Communication

Dissipation Factors of Spherical Current Modes on Multiple Spherical Layers

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Abstract—Radiation efficiencies of modal current densities distributed on a spherical shell are evaluated in terms of dissipation factor. The presented approach is rigorous, yet simple and straightforward, leading to closed-form expressions. The same approach is utilized for a two-layered shell and the results are compared with other models existing in the literature. Discrepancies in this comparison are reported and reasons are analyzed. Finally, it is demonstrated that radiation efficiency potentially benefits from the use of internal volume which contrasts with the case of the radiation Q-factor.

Index Terms—Radiation efficiency, Antenna theory, Optimization methods

I. INTRODUCTION

The fundamental bounds on radiation efficiency have become increasingly interesting in recent years [1]–[3] as low radiation efficiency, together with a high radiation Q-factor presents a serious performance bottleneck for all electrically small antenna designs [4].

Similar to fundamental bounds on radiation Q-factor, fundamental bounds on radiation efficiency were first approached using the example of a spherical shell. The reason is twofold. First, the mathematics of spherical modes is analytically tractable. Second, it has been assumed [1] that, analogous to radiation Q-factor, the best radiation efficiency belongs to surface spherical modes.

The major purpose of this communication is to extend the study presented in [1] by providing a full-wave treatment of multilayer scenarios and to provide evidence that the surface spherical currents do not form a lower bound to the dissipation factor of a general volumetric radiator. Considering the practical demand on resonance and the fact that lossless external tuning is unreachable [1]–[3], [5], here, attention is primarily paid to self-resonant current densities, and the externally tuned results are considered only as intermediate products. All analytical results are verified with full-wave numerical calculations.

The communication is organized as follows. In Section II the lowest dissipation factor for a single spherical shell is derived and compared with existing results. The model is generalized to two spherical layers in Section III and to multiple layers in Section IV. The communication is concluded in Section V.

II. DISSIPATION FACTOR OF A SINGLE SPHERICAL LAYER

This section reformulates the results presented in [1] by directly manipulating vector spherical waves [6]. Some discrepancies in the model used in [1] are also indicated.

It is possible to show [7] that, within a time-harmonic steady state, electric field \( \mathbf{E} \) and surface current density \( \mathbf{J} \), corresponding to the modes of a spherical layer of radius \( a \), read

\[
E_{mn}^{TE} = -Z_0 \frac{\psi_n'(ka) \zeta_n(ka) M_{mn}}{\eta}, \quad (1)
\]

\[
E_{mn}^{TM} = Z_0 \frac{\psi_n(ka) \zeta_n'(ka) N_{mn}}{\eta}, \quad (2)
\]

\[
J_{mn}^{TE} = \hat{r} \times N_{mn}, \quad (3)
\]

\[
J_{mn}^{TM} = \hat{r} \times M_{mn}, \quad (4)
\]

where

\[
\psi_n(x) = x j_n(x), \quad (5)
\]

\[
\chi_n(x) = -x y_n(x), \quad (6)
\]

\[
\zeta_n(x) = x h_n^{(2)}(x) = \psi_n(x) + j \chi_n(x), \quad (7)
\]

are Riccati-Bessel functions, the symbol \( \hat{\cdot} \) denotes differentiation, \( Z_0 \) is the free-space impedance, \( k \) is the free-space wavenumber, \( j_n, y_n \), and \( h_n^{(2)} \) are the spherical Bessel’s functions of order \( n \) [8]. Functions \( M \) and \( N \) are spherical vector waves defined in [7] with Bessel’s function \( j_n \) inserted, and \( \hat{r} \) is the unit vector pointing in the radial direction. Electric field \( E_{mn}^{TE/TM} \) and current density \( J_{mn}^{TE/TM} \) also depend on spherical angular variables, but this dependence is of no relevance in this paper.

