Dirac and Nonholonomic Reduction

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Abstract
Several aspects of Dirac reduction are compared and formulated from the same geometric point of view. A link with nonholonomic reduction is found. The theory of optimal momentum maps and reduction is extended from the category of Poisson manifolds to that of closed Dirac manifolds. An optimal reduction method for a class of nonholonomic systems is formulated. Several examples are studied in detail.

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7 Optimal reduction for nonholonomic systems

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A Push-down of distributions

1 Introduction

The equations of motion of nonholonomic mechanical systems and those in circuit theory can be geometrically described using a Dirac structure (introduced by Courant (1990b)) in taking either a Hamiltonian or Lagrangian point of view (see, e.g., Blankenstein (2000), Blankenstein and Ratiu (2004), Blankenstein and van der Schaft (2001), Yoshimura and Marsden (2006a, b, c, 2007)).

Dirac structures simultaneously generalize symplectic and Poisson structures and also form the correct setting for the description of implicit Hamiltonian and Lagrangian systems usually appearing as systems of algebraic-differential equations. In symplectic and Poisson geometry, as well as geometric mechanics, a major role is played by the reduction method since it creates, under suitable hypotheses or in categories weaker than smooth manifolds, new spaces with the same type of motion equations on them. Briefly put, it is a method that eliminates variables and hence yields systems on smaller dimensional manifolds. Due to the spectacular array of applications, reduction has been extensively studied in various settings, including that of Dirac manifolds. The present paper continues these investigations, connects Dirac and nonholonomic reduction, introduces optimal reduction, and presents several classical examples in the different settings considered in the rest of the paper.

A Dirac structure $D$ on a manifold $M$ is a subbundle of the Pontryagin bundle $TM \oplus T^*M$ which is Lagrangian relative to the canonical symmetric pairing on it. Dirac structures were introduced by Courant (1990b) to provide a geometric framework for the study of constrained mechanical systems. The easiest example of a Dirac structure is the graph of a 2-form $\omega \in \Omega^2(M)$. Closed or integrable Dirac structures have an additional integrability condition. They have been more intensively studied because they generalize, in a certain sense, Poisson structures. For example, if the Dirac structure is the graph of $\omega \in \Omega^2(M)$, then it is integrable if and only if $d\omega = 0$. Other examples of integrable Dirac structures include various foliated manifolds. In general, a closed Dirac structure determines a singular foliation on $M$ whose leaves carry a natural induced presymplectic structure.

In this paper we study several aspects of Dirac reduction. First we recall the necessary background on Dirac geometry in §2. We begin our investigations with the comparison of two different descriptions of Dirac reduction by symmetry groups in §3. It is known that under certain assumptions beyond the usual ones, the quotient manifold carries a natural Dirac structure. These hypotheses are formulated in the literature in two different manners: using sections (see Blankenstein and van der Schaft (2001)) or appealing to the theory of fiber bundles (see Bursztyn et al. (2007)). While each approach has its advantages and both lead to the same result, it turns out that the method using sections needs an additional technical hypothesis, discussed in detail in the appendix A. We show in §4 that Dirac reduction as presented in §3 coincides with the method of reduction for nonholonomic systems due to Bates and Sniatycki (1993). This is achieved by reformulating their Hamiltonian approach to nonholonomic systems in the context of Dirac structures.

The second aspect of reduction studied in §5 and §6 is the extension of the optimal point reduction for Poisson manifolds (see Ortega and Ratiu (2004)) to symmetric closed Dirac manifolds. The Dirac optimal reduction theorem has as corollary the stratification in presymplectic leaves of a closed Dirac manifold. The reduction is carried out in two steps. First, one restricts the Dirac structure to the leaves of an appropriately chosen distribution jointly defined by the symmetry group and the Dirac structure. The leaves of this generalized distribution are the level sets of the optimal momentum map. Second, one passes to the quotient and constructs on it the reduced Dirac structure. It is not possible to extend this result in
a naive manner to non-closed Dirac structures because the first consequence of non-closedness is the non-integrability of the distribution used in the previously described reduction process. However, under certain integrability assumptions imposed on another distribution, it is possible to extend the ideas in Marsden-Weinstein reduction to nonholonomic systems. This is achieved in (7). These integrability conditions are certainly strong since they imply that the nonholonomic Noether 1-forms that descend to the quotient are exact. This is not true in general but holds in the case of certain systems such as the vertical rolling disk or the constrained particle. In order to present this nonholonomic reduction method, we reformulate the nonholonomic Noether Theorem (see Bates and Sniatycki (1993), §6, Cushman et al. (1995), Theorem 2, and Bloch (2003)) on the Hamiltonian side and give an explanation for certain constants of motion that sometimes appear as a consequence of this theorem (see Fasso et al. (2007)).

Conventions. Throughout the paper M is a paracompact manifold, that is, it is Hausdorff and every open covering admits a locally finite refinement. The orientation preserving rotation group SO(2) of the plane R^2 is also denoted by S^1 and consists of matrices of the form

\[
\begin{bmatrix}
\cos \alpha & -\sin \alpha \\
\sin \alpha & \cos \alpha
\end{bmatrix}, \quad \alpha \in \mathbb{R}.
\]

2 Generalities on Dirac structures

This section briefly summarizes the key facts from the theory of Dirac manifolds needed in the rest of the paper. It also establishes notation, terminology, and conventions, since these are not uniform in the literature. The proofs of the statements below can be found in Courant (1990b), Blankenstein and van der Schaft (2001), Blankenstein and Ratnie (2004), Bursztyn et al. (2007).

Throughout this paper we shall use the following notation. If E → M is a smooth fiber bundle over a manifold M, the spaces of smooth global and local sections are denoted by Γ(global)(E) and Γ(E), respectively. For example, \( \mathfrak{X}(M) := \Gamma(TM) \) denotes the Lie algebra of smooth local vector fields endowed with the usual Jacobi-Lie bracket \([X, Y](f) = X[Y(f)] - Y[X(f)]\), where \( X, Y \in \mathfrak{X}(M) \), \( f \) is a smooth (possibly only locally defined) function on \( M \), and \( X[f] := \mathcal{L}_X f = df(X) \) denotes the Lie derivative of \( f \) in the direction \( X \). If \( \wedge^k M \to M \) denotes the vector bundle of exterior k-forms on \( M \) then \( \Omega^k(M) := \Gamma(\wedge^k M) \) is the space of local k-forms on the manifold \( M \).

2.1 Dirac structures

For a smooth manifold \( M \) denote by \((\cdot, \cdot)\) the duality pairing between the cotangent bundle \( T^*M \) and the tangent bundle \( TM \) or \( \Omega^1(M) \) and \( \mathfrak{X}(M) \). The Pontryagin bundle \( TM \oplus T^*M \) is endowed with a nondegenerate symmetric fiberwise bilinear form of signature \((\dim M, \dim M)\) given by

\[
\langle (u_m, \alpha_m), (v_m, \beta_m) \rangle := \langle \beta_m, u_m \rangle + \langle \alpha_m, v_m \rangle
\]

for all \( u_m, v_m \in TM \) and \( \alpha_m, \beta_m \in T^*_m M \). A Dirac structure (see Courant (1990b)) on \( M \) is a Lagrangian subbundle \( D \subset TM \oplus T^*M \), that is, \( D \) coincides with its orthogonal relative to \( \mathfrak{X}(M) \) and so its fibers are necessarily \( \dim M \)-dimensional.

The space \( \Gamma(TM \oplus T^*M) \) of local sections of the Pontryagin bundle is also endowed with a \( \mathbb{R} \)-bilinear skew-symmetric bracket (which does not satisfy the Jacobi identity) given by

\[
[(X, \alpha), (Y, \beta)] := \left([X, Y], \mathcal{L}_X \beta - \mathcal{L}_Y \alpha + \frac{1}{2} d (\alpha(Y) - \beta(X))\right)
\]

\[
= \left([X, Y], \mathcal{L}_X \beta - i_Y d \alpha - \frac{1}{2} d \langle (X, \alpha), (Y, \beta) \rangle\right)
\]

(see Courant (1990b)). The Dirac structure is closed or integrable if \( \Gamma(D), \Gamma(D) \subset \Gamma(D) \). Since \( \langle (X, \alpha), (Y, \beta) \rangle = 0 \) if \( (X, \alpha), (Y, \beta) \in \Gamma(D) \), closedness of the Dirac structure is often expressed in the literature relative to a non-skew-symmetric bracket that differs from (2) by eliminating in the second line the third term of the second component. This truncated expression which satisfies the Jacobi identity but is no longer skew-symmetric is called the Courant bracket (see Bursztyn et al. (2004), Bursztyn and Crainic (2005), Liu et al. (1997), Severa and Weinstein (2001)).

\( ^1 \)A somewhat restricted version of the momentum equation was given in Kozlov and Kolesnikov (1978); see also Arnold et al. (1988).
2.2 Distributions

We will need a few standard facts from the theory of generalized distributions on a smooth manifold $M$ (see Stefan (1974a, b, 1980), Sussmann (1973) for the original articles and Libermann and Marle (1987), Vaisman (1994), Pflaum (2001), or Ortega and Ratiu (2004), for a quick review of this theory).

A generalized distribution $\Delta$ on $M$ is a subset of the tangent bundle $TM$ such that $\Delta(m) := \Delta \cap T_m M$ is a vector subspace of $T_m M$. The number $\dim \Delta(m)$ is called the rank of $\Delta$ at $m \in M$. A local differentiable section of $\Delta$ is a smooth vector field $X \in \mathfrak{X}(M)$ defined on some open subset $U \subset M$ such that $X(u) \in \Delta(u)$ for each $u \in U$. In keeping with our previous notations, $\Gamma(\Delta)$ (respectively $\Gamma_{\text{glob}}(\Delta)$) denotes the space of local (respectively global) sections of $\Delta$. A generalized distribution is said to be differentiable or smooth if for every point $m \in M$ and every vector $v \in \Delta(m)$, there is a differentiable section $X \in \Gamma(\Delta)$ defined on an open neighborhood $U$ of $m$ such that $X(m) = v$. The distribution $\Delta$ is locally finite if for each point $m \in M$ there exists a neighborhood $U$ of $m$ and smooth vector fields $X_1, \ldots, X_k$ defined on $U$ such that for all $m'$ in $U$ we have

$$\Delta(m') = \text{span}\{X_1(m'), \ldots, X_k(m')\}.$$ 

Note that a locally finite distribution is necessarily smooth.

The term distribution is usually synonymous to that of a vector subbundle of $TM$. Since we shall work mostly with generalized distributions, we shall call below all generalized distributions simply distributions. If the generalized distribution happens to be a vector subbundle we shall always state this fact explicitly.

In all that follows, $\Delta$ is a smooth distribution. An integral manifold of $\Delta$ is an injectively immersed connected manifold $\iota_L : L \hookrightarrow M$, where $\iota_L$ is the inclusion, satisfying the condition $T_m \iota_L(T_m L) \subset \Delta(m)$ for every $m \in L$. The integral manifold $L$ is of maximal dimension at $m \in L$ if $T_m \iota_L(T_m L) = \Delta(m)$. The distribution $\Delta$ is completely integrable if for every $m \in M$ there is an integral manifold $L$ of $\Delta$, $m \in L$, everywhere of maximal dimension. The distribution $\Delta$ is involutive if it is invariant under the (local) flows associated to differentiable sections of $\Delta$. The distribution $\Delta$ is algebraically involutive if for any two smooth vector fields defined on an open set of $M$ which take values in $\Delta$, their bracket also takes values in $\Delta$. Clearly involutive distributions are algebraically involutive and the converse is true if the distribution is a subbundle. The analog of the Frobenius theorem (which deals only with vector subbundles of $TM$) for distributions is known as the Stefan-Sussmann Theorem. Its statement is the same except that one needs the distribution to be involutive and not just algebraically involutive: $\Delta$ is completely integrable if and only if $\Delta$ is involutive.

Recall that the Frobenius theorem states that a vector subbundle of $TM$ is (algebraically) involutive if and only if it is the tangent bundle of a foliation on $M$. The same is true for distributions: A smooth distribution is involutive if and only if it coincides with the set of vectors tangent to a generalized foliation. To give content to this statement and elaborate on it, we need to quickly review the concept and main properties of generalized foliations.

A generalized foliation on $M$ is a partition $\mathfrak{F} := \{\mathcal{L}_a\}_{a \in A}$ of $M$ into disjoint connected sets, called leaves, such that each point $m \in M$ has a generalized foliated chart $(U, \varphi : U \to V \subset \mathbb{R}^{\dim M})$, $m \in U$. This means that there is some natural number $p_\alpha \leq \dim M$, called the dimension of the leaf $\mathcal{L}_\alpha$, and a subset $S_\alpha \subset \mathbb{R}^{\dim M-p_\alpha}$ such that $\varphi(U \cap \mathcal{L}_\alpha) = \{(x^1, \ldots, x^{\dim M}) \in V \mid (x^{p_\alpha+1}, \ldots, x^{\dim M}) \in S_\alpha\}$. The key difference with the concept of foliation is that the number $p_\alpha$ can change from leaf to leaf. Note that each $(x^{p_\alpha+1}, \ldots, x^{\dim M}) \in S_\alpha$ determines a connected component $(U \cap \mathcal{L}_\alpha)_0$ of $U \cap \mathcal{L}_\alpha$, that is, $\varphi((U \cap \mathcal{L}_\alpha)_0) = \{(x^1, \ldots, x^{p_\alpha}, x^{p_\alpha+1}, \ldots, x^{\dim M}) \in V\}$. The generalized foliated charts induce on each leaf a smooth manifold structure that makes them into initial submanifolds of $M$.

Recall that a subset $N \subset M$ is an initial submanifold of $M$ if $N$ carries a manifold structure such that the inclusion $\iota : N \hookrightarrow M$ is a smooth immersion and satisfies the following condition: for any smooth manifold $P$ an arbitrary map $g : P \to N$ is smooth if and only if $\iota \circ g : P \to M$ is smooth. The notion of initial submanifold lies strictly between those of injectively immersed and embedded submanifolds.

A leaf $\mathcal{L}_\alpha$ is called regular if it has an open neighborhood that intersects only leaves whose dimension equals $\dim \mathcal{L}_\alpha$. If such a neighborhood does not exist, then $\mathcal{L}_\alpha$ is called a singular leaf. A point is called regular (singular) if it is contained in a regular (singular) leaf. The set of vectors tangent to the leaves of $\mathfrak{F}$ is defined by

$$T(M, \mathfrak{F}) := \bigcup_{a \in A} \bigcup_{m \in \mathcal{L}_a} T_m \mathcal{L}_a \subset TM.$$
Under mild topological conditions on $M$ a generalized foliation has very useful properties. Assume that $M$ is second countable. Then for each $p$-dimensional leaf $L_m$ and any generalized foliated chart $(U, \varphi : U \to V \in \mathbb{R}^{\dim M})$ that intersects it, the corresponding set $S_m$ is countable. The set of regular points is open and dense in $M$. Finally, any closed leaf is embedded in $M$. Note that this last property is specific to (generalized) foliations since an injectively immersed submanifold whose range is closed is not necessarily embedded.

Let us return now to the relationship between distributions and generalized foliations. As already mentioned, given an involutive (and hence a completely integrable) distribution $\Delta$, each point $m \in M$ belongs to exactly one connected integral manifold $L_m$ that is maximal relative to inclusion. It turns out that $L_m$ is an initial submanifold and that it is also the accessible set of $m$, that is, $L_m$ equals the subset of points in $M$ that can be reached by applying to $m$ a finite number of composition of flows of elements of $\Gamma(\Delta)$. The collection of all maximal integral submanifolds of $\Delta$ forms a generalized foliation $\mathfrak{F}_\Delta$ such that $\Delta = T(M, \mathfrak{F}_\Delta)$. Conversely, given a generalized foliation $\mathfrak{F}$ on $M$, the subset $T(M, \mathfrak{F}) \subset TM$ is a smooth completely integrable (and hence involutive) distribution whose collection of maximal integral submanifolds coincides with $\mathfrak{F}$. These two statements expand the Stefan-Sussmann Theorem cited above.

In the study of Dirac manifolds we will also need the concept of codistribution. A generalized codistribution $\Xi$ on $M$ is a subset of the cotangent bundle $T^*M$ such that $\Xi(m) := \Xi \cap T^*_mM$ is a vector subspace of $T^*_mM$. The notions of rank, differentiable section, and smooth codistribution are completely analogous to those for distributions.

If $\Delta \subset TM$ is a smooth distribution on $M$, its (smooth) annihilator $\Delta^\circ$ is defined by

$$\Delta^\circ(m) := \{\alpha(m) \mid \alpha \in \Omega^1(M), \langle \alpha, X \rangle = 0 \text{ for all } X \in \mathfrak{X}(U), m \in U, \text{ such that } X(u) \in \Delta(u) \text{ for all } u \in U\}.$$ 

We have the, in general strict, inclusion $\Delta \subset \Delta^{\circ\circ}$. A similar definition holds for smooth codistributions. Note that the annihilators are smooth by construction. If a distribution (codistribution) is a vector subbundle of $TM$ (respectively of $T^*M$), then its annihilator is also a vector subbundle of $T^*M$ (respectively of $TM$). If $\Delta$ is a subbundle then $\Delta = \Delta^{\circ\circ}$ and similarly for codistributions.

### 2.3 Characteristic equations

A Dirac structure defines two smooth distributions $G_0, G_1 \subset TM$ and two smooth codistributions $P_0, P_1 \subset T^*M$:

$$G_0(m) := \{X(m) \in T_mM \mid X \in \mathfrak{X}(M), (X, 0) \in \Gamma(D)\}$$

$$G_1(m) := \{X(m) \in T_mM \mid X \in \mathfrak{X}(M), \text{ there is an } \alpha \in \Omega^1(M), \text{ such that } (X, \alpha) \in \Gamma(D)\}$$

and

$$P_0(m) := \{\alpha(m) \in T^*_mM \mid \alpha \in \Omega^1(M), (0, \alpha) \in \Gamma(D)\}$$

$$P_1(m) := \{\alpha(m) \in T^*_mM \mid \alpha \in \Omega^1(M), \text{ there is an } X \in \mathfrak{X}(M), \text{ such that } (X, \alpha) \in \Gamma(D)\}.$$ 

The smoothness of $G_0, G_1, P_0, P_1$ is obvious since, by definition, they are generated by smooth local sections. In general, these are not vector subbundles of $TM$ and $T^*M$, respectively. It is also clear that $G_0 \subset G_1$ and $P_0 \subset P_1$. The distributions $G_0, G_1$ are related to the codistributions $P_0$ and $P_1$ through the operation of taking annihilators.

The characteristic equations of a Dirac structure are

(i) $G_0 = P_1^\circ, P_0 = G_1^\circ$.

(ii) $P_1 \subset G_0^\circ, G_1 \subset P_0^\circ$.

(iii) If $P_1$ has constant rank, then $P_1 = G_0^\circ$. If $G_1$ has constant rank, then $G_1 = P_0^\circ$. 

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The following facts are useful in the study of Dirac structures.

**A.** Let \( P \) be a constant rank codistribution on \( M \) and \( \flat : P^o \to (P^o)^* \) a skew-symmetric vector bundle map (in every fiber). Then \( D \subset TM \oplus T^*M \) defined for every \( m \in M \) by

\[
D(m) := \{(X(m), \alpha(m)) \in T_mM \oplus T^*_mM \mid X \text{ a smooth local section of } P^o, \alpha \in \Omega^1(M), \alpha|_{P^o} = X^\flat \} \quad (3)
\]

is a Dirac structure on \( M \).

Conversely, if \( D \) is a Dirac structure on \( M \) having the property that \( G_1 \subset TM \) is a constant rank distribution on \( M \), then there exists a skew-symmetric vector bundle map \( \flat : G_1 \to G_1 \) such that \( D \) is given by \( (2) \) with \( P := P_0 = G_1^\flat \). Also, \( \ker(\flat : G_1 \to G_1^\flat) = G_0 \).

**B.** Let \( G \) be a constant rank distribution on \( M \) and \( \sharp : G^o \to (G^o)^* \) a skew-symmetric vector bundle map (in every fiber). Then \( D \subset TM \oplus T^*M \) defined for every \( m \in M \) by

\[
D(m) := \{(X(m), \alpha(m)) \in T_mM \oplus T^*_mM \mid \alpha \text{ a smooth local section of } G^o, X \in \mathfrak{X}(M), X|_{G^o} = \alpha^\sharp \} \quad (4)
\]

is a Dirac structure on \( M \).

Conversely, if \( D \) is a Dirac structure on \( M \) having the property that \( P_1 \subset T^*M \) is a constant rank codistribution on \( M \), then there exists a skew-symmetric vector bundle map \( \sharp : P_1 \to P_1^\flat \) such that \( D \) is given by \( (3) \) with \( P := G_0 = P_1^\flat \). Also, \( \ker(\sharp : P_1 \to P_1^\flat) = P_0 \).

If \( D \) is a closed Dirac structure on \( M \) then \( G_0 \) and \( G_1 \) are algebraically involutive distributions. Hence, if \( G_1 \) is in addition a subbundle of \( TM \), it is integrable in the sense of Frobenius. Analogously, if the codistribution \( P_1 \) has constant rank, i.e., \( P_1 \subset T^*M \) is a subbundle, then \( G_0 = P_1^\flat \subset TM \) is an involutive subbundle and thus integrable.

A function \( f \in C^\infty(M) \) is called **admissible** if \( df \in \Gamma(P_1) \). If the Dirac structure \( D \) on \( M \) is closed, there is an induced Poisson bracket \( \{\cdot,\cdot\}_D \) on the admissible functions given by

\[
\{f,g\}_D = X_f[g] = -X_g[f],
\]

where \( X_f \in \mathfrak{X}(M) \) is such that \( (X_f, df) \in \Gamma(D) \). Note that \( X_f \in \mathfrak{X}(M) \) is not uniquely determined by this condition. If the Dirac structure is not closed, we get with the same definition an almost Poisson structure, that is, the Jacobi-identity doesn’t necessarily hold.

### 2.4 Integrable Dirac structures as Lie algebroids

The statement of integrability of \( G_1 \) in the preceding subsection can be extended to closed Dirac structures without the assumption of constant dimensionality of the fibers of \( G_1 \). To formulate this well-known result in detail, we need a short presentation of Lie algebroids.

A Lie algebroid \( E \to M \) is a smooth vector bundle over \( M \) with a vector bundle homomorphism \( \rho : E \to TM \), called the anchor, and a Lie algebra bracket \( [\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \) satisfying:

1. \( \rho \) is a Lie algebra homomorphism
2. for all \( f \in C^\infty(M) \) and \( X, Y \in \Gamma(E) \):

\[
[X, fY] = f[X,Y] + \rho(X)[f]Y.
\]

It is shown in Courant (1990b) that for an arbitrary Lie algebroid \( E \to M \), the smooth distribution \( \rho(E) \) is completely integrable.

Assume that \( D \) is a closed Dirac structure. Then, relative to the Courant bracket \( (2) \) and the anchor \( \pi_1 : D \to TM \) given by the projection on the first factor, \( D \) becomes a Lie algebroid over \( M \). The smooth distribution \( \pi_1(D) \subset TM \) coincides with \( G_1 \). Indeed, \( v_m \in \pi_1(D) \) if and only if there is some \( \alpha_m \in T^*M \) such that \( (v_m, \alpha_m) \in D(m) \). However, \( D \) is a vector bundle and hence admits local sections. Therefore, the point \( m \in M \) has an open neighborhood \( U \ni m \) and there are \( X \in \mathfrak{X}(U) \) and \( \alpha \in \Omega^1(U) \) such that \( v_m = X(m) \) and \( \alpha(m) = \alpha_m \), which is equivalent to \( v_m \in G_1(m) \). Furthermore, Theorem 2.3.6 in Courant (1990b) states the following result.
Theorem 2.1 An integrable Dirac structure has a generalized foliation by presymplectic leaves.

The presymplectic form $\omega_N$ on a leaf $N$ of the generalized foliation is given by

$$\omega_N(\tilde{X}, \tilde{Y})(p) = \alpha(Y)(p) = -\beta(X)(p)$$

(6)

for all $p \in N$ and $\tilde{X}, \tilde{Y} \in \mathfrak{X}(N)$, where $i_N : N \hookrightarrow M$ is the inclusion and $X, Y \in \Gamma(G_1)$ are $i_N$-related to $\tilde{X}, \tilde{Y}$, respectively; we shall denote $i_N$-relatedness by $X \sim_{i_N} Y$ and $\tilde{X} \sim_{i_N} \tilde{Y}$. The 1-forms $\alpha, \beta \in \Omega^1(M)$ are such that $(X, \alpha), (Y, \beta) \in \Gamma(D)$. Formula (6) is independent of all the choices involved. Note that there is an induced Dirac structure on $N$ given by the graph of the bundle map $\iota : TN \to T^*N$ associated to $\omega_N$ (see (2,3)).

2.5 Implicit Hamiltonian systems

Let $D$ be a Dirac structure on $M$ and $H \in C^\infty(M)$. The implicit Hamiltonian system $(M, D, H)$ is defined as the set of $C^\infty$ solutions $x(t)$ satisfying the condition

$$(\dot{x}, dH(x(t))) \in D(x(t)), \quad \text{for all } t.$$  

(7)

In this general situation, conservation of energy is still valid: $\dot{H}(t) = (dH(x(t)), \dot{x}(t)) = 0$, for all $t$ for which the solution exists. In addition, these equations contain algebraic constraints, namely, $dH(x(t)) \in \mathfrak{p}_1(x(t))$, for all $t$. Note that $\dot{x}(t) \in \mathcal{G}_1(x(t))$, so the set of admissible flows have velocities in the distribution $\mathcal{G}_1$. Thus, an implicit Hamiltonian system defines a set of differential and algebraic equations.

Note that if $\mathcal{G}_1$ is an involutive subbundle of $TM$, then there are $\dim \mathcal{G}_1 \leq \text{rank } \mathcal{G}_1$ independent conserved quantities for the Hamiltonian system (7). We want to emphasize that standard existence and uniqueness theorems do not apply to (7), even if all the distributions and codistributions are subbundles. The only general theorems that ensure the local existence and uniqueness of solutions for (7) are for the so-called implicit Hamiltonian systems of index one (see Blankenstein (2000), Blankenstein and van der Schaft (2001)).

2.6 Restriction of Dirac structures

First, we describe the restriction of Dirac structures to submanifolds. Let $D$ be a Dirac structure on $M$ and $N \subset M$ a submanifold of $M$. Define the map $\sigma(m) : T_mN \times T^*_mM \to T_mN \times T^*_mN$, $m \in N$, by $\sigma(m)(v_m, \alpha_m) = (v_m, \alpha_m|_{T_mN})$. Assume that the dimension of $\mathcal{G}_1(m) \cap T_mN$ is independent of $m \in N$ and that the rank of $\mathcal{G}_1$ is constant on $M$. Define the vector subbundle $D_N \subset TN \oplus T^*N$ by

$$D_N(m) = \sigma(m)(D(m) \cap (T_mN \times T^*_mM)), \quad m \in N.$$  

Then $D_N$ is a Lagrangian subbundle in the Pontryagin bundle $TN \oplus T^*N$ and is thus a Dirac structure on $N$.

Let $\iota : N \hookrightarrow M$ denote the inclusion map and define for all $m \in N$

$$E_s(m) := \{(X(m), \alpha(m)) \in T_mM \times T^*_mM \mid \alpha \in \Omega^1(M), X \in \mathfrak{X}(M) \text{ such that } X(n) \in T_nN \text{ for all } n \in N \text{ for which } X \text{ is defined}\}$$

(where the subscript $s$ stands for submanifold). This defines a smooth bundle $E_s = \cup_{m \in N} E_s(m)$ on $N$. Blankenstein and van der Schaft (2001) show that under the assumption that the fibers of $E_s \cap D$ have constant dimension on $M$, there is another way to give the induced Dirac structure, namely, $(\tilde{X}, \tilde{\alpha})$ is a local section of $D_N$ if and only if there exists a local section $(X, \alpha)$ of $D$ such that $\tilde{X} \sim_{\iota} X$ and $\tilde{\alpha} = \iota^* \alpha$. Otherwise stated,

$$\Gamma(D_N) = \{(\tilde{X}, \tilde{\alpha}) \in \mathfrak{X}(N) \oplus \Omega^1(N) \mid \text{there is } (X, \alpha) \in \Gamma(D) \text{ such that } \tilde{X} \sim_{\iota} X \text{ and } \tilde{\alpha} = \iota^* \alpha\}.$$  

(8)

Furthermore, if $D$ is closed, then $D_N$ is also closed. As stated in Blankenstein and van der Schaft (2001), if $\mathcal{G}_1$ is constant dimensional, the assumptions for both methods of restriction are equivalent.

Second, we recall the restriction construction for implicit Hamiltonian systems. Given is the implicit Hamiltonian system $(M, D, H)$ and $N \subset M$ an invariant submanifold under the integral curves of $(M, D, H)$
(if they exist). Define $H_N := H|_N = H \circ \iota$. Then every solution $x(t)$ of $(M, D, H)$ which leaves $N$ invariant (that is, $x(t) \subset N$) is a solution of $(N, D_N, H_N)$. The converse statement is not true, in general.

For example, assume that $N \subset M$ is such that every $X \in G_1$ is tangent to $N$, that is, $X(n) \in T_nN$, for all $n \in N$. Then the solutions of $(M, D, H)$ contained in $N$ are exactly the solutions of the implicit generalized Hamiltonian system $(N, D_N, H_N)$.

Another interesting example of the restriction construction appears under the following hypotheses. Assume that $D$ is closed and $G_1$ is a vector subbundle of $TM$. Recall that there exists a skew-symmetric vector bundle map $\flat : G_1 \to G_1^*$ with kernel $G_0$, such that

$$D(m) = \{ (v_m, \alpha_m) \in TM \times T^*M \mid \alpha_m - \flat(v_m) \in G_1(m)^\circ, \, v \in G_1(m) \}, \quad m \in M.$$

Since in this case $G_1$ is algebraically involutive and constant dimensional, it is integrable in the sense of Frobenius. Hence $G_1$ defines a foliation partitioning $M$ into integral submanifolds of $G_1$.

Restricting $D$ to such an integral submanifold $N$ yields

$$D_N(m) = \{ (\tilde{v}_m, \tilde{\alpha}_m) \in TN \times T^*_N \mid \tilde{\alpha}_m = \tilde{\flat}(\tilde{v}_m), \, \text{for all } m \in N \},$$

where $\tilde{\flat}$ is the restriction of $\flat$ to $N$. Then $\tilde{\flat}$ defines a closed 2-form on $N$ with kernel $G_0$. Hence $D_N$ is a presymplectic structure on $N$. This leads to a special case of Theorem 2.1. In particular, the restriction $(N, D_N, H_N)$ is a presymplectic Hamiltonian system on $N$.

## 3 Reduction of Dirac structures

In this section we introduce Lie group and Lie algebra symmetries of a Dirac manifold. Then we present two of the three symmetry reduction methods of Dirac structures found in the literature and show that they are equivalent.

### 3.1 Lie group and Lie algebra symmetries

Let $G$ be a Lie group and $\Phi : G \times M \to M$ a smooth left action. Then $G$ is called a symmetry Lie group of $D$ if for every $g \in G$ the condition $(X, \alpha) \in \Gamma(D)$ implies that $(\Phi^*_gX, \Phi^*_g\alpha) \in \Gamma(D)$. We say then that the Lie group $G$ acts canonically or by Dirac actions on $M$.

For any admissible $f \in C^\infty(M)$, i.e., a function such that $(X_f, df) \in \Gamma(D)$ for some $X_f \in \mathfrak{X}(M)$, this yields $(\Phi^*_gX_f, \Phi^*_gdf) \in \Gamma(D)$ or $(\Phi^*_gX_f, df(\Phi^*_gf)) \in \Gamma(D)$. Hence we have simultaneously the facts that $\Phi^*_gX_f$ is admissible and that $\Phi^*_gX_f - X_{\Phi^*_g f} =: Y \in \Gamma(G_0)$. This implies for the almost Poisson bracket on admissible functions (see [2.1]):

$$\{ \Phi^*_g f, h \}_D = -\Phi^*_g(X_f[h]) = -(\Phi^*_gX_f)[\Phi^*_g h] = -(Y + X_{\Phi^*_g f})[\Phi^*_g h]$$

$$= -d(\Phi^*_g h)(Y + X_{\Phi^*_g f}) = d(\Phi^*_g h)(X_{\Phi^*_g f}) = \{ \Phi^*_g f, \Phi^*_g h \}_D$$

since $\Phi^*_g h$ is an admissible function (and hence $d(\Phi^*_g h) \in \mathfrak{p}_1 \subset G_0^\circ$ and $Y \in \Gamma(G_0)$).

The Lie group $G$ is a symmetry Lie group of the implicit Hamiltonian system $(M, D, H)$ if, in addition, $H$ is $G$-invariant, that is, $H \circ \Phi_g = H$ for all $g \in G$.

Let $\mathfrak{g}$ be a Lie algebra and $\xi \in \mathfrak{g} \mapsto \xi_M : \mathfrak{X}(M)$ be a smooth left Lie algebra action, that is, the map $(x, \xi) \in M \times \mathfrak{g} \mapsto \xi_M(x) \in TM$ is smooth and $\xi \in \mathfrak{g} \mapsto \xi_M \in \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism. The Lie algebra $\mathfrak{g}$ is said to be a symmetry Lie algebra of $D$ if for every $\xi \in \mathfrak{g}$ the condition $(X, \alpha) \in \Gamma(D)$ implies that $(\mathcal{L}_\xi X, \mathcal{L}_\xi \alpha) \in \Gamma(D)$. If, in addition, $\mathcal{L}_\xi H = 0$ for all $\xi \in \mathfrak{g}$, then $\mathfrak{g}$ is a symmetry Lie algebra of the implicit Hamiltonian system $(M, D, H)$. Of course, if $\mathfrak{g}$ is the Lie algebra of $G$ and $\xi \mapsto \xi_M$ is the associated infinitesimal generator, then if $G$ is a symmetry Lie group of $D$ it follows that $\mathfrak{g}$ is a symmetry Lie algebra of $D$.

### 3.2 The reduction methods

There are three reduction procedures of Dirac structures in the literature. Two of them are standard and appear in various works, the third one is still in the stage of development and is considerably more
general (Cendra et al. (2008)). We shall review the two established procedures here and show that they are equivalent.

In all that follows we shall assume that $G$ is a symmetry Lie group of the Dirac structure $D$ on $M$ and that the action is free and proper. Thus, the projection on the quotient $\pi : M \to M/G := \hat{M}$ defines a left principal $G$-bundle. Note that the Dirac structure $D \subset TM \oplus T^*M$ is $G$-invariant as a subbundle since for all $g \in G$ and $(X,\alpha) \in \Gamma(D)$ we have $(\Phi^*_g X, \Phi^*_g \alpha) \in \Gamma(D)$. Recall that the infinitesimal generators $\xi_M$ for $\xi \in \mathfrak{g}$ are also $G$-equivariant: for all $\xi \in \mathfrak{g}$ we have

$$\Phi^*_g \xi_M = (Ad_g^{-1} \xi)_M \in \mathfrak{g}_M$$

(see, e.g., Marsden and Ratiu (1999), Lemma 9.3.1), where $\mathfrak{g}_M := \{ \eta_M \mid \eta \in \mathfrak{g} \} \subset \mathfrak{X}(M)$. Define for $m \in M$ the vector subspace $V(m) := \{ \xi_M(m) \mid \xi \in \mathfrak{g} \} \subseteq T_mM$ and the distribution $V := \cup_{m \in M} V(m)$. Since the $G$-action is free, $V$ is a vector subbundle of $TM$. The subbundle $V$ is $G$-invariant (see (9)). It is worth noting that the space of sections $\Gamma(V)$ coincides with the $C^\infty(M)$-module spanned by $\mathfrak{g}_M$. The identity $\Phi^*_g V^\circ = V^\circ$ for all $g \in G$ follows immediately. Note also that $\Gamma(V)$ is the tangent space at $m \in M$ of the $G$-orbit through $m$, and the orbit is endowed with the manifold structure that makes it diffeomorphic to $G$, using the freeness of the action.

For all $m \in M$ the map $T_m \pi : T_mM \to T_{\pi(m)}\hat{M}$ is surjective with kernel $V(m)$. This yields an isomorphism between $T_mM/V(m)$ and $T_{\pi(m)}\hat{M}$. The Lie group $G$ acts smoothly on the quotient vector bundle $TM/V$ by $g \cdot \dot{v} := T\Phi^*_g(v)$, where $\dot{v} \in TM/V$; this action is well defined by (9).

In what follows we shall need the following elementary observation: each section of $TM/V$ is the projection of a smooth vector field on $M$. Indeed, pick a $G$-invariant Riemannian metric on $M$ (whose existence is guaranteed by the paracompactness of $M$ and the properness of the $G$-action; Palais (1961), Theorem 4.3.1 or Duistermaat and Kolk (2000), Proposition 2.5.2), decompose $TM = V \oplus V^\perp$, and identify the vector bundles $TM/V$ and $V^\perp$. Thus sections of $TM/V$ are identified with smooth vector fields on $M$ taking values only in $V^\perp$.

For $X \in \mathfrak{X}(M)$, we will say that the section $\tilde{X} := X(\mod V)$ of $TM/V$ is $G$-equivariant, if there is a representative $X^G$ of $\tilde{X}$ that is $G$-equivariant, i.e., a smooth section $X^G \in \mathfrak{X}(\hat{M})^G$ with $X^G = X - X^G \in \Gamma(V)$. This is equivalent to the condition $[X, V] \in \Gamma(V)$ for all representatives $X$ of $\tilde{X}$ and for all $V \in \Gamma(V)$; this is the content of Corollary 9.2. In what follows we shall use these two equivalent definitions interchangeably.

