Covariant multi-galileons and their generalisation

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We find a covariant completion of the flat-space multi-galileon theory, preserving second-order field equations. We then generalise this to arrive at an enlarged class of second order theories describing multiple scalars and a single tensor, and conjecture that these are a multi-scalar version of Horndeski’s most general scalar-tensor theory.

I. INTRODUCTION

Whilst General Relativity is supremely successful at describing gravity at solar system scales, it can only be made compatible with observation at galactic and cosmological scales if we accept that 95% of the Universe is composed of stuff we know very little about, namely dark matter [1] and dark energy [2]. Dark energy is particularly poorly understood from a particle physics perspective, so for this reason it is natural to ask if General Relativity ought to be replaced by a modified theory of gravity at the relevant scales (see [3] for an extensive review). The simplest modifications of gravity are those that contain additional scalar fields. So called scalar-tensor theories date back to Scherrer [4], Jordan [5], and Thiry [6], and, most notably, to Brans and Dicke [7]. Higher dimensional gravity generically introduces additional scalar fields when reduced down to four dimensions, through either Kaluza-Klein or braneworld compactifications (for reviews, see, [8–10]).

In 1974, Horndeski derived the most general theory describing a single scalar and a single tensor in four dimensions, preserving second order field equations. This last condition is required in order to avoid Ostrogradski ghosts associated with higher derivatives [14]. Much more recently, DGSZ [12] developed a generalisation of covariant galileon theory [15, 16]. The DGSZ model was later shown to be equivalent to Horndeski’s theory in four dimensions [13], although it is written in a more elegant form and generalises to any number of dimensions. Recent interest in Horndeski’s theory has been considerable, ranging from applications to inflation [13, 17], a discussion of the Vainshtein mechanism [18], and the derivation of boundary terms and junction conditions [19]. We should also note that it contains a number of very interesting theories as a subset including k-essence [20], covariant-galileons [16], KGB gravity [21] and the self-tuning Fab-Four scenario [22].

In this paper we work towards a multi-scalar analogue of Horndeski’s theory, describing the most general theory of multiple scalars and a single tensor, admitting second order field equations. We begin with the multi-galileon theory described in \( D \) dimensional Minkowski space \( ^{1} \)[23–26]

\[
S_{\text{multi-gal}} = \int_M d^Dx \frac{1}{\sqrt{-g}} \sum_{m=1}^{D+1} \alpha^{i_1 \ldots i_m} \pi_{i_1} \partial^{a_2} \partial^{\pi_{i_2}} \ldots \partial^{a_m} \partial^{\pi_{i_m}}
\]  

(1)

where \( \alpha^{i_1 \ldots i_m} \) is symmetric. Here \( i, j, k \) labels the scalar, and \( a, b, c \) labels the spacetime indices. For \( N \) scalars, \( i, j, k \) run from \( 1 \ldots N \), and in \( D \) dimensions \( a, b, c \) run from \( 0 \ldots D - 1 \). Throughout this paper, antisymmetrization omits the usual factor of \( 1/n! \).

Covariantization of (1) is achieved by first minimally coupling the scalars to gravity which generically introduces higher order field equations. To restore the system to second order we add curvature dependent counter terms and arrive at the following

\[
S_{\text{cov-multi-gal}} = \int_M d^Dx \sqrt{-g} \sum_{m=1}^{D+1} \alpha^{i_1 \ldots i_m} \pi_{i_1} \nabla_{a_2} \nabla^{[a_2} \pi_{i_2} \ldots \nabla_{a_m} \nabla^{a_m]} \pi_{i_m} + \sum_{m=3}^{D+1} \sum_{n=1}^{m-1} C^m_n
\]  

(2)

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1 This is the unique action that (i) is invariant under so-called galilean symmetry \( \pi \rightarrow \pi + b_0 a^0 + c \) and (ii) has at most second derivatives in the field equations.
where
\[
C_n^m = \left(-\frac{1}{4}\right)^n \frac{(m-1)!}{(m-2n-1)!(n)!} \alpha_1 \cdots \alpha_m \pi_{i_1} X_{i_2 i_3} \cdots X_{i_{2n+2} i_{2n+3}} \nabla_{a_2} \cdots \nabla_{a_{2n+2}} \pi_{i_{2n+3}} \cdots \nabla_{a_m} \pi_{i_m} R_{b_1 c_1}^{b_2 c_2} \cdots \nabla_{b_n}^{c_n}
\]
for \( n > 0 \), and \( X_{ij} = \frac{1}{2} \nabla_\alpha \pi_j \nabla^\alpha \pi_i \).

The covariant multi-galileon theory (2) describes a multiple scalar-tensor theory, with potentially interesting applications, ranging from multi-field galileon inflation [27] to covariant self-tuning scenarios (see section IV). In the case of a single scalar field, our theory does not quite reduce to the covariant galileon theory presented in [16] owing to the fact that the flat-space Lagrangians differ by a total derivative and this affects the subsequent covariant completion. Of course, both versions of the covariant single galileon still correspond to a subset of Horndeski’s theory. The derivation of our covariant multi-galileon theory is presented in section II, with some details postponed to the appendix. The appendix also includes the resulting field equations.

