THE GEOMETRY OF THE GIBBS MEASURE OF PURE SPHERICAL SPIN GLASSES

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Abstract. We analyze the statics for pure \( p \)-spin spherical spin glass models with \( p \geq 3 \), at low enough temperature. With \( F_{N,\beta} \) denoting the free energy, we compute the second order (logarithmic) term of \( NF_{N,\beta} \) and prove that, for an appropriate centering \( c_{N,\beta} \), \( NF_{N,\beta} - c_{N,\beta} \) is a tight sequence. We establish the absence of temperature chaos and analyze the transition rate to disorder chaos of the Gibbs measure and ground state. Those results follow from the following geometric picture we prove for the Gibbs measure, of interest by itself: asymptotically, the measure splits into infinitesimal spherical ‘bands’ centered at deep minima, playing the role of so-called ‘pure states’. For the pure models, the latter makes precise the so-called picture of ‘many valleys separated by high mountains’ and significant parts of the TAP analysis from the physics literature.

1. Introduction

The Hamiltonian of the pure \( p \)-spin spherical spin glass model is given by

\[
H_N(\sigma) = H_{N,p}(\sigma) = \frac{1}{N^{(p-1)/2}} \sum_{i_1, \ldots, i_p=1}^N J_{i_1, \ldots, i_p} \sigma_{i_1} \cdots \sigma_{i_p}, \quad \sigma \in S^{N-1},
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_N) \), \( S^{N-1} \triangleq \{ \sigma \in \mathbb{R}^N : \|\sigma\|_2 = \sqrt{N} \} \), and \( (J_{i_1, \ldots, i_p}) \) are i.i.d standard normal variables. Unless said otherwise, in the sequel we will always assume that \( p \geq 3 \). In terms of the overlap function \( R(\sigma, \sigma') \), the covariance of \( H_N(\sigma) \) is expressed by

\[
\mathbb{E}\{H_N(\sigma)H_N(\sigma')\} = N \left(R(\sigma, \sigma')\right)^p \text{ with } R(\sigma, \sigma') = \frac{\sigma \cdot \sigma'}{\|\sigma\| \|\sigma'\|}.
\]

In this paper we carry out a rather thorough analysis of the statics for pure \( p \)-spin spherical models with \( p \geq 3 \), at low enough temperature. As is well known, their free energy is given by the spherical version of the Parisi formula discovered by Crisanti and Sommers [22], proved by Talagrand [39] and extended by Chen [13]. We shall compute the free energy by a different method, improve the latter by computing a logarithmic second order term, and prove that the fluctuations of the free energy are tight (Theorem 2). We will further prove the absence of temperature chaos (Theorem 5) and analyze the transition rate to disorder chaos of the Gibbs measure and ground state (see Section 12).

Those results will follow from the following geometric picture for the Gibbs measure, of interest by itself: asymptotically, the measure splits into infinitesimal spherical ‘bands’ centered at deep minima, playing the role of so-called ‘pure states’. On those bands (the restriction of) \( H_N(\sigma) \) will be interpreted as a replica symmetric mixed spherical model, closely related to the 2-spin model at an (effective) high temperature. We note that the disorder \( (J_{i_1, \ldots, i_p}) \) determines the locations of the bands through those of the minima, and the radius of the bands is determined by the temperature. This relates systems at different temperatures or with perturbed disorder, a fact that will be crucial for us when we investigate chaos phenomena.
We point out that the genericity of mixed $p$-spin models\footnote{A mixed model $H_{N}(\sigma) = \sum_{p \geq 2} \gamma_{p} H_{N,p}(\sigma)$, either with spherical or Ising spins, is generic if and only if $\sum p^{-1} 1(\gamma_{p} \neq 0) = \infty$.} (see \cite{31}) and the assumption that interactions are even have been found to be very useful properties and were essential ingredients in several recent works. The pure models we consider are ‘as far as can be’ from being generic and we do not assume $p$ is even. In view of the very recent proof by Panchenko \cite{32} that generic mixed models with even interactions exhibit temperature chaos,\footnote{His proof dealt with Ising spin models, but his method is expected to apply to spherical models as well.} the absence of chaos in pure models expresses a significant difference between the pure and generic models (and is somewhat surprising). On the other hand, the spherical pure models are known to exhibit 1-step replica symmetry breaking (RSB) in the low temperature phase \cite{33, Proposition 2.2}, and are simpler in this respect than general mixed models.

For measurable $B \subset \mathbb{S}^{N-1}$, define the relative partition function or relative mass and the Gibbs measure, respectively, by
\begin{equation} Z_{N,\beta}(B) = \int_{B} \exp\{-\beta H_{N}(\sigma)\} \, d\mu_{N}(\sigma) \quad \text{and} \quad G_{N,\beta}(B) = \frac{Z_{N,\beta}(B)}{Z_{N,\beta}(\mathbb{S}^{N-1})}, \end{equation}
where $\mu_{N}$ is the uniform probability measure on $\mathbb{S}^{N-1}$. We also denote by $Z_{N,\beta} \triangleq Z_{N,\beta}(\mathbb{S}^{N-1})$ the usual partition function. For a given point $\sigma_{0} \in \mathbb{S}^{N-1}$ and overlaps $-1 \leq q \leq q' \leq 1$, define the spherical band
\begin{equation} \text{Band}(\sigma_{0}, q, q') \triangleq \{ \sigma \in \mathbb{S}^{N-1} : q \leq R(\sigma, \sigma_{0}) \leq q' \}. \end{equation}
A point $\sigma_{0}$ is a critical point if $\nabla H_{N}(\sigma_{0}) = 0$ with respect to the standard differential structure on the sphere. For odd $p$, let $\sigma_{0}^{i}$, $i = 1, 2, \ldots$, be an enumeration of the critical points of $H_{N}(\sigma)$ ordered so that $H_{N}(\sigma_{0}^{i+1}) \leq H_{N}(\sigma_{0}^{i})$. When $p$ is even, for any critical point $\sigma_{0}$, $-\sigma_{0}$ is also a critical point with the same critical value. In this case, let $\sigma_{0}^{i}$, $i = \pm 1, \pm 2, \ldots$, be an enumeration such that $\sigma_{0}^{i} = -\sigma_{0}^{i}$ and $H_{N}(\sigma_{0}^{i})$ increases for $i \geq 1$.\footnote{We note that though we work with critical points, by Corollary \cite{9} we could have replaced everywhere in the results $\sigma_{0}$ by the corresponding enumeration of local minima instead of general critical points.} In Section 5 we will define the overlap value $q_{s} := q_{s}(\beta)$, see (5.10). We use it to define
\begin{equation} \text{Band}_{i}(\epsilon) := \text{Band}_{i,\beta}(\epsilon) = \text{Band}(\sigma_{0}^{i}, q_{s} - \epsilon, q_{s} + \epsilon). \end{equation}
We define the conditional measure of $G_{N,\beta}$ given $\text{Band}_{i}(cN^{-1/2})$, $G_{N,\beta}^{c, i}(\cdot) = G_{N,\beta} \left( \cdot \cap \text{Band}_{i} \left( cN^{-1/2} \right) \right) / G_{N,\beta} \left( \text{Band}_{i} \left( cN^{-1/2} \right) \right)$. Let $G_{N,\beta}^{c, i} \otimes G_{N,\beta}^{c, j} \{ (\sigma, \sigma') \in \cdot \}$ denote the product measure of $G_{N,\beta}^{c, i}$ and $G_{N,\beta}^{c, j}$. For odd $p$ define $[k]_{p} = \{1, \ldots, k\}$ and for even $p$ define $[k]_{p} = \{\pm 1, \ldots, \pm k\}$. By an abuse of notation, we will simply write $[k]$ in the sequel.

**Theorem 1.** (Geometry of the Gibbs measure) For large enough $\beta$ we have
\begin{enumerate}
\item Asymptotic support:
\begin{equation} \lim_{k, c \to \infty} \liminf_{N \to \infty} \mathbb{E} \left\{ G_{N,\beta} \left( \bigcup_{i \in [k]} \text{Band}_{i} \left( cN^{-1/2} \right) \right) \right\} = 1. \end{equation}
\item States are pure: for any $i$ and $c > 0$, for even $p$,
\begin{equation} \lim_{\rho \to \infty} \limsup_{N \to \infty} \mathbb{P} \left\{ G_{N,\beta}^{c, i} \otimes G_{N,\beta}^{c, \pm i} \left\{ \left| R(\sigma, \sigma') \right| > \rho N^{-1/2} \right\} \right\} = 0. \end{equation}
For odd $p$, the same holds with the $\pm$ signs removed.
\end{enumerate}
Orthogonality of states: for any \( i \neq \pm j \), \( c, \delta > 0 \)

\[
\lim_{N \to \infty} \mathbb{P} \left\{ G_{c,i}^{c,i} \otimes G_{c,j}^{c,j} \left| R(\sigma, \sigma') \right| > \delta \right\} = 0.
\]

The decomposition of Theorem 1 is closely related to the works of Talagrand [40] and Jagannath [28] who proved certain abstract ‘pure states’ decompositions, assuming that the Ghirlanda-Guerra identities are satisfied in a limiting sense. The critical points and values of the Hamiltonian \( H_N(\sigma) \) have been recently investigated by Auffinger, Ben Arous and Černý [5], the author [37], and Zeitouni and the author [38]; see Section 2.2. In particular, in [38] the law of the (deep) critical values \( H_N(\sigma_i^0) \) was described in terms of a limiting point process (see Theorem 11), which complements Theorem 1. Further, a key in proving (1.7) is that the deep critical points are either antipodal or approximately orthogonal as vectors in the Euclidean space, see Corollary 13 which builds on [37].

In the physics literature, pure states are often described by the so-called picture of ‘many valleys separated by high mountains’ (see e.g. [34]). Our results (see Corollary 13 and (8.5)) indeed allow one to interpret the neighborhoods of (exponentially many) critical points corresponding to critical values deeper than a certain fraction of the global minimum of \( H_N(\sigma) \) as valleys. Therefore, Theorem 1 validates the prediction of multiple valleys, at least in the current setting, and further identifies the valleys around the deepest critical points as the relevant ones and bands as the relevant regions inside them.

The Thouless-Anderson-Palmer (TAP) approach [41] suggests that the pure states are related to the solutions of the so-called TAP equations, and that by correctly attributing mass to states at a given energy and estimating how many states there are at any energy – i.e., computing the so-called TAP free energy and complexity, respectively – one can calculate the free energy. Kurchan, Parisi and Virasoro [29] and Crisanti and Sommers [23] carried out the TAP analysis of pure spherical models (their analysis is not rigorous, and neither is claimed to be). Interestingly, for pure spherical spin glasses, TAP solutions are nothing but the critical points of the Hamiltonian (see [5, Section 6]). One may therefore wonder whether the mass of bands we compute coincides with the TAP free energy of [29, 23]. As we shall see, this is indeed the case, at least in the relevant range of overlaps; see Remark 21.

As part of our investigation of the weights and structure of the Gibbs measure on the thin bands of (1.5) we will study the conditional law of the restriction of the Hamiltonian \( H_N(\sigma) \) to the sub-sphere \( \{ \sigma : R(\sigma, \sigma_0) = q_u \} \) of co-dimension 1, conditional on \( \nabla H_N(\sigma_0) = 0 \) and \( H_N(\sigma_0) = u \) for some level \( u \in \mathbb{R} \). The latter, we shall see, is identical in distribution to a certain mixed spherical model involving \( k \)-spin interactions with \( 2 \leq k \leq p \) only (see Corollary 17), shifted by a constant. The Onsager reaction term added in [29, 23] to what they call the ‘naive’ free energy will arise in our calculation as the free energy of this mixture. Moreover, we will see that the fluctuations of the free energy of the original pure \( p \)-spin model on the sphere are intimately related to those of the 2-spin component of the mixture. The convergence of the free energy for the spherical 2-spin model proved by Baik and Lee [7] (see Theorem 7) will be crucial to analyzing the fluctuations of the latter. Apart from the results stated in Section 2 (which include the convergence result of [7] and results on the critical points from [5, 37, 38]), our analysis is essentially self contained.

**The free energy.** The free energy is defined by \( F_{N,\beta} = \frac{1}{N} \log (Z_{N,\beta}) \). The following theorem gives the second order correction of \( NF_{N,\beta} \) and shows that, appropriately centered, \( NF_{N,\beta} \) is tight.
Theorem 2. For large enough $\beta$, with $\Lambda_Z(E,q)$, $E_0$ and $c_p$ defined by (5.6), (2.6) and (2.8),

$$(1.8) \lim_{t \to \infty} \limsup_{N \to \infty} \mathbb{P} \left\{ \left| N F_{N,\beta} - N \Lambda_Z(E_0,q_*) + \frac{\beta q^*_{\beta} \log N}{2c_p} \right| > t \right\} = 0.$$

Earlier results regarding fluctuations of the free energy of mean field spin glass models are surveyed in Section 2.1. In particular, the above is the first result proving that fluctuations are of order $O(1)$ in the low-temperature phase. Theorem 2 implies that $F_{N,\beta}$ converges in probability to $\Lambda_Z(E_0,q_*)$. Obviously, the latter must coincide with the expression given by the spherical version of the Parisi formula discovered by Crisanti and Sommers [22], proved by Talagrand [39] and extended by Chen [13]. We compare $\Lambda_Z(E_0,q_*)$ and the 1-step RSB Parisi functional by a direct calculation in Section 12. As $\beta \to \infty$ we obtain the limiting law of the free energy in the following proposition.

Proposition 3. There exist deterministic $a_{N,\beta}$ such that, for any $t \in \mathbb{R}$,

$$(1.9) \lim_{\beta \to \infty} \limsup_{N \to \infty} \mathbb{P} \left\{ \frac{1}{\beta} (N F_{N,\beta} - a_{N,\beta}) \leq t \right\} - \exp \left\{ -c_p^{-1} e^{-c_p t} \right\} = 0.$$

In fact, we will prove the above with $a_{N,\beta} = \log(\mathcal{U}_{N,\beta}(m_N))$ where $\mathcal{U}_{N,\beta}(u)$ and $m_N$ are defined by (6.2) and (2.7). We finish with the following representation for the free energy in the $N \to \infty$ limit.

Corollary 4. For large enough $\beta$, there exist random variables $Z_{N,i}$, such that each $Z_{N,i}$ is measurable with respect to $(H_N(\sigma_0^i), \nabla^2 H_N(\sigma_0^i))$, and a sequence $k_N \geq 1$ with $k_N \to \infty$, so that

$$\forall \epsilon > 0 : \lim_{N \to \infty} \mathbb{P} \left\{ \left| N F_{N,\beta} - \log \sum_{i \in [k_N]} Z_{N,i} \right| > \epsilon \right\} = 0.$$

See (10.9) for an explicit expression for $Z_{N,i}$ (as a conditional mass of a band).

Absence of temperature chaos. Chaos phenomena, discovered by Bray and Moore [11] and Fisher and Huse [27], have been studied extensively in the physics literature; see the recent survey [35] by Rizzo. They refer to the situation where a small perturbation in the parameters of the system results in a drastic change in macroscopic observables. In particular, we say that temperature chaos occurs if for any $\beta_1 \neq \beta_2$,

$$(1.10) \exists q_0 \in [-1,1] , \forall \epsilon > 0 : \lim_{N \to \infty} \mathbb{E} \left\{ G_{N,\beta_1} \otimes G_{N,\beta_2} \left\{ | R(\sigma,\sigma') - q_0 | > \epsilon \right\} \right\} = 0.$$

That is, we sample at two different temperatures with the disorder fixed. We prove that spherical pure $p$-spin models do not exhibit temperature chaos, verifying a prediction of Rizzo and Yoshino [36]. To the best of knowledge, this is the first example where absence of temperature chaos is proved rigorously for mean-field spin glass models.

Theorem 5. For large enough $\beta_1 < \beta_2$, (1.10) does not hold.

In contrast, very recently Panchenko [32] proved that generic Ising mixed even $p$-spin models exhibit temperature chaos (see also the related works of Chen [15] and Chen and Panchenko [19]). His methods are also expected to work for spherical models. For pure spherical $p$-spin models with even $p \geq 4$, Panchenko and Talagrand [33] proved that for any $\epsilon > 0$, $\lim_{N \to \infty} \mathbb{E} \left\{ G_{N,\beta_1} \otimes G_{N,\beta_2} \left\{ | R(\sigma,\sigma') \in A(\epsilon) \right\} \right\} = 1$ with $A(\epsilon)$ denoting the union of balls of radius $\epsilon$ around $0$, $q_{12}$ and $-q_{12}$, with $q_{12} = q_{*}(\beta_1)q_{*}(\beta_2)$, assuming $\beta_1$ and $\beta_2$ are in the low-temperature phase. By itself, this is not enough to conclude or rule out chaos. Our proof...
shows that in the $N \to \infty$ limit, any of those three values is charged. See Proposition 33 and Remark 34. The proof of Theorem 5 will be based on the fact that the centers of the bands that carry most of the mass do not change with temperature, assuming the temperature is low enough.

Theorem 1 and an investigation of the critical points of the Hamiltonian can be also used to study disorder chaos of the Gibbs measure and the ground state. We discuss this in Section 12.

Structure of the paper. In Section 2 we review earlier related works. Section 3 is dedicated to an outline of the proof of Theorem 1. In Section 4 we develop a certain decomposition for the Hamiltonian and study conditional laws related to it. Various overlap values of importance are defined in Section 5. In particular, they are used to define two ranges of overlaps for which conditional weights of corresponding bands around a single critical point are investigated by different methods in Sections 6 and 7. Those are used in Section 8 to derive bounds on contributions to the partition function of the corresponding bands around all (low enough) critical points. The latter are combined in Section 9 to prove Theorem 1. The proofs of Theorem 2, Proposition 3 and Corollary 4, which deal with the free energy, are given in Section 10. The proof of the absence of temperature chaos, Theorem 5, is given in Section 11. Section 12 is dedicated to concluding remarks including, in particular, a discussion about the transition to disorder chaos.

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2. Earlier and related works

The first section below surveys works related to Theorem 2. The second section is devoted to recent results about the critical points and values of the Hamiltonian $H_N(\sigma)$, directly related to Theorem 1 and frequently used in the sequel.

2.1. Fluctuations of the free energy. First we state two result regarding the free energy of the 2-spin model which will be crucial when we study the weights of bands. The first is obtained as a corollary from Talagrand’s proof [39] of the spherical version of the Parisi formula.

Corollary 6. [39] Theorem 1.1, Proposition 2.2 For the pure 2-spin spherical model, the limiting free energy is given by $\lim_{N \to \infty} F_{N,\beta} = \mathcal{P}_2(\beta)$ where

\begin{equation}
\mathcal{P}_2(\beta) = \begin{cases} 
\frac{1}{4} \beta^2 & \text{if } \beta \leq 1/\sqrt{2}, \\
\sqrt{2} \beta - \frac{3}{4} - \frac{1}{2} \log(\beta) - \frac{1}{4} \log 2 & \text{if } \beta > 1/\sqrt{2}.
\end{cases}
\end{equation}
The following convergence was recently proved by Baik and Lee \[7\] \footnote{Below we will apply \[7\] Theorem 1.2] for the 2-spin model defined by \[1.1\] with \(p = 2\) and the inverse-temperature \(\beta_{\text{eff}} = \beta_{\text{eff}} (N, q)\) defined in \[6.32\]. In the notation of \[7\] Theorem 1.2], this corresponds to centered Gaussian \(J_{ij} = J_{ji}\) with variance 2 if \(i = j\) and 1 if \(i \neq j\) and inverse-temperature \(\beta_{\text{eff}}/\sqrt{2}\). Since \(\beta_{\text{eff}} \in (0, 1/\sqrt{2})\), in our application we will only use (2.2).}

We denote by \(\rightarrow\) convergence in distribution and by \(\mathcal{N} (m, \sigma^2)\) the Gaussian distribution with mean \(m\) and variance \(\sigma^2\).

**Theorem 7.** \[7\] Theorem 1.2] For the pure 2-spin spherical model, with \(f = \frac{1}{4} \log (1 - 2\beta^2)\), \(\alpha = -2f\), and \(TW_1\) denoting the GOE Tracy-Widom distribution,

\[
\forall \beta \in \left(0, 1/\sqrt{2}\right) : N (F_{N, \beta} - \mathcal{P}_2 (\beta)) \xrightarrow{d_{N \to \infty}} \mathcal{N} (f, \alpha),
\]

\[
\forall \beta \in \left(1/\sqrt{2}, \infty\right) : \left( \frac{1}{\sqrt{2}} \beta - \frac{1}{2} \right)^{-1} N^{2/3} (F_{N, \beta} - \mathcal{P}_2 (\beta)) \xrightarrow{d_{N \to \infty}} TW_1.
\]

As for the fluctuations of the free energy in other spin glass models, we mention the following.

In the high-temperature phase, Aizenman, Lebowitz and Ruelle \[2\] proved the convergence of \(N(F_{N, \beta} - C_{N, \beta})\), where \(C_{N, \beta}\) is an appropriate centering, to a Gaussian variable for the Sherrington-Kirkpatrick (SK) model. Comets and Neveu \[21\] later proved a similar result by a different approach using martingale methods. For pure Ising \(p\)-spin models with even \(p \geq 4\), again in the high-temperature phase, Bovier, Kurkova and Löwe \[10\], proved similar convergence for \(N^{(p+2)/4}(F_{N, \beta} - C_{N, \beta})\) by adapting the method of \[21\].

With a different definition for the model, dropping the diagonal terms in \[1.1\], (but still working with Ising spins, \(\sigma \in \{\pm 1\}^N\) and at high-temperature) they show for \(p \geq 3\) the convergence of \(N^{p/2}(F_{N, \beta} - C_{N, \beta})\). At any temperature, Chatterjee \[12\] showed that for Ising mixed even \(p\)-spin models without an external field, \(\text{Var} (NF_{N, \beta}) \leq c_{\beta} N / \log N\). For Ising mixed \(p\)-spin models in the presence of an external field and at any temperature, Chen, Dey and Panchenko \[17\] showed that \(\text{Var} (NF_{N, \beta}) \leq c_{\beta} N\). When assuming in addition that there are no odd \(p\)-spin interactions, they also showed convergence to a Gaussian variable. As they remark, their approach should also work for spherical models.

### 2.2. Critical points and values.

For \(B \subset \mathbb{R}\) define

\[
\mathcal{C}_N (B) \triangleq \{ \sigma : \nabla H_N (\sigma) = 0, H_N (\sigma) \in B\}.
\]

By an abuse of notation we will also write \(\mathcal{C}_N (a, b)\) for \(\mathcal{C}_N ((a, b))\). In the seminal work \[5\] Auffinger, Ben Arous and Černý proved the following (see also \[4\] for the mixed case).

**Theorem 8.** \[5\] Theorem 2.8] Assume \(p \geq 3\). For any \(E \in \mathbb{R}\), there exists \(\Theta_p (E)\) (defined in \[12,17\] below) so that

\[
\lim_{N \to \infty} \frac{1}{N} \log (\mathbb{E} |\mathcal{C}_N (\infty, NE)|) = \Theta_p (E).
\]

Set \(E_\infty = 2\sqrt{(p-1)/p}\) and

\[
\text{let } E_0 > E_\infty \text{ be the unique number satisfying } \Theta_p (E) = 0.
\]

Critical points of a given index were also considered in \[5\] (the index of a critical point is the number of negative eigenvalues of the Hessian at that point). By Markov’s inequality one has the following.
Corollary 9. [3] Theorem 2.5] For $p \geq 3$, there exists a number $E_1 \in (E_\infty, E_0)$ such that for any $\delta > 0$ there exists $c(\delta) > 0$ for which

$$\mathbb{P} \left\{ \exists \sigma \in \mathcal{C}_N (\infty, -N (E_1 + \delta)) : \sigma \text{ is not a local min} \right\} < e^{-c(\delta)N}.$$  

By a second moment computation for $|\mathcal{C}_N (NB)|$ (where $NB = \{Nx : x \in B\}$), concentration of the number of critical points around its mean was proved in [37].

Theorem 10. [37] Corollary 2] For $p \geq 3$ and $E \in (-E_0, -E_\infty)$,

$$\lim_{N \to \infty} \mathbb{E} \left| \mathcal{C}_N (\infty, NE) \right| = 1, \text{ in } L^2.$$  

Set

$$m_N = -E_0N + \frac{1}{2c_p} \log N - K_0,$$

$$c_p = \frac{d}{dx} \left|_{x=-E_0} \right. \Theta_p (x) = E_0 - \frac{2}{E_\infty^2} \left( E_0 - \sqrt{E_0^2 - E_\infty^2} \right),$$  

where $K_0$ is given in [38] Eq. (2.6)]. Zeitouni and the author [38] proved the convergence of the extremal process of critical points defined by

$$\xi_N \triangleq (1 + \iota_p)^{-1} \sum_{C \in \mathcal{C}_N (\infty, \infty)} \delta_{C_N (\sigma) - m_N},$$  

where $\iota_p = (1 + (-1)^p)/2$ (normalizing the weights so that $\xi_N$ is a simple point process a.s. for even $p$). Let $PPP (\mu)$ denote the distribution of a Poisson point process with intensity measure $\mu$ and endow the space of point processes with the vague topology.

Theorem 11. [38] Theorem 1] For $p \geq 3$,

$$\xi_N \xrightarrow{d_{N \to \infty}} \xi_\infty \sim PPP (e^{c_p x} dx).$$  

Corollary 12. [38] Theorem 1, Corollary 2] For $p \geq 3$, $H_N (\sigma_0^1) = \min_{\sigma} H_N (\sigma)$ converges to the negative of a Gumbel variable, namely, $\mathbb{P} \{ H_N (\sigma_0^1) - m_N \geq x \} \to \exp \{ -c_p^{-1} e^{c_p x} \}$ as $N \to \infty$. Moreover,

$$\forall k \geq 1, \lim_{L \to \infty} \lim_{N \to \infty} \mathbb{P} \{ \mathcal{C}_N (m_N - L, m_N + L) \cap \{ \sigma_0^1 : i \in [k] \} \} = 1,$$

$$\forall L > 0, \lim_{k \to \infty} \lim_{N \to \infty} \mathbb{P} \{ \mathcal{C}_N (m_N - L, m_N + L) \cap \{ \sigma_0^1 : i \in [k] \} \} = 1.$$

Another consequence of the second moment calculation [37] and the bound on the minimum is the following bound on overlaps of critical points.$^{6}$

Corollary 13. For $p \geq 3$, for any $\epsilon > 0$ there exist $\delta (\epsilon), c (\epsilon) > 0$ such that

$$\mathbb{P} \{ \exists \sigma, \sigma' \in \mathcal{C}_N (N (-E_0 - \delta (\epsilon), -E_0 + \delta (\epsilon)), \sigma \neq \pm \sigma' : |R (\sigma, \sigma')| \geq \epsilon \} < e^{-c(\epsilon)N}.$$  

Moreover, there exists a sequence $\epsilon_N > 0$ with $\epsilon_N \to 0$ as $N \to \infty$ such that

$$\lim_{N \to \infty} \mathbb{P} \{ \exists \sigma, \sigma' \in \mathcal{C}_N (-m_N + \sqrt{N}, \sigma \neq \pm \sigma' : |R (\sigma, \sigma')| \geq \epsilon_N \} = 0.$$

$^6$Corollary 13 follows from [38] Eq. (5.2), (5.3)] and since it is shown in the proof of [38] Proposition 4] that [38] Eq. (5.2)] is negative.
Schematically, we think of an overlap-depth plane by defining the contribution of \( A \times B \subset [-1,1] \times \mathbb{R} \) to \( Z_{N,\beta} \) as

\[
\text{Cont}_{N,\beta} (A \times B) = Z_{N,\beta} \left( \bigcup_{\sigma_0 \in \mathcal{E}_N (B)} \{ \sigma : R(\sigma, \sigma_0) \in A \} \right),
\]

The basic picture we prove about (3.1) is that as one ‘scans’ possible depths \( u = EN \) for critical points \( \sigma_0 \) and overlaps \( q \) (in some range, as we explain below), the maximal value for the contribution (3.1) with \( A = (q, q + o(1)) \) and \( B = (u, u + o(1)) \) is obtained with high probability (w.h.p) when \( q = q_* \) and \( u = -E_0 N \). Of course, \( \text{Cont}_{N,\beta} (\cdot) \) is not additive, since bands can intersect each other, and exploring the whole range of \([-1,1] \times \mathbb{R}\) therefore makes no sense; it is additive, however, when restricted to small enough range of overlaps around 1 and low enough depths (clearly, at least when we allow those ranges to depend on the disorder).

The next section explains how we restrict to a small range of overlaps containing \( q_* \) and range of depths near \( u = -E_0 N \).

