LARGE AUTOMORPHISM GROUPS OF ORDINARY CURVES IN CHARACTERISTIC $p$

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Abstract. Let $X$ be a (projective, non-singular, geometrically irreducible) curve defined over an algebraically closed field $K$ of characteristic $p$. If the $p$-rank $\gamma(X)$ equals the genus $g(X)$, then $X$ is ordinary. In this paper, we deal with large automorphism groups $G$ of ordinary curves. On the one hand, for $p = 2$ and $G$ solvable, we prove that $|G| < 35(g(X) - 1)^{3/2}$. On the other hand, for $p > 2$ and $g(X)$ even, we prove that $|G| \leq 919g(X)^{7/4}$. For the sporadic cases Aut$(X) \cong \text{Alt}_7$ and Aut$(X) \cong M_{11}$, the classical Hurwitz bound $|\text{Aut}(X)| \leq 84(g(X) - 1)$ holds, unless $p = 3$, $g(X) = 26$ and Aut$(X) \cong M_{11}$. An important example here is given by the modular curve $X(11)$.

1. Introduction

Let $p > 0$ be a prime. By a curve $X$ we mean a projective, non-singular, geometrically irreducible curve defined over an algebraically closed field $K$ of characteristic $p$. Usually, the study of the geometry of $X$ is carried out through the study of its birational invariants, such as its genus $g(X)$, its $p$-rank (or Hasse-Witt invariant) $\gamma(X)$, and its automorphism group Aut$(X)$. The genus can be thought of as the dimension of the $K$-vector space of holomorphic differentials on $X$, while the $p$-rank is the dimension of the $K$-space of holomorphic logarithmic differentials. A curve $X$ is ordinary if $g(X) = \gamma(X)$. An automorphism of $X$ arises by defining the concept of a field automorphism $\sigma$ of the function field $K(X)$ fixing the ground field $K$ elementwise. From a purely geometric point of view, $X$ admits a non-singular model in some projective space $\text{PG}(r,K)$ for $r \geq g(X)$, and every automorphism of $X$ can be represented by a linear collineation in $\text{PGL}(r+1,K)$ leaving $X$ invariant.

Henceforth, we shall denote the full automorphism group of $X$ by Aut$(X)$. By a classical result, Aut$(X)$ is finite whenever $g(X) \geq 2$. Also, if the characteristic $p$ of $K$ divides $|\text{Aut}(X)|$, then the classical Hurwitz bound $|\text{Aut}(X)| \leq 84(g(X) - 1)$ no longer holds. A major achievement here, due to Henn, is the classification of all the curves admitting an automorphism group whose size exceeds $8g(X)^3$. On the one hand, as all such curves have zero $p$-rank, then a classical problem is to find a function $f(g)$ depending on the genus $g$ such that all curves $X$ such that $|\text{Aut}(X)| > f(g(X))$ have zero $p$-rank. Such a problem has gained much attention since the early 2000’s, and several recent results point out that such a function

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is \( f(g) = cg^2 \) for some constant \( c \); see for instance [2] for the case when \( g(\mathcal{X}) \) is even. Loosely speaking, the hypothesis that \( 2|g(\mathcal{X}) \) gives strong restrictions on the structure of a Sylow 2-subgroup of \( \text{Aut}(\mathcal{X}) \), whence the powerful tools from finite group theory can be exploited. On the other hand, a general curve is ordinary, so it is natural to seek for bound on the size of the automorphism group \( \text{Aut}(\mathcal{X}) \) of ordinary curves. If this is the case, a bound of the type \( |\text{Aut}(\mathcal{X})| \leq c(p)g(\mathcal{X})^{8/5} \) for some constant \( c(p) \) depending on \( p \) is expected; see for instance [12]. If \( \text{Aut}(\mathcal{X}) \) is solvable and \( p > 2 \), an even tighter bound

\[
|\text{Aut}(\mathcal{X})| \leq 34(g(\mathcal{X}) + 1)^{3/2}
\]

holds; see [10]. The latter bound is sharp up to the constant term; see [11].

The object of this paper is two-fold. In Section 3, we establish the characteristic 2 analogue of [11]. Two sporadic cases then arise: \( \text{Aut}(\mathcal{X}) \cong \text{Alt}_7, M_{11}, \) where \( M_{11} \) is the Mathieu group of degree 11. In these cases, the classical Hurwitz bound for \( |\text{Aut}(\mathcal{X})| \) holds, unless \( p = 3, g(\mathcal{X}) = 26, \text{Aut}(\mathcal{X}) = M_{11}, \) an example being given by the modular curve \( X(11) \), see Section 4.

2. Background and Preliminary Results

2.1. Automorphisms of algebraic curves. Our notation and terminology are standard. Well-known references for the theory of curves and algebraic function fields are [4] and [16]. Let \( \mathcal{X} \) be a curve defined over an algebraically closed field \( K \) of positive characteristic \( p \) for some prime \( p \). We denote by \( K(\mathcal{X}) \) the function field of \( \mathcal{X} \). By a point \( P \in \mathcal{X} \) we mean a point in a nonsingular model of \( \mathcal{X} \); in this way, we have a one-to-one correspondence between points of \( \mathcal{X} \) and places of \( K(\mathcal{X}) \). Let \( \text{Aut}_K(\mathcal{X}) \) denote the full automorphism group of \( \mathcal{X} \). For a subgroup \( S \) of \( \text{Aut}_K(\mathcal{X}) \), we denote by \( K(\mathcal{X})^S \) the fixed field of \( S \). A nonsingular model \( \mathcal{X} \) of \( K(\mathcal{X})^S \) is referred as the quotient curve of \( \mathcal{X} \) by \( S \) and denoted by \( \mathcal{X}/S \). Note that \( \mathcal{X}/S \) is defined up to birational equivalence. The field extension \( K(\mathcal{X}) : K(\mathcal{X})^S \) is Galois with Galois group \( S \). For a point \( P \in \mathcal{X}, \) \( S(P) \) is the orbit of \( P \) under the action of \( S \) on \( \mathcal{X} \) seen as a point-set. The orbit \( S(P) \) is said to be long if \( |S(P)| = |S| \), short otherwise. There is a one-to-one correspondence between short orbits and ramified points in the extension \( K(\mathcal{X}) : K(\mathcal{X})^S \). It might happen that \( S \) has no short orbits; if this is the case, the cover \( \mathcal{X} \to \mathcal{X}/S \) (or equivalently, the extension \( K(\mathcal{X}) : K(\mathcal{X})^S \) ) is unramified. On the other hand, \( S \) has a finite number of short orbits.

For \( P \in \mathcal{X} \), the subgroup \( S_P \) of \( S \) consisting of all elements of \( S \) fixing \( P \) is called the stabilizer of \( P \) in \( S \). For a non-negative integer \( i \), the \( i \)-th ramification group of \( \mathcal{X} \) at \( P \) is denoted by \( S_P^{(i)} \), and defined by

\[
S_P^{(i)} = \{\sigma \mid v_P(\sigma(t) - t) \geq i + 1, \sigma \in S_P\},
\]

where \( t \) is a local parameter at \( P \) and \( v_P \) is the respective discrete valuation. Here \( S_P = S_P^{(0)} \). Furthermore, \( S_P^{(1)} \) is the only Sylow \( p \)-subgroup of \( S_P^{(0)} \), and the factor group \( S_P^{(0)}/S_P^{(1)} \) is cyclic of
order prime to \( p \); see e.g. \cite{[13]} Theorem 11.74. In particular, if \( S_P \) is a \( p \)-group, then \( S_P = S_P^{(0)} = S_P^{(1)} \).

Let \( g \) and \( \bar{g} \) be the genus of \( X \) and \( \bar{X} = X/S \), respectively. The Hurwitz genus formula is

\[
2g - 2 = |S|(2\bar{g} - 2) + \sum_{\gamma \in G} \sum_{i \geq 0} (|S|_{\gamma}^{(i)} - 1);
\]

see \cite{[13]} Theorem 11.72. If \( \ell_1, \ldots, \ell_k \) are the sizes of the short orbits of \( S \), then \cite{[2]} yields

\[
2g - 2 \geq |S|(2\bar{g} - 2) + \sum_{\nu=1}^k (|S| - \ell_\nu),
\]

and equality holds if \( \gcd(|S_P|, p) = 1 \) for all \( P \in X \); see \cite{[4]} Theorem 11.57 and Remark 11.61.

Let \( \gamma = \gamma(X) \) denote the \( p \)-rank (equivalently, the Hasse-Witt invariant of \( X \)). If \( S \) is a \( p \)-subgroup of \( \text{Aut}_K(X) \) then the Deuring-Shafarevich formula, see \cite{[4]} Theorem 11.62, states that

\[
\gamma - 1 = |S|/\bar{\gamma} - 1 + \sum_{i=1}^k (|S| - \ell_i),
\]

where \( \bar{\gamma} = \gamma(X/S) \) is the \( p \)-rank of \( X/S \) and \( \ell_1, \ldots, \ell_k \) denote the sizes of the short orbits of \( S \). Both the Hurwitz and Deuring-Shafarevich formulas hold true for rational and elliptic curves provided that \( G \) is a finite subgroup. A subgroup of \( \text{Aut}_K(X) \) is a \( p' \)-group (or a prime to \( p \)) group if its order is prime to \( p \). A subgroup \( G \) of \( \text{Aut}_K(X) \) is tame if the 1-point stabilizer of any point in \( G \) is \( p' \)-group. Otherwise, \( G \) is non-tame (or wild). Every \( p' \)-subgroup of \( \text{Aut}_K(X) \) is tame, but the converse is not always true. If \( G \) is tame, then the classical Hurwitz bound \( |G| \leq 84(\gamma(X) - 1) \) holds, but for non-tame groups this is far from being true. A result by Henn, however, states that \( |G| \leq 8g(X)^3 \) unless \( X \) is isomorphic to one of four exceptional curves, see \cite{[4]} Theorem 11.127. It should be noted that all such exceptions have zero \( p \)-rank.

In this paper we deal with ordinary curves, that is, curves for which equality \( g(X) = \gamma(X) \) holds. We now collect some results regarding automorphism groups of ordinary curves.

**Theorem 2.1** \cite{[13]}, Theorem 3. Let \( X \) be an ordinary curve with \( g(X) \geq 2 \). Then the following inequality holds:

\[
|\text{Aut}(X)| \leq 84(g(X) - 1)g(X).
\]

**Theorem 2.2** \cite{[13]}, Theorem 2(i). Let \( X \) be ordinary, and let \( G \) be a finite subgroup of \( \text{Aut}(X) \). Then for every point \( P \) of \( X \), \( G_2(P) \) is trivial.

**Theorem 2.3** \cite{[13]}. Let \( X \) be an ordinary curve, \( G \) a finite subgroup of \( \text{Aut}(X) \) and \( Y = X/G \). If \( |G| > 84(\gamma(X) - 1) \), then one of the following holds.

(i) \( p \geq 3 \) and \( g(Y) = 0 \); \( e_Q = 1 \) if \( Q \neq Q_1, Q_2, Q_3 \in Y \), \( p|e_Q_1, \ e_Q_2 = e_Q_3 = 2 \);
(ii) \( g(Y) = 0 \); \( e_Q = 1 \) if \( Q \neq Q_1, Q_2 \in Y \); \( p|e_Q_1, e_Q_2 \);
(iii) \( g(Y) = 0 \); \( e_Q = 1 \) if \( Q \neq Q_1, Q_2 \in Y \); \( p|e_Q_1, p \nmid e_Q_2 \).

2.2. **Solvable automorphism groups of ordinary curves.** The next three results can be found in \cite{[10]} and give bounds as well as conditions on the structure of the Sylow \( p \)-subgroups of solvable automorphism groups of ordinary curves.
Lemma 2.7. Let $\mathcal{X}$ be an ordinary algebraic curve of genus $g$ and p-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. If $G$ is a solvable subgroup of $\text{Aut}(\mathcal{X})$ or $G$ admits an elementary abelian minimal normal subgroup, then $|G| \leq cg^{3/2}$ for some constant $c$.