In section III we begin to generalise this theory, with a view to deriving a multi-scalar version of Horndeski. Using methods similar to those presented in [12], we first introduce the following generalised multi-galileon theory\(^2\)

\[
S_{\text{multi-scalar}} = \int_{\mathcal{M}} d^D x \ A(\bar{X}_{ij}, \pi_i) + \sum_{m=1}^{D-1} A^{k_1 \cdots k_m} (\bar{X}_{ij}, \pi_i) \partial^{a_1} \cdots \partial^{a_m} \partial_{a_m} \pi_{k_m}
\]

where \( \bar{X}_{ij} = \frac{1}{2} \partial_\alpha \pi_i \partial^\alpha \pi_j \), and conjecture that this is the most general multi-scalar theory defined on Minkowski space, preserving second order field equations, provided \( \frac{\partial A^{i_1 \cdots i_m}}{\partial X_{k_l \cdots k_l}} \) is symmetric in all of its indices \( i_1, \ldots, i_m, k, l \) (a formal proof of this conjecture now appears in [28]). We can covariantise this theory in the way described earlier, thereby arriving at the following

\[
S^{\text{cov-multi-scalar}} = \int_{\mathcal{M}} d^D x \sqrt{-g} A(X_{ij}, \pi_i) + A^k (X_{ij}, \pi_i) \Box \pi_k
\]

\[\begin{align*}
+ \sum_{m=2}^{D-1} \frac{(4\pi)^m}{m!} A^{k_1 \cdots k_m} (\bar{X}_{ij}, \pi_i) & \nabla_{a_1} \cdots \nabla_{a_m} \pi_{k_m} + \sum_{m=2}^{D-1} \sum_{n=1}^{\tilde{n}} Q^m_n,
\end{align*}\]

where \( \tilde{n} = \left\lfloor \frac{D-1}{2} \right\rfloor \) and

\[
Q^m_n = \left(-\frac{4\pi}{m!(m-2\tilde{n})!} \frac{\partial^{\tilde{n}-n}}{\partial X_{k_1 k_2} \cdots \partial X_{k_{2(n-n)-1} k_{2(n-n)}}} B_{m}^{k_2(n-n)+1 \cdots k_{m-2n}} (X_{ij}, \pi_i) \right)
\]

\[
\nabla_{a_1} \cdots \nabla_{a_{m-2n}} \pi_{k_{m-2n}} R_{b_1 c_1}^{b_2 c_2} \cdots \nabla_{b_n}^{c_n}
\]

Note that it is convenient to rewrite \( A^{k_1 \cdots k_m} = \frac{(4\pi)^m}{m!} \frac{\partial^{\tilde{n}}}{\partial X_{k_1 k_2} \cdots \partial X_{k_{2(n-n)-1} k_{2(n-n)}}} B_{m}^{k_2(n-n)+1 \cdots k_{m-2n}} \) for \( m \geq 2 \), and we remind the reader that this should be taken to be symmetric in all of its indices, as should its higher derivatives with respect to \( X_{ij} \). This generalised theory of multiple scalars and a single tensor reduces to Horndeski’s theory for the case of a single scalar, and represents the maximal extension of the most general flat space theory in curved space. We conjecture that this theory would also be the most general multi-scalar tensor theory giving equations of motion of derivative order up to two, however a proof of this is not known yet. Again, the potential applications of this theory are likely to be considerable, from multi-field inflation to a possible multi-field extension of the *Fab-Four* [22]. These and other future directions are discussed in greater detail in section IV.

II. MULTI-GALILEONS AND COVARIANTIZATION

We begin with the action describing multiple galileon fields in Minkowski space [26],

\[
S_{\text{multi-gal}} = \int_{\mathcal{M}} d^D x \sum_{m=1}^{D+1} \alpha^{i_1 \cdots i_m} \pi_{i_1} \partial^{a_1} \cdots \partial^{a_m} \partial_{a_m} \pi_{i_m}
\]

\[\text{(7)}\]

\[\text{\(\text{\(\text{\(\footnote{Notice that, after some integration by parts, (1) is the subset of (4) with the functions } A \text{ and } A^{k_1 \cdots k_m} \text{ linear in } X_{ij}.\)}}\text{\footnote{Notice that, after some integration by parts, (1) is the subset of (4) with the functions } A \text{ and } A^{k_1 \cdots k_m} \text{ linear in } X_{ij}.\)}}\text{\footnote{Notice that, after some integration by parts, (1) is the subset of (4) with the functions } A \text{ and } A^{k_1 \cdots k_m} \text{ linear in } X_{ij}.\)}}\]

\[\text{\(\text{\(\text{\(\footnote{Notice that, after some integration by parts, (1) is the subset of (4) with the functions } A \text{ and } A^{k_1 \cdots k_m} \text{ linear in } X_{ij}.\)}}\text{\footnote{Notice that, after some integration by parts, (1) is the subset of (4) with the functions } A \text{ and } A^{k_1 \cdots k_m} \text{ linear in } X_{ij}.\)}}\text{\footnote{Notice that, after some integration by parts, (1) is the subset of (4) with the functions } A \text{ and } A^{k_1 \cdots k_m} \text{ linear in } X_{ij}.\)}}\]
where $\alpha^{i_1 ... i_m}$ is completely symmetric. Recall that antisymmetrization omits the usual factor of $1/n!$ and that the indices $i, j, k$ label the scalar field, while $a, b, c$ are spacetime indices. The first step towards covariantizing this theory is to couple gravity minimally, promoted partial derivatives to covariant ones, $\partial_a \rightarrow \nabla_a$, such that

$$S_{\text{multi-gal}} \rightarrow \int_M d^Dx \sqrt{-g} \sum_{m=1}^{D+1} \alpha i_1 i_2 ... i_m \pi_{i_1} \nabla_{a_2} \pi_{i_2} \ldots \nabla_{a_m} \pi_{i_m}$$