**Restriction to caps.** Observe that, with \( B = \{ \sigma : H_N(\sigma) \geq u \} \) being the super-level set of \( H_N(\sigma) \) with some level \( u \in \mathbb{R} \), we have that \( Z_{N,\beta} (B) \leq e^{-\beta u} \). In Lemma 29 we shall prove that for an appropriate choice of \( u_{LS} \) and \( q_{LS} \) (see (5.12) and (8.2)) the sub-level set of \( u_{LS} + \delta \) with small \( \delta > 0 \), is covered by \( \bigcup \text{Cap} (\sigma_0, q_{LS}) \), where the union goes over \( \sigma_0 \in \mathcal{E}_N (-\infty, u_{LS} + \delta) \) and we define the spherical caps

\[
\text{Cap} (\sigma_0, q) \triangleq \{ \sigma \in S^{N-1} : q \leq R(\sigma, \sigma_0) \}.
\]

We shall also prove a lower bound on the contribution coming from overlaps roughly \( q_* \) (see (8.9)) by which for the same choice of \( u_{LS} \) the free energy is w.h.p greater than \( e^{-\beta u_{LS} - o(N)} \). Hence, w.h.p the mass of the complement of the caps is negligible. That is, when analyzing contributions we may restrict to

\[
\text{Reg}_{LS} (\delta) = (q_{LS}, 1) \times (-\infty, u_{LS} + \delta) .
\]

(See Figure 3.1 for a graphical description.) In addition, from bounds on the global minimum of \( H_N(\sigma) \) (Corollary 12), with probability going to 1 as \( \kappa' \to \infty \), for large \( N \) the contribution of

\[
\text{Reg}_{GS} (\kappa') = (-1, 1) \times (-\infty, m_N - \kappa')
\]

is equal to 0 w.h.p, where \( m_N \) is defined in (2.7). Combining the above with bounds on the lowest critical values of \( H_N(\sigma) \) (see Corollary 12), we will see that (1.5) follows if we prove:

1. an appropriate lower bound on \( \text{Cont}_{N,\beta} (\text{Reg}_* (c, \kappa, \kappa')) \) where

\[
\text{Reg}_* (c, \kappa, \kappa') = (q_* - cN^{-1/2}, q_* + cN^{-1/2}) \times (m_N - \kappa', m_N + \kappa);
\]

2. upper bounds on \( \text{Cont}_{N,\beta} (A_i \times B_i) \) for some collection of sets \( A_i \times B_i \) covering

\[
\text{Reg}_{UB} (\delta, c, \kappa, \kappa') = \text{Reg}_{LS} (\delta) \setminus (\text{Reg}_* (c, \kappa, \kappa') \cup \text{Reg}_{GS} (\kappa'))
\]

\[
= (q_{LS}, 1) \times (m_N - \kappa', u_{LS} + \delta) \setminus \left( q_* - cN^{-1/2}, q_* + cN^{-1/2} \right) \times (m_N - \kappa', m_N + \kappa).
\]

In Section 8.2 we split the latter to several regions, dealt with separately, corresponding to critical values near and far from \( m_N \) and overlaps near and far from 1, see Figure 8.1.
The Kac-Rice formula. The basic tool we use for deriving bounds is the Kac-Rice formula; see Appendix I. The formula will allow us to bound the expected number of critical points $\sigma_0$ that satisfy various conditions by integrals involving the intensity of the empirical measure of critical values and conditional probabilities that the (arbitrarily chosen) point:

$$\hat{n} \triangleq (0, ..., 0, \sqrt{N})$$

satisfies the aforementioned conditions, given that

$$H_N(\hat{n}) = u \text{ and } \nabla H_N(\hat{n}) = 0.$$  

When combined with Markov’s inequality to upper bound corresponding probabilities, the formula can roughly be thought of as a variant of the union bound, where the intensity accounts for the number of events. In the sections below we explain in more detail how the formula is used to derive bounds on the contributions $\text{Cont}_{N,\beta}(A \times B)$. Prior to deriving those bounds (in Section 8), we will need to investigate the conditional probabilities mentioned above.

Conditional structure around critical points. The Kac-Rice formula allows us to transfer questions dealing with $\text{Cont}_{N,\beta}(A \times B)$ to ones about the conditional law of $\{H_N(\sigma)\}_\sigma$ given (3.5). A useful description of the latter will be obtained in Section 4 by decomposing $H_N(\sigma)$ as a sum of independent fields, see (4.10). For any overlap $q$, on $\{\sigma : R(\sigma, \hat{n}) = q\}$ the $k$-th of those fields is distributed like a pure spherical $k$-spin model multiplied by a factor that is a function of $q$, which can be thought of as an effective temperature (see (4.11)). The effect of conditioning on $H_N(\hat{n}) = u$ and $\nabla H_N(\hat{n}) = 0$ is equivalent to dropping the field corresponding to $k = 1$ and setting the field corresponding to $k = 0$ to be equal to some deterministic function of $\sigma$. On thin bands around overlap $q$, the conditional field is roughly a mixture of pure $k$-spin with $2 \leq k \leq p$, shifted by a constant.

Upper bounds on masses of bands. Denote the conditional law given (3.5) and expectation by $P_{u,0}$ and $E_{u,0}$ (see Remark 16). Abbreviate, only in this subsection, $Z(q) = Z_{N,\beta}(\text{Band}(\hat{n}, q, q + o(1)))$. Combining a variant of the Kac-Rice formula (see Lemmas 40 and 41) with a computation of $E_{u,0}Z(q)$, we will derive an upper bound on the expectation of contributions to the partition function (3.1). By Markov’s inequality, they yield upper bounds on the probabilities that the contributions are not small compared to the mass of the bands.
in Theorem 1. This will be good enough for overlaps sufficiently close to 1. Specifically, in Section 6 we will use such bounds for overlaps in the range \( q \in (q^{**}, 1) \) with \( q^{**} \) defined in (5.10) (see Figure 3.2).

In general, however, what describes the typical behavior (under \( P_{u,0} \)) is the corresponding free energy, i.e., \( \frac{1}{N} E_{u,0} \log (Z(q)) \). By Jensen’s inequality, this free energy is bounded from above by \( \frac{1}{N} \log (E_{u,0} Z(q)) \), which for \( u = -NE \) is asymptotically equal to the expression \( \Lambda_Z(E, q) \) of (5.6) (see Remark 21). In fact, by the methods we use in Section 6 one can check that this bound is tight asymptotically for \( q \in (q_c, 1) \), where \( q_c \) is given in (5.5).

We remark that the bound obtained from Jensen’s inequality is not good enough to bound contributions related to overlaps sufficiently close to \( q^{LS} \). We deal with the range \((q^{LS}, q^{**})\) in Section 7 where we upper bound the \( N \to \infty \) limit of the free energy \( \frac{1}{N} E_{u,0} \log (Z(q)) \) which, with \( u = -NE \), we denote by \( \Lambda_F(E, q) \) (see (7.2)). First, we will relate \( \Lambda_F(E, q) \) to a free energy \( \Lambda_{F,2}(E, q) \) (see (7.4)) which in a certain sense takes into account in a non-trivial way only the \( k = 2 \) component in the decomposition of \( H_N(\sigma) \), i.e., a pure 2-spin. Second, we shall bound the fluctuations (under \( P_{u,0} \)) of \( \frac{1}{N} \log (Z(q)) \) from its mean. Those will allow us to use the Kac-Rice formula to control the number of critical points whose corresponding bands have exceedingly high mass. Together with bounds on the number of critical points obtained from Theorem 8, this yields an upper bound on contribution related to overlaps in the range \((q^{LS}, q^{**})\) w.h.p.

![Figure 3.2. A qualitative graph of \( \Lambda_{F,2}(E_0, q) \) (black) and \( \Lambda_Z(E_0, q) \) (gray), at low temperature (on \((q_c, 1)\) both coincide). Vertical lines correspond to overlap values.](image)

**Lower bound on the mass of a band of overlap \( q_s \).** Conditional on (3.5), the behavior of the Hamiltonian on a thin band \( \text{Band}(\hat{n}, q_s - o(1), q_s + o(1)) \), is essentially described by its behavior on \( \{ \sigma : R(\sigma, \hat{n}) = q_s \} \). As we already mentioned, the latter is distributed like a spherical mixed model, where the coefficients are determined by \( \beta \). As \( \beta \) increases - and the band narrows, i.e., \( q_s \) increases to 1 - the 2-spin interaction becomes more dominant relative to the other spins. This will allow us to prove (see Proposition 19 and Lemmas 24 and 25) that the fluctuations of the free energy attributed to the band are roughly determined only by the 2-spin interaction. We remark that the 2-spin interaction is closely related to the Hessian of the Hamiltonian at the center of the band, which is key to Corollary 4. Also, this interaction will have an ‘effective’ high temperature which, based on the convergence result of Baik and Lee [7] (Theorem 7), implies Gaussian fluctuations for the free energy. Combined with the Kac-Rice formula this will be used to show that w.h.p there are no bands corresponding to (3.3) with relative mass which is too low. On the other hand, from Theorem 11 we know
that the probability that there are no critical points with values in the range corresponding to \((3.3)\) goes to 0 as \(N \to \infty\). In Proposition 30 we will this to conclude that there exist bands corresponding to \((3.3)\) with large enough relative mass and prove a lower bound on the contribution of \((3.3)\).

**Overlap distribution.** We shall compute the first and second moments of the mass of bands as in Theorem 1 (see Proposition 18 and Lemma 26) and see that: the contribution to the second moment coming from pairs of points whose overlap ‘inside the band’ is not approximately 0 is negligible; and that the ratio of second to first moment squared converges to a constant as \(N \to \infty\). Combined with the lower bound on the mass of corresponding bands which we discussed in the previous subsection, this will yield (1.6). To prove (1.7) from the latter, we will use Corollary 13, which states that deep critical points are either antipodal or essentially orthogonal, and a simple deterministic geometric argument (see Lemma 32).

4. Decomposition around (critical) points

A crucial ingredient in our analysis is a certain decomposition of \(H_N(\sigma)\) around a given point on the sphere, which we develop in this section. We shall use \(\hat{n}\) as the center point, but since the Hamiltonian is invariant under rotations of the sphere, similar results hold for an arbitrary point on the sphere. Let \((E_i)_{i=1}^{N-1} = (E_i(\sigma))_{i=1}^{N-1}\) be a smooth orthonormal frame field defined over a neighborhood of \(\hat{n}\) (relative to the standard Riemannian metric). With \(P_{\hat{n}} : (x_1, ..., x_N) \mapsto (x_1, ..., x_{N-1})\) denoting the projection from the hemisphere \(\{x \in S^{N-1}(\sqrt{N}) : x_N > 0\}\) to \(\mathbb{R}^{N-1}\), we shall assume that for any smooth function \(g : S^{N-1}(\sqrt{N}) \to \mathbb{R}\):

\[
E_ig(\hat{n}) = \frac{d}{dx_i} g \circ P_{\hat{n}}^{-1}(0), \quad E_iE_jg(\hat{n}) = \frac{d}{dx_i} \frac{d}{dx_j} g \circ P_{\hat{n}}^{-1}(0).
\]

Define \(F_k\) as the \(\sigma\)-algebra generated by

\[
\{E_{i_1} \cdots E_{i_j} H_N(\hat{n}) : 1 \leq i_1 \leq \cdots \leq i_j \leq N - 1, 0 \leq j \leq k\},
\]

that is, \(H_N(\hat{n})\) and all the derivatives (by \(E_i\)) of \(H_N(\sigma)\) at \(\hat{n}\) up to order \(k\). Setting \(\hat{H}_{N,k}^0(\sigma) = \mathbb{E}[H_N(\sigma) | F_k]\), define \(\hat{H}_{N,k}^0(\sigma) = \hat{H}_{N,k}^0(\sigma) - \hat{H}_{N,k-1}^0(\sigma)\) for \(k > 0\), and \(\hat{H}_{N}^0(\sigma) = \hat{H}_{N}^0(\sigma)\).

Define \(H_{N,\text{Eucl}}^\text{uc}(x)\) as the extension of \(H_N(\sigma)\) to \(\mathbb{R}^N\) defined by the formula (1.1) only with general \(x \in \mathbb{R}^N\) instead of \(\sigma\) from the sphere. Clearly, \(H_N(\hat{n})\) and the derivatives of \(H_N(\sigma)\) at \(\hat{n}\) up to order \(k\) are determined by \(H_{N,\text{Eucl}}^\text{uc}(\hat{n})\) and the Euclidean derivatives of \(H_N^\text{Eucl}(x)\) at

---

7. I.e., the overlap between the projections of the points to the orthogonal space to the center point \(\sigma_0 \in \mathbb{R}^N\).

8. The fact that such frame field exists can be seen from the following. If we let \(\{\frac{\partial}{\partial x_i}\}_{i=1}^{N-1}\) be the pullback of \(\{\frac{\partial}{\partial x_i}\}_{i=1}^{N-1}\) by \(P_{\hat{n}}\), then \(\{\frac{\partial}{\partial x_i}(\hat{n})\}_{i=1}^{N-1}\) is an orthonormal frame at the north pole. For any point in a small neighborhood of \(\hat{n}\) we can define an orthonormal frame as the parallel transport of \(\{\frac{\partial}{\partial x_i}(\hat{n})\}_{i=1}^{N-1}\) along a geodesic from \(\hat{n}\) to that point. This yields an orthonormal frame field on that neighborhood, say \(E_i(\sigma) = \sum_{j=1}^{N-1} a_{ij}(\sigma) \frac{\partial}{\partial x_j}(\sigma), i = 1, ..., N - 1\). Working with the coordinate system \(P_{\hat{n}}\) one can verify that at \(x = 0\) the Christoffel symbols \(\Gamma_{ij}^{k}\) are equal to 0, and therefore (see e.g., [29, Eq. (2), P. 53]) the derivatives \(\frac{\partial}{\partial x_i} a_{ij}(\hat{n})\) at \(x = 0\) are also equal to 0.
The advantage of this is that the Euclidean derivatives (4.3) are directly related to the disorder $J_{i_1, \ldots, i_p}$.

The following lemma directly follows. For $0 \leq k \leq p$, let $H^\text{pure}_N (\sigma)$ be independent pure $k$-spin models on the sphere $S^{N-2}$, where by 0-spin model we simply mean a constant centered Gaussian field with variance $\sqrt{N-1}$. Also, with $\sigma = (\sigma_1, \ldots, \sigma_N)$ define

$$\tilde{\sigma} = \sqrt{\frac{N-1}{N}} \frac{\sigma_1, \ldots, \sigma_{N-1}}{\sqrt{1-q^2(\sigma)}} \in S^{N-2} \text{ and } q(\sigma) = \frac{\sigma_N}{\sqrt{N}} = R(\sigma, \hat{n}).$$

Lastly, define

$$\alpha_k(q) \triangleq \sqrt{\left( \frac{p}{k} \right) (1-q^2)^k q^{p-k}},$$

where obviously $\sum_{k=0}^p \alpha_k(q) = (1-q^2 + q^2)^p = 1$. Note that $\alpha_k(q)$ can be negative. Denote by $\equiv$ equality in distribution.

**Lemma 14.** The Hamiltonian decomposes as

$$H_N(\sigma) = \sum_{k=0}^p \tilde{H}^k_N(\sigma),$$

where $\tilde{H}^k_N(\sigma)$ is the Hamiltonian with $k$-spin interactions.
where the Gaussian fields \( \{ \tilde{H}_{N}^{k} (\sigma) \} \), \( 0 \leq k \leq p \), are independent of each other and

\[
\{ \tilde{H}_{N}^{k} (\sigma) \} \sim d \left\{ \alpha_k (q (\sigma)) \sqrt{\frac{N}{N-1}} H_{N-1}^{\text{pure} k} (\sigma) \right\}.
\]

**Proof.** Equation (4.10) follows from (4.7) and (4.4). Independence follows from that of \( J_{i_1, \ldots, i_p} \) of each other. By simple algebra

\[
\tilde{H}_{N}^{k} (\sigma) = \sqrt{\frac{N}{N-1}} \left( \frac{p}{k} \right)^{1/2} \alpha_k (q (\sigma)) \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq N-1} J_{i_1, \ldots, i_k, N, \ldots, N} \bar{\sigma}_{i_1} \cdots \bar{\sigma}_{i_k},
\]

where \( \bar{\sigma}_i \) are the elements of \( \bar{\sigma} \). Using the fact that

\[
\text{Var} \left( J_{i_1, \ldots, i_k, N, \ldots, N} \right) = \left( \frac{p}{k} \right) \text{Var} \left( J_{i_1, \ldots, i_k} \right),
\]

(4.11) follows from (4.4) (with \( N \) instead of \( N \) in the latter).

The following lemma expresses \( \tilde{H}_{N}^{k} (\sigma) \), \( k \leq 2 \), in terms of the Hamiltonian \( H_{N} (\sigma) \) directly. Denote

\[
\nabla H_{N} (\hat{n}) = (E_i H_{N} (\hat{n}))_{i=1}^{N-1} \quad \text{and} \quad \nabla^2 H_{N} (\hat{n}) = (E_i E_j H_{N} (\hat{n}))_{i,j=1}^{N-1}.
\]

We define the random matrix

\[
G (\hat{n}) := G_{N-1} (\hat{n}) \triangleq \nabla^2 H_{N} (\hat{n}) + \frac{p}{N} H_{N} (\hat{n}) I,
\]

where \( I := I_{N-1} \) denotes the identity matrix of dimension \( N-1 \). A random matrix \( M \) from the \( N \times N \) Gaussian orthogonal ensemble, or for brevity an \( N \times N \) GOE matrix, is a real, symmetric matrix such that all elements are centered Gaussian variables which, up to symmetry, are independent with variance \( 2/N \) on the diagonal and \( 1/N \) off the diagonal.

**Lemma 15.** We have that

\[
\tilde{H}_{N}^{0} (\sigma) = q^0 (\sigma) \cdot H_{N} (\hat{n}),
\]

\[
\tilde{H}_{N}^{1} (\sigma) = q^0 (\sigma) \left( 1 - q^2 (\sigma) \right)^{1/2} \sqrt{\frac{N}{N-1}} \cdot \langle \nabla H_{N} (\hat{n}), \bar{\sigma} \rangle,
\]

\[
\tilde{H}_{N}^{2} (\sigma) = \frac{1}{2} q^{p-2} (\sigma) \left( 1 - q^2 (\sigma) \right) \frac{N}{N-1} \cdot \bar{\sigma}^T G_{N-1} (\hat{n}) \bar{\sigma},
\]

and \( H_{N} (\hat{n}), \nabla H_{N} (\hat{n}), \) and \( G_{N-1} (\hat{n}) \) are independent. Moreover, \( \nabla H_{N} (\hat{n}) \sim \mathcal{N} (0, p I_{N-1}) \) and with \( M \) being a GOE matrix of dimension \( N-1 \),

\[
G_{N-1} (\hat{n}) \sim d \sqrt{\frac{N-1}{N-1}} \frac{p}{(p-1) M}.
\]

**Proof.** Since \( H_{N} (\hat{n}) = N^{1/2} J_{N_N} \) and \( q (\sigma) = \sigma_N / \sqrt{N} \), (4.13) follows from (4.7) (with \( k = 0 \)). From (4.1), (4.4) and the fact that

\[
\frac{d}{dx} \bigg|_{x=0} \left( x_{i_1} \cdots x_{i_k} \left( N - \sum_{i=1}^{N-1} x_i^2 \right)^{p-1} \right) = \begin{cases} N^{p-1} / 2, & k = 1, i_1 = i, \\ 0, & \text{otherwise} \end{cases},
\]

we obtain that

\[
E_i H_{N} (\hat{n}) = J_{i,N_N} \sim \mathcal{N} (0, p).
\]
Thus, (4.14) follows from (4.7) (with $k = 1$) and some rearranging. The independence of $(J'_{i,j,N,...,N})$ implies that $\nabla H_N(\hat{n}) \sim \mathcal{N}(0, p I_{N-1})$.

Similarly, from (4.1), (4.4) and the fact that

$$\frac{d}{dx_i} \frac{d}{dx_j} \bigg|_{x=0} \left( \frac{x_{i1} \cdots x_{ik}}{N - \sum_{i=1}^{N-1} x_i^2} \right)^{\frac{k}{2}} = \begin{cases} (1 + \delta_{ij}) \frac{N^{p-2}}{2}, & k = 2, \{i_1, i_2\} = \{i, j\} \\ -\delta_{ij} p N^{p-2}, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

we obtain that

$$E_i E_j H_N(\hat{n}) = N^{-1/2} \left( (1 + \delta_{ij}) J'_{i,j,N,...,N} - \delta_{ij} p J'_{N,...,N} \right),$$

and thus

$$(4.18) \quad (G_{N-1}(\hat{n}))_{ij} = N^{-1/2} (1 + \delta_{ij}) J'_{i,j,N,...,N}.$$ 

Since $J'_{i,j,N,...,N} \sim \mathcal{N}(0, p (p - 1) / (1 + \delta_{ij}))$ are independent, (4.16) follows. Substituting this in (4.7) (with $k = 2$), after some algebra we obtain (4.15).

Lastly, the independence $H_N(\hat{n}), \nabla H_N(\hat{n})$, and $G_{N-1}(\hat{n})$ follows since they are measurable w.r.t to disjoint subsets of the random variables $\{J'_{1,...,i_p}\}$.

□

Denote

$$(4.19) \quad \tilde{H}^{n_{k+}}_N(\sigma) = \sum_{m=k}^{p} \tilde{H}^{n_m}_N(\sigma).$$

**Remark 16.** In many situations in the sequel we will have some random variable $X$ for which $H_N(\hat{n}), \nabla H_N(\hat{n})$ and $X$ have a continuous joint density and we will need to compute or estimate the probability that $X \in B$, for some measurable set $B$, conditional on $H_N(\hat{n}) = u, \nabla H_N(\hat{n}) = 0$. In this case, the notation $\mathbb{P}_{u,0}\{X \in B\}$ should be understood as the probability under the law determined the usual conditional density given by the ratio of the joint densities. That is, if $\varphi_1(v, w, x)$ and $\varphi_2(v, w)$ are the continuous densities of $(H_N(\hat{n}), \nabla H_N(\hat{n}), X)$ and $(H_N(\hat{n}), \nabla H_N(\hat{n}))$, respectively, then $\mathbb{P}_{u,0}\{X \in B\} = \int_B \varphi_1(u, 0, x) dx / \varphi_2(u, 0)$. We will also refer to $\mathbb{P}_{u,0}\{X \in \cdot\}$ as the conditional law of $X$ under $\mathbb{P}_{u,0}\{\cdot\}$. We define $\mathbb{P}_{u,0,A}\{X \in B\}$ similarly, assuming the corresponding joint density exists, as the conditional probability given

$$(4.20) \quad H_N(\hat{n}) = u, \nabla H_N(\hat{n}) = 0 \text{ and } G_{N-1}(\hat{n}) = A.$$ 

The conditional expectations corresponding to the above will denoted by $E_{u,0}\{\cdot\}$ and $E_{u,0,A}\{\cdot\}$.

**Corollary 17.** The conditional law of the field $H_N(\sigma)$ under $\mathbb{P}_{u,0}\{\cdot\}$ is identical to the (unconditional) law of

$$u q^p (\sigma) + \tilde{H}^{n_{2+}}_N(\sigma).$$

Similarly, the conditional law of the field $H_N(\sigma)$ under $\mathbb{P}_{u,0,A}\{\cdot\}$ is identical to the (unconditional) law of

$$u q^p (\sigma) + \frac{1}{2} q^{p-2} (\sigma) (1 - q^2(\sigma)) \frac{N}{N-1} \cdot \tilde{\sigma}^T A \tilde{\sigma} + \tilde{H}^{n_{3+}}_N(\sigma).$$
5. Important overlap values

Our analysis requires understanding the contribution to the partition function $Z_{N,\beta}$ coming from different distances, or equivalently overlaps, from critical points. In this section we define several important overlap values that will be used to define different regions in the overlap-depth plane of Section 3 (see Figure 3.1). It will be very useful for us to investigate the restriction of the random fields $\bar{H}_{N}^{h,k}(\sigma)$ (with $k = 2$ in particular) to

$$\{\sigma \in \mathbb{S}^{N-1} : R(\sigma, \hat{n}) = q\},$$

which for convenience we parametrize as random fields on $\mathbb{S}^{N-2}$. With $\theta_q : \mathbb{S}^{N-2} \to \mathbb{S}^{N-1}$ being the left inverse of $\sigma \mapsto \tilde{\sigma}$ (see (4.8)) given by

$$\theta_q ((\sigma_1, \ldots, \sigma_{N-1})) = \sqrt{\frac{N}{N-1}} (1 - q^2) (\sigma_1, \ldots, \sigma_{N-1}, 0) + q \hat{n},$$

for any function $h : \mathbb{S}^{N-1} \to \mathbb{R}$ define $h|_q : \mathbb{S}^{N-2} \to \mathbb{R}$ by

$$(5.1) h|_q (\sigma) = h \circ \theta_q (\sigma).$$

Note that by (4.11), $\bar{H}_{N}^{h,k}|_q$ is a pure $k$-spin for any $q$, up to a multiplicative factor. Specifically, we have that

$$(5.2) \mathbb{E} \left\{ \bar{H}_{N}^{h,2}|_q (\sigma) \bar{H}_{N}^{h,2}|_q (\sigma') \right\} = N (\alpha_2 (q) R (\sigma, \sigma'))^2,$$

where $\alpha_2 (q)$ was defined in (4.9). With $\beta$ fixed, we can think of the partition function corresponding to $\bar{H}_{N}^{h,2}|_q$ as that of the pure 2-spin model on $\mathbb{S}^{N-2}$ with an ‘effective’ temperature

$$(5.3) \beta |_{\alpha_2 (q)} \sqrt{\frac{N}{N-1}}.$$

By Corollary 6 with $\beta$ fixed, the limiting free energy of $\bar{H}_{N}^{h,2}|_q$ undergoes a transition at values of $q$ that satisfy

$$(5.4) \alpha_2 (q) = \left(\frac{p}{2}\right)^{1/2} q^{p-2} (1 - q^2) = \frac{1}{\beta \sqrt{2}} \triangleq \chi_2.$$

Define, assuming $\beta$ is large enough so that the set below is non-empty,

$$(5.5) q_c := q_c (\beta) = \max \{q \in (0, 1) : \alpha_2 (q) = \chi_2\}.$$

In our computations we will encounter the function

$$(5.6) \Lambda_{Z} (E, q) \triangleq \frac{1}{2} \log (1 - q^2) + \beta Eq^p + \frac{1}{2} \beta^2 (1 - q^{2p} - pq^{2p-2} (1 - q^2)),$$

which is related to the free energy of bands of overlap approximately $q$ around critical points of depth $-EN$ (see Lemma 20). The critical points of $\Lambda_{Z} (E_0, q)$ as a function of $q$ are the solutions of

$$(5.7) -\frac{q}{1 - q^2} + p \beta E_0 q^{p-1} - p (p - 1) \beta^2 q^{2p-3} (1 - q^2) = 0.$$

Viewing the latter as a quadratic equation in $\beta$ we have that the solutions with $q \neq 0$ are characterized by the relation

$$(5.8) \alpha_2 (q) = \frac{1}{\sqrt{2} \beta} \left( \frac{E_0}{E_\infty} \pm \sqrt{\frac{E_0^2}{E_\infty^2} - 1} \right).$$
From the fact that $z - \sqrt{z^2 - 1}$ decreases in $z > 1$ and since $E_0 > E_\infty$, 

\[(5.9) \quad \chi_1 \triangleq \frac{1}{\sqrt{2\beta}} \left( \frac{E_0}{E_\infty} - \sqrt{\frac{E_0^2}{E_\infty^2} - 1} \right) < \chi_2 < \frac{1}{\sqrt{2\beta}} \left( \frac{E_0}{E_\infty} + \sqrt{\frac{E_0^2}{E_\infty^2} - 1} \right) \triangleq \chi_3.\]

Thus, assuming $\beta$ is sufficiently large so that the sets below are non-empty, defining

\[(5.10) \quad q_* := q_*(\beta) = \max \{ q \in (0, 1) : \alpha_2(q) = \chi_1 \},
q_{**} := q_{**}(\beta) = \max \{ q \in (0, 1) : \alpha_2(q) = \chi_3 \},\]

which are in particular critical points of $\Lambda_Z(E, q)$ in $q$, we have that 

\[0 < q_{**} < q_c < q_* < 1.\]

(See Figure 3.2.) We also have

\[(5.11) \quad \lim_{\beta \to \infty} \frac{1}{2} (1 - q_*^2)^2 \frac{\partial^2}{\partial q^2} \Lambda_Z(E_0, q_*) = 2 (\beta \chi_1)^2 - 1 < 2 (\beta \chi_2)^2 - 1 = 0.\]

Thus, for large enough $\beta$, $\frac{\partial^2}{\partial q^2} \Lambda_Z(E_0, q_*) < 0$.

With an arbitrary constant which will be fixed $C_{LS} > (2p(E_0 - E_\infty))^{-1}$, the last overlap value we define is

\[(5.12) \quad q_{LS} = 1 - C_{LS} \frac{\log \beta}{\beta}.\]

This overlap value is the one we will use to define the caps which we restrict to, as described in the outline in Section 3.