Corollary 2.5. Let $\mathcal{X}$ be an ordinary algebraic curve of genus $g$ and p-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. If $G$ is a solvable subgroup of $\text{Aut}(\mathcal{X})$ or $G$ admits an elementary abelian minimal normal subgroup, then a Sylow $p$-subgroup of $G$ is also elementary abelian.

Remark 2.6. Let $\mathcal{X}$ be an ordinary algebraic curve of genus $g$ and p-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. If $G$ admits an elementary abelian minimal normal p-subgroup $Q$, $G$ has exactly two short orbits which are both non-tame, and $G = Q \times U$ where $U$ is an abelian tame subgroup then $|U| < \sqrt{4g+4}$.

2.3. Automorphism groups of curves of even genus. In this subsection, $K$ is an algebraically closed field of odd characteristic $p$, $\mathcal{X}$ is an algebraic curve whose genus $g$ is a (nonzero) even integer, and $G$ is an automorphism group of $\mathcal{X}$. The following results can be found in [2].

Lemma 2.7. Let $G$ be a solvable automorphism group of an algebraic curve of genus $g \geq 2$ containing a normal $d$-subgroup $Q$ of odd order such that $|Q|$ and $[G:Q]$ are coprime. Suppose that a complement $U$ of $Q$ in $G$ is abelian, and that $N_G(U) \cap Q = \{1\}$. If
\begin{equation}
|G| = c(g - 1) \quad \text{with} \quad c \geq 30,
\end{equation}
then $d = p$ and $U$ is cyclic. Moreover, the quotient curve $\tilde{\mathcal{X}} = \mathcal{X}/Q$ is rational and either
\begin{enumerate}[label=(\roman*)]
\item $\mathcal{X}$ has positive p-rank, $Q$ has exactly two (non-tame) short orbits, and they are also the only short orbits of $G$; or
\item $\mathcal{X}$ has zero p-rank and $G$ fixes a point.
\end{enumerate}

Lemma 2.8. $G$ contains no elementary abelian 2-subgroup of order 8.

Lemma 2.9. If $G$ has a non-abelian minimal normal subgroup $N$ then $N$ is a simple group and one of the following cases occurs.

\begin{enumerate}[label=(\roman*)]
\item $N \cong \text{PSL}(2, q)$ with $q \equiv 5 \text{ (mod 4)}$ odd, and a Sylow 2-subgroup of $N$ is dihedral;
\item $N \cong \text{PSL}(3, q)$ with $q \equiv 3 \text{ (mod 4)}$, and a Sylow 2-subgroup of $N$ is semidihedral;
\item $N \cong \text{PSU}(3, q)$ with $q \equiv 1 \text{ (mod 4)}$, and a Sylow 2-subgroup of $N$ is semidihedral;
\item $N \cong \text{Alt}_7$, and a Sylow 2-subgroup of $N$ is dihedral;
\item $N \cong M_{11}$, the Mathieu group on 11 letters, and a Sylow 2-subgroup of $N$ is semidihedral.
\end{enumerate}

Lemma 2.10. If $G$ is an odd core-free group with a non-abelian simple minimal normal subgroup, then one of the following cases occurs, up to group isomorphisms, where $q$ is a prime power $d^k$ with $k$ odd:

\begin{enumerate}[label=(\roman*)]
\item $\text{PSL}(2, q) \leq G \leq \text{PGL}(2, q)$ with $q \geq 5$;
\item $\text{PSL}(3, q) \leq G \leq \text{PGL}(3, q)$ with $q \equiv 3 \text{ (mod 4)}$;
\item $\text{PSU}(3, q) \leq G \leq \text{PGL}(3, q)$ with $q \equiv 1 \text{ (mod 4)}$;
\item $G = \text{Alt}_7$;
\item $G = M_{11}$.
\end{enumerate}

Remark 2.11. The subgroups of $\text{PGL}(2, q)$ containing $\text{PSL}(2, q)$ whose Sylow 2-subgroups have 2-rank 2 are $\text{PSL}(2, q), \text{PGL}(2, q)$ and, when $q = d^k$ with an odd prime $d$ and $k \geq 2$, the semidirect
Proof. We have as the quotient curve $\overline{U}$ then

$$\tag{7}$$

Furthermore, from the Hurwitz genus formula applied to $Q$ whicne $\Hurwitz$ applied to $Q$ such that $Q$ is ordinary and admits the Symmetric group $\Sym_5 \cong \PGL(2)$ as an automorphism group.

3. Solvable subgroups of $\Aut_{\K}(\mathcal{X})$ for $p = 2$

Throughout this Section, $\mathcal{X}$ is a curve defined over an algebraically closed field $K$ of characteristic 2.

**Lemma 3.1.** Let $H$ be a solvable automorphism group of $\Aut(\mathcal{X})$ containing a normal 2-subgroup $Q$ such that $|H : Q|$ is odd. Suppose that a complement $U$ of $Q$ in $H$ is abelian. If

$$|H| > 12(g - 1),$$

then $U$ is cyclic. Moreover, the quotient curve $\mathcal{X}/Q$ is rational and either

(i) $\mathcal{X}$ has positive $p$-rank, $Q$ has exactly two (non-tame) short orbits, and they are also the only short orbits of $H$; or

(ii) $\mathcal{X}$ has zero $p$-rank and $H$ fixes a point.

**Proof.** We have $H = Q \times U$. Set $|Q| = 2^k$, $|U| = u$. Three cases are treated separately according as the quotient curve $\mathcal{X} = \mathcal{X}/Q$ has genus $\bar{g}$ at least 2, or $\mathcal{X}$ is elliptic, or rational.

If $\bar{g} \geq 2$, then $\Aut(\mathcal{X})$ has a subgroup isomorphic to $U$, and [4] Theorem 11.79 implies $4\bar{g} + 4 \geq |U|$. Furthermore, from the Hurwitz genus formula applied to $Q$, $\bar{g} - 1 \geq |Q|/(\bar{g} - 1)$. Therefore,

$$(4\bar{g} + 4)|Q| \geq |U||Q| = |H| > 12(g - 1) \geq 12|Q|/(\bar{g} - 1),$$

whence

$$12 < 4 \frac{\bar{g} + 1}{\bar{g} - 1} \leq 12$$

contradicting (7).

If $\mathcal{X}$ is elliptic, then $\mathcal{X} \to \mathcal{X}$ ramifies, otherwise $\mathcal{X}$ itself would be elliptic. Thus, $Q$ has some short orbits. Take one of them together with its images $o_1, \ldots, o_m$ under the action of $H$. Since $Q$ is a normal subgroup of $H$, $o = o_1 \cup \ldots \cup o_m$ is a $H$-orbit of size $u_1 2^v$ where $2^v = |o_1| = \ldots |o_m|$ and $v < k$. Equivalently, the stabilizer $H_P$ of a point $P \in o$ has order $2^{k-v} u_1$, and it is the semidirect product $Q_1 \rtimes U_1$ where $|Q_1| = 2^{k-v}$ and $|U_1| = u_1$ for a subgroup $Q_1$ of $Q$ and $U_1$ of $U$ respectively. The point $P$ lying under $P$ in the covering $K(\mathcal{X}) \to \mathcal{X}$ is fixed by the factor group $\bar{U}_1 = U_1Q/Q$. Since $\mathcal{X}$ is elliptic, [4] Theorem 11.94 implies $|\bar{U}_1| \leq 3$ and so $u_1 \geq u/3$. From the Hurwitz genus formula applied to $Q$,

$$2g - 2 \geq 2^v u_1 d_P \geq 2^v u_1 \cdot 2(2^{k-v} - 1) \geq 2^v \frac{u}{3} \cdot 2(2^{k-v} - 1) \geq 2^v \frac{u}{3} (2^{k-v}) = \frac{2^v u}{3} = \frac{|H|}{3},$$

where the last inequality follows from $v < k$. Thus $|H| \leq 6(g - 1)$, contradicting (7).

If $\mathcal{X}$ is rational, then $Q$ has at least one short orbit. Furthermore, $\bar{U} = UQ/Q$ is isomorphic to a subgroup of $\PGL(2, K) \cong \Aut(\mathcal{X})$. Since $U \cong \bar{U}$ has odd order, the classification of finite subgroups of $\PGL(2, K)$, see [9], shows that $U$ is a cyclic group, $\bar{U}$ fixes two points $\bar{P}_0$ and $\bar{P}_\infty$ but no non-trivial element in $\bar{U}$ fixes a point other than $\bar{P}_0$ and $\bar{P}_\infty$. Let $o_\infty$ and $o_0$ be the $Q$-orbits lying over $\bar{P}_0$ and $\bar{P}_\infty$, respectively. Obviously, $o_\infty$ and $o_0$ are short orbits of $H$. We show that $Q$ has at most one or two short orbits, the candidates being $o_\infty$ and $o_0$. By way of contradiction,
there is a $Q$-orbit $o$ of size $2^m$ with $m < k$ which lies over a point $\bar{P} \in \bar{X}$ different from both $\bar{P}_\infty$ and $\bar{P}_\infty$. Since the orbit of $\bar{P}$ in $\bar{U}$ has length $u$, then the $H$-orbit of a point $P \in o$ has length $u2^m$. If $u > 3$, that is $u \geq 5$, the Hurwitz genus applied to $Q$ gives

$$2g - 2 \geq -2 \cdot 2^k + u2^m \cdot 2(2^{k-m} - 1) \geq -2 \cdot 2^k + u2^m(2^{k-m}) = 2^k(u - 2) > 2^k \frac{u}{2} = \frac{|H|}{2},$$

a contradiction to (7). If $u = 3$ then

$$2g - 2 \geq -2 \cdot 2^k + 32^m \cdot 2(2^{k-m} - 1) \geq -2 \cdot 2^k + 32^m(2^{k-m}) = 2^k = |Q|,$$

hence $|H| = 3|Q| \leq 6(g - 1)$ contradicting (7).

We proved that $H$ has exactly two short orbits $o_0$ and $o_\infty$. Assume that they are both short orbits of $Q$. If their lengths are $2^a$ and $2^b$ with $a, b < k$, the Deuring-Shafarevich formula applied to $Q$ gives

$$\gamma(X) - 1 = -2^k + (2^k - 2^a) + 2^k - 2^b,$$

whence $\gamma(X) = 2^k - (2^a + 2^b) + 1 \geq 2^k - 2 \cdot 2^{k-1} + 1 > 0$. The same argument shows that if $Q$ has just one short orbit, say $o_0$ of length $2^a$ with $a < k$, then $\gamma(X) = 0$ and the short orbit consists of a single point $P$. In fact in this case the Deuring-Shafarevich formula applied to $Q$ gives

$$\gamma(X) - 1 = -2^k + (2^k - 2^a) = -2^a,$$

which implies $\gamma(X) = 0$ and $a = 0$ and so $o_0 = \{P\}$. Since $Q$ is a normal subgroup of $H$, $P$ is also fixed by $H$. □

Lemma 3.2. Let $G$ be a solvable subgroup of $\text{Aut}(X)$ satisfying the following conditions.

(I) A Sylow 2-subgroup $G$ fixes a point of $X$.

(II) The number of non-tame short orbits comprising the fixed points of the Sylow 2-subgroups of $G$ is odd.

(III) At any point of $X$, the second ramification group of $G$ is trivial. Then $g(X)$ is even.

