Spin generalization of the Calogero–Moser hierarchy and the matrix KP hierarchy

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Abstract
We establish a correspondence between rational solutions to the matrix KP hierarchy and the spin generalization of the Calogero–Moser system on the level of hierarchies. Namely, it is shown that the rational solutions to the matrix KP hierarchy appear to be isomorphic to the spin Calogero–Moser system in a sense that the dynamics of poles of solutions to the matrix KP hierarchy in the higher times is governed by the higher Hamiltonians of the spin Calogero–Moser integrable hierarchy with rational potential.

Keywords: matrix KP hierarchy, Calogero–Moser hierarchy, pole dynamics

1. Introduction

In the paper [1] it was discovered that the motion of poles of rational solutions to the Korteweg–de Vries (KdV) and Boussinesq equations is given by dynamics of the many-body Calogero–Moser system of particles [2–4] with some additional restrictions in the configuration space. Subsequently, the celebrated isomorphism between dynamics of poles of rational solutions to the Kadomtsev–Petviashvili (KP) equation (which generalizes the KdV and Boussinesq equations) and solutions to equations of motion for the rational Calogero–Moser system was established in [5] (see also [6]). Namely, in [5] the general approach of constructing rational solutions in the variable \(t_1\) to the KP equation was proposed and it was found that the positions of the poles \(x_i\) change with time \(t_2\) in the same way as the particles of the rational Calogero–Moser system. This remarkable connection was further generalized by Krichever in [7], where the analogous results were obtained in the case of elliptic solutions.

The further development is Shiota’s work [8], where the correspondence between dynamics of poles of rational KP solutions and many-body integrable systems of particles was extended to the level of hierarchies. There it was proved that the evolution of poles with respect to the
higher times \( t_k \) of the infinite KP hierarchy is governed by higher Hamiltonians \( H_k \) of the integrable Calogero–Moser system.

In this note we generalize this result to the rational solutions of the matrix KP hierarchy. It should be noted that singular (in general, elliptic) solutions to the matrix KP equation were studied in [9]. It has been shown that the evolution of data of such solutions (positions of poles and some internal degrees of freedom) with respect to the time \( t_2 \) is isomorphic to the dynamics of a spin generalization of the Calogero–Moser system. This generalization is known as the Gibbons–Hermsen system [10]. It is a system of \( N \) particles with coordinates \( x_i \) with internal degrees of freedom given by \( N \)-dimensional column vectors \( \mathbf{a}_i, \mathbf{b}_i \) which pairwise interact with each other. The Hamiltonian is

\[
H = \sum_{i=1}^{N} b_i^2 - \sum_{i \neq k} \frac{(\mathbf{b}_i^T \mathbf{a}_k)(\mathbf{b}_i^T \mathbf{a}_j)}{(x_i - x_k)^2}
\]

(here \( \mathbf{b}_i^T \) is the transposed row-vector) with the non-vanishing Poisson brackets \( \{x_i, p_k\} = \delta_{ik}, \{\mathbf{a}_i, \mathbf{b}_j^T\} = \delta_{ij} \delta_{ik} \). The model is known to be integrable, with the higher Hamiltonians in involution being given by \( H_k = \text{tr} \ L_k \), where \( L \) is the Lax matrix of the model.

Here we extend this result to the level of hierarchies, i.e. we show that the evolution of the poles and the internal degrees of freedom with respect to the higher times \( t_k \) of the matrix KP hierarchy is governed by the higher Hamiltonians \( H_k \) of the Gibbons–Hermsen system.

The matrix extension of the KP hierarchy is closely related to the so-called multicomponent KP hierarchy [11, 12]. In section 2, we start with a short review of these hierarchies. We use the bilinear formalism. The main object (the dependent variable) is the tau-function \( \tau \) which obeys an infinite number of bilinear relations encoded by the universal bilinear identity (1). We also introduce the matrix Baker–Akhiezer functions \( \Psi, \Psi^\dagger \) which satisfy a system of linear equations. The compatibility conditions of this system give non-linear equations of the hierarchy. In section 3, we study the rational solutions of the matrix KP hierarchy in the time \( t_1 \). For such solutions, the tau-function is a polynomial in \( x = t_1 \) with roots \( x_i \), with the Baker–Akhiezer functions having simple poles at the points \( x_i \). Using the bilinear equations for the tau-function, we show that residues at these poles are matrices of rank 1. The internal degrees of freedom associated with \( x_i \) are expressed in terms of these residues. The dynamics of \( x_i \) and the internal degrees of freedom is derived using the linear problems for the Baker–Akhiezer functions. It should be noted that rational solutions to the multicomponent and matrix KP hierarchies and their relation to Calogero-like systems were studied in [13–17] from other perspectives and points of view.

2. The matrix KP hierarchy

2.1. The bilinear identity for the multicomponent KP hierarchy

First of all, consider the multicomponent KP hierarchy [11, 12]. In the bilinear formalism, it can be defined as follows (see, e.g. [18, 19]). Suppose there are \( N \) infinite sets of the independent continuous time variables:

\[
t = \{t_1, t_2, \ldots, t_N\}, \quad t_\alpha = \{t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, \ldots\}, \quad \alpha = 1, \ldots, N.
\]

Next, one introduces \( N \) discrete variables called charges
\( s = \{s_1, s_2, \ldots, s_N\}, \quad \sum_{\alpha=1}^{N} s_{\alpha} = 0 \)

(they are integer numbers). The \( N \)-component KP hierarchy is then defined by the infinite set of bilinear equations for the tau-function \( \tau(s; t) \) that follow from the condition (the bilinear identity)

\[
\sum_{\gamma=1}^{N} \epsilon_{\alpha\gamma}(s) \epsilon_{\beta\gamma}(s') \oint_{C_\infty} dz \, e^{z_{\gamma} - z'_{\gamma} + \delta_{\gamma\gamma} + \delta_{\beta\gamma} - 2e^{(t_{\gamma} - t'_{\gamma})}}
\]

\[
\cdot \tau(s + e_\alpha - e_\beta; t - [z^{-1}]_\gamma) \tau(s' + e_\gamma - e_\beta; t' + [z^{-1}]_\gamma) = 0, \quad \alpha, \beta = 1, \ldots, N, \quad (1)
\]

valid for any \( s, s', t, t' \). The notation is as follows: \( e_\alpha \) is the vector with 1 on the \( \alpha \)th place and with all other entries equal to zero,

\[
\epsilon_{\alpha\gamma}(s) = \begin{cases} 
(-1)^{s_{\alpha+1} + \cdots + s_{\gamma}} & \text{if } \alpha < \gamma \\
1 & \text{if } \alpha = \gamma \\
-(-1)^{s_{\gamma+1} + \cdots + s_{\alpha}} & \text{if } \alpha > \gamma 
\end{cases}
\]

and

\[
\xi(t, z) = \sum_{k \neq 1} t_{\gamma, k} z^k.
\]

The integration contour \( C_\infty \) around \( \infty \) is such that all singularities coming from the power of \( z \) and the exponential function \( e^{(t_{\beta} - t'_{\beta})} \) are inside it and all singularities coming from the \( \tau \)-factors are outside it. We remark that the sign factors \( \epsilon_{\alpha\beta}(s) \) satisfy the identities

\[
\epsilon_{\beta\alpha}(s) = -\epsilon_{\alpha\beta}(s), \quad \epsilon_{\alpha\beta}(-s) = \epsilon_{\alpha\beta}(s), \quad \epsilon_{\alpha\gamma}(s + e_\alpha - e_\beta) = \epsilon_{\beta\gamma}(s)\epsilon_{\alpha\beta}(s) \quad (2)
\]

for any distinct \( \alpha, \beta, \gamma \).

