Shock-free Solutions of the Compressible Euler Equations

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Abstract

We study the structure of shock-free solutions of the compressible Euler equations with large data. We describe conditions under which the Rarefactive/Compressive character of solutions changes, and conditions under which the vacuum is formed asymptotically. We present several new examples of shock-free solutions, which demonstrate a large variety of behaviors.

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1 Introduction

We consider hyperbolic systems of conservation laws in one space dimension,

\[ u_t + f(u)_x = 0, \quad u \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n. \]  (1.1)

It is well known that, due to the absence of dissipative effects, classical \( C^1 \) solutions cannot be sustained, and generically, gradients blow up in finite time. This is a physical effect which is manifested by the development of shock waves, at which the conserved variable \( u \) becomes discontinuous. Once a shock wave forms, one must study weak solutions, and the analysis becomes much more difficult.

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Breakdown of classical solutions for scalar equations is classical, going back to Bethe and Hopf \[12\], and was resolved for $2 \times 2$ systems by Lax \[14, 15\]. These results state that in the presence of genuine nonlinearity, nontrivial small data leads to gradient blowup in finite time. For larger systems, similar results are available, again provided the initial data is small \[10, 18, 21\]. For $2 \times 2$ systems, a pair of Riccati-type equations of the form $\dot{w} = w^2$ which blow up in finite time can be derived. In \[10\], John derives an analogous system of equations for the gradient variables, also with quadratic inhomogeneous part. These quadratic terms represent interactions between different nonlinear fields, including self-interaction terms. For small data, after an initial period of nonlinear wave interaction, the solution is essentially decoupled into $n$ waves, each propagating with its own wavespeed. Each of these waves can be approximately treated as scalar, and so breaks down in finite time. When the data is large, some results are available for particular systems \[4, 3, 5\], but in general the breakdown of solutions with large data remains an open problem. For results in higher dimensional settings, see \[24, 25, 26\].

In this paper, we study the $3 \times 3$ system of Euler equations of gas dynamics, which has one linearly degenerate field. The equations, representing conservation of mass, momentum and energy, respectively, are

$$
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x &= 0, \\
\left(\frac{1}{2} \rho u^2 + \rho e\right)_t + (\frac{1}{2} \rho u^3 + u p)_x &= 0.
\end{align*}
$$

We use a Lagrangian frame, co-moving with the fluid, given by $x = \int \rho \, dx'$. The equations become

$$
\begin{align*}
\tau_t - u_x &= 0, \\
u_t + p_x &= 0, \\
\left(\frac{1}{2} u^2 + e\right)_t + (u p)_x &= 0,
\end{align*}
$$

and these are equivalent to \[1.2\] \[3, 31\]. Here $\tau = 1/\rho$ is the specific volume, $p$ is pressure, $u$ is fluid velocity, and $e$ is the specific internal energy. By the Second Law of Thermodynamics, for classical solutions, the third (energy) equation can be replaced by the entropy equation,

$$
S_t = 0,
$$

see \[6\]. The system is closed by specifying a constitutive law; for convenience, we consider a polytropic ideal $\gamma$-law gas. For classical solutions, in regions of
constant entropy, the first two equations of (1.3) close, forming the p-system of isentropic gas dynamics [27].

In [4], the first author independently derived a set of Riccati type equations for the gradients of sound waves, used these to give a consistent definition of the rarefactive and compressive character (R/C character) of the nonlinear sound waves, and gave conditions which guarantee shock formation, for data of arbitrary size. By (1.4), the entropy is a linearly degenerate contact field, stationary for $C^1$ solutions. These equations are analogous to those of [10, 20], and can be generalized to other physical systems [5].

The presence of a stationary entropy profile of moderate strength means that nonlinear interactions between fields occur on the same scale as self-interactions, and can lead to surprising behavior [28]. In particular, a sound wave can change its R/C character across a contact discontinuity (entropy jump). In [4, 5], the entropy field is assumed to be continuous, and conditions which guarantee gradient blowup are given. In this paper, we study the growth of gradients of shock-free solutions with a varying entropy profile that can include jump discontinuities, and we consider data having large amplitudes. In particular, we will demonstrate the consistency of results of [4] for $C^2$ entropy profiles and [28] for contact discontinuities.

For our calculations, it is convenient to consider shock-free solutions, in which the velocity $u$ and pressure $p$ are $C^2$, while the entropy $S(x)$ is $C^2$ except at finitely many points, and one-sided limits of $S$ and $S_x$ exist everywhere. If the initial data are shock-free, the solution remains shock-free until the first derivatives of $u$ and $p$ blow up [23, 7]. The following theorem generalizes results of [28, 4].

**Theorem 1.** If the entropy is non-decreasing, a forward $R$ (resp. $C$) can change its character only if it crosses backward $C$ waves (resp. $R$ waves); a backward $C$ (resp. $R$) can change only if it crosses a forward $C$ (resp. $R$). Symmetric results hold if $m(x)$ is non-increasing, provided the character of the opposite simple wave is reversed.

We say that a solution is **eventually noninteracting** if all wave interactions end in finite time: that is, for $t$ large enough, the solution consists of three regions, defined by outgoing backward, stationary and forward waves respectively. The forward and backward waves are either rarefactions or a single shock; however the profiles of the rarefactions and contact field need not be explicitly known. A Riemann solution is the most obvious example of a solution that is eventually noninteracting, but we will present several other examples.
We define an **asymptotic vacuum** as a characteristic with a vertical asymptote in the \((x, t)\)-plane. Since the characteristic is determined by the equation \(\frac{dx}{dt} = \pm c\), we necessarily have \(\rho \to 0\) along this characteristic. Hence, along this characteristic, the \(\rho, c\) and \(p\) all vanish as \(t \to \infty\). Note that the uniqueness theorem for ODEs implies that the vacuum is not taken on in finite time unless it is present in the initial data, see [33, 16].

According to [28], it is possible to carefully choose an oscillating entropy profile which supports shock-free, space- and time-periodic solutions to the Euler equations. Thus we do not expect to be able to prove definitive gradient blowup results if the entropy is non-monotonic, so we largely restrict our attention to monotonic entropy profiles.

**Theorem 2.** Assume that the (variation of the) data is compactly supported and that the entropy is monotone. Then a globally defined shock-free solution is either eventually noninteracting or contains an asymptotic vacuum.

In other words, in a monotone entropy field, if a solution continues to interact for arbitrarily long times and does not contain an asymptotic vacuum (so in particular (1.5) fails), then a shock necessarily forms in finite time.

In some cases, we can predict the appearance of an asymptotic vacuum by a condition on the initial data: an asymptotic vacuum can occur in the solution only if the **Vacuum Condition** holds, namely

\[
u_0(\infty) - \nu_0(-\infty) \geq m(-\infty) z_0(-\infty) + m(\infty) z_0(\infty). \quad (1.5)
\]

Here \(m\) and \(z\) are canonical thermodynamic variables which are nonlinear transformations of the entropy and density, respectively, given in (2.2), (2.3) below. We show that if the vacuum condition (1.5) holds, then there are no eventually noninteracting shock-free solutions. Note that this vacuum condition is identical to the condition that predicts the existence of the vacuum in the solution of the Riemann problem [27], and is asymptotically equivalent to the condition for an embedded vacuum in general initial data [33].

We then provide several examples of shock-free solutions. In particular, if the entropy profile is piecewise constant and increasing and the data is rarefactive, we show that the boundary of the interaction region is necessarily a forward characteristic.

We also analyze a case in which the entropy is nonmonotonic. The entropy consists of two contacts of equal strength. By restricting data to be between the contacts, we show that shock-free solutions are everywhere rarefactive. Moreover, in this case we analyze the long-time behavior of the solution.
Theorem 3. There are three possible long-time behaviors: asymptotic vacuum; infinitely reflected waves which converge to the vacuum state; and infinitely reflected waves with non-vanishing wavespeed and density. The long-time behavior is determined explicitly by a bifurcation parameter $\zeta$ determined by the initial data.

In particular, we obtain interacting solutions which asymptotically approach vacuum at the specific rate

$$s - r = O(1) (1 + t)^{-\frac{\gamma-1}{\gamma+1}}, \quad \text{or} \quad \tau = O(1) (1 + t)^{2/(1+\gamma)},$$

where $s$ and $r$ are the Riemann invariants. As a consequence, we note that the presence of an asymptotic vacuum is a stronger condition than development of vacuum as $t \to \infty$, and in particular, (1.5) can hold even though no asymptotic vacuum is present.

We contrast the cases of monotonic and nonmonotonic entropies: when the entropy is monotonic, shock-free solutions must be eventually noninteracting or contain an asymptotic vacuum, while if the entropy is nonmonotonic, waves can be reflected between the contacts infinitely often.

Finally, we briefly discuss solutions containing a single shock. We treat this as a free boundary problem with specific conditions on either side of the shock, and discuss global solutions of this type.

