Note on the non-adjacent BCFW deformations

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Abstract: As the first part, we provide a proof of the consistency condition by Kleiss-
Kuijf relation and the bonus relation by BCJ relation for the non-adjacent BCFW
defformations. On the other hand, rather than appealing to field theory argument, we
provide an alternative proof of bonus relation for the non-adjacent BCFW deformations
by a purely S matrix analysis in the context of $\mathcal{N} = 4$ SYM theory.
1. Introduction

S matrix program is to try to construct the scattering amplitude based on some general principles such as Poincare invariance, analyticity, and unitarity. A great advance along this line is the recent discovery of BCFW recursion relation, which allows us to construct the tree level on-shell scattering amplitude from the lower ones\cite{1, 2}. In particular, by the consistency condition between the adjacent BCFW deformations, one can construct the tree level scattering amplitude for massless spin-1 particles in four dimensional Minkowski spacetime just under the good BCFW deformation on any adjacent pair without resorting to the underlying quantum theory of gauge fields\cite{3, 4, 5}.

However, there appears to be some additional information hidden in the non-adjacent BCFW deformations. For example, in the most recent derivation of Kleiss-Kuijf relation from BCFW recursion relation, the consistency condition imposed by the non-adjacent BCFW deformations has been employed, and it is also shown that BCJ relation can be obtained by the bonus relation from the non-adjacent BCFW deformations\cite{6}.

The purpose of this note is two-fold. Firstly, we provide a proof of the consistency condition by Kleiss-Kuijf relation and the bonus relation by BCJ relation for
the non-adjacent BCFW deformations reversely, which thus implies that such a consistency condition and bonus relation are essentially equivalent to Kleiss-Kuijf relation and BCJ relation respectively. On the other hand, rather than appealing to field theory argument[7], we reproduce the bonus relation for the non-adjacent BCFW deformation by a purely S matrix analysis, where $\mathcal{N} = 4$ SYM theory will be employed for convenience.

2. Proof of consistency condition by Kleiss-Kuijf relation

We firstly introduce the Kleiss-Kuijf relation[8], i.e.,

$$A_n(1, \{\alpha\}, i, \{\beta\}) = (-1)^{n_\beta} \sum_\sigma A_n(1, \sigma(\{\alpha\}, \{\beta^T\}), i),$$  \hspace{1cm} (2.1)$$

where the set $\{\beta^T\}$ denotes the reversed ordering of $\{\beta\}$, $n_\beta$ is the number of elements in $\{\beta\}$, and $\sigma$ takes over all the permutations which preserve the relative ordering in $\{\alpha\}$ and $\{\beta^T\}$. It is noteworthy that the Kleiss-Kuijf can actually include the color-order reversed relation and $U(1)$ decoupling equation as its special cases. Speaking specifically, when we set $\{\alpha\} = \emptyset$, the Kleiss-Kuijf relation gives us the color-order reversed relation, i.e.,

$$A_n(1, 2, \cdots, n) = (-1)^n A_n(n, n-1, \cdots, 1),$$  \hspace{1cm} (2.2)$$
on the other hand, when we set $n_\beta = 1$, the Kleiss-Kuijf relation follows the $U(1)$ decoupling equation, i.e.,

$$\sum_{\sigma \in \text{cyclic}} A_n(1, \sigma(2, 3, \cdots, n)) = 0. $$  \hspace{1cm} (2.3)$$