2.2. The Hirota equations for the multicomponent KP hierarchy

Choosing \( s' \) and \( t' \) in (1) in a specific way, one can obtain, after calculating the integral with the help of residues, a number of differential and difference Hirota bilinear equations for the tau-function (called Fay identities in \([18]\)). The full list of such equations is given in \([18]\). Here we give only the equations that are used in what follows.

For any distinct \( \alpha, \beta, \kappa \) it holds

\[
\partial_{e_{\alpha}} \tau(s; t) \cdot \tau(s + e_\alpha - e_\beta; t) = \partial_{e_{\kappa}} \tau(s + e_\alpha - e_\beta; t) \cdot \tau(s, t)
\]

\[
= -\frac{\epsilon_{\alpha\beta}(s)}{\epsilon_{\alpha\kappa}(s)} \tau(s + e_\alpha - e_\beta; t) \cdot \tau(s + e_\alpha - e_\beta; t). \quad (3)
\]

For any distinct \( \alpha, \beta \) it holds

\[
\partial_{e_\alpha} \tau(s + e_\alpha - e_\beta; t) \cdot \tau(s; t - [z^{-1}]_\alpha) = \partial_{e_\beta} \tau(s; t - [z^{-1}]_\alpha) \cdot \tau(s + e_\alpha - e_\beta; t)
\]

\[
= \tau(s; t - [z^{-1}]_\alpha) \cdot \tau(s + e_\alpha - e_\beta; t) - \tau(s; t) \cdot \tau(s + e_\alpha - e_\beta; t - [z^{-1}]_\alpha). \quad (4)
\]
Taking \( s' = s + e_\alpha - e_\lambda \), \( t' = t \) in the bilinear identity, one can see that for any distinct \( \alpha, \beta, \kappa, \lambda \) it holds
\[
\epsilon_{\beta\alpha}(s + e_\kappa - e_\lambda)\tau(s, t) \cdot \tau(s + e_\kappa - e_\lambda + e_\alpha - e_\beta; t)
+ \epsilon_{\alpha\beta}(s)\tau(s + e_\alpha - e_\beta; t) \cdot \tau(s + e_\kappa - e_\lambda; t)
+ \epsilon_{\alpha\lambda}(s)\epsilon_{\beta\lambda}(s + e_\kappa - e_\lambda)\tau(s + e_\alpha - e_\lambda; t) \cdot \tau(s + e_\kappa - e_\beta; t) = 0.
\] (5)

2.3. The Baker–Akhiezer functions

The Baker–Akhiezer function \( \Psi(s, t; z) \) and its adjoint \( \Psi^*(s, t; z) \) are \( N \times N \) matrices with components defined by the following formulae:
\[
\Psi_{\alpha\beta}(s, t; z) = \epsilon_{\alpha\beta}(s) \sum_{k \geq 1} \frac{w^{(k)}_{\alpha\beta}(s, t)}{z^k} e^{\xi(t, z)},
\]
\[
\Psi^*_{\alpha\beta}(s, t; z) = \epsilon_{\alpha\beta}(s) \sum_{k \geq 1} \frac{w^{(k)*}_{\alpha\beta}(s, t)}{z^k} e^{-\xi(t, z)}.
\] (6)

In terms of the Baker–Akhiezer functions, the bilinear identity (1) can be written as
\[
\int_{C_\infty} dz \Psi(s, t; z) \Psi^\dagger(s', t'; z) = 0, \quad \Psi^\dagger_{\alpha\beta} = \Psi^*_{\beta\alpha}
\] (here and below \( \Psi^\dagger \) does not mean the Hermitian conjugation). Around \( z = \infty \), the Baker–Akhiezer functions can be represented in the form of the series
\[
\Psi_{\alpha\beta}(s, t; z) = \left( \delta_{\alpha\beta} + \sum_{k \geq 1} \frac{w^{(k)}_{\alpha\beta}(s, t)}{z^k} \right) e^{\xi(t, z)},
\] (8)
\[
\Psi^*_{\alpha\beta}(s, t; z) = \left( \delta_{\alpha\beta} + \sum_{k \geq 1} \frac{w^{(k)*}_{\alpha\beta}(s, t)}{z^k} \right) e^{-\xi(t, z)}.
\] (9)

It follows from the bilinear identity in the form (7) taken at \( s' = s, t' = t \) that
\[
w^{(1)*}_{\alpha\beta}(s, t) = -w^{(1)}_{\alpha\beta}(s, t).
\] (10)

The multicomponent KP hierarchy can be understood as an infinite set of evolution equations in the times \( t \) for matrix functions of a variable \( x \). For example, one can consider the coefficients \( w^{(k)} \) of the Baker–Akhiezer function as such matrix functions, the evolution being \( w^{(k)}(x) \to w^{(k)}(x, t) \). In what follows we will denote \( \tau(x, t) \), \( w^{(k)}(x, t) \) simply as \( \tau(t) \), \( w^{(k)}(t) \). Let us introduce the (matrix pseudo-differential) wave operator
\[
W = I + \sum_{k \geq 1} w^{(k)}(t) \partial_x^{-k},
\]
where \( I \) is the unity \( N \times N \) matrix and \( w^{(k)}(t) \) are the same matrix functions as in (8). Writing this in matrix elements, we have
\[ W_{\alpha\beta} = \delta_{\alpha\beta} + \sum_{k \geq 1} W_{\alpha\beta}^{(k)}(t) \partial_x^{-k}. \]  

The Baker–Akhiezer function can be written as a result of action of the wave operator to the exponential function:

\[ \Psi(t; z) = W \exp \left( xz I + \sum_{\alpha=1}^{N} E_{\alpha} \xi(t_{\alpha}, z) \right), \]

where \( E_{\alpha} \) is the \( N \times N \) matrix with 1 on the \( (\alpha, \alpha) \) component and zero elsewhere. The adjoint Baker–Akhiezer function can be written as

\[ \Psi^\dagger(t; z) = \exp \left( -xz I - \sum_{\alpha=1}^{N} E_{\alpha} \xi(t_{\alpha}, z) \right) W^{-1}. \]

Here it is assumed that the operators \( \partial_x \) entering \( W^{-1} \) act to the left (i.e. we define \( f \partial_x = -\partial_x f \)).

