Logarithm corrections in the critical behavior of the Ising model on a triangular lattice modulated with the Fibonacci sequence

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(Dated: Received: date / Revised version: date)

Abstract

We investigated the critical behavior of the Ising model in a triangular lattice with ferro and anti-ferromagnetic interactions modulated by the Fibonacci sequence, by using finite-size numerical simulations. Specifically, we used a replica exchange Monte Carlo method, known as Parallel Tempering, to calculate the thermodynamic quantities of the system. We have obtained the staggered magnetization $q$, the associated magnetic susceptibility ($\chi$) and the specific heat $c$, to characterize the universality class of the system. At the low-temperature limit, we have obtained a continuous phase transition with a critical temperature around $T_c \approx 1.4116$ for a particular modulation of the lattice according to the Fibonacci letter sequence. In addition, we have used finite-size scaling relations with logarithmic corrections to estimate the critical exponents $\beta$, $\gamma$, and $\nu$, and the correction exponents $\hat{\beta}$, $\hat{\gamma}$, $\hat{\alpha}$ and $\hat{\lambda}$. Our results show that the system obeys the Ising model universality class and that the critical behavior has logarithmic corrections.

PACS numbers: 05.50.+q,64.60.F-,75.50.Kj
INTRODUCTION

At the beginning of the last century, with the experiments of Rutherford (awarded with the Nobel Prize in Chemistry in 1908) on the discovery and interpretation of the dispersion a beam of alpha particles directed to a fine gold leaf, there was a great effort in the scientific community in knowing the structure of matter, especially in its solid state. Since then, one aspect of the structure of matter that has become well known has been the symmetry of translation of atoms organized into crystals. However, in 1984, in their work entitled “Metallic Phase with Long-Range Orientational Order and No Translational Symmetry”, Shechtman et al., that was awarded the Nobel Prize in Chemistry in 2011, has shown new solid materials that exhibit a new symmetry in the structure of matter: the quasicrystals. Quasicrystals are a particular type of solid structure that can have unusual discrete point group symmetries, not expected from a translational symmetric Bravais lattice in two dimensions. It has also been shown that quasicrystals can have icosahedral symmetry in three dimensions. It had long been known that icosahedral symmetry is not allowed for periodic objects like crystals. It is forbidden in crystallography. However, systems like quasicrystals are not periodic but exhibit an exotic (forbidden) symmetry. They have a long-range order called quasiperiodicity, that characterize their unique and fascinating properties: they follow mathematical rules. Some metallic alloys, self-matter systems, supramolecular dendritic systems, and copolymers are examples of quasiperiodic systems. In common, these systems possess magnetic properties sensitive to the local atomic structure such as the atomic distance, coordination number and the kind of the nearest-neighbor atoms. Over of the last decades, the quasicrystals have to contribute to advance knowledge about the atomic scale structure. However, some questions remain open, such as magnetic properties of the physical systems that present these quasiperiodic structures. For example, a question unanswered is whether long-range antiferromagnetic (AFM) order can be sustained in real quasicrystalline systems.

In recent decades, research on quasicrystals has contributed to the advancement of knowledge about the structure of the matter at the atomic scale. However, some questions remain open. One is: what are the magnetic properties of the physical systems that have these atoms organized in quasi-periodic order? Specifically, another question unanswered is: a long-range antiferromagnetic (AFM) order can be sustained in real quasicrystalline systems?
The lack of translational symmetry in the quasiperiodic structure can induce, as consequence, anomalous properties different from a regular crystal. In the other hand, the quasicrystals are different from disordered materials, because they possess self-similar properties, i.e., any finite section of the structure represent exactly or approximately the structure of the quasicrystal in a distance of a degree of its linear (2D or 3D) scale. Also, the quasicrystal can present long-range correlations in sufficiently low temperature. If all these ingredients come into place in a simple lattice model, the physical observables obtained can be affected by the quasiperiodicity of the lattice itself. Instead of the antiferromagnetic periodic crystals, the antiferromagnetic arrangement in quasicrystals has shown a different behavior from usual crystals. For example, rare earth-containing quasicrystals \[11, 12\] exhibit an aperiodic ferrimagnet freezing phase at low-temperature.

