BLOW-UP PHENOMENA AND TRAVELLING WAVE SOLUTIONS TO THE PERIODIC INTEGRABLE DISPERSE HUNTER-SAXTON EQUATION

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(Communicated by Adrian Constantin)

Abstract. In this paper, we mainly study the Cauchy problem of an integrable dispersive Hunter-Saxton equation in periodic domain. Firstly, we establish local well-posedness of the Cauchy problem of the equation in $H^s(S)$, $s > \frac{3}{2}$, by applying the Kato method. Secondly, by using some conservative quantities, we give a precise blow-up criterion and a blow-up result of strong solutions to the equation. Finally, based on a sign-preserve property, we transform the original equation into the sinh-Gordon equation. By using the travelling wave solutions of the sinh-Gordon equation and a period stretch between these two equations, we get the travelling wave solutions of the original equation.

1. Introduction. In the paper we consider the Cauchy problem for the following periodic integrable dispersive Hunter-Saxton equation

$$
\begin{cases}
u_{xt} = \nu + 2\nu \nu_{xx} + \nu_x, \\
u(t, x)|_{t=0} = \nu_0(x), \\
u(t, x+1) = \nu(t, x).
\end{cases}
$$

(1.1)

Recently, Hone, Novikov and Wang in [28] present a classification of nonlinear partial differential equations of second order of the general form

$$
u_{xt} = \nu + c_0 \nu^2 + c_1 \nu \nu_x + c_2 \nu \nu_{xx} + c_3 \nu_x^2 + d_0 \nu^3 + d_1 \nu^2 \nu_x + d_2 \nu^2 \nu_{xx} + d_3 \nu \nu_x^2.
$$

(1.2)

It is known that the above form contains many interesting equations, especially some valuable integrable ones.

2010 Mathematics Subject Classification. Primary: 35G25; Secondary: 35L05, 35Q53.

Keywords and phrases. An integrable dispersive Hunter-Saxton equation, the Kato method, Blow-up, travelling wave solutions, the sinh-Gordon equation.

This work was partially supported by NNSFC (No.11671407), FDCT (No. 008/2013/A3), Guangdong Special Support Program (No. 8-2015), and the key project of NSF of Guangdong province (No. 2016A030311004).

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The first important integrable member of the form (1.2) is the short-pulse equation \[^47\] in case that \(c_j = 0\) for \(j = 0, 1, 2, 3\) and \(d_0 = d_1 = 0, d_3 = 2d_2\), or more simplified

\[ u_{xt} = u + (u^3)_{xx}. \]

The short-pulse equation is derived by Schäfer and Wayne in \[^47\] as an approximate model of Maxwell’s equations describing the propagation of ultra-short optical pulses in nonlinear media. The local well-posedness of the short-pulse equation in \(H^2\) on the line was showed by Schäfer and Wayne in their original paper \[^47\]. They also proved the non-existence of smooth traveling wave solutions \[^47\]. Later, by using some conserved quantities, Pelinovsky and Sakovich \[^46\] extended the local solution into a global solution. The blow-up phenomena both on the line and in the periodic domain were investigated by Liu, Pelinovsky and Sakovich in \[^40\].

The second important integrable member of (1.2) is the Ostrovsky-Hunter equation \[^3\]

\[ u_{xt} = u + (u^2)_{xx}. \]

This equation is known under different names, such as the Vakhnenko equation \[^42\], the short-wave equation \[^29\], and the reduced Ostrovsky equation \[^44\], which models small-amplitude long waves in rotating fluids of finite depth, under the assumption of no-high frequency dispersion. Local existence of solutions of the Ostrovsky-Hunter equation in \(H^s(\mathbb{R})\) for \(s > \frac{3}{2}\) was obtained in \[^48\]. But for sufficiently steep initial data \(u_0 \in C^1(\mathbb{R})\), local solutions break \[^39\,4\] in a finite time in the standard sense of finite-time wave breaking that occurs in the inviscid Burgers equation \(u_t + uu_x = 0\). Then, by using a new transformation of the reduced Ostrovsky equation to the integrable Tzitzéica equation, Grimshaw and Pelinovsky \[^26\] proved global existence of small-norm solutions in \(H^3(\mathbb{R})\).

Another series equation of the type (1.2) is the nonlinear Klein-Gordon equation. Considering a linear coordinate transform \(x = \xi + \tau^2, t = \xi - \tau^2\), which leads to \(\partial_{xt} = \partial_\xi^2 - \partial^2_\tau := \Box\). After this transform, and letting \(c_j = d_j = 0\) for \(j = 1, 2, 3\), we get the well-known nonlinear Klein-Gordon equation

\[ \Box u = u + c_0 u^2 + d_0 u^3. \] (1.3)

Some sufficient conditions on quadratic \((d_0 = 0)\) or cubic \((c_0 = 0)\) nonlinearities were given in \[^24\], to achieve the global existence and find sharp asymptotic of small solutions to the Cauchy problem with small and regular initial data having a compact support. Moreover it was proved that the asymptotic profile of solutions differs from that of the linear Klein-Gordon equation. Compactness condition on the data was removed in \[^27\] in the case of a real-valued solution. When the initial data are complex-valued, the global existence and \(L^\infty\) time decay estimates of small solutions to the Klein-Gordon equation with cubic nonlinearity were obtained in \[^50\].

