Reconciling a quantum gravity minimal length with lack of photon dispersion

Michael Bishop

Mathematics Department, California State University Fresno, Fresno, CA 93740

Joey Contreras, Jaeyeong Lee and Douglas Singleton

Physics Department, California State University Fresno, Fresno, CA 93740

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Abstract

Generic arguments lead to the idea that quantum gravity has a minimal length scale. A possible observational signal of such a minimal length scale is that photons should exhibit dispersion. In 2009 the observation of a short gamma ray burst seemed to bound the minimal length scale to distances smaller than the Planck length, implying that spacetime appeared continuous to distances below the Planck length. This poses a challenge for such minimal distance models. Here we propose a modification of the position and momentum operators, $\hat{x}$ and $\hat{p}$, which lead to a minimal length scale, but preserve the photon energy-momentum relationship $E = pc$. In this way there is no dispersion of photons with different energies. This can be accomplished without modifying the commutation relationship $[\hat{x}, \hat{p}] = i\hbar$. 

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*Electronic address: mibishop@mail.fresnostate.edu
†Electronic address: mkfetch@mail.fresnostate.edu
‡Electronic address: yeong0219@mail.fresnostate.edu
§Electronic address: dougs@mail.fresnostate.edu
I. GAMMA RAY BURST CONSTRAINTS ON MINIMAL LENGTH

Theories of quantum gravity often incorporate Lorentz invariance violation (LIV) at very high energy scales. Having an absolute minimum length scale is one possible consequence of LIV - see reviews in references [2] and [3]. This violation of Lorentz symmetry cannot be tested in the laboratory at present. However a proposal has been put forward [4–6] to test for a minimal length scale/LIV using observations of gamma ray bursts (GRB). Gamma ray bursts are emissions of extremely high energy photons that generally are detected after traveling large, cosmological distances. Gamma ray bursts fall into two categories: (i) short gamma ray bursts which are thought to come from neutron star mergers or neutron star-black hole mergers and (ii) long gamma ray bursts which are thought to come from supernova. It is the short gamma ray bursts which are most useful in potentially observing the effects of a minimal length scale. As photons travel from the GRB to Earth, they should generically exhibit dispersion due to a minimal length i.e. photons of different energies will have slightly different velocities.

In references [4–6] a generic quantum gravity modified energy-momentum relation for photons was proposed of the form

\[ p^2 c^2 = E^2 [1 + f(E/E_{QG})] \]  

where \( f(E/E_{QG}) \) is some arbitrary function associated with one’s theory of quantum gravity and with \( E_{QG} \) being the energy scale of quantum gravity. Often this is set to be the Planck scale \( (E_{QG} = E_{Pl} = \sqrt{\hbar c/G} \approx 10^{19} \text{ GeV}) \). When \( E \ll E_{QG} \), the Taylor expansion of (1) leads to

\[ p^2 c^2 = E^2 [1 + \xi (E/E_{QG}) + \mathcal{O}(E/E_{QG})^2]. \]  

The sign of \( \xi \) is model dependent. From Hamilton’s equation, the photon’s velocity is given as \( \frac{\partial E}{\partial p} = v \). Using (2) and series expanding gives an energy dependent photon velocity

\[ v = \frac{\partial E}{\partial p} \approx c \left( 1 - \xi \frac{E}{E_{QG}} \right). \]  

This energy dependence of photon velocity leads to a difference in the arrival time \( \delta t \) of the photons with energy difference \( \delta E \) which to first order in \( E \) is

\[ \delta t = \xi \frac{L}{c E_{QG}}. \]
where $L$ is the distance the photons traveled. This implies that the farther photons travel and the larger the energy differences are, the bigger the time delay will be. Photon velocity dispersion and its associated time delays, as given in (3) and (4) have been criticized [7] as leading to non-local effects, which may already rule out deformations like (1) (2) to first order in $E$. One of the goals of this work is to show that it is possible to have a minimal length scale while avoiding dispersion of the form in (3).

