STABILITY OF VOLUME COMPARISON FOR COMPLEX CONVEX BODIES

ALEXANDER KOLDOBSKY

Abstract. We prove the stability of the affirmative part of the solution to the complex Busemann-Petty problem. Namely, if \( K \) and \( L \) are origin-symmetric convex bodies in \( \mathbb{C}^n \), \( n = 2 \) or \( n = 3 \), \( \varepsilon > 0 \) and \( \text{Vol}_{2n-2}(K \cap H) \leq \text{Vol}_{2n-2}(L \cap H) + \varepsilon \) for any complex hyperplane \( H \) in \( \mathbb{C}^n \), then \( (\text{Vol}_{2n}(K))^{\frac{n-1}{n}} \leq (\text{Vol}_{2n}(L))^{\frac{n-1}{n}} + \varepsilon \), where \( \text{Vol}_{2n} \) is the volume in \( \mathbb{C}^n \), which is identified with \( \mathbb{R}^{2n} \) in the natural way.

1. Introduction

The Busemann-Petty problem, posed in 1956 (see [BP]), asks the following question. Suppose that \( K \) and \( L \) are origin symmetric convex bodies in \( \mathbb{R}^n \) such that

\[
\text{Vol}_{n-1}(K \cap H) \leq \text{Vol}_{n-1}(L \cap H)
\]

for every hyperplane \( H \) in \( \mathbb{R}^n \) containing the origin. Does it follow that

\[
\text{Vol}_n(K) \leq \text{Vol}_n(L) \spaceskip 0.5em
\]

The answer is affirmative if \( n \leq 4 \) and negative if \( n \geq 5 \). The solution was completed in the end of the 90’s as the result of a sequence of papers [LR], [Ba], [Gi], [Bo], [L], [Pa], [G1], [G2], [Z1], [Z2], [K1], [K2], [Z3], [GKS] ; see [K3, p. 3] or [G3, p. 343] for the history of the solution.

The complex version of the Busemann-Petty problem was solved in [KKZ], the answer is affirmative for convex bodies in \( \mathbb{C}^n \) when \( n \leq 3 \), and it is negative for \( n \geq 4 \). To formulate the complex version, we need several definitions.

For \( \xi \in \mathbb{C}^n \), \( |\xi| = 1 \), denote by

\[
H_\xi = \{ z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^{n} z_k \xi_k = 0 \}
\]

the complex hyperplane through the origin perpendicular to \( \xi \).
Origin symmetric convex bodies in $\mathbb{C}^n$ are the unit balls of norms on $\mathbb{C}^n$. We denote by $\| \cdot \|_K$ the norm corresponding to the body $K$:

$$K = \{ z \in \mathbb{C}^n : \| z \|_K \leq 1 \}.$$ 

In order to define volume, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ using the mapping

$$\xi = (\xi_1, \ldots, \xi_n) = (\xi_{11} + i\xi_{12}, \ldots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}).$$

Under this mapping the hyperplane $H_\xi$ turns into a $(2n-2)$-dimensional subspace of $\mathbb{R}^{2n}$.

Since norms on $\mathbb{C}^n$ satisfy the equality

$$\| \lambda z \| = |\lambda| \| z \|, \quad \forall z \in \mathbb{C}^n, \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies $K$ in $\mathbb{R}^{2n}$ that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $\xi = (\xi_{11}, \xi_{12}, \ldots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\| \xi \|_K = \| R_\theta(\xi_{11}, \xi_{12}), \ldots, R_\theta(\xi_{n1}, \xi_{n2}) \|_K,$$

where $R_\theta$ stands for the counterclockwise rotation of $\mathbb{R}^2$ by the angle $\theta$ with respect to the origin. We shall simply say that $K$ is invariant with respect to all $R_\theta$ if it satisfies (1).