The representative $X^G$ of $\tilde{X}$ uniquely induces a smooth vector field $\hat{\tilde{X}}$ on $M$, where $\hat{\tilde{X}}$ is defined by the condition $X^G \sim_{\pi} \hat{\tilde{X}}$, that is, $T\pi \circ X^G = \hat{\tilde{X}} \circ \pi$ (see also Proposition 9.2). Then, for any representative $Y$ of $\hat{\tilde{X}} = \hat{\tilde{X}}^G$ we have $Y - X^G =: V \in \Gamma(V)$, and hence $T\pi \circ Y = T\pi \circ X^G + T\pi \circ V = \hat{\tilde{X}} \circ \pi$, which shows that

$$\Pi : \Gamma(TM/V)^G \to \mathfrak{X}(\hat{M})$$

$$X(\mod V) \mapsto \hat{\tilde{X}},$$

where $\hat{\tilde{X}}$ is defined by the condition $X^G \sim_{\pi} \hat{\tilde{X}}$, that is, $T\pi \circ X^G = \hat{\tilde{X}} \circ \pi$, is a well defined homomorphism of $C^\infty(M)$-modules (note that $C^\infty(M) \simeq C^\infty(M)^G$ via $\tilde{f} \mapsto \pi^* \tilde{f}$). This map (10) is in fact bijective, hence an isomorphism. To prove injectivity, let $\hat{\tilde{X}}$ and $\hat{\tilde{Y}}$ be $G$-equivariant elements of $\Gamma(TM/V)$ with $\Pi(\hat{\tilde{X}}) = \Pi(\hat{\tilde{Y}}) = \hat{\tilde{X}}$. Then we necessarily have $X^G - Y^G \in \Gamma(V)$ and thus $\hat{\tilde{X}} = \hat{\tilde{X}}^G = \hat{\tilde{Y}}^G = \hat{\tilde{Y}}$. The proof of surjectivity uses the Tube Theorem (see, e.g., Palais (1961) or Ortega and Ratiu (2004) Theorem 2.3.28) which states in the case of free proper actions that for every point $m \in M$ one can find a $G$-invariant open neighborhood $U$ of $m$ and a $G$-equivariant diffeomorphism $\psi : U \to G \times B$, where $B$ is an open ball in the vector space $T_mM/T_m(G \cdot m)$; the $G$-action on $G \times B$ is left translation on the first factor. Thus $\psi$ induces a diffeomorphism $\psi : U/G \to B$ uniquely determined by the condition $\psi \circ \pi|_U = p_2 \circ \psi$, where $p_2 : G \times B \to B$ is the projection on the second factor. Now if $X \in \mathfrak{X}(M)$, then $\psi_*X \in \mathfrak{X}(B)$ so that $\hat{\tilde{X}} \in \mathfrak{X}(G \times B)$ defined by $\hat{\tilde{X}}(g, b) := (0, (\hat{\tilde{X}})_b(g))$, for $g \in G$ and $b \in B$, is $G$-equivariant. Therefore, $\psi^*\hat{\tilde{X}} \in \mathfrak{X}(U)$ is $G$-equivariant and we clearly have $\psi^*\hat{\tilde{X}} \sim_{\pi} \hat{\tilde{X}}$ by construction which show that the map $X(\mod V) \mapsto \hat{\tilde{X}}$ is surjective. (The construction of $G$-equivariant liffted vector fields on $M$ from vector fields on $M/G$ is done for compact groups in Bierstone (1975), Theorem D, and for general proper actions in Duistermaat, Theorem 6.10.)

In the same way, for all $\alpha \in \Omega^1(\hat{M})$, we have $\pi^*\alpha \in \Gamma(V^\circ)^G$. Note that if $\alpha \in \Gamma(V^\circ)^G$, then the 1-form $\tilde{\alpha} \in \Omega^1(\hat{M})$ defined by $\langle \tilde{\alpha}(\pi(m)), T_m\pi(v_m) \rangle := \langle \alpha(m), v_m \rangle$, for all $v_m \in T_mM$, is well defined and satisfies $\pi^*\tilde{\alpha} = \alpha$. This shows that the map $\tilde{\alpha} \in \Omega^1(\hat{M}) \mapsto \pi^*\tilde{\alpha} \in \Gamma(V^\circ)^G$ is an isomorphism of $C^\infty(M)$-modules.
We close these preliminary remarks by recording that the $G$-action on $(TM/V) \oplus V^\circ$

$$g \cdot (v_m, \alpha_m) := \left( T_m \Phi_g(v_m), T^{*}_g m \Phi_g^{-1} \alpha_m \right)$$

is free and proper.

**A. Dirac reduction as a particular instance of Courant algebroid reduction.** This method is introduced in Bursztyn et al. (2007) who describe a general procedure of reduction of Courant algebroids. This is then used to reduce Dirac structures in a Courant algebroid. A Courant algebroid over a manifold $M$ is a vector bundle $E \to M$ with a fiberwise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$ (not necessarily skew-symmetric), and a bundle map $\rho : E \to TM$ called the anchor, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$:

1. $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]],$
2. $\rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)],$
3. $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)(f))e_2,$
4. $\rho(e_1) \langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle,$
5. $\langle [e_1, e_1], \cdot \rangle = \frac{i}{2} \rho^* \mathfrak{d}(e_1, e_1)$

(see Bursztyn et al. (2007)). In the standard case of interest to us in this paper, this reduction procedure is very simple and can be described as follows.

The Courant algebroid $E$ is the Pontryagin bundle $TM \oplus T^*M$ with the Courant bracket (see the discussion following (2)):

$$[(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X \beta - i_Y \mathfrak{d} \alpha)$$

(11)

for all sections $(X, \alpha)$ and $(Y, \beta)$ of $TM \oplus T^*M$.

We apply the results of Bursztyn et al. (2007) to the vector subbundle $\mathcal{K} := \mathcal{V} \oplus \{0\} \subset TM \oplus T^*M$ of the Pontryagin bundle and its orthogonal complement $\mathcal{K}^\perp = TM \oplus \mathcal{V}^\circ$. Both vector subbundles are $G$-invariant and it is easy to show (in agreement with the more general results of Bursztyn et al. (2007)) that

$$\frac{\mathcal{K}^\perp}{\mathcal{K}} \bigg/ G = \frac{TM \oplus \mathcal{V}^\circ}{\mathcal{V} \oplus \{0\}} \bigg/ G = \frac{TM}{V} \oplus \mathcal{V}^\circ \bigg/ G$$

(12)

is a Courant algebroid over $\hat{M}$ with the symmetric bilinear 2-form that descends from the one on $\mathcal{K}^\perp/\mathcal{K}$ given by

$$\langle (\tilde{X}, \alpha), (\tilde{Y}, \beta) \rangle_{\mathcal{K}^\perp/\mathcal{K}} = \beta(X) + \alpha(Y)$$

(13)

for all $\alpha, \beta \in \Gamma(V^\circ)$ and $X, Y$ in $\mathfrak{X}(M)$; here $\tilde{X} := X \mod \mathcal{V}$, $\tilde{Y} := Y \mod \mathcal{V}$ denote local sections of $TM/\mathcal{V}$ induced by local vector fields on $M$.

We apply the above the following general fact that will be needed also in later arguments.

**Lemma 3.1** Let $\pi : E \to M$ be a smooth vector bundle over $M$. Assume that there are two free proper $G$-actions on $E$ and $M$, respectively, such that $\pi$ is equivariant. Then the induced map $\pi_G : E/G \to M/G$ defined by the commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\pi} & M \\
\downarrow{\pi_E} & & \downarrow{\pi_M} \\
E/G & \xrightarrow{\pi_G} & M/G
\end{array}
$$

is also a smooth vector bundle whose rank is equal to the rank of $E$.

**PROOF:** It is straightforward to check that the map $\pi_G$ is a smooth surjective submersion and that its fibers are vector spaces. To prove local triviality, choose $m \in M$ and an open neighborhood $\hat{U}$ of $\pi_M(m)$ such that there exists a diffeomorphism

$$\Psi : \pi_M^{-1}(\hat{U}) \to \hat{U} \times G$$

$$n \mapsto (\pi_M(n), \Psi_2(n))$$

10
with $\Psi(m) = (\pi_M(m), e)$. Now, since $E$ is a vector bundle over $M$, there exists an open set $U$ with $m \in U \subseteq \pi_M^{-1}(U)$, a diffeomorphism

$$
\Theta : \pi^{-1}(U) \rightarrow U \times \pi^{-1}(m)
$$

$$
v_n \mapsto (n, \Theta_2(v_n))
$$

(where $v_n$ is an element of $\pi^{-1}(n)$), and an open set $U'$ with $m \in U' \subseteq U$ such that $\Psi_{U'} : U' \rightarrow \pi_M(U') \times \Psi_2(U')$ is a diffeomorphism and hence $\Psi_2(U')$ an open neighborhood of $e$ in $G$. Define

$$
\Lambda : \pi_G^{-1}(\pi_M(U')) \rightarrow \pi_M(U') \times \pi_G^{-1}(\pi_M(m))
$$

$$
\pi_E(v_n) \mapsto \left( \pi_M(n), (\pi_E \circ \Theta_2 \circ \Phi^{G}_{\Psi_2^{-1}})(v_n) \right),
$$

where $\Phi^{G} : G \times E \rightarrow E$ denotes the $G$-action on $E$. Since, $\pi_M(U')$ is open in $M/G$ with $\pi(m) \in \pi_M(U')$, this is a smooth local trivialization for the vector bundle $\pi_G : E/G \rightarrow M/G$ around the point $\pi(m)$.

The rank of $E/G$ is computed to be

$$
\text{rank}(E/G) = \dim(E/G) - \dim(M/G) = \dim E - \dim G - \dim M + \dim G
$$

$$
\text{rank} E - \dim M = \text{rank} E.
$$

In fact, with the identifications given above of $\Gamma(V)^G$ with $\Omega^1(M)$ and $\Gamma(TM/V)^G$ with $\mathcal{X}(M)$, it is obvious that the $G$-equivariant sections of (12) are in one-to-one correspondence with those of $TM \oplus T^*\bar{M}$. Note that this says that we have a vector bundle isomorphism

$$
\frac{\mathcal{X}^{\perp}}{\mathcal{X}} \cong T\bar{M} \oplus T^*\bar{M}
$$

over $\bar{M} = M/G$. This vector bundle isomorphism preserves the symmetric pairing; indeed, for all $m \in M$ and $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(T\bar{M} \oplus T^*\bar{M})$ the bracket $(\cdot, \cdot)_{\bar{M}}$ satisfies

$$
\langle (\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \rangle_{\bar{M}}(\pi(m)) = \bar{\alpha}(\bar{Y})(\pi(m)) + \bar{\beta}(\bar{X})(\pi(m)) = (\pi^*\bar{\alpha})(Y)(m) + (\pi^*\bar{\beta})(X)(m)
$$

$$
= \langle (\bar{X}, \pi^*\bar{\alpha}), (\bar{Y}, \pi^*\bar{\beta}) \rangle_{\pi^{-1}(m)}(m) = \langle \bar{X}, \pi^*\bar{\alpha} \rangle, \langle \bar{Y}, \pi^*\bar{\beta} \rangle_{\pi^{-1}(m)}
$$

where $X$ and $Y$ are $G$-equivariant local vector fields such that $X \sim_{\pi} \bar{X}$, $Y \sim_{\pi} \bar{Y}$ and $\bar{X} = X(\text{mod } V)$, $\bar{Y} = Y(\text{mod } V)$. Since all chosen objects are $G$-equivariant and the vector bundle isomorphism (14) is equivalent to the one defined on the corresponding spaces of local sections, this relation proves the statement.

We shall prove below that the Courant bracket on $T\bar{M} \oplus T^*\bar{M}$ also descends from the Courant bracket on $TM \oplus T^*M$ in the following sense. Recall that if $(\bar{X}, \bar{\alpha})$ and $(\bar{Y}, \bar{\beta})$ are sections of $T\bar{M} \oplus T^*\bar{M}$, then the truncated Courant bracket on sections of $T\bar{M} \oplus T^*\bar{M}$ is given by (see 11)

$$
[(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta})] = \left( [\bar{X}, \bar{Y}], \mathcal{L}_{\bar{X}}\bar{\beta} - i_\bar{Y}d\bar{\alpha} \right).
$$

Let $X, Y \in \mathcal{X}(M)$ and $\alpha, \beta \in \Omega^1(M)$ be such that $\bar{X}, \bar{Y} \in \Gamma(TM/V)^G$ and $\bar{\alpha}, \bar{\beta} \in \Gamma(V)^G$. Thus, these define uniquely $\bar{X}, \bar{Y} \in \mathcal{X}(M)$, $\alpha, \beta \in \Omega^1(M)$ by the conditions $X^G \sim_{\pi} \bar{X}$, $Y^G \sim_{\pi} \bar{Y}$ and $\pi^*\bar{\alpha} = \alpha$, $\pi^*\bar{\beta} = \beta$, where $X^G, Y^G \in \mathcal{X}^G(M)$ are such that $X - X^G = V$, $Y - Y^G = W \in \Gamma(V)$. Since $X^G$ and $Y^G$ are $G$-equivariant, the last three terms in

$$
[X, Y] = [X^G + V, Y^G + W] = [X^G, Y^G] + [V, Y^G] + [X^G, W] + [V, W].
$$

(15)

are sections of $V$. Hence

$$
[X, \bar{Y}] := [X^G, Y^G] \underset{16}{=} [X, Y]
$$

defines a Lie bracket on the $G$-invariant sections of $\Gamma(TM/V)$ (where $\Gamma(TM/V)^G$ is considered with its $C^\infty(M)$-module structure). Since $[X^G, Y^G] \sim_{\pi} [X, Y]$, this shows that the image of $[X, \bar{Y}]$ under the isomorphism of sections given in (10) is exactly $[\bar{X}, \bar{Y}]$, the first component of $[(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta})]$. Since $\pi^*\bar{\alpha}, \pi^*\bar{\beta} \in \Gamma(V)^G$, we have for all $V, W \in \Gamma(V)$

$$
\mathcal{L}_{X^G + V}(\pi^*\bar{\beta}) - i_{Y^G + W}d(\pi^*\bar{\alpha}) = \mathcal{L}_{X^G}(\pi^*\bar{\beta}) - i_Y d(\pi^*\bar{\alpha}) = \pi^*(\mathcal{L}_{\bar{X}}\bar{\beta} - i_\bar{Y}d\bar{\alpha}).
$$

(17)
Thus, since for all $V,W \in \Gamma(V)$
\[
[(\mathbf{X}^G + V, \pi^*\alpha), (\mathbf{Y}^G + W, \pi^*\beta)] = [(\mathbf{X}^G, \mathbf{Y}^G)] + [V, \mathbf{Y}^G] + [\mathbf{X}^G, W] + [V, W],
\]
we conclude that
\[
(\mathbf{X}, \mathbf{Y}, \pi^*\mathbf{L}_X\tilde{\beta} - i_Y\tilde{\alpha})
\]
is exactly the $G$-equivariant section of $\mathcal{K}^\perp/\mathcal{K}$ corresponding to $[(\mathbf{X}, \tilde{\alpha}), (\mathbf{Y}, \tilde{\beta})]$. This discussion proves the following.

**Proposition 3.2** The Courant bracket on $TM \oplus T^*M$ induces a well-defined bracket on the $G$-invariant sections of $\mathcal{K}^\perp/\mathcal{K}$ characterized by the property that if $\mathbf{X}, \mathbf{Y} \in \Gamma(TM/V)^G$ and $\alpha, \beta \in \Gamma(V^G)$ correspond to $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ and $\tilde{\alpha}, \tilde{\beta} \in \Omega^1(M)$, respectively, then the bracket $[(\mathbf{X}, \alpha), (\mathbf{Y}, \beta)]$ corresponds to $[(\mathbf{X}, \tilde{\alpha}), (\mathbf{Y}, \tilde{\beta})]$. This bracket on $\Gamma(\mathcal{K}^\perp/\mathcal{K})^G$, also called Courant bracket, corresponds by the quotient map in (14) to the Courant bracket on $T\tilde{M} \oplus T^*\tilde{M}$.

Now assuming that $D \cap \mathcal{K}^\perp$ has constant rank, that is, $D \cap \mathcal{K}^\perp$ is a smooth vector subbundle of $TM \oplus T^*M$, it follows that $(D \cap \mathcal{K}^\perp)^\perp = D \oplus \mathcal{K}$ and $D \cap \mathcal{K}$ are vector subbundles of $TM \oplus T^*M$. The second conclusion follows from the fact that for all $m \in M$ we have $\dim(D(m) + \mathcal{K}(m)) = \dim D(m) + \dim \mathcal{K}(m) - \dim (D(m) \cap \mathcal{K}(m))$ which shows that $D \cap \mathcal{K}$ has constant dimensional fibers and is hence a vector subbundle. Form the pointwise quotient
\[
\frac{(D \cap \mathcal{K}^\perp) + \mathcal{K}}{\mathcal{K}} = \frac{(D \cap (TM \oplus V^G)) + (V \oplus \{0\})}{V \oplus \{0\}}
\]
with base $M$. At each point $m \in M$, one gets a subspace of the vector space $(T_mM/V(m)) \oplus V^G(m) \simeq \mathcal{K}^\perp(m)/\mathcal{K}(m)$ (see (12)).

**Proposition 3.3** Relative to the symmetric nondegenerate bilinear form (13) on $\mathcal{K}^\perp/\mathcal{K}$, the vector subspace
\[
\tilde{D}(m) := \frac{(D(m) \cap \mathcal{K}(m))^\perp + \mathcal{K}(m)}{\mathcal{K}(m)}
\]
satisfies $\tilde{D}(m) \supseteq \tilde{D}(m)^\perp$.

**Proof:** Let us prove that $\tilde{D}(m) \subseteq \tilde{D}(m)^\perp$. Let $(\mathbf{X}(m), \alpha(m)) \in \tilde{D}(m)$. If $(\mathbf{X}, \alpha) \in \Gamma(D)$ are local sections about $m$, then $\alpha \in \Gamma(V^G)$ and there are $X \in \mathfrak{X}(M)$ and $V \in \Gamma(V)$ such that $(X + V, \alpha) \in \Gamma(D)$ and $\mathbf{X} = X \bmod V$. For all $(\mathbf{Y}, \beta) \in \Gamma(D)$ we have analogously local vector fields $Y \in \mathfrak{X}(M)$ and $W \in \Gamma(V)$ such that $(Y + W, \beta) \in \Gamma(D)$ and $\mathbf{Y} = Y \bmod V$. This yields
\[
\langle(\mathbf{X}, \alpha), (\mathbf{Y}, \beta)\rangle_{\mathcal{K}^\perp/\mathcal{K}} \supseteq \langle(X + V, \alpha), (Y + W, \beta)\rangle = 0,
\]
since $(X + V, \alpha), (Y + W, \beta) \in \Gamma(D)$.

To prove the inclusion, $\tilde{D}(m)^\perp \subseteq \tilde{D}(m)$ let $(\mathbf{X}(m), \alpha(m)) \in \tilde{D}(m)^\perp$ be such that $(\mathbf{X}, \alpha) \in \Gamma(\mathcal{K}^\perp/\mathcal{K})$ and for all $(\mathbf{Y}, \beta) \in \Gamma(D)$ we have $\langle(\mathbf{X}, \alpha), (\mathbf{Y}, \beta)\rangle_{\mathcal{K}^\perp/\mathcal{K}} = 0$. Choose $X \in \mathfrak{X}(M)$ such that $\mathbf{X} = X \bmod V$. For all $(\mathbf{Y}, \beta) \in \Gamma(D \cap \mathcal{K}^\perp)$, $(\mathbf{Y}, \beta)$ lies in $\Gamma(D)$ and we get
\[
0 = \langle(\mathbf{X}, \alpha), (\mathbf{Y}, \beta)\rangle_{\mathcal{K}^\perp/\mathcal{K}} = \langle(X, \alpha), (Y, \beta)\rangle = \alpha(Y) + \beta(X).
\]
This yields $(X, \alpha) \in \Gamma((D \cap \mathcal{K}^\perp)^\perp)$. We have $(D_q \cap \mathcal{K}_q^\perp)^\perp = D_q^\perp + (\mathcal{K}_q^\perp)^\perp = D_q + \mathcal{K}_q$ for every $q$ in the domain of definition of $(X, \alpha)$. Thus, since $D$ and $\mathcal{K}$ are smooth vector bundles, there exists $X' \in \mathfrak{X}(M)$ and $W \in \Gamma(V)$ such that $(X', \alpha) \in \Gamma(D)$ and $X = X' + W$. Now recall that the 1-form $\alpha$ is in fact in $\Gamma(V^G)$ since $(\mathbf{X}, \alpha)$ was an element of $\Gamma(\mathcal{K}^\perp/\mathcal{K})$. The pair $(X', \alpha)$ is consequently in $\Gamma(D \cap \mathcal{K}^\perp)$ and, since $\mathbf{X} = (X' + W) \bmod V = X' \bmod V$, our $(\mathbf{X}, \alpha)$ is a local section of $\tilde{D}$, as required. \hfill \Box

This proposition immediately implies that $\dim \tilde{D}(m)$ is constant on $M$ and equal to
\[
\frac{\dim \mathcal{K}^\perp(m) - \dim \mathcal{K}(m)}{2} = \frac{\dim M + \dim (\dim M - \dim G) - \dim G}{2} = \dim M - \dim G.
\]
Thus $\tilde{D}$ is a smooth $G$-invariant subbundle of $\mathcal{K}^\perp/\mathcal{K}$. Its image by the isomorphism (14) gives a subbundle $D_{\text{red}} = \tilde{D}/G$ of

$$\frac{\mathcal{K}^\perp}{\mathcal{K}} \bigg/ G \simeq T\bar{M} \oplus T^*\bar{M},$$

whose rank is $(\dim M - \dim G)$, which is isotropic relative to the symmetric pairing on $T\bar{M} \oplus T^*\bar{M}$. Hence $D_{\text{red}}$ is a Dirac structure called the reduction of $D$ by $G$. This discussion and Proposition 3.3 yield the following consequence.

**Proposition 3.4** The sections of $D_{\text{red}}$ are in one-to-one correspondence with the $G$-equivariant sections of the quotient (15) via the isomorphisms $\mathfrak{x}(\bar{M}) \simeq \Gamma(TM/\mathcal{V})^G$ and $\Omega^1(\bar{M}) \simeq \Gamma(\mathcal{V})^G$ given at the beginning of this subsection.

It is customary to denote the “quotient” Dirac structure on $M/G$ by

$$D_{\text{red}} = \left(\frac{D \cap \mathcal{K}^\perp}{\mathcal{K}}\right)/G.$$ 

**Proposition 3.5** If the Dirac structure $D$ is closed then the reduced Dirac structure $D_{\text{red}}$ is also closed.

**Proof:** The proof is based on the fact that the Courant bracket on the $G$-invariant sections of $\mathcal{K}^\perp/\mathcal{K}$ descends to the Courant bracket on $T\bar{M} \oplus T^*\bar{M}$. Indeed, if $(\bar{X}, \bar{\alpha}), (\bar{Y}, \bar{\beta}) \in \Gamma(D_{\text{red}})$, consider the corresponding $G$-invariant sections $(\tilde{X}, \pi^*\bar{\alpha}), (\tilde{Y}, \pi^*\bar{\beta})$ of $((D \cap \mathcal{K}^\perp) + \mathcal{K})/\mathcal{K}$. Let now $X^G, Y^G \in \mathfrak{x}(M)^G$ be such that $X = X^G + V, Y = Y^G + W$, where $V, W \in \Gamma(\mathcal{V})$. Since $(X^G + V, \pi^*\bar{\alpha})$ and $(Y^G + W, \pi^*\bar{\beta})$ are sections of $(D \cap \mathcal{K}^\perp) + \mathcal{K}$ and $D$ is closed, we get as in (17),

$$[(X^G + V, \pi^*\bar{\alpha}), (Y^G + W, \pi^*\bar{\beta})] = [(X^G + V, Y^G + W), \pi^* (\mathcal{L} \bar{\beta} - \mathcal{i}_Y \mathcal{d}\bar{\alpha})] \in \Gamma(D \cap \mathcal{K}^\perp + \mathcal{K}).$$

Thus, from (15) and (16) we deduce that

$$\left(\tilde{[X, Y]}, \pi^* (\mathcal{L} \bar{\beta} - \mathcal{i}_Y \mathcal{d}\bar{\alpha})\right) \in \Gamma\left(\left(\frac{D \cap \mathcal{K}^\perp}{\mathcal{K}}\right)/G\right).$$

However, by Proposition 3.2, $\left(\tilde{[X, Y]}, \pi^* (\mathcal{L} \bar{\beta} - \mathcal{i}_Y \mathcal{d}\bar{\alpha})\right)$ descends precisely to the Courant bracket

$$[\tilde{(X, \bar{\alpha})}, \tilde{(Y, \bar{\beta})}] = \tilde{[X, Y]}, \mathcal{L} \bar{\beta} - \mathcal{i}_Y \mathcal{d}\bar{\alpha}].$$

Therefore, $[\tilde{(X, \bar{\alpha})}, \tilde{(Y, \bar{\beta})}] \in \Gamma(D_{\text{red}})$ which proves that $D_{\text{red}}$ is closed.

**B. Dirac reduction as an extension of Poisson reduction.** This was historically the first method to reduce Dirac structures and it is due to Blankenstein (2000) and Blankenstein and van der Schaft (2001) (see Blankenstein and Ratiu (2004) for the singular case).

Define for all $m \in M$ the vector subspace

$$E(m) = \{(X(m), \alpha(m)) \in T_m M \times T^*_m M \mid X \in \mathfrak{x}(M), \alpha = \pi^*\bar{\alpha} \text{ for some } \bar{\alpha} \in \Omega^1(\bar{M})\}.$$ 

Then $E := \cup_{m \in M} E(m) = TM \oplus \mathcal{V} = \mathcal{K}^\perp$ is a vector bundle and thus the assumption

$$D \cap E$$

is a vector subbundle of $TM \oplus T^*M$ of Blankenstein and van der Schaft (2001) is identical to the assumption

$$D \cap \mathcal{K}^\perp$$

is a vector subbundle of $TM \oplus T^*M$ of Bursztyn et al. (2007), which is in turn equivalent, as we have seen before, to the hypothesis that the fibers of $D \cap \mathcal{K}$ are constant dimensional. In Blankenstein and van der Schaft (2001), there is the additional assumption that $\mathcal{V} + G_0$ is constant dimensional on $M$. Their proof is based on results in Nijmeijer and van der Schaft (1990) and Isidori (1995). They also need $\mathcal{V}$ to be an involutive subbundle of $TM$, which holds in our case since the action of $G$ on $M$ is free and proper. The cited result of Isidori (1995) (and of Nijmeijer and van der Schaft (1990) with a stronger hypothesis) is exactly the statement of
Proposition 3.6 applied to the involutive subbundle $\mathcal{V}$ of $TM$ and the generalized distribution $\mathcal{G}_0$. Our proof of this proposition, inspired by [Cheng and Tarn (1989)], needs only that $\mathcal{G}_0$ is a locally finite smooth distribution and that $\mathcal{V}$ is an involutive vector subbundle of $TM$.

To summarize, the hypothesis needed for the two methods of reduction are not equivalent; in [Bursztyn et al. (2007)] one needs only that $D \cap \mathcal{K}^\perp$ is a subbundle of $TM \oplus T^*M$ and in [Blankenstein and van der Schaft (2001)] one needs the additional assumption that $\mathcal{G}_0$ is a locally finite smooth distribution.

The reduced Dirac structure of Blankenstein and van der Schaft (2001) is given by

$$\Gamma(\bar{D}) = \{(\bar{X}, \bar{\alpha}) \in \Gamma(TM \oplus T^*M) \mid \text{there is } X \in \mathfrak{X}(M) \text{ such that } X \sim_\pi \bar{X} \text{ and } (X, \pi^*\bar{\alpha}) \in \Gamma(D)\}. \quad (19)$$

**Proposition 3.6** The sections of $\bar{D}$ are exactly those of $D_{\text{red}}$ and the vector bundles $\bar{D}$ and $D_{\text{red}}$ are identical.

**Proof:** Choose $(\bar{X}, \bar{\alpha}) \in \Gamma(\bar{D})$. By (19), there exists $X \in \mathfrak{X}(M)$ such that $X \sim_\pi \bar{X}$ and $(X, \pi^*\bar{\alpha}) \in \Gamma(D)$. Since there exists $X^G \in \mathfrak{X}(M)^G$ such that $X^G \sim_\pi X$, we have with $\bar{X} = \bar{X}^G$ that $\bar{X}$ is a $G$-equivariant section of $TM/\mathcal{V}$. The 1-form $\alpha := \pi^*\bar{\alpha}$ lies in $\Gamma((\mathcal{V}^\circ)^G)$ and thus we have $(X, \alpha) \in \Gamma(D \cap \mathcal{K}^\perp)$ and

$$(\bar{X}, \alpha) \in \Gamma\left(\frac{(D \cap \mathcal{K}^\perp) + \mathcal{K}}{\mathcal{K}}\right)^G.$$ 

Thus this section corresponds to a unique section $(X_{\text{red}}, \alpha_{\text{red}})$ of $D_{\text{red}}$. Now this section is given by $X \sim_\pi X_{\text{red}}$, which yields $\bar{X} = X_{\text{red}}$. The equality $\pi^*\alpha_{\text{red}} = \alpha = \pi^*\bar{\alpha}$ implies that $\alpha_{\text{red}} = \bar{\alpha}$ because $\pi$ is a surjective submersion. \qed

The description of $D_{\text{red}}$ shows that the smooth distribution $\mathcal{G}_0/\mathcal{V}$ projects to $\mathcal{G}_0^{\text{red}}$ and that the smooth codistribution $\pi_2(D \cap (V \oplus V^\circ))$ projects to $\mathcal{P}_1^{\text{red}}$, where $\pi_2$ is the projection $\pi_2 : TM \oplus T^*M \to T^*M$. There is no analogous description as quotients of the distribution $\mathcal{G}_1^{\text{red}}$ and $\mathcal{P}_1^{\text{red}}$; they need to be computed from the definition on a case by case basis.

Depending on the example, one needs to choose which method of Dirac reduction is easier to implement. In the next section, we will present cases where we have global bases of sections for the Dirac structure and in that situation the first method is more convenient.

The third method of reduction alluded to at the beginning of this subsection is due to Yoshimura and Marsden (2006a, 2007). It is undergoing a major extension to encompass both the Lagrangian and Hamiltonian version of classical reduction (see Cendra et al. (2008)). Since this work is still in progress we shall not comment on it here.

## 4 Reduction of nonholonomic systems

### 4.1 Summary of the nonholonomic reduction method

Bates and Śniatycki (1993) propose a reduction method for constrained Hamiltonian systems. They start with the configuration space $Q$, a hyperregular Lagrangian $L : TQ \to \mathbb{R}$ taken as the kinetic energy of a Riemannian metric, and a constraint distribution $\mathcal{D}$ on $Q$ equal to the kernel of smooth 1-forms $\phi^1, \ldots, \phi^k \in \Omega^1(Q)$ satisfying pointwise $\phi^1 \wedge \ldots \wedge \phi^k \neq 0$, that is,

$$\mathcal{D} := \{v \in TQ \mid \phi^j(v) = 0, \ j = 1, \ldots, k\}.$$ 

The independence of the forms (which is equivalent to the hypothesis $\phi^1 \wedge \ldots \wedge \phi^k \neq 0$ at every point of $Q$) ensures that $\mathcal{D}$ is a smooth vector subbundle of $TQ$.

Denote by $\langle \cdot, \cdot \rangle : T^*Q \times TQ \to \mathbb{R}$ the duality pairing between 1-forms and tangent vectors. Let $FL : TQ \to T^*Q$,

$$\langle FL(v), w \rangle := \frac{d}{dt} \bigg|_{t=0} L(v + tw), \ v, w \in T_qQ,$$

be the Legendre transformation associated to $L$ which is a diffeomorphism since the Lagrangian is hyperregular. If $A(v) := \langle FL(v), v \rangle$ denotes the action of $L$, let $H(p) := A((FL)^{-1}(p)) - L((FL)^{-1}(p)), p \in T^*Q$,
be the associated Hamiltonian. The Hamiltonian vector field $X$ determined by $H$ and the constraint forms $\phi^1, \ldots, \phi^k \in \Omega^1(Q)$ is defined classically by

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = \frac{\partial H}{\partial q} + \lambda_j \phi^j
$$

or

$$
i_X \omega = dH + \lambda_j \pi^* \omega U \phi^j
$$

where $\pi_{T^*Q} : T^*Q \to Q$ is the cotangent bundle projection and $\lambda_1, \ldots, \lambda_k \in C^\infty(Q)$ are the Lagrange multipliers associated to the constraint forms $\phi^1, \ldots, \phi^k$, and the constraint equations

$$
\phi^j (\mathbb{F} L^{-1}(p)) = \phi^j (T \pi_{T^*Q} X) = 0.
$$

The counterpart of the constraint distribution $\mathcal{D}$ in phase space is the constraint manifold

$$
M := \mathbb{F} L(D) = \{ p \in T^*Q \mid \phi^j (\pi_{T^*Q}(p)) ((\mathbb{F} L)^{-1}(p)) = 0, j = 1, \ldots, k \} \subset T^*Q.
$$

Since we require that the solution be in the constraint submanifold $M$, it follows that $X$ is tangent to $M$.

Set $\omega_M := i^* \omega_{\text{can}},$ where $i : M \hookrightarrow T^*Q$ is the inclusion and $\omega_{\text{can}}$ is the canonical symplectic form on $T^*Q$. Define

$$
\mathcal{F} := \{ U \in TT^*Q \mid \pi^* \omega_{T^*Q} \phi^j(U) = 0, j = 1, \ldots, k \}
$$

and note that $\pi^* \omega_{T^*Q} \phi^1 \wedge \ldots \wedge \pi^* \omega_{T^*Q} \phi^k \neq 0$ on $T^*Q$. Therefore $\mathcal{F} \to T^*Q$ is a vector subbundle of $TT^*Q$. The nonholonomic horizontal distribution is defined by

$$
\mathcal{H} := \mathcal{F} \cap TM \to M.
$$

Bates and Śniatycki (1993) prove that the restriction $\omega_{\mathcal{H}}$ of $\omega_M$ to $\mathcal{H} \times \mathcal{H}$ is nondegenerate. Their proof uses the fact that the Lagrangian is the kinetic energy of a metric plus a potential. They also show that $\mathcal{H}$ is a vector subbundle of $TM$. With the condition (21) on $X$, we get for $j = 1, \ldots, k$,

$$
\pi^* \omega_{T^*Q} \phi^j(X) = \phi^j (T \pi_{T^*Q} X) = 0
$$

and thus the vector field $X$ is a section of $\mathcal{H}$. Hence it is easy to see that the pull back to $M$ of (20) subject to the constraints (21) is equivalent to $X \in \Gamma(\mathcal{H})$ and $i_X \omega_{\mathcal{H}} = dH|_{\mathcal{H}}$.

Assume that $G$ is a Lie group acting symplectically on $T^*Q$ (not necessarily the lift of an action on $Q$), leaves $M$ invariant, and preserves the Hamiltonian $H$. Assume that the quotient $\bar{M} = M/G$ is a smooth manifold with projection map $\pi : \bar{M} \to M$ a submersion. Since $G$ is a symmetry group of the nonholonomic system, all intrinsically defined vector fields and distributions push down to $\bar{M}$.

In particular, the vector field $X$ on $M$ pushes down to a vector field $\bar{X}$ with $X \sim \bar{X}$ and the distribution $\mathcal{H}$ pushes down to a distribution $\mathcal{H}_{\text{red}}$ on $\bar{M}$. However, $\omega_{\mathcal{H}}$ need not push down to a 2-form defined on $\mathcal{H}_{\text{red}}$ on $\bar{M}$, despite the fact that $\omega_{\mathcal{H}}$ is $G$-invariant. This is because there may be infinitesimal symmetries $\xi_M$ which are horizontal (that is, take values in $\mathfrak{h}$), but $i_{\xi_M} \omega_{\mathcal{H}} \neq 0$. Let $\mathbb{V}$ be the distribution on $M$ tangent to the orbits of $G$, that is, its fibers are $\mathbb{V}(m) := \{ \xi_M(m) \mid \xi \in \mathfrak{g} \}$ for all $m \in M \subseteq T^*Q$. Define the horizontal annihilator $\mathcal{U}$ of $\mathbb{V}$ by

$$
\mathcal{U} = (\mathbb{V} \cap \mathcal{H})^{\omega_M} \cap \mathcal{H} \subseteq TM \subseteq TT^*Q,
$$

where the superscript $\omega_M$ on a distribution denotes its fiberwise $\omega_M$-orthogonal complement in $TM$. Clearly, $\mathcal{U}$ and $\mathbb{V}$ are both $G$-invariant, project down to $\bar{M}$, and the image of $\mathbb{V}$ is $\{ 0 \}$. Define $\bar{\mathcal{H}} := T \pi (\mathcal{U}) \subseteq \bar{TM}$ to be the projection of $\mathcal{U}$ to $\bar{M}$. Bates and Śniatycki (1993) show that $X$ takes values in $\mathcal{U}$ and that the restriction $\omega_{\mathcal{H}}$ to $\mathcal{U} \times \mathcal{U}$ pushes down to a nondegenerate form $\omega_{\mathcal{H}}$ on $\bar{\mathcal{H}}$, i.e., $\pi^* \omega_{\mathcal{H}} = \omega_{\bar{\mathcal{H}}}$. In addition, the function $\bar{H} \in C^\infty(\bar{M})$ defined by $\pi^* \bar{H} = H|_{\bar{M}}$ and the induced vector field $\bar{X}$ on $\bar{M}$ are related by

$$
\bar{i}_X \omega_{\bar{\mathcal{H}}} = d\bar{H}|_{\bar{\mathcal{H}}}
$$

which can be interpreted as the definition of the reduced nonholonomic Hamiltonian vector field $\bar{X}$.

