ABSTRACT. This article introduces planar ribbons, Vergili ribbon complexes and ribbon nerves in Alexandroff-Hopf-Whitehead CW (Closure finite Weak) topological spaces. A planar ribbon (briefly, ribbon) in a CW space is the closure of a pair of nesting, non-concentric filled cycles that includes boundary but does not include the interior of the inner cycle. Each planar ribbon has its own distinctive shape determined by its outer and inner boundaries and the interior within its boundaries. A Vergili ribbon complex (briefly, ribbon complex) in a CW space is a non-void collection of countable planar ribbons. A ribbon nerve is a nonvoid collection of planar ribbons (members of a ribbon complex) that have nonempty intersection. A planar CW space is a non-void collection of cells (vertexes, edges and filled triangles) that may or may not be attached to other and which satisfy Alexandroff-Hopf-Whitehead containment and intersection conditions. In the context of CW spaces, planar ribbons, ribbon complexes and ribbon nerves are characterized by Betti numbers derived from standard Betti numbers $B_0$ (cell count), $B_1$ (cycle count) and $B_2$ (hole count), namely, $B_{rb}$ and $B_{rbNrv}$ introduced in this paper. Results are given for collections of ribbons and ribbon nerves in planar CW spaces equipped with an approximate descriptive proximity, division of the plane into three bounded regions by a ribbon and Brouwer fixed points on ribbons. In addition, the homotopy types of ribbons and ribbon nerves are introduced.

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1. Introduction

This paper introduces planar ribbons and ribbon complexes in a CW topological space $K$. A cell complex is a nonempty collection of cells. A cell in the Euclidean plane is either a 0-cell (vertex) or 1-cell (edge) or 2-cell (filled triangle). A nonvoid collection of cell complexes $K$ has a Closure finite Weak (CW) topology, provided $K$ is Hausdorff (every pair of distinct cells is contained in disjoint neighbourhoods [8, §5.1, p. 94]) and the collection of cell complexes in $K$ satisfy the Alexandroff-Hopf-Whitehead [1, §III, starting on page 124], [14, pp. 315-317], [15, §5, p. 223] conditions, namely, containment (the closure of each cell complex is in $K$) and intersection (the nonempty intersection of cell complexes is in $K$).

Definition 1. Planar Ribbon.
Let $cycA$, $cycB$ be nested filled cycles (with $cycB$ in the interior of $cycA$) defined on a collection of vertices $E$ on a finite, bounded, planar region in a CW space $K$. A planar ribbon $E$ (denoted by $rbE$) is defined by

$$rbE = \{cl(cycA) \setminus (cl(cycB) \setminus int(cycB)) : bdy(cl(cycB)) \subset cl(rbE)\}.$$ 

Example 1. A planar ribbon $rbE$ is shown in Fig. 1.1 on a pair of nested filled cycles $cycA$, $cycB$ with cycle $cycB$ in the interior of cycle $cycA$ and the interior $int(cl(cycB))$ is removed (not included) in the interior of cycle $cycA$. The white region in the interior of $cycA$ in Fig. 1.1 represents the interior of $cycB$ not included in $int(cycA)$. □

Definition 2. Vergili Ribbon Complex
Let $2^{VertK}$ denote the collection of subsets of vertices in a CW space $K$. A Vergili\(^1\) ribbon complex $K$ (denoted by $rbxK$) is a non-void collection of countable planar ribbons in a CW space, i.e.,

$$rbxK = \{rbE : E \in 2^{VertK}\}.$$ □

Example 2. Examples of Vergili ribbon complexes are given in 1° Fig. 1.1: $rbxK = \{rbE\}$

\(^1\)The structure of a ribbon complex was suggested by T. Vergili [13].
A filled planar cycle \( A \) (denoted by \( \text{cyc}A \)) has a nonempty interior with a boundary containing a non-void finite, collection \( E \) of path-connected vertices so that there is a path between any pair of vertices in \( \text{cyc}A \). The interior of the inner cycle in a ribbon is excluded from the ribbon. The outer and inner boundaries of a ribbon are simple, closed curves. A simple curve has no self-intersections (loops). A closed curve begins and ends in the same vertex for each vertex in the curve. A filled planar cycle includes that part of the plane inside the cycle boundary. A pair of cycles are nesting, provided one cycle contains the other cycle entirely within its interior. Let \( rbE \) be a planar ribbon. The closure of a ribbon \( E \) (denoted by \( \text{cl}(rbE) \)) includes its boundary (denoted by \( \text{bdy}(rbE) \)) and its interior (denoted by \( \text{int}(rbE) \)). The boundary of a filled cycle \( \text{cyc}A \) with the cycle interior excluded is denoted by \( \text{cl}(\text{cyc}A) \setminus \text{int}(\text{cyc}A) \).

