Emergent supersymmetry in a chain of Majorana Cooper pair boxes

Hiromi Ebisu, Eran Sagi, and Yuval Oreg
Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot, Israel 76100

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The charging energy of a small superconducting island containing Majorana zero modes - a Majorana Cooper-pair box - induces interactions between the Majorana zero modes. In this manuscript, we investigate a chain of Majorana Cooper-pair boxes, and theoretically demonstrate the emergence of supersymmetry in the strong charging energy regime. A mapping between the Majorana zero modes and spin-1 degrees of freedom results in an effective Blume-Capel model, known for exhibiting an emergent supersymmetry at a non-trivial critical point with central charge \( c = 7/10 \). We corroborate our findings by mapping the chain to a supersymmetric low energy field theory, exhibiting the same central charge at criticality. The microscopic model we propose consists of local tunneling of Majorana zero modes and local charging energy terms, which can be controlled by gate potentials, thus making its realization more feasible.

Introduction — Majorana zero modes (MZMs) consisting of an equal superposition of electrons and holes have been tremendously important in condensed matter physics. They shed light on new aspects of physics, such as topological phases of matter [1] and non-Abelian statistics [2]. In recent years, considerable theoretical and experimental efforts have been devoted to realizations of MZMs on the ends of topological nanowires [3–9]. Remarkably, signatures of MZMs has been reported in experiments exhibiting charging effect [10] (for a recent review see [11]).

When interactions between MZMs are present, exotic fractional phases may arise. Well-known examples are the \( Z_2 \) spin liquid state [12] and \( Z_8 \) topological phases [13, 14]. A fruitful playground for studying such interacting MZMs systems is the Majorana Cooper-pair box (MCB) [15]. A MCB comprises superconducting island (a.k.a. Cooper-pair box) and MZMs residing on the island.

The conventional Cooper-pair box has been intensively studied for theoretical and practical purposes (see [16] and references therein). One of the intriguing results is that a qubit state can be generated by tuning the charging energy and the Josephson coupling [17]. Moreover, this qubit state resembles a quantum state of spin-1/2, allowing one to map a one-dimensional array of Cooper-pair boxes to the XXZ spin chain [18]. Due to the presence of the MZMs, a MCB has additional degrees of freedom compared to conventional Cooper-pair boxes. Therefore, one may expect an even richer physics in the MCB case.

In Ref. [19], a MCB hosting six MZMs (the so-called hexon) was proposed as a basic element for constructing the one-dimensional transverse field Ising spin chain and the two-dimensional Yao-Kivelson model [20], in the strong charging energy regime. Furthermore, by tuning various parameters, it was possible to show that an Ising fixed point, described by a \( 1 + 1 \)-dimensional conformal field theory [21] (CFT) with central charge \( c = 1/2 \), is stabilized. The purpose of this paper is to go beyond the work presented in [19], and demonstrate that a similar chain of MCBs gives rise to a more exotic critical point, described by a superconformal field theory (SCFT) with central charge \( c = 7/10 \), and thus experiences an emergent supersymmetry (SUSY).

There are several models which manifest a fixed point described by SCFT with \( c = 7/10 \). One classic example is the Blume-Capel (BC) model [22, 23]. By a-classical-to-quantum mapping [24], the two-dimensional classical BC model is mapped onto a one-dimensional quantum model, given by the Hamiltonian

\[
H_{BC} = \sum_j \alpha S_j^z + \delta (S_j^z)^2 - J S_j^x S_{j+1}^x,
\]

where \( S_j^z(x) \) are spin-1 operators along the \( z \) (\( x \)) axis at site \( j \). The BC phase diagram is special as it has a first order transition line that meets a second order line at a tri-critical fixed point. We can understand the phase diagram using the following considerations. In the limit \( \delta \rightarrow -\infty \), the \( S_z = 0 \) state becomes highly excited and only the two states with \( S_z = \pm 1 \) contribute at low energies. These two degenerate states can be regarded as an effective spin-1/2 degree of freedom, resulting in a model that resembles the spin-1/2 transverse field Ising model. Indeed, a ferromagnetic order due to the third term, \( -J S_j^x S_{j+1}^x \) is competing with the transverse field, \( \alpha S_j^z \) which favors a disordered phase – giving rise to the second order phase transition. For \( \alpha = 0 \) and \( \delta > 0 \), there are two possible phases, a ferromagnetic one and the phase with \( S_j^z = 0 \). These phases are separated by a first order phase transition [23]. At the tri-critical fixed point connecting the first- and second-order lines, a CFT with \( c = 7/10 \) emerges [24]. This CFT is further known to possess \( N = 1 \) SUSY [25]. Indeed, numerical studies confirmed the existence of a tri-critical fixed point for finite \( \alpha \) and \( \delta \) [26, 27].

In this paper, we focus on an array of MCBs, and use...
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Previous works already discussed the possibility of the emergent SUSY in interacting MZM systems [31, 32]. The proposal here is based on a concrete microscopic model, requiring only local couplings of MZMs and charging energy, which can be controlled by gates and therefore may be more feasible in reality.

