Quantum Hyperdeterminants and Hyper-Pfaffians

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Abstract. A notion of the quantum hyper-monoid is introduced. It is proved that the quantum coordinate ring of the monoid can be lifted to a quantum hyper-algebra, in which the quantum determinant and the quantum Pfaffian are lifted to the quantum hyperdeterminant and quantum hyper-Pfaffian respectively. The quantum hyperdeterminant in even dimension is shown to be a $q$-analog of Cayley’s first hyperdeterminant.

1. Introduction

In mathematics and physics, one is often lead to consider $m$-dimensional hypermatrices $A = (a_{i_1 \cdots i_m})$ indexed by multi-indices, while the usual rectangular matrices are 2-dimensional [GKZ, HT, Mat]. An $m$-dimensional hypermatrix is said to have format $n^m = n \times \cdots \times n$ if its basic index $i_k$ runs through $1, \cdots, n$.

The space $\mathcal{A}_1$ of $m$-dimensional hypermatrices forms a representation of the group $\text{GL}_n^\otimes m \times \text{GL}_n^\otimes m$ from the left and the right action:

\begin{equation}
\text{GL}_n^\otimes m \times \mathcal{A}_1 \times \text{GL}_n^\otimes m \longrightarrow \mathcal{A}_1.
\end{equation}

The left action is defined by the multiplication rule $M_n \times \mathcal{A}_1 \overset{\circ_k}{\longrightarrow} \mathcal{A}_1$:

\begin{equation}
(B \circ_k A)_{i_1 \cdots i_m} = \sum_{j=1}^n b_{i_k,j}a_{i_1 \cdots i_{k-1}j_{k+1} \cdots i_m}.
\end{equation}

Similarly the right action is defined.

Cayley had laid the foundation of classical invariant theory and multilinear algebra, in which he introduced the notion of hyperdeterminants for studying hypermatrices and tensors, see [GKZ] for a modern account of Cayley’s theory. Cayley’s first hyperdeterminant $\text{Det} [C]$ is defined for any

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even-dimensional hypermatrix $A = (a_{i_1 \cdots i_{2m}})$, where $1 \leq i_j \leq n$. One remarkable property of $\text{Det}$ is the relative invariance under the action of $\text{GL}_n^{\otimes 2m}$:

$$\text{Det}(A \circ_k B) = \text{Det}(B \circ_k A) = \text{Det}(A) \text{det}(B),$$

where $B$ is an $n \times n$-matrix.

The aim of the present work is to introduce a notion of quantum hypermatrices and generalize the algebra $\text{Mat}_q(n)$ of quantum matrices to that of quantum hypermatrices. In $[\text{Ma1}, \text{Ma2}]$, Manin showed that the quantum matrix ring $\text{Mat}_q(n)$ can be formulated as the quantum transformations that preserve the quantum exterior algebra and the quantum Weyl algebra.

To generalize Manin’s idea we first show that the quantum Weyl algebra can be replaced by the quantum exterior algebra provided that one imposes the invariance for the dual quantum transformation $A^T$. This idea then works well for quantum hypermatrices when we require that all matrix realignments are transformed according to the rule of the quantum exterior algebra. We show that these transformation rules provide enough quantum symmetry to warrant that the hyperdeterminant is a quantum volume element. Moreover we introduce the notion of anti-symmetric hypermatrices and quantum hyper-Pfaffian. We also prove that the quantum hyper-Pfaffian is defined by a volume element of a quantum 2-form.

One important property and new feature of our quantum hyperdeterminants is that we are able to define them for any dimensional hypermatrices, removing the restriction that Cayley’s hyperdeterminant is only defined for even dimensional. Even more interesting is the fact that our hyperdeterminant also works for $q = 1$ for noncommutative Manin-type hypermatrices. These hyperdeterminants at odd dimension will only vanish when the matrix elements are commutative. This is in complete agreement with the general phenomena that classical singularity is often better regularized at the level of quantum deformation.

We remark that our guiding principle is to try to find minimum defining relations to ensure both quantum hyperdeterminants and quantum hyper-Pfaffians work. In other words, the relations that we have found will ensure that the diamond lemma is satisfied to rearrange the products in the determinant and Pfaffians.

2. Quantum hyperdeterminants and quantum hyper-Pfaffians

2.1. Quantum hyperdeterminants. Let $q$ be a non-zero complex number. The quantum exterior algebra $\Lambda_n = \Lambda$ is the quadratic algebra generated by $x_1, \cdots, x_n$ over the field $F$ subject to the following relations:

$$x_j \wedge x_i = -qx_i \wedge x_j,$$

$$x_i \wedge x_i = 0,$$
where \( i < j \). The algebra \( \Lambda \) is naturally \( \mathbb{Z}_{n+1} \)-graded and decomposes itself:

\[
\Lambda = \bigoplus_{k=0}^{n} \Lambda^k,
\]

where the \( k \)th homogeneous subspace \( \Lambda^k \) is spanned by \( x_{i_1} \cdots x_{i_k} \), \( 1 \leq i_1 < \cdots < i_k \leq n \). So \( \dim(\Lambda) = 2^n \).

The quantum monoid \( \text{Mat}_q(n) \) is a bialgebra generated by \( a_{ij} \) subject to certain quadratic relations. To present the relations we write \( A = (a_{ij}) \) and consider the tensor product \( \text{Mat}_q(n) \otimes \Lambda_n \), where we require that \( x_i \) and \( a_{ij} \) commute. We will simply write \( ax \) for the tensor product \( a \otimes x \in \Lambda \otimes A \), and the wedge products are similarly written as follows.

\[
(ax) \wedge (by) = ab(x \wedge y).
\]

Thus \( x_k a_{ij} = a_{ij} x_k \) for any admissible \( i, j, k \). The following is a reformulation of Manin’s result \([Ma1], [Ma2]\).

**Proposition 2.1.** The defining relations of the generators \( a_{ij} \) for the quantum monoid \( \text{Mat}_q(n) \) are equivalent to the following rule: suppose that \( x_i \)'s obey the relations (2.1-2.2), then \( y_i = \sum_k a_{ik} x_k \) and \( y'_i = \sum_k a_{ki} x_k \) both satisfy the relations (2.1-2.2).

Explicitly the relations given in Proposition 2.1 are as follows.

\[
\begin{align*}
(2.3) \quad a_{ik}a_{il} &= qa_{il}a_{ik}, \\
(2.4) \quad a_{ik}a_{jk} &= qa_{jk}a_{ik}, \\
(2.5) \quad a_{jk}a_{il} &= a_{il}a_{jk}, \\
(2.6) \quad a_{ik}a_{jl} - a_{jl}a_{ik} &= (q - q^{-1})a_{il}a_{jk},
\end{align*}
\]

where \( i < j \) and \( k < l \).

We define the quantum determinant \( \det_q(A) \) via a quantum volume element. Let \( y_i \) be defined as in Prop. 2.3. then

\[
(2.7) \quad y_1 \wedge \cdots \wedge y_n = \det_q(A)x_1 \wedge \cdots \wedge x_n.
\]

Using the quantum exterior relations \([JZ]\), \( \det_q \) is explicitly given by

\[
\begin{align*}
(2.8) \quad \det_q(A) &= \sum_{\sigma \in S_n} (-q)^{l(\sigma)} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \\
(2.9) \quad &= \sum_{\sigma \in S_n} (-q)^{l(\sigma)} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}.
\end{align*}
\]

where \( l(\sigma) \) is the number of inversions made by the permutation \( \sigma \). Here, the two expressions are respectively called the row determinant and column determinant.

Let \( B \) be another \( n \times n \)-quantum matrix commuting with \( A \), i.e. their entries commute with each other. Then

\[
(2.10) \quad \det_q(AB) = \det_q(A)\det_q(B).
\]
We remark that this holds even if $B$ is an ordinary permutation matrix, i.e., Eq. (2.10) contains the property that $\det_q(A)$ becomes $(-q)\det_q(A)$ when two rows (column) are interchanged.