**Remark 1.** A planar ribbon is analogous to a Brooks ribbon [2, §1.2], whose boundaries are a pair of simple, smoothly closed curves. A ribbon structure is also analogous to a vortex tube, which is a collection of lines that form a vortex surface or vector tube [5, §1.3, p. 7]. Although not considered here, it is possible to represent a non-void collection of convex ribbons in a ribbon complex as a Klee-Phelps convex groupoid [11].

A planar ribbon divides the plane into three disjoint three open sets and provides a boundary of each of the three planar regions. (see Theorem 9). In this paper, the focus is on ribbons on a finite, bounded region of the plane. In that case, a ribbon in the interior of a finite, bounded region of the plane divides the region into three disjoint bounded regions (see Theorem 8). In addition, a continuous map on a planar ribbon to itself has a fixed point (see the Ribbon Fixed Point Theorem 11), which is a straightforward consequence of the Brouwer Fixed Point Theorem.

**Definition 3.** A ribbon nerve \( E \) (denoted by \( \text{rbNrv}K \)) is a non-void collection of planar ribbons (in a ribbon complex \( rbxK \)) that have nonempty intersection, i.e.,

\[
\text{rbNrv}K = \left\{ rbE \in rbxK : \bigcap rbE \neq \emptyset \right\}.
\]

**Example 3.** Examples of ribbon nerves derived from a ribbon complex \( rbxK \) on a finite bounded planar region in a CW space are given in

1° Fig. 1.1: \( \text{rbNrv}K = \{ \{ rbE \} \} \)
Planar ribbon nerve complexes are examples of Edelsbrunner-Harer nerves.

**Definition 4.** Let $F$ be a finite, non-void collection of sets. An Edelsbrunner-Harer nerve \cite[§III.2, p. 59]{Edelsbrunner1994} consists of all nonempty subcollections of $F$ (denoted by $NrvF$) whose sets have nonempty intersection, i.e.,

$$NrvF = \{X \subseteq F : \bigcap X \neq \emptyset\}.$$

**Theorem 1.** A ribbon nerve is an Edelsbrunner-Harer nerve.

**Proof.** Let $F$ be a finite collection of ribbons in a CW space. Let the ribbon nerve $K$ (denoted by $rbNrvK$) be defined by

$$rbNrvK = \{rbE \in F : \bigcap rbE \neq \emptyset\}.$$

Hence, from Def. 4, $rbNrvK$ is an Edelsbrunner-Harer nerve. \qed

A partially filled planar ribbon interior contains planar holes. A planar hole is a finite planar region bounded by a simple, closed curve that has an empty interior. That is, the interior of a planar hole contains no cells. Holes in ribbons are analogous to surface jump discontinuities (gaps in a surface map) commonly found in vortex structures \cite{Peters2019}.

**Example 4.** Ribbon $rbE$ in Fig. 1(1.1) contains two holes in the interior of cycle cycA. In each case, a hole is represented by an opaque grey region. Again, for example, ribbon $rbB$ in Fig. 1(1.2) contains three holes (opaque gray regions) in its interior. \qed

Ribbons in a CW complex can be extracted from ordinary vortex nerves.

**Definition 5.** Let $K$ be a finite CW complex and let $2^K$ be the collection of all subsets of cells in $K$. A vortex nerve consists of a nonempty collection $E$ of nesting, usually non-concentric filled cycles $cycA$ in $K$ (denoted by $vNrvK$) that have have nonempty intersection and which have zero or more edges (called filaments) attached between pairs of cycles in $vNrvK$, i.e.,

$$vNrvK = \{cycA \in 2^K : \bigcap cycA \neq \emptyset\}.$$

**Example 5.** A collection $X$ of filled cycles $= \{cycA, cycA', cycB\}$ in a CW space $K$ is represented in Fig. 3. In this case, we have

$$vNrvK = \{cycA \in X : \bigcap cycA \neq \emptyset\}.$$

That is, the intersection of all cycles in the collection $X$ is nonempty. Hence, $vNrvK$ in Fig. 3 is an Edelsbrunner-Harer nerve. \qed