The theoretical works \[13–17\] suggest the possibility of a non-trivial magnetic ordering for the quasicrystals. Although no antiferromagnetic quasicrystal has not yet been discovered, these studies reveal that the behavior of some quasicrystals at low-temperature show magnetic topological order and frustration. The debate in question allows us to study new theoretical models to answer the questions that remain open. Amongst them, the change of the critical exponents is partially answered by the Harris-Luck criterion, valid for ferromagnetic systems \[18\].

On another hand, there are ways to control the disorder in a certain systems and obtain a transition from a long-range order to a quenched disorder \[19, 20\] or, alternatively, the quasiperiodic order by modifying the exchange strengths and signals. The types of the quasiperiodic order that can be used to model the quasicrystals are: 1) we can modulate the interactions; 2) we can change the geometry of the crystal lattice; We choose the second option by considering a triangular lattice with ferro and antiferromagnetic exchange interactions modulated by a quasiperiodic sequence.

Recently, we published two papers about quasiperiodic models based on Fibonacci and Octonacci sequence, respectively. The models were applied in a square lattice. We have shown that is possible obtained the critical behavior of the two models, being that both have presented second order transition, with critical temperature, \(T_c = 1.274\) and \(T_c = 1.413\). The square lattice was modulated in a way that generates frustrated plaquettes throughout the lattice. Therefore, depending on the rate of the frustrated plaquettes, the system is induced to an aperiodic diluted ferrimagnet phase \[21\]. In both the quasiperiodic models,
we found very interesting results. We can highlight the curves of the heat specific $c$ that are not collapsed with the inverse of the $\ln L$, where $L$ is the size of the lattice. We had to use the logarithmic scale corrections in order to achieve the collapse of the specific heat curves [22]. This new theory opened a promising field of research about systems that can display unusual (quasiperiodic or aperiodic) orderings at low-temperature.

In this work, we considered the triangular lattice and obtain the relevant thermodynamic properties of the Ising model in two dimensions with positive and negative exchange interactions with the same strength modulated by Fibonacci quasiperiodic sequence. On the triangular lattice, we have several ways to modulate the lattice and have some control over the frustration rate and the ratio of the antiferromagnetic interactions. By controlling the modulation of the lattice we can investigate the influence of quasiperiodic modulation in critical properties. In section 2 we present our model. In section 3 we show the results and discussion. Finally, in section 4 we present our conclusions.

MODEL AND SIMULATIONS

We consider the Ising model in a triangular lattice with only first neighbor interactions. The Hamiltonian of the model is given by [23]

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{ij} S_i S_j,$$

where $S_i$ and $S_j$ are the spin on sites $i$ and $j$, respectively and its values can be $\pm 1$. The exchange interactions $J_{ij}$ between first neighbor spins $S_i$ and $S_j$ are modulated according to an aperiodic letter sequence and they can have values $1$ and $-1$.

We have used the Fibonacci letters sequence, to investigate if a particular marginal quasiperiodic order can confirm a new the universality class or not. The Fibonacci sequence can be obtained from the substitution rules $A \rightarrow AB$ and $B \rightarrow A$ in 1D [24–26], or alternatively from the substitution ruler: $S_n = S_{n-1}S_{n-2}$ (for $n \geq 2$), with, $S_1 = A$ and $S_2 = AB$. Any generation of the aperiodic sequence can be constructed from the previous generation by replacing all letters $A$ with $AB$ and all letters $B$ with $A$. Starting with the letter $A$, by repetitive applications of the substitution rule we can obtain the successive iterations of the Fibonacci sequence in 1D.

The modulation of the exchange interactions in the triangular lattice was made by means
of the Fibonacci sequence. We consider the three bonds present in a unitary cell, named I, II, and III, which connects one site $i$ at the position $\mathbf{r}_i$ and the neighbors placed on positions I) $\mathbf{r}_i + \mathbf{a}_1$, II) $\mathbf{r}_i + \mathbf{a}_2$ and III) $\mathbf{r}_i + \mathbf{a}_1 + \mathbf{a}_2$, where $\mathbf{a}_j$ are the Bravais vectors of the lattice, given by

\begin{align}
\mathbf{a}_1 &= \frac{1}{2} \mathbf{i} + \frac{\sqrt{3}}{2} \mathbf{j} \\
\mathbf{a}_2 &= \frac{1}{2} \mathbf{i} - \frac{\sqrt{3}}{2} \mathbf{j}.
\end{align}

(2)

The modulation can be done in three ways in order to generate plaquette frustration, according to the unitary cell bonds and lattice directions. Lattice bonds are sorted as

I) Bonds I) are lattice bonds at $\mathbf{a}_1$ direction;

II) Bonds II) are lattice bonds at $\mathbf{a}_2$ direction;

III) Bonds III) are lattice bonds at $\mathbf{a}_1 + \mathbf{a}_2$ direction.

