Jack superpolynomials: physical and combinatorial definitions

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Abstract

Jack superpolynomials are eigenfunctions of the supersymmetric extension of the quantum trigonometric Calogero-Moser-Sutherland. They are orthogonal with respect to the scalar product, dubbed physical, that is naturally induced by this quantum-mechanical problem. But Jack superpolynomials can also be defined more combinatorially, starting from the multiplicative bases of symmetric superpolynomials, enforcing orthogonality with respect to a one-parameter deformation of the combinatorial scalar product. Both constructions turn out to be equivalent. This provides strong support for the correctness of the various underlying constructions and for the pivotal role of Jack superpolynomials in the theory of symmetric superpolynomials. 1

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The aim of this contribution is to highlight some aspects of our work [1, 2], focusing on the equivalence between two totally different approaches to the construction of Jack superpolynomials: a physical one, in terms of an eigenvalue problem in supersymmetric quantum mechanics [1] and a more mathematical definition linked to algebraic combinatorics [2], along the lines of [3]. Most references to original works are omitted here and can be found in these quoted articles. Moreover, the presentation is kept at a rather informal level.

1 Jack superpolynomials as eigenfunctions of the stCMS model

Jack superpolynomials are eigenfunctions of the supersymmetric extension of the quantum trigonometric Calogero-Moser-Sutherland (stCMS) Hamiltonian, without its ground state contribution. This model describes the interaction of \( N \) particles on a unit radius circle, with canonical variables \( x_i \) and \( p_i = -i\partial/\partial x_i \) (with \( x_j \) subsequently replaced by \( z_j = \exp(ix_j) \)), together with their fermionic partners, the Grassmannian variables \( \theta_i \) (with \( \theta_i \theta_j = -\theta_j \theta_i \)) and \( \theta_i^\dagger = \partial/\partial \theta_i^\dagger \):

\[
\mathcal{H} = \sum_i (z_i \partial_i)^2 + \beta \sum_{i<j} \frac{z_i + z_j}{z_{ij}}(z_i \partial_i - z_j \partial_j) - 2\beta \sum_{i<j} \frac{z_i z_j}{z_{ij}^2}(1 - \kappa_{ij}).
\]

The whole dependence upon fermionic variables is contained in the factor \( \kappa_{ij} \) which reads

\[
\kappa_{ij} \equiv 1 - \theta_i \theta_j^\dagger = 1 - (\theta_i - \theta_j)(\theta_i^\dagger - \theta_j^\dagger).
\]

Remarkably, this is a fermionic-exchange operator, i.e., \( \kappa_{12} \theta_1 \theta_2 = \theta_2 \theta_1 = -\theta_1 \theta_2 \). That the supersymmetric extension is fully captured by the introduction of a fermionic exchange operator is a key technical tool in the integrability analysis.

The Jack superpolynomials are thus eigenfunctions of the operator (1). More precisely, the Jack superpolynomials diagonalize the full set of \( 2N \) commuting conserved operators of the stCMS model. This readily implies their orthogonality with respect to the ‘physical’ scalar product:

\[
\langle A(z, \theta), B(z, \theta) \rangle_\beta = \prod_j \frac{1}{2\pi i} \oint \frac{dz_j}{z_j} \int d\theta_j \theta_j \prod_{k \neq l} \left( 1 - \frac{z_k}{z_l} \right)^\beta A(z, \theta)^* B(z, \theta),
\]

where the complex conjugation \( * \) is defined as

\[
z_j^* = 1/z_j \quad \text{and} \quad (\theta_{i_1} \cdots \theta_{i_m})^* \theta_{i_1} \cdots \theta_{i_m} = 1.
\]

The integration over the Grassmannian variable is the standard Berezin integration, i.e., \( \int d\theta = 0 \) and \( \int d\theta^\dagger = 1 \).

2 Jack superpolynomials as symmetric superfunctions

Ordinary Jack polynomials are symmetric polynomials, i.e., invariant under the action of the operator \( K_{ij} \) that exchanges the variables \( z_i \) and \( z_j \):

\[
K_{ij} f(z_i, z_j) = f(z_j, z_i) K_{ij}.
\]

Jack superpolynomials are invariant under a generalization of this condition, namely, under the simultaneous exchange of the bosonic and the fermionic variables generated by:

\[
K_{ij} = K_{ij} \kappa_{ij}.
\]

\[2\] We stress that Jack superpolynomials (also called Jack polynomials in superspace) do not appear to have any relation to the super-Jack polynomials of [4], based on Lie superalgebras.
where $\kappa_{ij}$ is defined in (8). Manifestly, $\tilde{H}$ leaves invariant the space of polynomials of a given degree in $z$ and a given degree in $\theta$, being homogeneous in both sets of variables. Eigenfunctions are thus of the form:

$$A^{(m)}(z, \theta; \beta) = \sum_{1 \leq i_1 < \ldots < i_m \leq N} \theta_{i_1} \cdots \theta_{i_m} A^{(i_1 \ldots i_m)}(z; \beta), \quad m = 0, 1, \ldots, N,$$