6. Mass of Bands under $\mathbb{P}_{u,0}$: The Range $(q_{**}, 1)$

In this section we evaluate the relative partition function of bands and caps. To shorten the notation, we will henceforth use the abbreviations

\[(6.1) \quad \text{Cap} \triangleq \text{Cap}(\hat{n}, q_{**}),
\text{Band}(\epsilon) \triangleq \text{Band}(\hat{n}, q_* - \epsilon, q_* + \epsilon).\]

The main results of this section are the two propositions below. In Proposition 18 we compute, under $\mathbb{P}_{u,0}$, the expected relative partition function of Band $(cN^{-1/2})$, and show that for large $c$ it is much larger than that of Cap $\setminus$ Band $(cN^{-1/2})$. The levels $u$ considered in the proposition are approximately equal to $m_N$ (in fact for higher levels the overlap that captures most of the mass is not $q_*$). However, because of the simple dependence of the conditional law in $u$ (see Corollary 17) we will be able to easily derive from Proposition 18 bounds for higher levels when needed (e.g., in the proof of Lemma 8.12). As mentioned in Section 3, bounds on the expected relative partition function will be sufficient for our analysis of overlaps close enough to 1. Specifically, the caps (6.1) cover the range $(q_{**}, 1)$. Define

\[(6.2) \quad \mathcal{M}_{N,\beta}(u) \triangleq (1 - q_*^2)^{-3/2} \left[ \frac{\partial^2}{\partial q^2} \Lambda_Z(E_0, q_*) \right]^{-1/2} \exp \left\{ N \Lambda_Z(E_0, q_*) - (\beta E_0 N + \beta u) q_*^p \right\}.\]
Proposition 18. For large enough $\beta$ we have the following. Let $a_N = o(\sqrt{N})$ be a sequence of positive numbers and set $J_N = (m_N - a_N, m_N + a_N)$. Then,

\[(6.3) \lim_{c \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \left| \frac{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) \right) \right\}}{E_{N,\beta} (u)} - 1 \right| = 0,\]

and

\[(6.4) \lim_{c \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \frac{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Cap} \setminus \text{Band} \left( cN^{-1/2} \right) \right) \right\}}{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) \right) \right\}} = 0.\]

With $\chi_1$ given by (5.9), define

\[(6.5) C_* = 1 - 2 (\beta \chi_1)^2 = 1 - \left( \frac{E_0}{E_{\infty}} - \sqrt{\frac{E_0^2}{E_{\infty}^2} - 1} \right)^2\]

and note that by (5.9) and (5.4), $C_* > 0$. Define

\[(6.6) Y_* \sim \mathcal{N} \left( \frac{1}{4} \log(C_*), -\frac{1}{2} \log(C_*) \right).\]

The following proposition is key to controlling the mass of the bands in (1.5).

Proposition 19. For $\beta > 0$ large enough we have the following. Let $a_N = o(N)$ and $\epsilon_N = o(1)$ be sequences of positive numbers and set $J_N = (m_N - a_N, m_N + a_N)$. Let $Y_*$ be defined as in (6.6). Then,

\[(6.7) \lim_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{N} \log \left( \frac{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( \epsilon_N \right) \right) \right\}}{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Cap} \setminus \text{Band} \left( \epsilon \right) \right) \right\}} \right) - \Lambda \left( E_0, q_* \right) \right\} = 0.\]

In Section 6.1 we prove Proposition 18. We proceed with a corresponding second moment computation in Section 6.2. In Section 6.3 we prove a version of Proposition 19 where the ‘3-and-above’ spins in the decomposition (4.10) are averaged out. Lastly, we prove Proposition 19 in Section 6.4 by essentially showing that this averaging has no effect in the scale we work in.

6.1. Proof of Proposition 18. The first step in the proof of the proposition, is the computation of expectations on exponential scale in the lemma below.

Lemma 20. For large enough $\beta$ we have the following. Let $a_N > 0$ be a sequence such that $a_N/N \to 0$ and set $J_N = (m_N - a_N, m_N + a_N)$. Then for any $c, \epsilon > 0$,

\[(6.8) \lim_{N \to \infty} \sup_{u \in J_N} \left| \frac{1}{N} \log \left( \frac{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) \right) \right\}}{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Cap} \setminus \text{Band} \left( \epsilon \right) \right) \right\}} \right) - \Lambda \left( E_0, q_* \right) \right| = 0,\]

and

\[(6.9) \lim_{N \to \infty} \sup_{u \in J_N} \left| \frac{1}{N} \log \left( \frac{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Cap} \setminus \text{Band} \left( \epsilon \right) \right) \right\}}{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( \epsilon \right) \right) \right\}} \right) \right| < \Lambda \left( E_0, q_* \right).\]

Proof. If $\{q_i\}_{i=0}^k$ is a finite sequence such that, with $j < k$,

\[(6.10) q_* = q_0 < q_1 < \cdots < q_j = q_* - \epsilon < q_* + \epsilon = q_{j+1} < \cdots < q_k = 1,\]

then
then
\begin{equation}
(6.11) \quad \mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Cap} \setminus \text{Band} (\epsilon)) \} = \sum_{i \neq j} \mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\hat{n}, q_i, q_{i+1})) \}.
\end{equation}

Using the co-area formula with the mapping \( \sigma \mapsto R(\sigma, \hat{n}) \) we obtain
\begin{equation}
(6.12) \quad \mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\hat{n}, q_i, q_{i+1})) \} = \int_{q_i}^{q_{i+1}} \Phi_{N,\beta,u}^{(1)} (q) \, dq,
\end{equation}
where, using Corollary [17] and (4.11),
\begin{equation}
(6.13) \quad \Phi_{N,\beta,u}^{(1)} (q) \triangleq \frac{\omega_{N-1}}{\omega_N} (1 - q^2) \frac{N-3}{2} \Xi_{N,\beta,u}^{(1)} (q),
\end{equation}
and,
\begin{equation}
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\hat{n}, q - s, q + s)) \} \right) = o(s).
\end{equation}

For small enough \( \delta > 0 \),
\begin{equation}
\sup_{u \in J_N} \frac{1}{N} \log \left( \mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Cap} (\hat{n}, 1 - \delta)) \} \right)
\end{equation}
is as negative as we wish. Hence, for small enough \( \delta \), denoting \( I = (q_*, 1 - \delta) \setminus [q_* - \epsilon, q_* + \epsilon] \), using (6.11) and refining the partition (6.10) if needed, we have that the left-hand side of (6.9) is bounded from above by
\begin{equation}
\limsup_{N \to \infty} \sup_{u \in J_N} \frac{1}{N} \log \left( \Phi_{N,\beta,u}^{(1)} (q) \right) \leq \sup_{q \in I} \Lambda_Z (E_0, q).
\end{equation}

In a similar manner, from the representation (6.12) with \( q_i = q_* - cN^{-1/2} \) and \( q_{i+1} = q_* + cN^{-1/2} \) and since \( \Lambda_Z (E_0, q) \) is continuous in \( q \) in a neighborhood of \( q_* \), we have that (6.8) holds.

Thus, in order to finish the proof it is enough to show that \( q_* \) is the unique maximum point of \( \Lambda_Z (E_0, q) \) in the interval \( q \in [q_*, 1] \). This follows since \( q_* \) is the only critical point (see Section 5) in the interior of the interval and since by (5.11), \( \frac{\partial^2}{\partial q^2} \Lambda_Z (E_0, q_* < 0 \), for large \( \beta \). \)

\begin{remark}
From (6.12) it also follows that for any \( E > 0 \) and \( q \in (-1, 1) \), if \( a_N, \epsilon_N > 0 \) are sequences such that \( a_N = o(N) \), \( \epsilon_N = o(N) \) and \( \log \epsilon_N = o(N) \), then, setting \( J_N = (-NE - a_N, -NE + a_N) \),
\begin{equation}
\limsup_{N \to \infty} \sup_{u \in J_N} \frac{1}{N} \log \left( \mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\hat{n}, q, q + \epsilon_N)) \} \right) - \Lambda_Z (E, q) = 0.
\end{equation}

We note that \( \Lambda_Z (E, q) \) coincides (when taking into account small differences in the definition of the model) with the TAP free energy computed by Kurchan, Parisi and Virasoro [29] Eq.
(20]) and Crisanti and Sommers [23, Eq. (7)] for a pure state at energy $E$ and such that the overlap inside the state is $q^2$. Recall that Theorem 1 states that if two samples are taken from a band of overlaps approximately $q_\ast$ as in (1.5), then the overlap of the samples is generically equal to $q^2$. The proof of this fact generalizes to any $q > q_\ast$ instead of $q_\ast$. The condition $q > q_\ast$ corresponds to [29, Eq. (25)] which is referred to in [29] as a condition for stability with respect to fluctuations inside a cluster.

We continue with the proof of Proposition 18. It follows from Lemma 20 that there exists some sequence $\epsilon_N$ such that $\epsilon_N \to 0$ as $N \to \infty$

$$\lim_{N \to \infty} \sup_{u \in J_N} \frac{\mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Cap} \setminus \text{Band} (\epsilon_N)) \}}{\mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (cN^{-1/2})) \}} = 0,$$

for any $c > 0$ (with no information provided, however, on the rate of convergence of $\epsilon_N$ to 0). In order to prove (6.14), what remains to show is that

$$\lim_{c \to \infty} \lim_{N \to \infty} \sup_{u \in J_N} \frac{\mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\epsilon_N) \setminus \text{Band} (cN^{-1/2})) \}}{\mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (cN^{-1/2})) \}} = 0.$$

By (6.12), with

$$A_1 := A_{1,N} = (q_\ast - \epsilon_N, q_\ast + \epsilon_N), \quad A_2 := A_{2,N} = \left( q_\ast - cN^{-1/2}, q_\ast + cN^{-1/2} \right),$$

the ratio of expectations in (6.14) is equal to

$$\frac{\int_{A_1 \setminus A_2} \Phi_{N,\beta,u}^{(1)} (q) \, dq}{\int_{A_2} \Phi_{N,\beta,u}^{(1)} (q) \, dq}.$$

Recall that in (5.11) we showed that for $\beta$ large, $\frac{\partial^2}{\partial q^2} \Lambda_Z (E_0, q_\ast) < 0$ and that $q_\ast$ was chosen such that $\frac{\partial}{\partial q} \Lambda_Z (E_0, q_\ast) = 0$ (see (5.7)-(5.9)). Therefore, as $q \to q_\ast$,

$$\Lambda_Z (E_0, q) = \Lambda_Z (E_0, q_\ast) + \frac{1}{2} \frac{\partial^2}{\partial q^2} \Lambda_Z (E_0, q_\ast) (q - q_\ast)^2 + o \left( (q - q_\ast)^2 \right).$$

From our assumption made in Proposition 18 that $a_N = o \left( \sqrt{N} \right)$, uniformly in $u \in J_N$ and $q \in q_\ast + (-\epsilon_N, \epsilon_N) \cup (-cN^{-1/2}, cN^{-1/2})$, with some $v (q, u, N) = o \left( \sqrt{N} \right)$,

$$\beta E_0 N + \beta u \right) q^p = \left( \beta E_0 N + \beta u \right) q_\ast^p + v (q, u, N) \cdot (q - q_\ast),$$

and, from (6.13),

$$\Phi_{N,\beta,u}^{(1)} (q) = \frac{\omega_{N-1}}{\omega_N} (1 - q^2)^{-3/2} \exp \{ N \Lambda_Z (E_0, q) - (\beta E_0 N + \beta u) q^p \}
= (1 + o(1)) \sqrt{\frac{N}{2\pi}} (1 - q_\ast^2)^{-3/2} \exp \{ N \Lambda_Z (E_0, q_\ast) - (\beta E_0 N + \beta u) q_\ast^p \}
\times \exp \left\{ \frac{1}{2} N \frac{\partial^2}{\partial q^2} \Lambda_Z (E_0, q_\ast) \cdot (q - q_\ast)^2 + N \cdot o \left( (q - q_\ast)^2 \right) + v (q, u, N) \cdot (q - q_\ast) \right\},$$

where we used the fact that $\sqrt{N} \omega_N / \omega_{N-1} \to \sqrt{2\pi}$ as $N \to \infty$ and the $o(1)$ term is with respect to taking $N \to \infty$. 
By the change of variables $\sqrt{N} (q - q_*) \mapsto s$ we obtain that, uniformly in $u \in J_N$,
\[
\lim_{c \to \infty} \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R} \setminus (-c,c)} \exp \left\{ \frac{1}{2} \frac{\partial^2}{\partial q^2} \Lambda_Z (q_*) s^2 \right\} ds = 0.
\]
Similarly, uniformly in $u \in J_N$,
\[
\lim_{c \to \infty} \lim_{N \to \infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R} \setminus (-c,c)} \exp \left\{ \frac{1}{2} \frac{\partial^2}{\partial q^2} \Lambda_Z (q_*) s^2 \right\} ds = 0.
\]
This proves (6.14) and therefore (6.4). By (6.12), the last equation also proves (6.3), which completes the proof of Proposition 18.

6.2. A second moment calculation. For $\sigma, \sigma', \sigma_0 \in S^{N-1}$ define the projective overlap of $\sigma$ and $\sigma'$ relative to $\sigma_0$ by
\[
R_{\sigma_0} (\sigma, \sigma') = R \left( \sigma - R (\sigma, \sigma_0) \sigma_0, \sigma' - R (\sigma', \sigma_0) \sigma_0 \right).
\]
For any Borel set $B \subset S^{N-1}$ and $I_R \subset [-1,1]$, define the subset
\[
T_{\sigma_0} (B; I_R) \triangleq \left\{ (\sigma, \sigma') \in B \times B : R_{\sigma_0} (\sigma, \sigma') \in I_R \right\}.
\]
For any Borel set $B_2 \subset (S^{N-1})^2$ define
\[
(Z \times Z)_{N,\beta} (B_2) \triangleq \int_{B_2} \exp \left\{ -\beta \left( H_N (\sigma) + H_N (\sigma') \right) \right\} d\mu_N \otimes \mu_N (\sigma, \sigma'),
\]
and, by an abuse of notation,
\[
(Z \times Z)_{N,\beta} (\text{Band} (\sigma_0, q, q') ; I_R) \triangleq (Z \times Z)_{N,\beta} (T_{\sigma_0} (\text{Band} (\sigma_0, q, q') ; I_R)).
\]
Note that $(Z \times Z)_{N,\beta} (\text{Band} (\sigma_0, q, q') ; [-1,1]) = (Z_{N,\beta} (\text{Band} (\sigma_0, q, q')))^2$.

Lemma 22. For large enough $\beta$ we have the following. Let $a_N > 0$ be a sequence such that $a_N/N \to 0$ and set $J_N = (m_N - a_N, m_N + a_N)$. Define $C_*$ by (6.5). Then:

1. For any $c > 0$,
\[
\lim_{N \to \infty} \sup_{u \in J_N} \frac{1}{N} \log \left( \mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} (\text{Band} (cN^{-1/2})) \right)^2 \right\} \right) - 2\Lambda_Z (E_0, q_*) = 0.
\]
2. For any $c > 0$ and $\rho_0 > 0$,
\[
\lim_{N \to \infty} \sup_{u \in J_N} \frac{1}{N} \log \left( \mathbb{E}_{u,0} \left\{ (Z \times Z)_{N,\beta} (\text{Band} (cN^{-1/2}) ; [-1,1] \setminus (-\rho_0, \rho_0)) \right\} \right) < 2\Lambda_Z (E_0q_*)\]
(3) For any $c > 0$,

\[
\lim_{\rho \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \frac{\mathbb{E}_{u,0} \left\{ (Z \times Z)_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) ; [-1,1] \setminus (-\rho N^{-1/2}, \rho N^{-1/2}) \right) \right\}}{\left( \mathbb{E}_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) \right) \right\} \right)^2} = 0.
\]

(4) For any $c > 0$,

\[
\lim_{N \to \infty} \sup_{u \in J_N} \left| \frac{\mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) \right) \right)^2 \right\}}{\left( \mathbb{E}_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) \right) \right\} \right)^2} - \frac{1}{\sqrt{C_\ast}} \right| = 0.
\]

Proof. Using the co-area formula with the mapping

\[
(\sigma, \sigma') \mapsto (q_1, q_2, \varrho) = (R(\sigma, \hat{n}), R(\sigma', \hat{n}), R_\alpha(\sigma, \sigma')),
\]

we have that, for $I_R = [-1, 1]$ or $I_R = [-1, 1] \setminus (-\rho_0, \rho_0)$,

\[
\mathbb{E}_{u,0} \left\{ (Z \times Z)_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) ; I_R \right) \right\} = \int_{q_*-cN^{-1/2}}^{q_*+cN^{-1/2}} dq_1 \int_{q_*-cN^{-1/2}}^{q_*+cN^{-1/2}} dq_2 \int_{I_R} d\varphi \Phi_{N,\beta,u}^{(2)}(q_1, q_2, \varrho),
\]

where, using Corollary 17 and (4.11).

\[
\Phi_{N,\beta,u}^{(2)}(q_1, q_2, \varrho) \triangleq \frac{\omega N^{-1} q_1^{-2}}{\omega_N} \left( 1 - q_1^2 \right)^{N-3/2} \left( 1 - q_2^2 \right)^{N-3/2} \left( 1 - \varrho^2 \right)^{N-3/2} \Xi_{N,\beta,u}^{(2)}(q_1, q_2, \varrho),
\]

\[
\Xi_{N,\beta,u}^{(2)}(q_1, q_2, \varrho) \triangleq \mathbb{E}_{u,0} \{ \exp \{ -\beta H_N(\sigma) - \beta H_N(\sigma') \} \} = \Xi_{N,\beta,u}^{(1)}(q_1) \cdot \Xi_{N,\beta,u}^{(1)}(q_2) \cdot \exp \left\{ \beta^2 N \left( (1 - q_1^2)^{1/2} (1 - q_2^2)^{1/2} \varrho + q_1 q_2 \right)^p - q_1^p q_2^p - p q_1^{p-1} q_2^{p-1} (1 - q_1^2)^{1/2} (1 - q_2^2)^{1/2} \varrho \right\},
\]

and where the relation between $(\sigma, \sigma')$ and $(q_1, q_2, \varrho)$ in the last equation is as in (6.24). It follows that

\[
\lim_{N \to \infty} \sup_{u \in J_N} \frac{1}{N} \log \mathbb{E}_{u,0} \left\{ (Z \times Z)_{N,\beta} \left( \text{Band} \left( cN^{-1/2} \right) ; I_R \right) \right\} - \sup_{\varphi \in I_R} \log \left( \Phi_{N,\beta,-NE_o}^{(2)}(q_*, q_*, \varrho) \right) = 0.
\]

Set

\[
A_0(\varrho) = \lim_{N \to \infty} \frac{1}{N} \log \left( \Phi_{N,\beta,-NE_o}^{(2)}(q_*, q_*, \varrho) \right)
\]

and note that for any $\delta > 0$ the convergence is uniform in $\varrho \in (-1 + \delta, 1 - \delta)$. Also, for small enough $\delta > 0$,

\[
\limsup_{N \to \infty} \sup_{\varrho \notin (-1 + \delta, 1 - \delta)} \frac{1}{N} \log \left( \Phi_{N,\beta,-NE_o}^{(2)}(q_*, q_*, \varrho) \right)
\]
is as negative as we wish. Therefore, part [2] of the lemma will follow if we show that $A_0 (\varrho)$ attains its maximum over $(-1,1)$ uniquely at $\varrho = 0$. Since $A_0 (0) = 2 \Lambda Z (E_0, q_*)$, the latter also implies part [1] of the lemma.

We note that
\[
\sup_{\rho \in I_R} A_0 (\varrho) = \log (1 - q_*^2) + 2 \beta E_0 q_*^p + \beta^2 \left( (1 - q_*^2)^{p-1} q_*^{2p} - p q_*^{2p-2} (1 - q_*^2) \right) + \sup_{\rho \in I_R} A_1 (\varrho),
\]
where
\[
A_1 (\varrho) = \frac{1}{2} \log \left( 1 - \varrho^2 \right) + \beta^2 \left( (1 - q_*^2) \varrho + q_*^2 \right)^p - q_*^{2p} - p q_*^{2p-2} (1 - q_*^2) \varrho.
\]

By expanding $\left( (1 - q_*^2) \varrho + q_*^2 \right)^p$, one can verify that for $\varrho \in (0,1)$, $A_0 (\varrho) > A_0 (-\varrho)$. Also,
\[
\frac{d}{d\varrho} A_1 (\varrho) = -\frac{\varrho}{1 - \varrho^2} + \frac{\varrho^2}{1 - \varrho^2} \varrho^p (1 - q_*^2) + q_*^2 \varrho^{p-1} - q_*^{2p-2} (1 - q_*^2) \varrho
\]
\[
= \frac{\varrho}{1 - \varrho^2} + p \beta^2 (1 - q_*^2) \sum_{k=1}^{p-1} \binom{p-1}{k} (1 - q_*^2) q_*^{2(p-1-k)}
\]
\[
< -\varrho C_\beta (1 + o(1)),
\]
as $\beta \to \infty$, uniformly in $\varrho$, where we relied on the fact that as $\beta \to \infty$, $q_* \to 1$. As noted immediately after its definition (6.5), $C_\beta^*$ is positive. Hence, if $\beta$ is large enough, the derivative $\frac{d}{d\varrho} A_1 (\varrho)$ is negative for $\varrho \in (0,1)$. This implies that the maximum of $A_1 (\varrho)$ in $(-1,1)$ is attained uniquely at $\varrho = 0$ and proves parts [1] and [2] of the lemma.

We move on to the next parts. First note that from parts [1] and [2] of the lemma there exists some sequence $\rho_N \in (0,1)$ with $\rho_N \to 0$, as $N \to \infty$, such that
\[
\lim_{N \to \infty} \sup_{u \in J_N} \left| \mathbb{E}_{u,0} \left\{ (Z \times Z)_{N,\beta} \left( \text{Band} (cN^{-1/2}) ; [-1,1] \setminus (-\rho_N, \rho_N) \right) \right\} \right| = 0,
\]
for any $c > 0$. In order to prove part [3] it is therefore enough to show that, with $I_{R,N} := I_{R,N} (\rho) \triangleq (-\rho_N, \rho_N) \setminus (-\rho_{N/2}, \rho_{N/2})$,
\[
\lim_{\rho \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \frac{\mathbb{E}_{u,0} \left\{ (Z \times Z)_{N,\beta} (\text{Band} (cN^{-1/2}) ; I_{R,N}) \right\}}{\left( \mathbb{E}_{u,0} \left\{ Z_{N,\beta} (\text{Band} (cN^{-1/2})) \right\} \right)^2} = 0.
\]

By substitution of (6.12) and (6.25), this is equivalent to
\[
\lim_{\rho \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \int_{q_*-cN^{-1/2}}^{q_*+cN^{-1/2}} dq_1 \int_{q_*-cN^{-1/2}}^{q_*+cN^{-1/2}} dq_2 \int_{I_{R,N}} d\rho \Phi_N^{(2)} (\varrho_{N,\beta}, u) (q_1, q_2, \varrho) = 0.
\]
Therefore, in order to prove (6.22) it is enough to show that
\[
\lim_{\rho \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \sup_{q_* \in \Omega_{-1/2}} \frac{\int_{I_{R,N}} d\rho \Phi_N^{(2)} (\varrho_{N,\beta}, u) (q_1, q_2, \varrho)}{\Phi_N^{(1)} (\varrho_{N,\beta}, u) (q_1, q_2, \varrho)} = 0,
\]
which from (6.13), (6.26), and the fact that $\sqrt{N \omega_N / \omega_{N-1}} \to \sqrt{2 \pi}$ as $N \to \infty$, is equivalent to
\[
(6.30) \lim_{\rho \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \sup_{q_* \in \Omega_{-1/2}} \frac{N}{2\pi} \int_{I_{R,N}} d\varrho \left( 1 - \varrho^2 \right)^{N-4} \frac{\Xi_N^{(2)} (\varrho_{N,\beta}, u) (q_1, q_2, \varrho)}{\Xi_N^{(1)} (\varrho_{N,\beta}, u) (q_1, q_2, \varrho)} = 0.
\]
From (6.27),
\[ (1 - \varrho^2) \frac{\Xi^{(2)}_{N,\beta,u}(q_1, q_2, \varrho)}{\Xi^{(1)}_{N,\beta,u}(q_1) \Xi^{(1)}_{N,\beta,u}(q_2)} = (1 - \varrho^2)^{-2} \exp \left\{ -\frac{1}{2} C_* N \varrho^2 + N \beta^2 \sum_{i=2}^{p} K_{i,N} (q_1, q_2) \varrho^i \right\} , \]
where the coefficients \( K_{i,N} := K_{i,N} (q_1, q_2) \) satisfy
\[ \lim_{N \to \infty} K_{2,N} (q_1, q_2) = 0 \quad \text{and} \quad |K_{i,N} (q_1, q_2)| < K, \quad \forall i \leq p, \]
for some appropriate constant \( K > 0 \), uniformly in \( q_1, q_2 \in (q_* - cN^{-1/2}, q_* + cN^{-1/2}) \) and independently of \( u \). Hence, by a change of variables, the left-hand side of (6.30) is bounded from above by
\[ \lim_{\varrho \to \infty} \limsup_{N \to \infty} \int_{\sqrt{N} I_{R,N}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} C_* \varrho^2} d\varrho \leq \lim_{\varrho \to \infty} \int_{(-\varrho,\varrho)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} C_* \varrho^2} d\varrho, \]
where we have used the fact that \( \rho_N \to 0 \) as \( N \to \infty \) to neglect high powers of \( \varrho \) in the limit. Part (3) of the lemma follows.

From (6.22), in order to prove (6.23) it is enough to prove that, with \( I'_{R,N} := I'_{R,N} (\rho) \triangleq (-\rho N^{-1/2}, \rho N^{-1/2}) \),
\[ \lim_{\varrho \to \infty} \limsup_{N \to \infty} \sup_{u \in J_{N}} \left| \frac{E_{u,0} \left\{ (Z \times Z)_{N,\beta} \left( \text{Band} (e_N) ; I'_{R,N} \right) \right\}}{E_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} (cN^{-1/2}) \right) \right\}^2} - \frac{1}{\sqrt{C_*}} \right| = 0. \]
Thus, by similar arguments to the above, it is sufficient to show that
\[ \lim_{\varrho \to \infty} \sup_{N \to \infty} \sup_{u \in J_{N}} \left| \sqrt{N} \int_{I'_{R,N}} d\varrho \ (1 - \varrho^2)^{\frac{N-4}{2}} \frac{\Xi^{(2)}_{N,\beta,u}(q_1, q_2, \varrho)}{\Xi^{(1)}_{N,\beta,u}(q_1) \Xi^{(1)}_{N,\beta,u}(q_2)} - \frac{1}{\sqrt{C_*}} \right| = 0. \]
Using (6.31) and a change of variables, this is equivalent to
\[ \lim_{\varrho \to \infty} \left| \int_{(-\varrho,\varrho)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} C_* \varrho^2} d\varrho - \frac{1}{\sqrt{C_*}} \right| = 0, \]
which completes the proof of part (4) of the lemma. \( \square \)

We finish the subsection with the following corollary.

**Corollary 23.** Assume that \( \epsilon_N, \epsilon'_N \to 0 \) and \( N^{-1} \log (\epsilon'_N + \epsilon_N) \to 0 \), as \( N \to \infty \). Then Lemma 22 is also true if we replace everywhere \( \text{Band} (cN^{-1/2}) \) by \( \text{Band} (\hat{n}, q_* - \epsilon_N, q_* + \epsilon'_N) \).