The Majorana Cooper-pair Box (MCB) chain – Throughout this paper, we focus on the case of MCBs consisting of six MZMs in a Cooper-pair box - an hexon. For other configurations of MZMs on a MCB, such as ‘tetrion’ (four MZMs), see for example Refs. [12, 33]. We consider placing three semiconducting nanowires, labeled by the index \( p = x, y, z \), on top of the superconducting island, see Fig. 1(a). Assuming these nanowires are proximity coupled to the superconducting island, under application of a magnetic field parallel to the wires, each nanowire hosts two MZMs on its ends. These two MZMs can accommodate a fermion. Introducing the six Majorana operators \( a_p, b_p \), obeying the anti-commutation relations \( \{a_p, a_q\} = \{b_p, b_q\} = \delta_{p,q} \), we denote the annihilation operator of these fermions by \( \frac{1}{2}(a_p + i b_p) \). For each pair of MZMs, the \( \mathbb{Z}_2 \) fermion parity takes the values \( i a_p b_p = \pm 1 \).

We describe a system consisting of a one-dimensional array of MCBs with six MZMs each using the Hamiltonian \( H = H_U + H_0 \), with

\[
H_U = \sum_j U(2\tilde{N}_j^+ + \tilde{n}_j^+ + \tilde{n}_j^-)^2, 
\]

\[
H_0 = i \sum_{j,j',p} t_{jj'}^p a_j^p a_{j'}^p + i \sum_{j,j',p} h_{jj'}^{p,q} a_j^p a_{j'}^q + h_{jj'}^{p,b} b_j^p b_{j'}^p, 
\]

Here, the superscript \( j \) labels the \( j \)-th MCB, \( U \) is the charging energy of the box, \( \tilde{n}_j \) represents the number of a charge which takes continuous value controlled by a gate potential on each box, \( \tilde{N}_j \) is the number operator of Cooper-pairs in the box, and finally \( \tilde{\Delta}_j = \sum_p (1 - i a_j^p b_j^p)/2 \) is the number of fermions occupying the MZMs. The Hamiltonian \( H_0 \) describes a generic local (real) couplings between the MZMs whose sign and magnitude are controlled by tunable physical parameters such as gate potentials.

Construction of the Blume-Capel (BC) model – We now present the first approach to obtain the emergent SUSY, in which we construct the BC model given in Eq. (1). We begin with the Hamiltonian \( H = H_U + H_0 \) defined in Eqs. (2) and (3), and set the (real) couplings in \( H_0 \) as \( t_{jj'}^p = \lambda \), \( t_{jj'}^p = \lambda - 2\delta \), \( h_{jj'}^{p,q} = h^{b}_{jj'} = \alpha \), \( t_{jj'}^p = -t_{jj'}^p = t' \), with all other couplings set to zero (See Fig. 1(b)). Below, we will see how this Hamiltonian reproduces the BC model when \( U \gg t_{jj'}^p, h^{a/b}_{jj'} \).

The six MZMs in each hexon define spin-1/2 operators \( \hat{s}_x\hat{s}_y\hat{s}_z \):

\[
\hat{s}_x = ia_x^a a_y^b, \quad \hat{s}_y = ia_x^b a_y^a, \quad \hat{s}_z = ia_x^a a_y^a, \quad \hat{s}_y = ib_x^a b_y^b. 
\]

It is straightforward to check that Eq. 4 satisfies the spin-1/2 algebra. That is, \((\hat{s}_p^{a/b})^2 = 1, [\hat{s}_p^{a}, \hat{s}_q^{b}] = i\epsilon_{pqk}\hat{s}_k^{a/b}\) (and similarly for \( \{s_p^{a/b}\} \)), with \( \epsilon_{pqk} \) being the anti-symmetric tensor. Due to the strong charging energy, \( U \), the number of pairs in each MCB is fixed, leading to a constraint on the total \( \mathbb{Z}_2 \) fermion parity of each hexon. Such constraint reads as

\[
(ia_x^a b_y^b)(ia_x^b b_y^a)(ia_x^a b_y^a) = 1. 
\]

This constraint ensures that the total number of states in each hexon is four, which is identical to the number of states of two spin-1/2 degrees of freedom. A key step of our construction is to project out singlet state of the total spin \( S_j^2 = s_j^a + s_j^b \) of each MCB. This allows us to obtain spin-1 (spin-triplet) states. Such a projection can be implemented by introducing couplings of the MZMs as

\[
H_{\lambda} = i\lambda \sum_{j,j'=x,y,z} a_j^b b_{j'}^p + b_{j'}^b a_j^p 
\]

and setting \( \lambda > 0 \). To see this, we note that the norm of the total spin is written as \( S_j^2 \cdot S_j = 2 + 2s_j^2 \cdot s_j^2 \), thus
using Eq. (4) and the constraint in Eq. (5), we find:

\[ S^a \cdot S^b = -i \sum_p a_p^b b_p^a. \]

Therefore, we have \( H_\lambda = -\frac{\lambda}{2} \sum_j S_j^z \cdot S_j^z + \lambda \), with \( \lambda > 0 \). Increasing \( \lambda \), the spin-1 (spin-triplet) states become energetically favored. Thus, focusing on low energies, the spin-singlet state is projected out. Below, we use \( S^z \) to denote spin-1 operators.