(7)

where $A^{(i_1 \ldots i_m)}$ is a homogeneous polynomial in $z$. Note the simple dependence upon the fermionic variables, which factorizes in monomial prefactors, a manifest consequence of their anticommuting nature. Since the eigenfunctions $A^{(m)}$ are assumed to be invariant under the action of the exchange operators $K_{ij}$ and given that the $\theta$ products are antisymmetric, i.e.,

$$\kappa_{jk} \theta_{i_1} \cdots \theta_{i_m} = -\theta_{i_1} \cdots \theta_{i_m} \quad \text{if} \quad j, k \in \{i_1, \ldots, i_m\},$$

(8)

the superpolynomials $A^{(i_1 \ldots i_m)}$ must be partially antisymmetric to ensure the complete symmetry of $A^{(m)}$. In fact, each polynomial $A^{(i_1 \ldots i_m)}$ is completely antisymmetric in the variables $\{z_{i_1}, \ldots, z_{i_m}\}$, and totally symmetric in the remaining variables $z \setminus \{z_{i_1}, \ldots, z_{i_m}\}$.

In the same way as symmetric polynomials are indexed by partitions, symmetric superpolynomials are indexed by superpartitions. A superpartition is a sequence of non-negative integers that generates two partitions separated by a semicolon:

$$\Lambda = (\Lambda_1, \ldots, \Lambda_m; \Lambda_{m+1}, \ldots, \Lambda_N) = (\Lambda^a; \Lambda^s),$$

(9)

$\Lambda^a$ being associated to an antisymmetric function of the variables $\{z_{i_1}, \ldots, z_{i_m}\}$ (so that $\Lambda_i > \Lambda_{i+1}$ for $i = 1, \ldots, m - 1$) and $\Lambda^s$, to a symmetric function of the variables $\{z_{i_{m+1}}, \ldots, z_{i_N}\}$, (i.e., $\Lambda_i \geq \Lambda_{i+1}$ for $i \geq m + 1$). For $m = 0$, the semicolon disappears and we recover the partition $\Lambda^s$. The number $m$ is called the fermionic degree. The bosonic degree of a superpartition is simply the sum of its parts. For instance, the superpartitions of bosonic and fermionic degrees 3 and 1 respectively are:

$$(3; 0), \quad (2; 1), \quad (1; 2), \quad (1; 1, 1), \quad (0; 3), \quad (0; 2, 1), \quad (0; 1, 1, 1).$$

(10)

A superpartition $\Lambda = (\Lambda^a; \Lambda^s)$ can also be viewed as a partition (by reordering its parts) in which every part of $\Lambda^a$ is circled. If a part $\Lambda^a_j = b$ is equal to at least one part of $\Lambda^s$, then we circle the leftmost $b$. For instance, $\Lambda = (3, 0; 4, 3) \equiv (4, \overline{3}, 3, \overline{0})$.

(11)

This allows us to associate to each $\Lambda$ a unique superdiagram, denoted by $D[\Lambda]$ in which the ‘fermionic rows’ (circled parts) have an additional circle at the end. We will also need to introduce the transposition $\Lambda'$ of $\Lambda$, which is defined by the transposition of the corresponding superdiagram, an operation that manifestly preserves the fermionic degree (number of circles). For instance, we have

$$D[(3, 0; 4, 3)] = \begin{array}{|c|c|c|} \hline \circ & \circ & \circ \\ \hline \end{array}, \quad D[(3, 0; 4, 3)'] = \begin{array}{|c|c|c|} \hline \circ & \circ & \circ \\ \hline \end{array},$$

(12)

meaning that $(3, 0; 4, 3)' = (3; 1; 3; 3)$.