Now let us do the BCFW deformation on the scattering amplitude $A(1, \{\alpha\}, i, \{\beta\})$ by shifting the non-adjacent pair denoted by $[1, i]$, i.e.,

$$A_{[1,i]}(1, \{\alpha\}, i, \{\beta\}) = \sum A_L(\hat{P}, \{\beta_2\}, \hat{1}, \{\alpha_1\}) \frac{1}{P^2} A_R(\hat{i}, \{\beta_1\}, -\hat{P}, \{\alpha_2\}),$$  \hspace{1cm} (2.4)$$

where $\sum$ represents the summation over internal helicities, and all possible divisions of $\{\alpha\} = \{\alpha_1\} \cup \{\alpha_2\}$ and $\{\beta\} = \{\beta_1\} \cup \{\beta_2\}$. Applying the Kleiss-Kuijf relation to both of the left and right hand side amplitudes, we obtain

$$A_{[1,i]}(1, \{\alpha\}, i, \{\beta\}) = \sum (-1)^{\alpha_1+\alpha_2} A_L(\hat{P}, \sigma^L(\{\beta_2\}, \{\alpha_1^T\}), \hat{1}) \frac{1}{P^2} A_R(\hat{i}, \sigma^R(\{\beta_1\}, \{\alpha_2^T\}), -\hat{P})$$

$$= \sum (-1)^{\beta_1} A_L(\hat{P}, \sigma^L(\{\beta_2\}, \{\alpha_1^T\}), \hat{1}) \frac{1}{P^2} A_R(\hat{i}, \sigma^R(\{\beta_1\}, \{\alpha_2^T\}), -\hat{P}).$$  \hspace{1cm} (2.5)$$
Obviously, now the BCFW deformation on the pair becomes an adjacent one, denoted by \((1, i)\), i.e.,

\[
A_{[1,i]}(1, \{\alpha\}, i, \{\beta\}) = \sum (-1)^\alpha A_{(1,i)}(i, \sigma(\{\beta\}, \{\alpha^T\}), 1) = A(1, \{\alpha\}, i, \{\beta\}),
\] (2.6)

where we have used the fact that \(\sigma^L(\{\beta_2\}, \{\alpha^T\})\) together with \(\sigma^R(\{\beta_1\}, \{\alpha^T\})\) is the same as \(\sigma(\{\beta\}, \{\alpha^T\})\) in the first step, and employed the Kleiss-Kuijf relation in the last step. Therefore, if the Kleiss-Kuijf relation holds for the scattering amplitude constructed from the adjacent BCFW deformation, then the non-adjacent BCFW deformation produces the same scattering amplitude as the adjacent one, which completes our proof.

3. Proof of bonus relation by BCJ relation

Let us start with BCJ relation, i.e.,

\[
A_n(1, 2, \{\alpha\}, 3, \{\beta\}) = \sum_{\sigma} A_n(1, 2, 3, \sigma(\{\alpha\}, \{\beta\})) \prod_{k=4}^{m} \frac{\mathcal{F}(3, \sigma(\{\alpha\}, \{\beta\}), 1\vert k)}{s_{2,4,\ldots,k}}. \] (3.1)

Here without loss of generality, we set \(\{\alpha\} \equiv \{4, 5, \ldots, m - 1, m\}\) and \(\{\beta\} \equiv \{m + 1, m + 2, \ldots, n - 1, n\}\). In addition, \(\sigma\) denotes all the permutations of the set \(\{\alpha\} \cup \{\beta\}\) that maintains the order of elements in \(\{\beta\}\). The function \(\mathcal{F}\) associated with \(k\) is given by

\[
\mathcal{F}(3, \sigma(\{\alpha\}, \{\beta\}), 1\vert k) \equiv \mathcal{F}(\rho\vert k) = \begin{cases} 
\sum_{l=t_k}^{n-1} G(k, \rho_l) & \text{if } t_{k-1} < t_k \\
- \sum_{l=1}^{t_k} G(k, \rho_l) & \text{if } t_{k-1} > t_k \\
+ \begin{cases} 
 s_{2,4,\ldots,k} & \text{if } t_{k-1} < t_k < t_{k+1} \\
- s_{2,4,\ldots,k} & \text{if } t_{k-1} > t_k > t_{k+1} \\
0 & \text{else}
\end{cases} 
\end{cases}. \] (3.2)

