THE MORSE INDEX THEOREM IN SEMI-RIEMANNIAN GEOMETRY

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Abstract. We prove a semi-Riemannian version of the celebrated Morse Index Theorem for geodesics in semi-Riemannian manifolds; we consider the general case of both endpoints variable on two submanifolds. The key role of the theory is played by the notion of the Maslov index of a semi-Riemannian geodesic, which is a homological invariant and it substitutes the notion of geometric index in Riemannian geometry. Under generic circumstances, the Maslov index of a geodesic is computed as a sort of algebraic count of the conjugate points along the geodesic. For non positive definite metrics the index of the index form is always infinite; in this paper we prove that the space of all variations of a given geodesic has a natural splitting into two infinite dimensional subspaces, and the Maslov index is given by the difference of the index and the coindex of the index form to these subspaces. In the case of variable endpoints, two suitable correction terms, defined in terms of the endmanifolds, are added to the equality. Using appropriate change of variables, the theory is entirely extended to the more general case of symplectic differential systems, that can be obtained as linearizations of the Hamilton equations. The main results proven in this paper were announced in [23].
1. Introduction

Let \((M, g)\) be a Riemannian manifold; the classical Morse Index Theorem states that the number of conjugate points along a geodesic \(\gamma : [a, b] \to M\) counted with multiplicities (the geometric index of \(\gamma\)) is equal to the index of the second variation of the Riemannian action functional \(E(z) = \frac{1}{2} \int_a^b g(\dot{z}, \dot{z}) \, dt\) at the critical point \(\gamma\). Such second variation is called the index form, and it will be denoted by \(I_\gamma\). The theorem has later been extended in several directions (see [3, 4, 9, 10, 11, 12, 15, 19, 20, 27] for versions of this theorem in different contexts). In Lorentzian geometry, the theorem holds in the case of causal (i.e., nonspacelike) geodesics, provided that one considers the restriction of \(I_\gamma\) to the space of variations that are everywhere orthogonal to the geodesic. However, when one considers the case of spacelike Lorentzian geodesics or geodesics in semi-Riemannian manifolds with metric of arbitrary index, there is no hope to extend the original formulation of the theorem, due mainly to the following phenomena:

- the set of conjugate points along a geodesic may fail to be discrete (see [14, 25]);
- the index of \(I_\gamma\) is always infinite, even when restricted to the space of variations orthogonal to \(\gamma\) (see Proposition 2.3).

The case of spacelike Lorentzian geodesics has been studied in [13], where the authors consider a stationary metric \(g\), i.e., a metric admitting a timelike Killing vector field \(Y\). The Killing field \(Y\) gives a conservation law for geodesics \(\gamma\): \(g(\dot{\gamma}, Y) = \) constant; the main result of the paper is that, if one restricts the index form to the space of variational vector fields along \(\gamma\) corresponding to variations of \(\gamma\) by curves that satisfy such conservation law, then the index of this restriction is finite, and it is equal to a homological invariant of the geodesic called the Maslov index. The notion of Maslov index associated to curves in a Lagrangian submanifold of \(H^{2n}\) appeared originally in the Russian literature (see for instance [2] and the references therein). Some interesting applications in Variational Calculus of the Maslov index were shown by Duistermaat in [10], where it is proven an index theorem for solutions of convex Hamiltonian systems. An index theorem for solutions of non convex Hamiltonian systems is proven in [22]; the result of [22] is a weak form of the index theorem proven in this paper in a sense clarified below.

There is nowadays quite an extensive literature concerning applications of the Maslov index to the theory of Hamiltonian systems (see for instance [8, 16, 26]); in the context of semi-Riemannian geodesics the Maslov index was introduced by Helfer in [14]. Under a suitable nondegeneracy assumption, that holds generically, one proves that each conjugate point along a semi-Riemannian geodesic is isolated, and that the Maslov index of the geodesic is given by the sum of the signatures of the conjugate points (see Definition 2.1). The Maslov index is defined in general as the intersection number of a curve \(\ell\) in the Lagrangian Grassmannian \(\Lambda\) of a symplectic space with the codimension one, transversally oriented subvariety of \(\Lambda\), consisting of those Lagrangians that are not transverse to a fixed one. The curve \(\ell\) is obtained from the flow of the Jacobi equation along \(\gamma\).

The main purpose of this paper is to determine the relations between the Maslov index of a semi-Riemannian geodesic \(\gamma\) and the index form \(I_\gamma\), obtaining a general version of the Morse index theorem in semi-Riemannian geometry. More precisely, generalizing the ideas in [13, 22], we prove that the choice of a maximal negative distribution along \(\gamma\) determines a natural splitting of the space of all variations of \(\gamma\) into two \(I_\gamma\)-orthogonal infinite dimensional subspaces \(K_\gamma, S_\gamma\) such that the Maslov index is given by the difference of the index of \(I_\gamma|_{K_\gamma}\) and the coindex of \(I_\gamma|_{S_\gamma}\) (i.e., the index of \(-I_\gamma|_{S_\gamma}\)). This kind of
result aims to a generalized Morse theory for strongly indefinite functionals on Hilbert manifolds (see [1]).

A different index theory for semi-Riemannian geodesics is presented in [14], where, under a suitable nondegeneracy assumption, the author proves an equality between the Maslov index and the spectral index of the geodesic, which is an integer number defined in terms of the spectral properties of the Jacobi differential operator. Also, in [14] there is an attempt to relate the spectral index with the difference between the index and the coindex of restrictions of $I_\gamma$. However, the construction discussed by Helfer has no geometrical interpretation, and, as a matter of facts, it is not hard to prove that, by minor modifications of this construction, one can produce any integer number as a difference of the index and the coindex of restrictions of $I_\gamma$. A further discussion of Helfer’s results can be found in references [18, 24].

In order to motivate the main result of this paper, we can consider the following simple but instructive example. Consider the case of a product semi-Riemannian manifold $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$, endowed with the metric $g = g_1 \oplus (-g_2)$, where $g_1$, $g_2$ are Riemannian metrics on $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. If $\gamma = (\gamma_1, \gamma_2)$ is a geodesic in $\mathcal{M}$, the set of conjugate points along $\gamma$ is given by the union of the set of conjugate points along $\gamma_1$ and the set of conjugate points along $\gamma_2$. Using the Riemannian Morse Index Theorem it is easily seen that the index of the restriction of $I_\gamma$ to the space $K_\gamma$ of variational vector fields along $\gamma_1$ equals the number of conjugate points along $\gamma_1$, while the coindex of the restriction of $I_\gamma$ (i.e., the index of $-I_\gamma$) to the space $S_\gamma$ of variational vector fields along $\gamma_2$ equals the number of conjugate points along $\gamma_2$. In this case, the Maslov index of $\gamma$ equals the difference between the geometric indexes of $\gamma_1$ and $\gamma_2$.

The idea of the construction of the spaces $K_\gamma$ and $S_\gamma$ for the general case is the following. One considers a maximal distribution $D$ of subspaces along the geodesic $\gamma$ on which the metric is negative definite; in the above example, $D$ would be given by $T\mathcal{M}_2$. The space $S_\gamma$ is defined as the space of variational vector fields along $\gamma$ taking values in $D$. The space $K_\gamma$ is defined as the space of variational vector fields along $\gamma$ that are Jacobi in the directions of $D$, that is, vector fields whose image by the Jacobi differential operator is orthogonal to the distribution $D$. One proves that the restrictions of $I_\gamma$ to $S_\gamma$ and $K_\gamma$ are represented by a compact perturbation of a negative and a positive isomorphism, respectively, and therefore $n_+ (I_\gamma |_{S_\gamma})$ and $n_- (I_\gamma |_{K_\gamma})$ are finite natural numbers. Here, by $n_-$ and $n_+$, we mean respectively the index and the coindex of a symmetric bilinear form.

The spaces $K_\gamma$ and $S_\gamma$ are naturally associated to the quadruple $(\mathcal{M}, g, \gamma, D)$ in the following categorical sense. If $F : (\mathcal{M}, g) \to (\mathcal{M}, \tilde{g})$ is an isometry sending $\gamma$ to $\gamma$ and $D$ onto $\tilde{D}$, then $F$ also sends the spaces $K_\gamma, S_\gamma$ corresponding to $(\mathcal{M}, g, \gamma, D)$ to the spaces $\tilde{K}_{\gamma}, \tilde{S}_{\gamma}$ corresponding to $(\mathcal{M}, \tilde{g}, \gamma, \tilde{D})$.

Let us now give a brief description of the technique used to prove our main result.

The computation of $n_+ (I_\gamma |_{S_\gamma})$ is done by proving that $-I_\gamma |_{S_\gamma}$ corresponds to the index form of a positive definite symplectic system (Subsections 5.2 and 5.3); in this case the classical Morse Index Theorem applies.

The computation of the index $n_- (I_\gamma |_{K_\gamma})$ is done by considering the evolution of the function $i(t) = n_- (I_\gamma(t) |_{K_{\gamma(t)}})$, where $I_\gamma(t)$ is the index form of the restriction $\gamma|_{[a,t]}$ and $K_{\gamma(t)}$ is the corresponding restricted version of $K_{\gamma(t)}$. By a perturbation argument, one can assume that there is only a finite number of conjugate points along $\gamma$, in which case $i$ is piecewise constant (although not necessarily monotonic). The jumps of $i$ occur at those instants $t$ for which $\gamma(t)$ is conjugate and also when $K_{\gamma(t)} \cap S_{\gamma(t)} \neq \{0\}$; here, by $S_{\gamma(t)}$ we mean the restricted version of the space $S_{\gamma(t)}$.
function $i$, a technical problem arises due to the fact that the family $K_{\gamma}(t)$ does not vary smoothly with respect to $t$; indeed, the family may have singularities at those instants $t$ when $K_{\gamma}(t) \cap S_{\gamma}(t) \neq \{0\}$. In order to overcome this problem, we introduce an auxiliary extension $I^\#_\gamma(t)$ of the index form and an auxiliary extension $K^\#_\gamma(t)$ of $K_{\gamma}(t)$ such that:

- $K^\#_\gamma(t)$ varies smoothly with $t$;
- for $t \neq t_0$, the indexes of $I^\#_\gamma(t)|_{K^\#_\gamma(t)}$ and of $I_{\gamma}(t)|_{K_{\gamma}(t)}$ are easily related;
- $I_{\gamma}(t_0)$ is nondegenerate on $K^\#_\gamma(t)$, therefore its index is constant around $t = t_0$.

Using a symplectic geometry result (Lemma 3.3), we conclude that the jump of $i$ at each conjugate point coincides with its contribution to the Maslov index of the geodesic. It is a surprising fact that virtually all the previous versions of the Morse Index Theorem can be deduced as a simple consequence of this Lemma. As to the jumps of $i$ corresponding to those $t$'s for which $K_{\gamma}(t) \cap S_{\gamma}(t) \neq \{0\}$, we employ a functional analytical technique which says essentially that the jump of the index of a $C^1$ curve of symmetric bilinear forms passing through a degenerate instant is given by the signature of the derivative restricted to the kernel.

For the sake of completeness, in the paper we will consider the more general case that the initial endpoint of the geodesic is left free to move in a nondegenerate submanifold $P$ of $M$, and the notion of conjugate point is replaced by that of $P$-focal point. For this case, the theory is perfectly analogous to the case of a fixed initial point, with the only exception that the initial value of the function $i$ is in general non zero, but it is given by the index of the restriction of the metric $g$ to $T_{\gamma(a)}P$. This is an entirely new phenomenon, that can only occur in manifolds with a nonpositive definite metric.

The index theorem in the even more general case of a geodesic with final endpoint varying in a submanifold $Q$ of $M$ is then easily obtained by a simple observation, that appears already in [21]. What is interesting to remark here is that this observation led the authors to the idea of considering the auxiliary extension of the index form $I^\#_\gamma$ that was mentioned above. Namely, $I^\#_\gamma$ can be thought of as the index form corresponding to the geodesic $\gamma$ when the final endpoint varies in a fictitious submanifold.

We outline briefly the structure of the paper.

In Section 2 we give the basic definitions concerning focal points and the index form and in Section 3 we define the Maslov index. In Subsection 3.1 we study curves in the Lagrangian Grassmannian of a symplectic space and give a few technical lemmas in symplectic geometry. In Subsection 3.2 we define the Maslov index of a semi-Riemannian geodesic.

In Section 4 we give some abstract functional analytical results concerning the variation of the index of a curve of symmetric bilinear forms on a Hilbert space.

Our main results are stated in Section 5; the proofs are spread throughout the following subsections. In Subsection 5.1, by means of a parallel trivialization of the tangent bundle along the geodesic, we reduce the problem to the theory of Morse–Sturm systems in $B^n$. In Subsection 5.2 we introduce the class of symplectic differential systems needed in the computation of the coindex $n_+(I_{\gamma}|_S)$; the class of symplectic differential systems extends naturally the class of Morse–Sturm systems. In Subsection 5.3 we introduce the reduced symplectic system, which is naturally associated to the choice of the maximal negative distribution $D$. In Subsection 5.4 we define the auxiliary extension $I^\#_\gamma$ and we discuss its properties. The index function $i(t)$ is introduced in Subsection 5.5 and in Subsection 5.6 we conclude the proof of our main theorem.
In Section 6, using the fact that every symplectic system is isomorphic to a Morse–Sturm system, we extend the theory to this context and we obtain an Index Theorem for solutions of Hamiltonian systems. A preliminary version of this theorem appears in [22], where the result is proven under the restrictive assumption that \( I_\gamma \) is negative definite in \( \mathcal{S}_\gamma \).

2. Semi-Riemannian Geodesics

In this section we give the basic definitions concerning the geometry of semi-Riemannian manifolds and their geodesics.

We start with some general definitions concerning symmetric bilinear forms for later use. Let \( V \) be any real vector space and \( B : V \times V \to \mathbb{R} \) a symmetric bilinear form; given a subspace \( W \subset V \), we will denote with \( B|_W \) the restriction of \( B \) to \( W \times W \). The negative type number (or index) \( n_-(B) \) of \( B \) is the possibly infinite number defined by

\[
(2.1) \quad n_-(B) = \sup \left\{ \dim(W) : \text{subspace of } V \text{ such that } B|_W \text{ is negative definite} \right\}.
\]

The positive type number \( n_+(B) \) (or coindex) is given by \( n_+(B) = n_-(B); \) if at least one of these two numbers is finite, the signature \( \text{sgn}(B) \) is defined by:

\[
\text{sgn}(B) = n_+(B) - n_-(B).
\]

The kernel of \( B \), \( \text{Ker}(B) \), is the set of vectors \( v \in V \) such that \( B(v, w) = 0 \) for all \( w \in V \); the degeneracy \( \text{dgn}(B) \) of \( B \) is the (possibly infinite) dimension of \( \text{Ker}(B) \). If \( V \) is finite dimensional, then the numbers \( n_+(B) \), \( n_-(B) \) and \( \text{dgn}(B) \) are respectively the number of 1’s, -1’s and 0’s in the canonical form of \( B \) as given by the Sylvester’s Inertia Theorem. In this case, \( n_+(B) + n_-(B) \) is equal to the codimension of \( \text{Ker}(B) \), and it is also called the rank of \( B \), \( \text{rk}(B) \).

Let \( (\mathcal{M}, g) \) be an \( n \)-dimensional semi-Riemannian manifold, with \( g \) a metric tensor of (constant) index \( k \):

\[
(2.2) \quad n_-(g) = k.
\]

Let \( \nabla \) denote the Levi–Civita connection of \( g \) and let \( \mathcal{R} \) be the corresponding curvature tensor, chosen with the following sign convention:

\[
\mathcal{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.
\]

Let \( \mathcal{P} \subset \mathcal{M} \) be a smooth submanifold and \( \gamma : [a, b] \to \mathcal{M} \) be a geodesic with \( \gamma(a) \in \mathcal{P} \) and \( \dot{\gamma}(a) \in T_{\gamma(a)}\mathcal{P}^\perp \), where \( \perp \) denotes the orthogonal complement with respect to \( g \).