Having a spin-1 degree of freedom on each box, we can reproduce the BC model of Eq. (1) using a chain of MCBs. The first term of the BC model is obtained by

\[ H_0 = i\alpha \sum_j (a_j^\dagger a_j^1 + b_j^\dagger b_j^1) = \sum_j \alpha S_j^1. \]

Indeed, \( i a_j^1 a_j^1 + i b_j^1 b_j^1 = s_j^z a_j^1 + s_j^z b_j^1 = S_j^1 \). The second term of the BC model, \( \delta(S_j^z)^2 \) is realized by

\[ H_\delta = -2\delta \sum_j i a_j^1 b_j^1 = \sum_j \delta(S_j^z)^2 - 2\delta, \]

as \( \frac{1}{2}(S_j^z)^2 - 1 = s_j^x a_j^1 s_j^z b_j^1 = (i a_j^1 a_j^1)(i b_j^1 b_j^1) = -i a_j^1 b_j^1 \), where the last equality follows from the \( \mathbb{Z}_2 \) parity constraint in Eq. (5).

To reproduce the last term of the BC model, \( S_j^x S_j^z \), we consider small magnitude of couplings of MZMs between adjacent islands:

\[ i \sum_j t'(b_j^a a_j^z - b_j^1 a_j^z). \]

Due to the strong charging energy, these couplings are regarded as a perturbation. Using a Schrieffer-Wolff transformation, we obtain the following Hamiltonian, which is further mapped to the last term of the BC model after the projection to the spin-1 states:

\[ H_J = J \sum_j (i b_j^1 a_j^{z+1})(i b_j^1 a_j^{z+1}) = -J \sum_j s_j^x s_j^z \sum_j -JS_j^x S_j^{z+1}, \]

where \( J = t'^2/2U \). For derivation of the last relation, see supplemental material.

The terms in Eqs. (7), (8), and (9) establish the mapping between the chain of MCBs and the BC spin-1 model. We emphasize again that the parameters \( \alpha, \delta, \) and \( J = t'^2/(2U) \) in Eq. (1) are tunable by local gates controlling the MZMs couplings in our model. It was found numerically that for \( \alpha/J \approx 0.9 \) and \( \delta/J \approx 0.4 \), the phase of the MCBs reaches the tri-critical fixed point.

At this point, SUSY emerges and the long distance behavior of the BC model is characterized by SCFT with central charge \( c = 7/10 \).
Setting the velocity to be $t = 1$, and defining the Majorana spinor $\psi_p = (\eta_{pR}, \eta_{pL})^T$, and the Dirac gamma matrices $\gamma_0 = \sigma_y$, $\gamma_1 = i\sigma_x$ (where $\sigma_{x,y}$ denotes the $2 \times 2$ Pauli matrices), we get the 1 + 1 dimensional Lagrangian density

$$\mathcal{L} = \sum_{p} \left[ \frac{1}{2} \bar{\psi}_p i\gamma_0 \partial_\mu \psi_p + i h(\eta_{pR} \eta_{-L} + \eta_{pL} \eta_{-R}) \right],$$

(12)

with $\bar{\psi}_p = \psi_p^\dagger \gamma_0$, $\partial_\mu = \partial_\mu \gamma^\mu$.

For $U \gg \ell$ we can perform the Villain approximation, yielding the following interacting term in the Lagrangian:

$$g \left( \sum_p \bar{\psi}_p \psi_p \right)^2,$$

(13)

where $g$ relates to $U$ in Eq. (2) by $g \approx \frac{1}{32} U$. The derivation of Eq. (13) is given in the supplemental material [35]. We note that if $h = 0$, Eq. (12) and Eq. (13) form the Gross-Neveu model [37].

To analyze the effect of the term proportional to $h$ on the Gross-Neveu model, we implement the bosonization procedure. To do so, we form a fermion out of $\eta_{pR/L}$ and $\eta_{xR/L}$, which is then bosonized:

$$\Psi_{R/L} = \eta_{pR/L} + i\gamma_{xR/L} \approx e^{\pm i\sqrt{4\pi} \phi_{R/L}},$$

where $\Psi_{R/L}$ and $\phi_{R/L}$ indicate a right/left moving Dirac and boson fields, respectively. An important consequence of the bosonization formulation is that we obtain one bosonic field and one Majorana (real fermion) field. This hints at the possibility of SUSY, where the number of bosonic degrees of freedom is equal to the fermionic one. Denoting $\psi_x \rightarrow \psi$, we find the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} \bar{\psi} i \gamma_0 \partial_\mu \psi + g (\bar{\psi} \psi + \cos \sqrt{4\pi} \varphi)^2 + h \sin \sqrt{4\pi} \varphi$$

(14)

with $\varphi = \phi_R + \phi_L$.