**Remark 4.1** Note that we have no information about the dimensions of the fibers of $\mathcal{U}$. In general, $\mathcal{U}$ is not a vector subbundle of $TM$. 

\[\triangle\]
4.2 Link with Dirac reduction

Let \( M, \omega_M, \pi_{TQ}, \mathcal{H}, \bar{M}, \) and \( \pi : M \rightarrow \bar{M} \) be as in the preceding subsection. An easy verification shows that

\[
\mathcal{H} = (T(\pi_{TQ}|_M))^{-1}(\mathcal{D}) \subseteq TM \subseteq T^*Q,
\]

where

\[
\mathcal{D} := \{ v \in TQ \mid \langle \phi^j, v \rangle = 0, j = 1, \ldots, k \} \subseteq TQ
\]
is the constraint distribution on \( Q \).

We introduce the Dirac structure \( D \) on \( M \) as in Y osimura and Marsden (2006b): for all \( m \in M \) define

\[
D(m) = \{(X(m), \alpha_m) \in TM \oplus T^*M \mid X \in \Gamma(\mathcal{H}), \alpha - \iota_X \omega_M \in \Gamma(\mathcal{H})\}
\]

and let \( D := \bigcup_{m \in M} D(m) \subseteq TM \).

The Lie group \( G \) acts on \( M \) and leaves \( \mathcal{H}, \omega_M \), and thus the Dirac structure \( D \) invariant. Define \( \mathcal{K} := \mathcal{V} \oplus \{0\} \subseteq TM \oplus T^*M \) and its orthogonal complement \( \mathcal{X}^\perp = TM \oplus V^o \) as in (3.2). Assume, as in (3.2) that \( D \cap \mathcal{X}^\perp \) is a vector subbundle of \( TM \oplus T^*M \) and consider the reduced Dirac manifold \((\bar{M}, D_{\text{red}})\). The next proposition shows that, if \( \mathcal{H} \) is constant dimensional, the reduced Dirac structure is given by the formula

\[
D_{\text{red}} = \{ (X, \alpha) \in \Gamma(TM \oplus T^*\bar{M}) \mid X \in \Gamma(\bar{\mathcal{H}}), \alpha \vert_{\mathcal{K}} = \iota_X \omega_{\mathcal{K}} \}
\]

where \( \bar{\mathcal{H}} \) and \( \omega_{\bar{\mathcal{H}}} \) are defined as in the preceding subsection.

Proposition 4.2

(i) The generalized distribution \( G_0 \) is trivial and the codistribution \( P_1 \) is given by \( P_1 = T^*M \).

(ii) Let \( \mathcal{U} = \mathcal{H} \cap (\mathcal{V} \cap \mathcal{H})^{\omega_M} \) (see (28)). Then

\[
X \in \Gamma(\mathcal{U}) \iff \text{there exists } \alpha \in \Gamma(\mathcal{V}^o) \text{ such that } (X, \alpha) \in \Gamma(D \cap \mathcal{X}^\perp).
\]

With the additional assumption that \( \mathcal{V} + \mathcal{H} = TM \), the section \( \alpha \) in (29) is unique.

(iii) The reduced distributions \( G_1^{\text{red}} \) and \( G_0^{\text{red}} \) are given by

\[
G_1^{\text{red}} = \bar{\mathcal{H}} \quad \text{and} \quad G_0^{\text{red}} = \{0\}.
\]

(iv) For each \( \alpha \in \Gamma(\mathcal{V}^o) \) there exists exactly one section \( X \in \Gamma(\mathcal{U}) \) such that \( (X, \alpha) \in \Gamma(D) \). Hence, we have \( \pi_2(D \cap \mathcal{X}^\perp) = V^o \) and the reduced codistribution \( P_1^{\text{red}} \) is equal to \( T^*(M/G) \).

(v) Assume that \( G_1^{\text{red}} = \bar{\mathcal{H}} \) is constant dimensional. The 2-form defined on \( G_1^{\text{red}} = \bar{\mathcal{H}} \) by the Dirac structure \( D_{\text{red}} \) (see (3)) is nondegenerate and is equal to \( \omega_{\bar{\mathcal{H}}} \).

Proof: (i) If \( X \) is a section of \( G_0 \), we have \( \iota_X \omega_M \in \Gamma(\mathcal{H}) \) and \( X \in \Gamma(\mathcal{H}) \). Hence, since \( \omega_{\mathcal{H}} \) is nondegenerate, the vector field \( X \) has to be the zero section. Thus \( G_0 = \{0\} \).

Since the 2-form \( \omega_{\mathcal{H}} \) is nondegenerate an arbitrary \( \alpha \in \Omega^1(M) \) determines a unique section \( X \) of \( \mathcal{H} \) by the equation \( \iota_X \omega_{\mathcal{H}} = \alpha \vert_{\mathcal{K}} \). Therefore, \( P_1 = T^*M \).

(ii) If \( (X, \alpha) \) is a local section of \( D \cap \mathcal{X}^\perp \), then we have \( X \in \Gamma(\mathcal{H}), \alpha \in \Gamma(\mathcal{V}^o) \), and \( \alpha = \iota_X \omega_M \) on \( \mathcal{H} \). Hence, \( (\iota_X \omega_M) \vert_{\mathcal{U} + \mathcal{V}} = 0 \) and thus we have

\[
X \in \Gamma(\mathcal{H} \cap (\mathcal{V} \cap \mathcal{H})^{\omega_M}) = \Gamma(\mathcal{U}).
\]

Conversely, if \( X \in \Gamma(\mathcal{U}) \), we have \( \iota_X \omega_M = 0 \) on \( \mathcal{V} \cap \mathcal{H} \) and we can find a section \( \alpha \in \Gamma(\mathcal{V}^o) \) such that the restriction of \( \alpha \) and \( \iota_X \omega_M \) to \( \mathcal{H} \) are equal.

If, in addition, we make the usual assumption \( \mathcal{V} + \mathcal{H} = TM \), we have for each \( X \in \Gamma(\mathcal{U}) \) exactly one \( \alpha \in \Omega^1(M) \) such that \( \alpha \vert_{\mathcal{K}} = \iota_X \omega_M \) and \( \alpha \vert_{\mathcal{V}} = 0 \).

(iii) By construction, the constraint distribution \( G_1^{\text{red}} \) associated to the Dirac structure \( D_{\text{red}} \) on \( \bar{M} \) is given by

\[
\frac{\mathcal{U} + \mathcal{V}}{\mathcal{V}}/G.
\]
This can obviously be identified with \( \mathcal{H} = T\pi(\mathcal{U}) \).

If we have \( \bar{X} \in \Gamma(G_0^{\text{red}}) \), then \( (\bar{X}, 0) \in \Gamma(D_{ \text{red}}) \) and there exists \( X \in \mathfrak{X}(M) \) with \( \sim_x \bar{X} \) and \( (X, 0) \in \Gamma(D) \). Hence we have \( X \in \Gamma(G_0) \) and since \( G_0 = \{ 0 \} \), we get \( X = 0 \). This shows that \( G_0^{\text{red}} = \{ 0 \} \).

(iv) This follows directly from (i) and (ii).

(v) Let \( \omega_{D_{ \text{red}}} \) be the 2-form defined on \( G_0^{\text{red}} = \mathcal{H} \) by the Dirac structure \( D_{ \text{red}} \) (see 3). If \( X \in \Gamma(\mathcal{H}) \) is such that \( \omega_{D_{ \text{red}}}(X, Y) = 0 \) for all \( Y \in \Gamma(\mathcal{H}) \), then \( (X, 0) \) is a section of \( D_{ \text{red}} \) and hence we have by (iii) \( X = 0 \). Thus \( \omega_{D_{ \text{red}}} \) is nondegenerate on \( \mathcal{H} \). Let \( \bar{X} \) and \( \bar{Y} \) be sections of \( \mathcal{H} \). We show now that \( \omega_{D_{ \text{red}}}(\bar{X}, \bar{Y}) = \omega_{\mathcal{H}}(\bar{X}, \bar{Y}) \). Indeed, by definition, we have \( \omega_{D_{ \text{red}}}(X, Y) = \alpha(Y) \), where \( \alpha, \beta \in \Omega^1(M/G) \) are such that \( (\bar{X}, \alpha), (\bar{Y}, \beta) \in \Gamma(D_{ \text{red}}) \). Choose \( X, Y \in \Gamma(\mathcal{U}) \) with \( \sim_x \bar{X} \), \( \sim_x \bar{Y} \) and \( (X, \pi^*\alpha), (Y, \pi^*\beta) \in \Gamma(D \cap K^\perp) \). Then we have
\[
\omega_{D_{ \text{red}}}(\bar{X}, \bar{Y}) = \alpha(\bar{Y}) = (\pi^*\alpha)(Y) = \omega_{\mathcal{H}}(X, Y) = \omega_{\mathcal{H}}(\bar{X}, \bar{Y}),
\]
where the last equality follows simply from the definition of \( \omega_{\mathcal{H}} \).

We shall use part (ii) of this proposition to simplify certain computations in the examples that follow.

**Remark 4.3** Note that if \( \mathcal{H} + \mathcal{V} \) has constant rank on \( M \), we have automatically that \( D \cap K^\perp \) has constant dimensional fibers on \( M \).

Since \( \mathcal{H}, \mathcal{V}, \mathcal{H} + \mathcal{V} \) are vector subbundles of \( TM \), \( \mathcal{H} \cap \mathcal{V} \) is also a subbundle of \( TM \). By the nondegeneracy of \( \omega_{\mathcal{H}} \), we get that \( \mathcal{U} = (\mathcal{H} \cap \mathcal{V})^\omega \cap \mathcal{H} = (\mathcal{H} \cap \mathcal{V})^\omega \cap \mathcal{H} \) has also constant dimensional fibers on \( M \) and is in particular a vector subbundle of \( \mathcal{H} \). Let \( \mathcal{U} \) be the dimension of the fibers of \( \mathcal{U} \), \( r \) the dimension of the fibers of \( \mathcal{H} \). Then, if \( n = \dim M \), \( n - r \) is the rank of the codistribution \( \mathcal{H}^\circ \). Let finally \( l \) be the rank of the codistribution \( \mathcal{H}^\circ \cap \mathcal{V}^\circ = (\mathcal{V} + \mathcal{H})^\circ \subseteq \mathcal{H}^\circ \). Choose local basis vector fields \( H_1, \ldots, H_r \) for \( \mathcal{H} \) such that \( H_1, \ldots, H_u \) are basis vector fields for \( \mathcal{U} \). In the same way, choose basis 1-forms \( \beta_1, \ldots, \beta_{n-r} \) for \( \mathcal{H}^\circ \) such that \( \beta_1, \ldots, \beta_l \) are basis 1-forms for \( \mathcal{V}^\circ \cap \mathcal{H}^\circ \). Then a local basis of sections of \( D \) is
\[
\{ (H_1, 1_H, \omega_M), \ldots, (H_r, 1_H, \omega_M), (0, \beta_1), \ldots, (0, \beta_{n-r}) \}.
\]
The considerations above show that \( D \cap K^\perp \) is then spanned by the sections
\[
\left\{ \left( H_1, 1_H, \omega_M + \sum_{i=1}^{n-r} a_i^1 \beta_i \right), \ldots, \left( H_u, 1_H, \omega_M + \sum_{i=1}^{n-r} a_i^u \beta_i \right), (0, \beta_1), \ldots, (0, \beta_l) \right\},
\]
where \( a_i^j \) are smooth functions chosen such that \( 1_H, \omega_M + \sum_{i=1}^{n-r} a_i^j \beta_i \in \Gamma(\mathcal{V}^\circ) \) for \( j = 1, \ldots, u \). Since these sections are linearly independent, they are smooth local basis sections for \( D \cap K^\perp \). \qed

### 4.3 Example: the constrained particle in space

Bates and Śniatycki (1992) study the motion of the constrained particle in space. The configuration space of this problem is \( \mathcal{Q} := \mathbb{R}^3 \) whose coordinates are denoted by \( \mathbf{q} := (x, y, z) \). They take the following concrete constraints on the velocities:
\[
\mathcal{D} := \ker(dz - ydx) = \{ v_x \partial_x + v_y \partial_y + v_z \partial_z \mid v_x - yv_x = 0 \} \subseteq TQ.
\]
The Lagrangian is hyperregular and taken to be the kinetic energy of the Euclidean metric, that is, \( L(\mathbf{q}, \mathbf{v}) := \frac{1}{2} \| \mathbf{v} \|^2 \). Hence the constraint manifold (22) is five dimensional and given by
\[
\mathcal{M} := \{ (x, y, z, p_x, p_y, p_z) \mid p_z = yp_x \} \subseteq T^*\mathcal{Q},
\]
where \( (x, y, z, p_x, p_y, p_z) \) are the coordinates of \( T^*\mathcal{Q} \). The global coordinates on \( \mathcal{M} \) are thus \( (x, y, z, p_x, p_y) \). The pull back \( \omega_M \) of the canonical 2-form \( \omega \) on \( T^*\mathcal{Q} \) to \( M \) has hence the expression
\[
\omega_M = dx \wedge dp_x + dy \wedge dp_y + dz \wedge (p_x dy + y dp_x).
\]
The Dirac structure \( D \) modeling this problem is given by (25). Formula (27) gives the vector subbundle
\[
\mathcal{K} := (T(\pi_{T^*\mathcal{Q}}|\mathcal{M}))^{-1}(\mathcal{D}) = \text{span}\{ \partial_x + y \partial_z, \partial_y, \partial_{p_x}, \partial_{p_y} \} \subseteq TM,
\]
and consequently
\[ \mathcal{K}^\circ = \text{span}\{dz - ydx\}. \]

A computation yields
\[
\begin{align*}
i_{\partial_x+y\partial_z}\omega_M &= (1+y^2)dp_x + yp_xdy \\
i_{\partial_y}\omega_M &= dp_y - p_xdz \\
i_{\partial_{py}}\omega_M &= -dy \\
i_{\partial_{px}}\omega_M &= -ydz - dx.
\end{align*}
\]

Hence
\[
\{ (\partial_x + y\partial_z, (1+y^2)dp_x + yp_xdy); (\partial_y, dp_y - p_xdz); (\partial_{py}, -dy); (\partial_{px}, -ydz - dx); (0, dz - ydx) \} \tag{30}
\]
is a smooth global basis for \( D \).

We consider the action of the Lie group \( G = \mathbb{R}^2 \) on \( M \) given by
\[
\Phi : G \times M \to M, \quad \Phi((r,s),m) = (x+r, y+z+s, px, py),
\]
where \( m := (x, y, z, px, py) \in M \). This \( \mathbb{R}^2 \)-action is the restriction to \( M \) of the cotangent lift of the action \( \phi : G \times Q \to Q, \phi((r,s),(x,y,z)) = (x+r, y+z+s) \). It obviously leaves the Hamiltonian \( H(m) = \frac{1}{2}((1+y^2)p_x^2 + p_y^2) \) on \( M \) invariant. Note that if \( (X, \alpha) \in \Gamma(D) \) we have
\[
(\mathcal{L}_\xi X, \mathcal{L}_\xi \alpha) \in \Gamma(D) \quad \text{for all} \quad \xi \in \mathfrak{g} = \mathbb{R}^2.
\]

Since the vertical bundle in this example is \( V = \text{span}\{\partial_x, \partial_z\} \), we have
\[
\mathcal{K} = V \oplus \{0\} = \text{span}\{(\partial_x, 0), (\partial_z, 0)\} \subset TM \oplus T^*M
\]
and thus
\[
\mathcal{K}^\perp = TM \oplus V^o = \text{span}\{(\partial_x, 0), (\partial_y, 0), (\partial_z, 0), (\partial_{py}, 0), (\partial_{px}, 0), (0, dy), (0, dp_x), (0, dp_y)\} \tag{31}
\]
A direct computation using (30) and (31) yields
\[
D \cap \mathcal{K}^\perp = \text{span}\{(\partial_{py}, -dy), (\partial_x + y\partial_z, (1+y^2)dp_x + yp_xdy), ((1+y^2)\partial_y - yp_x\partial_{px}, (1+y^2)dp_y)\}
\]
and
\[
(D \cap \mathcal{K}^\perp) + \mathcal{K} = \text{span}\{(\partial_{py}, -dy), (\partial_x, 0), (\partial_z, 0), (0, (1+y^2)dp_x + yp_xdy), ((1+y^2)\partial_y - yp_x\partial_{px}, (1+y^2)dp_y)\}
\]
since in this case \( (D \cap \mathcal{K}^\perp) \cap \mathcal{K} = \{0\} \).

Note that there is an easier way to compute the spanning sections of \( D \cap \mathcal{K}^\perp \) by using (29). First, one determines spanning sections of \( U \). Second, for each spanning section \( X \in \Gamma(U) \) we find \( \lambda \in C^\infty(M) \) such that
\[
i_X\omega_M + \lambda(dz - ydx) \in \Gamma(V^o).
\]
Third, setting \( \alpha := i_X\omega_M + \lambda(dz - ydx) \) we have found a spanning section \( (X, \alpha) \in \Gamma(D \cap \mathcal{K}^\perp) \). In the following examples, we will proceed like this.

We get the reduced Dirac structure
\[
D_{\text{red}} = \frac{(D \cap \mathcal{K}^\perp) + \mathcal{K}}{\mathcal{K}}/G = \text{span}\left\{ \begin{array}{l}
(\partial_{py}, -dy), (0, (1+y^2)dp_x + yp_xdy), \\
((1+y^2)\partial_y - yp_x\partial_{px}, (1+y^2)dp_y)\
\end{array} \right\}
\]
on the three dimensional manifold \( \tilde{M} := M/G \) with global coordinates \( (y, py, px) \).
Since $\partial_x + y\partial_z$ is a spanning section of $\mathcal{H} \cap V$, the distribution $\mathcal{U} \subset TM$ (see (23)) is given by

$$\mathcal{U} = (V \cap \mathcal{H})^{\omega_M} \cap \mathcal{H} = \ker\{1_{\mathcal{U}} + y\partial_z, \omega_M\} \cap \mathcal{H}$$

$$= \ker\{(1 + y^2)d_{pz} + yp_zdy\} \cap \mathcal{H}$$

$$= \text{span}\{(1 + y^2)\partial_y - yp_x\partial_{pz}, \partial_x + y\partial_z, \partial_{pz}\}.$$

Thus

$$\mathcal{H} = T\pi(\mathcal{U}) = \text{span}\{\partial_{pz}, (1 + y^2)\partial_y - yp_x\partial_{pz}\}$$

recovering the result in Bates and Śniatycki (1993). Note that, as discussed in §3.2, the distribution $\mathcal{H} \subset TM$ coincides with the projection on the first factor of the reduced Dirac structure (22). As in Bates and Śniatycki (1993), $\mathcal{H}$ is an integrable subbundle of $TM$; in fact $[\partial_{pz}, (1 + y^2)\partial_y - yp_x\partial_{pz}] = 0$. The 2-form $\omega_\mathcal{H}$ is easily computed to equal

$$\omega_\mathcal{H}(\partial_{pz}, (1 + y^2)\partial_y - yp_x\partial_{pz}) = -dy((1 + y^2)\partial_y - yp_x\partial_{pz}) = -(1 + y^2).$$

As predicted by the general theory in §4.1, $\omega_\mathcal{H}$ is nondegenerate.

It is easy to check that the reduced manifold $\bar{M}$ is Poisson relative to the 2-tensor

$$-\partial_{\theta} \wedge \partial_{p_{\theta}} + \frac{yp_x}{1 + y^2}\partial_{p_{\theta}} \wedge \partial_{p_{\phi}},$$

or with Poisson bracket determined by $\{y, p_y\} = -1, \{y, p_z\} = 0, \{p_{\theta}, p_x\} = yp_x/(1 + y^2)$, and that $\mathcal{D}_{\text{red}}$ given by (32) is the graph of the vector bundle homomorphism $\beta : T^*\bar{M} \to \bar{M}$ associated to the Poisson structure.

### 4.4 Example: the vertical rolling disk

This example is standard in the theory of nonholonomic mechanical systems; it can be found for example in Bloch (2003). Consider a vertical disk of zero width rolling on the $xy$-plane and free to rotate about its vertical axis. Let $x$ and $y$ denote the position of contact of the disk in the $xy$-plane. The remaining variables are $\theta$ and $\varphi$, denoting the orientation of a chosen material point $P$ with respect to the vertical and the “heading angle” of the disk. Thus, the unconstrained configuration space for the vertical rolling disk is $Q := \mathbb{R}^2 \times S^1 \times S^1$. The Lagrangian for the problem is taken to be the kinetic energy

$$L(x, y, \theta, \varphi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\varphi}) = \frac{1}{2}\mu(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2,$$

where $\mu$ is the mass of the disk, and $I, J$ are its moments of inertia. Hence, the Hamiltonian of the system is

$$H(x, y, \theta, \varphi, p_x, p_y, p_{\theta}, p_{\varphi}) = \frac{1}{2\mu}(p_x^2 + p_y^2) + \frac{1}{2I}p_{\theta}^2 + \frac{1}{2J}p_{\varphi}^2.$$

The rolling constraints may be written as $\dot{x} = R\dot{\theta}\cos \varphi$ and $\dot{y} = R\dot{\theta}\sin \varphi$, where $R$ is the radius of the disk, that is,

$$\mathcal{D} := \{(x, y, \theta, \varphi, R\dot{\theta}\cos \varphi, R\dot{\theta}\sin \varphi, \dot{\theta}, \dot{\varphi}) \mid x, y \in \mathbb{R}, \theta, \varphi \in S^1\} \subset TQ.$$

Note that the 1-forms defining this distribution $\mathcal{D}$ are $\phi_1 := dx - R\cos \varphi d\theta$ and $\phi_2 := dy - R\sin \varphi d\theta$.

The constraint manifold (22)

$$M := \left\{(x, y, \theta, \varphi, p_x, p_y, p_{\theta}, p_{\varphi}) \in T^*Q \mid p_x = \frac{\mu R}{I}p_{\theta}\cos \varphi, p_y = \frac{\mu R}{I}p_{\theta}\sin \varphi \right\} \subseteq T^*Q$$

is in this example a graph over the coordinates $(x, y, \theta, \varphi, p_{\theta}, p_{\varphi})$ and is hence six dimensional. The induced 2-form $\omega_M = i^\ast\omega_{\text{can}}$ is given by the formula

$$\omega_M = dx \wedge \left(\frac{\mu R}{I}\cos \varphi dp_{\theta} - \frac{\mu R}{I}\sin \varphi dp_{\varphi}\right) + dy \wedge \left(\frac{\mu R}{I}\sin \varphi dp_{\theta} + \frac{\mu R}{I}\cos \varphi dp_{\varphi}\right)$$

$$+ d\theta \wedge dp_{\theta} + d\varphi \wedge dp_{\varphi}.$$
and the distribution \( \mathcal{H} = \ker \{ dx - R \cos \varphi d\theta, \ dy - R \sin \varphi d\theta \} \subseteq TM \) is in this case

\[
\mathcal{H} = \text{span}\{ \partial_\varphi, \partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \partial_{p_\varphi}, \partial_{p_\theta} \} \subset TM.
\] (32)

Therefore its annihilator is

\[
\mathcal{H}^\perp = \text{span}\{ dx - R \cos \varphi d\theta, dy - R \sin \varphi d\theta \} \subset T^* M.
\]

The Dirac structure on \( M \) describing the nonholonomic mechanical system is again given by (28). Since

\[
\mathcal{H} = \text{span}\{ \partial_\varphi, \partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \partial_{p_\varphi}, \partial_{p_\theta} \} \subset TM.
\]

Therefore its annihilator is

\[
\mathcal{H}^\perp = \text{span}\{ dx - R \cos \varphi d\theta, \ dy - R \sin \varphi d\theta \} \subset T^* M.
\]

The case

1. \( H \) and clearly leaves the Hamiltonian \( H \) invariant. The distribution \( \mathcal{V} \) on \( M \) is in this case \( \mathcal{V} = \text{span}\{ \partial_x, \partial_y \} \), so that \( \mathcal{V} \cap \mathcal{H} = \{ 0 \} \) by (32). Therefore, in this example, \( \mathcal{U} = \mathcal{H} \). We have

\[
\mathcal{H} = \mathcal{V} \oplus \{ 0 \} = \text{span}\{ (\partial_x, 0), (\partial_y, 0) \} \subset TM \oplus T^* M
\]

and

\[
\mathcal{H}^\perp = TM \oplus \mathcal{V}^\perp
\]

\[
= \text{span}\{ (\partial_x, 0), (\partial_y, 0), (\partial_\theta, 0), (\partial_\varphi, 0), (\partial_{p_\varphi}, 0), (\partial_{p_\theta}, 0), (0, d\varphi), (0, d\theta), (0, dp_\varphi), (0, dp_\theta) \}.
\]
By \(29\) and the fact that \(\mathcal{V} + \mathcal{H} = TM\), we know that for each spanning section \(X\) of \(\mathcal{H}\), there exists exactly one \(\alpha \in \Gamma(\mathcal{V})\) such that the pair \((X, \alpha)\) is a section of \(\mathcal{D} \cap \mathcal{X}^\perp\). Using \(33\) and the equalities

\[
\begin{align*}
    i_{\partial_\varphi} \omega_M &= -\frac{\mu R \sin \varphi}{T} p_\theta (dx - R \cos \varphi \theta) + \frac{\mu R \cos \varphi}{T} (dy - R \sin \varphi \theta) = dp_\varphi \\
    i_{\partial_{p_\varphi}} \omega_M &= \frac{\mu R \cos \varphi}{T} p_\theta (dx - R \cos \varphi \theta) \\
    &+ \frac{\mu R \sin \varphi}{T} (dy - R \sin \varphi \theta) = -\left(1 + \frac{\mu R^2}{T}\right) d\theta
\end{align*}
\]

we find

\[
D \cap \mathcal{X}^\perp = \text{span}\left\{ (\partial_\varphi, dp_\varphi), \left( \partial_{p_\varphi}, -\left(1 + \frac{\mu R^2}{T}\right) d\theta \right), \left( \partial_{p_\varphi}, -d\varphi \right), \left( \partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \left(1 + \frac{\mu R^2}{T}\right) dp_\theta \right) \right\}.
\]

Hence

\[
(D \cap \mathcal{X}^\perp) + \mathcal{X} = \text{span}\left\{ (\partial_\varphi, dp_\varphi), \left( \partial_{p_\varphi}, -\left(1 + \frac{\mu R^2}{T}\right) d\theta \right), \left( \partial_{p_\varphi}, -d\varphi \right), \left( \partial_\theta, \left(1 + \frac{\mu R^2}{T}\right) dp_\theta \right), \left( \partial_x, 0 \right), \left( \partial_y, 0 \right) \right\}
\]

and finally we get the reduced Dirac structure

\[
D_{\text{red}} = \frac{(D \cap \mathcal{X}^\perp) + \mathcal{X}}{\mathcal{K}} = \text{span}\left\{ (\partial_\varphi, dp_\varphi), \left( \partial_{p_\varphi}, -(1 + \frac{\mu R^2}{T}) d\theta \right), \left( \partial_{p_\varphi}, -d\varphi \right), \left( \partial_\theta, (1 + \frac{\mu R^2}{T}) dp_\theta \right) \right\}
\]

on the four dimensional manifold \(\tilde{M} = M/G\) with coordinates \((\varphi, \theta, p_\varphi, p_\theta)\). Thus, \(D_{\text{red}}\) is the graph of the symplectic form on \(\tilde{M}\) given by \(\omega_{\text{red}} = dp_\varphi \wedge dp_\theta + (1 + \frac{\mu R^2}{T}) d\theta \wedge dp_\theta\).

As already mentioned, in this example, \(\mathcal{H} = \mathcal{K}\) and hence \(\tilde{\mathcal{H}} = T\pi(\mathcal{H}) = \text{span}\{\partial_\varphi, \partial_{p_\varphi}, \partial_\theta, \partial_{p_\theta}\}\) by \(32\) which coincides with the projection on the first factor of the reduced Dirac structure \(34\). In this case \(\tilde{\mathcal{H}} = TM\) and so \(\omega_{\tilde{\mathcal{H}}} = \omega_{\text{red}}\) is of course nondegenerate.

2. The case \(G = \text{SE}(2)\) (Bloch (2003)).

The Lie group \(\text{SE}(2) := \mathbb{S}^1 \otimes \mathbb{R}^2\) is the semidirect product of the circle \(\mathbb{S}^1\) identified with matrices of the form

\[
\begin{bmatrix}
    \cos \alpha & -\sin \alpha \\
    \sin \alpha & \cos \alpha
\end{bmatrix}
\]

and acting on \(\mathbb{R}^2\) by usual matrix multiplication. Denote elements of \(\text{SE}(2)\) by \((\alpha, r, s)\) where \(\alpha \in \mathbb{S}^1\) and \(r, s \in \mathbb{R}\). Define the action of the Lie group \(\text{SE}(2)\) on \(M\) by

\[
(\alpha, r, s) \cdot (x, y, \theta, \varphi, p_\theta, p_\varphi) = (x \cos \alpha - y \sin \alpha + r, x \sin \alpha + y \cos \alpha + s, \theta, \varphi + \alpha, p_\theta, p_\varphi)
\]

and note that the Hamiltonian \(H\) is invariant by this action. The distribution \(\mathcal{V}\) on \(M\) is in this case \(\mathcal{V} = \text{span}\{\partial_\varphi, \partial_\theta, \partial_x\}\) and we get

\[
\mathcal{K} = \mathcal{V} \oplus \{0\} = \text{span}\{\partial_x, 0\}, \partial_y, 0\}, \partial_x, 0\}\}.
\]

Thus

\[
\mathcal{K}^\perp = TM \oplus \mathcal{V}^\circ = \text{span}\{\partial_x, 0\}, \partial_y, 0\}, \partial_\theta, 0\}, \partial_x, 0\}, \partial_\theta, 0\}, (0, dp_\varphi), (0, dp_\theta), (0, d\theta)\}.
\]
We have $\mathcal{V} \cap \mathcal{H} = \text{span}\{\partial_\varphi\}$ (see (32)) and hence
\[
(\mathcal{V} \cap \mathcal{H})^{\omega_M} = \ker \left( dp_\varphi + \frac{\mu R \sin \varphi}{I} p_0 dx - \frac{\mu R \cos \varphi}{I} p_0 dy \right)
\]
so that
\[
\mathcal{U} = \mathcal{H} \cap \ker \left( dp_\varphi + \frac{\mu R \sin \varphi}{I} p_0 dx - \frac{\mu R \cos \varphi}{I} p_0 dy \right) = \text{span}\{\partial_\varphi, \partial_\vartheta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \partial_{p_\varphi}\}.
\]
Using (33), (34), and (35), we get
\[
D \cap \mathcal{K}^\perp = \text{span}\left\{\begin{array}{l}
(\partial_\varphi, dp_\varphi), \\
(\partial_{p_\vartheta}, -(1 + \frac{\mu R^2}{I}) d\vartheta), \\
(\partial_\vartheta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, (1 + \frac{\mu R^2}{I}) dp_\vartheta)\end{array}\right\}.
\]
Thus,
\[
D_{\text{red}} = \frac{(D \cap \mathcal{K}^\perp) + \mathcal{K}}{\mathcal{K}} \bigg/ G = \text{span}\left\{\begin{array}{l}
(0, dp_\varphi), \\
(\partial_{p_\vartheta}, -(1 + \frac{\mu R^2}{I}) d\vartheta), \\
(\partial_\vartheta, (1 + \frac{\mu R^2}{I}) dp_\vartheta)\end{array}\right\}
\]
is the graph of the Poisson tensor
\[
\frac{I}{\mu R^2 + I} \partial_{p_\vartheta} \wedge \partial_\vartheta
\]
defined on the manifold $\bar{M} := M / G$ with coordinates $(\theta, p_\theta, p_\varphi)$.

In addition,
\[
\mathcal{H} = T\pi(\mathcal{U}) = \text{span}\{\partial_\theta, \partial_{p_\varphi}\}
\]
is an integrable subbundle of $T\bar{M}$ (since $[\partial_\theta, \partial_{p_\varphi}] = 0$). Note that the projection on the first factor of $D_{\text{red}}$ equals $\mathcal{H}$. Finally, the 2-form $\omega_{\mathcal{H}}$ is easily computed to be
\[
\omega_{\mathcal{H}} (\partial_\theta, \partial_{p_\varphi}) = 1 + \frac{\mu R^2}{I}
\]
and, as predicted by the general theory, it is nondegenerate on $\mathcal{H}$.

3. The case $G = S^1 \times \mathbb{R}^2$ (Bloch (2003)). The direct product Lie group $S^1 \times \mathbb{R}^2$ acts on $M$ by
\[
(\alpha, r, s) \cdot (x, y, \theta, \varphi, p_\theta, p_\varphi) = (x + r, y + s, \theta + \alpha, \varphi, p_\theta, p_\varphi).
\]
The distribution $\mathcal{V}$ on $M$ is in this case $\mathcal{V} = \text{span}\{\partial_x, \partial_y, \partial_\vartheta\}$,
\[
\mathcal{K} = \mathcal{V} \oplus \{0\} = \text{span}\{(\partial_x, 0), (\partial_y, 0), (\partial_\vartheta, 0)\},
\]
and thus
\[
\mathcal{K}^\perp = TM \oplus \mathcal{V}^\circ = \text{span}\{\begin{array}{l}
(\partial_x, 0), \\
(\partial_y, 0), \\
(\partial_\vartheta, 0), \\
(\partial_{p_\vartheta}, 0), \\
(\partial_{p_\varphi}, 0), \\
(0, dp_\varphi), \\
(0, dp_\theta), \\
(0, d\varphi)\end{array}\}
\]
Using (32) we get $\mathcal{V} \cap \mathcal{H} = \text{span}\{\partial_\vartheta + R \cos \varphi \partial_x + R \sin \varphi \partial_y\}$ and hence
\[
(\mathcal{V} \cap \mathcal{H})^{\omega_M} = \ker \left\{\begin{array}{l}
1 + \frac{\mu R^2}{I} dp_\theta\end{array}\right\}.
\]
Therefore, again by \(\mathfrak{H}\) we conclude
\[
\mathfrak{U} = \mathfrak{H} \cap (\mathfrak{V} \cap \mathfrak{H})^{\omega M} = \mathfrak{H} \cap \ker \left\{ \left( 1 + \frac{\mu R^2}{I} \right) d\varphi \right\}
\]
\[
= \text{span}\{ \partial_\varphi, \partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \partial_{p_x} \}.
\]

Using \(\mathfrak{H}\) and \(\mathfrak{K}\), we obtain
\[
D \cap \mathfrak{K}^\perp = \text{span} \left\{ (\partial_\varphi, dp_\varphi), (\partial_{p_x}, -d\varphi), (\partial_\theta + R \cos \varphi \partial_x + R \sin \varphi \partial_y, \left( 1 + \frac{\mu R^2}{I} \right) dp_\theta) \right\}
\]
and hence
\[
D_{\text{red}} = \left( \frac{D \cap \mathfrak{K}^\perp + \mathfrak{K}}{\mathfrak{K}} \right) G = \text{span} \left\{ (\partial_\varphi, dp_\varphi), (\partial_{p_x}, -d\varphi), (0, dp_\theta) \right\},
\]
which is the graph of the Poisson tensor
\[
\partial_{p_x} \wedge \partial_\varphi
\]
on the three dimensional reduced manifold \(\tilde{M} = M/G\) with coordinates \((\varphi, p_\varphi, p_\theta)\). We have
\[
\tilde{\mathfrak{H}} = T\pi(\mathfrak{U}) = \text{span}\{ \partial_\varphi, \partial_{p_x} \}
\]
which is an integrable subbundle of \(T\tilde{M}\) (since \([\partial_\varphi, \partial_{p_x}] = 0\)). As before, the projection on the first factor of \(D_{\text{red}}\) equals \(\tilde{\mathfrak{H}}\). The 2-form \(\omega_{\tilde{\mathfrak{H}}}\) has the expression
\[
\omega_{\tilde{\mathfrak{H}}} (\partial_\varphi, \partial_{p_x}) = 1
\]
and, as the general theory states, it is nondegenerate on \(\tilde{\mathfrak{H}}\).