**Proof.** The corollary follows by replacing everywhere \(-cN^{-1/2}\) and \(cN^{-1/2}\) by \(-\epsilon_N\) and \(\epsilon'_N\), respectively, in the proof of Lemma 22. \( \square \)

6.3. **Convergence with ‘3-and-above’ spins averaged.** Recall the definition of the conditional probability \( \mathbb{P}_{u,0,A} \) and random matrix \( G (\hat{n}) := G_{N-1} (\hat{n}) \) given in Remark 16 and (4.12), respectively. If \( X \) is some random variable, viewing \( E_{u,0,A} \{ X \} \) as a (deterministic) function of \( A, E_{u,0,A} \{ X \} \) is a random variable (measurable with respect to \( G (\hat{n}) \)). In this subsection we prove the following lemma.
Lemma 24. For large enough $\beta$ we have the following. Let $\epsilon_N > 0$ be a sequence such that $\epsilon_N \to 0$, as $N \to \infty$. With $Y_*$ given by (6.6),

$$
\lim_{N \to \infty} \sup_{u \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{\mathbb{E}_{u,0,G_0} \{ Z_{N,\beta} (\text{Band} (\epsilon_N)) \}}{\mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\epsilon_N)) \}} \leq t \right\} - \mathbb{P} \{ e^{Y_*} \leq t \} \right| = 0.
$$

Proof. Assume throughout the proof that $\beta$ is fixed, but large enough. We define

$$
c_{\text{eff}} (N, q) = \sqrt{\frac{1}{2}} \frac{N}{N - 1} \alpha_2 (q) \quad \text{and} \quad \beta_{\text{eff}} (N, q) = \beta \cdot c_{\text{eff}} (N, q).
$$

Define the matrix $G_0 := G_{0,N-1} = \sqrt{\frac{N}{p(p-1)}} G_{N-1} (\hat{n})$ and random field

$$
\forall \sigma \in S^{N-2} : \quad \bar{H}_{N-1}^G (\sigma) = \frac{1}{\sqrt{N-1}} \sigma^T G_0 \sigma,
$$

and note that (see (5.1) for the definition of the restriction)

$$
\bar{H}_{N-1}^g (q) = c_{\text{eff}} (N, q) H_{G_0}^G (\sigma) \quad \text{and} \quad Z_{N-1,\beta} \left( \bar{H}_{N-1}^g (q) \right) = Z_{N-1,\beta_{\text{eff}} (N, q)} \left( H_{G_0}^G (\sigma) \right),
$$

where for a real function $f$ on $S^{N-1}$, by abuse of notation, $Z_{N,\beta} (f)$ denotes the corresponding partition function

$$
Z_{N,\beta} (f) = \int \exp \{-\beta f (\sigma)\} d\mu_N (\sigma).
$$

Similarly to (6.12), from Lemmas 14 and 15 we have that

$$
\frac{\mathbb{E}_{u,0,G_0} \{ Z_{N,\beta} (\text{Band} (\epsilon_N)) \}}{\mathbb{E}_{u,0} \{ Z_{N,\beta} (\text{Band} (\epsilon_N)) \}} = \frac{\int_{q_* - \epsilon_N}^{q_* + \epsilon_N} \frac{1}{2} \omega_{N-1}^{-1} e^{-\beta u q^p} (1 - q^2)^{\frac{N-3}{2}} Z_{\bar{H}_{N-1}^g (q)} \mathbb{E} \left\{ e^{-\beta \bar{H}_{N-1}^g (\sigma)} \right\} dq}{\int_{q_* - \epsilon_N}^{q_* + \epsilon_N} \frac{1}{2} \omega_{N-1}^{-1} e^{-\beta u q^p} (1 - q^2)^{\frac{N-3}{2}} \mathbb{E} \left\{ Z_{\bar{H}_{N-1}^g (q)} \right\} \mathbb{E} \left\{ e^{-\beta \bar{H}_{N-1}^g (\sigma)} \right\} dq},
$$

where $\sigma_q$ is an arbitrary point such that $R (\sigma_q, \hat{n}) = q$. From the relation (6.33), since $\beta_{\text{eff}} (N, q_*) \to \frac{1}{2} \sqrt{1 - C_*}$ and $\epsilon_N \to 0$ as $N \to \infty$, in order to prove the lemma it is sufficient to show that:

1. $Y^G_N (\hat{n}) \xrightarrow{d} Y_*$ as $N \to \infty$, where we define, for any $\beta_0 > 0$, the random variable

$$
Y^G_N (\beta_0) \triangleq \log \left( Z_{N-1,\beta_0} (H_{G_0}^G (\sigma)) \right) - \log \left( \mathbb{E} \left\{ Z_{N-1,\beta_0} (H_{G_0}^G (\sigma)) \right\} \right)
$$

$$
= \log \left( Z_{N-1,\beta_0} (H_{G_0}^G (\sigma)) \right) - (N - 1) \beta_0^2.
$$

2. For any small enough $\tau > 0$, the random process $\left\{ Y^G_N (\beta_0) \right\}_{\beta_0 \in D (\tau)}$ on the interval $D (\tau) := \left\{ \frac{1}{2} \sqrt{1 - C_*^\tau} + [-\tau, \tau] \right\}$ converges in distribution as $N \to \infty$ (or in fact, even just tight in $N \geq 1$) in the space of continuous functions on $D (\tau)$, equipped with the supremum norm.

The Gaussian random matrix $G_0$ is an $N - 1 \times N - 1$ symmetric matrix whose on-or-above-diagonal entries are independent with variance 1 off the diagonal and 2 on the diagonal. Thus, the first item above follows from Theorem 4 since $H_{G_0}^G (\sigma) \overset{d}{=} \sqrt{2} H_{N-1}^{\text{pure}} (\sigma)$ and $\frac{1}{2} \sqrt{1 - C_*^\tau} < \frac{1}{2}$. In order to prove the second item we rely on certain calculations from the proof of [7] Theorem 2.10] and a result from [6] concerning convergence of linear statistics of
Wigner matrices. In [7, Eq. (2.14)] the quantity \( \hat{\gamma}(\beta_0) \) is defined. In the sentence following this definition, it is noted that \( \hat{\gamma}(\beta_0) \) is a decreasing function of \( \beta_0 \in (0, \beta_c) \) and as \( \beta_0 \to \beta_c \), \( \hat{\gamma}(\beta_0) \to C_+ \) where \( C_+ \) and \( \beta_c \) are given in [7, Condition 2.3, Definition 2.8] and in our case (of Wigner matrices) \( C_+ = 2 \) and \( \beta_c = 1/2 \). Hence, if \( \tau \) is small enough, then
\[
\inf_{\beta_0 \in D(\tau)} \hat{\gamma}(\beta_0) > C_+.
\]
In the proof of [7, Corollary 5.2], the variable \( \delta \) is chosen so that \( \delta \in (0, (\hat{\gamma}(\beta_0) - C_+)/2) \) for some fixed \( \beta_0 \). Replacing this requirement on \( \delta \) by
\[
\delta \in \left(0, \inf_{\beta_0 \in D(\tau)} (\hat{\gamma}(\beta_0) - C_+)/2\right),
\]
and using the fact that the derivatives up to order 3 of \( \hat{G}(z) \) (defined in [7, Eq. (5.1)]) are bounded uniformly on \([C_+ + \delta, K]\) for any \( \delta, K > 0 \), all the statements and proofs from [7, Corollary (5.2)] hold with the following changes, assuming \( \tau \) is small enough. First, any equation that depends on \( \beta_0 \) (either directly, or implicitly by depending on \( \hat{\gamma} = \hat{\gamma}(\beta_0) \) or \( \gamma = \gamma(\beta_0) \); where the latter of the two is defined in [7, Lemma 4.1]) holds simultaneously for all \( \beta_0 \in D(\tau) \). In particular, equations that hold with probability that goes to 1 as \( N \to \infty \) for fixed \( \beta_0 \) also hold simultaneously for all \( \beta_0 \in D(\tau) \) with such probability.

Second, any of the estimates of the form \( O(t_N) \) hold uniformly in \( \beta_0 \in D(\tau) \); that is, in any estimate where a term of the form \( O(t_N) \) appears, we can replace it by a sequence \( \alpha_N(\beta_0) \) satisfying \( \sup_{\beta_0 \in D(\tau)} |\alpha_N(\beta_0)| = O(t_N) \).

Hence, from (the modified) [7, Eq. (5.33)] (using Definition 2.13, (3.11), (A.4) and (A.7) of [7]), we have that with probability tending to 1 as \( N \to \infty \),
\[
Y_N^G(\beta_0) = -\frac{1}{2} N_\varphi + \log (2\beta_0) - \frac{1}{2} \log (f_2(\beta_0)) - \frac{1}{2} \log \left(1 + \frac{N_\psi}{(N-1)f_2(\beta_0)}\right) + \alpha_N(\beta_0),
\]
where \( |\alpha_N(\beta_0)| = O(N^{-1+a}) \) with arbitrarily chosen \( a > 0 \), and where for any function \( \phi(\beta_0, x) \) with \( \psi_0 \) denoting the eigenvalues of \( G_0/\sqrt{N-1} \), we abbreviate \( N_\phi := N_\phi\left(\beta_0, (\psi_0)^{N-1}\right) \) where
\[
N_\phi \left(\beta_0, (\psi_i)^{N-1}_{i=1}\right) = \sum_{i=1}^{N-1} \phi(\beta_0, \psi_i) - (N-1) \int_{-2}^{2} \phi(\beta_0, x) d\nu(x),
\]
\( \nu = \nu^* \) is the semicircle law given in (12.9), and with \( \hat{\gamma} = 2\beta_0 + (2\beta_0)^{-1} \), the functions \( \varphi, \psi \) and \( f_2 \) are given by
\[
\varphi(\beta_0, x) = \log (\hat{\gamma} - x), \quad \psi(\beta_0, x) = (\hat{\gamma} - x)^{-2}, \quad f_2(\beta_0) = -\frac{1}{2} + \frac{\hat{\gamma}}{2\sqrt{\hat{\gamma}^2 - 4}}.
\]
In particular, \( f_2(\beta_0) \) is continuously differentiable in a neighborhood of \( \frac{1}{2}\sqrt{1-C_+^*} \), and bounded away from 0 uniformly in \( \beta_0 \in D(\tau) \).

Finally, assuming \( \tau \) is small enough, the convergence in distribution as \( N \to \infty \) of the processes
\[
\{N_\phi(\beta_0, \lambda_i)\}_{\beta_0 \in D(\tau)} \quad \text{and} \quad \{N_\psi(\beta_0, \lambda_i)\}_{\beta_0 \in D(\tau)}
\]
in the space of continuous function with the supremum norm, follows from [6, Example 9.3]. Combined with (6.38), this concludes the proof.\footnote{We note that, as verified with the authors of [7], in the version currently on the arXiv, there is a small typo in [7, Eq. (5.33)] and the \( O(N^{-1+a}) \) should be replaced by \( O(N^{-2+c}) \).}
6.4. **Proof of Proposition 19** In view of Lemma 24, Proposition 19 will follow if we prove the following lemma.

**Lemma 25.** For large enough $\beta$ we have the following. Let $a_N, \epsilon_N > 0$ be sequences such $a_N = o(N), \epsilon_N = o(1)$, and set $J_N = (m_N - a_N, m_N + a_N)$. Then

$$\forall \delta > 0 : \lim_{N \to \infty} \sup_{u \in J_N} \left\{ \left| \frac{Z_{N,\beta} (\text{Band} (\epsilon_N))}{E_{u,0,G(\tilde{\bar{u}})} \{ Z_{N,\beta} (\text{Band} (\epsilon_N)) \}} - 1 \right| \geq \delta \right\} = 0.$$  

**Proof.** Denote

$$X(u) = \frac{Z_{N,\beta} (\text{Band} (\epsilon_N))}{E_{u,0,G(\tilde{\bar{u}})} \{ Z_{N,\beta} (\text{Band} (\epsilon_N)) \}}.$$  

It is enough to show that there exist subsets $\mathcal{A} := \mathcal{A}_{N,u}$ of the space of real symmetric $N - 1 \times N - 1$ matrices such that

$$\lim_{N \to \infty} \inf_{u \in J_N} \mathbb{P} \{ \mathcal{G}_{N-1} (\tilde{\bar{u}}) \in \mathcal{A}_{N,u} \} = 1,$$  

(6.41)  

(6.41) (which is equivalent to the same with $\mathbb{P}_{u,0}$, by Lemma 15) and

$$\forall \delta > 0 : \lim_{N \to \infty} \sup_{u \in J_N} \sup_{A \in \mathcal{A}_{N,u}} \{ |X(u) - 1| \geq \delta \} = 0.$$  

(6.42)  

Since $E_{u,0,A} \{ X(u) \} = 1$ by definition, (6.42) follows by Chebyshev’s inequality if we show that

$$\lim_{N \to \infty} \sup_{u \in J_N} \sup_{A \in \mathcal{A}_{N,u}} \left\{ (X(u))^2 \right\} \leq 1.$$  

(6.43)  

Let $\rho_N \in (0,1)$ be an arbitrary sequence such that $\rho_N \to 0$ and $\rho_N \sqrt{N} \to \infty$ as $N \to \infty$. Define the quantities $Q_i := Q_{i,u,A}$ by

$$Q_1 \triangleq E_{u,0,A} \left\{ (Z \times Z)_{N,\beta} (\text{Band} (\epsilon_N); (-\rho_N, \rho_N)) \right\},$$  

$$Q_2 \triangleq E_{u,0,A} \left\{ (Z \times Z)_{N,\beta} (\text{Band} (\epsilon_N); (-\rho_N, \rho_N)^C) \right\},$$  

$$Q_3 \triangleq \int_{T_{\beta}(\text{Band}(\epsilon_N);(-\rho_N, \rho_N))} \prod_{i=1,2} E_{u,0,A} \left\{ e^{-\beta H_N(\sigma_i)} \right\} d\mu_N \otimes \mu_N (\sigma_1, \sigma_2),$$  

$$Q_4 \triangleq \int_{T_{\beta}(\text{Band}(\epsilon_N);(-\rho_N, \rho_N)^C)} \prod_{i=1,2} E_{u,0,A} \left\{ e^{-\beta H_N(\sigma_i)} \right\} d\mu_N \otimes \mu_N (\sigma_1, \sigma_2),$$  

where $(-\rho_N, \rho_N)^C = [-1,1] \setminus (-\rho_N, \rho_N)$ and $T_{\beta} (B; I)$ and $(Z \times Z)_{N,\beta} (B; I)$ are defined in (6.17) and (6.18). Since

$$E_{u,0,A} \left\{ (X(u))^2 \right\} = \frac{Q_1 + Q_2}{Q_3 + Q_4} \leq \frac{Q_1}{Q_3} + \frac{Q_2}{Q_3 + Q_4},$$  

(6.44)  

(6.43) will follow if we show that

$$\lim_{N \to \infty} \sup_{u \in J_N} \sup_{A \in \mathcal{A}_{N,u}} \frac{Q_2}{Q_3 + Q_4} = 0,$$  

and

$$\lim_{N \to \infty} \sup_{u \in J_N} \sup_{A \in \mathcal{A}_{N,u}} \frac{Q_1}{Q_3} \leq 1,$$  

(6.45)  

(6.44)
where in the last equality the supremum in $A$ is over all real, symmetric $N - 1 \times N - 1$ matrices.

By part (3) of Lemma 22 and Corollary 23, there exist sets $A_{N,u}^{(1)}$ such that (6.41) holds with $A_{N,u}^{(1)}$ instead of $A_{N,u}$ and

$$
\lim_{N \to \infty} t_N = 0, \quad \text{with } t_N = \sup_{u \in \mathcal{F}} \sup_{A \in A_{N,u}^{(1)}} \frac{Q_2}{(E_{u,0} \{ Z_{N,\beta} (\text{Band} (\epsilon N)) \})^2} = 0.
$$

By Lemma 24, since

$$
Q_3 + Q_4 = (E_{u,0,A} \{ Z_{N,\beta} (\text{Band} (\epsilon N)) \})^2,
$$

we have that

$$
\lim_{N \to \infty} \inf_{u \in \mathbb{R}} P \left\{ \frac{E_{u,0,A} \{ Z_{N,\beta} (\text{Band} (\epsilon N)) \}}{E_{u,0,A} \{ Z_{N,\beta} (\text{Band} (\epsilon N)) \}} \geq t_N^{1/4} \right\} = 1.
$$

In other words, there exist sets $A_{N,u}^{(2)}$ such that (6.41) holds with $A_{N,u}^{(2)}$ instead of $A_{N,u}$ and

$$
\inf_{u \in \mathbb{R}} \inf_{A \in A_{N,u}^{(2)}} \frac{Q_3 + Q_4}{(E_{u,0,A} \{ Z_{N,\beta} (\text{Band} (\epsilon N)) \})^2} \geq t_N^{1/2}.
$$

Setting $A_{N,u} = A_{N,u}^{(1)} \cap A_{N,u}^{(2)}$, we have that (6.41) and (6.44) hold.

Lastly, note that for any real, symmetric $N - 1 \times N - 1$ matrix $A$, with all suprema taken over $(\sigma_1, \sigma_2) \in T_n (\text{Band} (\epsilon_N); (-\rho_N, \rho_N))$,

$$
\frac{Q_1}{Q_3} \leq \sup \frac{E_{u,0,A} \{ \prod_{i=1,2} e^{-\beta H_N (\sigma_i)} \}}{\prod_{i=1,2} E_{u,0,A} \{ e^{-\beta H_N (\sigma_i)} \}}
= \sup \frac{E_{u,0,A} \{ \prod_{i=1,2} e^{-\beta H_N^{3+} (\sigma_i)} \}}{\prod_{i=1,2} E_{u,0,A} \{ e^{-\beta H_N^{3+} (\sigma_i)} \}}
= \sup \exp \left\{ \beta^2 \text{Cov}_{N}^{3+} (\sigma_1, \sigma_2) \right\} \leq \exp \left\{ N \beta^2 \sum_{k=3}^{p} \left( \frac{p}{k} \right)^k \rho_N^k \right\},
$$

where the inequality in the first line follows since $\int_T f(t) \mu(t) \leq \sup_{t \in T} f(t)$ for positive functions, the equality in the second line follows from Lemmas 14 and 15, the equality in the third line follows since the law of $H_N^{3+}$ under $P_{u,0,A}$ is the same as its unconditional law, and the inequality in the third line follows from

$$
(6.46) \quad \text{Cov}_{N}^{3+} (\sigma_1, \sigma_2) \triangleq \mathbb{E} \left\{ H_N^{3+} (\sigma_1) H_N^{3+} (\sigma_2) \right\}
= N \sum_{k=3}^{p} \binom{p}{k} \left( \sqrt{1 - q_1^2} \sqrt{1 - q_2^2} R_n (\sigma_1, \sigma_2) \right)^k q_1^{p-k} q_2^{p-k},
$$

which follows, with $q_i = R(\sigma_i, \hat{n})$, from (4.11) and (4.10). If we choose $\rho_N$ such that $\rho_N N^{1/3} \to 0$ as $N \to \infty$, then (6.45) holds. This completes the proof. \qed
7. MASS OF BANDS UNDER $\mathbb{P}_{u,0}$: THE RANGE $(q_{LS}, q_{**})$

As pointed out in Section 3, in order to bound contributions to the partition function related to overlap range $(q_{LS}, q_{**})$ we derive upper bounds for the corresponding free energy of bands. Throughout the section we shall use

$$(7.1) \text{Band} = \text{Band} (\hat{n}, q_1, q_2)$$

as an abbreviation for a band with general overlaps. Define

$$(7.2) \Lambda_F (E, q) = \frac{1}{2} \log (1 - q^2) + \beta E q^p + \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left\{ \log \left( Z_{N-1, \beta} \left( \bar{H}_N^{R,2+} | q \right) \right) \right\},$$

where $\bar{H}_N^{R,2+} | q$, defined by (4.19) and (5.1), is a mixed spherical models on $\mathbb{S}^{N-2}$, and where $Z_{N, \beta} (f)$ is given by (6.34). From a standard concentration argument we have the following.

**Lemma 26.** Let $E > 0$ be some positive number and set $J_N = [\pm EN, EN]$. Then, for any $\beta$,

$$(7.3) \limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{N} \mathbb{E}_{u,0} \left\{ \log (Z_{N, \beta} (\text{Band})) \right\} - \sup_{q \in (q_1, q_2)} \Lambda_F \left( -\frac{u}{N}, q \right) \right\} \leq 0.$$

Define

$$(7.4) \Lambda_{F,2} (E, q) = \frac{1}{2} \log (1 - q^2) + \beta E q^p + \frac{1}{N} \log \mathbb{E} \left\{ Z_{N-1, \beta} \left( \bar{H}_N^{R,3+} | q \right) \right\} + \lim_{N \to \infty} \frac{1}{N} \mathbb{E} \left\{ \log \left( Z_{N-1, \beta} \left( \bar{H}_N^{R,2} | q \right) \right) \right\}.$$  

(Where the term involving $\bar{H}_N^{R,3+}$ above does not depend on $N$.)

**Lemma 27.** For any $\beta$, $E$ and $q$, with $\mathcal{P}_2 (\beta)$ defined by (2.1),

$$(7.5) \Lambda_F (E, q) \leq \Lambda_{F,2} (E, q) = \frac{1}{2} \log (1 - q^2) + \beta E q^p + \mathcal{P}_2 (|\alpha_2 (q)| \beta) + \frac{1}{2} \beta^2 \sum_{k=3}^p \alpha_k^2 (q).$$

We also derive bounds on the fluctuations.

**Lemma 28.** For any $\beta$, $u$ and $x > 0$,

$$(7.6) \mathbb{P}_{u,0} \left\{ | \log (Z_{N, \beta} (\text{Band})) - \mathbb{E}_{u,0} \log (Z_{N, \beta} (\text{Band})) | > Nx \right\} \leq e^{-\frac{Nx^2}{2V \beta^2}},$$

where, with $\alpha_k (q)$ defined in (4.9),

$$V := V (q_1, q_2) = \sum_{k=2}^p \sup_{q \in (q_1, q_2)} (\alpha_k (q))^2.$$  

The rest of the section is taken up with the proofs of the three lemmas.

---

\(^{10}\)The fact that the limit in (7.2) exists follows from the spherical Parisi formula \cite{39, 13}. We could avoid relying on the formula by simply replacing the limit with the supremum limit and modify the next result appropriately.
Proof of Lemma 26. Using the fact that as $|q| \to 1$, $\Lambda_F \left( \frac{x}{N}, q \right) \to -\infty$ uniformly in $u \in J_N$, we may, as we will, make the assumption that $-1 < q_1, q_2 < 1$. By partitioning $(q_1, q_2)$ to finitely many intervals and letting the maximal length of an interval approach 0, it is enough to show that the left-hand side of (7.3) is smaller than $c(q_2 - q_1)$ for some $c > 0$ independent of $N$.

By Corollary 17, similarly to (6.12), under $P_{u,0}$, $Z_{N,\beta}$ (Band) has the same distribution as

$$\int_{q_1}^{q_2} \frac{\omega_{N-1}}{\omega_N} \left( 1 - q^2 \right)^{N-3} e^{-\beta u q^q} Z_{N-1,\beta} \left( \tilde{H}_{N}^{\beta,2+} | q \right) dq.$$  

Therefore, the proof is concluded if we show that

$$X_0 \triangleq \sup_{q \in (q_1, q_2)} \frac{1}{N} \log \left( Z_{\beta} \left( \tilde{H}_{N}^{\beta,2+} | q \right) \right) \leq \frac{1}{N} E \left\{ \log \left( Z_{\beta} \left( \tilde{H}_{N}^{\beta,2+} | q \right) \right) \right\}$$  

satisfies

$$\limsup_{N \to \infty} \frac{1}{N} \log (P \{|X_0| \geq t + (q_2 - q_1) c_1\}) \leq -c_2 t^2,$$  

for some $c_1, c_2 > 0$.

Using (4.11) one has (for example, by differentiating by $q$ the first logarithm in (7.7)) that (7.8) follows if we prove that for any $2 \leq k \leq p$ and large enough $N$,

$$P \left\{ \max_{\sigma \in \mathcal{S}_{N-2}} \left| H_{N-1}^{\text{pure } k} (\sigma) \right| \geq tN + c_{1} N \right\} \leq e^{-c_{2} N t^2}$$

for appropriate constants $c_{1}, c_{2} > 0$. The bound of (7.9) follows by Corollary 12 for $k \geq 3$ and the connection to GOE matrices as in (6.33) for $k = 2$ and the Borell-TIS inequality [1, Theorem 2.1.1].

Proof of Lemma 27. Let $E_{G_{\hat{n}}} \{ \cdot \}$ denote the conditional expectation given $G_{\hat{n}}$ (defined in (4.12)). By (4.15) and the independence of $\tilde{H}_{N}^{\text{pure } k}$ for different $k$,

$$E_{G_{\hat{n}}} \left\{ Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right\} = E \left\{ e^{\beta \tilde{H}_{N}^{\text{pure } 2+} (\sigma_q)} \right\} \cdot Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right),$$

where $\sigma_q$ is an arbitrary point such that $R(\sigma_q, \hat{n}) = q$. Thus, by Jensen’s inequality and (4.11),

$$E \left\{ \log \left( Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right) \right\} \leq E \left\{ \log \left( E_{G_{\hat{n}}} \left\{ Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right\} \right) \right\}$$

$$= \frac{1}{2} N \beta^2 \sum_{k=3}^{p} (\alpha_k (q))^2 + E \left\{ \log \left( Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right) \right\}$$

$$= \log \left( E \left\{ Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right\} \right) + E \left\{ \log \left( Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right) \right\}.$$  

By (4.11), $Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right)$ is equal to the partition function of $H_{N-1}^{\text{pure } 2} (\sigma)$ at temperature $\beta |c_2 (q)| \cdot \sqrt{\frac{N}{N-1}}$. Thus, by Corollary 6 (and using the monotonicity in temperature of the partition function) we have that

$$\lim_{N \to \infty} \frac{1}{N} E \left\{ \log \left( Z_{N-1,\beta} \left( \tilde{H}_{N}^{\text{pure } 2+} | q \right) \right) \right\} = \mathcal{P}_2 (\beta).$$

This completes the proof.
Proof of Lemma 28. We will show that \( \frac{1}{N} \log Z_{N,\beta} \) (Band) is a Lipschitz function of a set of i.i.d standard Gaussian variables with Lipschitz constant bounded from above by \( \beta \sqrt{V/N} \). The required bound will follow by standard concentration inequalities (see e.g. [3, Lemma 2.3.3]). By Corollary [17] (4.11) and (1.1) we have the following. The law of the field \( \{H_N(\sigma)\}_\sigma \) under \( \mathbb{P}_{u,0} \) is identical to that of \( \{h(\sigma, u, J)\}_\sigma \) where \( J = (J_{i_1, \ldots, i_k}) \) is an array of random variables with \( 1 \leq i_j \leq N, \ 2 \leq k \leq p \), whose elements are i.i.d standard Gaussian variables, and where the (deterministic) function \( h \) is given by

\[
(7.11) \quad h(\sigma, u, x) = u \left( R(\sigma, \hat{u}) \right)^p + \alpha_k(q(\sigma)) \sum_{k=2}^p \frac{N^{1/2}}{(N-1)^{k/2}} \sum_{i_1, \ldots, i_k=1}^N x_{i_1, \ldots, i_k} \sigma_{i_1} \cdots \sigma_{i_k},
\]

where \( x = (x_{i_1, \ldots, i_k}) \) is an array of real numbers as above, and \( \hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_{N-1}) \) is the vector of norm \( \sqrt{N-1} \) defined in (4.8).

For any \( i_1, \ldots, i_k \),

\[
D_{i_1, \ldots, i_k} \triangleq \frac{d}{d \sigma_{i_1, \ldots, i_k}} \log \left( \int_{\text{Band}} \exp \left\{ -\beta h(\sigma, u, x) \right\} d\mu_N(\sigma) \right)
\]

\[
= -\beta \frac{N^{1/2}}{(N-1)^{k/2}} \int_{\text{Band}} \alpha_k(q(\sigma)) \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_k} \exp \left\{ -\beta h(\sigma, u, x) \right\} d\mu_N(\sigma) / \int_{\text{Band}} \exp \left\{ -\beta h(\sigma, u, x) \right\} d\mu_N(\sigma).
\]

The ratio of integrals in the last equation can be viewed as an expectation under a Gibbs measure on the band which corresponds to (7.11). Denote expectation by this measure by \( \langle \cdot \rangle_h \), so that the ratio is simply \( \langle \alpha_k(q(\sigma)) \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_k} \rangle_h \). We then have

\[
(7.12) \quad \sum_{k=2}^p \sum_{i_1, \ldots, i_k=1}^N (D_{i_1, \ldots, i_k})^2 = \sum_{k=2}^p \beta^2 \frac{N}{(N-1)^k} \sum_{i_1, \ldots, i_k=1}^N \langle \alpha_k(q(\sigma)) \hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_k} \rangle_h^2 \leq \sum_{k=2}^p \beta^2 \frac{N}{(N-1)^k} \sup_{q \in (q_1, q_2)} \langle \alpha_k(q) \rangle_h^2 \sum_{i_1, \ldots, i_k=1}^N \langle (\hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_k})^2 \rangle_h.
\]

Note that

\[
\sum_{i_1, \ldots, i_k=1}^N \langle (\hat{\sigma}_{i_1} \cdots \hat{\sigma}_{i_k})^2 \rangle_h = \langle \|\hat{\sigma}\|_2^2 \rangle_h = (N-1)^k.
\]

Since the Lipschitz constant mentioned in the beginning is equal to \( N^{-1} \) times the square root of (7.12), the proof is completed. \( \square \)

8. Bounds on contributions to \( Z_{N,\beta} \)

In this section we derive the bounds on contributions required for the proof of Theorem 1. In Section 8.1 we make precise the argument about restriction to caps outlined in Section 3. In Section 8.2 we define various overlap-depth regions and state the bounds we shall need for each. We prove them in Section 8.3 using the results on the conditional law of masses derived in Sections 6 and 7 and corollaries of the Kac-Rice formula from Appendix I.