Proof. Let $o_1, \ldots, o_k$ denote the non-tame short orbits of $G$. Choose one point $P_i$ from $o_i$, and denote by $S_{P_i}$ the Sylow 2-subgroup of the stabilizer of $P_i$ in $G$. From (III), $S_{P_i}^{(2)}$ is trivial. If $G$ also has tame short orbits, say $\theta_1, \ldots, \theta_m$, choose one point $P_j$ from each of them. From the Hurwitz genus formula applied to $G$,

$$2(g(X) - 1) = 2|G|((g(X)/G) - 1) + \sum_{i=1}^{k} |a_i|(|G_{P_i}| - 1 + |S_{P_i}| - 1) + \sum_{j=1}^{m} |\theta_j|(|G_{Q_j}| - 1).$$

The stabilizer of a point in any tame short orbit is odd while the length of any tame short orbits is even. Therefore, $\sum_{j=1}^{m} |\theta_j|(|G_{Q_j}| - 1)$ is divisible by 4. Furthermore, if $S_{P_i}$ for some $1 \leq i \leq k$ is not a Sylow 2-subgroup of $G$ then $|a_i| = |G|/|G_{P_i}|$ is even, and hence $|a_i|(|G_{P_i}| - 1 + |S_{P_i}| - 1)$ is also divisible by 4. By way of contradiction, if $S_{P_i}$ for some $1 \leq i \leq k$ is a Sylow 2-subgroup of $G$ then $|a_i| = |G|/|G_{P_i}|$ is odd whereas

$$|G_{P_i}| - 1 + |S_{P_i}| - 1 = |S_{P_i}| \left(\frac{|G_{P_i}|}{|S_{P_i}|} + 1\right) - 2 \equiv 2 \pmod{4}.$$

Therefore, (II) yields that the positive integer in the right hand side of (8) is congruent to 2 (mod 4) whence $g(X) - 1$ is odd. □
Lemma 3.3. Let $\mathcal{X}$ be a curve defined over an algebraically closed field of characteristic $p$ with $g(\mathcal{X}) \geq 2$ and $\gamma(\mathcal{X}) > 0$. Let $G$ be a solvable subgroup of $\text{Aut}(\mathcal{X})$ satisfying the following condition.

(IV) $G$ has two non-tame short orbits both consisting of fixed points of Sylow $2$-subgroups of $G$. Then either $|G| \leq 84(g(\mathcal{X}) - 1)$, or $G = S_2 \times U$ where $S_2$ is an (elementary abelian minimal normal) Sylow $2$-subgroup with a cyclic complement $U$, and $\mathcal{X}/S_2$ is rational. Moreover, in the latter case $g(\mathcal{X})$ is even.

Proof. By contradiction. We assume that $|G| > 84(g(\mathcal{X}) - 1)$ and that the second claim does not hold. Take for $\mathcal{X}$ a curve that is a minimal counterexample to Lemma 3.3 with respect to the genus. From $|G| > 84(g(\mathcal{X}) - 1)$ and (IV), $G$ has exactly two short orbits, say $o$ and $\theta$, both non-tame; see [4, Theorem 11.56]. Choose a minimal normal subgroup $T$ of $G$, and consider the quotient curve $\mathcal{X}/T$ and its automorphism group $\bar{G} = G/T$. Three cases are separately investigated according to the value of $g(\mathcal{X})$.

If $g(\mathcal{X}/T) \geq 2$, then the Hurwitz genus formula applied to $Q$ implies $g(\mathcal{X}) - 1 \geq |T|(g(\mathcal{X}/T) - 1)$. This together with $|G| > 84(g(\mathcal{X}) - 1)$ yields $|G| > 84(g(\mathcal{X}/T) - 1)$. Furthermore, $G$ has two non-tame short orbits. In fact, the points lying under $o$, as well as those lying under $\theta$, form two orbits of $G$, say $\bar{o}$ and $\bar{\theta}$, respectively. To prove that they consist of fixed points of some Sylow $2$-subgroups of $G$, take a point $\bar{P} \in \bar{o} \cup \bar{\theta}$ of $\mathcal{X}/T$ together with a point $P \in o$ of $\mathcal{X}$. Let $S_2$ be a Sylow $2$-subgroup of $G$ fixing $P$. Then $\bar{S}_2 = S_2T/T$ is a subgroup of $G$ fixing $\bar{P}$. If $|T|$ is odd, then $S_2 \cong S_2$ and $\bar{S}_2$ is a Sylow $2$-subgroup of $\bar{G}$. The latter claim holds true for $|T|$ even, that is for $|T| = 2^h$, since $|\bar{S}_2| = |S_2/T|$ and

$$\frac{|\bar{G}|}{|\bar{S}_2|} = \frac{|G/T|}{|S_2/T|} = \frac{|G|}{|S_2|}$$

is an odd number. Observe that $\bar{S}_2$ is non trivial, otherwise $S_2$ would be a Sylow $2$-subgroup, but then $\bar{G}$ would be a tame subgroup whose order cannot exceed $84(g(\mathcal{X}/T) - 1)$. From [4, Theorem 11.56], $\bar{o}$ and $\bar{\theta}$ are the only short orbits of $\bar{G}$. Therefore $\bar{G}$ satisfies Condition (IV). Since $\mathcal{X}$ is taken as a minimal counterexample to Lemma 3.3, $\bar{G} = S_2 \times U$, where $U$ is tame and cyclic. This implies that $\bar{G}$ fixes a point $\bar{P}$ of $\mathcal{X}$ and hence that one of the two non-tame short orbits of $G$, say $\bar{o}$, lies under a $T$-orbit $|o| \leq |T|$. If $|T|$ is odd then $G = S_2 \times U$ where $|U|$ is odd. The quotient curve $\mathcal{X}/S_2$ is either rational or elliptic because otherwise the tame quotient group $G/S_2$ would satisfy $|G/S_2| > 84(g(\mathcal{X}/S_2) - 1)$, a contradiction. If $\mathcal{X}/S_2$ is rational then $U$ is cyclic from [2], a contradiction. Therefore $\mathcal{X}/S_2$ is elliptic. Since $|o| \leq |T|$ the quotient group $G/S_2$ contains a subgroup of order at least $|U|/|T|$ fixing a point $P'$ of $\mathcal{X}/S_2$. From [4, Theorem 11.94 (ii)] we have that either $U = T$ or $|U| = 3|T|$. In the first case, as $U$ is abelian, from Lemma 3.3 $U$ is cyclic, a contradiction. The subgroup $H = \langle S_2, T \rangle = S_2 \times T$ is a subgroup of $G$ of index $3$. From Lemma 3.1 $H$ fixes a point which is also a fixed point of $G$. Since this implies that $U$ is cyclic from [4, Theorem 11.49] we have a contradiction. Assume that $|T|$ is even. Since $U$ is isomorphic to $U$ which is cyclic, we have a contradiction.

If $g(\mathcal{X}/T) = 1$, then the Hurwitz genus formula applied to $T$ gives

$$2(g(\mathcal{X}) - 1) \geq |o|(|TP| - 1) + |\theta|(|TR| - 1) \geq \frac{|T|(|o| + |\theta|)}{2}$$

where $P \in o$ and $R \in \theta$. Let $\bar{P}$ be the point of $\mathcal{X}/T$ lying under $P$ in the covering $\mathcal{X} \to \mathcal{X}/T$. Then $G_{\bar{P}}/T = G_P T/T$ is isomorphic to a subgroup of $\text{Aut}(\mathcal{X}/T)$ fixing $\bar{P}$. From [4, Theorem
11.94, \(|G_Pr/T| \leq 24\). Therefore, \(|o| = |G|/|G_Pr| \geq \frac{1}{24(t)}|G|\). The same holds for \(R\). Therefore, 
\(2(g(\mathcal{X}) - 1) \geq \frac{1}{24}|G|\) but this contradicts our assumption.

If \(g(\mathcal{X}/T) = 0\) then \(\mathcal{X}/T\) is rational. Take a point \(P \in o\) of \(\mathcal{X}\) fixed by a Sylow 2-subgroup of \(S_2\) of \(G\). Assume that \(S_2 \neq T\). Then \(\bar{S} = S_2T/T\) fixes the point \(\bar{P}\) of \(\mathcal{X}/T\) lying under \(P\) in the covering \(\mathcal{X} \to \mathcal{X}/T\). Now take a point \(R \in \theta\) of \(\mathcal{X}\) fixed by a Sylow 2-subgroup \(S_2\) of \(G\). Since \(\theta\) is a \(G\)-orbit and \(S_2\) is conjugate to \(S_2^*\) in \(G\), there exists a point \(S \in \theta\) of \(\mathcal{X}\) fixed by \(S_2\). Hence the point \(\bar{S}\) of \(\mathcal{X}/T\) lying under \(S\) in the covering \(\mathcal{X} \to \mathcal{X}/T\) is also fixed by \(\bar{S}_2\). Since \(\bar{P} \neq \bar{S}\), \(\bar{S}_2\) has two fixed points, but this is impossible since \(S_2\) has order of characteristic of \(K\). Therefore, \(S_2 = T\) and hence \(G = S_2 \times U\) with a subgroup \(U\) of odd order. Furthermore \(G \cong U\) has odd order. From the classification of finite subgroups of \(\text{PGL}(2, K)\), \(G\) is a cyclic group. This shows that \(G = S_2 \times U\) with a cyclic subgroup of odd order, a contradiction. Finally, from Lemma 3.1, \(\mathcal{X}\) has even genus. \(\square\)

**Proposition 3.4.** Let \(\mathcal{X}\) be a curve defined over an algebraically closed field of characteristic 2. Assume that the following hold.

(i) \(\mathcal{X}\) is ordinary.
(ii) \(g(\mathcal{X}) \geq 2\) is even.

Furthermore, let \(G\) be a subgroup of \(\text{Aut}(\mathcal{X})\) satisfying the following conditions.

(iii) \(G\) has a nontrivial normal 2-subgroup \(Q\), and \(Q\) is not a Sylow 2-subgroup of \(G\).
(iv) \(\mathcal{X}/Q\) is neither rational nor elliptic.
(v) \(|G| > 84(g(\mathcal{X}) - 1)\).

Then \(g(\mathcal{X}/Q)\) is even.

**Proof.** We show that the subgroup \(\bar{G} = G/Q\) of \(\text{Aut}(\mathcal{X}/Q)\) satisfies the conditions (I), (II) and (III). From our previous result, (i) and (ii) yield that a Sylow 2-subgroup \(S_2\) of \(G\) has a fixed point. Since \(Q\) is a normal subgroup of \(G\), \(S_2\) contains \(Q\). Therefore, the factor group \(S_2/Q\) viewed as a subgroup of \(\text{Aut}(\mathcal{X}/Q)\) is a Sylow 2-subgroup and fixes a point of \(\mathcal{X}/Q\). This shows that Condition (I) is satisfied. Since \(\mathcal{X}\) is ordinary, so is \(\mathcal{X}/Q\), as \(Q\) is a 2-subgroup of \(G\). Therefore Condition (III) is satisfied. It remains to investigate Condition (II). By (ii) and (v), \(G\) has either one or two non-tame short orbits and in the latter case \(G\) has no tame short orbits; see [3] Theorem 11.56.