A modulation can be done by changing bonds I), II) and III) at one defined lattice direction according to the Fibonacci letter sequence. The three modulation schemes considered here are

A) Modulating bond I) at the lattice line defined by $\mathbf{a}_2$ direction;

B) Modulating bonds I) and II) at the lattice lines defined by $\mathbf{a}_2$ and $\mathbf{a}_1$ directions, respectively;

C) Modulating bonds I), II) and III) at the lattice lines defined by the $\mathbf{a}_2$, $\mathbf{a}_1$ and $\mathbf{a}_1$ directions, respectively.

We performed simulations in the triangular lattice for each one of the modulation schemes A), B) and C) with the Fibonacci sequence in order to characterize the magnetic ordering of the spins at low temperature. We show an example of such lattice modulated according to way C) in Fig. (1). The Fig. (2) displays the possibilities we will have a plaquette frustrated throughout the lattice. In the triangular lattice, each plaquette is composed of the three sites, the links are made by $J_{ij}$ exchange interaction strength. The plaquette frustrates if there are two bonds with $J_A = 1$ and one bond $J_B = -1$ or the three bonds with $J_B = -1$. 
FIG. 1: (Color online) Example of a lattice with exchange interactions modulated by the Fibonacci sequence. The black and red lines stand for exchange interaction strengths $J_A = 1$ (ferromagnetic) and $J_B = -1$ (anti-ferromagnetic) respectively. We used the Fibonacci letter sequence, which is obtained from the substitution rules $A \rightarrow AB$ and $B \rightarrow A$ which means that any generation of the lattice can be constructed from the previous generation by replacing all letters $A$ with $AB$ and all letters $B$ with $A$. The modulation of the bonds between sites of the triangular lattice was made by means of the Fibonacci sequence at the lattice directions specified by Bravais vectors $a_1$ and $a_2$, according to way C) described in the text.

We focused on modulation C) because of its distinctive critical behavior as we present in next section.

By using the Replica Exchange Monte Carlo technique (also known as Parallel Tempering) [27-30], which is suited to find the ground state of such systems with alternating interactions, we obtained the staggered magnetization order parameter $\langle q \rangle$, the associated
FIG. 2: Frustrated plaquettes of the triangular lattice. The black and red lines stand for exchange interaction strengths $J_A = 1$ (Ferromagnetic) and $J_B = -1$ (Anti-ferromagnetic) respectively. If a given site, represented by the black circle, has an spin up and interacts with a neighbor site through of a $J_A$ exchange interaction, the spin of this site remains up. Otherwise, if the interaction is through of a $J_B$ exchange, the spin orientation of the neighboring site changes to down for minimal energy. For the two possibilities of a frustrated plaquette in the triangular lattice, the spin orientation on right corner can be either up or down for minimal energy.

susceptibility $\chi$, the specific heat $c$ and Binder cumulant $g$

$$q = \frac{1}{N} \sum_{i}^N S^0_{i,j} S_i,$$

$$\chi = N \left( \langle q^2 \rangle - \langle q \rangle^2 \right) / T,$$

$$c = N \left( \langle H^2 \rangle - \langle H \rangle^2 \right) / T^2,$$

$$g = 1 - \frac{\langle q^4 \rangle}{3\langle q^2 \rangle^2}.$$

where $\langle ... \rangle$ stands for a thermal average over sufficiently many independent steady state system configurations, $q$ is the staggered magnetization of the system, corresponding to a ferrimagnet phase where $S_{i,j}^0$ is the ground state, and $L$ and $T$ are the lattice size and the absolute temperature, respectively. We used the following values of the lattice size $L$: 34, 55, 89, 144 and 233, which are Fibonacci’s numbers $F_n$, given by the recursion rule:

$$F_n = F_{n-2} + F_{n-1},$$

where $P_0 = 1$ and $P_1 = 1$. The total number of spins for each lattice size is $N = L^2$.

