Measures of correlations in infinite-dimensional quantum systems

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Abstract. Several important measures of correlations of the state of a finite-dimensional composite quantum system are defined as linear combinations of marginal entropies of this state. This paper is devoted to infinite-dimensional generalizations of such quantities and to an analysis of their properties.

We introduce the notion of faithful extension of a linear combination of marginal entropies and consider several concrete examples, the simplest of which are quantum mutual information and quantum conditional entropy. Then we show that quantum conditional mutual information can be defined uniquely as a lower semicontinuous function on the set of all states of a tripartite infinite-dimensional system possessing all the basic properties valid in finite dimensions. Infinite-dimensional generalizations of some other measures of correlations in multipartite quantum systems are also considered. Applications of the results to the theory of infinite-dimensional quantum channels and their capacities are considered. The existence of a Fawzi-Renner recovery channel reproducing marginal states for all tripartite states (including states with infinite marginal entropies) is shown.

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§ 1. Introduction

The quantitative description of correlations in composite quantum systems is one of the main mathematical problems in quantum information theory which have attracted attention since the middle of the 20th century [1], [2]. Notable progress in this direction has been achieved in the last two decades when several quantities characterizing special forms of quantum correlations were found and investigated. Initially, all these characteristics were studied in finite-dimensional settings, which have none of the problems arising in the analysis of infinite-dimensional quantum systems (such as the von Neumann entropy being discontinuous or taking infinite values, the state space being noncompact, and so on). Nevertheless, keeping in mind applications it seems reasonable to construct infinite-dimensional generalizations of the commonly used information quantities and to study their properties.
One of the main problems in infinite-dimensional generalization of information quantities is the appearance of the uncertainty $\infty - \infty$ in their original definitions. For example, many important characteristics of a state $\omega$ of a multipartite finite-dimensional system $A_1 \ldots A_n$ are defined as real linear combinations of the marginal entropies

$$\sum_k \alpha_k H(\omega_{X_k}),$$  \hspace{1cm} (1.1)

where $\omega_{X_k}$ is the partial (marginal) state of $\omega$ corresponding to a subsystem $X_k$ of $A_1 \ldots A_n$ and $H(\cdot)$ is the von Neumann entropy. The linear combination (1.1) gives a well-defined value in $[-\infty, +\infty]$ only for states $\omega$ of an infinite-dimensional system $A_1 \ldots A_n$ for which all the terms in (1.1) are either distinct from $-\infty$ or from $+\infty$. Since the states with finite von Neumann entropy form a first-category subset of the set of all states \cite{3}, a direct translation of the definition (1.1) to the case of infinite-dimensional multipartite systems makes the corresponding quantity undefined for ‘almost all’ states.

The above problem can be (partially or completely) solved by using alternative expressions for (1.1), consisting of terms which are more ‘stable’ when passing to infinite dimensions. The simplest example uses the expression

$$H(\omega_{AB} \parallel \omega_A \otimes \omega_B),$$  \hspace{1cm} (1.2)

where $H(\cdot \parallel \cdot)$ is the quantum relative entropy, instead of the linear combination

$$H(\omega_A) + H(\omega_B) - H(\omega_{AB})$$  \hspace{1cm} (1.3)

in the definition of the quantum mutual information of the state $\omega_{AB}$ of a finite-dimensional bipartite system \cite{4}. Properties of the relative entropy show that (1.2) gives an adequate definition of the quantum mutual information for any state $\omega_{AB}$ of an infinite-dimensional bipartite system, which inherits all basic characteristics of this quantity (nonnegativity, monotonicity and so on; see §4.1).

Using the fact that (1.2) and (1.3) coincide, the quantum conditional entropy $H(A|B)_\omega = H(\omega_{AB}) - H(\omega_B)$ was extended in \cite{5} to the set $\{\omega_{AB} \mid H(\omega_A) < +\infty\}$ (containing states with $H(\omega_{AB}) = H(\omega_B) = +\infty$) by the formula

$$H(A|B)_\omega = H(\omega_A) - H(\omega_{AB} \parallel \omega_A \otimes \omega_B),$$  \hspace{1cm} (1.4)

which preserves all the basic properties of quantum conditional entropy (monotonicity, concavity, subadditivity). This extension has several important applications (see §5).

We begin by introducing the notion of a faithful extension ($F$-extension) of the linear combination (1.1) as an extension satisfying certain requirements (Definition 2), which seem reasonable from both the mathematical and physical points of view.

Then we consider infinite-dimensional generalizations of several entropic quantities having the form (1.1), starting with an analysis of the continuity properties of the quantum mutual information $I(A:B)_\omega$ defined by (1.2) as a function of $\omega$ (Theorem 1). We also describe the extension (1.4) of the quantum conditional entropy and its applications.
We pay special attention to a generalization of the conditional mutual information

$$I(A:C|B) = H(\omega_{AB}) + H(\omega_{BC}) - H(\omega_{ABC}) - H(\omega_B),$$

because of its numerous applications in quantum information theory. It is shown that $I(A:C|B)$ has a unique lower semicontinuous extension to the whole set of states of an infinite-dimensional tripartite system $ABC$, and that this has all the basic properties of the conditional mutual information valid in finite dimensions (Theorem 2).

Infinite-dimensional generalizations of some other measures of correlations in multipartite quantum systems (including amongst others topological entanglement entropy, and unconditional and conditional secrecy monotones $S_n$) are also considered.

In studying measures of correlations we pay special attention to their continuity properties. In finite dimensions any quantity (1.1) is obviously continuous on the whole set of states$^1$. In infinite dimensions global continuity is a very strong requirement, but one can attempt to obtain conditions for local continuity, that is, continuity with respect to the variation of a state within a particular subset. Intuitively, local continuity of some measure of correlations can be understood as stability with respect to local perturbations of a state (which are unavoidable due to the finite accuracy of the state preparation procedure).

In the paper we show that for all commonly used correlation measures (defined by formula (1.1) in finite dimensions) there exist simple sufficient conditions for local continuity expressed via the local continuity of one or several marginal entropies (not necessarily involved in (1.1)). In some cases local continuity of just one marginal entropy implies local continuity of the whole linear combination (1.1) consisting of $n$ terms (see, for instance, Proposition 6).

We also obtain some general results describing when local continuity is preserved when we go over to conditional quantities or take the partial trace (Proposition 2 and Corollary 6).

In §8 we consider applications of the general results in §§1–7 to the theory of infinite-dimensional quantum channels and their capacities. In particular, we show that the classical entanglement-assisted capacity with the constraint defined by the linear inequality $\text{Tr} F \rho \leq E$ is continuous on the set of all channels equipped with the strong convergence topology, provided the von Neumann entropy is continuous on the set of states satisfying this inequality (Proposition 11).

We also prove that a Fawzi-Renner recovery channel exists, that exactly reproduces marginals for all tripartite states (including states with infinite marginal entropies), starting with the corresponding finite-dimensional result in [9] and using the extended conditional mutual information (Proposition 13).

$^1$There exist discontinuous measures of correlations in finite-dimensional multipartite quantum systems and their discontinuity has a physical meaning. Such a measure (called an irreducible tripartite correlation) is considered in [6] and [7]. Its discontinuity is a consequence of the discontinuity of the maximal entropy inference — an interesting purely quantum effect discovered by Weis and Knauf [8].
\section{Preliminaries}

Let $\mathcal{H}$ be a separable Hilbert space, let $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{T}(\mathcal{H})$ be Banach spaces of all bounded operators and of all trace-class operators in $\mathcal{H}$, respectively, let $\mathfrak{T}_+ (\mathcal{H})$ be the cone of positive operators in $\mathfrak{T}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ the set of quantum states (operators with unit trace in $\mathfrak{T}_+ (\mathcal{H})$) \cite{1}, \cite{2}.

Trace class operators (not only states) will be denoted by the Greek letters $\rho$, $\sigma$, $\omega$, \ldots. All others linear operators (in particular, unbounded operators) will be denoted by the Latin letters $A$, $B$, $F$, $H$, \ldots.

We denote the unit operator in a Hilbert space $\mathcal{H}$ by $I_\mathcal{H}$ and the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$ by $\text{Id}_\mathfrak{T}(\mathcal{H})$.

A \textit{quantum operation} $\Phi$ from a system $A$ to a system $B$ is a completely positive trace non-increasing linear map $\mathfrak{T}(\mathcal{H}_A) \rightarrow \mathfrak{T}(\mathcal{H}_B)$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces associated with the systems $A$ and $B$, respectively. A trace-preserving quantum operation is called a \textit{quantum channel} \cite{1}, \cite{2}.

The \textit{von Neumann entropy} $H(\rho) = \text{Tr} \eta(\rho)$ of a state $\rho \in \mathcal{S}(\mathcal{H})$, where $\eta(x) = -x \log x$, has the natural extension

$$H(\rho) = \text{Tr} \rho H\left(\frac{\rho}{\text{Tr} \rho}\right) = \text{Tr} \eta(\rho) - \eta(\text{Tr} \rho), \quad \rho \in \mathfrak{T}_+ (\mathcal{H}),$$

(2.1)

to the cone $\mathfrak{T}_+ (\mathcal{H})$ (see \cite{10}).

The nonnegativity, concavity and lower semicontinuity of the von Neumann entropy on the cone $\mathfrak{T}_+ (\mathcal{H})$ follow from the corresponding properties of this function on the set $\mathcal{S}(\mathcal{H})$ \cite{3}, \cite{10}. By definition

$$H(\lambda \rho) = \lambda H(\rho), \quad \lambda \geq 0.$$

(2.2)

The concavity of the von Neumann entropy is supplemented by the inequality

$$H(\lambda \rho + (1 - \lambda) \sigma) \leq \lambda H(\rho) + (1 - \lambda) H(\sigma) + \max\{\text{Tr} \rho, \text{Tr} \sigma\} h_2(\lambda),$$

(2.3)

where $h_2(\lambda) = \eta(\lambda) + \eta(1 - \lambda)$, which is valid for any operators $\rho, \sigma \in \mathfrak{T}_+ (\mathcal{H})$.

The \textit{quantum relative entropy} for two operators $\rho$ and $\sigma$ in $\mathfrak{T}_+ (\mathcal{H})$ is defined as follows (see \cite{10}):

$$H(\rho \Vert \sigma) = \sum_{i=1}^{+\infty} \langle i | \rho \log \rho - \rho \log \sigma + \sigma - \rho | i \rangle,$$

where $\{|i\rangle\}_{i=1}^{+\infty}$ is the orthonormal basis of eigenvectors of the operator $\rho$ and it is assumed that $H(\rho \Vert \sigma) = +\infty$ if the support of $\rho$ is not contained in the support of $\sigma$. This definition implies that

$$H(\lambda \rho \Vert \lambda \sigma) = \lambda H(\rho \Vert \sigma), \quad \lambda \geq 0.$$

(2.4)

We will use the following result from purification theory.

\footnote{Here and in what follows log denotes the natural logarithm.}
Lemma 1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces such that $\dim \mathcal{H} = \dim \mathcal{K}$. For an arbitrary pure state $\omega_0$ in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ and an arbitrary sequence $\{\rho_k\}$ of states in $\mathcal{S}(\mathcal{K})$ converging to the state $\rho_0 = \Tr:\mathcal{K} \omega_0$ there exists a sequence $\{\omega_k\}$ of pure states in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ converging to the state $\omega_0$ such that $\rho_k = \Tr:\mathcal{K} \omega_k$ for all $k$.

The assertion of Lemma 1 can be proved by noting that the infimum in the definition of the Bures distance between two quantum states can be taken over all purifications of one state for a fixed purification of the other state and that the convergence of a sequence of states in the trace norm distance implies its convergence in the Bures distance; see [1], [2] and [9].

We will use the following property of the von Neumann entropy.

Lemma 2. Let $\omega_{A_1 \ldots A_n}$ be a state of $n$-partite system $A_1 \ldots A_n$ and suppose that $\{P_{A_1}^k\}_k \subset \mathcal{B}(\mathcal{H}_{A_1})$, $\ldots$, $\{P_{A_n}^k\}_k \subset \mathcal{B}(\mathcal{H}_{A_n})$ are sequences of projections strongly converging to the identity operators $I_{A_1}, \ldots, I_{A_n}$. Let $\omega_{A_1 \ldots A_n}^k = \lambda_k^{-1} Q_k \omega_{A_1 \ldots A_n} Q_k$, where $Q_k = P_{A_1}^k \otimes \ldots \otimes P_{A_n}^k$, $\lambda_k = \Tr Q_k \omega_{A_1 \ldots A_n}$ and let $A_{i_1} \ldots A_{i_m}$, $m \leq n$, be a subsystem of $A_1 \ldots A_n$. Then

$$\lim_{k \to \infty} H(\omega_{A_{i_1} \ldots A_{i_m}}^k) = H(\omega_{A_{i_1} \ldots A_{i_m}}) \leq +\infty.$$  

Proof. Noting that

$$\lambda_k \omega_{A_{i_1} \ldots A_{i_m}}^k \leq P_{A_{i_1}}^k \otimes \ldots \otimes P_{A_{i_m}}^k \omega_{A_{i_1} \ldots A_{i_m}} P_{A_{i_1}}^k \otimes \ldots \otimes P_{A_{i_m}}^k \quad \forall k,$$

this assertion can be proved using Simon’s convergence theorems for the von Neumann entropy [11]; see Appendix.

Remark 1. Throughout this paper we assume that continuous functions on a metric space are finite on this space (in contrast to lower (upper) semicontinuous functions, which can take infinite values). We will say that the local continuity of a function $f$ implies the local continuity of a function $g$ if

$$\lim_{k \to \infty} f(x_k) = f(x_0) \neq \pm \infty \quad \implies \quad \lim_{k \to \infty} g(x_k) = g(x_0) \neq \pm \infty$$

for any sequence $\{x_k\}$ converging to $x_0$.

We use the following simple fact.

Lemma 3. Let $f_1, \ldots, f_n$ be nonnegative lower semicontinuous functions on a metric space $X$. Then the local continuity of $\sum_{k=1}^n f_k$ implies the local continuity of all the functions $f_1, \ldots, f_n$.

Remark 2. Throughout the paper we assume that any function $F$ on the set $\mathcal{S}(\mathcal{H})$ of quantum states is extended to the cone $\mathcal{T}_+(\mathcal{H})$ by

$$F(\rho) = [\Tr \rho] F\left(\frac{\rho}{\Tr \rho}\right).$$

Note that the extension of a function $F(\rho) = \sum_k \alpha_k H(\Phi_k(\rho))$, where the $\Phi_k$ are positive linear maps, is determined by the same formula, provided that the extension (2.1) of the von Neumann entropy is used.
§ 3. Faithful extension of entropic quantities

Correlations of the state of a finite-dimensional \( n \)-partite quantum system \( A_1 \ldots A_n \) are described by various entropic quantities (such as quantum mutual information, quantum conditional entropy, conditional mutual information, topological entanglement entropy and so on), defined as real linear combinations of marginal entropies, that is, functions

\[
F(\omega_{A_1 \ldots A_n}) = \sum_k \alpha_k H(\omega_{X_k})
\]

on the set of all states of the system, where \( \omega_{X_k} \) is a partial state of \( \omega_{A_1 \ldots A_n} \) corresponding to a subsystem \( X_k \) of \( A_1 \ldots A_n \).

In infinite dimensions such an entropic quantity is well defined if all the marginal entropies \( H(\omega_{X_k}) \) involved in the corresponding linear combination (3.1) are finite (or at least this linear combination does not contain the uncertainty \( \infty - \infty \)). This narrow domain of definition can be extended by using alternative expressions, which are more ‘stable’ under passing to infinite dimensions than the linear combination in (3.1). Examples of such extensions for quantum mutual information and for quantum conditional entropy were mentioned in §1. These examples motivate questions about the possibility of extending the linear combination (3.1) to states at which it is not correctly defined and about the requirements on these extensions in the general settings.

In general the function \( F \) in (3.1) is not lower or upper semicontinuous on the set of states at which it is well defined, that is, the linear combination does not contain the uncertainty \( \infty - \infty \), but Lemma 2 shows that for any such state this function possesses the following property:

\[
\lim_{k \to \infty} F(\omega_{A_1 \ldots A_n}^k) = F(\omega_{A_1 \ldots A_n}^k) \in [-\infty, +\infty]
\]

for any sequence of ‘truncated’ states

\[
\omega_{A_1 \ldots A_n}^k = \lambda_k^{-1} Q_k \omega_{A_1 \ldots A_n} Q_k, \quad Q_k = P_A^k_1 \otimes \cdots \otimes P_A^k_n, \quad \lambda_k = \text{Tr} Q_k \omega_{A_1 \ldots A_n},
\]
determined by sequences of projections \( \{P_A^k\}_k \subset \mathcal{B}(\mathcal{H}_A), \ldots, \{P_A^k\}_k \subset \mathcal{B}(\mathcal{H}_A) \) converging strongly to the identity operators

\[
I_{A_1}, \ldots, I_{A_n}.
\]

This property can be treated as the self-consistency or stability of \( F \) with respect to state truncation. It seems to be a reasonable requirement for any measure of quantum correlations from both the analytic and physical points of view. This motivates the following definition.

**Definition 1.** Let \( \mathcal{A} \) be a subset of states of a multipartite system \( A_1 \ldots A_n \) such that

\[
\omega_{A_1 \ldots A_n} \in \mathcal{A} \iff [\text{Tr} Q \omega_{A_1 \ldots A_n}]^{-1} Q \omega_{A_1 \ldots A_n} Q \in \mathcal{A},
\]

where \( Q = P_A \otimes \cdots \otimes P_A \), for any projections \( P_A \in \mathcal{B}(\mathcal{H}_A), \ i = 1, \ldots, n \). A function \( F \) taking values in \([-\infty, +\infty]\) is said to be **faithful** on \( \mathcal{A} \) if condition (3.2) holds for any state \( \omega_{A_1 \ldots A_n} \in \mathcal{A} \).

\[\text{If } F(\omega_{A_1 \ldots A_n}) \text{ is well defined then } F(\omega_{A_1 \ldots A_n}^k) \text{ is well defined for all } k \text{ (since if } H(\rho) \text{ is finite, then so is } H(P \rho P) \text{ for any projection } P).\]
It is clear that if $F$ is continuous, then it is faithful, but the converse is not true. The simplest example is given by the von Neumann entropy of a marginal state $F(\omega_{A_1 \ldots A_n}) = H(\omega_X)$, $X \subseteq A_1 \ldots A_n$, which is a faithful function on the whole set of states of $A_1 \ldots A_n$ by Lemma 2.

The faithfulness property seems to be a reasonable replacement for continuity in infinite dimensions. Many characteristics of the states of bi- and multipartite infinite-dimensional systems are globally faithful but not continuous. For example, bi- and multipartite quantum mutual information, entanglement monotones obtained by the universal extension from finite-dimensional ones (see §3 in [12]) and so on. The faithfulness of these characteristics follows because they are lower semicontinuous and monotonic under local operations.

**Lemma 4.** If $F$ is a lower semicontinuous function on a subset $\mathcal{A}$ of $\mathcal{F}(\mathcal{H}_1 \ldots \mathcal{H}_n)$ satisfying condition (3.3) and $F(\Phi_1 \otimes \cdots \otimes \Phi_n(\omega)) \leq F(\omega)$ for any $\omega \in \mathcal{A}$ and arbitrary quantum operations $\Phi_1 : A_1 \to A_1, \ldots, \Phi_n : A_n \to A_n$ then $F$ is faithful on $\mathcal{A} \cap \mathcal{G}(\mathcal{H}_1 \ldots \mathcal{H}_n)$.

Faithfulness is a very convenient property from the analytic point of view. Faithfulness of the bipartite quantum mutual information was implicitly used in [5] to prove that concavity and other basic properties of quantum conditional entropy hold for its extension (1.4).

The above arguments show that a function $\hat{F}$ can be treated as an adequate extension of a particular information quantity $F$ to a subset $\mathcal{A}$ of $\mathcal{G}(\mathcal{H}_1 \ldots \mathcal{H}_n)$ if it is faithful on this subset.

**Definition 2.** A function $\hat{F}$ is called a faithful extension (briefly, an $\mathcal{F}$-extension) of the function $F$ in (3.1) to a subset $\mathcal{A} \in \mathcal{G}(\mathcal{H}_1 \ldots \mathcal{H}_n)$ satisfying condition (3.3) if it is faithful on $\mathcal{A}$ and $\hat{F}(\omega) = F(\omega)$ for any state $\omega \in \mathcal{A}$ for which $F(\omega)$ is well defined.

Definition 2 implies the following two simple observations used below.

**Lemma 5.** If an $\mathcal{F}$-extension of a linear combination of marginal entropies to a particular subset exists then it is uniquely defined.

**Lemma 6.** If $\hat{F}_1$ and $\hat{F}_2$ are $\mathcal{F}$-extensions of quantities $F_1$ and $F_2$ to a particular set $\mathcal{A}$ such that $c_1 \hat{F}_1 + c_2 \hat{F}_2$, $c_1, c_2 \in \mathbb{R}$, is well defined on $\mathcal{A}$ then $c_1 \hat{F}_1 + c_2 \hat{F}_2$ is a $\mathcal{F}$-extension of the quantity $c_1 F_1 + c_2 F_2$ to the set $\mathcal{A}$.

Now consider a general result concerning the case when the linear combination (3.1) is bounded on the set where it is well defined. This result (which is proved using Winter’s modification of the Alicki-Fannes techniques [13], [14]) shows that the boundedness of (3.1) implies its uniform continuity and gives estimates for its variation.