The complex Busemann-Petty problem can be formulated as follows: suppose $K$ and $L$ are origin symmetric invariant with respect to all $R_\theta$ convex bodies in $\mathbb{R}^{2n}$ such that

$$\text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi)$$

for each $\xi$ from the unit sphere $S^{2n-1}$ of $\mathbb{R}^{2n}$. Does it follow that

$$\text{Vol}_{2n}(K) \leq \text{Vol}_{2n}(L)?$$

As mentioned above, the answer is affirmative if and only if $n \leq 3$. In this article we prove the stability of the affirmative part of the solution:

**Theorem 1.** Suppose that $\varepsilon > 0$, $K$ and $L$ are origin-symmetric invariant with respect to all $R_\theta$ convex bodies bodies in $\mathbb{R}^{2n}$, $n = 2$ or $n = 3$. If for every $\xi \in S^{2n-1}$

$$\text{Vol}_{2n-2}(K \cap H_\xi) \leq \text{Vol}_{2n-2}(L \cap H_\xi) + \varepsilon,$$

then

$$\text{Vol}_{2n}(K)^{\frac{1}{n-1}} \leq \text{Vol}_{2n}(L)^{\frac{1}{n-1}} + \varepsilon.$$

The result does not hold for $n > 3$, simply because the answer to the complex Busemann-Petty problem in these dimensions is negative; see [KKZ].

It immediately follows from Theorem 1 that
Corollary 1. If \( n = 2 \) or \( n = 3 \), then for any origin-symmetric invariant with respect to all \( R_\theta \) convex bodies \( K, L \) in \( \mathbb{R}^{2n} \),

\[
\left| \mathrm{Vol}_{2n}(K)^{\frac{n-1}{n}} - \mathrm{Vol}_{2n}(L)^{\frac{n-1}{n}} \right| \leq \max_{\xi \in S^{2n-1}} |\mathrm{Vol}_{2n-2}(K \cap H_\xi) - \mathrm{Vol}_{2n-2}(L \cap H_\xi)|.
\]

Note that stability in comparison problems for volumes of convex bodies was studied in [K5], where it was proved for the original (real) Busemann-Petty problem.

For other results related to the complex Busemann-Petty problem see [R], [Zy1], [Zy2].

2. Proofs

We use the techniques of the Fourier approach to sections of convex bodies; see [K3] and [KY] for details.

The Fourier transform of a distribution \( f \) is defined by \( \langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle \) for every test function \( \phi \) from the Schwartz space \( S \) of rapidly decreasing infinitely differentiable functions on \( \mathbb{R}^n \).

If \( K \) is a convex body and \( 0 < p < n \), then \( \| \cdot \|_K^{-p} \) is a locally integrable function on \( \mathbb{R}^n \) and represents a distribution. Suppose that \( K \) is infinitely smooth, i.e. \( \| \cdot \|_K \in C^\infty(S^{n-1}) \) is an infinitely differentiable function on the sphere. Then by [K3, Lemma 3.16], the Fourier transform of \( \| \cdot \|_K^{-p} \) is an extension of some function \( g \in C^\infty(S^{n-1}) \) to a homogeneous function of degree \( -n + p \) on \( \mathbb{R}^n \). When we write \( (\| \cdot \|_K^{-p})^\wedge(\xi) \), we mean \( g(\xi), \xi \in S^{n-1} \). If \( K, L \) are infinitely smooth star bodies, the following spherical version of Parseval’s formula was proved in [K4] (see [K3, Lemma 3.22]): for any \( p \in (-n, 0) \)

\[
\int_{S^{n-1}} (\| \cdot \|_K^{-p})^\wedge(\xi) (\| \cdot \|_L^{-n+p})^\wedge(\xi) = (2\pi)^n \int_{S^{n-1}} \| x \|_K^{-p} \| x \|_L^{-n+p} \, dx.
\]  

(3)

A distribution is called positive definite if its Fourier transform is a positive distribution in the sense that \( \langle \hat{f}, \phi \rangle \geq 0 \) for every non-negative test function \( \phi \).

The Fourier transform formula for the volume of complex hyperplane sections was proved in [KKZ]:

Proposition 1. Let \( K \) be an infinitely smooth origin symmetric invariant with respect to \( R_\theta \) convex body in \( \mathbb{R}^{2n}, n \geq 2 \). For every \( \xi \in S^{2n-1} \), we have

\[
\mathrm{Vol}_{2n-2}(K \cap H_\xi) = \frac{1}{4\pi(n-1)} (\| \cdot \|_K^{-2n+2})^\wedge(\xi).
\]  

(4)
We also use the result of Theorem 3 from [KKZ]. It is formulated in [KKZ] in terms of embedding in $L^{-p}$, which is equivalent to our formulation below. However, the reader does not need to worry about embeddings in $L^{-p}$, because the proof of Theorem 3 in [KKZ] directly establishes the following:

**Proposition 2.** Let $n \geq 3$. For every origin symmetric invariant with respect to $R_\theta$ convex body $K$ in $\mathbb{R}^{2n}$, the function $\| \cdot \|_{K^{-2n+4}}$ represents a positive definite distribution.