In order to evaluate radiation efficiency \( \eta \) of modal current distributions, this paper uses dissipation factor \( \delta \) [9] defined via \( \eta = 1/(1 + \delta) \). Dissipation factor is thus the ratio of the cycle mean power lost by conduction and cycle mean power lost by radiation.

In order to evaluate dissipation factors of surface current distributions, the complex power [10]

\[
P_{rad} + j P_{react} = -\frac{1}{2} \int J^* \cdot E \, dS \quad (8)
\]

and cycle mean lost power

\[
P_{lost} = \frac{R_s}{2} \int J^* \cdot J \, dS \quad (9)
\]

are needed, where \( J^* \) denotes complex conjugation, and \( R_s \) denotes surface resistance (homogeneously distributed over
the surface $S$). For current densities flowing on highly conducting bodies, a surface resistance model $R_s = 1/(\sigma d)$ can be assumed, with $d$ being an effective penetration distance of the field into the conductor. For electrically thick conductors, $d$ can be put equal to the penetration depth [11].

In line with [1] and considering a major cost of resonance tuning to radiation efficiency [3] let us also prepare the grounds to form a resonant combination of selected spherical modes. To that point suppose a current density

$$\mathbf{J} = \mathbf{J}^c + \alpha \mathbf{J}^m$$

(10)

with tuning coefficient

$$|\alpha|^2 = -\frac{P_{\text{react}}^c}{P_{\text{react}}^m}$$

(11)

is formed with $\mathbf{J}^c$ and $\mathbf{J}^m$ being capacitative and inductive (excess electric or magnetic energy) spherical modes [5, 4], and $P_{\text{react}}^c$, $P_{\text{react}}^m$ being the corresponding reactive powers [8].

Owing to the orthogonality of spherical modes [7], the current density (10) is self-resonant with $P_{\text{react}} = 0$. The dissipation factors $\delta$, corresponding to the current density (10), reads

$$\delta = \frac{P_{\text{lost}}}{P_{\text{rad}}} = \frac{P_{\text{lost}}^c + |\alpha|^2 P_{\text{lost}}^m}{P_{\text{rad}}^c + |\alpha|^2 P_{\text{rad}}^m} = \frac{\delta^c - \frac{\chi^c}{\chi^m} \delta^m}{1 - \frac{\chi^c}{\chi^m}},$$

(12)

where mode orthogonality has once more been employed and where normalized reactances

$$\chi^{c/m} = \frac{P_{\text{react}}^{c/m}}{P_{\text{rad}}^{c/m}}$$

(13)

were defined.

At small electrical sizes, where bounds on dissipation are of interest, the capacitive modes $^c$ are the spherical TM modes, while inductive modes $^m$ are the spherical TE modes. The last step prior to evaluation of (12) is thus to find the dissipation factors $\delta^c_{\text{TE/TM}}$. The substitution of (1)–(4) into (8) and (9) leads to

$$\delta^\text{TE}_{0,n} = \frac{R_s}{Z_0} \left(\psi_n^c(ka)\right)^2,$$

(14)

$$\delta^\text{TM}_{0,n} = \frac{R_s}{Z_0} \left(\psi_n^m(ka)\right)^2,$$

(15)

and to the expressions for the normalized reactances

$$\chi^\text{TE}_{0,n} = \frac{\chi_n^c(ka)}{\psi_n^c(ka)},$$

(16)

$$\chi^\text{TM}_{0,n} = \frac{\chi_n^m(ka)}{\psi_n^m(ka)},$$

(17)

which are equal to the characteristic numbers of a perfectly conducting spherical layer [12]. Subindex $\circ$ in (14)–(17) denotes quantities corresponding to a single spherical layer.

The dissipation factor (12) of any resonant combination of two spherical modes on a single spherical layer can easily be evaluated by substituting (14)–(17) into (12).