8.1. Restriction to caps. In (8.9) below we shall prove that for any \( \delta > 0 \), w.h.p \( Z_{N,\beta} \geq \exp \{ N (\Lambda_Z(E_0, q_s) - \delta) \} \). Observe that

\[
(8.1) \quad Z_{N,\beta} \left( \{ \sigma : H_N(\sigma) \geq u \} \right) \leq \mu_N \left( \{ \sigma : H_N(\sigma) \geq u \} \right) e^{-\beta u} \leq e^{-\beta u}.
\]

Hence, setting

\[
(8.2) \quad u_{LS} := u_{LS}(\beta) = -\Lambda_Z(E_0, q_s) N / \beta,
\]

...
for any $\delta > 0$

\[(8.3) \quad Z_{N,\beta} (\{ \sigma : H_N (\sigma) \geq u_{LS} + \delta N \}) / Z_{N,\beta} \overset{N \to \infty}{\longrightarrow} 0.\]

The (random) set in \((8.3)\) is related to critical points by the following lemma. Recall that $q_{LS} = 1 - C_{LS} \log_\beta \beta$ was defined in \((5.12)\) with an arbitrary $C_{LS} > (2p (E_0 - E_{\infty}))^{-1}$.

**Lemma 29.** For large enough $\beta$, for small enough $\delta := \delta(\beta) > 0$,

\[(8.4) \quad \lim_{N \to \infty} \mathbb{P} \left\{ \{ \sigma : H_N (\sigma) < u_{LS} + \delta N \} \subset \bigcup_{\sigma_0 \in \mathcal{C}_N (-\infty, u_{LS} + \delta N)} \text{Cap}_N (\sigma_0, q_{LS}) \right\} = 1.\]

**Proof.** Any connected component of $\{ \sigma : H_N (\sigma) < u_{LS} + \delta N \}$ contains at least one critical point $\sigma_0 \in \mathcal{C}_N (-\infty, u_{LS} + \delta N)$ (one local minimum point, in particular). Thus, it will be enough to show that a.h.p for any critical $\sigma_0 \in \mathcal{C}_N (-\infty, u_{LS} + \delta N)$ the connected component of $\sigma_0$ is contained in $\text{Cap}_N (\sigma_0, q_{LS})$. This will follow if we show that

\[(8.5) \quad \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left\{ \sigma_0 \in \mathcal{C}_N (-E_0 N, u_{LS} + \delta N) : \inf_{\sigma : H_N (\sigma) = q_{LS}} H_N (\sigma) < u_{LS} + \delta N \right\} < 0,
\]

since by Corollary 12 $\mathbb{P} \{ \mathcal{C}_N (-\infty, -E_0 N) = \emptyset \} \to 1$ as $N \to \infty$. We recall that $-\Lambda_Z (E_0, q_*) / \beta = u_{LS} / N \to -E_0$ as $\beta \to \infty$, $\Theta_p (x)$ is continuous, and $\Theta_p (-E_0) = 0$. Also,

$$
\mathbb{P}_{u,0} \left\{ \inf_{\sigma : H_N (\sigma) \leq u_{LS} + \delta N} H_N (\sigma) \leq u_{LS} + \delta N \right\} = \mathbb{P}_{u,0} \left\{ \inf_{\sigma \in \mathbb{S}^{N-2}} H_{N|q_{LS}} (\sigma) \leq u_{LS} + \delta N \right\}
$$

is decreasing with $u$. By Lemma 37, in order to finish the proof it will be enough to show that

\[(8.6) \quad \lim_{\beta \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{E_0,0} \left\{ \inf_{\sigma \in \mathbb{S}^{N-2}} H_{N|q_{LS}} (\sigma) \leq u_{LS} + \delta N \right\} < 0.\]

By Corollary 17, Lemma 14 and \((4.15)\), the probability in \((8.6)\) is bounded from above by

$$
\mathbb{P} \left\{ \frac{1}{\sqrt{2}} \alpha_2 (q_{LS}) N \lambda_N + \sum_{k=3}^{p} \alpha_k (q_{LS}) \sqrt{\frac{N}{N - 1}} \inf_{\sigma} H^\text{pure}_{N-1} (\sigma) \leq u_{LS} + \delta N + E_0 (q_{LS})^p N \right\},
$$

where $\lambda_N$ is the minimal eigenvalue of an $N - 1$ dimensional GOE matrix and $\lambda_N$ and $\lambda_{N-1}$, $k \geq 3$, are independent of each other.

From Theorem 8 and since $\alpha_k (q_{LS}) = O \left( (\log \beta / \beta)^{k/2} \right)$, for any $t > 0$,

\[(8.7) \quad \lim_{\beta \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P} \left\{ \sum_{k=3}^{p} \alpha_k (q_{LS}) \sqrt{\frac{N}{N - 1}} \inf_{\sigma} H^\text{pure}_{N-1} (\sigma) \leq t \log \beta / \beta - N \right\} < 0.\]

From some calculus and the definitions \((5.10)\), \((5.9)\), \((5.12)\) and \((4.9)\), one has that $q_* = 1 - \frac{\alpha_2}{\sqrt{2p (p-1)}} + O (\beta^{-2})$ and

$$
\lim_{\beta \to \infty} \frac{1}{\log \beta} \frac{1}{N} (u_{LS} + E_0 q_{LS}^p N) = \frac{1}{2} - E_0 p C_{LS}, \quad \lim_{\beta \to \infty} \frac{1}{\log \beta} \frac{1}{\sqrt{2}} \alpha_2 (q_{LS}) = \frac{1}{2} p E_\infty C_{LS}.
$$

Therefore, based on \((8.7)\), the left-hand side of \((8.6)\) is bounded from above by

$$
\limsup_{\beta \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log \left\{ \frac{1}{2} p E_\infty C_{LS} \lambda_N \leq \frac{1}{2} - E_0 p C_{LS} + \delta / 2 \right\}.\]
Since for any $\epsilon > 0$, \(\limsup_{N \to \infty} \frac{1}{N} \log \mathbb{P} \{\lambda_N \leq -2 - \epsilon\} < 0\) (see e.g. [8, Lemma 6.3]) and since we assumed that \(C_{LS} \geq (2p (E_0 - E_\infty))^{-1}\), the proof is completed. \(\square\)

8.2. **Overlap-depth regions and bounds.** For the convenience of the reader, we recall that

\[
\text{Cont}_{N,\beta} (A \times B) = \mathbb{Z}_{N,\beta} (\cup_{\sigma_0 \in \mathcal{E}_{N}(B)} \{\sigma : R(\sigma, \sigma_0) \in A\}),
\]

\[
\text{Reg}_* (c, \kappa, \kappa') = \left( q_* - cN^{-1/2}, q_* + cN^{-1/2} \right) \times (m_N - \kappa, m_N + \kappa).
\]

Let $\tau_N > 0$ be a sequence such that $\tau_N \to 0$ as $N \to \infty$, required to satisfy a certain relation which will be specified shortly (see Remark [31]). In order to bound the contribution of $\text{Reg}_{UB} (\delta, c, \kappa, \kappa')$ (see 3.4) we define the following regions,

\[
\text{Reg}_{I} (c, \kappa') = \left( (q_{**}, 1) \setminus \left( q_* - cN^{-1/2}, q_* + cN^{-1/2} \right) \right) \times (m_N - \kappa', -E_0 N + \tau_N N),
\]

\[
\text{Reg}_{II} (c, \kappa) = \left( q_* - cN^{-1/2}, q_* + cN^{-1/2} \right) \times (m_N + \kappa, -E_0 N + \tau_N N),
\]

\[
\text{Reg}_{III} (\tau, \delta) = (q_{**}, 1) \times (-E_0 N + \tau N, u_{LS} + \delta N),
\]

\[
\text{Reg}_{IV} (\delta) = (q_{LS}, q_{**}) \times (m_N - \kappa', u_{LS} + \delta N).
\]

**Figure 8.1.** Regions diagram. The letters correspond to subscripts in the definitions of the regions. (for $\text{Reg}_{III} (\tau, \delta)$, above we take $\tau = \tau_N$)

We now state the bounds we need on the regions above. Let $\eta_0 : (0, \infty) \to (0, \infty)$ be a function, which will be fixed henceforth, such that (with $c_p$ defined in (2.8))

\[
\lim_{\kappa \to \infty} \left( \mathbb{P} \left\{ e^{y_*} \leq \eta_0 (\kappa) \right\} \right)^{1/2} \int_0^\kappa dv \cdot e^{cv} = 0.
\]

We remind the reader that the definitions of $\Lambda_Z (E, q)$ and $\mathcal{V}_{N,\beta} (u)$ are given in (5.6) and (6.2) respectively.
Proposition 30. For large enough $\beta$, for any positive $c$, $\tau$, $\kappa'$ and small enough $\delta$,

\begin{align}
\lim_{\kappa,\kappa' \to \infty} \liminf_{N \to \infty} \Pr \{ \text{Cont}_{N,\beta} (\text{Reg}_i (c, \kappa, \kappa')) \geq \eta_0 (\kappa) \mathcal{Y}_{N,\beta} (m_N + \kappa) \} &= 1, \tag{8.9} \\
\lim_{\kappa \to \infty} \limsup_{N \to \infty} \mathbb{E} \left\{ \text{Cont}_{N,\beta} (\text{Reg}_i (c, \kappa')) \right\} / \mathcal{Y}_{N,\beta} (m_N) &= 0, \tag{8.10} \\
\lim_{\kappa \to \infty} \limsup_{N \to \infty} \mathbb{E} \left\{ \text{Cont}_{N,\beta} (\text{Reg}_i (c, \kappa)) \right\} / \mathcal{Y}_{N,\beta} (m_N) &= 0, \tag{8.11} \\
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \text{Cont}_{N,\beta} (\text{Reg}_i (\tau, \delta)) \right\} \right) < \Lambda_Z (E_0, q_*) \tag{8.12}, \\
\limsup_{N \to \infty} \frac{1}{N} \log \left( \Pr \left\{ \frac{1}{N} \log (\mathbb{E} \left\{ \text{Cont}_{N,\beta} (\text{Reg}_i (\tau, \delta)) \right\}) < \Lambda_Z (E_0, q_*) - \delta \right\} \right) < 0. \tag{8.13}
\end{align}

Remark 31. We assume that $\tau_N$ satisfies $\mathbb{E} \left\{ \text{Cont}_{N,\beta} (\text{Reg}_i (\tau_N, \delta)) \right\} / \mathcal{Y}_{N,\beta} (m_N) \to 0$, as $N \to \infty$. The existence of such sequence follows from (8.12) and the fact that $N^{-1} \log (\mathcal{Y}_{N,\beta} (m_N)) \to \Lambda_Z (E_0, q_*)$.

8.3. Proof of Proposition 30

8.3.1. Proof of (8.9). Denoting

\begin{align*}
\text{Band}_* (\sigma_0) &= \text{Band} (\sigma_0, q_* - c N^{-1/2}, q_* + c N^{-1/2}). 
\end{align*}

to finish the proof it will certainly be enough to show that

\begin{align}
\lim_{\kappa,\kappa' \to \infty} \liminf_{N \to \infty} \Pr \{ \exists \sigma_0 \in \mathcal{C}_N (m_N, m_N + \kappa) : Z_{N,\beta} (\text{Band}_* (\sigma_0)) \geq \eta_0 (\kappa) \mathcal{Y}_{N,\beta} (m_N + \kappa) \} &= 1. \tag{8.14}
\end{align}

This will follow if we show that

\begin{align}
\lim_{\kappa \to \infty} \liminf_{N \to \infty} \Pr \{ |\mathcal{C}_N (m_N, m_N + \kappa)| \geq 1 \} &= 1, \tag{8.15}
\end{align}

and

\begin{align}
\lim_{\kappa,\kappa' \to \infty} \limsup_{N \to \infty} \mathbb{E} \left\{ \sigma_0 \in \mathcal{C}_N (m_N, m_N + \kappa) : \frac{Z_{N,\beta} (\text{Band}_* (\sigma_0))}{\mathcal{Y}_{N,\beta} (m_N + \kappa)} < \eta_0 (\kappa) \right\} = 0. \tag{8.16}
\end{align}

Equation (8.14) follows from Theorem 11. By Lemma 38, the left-hand side of (8.15) is bounded from above by

\begin{align}
\lim_{\kappa,\kappa' \to \infty} \limsup_{N \to \infty} C \cdot \int_0^\kappa d\nu \cdot e^{\nu v} \left( \mathbb{P}_{m_N + v, 0} \left\{ \frac{Z_{N,\beta} (\text{Band}_* (\sigma_0))}{\mathcal{Y}_{N,\beta} (m_N + \kappa)} < \eta_0 (\kappa) \right\} \right)^{1/2}. \tag{8.17}
\end{align}

By Propositions 18, 19 and the fact that $\mathcal{Y}_{N,\beta} (m_N + v)$ decreases with $v$, (8.15) is bounded from above by

\begin{align}
\lim_{\kappa,\kappa' \to \infty} \limsup_{N \to \infty} C \cdot \int_0^\kappa d\nu \cdot e^{\nu v} \left( \mathbb{P}_{m_N + v, 0} \left\{ \frac{Z_{N,\beta} (\text{Band}_* (\sigma_0))}{\mathcal{Y}_{N,\beta} (m_N + \kappa)} < \eta_0 (\kappa) \right\} \right)^{1/2} \leq \lim_{\kappa \to \infty} C \left( \mathbb{P} \{ e^{Y_\kappa} \leq \eta_0 (\kappa) \} \right)^{1/2} \int_0^\kappa d\nu \cdot e^{\nu v}.
\end{align}

Therefore the lemma follows from our assumption on $\eta_0 (\kappa)$. □
8.3.2. Proof of (8.10). Set \( J_N = (m_N - \kappa', -E_0N + \tau_NN) \) and
\[
\text{Band}_I(\sigma_0) = \text{Band}(\sigma_0, q_{\ast\ast}, 1) \setminus \text{Band}(\sigma_0, q_{\ast} - cN^{-1/2}, q_{\ast} + cN^{-1/2}).
\]
Of course,
\[
\sum \text{Cont}_{N, \beta}(\text{Reg}_I(c, \kappa', \tau_NN)) \leq \sum_{\sigma_0 \in \mathcal{E}_N(J_N)} Z_{N, \beta}(\text{Band}_I(\sigma_0)),
\]
so an appropriate bound on the expectation of the right-hand side of (8.17) is sufficient.

We will first bound the expectation of the same with \( \text{Band}_I(\sigma_0) \) replaced by
\[
\text{Band}'_I(\sigma_0, \epsilon) = \text{Band}(\sigma_0, q_{\ast\ast}, 1) \setminus \text{Band}(\sigma_0, q_{\ast} - \epsilon, q_{\ast} + \epsilon).
\]
By Lemma 20 and Corollary 17, with \( \varphi(x) = \Lambda_Z(E_0, q_{\ast}) - a(\epsilon) \) (i.e., independent of \( x \)) and small enough \( a(\epsilon) > 0 \), with \( I(\delta) = (-E_0, -E_0 + \delta) \),
\[
\limsup_{N \to \infty} \sup_{u \in NI(\delta)} \left\{ \frac{1}{N} \log \left( \mathbb{E}_{u, 0} \left\{ Z_{N, \beta}(\text{Band}'_I(\hat{n}, \epsilon)) \right\} \right) - \varphi \left( \frac{u}{N} \right) \right\} 
\leq \limsup_{N \to \infty} \left\{ \frac{1}{N} \log \left( \mathbb{E}_{-E_0, N, 0} \left\{ Z_{N, \beta}(\text{Band}'_I(\hat{n}, \epsilon)) \right\} \right) - \varphi \left( \frac{u}{N} \right) \right\} \leq 0.
\]
Therefore, by Lemma 40 for any \( \delta, \epsilon > 0 \),
\[
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}_N(NI(\delta))} Z_{N, \beta}(\text{Band}'_I(\sigma_0, \epsilon)) \right\} \right) 
\leq \sup_{\epsilon \in I(\delta)} \left\{ \Theta_p(E) + \Lambda_Z(E_0, q_{\ast}) - a(\epsilon) \right\}.
\]
Therefore, from the fact that \( \Theta_p \) is continuous and \( \Theta_p(-E_0) = 0 \) and \( J_N \subset I(\delta) \) for large \( N \) and any \( \delta \),
\[
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}_N(J_N)} Z_{N, \beta}(\text{Band}'_I(\sigma_0, \epsilon)) \right\} \right) \leq \Lambda_Z(E_0, q_{\ast}) - a(\epsilon).
\]
Since this holds true for any \( \epsilon \) and \( N^{-1} \log(\mathcal{W}_{N, \beta}(m_N)) \xrightarrow{N \to \infty} \Lambda_Z(E_0, q_{\ast}) \), there exists a sequence \( \epsilon_N > 0 \) such that \( \epsilon_N \xrightarrow{N \to \infty} 0 \) for which
\[
\limsup_{N \to \infty} \frac{\mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}_N(J_N)} Z_{N, \beta}(\text{Band}'_I(\sigma_0, \epsilon_N)) \right\}}{\mathcal{W}_{N, \beta}(m_N)} \leq 0.
\]
What remains is to prove an appropriate upper bound for the expectations of the masses
\[
\sum_{\sigma_0 \in \mathcal{C}(J_N)} Z_{N, \beta}(\text{Band}'_I(\sigma_0)), \quad i = 1, 2,
\]
with
\[
\text{Band}'_I(1)(\sigma_0) = \text{Band}_N(\sigma_0, q_{\ast} - \epsilon_N, q_{\ast} - cN^{-1/2}),
\]
\[
\text{Band}'_I(2)(\sigma_0) = \text{Band}_N(\sigma_0, q_{\ast} + cN^{-1/2}, q_{\ast} + \epsilon_N),
\]
where we assume without loss of generality that \( \epsilon_N > cN^{-1/2} \).

By Corollary 17
\[
\mathbb{E}_{u, 0} \left\{ \left( Z_{N, \beta}(\text{Band}'_I(\hat{n})) \right)^2 \right\} = e^{-2\beta \theta u(u-m_N)(1+o(1))} \mathbb{E}_{m_N, 0} \left\{ \left( Z_{N, \beta}(\text{Band}'_I(\hat{n})) \right)^2 \right\},
\]
uniformly in \( u \in J_N \), as \( N \to \infty \). From Corollary 23, (6.23), (6.3), and (6.4), uniformly in \( u \in J_N \), as \( N \to \infty \),

\[
\mathbb{E}_{m_N,0} \left\{ \left( Z_{N,\beta} \left( \text{Band}_1^{(1)} (\hat{n}) \right) \right)^2 \right\} \\
\leq (1 + o(1)) \frac{1}{\sqrt{C_*}} \left( \mathbb{E}_{m_N,0} \left\{ Z_{N,\beta} \left( \text{Band}_1^{(1)} (\hat{n}) \right) \right\} \right)^2 \\
\leq (1 + o(1)) (\mathfrak{V}_{N,\beta} (m_N) r(c))^2,
\]

for some \( r(c) \xrightarrow{c \to \infty} 0 \).

Now, Lemma 41 yields, for large enough \( N \),

\[
\mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{C}(J_N)} Z_{N,\beta} \left( \text{Band}_1^{(1)} (\sigma_0) \right) \right\} \\
\leq C \mathfrak{W}_{N,\beta} (m_N) r(c) \int_{J_N} du \cdot e^{c_p (u-m_N) - \beta q^*_p (u-m_N)(1+o(1))} \\
\leq C \mathfrak{W}_{N,\beta} (m_N) r(c) g(\beta),
\]

for some \( g(\beta) < \infty \), assuming \( \beta \) is large enough so \( \beta q^*_p > c_p \). By similar arguments, a similar inequality holds for \( B_{1}^{(2)} (\sigma_0, \epsilon) \). Combined with (8.18) and (8.17) this proves the lemma. \( \square \)

8.3.3. Proof (8.11). Set \( J_N = (m_N + \kappa, -E_0 N + \tau N) \) and

\[
\text{Band}_1 (\sigma_0) = \text{Band} \left( \sigma_0, q_* - cN^{-1/2}, q_* + cN^{-1/2} \right).
\]

We have that

\[
\text{Cont}_{N,\beta} (\text{Reg}_I (c, \kappa)) \leq \sum_{\sigma_0 \in \mathcal{E}(J_N)} Z_{N,\beta} (\text{Band}_1 (\sigma_0)).
\]

By similar arguments to those used in the proof of (8.10), we have here that for large \( N \) (8.19)

\[
\mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}(J_N)} Z_{N,\beta} (\text{Band}_1 (\sigma_0)) \right\} \leq C \mathfrak{W}_{N,\beta} (m_N) \int_{J_N} du \cdot e^{c_p (u-m_N) - \beta q^*_p (u-m_N)(1+o(1))},
\]

where \( C > 0 \) is a universal constant (in particular, independent of \( \beta \), which will be useful in the proof of Proposition 3). In this case however we are taking the limit in \( \kappa \) instead of \( c \), and since for large \( \beta \)

\[
\lim_{\kappa \to \infty} \limsup_{N \to \infty} \int_{J_N} du \cdot e^{c_p (u-m_N) - \beta q^*_p (u-m_N)(1+o(1))} = 0,
\]

the proof is completed. \( \square \)

8.3.4. Proof of (8.12). Recall that \( u_{LS} = -\Lambda_Z (E_0, q_*) N/\beta \) and set

\[
J_N = NJ = ((-E_0 + \tau) N, -\Lambda_Z (E_0, q_*) N/\beta + \delta N),
\]

\[
\text{Cap}_II (\sigma_0) = \text{Cap} (\sigma_0, q_*). \]

We have that

\[
\text{Cont}_{N,\beta} (\text{Reg}_II (\tau, \delta)) \leq \sum_{\sigma_0 \in \mathcal{E}(J_N)} Z_{N,\beta} (\text{Cap}_II (\sigma_0)).
\]
By Corollary [17] uniformly in $u \in J_N$, 
$$
\mathbb{E}_{u,0} \{Z_{N,\beta} (\text{Cap}_{III} (\hat{n}))\} \leq e^{-\beta q^p_{**}(u+E_0)N} \mathbb{E}_{-E_0,0} \{Z_{N,\beta} (\text{Cap}_{III} (\hat{n}))\},
$$
and therefore by Proposition [18]
$$
\limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{N} \log (\mathbb{E}_{u,0} \{Z_{N,\beta} (\text{Cap}_{III} (\hat{n}))\}) - \left( \Lambda_Z (E_0, q_*) - \beta q^p_{**} \left(\frac{u}{N} + E_0\right)\right) \right\} \leq 0.
$$

From Lemma [40]
$$
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}(J_N)} Z_{N,\beta} (\text{Cap}_{III} (\hat{n})) \right\} \right) \leq \sup_{E \in J} \{ \Theta_p (E) + \Lambda_Z (E_0, q_*) - \beta q^p_{**} (E + E_0) \}.
$$
For large enough $\beta$, the supremum is equal to
$$
\Theta_p (E_0 + \tau) + \Lambda_Z (E_0, q_*) - \tau \beta q^p_{**}
$$
and is strictly less than $\Lambda_Z (E_0, q_*)$. This completes the proof. \hfill \Box

8.3.5. Proof of (8.14). Let $\delta > 0$ and set
$$
J_N = NJ = N (-E_0, -\Lambda_Z (E_0, q_*) / \beta + \delta)
$$
and note that (see (8.2), (2.7)) for large $N$, $(m_N - \kappa', u_{LS} + \delta N) \subset J_N$. Let $q_1 < q_2$ be some overlaps in $(-1, 1)$ and denote $W_{N,\beta} (\sigma_0) = \frac{1}{N} \log Z_{N,\beta} (\text{Band}(\sigma_0, q_1, q_2))$. Let $\epsilon > 0$ be a number such that
$$
\forall q \in (q_{LS}, q_{**}) : \Lambda_Z (E_0, q_*) - \Lambda_{F,2} (E_0, q) > \epsilon.
$$

By Lemmas [26, 27] and [28] as $N \to \infty$,
$$
\sup_{u \in J_N} \mathbb{P}_{u,0} \{ W_{N,\beta} (\hat{n}) \geq \Lambda_Z (E_0, q_*) - \epsilon \} \leq \mathbb{P}_{-N,0} \{ W_{N,\beta} (\hat{n}) \geq \Lambda_Z (E_0, q_*) - \epsilon \}
$$
$$
\leq \mathbb{P}_{-N,0} \left\{ W_{N,\beta} (\hat{n}) - \mathbb{E}_{-N,0} W_{N,\beta} (\hat{n}) \geq \Lambda_Z (E_0, q_*) - \epsilon - \sup_{q \in (q_1, q_2)} \Lambda_{F,2} (E_0, q) - o(1) \right\}
$$
$$
\leq \exp \left\{ -N (L_\epsilon (q_1, q_2) - o(1)) \right\},
$$
where
$$
L_\epsilon (q_1, q_2) \triangleq \inf_{q \in (q_1, q_2)} \left( \Lambda_Z (E_0, q_*) - \epsilon - \Lambda_{F,2} (E_0, q) \right)^2.
$$

By Lemma [37]
$$
(8.21)
\limsup_{N \to \infty} \frac{1}{N} \log (\mathbb{E} \{ \{ \sigma_0 \in \mathcal{E}_N (J_N) : W_{N,\beta} (\sigma_0) \geq \Lambda_Z (E_0, q_*) - \epsilon \} \}) \leq \sup_{E \in J} \Theta_p (E) - L_\epsilon (q_1, q_2).
$$

Recall that $\Lambda_Z (E_0, q_*) / \beta \to E_0$ as $\beta \to \infty$, and note that $\Theta_p (E)$ increases in $E < 0$. Thus, for any given $\tau > 0$, if $\beta$ is large enough and $\delta$ is small enough, then by Theorem [8]
$$
(8.22)
\limsup_{N \to \infty} \frac{1}{N} \log (\mathbb{E} \{ \{ \sigma_0 \in \mathcal{E}_N (J_N) \} \}) = \sup_{E \in J} \Theta_p (E) < \tau.
$$

We claim that in order to finish the proof it will be sufficient to show that there exist positive $\epsilon, c$ and $\beta_0$ such that (8.20) holds and for any $\beta > \beta_0$ and there exists $l := l (\beta)$ such that for any $q_1, q_2 \in [q_{LS}, q_{**}]$ such that $0 < q_2 - q_1 < l$ we have that
$$
L_\epsilon (q_1, q_2) > c.$$

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To see this, note that if the above holds and we choose $\beta_0$ large enough and $\delta$ small enough then, first we have from (8.22) that

$$\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{P} \left\{ \frac{1}{N} \log (|\{\sigma_0 \in \mathcal{C}_N (J_N)\}|) \geq \epsilon/2 \right\} \right) \leq -\epsilon/4,$$

and second

$$\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{P} \left\{ \exists \sigma_0 \in \mathcal{C}_N (J_N) : \frac{1}{N} \log (Z_{N,\beta} (\text{Band}(\sigma_0, q_{LS}, q_{**}))) \geq \Lambda_Z (E_0, q_*) - \epsilon \right\} \right) \leq -\epsilon/2.$$

With $\beta$ fixed,

$$L_\epsilon (q_1, q_2) \xrightarrow{q_2 \to q_1} L_\epsilon (q_1) \triangleq \frac{(\Lambda_Z (E_0, q_*) - \epsilon - \Lambda_{F,2} (E_0, q_1))^2}{2\beta^2 \sum_{k=2}^p (\alpha_k (q_1))^2}$$

uniformly in $q_1 \in [q_{LS}, q_{**}]$. Hence, it will be enough to show that, for large enough $\beta$,

$$\sup_{q \in (q_{LS}, q_{**})} L_\epsilon (q) > c.$$

Using the approximations

$$q_* = 1 - \frac{\lambda_1}{\sqrt{2p(p-1)}} + O (\beta^{-2}), \quad q_{**} = 1 - \frac{\lambda_3}{\sqrt{2p(p-1)}} + O (\beta^{-2}),$$

one can verify that, since $\Lambda_Z (E, q) \geq \Lambda_{F,2} (E, q)$ (e.g., from (7.5) and the fact that $\sum_{k=3}^p \alpha_k^2 (q) = 1 - \sum_{k=0}^2 \alpha_k^2 (q)$),

$$\liminf_{\beta \to \infty} \{\Lambda_Z (E_0, q_*) - \Lambda_{F,2} (E_0, q_*)\} \geq \lim_{\beta \to \infty} (\Lambda_Z (E_0, q_*) - \Lambda_Z (E_0, q_{**})) = M - m > 0,$$

where $M$ and $m$ are the unique local maximum and minimum in $(0, \infty)$, respectively, of the function $2^{-1} \log t - \sqrt{2t} E_0 / E_\infty + 2^{-1} t^2$.