As is proved in [18], the Baker–Akhiezer function and its adjoint satisfy the linear equations

\[ \partial_{t_{\alpha}, m} \Psi(t; z) = B_{\alpha m} \Psi(t; z), \]

\[ -\partial_{t_{\alpha}, m} \Psi^\dagger(t; z) = \Psi^\dagger(t; z) B_{\alpha m}, \]

where \( B_{\alpha m} \) is the differential operator

\[ B_{\alpha m} = \left( WE_{\alpha} \partial_x^{m} W^{-1} \right) \delta. \]

Here \((\ldots)_{x}\) denotes the differential part of a pseudo-differential operator, i.e. the sum of terms with \( \partial_x^{k} \), where \( k \geq 0 \). In particular,

\[ \sum_{\alpha=1}^{N} \partial_{t_{\alpha}, m} \Psi(t; z) = \partial_{x} \Psi(t; z), \]

so the vector field \( \partial_{x} \) can be identified with the vector field \( \sum_{\alpha} \partial_{t_{\alpha}, r} \).

### 2.4. The matrix KP hierarchy and linear problems for the Baker–Akhiezer functions

Let us proceed to the specification to the matrix KP hierarchy. This hierarchy results from the multicomponent KP one after a restriction of the times and the charge variables. For each set of the times \( t_{\alpha} \) we fix the ‘initial values’ \( t_{\alpha}^{(0)} \) and suppose that the times change in the following manner:

\[ t_{\alpha, m} = t_{\alpha}^{(0)} + t_{m} \quad \text{for each } \alpha \text{ and } m. \]

In other words, for any fixed \( m \), the time evolution with respect to each \( t_{\alpha, m} \) is the same and is defined by \( t_{m} \) only. The corresponding vector fields are related as \( \partial_{t_{\alpha}} = \sum_{\alpha=1}^{N} \partial_{t_{\alpha, m}} \). The charge variables are supposed to be fixed. It is convenient to put \( s = 0 \). In what follows we omit them in the notation for the tau-function and the Baker–Akhiezer functions and put \( s = s' = 0 \) in the bilinear identity. Accordingly, the bilinear identity for the matrix KP hierarchy acquires the form

\[ \sum_{\gamma=1}^{N} \epsilon_{\alpha\gamma} \epsilon_{\beta\gamma} \int_{C_{\infty}} d\zeta \zeta^{\delta_{\alpha\gamma} + \delta_{\beta\gamma} - 2\epsilon_{\gamma} (t_{\alpha} - t_{\beta}, z)} \tau_{\alpha\gamma}^{(t - [z^{-1}]_{\gamma})} \tau_{\gamma\beta}^{(t' + [z^{-1}]_{\gamma})} = 0, \]  

\[ (14) \]
where \( \epsilon_{\alpha\gamma} = 1 \) if \( \alpha \leq \gamma \), \( \epsilon_{\alpha\gamma} = -1 \) if \( \alpha > \gamma \) and

\[
\tau_{\alpha\beta}(t) = \tau(e_{\alpha} - e_{\beta}; t).
\]

(15)

The Baker–Akhiezer function and its adjoint have the expansions

\[
\Psi_{\alpha\beta}(t; z) = \left( \delta_{\alpha\beta} + w_{\alpha\beta}^{(1)}(t)z^{-1} + O(z^{-2}) \right) e^{z^\tau(t)}
\]

\[
\Psi^{*}_{\alpha\beta}(t; z) = \left( \delta_{\alpha\beta} - w_{\alpha\beta}^{(1)}(t)z^{-1} + O(z^{-2}) \right) e^{-z^\tau(t)},
\]

(16)

where \( \xi(t, z) = \sum_{k \geq 1} t_k z^k \). It is easy to see from (6) that

\[
w_{\alpha\beta}^{(1)}(t) = \begin{cases} 
\epsilon_{\alpha\beta} \tau_{\alpha\beta}(t) & \text{if } \alpha \neq \beta \\
-\frac{\partial_{\alpha\beta}(t)}{\tau(t)} & \text{if } \alpha = \beta.
\end{cases}
\]

(17)

Let us derive a useful corollary of the bilinear identity which will be used for analysis of the rational solutions. In order to obtain it, we differentiate the bilinear identity with respect to \( t_m \) and put \( t' = t \) after this. It is not difficult to see that the result is

\[
\frac{1}{2\pi i} \sum_{\gamma = 1}^{N} \oint_{C_{\infty}} dz z^{m} \Psi_{\alpha\gamma}(t; z) \Psi^{*}_{\alpha\gamma}(t; z) = -\epsilon_{\alpha\beta} \partial_{\alpha} \left( \frac{\tau_{\alpha\beta}(t)}{\tau(t)} \right)
\]

(18)

for \( \alpha \neq \beta \) and

\[
\frac{1}{2\pi i} \sum_{\gamma = 1}^{N} \oint_{C_{\infty}} dz z^{m} \Psi_{\alpha\gamma}(t; z) \Psi^{*}_{\alpha\gamma}(t; z) = \partial_{\alpha} \partial_{\alpha,1} \log \tau(t).
\]

(19)

Comparing with (17), we conclude that

\[
\frac{1}{2\pi i} \sum_{\gamma = 1}^{N} \oint_{C_{\infty}} dz z^{m} \Psi_{\alpha\gamma}(t; z) \Psi^{*}_{\alpha\gamma}(t; z) = -\partial_{\alpha} w_{\alpha\beta}^{(1)}(t)
\]

(20)

for any \( \alpha, \beta \). Note also that summing (19) over \( \alpha \) from 1 to \( N \), we get

\[
\frac{1}{2\pi i} \sum_{\alpha, \beta = 1}^{N} \oint_{C_{\infty}} dz z^{m} \Psi_{\alpha\beta}(t; z) \Psi^{*}_{\alpha\beta}(t; z) = \frac{1}{2\pi i} \oint_{C_{\infty}} dz z^{m} \text{tr} (\Psi(t; z) \Psi^\dagger(t; z)) = \partial_{\alpha} \partial_{\beta} \log \tau(t).
\]

(21)

Recalling (13), we can identify

\[
\partial_{x} = \partial_{t} = \sum_{\alpha = 1}^{N} \partial_{\alpha,1}.
\]

As it follows from (12), the Baker–Akhiezer function and its adjoint satisfy the linear equations

\[
\partial_{\alpha} \Psi(t; z) = B_{m} \Psi(t; z), \quad m \geq 1,
\]

\[
-\partial_{\alpha} \Psi^\dagger(t; z) = \Psi^\dagger(t; z) B_{m}, \quad m \geq 1,
\]

(22)

where \( B_{m} \) is the differential operator
\[ B_m = \left( W \partial^m W^{-1} \right)_+. \]

At \( m = 1 \) we have \( \partial_t \Psi = \partial_x \Psi \), so the evolution in \( t_1 \) is simply a shift of the variable \( x \):

\[ w^{(k)}(x, t_1, t_2, \ldots) = w^{(k)}(x + t_1, t_2, \ldots). \quad (23) \]

At \( m = 2 \) we have the linear problems

\[ \partial_t \Psi = \partial_x^2 \Psi + V(t) \Psi, \quad (24) \]
\[ -\partial_t \Psi^\dagger = \partial_x^2 \Psi^\dagger + \Psi^\dagger V(t) \quad (25) \]

which have the form of the non-stationary matrix Schrödinger equations with the potential

\[ V(t) = -2 \partial_x w^{(1)}(t). \quad (26) \]

### 3. Rational solutions to the matrix KP hierarchy

In this section we study solutions to the matrix KP hierarchy which are rational functions of the variable \( x \) (and, therefore, \( t_1 \)). First of all we find the form of the Baker–Akhiezer functions for the rational solutions.