2 Equations and Wave Curves

We restrict our attention to a polytropic ideal gas, with equation of state

$$e = c_v T = \frac{p \tau}{\gamma - 1} \quad \text{and} \quad p \tau = RT,$$

so that

$$p = Ke^{S/c_v} \tau^{-\gamma}. \quad (2.1)$$

Here $S$ is the entropy, $T$ is the temperature, $R$, $K$, $c_v$ are positive constants, and $\gamma > 1$ is the adiabatic gas constant, c.f. [6]. The Lagrangian sound speed is given by

$$c = \sqrt{-p \tau} = \sqrt{K \gamma \tau^{-(\gamma+1)/2} e^{S/2c_v}}.$$

We use the coordinates of [28]: that is, we define variables $m$ and $z$ so that

$$c = mz^d, \quad p = \int mc \, dz \quad \text{and} \quad \tau = -\int \frac{m}{c} \, dz, \quad (2.2)$$
for some $d$. In fact, it suffices to take $d = \frac{\gamma + 1}{\gamma - 1}$ and set

$$m = C_m e^{S/2e_v} \quad \text{and} \quad z = C_z \tau^{-(\gamma - 1)/2},$$

with constants

$$C_z = (d - 1)^{\frac{1}{1 - d}} \quad \text{and} \quad C_m = \sqrt{K \gamma C_z^{-d}},$$

and it is easy to check that (2.2) becomes

$$c = m z^d, \quad p = \frac{m^2 z^{1+d}}{d + 1}, \quad \text{and} \quad \tau = \frac{z^{1-d}}{d - 1}. \quad (2.3)$$

We shall continue to refer to $m$ as the entropy variable.

In these coordinates, for $C^1$ solutions, (1.3) are equivalent to

$$z_t + \frac{c}{m} u_x = 0,$$
$$u_t + mcz_x + 2\frac{p}{m} m_x = 0,$$
$$m_t = 0,$$  \quad (2.4)

the last equation being (1.4) replacing the energy equation, valid for smooth solutions.

### 2.1 Hugoniot curves

We describe the shock waves (and contact discontinuities) by the Rankine-Hugoniot conditions,

$$\xi [\tau] = -[u],$$
$$\xi [u] = [p],$$
$$\xi \left[ \frac{1}{2} u^2 + e \right] = [up], \quad (2.5)$$

where $\xi$ is the shock speed, and the brackets denote the change across the discontinuity, as usual. Using the identity $[a b] = \overline{a} [b] + [a] \overline{b}$, where $\overline{7} = (q_0 + q_1)/2$, the third equation of (2.5) simplifies to

$$\xi \left( [e] + \overline{p} [\tau] \right) = 0. \quad (2.6)$$

Shock waves correspond to solutions of (2.5) with $\xi \neq 0$: to describe these fully, we solve (2.6) and use (2.5) to determine $\xi$ and $[u]$. Using

$$e = \frac{p \tau}{\gamma - 1} \quad \text{and} \quad \gamma = \frac{d + 1}{d - 1},$$
equation (2.6) becomes
\[(d + 1) \overline{p} \tau + (d - 1) |p| \overline{\tau} = 0.\]

We now use (2.3) and solve to get
\[
\left(\frac{m_1}{m_0}\right)^2 Z^{d+1} = \frac{d Z^{d-1} - 1}{d - Z^{d-1}}, \quad \text{where} \quad Z = \frac{z_1}{z_0},
\]
which we write as
\[
\frac{m_1}{m_0} = f\left(\frac{z_1}{z_0}\right),
\]
having defined
\[
f(Z) := \sqrt{\frac{Z^{d-1} - \frac{1}{d}}{Z^{d+1} - \frac{1}{d} Z^{2d-1}}} \quad \text{for} \quad d^{\frac{1}{d-3}} < Z < d^{\frac{1}{d-1}}.
\]

We now use (2.5) to describe \(u\) and \(\xi\), namely
\[
[u] = \pm \sqrt{[p] \tau} \quad \text{and} \quad \xi = \pm \sqrt{[p] \tau}.
\]

Substituting and simplifying as above, we obtain
\[
u_1 - u_0 = \pm m_0 z_0 g(Z) \quad \text{and} \quad \xi = \pm m_0 z_0^d h(Z),
\]
where \(g\) and \(h\) are respectively defined by
\[
g(Z) := \frac{1}{\sqrt{d^2 - 1}} \sqrt{\left(f^2(Z) Z^{1+d} - 1\right)} \left(1 - Z^{1-d}\right),
\]
\[
h(Z) := \sqrt{\frac{d - 1}{d + 1}} \sqrt{\frac{f^2(Z) Z^{1+d} - 1}{1 - Z^{1-d}}}.
\]

Here we have labelled the states on opposite sides of the shock with the subscripts 0 and 1, without explicitly referring to the left and right states, and these can be interchanged in this description. We choose the signs in (2.8) and the value of \(Z\) by referring to Lax’s entropy condition: that is, we require that the (absolute) wavespeed be larger behind the shock, which implies that \(Z > 1\) if \(z_1\) is behind the shock, and \(Z < 1\) if \(z_1\) is ahead of the shock; if \(z_1\) is the right state, say, we require \(Z > 1\) for a backward shock and \(Z < 1\) for a forward shock. In this way we can describe both forward and backward shocks with similar equations, as in [34]. For a further analysis of the structure of the wave curves and the functions \(f, g\) and \(h\), see [3].
2.2 Stationary Solutions

The contact discontinuities provide another class of weak solutions, being solutions of (2.5) with wavespeed $\xi = 0$. It is clear that contacts should satisfy $[u] = [p] = 0$, while the entropy variable $m$ can vary: in this case, by (2.3) we have

$$m_0^2 z_0^{1+d} = m_1^2 z_1^{1+d},$$

which in turn determines the jump in $z$. It is well-known that the entropy is linearly degenerate and these waves are contacts which are stationary in a Lagrangian frame [27]. More generally, we can obtain stationary waves by allowing $m$ (or $z$) to vary while fixing $u$ and $p$ as constants: this is easily seen by direct substitution into equations (1.3). That is, any time-independent states $(z, u, m)$ given in some region by

$$m = m(x), \quad u = U, \quad m(x)^2 z(x)^{1+d} = P,$$

with $U$ and $P$ constants, form a stationary wave solution to (1.3).

2.3 Isentropic Flow

A simpler $2 \times 2$ system, known as isentropic flow, is obtained when the entropy $S$ (or $m$) is taken to be identically constant, and the third (energy) equation of (1.3) is dropped, to give

$$\tau_t - u_x = 0,$$
$$u_t + p_x = 0,$$

with $p = p(\tau)$, also known as the $p$-system.

The $p$-system is considerably simpler than the full Euler system because the equations weakly decouple. Indeed, for $C^1$ solutions, (2.4) becomes

$$z_t + \frac{c}{m} u_x = 0,$$
$$u_t + m c z_x = 0,$$

so that the Riemann invariants, given by

$$r = u - mz \quad \text{and} \quad s = u + mz,$$

respectively, satisfy

$$r' = 0 \quad \text{and} \quad s' = 0.$$

Here $'$ and $'$ denote differentiation along backward and forward characteristics, respectively,

$$'= \partial_t - c\partial_x \quad \text{and} \quad '= \partial_t + c\partial_x.$$
3 \( R/C \) Character of Solutions

We briefly recall results from the authors’ previous papers \[28, 4\] describing the local rarefactive and compressive nature of solutions.

In a constant entropy field For isentropic flow (2.11), the Riemann invariants \( r \) and \( s \) are constant along characteristics, and simple waves are described by

\[
m_r = m_l, \quad u_r - u_l = m_l(z_a - z_b),
\]

where the subscripts \( l, r, a, b \) denote the states to the left, right, ahead of and behind the wave, respectively. Noninteracting simple waves are classified as rarefactive or compressive according to whether the characteristics diverge or converge, respectively. For isentropic flow, this is determined by the profile of the Riemann invariants. Since in the full system, entropy is stationary before shock formation, the following results extend immediately to isentropic regions in full \(3 \times 3\) flows:

**Definition 3.1.** [28] In a constant entropy field, the local \( R/C \) character of a \( C^1 \) solution is:

- **Forward R** iff \( s_t < 0 \),
- **Forward C** iff \( s_t > 0 \),
- **Backward R** iff \( r_t > 0 \),
- **Backward C** iff \( r_t < 0 \).

**Lemma 3.2.** [34] In a constant entropy field, if an interacting solution is \( C^2 \), the \( R/C \) character of each wave is preserved along characteristics.

**Lemma 3.3.** [14, 35] Assume the initial data \( r_0(x) \) and \( s_0(x) \) of \( r \) and \( s \) are \( C^2 \), and the initial entropy is constant. If \(-s_t\) or \(r_t\) is negative somewhere in the initial data, then \(|u_x|\) and/or \(|p_x|\) blow up in finite time.

In [14], Lemma 3.3 relies on an *a priori* assumption that the solution stays away from vacuum; this assumption is removed in [35]. Thus gradients will blow up (shocks will form) in finite time if and only if there are compressive waves in the data; see also [16].