(8)

Here we use the notation $\nabla_a b \equiv \nabla_a \nabla b$ and repeated indices are summed over. Indeed, let us summarize the notation we will adopt for the remainder of this paper in the following table. It is also convenient define the following scalars

| Notation | Description | Definition/Example |
|----------|-------------|-------------------|
| $i, j, k \ldots$ | Internal indices of the field | $\pi_i, \pi_j$ etc., $i, j, k \in \{1 \ldots N\}$ |
| $a, b, c \ldots$ | Space-time indices | $\nabla_a, \nabla_b, \nabla_c \in \{0 \ldots D-1\}$ |
| $\nabla_{ab}, \nabla^a$ | Double covariant derivative | $\nabla_{ab} \equiv \nabla_a \nabla_b, \nabla^a \equiv \nabla_a \nabla^b$ |
| $I_{2p}, J_q$ | Collective unordered internal index | $I_{2p} \equiv \{r_1 \ldots r_{2p}\}, J_q \equiv \{s_1 \ldots s_q\}$ |
| $\hat{a}, \hat{b}, \hat{c} \ldots$ | Antisymmetrized space-time index | $X^{\hat{a} \hat{b}} \times Y^{\hat{c} \hat{d}} \times Z^{\hat{f} \hat{g}} \equiv X^{[ab} Y^{cd]} Z^{fg]}$ |

Table I: Notations

for the sake of brevity,

$$E_{I_{2p}} = (\nabla_{a_1} \pi_{r_1} \nabla_{a_2} \pi_{r_2}) \ldots (\nabla_{a_{2p}} \pi_{r_{2p-1}} \nabla_{a_{2p}} \pi_{r_{2p}})$$

$$F_{J_q} = (\nabla_{a_1} \hat{a}_{s_1}) \ldots (\nabla_{a_q} \hat{a}_{s_q})$$

$$G_r = R^{\hat{a} \hat{b}}_{ \hat{a}_1 \hat{b}_1} \ldots R^{\hat{a} \hat{b}}_{ \hat{a}_{r-1} \hat{b}_{r-1}}$$

Here we take $E_{I_0} = F_{J_0} = G_0 = 1$ and $E_{I_{2p}} = F_{J_q} = G_r = 0$ when $p, q, r$ are negative. According to our notations we can write the $m$th order Lagrangian term as

$$C_{0}^m = \alpha^{i_1 i_2 \ldots i_m} \pi_{i_1} \nabla_{a_2} \pi_{i_2} \ldots \nabla_{a_m} \pi_{i_m} = \alpha^{i_1 J_{m-1}} \pi_{i_1} F_{J_{m-1}}$$

(10)

Variation of this term induced by $\pi_k \rightarrow \pi_k + \delta \pi_k$ where $k$ is an arbitrary integer between 1 and $N$ is given by,

$$\delta C_{0}^m = \alpha^{k J_{m-1}} F_{J_{m-1}} \delta \pi_k + (m-1)\alpha^{i_1 k J_{m-2}} \pi_{i_1} \nabla_{a} \hat{a} \delta \pi_k F_{J_{m-2}}$$

(11)

and after integrating by parts we get,

$$\delta C_{0}^m = \left\{ \alpha^{k J_{m-1}} F_{J_{m-1}} + (m-1)\alpha^{i_1 k J_{m-2}} \pi_{i_1} \nabla_{a} \hat{a} \pi_{i_1} F_{J_{m-2}} \right. \right.$$  

$$+ (m-1)(m-2)\alpha^{i_1 k J_{m-3}} (2 \nabla_{a} \pi_{i_1} \nabla_{b} \pi_{i_2} F_{J_{m-3}} + \pi_{i_1} \nabla_{a} \nabla_{b} \pi_{i_2} F_{J_{m-3}}) \right\} \delta \pi_k$$  

$$= \left\{ \alpha^{k J_{m-1}} F_{J_{m-1}} + (m-1)\alpha^{i_1 k J_{m-2}} \pi_{i_1} \nabla_{a} \hat{a} \pi_{i_1} F_{J_{m-2}} + (m-1)(m-2)\alpha^{i_1 k J_{m-3}} (\nabla_{a} \pi_{i_1} \nabla_{b} \pi_{i_2} R^{\hat{a} \hat{b}}_{bc} F_{J_{m-3}} + \right.$$  

$$- \frac{1}{4} \pi_{i_1} \nabla_{i_3} \nabla_{c} \nabla_{e} R^{\hat{a} \hat{b}}_{\hat{a} \hat{b}} F_{J_{m-3}} + \frac{1}{2} \pi_{i_1} R^{\hat{a} \hat{b}}_{\hat{a} \hat{b}} \nabla_{a} \nabla_{i} \pi_{i_2} F_{J_{m-3}}) \right\} \delta \pi_k$$

(12)

Here we have used the Riemann and Bianchi identities in the second step. To remove the term containing third derivatives in the metric we add the following counter term to the action.