![Fig. 1. Comparison of results (14), (16) and (19) from [1] with corresponding results of this paper. A comparison of the asymptotic (solid line) and full-wave (solid line with marks) expressions derived in this paper is also shown. The results correspond to a single spherical layer. The dissipation factors originating from [1] were multiplied by a factor of two, since [1] originally assumed two infinitesimally spaced resistive layers.](image)
these analytical results can be used to validate more general dissipation bounds, such as those presented in [13]. In particular, for a single spherical surface the results shown in [13] Eq. 18, version 5] suggest \((Z_0/R_s) \delta = 6/(2ka)^2\), while the first-order asymptotic expansion of [15] gives \((Z_0/R_s) \delta = 9/(2ka)^2\) for the lowest TM mode. The bound presented in [13] is thus rather conservative for a spherical shell. It is also important to notice that, for small electrical sizes, non-resonant electric-dipole-like dissipation factors scale as \(1/(ka)^2\), while resonant dissipation factors scale as \(1/(ka)^4\), see [3] for a more general exposition of this phenomenon.

III. DISSIPATION FACTOR OF TWO SPHERICAL LAYERS

The reduction of the dissipation factor by the specific composition of two resistive layers evokes the question of the general behavior of this setup. Specifically, assume that when forming a resonant current distribution (10), its constituents are yet another combination of spherical modes on two distinct layers of radius \(a\) and radius \(b < a\). The capacitive current will be formed as

\[
J^c = J^c_a + \beta \psi^c J^m_b ,
\]

and the inductive current will be formed as

\[
J^m = J^m_a + \beta \psi^m J^m_b ,
\]

where it is assumed that currents of the same type (capacitive or inductive) are always formed by the same spherical mode. On the contrary, currents \(J^c\) and \(J^m\) are always formed by two distinct spherical modes and are thus orthogonal with respect to complex power as well as lost power. Therefore, formula (12) also remains valid in this case.

Dissipation factors and normalized reactances correspond-

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
(Z_0/R_s) \delta & \text{Paper [1]} & \text{multiplied by 2} & \text{This paper} \\
\hline
\text{TE}_{10} & \frac{10}{(ka)^4} + \frac{22}{5(ka)^2} & \frac{9}{(ka)^4} + \frac{9}{5(ka)^2} \\
\text{TM}_{10} \cdot \text{TE}_{10} & \frac{10}{3(ka)^4} + \frac{34}{30(ka)^2} & \frac{3}{(ka)^4} + \frac{3}{10(ka)^2} \\
\text{TM}_{10} \cdot \text{TE}_{20} & \frac{78}{10(ka)^4} - \frac{94}{140(ka)^2} & \frac{15}{2(ka)^4} - \frac{27}{28(ka)^2} \\
\end{array}
\]

Fig. 2. Normalized dissipation factor corresponding to a TE_{10} mode distributed on two spherical layers of radius \(a\) and radius \(b < a\). The results correspond to electrical size \(ka = 0.1\). A curve showing the minima of the dissipation factors is also shown.

\[
\begin{align*}
\delta_{\odot,n}^{\text{TE}} &= \frac{AB + |\beta_{\text{TE}}|^2 A}{AB + 2\Re[\beta_{\text{TE}}]} \delta_{\odot,n}^{\text{TE}}, \\
\delta_{\odot,n}^{\text{TM}} &= \frac{AB + |\beta_{\text{TM}}|^2 B}{AB + 2\Re[\beta_{\text{TM}}]} \delta_{\odot,n}^{\text{TM}}, \\
\lambda_{\odot,n}^{\text{TE}} &= \frac{AB + 2\Re[\beta_{\text{TE}}]^2 C}{AB + 2\Re[\beta_{\text{TE}}]} \lambda_{\odot,n}^{\text{TE}}, \\
\lambda_{\odot,n}^{\text{TM}} &= \frac{AB + 2\Re[\beta_{\text{TM}}]^2 D}{AB + 2\Re[\beta_{\text{TM}}]} \lambda_{\odot,n}^{\text{TM}},
\end{align*}
\]

where

\[
A = \frac{\psi_n(ka)}{\psi_n(kb)}, \quad B = \frac{\psi'_n(ka)}{\psi'_n(kb)}, \\
C = \frac{\chi_n(ka)}{\chi_n(kb)}, \quad D = \frac{\chi'_n(ka)}{\chi'_n(kb)},
\]

and where the \(\odot\) symbol denotes quantities corresponding to two spherical layers.