In this paper, we study the equation \((1.1)\), which is one of the integrable generalized short-pulse equation of the form \((1.2)\), that means it possesses an infinite hierarchy of local higher symmetries \[^28\]. We can consider \((1.1)\) as the Hunter-Saxton (HS) equation with a dispersion term \(u\). In fact, dropping the dispersion
term $u$ from right-hand side of (1.1) and replacing $u$ with $\frac{1}{2}u$, we can get the following HS equation

$$ (u_t + uu_x)_x = \frac{1}{2}u_x^2. \quad (1.4) $$

The equation (1.4) was derived by Hunter and Saxton as an asymptotic model of liquid crystals [30]. The $x$ derivative of the HS equation corresponds to geodesic flow on an infinite-dimensional homogeneous space with constant positive curvature [36]. The HS equation also has a bi-Hamiltonian structure [30, 43] and is completely integrable [11, 31]. The initial value problems for the HS equation on the line and the circle were studied in [30, 54]. Global solutions of the HS equation was investigated in [10, 13, 14]. The local well-posedness for the Cauchy problem of the equation (1.1) has not been studied yet. In the paper we first establish the local well-posedness of the Cauchy problem (1.1) in $H^1$ laws of the equation, and use these obtained conservative quantities to control the existence of strong solutions were discussed in [10, 13, 14]. The global weak solutions were studied in [11, 12, 15, 16, 41, 51]. The local well-posedness for the Cauchy problem of the equation (1.1) has not been studied yet. In the paper we first establish the local well-posedness of the Cauchy problem (1.1) in $H^s(\mathbb{S})$, $s > \frac{3}{2}$, by using the Kato method. Then, we deduce some useful conservation laws of the equation, and use these obtained conservative quantities to control the $H^1$ norm of the solution. Next, we give a precise blow-up criterion, and apply this precise criterion to obtain a blow-up result and the precise blow-up rate for strong solutions to the equation (1.1). Moreover, we study the travelling wave solutions. Based on a sign-preserve property, we transform the original equation into the sinh-Gordon equation, using the travelling wave solutions of the sinh-Gordon equation and a period stretch between these two equation, we then get the travelling wave solutions of the original equation (1.1).

2. Local well-posedness. In order to establish local well-posedness of the Cauchy problem (1.1), we first consider the following Cauchy problem

$$ \begin{cases} u_t - 2u u_x = u - u^2 - \int_0^1 (u - u_x^2) dy, & t > 0, \ x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), & t \geq 0, \ x \in \mathbb{R}, \\ u(t, x)|_{t=0} = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (2.1) $$

Comparing (2.1) with the original equation (1.1), here we minus the average of $(u - u_x^2)$ to insure that the integral $\int_0^1$ of the right hand side of (2.1) is a continuous periodic function when $u$ belongs to some Sobolev spaces. Integrating both sides of (2.1) with respect to $x$, and by choosing a specific boundary term, we can transform (2.1) into a transport-like equation

$$ \begin{cases} u_t - 2u u_x = \partial_x^{-1}(u - u_x^2) - \int_0^1 \partial_x^{-1}(u - u_x^2)(y) dy, & t > 0, \ x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), & t \geq 0, \ x \in \mathbb{R}, \\ u(t, x)|_{t=0} = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (2.2) $$

Here $\partial_x^{-1}f(x) := \int_0^1 (f(y) - f(z)) dz dy$. The boundary term $\int_0^1 \partial_x^{-1}(u - u_x^2)(y) dy$
is chosen to insure that the quantity $\int_0^1 (u - u_x^2)(t, y)dy$ is conserved, which will be shown a posteriori.

Let $S$ denote a circle of unit length. The usual Sobolev norm of a function $f(x)$ on $S$ is defined based on its Fourier series $\hat{f}(n), n \in \mathbb{Z}$, more precisely, we have

$$||f||^2_{H^s(S)} = \sum_{n=-\infty}^{\infty} (1 + n^2)^s |\hat{f}(n)|^2.$$  

For simplicity, sometimes we will denote the $H^s(S)$ norm by $|| \cdot ||_s$. Local well-posedness of the Cauchy problem (2.2) with initial data $u_0 \in H^s(S), s > \frac{3}{2}$ can be obtained by using the framework presented in [54] for the following Hunter-Saxton equation,

$$\begin{align*}
\begin{cases}
  u_{xt} = a - uu_{xx} - \frac{1}{2} u_x^2, & t > 0, x \in \mathbb{R}, \\
  u(t,x)|_{t=0} = u_0(x), & x \in \mathbb{R}, \\
  u(t, x + 1) = u(t, x) & t > 0, x \in \mathbb{R},
\end{cases}
\end{align*}$$

(2.3)

where $a = -\frac{1}{2} \int_0^1 u_x^2 dx = -\frac{1}{2} \int_0^1 u_{xx} dx$ is a constant. Similar to the proof of the well-posedness of the above HS equation, we have the following local well-posedness result for (2.2) by the Kato method.