**Theorem 2.** A ribbon is a vortex nerve.

**Proof.** Let $rbE$ be a ribbon in planar CW space $K$ containing a pair of nesting, non-concentric cycles $cycA, cycB$ such that the boundary of $cycB$ is in the interior of $cycA$ and the interior of $cycB$ is not included in the interior of $cycA$, i.e., $\text{int}(\text{cl}(cycA)) \supset \text{bdy}(\text{cl}(cycB) \setminus \text{int}(cycB))$. Consequently, $cycA \cap cycB \neq \emptyset$. Then define nerve $Nrv$ to be

$$NrvK = \{cycA \in rbE : \bigcap cycA \neq \emptyset\}.$$

Hence, $NrvK$ is a vortex nerve. \qed
In general, a vortex nerve $vNrvK$ contains $k$ nesting, non-concentric cycles. The number of cycles $k$ in $vNrvE$ can be either an even or odd number. By definition, each pair of cycles in $vNrvK$ closest to each other is a ribbon complex. It is always the case that ribbons $rbA, rbA'$ in ascending order and next to each other in nerve $vNrvE$ have a common cycle, i.e., the outer boundary $bdy(rbA')$ is also the inner boundary of $rbA$. These observations lead to the following result.

**Theorem 3.** A vortex nerve with $k > 1$ cycles contains $k - 1$ ribbons.

**Example 6.** A sample vortex nerve $vNrvK$ containing 3 nesting, non-centric cycles $cycA, cycA', cycB$ (in ascending order) and 2 ribbons $rbE, rbE'$ (also in ascending order) is represented in Fig. 3. In this particular nerve, cycle $cycA'$ is the outer boundary of ribbon $rb$ and the inner boundary of ribbon $rbE'$.

**Theorem 4.** A vortex nerve with $k > 2$ cycles contains $k - 2$ ribbon nerves.

**Proof.** Immediate from Theorem 3 and Def. 3 (ribbon nerve).

**Example 7.** From Theorem 4, a vortex nerve $vNrvK$ containing $k = 3$ cycles has $k - 2 = 1$ ribbon nerve $rbNrvK = \{rbE, rbE' \in vNrvK : rbE \cap rbE' \neq \emptyset\}$, which is represented in Fig. 3.

## 2. Preliminaries

This section briefly introduces the approximate descriptive closeness of cell complexes in a CW space. For the axioms for an approximate descriptive proximity of nonempty sets $A, B$ (denoted by $A \delta_{\|\Phi\|} B$), the usual set intersection $\cap$ for a traditional spatial proximity [9, §1, p. 7] is replaced by an approximate descriptive intersection $\cap_{\|\Phi\|}$ [10, §7.2, p. 303], which is an extension of ordinary descriptive intersection [4] (denoted by $\cap$).

Approximate closeness of nonempty sets $A, B$ is measured in terms of the Euclidean distance between feature vectors $\vec{a}, \vec{b}$ (denoted by $\|\vec{a} - \vec{b}\|$) in $n$-dimensional Euclidean space $\mathbb{R}^n$. In this context, $\mathbb{R}^n$ is a feature space in which each feature vector is a description of a nonempty set in a space $X$. Let $2^X$ denote the collection of subsets in $X$. A probe function $\Phi : 2^X \rightarrow \mathbb{R}^n$ maps each a nonempty subset $A$ in $2^X$ to a feature vector.
that describes $A$. The mapping $\Phi : 2^X \rightarrow \mathbb{R}^n$ is defined by

$$\Phi(A) = \left( R_1, \ldots, R_n \right).$$

Let $A \delta_{\Phi} B$ denote the descriptive proximity of $A$ and $B$, i.e., the description $\Phi(A)$ matches the description $\Phi(B)$. Recall that the descriptive intersection of cell complexes in a space $K$ (denoted by $\bigcap_{\Phi} K$) is the set of all descriptively close cell complexes defined by

$$i.e., A \delta_{\Phi} B \quad \bigcap_{\Phi} K = \{ A, B \in K : \Phi(A) = \Phi(B) \}.$$

This form of descriptive proximity is not very useful, since it often the case that nonempty sets $A, B$ are close descriptively and yet $\Phi(A) \neq \Phi(B)$. For this reason, we consider defining descriptive closeness in terms of some threshold $th$, ushering in approximate descriptive closeness of sets $A, B \subset K$ (denoted by $A \parallel_{\Phi} B$), defined by

$$i.e., \text{Euclidean norm } \| \Phi(A) - \Phi(B) \| \text{ less than threshold } th > 0$$