#### 3.1. Baker–Akhiezer functions for rational solutions

For the rational solutions, the tau-function should be a polynomial in \( x \) (possibly multiplied by an exponential function):

\[ \tau(t) = C e^{\alpha_k} \prod_{i=1}^{\mathcal{N}} (x - x_i(t)). \quad (27) \]

Here \( \mathcal{N} \) is the number of roots \( x_i \) of the polynomial and the roots depend on the times \( t \). We assume that all the roots are distinct. Let us use the notation

\[ \tau' \left( x_i \right) = \lim_{x \to x_i} \frac{\tau(t)}{x - x_i} = C e^{\alpha_k} \prod_{j \neq i} (x_i - x_j). \]

\[ \tau_{\alpha\beta} = \tau_{\alpha\beta} \bigg|_{x=x_i}, \quad \partial_{\alpha,1} \tau \bigg|_{x=x_i} = \partial_{\beta,1} \tau \bigg|_{x=x_i}. \]

It is clear from (6) that the Baker–Akhiezer functions \( \Psi, \Psi^\dagger \), as functions of \( x \), have simple poles at \( x = x_i \). From (17) we see that the residue of \( w^{(1)}_{\alpha\beta} \) (as a function of \( x \)) at the pole \( x_i \) is given by

\[ \text{res}_{x=x_i} w^{(1)}_{\alpha\beta} = \begin{cases} \epsilon_{\alpha\beta} \frac{\tau_{\alpha\beta}(t)}{\tau'(t)} & \text{if } \alpha \neq \beta, \\ -\frac{\partial_{\alpha,1} \tau(x_i)}{\tau'(x_i)} & \text{if } \alpha = \beta. \end{cases} \quad (28) \]

We are going to show, using the Hirota equations (3)–(5), that the dependence on \( \alpha \) and \( \beta \) in \( \text{res}_{x=x_i} w^{(1)}_{\alpha\beta} \) actually factorizes, i.e.
\( \text{res}_{x=x_i} w^{(1)}_{\alpha \beta} = -a_i^\alpha b_i^\beta \) or \( \text{res}_{x=x_i} w^{(1)} = -a_i b_i^T \) (29) for some column vectors \( a_i = (a_i^1, a_i^2, \ldots, a_i^N)^T \), \( b_i = (b_i^1, b_i^2, \ldots, b_i^N)^T \) (\( T \) means transposition), so the matrix \( \text{res}_{x=x_i} w^{(1)} \) is of rank 1. Note that in [9] the form (29) was derived from some algebro-geometric considerations using analytic properties of the Baker–Akhiezer function on the algebraic curve.

Setting in (5) \( s = 0 \) and taking it at \( x = x_i \) (so that the first term vanishes), we arrive at the relation

\[
\epsilon_{\alpha \beta} \tau_{\alpha \beta}(x_i) = \epsilon_{\alpha \lambda} \epsilon_{\kappa \lambda} \frac{\tau_{\alpha \lambda}(x_i) \tau_{\kappa \beta}(x_i)}{\tau_{\kappa \lambda}(x_i)}
\]

for distinct \( \alpha, \beta, \kappa, \lambda \), where (2) was used for the transformation of \( \epsilon \)-factors. Consider now (3), put \( s = 0 \) there, change \( \alpha \rightarrow \kappa \), \( \kappa \rightarrow \lambda \) and substitute \( x = x_i \) (so that the second term in the left hand side vanishes). We get

\[
\epsilon_{\alpha \kappa} \tau_{\alpha \kappa}(x_i) = -\epsilon_{\alpha \lambda} \epsilon_{\kappa \lambda} \frac{\partial_{\kappa,1} \tau(x_i) \tau_{\kappa \beta}(x_i)}{\tau_{\kappa \lambda}(x_i)}.
\]

Similarly, changing in (3) \( \beta \rightarrow \lambda \) and putting \( x = x_i \) (the second term in the left hand side vanishes), we get

\[
\epsilon_{\alpha \kappa} \tau_{\alpha \kappa}(x_i) = -\epsilon_{\alpha \lambda} \epsilon_{\kappa \lambda} \frac{\partial_{\kappa,1} \tau(x_i) \tau_{\alpha \lambda}(x_i)}{\tau_{\alpha \lambda}(x_i)}.
\]

Altogether, these formulae mean that

\[
\epsilon_{\alpha \beta} \tau_{\alpha \beta}(x_i) = \frac{A_\alpha B_\beta}{\epsilon_{\kappa \lambda} \tau_{\kappa \lambda}(x_i)}, \quad \alpha \neq \beta,
\]

i.e. the factorization holds for \( \alpha \neq \beta \) with

\[
A_\alpha = \begin{cases} 
\epsilon_{\alpha \lambda} \tau_{\alpha \lambda}(x_i), & \alpha \neq \lambda \\
-\partial_{\alpha,1} \tau(x_i), & \alpha = \lambda,
\end{cases}
\]

\[
B_\beta = \begin{cases} 
\epsilon_{\kappa \beta} \tau_{\kappa \beta}(x_i), & \beta \neq \kappa \\
-\partial_{\kappa,1} \tau(x_i), & \beta = \kappa.
\end{cases}
\]

Moreover, at \( \alpha = \beta \) we use (3) with the changes \( \kappa \rightarrow \alpha, \alpha \rightarrow \kappa, \beta \rightarrow \lambda \). At \( x = x_i \) (the second term in the left hand side vanishes) we get

\[
-\partial_{\kappa,1} \tau(x_i) = \epsilon_{\alpha \lambda} \epsilon_{\kappa \alpha} \frac{\tau_{\alpha \lambda}(x_i) \tau_{\kappa \alpha}(x_i)}{\tau_{\kappa \lambda}(x_i)},
\]

which means, together with (30), that

\[
\text{res}_{x=x_i} w^{(1)}_{\alpha \beta} = \epsilon_{\alpha \lambda} \frac{A_\alpha B_\beta}{\tau_{\kappa \lambda}(x_i) \tau'(x_i)} \quad \text{for any } \alpha, \beta,
\]

so the representation (29) is valid.