**Proposition 1.** If a function $F(\omega_{A_1 \ldots A_n}) = \sum_k \alpha_k H(\omega_{X_k})$ is bounded on the set $\mathcal{S}_f = \{\omega_{A_1 \ldots A_n} | \max_k \text{rank} \omega_{A_k} < +\infty\}$ then it has a unique continuous extension $\hat{F}$ to the set $\mathcal{G}(\mathcal{H}_1 \ldots \mathcal{H}_n)$ such that

$$|\hat{F}(\omega^1) - \hat{F}(\omega^2)| \leq \varepsilon \sup_{\omega, \omega' \in \mathcal{S}_f} |F(\omega) - F(\omega')| + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)\sum_k |\alpha_k|$$

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Examples corresponding to this case are considered below (see Corollaries 1, 5, 8 and 11).
for any $\omega^1, \omega^2 \in \mathcal{S}(\mathcal{H}_{A_1...A_n})$ such that $\epsilon = \frac{1}{2}\|\omega^1 - \omega^2\|_1$, where $h_2$ is the binary entropy.

If $F$ is a concave (convex) function on $\mathcal{S}_f$ then $\hat{F}$ is a concave (convex) function on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ and $\sum_k |\alpha_k|$ in (3.4) can be replaced by $\sum_{k:\alpha_k>0} |\alpha_k|$ and $\sum_{k:\alpha_k<0} |\alpha_k|$, respectively.

**Proof.** It suffices to show that (3.4) holds with $\hat{F} = F$ for any states $\omega^1, \omega^2 \in \mathcal{S}_f$, since this would imply that $F$ is a uniformly continuous function on the dense subset $\mathcal{S}_f$ of $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ which can be uniquely extended, in the standard way, to a continuous function $\hat{F}$ on $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ satisfying (3.4).

The concavity of the von Neumann entropy and inequality (2.3) imply

$$\lambda F(\omega^1) + (1 - \lambda) F(\omega^2) + C_- h_2(\lambda) \leq F(\lambda \omega^1 + (1 - \lambda) \omega^2) \leq \lambda F(\omega^1) + (1 - \lambda) F(\omega^2) + C_+ h_2(\lambda) \tag{3.5}$$

for any states $\omega^1, \omega^2 \in \mathcal{S}_f$, where $C_- = \sum_{k:\alpha_k<0} |\alpha_k|$ and $C_+ = \sum_{k:\alpha_k>0} |\alpha_k|$.

Following [14] we introduce the state $\omega^* = (1 + \epsilon)^{-1}(\omega^1 + [\omega^2 - \omega^1]^+)$ Then

$$\frac{1}{1+\epsilon} \omega^1 + \frac{\epsilon}{1+\epsilon} \tilde{\omega}^1 = \omega^* = \frac{1}{1+\epsilon} \omega^2 + \frac{\epsilon}{1+\epsilon} \tilde{\omega}^2, \tag{3.6}$$

where $\tilde{\omega}^1 = \epsilon^{-1} [\omega^2 - \omega^1]^+$ and $\tilde{\omega}^2 = \epsilon^{-1} (1 + \epsilon) \omega^* - \omega^2$ are states in $\mathcal{S}_f$. Applying (3.5) to the convex decompositions (3.6) of $\omega^*$ we obtain

$$\frac{1}{1+\epsilon} (F(\omega^1) - F(\omega^2)) \leq \frac{\epsilon}{1+\epsilon} (F(\tilde{\omega}^1) - F(\tilde{\omega}^2)) + (C_+ - C_-) h_2 \left( \frac{\epsilon}{1+\epsilon} \right),$$

$$\frac{1}{1+\epsilon} (F(\omega^2) - F(\omega^1)) \leq \frac{\epsilon}{1+\epsilon} (F(\tilde{\omega}^1) - F(\tilde{\omega}^2)) + (C_+ - C_-) h_2 \left( \frac{\epsilon}{1+\epsilon} \right).$$

These inequalities imply (3.4) with $\hat{F} = F$ since $C_+ - C_- = \sum_k |\alpha_k|$.

The last assertion of the proposition follows from the above proof. Proposition 1 is proved.

If $d_A \doteq \dim \mathcal{H}_A < +\infty$ then Proposition 1 gives the following continuity bounds for the von Neumann entropy and the conditional entropy

$$|H(\omega^1) - H(\omega^2)| \leq \epsilon \log d_A + (1 + \epsilon) h_2 \left( \frac{\epsilon}{1+\epsilon} \right), \tag{3.7}$$

$$|H(A|B)\omega^1 - H(A|B)\omega^2| \leq 2\epsilon \log d_A + (1 + \epsilon) h_2 \left( \frac{\epsilon}{1+\epsilon} \right) \tag{3.8}$$

(the concavity of both quantities has been taken into account).

The continuity bound (3.7) is close (and asymptotically equivalent) to the sharpest continuity bound for the von Neumann entropy

$$|H(\omega^1) - H(\omega^2)| \leq \epsilon \log (d_A - 1) + h_2(\epsilon),$$

which was obtained by Audenaert [15]. The continuity bound (3.8) coincides with the tight continuity bound for the conditional entropy proved by Winter using (3.6) and the special representation for the conditional entropy [14], Lemma 2. So, we have some reason to expect that Proposition 1 gives quite a sharp continuity bound despite its universality.
§ 4. Quantum mutual information and its applications

4.1. The bipartite case. Correlations of a state \( \omega_{AB} \) of a finite-dimensional bipartite quantum system \( AB \) are characterized by the value

\[
I(A:B)_{\omega} = H(\omega_A) + H(\omega_B) - H(\omega_{AB}),
\]

(4.1)
called the quantum mutual information of this state \([2], [16], [17]\).

In infinite dimensions the linear combination of marginal entropies in (4.1) may contain the uncertainty \( \infty - \infty \), but it can be well defined for any state \( \omega_{AB} \) (as a value in \([0, +\infty)\)) by the expression

\[
I(A:B)_{\omega} = H(\omega_{AB} | \omega_A \otimes \omega_B),
\]

(4.2)
which coincides with (4.1) for any state \( \omega_{AB} \) with finite marginal entropies.

The right-hand side of (4.2) plays the role of a ‘building block’ in the construction of many characteristics of infinite-dimensional quantum systems and channels. It is used in the extension (1.4) of quantum conditional entropy, and in the definitions of quantum mutual and coherent information of an infinite-dimensional quantum channel (see §§ 8.1 and 8.2 below) etc.

In some applications the quantity (4.1) is used with arbitrarily positive trace class operators \( \omega_{AB} \) (not only states). In this case (2.2) and (2.4) imply

\[
I(A:B)_{\omega} = H(\omega_{AB} | \omega_A \otimes \omega_B) - \text{Tr} \omega_{AB}.
\]

The basic properties of the relative entropy show that \( \omega \mapsto I(A:B)_{\omega} \) is a lower semicontinuous function on the cone \( T^+ (\mathcal{H}_{AB}) \) and has the following properties:

A1) \( I(A:B)_{\omega} \geq 0 \) for any operator \( \omega_{AB} \) and \( I(A:B)_{\omega} = 0 \) if and only if \( \omega_{AB} \) is a product operator, that is, \( [\text{Tr} \omega_{AB}] \omega_{AB} = \omega_A \otimes \omega_B \);

A2) monotonicity under local reduction: \( I(A:BC)_{\omega} \geq I(A:B)_{\omega} \);

A3) monotonicity under local operations: \( I(A:B)_{\omega} \geq I(A':B')_{\Phi_A \otimes \Phi_B(\omega)} \)

for arbitrary quantum operations\(^5\) \( \Phi_A : A \rightarrow A' \) and \( \Phi_B : B \rightarrow B' \);

A4) additivity: \( I(A':BB')_{\omega \otimes \omega'} = I(A:B)_{\omega} + I(A':B')_{\omega'} \).

By Lemma 4 the lower semicontinuity of \( I(A:B)_{\omega} \) and property A3) show that \( I(A:B)_{\omega} \) defined by (4.2) is the \( \mathfrak{F} \)-extension of (4.1) to the set \( \mathfrak{F} (\mathcal{H}_{AB}) \).

We will use the following upper bound, mentioned in [16] in the finite-dimensional case:

\[
I(A:B)_{\omega} \leq 2 \min \{ H(\omega_A), H(\omega_B) \}.
\]

(4.3)
It directly follows from identity (9.3).

The following theorem gives local continuity conditions for the bipartite quantum mutual information (as a function on the cone \( T^+ (\mathcal{H}_{AB}) \)), which we will make key use of below.

**Theorem 1.** A) The limit relation

\[
\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0}
\]

(4.4)

\(^5\)If \( \Phi_A \) and \( \Phi_B \) are quantum channels then this property follows directly from the monotonicity of the relative entropy, if either \( \Phi_A \) or \( \Phi_B \) is a non-trace-preserving quantum operation then additional arguments are required, for example, (9.6) can be used.
holds for a sequence \( \{\omega^k\} \subset \mathcal{F}_+ (\mathcal{H}_{AB}) \) converging to an operator \( \omega^0 \) if one of the following conditions holds:

(a) \( \lim_{k \to \infty} H(\omega^k_X) = H(\omega^0_X) < +\infty \), where \( X \) is one of the systems \( A \) and \( B \) (in this case the limit in (4.4) is finite by (4.3));

(b) \( \lambda_k \omega^k \leq \Phi_A^k \otimes \Phi_B^k (\omega^0) \) for some sequences \( \{\Phi_A^k\} \) and \( \{\Phi_B^k\} \) of local quantum operations and some sequence \( \{\lambda_k\} \subset [0, 1] \) converging to 1.

B) If (4.4) holds for a sequence \( \{\omega^k\} \subset \mathcal{F}_+ (\mathcal{H}_{AB}) \) as a finite limit then

\[
\lim_{k \to \infty} I(A':B')_{\Phi_A \otimes \Phi_B(\omega^k)} = I(A':B')_{\Phi_A \otimes \Phi_B(\omega^0)} < +\infty
\]

for arbitrary quantum operations \( \Phi_A : A \to A' \) and \( \Phi_B : B \to B' \).

Assertion B) of Theorem 1 means that the local continuity of the quantum mutual information is preserved by local operations.

Theorem 1 (proved in § 9) shows, in particular, that the quantum mutual information \( I(A : B)_\omega \) is continuous on the set \( \mathcal{G}(\mathcal{H}_{AB}) \) if and only if either \( A \) or \( B \) is a finite-dimensional system. Proposition 1 and the upper bound (4.3) give the continuity bound for \( I(A : B)_\omega \) in this case.

**Corollary 1.** If one of the systems \( A \) and \( B \), say \( A \), is finite-dimensional then \( I(A : B)_\omega \) is a continuous bounded function on the set \( \mathcal{G}(\mathcal{H}_{AB}) \) and

\[
|I(A : B)_{\omega^1} - I(A : B)_{\omega^2}| \leq 2\varepsilon \log \dim \mathcal{H}_A + 3(1 + \varepsilon) h_2 \left( \frac{\varepsilon}{1 + \varepsilon} \right)
\]

for any \( \omega^1, \omega^2 \in \mathcal{G}(\mathcal{H}_{AB}) \), where \( \varepsilon = \frac{1}{2} \|\omega^1 - \omega^2\|_1 \) and \( h_2 \) is the binary entropy.

**Remark 3.** The continuity condition (a) for \( I(A : B)_\omega \) in part A) of Theorem 1 coincides with the continuity condition for the entanglement of formation \( E_F(\omega) \) and for the squashed entanglement \( E_{sq}(\omega) \) of the state \( \omega \) in an infinite-dimensional bipartite system [12], § 6.

The following example shows that condition (a) in part A) of Theorem 1 is not necessary for limit relation (4.4) to hold.

**Example 1.** Consider a sequence \( \{\rho_k\} \subset \mathcal{G}(\mathcal{H}_A) \) converging to a state \( \rho_0 \) such that

\[
\lim_{k \to \infty} H(\rho_k) \neq H(\rho_0).
\]

By Lemma 1 there exists a sequence \( \{\omega^k\} \subset \mathcal{G}(\mathcal{H}_{AB}) \), \( \mathcal{H}_B \cong \mathcal{H}_A \), converging to a state \( \omega^0 \in \mathcal{G}(\mathcal{H}_{AB}) \) such that \( \omega^k_A = \rho_k \) for all \( k \geq 0 \). Let \( \sigma_k = \omega^k_B \) and \( \tilde{\omega}^k = p_k \omega^k + (1 - p_k) \rho_0 \otimes \sigma_k \), where \( \{p_k\} \) is a sequence of positive numbers such that \( \lim_{k \to \infty} p_k H(\rho_0) = 0 \). The sequence \( \{\tilde{\omega}^k\} \) converges to the state \( \tilde{\omega}^0 = \rho_0 \otimes \sigma_0 \).

Using the convexity of the relative entropy it is easy to see that

\[
\lim_{k \to \infty} I(A : B)_{\tilde{\omega}^k} = 0 = I(A : B)_{\tilde{\omega}^0},
\]

while

\[
\lim_{k \to \infty} H(\tilde{\omega}^k_A) \neq H(\tilde{\omega}^0_A) \quad \text{and} \quad \lim_{k \to \infty} H(\tilde{\omega}^k_B) \neq H(\tilde{\omega}^0_B),
\]

since \( \tilde{\omega}^k_A = \rho_k \) and \( \tilde{\omega}^k_B = \sigma_k \) for all \( k \geq 0 \).
Condition (b) in part A) of Theorem 1 implies the following important result, which we will use below.

**Corollary 2.** Let $X$ and $Y$ be disjoint subsystems of $A_1 \ldots A_n$. The function \( \omega_{A_1 \ldots A_n} \mapsto I(X:Y)_\omega \) is faithful on \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \).

**Proof.** We may assume that $X = A_1$ and $Y = A_2$. That (3.2) holds for the function \( \omega_{A_1 \ldots A_n} \mapsto I(A_1:A_2)_\omega \) follows from the inequality

\[
\lambda_k \omega_{A_1 A_2}^k \leq P_{A_1}^k \otimes P_{A_2}^k \omega_{A_1 A_2} P_{A_1}^k \otimes P_{A_2}^k \ \forall \ k
\]

(where \( \lambda_k \) is defined in (3.2)) and condition (b) in part A) of Theorem 1.

**Remark 4.** By Lemma 6, Corollary 2 and Lemma 2 show that the function

\[
\omega_{A_1 \ldots A_n} \mapsto \sum_k \alpha_k I(X_k:Y_k)_\omega + \sum_k \beta_k H(\omega_{Z_k}), \tag{4.5}
\]

where the \( X_k, Y_k \) and the \( Z_k \) are subsystems of \( A_1 \ldots A_n \), is faithful on the set of all states at which it is well defined (the linear combination in (4.5) does not contain the uncertainty \( \infty - \infty \)).

### 4.2. The multipartite case.

The quantum mutual information of a state \( \omega_{A_1 \ldots A_n} \) of a finite-dimensional \( n \)-partite quantum system \( A_1 \ldots A_n \) is defined as follows (see [2], [4], [18]–[20])

\[
I(A_1: \ldots : A_n)_\omega = \sum_{i=1}^{n} H(\omega_{A_i}) - H(\omega_{A_1 \ldots A_n}). \tag{4.6}
\]

In infinite dimensions it can be well defined for any state \( \omega_{A_1 \ldots A_n} \) (as a value in \([0, +\infty]\)) by the expression

\[
I(A_1: \ldots : A_n)_\omega = H(\omega_{A_1 \ldots A_n} \parallel \omega_{A_1} \otimes \cdots \otimes \omega_{A_n}), \tag{4.7}
\]

coinciding with (4.6) for any state \( \omega_{A_1 \ldots A_n} \) with finite marginal entropies.

Now, (4.6) has a homogeneous extension to the cone \( \mathfrak{T}_+(\mathcal{H}_{A_1 \ldots A_n}) \) given by

\[
I(A_1: \ldots : A_n)_\omega = H\left(\omega_{A_1 \ldots A_n} \parallel \frac{\omega_{A_1} \otimes \cdots \otimes \omega_{A_n}}{\text{Tr} \omega_{A_1 \ldots A_n}}\right)^{n-1}. \tag{4.8}
\]

Basic properties of the relative entropy show that \( I(A_1: \ldots : A_n)_\omega \) is a lower semicontinuous function on the cone \( \mathfrak{T}_+(\mathcal{H}_{A_1 \ldots A_n}) \) which takes values in \([0, +\infty]\) and has analogues of properties A1)–A4). This can be shown using the identity

\[
I(A_1: \ldots : A_n)_\omega = I(A_2:A_1)_\omega + I(A_3:A_1 A_2)_\omega + \cdots + I(A_n:A_1 \ldots A_{n-1})_\omega, \tag{4.8}
\]

which is directly verified for a state \( \omega \) with finite marginal entropies and can be extended to arbitrary states by means of approximation, using Corollary 2. It follows from Lemma 4 that \( I(A_1: \ldots : A_n)_\omega \) defined by (4.7) is an \( \mathfrak{T} \)-extension of the linear combination (4.6) to the set \( \mathcal{S}(\mathcal{H}_{A_1 \ldots A_n}) \).

---

6See Definition 1. In general, the faithfulness of the function \( \omega_{XY} \mapsto F(\omega_{XY}) \) does not imply the faithfulness of the function \( \omega_{A_1 \ldots A_n} \mapsto F(\omega_{XY}) \).
Identity (4.8) and the upper bound (4.3) imply that
\[ I(A_1 : \ldots : A_n)_\omega \leq 2 \min_{1 \leq j \leq n} \sum_{i \neq j} H(\omega_{A_i}). \tag{4.9} \]

Identity (4.8) also makes it possible to derive an \( n \)-partite version of Theorem 1.

**Corollary 3.** A) The limit relation
\[ \lim_{k \to \infty} I(A_1 : \ldots : A_n)_{\omega^k} = I(A_1 : \ldots : A_n)_{\omega^0} \tag{4.10} \]
holds for a sequence \( \{\omega^k\} \subset \mathcal{T}_+(\mathcal{H}_{A_1 \ldots A_n}) \) converging to an operator \( \omega^0 \) if one of the following conditions holds:

(a) \( \lim_{k \to \infty} H(\omega^k_{A_i}) = H(\omega^0_{A_i}) < +\infty \) for at least \( n-1 \) values\(^7\) of \( i \) (in this case the limit in (4.10) is finite according to (4.9));

(b) \( \lambda_k \omega^k \leq \Phi^k_{A_1} \otimes \ldots \otimes \Phi^k_{A_n}(\omega^0) \) for some sequences \( \{\Phi^k_{A_1}\}, \ldots, \{\Phi^k_{A_n}\} \)
of local quantum operations and some sequence \( \{\lambda_k\} \subset [0,1] \) converging to 1.

B) If (4.10) holds for a sequence \( \{\omega^k\} \subset \mathcal{T}_+(\mathcal{H}_{A_1 \ldots A_n}) \) as a finite limit then
\[ \lim_{k \to \infty} I(A'_1 : \ldots : A'_n)_{\Phi_{A_1} \otimes \ldots \otimes \Phi_{A_n}(\omega^k)} = I(A'_1 : \ldots : A'_n)_{\Phi_{A_1} \otimes \ldots \otimes \Phi_{A_n}(\omega^0)} \]
for arbitrary quantum operations \( \Phi_{A_1} : A_1 \to A'_1, \ldots, \Phi_{A_n} : A_n \to A'_n \).

Assertion B) of Corollary 3 means that, as in the bipartite case, local operations do not destroy the continuity of the multipartite quantum mutual information.

Using condition (b) in part A) of Corollary 3 we can obtain a generalization of Corollary 2, that is, we can show the global faithfulness of the function \( \omega_{A_1 \ldots A_n} \mapsto I(A_1 : \ldots : A_{i_m})_{\omega} \) for any subsystems \( A_{i_1}, \ldots, A_{i_m} \) of \( A_1 \ldots A_n \).

Proposition 1 and the upper bound (4.9) imply the following result.

**Corollary 4.** If \( n-1 \) subsystems, say \( A_1, \ldots, A_{n-1} \), are finite-dimensional then \( I(A_1 : \ldots : A_n)_{\omega} \) is a continuous bounded function on the set \( \mathcal{G}(\mathcal{H}_{A_1 \ldots A_n}) \) and
\[ |I(A_1 : \ldots : A_n)_{\omega^1} - I(A_1 : \ldots : A_n)_{\omega^2}| \leq 2\epsilon C + (n+1)(1+\epsilon)h_2 \left( \frac{\epsilon}{1+\epsilon} \right) \]
for any \( \omega^1, \omega^2 \in \mathcal{G}(\mathcal{H}_{A_1 \ldots A_n}) \), where \( \epsilon = \frac{1}{2} \|\omega^1 - \omega^2\|_1 \), \( h_2 \) is the binary entropy and \( C = \log \dim \mathcal{H}_{A_1 \ldots A_{n-1}} \).

### 4.3. General relations between conditional and unconditional quantities.

Given a quantity \( F(\omega_{A_1 \ldots A_n}) = \sum_k \alpha_k H(\omega_{X_k}) \) we introduce the corresponding conditional quantity
\[ F_{|B}(\omega_{A_1 \ldots A_nB}) = \sum_k \alpha_k \left[ H(\omega_{X_kB}) - H(\omega_B) \right]. \tag{4.11} \]

Regarding \( F \) as a function of \( \omega_{A_1 \ldots A_nB} \) we have
\[ [F_{|B} - F](\omega_{A_1 \ldots A_nB}) = -\sum_k \alpha_k \left[ H(\omega_{X_k}) + H(\omega_B) - H(\omega_{X_kB}) \right] \]

\(^7\)It is easy to see that if this relation holds for \( n-2 \) values of \( i \) it does not imply (4.10).
for any state with finite marginal entropies. Hence, Remark 4, (4.3) and Theorem 1 imply the following observation.