2.2. Quantum hyperdeterminants. We now generalize $\det_q$ to higher dimensional quantum matrices. For natural number $n$ we denote the index set $\{1,\ldots,n\}$ by $[1,n]$. Fix $m,n \geq 1$. Let $A = (a_{i_1\ldots i_m})$ be a hypermatrix of format $n \times \cdots \times n = n^m$. We will use the matrix realignment to simplify our representation. For each hypermatrix $A$ of format $n^m$, one associates $m$ realigned or unfolding rectangular matrices $A^{(1)},\ldots,A^{(m)}$. The $k$th matrix realignment $A^{(k)} = (a^{(k)}_{i\alpha})$ is a $n \times n^{m-1}$-rectangular matrix with entries

$$a^{(k)}_{i\alpha} = a_{j_1\ldots j_{k-1}i j_{k+1}\ldots j_m}, \quad \alpha = (j_1,\ldots,j_{k-1},i,j_{k+1},\ldots,j_{m}).$$

Here, $i \in [1,n]$ and $\alpha$ runs through the set $[1,n]^{m-1}$ in the lexicographic order. For example, the $2 \times 2 \times 2 \times 2$ matrix $A$ has the following unfoldings with respect to the second and third indices:

$$A^{(2)} = \begin{pmatrix} a_{1111} & a_{1112} & a_{1121} & a_{1122} & a_{2111} & a_{2112} & a_{2121} & a_{2122} \\ a_{1211} & a_{1212} & a_{1221} & a_{1222} & a_{2211} & a_{2212} & a_{2221} & a_{2222} \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} a_{1111} & a_{1112} & a_{1121} & a_{1122} & a_{2111} & a_{2112} & a_{2121} & a_{2122} \\ a_{1211} & a_{1212} & a_{1221} & a_{1222} & a_{2211} & a_{2212} & a_{2221} & a_{2222} \end{pmatrix}. $$

We consider the $n^{m-1}$-dimensional column vector $X = [x_{\alpha}]$ with entries $x_{\alpha} = x_{\alpha_1} \otimes x_{\alpha_2} \otimes \cdots \otimes x_{\alpha_{m-1}} \in (\Lambda^1)^{\otimes m-1}$, $\alpha_k \in [1,n]$. Here, the entries of $X$ are also ordered in the lexicographic order. For example, if $m = n = 2$,

$$X = (x_{1111}, x_{1112}, x_{1211}, x_{1212}, x_{2111}, x_{2112}, x_{2121}, x_{2221}, x_{2222})^T.$$

In the following, we assume $q^2 \neq -1$. Suppose $a_{i_1,\ldots,i_m}$ commute with the $x_i$. Define $x^{(k)}_i$'s by

$$\begin{pmatrix} x^{(k)}_1 \\ x^{(k)}_2 \\ \vdots \\ x^{(k)}_n \end{pmatrix} = A^{(k)} X.$$

In other words, $x^{(k)}_i = \sum \alpha a^{(k)}_{i\alpha} x_{\alpha}$. For any subset $J = \{j_1,\ldots,j_r\} \subset [1,n]$ we denote $x_J = x_{j_1} \wedge \cdots \wedge x_{j_r}$. For $n \in \mathbb{N}$, the $q$-number $[n]_q = 1+q+\cdots+q^{n-1}$ and the quantum factorial $[n]_q! = [1]_q [2]_q \cdots [n]_q$.

We remark that $x_{\alpha}$ denotes an element from the tensor product $(\Lambda^1)^{\otimes (m-1)}$ to keep notation under control, while $x_J$ represents elements in the quantum Weyl algebra $\Lambda$. Since the subset $J$ has no repetition no confusion will arise in general. When we consider products of $x_{\alpha}$'s, we will use multi-components as superscripts to distinguish the elements.
Theorem 2.2. The following two statements are equivalent.

(i) $x_j^{(k)} x_i^{(k)} = -q x_i^{(k)} x_j^{(k)}$ for $i < j$, $x_i^{(k)} x_i^{(k)} = 0$.

(ii) For any fixed $J = I_1 \times \cdots \times I_{m-1} \subset [1, n]^{m-1}$ with $|I_i| = 2$ we have

\begin{equation}
\sum_{\alpha \beta = J} (-q)^{\text{inv}(\alpha, \beta)} a_{i \alpha}^{(k)} a_{i \beta}^{(k)} = 0,
\end{equation}

\begin{equation}
\sum_{\alpha \beta = J} (-q)^{\text{inv}(\alpha, \beta)} a_{j \alpha}^{(k)} a_{j \beta}^{(k)} = -q \sum_{\alpha \beta = J} (-q)^{\text{inv}(\alpha, \beta)} a_{i \alpha}^{(k)} a_{j \beta}^{(k)},
\end{equation}

where $\text{inv}(\alpha, \beta) = |\{(i_s > j_s) | \alpha = (i_1 \cdots i_{m-1}), \beta = (j_1 \cdots j_{m-1})\}|$, the total number of inversions from the natural ordering of $J$.

Proof. Note that

\begin{equation}
x_i^{(k)} x_j^{(k)} = \sum_{\alpha \beta} a_{i \alpha}^{(k)} a_{j \beta}^{(k)} x_\alpha x_\beta = \sum_J \left( \sum_{\alpha \beta = J} (-q)^{\text{inv}(\alpha, \beta)} a_{i \alpha}^{(k)} a_{j \beta}^{(k)} \right) x_J
\end{equation}

where $J = \{i_1, j_1\} \cup \cdots \cup \{i_{m-1}, j_{m-1}\}$ is written in the natural lexicographic order. Then the two relations are obtained by comparing coefficients. \(\square\)

Let $\mathcal{A} = \mathcal{A}_q^{[m]}$ be the associative algebra generated by $a_{i_1 \cdots i_m}$, $1 \leq i_k \leq n$, subject to the relations (2.15, 2.16). We arrange the generators into a hypermatrix $A = (a_{i_1 \cdots i_m})$ of format $n \times \cdots \times n = n^m$, and we often call $\mathcal{A}$ a quantum hypermatrix for simplicity.

When the hypermatrix is even-dimensional, the algebra becomes a bialgebra. In this case, we denote the generators of $\mathcal{A}_q^{[2m]}$ by $a_{\alpha \beta}$, $\alpha, \beta \in [1, n]^m$. The comultiplication is defined by

\begin{equation}
\Delta(a_{\alpha \beta}) = \sum_{\gamma} a_{\alpha \gamma} \otimes a_{\gamma \beta},
\end{equation}

where $\gamma$ runs through all $n$-element subsets of $[1, n]^m$. Under this comultiplication, $\mathcal{A}_q^{[2m]}$ becomes a bialgebra and is called the quantum coordinate algebra of the monoid of hypermatrices, generalizing the quantum monoid $\text{Mat}_q(n)$, which is $\mathcal{A}_q^{[2]}$ in our notation. Moreover, we will prove that $\mathcal{A}_q^{[2m]}$ also has a distinguished group-like element similar to $\det_q$.

When $q^2 \neq -1$, by subtracting two relations in Theorem 2.2 (ii) we obtain that for a fixed product $I$ of 2-element subsets in $[1, n]^{m-1}$ and any $s, t$ and $i < j, k < l$

\begin{equation}
\sum_{\alpha \beta = J} (-q)^{\text{inv}(\alpha, \beta)} a_{i \alpha}^{(s)} a_{j \beta}^{(s)} = \sum_{\alpha' \beta' = J} (-q)^{\text{inv}(\alpha', \beta')} a_{k \alpha}^{(t)} a_{l \beta'}^{(t)}.
\end{equation}

Under the assumption that $q^2 \neq -1$, using (2.15−2.18) we can derive that for any $k, l$

\begin{equation}
x_1^{(k)} x_2^{(k)} \cdots x_n^{(k)} = x_1^{(l)} x_2^{(l)} \cdots x_n^{(l)}.
\end{equation}

From now on we impose no condition on $q$ and consider arbitrary $q$. Let us list some simple properties of the quantum algebra $\mathcal{A}$. First of all, it is
straightforward to verify that \( f^{(k)} : \Lambda \to (\mathcal{A}) \otimes \Lambda^{\otimes m-1} \) given by \( x_i \mapsto x_i^{(k)} \) defines an algebra homomorphism.