To determine the critical behavior, we have used the following Finite Size Scaling (FSS)
relations\[31\], with logarithmic corrections\[22, 32–34\]

\[ q \propto L^{-\beta/\nu} (\ln L)^{-\beta \lambda} f_q(\vartheta), \]  
\[ (8) \]

\[ \chi \propto L^{\gamma/\nu} (\ln L)^{-\gamma + \gamma \lambda} f_\chi(\vartheta), \]  
\[ (9) \]

\[ c \propto (\ln L)^{\hat{\alpha}} f_c(\vartheta), \]  
\[ (10) \]

\[ g \propto f_g(\vartheta), \]  
\[ (11) \]

where \( \beta = 1/8, \gamma = 7/4, \alpha = 0 \) (logarithmic divergence) and \( \nu = 1 \) are the critical exponents (the Ising 2d ones). The \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) and \( \hat{\lambda} \) are the logarithmic correction exponents. The \( f_i(\vartheta) \) are the FSS functions with a logarithmic corrected scaling variable

\[ \vartheta = L^{1/\nu} (T - T_c) |\ln |T - T_c||^{-\hat{\lambda}}. \]  
\[ (12) \]

The correction exponents \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) and \( \hat{\lambda} \) obey the following scaling relations\[22\]

\[ \hat{\alpha} = 1 - d\nu \hat{\lambda} \]  
\[ (13) \]

\[ 2\hat{\beta} - \hat{\gamma} = -d\nu \hat{\lambda}, \]  
\[ (14) \]

where \( d \) is the dimensionality of the system. The scaling relation \[13\] is valid only for \( \alpha = 0 \) (logarithmic divergences), in the general case, \( \hat{\alpha} = -d\nu \hat{\lambda} \). For \( \alpha = 0 \) and \( \hat{\alpha} = 0 \), we have the double logarithmic divergence \((\ln \ln L)\) of the specific heat as seen for the 2d diluted Ising model\[22\].

We used \( 1 \times 10^5 \) MCM (Monte-Carlo Markov) steps to make the \( N_t = 600 \) system replicas (each system replica has a different temperature) reach the equilibrium state and the independent steady-state system configurations are estimated in the next \( 2 \times 10^6 \) MCM steps with 10 MCM steps between one system state and another one to avoid self-correlation effects. Every MCM steps are composed of two parts, a sweep, and a swap. One sweep is accomplished when all \( N \) spins were investigated if they flip or not and one swap is accomplished if all the \( N_t \) lattices are investigated if they exchange or not their temperatures (swap part). We carried out \( 10^5 \) independent steady-state configurations to calculate the needed thermodynamic averages.

RESULTS AND DISCUSSION

We focus on the modulation scheme C) in all results with the exception of specific heat because of its distinctive critical behavior as we will show in the following. First, we estimate
the critical temperature by using the Binder cumulant \( g \) given by Eq. (6). We show the Binder cumulant in the inset of Fig.(3) for the modulation scheme C). The critical temperature \( T_c \) is estimated at the point where the curves for different size lattices intercept each other. From Fig.(3), we obtained \( T_c \approx 1.4116 \) for the modulation C). For modulations A) and B), we obtained \( T_c \approx 2.6284 \) and \( T_c \approx 2.1540 \), respectively.

\[ g \]

\[ L = 34 \]
\[ L = 55 \]
\[ L = 89 \]
\[ L = 144 \]
\[ L = 233 \]

\[ k_B T/J \]

\[ T_c \approx 1.4116 \]

**FIG. 3:** (Color Online) Data collapse of the Binder Cumulant \( g \) versus the scaling parameter \( L^{1/\nu}(T - T_c) \ln|T - T_c|^{-\hat{\lambda}} \) for different lattice sizes \( L \) where we considered the modulation C) of the triangular lattice (discussed in the text). Inset: Binder Cumulant versus the temperature for different lattice sizes. The values of \( L \) obey the Fibonacci sequence. We estimated the critical temperature \( T_c \approx 1.4116 \), as shown in the inset, by averaging the numerical values of the temperatures where the curves intersect each other, identified by a circle with a dashed line. The best collapse was done by using the logarithmic correction exponent \( \hat{\lambda} = 0.075 \). The model is in the Ising universality class with logarithmic corrections.