$A \parallel_{\Phi} B$ implies

$$\| \Phi(A) - \Phi(B) \| < th.$$

In other words, we have two possible forms of approximate descriptive closeness to consider, namely,

1° Approximate descriptive closeness of cell complexes $cxE \in K, cxE' \in K$ on a single space $K$, defined by

$$\text{Descriptions } \Phi(cxE), \Phi(cxE') \text{ are } \delta_{\Phi} \text{ close in } K$$

$$\bigcap_{\| \Phi \|} K = \{ cxE, cxE' \in K : \| \Phi(cxE) - \Phi(cxE') \| < th \}.$$

2° Approximate descriptive closeness of cell complexes $cxE, cxE'$ on separated (disjoint) spaces $K, K'$, defined by

$$\text{Descriptions } \Phi(cxE), \Phi(cxE') \text{ are } \delta_{\| \Phi \|} \text{ close in } K \cup K'$$

$$\bigcap_{\| \Phi \|} (K \cup K') = \{ cxE, cxE' \in K \cup K' : \| \Phi(cxE) - \Phi(cxE') \| < th \}.$$

An approximate intersection of distinct CW complexes $K, K'$ is easily derived from ordinary descriptive intersection $\bigcap_{\Phi}$ by considering the norm of the difference between the feature vectors $\Phi(cxA), \Phi(cxB)$, which is less than some chosen threshold $th > 0$, i.e.,

$$\text{Approximation threshold: } th > 0.$$  

$$cxA \in K, cxB \in K'$$

$$cxA \delta_{\| \Phi \|} cxB, \text{ provided } \| \Phi(cxA) - \Phi(cxB) \| < th :$$

$$K \bigcap_{\| \Phi \|} K' =$$

$$\text{Approx. descriptive intersection of cell complexes}$$

$$\{ cxA, cxB \in K \cup K' : \| \Phi(cxA) - \Phi(cxB) \| < th \}.$$
Example 8. Let $K, K'$ be CW complexes, ribbons $rbE \in K, rbA \in K'$ and let $B_i$ be the Betti number, which is a count of the number of cycles in a cell complex, threshold $th = 1$ and let $\Phi(rbE) = B_1(rbE)$, $\Phi(rbA) = B_1(rbA)$. Then, for instance, we have $B_1(rbE \in K) = 2$ in Fig. 1.1.

$B_1(rbA \in K') = 2$ in Fig. 1.2. Hence, $rbE \delta_{||\Phi||} rbA$, since $||\Phi(rbE) - \Phi(rbA)|| < th$, and

$K \cap ||\Phi|| K' \neq \emptyset$. ■

Many other instances of non-void approximate descriptive intersection are possible.

Example 9. Let $K, K'$ be a pair of cell complexes on a finite, bounded planar region. Then consider sub-complexes $cxA \in K, cxB \in K'$ that are close to each other for some threshold $th > 0$. That is,

Collection of cell complexes that are $\delta_{||\Phi||}$ close

$$K \cap ||\Phi|| K' = \{cxA, cxB \in K \cup K': ||\Phi(cxA) - \Phi(cxB)|| < th\}. \quad ■$$

Example 10. Consider isolating those ribbon complexes $rbA \in K, rbB \in K'$ that are close to each other within some chosen threshold $th > 0$, i.e.,

Ribbon shapes that are $\delta_{||\Phi||}$ close

$$K \cap ||\Phi|| K' = \{rbA \in K, rbB \in K': ||\Phi(rbA) - \Phi(rbB)|| < th\}. \quad ■$$

Let $B_2(cxE)$ be the Betti number which is a count of the number of holes in a cell complex $cxE$. In Fig. 1.1, $B_2(rbE) = 2$, i.e., ribbon $rbE$ has two holes, which appear as opaque regions $\emptyset$ on $rbE$. In Fig. 1.2, $B_2(rbNrvK) = 5$, $B_2(rbA) = 2$, and $B_2(rbB) = 3$. Let threshold $th = 1$, $\Phi(cxE) = B_2(cxE)$ and require $||\Phi(cxA) - \Phi(cxB)|| < th$ for a pair of CW cell complexes $cxA, cxB$ in a CW complex $K$. Then, we have $th = 1$.

$$||\Phi(rbE) - \Phi(rbA)|| = 0 < th \Rightarrow rbE \delta_{||\Phi||} rbA.$$

$$||\Phi(rbE) - \Phi(rbB)|| \neq th \Rightarrow rbE \delta_{||\Phi||} rbB.$$

$$||\Phi(rbE)) - \Phi(rbNrvK)|| \neq th \Rightarrow rbE \delta_{||\Phi||} rbNrvE.$$