Next the symmetric group \( S_m \) acts on the algebra \( \mathcal{A} \) via

\[
\sigma a_{i_1,i_2,\ldots,i_m} = a_{i_{\sigma^{-1}(1)},i_{\sigma^{-1}(2)},\ldots,i_{\sigma^{-1}(m)}}.
\]

This can be easily seen as \( (\sigma \sigma') a_{i_1,i_2,\ldots,i_m} = \sigma (\sigma' a_{i_1,i_2,\ldots,i_m}) \). We denote the homomorphism by \( \Theta : S_m \to \text{Aut} \mathcal{A} \). This property generalizes the duality of rows and columns in the usual quantum monoid \( \text{Mat}_{q} \).

We now define the quantum analog of Cayley’s first hyperdeterminant by (when \( q \) is generic)

\[
(2.20) \quad \text{Det}^m_q(\mathcal{A}) = \frac{1}{[n]_q!} \sum_{\sigma_1,\ldots,\sigma_m \in S_n} (-q)^{\sum_{i=1}^m l(\sigma_i)} \prod_{i=1}^n a_{\sigma_1(i),\sigma_2(i),\ldots,\sigma_m(i)}.
\]

The quantum hyperdeterminant is invariant under the action of \( S_m \):

\[
\sigma \text{Det}^m_q(\mathcal{A}) = \text{Det}^m_q(\mathcal{A})
\]

for any \( \sigma \in S_m \). This invariance generalizes the usual property that \( \det_q(A) = \det_q(A^T) \). Sometimes we will also use \( \text{Det}_q(A) \) to denote the quantum hyperdeterminant if there is no confusion.

**Example 2.3.** The hyperdeterminant of \( A = (a_{ijk}) \) of format \( 2^3 \) is given by

\[
(2.21) \quad \text{Det}_q = a_{111}a_{222} - qa_{112}a_{221} - qa_{121}a_{212} + q^2a_{122}a_{211},
\]

which is not 0 at \( q = 1 \). When one imposes extra conditions that all entries \( a_{ijk} \) commute with each other then \( \text{det}_1 = 0 \). Note that Cayley’s first hyperdeterminant vanishes at odd dimension.

Consider now the algebra \( A^{\otimes 2m} \). Let \( \varphi \) be the linear map : \( \mathcal{A} \to A^{\otimes 2m} \) defined by

\[
\varphi(a_{i_1,i_2,\ldots,i_{2m}}) = a_{i_1,i_2} \otimes a_{i_3,i_4} \otimes \cdots \otimes a_{i_{2m-1},i_{2m}}.
\]

Then the following result can be easily seen.

**Proposition 2.4.** The map \( \varphi \) is an algebra homomorphism and

\[
\varphi \text{Det}^m_q(\mathcal{A}) = ([n]_q!)^{2m-1} \text{det}_q(A)^{\otimes 2m}.
\]

Moreover for any \( \sigma \in S_m \) one has that \( \varphi \sigma \in \text{Hom} \mathcal{A}, A^{\otimes 2m} \).

For a fixed \( k \in [1,m] \), let \( \eta_{i}^{(k)} = \sum_\alpha a_{i_{\alpha}}^{(k)} \otimes x_{i_{\alpha}} \), then \( \eta_{j}^{(k)} \eta_{i}^{(k)} = q^2 \eta_{i}^{(k)} \eta_{j}^{(k)} \) for \( i < j \). Consider \( \Omega_k = \sum_{i=1}^n \eta_{i}^{(k)} \), we have

\[
\wedge^n \Omega_k = [n]_q! \eta_1^{(k)} \wedge \eta_2^{(k)} \wedge \cdots \wedge \eta_n^{(k)},
\]

where \( [n]_q! = \sum_{\sigma \in S_n} q^{2l(\sigma)} \).

Writing out the exterior product, we have that

\[
\wedge^n \Omega_k = [n]_q! \text{Det}^m_q(\mathcal{A})(x_1 \wedge \cdots \wedge x_n)^{\otimes m}.
\]
Therefore
\[ \eta_1^{(k)} \wedge \eta_2^{(k)} \wedge \cdots \wedge \eta_n^{(k)} = \text{Det}_q^{[m]}(\mathcal{A})(x_1 \wedge \cdots \wedge x_n)^\otimes m. \]

On the other hand, an explicit computation gives that
\[
\eta_1^{(k)} \wedge \cdots \wedge \eta_n^{(k)} = \sum_{\sigma_1 \in S_n, i \neq k} (-q)^{\sum_{i \neq k} l(\sigma_i)} \prod_{i=1}^{m} a_{\sigma_1(i)} \cdots \hat{a}_{k(i)} \cdots a_{\sigma_m(i)}(x_1 \wedge \cdots \wedge x_n)^\otimes m.
\]

So we also have for any fixed \( k \)
\[
\text{Det}_q^{[m]}(\mathcal{A}) = \sum_{\sigma_1 \in S_n, i \neq k} (-q)^{\sum_{i \neq k} l(\sigma_i)} \prod_{j=1}^{m} a_{\sigma_1(j)} \cdots \hat{a}_{k(j)} \cdots a_{\sigma_m(j)},
\]
which can be used to define the quantum hyperdeterminant for any \( q \).

Let \( I_1, I_2, \cdots, I_m \) be \( m \) subsets of \([1,n]\) with \( |I_k| = r \). The quantum \( r \)-minor hyperdeterminants are defined as
\[
\xi(I_1, I_2, \cdots, I_m) = \sum_{\sigma_2, \cdots, \sigma_m \in S_r} (-q)^{\sum_{i=2}^{m} l(\sigma_i)} \prod_{i=1}^{r} a_{i, \sigma_2(i), \cdots, \sigma_m(i)}.
\]

For \( m, l \) we can also write the generators of \( \mathcal{A}^{[m+l]} \) as \( a_{\alpha \beta} \), where \( \alpha \in [1,n]^m \) and \( \beta \in [1,n]^l \). More general we can use any composition of \( m \) to parametrize the generators of \( \mathcal{A} \). In particular matrix realignments of \( \mathcal{A}^{[m]} \) are such examples. The following result is proved by direct computation.

**Theorem 2.5.** For any fixed \( r \), the following map is an algebra homomorphism
\[
\mathcal{A}^{[n+l]} \xrightarrow{\Delta_r} \mathcal{A}^{[n+r]} \otimes \mathcal{A}^{[r+l]},
\]
\[
\Delta_r(a_{\alpha \beta}) = \sum_{\gamma} a_{\alpha \gamma} \otimes a_{\gamma \beta},
\]
where \( \gamma \) runs through all elements of \([1,n]^r\). One has that
\[
\Delta_r(\text{Det}_q(\mathcal{A}^{[m+l]})) = \text{Det}_q(\mathcal{A}^{[m+r]}) \otimes \text{Det}_q(\mathcal{A}^{[r+l]}).
\]
In particular, \( \text{Det}_q(\mathcal{A}) \) is a group-like element in the bialgebra \( \mathcal{A}^{[2m]} \). Moreover, we have an analogous Laplace expansion:
\[
\Delta_r(\xi(I \times J)) = \sum_K \xi(I \times K) \otimes \xi(K \times J).
\]

**Example 2.6.** Let \( \{e_\alpha\} \) be the standard basis of \( V^\otimes m \) with \( \dim V = n \). Then any linear operator \( A \) of \( V^\otimes m \) is a hypermatrix given by \( A(e_\alpha) = \sum_\beta A_{\beta \alpha} e_\beta \). Then \( AB \) gives rise to a multiplication of two hypermatrixes:
\[
(AB)_{\alpha \beta} = \sum_\gamma A_{\alpha \gamma} B_{\gamma \beta}
\]
The correspondingly quantum comultiplication of \( \mathcal{A}^{[2m]} \) is exactly
\[
(2.27) \quad \Delta(\text{Det}_q(\mathcal{A})) = \text{Det}_q(\mathcal{A}) \otimes \text{Det}_q(\mathcal{A}).
\]

2.3. Laplace expansions of \( q \)-hyperdeterminants and Plücker relations. As in the classical case the quantum hyperdeterminant can also be expanded into a sum of quantum hyper-minors and the complement hyper-minors. The Laplace expansion is also proved by the exterior products. In the following we only state the results for the first folding, while it is obvious that there are similarly \( m \) ways to expand just like there are two equivalent ways to expand in rows and columns for the rectangle matrices.