In 2009 [8], the Fermi Gamma-Ray space telescope detected a powerful short GRB (GRB090510). The observations of GRB090510 allowed one to constrain the quantum gravity scale from 1.2 to 100 times the Planck scale depending on whether one made conservative or liberal assumptions (i.e. $M_{QG} > 1.2M_{Pl}$ to $M_{QG} > 100M_{Pl}$). One of the assumptions that went into setting these bounds is that the lowest order correction to the dispersion relationship from (1) was linear in $E$ as in (2). If the lowest order correction were of order $E^2$ or higher then the constraints were much weaker. This result appeared to contradict the expectation that one should find evidence for LIV/a minimal length scale well before reaching the Planck scale. If taken at face value these results present a challenge to the ideas of LIV/minimal distance scales emerging from quantum gravity.

Below we will present a model of a minimal length scale that respects the above constraints coming from the GRB observations. We propose modified position and momentum operators which lead to a minimal length but do so without leading to dispersion of photons of different energies. This can be accomplished either with or without modifying the commutation relationship between position and momentum.

II. MODIFIED ENERGY AND MOMENTUM WITHOUT PHOTON DISPERSION

In this section, we propose modified momentum operators which do not lead to dispersion of photons of different energies while, at the same time, having a minimal length scale. Two variants of modified momentum operators are

$$p' = p_0 \tanh \left( \frac{p}{p_0} \right) \quad \text{and} \quad p' = p_0 \arctan \left( \frac{p}{p_0} \right).$$

These two modified momentum operators have the common feature of being bounded and going over to the ordinary momentum at small $p$ ($p' \to p_0$ as $p \to \infty$ and $p' \to p$ for $p \ll p_0$).
The parameter $p_0$ is the maximum momentum which characterizes the quantum gravity scale; in the following section we show that it is connected to the minimum distance. These modified momentum operators are loosely motivated by Born-Infeld electrodynamics [11] where Maxwell’s equations are modified so as to have a maximum electric field strength. It is also worth noting that $p' = p_0 \arctan(p/p_0)$ recalls the forms of the sine-Gordon soliton which does not exhibit dispersion.

In Minkowski spacetime, one gets the dispersion relationship for a particle of mass $m$ from the energy-momentum relation which is $E^2 = p^2 c^2 + m^2 c^4$. Since we are interested in massless photons this energy-momentum relationship becomes $E = pc$. Using Hamilton’s equation one finds $\frac{\partial E}{\partial p} = c$ so that the velocity does not depend on $E$ or $p$ and there is no dispersion. For the modified momenta, $p'$, in (5) we want to write down an associated modified energy, $E'$, so that in terms of $p'$ and $E'$ one has the standard relationship $E' = p'c$. Then by Hamilton’s equation in terms of these modified energy and momentum one again has $\frac{\partial E'}{\partial p'} = c$ and there is no dispersion. Since the modified momentum $p'$ are bounded by $p_0$ the modified energy should also bounded by a maximum energy $E_0$. As $p \to \infty$ we want a modified energy that satisfies

$$E' = p'c \to E_0 = p_0c \quad \text{as} \quad p \to \infty.$$  \hfill (6)

Using the two modified momentum operators from (5) in the energy-momentum relationship of (6) one finds that the modified energies are

$$E' = p_0 c \tanh \left( \frac{p}{p_0} \right) = E_0 \tanh \left( \frac{E}{E_0} \right),$$  \hfill (7)

and

$$E' = p_0 c \arctan \left( \frac{p}{p_0} \right) = E_0 \arctan \left( \frac{E}{E_0} \right).$$  \hfill (8)

To obtain the last expressions in (7) and (8) we used $p = \frac{p}{c}$ and $p_0 = \frac{E_0}{c}$.

The energy-momentum relationship between the modified energy and modified momentum as given in equations (5), (7), and (8) is the same as the standard energy-momentum relationship. These modified energies and momenta do not lead to dispersion since the energy-momentum relationship in (6) is unchanged. This lack of dispersion can be seen either from the phase velocity, $\frac{E'}{p'} = c$, or the group velocity obtained via Hamilton’s equation, $\frac{\partial E'}{\partial p'} = c$. In contrast, several common phenomenological models of quantum gravity [4–6, 9, 10] give a modified energy-momentum relationship. This modified energy-momentum
The relationship of (1) and (2) leads to an energy dependence of the photon velocity given in (3).