Next we show the order parameter behavior \( q \) as a function of temperature \( T \), for the modulation scheme C). The result of the inset in the Fig.(4) suggests the presence of a second-order phase transition in the system. Also, the Fig.(4) presents the data collapse using the FSS with logarithmic relation written on Eq.(8). The best collapse was obtained by using the values for the critical exponents \( \nu = 1 \), \( \beta = 1/8 \) and the correction exponents \( \hat{\beta} = -0.06 \) and \( \hat{\lambda} = 0.075 \).
FIG. 4: (Color Online) Data collapse of the order parameter $q$, rescaled by $L^{\beta/\nu} (\ln L)^{\tilde{\beta}+\tilde{\lambda}}$ versus the scaling parameter $L^{1/\nu} (T - T_c) |\ln |T - T_c||^{-\tilde{\lambda}}$ for different lattice sizes $L$. Here, we considered the modulation scheme C) (discussed in text). Inset: order parameter $q$ as a function of temperature $T$ for different lattice sizes $L$. The values of $L$ obey the Fibonacci sequence. The curves suggest a second order phase transition. The best collapse is done by using the values for the logarithmic correction exponents: $\tilde{\beta} = -0.06$ and $\tilde{\lambda} = 0.075$. The model is in the Ising universality class with logarithmic corrections.

Continuing the analysis of the behavior critical of the system, the inset of the Fig.5 show the susceptibility $\chi$ as a function of temperature $T$ for the modulation scheme C). In the large lattice size limit, the susceptibility diverges at $T_c \approx 1.4116$. The Fig.5 also show the data collapse of the susceptibilities for different lattice sizes according to FSS with logarithmic correction relation given in the Eq.(9). All maxima are well fitted by using the scale relation with logarithmic correction ($\tilde{\gamma} = 0.03$ and $\tilde{\lambda} = 0.075$) and the 2D Ising critical exponents $\gamma = 1.75$ and $\nu = 1$.

Finally, we show the specific heat $c$, given by the Eq.(5), at the inset of the Figs.(6), (7) for the modulation schemes A) and B), respectively. We see that the usual scaling relation with a pure logarithm divergence fits all data, being consistent with no logarithm corrections for modulation schemes A) and B). We note in the Figs. (6) and (7) for modulation schemes A) and B), the maxima of the specific heat scales with $1/\ln L$, like the Ising ferromagnetic
FIG. 5: (Color Online) Data collapse of the susceptibility $\chi$, rescaled by $L^{-\gamma/\nu} (\ln L)^{\hat{\gamma}-\gamma\hat{\lambda}}$ versus the scaling parameter $L^{1/\nu} (T - T_c) |\ln |T - T_c||^{-\hat{\lambda}}$ for different lattice sizes $L$. Here, we considered the modulation scheme C) (discussed in text). Inset: Susceptibility $\chi$ as a function of temperature $T$ for different lattice sizes $L$. The values of $L$ obey the Fibonacci sequence. The susceptibility diverges at $T_c$ in the large lattice size limit suggesting a second order phase transition. The best collapse is done by using the values for the logarithmic correction exponents: $\hat{\gamma} = 0.03$ and $\hat{\lambda} = 0.075$. The model is in the Ising universality class with logarithmic corrections.

However, the modulation scheme C) have a different critical behavior from the other previous cases, as we anticipated. The maxima of the specific heat $c$ do not collapse with the $1/\ln L$, as emphasized by a circle with the dashed line in the Fig. S. Therefore, using the scaling relation without logarithmic corrections does not collapse our numerical data for the specific heat $c$. However, we obtain a good collapse by using the scaling relations written on Eq. (13) and our best estimates for the logarithm correction exponents are $\hat{\alpha} = 0.85$ and $\hat{\lambda} = 0.075$ which obeys the scaling relations for the logarithmic correction exponents given in Eq. (13). The Fig. (S) show the collapse of the maxima of the specific heat, as indicated by the circle with the dashed line.
FIG. 6: (Color Online) Specific Heat $c$, rescaled by $1/\ln L$ versus the scaling parameter $L^{1/\nu}(T-T_c)$ for different lattice sizes $L$ and for modulation scheme A) of the triangular lattice. Inset: Specific Heat $c$ as a function of temperature $T$ for different lattice sizes $L$. The values of $L$ obey the Fibonacci sequence. We can see that the usual FSS relation without logarithmic corrections does collapse our numerical data obtained from the simulations of the triangular lattice with modulation scheme A).