**Theorem 2.1.** Given $u_0 \in H^r(S), r > \frac{3}{2}$. Then there exists a maximal $T = T(u_0) > 0$, and a unique solution $u$ to (2.2), such that

$$u = u(\cdot, u_0) \in C([0, T); H^r(S)) \cap C^1([0, T); H^{r-1}(S)).$$

And, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \mapsto u(\cdot, u_0) : H^r(S) \rightarrow C([0, T); H^r(S)) \cap C^1([0, T); H^{r-1}(S)),$$

is continuous. Moreover, for $t \in [0, T)$ we have the following conserved quantity

$$\int_0^1 (u - u_x^2)(t, x)dx = \int_0^1 (u_0 - (\partial_x u_0)^2)dx.$$  

**Remark 2.2.** The conservative quantity of Theorem 2.1 implies the local well-posedness of the equation (1.1). In fact, for the initial data $u_0 \in H^r(S), s > \frac{3}{2}$ satisfying $\int_0^1 (u_0 - u_{xx})dx = 0$, the equation (1.1) and the equation (2.2) are equivalent. On the one hand, for $u_0$ satisfying $\int_0^1 (u_0 - u_{xx})dx = 0$, by Theorem 2.1 the corresponding solution of (2.2) satisfies $\int_0^1 (u - u_x^2)(t, x)dx = 0$. Differentiating (2.2) with respect to $x$, we deduce that $u$ satisfies the equation (1.1). On the other hand, any solution $u(t, x)$ of (1.1) entails that $\int_0^1 u(t, x)dx = \int_0^1 u_x^2(t, x)dx = \int_0^1 u_0(x)dx = \int_0^1 u_x^2(0, x)dx$, see Lemma 3.1 below. Thus, by integrating (1.1) with respect to $x$, one can easily see that $u(t, x)$ is also the solution of (2.2).

Applying Theorem 2.1 and Remark 2.2, we have the following local well-posedness result for (1.1).

**Theorem 2.3.** Given $u_0 \in H^r(S), r > \frac{3}{2}$. Suppose that $u_0$ satisfies the following condition:

$$\int_0^1 (u_0 - (\partial_x u_0)^2)dx = 0.$$

Then there exists a maximal $T = T(u_0) > 0$, and a unique solution $u$ to (1.1), such that

$$u = u(\cdot, u_0) \in C([0, T); H^r(S)) \cap C^1([0, T); H^{r-1}(S)).$$
And, the solution depends continuously on the initial data, i.e., the mapping
\[ u_0 \mapsto u(\cdot, u_0) : H^r(S) \to C([0, T); H^r(S)) \cap C^1([0, T); H^{r-1}(S)), \]
is continuous. Moreover, for \( t \in [0, T) \) we have the following conserved quantity
\[ \int_0^1 (u - u_x^2)(t, x)dx = \int_0^1 (u_0 - (\partial_x u_0)^2)dx = 0. \]

To prove Theorem 2.1 we first recall the Kato method for the Cauchy problem for abstract quasi-linear equations of evolution. For convenience we state the relevant theorem in the simplest form sufficient for the present purpose. Consider the Cauchy problem for the quasi-linear equation of evolution
\[
\begin{aligned}
\frac{dv}{dt} + A(v)v &= f(t, v), \quad t \geq 0, \\
v(0) &= \phi,
\end{aligned}
\] (2.4)

Let \( X, Y \) be reflexive Banach spaces with \( Y \) continuously and densely imbedded in \( X \). Let \( Q \) be an isomorphism (bi-continuous linear map) of \( Y \) onto \( X \). Assume that the function \( A \), defined on \( Y \), and \( f(t, \cdot) \) satisfies the following conditions:
(i) \( A(y) \in L(Y, X) \) for \( y \in X \) with
\[ \| (A(y) - A(z))w \|_X \leq \mu_1 \| y - z \|_Y \| w \|_Y, \quad y, z, w \in Y, \]
and \( A(y) \in G(X, 1, \beta) \) (i.e., \( A(y) \) is quasi-\( m \)-accretive), uniformly on bounded sets in \( Y \).
(ii) \( QA(y)Q^{-1} = A(y) + B(y), \) where \( B(y) \in L(X) \) is bounded, uniformly on bounded sets in \( Y \). Moreover,
\[ \| (B(y) - B(z))w \|_X \leq \mu_2 \| y - z \|_Y \| w \|_X, \quad y, z \in Y, \quad w \in X. \]
(iii) For each \( y \in Y \), \( t \to f(t, y) \) is continuous on \([0, \infty)\) to \( X \). For each \( t \in [0, \infty) \), \( f(t, y) : Y \to Y \) and extends also to a map from \( X \) into \( X \). All \( t \in [0, \infty) \), \( f \) is uniformly bounded on bounded sets in \( Y \), and
\[ \| f(t, y) - f(t, z) \|_Y \leq \mu_3 \| y - z \|_Y, \quad t \in [0, \infty), \quad y, z \in Y, \]
\[ \| f(t, y) - f(t, z) \|_X \leq \mu_4 \| y - z \|_X, \quad t \in [0, \infty), \quad y, z \in X. \]

Here \( \mu_1, \mu_2, \mu_3, \) and \( \mu_4 \) depend only on \( \| y \|_Y, \| z \|_Y \).

