi^k))/X_i^k, & \text{if } \theta = b. \end{cases}$$

we have $\hat{\vartheta}_i^k = \theta^* + \xi^k$ for all $i$. As a result, after $k$ iterative fixed-point steps, we obtain $k$ samples $\{\theta^* + \xi^1, \ldots, \theta^* + \xi^k\}$ of the estimated parameter. Since for all $k \geq 0$ and all $i$,

$$\bar{\vartheta}_i^k = (k-1)\bar{\vartheta}_i^{k-1} + \hat{\vartheta}_i^k/k,$$

the sample mean of the estimated parameter is given by $\bar{\vartheta}_i^k$, i.e.,

$$\bar{\vartheta}_i^k = \frac{\sum_{l=1}^{k} (\theta^* + \xi^l)}{k}.$$

Therefore, $\bar{\vartheta}_i^k \to \theta^*$ a.s. as $k \to \infty$, which implies by the boundedness of $\{\theta_i^k\}$ that for all $i$

$$\hat{\theta}_i^{k+1} = \frac{1}{k+1} \theta_i^{k+1} + \frac{k}{k+1} \bar{\vartheta}_i^k \to \theta^* \quad \text{a.s. as } k \to \infty,$$

by the strong law of large numbers. By Proposition 4, $x_i^{k+1} = x_i^{k+1}(\hat{\theta}_i^{k+1}, \epsilon^k)$ is a continuous function of $(\hat{\theta}_i^{k+1}, \epsilon^k)$, and $x_i^{k+1}(\theta^*, 0) = x^*$. Therefore, $x_i^{k+1} \to x^*$ a.s. as $k \to \infty$.

### D. Noise-free prices and nonlinear price functions

In this section, we specialize the result of the previous subsection to a regime where accurate (noise-free) price observations are available; specifically, if $\xi^k = 0$ for all $k$. As a consequence, price at the $k$th step is given by

$$p(X^k; \theta) \triangleq a^* - b^* X^k,$$  \hspace{1cm} (26)

Based on this specification of price, agent $i$ uses its estimated aggregate output $X_i$ to calculate $\theta^*$, and is given by

$$\vartheta_i^k = \begin{cases} p(X^k; \theta) + b^* X_i^k, & \text{if } \theta = a, \\ (a^* - p(X^k; \theta))/X_i^k, & \text{if } \theta = b. \end{cases}$$  \hspace{1cm} (27)
We begin the analysis of Algorithm II by noting that, given \( p(X^k; \theta, 0) \), the fixed-point problem (Fixed point) can be decomposed into \( N \) subproblems, in which the \( i \)th agent solves the following problem,

\[
z_i^{k+1} = \Pi_{\hat{K}} \left( z_i^{k+1} - \gamma \left( F^k(z_i^{k+1}; p(X^k; \theta, 0)) + \epsilon z_i^{k+1} \right) \right),
\]

where \( \hat{K} \equiv \prod_{j=1}^{N} K_j \times \Theta \), and \( F^k(z_i; p) \) is defined as

\[
F^k(z_i; p^k) = (\nabla_{x_i} f_1(x_i; \theta_i), \ldots, \nabla_{x_i} f_N(x_i; \theta_i), \tilde{p}_i(p^k))^T,
\]

where \( z_i = (x_i, \theta_i) = (x_{i1}, \ldots, x_{iN}, \theta_i)^T \). By analyzing the properties of \( F^k \), the uniqueness of the solution to the fixed-point problem may be guaranteed.

**Lemma 7:** Suppose (A5), (A6) and (A7) hold. If \( k \geq 0 \) and \( \epsilon^k < 0 \), then the following hold:

(a) Suppose \( \theta = a \). Then given \( p(X^k; \theta, 0) \), the solution to (Fixed point) is a singleton.

(b) Suppose \( \theta = b \). Then given \( p(X^k; \theta, 0) \), the solution to (Fixed point) is a singleton.

**Proof:** This follows from Prop. 3 when \( \xi^k = 0 \) for all \( k \).

Since this scheme is a special instance of that proposed in the previous section, almost-sure convergence follows directly. But, much more can be said about the sequence of iterates in such regimes. In particular, we show that after a **single** fixed-point step, the agents in Algorithm II learn the correct parameter while after an additional step, under the common knowledge assumption, every firm learns the true Nash-Cournot equilibrium.

**Theorem 3 (Finite termination of iterative fixed-point scheme):** Suppose (A5), (A6), (A7) and (A8) hold. Let \( \{x_i^1, \hat{\theta}_i^1\} \) be computed via Algorithm II for \( i = 1, \ldots, N \). Then \( \hat{\theta}_i^1 = \theta^* \) for \( i = 1, \ldots, N \) and \( x_i^2 = x^* \) for \( i = 1, \ldots, N \), where \( x^* \) is a solution of the variational inequality (2).

**Proof:** Suppose \( k = 0 \). Given \( p(X^0; \theta) \), the solution \( (x_i^1, \theta_i^1) \) to (Fixed point) satisfies \( x_{ij}^1 = x_{jj}^1 \) for all \( i, j \), since the solution to (Fixed point) is unique by Lemma 7. Thus, for \( k = 1 \) and all \( i \), we have that

\[
p(X^k; \theta) = a^* - b^* \sum_{j=1}^{N} x_{jj}^k = a^* - b^* \sum_{j=1}^{N} x_{ij}^k.
\]

Since for \( k = 1 \) and all \( i \),

\[
\hat{\theta}_i = \begin{cases} 
p(X^k; \theta) + b^* \sum_{j=1}^{N} x_{ij}^k, & \text{if } \theta = a, \\
(a^* - p(X^k; \theta))/\sum_{j=1}^{N} x_{ij}^k, & \text{if } \theta = b,
\end{cases}
\]
we have $\hat{\theta}_i = \hat{\theta}_i^k = \theta^*$ for $i = 1, \ldots, N$. Then, it suffices for each agent to solve the strongly monotone variational inequality problem corresponding to the following fixed-point problem:

$$x_{ij} = \Pi_{K_j} \left( x_{ij} - \gamma \left( c_j'(x_{ij}) + b^* x_{ij} + b^* \sum_{j=1}^{N} x_{ij} - a^* \right) \right), \quad \forall i, j = 1, \ldots, N.$$  \hspace{1cm} (30)

But this is a solution to the original Nash-Cournot equilibrium problem.

E. Extensions to nonlinear price functions

While our discussion has been restricted to affine price functions up to this point, we now consider a generalization of the following nature:

$$p(X; \theta^*, \xi) \triangleq \begin{cases} a^* - b^* X^\sigma + \xi, \\ a^* - (b^* + \xi) X^\sigma. \end{cases}$$  \hspace{1cm} (31)

This nonlinear price function has been examined by [32] where a discussion of the strict monotonicity of the associated mapping is presented (Lemma 8(a)). Specifically, the equilibrium of the Nash-Cournot game are captured by $\text{VI}(K, F)$ where $F(x)$ is defined as

$$F(x) \triangleq \begin{pmatrix} c_1'(x_1) - (a^* - b^* X^\sigma) + \sigma b^* X^{\sigma-1} x_1 \\ \vdots \\ c_N'(x_N) - (a^* - b^* X^\sigma) + \sigma b^* X^{\sigma-1} x_N \end{pmatrix}.$$  \hspace{1cm} (32)

The mapping $F(x)$ is strongly monotone for all $x \in K$ if $\nabla F(x)$ is a diagonally dominant matrix for all $x \in K$. In effect, this requires ascertaining the conditions under which the following matrix is positive definite:

**Lemma 8:** Consider the mapping $F(x)$ defined in (32). Suppose (A7) holds, $N < \frac{3\sigma - 1}{\sigma - 1}$ and $\sigma > 1$. Then the following hold:

(a) $F(x)$ is a strictly monotone mapping over $K$;

(b) Suppose $X \geq \eta$ for some $\eta > 0$, then $F(x)$ is a strongly monotone mapping over $K$.