At a contact discontinuity Simple waves preserve their character in isentropic regions, as the (derivatives of) Riemann invariants propagate with the wave. However, waves may change type when crossing a contact discontinuity which separates different isentropic regions. According to (2.19), the change in variables is

\[
u_r = u_l \quad \text{and} \quad m_r z_r = m_l z_l Q,
\]

(3.2)
where we have set

\[ Q = \left( \frac{m_r}{m_l} \right)^{\frac{d-1}{d+1}}, \quad \text{so also} \quad \frac{z_r}{z_l} = Q^{\frac{2}{d-1}}. \quad (3.3) \]

with corresponding changes in the derivatives of Riemann invariants \( r_t \) and \( s_t \) by (2.13). It follows that if forward and backward simple waves cross the jump simultaneously, then one of the waves could change character.

Following [3], we will call the contact discontinuity a 1-contact if the entropy decreases, \( m_l > m_r \) (so \( Q < 1 \)), and a 3-contact if \( m_l < m_r \) (\( Q > 1 \)).

**Lemma 3.4.** [28] A nonlinear wave changes its R/C value at a contact discontinuity when and only when one of the following inequalities hold:

\[
\begin{align*}
R_{in}^- & \rightarrow C_{out}^- \iff Qm_l \dot{z}_l < \dot{u}_l < m_l \dot{z}_l, \\
C_{in}^- & \rightarrow R_{out}^- \iff m_l \dot{z}_l < \dot{u}_l < Qm_l \dot{z}_l, \\
R_{in}^+ & \rightarrow C_{out}^+ \iff -Qm_l \dot{z}_l < \dot{u}_l < -m_l \dot{z}_l, \\
C_{in}^+ & \rightarrow R_{out}^+ \iff -m_l \dot{z}_l < \dot{u}_l < -Qm_l \dot{z}_l,
\end{align*}
\]

where \( \dot{y} := y_t \) denotes the time derivative, the subscripts denote incoming and outgoing waves (or the side of the jump), and the superscripts indicate the direction of the wave: \(-\) is backward, \( +\) is forward.

Note that the conditions of the lemma are mutually exclusive, so only one wave can change its character at any time. Moreover, for a fixed jump, a change in one wave is possible only if the opposite wave has the right character.

**Corollary 3.5.** At a 3-contact (\( Q > 1 \)), the backward wave can change from \( R \) to \( C \) (resp. \( C \) to \( R \)) only if both the incoming and outgoing forward waves are \( R \) (resp. \( C \)); the forward wave can change from \( C \) to \( R \) (resp. \( R \) to \( C \)) only if both backward waves are \( R \) (resp. \( C \)). Similar conclusions hold for a 1-contact, but the character of the incoming opposite wave changes.

For later use, we record the change of Riemann invariants across the jump: it follows easily from (2.13), (3.2) that

\[ r_r = \frac{1+Q}{2} r_l + \frac{1-Q}{2} s_l, \quad \text{and} \quad s_r = \frac{1-Q}{2} r_l + \frac{1+Q}{2} s_l. \quad (3.4) \]

**For non-isentropic smooth solutions** In [4], the first author provides an appropriate definition of the R/C character for the full (non-isentropic) Euler equations. Recalling (2.15), define the quantities

\[
\begin{align*}
\alpha := -p'/c^2 &= u_x + mz_x + \frac{2}{d+1} m_x z \quad \text{and} \\
\beta := -p'/c^2 &= u_x - mz_x - \frac{2}{d+1} m_x z;
\end{align*}
\]

(3.5)
these are multiples of derivatives of Riemann invariants.

**Definition 3.6.** [4] [5] The local \( R/C \) character in a \( C^1 \) solution is

- **Forward** \( R \) iff \( \alpha > 0 \),
- **Forward** \( C \) iff \( \alpha < 0 \),
- **Backward** \( R \) iff \( \beta > 0 \),
- **Backward** \( C \) iff \( \beta < 0 \).

For \( C^1 \) solutions, it is easy to show using (2.3), (2.4) and (2.13), that

\[
s_t + c\alpha = 0, \quad r_t - c\beta = 0,
\]

so this definition agrees with and extends Definition 3.1.

The following Riccati type equations describe the growth of gradients:

**Lemma 3.7.** [4] If the solution of (1.3) is \( C^2 \), then

\[
\alpha' = k_1(k_2(3\alpha + \beta) + \alpha\beta - \alpha^2) \quad \text{and} \quad \\
\beta' = k_1(-k_2(\alpha + 3\beta) + \alpha\beta - \beta^2),
\]

where

\[
k_1 := \frac{\gamma + 1}{2(\gamma - 1)} \frac{1}{z^{\frac{2}{\gamma + 1}}} \quad \text{and} \quad k_2 := \frac{\gamma - 1}{\gamma(\gamma + 1)} z_m x.
\]

Moreover,

\[
|\alpha| \text{ or } |\beta| \to \infty \quad \text{iff} \quad |u_x| \text{ or } |p_x| \to \infty.
\]

We note that these equations are similar to those derived by F. John in [10], but were independently derived from a different point of view by the first author in [4]. Condition (3.8) coincides exactly with formation of a shock wave.

### 3.1 Global \( R/C \) Structure

In a constant entropy field, the \( R/C \) character of waves is preserved, but in a varying entropy field, it may change. We describe conditions under which the \( R/C \) character of an interacting wave changes. Essentially, the only way a wave can change is if it is nonlinearly superposed with reflections from the interaction of opposite waves with the background entropy field; this is consistent with the changes across a contact described above.
Theorem 1. Suppose a solution satisfies the Shock-free Condition. If the entropy \( m(x) \) (i.e. \( S(x) \)) is non-decreasing, a forward \( R \) (resp. \( C \)) can change its character only if it crosses backward \( C \) waves (resp. \( R \) waves); a backward \( C \) (resp. \( R \)) can change only if it crosses a forward \( C \) (resp. \( R \)). Symmetric results hold if \( m(x) \) is non-increasing, provided the character of the opposite simple wave is reversed.

It follows from the proof that this statement includes changes of type from zero strength waves to \( C \) or \( R \).

Proof. First suppose the entropy is \( C^2 \), and consider a forward rarefaction. We consider the evolution of \( \alpha \) along the forward characteristic, propagating through a field of non-decreasing entropy. Also, suppose that \( \beta \geq 0 \) along this characteristic, so that our forward wave crosses no backward compressions. Let \( \Gamma \) denote the forward characteristic, parameterized by \( t_0 \leq t \).

We prove by contradiction that \( \alpha(t) > 0 \) on \( \Gamma \). Suppose not, and let \( t_* \) be the first time for which \( \alpha(t) = 0 \) along \( \Gamma \). Since \( \beta \geq 0 \) along \( \Gamma \), \( k_1 \) and \( k_2 \) are non-negative, and \( \alpha(t) > 0 \) for \( t_0 < t < t_* \), by (3.6) we have

\[
\alpha' \geq 3k_1k_2\alpha - k_1\alpha^2 \quad \text{for} \quad t_0 < t < t_*.
\]

(3.9)

Denote

\[
\tilde{\alpha}(t) := e^{-\int_{t_0}^{t} 3k_1k_2 dt} \quad \text{and} \quad k_+ := k_1 e^{\int_{t_0}^{t} 3k_1k_2 dt},
\]

where the integral is along \( \Gamma \). Using the integrating factor \( e^{-\int_{t_0}^{t} 3k_1k_2 dt} \), (3.9) yields

\[
\tilde{\alpha}' \geq -k_+\tilde{\alpha}^2.
\]

Dividing by \( \tilde{\alpha}^2 \) and integrating along \( \Gamma \), we get

\[
\frac{1}{\tilde{\alpha}(t)} \leq \int_{t_0}^{t} k_+ dt + \frac{1}{\tilde{\alpha}(t_0)}.
\]

Since \( \tilde{\alpha}(t_0) > 0 \) and \( \tilde{\alpha}(t_*) = 0 \), we must have

\[
\lim_{t \to t_*^-} \int_{t_0}^{t} k_+ dt = +\infty,
\]

which contradicts the Shock-free condition. We conclude that \( \alpha > 0 \) on \( \Gamma \), with the lower bound

\[
\alpha(t) \geq \frac{e^{\int_{t_0}^{t} 3k_1k_2 dt} \alpha(t_0)}{1 + \alpha(t_0) \int_{t_0}^{t} k_+ dt}.
\]
Now consider a point at which the entropy is not $C^2$, (actually a contact, since entropy is stationary). By the Shock-free condition, one-sided limits of $m$ and $m_x$ exist and $u$ and $p$ are $C^2$, so our $R/C$ variables $\alpha$ and $\beta$ and characteristic $\Gamma$ are defined up to $x = x_*$ with well-defined one-sided limits. If $m(x)$ is continuous at $x_*$, then $\alpha = -\frac{rt}{c}$ and $\beta = \frac{st}{c}$ are also continuous, so the above argument yields $\alpha(t_++0) > 0$, and we continue the characteristic forward in time. If the entropy has a jump at $x_*$, then Corollary 3.4 applies, and the conclusion of the theorem follows.