We assume that \( \mathcal{P} \) is nondegenerate at \( \gamma(a) \), i.e., that the restriction of \( g \) to \( T_{\gamma(a)}\mathcal{P} \) is nondegenerate. For \( p \in \mathcal{P} \) and \( n \in T_p\mathcal{P}^\perp \), the second fundamental form \( \mathcal{S}^\mathcal{P}_n \) is the symmetric bilinear form on \( T_p\mathcal{P} \) defined by:

\[
\mathcal{S}^\mathcal{P}_n(v_1, v_2) = g(\nabla_{v_1} V_2, n),
\]

where \( V_2 \) is any smooth vector field in \( \mathcal{P} \) with \( V_2(p) = v_2 \). Since \( \mathcal{P} \) is nondegenerate at \( \gamma(a) \), then \( \mathcal{S}^\mathcal{P}_n \) can be thought of as a \( g \)-symmetric linear endomorphism of \( T_{\gamma(a)}\mathcal{P} \).

A Jacobi field along \( \gamma \) is a smooth vector field \( J \) along \( \gamma \) satisfying the second order linear differential equation:

\[
J'' = \mathcal{R}(\dot{\gamma}, J) \dot{\gamma},
\]
where the prime means covariant derivative along $\gamma$. A $\mathcal{P}$-Jacobi field is a Jacobi field satisfying the initial conditions:

\begin{equation}
J(a) \in T_{\gamma(a)}\mathcal{P}, \quad \text{and} \quad J'(a) + \mathcal{S}_\gamma^P(J(a)) \in T_{\gamma(a)}\mathcal{P}^\perp.
\end{equation}

We denote by $\mathfrak{J}$ the vector space of all $\mathcal{P}$-Jacobi fields along $\gamma$:

\begin{equation}
\mathfrak{J} = \left\{J : J \text{ is } \mathcal{P}\text{-Jacobi along } \gamma\right\}.
\end{equation}

$\mathfrak{J}$ is an $n$-dimensional vector space; for all $t \in [a, b]$, we set:

\begin{equation}
\mathfrak{J}[t] = \left\{J(t) : J \in \mathfrak{J}\right\} \subset T_{\gamma(t)}\mathcal{M}.
\end{equation}

A point $\gamma(t)$, with $t \in ]a, b]$, is said to be $\mathcal{P}$-focal if there exists a non zero $J \in \mathfrak{J}$ such that $J(t) = 0$. We have that $\gamma(t)$ is $\mathcal{P}$-focal if and only if $\mathfrak{J}[t] \neq T_{\gamma(t)}\mathcal{M}$. The multiplicity $\text{null}(t)$ of the $\mathcal{P}$-focal point $\gamma(t)$ is the dimension of the space of those $J \in \mathfrak{J}$ such that $J(t) = 0$; the multiplicity of $\gamma(t)$ coincides with the codimension of $\mathfrak{J}[t]$ in $T_{\gamma(t)}\mathcal{M}$.

For non positive definite metrics, we have a more appropriate notion of “size” for a $\mathcal{P}$-focal point:

**Definition 2.1.** The *signature* $\text{sgn}(t)$ of a $\mathcal{P}$-focal point $\gamma(t)$ is the signature of the restriction of $\mathfrak{J}$ to $\mathfrak{J}[t]^\perp$:

\[
\text{sgn}(t) = \text{sgn} \left( \mathfrak{J}[t]^\perp \right).
\]

The $\mathcal{P}$-focal point $\gamma(t)$ is said to be *nondegenerate* if such restriction is nondegenerate. If there are only a finite number of $\mathcal{P}$-focal points along $\gamma$, then we define the *focal index* $i_{\text{foc}}(\gamma)$ of $\gamma$ as the sum of the signatures of all the $\mathcal{P}$-focal points along $\gamma$:

\[
\text{i}_{\text{foc}}(\gamma) = \sum_{t \in [a, b]} \text{sgn}(t).
\]

For instance, if $(\mathcal{M}, g)$ is Riemannian ($k = 0$), or if $(\mathcal{M}, g)$ is Lorentzian ($k = 1$) and $\gamma$ is causal, i.e., $g(\dot{\gamma}, \dot{\gamma}) \leq 0$, then all the $\mathcal{P}$-focal points are nondegenerate, and their signatures coincide with their multiplicity. Namely, in this case $g$ is positive definite in $\mathfrak{J}[t]^\perp$.

In the Riemannian or in the causal Lorentzian case it is well known that the set of $\mathcal{P}$-focal points along a geodesic is discrete; in the general semi-Riemannian case, focal points may indeed accumulate (see [14, 25]) even in the case that $\mathcal{P}$ is a point. We have the following result concerning the distribution of $\mathcal{P}$-focal points:

**Proposition 2.2.** There are no $\mathcal{P}$-focal points $\gamma(t)$ for $t$ near $a$. Nondegenerate $\mathcal{P}$-focal points are isolated. Moreover, if $(\mathcal{M}, g)$ is real analytic, then the set of $\mathcal{P}$-focal points along $\gamma$ is finite.

**Proof.** See for instance [18, Proposition 2.5.1, Remark 2.5.3].

We consider the following symmetric bilinear form:

\begin{equation}
I_\gamma^P(v, w) = \int_a^b \left[ g(v', w') + g(R(\gamma, v) \dot{\gamma}, w) \right] \, dt - \mathcal{S}_\gamma^P(v(a), w(a)),
\end{equation}

defined on the space $\mathcal{H}_\gamma^P$ of all vector fields $v$ along $\gamma$ of $H^1$-Sobolev regularity\footnote{This means that $v : [a, b] \to T\mathcal{M}$ is absolutely continuous, and the covariant derivative $v'$ is square-integrable with respect to some positive definite inner product along $\gamma$.} with $v(a) \in T_{\gamma(a)}\mathcal{P}$ and $v(b) = 0$. The space $\mathcal{H}_\gamma^P$ has the topology of a Hilbertable space, and $I_\gamma^P$ is continuous in this topology. The set $\Omega_{\mathcal{P}, \gamma(b)}$ of all curves of $H^1$-regularity in $\mathcal{M}$
joining \( \mathcal{P} \) and \( \gamma(b) \) can be given the structure of an infinite dimensional Hilbert manifold, and the semi-Riemannian action functional \( f(z) = \frac{1}{2} \int_a^b g(z, z) \, dt \) is smooth on \( \Omega_{\mathcal{P}, \gamma(b)} \). The geodesic \( \gamma \) is a critical point of \( f \) in \( \Omega_{\mathcal{P}, \gamma(b)} \), \( \mathcal{H}_{\gamma}^P \) is the tangent space \( T_{\gamma, \Omega_{\mathcal{P}, \gamma(b)}} \) and the symmetric bilinear form \( I_{\gamma}^P \) is the Hessian of \( f \) at \( \gamma \).

We now consider another smooth submanifold \( \mathcal{Q} \subset M \) with \( \gamma(b) \in \mathcal{Q} \) and \( \dot{\gamma}(b) \in T_{\gamma(b)} \mathcal{Q} \). In this situation, \( \gamma \) is also a critical point for the action functional \( f \) defined in the Hilbert manifold \( \Omega_{\mathcal{P}, \mathcal{Q}} \) of all \( H^1 \)-curves joining \( \mathcal{P} \) and \( \mathcal{Q} \). The tangent space \( T_{\gamma, \Omega_{\mathcal{P}, \mathcal{Q}}} \) will be denoted by \( \mathcal{H}_{\mathcal{P}, \mathcal{Q}} \), and it consists of all \( H^1 \)-vector fields \( v \) along \( \gamma \) with \( v(a) \in T_{\gamma(a)} \mathcal{P} \) and \( v(b) \in T_{\gamma(b)} \mathcal{Q} \). The Hessian of \( f \) at \( \gamma \) in the space \( \Omega_{\mathcal{P}, \mathcal{Q}} \) is given by the following bounded symmetric bilinear form in \( \mathcal{H}_{\mathcal{P}, \mathcal{Q}} \):

\[
I_{\gamma}^{\mathcal{P}, \mathcal{Q}}(v, w) = \int_{a}^{b} \left[ g(v', w') + g(R(\dot{\gamma}, v) \dot{\gamma}, w) \right] \, dt + S_{\gamma(b)}^\mathcal{Q}(v(b), w(b)) - S_{\gamma(a)}^\mathcal{P}(v(a), w(a)).
\]

If \( k > 0 \), then \( I_{\gamma}^{\mathcal{P}} \) has infinite index, and so does \( I_{\gamma}^{\mathcal{P}, \mathcal{Q}} \):

**Proposition 2.3.** If \( k > 0 \), then \( I_{\gamma}^{\mathcal{P}} \) has infinite index in \( \mathcal{H}_{\mathcal{P}}^P \). If \( k \geq 2 \) or if \( k = 1 \) and \( g(\dot{\gamma}, \dot{\gamma}) > 0 \), then \( I_{\gamma}^{\mathcal{P}} \) has infinite index in the space of all vector fields in \( \mathcal{H}_{\gamma}^P \) that are everywhere orthogonal to \( \dot{\gamma} \).

**Proof.** If \( Y \) is a Jacobi field along \( \gamma \) and \( f : [a, b] \to \mathbb{R} \) is a smooth function vanishing at the endpoints, it is easily computed:

\[
I_{\gamma}^{P}(fY, fY) = \int_{a}^{b} \left[ f'^2 g(Y, Y) + \frac{d}{dt} (f^2 g(Y', Y')) \right] \, dt = \int_{a}^{b} f'^2 g(Y, Y) \, dt.
\]

Let \( t_0 \in ]a, b[; \) if \( k > 0 \), then we can find a Jacobi field \( Y \) with \( g(Y, Y) < 0 \) in a neighborhood \( V \) of \( t_0 \). If \( k \geq 2 \) or if \( k = 1 \) and \( g(\dot{\gamma}, \dot{\gamma}) > 0 \), then the field \( Y \) can also be chosen orthogonal to \( \dot{\gamma} \) everywhere. From (2.8), it follows that \( I_{\gamma}^{P} \) is negative definite in the space of fields \( fY \), where \( f \) is supported in \( V \).

Obviously, the result of Proposition 2.3 holds for the bilinear form \( I_{\gamma}^{P, \mathcal{Q}} \).

### 3. The Maslov Index

In this section we present some techniques of symplectic spaces and we discuss the notion of *Maslov index* that will be used to define an integer valued invariant for semi-Riemannian geodesics.

#### 3.1. The Maslov Index of a curve of Lagrangians

Let \( (V, \omega) \) be a finite dimensional symplectic space, i.e., \( V \) is a \( 2n \)-dimensional real vector space and \( \omega \) is a nondegenerate skew-symmetric bilinear form in \( V \). A subspace \( L \) of \( V \) is Lagrangian if \( \dim(L) = n \) and \( \omega \) vanishes on \( L \times L \). The set \( \Lambda \) of all Lagrangian subspaces of \( V \) is called the Lagrangian Grassmannian of \( (V, \omega) \); \( \Lambda \) is a compact, connected real analytic \( \frac{1}{2} n(n+1) \)-dimensional embedded submanifold of the Grassmannian \( G_n(V) \) of all \( n \)-dimensional subspaces of \( V \). We will use several well known facts about the geometry of the Lagrangian Grassmannian of a symplectic space (see for instance [2, 10, 18]); in particular, we will make full use of the notations and of the results proven in Reference [18].

For our purposes, we need the following description of an atlas of charts on \( \Lambda \). Given a pair \( L_0, L_1 \) of complementary Lagrangian subspaces of \( V \), i.e., \( V = L_0 \oplus L_1 \), we define
an isomorphism $D_{L_0,L_1} : L_1 \to L_0^*$ by:

$$D_{L_0,L_1}(v) = \omega(v, \cdot)|_{L_0}, \quad v \in L_1.$$  

We observe that, by the anti-symmetry of $\omega$, the following identity holds:

$$D_{L_0,L_1} = -(D_{L_1,L_0})^*.$$  

Let $L \in \Lambda$ be fixed; we define the following subsets of $\Lambda$:

$$\Lambda_k(L) = \left\{ L' \in \Lambda : \dim(L' \cap L) = k \right\}, \quad k = 0, \ldots, n.$$  

Each $\Lambda_k(L)$ is a connected embedded real analytic submanifold of $\Lambda$ having codimension $\frac{1}{2} k(k+1)$ in $\Lambda$; $\Lambda_0(L)$ is a dense open subset of $\Lambda$, while its complementary set:

$$\Lambda_{\geq 1}(L) = \bigcup_{k=1}^{n} \Lambda_k(L)$$

is not a regular submanifold of $\Lambda$, but only an analytic subset. Its regular part is given by $\Lambda_1(L)$, which is a dense open subset of $\Lambda_{\geq 1}(L)$.

Given a pair $L_0, L_1$ of complementary Lagrangians in $V$, it is defined a chart

$$\phi_{L_0,L_1} : \Lambda_0(L_1) \to B_{\text{sym}}(L_0, \mathbb{R}),$$

where $B_{\text{sym}}(L_0, \mathbb{R})$ is the vector space of symmetric bilinear forms on $L_0$. For $L \in \Lambda_0(L_1)$, we have:

$$\phi_{L_0,L_1}(L) = D_{L_0,L_1} \circ T,$$

where $T : L_0 \to L_1$ is the unique linear map whose graph in $L_0 \oplus L_1 = V$ is equal to $L$. In equality (3.5) we are identifying a linear map $L_0 \to L_0^*$ with a bilinear form on $L_0$; such identifications of linear maps from a space to its dual and bilinear forms on the space will be used throughout in the rest of the section.

Observe that, given $L \in \Lambda_0(L_1)$, the bilinear form $\phi_{L_0,L_1}(L)$ is nondegenerate (i.e., the corresponding linear map $L_0 \to L_0^*$ is invertible) if and only if $L \in \Lambda_0(L_0)$.

The map $\phi_{L_0,L_1}$ defined in (3.5) is a diffeomorphism, and it follows in particular that $\Lambda_0(L_1)$ is contractible for all $L_1 \in \Lambda$. The Lagrangian Grassmannian $\Lambda$ is diffeomorphic to the homogeneous space $U(n)/O(n)$ ([18, Proposition 3.2.5]), and using such diffeomorphism one computes the fundamental group $\pi_1(\Lambda) \simeq \mathbb{Z}$ ([18, Corollary 4.1.2]). It follows that the first singular homology group $H_1(\Lambda; \mathbb{Z})$ is also isomorphic to $\mathbb{Z}$; for a given Lagrangian $L_0 \in \Lambda$, since $\Lambda_0(L_0)$ is contractible, we compute the first relative homology group of the pair $(\Lambda, \Lambda_0(L_0))$ as:

$$H_1(\Lambda, \Lambda_0(L_0); \mathbb{Z}) \simeq \mathbb{Z}.$$  

The choice of the above isomorphism is related to the choice of a transverse orientation of $\Lambda_1(L_0)$ in $\Lambda$, which is canonically associated to the symplectic form ([18, Proposition 3.2.10]). Every continuous curve $l$ in $\Lambda$ with endpoints in $\Lambda_0(L_0)$ defines an element in $H_1(\Lambda, \Lambda_0(L_0); \mathbb{Z})$, and we denote by

$$\mu_{L_0}(l) \in \mathbb{Z}$$

the integer number corresponding to the homology class of $l$ by the isomorphism (3.6). This number, which is additive by concatenation and invariant by homotopies with endpoints in $\Lambda_0(L_0)$, can be interpreted as an intersection number of the curve $l$ with $\Lambda_{\geq 1}(L_0)$.

**Definition 3.1.** Given a continuous curve $l : [a, b] \to \Lambda$ with $l(a), l(b) \in \Lambda_0(L_0)$, the integer number $\mu_{L_0}(l)$ of (3.7) is called the Maslov index of $l$ relative to $L_0$. 
The Maslov index of a continuous curve in $\Lambda$ can be computed in terms of the coordinate charts $\phi_{L_0, L_1}$.