To complete our analysis, we assume further that $h$ and $g$ are positive and $h \sim g \gg 1$. Focusing on low energies, we can thus expand the boson field around the minimum of the Hamiltonian (or the maximum of the Lagrangian), $\varphi \approx \frac{1}{2} \frac{1}{\sqrt{4\pi}} + \bar{\varphi}$, resulting in the expanded Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \bar{\varphi})^2 + \frac{1}{2} \bar{\psi} i \gamma_0 \partial_\mu \psi + 2g(\sqrt{4\pi} \bar{\varphi}) \bar{\psi} \psi - g \left( 1 - \frac{4\pi}{2} \varphi^2 \right)^2 + h \left( 1 - \frac{4\pi}{2} \varphi^2 \right).$$

(15)

Tuning $g = \pi/2$ and $h = \pi$, $\mathcal{L}$ is further simplified to

$$\mathcal{L} \simeq \frac{1}{2} (\partial_\mu \bar{\varphi})^2 + \frac{1}{2} \bar{\psi} i \gamma_0 \partial_\mu \psi - \frac{1}{2} u \bar{\varphi} \psi - \frac{1}{8} u^2 \varphi^4,$$

(18)

where $v = 2\pi \sqrt{4\pi}$. Remarkably, Eq. (18) is identical to the $N = 1$ super LG action. The relation to the super LG action can be obtained explicitly by considering the SUSY model

$$S_{\text{SUSY}} = \int dtd\theta d\bar{\theta} \left[ \frac{1}{4} (\bar{\Phi} D \Phi)(\Phi) + W(\Phi), \right],$$

(19)

where $\Phi$ is the superfield defined by $\Phi = \bar{\varphi} + \bar{\theta} \psi + \frac{1}{2} \bar{\theta} \theta F$, $D$ represents covariant derivative in superspace, and $W(\Phi)$ describes superpotential which is a polynomial function of $\Phi$ [38]. In our case, $W(\Phi)$ is given by $W(\Phi) = \frac{3}{4} \Phi^3$.

Ref. [29] showed that at long distances, the super LG action with a super potential $W(\Phi) \simeq \Phi^m$ ($m = 2, 3, \cdots$) exhibits a supersymmetric analog of the minimal models, characterized by central charge $c = \frac{3}{2} - \frac{12}{m(m+2)}$. Since our case corresponds to $m = 3$, the continuum theory given in Eq. (18) effectively manifests an emergent SUSY described by a SCFT with $c = 7/10$.

While we chose $g = \pi/2$ and $h = \pi$, above, we can instead follow Ref. [39] and realize an identical SCFT for generic values of $g$ by tuning $h$ properly. Indeed, redefining $\bar{\sigma} = \sqrt{K} \bar{\varphi}, u = 4g \sqrt{\frac{3}{4\pi}}, K = 1 - \frac{4\pi}{2} \rho, \rho = \frac{1-2g/\pi}{1-8g/\pi}$, $h = 2(1-\rho)g$, the theory in Eq. (17) takes the form

$$\mathcal{L} \simeq \frac{1}{2} (\partial_\mu \bar{\sigma})^2 + \frac{1}{2} \bar{\psi} i \gamma_0 \partial_\mu \psi - \frac{1}{2} u \bar{\sigma} \psi - \frac{1}{8} u^2 \bar{\sigma}^4,$$

(20)

which is again equivalent to the super LG theory with the superpotential $W(\Phi) = u \Phi^3$.

FIG. 3: A schematic picture of the phase diagram of the BC model. The red star depicts the tricritical fixed point and the solid (dashed) line represents the first (second) order transition line. Two red arrows indicate perturbations in the phase transition lines, whose conformal dimensions are given by $(\frac{3}{5}, \frac{3}{5})$ and $(\frac{3}{10}, \frac{1}{10})$, respectively.
Stability of the emergent SUSY to local perturbations – We can analyze the stability of the emergent SUSY using knowledge of the operator content of the SCFT. We focus on the case of the BC model. Suppose the BC model (constructed from the MCBs) is tuned to the tri-critical fixed point, i.e., \((\alpha, \delta) = (\alpha_c, \delta_c)\) (\(\alpha_c \approx 0.9, \delta_c \approx 0.4\) with \(J\) being unity). We introduce a small deviation of \((\alpha, \delta)\) from the critical value, \((\alpha + \kappa_\alpha, \delta + \kappa_\delta)\) at specific site of the MCBs, \(j = j_0\). Here, \(\kappa_\alpha/\delta\) represents an infinitesimal deviation. Consider shifting the parameters \((\alpha, \delta)\) tangentially (orthogonally) to the phase transition line at the tricritical fixed point, see, Fig. 3. Such a deviation can be done through a linear combination of \(\delta_0\) and \((\delta_0)\), as these two terms are realized by local couplings of MZMs in the MCBs.

At low energies, this situation can be described by a \(c = 7/10\) CFT which is perturbed by its primary fields. Moreover, the deviation of the parameters in the tangential (orthogonal) direction yields a perturbation given by a product of holomorphic and antiholomorphic primary fields of the form \(\varepsilon_{\mathcal{E}}|\varepsilon_{\mathcal{E}}|\), with conformal dimension \((3, \frac{3}{2})(\frac{1}{1}, \frac{1}{1})\) \([41, 42]\).

Our consideration here is reminiscent of the localization problem of a single local impurity in a Luttinger liquid \([42]\). Similarly to this problem, we can judge whether the perturbation is relevant or not by following renormalization group equation:

\[
\frac{d\mathcal{O}}{dl} = (1 - \Delta_O)\mathcal{O},
\]

where \(\mathcal{O}\) is either \(\varepsilon_{\mathcal{E}}\) or \(\varepsilon_{\mathcal{E}}\), \(l\) represents the logarithmic rescaling factor, and \(\Delta_O\) is the scaling dimension of \(\mathcal{O}\). Since the scaling dimension of \(\varepsilon_{\mathcal{E}}\) \((\varepsilon_{\mathcal{E}})\) is \((3, \frac{3}{2})\), the \(c = 7/10\) fixed point, and thus the emergent SUSY, is robust (sensitive) with respect to the tangential (orthogonal) perturbation.