The proofs of other cases are entirely similar, and omitted.

**Boundary $R/C$ character structure** Finally we consider the $R/C$ structure at the edge of the support of the entropy profile. The main case of interest is that of no incoming wave and an outgoing rarefaction, corresponding to initial waves being compactly supported and no shocks outside the support of the entropy, respectively.

**Lemma 3.8.** Suppose that the solution satisfies the Shock-free condition. A wave emerging from a region of varying entropy keeps its $R/C$ character as long as there are no incoming waves of the other family. If $m(x)$ is non-decreasing, a forward $R$ (resp. $C$) emerging to the right reflects a backward $C$ (resp. $R$) back into the region of varying entropy. A backward $R$ (resp. $C$) emerging to the left of the varying entropy reflects a forward $R$ (resp. $C$) into the region of varying entropy. Similar results hold if the entropy $m(x)$ is non-increasing, but the character of the reflected wave is reversed.

**Proof.** We consider only the first case: a forward rarefaction emerging from the right edge $x_1$ of the varying entropy, which is non-decreasing. All other cases are similar.

First suppose there is a jump at $x_1$: by Def. 3.4 and (2.13), the states to the right of the jump satisfy

$$\dot{r}_r = \dot{u}_r - m_r \dot{z}_r = 0 \quad \text{and} \quad \dot{s}_r = \dot{u}_r + m_r \dot{z}_r < 0,$$

so that $\dot{z}_r < 0$, where $\dot{y} := \frac{dy}{dt}$ denotes the time derivative. Applying (3.2) and using $Q > 1$, we see that

$$\dot{r}_l = \dot{u}_l - m_l \dot{z}_l = m_r \dot{z}_r (1 - 1/Q) < 0 \quad \text{and} \quad \dot{s}_l = m_r \dot{z}_r (1 + 1/Q) < 0,$$

so that the waves on the left are a forward rarefaction and backward compression.
Now suppose the entropy is smooth \((C^1)\) and increasing up to \(x_1\), but constant for \(x \geq x_1\). Then also, for \(x \geq x_1\), we have
\[
\alpha > 0 \quad \text{and} \quad \beta = 0,
\]
since there are no incoming backward waves and the emerging forward wave is a rarefaction. By continuity, for \(x\) near \(x_1\) and \(x < x_1\) we have \(\alpha > 0\), \(\beta \approx 0\), and, by (3.7), \(k_2 > 0\). Thus the forward wave is a rarefaction for \(x\) near \(x_1\), and moreover, in this neighborhood,
\[
\beta^0 < 0, \quad \text{so also} \quad \beta < 0,
\]
so the backward characteristic reflected back into the varying entropy region is compressive.

\section{Shock-free solutions with a single contact}

Our main results refer to interacting solutions which interact only for finite times or contain asymptotic vacuums. When there is a single entropy jump, say at \(x = 0\), then these are the only possibilities for shock-free solutions. By first treating a single contact discontinuity, we avoid issues of multiple reflections of waves considered in later sections.

Consider the interaction of smooth isentropic waves at a contact discontinuity separating constant entropy values \(m_l\) and \(m_r\). We describe the states on either side of the contact by
\[
z_l(t) = z(0-, t), \quad z_r(t) = z(0+, t) \quad \text{and} \quad u_l(t) = u_r(t) = u(0, t).
\]
We find the \(R/C\) character of the incoming and outgoing waves by using (2.13) to calculate the Riemann invariants, and differentiating these in time.

\begin{lemma}
Suppose the interacting solution contains no shocks. The solution is eventually noninteracting if and only if the Vacuum Condition (1.5) is not satisfied. If the Vacuum Condition holds, the solution contains an asymptotic vacuum.
\end{lemma}

\begin{proof}
Suppose that interaction ends in finite time \(T \gg 1\), and use subscripts \(g\) and \(d\) to denote the constant states on the left and right of the contact after the interaction has completed, respectively, so that \(u_g = u_l(T)\), etc. Since there are no shocks, the outgoing waves leaving the interaction region
must be rarefactions. Using (3.1), we relate these states to the extreme states (subscripted by $\pm \infty$) as

\begin{align*}
    u_g - u_{-\infty} &= m_{-\infty} (z_{-\infty} - z_g), \\
    u_{\infty} - u_d &= m_\infty (z_\infty - z_d).
\end{align*}

(4.1)

We now use (3.2) to relate the states across the jump,

\begin{align*}
    u_d &= u_g \quad \text{and} \quad m_d z_d = m_g z_g Q.
\end{align*}

Now since $m_g = m_{-\infty}$ and $m_d = m_\infty$, we get

\begin{align*}
    u_\infty - u_{-\infty} - m_{-\infty} z_{-\infty} - m_\infty z_\infty &= -(1 + Q) m_{-\infty} z_g < 0,
\end{align*}

so that (1.5) fails.

Now suppose that the vacuum condition holds. Then the interaction persists for all time, and because there are no shocks, the outgoing waves must both be rarefactions. For $t > 0$, trace forward and backward characteristics back from the contact at $x = 0$ to the initial time $t = 0$, to define functions $x_-(t) < 0$ and $x_+(t) > 0$, respectively. Thus the forward rarefaction starting at $x_-(t)$ meets the contact at $t$, etc., and since characteristics cannot intersect, we have $\dot{x}_- \leq 0$ and $\dot{x}_+ \geq 0$. Now, since Riemann invariants are preserved on characteristics, we have

\begin{align*}
    u_l(t) + m_l z_l(t) &= u_0(x_-(t)) + m_l z_0(x_-(t)) \quad \text{and} \\
    u_r(t) - m_r z_r(t) &= u_0(x_+(t)) - m_r z_0(x_+(t)),
\end{align*}

where $(z_0(x), u_0(x))$ is the initial data. Using (3.2), we conclude that

\begin{align*}
    u_0(x_-) - u_0(x_+) + m_l z_0(x_-) + m_r z_0(x_+) \\
    &= (1 + Q) m_l z_l(t) > 0.
\end{align*}

Now since (1.5) holds, at least one of $x_-(t)$ or $x_+(t)$ must converge to some finite $x_*$ as $t \to \infty$. It follows that the incoming characteristic beginning at $x_*$ (and also those starting further out) cannot meet the contact in finite time, so remains bounded for all time. Thus this characteristic has a vertical asymptote, and the solution contains an asymptotic vacuum. \hfill \Box

Corollary 4.2. If a shock-free solution is eventually noninteracting, then the vacuum condition (1.5) fails, whatever the entropy profile.
Proof. The proof proceeds as above and (4.1) continues to hold, provided subscripts $g$ and $d$ refer to the states on either side of the varying entropy. For large times $t > T$, the solution restricted to the entropy profile is a stationary entropy wave, across which velocity $u$ and pressure $p$ are constant. Thus $u_g = u_d$ and (4.1) yields
\[ u_\infty - u_{-\infty} - m_{-\infty} z_{-\infty} - m_\infty z_\infty = -m_{-\infty} z_d - m_\infty z_g < 0, \]
so the vacuum condition fails.

In isentropic flow, the asymptotic vacuum is produced only by the interaction of two opposite rarefaction waves [19, 33]. When a contact is present, a single strong rarefaction can produce an asymptotic vacuum, since the leading edge of the rarefaction crossing the contact may produce a reflected rarefaction, and the interaction of the initial strong rarefaction with the reflected rarefaction can lead to the vacuum. According to Corollary 3.5 one of the incoming waves must be a pure rarefaction, while the other can contain compressive regions which are changed by the interaction at the contact, or there could be no opposite incoming wave.

To be specific, consider a backward rarefaction (initially compactly supported in $(0, \infty)$) interacting with a 3-contact (increasing entropy jump), $m_r > m_l$, at $x = 0$, with no incoming forward wave. The profile of the outgoing waves is determined by the traces of the states on either side of the contact discontinuity. By (3.2) and (3.1), the initial data consists of constant states $(z_l, u_l, m_l)$ and $(z_m, u_l, m_r)$ satisfying
\[ m_r z_m = m_l z_l Q, \]
Together with a backward rarefaction given by
\[ u_0(x) - u_l = m_r z_m - m_r z_0(x), \]
for a decreasing function $z_0(x)$ with $z_0(0) = z_m$. If there is some $x_*$ such that
\[ u_0(x_*) - m_r z_0(x_*) = u_l + m_l z_l, \]
that is,
\[ m_r z_0(x_*) = \frac{Q - 1}{2} m_l z_l, \] or \[ z_0(x_*) = \frac{Q - 1}{2 Q} z_m, \]
then all backward characteristics beginning to the left of $x_*$ cross the contact, while all those starting at $x \geq x_*$ asymptote, with corresponding states approaching vacuum. Note that the support of the vacuum in the limit \( t \to \infty \) is some interval of the form \([0, x_#]\).
5 Shock-free solutions with Monotone Entropy

We now prove our main theorem, which describes the structure of shock-free solutions with a monotone entropy profile.