**Theorem 2.4.** (Kato [32]) Assume that (i), (ii), and (iii) hold. Given \( \phi \in Y \), there is a maximal \( T > 0 \) depending only on \( \| \phi \|_Y \) and a unique solution \( v \) to (2.4) such that
\[ v = v(\cdot, \phi) \in C([0, T); Y) \cap C^1([0, T); X). \]
Moreover, the map \( \phi \to v(\cdot, \phi) \) is continuous from \( Y \) to \( C([0, T); Y) \cap C^1([0, T); X). \)

Set \( A(u) = -2u\partial_x, \) \( f(t, u) = \partial_x^{-1}(u-u_x^2) - \int_0^1 \partial_x^{-1}(u-u_x^2)(y)dy, \) \( Y = H^r(S), X = H^{r-1}(S), \) and \( Q = \Lambda = (1 - \partial_x^2)^{3/2} \). Obviously, \( Q \) is an isomorphism of \( H^r(S) \) into \( H^{r-1}(S) \). In order to prove Theorem 2.1 by applying above theorem, we only need to verify that \( A(u) \) and \( f(t, u) \) satisfy the conditions (i)-(iii).

To pursue our goal, we need the following lemmas.

**Lemma 2.5.** [32] Let \( s, t \) be real numbers such that \(-s < t \leq s, \) and \( f \in H^s(S), g \in H^t(S) \). Then
\[
\| fg \|_t \leq C \| f \|_s \| g \|_t, \quad \text{if} \quad s > \frac{1}{2},
\]
\[
\| fg \|_{s+t-\frac{1}{2}} \leq C \| f \|_s \| g \|_t, \quad \text{if} \quad s < \frac{1}{2},
\]
Lemma 2.6. [54] Assume that $u \in H^r(S)$, $r > \frac{3}{2}$. Then, the operator $A(u) = -2u \partial_x$ is quasi-m-accretive in $L^2$ and $H^{r-1}$, i.e.,

$$A(u) \in G(L^2(S), 1, \beta_1) \cap G(H^{r-1}(S), 1, \beta_2),$$

with some $\beta_1, \beta_2 \in \mathbb{R}$. Moreover, $A(u) \in L(H^r(S), H^{r-1}(S))$, and we have the following estimate,

$$\| (A(u) - A(v))w \|_{r-1} \leq \mu_1 \| u - v \|_r \| w \|_r, \quad \forall \ u, v, w \in H^r(S).$$

Lemma 2.7. [54] $B(u) = [A, -2u \partial_x]A^{-1} \in L(H^{r-1}(S))$ for $u \in H^r(S)$. Moreover,

$$\| (B(u) - B(v))w \|_{r-1} \leq \mu_2 \| u - v \|_r \| w \|_r - 1,$n

for all $u, v \in H^r(S)$ and $w \in H^{r-1}(S)$.

The above two lemmas can be proved in the same way as in Lemmas (2.6)-(2.9) in [54] by replacing $u \partial_x$ with $-2u \partial_x$. At last, we have the following estimate for the right-hand side term $f(t, u)$ of (2.2).

Lemma 2.8. Let $f(t, u) = \partial_x^{-1}(u - u_x^2) - \int_0^1 \partial_x^{-1}(u - u_x^2)(y)dy$. Then $f(t, u)$ is uniformly bounded on bounded sets in $H^r(S)$ and satisfies

1. $\| f(t, v) - f(t, w) \|_r \leq \mu_3 \| v - w \|_r, \quad v, w \in H^r(S),$
2. $\| f(t, v) - f(t, w) \|_{r-1} \leq \mu_4 \| v - w \|_{r-1}, \quad v, w \in H^{r-1}(S).$

Proof. Let $v, w \in H^r(S)$, $r > \frac{3}{2}$, and denote $M(v, w) := \int_S \partial_x^{-1}(v - w)(y)dy + \int_S \partial_x^{-1}(v_x^2 - w_x^2)(y)dy$. Since $M(v, w)$ is a constant function of $x$, by the embedding $H^{r-1} \hookrightarrow L^\infty$ and Lemma [2.5] for $\forall \ p \in \mathbb{R}$ we have

$$\| M(v, w) \|_p = |M(v, w)| \leq \| \partial_x^{-1}(v - w) \|_{L^\infty(S)} + \| \partial_x^{-1}(v_x^2 - w_x^2) \|_{L^\infty(S)}$$

$$\leq C\| \partial_x^{-1}(v - w) \|_{r-1} + C\| \partial_x^{-1}(v_x^2 - w_x^2) \|_{r-1}$$

$$\leq C\| v - w \|_{r-2} + C\| (v_x + w_x)(v_x - w_x) \|_{r-2}$$

$$\leq C\| v - w \|_{r-2} + C\| v + w \|_r \| v - w \|_{r-1}$$

$$\leq C(1 + \| v \|_r + \| w \|_r) \| v - w \|_{r-1} \quad \text{(2.7)}$$

Note that $H^{r-1}(S)$ is a Banach algebra. Using (2.7) we can get

$$\| f(t, v) - f(t, w) \|_r = \| \partial_x^{-1}(v - w) - \partial_x^{-1}(v_x^2 - w_x^2) - M(v, w) \|_r$$

$$\leq C\| v - w \|_{r-1} + C\| v_x^2 - w_x^2 \|_{r-1} + \| M(v, w) \|_r$$

$$\leq C\| v - w \|_r + C\| v_x + w_x \|_{r-1} \| v_x - w_x \|_{r-1} + \| M(v, w) \|_r$$

$$\leq C(1 + \| v \|_r + \| w \|_r) \| v - w \|_r$$

$$= \mu_3 \| v - w \|_r.$$

This proves (1). Let $w = 0$ in the above inequality, we obtain that $f$ is uniformly bounded on bounded set in $H^r(S)$. To prove (2), we have

$$\| f(t, v) - f(t, w) \|_{r-1} = \| \partial_x^{-1}(v - w) - \partial_x^{-1}(v_x^2 - w_x^2) - M(v, w) \|_{r-1}$$

$$\leq C\| v - w \|_{r-2} + C\| v_x^2 - w_x^2 \|_{r-2} + \| M(v, w) \|_{r-1}$$

$$\leq C\| v - w \|_{r-1} + C\| \partial_x(v + w) \|_{r-1} \| v_x - w_x \|_{r-2} + \| M(v, w) \|_r$$

$$\leq C(1 + \| v \|_r + \| w \|_r) \| v - w \|_{r-1}.$$
\[= \mu_4 \|v - w\|_{r-1},\]

where we applied Lemma 2.5 with \(s = r - 1, \ t = r - 2.\)