$$C_{1}^m = -\frac{1}{8} \alpha^{i_1 i_2 J_{m-3}} \pi_{i_1} i_{I_{2}} F_{J_{m-3}} G_{1}$$

(13)

Although the variation of $E_{I_{2}}$ would generate the correct term to cancel the higher derivative term in (12), a further higher order term would be generated through the variation of $F_{J_{m-3}}$. Thus it is clear that a finite number of counter
terms should be added recursively at each order in $\pi$. We find that the counter term needed at the $n^{th}$ step is given by (see appendix for details),

$$
C^m_n = T_n^{i_1 i_2 \ldots i_{m-2-n}} \pi_{i_1} E_{i_2} \ldots E_{i_{m-2-n}} G_n
$$

(14)

where,

$$
T_n^{i_1 \ldots i_m} = \left( \frac{1}{8} \right)^n \frac{(m-1)!}{(m-2n-1)! (n!)^2} \alpha^{i_1 \ldots i_m} \quad m \geq 3
$$

(15)

It turns out that these counter terms are also sufficient to remove higher derivative terms generated in the $g_{ab}$ equation of motion (see appendix). Thus a covariant generalization of multi-galileon theory, preserving second order field equations, is given by

$$
S_{\text{cov-muti-gal}} = \int d^D x \sqrt{-g} \sum_{m=1}^{D+1} \sum_{n=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} C^m_n
$$

(16)

Of course, this was already expressed using more familiar notation in equation (2). The corresponding field equations are given by equations (A9) and (A15) in the appendix. For a single scalar field this theory does not quite reduce to the one presented in [16], although it does still correspond to a subset of Horndeski’s theory [11, 12]. The reason for the slight discrepancy is that our starting point in Minkowski space differs from that in [16] by a total derivative and this affects the details of the subsequent covariantisation.

### III. TOWARDS MULTI-SCALAR HORNDESKI

Having derived the covariant multi-galileon theory, it is natural to ask if we can go a stage further and find a multi-scalar generalisation of Horndeski’s panoptic theory [11]. Recall that Horndeski’s theory was rediscovered by DGSZ [12] using the following method: find the most general theory of a scalar in Minkowski space, with second order field equations, and then covariantise the resulting theory. Here we will conjecture the form of the most general multi-scalar theory in Minkowski space, with second order equations of motion, and covariantise the result in order to give a generalised multi-scalar tensor theory of gravity. We do not attempt to prove the generality of our theory here, and leave this question as a future project.

To arrive at our proposed multi-scalar theory in Minkowski we begin by performing an integration by parts on the multi-galileon action (1), and some relabelling, to arrive at the following

$$
S_{\text{multi-gal}} = \int \mathcal{M} d^D x \left[ \frac{1}{2} \alpha^i \pi_i - \alpha_{ij} \tilde{X}_{ij} \right] - \frac{1}{2} \sum_{m=1}^{D-1} \frac{m + 2}{2} \alpha_{ijkl} \ldots \pi_{k_m} \pi_{l_m} \sum_{l=1}^{D-m} \pi_{l_m} \pi_{k_m}
$$

(17)

where we recall that $\tilde{X}_{ij} = \frac{1}{2} \partial_i \pi_j \partial^a \pi_j$. An obvious generalisation of this, consistent with the one for a single scalar presented in [12], is given by

$$
S_{\text{multi-scalar}} = \int \mathcal{M} d^D x \left[ \tilde{X}_{ij} (\pi_i, \pi_l) + \sum_{m=1}^{D-1} \bar{A}_l \ldots \pi_{k_m} \pi_{l_m} \sum_{l=1}^{D-m} \pi_{l_m} \pi_{k_m}
$$

(18)

Taken at face value, this action will yield higher order equations of motion. However, this can be avoided by imposing the condition that $\frac{\partial A^{i_1 \ldots i_m}}{\partial x_{i_1 \ldots i_m}}$ is symmetric in all of its indices $i_1, \ldots, i_m, k, l$. We conjecture that this theory is the most general multi-scalar theory in Minkowski space, with second order equations of motion. This is certainly true for the case of a single scalar for which (18) reduces to the general theory presented in [12].

The next step is to covariantise the theory (18). As before we begin by minimally coupling to gravity, promoting partial derivatives to covariant ones, $\partial_a \rightarrow \nabla_a$ yielding

$$
S_{\text{multi-scalar}} \rightarrow \int \mathcal{M} d^D x \sqrt{-g} \sum_{m=0}^{D-1} A(X_{ij}, \pi_l) \bar{A}^{i_1 \ldots i_m} \nabla_{a_1} \pi_{i_1} \ldots \nabla_{a_m} \pi_{i_m}
$$

(19)

with $X_{ij} := \frac{1}{2} \nabla_a \pi_i \nabla^a \pi_j$. Analogous to the covariantized multi-galileons, this action would yield equations of motion of derivative order greater than two. Thus we introduce the following counter terms at each order in $\pi_i$ to cancel those higher derivatives.