In summary, we have introduced a chain of Majorana Cooper-pair Boxes (MBCs) each of which has six Majorana zero modes (MZM), forming an ‘hexon’. We mapped the system onto a spin-1 (Blume-Capel) model and a generalized Gross-Neveu model, and demonstrated that a supersymmetric critical point with central charge \(c = 7/10\) emerges. It would be interesting to extend our considerations to the two-dimensional case through a wire construction. One can expect a two-dimensional topological phase with a chiral edge mode carrying central charge \(c = 7/10\). Such a phase supports universal quantum computation \([2]\). This analysis is left to future projects.

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This supplemental material consists of five parts: In Sec. A, we derive the condition of the \( \mathbb{Z}_2 \) fermion parity demonstrated in Eq. (5) in the main text by analyzing the charging energy of the Majorana Cooper-pair box (MCB). In Sec. B, we prove the last relation in Eq. (9). In Sec. C, we see that the charging energy yields the interacting term of Majorana fields in Eq. (13) in the continuum low energy regime. In Sec. D, we give a brief explanation about the superspace and superfield introduced in the main text to derive Eq. (18) and Eq. (19). Finally, in Sec. E, we derive Eq. (20).

### A: CHARGING ENERGY

In this section, we see how the condition of the \( \mathbb{Z}_2 \) fermion parity, Eq. (5) in the main text is derived in the strong charging energy regime. Consider an array of MCBs and start with the Hamiltonian \( H = H_U + H_0 \) defined in Eqs. (2)(3). For convenience, we show \( H_U \) here again:

\[
H_U = U \sum_j (2 \tilde{N}^j + \tilde{n}_M^j + \tilde{n}_g^j)^2 \tag{21}
\]

where \( U \) is the charging energy of the box, \( \tilde{n}_j \) represents the number of a charge which takes continuous value controlled by a gate potential on each box, \( \tilde{N}^j \) is the number operator of Cooper-pairs in the box, and \( \tilde{n}_M^j = \sum_p (1 - i a_j b_p^p) / 2 \) is the number of fermions occupying the MZMs. In the following, we sum up the degree of freedom of \( \tilde{N}^j \) to yield an effective Hamiltonian which includes only MZMs terms. To do so, we write a partition function of the chain of the MCBs as \( Z = \sum \{ \tilde{N}^j \} \sum' e^{-\beta H} \) with \( \beta \) being inverse temperature. The symbol \( \sum' \) represents summation over the degree of freedom of the number of Cooper-pairs, whereas \( \sum' \) indicates summation over other degrees of freedom, i.e., the ones of \( \tilde{n}_j \) and \( \tilde{n}_M^j \). For simplicity, we set \( \beta \) to be unity. In the case of the strong charging energy, \( U \), we use the Villain approximation \[43\] to find

\[
\sum \tilde{N}^j \sum' e^{-H_U + H_0} = \sum \tilde{N}^j \sum' e^{-\sum_j (2 \tilde{N}^j + \tilde{n}_M^j + \tilde{n}_g^j)^2 - H_0} \approx \sum \sum' e^{-\sum_j \frac{U}{\pi^2} \cos(\pi (\tilde{n}_M^j + \tilde{n}_g^j))} e^{-H_0} \tag{22}
\]

Therefore, we obtain an effective Hamiltonian, reading

\[
H_{\text{eff}} = H'_U + H_0, \tag{23}
\]

with

\[
H'_U = \frac{U}{\pi^2} \sum_j \cos(\pi (\tilde{n}_M^j + \tilde{n}_g^j)). \tag{24}
\]
Setting \( \{ \tilde{n}_j^2 \} = 0 \) and using \( \tilde{n}_j^2 = \sum_{p=x,y,z} \frac{1}{2} (1 - i a_p^i b_p^j) \), \( H_U \) becomes

\[
H_U = \frac{U}{\pi^2} \sum_j \cos \left[ \frac{3\pi}{2} - \frac{\pi}{2} \sum_p (ia_p^i b_p^j) \right] = \frac{U}{\pi^2} \sum_j \sin \left[ \frac{\pi}{2} \sum_p (ia_p^i b_p^j) \right] = -\frac{U}{\pi^2} \prod_p (ia_p^i b_p^j).
\]  

(25)

Since \( U \) is large, Eq. (25) leads to a constraint on the \( \mathbb{Z}_2 \) fermion parity. Indeed, taking the limit \( U \rightarrow \infty \), the constraint \( \prod_p (ia_p^i b_p^j) = 1 \) is enforced strictly.

**B: PROJECTION OPERATOR**

In this section, we explicitly write the operator \( P_j^t \) that projects the states on the \( j \)-th MCB to the spin-1 states defined in the main text, and check its action on spin operators. The goal of this section is to prove the relation, 

\[
P_j^{tot} s_j^x s_j^{x+1} P_j^{tot} = S_j^z S_j^z+1
\]  

(Eq. (9)) which plays an important role for realization of the last term of the BC model (Eq. (1)).