Directly deriving a Lipschitzian statement on $F(x; \theta)$ in terms of $\theta$ is not easy when the price function has the prescribed nonlinear form; instead, by noting that $\nabla F(x)$ is bounded when $x$ is bounded, allows for proving such a statement. Next, we provide a corollary of Proposition 4 where such a property is derived.

**Corollary 1:** Consider a $\text{VI}(K, F(: \theta))$ where $F(x; \theta)$ is strongly monotone in $x$ over $K$ for all $\theta \in \Theta$, and Lipschitz continuous in $\theta$ for all $x \in K$. Also, there is a constant $R > 0$, such that $\|\nabla F(x; \theta)\| \leq R$ for all $x \in K$ and $\theta \in \Theta$. Then, the following hold:
(a) If \( x(\theta) \) denotes the solution of \( \text{VI}(K,F(\cdot;\theta)) \), then \( x(\theta) \) is Lipschitz continuous in \( \theta \) for all \( \theta \in \Theta \).

(b) Given an \( \epsilon > 0 \), if \( x(\theta,\epsilon) \) denotes the solution of \( \text{VI}(K,F(\cdot;\theta) + \epsilon I) \), then \( x(\theta,\epsilon) \) is Lipschitz continuous in \( \theta \) and \( \epsilon \).

**Proof:** By Proposition 4, it suffices to show that \( F(x;\theta) \) is Lipschitz continuous in \( x \) for all \( \theta \in \Theta \). For \( \theta \in \Theta \), and \( x,y \in K \), we have that

\[
\|F(x;\theta) - F(y;\theta)\| = \left\| \int_0^1 \nabla F(y + \alpha(x - y);\theta)(x - y)d\alpha \right\| \leq \int_0^1 \|\nabla F(y + \alpha(x - y);\theta)\| \|x - y\|d\alpha \leq \int_0^1 r\|x - y\|d\alpha = r\|x - y\|,
\]

which implies the Lipschitz continuity in \( x \) of the mapping \( F \). 

**Proposition 5:** Let \( \theta = a \). Consider the mapping \( F(x) \) defined in (32) and suppose (A7) and (A8) hold. Suppose \( X \geq \eta \) for some \( \eta > 0 \) and all \( x \in K \), where \( X = \sum_{i=1}^N x_i \). If \( N < \frac{3\sigma - 1}{\sigma - 1} \) and \( \sigma > 1 \), then the following hold:

(a) If \( x(\theta) \) denotes the solution of \( \text{VI}(K,F(\cdot;\theta)) \), then \( x(\theta) \) is Lipschitz continuous in \( \theta \) for all \( \theta \in \Theta \).

(b) Given an \( \epsilon > 0 \), if \( x(\theta,\epsilon) \) denotes the solution of \( \text{VI}(K,F(\cdot;\theta) + \epsilon I) \), then \( x(\theta,\epsilon) \) is Lipschitz continuous in \( \theta \) and \( \epsilon \).

**Proof:** By Lemma 8, \( F(x;\theta) \) is a strongly monotone mapping over \( K \) for all \( \theta \in \Theta \). By definition of \( F \), \( F(x;\theta) \) is Lipschitz continuous in \( \theta \) for all \( x \in K \). By definition of \( \nabla F \) and boundedness of \( x \in K \), \( \nabla F(x;\theta) \) is bounded for \( x \in K \) and \( \theta \in \Theta \). Then, the conclusion follows from Corollary 1.

We may now show that the fixed-point problem yields a unique solution.

**Proposition 6:** Suppose (A6) and (A7) hold. Let the price be given by (31). If \( N < \frac{3\sigma - 1}{\sigma - 1} \) and \( \sigma > 1 \), then given \( p^k(\xi^k) \) and \( \{\hat{\theta}^k\}_{i=1}^N \), the solution to (Fixed point) is a singleton.

By leveraging Propositions 5 and 6, the convergence of the iterative fixed-point scheme can be claimed under the caveat that the aggregate output is always bounded away from zero, as stated by the next result. Note that the proof follows in a fashion similar to Theorem 2 and is omitted.

**Theorem 4:** Suppose (A6), (A7) and (A8) hold. Suppose \( X \geq \eta \) for some \( \eta > 0 \) and all \( x \in K \), where \( X = \sum_{i=1}^N x_i \). Let \( \{x^k_i, \hat{\theta}^k_i\} \) be computed via Algorithm II for \( i = 1, \ldots, N \). Suppose a
unique solution to the fixed-point problem (Fixed point) can be obtained, given \( p_k(x^k) \) and \( \{\bar{\theta}_i^k\}_{i=1}^N \) for each \( k \geq 0 \). Then, \( \hat{\theta}_i^k \to \theta^* \) almost surely for \( i = 1, \ldots, N \) and \( x_i^k \to x^* \) almost surely for \( i = 1, \ldots, N \), where \( x^* \) is a solution of the variational inequality (2).

We conclude this section with an observation. If one used a more widely used estimation technique such as a least-squares estimation then it remains unclear if almost-sure convergence statements can always be claimed since least-squares estimators generally converge in a weaker-sense while stronger statements may be available for linear regression (see [39]). In effect, a scheme that combines a least-squares estimation technique with a strategy update, while convergent, may not possess desirable almost-sure convergence properties. While, we examine nonlinear Nash-Cournot games in this section, we also show that such claims hold for more general aggregative Nash games. However, it should be emphasized that extending this avenue to Nash games where the associated variational map is non-monotone may lead to challenges. In particular, what are perfectly reasonable schemes for a subclass of Nash games may not be supported by similar asymptotic guarantees when the structural properties of the problem do not satisfy some key requirements.

**IV. PROOFS OF SUPPORTING STATEMENTS**

*Proof of Lemma 3:* By employing the non-expansivity of the Euclidean projector, (5) may be expressed as follows:

\[
\|\Pi_K(x - \gamma H(x)) - \Pi_K(y - \gamma H(y))\|^2 \leq \|(x - \gamma H(x)) - (y - \gamma H(y))\|^2
\]

\[
= \|(x - y) - \gamma(H(x) - H(y))\|^2 = \|x - y\|^2 - 2\gamma(x - y)^T(H(x) - H(y)) + \gamma^2\|H(x) - H(y)\|^2.
\]

The result follows by invoking the strong monotonicity and Lipschitz continuity of \( H \) over \( K \):

\[
\|x - y\|^2 - 2\gamma(x - y)^T(H(x) - H(y)) + \gamma^2\|H(x) - H(y)\|^2 \leq (1 - 2\mu\gamma + \gamma^2L^2)\|x - y\|^2.
\]