**Proposition 3.2.** Let $L_0 \in \Lambda$ and let $l : [a, b] \to \Lambda$ be any continuous curve with endpoints in $\Lambda_0(L_0)$. Suppose that there exists a Lagrangian subspace $L_1$ complementary to $L_0$ such that the image of $l$ is entirely contained in the domain $\Lambda_0(L_1)$ of the chart $\phi_{L_0, L_1}$. Then, the Maslov index $\mu_{L_0}(l)$ is given by:

$$
\mu_{L_0}(l) = n_+ \left( \phi_{L_0, L_1} \left( l(b) \right) \right) - n_+ \left( \phi_{L_0, L_1} \left( l(a) \right) \right).
$$

**Proof.** See [18, Proposition 4.3.1].

To our purposes, we will need to extend the result of Proposition 3.2 to the case that the image of the curve $l$ fails to be contained in the domain of the chart $\phi_{L_0, L_1}$ at an isolated instant $t_0 \in ]a, b[$. We need first a technical Lemma:

**Lemma 3.3.** Let $L, L_s, L_0, L_1$ be four Lagrangian subspaces of $V$, with $L_0$ and $L_1$ complementary to each other, $L$ complementary to $L_0$ and with $L_s$ complementary to both $L$ and $L_0$. Then,

$$
\phi_{L_0, L_1}(L_s) - \phi_{L_1, L_0}(L) = (D_{L_0, L_1})^* \circ \phi_{L_0, L_s}(L) \circ (D_{L_0, L_1})^{-1} \circ D_{L_0, L_1}.
$$

**Proof.** Let $T, S : L_1 \to L_0$ be linear maps whose graphs in $V = L_1 \oplus L_0$ are equal to $L_s$ and $L$ respectively; moreover let $U : L_0 \to L_s$ be the linear map whose graph in $V = L_0 \oplus L_s$ is $L$. Observe that $U$ is invertible; it is easily computed:

$$
Sv = U^{-1}(v +Tv) +Tv, \quad \forall v \in L_1.
$$

From (3.5), we have:

$$
\phi_{L_0, L_s}(L) = D_{L_0, L_s} \circ U, \quad \phi_{L_1, L_0}(L) = D_{L_1, L_0} \circ S, \quad \phi_{L_1, L_0}(L_s) = D_{L_1, L_0} \circ T.
$$

From (3.1), we compute:

$$
(D_{L_0, L_s})^{-1} \circ D_{L_0, L_1}(v) = v +Tv, \quad \forall v \in L_1.
$$

Using (3.10), (3.11) and (3.12), it follows

$$
\phi_{L_1, L_0}(L) - \phi_{L_1, L_0}(L_s) = D_{L_1, L_0} \circ \phi_{L_0, L_s}(L) \circ (D_{L_0, L_1})^{-1} \circ D_{L_0, L_1}.
$$

The conclusion follows from (3.2) and (3.13). □

**Corollary 3.4.** Under the assumptions of Lemma 3.3, we have:

$$
n_+ \left( \phi_{L_1, L_0}(L_s) - \phi_{L_1, L_0}(L) \right) = n_+ \left( \phi_{L_0, L_s}(L) \right).
$$

In addition, $\phi_{L_1, L_0}(L_s) - \phi_{L_1, L_0}(L)$ is nondegenerate.

**Proof.** It follows immediately from (3.9), considering that:

$$
n_+ \left( (D_{L_0, L_1})^* \circ \phi_{L_0, L_s}(L) \circ (D_{L_0, L_1})^{-1} \circ D_{L_0, L_1} \right) = n_+ \left( \phi_{L_0, L_s}(L) \right).
$$

We can now prove the aimed extension of Proposition 3.2:

**Proposition 3.5.** Let $L_0, L_1 \in \Lambda$ be given, with $L_0 \cap L_1 = \{0\}$, and let $l : [a, b] \to \Lambda$ be a continuous curve such that $l(t) \in \Lambda_0(L_0)$ except possibly for $t = t_0 \in ]a, b[$. Let $L_s \in \Lambda$ be complementary to both $l(t_0)$ and $L_0$; then, for $\varepsilon > 0$ sufficiently small, we have:

$$
\mu_{L_0}(l) = n_- \left( \phi_{L_1, L_0} \left( l(t_0 + \varepsilon) \right) - \phi_{L_1, L_0}(L_s) \right) - n_- \left( \phi_{L_1, L_0} \left( l(t_0 - \varepsilon) \right) - \phi_{L_1, L_0}(L_s) \right).
$$
Proof. Let $\varepsilon > 0$ be small enough so that $l(t) \in \Lambda_0(L_\varepsilon)$ for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$; since $t_0$ is the unique instant where $l$ passes through $\Lambda_{\geq 1}(L_0)$, then $\mu_{L_0}(l) = \mu_{L_\varepsilon}(l|_{[t_0 - \varepsilon, t_0 + \varepsilon]})$. Using Proposition 3.2, we obtain

$$\mu_{L_\varepsilon}(l) = n_+ \left( \phi_{L_\varepsilon, L_\varepsilon} \left( l(t_0 + \varepsilon) \right) \right) - n_+ \left( \phi_{L_\varepsilon, L_\varepsilon} \left( l(t_0 - \varepsilon) \right) \right).$$

The conclusion follows by applying twice Corollary 3.4 to the above equation, once by taking $L = l(t_0 + \varepsilon)$ and again by taking $L = l(t_0 - \varepsilon)$. □

3.2. The Maslov index of a semi-Riemannian geodesic. We now consider a semi-Riemannian setup as in Section 2, consisting of a semi-Riemannian manifold $(M, g)$, a nondegenerate smooth submanifold $P$ of $M$ and a geodesic $\gamma : [a, b] \to M$ starting orthogonally to $P$. We start by observing that, for $J_1, J_2 \in \mathfrak{J}$, we have:

$$(3.16) \quad g(J_1'(t), J_2'(t)) = g(J_1(t), J_2'(t)), \quad \forall t \in [a, b].$$

We choose a parallel trivialization of the tangent bundle $TM$ along $\gamma$; we may then identify vector fields along $\gamma$ with curves in $\mathbb{R}^n$ and the metric tensor $g$ along $\gamma$ with a fixed nondegenerate symmetric bilinear form $g$ in $\mathbb{R}^n$. The space $\mathfrak{J}$ will then correspond to a space of smooth curves in $\mathbb{R}^n$.

Let us consider the canonical symplectic structure $\omega$ on the vector space $V = \mathbb{R}^n \oplus \mathbb{R}^{n\ast}$ given by:

$$(3.17) \quad \omega((v_1, \alpha_1), (v_2, \alpha_2)) = \alpha_2(v_1) - \alpha_1(v_2).$$

For all $t \in [a, b]$, we define an $n$-dimensional subspace $\ell(t) \subset V$ by:

$$(3.18) \quad \ell(t) = \left\{ (J(t), gJ'(t)) : J \in \mathfrak{J} \right\};$$

here $g$ is thought of as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^{n\ast}$. By (3.16), $\ell(t)$ is a Lagrangian subspace of $(V, \omega)$ for all $t \in [a, b]$, and we therefore obtain a smooth curve $\ell : [a, b] \to L$.

We fix the following Lagrangian subspace $L_0$ of $V$:

$$(3.19) \quad L_0 = \{0\} \oplus \mathbb{R}^{n\ast}.$$ 

Observe that, for $t \in [a, b]$, $\ell(t) \in \Lambda_{\geq 1}(L_0)$ if and only if $\gamma(t)$ is a $P$-focal point; moreover, the multiplicity of $\gamma(t)$ coincides with the dimension of $\ell(t) \cap L_0$. In particular, if $\gamma(b)$ is not a $P$-focal point, then the curve $\ell$ has final endpoint in $\Lambda_0(L_0)$. On the other hand, $\ell(a) \in \Lambda_{\geq 1}(L_0)$; however, since there are no $P$-focal points near $a$ (Proposition 2.2), in order to define the Maslov index of the geodesic $\gamma$ we can consider a restriction $\ell|_{[a+\varepsilon, b]}$ with $\varepsilon > 0$ small.

Definition 3.6. Suppose that $\gamma(b)$ is not $P$-focal. The Maslov index $i_{\text{maslov}}(\gamma)$ of the geodesic $\gamma$ is defined as:

$$(3.20) \quad i_{\text{maslov}}(\gamma) = \mu_{L_0}(\ell|_{[a+\varepsilon, b]}),$$

where $\varepsilon > 0$ is chosen such that $\gamma(t)$ is not $P$-focal for $t \in [a, a + \varepsilon]$.

Clearly, the right hand side of (3.20) does not depend on the choice of $\varepsilon$; moreover, in order to make rigorous the above definition we need the following:

Proposition 3.7. The term on the right hand side of equality (3.20) does not depend on the choice of a parallel trivialization of $TM$ along $\gamma$.

Proof. If $\hat{\ell} : [a, b] \to \Lambda$ is the curve of Lagrangians corresponding to a different choice of a parallel trivialization of $TM$ along $\gamma$, then the relation between $\ell$ and $\hat{\ell}$ is given by:

$$\hat{\ell} = \sigma \circ \ell,$$
where \( \sigma : V \to V \) is a fixed symplectomorphism that preserves \( L_0 \). Namely, \( \sigma \) is given by:
\[
\sigma(v, \alpha) = (s(v), s^*\alpha),
\]
where \( s : \mathbb{R}^n \to \mathbb{R}^n \) is the isomorphism that relates the two trivializations. The conclusion follows from the fact that composition with a fixed symplectomorphism that preserves \( L_0 \) induces the identity in the relative homology group \( H_1(\Lambda, \Lambda_0(L_0)) \). \( \square \)

We have the following relation between the Maslov index and the focal index of a semi-Riemannian geodesic:

**Proposition 3.8.** Suppose that \( \gamma(b) \) is not \( \mathcal{P} \)-focal and that all the \( \mathcal{P} \)-focal points are nondegenerate. Then,
\[
i_{\text{maslov}}(\gamma) = i_{\text{foc}}(\gamma).
\]

**Proof.** See [18, Theorem 5.1.2]. \( \square \)

We remark that the thesis of Proposition 3.8 is false without the nondegeneracy assumption on the focal points, even for real analytic manifolds (see [18, Subsection 7.4]).

It is easy to see that, due to its topological nature, the Maslov index is invariant by uniformly small perturbations of the data of the geometric problem; on the other hand, the focal index is unstable. The stability property is a first indication that the Maslov index is the correct generalization of the notion of geometric index to semi-Riemannian geometry.

4. **Abstract Results of Functional Analysis**

The goal of this section is to provide a method of computing the change of index of a smooth family of symmetric bilinear forms on a Hilbert space; we will use the notations and several results from [13] that will be restated for the reader’s convenience. All Hilbert spaces of the Section will be assumed real.

Given Hilbert spaces \( \mathcal{H}, \mathcal{H}' \), we will denote by \( \mathcal{L}(\mathcal{H}, \mathcal{H}') \) the space of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{H}' \); by \( \mathcal{L}(\mathcal{H}) \) we will mean \( \mathcal{L}(\mathcal{H}, \mathcal{H}) \). By \( B_{\text{sym}}(\mathcal{H}, \mathcal{H}) \) we will now mean the set of symmetric bounded bilinear forms on \( \mathcal{H} \). Let \( \langle \cdot, \cdot \rangle \) be a Hilbert space inner product on \( \mathcal{H} \); to any bounded bilinear form \( B : \mathcal{H} \times \mathcal{H} \to \mathcal{R} \) by Riesz’s theorem there corresponds a bounded linear operator \( T_B : \mathcal{H} \to \mathcal{H} \), which is related to \( B \) by:
\[
B(x, y) = \langle T_B(x), y \rangle, \quad \forall x, y \in \mathcal{H}.
\]

We say that \( T_B \) is the linear operator that represents \( B \) with respect to the inner product \( \langle \cdot, \cdot \rangle \). Clearly, \( B \) is symmetric if and only if \( T_B \) is self-adjoint. We say that \( B \) is nondegenerate if \( T_B \) is injective; \( B \) will be said to be strongly nondegenerate if \( T_B \) is an isomorphism. If \( T_B \) is a Fredholm operator of index 0 (for instance if \( T_B \) is a compact perturbation of an isomorphism), then \( B \) is nondegenerate if and only if it is strongly nondegenerate. Observe that strong nondegeneracy is stable by small perturbations, since the set of isomorphisms of \( \mathcal{H} \) is open in \( \mathcal{L}(\mathcal{H}) \).

We will consider 1-parameter families of bilinear forms defined on a variable domain, and we need the following notion of \( C^1 \)-family of closed subspace of a Hilbert space:

**Definition 4.1.** Let \( \mathcal{H} \) be a Hilbert space, \( I \subset \mathcal{R} \) an interval and \( \{D_t\}_{t \in I} \) be a family of closed subspaces of \( \mathcal{H} \). We say that \( \{D_t\}_{t \in I} \) is a \( C^1 \)-family of subspaces if for all \( t_0 \in I \) there exists a \( C^1 \)-curve \( \alpha : [t_0 - \varepsilon, t_0 + \varepsilon] \cap I \to \mathcal{L}(\mathcal{H}) \) and a closed subspace \( \overline{D} \subset \mathcal{H} \) such that \( \alpha(t) \) is an isomorphism and \( \alpha(t)(D_t) = \overline{D} \) for all \( t \).
We have the following criterion to establish the regularity of a family of closed subspaces:

**Lemma 4.2.** Let \( I \subset \mathbb{R} \) be an interval, \( \mathcal{H}, \tilde{\mathcal{H}} \) be Hilbert spaces and \( F : I \to \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}}) \) be a \( C^1 \)-map such that each \( F(t) \) is surjective. Then, the family \( \mathcal{D}_t = \text{Ker}(F(t)) \) is a \( C^1 \)-family of closed subspaces of \( \mathcal{H} \).

**Proof.** See [13, Lemma 2.9]. \( \square \)

The next Proposition, also proven in [13], gives a method for computing the change of the index of a smooth family of closed subspaces that are represented by a compact perturbation of a positive isomorphism. Recall that a self-adjoint linear operator \( T \) in \( \mathcal{H} \) is a compact perturbation of a positive (negative) isomorphism of \( \mathcal{H} \) if it is of the form \( L + K \), where \( L \) is a self-adjoint positive (negative) isomorphism of \( \mathcal{H} \) and \( K \) is a compact self-adjoint operator on \( \mathcal{H} \).

**Proposition 4.3.** Let \( \mathcal{H} \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \), and let \( B : [t_0, t_0 + r] \to \text{Bsym}(\mathcal{H}, \mathbb{R}) \), \( r > 0 \), be a map of class \( C^1 \). Let \( \{\mathcal{D}_t\}_{t \in [t_0, t_0 + r]} \) be a \( C^1 \)-family of closed subspaces of \( \mathcal{H} \), and denote by \( \overline{B}(t) \) the restriction of \( B(t) \) to \( \mathcal{D}_t \times \mathcal{D}_t \). Assume that the following three hypotheses are satisfied:

1. \( \overline{B}(t_0) \) is represented by a compact perturbation of a positive isomorphism of \( \mathcal{D}_t \);
2. the restriction \( \overline{B} \) of the derivative \( B'(t_0) \) to \( \text{Ker}(\overline{B}(t)) \times \text{Ker}(\overline{B}(t)) \) is nondegenerate;
3. \( \text{Ker}(\overline{B}(t_0)) \subseteq \text{Ker}(B(t_0)) \).

Then, for \( t > t_0 \) sufficiently close to \( t_0 \), \( \overline{B}(t) \) is nondegenerate, and we have:

\[
(4.2) \quad n_-(\overline{B}(t)) = n_-(\overline{B}(t_0)) + n_-(\overline{B}),
\]

all the terms of the above equality being finite natural numbers.

**Proof.** See [13, Proposition 2.5]. \( \square \)

**Remark 4.4.** Observe that, by Proposition 4.3, if \( \overline{B}(t_0) \) is nondegenerate on \( \mathcal{D}_t \), then \( n_-(\overline{B}(t)) \) is constant for \( t \) near \( t_0 \). Actually, we have the following stronger continuity property for the positive and the negative type numbers of symmetric bilinear forms. If \( B_n \to B \) in \( \text{Bsym}(\mathcal{H}, \mathbb{R}) \), \( D_n \) converges\(^2\) to \( D \), if \( B|_D \) is nondegenerate and it is represented by a compact perturbation of a positive (resp., negative) isomorphism of \( D \), then for \( n \) sufficiently large, it is \( n_-(B_n|_{\mathcal{D}_n}) = n_-(B|_D) \) (resp., \( n_+(B_n|_{\mathcal{D}_n}) = n_+(B|_D) \)).

For the purposes of this article, we need an extension of the result of Proposition 4.3 that holds in the more general situation in which hypothesis (3) is not satisfied. To this aim, we need to define a notion of derivative of the family \( B(t) \) that takes into consideration the variation of the domain \( \mathcal{D}_t \).