In the diagonal basis of \( z \)-component of the spin operator, \( S_z \), which we denote \( |1\rangle \), \( |0\rangle \), and \( |-1\rangle \), \( S_z \) is described by

\[
S_z = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]  

(26)

Since the spin-1 states is equivalent to the spin-triplet states of spin-1/2, the basis \( |1\rangle \), \( |0\rangle \), and \( |-1\rangle \) can be rewritten as

\[
|1\rangle = |\uparrow\uparrow\rangle, \ |0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \ |-1\rangle = |\downarrow\downarrow\rangle,
\]  

(27)

where we have defined a tensor product state of two spin-1/2 states: \( |s_j^x s_j^y\rangle = |s_j^x\rangle \otimes |s_j^y\rangle \ (s_j^x, s_j^y = \uparrow, \downarrow) \). In the basis of \( |s_j^x s_j^y\rangle \), that is, in the basis of \( |\uparrow\uparrow\rangle \), \( |\uparrow\downarrow\rangle \), \( |\downarrow\uparrow\rangle \), \( |\downarrow\downarrow\rangle \), \( S_z \) is rewritten as

\[
S_z = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]  

(28)

A projection operator to the spin-1 states is defined by

\[
P_j = |1\rangle \langle 1| + |0\rangle \langle 0| + |-1\rangle \langle -1|.
\]  

(29)

Using Eq. (25), Eq. (29) is rewritten as

\[
P_j = |\uparrow\uparrow\rangle \langle \uparrow\uparrow| + \frac{1}{2} \left( |\uparrow\downarrow\rangle \langle \uparrow\downarrow| + |\uparrow\downarrow\rangle \langle \downarrow\uparrow| + |\downarrow\uparrow\rangle \langle \uparrow\downarrow| + |\downarrow\downarrow\rangle \langle \downarrow\downarrow| \right)
\]  

(30)

In the matrix form, \( P_j \) is given by

\[
P_j = \begin{pmatrix}
1/2 & 1/2 \\
1/2 & 1/2
\end{pmatrix}.
\]  

(31)

Using spin-1/2 operators, \( s_p^a \) and \( s_p^b \) (\( p = x, y, z \)), it is straightforward to check that \( P_j = \frac{3}{4} I \otimes I + \frac{1}{4} \sum_p s_p^a \otimes s_p^b \) which acts on the state \( |s_j^x s_j^y\rangle \). With the considerations above in mind, we consider a one-dimensional array of the MCBs. There are two spin-1/2 algebras on each MCB as demonstrated in Eq. (4) in the main text. Let operators of these two spin-1/2 algebra be \( s_p^{aj} \) and \( s_p^{bj} \), where the superscript \( j \) labels the \( j \)-th MCB. We investigate how the projection operator to the triplet states acts on the operator \( s_j^x s_{j+1}^x \) given in Eq. (9). We introduce the projection operator to
spin triplet states, \( \mathcal{P}_t^j = \frac{3}{2} \mathbb{I} \otimes \mathbb{I} + \frac{1}{2} \sum_p s_p^{a_j} \otimes s_p^{b_j} \) on each MCB. Thus, the total projection operator is then given by \( \mathcal{P}_t^{\text{tot}} = \sum_j \mathcal{P}_t^j \). In the basis of \( |s_z^{a_j} s_z^{b_j}\rangle \), the operator \( s_z^{b_j} \) is represented by

\[
s_z^{b_j} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

Similarly, \( s_z^{a_j+1} \) is expressed by

\[
s_z^{a_j+1} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}
\]

in the basis of \( |s_z^{a_j+1} s_z^{b_j+1}\rangle \). By simple calculations, we obtain

\[
\mathcal{P}_t^j s_z^{b_j} \mathcal{P}_t^j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\mathcal{P}_t^{j+1} s_z^{a_j+1} \mathcal{P}_t^{j+1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Eqs. (32) and (33) yields \( \mathcal{P}_t^j s_z^{b_j} \mathcal{P}_t^j = S_z^j \) and \( \mathcal{P}_t^{j+1} s_z^{a_j+1} \mathcal{P}_t^{j+1} = S_z^{j+1} \), implying \( \mathcal{P}_t^{\text{tot}} s_z^{b_j} s_z^{a_j+1} \mathcal{P}_t^{\text{tot}} = S_z^j S_z^{j+1} \). This completes the derivation of Eq. (9).

**C: GROSS-NEVEU INTERACTION**

In this section, we see how the interacting terms of the Majorana fields given in Eq. (13) is derived by analyzing the charging energy term. We start by the Hamiltonian \( H = H_U + H_0 = H_U + H_t + H_h \), where \( H_U \), \( H_t \), and \( H_h \) is defined in Eqs. (2), (10) and (11) in the main text, respectively. Similarly to the Sec. A, we focus on the strong charging energy regime, which enables us to utilize the approximation in Eq. (22). Therefore, we replace \( H_U \) with \( H_U' \) defined in Eq. (24).