Now we turn to the residues of the Baker–Akhiezer functions. From (6) at \( s = 0 \) and \( t_{\alpha,m}^{(0)} = 0 \) we have:
\[ \text{res}_{x_i} \psi_{\alpha\beta} = e^{\xi z + \xi(t_z)} e^{\delta_{\alpha\beta} - 1} t_{\alpha\beta} \left[ x_i; t - z^{-1} \right]_{\beta} \frac{\tau_{\alpha\beta} \left( x_i; t - z^{-1} \right)_{\beta}}{\tau'(x_i)}, \] 

(31)

\[ \text{res}_{x_i} \psi_{\alpha\beta}^\dagger = -e^{-\xi z - \xi(t_z)} e^{\delta_{\alpha\beta} - 1} t_{\alpha\beta} \left[ x_i; t + z^{-1} \right]_{\alpha} \frac{\tau_{\alpha\beta} \left( x_i; t + z^{-1} \right)_{\alpha}}{\tau'(x_i)}. \] 

(32)

for any \( \alpha, \beta \). In order to transform these expressions, we use the Hirota equation (4). First, we set \( s = e_\beta - e_\alpha \), change \( \alpha \leftrightarrow \beta \) and substitute \( x = x_\alpha \), so that the second term in the left hand side and the first term in the right hand side vanish. The result is

\[ z^{-1} \tau_{\alpha\beta} (x, t - z^{-1})_{\beta} = -\tau_{\alpha\beta} (x, t) \frac{\tau(x, t - z^{-1})_{\beta}}{\partial_{\beta, i} \tau(x, t)}. \]

Similarly, changing \( t \rightarrow t + z^{-1} \) in (4) and putting \( x = x_\beta \), we get

\[ z^{-1} \tau_{\alpha\beta} (x, t + z^{-1})_{\alpha} = \tau_{\alpha\beta} (x, t) \frac{\tau(x, t + z^{-1})_{\alpha}}{\partial_{\alpha, i} \tau(x, t)}. \]

Using these formulae, it is easy to see that equations (31) and (32) can be written in the form

\[ \text{res}_{x_i} \psi_{\alpha\beta} = -\left( \text{res}_{x_i} w_{\alpha\beta}^{(1)} \right) e^{\xi z + \xi(t_z)} \frac{\tau(x_i; t - z^{-1})_{\beta}}{\partial_{\beta, i} \tau(x_i; t)}, \]

\[ \text{res}_{x_i} \psi_{\alpha\beta}^\dagger = -\left( \text{res}_{x_i} w_{\alpha\beta}^{(1)} \right) e^{-\xi z - \xi(t_z)} \frac{\tau(x_i; t + z^{-1})_{\alpha}}{\partial_{\alpha, i} \tau(x_i; t)} \]

for any \( \alpha, \beta \). Therefore, plugging here (29), we conclude that \( \text{res}_{x_i} \psi, \text{res}_{x_i} \psi^\dagger \) are matrices of rank 1:

\[ \text{res}_{x_i} \psi_{\alpha\beta} = e^{\xi z + \xi(t_z)} d_{\alpha\beta} e_i^\beta, \quad \text{res}_{x_i} \psi_{\alpha\beta}^\dagger = e^{-\xi z - \xi(t_z)} c_i^\alpha c_i^\beta, \]

(33)

where \( c_i^\alpha, c_i^\alpha \) are components of some vectors \( e_i = (c_i^1, \ldots, c_i^N)^T, c_i^* = (c_i^{*1}, \ldots, c_i^{*N})^T \).

Summing up, we have the following representation of the Baker–Akhiezer functions:

\[ \psi(t; z) = e^{\xi z + \xi(t_z)} \left( I + \sum_{i=1}^N \frac{a_i c_i^T}{x - x_i(t)} \right). \] 

(34)

\[ \psi^\dagger(t; z) = e^{-\xi z - \xi(t_z)} \left( I + \sum_{i=1}^N \frac{c_i^* b_i^T}{x - x_i(t)} \right). \] 

(35)

or, in components,

\[ \psi_{\alpha\beta}(t; z) = e^{\xi z + \xi(t_z)} \left( \delta_{\alpha\beta} + \sum_{i=1}^N \frac{a_i^\alpha c_i^\beta}{x - x_i(t)} \right). \] 

(36)

\[ \psi_{\alpha\beta}^\dagger(t; z) = e^{-\xi z - \xi(t_z)} \left( \delta_{\alpha\beta} + \sum_{i=1}^N \frac{c_i^{\alpha*} b_i^\beta}{x - x_i(t)} \right). \] 

(37)

Here the vectors \( a_i, b_i \) depend on the times \( t_k \) with \( k \geq 2 \) while the vectors \( c_i, c_i^* \) depend on the same set of times and on \( z \). For the matrices \( w_{\alpha\beta}^{(1)} \) and \( V = -2\partial_t w_{\alpha\beta}^{(1)} \) we have
\[ w^{(1)}(t) = s = \sum_{i=1}^{N} \frac{a_i b_i^T}{x - x_i(t)}, \quad V(t) = -2 \sum_{i=1}^{N} \frac{a_i b_i^T}{(x - x_i(t))^2}, \] (38)

where \( s \) does not depend on \( t \), or, in components,
\[ w^{(1)}_{\alpha\beta}(t) = s_{\alpha\beta} - \sum_{i=1}^{N} \frac{a^\alpha_i b^\beta_i}{x - x_i(t)}, \quad V_{\alpha\beta}(t) = -2 \sum_{i=1}^{N} \frac{a^\alpha_i b^\beta_i}{(x - x_i(t))^2}. \] (39)

3.2. Equations of motion with respect to \( t_2 \)

According to the Krichchev approach [5], the strategy is to substitute the pole ansatz for the Baker–Akhiezer functions (36) and (37) into the linear problems (24) and (25):

\[ \partial_{t_2} \Psi_{\alpha\beta} = \partial^2_x \Psi_{\alpha\beta} - 2 \sum_{i=1}^{N} \sum_{\gamma} \frac{a^\alpha_i b^\beta_i}{(x - x_i)^2} \Psi_{\gamma\beta}, \]
\[- \partial_{t_2} \Psi^\dagger_{\alpha\beta} = \partial^2_x \Psi^\dagger_{\alpha\beta} - 2 \sum_{\gamma} \Psi^\dagger_{\alpha\gamma} \sum_{i=1}^{N} \frac{a^\gamma_i b^\beta_i}{(x - x_i)^2}. \]