**Proposition 2.** The difference \([F_\cdot | B] - F\) has the finite \(\mathcal{F}\)-extension

\[
[F_\cdot | B] - F)(\omega_{A_1...A_n}B) = -\sum_k \alpha_k I(X_k : B)\omega
\]

(4.12)

to the set \(\{\omega_{A_1...A_n}B \mid \min \{H(\omega_B), \sum_k H(\omega_{X_k})\} < +\infty\}\), with the following properties:

1) \(|[F_\cdot | B] - F)(\omega_{A_1...A_n}B)| \leq \min \{H(\omega_B) \mid [\sum_k \alpha_k + \sum_k \alpha_k] / 2, \sum_k \alpha_k H(\omega_{X_k})\};

2) the function \(\omega_{A_1...A_n}B \mapsto [F_\cdot | B] - F)(\omega_{A_1...A_n}B)\) is continuous on a subset \(\mathcal{F} \subset \mathcal{F}_+(\mathcal{H}_{A_1...A_n}B)\) if one the following conditions holds:

(a) the function \(\omega_{A_1...A_n}B \mapsto H(\omega_B)\) is continuous on \(\mathcal{F}\);

(b) the function \(\omega_{A_1...A_n}B \mapsto H(\omega_{X_k})\) is continuous on \(\mathcal{F}\) for each \(k\).

Condition (a) in Proposition 2 shows, roughly speaking, that conditioning a system with finite (continuous) entropy does not destroy the finiteness (continuity) of the quantity \(F(\omega_{A_1...A_n}) = \sum_k \alpha_k H(\omega_{X_k})\).

Condition (b) in Proposition 2 shows that the finiteness (continuity) of all marginal entropies in the linear combination \(F(\omega_{A_1...A_n}) = \sum_k \alpha_k H(\omega_{X_k})\) guarantees the finiteness (continuity) of the conditional quantity \(F_\cdot | B\) regardless of the system \(B\).

By Proposition 2 the formula

\[
F_\cdot | B)(\omega_{A_1...A_n}B) = \sum_k \alpha_k [H(\omega_{X_k}) - I(X_k : B)\omega]
\]

(4.13)
defines an \(\mathcal{F}\)-extension of (4.11) to the set \(\{\omega_{A_1...A_n}B \mid \max \{H(\omega_{X_k})\} < \infty\}\).

**Remark 5.** Proposition 2 does not assert the existence of an \(\mathcal{F}\)-extension of the quantity (4.11) to the set \(\{\omega_{A_1...A_n}B \mid H(\omega_B) < +\infty\}\).

§ 5. Extended quantum conditional entropy

The quantum conditional entropy

\[
H(A | B)\omega = H(\omega_{AB}) - H(\omega_B)
\]

(5.1)
of a state of a finite-dimensional bipartite system \(AB\) has a key role in the analysis of quantum systems and channels even though it might be negative [1], [2]. It has the following basic properties:

B1) concavity: \(H(A | B)_{p\omega_1 + (1-p)\omega_2} \geq pH(A | B)_{\omega_1} + (1-p)H(A | B)_{\omega_2}\);

B2) monotonicity: \(H(A | B)_{\omega} \geq H(A | BC)_{\omega}, \omega = \omega_{ABC}\);

B3) subadditivity: \(H(\omega A' | BB')_{\omega} \leq H(\omega A | BB)_{\omega} + H(A' | B')_{\omega}, \omega = \omega_{A' A'' BB'}\).

In infinite dimensions formula (5.1) defines a finite quantity possessing properties B1)–B3) on the set \(\mathcal{G}_f = \{\omega_{AB} \mid \max \{H(\omega_A), H(\omega_B)\} < +\infty\}\). This narrow domain of definition of \(H(A | B)_{\omega}\) is extended in [5], where it is shown that the quantity

\[
H_e(A | B)\omega = H(\omega_A) - H(\omega_{AB} \| \omega_A \otimes \omega_B) = H(\omega_A) - I(A : B)\omega,
\]

(5.2)
which coincides with (5.1) on \( \mathcal{S}_f \), has properties B1–B3) on the convex set \( \{ \omega_{AB} \mid H(\omega_A) < +\infty \} \) containing states with \( H(\omega_{AB}) = H(\omega_B) = +\infty \). This extension turns out to be very useful\(^8\) in analyzing infinite-dimensional quantum systems and channels [21], [22]. Note that (5.2) is a partial case of (4.13) for \( F(\omega_A) = H(\omega_A) \).

Proposition 1 in [5] and Proposition 2 imply the following observations.

**Proposition 3.** A) The quantity \( H_e(A|B)_\omega \) defined by (5.2) is an \( \mathcal{F} \)-extension of the quantum conditional entropy (5.1) to the set \( \{ \omega_{AB} \mid H(\omega_A) < +\infty \} \); it has properties B1–B3) and satisfies \( |H_e(A|B)_\omega| \leq H(\omega_A) \).

B) The local continuity of \( H(\omega_A) \) implies the local continuity of \( H_e(A|B)_\omega \).

The continuity bound for the conditional entropy defined by (5.1) under the condition \( \dim \mathcal{H}_A < +\infty \) was originally obtained by Alicki and Fannes in [13] and was then strengthened by Winter [14]. Proposition 1 gives a continuity bound for the extended quantum conditional entropy which coincides with Winter’s\(^9\).

**Corollary 5.** If the system \( A \) is finite dimensional then \( H_e(A|B)_\omega \) is a continuous bounded function on the set \( \mathcal{S}(\mathcal{H}_{AB}) \) and

\[
|H_e(A|B)_{\omega^1} - H_e(A|B)_{\omega^2}| \leq 2\varepsilon \log \dim \mathcal{H}_A + (1 + \varepsilon) h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)
\]

for any \( \omega^1, \omega^2 \in \mathcal{S}(\mathcal{H}_{AB}) \), where \( \varepsilon = \frac{1}{2} \|\omega^1 - \omega^2\|_1 \) and \( h_2 \) is the binary entropy.

In addition to the above-mentioned direct applications the function \( H_e(A|B)_\omega \) can be used to construct \( \mathcal{F} \)-extensions for linear combinations of marginal entropies of a special form. By Remark 4 the function

\[
\omega_{A_1...A_n} \mapsto \sum_k \alpha_k H_e(X_k|Y_k)_\omega,
\]

where the \( X_k \) and the \( Y_k \) are disjoint subsystems of \( A_1...A_n \), is faithful on the set of all states for which \( H(\omega_{X_k}) < +\infty \) for all \( k \). Therefore, Proposition 3 implies the following.

**Proposition 4.** If a quantity \( F(\omega_{A_1...A_n}) = \sum_k \alpha_k H(\omega_{X_k}) \) can be represented as follows:

\[
F(\omega_{A_1...A_n}) = \beta_1 H(\omega_{A_{10}}) + \sum_{k > 1} \beta_k [H(\omega_{A_{i0}Y_k}) - H(\omega_{Y_k})],
\]

where \( Y_k \) is a subsystem of \( A_1...A_n\backslash A_{i0} \) for each \( k \), then \( F \) has the \( \mathcal{F} \)-extension

\[
\hat{F}(\omega_{A_1...A_n}) = \beta_1 H(\omega_{A_{10}}) + \sum_{k > 1} \beta_k H_e(A_{i0}|Y_k)_\omega
\]

to the set \( \{ \omega_{A_1...A_n} \mid H(\omega_{A_{10}}) < +\infty \} \), which has the following properties:

---

\(^8\)This extension of the quantum conditional entropy plays a significant role in the proof of the generalized version of the Bennett-Shor-Smolin-Thaplyal theorem given in [22].

\(^9\)So Corollary 5 generalizes Winter’s continuity bound to the case when \( H(\omega_B) = +\infty \). It can be derived directly from Lemma 2 in [14] by means of approximation.
Proposition 4, it has an
Remark 6.
possesses the above properties 1) and 2) (with form (5.3) for some index $\omega$)
of a tripartite finite-dimensional system

Remark 7. continuous entropy does not destroy
{

This quantity plays an important role in different branches of quantum information

Corollary 6. shows, roughly speaking, that reducing a system with finite (con-

Corollary 6 shows, roughly speaking, that reducing a system with finite (continuous) entropy does not destroy the finiteness (continuity, respectively) of any quantity $F$ defined as a linear combination of marginal entropies.

Remark 7. Corollary 6 does not assert that there exist separate $\mathcal{F}$-extensions to

§ 6. Conditional mutual information

6.1. Tripartite systems. The conditional mutual information of a state $\omega_{ABC}$
of a tripartite finite-dimensional system $ABC$ is defined by

$$I(A:C|B)_\omega \doteq H(\omega_{AB}) + H(\omega_{BC}) - H(\omega_{ABC}) - H(\omega_{B}).$$

This quantity plays an important role in different branches of quantum information theory [9], [17], [19], [20], [23]–[25]. It has the following basic properties:
C1) \( I(A:C|B)_\omega \geq 0 \) for any state \( \omega_{ABC} \) and \( I(A:C|B)_\omega = 0 \) if and only if there exists a channel \( \Phi: B \to BC \) such that \( \omega_{ABC} = \text{Id}_A \otimes \Phi(\omega_{AB}) \) (see [26]);

C2) monotonicity under local conditioning: \( I(AB:C)_\omega \geq I(A:C|B)_\omega \);

C3) monotonicity under local operations\(^{10}\):

\[
I(A:C|B)_\omega \geq I(A':C'|B)_\Phi_A \otimes \text{Id}_B \otimes \Phi_C(\omega)
\]

for arbitrary quantum operations \( \Phi_A: A \to A' \) and \( \Phi_C: C \to C'; \)

C4) additivity: \( I(AA':CC'|BB')_{\omega \otimes \omega'} = I(A:C|B)_\omega + I(A':C'|B')_{\omega'} ; \)

C5) duality: \( I(A:C|B)_\omega = I(A:C|D)_\omega \) for any pure state \( \omega_{ABCD} \) (see [24]).

The nonnegativity of \( I(A:C|B)_\omega \) is a basic result in quantum information theory, known as the strong subadditivity of von Neumann entropy [11].

The conditional mutual information (6.1) can be represented by one of the formulae

\[
I(A:C|B)_\omega = I(A:BC)_\omega - I(A:B)_\omega , \tag{6.2}
\]

\[
I(A:C|B)_\omega = I(AB:C)_\omega - I(B:C)_\omega . \tag{6.3}
\]

Using these representations, the nonnegativity of \( I(A:C|B) \) is a direct corollary of the monotonicity of relative entropy under partial trace\(^ {11} \).

Formula (4.12) in this case implies

\[
I(A:C|B)_\omega = I(A:C)_\omega - I(A:B)_\omega - I(C:B)_\omega + I(AC:B)_\omega . \tag{6.4}
\]

The quantity \( I(A:C|B)_\omega \) defined in (6.1) can be also represented as follows

\[
I(A:C|B)_\omega = I(A:C)_\omega + I(AB:D)_{\tilde{\omega}} + I(BC:D)_{\tilde{\omega}} + I(AC:D)_{\tilde{\omega}} - 4H(\omega_{ABC}) , \tag{6.5}
\]

where \( \tilde{\omega} = \tilde{\omega}_{ABCD} \) is any purification of the state \( \omega_{ABC} \).

In infinite dimensions the quantity \( I(A:C|B)_\omega \) is well defined by (6.1) as a faithful function (see Definition 1) on the set

\[
\mathcal{G}_0 = \{ \omega_{ABC} \mid H(\omega_{ABC}) < +\infty , H(\omega_B) < +\infty \}.
\]

By Remark 4 formulæ (6.2)–(6.5) define \( \mathfrak{F} \)-extensions of (6.1) to the sets

\[
\mathcal{G}_1 = \{ \omega_{ABC} \mid I(A:B)_{\omega} < +\infty \} , \quad \mathcal{G}_2 = \{ \omega_{ABC} \mid I(B:C)_{\omega} < +\infty \} ,
\]

\[
\mathcal{G}_3 = \{ \omega_{ABC} \mid H(\omega_B) < +\infty \} \quad \text{and} \quad \mathcal{G}_4 = \{ \omega_{ABC} \mid H(\omega_{ABC}) < +\infty \} ,
\]

respectively.

\(^{10}\)If either \( \Phi_A \) or \( \Phi_C \) is a trace nonpreserving operation then \( I(A':C'|B)_{\Phi_A \otimes \text{Id}_B \otimes \Phi_C(\omega)} \) is defined by (6.1), where \( H \) is the extended von Neumann entropy (2.1); see Remark 2.

\(^{11}\)The monotonicity of quantum relative entropy and the strong subadditivity of von Neumann entropy are globally equivalent [1].
The following theorem shows that formulae (6.1)–(6.5) agree with one another (coincide on the sets $\mathcal{S}_i \cap \mathcal{S}_j$) and can be extended to a unique lower semicontinuous function on the set $\mathcal{S}(\mathcal{H}_{ABC})$ which possesses the basic properties of conditional mutual information.

**Theorem 2.** There exists a unique lower semicontinuous function $I_e(A:C|B)_\omega$ on the set $\mathcal{S}(\mathcal{H}_{ABC})$ such that:

1) $I_e(A:C|B)_\omega$ coincides with $I(A:C|B)_\omega$ given by (6.1)–(6.5) on the sets $\mathcal{S}_0 \cap \mathcal{S}_4$, respectively;

2) $I_e(A:C|B)_\omega$ has properties C1–C5 of conditional mutual information stated above.

This function can be defined by one of the equivalent expressions\(^{12}\)

\[
I_e(A:C|B)_\omega = \sup_{P_A} [I(A:BC)_{\omega Q} - I(A:B)_{\omega Q}], \quad Q = P_A \otimes I_B \otimes I_C, \quad (6.6)
\]

\[
I_e(A:C|B)_\omega = \sup_{P_C} [I(AB:C)_{\omega Q} - I(B:C)_{\omega Q}], \quad Q = I_A \otimes I_B \otimes P_C, \quad (6.7)
\]

where the suprema are taken over all finite-rank projections $P_X \in \mathcal{B}(\mathcal{H}_X), X = A, C$.

The function $I_e(A:C|B)_\omega$ satisfies the condition for an $\mathcal{S}$-extension (3.2) for any state $\omega_{ABC}$ such that\(^{13}\) min\{$I(A:B)_\omega, I(B:C)_\omega, H(\omega_{ABC}), H(\omega_B)$\} < $+\infty$. For an arbitrary state $\omega \in \mathcal{S}(\mathcal{H}_{ABC})$ the following weaker property is valid:

\[
I_e(A:C|B)_\omega = \lim_{k \to \infty} \lim_{l \to \infty} I_e(A:C|B)_{\omega_{kl}}, \quad (6.8)
\]

where

\[
\omega_{kl} = \lambda_{kl}^{-1} Q_{kl} \omega_{kl}, \quad Q_{kl} = P_A^k \otimes P_B^l \otimes P_C^k, \quad \lambda_{kl} = \text{Tr} Q_{kl} \omega,
\]

and $\{P_A^k\}_k \subset \mathcal{B}(\mathcal{H}_A), \{P_B^l\}_l \subset \mathcal{B}(\mathcal{H}_B)$ and $\{P_C^k\}_k \subset \mathcal{B}(\mathcal{H}_C)$ are sequences of projections converging strongly to the identity operators $I_A, I_B$ and $I_C$ such that min\{rank $P_A^k$, rank $P_C^k$\} < $+\infty$ for all $k$.

Theorem 2 (proved in §9) shows that the function $I_e(A:C|B)_\omega$ can be regarded as an extension of the conditional mutual information to the set of all states of infinite-dimensional tripartite system. So, in what follows we will denote it by $I(A:C|B)_\omega$ (omitting the subscript e).

**Remark 8.** Using formulae (6.2)–(6.5), the upper bound (4.3) and Proposition 2 it is easy to show that $\frac{1}{2} I(A:C|B)_\omega$ is bounded above by each of the quantities

\[
H(\omega_A), H(\omega_C), H(\omega_{AB}), H(\omega_{BC}), H(\omega_B) + \frac{1}{2} I(A:C), H(\omega_{ABC}) + \frac{1}{2} I(A:C).
\]

The following corollary shows that the local continuity of at least one of these ‘upper bounds’ implies the local continuity of $I(A:C|B)_\omega$.

---

\(^{12}\) In accordance with Remark 2 we regard the mutual information $I(X:Y)_\omega$ as a function on the cone $C_{+}(\mathcal{H}_{XY})$, so that $I(X:Y)_{Q_{\omega Q}} = [\text{Tr} Q_{\omega} I(X:Y)_{Q_{\omega Q}}/\text{Tr} Q_{\omega}$.

\(^{13}\) It follows from (4.3) that this condition can be replaced by the stronger but more explicit condition min\{$H(\omega_A), H(\omega_B), H(\omega_C), H(\omega_{AB}), H(\omega_{BC}), H(\omega_{ABC})$\} < $+\infty$.\n
Corollary 7. A) The local continuity of one of the marginal entropies $H(\omega_A)$, $H(\omega_C)$, $H(\omega_{AB})$ and $H(\omega_{BC})$ implies the local continuity of the function $I(A;C|B)_\omega$. The local continuity of one of the marginal entropies $H(\omega_B)$ and $H(\omega_{ABC})$ implies the local continuity of the difference $I(A:C|B)_\omega - I(A:C)_\omega$.

B) If $\{\omega^k\}$ is a sequence of states in $\mathcal{S}(\mathcal{H}_{ABC})$ converging to a state $\omega^0$ such that $\lambda_k \omega^k \leq \Phi_A^k \otimes \text{Id}_B \otimes \Phi_C^k(\omega^0)$ for some sequences $\{\Phi_A^k\}$ and $\{\Phi_C^k\}$ of local quantum operations and some sequence $\{\lambda_k\} \subset [0,1]$ converging to 1 then

$$\lim_{k \to \infty} I(A:C|B)_{\omega^k} = I(A:C|B)_{\omega^0} \leq +\infty.$$ 

Proof. A) If either $H(\omega_A)$ or $H(\omega_C)$ is continuous on a subset $\mathcal{A}$ of $\mathcal{S}(\mathcal{H}_{ABC})$ then the continuity of $I(A:C|B)_\omega$ on $\mathcal{A}$ follows from part A) of Theorem 1 and formulae (6.2) and (6.3), respectively.

If $H(\omega_{AB})$ or $H(\omega_{BC})$ is continuous on a subset $\mathcal{A}$ then part A) of Theorem 1 implies the first term in (6.3) or in (6.2), respectively, is continuous on $\mathcal{A}$. By Lemma 3 the continuity of $I(A:C|B)_\omega$ on $\mathcal{A}$ follows because it is lower semicontinuous (Theorem 2), as are $I(A:B)_\omega$ and $I(B:C)_\omega$.

If $H(\omega_B)$ or $H(\omega_{ABC})$ is continuous on a subset $\mathcal{A}$ then the continuity of the difference $I(A:C|B)_\omega - I(A:C)_\omega$ on $\mathcal{A}$ follows from part A) of Theorem 1 and formulae (6.4) or (6.5), respectively (in the second case Lemma 1 and the equality $H(\omega_{ABC}) = H(\omega_D)$ are also used).

B) We will use the inequality

$$\lambda I(A:C|B)_\rho + (1 - \lambda)I(A:C|B)_\sigma \leq I(A:C|B)_{\lambda \rho + (1 - \lambda) \sigma} + 2h_2(\lambda), \quad (6.9)$$

where $h_2(\lambda)$ is the binary entropy; this holds for any operators $\rho, \sigma \in \mathfrak{S}_+(\mathcal{H}_{ABC})$ such that $\max\{\text{Tr} \rho, \text{Tr} \sigma\} \leq 1$. If all the marginal entropies of the operators $\rho$ and $\sigma$ are finite then (6.9) follows from (6.1) and (2.3). In the general case (6.9) can be proved by approximating the operators $\rho$ and $\sigma$ by the double sequences of operators

$$\rho_{kl} = Q_{kl} \rho Q_{kl} \quad \text{and} \quad \sigma_{kl} = Q_{kl} \sigma Q_{kl}, \quad Q_{kl} = P^A_k \otimes P^B_l \otimes P^C_l,$$

where $\{P^A_k\} \subset \mathfrak{B}(\mathcal{H}_A)$, $\{P^B_l\} \subset \mathfrak{B}(\mathcal{H}_B)$ and $\{P^C_l\} \subset \mathfrak{B}(\mathcal{H}_C)$ are sequences of finite-rank projections converging strongly to the identity operators $I_A, I_B$ and $I_C$. Since (6.9) holds for the operators $\rho_{kl}$ and $\sigma_{kl}$ for all $k$ and $l$, using property (6.8) we can show that (6.9) is valid for the operators $\rho$ and $\sigma$.

Inequality (6.9) and the nonnegativity and monotonicity of the conditional mutual information under local operations show that

$$\lambda_k I(A:C|B)_{\omega^k} \leq I(A:C|B)_{\Phi_A^k \otimes \text{Id}_B \otimes \Phi_C^k(\omega^0)} + 2h_2(\lambda_k') \leq I(A:C|B)_{\omega^0} + 2h_2(\lambda_k'),$$

where $\lambda_k' = \lambda_k [\text{Tr} \Phi_A^k \otimes \text{Id}_B \otimes \Phi_C^k(\omega^0)]^{-1} \geq \lambda_k$. This inequality and the lower semicontinuity of the conditional mutual information (Theorem 2) imply the required limit relation.

The corollary is proved.

\textsuperscript{14}This means that

$$\lim_{k \to \infty} H(\omega_X^k) = H(\omega_X^0) < +\infty \implies \lim_{k \to \infty} I(A:C|B)_{\omega^k} = I(A:C|B)_{\omega^0} < +\infty$$

for any sequence $\{\omega^k\}$ converging to a state $\omega^0$, where $X$ is one of the systems $A$, $C$, $AB$ and $BC$. 