**Definition 4.5.** Let \( \mathcal{H} \) be a Hilbert space and let \( B : [t_0, t_0 + r] \to \text{Bsym}(\mathcal{H}, \mathbb{R}) \), \( r > 0 \), be a map of class \( C^1 \). Let \( \{\mathcal{D}_t\}_{t \in [t_0, t_0 + r]} \) be a \( C^1 \)-family of closed subspaces of \( \mathcal{H} \), and denote by \( \overline{B}(t) \) the restriction of \( B(t) \) to \( \mathcal{D}_t \times \mathcal{D}_t \). We define the symmetric bilinear form

\(^{2}\)in the sense that \( F_n \to F \) in \( \mathcal{L}(\mathcal{H}, \tilde{\mathcal{H}}) \), where \( \tilde{\mathcal{H}} \) is any Hilbert space, \( F \) is surjective and \( \mathcal{D}_n = \text{Ker}(F_n) \), \( D = \text{Ker}(F) \).
\(B_0(t_0)\) in \(\text{Ker}(B(t_0))\) by:

\[
(4.3) \quad \overline{B}'(t_0)(v, w) = \frac{d}{dt} B(t)(v(t), w(t)) \bigg|_{t=t_0} = \]

\[= B'(t_0)(v, w) + B(t_0)(v'(t_0), w) + B(t_0)(v, w'(t_0)), \quad \forall v, w \in \text{Ker}(B(t_0)),\]

where \(v(t)\) and \(w(t)\) are \(C^1\)-curves in \(\mathcal{H}\) with \(v(t_0) = v, w(t_0) = w, v(t) \in D_t\) and \(w(t) \in D_t\) for all \(t\).

**Remark 4.6.** Note that formula (4.3) defines \(\overline{B}'(t_0)(v, w)\) independently of the extensions \(v(t)\) and \(w(t)\) chosen. To see this, simply observe that the classes of the derivatives \(v'(t_0)\) and \(w'(t_0)\) modulo \(D_{t_0}\) are independent of the extensions.

This is the aimed extension of Proposition 4.3:

**Proposition 4.7.** Let \(\mathcal{H}\) be a Hilbert space with inner product \((\cdot, \cdot)\), and let \(B : [t_0, t_0 + r] \to B_{\text{sym}}(\mathcal{H}, \mathbb{R}), r > 0\), be a map of class \(C^1\). Let \(\{D_t\}_{t \in [t_0, t_0 + r]}\) be a \(C^1\)-family of closed subspaces of \(\mathcal{H}\), and denote by \(\overline{B}(t)\) the restriction of \(B(t)\) to \(D_t \times D_t\). Assume that the following two hypotheses are satisfied:

1. \(\overline{B}(t_0)\) is represented by a compact perturbation of a positive isomorphism of \(D_{t_0}\);
2. the symmetric bilinear form \(\overline{B}'(t_0)\) is nondegenerate.

Then, for \(t > t_0\) sufficiently close to \(t_0\), \(\overline{B}(t)\) is nondegenerate, and we have:

\[
(4.4) \quad n_-(\overline{B}(t)) = n_-\left(\overline{B}(t_0)\right) + n_-(\overline{B}'(t_0)),
\]

all the terms of the above equality being finite natural numbers.

**Proof.** By possibly passing to a smaller \(r\), we can assume the existence of a \(C^1\)-curve \(\alpha(t)\) of isomorphisms of \(\mathcal{H}\) such that \(\alpha(t)\) carries \(D_t\) to a fixed subspace \(\overline{D}\) of \(\mathcal{H}\). Define \(C(t) = B(t)(\alpha(t)^{-1}, \alpha(t)^{-1} \cdot \cdot )\) as a bilinear form on the fixed space \(\overline{D}\). Then, \(C(t)\) is a push-forward of \(\overline{B}(t)\) and the restriction of \(C'(t_0)\) to \(\text{Ker}(C(t_0))\) is a push-forward of \(\overline{B}'(t_0)\).

The conclusion follows by applying Proposition 4.3 to the curve \(C(t)\) in \(B_{\text{sym}}(\overline{D}, \mathbb{R})\).

**Corollary 4.8.** Let \(B : [t_0 - r, t_0 + r] \to B_{\text{sym}}(\mathcal{H}, \mathbb{R})\) and \(\{D_t\}_{t \in [t_0 - r, t_0 + r]}\) satisfy the same hypotheses of Proposition 4.7. Then, in the notations of Proposition 4.7, for \(\varepsilon > 0\) small enough, we have:

\[
(4.5) \quad n_-(B(t_0 - \varepsilon)) - n_-(B(t_0 + \varepsilon)) = \text{sgn}(\overline{B}'(t_0)).
\]

**Proof.** Use Proposition 4.7 twice, once to \(B_{1\mid [t_0, t_0 + r]}\) and once to a backwards reparameterization of \(B_{1\mid [t_0 - r, t_0]}\). \(\square\)

5. The Morse Index Theorem

In this section we go back to the geometrical setup of Section 2 and we state and prove an extension of the Morse Index Theorem for geodesics in semi-Riemannian manifolds with metric tensor of arbitrary index. As we have seen in Proposition 2.3, if \((\mathcal{M}, g)\) is not Riemannian then the index of \(I_\gamma^P\) is always infinite. However, we show that it is possible to split the Hilbert space of all variations of a given geodesic into two subspaces such that \(I_\gamma^P\) has finite index on the first and finite coindex on the second.

The definition of these spaces of variations depend on the choice of a distribution of maximal negative subspaces along the geodesic \(\gamma\):
Definition 5.1. We say that a family of subspaces $D_t \subset T_{\gamma(t)}M$, $t \in [a, b]$, along the geodesic $\gamma$ is smooth if there exist a family $Y_1, \ldots, Y_r$ of smooth vector fields along $\gamma$ which forms a pointwise basis for $D$; such a family $Y_1, \ldots, Y_r$ is called a frame for $D$. A maximal negative distribution along $\gamma$ is a smooth family of $k$-dimensional subspaces $D$ along $\gamma$ such that $g$ is negative definite on $D_t$ for all $t$ (recall (2.2)).

Obviously, maximal negative distributions along any geodesic always exist; for instance, one can obtain such distributions by considering the parallel transport of any maximal subspace of $T_{\gamma(a)}M$ on which $g$ is negative definite.

Given a maximal negative distribution $D$ along $\gamma$, we define the following closed subspaces of $H^p_{\gamma}$:

$$K_{\gamma, P}^D = \left\{ v \in H^p_{\gamma} : g(v', Y_i) \right\}$$

(5.1)

$$\mathcal{S}_\gamma^D = \left\{ v \in H^p_{\gamma} : v(a) = 0, \; \forall t \in [a, b] \right\},$$

where $Y_1, \ldots, Y_k$ is a frame for $D$. It is easy to check that the space $K_{\gamma, P}^D$ does not actually depend on the choice of the frame $Y_1, \ldots, Y_k$. The space $\mathcal{S}_\gamma^D$ can be roughly described as the space of vector fields along $\gamma$ that are “Jacobi in the directions of $D$”; observe indeed that if $v \in H^p_{\gamma}$ is a vector field of class $C^2$, then $v \in K_{\gamma, P}^D$ if and only if:

$$v'' - R(\dot{\gamma}, v) \dot{\gamma} \in D^\perp.$$

We are ready to state the main result of the paper:

**Theorem 5.2** (Semi-Riemannian Morse Index Theorem). Let $(M, g)$ be a semi-Riemannian manifold, $P$ a smooth submanifold of $M$, $\gamma : [a, b] \to M$ a geodesic such that:

- $\gamma(a) \in P$ and $\dot{\gamma}(a) \in T_{\gamma(a)}P^\perp$;
- $P$ is nondegenerate at $\gamma(a)$;
- $\gamma(b)$ is not a $P$-focal point.

Let $D$ be a maximal negative distribution along $\gamma$; let $K_{\gamma, P}^D$ and $\mathcal{S}_\gamma^D$ be the corresponding subspaces of $H^p_{\gamma}$ defined in (5.1). Then,

$$i_{\text{maslov}}(\gamma) = n - \left( I^p_{\gamma} \big|_{K_{\gamma, P}^D} \right) - n_+ \left( I^p_{\gamma} \big|_{\mathcal{S}_\gamma^D} \right) - n_0 \left( g \big|_{T_{\gamma(a)}P} \right),$$

(5.2)

where all the terms in the above formula are finite integer numbers.

The proof of Theorem 5.2 requires some work and it is spread along the remaining subsections of this section.

The spaces $K_{\gamma, P}^D$ and $\mathcal{S}_\gamma^D$ are $I^p_{\gamma}$-orthogonal (Lemma 5.30); moreover, under generic circumstances, they are complementary in $H^p_{\gamma}$ (Corollary 5.18 and Corollary 5.28). An explicit formula to compute the term $n_+ \left( I^p_{\gamma} \big|_{\mathcal{S}_\gamma^D} \right)$ that appears in (5.2) is given in Corollary 5.22.

The last term of equality (5.2) is the contribution given by the initial submanifold $P$; in the case of Riemannian or causal Lorentzian geodesics, $P$ is spacelike at $\gamma(a)$, and therefore the last term of (5.2) vanishes.

Let’s take a closer look at some special examples to get a better feeling of the result of Theorem 5.2.
Example 5.3. If $(\mathcal{M}, g)$ is Riemannian, then $\mathcal{D} = 0$, the space $\mathcal{K}_\gamma^D \mathcal{P}$ coincides with $\mathcal{H}_\gamma^D$, $\mathcal{E}_\gamma^D = \{0\}$ and the Maslov index of $\gamma$ is equal to the sum of the multiplicities of the $\mathcal{P}$-focal points along $\gamma$.

For simplicity, in our next example we will assume that the initial submanifolds to the given geodesics reduce to a point; in this case we will omit the subscripts and the superscripts $\mathcal{P}$ in our notation.

Example 5.4. Let $(\mathcal{M}_1, g_1)$, $(\mathcal{M}_2, g_2)$ be Riemannian manifolds and consider the product $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$ endowed with the semi-Riemannian metric $g = g_1 \oplus (-g_2)$. It is easily seen that $\gamma = (\gamma_1, \gamma_2): [a, b] \to \mathcal{M}$ is a geodesic iff $\gamma_1$ and $\gamma_2$ are geodesics; the space $\mathcal{H}_\gamma$ of vector fields along $\gamma$ vanishing at both endpoints is identified with the direct sum $\mathcal{H}_{\gamma_1} \oplus \mathcal{H}_{\gamma_2}$, where $\mathcal{H}_{\gamma_i}$ is the space of vector fields along $\gamma_i$ vanishing at both endpoints, $i = 1, 2$. It is easily seen that the index form $I_\gamma$ is given by $I_\gamma = I_{\gamma_1} \oplus (-I_{\gamma_2})$ where $I_{\gamma_i}$ is the index form corresponding to $\gamma_i$ in the Riemannian manifold $(\mathcal{M}_i, g_i)$, $i = 1, 2$; assuming that $\gamma(b)$ is not conjugate to $\gamma(a)$ one easily sees that the spaces $\mathcal{K}_\gamma^D$ and $\mathcal{S}_\gamma^D$ corresponding to the distribution $D_t = T_{\gamma_1(t)} \mathcal{M}_1 \subset T_{\gamma(t)} \mathcal{M}$ are given respectively by $\mathcal{H}_{\gamma_1}$ and $\mathcal{H}_{\gamma_2}$. In this case, Theorem 5.2 is an easy consequence of the Riemannian Morse index theorem applied to each geodesic $\gamma_i$.

Example 5.5. If $(\mathcal{M}, g)$ is Lorentzian and $\gamma$ is timelike, i.e., $g(\dot{\gamma}, \dot{\gamma}) < 0$, then we can consider the distribution $\mathcal{D}$ spanned by $\dot{\gamma}$. In this case, $\mathcal{K}_\gamma^D \mathcal{P}$ correspond to the space of variational vector fields that are everywhere orthogonal to $\dot{\gamma}$ and $n_+ \left( I_\gamma^P \big| \mathcal{E}_\gamma^P \right) = 0$. Also in this case, the Maslov index of $\gamma$ equals the sum of the multiplicities of the $\mathcal{P}$-focal points along $\gamma$.

Example 5.6. Suppose that, in the general semi-Riemannian case, we can find $Y_1, \ldots, Y_k$ Jacobi fields along $\gamma$, with $k = n_-(g)$, that form a frame for a $k$-dimensional distribution $\mathcal{D}$ on which $g$ is negative definite. If $g(Y_i', Y_j')_{ij}$ is symmetric, then also in this situation $n_+ \left( I_\gamma^P \big| \mathcal{E}_\gamma^P \right) = 0$ (this will follow from Corollary 5.23 ahead); observe that we can always find such a family of Jacobi fields on sufficiently small segments of a geodesic. In this context, the space $\mathcal{K}_\gamma^D \mathcal{P}$ is given by the set of vectors fields $v \in \mathcal{H}_\gamma^D$ such that the quantities $g(v, Y_i') - g(v, Y_i')$ are constant for every $i$. In the Lorentzian case, $k = 1$ and this observation applies when the geodesic $\gamma$ admits a timelike Jacobi field along it.

Example 5.7. Suppose that $G$ is a $k$-dimensional Lie group acting on $\mathcal{M}$ by isometries with no fixed points, or more in general, having only discrete isotropy groups. Suppose that $g$ is negative definite on the orbits of $G$. If $\dot{\gamma}(a)$ is orthogonal to the orbit of the commutator subgroup $[G, G]$ for instance if $G$ is abelian, then we can consider the distribution $\mathcal{D}$ tangent to the orbits of $G$. Observe that $\mathcal{D}$ is generated by $k$ linearly independent Killing vector fields $Y_1, \ldots, Y_k$ on $\mathcal{M}$, which therefore restrict to Jacobi fields along any geodesic. Then, one falls into the case of Example 5.6 by observing that the symmetry of $g(Y_i', Y_j')$ follows from the orthogonality of $\dot{\gamma}(a)$ with the orbits of $[G, G]$: $g(Y_i', Y_j') - g(Y_i', Y_j') = -g(\nabla_{Y_i} Y_j, \dot{\gamma}) + g(\nabla_{Y_j} Y_i, \dot{\gamma}) = g([Y_i, Y_j], \dot{\gamma}) = 0$.

In this situation, the space $\mathcal{K}_\gamma^D \mathcal{P}$ can be described as the space of variational vector fields along $\gamma$ corresponding to variations of $\gamma$ by curves that are geodesics along $\mathcal{D}$, i.e., whose second derivatives are orthogonal to $\mathcal{D}$. 
Example 5.8. Another situation in which the term \( n_+ \left( I^T_\gamma |_{\mathcal{D}^\perp} \right) \) vanishes occurs when the bilinear form \( g(\mathcal{R}(\dot{\gamma}, \cdot) \dot{\gamma}, \cdot) \) is negative semi-definite along the geodesic \( \gamma \) and \( \mathcal{D} \) is parallel (again, this will follow from Corollary 5.23).

We conclude this subsection by showing that Theorem 5.2 can be easily generalized to the case of geodesics with both endpoints variable.

In order to give a statement of this extension we need to introduce the following objects.