III. MINIMAL LENGTH VIA MODIFIED OPERATORS

Many approaches to phenomenological quantum gravity with a minimum length scale \[1, 12–17\] use a generalized uncertainty principle (GUP). Here we review a common approach to a quantum gravity inspired minimal length given in \[1\]. This work proposed a GUP by adding the term $\beta \Delta p^2$ to the usual Heisenberg relationship

$$\Delta x \Delta p \geq \frac{\hbar}{2} \rightarrow \Delta x' \Delta p \geq \frac{\hbar}{2} (1 + \beta \Delta p^2).$$

(9)

The phenomenological parameter $\beta$ characterizes the effect of quantum gravity and the primed quantities are modified operators. Thus in this approach to a minimal length scale the position operator is modified while the momentum operator is not.

For the usual Heisenberg uncertainty relationship $\Delta x \propto 1/\Delta p$, so that $\Delta x \rightarrow 0$ as $\Delta p \rightarrow \infty$ i.e. there is no absolute, minimum length. However, from (9) one sees that $\Delta x' = \frac{\hbar}{2} \left( \frac{1}{\Delta p} + \beta \Delta p \right)$. Because of this $\Delta x'$ now has a local minimum, which occurs at $\Delta p = 1/\sqrt{\beta}$, and gives a minimum length of $\Delta x_0 = \hbar \sqrt{\beta}$ as shown in Fig. 1.

The addition of the $\beta \Delta p^2$ term in (9) can be motivated in the following way: as one probes shorter distances via higher energy/momentum collisions at some point the center...
of mass energy becomes great enough to create a micro black hole, whose Schwarzschild
radius is set by the mass-energy in the collision \(i.e. \Delta x \sim R_{sch} \sim E\). As one goes to
even higher energy/momentum in the collision the resulting Schwarzschild radius will grow
linearly preventing one from probing smaller distances.

Having a modified uncertainty relationship like (9) implies that there is modification of
the commutator between the position and momentum operators. The usual commutator
\([\hat{x}, \hat{p}] = i\hbar\) leads to the following connection between the commutator and the HUP
\[
\Delta x \Delta p \geq \frac{1}{2} |\langle [\hat{x}, \hat{p}] \rangle| \to \Delta x \Delta p \geq \frac{\hbar}{2}.
\] (10)

A GUP like that in (9) implies a modified commutator of the form
\[
[\hat{x}', \hat{p}] = i\hbar(1 + \beta p^2),
\]
where the primes indicate that the operators are modified. In [1], the position operator was
modified while the momentum operator was not, and this is the reason that the \(\Delta p\) and \(\hat{p}\)
are not primed in (9) and (11).

Modifying the commutator as in (11) comes with some challenges as reviewed in [18].
These challenges are: (i) violation of the equivalence principle; (ii) the “soccer” ball problem
[19, 20] (i.e. the difficulty in constructing consistent multi-particle states); (iii) velocity
dependence of position and momentum uncertainties. In the present work we avoid these
problems as well as the constraints coming from GRB observations. We do this by modifying
the position and momentum operators so as to give a minimal length, while at the same
time preserving the usual commutator and usual energy-momentum relationship for these
modified operators.