Let \(o\) be a non-tame short orbit consisting of fixed points of Sylow 2-subgroups of \(G\). Choose one of these Sylow 2-subgroup, say \(S_2\), and look at the set \(\rho\) of all fixed points of \(S_2\) in \(o\). Obviously, the normalizer \(N = N_G(S_2)\) leaves \(\rho\) invariant. Actually, \(N\) is transitive on \(\rho\). In fact, if \(P, R \in \rho\) are fixed by \(S_2\), there exists \(g \in G\) that takes \(P\) to \(R\). Then the Sylow 2-subgroup \(gS_2g^{-1}\) of \(G\) fixes \(R\). Therefore, \(R\) is fixed by both \(S_2\) and \(gS_2g^{-1}\). Hence \(S_2 = gS_2g^{-1}\) as \(G\) has a unique maximal 2-subgroup. Thus \(g \in N\). This shows that \(|\rho| = |N|/|NP|\) is odd. By our previous result, (i) and (ii) yield that \(S_2\) has an odd number of fixed points. This rules out the possibility that \(G\) has two non-tame short orbits both consisting of fixed points of Sylow 2-subgroups of \(G\). In fact, since \(|\rho|\) is odd, if both non-tame short orbits contained fixed points from the same Sylow 2-subgroup then \(S_2\) would have an even number of fixed points. Otherwise, there would exist two Sylow 2-subgroups of \(G\) whose fixed points are in different (non-tame short) orbits of \(G\), respectively. But this is not possible, as all the Sylow 2-subgroups of \(G\) are pairwise conjugate. Therefore, just one \(G\)-orbit, say \(o\), comprises the fixed points of the Sylow 2-subgroups of \(G\). Assume that \(G\) has another non-tame short orbit, say \(\theta\). Since \(Q\) is a normal subgroup of \(G\), \(\theta\) is partitioned in \(Q\)-orbits with the same length \(t\). Obviously, \(t > 1\) is odd if and only if \(Q\) does not fix any point in \(\theta\). We show that this
case actually occurs. By contradiction, \( t = 1 \). Take a point \( P \) from \( \theta \). Since \( Q \) is contained in all Sylow 2-subgroups, the Hurwitz genus formula applied to \( Q \) reads
\[
2(g(\mathcal{X}) - 1) = 2|Q|(g(\mathcal{X}/Q) - 1) + 2|o|(|Q| - 1) + 2|\theta|(|Q| - 1).
\]
whence \( \theta \) is even.

Let \( \Delta \neq \{P\} \) be a \( Q \)-orbit contained in \( \theta \) and let \( \bar{\theta} \) be the set of the points of \( \mathcal{X}/Q \) lying under those in \( \theta \). Now, we are in a position to prove that the factor group \( \bar{G} = G/Q \) viewed as a subgroup of \( \text{Aut}(\mathcal{X}/Q) \) satisfies Condition (II). The Sylow 2-subgroups of \( \bar{G} \) are the factor groups \( S_2/Q \) where \( S_2 \) ranges over the Sylow 2-subgroups of \( G \). Furthermore, the covering \( \mathcal{X} \to \mathcal{X}/Q \) is totally ramified at the points lying under the points in \( o \). Those points of \( \mathcal{X}/Q \) form a set \( \bar{o} \) with \(|\bar{o}| = |o|\), and \( \bar{o} \) is a set of fixed points of Sylow 2-subgroups of \( \bar{G} \).

Now either \( S_2 \) preserves \( \Delta \) or not. In the first case, \( \bar{o} \) and \( \theta \) are exactly the non-tame short orbits of the quotient group \( \bar{G} \) and they are both composed by fixed points of Sylow 2-subgroups of \( \bar{G} \). Thus, the claim follows from Lemma 3.5. In the second case, \( S_2 \) has no fixed points on \( \theta \) and thus \( \bar{G} \) satisfies Condition (III). Now the claim follows from Lemma 3.2.

For the rest of this Section, \( \mathcal{X} \) is an ordinary curve.

**Lemma 3.5.** If \( S \) is a Sylow 2-subgroup of \( \text{Aut}(\mathcal{X}) \) then \( S \) fixes at least a point \( P \) of \( \mathcal{X} \).

**Proof.** Let \( \mathcal{X}' \) be the quotient curve \( \mathcal{X}/S \) and \( g' = g(\mathcal{X}') \). From the Hurwitz genus formula applied to \( S \) and [13] Theorem 2 (i)
\[
2g - 2 = |S|(2g' - 2) + 2 \sum_{P \in \mathcal{X}} (|SP| - 1) + 2 \sum_{i=1}^{k} \ell_i(|S|/\ell_i - 1),
\]
where \( \ell_1, \ldots, \ell_k \) are the short orbits of \( S \). Hence,
\[
g - 1 = |S|(g' - 1) + \sum_{P \in \mathcal{X}} (|SP| - 1) + \sum_{i=1}^{k} \ell_i(|S|/\ell_i - 1).
\]
Since \( |S|(g' - 1) \) and \( \sum_{i=1}^{k} \ell_i(|S|/\ell_i - 1) \) are both either even or zero while \( g - 1 \) is odd, \( \sum_{P \in \mathcal{X}} (|SP| - 1) \) cannot vanish. Thus, the set
\[
\Delta = \{P \in \mathcal{X}: SP = S\},
\]
has odd length, which is in particular at least equal to 1.

The following corollary follows from the proof of Lemma 3.5 and 13 Corollary of Theorem 2.

**Corollary 3.6.** Let \( S \) be a Sylow 2-subgroup of \( \text{Aut}(\mathcal{X}) \). Then \( S \) is an elementary abelian 2-group and it fixes an odd number of points of \( \mathcal{X} \).

**Lemma 3.7.** Let \( G \leq \text{Aut}(\mathcal{X}) \) be solvable and such that \( 2 \mid |G| \). If the odd-core \( O(G) \) of \( G \) is not trivial and \( \bar{g} = g(\mathcal{X}/O(G)) \), then \( \bar{g} \) is even.

**Proof.** From the Hurwitz genus formula applied to \( O(G) \),
\[
g - 1 = |O(G)|(\bar{g} - 1) + \frac{1}{2} \sum_{P \in \mathcal{X}} (|O(G)P| - 1).
\]
Let $\Gamma = \{ P \in \mathcal{X} \mid |O(G)_P| > 1 \}$ and $D = \sum_{P \in \mathcal{X}} (|O(G)_P| - 1)$. If either $D = 0$ or $|\Gamma|$ is even then the claim follows as both $g - 1$ and $|O(G)|$ are odd. Therefore we assume that $|\Gamma|$ is odd and let $S \in Syl_2(G)$. From Corollary 3.8, $S$ is an elementary abelian 2-group fixing at least a point on $\mathcal{X}$. Since $O(G)$ is normal in $G$, $S$ acts on $\Gamma$ and hence there exists a point $P \in \Gamma$ such that $S_P = S$. As both $O(G)_P$ and $S$ are normal in $G_P$ we have that $G_P = O(G)_P \times S$ and a contradiction is obtained from [4] Lemma 11.75 and [12] Theorem 2 (i). \hfill $\Box$

The following corollary follows from the proof of Lemma 3.7.

**Corollary 3.8.** Let $G \leq \text{Aut}(\mathcal{X})$ be solvable and such that $2 \mid |G|$. If the odd-core $O(G)$ of $G$ is not trivial and $\Gamma = \{ P \in \mathcal{X} \mid |O(G)_P| > 1 \}$, then $|\Gamma|$ is even.

**Lemma 3.9.** Let $G \leq \text{Aut}(\mathcal{X})$ be solvable of even order such that $O(G)$ is not trivial and let $P \in \mathcal{X}$. If $P$ is fixed by a 2-subgroup $S$ of $G$, then $O(G)_P$ is trivial.

**Proof.** By contradiction, assume that $\alpha \in O(G)$ fixes $P$. Then $S$ and $\alpha$ commute. From [4] Lemma 11.75 (i), this is a contradiction to [12] Theorem 2. \hfill $\Box$

**Lemma 3.10.** Let $G \leq \text{Aut}(\mathcal{X})$ be solvable and such that $2 \mid |G|$. If the odd-core $O(G)$ of $G$ is not trivial and $G = G/O(G)$, then $\bar{G}_P^{(2)}$ is trivial for any $\bar{P} \in \mathcal{X}/O(G)$.

**Proof.** Let $\bar{P} \in \mathcal{X}/O(G)$ and $S$ be the Sylow 2-subgroup of the stabilizer $\bar{G}_\bar{P} = (G/O(G))_\bar{P}$. Since $|O(G)|$ is odd, the automorphism group $S$ is induced by some 2-subgroup $S \cong S_3$ of $\text{Aut}(\mathcal{X})$, and $S$ acts on the $O(G)$-orbit $O$ which lies over the point $\bar{P}$. As $|O|$ is odd, $S$ fixes at least a point $P \in O$. From Lemma 3.9, $O$ is a long orbit of $O(G)$. Therefore each local parameter $t \in K(\mathcal{X}/O(G))$ at $\bar{P}$, is also a local parameter at $P$ in $F$. Let $\bar{\alpha} \in \bar{S}$ and $\alpha \in S$ which induces $\bar{\alpha}$. Then

$$v_P(\alpha(t) - t) = v_P(\bar{\alpha}(t) - t) = v_P(\bar{\alpha}(t) - t).$$

Hence, $G^{(i)}_P = \bar{G}^{(i)}_{\bar{P}}$ for any $i \geq 1$. In particular, $\bar{G}_\bar{P}^{(2)}$ is trivial. \hfill $\Box$

**Theorem 3.11.** Let $X$ be an ordinary curve of even genus $g \geq 2$ defined over an algebraically closed field $K$ of characteristic 2. Let $G \leq \text{Aut}(\mathcal{X})$ be solvable and such that $2 \mid |G|$. If the odd-core $O(G)$ of $G$ is not trivial then $|G| < 35(g - 1)^{3/2}$.

**Proof.** By way of contradiction, $X$ is taken as minimal counterexample with respect to the genera so that for any solvable subgroup $G' \leq \text{Aut}(\mathcal{X}')$ with non trivial odd core of the ordinary curve $\mathcal{X}'$ with even genus $g'$ we have $|G'| \leq 35(g' - 1)^{3/2}$.

Let $X = \mathcal{X}/O(G)$, $\bar{G} = G/O(G)$ and $\bar{g} = g(\bar{X})$. From Lemma 3.7, either $\bar{g} \geq 2$ and $\bar{g}$ is even, or $\bar{g} = 0$. Assume that $\bar{g} \geq 2$. The quotient group $\bar{G}$ is a subgroup of $\text{Aut}((\mathcal{X})$. From the Hurwitz genus formula $2g - 2 \geq |O(G)|(2\bar{g} - 2)$, which implies that $|\bar{G}| > 35(\bar{g} - 1)^{3/2}$.

By Lemma 3.10, $\bar{G}^{(1)}_\bar{P} = G^{(1)}_P$ and $\bar{G}^{(2)}_\bar{P} = \{1\}$, for any point $\bar{P}$ of $X$ and $P|\bar{P}$ of $X$. Also, any Sylow 2-subgroup $S_2$ of $G$ is induced by a Sylow 2-subgroup $S_2$ of $G$ and fixes an odd number of points of $\mathcal{X}$ from Corollary 3.6. Let $S_2 \in Syl_2(G)$, $\mathcal{X}'$ the quotient curve $X/S_2$ and $g' = g(X')$. Assume $g' = 0$. From the Hurwitz genus formula and the Deuring-Shafarevich formula applied to $X \to X'$, $X$ is ordinary, a contradiction to the minimality of $X$. Assume $g' = 1$. If $X'$ is ordinary, a contradiction to the minimality of $X$ follows as above. Thus $X'$ has zero 2-rank. From Theorem 4.11 Theorem 11.137 either $\bar{G}$ fixes a point $\bar{R}$ of $X$ or $|S_2| = 2$. In the latter case $G = O(G) \times S_2$, and hence $|G| \leq 168(g - 1)$, a contradiction. In the former case $\bar{G} = \bar{S}_2 \times \bar{U}$ with $\bar{U}$ tame and
cyclic. From Lemma \[3.4\], \( \mathcal{X}/S_2 \) is rational and hence \( \mathcal{X} \) is ordinary, a contradiction. Therefore \( g' \geq 2 \). Any minimal normal subgroup of \( \bar{G} \) is a 2-group because \( \bar{G} \) is a solvable odd-core free group. Let \( \bar{Q} \triangleleft \bar{G} \) be a minimal normal subgroup. If \( \bar{Q} \in Syl_2(\bar{G}) \) then from the Hurwitz genus formula
\[
2\bar{g} - 2 \geq |\bar{Q}|(2g' - 2),
\]
which implies that \( |\bar{G}/\bar{Q}| > 35(\bar{g} - 1)^{3/2} \). On the other hand \( \bar{G}/\bar{Q} \) is tame, and hence \( |\bar{G}/\bar{Q}| \leq 84(\bar{g}' - 1) \), a contradiction. Thus \( \bar{Q} \notin Syl_2(\bar{G}) \) and \( \bar{Q} = \bigcap\{S_2 \mid S_2 \in Syl_2(\bar{G})\} \).