$$
Q^m_n = A_n (X_{ij}, \pi_l)_{j_{m-2n}} F_{j_{m-2n}} G_n
$$

(20)
Note that $Q_0^m = A(X_{ij}, \pi_i)^{i_1 \cdots i_m} \nabla_{a_{[i_1}} \cdots \nabla_{a_{i_m]}} \pi_{i_m}}$. In order to impose the constraint that the equations of motion are second order, we take the variation of $Q_0^m$ induced by $\pi_k \to \pi_k + \delta \pi_k$. We focus only on those terms that contain higher derivatives, and using the following short-hand,

$$A_{ij} = A_a(X_{ij}, \pi_i)^{ij}a, \quad \partial^{ij} A_{ij}^m = \frac{\partial A_{ij}}{\partial X_{ij}}$$

we obtain

$$\delta Q_0^m = \frac{\partial A_{ij}}{\partial X_{ij}} \delta J_{ij} = \delta J_{ij} A_n^m = \frac{\partial A_{ij}}{\partial X_{ij}} \frac{\partial A_{ij}}{\partial X_{ij}}$$

we find that

$$\delta Q_0^m = \delta A_{ij}^m \delta J_{ij} = \frac{\partial A_{ij}^m}{\partial X_{ij}} \frac{\partial A_{ij}^m}{\partial X_{ij}}$$

After performing an integration by parts, and using both the Riemann and Bianchi identities, we find that

$$\delta Q_0^m \cong \left\{ - (m-2n) \partial^{ij} A_{ij}^m \delta J_{ij} \delta X_{ij} \frac{\partial A_{ij}^m}{\partial X_{ij}} \frac{\partial A_{ij}^m}{\partial X_{ij}} \right\}$$

where the bracket $(ij \ldots)$ stands for the symmetrization of the indices. We should also note that subsequent derivatives preserve this property of the function i.e. $\partial^{ij} A_{ij}^m = \partial^{ij} A_{ij}^m$. The remaining higher order terms are now

$$\delta Q_0^m \cong \left\{ - n \partial^{ij} A_{ij}^m \delta J_{ij} \delta X_{ij} \frac{\partial A_{ij}^m}{\partial X_{ij}} \frac{\partial A_{ij}^m}{\partial X_{ij}} \right\}$$

These can be cancelled off by successive counter terms, $Q_{n+1}$, provided the following recursive relationship holds.

$$(s + 1) \partial^{ij} A_{ij}^m = \frac{(m-2n)(m-2s-1)}{4} A_{ij}^m \delta J_{ij}$$

Operating with $(\partial^{ij})^s \cdots$ on both sides and taking the following product, we find that

$$\prod_{s=0}^{n-1} \frac{(\partial^{ij})^s \cdots A_{ij}^m \delta J_{ij} \delta X_{ij}}{(\partial^{ij})^s \cdots \partial^{ij} A_{ij}^m \delta X_{ij}} = \frac{A_{ij}^m \delta J_{ij} \delta X_{ij}}{(\partial^{ij})^n \cdots \partial^{ij} A_{ij}^m \delta X_{ij}} = \frac{(-4)^{n-\bar{n}} n! (m-2n)!}{n! (m-2n)!}$$

Here $\bar{n} = \left\lfloor \frac{m}{2} \right\rfloor$ denotes the last counter term. We take $A_{ij}^m = B_{ij}^m$ to define an arbitrary function for each $m$, giving,

$$A_{ij}^m = \frac{(-4)^{n-\bar{n}} n! (m-2\bar{n})!}{n! (m-2n)!} (\partial^{ij})^n \cdots \partial^{ij} A_{ij}^m \delta X_{ij} \delta J_{ij} \delta X_{ij} \cdots \delta J_{ij} \delta X_{ij}$$

We conclude that the following generalized multi-scalar tensor theory is second order field equations from variation of the scalars

$$S_{\text{cov-multiscalar}} = \int d^D x \sqrt{-g} \sum_{m=0}^{D-1} \sum_{n=0}^{|\bar{n}|} Q_0^m$$

This action is written using more familiar notation in equation (5). It remains to show that it does not give rise to higher derivatives in the $g_{ab}$ equations of motion. Variation of the metric gives

$$\delta Q_0^m \cong A_{ij}^m \delta J_{ij} \delta X_{ij} F_{ij}^{m-2n} G_n + A_{ij}^m \delta J_{ij} \delta X_{ij} F_{ij}^{m-2n} G_n$$

$$= (m-2n) A_{ij}^m \delta J_{ij} \delta X_{ij} \left( \nabla^{a_b} \pi_{a} \delta g_{ab} - \nabla^{a_{[i_1}} \pi_{i_2} \delta g_{ab} \cdots + \frac{1}{2} g^{a_b} \nabla^a \pi_{a_b} \delta g_{ab} \cdots \right) F_{ij}^{m-2n} G_n$$

$$+ n A_{ij}^m \delta J_{ij} \delta X_{ij} \left( R_{c}^{b_a} \delta g_{ab} - 2 g^{a_b} \delta g_{abc} \right) G_n$$

(29)
where we again focus on terms that yield higher derivatives. Integrating by parts and making use of the geometric identities, we find that

$$\delta Q_n^m \supset \left\{ -\frac{(m-2n)(m-2n-1)}{2} A_n^{ij} A_{m-2n}^{j,\cdots d} \nabla_{\pi} \nabla_{\pi} \pi_{i} \nabla_{\pi}^{d} F^{F_{j,m-2n} G_n \delta^{ab}} ight. \left. -2n \partial_{ij} A_{n}^{m-2n} \nabla_{\pi} \nabla_{\pi}^{d} F^{F_{j,m-2n} G_n \delta^{ab}} \right\} \delta g^{ab} \right.$$  

(30)

It is clear that these higher derivative terms would cancel if the same recursive relationship (25) holds. We therefore conclude that our generalised theory (5) gives at most second order field equations under variation of all fields. As a consistency check of our work, it is reassuring to see that (5) does indeed reduce to Horndeski’s theory [11, 12] in the case of a single scalar.