Assume that we are given a smooth submanifold \( \mathcal{Q} \) of \( \mathcal{M} \) such that \( \gamma(b) \in \mathcal{Q} \) and \( \dot{\gamma}(b) \in T_{\gamma(b)}\mathcal{Q}^\perp \). In analogy with (5.1), we define the space \( K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D} \) by:

\[
K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D} = \left\{ v \in \mathcal{H}^\mathcal{P}_{\mathcal{Q}} : g(v', Y_i) \text{ is of Sobolev regularity } H^1, \text{ and} \right\}
\]

\[
\forall i = 1, \ldots, k
\]

(5.3)

Suppose that \( \gamma(b) \) is not \( \mathcal{P} \)-focal; let \( S_\gamma \) be the linear endomorphism of \( T_{\gamma(b)}\mathcal{M} \) defined by:

\[
S_\gamma(J(b)) = -J'(b),
\]

for all \( J \in \mathfrak{J} \). Observe that the assumption of non focality for \( \gamma(b) \) implies that \( \mathfrak{J} \ni J \rightarrow J(b) \in T_{\gamma(b)}\mathcal{M} \) is an isomorphism, and therefore \( S_\gamma \) is well defined. Observe also that, by (3.16), \( S_\gamma \) is \( g \)-symmetric; we denote by \( S_\gamma \) also the corresponding symmetric bilinear form on \( T_{\gamma(b)}\mathcal{M} \), which is given by:

\[
S_\gamma(J_1(b), J_2(b)) = -g(J_1(b), J'_2(b)), \quad \forall J_1, J_2 \in \mathfrak{J}.
\]

Theorem 5.9. Under the hypotheses of Theorem 5.2, assume also that we are given a smooth submanifold \( \mathcal{Q} \) of \( \mathcal{M} \) such that \( \gamma(b) \in \mathcal{Q} \) and \( \dot{\gamma}(b) \in T_{\gamma(b)}\mathcal{Q}^\perp \). Then,

\[
i_{\text{maslov}}(\gamma) = n_+ \left( I^T_\gamma |_{K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D}} \right) - n_+ \left( I^T_\gamma |_{\mathcal{D}^\perp} \right) - n_- \left( \mathfrak{g} |_{T_{\gamma(b)}\mathcal{P}} \right) - n_-(S_\gamma |_{T_{\gamma(b)}\mathcal{Q}}).
\]

(5.4)

Proof. Recalling (2.4), we define:

\[
\mathfrak{J}_{\mathcal{Q}} = \left\{ J \in \mathfrak{J} : J(b) \in T_{\gamma(b)}\mathcal{Q} \right\}.
\]

Since \( \gamma(b) \) is not \( \mathcal{P} \)-focal and \( \mathfrak{J}_{\mathcal{Q}} \subset K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D} \), it follows easily that

\[
K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D} = K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D} \oplus \mathfrak{J}_{\mathcal{Q}}.
\]

Integration by parts in (2.7) shows that the direct sum in (5.5) is \( I^T_\gamma \mathcal{Q} \)-orthogonal, hence

\[
n_- \left( I^T_\gamma |_{K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D}} \right) = n_- \left( I^T_\gamma |_{K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D}} \right) + n_- \left( I^T_\gamma |_{\mathfrak{J}_{\mathcal{Q}}} \right).
\]

(5.6)

The restriction of \( I^T_\gamma \mathcal{Q} \) to \( K_{\gamma, \mathcal{P}, \mathcal{Q}}^\mathcal{D} \) is obviously equal to the restriction of \( I^T_\gamma \mathcal{Q} \) to the same space, hence the first term on the right hand side of equality (5.6) is computed in Theorem 5.2.

The conclusion follows by observing that the isomorphism \( \mathfrak{J}_{\mathcal{Q}} \ni J \rightarrow J(b) \in T_{\gamma(b)}\mathcal{Q} \) carries the restriction of \( I^T_\gamma \mathcal{Q} \) to \( S_{\gamma(b)} - S_\gamma |_{T_{\gamma(b)}\mathcal{Q}} \). □

Observe that the last term in equality (5.4) is the contribution of the final manifold \( \mathcal{Q} \); it already appears in the Riemannian Morse Index Theorem for variable endpoints ([15]).

We now pass to the proof of Theorem 5.2.
5.1. Reduction to a Morse–Sturm system in $\mathbb{R}^n$. A Morse–Sturm system in $\mathbb{R}^n$ is a second order linear differential system of the form:

$$v''(t) = R(t) v(t), \quad t \in [a, b], \quad v(t) \in \mathbb{R}^n,$$

where $R(t)$ is a continuous map of linear endomorphisms of $\mathbb{R}^n$ that are symmetric with respect to a fixed nondegenerate symmetric bilinear form $g$ on $\mathbb{R}^n$.

Morse–Sturm systems arise from the Jacobi equation along a geodesic $\gamma$ in a semi- Riemannian manifold $(\mathcal{M}, g)$ by means of a parallel trivialization of the tangent bundle $T\mathcal{M}$ along $\gamma$. Using such a trivialization, we may then identify vector fields along $\gamma$ with curves in $\mathbb{R}^n$ and the metric tensor $g$ along $\gamma$ with a fixed nondegenerate symmetric bilinear form $g$ in $\mathbb{R}^n$. For all $t \in [a, b]$, the endomorphism $v \mapsto R(\gamma(t), v) \dot{\gamma}(t)$ of $T_{\gamma(t)}\mathcal{M}$ is identified with a $g$-symmetric endomorphism $R(t)$ of $\mathbb{R}^n$. Since covariant derivative along $\gamma$ corresponds to the usual derivative of curves in $\mathbb{R}^n$, the Jacobi equation along $\gamma$ becomes the Morse–Sturm system (5.7).

If $\mathcal{P}$ is a smooth submanifold of $\mathcal{M}$ such that $\gamma(a) \in \mathcal{P}$ and $\dot{\gamma}(a) \in T_{\gamma(a)}\mathcal{P}^\perp$, then the tangent space $T_{\gamma(a)}\mathcal{P}$ is identified with a subspace $P$ of $\mathbb{R}^n$, and the second fundamental form $S^P_{\gamma(a)}$ is identified with a symmetric bilinear form $S$ on $P$. We assume that $\mathcal{P}$ is nondegenerate at $\gamma(a)$, so that $g$ is nondegenerate on $P$.

The space $\mathfrak{J}$ of $\mathcal{P}$-Jacobi fields corresponds to the space $\mathfrak{J}$ of solutions of (5.7) satisfying the initial conditions:

$$v(a) \in P, \quad v'(a) + S\{v(a)\} \in P^\perp,$$

where $\perp$ denotes the orthogonal complements with respect to $g$ and $S$ is seen as a $g$-symmetric linear endomorphism of $P$.

We denote by $L^2([a, b]; \mathbb{R}^m)$ the Hilbert space of square integrable $\mathbb{R}^m$-valued functions on $[a, b]$, by $H^1([a, b]; \mathbb{R}^m)$ the Sobolev space of absolutely continuous maps with derivative in $L^2([a, b]; \mathbb{R}^m)$, and by $H^1_{\text{loc}}([a, b]; \mathbb{R}^m)$ the subspace of $H^1([a, b]; \mathbb{R}^m)$ consisting of functions vanishing at $a$ and at $b$. We also denote by $C^0([a, b]; \mathbb{R}^m)$ the Banach space of continuous functions from $[a, b]$ to $\mathbb{R}^m$. It is well known that the inclusion maps $H^1([a, b]; \mathbb{R}^m) \hookrightarrow C^0([a, b]; \mathbb{R}^m)$ and $H^1([a, b]; \mathbb{R}^m) \hookrightarrow L^2([a, b]; \mathbb{R}^m)$ are compact operators (see for instance [6]).

The Hilbert space $\mathcal{H}^\gamma_\mathcal{P}$ corresponds by the parallel trivialization to the subspace $\mathcal{H} \subset H^1([a, b]; \mathbb{R}^n)$ given by:

$$\mathcal{H} = \left\{ v \in H^1([a, b]; \mathbb{R}^n) : v(a) \in P, \ v(b) = 0 \right\};$$

moreover, the index form $I^\mathcal{P}_{\mathcal{H}}$ defines a bounded symmetric bilinear form $I$ on $\mathcal{H}$ by:

$$I(v, w) = \int_a^b \left[ g(v', w') + g(Rv, w) \right] dt - S\{v(a), w(a)\}.$$

Observe that the kernel of $I$ in $\mathcal{H}$ is the space:

$$\text{Ker}(I) = \mathcal{H} \cap \mathfrak{J}.$$

The notions of focal instants, multiplicity, signature, focal index and Maslov index may be defined for Morse–Sturm systems (5.7) with initial conditions (5.8) in the obvious way.

**Definition 5.10.** An instant $t \in [a, b]$ is said to be focal for the Morse–Sturm system (5.7) (with initial conditions (5.8)) if there exists a non zero solution $v \in \mathfrak{J}$ such that $v(t) = 0$. The dimension of the space of such solutions is the multiplicity of the focal instant. The
signature of the focal instant \( t \) is defined to be the signature of the restriction of \( g \) to \( J[t] \), where:

\[
J[t] = \left\{ J(t) : J \in J \right\}.
\]

A focal instant is nondegenerate if \( g \) is nondegenerate on \( J[t] \). If there are only a finite number of focal instants, we define the focal index of the Morse–Sturm system to be the sum of the signatures of the focal instants in \( [a, b] \). If \( t = b \) is not a focal instant, we define the Maslov index of the Morse–Sturm system to be the number \( \mu_{L_0}(\ell_{[a+\epsilon, b]}) \), where \( \epsilon > 0 \) is such that there are no focal instants in \( [a, a+\epsilon] \) and \( \ell, L_0 \) are defined in (3.18) and (3.19).

Proposition 3.8 generalizes in an obvious way to Morse–Sturm systems.

We are going to prove a version of Theorem 5.2 for such systems, which in particular implies that the result holds in the geometrical context.

As a matter of facts, it is not hard to prove that every Morse–Sturm system (5.7) with smooth coefficients arises from the Jacobi equation along a semi-Riemannian geodesic, provided that one considers a parallel trivialization of the normal bundle along the geodesic. Details are found in [18, Proposition 2.3.1].

Let us consider now a maximal negative distribution \( D \) along \( \gamma \); each subspace \( D_t \subset T_{\gamma(t)}M \) corresponds to a subspace \( D_t \subset \mathbb{R}^n \) by means of the parallel trivialization of \( T_{\gamma(t)}M \) along \( \gamma \). Obviously, each \( D_t \) is a maximal negative subspace for the bilinear form \( g \).

The subspaces \( K_{\gamma, p}^D \) and \( S_{\gamma}^D \) of \( H_{\gamma}^D \) correspond to the closed subspaces \( K \) and \( S \) of \( H \) given by:

\[
K = \left\{ v \in H : g(v', Y_i) \in H^1([a, b]; \mathbb{R}), \quad g(v', Y_i)' = g(v', Y_i') + g(Rv, Y_i), \ i = 1, \ldots, k \right\},
\]

\[
S = \left\{ v \in H : v(a) = 0, \ v(t) \in D_t, \ \forall t \in [a, b] \right\},
\]

where \( Y_1, \ldots, Y_k \) is a frame for \( D \), i.e., each \( Y_i : [a, b] \to \mathbb{R}^n \) is a smooth curve and \( \{Y_1(t), \ldots, Y_k(t)\} \) is a basis of \( D_t \) for all \( t \).

We are interested in determining the elements of the intersection \( K \cap S \); such elements are characterized as solutions of a second order linear differential equation in \( \mathbb{R}^n \) which is in general not a Morse–Sturm system. This equation belongs to the more general class of symplectic differential systems, that will be discussed in the next subsection.

5.2. Symplectic differential systems in \( \mathbb{R}^n \). A Morse–Sturm system (5.7) can be written as the following first order linear system in \( \mathbb{R}^n \oplus \mathbb{R}^n^* \):

\[
\begin{pmatrix}
  v \\
  \alpha
\end{pmatrix}' = \begin{pmatrix}
  0 & g^{-1} \\
  gR & 0
\end{pmatrix} \begin{pmatrix}
  v \\
  \alpha
\end{pmatrix}, \quad v(t) \in \mathbb{R}^n, \ \alpha(t) \in \mathbb{R}^n^*,
\]

where again the bilinear form \( g \) is seen as a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^n^* \).

We denote by \( \text{Sp}(2n, \mathbb{R}) \) the Lie group of symplectic transformations of the space \( (\mathbb{R}^n \oplus \mathbb{R}^n^*, \omega) \), where \( \omega \) is the symplectic form defined in (3.17), and by \( \text{sp}(2n, \mathbb{R}) \) its Lie algebra. Recall that an element \( X \in \text{sp}(2n, \mathbb{R}) \) is a linear endomorphism of \( \mathbb{R}^n \oplus \mathbb{R}^n^* \) such that \( \omega(X \cdot, \cdot) \) is symmetric; in block matrix form, \( X \) is given by:

\[
X = \begin{pmatrix}
  A & B \\
  C & -A^*
\end{pmatrix},
\]

where \( A : \mathbb{R}^n \to \mathbb{R}^n \) is an arbitrary linear map, and \( B : \mathbb{R}^n^* \to \mathbb{R}^n \), \( C : \mathbb{R}^n \to \mathbb{R}^n^* \) are symmetric when regarded as bilinear forms.
We observe that the coefficient matrix of the Morse–Sturm system (5.13) is of the form (5.14) with $A = 0$, $B = g^{-1}$ and $C = gR$; we call a symplectic differential system in $\mathbb{R}^n$ a first order linear differential system in $\mathbb{R}^n \oplus \mathbb{R}^{n*}$ whose coefficient matrix $X(t)$ is a continuous curve in $\text{sp}(2n, \mathbb{R})$, where the blocks $A$ and $B$ are of class $C^1$, and $B(t)$ is invertible for all $t \in [a, b]$:

\begin{equation}
\begin{aligned}
\begin{cases}
v'(t) = A(t)v(t) + B(t)\alpha(t); \\
\alpha'(t) = C(t)v(t) - A^*(t)\alpha(t),
\end{cases}
\end{aligned}
\tag{5.15}
\end{equation}

Morse–Sturm systems are special cases of symplectic differential systems with $A = 0$ and $B$ constant; the index theory for Morse–Sturm systems extends naturally to the class of symplectic differential systems (see [22]). Such systems appear naturally as linearizations of the Hamilton equations, and also as the Jacobi equations along geodesics when a non parallel trivialization of the tangent bundle is chosen. Moreover, the class of symplectic differential systems is the more natural class for which it is possible to define the notion of Maslov index (see Section 6).

We will need the extension of the index theory to symplectic systems in order to calculate the term $n_+ \left( \bar{T}^\gamma \big|_{\epsilon=2} \right)$ that appears in equation (5.2). Namely, the restriction $\bar{T}^\gamma \big|_{\epsilon=2}$ can be thought as the index form associated to a symplectic system which is determined by the Jacobi equation along the geodesic and by the choice of the distribution $D$.

To clarify the situation, we outline briefly the basics of the index theory for symplectic differential systems.

Consider the symplectic differential system (5.15) with coefficient matrix $X$ given by (5.14); we say that a $C^1$-curve $v : [a, b] \to \mathbb{R}^n$ is an $X$-solution if there exists $\alpha : [a, b] \to \mathbb{R}^{n*}$ of class $C^1$ such that the pair $(v, \alpha)$ is a solution of (5.15). It is easy to see that an $X$-solution $v$ is of class $C^2$, and that, since $B$ is invertible, the unique $\alpha = \alpha_v$ such that $(v, \alpha)$ is a solution of (5.15) is given by:

\begin{equation}
\alpha_v = B^{-1}(v' - Av).
\end{equation}

We denote by $\mathbb{V}$ the set of all $X$-solutions vanishing at $t = a$:

\begin{equation}
\mathbb{V} = \left\{ v : v \text{ is an } X\text{-solution, with } v(a) = 0 \right\}.
\end{equation}

Using the symmetry of $B$ and $C$ and (5.15), it is easy to see that the following equality holds:

\begin{equation}
\alpha_v(w) = \alpha_w(v), \quad \forall v, w \in \mathbb{V}.
\end{equation}

For $t \in [a, b]$, we set

\begin{equation}
\mathbb{V}[t] = \left\{ v(t) : v \in \mathbb{V} \right\}.
\end{equation}

From (5.18) and a simple dimension counting argument, the annihilator of $\mathbb{V}[t]$ is given by:

\begin{equation}
\mathbb{V}[t]^0 = \left\{ \alpha_v(t) : v \in \mathbb{V}, \ v(t) = 0 \right\}, \quad t \in [a, b].
\end{equation}

**Definition 5.11.** An instant $t \in [a, b]$ is said to be focal if there exists a non zero $v \in \mathbb{V}$ such that $v(t) = 0$, i.e., if $\mathbb{V}[t] \neq \mathbb{R}^n$. The multiplicity $\text{mul}(t)$ of the focal instant $t$ is defined to be the dimension of the space of those $v \in \mathbb{V}$ vanishing at $t$, or, equivalently, the codimension of $\mathbb{V}[t]$ in $\mathbb{R}^n$. The signature $\text{sgn}(t)$ of the focal instant $t$ is the signature of the restriction of the bilinear form $B(t)$ to the space $\mathbb{V}[t]^0$, or, equivalently, the signature of the restriction of $B(t)^{-1}$ to the $B(t)^{-1}$-orthogonal complement $\mathbb{V}[t] \perp$ of $\mathbb{V}[t]$ in $\mathbb{R}^n$. 

THE SEMI-RIEMANNIAN MORSE INDEX THEOREM

19
The focal instant $t$ is said to be nondegenerate if such restriction is nondegenerate. If there is only a finite number of focal instants in $[a, b]$, we define the focal index $i_{\text{foc}} = i_{\text{foc}}(X)$ to be the sum:

$$i_{\text{foc}} = \sum_{t \in [a, b]} \text{sgn}(t).$$

In the special situation that the bilinear form $B(t)$ is positive definite, then the focal instants are obviously nondegenerate, and their signatures coincide with their multiplicities. Moreover, as in Proposition 2.2, it can be proven that nondegenerate focal instants are isolated.