**Theorem 2.** Assume that the (variation of the) data is compactly supported and that the entropy is monotone. Then a globally defined shock-free solution is either eventually noninteracting or contains an asymptotic vacuum.

**Proof.** Assume without loss of generality that the entropy is non-decreasing. We assume that the solution is shock-free, has no asymptotic vacuum, and is not eventually noninteracting, and derive a contradiction. Denote the interval on which the entropy varies by \([x_0, x_1]\). Since the data is compactly supported, there is some \(T \gg 1\) such that

\[
\beta(x_1, t) = 0 \quad \text{and} \quad \alpha(x_0, t) = 0 \quad \text{for every} \quad t \geq T.
\]

Next, since there are no shocks, by Lemmas 3.2 and 3.3, the outgoing waves must be rarefactions, so that

\[
\alpha(x_1, t) \geq 0 \quad \text{and} \quad \beta(x_0, t) \geq 0 \quad \text{for each} \quad t \geq 0.
\]

Because there is no asymptotic vacuum, all forward and backward characteristics pass through the interval \([x_0, x_1]\) of varying entropy in finite time.

It follows from Lemma 3.8 that

\[
\alpha(x_1, t) \geq 0 \quad \text{and} \quad \beta(x_0, t) \leq 0 \quad \text{for} \quad t \geq T.
\]

For each \(t \geq T\), denote the backward characteristic starting from the point \((x_1, t)\) by \(\Gamma_-(t)\), and define

\[
x_*(t) = \min\{x_0 \leq x \leq x_1 | \beta(\xi, \tau) \leq 0, \ \forall \xi \geq x, \ (\xi, \tau) \in \Gamma_-(t)\},
\]

so \(x_*(t)\) is the first possible point on the characteristic that \(\beta\) becomes positive. The curve \(x_*(t)\) is continuous, since \(\beta\) is continuous away from contacts, and if \(\beta\) first changes sign at a contact, \(\beta(x_c^-) \leq 0 \leq \beta(x_c^+)\), then we have \(x_*(t) = x_c\). In this case, continuity of \(x_*(t)\) follows since \(\beta(x_c^-)\) is a continuous function of \(\alpha(x_c^+)\) and \(\beta(x_c^+)\).

By definition, we have \(\beta \leq 0\) on the right of the curve \(x_*(t)\), while also \(\beta = 0\) on the curve (except possibly at a contact). Away from a contact, we must have \(\beta' \geq 0\), so that \(\alpha \leq 0\) at \(x_*(t)\), by (3.6). At a contact, Corollary 3.5 also implies that the forward wave is \(C\), so again \(\alpha \leq 0\).
Now, since the solution is not eventually noninteracting, there is some $t_#$ such that
\[ \alpha(x_1-, t_#) > 0, \quad \text{and} \quad \beta(x_1-, t_#) < 0, \]
by Lemma 3.8. We now trace the forward characteristic $\Gamma_+$ back from $(x_1-, t_#)$ until it first meets the curve $x_*(t)$ at the point $(x_\dagger, t_\dagger)$, say. We now have
\[ \alpha(x_\dagger, t_\dagger) \leq 0 \quad \text{and} \quad \alpha(x_1-, t_#) > 0, \]
while also
\[ \beta(x, t) \leq 0 \quad \text{for} \quad x_\dagger \leq x \leq x_\#, \quad (x, t) \in \Gamma_+, \]
which together contradict Theorem 1. \hfill \Box

5.1 Piecewise Constant Entropy

We can describe the structure of solutions in more detail if we make the further simplifying assumption that the entropy is piecewise constant and monotone non-decreasing. By this we mean that the entropy has finitely many jumps, while $u$ and $p$ remain $C^2$. Our results apply directly to monotone non-increasing piecewise constant entropy with appropriate modification, and we expect that the extension to general shock-free solutions with varying monotone entropy holds with technical changes in the proofs.

**Lemma 5.1.** Suppose the entropy is piecewise constant non-increasing. If the initial data are never forward compressive but somewhere backward compressive, then there are no shock-free solutions.

**Proof.** If a backward $C$ leaves the varying entropy field, then shocks form in finite time by Lemma 3.3. If not, because there are only finitely many

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Figure 1: Proof of Lemma 5.1
isentropic blocks, we can isolate the left-most isentropic block $B$ containing some backward $C$. By Corollary 3.5, forward $C$ can form only at a 3-contact when the crossing backward waves are $C$. Thus there is no forward $C$ in block $B$ or the blocks to the left of it. Hence the backward $C$ in this block cannot be cancelled. Thus the backward $C$ must be obstructed by a backward asymptotic vacuum in (isentropic) block $B$. But in this case, a singularity forms in finite time by Lemma 3.3.

**Lemma 5.2.** Suppose the entropy is piecewise constant non-increasing. If the initial data are never compressive, then shock-free solutions are never compressive. For such solutions, we also have:

(a) If the data is constant to the right of the entropy jumps, and if there is some initial forward $R$, then there are no shock-free solutions.

(b) If the solution is shock-free and eventually noninteracting, then the upper boundary of the interaction region is a forward characteristic which does not end at an interior contact.

![Figure 2: Shock-free eventually noninteracting solution with rarefaction data](image)

**Proof.** Recall from Corollary 3.5 that a forward wave can change from $R$ to $C$ at a 3-contact, only when the crossing backward waves are $C$. Thus, any backward $C$ must appear earlier than forward $C$. Since there are no $C$ in the initial data, by Lemma 5.1 shock-free solutions are always non-compressive. This proves the first statement.

We now note the following consequences of Corollary 3.5:

1. If there are no incoming backward waves while the incoming forward waves are rarefactive on some piece of the right-most 3-contact, then the reflected backward waves are compressive. If this happens, a shock necessarily forms in finite time, by above.
2. In a shock-free solution, a forward $R$ cannot be cancelled since the backward waves are nowhere $C$.

Statement (a) now follows from 1 and 2, because the initial forward $R$ remains $R$ until it meets the last contact, and so reflects a backward $C$.

Finally assume the solution is shock-free and eventually noninteracting. Let $T$ denote the maximum time at which waves cross the right-most entropy jump (at $x_1$). We claim the forward characteristic traced back from $(x_1, T)$ is the upper bound of the interaction region. Since our shock-free solution is nowhere $C$, any backward $R$ above this characteristic would reflect a forward $R$, which in turn reflects a backward $C$ at $x_1$. If this characteristic ended at an interior contact, a backward rarefaction would emerge, and later reflect another forward wave, a contradiction.

**Lemma 5.3.** Suppose the entropy is piecewise constant non-increasing. If the initial data are nowhere compressive, then shock-free solutions contain an asymptotic vacuum if and only if (1.5) holds.

**Proof.** By Corollary 4.2 and Theorem 2, we only need to show that if there is an asymptotic vacuum, then (1.5) holds.

First, we claim that the right-most backward characteristic, denoted by $x = \Phi(t)$ forms an asymptotic vacuum to the right of the entropy field, so that $\Phi(\infty) \geq x_1$. If not, the interaction must end in finite time by Lemma 5.2(b). Next, for any time $T$, there exists a forward characteristic $\Psi$ which crosses all 3-contacts and which meets $x_1$ at some $t_* > T$, so that

$$\Psi(0) < x_0 \quad \text{and} \quad \Psi(t_*) = x_1, \quad t_* > T.$$  

Moreover, by choosing $T$ large enough, we can assume $\Psi(0)$ is to the left of the support of the initial data. We denote by $\psi_0, \ldots, \psi_n$ the points at which $\Psi$ crosses the $x$-axis and contacts, respectively, as in Figure 3. Also set $\phi_0 = (\Phi(0), 0)$ and $\phi_1 = (\Phi(t_*), t_*)$.

Since there are no compressions, we have for any $(x, t)$,

$$u_x = \frac{r_x + s_x}{2} \geq 0,$$

and recall that $u$ is $C^2$. We write

$$u_0(\infty) - u_0(-\infty) = u(\phi_0) - u(\phi_1) + u(\phi_1) - u(\psi_n) + u(\psi_n) - u(\psi_0)$$

$$\geq u(\phi_0) - u(\phi_1) + u(\psi_n) - u(\psi_0). \quad (5.1)$$
Now we use (3.1) to write
\[ u(\phi_0) - u(\phi_1) = m_\infty z_0(\infty) - m_\infty z(\phi_1), \]
and, telescoping, we write
\[ u(\psi_n) - u(\psi_0) = u(\psi_n) - u(\psi_{n-1}) + \cdots + u(\psi_1) - u(\psi_0) = m_n z(\psi_{n-1}) - m_n z(\psi_n) + \cdots + m_\infty z(\psi_0) - m_\infty z(\psi_1) > -m_n z(\psi_n) + m_\infty z_0(-\infty), \]
where we have used
\[ m_{k+1} z(\psi_k) = Q_k m_k z(\psi_k) > m_k z(\psi_k), \]
by (3.2). Here the \( m_k \) are the intermediate entropy levels (with \( m_0 = m_\infty \)), and \( Q_k > 1 \) the corresponding entropy jumps. Equation (5.1) thus becomes
\[ u_0(\infty) - u_0(-\infty) \geq m_\infty z_0(\infty) + m_\infty z_0(-\infty) - m_\infty z(\phi_1) - m_n z(\psi_n), \]
and the last two terms vanish in the limit as \( t_* \to \infty \), yielding (1.5).