### IV. DISCUSSION

In this paper, we have shown how gravity may be coupled to multi-galileons [24-26] without introducing higher order field equations, and generalised our result, proposing a multi-scalar version of Horndeski’s panoptic scalar-tensor theory [11, 12]. The actions for these theories are given by equations (2) and (5) respectively, and both may have interesting applications in multi-field inflation and quintessence scenarios. Indeed, the renaissance of Horndeski’s single scalar-tensor theory has prompted a scan of generalised single field inflation models [13]. Already there we see a variety of unexpected observational signatures eg. galileon inflation can give rise to observable 4-point functions even when the 3-point function is small [31]. The multi-scalar tensor theory proposed here now opens up the possibility of scanning the properties of generalised multi-field inflation models.

For the case of two galileons, the covariant theory (2) may be particularly relevant in the context of the cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant by the self-adjusting galileon fields. This is not in contradiction to Weinberg’s famous no-go theorem [29] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [24] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant by the self-adjusting galileon fields. This is not in contradiction to Weinberg’s famous no-go theorem [29] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [24] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant by the self-adjusting galileon fields. This is not in contradiction to Weinberg’s famous no-go theorem [29] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [24] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant by the self-adjusting galileon fields. This is not in contradiction to Weinberg’s famous no-go theorem [29] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [24] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant by the self-adjusting galileon fields. This is not in contradiction to Weinberg’s famous no-go theorem [29] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [24] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can give rise to self-tuning, with the physical spacetime screened from the cosmological constant by the self-adjusting galileon fields. This is not in contradiction to Weinberg’s famous no-go theorem [29] as Poincaré invariance is broken in the galileon sector. However, the proposal presented in [24] was not entirely satisfactory for the following reason. Self-tuning a large cosmological constant problem. In [24] it was shown that certain classes of bigalileon theories can gi...
Staying on the subject of self-tuning, we note that our proposed multi-scalar version of Horndeski's theory (5), puts us in a good position to generalise the so-called Fab Four theory [22] to multiple fields. The Fab Four Lagrangians were obtained by asking which subset of Horndeski's theory can "solve" the cosmological constant problem in that they screen the curvature from the vacuum energy. Given the generality of Horndeski, this enables one to say that a self-tuning single scalar-tensor theory in four dimensions must correspond to a Fab Four theory. This is a rather powerful statement, but we are now in position to make it even more powerful by generalising to multiple fields. Multiple fields will open up new possibilities as well, allowing for greater flexibility in deriving stable, phenomenologically consistent solutions.

We close our discussion by drawing attention to an interesting two-scalar tensor theory which can now be seen as a subset of the two-scalar version of our generalised theory (5). This so-called Fab Five theory is an extension of certain Fab Four Lagrangians[32] and is given by the following

\[
S_{\text{Fab5}} = \int d^4x \sqrt{-g} \left[ \frac{M^{pl}_2}{2} R - \frac{c_1}{2} (\nabla \pi)^2 + f \left( \frac{c_2}{2} (\nabla \pi)^2 + \frac{c_G}{M^2} G^{ab} \nabla_a \pi \nabla_b \pi \right) \right]
\]  

(33)

When the function \(f\) is the identity, this corresponds to a theory built from John and George from the Fab Four, along with a canonical kinetic term for the scalar. Generalising \(f\) introduces an additional scalar degree of freedom, and the theory can be written as [32]

\[
S_{\text{Fab5}} = \int d^4x \sqrt{-g} \left[ \frac{M^{pl}_2}{2} R - \frac{1}{2} (c_1 + c_2 f'(\xi))(\nabla \pi)^2 + \frac{c_G}{M^2} f'(\xi) G^{ab} \nabla_a \pi \nabla_b \pi + f(\xi) - \xi f'(\xi) \right]
\]  

(34)

This corresponds to a particular two scalar-tensor theory contained within (5).

Whilst we have alluded to a few, at this stage it is impossible to envisage all the potential applications of our generalised multi-scalar tensor theory. Horndeski’s theory is currently the focus of plenty of research, and to that we can now add its generalisation. See [33] for some interesting recent use of Horndeski’s theory that may now be extended to include the theory presented in this paper.

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Appendix A: Recursive cancellation of higher order terms via counter terms

Here demonstrate how the counter-terms defined for the covariant multi-galileon theorem, \(C^m_n\), give rise to recursive cancellation of higher order derivatives upon variation. The methods described here were also applied to the generalised Horndeski theory, although the details are slightly different. Note that in passing we will present the field equations for the covariant multi-galileon theory (2)

1. \(\pi_k\) equation of motion

Let us begin with the scalar equations of motion. The minimally coupled Lagrangian term at \(m^{th}\) order in \(\pi_r\) is given by,

\[
C^m_0 = \alpha^{1\ldots i_m} \pi_{i_1} \nabla_a \ldots \nabla_a a^{m} \pi_{i_m} \tag{A1}
\]