CONCLUSIONS

We have presented a theoretical model on a triangular lattice with quasiperiodic long-ranged order based on Fibonacci quasiperiodic sequence, with competing interactions, and we have obtained a critical behavior of a second order phase transition, driven by the temperature at the triangular lattice modulated by three different ways A), B), and C), dependent on the number of interactions in each lattice plaquette is changed according to the quasiperiodic sequence.

Note that the three modulating schemes A), B) and C) can be sorted as in increasing order of modulating strength, where we modulate an increasing number of lattice bonds and introduce increasing corrections on partition function in order to observe a change in the critical behavior. In fact, modulation scheme C) is sufficient to introduce logarithm corrections in the thermodynamic properties. This is a signal of a marginal behavior depending
FIG. 7: (Color Online) Specific Heat $c$, rescaled by $1/\ln L$ versus the scaling parameter $L^{1/\nu}(T - T_c)$ for different lattice sizes $L$ and for modulation scheme B) of the triangular lattice. Inset: Specific Heat $c$ as a function of temperature $T$ for different lattice sizes $L$. The values of $L$ obey the Fibonacci sequence. We can see that the usual FSS relation without logarithmic corrections does collapse our numerical data obtained from the simulations of the triangular lattice with modulation scheme B).

on the modulation, placed between the case where we have a Ising ferromagnetic long-range order at lower temperatures and the situation for a sufficiently strong modulation where we have only a paramagnetic phase for any finite temperature. In the marginal modulation we have a ferrimagnetic ordering and logarithm corrections in the critical behavior.

In the case when we choose only one interaction and two interactions to be modulated, the system obeys the same critical behavior of the pure Ising ferromagnetic model, as shown in Figs. (6) and (7) where the specific heat have a logarithm divergence. However, the case with the three plaquette interactions chosen to be modulated, the system deviated from the pure model where the system retains the same universality class, but with logarithm corrections in its critical behavior. In this case, at the low-temperature limit, we obtained an aperiodic ferrite phase with critical temperature $T_c \approx 1.4116$, which is different from the critical temperatures of the model on the square lattice, modulated by using Fibonacci[21] and Octonacci[35] sequences.
Specifically, we have obtained the critical exponents $\beta = 1/8$, $\gamma = 7/4$ and $\nu = 1$ (Ising universality class) and the estimates for logarithmic correction exponents given by $\hat{\alpha} = 0.85$, $\hat{\beta} = -0.06$, $\hat{\gamma} = 0.03$ and $\lambda = 0.075$ in the case of equal antiferromagnetic and ferromagnetic strengths. The critical exponents of the logarithmic correction to the triangular lattice is not the same as the square lattice. Therefore, the quasiperiodic ordering is marginal in the sense of introducing logarithmic corrections as in seen for 4-state 2D Potts model, Fibonacci sequence[21] and Octonacci sequence ([35]) in the square lattice.

Acknowledgments

We would like to thank CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) and FAPEPI (Fundação de Amparo a Pesquisa do Estado do Piauí) for the financial support. We acknowledge the use of Dietrich Stauffer Computational Physics Lab - UFPI, Teresina, Brazil where the numerical simulations were performed.
FIG. 9: (Color Online) Data collapse of the Specific Heat $c$, rescaled by $(\ln L)^{-\hat{\alpha}}$ versus the scaling parameter $L^{1/\nu}(T - T_c)\ln |T - T_c|^{-\hat{\lambda}}$ for different lattice sizes $L$ and for modulation scheme C) of the triangular lattice. Inset: Specific Heat $c$ as a function of temperature $T$ for different lattice sizes $L$. The values of $L$ obey the Fibonacci sequence. The best collapse is done by using the values for the logarithmic correction exponents: $\hat{\alpha} = 0.85$ and $\hat{\lambda} = 0.075$, as is identified by a circle with dashed line. The model with modulation scheme C) is in the Ising universality class with logarithmic corrections.