*Proof of Lemma 6:* Let \( g(x) = (c_1'(x_1), \ldots, c_N'(x_N))^T \) and \( e = (1, \ldots, 1)^T \). Then, we have \( F(x) = g(x) + b^*(x + Xe) - a^*e \), where \( X = \sum_{i=1}^N x_i \). Note that \( g(x) \) is monotone in \( x \). Thus, we have for \( x, y \in K \)

\[
(F(x) - F(y))^T(x - y) = (g(x) - g(y))^T(x - y) + b^*(x - y)^T(x - y) + b^*(X - Y)e^T(x - y)
\]

\[
\geq b^*(x - y)^T(x - y) + b^*(X - Y)^T(X - Y) \geq b^*\|x - y\|^2.
\]
This implies that \( F(x) \) is strongly monotone in \( x \) with constant \( b^* \). Note that \( g(x) \) is Lipschitz continuous on \( K \) with constant \( M \), where \( M \triangleq \max_i \{M_i\} \). The Lipschitz continuity of \( F(x) \) is easily shown:

\[
\|F(x) - F(y)\| = \|g(x) - g(y)\| + b^*\|x - y\| + b^*\|(X - Y)e\|
\leq M\|x - y\| + b^*\|x - y\| + b^*\|ee^T\|\|x - y\| = L\|x - y\|,
\]

where \( L = M + b^* + b^*\|ee^T\| \). It follows that \( F(x) \) is Lipschitz continuous with constant \( L \).

**Proof of Proposition 3:** It suffices to show that given \( p(X^k; \theta, \xi^k) \) and \( \{\tilde{\theta}_i^k\}_{i=1}^N \), the fixed-point problem (24) has a unique solution for each \( i \). Given \( p, \tilde{\theta}_i, i \) and \( k \), let \( F(z_i) = F^k(z_i; p, \tilde{\theta}_i) \) and let \( H(z_i) \) denote the Jacobian matrix \( \nabla F(z_i) \) of \( F \) at \( z_i \in \tilde{K} \). We will proceed to show that \( H(z_i) \) is a \( \mathbf{P} \)-matrix for all \( z_i \in \tilde{K} \) in part (a) and a \( \mathbf{P}_0 \)-matrix for all \( z_i \in \tilde{K} \) in part (b) where \( \tilde{K} \subset \tilde{K} \) and \( \tilde{K} \) is a rectangle. Then, by invoking Proposition 3.5.9 in [37], the associated mapping \( F \) is \( \mathbf{P} \)-mapping on \( \tilde{K} \) in part (a) and a \( \mathbf{P}_0 \)-mapping on \( \tilde{K} \) in part (b). Consequently, by Theorem 3.5.15 in [37], the regularized fixed-point problem (24) has a unique solution in both parts (a) and (b). Specifically, we employ a rectangular \( \tilde{K} \) defined as

\[
\tilde{K} \triangleq [0, \infty)^N \times \Theta,
\]

where \( \Theta \) is a compact set in \( (0, \infty) \). (a) Given \( z_i \in \tilde{K} \), let \( H_i \) denote \( H(z_i) \). Then,

\[
H_i = \begin{pmatrix} A_i & B \\ C & D \end{pmatrix}, \tag{33}
\]

where \( A_i = b^*(I + ee^T) + E_i \), \( B = -\frac{1}{k+1}e \), \( C = -b^*e^T \), \( D = \frac{1}{k+1} \), \( e \) denotes the column of ones in \( \mathbb{R}^N \), \( E_i \) is an \( N \times N \) diagonal matrix with \( e_{ij}^* \) as its \( j \)th diagonal entry. Since, the nonnegativity of \( e_{ij}^* \) follows from the convexity of costs, \( E_i \) is a nonnegative diagonal matrix and is therefore positive semidefinite. Recall that the sum of a diagonal positive semidefinite matrix and a \( \mathbf{P} \)-matrix is a \( \mathbf{P} \)-matrix and it suffices to show that \( H_i \) is a \( \mathbf{P} \)-matrix when \( E_i = 0 \). This amounts to showing that the principal minors of \( H \) are positive.

Since \( A_i \) and \( D \) are \( \mathbf{P} \)-matrices, we only consider the principal submatrix \( H_\alpha \) of \( H_i \), where \( \alpha \subseteq \{1, \ldots, N\} \) is a nonempty index set, and \( H_\alpha \) is given by

\[
H_\alpha \triangleq \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D \end{pmatrix},
\]
where \( A_\alpha = b^*(I_{n_\alpha} + e^{n_\alpha}(e^{n_\alpha})^T), \) \( B_\alpha = -\frac{1}{k+1}e^{n_\alpha}, \) \( C_\alpha = -b^*(e^{n_\alpha})^T, \) and \( I_{n_\alpha} \) and \( e^{n_\alpha} \) denote the identity matrix and the column of ones in \( \mathbb{R}^{n_\alpha \times n_\alpha} \) and \( \mathbb{R}^{n_\alpha}, \) respectively, with \( n_\alpha = |\alpha|. \) Since \( A_\alpha^{-1} = \frac{1}{b^*}(I_{n_\alpha} - \frac{1}{n_\alpha+1}e^{n_\alpha}(e^{n_\alpha})^T), \) we have

\[
C_\alpha A_\alpha^{-1} B_\alpha = \frac{1}{k+1} (e^{n_\alpha})^T \left( I_{n_\alpha} - \frac{1}{n_\alpha+1} e^{n_\alpha}(e^{n_\alpha})^T \right) e^{n_\alpha} = \frac{1}{k+1} \left( n_\alpha - \frac{n_\alpha^2}{n_\alpha+1} \right) = \frac{1}{k+1} \left( \frac{n_\alpha}{n_\alpha+1} \right).
\]

It follows that \( D - C_\alpha A_\alpha^{-1} B_\alpha = \frac{1}{k+1} - \frac{1}{k+1} \left( \frac{n_\alpha}{n_\alpha+1} \right) = \frac{1}{k+1} \left( \frac{1}{n_\alpha+1} \right) > 0. \) Since \( \det(A_\alpha) > 0, \) we have \( \det(H_\alpha) = \det(A_\alpha) \det(D - C_\alpha A_\alpha^{-1} B_\alpha) > 0 \) for all \( \alpha \subseteq \{1, \ldots, N\} \) with \( \alpha \neq \emptyset. \) Therefore, \( H \) is a \( P \)-matrix.

(b) Analogous to our approach for (a), we consider a matrix \( H_i, \) given by \( H_i = \nabla F(z_i). \) Then,

\[
H_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix},
\]

where \( A_i = \hat{b}_i(I + ee^T) + E_i, \) \( B_i = \frac{1}{k+1}(x_i + (e^T x_i)e), \) \( C_i = \hat{b}_i e^T, \) and \( D_i = \frac{1}{k+1}(e^T x_i), \) where \( \hat{b}_i = \frac{1}{k+1}b_i + \frac{k}{k+1}\bar{b}_i, \) \( x_i = (x_{i1}, \ldots, x_{iN})^T, \) \( e \) denotes the column of ones in \( \mathbb{R}^N, \) and \( E_i \) is an \( N \times N \) diagonal matrix with \( c''_j(x_{ij}) \) as its \( j \)th diagonal entry. Recall that the sum of a diagonal positive semidefinite matrix and a \( P_0 \)-matrix is a \( P_0 \)-matrix. As in (a), it suffices to show that \( H \) is a \( P_0 \)-matrix when \( E_i = 0. \)