\section{Examples of shock-free solutions}

When the entropy is monotonic, a nontrivial shock-free solution must contain an asymptotic vacuum or must be eventually non-interacting. Here we
describe a general method for constructing eventually noninteracting shock-free solutions.

Begin by specifying a piecewise smooth \((C^2)\) compactly supported entropy profile. Fix some constant \(x_0\) (say the location of an entropy jump), and specify data \((z,u)\) or \((r,s)\) on a compact \(t\)-interval at this \(x_0\). Treat this as Cauchy data and evolve it \textit{spatially} in both forward and backward directions. The equations for spatial evolution form a \(2 \times 2\) system with varying coefficients, while the entropy is smooth. At an entropy jump, the jump is resolved by the Hugoniot conditions (2.9), which for a \(\gamma\)-law gas is simply a \(2 \times 2\) linear map (3.2).

In order to obtain global existence, we require only an \textit{a priori} \(C^1\) estimate \([16\text{, }17\text{, }23]\). That is, we obtain a shock-free solution as long as \(\alpha\) and \(\beta\) remain bounded in the half-plane \(t > 0\). Taking the trace of the solution on \(t = 0\) then yields non-trivial initial data which generates this nontrivial interacting solution. By finite speed of propagation, for any fixed \(x\), the solution will be stationary for \(t\) large enough, so the solution is eventually noninteracting.

Because of Lemma 5.2(b), all non-compressive shock-free solutions with piecewise constant monotone entropy profile can be generated in this way, and we expect that all such non-compressive shock-free solutions have this structure as long as the entropy profile is monotonic.

**Rarefactions with two monotonic contacts**  By way of example, we explicitly construct an eventually noninteracting solution consisting of rarefactions with two increasing entropy jumps (3-contacts). For convenience we set \(m_{-\infty} = 1\) and define the entropy profile by jumps \(Q_0 > 1\) and \(Q_1 > 1\) at \(x_0\) and \(x_1\), respectively, see (3.3).

We specify Cauchy data by \(r(t,x_0-)\) and \(s(t,x_0-) = 0\), with

\[
\dot{r}(t,x_0-) := r_1(t,x_0-) > 0, \quad \text{supported on } \quad t \in [0,T].
\]

This means that there is a simple backward rarefaction wave emerging from \(x_0-\), and no incoming forward wave. Equivalently, by (2.13), we choose

\[
z(t,x_0-) = Z(t), \quad u(t,x_0-) = -Z(t) \quad \text{for } \quad t \in [0,T],
\]

where \(Z(t)\) is a positive monotone decreasing function.

By applying Corollary 3.5 at both jumps, it follows that all waves in the solution are rarefactions, so that \(\dot{s} \leq 0\) and \(\dot{r} \geq 0\) everywhere. In particular, the maximum and minimum values of \(z\) are taken on at \((x_0-,0)\)
and \((x_1, +\infty)\), respectively. Since the solution is eventually noninteracting, by \((3.3)\), it follows that we have the uniform global bounds

\[
Z_* \leq z(x,t) \leq Z^*,
\]

where the bounds are given by

\[
Z^* \geq Z(0) \quad \text{and} \quad Z_* \leq Q_{0}^{\frac{2}{m+1}} Q_{1}^{\frac{2}{m+1}} Z(T).
\]

We recall Lax’s estimate of gradient growth in the \(p\)-system \([14, 15]\).

**Lemma 6.1.** In a constant entropy field, the gradients of Riemann invariants of \(C^2\) solutions satisfy

\[
\frac{1}{-\dot{s}(B)} = \frac{1}{-\dot{s}(A)} R(A,B) + \int K(x,B) \, dx,
\]

\[
\frac{1}{\dot{r}(B)} = \frac{1}{\dot{r}(A)} R(A,B) - \int K(x,B) \, dx,
\]

where the integrations are along the forward and backward characteristics connecting the points \(A = (x_A, t_A)\) and \(B = (x_B, t_B)\), respectively, and where

\[
R(A,B) := \left( \frac{z(A)}{z(B)} \right)^{d/2} \quad \text{and} \quad K(A,B) := \frac{d R(A,B)}{2 m^2 z^{d+1}(A)}.
\]

For convenience we choose \(x_1\) large enough (or \(T\) small enough) that the backward wave to the right of \(x_0\) meets the entropy jump at \(x_1\) in negative time \(t < 0\), so that only the forward wave crosses the second jump in our region of interest. This is clearly possible because we have uniform bounds \((6.1)\) for \(z\), and hence for the wavespeed. For the same reason, the backward wave to the right of \(x_1\) meets the \(t\)-axis in some bounded interval \([x_1, X^*]\), so that the support of the data for the corresponds initial value problem is \([x_0, X^*]\). By the mean value theorem, integrating the characteristics from \((x_0+, T)\), we get

\[
\begin{align*}
  x_1 - x_0 &\geq \bar{c}_1 T \\
  x_1 - x_0 &= \bar{c}_2 (T_* - T), \quad \text{and} \\
  X_* - x_1 &= \bar{c}_3 T_*,
\end{align*}
\]

for some values \(\bar{c}\), where \(T_*\) is the time at which the forward characteristic from \((x_0+, T)\) meets \(x_1\). In fact, since the forward wave is simple, \(\bar{c}_2\) is exactly \(c(z(T, x_0+))\).
It remains to show that the backward rarefactions focus in the half-plane $t < 0$. It is convenient to describe the Riemann invariants in terms of the data $Z(t)$, as follows. First, note that
\[
\dot{r}(x_0-, t) = -\dot{s}(x_0-, t) = -2\dot{Z}(t) \geq 0,
\]
so that, by (3.4),
\[
\dot{r}(x_0+, t) = -(1 + Q_0) \dot{Z}(t) \quad \text{and} \quad
-\dot{s}(x_0+, t) = -(Q_0 - 1) \dot{Z}(t).
\]
(6.5)

By construction, there are two cases to consider, namely, a backward characteristic from $x_0+$, and a forward characteristic from $x_0+$ to $x_1-$ followed by a backward characteristic from $x_1+$. For the first case, we apply (6.3) with $A = (x_0+, t)$, and $B = (x, 0)$, with $t \leq T$ and $x_0 \leq x \leq x_1$. We require
\[
\frac{1}{\dot{r}(B)} = \frac{1}{\dot{r}(A)} R(A, B) - \int K(x, B) \, dx > 0,
\]
which reduces to
\[
\dot{r}(A) < \frac{R(A, B)}{\int K(x, B) \, dx},
\]
and which clearly follows if
\[
-\dot{Z}(t) < \frac{1}{(x_1 - x_0) (1 + Q_0)} \min R \max K.
\]
(6.6)

Now consider a forward characteristic from $A$ to $C = (x_1, t_1)$, followed by a backward characteristic from $C$ to $D = (x, 0)$. Since $-\dot{s} \geq 0$, (6.3) implies that
\[
-\dot{s}(C-) \leq -\dot{s}(A) \frac{1}{R(A, C-)},
\]
and again applying (3.4), we get
\[
\dot{r}(C+) = -\dot{s}(C-) \frac{Q_1 - 1}{2} \leq -\dot{s}(A+) \frac{Q_1 - 1}{2 R(A, C-)}
\leq -\dot{Z}(t) \frac{(Q_1 - 1)(Q_0 - 1)}{2 R(A, C-)}.
\]
As in the first case, the backward rarefaction focusses at $t < 0$ provided
\[
\dot{r}(C+) < \frac{R(C+, D)}{\int K(x, D) \, dx},
\]
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which certainly holds if
\[-\dot{Z}(t) < \frac{2}{(Q_1 - 1) (Q_0 - 1)} \frac{(\min R)^2}{(X_* - x_1) \max K}. \tag{6.7}\]

It follows that if (6.6) and (6.7) hold, then our solution satisfies the required properties. We now further estimate (6.6) and (6.7). We fix the bounds $Z_*$ and $Z^* \geq Z(0)$ for the data $Z(t)$, use (6.4) to eliminate $X_* - x_1$, and simplify, to find constants $K_i$ depending only on $Z_*$, $Z^*$ and $Q_i$, such that, if
\[-\dot{Z}(t) < \min \left\{ \frac{K_1}{x_1 - x_0}, \frac{K_2}{x_1 - x_0 + c_2 T} \right\}, \tag{6.8}\]
then (6.6) and (6.7) hold.