We have:
\[ \partial_{t_2} \Psi_{\alpha\beta} = \frac{d^2}{dx^2} \Psi_{\alpha\beta} + e^{\xi + \zeta(x)} \sum_{i=1}^{N} \left( \partial_{t_2} (a^\alpha_i c^\beta_i) \right) \frac{x - x_i}{(x - x_i)^2} + \frac{a^\alpha_i c^\beta_i x_i}{(x - x_i)^2}, \]

where \( x_k = \partial_{t_2} x_k \) and
\[ \partial^2_x \Psi_{\alpha\beta} = \frac{d^2}{dx^2} \Psi_{\alpha\beta} - 2 e^{\xi + \zeta(x)} \sum_{i=1}^{N} \frac{a^\alpha_i c^\beta_i}{x - x_i} + 2 e^{\xi + \zeta(x)} \sum_{i=1}^{N} \frac{a^\alpha_i c^\beta_i}{(x - x_i)^2}, \]
\[ \sum_{i=1}^{N} \frac{a^\alpha_i b^\gamma_i}{(x - x_i)^2} \left( \delta_{\gamma\beta} + \sum_{k=1}^{N} \frac{a^\gamma_k c^\beta_k}{x - x_k} \right) = \sum_{i=1}^{N} \left( \frac{a^\alpha_i b^\beta_i}{(x - x_i)^2} + \frac{a^\alpha_i b^\gamma_i a^\gamma_i c^\beta_i}{(x - x_i)^3} \right) \]
\[ + \sum_{i \neq k} \frac{a^\alpha_i b^\gamma_i a^\gamma_i c^\beta_i}{(x - x_i)^2 (x - x_k)} + \sum_{i \neq k} \frac{a^\alpha_i b^\gamma_i a^\gamma_i c^\beta_i}{(x - x_i)(x - x_k)^2}, \]

where summation over repeated index \( \gamma \) is implied. Substituting these expressions into the linear problem and equating coefficients at the poles at \( x = x_i \) of different orders, we get the following conditions:

- At \( \frac{1}{(x - x_i)} \): \( b^\alpha_i a^\beta_i = 1 \) or \( b^\alpha_i a^\beta_i = 1 \);
- At \( \frac{1}{(x - x_i)^2} \): \( a^\alpha_i c^\beta_i x_i = -2a^\alpha_i c^\beta_i - 2a^\alpha_i b^\beta_i - 2 \sum_{k \neq i} a^\alpha_i b^\gamma_i a^\gamma_i c^\beta_i \);
- At \( \frac{1}{x - x_i} \): \( \partial_{t_2} (a^\alpha_i c^\beta_i) = -2 \sum_{k \neq i} a^\alpha_i b^\gamma_i a^\gamma_i c^\beta_i - a^\alpha_i b^\gamma_i a^\gamma_i c^\beta_i \).
The conditions coming from the second order poles can be written in the matrix form:

\[ \sum_{k=1}^{N} (zI - L)_{k} c_{k}^{\alpha} = -b_{i}^{\alpha}, \quad L_{k} = -\frac{x_{i}}{2} \delta_{k} - (1 - \delta_{k}) \frac{b_{i}^{T} a_{k}}{x_{i} - x_{k}}, \]  

(40)

where \( I \) is the \( N \times N \) unity matrix. As for the conditions at the first order poles, we write \( \delta_{k} (a_{i}^{\alpha} c_{k}^{\beta}) = \dot{a}_{i}^{\alpha} c_{k}^{\beta} + a_{i}^{\alpha} \dot{c}_{k}^{\beta} \), and equate the two terms separately to the two terms in the right hand side, thus obtaining sufficient conditions for cancellation of the poles:

\[ \dot{c}_{i}^{\alpha} = \sum_{k=1}^{N} M_{ik} c_{k}^{\alpha}, \quad \dot{a}_{i}^{\alpha} = -\sum_{k=1}^{N} a_{k}^{\alpha} M_{ik}, \quad M_{ik} = 2(1 - \delta_{k}) \frac{b_{i}^{T} a_{k}}{(x_{i} - x_{k})^{2}}. \]  

(41)

Similar calculations with the linear problem for \( b_{i} \) lead to the same condition \( b_{i}^{T} a_{i} = 1 \) and to the equations

\[ \sum_{k=1}^{N} c_{k}^{\alpha} (zI - L)_{i k} = a_{i}^{\alpha}, \]  

(42)

\[ \dot{c}_{i}^{+\alpha} = -\sum_{k=1}^{N} c_{k}^{+\alpha} M_{ik}, \quad \dot{b}_{i}^{\alpha} = \sum_{k=1}^{N} M_{ik} b_{k}^{\alpha}. \]  

(43)

Note that the second equations in (41) and (43) give equations of motion for the vectors \( a_{i} \) and \( b_{i} \).

Therefore, we have the following over-determined linear problems for the \( N \)-component vectors \( C^{\alpha} = (c_{1}^{\alpha}, \ldots, c_{N}^{\alpha})^{T} \) and \( C^{+\alpha} = (c_{1}^{+\alpha}, \ldots, c_{N}^{+\alpha})^{T} \):

\[ \begin{cases} (zI - L) C^{\alpha} = -B^{\alpha} \\ C^{\alpha} = MC^{\alpha}, \quad \{ C^{+\alpha T} (zI - L) = A^{\alpha} \\ C^{+\alpha T} = -C^{\alpha T} M, \end{cases} \]

where \( A^{\alpha} = (a_{1}^{\alpha}, \ldots, a_{N}^{\alpha})^{T} \), \( B^{\alpha} = (b_{1}^{\alpha}, \ldots, b_{N}^{\alpha})^{T} \). The consistency of these linear problems implies (after applying \( \delta_{k} \) to the first one in the first pair):

\[-LC^{\alpha} + (zI - L) C^{\alpha} = -B^{\alpha} = -MB^{\alpha} = M(zI - L)C^{\alpha},\]

i.e. \( \left( L - [M, L] \right) C^{\alpha} = 0 \). The second pair of the linear problems yields, in a similar way, \( C^{+\alpha T} \left( L - [M, L] \right) = 0 \). Therefore, the consistency condition for the linear problems is

\[ L = [M, L] \]  

(44)

which is the Lax equation for our model. Using equations of motion for the vectors \( a_{i} \) and \( b_{i} \), one can check that non-diagonal parts of the Lax equation are satisfied identically while the diagonal parts yield equations of motion for the poles \( x_{i} \):

\[ \ddot{x}_{i} = -8 \sum_{k \neq i} \frac{(b_{i}^{T} a_{i}) (b_{k}^{T} a_{k})}{(x_{i} - x_{k})^{2}}. \]  

(45)

Together with the equations for the vectors \( a_{i}, b_{i} \) (see (41) and (43)),

\[ \dot{a}_{i} = -2 \sum_{k \neq i} \frac{(b_{i}^{T} a_{i}) a_{k}}{(x_{i} - x_{k})^{2}}, \quad \dot{b}_{i} = 2 \sum_{k \neq i} \frac{(b_{i}^{T} a_{i}) b_{k}}{(x_{i} - x_{k})^{2}}, \]  