The group \( \bar{Q} \) is induced by a 2-subgroup \( \bar{Q} \cong Q \) of \( \bar{G} \), which is normal in \( \bar{G} \) and satisfies \( \bar{Q} \notin Syl_2(\bar{G}) \) and \( \bar{Q} = \bigcap\{S_2 \mid S_2 \in Syl_2(\bar{G})\} \). Let \( \bar{X} = \mathcal{X}/\bar{Q} \) with genus \( \bar{g} \). From Lemma \[3.4\] either \( \bar{g} = 0 \), or \( \bar{g} = 1 \), or \( \bar{g} \geq 2 \) is even. Since \( g' \geq 2 \), the first and the second cases do not occur. Therefore \( \bar{X} \) has even genus, it is ordinary and from the Hurwitz genus formula \( |\bar{G}/\bar{Q}| \geq 35(\bar{g} - 1)^{3/2} \), a contradiction to the minimality of \( \mathcal{X} \).

Thus \( \bar{g} = 0 \) and the covering \( \mathcal{X} \rightarrow \mathcal{X} \) ramifies. As \( \bar{G} \) is isomorphic to a solvable subgroup of \( \text{PGL}(2, K) \), from \[9\] \( \bar{G} = E_{2h} \rtimes C \), where \( E_{2h} \) is an elementary abelian 2-group and \( C \) is a cyclic group of order \( c \). Moreover, \( \bar{G} \) fixes a point \( \bar{P}_0 \) of \( \mathcal{X} \) and \( S \) acts semiregularly on \( \mathcal{X} \setminus \{\bar{P}_0\} \), while \( C \) fixes also another point \( \bar{P}_\infty \) acting semiregularly on \( \mathcal{X} \setminus \{\bar{P}_0, \bar{P}_\infty\} \).

This implies that the \( G \)-orbit of \( \bar{P}_0 \) has length \( q = 2^h \). Since \( O(G) \) has odd order, the group \( E_{2h} \) is induced by a Sylow 2-subgroup \( S \) of \( \text{Aut}(\mathcal{X}) \). Denote by \( O \) the \( O(G) \)-orbit lying over \( \bar{P}_\infty \) in \( \mathcal{X} \). We prove that \( O \) is a long orbit for \( O(G) \). As \( S \) acts on \( O \), there exists a point \( P \in O \) such that \( S_P = S \). If \( O \) is a short orbit of \( \bar{G} \), then \( G_P = O(G)_P \rtimes S \), and hence from \[11\] Theorem 11.75 we have a contradiction to \[13\] Theorem 2 (i). Thus, for \( P \in O \) we have \( G_P = S \rtimes U \), where \( |U| = c \). Denote by \( O_1, \ldots, O_q \) the \( O(G) \)-orbits lying over the \( q \) distinct points in \( F \) of the \( G \)-orbit of \( \bar{P}_0 \), and let \( \ell = |O_1| = \cdots = |O_q| \). Assume that there exists a short orbit \( \Sigma \) of \( O(G) \) such that \( \Sigma \neq O_i \) for any \( i = 1, \ldots, q \) and let \( T \in \Sigma \). From the Hurwitz genus formula
\[
2g_2 - 2 \geq -2|O(G)| + q\sqrt{|O(G)|/|O(G)|T}(|O(G)|T - 1),
\]
as \( \bar{G} \) acts on \( \mathcal{X} \setminus \{\bar{P}_0, \bar{P}_\infty\} \). Hence,
\[
2g_2 - 2 \geq |O(G)|(-2 + q\sqrt{2}) \geq \frac{|O(G)|q\sqrt{2}}{4} \geq \frac{|G|}{4},
\]
a contradiction. Since \( \mathcal{X} \rightarrow \mathcal{X} \) ramifies this proves that \( O_1, \ldots, O_q \) are short orbits for \( O(G) \). Write \( |G| = |O(G)|q\ell \) and let \( \mathcal{O} = O_1 \cup \cdots \cup O_q \). Clearly, \( \mathcal{O} \) is a short orbit of \( G \). Moreover, for every point \( R \in O_1 \),
\[
|\mathcal{O}| = q|O_1| = \frac{q|O(G)|}{|O(G)|_R}, \quad \text{and} \quad |G| = |G_R||\mathcal{O}|,
\]
and hence \( G_R \) is a cyclic tame group with \( |G_R| = |O(G)|_R/c \). From the Hurwitz genus formula
\[
2g_2 - 2 = -2|O(G)| + q|O_1|(|O(G)|_R - 1) \geq -2|O(G)| + \frac{q|O_1||O(G)|_R}{2}
\]
\[
= |O(G)|(q/2 - 2) = |O(G)|R|O_1|(q/2 - 2) \geq \frac{|O(G)|_R|O_1|}{5} \geq \frac{|G|}{5c}.
\]
Thus,
\[
(10) \quad |G| \leq 10c(g - 1)
\]
and since \( |G| = |O(G)|q\ell \),
\[
(11) \quad |O(G)|q \leq 10(g - 1).
\]
From [13, Proposition 1], \( c \leq q - 1 \) and hence \( c|O(G)| \leq 10(g - 1) \) holds. From Lemma 3.11 either \( qc \leq 12(g - 1) \) or \( qc > 12(g - 1) \) and in the latter case \( S \) and \( G \) fix a point having only another short orbit \( \Omega \) of length \( 2^k = q/\ell \). If \( qc \leq 12(g - 1) \), then \( c \leq 2\sqrt[3]{3}\sqrt{g - 1} \) from [13, Proposition 2]. Combining with Equation (11)
\[
|G| \leq (2\sqrt[3]{3}\sqrt{g - 1}) \cdot 10(g - 1) = 20\sqrt[3]{3}(g - 1)^{3/2} < 35(g - 1)^{3/2},
\]
a contradiction. Thus \( qc > 12(g - 1) \), the point \( P \in O \) is fixed by \( G \) and \( G \) has only another short orbit \( \Omega \) with \( |\Omega| = 2^k = q/\ell > 1 \). From Equation (11)
\[
c|O(G)| < q|O(G)| \leq 10(g - 1) < 12(g - 1) < qc,
\]
and \( |O(G)| < q \). Since this implies that \( O \) cannot contain a long orbit of \( S \)
\[
|O(G)| = 1 + |\Omega| = 1 + q/\ell = 1 + 2^k,
\]
and \( G/Ker\varphi \) acts sharply 2-transitively on \( O \) where \( Ker\varphi \) denotes the Kernel of the permutation representation of \( G \) on \( O \). From [8] as \( G/Ker\varphi \) is a solvable 2-transitive group, \( 1 + 2^k = |O| = p^t \) for some prime \( p \) and \( t \geq 1 \) and so either \( p = 3, t = 2 \) and \( k = 3 \) or \( t = 1 \). If \( |O(G)| = 9 \), the Sylow 2-subgroup of \( G/Ker\varphi \) is isomorphic to the multiplicative group of a near-field \( F \) of order 9; see [3, Theorem 20.7.1]. Moreover, the Sylow 2-subgroup of \( G/Ker\varphi \) has order 8 and must be isomorphic either to the quaternion group \( Q_8 \) (if \( F \) is the Dickson near-field) or to a cyclic group (if \( F \) is a field), a contradiction. Assume that \( |O(G)| = p \). From the Deuring-Shafarevich formula applied to \( S, g - 1 = \gamma - 1 = q - p, \) and so \( q = g + |O(G)| - 1 \). Combining with Equation (11)
\[
(g - 1)|O(G)| < (g + |O(G)| - 1)|O(G)| = q|O(G)| \leq 10(g - 1).
\]
Since this implies that \( 1 + g^k = p = |O(G)| < 8 \) we have either \( p = 3 \) or \( p = 5 \). If \( p = 3, U \) and a subgroup \( S' \) of \( S \) of index 2 fix \( O \) pointwise. Let \( \mathcal{X}' = \mathcal{X}/S' \) and \( g' = g(\mathcal{X}') \). From the Riemann-Hurwitz formula applied to \( S' \)
\[
2(q - 3) = 2(g - 1) = 2\left(\frac{g}{2} - 1\right) + 2 \cdot 3\left(\frac{g}{2} - 1\right),
\]
and hence \( g' = 0 \). The quotient group \( S'U/S' \cong U \) is isomorphic to a subgroup of \( \text{PGL}(2, K) \) fixing three points, a contradiction to [9]. Let \( p = 5 \). If \( q \geq 16 \) then a Sylow 2-subgroup of \( G/Ker\varphi \) has order at least 4 and it must be cyclic being isomorphic to the multiplicative group of a field \( F \), a contradiction. The cases \( q = 4 \) or \( q = 2 \) are impossible as \( \Omega \) is a short orbit of \( S \). Assume that \( q = 8 \). In this case either \( c = 1 \) or \( c = 7 \) from [13, Proposition 1] and \( g = 4 \). If \( c = 1 \) then \( |G| = 40 < 35(g - 1)^{3/2} \), a contradiction. If \( c = 7 \) then \( |G| = 280 \). Moreover, \( U \) and a subgroup \( S' \) of \( S \) of order 2 fixes \( O \) pointwise. Let \( \mathcal{X}' = \mathcal{X}/S' \) and \( g' = g(\mathcal{X}') \). From the Riemann-Hurwitz formula applied to \( S' \)
\[
2(q - p) = 6 = 2(g - 1) = 2 \cdot (g' - 1) + 2 \cdot 5(2 - 1) = 4(g' - 1) + 10,
\]
and hence \( g' = 0 \). The quotient group \( S'U/S' \cong U \) is isomorphic to a non trivial subgroup of \( \text{PGL}(2, K) \) fixing 5 points, a contradiction to [9].

\[\square\]

**Proposition 3.12.** Let \( \mathcal{X} \) be an ordinary curve of even genus \( g \geq 2 \) defined over an algebraically closed field \( K \) of characteristic 2. Let \( G \leq \text{Aut}(\mathcal{X}) \) be solvable and such that \( 2 \mid |G| \). If the odd-core \( O(G) \) of \( G \) is trivial then \( |G| < 3\sqrt{2}g^{3/2} \).
Proof. By way of contradiction, $\mathcal{X}$ is taken as minimal counterexample with respect to the genus so that for any solvable subgroup $G' \leq \text{Aut}(\mathcal{X}')$ with trivial odd core of ordinary curve $\mathcal{X}'$ with even genus we have $|G'| \leq 3\sqrt{2}g^{3/2}$.

Let $Q$ be the largest normal 2-subgroup of $G$, $\bar{g} = g(\mathcal{X}/Q)$ and $\bar{G} = G/Q$. Three cases are distinguished according as $\mathcal{X}/Q$ is elliptic, rational or $\bar{g} \geq 2$.

Assume that $\bar{g} = 1$. Thus $\mathcal{X} \to \mathcal{X}/Q$ ramifies otherwise $\mathcal{X}$ itself would be elliptic. Let $\Delta_1$ be a short orbit of $Q$ and let $\Gamma$ be the short orbit of $G$ containing $\Delta_1$. Thus $\Gamma = \Delta_1 \cup \ldots \cup \Delta_k$ where $\Delta_i$ is a short orbit of $Q$ with $|\Delta_i| = |\Delta_1|$ for every $i = 1, \ldots, k$. For a point $P \in \Delta_1$, $|G_P| = |G'|/k|\Delta_1|$ and the quotient group $G_PQ/Q$ fixes the point $P$ of $\mathcal{X}/Q$ which lies under the orbit $\Delta_1$. From [13, Theorem 11.94]

$$|G_PQ/Q| = |G_P|/|Q_P| \leq 24.$$  

Moreover, from the Hurwitz genus formula applied to $Q$,

$$2g - 2 \geq 2k|\Delta_1|(|Q_P| - 1) \geq \frac{2k|\Delta_1||Q_P|}{2} \geq \frac{k|\Delta_1||(G_P|}{24} = \frac{|G|}{24},$$

a contradiction.