The index form $I_X$ associated to (5.15) is the bounded symmetric bilinear form on the Hilbert space $H^1_0([a, b]; \mathbb{R}^n)$ given by:

$$I_X(v, w) = \int_a^b \left[ B(\alpha_v, \alpha_w) + C(v, w) \right] \, dt.$$

Observe that, for a symplectic system (5.13) coming from a Morse–Sturm system (5.7), the index form $I_X$ coincides with the index form $I$ of formula (5.10) when the subspace $P$ is chosen equal to $\{0\}$.

Integration by parts in (5.21) shows that the kernel of $I_X$ is given by:

$$\text{Ker}(I_X) = \left\{ v \in \mathbb{V} : v(b) = 0 \right\}.$$

**Remark 5.12.** Observe that, from (5.22) we obtain easily that if $B$ is positive definite and $C$ is positive semi-definite, then the symplectic differential system (5.15) has no focal instants.

There is a natural notion of isomorphism in the class of symplectic systems. Let $L_0$ be the Lagrangian subspace $\{0\} \oplus \mathbb{R}^n$ of $(\mathbb{R}^n \oplus \mathbb{R}^n^*, \omega)$; we denote by $\text{Sp}(2n, \mathbb{R}; L_0)$ the closed subgroup of $\text{Sp}(2n, \mathbb{R})$ consisting of those symplectomorphisms $\phi_0$ such that $\phi_0(L_0) = L_0$. It is easily seen that any such symplectomorphism is given in block matrix form by:

$$\phi_0 = \begin{pmatrix} Z & 0 \\ Z^* - W & Z^* - W \end{pmatrix},$$

with $Z : \mathbb{R}^n \to \mathbb{R}^n$ an isomorphism and $W$ a symmetric bilinear form in $\mathbb{R}^n$.

We give the following:

**Definition 5.13.** The symplectic differential systems with coefficient matrices $X$ and $\tilde{X}$ are said to be isomorphic if there exists a $C^1$-map $\phi_0 : [a, b] \to \text{Sp}(2n, \mathbb{R}; L_0)$ whose upper-left $n \times n$ block is of class $C^2$ and such that:

$$\tilde{X} = \phi_0^{-1} + \phi_0 X \phi_0^{-1}.$$

We call the map $\phi_0$ an isomorphism between $X$ and $\tilde{X}$.

The motivation of such notion of isomorphism is that, for isomorphic systems $X$ and $\tilde{X}$, a pair $(v, \alpha)$ is an $X$-solution if and only if $\phi_0(v, \alpha)$ is an $\tilde{X}$-solution. More precisely, we have the following relations between isomorphic symplectic systems:

**Proposition 5.14.** Let $X$ and $\tilde{X}$ be the coefficient matrices of isomorphic symplectic systems, and let $\phi_0$ as in formula (5.23) be an isomorphism between $X$ and $\tilde{X}$.

Then, the focal instants corresponding to the systems associated to $X$ and $\tilde{X}$ are the same, and they have the same multiplicities and signatures. Moreover, the isomorphism $v \mapsto Zv$ of $H^1_0([a, b]; \mathbb{R}^n)$ carries the index form $I_X$ into the index form $I_{\tilde{X}}$. 


Proof. See [22, Subsection 2.10, Proposition 2.10.3]. □

Although an index theory for symplectic systems may be developed directly, the easiest way to extend the Morse Index theorem to this class of systems is given by considering the following result:

**Proposition 5.15.** Every symplectic system (5.15) such that $B$ is a map of class $C^2$ is isomorphic to a Morse–Sturm system (5.13).

Proof. See [22, Proposition 2.11.1]. □

Observe that the index $n_- (B)$ is invariant by isomorphisms of symplectic systems. Hence we have the following:

**Corollary 5.16.** Consider the symplectic system (5.15), with $B$ a map of class $C^2$ and positive definite. Then, there are only finitely many focal instants, the index of $I_X$ in $H^1_0([a, b]; \mathbb{R}^n)$ is finite, and it is equal to the sum of the multiplicities of the focal instants in $[a, b]$.

Proof. The result is well known for Morse–Sturm systems (see for instance [13, Corollary 3.7]). The conclusion follows from Proposition 5.14 and Proposition 5.15. □

5.3. The reduced symplectic system. We now go back to the setup of Subsection 5.1 and we study the intersection of the spaces $\mathcal{K}$ and $\mathcal{S}$.

**Lemma 5.17.** Let $v \in \mathcal{S}$; write $v = \sum_{i=1}^{k} f_i Y_i$. Then, $v \in \mathcal{K}$ if and only if $f = (f_1, \ldots, f_k)$ is a solution of the following symplectic differential system:

$$
\begin{cases}
    f' = -B^{-1} C f - B^{-1} \varphi, \\
    \varphi' = (C^* B^{-1} C - \mathcal{I}) f + C^* B^{-1} \varphi,
\end{cases}
$$

(5.25)

where $B$, $\mathcal{I}$ are bilinear forms in $\mathbb{R}^k$, and $C$ is a linear map from $\mathbb{R}^k$ to $\mathbb{R}^k^*$, whose matrices in the canonical basis are given by:

$$
\begin{align*}
    B_{ij} &= g(Y_i, Y_j), \\
    C_{ij} &= g(Y'_j, Y_i), \\
    \mathcal{I}_{ij} &= g(Y'_j, Y'_i) + g(RY_i, Y_j).
\end{align*}
$$

(5.26)

Proof. It is a simple calculation based on the definition of the space $\mathcal{K}$ given in (5.12). □

**Corollary 5.18.** The dimension of the intersection $\mathcal{K} \cap \mathcal{S}$ is equal to the multiplicity of $t = b$ as a focal instant for the symplectic differential system (5.17). □

**Definition 5.19.** The system (5.25) is called the reduced symplectic system associated to the Morse–Sturm system (5.7), the maximal negative distribution $D$ and the frame $Y_1, \ldots, Y_k$.

It is not hard to prove that different choices of a frame for the distribution $D$ produce isomorphic reduced symplectic systems (see [22, Proposition 2.10.4]).

**Remark 5.20.** It is easily seen that the following symplectic differential system is isomorphic to (5.25):

$$
\begin{cases}
    f' = -B^{-1} C_a f + B^{-1} \varphi; \\
    \varphi' = (\mathcal{I} - C_a^* + C_a B^{-1} C_a) f - C_a B^{-1} \varphi,
\end{cases}
$$

(5.27)

where $C_a$ and $C_s$ are given by:

$$
C_a = \frac{1}{2} (C - C^*), \quad C_s = \frac{1}{2} (C + C^*).
$$

The index form of the reduced symplectic system (5.25) corresponds to the restriction of $-I$ to the space $\mathcal{S}$:
Proposition 5.21. The Hilbert space isomorphism

\[ H_0^1([a, b]; \mathbb{R}^k) \ni f = (f_1, \ldots, f_k) \mapsto \sum_{i=1}^k f_i \cdot Y_i \in \mathcal{S} \]

carries the index form of the reduced symplectic system (5.25) to the restriction of \(-I\) to \(\mathcal{S}\), where \(I\) is the index form of the original Morse–Sturm system defined in (5.10).

Proof. It is an easy calculation that uses (5.21).

Corollary 5.22. The coindex \(n_+(I|_{\mathcal{S}})\) of the restriction of \(I\) to \(\mathcal{S}\) is finite, and it is equal to the sum of the multiplicities of the conjugate instants of the reduced symplectic system (5.25) in \([a, b]\).

Proof. Observe that the coefficient of \(\varphi\) in the first equation of (5.25) is positive definite. The conclusion follows from Corollary 5.16 and Proposition 5.21.

We now give a criterion for the vanishing of the number \(n_+(I|_{\mathcal{S}})\):

Corollary 5.23. Suppose that either one of the following symmetric bilinear forms (see (5.26) and (5.27)):

\[ C^*B^{-1}C - \mathcal{I}, \quad C'_\alpha - C\alpha B^{-1}C_\alpha - \mathcal{I} \]

is positive semi-definite on \([a, b]\). Then \(n_+(I|_{\mathcal{S}}) = 0\).

Proof. It follows directly from Remark 5.12, Remark 5.20 and Corollary 5.22.

Corollary 5.24. The restriction of \(I\) to \(\mathcal{S}\) is represented by a self-adjoint operator on \(\mathcal{S}\) which is a compact perturbation of a negative isomorphism of \(\mathcal{S}\).

Proof. The index form of any symplectic differential system (5.15) with the coefficient \(B\) positive definite is represented by a compact perturbation of a positive isomorphism of \(H_0^1([a, b]; \mathbb{R}^n)\) (see [22, Lemma 2.6.6]). The conclusion follows from Proposition 5.21.

5.4. An extension of the index form. The strategy for proving Theorem 5.2 will be to apply Proposition 4.7 to a family \(I_t\) of symmetric bilinear forms on Hilbert spaces \(K_t\) obtained by considering restrictions of the Morse–Sturm system (5.7) to the interval \([a, t]\). Unfortunately, we run into the annoying technical problem that the family \(K_t\) fails to be \(C^1\) around the focal instants of the reduced symplectic system (5.25).

In this subsection we describe a trick to overcome this problem by introducing an artificial extension \(I^\#\) of the index form \(I\) to a space \(K^\#\) so that the corresponding family \(K^\#_t\) will be of class \(C^1\).

Let us introduce the “sharped” versions of the objects of our theory: \(\mathcal{H}^\#, \mathcal{K}^\#, \mathcal{G}^\#\) and \(I^\#\). Set:

\[
\mathcal{H}^\# = \left\{ v \in H^1([a, b]; \mathbb{R}^n) : v(a) \in P \right\},
\]

\[
\mathcal{K}^\# = \left\{ v \in \mathcal{H}^\# : g(v', Y_i) \in H^1([a, b]; \mathbb{R}), \quad \begin{array}{l}
g(v', Y_i)' = g(v', Y_i') + g(Rv, Y_i), & i = 1, \ldots, k, \\
g(v', Y_i) = v(t), & \forall t \in [a, b]\end{array} \right\},
\]

\[
\mathcal{G}^\# = \left\{ v \in \mathcal{H}^\# : v(a) = 0, v(t) \in D_t, & \forall t \in [a, b] \right\}.
\]

(5.28)
Throughout this subsection we consider a fixed symmetric bilinear form $\Theta$ in $\mathbb{R}^n$. The extended bilinear form $I^\#$ is defined using $\Theta$ by:

\[(5.29) \quad I^\#(v, w) = \int_a^b \left[ g(v'(t), w'(t)) + g(Rv(t), w(t)) \right] \, dt + \Theta(v(a), w(a)). \]

Recall that $\mathbb{J}$ denotes the space of solutions of (5.7) satisfying the initial conditions (5.8); we can characterize the kernel of $I^\#$ as:

\[(5.30) \quad \ker(I^\#) = \left\{ v \in \mathbb{J} : g(v'(b) + \Theta(v(b)) = 0 \right\}, \]

where $g$ and $\Theta$ are considered as linear maps from $\mathbb{R}^n$ to $\mathbb{R}^{n*}$.

Observe that $K = K^* \cap H$, $\mathcal{S} = \mathcal{S}^* \cap H$ and that $I$ is the restriction of $I^\#$ to $H$.

We define a bounded linear map $F : H^\# \rightarrow L^2([a, b]; \mathbb{R}^{k*})$:

\[(5.31) \quad [F(v)(t)]_i = g(v'(t), Y_i(t)) - \int_a^t \left[ g(v', Y'_i) + g(Rv(t), Y_i(t)) \right] \, ds, \quad i = 1, \ldots, k. \]

Obviously, $K^\#$ is the inverse image by $F$ of the subspace $\mathcal{C}$ of $L^2([a, b]; \mathbb{R}^{k*})$ consisting of constant functions.

**Lemma 5.25.** The restriction of $F$ to $\mathcal{S}^\#$ is an isomorphism.

**Proof.** We identify the space $\mathcal{S}^\#$ with the space $\mathcal{X} = \{ f \in H^1([a, b]; \mathbb{R}^k) : f(a) = 0 \}$ by the map $v = \sum_i f_i Y_i \mapsto f = (f_1, \ldots, f_k)$; then using (5.26), the map $F$ on $\mathcal{S}^\#$ can be written as:

\[(5.32) \quad F(v)(t) = B(f')(t) + C(f)(t) - \int_a^t \left[ C^*(f') + \mathcal{I}(f) \right] \, ds. \]

Using the fact that the inclusion of $H^1$ in $L^2$ is compact, it is easy to see from (5.32) that the restriction of $F$ to $\mathcal{S}^\#$ is a compact perturbation of the isomorphism $\mathcal{X} \ni f \mapsto B(f') \in L^2([a, b]; \mathbb{R}^{k*})$. Hence, the restriction of $F$ to $\mathcal{S}^\#$ is a Fredholm operator of index zero, and to prove the Lemma it suffices to show that $F$ is injective on $\mathcal{S}^\#$. To this aim, observe that if $f \in \ker(F|_{\mathcal{S}^\#})$, then using (5.32) we see that $f$ is a solution of a second order homogeneous linear differential equation, and that $f(a) = f'(a) = 0$. This concludes the proof. \(\square\)

**Corollary 5.26.** The map $F$ is surjective, $\dim(K^\# \cap \mathcal{S}^\#) = k$ and $\mathcal{H}^\# = K^\# + \mathcal{S}^\#$.

**Proof.** It follows easily from Lemma 5.25 and the fact that $K^\#$ is the inverse image by $F$ of the space $\mathcal{C}$ of constant functions, which is $k$-dimensional. \(\square\)

**Lemma 5.27.** Let $q : L^2([a, b]; \mathbb{R}^{k*}) \rightarrow L^2([a, b]; \mathbb{R}^{k*})/\mathcal{C}$ be the quotient map. Suppose that $K \cap \mathcal{S} = \{0\}$; then, $q \circ F$ maps $\mathcal{S}$ isomorphically onto $L^2([a, b]; \mathbb{R}^{k*})/\mathcal{C}$.

**Proof.** The proof is essentially the same as the proof of Lemma 5.25. Namely, using (5.32) we show that the restriction of $q \circ F$ to $\mathcal{S}$ is a Fredholm operator of index zero. The injectivity of this restriction is obviously equivalent to $K \cap \mathcal{S} = \{0\}$. \(\square\)

**Corollary 5.28.** If $K \cap \mathcal{S} = \{0\}$, then $q \circ F$ is surjective and $\mathcal{H} = K \oplus \mathcal{S}$.

**Proof.** It follows from Lemma 5.27 and the fact that $K = \ker(q \circ F|_{\mathcal{H}})$. \(\square\)

**Lemma 5.29.** The space $(K^\# + \mathcal{S})$ is closed in $\mathcal{H}^\#$, and its codimension is equal to $\dim(K \cap \mathcal{S})$. 


Proof. The fact that \((K^\# + \mathcal{S})\) is closed follows easily from \(\mathcal{H}^\# = K^\# + \mathcal{S}^\#\) (Corollary 5.26). Now, we observe:

\[
\text{codim } (K^\# + \mathcal{S}) = \dim \left( \frac{\mathcal{H}^\#}{K^\# + \mathcal{S}} \right)
\]

We have a surjective map:

\[
\iota: \frac{\mathcal{S}^\#}{\mathcal{S}} \to \frac{K^\# + \mathcal{S}^\#}{K^\# + \mathcal{S}} = \frac{\mathcal{H}^\#}{K^\# + \mathcal{S}}
\]

induced by inclusion, and

\[
\text{Ker}(\iota) = \frac{\mathcal{S}^\# \cap (K^\# + \mathcal{S})}{\mathcal{S}} = \frac{(\mathcal{S}^\# \cap K^\#) + \mathcal{S}}{\mathcal{S}} \simeq \frac{\mathcal{S}^\# \cap K^\#}{K \cap \mathcal{S}}.
\]

Finally, using Corollary 5.26, (5.33), (5.34) and (5.35) we compute:

\[
\text{codim}(K^\# + \mathcal{S}) = \dim(\text{Im}(\iota)) = \dim \left( \frac{\mathcal{S}^\#}{\mathcal{S}} \right) - \dim \left( \frac{\mathcal{S}^\# \cap K^\#}{K \cap \mathcal{S}} \right) = k - (k - \dim(K \cap \mathcal{S})) = \dim(K \cap \mathcal{S}). \quad \square
\]

Lemma 5.30. The spaces \(K^\#\) and \(\mathcal{S}\) are \(I^\#\)-orthogonal, i.e., \(I^\#(v, w) = 0\) for all \(v \in K^\#\) and all \(w \in \mathcal{S}\).