Corollary 7 shows, in particular, that the conditional mutual information $I(A:C|B)_\omega$ is continuous on the set $\mathcal{S}(|\mathcal{H}_{ABC})$ if either $A$ or $C$ is a finite-dimensional system. Proposition 1 and Remark 8 give a continuity bound for $I(A:C|B)_\omega$ in this case.

**Corollary 8.** If one of the systems $A$ and $C$ (for instance, $A$) is finite-dimensional then $I(A:C|B)_\omega$ is a continuous bounded function on the set $\mathcal{S}(|\mathcal{H}_{ABC})$ and

$$|I(A:C|B)_{\omega^1} - I(A:C|B)_{\omega^2}| \leq 2\varepsilon \log \dim \mathcal{H}_A + 4(1+\varepsilon)h_2\left(\frac{\varepsilon}{1+\varepsilon}\right)$$

for any $\omega^1, \omega^2 \in \mathcal{S}(|\mathcal{H}_{ABC})$, where $\varepsilon = \frac{1}{2}\|\omega^1 - \omega^2\|_1$ and $h_2$ is the binary entropy. If the system $B$ is finite-dimensional then the difference $I(A:C|B)_\omega - I(A:C)_\omega$ is uniformly continuous and bounded on the set $\mathcal{S}(|\mathcal{H}_{ABC})$.

We will use the following analogue of Corollary 2.

**Corollary 9.** Let $X$, $Y$ and $Z$ be disjoint subsystems of $A_1 \ldots A_n$ with an empty intersection, and let $R = A_1 \ldots A_n \setminus X Y Z$. Then the function $\omega_{A_1 \ldots A_n} \mapsto I(X:Z|Y)_\omega$ satisfies the condition for an $\mathcal{S}$-extension (3.2) for any state $\omega_{A_1 \ldots A_n}$ such that

$$\min\{H(\omega_X), H(\omega_Y), H(\omega_Z), H(\omega_{XY}), H(\omega_{YZ}), H(\omega_{XYZ})\} < +\infty. \quad (6.10)$$

For an arbitrary state $\omega \in \mathcal{S}(|\mathcal{H}_{A_1 \ldots A_n})$ the following weaker property is valid:

$$I(X:Z|Y)_\omega = \lim_{k \to \infty} \lim_{l \to \infty} I(X:Z|Y)_{\omega_{klt}} \quad (6.11)$$

for $t = k$ and for $t = l$, where

$$\omega_{klt} = \lambda_{klt}^{-1} Q_{klt} \omega Q_{klt}, \quad Q_{klt} = P_X^k \otimes P_Y^l \otimes P_Z^k \otimes P_R^l, \quad \lambda_{klt} = \text{Tr} Q_{klt} \omega,$$

and $\{P_X^k\}_k \subset \mathcal{B}(|\mathcal{H}_X), \{P_Y^l\}_l \subset \mathcal{B}(|\mathcal{H}_Y), \{P_Z^k\}_k \subset \mathcal{B}(|\mathcal{H}_Z)$ and $\{P_R^l\}_l \subset \mathcal{B}(|\mathcal{H}_R)$ are sequences of projections converging strongly to the identity operators $I_X, I_Y, I_Z$ and $I_R$ such that $\min\{\text{rank} P_X^k, \text{rank} P_Z^k\} < +\infty$ for all $k$.

**Proof.** We will consider the cases $t = k$ and $t = l$ simultaneously. Since

$$\lambda_{klt} \omega_{XYZ}^{klt} \leq \bar{\omega}_{XYZ} \leq P_X^k \otimes P_Y^l \otimes P_Z^k \cdot \omega_{XYZ} \cdot P_X^k \otimes P_Y^l \otimes P_Z^k, \quad (6.12)$$

inequality (6.9) and the nonnegativity of $I(X:Z|Y)$ imply that

$$\lambda_{klt} I(X:Z|Y)_{\omega_{klt}} \leq I(X:Z|Y)_{\bar{\omega}} + 2h_2(\lambda_{klt}[\text{Tr} \bar{\omega}^{kl}]^{-1}). \quad (6.13)$$

Since Theorem 2 shows that $\lim_{k \to \infty} I(X:Z|Y)_{\bar{\omega}^{kk}} = I(X:Z|Y)_\omega$ for any state $\omega_{A_1 \ldots A_n}$ satisfying (6.10), the first assertion of the corollary follows from (6.13) and the lower semicontinuity of the function $\omega_{A_1 \ldots A_n} \mapsto I(X:Z|Y)_\omega$. To prove property (6.11) we set $\omega^{kl} = \lim_{l \to \infty} \omega_{klt}$ and $\bar{\omega}^{kl} = \lim_{l \to \infty} = \bar{\omega}^{kl}$.

By the condition $\min\{\text{rank} P_X^k, \text{rank} P_Z^k\} < +\infty$ Corollary 8 implies

$$\lim_{l \to \infty} I(X:Z|Y)_{\omega_{klt}} = I(X:Z|Y)_{\omega^{kl}}.$$
Because the conditional mutual information is lower semicontinuous and monotonic under local operations, we have

\[
\lim_{k \to \infty} I(X:Z|Y)_{\omega^k} = I(X:Z|Y)_\omega.
\]

Since inequalities (6.12) and (6.13) hold with \( l = * \) if we set \( P^*_Y = I_Y \) and \( \lambda_{k*} = \lim_{l \to \infty} \lambda_{klt} \), the above limit relation and the lower semicontinuity of the function \( \omega_{A_1 \ldots A_n} \mapsto I(X:Z|Y)_{\omega} \) imply that \( \lim_{k \to \infty} I(X:Z|Y)_{\omega^k} = I(X:Z|Y)_\omega \).

The lower semicontinuity of \( I(A:C|B)_\omega \) implies the following observations concerning quantum mutual information.

**Corollary 10.** A) The function \( \omega \mapsto [I(A:BC)_\omega - I(A:B)_\omega] \) is lower semicontinuous on the set \( \{ \omega \in \mathcal{T}_+(\mathcal{H}_{ABC}) \mid I(A:B)_\omega < +\infty \} \).

B) The local continuity of the function \( \omega_{ABC} \mapsto I(A:BC)_\omega \) implies the local continuity of the function \( \omega_{ABC} \mapsto I(A:B)_\omega \).

C) The local continuity of the function \( \omega_{ABC} \mapsto H(\omega_{BC}) \) implies the local continuity of function \( \omega_{ABC} \mapsto I(A:B)_\omega \).

**Proof.** Assertion A) follows from Theorem 2 and (6.2). By Lemma 3 assertion B) follows from A) and the lower semicontinuity of quantum mutual information. Assertion C) follows from B) and part A) of Theorem 1.

### 6.2. Multipartite systems.

The conditional mutual information of a state \( \omega_{A_1 \ldots A_nB} \) of a finite-dimensional multipartite system \( A_1 \ldots A_nB \) is defined by

\[
I(A_1: \ldots :A_n|B)_\omega = \sum_{i=1}^{n} H(A_i|B)_\omega - H(A_1 \ldots A_n|B)_\omega
= \sum_{i=1}^{n} H(\omega_{A_iB}) - H(\omega_{A_1 \ldots A_nB}) - (n-1)H(\omega_B). \tag{6.14}
\]

The analogues of properties C1)–C4), given above, for the tripartite conditional mutual information can be proved for \( I(A_1: \ldots :A_n|B)_\omega \) using the representation

\[
I(A_1: \ldots :A_n|B)_\omega = I(A_2:A_1|B)_\omega + I(A_3:A_1A_2|B)_\omega + \ldots + I(A_n:A_1 \ldots A_{n-1}|B)_\omega \tag{6.15}
\]

(see [20]) and modifications of it obtained by permuting the indices on the right-hand side.

In this case formula (4.12) shows that

\[
I(A_1: \ldots :A_n|B)_\omega - I(A_1: \ldots :A_n)_\omega = I(A_1 \ldots A_n:B)_\omega - \sum_{i=1}^{n} I(A_i:B)_\omega. \tag{6.16}
\]

Since (6.15) is valid with arbitrarily permuted indices 1, \ldots , n on the right-hand side, Remark 8 implies that \( I(A_1: \ldots :A_n|B)_\omega \) is bounded above by the quantity

\[
2 \min_{1 \leq j \leq n} \sum_{i \neq j} \min\{ H(\omega_{A_i}), H(\omega_{A_iB}) \}. \tag{6.17}
\]
Using the representation (6.15) and the extended conditional mutual information described in Theorem 2 we can define $I(A_1 : \ldots : A_n|B)_\omega$ for any state of an infinite-dimensional system $A_1 \ldots A_n B$.

**Proposition 5.** A) The quantity $I(A_1 : \ldots : A_n|B)_\omega$ in (6.14) has a lower semi-continuous extension to the set $\mathcal{G}(\mathcal{H}_{A_1 \ldots A_n B})$ which has analogues of properties (C1)–(C4), above, of the conditional mutual information and is bounded above by the quantity (6.17). This extension can be defined by (6.15) in which each term $I(X:Y|B)_\omega$ coincides with the function $I_e(X:Y|B)_\omega$ described in Theorem 2.

B) This extension (also denoted $I(A_1 : \ldots : A_n|B)_\omega$) satisfies condition (3.2) for an $\mathfrak{F}$-extension for any state $\omega_{A_1 \ldots A_n B}$ such that either (6.17) or $H(\omega_B)$ is finite. For an arbitrary state $\omega \in \mathcal{G}(\mathcal{H}_{A_1 \ldots A_n B})$ the following weaker property holds:

$$I(A_1 : \ldots : A_n|B)_\omega = \lim_{k \to \infty} \lim_{l \to \infty} I(A_1 : \ldots : A_n|B)_{\omega^{kl}}, \quad (6.18)$$

where

$$\omega^{kl} = \lambda_{kl}^{-1} Q_{kl}\omega Q_{kl}, \quad Q_{kl} = P^{k}_{A_1} \otimes \cdots \otimes P^{k}_{A_n} \otimes P^{l}_{B}, \quad \lambda_{kl} = \text{Tr} Q_{kl}\omega,$$

and $\{P^{k}_{A_i}\}_{k} \subset \mathfrak{B}(\mathcal{H}_{A_i})$, $i = 1, \ldots, n$, and $\{P^{l}_{B}\}_{l} \subset \mathfrak{B}(\mathcal{H}_{B})$ are sequences of projections which converge strongly to the identity operators $I_{A_i}$ and $I_B$, and are such that $\min_{1 \leq j \leq n} \sum_{i \neq j} \text{rank} P^{k}_{A_i} < +\infty$ for all $k$.

C) The representation (6.15) is valid for $I(A_1 : \ldots : A_n|B)_\omega$ for any permutation of the indices $1, \ldots, n$ on the right-hand side (provided each summand $I(X:Y|Z)_\omega$ coincides with $I_e(X:Y|Z)_\omega$).

D) The local continuity of the $n-1$ marginal entropies $H(\omega_{X_{i_k}}), \ldots, H(\omega_{X_{i_{n-1}}})$, where $X_{i_k}$ is either $A_{i_k}$ or $A_{i_k} B$, implies the local continuity of $I(A_1 : \ldots : A_n|B)_\omega$. The local continuity of the marginal entropy $H(\omega_B)$ implies the local continuity of the difference $I(A_1 : \ldots : A_n|B)_\omega - I(A_1 : \ldots : A_n)_\omega$, which has the representation (6.16).

E) If $n-1$ subsystems, for instance, $A_1, \ldots, A_{n-1}$, are finite-dimensional then $I(A_1 : \ldots : A_n|B)_\omega$ is a continuous bounded function on the set $\mathcal{G}(\mathcal{H}_{A_1 \ldots A_n B})$ and

$$|I(A_1 : \ldots : A_n|B)_{\omega^1} - I(A_1 : \ldots : A_n|B)_{\omega^2}| \leq 2\varepsilon C + 2n(1 + \varepsilon)h_2 \left(\frac{\varepsilon}{1 + \varepsilon}\right)$$

for any $\omega^1, \omega^2 \in \mathcal{G}(\mathcal{H}_{A_1 \ldots A_n B})$, where $\varepsilon = \frac{1}{2} \|\omega^1 - \omega^2\|_1$, $h_2$ is the binary entropy and $C = \log \dim \mathcal{H}_{A_1 \ldots A_{n-1}}$.

**Proof.** Assertion A) follows from Theorem 2, Remark 8 and assertion C), which is proved below.

In the case when $\sum_{i=2}^{n} \text{rank} P^{k}_{A_i} < +\infty$ for all $k$, property (6.18) follows from the second assertion of Corollary 9, that is, limit relation (6.11), applied to each term on the right-hand side of (6.15). This property and (6.11) make it possible to prove assertion C) via approximation by noting that it holds for any state $\omega_{A_1 \ldots A_n B}$ such that rank $\omega_{A_i} < +\infty$ for all $i$ and rank $\omega_B < +\infty$.

Now using assertion C) we can prove B) by applying Corollary 9 to each term on the right-hand side of (6.15) with appropriately permuted indices $1, \ldots, n$ (using Lemmas 5 and 6).
The first part of assertion D) follows from part A) of Corollary 7, applied to each term on the right-hand side of (6.15) with appropriately permuted indices 1, . . . , n. The second part of assertion D) follows from Proposition 2.

Assertion E) follows from Proposition 1 and upper bound (6.17).

§ 7. Other information measures in multipartite systems

7.1. A quantum version of the interaction information. Consider the following characteristic of the state \( \omega_{A_1 \ldots A_n} \) of an \( n \)-partite quantum system

\[
I_n(\omega_{A_1 \ldots A_n}) = \sum_i H(\omega_{A_i}) - \sum_{i<j} H(\omega_{A_iA_j}) + \sum_{i<j<k} H(\omega_{A_iA_jA_k}) - \cdots - (-1)^n H(\omega_{A_1 \ldots A_n}), \tag{7.1}
\]

which (up to sign) can be regarded as a noncommutative version of the interaction information of a \( n \)-partite classical system [27], [28].

Note that \( I_1(\omega_A) \) is the von Neumann entropy of a one-partite state \( \omega_A \), \( I_2(\omega_{AB}) \) is the quantum mutual information of a bipartite state \( \omega_{AB} \), while

\[
I_3(\omega_{ABC}) = H(\omega_A) + H(\omega_B) + H(\omega_C) - H(\omega_{AB}) - H(\omega_{AC}) - H(\omega_{BC}) + H(\omega_{ABC})
\]

is the topological entanglement entropy of a tripartite state \( \omega_{ABC} \) typically denoted by \( H_{\text{topo}}(\omega_{ABC}) \) (or \( S_{\text{topo}}(\omega_{ABC}) \)) [29]. Even though it may be negative, this quantity is also used as a special measure for quantum correlations [30], [31].

An interesting feature of the linear combinations of marginal entropies in (7.1) consists in the fact that for any \( n \) the finiteness of just one marginal entropy \( H(\omega_{A_i}) \) ‘eliminates’ all possible uncertainties \( -\infty < \infty \) in (7.1), while the continuity of \( H(\omega_{A_i}) \) guarantees the continuity of \( I_n(\omega_{A_1 \ldots A_n}) \).

**Proposition 6.** A) The quantity \( I_n(\omega_{A_1 \ldots A_n}) \) in (7.1) has a finite \( \mathcal{F} \)-extension\(^\text{15} \) to the set \( \{\omega_{A_1 \ldots A_n} \mid \exists i : H(\omega_{A_i}) < +\infty\} \) such that

\[
|I_n(\omega_{A_1 \ldots A_n})| \leq 2^{n-1} \min\{H(\omega_{A_1}), \ldots, H(\omega_{A_n})\}.
\]

B) If \( H(\omega_{A_i}) < +\infty \) then this \( \mathcal{F} \)-extension is given by

\[
I_n(\omega_{A_1 \ldots A_n}) = H(\omega_{A_i}) - \sum_j H_\phi(A_i|A_j)_\omega + \sum_{j<k} H_\phi(A_i|A_kA_j)_\omega - \sum_{j<k<l} H_\phi(A_i|A_jA_kA_l)_\omega + \cdots + (-1)^{n-1} H_\phi(A_i|A_1 \ldots A_{i-1}A_{i+1} \ldots A_n)_\omega, \tag{7.2}
\]

where all the indices \( k, j, l, \ldots \) in each sum run over the set \( \{1, \ldots, n\} \setminus \{i\} \) and \( H_\phi(X|Y)_\omega \) is the extended quantum conditional entropy defined by (5.2).

C) The local continuity of one of the marginal entropies \( H(\omega_{A_1}), \ldots, H(\omega_{A_n}) \) implies the local continuity of \( I_n(\omega_{A_1 \ldots A_n}) \).

\(^\text{15} \)We denote this \( \mathcal{F} \)-extension of \( I_n(\omega_{A_1 \ldots A_n}) \) by the same symbol.
Proof. A) Assume that $H(\omega_{A_n})$ is finite. Consider the quantity
\[
F(\omega_{A_1...A_n}) = H(\omega_{A_n}) + I_{n-1}(\omega_{A_1...A_{n-1}}) - I_n(\omega_{A_1...A_n}) = \sum_{i<n} H(\omega_{A_iA_n})
- \sum_{i<j<n} H(\omega_{A_iA_jA_n}) + \sum_{i<j<k<n} H(\omega_{A_iA_jA_kA_n}) - \cdots - (-1)^{n-1} H(\omega_{A_1...A_n}).
\]
In terms of Corollary 6 we have $F_{\setminus A_n} = I_{n-1}$ and hence
\[
I_n(\omega_{A_1...A_n}) = H(\omega_{A_n}) + [F_{\setminus A_n} - F](\omega_{A_1...A_n}). \quad (7.3)
\]
So Corollary 6 implies that an $\mathfrak{F}$-extension of the quantity $I_n(\omega_{A_1...A_n})$ defined in (7.1) to the set $\{\omega_{A_1...A_n} \mid H(\omega_{A_n}) < +\infty\}$ exists, and is determined by formula (7.2) with $i = n$ such that
\[
|I_n(\omega_{A_1...A_n})| \leq 2^n H(\omega_{A_n}).
\]
To complete the proof of A) it suffices to note that the $\mathfrak{F}$-extensions of $I_n$ to the sets $\{\omega_{A_1...A_n} \mid H(\omega_{A_i}) < +\infty\}$ and $\{\omega_{A_1...A_n} \mid H(\omega_{A_j}) < +\infty\}$ agree with each other by Lemma 5.

B) If the function $\omega_{A_1...A_n} \mapsto H(\omega_{A_n})$ is continuous on a set $\mathcal{A}$ then Corollary 6 and (7.3) imply that $\omega_{A_1...A_n} \mapsto I_n(\omega_{A_1...A_n})$ is continuous on $\mathcal{A}$.

The proposition is proved.

Remark 9. For given $n$ the general formula (7.2) can be simplified. For example, the $\mathfrak{F}$-extension of the topological entanglement entropy $I_3(\omega_{A_1A_2A_3})$ to the set $\{\omega_{A_1A_2A_3} \mid H(\omega_{A_1}) < +\infty\}$ can be expressed as
\[
I_3(\omega_{A_1A_2A_3}) = I(A_1:A_2)_\omega - I(A_1:A_2|A_3)_\omega,
\]
where $I(A_1:A_2|A_3)_\omega$ is the extended conditional mutual information.

Propositions 1 and 6 imply the following result.

Corollary 11. If one of the systems $A_1 \ldots A_n$, say $A_i$, is finite-dimensional then $I_n(\omega_{A_1...A_n})$ is a continuous bounded function on the set $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ and
\[
|I_n(\omega^1) - I_n(\omega^2)| \leq 2^n \varepsilon \log \dim \mathcal{H}_{A_i} + (2^n - 1)(1 + \varepsilon) h_2 \left( \frac{\varepsilon}{1 + \varepsilon} \right)
\]
for any $\omega^1, \omega^2 \in \mathcal{G}(\mathcal{H}_{A_1...A_n})$, where $\varepsilon = \frac{1}{2} \|\omega^1 - \omega^2\|_1$ and $h_2$ is the binary entropy.

7.2. The secrecy monotone $S_n$ (unconditional and conditional). Along with the quantum mutual information $I(A_1: \ldots : A_n)$ the secrecy monotone
\[
S_n(A_1: \ldots : A_n)_\omega = \sum_{i=1}^n H(\omega_{A_1...A_{i-1}A_{i+1}...A_n}) - (n - 1) H(\omega_{A_1...A_n})
\]
was proposed in [18] as a measure of quantum correlations of a state $\omega_{A_1...A_n}$ of a finite-dimensional $n$-partite system. Note that $S_2(A_1:A_2)_\omega = I(A_1:A_2)_\omega$, so

\[\text{16} \text{The same quantity was independently proposed and analyzed in [32], where it was called the ‘operational quantum mutual information’.}\]
that the quantity $S_n$ can be considered as a particular $n$-partite generalization of the bipartite quantum mutual information. It can be expressed as follows:

$$
S_n(A_1: \ldots : A_n) = I(A_1:A_2 \ldots A_n) + I(A_2:A_3 \ldots A_n|A_1) + \ldots + I(A_n-1:A_n|A_1 \ldots A_{n-2}).
$$

(7.4)

The conditional version of $S_n$, that is, the characteristic

$$
S_n(A_1: \ldots : A_n|B) = \sum_{i=1}^{n} H(\omega_{A_1 \ldots A_{i-1} A_{i+1} \ldots A_n B}) - (n - 1) H(\omega_{A_1 \ldots A_n B}) - H(\omega_B)
$$

(7.5)

of a state $\omega_{A_1 \ldots A_n B}$, is also used in applications [20]. So we will consider an infinite-dimensional generalization of the quantity $S_n(A_1: \ldots : A_n|B)$, bearing in mind that $S_n(A_1: \ldots : A_n)$ is a particular case of $S_n(A_1: \ldots : A_n|B)$ for trivial $B$.