Let us formulate precisely what we are going to use later. The case $n = 2$ follows from Proposition 1 (obviously, the volume is positive), the case $n = 3$ is immediate from Proposition 2.

**Corollary 2.** If $n = 2$ or $n = 3$, then for every origin symmetric infinitely smooth invariant with respect to $R_\theta$ convex body $K$ in $\mathbb{R}^{2n}$, $(\| \cdot \|_{K^{-2}})^\wedge$ is a non-negative infinitely smooth function on the sphere $S^{2n-1}$.

We need the following simple fact:

**Lemma 1.** For every $n \in \mathbb{N}$,

$$\left(\Gamma(n)\right)^\frac{1}{n} \leq n^\frac{n-1}{n}.$$

**Proof:** By log-convexity of the $\Gamma$-function (see [K3, p.30]),

$$\frac{\log(\Gamma(n+1)) - \log(\Gamma(1))}{n} \geq \frac{\log(\Gamma(n)) - \log(\Gamma(1))}{n - 1},$$

so

$$(\Gamma(n+1))^{\frac{n-1}{n}} \geq \Gamma(n).$$

Now note that $\Gamma(n+1) = n\Gamma(n)$.

The polar formula for the volume of a convex body $K$ in $\mathbb{R}^{2n}$ reads as follows (see [K3, p.16]):

$$\text{Vol}_{2n}(K) = \frac{1}{2n} \int_{S^{2n-1}} \|x\|_{K^{-2n}}^{-2n} dx. \quad (5)$$

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** By the approximation argument of [S, Th. 3.3.1] (see also [GZ]), we may assume that the bodies $K$ and $L$ are infinitely smooth. Using [K3, Lemma 3.16] we get in this case that the Fourier transforms $(\| \cdot \|_{K^{-2n+2}})^\wedge$, $(\| \cdot \|_{L^{-2n+2}})^\wedge$, $(\| \cdot \|_{K}^{-2})^\wedge$ are the extensions of infinitely differentiable functions on the sphere to homogeneous functions on $\mathbb{R}^{2n}$.
By (4), the condition (2) can be written as
\[
(\| \cdot \|^2_n)^{\wedge} (\xi) \leq (\| \cdot \|^2_{L_n})^{\wedge} (\xi) + 4\pi(n - 1)\varepsilon
\]
for every \( \xi \in S^{2n-1} \). Integrating both sides with respect to a non-negative (by Corollary 2) density, we get
\[
\int_{S^{2n-1}} (\| \cdot \|^2_n)^{\wedge} (\xi) (\| \cdot \|^2_{L_n})^{\wedge} (\xi) d\xi 
\leq \int_{S^{4n-1}} (\| \cdot \|^2_{L_n})^{\wedge} (\xi) (\| \cdot \|^2_{K_n})^{\wedge} (\xi) d\xi
+ 4\pi(n - 1)\varepsilon \int_{S^{2n-1}} (\| \cdot \|^2_{K_n})^{\wedge} (\xi) d\xi.
\]
By the Parseval formula (3) applied twice,
\[
(2\pi)^n \int_{S^{2n-1}} \|x\|^2_K dx \leq (2\pi)^n \int_{S^{2n-1}} \|x\|^2_{L_n} \|x\|^2_K dx
+ 4\pi(n - 1)\varepsilon \int_{S^{2n-1}} (\| \cdot \|^2_{K_n})^{\wedge} (\xi) d\xi.
\]
Estimating the first summand in the right-hand side of the latter inequality by Hölder’s inequality,
\[
(2\pi)^n \int_{S^{2n-1}} \|x\|^2_K dx \leq (2\pi)^n \left( \int_{S^{2n-1}} \|x\|^2_{L_n} dx \right)^{\frac{n-1}{n}} \left( \int_{S^{2n-1}} \|x\|^2_K dx \right)^{\frac{1}{n}}
+ 4\pi(n - 1)\varepsilon \int_{S^{2n-1}} (\| \cdot \|^2_{K_n})^{\wedge} (\xi) d\xi.
\]
and using the polar formula for the volume (5),
\[
(2\pi)^n(2n)\text{Vol}_{2n}(K) \leq (2\pi)^n(2n) (\text{Vol}_{2n}(L))^{\frac{n-1}{n}} (\text{Vol}_{2n}(K))^{\frac{1}{n}}
+ 4\pi(n - 1)\varepsilon \int_{S^{2n-1}} (\| \cdot \|^2_{K_n})^{\wedge} (\xi) d\xi.
\]
We now estimate the second summand in the right-hand side. First we use the formula for the Fourier transform (in the sense of distributions; see [GS, p.194])
\[
(| \cdot |^{2n+2}_n)^{\wedge} (\xi) = \frac{4\pi^n}{\Gamma(n - 1)}
\]
where \(| \cdot |_2\) is the Euclidean norm in \( \mathbb{R}^{2n} \) and \( \xi \in S^{2n-1} \). We get
\[
4\pi(n - 1)\varepsilon \int_{S^{2n-1}} (\| \cdot \|^2_{K_n})^{\wedge} (\xi) d\xi
= \frac{4\pi(n - 1)\Gamma(n - 1)\varepsilon}{4\pi^n} \int_{S^{2n-1}} (\| \cdot \|^2_{K_n})^{\wedge} (\xi) (| \cdot |^{2n+2}_n)^{\wedge} (\xi) d\xi,
\]
and by Parseval’s formula (3) and Hölder’s inequality,