Here \(t_k\) is the position of \(k\) in the set \(\{\rho\}\), except that we set \(t_3 \equiv t_5\) and \(t_{m+1} \equiv 0\) as the boundary condition once and for all. The function \(G\) is given by

\[
G(i, j) = \begin{cases} 
 s_{i,j} & \text{if } i < j \text{ or } j = 1, 3 \\
0 & \text{else}
\end{cases}. \] (3.3)

Finally, the kinematic invariants are defined as

\[
s_{i,j} = (k_i + k_j)^2,
\]

\[
s_{2,4,\ldots,i} = (k_2 + k_4 + \ldots + k_i)^2. \] (3.4)
Apparently, to prove the bonus relation for the non-adjacent BCFW deformations is equivalent to show the corresponding large $z$ behavior goes like $\frac{1}{z^2}$. Note that BCJ relation is trivial in the case of $\{\alpha\} = \emptyset$. Thus in what follows, we shall assume the set $\{\alpha\} \neq \emptyset$, which means the BCFW deformation of the pair $[2,3]$ on the left hand side of Eq.(3.1) is always a non-adjacent one. So it is sufficient for us to show the whole product part of right hand side of Eq.(3.1) is of order $\frac{1}{z}$ when $z$ goes to infinity, as the same deformation on the adjacent pair $(2,3)$ of the amplitude part gives the order $\frac{1}{z}$ by BCFW recursion relation.

It is noteworthy that in each $k$th term of the product part, the denominator always contributes order of $z$ due to the deformation on particle 2, while the numerator contributes order of $z$ or order of 1, depending on the specific condition on $t_k$. So it is only necessary for us to show that there exists at least one numerator which contributes order of 1.

Firstly, let us check the case of $m = 4$ where $t_3 = t_5 = 0$ due to the boundary condition. So by Eq.(3.2), the contribution from the numerator gives the desired order of 1. Similarly, for the case of $m = 5$, if $t_3 < t_4$, then the contribution from the 4th term follows order of 1; on the other hand, if $t_3 > t_4$, then the contribution from the 5th term yields order of 1.

Now let us move on to the more general cases, i.e., $m > 5$. Here we assume that each $k$th term contributes order of $z$ if only $k \neq m$, otherwise the proof is automatically completed by definition. By Eq.(3.2), such an assumption implies that $t_{k-1} < t_k < t_{k+1}$ should hold for $k = 4, 5, \ldots, m - 1$. In particular, we thus have $t_{m-1} < t_m > t_{m+1} = 0$. Then it follows from Eq.(3.2) that the contribution from the $m$th term produces the expected order of 1, which thus completes our proof.

4. Proof of bonus relation by a purely S matrix analysis in the context of $\mathcal{N} = 4$ SYM theory

In what follows, to eschew the cumbersome helicity analysis, we shall work in the context of $\mathcal{N} = 4$ SYM theory, which is well known to produce the same result for the tree amplitude in purely gauge theory.

4.1 Super-BCFW recursion relation in $\mathcal{N} = 4$ SYM theory

In $\mathcal{N} = 4$ SYM theory, we can group all on-shell states into a super-wavefunction as\cite{10, 11}

$$
\Phi(p, \eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2} \eta^A \eta^B S_{AB}(p)
$$
\[
\frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{F}^D(p) + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD} G^-(p),
\] (4.1)

where \( \eta^A \) is the Grassmann variable with \( A = 1, 2, 3, 4 \). Whence the corresponding super-amplitude can be written as a function of \((\lambda, \bar{\lambda}, \eta)\). For example, the super-MHV amplitude is given by [12]

\[
A_n = \frac{\delta^4(\sum_{i=1}^n \lambda_i \bar{\lambda}_i) \delta^8(\sum_{i=1}^n \lambda_i \eta_i)}{\langle 1|2|3|\ldots|n|1 \rangle} = \frac{\delta^4(\sum_{i=1}^n \lambda_i \bar{\lambda}_i) \delta^8(\sum_{i=1}^n \lambda_i \eta_i)}{\text{cyc}(1, 2, \ldots, n)}.
\] (4.2)

The purely gluon amplitude can be obtained by integral over the grassmann variable or setting it to be zero, depending the specific helicity of gluon.