The conditional secrecy monotone $S_n$ can be represented by conditioning (7.4) as follows

$$
S_n(A_1: \ldots : A_n|B) = I(A_1:A_2 \ldots A_n|B) + I(A_2:A_3 \ldots A_n|A_1 B) + \ldots + I(A_n-1:A_n|A_1 \ldots A_{n-2} B).
$$

(7.6)

Basic properties of the conditional quantum mutual information show that the quantity $S_n(A_1 : \ldots : A_n|B)$ is nonnegative and does not increase under local operations $\Phi_{A_1} : A_1 \rightarrow A_1$, \ldots, $\Phi_{A_n} : A_n \rightarrow A_n$, that is,

$$
S_n(A_1 : \ldots : A_n|B) \geq S_n(A_1: \ldots : A_n|B)_{\Phi_{A_1} \otimes \cdots \otimes \Phi_{A_n} \otimes \text{Id}_B(\omega)}.
$$

(7.7)

Formula (4.12) in this case has the form

$$
S_n(A_1: \ldots : A_n|B) - S_n(A_1: \ldots : A_n)
$$

$$
= (n - 1) I(A_1 \ldots A_n|B) - \sum_{i=1}^{n} I(A_1 \ldots A_{i-1} A_{i+1} \ldots A_n|B).
$$

(7.8)

Since (7.6) is valid for arbitrary permutations of the indices 1, \ldots, $n$ on the right-hand side, Remark 8 implies that $S_n(A_1: \ldots : A_n|B)$ is bounded above by the value

$$
2 \min_{1 \leq j \leq n, i \neq j} H(\omega_{A_i}).
$$

(7.9)

Using the representation (7.6) and the extended conditional quantum mutual information described in § 6 we can define $S_n(A_1: \ldots : A_n|B)$ for any state of an infinite-dimensional system $A_1 \ldots A_n B$.

**Proposition 7. A)** The quantity $S_n(A_1 : \ldots : A_n|B)$ defined in (7.5) has a lower semicontinuous nonnegative extension to the set $\mathcal{G}$. **This has property (7.7)** and is bounded above by the quantity (7.9). This extension can be defined by formula (7.6) in which each term $I(X:Y|Z)$ coincides with the function $I_e(X:Y|Z)$ described in Theorem 2.
B) This extension (also denoted \( S_n(A_1 : \ldots : A_n|B)_\omega \)) satisfies condition (3.2) for an \( \mathcal{F} \)-extension for any state \( \omega_{A_1 \ldots A_nB} \) such that (7.9) is finite. For an arbitrary state \( \omega \in \mathcal{G}(\mathcal{H}_{A_1 \ldots A_nB}) \) the following weaker holds:

\[
S_n(A_1 : \ldots : A_n|B)_\omega = \lim_{k_n \to \infty} \cdots \lim_{k_1 \to \infty} \lim_{l \to \infty} S_n(A_1 : \ldots : A_n|B)_{\omega^{k_1 \ldots k_n l}}, \quad (7.10)
\]

where

\[
\omega^{k_1 \ldots k_n l} = \lambda^{-1}Q\omega Q, \quad Q = P_{A_1}^{k_1} \otimes \cdots \otimes P_{A_n}^{k_n} \otimes P_B^l, \quad \lambda = \text{Tr} Q\omega,
\]

and \( \{P_{A_i}^{k_i}\} \subset \mathcal{B}(\mathcal{H}_{A_i}), i = 1, \ldots, n \), and \( \{P_B^l\} \subset \mathcal{B}(\mathcal{H}_B) \) are sequences of finite-rank projections converging strongly to the identity operators \( I_{A_i} \) and \( I_B \).

C) The representation (7.6) is valid for \( S_n(A_1 : \ldots : A_n|B)_\omega \) with arbitrarily permuted indices \( 1, \ldots, n \) on the right-hand side (provided each term \( I(X : Y|Z)_{\omega} \) coincides with \( I(\epsilon(X : Y|Z)_{\omega}) \)).

D) The local continuity of the \( n - 1 \) marginal entropies \( H(\omega_{A_1}), \ldots, H(\omega_{A_{n-1}}) \) implies the local continuity of \( S_n(A_1 : \ldots : A_n|B)_\omega \). The local continuity of \( H(\omega_B) \) implies the local continuity of the difference \( S_n(A_1 : \ldots : A_n|B)_\omega - S_n(A_1 : \ldots : A_n)_\omega \), which has the representation (7.8).

E) If \( n - 1 \) subsystems, for instance, \( A_1, \ldots, A_{n-1} \), are finite-dimensional then \( S_n(A_1 : \ldots : A_n|B)_\omega \) is a bounded continuous function on the set \( \mathcal{G}(\mathcal{H}_{A_1 \ldots A_nB}) \) and

\[
|S_n(A_1 : \ldots : A_n|B)_{\omega^1} - S_n(A_1 : \ldots : A_n|B)_{\omega^2}| \leq 2\varepsilon C + 2n(1 + \varepsilon)h_2 \left( \frac{\varepsilon}{1 + \varepsilon} \right)
\]

for any \( \omega^1, \omega^2 \in \mathcal{G}(\mathcal{H}_{A_1 \ldots A_nB}) \), where \( \varepsilon = \frac{1}{2}\|\omega^1 - \omega^2\|_1 \), \( h_2 \) is the binary entropy and \( C = \log \dim \mathcal{H}_{A_1 \ldots A_{n-1}} \).

Proof. Assertion A) follows from Theorem 2, Remark 8 and assertion C), which is proved below.

Property (7.10) follows from Corollary 8 and the second assertion of Corollary 9, that is, the limit relation (6.11), applied to each term on the right-hand side of (7.6). This property and (6.11) make it possible to prove assertion C) via approximation by noting that it is valid for any state \( \omega_{A_1 \ldots A_nB} \) such that rank \( \omega_{A_i} < +\infty \) for all \( i \) and rank \( \omega_B < +\infty \).

Now using assertion C), the first part of assertion B) can be proved by applying Corollary 9 to each term on the right-hand side of (7.6) for appropriately permuted indices \( 1, \ldots, n \) (using Lemmas 5 and 6).

The first part of assertion D) follows from part A) of Corollary 7 applied to each term on the right-hand side of (7.6) for appropriately permuted indices \( 1, \ldots, n \). The second part of D) follows from Proposition 2.

 Assertion E) follows from Proposition 1 and the upper bound (7.9).

The proposition is proved.

Other properties of the extension \( S_n(A_1 : \ldots : A_n|B)_\omega \) defined in Proposition 7 (monotonicity under local conditioning, additivity, see [20]) can be derived from the corresponding properties in the finite-dimensional settings using the approximation property (7.10).

\footnote{The limits over \( k_1, \ldots, k_n \) in (7.10) can be taken in an arbitrary order. This follows from assertion C) of this proposition. The projections in the sequence \( \{P_B^l\} \) can be arbitrary.}
7.3. The gap $I(A_1A'_1: \ldots : A_nA'_n) - I(A'_1: \ldots : A'_n)$ and Wilde’s inequality.

In the construction of entanglement measures in a multipartite finite-dimensional quantum system $A_1 \ldots A_n$ the difference

$$\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n}) \doteq I(A_1A'_1: \ldots : A_nA'_n)_\omega - I(A'_1: \ldots : A'_n)_\omega$$  \hspace{1cm} (7.11)

between mutual informations of the state of the extended system $A_1A'_1 \ldots A_nA'_n$ is used [33]. Basic properties of the quantum mutual information show that the gap $\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n})$ is nonnegative and does not increase under local operations $\Phi_{A_1}: A_1 \to A_1', \ldots, \Phi_{A_n}: A_n \to A_n'$, that is,

$$\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n}) \geq \Delta I(\Phi_{A_1} \otimes \cdots \otimes \Phi_{A_n} \otimes \text{Id}_{A_1' \ldots A_n'})(\omega_{A_1A'_1 \ldots A_nA'_n})).$$  \hspace{1cm} (7.12)

Recently Wilde proved that

$$\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n}) \geq \frac{1}{4n^2} \left\| \omega_{A_1A'_1 \ldots A_nA'_n} - \Phi_1 \otimes \cdots \otimes \Phi_n(\omega_{A'_1 \ldots A'_n}) \right\|^2_1$$  \hspace{1cm} (7.13)

for particular recovery channels $\Phi_1: A'_1 \to A_1A'_1$, $\ldots$, $\Phi_n: A'_n \to A_nA'_n$ (see [34]). This result shows that if the gap $\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n})$ is close to zero then we can recover the state $\omega_{A_1A'_1 \ldots A_nA'_n}$ approximately from its marginal state $\omega_{A'_1 \ldots A'_n}$ using the local recovery channels $\Phi_1: A'_1 \to A_1A'_1$, $\ldots$, $\Phi_n: A'_n \to A_nA'_n$.

This result has several applications to quantum information theory [34]. It can be generalized to all states of infinite-dimensional quantum system $A_1A'_1 \ldots A_nA'_n$ by constructing an appropriate extension of the information gap $\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n})$ to the set $\mathcal{G}(\mathcal{H}_{A_1A'_1 \ldots A_nA'_n})$ (see Corollary 12 below).

To construct this extension we will use the representation

$$\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n}) = I(A_1:A'_2 \ldots A'_n|A'_1)_{\omega}$$

$$+ \sum_{i=2}^{n} I(A_i:A_1 \ldots A_{i-1}A'_1 \ldots A'_{i-1}A'_{i+1} \ldots A'_n|A'_1)_{\omega},$$  \hspace{1cm} (7.14)

which is valid for any state of the finite-dimensional system $A_1A'_1 \ldots A_nA'_n$ (see [34], Lemma 1), and the extended conditional mutual information defined in §6.

**Proposition 8.** A) The quantity $\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n})$ defined in (7.11) has a lower semicontinuous nonnegative extension to the set $\mathcal{G}(\mathcal{H}_{A_1A'_1 \ldots A_nA'_n})$, which possesses property (7.12) and is bounded above by $2 \sum_{i=1}^{n} H(\omega_{A_i})$. This extension can be defined by formula (7.14) in which each term $I(X:Y|Z)_{\omega}$ coincides with the function $I_{\omega}(X:Y|Z)_{\omega}$ described in Theorem 2.

B) This extension (also denoted by $\Delta I(\omega_{A_1A'_1 \ldots A_nA'_n})$) satisfies the condition for an $\mathcal{F}$-extension (3.2) for any state $\omega_{A_1A'_1 \ldots A_nA'_n}$ such that $H(\omega_{A_i}) < +\infty$ for all $i$.

For an arbitrary state $\omega \in \mathcal{G}(\mathcal{H}_{A_1A'_1 \ldots A_nA'_n})$ the following weaker property holds:

$$\Delta I(\omega) = \lim_{k \to \infty} \lim_{k' \to \infty} \Delta I(\omega^{kk'}),$$  \hspace{1cm} (7.15)

where

$$\omega^{kk'} = \lambda^{-1} Q_{\omega} Q, \quad Q = P_{A_1}^k \otimes \cdots \otimes P_{A_n}^k \otimes P_{A'_1}^{k'} \otimes \cdots \otimes P_{A'_n}^{k'}, \quad \lambda = \text{Tr} Q_{\omega},$$
and \( \{P_{A_i}^k\} \subset \mathcal{B}(\mathcal{H}_{A_i}) \) and \( \{P_{A_i'}^k\} \subset \mathcal{B}(\mathcal{H}_{A_i'}) \) are arbitrary sequences of finite-rank projections converging strongly to the identity operators \( I_{A_i} \) and \( I_{A_i'} \), \( i = 1, \ldots, n \).

C) The representation (7.14) is valid for \( \Delta I(\omega_{A_1A'_1} \ldots A_nA'_n) \) with arbitrarily permuted indices 1, \ldots, \( n \) on the right-hand side (provided that each term \( I(X:Y|Z)_{\omega} \) coincides with \( I_e(X:Y|Z)_{\omega} \)).

D) The local continuity of the marginal entropies \( H(\omega_{A_1}), \ldots, H(\omega_{A_n}) \) implies the local continuity of \( \Delta I(\omega_{A_1A'_1} \ldots A_nA'_n) \).

E) If the subsystems \( A_1, \ldots, A_n \) are finite-dimensional then \( \Delta I(\omega_{A_1A'_1} \ldots A_nA'_n) \) is a continuous bounded function on the set \( \mathcal{S}(\mathcal{H}_{A_1A'_1} \ldots A_nA'_n) \) and

\[
|\Delta I(\omega^1) - \Delta I(\omega^2)| \leq 2\varepsilon C + 2(n + 1)(1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)
\]

for any \( \omega^1, \omega^2 \in \mathcal{S}(\mathcal{H}_{A_1A'_1} \ldots A_nA'_n) \), where \( \varepsilon = \frac{1}{2}\|\omega^1 - \omega^2\|_1 \), \( h_2 \) is the binary entropy and \( C = \log \dim \mathcal{H}_{A_1} \ldots A_n \).

**Proof.** Assertion A) is proved using Theorem 2 and Remark 8.

Assertion B) is proved by applying Corollaries 8 and 9 to each term on the right-hand side of (7.14).

By Lemma 1 in [34], assertion C) is valid for any state \( \omega_{A_1A'_1} \ldots A_nA'_n \) such that rank \( \omega_{A_i} < +\infty \) and rank \( \omega_{A_i'} < +\infty \) for all \( i \). Its validity for an arbitrary state can be shown via approximation by using properties (6.11) and (7.15).

Assertion D) follows from part A) of Corollary 7 applied to each term on the right-hand side of (7.14).

Assertion E) follows from Proposition 1 and the upper bound for the quantity \( \Delta I(\omega_{A_1A'_1} \ldots A_nA'_n) \) mentioned in assertion A).

The proposition is proved.

Inequality (7.13) was proved in [34] using the following two facts:

1) for any state \( \omega_{ABC} \) of a finite-dimensional tripartite system there exists a channel \( \Phi : B \rightarrow BC \) (the Fawzi-Renner recovery map) such that

\[
2 - \frac{1}{2}I(A;C|B)_{\omega} \leq F(\omega_{ABC}, \text{Id}_A \otimes \Phi(\omega_{AB})),
\]

where \( F(\rho, \sigma) \doteq \|\sqrt{\rho} \sqrt{\sigma}\|_1 \) is the quantum fidelity [9];

2) the representation (7.14) holds with arbitrarily permuted indices 1, \ldots, \( n \) on the right-hand side, and thus shows that for \( i = 1, \ldots, n \)

\[
\Delta I(\omega_{A_1A'_1} \ldots A_nA'_n) \\
\geq I(A_i : A_1 \ldots A_{i-1}A_{i+1} \ldots A_nA'_1 \ldots A_{i-1}'A_{i+1}' \ldots A'_n|A'_i)_{\omega}.
\]

The first fact is valid for all states of an infinite-dimensional tripartite system provided that \( I(A: C|B)_{\omega} = I_e(A: C|B)_{\omega} \), the extended conditional mutual information. For states with finite marginal entropies this was proved in [9], for arbitrary states this follows from Proposition 13 in §8.4 below.

By part C) of Proposition 8, the second fact is valid for all states of the infinite-dimensional system \( A_1A'_1 \ldots A_nA'_n \). So, repeating the arguments in [34], we can prove that inequality (7.13) holds for all states of the infinite-dimensional system \( A_1A'_1 \ldots A_nA'_n \).
Corollary 12. Wilde’s inequality (7.13) holds for all states of the infinite-dimensional system $A_1A'_1\ldots A_nA'_n$ provided that $\Delta I(\omega; A_1A'_1\ldots A_nA'_n)$ is the extension of the gap $I(A_1A'_1; \ldots; A_nA'_n)\omega - I(A_1'; \ldots; A'_n)\omega$ described in Proposition 8.

§ 8. Some applications

8.1. Continuity relations for quantum channels. Let $\Phi: A \to B$ be a quantum channel, that is, a completely positive trace preserving linear map $\mathcal{F}(\mathcal{H}_A) \to \mathcal{F}(\mathcal{H}_B)$. Stinespring’s theorem implies the existence of a Hilbert space $\mathcal{H}_E$ and an isometry $V: \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = \text{Tr}_E V \rho V^*,$$

\(\rho \in \mathcal{F}(\mathcal{H}_A).\)  

(8.1)

The quantum channel

$$\mathcal{F}(\mathcal{H}_A) \ni \rho \mapsto \Phi(\rho) = \text{Tr}_E V \rho V^* \in \mathcal{F}(\mathcal{H}_E)$$

(8.2)

is called complementary to the channel $\Phi$ [1], Ch. 6.

It is well known that for a finite-dimensional channel $\Phi$ and a state $\rho$ in $\mathcal{F}(\mathcal{H}_A)$ the input entropy $H(\rho)$, the output entropy $H(\Phi(\rho))$ and the entropy exchange $H(\Phi, \rho) = H(\Phi(\rho))$ (in what follows the last two quantities will be denoted by $H_\Phi(\rho)$ and $H_\Phi(\rho)$, respectively) satisfy the ‘triangle inequalities’:

$$|H_\Phi(\rho) - H_\Phi(\rho)| \leq H(\rho),$$

$$|H_\Phi(\rho) - H(\rho)| \leq H_\Phi(\rho),$$

$$|H_\Phi(\rho) - H(\rho)| \leq H_\Phi(\rho).$$

(8.3)

The quantity $I_c(\Phi, \rho) = H_\Phi(\rho) - H_\Phi(\rho)$, called the coherent information of the channel $\Phi$ at the state $\rho$, is an important characteristic of a quantum channel related to its quantum capacity [1], [2].

The quantity $EG(\Phi, \rho) = H_\Phi(\rho) - H(\rho)$, called the entropy gain of the channel $\Phi$ at the state $\rho$, is a convex function of $\rho$ which is also used in the analysis of the information properties of a quantum channel [36], [37].

If $\Phi$ is an infinite-dimensional quantum channel, then inequalities (8.3) also hold provided all the terms are finite. Moreover, if we use the extension of the coherent information $I_c(\Phi, \rho) = H_\Phi(\rho) - H_\Phi(\rho)$ to $\{\rho \in \mathcal{F}(\mathcal{H}_A) \mid H(\rho) < +\infty\}$ given by\(^{18}\)

$$I_c(\Phi, \rho) = H(\Phi \otimes \text{Id}_R(|\varphi_\rho\rangle\langle\varphi_\rho|))H(\rho) - H(\rho),$$

(8.4)

where $|\varphi_\rho\rangle$ is a purification of the state $\rho$ in $\mathcal{H}_A \otimes \mathcal{H}_R$ and $\varrho = \text{Tr}_A |\varphi_\rho\rangle\langle\varphi_\rho|$, then (4.3) implies that the first inequality in (8.3) becomes valid for an arbitrary state $\rho$ (with the possible value $+\infty$ on both sides).

The quantities $EG(\Phi, \rho) = H_\Phi(\rho) - H(\rho)$ and $EG(\Phi, \rho) = H_\Phi(\rho) - H(\rho)$ can be extended by the formulae

$$EG(\Phi, \rho) = H(V \rho V^*|\Phi(\rho) \otimes \Phi(\rho)) - H_\Phi(\rho),$$

$$EG(\Phi, \rho) = H(V \rho V^*|\Phi(\rho) \otimes \Phi(\rho)) - H_\Phi(\rho),$$

(8.5)

(8.6)

\(^{18}\)That this extension makes sense was shown in [38].
to the sets \( \{ \rho \in \mathcal{S}(\mathcal{H}_A) \mid H_\Phi(\rho) < +\infty \} \) and \( \{ \rho \in \mathcal{S}(\mathcal{H}_A) \mid H_\Phi(\rho) < +\infty \} \), respectively, where \( V \) is any Stinespring isometry for \( \Phi \) and \( \widehat{\Phi} \) is the version of complementary channel corresponding to this isometry via (8.2). The right-hand sides of (8.5) and (8.6) can be written as \(-H_e(E|B)_{V\rho V^*}\) and \(-H_e(B|E)_{V\rho V^*}\), respectively, where \( H_e(A|B) \) is the extended quantum conditional entropy given in [5] and described in §5 of this paper. So the concavity of \( H_e(A|B) \) means that \( EG(\Phi, \rho) \) and \( EG(\widehat{\Phi}, \rho) \) defined by (8.5) and (8.6) are convex as functions of \( \rho \). The upper bound (4.3) shows that

\[
|EG(\Phi, \rho)| \leq H_\Phi(\rho) \quad \text{and} \quad |EG(\widehat{\Phi}, \rho)| \leq H_\Phi(\rho).
\]

So the second and third inequalities in (8.3) are also valid for an arbitrary state \( \rho \) (with the possible value +\( \infty \) on both sides) provided the values \( H_\Phi(\rho) - H(\rho) \) and \( H_\Phi(\rho) - H(\rho) \) are defined by formulae (8.5) and (8.6), respectively.

Theorem 1 makes it possible to show that the continuity of one of the functions \( H(\rho), H_\Phi(\rho) \) and \( H_\Phi(\rho) \) implies the continuity of the difference between the other two.

**Proposition 9.** Let \( \Phi: A \to B \) be a quantum channel and \( \widehat{\Phi}: A \to E \) its complementary channel.

A) The local continuity of \( H(\rho) \) implies the local continuity of the coherent information \( I_c(\Phi, \rho) = [H_\Phi(\rho) - H_\Phi(\rho)] \) defined by (8.4).

B) The local continuity of \( H_\Phi(\rho) \) implies the local continuity of the entropy gain \( EG(\Phi, \rho) = [H_\Phi(\rho) - H(\rho)] \) defined by (8.5).