In effect, only $rbE$ in Fig. 1.1 and $rbA$ in Fig. 1.2 have approximate descriptive proximity, i.e. only these ribbon complexes are approximately close descriptively relative to the chosen threshold and feature vector defined in terms of the Betti number $B_2(rbE)$ for a ribbon complex $rbE$ in the CW space $K$ and $B_2(rbA)$ for a ribbon complex $rbA$ in the CW space $K'$ represented in Fig. 1. ■

Let $K$ be a finite, bounded, planar nonempty space, $A, B \in K$. If $A \cap B$ is nonempty, there is at least one element of $A$ with a description that approximately matches the description of an element of $B$. It is entirely possible to identify a pair of nonempty sets $A, B$ separated spatially (i.e., $A$ and $B$ have no members in common) and yet $A, B$ have approximate matching descriptions. The pair $(K, \delta_{||\Phi||})$ is an approximate descriptive proximity space, provided the following axioms are satisfied.

$$(xdP0): \emptyset \delta_{||\Phi||} A, \forall A \subset K.$$
(xdP1): \( A \delta_{||\Phi||} B \Leftrightarrow B \delta_{||\Phi||} A. \)

(xdP2): \( A \bigcap B \neq \emptyset \Rightarrow A \delta_{||\Phi||} B. \)

(xdP3): \( A \delta_{||\Phi||} (B \cup C) \Leftrightarrow A \delta_{||\Phi||} B \) or \( A \delta_{||\Phi||} C. \)

The converse of axiom xdP2 also holds.

**Lemma 1.** Let \( K \) be a space equipped with the relation \( \delta_{||\Phi||} \), \( A, B \subset K \). Then \( A \delta_{||\Phi||} B \) implies \( A \bigcap B \neq \emptyset \).

**Proof.** Let \( \theta > 0 \), \( A, B \subset K \). By definition, \( A \delta_{||\Phi||} B \) implies \( ||\Phi(A) - \Phi(B)|| < \theta \). Consequently, \( A, B \in \bigcap K \). Hence, \( \bigcap K \neq \emptyset \) and the converse of (xdP2) follows. \( \square \)

**Theorem 5.** Let \( K \) be a collection of planar ribbon complexes equipped with the proximity \( \delta_{||\Phi||} \), \( rbA, rbB \in K \). Then \( rbA \delta_{||\Phi||} rbB \) implies \( rbA \bigcap rbB \neq \emptyset \).

**Proof.** Immediate from Lemma 1. \( \square \)

**Corollary 1.** Let \( K \) be a collection of planar ribbon nerves equipped with the proximity \( \delta_{||\Phi||} \), \( rbNrvA, rbNrvB \in K \). Then \( rbNrvA \delta_{||\Phi||} rbNrvB \) if and only if \( rbNrvA \bigcap rbNrvB \neq \emptyset \).

**Proof.** Let \( K \) be equipped with \( \delta_{||\Phi||} \).

\[ \Rightarrow: \text{From Lemma 1, } rbNrvA \delta_{||\Phi||} rbNrvB \text{ implies } rbNrvA \bigcap rbNrvB \neq \emptyset. \]

\[ \leftarrow: \text{From Axiom xdP2, } rbNrvA \bigcap rbNrvB \neq \emptyset \text{ implies } rbNrv \delta_{||\Phi||} rbNrv. \] \( \square \)

**Lemma 2.** Let \( K \) be a collection of ribbon complexes equipped with the relation \( \delta_{||\Phi||} \), ribbons \( rbA, rbB \subset K \). Then \( K \) is an approximate descriptive proximity space.

**Proof.** Let \( K \) be a collection of ribbon complexes equipped with \( \delta_{||\Phi||} \) and threshold \( \theta > 0 \). Then (dP0): The empty set contains ribbons. Hence, \( \emptyset \delta_{||\Phi||} rbA, \forall rbA \in K \).

(dP1): Assume \( rbA \delta_{||\Phi||} rbB \), if and only if \( ||\Phi(rbA) - \Phi(rbB)|| < \theta \). Then \( rbA \bigcap rbB \neq \emptyset \), if and only if \( rbB \bigcap rbA \neq \emptyset \), if and only if \( rbB \delta_{||\Phi||} rbA \).