\[
\frac{(2\pi)^n \varepsilon \Gamma(n)}{\pi^{n-1}} \int_{S^{2n-1}} \|x\|^2_K \, dx \\
\leq \frac{(2\pi)^n \varepsilon \Gamma(n)}{\pi^{n-1}} \left( \int_{S^{2n-1}} \|x\|^{-2n}_K \, dx \right)^{\frac{1}{n}} |S^{2n-1}|^{\frac{n-1}{n}},
\]

where \(|S^{2n-1}| = (2\pi^n)/\Gamma(n)\) is the surface area of the unit sphere in \(\mathbb{R}^{2n}\). By the polar formula for the volume, the latter is equal to

\[
(2\pi)^n (2n) \varepsilon (\text{Vol}_{2n}(K))^\frac{1}{n} \leq (2\pi)^n (2n) \varepsilon (\text{Vol}_{2n}(K))^\frac{1}{n}
\]

by Lemma 1. Combining this with (6), we get the result. □

We finish with the following “separation” property (see [K5] for more results of this kind). Note that for any \(x \in S^{2n-1}\), \(\|x\|^{-1}_K = \rho_K(x)\) is the radius of \(K\) in the direction \(x\), and denote by \(r_K = \min_{x \in S^{2n-1}} \rho_K(x)\) the normalized inradius of \(K\). Clearly, for every \(x \in S^{2n-1}\) we have

\[
\|x\|^{-1}_K \geq r_K (\text{Vol}_{2n}(K))^{\frac{1}{n}}.
\]

**Theorem 2.** Suppose that \(\varepsilon > 0\), \(K\) and \(L\) are origin-symmetric invariant with respect to all \(R_\theta\) convex bodies bodies in \(\mathbb{R}^{2n}\), \(n = 2\) or \(n = 3\). If for every \(\xi \in S^{2n-1}\)

\[
\text{Vol}_{2n-2}(K \cap H_{\xi}) \leq \text{Vol}_{2n-2}(L \cap H_{\xi}) - \varepsilon,
\]

then

\[
\text{Vol}_{2n}(K)^{\frac{n-1}{n}} \leq \text{Vol}_{2n}(L)^{\frac{n-1}{n}} - \pi r^2(K) \frac{\varepsilon}{n}.
\]

**Proof:** We follow the lines of the proof of Theorem 1 to get

\[
(2\pi)^n (2n) \text{Vol}_{2n}(K) \leq (2\pi)^n (2n) (\text{Vol}_{2n}(L))^{\frac{n-1}{n}} (\text{Vol}_{2n}(K))^{\frac{1}{n}}
\]

\[ - 4\pi(n-1)\varepsilon \int_{S^{2n-1}} (\|x\|^{-2}_K)^\wedge (\xi) \, d\xi. \tag{7}
\]

We now need a lower estimate for

\[
4\pi(n-1)\varepsilon \int_{S^{2n-1}} (\|x\|^{-2}_K)^\wedge (\xi) \, d\xi.
\]

Similarly to how it was done in Theorem 1, we write the latter as

\[
\frac{(2\pi)^n \varepsilon \Gamma(n)}{\pi^{n-1}} \int_{S^{2n-1}} \|x\|^{-2}_K \, dx \geq \frac{(2\pi)^n \varepsilon \Gamma(n) r^2(K) (\text{Vol}_{2n}(K))^{\frac{1}{n}}}{\pi^{n-1}} |S^{2n-1}|. \tag{□}
\]
Acknowledgement. The author wishes to thank the US National Science Foundation for support through grants DMS-0652571 and DMS-1001234.