To guarantee the supersymmetric counterpart of momentum conservation, besides the ordinary deformation on the pair \((k, l)\), i.e.,

\[
\lambda_k(z) = \lambda_k + z \lambda_l,
\]
\[
\bar{\lambda}_l(z) = \bar{\lambda}_l - z \bar{\lambda}_k,
\] (4.3)

one need also do the additional deformation for \( \eta \) as

\[
\eta_l(z) = \eta_l - z \eta_k.
\] (4.4)

With such a deformation, the super-BCFW recursion relation can be expressed as [11]

\[
A_{k,l} = \sum_{L,R} \int d^4 \eta P A_L[\lambda_k(z_0), -\lambda_P(z_0), \bar{\lambda}_P(z_0), \eta_P] \frac{1}{P_2} A_R[\lambda_P(z_0), \bar{\lambda}_P(z_0), \eta_P, \bar{\lambda}_l(z_0), \eta_l(z_0)].
\] (4.5)

Note that the minus sign is judiciously chosen on the left hand side \( \lambda_P \) such that the left hand side momentum has the opposite sign as the right hand side one. Such a choice also ensures the supersymmetry.

4.2 Super-MHV expansion in \( \mathcal{N} = 4 \) SYM theory

Starting from the amplitude constructed essentially by any adjacent BCFW deformation, it has been shown without any other assumption that MHV vertex expansion is valid for all tree amplitudes in \( \mathcal{N} = 4 \) SYM theory [13, 14]. Such a result has also been generalized to super-MHV vertex expansion [13]. Speaking specifically, the large \( z \) behavior of \( N^k \)MHV super-amplitude goes like \( \frac{1}{z^k} \) under the all-line supershifts, i.e.,

\[
\tilde{i}(z) = \tilde{i} + z c_i \tilde{X},
\]
\[
\eta_l(z) = \eta_l + z c_i \eta_X
\] (4.6)
Here $c_i$ satisfies $\sum_{i=1}^n c_i i = 0$ but $\sum_{i \in \{ \alpha \}} c_i i \neq 0$ with $\{ \alpha \}$ all proper subsets of consecutive external lines. In addition, $\hat{X}$ and $\eta_X$ are the arbitrary reference spinor and Grassmann variable respectively. Thus it follows from the corresponding recursion relation that the $N^k$MHV super-amplitude can be expressed as the super-MHV expansion, i.e.,

$$A_n = \sum_{\{ \alpha_1, \alpha_2, \ldots, \alpha_k \}} \frac{\delta^4(\sum_{i=1}^n \lambda_i \hat{\lambda}_i) \delta^8(\sum_{i=1}^n \lambda_i \eta_i)}{\text{cyc}(I_1) \text{cyc}(I_2) \ldots \text{cyc}(I_{k+1})} \prod_{l=1}^k \frac{1}{P_{\alpha_l}^2} \prod_{A=1}^4 \left[ P_{\alpha_l}^2 \eta_X^A + 2 \sum_{i \in \{ \alpha_l \}} (i \hat{X} \cdot P_{\alpha_l} \eta_i^A) \right],$$

(4.7)

where those internal line spinors implicit in cyc are given by $\hat{X} \cdot P_{\alpha_l}$.  