Assume that $\bar{g} = 0$. The quotient group $\bar{G}$ has no non trivial normal 2-subgroup and it is isomorphic to a subgroup of $\text{PGL}(2, K)$. From [9], either $\bar{G}$ is a tame cyclic group or $\bar{G}$ is isomorphic to the alternating group $\text{Alt}_4$ or $\bar{G}$ is isomorphic to the symmetric group $\text{Sym}_4$. In the last two cases from [13, Theorem 1]

$$|G| \leq 24|Q| \leq 96(g - 1),$$

a contradiction. Therefore $\bar{G}$ is a tame cyclic group and hence $Q \in Syl_2(G)$. From the Schur-Zassenhaus Theorem $G = Q \times U$ where $U \cong \bar{G}$ is tame and cyclic. Since $|G| > 12(g - 1)$, from Lemma 3.1 $Q$ has a fixed point $P$ on $\mathcal{X}$ and exactly another short orbit $\Omega$ which is not trivial. Moreover $G = G_P$ and $g = |Q| - |\Omega|$.

Let $V$ be the subgroup of $Q$ fixing a point of $\Omega$. Then $V$ fixes $\Omega$ pointwise and since $U$ normalizes $Q$, $U$ normalizes $V$. Therefore the representation of the action of $U$ on $Q$ by conjugation is completely reducible from Maschke’s Theorem; see [8, Theorem 6.1]. This means that $Q = V \times M$, where $M$ acts sharply transitively on $\Omega$. Let $g'$ be the genus of the quotient curve $\mathcal{X}/M$. Since the quotient group $G/M$ has even order and fixes two points in $\mathcal{X}/M$, $g' \neq 0$ from [9]. Assume that $g' = 1$. From the Hurwitz genus formula

$$2(|Q| - |M| - 1) = 2(g - 1) = 2(|M| - 1),$$

and hence

$$|Q| = 2|M|.$$  

Furthermore, $|G/M| = 2|U|$ and $G/M$ fixes two points in $\mathcal{X}/M$. From [13, Theorem 11.94], $|U| = 3$ and hence from [13, Theorem 1]

$$|G| = |Q||U| \leq 12(g - 1),$$

a contradiction. Assume that $g' \geq 2$. From the Hurwitz genus formula

$$2(|Q| - |M| - 1) = 2(g - 1) = 2|M|(g' - 1) + 2(|M| - 1),$$

and thus

$$g' = |Q|/|M| - 1 = |V| - 1.$$
The quotient group $G/M \cong V \times U$ has two fixed points of $X/M$, and $(X/M)/K$ is rational. From the proof of Theorem 3 in [13] (see Equation (4.16) and page 606), $|U| \leq \sqrt{|V|} + 1 = \sqrt{g/|M|} + 1 + 1$. Thus,

$$|G| = |Q||U| \leq (g + |M|)(\sqrt{g/|M|} + 1 + 1) \leq (g + |M|)(2\sqrt{2}\sqrt{g/|M|}).$$

From [4] Theorem 11.78 (i) $|M| \leq g$ and

$$|G| \leq 2\sqrt{2}(g + 1)\sqrt{g} \leq 3\sqrt{2}g^{3/2},$$

a contradiction.

Assume that $\bar{g} \geq 2$. From the Hurwitz genus formula, $|G/Q| > 3\sqrt{2}\bar{g}^{3/2}$. If $Q \not\in \text{Syl}_2(G)$ then $\bar{g}$ is even from Lemma 3.3. Since $X/Q$ is ordinary we have a contradiction to the minimality of $X$. The case $Q \in \text{Syl}_2(G)$ cannot occur since $G/Q$ is tame and hence $|G/Q| \leq 84(\bar{g} - 1)$, a contradiction.

### 4. On ordinary curves of even genus

In this section, $K$ is an algebraically closed field of odd characteristic $p$, $X$ is an ordinary algebraic curve, whose genus $g = g(X)$ is a (nonzero) even integer, and $G$ is an automorphism group of $X$. Because of Theorem 2.4 some conditions on $G$ can be assumed without loss of generality.

**Assumption 4.1.** *The automorphism group $G$ is odd core-free. Moreover, $G$ has no solvable normal subgroups.*

Assumption 4.1 can be justified as follows. Assume that $O(G)$ is not trivial. Thus, since $|O(G)|$ is odd, by the Feit-Thompson Theorem $O(G)$ is solvable. Then, by definition of minimality for normal subgroups, there exists a minimal normal subgroup $N$ of $G$ which is contained in $O(G)$ and then $|G| \leq cg^{3/2}$ for some constant $c$ by Theorem 2.4.

The main result in this Section is the following Theorem.

**Theorem 4.2.** $|G| \leq 919g^{7/4}$. If $G \cong \text{Alt}_7$, $M_{11}$, the bound can be refined to $|G| \leq 84(g - 1)$ unless $p = 3$, $g(X) = 26$, $G \cong M_{11}$.

Since according to Remark 2.11 there are only a few possibilities for the structure of $G$, the proof of Theorem 4.2 is divided into several steps.

**Proposition 4.3.** Let $X$ be an ordinary algebraic curve of even genus $g$ and $p$-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. Let $G$ be a subgroup of $\text{Aut}(X)$. If $G$ admits a minimal normal subgroup $N$ which is isomorphic to $\text{PSL}(2, q)$ for some $q = d^k$, where $d$ is a prime, then $|G| \leq 330g^{7/4}$.

**Proof.** By contradiction, $X$ is taken for a minimal counterexample with respect to $g$ so that $\text{Aut}(X)$ has a subgroup $G$ with a minimal normal subgroup $N \cong \text{PSL}(2, q)$ but its order $|G|$ exceeds $330g^{7/4}$. According to Remark 2.11 there are a few possibilities for the structure of $G$:

1. $G \cong \text{PSL}(2, q)$,
2. $G \cong \text{PGL}(2, q)$,
3. $G \cong \text{PSL}(2, q) \rtimes C_r$,
4. $G \cong \text{PGL}(2, q) \rtimes C_r$, where $r$ is a product of primes.
where $r$ is an odd divisor of $k$. We recall that $|N| = q(g-1)(q+1)$ and $N$ admits a solvable subgroup $H = S_q \rtimes C_{q-1}$, where $S_q$ is a Sylow $p$-subgroup of $N$ and $C_{q-1}$ is a cyclic group of order $q - 1$. In order to apply Lemma 2.7, two cases are distinguished. Assume that $|H| < 30(g-1)$, so that $q \leq \sqrt{30g}$. By direct computation we get that $|N| \leq \sqrt{30g}^2(\sqrt{30g}+1)(\sqrt{30g}-1) = \sqrt{30g}(30g-1) < 165g^{3/2}$. Since $|PGL(2,q) : PSL(2,q)| \leq 2$, we can exclude cases 1 and 2. Assume that case 3 holds. Since $r$ divides $k$ and $k = \log q / \log d$, we have that $r \leq k \leq \log q \leq \log \sqrt{g} \leq (\sqrt{g}+1)/(g^{1/4}) \leq g^{1/4}$. Thus,

$$\tag{12} |G| < 165g^{3/2} \cdot g^{1/4} = 165g^{7/4}. $$

Analogously, if case 4 holds then

$$\tag{13} |G| < 330g^{3/2} \cdot g^{1/4} = 330g^{7/4}. $$

Trivially, both Inequalities (12) and (13) give rise to a contradiction.

Thus, we can assume that $|H| \geq 30(g-1)$ and so Lemma 2.7 applies to $H$. Since $\gamma(\mathcal{X}) > 0$, we have that $d=p$ and $S_q$ has exactly two short orbits with $H$ are also the unique short orbits of $H$. By Remark 2.6, we have that $q - 1 = |C_{q-1}| < 3g + 4 = 2\sqrt{g} + 1$ and so $q \leq 2\sqrt{g} + 1$. As before, we get that $|PSL(2,q)| \leq (2\sqrt{g} + 1)(2\sqrt{g} + 1 - 1)(2\sqrt{g} + 1 + 1) = (2\sqrt{g} + 1)(4g + 3) < 8(g+1)^{3/2}$, so that both case 1 and 2 can be excluded. Assume that case 3 holds. Then, as in the previous case,

$$|G| \leq 16(g+1)^{3/2} \cdot (2\sqrt{g} + 1 + 1)/(2(g+1)^{1/4}) \leq 16(g+1)^{3/2} \cdot (g+1)^{1/4} = 16(g+1)^{7/4}. $$

By observing that $330g^{7/4} > 16(g+1)^{7/4}$, we get a contradiction. \hfill $\Box$

**Proposition 4.4.** Let $\mathcal{X}$ be an ordinary algebraic curve of even genus $g$ and $p$-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. Let $G$ be a subgroup of $\text{Aut}(\mathcal{X})$. If $G$ admits a minimal normal subgroup $N$ which is isomorphic to $PSU(3,q)$ for some $q = d^k$, where $d$ is a prime, then $|G| \leq 919g^{7/4}$. 

**Proof.** By contradiction, $\mathcal{X}$ is taken for a minimal counterexample with respect to $g$ so that $\text{Aut}(\mathcal{X})$ has a subgroup $G$ with a minimal normal subgroup $N \cong PSU(3,q)$ but its order $|G|$ exceeds $919g^{7/4}$. According to Remark 2.11, there are few possibilities for the structure of $G$:

1. $G \cong PSU(3,q), q \equiv 1 \mod 4$,
2. $G \cong PGU(3,q), q \equiv 1 \mod 4$,
3. $G \cong PSU(3,q) \rtimes C_r, q \equiv 1 \mod 4$,
4. $G \cong PGU(3,q) \times C_r, q \equiv 1 \mod 4$,

where $r$ is an odd divisor of $k$. Let $\delta = (3, q+1)$. Then $|N| = q^3(q^2-1)(q^3+1)/\delta \geq |PSU(3,5)| = 126000$, which exceeds $84(g^2-g)$ for $g \leq 38$, whence we may assume $g \geq 40$. Also, $N$ admits a solvable subgroup $H$, which is also maximal, with $H = S_{q^3} \rtimes C_{(q^2-1)/\delta}$, where $S_{q^3}$ is a Sylow $d$-subgroup of $N$ and $C_{(q^2-1)/\delta}$ is a cyclic group of order $(q^2-1)/\delta$. In order to apply Lemma 2.7, two cases are distinguished. Assume that $|H| < 30(g-1)$, so that $q^3(q^2-1) \leq 30\delta(g-1) \leq 90(g-1)$. This in particular implies that $q \leq (90g)^{1/5}$. Thus, $|N| \leq 90(g-1) \cdot ((90g)^{3/5} + 1) \leq 90g((90g)^{3/5} + 1) < 1,003 \cdot (90g)^{8/5}$, since $g \geq 40$ and $((90g)^{3/5} + 1) < 1,003(90g)^{3/5}$; a contradiction since $|G| < 779g^{7/4}$. 