Since \( A_i \) and \( D_i \) are \( P_0 \)-matrices, we restrict our attention to the principal submatrix \( H_\alpha \) of \( H_i, \) where \( \alpha \subseteq \{1, \ldots, N\} \) is a nonempty index set, and \( H_\alpha \) is given by

\[
H_\alpha \triangleq \begin{pmatrix} A_\alpha & B_\alpha \\ C_\alpha & D_\alpha \end{pmatrix},
\]

where \( A_\alpha = \hat{b}_i(I_{n_\alpha} + e^{n_\alpha}(e^{n_\alpha})^T), \) \( B_\alpha = \frac{1}{k+1}(x_\alpha + (e^T x_\alpha)e^{n_\alpha}), \) \( C_\alpha = \hat{b}_i(e^{n_\alpha})^T, \) and \( I_{n_\alpha} \) and \( e^{n_\alpha} \) denote the identity matrix and the column of ones in \( \mathbb{R}^{n_\alpha \times n_\alpha} \) and \( \mathbb{R}^{n_\alpha}, \) respectively, with \( n_\alpha = |\alpha|. \) Then, the following hold:

1. If \( \hat{b}_i = 0, \) then \( A_\alpha = 0 \) and \( C_\alpha = 0, \) which implies \( \det(H_\alpha) = 0. \)
(2) If \( \hat{b}_i > 0 \), then \( A_{\alpha}^{-1} = \frac{1}{b_i}(I_{n_{\alpha}} - \frac{1}{n_{\alpha}+1}e^{n_{\alpha}}(e^{n_{\alpha}})^T) \). So, we have
\[
C_{\alpha}A_{\alpha}^{-1}B_{\alpha} = \frac{1}{k+1}(e^{n_{\alpha}})^T(I_{n_{\alpha}} - \frac{1}{n_{\alpha}+1}e^{n_{\alpha}}(e^{n_{\alpha}})^T)(x_{\alpha} + (e^Tx_i)e^{n_{\alpha}})
\]
\[
= \frac{1}{k+1}((e^{n_{\alpha}})^T - \frac{n_{\alpha}}{n_{\alpha}+1}(e^{n_{\alpha}})^T)(x_{\alpha} + (e^Tx_i)e^{n_{\alpha}})
\]
\[
= \frac{1}{(k+1)(n_{\alpha}+1)}((e^{n_{\alpha}})^Tx_{\alpha} + n_{\alpha}(e^Tx_i)) .
\]

It follows that
\[
D_i - C_{\alpha}A_{\alpha}^{-1}B_{\alpha} = \frac{1}{k+1}e^Tx_i - \frac{1}{(k+1)(n_{\alpha}+1)}((e^{n_{\alpha}})^Tx_{\alpha} + n_{\alpha}(e^Tx_i))
\]
\[
= \frac{1}{(k+1)(n_{\alpha}+1)}(e^Tx_i - (e^{n_{\alpha}})^Tx_{\alpha}) \geq 0.
\]

Since \( \det(A_{\alpha}) > 0 \), we have \( \det(H_{\alpha}) = \det(A_{\alpha})\det(D_i - C_{\alpha}A_{\alpha}^{-1}B_{\alpha}) \geq 0. \)
Therefore, \( \det(H_{\alpha}) \geq 0 \) for all \( \alpha \subseteq \{1, \ldots, N\} \) with \( \alpha \neq \emptyset \), implying that \( H_i \) is a \( P_0 \)-matrix.

Proof of Proposition 4: Consider \( \theta_1, \theta_2 \in \Theta \) and let \( F_i(\cdot) := F(\cdot, \theta_i), \ i = 1, 2. \) Let \( x_i \) be a solution of \( \text{VI}(K, F_i) \) for \( i = 1, 2. \) By the assumption of strong monotonicity on the map, we have that
\[
(x_1 - x_2)^T(F_1(x_1) - F_1(x_2)) \geq c\|x_2 - x_1\|^2,
\]
for some constant \( c > 0 \) (assumed to be independent of \( \theta_1 \)). Since \( x_1 \) is a solution of \( \text{VI}(K, F_1) \), it follows that \( (x_2 - x_1)^TF_1(x_1) \geq 0 \), which together with (35) implies
\[
(x_2 - x_1)^TF_1(x_2) \geq c\|x_2 - x_1\|^2.
\]
We may express (36) as \( (x_2 - x_1)^TF_1(x_2) - F_2(x_2) + F_2(x_2)) \geq c\|x_2 - x_1\|^2. \) Now since \( x_2 \) is the solution of \( \text{VI}(K, F_2) \), it follows that \( (x_2 - x_1)^TF_2(x_2) \leq 0. \) Consequently we obtain
\[
\|x_2 - x_1\|\|F_1(x_2) - F_2(x_2)\| \geq (x_2 - x_1)^T(F_1(x_2) - F_2(x_2)) \geq c\|x_2 - x_1\|^2.
\]
By Lipschitz continuity of \( F(x, \theta) \) (assuming it is uniform in \( x \)), we have that \( \|F_1(x_2, \theta_1) - F_2(x_2, \theta_2)\| \leq L_{\theta}\|\theta_2 - \theta_1\|, \) and hence by (37) \( L_{\theta}\|x_2 - x_1\|\|\theta_2 - \theta_1\| \geq c\|x_2 - x_1\|^2. \) It follows that \( \|x_2 - x_1\| \leq L_{\theta}c^{-1}\|\theta_2 - \theta_1\|. \)

To show (b), let \( x(\theta_i, \epsilon_i) \) be the solution of \( \text{VI}(K, G_{ij}(\cdot)) \), where \( G_{ij}(\cdot) = F(\cdot, \theta_i) + \epsilon_jI. \) We begin by applying the triangle inequality to obtain that \( \|x(\theta_1, \epsilon_1) - x(\theta_2, \epsilon_2)\| \leq \|x(\theta_1, \epsilon_1) - x(\theta_1, \epsilon_2)\| + \|x(\theta_1, \epsilon_2) - x(\theta_2, \epsilon_2)\|.
\]
\[ x(\theta_2, \epsilon_1) + \|x(\theta_2, \epsilon_1) - x(\theta_2, \epsilon_2)\|. \] Since \( G_{i1} \) is strongly monotone in \( x \) with constant \( c + \epsilon_1 \) and Lipschitz continuous in \( \theta \) with constant \( L_\theta \), respectively, we have that the first term is bounded by \( L_\theta (c + \epsilon_1)^{-1}\|\theta_2 - \theta_1\| \) as a result from part (a). Before proceeding, the Lipschitz continuity of \( F(x; \theta) + \epsilon I \) with respect to \( \epsilon \) can be obtained as

\[ \|\epsilon_1 - \epsilon_2\| \leq \sqrt{\epsilon_1 - \epsilon_2} = D(\epsilon_1 - \epsilon_2) \]

Since \( G_{2j} \) is strongly monotone in \( x \) with constant \( c + \epsilon_j \) and Lipschitz continuous in \( \epsilon \) with constant \( D \), respectively, we have that the second term is bounded by \( D(c + \epsilon_1)^{-1}\|\epsilon_2 - \epsilon_1\| \) as a result from part (a). Consequently, we obtain that

\[ \|x(\theta_1, \epsilon_1) - x(\theta_2, \epsilon_2)\| \leq L_\theta (c + \epsilon_1)^{-1}\|\theta_2 - \theta_1\| + D(c + \epsilon_1)^{-1}\|\epsilon_2 - \epsilon_1\| \]

The Lipschitz continuity of \( x(\theta, \epsilon) \) with respect to its parameters follows.