Proof. Let \(v \in K^\#\) and \(w = \sum_i f_iY_i \in \mathcal{S}\) be given, with \(f_i \in H^1_0([a, b]; \mathbb{R})\). Using the definition of \(K^\#\), we compute:

\[
I^\#(v, w) = \sum_{i=1}^k \int_a^b \frac{d}{dt} [f_i g(v', Y_i)] \ dt = 0. \quad \square
\]

Proposition 5.31. If \(K \cap \mathcal{S} = \{0\}\), then the kernel of the restriction of \(I\) to \(K\) is equal to the kernel of \(I\) in \(\mathcal{H}\), given in formula (5.11).

Proof. From Lemma 5.30 it follows that \(K\) and \(\mathcal{S}\) are \(I\)-orthogonal; the conclusion follows from Corollary 5.28 and the observation that the kernel of \(I\) in \(\mathcal{H}\) is contained in \(K\). \quad \square

Proposition 5.32. Suppose that \(I^\#\) is nondegenerate on \(\mathcal{H}^\#\). Then, the kernel of the restriction of \(I^\#\) to \(K^\#\) is given by \(K \cap \mathcal{S}\).

Proof. The nondegeneracy assumption of \(I^\#\) on \(\mathcal{H}^\#\) means that it is represented by an injective operator on \(\mathcal{H}^\#\). Using the compact inclusion of \(H^1 \subset C^0\), it is easily seen that formula (5.29) defines a bilinear form which is represented by a compact perturbation of an isomorphism of \(H^1([a, b]; \mathbb{R}^n)\). Using the additivity of the Fredholm index of operators it is easily proven that \(I^\#\) is represented by a Fredholm operator of index zero in \(\mathcal{H}^\#\); hence it follows that \(I^\#\) is indeed represented by an isomorphism of \(\mathcal{H}^\#\).

Using Lemma 5.30, we have inclusions:

\[
K \cap \mathcal{S} \subset \text{Ker } (I^\#|_{K^\#}) \subset \{ v \in \mathcal{H}^\# : I^\#(v, K^\# + \mathcal{S}) = 0 \}.
\]

Since \(I^\#\) is an isomorphism, the dimension of the third member in (5.37) equals the dimension of the annihilator of \(K^\# + \mathcal{S}\) in \((\mathcal{H}^\#)^*\). The dimension of this annihilator coincides with the codimension of \(K^\# + \mathcal{S}\) in \(\mathcal{H}^\#\); by Lemma 5.29, it follows that the inclusions in (5.37) are equalities, which concludes the proof. \quad \square

Proposition 5.33. The restriction of \(I^\#\) to \(K^\#\) (respectively, of \(I\) to \(K\)) is represented by a compact perturbation of a positive isomorphism of \(K^\#\) (respectively, of \(K\)).
Proof. It is essentially identical to the proof of [22, Lemma 2.6.6]. ∎

Proposition 5.34. Suppose that \( t = b \) is not a focal instant for the Morse–Sturm system (5.7). Then,

\[
(5.38) \quad n_-(I^#|_{K^s}) = n_-(I|_{K}) + n_-(\Theta - \phi_{L_1,L_0}(\ell(b))),
\]

where \( L_0 = \{0\} \oplus \mathbb{R}^n, L_1 = \mathbb{R}^n \oplus \{0\}, \phi_{L_1,L_0} \) is the chart of the Lagrangian Grassmannian \( \Lambda \) defined in (3.5) and \( \ell : [a, b] \to \Lambda \) is the curve of Lagrangians defined in (3.18).

Proof. Let us denote by \( \beta \) the symmetric bilinear form on \( L_1 \) given by \( \phi_{L_1,L_0}(\ell(b)) \); we regard \( \beta \) as a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^{n^2} \) by identifying \( L_1 \simeq \mathbb{R}^n \). By the definition of the chart \( \phi_{L_1,L_0} \), we have:

\[
(5.39) \quad \beta(v(b)) = -g'v(b), \quad \forall v \in J.
\]

It is an easy observation that, since \( t = b \) is not a focal instant, we have:

\[
(5.40) \quad n_-(I^#|_{K^s}) = n_-(I|_{K}) + n_-(I^#|_{J}).
\]

The conclusion follows from the fact that, using (5.39), it is easily seen that the isomorphism \( J \ni v \mapsto v(b) \in \mathbb{R}^n \) carries the restriction of \( I^# \) to the bilinear form \( \Theta - \beta \). ∎

5.5. The index function \( i(t) \). In this subsection we consider the restriction of the Morse–Sturm system (5.7) to the interval \( [a, t] \), with \( t \in ]a, b[ \). We define the objects \( \mathcal{H}_t, \mathcal{H}^#_t, I_t, I^#_t, K_t, K^s_t, F_t, K^#_t, G_t, G^#_t \) as in formulas (5.9), (5.10), (5.12), (5.28), (5.29) and (5.31) by replacing \( b \) with \( t \). The definition of the bilinear form \( I^#_t \) depends on the choice of a symmetric bilinear form \( \Theta \); such choice will be made appropriately when needed.

Clearly, all the results of the previous subsections remain valid when the Morse–Sturm system is restricted to the interval \( [a, t] \).

We study the evolution of the index function:

\[
(5.41) \quad i(t) = n_-(I|_{K_t}), \quad t \in ]a, b[;
\]

obviously,

\[
(5.42) \quad n_-(I|_{K_t}) = i(b).
\]

We will use the isomorphisms \( \Phi_t : \mathcal{H}^# \to \mathcal{H}^#_t \) defined by: \( \Phi_t(\tilde{v}) = v \), where

\[
(5.43) \quad v(s) = \tilde{v}(u_s), \quad u_s = a + \frac{b - a}{t - a}(s - a), \quad \forall s \in [a, t];
\]

observe that \( \Phi_t \) carries \( \mathcal{H} \) onto \( \mathcal{H}_t \).

We get families of closed subspaces of \( \mathcal{H}^# \) given by:

\[
K_t = \Phi_t^{-1}(K_t^s), \quad K^#_t = \Phi_t^{-1}(K^#_t), \quad G_t = \Phi_t^{-1}(G_t), \quad G^#_t = \Phi_t^{-1}(G^#_t),
\]
we also get curves \( \hat{I} : [a, b] \to B_{\text{sym}}(H, R) \), \( \hat{I}_t : [a, b] \to B_{\text{sym}}(H^#, R) \) of symmetric bilinear forms and a curve \( \hat{F} : [a, b] \to \mathcal{L}(H^#, L^2([a, b]; \mathbb{R}^{k^*})) \) of maps, defined by:

\[
\hat{I}_t = I(\Phi_t \cdot , \Phi_t \cdot ), \quad \hat{I}^#_t = I^#(\Phi_t \cdot , \Phi_t \cdot ), \quad \hat{F}_t = \Phi_t^{-1} \circ F_t \circ \Phi_t.
\]

We are also denoting by \( \Phi_t \) the isomorphism from \( L^2([a, b]; \mathbb{R}^{k^*}) \) to \( L^2([a, t]; \mathbb{R}^{k^*}) \) defined by formula (5.42).

An explicit formula for \( \hat{I}^#_t \) is given by:

\[
\hat{I}^#_t(\hat{v}, \hat{w}) = \int_a^t \left[ \left( \frac{b-a}{t-a} \right)^2 g(\hat{v}'(u), \hat{w}'(u)) + g(R(s)\hat{v}(u), \hat{w}(u)) \right] ds
\]

\[+ \Theta(\hat{v}(b), \hat{w}(b)) - S(\hat{v}(a), \hat{w}(a)),
\]

for all \( \hat{v}, \hat{w} \in H^# \).

As to the initial value \( i(a) \) of the index function, we need to consider suitable extensions to \( t = a \) of the objects \( \hat{I}_t, \hat{K}_t \) and \( \hat{F}_t \). We set:

\[
\mathcal{I}_t = (t-a)\hat{I}_t, \quad \mathcal{F}_t = (t-a)\hat{F}_t, \quad t \in [a, b].
\]

A change of variable in (5.43) gives the following explicit formula for \( \mathcal{I}_t \):

\[
\mathcal{I}_t(\hat{v}, \hat{w}) = \int_a^b \left[ \left( \frac{b-a}{t-a} \right) g(\hat{v}'(u), \hat{w}'(u)) + \frac{(t-a)^2}{b-a} g(R(s)\hat{v}(u), \hat{w}(u)) \right] du
\]

\[+ (t-a) \left( \Theta(\hat{v}(b), \hat{w}(b)) - S(\hat{v}(a), \hat{w}(a)) \right),
\]

for all \( \hat{v}, \hat{w} \in H \), where \( s_a = a + \frac{t-a}{b-a}(u-a) \). Setting \( t = a \) in (5.44), we define \( \mathcal{I}_a \) as:

\[
\mathcal{I}_a(\hat{v}, \hat{w}) = (b-a) \int_a^b g(\hat{v}'(u), \hat{w}'(u)) du.
\]

Observe that:

\[
\hat{K}_t = \text{Ker}(q \circ \hat{F}_t|_H) = \text{Ker}(q \circ \mathcal{F}_t|_H), \quad \forall t \in [a, b],
\]

where \( q : L^2([a, t]; \mathbb{R}^{k^*}) \to L^2([a, t]; \mathbb{R}^{k^*})/\mathbb{C} \) is the quotient map and \( \mathbb{C} \) is the space of constant functions.

An explicit formula for \( \mathcal{F}_t \) is given by:

\[
\left[ \mathcal{F}_t(\hat{v})(u) \right]_i = (b-a) g(\hat{v}'(u), Y_i(s_u))
\]

\[ - \int_a^u \left[ (t-a) g(\hat{v}'(x), Y_i'(r_x)) + \frac{(t-a)^2}{b-a} g(R(r_x)\hat{v}(x), Y_i(r_x)) \right] dx,
\]

\[ i = 1, \ldots, k, \text{ for all } \hat{v} \in H^# \text{, where } r_x = a + \frac{t-a}{u-a}(x-a). \]

Setting \( t = a \) in (5.46) gives the following definition for \( \mathcal{F}_a \):

\[
\left[ \mathcal{F}_a(\hat{v})(u) \right]_i = (b-a) g(\hat{v}'(u), Y_i(a)), \quad \forall \hat{v} \in H^#.
\]

We also set \( \hat{K}_a = \text{Ker}(q \circ \mathcal{F}_a|_H) \), namely,

\[
\hat{K}_a = \left\{ \hat{v} \in H : g(\hat{v}'(u), Y_i(a)) = \text{constant}, \quad i = 1, \ldots, k \right\}.
\]
Proposition 5.35. Suppose that \( R \) is a map of class \( C^1 \). Then, \( \hat{I}^\# : [a, b] \to B_{\text{sym}}(H^#, R) \) and \( I : [a, b] \to B_{\text{sym}}(H, R) \) are maps of class \( C^1 \). Moreover, \( \{ \hat{K}^\#_t \}_{t \in [a, b]} \) is a \( C^1 \)-family of closed subspaces of \( H^# \) and, provided that there are no focal instants for the reduced symplectic system (5.25) in the interval \( [c, d] \subset [a, b] \), \( \{ K_t \}_{t \in [c, d]} \) is a \( C^1 \)-family of closed subspaces of \( H \).

Proof. By standard regularity arguments (see [13, Lemma 2.3, Proposition 3.3 and Lemma 4.3]), formula (5.44) shows that \( I^# \) and \( I^\# \) are \( C^1 \) in \( [a, b] \), which obviously implies that \( I \) is \( C^1 \) in \( [a, b] \). Similarly, formula (5.46) shows that \( \mathcal{F} \) is of class \( C^1 \) on \( [a, b] \); from Corollary 5.26 we deduce that \( \mathcal{F}_t \) is surjective for \( t \in [a, b] \). The regularity of the family \( \{ \hat{K}^\#_t \}_{t \in [a, b]} \) follows then from Lemma 4.2.

As to the regularity of the family \( \{ \hat{K}_t \}_{t \in [c, d]} \), we have to show that \( q \circ \mathcal{F}_t \mid H \) is surjective for \( t \in [c, d] \). For \( t = a \) it follows directly from the definition of \( \mathcal{F}_a \) in (5.47). For \( t > a \), the surjectivity follows from Corollary 5.18 and Corollary 5.28.

Corollary 5.36. Suppose that \( R \) is a map of class \( C^1 \). If there are no focal instants of the Morse–Sturm system (5.7) and also of the reduced symplectic system (5.25) in the interval \( [c, d] \subset [a, b] \), then the index function \( i \) is constant on \( [c, d] \).

Proof. Let \( t \in [c, d] \) be fixed. Using formula (5.11), Corollary 5.18 and Proposition 5.31 we conclude that \( I_t \) is nondegenerate on \( K_t \), and therefore \( \hat{I}_t \) is nondegenerate on \( \hat{K}_t \). Keeping in mind the result of Proposition 5.33 and Proposition 5.35, the conclusion follows by applying Remark 4.4.

5.6. Proof of Theorem 5.2. We start with the following:

Lemma 5.37. Let \( t_0 \in [a, b] \) and \( \hat{v}_0, \hat{w}_0 \in \hat{K}_{t_0} \cap \hat{S}_{t_0} \), be fixed. Let \( \hat{v}, \hat{w} : [a, b] \to H^# \) be \( C^1 \)-curves with \( \hat{v}_t, \hat{w}_t \in \hat{K}^\#_t \) for all \( t \in [a, b] \) and with \( \hat{v}_t = \hat{v}_0, \hat{w}_t = \hat{w}_0 \). Then:

\[
\frac{d}{dt} \hat{I}^\#(\hat{v}_t, \hat{w}_t) \bigg|_{t=t_0} = g(v'_0(t_0), w'_0(t_0)),
\]

where \( v_0 = \Phi_{t_0}(\hat{v}_0) \) and \( w_0 = \Phi_{t_0}(\hat{w}_0) \).

Proof. By Remark 4.6, the term on the left hand side of (5.49) does not depend on the choice of \( \hat{v} \) and \( \hat{w} \). In order to facilitate the computation, we make a suitable choice of the curves \( v_t \) and \( w_t \), as follows. Write \( v_0(s) = \sum_{i=1}^{k} f_i^{(1)}(s) Y_i(s) \) and \( w_0(s) = \sum_{i=1}^{k} f_i^{(2)}(s) Y_i(s) \), with \( s \in [a, t_0] \); by Lemma 5.17, the maps \( \{ f_i^{(1)} \}_1 \) and \( \{ f_i^{(2)} \}_1 \), are solutions of the reduced symplectic system (5.25), hence they define maps of class \( C^2 \) on the entire interval \( [a, b] \). We set

\[
v_t(s) = \sum_{i=1}^{k} f_i^{(1)}(s) Y_i(s), \quad w_t(s) = \sum_{i=1}^{k} f_i^{(2)}(s) Y_i(s), \quad s \in [0, t], \quad t \in [a, b];
\]

again by Lemma 5.17, \( v_t \) and \( w_t \) are in \( K^\#_t \) for all \( t \), and so the maps \( \hat{v}_t \) and \( \hat{w}_t \) defined by:

\[
\hat{v}_t = \Phi^{-1}_t(v_t), \quad \hat{w}_t = \Phi^{-1}_t(w_t)
\]

are in \( K^\#_t \). Obviously, the maps \( (t, u) \mapsto \hat{v}_t(u) \) and \( (t, u) \mapsto \hat{w}_t(u) \) are of class \( C^2 \), and therefore they define \( H^# \)-valued \( C^1 \)-maps.