More generally we define the corresponding counter term required at the \(n^{th}\) recursive step at the same order to be,

\[
C^m_n = T^{i_1 i_2 
abla \pi_{i_1} E_{i_2} F_{i_3} F_{i_m-2n-1} G_n \tag{A2}
\]

where \(T^{i_1 i_2 J_m-2n-1} E_{i_2} F_{i_3} F_{i_m-2n-1} G_n \) is symmetric in the last \((m-1)\) indices and to be determined. Variation of \(C^m_n\) induced by the variation in \(\pi_k\) is,

\[
\delta C^m_n = T^{i_1 i_2 J_m-2n-1} E_{i_2} F_{i_m-2n-1} \delta \pi_k + 2n T^{i_1 k i_2 J_m-2n-1} \pi_1 \nabla_a \pi_{i_1} E_{i_2} F_{i_m-2n-1} G_n \nabla_a \delta \pi_k \\
+ (m-2n-1) T^{i_1 k i_2 J_m-2n-2} \pi_{i_1} E_{i_2} F_{i_m-2n-2} G_n \nabla_a \delta \pi_k \tag{A3}
\]
and subsequent integration by parts yields,

\[
\delta C^m_n = \left\{ T_{n}^{k_{1}l_{2}n}J_{m-2n-1}E_{l_{2}n}F_{J_{m-2n-1}G_{n}} \\
-2n T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-1}\nabla_{a}(\pi_{i_{1}}\nabla^{a}\pi_{i_{3}}E_{l_{2}n-2})F_{J_{m-2n-1}G_{n}} \\
+2n(m - 2n - 1) T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-2}\left[ -\pi_{i_{1}}\nabla_{b}\pi_{i_{3}}R^{bc}_{\quad ac}\nabla^{c}\pi_{i_{4}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n}} \\
+2\nabla_{a}\pi_{i_{1}}\nabla^{b}\pi_{i_{3}}\nabla_{b}\pi_{i_{4}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n}} + \pi_{i_{1}}(\nabla_{ab}\pi_{i_{3}}\nabla^{ab}\pi_{i_{4}})E_{l_{2}n-2}F_{J_{m-2n-2}G_{n}} \right] \\
+T_{n}^{k_{1}l_{2}n}J_{m-2n-2}(m - 2n - 1)\nabla_{a}\pi_{i_{1}}E_{l_{2}n}F_{J_{m-2n-2}G_{n}} \\
+2\left(\frac{n}{2}\right)(m - 2n - 1) T_{n}^{k_{1}l_{3}i_{3}l_{4}i_{4}l_{2}n-4}J_{m-2n-2}^{2}\pi_{i_{1}}(\nabla_{ab}\pi_{i_{3}}\nabla^{b}\pi_{i_{4}}\nabla^{c}\pi_{i_{5}}E_{l_{2}n-4}F_{J_{m-2n-2}G_{n}} \\
-2 T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-3}\pi_{i_{1}}\nabla^{c}\pi_{i_{3}}R^{bc}_{\quad dc}\pi_{i_{4}}E_{l_{2}n-4}F_{J_{m-2n-3}G_{n}} \\
-\left(\frac{m - 2n - 1}{4}\right) T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-3}(\pi_{i_{1}}\nabla^{c}\pi_{i_{3}}\nabla^{c}\pi_{i_{4}}R^{ab}_{\quad dc}\pi_{i_{5}}E_{l_{2}n-4}F_{J_{m-2n-3}G_{n}} \\
-2 T^{2}_{n}T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-2}\pi_{i_{1}}\nabla^{c}\pi_{i_{3}}(\nabla^{c}R^{ab}_{\quad dc}\pi_{i_{5}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n-1}} \right) \delta\pi_{k} \right) \tag{A4}
\]

Notice that the last two terms contain third derivative terms in the metric,

\[
\delta C^m_n \supset -\left\{ \left(\frac{m - 2n - 1}{4}\right) T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-3}(\pi_{i_{1}}\nabla^{c}\pi_{i_{3}}\nabla^{c}\pi_{i_{4}}R^{bc}_{\quad dc}\pi_{i_{5}}E_{l_{2}n-4}F_{J_{m-2n-3}G_{n}} \\
-2 T^{2}_{n}T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-2}\pi_{i_{1}}\nabla^{c}\pi_{i_{3}}(\nabla^{c}R^{ab}_{\quad dc}\pi_{i_{5}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n-1}} \right) \delta\pi_{k} \right) \tag{A5}
\]

It is clear that these terms can be absorbed into each other recursively if the following relationship holds,

\[
\frac{T_{n}^{i_{1}i_{2}...i_{m}}}{T_{n}^{i_{s+1}...i_{m}}} = -\frac{1}{8} \left(\frac{m - 2s - 1}{s}\right) \left(m - 2s - 2\right) \tag{A6}
\]

which implies that

\[
\prod_{s=0}^{n-1} \left[ \frac{T_{n}^{i_{1}i_{2}...i_{m}}}{T_{n}^{i_{s+1}...i_{m}}} \right] = \frac{T_{n}^{i_{1}i_{2}...i_{m}}}{\alpha^{i_{1}...i_{m}}} = \left(\frac{1}{8}\right)^{n} \frac{(m - 1)!}{(n!)^{2}} \frac{1}{(m - 2n - 1)!} \qquad m > 2 \tag{A7}
\]