Finally, we show consistency of the construction, as follows. First, fix $Z_*$, $Z^*$ and $Q_i$, and choose $x_0$, $x_1$ and $T$ such that
\[x_1 - x_0 \geq C_1 T,\]
where $C_1$ is an upper bound for the wave speed in the region $(x_0, x_1)$. Now choose $Z(t)$ such that (6.8) holds and such that
\[Z^* \geq Z(0) \geq Z(T) \geq Q_0^{\frac{2}{\gamma - 1}} Q_1^{\frac{2}{\gamma - 1}} Z_* .\]
It is evident that these conditions are consistent by further requiring, say,
\[0 \leq -\dot{Z}(t) \leq \frac{Z^* - Q_0^{\frac{2}{\gamma - 1}} Q_1^{\frac{2}{\gamma - 1}} Z_*}{T} .\]

7 Non-monotonic Contact Discontinuities

We have shown that if the entropy profile is monotonic, then shock-free solutions are eventually noninteracting or contain an asymptotic vacuum. Here, by example, we show that this need not be true when the entropy profile is non-monotonic. Our entropy profile is piecewise constant with two contacts. As in earlier sections, we characterize the jumps using (3.3). Thus, we place a 3-contact $Q_0 > 1$ at $x_0$ and a 1-contact $Q_1 < 1$ at $x_1$; without loss of generality we assume $Q_0 = Q = 1/Q_1$. We assume also that there are no incoming waves on either side of the interaction region, so the interactions are confined to the strip $x_0 < x < x_1$.

We consider the initial value problem with $C^2$ data prescribed in the interval $(x_0, x_1)$, and constant outside that interval. We note that if the data is anywhere compressive, shocks must form in finite time, as the compressions cannot be cancelled at the entropy jumps.
Lemma 7.1. Fix the entropy profile as described above, and assume the initial data is constant outside the interval \([x_0, x_1]\). Then the solution is globally shock-free if and only if the data is nowhere compressive.

Proof. It suffices to show that rarefactive data produces global shock-free solutions. We first solve the initial boundary value problem in the region

\[ \Omega = \{(x, t) \mid x_0 < x < x_1, \ t > 0\}, \]

with boundary data prescribed by the requirement that there are no incoming waves. We then resolve the states across the entropy jumps and propagate the solution outwards as simple (rarefaction) waves.

We obtain the boundary conditions by setting

\[ s(x_0 - , t) = s(x_0 -, 0) \quad \text{and} \quad r(x_1 + , t) = r(x_1 +, 0), \]

and solving the Hugoniot conditions (3.4). After simplification, we write the boundary conditions as

\[
\begin{align*}
  s(x_0 +, t) &= 1 - \frac{Q}{1 + Q} r(x_0 +, t) + \frac{2Q}{1 + Q} s(x_0 -, 0), \\
  r(x_1 -, t) &= 1 - \frac{Q}{1 + Q} s(x_1 -, t) + \frac{2Q}{1 + Q} r(x_1 +, 0). \tag{7.1}
\end{align*}
\]

According to Corollary 3.5, the waves that are reflected back into the domain are always rarefactive, so we obtain global bounds on the derivatives of Riemann invariants.

We now apply the existence theorem of [16], Chap. 6, Thm. 3.1, which states that there is a unique global \(C^1\) solution in \(\Omega\) which is nowhere compressive, provided \(a\ priori\) bounds are satisfied. We obtain explicit time-dependent bounds below, which suffice for application of the theorem.

Finally, having solved the IBVP inside the domain \(\Omega\), we again apply (3.4) to obtain the Riemann invariants outside the domain, which are rarefactions by construction. These thus propagate for all times without forming shocks.

It is clear that these solutions are not eventually noninteracting, as there is always reflected rarefaction. It follows that either the solution forms an asymptotic vacuum, or waves continue to reflect back and forth between \(x_0\) and \(x_1\) for all times.

We analyze a general rarefactive solution, as follows. Starting from the corner \((x_1, 0)\), we trace the reflected characteristic through the solution. Let
Figure 4: Infinite reflection of characteristics

t_k denote the time of the k-th intersection of this characteristic with the boundary, as in Figure 4, and use the subscript to denote the corresponding interior state, so \( z_{2k} = z(x_1-, t_{2k}) \) and \( z_{2k+1} = z(x_0+, t_{2k+1}) \). We use (3.1) to describe the states at either end of the backward and forward characteristics, respectively, by

\[
\begin{align*}
  u_{2k} - u_{2k+1} &= m (z_{2k} - z_{2k+1}), \\
  u_{2k} - u_{2k-1} &= m (z_{2k-1} - z_{2k}).
\end{align*}
\]

Since the waves outside \( \Omega \) are simple, we obtain

\[
\begin{align*}
  u_{2k+2} - u_{2k} &= \frac{m}{Q} (z_{2k+2} - z_{2k}), \\
  u_{2k+1} - u_{2k-1} &= \frac{m}{Q} (z_{2k-1} - z_{2k+1}),
\end{align*}
\]

where we have used (3.2) to express the outside states in terms of the interior states. Eliminating \( u_n \), we get the same equation for even and odd \( n \),

\[
(1 + Q) z_{n+1} = 2Q z_n + (1 - Q) z_{n-1}, \quad (7.2)
\]

a linear difference equation for \( z_n \). We obtain a linear equation because the simple wave description (3.1) and the jump conditions (3.2) are linear in \( u \) and \( z \), for \( m \) constant.

Setting \( z_n = \lambda^n \), we get

\[
\lambda^2 - (1 + \eta) \lambda + \eta = (\lambda - 1)(\lambda - \eta) = 0,
\]

where we have set

\[
\eta := \frac{Q - 1}{Q + 1} \in (0, 1).
\]

It follows that the general solution of (7.2) is

\[
z_n = \frac{z_1 - \eta z_0}{1 - \eta} + \eta^n \frac{z_0 - z_1}{1 - \eta}, \quad (7.3)
\]

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Theorem 3. For entropy profile as given above, and rarefactive initial data prescribed on \((x_0, x_1)\), there are three possible long-time behaviors: asymptotic vacuum; infinitely reflected waves which converge to the vacuum state; and infinitely reflected waves with non-vanishing wavespeed and density. The long-time behavior is determined by the expression \(\zeta = z_1 - \eta z_0\), with a bifurcation at \(\zeta = 0\).

Proof. Since the data is rarefactive, \(z_1 < z_0\), so the second term of (7.3) is positive and decreasing with \(n\). If \(\zeta > 0\), then \(z_n\) is defined for all \(n\) and approaches \(\zeta / (1 - \eta) > 0\) as \(n \to \infty\). This implies that the waves are reflected infinitely often with uniformly bounded wavespeed.

If \(\zeta = 0\), then \(z_n = \eta^n z_0\), which clearly converges to the vacuum as \(n \to \infty\). Moreover, since \(z_n\) is defined for all \(n\), the waves interact with the entropy jumps infinitely often, and the \(n\)-th characteristic has nonzero speed, so the solution does not contain an asymptotic vacuum.

Finally, if \(\zeta < 0\), then for some \(N\), (7.3) yields \(z_N < 0\), which contradicts the physical requirement that \(z > 0\). We conclude that \(z_N\) cannot be defined, which means that the \(N\)-th characteristic never meets the boundary, and we therefore have an asymptotic vacuum.

Corollary 7.2. Solutions with nonmonotic entropy may converge to vacuum as \(t \to \infty\) even though they contain no asymptotic vacuum. In particular, the condition of an asymptotic vacuum is stronger than the vacuum condition (1.5).

Proof. We have seen that \(\zeta = 0\) yields a vacuum in the limit as \(t \to \infty\) for all \(x_0 \leq x \leq x_1\). We show that in case \(\zeta = 0\), (1.5) holds as an equality. By (3.2), at \(x_1\) and \(x_0\), respectively, we have
\[
\begin{align*}
&u_0(\infty) = u(x_1-, 0+), \quad m(\infty) z_0(\infty) = m Q z_0 \\
&u(x_0-, t_1) = u(x_0+, t_1), \quad m z_1 = Q m(-\infty) z(x_0-, t_1),
\end{align*}
\]
where \(z_0 = z(x_1-, 0+)\) and \(z_1 = z(x_0+, t_1)\). Now, by (3.1) we also have
\[
\begin{align*}
&u(x_0-, t_1) - u_0(-\infty) = m(-\infty) (z_0(-\infty) - z(x_0-, t_1)), \\
&u(x_1-, 0+) - u(x_0+, t_1) = m (z_0 - z_1).
\end{align*}
\]
Now, \(m(-\infty) = m(\infty)\), and we calculate
\[
\begin{align*}
u_0(\infty) - u_0(-\infty) &= m(\infty) (z_0(-\infty) - z_0(x_0-, t_1)) + m(z_0 - z_1) \\
&= m(\infty) z_0(-\infty) - m Q z_1 - m z_1 + Q m(\infty) z_0(\infty),
\end{align*}
\]
which yields
\[
\begin{align*}
    u_0(\infty) - u_0(-\infty) - m(-\infty) z_0(-\infty) - m(\infty) z_0(\infty) \\
    = \frac{m}{Q} z_1 - m z_1 + (Q - 1) \frac{m}{Q} z_0.
\end{align*}
\]
This vanishes since \( z_1 = \eta z_0 = \frac{Q - 1}{Q + 1} z_0. \)