The usually inference drawn from the above arguments is that it is the modification of
the commutator which determines the existence or not of a minimal distance. However, it
was shown in reference [21] that this is not the case; there are a host of different ways to alter
the position and/or momentum operators which lead to the same commutator (11) but may
or may not give a minimal length. If instead of (9) one modifies the momentum operator
then one would find that \(\Delta x' \Delta p' \geq \frac{\hbar}{2} \left(1 + \beta \Delta p^2\right)\). In this case \(\Delta x'\) would be proportional to
\(1/\Delta p' + \beta \Delta p^2/\Delta p'\), and whether or not this has a minimum would depend on the detailed
behavior of \(\Delta p'\). The conclusion of [21] was that it was the specific modification of the
position and momentum operators which determined the existence of a minimal length
rather than the way in which the commutator was modified.
Here we show that we can use the modified momentum operators of (5) to obtain a minimum length scale but without modifying the fundamental commutator between the new, modified position and momentum operator. The position operator must be modified if the commutator, in terms of the new operators, is to remain unchanged, since the momentum operators in (5) are changed relative to the standard momentum. In other words, we want
\[
[\hat{x}', \hat{p}'] = i\hbar \rightarrow \Delta x' \Delta p' \geq \frac{\hbar}{2},
\] (12)

We call (12) the Modified Heisenberg Uncertainty principle (MHUP) since we modify the operators but keep the commutation relationship the same, unlike the GUP where the commutator is modified. For the usual HUP, written in terms of \(\Delta x\) and \(\Delta p\), there is no lower bound on \(\Delta x\). This is because \(\Delta x \propto \frac{1}{\Delta p}\) and since \(\Delta p\) can go to \(\infty\), \(\Delta x\) can go to zero \(i.e.\) it has no lower limit. However, for the modified momentum operators in (5), \(\Delta p'\) is bounded above by \(p_0\) and so \(\Delta x'\) will be bounded below by \(\Delta x' > \frac{\hbar}{2p_0}\). To see the upper bound on uncertainty in modified momentum, we note that the uncertainty of the modified momentum is \(\Delta p' = \sqrt{\langle p'^2 \rangle - \langle p' \rangle^2}\). From \(\langle p' \rangle^2 \geq 0\) one has \(\Delta p' \leq \sqrt{\langle p'^2 \rangle}\). Since the magnitude of the momentum operator \(p'\) is bounded above by \(p_0\), we find \(\langle p'^2 \rangle < p_0^2\) and thus \(\Delta p' < p_0\).

Fig. 2 shows the behavior of the MHUP as given by (12) with \(\Delta p' < p_0\) and thus \(\Delta x' > \frac{\hbar}{2p_0}\). In comparing Fig. 1 with Fig. 2 one sees that although both have a non-zero minimum for \(\Delta x'\) the way in which the minimum is obtained is different.

In order to preserve the standard commutator of (12), we need to modify the standard position which in momentum space is given by \(\hat{x} = i\hbar \partial_p\) in momentum space. We assume that our modified position operator has the form
\[
\hat{x}' = i\hbar f(p) \partial_p,
\] (13)
and we will choose the function \(f(p)\) so as to make (12) true for either modified momentum in (5).
FIG. 2: The relationship between $\Delta x'$ and $\Delta p'$ from the Modified Heisenberg Uncertainty Principle where $\Delta p_{\text{max}} = p_0$ and $\Delta x' > \frac{\hbar}{2p_0} = \Delta x_0$.

Inserting the modified position (13) and modified momenta (5) into $[\hat{x}', \hat{p}']$ we find

$$\left[\hat{x}', p_0 \tanh \left( \frac{p}{p_0} \right) \right] = i\hbar f(p) \text{sech}^2 \left( \frac{p}{p_0} \right),$$  \hspace{1cm} (14)

and

$$\left[\hat{x}', p_0 \arctan \left( \frac{p}{p_0} \right) \right] = i\hbar f(p) \left( 1 + \left( \frac{p}{p_0} \right)^2 \right)^{-1}.$$  \hspace{1cm} (15)

Requiring that the right hand sides of (14) and (15) equal $i\hbar$ leads to the following modified position operators

$$\hat{x}' = i\hbar \cosh^2 \left( \frac{p}{p_0} \right) \partial_p \text{ for } \hat{p}' = p_0 \tanh \left( \frac{p}{p_0} \right)$$ \hspace{1cm} (16)

and

$$\hat{x}' = i\hbar \left[ 1 + \left( \frac{p}{p_0} \right)^2 \right] \partial_p \text{ for } \hat{p}' = p_0 \arctan \left( \frac{p}{p_0} \right).$$ \hspace{1cm} (17)

Note that the position operator in (17) is that same as the modified position operator used in [1] if one sets $\beta = 1/p_0^2$.