REFERENCES

[Ba] K. Ball, Some remarks on the geometry of convex sets, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math. 1317, Springer-Verlag, Berlin-Heidelberg-New York, 1988, 224–231.

[Bo] J. Bourgain, On the Busemann-Petty problem for perturbations of the ball, Geom. Funct. Anal. 1 (1991), 1–13.

[BP] H. Busemann and C. M. Petty, Problems on convex bodies, Math. Scand. 4 (1956), 88–94.

[G1] R. J. Gardner, Intersection bodies and the Busemann-Petty problem, Trans. Amer. Math. Soc. 342 (1994), 435–445.

[G2] R. J. Gardner, A positive answer to the Busemann-Petty problem in three dimensions, Annals of Math. 140 (1994), 435–447.

[G3] R. J. Gardner, Geometric tomography, Second edition, Cambridge University Press, Cambridge, 2006.

[GKS] R. J. Gardner, A. Koldobsky and Th. Schlumprecht, An analytic solution to the Busemann-Petty problem on sections of convex bodies, Annals of Math. 149 (1999), 691–703.

[GS] I. M. Gelfand and G. E. Shilov, Generalized functions, vol. 1. Properties and operations, Academic Press, New York, 1964.

[Gi] A. Giannopoulos, A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies, Mathematika 37 (1990), 239–244.

[GZ] E. Grinberg and Gaoyong Zhang, Convolutions, transforms, and convex bodies, Proc. London Math. Soc. (3) 78 (1999), 77–115.

[K1] A. Koldobsky, Intersection bodies, positive definite distributions and the Busemann-Petty problem, Amer. J. Math. 120 (1998), 827–840.

[K2] A. Koldobsky, Intersection bodies in $\mathbb{R}^4$, Adv. Math. 136 (1998), 1–14.

[K3] A. Koldobsky, Fourier analysis in convex geometry, Amer. Math. Soc., Providence RI, 2005.

[K4] A. Koldobsky, A generalization of the Busemann-Petty problem on sections of convex bodies, Israel J. Math. 110 (1999), 75–91.

[K5] A. Koldobsky, Stability in the Busemann-Petty and Shephard problems, preprint.

[KKZ] A. Koldobsky, H. Kö nig and M. Zymonopoulou, The complex Busemann-Petty problem on sections of convex bodies, Adv. Math. 218 (2008), 352–367.

[KY] A. Koldobsky and V. Yaskin, The interface between convex geometry and harmonic analysis, CBMS Regional Conference Series in Mathematics, 108, American Mathematical Society, Providence, RI, 2008.

[LR] D. G. Larman and C. A. Rogers, The existence of a centrally symmetric convex body with central sections that are unexpectedly small, Mathematika 22 (1975), 164–175.

[L] E. Lutwak, Intersection bodies and dual mixed volumes, Adv. Math. 71 (1988), 232–261.
[Pa] M. Papadimitrakis, *On the Busemann-Petty problem about convex, centrally symmetric bodies in* \( \mathbb{R}^n \), Mathematika 39 (1992), 258–266.

[R] B. Rubin, *Comparison of volumes of convex bodies in real, complex, and quaternionic spaces*, Adv. Math. 225 (2010), 1461–1498.

[S] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge, 1993.

[Z1] Gaoyong Zhang, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. 345 (1994), 777–801.

[Z2] Gaoyong Zhang, *Intersection bodies and Busemann-Petty inequalities in* \( \mathbb{R}^4 \), Annals of Math. 140 (1994), 331–346.

[Z3] Gaoyong Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Annals of Math. 149 (1999), 535–543.

[Zy1] M. Zymonopoulou, *The modified complex Busemann-Petty problem on sections of convex bodies*, Positivity 13 (2009), no. 4, 717–733.

[Zy2] M. Zymonopoulou, *The complex Busemann-Petty problem for arbitrary measures*, Arch. Math. (Basel) 91 (2008), no. 5, 436–449.

Department of Mathematics, University of Missouri, Columbia, MO 65211

E-mail address: koldobskiya@missouri.edu