C) The local continuity of \( H_\Phi(\rho) \) implies the local continuity of the entropy gain \( EG(\widehat{\Phi}, \rho) = [H_\Phi(\rho) - H(\rho)] \) defined by (8.6).

While the triangle inequalities (8.3) show that a triangle with sides of length \( H(\rho), H_\Phi(\rho) \) and \( H_\Phi(\rho) \) exists, Proposition 9 states that a small deformation of any side of this triangle leads to a small deformation of the difference between the lengths of the other sides (despite the possibility of a ‘large’ deformation of each of these sides).

**Proof of Proposition 9.** Assertions B) and C) follow directly from part A) of Theorem 1. Assertion A) is also derived from part A) of Theorem 1, if we use Lemma 1 and note that \( H(\rho) = H(\rho) \) for the state \( \rho \) in (8.4).

Since \( EG(\Phi, \rho) = -H_e(E|B)_{V\rho V^*} \), applying Corollary 5 we can improve assertion B) of Proposition 9 in the case when \( \dim \mathcal{H}_E = k < +\infty \).

**Corollary 13.** If \( \Phi: A \to B \) is a quantum channel with finite Choi rank\(^{19}\) \( k \) then the entropy gain (8.5) is a bounded continuous function on the set \( \mathcal{S}(\mathcal{H}_A) \) and

\[
|EG(\Phi, \rho_1) - EG(\Phi, \rho_2)| \leq 2\varepsilon \log k + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)
\]

for any \( \rho_1, \rho_2 \in \mathcal{S}(\mathcal{H}_A) \), where \( \varepsilon = \frac{1}{2}\|\rho_1 - \rho_2\|_1 \) and \( h_2 \) is the binary entropy.

\(^{19}\)The Choi rank of a channel \( \Phi \) is the minimum dimension of the space \( \mathcal{H}_E \) in the Stinespring representation (8.1) [1].
Applying Corollary 1 and noting that $\dim \mathcal{H}_R = \dim \mathcal{H}_A$ we can obtain continuity bounds for the coherent information (8.4) and for the quantum mutual information (8.8) in the case when $\dim \mathcal{H}_A < +\infty$.

8.2. Continuity of the functions $(\Phi, \rho) \mapsto I(\Phi, \rho)$ and $(\Phi, \rho) \mapsto I_c(\Phi, \rho)$. The quantum mutual information is an important characteristic of a quantum channel, related to its entanglement-assisted classical capacity (see [39], [40], [1] and [2]). For a finite-dimensional channel $\Phi: A \rightarrow B$ it can be defined by

$$I(\Phi, \rho) = H(\rho) + H(\Phi(\rho)) - H(\Phi(\rho \otimes \varrho)).$$

(8.7)

In infinite dimensions this definition may contain the uncertainty $\infty - \infty$, but it can be modified to avoid this problem as follows

$$I(\Phi, \rho) = H(\Phi \otimes \text{Id}_R(\varphi_\rho)\langle \varphi_\rho | \varphi_\rho \rangle \| \Phi(\rho \otimes \varrho),$$

(8.8)

where $|\varphi_\rho \rangle$ is a purification of the state $\rho$ in $\mathcal{H}_A \otimes \mathcal{H}_R$ and $\varrho = \text{Tr}_A |\varphi_\rho \rangle\langle \varphi_\rho |$. For an arbitrary quantum channel $\Phi$ the nonnegative function $\rho \mapsto I(\Phi, \rho)$ defined by (8.8) is concave and lower semicontinuous on the set $\mathcal{S}(\mathcal{H}_A)$ [38].

In studying the entanglement-assisted classical capacity of an infinite-dimensional channel and, in particular, in analyzing its continuity as a function of a channel (which can be interpreted as robustness or the stability of the capacity with respect to perturbations of the channel) it is necessary to explore the continuity properties of the quantum mutual information $I(\Phi, \rho)$ with respect to the simultaneous variation of $\Phi$ and $\rho$. This means that we have to consider $I(\Phi, \rho)$ as a function of the pair $(\Phi, \rho)$, that is, as a function on the Cartesian product of the set $\mathcal{F}_{AB}$ of all quantum channels from $A$ to $B$ equipped with an appropriate topology and the set $\mathcal{S}(\mathcal{H}_A)$ of input states.

Since a quantum channel $\Phi$ is a completely bounded map, the set $\mathcal{F}_{AB}$ is typically equipped with the norm of complete boundedness [41], which can be defined as the upper bound of the operator norms of the maps $\Phi \otimes \text{Id}_C^n$, $n \in \mathbb{N}$. In infinite dimensions, along with the topology induced by the norm of complete boundedness, we can consider weaker topologies on the set of quantum channels, in particular, the strong convergence topology generated by the strong operator topology on the set of all bounded linear operators between the Banach spaces $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$ (see [42]). Strong convergence of the sequence $\{\Phi_n\} \subset \mathcal{F}_{AB}$ to the quantum channel $\Phi_0 \in \mathcal{F}_{AB}$ means that

$$\lim_{n \rightarrow +\infty} \Phi_n(\rho) = \Phi_0(\rho) \quad \forall \rho \in \mathcal{S}(\mathcal{H}_A).$$

The use of the strong convergence topology in infinite dimensions seems preferable for the following reason. It was shown in [43] that the closeness of two quantum channels in the norm of complete boundedness means, roughly speaking, that the corresponding Stinespring isometries are close in the operator norm. So, if we use the norm of complete boundedness then we can only take into account perturbations of the channel corresponding to uniform deformations of the Stinespring isometry (that is, deformations with small operator norm). Physically, it seems reasonable to consider a wider class of perturbations of the channel, including perturbations...
corresponding to deformations of the Stinespring isometry in the strong operator topology.

Thus, we will assume in what follows that $\mathfrak{F}_{AB}$ is the set all channels $\Phi: A \to B$ equipped with the strong convergence topology. As the set $\mathcal{S}(\mathcal{H}_A)$ is separable, the strong convergence topology on $\mathfrak{F}_{AB}$ is metrizable (can be induced by a metric). Note also that it is in the strong convergence topology that the set $\mathfrak{F}_{AB}$ is topologically isomorphic to a subset of states of a composite system (the generalized Choi-Jamiolkowski isomorphism; [42], Proposition 3).

Using Theorem 1 the continuity condition for the function $(\Phi, \rho) \mapsto I(\Phi, \rho)$ obtained in [38], Proposition 5 can be strengthened considerably.

**Proposition 10.** A) The continuity of $H(\rho)$ on a set $\mathcal{A} \subset \mathcal{S}(\mathcal{H}_A)$ implies the continuity of the function $(\Phi, \rho) \mapsto I(\Phi, \rho)$ on the set $\mathfrak{F}_{AB} \times \mathcal{A}$.

B) The local continuity of the function $(\Phi, \rho) \mapsto H(\Phi(\rho))$ implies the local continuity of the function $(\Phi, \rho) \mapsto I(\Phi, \rho)$.

C) The local continuity of the function $(\Phi, \rho) \mapsto I(\Phi, \rho)$ implies the local continuity of the function $(\Phi, \rho) \mapsto I(\psi \circ \Phi, \rho)$ for any channel $\psi: B \to C$.

Assertion A) of Proposition 10 states that

$$\lim_{n \to \infty} H(\rho_n) = H(\rho_0) < +\infty \implies \lim_{n \to \infty} I(\Phi_n, \rho_n) = I(\Phi_0, \rho_0) < +\infty,$$

(8.9)

where $\rho_0 = \lim_{n \to \infty} \rho_n$, for an arbitrary sequence $\{\Phi_n\}$ of channels which converge strongly to a channel $\Phi_0$. In contrast to Proposition 5 in [38] the existence of a sequence $\{\Phi_n\}$ converging to the channel $\hat{\Phi}_0$ is not required in (8.9).

Assertions B) and C) can be formulated as the implications

$$\lim_{n \to \infty} H(\Phi_n(\rho_n)) = H(\Phi_0(\rho_0)) < +\infty \implies \lim_{n \to \infty} I(\Phi_n, \rho_n) = I(\Phi_0, \rho_0) < +\infty$$

and

$$\lim_{n \to \infty} I(\Phi_n, \rho_n) = I(\Phi_0, \rho_0) < +\infty \implies \lim_{n \to \infty} I(\psi \circ \Phi_n, \rho_n) = I(\psi \circ \Phi_0, \rho_0) < +\infty,$$

for sequences $\{\rho_n\} \subset \mathcal{S}(\mathcal{H}_A)$ and $\{\Phi_n\} \subset \mathfrak{F}_{AB}$ converging to a state $\rho_0$ and a channel $\Phi_0$, respectively, and an arbitrary channel $\psi \in \mathfrak{F}_{BC}$.

**Proof of Proposition 10.** If the sequence $\{\Phi_n\}$ converges strongly to the channel $\Phi_0$ then $\{\Phi_n \otimes \text{Id}_R\}$ converges strongly to the channel $\Phi_0 \otimes \text{Id}_R$ [42], and so assertions A) and B) follow from part A) of Theorem 1 and Lemma 1 (since $H(\varrho) = H(\rho)$ for the state $\varrho$ in (8.8)) while assertion C) follows from part B) of Theorem 1.

Assertion A) in Proposition 10 gives a continuity condition for the coherent information $I_c(\Phi, \rho)$ defined by (8.4), which means that $I_c(\Phi, \rho) = I(\Phi, \rho) - H(\rho)$.

**Corollary 14.** If $H(\rho)$ is continuous on a set $\mathcal{A} \subset \mathcal{S}(\mathcal{H}_A)$ then the function $(\Phi, \rho) \mapsto I_c(\Phi, \rho)$ is continuous on the set $\mathfrak{F}_{AB} \times \mathcal{A}$.

Corollary 14 states that (8.9) holds with $I_c(\Phi, \rho)$ instead of $I(\Phi, \rho)$; it can be regarded as a generalized version of assertion A) of Proposition 9.

It is easy to see that the analogue of assertion B) of Proposition 10 is not valid for the function $(\Phi, \rho) \mapsto I_c(\Phi, \rho)$. 


8.3. The continuity of the entanglement-assisted classical capacity as a function of the channel. Conditions for the continuity\(^{20}\) of the entanglement-assisted classical capacity of an infinite-dimensional quantum channel with linear constraint (as a function of the channel) were obtained in [22]. In this section we strengthen these substantially.

The rate of transmission of classical information over a quantum channel can be increased by using an entangled state as an additional resource. A detailed description of the corresponding protocol can be found in [1] and [2]. The ultimate rate of information transmission by this protocol is called the **entanglement-assisted classical capacity** of a quantum channel.

If \(\Phi: A \to B\) is a finite-dimensional quantum channel then using the Bennett-Shor-Smolin-Thaplyal (BSST) theorem [40] we have the following expression for its entanglement-assisted classical capacity

\[
C_{ea}(\Phi) = \sup_{\rho \in \mathcal{S}(\mathcal{H}_A)} I(\Phi, \rho),
\]

where \(I(\Phi, \rho)\) is the quantum mutual information defined by (8.7).

The continuity of this capacity as a function of the channel follows directly from the continuity of the function \((\Phi, \rho) \mapsto I(\Phi, \rho)\) and the compactness of the space of input states [44].

If \(\Phi\) is an infinite-dimensional quantum channel then we have to impose constraints on the states used for coding information. Typically, these are linear constraints determined by the inequality \(\text{Tr} F \rho \leq E\), where \(F\) is a positive operator and \(E > 0\). An operational definition of the entanglement-assisted classical capacity \(C_{ea}(\Phi, F, E)\) of an infinite-dimensional quantum channel \(\Phi\) with the above linear constraint was given in [45], where a generalization of the BSST theorem was proved under special restrictions on the channel \(\Phi\) and the constraint operator \(F\). A general version of the BSST theorem for an infinite-dimensional channel with linear constraints without any simplifying restrictions was proved in [22]; it states that

\[
C_{ea}(\Phi, F, E) = \sup_{\text{Tr} F \rho \leq E} I(\Phi, \rho) \leq +\infty
\]

for an arbitrary channel \(\Phi\) and any positive constraint operator \(F\), where \(I(\Phi, \rho)\) is the quantum mutual information defined by (8.8). Noting that the function \((\Phi, \rho) \mapsto I(\Phi, \rho)\) is lower semicontinuous on \(\mathcal{F}_{AB} \times \mathcal{S}(\mathcal{H}_A)\) (see [38]) it is easy to show that the function \(\Phi \mapsto C_{ea}(\Phi, F, E)\) is lower semicontinuous on \(\mathcal{F}_{AB}\), that is,

\[
\liminf_{n \to +\infty} C_{ea}(\Phi_n, F, E) \geq C_{ea}(\Phi_0, F, E) \quad (\leq +\infty)
\]

for any sequence \(\{\Phi_n\}\) of channels converging strongly to a channel \(\Phi_0\). But this function is not continuous in general (see Example 2 below).

Assertion A) of Proposition 10 implies the following sufficient condition for the global continuity of the function \(\Phi \mapsto C_{ea}(\Phi, F, E)\).

\(^{20}\)The importance of studying continuity properties of quantum channel capacities was discussed in [44], [42] and [22]. Briefly, it is explained by unavoidable perturbations of the channel used for information transmission.
Proposition 11. Let $\mathcal{K}_{F,E}$ be the set of input states $\rho$ such that $\text{Tr} F \rho \leq E$. If $H(\rho)$ is continuous on $\mathcal{K}_{F,E}$ then the function $\Phi \mapsto C_{ea}(\Phi, F, E)$ is continuous on the set $\mathfrak{F}_{AB}$ of all channels (and bounded above by $\sup_{\rho \in \mathcal{K}_{F,E}} 2H(\rho)$).

The assumption that $H(\rho)$ is continuous on $\mathcal{K}_{F,E}$ holds for any $E > 0$ if the operator $F$ satisfies $\text{Tr} \exp(-\lambda F) < +\infty$ for all $\lambda > 0$ (see [3]). This condition holds for

$$F = R^\top \epsilon R,$$

the Hamiltonian of a many-mode Bosonic quantum system, where $\epsilon$ is a nondegenerate energy matrix and $R$ represents the canonical variables of the system (see [1], Ch. 12, for details). In this case $C_{ea}(\Phi, F, E)$ is the entanglement-assisted classical capacity of a channel $\Phi$ under the condition that the mean energy of the states used for coding information is $\leq E$. Proposition 11 implies the following observation.

Corollary 15. Let $F = R^\top \epsilon R$ be the Hamiltonian of a many-mode Bosonic quantum system $A$ and let $E > 0$. Then the function $\Phi \mapsto C_{ea}(\Phi, F, E)$ is continuous on the set $\mathfrak{F}_{AB}$ of all channels from the system $A$ to an arbitrary system $B$.

This shows that the entanglement-assisted classical capacity of a Bosonic Gaussian channel with energy constraint varies continuously under any perturbations of this channel.

Proof of Proposition 11. Since the set $\mathcal{K}_{F,E}$ is closed and convex, the continuity of the concave function $H(\rho)$ on $\mathcal{K}_{F,E}$ implies it is bounded\(^2\) on $\mathcal{K}_{F,E}$. So the set $\mathcal{K}_{F,E}$ is compact by Corollary 5 in [46].

Now the function $\Phi \mapsto C_{ea}(\Phi, F, E)$ is lower semicontinuous on the set $\mathfrak{F}_{AB}$, so it suffices to show that it is finite and upper semicontinuous.

Assume that there exists a sequence $\{\Phi_n\}$ of channels in $\mathfrak{F}_{AB}$ converging strongly to a channel $\Phi_0$ and such that

$$\lim_{n \to +\infty} C_{ea}(\Phi_n, F, E) \geq C_{ea}(\Phi_0, F, E) + \varepsilon$$

for some $\varepsilon > 0$. It follows from (8.10) that for each $n$ there exists a state $\rho_n \in \mathcal{K}_{F,E}$ such that

$$C_{ea}(\Phi_n, F, E) < I(\Phi_n, \rho_n) + \frac{\varepsilon}{2}.$$

Since the set $\mathcal{K}_{F,E}$ is compact, we can assume (by passing to a subsequence) that the sequence $\{\rho_n\}$ converges to a particular state $\rho_0 \in \mathcal{K}_{F,E}$. Assertion A) of Proposition 10 implies

$$\lim_{n \to +\infty} I(\Phi_n, \rho_n) = I(\Phi_0, \rho_0).$$

Since (8.10) shows that $C_{ea}(\Phi_0, F, E) \geq I(\Phi_0, \rho_0)$, this contradicts (8.11).

The proposition is proved.

Using assertion B) of Proposition 10 and the same arguments we can prove the following ‘local’ continuity condition.

\(^2\)If for each $n$ there exists a state $\rho_n \in \mathcal{K}_{F,E}$ such that $H(\rho_n) \geq 2^n$ then $\sum_{n=1}^{+\infty} 2^{-n} \rho_n \in \mathcal{K}_{F,E}$ and $H(\sum_{n=1}^{+\infty} 2^{-n} \rho_n) \geq \sum_{n=1}^{+\infty} 2^{-n} H(\rho_n) = +\infty$, contradicting the continuity of $H$.\)
Proposition 12. If \( \mathcal{K}_{F,E} = \{ \rho \in \mathcal{S}(\mathcal{H}_A) \mid \text{Tr} F \rho \leq E \} \) is a compact set and \( \{ \Phi_n \} \) is a sequence of channels converging strongly to a channel \( \Phi_0 \) such that
\[
\lim_{n \to +\infty} H(\Phi_n(\rho_n)) = H(\Phi_0(\rho_0)) < +\infty
\]
for any sequence \( \{ \rho_n \} \subset \mathcal{K}_{F,E} \) converging to a state \( \rho_0 \) then
\[
\lim_{n \to +\infty} C_{\text{ea}}(\Phi_n, F, E) = C_{\text{ea}}(\Phi_0, F, E) < +\infty. \tag{8.12}
\]

Remark 10. A similar continuity condition holds for the Holevo capacity of an infinite-dimensional channel with linear constraints ([42], Proposition 7).

The following example shows that the compactness of the set \( \mathcal{K}_{F,E} \) and the continuity of the output entropies of all the channels \( \Phi_n \) do not imply (8.12).

Example 2. Let \( \{ |k\rangle \}_{k \geq 0} \) be an orthonormal basis in a separable Hilbert space \( \mathcal{H}_A = \mathcal{H}_B \) and let \( P_n = \sum_{k=1}^{n} |k\rangle\langle k| \) be a projection of rank \( n \). Consider the sequence of channels
\[
\Phi_n(\rho) = [\text{Tr} (I_A - q_n P_n) \rho]|0\rangle\langle 0| + q_n P_n \rho P_n
\]
with finite-dimensional output space, where \( \{ q_n \} \) is a sequence of positive numbers specified below.

Let \( F = \sum_{k=0}^{+\infty} \log(\log(k+3)) |k\rangle\langle k| \) be a positive operator. By the lemma in [45] the corresponding set \( \mathcal{K}_{F,E} \) is compact for any \( E > 0 \). Since \( \text{Tr} \exp(-\lambda F) = +\infty \) for any \( \lambda > 0 \), Proposition 1 in [46] and its proof imply the existence of a sequence \( \{ \rho_n \} \subset \mathcal{K}_{F,E} \) such that \( \rho_n = P_n \rho_n P_n \) and \( \lim_{n \to \infty} H(\rho_n) = +\infty \).

It is easy to show that \( I(\Phi_n, \rho_n) \geq 2q_n H(P_n \rho_n P_n) = 2q_n H(\rho_n) \). Since \( \{ \rho_n \} \subset \mathcal{K}_{F,E} \), this implies that \( C_{\text{ea}}(\Phi_n, F, E) \geq 2q_n H(\rho_n) \). So, for any sequence \( \{ q_n \} \) such that \( \lim_{n \to \infty} q_n = 0 \) and \( \lim_{n \to \infty} q_n H(\rho_n) = C > 0 \) we have
\[
\liminf_{n \to +\infty} C_{\text{ea}}(\Phi_n, F, E) \geq 2C
\]
while the sequence \( \{ \Phi_n \} \) converges strongly to the completely depolarizing channel \( \Phi_0(\rho) = [\text{Tr} \rho]|0\rangle\langle 0| \) for which \( C_{\text{ea}}(\Phi_0, F, E) = 0 \).

8.4. Existence of the Fawzi-Renner recovery channel for an arbitrary tripartite state. A fundamental strong subadditivity property of the von Neumann entropy, which means that \( I(A : C | B)_{\omega} \) is nonnegative, has recently been specified by Fawzi and Renner [9], who proved that for any state \( \omega_{ABC} \) there exists a recovery channel\(^{22} \) \( \Phi : B \to BC \) such that
\[
2^{-\frac{1}{2}} I(A : C | B)_{\omega} \leq F(\omega_{ABC}, \text{Id}_A \otimes \Phi(\omega_{AB})), \tag{8.13}
\]
where \( F(\rho, \sigma) = \| \sqrt{\rho} \sqrt{\sigma} \|_1 \) is the quantum fidelity between states \( \rho \) and \( \sigma \). This result can be regarded as an \( \varepsilon \)-version of the well-known characterization of states

\(^{22}\) It has recently been shown that the recovery channel \( \Phi : B \to BC \) satisfying (8.13) can be chosen independent of \( A \) [47].
\( \omega_{ABC} \) for which \( I(A:C|B)_\omega = 0 \) as Markov chains (that is, as states reconstructed from their marginal states \( \omega_{AB} \) by a channel \( \mathrm{Id}_A \otimes \Phi \)). It has several important applications in quantum information theory [9], [34].

The existence of a channel \( \Phi \) satisfying (8.13) was proved in the finite-dimensional setting in [9], by means of a quasi-explicit construction. Then, using approximation techniques, this result was extended in [9] and [47] to states \( \omega_{ABC} \) of an infinite-dimensional system such that \( I(A:C|B)_\omega = H(A|B)_\omega - H(A|BC)_\omega \), that is, assuming that the marginal entropies of \( \omega_{ABC} \) are finite.