Thus we have found modified position operators, modified momentum operators (summarized in equations (16) and (17)), and modified energies (given in (7) and (8)) which preserve the usual position-momentum commutator and the usual energy-momentum relationship in terms of these modified operators. Nevertheless the modified operators lead a minimum distance as illustrated in Fig. 2, while the energy-momentum relationship does not lead
to dispersion. Additionally, since both the commutator and energy-momentum relationship of these modified operators and quantities are the same as for the standard operators and quantities, the problems [7, 18, 20] connected with a GUP, as embodied by equations (9) and (11), are avoided.

IV. MODIFIED GENERALIZED UNCERTAINTY PRINCIPLE

In the previous section we accomplished the task of finding a modified position, momentum and energy which gave a minimum length but did not lead to dispersion of photons of different energies, thus evading the constraints on a minimum length coming from the Fermi Gamma-Ray space telescope observations [8]. However, as already mentioned the way in which the minimum distance was obtained via the GUP is different than how it was obtained via the MHUP. This is seen graphically by comparing Fig. 1 (GUP) with Fig. 2 (MHUP). For the GUP $\Delta x'$ increases linearly after reaching its minimum, whereas for the MHUP $\Delta x'$ approaches its minimal value $\hbar \sqrt{\beta}$ as $p \to \infty$ and $p' \to p_0$.

It is possible to retain the linear increase of $\Delta x'$ as $\Delta p'$ increases, at least for some range of $\Delta p'$. This is accomplished by picking modified position and momentum so that instead of the commutator remaining unchanged, as in (12), that it takes the form

$$[\hat{x}', \hat{p}'] = i\hbar \left(1 + \beta p'^2\right),$$  \hspace{1cm} (18)

which is similar to the modified commutator in (11) except here the momentum on the right hand side of (18) is the modified momentum rather than the usual momentum. Since in the commutator (11) we use the same, unmodified momentum on both sides we do the same in the commutator (18) except now the momentum operator is modified on both sides. One could use the ordinary momentum on the right hand side of (18), but this would lead to more complicated calculations without changing the overall results and conclusions we draw from (18).

The modified GUP (MGUP) \footnote{Modified GUP means modifying the position and momentum operators, and the commutator as in (18).} associated with (18) is $\Delta x' = \frac{\hbar}{2} \left(\frac{1}{\Delta p'} + \beta \Delta p'\right)$. Since the relationship between $\Delta x'$ and $\Delta p'$ coming from (18) is exactly the same as between $\Delta x'$ and $\Delta p$ coming from (9), one might expect that the behavior will be identical to Fig. 1. However
since (18) has two parameters \( i.e. \ p_0 \) and \( \beta \) relative to (9) there will be two different regimes. First, when \( \frac{1}{\sqrt{\beta}} \geq p_0 \) the local minimum (which in Fig. 1 is at \( \Delta p = \frac{1}{\sqrt{\beta}} \)) will occur after the modified momentum uncertainties have reached their maximum value of \( p_0 \). This case is shown in Fig. 3.

![Graph showing the modified GUP for the case when \( \frac{1}{\sqrt{\beta}} \geq p_0 = \Delta p_{max} \).](image)

FIG. 3: The modified GUP for the case when \( \frac{1}{\sqrt{\beta}} \geq p_0 = \Delta p_{max} \). The maximum value of the modified momentum operators, \( p' = p_0 \tanh(p/p_0) \) or \( p' = p_0 \arctan(p/p_0) \), is reached before the local minimum is reached.

Second, when \( \frac{1}{\sqrt{\beta}} < p_0 \) the local minimum in \( \Delta x' \) coming from the \( \beta \) term in (18) will occur for values of \( \Delta p' \) less than \( p_0 \). Thus \( \Delta x' \) will have a local minimum before \( \Delta p' \) reaches \( p_0 \) as shown in Fig. 4.