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Hence, cases 1 and 2 can be excluded. Next, assume that case 3 or 4 holds. If this is the case, then $|N| \geq \text{PSU}(3,125)$, whence $g \geq 15378928$ can be assumed. Then as before, $r \leq k \leq \log_q \leq \log(90g)^{1/5} \leq ((90g)^{1/5} + 1)/(90g)^{1/10} \leq (90g)^{1/10}$. Thus, arguing as in the previous case,

$$|G| \leq 1,000000000032 \cdot (90g)^{8/5} \cdot (90g)^{1/10} = 1,000000000032 \cdot (90g)^{17/10}.$$  

Since this implies that $|G| < 919g^{7/4}$, we get a contradiction. Thus, we can assume that $|H| \geq 30(g - 1)$ and so Lemma 2.7 applies to $H$. Since $\gamma(X) > 0$, we have that $d = p$ and $S_p$ has exactly two short orbits with are also the unique short orbits of $H$. Since $S_p$ is not abelian and a Sylow $p$-subgroup of $H$ must be elementary abelian by Corollary 2.5, we get a contradiction. □

**Proposition 4.5.** Let $X$ be a general algebraic curve of even genus $g$ and $p$-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. Let $G$ be a subgroup of $\text{Aut}(X)$. If $G$ admits a minimal normal subgroup $N$ which is isomorphic to $\text{PSL}(3,q)$ for some $q = d^k$, where $d$ is a prime, then $|G| \leq 919g^{7/4}$.

**Proof.** By way of contradiction, $X$ is taken for a minimal counterexample with respect to $g$ so that $\text{Aut}(X)$ has a subgroup $G$ with a minimal normal subgroup $N \cong \text{PSL}(3,q)$ but its order $|G|$ exceeds $919g^{7/4}$. According to Remark 2.11, there are few possibilities for the structure of $G$:

1. $G \cong \text{PSL}(3,q), q \equiv 3 \text{ mod } 4$,
2. $G \cong \text{PGL}(3,q), q \equiv 3 \text{ mod } 4$,
3. $G \cong \text{PSL}(3,q) \rtimes C_r, q \equiv 3 \text{ mod } 4$,
4. $G \cong \text{PGL}(3,q) \rtimes C_r, q \equiv 3 \text{ mod } 4$,

where $r$ is an odd divisor of $k$. As the smallest $\text{PSL}(3,q)$ that can occur here is $\text{PSL}(3,3)$ whose size is equal to 5616, $g \geq 10$ can be assumed. Let $Q$ be a Sylow d-subgroup of $N$ and let $X = X/Q$, where $\bar{g} = g(X)$. Assume that $\bar{g} \geq 2$. Then $|Q| = q^3$ and by Theorem 2.4] the normalizer $N_{\text{PSL}(3,q)}(Q)$ in $\text{PSL}(3,q)$ has size equal to $q^3(q - 1)^2(q + 1)/(3,q - 1)$. In particular, for each case listed above $G$ admits a solvable subgroup of size $|N_{\text{PSL}(3,q)}(Q)|$. By Lemma 2.7 we have that $q^3(q - 1)^2(q + 1)/(3,q - 1) < 30(g - 1)$ whence $q \leq (90g)^{1/6}$. From $|\text{PGL}(3,q)| = q^3(3^3 - 1)(q^2 - 1) \geq 3|\text{PSL}(3,q)|$ we get

$$|G| \leq rq^3(q^3 - 1)(q^2 - 1) \leq r404g^{4/3}.$$  

Note that

$$r \leq k \leq \log_q \leq \frac{(2 + (90g)^{1/3})}{(90g)^{1/6}} < 3g^{1/6},$$  

as $g \geq 10$. Then clearly,

$$|G| < 919g^{7/4}.$$  

Next, assume $|N_{\text{PSL}(3,q)}(Q)| \geq 30(g - 1)$. Then by Lemma 2.7 we have that $d = p$ and $\bar{g} = 0$. Then $N_{\text{PSL}(3,q)}(Q)$ is isomorphic to a subgroup of $\text{PGL}(2,K)$. Since $N_{\text{PSL}(3,q)}(Q)$ contains a subgroup $R \cong Q \rtimes V$ where $V$ is tame and elementary abelian which is not cyclic, we obtain an absurd using the complete classification of subgroups of $\text{PGL}(2,K)$. □

**Proposition 4.6.** Let $X$ be an ordinary curve of even genus $g$ and $p$-rank $\gamma = g$ defined over a field $K$ of odd characteristic $p > 0$. Let $G$ be a subgroup of $\text{Aut}(X)$. If $G = \text{Alt}_7$, then $|G| \leq 84(g - 1)$. 

Proof. If \( p > 7 \), then \( G \) is a tame automorphism group of \( \mathcal{X} \) and the claim follows by the classical Hurwitz bound. If \( p = 3, 5, 7 \), a careful case-by-case analysis is needed. As \( |G| = 2520 \), the Riemann-Hurwitz formula applied to the covering \( \mathcal{X} \to \mathcal{X}/G = \mathcal{Y} \) reads

\[
\frac{2g - 2}{2520} = -2 + \sum_{Q \in \mathcal{Y}} \frac{d_Q}{e_Q},
\]

as \( g(\mathcal{Y}) = 0 \). By Theorem 2.2, we have only three possibilities, namely

(i) \( e_Q = 1 \) if \( Q \neq Q_1, Q_2, Q_3 \in \mathcal{Y} \), \( p|e_{Q_1}, e_{Q_2} = e_{Q_3} = 2 \);

(ii) \( e_Q = 1 \) if \( Q \neq Q_1, Q_2 \in \mathcal{Y} \); \( p|e_{Q_1}, e_{Q_2} \);

(iii) \( e_Q = 1 \) if \( Q \neq Q_1, Q_2 \in \mathcal{Y} \); \( p|e_{Q_1}, p \nmid e_{Q_2} \).

Let \( p = 7 \). Then the 7-Sylow of \( G \) is a cyclic group \( C_7 \) of order 7 whose normalizer is the semidirect product \( C_7 \rtimes C_3 \) with \( C_3 \) a cyclic 3-group. Assume that (i) holds. Then (14) reads

\[
\frac{g - 1}{1260} = -1 + \frac{d_{Q_1}}{e_{Q_3}} = \frac{5}{e_{Q_3}}.
\]

By Theorem 2.2, for any point \( P \) lying over \( Q_3 \), we have \( |G_2(P)| = 1 \). Thus, there are only two possibilities: either \((e_{Q_2}, d_{Q_2}) = (7, 12)\) or \((e_{Q_3}, d_{Q_3}) = (21, 26)\); in any case we get an odd value for the genus, a contradiction.

Next, assume (ii) holds. Then (14) reads

\[
\frac{g - 1}{1260} = -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}}.
\]

This time, we are left with three possibilities: \((e_{Q_1}, d_{Q_1}) = (e_{Q_2}, d_{Q_2}) = (7, 12)\), \((e_{Q_1}, d_{Q_1}) = (e_{Q_2}, d_{Q_2}) = (21, 26)\), or \((e_{Q_1}, d_{Q_1}) = (7, 12)\) and \((e_{Q_2}, d_{Q_2}) = (21, 26)\). Again, a computation shows that all the three corresponding values for \( g \) are odd.

If (iii) holds, then (14) reads

\[
\frac{g - 1}{1260} = -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}},
\]

with either \((e_{Q_1}, d_{Q_1}) = (7, 12)\), or \((e_{Q_1}, d_{Q_1}) = (21, 26)\), while

\[(e_{Q_2}, d_{Q_2}) \in \{(2, 1), (3, 2), (4, 3), (5, 4), (6, 5)\}.
\]

Note that, in order to get \( g \) even, we must have \( 1260(-2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}}) \) odd. As \( \gcd(e_{Q_1}, e_{Q_2}) = 1 \), this can happen only if \( e_{Q_2} = 4 \). A computation shows that even this case is not possible.

Now let \( p = 3 \). The 3-Sylow of \( G \) is an elementary abelian group \( E \) of order 9 with normalizer \( C_3 \rtimes S_3 \cdot 2 \). Assume that (i) holds. Then (14) reads

\[
\frac{g - 1}{1260} = \frac{d_{Q_2}}{e_{Q_3}} - 1.
\]

Again, in order to get an even genus we need a non-tame one-point stabilizer whose order is divided by 4, that is, we need subgroups \( G' \) of \( G \) that are semidirect products of a 3-group by a cyclic tame group whose order is divided by 4. Looking at the subgroup lattice of \( \text{Alt}_7 \) we see that there is no such group. This allows us to discard case (ii) as well. Next, assume that case (iii) holds. By the above discussion it is clear that the tame one-point stabilizer must have order 4 as there are no cyclic subgroups of \( G \) of order greater than 4 whose order is divided by 4. Then, (14) reads
\begin{align*}
\frac{g - 1}{1260} &= -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{3}{4},
\end{align*}

Also, recall that we are assuming $|G| \geq 84(g-1)$, that is $g \leq 31$. An easy computation shows that the above formula yields a number smaller than or equal to 31 whenever \(\frac{d_{Q_1}}{e_{Q_1}} \leq \frac{65}{84}\), a contradiction since for non-tame groups in odd characteristic \(\frac{d_{Q_1}}{e_{Q_1}} > 1\).

Let \(p = 5\). The The 5-Sylow of \(G\) is a cyclic group \(C_5\) of order 5 with normalizer \(D_{10} \cdot 2\), which is isomorphic to the general affine group \(GA(1,5)\). Here, there are three possibilities for the size of a non-tame one-point stabilizer, namely 5, 10, 20.

Assume (i) holds. Then (14) reads
\[\frac{g - 1}{1260} = \frac{3}{e_{Q_3}},\]
with \((e_{Q_1}, d_{Q_3}) \in \{(5, 8), (10, 13), (20, 23)\}\). It is immediately seen that we can get an even value for \(g\) if and only if \(e_{Q_3} = 20\), which yields \(g = 190\).

Next, assume (ii) holds. Then (14) reads
\[\frac{g - 1}{1260} = -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}}.\]

Arguing as before, we see that we can obtain an even genus if and only if exactly one of the \(e_{Q_i}\) equals 20. Then, we are left with two possibilities: \((e_{Q_1}, d_{Q_2}) = (20, 23)\), and \((e_{Q_2}, d_{Q_2}) = (5, 8)\) or \((e_{Q_1}, d_{Q_1}) = (20, 23)\) and \((e_{Q_2}, d_{Q_2}) = (10, 13)\). In the former case we have \(g = 946\), in the latter \(g = 568\).

If (iii) holds, then (14) reads
\[\frac{g - 1}{1260} = -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}}.\]
with \((e_{Q_1}, d_{Q_1}) \in \{(5, 8), (10, 13), (20, 23)\}\) while \((e_{Q_2}, d_{Q_2}) \in \{(2, 1), (3, 2), (4, 3), (6, 5), (7, 6)\}\). Again, we are left only with the cases where \((e_{Q_1}, d_{Q_1}) = (20, 23)\). The only case which yields a nonnegative value for \(g\) is \((e_{Q_2}, d_{Q_2}) = (7, 6)\), which gives \(g = 10\). We show that there exist no ordinary genus 10 algebraic curve defined over an algebraically closed field of characteristic 5 with \(\text{Alt}_7\) as an automorphism group. In fact, assume that such a curve exists, and denote by \(X'\) the quotient curve \(X/C_5\). As \(C_5\) fixes just one point, the Riemann-Hurwitz formula applied to the covering \(X \rightarrow X'\) yields \(g(X') = 2\), and via the Deuring-Shafarevic formula we see that \(X'\) must be an ordinary hyperelliptic curve. Also, because of Galois Theory \(X'\) must have an automorphism group isomorphic to \(C_4\) fixing a point. Again by Riemann-Hurwitz, we see that \(g(X'/C_4) \in \{0, 1\}\). A simple computation on the Riemann-Hurwitz formula shows that we can have neither the former nor the latter.

\[\square\]

**Proposition 4.7.** Let \(X\) be an ordinary curve of even genus \(g\) and \(p\)-rank \(\gamma = g\) defined over a field \(K\) of odd characteristic \(p > 0\). Let \(G\) be a subgroup of \(\text{Aut}(X)\). If \(G = M_{11}\), then \(|G| \leq 84(g - 1)\) unless \(p = 3\) and \(g = 26\).