### 4.3 Proof of bonus relation for non-adjacent BCFW deformations

For simplicity but without loss of generality, we shall focus on the proof of bonus relation for the BCFW deformation on the non-adjacent pair $[1, i]$. To achieve our goal, we firstly do expand such a deformed amplitude on the basis of the BCFW recursion relation for the pair $(1, 2)$. Obviously, there are only two kinds of diagrams contributing to the recursion relation, i.e., the diagrams with $i$ staying with 1 on the left or the diagrams with $i$ staying with 2 on the right. For the former case, the large $z$ behavior is completely determined by the right hand side lower point amplitude since the $z$ dependence comes only from this lower point amplitude. What’s more, such a $z$ dependence can be regarded as the effect of the secondary BCFW deformation on the the non-adjacent pair $[1, i]$ in the lower point on-shell amplitude. On the other hand, for the latter case, the situation becomes a little bit cumbersome, because the $z$ dependence comes from the three parts, i.e., the propagator, the right and left hand side lower point amplitudes. However, by the high school spinor analysis, one can show the right and left hand side parts can be considered effectively as the secondary super-BCFW deformation on the adjacent pair $(1, P)$ and super-Risager deformation on the triple $\{ i, P, 2 \}$ individually(Please refer to Appendix for explicit calculations).

If we choose $\hat{I}$ and $\eta_1$ as the reference spinor and Grassmann variable for the all-line supershifts (4.6), then by the corresponding super-MHV vertex expansion (4.7), the $z$ dependence of super-amplitude comes only from the propagators under our triple super-Risager deformation. Therefore the worst possible large $z$ behavior comes from the case where our triple super-Risager deformation occurs on the same super-MHV vertex, which gives us the order of $z^0$.

Now taking into account the fact that the large $z$ behavior for the adjacent super-BCFW deformation and triple super-Risager deformation go like $z^{-1}$ and $z^0$ respectively, we thus can prove that the large $z$ behavior goes like $z^{-2}$ for the non-adjacent
super-BCFW deformation by induction. Note that the four point scattering amplitude is just the MHV amplitude Eq. (4.2), which satisfies the $z^{-2}$ behavior for the non-adjacent super-BCFW deformation, we thus complete our proof.

5. Conclusion

Along the line of S matrix program for massless spin-1 particles, the consistency condition between the BCFW deformations on various adjacent pairs gives us nothing but Lie algebra structure for the coupling constant [3, 4, 5], which thus allows us to construct the color stripped scattering amplitude by the BCFW deformation on any adjacent pair.

On the other hand, the consistency condition and bonus relation associated with the non-adjacent BCFW deformations give Kleiss-Kuijf relation and BCJ relation on the scattering amplitude [6, 16]. It is now shown in this note that Kleiss-Kuijf relation and BCJ relation follow the consistency condition and bonus relation for the non-adjacent BCFW deformations respectively, which implies that such a consistency condition and Kleiss-Kuijf relation are essentially equivalent to each other, the same for the bonus relation and BCJ relation.

Note that both the bonus relation and BCJ relation imply the consistency condition and Kleiss-Kuijf relation. Therefore, we would obtain a purely S matrix construction of these objects once we could provide a proof of bonus relation or BCJ relation by the above constructed scattering amplitude through any adjacent pair rather than involve any field theory argument [6, 16]. As shown in this note, we have virtually accomplished this task by reproducing the bonus relation through a purely S matrix analysis in the context of $\mathcal{N} = 4$ SYM theory.

We conclude with one simple observation and two open problems. Applying the same strategy given in Section 4 to the adjacent BCFW deformation on the pair $(1, n)$, it is obvious to obtain by induction that the large $z$ behavior goes like $z^{-1}$ for such a deformation, which thus provides an alternative way to argue for the consistency condition among those adjacent BCFW deformations [4, 5].

In addition, parallel to the bonus relation and BCJ relation in gauge theory, gravity has the similar bonus relation and KLT relation [7, 17, 18, 19, 20, 21], so it is interesting to investigate how the bonus relation and KLT relation are explicitly related to each other in gravity theory. In particular, it is also tempting to explore whether the bonus relation or KLT relation can be obtained by a purely S matrix analysis.
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APPENDIX

Before we do the BCFW deformation on the non-adjacent pair $[1, i]$, the deformation on the adjacent pair $(1, 2)$ imposes the on-shell condition on the internal line as

$$P^2 - 2z_0(2\bar{1}) \cdot P = 0.$$  \hfill (1)