**Proof of Lemma 8:**

(a) Strict monotonicity of \( F(x) \) is implied by the positive definiteness of the Jacobian \( \nabla F(x) \).

This is given by \( \nabla F(x) = J_1 + J_2 + J_3 \), where

\[
J_1 = \begin{pmatrix}
c_i'(x_1) \\
\vdots \\
c_N'(x_N)
\end{pmatrix}, \quad J_2 = 2b^* \sigma X^{\sigma - 1} \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix}, \quad J_3 = b^* \sigma (\sigma - 1) X^{\sigma - 2} \begin{pmatrix}
x_1 \\
\vdots \\
x_1
\end{pmatrix}.
\]

Since \( c_i(x_i) \) is a convex function in \( x_i \) for all \( i \), \( J_1 \) is a positive semidefinite matrix. \( J_2 \), compactly stated as \( 2b^* \sigma X^{\sigma - 1}ee^T \), is also a positive semidefinite matrix. As a consequence, positive definiteness of \( \nabla F(x) \) follows from the diagonal dominance of the following matrix:

\[
b^* \sigma (\sigma - 1) X^{\sigma - 2} \begin{pmatrix}
x_1 \\
\vdots \\
x_1
\end{pmatrix}.
\]
By a minor rearrangement, it suffices to show the diagonal dominance of the following:

\[
b^* \sigma (\sigma - 1) X^{\sigma - 2} \begin{pmatrix}
\frac{x_1}{\sigma - 1} + (1 + \frac{1}{(\sigma - 1)}) x_1 & \cdots & \frac{1}{2} (x_1 + x_N) \\
\vdots & \ddots & \vdots \\
\frac{1}{2} (x_N + x_1) & \cdots & \frac{x_N}{\sigma - 1} + (1 + \frac{1}{(\sigma - 1)}) x_N
\end{pmatrix},
\]

where \( X_j \triangleq \sum_{i \neq j} x_i \). The result follows by noting that

\[
\left( 1 + \frac{1}{(\sigma - 1)} \right) > \frac{(N - 1)}{2} \quad \text{or} \quad \frac{2\sigma}{\sigma - 1} > N - 1 \quad \text{or} \quad N < \frac{3\sigma - 1}{\sigma - 1}.
\]

(b) For \( x, y \in K \),

\[
(x - y)^T (F(x) - F(y)) = \int_0^1 (x - y)^T \nabla F(y + \alpha (x - y))(x - y) d\alpha.
\]

Let \( \tilde{x} = y + \alpha (x - y) \) and \( \tilde{X} = \sum_{i=1}^N \tilde{x}_i \). Akin to \( \nabla F(x) \), \( \nabla F(y + \alpha (x - y)) = \tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 \),

where \( \tilde{J}_1 \) and \( \tilde{J}_2 \) are positive semidefinite, and \( \tilde{J}_3 = b^* \sigma (\sigma - 1) \tilde{X}^{\sigma - 2} \tilde{J}_4 \), where

\[
\tilde{J}_4 = \begin{pmatrix}
\frac{x_1}{\sigma - 1} + (1 + \frac{1}{(\sigma - 1)}) x_1 & \cdots & \frac{1}{2} (x_1 + x_N) \\
\vdots & \ddots & \vdots \\
\frac{1}{2} (x_N + x_1) & \cdots & \frac{x_N}{\sigma - 1} + (1 + \frac{1}{(\sigma - 1)}) x_N
\end{pmatrix} + \left( 1 + \frac{1}{(\sigma - 1)} - \frac{N - 1}{2} \right) I_N
\]

\( \triangleq \tilde{J}_5 + \rho I_N \),

where \( \tilde{J}_5 \) is a positive semidefinite matrix and \( \rho = (1 + \frac{1}{(\sigma - 1)} - \frac{N - 1}{2}) > 0 \). Therefore,

\[
(x - y)^T (F(x) - F(y)) \geq \int_0^1 (x - y)^T \tilde{J}_3 (x - y) d\alpha
\]

\[
\geq b^* \sigma (\sigma - 1) \int_0^1 (x - y)^T \tilde{X}^{\sigma - 2} \tilde{J}_4 (x - y) d\alpha
\]

\[
\geq b^* \sigma (\sigma - 1) \eta^{\sigma - 2} \int_0^1 (x - y)^T (\tilde{J}_5 + \rho I_N) (x - y) d\alpha
\]

\[
\geq b^* \sigma (\sigma - 1) \eta^{\sigma - 2} \rho \| x - y \|^2,
\]

implying the strong monotonicity of \( F \). \( \blacksquare \)
that JACOBian matrix \( \nabla F(z_i) \) of the mapping \( F \) at \( z_i \in \tilde{K} \). Then, as in Lemma 3, it suffices to show that \( H(z_i) \) is a P-matrix for all \( z_i \in \tilde{K} \). Given \( z_i \in \tilde{K} \), let \( H = H(z_i) \). Then,

\[
H = H(z_i) = \begin{pmatrix} A_i & B \\ C_i & D \end{pmatrix},
\]

where \( A_i = \sigma b^*(X_i)^{\sigma - 2} \left[ X_i (I + ee^T) + (\sigma - 1) x_i e^T \right] + E_i \), \( B = -\frac{1}{k+1} e \), \( C_i = -\sigma b^*(X_i)^{\sigma - 1} e^T \), and \( D = \frac{1}{k+1} \), where \( X_i = \sum_{j=1}^{N} x_{ij}, x_i = (x_{i1}, \ldots, x_{iN})^T \), and \( E_i \) is an \( N \times N \) diagonal matrix with \( c_j''(x_{ij}) \) as its \( j \)th diagonal entry. It suffices to show that \( H \) is a P-matrix when \( E_i = 0 \).

If \( N < \frac{3\sigma - 1}{\sigma - 1} \), then \( A_i \) is positive semidefinite by Lemma 8. Therefore, we only consider the principal submatrix \( H_\alpha \) of \( H \), where \( \alpha \subseteq \{1, \ldots, N\} \) is a nonempty index set, and \( \alpha_0 = A_\alpha = \sigma b^*(X_i)^{\sigma - 1} \left[ I_{n_\alpha} + e^{n_\alpha} (e^{n_\alpha})^T \right] + \sigma (\sigma - 1) b^*(X_i)^{\sigma - 2} x_\alpha (e^{n_\alpha})^T, \) \( B_\alpha = -\frac{1}{k+1} e^{n_\alpha} \), \( C_\alpha = -\sigma b^*(X_i)^{\sigma - 1} (e^{n_\alpha})^T \), and \( I_{n_\alpha} \) and \( e^{n_\alpha} \) denote the identity matrix and the column of ones in \( \mathbb{R}^{n_\alpha \times n_\alpha} \) and \( \mathbb{R}^{n_\alpha} \), respectively, with \( n_\alpha = |\alpha| \). Since

\[
B_\alpha D^{-1} C_\alpha = \frac{1}{k+1} e^{n_\alpha} (k + 1) \sigma b^*(X_i)^{\sigma - 1} (e^{n_\alpha})^T = \sigma b^*(X_i)^{\sigma - 1} e^{n_\alpha} (e^{n_\alpha})^T,
\]

it follows that \( A_\alpha - B_\alpha D^{-1} C_\alpha = \sigma b^*(X_i)^{\sigma - 1} I_{n_\alpha} + \sigma (\sigma - 1) b^*(X_i)^{\sigma - 2} x_\alpha (e^{n_\alpha})^T \), which is a sum of a diagonal positive definite matrix and a P_0-matrix, and thus is a P-matrix. Therefore, \( \det(H_\alpha) = \det(D) \det(A_\alpha - B_\alpha D^{-1} C_\alpha) > 0 \) for all \( \alpha \subseteq \{1, \ldots, N\} \) with \( \alpha \neq \emptyset \), which implies that \( H \) is a P-matrix. 