### 4.5 Example: the Chaplygin skate

**The standard Chaplygin skate.** This example can be found in \cite{Rosenberg}. It describes the motion of a hatchet on a hatchet planimeter, that behaves like a curved knife edge. It is now commonly known under the name of “Chaplygin skate”. Let the contact point of the knife edge have the coordinates \(x, y \in \mathbb{R}^2\), let its direction relative to the positive \(x\)-axis be \(\theta\), and let its center of mass be at distance \(s\) from the contact point. Denote the total mass of the knife edge by \(m\). Thus the moment of inertia about an axis through the contact point normal to the \(xy\) plane is \(I = ms^2\). The configuration space of this problem is the semidirect product \(Q := \text{SE}(2) = \mathbb{S}^1 \ltimes \mathbb{R}^2\) whose coordinates are denoted by \(q := (\theta, x, y)\). We have the following concrete constraints on the velocities:
\[
\mathcal{D} := \ker(\sin \theta dx - \cos \theta dy) = \text{span} \{ \cos \theta \partial_x + \sin \theta \partial_y, \partial_\theta \} \subset TQ.
\]
The Lagrangian is hyperregular and taken to be the kinetic energy of the knife edge, namely,
\[
L(\theta, x, y, \dot{\theta}, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x} - s \dot{\theta} \sin \theta)^2 + \frac{1}{2} m (\dot{y} + s \dot{\theta} \cos \theta)^2
\]
\[
= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} ms^2 \dot{\theta}^2 + ms \dot{\theta}(\dot{y} \cos \theta - \dot{x} \sin \theta),
\]
where we have used that the \(x\) and \(y\) components of the velocity of the center of mass are, respectively,
\[
\dot{x} = s \dot{\theta} \sin \theta \quad \text{and} \quad \dot{y} + s \dot{\theta} \cos \theta.
\]
Compute
\[
p_x = \frac{\partial L}{\partial \dot{x}} = m \dot{x} - ms \dot{\theta} \sin \theta
\]
\[
p_y = \frac{\partial L}{\partial \dot{y}} = m \dot{y} + ms \dot{\theta} \cos \theta
\]
\[
p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ms^2 \dot{\theta} + ms(\dot{y} \cos \theta - \dot{x} \sin \theta).
\]
In $\mathcal{D}$ we have $\dot{y}\cos\theta - \dot{x}\sin\theta = 0$ and hence we get for $(\theta, x, y, p_\theta, p_x, p_y)$ in the constraint submanifold $M \subseteq T^*Q$:

\[
p_\theta = ms^2\dot{\theta} \quad \text{and} \quad p_x \sin\theta = m\dot{x}\sin\theta - ms\dot{\theta}\sin^2\theta
\]
\[
= m\dot{y}\cos\theta - ms\dot{\theta}(1 - \cos^2\theta)
\]
\[
= m\dot{y}\cos\theta + ms\dot{\theta}\cos^2\theta - ms\dot{\theta}
\]
\[
= p_y\cos\theta - \frac{1}{s}p_x.
\]

Hence the constraint manifold $M$ is five dimensional and given by

\[
M := \{ (\theta, x, y, p_\theta, p_x, p_y) \mid p_\theta = sp_y\cos\theta - sp_x\sin\theta \} \subseteq T^*Q.
\]

The global coordinates on $M$ are thus $(\theta, x, y, p_x, p_y)$. The pull back $\omega_M$ of the canonical 2-form $\omega$ on $T^*Q$ to $M$ has hence the expression

\[
\omega_M = dx \wedge dp_x + dy \wedge dp_y + d(\theta d\theta - dp_y) - s\sin\theta d\theta \wedge dp_x.
\]

The Dirac structure $D$ modeling this problem is given by (28). Formula (27) gives the vector subbundle

\[
\mathcal{H} := (T(\pi_{T^*Q}|M))^{-1}(\mathcal{D}) = \text{span}\{\cos\theta \partial_x + \sin\theta \partial_y, \partial_\theta, \partial_{p_x}, \partial_{p_y}\} \subset TM,
\]

or equivalently

\[
\mathcal{H}^c = \text{span}\{\sin\theta dx - \cos\theta dy\}.
\]

A computation yields

\[
i_{\cos\theta \partial_x + \sin\theta \partial_y} \omega_M = \cos\theta dp_x + \sin\theta dp_y
\]
\[
i_{\partial_\theta} \omega_M = s \cos\theta dp_y - s \sin\theta dp_x
\]
\[
i_{\partial_{p_y}} \omega_M = -dy - s \cos\theta d\theta
\]
\[
i_{\partial_{p_x}} \omega_M = -dx + s \sin\theta d\theta.
\]

Hence

\[
\{ (\cos\theta \partial_x + \sin\theta \partial_y, \cos\theta dp_x + \sin\theta dp_y) ; (\partial_\theta, s \cos\theta dp_y - s \sin\theta dp_x) ; (\partial_{p_y}, -dy - s \cos\theta d\theta) ; (\partial_{p_x}, -dx + s \sin\theta d\theta) ; (0, \sin\theta dx - \cos\theta dy) \}
\]

is a smooth global basis for $D$.

We consider the action of the Lie group $G = \text{SE}(2)$ on $Q$, given by

\[
\phi : G \times Q \rightarrow Q, \quad \phi((\alpha, r, s), (\theta, x, y)) = (\theta + \alpha, \cos\alpha x - \sin\alpha y + r, \sin\alpha x + \cos\alpha y + s).
\]

Thus, the induced action on $\Phi : G \times T^*Q \rightarrow T^*Q$ is given by

\[
\Phi((\alpha, r, s), (\theta, x, y, p_\theta, p_x, p_y))
\]
\[
= (\theta + \alpha, \cos\alpha x - \sin\alpha y + r, \sin\alpha x + \cos\alpha y + s, p_\theta, \cos\alpha p_x - \sin\alpha p_y, \sin\alpha p_x + \cos\alpha p_y).
\]

The action on $Q$ obviously leaves the Lagrangian invariant. We show that the induced action on $T^*Q$ leaves the manifold $M$ invariant: we denote with $\theta', x', y', p_x', p_y'$ the coordinates of $\Phi((\alpha, r, s), (\theta, x, y, p_\theta, p_x, p_y))$ and compute

\[
s \cos\theta' p_y' - s \sin\theta' p_x' = s \cos(\theta + \alpha)(\sin\alpha p_x + \cos\alpha p_y) - s \sin(\theta + \alpha)(\cos\alpha p_x - \sin\alpha p_y)
\]
\[
= s(\cos\theta \cos\alpha - \sin\theta \sin\alpha)(\sin\alpha p_x + \cos\alpha p_y)
\]
\[
- s(\sin\theta \cos\alpha + \cos\theta \sin\alpha)(\cos\alpha p_x - \sin\alpha p_y)
\]
\[
= s \cos\theta p_y - s \sin\theta p_x = p_\theta' = p_\theta.'
Since the vertical bundle in this example is $V = \text{span}\{\partial_x, \partial_y, \partial_\vartheta\}$, we have $V \cap H = \text{span}\{\partial_\vartheta, \cos \vartheta \partial_x + \sin \vartheta \partial_y\}$ and $(V \cap H)^\ast = \ker\{\cos \vartheta \partial_x + \sin \vartheta \partial_y, s \cos \vartheta \partial_y - s \sin \vartheta \partial_x\} = \ker\{\partial_\vartheta, \partial_x, \partial_y\}$. Hence the distribution $\mathcal{U} = (V \cap H)^\ast \cap \mathcal{H}$ is given by $\text{span}\{\partial_\vartheta, \cos \vartheta \partial_x + \sin \vartheta \partial_y\}$ and

$$D \cap \mathcal{X}^\perp = \text{span}\{\cos \vartheta \partial_x + \sin \vartheta \partial_y, \cos \vartheta \partial_x + \sin \vartheta \partial_y\}, \partial_\vartheta, s \cos \vartheta \partial_y - s \sin \vartheta \partial_x\}.$$

We get the reduced Dirac structure

$$D_{\text{red}} = \frac{(D \cap \mathcal{X}^\perp) + \mathcal{X}}{G} = \text{span}\{\cos \vartheta \partial_x + \sin \vartheta \partial_y, \partial_\vartheta, s \cos \vartheta \partial_y - s \sin \vartheta \partial_x\}$$

on the two dimensional manifold $\bar{M} := \bar{M}/G$ with global coordinates $(p_x, p_y)$. Note that this is the graph of the trivial Poisson tensor on $\bar{M}$.

**The Chaplygin skate with a rotor on it.** We propose here a variation of the previous example by considering the Chaplygin skate with a disk attached to the center of mass of the skate that is free to rotate about the vertical axis. Again, let the contact point of the knife edge have the coordinates $x, y \in \mathbb{R}^2$, let its direction relative to the positive $x$-axis be $\vartheta$, and let its center of mass be at distance $s$ from the contact point. Denote by $m$ the mass of the knife edge. Thus its moment of inertia about an axis through the contact point normal to the $xy$ plane is $I = ms^2$. Let $\phi$ be the angle between a fixed point on the disk and the positive $x$-axis and $J$ be the moment of inertia of the disk about the vertical axis. The configuration space of this problem is $Q := S^1 \times S^1 \times \mathbb{R}^2$ whose points are denoted by $q := (\phi, \theta, x, y)$. We have again the following concrete constraints on the velocities:

$$\mathcal{D} := \ker\{\sin \vartheta \partial_x - \cos \vartheta \partial_y\} = \text{span}\{\cos \vartheta \partial_x + \sin \vartheta \partial_y, \partial_\vartheta\} \subset TQ.$$

The Lagrangian is the kinetic energy of the knife edge:

$$L(\phi, \theta, x, y, \dot{\phi}, \dot{\theta}, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x} - s\dot{\vartheta}\sin \theta)^2 + \frac{1}{2}m(\dot{y} + s\dot{\vartheta}\cos \theta)^2 + \frac{1}{2}J(\dot{\theta} + \dot{\vartheta})^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(I + J)\dot{\theta}^2 + ms(\dot{\vartheta}\cos \theta - \dot{x}\sin \theta) + \frac{1}{2}J\dot{\vartheta}^2 + J\dot{\varphi}\vartheta.$$ Computex

\begin{align*}
p_x &= m\dot{x} - ms\dot{\vartheta}\sin \theta \\
p_y &= m\dot{y} + ms\dot{\vartheta}\cos \theta \\
p_\theta &= (I + J)\dot{\theta} + ms(\dot{\vartheta}\cos \theta - \dot{x}\sin \theta) + J\dot{\varphi} \\
p_\phi &= J(\dot{\varphi} + \dot{\theta}).
\end{align*}

Again, if we have $\dot{y}\cos \theta - \dot{x}\sin \theta = 0$, we compute:

$$p_\theta = (I + J)\dot{\theta} + J\dot{\varphi} = I\dot{\theta} + J(\dot{\vartheta} + \dot{\phi}) = I\dot{\theta} + p_\phi = ms^2\dot{\vartheta} + p_\phi$$

and

$$p_x \sin \theta = m\dot{x}\sin \theta - ms\dot{\vartheta}\sin^2 \theta = m\dot{y}\cos \theta - ms\dot{\vartheta}(1 - \cos^2 \theta)$$

$$= m\dot{y}\cos \theta + ms\dot{\vartheta}\cos^2 \theta - ms\dot{\vartheta} = p_y \cos \theta + \frac{1}{s}(p_\phi - p_\theta).$$

Hence the constraint manifold $M$ is seven dimensional and given by

$$M := \{(\phi, \theta, x, y, p_\vartheta, p_x, p_y) \mid p_\theta = sp_y \cos \theta - sp_x \sin \theta + p_\phi\} \subset T^*Q.$$ The global coordinates on $M$ are thus $(\phi, \theta, x, y, p_\vartheta, p_x, p_y)$. The pull back $\omega_M$ of the canonical 2-form $\omega$ on $T^*Q$ to $M$ has hence the expression

$$\omega_M = dx \wedge dp_x + dy \wedge dp_y + d\theta \wedge d(sp_y \cos \theta - sp_x \sin \theta + p_\phi) + d\phi \wedge dp_\phi$$

$$= dx \wedge dp_x + dy \wedge dp_y + s \cos \theta dp \wedge dp_y - s \sin \theta dp \wedge dp_x + (d\theta + d\phi) \wedge dp_\phi.$$
The Dirac structure $D$ modeling this problem is given by (28). Formula (27) gives the vector subbundle 

$$ \mathcal{H} := (T(\pi_{T^*Q}|_M))^{-1}(D) = \text{span}\{\partial_\phi, \partial_\theta, \cos \theta \partial_x + \sin \theta \partial_y, \partial_{p_\phi}, \partial_{p_x}, \partial_{p_\theta}\} \subset TM,$$

or equivalently

$$ \mathcal{H}^c = \text{span}\{\sin \theta dx - \cos \theta dy\}.$$

A computation yields

\begin{align*}
\mathfrak{i}_{\partial_\phi} \omega_M &= d\phi \\
\mathfrak{i}_{\partial_\theta} \omega_M &= s \cos \theta dp_y - s \sin \theta dp_x + d\phi \\
\mathfrak{i}_{\cos \theta \partial_x + \sin \theta \partial_y} \omega_M &= \cos \theta dp_x + \sin \theta dp_y \\
\mathfrak{i}_{\partial_{p_\phi}} \omega_M &= -d\theta - d\phi \\
\mathfrak{i}_{\partial_{p_\theta}} \omega_M &= -dy - s \cos \theta d\theta \\
\mathfrak{i}_{\partial_{p_\theta}} \omega_M &= -dx + s \sin \theta d\theta.
\end{align*}

We get

$$ D = \text{span}\{ (\partial_\phi, dp_\phi) ; (\partial_\theta, s \cos \theta dp_y - s \sin \theta dp_x + dp_\phi) ; (\cos \theta \partial_x + \sin \theta \partial_y, \cos \theta dp_x + \sin \theta dp_y) ; \\
(\partial_{p_\phi}, -d\theta - d\phi) ; (\partial_{p_y}, -dy - s \cos \theta d\theta) ; (\partial_{p_x}, dx + s \sin \theta d\theta) \}.$$

We consider the action of the Lie group $G = S^1 \times \text{SE}(2)$ on $Q$, given by

$$ \phi : G \times Q \rightarrow Q, \quad \phi((\beta, \alpha, r, s), (\phi, \theta, x, y)) = (\phi + \beta, \theta + \alpha, \cos \alpha x - \sin \alpha y + r, \sin \alpha x + \cos \alpha y + s).$$

Thus, the induced action $\Phi : G \times T^*Q \rightarrow T^*Q$ on $T^*Q$ is given by

$$ \Phi((\beta, \alpha, r, s), (\phi, \theta, x, y, p_\phi, p_\theta, p_y)) \quad = \quad (\phi + \beta, \theta + \alpha, \cos \alpha x - \sin \alpha y + r, \sin \alpha x + \cos \alpha y + s, p_\theta, \cos \alpha p_x - \sin \alpha p_y, \sin \alpha p_x + \cos \alpha p_y).$$

The Lagrangian is invariant under the lift to $TQ$ of $\phi$ and it is easy to see, with the considerations in the previous example, that the induced action $\Phi$ on $T^*Q$ leaves the manifold $M$ invariant.

Since the vertical bundle in this example is $V = \text{span}\{\partial_\phi, \partial_\theta, \partial_x, \partial_y\}$, we have $V \cap \mathcal{H} = \text{span}\{\partial_\phi, \partial_\theta, \cos \theta \partial_x + \sin \theta \partial_y\}$ and $(V \cap \mathcal{H})^\perp = \ker\{dp_\phi, \cos \theta dp_x + \sin \theta dp_y, s \cos \theta dp_y - s \sin \theta dp_x + dp_\phi\} = \ker\{dp_\phi, dp_x, dp_y\}$. Hence the distribution $\mathcal{U} = (V \cap \mathcal{H})^\perp \cap \mathcal{H}$ is given by $\text{span}\{\partial_\phi, \partial_\theta, \cos \theta \partial_x + \sin \theta \partial_y\}$ and

$$ D \cap \mathcal{K}^\perp = \text{span}\{ (\partial_\phi, dp_\phi) , (\cos \theta \partial_x + \sin \theta \partial_y, \cos \theta dp_x + \sin \theta dp_y) , (\partial_\theta, s \cos \theta dp_y - s \sin \theta dp_x + dp_\phi) \}.$$

We get the reduced Dirac structure

\begin{align*}
D_{\text{red}} &= \frac{(D \cap \mathcal{K}^\perp) + \mathcal{K}}{\mathcal{K}} \\
&= \text{span}\{ (0, dp_\phi) , (0, \cos \theta dp_x + \sin \theta dp_y) , (0, s \cos \theta dp_y - s \sin \theta dp_x + dp_\phi) \} \\
&= \text{span}\{ (0, dp_\phi) , (0, dp_x) , (0, dp_y) \}
\end{align*}

on the three dimensional manifold $\bar{M} := M/G$ with global coordinates $(p_\phi, p_x, p_y)$. This is again the graph of the trivial Poisson tensor on $\bar{M}$.

In these six examples we get integrable Dirac structures after reduction. We shall come back to this remark in the last section of the paper.

5 The optimal momentum map for closed Dirac manifolds

5.1 Definition of the optimal momentum map

Let $(M, D)$ be a closed Dirac manifold, $G$ a symmetry Lie group of $D$ acting freely and properly on $M$. Assume in the following that $D \cap \mathcal{K}^\perp$ is a vector bundle, where $\mathcal{K} = V \oplus \{0\} \subset TM \oplus T^*M$ and $\mathcal{K}^\perp = TM \times V^\circ$.
(see [3.2]). To define the optimal momentum map (as in Ortega and Ratiu 2004) we need to introduce an additional smooth distribution. Define
\[ \mathcal{D}_G(m) := \{ X(m) \mid \text{there is } \alpha \in \Gamma(V) \subseteq \Omega^1(M) \text{ such that } (X, \alpha) \in \Gamma(D) \} \subseteq G_1(m) \]
for all \( m \in M \). Then \( \mathcal{D}_G = \cup_{m \in M} \mathcal{D}_G(m) \) is a smooth distribution on \( M \).

If the manifold \( M \) is Poisson and the Dirac structure is the graph of the Poisson map \( \xi : T^*M \to TM \), then \( \mathcal{D}_G(p) = \{ X_f(p) \mid \text{there is } f \in C^\infty(M)\mathbb{G} \text{ such that } X_f = \xi(d_f) \} \), which recovers the definition in Ortega and Ratiu (2004).

Returning to the general case of Dirac manifolds, note that \( \mathcal{D}_G \) is integrable, the space of local sections of the intersection of vector bundles \( D \cap X^\perp \) is a vector bundle, \( \mathcal{D}_G \) is integrable in the sense of Stefan-Sussmann.

Lemma 5.1 Let \( (X, \alpha), (Y, \beta) \in \Gamma(D \cap X^\perp) \), i.e., \( X, Y \in \Gamma(D_G) \). Then the 1-form \( \mathcal{L}_X \beta - i_Y \mathcal{d} \alpha \) is a local section of of \( V \).

Proof: It suffices to show that \( (\mathcal{L}_X \beta - i_Y \mathcal{d} \alpha)(\xi_M) = 0 \) for all \( \xi \in \mathfrak{g} \). Since \( D \) is \( G \)-invariant, we have \( (\mathcal{L}_{\xi_M} X, \mathcal{L}_{\xi_M} \alpha) \in \Gamma(D) \) for all \( \xi \in \mathfrak{g} \). Since \( (Y, \beta) \in \Gamma(D) \), we conclude \( \beta(\mathcal{L}_{\xi_M} X) + (\mathcal{L}_{\xi_M} \alpha)(Y) = 0 \) or \( \beta(\mathcal{L}_{\xi_M} X) = -(\mathcal{L}_{\xi_M} \alpha)(Y) \). Thus we get
\[
(\mathcal{L}_X \beta - i_Y \mathcal{d} \alpha)(\xi_M) = \mathcal{L}_X(\beta(\xi_M)) - \beta(\mathcal{L}_X \xi_M) - \mathcal{d} \alpha(Y, \xi_M)
= \mathcal{L}_X(0) + \mathcal{L}_X \xi_M - Y[\alpha(\xi_M)] + \xi_M[\alpha(Y)] - \mathcal{d} \alpha(Y, \xi_M)
= -(\mathcal{L}_{\xi_M} \alpha)(Y) - \mathcal{L}_Y(0) + \mathcal{L}_{\xi_M} \alpha(Y) - \mathcal{d} \alpha(\xi_M, Y) = 0,
\]
where we used \( \beta(\xi_M) = 0 \) since \( \alpha, \beta \in \Gamma(V) \).

Lemma 5.2 If \( D \) is integrable, the space of local sections of the intersection of vector bundles \( D \cap X^\perp \) is closed under the Courant bracket. Hence, under the assumption that \( D \cap X^\perp \) has constant dimensional fibers, this vector bundle inherits a Lie algebroid structure relative to the truncated Courant bracket on \( \Gamma(D \cap X^\perp) \) and the anchor map \( \pi_1 : D \cap X^\perp \to TM \). Thus, the distribution \( \mathcal{D}_G = \pi_1(D \cap X^\perp) \) is integrable in the sense of Stefan-Sussmann.

Proof: Since \( D \) is integrable, the space of its local sections is closed under the Courant bracket, and hence for all \( (X, \alpha), (Y, \beta) \in \Gamma(D \cap X^\perp) \) we have (see (2))
\[
[(X, \alpha), (Y, \beta)] = ([X, Y], \mathcal{L}_X \beta - i_Y \mathcal{d} \alpha) \in \Gamma(D).
\]
Lemma 5.1 implies that \( \mathcal{L}_X \beta - i_Y \mathcal{d} \alpha \in \Gamma(V) \). Thus, \( [(X, \alpha), (Y, \beta)] \in \Gamma(D \cap X^\perp) \) and hence \( [X, Y] \in \Gamma(D_G) \).

The remaining statements follow immediately.

Thus, if \( D \cap X^\perp \) is a vector bundle, \( M \) admits a generalized foliation by the leaves of the generalized distribution \( \mathcal{D}_G \). The optimal momentum is now defined like in Ortega and Ratiu (2004).

Definition 5.3 Assume that \( D \cap X^\perp \) is a vector subbundle of \( TM \oplus T^*M \). The projection
\[
\beta : M \to M/\mathcal{D}_G
\]
on the leaf space of \( \mathcal{D}_G \) is called the (Dirac) optimal momentum map.

In order to formulate in the next section the reduction theorem with this optimal momentum map, we need an induced action of \( G \) on the leaf space of \( \mathcal{D}_G \). This doesn’t follow, as usual, from the \( G \)-equivariance of the vector fields spanning \( \mathcal{D}_G \) because, in this case, they are not necessarily \( G \)-equivariant.

Lemma 5.4 If \( m \) and \( m' \) are in the same leaf of \( \mathcal{D}_G \) then \( \Phi_g(m) \) and \( \Phi_g(m') \) are in the same leaf of \( \mathcal{D}_G \) for all \( g \in G \). Hence there is a well defined action \( \bar{\Phi} : G \times M/\mathcal{D}_G \to M/\mathcal{D}_G \) given by
\[
\bar{\Phi}_g(\beta(m)) := \beta(\Phi_g(m))
\]
PROOF: Let $m$ and $m'$ be in the same leaf of $\mathcal{D}_G$. Without loss of generality we can assume that there exists $X \in \Gamma(\mathcal{D}_G)$ with flow $F^X_t$ such that $F^X_t(m) = m'$ for some $t$ (in reality, $m$ and $m'$ can be joined by finitely many such curves). Since $(X, \alpha) \in \Gamma(D)$ for some $\alpha \in \Gamma(V^\circ)$ and $D$ is $G$-invariant, it follows that $(\Phi^*_g X, \Phi^*_g \alpha) \in \Gamma(D)$ for all $g \in G$. Because $\alpha \in \Gamma(V^\circ)$, for all $\xi \in \mathfrak{g}$ we have

$$\langle (\Phi^*_g \alpha)(p), \xi_M(p) \rangle = \langle \alpha(\Phi_g(p)), (\text{Ad}_{g^{-1}} \xi)M(\Phi_g(p)) \rangle = 0$$

which shows that $\Phi^*_g \alpha \in \Gamma(V^\circ)$. Hence, $\Phi^*_g X \in \Gamma(\mathcal{D}_G)$ for all $g \in G$. For all $s \in [0, t]$ we have

$$\frac{d}{ds} (\Phi_g \circ F^X_s)(m) = T_{F^X_t(m)} \Phi_g \big( X(F^X_s(m)) \big) = (\Phi_g)^{-1} X(\Phi_g(F^X_s(m))) \in \mathcal{D}_G(\Phi_g(F^X_s(m))).$$

Thus the curve $c(s) = (\Phi_g \circ F^X_s)(m)$ connecting $c(0) = \Phi_g(m)$ to $c(t) = \Phi_g(F^X_t(m)) = \Phi_g(m')$ has all its tangent vectors in the distribution $\mathcal{D}_G$ and hence it lies entirely in the leaf of $\mathcal{D}_G$ through the point $\Phi_g(m)$.

Denote by $G_\rho$ the isotropy subgroup of $\rho \in M/\mathcal{D}_G$ for this induced action. If $g \in G_\rho$ and $m \in \mathfrak{g}^{-1}(\rho)$, then

$$\mathfrak{g}(\Phi_g(m)) = \Phi_g(\mathfrak{g}(m)) = \Phi_g(\rho) = \mathfrak{g}(m)$$

and we get the usual fact that $G_\rho$ leaves $\mathfrak{g}^{-1}(\rho)$ invariant. Thus we get an induced action of $G_\rho$ on $\mathfrak{g}^{-1}(\rho)$, which is free if the original $G$-action on $M$ is free.

Also, $\mathfrak{g}^{-1}(\rho)$ is an initial submanifold of $M$ since it is a leaf of the generalized foliation defined by the integrable distribution $\mathcal{D}_G$. By Proposition 3.4.4 in [Ortega and Ratiu (2004)], there is a unique smooth structure on $G_\rho$ with respect to which this subgroup is an initial Lie subgroup of $G$ with Lie algebra

$$\mathfrak{g}_\rho = \{ \xi \in \mathfrak{g} \mid \xi_M(m) \in T_m \mathfrak{g}^{-1}(\rho), \text{ for all } m \in \mathfrak{g}^{-1}(\rho) \}.$$

In general, $G_\rho$ is not closed in $G$.

### 5.2 The universality of the optimal momentum map

**Definition 5.5** Let $(M, D)$ be a Dirac manifold with integrable Dirac structure $D$ and $G$ a Lie group acting canonically on it. Let $P$ be a set and $J : M \to P$ a map. We say that $J$ has the Noether property for the $G$-action on $(M, D)$ if the flow $F_t$ of any implicit Hamiltonian vector field associated to any $G$-invariant admissible function $h \in C^\infty(M)$ preserves the fibers of $J$, that is,

$$J \circ F_t = J |_{\text{Dom}(F_t)}$$

where $\text{Dom}(F_t)$ is the domain of definition of $F_t$.

Like in the Poisson case (see [Ortega and Ratiu (2004)]), one gets the following universality property.

**Theorem 5.6** Let $G$ be a symmetry Lie group of the Dirac manifold $(M, D)$ and $J : M \to P$ a function with the Noether property. Then there exists a unique map $\phi : M/\mathcal{D}_G \to P$ such that the following diagram commutes:

$$\begin{array}{ccc}
M & \xrightarrow{\mathfrak{g}} & P \\
\downarrow{\phi} & & \downarrow{\phi} \\
M/\mathcal{D}_G & \xrightarrow{J} & P
\end{array}$$

If $J$ is smooth and $G$-equivariant with respect to some $G$-action on $P$, then $\phi$ is also smooth and $G$-equivariant.

**Proof:** The proof is the same as for Poisson manifolds (see [Ortega and Ratiu (2004)]). Define $\phi : M/\mathcal{D}_G \to P$ by $\phi(\rho) := J(m)$, where $\rho = \mathfrak{g}(m)$. The map $\phi$ is well defined since if $m' \in \mathfrak{g}^{-1}(\rho)$ then there is a finite composition $T_\rho$ of flows associated to sections of $\mathcal{D}_G$ such that $m' = T_\rho(m)$. Since $J$ is a Noether momentum map we have

$$J(m') = J(T_\rho(m)) = J(m) = \phi(\rho).$$
The definition immediately implies that the diagram commutes. Uniqueness of \( \phi \) follows from the requirement that the diagram commutes and the surjectivity of \( \mathcal{J} \). Equivariance of \( \phi \) is a direct consequence of the definition \( \text{[11]} \) of the \( G \)-action on \( M/D_G \). Finally, if all objects are smooth manifolds and \( \mathcal{J}, \mathcal{J} \) are smooth maps then \( \phi \) is a smooth map as the quotient of the smooth map \( \mathcal{J} \) by the projection \( \mathcal{J} \) (see \text{[Bourbaki 1967]}).

6 Optimal reduction for closed Dirac manifolds

In this section we generalize the optimal reduction procedure from Poisson manifolds (see \text{[Ortega and Ratiu 2004]} to closed Dirac manifolds. As we shall see, with appropriately extended definitions this important desingularization method works also for Dirac manifolds.

6.1 The reduction theorem

Theorem 6.1 (Optimal point reduction by Dirac actions) Let \( (M, D) \) be an integrable Dirac manifold and \( G \) a Lie group acting freely and properly on \( M \) and leaving the Dirac structure invariant. Assume that \( D \cap \mathcal{X}^\perp \) is constant dimensional and let \( \mathcal{J} : M \to M/D_G \) be the optimal (Dirac) momentum map associated to this action. Then, for any \( \rho \in M/D_G \) whose isotropy subgroup \( G_\rho \) acts properly on \( \mathcal{J}^{-1}(\rho) \), the orbit space \( M_\rho = \mathcal{J}^{-1}(\rho)/G_\rho \) is a smooth presymplectic regular quotient manifold with presymplectic form \( \omega_\rho \in \Omega^2(M_\rho) \) defined by

\[
(\pi_\rho \omega_\rho)(m)(X(m), Y(m)) = \alpha_m(Y(m)) - \beta_m(X(m))
\]

for any \( m \in \mathcal{J}^{-1}(\rho) \) and any \( X,Y \in \Gamma(D_G) \) defined on an open set around \( \mathcal{J}(\rho) \), where \( \alpha, \beta \in \Gamma(\mathcal{F}) \) are such that \( (X, \alpha), (Y, \beta) \in \Gamma(D \cap \mathcal{X}^\perp) \), and \( \pi_\rho : \mathcal{J}^{-1}(\rho) \to M_\rho \) is the projection. The pair \( (M_\rho, D_\rho) \) is called the (Dirac optimal) point reduced space of \( (M, D) \) at \( \rho \), where \( D_\rho \) is the graph of the presymplectic form \( \omega_\rho \).

Note that if \( D \) is the graph of a Poisson structure on \( M \), the distribution \( G \) is \{0\}, all functions in \( C^\infty(M) \) are admissible, and we are in the setting of the Optimal point reduction by Poisson actions Theorem (see \text{[Ortega and Ratiu 2004]} Theorem 9.1.1).

Proof: Denote by \( \Phi^\rho : G_\rho \times \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho) \) the restriction of the original \( G \)-action on \( M \) to the Lie subgroup \( G_\rho \) and the manifold \( \mathcal{J}^{-1}(\rho) \). Since, by hypothesis, the \( G \)-action on \( \mathcal{J}^{-1}(\rho) \) is proper, the quotient \( \mathcal{J}^{-1}(\rho)/G_\rho \) is a regular quotient manifold and hence the projection \( \pi_\rho : \mathcal{J}^{-1}(\rho)/G_\rho \to \mathcal{J}^{-1}(\rho)/G_\rho \) is a smooth surjective submersion. We show that \( \omega_\rho \) given by \( \text{[11]} \) is well-defined. Let \( m, m' \in \mathcal{J}^{-1}(\rho) \) be such that \( \pi_\rho(m) = \pi_\rho(m') \), and let \( v, w \in T_m \mathcal{J}^{-1}(\rho) \), \( v', w' \in T_{m'} \mathcal{J}^{-1}(\rho) \) be such that \( T_m \pi_\rho(v) = T_{m'} \pi_\rho(v'), T_m \pi_\rho(w) = T_{m'} \pi_\rho(w') \). Let \( (X, \alpha), (X', \alpha'), (Y, \beta), (Y', \beta') \) be sections of \( D \cap \mathcal{X}^\perp \) such that \( X(m) = T_m \pi_\rho(v), X'(m') = T_{m'} \pi_\rho(v'), Y(m) = T_m \pi_\rho(w), Y'(m') = T_{m'} \pi_\rho(w') \). The condition \( \pi_\rho(m) = \pi_\rho(m') \) implies the existence of an element \( k \in G_\rho \) such that \( m' = \Phi^\rho_k(m) \). We have then \( \pi_\rho = \pi_\rho \circ \Phi^\rho_k \) and thus \( T_m \pi_\rho = T_m \pi_\rho \circ T_m \Phi^\rho_k \). Furthermore, because of the equalities \( T_m \pi_\rho(v) = T_{m'} \pi_\rho(v'), T_m \pi_\rho(w) = T_{m'} \pi_\rho(w') \), we have

\[
T_{m'} \pi_\rho (T_m \Phi^\rho_k(v) - v') = 0 \quad \text{and} \quad T_{m'} \pi_\rho (T_m \Phi^\rho_k(w) - w') = 0
\]

and there exist elements \( \xi^1, \xi^2 \in g_\rho \) and sections \( (V_1, \eta_1), (V_2, \eta_2) \in \Gamma(D \cap \mathcal{X}^\perp) \), such that

\[
X'(m') - T_m \Phi_k(X(m)) = \xi^1_m(m') = V_1(m') \quad \text{and} \quad Y'(m') - T_m \Phi_k(Y(m)) = \xi^2_m(m') = V_2(m').
\]

This yields

\[
X'(m') = (\Phi^\rho_{k^{-1}}X)(m') + V_1(m') \quad \text{and} \quad Y'(m') = (\Phi^\rho_{k^{-1}}Y)(m') + V_2(m').
\]

Because \( (X', \alpha') \) and \( (\Phi^\rho_{k^{-1}}Y, \Phi^\rho_{k^{-1}}\beta) \) are sections of \( D \cap \mathcal{X}^\perp \) in a neighborhood of the point \( m' \), we have

\[
(\Phi^\rho_{k^{-1}}\beta)(X') = -\alpha' (\Phi^\rho_{k^{-1}}Y),
\]

(42)
and thus we conclude
\[
\omega_\rho(\pi_\rho(m'))(T_m\pi_\rho(v'), T_m\pi_\rho(w')) = (\pi_\rho^\ast\omega_\rho)(m')(X'(m'), Y'(m'))
\]
\[
= \alpha'(m')(Y'(m')) = \alpha'(m')(\Phi_{k-1}X)(m') + V_2(m')
\]
\[
= \alpha'(m')(\Phi_{k-1}Y)(m')
\]
\[
\equiv -(\Phi_{k-1}\beta)(m')(X'(m'))
\]
\[
= -(\Phi_{k-1}\beta)(m')(\Phi_{k-1}X)(m') + V_4(m')
\]
\[
= -(\Phi_{k-1}\beta)(m')(\Phi_{k-1}X)(m')
\]
\[
= -\beta(m)(X(m))
\]
\[
= \omega_\rho(\pi_\rho(m))(T_m\pi_\rho(v), T_m\pi_\rho(w)).
\]

Finally, we show that \(\omega_\rho\) is closed. Let \(m \in B^{-1}(\rho)\), \(v, w, u \in T_mB^{-1}(\rho)\). Again, choose sections \((X, \alpha), (Y, \beta), (Z, \gamma) \in \Gamma(D \cap X^\perp)\) such that \(v = X(m), w = Y(m), u = Z(m)\). Now write
\[
(\mathbf{d}(\pi_\rho^\ast\omega_\rho))_m(v, w, u) = (\mathbf{d}(\pi_\rho^\ast\omega_\rho))_m(X(m), Y(m), Z(m))
\]
and compute, recalling the definition (11) and formula (38),
\[
\mathbf{d}(\pi_\rho^\ast\omega_\rho)(X, Y, Z) = X ([\pi_\rho^\ast\omega_\rho](Y, Z)] - Y ([\pi_\rho^\ast\omega_\rho](X, Z])
\]
\[
+ Z ([\pi_\rho^\ast\omega_\rho](X, Y] - (\pi_\rho^\ast\omega_\rho))(X, Y, Z)
\]
\[
+ (\pi_\rho^\ast\omega_\rho)((X, Z], Y) - (\pi_\rho^\ast\omega_\rho)(Y, Z, X)
\]
\[
\equiv X [\gamma(Z)] + Y [\gamma(X)] + Z [\alpha(Y)]
\]
\[
+ \gamma([X, Y]) + (\mathbf{L}_X\gamma - 1_Zd\alpha)(Y) + [\alpha([Y, Z])]
\]
\[
\equiv X [\beta(Z)] + Y [\gamma(X)] + Z [\alpha(Y)] + \gamma([X, Y]) - \gamma([X, Y])
\]
\[
+ X [\alpha(Z)] - Z [\alpha(Y)] + Y [\alpha(Z)] + \alpha([Z, Y]) + \alpha([Y, Z])
\]
\[
\equiv X [\beta(Z)] + Y [\gamma(X)] + X [\gamma(Y)] + Y [\alpha(Z)]
\]
\[
= X [\beta(Z)] + Y [\gamma(X)] - X [\beta(Z)] - Y [\gamma(X)] = 0,
\]
where we used the fact that \(\gamma(X) + \alpha(Z) = 0\) and \(\gamma(Y) + \beta(Z) = 0\) (this follows directly from \((X, \alpha), (Y, \beta), (Z, \gamma) \in \Gamma(D)\)). Thus, \(\pi_\rho^\ast\mathbf{d}\omega_\rho = \mathbf{d}(\pi_\rho^\ast\omega_\rho) = 0\) and, because \(\pi_\rho\) is a surjective submersion, it follows that \(\mathbf{d}\omega_\rho = 0\). Therefore, \(\omega_\rho\) is a well-defined presymplectic form on \(M_\rho\). \(\square\)

Recall that, since \(D \cap X^\perp\) is assumed to have constant dimensional fibers, one can build the reduced Dirac manifold \((\bar{M}, D_{\text{red}})\) as in (33). The following theorem gives the relation between the reduced manifold \(M\) and the reduced manifolds \(M_\rho\) given by the optimal reduction theorem.

**Theorem 6.2** If \(m \in B^{-1}(\rho) \subseteq M\), the reduced manifold \(M_\rho\) is diffeomorphic to the presymplectic leaf \(\bar{N}\) through \(\pi(m)\) of the reduced Dirac manifold \((\bar{M}, D_{\text{red}})\) via the map \(\Theta : M_\rho \rightarrow \bar{N}, \pi_\rho(x) \mapsto (\pi \circ i_\rho)(x)\). Furthermore, \(\Theta^\ast\omega_{\bar{N}} = \omega_\rho\), where \(\omega_{\bar{N}}\) is the presymplectic form on \(\bar{N}\).