Once the choice of \( \hat{v}_t \) and \( \hat{w}_t \) is made, we compute as follows:

\[
\frac{d}{dt} \hat{I}^\#(\hat{v}_t, \hat{w}_t) \bigg|_{t=t_0} = \frac{d}{dt} I^\#(v_t, w_t) \bigg|_{t=t_0} = g(v'_0(t_0), w'_0(t_0)),
\]
Corollary 5.38. Let \( t_0 \in [a, b] \) and suppose that \( I^\#_{t_0} \) is nondegenerate in \( \mathcal{H}^\#_{t_0} \). Setting \( B(t) = I^\#_t \) and \( D_t = \mathcal{K}^\#_t \), then the symmetric bilinear form \( \overline{B}(t_0) \) on \( \text{Ker}(\overline{B}(t_0)) = \mathcal{K}_{t_0} \cap \mathcal{E}_{t_0} \) introduced in Definition 4.5 is negative definite.

Proof. Recall that the kernel of \( \overline{B}(t_0) \) is given in Proposition 5.32. For \( v_0, w_0 \in \mathcal{E}_{t_0} \), we have \( v'_0(t_0), w'_0(t_0) \in D_{t_0} \), and \( g \) is negative definite in \( D_{t_0} \). From Lemma 5.37 it follows that \( \overline{B}(t_0) \) is negative semi-definite. To conclude the proof we have to show that, if \( v_0 \in \mathcal{K}_{t_0} \cap \mathcal{E}_{t_0} \) and \( v'_0(t_0) = 0 \), then \( v_0 = 0 \). This follows easily from Lemma 5.17.

We now determine the initial value of the index function \( i(t) \).

Lemma 5.39. The restriction of the symmetric bilinear form \( \mathcal{J}_a \) to \( \mathcal{K}_a \) is represented by a compact perturbation of a positive isomorphism. Moreover, it is nondegenerate, and its index equals the index of the restriction of \( g \) to \( P \).

Proof. See [22, Lemma 2.7.8].

Corollary 5.40. For \( t \in [a, b] \) sufficiently close to \( a \), we have \( i(t) = n_-(g|_P) \).

Proof. Obviously, for \( t \in [a, b] \), \( i(t) = n_-(\mathcal{J}_a|_{\mathcal{K}_a}) \). The conclusion follows from Remark 4.4 and Lemma 5.39.

We are finally ready for:

Proof of Theorem 5.2. The proof will be done for Morse–Sturm systems (see Subsection 5.1) with coefficients of class \( C^1 \); the geometrical version of the theorem is an immediate corollary.

We first consider the case that there are only a finite number of focal instants for the Morse–Sturm system (5.7) and that \( t = b \) is not a focal instant for the reduced symplectic system (5.25). Observe that, by Corollary 5.16, the number of focal instants for the reduced symplectic system is finite.

By Corollary 5.36, the function \( i(t) \) is piecewise constant, with jumps at the focal instants of either the Morse–Sturm system or the reduced symplectic system. By Corollary 5.40, \( i(t) = n_-(g|_P) \) for \( t \) sufficiently close to \( a \).

Let \( t_0 \in ]a, b[ \) be a focal instant for either the Morse–Sturm or the reduced symplectic system; we compute the jump of \( i \) at \( t_0 \). Choose a Lagrangian \( L_a \) of \( (\mathbb{R}^n \oplus \mathbb{R}^n, \omega) \) which is complementary to both \( \ell(t_0) \) and \( L_0 = \{0\} \oplus \mathbb{R}^n \); such Lagrangian always exists (see for instance [18, Corollary 3.2.9]). Consider the extended index form \( I^\#_t \) defined in (5.29) corresponding to the choice of the bilinear form \( \Theta = \phi_{L_1, L_0}(L_a) \), where \( L_1 = \mathbb{R}^n \oplus \{0\} \). With such a choice, we have that \( I^\#_t \) is nondegenerate on \( \mathcal{H}^\#_t \) for \( t \) near \( t_0 \) (see formula (5.30)).

Using Proposition 5.34, for \( t \neq t_0 \) sufficiently close to \( t_0 \) we have:

\[
i(t) = n_-(I^\#_t|_{\mathcal{K}^\#_t}) - n_-(\phi_{L_1, L_0}(L_a) - \Phi_{L_1, L_0}(\ell(t))).
\]

Using Corollary 4.8, Proposition 5.33 and Corollary 5.38, the jump of the function \( n_-(I^\#_t|_{\mathcal{K}^\#_t}) \) as \( t \) passes through \( t_0 \) equals the dimension of \( \mathcal{K}_{t_0} \cap \mathcal{E}_{t_0} \), which by Corollary 5.18 is equal to the multiplicity of \( t_0 \) as a focal instant for the reduced symplectic system. The sum of these multiplicities as \( t_0 \) varies in \( ]a, b[ \) equals \( n_+(I|_\Theta) \) by Corollary 5.22.
By Proposition 3.5, the jump of the function \( n_-(\phi_{L_1,t_0}(L_s) - \Phi_{L_1,t_0}(\ell(t))) \) as \( t \) passes through \( t_0 \) is equal to \( -\mu_{L_0}(\ell|_{[t_0-\varepsilon,t_0+\varepsilon]}(t)) \) for \( \varepsilon > 0 \) sufficiently small. Since \( \mu_{L_0} \) is additive by concatenation, the sum of these jumps equals minus the Maslov index of the Morse–Sturm system.

This concludes the proof for the case of a Morse–Sturm system (5.7) whose focal instants are isolated and such that \( t = b \) is not focal for the reduced symplectic system (5.25).

Consider now the more general case of a Morse–Sturm system for which \( t = b \) is not focal for the associated reduced symplectic system. Let \( R_n : [a, b] \to \mathcal{L}(\mathbb{R}^n) \) be a sequence of real analytic curves of \( g \)-symmetric linear endomorphisms of \( \mathbb{R}^n \) that converges uniformly to \( R \) on \([a, b]\). Let \( I^{(n)} \) be the index form of the corresponding Morse–Sturm problem and denote by \( K^{(n)} \) the associated space defined as in (5.1). Then, \( I^{(n)} \) converges to \( I \) in the operator norm topology. Since \( I|_{\mathcal{K}} \) is nondegenerate (see formula (5.22) and Proposition 5.21) and it is represented by a compact perturbation of a negative isomorphism of \( \mathcal{K} \) (Corollary 5.24), it follows that, for \( n \) sufficiently large, \( n_+(I^{(n)}|_{\mathcal{K}}) = n_+(I|_{\mathcal{K}}) \) (Remark 4.4). Moreover, since \( I|_{\mathcal{K}} \) is nondegenerate, \( I|_{\mathcal{K}} \) is represented by a compact perturbation of a positive isomorphism of \( \mathcal{K} \) and \( K^{(n)} \) converges\(^3\) to \( K \), by Remark 4.4 we have \( n_-(I^{(n)}|_{K^{(n)}}) = n_-(I|_{K}) \) for \( n \) sufficiently large.

The conclusion in the case that \( t = b \) is not focal for the reduced symplectic system follows from the stability of the Maslov index by uniformly small perturbations of the coefficient \( R \) in (5.7) (see [18, Theorem 5.2.1]).

In the general case that \( t = b \) may be focal for the reduced symplectic system, the conclusion follows from the fact that the functions \( i(t) \) and \( n_+|_{\mathcal{E}_t} \) are left-continuous at \( t = b \). The left-continuity of \( n_+(I_t|_{\mathcal{E}_t}) \) follows from Corollary 5.22; the left-continuity of \( i(t) \) follows from Corollary 5.38 and from formula (5.51). \( \square \)

### 6. The Index Theorem for Symplectic Differential Systems

In Subsection 5.2 we have defined the notion of symplectic differential system, and we have seen that every such system is isomorphic to a Morse–Sturm system (Proposition 5.15). Moreover, we have seen that the notions of focal instant, multiplicity, signature and index form are invariant by isomorphisms (Proposition 5.14). This suggests that it is possible to give a general version of the index theorem for symplectic differential systems; the purpose of this section is to give the main definitions and to state the generalized index theorem for symplectic systems with initial conditions. We will use most of the notations introduced in Subsection 5.2; the details of many of the results presented in this section may be found in [22, Section 2].

We consider a symplectic differential system in \( \mathbb{R}^n \) of the form (5.15), and we consider the initial conditions:

\[
(6.1) \quad v(a) \in P, \quad \alpha(a)|_P + S(v(a)) = 0,
\]

where \( P \subset \mathbb{R}^n \) is a subspace and \( S \) is a symmetric bilinear form on \( P \), considered as a map from \( P \) to \( P^* \). The set \( \ell_0 \subset \mathbb{R}^n \oplus \mathbb{R}^{n*} \) defined by:

\[
(6.2) \quad \ell_0 = \left\{ (v, \alpha) : v \in P, \alpha|_P + S(v) = 0 \right\}
\]

is a Lagrangian subspace of \((\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)\); conversely, every Lagrangian subspace \( \ell_0 \) of \((\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)\) defines uniquely a subspace \( P \subset \mathbb{R}^n \) and a symmetric bilinear form on

---

\(^3\)here we use the fact that \( t = b \) is not focal for the reduced symplectic system, as well as Corollary 5.28.
P such that (6.2) holds. We will say that \( v \) is an \((X, \ell_0)\)-solution if \( v \) is an \( X \)-solution such that \((v(a), \alpha_v(a)) \in \ell_0\). In analogy with (5.17), we now define:

\[
\mathcal{V} = \left\{ v : v \text{ is an } (X, \ell_0)\text{-solution} \right\}.
\]

The notions of focal instant, multiplicity, signature and focal index for the pair \((X, \ell_0)\) are given in Definition 5.11, where the space \( \mathcal{V} \) is now redefined in (6.3).

As in the case of semi-Riemannian geodesics, we need the following nondegeneracy assumption on the initial conditions for the symplectic differential system:

**Definition 6.1.** A pair \((X, \ell_0)\) where \( X \) is the coefficient matrix of a symplectic differential system and \( \ell_0 \) is a Lagrangian subspace of \((\mathbb{R}^n \oplus \mathbb{R}^{n*}, \omega)\) is said to be a set of data for the symplectic differential problem if the symmetric bilinear form \( B(a)^{-1} \) in \( \mathbb{R}^n \) is nondegenerate on the subspace \( P \) associated to \( \ell_0 \).

Let \((X, \ell_0)\) be a set of data for the symplectic differential problem; for each \( t \in [a, b] \), the subspace \( \ell(t) \subset \mathbb{R}^n \oplus \mathbb{R}^{n*} \) given by:

\[
\ell(t) = \left\{ (e(t), \alpha_v(t)) : v \in \mathcal{V} \right\}
\]

is Lagrangian. So, we get a \( C^1 \)-curve \( \ell \) in the Lagrangian Grassmannian \( \Lambda \); recalling the notations of Subsection 3.1 and setting \( L_0 = \{0\} \oplus \mathbb{R}^{n*} \), it is easily seen that \( \ell(t) \in \Lambda_{\geq 1}(L_0) \) if and only if \( t \) is a focal instant.

Given the nondegeneracy assumption in Definition 6.1, it is possible to see that there exists \( \varepsilon > 0 \) such that there are no focal instants in \( [a, a + \varepsilon] \). We can therefore give the following definition:

**Definition 6.2.** If \( t = b \) is not a focal instant, the Maslov index \( i_{\text{maslov}}(X, \ell_0) \) of the pair \((X, \ell_0)\) is defined as:

\[
i_{\text{maslov}}(X, \ell_0) = \mu_{L_0}(\ell|_{[a+\varepsilon,b]}),
\]

where \( \varepsilon > 0 \) is chosen in such a way that there are no focal instants in \( [a, a + \varepsilon] \).

Proposition 3.8 can be generalized to symplectic systems.

The index form \( I_{(X, \ell_0)} \) associated to the symplectic differential problem is the bounded symmetric bilinear form on the Hilbert space \( \mathcal{H} \) given in (5.9) defined by:

\[
I_{(X, \ell_0)}(v, w) = \int_a^b \left[ B(\alpha_v, \alpha_w) + C(v, w) \right] \, dt - S(v(a), w(a)).
\]

Recalling Definition 5.13, we now give the following definition of isomorphisms for symplectic differential systems with initial data:

**Definition 6.3.** The pairs \((X, \ell_0)\) and \((\tilde{X}, \tilde{\ell}_0)\) of data for the symplectic differential problem are said to be isomorphic if there exists an isomorphism \( \phi_0 \) between \( X \) and \( \tilde{X} \) such that \( \phi_0(a)(\ell_0) = \tilde{\ell}_0 \).

Proposition 5.14 generalizes *mutatis mutandis* to the case of isomorphisms of pairs \((X, \ell_0)\); moreover, isomorphic pairs have the same Maslov index (see [22, Proposition 2.10.2]).

Using Proposition 5.15, we have the following index theorem for symplectic systems:

**Theorem 6.4.** Let \((X, \ell_0)\) be a smooth set of data for the symplectic differential problem in \( \mathbb{R}^n \), with \( k = n_-(B) \). Let \( Y_1, \ldots, Y_k : [a, b] \to \mathbb{R}^n \) be smooth maps such that, for each \( t \in [a, b] \), \( Y_1(t), \ldots, Y_k(t) \) for a basis of a subspace \( D_t \subset \mathbb{R}^n \) on which \( B(t)^{-1} \) is
negative definite. Consider the following two closed subspaces of $\mathcal{H}$ (see (5.9)):

$$\mathcal{K} = \left\{ v \in \mathcal{H} : \alpha_v(Y_i) \in H^1([a, b]; \mathbb{R}) \text{ and} \right\}$$

(6.6) $$\alpha_v(Y_i)' = B(\alpha_v, \alpha_{Y_i}) + C(v, Y_i), \ \forall i = 1, \ldots, k;$$

$$\mathcal{S} = \left\{ v \in H^1_0 ([a, b]; \mathbb{R}^n) : v(t) \in D_i, \ \forall t \in [a, b] \right\}.$$

Then, if $t = b$ is not focal, we have:

(6.7) $$i_{\text{maslov}}(X, t_0) = n_-(I(X, t_0)|{\mathcal{K}}) - n_+(I(X, t_0)|{\mathcal{S}}) - n_-(B(a)^{-1}|{\mathcal{S}}),$$

where all the terms in the above equality are finite integer numbers.

Proof. It follows directly from Proposition 5.15 and the proof of Theorem 5.2. $\square$

It is easy to see that the space $\mathcal{K}$ depends only on the family of subspaces \{D_i\}_{t \in [a, b]}, and not on the particular choice of a basis $Y_1, \ldots, Y_k$. Moreover, the spaces $\mathcal{K}$ and $\mathcal{S}$ are $I(X, t_0)$-orthogonal.

Also in this context it is possible to determine a reduced symplectic system associated to the choice of the vector fields $Y_i$. The formula of this reduced system is the same as (5.25), where the matrices $B$, $C$ and $I$ are now given by:

(6.8) $$B_{ij} = B^{-1}(Y_i, Y_j), \quad C_{ij} = \alpha_{Y_j}(Y_i), \quad I_{ij} = B(\alpha_{Y_i}, \alpha_{Y_j}) + C(Y_i, Y_j).$$

Observe that the reduced symplectic system is always considered with initial conditions $f(a) = 0$ regardless of the initial conditions considered for the original symplectic system.

Many of the results of Subsection 5.3 (Lemma 5.17, Corollary 5.18, Proposition 5.21, Corollaries 5.22 and 5.24) generalize to this context. In particular, if $t = b$ is not focal for the reduced symplectic system, then $\mathcal{H} = \mathcal{K} \oplus \mathcal{S}$ and the term $n_+(I(X, t_0)|{\mathcal{S}})$ in formula (6.7) can be computed as the sum of the multiplicities of the focal instants of the reduced symplectic system in $[a, b]$.

Remark 6.5. Observe that the proof of Theorem 6.4 is valid under a weaker assumption on the regularity of the coefficients of the symplectic system and of the fields $Y_i$. More precisely, if $t = b$ is not focal for the reduced symplectic system, our proof works in the case that $A$ is of class $C^1$, $B$ is of class $C^2$, $C$ is continuous and the $Y_i$’s are of class $C^2$. In the general case, one has to assume that $A$ is of class $C^2$, $B$ is of class $C^3$, $C$ is of class $C^1$ and the $Y_i$’s are of class $C^3$. It is known to the authors that a direct proof of Theorem 6.4 (that does not use Proposition 5.15), technically more involved than the one presented in this paper, shows that the regularity assumption can be weakened even further. Namely, if $t = b$ is not focal for the reduced symplectic system, Theorem 6.4 is valid under the assumption that $A$ and $B$ are of class $C^1$, $C$ is continuous and the $Y_i$’s are of class $C^2$; in the general case one needs the assumption that also $C$ is of class $C^1$.

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