By introducing slowly varying Majorana fields, we move to continuum low energy regime. Accordingly, the Hamiltonian density corresponding to \( H_U' \) takes the form

\[
\mathcal{H}_U' \simeq \frac{U}{\pi^2} \cos \left[ \frac{\pi}{4} \sum_p \bar{\psi}_p \psi_p \right],
\]

where the Majorana spinor is defined by \( \psi_p = (\eta_{R_p}, \eta_{L_p})^T \) \( (p = x, y, z) \), and we have set \( \{ \hat{n}_g^j \} \) as \( \{ \hat{n}_g^j \} = 3/2 \). Eq. (36) can be expanded as

\[
\mathcal{H}_U' \simeq \frac{U}{\pi^2} \left\{ 1 - \frac{1}{2} \left[ \frac{\pi}{4} \right]^2 \left( \sum_p \bar{\psi}_p \psi_p \right)^2 \right\}.
\]

Thus, the interacting term that enters in the Lagrangian is given by \( g \left( \sum_p \bar{\psi}_p \psi_p \right)^2 \) with \( g = \frac{U}{32} \), which coincides with Eq. (13).
D: SUPERSYMMETRY

In this section, we briefly review the superfield and superspace. For more detailed explanation about these subjects, the readers should consult with a standard textbook of supersymmetry, e.g., Ref. [44].

We focus on the two-dimensional $\mathcal{N} = 1$ superspace formalism, where $\mathcal{N}$ labels the number of superpartners, i.e., the number of pairs of bosons and fermions. In this formalism, superspace is a collection of points described by coordinate $(x^\mu, \theta_\alpha, \bar{\theta}_\beta)$, where $x^\mu$ ($\mu = 0, 1$) denotes two-dimensional bosonic coordinates, $\theta_\alpha$ and $\bar{\theta}_\beta$ ($\alpha = 1, 2$) represents two-dimensional the Majorana spinor and its conjugate. Supersymmetry transformation which exchanges boson and fermion can be implemented by a translation in the superspace:

$$x^\mu \to x^\mu + i\varepsilon \gamma^\mu \theta, \quad \theta \to \theta + \varepsilon,$$

where $\varepsilon$ is an infinitesimal Majorana spinor and $\gamma^\mu$ is the Dirac Gamma matrix in two-dimension. A generator of this translation is given by

$$Q_\alpha = i\frac{\partial}{\partial \theta_\alpha} + i(\gamma^\mu \theta)_\alpha \partial_\mu.$$

Eq. (38) is called supercharge satisfying supersymmetric algebra: $\{Q_\alpha, \bar{Q}_\beta\} = -2i\gamma^\mu \partial_\mu$. A superfield $\Phi$ is defined in the superspace which is a power series of $\theta$ and $\bar{\theta}$. A superfield $\Phi$ is defined in the superspace which is a power series of $\theta$ and $\bar{\theta}$:

$$\Phi = \phi(x) + \bar{\theta}\chi(x) + \frac{1}{2}\bar{\theta}\theta F(x),$$

where $\phi, \chi$, and $F$ indicates a scalar, the Majorana spinor, and an auxiliary field to be eliminated by an equation of motion, respectively. Note that in the power series expansion, terms proportional to $\sim \theta^A \bar{\theta}^B$ ($A, B \geq 2$) vanish due to the fermi statistics. The supercharge defined in Eq. (39) acts on the superfield $\Phi$ by

$$\delta \Phi = \varepsilon Q \Phi.$$ (41)

If we set $F = 0$, corresponding to a free theory of one boson and one fermion, then the transformation of (11) gives in component

$$\delta \phi = \varepsilon \chi$$

$$\delta \chi = -i\gamma^\mu \varepsilon \partial_\mu \phi,$$

implying boson and fermion are interchanged after the transformation.

To write a supersymmetric Lagrangian, we introduce a covariant derivative in the superspace as $D_\alpha = \frac{\partial}{\partial \theta_\alpha} - i\partial_\mu (\gamma^\mu \theta)_\alpha$ which preserves the supersymmetry transformation due to the relation $D_\alpha (\varepsilon Q) = (\varepsilon Q)D_\alpha$. Generic form of the supersymmetric Lagrangian is described by

$$L_{\text{SUSY}} = \int d^2xd^2\theta \left[\frac{1}{4}(D\Phi)(D\Phi) + W(\Phi)\right],$$ (44)

where $W(\Phi)$ is a superpotential which is an arbitrary function of the superfield $\Phi$, and $\int d^2\theta$ represents the Grassmann integration satisfying $\int d^2\theta (\bar{\theta}\theta) = 2$. After the Grassmann integration, $L_{\text{SUSY}}$ is rewritten as

$$L_{\text{SUSY}} = \int d^2x \left\{\frac{1}{2}[(\partial_\mu \phi)^2 + \bar{\chi}i\bar{\phi}\chi + F^2] + F \frac{\partial W(\Phi)}{\partial \Phi} \bigg|_{\Phi = \phi} - \frac{1}{2} \frac{\partial^2 W(\Phi)}{\partial \Phi^2} \bigg|_{\Phi = \phi} \bar{\chi}\chi\right\}. $$ (45)

To see this, expand $W(\Phi)$ as

$$W(\Phi) = \phi(x) + \bar{\theta}\chi(x) + \frac{1}{2}\bar{\theta}\theta F(x)$$

$$\simeq W(\phi) + \frac{\partial W}{\partial \phi} \left(\cdots + \frac{1}{2}\bar{\theta}\theta F(x)\right) + \frac{1}{2} \frac{\partial^2 W}{\partial \phi^2} \left(\cdots + (\bar{\theta}\chi)(\bar{\theta}\theta) + \cdots\right) + \cdots$$