**Proof:** First of all, we will show that the distribution \(D_G\) is spanned by \(G\)-sections that “descend” to \(\bar{M}\). Let \(X\) be an arbitrary section of \(D_G\). Then we find \(\alpha \in \Gamma(V^\nu)\) such that \((X, \alpha) \in \Gamma(D \cap X^\perp)\). Since \(G\) acts on \(M\) by Dirac actions, we have \((\mathbf{L}_{\xi_M}X, \mathbf{L}_{\xi_M}\alpha) \in \Gamma(D)\) for all \(\xi \in \mathfrak{g}\). If \(\eta \in \mathfrak{g}\), we get
\[
(\mathbf{L}_{\xi_M}(\alpha(\eta_M))) = \mathbf{L}_{\xi_M}(\alpha(\eta_M)) - \alpha([\xi_M, \eta_M]) = \mathbf{L}_{\xi_M}(0) + \alpha([\xi, \eta_M]) = 0,
\]
because \(\alpha\) annihilates the infinitesimal generators of the \(G\)-action. Hence, we have \((\mathbf{L}_{\xi_M}X, \mathbf{L}_{\xi_M}\alpha) \in \Gamma(D \cap X^\perp)\) and consequently \(\mathbf{L}_{\xi_M}X \in \Gamma(D_G)\). Now, if \(V\) is an arbitrary section of \(V\), it can be written \(V = \sum_{i=1}^k f_i \xi_M\) where \(f_1, \ldots, f_k\) are smooth locally defined functions on \(M\) and \(\{\xi^1, \ldots, \xi^k\}\) is a basis for \(\mathfrak{g}\). Therefore, \([V, X] = \sum_{i=1}^k f_i \mathbf{L}_{\xi_M}X - \sum_{i=1}^k X f_i \xi_M \in \Gamma(D_G + V)\). Thus we have \([\Gamma(D_G), \Gamma(V)] \subseteq \Gamma(D_G + V)\) and, since \(D_G = \pi_1(D \cap X^\perp)\) is the image of the vector bundle \(D \cap X^\perp\), it is locally finite. Proposition A.1 then guarantees that \(D_G\) is spanned by sections \(X\) satisfying \([X, \Gamma(V)] \subseteq \Gamma(V)\). A vector field satisfying this
condition will be called \textit{descending} in the following, because, using \textit{A.3} we can write each such vector field locally as a sum \( X = X^G + X^V \) with \( X^G \in \mathfrak{X}(M)^G \) and \( X^V \in \Gamma(V) \).

Hence we can choose \( U \subseteq M \) open with \( m \in U \) and smooth local descending vector fields \( X_1, \ldots, X_r \), such that for all \( q \in U \) we have \( D_M(q) = \text{span}\{X_1(q), \ldots, X_r(q)\} \). Let \( F \) be the set of all such sections of \( D_M \). Then \( F \) is an everywhere defined family of descending vector fields on \( M \) that span the integrable generalized distribution \( D_M \). Denote by \( A_F \) the pseudogroup of local diffeomorphisms associated to the flows of the family \( F \), i.e.,

\[ A_F = \{ \text{Id} \} \cup \{ F^1_t \circ \cdots \circ F^n_n \mid n \in \mathbb{N} \text{ and } F^n_n \text{ or } (F^n_n)^{-1} \text{ flow of } X^n \in F \}. \]

This pseudogroup is integrable, that is, its orbits define a generalized foliation on \( M \). The leaves of this foliation are also called the accessible sets of \( A_F \). Since \( D_M \) is also integrable, the accessible sets of \( A_F \) are the integral leaves of \( D_M \). The proofs of these statements can be found in \textit{Stefani (1974a,b); Ortega and Ratiu (2004)} gives a quick summary of this theory.

Now we turn to the proofs of the claims in the theorem. We begin by showing that the map \( \Theta \) is well-defined. Let \( x, y, z \in J^{-1}(\rho) \) be such that \( \pi_* X(x) = \pi_* Y(y) \). Then there exists \( g \in G_\rho \subseteq G \) such that \( \Phi_g(x) = y \) which implies that \( \Phi_g(i_\rho(x)) = i_\rho(y) \) and \( \pi(i_\rho(x)) = \pi(i_\rho(y)) \). Thus, it remains to show that \( \pi(i_\rho(x)) \in \tilde{N} \).

Since \( x \in J^{-1}(\rho) \) and the integral leaves of \( D_M \) coincide with the \( A_F \)-orbits, it follows that \( \pi(i_\rho(z)) \in \tilde{N} \). Moreover, we claim that the topology of \( \tilde{N} \) is well defined.

To prove that \( \Theta \) is injective, let \( \pi(x), \pi(y) \in M_\rho \) be such that \( \pi(x) = \pi(y) \). Then \( x, y \in J^{-1}(\rho) \) and there exists \( g \in G \) satisfying \( \Phi_g(x) = y \). This shows that \( g \in G_\rho \) and \( \Phi_g(x) = y \), so we get \( \pi(x) = \pi(y) \).

For the surjectivity of \( \Theta \) choose \( \pi(x) \in \tilde{N} \) and assume, again without loss of generality, that \( \pi(x) = \tilde{F}^{X_\rho}(\pi(y)) \), where \( X \) is a section of \( G_1 \) and \( \tilde{F}^{X} \) is its flow. Choose a vector field \( X \in \Gamma(X \cap \mathcal{X}^+) \) such that \( X \sim X \). Then the flow \( \tilde{F}^{X} \) of \( X \) satisfies \( \pi \circ \tilde{F}^{X} = \tilde{F}^{X} \circ \pi \) for all \( s \) and restricts to \( J^{-1}(\rho) \). If we define \( x' = F^X_{t_1}(m) \) we get \( \pi(x') = \tilde{F}^{X_\rho}(\pi(m)) = \pi(x) \) and hence \( \Theta(\pi(x')) = \pi(x) \). Note that we have simultaneously shown that \( \pi(J^{-1}(\rho)) \subseteq M \) is equal as a set to \( \tilde{N} \). Moreover, we claim that the topology of \( \tilde{N} \) (which is in general \textit{not} the relative topology induced from the topology on \( M \)) is the quotient topology of \( J^{-1}(\rho) \), that is, a set is open in \( \tilde{N} \) if and only if its preimage under \( \pi|_{J^{-1}(\rho)} \) is open in \( J^{-1}(\rho) \).

The topology on \( \tilde{N} \) is the relative topology induced on \( \tilde{N} \) by a topology we call the \( G_1 \)-topology on \( M \): this is the strongest topology on \( M \) such that all the maps

\[
\begin{align*}
U & \to M \\
(t_1, \ldots, t_k) & \mapsto (F^{X_1}_{t_1} \circ \cdots \circ F^{X_k}_{t_k})(m)
\end{align*}
\]

are continuous, where \( m \in M \), \( F^{X_i}_{t_i} \) is the flow of a section \( X_i \) of \( G_1 \) for \( i = 1, \ldots, k \), and \( U \subseteq \mathbb{R}^k \) is an appropriate open set in \( \mathbb{R}^k \). In the same manner, because \( J^{-1}(\rho) \) is an accessible set of the family

\[
\{ X \in \mathfrak{X}(M) \mid \exists \alpha \in \Omega^1(M) \text{ such that } (X, \alpha) \text{ is a descending section of } D \cap \mathcal{K}^+ \},
\]

the topology on \( J^{-1}(\rho) \) is the relative topology induced on \( J^{-1}(\rho) \) by the topology we call the \( \pi_1(D \cap \mathcal{K}^+) \)-topology on \( M \): this is the strongest topology on \( M \) such that all the maps

\[
\begin{align*}
U & \to M \\
(t_1, \ldots, t_k) & \mapsto (F^{X_1}_{t_1} \circ \cdots \circ F^{X_k}_{t_k})(m)
\end{align*}
\]

are continuous, where \( m \in M \), \( F^{X_i}_{t_i} \) is the flow of a vector field \( X_i \) on \( M \) such that there exists \( \alpha_i \in \Omega^1(M) \) such that \( (X_i, \alpha_i) \) is a descending section \( D \cap \mathcal{K}^+ \) for \( i = 1, \ldots, k \), and \( U \subseteq \mathbb{R}^k \) is an appropriate open set in \( \mathbb{R}^k \). Now our claim is easy to show, using the fact that for each section \( X \) of \( G_1 \), there exists a descending section \( (X, \alpha) \) of \( D \cap \mathcal{K}^+ \) such that \( X \sim X \) and hence \( F^{X}_{t_i} \circ \pi = \pi \circ F^{X}_{t} \). Conversely, for each
descending section \((X, \alpha)\) of \(D \cap X^\perp\), the vector field \(\bar{X}\) satisfying \(X \sim_\pi \bar{X}\) is a section of \(G_1\) and we have \(F^X_\pi \circ \pi = \pi \circ F^\pi_\bar{X}\). Hence, a map \(f : \bar{N} \to P\) is smooth if and only if \(f \circ \pi|_{\beta^{-1}(\rho)} : \beta^{-1}(\rho) \to P\) is smooth.

Thus, the smoothness of \(\Theta\) and of its inverse \(\Theta^{-1} : \bar{N} \to M_\rho\), \(\pi(x) \mapsto \pi_\rho(x)\) follow from the following commutative diagrams:

\[
\begin{array}{c}
\beta^{-1}(\rho) \downarrow \pi \\
\bar{M}_{\pi} \downarrow i_\pi \\
M_{\rho} \downarrow \pi_\rho \\
\end{array}
\]

Consider the first diagram: since \(\pi \circ i_\rho\) is smooth, we have automatically (by the quotient manifold structure on \(M_\rho\)) that \(i_\pi \circ \Theta\) is smooth. Since \(\bar{N}\) is an initial submanifold of \(\bar{M}\), the smoothness of \(\Theta\) follows. With the considerations above and, because \(\pi_\rho\) is smooth, we get the smoothness of \(\Theta^{-1}\) with the second diagram.

Now we show that \(\Theta\) is a *presymplectomorphism*, i.e., \(\Theta^* \omega_{\bar{N}} = \omega_{\rho}\). Let \(\pi_\rho(x) \in M_\rho, x \in \beta^{-1}(\rho)\), and \(v, w \in T_x\beta^{-1}(\rho)\), i.e., we have \(T_x i_\rho v, T_x i_\rho w \in \mathcal{D}_G(i_\rho(x))\). Then find \(X, Y \in \Gamma(G_1)\) and \(\bar{\alpha}, \bar{\beta} \in \Omega^1(M)\) such that \((X, \bar{\alpha}), (Y, \bar{\beta}) \in \Gamma(D_{\text{red}})\), \(T_x(\pi \circ i_\rho) v = T_x(\pi \circ i_\rho) w = \bar{X}(i_\rho(x))\), and \(T_x(\pi \circ i_\rho) w = T_x(\pi \circ i_\rho) w = \bar{Y}(i_\rho(x))\). Choose \(X, Y \in \mathfrak{X}(M)\) such that \(X \sim_\pi \bar{X}, Y \sim_\pi \bar{Y}\), and \((X, \pi^* \bar{\alpha}), (Y, \pi^* \bar{\beta}) \in \Gamma(D \cap X^\perp)\). Then we get

\[
(\Theta^* \omega_{\bar{N}})(\pi_\rho(x)) (T_x \pi_\rho v, T_x \pi_\rho w) = \omega_{\rho}((\Theta \circ \pi_\rho)(x)) (T_x(\Theta \circ \pi_\rho) v, T_x(\Theta \circ \pi_\rho) w)
\]

where the last equality is the definition of \(\omega_{\rho}\).

\[\square\]

### 6.2 Induced Dirac structure on a leaf of \(D_G\)

In order to check the power of the Optimal Point Reduction Theorem [6.3], we shall implement it in the case of the trivial symmetry group \(G = \{e\}\). For this and also for the reduction of dynamics in the next subsection, we need to describe the induced Dirac structure on a leaf \(\beta^{-1}(\rho)\) of \(D_G\). Of course, we could use the fact that since \(D_G\) is a subdistribution of \(G_1\), each leaf of \(D_G\) is an immersed submanifold of a leaf of \(G_1\). Knowing this, the induced Dirac structure on a leaf of \(D_G\) is the graph of the pullback of the presymplectic 2-form on the corresponding leaf of \(G_1\). But we want to get the stratification in presymplectic leaves of \(M\) as a corollary of Theorem [6.3] and so we have to derive directly the induced Dirac structure on \(\beta^{-1}(\rho)\) from the definition of the map \(\beta\), which is what we do next.

Let \(i_\rho : \beta^{-1}(\rho) \hookrightarrow M\) be the inclusion. Define the smooth 2-form on \(\beta^{-1}(\rho)\) by

\[
\omega_{\beta^{-1}(\rho)}(m)(\bar{X}(m), \bar{Y}(m)) = \alpha_m(Y(m)) = -\beta_m(X(m))
\]

for all \(\bar{X}, \bar{Y} \in \mathfrak{X}(\beta^{-1}(\rho))\) and \(m \in \beta^{-1}(\rho)\), where \(X, Y \in \Gamma(D_G)\) are such that \(X \sim_{i_\rho} X\) and \(Y \sim_{i_\rho} Y\) and \(\alpha, \beta\) are sections of \(\mathfrak{V}^\rho\) such that \((X, \alpha)\) and \((Y, \beta) \in \Gamma(D \cap X^\perp)\). Note that in the proof of the closedness of \(\omega_{\rho}\), we have shown that \(\omega_{\beta^{-1}(\rho)} = \pi_{\rho*} \omega_{\rho}\) is a smooth closed 2-form on \(\beta^{-1}(\rho)\).

The induced Dirac structure on \(\beta^{-1}(\rho)\) is given by

\[
\Gamma(D_{\beta^{-1}(\rho)}) = \left\{(\bar{X}, \bar{\alpha}) \in \Gamma(T\beta^{-1}(\rho) \oplus T^* \beta^{-1}(\rho)) \mid i_{\bar{X}} \omega_{\beta^{-1}(\rho)} = \bar{\alpha}\right\}.
\]

Let \((\bar{X}, \bar{\alpha}) \in \Gamma(D_{\beta^{-1}(\rho)})\). Let \(X \in \mathfrak{X}(M)\) be such that \(X \in \Gamma(D_G)\) and \(\bar{X} \sim_{i_\rho} X\). Hence, there exists \(\alpha \in \Gamma(\mathfrak{V}^\rho)\) such that \((X, \alpha) \in \Gamma(D)\). Then, choosing for each \(\bar{Y} \in \mathfrak{X}(\beta^{-1}(\rho))\) a section \(Y \in \Gamma(D_G)\) with \(\bar{Y} \sim_{i_\rho} Y\) and a section \(\beta \in \Gamma(\mathfrak{V}^\rho)\) such that \((Y, \beta) \in \Gamma(D)\), this yields

\[
\bar{\alpha}(m)(\bar{Y}(m)) = (i_{\bar{X}} \omega_{\beta^{-1}(\rho)})(m)(\bar{Y}(m)) = \alpha(m)(Y(m)) = (i_{\rho*} \alpha)(m)(\bar{Y}(m))
\]
and we get $\tilde{\alpha} = i_\rho^*\alpha$.

Choose now an arbitrary $\alpha' \in \Omega^1(M)$ (not necessarily in $\Gamma(V^\circ)$) such that $(X,\alpha) \in \Gamma(D)$. We get $\alpha - \alpha' \in \Gamma(P_0)$ and hence $\alpha(Y) = \alpha'(Y)$ for all $Y \in \Gamma(G_1)$. In view of the considerations above, this yields $\tilde{\alpha} = i_\rho^*\alpha'$. Now recall that each $X \in \mathfrak{X}_0(M)$ satisfying $\tilde{X} \sim_{i_\rho} X$ is necessarily a section of $\mathcal{D}_G$. We have proved the following result.

**Proposition 6.3** The induced Dirac structure $D_{\mathcal{J}^{-1}(\rho)}$ is given equivalently by

$$D_{\mathcal{J}^{-1}(\rho)}(m) = \left\{ (\tilde{X}(m), \tilde{\alpha}(m)) \in T_m\mathcal{J}^{-1}(\rho) \oplus T_m^*\mathcal{J}^{-1}(\rho) \mid \tilde{X} \in \mathfrak{X}(\mathcal{J}^{-1}(\rho)), \right.$$

$$\tilde{\alpha} \in \Omega^1(\mathcal{J}^{-1}(\rho)), \text{ there exists } X \in \mathfrak{X}(M) \text{ and } \alpha \in \Omega^1(M)$$

$$\text{such that } \tilde{\alpha} = i_\rho^*\alpha, \tilde{X} \sim_{i_\rho} X, \text{ and } (X,\alpha) \in \Gamma(D) \left\} \right..$$

This formula was found by Blankenstein and van der Schaft (2001) in the case of submanifolds. The proposition above extends it to the important case of the level sets $\mathcal{J}^{-1}(\rho)$ of the optimal momentum map $\mathcal{J}$ which are only initial submanifolds.

Now we go back and apply Theorem 6.1 to the case $G = \{e\}$. This condition implies that $\mathcal{D}_G = G_1$ and so the leaves of the generalized foliation are the presymplectic leaves

$$N := M_\rho = \mathcal{J}^{-1}(\rho)/G_\rho = \mathcal{J}^{-1}(\rho)/\{e\} = \mathcal{J}^{-1}(\rho).$$

Thus, if $G = \{e\}$, the presymplectic form $\omega_N$ given in (43) is equal to $\omega_\rho$ in the Optimal Point Reduction Theorem 6.1. Hence the Dirac structure on the presymplectic leaf $N$ is given by

$$D_N = \left\{ (\tilde{X}, \tilde{\alpha}) \in \Gamma(TN \oplus T^*N) \mid i_N \omega_N = \tilde{\alpha} \right\}$$

$$= \left\{ (\tilde{X}, \tilde{\alpha}) \in \Gamma(TN \oplus T^*N) \mid \text{there exists } X \in \mathfrak{X}(M) \text{ and } \alpha \in \Omega^1(M) \right.$$

$$\text{such that } \tilde{\alpha} = i_N^*\alpha, \tilde{X} \sim_{i_N} X \text{ and } (X,\alpha) \in \Gamma(D) \left\} \right..$$

where $i_N : N \to M$ is the inclusion. This is exactly the induced Dirac structure given by (43).

Thus, the theorem stating that each closed Dirac manifold has a generalized foliation by presymplectic leaves, each leaf having the induced Dirac structure, is the trivial case of the Optimal Point Reduction Theorem 6.1.

Note that Theorems 6.1 and 6.2 extend this result by characterizing the presymplectic leaves of $\mathcal{M}$ if $G \neq \{e\}$.

**Remark 6.4** Assume that $M$ is a Poisson manifold and $D$ is the graph of the induced map $\mathcal{J} : T^*M \to TM$. We shall prove that the reduced spaces $M_\rho$ are symplectic manifolds, in agreement with the result of Ortega (2002) (see also Ortega and Ratiu (2004), Theorem 9.1.1). Indeed, the reduced distributions $G_\rho^0$ are all trivial. To see this, recall that $G_\rho^0/V_\rho$ descends to $G_\rho^0,$ where $G_\rho^0$ is the distribution defined by the Dirac structure on $\mathcal{J}^{-1}(\rho)$. Let $\tilde{X}$ be a section of $G_\rho^0$, i.e., there exists $X \in \Gamma(\mathcal{D}_G)$ and $\alpha \in \Gamma(V^\circ)$ such that $\tilde{X} \sim_{i_\rho} X, i_\rho^*\alpha = 0$ and $(X,\alpha) \in \Gamma(D)$. Choose an arbitrary section $\beta$ of $V^\circ$. Since $D$ is the graph of the map $\mathcal{J}$ associated to the Poisson structure, $P_1 = TM$ and there exists $Y \in \Gamma(\mathcal{D}_G)$ such that $(Y,\beta) \in \Gamma(D)$. Let $\tilde{Y}$ be the vector field on $\mathcal{J}^{-1}(\rho)$ such that $\tilde{Y} \sim_{i_\rho} Y$. Then $(\tilde{Y}, i_\rho^*\beta)$ is a section of $D_{\mathcal{J}^{-1}(\rho)}$ and we can compute for all $m \in \mathcal{J}^{-1}(\rho)$:

$$\beta_{i_\rho(m)}(X(i_\rho(m))) = (i_\rho^*\beta)_m(\tilde{X}(m)) = (i_\rho^*\alpha)_m(\tilde{Y}(m)) = 0.$$

Hence, we have $X(i_\rho(m)) \in (\mathcal{D}_G \cap V)(i_\rho(m))$ and hence $\tilde{X}(m) \in V_\rho(m)$. \(\triangle\)

### 6.3 Reduction of dynamics

In this subsection we study the dynamic counterpart of the geometric Theorem 6.1.

**Definition 6.5** Let $\mathcal{D}_h \subseteq TM$ denote the affine distribution whose smooth sections are the solutions of the implicit Hamiltonian system $(X, \mathcal{D}_h) \in \Gamma(D)$ for an admissible function $h \in C^\infty(M)$, i.e., the vector fields $X \in \mathfrak{X}(M)$ satisfying $(X, \mathcal{D}_h) \in \Gamma(D)$. 

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If $X_h$ is a solution of the implicit Hamiltonian system $(X, dh) \in \Gamma(D)$, then $\Gamma(D_h) = X_h + \Gamma(G_0)$. Indeed, if $X$ is another solution, we have $(X, dh), (X_h, dh) \in (D)$ and thus $(X - X_h, 0) \in \Gamma(D)$ which is equivalent to $X - X_h \in \Gamma(G_0)$. This shows that $D_h \subseteq TM$ is really a smooth affine distribution.

If $h \in C^\infty(M)^G$ and $X_h \in \Gamma(D_h)$, we have $X_h \in \Gamma(D_G)$ since $dh \in \Gamma(V^0)$. This also shows that $D_h \subseteq D_G$.

According to the considerations in Section 2 if $\tilde{X}_h \in \mathcal{X}(\mathbb{R}^1)$ satisfies $\tilde{X}_h \sim_{i_p} X_h$ then we necessarily have $(\tilde{X}_h, d(i_p^*\rho)) \in \Gamma(D_{\mathbb{R}^1})$. Hence $\tilde{X}_h + \Gamma(G_0) = \Gamma(D_{\text{hor}}, \rho)$, where $G_0$ is, as above, the distribution defined by the Dirac structure on $\mathbb{R}^1$. Denote by $G_0^\rho$ the smooth distribution associated to the Dirac structure on $M_\rho$; hence $G_0^\rho$ is the kernel of the presymplectic form $\omega_\rho$.

**Theorem 6.6** Assume that $G_0^\rho$ is a locally finite smooth distribution on $\mathbb{R}^1$.

(i) Let $h \in C^\infty(M)^G$ be an admissible function. The flow $F_t$ of each $X_h \in \Gamma(D_h)$ leaves $\mathbb{R}^1$ invariant.

The set $D_h$ is $G$-invariant in the sense that $\Phi_g^*(D_h) = D_h$ for all $g \in G$. Therefore, the set $D_{\text{hor},\rho} \subseteq T\mathbb{R}^1(\rho)$ is $G_{\rho}$-invariant.

(ii) The affine distribution $D_{\text{hor},\rho} \subseteq T\mathbb{R}^1(\rho)$ projects to an affine distribution $D_{\text{hor},\rho}$ on $M_\rho$.

(iii) Since $h \in C^\infty(M)^G$, the function $h_\rho$ given by the equality $h_\rho \circ \rho_\pi = h \circ i_\rho$ is well defined. It is admissible and the corresponding affine distribution $D_\rho$ is equal to $D_{\text{hor},\rho}$.

(iv) Let $k \in C^\infty(M)^G$ be another admissible $G$-invariant function on $M$ and denote by $\{\cdot, \cdot\}_\rho$ the Poisson bracket on admissible functions associated to the Dirac structure $D_\rho$ on $M_\rho$. Then $\{h, k\}_\rho = \{h_\rho, k_\rho\}_\rho$, where $\{\cdot, \cdot\}_\rho$ is the Poisson bracket on admissible functions on $M_\rho$.

The proof of (ii) requires the technical result given in Proposition 4.1 of the appendix.

**Proof:** (i) The first statement is obvious: since $(X_h, dh) \in \Gamma(D)$ and $dh \in \Gamma(V^0)$ we have $(X_h, dh) \in \Gamma(D \cap \mathcal{X}^\perp)$ and hence $X_h \in \Gamma(D_G)$ which implies that the flow of $X_h$ leaves the leaves of the generalized foliation defined by the distribution $D_G$ invariant. However, these leaves are precisely the level sets $\mathbb{R}^1(\rho)$.

To prove the second statement, we first show that $\Phi_g^*(X_h + Y) \in X_h + \Gamma(G_0)$ for all $g \in G$ and $Y \in \Gamma(G_0)$. Indeed, we have $\Phi_g^*(X_h + Y, dh) \in \Gamma(D)$. Since $\rho$ is $G$-invariant, this yields $\Phi_g^*(X_h + Y, dh) \in \Gamma(D)$ and consequently $\Phi_g^*(X_h + Y) - (X_h + Y, 0) \in \Gamma(D)$. Thus we have $\Phi_g^*(X_h + Y) \in X_h + Y + \Gamma(G_0) = X_h + \Gamma(G_0)$. This shows that $\Phi_g^*(D_h) \subseteq D_h$ for all $g \in G$ since $\Gamma(D_h) = X_h + \Gamma(G_0)$. The reverse inclusion is obtained in the following way: $X_h + \Gamma(G_0) = (\Phi_g \circ \Phi_g^{-1})(X_h + \Gamma(G_0)) \subseteq \Phi_g^*(X_h + \Gamma(G_0))$.

To prove the third statement we note that for all $g \in G_{\rho}$ we have $i_\rho \circ \Phi_g^\rho = \Phi_g \circ i_\rho$. This easily implies that $\Phi_g^\rho \tilde{X}_h \sim_{i_\rho} \Phi_g^\rho X_h$ and $\Phi_g^\rho dh \circ i_\rho = i_\rho^* (\Phi_g^\rho dh) = i_\rho^* dh$. The last statement follows now by repeating the method of the proof above.

(ii) For all $\xi \in g$ we have $(\xi_{\mathbb{R}^1} X_h, \xi_{\mathbb{R}^1} dh) = (\xi_{\mathbb{R}^1} X_h, 0) \in \Gamma(D)$. Because $\xi_{\mathbb{R}^1(\rho)}$ is $i_\rho$-related to $\xi_{\mathbb{R}^1}$ for all $\xi \in g_{\rho}$, we conclude that $\xi_{\mathbb{R}^1(\rho)} \tilde{X}_h$ is $i_\rho$-related to $\xi_{\mathbb{R}^1} X_h$ and with the formula (23) for $D_{\mathbb{R}^1(\rho)}$, this yields

$$
(\xi_{\mathbb{R}^1(\rho)} \tilde{X}_h, \xi_{\mathbb{R}^1(\rho)} d(i_\rho^* h)) = (\xi_{\mathbb{R}^1(\rho)} \tilde{X}_h, \xi_{\mathbb{R}^1(\rho)} (i_\rho^* dh)) = (\xi_{\mathbb{R}^1(\rho)} \tilde{X}_h, i_\rho^* (\xi_{\mathbb{R}^1} (dh)))
$$

Thus, $\xi_{\mathbb{R}^1(\rho)} \tilde{X}_h \in \Gamma(G_0^\rho)$ and the inclusion $[\tilde{X}_h, \Gamma(V_\rho)] \subseteq \Gamma(V_\rho + G_0^\rho)$ follows. Furthermore, since $G_0^\rho = D_0$ with $0 \in C^\infty(\mathbb{R}^1(\rho))^G$, an analogous argument shows that $[\Gamma(G_0^\rho), \Gamma(V_\rho)] \subseteq \Gamma(V_\rho + G_0^\rho)$. This shows that all hypotheses of Proposition 4.1 are satisfied for the involutive subbundle $V_\rho$ of $T\mathbb{R}^1(\rho)$ and the locally finite generalized distribution $G_0^\rho$. Thus there exist $Z \in \Gamma(G_0^\rho)$ and $X_h \in \mathcal{X}(M_\rho)$ such that $\tilde{X}_h + Z \sim_{i_\rho} \tilde{X}_h$. In addition, Proposition 4.1 ensures the existence of spanning vector fields $Z_1, \ldots, Z_k$ for $\Gamma(G_0^\rho)$ such that $Z_i \sim_{i_\rho} Z_i$ for vector fields $Z_1, \ldots, Z_k \in \mathcal{X}(M_\rho)$, $i = 1, \ldots, k$. Hence $D_{\text{hor},\rho}$ projects to the affine distribution $D_{\text{hor},\rho}$ on $M_\rho$ defined by $\Gamma(D_{\text{hor},\rho}) = \tilde{X}_h + \text{span}_{\mathbb{R}^1(\rho)} \{Z_1, \ldots, Z_k\}$.

(iii) We continue to use the vector fields defined in the proofs of the first two statements. First show that $h_\rho$ is admissible, i.e., $(\tilde{X}_h, dh_\rho) \in \Gamma(D)$. To see this, note that for $m \in \mathbb{R}^1(\rho)$ and $v = \tilde{Y}(m) \in T_m \mathbb{R}^1(\rho)$, we have

$$
(i_{\tilde{X}_h} \omega_\rho)(m)(T_m h_\rho v) = (i_{\tilde{X}_h} \omega_\rho)(m)((\tilde{X}_h + Z)(m), \tilde{Y}(m)) = (i_{\tilde{X}_h} \omega_\rho)(m)(d\tilde{h}_\rho(\tilde{Y}(m)) = (d\tilde{h}_\rho)(\tilde{Y}(m)) (i_{\tilde{X}_h} \omega_\rho)(m)(T_m h_\rho v)
$$

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Thus, we have $i_{X_h} \omega_\rho = dh_\rho$ and hence $(\bar{X}_h, dh_\rho) \in \Gamma(D_\rho)$ which shows that $h_\rho$ is admissible. By an analogous argument with $h_\rho$ replaced by the zero function on $M_\rho$ (and $h$ by the zero function on $M$) we get that $\bar{Z}_i \in \Gamma(G_0)$ for $i = 1, \ldots, k$, and hence that $D_{h\omega_i} \subseteq D_{h_\rho}$. Denote $\bar{X}_h = X_{h_\rho}$, since $X_h$ is a solution of the implicit Hamiltonian system $(\bar{X}_h, dh_\rho) \in \Gamma(D_\rho)$.

For the converse inclusion, it is sufficient to show that $X_{h_\rho} + \mathcal{G}_0 \subseteq D_{h\omega_i}$. So we need to prove that if $\bar{Y} \in \Gamma(G_0)$, then $X_{h_\rho} + \bar{Y} \in \Gamma(D_{h\omega_i})$. Indeed, since $\bar{Y} \in \Gamma(G_0) \subseteq \mathcal{X}(\mathcal{M})$, there exists $\bar{Y} \in \mathcal{X}(\mathcal{M})$ such that $\bar{Y} \sim_{i_\rho} \bar{Y}$. The existence of $Y \in \Gamma(D_G)$, $\alpha \in \Gamma(V^\circ)$ with $Y \sim_{i_\rho} Y$ and $(Y, \alpha) \in \Gamma(D)$ follows by the construction of the leaf $\mathcal{M}^{-1}(\rho)$. For all $m \in \mathcal{M}^{-1}(\rho)$ and $v \in D_G(m) \subseteq T_M M$ we get

$$\alpha(m)(v) = \omega_{\mathcal{M}^{-1}(\rho)}(m)(\bar{Y}(m), v) = (\pi^*_\rho \omega_\rho)(\bar{Y}(m), v) = \omega_\rho(\pi_\rho(m))(\bar{Y}(\pi_\rho(m)), T_m \pi_\rho(v)) = 0$$

where the last equality holds because $\bar{Y} \in \Gamma(G_0)$. Now (44) leads to $(\bar{Y}, i_\rho \alpha) \in \Gamma(D_{\mathcal{M}^{-1}(\rho)})$ and from the computation above we conclude hence that $i_\rho \alpha = 0$. Therefore $\bar{Y} \in \Gamma(G_0)$. Since $\bar{Y} + \bar{X}_h + Z \sim_{i_\rho} \bar{Y} + X_{h_\rho}$, the assertion is shown.

(iv) This last statement is a straightforward computation which follows from the considerations above. Indeed, for all $m \in \mathcal{M}^{-1}(\rho)$ we have

$$(\{h, k\})_\rho (\pi_\rho(m)) = (\{h, k\} \circ i_\rho)(m) = (dh)(i_\rho(m)) (X_k(i_\rho(m))) = (i^*_\rho dh)(m) (\bar{X}_k(m))$$

$$= (\pi^*_\rho dh_\rho)(m) (\bar{X}_k(m)) = (dh_\rho)(\pi_\rho(m)) (T_m \pi_\rho (\bar{X}_k(m)))$$

$$= (-d(k\rho)(\pi_\rho(m)) (X_{h_\rho}(\pi_\rho(m))) = \{h_\rho, k\}_\rho (\pi_\rho(m)),$$

where the sixth equality follows from $T_m \pi_\rho (\bar{X}_k(m)) \in \mathcal{D}_{h_\rho}(m)$ which holds since $\bar{X}_k(m) \in \mathcal{D}_{h\omega_k}(m)$. □

7 Optimal reduction for nonholonomic systems

Recall the setting of $\mathcal{H}$. $Q$ is a configuration space which is a smooth Riemannian manifold, $\mathcal{D} \subseteq TQ$ is the constraints distribution given as the intersection of the kernels of $k$ linearly independent 1-forms on $Q$ and is hence a vector subbundle of $TQ$. $L$ is a classical Lagrangian equal to the kinetic energy of the given Riemannian metric on $Q$ minus a potential, $M := F L(\mathcal{D}) \subset T^* Q$ is a submanifold and represents the constraints in phase space $T^* Q$, and $\omega_M := i^* \omega_{\mathcal{H}an} \in \Omega^2(M)$ is the induced 2-form on $M$, where $i : M \hookrightarrow T^* Q$ is the inclusion and $\omega_{\mathcal{H}an}$ the canonical symplectic form on $T^* Q$. The distribution $\mathcal{H} := TM \cap (T \pi_{T^* Q})^{-1}(\mathcal{D})$ is not integrable but has the property that the restriction $\omega_M$ of $\omega_M$ on $\mathcal{H} \times \mathcal{H}$ is nondegenerate. The Dirac structure $D$ associated to this nonholonomic system has fibers

$$D(m) = \{(X(m), \alpha_m) \in T_m M \oplus T^*_m M \mid X \in \Gamma(H), \alpha - i_X \omega_M \in \Gamma(\mathcal{H}_m)\}$$

for all $m \in M$ and is, in general, not integrable. Recall from Proposition (42) (i) that $G_0 = \{0\}$ and $P_1 = T^* M$ and hence all functions are admissible.

Consider a $G$-action $\phi : G \times Q \to Q$ on $Q$ that leaves the constraints and the Lagrangian invariant. The lift $\Phi : G \times T^* Q \to T^* Q$ of the action is defined by $\Phi_g = (T \phi_{g^{-1}})^*$; this is a symplectic action on $T^* Q$ that leaves $M$ invariant. Thus we get a canonical $G$-action on the Dirac manifold $(M, D)$ and we have for all $g \in G$,

$$\Phi_g^* \omega_M = \Phi_g^* (i^* \omega_{\mathcal{H}an}) = i^* (\Phi_g^* \omega_{\mathcal{H}an}) = i^* \omega_{\mathcal{H}an} = \omega_M$$

since the $G$-action commutes with the inclusion. Note that in this section the $G$-action on $T^* Q$ is a lift, whereas in $\mathcal{H}$ we needed only that it is a symplectic action.

In this section we shall define a distribution on $M$ that yields the equations of motion and the conserved quantities given by the Nonholonomic Noether Theorem (see Cushman et al. (1995), Theorem 2 and also Bloch (2003), Chapter 5 and the corresponding internet supplement). If this distribution is integrable, we will prove a Marsden-Weinstein reduction theorem that gives a reduced Dirac structure which is the graph of a nondegenerate 2-form (not necessarily closed). This reduction procedure is done from an “optimal” point of view as in $\mathcal{H}$ although these to sections are completely independent of each other.
7.1 The nonholonomic Noether theorem

We recall in this subsection the Hamiltonian formulation of the Nonholonomic Noether Theorem. Let $\mathbf{J} : T^*Q \to \mathfrak{g}^*$ be the canonical momentum map associated to the action of $G$ on $T^*Q$ (see, e.g., Marsden and Ratiu (1999))

$$\mathbf{J}(p)(\xi) = \langle p, \xi_Q(\pi(p)) \rangle$$

(45)

for all $p \in T^*Q$, where $\pi : T^*Q \to Q$ is the projection. For all $\xi \in \mathfrak{g}$, the $\xi$-component of $\mathbf{J}$ is the map

$$J^\xi : T^*Q \to \mathbb{R}$$

is defined by

$$J^\xi(p) := \langle J(p), \xi \rangle$$

(46)

for all $p \in T^*Q$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between elements of $\mathfrak{g}^*$ and elements of $\mathfrak{g}$. We shall denote by the same symbol $J^\xi$ its restriction to the manifold $M$. For an arbitrary $\xi \in \mathfrak{g}$ we have therefore

$$i_{\xi_{T\cdot Q}^\omega_{can}} = dJ^\xi.$$  

(47)

Since the action of $G$ on $T^*Q$ leaves the submanifold $M$ invariant, we have $\xi_{T\cdot Q}(m) \in T_mM$ for all $m \in M$ and hence the fundamental vector field $\xi_{T\cdot Q}$ is $i$-related to $\xi_M$, i.e., $Ti \circ \xi_M = \xi_{T\cdot Q} \circ i$, where $i : M \hookrightarrow T^*Q$ is the inclusion. Choosing for each vector field $X \in \mathfrak{X}(M)$ an arbitrary extension $X' \in \mathfrak{X}(T^*Q)$ (and hence $X \sim X'$) we get for all $m \in M$,

$$i_{\xi_M} \omega_M(X)(m) = i_{\xi_M} (i^* \omega_{can})(X)(m) = i_{\xi_{T\cdot Q}^\omega_{can}}(X')(i(m)) = (dJ^\xi(X'))(i(m))$$

$$= (i^*dJ^\xi)(X)(m) = (dJ^\xi(X))(m)$$

which shows that (47) naturally restricts to $M$

$$i_{\xi_M} \omega_M = dJ^\xi.$$  

(48)

Define for all $p \in M$ the vector subspace $\mathfrak{g}^p := \{ \xi \in \mathfrak{g} \mid \xi_M(p) \in (\mathcal{V} \cap \mathcal{H})(p) \} \subseteq \mathfrak{g}$. Then

$$\mathfrak{g}^{\mathcal{H}} := \bigcup_{p \in M} \mathfrak{g}^p$$

is a smooth (not necessarily trivial) vector subbundle of the trivial bundle $M \times \mathfrak{g}$ if and only if $\mathcal{H} + \mathcal{V}$ has constant rank on $M$, for instance if $\mathcal{H} + \mathcal{V} = TM$. Indeed, note first that $\mathfrak{g}^{\mathcal{H}} = \Lambda^{-1}(\mathcal{V} \cap \mathcal{H})$, where $\Lambda : M \times \mathfrak{g} \to \mathcal{V}$ is the vector bundle isomorphism over $M$ given by $\Lambda(m, \xi) := \xi_M(m)$. However, since $\mathcal{H} + \mathcal{V}$, $\mathcal{H}$, and $\mathcal{V}$ are subbundles of $TM$, it follows that $\mathcal{V} \cap \mathcal{H}$ is also a subbundle of both $TM$ and $\mathcal{V}$. Consequently, $\mathfrak{g}^{\mathcal{H}} = \Lambda^{-1}(\mathcal{V} \cap \mathcal{H})$ is a subbundle of the trivial vector bundle $M \times \mathfrak{g}$. Thus, if $\mathfrak{g}^{\mathcal{H}}$ is a vector bundle over $M$, then $\mathcal{V} \cap \mathcal{H}$ is also a vector bundle and hence its fibers have constant dimension on $M$. It follows immediately that the rank of $\mathcal{H} + \mathcal{V}$ is also constant on $M$.