This yields the result given by equation (15). Finally, to express the \(\pi_{k}\) equation of motion, we collect terms that at most second order in \(\delta C^m_n\). These are given by

\[
\epsilon^{(k)}_m = T_{n}^{k_{1}l_{2}n}J_{m-2n-1}E_{l_{2}n}F_{J_{m-2n-1}G_{n}} \\
-2n T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-1}\nabla_{a}(\pi_{i_{1}}\nabla^{a}\pi_{i_{3}}E_{l_{2}n-2})F_{J_{m-2n-1}G_{n}} \\
+2n(m - 2n - 1) T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-2}\left[ -\pi_{i_{1}}\nabla_{b}\pi_{i_{3}}R^{bc}_{\quad ac}\nabla^{c}\pi_{i_{4}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n}} \\
+2\nabla_{a}\pi_{i_{1}}\nabla^{b}\pi_{i_{3}}\nabla_{b}\pi_{i_{4}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n}} + \pi_{i_{1}}(\nabla_{ab}\pi_{i_{3}}\nabla^{ab}\pi_{i_{4}})E_{l_{2}n-2}F_{J_{m-2n-2}G_{n}} \right] \\
+T_{n}^{k_{1}l_{2}n}J_{m-2n-2}(m - 2n - 1)\nabla_{a}\pi_{i_{1}}E_{l_{2}n}F_{J_{m-2n-2}G_{n}} \\
+2\left(\frac{n}{2}\right)(m - 2n - 1) T_{n}^{k_{1}l_{3}i_{3}l_{4}i_{4}l_{2}n-4}J_{m-2n-2}^{2}\pi_{i_{1}}(\nabla_{ab}\pi_{i_{3}}\nabla^{b}\pi_{i_{4}}\nabla^{c}\pi_{i_{5}}E_{l_{2}n-4}F_{J_{m-2n-2}G_{n}} \\
-2 T^{2}_{n}T_{n}^{k_{1}l_{3}i_{3}l_{2}n}J_{m-2n-2}\pi_{i_{1}}\nabla^{c}\pi_{i_{3}}(\nabla^{c}R^{ab}_{\quad dc}\pi_{i_{5}}E_{l_{2}n-2}F_{J_{m-2n-2}G_{n-1}} \right) \delta\pi_{k} \right) \tag{A8}
\]
It follows that the $\pi_k$ equation of motion is given by,

$$ \sum_{n=1}^{D+1} \sum_{m=0}^{D+1} \epsilon^{(k)}_n m = 0 $$

(9)

which is, of course, at most second order in derivatives, as desired.

2. $g_{ab}$ equation of motion

We now verify that our chosen counter-terms (14) also guarantee second order equations of motion from metric variation. To this end, we first note the following identities

$$ \delta K_{ab}^{\hat{\phi}} \pi^{\hat{\phi} d} = (\delta \hat{R}_{ab}^{\hat{\phi}} - 2g_{ab} \delta g_{bc}^{\hat{\phi}} \hat{\delta} \hat{\pi}^{\hat{\phi} d}) \pi^{\hat{\phi} d} $$

(A10)

$$ \delta \nabla_a \pi_s = -\nabla^b \pi_s \delta g_{ab} - \nabla^a \pi_s \delta g_{ab} \hat{\delta} \hat{\pi}^{\hat{\phi} d} \pi_d + \frac{1}{2} g_{ab} \nabla^c \pi_s \delta g_{ac} $$

(A11)

The variation of the counter-term induced by the metric variation is,

$$ \delta C_{nm} = T^{i_1 i_2 \cdots i_{m-1}}_{n} \left[ \right] - T^{i_1 i_2 \cdots i_{m-1}}_{n} \left[ \right] - n \pi_{i_1} \nabla^b \pi_{i_2} \delta g_{ab} - n \pi_{i_1} \nabla^b \pi_{i_2} \hat{\delta} \hat{\pi}^{\hat{\phi} d} \pi_d + \frac{1}{2} g_{ab} \nabla^c \pi_s \delta g_{ac} $$

so that after integration by parts we obtain,

$$ \delta C_{nm} = \left\{ \right. $$

(A12)

Again, focussing on the third derivative terms,

$$ \delta C_{nm} \supset \left\{ \right. $$

(A13)

(A14)
we see that they can be recursively cancelled if the same relationship (A6) holds. As before, to express the metric equation of motion we collect terms up to second order in derivatives, remembering to include the term generated by the variation of the metric determinant $\sqrt{-g}$. We find that the $g_{ab}$ equations of motion are given by,

$$
\sum_{m=1}^{D+1} \sum_{n=0}^{\frac{2n+1}{2}} \mathcal{E}_n^{(m)ab} = 0
$$

(A15)

where $\mathcal{E}_n^{(m)ab} = \frac{1}{2} (\epsilon_n^{(m)ab} + \epsilon_n^{(m)ba})$ and

$$
\epsilon_n^{(m)ab} = -n T_n^{i_1 i_2 j_2 m-2} \pi_{i_1} \nabla^a \pi_{i_2} E_{j_2 m-2} F_{j_2 m-2} G_n - 4n^2 \pi_{i_1} \nabla_c E_{j_2 m-2} F_{j_2 m-2} E_{i_1} G_n - (m-2n+1) T_n^{i_1 i_2 j_2 m-2} \pi_{i_1} \nabla^b \pi_{i_2} E_{j_2 m-2} F_{j_2 m-2} G_n + (\pi_{i_1} \nabla^b \pi_{i_2} E_{j_2 m-2})^a F_{j_2 m-2} G_n
$$

(A16)

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