It was also shown in Remark 5.3 in [9] that in finite dimensions a channel \( \Phi: B \to BC \) satisfying (8.13) can be chosen so that

\[
[\Phi(\omega_B)]_B = \omega_B \quad \text{and} \quad [\Phi(\omega_B)]_C = \omega_C, \tag{8.14}
\]

that is, a recovery channel \( \Phi \) can reproduce the marginal states exactly\(^23\).

The properties of the extended conditional mutual information (described in §6) make it possible to show the existence of a recovery channel satisfying (8.13) and (8.14) for all the states of an infinite-dimensional tripartite system, starting with the above-mentioned finite-dimensional result.

**Proposition 13** (a generalization of Remark 5.3 in [9]). For an arbitrary state \( \omega_{ABC} \) of an infinite-dimensional tripartite system there exists a channel \( \Phi: B \to BC \) satisfying (8.13) and (8.14) provided that \( I(A:C|B)_\omega \) is the extended conditional mutual information (described in Theorem 2).

We will use the following corollary of the compactness criterion for families of quantum operations in the strong convergence topology ([42], Corollary 2)\(^24\).

**Lemma 7.** Let \( \rho_A \) be a full rank state in \( \mathcal{G}(\mathcal{H}_A) \) and \( \{\Phi_n\} \) a sequence of quantum operations from \( A \) to \( BC \) such that

\[
[\Phi_n(\rho_A)]_B \leq \rho_B \quad \text{and} \quad [\Phi_n(\rho_A)]_C \leq \rho_C \quad \forall n
\]

for some operators \( \rho_B \in \mathcal{T}_+(\mathcal{H}_B) \) and \( \rho_C \in \mathcal{T}_+(\mathcal{H}_C) \). Then the sequence \( \{\Phi_n\} \) is relatively compact in the strong convergence topology.

**Proof.** It suffices to note that the set

\[
\{ \omega \in \mathcal{T}_+(\mathcal{H}_{BC}) \mid \omega_B \leq \rho_B, \omega_C \leq \rho_C \}
\]

is compact (see Corollary 6 in [42]) and to apply Corollary 2 in [42].

**Proof of Proposition 13.** By Remark 5.3 in [9] the assertion of the proposition is valid if \( \omega_A, \omega_B \) and \( \omega_C \) are finite-rank states. We will extend the class of states for which this assertion is valid to the whole set \( \mathcal{G}(\mathcal{H}_{ABC}) \) in several steps.

To simplify our notation we will assume at each step that \( \mathcal{H}_X = \text{supp} \omega_X \) for \( X = A, B, C \), so that \( \dim A = \dim \mathcal{H}_A = \text{rank} \omega_A \) and so on.

\(^{23}\) In Remark 5.3 in [9] the existence of a quantum operation \( \Phi' \) satisfying (8.13) such that \( [\Phi'(\omega_B)]_B \leq \omega_B \) and \( [\Phi'(\omega_B)]_C \leq \omega_C \) was established. A quantum channels \( \Phi \) satisfying (8.13) and (8.14) can be obtained from \( \Phi' \) by \( \Phi(\rho) = \Phi'(\rho) + [\text{Tr} \rho - \text{Tr} \Phi'(\rho)] \sigma \), where \( \sigma \) is the normalized positive operator \( (\omega_B - [\Phi'(\omega_B)]_B) \otimes (\omega_C - [\Phi'(\omega_B)]_C) \).

\(^{24}\) The strong convergence topology was described in §8.2.
Throughout the proof we will assume that $P^n_X$ is the spectral projection of a state $\omega_X$ corresponding to its $n$ maximal eigenvalues, $X = A, B, C$. When we talk about the compactness and convergence of a sequence of quantum operations we will bear in mind that the strong convergence topology is used.

**Step 1.** Assume that $\dim A \leq +\infty$ but $\dim B < +\infty$ and $\dim C < +\infty$. Let

$$\omega_{ABC}^n = \lambda_n^{-1} Q_n \omega_{ABC} Q_n, \quad Q_n = P^n_A \otimes I_B \otimes I_C, \quad \lambda_n = \text{Tr} Q_n \omega_{ABC}. \tag{8.15}$$

By Remark 5.3 in [9] for each $n$ there exists a channel $\Phi_n : B \to BC$ such that

$$2^{-\frac{1}{2}} I(A:C|B)_{\omega^n} \leq F(\omega_{ABC}^n, \text{Id}_A \otimes \Phi_n(\omega_{AB}^n)), \tag{8.16}$$

$$[\Phi_n(\omega_B^n)]_B = \omega_B^n, \quad [\Phi_n(\omega_B^n)]_C = \omega_C^n. \tag{8.17}$$

By Step 1 for each $n$ there exists a channel $\Phi_n : B_n \to B_n C$, where $B_n$ corresponds to the subspace $P^n_B(H_B)$, such that (8.15) and (8.16) hold. Consider the quantum operation $\Psi_n = \Phi_n \circ \Pi_n$ from $B$ to $BC$, where $\Pi_n(\cdot) = P^n_B(\cdot) P^n_B$. Since $P^n_B$ is a spectral projection of $\omega_B$, it follows from (8.16) that

$$[\Psi_n(\omega_B^n)]_B = \lambda_n \omega_B^n \leq \omega_B \quad \text{and} \quad [\Psi_n(\omega_B^n)]_C = \lambda_n \omega_C^n \leq \omega_C \quad \forall n. \tag{8.18}$$

It follows from (8.15)–(8.18) that the channel $\Phi_*$ satisfies (8.13) and (8.14).

**Step 2.** Assume that $\dim A \leq +\infty$ and $\dim B \leq +\infty$ but $\dim C < +\infty$. Let

$$\omega_{ABC}^n = \lambda_n^{-1} Q_n \omega_{ABC} Q_n, \quad Q_n = I_A \otimes P^n_B \otimes I_C, \quad \lambda_n = \text{Tr} Q_n \omega_{ABC}. \tag{8.19}$$

By Step 1 for each $n$ there exists a channel $\Phi_n : B \to BC$, where $B$ corresponds to the subspace $P^n_B(H_B)$, such that (8.15) and (8.16) hold. Consider the quantum operation $\Psi_n = \Phi_n \circ \Pi_n \circ \Pi_n$ from $BC$ to $BC$, where $\Pi_n \circ \Pi_n$ is a channel $\Phi_n : B \to BC$ such that (8.15) and (8.16) hold. Consider the quantum operation $\Psi_n = \Phi_n \circ \Theta \circ \Pi_n$ from $BC$ to $BC$, where
So we can assume that there exists \( \omega \). Since

\[
\Theta(\cdot) = \text{Tr}_C(\cdot) \quad \text{and} \quad \Pi_n(\cdot) = I_B \otimes P^n_C(\cdot) I_B \otimes P^n_C.
\]

Since \( P^n_C \) is a spectral projection of \( \omega_C \), it follows from (8.17) that

\[
[\Psi_n(\omega_{BC})]_B = \lambda_n \omega^n_B \leq \omega_B \quad \text{and} \quad [\Psi_n(\omega_{BC})]_C = \lambda_n \omega^n_C \leq \omega_C \quad \forall n.
\]

Since \( \omega_{BC} \) is a full rank state, the sequence \( \{\Psi_n\} \) is relatively compact by Lemma 7. So we can assume that there exists \( \lim_{n \to \infty} \Psi_n = \Psi_* \). It is easy to see that \( \Psi_* \) is a channel from \( BC \) to \( BC \).

Let \( \Lambda(\rho) = \rho \otimes \sigma \) be a channel from \( B \) to \( BC \), where \( \sigma \) is a given state in \( \mathcal{S}(\mathcal{H}_C) \).

Consider the channel \( \Phi_* = \Psi_* \circ \Lambda \) from \( B \) to \( BC \). Since

\[
\Psi_*(\rho_{BC}) = \lim_{n \to \infty} \Psi_n(\rho_{BC}) = \lim_{n \to \infty} \Psi_n(\rho_B \otimes \sigma) = \Psi_*(\rho_B \otimes \sigma) = \Phi_*(\rho_B)
\]

for any state \( \rho_{BC} \in \mathcal{S}(\mathcal{H}_{BC}) \), we have \( \Psi_* = \Phi_* \circ \Theta \).

Noting that

\[
\Phi_*(\omega_B) = \lim_{n \to \infty} \Psi_n(\omega_B \otimes \sigma) = \lim_{n \to \infty} [\text{Tr} P^n_C \sigma] \Phi_n(\omega_B),
\]

from (8.16) we find that the channel \( \Phi_* \) satisfies (8.14). Since

\[
\text{Id}_A \otimes \Phi_*(\omega_{AB}) = \text{Id}_A \otimes \Psi_*(\omega_{AB}) = \lim_{n \to \infty} \text{Id}_A \otimes \Psi_n(\omega_{AB}) = \lim_{n \to \infty} \text{Id}_A \otimes \Phi_n(\omega_{AB}^n),
\]

it follows from (8.15), (8.17) and (8.18) that \( \Phi_* \) satisfies (8.13). In this case (8.17) follows from the lower semicontinuity of \( I(A:C|B)_\omega \) and the fact that it is monotonic under local operations (Theorem 2).

Step 4. To relax the full rank condition for \( \omega_{BC} \) consider the sequence of states

\[
\omega^n_{ABC} = (1 - \varepsilon_n)\omega_{ABC} + \varepsilon_n \omega_{AB} \otimes \omega_C,
\]

where \( \varepsilon_n = 1/n \). Since \( \omega_B \otimes \omega_C \) is a full rank state, \( \omega^n_{BC} \) is a full rank state for each \( n \). By Step 3, for each \( n \) there exists a channel \( \Phi_n : B \to BC \) such that (8.15) and (8.16) hold. Since \( \omega^n_B = \omega_B \) and \( \omega^n_C = \omega_C \) for all \( n \), it follows from (8.16) and Lemma 7 that the sequence \( \{\Phi_n\} \) is relatively compact. So we may assume that \( \lim_{n \to \infty} \Phi_n = \Phi_* \) exists. It follows from (8.15)–(8.18) that the channel \( \Phi_* \) satisfies (8.13) and (8.14). In this case (8.17) follows from Lemma 8 below.

Proposition 13 is proved.

**Lemma 8.** Let \( \omega_{ABC} \) be an arbitrary state of a tripartite system and let \( \omega^\varepsilon_{ABC} = (1 - \varepsilon)\omega_{ABC} + \varepsilon \omega_{AB} \otimes \omega_C \), where \( \varepsilon \in (0,1) \). Then

\[
\lim_{\varepsilon \to +0} I(A:C|B)_{\omega^\varepsilon} = I(A:C|B)_\omega.
\]

**Proof.** First assume that \( I(A:C|B)_\omega \) is well defined by (6.2), that is,

\[
I(A:C|B)_\omega = H(\omega_{ABC} \parallel \omega_A \otimes \omega_{BC}) - H(\omega_{AB} \parallel \omega_A \otimes \omega_B).
\]

Since \( \omega^\varepsilon_X = \omega_X \) for \( X = A, B, C, AB \) and \( \omega^\varepsilon_{BC} = (1 - \varepsilon)\omega_{BC} + \varepsilon \omega_B \otimes \omega_C \), the joint convexity of the quantum relative entropy implies

\[
I(A:C|B)_{\omega^\varepsilon} = H(\omega^\varepsilon_{ABC} \parallel \omega_A \otimes \omega^\varepsilon_{BC}) - H(\omega_{AB} \parallel \omega_A \otimes \omega_B)
\]

\[
\leq (1 - \varepsilon)H(\omega_{ABC} \parallel \omega_A \otimes \omega_{BC}) + \varepsilon H(\omega_{AB} \parallel \omega_C \parallel \omega_A \otimes \omega_B \otimes \omega_C)
\]

\[
- H(\omega_{AB} \parallel \omega_A \otimes \omega_B) = (1 - \varepsilon)I(A:C|B)_\omega.
\]
Using approximation property (6.8) it is easy to show that
\[
I(A:C|B)_{\omega^\varepsilon} \leq (1 - \varepsilon)I(A:C|B)_{\omega}
\]
for any state \(\omega_{ABC}\). The assertion of the lemma follows from this inequality and the lower semicontinuity of \(I(A:C|B)_{\omega}\) (Theorem 2).

§ 9. Appendix: The proofs of Theorems 1 and 2

Proof of Theorem 1. A) We will first prove that
\[
\lim_{k \to \infty} I(A:B)_{\omega^k} = I(A:B)_{\omega^0}
\]
if condition (b) is valid, that is, if \(\lambda_k \omega^k \leq \Phi^k_A \otimes \Phi^k_B(\omega^0)\) for some sequences \(\{\Phi^k_A\}\) and \(\{\Phi^k_B\}\) of quantum operations and some sequence \(\{\lambda_k\}\) converging to 1.

We use the inequality
\[
\lambda I(A:B)_{\rho} + (1 - \lambda)I(A:B)_{\sigma} \leq I(A:B)_{\lambda \rho + (1 - \lambda)\sigma} + h_2(\lambda),
\]
where \(h_2(\lambda)\) is the binary entropy, which is valid for arbitrary operators \(\rho, \sigma \in \mathfrak{F}_+(\mathcal{H}_{AB})\) such that \(\max\{\text{Tr} \rho, \text{Tr} \sigma\} \leq 1\). If all the marginal entropies of the operators \(\rho\) and \(\sigma\) are finite then (9.2) follows directly from (4.1) and (2.3). In the general case (9.2) can be proved by approximating the operators \(\rho\) and \(\sigma\) by the sequences of operators
\[
\rho_k = P^k_A \otimes P^k_B \cdot \rho \cdot P^k_A \otimes P^k_B \quad \text{and} \quad \sigma_k = P^k_A \otimes P^k_B \cdot \sigma \cdot P^k_A \otimes P^k_B,
\]
where \(\{P^k_A\} \subset \mathfrak{B}(\mathcal{H}_A)\) and \(\{P^k_B\} \subset \mathfrak{B}(\mathcal{H}_B)\) are sequences of finite-rank projections converging strongly to the identity operators \(I_A\) and \(I_B\), respectively.

Now, (9.2) holds for the operators \(\rho_k\) and \(\sigma_k\) for all \(k\), and so it holds for the operators \(\rho\) and \(\sigma\) due to the relations
\[
\lim_{k \to +\infty} I(A:B)_{\omega^k} = I(A:B)_{\omega} \leq +\infty, \quad \omega = \rho, \sigma, \lambda \rho + (1 - \lambda)\sigma,
\]
which follow directly from the lower semicontinuity of the quantum mutual information and its monotonicity under local operations.

Inequality (9.2), and the nonnegativity and monotonicity of the quantum mutual information under local operations show that
\[
\lambda_k I(A:B)_{\omega^k} \leq I(A:B)_{\Phi^k_A \otimes \Phi^k_B(\omega^0)} + \gamma_k h_2(\lambda'_k) \leq I(A:B)_{\omega^0} + \gamma_k h_2(\lambda'_k),
\]
where \(\gamma_k = \text{Tr} \Phi^k_A \otimes \Phi^k_B(\omega^0) \leq \text{Tr} \omega^0\) and \(\lambda'_k = \gamma_k^{-1} \lambda_k \text{Tr} \omega_k \geq \lambda_k \text{Tr} \omega_k / \text{Tr} \omega^0\). This inequality and the lower semicontinuity of \(I(A:B)_{\omega}\) imply (9.1).

In the next part of the proof we use the identity
\[
I(A:B)_{\omega} + I(B:C)_{\omega} = 2H(\omega_B),
\]
which holds for any 1-rank operator \(\omega \in \mathfrak{F}_+(\mathcal{H}_{ABC})\) (the value \(+\infty\) is possible on both sides). If \(H(\omega_A), H(\omega_B)\) and \(H(\omega_C)\) are finite then (9.3) is easily verified by noting that \(H(\omega_A) = H(\omega_{BC}), H(\omega_B) = H(\omega_{AC})\) and \(H(\omega_C) = H(\omega_{AB})\). In the
general case (9.3) can be proved by approximating the operator $\omega$ by the sequence of operators

$$\omega^k = P^k_A \otimes P^k_B \otimes P^k_C \cdot \omega \cdot P^k_A \otimes P^k_B \otimes P^k_C,$$

where \{\(P^k_X\)\} \(\subset \mathfrak{B}(\mathcal{H}_X)\) is a sequence of finite-rank projections converging strongly to the identity operator \(I_X\), \(X = A, B, C\). Since (9.3) holds for each operator \(\omega^k\), it also holds for the operator \(\omega\) in view of the relations

$$\lim_{k \to +\infty} I(X:Y)_{\omega^k} = I(X:Y)_\omega \leq +\infty, \quad XY = AB, BC, \quad (9.4)$$

$$\lim_{k \to +\infty} H(\omega^k_B) = H(\omega_B) \leq +\infty. \quad (9.5)$$

Since \(\omega^k_{XY} \leq P^k_X \otimes P^k_Y \omega_{XY} P^k_X \otimes P^k_Y\), relations (9.4) follow from the continuity condition (b) proved above. Relation (9.5) follows from Lemma 2.

To prove condition (a), by symmetry it suffices to prove (9.1) assuming that the limit

$$\lim_{k \to +\infty} H(\omega^k_B) = H(\omega^0_B) < +\infty$$

exists. By Lemma 1 there exists a sequence \{\(\tilde{\omega}^k\)\} of rank-1 operators in \(\mathfrak{S}_+(\mathcal{H}_{AB})\) converging to an operator \(\tilde{\omega}^0\) such that \(\omega^k_{AB} = \omega^k\) for all \(k \geq 0\). Since the sequence \{\(\tilde{\omega}^k_{BC}\)\} converges to the state \(\tilde{\omega}^0_{BC}\), the lower semicontinuity of the quantum mutual information shows that

$$\liminf_{k \to +\infty} I(A:B)_{\tilde{\omega}^k} \geq I(A:B)_{\tilde{\omega}^0} \quad \text{and} \quad \liminf_{k \to +\infty} I(B:C)_{\tilde{\omega}^k} \geq I(B:C)_{\tilde{\omega}^0},$$

while \(\lim_{k \to +\infty} H(\tilde{\omega}^k_B) = H(\tilde{\omega}^0_B) < +\infty\) by assumption (as \(\tilde{\omega}^k_B = \omega^k_B\)). So identity (9.3) and Lemma 3 imply (9.1).

B) It is enough to look at the case when \(\Phi_A\) is an arbitrary operation and \(\Phi_B = \text{Id}_B\). We have to show that the continuity of the function \(I(A:B)_\omega\) on a subset \(\mathcal{A} \subset \mathfrak{S}_+(\mathcal{H}_{AB})\) implies that \(I(A':B)_{\Phi_A \otimes \text{Id}_B(\omega)}\) is continuous on \(\mathcal{A}\).

If \(\Phi_A\) is a quantum channel then this implication follows directly from part B) of Corollary 10, since by the Stinespring representation the channel \(\Phi_A\) is isomorphic to a subchannel of partial trace.

If \(\Phi_A\) is a trace nonpreserving operation then consider the channel \(\Psi_A = \Phi_A \oplus \Delta\) from \(A\) to \(A'' = A' \oplus A^c\), where \(\Delta(\rho) = [\text{Tr} \rho - \text{Tr} \Phi_A(\rho)]\sigma\) is a quantum operation from \(A\) to \(A^c\) determined by a fixed state \(\sigma \in \mathcal{S}(\mathcal{H}_{A^c})\). We have\(^{25}\)

$$I(A'' : B)_{\Psi_A \otimes \text{Id}_B(\omega_{AB})} \leq H(\Psi_A \otimes \text{Id}_B(\omega_{AB}) \| \Psi_A(\omega_A) \otimes \omega_B)$$

$$= H(\Phi_A \otimes \text{Id}_B(\omega_{AB}) \| \Phi_A(\omega_A) \otimes \omega_B) + H(\Delta \otimes \text{Id}_B(\omega_{AB}) \| \Delta(\omega_A) \otimes \omega_B)$$

$$= I(A' : B)_{\tilde{\omega}} + H(\tilde{\omega}_B \| \lambda \omega_B) + H(\Delta \otimes \text{Id}_B(\omega_{AB}) \| \Delta(\omega_A) \otimes \omega_B), \quad (9.6)$$

where \(\tilde{\omega}_{A'B} = \Phi_A \otimes \text{Id}_B(\omega_{AB})\) and \(\lambda = \text{Tr} \tilde{\omega}_{A'B}\). Since \(\Psi_A\) is a channel, by the above remark the continuity of \(I(A : B)_\omega\) on \(\mathcal{A}\) implies the continuity of the left-hand side of (9.6) on \(\mathcal{A}\). Since all the terms on the right-hand side of (9.6) are lower semicontinuous functions, Lemma 3 shows that all these terms are continuous on \(\mathcal{A}\).

Theorem 1 is proved.

\(^{25}\)Here we use the following property of the relative entropy \(H(\rho_1 + \rho_2 \| \sigma_1 + \sigma_2) = H(\rho_1 \| \sigma_1) + H(\rho_2 \| \sigma_2)\) if \(\rho_1 \rho_2 = \sigma_1 \sigma_2 = \rho_1 \sigma_2 = \sigma_1 \rho_2 = 0\) [10].
Proof of Theorem 2. To prove that the function with the stated properties is unique it suffices to assume that $F(\omega)$ is a lower semicontinuous function on the set $\mathcal{S}(\mathcal{H}_{ABC})$ which possesses property C3) and coincides with $I(A : C|B)_\omega$ given by formula (6.2) on the set of states with finite $I(A : B)_\omega$.