If $q \in (q_{LS}, q_{**})$, then $0 < q < q_c$ (assuming $\beta$ is large and all of those overlaps are defined) and thus $|\alpha_2 (q) \beta| > \sqrt{\frac{1}{2}}$. By some calculus

$$\sup_{q \in (q_{LS}, q_{**})} \left| \alpha_k (q) \cdot \frac{d}{dq} \alpha_k (q) \right| \leq t_k \left( \frac{\beta}{\log \beta} \right)^{-k+1},$$

for some $t_k > 0$ and, using the fact that $q_{LS} \to 1$ as $\beta \to \infty$,

$$\lim_{\beta \to \infty} \sup_{q \in (q_{LS}, q_{**})} \left| \frac{d}{dq} \Lambda_{F,2} (E_0, q) - p (E_0 - E_\infty) \right| = 0.$$

It follows from (8.24) and (8.25) that for large enough $\beta$ and $q \in (q_{LS}, q_{**})$, for some $c_1, c_2 > 0$,

$$\Lambda_Z (E_0, q_*) - \Lambda_{F,2} (E_0, q) \geq 2c_1 + c_2 \beta (q_{**} - q).$$

Since $q_{LS} \to 1$ as $\beta \to \infty$, from the definition 4.9 of $\alpha_k (q)$, for any $q \in [q_{LS}, q_{**}]$,

$$\sum_{k=2}^p (\alpha_k (q))^2 \leq c_3 (\alpha_2 (q))^2,$$

for some $c_3 > 1$ for any $\beta$ large enough. Hence, with $\epsilon = c_1$, if $1 - 2 (1 - q_{**}) < q \leq q_{**}$, then $\alpha_2 (q) \leq 2 \alpha_2 (q_{**})$ and therefore

$$L_{c_1} (q) \geq \frac{c_1^2}{2\beta^2 c_3 (2\alpha_2 (q_{**}))^2}.$$
and if \( q_{LS} \leq q \leq 1 - 2(1 - q_{**}) \) then \( q_{**} - q \geq (1 - q)/2 \) and, for an appropriate \( c_4 > 0 \) assuming \( \beta \) is large,

\[
L_{c_1}(q) \geq \frac{(c_2\beta_2^2(1 - q))^2}{2\beta_2^2c_3(\alpha_2(q))^2} \geq c_4.
\]

Since \( \beta \cdot \alpha_2(q_{**}) \) is independent of \( \beta \), the proof is completed. \( \square \)

9. Proof of Theorem \( \text{(1)} \)

9.1. Proof of part \( \text{(1)} \). For the definitions of the various regions used below see Section \( \text{8.2 and (3.4)} \). Throughout the proof \( \beta \) and the positive constants \( c, \kappa, \kappa' \) are assumed to be large enough and \( \delta \) is assumed to be small enough whenever needed. Let \( \epsilon > 0 \) be an arbitrary small number. For given \( \kappa \) and \( \kappa' \), assume \( k \) is large enough so that, by Corollary \( \text{12} \) with probability at least \( 1 - \epsilon \) for large \( N \)

\[
Z_{N,\beta} \left( \bigcup_{i \in [k]} \text{Band}_i (cN^{-1/2}) \right) \geq \text{Cont}_{N,\beta} \left( \text{Reg}_a (c, \kappa, \kappa') \right).
\]

From Lemma \( \text{29} \) and Corollary \( \text{12} \) with probability at least \( 1 - \epsilon \) for large \( N, H_N(\sigma^1_0) > m_N - \kappa' \)

\[
Z_{N,\beta} \left( \mathbb{S}^{N-1} \setminus \bigcup_{i \in [k]} \text{Band}_i (cN^{-1/2}) \right) \leq Z_{N,\beta} \left( \{ \sigma : H_N(\sigma) \geq u_{LS} + \delta N \} \right) + \text{Cont}_{N,\beta} \left( \text{Reg}_{UB} (\delta, c, \kappa, \kappa') \right),
\]

where \( \text{Reg}_{UB} (\delta, c, \kappa, \kappa') \) is contained, up to a set of Lebesgue measure zero, in

\[
\text{Reg}_1 (c, \kappa') \cup \text{Reg}_2 (c, \kappa) \cup \text{Reg}_3 (\tau_N, \delta) \cup \text{Reg}_4 (\delta).
\]

By Proposition \( \text{30} \) \((8.1)\) and \((8.2)\), Remark \( \text{31} \) and the facts that

\[
N^{-1} \log (\mathbb{Z}_{N,\beta} (m_N)) \xrightarrow{N \to \infty} \Lambda_Z (E_0, q_*)
\]

and \( \eta_0(\kappa) \mathbb{Z}_{N,\beta} (m_N + \kappa) / \mathbb{Z}_{N,\beta} (m_N) > C \) for some \( C > 0 \) independent of \( N \), we have that with probability at least \( 1 - 3\epsilon \) for large \( N \)

\[
Z_{N,\beta} \left( \mathbb{S}^{N-1} \setminus \bigcup_{i \in [k]} \text{Band}_i (cN^{-1/2}) \right) \geq Z_{N,\beta} \left( \bigcup_{i \in [k]} \text{Band}_i (cN^{-1/2}) \right) \leq \epsilon,
\]

and the proof is completed. \( \square \)

9.2. Proof of part \( \text{(2)} \). Let \( \epsilon, \delta > 0 \) and \( k \geq 1 \) be arbitrary. Choose \( \kappa := \kappa(k, \epsilon) > 0 \) large enough such that by Corollary \( \text{12} \) setting \( J_N = (m_N - \kappa, m_N + \kappa) \), with probability at least \( 1 - \epsilon \) for large \( N \)

\[
\mathcal{E}_N (J_N) \supset \{ \sigma^1_i : i \in [k] \}.
\]

Define (see definition \( \text{(6.19)} \))

\[
P_{\sigma}^{(1)} = \frac{(Z \times Z)_{N,\beta} \left( \text{Band} (\sigma, cN^{-1/2}) ; [-1,1] \setminus (-\rho'N^{-1/2}, \rho'N^{-1/2}) \right)}{\text{Cont}_{u,0} \left( \mathbb{Z}_{N,\beta} \left( \text{Band} (\hat{n}, cN^{-1/2}) \right) \right)^2}
\]

with

\[
\text{Band} (\sigma, cN^{-1/2}) = \text{Band} (\sigma, q_* - cN^{-1/2}, q_* + cN^{-1/2}).
\]

Using part \( \text{B} \) of Lemma \( \text{22} \), assume \( \rho' \cdot \rho' (\delta, \epsilon, \kappa) > 0 \) is large enough such that for large \( \epsilon \), \( \sup_{u \in J_N} \mathbb{E}_{u,0} D_{\sigma}^{(1)} \) is small enough so that by Lemma \( \text{38} \)

\[
\mathbb{P} \left\{ \exists \sigma_0 \in \mathcal{E}_N (J_N) : D_{\sigma_0}^{(1)} > \delta \right\} < \epsilon.
\]
Define
\[ D^{(2)}_{\sigma} = \left( \frac{Z_{N,\beta} \left( \text{Band } (\sigma, cN^{-1/2}) \right)}{\mathbb{E}_{u,0} \left( Z_{N,\beta} \left( \text{Band } (\hat{n}, cN^{-1/2}) \right) \right)} \right)^2. \]

Assume \( \delta' := \delta'(\epsilon, \kappa) \) is small enough so that by Proposition 19 for large \( N, \sup_{u \in J_N} \mathbb{P}_{u,0} \{ D^{(2)}_{\sigma} < \delta' \} \) is small enough so that by Lemma 38
\[
\mathbb{P} \left\{ \exists \sigma_0 \in \mathcal{G}_N (J_N) : D^{(2)}_{\sigma_0} < \delta' \right\} < \epsilon.
\]
Combining the above we have that with probability at least \( 1 - 3\epsilon \), for large \( N \) and any \( i \in [k] \),
\[
D^{(1)}_{\sigma_0} / D^{(2)}_{\sigma_0} \leq \delta' / \delta'.
\]
Since (see (6.16) for the definition of \( R_{\sigma_0}(\sigma, \sigma') \))
\[
\begin{align*}
R(\sigma, \sigma') &= R(\sigma, \sigma_0)R(\sigma', \sigma_0) \\
&+ R_{\sigma_0}(\sigma, \sigma')N^{-1} \| \sigma - \sigma_0 \| \| \sigma' - \sigma_0 \| R(\sigma', \sigma_0),
\end{align*}
\]
if \( \sigma, \sigma' \in \text{Band } (\sigma_0, cN^{-1/2}) \) and \( |R_{\sigma_0}(\sigma, \sigma')| < \rho' N^{-1/2} \), then
\[
|R(\sigma, \sigma') - q_i^2| < \rho N^{-1/2}
\]
for some \( \rho := \rho(c, \rho') \) independent of \( N \). Thus, under (9.1)
\[
G^{\epsilon,i}_{N,\beta} \otimes G^{\epsilon,i}_{\tilde{N},\beta} \left\{ |R(\sigma, \sigma') \pm q_i^2| > \rho N^{-1/2} \right\} < \delta' / \delta'.
\]
By choosing \( \delta \) small enough compared to \( \delta' \), (1.6) follows for the case \( \pm i = i \). For even \( p \), the case \( \pm i = -i \) follows from the fact that \( H_N(\sigma) = H_N(-\sigma) \).

\[ \square \]

9.3. **Proof of part (3).** We first prove the following lemma.

**Lemma 32.** For any \( q \in (-1, 1) \) there exists a function \( \delta_q(\epsilon) > 0 \) with \( \lim_{\epsilon \to 0^+} \delta_q(\epsilon) = 0 \) such that the following holds. Let \( q \in (-1, 1), 0 < \epsilon < |q|, 1 - |q|, \) let \( M \) be a fixed deterministic measure on \( \text{Band } (\sigma_0, \epsilon, q + \epsilon) \) (where the dimension \( N \) is fixed) and assume that
\[
(9.3) \quad M \otimes M \left\{ |R(\sigma, \sigma') - q_i^2| > \epsilon \right\} < \epsilon.
\]
Then, denoting \( \sigma_\perp = \sigma - \sigma_0 R(\sigma, \sigma_0), \)
\[
(9.4) \quad \sup_{\tau \in S^{N-1}} \mathbb{M} \left\{ |R(\sigma_\perp, \tau)| > \delta_q(\epsilon) \right\} \leq \delta_q(\epsilon).
\]

**Proof.** Assume towards contradiction that there exist \( q \in (-1, 1), \delta > 0 \), which will be fixed from now on, and \( \epsilon > 0 \) as small as we wish such that (9.3) holds and for some \( \tau \in S^{N-1}, \)
\[
(9.5) \quad \mathbb{M} \left\{ |R(\sigma_\perp, \tau)| > \delta \right\} \geq \delta.
\]
Let \( \sigma_j, 1 \leq j, \) be distributed according to \( M^{\otimes\infty} \) (i.e., be an i.i.d sequence of samples from \( M \)), and define \( \sigma_{j,\perp} \) similarly to \( \sigma_\perp \). By (9.3) and (9.2) there exists a deterministic function \( \rho_q(\epsilon) \) such that \( \lim_{\epsilon \to 0^+} \rho_q(\epsilon) = 0 \) and, for any \( k \geq 1, \)
\[
M^{\otimes\infty} \left\{ \max_{i,j \leq k} |R(\sigma_{i,\perp}, \sigma_{j,\perp})| > \rho_q(\epsilon) \right\} \leq M^{\otimes\infty} \left\{ \max_{i,j \leq k} |R(\sigma_i, \sigma_j) - q_i^2| > \epsilon \right\} \leq \frac{k(k-1)}{2} \epsilon.
\]
From (9.5),

$$M^\otimes\infty \left\{ \min_{i \leq k} |R(\sigma_{i,\perp}, \tau)| > \delta \right\} \geq \delta^k.$$  

For arbitrary \( k \geq 1 \), assuming \( \epsilon \) is such that \( \delta^k > k(k-1)\epsilon/2 \), we conclude that there exists deterministic vectors, related by \( v_0 = \tau/\|\tau\|, v_1 = \sigma_{1,\perp}/\|\sigma_{1,\perp}\|, \ldots, v_k = \sigma_{k,\perp}/\|\sigma_{k,\perp}\| \) to realizations of the vectors above, such that the matrix \((v_i \cdot v_j)\) is of the from

$$A = \begin{pmatrix} 1 & \delta_1 & \delta_2 & \cdots & \delta_k \\ \delta_1 & 1 & \rho_{1,2} & \cdots & \rho_{1,k} \\ \delta_2 & \rho_{1,2} & \ddots & \cdots & \rho_{k-1,k} \\ \vdots & \vdots & \ddots & \ddots & \rho_{k-1,k} \\ \delta_k & \rho_{1,k} & \cdots & \rho_{k-1,k} & 1 \end{pmatrix}$$

with \( |\delta_i| > \delta \) and \( |\rho_{i,j}| \leq \rho q(\epsilon) \). With \( x = (1, -\delta_1, \ldots, -\delta_k) \), \( \|v_0 - \delta_1 v_1 - \cdots - \delta_k v_k\|^2 = xA^T x \). However, by a direct computation one sees that \( xA^T x \leq 1 - k\delta^2 + k^2\delta^2(\rho_{i,j})(\epsilon) \), which is negative assuming \( k > 1/\delta^2 \) and \( \epsilon \) is small enough. We conclude that our initial assumption is false and therefore the proof is completed. \( \square \)

Denote \( q_i = R(\sigma, \sigma_{i,0}) \) and \( q_j = R(\sigma', \sigma_{i,0}') \). From part (2) of Theorem 1 and Lemma 32, applied once with \( \tau = \sigma \) and \( M = G^c_{N,\beta} \) and once with \( \tau = \sigma_{i,0}' \) and \( M = G^c_{N,\beta} \), for arbitrary \( \delta > 0 \), with probability that goes to 1 as \( N \to \infty \),

$$G^c_{N,\beta} \otimes G^c_{N,\beta} \left\{ \frac{1}{N} \left| \left\langle \sigma, \sigma' - q_j \sigma_{i,0}' \right\rangle \right| \leq \delta \right\} \geq 1 - \delta,$$

(9.6)

$$G^c_{N,\beta} \otimes G^c_{N,\beta} \left\{ \frac{1}{N} \left| \left\langle \sigma_{i,0}', \sigma - q_i \sigma_{i,0} \right\rangle \right| \leq \delta \right\} \geq 1 - \delta.$$

Whenever the two events above occur

$$\frac{1}{N} \left| \left\langle \sigma, \sigma' \right\rangle - q_j \left( \left\langle \sigma, \sigma_{i,0}' \right\rangle \right) + q_j \left( \left\langle \sigma_{i,0}', \sigma \right\rangle - q_i \left( \left\langle \sigma_{i,0}', \sigma_{i,0}' \right\rangle \right) \right) \right| \leq 2\delta,$$

and since \( |q_i - q_*|, |q_j - q_*| < cN^{-1/2} \), for large \( N \),

$$\left| R(\sigma, \sigma') - q_j^2 R(\sigma_{i,0}', \sigma_{i,0}') \right| \leq 3\delta.$$

The proof is completed by Corollaries 12 and 13. \( \square \)

10. PROOFS OF THEOREM 2, PROPOSITION 3 AND COROLLARY 4

10.1. Proof of Theorem 2. Let \( \epsilon > 0 \) be an arbitrary number. For large enough \( \kappa, c > 0 \), by (1.5) and Corollary 12, for large \( N \)

(10.1)

$$\mathbb{P}\left\{ \left| NF_{N,\beta} - \log \left( C^c_{N,\beta} \right) \right| > \epsilon \right\} < \epsilon,$$

with \( C^c_{N,\beta} = \text{Cont}_{N,\beta}(\text{Reg}, (c, \kappa, \kappa)) \). Hence to finish the proof it is sufficient to prove that for any \( \delta > 0 \), there exist \( c_0(\delta), \kappa_0(\delta) > 0 \) such that for any \( c > c_0(\delta), \kappa > \kappa_0(\delta) \) there exists \( t = t(\delta, c, \kappa) > 0 \) such that for large \( N \),

$$\mathbb{P}\left\{ \left| \log \left( C^c_{N,\beta} \right) - N\Lambda \left( E_0, q_* \right) + \frac{\beta d^p}{2c_\beta} \log N \right| > t \right\} < \delta.$$

This follows from (8.9) and (8.19). \( \square \)
10.2. Proof of Proposition 3. In Corollary 12 it is stated that $\mathbb{P}\{H_N(\sigma_0) - m_N \geq x\} \to \exp\{-c_p^{-1}e^{\delta x}\}$, as $N \to \infty$. In view of this and the fact that $\lim_{\beta \to \infty} q_\ast = 1$, to finish the proof it is enough to prove that for any $\delta > 0$,

$$
\limsup_{\beta \to \infty} \limsup_{N \to \infty} \mathbb{P}\left\{ \frac{1}{\beta q_\ast} \left( NF_{N,\beta} - \log (\mathfrak{W}_{N,\beta} (m_N)) + H_N (\sigma_0^1) - m_N \right) \geq \delta \right\} \leq \delta. 
$$

From part (1) of Theorem 1, the above will follow if we show that for any $\delta > 0$,

$$
\limsup_{\beta \to \infty} \limsup_{c, k \to \infty} \limsup_{N \to \infty} \mathbb{P}\left\{ \frac{1}{\beta q_\ast} \left[ \log \left( \sum_{i \in [k]} Z_{N,\beta} \left( \text{Band}_i \left( cN^{-1/2} \right) \right) \right) - \log \left( Z_{N,\beta} \left( \text{Band}_1 \left( cN^{-1/2} \right) \right) \right) \right] \geq \delta \right\} \leq \delta,
$$

and

$$
\limsup_{\beta \to \infty} \limsup_{c, k \to \infty} \limsup_{N \to \infty} \mathbb{P}\left\{ \frac{1}{\beta q_\ast} \left[ \log \left( Z_{N,\beta} \left( \text{Band}_1 \left( cN^{-1/2} \right) \right) \right) \right] - \log (\mathfrak{W}_{N,\beta} (m_N)) + H_N (\sigma_0^1) - m_N \geq \delta \right\} \leq \delta.
$$

(Where we used the fact that the bands above are disjoint for large enough $\beta$ and $N$, since $\lim_{\beta \to \infty} q_\ast = 1$ and by Corollary 13.

Fix $\beta > 0$ which will be assumed to be large enough wherever needed. Let $\delta > 0$ be arbitrary. First, we prove (10.3) by showing that, with high probability, $H_N (\sigma_0^1)$ is not far from $m_N$ and that for any critical value close enough to $m_N$ the fluctuation of the mass of the corresponding band from its expectation is not large. Choose $\kappa = \kappa(\delta)$ large enough so that by Corollary 12.

$$
\limsup_{N \to \infty} \mathbb{P}\{ |H_N(\sigma_0^1) - m_N| \geq \kappa \} < \delta/8.
$$

Denote $Z_{N,\beta,c}(\sigma_0) = Z_{N,\beta} \left( \text{Band}(\sigma_0, q_\ast - cN^{-1/2}, q_\ast + cN^{-1/2}) \right)$. Choose $t = t(\delta, \kappa)$ sufficiently large so that by Propositions 18 and 19 and Lemma 38.

$$
\limsup_{c \to \infty} \limsup_{N \to \infty} \mathbb{P}\{ \exists \sigma_0 \in \mathcal{C}_N \left( m_N - \kappa, m_N + \kappa \right) : \left| \log (Z_{N,\beta,c}(\sigma_0)) - \log (\mathfrak{W}_{N,\beta} (H_N(\sigma_0))) \right| \geq t \} < \delta/8.
$$

Since (by the definition of $\mathfrak{W}_{N,\beta}(\sigma_0)$ (6.2))

$$
\log (\mathfrak{W}_{N,\beta} (H_N(\sigma_0))) - \log (\mathfrak{W}_{N,\beta} (m_N)) = -H_N (\sigma_0) + m_N,
$$

(10.4) and (10.5) imply (10.3). They also imply a lower bound on the mass corresponding to $H_N (\sigma_0^1)$. Namely,

$$
\limsup_{c \to \infty} \limsup_{N \to \infty} \mathbb{P}\{ \log (Z_{N,\beta} \left( \text{Band}_1 \left( cN^{-1/2} \right) \right)) \leq \log (\mathfrak{W}_{N,\beta} (m_N - \kappa)) - t \} < \delta/4.
$$

\footnote{We remark that $\kappa$ is independent of $\beta$, as will be all other parameters we choose in the rest of the proof. This is of course crucial to the proof.}
Therefore, using (8.19) we can choose \( \kappa' = \kappa'(\delta, \kappa, t) \) such that for any fixed \( \kappa'' > 0 \),

\[
\limsup_{c \to \infty} \limsup_{N \to \infty} \mathbb{P} \left\{ \frac{\sum_{\sigma_0 \in \mathcal{C}_N (m_N - \kappa', m_N - \kappa'')} Z_{N, \beta, c}(\sigma_0)}{Z_{N, \beta} \left( \text{Band}_1 \left( cN^{-1/2} \right) \right)} \geq \frac{\delta}{4} \right\} < \delta/2,
\]

uniformly in \( \beta \geq \beta_0 \) for some large enough \( \beta_0 \) (which is independent of our choice of all other parameters). That is, we have an upper bound on the masses of bands corresponding to critical values far enough from \( m_N \) (within a microscopic distance).

Lastly, we need to bound the weights of the bands corresponding to critical values larger than \( m_N - \kappa' \) that are not equal to \( \hat{H}_N (\sigma_0) \). This is done by bounding the number of such points and the fluctuations of the corresponding masses from their expectation. Choose large enough \( k = k(\delta, \kappa') \) so that from Corollary 12

\[
\limsup_{N \to \infty} \mathbb{P} \left\{ \left[ \mathcal{C}_N \left( -\infty, m_N - \kappa' \right) \right] \geq k \right\} < \delta/8.
\]

Similarly to (10.5), choose \( t' = t'(\delta, \kappa) \) large enough so that

\[
\limsup_{c \to \infty} \limsup_{N \to \infty} \mathbb{P} \left\{ \exists \sigma_0 \in \mathcal{C}_N \left( -\infty, m_N - \kappa' \right) : \right\}
\]

\[
\left\{ \log \left( Z_{N, \beta, c}(\sigma_0) \right) - \log \left( \mathcal{W}_{N, \beta} (\hat{H}_N(\sigma_0)) \right) \geq t' \right\} < \delta/8.
\]

Since \( \mathcal{W}_{N, \beta}(u) \) is decreasing, it follows that

\[
\limsup_{c \to \infty} \limsup_{N \to \infty} \mathbb{P} \left\{ \frac{\sum_{\sigma_0 \in \mathcal{C}_N (m_N - \kappa', m_N - \kappa'')} Z_{N, \beta, c}(\sigma_0)}{\mathcal{W}_{N, \beta} (\hat{H}_N(\sigma_0))} \geq ke^{t'} \right\} < \delta/2.
\]

Since the bounds in (10.6) and (10.8) are independent of \( \beta \) (assuming it is large enough) and since the difference in (10.2) is divided by \( \beta q_o^k \), (10.2) follows from the above and the proof is completed.

10.3. Proof of Corollary 4. Let \( 0 < a \) and set \( J_N = (m_N - a, m_N + a) \). For any \( c, \delta > 0 \), by Lemmas 25 and 38

\[
\lim_{N \to \infty} \mathbb{E} \left\{ \sigma_0 \in \mathcal{C}_N (J_N) : \left| \frac{Z_{N, \beta, c}(\sigma_0)}{\mathbb{E}_{\hat{H}_N(\sigma_0), \mathcal{G}(\sigma_0)} \{ Z_{N, \beta, c}(\hat{n}) \}} - 1 \right| \geq \delta \right\} = 0,
\]

where \( Z_{N, \beta, c}(\sigma_0) = Z_{N, \beta} \left( \text{Band} \left( \sigma_0, q_* - cN^{-1/2}, q_* + cN^{-1/2} \right) \right) \). It therefore also follows that there exist positive sequences \( a_N > 0, \delta_N = o(1), c_N = o(N^{1/2}) \) with \( a_N, c_N \to \infty \), as \( N \to \infty \), such that the above holds with them instead of the corresponding constants. By Corollary 12 there exists a sequence \( k_N \geq 1 \) such that \( k_N \to \infty \) and

\[
\lim_{N \to \infty} \mathbb{P} \left\{ \forall i \in [k_N] : \left| Z_{N, \beta, c_N}(\sigma_0^i)/Z_{N, i} - 1 \right| < \delta_N \right\} = 1,
\]

where we define

\[
Z_{N, i} \equiv \mathbb{E}_{\hat{H}_N(\sigma_0^i), \mathcal{G}(\sigma_0^i)} \{ Z_{N, \beta, c_N}(\hat{n}) \}.
\]

By Corollaries 12 and 13 by choosing \( k_N \) that increases slower if needed, we also have that

\[
\lim_{N \to \infty} \mathbb{P} \left\{ \forall i, j \in [k_N], i \neq \pm j : \left| R \left( \sigma_0^i, \sigma_0^j \right) \right| \leq \tau_N \right\} = 1,
\]

for some \( \tau_N = o(1) \). For large \( \beta \), since \( q_* (\beta) \to 1 \) as \( \beta \to \infty \), for \( i \) and \( j \) with \( \left| R \left( \sigma_0^i, \sigma_0^j \right) \right| \leq \tau_N \), assuming \( N \) is large enough,

\[
\cap_{a = i, j} \text{Band} \left( \sigma_0^a, q_* - cN^{-1/2}, q_* + cN^{-1/2} \right) = \emptyset.
\]
implying that
\[
\lim_{N \to \infty} \mathbb{P}\left\{ \sum_{i \in [k_N]} Z_{N,\beta,c_N}(\sigma_0^i) \leq Z_{N,\beta} \right\} = 1.
\]
Combined with (1.5) this completes the proof. \(\square\)

11. Proof of Theorem 5

Define the overlap distribution
\[
\zeta_N(\cdot) = \mathbb{E}\left\{ M_{N,1} \otimes M_{N,2} \left\{ R(\sigma, \sigma') \in \cdot \right\} \right\}
\]
and let \(\mathcal{B}(q, \epsilon) \subset \mathbb{R}\) denote the ball of radius \(\epsilon\) centered at \(q\). For \(\beta_1, \beta_2 > 0\) set
\[
q_{12} = q_*(\beta_1) q_*(\beta_2) \quad \text{and} \quad Q = \left\{ 0, q_{12}, (-1)^{p+1} q_{12} \right\}.
\]
Theorem 5 follows directly from the following proposition, which also contains information about the overlap distribution.

**Proposition 33.** For large enough \(\beta_1, \beta_2 > 0\), either different or equal, with \(M_{N,i} = G_{N,\beta_i}\), there exists \(v > 0\) such that the following holds. For any \(\epsilon > 0\),
\[
\lim_{N \to \infty} \inf \zeta_N(\mathcal{B}(q_0, \epsilon)) > v,
\]
and
\[
\lim_{N \to \infty} \zeta_N((\cup_{q \in Q} \mathcal{B}(q, \epsilon))^c) = 0.
\]

**Remark 34.** For even \(p \geq 4\), (11.4) was already known – it was proved by Panchenko and Talagrand [33].

In the next subsections we relate the overlap distribution (11.1) to critical points and prove Proposition 33.

11.1. Chaotic behavior and critical points. This section is devoted to an auxiliary result which reduces questions about chaos to ones about the behavior of the critical points and values of the Hamiltonian \(H_N(\sigma)\) under perturbations. Suppose \(H_{N}^{(1)}(\sigma)\) and \(H_{N}^{(2)}(\sigma)\) are two copies of the Hamiltonian \(H_N(\sigma)\) defined on the same probability space, but not necessarily independent, and let \(G_{N,\beta}^{(1)}\) and \(G_{N,\beta}^{(2)}\) denote the corresponding Gibbs measures. Similarly, let \(\sigma_{0}^{(1)}\) and \(\sigma_{0}^{(2)}\) be the enumerations of the critical points of \(H_{N}^{(1)}(\sigma)\) and \(H_{N}^{(2)}(\sigma)\), respectively, and with \(W_i^{(n)} = Z_{N,\beta_n} \left( \text{Band}_{i,\beta_n}^{(n)} (cN^{-1/2}) \right) \) (defined accordingly by (1.4) and (1.2)) define
\[
\zeta_{N,c,k}^{\text{crit}}(\cdot) \triangleq \mathbb{E}\left\{ \sum_{i,j \in [k]} W_i^{(1)} W_j^{(2)} \delta_{q_{12} R(\sigma_{0}^{(1)}, \sigma_{0}^{(2)})}(\cdot) \right\},
\]
where \(q_{12}\) is given in (11.2). Finally, let \(d_{BL}\) denote the bounded Lipschitz metric (see e.g. [24, Appendix D]).