The following generalizes a result of [TT].

**Proposition 2.7.** For any \( 1 \leq j_1, \cdots, j_n \leq n \), one has
\[
\sum_{\sigma_2, \cdots, \sigma_m \in S_n} (-q)^{\sum_{i=2}^{m} l(\sigma_i)} \prod_{i=1}^{m} a_{j_i, \sigma_2(i), \cdots, \sigma_m(i)}
\]
\[
\begin{cases}
0, & \text{if two } k \text{'s coincide}, \\
(-q)^{l(\pi)} \text{Det}_q^{[m]}(\mathcal{A}) & \text{if } k \text{'s are distinct},
\end{cases}
\]
where \( \pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} \).

**Proof.** In the first folding, we simply write \( x_j^{(1)} = \omega_j \). It is clear that \( \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_n} = 0 \) whenever two indices coincide. For any permutation \( \pi \) one has that
\[
\omega_{\pi_1} \wedge \omega_{\pi_2} \wedge \cdots \wedge \omega_{\pi_n} = (-q)^{l(\pi)} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n.
\]
For any composition \( (k_1 k_2 \cdots k_n) \) we can compute that
\[
\omega_{k_1} \wedge \omega_{k_2} \wedge \cdots \wedge \omega_{k_n}
\]
\[
= (-q)^{l(\pi)} \sum_{\sigma_2, \cdots, \sigma_m \in S_n} (-q)^{\sum_{i=2}^{m} l(\sigma_i)} \prod_{i=1}^{m} a_{j_i, \sigma_2(i), \cdots, \sigma_m(i)} (x_1 \wedge \cdots \wedge x_n)^{\otimes m-1}
\]
So the proposition is proved. \( \square \)

Now we discuss the Laplace expansion of quantum hyper-determinant. We first choose \( r \) indices \( i_1 < i_2 < \cdots < i_r \) from \( 1, 2, \cdots, n \) and let the remaining ones be \( i_{r+1} < i_{r+2} < \cdots < i_n \). We have
\[
(2.28) \quad \omega_{i_1} \wedge \omega_{i_2} \wedge \cdots \wedge \omega_{i_n} = (-q)^{i_1 + \cdots + i_r - \frac{(r+1)}{2}} \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n
\]
since \( \omega_i \wedge \omega_j = -q \omega_j \wedge \omega_i \) if \( i < j \). For any \( r \)-element subset \( J \), we define \( l(J) = \sum_{i \in J} i - |J|(|J| + 1)/2 \).

Now let \( I_1, \cdots, I_m \) be \( r \)-element subsets of \([1, n]\), then \( I'_i = [1, n] - I_i \) are \((n-r)\)-element subsets. We compute that
Comparing two equations, it follows that

\[(2.31)\]

\[
\sum_{I_2,\ldots,I_m \in PM} \xi(I_1,\ldots,I_m)x_1 \otimes \cdots \otimes x_t, \quad \xi(I_1,\ldots,I_m)x_1 \otimes \cdots \otimes x_I \n \]

\[
= \sum_{I_2,\ldots,I_m \in PM} (-q)^{\sum_{j=2}^m l(I_j)} \xi(I_1,\ldots,I_m)\xi(I'_1,\ldots,I'_m)(x_1 \wedge \cdots \wedge x_n)^{m-1}.
\]

Proposition 2.8. For any \(r\)-element subsets \(I_1,\ldots,I_m\) of \([1,n]\), we have

\[(2.30)\]

\[
\text{Det}_q^{[m]}(\mathcal{A}) = \sum_{I_2,\ldots,I_m \in PM} (-q)^{\sum_{j=2}^m l(I_j)} \xi(I_1,\ldots,I_m)\xi(I'_1,\ldots,I'_m)
\]

Now let \(K\) be an \(n\)-element subset of \([1,2n]\), in particular \(I = [1,n] \subseteq [1,2n]\). We define the permutation \(\sigma_K\) to be the permutation \(i_1 \cdots i_n j_1 \cdots j_n\) of \(S_{2n}\), where the first \(n\) elements of \(K\) and the last \(n\) elements of \(K' = [1,2n] - K\) are listed in the natural order.

Proposition 2.9. Let \(K_1\) be an \(n\)-element subset of \([1,2n]\) such that its elements \(i_k = k\) for \(1 \leq k \leq r < n\), \(i_{r+1} < i_{r+2} < \cdots < i_n\). Let \(K_2,\ldots,K_m\) be arbitrary \(n\)-element subsets of \([1,2n]\). Then

\[(2.31)\]

\[
\sum_{I_k} (-q)^{\sum_{\ell=2}^m l(\sigma_i)} \xi(I, K_2, \ldots, K_m)\xi(I, K'_2, \ldots, K'_m) = 0,
\]

\[(2.32)\]

\[
\sum_{I_k} (-q)^{-\sum_{\ell=2}^m l(\sigma_i)} \xi(I, K'_2, \ldots, K'_m)\xi(I, K_2, \ldots, K_m) = 0,
\]

where \(\sigma_i = \sigma_{K_i}\).

Proof. For any \(i, r \in I = [1,n]\), let

\[
\omega_i = \sum_{\alpha} a_{i} x_{\alpha} = \sum_{i_{r+1} < \cdots < i_n} a_{i_{r+1} < \cdots < i_n} x_{i_{r+1} < \cdots < i_n},
\]

\[
\omega_{ir} = \sum_{\beta} a_{i} x_{\beta} = \sum_{i_{r+1} < \cdots < i_n} a_{i_{r+1} < \cdots < i_n} x_{i_{r+1} < \cdots < i_n},
\]

and let \(\omega_{ir}' = \omega_i - \omega_{ir}\). Since the second running index is from 1 to \(r\) (\(r < n\)), one has that \(\omega_{ir} \wedge \cdots \wedge \omega_{ir} = 0\). Also it is easy to see that for any \(i < j\)

\[
\omega_{jr} \wedge \omega_{ir} = -q\omega_{ir} \wedge \omega_{jr}, \quad \omega_{jr} \wedge \omega_{ir}' = -q\omega_{ir}' \wedge \omega_{jr},
\]

\[
\omega_{ir} \wedge \omega_{jr} = -q\omega_{jr} \wedge \omega_{ir}, \quad \omega_{ir}' \wedge \omega_{jr} = -q\omega_{jr} \wedge \omega_{ir}',
\]

\[
\omega_i \wedge \omega_{jr} = -\omega_{jr} \wedge (q^{-1}\omega_{ir} + q\omega_{ir}').
\]
Now for \( r < n \), we have that
\[
\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \wedge \omega_{1r} \wedge \cdots \wedge \omega_{nr}
\]
\[= (\omega_{1r} + \omega_{1r}') \wedge \cdots \wedge (\omega_{nr} + \omega_{nr}') \wedge \omega_{1r} \wedge \cdots \wedge \omega_{nr}'
\]
\[= \omega_{1r} \wedge \cdots \wedge \omega_{nr} \wedge \omega_{1r}' \wedge \cdots \wedge \omega_{nr}' = 0
\]

On the other hand,
\[
\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n \wedge \omega_{1r} \wedge \cdots \wedge \omega_{nr}
\]
\[= \sum_{K_t} (-q)^{\sum_{i=2}^{m} l(I_i)} \xi(I, K_2, \cdots, K_m) \xi(I, K_2', \cdots, K_m') (x_1 \wedge \cdots \wedge x_{2n})^{\otimes (m-1)}
\]

Comparing Eqs. (2.33) and (2.34), we prove the first equation. The second equation follows similarly from (2.34).