(dP2): Assume \( rbA \bigcap rbB \neq \emptyset \). Then, by definition of \( \bigcap \), \( ||\Phi(rbA) - \Phi(rbB)|| < \theta \).

Hence, \( \bigcap rbA \delta_{||\Phi||} rbB \).

(dP3): \( rbA \delta_{||\Phi||} (rbB \cup rbC) \) if and only if \( ||\Phi(rbA) - \Phi(rbB)|| < \theta \), implying \( rbA \delta_{||\Phi||} rbB \) or, by definition of \( \bigcup \), \( ||\Phi(rbA) - \Phi(rbC)|| < \theta \), implying \( rbA \delta_{||\Phi||} rbB \).

Hence, \( (K, \delta_{||\Phi||}) \) is an approximate descriptive proximity space. \( \square \)

**Theorem 6.** Let \( K \) be a collection of planar ribbon nerves equipped with the proximity \( \delta_{||\Phi||} \). Then \( (K, \delta_{||\Phi||}) \) is an approximate descriptive proximity space.

**Proof.** Immediate from Lemma 2, since each planar ribbon nerve is a collection of ribbons equipped with \( \delta_{||\Phi||} \).

**Example 11.** Let \( K \) in Fig. 4.1 be a collection of planar ribbons equipped with \( \delta_{||\Phi||} \), forming a ribbon nerve \( rbNrvE \) in an approximate descriptive space \( (K, \delta_{||\Phi||}) \). Similarly, let \( K' \) in Fig. 4.2 be a collection of planar ribbons equipped with \( \delta_{||\Phi||} \), forming a ribbon nerve \( rbNrvE' \).
an approximate descriptive space \((K', \delta_{||\cdot||})\). Also let \((K \cup K', \delta_{||\cdot||})\) be an approximate descriptive space.

Recall that Betti number \(B_2\) is a count of the number of holes in a cell complex. Then let \(\Phi(rbNrvE) = B_2\) and let \(\Phi(rbNrvE') = B_2\), i.e., each of the ribbon nerves in Fig. 4 has description equal to the number of ribbon surface holes. Ribbon nerve \(rbNrvE'\) is described in a similar fashion. Further, let threshold \(th > 1\). Then we have

\[
rbNrvE \delta_{||\cdot||} rbNrvE', \text{ since, for } th > 0, \quad \Phi(rbNrvE) = (B_2(rbNrvE)) = \Phi(rbNrvE') = 6.
\]

That is, \(||\Phi(rbNrvE) - \Phi(rbNrvE')|| < th\) for every choice of \(th > 0\). \(\blacksquare\)

**Theorem 7.** Let \(K\) be a collection of planar ribbon nerves equipped with the proximity \(\delta_{||\cdot||}, (K, \delta_{||\cdot||})\) an approximate descriptive proximity space, \(rbNrvE, rbNrvE' \in K, th > 0\). \(rbNrvE \delta_{||\cdot||} rbNrvE'\) if and only if \(\Phi(rbNrvE) = \Phi(rbNrvE')\).

**Proof.** Immediate from the definition \(\delta_{||\cdot||}\) on pairs of ribbon nerves in \(K\). \(\square\)

### 3. Main Results

This section gives some main results for ribbon complexes.

#### 3.1. Ribbon division of the plane into three bounded regions and Brouwer fixed points on ribbons.

L.E.J. Brouwer [3] introduced a curve which divides the plane into three open sets and provides a boundary of each of the regions. An important result for a ribbon complex in a CW space is the division of a finite bounded region of the plane into three bounded regions.

**Theorem 8.** A ribbon in the interior of a finite, bounded region of the plane divides the region into three disjoint bounded regions.

**Proof.** Let \(rbE\) be a ribbon containing nesting cycles \(cycA, cycB\) with \(\text{bdy}(\text{cl}(cycB)) \subset \text{int}(\text{cl}(cycA))\) on \(\pi\), a finite, bounded region of the plane \(\pi\). The planar region \(\pi_1 = \ldots\)
\( \pi \setminus \text{cl}(rbE) \) is that part of \( \pi \) outside the ribbon. A second region of the plane \( \pi_2 \subset \pi \) is the closure of \( rbE \) minus the closure of its inner cycle \( \text{cl}(cycB) \), defined by

\[ \pi_2 = \text{cl}(rbE) \setminus \text{cl}(cycB). \]

\( \pi_1 \cap \pi_2 = \emptyset \), since ribbon \( rbE \) is not included in \( \pi_1 \). A third region \( \pi_3 \subset \pi \) is the closure of \( cycB \), i.e., \( \pi_3 = \text{cl}(cycB) \). \( \pi_2 \cap \pi_3 = \emptyset \), since \( \text{cl}(cycB) \) is not included in \( \pi_2 \). Hence, these three planar regions are disjoint.