**Lemma 35.** Assume \(\beta_1, \beta_2\) are large enough and let \(M_{N,1} = G_{N,\beta_1}^{(1)}\) and \(M_{N,2} = G_{N,\beta_2}^{(2)}\). Then
\[
\lim_{c,k \to \infty} \lim_{N \to \infty} d_{BL}(\zeta_N(\cdot), \zeta_{N,c,k}^{\text{crit}}(\cdot)) = 0.
\]
Proof. Define the conditional measure
\[ G^{c,i}_{N,\beta_1}(\cdot) = G^{(n)}_{N,\beta_1}(\cdot \cap \text{Band}_{i,\beta_2}^{(n)}(cN^{-1/2}))/G^{(n)}_{N,\beta_1}(\text{Band}_{i,\beta_2}^{(n)}(cN^{-1/2})). \]

Similarly to \([9.6]\) and the discussion surrounding it, by Lemma \([32]\) and using part \((2)\) of Theorem \([1]\) for any fixed \(i\) and \(j\), for any \(\delta > 0\), with probability that goes to 1 as \(N \to \infty\),
\[ G^{c,i}_{N,\beta_1} \otimes G^{c,j}_{N,\beta_2} \left\{ \left| R(\sigma, \sigma') - q_{12}R(\sigma_0^{i,(1)}, \sigma_0^{j,(2)}) \right| < \delta \right\} > 1 - \delta. \]
The proof is completed by part \((1)\) of Theorem \([1]\). \(\square\)

11.2. Proof of Proposition \([33]\). The key to the proof is that both Gibbs measures \(G_{N,\beta_1}\) are concentrated around the critical points of the same random field \(H_N(\sigma)\) which through \((11.6)\) determine the overlap distribution. More precisely, in view of Lemmas \([35, 12, 13]\) we have that \((11.4)\) holds and that in order to prove \((11.3)\) we only need to show that there exist \(\kappa > 0\) and \(a, a' \in (0, 1)\) such that for large enough \(c\),
\[(11.7) \{3\sigma_1, \sigma_2 \in \mathcal{C}_N(m_N - \kappa, m_N + \kappa) : \sigma_1 \neq \pm \sigma_2, G_{N,\beta_1}(\text{Band}_{\beta_j,c}(\sigma_1)) > a, \forall i, j = 1, 2\} \]
occur with probability at least \(a'\), for large enough \(N\), with
\[ \text{Band}_{\beta_j,c}(\sigma) = \text{Band}(\sigma, q_*(\beta_j) - cN^{-1/2}, q_*(\beta_j) + cN^{-1/2}). \]

Let \(\epsilon > 0\) be an arbitrarily small number and choose large enough \(\kappa = \kappa(\epsilon)\) such that the following three hold. Firstly, for any large enough \(c > 0\), by \([8.15]\), for \(j = 1, 2\),
\[ \limsup_{N \to \infty} \mathbb{E} \left\{ \left| \sigma_0 \in \mathcal{C}_N(m_N, m_N + \kappa) : \frac{Z_{N,\beta_1}(\text{Band}_{\beta_j,c}(\sigma))}{\mathfrak{Z}_{N,\beta_1}(m_N + \kappa)} < \eta_0(\kappa) \right| \right\} < \epsilon, \]
where \(\eta_0\) satisfies \((8.8)\). Secondly, by Theorem \([2]\) \((6.2)\) and \((2.7)\), for some large enough \(t = t(\kappa, \epsilon)\)
\[(11.8) \limsup_{N \to \infty} \mathbb{P} \{ NF_{N,\beta_j} - \log(\mathfrak{Z}_{N,\beta_j}(m_N + \kappa)) > t \} < \epsilon. \]
And thirdly, by Theorem \([11]\)
\[ \liminf_{N \to \infty} \mathbb{P} \{ \mathcal{C}_N(m_N, m_N + \kappa) < 4 \} < \epsilon. \]

For this choice of parameters, we have that with probability at least \(a' = 1 - 3\epsilon\), there exist \(\sigma_1, \sigma_2\) as in \((11.7)\) with the lower bound satisfied with \(a = \eta_0(\kappa)e^{-t}\) for large \(c\). This completes the proof. \(\square\)

12. Concluding remarks

The process of log-masses. One can associate a point process to the log-masses of the bands (say with some \(c_N = o(N^{1/2})\), going to \(\infty\)),
\[ \xi^W_{N,\beta} = \sum \delta_{W_{N,\beta}^i}, \quad W_{N,\beta}^i = \log \left( Z_{N,\beta} \left( \text{Band}_i \left( c_N N^{-1/2} \right) \right) \right). \]

Similarly to the point process of extremal critical values \((2.9)\), we predict that \((1 + \epsilon_p)^{-1} \xi^W_{N,\beta}\), shifted by and appropriate term of \(m^W_{N} = \log(\mathfrak{Z}_{N,\beta}(m_N)) + O(1)\), converges as \(N \to \infty\) to a Poisson point process, however with intensity \(\exp \left\{ \frac{c_p - x}{\beta_p} \right\} dx\); this would also determine the limiting law of fluctuations of the free energy. Evidence for this can be found in Corollary \([17]\) and the proof of Corollary \([4]\) and through the connection of Poisson point processes of exponential intensities, the Poisson-Dirichlet distribution, and the distribution of pure state
masses; see [31, 30, 28]. The convergence of $\xi_{N,\beta}^W$ should be possible to prove by establishing invariance under perturbations of the disorder and using Liggett’s characterization of shifted Poisson point processes [30], as done for the point process of critical points in [38]. This will be studied in future work.

\( \Lambda_Z(E_0, q_*) \) and the Parisi functional.\(^\text{12} \) Theorem 2 implies that \( F_{N,\beta} \) converges in distribution to \( \Lambda_Z(E_0, q_*) \) which must coincide with the spherical variant of the Parisi formula [22, 39, 13]. We recall that pure spherical models are known to exhibit 1-step replica symmetry breaking [39, Proposition 2.2]. The Parisi functional corresponding to \( m\delta_0 + (1 - m)\delta_q \) (see [32, (1.16)]), where in our case \( \xi(q) = q^2 \) is given by

\[
(12.1) \quad P(m, q) = \frac{1}{2} \left( \beta^2 (1 - q^p + mq^p) + \log (1 - q) + \frac{1}{m} \log \left( 1 + \frac{mq}{1 - q} \right) \right).
\]

Reassuringly, Theorem 1 implies that the limiting overlap distribution function (restricted to \([0, 1]\)) is of the from \( m\delta_0 + (1 - m)\delta_q \) with \( q = q_*^2 \). Our prediction regarding the point process of log-masses suggests, using the connection to the Poisson-Dirichlet distribution, that \( m \) should be equal to \( m_* = c_p / (\beta q_*^2) \). Therefore, one would expect that

\[
(12.2) \quad \Lambda_Z(E_0, q_*) = P(m_*, q_*^2).
\]

This can be verified by a direct calculation.\(^\text{12} \)

\textbf{Temperature chaos in mixed models.} Suppose \( H_N^m(\sigma) = \sum_{p \geq 2} \gamma_p H_{N,p}(\sigma) \), where \( H_{N,p}(\sigma) \) are independent pure spherical models. We recall that the mixed model \( H_N^m(\sigma) \) is called generic if and only if \( \sum_{p \geq 2} p^{-1} 1(\gamma_p \neq 0) = \infty \). Panchenko [32] proved that mixed even \( p \)-spin Ising spin glass models (with or without external field) exhibit temperature chaos. His methods are expected to apply to spherical spin glasses as well. In contrast, we have seen that if \( \gamma_{p_0} = c > 0 \) for some \( p_0 \geq 3 \) and \( \gamma_p = 0 \) for all \( p \neq p_0 \) then chaos does not occur. The transition in chaotic behavior as soon as one moves from a pure model to a generic model, even if \( \gamma_p, p \neq p_0, \) are extremely small, may seem surprising.

First moment calculations for the critical points as in Theorem 8 were carried out for the mixed case by Auffinger and Ben Arous in [4]. Assuming that \( \gamma_{p_0} = c \) for some \( p_0 \geq 3 \) and \( \gamma_p, p \neq p_0, \) are sufficiently small, we have checked that the second moment computation of [37] works for \( H_N^m(\sigma) \) (using a certain truncation argument), at least on the exponential level (i.e., a result similar to [37, Theorem 3]) can be proved). We expect that, under the same assumption, the proof of the concentration result, Theorem 10 and the convergence of the extremal process, Theorem 11, proved in [37, 38], also carry over to the mixed case.

One significant difference, however, between the pure and mixed models is related to the decomposition of Section 4. In the mixed case, instead of just one variable, in the right-hand side of (4.5) and (4.6) we have a sum over \( p \) of the expressions corresponding to the case of pure \( p \)-spin, multiplied by \( \gamma_p \). Consequently, upon conditioning on \( H_N^m(\hat{n}) = u, \nabla H_N^m(\hat{n}) = 0 \) and \( \nabla^2 H_N^m(\hat{n}) = A \), the terms in the decomposition that correspond to 0, 1 and 2 are not deterministic, in contrast to the pure case. Therefore, we expect that in the mixed case it

\[ c_p + \frac{1}{c_p} \log \left( \frac{1 - q_*^2(1 - c_p / (\beta q_*^2))}{1 - q_*^2} \right) - 2E_0 + \beta pq_*^{-2}(1 - q_*^2) = 0. \]

Using (2.8), (5.10) and (5.8), by straightforward algebra, one can see that the left-hand side of the above is equal to \( 2\Theta_p(\beta E_0) \) (see (12.11)), which by the definition of \( E_0 \) (2.6), verifies (12.2).
is possible that the contribution related to critical values which are roughly equal to some $u = m_N + v$ is not monotone in $v$.

Suppose that the convergence of the extremal process $\xi_N$ of (2.9) and the fact that for large $\beta$ the Gibbs measure is supported on bands around critical points with critical values approximately $m_N$ (defined appropriately) do carry over to some mixed model $H^W_N(\sigma)$. Even if this is the case, the occurrence of chaos could be explained by the following mechanism, related to the above: the bands that carry most of the Gibbs mass correspond to critical values which are approximately equal to $m_N + v_N(\beta)$ where $\lim_{N \to \infty} |v_N(\beta) - v_N(\beta')| = \infty$, for large $\beta \neq \beta'$, and therefore the centers of the relevant bands for $\beta$ are orthogonal to those corresponding to $\beta'$ (assuming an equivalent of Corollary 13 holds for $H^W_N(\sigma)$). In the language of the extremal point process of critical values $\xi_N$ and the point process of log-masses $\xi^W_N,\beta$ the corresponding picture is that the leading particles of $\xi^W_N,\beta$ correspond (by being related to the same critical point of the Hamiltonian) to points of $\xi_N$ of depth that depends on $\beta$.

**Transition to disorder chaos of the Gibbs measure and ground state.** Let $H_N(\sigma)$ be an i.i.d copy of $H_N(\sigma)$ and set, for $t \in [0,1]$, 

\[(12.3) \quad H_{N,t}(\sigma) = (1-t)H_N(\sigma) + \sqrt{2t-t^2}H'_N(\sigma) . \]

Denote the Gibbs measure of $H_{N,t}$ by $G_{N,t,\beta}$. For $\beta = \infty$ and odd $p$ let $G_{N,\infty}$ be the probability measure concentrated at the global minimum point of $H_N(\sigma)$ and define $G_{N,t,\infty}$ similarly. For even $p$, for which there are two global minimum points a.s., assume that each is sampled such that $N$ is odd and odd $p$ let $G_{N,\infty}$ be the probability measure concentrated at the global minimum point of $H_N(\sigma)$ and define $G_{N,t,\infty}$ similarly. For even $p$, for which there are two global minimum points a.s., assume that each is sampled with probability 1/2. We say that disorder chaos occurs if 

\[(12.4) \quad \exists q_0 \in [-1,1], \forall \epsilon > 0 : \lim_{N \to \infty} \mathbb{E} \{ M_{N,1} \otimes M_{N,2} \{ |R(\sigma,\sigma') - q_0| > \epsilon \} \} = 0 \]

holds with $M_{N,1} = G_{N,\beta}$ and $M_{N,2} = G_{N,t,\beta}$, for any $t \in (0,1)$. Disorder chaos of the ground state, or ground state chaos for short, is defined similarly with $\beta = \infty$. Chen, Hsieh, Hwang and Sheu [18] proved disorder chaos for spherical mixed (and in particular, pure as we consider) even $p$-spin model, with or without external fields. For the same models, Chen and Sen [20] proved ground state chaos. See also [13] and [16] by Chen for related results for models with Ising spins.

Once it is known that chaos occurs, it is natural to ask whether chaos can be seen on a finer scale, i.e., when we take $t_N \to 0$. In this regard, we mention that Chatterjee [12] proved for Ising mixed even $p$-spin models without an external field and $\beta < \infty$ that (12.4) holds with $M_{N,1} = G_{N,\beta}$ and $M_{N,2} = G_{N,t,\beta}$ for any $t_N$ such that $t_N \log N \to \infty$. For the pure spherical models the following precise transition holds, for large enough $\beta \in (0,\infty)$ or $\beta = \infty$:

(1) for any $s > 0$, with $t_N = s/N$, (12.4) does not hold with $M_{N,1} = G_{N,\beta}$ and $M_{N,2} = G_{N,t,\beta}$:

(2) in contrast, it does hold if $t_N = s_N/N + o(1)$ assuming $s_N \to \infty$ as $N \to \infty$.

In Appendix II we prove the first of the two statements based on a certain mixing property for the deepest critical values under perturbations of the disorder, which relies on tools developed by Zeitouni and the author in [38]. We also explain in Appendix II how the second statement can be proved by an extension of certain results of [38]. A full proof of this extension would require a rewrite of a significant part of [38]. Instead of doing so, we will refer to the original text and detail the required changes.
APPENDIX I: THE KAC-RICE FORMULA AND RELATED AUXILIARY RESULTS

The Kac-Rice formula \[1\] Theorem 12.1.1] is a basic tool in our analysis, allowing us to relate several quantities of interest to the conditional probability \( \mathbb{P}_{\sigma_0} \). Below we prove a number of simple derivatives of the formula; some using the results of Section 2.2 on critical points and values.

The Kac-Rice formula can be used to express the mean number of critical points \( \sigma_0 \) on the sphere at which the values of some other fields belong to some target set. For us, those other fields will be usually related to masses of bands around \( \sigma_0 \). The variant of the Kac-Rice formula that we use is \[1\] Theorem 12.1.1]. In the notation of \[1\] Theorem 12.1.1] we will consider situations where, with some function \( g_N(\sigma) \)

\[
M = S^{N-1}, \quad f(\sigma) = \nabla H_N(\sigma), \quad u = 0 \in \mathbb{R}^{N-1}, \quad h(\sigma) = (H_N(\sigma), g_N(\sigma)),
\]

where we recall that, with \( E = (E_i)_{i=1}^{N-1} \) being an orthonormal frame field on the sphere, we denote

\[
\nabla H_N(\hat{n}) = (E_i H_N(\hat{n}))_{i=1}^{N-1} \quad \text{and} \quad \nabla^2 H_N(\hat{n}) = (E_i E_j H_N(\hat{n}))_{i,j=1}^{N-1}.
\]

The application of \[1\] Theorem 12.1.1] requires \( h(\sigma) \) to satisfy certain non-degeneracy conditions - namely, conditions (a)-(g) in \[1\] Theorem 12.1.1]. We will say that \( g_N(\sigma) \) is tame if the conditions are satisfied and if \( \{h(\sigma)\} \sigma \) is a stationary random field. Using \[1,1\], the conditions are easy to check in any case we will apply the formula and this will be left to the reader. Let

\[
\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}
\]

denote the surface area of the \( N-1 \)-dimensional unit sphere. The following is obtained from a direct application of the formula.

**Lemma 36.** Suppose that \( g_N(\sigma) \) is tame and \( D_N \) and \( J_N \) are some intervals, then

\[
\mathbb{E} \left| \{ \sigma_0 \in \mathcal{C}_N(J_N) : g_N(\sigma_0) \in D_N \} \right| = \omega_N \left( (N-1) \frac{p-1}{2\pi} \right)^{N-1/2}
\]

\[
\times \int_{J_N} \frac{du}{\sqrt{2\pi N}} e^{-\frac{u^2}{2N}} \mathbb{E}_{u,0} \left\{ \left| \det \left( \frac{\nabla^2 H_N(\hat{n})}{\sqrt{p(p-1)(N-1)/N}} \right) \right| 1 \{g_N(\hat{n}) \in D_N \} \right\}.
\]

**Proof.** Applying \[1\] Theorem 12.1.1] with \[12.5\] yields the integral formula

\[
\mathbb{E} \left| \{ \sigma_0 \in \mathcal{C}_N(J_N) : g_N(\sigma_0) \in D_N \} \right| = \int_{S^{N-1}} d\nu(\sigma) \phi_{\nabla H_N(\sigma)}(0)
\]

\[
\times \mathbb{E} \left\{ \left| \det (\nabla^2 H_N(\sigma)) \right| 1 \{H_N(\sigma) \in J_N, g_N(\sigma) \in D_N \} \right\} \nabla H_N(\sigma) = 0 \}
\]

where \( \phi_{\nabla H_N(\sigma)}(x) \) is the Gaussian density of \( \nabla H_N(\sigma) \) and \( \nu \) is the standard measure on the sphere (not normalized). Since the integrand above is independent of \( \sigma \)[13] we can replace the integral with the value of the integrand evaluated at \( \sigma = \hat{n} \) and multiply by \( \omega_N N^{-1} \), the volume of \( S^{N-1} \). By Lemma \[15\] \( \nabla H_N(\hat{n}) \sim N(0, pI_{N-1}) \), so that \( \phi_{\nabla H_N(\sigma)}(0) = (2\pi p)^{-N-1} \).

\[13\] If we apply the Kac-Rice formula to compute the expectation as is \[12.8\] with the additional restriction that \( \sigma_0 \) belongs to some measurable subset \( B \subset S^{N-1} \), then what changes in the integral formula on the right-hand side of \[12.8\] is that the domain of integration \( S^{N-1} \) is replaced by \( B \). Thus, the corresponding integrand is a continuous Radon-Nikodym derivative w.r.t the Lebesgue measure which, since \( (H_N(\sigma), g_N(\sigma)) \) is stationary, is invariant to rotations of the sphere. It is therefore a constant function.
and $H_N (\mathbf{n})$ and $\nabla H_N (\mathbf{n})$ are independent. The proof is completed by conditioning on $H_N (\mathbf{n})$ in addition to $\nabla H_N (\sigma) = 0$ and some calculus.

Several derivatives of (12.7) are of use to us. Their proofs will involve the rate function $\Theta_p (E)$ of Theorem 8 which we now define explicitly. Let $\nu^*$ denote the semicircle measure, the density of which with respect to Lebesgue measure is

$$\frac{d\nu^*}{dx} = \frac{1}{2\pi} \sqrt{4 - x^2} 1_{|x|\leq 2},$$

and define the function (see, e.g., [26, Proposition II.1.2])

$$\Omega(x) \triangleq \int_\mathbb{R} \log |\lambda - x| d\nu^* (\lambda)$$

$$= \left\{ \begin{array}{ll}
\frac{x^2}{4} - \frac{1}{2}, & \text{if } 0 \leq |x| \leq 2,
\frac{x^2}{4} - \frac{1}{2} - \frac{1}{\pi} \sqrt{x^2 - 4} - \log \left( \sqrt{\frac{x^2}{4} - 1} + \frac{|x|}{2} \right), & \text{if } |x| > 2.
\end{array} \right.$$  

The exponential growth rate function of (2.5) is given [5] by

$$\Theta_p (E) = \left\{ \begin{array}{ll}
\frac{1}{2} + \frac{1}{2} \log (p - 1) - \frac{E^2}{2} + \Omega (\gamma_p E), & \text{if } u < 0,
\frac{1}{2} \log (p - 1), & \text{if } u \geq 0,
\end{array} \right.$$  

where $\gamma_p = \sqrt{p / (p - 1)}$.

**Lemma 37.** Assume the conditions of Lemma 36 with $J_N = NJ$ where $J \subset (-\infty, 0)$ is a fixed finite interval. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that

$$\limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{N} \log (\mathbb{P}_{u,0} \{ g_N (\mathbf{n}) \in D_N \}) - \varphi \left( \frac{u}{N} \right) \right\} \leq 0.$$  

Then

$$\limsup_{N \to \infty} \frac{1}{N} \log (\mathbb{E} |\{ \sigma_0 \in \mathcal{C}_N (J_N) : g_N (\sigma_0) \in D_N \}|) \leq \sup_{E \in J} \{ \Theta_p (E) + \varphi (E) \}.$$  

**Proof.** From (12.7) and Hölder’s inequality,

$$\mathbb{E} |\{ \sigma_0 \in \mathcal{C}_N (J_N) : g_N (\sigma_0) \in D_N \}| \leq \omega_N \left( (N - 1) \frac{p - 1}{2\pi} \right)^{N - 1 \over 2}$$

$$\times \int_{J_N} du \frac{1}{\sqrt{2\pi N}} e^{-{u^2 \over 2\pi}} \mathbb{E}_{u,0} \{| \det (V) |^a \}^{1/a} \mathbb{P}_{u,0} \{ g_N (\mathbf{n}) \in D_N \}^{1/b},$$  

where

$$V \triangleq \frac{\nabla^2 H_N (\mathbf{n})}{\sqrt{p (p - 1) (N - 1) / N}},$$  

$a > 1$ is an arbitrary number, and $b := b(a) = a / (a - 1)$.

First,

$$\lim_{N \to \infty} \omega_N \left( (N - 1) \frac{p - 1}{2\pi} \right)^{N - 1 \over 2} = \frac{1}{2} + \frac{1}{2} \log (p - 1).$$  

Second, by a change of variables $u \mapsto Nv$,

$$\limsup_{N \to \infty} \frac{1}{N} \log \left( \int_{J_N} du \frac{1}{\sqrt{2\pi N}} e^{-{u^2 \over 2\pi}} \exp \left\{ N \left( \Theta_p (\frac{u}{N}) + \varphi \left( \frac{u}{N} \right) \right) \right\} \right) \leq \sup_{x \in J} \left\{ \Theta_p (x) - \frac{x^2}{2} + \varphi (x) \right\}.$$
where \( \gamma_p = \sqrt{p/(p-1)} \) and \( \Omega \) is defined in \((12.10)\). Thus, by \((12.11)\), the lemma follows if we show that for any \( a > 1 \),

\[
(12.18) \quad \limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{Na} \log \left( \mathbb{E}_{u,0} \{ |\det \mathbf{V}|^a \} \right) - \Omega \left( \gamma_p \frac{u}{N} \right) \right\} \leq 0,
\]

and that

\[
(12.19) \quad \limsup_{a \to \infty} \limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{Nb(a)} \log \left( \mathbb{E}_{u,0} \{ g_N (\mathbf{n}) \in D_N ) \} \right) - \varphi \left( \frac{u}{N} \right) \right\} \leq 0.
\]

The inequality \((12.19)\) obviously follows from \((12.12)\), since \( \lim_{a \to \infty} b(a) = 1 \).

From \((4.12)\) and \((4.16)\), the conditional law of \( \mathbf{V} \) under \( \mathbb{P}_{u,0} \{ \cdot \} \) is identical to that of 

\[
(12.20) \quad \mathbf{M}_u := \mathbf{M}_{u,N-1} \triangleq \mathbf{M} - \gamma_p \frac{u}{\sqrt{N(N-1)}} \mathbf{I},
\]

where \( \mathbf{M} \) is a GOE matrix and \( \mathbf{I} \) is the identity matrix, both of dimension \( N - 1 \). For any \( 0 < \epsilon < 1 < \kappa \), with \( \lambda_i \) denoting the eigenvalues of \( \mathbf{M} \) and \( \lambda_*(u) = \max \{ \lambda_i - \gamma_p \frac{u}{\sqrt{N(N-1)}} \} \),

\[
\mathbb{E} \left\{ |\det (\mathbf{M}_u)|^a \right\} \leq \mathbb{E} \left\{ \exp \left\{ a \sum_i \log^a \left( \left| \lambda_i - \gamma_p \frac{u}{\sqrt{N(N-1)}} \right| \right) \right\} \right\}
\]

\[
+ \mathbb{E} \left\{ (\lambda_*(u))^{a(N-1)} \mathbf{1} \{ (\lambda_*(u)) \geq \kappa \} \right\},
\]

where

\[
(12.22) \quad \log^a \epsilon (x) = \begin{cases} \log (\epsilon) & \text{if } x \leq \epsilon, \\ \log x & \text{if } x \in (\epsilon, \kappa), \\ \log \kappa & \text{if } x \geq \kappa. \end{cases}
\]

From the upper bound on the maximal eigenvalue of \([8\text{ Lemma 6.3}]\),

\[
(12.23) \quad \lim_{\kappa \to \infty} \limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ (\lambda_*(u))^{a(N-1)} \mathbf{1} \{ (\lambda_*(u)) \geq \kappa \} \right\} \right) = -\infty,
\]

uniformly for \( u \in J_N \). The empirical measure of eigenvalues of GOE matrices \( L_N = \frac{1}{N-1} \sum_{i=1}^{N-1} \lambda_i \) satisfies a large deviation principle with speed \( N^2 \) and a good rate function \( J_0 (\nu) \), in the space of Borel probability measures on \( \mathbb{R} \) equipped with the weak topology, which is compatible with the Lipschitz bounded metric; see \([9\text{ Theorem 2.1.1}]\). The good rate function \( J_0 (\nu) \) satisfies \( J_0 (\nu) = 0 \) if and only if \( \nu = \nu^* \) is the semicircle law \((12.9)\). Combined with the fact that the functions \( \log^a \epsilon (| \cdot - u' |) \) have the same Lipschitz constant and bound for all \( u' \in \mathbb{R} \), this implies that for the event

\[
F (u, \delta) = \left\{ \frac{1}{N-1} \sum_i \log^a \epsilon \left( \left| \lambda_i - \gamma_p \frac{u}{\sqrt{N(N-1)}} \right| \right) - \int \log^a \epsilon \left( | \lambda - \gamma_p \frac{u}{N} | \right) d\mu^* (\lambda) > \delta \right\},
\]

we have for any \( \delta > 0 \) some positive number \( d (\delta) \) such that

\[
\limsup_{N \to \infty} \sup_{u \in J_N} \frac{1}{N^2} \log (\mathbb{P} \{ F (u, \delta) \}) < -d (\delta).
\]

By taking \( \kappa \) large enough and \( \epsilon \) small enough, combining this with \((12.21)\) and \((12.23)\) proves \((12.18)\) and completes the proof. \( \square \)

For intervals \( J_N \) of length \( o (N) \) around \( -NE_0 \) the following is also useful for us.
Lemma 38. Assume the conditions of Lemma 36 with \( J_N = (m_N - a_N, m_N + a_N) \) for some sequences \( a_N, a_N' = O(N) \). Let \( c_p \) be as defined in (12.8). Then for large enough \( N \),

\[
\mathbb{E} \{ \sigma_0 \in \mathcal{C}_N(J_N): g_N(\sigma_0) \in D_N \} \leq C \int_{J_N} du \cdot e^{c_p(u-m_N)} \left( \mathbb{P}_{u,0} \{ g_N(\hat{n}) \in D_N \} \right)^{1/2},
\]

where \( C > 0 \) is an appropriate constant.

Proof. We shall use the notation (12.15) introduced in the proof of Lemma 37. Since the conditional law of \( V \) under \( \mathbb{P}_{u,0} \{ \cdot \} \) is identical to that of the shifted GOE matrix (12.20), as a particular case of [37, Corollary 23],

\[
\left( \mathbb{E}_{u,0} \{ |\det(V)|^2 \} \right)^{1/2} \leq C \mathbb{E}_{u,0} \{ |\det(V)| \},
\]

uniformly in \( u \in J_N \), for some constant \( C > 0 \). As in the proof of Lemma 37, (12.14) holds and therefore, taking \( a = 2 \) and using (12.25) we obtain that

\[
\int_{J_N} du \cdot \frac{1}{\sqrt{2\pi N}} e^{-\frac{u^2}{2\pi}} \mathbb{E}_{u,0} \{ |\det(V)| \} \left( \mathbb{P}_{u,0} \{ g_N(\hat{n}) \in D_N \} \right)^{1/2},
\]

By Lemma 36,

\[
\mathbb{E} \{ |\mathcal{C}_N(J_N)| \} = \int_{J_N} du \omega_N \left( (N-1) \frac{p-1}{2\pi} \right)^{1/2} \frac{1}{\sqrt{2\pi N}} e^{-\frac{u^2}{2\pi}} \mathbb{E}_{u,0} \{ |\det(V)| \} \cdot
\]

By definition, up to a factor of 2 in the even \( p \) case (related to the normalization factor in (2.9)), the integrand above is equal to the density of the intensity measure of the extremal point process of critical points \( \xi_N \), defined in (2.9), shifted by \( m_N \). Thus, by [38, Proposition 3] it converges uniformly to \( e^{c_p(u-m_N)} \) (see also Theorem 11 above). Combined with (12.26) this yields (12.24).