**Proof of Theorem 2.1.** Combining Theorem 2.4 and Lemmas 2.6-2.8, we can get the local well-posedness result of Theorem 2.1. For the conservative property, multiplying (2.1) by \(u_x\), and integrating by parts in the unit circle \(S\), we have

\[
\frac{1}{2} \frac{d}{dt} \int_S u_x^2(t, x) dx = \int_S u_x u_{xt} dx
\]

\[= \int_S u_x(u + 2uu_{xx} + u_x^2 - \int_S (u - u_x^2) dy) dx\]

\[= \int_S (uu_x - u_x^3 + u_x^3) dx\]

\[= 0,\]

Then, integrating (2.2) directly over \(S\) yields

\[
\frac{d}{dt} \int_S u(t, x) dx = \int_S (2uu_x + \partial_x^{-1}(u - u_x^2) - \int_S \partial_x^{-1}(u - u_x^2) dy) dx
\]

\[= 0.\]

This completes the proof of Theorem 2.1.

**3. Blow-up.** In this section, we study the wave breaking phenomena of Eq. (1.1). It turns out that there are some smooth initial data for which the slope of corresponding solutions will blow up in finite time.

The following conserved quantities are key to our main blow-up result.

**Lemma 3.1.** Let \(u_0 \in H^s, \ s > \frac{3}{2}\). Then the corresponding solution \(u\) to (1.1) with the initial data \(u_0\) satisfies the following conservation laws:

\[\|u_x(\cdot, t)\|_{L^2(S)} = \|\partial_x u_0\|_{L^2(S)}, \ \int_S u(x, t) dx = \int_S u_0(x) dx.\]

Or, more precisely, we have

\[\int_S u_x^2 dx = \int_S u dx = K,\]

here \(K \geq 0\) is a constant only depending on \(u_0\), and as a consequence

\[K - \sqrt{K} \leq u(x, t) \leq K + \sqrt{K},\]

for any \((x, t) \in S \times [0, T).\)

**Proof.** Multiplying (1.1) by \(u_x\), and integrating by parts in the unit circle \(S\), we get

\[
\frac{1}{2} \frac{d}{dt} \|u_x\|_{L^2(S)}^2 = \int_S u_x u_{xt} dx
\]

\[= \int_S u_x(u + 2uu_{xx} + u_x^2) dx\]

\[= \int_S (uu_x - u_x^3 + u_x^3) dx\]

\[= 0,\]
which implies the desired conserved quantity for \( \|u_x\|_{L^2} \). On the other hand, integrating (1.1) directly over \( S \) yields

\[
\int_S u_{xt} \, dx = \int_S (u + 2uu_x + u_x^2) \, dx = \int_S u \, dx - \int_S u_x^2 \, dx = \partial_t \int_S u_x \, dx = 0.
\]

By the conserved quantity \( \|u_x\|_{L^2} \), we obtain

\[
\int_S u(x,t) \, dx = \int_S u_x^2(x,t) \, dx = \int_S (\partial_x u_0)^2 \, dx = K. \quad (3.1)
\]

Obviously, we have \( K \geq 0 \) only depending on \( u_0 \). Then we have

\[
\left| u(x,t) - \int_S u(y,t) \, dy \right| = \left| \int_S (u(x,t) - u(y,t)) \, dy \right| = \left| \int_S \int_y^x u_x(z,t) \, dz \, dy \right| \leq \sup_{y \in S} \int_y^x |u_x(z,t)| \, dz \leq \int_S |u_x(z,t)| \, dz \leq \sqrt{\int_S |u_x(z,t)|^2 \, dz}.
\]

According to the equality (3.1), we finally get

\[
\left| u(x,t) - K \right| \leq \sqrt{K},
\]

for any \((x,t) \in S \times [0,T)\). This completes the proof. \( \square \)

In fact, by the above conservation law of \( \|u_x\|_{L^2} \) and the boundedness of \( \|u\|_{L^\infty} \), the \( H^1 \) norm of \( u \) can be controlled in the periodic case. We now present a precise blow-up criterion for (1.1) as follows.