For the rest of this subsection we assume that $\mathcal{H} + \mathcal{V}$ has constant rank on $M$ and hence that $\mathfrak{g}^{\mathcal{H}}$ is a vector subbundle of the trivial vector bundle $M \times \mathfrak{g}$. If $\xi^{\mathcal{H}}$ is a smooth section of $\mathfrak{g}^{\mathcal{H}}$, then $\xi(p) := (\xi^{\mathcal{H}}(p))_M(p)$ defines a smooth section of $\mathcal{V} \cap \mathcal{H}$. Conversely, if $\{\xi_1, \ldots, \xi_k\}$ is a chosen basis for the Lie algebra $\mathfrak{g}$, then the vector fields $\xi_1^M, \ldots, \xi_k^M$ are global vector fields on $M$ that don’t vanish and are everywhere linearly independent. Hence, $\xi_1^M, \ldots, \xi_k^M$ are smooth basis vector fields for the bundle $\mathcal{V}$. Every section $\xi$ of $\mathcal{V} \cap \mathcal{H}$ can hence be written $\xi = \sum_{i=1}^k f_i \xi_i^M$ with smooth (local) functions $f_1, \ldots, f_k$, and corresponds exactly to the section $\xi^{\mathcal{H}} = \sum_{i=1}^k f_i \xi_i$ of $\mathfrak{g}^{\mathcal{H}}$.

Note that since $\mathcal{V} \cap \mathcal{H}$ is a subbundle of $TM$ it is a locally finite smooth distribution. Hence, since we have

$$[\Gamma(\mathcal{V} \cap \mathcal{H}), \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V}) = \Gamma((\mathcal{V} \cap \mathcal{H}) + \mathcal{V}),$$

we get with Propositions 1 and 2 for each $p \in M$ there exists a neighborhood $U$ of $p$ and $G$-equivariant spanning sections of $\mathcal{V} \cap \mathcal{H}$ on $U$.

Let $\xi^{\mathcal{H}}$ be a smooth section of $\mathfrak{g}^{\mathcal{H}}$. For all $p \in M$ and all $X \in \mathfrak{X}(M)$ the definition of the corresponding $\xi$ and (38) yield

$$\omega_M(p)(\xi(p), X(p)) = dJ^{\mathcal{H}}(p)(X).$$

(49)
As above, write $\xi^{2c} = \sum_{i=1}^k f_i \xi^i$ with smooth functions $f_1, \ldots, f_k$ and the chosen basis $\{\xi^1, \ldots, \xi^k\}$ of $\mathfrak{g}$. Define the smooth map

$$J^{\xi^2} : M \rightarrow \mathbb{R}
 p \mapsto J^{\xi^2}(p) = \langle J(p), \xi^{2c}(p) \rangle.$$ 

Using (15) and (40) we get

$$J^{\xi^2}(p) = J \sum_{i=1}^k f_i(p) \xi^i = \sum_{i=1}^k f_i(p) J^\xi(p).$$

If $c : (-\varepsilon; \varepsilon) \rightarrow M$ is a solution curve of a vector field $X \in \mathfrak{X}(M)$ with $c(0) = p$, we have

$$dJ^{\xi^{2c}}(p)(X) = \left. \frac{d}{dt} \right|_{t=0} J^{\xi^{2c}}(c(t)) = \frac{d}{dt} \left. \sum_{i=1}^k f_i(c(t)) J^\xi(c(t)) \right|_{t=0}
 = \sum_{i=1}^k d f_i(p) (\dot{c}(0)) J^\xi(p) + \sum_{i=1}^k f_i(p) dJ^\xi(p)(\dot{c}(0))
 = J \sum_{i=1}^k d f_i(p) X^\xi(p) + dJ^{\xi^2}(p)(p)(X)
 = J X^{[\xi^2]}(p) + dJ^{\xi^2}(p)(p)(X),$$

where we write $X^{[\xi^2]} := \sum_{i=1}^k X[f_i] \xi^i$. Thus (49) becomes for all $p \in M$ and all $X \in \mathfrak{X}(M)$,

$$\omega_M(p)(\xi(p), X(p)) = dJ^{\xi^2}(p)(X) - J X^{[\xi^2]}(p).$$

Hence, if the one form $\alpha^\xi \in \Omega^1(M)$ is defined by

$$\alpha^\xi(X) := dJ^{\xi^2}(X) - J X^{[\xi^2]}$$

for all $X \in \mathfrak{X}(M)$, we have $i_X \omega_M = \alpha^\xi$ and so the pair $(\xi, \alpha^\xi)$ is a section of $D$.

Let $h$ be a $G$-invariant Hamiltonian and $X_h \in \Gamma(\mathfrak{h})$ the solution of the implicit Hamiltonian system $(X, dh) \in \Gamma(D)$. Then

$$dJ^{\xi^2}(X_h) - J X_h^{[\xi^2]} = \alpha^\xi(X_h) = \omega_M(\xi, X_h) = -dh(\xi) = 0$$

since $dh(\xi)(p) = \langle dh(p), \xi(p) \rangle = \langle dh(p), \xi^{2c}(p) \rangle_M(p) = 0$ by $G$-invariance of $h$. Thus, we have proved the following result.

**Theorem 7.1** Let $\xi^{2c}$ be a section of $\mathfrak{g}^{2c}$ and $X_h \in \Gamma(\mathfrak{h})$ the solution of the implicit Hamiltonian system $(X, dh) \in \Gamma(D)$, where $h$ is a $G$-invariant Hamiltonian. Then $X_h$ satisfies the Nonholonomic Noether Momentum Equation:

$$dJ^{\xi^2}(X_h) - J X_h^{[\xi^2]} = 0. \quad \text{(50)}$$

Recall from (22) and (23) that $\mathfrak{h}$ is defined in terms of the given Lagrangian $L : TQ \rightarrow \mathbb{R}$ and hence, only the dynamics defined by the corresponding Hamiltonian $H$ is the object of interest. Thus, for each other Lagrangian $L'$ we obtain another distribution $\mathfrak{h}'$.

**Remark 7.2** In [Bloch 2003], Theorem 5.5.4, the Nonholonomic Noether Theorem is formulated in terms of a Lagrangian of a classical mechanical systems (hence equal to the kinetic energy of a metric minus a potential). Let $\mathcal{V}_Q \subseteq TQ$ be the vertical subbundle of the action $\phi : G \times Q \rightarrow Q$. Under the Dimension Assumption $\mathcal{D} + \mathcal{V}_Q = TQ$, the distribution $\mathcal{D} \cap \mathcal{V}_Q$ is a smooth subbundle of $TQ$. Note that this assumption leads automatically to $\mathfrak{f} + \mathcal{V}_{T^*Q} = T^*Q$ and hence to $\mathfrak{h} + \mathcal{V} = TM$.

The smooth vector bundle $\mathfrak{g}^D := \bigcup_{p \in M} \mathfrak{g}^D(q)$ is defined pointwise by

$$\mathfrak{g}^D(q) := \{ \xi \in \mathfrak{g} | \xi_Q(q) \in (\mathcal{V}_Q \cap \mathcal{D})(q) \} \subseteq \mathfrak{g}.$$
Let \((g^D)^*\) be the dual bundle, that is, its fibers are \((g^D)^*(q) := (g^D(q))^*\) for all \(q \in Q\). The nonholonomic momentum map \(J^{nhc} : TQ \to (g^D)^*\) is the vector bundle map over \(Q\) defined by

\[
(J^{nhc}(v_q), \xi) = \langle \mathcal{F}L(v_q), \xi_Q(q) \rangle = \frac{\partial L}{\partial \dot{q}^i}(\xi_Q)^i(q) =: J^{nhc}(\xi)(v_q)
\]

where \(\xi \in g^D(q)\). Let \(\xi^D\) be a section of the bundle \(g^D\). Theorem 5.5.4 in [Bloch (2003)] states that any solution \(c(t) = (q(t), \dot{q}(t))\) of the Lagrange-d’Alembert equations for a nonholonomic system must satisfy, in addition to the given kinematic constraints, the momentum equation

\[
\frac{d}{dt} J^{nhc}(\xi^D(q(t))) (c(t)) = \frac{\partial L}{\partial \dot{q}^i} \left[ \frac{d}{dt}(\xi^D(q(t))) \right]^i.
\]

Since for all \(\xi \in g\) the vector field \(\xi_{T^*Q}\) is the cotangent lift of \(\xi_Q\), we have in local charts

\[
\xi_{T^*Q} = \xi_Q \frac{\partial}{\partial q^i} - \frac{\partial \xi_Q}{\partial p^i} \frac{\partial}{\partial p^j}.
\]

Hence, if \(\xi_Q(q) \in \mathcal{D}(q)\), we get \(\xi_{T^*Q}(\alpha_q) \in \mathcal{F}(\alpha_q)\) for all \(\alpha_q \in T^*_Q Q\) (see [28] for the definition of \(\mathcal{F}\)).

Consequently \(\xi(p) := (\xi^D(\pi_{T^*Q}(p)))_M(p)\) for all \(p \in M\) defines a smooth section \(\xi\) of \(\mathcal{X} \cap \mathcal{K}\) and hence a smooth section \(\xi^{\mathcal{K}}\) of \(g^\mathcal{K}\). Note that \(\xi^{\mathcal{K}}(p) = \xi^D(\pi_{T^*Q}(p))\) for all \(p \in M\) and if \(g^D = \sum_{i=1}^k f_i \xi_Q^i\) with smooth functions \(f_1, \ldots, f_k\), then \(g^{\mathcal{K}} = \sum_{i=1}^k f_i \xi_M^i\) with the smooth functions \(f_i\) defined by \(f_i = i_M^{*} \pi_{T^*Q} f_i\), where \(i_M : M \hookrightarrow T^* Q\) is the inclusion. Let \(X_H\) be a solution of the implicit Hamiltonian system \((X, dH) \in \Gamma(D)\), where \(H\) is the \(G\)-invariant Hamiltonian on \(M\) associated to the Lagrangian \(L\) by the Legendre transformation, and \(p(t)\) an integral curve of \(X_H\). Then \(c(t) := (q(t), \dot{q}(t)) = (\mathcal{F}L)^{-1}(p(t))\) is a solution of the Lagrange-d’Alembert equations. We have for all \(t\)

\[
0 = \left( dJ^{\mathcal{K}}(X_H) - J^{\mathcal{K}}(\xi^{\mathcal{K}}) \right)(p(t)) = \frac{d}{dt} J^{\mathcal{K}}(p(t)) - \sum_{i=1}^k \frac{d}{dt}(F_i(p(t))) J^{\mathcal{K}}(p(t))
\]

\[
= \frac{d}{dt} \left( \mathcal{F}L(c(t)), (\xi^D(q(t)))_Q(q(t)) \right) - \left( p(t), \left( \frac{d}{dt}(F_i(p(t))) \xi^i \right)_Q(q(t)) \right)
\]

\[
= \frac{d}{dt} \left( \mathcal{F}L(c(t)), (\xi^D(q(t)))_Q(q(t)) \right) - \left( \mathcal{F}L(c(t)), \left( \frac{d}{dt}(\xi^D(q(t))) \right)_Q(q(t)) \right)
\]

\[
= \frac{d}{dt} J^{nhc}(\xi^D(q(t)))(c(t)) - \frac{\partial L}{\partial \dot{q}^i} \left[ \frac{d}{dt}(\xi^D(q(t))) \right]^i.
\]

Hence our Nonholonomic Noether Theorem [7.3] is the Hamiltonian version of Theorem 5.5.4 in [Bloch (2003)], that is, [30] and [51] are equivalent.

**Proposition 7.3** Assume that \(\mathcal{V} + \mathcal{K} = TM\). Let \(\xi^{\mathcal{K}}\) be a \(G\)-equivariant section of \(g^{\mathcal{K}}\). Then the corresponding section \((\xi, \alpha^\xi)\) of \(D\) is also \(G\)-equivariant. There are two possibilities:

(i) \(\alpha^\xi = i_{\xi} \omega_M = 0\) on \(\mathcal{V} \cap \mathcal{K}\). Then there exist \(\alpha' \in \Gamma(V^o)\) such that \((\xi, \alpha')\) is a \(G\)-equivariant section of \(D \cap \mathcal{K}^\perp\) and exactly one section \(\alpha \in \Gamma(P_0^{\mathcal{K}d})\) such that \(\pi^* \alpha = \alpha'\). Conversely, each section of \(\Gamma(P_0^{\mathcal{K}d})\) pulls back to a section \(\alpha'\) defined as above and satisfying this condition.

(ii) \(\mathcal{V} \cap \mathcal{K} \not\subseteq \xi_{\omega_M}\) and hence \(\alpha^\xi \neq 0\) on \(\mathcal{V} \cap \mathcal{K}\). Then \(\alpha^\xi\) leads to a momentum equation that doesn’t appear in the reduced implicit Hamiltonian system.

**Proof:** If \(\xi^{\mathcal{K}}\) is \(G\)-equivariant, we have \(\xi^{\mathcal{K}}(g \cdot p) = \text{Ad}_g \xi^{\mathcal{K}}(p)\) for all \(p \in M\) and hence for the corresponding \(\xi\) we get using (9),

\[
(\Phi_g^*(\xi))(p) = T_{g \cdot p} \Phi_{g^{-1}}(\xi)(g \cdot p) = T_{g \cdot p} \Phi_{g^{-1}}(\xi^{\mathcal{K}}(g \cdot p))_M(g \cdot p)
\]

\[
= T_{g \cdot p} \Phi_{g^{-1}}(\text{Ad}_g(\xi^{\mathcal{K}}(p)))_M(g \cdot p)
\]

\[
= (\text{Ad}_{g^{-1}} \circ \text{Ad}_g(\xi^{\mathcal{K}}(p)))_M(p) = (\xi^{\mathcal{K}}(p))_M(p) = \xi(p).
\]
Note that conversely, if $\xi$ is equivariant, then the corresponding section $\xi^{\mathcal{C}}$ of $\mathfrak{g}^*\mathcal{K}$ is $G$-equivariant. Since $\Phi_g^*\omega_M = \omega_M$ for all $g \in G$, the section $(\xi, \alpha^\xi)$ is $G$-equivariant. Since $\mathcal{V} + \mathcal{H} = TM$, if $\alpha^\xi = 0$ on $\mathcal{V} \cap \mathcal{H}$ there exists as in Proposition 4.2(ii) a unique section $\beta \in \Gamma(\mathcal{H}^c)$ such that $\alpha^\xi + \beta \in \Gamma(\mathcal{V}^\circ)$ and hence $(\xi, \alpha^\xi + \beta) \in \Gamma(D \cap \mathcal{K}^\perp)$.

But since $\Phi_g^*D = D$ and $\Phi_g^*\mathcal{K}^\perp = \mathcal{K}^\perp$ we have also

$$(\xi, \alpha^\xi + \Phi_g^*\beta) = (\Phi_g^*\xi, \Phi_g^*\alpha^\xi + \Phi_g^*\beta) \in \Gamma(D \cap \mathcal{K}^\perp)$$

for all $g \in G$ and, because $\beta$ is unique, we get $\Phi_g^*\beta = \beta$. Hence the first statement of (i) holds with $\alpha' := \alpha^\xi + \beta$. But because $\xi \in \Gamma(\mathcal{V})$, the section of $D_{\text{red}}$ corresponding to $(\xi, \alpha')$ will be $(0, \tilde{\alpha})$ with $\tilde{\alpha} \in \Omega^1(M)$ such that $\pi^*\tilde{\alpha} = \alpha'$.

On the other hand, if we choose a non-zero section $\tilde{\alpha}$ of $P_0^0$, the codistribution associated to the reduced Dirac structure on $M$, we have $(0, \tilde{\alpha}) \in \Gamma(D_{\text{red}})$ and we find $X \in \Gamma(\mathcal{H})$ such that $X \sim 0$ and $(X, \pi^*\tilde{\alpha}) \in \Gamma(D \cap \mathcal{K}^\perp)$. If $X = 0$, then we have $\pi^*\tilde{\alpha} = 0$ on $\mathcal{V} + \mathcal{H} = TM$, contradicting the fact that $\tilde{\alpha}$ is a non-zero section of $P_0^0$. Therefore $X$ is a non-zero vector field lying in $\Gamma(\mathcal{H} \cap \mathcal{V})$ with $1_X \omega_M = 0$ on $\mathcal{H} \cap \mathcal{V}$. We conclude from this that the sections of $P_0^0$ pull back exactly to the $G$-equivariant sections $\alpha^\xi + \beta \in \Gamma(\mathcal{V}^\circ)$ induced by sections $\xi$ of $(\mathcal{V} \cap \mathcal{H}) \cap (\mathcal{V} \cap \mathcal{H})^\omega_M$.

If $\mathcal{V} \cap \mathcal{H} \not\subseteq \xi^{\mathcal{C}}$, and hence $\xi^\xi \neq 0$ on $\mathcal{V} \cap \mathcal{H}$, then there is no $\beta \in \Gamma(\mathcal{H}^c)$ such that $(\xi, \alpha^\xi + \beta) \in \Gamma(D \cap \mathcal{K}^\perp)$ and (ii) follows immediately.

**Definition 7.4** We will call Nonholonomic Noether Equation a section $\alpha^\xi$ corresponding to a smooth section $\xi$ of $\mathcal{V} \cap \mathcal{H}$. A $\mathcal{H}$-modified Nonholonomic Noether Equation is a 1-form $\alpha^\xi \in \Omega^1(M)$ that can be written $\alpha^\xi = \alpha^\xi + \beta$ with a nonholonomic Noether equation $\alpha^\xi$ and $\beta \in \Gamma(\mathcal{H}^c)$. A Descending (\mathcal{H}-Modified) Nonholonomic Noether Equation is a (\mathcal{H}-modified) nonholonomic Noether equation as in Proposition 7.3(i).

Note that because of the $\beta$-part of a descending $\mathcal{H}$-modified nonholonomic Noether equation, sections of $P_0^0$ don’t pull back exactly to sections $\alpha^\xi$ associated to sections $\xi^{\mathcal{C}}$ as in Theorem 7.1 (the nonholonomic Noether equations). It is possible that they pull back to one-forms that coincide only on $\mathcal{H}$ with some $\alpha^\xi$.

**Proposition 7.5** The codistribution spanned by the Noether equations which descends to the quotient $M/G$ is given by

$$\pi_2(D \cap (\mathcal{V} \oplus \mathcal{V}^c)) = (\mathcal{V} \cap \mathcal{H}^c) \cap \mathcal{V}^c$$

where $b : TM \to T^*M$ is associated to $\omega_M$.

**Proof:** We have seen that a descending (\mathcal{H}-modified) nonholonomic Noether equation $\alpha$ is a $G$-invariant section of $\mathcal{V}^\circ$ such that there exists a $G$-equivariant section $X$ of $\mathcal{V} \cap \mathcal{H}$ with $(X, \alpha) \in \Gamma(D \cap \mathcal{K}^\perp)$. So we only have to show equality (52). Let $\alpha$ be a section of the left-hand side. Then there exists $X \in \Gamma(\mathcal{V} \cap \mathcal{H})$ such that $(X, \alpha) \in \Gamma(D \cap (\mathcal{V} \oplus \mathcal{V}^c))$ and hence there exists $\beta \in \Gamma(\mathcal{H}^c)$ such that $\alpha = 1_X \omega_M + \beta$. Therefore $\alpha \in \Gamma(\mathcal{V} \cap \mathcal{H}^c \cap \mathcal{V}^c)$ and hence $\alpha \in \Gamma(\mathcal{V}^c)$ with $X \in \Gamma(\mathcal{V} \cap \mathcal{H})$ and $\beta \in \Gamma(\mathcal{H}^c)$. But this means that $(X, \alpha)$ is a section of $D \cap (\mathcal{V} \oplus \mathcal{V}^c)$.

**Example 7.6** We compute $\alpha$ for the constrained particle (see [1,3]). In this example, $Q = \mathbb{R}^3$, $M := \{(x, y, z, p_x, p_y, p_z) \mid p_x = y p_y \} \subseteq T^*Q = \mathbb{R}^3 \times \mathbb{R}^3$, and $\mathcal{V} \cap \mathcal{H} = \text{span}\{\partial_x + y \partial_z\}$. If $\xi^1 := (1, 0)$ and $\xi^2 := (0, 1)$ is the chosen basis of $\mathfrak{g} = \mathbb{R}^2$, then $(1, 0)_M = \partial_x$ and $(0, 1)_M = \partial_y$ so that $\xi^f := \text{span}\{(1, 0) + y(0, 1)\}$ is the fiber of the vector bundle $\mathfrak{g}^\mathcal{K}$ at the point $(x, y, z, p_x, p_y) \in M$. Therefore, any section $\xi^f$ of $\mathfrak{g}^\mathcal{K}$ has the form $\xi^f(x, y, z, p_x, p_y) = f(x, y, z, p_x, p_y) (1, 0) + y(0, 1)$, where $f \in C^\infty(M)$. Consequently

$$\xi(x, y, z, p_x, p_y) = \xi^f(x, y, z, p_x, p_y) \big|_{\mathcal{M}} = f(x, y, z, p_x, p_y) (\partial_x + y \partial_z).$$

The components of the momentum map $J : T^*Q \to \mathfrak{g}^*$ are $J^{(1,0)} = p_x$ and $J^{(0,1)} = p_y$, so the restrictions to $M$ of these functions are $J^{(1,0)} = p_x$ and $J^{(0,1)} = y p_y$. Therefore $J^{\xi^f}(x, y, z, p_x, p_y) = f(x, y, z, p_x, p_y) p_x (1 + y^2)$ and if $X \in \mathfrak{X}(M)$, then $X[f(1, 0) + y f(0, 1)] = X[f](1, 0) + X[y f](0, 1)$ and so

$$J^X[f(1,0)+yf(0,1)] = X[f]p_x + X[yf]yp_x = p_x (1 + y^2) X[f] + yp_x f X[y].$$
The 1-form on $M$ which applied to $X$ yields the right hand side is $p_x(1 + y^2) df + p_x fdy$ and hence
\[\alpha^\xi(x, y, z, p_x, p_y) = \mathbf{d}J^\xi(x, y, z, p_x, p_y) - p_x(1 + y^2) df(x, y, z, p_x, p_y) - p_x f(x, y, z, p_x, p_y) dy\]
\[= f(x, y, z, p_x, p_y) ((1 + y^2) dp_x + y p_y dy).\]

So the section spanning the codistribution $\mathcal{P}^\text{red}_0$ in this example is $(1 + y^2) dp_x + y p_y dy$, as (32). It is easy to see that in this case $V \subseteq (V \cap \mathcal{H})^{\omega_M}$ and hence the nonholonomic Noether equation descends to the quotient.

\[\Diamond\]

### 7.2 The reaction-annihilator distribution

An important problem is to decide when the Nonholonomic Noether momentum equation gives a constant of motion rather than an equation of motion. We have to distinguish between two cases:

(i) The section $\xi^\eta$ is constant, i.e., $\xi^\eta(p) = \xi$ for all $p \in M$, where $\xi \in \mathfrak{g}$. Then $\xi(p) = \xi_M(p)$ and so we have $\alpha^\xi = \mathbf{d}J^\xi$, so $J^\xi$ is a constant of the motion. We will see below that sometimes one can find $\eta \in \mathfrak{g}$ such that $J^\eta$ is a constant of motion for all solutions of $G$-invariant Hamiltonians, but $\eta_M$ is not a section of $\mathcal{V} \cap \mathcal{H}$ (see also Fasso et al. (2007)).

(ii) The other case is that of gauge symmetries, that is, non-constant sections of $\mathfrak{g}^\mathcal{H}$ that yield constants of motion (see Fasso et al. (2007)). Note that if $\xi^\eta = \sum_{i=1}^k f_i \xi_i$ then it leads to a constant of motion if one of the corresponding forms $\alpha^\xi + \beta$ is exact, that is, we can find $f \in C^\infty(M)$ such that $df = \beta + \sum_{i=1}^k f_i df(\xi_i)$. However, we do not know of any other characterization of the section so that the momentum equation gives constants of motion rather than an equation of motion.

In the reduction method for nonholonomic systems, the first step is to compute the horizontal annihilator $\mathbb{U}$ of $\mathcal{V}$, that is, the distribution $\mathbb{U} = (\mathcal{V} \cap \mathcal{H})^{\omega_M} \cap \mathcal{K} \subseteq TM \subseteq TT^* Q$ (see (25)). We have seen in Proposition 4.2(ii) that any section of $\mathbb{U}$ corresponds to one section of $D \cap \mathcal{K}^\perp$: for each $X \in \Gamma(\mathbb{U})$ there exists $\alpha \in \Gamma(\mathcal{V})$ such that $(X, \alpha) \in \Gamma(D)$ and hence $\alpha - i_X \omega_M \in \Gamma(\mathcal{H}^C)$. So the method of finding a section $\alpha \in \Gamma(\mathcal{H}^C)$ associated to $X \in \Gamma(\mathbb{U})$ is the same as determining $\beta \in \Gamma(\mathcal{H}^C)$ such that $i_X \omega_M + \beta =: \alpha \in \Gamma(\mathcal{V})$. As we have seen in (14.4) case 3, sometimes not the whole of $\mathcal{H}^C$ is needed in this construction. This is why we introduce the new codistribution $\mathcal{R}$ on $M$ whose fiber at $p \in M$ equals

\[\mathcal{R}(p) = \{\beta(p) \mid \beta \in \Gamma(\mathcal{H}^C)\}
\text{ and there is some } X \in \Gamma(\mathbb{U}) \text{ such that } \beta + i_X \omega_M \in \Gamma(\mathcal{V}) \} \subseteq \mathcal{H}^C.\] (53)

In general, $\mathcal{R}$ is strictly included in $\mathcal{H}^C$.

If $h \in C^\infty(M)^G$ is an admissible function, then there exists $X_h \in \Gamma(\mathcal{H})$ such that $(X_h, dh) \in \Gamma(D)$. Recall that $X_h$ is unique since $G_0 = \{0\}$. In addition, since $dh \in \Gamma(\mathcal{V})$, we have $(X_h, dh) \in \Gamma(D \cap K^\perp)$ and hence $X_h \in \Gamma(\mathbb{U})$. Thus, there exists $\beta \in \Gamma(\mathcal{H}^C)$ such that $dh = i_{X_h} \omega_M + \beta$. This is exactly the Hamilton equation for the given nonholonomic system (with the Hamiltonian $h$) and we have $\beta \in \Gamma(\mathcal{R})$, often interpreted as the reaction force. In fact $\mathcal{R}^c \subset TM$ is the analogue of the reaction-annihilator distribution of Fasso et al. (2007).

**Proposition 7.7** We have

\[\mathbb{b}(\mathbb{U}) \oplus \mathcal{R} = \mathcal{V}^\text{c} + \mathcal{R},\]

where $\mathbb{b} : TM \rightarrow T^* M$ corresponds to $\omega_M$.

**Proof:** The sum on the left hand side is direct since if $X \in \Gamma(\mathbb{U})$, then $X \in \Gamma(\mathcal{H})$ and hence $i_X \omega_M \notin \Gamma(\mathcal{H}^C)$ because $\omega_M|_{\mathcal{H} \cap \mathcal{H}^C}$ is nondegenerate. Thus $i_X \omega_M \notin \Gamma(\mathcal{R}) \subseteq \Gamma(\mathcal{H}^C)$. Second, recall that $P_1 = T^* M$ (see Proposition 4.2(i)) so for all $\alpha \in \Gamma(\mathcal{V}^\text{c})$ we find $X \in \Gamma(\mathcal{H})$ (actually $X \in \Gamma(\mathbb{U})$) such that $(X, \alpha) \in \Gamma(D)$. Thus $\pi_2(D \cap K^\perp) = \mathcal{V}^\text{c}$.

Now we are ready to prove the formula in the statement. If $X \in \Gamma(\mathbb{U})$, the considerations in (14.2) show that there exists $\beta \in \Gamma(\mathcal{H}^C)$ such that $i_X \omega_M + \beta \in \Gamma(\mathcal{V}^\text{c})$. The definition (53) of $\mathcal{R}$ yields directly that $\beta \in \Gamma(\mathcal{R})$. This shows $\mathbb{b}(\mathbb{U}) \oplus \mathcal{R} \subseteq \mathcal{V}^\text{c} + \mathcal{R}$. For the other inclusion, choose $\alpha \in \Gamma(\mathcal{V}^\text{c})$ and $X \in \Gamma(\mathbb{U})$ such that $(X, \alpha) \in \Gamma(D \cap K^\perp)$. Then the definition of $D$ yields $\beta := \alpha - i_X \omega_M \in \Gamma(\mathcal{H}^C)$ and again, using (53), we conclude that $\beta \in \Gamma(\mathcal{R})$. \[\square\]
The last Lemma leads directly to the equality
\[ U^\omega_M \cap \mathcal{R}^o = \mathcal{V} \cap \mathcal{R}^o. \]

Note that \( U^\omega_M = ((\mathcal{H} \cap \mathcal{V})^\omega_M \cap \mathcal{H})^\omega_M = (\mathcal{H} \cap \mathcal{V}) + \mathcal{H}^\omega_M \) since the kernel of \( \omega_M \) lies in \( \mathcal{H}^\omega_M \).

Now we are able to state the main theorem of this subsection which is the Hamiltonian analogue of the main statement of Fassò et al. (2007).

**Theorem 7.8** Let \( \xi \in \mathfrak{g} \). Then the function \( J^\xi \) is a constant of motion for every \( G \)-invariant Hamiltonian \( h \) if and only if \( \xi \in \Gamma(\mathcal{V} \cap \mathcal{R}^o) \).

**Proof:** Choose \( \xi \in \mathfrak{g} \) such that \( \xi_M \in \Gamma(\mathcal{V} \cap \mathcal{R}^o) \). We have seen in the preceding section that \( \mathbf{i}_{\xi_M} \omega_M = dJ^\xi \). For an arbitrary \( X \in \Gamma(\mathcal{U}) \) choose \( \beta \in \Gamma(\mathcal{H}) \) with \( \mathbf{i}_X \omega_M + \beta =: \alpha \in \Gamma(\mathcal{V}^o) \) and get
\[ dJ^\xi(X) = \omega_M(\xi_M, X) = \beta(\xi_M) - \alpha(\xi_M) = 0. \]
This yields the statement since for all \( G \)-invariant Hamiltonian \( h \) the (unique) solution \( X_h \) of the implicit Hamiltonian system \((X, dh) \in \Gamma(D)\) is a section of \( \mathcal{U} \) (with \( \alpha = dh \) the corresponding section of \( \mathcal{V}^o \) and \( \beta = dh - \mathbf{i}_X \omega_M \)). For the converse implication, choose \( \xi \in \mathfrak{g} \) such that \( J^\xi \) is a constant of the motion for the solution curves of every \( G \)-invariant Hamiltonian. Note that since \( \mathcal{V} \) is an involutive subbundle of \( TM \), the exterior derivatives of all \( G \)-invariant functions span pointwise \( \mathcal{V}^o \) and hence the corresponding solutions span \( \mathcal{U} \). This yields \( dJ^\xi = 0 \) on \( \mathcal{U} \). If we choose \( \beta \in \Gamma(\mathcal{H}) \), there exists \( X \in \Gamma(\mathcal{U}) \) such that \( \mathbf{i}_X \omega_M + \beta =: \alpha \in \Gamma(\mathcal{V}^o) \). Hence we get
\[ 0 = (\mathbf{i}_X \omega_M + \beta)(\xi_M) = \omega_M(X, \xi_M) + \beta(\xi_M) = -dJ^\xi(X) + \beta(\xi_M) = 0 + \beta(\xi_M) \]
and therefore \( \xi_M \in \Gamma(\mathcal{R}^o \cap \mathcal{V}) \). \( \square \)

**Corollary 7.9** Assume that \( \mathcal{H} + \mathcal{V} \) has constant rank on \( M \). If \( dJ^\xi = 0 \) on \( \mathcal{V} \cap \mathcal{H} \) there exist \( \beta \in \Gamma(\mathcal{H}^o) \) and \( \eta \in \Gamma(\mathcal{V} \cap \mathcal{H}) \) such that \( \alpha^o = dJ^\xi + \beta \).

**Proof:** Since \( P_1 = T^*M \) (see Proposition 4.1(i)) there exists \( X \in \Gamma(\mathcal{H}) \) such that \( (X, dJ^\xi) \in \Gamma(D) \). Hence we have \( dJ^\xi = \mathbf{i}_X \omega_M + \beta' \) with \( \beta' \in \Gamma(\mathcal{H}^o) \) and since \( dJ^\xi = 0 \) on \( \mathcal{U} \) we get \( X \in \mathcal{U}^\omega_M = (\mathcal{H} \cap \mathcal{V}) + \mathcal{H}^\omega_M \). Write \( X = V + Y \) with \( V \in \Gamma(\mathcal{H} \cap \mathcal{V}) \) and \( Y \in \Gamma(\mathcal{H}^\omega_M) \). Since \( X \) and \( V \) are sections of \( \mathcal{H} \), then so is \( Y \). But since \( \mathcal{H} \cap \mathcal{H}^\omega_M = \{0\} \), this yields \( Y = 0 \) and hence \( X \in \Gamma(\mathcal{H} \cap \mathcal{V}) \). We find \( \eta^o \in \Gamma(\mathcal{g}^H) \) such that the corresponding section \( \eta \in \Gamma(\mathcal{V} \cap \mathcal{H}) \) is equal to \( X \) and therefore \( (\eta, dJ^\xi) \in \Gamma(D) \). We get \( \alpha^o = dJ^\xi + \beta \) with \( \beta \in \Gamma(\mathcal{H}^o) \), a nonholonomic Noether equation corresponding to the section \( \eta^o \in \Gamma(\mathcal{g}^H) \). \( \square \)

### 7.3 Optimal momentum map for nonholonomic mechanical systems

In this and the next subsection we assume that \( \mathcal{H} + \mathcal{V} \) has constant rank on \( M \). Recall from Remark 4.8 that this implies that \( \mathcal{V} \cap \mathcal{H} \) and \( \mathcal{U} \) also have constant rank on \( M \).

We show in this subsection that, under certain integrability assumptions, it is possible to restrict the system to “level sets” given by the nonholonomic momentum equations and then perform reduction.