[1] Sir E. Rutherford F.R.S., The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 27, 488 (1914).
[2] D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. 53, 1951 (1984).
[3] P. Steinhard and S. Ostlun, The Physics of Quasicrystals (World Scientific, 1987).
[4] W. Steurer, Chem. Soc. Rev. 41 (2012).
[5] E. Maci, Rep. Prog. Phys. 69 (2006).
[6] S. Fischer, A. Exner, K. Zielske, J. Perlich, S. Deloudi, W. Steurer, P. Lindner, , and S. Frster, Proc. Natl. Acad. Sci. USA 108, 1810 (2011).
[7] X. Zeng, G. Ungar, Y. Liu, V. Percec, A. E. Dulcey, , and J. K. Hobbs, Nature (London) 428,
[8] X. Zeng, Curr. Opin. Colloid Interface Sci. 9, 384 (2005).
[9] A. Takano, W. Kawashima, A. Noro, Y. Isono, N. Tanaka, T. Dotera, and Y. Matsushita, J. Polym. Sci., Part B: Polym. Phys. 43, 2427 (2005).
[10] K. Hayashida, T. Dotera, A. Takano, and Y. Matsushita, J. Polym. Sci., Part B: Polym. Phys. 98, 195502 (2007).
[11] Z. Islam, I. R. Fisher, J. Zarestky, P. C. Canfield, C. Stassis, and A. I. Goldman, Phys. Rev. B 57, R11047 (1998).
[12] M. Scheffer and J.-B. Suck, Materials Science and Engineering: A 294, 629 (2000).
[13] A. Jagannathan, Phys. Rev. Lett. 92, 047202 (2004).
[14] E. Y. Vedmedenko, U. Grimm, and R. Wiesendanger, Phys. Rev. Lett. 93, 076407 (2004).
[15] S. Matsuo, S. Fujiwara, H. Nakano, and T. Ishimasa, J. Non-Cryst. Solids 334&335, 421 (2004).
[16] A. Jagannathan, Phys. Rev. B 71, 115101 (2005).
[17] E. Y. Vedmedenko, U. Grimm, and R. Wiesendanger, Philos. Mag. 86, 733 (2006).
[18] J. M. Luck, Europhys. Lett. 24, 359 (1993).
[19] Z.-Y. Li, X. Zhang, and Z.-Q. Zhang, Phys. Rev. B 61, 15738 (2000).
[20] V. A. Shchukin and D. Bimberg, Rev. Mod. Phys. 71, 1125 (1999).
[21] E. Ilker and N. Berker, Phys. Rev. E 89, 042139 (2014).
[22] R. Kenna, Universal scaling relations for logarithmic-correction exponents (World Scientific Publishing Co. Pte. Ltd., London, 2012), vol. 3, chap. 1, p. 1.
[23] E. Ising, Z. Physik 31, 253 (1925).
[24] M. S. Vasconcelos, F. F. Medeiros, E. L. Albuquerque, and G. A. Farias, Surface Science 600, 4337 (2006).
[25] M. S. Vasconcelos, F. F. Medeiros, and E. L. Albuquerque, Surface Science 601, 4492 (2007).
[26] G. A. Alves, M. S. Vasconcelos, and T. F. A. Alves, Phys. Rev. E 93, 042111 (2016).
[27] D. J. Earl and M. W. Deem, Phys. Chem. Chem. Phys. 7, 3910 (2005).
[28] R. H. Swendsen and J.-S. Wang, Phys. Rev. Lett. 57, 2607 (1986).
[29] K. Hukushima and K. Nemoto, J. Phys. Soc. Jpn. 65, 1604 (1996).
[30] C. J. Geyer, in Computing Science and Statistics Proceedings of the 23rd Symposium on the Interface (American Statistical Association, 1991), p. 156.
[31] F. W. S. Lima, Physica A 391, 1753 (2012).
[32] R. Kenna, D. A. Johnston, and W. Janke, Phys. Rev. Lett. 96, 115701 (2006).
[33] R. Kenna, D. A. Johnston, and W. Janke, Phys. Rev. Lett. 97, 155702 (2006).
[34] V. Palchykov, C. von Ferber, R. Folk, Y. Holovatch, and R. Kenna, Phys. Rev. E 82, 011145 (2010).
[35] G. A. Alves, M. S. Vasconcelos, and T. F. A. Alves, J. Stat. Mech. 2017, 123302 (2017).