**Proof.** If \(p \neq 3, 5, 11\), then \(G\) is a tame automorphism group of \(X\) and the claim follows by the classical Hurwitz bound. If \(p = 3, 5, 11\), a careful case-by-case analysis is needed. As \(|G| = 7920\),
the Riemann-Hurwitz formula applied to the covering $\mathcal{X} \to \mathcal{X}/G = \mathcal{Y}$ reads

\begin{equation}
\frac{g - 1}{3960} = -2 + \sum_{Q \in \mathcal{Y}} \frac{d_Q}{e_Q}.
\end{equation}

Let $p = 11$. Then the 11-Sylow of $G$ is a cyclic group $C_{11}$ of order 11 whose normalizer is the semidirect product $C_{11} \rtimes C_5$ with $C_5$ a cyclic 5-group. Assume that (i) holds. Then (15) reads

\[
\frac{g - 1}{3960} = \frac{d_{Q_3}}{e_{Q_3}} - 1 = \frac{9}{e_{Q_3}},
\]

which is immediately seen to yield an odd $g$. The same argument allows us to discard case (ii). For case (iii), (15) reads

\[
\frac{g - 1}{3960} = -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}}.
\]

As before, we need $3960(-2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{d_{Q_2}}{e_{Q_2}})$ to be odd, whence $(e_{Q_2}, d_{Q_2}) = (8, 7)$ follows. Then $(e_{Q_1}, d_{Q_1}) = (5, 8)$ yields $g = 1882$ while $(e_{Q_1}, d_{Q_1}) = (55, 64)$ is impossible as it would give $g < 0$.

Let $p = 5$. The 5-Sylow of $G$ is a cyclic group $C_5$ of order 5 with normalizer $D_{10} \cdot 2$, which is isomorphic to the general affine group $GA(1, 5)$. Here, there are three possibilities for the size of a non-tame one-point stabilizer, namely 5, 10, 20. Hence, we see at once that cases (i)-(ii) cannot occur as 8 is the highest power of 2 dividing 3960. Thus, we are left only with sub-cases of (iii) where the size of the tame one-point stabilizer equals 8. Then (15) reads

\[
\frac{g - 1}{3960} = -2 + \frac{d_{Q_1}}{e_{Q_1}} + \frac{7}{8}.
\]

with $(e_{Q_1}, d_{Q_1}) \in \{(5, 8), (10, 13), (20, 23)\}$. The corresponding values for $g$ are 1882, 694 and 100 respectively.

Let $p = 3$. Then the 3-Sylow of $G$ is an elementary abelian group $E$ of order 9 whose normalizer is the semidirect product $E \rtimes SD_{16}$ where $SD_{16}$ is a semidihedral group of order 16. Assume that (i) holds. Then (15) reads

\[
\frac{g - 1}{3960} = \frac{d_{Q_3}}{e_{Q_3}} - 1.
\]

Then $(e_{Q_3}, d_{Q_3}) \in \{(3, 4), (6, 7), (9, 16), (18, 25), (72, 79)\}$. We can discard the case $e_{Q_3} = 36$ as the two classes of subgroups of order 36 in $M_{11}$ cannot be one-point stabilizers of the automorphism group of an algebraic curve as they are not isomorphic to the semidirect product of $E$ by a cyclic group of order 4. Anyway, we get an even genus only if $e_{Q_3} = 72$; in this case, $g = 386$, a contradiction.

Next, assume (ii) holds. Then we see that to get an even value for the genus we must have $e_{Q_1} = 72$ and $e_{Q_2} \neq 72$. Then (15) reads

\[
\frac{g - 1}{3960} = -2 + \frac{79}{72} + \frac{d_{Q_2}}{e_{Q_2}}.
\]

Recall that we are assuming $|G| \geq 84(g - 1)$, that is, $g \leq 95$. This is possible if and only if $\frac{d_{Q_2}}{e_{Q_2}} \leq 1223/1320$, a contradiction.
We are left with case (iii). In this case, \((e_{Q_2}, d_{Q_2}) \in \{(2, 1), (4, 3), (5, 4), (8, 7), (11, 10)\}\). Note that, apart from the case \((8, 7)\), in all the other cases we get an even genus if and only if \((e_{Q_2}, d_{Q_2}) = (72, 79)\). If this is the case, the above formula yields a nonnegative value if and only if \((e_{Q_2}, d_{Q_2}) = (11, 10)\), which yields \(g = 26\). The case \((e_{Q_2}, d_{Q_2}) = (8, 7)\) is impossible as there is a unique conjugacy class of cyclic groups of order 8 in \(M_{11}\) which is contained in the normalizer of \(E\), whence any of such groups fixes the same point as \(E\). \(\square\)

The problem of existence of ordinary curves of genus 26 admitting \(M_{11}\) as an automorphism group will be discussed in Section 5.

5. ON ORDINARY CURVES OF GENUS 26 ADMITTING \(M_{11}\) WHEN \(p = 3\)

In this Section, the characteristic of the ground field \(K\) is assumed to be \(p = 3\). We deal with the existence of ordinary genus 26 curves with \(M_{11}\) as an automorphism group. Adler [10] showed that in this case, the modular curve \(X(11)\) is a genus 26 curve such that \(M_{11} \subset \text{Aut}_K(X(11))\); later Rajan [2] showed that actually \(M_{11} \cong \text{Aut}_K(X(11))\). A plane model for \(X(11)\) is given by

\[
X(11) : y^{10}(y + 1)^9 = x^{22} - y(y + 1)^4x^{11}(y^3 + 2y + 1),
\]

see [19, Section 4.3].

**Remark 5.1.** By direct MAGMA computation it can be checked that if \(\gamma(X(11))\) is the 3-rank of \(X(11)\), then \(\gamma(X(11)) = g(X(11)) = 26\).

It is now natural to ask whether \(X(11)\) is or not the only ordinary genus 26 curve in characteristic 3 admitting \(M_{11}\) as an automorphism group. Let \(\mathcal{X}\) be any such curve. Then, by the proof of Proposition 4.7, \(M_{11}\) can have only two short orbits on \(\mathcal{X}\): a non-tame orbit \(O_1\), formed by 110 points, each with ramification index 72 and a tame orbit \(O_2\) consisting of 720 points each with ramification index 11. Further, the following holds.

**Proposition 5.2.** Let \(\mathcal{X}\) be an ordinary curve of genus 26 such that \(M_{11} \subset \text{Aut}_K(\mathcal{X})\). Denote by \(E\) the Sylow 3-subgroup of \(M_{11}\). Then the quotient curve \(\mathcal{X}/E\) is isomorphic to the hyperelliptic genus 2 ordinary curve

\[
\mathcal{Y} : y^2 = x^5 - x.
\]

**Proof.** Recall that \(E\) is an elementary abelian 3-group of order 9. There are exactly 55 3-Sylow in \(M_{11}\) which form a unique conjugacy class. The ramified points of \(\mathcal{X} \rightarrow \mathcal{X}/E\) are contained in \(O_1\). Moreover, \(O_1\) can be partitioned into short orbits of the 3-Sylows. A trivial counting argument shows that the only possibility is that each 3-Sylow fixes two points and acts semi-regularly on the rest of \(X(11)\). Let \(P_1, P_2\) be the two points that are fixed by \(E\). For \(j = 1, 2\), we have \(|G^0_{P_j}| = |G^1_{P_j}| = 9\), as \(G^0_{P_j} = S\) is the stabilizer of \(P_j\) and \(G^1(P_j)\) is the unique maximal normal 3-subgroup of \(G^0_{P_j}\). Also, as \(\mathcal{X}\) is ordinary, \(G^2_{P_j}\) is trivial. Hence, the Riemann-Hurwitz formula yields \(g(\mathcal{X}/E) = 2\).

By [18, Chapter 5.3.8] the normalizer of \(E\) in \(M(11)\) is the semidirect product \(E \rtimes SD_{16}\), where the semi-dihedral group \(SD_{16}\) of order 16 is a Sylow 2-subgroup of \(M(11)\). Then clearly \(\mathcal{X}/E\) has an automorphism group isomorphic to \(SD_{16}\). In [15, Section 3.2] it is shown that over a field of characteristic \(p \neq 2\) there is up to isomorphism only one curve with such an automorphism group, which is isomorphic to \(\mathcal{Y}\). We point out that the full automorphism group of \(\mathcal{Y}\) is \(S_4\), a double cover of the Symmetric Group on 4 letters. \(\square\)
Things being so, any of the curves we are looking for are exactly the curves whose function field is an elementary abelian extension of $K(Y)$ which ramifies at two points. It can be shown that these points coincide with the only two fixed points of the cyclic group $C_8$ acting on $Y$. Computing the actual number number of non-isomorphic such curves and deciding whether or not they admit $M_{11}$ as an automorphism group seems a rather challenging task.

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References

[1] A. Adler, The Mathieu Group $M_{11}$ and the modular curve $X(11)$, Proc. London Math. Soc. (3) 74 (1997) 1-28.
[2] M. Giulietti, G. Korchmáros, Algebraic curves with many automorphisms, [arXiv:1702.08812].
[3] M. Hall, The Theory of Groups, Macmillan, New York, (1959).
[4] J.W.P. Hirschfeld, G. Korchmáros and F. Torres, Algebraic Curves Over a Finite Field, Princeton Univ. Press, Princeton and Oxford 2008.
[5] B. Huppert, Endliche Gruppen. I, Grundlehren der Mathematischen wissenschaften 134, Springer, Berlin, 1967, xii+793 pp.
[6] B. Huppert, Zweifach transitive, auflösbare Permutationsgruppen, Math., 68 (1957), 126-150.
[7] King, Oliver H. The subgroup structure of finite classical groups in terms of geometric configurations. Surveys in combinatorics 2005, 2956, London Math. Soc. Lecture Note Ser., 327, Cambridge Univ. Press, Cambridge, 2005.
[8] A. Machì, Groups, An introduction to ideas and methods of the theory of groups, Unitext, 58, Springer, Milan, 2012. xiv+371 pp.
[9] R.C. Valentinì and M.L. Madan, A Hauptsatz of L.E. Dickson and Artin–Schreier extensions, J. Reine Angew. Math. 318 (1980), 156–177.
[10] G. Korchmáros, M. Montanucci, Ordinary algebraic curves with many automorphisms in positive characteristic, [arXiv:1610.05252].
[11] G. Korchmáros, M. Montanucci, P. Speziali, Transcendence Degree One Function Fields Over a Finite Field with Many Automorphisms. To appear in Journal of Pure and Applied Algebra, 2017.
[12] A. Kontogeorgis and V. Rotger, On abelian automorphism groups of Mumford curves, Bull. London Math. Soc. 40 (2008), 353-362.
[13] S. Nakajima, $p$-ranks and automorphism groups of algebraic curves, Trans. Amer. Math. Soc. 303 (1987), 595-607.
[14] C.S. Rajan, Automorphisms of $X(11)$ over characteristic 3, and the Mathieu Group $M_{11}$, J. Ramanujan Math. Soc. 13 (1998), no 1, 63-72.
[15] Shaska, T., Völklein, H., Elliptic subfields and automorphisms of genus 2 function fields. Algebra, arithmetic and geometry with applications (West Lafayette, IN, 2000), 703-723, Springer, Berlin, 2004.
[16] H. Stichtenoth, Algebraic function fields and codes, Springer-Verlag, Berlin and Heidelberg, 1993. vii+260 pp.
[17] H. Stichtenoth, Über die Automorphismengruppe eines algebraischen Funktionenkörpers von Primzahlcharakteristik. II. Ein spezieller Typ von Funktionenkörpern, Arch. Math. 24 (1973), 615–631.
[18] R.A. Wilson, The Finite Simple Groups, Springer-Verlag, London 2009.
[19] Yang, Y. Defining equations of modular curves, Adv. Math. 204 (2006), no. 2, 481-508.