V. NETWORKED NASH-COURNOT GAMES

In this section, we apply the developed algorithms on a class of networked Nash-Cournot games described in Section V-A. In Section V-B, we apply the distributed gradient-based schemes for purposes of learning equilibria and the misspecified parameters when aggregate output is observable, while in Section V-C, we apply the proposed iterative fixed-point schemes when aggregate output is unobservable. Note that the simulations were carried out on Matlab R2009a on a laptop with Intel Core 2 Duo CPU (2.40GHz) and 2GB memory. The complementarity solver PATH, developed by [40], was utilized for solving the variational inequality problems that arose in implementing the algorithms.
A. Problem description

We consider a setting where there are $N$ firms competing over a $W$-node network. Firm $f$ may produce and sell its good at node $i$ (denoted by $g_{fi}$ and $s_{fi}$, respectively), where $f = 1, \ldots, N$ and $i = 1, \ldots, W$. We assume that for a given firm $f$, the cost of generating $g_{fi}$ units of power at node $i$ is linear and is given by $c_{fi}g_{fi}$. Furthermore, the generation level associated with firm $f$ is bounded by its production capacity, which is denoted by $\text{cap}_{fi}$. The aggregate sales of all firms at node $i$ is denoted by $S_i$, and the nodal price of power at node $i$, assumed to be a linear function of $S_i$, is defined as $p_i(S_i) \triangleq \alpha_i^* - \beta_i^*S_i$, where $\alpha_i^*$ and $\beta_i^*$ are node-specific positive price function parameters. A given firm can produce at any node and then sell at different nodes, provided that the aggregate production at all nodes matches the aggregate sales at all nodes for each firm. For simplicity, we assume that there is no transportation cost between any two nodes, and that there is no limit of sales at any node. Then, the resulting problem faced by firm $f$ can be stated as

$$\max_{s_{fi} \geq 0} \sum_{i=1}^{W} (p_i(S_i)s_{fi} - c_{fi}g_{fi}) : \sum_{i=1}^{W} (s_{fi} - g_{fi}) = 0.$$  \hspace{1cm} (39)

The resulting Nash-Cournot equilibrium is given by $\{x_f^*\}_{f=1}^{N}$ where $x_f^*$ is a solution to (Firm($x_{-f}$)) for $f = 1, \ldots, N$. Prices are assumed to be corrupted by noise, in one of two ways:

$$p_i(S_i; \xi_i) = (\alpha_i^* + \xi_i) - \beta_i^*S_i$$ \hspace{1cm} (40)

$$p_i(S_i; \xi_i) = \alpha_i^* - (\beta_i^* + \xi_i)S_i. \hspace{1cm} (41)$$

Note that firm $f$ either has to learn $\theta^* \triangleq (\alpha_i^*)_{i=1}^{W}$ when prices are given by (40) or learn $\theta^* \triangleq (\beta_i^*)_{i=1}^{W}$ when prices are given by (41). In the remainder of this section, let $a^* \triangleq (a_1^*, \ldots, a_W^*)^T$, $b^* \triangleq (b_1^*, \ldots, b_W^*)^T$, $\theta^* \triangleq (\theta_1^*, \ldots, \theta_W^*)^T$, $\xi^* \triangleq (\xi_1^*, \ldots, \xi_W^*)^T$, $x_i \triangleq (s_{1i}, s_{2i}, \ldots, s_{Ni}, g_{1i}, g_{2i}, \ldots, g_{Ni})^T$, and $x \triangleq (x_1^T, \ldots, x_W^T)^T$.

Note that this problem is employed as a motivating example since Cournot-based models have been used extensively in their analysis (cf. [16], [17]). Naturally, a range of rationality assumptions can be imposed on firms in power markets, but given the sheer size of the problem and the repeated nature of competition (in most power markets, firms compete as many as 5–6 times every hour in the setting of prices) with relatively minor changes occurring in demand/availability over a short period.
B. Learning with observation of the aggregate output

In this subsection, we assume that every firm knows the aggregate output at each node, and employ the learning schemes proposed in Section II-A. Suppose, the nodal price function is given by (40) and suppose Algorithm I (the gradient-based distributed learning scheme), proposed in Section II-A, is employed for learning parameters and computing equilibria. Suppose, the noise \( \xi \) is distributed as per a uniform distribution and is specified by \( \xi \sim U[-a^*/2, a^*/2] \). Suppose the steplength sequence \( \{\gamma^k\} \) is chosen according to Lemma 5: \( \gamma^k_i = \frac{1}{(k+N_i)\alpha} \), where \( \alpha = 0.8 \) and \( N_i \) is randomly chosen from an interval \([1, 200]\). The algorithm was terminated at \( k = 10000 \). Table I shows the scaled errors of the learning scheme.

We can also see from Table I that Grenander’s conditions appear to hold. By using notation in Def. 1, we define \( x(i) = (-S_1^k, \ldots, -S_W^k)^T \), \( d_k(i) = x(i)^T x(i) \), and \( R_k(i) \) the sample covariance of \( x(i) \) for all \( i = 1, \ldots, W \) and \( k \geq 1 \). When the algorithm was terminated at \( k = 10000 \), we can see from Table I that for all \( i = 1, \ldots, W \): (a) \( d_k(i) \) converges to infinity; (b) \( \max_1 \leq \ell \leq k \frac{(S_i)^2}{d_k(i)} \) converges to 0; (c) \( R_k(i) \) is positive. In Table II, we examine a sequential scheme where we first estimate \( \theta^* \) via stochastic approximation (\( \ell \) steps) and then compute the equilibrium (10000 steps). Premature termination of the learning scheme leads to an error in estimating \( \theta^* \) and consequently \( x^* \). While it is obvious that the learning effort should be increased, in stochastic regimes, one may not have a priori knowledge regarding how long one should learn for.

### Table I

**Distributed gradient scheme for learning \( a^* \) and \( b^* \) in a stochastic regime: \( \xi \sim U[-a^*/2, a^*/2] \)**

| \( N \) | \( W \) | \( x^2 \)x | \( b^2 \)b | \( min \_d_k(i) \) | \( max \_d_k(i) \) | \( \frac{(S_i)^2}{d_k(i)} \) | \( max \_i \_R_k(i) \) |
|---|---|---|---|---|---|---|---|
| 5 | 1 | 59x10^{-3} | 1.5x10^{-2} | 1.3x10^{-2} | 1x10^{-7} | 2.6x10^{-4} | 9.2x10^{-1} |
| 5 | 2 | 29x10^{-2} | 2.6x10^{-2} | 5.5x10^{-2} | 2.7x10^{-5} | 2.4x10^{-3} | 7.3x10^{-2} |
| 5 | 3 | 13x10^{-2} | 7.8x10^{-2} | 6.3x10^{-2} | 2.9x10^{-5} | 1.7x10^{-3} | 1.0x10^{-2} |
| 5 | 4 | 8.6x10^{-2} | 4.3x10^{-2} | 4.2x10^{-2} | 3.3x10^{-5} | 2.1x10^{-3} | 4.2x10^{-2} |
| 5 | 5 | 5.4x10^{-2} | 4.9x10^{-2} | 3.4x10^{-2} | 2.4x10^{-5} | 1.3x10^{-3} | 1.5x10^{-2} |