(46)
they form the closed set of equations of motion for the model. Note that the equations (46) are compatible with the constraints $b_i^T a_i = 1$. We see that our dynamical system is the spin generalization of the Calogero system (the Gibbons–Hermsen model). It is a Hamiltonian system with the Hamiltonian

$$H = \sum_{i=1}^N p_i^2 - \sum_{i \neq k} \frac{(b_i^T a_i)(b_k^T a_k)}{(x_i - x_k)^2}$$

and the non-vanishing Poisson brackets $\{x_i, p_k\} = \delta_{ik}$, $\{a_i^\alpha, b_i^\beta\} = \delta_{\alpha\beta} \delta_{ik}$. The Hamiltonian equations of motion

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}, \quad \dot{a}_i^\alpha = \frac{\partial H}{\partial b_i^\alpha}, \quad \dot{b}_i^\alpha = -\frac{\partial H}{\partial a_i^\alpha}$$

are equivalent to (45) and (46). Taking into account that $\dot{x}_i = 2p_i$, we see that

$$H = H_2 = \text{tr} \ L^2.$$  

(48)

The spin generalization of the Calogero system is an integrable model. The higher Hamiltonians in involution are given by

$$H_k = \text{tr} \ L^k, \quad k \geq 1.$$  

(49)

The Hamiltonian $H_1 = -\sum_i p_i$ is the (minus) total momentum.

### 3.3. Dynamics in the higher times

The main tool for investigating the dynamics in higher times is the relation (20) which we write here in the form

$$\text{res}_\infty (z^m \Psi_{\alpha\gamma} \Psi_{\gamma\beta}^\dagger) = -\partial_{\alpha\beta} w^{(1)}_\alpha.$$  

(50)

(We use the notation $\text{res}_\infty f(z) = \frac{1}{2\pi i} \oint_C \infty f(z) dz = f^{-1}$, where $f^{-1}$ is the coefficient in front of $z^{-1}$ in the Laurent expansion $f(z) = \sum_k f_k z^k$.) In order to use it, we prepare the following expressions:

$$\Psi_{\alpha\gamma} \Psi_{\gamma\beta}^\dagger = \delta_{\alpha\beta} + \sum_i \frac{c_i^\alpha c_i^\beta}{x - x_i} + \sum_i \frac{\partial^\alpha c_i^\beta}{x - x_i} + \sum_{i \neq k} \frac{\partial^\alpha c_i^\gamma c_k^\gamma b_i^\beta}{(x_i - x_k)(x - x_k)} + \sum_i \frac{\partial^\alpha c_k^\gamma c_i^\gamma b_i^\beta}{(x - x_k)^2},$$

$$-\partial_{\alpha\beta} w^{(1)}_\alpha = \sum_i \frac{\partial_{\alpha\beta} (a_i^\alpha b_i^\beta)}{x - x_i} + \sum_i \frac{\partial^\alpha b_i^\beta \partial_{\alpha\beta} x_i}{(x - x_i)^2}.$$  

Comparing the second order poles at $x = x_i$ in (50), we obtain

$$\partial_{\alpha\beta} x_i = \text{res}_\infty (z^m c_i^\gamma c_i^{\gamma\dagger}).$$

(51)

Now we solve the linear equations (40) and (42) for $c_i^\gamma, c_i^{\gamma\dagger}$:

$$c_i^\gamma = -\sum_k (zI - L)_{ik}^{-1} b_k^\gamma, \quad c_k^{\gamma\dagger} = \sum_k (zI - L)_{ki}^{-1} a_k^\gamma$$

(52)

and substitute this into (51). We get:
\[ \partial_{\mu} x_i = -\text{res}_{\infty} \left( z^m \sum_{k,l} a_k^i (zI - L)^{-1}_{ki} (zI - L)^{-1}_{jl} b_l^j \right). \]

Recalling that \( p_k = \dot{x}_k / 2 \) and using the obvious relation \( \partial_{\mu} \frac{\partial}{\partial p_k} = -\delta_{ij} \delta_{mn} \), we can write this as

\[ \partial_{\mu} x_i = \text{res}_{\infty} \left( z^m \sum_{k,l,j,n} a_k^i (zI - L)^{-1}_{ki} \frac{\partial L_{jm}}{\partial p_i} (zI - L)^{-1}_{jl} b_l^j \right) \]

\[ = \frac{\partial}{\partial p_i} \text{res}_{\infty} \left( z^m \sum_{k,l} a_k^i (zI - L)^{-1}_{ki} b_l^l \right) = \frac{\partial}{\partial p_i} \sum_{k,l} (L^m)_{il} b_l^i = \frac{\partial}{\partial p_i} \text{tr} \left( L^m R \right), \]

where \( R \) is the \( N \times N \) matrix with matrix elements \( R_{ij} = b_j^i a_i \). Introduce the matrix \( X = \text{diag} (x_1, \ldots, x_N) \), then it is easy to check that \( R = I + [L, X] \). Therefore,

\[ \text{tr} \left( L^m R \right) = \text{tr} \left( L^m + L^m (LX - XL) \right) = \text{tr} L^m = H_m \]

and we obtain one set of the Hamiltonian equations for the higher flow \( t_m \):

\[ \partial_{\mu} x_i = \frac{\partial H_m}{\partial p_i}, \quad m \geq 2. \quad (53) \]

Note that formally these equations hold also for \( m = 1 \) yielding \( \partial_{\mu} x_i = -1 \) which is true because of (23).

In order to obtain the second set of Hamiltonian equations, we apply \( \partial_{\nu} \) to the both sides of equation (51):

\[ \partial_{\nu} \dot{x}_i = \text{res}_{\infty} \left( z^m \left( c_i^\gamma c_i^{\gamma \gamma} + c_i^{\gamma \gamma} c_i^\gamma \right) \right) = \text{res}_{\infty} \left( z^m \sum_k \left( c_i^\gamma M_k c_k^\gamma - c_k^\gamma M_k c_i^\gamma \right) \right). \]

Next we substitute (52):

\[ \partial_{\nu} \dot{x}_i = -\text{res}_{\infty} \left( z^m \sum_{k,l,j,n} \left( a_k^i (zI - L)^{-1}_{ki} M_{jk} (zI - L)^{-1}_{kl} b_l^l - a_k^i (zI - L)^{-1}_{ki} M_{jk} (zI - L)^{-1}_{nl} b_l^l \right) \right) \]

\[ = -\text{res}_{\infty} \left( z^m \sum_{k,l,j,n} \left( a_k^i (zI - L)^{-1}_{ki} (E_{il})_{j} M_{jk} (zI - L)^{-1}_{kl} b_l^l \right) \right) \]

\[ -a_k^i (zI - L)^{-1}_{ki} M_{kr} (E_{il})_{j} (zI - L)^{-1}_{jr} b_r^l \right) \]

\[ = -\text{res}_{\infty} \left( z^m \sum_{k,l,n,j} a_k^i (zI - L)^{-1}_{ki} [E_{il}, M]_{jk} (zI - L)^{-1}_{kl} b_l^l \right). \]