It is also the case that a planar ribbon has the division property of the curve discovered by Brouwer.

**Theorem 9.** A planar ribbon divides the plane into three open sets and provides a boundary of each of the three planar regions.

**Proof.** Let \( rbE \) be a ribbon on \( \mathbb{R}^2 \). From Theorem 8, \( rbE \) divides the plane into three open sets and provides a boundary of each of the three planar regions.

**Remark 2.** Approximate Descriptive Proximity of Brouwer Planar Regions. Let \( B = \{ \pi_1, \pi_2, \pi_3 \} \) be collection of planar regions from the proof of Theorem 8, equipped with the proximity \( \delta_{|\Phi|} \). Also, let \( B_1 \) be the Betti number with is a count of the number cycles in a cell complex and introduce threshold \( 0 < th \leq 1 \). For \( E \in B \), let \( \Phi(E) = B_1(E) \) and choose \( th = 1 \). From this, we obtain

1. \( \pi_1 \delta_{|\Phi|} \pi_2 \), since \( \| \Phi(\pi_1) - \Phi(\pi_2) \| \leq 1 \).
2. \( \pi_1 \delta_{|\Phi|} \pi_3 \), since \( \| \Phi(\pi_1) - \Phi(\pi_3) \| \leq 1 \).
3. \( \pi_2 \delta_{|\Phi|} \pi_3 \), since \( \| \Phi(\pi_2) - \Phi(\pi_3) \| < 1 \).

For \( th > 1 \), all three Brouwer planar regions from theorem 8 do have approximate descriptive proximity.

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**Figure 5.** \( \bullet \) = ribbon hole, and \( f(p \in \overline{pq}) = p \in \text{bdy}(\text{cl}(cycB)) \).

**Theorem 10.** Brouwer Fixed Point Theorem [12, §4.7, p. 194]

Every continuous map from \( \mathbb{R}^n \) to itself has a fixed point.

**Theorem 11.** Ribbon Fixed Point Theorem

A continuous map on a ribbon on \( \mathbb{R}^2 \) to itself has a fixed point.

**Proof.** Let \( rbE \) be a ribbon on \( \mathbb{R}^2 \). From Theorem 10, each continuous map \( f : rbE \to rbE \) has a fixed point.

**Example 12.** Let \( \overline{pq} \) be an edge attached between a vertex \( q \) on \( \text{bdy}(\text{cl}(cycA)) \) and vertex \( p \) on \( \text{bdy}(\text{cl}(cycB)) \) on a planar ribbon \( rbE \), cycles \( cycA, cycB \subset rbE \), \( \text{int}(cycA) \supset \text{bdy}(cycB) \) be represented in Fig. 5, i.e., the boundary of cycle \( cycB \) is a subset of the interior of cycle \( cycA \). Then let the continuous map \( f : rbE \to rbE \) be defined by

\[ f((x, y) \in rbE) = (x, y) \in \text{bdy}(\text{cl}(cycB)). \]
From Theorem 11, the mapping \( f \) has a fixed point. Let \( \overrightarrow{pq} \) be an edge with vertex \( p \in \text{bdy}(c\text{l}(\text{cyc} B)) \), \( q \in \text{bdy}(c\text{l}(\text{cyc} A)) \) as shown in Fig. 5. The mapping \( f \) maps vertex \( p \in \overrightarrow{pq} \) to \( p \in \text{bdy}(c\text{l}(\text{cyc} B)) \), a fixed point on the boundary of cyc \( B \). That is, \( f \) maps vertex \( p \in \overrightarrow{pq} \) to a fixed point on the inner boundary \( \text{rb} E \), namely, \( p \). Hence, \( f(p \in \mathbb{R}^2) = p \in \mathbb{R}^2 \), which is the desired result.

