GALOIS REPRESENTATIONS MODULO $p$ AND COHOMOLOGY OF HILBERT MODULAR VARIETIES

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ABSTRACT. The aim of this paper is to extend some arithmetic results on elliptic modular forms to the case of Hilbert modular forms. Among these results let’s mention:

– the control of the image of the Galois representation modulo $p$ [37][35],
– Hida’s congruence criterion outside an explicit set of primes $p$ [21],
– the freeness of the integral cohomology of the Hilbert modular variety over certain local components of the Hecke algebra and the Gorenstein property of these local algebras [30][16].

We study the arithmetic of the Hilbert modular forms by studying their modulo $p$ Galois representations and our main tool is the action of the inertia groups at the primes above $p$. In order to determine this action, we compute the Hodge-Tate (resp. the Fontaine-Laffaille) weights of the $p$-adic (resp. the modulo $p$) étale cohomology of the Hilbert modular variety. The cohomological part of our paper is inspired by the work of Mokrane, Polo and Tilouine [31, 33] on the cohomology of the Siegel modular varieties and builds upon the geometric constructions of [10, 11].

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Date: December 17, 2021.
Introduction

Let \( F \) be a totally real number field of degree \( d \), ring of integers \( \mathfrak{o} \) and different \( \mathfrak{d} \). Denote by \( \tilde{F} \) the Galois closure of \( F \) in \( \mathbb{Q} \) and by \( J_F \) the set of all embeddings of \( F \) into \( \mathbb{Q} \subset \mathbb{C} \).

We fix an ideal \( n \subset \mathfrak{o} \) and we put \( \Delta = N_{\mathbb{F}/\mathbb{Q}}(n \mathfrak{d}) \).

For a weight \( k = \sum_{\tau \in J_F} k_\tau \tau \in \mathbb{Z}[J_F] \) as in Def.1.1 we put \( k_0 = \max\{k_\tau | \tau \in J_F\} \). If \( \psi \) is a Hecke character of \( F \) of conductor dividing \( n \) and type \( 2 - k_0 \) at infinity, we denote by \( S_k(n, \psi) \) the corresponding space of Hilbert modular cuspforms (see Def.1.3).

Let \( f \in S_k(n, \psi) \) be a newform (that is a primitive normalized eigenform). For all ideals \( \mathfrak{a} \subset \mathfrak{o} \), we denote by \( c(f, \mathfrak{a}) \) the eigenvalue of the standard Hecke operator \( T_{\mathfrak{a}} \) on \( f \).

Let \( p \) be a prime number and let \( \iota_p : \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}}_p \) be an embedding.

Denote by \( E \) a sufficiently large \( p \)-adic field, of ring of integers \( \mathcal{O} \), maximal ideal \( \mathcal{P} \) and residue field \( \kappa \).

0.1. Galois image results. The absolute Galois group of a field \( L \) is denoted by \( \mathcal{G}_L \).

By results of Taylor [40] and Blasius-Rogawski [1] there exists a continuous representation \( \rho = \rho_{f, p} : \mathcal{G}_F \rightarrow \text{GL}_2(E) \) which is absolutely irreducible, totally odd, unramified outside \( \mathfrak{p} \mathfrak{n} \) and such that for each prime ideal \( v \) of \( \mathfrak{o} \), not dividing \( pn \), we have:

\[
\text{tr}(\rho(\text{Frob}_v)) = \iota_p(c(f, v)), \quad \det(\rho(\text{Frob}_v)) = \iota_p(\psi(v)) N_{\mathbb{F}/\mathbb{Q}}(v),
\]

where \( \text{Frob}_v \) denotes a geometric Frobenius at \( v \).

By taking a Galois stable \( \mathcal{O} \)-lattice, we define \( \mathfrak{p} = \rho \mod \mathcal{P} : \mathcal{G}_F \rightarrow \text{GL}_2(\kappa) \), whose semi-simplification is independent of the particular choice of a lattice.