Whence we can fix $z_0$ and the corresponding on-shell internal momentum denoted by $(\lambda, \tilde{\lambda})$. Now for the later convenience but without making difference in analysis of large $z$ behavior, we firstly do the BCFW deformation on the non-adjacent pair $[1, i]$ as follows

$$1(z) = 1 + z\langle \lambda | 2 \rangle i,$$
$$\tilde{i}(z) = \tilde{i} - z\langle \lambda | 2 \rangle \bar{1}. \hfill (2)$$

Now by the BCFW deformation on the adjacent pair $(1, 2)$, the on-shell condition of the internal line gives us

$$P^2 - 2[z'_02 + z\langle \lambda | 2 \rangle i \bar{1}] \cdot P = 0.$$  \hfill (3)

Setting $z'_0 = z_0 + z'$ and plugging Eq.\((1)\) into Eq.\((3)\), we have

$$z'2 + z\langle \lambda | 2 \rangle i = 0.$$  \hfill (4)

Then multiplying $\lambda$ yields

$$z' = z\langle i | \lambda \rangle.$$  \hfill (5)

Furthermore, the momentum conservation for the right hand side sub-amplitude also requires the shift of $\bar{\lambda}$, i.e.,

$$\bar{\lambda}(z) = \bar{\lambda} - z\langle 2 | i \rangle \bar{1}. \hfill (6)$$

By the same token, the momentum conservation for the left hand side sub-amplitude implies

$$1(z'_0) = 1 + z\langle \lambda | 2 \rangle i + z'_02 = 1 + z_02 + z\langle 2 | i \rangle (-\lambda), \hfill (7)$$
which can also be obtained by Schouten identity indeed.

Note that the corresponding Grassmann variables are shifted in accordance with the tilde spinors. Thus the dependence on the sub-amplitudes can be regarded as the secondary super-BCFW deformation on the adjacent pair \((1, P)\) on the left hand side sub-amplitude and kind of super-Risager deformation on the triple \(\{i, P, 2\}\) on the right hand side one.

References

[1] R. Britto, F. Cachazo, and B. Feng, Nucl. Phys. B715: 499(2005).
[2] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. 94: 181602(2005).
[3] P. Benincasa and F. Cachazo, arXiv:0705.4305[hep-th].
[4] S. He and H. Zhang, arXiv:0811.3210[hep-th].
[5] P. C. Schuster and N. Toro, JHEP 0906: 079(2009).
[6] B. Feng, R. Huang, and Y. Jia, arXiv:1004.3417[hep-th].
[7] N. Arkani-Hamed and J. Kaplan, JHEP 0804: 076(2008).
[8] R. Kleiss and H. Kuijf, Nucl. Phys. B312: 616(1989).
[9] Z. Bern, J. J. M. Carrasco, and H. Johansson, Phys. Rev. D78: 085011(2008).
[10] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. B828: 317(2010).
[11] N. Arkani-Hamed, F. Cachazo, and J. Kaplan, arXiv:0808.1446[hep-th].
[12] V. P. Nair, Phys. Lett. B214: 215(1988).
[13] H. Elvang, D. Freedman, and M. Kiermaier, JHEP 0904: 009(2009).
[14] H. Elvang, D. Freedman, and M. Kiermaier, JHEP 0906: 068(2009).
[15] M. Kiermaier and S. G. Naculich, JHEP 0905: 072(2009).
[16] Y. Jia, R. Huang, and C. Liu, arXiv:1005.1821.
[17] H. Kawai, D. C. Lewellen, and S. H. H. Tye, Nucl. Phys. B269: 1(1986).
[18] S. H. H. Tye and Y. Zhang, arXiv:1003.1732[hep-th].
[19] N. E. J. Bjerrum-Bohr and P. Vanhove, arXiv:1003.2396[hep-th].
[20] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard, and P. Vanhove, arXiv:1003.2403[hep-th].

[21] Z. Bern, T. Dennen, Y. Huang, and M. Kiermaier, arXiv:1004.0693[hep-th].

[22] K. Risager, JHEP 0512: 003(2005).