Grassmann integration

$$F \frac{\partial W(\Phi)}{\partial \Phi} \bigg|_{\Phi = \phi} - \frac{1}{2} \frac{\partial^2 W(\Phi)}{\partial \Phi^2} \bigg|_{\Phi = \phi} \bar{\chi}\chi.$$ (46)
which is identical to the last two terms in Eq. \((45)\). The first three terms in Eq. \((45)\) are similarly derived from Eq. \((44)\) by implementing the Grassmann integration. Eliminating \(F\) by an equation of motion, Eq. \((45)\) becomes

\[
S_{\text{susy}} = \int dx dt \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \tilde{\chi} \tilde{\phi} \chi - \frac{1}{2} \left. \frac{\partial^2 W(\Phi)}{\partial \Phi} \right|_{\Phi = \phi} - \frac{1}{2} \left( \frac{\partial W(\Phi)}{\partial \Phi} \right)_{\Phi = \phi}^2 \right].
\]

If we set \(W(\Phi) = \frac{\sqrt{2}}{\pi} \Phi^3\), the Lagrangian is then

\[
S_{\text{susy}} = \int dx dt \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \tilde{\chi} \tilde{\phi} \chi - \frac{1}{2} v \chi \chi - \frac{1}{8} v^2 \phi^4 \right].
\]

If we set \(\phi = \tilde{\varphi}\) and \(\chi = \psi\), Eq. \((48)\) matches with Eq. \((18)\) in the main text.

E: DERIVATION OF EQ. \((20)\)

In this section, we derive Eq. \((20)\). We use the trick invented in Ref. \([39]\) which enables us to tune a compactification radius of a bosonic field at will. We begin with Lagrangian given in Eq. \((14)\) which we show here:

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} \bar{\psi} i \bar{\sigma} \psi + g (\bar{\psi} \psi + \cos \sqrt{4\pi} \varphi)^2 + h \sin \sqrt{4\pi} \varphi.
\]

Rewrite the third term as

\[
g \cos^2 \sqrt{4\pi} \varphi + 2g \cos \sqrt{4\pi} \varphi \bar{\psi} \psi.
\]

Using identification \(\cos \sqrt{4\pi} \varphi \simeq -\frac{2}{\pi} (\partial_\mu \varphi)^2\), which is called the Fierz identity \([39]\), Eq. \((50)\) is then modified to

\[
-\frac{2\rho g}{\pi} (\partial_\mu \varphi)^2 + (1 - \rho) g \cos^2 \sqrt{4\pi} \varphi + 2g \cos \sqrt{4\pi} \varphi \bar{\psi} \psi.
\]

For subsentence use, we have set \(\rho = \frac{1 - 2g/\pi}{1 - 8g^2/\pi^2}\). Accordingly, Lagrangian \((49)\) becomes

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} \bar{\psi} i \bar{\sigma} \psi - (1 - \rho) g \sin^2 \sqrt{4\pi/K} \sigma + 2g \cos \sqrt{4\pi/K} \sigma \bar{\psi} \psi + h \sin \sqrt{4\pi/K} \sigma,
\]

with \(\sigma = \sqrt{K} \varphi, K = 1 - \frac{4\rho g}{\pi}\). Following the similar argument in the main text, we expand the bosonic field in order for \(h \sin \sqrt{4\pi/K} \sigma\) to become maximum, i.e., \(\sigma \simeq \frac{2}{\sqrt{4\pi/\pi}} + \tilde{\sigma}\). With this expansion, we find

\[
\mathcal{L} \simeq \frac{1}{2} (\partial_\mu \tilde{\sigma})^2 + \frac{1}{2} \bar{\psi} i \bar{\tilde{\sigma}} \psi - (1 - \rho) g \left( 1 - \frac{4\pi}{\sqrt{2} K} \tilde{\sigma}^2 \right) + 2g \sqrt{\frac{4\pi}{K}} \bar{\psi} \psi + h \left( 1 - \frac{4\pi}{\sqrt{2} K} \tilde{\sigma}^2 \right)
\]

\[
\simeq \frac{1}{2} (\partial_\mu \tilde{\sigma})^2 + \frac{1}{2} \bar{\psi} i \bar{\tilde{\sigma}} \psi + (\rho - 1) g \left( \frac{4\pi}{\sqrt{4} K} \tilde{\sigma}^2 \right) - 2g \sqrt{\frac{4\pi}{K}} \bar{\psi} \psi - \left( \frac{2\pi}{K} \right) (2(\rho - 1) g + h) \tilde{\sigma}^2.
\]

Setting \(h = 2(1 - \rho) g\) and together with \(\rho = \frac{1 - 2g/\pi}{1 - 8g^2/\pi^2}\), the Lagrangian finally becomes

\[
\mathcal{L} \simeq \frac{1}{2} (\partial_\mu \tilde{\sigma})^2 + \frac{1}{2} \bar{\psi} i \bar{\tilde{\sigma}} \psi - \frac{u^2}{8} \tilde{\sigma}^4 - \frac{u}{2} \bar{\tilde{\sigma}} \psi \bar{\psi},
\]

with \(u = 4g \sqrt{4\pi/K}\). Eq. \((54)\) is equivalent to Eq. \((20)\).