Chose any sequence of channels $\Phi^k_A : \mathfrak{T}(\mathcal{H}_A) \to \mathfrak{T}(\mathcal{H}_A)$ with finite output entropy such that $\lim_{k \to \infty} \Phi^k_A(\rho) = \rho$ for any $\rho \in \mathcal{S}(\mathcal{H}_A)$. Then the lower semicontinuity of $F$ and property C3) imply that

$$F(\omega) = \lim_{k \to \infty} F(\Phi^k_A \otimes \text{Id}_{BC}(\omega)) = \lim_{k \to \infty} I(A : C|B)_{\Phi^k_A \otimes \text{Id}_{BC}(\omega)}$$

for any state $\omega \in \mathcal{S}(\mathcal{H}_{ABC})$, where $I(A : C|B)_{\Phi^k_A \otimes \text{Id}_{BC}(\omega)}$ is defined by (6.2) since the upper bound (4.3) shows that $I(A : B)_{\Phi^k_A \otimes \text{Id}_{BC}(\omega)} < +\infty$ for all $k$. So $F(\omega)$ is uniquely defined.

By Remark 4 formulae (6.2)–(6.5) determine the $\mathfrak{T}$-extensions of the quantity $I(A : C|B)_\omega$ defined in (6.1) to the sets

$$\mathcal{S}_1 = \{ \omega_{ABC} \mid I(A : B)_\omega < +\infty \}, \quad \mathcal{S}_2 = \{ \omega_{ABC} \mid I(B : C)_\omega < +\infty \},$$

$$\mathcal{S}_3 = \{ \omega_{ABC} \mid H(\omega_B) < +\infty \} \quad \text{and} \quad \mathcal{S}_4 = \{ \omega_{ABC} \mid H(\omega_{ABC}) < +\infty \},$$

respectively.

By the uniqueness of the $\mathfrak{T}$-extension (Lemma 5) any pair of formulae (6.2)–(6.5) coincide on the set $\mathcal{S}_i \cap \mathcal{S}_j$ where both of them are well defined. So (6.2)–(6.5) give a well-defined $\mathfrak{T}$-extension of the quantity $I(A : C|B)$ defined in (6.1) to the set

$$\mathcal{S}_* = \bigcup_{i=1}^4 \mathcal{S}_i = \{ \omega_{ABC} \mid \min\{I(A : B)_\omega, I(B : C)_\omega, H(\omega_{ABC}), H(\omega_B)\} < +\infty \}$$

of states for which at least one of these formulae is well defined. It follows, in particular, that (6.2) and (6.3) coincide on the set

$$\mathfrak{T}_0 = \{ \omega \in \mathfrak{T}_+(\mathcal{H}_{ABC}) \mid I(A : B)_\omega < +\infty, I(B : C)_\omega < +\infty \}, \quad (9.7)$$

which contains the set $\mathfrak{T}_I = \{ \omega \in \mathfrak{T}_+(\mathcal{H}_{ABC}) \mid \text{rank} \omega_A < +\infty, \text{rank} \omega_C < +\infty \}$.

First we will prove the stated properties of $I_\omega(A : C|B)_\omega$ for the function

$$F(\omega) = \sup_{P_A, P_C} I(A : C|B)_\omega Q, \quad Q = P_A \otimes I_B \otimes P_C, \quad (9.8)$$

on the cone $\mathfrak{T}_+(\mathcal{H}_{ABC})$, where

$$I(A : C|B)_\omega Q = I(A : B)_\omega Q - I(A : B)_\omega Q = I(AB : C)_\omega Q - I(B : C)_\omega Q$$

(since $Q_\omega Q \in \mathfrak{T}_I$) and the supremum is taken over all finite-rank projections $P_A$ in $\mathfrak{B}(\mathcal{H}_A)$ and $P_C$ in $\mathfrak{B}(\mathcal{H}_C)$. Then we will show that $F(\omega)$ coincides with the function $I_\omega(A : C|B)_\omega$ defined by (6.6). By symmetry this will imply that (6.6) and (6.7) coincide.

By Corollary 1 the function $\omega \mapsto I(A : C|B)_\omega Q$, where $Q = P_A \otimes I_B \otimes P_C$, is continuous on the cone $\mathfrak{T}_+(\mathcal{H}_{ABC})$ for any finite-rank projections $P_A$ and $P_C$. Hence $F(\omega)$ is a lower semicontinuous function on $\mathfrak{T}_+(\mathcal{H}_{ABC})$. 
Lemma 9 below shows, by symmetry, that the set $\mathfrak{T}_0$ defined in (9.7) is invariant under local operations $\Phi_A : A \to A$ and $\Phi_C : C \to C$ and that

$$I(A:C|B)_\omega \geq I(A:C|B)_{\Phi_A \otimes \text{Id}_B \otimes \Phi_C(\omega)} \quad \text{for any } \omega \in \mathfrak{T}_0.$$  \hfill (9.9)

Hence

$$I(A:C|B)_\omega \geq I(A:C|B)_{Q_\omega Q} \quad \text{for any } \omega \in \mathfrak{T}_0,$$

where $Q = P_A \otimes I_B \otimes P_C$, for any finite-rank projections $P_A$ and $P_C$. It follows that

$$F(\omega) = I(A:C|B)_\omega \quad \text{for any } \omega \in \mathfrak{T}_I,$$

where $I(A:C|B)_\omega$ is given by either of (6.2) and (6.3).

The lower semicontinuity of $F$ implies

$$F(\omega) = \lim_{k \to \infty} F(Q_k \omega Q_k), \quad Q_k = P_A^k \otimes I_B \otimes P_C^k,$$  \hfill (9.11)

for an arbitrary operator $\omega \in \mathfrak{T}_+(\mathcal{H}_{ABC})$ and any sequences $\{P_A^k\} \subset \mathfrak{B}(\mathcal{H}_A)$ and $\{P_C^k\} \subset \mathfrak{B}(\mathcal{H}_C)$ of finite-rank projections converging strongly to the identity operators $I_A$, $I_C$, since (9.8) and (9.10) show that

$$F(\omega) \geq I(A:C|B)_{Q_k \omega Q_k} = F(Q_k \omega Q_k) \quad \text{for all } k.$$

By the definition of an $\mathfrak{F}$-extension (9.11) implies $F(\omega) = I(A : C|B)_\omega$ for any state $\omega$ in the set $\mathfrak{S}_*$ defined above.

The nonnegativity of the function $F$ (the first part of C1) follows from its definition (as the relative entropy is monotonic under partial trace).

To prove C2) note that for any state $\omega$ formula (6.3) implies that C2) holds for all the operators $Q_k \omega Q_k$ in (9.11). So using the faithfulness of the quantum mutual information and (9.11) we obtain

$$I(AB:C)_\omega = \lim_{k \to \infty} I(AB:C)_{Q_k \omega Q_k} \geq \lim_{k \to \infty} F(Q_k \omega Q_k) = F(\omega).$$

To prove C3) note that for any state $\omega$ all the operators $Q_k \omega Q_k$ in (9.11) belong to the set $\mathfrak{T}_I \subset \mathfrak{T}_0$. So using (9.9)–(9.11) and the lower semicontinuity of $F$, we obtain

$$F(\omega) = \lim_{k \to \infty} F(Q_k \omega Q_k) \geq \liminf_{k \to \infty} F(\Psi(Q_k \omega Q_k)) \geq F(\Psi(\omega)),$$

where $\Psi = \Phi_A \otimes \text{Id}_B \otimes \Phi_C$.

That property (3.2) holds for any state in the set $\mathfrak{S}_*$ follows because an $\mathfrak{F}$-extension of $I(A:C|B)$ to this set exists and coincides with $F$ as we proved above. Property (6.8) means that

$$\lim_{l \to \infty} F(\omega^{kl}) = F(\omega^k) \quad \text{and} \quad \lim_{k \to \infty} F(\omega^k) = F(\omega),$$

\[\text{To simplify our notation we consider quantum operations } A \to A \text{ and } C \to C. \text{ The generalization to quantum operations } A \to A' \text{ and } C \to C' \text{ is obvious.}\]
where \( \omega^k = \lim_{l \to \infty} \omega^{kl} \). Since the state \( \omega^k \) belongs to \( \mathcal{G}_* \) for each \( k \), \((3.2)\) holds for this state, which implies the first of these limit relations. The second follows from the lower semicontinuity of \( F \) and property C3) proved above.

Now we can show that the function \( I_e(A:C|B) \) defined by \((6.6)\) coincides with \( F \) defined by \((9.8)\).

Let \( Q_1 = P_A \otimes I_B \otimes I_C \) and \( Q_2 = P_A \otimes I_B \otimes P_C \). The properties of \( F \) that we have proved imply that
\[
I(A:C|B)_{Q_1} = F(Q_1 \omega_{Q_1}) \geq F(Q_2 \omega_{Q_2}) = I(A:C|B)_{Q_2 \omega_{Q_2}},
\]
\[
I(A:C|B)_{Q_1} = F(Q_1 \omega_{Q_1}) \leq F(\omega)
\]
for any state \( \omega \). The first of these inequalities implies that \( I_e(A:C|B)_\omega \geq F(\omega) \), while the second shows that \( I_e(A:C|B)_\omega \leq F(\omega) \).

Properties C4) and C5) of \( I_e(A:C|B)_\omega \) can be derived by approximation from the same properties of the conditional mutual information in the finite-dimensional setting. In the case of C4) it suffices to use property \((6.8)\). In the case of C5) the second part\(^{27}\) of Corollary 9 (that is, property \((6.11)\)) is necessary.

To prove the second part of C1) note that
\[
I_e(A:C|B)_\omega = 0
\]
implies, by Proposition 13 (proved independently), that there exists a channel \( \Phi : B \to BC \) such that \( \omega_{ABC} = \text{Id}_A \otimes \Phi(\omega_{AB}) \). The converse statement follows from the definition \((6.6)\) of \( I_e(A:C|B)_\omega \).

Theorem 2 is proved.

**Lemma 9.** For an arbitrary quantum operation \( \Phi_A : A \to A \) the set \( \mathfrak{T}_0 \) is invariant under the map \( \Phi_A \otimes \text{Id}_{BC} \) and \( I(A:C|B)_\omega \geq I(A:C|B)_{\Phi_A \otimes \text{Id}_{BC}(\omega)} \) for any \( \omega \in \mathfrak{T}_0 \).

**Proof.** If \( \Phi_A \) is a channel then the assertion of the lemma follows directly from the monotonicity of the quantum mutual information and formula \((6.3)\), since in this case \( I(B:C)_{\Phi_A \otimes \text{Id}_{BC}(\omega)} = I(B:C)_\omega \) for any state \( \omega_{ABC} \).

If \( \Phi_A \) is a trace nonpreserving quantum operation then consider the channel \( \Psi_A = \Phi_A \oplus \Delta \) from \( A \to A' = A \oplus A' \), where \( \Delta(\rho) = [\text{Tr} \rho - \text{Tr} \Phi_A(\rho)]\sigma \) is a quantum operation from \( A \to A' \) determined by a fixed state \( \sigma \in \mathcal{S}(\mathcal{H}_{A'}) \).

Let \( \tilde{\omega}_{ABC} = \Phi_A \otimes \text{Id}_{BC}(\omega_{ABC}) \), \( \omega_{ABC}^c = \Delta \otimes \text{Id}_{BC}(\omega_{ABC}) \) and let \( \lambda = \text{Tr} \tilde{\omega}_{ABC} \).

To prove that \( \mathfrak{T}_0 \) is invariant it suffices to note that \( I(A:B)_\omega \leq I(A:B)_{\omega} \) as the quantum mutual information is monotonic and that inequality \((9.2)\) implies that \( I(B:C)_{\tilde{\omega}} \leq I(B:C)_{\tilde{\omega}} + h_2(\lambda) = I(B:C)_\omega + h_2(\lambda) \) (the last equality holds since \( \Psi_A \) is a quantum channel).

Similarly to \((9.6)\),
\[
I(A'B:C)_{\tilde{\omega}} + h_2(\lambda) = I(AB:C)_{\tilde{\omega}} + H(\tilde{\omega}_C \parallel \lambda \omega_C) + H(\omega_{ABC}^c \parallel \omega_{AB}^c \otimes \omega_C), \tag{9.12}
\]
while the joint convexity of the relative entropy and relation \((2.4)\) imply that
\[
I(B:C)_{\tilde{\omega}} + h_2(\lambda) = H(\tilde{\omega}_C \parallel \lambda \omega_C) + H(\omega_{ABC}^c \parallel \omega_{AB}^c \otimes \omega_C).
\]

\(^{27}\)Property C5) of \( I_e(A:C|B)_\omega \) is not used in the proof of Corollary 9.
We have $I(A'B'C)\omega+\omega^c \leq I(AB:C)\omega$ and $I(B:C)\omega+\omega^c = I(B:C)\omega$ (since $\Psi_A$ is a channel). Hence it follows from (9.12) and (9.13) that

$$I(AB:C)\omega - I(B:C)\omega \leq I(AB:C)\omega - I(B:C)\omega - \delta,$$

where $\delta = H(\omega^c_{ABC} || \omega^c_{AB} \otimes \omega_C) - H(\omega^c_{BC} || \omega^c_{B} \otimes \omega_C) > 0$ by the monotonicity of the relative entropy.

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**Bibliography**

[1] A.S. Holevo, *Quantum systems, channels, information*, MCCME, Moscow 2010, 328 pp.; English transl. *Quantum systems, channels, information. A mathematical introduction*, De Gruyter Stud. Math. Phys., vol. 16, De Gruyter, Berlin 2012, xiii+349 pp.

[2] M.A. Nielsen and I.L. Chuang, *Quantum computation and quantum information*, Cambridge Univ. Press, Cambridge 2000, xxvi+676 pp.

[3] A. Wehrl, “General properties of entropy”, *Rev. Modern Phys.* 50:2 (1978), 221–260.

[4] G. Lindblad, “Entropy, information and quantum measurements”, *Comm. Math. Phys.* 33:4 (1973), 305–322.

[5] A.A. Kuznetsova, “Conditional entropy for infinite-dimensional quantum systems”, *Teor. Veroyatnost. i Primenen.* 55:4 (2010), 782–790; English transl. in *Theory Probab. Appl.* 55:4 (2011), 709–717.

[6] Jianxin Chen, Zhengfeng Ji, Chi-Kwong Li, Yiu-Tung Poon, Yi Shen, Nengkun Yu, Bei Zeng and Duanlu Zhou, “Discontinuity of maximum entropy inference and quantum phase transitions”, *New J. Phys.* 17 (2015), 083019, 18 pp.

[7] L. Rodman, I.M. Spitkovsky, A. Szkola and S. Weis, “Continuity of the maximum-entropy inference: convex geometry and numerical ranges approach”, *J. Math. Phys.* 57:1 (2016), 015204, 17 pp.; arXiv:1502.02018.

[8] S. Weis and A. Knauf, “Entropy distance: new quantum phenomena”, *J. Math. Phys.* 53:10 (2012), 102206, 25 pp.

[9] O. Fawzi and R. Renner, “Quantum conditional mutual information and approximate Markov chains”, *Comm. Math. Phys.* 340:2 (2015), 575–611; arXiv:1410.0664.

[10] G. Lindblad, “Expectations and entropy inequalities for finite quantum systems”, *Comm. Math. Phys.* 39:2 (1974), 111–119.

[11] E.H. Lieb and M.B. Ruskai, “Proof of the strong subadditivity of quantum mechanical entropy”, *J. Mathematical Phys.* 14 (1973), 1938–1941.

[12] M.E. Shirokov, “Squashed entanglement in infinite dimensions”, *J. Math. Phys.* 57:3 (2016), 032203, 22 pp.; arXiv:1507.08964.

[13] R. Alicki and M. Fannes, “Continuity of quantum conditional information”, *J. Phys. A* 37:5 (2004), L55–L57.

[14] A. Winter, “Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints”, *Comm. Math. Phys.* (to appear).
[15] K. M. R. Audenaert, “A sharp continuity estimate for the von Neumann entropy”, 
J. Phys. A 40:28 (2007), 8127–8136.

[16] Nan Li and Shunlong Luo, “Classical and quantum correlative capacities of 
quantum systems”, Phys. Rev. A 84:4 (2011), 042124.

[17] R. R. Tucci, Entanglement of distillation and conditional mutual information, 
arXiv:quant-ph/0202144.

[18] N. J. Cerf, S. Massar and S. Schneider, “Multiparty classical and quantum secrecy 
monotones”, Phys. Rev. A 66:4 (2002), 042309, 13 pp.

[19] F. Herbut, “On mutual information in multipartite quantum states and equality in 
strong subadditivity of entropy”, J. Phys. A 37:10 (2004), 3535–3542.

[20] Dong Yang, K. Horodecki, M. Horodecki, P. Horodecki, J. Oppenheim and 
Wei Song, “Squashed entanglement for multipartite states and entanglement 
measures based on the mixed convex roof”, IEEE Trans. Inform. Theory 55:7 
(2009), 3375–3387.

[21] F. Furrer, J. Aberg and R. Renner, “Min- and max-entropy in infinite dimensions”, 
Comm. Math. Phys. 306:1 (2011), 165–186.

[22] A. S. Holevo and M. E. Shirokov, “On classical capacities of infinite-dimensional 
quantum channels”, Probl. Peredachi Inf. 49:1 (2013), 19–36; English transl. in 
Problems Inform. Transmission 49:1 (2013), 15–31.

[23] M. Christandl and A. Winter, ““Squashed entanglement”: an addtive entanglement 
measure”, J. Math. Phys. 45:3 (2004), 829–840.

[24] I. Devetak and J. Yard, “Exact cost of redistributing multipartite quantum states”, 
Phys. Rev. Lett. 100:23 (2008), 230501.

[25] Lin Zhang, “Conditional mutual information and commutator”, Internat. 
J. Theoret. Phys. 52:6 (2013), 2112–2117.

[26] P. Hayden, R. Jozsa, D. Petz and A. Winter, “Structure of states which satisfy 
strong subadditivity of quantum entropy with equality”, Comm. Math. Phys. 246:2 
(2004), 359–374.

[27] A. Jakulin and I. Bratko, Quantifying and visualizing attribute interactions, arXiv: 
cs/0308002.

[28] W. J. McGill, “Multivariate information transmission”, Psychometrika 19:2 (1954), 
97–116.

[29] A. Kitaev and J. Preskill, “Topological entanglement entropy”, Phys. Rev. Lett. 
96:11 (2006), 110404, 4 pp.

[30] H. Casini and M. Huerta, “Remarks on the entanglement entropy for disconnected 
regions”, J. High Energy Phys., 2009, no. 3, 048, 19 pp.; arXiv:0812.1773.

[31] P. Hayden, M. Headrick and A. Maloney, “Holographic mutual information is 
monogamous”, Phys. Rev. D 87:4 (2013), 046003.

[32] A. Kumar, Multiparty operational quantum mutual information, arXiv:1504.07176.

[33] Dong Yang, M. Horodecki and Z. D. Wang, “An additive and operational 
entanglement measure: conditional entanglement of mutual information”, Phys. 
Rev. Lett. 101:14 (2008), 140501.

[34] M. M. Wilde, “Multipartite quantum correlations and local recoverability”, Proc. 
Roy. Soc. A 471:2177 (2015), 20140941, 20 pp.

[35] M. M. Wilde, “Recoverability in quantum information theory”, Proc. Roy. Soc. A 
471:2182 (2015), 20150338, 19 pp.; arXiv:1505.04661.

[36] R. Alicki, Isotropic quantum spin channels and additivity questions, arXiv: 
quant-ph/0402080.
A. S. Holevo, “The entropy gain of infinite-dimensional quantum evolutions”, *Dokl. Ross. Akad. Nauk* 434:2 (2010), 173–174; English transl. in *Dokl. Math.* 82:2 (2010), 730–731.

A. S. Holevo and M. E. Shirokov, “Mutual and coherent information for infinite-dimensional quantum channels”, *Probl. Peredachi Inf.* 46:3 (2010), 3–21; English transl. in *Problems Inform. Transmission* 46:3 (2010), 201–218.

C. Adami and N. J. Cerf, “Von Neumann capacity of noisy quantum channel”, *Phys. Rev. A* 56:5 (1997), 3470–3483.

C. H. Bennett, P. W. Shor, J. A. Smolin and A. V. Thapliyal, “Entanglement-assisted classical capacity of noisy quantum channel”, *Phys. Rev. Lett.* 83 (1999), 3081–3084.

V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Stud. Adv. Math., vol. 78, Cambridge Univ. Press, Cambridge 2002, xii+300 pp.

M. E. Shirokov and A. S. Holevo, “On approximation of infinite-dimensional quantum channels”, *Probl. Peredachi Inf.* 44:2 (2008), 3–22; English transl. in *Problems Inform. Transmission* 44:2 (2008), 73–90.

D. Kretschmann, D. Schlingemann and R. F. Werner, *A continuity theorem for Stinespring’s dilation*, arXiv:0710.2495.

D. Leung and G. Smith, “Continuity of quantum channel capacities”, *Comm. Math. Phys.* 292:1 (2009), 201–215.

A. S. Holevo, “Entanglement-assisted capacities of constrained quantum channels”, *Teor. Veroyatnost. i Primenen.* 48:2 (2003), 359–374; English transl. in *Theory Probab. Appl.* 48:2 (2004), 243–255.

M. E. Shirokov, “Entropy characteristics of subsets of states. І”, *Izv. RAN. Ser. Mat.* 70:6 (2006), 193–222; English transl. in *Izv. Math.* 70:6 (2006), 1265–1292.

D. Sutter, O. Fawzi and R. Renner, “Universal recovery map for approximate Markov chains”, *Proc. Roy. Soc. A* 472:2186 (2016), 20150623, 26 pp.; arXiv: 1504.07251.

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