**Lemma 3.2.** Let \( u_0(x) \in H^s \), \( s \geq 2 \), and let \( T \) be the maximal existence time of the solution \( u(x,t) \) to (1.1) with the initial data \( u_0(x) \). Then the corresponding solution blows up in finite time if and only if

\[
\limsup_{t \uparrow T} \sup_{x \in S} u_x(x,t) = +\infty.
\]

**Proof.** Applying Theorem 2.3 and a simple density argument, it’s sufficient to consider the case where \( u \in C_0^\infty \). To begin with, for the \( H^1 \) norm of \( u \), we have

\[
\|u(\cdot,t)\|_{H^1(S)}^2 = \int_S (u^2 + u_x^2) \, dx \leq \|u\|_{L^\infty}^2 + \int_S u_x^2 \, dx \leq (K + \sqrt{K})^2 + K \leq C. \quad (3.2)
\]

Differentiating both sides of (1.1) with respect to \( x \), taking \( L^2 \) inner product with \( u_{xx} \), and then integrating by parts, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_S u_{xx}^2 \, dx = \int_S u_{xx} \partial_t u_{xx} \, dx
\]

\[
= \int_S u_{xx}(u_x + 4u_x u_{xx} + 2u u_{xxx}) \, dx
\]

\[
= \int_S (u_x u_{xx} + 4u_x u_{xx}^2 + 2u u_{xxx}) \, dx
\]
Suppose that $u_x$ is bounded from above on $[0, T)$ and $T < \infty$, we get
\[
\frac{d}{dt} \int_S u_{xx}^2 \, dx \leq 6M \int_S u_{xx}^2 \, dx.
\]
An application of Gronwall’s inequality yields
\[
\int_S u_{xx}^2(x, t) \, dx \leq e^{6Mt} \int_S (\partial_{xx} u_0(x))^2 \, dx.
\]
By (3.2) and the above inequality, we obtain
\[
\|u(\cdot, t)\|_{H^2(S)}^2 \leq C + e^{6Mt} \|u_0\|_{H^2(S)}^2.
\]
This contradicts the assumption that $T < \infty$ is the maximal existence time.

In our next study, we will use the following result.

Lemma 3.3. Suppose that $v \in C^1([0, T); H^2(\mathbb{R}))$ for some $T > 0$, then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with
\[
m(t) := \inf_{x \in \mathbb{R}} v_x(x, t) = v_x(\xi(t), t).
\]
The function $m(t)$ is absolutely continuous on $[0, T)$ with
\[
\frac{dm(t)}{dt} = v_{x,t}(\xi(t), t) \text{ a.e. on } [0, T).
\]

Now we can state the following main theorem, which shows that there is some initial data such that the corresponding solution to (1.1) will blow up in finite time.

Theorem 3.4. Given $u_0 \in H^s$, $s > 3$, satisfies the following condition
\[
\int_S (\partial_{xx} u_0)^2 \, dx = \int_S u_0 \, dx = K \geq 1.
\]
Then the corresponding solution $u(t, x)$ of (1.1) blows up in finite time.

Proof. Let $T > 0$ be the maximal existence time of the solution $u(\cdot, t)$ of (1.1) with initial data $u_0 \in H^3$. By the local well-posedness of Theorem 2.3 and Lemma 3.3, there are exist a function $\xi(t)$ on $[0, T)$ such that $u_x(\xi(t), t) = \sup_{x \in S} u_x(x, t)$. Directly from equation (1.1), we have
\[
u_{xt}(\xi(t), t) = u(\xi(t), t) + 2uu_{xx}(\xi(t), t) + u_x^2(\xi(t), t).
\]
Since we deal with a minimum of $u_x(x, t)$, that means $u_{xx}(\xi(t), t) = 0$ for all $t \in [0, T)$. Defining now $m(t) := u_x(\xi(t), t)$, again, by the Lemma 3.3, we obtain
\[
\frac{dm(t)}{dt} = u(\xi(t), t) + m^2(t) \text{ a.e. } t \in [0, T).
\]
As we have assumed that $\int_S u_0 \, dx = K \geq 1$, then, from Lemma 3.1, we know
\[
u(\xi(t), t) \geq \inf_{x \in S} u(x, t) \geq K - \sqrt{K} := M^2 \geq 0.
\]
Considering that in the periodic case, \(u_0(\cdot)\) can’t be monotonic function or a constant over \(S\) (this contradicts the assumption \(3.3\)). Therefore, \(m_0 := \sup_{x \in S} \partial_x u_0(x) > 0\). By the above inequality, we can get
\[
\frac{dm(t)}{dt} \geq M^2 + m^2(t), \quad t \geq 0,
\]
which leads to
\[
m(t) \geq \frac{M \tan Mt + m_0}{1 - \frac{m_0}{M} \tan Mt}, \quad t \geq 0, \quad \text{when } K > 1 \quad (M > 0),
\]
or
\[
m(t) \geq \frac{m_0}{1 - m_0 t}, \quad t \geq 0, \quad \text{when } K = 1 \quad (M = 0).
\]

As we have mentioned that \(m_0 > 0\), there exist
\[
0 < T_1 \leq \frac{1}{M} \arctan \frac{M}{m_0}, \quad \text{and } 0 < T_2 \leq \frac{1}{m_0},
\]
such that \(m(t) \to +\infty\), as \(t \to T_1 \) for \( K > 1 \) (or \( t \to T_2 \) for \( K = 1 \)).

According to the previous blow-up criterion in Lemma 3.2, this proves the wave breaking theorem. \(\square\)

Finally, we give the exact blow-up rate for blowing-up solutions. The proof is trivial and similar to Theorem 4.7 in [38], here we omit it.