The gradient (angle \( \theta_p \) of the tangent) of a fixed point \( p \) on a ribbon cycle boundary is a useful source of a distinguishing characteristic of a ribbon, defined by

\[
\theta_p = \tan^{-1} \left( \frac{\partial f}{\partial x} \right) \left. \right|_{\overrightarrow{pq}}.
\]

Example 13. Let \( \text{rb} E \) be a ribbon complex with a fixed point \( f(p) = p \) on a ribbon boundary with gradient angle \( \theta_p \) in a CW space \( K \) and let \( \text{rb} E' \) be a ribbon complex with a fixed point \( g(q) = q \) on a ribbon boundary with gradient angle \( \theta_q \) in a CW space \( K' \). Let \( \Phi(\text{rb} E) = \theta_p \), \( \Phi(\text{rb} E') = \theta_q \), and threshold \( th > 0 \). Then \( \text{rb} E \) \( \Phi || \text{rb} E' \) if and only if \( \| \Phi(\text{rb} E) - \Phi(\text{rb} E') \| < th \), i.e., the fixed points \( p, q \) have close gradient angles.

3.2. Ribbon and Ribbon Nerve Betti numbers.

There are three basic types of Betti numbers, namely, \( B_0 \) (number of cells in a complex), \( B_1 \) (number cycles in a complex) and \( B_2 \) (number holes in a complex) \([16, \S 4.3.2, \text{p. 57}]\). In terms of ribbons and ribbon nerves in CW spaces, Betti numbers that enumerate fundamental shape structures are useful, namely,

**Ribbon Betti number**

Denoted by \( B_{\text{rb}} \), which is a count of the number of filaments (edges attached between ribbon cycles) + number of ribbon holes + 2 cycles. Let \( \text{rb} E \) be a planar ribbon, which is a pair of nesting, non-concentric filled cycles.

**Example 14.** In Fig. 5, the structure of ribbon \( \text{rb} E \) contains a pair of nesting cycles \( \text{cyc} A, \text{cyc} B \), a single filament \( \overrightarrow{pq} \) attached to the cycles and 3 holes (represented by \( \bullet \)). Hence, \( B_{\text{rb}}(\text{rb} E) = B_0(\text{rb} E) + B_1(\text{rb} E) + B_2(\text{rb} E) = 1 + 2 + 3 = 6 \).

**Vergili Ribbon complex Betti number**

Denoted by \( B_{\text{rbx}} \), which is a count of the number of ribbons in a Vergili ribbon complex.

**Example 15.** In Fig. 2, the structure of Vergili ribbon complex \( \text{rbx} K \) contains 5 ribbons. Hence, \( B_{\text{rbx}}(\text{rbx} K) = 5 \).

**Ribbon nerve Betti number**

Denoted by \( B_{\text{rbNrv}} \), which is a count of the number of filaments (edges attached between adjacent pairs of nerve cycles) + number of nerve holes + number of overlapping (intersecting) ribbons.

**Example 16.** In Fig. 4.1, the structure of a ribbon nerve \( \text{rbNrv} E \) contains 3 intersecting ribbons \( \text{rb} A, \text{rb} B, \text{rb} B' \), zero filaments and 6 holes (represented by \( \bullet \)). Hence, \( B_{\text{rbNrv}} = 0 + 3 + 3 = 6 \).

**Lemma 3.** Let \( B_0, B_1, B_2 \) be Betti numbers that count the number of cells, number of cycles and number of holes in a planar CW complex, respectively. Then

\[
1^o \quad B_{\text{rb}}(\text{rb} E) = B_0(\text{rb} E) + B_2(\text{rb} E) + 2 \quad \text{for a ribbon \text{rb} E}.
\]
For homotopy types.

**Theorem 13.** Let \( \mathcal{V} \) be a finite collection of closed, convex sets in Euclidean space. Then the nerve \( NrvK = \{ rbNrvK \in K : \cap rbNrvK \neq \emptyset \} \) of \( K \) and the union of the ribbon nerves \( rbNrvK \) in nerve \( NrvK \) have the same homotopy type.

**Proof.** From Theorem 13, we have that the union of the Vergili ribbon complexes \( rbxK \) in \( rbNrvK \) and ribbon nerve \( rbNrvK \) have the same homotopy type. \( \square \)

From Theorem 13, we obtain a fundamental results for ribbon nerves.

**Theorem 15.** Let \( K \) be a collection of ribbon nerves \( rbNrvK \) that are closed, convex complexes in Euclidean space. Then the nerve \( NrvK = \{ rbNrvK \in K : \cap rbNrvK \neq \emptyset \} \) of \( K \) and the union of the ribbon nerves \( rbNrvK \) in nerve \( NrvK \) have the same homotopy type.
Proof.
From Theorem 13, we have that the union of the ribbon nerves $\text{rbNrv}K$ in $K$ and nerve $\text{Nrv}K$ have the same homotopy type. □

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