PROPOSITION 2.10. Under the same hypothesis of Prop. 2.9 one has that
\[
\sum_{K_t} (-q)^{\sum_{i=2}^{m} l(I_i)} \xi(I, K_2, \cdots, K_m) \xi(I, K_2', \cdots, K_m') (x_1 \wedge \cdots \wedge x_{2n})^{\otimes (m-1)}
\]

PROOF. It is clear that for any \( i < j \)
\[
\omega_j \wedge \omega_{jr} = \omega_{jr} \wedge (-q)^{-1} \omega_{jr} + (-q) \omega_{jr}.
\]

So we have
\[
\omega_{n+1} \wedge \cdots \wedge \omega_{2n} \wedge \omega_{1r} \wedge \omega_{2r} \wedge \cdots \wedge \omega_{nr}
\]
\[= \omega_{1r} \wedge \omega_{2r} \wedge \cdots \wedge \omega_{nr} \wedge ((-q)^{-n} \omega_{n+1,r} + (-q)^{n} \omega_{n+1,r}')
\]
\[\wedge \cdots \wedge ((-q)^{-n} \omega_{2n,r} + (-q)^{n} \omega_{2n,r}')
\]
\[= (-q)^{n^2} \omega_{1r} \wedge \omega_{2r} \wedge \cdots \wedge \omega_{nr} \wedge ((-q)^{-2n} \omega_{n+1,r} + \omega_{n+1,r}') \wedge \cdots
\]
\[\wedge ((-q)^{-2n} \omega_{2n,r} + \omega_{2n,r}')
\]
\[= (-q)^{n^2-2nr} \omega_{1r} \wedge \omega_{2r} \wedge \cdots \wedge \omega_{nr} \wedge \omega_{n+1} \wedge \cdots \wedge \omega_{2n}.
\]

On the other hand, we can expand the wedge product of \( \omega ' s \) as follows.
\[
\sum_{K_t} (-q)^{\sum_{i=2}^{m} l(I_i)} \xi(I, K_2, \cdots, K_m) \xi(I, K_2', \cdots, K_m') (x_1 \wedge \cdots \wedge x_{2n})^{\otimes m-1},
\]
Comparing Eqs. (2.36-2.38), we obtain the proposition. 

\[ \omega_1' \wedge \omega_2' \wedge \cdots \wedge \omega_n' \wedge \omega_{n+1}' \wedge \cdots \wedge \omega_{2n} = \sum_{K_1} (-q)^{\sum_{j=2}^{m} n^2 - l(\sigma)} \xi(I, K_2', \cdots, K_m') \xi(I', K_2, \cdots, K_m) (x_1 \wedge \cdots \wedge x_{2n})^{\otimes m-1}, \]

2.4. Coaction of \( A \) on \( \mathcal{A} \). We can directly verify that \( \mathcal{A} \) has a left \( A \)-comodule structure given by

\[ L_G^{(k)}(a_{i\alpha}^{(k)}) = \sum_{j=1}^{n} a_{ij} \otimes a_{j\alpha}^{(k)}. \]

Moreover \( L_A \) is an algebra homomorphism, so \( \mathcal{A} \) is a \( A \)-comodule-algebra. Similarly, \( \mathcal{A} \) has a right \( A \)-comodule-algebra structure given by

\[ R_G^{(k)}(a_{i\alpha}^{(k)}) = \sum_{j=1}^{n} a_{j\alpha}^{(k)} \otimes a_{ji}. \]

Here the quantum algebra \( \text{Mat}_q(n) \) is endowed with the comultiplication \( \Delta(a_{ij}) = \sum_{l} a_{il} \otimes a_{lj} \). The following result follows easily from definition.

**Proposition 2.11.** For any \( \sigma \in S_m \)

\[ (1 \otimes \sigma)L_G^{(k)}(a_{i\alpha}^{(k)})^{-1} = L_G^{(\sigma(k))}, \]

\[ (\sigma \otimes 1)R_G^{(k)}(a_{i\alpha}^{(k)})^{-1} = R_G^{(\sigma(k))}. \]

Due to this symmetry, we will be focused on \( k = 1 \) for \( L_G \) and \( k = m \) for \( R_G \) in the remaining part. If \( \xi(I_1 \cdots I_m) \) is an \( r \)-minor hyperdeterminant, it is easy to verify that

\[ L_G(\xi(I_1 \cdots I_m)) = \sum_{|J|=r} \xi_{I_1}^J \otimes \xi(J, I_2, \cdots, I_m), \]

\[ R_G(\xi(I_1 \cdots I_m)) = \sum_{|J|=r} \xi(I_1, \cdots, I_{m-1}, J) \otimes \xi_{I_m}^J. \]

In particular, if \( I_1 = I_2 = \cdots = I_m = [1, n] \), we have that \( L_G(\text{Det}_q(\mathcal{A})) = \text{det}_q(A) \otimes \text{Det}_q(\mathcal{A}) \), and \( R_G(\text{Det}_q(\mathcal{A})) = \text{Det}_q(\mathcal{A}) \otimes \text{det}_q(A) \).

We recall the connection between the quantum group \( \text{GL}_q(n) \) and the quantum universal enveloping algebra \( U_q(\mathfrak{gl}(n)) \) or rather \( U_q(\mathfrak{sl}(n)) \) [FRT, NYM, JR]. Let \( P \) be the free \( \mathbb{Z} \)-lattice of rank \( n \) with the canonical basis \( \{ \varepsilon_1, \cdots, \varepsilon_n \} \), i.e. \( P = \bigoplus_{i=1}^{n} \mathbb{Z} \varepsilon_i \), endowed with the symmetric bilinear form \( \langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij} \). Now we define \( U_q(\mathfrak{g}) \) as the associative algebra with generators
\(e_i, f_i (1 \leq i \leq n)\) and \(q^\lambda (\lambda \in \frac{1}{2} P)\) with the following relations:

\[
\begin{align*}
q^0 &= 1, \quad q^\lambda q^\mu = q^{\lambda+\mu} \quad (\lambda, \mu \in \frac{1}{2} P), \\
q^\lambda e_k q^{-\lambda} &= q^{(\lambda, \varepsilon_k - \varepsilon_{k+1})} e_k \quad (\lambda \in \frac{1}{2} P, 1 \leq k \leq n), \\
q^\lambda f_k q^{-\lambda} &= q^{-(\lambda, \varepsilon_k - \varepsilon_{k+1})} f_k \quad (\lambda \in \frac{1}{2} P, 1 \leq k \leq n), \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{q^{\varepsilon_i - \varepsilon_{i+1}} - q^{-\varepsilon_i + \varepsilon_{i+1}}}{q - q^{-1}} \quad (1 \leq i, j < n), \\
e_i^2 e_i - (q + q^{-1}) e_i e_i + e_i e_i^2 &= 0 \quad (|i - j| = 1), \\
f_i^2 f_i - (q + q^{-1}) f_i f_i + f_i f_i^2 &= 0 \quad (|i - j| = 1), \\
e_i e_j &= e_j e_i, \quad f_i f_j = f_j f_i \quad (|i - j| > 1).
\end{align*}
\]

This algebra also has a structure of Hopf algebra with the following coproduct \(\Delta\), counit \(\varepsilon\), and antipode \(S\):

\[
\begin{align*}
\Delta(q^\lambda) &= q^\lambda \otimes q^\lambda, \quad \varepsilon(q^\lambda) = 1, \quad S(q^\lambda) = q^{-\lambda}, \\
\Delta(e_k) &= e_k \otimes q^{-(\varepsilon_k - \varepsilon_{k+1})/2} + q^{-(\varepsilon_k - \varepsilon_{k+1})/2} \otimes e_k, \\
\varepsilon(e_k) &= 0, \quad S(e_k) = q^{-1} e_k, \\
\Delta(f_k) &= f_k \otimes q^{-(\varepsilon_k - \varepsilon_{k+1})/2} + q^{-(\varepsilon_k - \varepsilon_{k+1})/2} \otimes f_k, \\
\varepsilon(f_k) &= 0, \quad S(f_k) = -q f_k.
\end{align*}
\]