Consider the distribution where all \( \alpha^\xi + \alpha' \) defined as in Proposition 4.8 vanish, namely
\[ D_G := \left[ \pi_2(D \cap (\mathcal{V} \oplus \mathcal{V}^o)) \right]^{\omega_G} = \left[ (\mathcal{g}(\mathcal{V} \cap \mathcal{H}) + \mathcal{V}) \cap \mathcal{V}^o \right]^o = (\mathcal{V} \cap \mathcal{H})^\omega_M \cap \mathcal{H} + \mathcal{V} = \mathcal{U} + \mathcal{V}, \]
where \( \mathcal{U} := (\mathcal{V} \cap \mathcal{H})^\omega_M \cap \mathcal{H} \subseteq TM \subseteq TT^*Q \) is the horizontal annihilator of \( \mathcal{V} \) (see (25)). Note that \( D_G \cap \mathcal{H} = \mathcal{U} + (\mathcal{V} \cap \mathcal{H}) \). If \( D_G \) is integrable, its leaves are the level sets of the constants of motion and equations of motion given by the Nonholonomic Noether Theorem 7.1 for sections \( \xi \) of \( (\mathcal{V} \cap \mathcal{H}) \cap (\mathcal{V} \cap \mathcal{H})^\omega_M \): the fiber at \( m \in M \) of the distribution \( \pi_2(D \cap (\mathcal{V} \oplus \mathcal{V}^o)) \) equals
\[ \{ \alpha(m) \mid \alpha \in \Gamma(\mathcal{V}^o) \text{ and there exists } X \in \Gamma(\mathcal{V}) \text{ such that } (X, \alpha) \in \Gamma(D) \}. \]
Note that if this distribution is spanned by closed 1-forms, hence locally exact 1-forms, then it can be written as
\[ \{ (df)(m) \mid f \in C^\infty(M)^G \text{ and there exists } X_f \in \Gamma(\mathcal{V}) \text{ such that } (X_f, df) \in \Gamma(D) \}. \]
For every \( m \in M \) we have
\[
\dim [(\mathcal{V}(m) \cap \mathcal{H}(m))^{\omega_N} \cap (\mathcal{V}(m) \cap \mathcal{H}(m))] = \dim [(\mathcal{U}(m) \cap (\mathcal{V}(m) \cap \mathcal{H}(m))] = \dim [(\mathcal{U}(m) \cap \mathcal{V}(m)] = \dim \mathcal{U}(m) + \dim \mathcal{V}(m) - \dim \mathcal{D}_G(m).
\]

Recall that \( \mathcal{U} \) and \( \mathcal{H} \cap \mathcal{V} \) are vector subbundles of \( TM \). If, in addition, \( \mathcal{D}_G \) is integrable, then its fibers \( \mathcal{D}_G(m) \) have constant dimension along the leaves of the leaves of the generalized foliation determined by \( \mathcal{D}_G \) and so the computation above shows that the fibers of \( (\mathcal{V} \cap \mathcal{H})^{\omega_N} \cap (\mathcal{V} \cap \mathcal{H}) \) along a leaf of \( \mathcal{D}_G \) are constant. Thus, the same is true for the fibers of \( \mathcal{D}_G \cap \mathcal{V} = \mathcal{U} + (\mathcal{V} \cap \mathcal{H}) \) since \( \mathcal{V} \cap \mathcal{H} = (\mathcal{V} \cap \mathcal{H})^{\omega_N} \cap (\mathcal{V} \cap \mathcal{H}) \). We shall use this fact in the next subsection where we describe the induced Dirac structure on a leaf.

In order to restrict the system to the leaves of the distribution \( \mathcal{D}_G \) and then perform reduction, we have to show several statements, the analogues of those needed for the Dirac optimal reduction. Since \( \Phi_g^* \omega_M = \omega_M \) for all \( g \in G \), the proof of the following proposition follows easily.

**Proposition 7.10** The distribution \( (\mathcal{V} \cap \mathcal{H})^{\omega_M} \) is \( G \)-invariant in the sense that
\[
\Phi_g^* ((\mathcal{V} \cap \mathcal{H})^{\omega_M}) = (\mathcal{V} \cap \mathcal{H})^{\omega_M}
\]
for all \( g \in G \). Since \( \mathcal{V} \) and \( \mathcal{H} \) are also \( G \)-invariant, it follows that the distribution \( \mathcal{D}_G = ((\mathcal{V} \cap \mathcal{H})^{\omega_M} \cap \mathcal{H}) + \mathcal{V} \) is \( G \)-invariant.

If \( \mathcal{D}_G \) is integrable, define like in the nonholonomic optimal momentum map
\[
\mathcal{J} : M \rightarrow M/\mathcal{D}_G.
\]

We have a result analogous to Lemma 5.3.

**Lemma 7.11** If \( m \) and \( m' \) are in the same leaf of \( \mathcal{D}_G \), i.e., if there is a \( X \in \Gamma(\mathcal{D}_G) \) with flow \( F^X \) such that \( F^X_t(m) = m' \) for some \( t > 0 \), then \( \Phi_g(m) \) and \( \Phi_g(m') \) are in the same leaf of \( \mathcal{D}_G \) for all \( g \in G \). Hence there is a well defined action of \( G \) on \( M/\mathcal{D}_G \):
\[
\bar{\Phi} : G \times M/\mathcal{D}_G \rightarrow M/\mathcal{D}_G
\]
\[
\bar{\Phi}_g(\mathcal{J}(m)) = \mathcal{J}(g \cdot m)
\]

For all \( \rho \in M/\mathcal{D}_G \), the isotropy subgroup of \( \rho \) contains \( G^0 \) (the connected component of the identity in \( G \)).

**Proof:** Let \( g \in G \), \( m, m' \in M \) be in the same leaf of \( \mathcal{D}_G \), \( X \in \Gamma(\mathcal{D}_G) \), and \( F^X \) the flow of \( X \). For all \( s \in [0, t] \) we have
\[
\frac{d}{ds} (\Phi_g \circ F^X)(m) = T_{F^X(m)} \Phi_g (X(F^X_s(m))) = (\Phi_g \circ F^X_s)(g \cdot F^X_s(m)) \in \mathcal{D}_G(g \cdot F^X_s(m)).
\]

Hence the curve \( c(s) = (\Phi_g \circ F^X_s)(m) \) connecting \( c(0) = \Phi_g(m) \) and \( c(t) = \Phi_g(m') \) lies entirely in the leaf of \( \mathcal{D}_G \) through the point \( \Phi_g(m) \) and the assertion follows.

The Lie group \( G^0 \) is generated as a group by the exponential of an open neighborhood of \( 0 \in \mathfrak{g} \). Thus, we can assume without loss of generality, that for any \( g \in G^0 \) and \( m \in M \), there exists some \( \xi \in \mathfrak{g} \) such that the curve \( \gamma : [0, t] \rightarrow M, \gamma(s) = \Phi_{\exp(\xi s)}(m) \), has endpoints \( m \) and \( g \cdot m \) (in reality, the points \( m \) and \( g \cdot m \) can be joined with finitely many such curves). For all \( s \in [0, t] \), we have \( \dot{\gamma}(s) = \xi g(\gamma(s)) \in \mathcal{D}_G(\gamma(s)) \) and, arguing as above, we conclude that the whole curve \( \gamma([0, t]) \) lies in the leaf of \( \mathcal{D}_G \) through \( m \). Hence, if \( \rho = \mathcal{J}(m) \), the equality \( \Phi_g(\mathcal{J}(m)) = \mathcal{J}(g \cdot m) = \mathcal{J}(m) \) proves the statement. \( \square \)

**Remark 7.12** The last statement shows that for all \( \rho \in M/\mathcal{D}_G \), the isotropy subgroup \( G_\rho \) is the union of connected components of \( G \) and is therefore closed in \( G \). This implies that the Lie group \( G_\rho \) acts properly on the leaf \( \mathcal{J}^{-1}(\rho) \). It is obvious that this action is also free. Recall that in the Optimal Point Reduction Theorem 6.1 the properness of the \( G_\rho \)-action on \( \mathcal{J}^{-1}(\rho) \) was not guaranteed. The reason why in the nonholonomic case this action is always proper is the inclusion \( \mathcal{V} \subset \mathcal{D}_G \). \( \triangle \)

**Remark 7.13** Note that if the nonholonomic system satisfies \( \mathcal{H} \oplus \mathcal{V} = TM \), then the bundle \( \mathcal{U} \) is given by \( \mathcal{U} = \{0\}^{\omega_M} \cap \mathcal{H} = \mathcal{H} \) and hence \( \mathcal{D}_G = \mathcal{U} + \mathcal{V} = TM \) is trivially integrable with the connected components of \( M \) as integral leaves. Hence, if \( M \) is connected, the method of reduction presented in the next subsection leads to the same reduced Dirac manifold as the Dirac reduction method of \( [\mathbb{L}2] \). \( \triangle \)
7.4 Optimal reduction for nonholonomic systems

Restrict the vector subbundles $\mathcal{V}, \mathcal{H} := TM \cap (T\pi_T^{-1}(\mathcal{D}))$, and $\mathcal{U} = (\mathcal{V} \cap \mathcal{H})^\perp \cap \mathcal{H}$ of $TM$ to vector bundles $\mathcal{V}_\rho, \mathcal{K}_\rho,$ and $\mathcal{U}_\rho$ on the manifold $J^{-1}(\rho)$. Since the distribution $(\mathcal{H} \cap (\mathcal{V} \cap \mathcal{H})^\perp) \cap \mathcal{H} = \mathcal{U} \cup (\mathcal{V} \cap \mathcal{H}) \subseteq D_G$ is constant dimensional on the leaves of $D_G$, the Dirac structure on a leaf $J^{-1}(\rho)$ of $D_G$ is given by

$$\mathcal{H}_{J^{-1}(\rho)}(m) = \{(X(m), \alpha) \in T\mathcal{H}_{J^{-1}(\rho)}(\rho) \cap T^*J^{-1}(\rho) \mid X \in \Gamma(\mathcal{U}_\rho + (\mathcal{V}_\rho \cap \mathcal{K}_\rho)),$$

$$\alpha - i_X\omega_{J^{-1}(\rho)}(\rho) \in \Gamma((\mathcal{U}_\rho + (\mathcal{V}_\rho \cap \mathcal{K}_\rho))^\circ) \}$$

for all $m \in J^{-1}(\rho)$ (see Blankenstein and van der Schaft (2001)); here $i_\rho : J^{-1}(\rho) \hookrightarrow M$ is the inclusion and $\omega_{J^{-1}(\rho)} := i_\rho^*\omega_M$. Since for all $\rho \in M/D_G$, the isotropy subgroup $G_\rho$ contains $G^*$, the distribution $\mathcal{V}_\rho$ spanned by the fundamental vector fields of the action of $G$ on $J^{-1}(\rho)$ is $\mathcal{V}_\rho = \mathcal{V}|_{J^{-1}(\rho)}$.

Lemma 7.14 Let $\mathcal{K}_\rho = \mathcal{V}_\rho \oplus \{0\}$ and $\mathcal{K}_{J^{-1}(\rho)} = T\mathcal{H}_{J^{-1}(\rho)}(\rho) \cap \mathcal{V}_{\rho}^\perp$ as in (53). Then $\mathcal{D}_{J^{-1}(\rho)} \cap \mathcal{K}_{J^{-1}(\rho)}$ is a vector bundle over $J^{-1}(\rho)$.

Proof: Since $\mathcal{H} + \mathcal{V}$ has constant rank on the $n$-dimensional manifold $M$, recall from Remark 4.3 that $D \cap \mathcal{K}^\perp$ is a vector bundle on $M$. We denote $r = \text{rank} \mathcal{H},$ $r = \text{rank} \mathcal{H},$ $l = \text{rank} \mathcal{V} \cap \mathcal{K}^\perp$, $u = \text{rank} \mathcal{U}$, and $s = \text{rank} \mathcal{U} \cap (\mathcal{V} \cap \mathcal{K})^\perp J^{-1}(\rho)$. Let $m \in J^{-1}(\rho)$. As in Remark 4.3, choose local basis fields $H_1, \ldots, H_s$ for $\mathcal{K}$ and local basis $1$-forms $\beta_1, \ldots, \beta_{n-r}$ for $\mathcal{K}^\perp$ defined on a neighborhood $U$ of $m$ in $M$. Assume that $H_1, \ldots, H_s$ are local basis fields for $\mathcal{V}_\rho, \mathcal{K}_\rho,$ with $u \leq s \leq r$, are basis fields for $\mathcal{U} \cup (\mathcal{V} \cap \mathcal{K})$, and $\beta_1, \ldots, \beta_l$ is a basis of $\mathcal{V} \cap \mathcal{K}^\perp = (\mathcal{V} \cap \mathcal{K})^\circ$. Note that the $1$-forms $\beta_1, \ldots, \beta_l$ vanish on $\mathcal{U} \cap \mathcal{V} + \mathcal{K}$ and that $\beta_{l+1}, \ldots, \beta_{n-r}$ don’t vanish on $\mathcal{U} \cap \mathcal{V}$ (otherwise we would have $\beta_1 \in \Gamma(\mathcal{U} \cap \mathcal{V} \cap \mathcal{K})^\perp \cap \mathcal{V} \cap \mathcal{K}^\perp \cap \mathcal{V} \cap \mathcal{K}^\perp$ for $j = l + 1, \ldots, n - r$, in contradiction to the choice of $\beta_1, \ldots, \beta_{n-r}$). The Dirac structure $\mathcal{D}_{J^{-1}(\rho)}$ is then given on $U \cap J^{-1}(\rho)$ by (see (53) or Blankenstein (2001))

$$\text{span} \left\{ (\tilde{H}_1, \tilde{i}_\rho^*H_1 |_M, \ldots, \tilde{H}_s, \tilde{i}_\rho^*H_1 |_M, (0, \tilde{i}_\rho^*\beta_1), \ldots, (0, \tilde{i}_\rho^*\beta_{n-r}) \right\},$$

(54)

where $\tilde{H}_1, \ldots, \tilde{H}_s$ are vector fields on $U \cap J^{-1}(\rho)$ such that $\tilde{H}_i \sim \rho_i H_i$ for $i = 1, \ldots, s$. Note that $\tilde{i}_\rho^*H_i |_M = \tilde{i}_\rho^*H_i |_\mathcal{H}$.

If $(X, \alpha) \in \mathcal{D}_{J^{-1}(\rho)} \cap (T\mathcal{H}_{J^{-1}(\rho)}(\rho) \cap \mathcal{V}_{\rho}^\perp)$ then $X \in \Gamma(\mathcal{U}_\rho + (\mathcal{V}_\rho \cap \mathcal{K}_\rho)), \alpha \in \Gamma(\mathcal{V}_{\rho}^\perp)$, and $i_X\omega_{J^{-1}(\rho)} - \alpha \in \Gamma((\mathcal{U}_\rho + (\mathcal{V}_\rho \cap \mathcal{K}_\rho))^\circ)$. This is only possible if $i_X\omega_{J^{-1}(\rho)} = 0$ on

$$\mathcal{V}_\rho \cap (\mathcal{U}_\rho + (\mathcal{V}_\rho \cap \mathcal{K}_\rho)) = (\mathcal{V}_\rho \cap \mathcal{U}_\rho) + (\mathcal{V}_\rho \cap \mathcal{K}_\rho) = \mathcal{L}_\rho \quad \text{where} \quad \mathcal{L} := [(\mathcal{V} \cap \mathcal{K}) \cap (\mathcal{V} \cap \mathcal{K})^\perp] + (\mathcal{V} \cap \mathcal{K}).$$

We have two different cases. First, if $X \in \Gamma(\mathcal{U}_\rho)$ then for all $m \in J^{-1}(\rho)$ and $V(m) \in \mathcal{L}(m)$ we have necessarily

$$(i_X\omega_{J^{-1}(\rho)}(m))V(m) = (i_\rho^*\omega_M)(m)(X(m), V(m)) = \omega_M(i_\rho(m))(T_m i_\rho X(m), T_m i_\rho V(m)) = 0$$

where we have used

$$T_m i_\rho X(m) \in (\mathcal{K} \cap (\mathcal{V} \cap \mathcal{K})^\perp)(i_\rho(m)) \quad \text{and} \quad T_m i_\rho V(m) \in \mathcal{L}(i_\rho(m))$$

and the definition of $\mathcal{L}$. Hence, for all $X \in \Gamma(\mathcal{U}_\rho)$ we have $i_X\omega_{J^{-1}(\rho)} |_{\mathcal{L}_\rho} = 0$ and hence we find $\alpha \in \Gamma(\mathcal{V}_{\rho}^\perp)$ such that $(X, \alpha) \in \Gamma(D_{J^{-1}(\rho)})$. Second, for a section $X$ of $\mathcal{V} \cap \mathcal{K}$ that doesn’t take values in $\mathcal{U}_\rho$, the $1$-form $i_X\omega_{J^{-1}(\rho)}$ doesn’t vanish on $\mathcal{V}_\rho \cap \mathcal{K}_\rho$ and thus neither on $\mathcal{L}_\rho$. Consequently, the sections of $D \cap \mathcal{K}_{J^{-1}(\rho)}^\perp$ have as first component a section of $\mathcal{U}_\rho$. Since for $i = l + 1, \ldots, n - r$ we have $i_\rho^*\beta_i \notin \Gamma(\mathcal{V}_{\rho}^\perp)$, we get

$$D \cap \mathcal{K}_{J^{-1}(\rho)}^\perp = \text{span} \left\{ (\tilde{H}_1, \tilde{i}_\rho^*H_1 |_M + \sum_{i=l+1}^{n-r} a_i^1 \tilde{i}_\rho^*\beta_i), \ldots, (\tilde{H}_s, \tilde{i}_\rho^*H_1 |_M + \sum_{i=l+1}^{n-r} a_i^s \tilde{i}_\rho^*\beta_i) \right\},$$

where the functions $a_i^j$ are chosen such that $\tilde{i}_\rho^*H_1 |_M + \sum_{i=l+1}^{n-r} a_i^j \tilde{i}_\rho^*\beta_i$ are sections of $\mathcal{V}_{\rho}$ for $i = j, \ldots, u$ and $i = l + 1, \ldots, n - r$. Since the vector fields $\tilde{H}_1, \ldots, \tilde{H}_u$ are linearly independent, we have found basis fields for $D_{J^{-1}(\rho)}$ on $U$. □
Hence, the reduced Dirac structure $D_\rho$ on $\mathcal{J}^{-1}(\rho)/G_\rho$ is given, according to the general considerations in §3.2 (or see Bursztyn et al. (2007)) by

$$D_\rho = \frac{[D_{\mathcal{J}^{-1}(\rho)} \cap (T\mathcal{J}^{-1}(\rho) \oplus \mathcal{V}_\rho^\ast)] + (\mathcal{V}_\rho \oplus \{0\})}{G_\rho}$$

(55)

The next theorem gives an easier description of this reduced Dirac structure.

**Theorem 7.15 (Nonholonomic optimal point reduction by Dirac actions)** Assume that the Lie group $G$ acts freely and properly on $M$ by Dirac actions. If $D_G$ is an integrable subbundle of $TM$ then for any $\rho \in M/D_G$ we have the following results.

(i) The orbit space $M_\rho = \mathcal{J}^{-1}(\rho)/G_\rho$ is a smooth regular Dirac quotient manifold whose Dirac structure $D_\rho$ is given by the graph of a nondegenerate (not necessarily closed) 2-form $\omega_\rho$. Denote by $i_\rho : \mathcal{J}^{-1}(\rho) \hookrightarrow M$ the inclusion and by $\pi_\rho : \mathcal{J}^{-1}(\rho) \to M_\rho$ the projection.

(ii) Let $h \in C^\infty(M)^G$ be an admissible and $G$-invariant Hamiltonian and $X_h$ the (unique) solution of the implicit Hamiltonian system $(X_h, dh) \in \Gamma(D)$. Then $X_h \in \Gamma(\mathcal{U})$ and we have $(X_h|_{\mathcal{J}^{-1}(\rho)}, i_\rho^*dh) \in \Gamma(D_{\mathcal{J}^{-1}(\rho)})$.

(iii) The flow $F_t$ of $X_h$ leaves $\mathcal{J}^{-1}(\rho)$ invariant, commutes with the $G$-action, and therefore induces a flow $F_t^n$ on $M_\rho$ uniquely determined by the relation $\pi_\rho \circ F_t = F_t^n \circ \pi_\rho$.

(iv) The flow $F_t^n$ is the flow of a vector field $X_{\bar{h}}_\rho$ in $\mathfrak{X}(M_\rho)$ that is the solution of the Hamiltonian system $1_{X_{\bar{h}}_\rho} \omega_\rho = dh_\rho$, where the function $h_\rho \in C^\infty(M_\rho)$ is given by the equality $h_\rho \circ \pi_\rho = h \circ i_\rho$.

**Proof:** According to Remark 7.14 the $G_\rho$-action on $\mathcal{J}^{-1}(\rho)$ is free and proper. Thus, the quotient $\mathcal{J}^{-1}(\rho)/G_\rho$ is a regular quotient manifold and the projection $\pi_\rho : \mathcal{J}^{-1}(\rho) \to M_\rho$ is a smooth surjective submersion. We denote from now on by $\omega_{\mathcal{J}^{-1}(\rho)} := i_\rho^*\omega_\rho$ the pull back of $\omega_\rho$ to $\mathcal{J}^{-1}(\rho)$.

(i) With Lemma 7.13 get

$$\frac{(D_{\mathcal{J}^{-1}(\rho)} \cap \mathcal{K}_\rho)}{\mathcal{K}_\rho} + \mathcal{K}_\rho = \left\{ (\bar{X}, \alpha) \in \Gamma(((T\mathcal{J}^{-1}(\rho)/\mathcal{V}_\rho) \oplus T^*\mathcal{J}^{-1}(\rho)) \mid \bar{X} = X \bmod \mathcal{V}_\rho \right\} \in \Gamma(\mathcal{U}_\rho),$$

$$\alpha \in \Gamma(\mathcal{V}_\rho^\ast), \text{ and } \alpha - i_X\omega_{\mathcal{J}^{-1}(\rho)} \in \Gamma((\mathcal{U}_\rho + (\mathcal{V}_\rho \cap \mathcal{K}_\rho))^\ast).$$

(56)

The $G_\rho$-quotient of this bundle defines the reduced Dirac structure $D_\rho$ on $M_\rho$. Note the fibers

$$(\mathcal{U}_\rho/\mathcal{V}_\rho)(m) := (\mathcal{U}_\rho + \mathcal{V}_\rho)(m)/\mathcal{V}(m) = T_m\mathcal{J}^{-1}(\rho)/\mathcal{V}_\rho(m), \quad m \in \mathcal{J}^{-1}(\rho)$$

of the vector bundle $\mathcal{U}_\rho/\mathcal{V}_\rho$, project surjectively to $T_{\pi_\rho(m)}M_\rho$. Like in §3 for each $G$-invariant $X \in \Gamma(\mathcal{U}_\rho)$ we identify $\bar{X} = X \bmod \mathcal{V}_\rho$ with the section $\bar{X}$ of $M_\rho$ such that $T\pi_\rho \circ X = \bar{X} \circ \pi_\rho$. Write each $G$-invariant $\alpha \in \Gamma(\mathcal{V}_\rho^\ast)$ as $\alpha = \pi_\rho^*\bar{\alpha}$ for some $\bar{\alpha} \in \Omega^1(M_\rho)$.

Next we show that $D_\rho$ is the graph of a nondegenerate 2-form. We begin by giving a formula for this 2-form $\omega_\rho$. Let $\bar{X}, \bar{Y} \in \mathfrak{X}(M_\rho)$ and choose $G$-invariant $X, Y \in \mathfrak{X}(\mathcal{J}^{-1}(\rho))$ that are $\pi_\rho$-related to $\bar{X}$ and $\bar{Y}$, respectively. Write $X = X + V$ and $Y = Y + W$ with $X, Y \in \Gamma(\mathcal{U}_\rho)^G$ and $V, W \in \Gamma(\mathcal{V}_\rho)^G$. Then $\bar{X}$ and $\bar{Y}$ are also $\pi_\rho$-related to $X$ and $Y$ and we can write, using the existence of $\bar{\alpha} \in \Omega^1(M_\rho)$ such that $(X, \bar{\alpha}) \in \Gamma(D_\rho)$,

$$\omega_\rho(X, Y) = \bar{\alpha}(\bar{Y}) = (\pi_\rho^*\bar{\alpha})(\bar{Y}) = \omega_{\mathcal{U}_\rho}(\bar{X}, \bar{Y}),$$

since $\bar{X}$ has to be the (unique) section of $\mathcal{U}_\rho$ associated to the 1-form $\pi_\rho^*\bar{\alpha}$ (see 7.14) and where $\omega_{\mathcal{U}_\rho}$ is the restriction of $\omega_{\mathcal{J}^{-1}(\rho)}$ to $\mathcal{U}_\rho \times \mathcal{U}_\rho$.

We prove that $\omega_\rho$ is nondegenerate. Let $\bar{X} \in \mathfrak{X}(M_\rho)$ with $\omega_{\mathcal{U}_\rho}(\bar{X}, \bar{Y}) = 0$ for all $\bar{Y} \in \mathfrak{X}(M_\rho)$. Choose a $G$-invariant section $X \in \Gamma(\mathcal{U}_\rho)$ as above. Extend $\bar{X}$ to a local vector field $X$ on $M$, that is, $X \in \Gamma(\mathcal{U}) \subseteq \mathfrak{X}(M)$ satisfies $\bar{X} \sim_{\pi_\rho} X$. For $m \in \mathcal{J}^{-1}(\rho)$ and $v \in \mathcal{U}(m) \subseteq T_mM$ we have

$$\omega_{\mathcal{U}_\rho}(X(m), v) = \omega_{\mathcal{U}_\rho}(\bar{X}(m), v) = \omega_{\mathcal{U}_\rho}(\pi_\rho(m))(\bar{X}(\pi_\rho(m)), T_m\pi_\rho(v)) = 0.$$
Thus, the vector $X(m)$ is an element of $\mathcal{U}(m) = (\mathcal{H} \cap \mathcal{V})^\omega (\mathcal{M})$ that is $\omega_\mathcal{H}(m)$-orthogonal to all $v \in \mathcal{U}(m)$ and hence lies in $((\mathcal{H} \cap \mathcal{V})^\omega \mathcal{M})$. Since $\omega_\mathcal{H}$ is nondegenerate, we have $((\mathcal{H} \cap \mathcal{V})^\omega \mathcal{M}) = \mathcal{H} \cap \mathcal{V}$. This yields $X(m) = X(m) \in (\mathcal{H} \cap \mathcal{V})(m)$ and thus the vector $X(m)$ is zero in $T_m M$. 

(ii) Recall that, since $G_0 = \{0\}$, the solution $X_h$ of the implicit Hamiltonian system $(X, dh) \in \Gamma(D)$ is unique if $Y$ is another solution, then $X - X_h \in \Gamma(G_0) = \{0\}$. We know already that $X_h \in \Gamma(\mathcal{U}_G)$. Furthermore, we have for all $Y \in \Gamma(\mathcal{U}_G)$, $V \in \Gamma(\mathcal{V}_G \cap \mathcal{H}_G)$ and all $m \in \mathcal{J}^{-1}(\rho)$

$$\omega_{\mathcal{J}^{-1}(\rho)}(m)(X_h(m), Y(m) + V(m)) = \omega_{\mathcal{M}}(m)(X_h(m), Y(m) + V(m)) = dh_m(Y(m) + V(m)) = (i^*_\rho dh)(m)(Y(m) + V(m))$$

and the assertion follows.

(iii) The fact that the flow of $X_h$ leaves $\mathcal{J}^{-1}(\rho)$ invariant follows from the preceding statement since we have $X_h \in \Gamma(D_G)$. By $G$-invariance of $D$ we have $(\Phi^*_g X_h, \Phi^*_g dh) \in \Gamma(D)$ for all $g \in G$. Since $h$ is $G$-invariant, the equality $\Phi^*_g dh = dh \Phi^*_g h = dh$ holds and thus we have $\Phi^*_g X_h - X_h \in \Gamma(G_0) = \{0\}$. The vector field $X_h$ is consequently $G$-equivariant and its flow commutes with the $G$-action.

(iv) Since $X_h \in \Gamma(\mathcal{U}_G)$ and $i^*_\rho dh \in V^0_\rho$, we have

$$(X_h, dh) \in \Gamma(D_{\mathcal{J}^{-1}(\rho)} \cap (T_{\mathcal{J}^{-1}(\rho)} \oplus V^0_\rho)).$$

The flow $F^\rho_\tau$ on $M$ induces a vector field $X_{\rho} \in \mathcal{X}(M_\rho)$. Therefore, taking the $t$-derivative of the relation in (iii) we get

$$X_{\rho}(\pi_\rho(m)) = \frac{d}{dt} \bigg|_{t=0} F^\rho_\tau(\pi_\rho(m)) = \frac{d}{dt} \bigg|_{t=0} (\pi_\rho \circ F_\tau)(m) = T_m \pi_\rho X_h(m),$$

that is, $X_{\rho} \sim_{\pi_\rho} X_h$. Choose $\bar{V} \in \mathcal{X}(M_\rho)$, $Y \in \Gamma(\mathcal{U}_G)^G$, and $V \in \Gamma(\mathcal{V}_G)^G$ such that $T \pi_\rho \circ (Y + V) = \bar{V} \circ \pi_\rho$. Then, for all $m \in \mathcal{J}^{-1}(\rho)$ we get

$$\omega_{\mathcal{J}^{-1}(\rho)}(m)(X_h(m), Y(m)) = (i^*_\rho dh)_m(Y(m)) = (i^*_\rho dh)_m(Y(m) + V(m)) = (dh_m)(Y(m) + V(m)) = (dh_m)(Y(\pi_\rho(m))),$$

so we have $i_{X_{\rho}}(\omega_{\rho}) = dh_{\rho}$, as claimed. $\square$

7.5 Example: the constrained particle in space

We return to the example treated in [13] and use the same notations and conventions. The distribution $\mathcal{V} \cap \mathcal{H}$ is pointwise the span of the vector field $\partial_x + y \partial_z$. Since $\mathcal{V}^0$ is spanned by the covector fields $dy$, $dp_x$, and $dp_y$, the considerations in [13] yield $i_{\partial_x + y \partial_z} \omega_M = -(1 + y^2) dp_x - yp_y dy \in \Gamma(\mathcal{V}^0)$ and hence $(\partial_x + y \partial_z, -(1 + y^2) dp_x - yp_y dy) \in \Gamma(D \cap (\mathcal{V} \oplus \mathcal{V}^0))$. Hence the distribution $D_G$ is in this case $\ker\{(1 + y^2) dp_x - yp_y dy\} = \ker\{df\}$, where $f(x, y, z, p_x, p_y) = \sqrt{1 + y^2} p_x$ is the constant of motion (in agreement with [19, 20]). Note that by Example 7.6 $df$ is the 1-form giving the Nonholonomic Noether Theorem. Hence

$$D_G = \text{span}\{dp_x, \partial_x, \partial_z, yp_x \partial_p_x - (y^2 + 1) \partial_y\}$$

is obviously involutive and constant dimensional (and consequently integrable). This shows that $M/D_G = \mathbb{R}$. The Dirac structure on a leaf $f^{-1}(\mu)$, $\mu \in M/D_G = \mathbb{R}$, of this distribution is given by

$$D_{f^{-1}(\mu)} = \{(X, \alpha) \in (T f^{-1}(\mu) \oplus T^* f^{-1}(\mu)) \mid X \in \Gamma(\mathcal{H} \cap D_G), \alpha = i_X \pi_\rho \omega_M \in \Gamma((\mathcal{H} \cap D_G)^*\mathcal{H})\}$$

and a computation yields

$$D_{f^{-1}(\mu)} = \text{span}\{(\partial_{p_x}, -dy), (\partial_x + y \partial_z, 0) \{0, dx - y dy, (1 + y^2) \partial_y - yp_x \partial_p_x, (1 + y^2) dp_y\}\}$$

because the 1-form $(1 + y^2) dp_x + yp_y dy$ vanishes on $T\{f^{-1}(\mu)\}$. Since $G$ is in this case connected, we have $G_\mu = G$ (see Remark 7.12). Consider the codistribution $\mathcal{V}^0$ on $f^{-1}(\mu)$ and get

$$D_{f^{-1}(\mu)} \cap (T f^{-1}(\mu) \oplus \mathcal{V}^0) = \text{span}\{(\partial_{p_x}, -dy), (\partial_x + y \partial_z, 0) \{1 + y^2 \partial_y - yp_x \partial_p_x, (1 + y^2) dp_y\}\}.$$
Hence the reduced Dirac structure $D_\mu$ on $M_\mu = f^{-1}(\mu)\cap G$ is given by the formula

$$D_\mu = \frac{[D_{f^{-1}(\mu)} \cap (T(f^{-1}(\mu)) \oplus \mathcal{V}^o)] + \mathcal{V} \oplus \{0\}}{G}$$

$$= \text{span} \{(\partial_{p_x}, -dy), (1 + y^2)\partial_y - yp_x \partial_{p_x}, (1 + y^2)\partial_{p_y}\}.$$ 

This corresponds exactly to a symplectic leaf (with its associated Dirac structure) of the Poisson structure (32) obtained in the first part of this example (see §4.3). Finally we compute $\mathcal{R}$ for this example. Since $\mathcal{H}^o$ is one-dimensional, we get $\mathcal{R} = \mathcal{H}^o$ or $\mathcal{R} = \{0\}$. Recall that $D$ is the span of

$$\{(\partial_{p_y}, -dy), (\partial_x + y\partial_z, (1 + y^2)dpx + yp_ydy), (0, dz - ydx), (\partial_y, dp_y - p_zdz), (\partial_{p_x}, -ydz - dx)\}$$

where we have computed:

$$i_{\partial_x + y\partial_z} \omega_M = (1 + y^2)dpx + yp_ydy$$

$$i_{\partial_p} \omega_M = dp_y - p_xdz$$

$$i_{\partial_y} \omega_M = -dy$$

$$i_{\partial_{p_y}} \omega_M = -ydz - dx.$$ 

Since $\mathcal{U} = \text{span} \{(\partial_{p_y}, (1 + y^2)\partial_y - yp_x \partial_{p_x}, \partial_x + y\partial_z)\}$, we conclude from

$$i_{(1+y^2)\partial_y - yp_x \partial_{p_x}} \omega_M = (1 + y^2)(dp_y - p_zdz) - yp_x(-ydz - dx)$$

$$= (1 + y^2)dp_y - p_z(dz - ydx),$$

that the distribution $\mathcal{R}$ is equal to $\mathcal{H}^o$. The constant of motion we have found above is a gauge constant of motion.

### 7.6 Example: the vertical rolling disk

In this subsection we shall determine the Nonholonomic Momentum Equations for the example of the vertical rolling disk studied in §4.3. The Dirac structure for this nonholonomic system is given by

$$D = \text{span} \left\{(\partial_\phi, dp_\phi + \frac{\mu R \sin \phi}{I} p_\theta dx - \frac{\mu R \cos \phi}{I} p_\theta dy), (\partial_{p_\theta}, -dy), (\partial_y, dp_y - p_zdz), (0, dx - R \cos \phi \theta), (0, d\phi - R \sin \phi \theta), (\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y, (1 + \frac{\mu R^2}{I})dp_\theta)\right\}$$

where we have computed

$$i_{\partial_\phi} \omega_M = dp_\phi + \frac{\mu R \sin \phi}{I} p_\theta dx - \frac{\mu R \cos \phi}{I} p_\theta dy$$

$$i_{\partial_{p_\theta}} \omega_M = -\frac{\mu R}{I} \cos \phi dx - \frac{\mu R}{I} \sin \phi dy - d\theta$$

$$i_{\partial_{p_\phi}} \omega_M = -d\phi$$

$$i_{\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y} \omega_M = dp_\theta + R \cos \phi \left(\frac{\mu R \cos \phi}{I} dp_\theta - \frac{\mu R \sin \phi}{I} d\phi\right)$$

$$+ R \sin \phi \left(\frac{\mu R \sin \phi}{I} dp_\theta + \frac{\mu R \cos \phi}{I} d\phi\right)$$

$$= \left(1 + \frac{\mu R^2}{I}\right) dp_\theta.$$ 

We consider again the three possible Lie groups:
1. The case $G = \mathbb{R}^2$ \cite{Cantrijn et al. 1998}

Here, $\mathcal{V}^\circ = \text{span}\{d\rho_\phi, d\phi, dp_\theta, d\theta\}$ but there are no nontrivial horizontal symmetries and hence the distribution $\mathcal{D}_G$ is simply the whole bundle $TM$. We next compute $\mathcal{R}$. The vector bundle $D \cap \mathcal{K}^\perp$ is given in this case by

$$\text{span}\left\{ (\partial_\phi, d\phi), (\partial_{p_\theta}, -1 + \frac{\mu R^2}{T} d\theta), (\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y, 1 + \frac{\mu R^2}{T} d\theta) \right\}.$$  

To get this, we have added

$$\frac{\mu R}{T} \rho_\theta \cos \phi (dx - R \cos \phi d\theta) + \frac{\mu R}{T} \sin \phi (dy - R \sin \phi d\theta) \tag{57}$$

to $i_{\partial_\theta} \omega_M$ and

$$-\frac{\mu R}{T} \rho_\theta \sin \phi (dx - R \cos \phi d\theta) + \frac{\mu R}{T} \cos \phi (dy - R \sin \phi d\theta) \tag{58}$$

to $i_{\partial_\phi} \omega_M$. This yields $\mathcal{R} = \mathcal{K}^\circ$ and thus the distribution $\mathcal{R}^\circ \cap \mathcal{V}$ is equal to $\mathcal{K} \cap \mathcal{V}$ and hence trivial.

2. The case $G = \text{SE}(2)$ \cite{Bloch 2003}

In this case, we have $\mathcal{V}^\circ = \text{span}\{d\rho_\phi, d\phi, dp_\theta, d\theta\}$ and $\mathcal{K} \cap \mathcal{V} = \text{span}\{\partial_\phi\}$. We get $\mathcal{K} \cap \mathcal{V} \subseteq (\mathcal{K} \cap \mathcal{V})^\omega_M$. A direct computation gives

$$i_{\partial_\phi} \omega_M = d\rho_\phi + \frac{\mu R \sin \phi}{T} p_\theta dx - \frac{\mu R \cos \phi}{T} p_\theta dy.$$  

Adding

$$\frac{\mu R}{T} \rho_\theta (\cos \phi dy - \sin \phi dx) \in \Gamma(\mathcal{K}^\circ)$$

to this expression we see that $\Gamma(D \cap (\mathcal{V} \oplus \mathcal{V}^\circ))$ is spanned by $\partial_\phi$ and $dp_\phi$ and the distribution $\mathcal{D}_G = \text{ker} dp_\phi$ is obviously integrable. For a value $\rho \in \mathbb{R}$ of the map $p_\phi$, the reduced Dirac structure on $M_\rho$ is spanned by

$$\left( \partial_\theta, 1 + \frac{\mu R^2}{T} d\theta \right) \quad \text{and} \quad \left( \partial_{p_\theta}, - \left( 1 + \frac{\mu R^2}{T} \right) d\theta \right).$$

The Nonholonomic Noether Theorem yields a constant of motion but this constant doesn’t arise from an element of $\mathcal{g}$ whose corresponding fundamental vector field is lying in $\Gamma(\mathcal{V} \cap \mathcal{K}^\circ)$: we have computed in \cite{Cantrijn et al. 1998} that, in this case, $\mathcal{U}$ is the span of the three vector fields $\partial_\phi, \partial_{p_\theta}$, and $\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y$. Again, we have to add \cite{57} to $i_{\partial_{p_\theta}} \omega_M$ and \cite{58} to $i_{\partial_\phi} \omega_M$ in order to get sections of $\mathcal{V}^\circ$. Thus, we need the whole of $\mathcal{K}^\circ$ in the construction of $D \cap \mathcal{K}^\perp$.