As an example, the results of (20) for a TE_{10} mode are depicted in Fig. 2 and Fig. 3. A comparison of the curves in Fig. 2 and Fig. 3 shows that irrespective of ratio \(b/a\), the TE_{10} current distribution on two spherical layers always results (for a specific \(\beta_{\text{TE}}\)) in a lower dissipation factor than that of a single spherical layer\(^3\). The optimal values of

\[^3\text{Notice that two infinitesimally spaced spherical layers assumed in [1] correspond to } A = B = C = D = \beta_{\text{TE/TM}} = 1 \text{ and thus exactly to two times lower dissipation factors in comparison to a single layer scenario.} \]
of coupling parameters $\beta$ will attain more dimensions. Last, but not least, it is important to realize that we did not prove that the resonant combination of $TM_{10} : TE_{10}$ modes on multiple spherical layers is the global minimizer to the resonant dissipation factor within spherical geometry. Due to the preceding reasons, this Section will compare a purely numerical approach with the analytical treatment.

The numerical method used here, and described in [14–16], is able to find the global minimizer for an arbitrary surface current support. The results for one, two, and three spherical shells are shown in Fig. 5 and compared to the analytical results in the preceding Section. Good agreement of the numerical and analytical results in Fig. 5 can be observed. A slight discrepancy can be attributed to the problem of comparing data corresponding to a perfect spherical surface with its triangularized (570 triangles per layer) counterpart. This allows us to finish this communication without the need to extend the analysis shown in this section, inductively, to the two-layer scenario. The attention is now turned to the lowest dissipation factor of multiple layers.

IV. DISSIPATION FACTOR OF MULTIPLE LAYERS

The case of more than two spherical layers is a straightforward extension of (20)–(25). The number of terms in complex power (8) increases and explicit relations become too long. It is also important to realize that the optimization of coupling parameters $\beta$ will attain more dimensions. Last,
with the following statements:

- The addition of more spherical layers systematically reduces the dissipation factor, although with significantly diminishing returns;
- The resonant combination of $\text{TM}_1 \text{TE}_1$ modes seems to give the lowest dissipation factor from all resonant current distributions, even in the multilayer scenario;
- The hypothesis from [1] that the bound on the tuned dissipation factor is presented by a resonant combination of $\text{TM}_1$ and $\text{TE}_1$ spherical currents distributed on a single spherical surface is not valid.

It is worth noting that the last point is strongly connected to the optimization task addressed in [17] and [18] in which it is shown that a volumetric current density with the angular distribution of the dominant spherical mode and radial dependence of the spherical Bessel function exhibits a lower dissipation factor than the purely surface current distribution of the same angular dependence.

V. Conclusion

Minimum dissipation factors corresponding to current densities distributed on multiple spherical layers have been found in an analytic or semi-analytic manner and have been proven to be valid by using a full-wave numerical method. Results corresponding to one and two spherical layers were also compared with existing works.

It has been demonstrated that spherical modes can always be distributed on two spherical layers so as to lead to a smaller dissipation factor than that offered by a single spherical layer. This holds irrespective of electrical size or the ratio of the layer radii and does not depend whether a non-resonant or resonant combination of modes is formed. Moreover, the addition of more layers reduces the dissipation factor even further which indicates that a volumetric current density with the angular distribution of characteristic modes solvers,” IEEE Trans. Antennas Propag., vol. 65, no. 8, pp. 4134–4145, 2017.

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