The Jack polynomials are most naturally defined in an expansion in terms of monomial symmetric functions. This is also true for their superextensions. The monomial symmetric superpolynomials (or supermonomials) are defined as:

$$m_\Lambda(z, \theta) = \sum'_{\sigma \in S_N} \theta_{\sigma(1)} \cdots \theta_{\sigma(m)} z_{\sigma(1)}^{\Lambda_{\sigma(1)}} \cdots z_{m}^{\Lambda_{\sigma(m)}} z_{m+1}^{\Lambda_{\sigma(m+1)}} \cdots z_{N}^{\Lambda_{\sigma(N)}},$$

(13)
where $S_N$ is the symmetric group, the prime indicates that the summation is restricted to distinct terms. For instance, for $N = 3$ and $\Lambda = (3,1;2)$, we have

$$m_{(3,1;2)} = \theta_1\theta_2(z_1^3z_2 - z_1z_2^3) + \theta_1\theta_3(z_1^3z_3 - z_1z_3^3)z_2^2 + \theta_2\theta_3(z_2^3z_3 - z_2z_3^3)z_1^2.$$  

(14)

We can now define the Jack superpolynomials as the unique eigenfunctions of the supersymmetric Hamiltonian $\mathcal{H}$,

$$\mathcal{H} J_{\Lambda}^{(\beta)}(z, \theta) = \varepsilon_\Lambda J_{\Lambda}^{(\beta)}(z, \theta),$$

(15)

(for some eigenvalue $\varepsilon_\Lambda$ that do not need to be specified) that can be decomposed triangularly in terms of the supermonomials:

$$J_{\Lambda}^{(\beta)}(z, \theta) = m_\Lambda(z, \theta) + \sum_{\Omega: \Omega < \Lambda} c_{\Lambda,\Omega}(\beta)m_\Omega(z, \theta).$$

(16)

and which are orthogonal with respect to the physical scalar product:

$$\langle J_{\Lambda}^{(\beta)}(z, \theta), J_{\Omega}^{(\beta)}(z, \theta) \rangle_\beta \propto \delta_{\Lambda,\Omega}.$$  

(17)

Observe that if we delete the semi-colon of a superpartition, it becomes a composition (i.e., a non-ordered sequences of non-negative integers). These are naturally ordered by the Bruhat ordering.

Here is a Jack superpolynomial of bosonic degree 2 and fermionic degree 1:

$$J_{(2,0)}^{(\beta)} = m_{(2,0)} + \frac{\beta}{2 + \beta} m_{(0,2)} + \frac{2\beta}{(1 + \beta)(2 + \beta)} m_{(0,1,1)}.$$  

(18)

Obviously, in the limit $\beta \to 0$, the Jack superpolynomial $J_{\Lambda}^{(\beta)}(z, \theta)$ reduce to supermonomial $m_\Lambda(z, \theta)$. Note also that for $m = 0$, $J_{\Lambda}^{(\beta)}(z, \theta)$ is the ordinary Jack polynomial $J_{\Lambda}^{(\beta)}(z)$.

### 3 Multiplicative bases of symmetric superpolynomials

The Jack superpolynomials $J_{\Lambda}^{(\beta)}$’s and the supermonomials $m_\Lambda$’s provide two bases for the space of symmetric superpolynomials. There are other natural candidate bases, namely those that would result from the superextension of the elementary $e_n$, homogeneous $h_n$, and power sum $p_n$ symmetric functions, naturally defined by their generating functions:

$$\sum_{n \geq 0} e_n t^n = \prod_{i \geq 1} (1 + z_i t), \quad \sum_{n \geq 0} h_n t^n = \prod_{i \geq 1} \frac{1}{(1 - z_i t)}, \quad \sum_{n \geq 1} p_n t^n = \prod_{i \geq 1} \frac{z_i t}{(1 - z_i t)}.$$  

(19)

The natural way of extending these symmetric functions is to make the following replacement in their generating functions:

$$tz_i \to tz_i + \tau \theta_i,$$  

(20)

where $\tau$ is a constant anticommuting parameter ($\tau^2 = 0$). That makes the resulting functions manifestly invariant under the action of $K_{ij}$. Denote by $\tilde{e}_n$, $\tilde{h}_n$ and $\tilde{p}_n$ the coefficient of $\tau t^n$ in each modified generating function. For instance, we have

$$\prod_{i \geq 1} (1 + z_i t + \tau \theta_i) = \sum_{n \geq 0} t^n [e_n + \tau \tilde{e}_n]$$  

(21)

with

$$e_n = \sum_{1 \leq i_1 < \cdots < i_n \leq N} z_{i_1} \cdots z_{i_n}, \quad \tilde{e}_{n-1} = \sum_{1 \leq j \neq i_1, \ldots, i_{n-1} \leq N} \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq N} \theta_j z_{i_1} \cdots z_{i_{n-1}}.$$  

(22)
with $e_0 = 1$ and $e_n = \hat{e}_{n-1} = 0$  $\forall n > N$. Similarly, we find

$$p_n = \sum_{i=1}^{N} z_i^n, \quad \hat{p}_n = \sum_{i=1}^{N} \theta_i z_i^{n-1}. \quad (23)$$