The explicit position operators from the two modified momentum operators are generalizations of (16) and (17) and have added terms due to the \( \beta \) term in (18). These modified position operators are

\[
\hat{x}' = i\hbar \left[ \cosh^2 \left( \frac{p}{p_0} \right) + \beta p_0^2 \sinh^2 \left( \frac{p}{p_0} \right) \right] \partial_p \quad \text{for} \quad p' = p_0 \tanh \left( \frac{p}{p_0} \right),
\]

and

\[
\hat{x}' = i\hbar \left[ 1 + \left( \frac{p}{p_0} \right)^2 \right] \left[ 1 + \beta p_0^2 \arctan^2 \left( \frac{p}{p_0} \right) \right] \partial_p \quad \text{for} \quad p' = p_0 \arctan \left( \frac{p}{p_0} \right).
\]

(19)  

(20)
The modified GUP for $\frac{1}{\sqrt{\beta}} < p_0 = \Delta p_{\text{max}}$. The maximum value of the modified momentum operators, $p' = p_0 \tanh(p/p_0)$ or $p' = p_0 \arctan(p/p_0)$, is reached after the local minimum is reached.

The motivation to add the extra $\beta$ terms comes from the heuristic arguments that $\Delta x'$ should behave linearly as the energy/momentum becomes large. There are now two phenomenological parameters -- $p_0$ and $\beta$ -- which are associated with the quantum gravity scale. Recent work on modifying the uncertainty principle [18] also proposes two different scales. However, in [18] the two scales were associated with a low energy/momentum scale coming from the Cosmological constant/de Sitter scale, and a high energy/momentum scale coming from the Planck/quantum gravity scale. In the above $p_0$ and $\beta$ are both associated with a high energy/momentum scale.

Adding the extra $\beta$ term as in the MGUP of equation [18] opens the door to the problems associated with modifying the position-momentum commutator [18], namely violation of the equivalence principle, the “soccer ball” problem, and velocity dependence of position and momentum uncertainties. This may be an argument to prefer the MHUP over the MGUP since the former does not run into these issues.

V. SUMMARY AND CONCLUSIONS

Phenomenological, bottom up approaches to quantum gravity, like double special relativity [9, 10] generically have as part of their structure the idea of an absolute minimal distance scale. Top down approaches to quantum gravity, such as string theory or loop
quantum gravity, also incorporate the idea of a minimal distance scale.

Generally, this absolute minimal length scale is thought to be around the Planck scale and thus hard to test. However, in [4] the proposal was made that one could test for this minimal length scale using short GRBs, since the minimal distance scale would lead to an energy dependent dispersion of gamma rays which could be detected through differences in arrival times of different energy photons. Observations of one such GRB (GRB090510) by the Fermi-Gamma ray observatory [8] have placed constraints on this minimal distance scale to be sub-Planckian. This constraint assumed that the relationship between energy and momentum was of the general form (1) and (2) leading to an energy dependence of the photon’s velocity as given in (3) and a time delay (4). Also assumed was that the lowest order correction to the photon velocity was linear in energy $E$.

In this work we constructed modified momenta, (5), and associated modified energies, (7) and (8). The modified momenta and energies has a maximum cut off (inspired by Born-Infeld electrodynamics [11] with its maximum electric field). However, the relationship between the modified momenta and energies, given in (7) and (8), are different from the relationships given in (1) and (2). The energy-momentum relationship in (7) and (8) do not lead to an energy-dependent photon velocity and thus no dispersion. This avoids the constraints coming from the GRB observations. In contrast the energy-momentum relationship in (1) and (2) do lead to an energy dependent photon velocity as given in (3).

We also obtained a minimal distance via a modified Heisenberg Uncertainty Principle (i.e. modification of the operators but not the commutator) in contrast to obtaining a minimal distance via the standard Generalized Uncertainty Principle (i.e. modification of the commutator and operators). In conjunction with the modified momenta from (5), we also introduced appropriately modified position operators, that led to a minimal distance. This was accomplished without modifying the fundamental position -momentum commutation relationship. This avoids many of the problems and issues associated with modifying the position-momentum commutator [18–20].

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