where \( E_{ij} \) is the matrix with matrix elements \( (E_{ij})_{jl} = \delta_{ij} \delta_{ln} \) (1 in the place \( ii \) and zeros elsewhere). It is easy to check that

\[ [E_{ij}, M]_{jk} = 2 \frac{\partial L_{jk}}{\partial x_i}. \]
Therefore, it holds
\[
\partial_a p_i = - \text{res} \left( z^m \sum_{k,l,n,j} a_j^k (zI - L)^{-1}_j \frac{\partial L^k_{ij}}{\partial x_i} (zI - L)^{-1}_n b_n^j \right)
\]
\[
= - \frac{\partial}{\partial x_i} \text{res} \left( z^m \sum_{l,n} a_l^n (zI - L)^{-1}_n b_n^i \right)
\]
\[
= - \frac{\partial}{\partial x_i} \sum_{l,n} a_l^n (L^m)_{ln} b_n^i = - \frac{\partial}{\partial x_i} \text{tr} \left( L^m R \right) = - \frac{\partial}{\partial x_i} \text{tr} \left( L^m \right)
\]
and we obtain the second set of Hamiltonian equations
\[
\partial_a p_i = - \frac{\partial H_m}{\partial x_i}, \quad m \geq 1.
\]

Let us turn to the first order poles in (50). Equating the coefficients at the first order poles in the both sides, we obtain the equation
\[
\partial_a (a^\alpha_i b^\beta_i) = \text{res} \left( z^m \left( c_i^{\alpha*} b^\beta_i + a_i^\alpha c^\beta_i + \sum_{k \neq i} a_k^\alpha c_k^{\gamma*} b_k^\beta + a_k^\alpha c_k^{\gamma} b_k^\beta \right) \right)
\]
which can be rewritten as
\[
b_i^\beta \left[ \partial_a a_i^\alpha - \text{res} \left( z^m \left( c_i^{\alpha*} + \sum_{k \neq i} a_k^\alpha c_k^{\gamma*} \right) \right) \right]
\]
\[
\quad + a_i^\alpha \left[ \partial_a b_i^\beta - \text{res} \left( z^m \left( c_i^{\beta} + \sum_{k \neq i} c_k^{\gamma} b_k^\beta \right) \right) \right] = 0.
\]
Equating to zero expressions in both square brackets separately (which gives sufficient conditions for cancellation of first order poles), we get a system of evolutionary equations for \(a_i^\alpha\) and \(b_i^\beta\):
\[
\partial_a a_i^\alpha = \text{res} \left( z^m \left( c_i^{\alpha*} + \sum_{k \neq i} a_k^\alpha c_k^{\gamma*} \right) \right),
\]
\[
\partial_a b_i^\beta = \text{res} \left( z^m \left( c_i^{\beta} + \sum_{k \neq i} c_k^{\gamma} b_k^\beta \right) \right).
\]
Consider the first equation. Substituting (52) for \(c_i^\gamma\), \(c_i^{\gamma*}\), we have:
\[
\partial_a a_i^\alpha = \text{res} \left[ z^m \left( \sum_k a_k^\alpha (zI - L)^{-1}_k - \sum_{k,l,n} a_l^n (zI - L)^{-1}_n \frac{(1 - \delta_k) a_k^\alpha}{x_i - x_k} (zI - L)^{-1}_n b_n^j \right) \right]
\]
so we obtain the Hamiltonian equations

\[ \partial_t a_i^\alpha = \frac{\partial H_m}{\partial b_i^\alpha}, \quad m \geq 1. \tag{57} \]

In a similar way, from (56) we obtain the Hamiltonian equations

\[ \partial_t b_i^\beta = - \frac{\partial H_m}{\partial a_i^\beta}, \quad m \geq 1. \tag{58} \]

Therefore, we have shown that the evolution of \( x_i, a_i^\alpha, b_i^\beta \) with respect to the higher times \( t_m \) of the matrix KP hierarchy is governed by the higher Hamiltonians \( H_m \) of the spin Calogero–Moser system.

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**References**

[1] Airault H, McKea H P and Moser J 1977 Rational and elliptic solutions of the Korteweg–De Vries equation and a related many-body problem *Commun. Pure Appl. Math.* **30** 95–148

[2] Calogero F 1971 Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials *J. Math. Phys.* **12** 419–36

[3] Calogero F 1975 Exactly solvable one-dimensional many-body systems *Lett. Nuovo Cimento* **13** 411–5

[4] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220

[5] Krichever I M 1978 Rational solutions of the Kadomtsev–Petviashvili equation and integrable systems of N particles on a line *Funct. Anal. Appl.* **12** 59–61

[6] Chudnovsky D V and Chudnovsky G V 1977 Pole expansions of non-linear partial differential equations *Nuovo Cimento* **40B** 339–50

[7] Krichever I M 1980 Elliptic solutions of the Kadomtsev–Petviashvili equation and integrable systems of particles *Funct. Anal. Ego Pril.* **14** 45–54 (in Russian) Krichever I M 1980 *Funct. Anal. Appl.* **14** 282–90 (Engl. transl.)
[8] Shiota T 1994 Calogero–Moser hierarchy and KP hierarchy J. Math. Phys. 35 5844–9
[9] Krichever I, Babelon O, Billey E and Talon M 1995 Spin generalization of the Calogero–Moser
system and the matrix KP equation Am. Math. Soc. Transl. 2 170 83–119
[10] Gibbons J and Hermsen T 1984 A generalization of the Calogero–Moser system Physica D 11 337–48
[11] Date E, Jimbo M, Kashiwara M and Miwa T 1981 Transformation groups for soliton equations III
J. Phys. Soc. Japan 50 3806–12
[12] Kac V and van de Leur J 1993 The n-component KP hierarchy and representation theory Important
Developments in Soliton Theory ed A S Fokas and V E Zakharov (Berlin: Springer)
[13] Tacchella A 2011 On rational solutions of multicomponent and matrix KP hierarchies J. Geom.
Phys. 61 1319–28
[14] Bergvelt M, Gekhtman M and Kasman A 2009 Spin Calogero particles and bispectral solutions of
the matrix KP hierarchy Math. Phys. Anal. Geom. 12 181–200
[15] Ben-Zvi D and Nevins T 2011 D-bundles and integrable hierarchies J. Eur. Math. Soc. 13 1505–67
[16] Ben-Zvi D and Nevins T 2008 From solitons to many-body systems Pure Appl. Math. Q. 4 319–61
(Special Issue in honor of Fedor Bogomolov, Part 1)
[17] Chalykh O and Silantyev A 2017 KP hierarchy for the cyclic quiver J. Math. Phys. 58 071702
[18] Teo L-P 2011 The multicomponent KP hierarchy: differential Fay identities and Lax equations J.
Phys. A: Math. Theor. 44 225201
[19] Takasaki K and Takebe T 2007 Integrable hierarchies and dispersionless limit Physica D 235 109–25