The non-negative random variable

\[
\sum_{\sigma_0 \in \mathcal{C}_N(J_N)} Z_{N,\beta} \left( \text{Band} \left( \sigma_0, q, q' \right) \right)
\]

can be approximated by

\[
\sum_{\sigma_0 \in \mathcal{C}_N(J_N)} \sum_{i \leq k} t_i 1 \{ Z_{N,\beta} \left( \text{Band} \left( \sigma_0, q, q' \right) \right) \in [t_i, t_{i+1}] \},
\]

where \( [t_i, t_{i+1}], i \leq k \), form a finite partition of \([0, \infty)\). Combining this with the monotone convergence theorem and (12.27) one obtains the following.

Corollary 39. We have that

\[
\mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{C}_N(J_N)} Z_{N,\beta} \left( \text{Band} \left( \sigma_0, q_N, q'_N \right) \right) \right\} = \omega_N \left( (N-1) \frac{p-1}{2\pi} \right)^{N-1/2}
\]

\[
\times \int_{J_N} du \frac{1}{\sqrt{2\pi N}} e^{-\frac{u^2}{2\pi}} \mathbb{E}_{u,0} \left\{ \left| \det \left( \frac{\nabla^2 H_N(\hat{n})}{\sqrt{p(p-1)}(N-1)/N} \right) \right| Z_{N,\beta} \left( \text{Band} \left( \hat{n}, q_N, q'_N \right) \right) \right\}.
\]
We have the following exponential bound on the expectation above.

**Lemma 40.** Let \( J_N = NJ \) where \( J \subset \mathbb{R} \) is a fixed interval. Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a continuous function and suppose that

\[
\limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{N} \log \left( \mathbb{E}_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( \hat{n}, q_N, q'_N \right) \right) \right\} \right) - \varphi \left( \frac{u}{N} \right) \right\} \leq 0.
\]

Then

\[
\limsup_{N \to \infty} \frac{1}{N} \log \left( \mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}_N(J_N)} Z_{N,\beta} \left( \text{Band} \left( \sigma_0, q_N, q'_N \right) \right) \right\} \right) \leq \sup_{x \in J} \left\{ \Theta_p(x) + \varphi(x) \right\}.
\]

**Proof.** Abbreviate \( \text{Band} \left( \sigma_0 \right) = \text{Band} \left( \sigma_0, q_N, q'_N \right) \). By Hölder’s inequality,

\[
b \mapsto \log \left( \mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right)^b \right\} \right)
\]

is a convex function. Hence, for \( b \in (1, 2) \), for any \( u \in J_N \),

\[
\frac{1}{Nb} \log \left( \mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right)^b \right\} \right) \leq \frac{1}{Nb} \left( (2 - b) \log \left( \mathbb{E}_{u,0} \left\{ Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right\} \right) + (b - 1) N\bar{C} \right),
\]

where we define

\[
\bar{C} := 2\beta^2 + \frac{2\beta}{N} \sup_{u \in J_N} |u| = 2\beta^2 + \frac{2\beta}{N} \sup_{x \in J} |x|
\]

and use the fact that

\[
\log \left( \mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right)^2 \right\} \right) \leq \log \left( \max_{\sigma, \sigma' \in \mathbb{S}^{N-1}} \mathbb{E}_{u,0} \left\{ e^{-\beta H_N(\sigma) - \beta H_N(\sigma')} \right\} \right) \leq 2\beta|u| + 2\beta^2 N.
\]

For any \( b > 1 \), with \( \alpha := a(b) = b/(b - 1) \), we have, using the notation (12.15),

\[
\mathbb{E}_{u,0} \left\{ |\text{det}(V)| Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right\} \leq \left( \mathbb{E}_{u,0} \left\{ |\text{det}(V)|^\alpha \right\} \right)^{1/\alpha} \left( \mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right)^b \right\} \right)^{1/b}.
\]

Using (12.30) and (12.18) and letting \( b \searrow 1 \) we obtain

\[
\limsup_{N \to \infty} \sup_{u \in J_N} \left\{ \frac{1}{N} \log \left( \mathbb{E}_{u,0} \left\{ |\text{det}(V)| Z_{N,\beta} \left( \text{Band} \left( \hat{n} \right) \right) \right\} \right) - \Omega \left( \gamma_p \frac{u}{N} \right) - \varphi \left( \frac{u}{N} \right) \right\} \leq 0.
\]

From (12.27), (12.16) and (12.17), we conclude that (12.29) follows. \( \square \)

We finish with the following lemma

**Lemma 41.** Let \( J_N = (m_N - a_N, m_N + a'_N) \) with some sequences \( a_N, a'_N = o(N) \), and define \( c_p \) as in (2.8). Then, for large enough \( N \),

\[
\mathbb{E} \left\{ \sum_{\sigma_0 \in \mathcal{E}_N(J_N)} Z_{N,\beta} \left( \text{Band} \left( \sigma_0, q_N, q'_N \right) \right) \right\} \leq C \cdot \int_{J_N} du \cdot e^{c_p(u - m_N)} \left( \mathbb{E}_{u,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} \left( \hat{n}, q_N, q'_N \right) \right) \right)^2 \right\} \right)^{1/2}.
\]
where \( C > 0 \) is an appropriate constant.

Proof. By (12.27), (12.31) and (12.25) it follows that, for an appropriate \( C > 0 \), for large \( N \),

\[
\mathbb{E} \left\{ \sum_{\sigma_0 \in C(N)} Z_{N,\beta} \left( \text{Band} (\sigma_0, q_N, q'_N) \right) \right\} \leq C \cdot \omega_N \left( (N - 1) \frac{B - 1}{2\pi} \right)^{N/2} \times \int_{J_N} du \frac{1}{\sqrt{2\pi N}} e^{-\frac{u^2}{2N}} \mathbb{E}_{a,0} \left\{ \left| \det(\mathbf{V}) \right| \right\} \left( \mathbb{E}_{a,0} \left\{ \left( Z_{N,\beta} \left( \text{Band} (\hat{n}, q_N, q'_N) \right) \right)^2 \right\} \right)^{1/2},
\]

where \( \mathbf{V} \) is defined in (12.15). The proof now follows from the argument used in the proof of Lemma 38 right after (12.26). \( \square \)

APPENDIX II: TRANSITION TO DISORDER CHAOS

The two statements with which we concluded the discussion about transition to disorder chaos in Section 12 follow from the two proposition below. Recall the definition (11.1) of \( \zeta_N (\cdot) \) and, similarly to (11.2), set in this section

\[
q_{12} = q_*(\beta) q_*(\beta) \quad \text{and} \quad Q = \left\{ 0, q_{12}, (-1)^{p+1} q_{12} \right\}.
\]

**Proposition 42.** For large enough \( \beta \in (0, \infty) \) or \( \beta = \infty \) and any \( s > 0 \) we have the following. With \( t_N = s/N \), let \( M_{N,1} = G_{N,\beta} \) be the Gibbs measure of \( H_N (\sigma) \) and \( M_{N,2} = G_{N,t_N,\beta} \) be the Gibbs measure of \( H_{N,t_N} (\sigma) \), see (12.3). Then there exists \( v = v(s, \beta) \) such that for any \( \epsilon > 0 \),

\[
\begin{align*}
\text{any } q_0 \in Q \text{ is charged:} & \quad \liminf_{N \to \infty} \zeta_N (\mathcal{B} (q_0, \epsilon)) > v, \\
\text{any } q \notin Q \text{ vanish:} & \quad \lim_{N \to \infty} \zeta_N (\left( \cup_{q \in Q} \mathcal{B} (q, \epsilon) \right)^c) = 0.
\end{align*}
\]

**Proposition 43.** For large enough \( \beta \in (0, \infty) \) or \( \beta = \infty \) and any sequence \( t_N = s_N/N = o(1) \) with \( s_N \to \infty \) as \( N \to \infty \), with \( M_{N,1} = G_{N,\beta} \) and \( M_{N,2} = G_{N,t_N,\beta} \),

\[
\forall \epsilon > 0 : \quad \lim_{N \to \infty} \zeta_N (\mathcal{B} (0, \epsilon)) = 1.
\]

The rest of the appendix is devoted to the proof of Proposition 42 and a sketch of a proof of Proposition 43.

**Proof of Proposition 42.** The main tool we use in the proof is the description of the perturbations of critical points and values under perturbations of the disorder developed by Zeitouni and the author in [38]. Roughly speaking, when perturbations of the disorder are as in the statement of Proposition 42: (i) the position of deep critical points does not change on the macroscopic level, and (ii) the change in corresponding critical values is of order \( O(1) \). Hence, typically the collections of critical points around which the original and perturbed Gibbs measures are concentrated are the same, up to microscopic changes in their positions. Relying on this, by Lemmas 35 and 13 we will identify the atoms of the overlap distribution as \( N \to \infty \) limit.

Fix \( s > 0 \) throughout the proof and let \( t_N = s/N \). Set

\[
e_N := e_N (s) = \frac{s^3}{N} \left( \frac{3}{2} - \frac{s}{N} \right) / \left( 1 - \frac{s}{N} \right)^2 = O(1/N)
\]
and note that with $H_{N,t}^\dagger (\sigma)$ being an independent copy of $H_N (\sigma)$ in the definition (12.3) of $H_{N,t} (\sigma)$ we have that

\begin{equation}
\frac{N}{N-s} H_{N,tN} (\sigma) = H_{N,s,eN}^\dagger (\sigma),
\end{equation}

where

\begin{equation}
H_{N,s,r}^\dagger (\sigma) = H_N (\sigma) + \sqrt{\frac{2s}{N}} (1 + r) H_N^\dagger (\sigma).
\end{equation}

In [38] the critical points and values of $H_N^\dagger (\sigma) = H_{N,s,r}^\dagger (\sigma)$ with $s = 1/2$ and $r_N = 0$ were related to those of the original model $H_N (\sigma)$, for large $N$. By a simple modification the proofs carry over to the more general case with fixed $s$ and any $r_N = o(1)$, and this variant is what we use below. Define $\mathcal{C}(B)$ similarly to $\mathcal{C}(B)$ using the random field $H_N (\sigma)$. By [38, Lemma 8] (modified), for $C_0 = (E_0 - c_0)/2$ (see (2.6) and (2.8)) there exists a sequence $\delta_N > 0$ converging to 0 as $N \to \infty$, such that for any $\kappa > 0$, with probability that goes to 1 as $N \to \infty$, there exists a mapping $\mathcal{\mathcal{G}}_{N,k} : \mathcal{C} (m_N - \kappa, m_N + \kappa) \to \mathcal{C}( \mathcal{\mathcal{G}}_{N,tN} (\mathbb{R})$ such that, if we denote $\sigma^i = \mathcal{\mathcal{G}}_{N,k} (\sigma)$, then for any $\sigma \in \mathcal{C} (m_N - \kappa, m_N + \kappa)$,

\begin{equation}
\text{val. pert.}: \left| \frac{N}{N-s} H_{N,tN} (\sigma^i) - \left( H_N (\sigma) + \sqrt{\frac{2s}{N}} H_N^\dagger (\sigma) - 2sC_0 \right) \right| \leq \delta_N,
\end{equation}

\begin{equation}
\text{loc. pert.}: R \left( \sigma, \sigma^i \right) \geq 1 - \delta_N.
\end{equation}

Moreover, from [38, Lemma 10] (modified) for any $\kappa^i > 0$,

\begin{equation}
\lim_{\kappa \to \infty} \lim_{N \to \infty} \mathbb{P} \{ \mathcal{\mathcal{C}} (m_N - \kappa^i, m_N + \kappa^i) \subset \mathcal{\mathcal{G}}_{N,k} (\mathcal{C} (m_N - \kappa, m_N + \kappa)) \} = 1.
\end{equation}

We note that $H_N^\dagger$ and $\mathcal{C} (m_N - \kappa, m_N + \kappa)$ are independent. Thus, from (12.40), Corollary 12 and an application of the union bound to bound $|H_N^\dagger (\sigma_0)|$ uniformly in $\sigma_0 \in \mathcal{C} (m_N - \kappa, m_N + \kappa)$, we also have

\begin{equation}
\lim_{\kappa \to \infty} \lim_{N \to \infty} \mathbb{P} \{ \mathcal{\mathcal{C}} (m_N - \kappa, m_N + \kappa) \supset \mathcal{\mathcal{G}}_{N,k} (\mathcal{C} (m_N - \kappa^i, m_N + \kappa^i)) \} = 1.
\end{equation}

Denote by $\sigma_{0,lN}^i$ the critical points of $H_{N,tN} (\sigma)$, similarly to $\sigma_{0,lN}^j$. Let $\rho > 0$ be an arbitrary number and fix an integer $i \neq 0$, assumed to be positive if $p$ is odd. From the above, we have that for large enough real $\kappa > 0$ and integer $k \geq 1$, for large $N$ with probability at least $1 - \rho$, there exists $\mathcal{\mathcal{G}}_{N,k}$ as above, $\sigma_{0,lN}^i \in \mathcal{C} (m_N - \kappa, m_N + \kappa)$, $\mathcal{\mathcal{G}}_{N,k} (\sigma_{0,lN}^j) = \sigma_{0,lN}^j$ for some $j \in [k]$, and $R(\sigma_{0,lN}^i, \sigma_{0,lN}^j) \geq 1 - \delta_N$. By Corollaries 12 and 13 for large $N$ with probability at least $1 - \rho$, for any $l \in [k]$, $l \neq \pm j$, $|R(\sigma_{0,lN}^i, \sigma_{0,lN}^j)| \leq \delta_N$, for some sequence $\delta_N = o(1)$. Thus, by Theorem 11 applied both to $H_N (\sigma)$ and $H_{N,tN} (\sigma)$ (which are identically distributed), and by Lemma 35 we have that (12.35) holds, in the case of large finite $\beta$. By similar arguments to those in the proof of Proposition 33 (12.34) can also be proved.

Assuming $p$ is odd, let $\sigma^*_{iN}$ and $\sigma_{*}^{iN}$ be the global minimum of $H_N (\sigma)$ and of $H_{N,tN} (\sigma)$, respectively. From the above and Corollaries 12 and 13 two conclusions follow. First, for any $\delta > 0$,

\begin{equation}
\lim_{N \to \infty} \mathbb{P} \{ R(\sigma_{*}^{iN}, \sigma_{*}^{iN}) \in (\delta, \delta) \cup (1 - \delta, 1) \} = 1.
\end{equation}

Second, using Theorem 11 and the fact that the Hamiltonian $H_N^\dagger (\sigma)$ is independent of $\mathcal{C} (\mathbb{R})$, there exists $\rho > 0$ small enough such that both the probability that $R(\sigma_{*}^{iN}, \sigma_{*}^{iN}) \geq 1 - \delta$, and the probability that $|R(\sigma_{*}^{iN}, \sigma_{*}^{iN})| \leq \delta$ are larger than $\rho$. This completes the proof of
Proposition 42 for the case where $\beta = \infty$ and $p$ is odd. The case with even $p$ follows similarly, by taking into account the fact that $H_N(\sigma) = H_N(-\sigma)$.

**Sketch of proof of Proposition 43.** As in the proof of Proposition 42, here too the perturbations of critical points and values will play an important role. However, in contrast to the setting of Proposition 42, perturbations of the critical values will diverge as $N \to \infty$ in the current setting.

Suppose that $H_{N,t}(\sigma)$ is defined by (12.3) with $H_N'(\sigma)$ being an independent copy of $H_N(\sigma)$. Denote by $\mathcal{C}_{N,t}(\mathbb{R})$ the set of critical points of $H_{N,t}(\sigma)$ and set $C_0 = (E_0 - c_p)/2$ (see (2.6) and (2.8)). Below we will explain how the proof of Lemma 8 of [38] can be modified to obtain the following.

**Lemma 44.** Suppose $s_N > 0$ is a sequence such that $s_N = o(N)$ and $s_N \to \infty$ as $N \to \infty$ and set $t_N = s_N/N$. Then there exists a sequence $\delta_N > 0$ converging to 0 as $N \to \infty$, such that for any $\kappa > 0$, with probability that goes to 1 as $N \to \infty$, there exists a mapping $\mathcal{G}_{N,\kappa}: \mathcal{C}_N(m_N - \kappa, m_N + \kappa) \to \mathcal{C}_{N,t_N}(\mathbb{R})$ such that, denoting $\sigma' = \mathcal{G}_{N,\kappa}(\sigma)$, for any $\sigma \in \mathcal{C}_N(m_N - \kappa, m_N + \kappa)$ we have that

\begin{equation}
(12.43) \quad \text{val. pert.:} \quad \left| \frac{N}{N-s_N} H_{N,t_N}(\sigma') - \left( H_N(\sigma) + \sqrt{\frac{2s_N}{N}} H_N'(\sigma) - 2s_N C_0 \right) \right| \leq \delta_N s_N,
\end{equation}

\begin{equation}
(12.44) \quad \text{loc. pert.:} \quad R(\sigma, \sigma') \geq 1 - \delta_N.
\end{equation}

If for some $a \in (0,1/2)$, $\sigma$ and $\sigma'$ are two points that satisfy (12.43) and

\begin{equation}
(12.45) \quad \left| H_N(\sigma) - m_N \right|, \left| N^{-1/2} H_N'(\sigma) \right| < s_N^a,
\end{equation}

then

$$H_{N,t_N}(\sigma') = \left( 1 - \frac{s_N}{N} \right) \left( H_N(\sigma) + \sqrt{\frac{2s_N}{N}} H_N'(\sigma) - 2s_N C_0 \right) + o(s_N)$$

$$= m_N + (E_0 - 2C_0) s_N + o(s_N) = m_N + c_p s_N + o(s_N),$$

where we used the fact that $m_N/N \to -E_0$. Thus, from Corollary 12 and the fact that $H_N'$ and $H_N$ are independent, (12.45) holds for any $\sigma \in \mathcal{C}_N(m_N - \kappa, m_N + \kappa)$ with high probability, and for any $k \geq 1$,

$$\lim_{\kappa \to \infty} \lim_{N \to \infty} \mathbb{P} \left\{ \forall i \in [k] : \sigma^i_{0,t_N} \in \mathcal{C}_N(m_N - \kappa, m_N + \kappa), \right. \right.$$

\begin{equation}
(12.46) \quad \left. \left. \left| H_{N,t_N}(\mathcal{G}_{N,\kappa}(\sigma^i_0)) - m_N - c_p s_N \right| \leq \tau s_N \right\} = 1,
\end{equation}

where $\tau \in (0, c_p)$ is an arbitrary constant.

Now, let $\epsilon' > 0$, fix an arbitrary $k \geq 1$ and let $i, j \in [k]$. Let $\sigma^i_{0,t_N}$ denote the critical points of $H_{N,t_N}(\sigma)$, defined similarly to $\sigma^i_0$. From the above, for large enough $\kappa > 0$, for large $N$ with probability at least $1 - \epsilon'$, there exists $\mathcal{G}_{N,\kappa}$ as above, $\sigma^i_0 \in \mathcal{C}_N(m_N - \kappa, m_N + \kappa)$, and $H_{N,t_N}(\mathcal{G}_{N,\kappa}(\sigma^i_0)) \geq m_N + (c_p - \tau)s_N$, and $R(\sigma^i_0, \mathcal{G}_{N,\kappa}(\sigma^j_0)) \geq 1 - \delta_N$ (where we recall that $c_p - \tau > 0$). By Corollaries 12 and 13 since for large $N$, $(c_p - \tau)s_N > \kappa$ and since $s_N = o(1)$, we have that with probability at least $1 - \epsilon$, $|R(\sigma^j_{0,t_N}, \mathcal{G}_{N,\kappa}(\sigma^i_0))| \leq \delta_N'$, where $\delta_N'$ is some sequence with $\delta_N' = o(1)$. Therefore, with probability at least $1 - 2\epsilon'$, for large $N$, $|R(\sigma^j_{0,t_N}, \sigma^i_0)| \leq \delta_N'$, for some sequence $\delta_N' > 0$ with $\delta_N' = o(1)$. By Theorem 1 and Lemma 35 (12.36) follows. What remains is to explain how the proof of [38, Lemma 8] can be modified to prove Lemma 44.
Sketch of proof of Lemma 44. Suppose $s'_N > 0$ is an arbitrary sequence such that $s'_N = o(N)$ and $s'_N \to \infty$ as $N \to \infty$ and set $t'_N = s'_N/N$. Define $H^+_{N,s,N} (\sigma)$ as in (12.39) and note that $H^+_{N,s(N+1r),0} (\sigma) = H^+_{N,s,N} (\sigma)$. Therefore, setting $s_N = s'_N (1 + \epsilon_N)$ where $\epsilon_N$ are defined in (12.37), by (12.38) we have that
\[
(12.47) \quad \frac{N}{N - s_N} H_{N,t'_N} (\sigma) = H^+_{N,s,n,0} (\sigma) = H_N (\sigma) + \sqrt{\frac{2s_N}{N}} H'_{s,N} (\sigma) =: H^+_{N,s,N} (\sigma).
\]

Denote the set of critical points of $H^+_{N,s,N} (\sigma)$ by $C_{N,s,N} (\mathbb{R})$ and note that it coincides with the set of critical points of $H_{N,t'_N} (\sigma)$. Thus, from (12.47), by increasing $\delta_N$ if needed, we have that in order to prove Lemma 44 it is enough to show the following.

**Lemma 45.** For any $s_N > 0$ such that $s_N = o(N)$ and $s_N \to \infty$ as $N \to \infty$, we have the following. There exists a sequence $\delta_N > 0$ converging to 0 as $N \to \infty$, such that for any $\kappa > 0$, with probability that goes to 1 as $N \to \infty$, there exists a mapping $\mathcal{C}_{N,\kappa} : \mathcal{C}_N (m_N - \kappa, m_N + \kappa) \to \mathcal{C}^+_{N,s,N} (\mathbb{R})$ such that, denoting $\sigma' = \mathcal{C}_{N,\kappa} (\sigma)$, for any $\sigma \in \mathcal{C}_N (m_N - \kappa, m_N + \kappa)$ we have that
\[
(12.48) \quad \text{val. pert.: } \left| H^+_{N,s,n} (\sigma') - \left( H_N (\sigma) + \sqrt{\frac{2s_N}{N}} H'_{s,N} (\sigma) - 2s_N C_0 \right) \right| \leq \delta_N s_N,
\]
\[
(12.49) \quad \text{loc. pert.: } R (\sigma, \sigma') \geq 1 - \delta_N.
\]

In [38, Lemma 8] the statement of Lemma 45 is proved with $H^+_{N} (\sigma) := H^+_{N,1/2} (\sigma')$, i.e., with constant $s_N = 1/2$. We now sketch how the proof of [38, Lemma 8] can be modified to obtain Lemma 45. In the proof of [38, Lemma 8], ‘good’ critical points are defined by several conditions related to the perturbed $H^+_{N} (\sigma)$ and unperturbed fields $H_N (\sigma)$ and approximations of them (these are defined through the functions $g_i (\sigma)$ and sets $B_i$). In [38, Lemmas 11, 12] it is proved that the probability that all critical points in $\mathcal{C}_N (m_N - \kappa, m_N + \kappa)$ are good goes to 1 as $N \to \infty$ and that on this event $\mathcal{C}_{N,\kappa}$ can be defined with the desired properties. The structure of the proof for the modified case with $H^+_{N,s,N} (\sigma)$ instead of $H^+_{N,1/2} (\sigma')$ and all the main arguments in the proof remain the same. What needs to be modified are the definitions of the functions $g_i (\sigma)$ and sets $B_i$ through which good points are defined and several inequalities involving them. The functions $g_i (\sigma)$ depend on additional functions and we also explain how these should be modified.

First, anywhere $f^+_{\sigma} (x)$, $f_{\sigma,lin}^+ (x)$ appear in the definitions of $g_i (\sigma)$ in [38, Section 7.3] and the definitions of $f^+_{\sigma,appx} (x), Y_{\sigma}, V_{\sigma}$ and $\Delta_{\sigma}$ in [38, Section 7.2], all of them should be multiplied by $\sqrt{2s_N}$. Additionally, in the definitions of $g_i (\sigma)$ with $i = 3, 4$, the terms involving the trace should be multiplied by $2s_N$. Lastly, the radii of balls and spheres in the suprema and infima in the definitions of $g_i (\sigma)$ with $i = 5, \ldots, 8$ should be changed from $N^{-\alpha}$ to $w_0 N^{-1/2} \sqrt{s_N}$, where we assume that $w_0 > 0$ is some constant such that $2^{-3/2} K_{p,\delta} w_0^2 > w_0 > 0$ where $K_{p,\delta}$ is the original constant in the definition of $B_7$. After those changes $g_i (\sigma)$ are defined appropriately.

The sets $B_i$ should be modified as follows. Let $\hat{s}_N > 0$ be some sequence such that $s_N = o(\hat{s}_N)$ and $(\hat{s}_N)^{3/2} = o(N^{1/2} s_N)$ (and therefore, in particular $s_N = o(N)$). Let $B_N (\epsilon) \subset \mathbb{R}^N$ denote the Euclidean ball centered at 0 with radius $\epsilon$. Set $B_2 = B_{N-1} (N^{-1/2} \sqrt{s_N})$, $B_3 = B_1 (N^{-1/2} \hat{s}_N)$, $B_4 = \hat{s}_N (C_0 - \tau_0 (N), C_0 + \tau_0 (N))$, $B_5 = B_1 (CN^{-1/2} \hat{s}_N \sqrt{s_N})$, $B_6 = B_1 (CN^{-1/2} (\hat{s}_N)^{3/2})$, $B_7 = (\frac{1}{2} K_{p,\delta} w_0^2 \hat{s}_N, \infty)$, $B_8 = (-\sqrt{s_N s_N w_0}, \infty)$, where $\tau_0 (N)$ is a sequence which is assumed to be large as we need in the proofs but that still goes to 0 as
$N \to \infty$, $\delta$ is small enough such that $K_{p,\delta}$ defined in [38, Eq. (7.10)] is positive, and $C > 0$ is a constant that is assumed to be large whenever needed in the proofs.

Once all the changes above have been applied, we can define the good critical points of $H_N(\sigma)$ as critical points $\sigma_0$ such that $g_i(\sigma_0) \in B_i$ for any $i = 1, \ldots, 8$, exactly like in the original proof of [38, Lemma 8]. Then, an equivalent of [38, Lemmas 11, 12] needs to proved; i.e., we need to show for $\kappa > 0$ fixed that with probability going to 1 as $N \to \infty$ all points in $\mathcal{C}_N(m_N - \kappa, m_N + \kappa)$ are good, and that on this event $\mathcal{G}_{N,\kappa}$ can be defined appropriately. The proof of [38, Lemma 12] follows from [38, Lemmas 14, 15, 16]. The statements of [38, Lemmas 11, 12] needs to be proved; with the modified $g_i(\sigma)$ and $B_i$, imply that for $\kappa > 0$ fixed that with probability going to 1 as $N \to \infty$ all points in $\mathcal{C}_N(m_N - \kappa, m_N + \kappa)$ are good. The proofs of [38, Lemmas 14, 15, 16] with the modified $g_i(\sigma)$ and $B_i$ require minor changes from the the original case. Those changes are only in formulas and all the principal arguments should be left as they are. For example, in the case involving $g_2(\sigma)$ in the proof of [38, Lemma 16] one needs to replace the expressions

$$\|Y_n\| \leq \frac{c}{N} \|\nabla f_n^+ (0)\| \quad \text{and} \quad \mathbb{P} \left\{ Q_{N-1} \geq N^{1-\alpha} \frac{c}{\sqrt{p}} \right\}$$

by

$$\|Y_n\| \leq \frac{c\sqrt{2sN}}{N} \|\nabla f_n^+ (0)\| \quad \text{and} \quad \mathbb{P} \left\{ Q_{N-1} \geq N^{-1/2} \sqrt{\frac{s_N^o}{s_N} \frac{c}{\sqrt{p}}} \right\},$$

but other that this the proof remains the same.

Lastly, we point out what needs to be changed in the proof of [38, Lemma 11] that states that if all points in $\mathcal{C}_N(m_N - \kappa, m_N + \kappa)$ are good, then $\mathcal{G}_{N,\kappa}$ can be defined appropriately. The last line in each of the first three displayed formulas in the proof should be replaced by the following, respectively,

$$\sqrt{N}f_{\sigma,\text{apx}}^+(Y_\sigma) = H_N\left(\sqrt{N}\sigma\right) + \frac{2sN}{N}H_N'\left(\sqrt{N}\sigma\right) - 2sNC_0 \left(1 + o\left(1\right)\right),$$

$$\sqrt{N}f_\sigma^+(Y_\sigma) \leq \sqrt{N}f_{\sigma,\text{apx}}^+(Y_\sigma) + O\left(N^{-1/2}(s_N^o)^{3/2}\right)$$

$$\inf_{x: \|x\|=w_0N^{-1/2}\sqrt{N}} \sqrt{N}f_\sigma^+(x) \geq H_N\left(\sqrt{N}\sigma\right) + \sqrt{\frac{2sN}{N}H_N'\left(\sqrt{N}\sigma\right)}$$

$$+ \left(\frac{1}{4}K_{p,\delta}w_0^2 - \frac{1}{\sqrt{2}}w_0\right)s_N^o \left(1 + o\left(1\right)\right).$$

Other changes that are required are straightforward. \qed

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