Corollary 7.3. At the bifurcation point \( \zeta = 0 \), the solution asymptotically approaches vacuum at the rate
\[
    z_0 \left(1 + \frac{t_n c_0}{x_1 - x_0} (\eta^{-d} - 1)\right)^{-1/d} \leq z_n \leq z_0 \left(1 + \frac{t_n c_0}{x_1 - x_0} (1 - \eta^d)\right)^{-1/d},
\]
so, in particular, \( z(t) \sim O(1) (1 + t)^{-1/d} \), which implies
\[
    \tau = O(1) (1 + t)^{1-1/d}, \quad \text{or} \quad \rho = O(1) (1 + t)^{1/d-1}.
\]

Proof. Since \( z \) is monotonic along characteristics, we estimate the interaction times \( t_n \) by
\[
    \frac{x_1 - x_0}{c(z_n)} \leq t_{n+1} - t_n \leq \frac{x_1 - x_0}{c(z_{n+1})},
\]
so that
\[
    \frac{x_1 - x_0}{m} \sum_{k=0}^{n-1} z_k^{-d} \leq t_n \leq \frac{x_1 - x_0}{m} \sum_{k=1}^{n} z_k^{-d}.
\]
If \( \zeta > 0 \), then the terms \( z_k^{-d} \) converge to \( \kappa = (1 - \eta)^d / \zeta^d \neq 0 \), so the times do not converge and \( t_n \sim \kappa n \) for \( n \) large.

When \( \zeta = 0 \), we get
\[
    \frac{x_1 - x_0}{m z_0} \sum_{k=0}^{n-1} \eta^{-kd} \leq t_n \leq \frac{x_1 - x_0}{m z_0} \sum_{k=1}^{n} \eta^{-kd},
\]
which simplifies to
\[
    \frac{\eta^{-nd} - 1}{\eta^{-d} - 1} \leq \frac{t_n m z_0^d}{x_1 - x_0} \leq \eta^{-d} \frac{\eta^{-nd} - 1}{\eta^{-d} - 1},
\]
which in turn yields
\[
    1 + \frac{t_n c_0}{x_1 - x_0} (1 - \eta^d) \leq \eta^{-nd} \leq 1 + \frac{t_n c_0}{x_1 - x_0} (\eta^{-d} - 1),
\]
where \( c_0 = c(z_0) = m z_0^d \). Since \( z_n = \eta^n z_0 \), the corollary follows. \( \square \)
8 Solutions with One Shock

We briefly analyze the evolution of a single shock, as follows. The shock curve and states on either side satisfy the Rankine-Hugoniot relations (2.5). We analyze the interacting shock by treating it as a free boundary problem and imposing the conditions that the flow on either side of the shock is either isentropic or stationary. This yields a nontrivial interacting shock which is however not difficult to analyze.

We parameterize the shock curve, $\Sigma = (x(a), t(a))$, together with Cauchy data on either side of $\Sigma$, subject to the Rankine-Hugoniot conditions. According to (2.7) and (2.8), we have

$$
\begin{align*}
\frac{z_1}{z_0} &= a, \\
\frac{m_1}{m_0} &= f(a), \\
\frac{p_1}{p_0} &= a^{d+1} f(a)^2, \\
\xi &= \pm m \overline{z}' g(a), \\
u_1 - u_0 &= \pm \overline{m} \overline{z}' h(a), \\
t(a) - t_0 &= \pm \int_0^a \frac{\dot{x}(\tilde{a})}{\overline{m} \overline{z}' g(\tilde{a})} d\tilde{a},
\end{align*}
$$

(8.1)

where

$$
\overline{m} = m_0 \frac{1 + f(a)}{2}, \quad \overline{z} = \left[ \frac{(d + 1) p_1}{m_0^2} \right]^{1/(d+1)}
$$

are the averages, and subscripts refer to opposite sides of the shock; here the parameter satisfies $1 < a < d^{1/(d-1)}$.

Once we specify consistent Cauchy data, we can extend the solution to a neighborhood around the shock by solving the Cauchy problem locally on either side of the shock. Because we implicitly assume the Lax entropy conditions, the Cauchy problem is non-characteristic as long as the shock has nonzero strength, and so general existence theorems apply [17, 16, 7]. Moreover, if we establish a priori estimates for gradients, global existence follows.

We specify the Cauchy data by assuming that the flow on one side of the shock is stationary, while the flow on the other side is isentropic. We do this by specifying that

$$
u_1(a) = U_1, \quad p_1(a) = P_1, \quad \text{and} \quad m_0(a) = M_0,
$$

(8.2)
so that 1 refers to the stationary solution and 0 to the isentropic solution. This leaves a single free function, namely $x(a)$, all other quantities being determined by (8.1), (8.2).

**Lemma 8.1.** There are globally defined interacting solutions containing one shock which separates the plane into two regions, such that the solution is isentropic in one region and stationary in the other.

**Proof.** Local existence follows because the Cauchy problem is not characteristic. Assume the shock is backward, with stationary solution behind the shock, so $u_1 = u_r$ is constant, and the sign choices in (8.1) are all $-$. It is clear that the stationary solution behind the shock is globally defined. To verify existence in the isentropic region, we must estimate the derivatives of the Riemann invariants.

To calculate the Riemann invariants, we differentiate the invariant along $\Sigma$, to get

$$
\frac{dr}{da} = r_x \frac{dx}{da} + r_t \frac{dt}{da} = r_x \frac{dx}{da} \left(1 - \frac{c(a)}{\xi(a)}\right), \quad (8.3)
$$

where we have used $r_t - cr_x = 0$ and $\frac{dx}{dt} = -\xi$, and similarly

$$
\frac{ds}{da} = s_x \frac{dx}{da} + s_t \frac{dt}{da} = s_x \frac{dx}{da} \left(1 + \frac{c(a)}{\xi(a)}\right). \quad (8.4)
$$

We now calculate from (8.1) and (2.3) that

$$
m_1(a) = M_0 f(a), \quad P_1 = \frac{M_0^2}{d+1} f(a)^2 (a z_0(a))^{d+1}, \quad \text{and}
$$

$$
U_1 - u_0(a) = -M_0 \left[\frac{(d+1) P_1}{M_0^2}\right]^{1/(d+1)} \frac{1 + f(a)}{2} h(a).
$$

We use these to solve for $z_0(a)$ and $u_0(a)$, and plug in to (8.3) and (8.4) to get

$$
r_x = \frac{da}{dx} R(a) \quad \text{and} \quad s_x = \frac{da}{dx} S(a), \quad (8.5)
$$

for explicit functions $R$ and $S$.

We now restrict $a$ to a compact subinterval of $(1, d^{1/(d-1)})$, which implies uniform bounds on all thermodynamic variables, and so also $R(a)$ and $S(a)$, and choose $x(a)$ so that $r_x$ and $s_x$ are small enough on $\Sigma$ that (6.3) yields finite bounds for $t \geq 0$.

The other case, of stationary solutions before the shock, is similar. Proceeding as above, we obtain (8.5) (with slightly different $R$ and $S$), and we
now must further restrict our data to ensure that $s_x \geq 0$, so the outgoing forward wave is a rarefaction, and we must ensure that the backward wave does not focus in the halfplane $t > 0$.

It is interesting to ask whether the shock persists. Disappearance of the backward shock occurs in the limit $a \to 1^+$, so we must demonstrate that this limit cannot occur for finite $(x(a), t(a))$. It is well known that in the limit of vanishing wave strength, we have

$$1 - \frac{c(a)}{\xi(a)} = O(a - 1) \text{ as } a \to 1^+,$$

see [27]. The functions $R(a)$ and $S(a)$ remain bounded away from 0 in the limit $a \to 1$, so we obtain

$$|r_x| = \frac{O(1)}{a - 1} \frac{d a}{d x},$$

and in particular, if $r_x$ remains finite, by [8,3], there exist $\nu > 0$ and $\delta > 0$ such that

$$(a - 1) \frac{d x}{d a} \geq \nu \text{ for } 1 < a < 1 + \delta.$$ 

Now, for $0 < \epsilon < \delta$, we have

$$x(1 + \delta) - x(1 + \epsilon) = \int_{1+\epsilon}^{1+\delta} \frac{d x}{d a} \frac{d a}{\hat{a}} \geq \int_{1+\epsilon}^{1+\delta} \frac{\nu}{\hat{a} - 1} \frac{d \hat{a}}{\hat{a}} = \nu \log \frac{\delta}{\epsilon} \to \infty,$$

as $\epsilon \to 0$, so $x \to -\infty$ as $a \to 1^+$. This implies that the shock strength can vanish only at infinity.

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