The sets $\{g_n, \hat{g}_n\}$ for $g_n \in \{e_n, h_n, p_n\}$ form new multiplicative bases for symmetric superpolynomials, e.g.:

$$p_{\Lambda} = \hat{p}_{\Lambda_1} \cdots \hat{p}_{\Lambda_n} p_{\Lambda_{n+1}} \cdots p_{\Lambda_N}. \quad (24)$$

Recall also the Cauchy formula, from which a ‘combinatorial’ scalar $\langle \langle \mid \rangle \rangle$ product can be defined:

$$\prod_{i,j} \frac{1}{1 - z_i w_j} = \sum_{\lambda} a_{\lambda}^{-1} p_{\lambda}(z) p_{\lambda}(w) \Rightarrow \langle \langle p_{\lambda} \mid p_{\mu} \rangle \rangle = a_{\lambda} \delta_{\lambda,\mu}, \quad (25)$$

where for $\lambda = (1^{m_1} 2^{m_2} \cdots), a_{\lambda} = \prod_i i^{m_i} m_i!$. Again, this can be extended naturally as follows:

$$\prod_{i,j} \frac{1}{1 - z_i w_j - \theta_i \phi_j} = \sum_{\lambda} a_{\lambda}^{-1} p_{\lambda}(z, \theta) p_{\lambda}(w, \phi) \Rightarrow \langle \langle p_{\lambda} \mid p_{\mu} \rangle \rangle = a_{\lambda} \delta_{\lambda,\phi}, \quad (26)$$

where $a_{\lambda} = (-1)^{m(m-1)/2} \delta_{\lambda,\mu}$, $m$ being the fermionic degree of $\Lambda$. This product can be deformed by considering the $\beta$-th power of the product in (26):

$$\langle \langle p_{\lambda} \mid p_{\mu} \rangle \rangle_{\beta} = \beta^{-\ell(\Lambda)} a_{\lambda} \delta_{\lambda,\Omega}, \quad (27)$$

where $\ell(\Lambda)$ is the length of the superpartitions: $\ell(\Lambda) = m + \ell(\Lambda')$. It is known that the Jack polynomials are orthogonal with respect to both the physical scalar product (3) and the above $\beta$-deformed combinatorial product, evaluated at $m = 0$. This turns out to be also true for the Jack superpolynomials. To be precise, if we expand $J^{(\beta)}_{\Lambda}$ in the basis of the super power-sums $p_{\Lambda}$’s, then we check that

$$\langle \langle J^{(\beta)}_{\Lambda} \mid J^{(\beta)}_{\Omega} \rangle \rangle_{\beta} \propto \delta_{\Lambda,\Omega}. \quad (28)$$

Actually, the Jack superpolynomials can be uniquely reconstructed in this way, together with the usual triangularity requirement.

Another contact between $J^{(\beta)}_{\Lambda}$ and the combinatorial bases is the following relation that also generalizes a well-known limiting case of the Jack polynomials

$$(-1)^{m(m-1)/2} \lim_{\beta \to \infty} J^{(\beta)}_{\Lambda}(z, \theta) = e_{\Lambda'}(z, \theta). \quad (29)$$

Here is a simple illustration of (24) for $N = 3$:

$$J^{(\beta)}_{(2,0,0)} = m_{(2,0,0)} + \frac{\beta}{1 + \beta} m_{(1,0,1)} \Rightarrow \lim_{\beta \to \infty} J^{(\beta)}_{(2,0,0)} = m_{(2,0,0)} + m_{(1,0,1)}. \quad (30)$$

We have $(2,0;0)' = (1,0;1)$ and

$$e_{(1,0;1)} = [\theta_1 (z_2 + z_3) + \theta_2 (z_1 + z_3) + \theta_3 (z_1 + z_2)] [\theta_1 + \theta_2 + \theta_3] [z_1 z_2 + z_1 z_3 + z_2 z_3], \quad (31)$$

where $e_{(1,0;1)} = \hat{e}_1 \hat{e}_2 e_1$. It is simple to see that the last two expressions are equivalent: for this it suffices to check the coefficient of $\theta_1 \theta_2$ which is $-(z_1 - z_2)(z_1 + z_2 + z_3)$. 
4 Conclusion

We should point out that the first approach (the physical) presented has already been extended to the construction of the Hermite generalized superpolynomials, eigenfunctions of the rational supersymmetric CMS model with confinement [5]. However, the supersymmetric version of the Ruijsenaars model is still missing. The Macdonald superpolynomials, their would-be eigenfunctions, can nevertheless be constructed combinatorially, along the above lines.

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