3. The case $G = S^1 \times \mathbb{R}^2$ \cite{Bloch 2003}

Here, we have $\mathcal{V}^\circ = \text{span}\{d\rho_\phi, dp_\theta, d\phi\}$ and $\mathcal{K} \cap \mathcal{V}$ is again one-dimensional: this time it is the span of the vector field $\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y$. Thus, $\Gamma(D \cap (\mathcal{V} \oplus \mathcal{V}^\circ))$ is spanned by

$$\left( \partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y, 1 + \frac{\mu R^2}{T} d\theta \right)$$

and the distribution $\mathcal{D}_G = \text{ker}\{1 + \frac{\mu R^2}{T} d\theta\}$ is again integrable. For a value $\rho \in \mathbb{R}$ of the map $p_\theta$, the reduced Dirac structure on $M_\rho$ is spanned by $\partial_\phi$ and $dp_\phi$ and $(\partial_{p_\theta}, -d\phi)$. In this case, we have $\mathcal{U} = \text{span}\{\partial_\phi, \partial_{p_\theta}, \partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y\}$. We get $\mathcal{R} = \text{span}\{\mu R \rho_\theta (\sin \phi dx - \cos \phi dy)\}$ from the considerations for the second case. Thus we have $\mathcal{R}^\circ = \text{span}\{\partial_\phi, \partial_{p_\theta}, \partial_{\theta}, \cos \phi \partial_x + \sin \phi \partial_y\}$ and our constant of motion $p_\theta$ really arises from a fundamental vector field lying in $\mathcal{R}^\circ$. 

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4. The case $G = \text{SE}(2) \times S^1$ (Bloch, 2003)

In this last case, we have $\mathcal{V} = \text{span}\{d\rho_x, d\rho_y\}$ and $\mathcal{H} \cap \mathcal{V}$ is this time two-dimensional: it is the span of the vector fields $\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y$ and $\partial_\phi$. Thus, $\Gamma(D \cap (\mathcal{V} \oplus \mathcal{V}^o))$ is spanned by $(\partial_\theta + R \cos \phi \partial_x + R \sin \phi \partial_y, (1 + \mu R^2) d\rho_y)$ and $(\partial_\phi, d\rho_x)$ and the distribution $D_G = \ker\{(1 + \mu R^2) d\rho_y, d\rho_\phi\}$ is integrable. Here, the reduced manifolds are single points.

We have $\mathcal{U} = \text{span}\{\partial_\theta, \partial_\phi + R \cos \phi \partial_x + R \sin \phi \partial_y\}$ and we get as above $\mathcal{R}^o = \text{span}\{\partial_\theta, \partial_{p_x}, \partial_{p_y}, \partial_\phi, \cos \phi \partial_x + \sin \phi \partial_y\}$.

7.7 Example: the Chaplygin skate

We continue here the examples of 7.5.

The standard Chaplygin skate. We have seen that $\mathcal{V} \cap \mathcal{H} = \text{span}\{\partial_\theta, \cos \theta \partial_x + \sin \theta \partial_y\}$. If we choose the basis $\xi^1 := (1, 0, 0), \xi^2 := (0, 1, 0), \xi^3 := (0, 0, 1)$ of the Lie algebra $\mathfrak{so}(2)$ we get $\xi^1_M = \partial_\theta - y \partial_x + x \partial_y, \xi^2_M = \xi^3_M = \partial_y$, and $\xi^1_M = \partial_y$. Hence, the sections $\xi^1, \xi^2 + y \xi^3 - x \xi^4, \cos \theta \xi^3 \in \Gamma(\mathfrak{g}^{\mathfrak{h}})$ are spanning sections of $\mathfrak{g}^{\mathfrak{h}}$ and the corresponding Nonholonomic Noether equations are $s \cos \theta d\rho_y - s \sin \theta d\rho_x$ and $\cos \theta d\rho_x + \sin \theta d\rho_y$ respectively. Consequently, the two spanning sections $-s^1 \sin \theta \xi^1 + (\cos^2 \theta - s^1 y \sin \theta) \xi^2 + \sin \theta (\cos \theta + s^{-1} x) \xi^3$ and $-s^1 \cos \theta \xi^1 + \cos \theta (\sin \theta + s^{-1} y) \xi^2 + (\sin^2 \theta - s^{-1} x \cos \theta) \xi^3$ of $\mathfrak{g}^{\mathfrak{h}}$ lead to the nonholonomic Noether equations $d\rho_x$ and $d\rho_y$ respectively. Thus, $D_G = \mathcal{U} + \mathcal{V} = \text{span}\{\partial_\theta, \partial_\phi, \partial_y\}$ is found easily because $D_G$ is the kernel of $\{d\rho_x, d\rho_y\}$. This is obviously integrable. The induced Dirac structure on a leaf $f^{-1}(a, b)$ (where $f$ is the projection on $(p_x, p_y)$) of $D_G$ is given by

$$D_{f^{-1}(a,b)} = \text{span}\{(\cos \theta \partial_x + \sin \theta \partial_y, 0), (\partial_\theta, 0), (0, \sin \theta d\rho_x - \cos \theta d\rho_y)\}.$$ 

Here the reduced space $M_{(a,b)}$ is a single point. The reduced Dirac structure is hence trivial, as can also be seen from the formula $\frac{(D_{f^{-1}(a,b)} \cap \chi_{(a,b)}) + \chi_{(a,b)}}{\chi_{(a,b)}} / G_{(a,b,c)}$.

We could also consider the action of $S^1$ on $M$ given by $\Phi : S^1 \times M \to M, (\alpha, \theta, x, y, p_x, p_y) \mapsto (\theta + \alpha, x \cos \alpha - y \sin \alpha, x \sin \alpha + y \cos \alpha, p_x \cos \alpha - p_y \sin \alpha, p_x \sin \alpha + p_y \cos \alpha)$. Here, we would have $\mathcal{V} \cap \mathcal{H} = \{0\}$ except for the points satisfying $x = -\sin \theta$ and $y = -\cos \theta$, so the condition that $\mathcal{V} \cap \mathcal{H}$ has constant rank is not satisfied (we have also $\mathcal{V} \cap \mathcal{H} \neq TM$).

If we consider the action of $\mathbb{R}^2$ on $M$ given by $\Phi : \mathbb{R}^2 \times M \to M, (r, s, \theta, x, y, p_x, p_y) \mapsto (\theta + r + s, x + r y, s + p_x, p_y)$, we have $\mathcal{V} = \text{span}\{\partial_x, \partial_y\}$. Hence, $\mathcal{V} \cap \mathcal{H} = TM$ and $\mathcal{V} \cap \mathcal{H} = \{\cos \theta \partial_x + \sin \theta \partial_y\}$ has constant rank on $M$. The distribution $D_G$ is given by $D_G = \ker\{\cos \theta d\rho_x + \sin \theta d\rho_y\}$ and has constant rank on $M$. This vector bundle is not involutive and it is not integrable. Since $\ker\{\cos \theta d\rho_x + \sin \theta d\rho_y\} = \text{span}\{\sin \theta \partial_x - \cos \theta \partial_y, \partial_y, \cos \theta \partial_x + \sin \theta \partial_y\}$, it is easy to see that $\mathcal{R} = \mathcal{R}^o$ and hence $\mathcal{R}^o \cap \mathcal{V} = \mathcal{V} \cap \mathcal{H} = \{\cos \theta \partial_x + \sin \theta \partial_y\}$, which confirms the fact that the nonholonomic Noether equation yields in this case no constant of motion.

The Chaplygin skate with a rotor on it. We have $\mathcal{V} \cap \mathcal{H} = \text{span}\{\partial_\phi, \partial_\theta, \cos \theta \partial_x + \sin \theta \partial_y\}$. If we choose the basis $\xi^1 := (1, 0, 0, 0), \xi^2 := (0, 1, 0, 0), \xi^3 := (0, 0, 1, 0), \xi^4 := (0, 0, 0, 1)$ of the Lie algebra $\mathbb{R} \times \mathfrak{so}(2)$ we get $\xi^1_M = \partial_\phi, \xi^2_M = \partial_\theta - y \partial_x + x \partial_y, \xi^3_M = \xi^4_M = \partial_y$. Hence, the sections $\xi^1, \xi^2 + y \xi^3 - x \xi^4, \cos \theta \xi^3 + \sin \theta \xi^4 \in \Gamma(\mathfrak{g}^{\mathfrak{h}})$ are spanning sections of $\mathfrak{g}^{\mathfrak{h}}$ and the corresponding Nonholonomic Noether equations are $d\rho_{\phi}, s \cos \theta d\rho_y - s \sin \theta d\rho_x + d\rho_{\phi}, \cos \theta d\rho_x + \sin \theta d\rho_y$, respectively. Thus, the three spanning sections $s^{-1} \sin \theta \xi^1 - s^{-1} \sin \theta \xi^2 + (\cos^2 \theta - s^{-1} y \sin \theta) \xi^3 + \sin \theta (\cos \theta + s^{-1} x) \xi^4$ and $-s^{-1} \cos \theta \xi^1 + s^{-1} \cos \theta \xi^2 + \cos \theta (\sin \theta + s^{-1} y) \xi^3 + (\sin^2 \theta - s^{-1} x \cos \theta) \xi^4$ of $\mathfrak{g}^{\mathfrak{h}}$ lead to the nonholonomic Noether equations $d\rho_{\phi}, d\rho_x$, and $d\rho_y$, respectively. Thus, $D_G = \mathcal{U} + \mathcal{V} = \text{span}\{\partial_\phi, \partial_\theta, \partial_\phi, \partial_\phi\}$ is found easily because $D_G$ is the kernel of $\{d\rho_{\phi}, d\rho_x, d\rho_y\}$. This is obviously integrable. The induced Dirac structure on a leaf $f^{-1}(a, b, c)$ (where $f$ is the projection on $(p_x, p_y, p_\theta)$ of $D_G$ is given by

$$D_{f^{-1}(a,b,c)} = \text{span}\{(\cos \theta \partial_x + \sin \theta \partial_y, 0), (\partial_\theta, 0), (0, \sin \theta d\rho_x - \cos \theta d\rho_y)\}.$$ 

Here the reduced space $M_{(a,b,c)}$ is a single point. The reduced Dirac structure is hence trivial, as can also be seen from the formula $\frac{(D_{f^{-1}(a,b,c)} \cap \chi_{(a,b,c)}) + \chi_{(a,b,c)}}{\chi_{(a,b,c)}} / G_{(a,b,c)}$.

Finally, note that in the last two examples, we have $\mathcal{R} = \{0\}$ and hence $\mathcal{R}^o \cap \mathcal{V} = \mathcal{V}$. This is why we get in the first of the two examples the three constants of motion $p_x, p_y$, and $p_\theta$ belonging to the three elements
\(\xi^2, \xi^3, \) and \(\xi^4\) of \(g\) and in the second example the constants \(p_x, p_y, p_\theta, \) and \(p_\phi\) belonging to the four elements \(\xi^3, \xi^4, \) and \(\xi^4\) of \(g\). Note that in this case, the constancy of \(p_\phi\) follows already from the existence of the constant section \(\xi^4\) of \(g^0\).

Like in the previous example, the other symmetry groups of the system (the “\(\theta\)-symmetry” \(S^1\), “the \(\phi\)-symmetry” \(S^1 \times S^1\), \(S^1 \times \mathbb{R}^2\), \(SE(2)\)) are not interesting for the method of reduction presented in this section.

7.8 Example: the Heisenberg particle

At last, we present an example where the reduced form is not closed. It can be found in [Bloch (2003)]. The configuration space \(Q\) is \(\mathbb{R}^3\) with coordinates \((x, y, z)\) subject to the constraint \(\dot{z} = y\dot{x} - x\dot{y}\). The Lagrangian on \(TQ\) is given by \(L(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\) and hence the Legendre transformation yields

\[
p_x = \dot{x}, \quad p_y = \dot{y}, \quad p_z = \dot{z}.
\]

For \((x, y, z, p_x, p_y, p_z)\), we have \(p_z = yp_x - xp_y\). Hence, we have the global coordinates \((x, y, z, p_x, p_y)\) for \(M\) and the 2-form \(\omega_M\) is given by

\[
\omega_M = dx \wedge dp_x + dy \wedge dp_y + dz \wedge (yp_x - xp_y)
= dx \wedge (dp_x + p_y dz) + dy \wedge (dp_y - p_x dz) + dz \wedge (yd p_x - xd p_y).
\]

The vector bundle \(\mathcal{H}\) is given by \(\mathcal{H} = \ker \{dz - ydx + xdy\} = \text{span}\{\partial_z + \partial_x, x\partial_z - y, \partial_{p_x}, \partial_{p_y}\}\) and we compute

\[
i_y \partial_z - \partial_x \omega_M = y(dp_x - x dp_y) - yp_y dx + yp_x dy + (dp_x + p_y dz)
\]

\[
i_x \partial_z - \partial_y \omega_M = x(dy dp_x - xd dp_y) - x p_y dx + x p_x dy - (dp_x - p_y dz)
\]

\[
i_{\partial_{p_x}} \omega_M = -dx - y dz
\]

\[
i_{\partial_{p_y}} \omega_M = -dy + x dz.
\]

Thus, we get the smooth global spanning sections

\[
\{(y \partial_z + \partial_x, (y^2 + 1) dp_x - xy dp_y - yp_y dx + yp_x dy + p_y dz) , (\partial_{p_x}, -dx - y dz) ,
(x \partial_z - \partial_y, -(x^2 + 1) dp_y + xy dp_x - xp_y dx + xp_x dy + p_x dz) , (\partial_{p_y}, -dy + x dz) , (0, dz - ydx + xdy) \}
\]

for the Dirac structure \(D\).

Consider the action \(\phi : \mathbb{R} \times Q \to Q\) of the Lie group \(G = \mathbb{R}\) on \(Q\), given by \(\phi(r, x, y, z) := (x, y, z + r)\). This action obviously leaves the Lagrangian and the constraints invariant. The induced action \(\Phi : G \times M \to M\) is given by \(\Phi(r, x, y, z, p_x, p_y, p_z) := (x, y, z + r, p_x, p_y)\) and hence the vertical bundle in this example equals \(\mathcal{V} = \text{span}\{\partial_z\}\). We get \(\mathcal{H} \cap \mathcal{V} = \{0\}\) and hence \(\mathcal{U} = \mathcal{H}\). The two methods of reduction (in (7.2) and (7.4)) lead in this case to the same result since the distribution \(D_G = \mathcal{U} + \mathcal{V} = \mathcal{H} + \mathcal{V} = TM\) is trivially integrable with \(M\) as single leaf. The reduced Dirac structure \(D_{\text{red}}\) on \(\tilde{M}\) with coordinates \((x, y, p_x, p_y)\) is thus given by

\[
D_{\text{red}} = \left(\frac{(D \cap \mathcal{X}^+) + \mathcal{K}}{G}\right)
= \text{span} \left\{ (\partial_x, (y^2 + 1) dp_x - xy dp_y - yp_y dx + yp_x dy + p_y (yd x - xdy)) , (\partial_{p_x}, -dx - y (yd x - xdy)) ,
(\partial_y, -(x^2 + 1) dp_y + xy dp_x - xp_y dx + xp_x dy + p_x (yd x - xdy)) , (\partial_{p_y}, -dy + x (yd x - xdy)) \right\}
= \text{span} \left\{ (\partial_x, (2y + 1) dp_x - xy dp_y + (yp_x - xp_y) dy) , (\partial_{p_x}, -(1 + y^2) dx + xy dy) ,
(\partial_y, -(x^2 + 1) dp_y + xp dp_x + (yp_y - xp_y) dx) , (\partial_{p_y}, -(1 + x^2) dy + xy dx) \right\}.
\]

Note that this is the graph of the 2-form

\[
\omega_{\text{red}} = (1 + y^2) dx \wedge dp_x + (1 + x^2) dy \wedge dp_y + (yp_y - xp_y) dx \wedge dp_y + dy \wedge -(dx \wedge dp_y + dy \wedge dp_x).
\]

A direct computation shows that the determinant of \(\omega_{\text{red}}\) equals \((1 + x^2 + y^2)^2 \neq 0\) on \(\tilde{M}\) which shows that the form \(\omega_{\text{red}}\) is nondegenerate. The equalities

\[
d\omega_{\text{red}}(\partial_x, \partial_y, \partial_{p_x}) = -2y \quad \text{and} \quad d\omega_{\text{red}}(\partial_x, \partial_y, \partial_{p_y}) = 2x
\]

show that \(\omega_{\text{red}}\) is not closed.

Note also that in this example we have \(\mathcal{R} = \mathcal{H}^0\) and hence \(\mathcal{R}^0 \cap \mathcal{V} = \mathcal{H} \cap \mathcal{V} = \{0\}\).
A Push-down of distributions

In the main text of this paper we have used twice a rather technical proposition on “control” of distributions (see [Nijmeijer and van der Schaft 1990]). Due to its importance we present here a complete proof which is inspired by the work of Cheng and Tarn (1989).

Recall that a distribution $D \subset TM$ is said to be locally finite if for each point $m \in M$ there are an open neighborhood $U \subset M$ of $m$ and smooth vector fields $X_1, \ldots, X_r \in \mathfrak{X}(U)$ such that at each point $x \in U$ we have $\text{span}\{X_1(x), \ldots, X_r(x)\} = D(x)$. Note that locally finite distributions are necessarily smooth.

**Proposition A.1** Let $V \subset TM$ be an involutive vector subbundle of $TM$ and $D$ a locally finite smooth generalized distribution on $M$. Assume that

$$[\Gamma(D), \Gamma(V)] \subseteq \Gamma(V + D). \tag{59}$$

Let $X$ be a vector field on $M$ satisfying

$$[X, \Gamma(V)] \subseteq \Gamma(V + D). \tag{60}$$

Then for each $p \in M$ there is an open set $U \subseteq M$ with $p \in U$ and smooth $D$-valued vector fields $Z, Z_1, \ldots, Z_r$ on $U$ satisfying

(i) $D(q) = \text{span}\{Z_1(q), \ldots, Z_r(q)\}$ for all $q \in U$,

(ii) $[Z_i, \Gamma(V)] \subseteq \Gamma(V)$ on $U$ for all $i = 1, \ldots, r$, and

(iii) $[X + Z, \Gamma(V)] \subseteq \Gamma(V)$ on $U$.

**Proof:** Let $n := \dim M$ and $k := \dim V(x)$, for $x \in M$. Since the vector subbundle $V$ is involutive, it is integrable by the Frobenius Theorem and thus any $p \in M$ lies in a foliated chart domain $U_1$ described by coordinates $(x^1, \ldots, x^n)$ such that the first $k$ among them define the local integral submanifold containing $p$ (see §2.2 for a review of these notions). Thus, for any $q \in U_1$ the basis vector fields $\partial_1, \ldots, \partial_k$ evaluated at $q$ span $V(q)$.

Because $D$ is locally finite, we can find on a sufficiently small neighborhood $U \subseteq U_1$ of $p$ smooth vector fields $X_1, \ldots, X_r$ spanning $D$, i.e., for all $q \in U$ we have

$$D(q) = \text{span}\{X_1(q), \ldots, X_r(q)\}.$$  

Write, for $i = 1, \ldots, r$

$$X_i = \sum_{j=1}^n X^j_i \partial_{x^j},$$

with $X^j_i$ local smooth functions defined on $U$ for $j = 1, \ldots, n$. By hypothesis (59) we get for all $i = 1, \ldots, r$ and $l = 1, \ldots, k$:

$$\partial_{x^l}(X_i) := [\partial_{x^l}, X_i] = \sum_{j=1}^n \partial_{x^l}(X^j_i) \partial_{x^j} \in \Gamma(V + D).$$

Hence we can write

$$\partial_{x^l}(X_i) = \sum_{j=1}^n \partial_{x^l}(X^j_i) \partial_{x^j} = \sum_{j=1}^k A^j_{li} \partial_{x^j} + \sum_{j=1}^r B^j_{li} X_j,$$

with $A^j_{li} \in C^\infty(U)$ for $i = 1, \ldots, r$, $j, l = 1, \ldots, k$ and $B^j_{li} \in C^\infty(U)$ for $i, j = 1, \ldots, r$, $l = 1, \ldots, k$. Set

$$X_i := \sum_{j=k+1}^n X^j_i \partial_{x^j}$$

for $i = 1, \ldots, r$ and get

$$\partial_{x^l}(X_i) := [\partial_{x^l}, X_i] = \sum_{j=k+1}^n \partial_{x^l}(X^j_i) \partial_{x^j} = \sum_{j=1}^r B^j_{li} \bar{X}_j,$$
We rewrite this system as

\[
(\partial_x(\tilde{X}_1), \ldots, \partial_x(\tilde{X}_r)) = \left(\tilde{X}_1, \ldots, \tilde{X}_r\right) B_1,
\]

where \(B_1 = [B_1^i]_{i=1}^r\) is an \(r \times r\) matrix with entries \(B_1^i = B_1^i j \in C^\infty(U)\), \(i, j = 1, \ldots, r\).

Now fix \(j \in \{1, \ldots, k\}\), think of \(x^j\) as a time variable and all the other \(x^i\) as parameters, and consider the following ordinary differential equation

\[
\partial_{x^j} Y = B_1^j Y.
\]  

(61)

Let \(Y_j^1, \ldots, Y_j^r\) be \(r\) linearly independent solutions of \((61)\). Set \(W_j = (Y_j^1, \ldots, Y_j^r)\) which is an invertible matrix. Since the rows of \((\tilde{X}_1, \ldots, \tilde{X}_r)\) (where we think of this as a \((n-k) \times r\)-matrix with columns \((X_i^{k+1}, \ldots, X_i^n)\) for \(i = 1, \ldots, r\)) are also solutions of \((61)\), we know that there exists a \(r \times (n-k)\) matrix \(L_j\) such that

\[
(\tilde{X}_1, \ldots, \tilde{X}_r)^\top = W_j L_j, \quad j = 1, \ldots, k
\]

where \(L_j\) is independent of \(x_j\) (which is the independent variable of differential equation \((61)\)). Therefore, we have

\[
W_1 L_1 = W_2 L_2 = \cdots = W_k L_k.
\]  

(62)

Because \(W_2\) is nonsingular, we have \(L_2 = W_2^{-1} W_1 L_1\). Set \(x_2 = 0\) on both sides of this equation and get \(L_2 = W_2^{-1} W_1 |_{x_2=0} L_1(0, x_3, \ldots, x_n)\) since \(L_2\) is independent of \(x_2\). The matrix \(H_2 := W_2^{-1} W_1 |_{x_2=0}\) is smooth and nonsingular and \(L_2 = H_2 L_1(0, x_3, \ldots, x_n)\). Recursively, assume \(H_i\) is a well-defined smooth nonsingular matrix and \(L_i(x) = H_i L_1(0, \ldots, 0, x_{i+1}, \ldots, x_n)\). Using \((62)\), get

\[
L_{i+1}(x) = W_{i+1}^{-1} W_i H_i L_1(0, \ldots, 0, x_{i+1}, \ldots, x_n) \quad \text{and let} \quad H_{i+1} := W_{i+1}^{-1} W_i H_i |_{x_{i+1}=0}.
\]

Since \(L_{i+1}\) is independent of \(x_{i+1}\), we have \(L_{i+1}(x) = H_{i+1} L_1(0, \ldots, 0, x_{i+2}, \ldots, x_n)\). Finally, get \(H_k\) and \(L_k(x) = H_k L_1(0, \ldots, 0, x_{k+1}, \ldots, x_n)\). Define the smooth nonsingular matrix \(H := W_k H_k\), and \(L := L_1(0, \ldots, 0, x_{k+1}, \ldots, x_n)\) which is independent of \(x_1, \ldots, x_k\). Then \((\tilde{X}_1, \ldots, \tilde{X}_r)^\top = H L\). Define

\[
B = (H^\top)^{-1} = (H^{-1})^\top.
\]  

(63)

Then \((\tilde{X}_1, \ldots, \tilde{X}_r) B = L^\top\) is independent of \(x_1, \ldots, x_k\). This yields

\[
0 = \left[\partial_{x^l}, \left((\tilde{X}_1, \ldots, \tilde{X}_r) B\right)_i\right]
\]

(64)

for \(l = 1, \ldots, k\) and \(i = 1, \ldots, r\). In this formula, we denote by \((\tilde{X}_1, \ldots, \tilde{X}_r) B\) the \(i\)th column of the matrix \((\tilde{X}_1, \ldots, \tilde{X}_r) B\), considered as the representation of a local vector field in the basis \(\{\partial_x^{k+1}, \ldots, \partial_x^n\}\), i.e. if

\[
(\tilde{X}_1, \ldots, \tilde{X}_r) B = [C_{jl}]_{j=k+1}^n_{l=1}^{r}
\]

we write \(((\tilde{X}_1, \ldots, \tilde{X}_r) B)_i\) for the smooth vector field

\[
((\tilde{X}_1, \ldots, \tilde{X}_r) B)_i := \sum_{j=k+1}^n C_{jl} \partial_x^j.
\]  

(65)

Let \(Z_1, \ldots, Z_r\) be the local vector fields defined by \(Z_i = ((X_1, \ldots, X_r) B)_i\) for \(i = 1, \ldots, r\), where again, if we have

\[
(X_1, \ldots, X_r) B = [C_{jl}]_{j=1}^n_{l=1}^{r}
\]

we write \(((X_1, \ldots, X_r) B)_i\) for the smooth vector field

\[
\sum_{j=1}^n C_{ji} \partial_x^j = \sum_{j=1}^k C_{ji} \partial_x^j + \sum_{j=k+1}^n C_{ji} \partial_x^j.
\]

We get (with the identification \((65)\) above)

\[
Z_i = \xi_i + ((\tilde{X}_1, \ldots, \tilde{X}_r) B)_i
\]
where
\[ \xi_i = \sum_{j=1}^{k} C_{ji} \partial x_j \in \Gamma(V). \]

Thus, we have for \( l = 1, \ldots, k, \)
\[ [\partial x^l, Z_i] = [\partial x^l, \xi_i] + \left[ \partial x^l, \left( \tilde{X}_1, \ldots, \tilde{X}_r \right) \cdot B \right] \in [\partial x^l, \xi_i] + 0 \in \Gamma(V) \]
for \( i = 1, \ldots, r. \) Hence, if we write an arbitrary section \( \eta \in \Gamma(V) \) as
\[ \eta = \sum_{j=1}^{k} \eta_j \partial x_j \]
with smooth local functions \( \eta_1, \ldots, \eta_k, \) we get for \( i = 1, \ldots, r: \)
\[ [\eta, Z_i] = \left[ \sum_{j=1}^{k} \eta_j \partial x_j, Z_i \right] = \sum_{j=1}^{k} \eta_j [\partial x_j, Z_i] + Z_i [\eta_j \partial x_j] \in \Gamma(V). \]

Thus, since by construction, \( Z_1, \ldots, Z_r \) also span \( D \) on \( U \) these vectors fields satisfy the first two statements of the proposition.

For the third statement, note that if \( X = \sum_{j=1}^{n} \alpha^j \partial x_j \) with \( C^\infty \)-functions \( \alpha^1, \ldots, \alpha^n, \) \( (63) \) yields
\[ [\partial x^j, X] = \sum_{j=1}^{n} \partial x^j (\alpha^j) \partial x_j \in \Gamma(V + D) \]
for \( l = 1, \ldots, k. \) This leads to \( \sum_{j=1}^{n} \partial x^j (\alpha^j) \partial x_j = \sum_{j=1}^{k} \sigma^j \partial x_j + \sum_{j=1}^{r} \beta^j X_j \) with \( C^\infty \)-functions \( \beta^1, \ldots, \beta^r, \) \( C^\infty \)-functions \( \sigma_1, \ldots, \sigma_k, \) and \( \tilde{X}_1, \ldots, X_r \) as above. Hence, if we define
\[ \tilde{X} := \sum_{j=k+1}^{n} \alpha^j \partial x_j, \]
we get for \( l = 1, \ldots, k \)
\[ [\partial x^l, \tilde{X}] = \sum_{j=k+1}^{n} \partial x^l (\alpha^j) \partial x_j = \sum_{j=1}^{r} \beta^j \tilde{X}_j = (\tilde{X}_1, \ldots, \tilde{X}_r) \beta_l \]
where \( \beta_l \) is the \( (r \times 1) \)-matrix with the entries \( \beta^1_l, \ldots, \beta^r_l. \) From the definition \( (63) \) of the matrix \( B \) we get for \( j, l = 1, \ldots, k: \)
\[ \begin{aligned}
(\tilde{X}_1, \ldots, \tilde{X}_r) B \left[ \partial x^j, (H^T \beta_j) \right] &= \left[ \partial x^j, (\tilde{X}_1, \ldots, \tilde{X}_r) BH^T \beta_j \right] \\
&= \left[ \partial x^j, \left[ \partial x^j, \tilde{X} \right] \right] = (\tilde{X}_1, \ldots, \tilde{X}_r) B \left[ \partial x^j, (H^T \beta_j) \right] \quad (66)
\end{aligned} \]
where we have used the Jacobi identity and \( \left[ \partial x^j, \partial x^j \right] = 0 \) in the third identity above.

Define now the \((r \times 1)\)-matrix with \( C^\infty \)-entries
\[ \gamma = (\gamma_1, \ldots, \gamma_r)^T := -B \left[ \int_0^x (H^T \beta_k)(x^1, \ldots, x^{k-1}, \tau, x^{k+1}, \ldots, x^n) d\tau \right. \]
\[ + \int_0^{x^{k-1}} (H^T \beta_{k-1})(x^1, \ldots, x^{k-2}, \tau, 0, x^{k+1}, \ldots, x^n) d\tau \]
\[ + \cdots + \int_0^{x^1} (H^T \beta_1)(\tau, 0, \ldots, 0, x^{k+1}, \ldots, x^n) d\tau \]
Then for \( l = 1, \ldots, k \) a computation using (67) and the definition of \( \gamma \) leads to
\[
[\partial_{x_l}, (\bar{X} + (\bar{X}_1, \ldots, \bar{X}_r)\gamma)] = (\bar{X}_1, \ldots, \bar{X}_r)\beta_l - (\bar{X}_1, \ldots, \bar{X}_r)\beta_l = 0.
\]
Thus the desired vector field satisfying the third condition in the statement of the proposition is
\[
Z = (X_1, \ldots, X_r)\gamma = \sum_{k=1}^{r} X_k\gamma_k \in \Gamma(\mathcal{D}),
\]
which we can also write as:
\[
Z = (\bar{X}_1, \ldots, \bar{X}_r)\gamma + (\bar{X}_1, \ldots, \bar{X}_r)\gamma = \sum_{k=1}^{r} \bar{X}_k\gamma_k + \sum_{k=1}^{r} \bar{X}_k\gamma_k,
\]
where \( \bar{X} := \sum_{j=1}^{k} x_j^\gamma \partial_{x_j} \) and \( \bar{X}_i = \sum_{j=1}^{k} X_i^j \partial_{x_j} \in \Gamma(\mathcal{V}) \) for \( i = 1, \ldots, r \).
Indeed, for any \( l = 1, \ldots, k \), since \( \partial_{x_1}, \ldots, \partial_{x_k} \) is a basis of the space of sections of \( \mathcal{V} \) over \( U \), we get
\[
[X + Z, \partial_{x_l}] = \left[ \bar{X} + \sum_{k=1}^{r} \bar{X}_k\gamma_k, \partial_{x_l} \right] + \left[ \bar{X} + \sum_{k=1}^{r} \bar{X}_k\gamma_k, \partial_{x_l} \right] = 0 \in \Gamma(\mathcal{V})
\]
since it follows from the definition of \( \bar{X} \) and \( \bar{X}_i \), \( i = 1, \ldots, r \), that \( \bar{X} + \sum_{k=1}^{r} \bar{X}_k\gamma_k \in \Gamma(\mathcal{V}) \). As in the first part of the proof, we get \([X + Z, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V}) \) on \( U \) and the proposition is proved. \( \square \)

**Proposition A.2** Assume that the Lie group \( G \) acts freely and properly on the smooth manifold \( M \), let \( \bar{M} := M/G \) be the orbit space, and denote by \( \pi : M \to \bar{M} \) the principal \( G \)-bundle projection. Let \( \mathcal{V} \subset TM \) be the vertical subbundle of this action. If \( X \in \mathfrak{X}(M) \) is a smooth vector field satisfying \([X, \Gamma(\mathcal{V})] \subseteq \Gamma(\mathcal{V}) \), then there exists \( \bar{X} \in \mathfrak{X}(\bar{M}) \) such that \( X \sim_\pi \bar{X} \).

**Proof:** If necessary, shrink the domain \( U \) of definition of \( X \) such that \( U \) is contained in a tube for the action of \( G \) on \( M \) (see, e.g., Palais [1961] or Ortega and Ratiu [2004] Theorem 2.3.28). Hence, since the action is free, we can find smooth coordinates \( \{g_1, \ldots, g_k, x_1, \ldots, x_{n-k}\} \) on \( U \) such that the projection map \( \pi \) is given in this chart by
\[
\pi : (g_1, \ldots, g_k, x_1, \ldots, x_{n-k}) \mapsto (x_1, \ldots, x_{n-k})
\]
and the vertical space \( \mathcal{V} \) is spanned by the sections \( \partial_{g_1}, \ldots, \partial_{g_k} \). Write the smooth vector field \( X \) as
\[
X = \sum_{j=1}^{k} a_j \partial_{g_j} + \sum_{j=1}^{n-k} b_j \partial_{x_j}
\]
with smooth functions \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_{n-k} \) defined on \( U \). For \( l = 1, \ldots, k \) we get
\[
[\partial_{g_l}, X] = \sum_{j=1}^{k} a_j [\partial_{g_l}, \partial_{g_j}] + \sum_{j=1}^{k} \partial_{g_l} (a_j) \partial_{g_j} + \sum_{j=1}^{n-k} b_j [\partial_{g_l}, \partial_{x_j}] + \sum_{j=1}^{n-k} \partial_{g_l} (b_j) \partial_{x_j} = \sum_{j=1}^{k} \partial_{g_l} (a_j) \partial_{g_j} + \sum_{j=1}^{n-k} \partial_{g_l} (b_j) \partial_{x_j}.
\]
Since this is an element of \( \Gamma(\mathcal{V}) \), we conclude that \( \partial_{g_l} (b_j) = 0 \) for \( l = 1, \ldots, k \) and \( j = 1, \ldots, n-k \). This means that the functions \( b_1, \ldots, b_{n-k} \) are independent of the variables \( g_1, \ldots, g_k \) and we can define \( \bar{X} = \sum_{j=1}^{n-k} b_j \partial_{x_j} \in \mathfrak{X}(M) \). We have then for all \( p = (g_1, \ldots, g_k, x_1, \ldots, x_{n-k}) \in U \) and \( \bar{p} = \pi(p) = (x_1, \ldots, x_{n-k}) \in \pi(U) \):
\[
T_p \pi(X(p)) = T_p \pi \left( \sum_{j=1}^{k} a_j (p) \partial_{g_j} |_p + \sum_{j=1}^{n-k} b_j (p) \partial_{x_j} |_p \right) = \sum_{j=1}^{n-k} b_j (\bar{p}) \partial_{x_j} |_{\bar{p}} = \bar{X}(\bar{p}),
\]
as required. \( \square \)
Corollary A.3 The local vector field $X \in \mathfrak{X}(M)$ satisfies $[X, V] \in \Gamma(V)$ for every $V \in \Gamma(V)$ if and only if there exists $X^G \in \mathfrak{X}(M)^G$ such that $X - X^G \in \Gamma(V)$.

Proof: We continue using the notations in the proof of Proposition A.2. Note that $X^G := \sum_{j=1}^{n-k} b_j \partial_{x_j}$ is $G$-equivariant and that $X - X^G = \sum_{j=1}^{k} a_j \partial_{v_j} \in \Gamma(V)$. Conversely, let $X \in \mathfrak{X}(M)$ be such that there exists $X^G \in \mathfrak{X}(M)^G$ satisfying $W := X - X^G \in \Gamma(V)$. Let us show that for any $V \in \Gamma(V)$ we have $[X, V] = [X^G, V] + [W, V] \in \Gamma(V)$. Since $V$ is involutive, we conclude that $[W, V] \in \Gamma(V)$. To see that $[X^G, V] \in \Gamma(V)$ write $X^G = \sum_{j=1}^{n-k} b_j \partial_{x_j}$ and note that since $X^G$ is $G$-equivariant, we have $\partial_{g_j}(a'_i(b'_j)) = 0$ for $j = 1, \ldots, k$, $i = 1, \ldots, k$ and $l = 1, \ldots, n - k$. For any local section $V$ of $\Gamma(V)$, write $V = \sum_{j=1}^{k} v_j \partial_{g_j}$ and compute:

$$[X^G, V] = \sum_{l=1}^{k} X^G(v_l) \partial_{g_l} - \sum_{l=1}^{k} v_l \left( \sum_{j=1}^{n-k} \partial_{g_j}(b'_j) \partial_{x_j} + \sum_{j=1}^{k} \partial_{g_j}(a'_j) \partial_{g_j} \right) = \sum_{l=1}^{k} X^G(v_l) \partial_{g_l} \in \Gamma(V).$$

□

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