For any fixed \(k\), there is a unique pairing of Hopf algebras

\[
(\ , \ ) : U_q(g) \times \text{GL}_q(n) \to \mathbb{C}
\]

satisfying the following relations

\[
\begin{align*}
q^\lambda (a_{ij}) &= \delta_{ij} q^{(\lambda, \varepsilon_i)}, \\
e_k(a_{ij}) &= \delta_{ik} \delta_{j, k+1} f_k(a_{ij}) = \delta_{i, k+1} \delta_{j, k}, \\
q^\lambda (\text{det}_q(A)^t) &= q^{(\lambda, \varepsilon_1 + \cdots + \varepsilon_n)} \quad (t \in \mathbb{Z}), \\
e_k(\text{det}_q(A)^t) &= f_k(\text{det}_q(A)^t) = 0 \quad (t \in \mathbb{Z})
\end{align*}
\]

We can regard the element of \(U_q(g)\) as a linear function of \(\text{GL}_q(n)\). If \(V\) is a right \(\text{GL}_q(n) - \text{comodule}\) (resp. left \(\text{GL}_q(n) - \text{comodule}\)) with structure map \(L_G : V \to V \otimes \text{GL}_q(n)\) (resp. \(L_G : V \to \text{GL}_q(n) \otimes V\), then \(V\) has a left (resp. right) module structure over \(U_q(g)\) defined by

\[
x v = (id \otimes x)R_G(v) \quad (\text{resp.} \ v x = (x \otimes id)L_G(v)),
\]

for all \(x \in U_q(g)\) and \(v \in V\).
The algebra $\mathcal{A}$ becomes a bimodule for $U_q(\mathfrak{g})$. We can describe the left action of its generators on $\mathcal{A}$ as follows:

\begin{align}
(2.57) \quad q^\lambda.a_{\alpha i} &= q^{(\lambda,e_i)}a_{\alpha i}, \\
(2.58) \quad e_ka_{\alpha i} &= \delta_{i,k+1}a_{\alpha,i-1}, \\
(2.59) \quad f_ka_{\alpha i} &= \delta_{i,k}a_{\alpha,i+1}.
\end{align}

Similarly the right module action is given by

\begin{align}
(2.60) \quad a_{\alpha i}q^\lambda &= q^{(\lambda,e_i)}a_{\alpha i}, \\
(2.61) \quad a_{\alpha i}e_k &= \delta_{k,i}a_{i+1,\alpha}, \\
(2.62) \quad a_{\alpha i}f_k &= \delta_{i,k+1}a_{i-1,\alpha}.
\end{align}

If $x \in U_q(\mathfrak{g})$, $\varphi, \psi \in \mathcal{A}$, and $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ then one has $x. (\varphi \psi) = \sum (x_{(1)} \varphi)(x_{(2)} \psi)$ and $(\varphi \psi).x = \sum (\varphi(x_{(1)}) \psi)(x_{(2)})$. In particular, $e_k.Det_q(\mathcal{A}) = f_k.Det_q(\mathcal{A}) = 0$, and $q^\lambda.Det_q(\mathcal{A}) = q^{(\lambda,e_1+\cdots+e_n)}Det_q(\mathcal{A})$.

### 3. Quantum hyper-Pfaffians and Generalizations

#### 3.1. Quantum hyper-Pfaffians

We define a partial order $\prec$ on $(m-1)$-element subsets of $[1,n]$. For $\alpha = (\alpha_1 \alpha_2 \cdots \alpha_{m-1}), \beta = (\beta_1 \beta_2 \cdots \beta_{m-1}) \in [1,n]^{m-1}$ with their elements arranged lexicographically, we say that $\alpha \prec \beta$ if $\alpha_1 < \beta_1, \cdots, \alpha_{m-1} < \beta_{m-1}$.

The algebra $\mathcal{B}$ of the quantum hyper-antisymmetric matrices is an associative algebra generated by $b_{\alpha \beta}$, where $\alpha, \beta \in [1,n]^{m-1}$ and $\alpha \prec \beta$, subject to the following relations:

For any fixed subset $I = I_1 \times I_2 \times \cdots \times I_{m-1} \subset [1,n]^{m-1}$ with $|I_1| = 4$, and any decomposition $I = \alpha(1) \sqcup \alpha(2) \sqcup \alpha(3) \sqcup \alpha(4)$ into $(m-1)$-element subsets such that $\alpha(1) \prec \alpha(2), \alpha(3) \prec \alpha(4)$, and $\alpha(1) \prec \alpha(3)$, one has that

\begin{equation}
\sum_{I=\cup \alpha(i)} (-q)^{l(\alpha)}b_{\alpha(1)\alpha(2)}b_{\alpha(3)\alpha(4)}
= q^{4(m-2)} \sum_{I=\cup \alpha(i)} (-q)^{-l(\alpha)}b_{\alpha(3)\alpha(4)}b_{\alpha(1)\alpha(2)},
\end{equation}

where $l(\alpha) = \sum l(\alpha_1^{(i)}, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)})$ and the sum runs through all decompositions $I = \sqcup \alpha(i)$ such that $\alpha(1) \prec \alpha(2), \alpha(3) \prec \alpha(4)$ and $\alpha(1) \prec \alpha(3)$.

Let $\Omega = \sum_{\alpha \prec \beta} b_{\alpha \beta}x_{\alpha} \wedge x_{\beta}$. We define the quantum hyper-Pfaffian $Pf_q^{[2,m-1]}$ as a special volume element:

\[
Pf_q^{[2,m-1]}(x_1 \wedge \cdots \wedge x_n)^{m-1} = \frac{1}{[n]q^1} \wedge^n \Omega
\]

Explicitly we have

\begin{equation}
Pf_q = \frac{1}{[n]q^1} \sum_{\alpha^{(1)} \prec \alpha^{(2)}} (-1)^{l(\alpha)}b_{\alpha^{(1)}\alpha^{(2)}} \cdots b_{\alpha^{(m-1)}\alpha^{(m)}}.
\end{equation}
The quantum hyper-Pfaffian generalizes the notion of the quantum Pfaffian [JZ], which has given a new definition of the quantum Pfaffian arisen from quantum invariant theory [S]. See also [JR] for a representation theoretic interpretation.

There is another way to define the quantum hyper-Pfaffian inductively as follows. First for any subset $I$ of $[1, n]^{m-1}$ of cardinality $2^{m-1}$, there is only one pair of $\alpha, \beta \in I$ such that $\alpha \prec \beta$ and $I = \alpha \sqcup \beta$. We define the hyper-Pfaffian of $I$ by $\text{Pf}^*_q(I) = [\alpha, \beta] = b_{\alpha, \beta}$. Suppose that the Pfaffian of $I$ of format $(2n-2)^{m-1}$ is defined, we then have

$$\text{Pf}^*_q([1, 2n]^{m-1}) = \sum_{\alpha < \beta} (-q)^{|\alpha|+|\beta|-3(m-1)} [\alpha, \beta] \text{Pf}^*_q([1, 2n]^{m-1}\setminus(\alpha \sqcup \beta)).$$

where the sum runs through $(m-1)$-element subsets $\alpha < \beta$ of $[1, 2]^{m-1}$ such that $\alpha_1 = 1$.

Using the inductive expression we obtain a formula of quantum hyper-Pfaffian that works for any $q$.

**Theorem 3.1.** The quantum hyper-Pfaffian is given by

$$\text{Pf}^*_q = \sum_{\alpha^{(1)} < \beta^{(1)}} (-q)^{l(\alpha, \beta)} [\alpha^{(1)}, \beta^{(1)}] \ldots [\alpha^{(n)}, \beta^{(n)}]$$

$$= \sum_{\alpha^{(1)} < \beta^{(1)}} (-q)^{l(\alpha, \beta)} b_{\alpha^{(1)}, \beta^{(1)}} \ldots b_{\alpha^{(n)}, \beta^{(n)}},$$

where the sum runs through all $(m-1)$-element subsets $\alpha^{(i)}, \beta^{(i)}$ such that $\alpha^{(i)} < \beta^{(i)}$ and $\alpha^{(1)} < \cdots < \alpha^{(n)}$. Here $l(\alpha, \beta) = \sum_k l(\alpha_k^{(1)}, \beta_k^{(1)}), \ldots, \alpha_k^{(n)}, \beta_k^{(n)}$.