### Table II

**Distributed sequential gradient scheme for learning \( a^* \) and \( b^* \) when \( N = 5 \) and \( W = 1 \), stopping at step \( k = 10000 \)**

| \( t \) | \( x^2 \)x | \( b^2 \)b | \( \frac{(S_i)^2}{d_k(i)} \) |
|---|---|---|---|
| 200 | 3.8x10^{-1} | 1.2x10^{-1} | 1.1x10^{-1} |
| 400 | 2.8x10^{-2} | 2.9x10^{-2} | 9.7x10^{-3} |
| 600 | 1.3x10^{-2} | 4.2x10^{-2} | 8.4x10^{-3} |
| 800 | 1.7x10^{-2} | 6.1x10^{-2} | 4.4x10^{-2} |
| 1000 | 4.6x10^{-2} | 6.4x10^{-2} | 5.1x10^{-2} |
C. Learning without observing the aggregate output

In this subsection, we examine how the schemes perform when firms are ignorant of aggregate output at each node while a common knowledge assumption is assumed to hold.

| TABLE III |
|---|

Noise-corrupted learning scheme for learning \(a^*\) and \(b^*\) (\(\xi \sim U[-\theta^*/2, \theta^*/2]\))

| (a) Learning \(a^*\) | (b) Learning \(b^*\) |
|---|---|
| \(N\) | \(W\) | \(\max \|x_f^k - x^*\|\) | \(\max \|x_f^k - a^*\|\) | \(\max \|x_f^k - b^*\|\) |
| 5 | 1 | \(6.0 \times 10^{-3}\) | \(5.4 \times 10^{-3}\) |  |
| 5 | 2 | \(1.9 \times 10^{-3}\) | \(1.6 \times 10^{-3}\) |  |
| 5 | 3 | \(1.4 \times 10^{-3}\) | \(2.7 \times 10^{-3}\) |  |
| 5 | 4 | \(7.8 \times 10^{-3}\) | \(2.8 \times 10^{-3}\) |  |
| 5 | 5 | \(1.0 \times 10^{-3}\) | \(2.5 \times 10^{-3}\) |  |
| 10 | 2 | \(2.0 \times 10^{-3}\) | \(1.9 \times 10^{-3}\) |  |
| 10 | 4 | \(1.1 \times 10^{-2}\) | \(4.2 \times 10^{-3}\) |  |
| 10 | 6 | \(1.8 \times 10^{-3}\) | \(0.8 \times 10^{-3}\) |  |
| 10 | 8 | \(2.0 \times 10^{-3}\) | \(2.7 \times 10^{-3}\) |  |
| 10 | 10 | \(1.1 \times 10^{-3}\) | \(3.5 \times 10^{-3}\) |  |

1) Linear price functions: Suppose, the nodal price function is given by (40) or (41) and suppose Algorithm II (the iterative fixed-point scheme), proposed in Section III-A, is employed for learning parameters and computing equilibria. Suppose, the noise \(\xi\) is distributed as per a uniform distribution and is specified by \(\xi \sim U[-\theta^*/2, \theta^*/2]\). Each run comprised of 10000 steps learning \(a^*\) and 50000 steps for learning \(b^*\). Table III shows the scaled errors of the learning scheme while Figures 1(a) and 1(b) illustrate the scaled errors of the learning scheme when the number of steps, denoted by \(k\), increases for learning \(x^*\) and \(a^*\), respectively. Analogous figures for learning \(x^*\) and \(b^*\) are provided (see Figures 2(a) and 2(b)).

![Fig. 1. Computing \(x^*\) and learning \(a^*\) (\(\xi \sim U[-\theta^*/2, \theta^*/2]\), \(N = 10\))](image)

In Table IV(a), we raise the upper bounds of the strategy sets of all agents and compare a sequential scheme with our iterative fixed-point scheme. In the sequential counterpart, we...
employ 10,000 steps of stochastic approximation-based learning followed by 10,000 steps of computation. It is seen that the error from the sequential scheme increases proportionally to the bound, while the error associated with our simultaneous scheme does not change significantly. Table IV(b) shows that when increasing the variance of the noise makes the difference in errors between the sequential and simultaneous schemes more pronounced. Consequently, for the same effort, it can be seen that the simultaneous scheme performs far better to the sequential scheme, particularly when the variance of the noise grows.

**TABLE IV**

Learning $x^*$ and $b^*$ in a stochastic regime when $N = 5$ and $W = 1$, stopping at step $k = 10000$

| Bound  | Sequential | Simultaneous | Sequential | Simultaneous |
|--------|------------|--------------|------------|--------------|
| 32.3664 | $2.1 \times 10^{-1}$ | $1.2 \times 10^{-2}$ | $4.9 \times 10^{-3}$ | $3.3 \times 10^{-3}$ |
| 64.7393 | $1.2 \times 10^{-1}$ | $1.0 \times 10^{-2}$ | $5.0 \times 10^{-3}$ | $3.3 \times 10^{-3}$ |
| 97.0993 | $5.5 \times 10^{-1}$ | $8.8 \times 10^{-2}$ | $5.0 \times 10^{-3}$ | $3.3 \times 10^{-3}$ |
| 129.4658 | $7.4 \times 10^{-1}$ | $1.1$ | $5.1 \times 10^{-3}$ | $3.4 \times 10^{-3}$ |
| 161.8322 | $1.2$ | $7.9 \times 10^{-1}$ | $5.1 \times 10^{-3}$ | $3.4 \times 10^{-3}$ |

VI. CONCLUDING REMARKS

Nash games, a broadly applicable paradigm for modeling strategic interactions in noncooperative settings, have emerged as immensely useful in the context of distributed control problems. Yet, the development of distributed protocols for learning equilibria may be complicated by several challenges: (i) Agents may have an incomplete specification of payoffs; (ii) Agents may be unavailable to observe the actions of their counterparts; and finally, (iii) Observations may be corrupted by noise. Accordingly, this paper is motivated by developing schemes for learning equilibria and price functions. We consider two specific settings as part of our investigation.
and apply these techniques on a class of networked Nash-Cournot games. First, we consider convex static stochastic Nash games characterized by a suitable monotonicity property in which agent payoffs are parameterized by an unknown vector. We consider a framework that combines (stochastic) gradient steps with a learning step that employs observations. Notably, we do not assume the common knowledge assumption, often a strong requirement in large-scale networked regimes. In such settings, we provide asymptotic statements that show that agents may learn equilibria (in the mean square sense) and price function parameters (in probability) as well as constant steplength error bounds. Second, we refine our statements to a Cournot regime where we assume common knowledge holds but aggregate output is unobservable. In such a setting, we construct a learning scheme in which firms maintain a belief of the aggregate output and the misspecified price function parameter. After each step, these beliefs are updated by employ fixed-point steps and by leveraging the disparity between estimated and (noisy) observed prices. We proceed to show that in the limit, every firm learns the true Nash-Cournot equilibrium strategy in an almost-sure sense. Additionally, every firm learns the correct value of the misspecified parameter in an almost-sure sense. Yet much remains to be studied, including weakening monotonicity requirements on the map and boundedness requirements on the strategy sets. It also remains to be investigated as to whether learning can allow for weakening the common knowledge assumption.