**Theorem 3.5.** Suppose \(u_0 \in H^s(S)\), \(\int_S (\partial_x u_0)^2 dx = K \geq 1\) and let \(T\) be the blow-up time of the corresponding solution \(u(x,t)\) to (1.1). Then, the following blow-up rate holds:
\[
\lim_{t \to T} \sup_{x \in \mathbb{R}} u_x(x,t)(T-t) = 1.
\]

4. **Traveling wave solutions.** In this section, we investigate the traveling wave solutions of (1.1). First of all, we’ll transform (1.1) into a Gordon-type equation. In order to simplify the problem and make all the transforms meaningful and invertible, we’ll omit some regularity discussion at first. For the original ideal of this transformation, the readers can refer to Hone, Novikov and Wang [28].

**Reciprocal transformation:** Taking the \(x\) derivative of (1.1), we can get the following equations readily,
\[
m_t = 2u m_x + 4u_x m, \quad m = 1 + 4u_{xx}.
\]

This equation shares the same form with the Camassa-Holm (CH) equation (in case \(m = u - u_{xx}\)). Similar to the CH equation we have the sign preserve property.

**Lemma 4.1.** Given \(u_0(x) \in H^3(S)\). Viewing \(u_0(x)\) as a periodic function over \(\mathbb{R}\). Assume that \(m_0(x) = 1 + 4\partial_x u_0(x) > 0\) for any \(x \in \mathbb{R}\), and let \(T\) be the maximal existence time of the solution \(u(x,t)\) to (1.1) with the initial data \(u_0(x)\). Then
\[
m(x,t) = 1 + 4u_{xx}(x,t) > 0, \quad \forall (x,t) \in \mathbb{R} \times [0,T).
\]

**Proof.** We prove the lemma in Lagrangian coordinates. Consider the following initial value problem:
\[
\begin{cases}
\frac{dq}{dt} = -2u(q(x,t),t), & t \in [0,T), x \in \mathbb{R}, \\
q(x,0) = x, & x \in \mathbb{R}.
\end{cases}
\]
According to the standard ODE theory, we infer that (4.2) has a unique solution \( q \in C^1(\mathbb{R} \times [0, T]; \mathbb{R}) \). Moreover, the map \( q(\cdot, t) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
q_x(x, t) = \exp \left( -2 \int_0^t u_x(q(x, s), s) ds \right) > 0, \quad \forall (x, t) \in \mathbb{R} \times [0, T).
\]

By (4.2) and (4.1), we deduce that for any fixed \( x \in \mathbb{R} \),

\[
\frac{d}{dt} m(q(x, t), t) = m_t(q(x, t), t) + m_x(q(x, t), t)q_t(x, t) = 4u_x(q(x, t), t)m(q(x, t), t).
\]

This implies

\[
m(q(x, t), t) = m_0(x) \exp \int_0^t (4u_x(q(x, \tau), \tau)) d\tau).
\]

Since \( m_0 > 0 \), and the map \( q(\cdot, t) \) is an increasing diffeomorphism, we obtain that \( m(x, t) > 0 \) for any \( (x, t) \in \mathbb{R} \times [0, T) \). \( \square \)

Hereafter we shall assume that \( m_0(x) > 0, \forall x \in S \). By Lemma 4.1 we have \( m(x, t) > 0, \forall (x, t) \in S \times [0, T) \). Under this assumption, the original equation (1.1) can be written in the conservative form

\[
p_t = (2up)_x, \quad p(x, t) = \sqrt{m(x, t)} = \sqrt{1 + 4u_{xx}}.
\]

We now introduce a coordinate transformation \((x, t) \rightarrow (\xi, t)\) according to

\[
\frac{d\xi}{p(x(\xi, t), t)} = dx + 2(up)(x(t), t) dt.
\]

Thanks to the equality (4.3), and \( p(x, t) > 0 \) for \( \forall (x, t) \in \mathbb{R} \times [0, T) \), it is easy to see that the above transformation is one-one and on-to from \( \mathbb{R} \) to \( \mathbb{R} \) with a period stretch from 1 to :

\[
\int_0^1 p(x, t) dx = \int_0^1 p(x, 0) dx = \int_0^1 \sqrt{1 + 4\partial_{xx}u_0(x)} dx = k.
\]

This new period is independent of \( t \), that means the map (4.4) is also a bijective from \( S = \mathbb{R} / \mathbb{Z} \) to \( S' = \mathbb{R} / k\mathbb{Z} \) for each \( t \in [0, T) \), and has an inverse map satisfying

\[
\frac{dx}{p(x(\xi, t), t)} = \frac{1}{p(x(\xi, t), t)} d\xi - 2u(x(\xi, t), t) dt.
\]

By this relation, we can easily get that

\[
\partial_\xi p(x(\xi, t), t) = \frac{p_x(x(\xi, t), t)}{p}, \quad \partial_t p(x(\xi, t), t) = (2u_x p)(x(\xi, t), t),
\]

\[
\partial_{\xi t} p(x(\xi, t), t) = 2u_{xx}(x(\xi, t), t) + \frac{u_x p_x}{p}(x(\xi, t), t).
\]

Now, define that \( v(\xi, t) = \ln p(x(\xi, t), t) \). By the above relation we can deduce that \( v \) satisfies the following Gorden-type equation :

\[
v_{\xi t} = \sinh v.
\]

This is the Sinh-Gordon equation, which has been studied by many authors, its traveling wave solution over \( \mathbb{R} \) was constructed in [52] with a complicated expression. As we mainly pay attention to its periodic solution, we’ll get a more natural and simpler form.