The following technical lemma is proved by induction on $n$ and using Eqs. (3.1).  

**Lemma 3.2.** One has that

$$\sum_{\beta^{(1)} < \beta^{(2)}} (-q)^{|\alpha|+|\beta|-3(m-1)} [\alpha, \beta] \text{Pf}^*_q([1, 2n]^{m-1}\setminus(\alpha \sqcup \beta)).$$

$$=[n]_q \text{Pf}^*_q([1, 2n]^{m-1})$$

where the sum runs through $(m-1)$-element subsets $\alpha < \beta$ of $[1, 2]^{m-1}$.

**Theorem 3.3.** The two definitions of quantum Pfaffian are equivalent, i.e.

$$\text{Pf}^*_{q^{2^{m-1}}} = \text{Pf}^*_q \sum_{\alpha^{(i)} < \beta^{(i)}} (-q)^{l(\alpha, \beta)} b_{\alpha^{(1)}, \beta^{(1)}} \ldots b_{\alpha^{(n)}, \beta^{(n)}},$$

where the sum runs through all $(m-1)$-element subsets $\alpha^{(i)}, \beta^{(i)}$ such that $\alpha^{(i)} < \beta^{(i)}$ and $\alpha_1^{(1)} < \cdots < \alpha_1^{(n)}$. 

Proof. This is again proved by induction on $n$ with help of Lemma 3.2. The case of $n = 1$ is trivial. Then (note that $\alpha^{(i)}$ are not required to be increasing in the following)

$$
\sum_{\alpha^{(i)} \prec \beta^{(i)}} (-q)^{l(\alpha,\beta)} b_{\alpha^{(1)}\beta^{(1)}} \cdots b_{\alpha^{(n)}\beta^{(n)}}
$$

$$
= \sum_{\alpha^{(i)} \prec \beta^{(i)}} (-q)^{|\alpha^{(i)}|+|\beta^{(i)}|-3(m-1)} b_{\alpha^{(1)}\beta^{(1)}} \sum_{\alpha^{(i)} \prec \beta^{(i)}} (-q)^{l(\alpha',\beta')} b_{\alpha^{(2)}\beta^{(2)}} \cdots b_{\alpha^{(n)}\beta^{(n)}}
$$

$$
= \sum_{\alpha^{(i)} \prec \beta^{(i)}} (-q)^{|\alpha^{(i)}|+|\beta^{(i)}|-3(m-1)} b_{\alpha^{(1)}\beta^{(1)}} [n - 1]_q^4 !\text{Pf}_q^*([1, 2n]^{m-1}\setminus (\alpha^{(1)} \cup \beta^{(1)}))
$$

$$
= [n]_q^4 \text{Pf}_q^*([1, 2n]^{m-1})
$$

which is exactly the left-hand side by Lemma 3.2. □

Now we consider the two-form

$$
(3.3) \quad \Omega = \omega_1 \wedge \omega_2 + \omega_3 \wedge \omega_4 + \cdots + \omega_{2n-1} \wedge \omega_{2n},
$$

where $\omega_i = \sum a_{i\alpha} x_\alpha$ can be easily seen to satisfy the relations of the quantum exterior algebra (2.1-2.2). Taking wedge products it follows that

$$
\bigwedge^n \Omega = [n]_q^4 !\omega_1 \wedge \omega_2 \cdots \wedge \omega_{2n-1} \wedge \omega_{2n}
$$

$$
= [n]_q^4 !\text{Det}_q^m(\mathcal{A})(x_1 \wedge \cdots \wedge x_{2n})^\otimes (m-1)
$$

by definition of the quantum determinant.

On the other hand, direct computation gives that

$$
\Omega = \sum_{\alpha < \beta} b_{\alpha\beta} \otimes x_\alpha x_\beta
$$

where $b_{\alpha\beta} = \sum_{t=1}^n \xi(I_t^1, \cdots, I_{m-1}), \; I_k = \{\alpha_k, \beta_k\}$ \,(1 \leq k \leq m - 1) and $I_0^t = \{2t - 1, 2t\}$.

We now come to one of our main results.

**Theorem 3.4.** On the quantum coordinate ring $\mathcal{A}$ one has that

$$
\text{Pf}_q^{[2, m-1]}(\mathcal{B}) = \text{Det}_q^m(\mathcal{A}),
$$

where $b_{\alpha\beta} = \sum_{t=1}^n \xi(J_t^0, J_t^1, \cdots, J_t^{m-1})$, where $J_k = \{\alpha_k, \beta_k\}$ \, $k = 1, \cdots, m - 1$ and $J_0^t = \{2t - 1, 2t\}$.

Proof. Recall that $\xi(J_t^0, J_t^1, \cdots, J_t^{m-1})$ is a quantum hyper-minor, and it is enough to check that $b_{\alpha\beta}$ satisfy the relations (3.1). Let $n = 2$, and...
where \( I \) satisfies the relations (3.1).

by quantum hyper-antisymmetric matrices is an associative algebra generated relations: for any \( \alpha \) \( \alpha \) and \( \alpha \) \( \alpha \) \( I \) where the sum runs through all decompositions \( \sum \). Then the \( \alpha \geq 1 \) in Propositions 2

3.2. Generalized quantum hyper-Pfaffians. The algebra \( \mathcal{B} \) of the quantum hyper-antisymmetric matrices is an associative algebra generated by \( b_{\alpha(1)\alpha(2)\ldots\alpha(k)} \), where \( \alpha(1) \prec \alpha(2) \prec \cdots \prec \alpha(k) \) subject to the following relations: for any \( \alpha(1), \alpha(2), \ldots, \alpha(2k) \in [1, n]^{m-1} \) such that \( \alpha(1) \prec \alpha(2) \prec \cdots \prec \alpha(k) \prec \alpha(k+1) \prec \alpha(k+2) \prec \cdots \prec \alpha(2k) \) and \( \alpha(1) \prec \alpha(1) \prec \alpha(1) \prec \alpha(1) \),

\[
(-q)^{k^2} \sum_{I=1}^{\mathcal{B}(i)} (-q)^{l(\alpha)} b_{\alpha(1)\ldots\alpha(k)} b_{\alpha(k+1)\ldots\alpha(2k)}
\]

(3.4)

where the sum runs through all decompositions \( I = I_1 \times I_2 \times \cdots \times I_{m-1} = \cup \mathcal{B}(i) \) such that \( \alpha(1) \prec \alpha(2) \prec \cdots \prec \alpha(k) \prec \alpha(k+1) \prec \alpha(k+2) \prec \cdots \prec \alpha(2k) \) and \( \alpha(1) \prec \alpha(1) \prec \alpha(1) \prec \alpha(1) \), \( l(\alpha') = \sum_i l(\alpha(i)) \), \( \alpha(i) \), \( l(\alpha) = \sum_i l(\alpha(i)) \), \( \alpha(i) \).

We define the quantum hyper-Pfaffian inductively as follows. Let

\[
[a^{(1)}, a^{(2)}, \ldots, a^{(k)}] = b_{a^{(1)}a^{(2)}\ldots a^{(k)}}.
\]

Then the \( \alpha \)th order \( q \)-Pfaffian is defined to be

\[
Pf_q(\mathcal{B}) = \sum_{\alpha(1) \prec \alpha(2) \prec \cdots \prec \alpha(k), \alpha(1) = 1} (-q)^{\sum_{i=1}^{km} k \frac{m}{2} b_{a^{(1)}a^{(2)}\ldots a^{(k)}}}
\times \{[1, kn]^{m-1} \setminus \{a^{(1)} \cup \alpha^{(2)} \cup \cdots \cup \alpha^{(k)}\}}.
\]

Similar to section 2, we have the following results.