REFERENCES

[1] H. Jiang and U. V. Shanbhag, “On the convergence of joint schemes for online computation and supervised learning,” in Proceedings of the IEEE Conference on Decision and Control (CDC), 2012, pp. 4462–4467.

[2] H. Jiang, U. V. Shanbhag, and S. P. Meyn, “Learning equilibria in constrained nash-cournot games with misspecified demand functions,” in Proceedings of the IEEE Conference on Decision and Control (CDC-ECE), 2011, pp. 1018–1023.

[3] H. P. Young and S. Z. (eds), Eds., Game theory and distributed control, vol. 4. Elsevier, 20xx.

[4] N. Li and J. R. Marden, “Designing games to handle coupled constraints,” in Proceedings of the IEEE Conference on Decision and Control (CDC). IEEE, 2010, pp. 250–255.

[5] ———, “Designing games for distributed optimization,” in Proceedings of the IEEE Conference on Decision and Control (CDC), 2011, pp. 2434–2440.

[6] D. Fudenberg and D. K. Levine, The theory of learning in games, ser. MIT Press Series on Economic Learning and Social Evolution. Cambridge, MA: MIT Press, 1998, vol. 2.

[7] H. P. Young, Strategic Learning and its Limits. Oxford University Press, 2004.

[8] S. Hart, “Adaptive heuristics,” Econometrica, vol. 73, no. 5, pp. 1401–1430, 2005.
[9] T. Başar, “Control and game-theoretic tools for communication networks,” *Appl. Comput. Math.*, vol. 6, no. 2, pp. 104–125, 2007.
[10] T. Alpcan and T. Başar, “Distributed algorithms for Nash equilibria of flow control games,” *Annals of Dynamic Games*, vol. 7, 2003.
[11] Y. Pan and L. Pavel, “Games with coupled propagated constraints in optical network with multi-link topologies,” *Automatica*, vol. 45, pp. 871–880, 2009.
[12] H. Yin, U. V. Shanbhag, and P. G. Mehta, “Nash equilibrium problems with scaled congestion costs and shared constraints,” *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1702–1708, 2011.
[13] F. Facchinei and J. S. Pang, “Nash Equilibria: The Variational Approach,” *Convex Optimization in Signal Processing and Communication, Cambridge University Press*, 2009.
[14] G. Scutari and J. S. Pang, “Joint sensing and power allocation in nonconvex cognitive radio games: Nash equilibria and distributed algorithms,” *CoRR*, vol. abs/1212.6437, 2012.
[15] G. Scutari, F. Facchinei, J. S. Pang, and D. P. Palomar, “Real and complex monotone communication games,” *CoRR*, vol. abs/1212.6235, 2012.
[16] B. Hobbs, “Linear complementarity models of nash-cournot competition in bilateral and poolco power markets,” *IEEE Transactions on Power Systems*, vol. 16, no. 2, pp. 194–202, 2001.
[17] B. F. Hobbs and J. S. Pang, “Nash-Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints,” *Oper. Res.*, vol. 55, no. 1, pp. 113–127, 2007.
[18] J. S. Shamma and G. Arslan, “Dynamic fictitious play, dynamic gradient play, and distributed convergence to Nash equilibria,” *IEEE Trans. Automat. Control*, vol. 50, no. 3, pp. 312–327, 2005.
[19] J. R. Marden, H. P. Young, G. Arslan, and J. S. Shamma, “Payoff-based dynamics for multiplayer weakly acyclic games,” *SIAM J. Control Optim.*, vol. 48, no. 1, pp. 373–396, 2009.
[20] A. P. Kirman, “Learning by firms about demand conditions,” in *Adaptive economic models (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974)*. New York: Academic Press, 1975, pp. 137–156. Math. Res. Center, Univ. Wisconsin, Publ. No. 34.
[21] G. I. Bischi, A. Naimzada, and L. Sbragia, “Oligopoly games with local monopolistic approximation,” *Journal of Economic Behavior and Organization*, vol. 62, pp. 371–388, 2007.
[22] G. I. Bischi, L. Sbragia, and F. Szidarovszky, “Learning the demand function in a repeated Cournot oligopoly game,” *Internat. J. Systems Sci.*, vol. 39, no. 4, pp. 403–419, 2008.
[23] F. Szidarovszky, “Global stability analysis of a special learning process in dynamic oligopolies,” *Journal of Economic and Social Research*, vol. 9, pp. 175–190, 2004.
[24] F. Szidarovszky and J. B. Krawczyk, “On stable learning in dynamic oligopolies,” *Pure Math. Appl.*, vol. 15, no. 4, pp. 453–468, 2004.
[25] D. Léonard and K. Nishimura, “Nonlinear dynamics in the Cournot model without full information,” *Ann. Oper. Res.*, vol. 89, pp. 165–173, 1999, nonlinear dynamical systems and adaptive methods (Vienna, 1997).
[26] W. L. Cooper, T. Homem-de Mello, and A. J. Kleywegt, “Models of the spiral-down effect in revenue management,” *Oper. Res.*, vol. 54, pp. 968–987, September 2006.
[27] L. Pavel, “A noncooperative game approach to OSNR optimization in optical networks,” *IEEE Trans. Automat. Control*, vol. 51, no. 5, pp. 848–852, 2006.
[28] W. H. Greene, *Econometric Analysis*. New Jersey: Prentice Hall, 2011.
[29] J. Hofbauer and W. H. Sandholm, “Stable games and their dynamics,” *J. Economic Theory*, vol. 144, no. 4, pp. 1665–1693, 2009.

[30] M. J. Fox and J. S. Shamma, “Population games, stable games, and passivity,” in *CDC*, 2012, pp. 7445–7450.

[31] B. T. Polyak, *Introduction to optimization*. New York: Optimization Software, Inc., 1987.

[32] A. Kannan and U. V. Shanbhag, “Distributed computation of equilibria in monotone Nash games via iterative regularization techniques,” *SIAM Journal of Optimization*, vol. 22, no. 4, pp. 1177–1205, 2012.

[33] G. I. Bischi, C. Chiarella, M. Kopel, and F. Szidarovszky, *Nonlinear oligopolies*. Berlin: Springer-Verlag, 2010, stability and bifurcations.

[34] R. J. Aumann, “Agreeing to disagree,” *The Annals of Statistics*, vol. 4, no. 6, pp. pp. 1236–1239, 1976.

[35] J. Littlewood, *Mathematical Miscellany*, B. Bollabos, Ed., 1953.

[36] T. Shelling, *The Strategy of Conflict*. Harvard University Press, Cambridge, Massachusetts, 1960.

[37] F. Facchinei and J. S. Pang, *Finite-dimensional variational inequalities and complementarity problems. Vol. I*, ser. Springer Series in Operations Research. New York: Springer-Verlag, 2003.

[38] S. Dafermos, “Sensitivity analysis in variational inequalities,” *Math. Oper. Res.*, vol. 13, no. 3, pp. 421–434, 1988.

[39] T. W. Anderson and J. B. Taylor, “Strong consistency of least squares estimates in dynamic models,” *The Annals of Statistics*, vol. 7, no. 3, pp. 484–489, 1979.

[40] M. C. Ferris and T. S. Munson, “Complementarity problems in gams and the path solver,” *Journal of Economic Dynamics and Control*, vol. 24, no. 2, pp. 165–188, 2000.