We look for its travelling wave solutions of the following form:

\[
v(\xi, t) = v(z), \quad z = \xi - ct.
\]
Applying (4.8) into (4.7) leads to the following ordinary differential equation for \( v(z) \):

\[
- cv''(z) = \sinh v(z).
\] (4.9)

The main results are included in the following theorem.

**Theorem 4.2.** For every \( c > 0 \), there exists a smooth periodic solution with the following form:

\[
\sqrt{c} \int_{-v_0}^{v(z)} \frac{dv}{\sqrt{\cosh v_0 - \cosh v}} = |z|, \quad -\frac{1}{2} T_c \leq z < -\frac{1}{2} T_c,
\] (4.10)

where

\[
T_c = \sqrt{2c} \int_{-v_0}^{v_0} \frac{dv}{\sqrt{\cosh v_0 - \cosh v}}
\]

is the period, and \( v_0 > 0 \) is an amplitude parameter. Any other non-trivial periodic solution is a parallel transport of this solution. Moreover, getting \( u(x,t) \) from the inverse relations (4.1), (4.3), (4.6) and (4.8), then \( u(x,t) \) is a travelling wave solution of (1.1) if and only if \( v_0, c \) satisfy

\[
\sqrt{2c} \int_{-v_0}^{v_0} \frac{cosh v}{\sqrt{cosh v_0 - cosh v}} dv = \frac{1}{n},
\] (4.11)

for some \( n \in \mathbb{N}^+ \).

**Proof.** Multiplying both sides of (4.9) with \( v'(z) \), we obtain

\[
\frac{d}{dz} \left( \frac{1}{2} c(v')^2 + \cosh v \right) = 0.
\]

Since we are looking for a periodic smooth solution, there exists at least one minimum \( v(z_0) \) in a period such that \( v'(z_0) = 0 \) and \( v''(z_0) \geq 0 \). By (4.9) we know \( v(z_0) \leq 0 \). Without losing generality, set \( z_0 = 0 \) and denote \( v(0) = -v_0 \leq 0 \). Then, we get the following equality

\[
\frac{1}{2} c(v'(z))^2 + \cosh v(z) = \cosh v_0.
\] (4.12)

Solving (4.12), as \( v(z) \) is increasing from \(-v_0\) to a maximum, more precisely, when

\[
0 \leq z \leq \int_{-v_0}^{v(z)} \frac{dv}{\sqrt{2c (\cosh v_0 - \cosh v)}} := \frac{1}{2} T_c,
\]

we have \( v'(z) \geq 0 \), and

\[
\int_{-v_0}^{v(z)} \frac{dv}{\sqrt{2c (\cosh v_0 - \cosh v)}} = z.
\] (4.13)

On the other hand, \( v(z) \) is decreasing when

\[
-\frac{1}{2} T_c \leq z \leq 0,
\]

we have \( v'(z) \leq 0 \), and

\[
\int_{-v_0}^{v(z)} \frac{dv}{\sqrt{2c (\cosh v_0 - \cosh v)}} = -z.
\] (4.14)
By the definition of $T_c$,
\[ v(-\frac{1}{2}T_c) = v(\frac{1}{2}T_c) = v_0. \]
we can continuously extend $v(z)$ periodically over $\mathbb{R}$. It’s easy to check that $v(z)$ is a periodic continuous solution of (4.9). In other words, $v(z)$ is a continuous solution of (4.9) over $S = \mathbb{R}/T_c \mathbb{Z}$. By virtue of 1D Laplace equation over circle, $v(z)$ is automatically smooth. This proves the first part of the theorem.

As for the period scaling, by the equality (4.6), we can calculate the minimum positive period of $p(x,t)$ by
\[ T^* = \int_{-T_c/2}^{T_c/2} p^{-1}(x(\xi,t),t) d\xi = \int_{-T_c/2}^{T_c/2} e^{-v(z)} dz. \quad (4.15) \]
As the solution of original equation (1.1) has a period equal to 1, so does $p(x,t)$.

In order to suit this periodic condition, there must exist an positive integer $n$, such that
\[ nT^* = 1. \quad (4.16) \]
Combining (4.15) and (4.16), we have
\[
\begin{align*}
\int_{-T_c/2}^{T_c/2} e^{-v(z)} dz &= \int_{-T_c/2}^{T_c/2} \left( \cosh v(z) - \sinh v(z) \right) dz \\
&= \int_{-T_c/2}^{T_c/2} \left( \cosh v(z) + cv''(z) \right) dz \\
&= 2 \int_{-v_0}^{v_0} \cosh v(z) \frac{dz}{dv} + c \left( v'(\frac{T_c}{2}) - v'(-\frac{T_c}{2}) \right) \\
&= 2 \int_{-v_0}^{v_0} \frac{\cosh v(z)}{\sqrt{\frac{2}{c} (\cosh v_0 - \cosh v)}} dv \\
&= \frac{1}{n}.
\end{align*}
\]
This completes the proof.

Remark 4.3. If $c < 0$, we can infer from (4.12) that $v'(z)$ can’t change sign, so $v(z)$ is a monotonic function then it must be 0, i.e., we can’t get non-trivial periodic solutions if $c < 0$. This means the waves only travel from left to right (or anti-clock direction in the circle case) and not the inverse.

Acknowledgments. The authors thank the referees for their valuable comments and suggestions.

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Received for publication June 2017.