**Proposition 3.5.** One has that

\[
Pf_q(\mathcal{B}) = \sum_{\alpha} (-q)^{l(\alpha)} b_{a^{(1)}a^{(2)}\ldots a^{(k)}} \cdots b_{a^{(k(n-1)+1)}a^{(kn)}}
\]
where the sum runs through all \((m - 1)\)-element subsets \(\alpha^{(km+1)} < \alpha^{(km+2)}< \cdots < \alpha^{(k(m+1))}\) such that \(\alpha^{(1)}_1 < \alpha^{(k+1)}_1 < \cdots < \alpha^{(k(m-1)+1)}_1\).

**Lemma 3.6.** One has that
\[
\sum_{\alpha^{(1)} < \alpha^{(2)} < \cdots < \alpha^{(k)}} (-q)^{\sum_{t=1}^{km} i_t - \frac{k(k+1)(m-1)}{2}} b_{\alpha^{(1)}_1 \alpha^{(2)}_2 \cdots \alpha^{(k)}_k} \times \left[ [1, km]^{m-1} \{ \alpha^{(1)}_1 \sqcup \alpha^{(2)}_1 \sqcup \cdots \sqcup \alpha^{(k)}_1 \} \right] = [n]_{q^k} \text{Pf}_q(\mathcal{B}).
\]

**Theorem 3.7.** We have that
\[
\sum_{\alpha^{(km+1)} < \alpha^{(km+2)} < \cdots < \alpha^{(k(m+1))}} (-q)^{s(\alpha)} b_{\alpha^{(1)}_1 \alpha^{(2)}_2 \cdots \alpha^{(k)}_k} \cdot \cdots \cdot b_{\alpha^{((k(n-1)+1)} \cdots \alpha^{(kn)}_k} = [n]_{q^k} \text{Pf}_q(\mathcal{B}).
\]

The following theorem shows that the quantum hyper-Pfaffian is a volume element.

**Theorem 3.8.** Let
\[
\Omega = \sum_{\alpha} b_{\alpha^{(1)}_1 \alpha^{(2)}_2 \cdots \alpha^{(k)}_k} x_{\alpha^{(1)}_1} x_{\alpha^{(2)}_2} \cdots x_{\alpha^{(k)}_k}
\]
then one has
\[
\land^n \Omega = [n]_{q^k} \text{Pf}_q^{[k,m]}(\mathcal{B})(x_1 \land \cdots \land x_{km}) \otimes (m-1).
\]

In the following we assume that \(q^{s(\xi)} = 1\), and \(k = 2s\) for \(s \in \mathbb{Z}^+\). Similar to the case of Pfaffians, a hyperdeterminant can be expressed as \(\text{Pf}(\mathcal{B})^{[k,m-1]}\), this is proved by the following theorem.

**Theorem 3.9.** If \(b_{\alpha^{(1)}_1 \alpha^{(2)}_2 \cdots \alpha^{(k)}_k} = \sum_{t=1}^n \xi(I^t_0, I^t_1, \cdots, I^t_{m-1})\) then
\[
\text{Pf}_q^{[k,m-1]}(\mathcal{B}) = \text{Det}_q^{[m]}(\mathcal{A}),
\]
where \(J^t_i = \{k(t-1) + 1, k(t-1) + 2, \cdots, kt\}\) and \(J_i = \{\alpha^1_i, \alpha^2_i, \cdots, \alpha^k_i\}\) for \(i = 1, \cdots, m - 1\).

**Proof.** We consider the special \(m\)-form \(\Omega = \omega_1 \land \omega_2 \land \cdots \land \omega_k + \omega_{k+1} \land \omega_{k+2} \land \cdots \land \omega_{2k} + \cdots + \omega_{k(n-1)+1} \land \cdots \land \omega_{kn}\), where \(\omega_i = \sum a_{i_1,i_2,\cdots,i_m} x_{i_1} \otimes \cdots \otimes x_{i_m}\).

For this \(\Omega\),
\[
\Omega^n = \sum_{\sigma \in S(n)} (((-q)^{k(\sigma)}) \omega_1 \land \omega_2 \land \cdots \land \omega_{kn}) = [n]_{q^k} \text{Det}_q^{[m]}(\mathcal{A})(x_1 \land \cdots \land x_{kn}) \otimes (m-1)
\]

On the other hand,
\[
\Omega = \sum_{\alpha} b_{\alpha^{(1)}_1 \alpha^{(2)}_2 \cdots \alpha^{(k)}_k} x_{\alpha^{(1)}_1} x_{\alpha^{(2)}_2} \cdots x_{\alpha^{(k)}_k}
\]
where $b_{\alpha_1(1)\alpha_2(2)\cdots\alpha_k(k)} = \sum_{t=1}^{n} \xi(I_{t0}, I_{1}, \cdots, I_{m-1})$.

Similar to theorem (3.4), it can be verified that $b_{\alpha_1(1)\alpha_2(2)\cdots\alpha_k(k)}$ satisfy the relation (3.4). Therefore we have that

$$\wedge^{n} \Omega = [n]_{q^2}! Pf_{q}\left[ k,m-1 \right]Q(B)\left( x_1 \wedge \cdots \wedge x_{kn} \right)^{\otimes m-1}. \tag{3.6}$$

Comparing Eq. (3.5) and Eq. (3.6), we deduce that

$$Pf_{q}\left[ k,m-1 \right]Q(A) = Det_{q}[m](\mathcal{A}). \tag{4.1}$$

We remark that the quantum hyper-Pfaffian is different from the hyper-Pfaffian studied in [JZ], while the latter generalized and deformed Barvinok’s hyperpfaffian [B], see also [LT].

4. Conclusion and discussion

We have defined a notion of quantum hypermatrices and introduce the quantum hyperdeterminant and quantum hyper-Pfaffians. The quantum hyperdeterminant has quantized Cayley’s first hyperdeterminant and provided a quantum invariant for the space of the quantum matrices. In particular we have obtained the formula:

$$\det_q(B \circ_k A) = \det_q(B)\det_q(A). \tag{4.1}$$

We also proved that the space $\mathcal{A}$ is a co-bimodule for a tensor product of the quantum coordinate ring $\text{Mat}_q(n)$:

$$\mathcal{A} \rightarrow \text{Mat}_q(n)^{\otimes m} \otimes \mathcal{A} \otimes \text{Mat}_q(n)^{\otimes m}. \tag{4.2}$$

Using the dual co-algebra structure, we also obtain that $\mathcal{A}$ is a bi-module for $U_q(sl(n))^{\otimes m}$:

$$U_q(sl(n))^{\otimes m} \otimes \mathcal{A} \otimes U_q(sl(n))^{\otimes m} \rightarrow \mathcal{A}. \tag{4.3}$$

Using this map we have shown that the image of $\det_q$ is exactly $\det_q^{\otimes m}$ in the space $\text{GL}_q(n)^{\otimes m}$. However our hyperdeterminant $\det_q$ is not a central element in $\mathcal{A}$. This means that the algebra $\mathcal{A}$ is a very general uplift of the algebra $\text{GL}_q(n)^{\otimes m}$.

We conjecture that there exists an intermediate quotient algebra $\overline{\mathcal{A}} = \mathcal{A}/I$, where the ideal $I$ contains some relations that make the quantum hyperdeterminant $\det_q$ a central element. This ideal is trivial in the case of 2-dimensional quantum matrices (i.e. $m = 1$ case). We note that quantum analogs of Cayley’s other hyperdeterminants may provide a solution, as indicated by the classical case (cf. [GKZ]). On the other hand, this also means that we have obtained some optimal relations to define both quantum hyperdeterminant and the quantum hyper-Pfaffian.

For $m = 4$, if one uses the quantum Weyl algebra to define the quadratic relations for quantum hypermatrices, there are more relations than our current approach of using the quantum exterior algebra. However, it seems that the quantum Weyl algebra is also not enough to make $\det_q$ central.
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