The following proposition is a generalization to the Hilbert modular case, of results of Serre [37] and Ribet [35] on elliptic modular forms (see Prop.3.1, Prop.3.8 and Prop.3.17).

**Proposition 0.1.** (i) For all but finitely many primes \( p \),

\( (\text{Irr}_{\mathfrak{p}}) \mathfrak{p} = \mathfrak{p}_{f, p} \) is absolutely irreducible.

(ii) If \( f \) is not a theta series, then for all but finitely many primes \( p \),

\( (\text{LI}_{\mathfrak{p}}) \) there exists a power \( q \) of \( p \) such that \( \text{SL}_2(\mathbb{F}_q) \subset \text{im}(\mathfrak{p}) \subset \kappa^\times \text{GL}_2(\mathbb{F}_q) \).

(iii) Assume that \( f \) is not a twist by a character of any of its \( d \) internal conjugates and is not a theta series. Then for all but finitely many primes \( p \), there exist a power \( q \) of \( p \), a partition \( J_F = \coprod_{i \in I} J_{F_i} \) and \( \sigma_{i, \tau} \in \text{Gal}(\mathbb{F}_q / \mathbb{F}_p) \), \( \tau \in J_{F_i} \), such that \( (\tau \neq \tau' \Rightarrow \sigma_{i, \tau} \neq \sigma_{i, \tau'}) \) and

\( (\text{LI}_{\text{Ind}_{\mathfrak{p}}}) \text{Ind}_{\mathcal{F}}^{\mathfrak{p}}(\mathfrak{p}) : \mathcal{G}_{\mathcal{F}^n} \rightarrow \text{SL}_2(\mathbb{F}_q)^{J_F} \) factors as a surjection \( \mathcal{G}_{\mathcal{F}^n} \rightarrow \text{SL}_2(\mathbb{F}_q)^I \) followed by the map \( (M_i)_{i \in I} \mapsto (M_i^{\sigma_{i, \tau}})_{i \in I, \tau \in J_{F_i}} \), where \( \mathcal{F}^n \) denotes the compositum of \( \mathcal{F} \) and the fixed field of \( (\text{Ind}_{\mathcal{F}}^{\mathfrak{p}}(\mathfrak{p}))^{-1}(\text{SL}_2(\mathbb{F}_q)^{J_F}) \).

0.2. Cohomological results. Let \( Y_{/\mathbb{Z}[[p]]} \) be the Hilbert modular variety of level \( K_1(n) \) (see §1.4). Consider the \( p \)-adic étale cohomology \( H^*(Y_{/\mathbb{Q}}, \mathcal{V}_n(\overline{\mathbb{Q}}_p)) \), where \( \mathcal{V}_n(\overline{\mathbb{Q}}_p) \) denotes the local system of weight \( n = \sum_{\tau \in J_F} (k_\tau - 2) \tau \in \mathbb{N}[J_F] \) (see §2.1). By a result of Brylinski and Labesse [3] the subspace \( W_f := \bigcap_{\mathfrak{a} \subset \mathfrak{o}} \ker(T_{\mathfrak{a}} - c(f, \mathfrak{a})) \) of \( H^d(Y_{/\mathbb{Q}}, \mathcal{V}_n(\overline{\mathbb{Q}}_p)) \) is isomorphic, as \( G_{\mathcal{F}} \)-module and after semi-simplification, to the tensor induced representation \( \otimes \text{Ind}_{\mathcal{F}}^{\mathfrak{p}}(\mathfrak{p}) \).
Assume that

(I) $p$ does not divide $\Delta$.

Then $Y$ has smooth toroidal compactifications over $\mathbb{Z}_p$ (see [10]). For each $J \subset J_F$, we put $|p(J)| = \sum_{\tau \in F(J)} (k_0 - m_\tau - 1) + \sum_{\tau \in J_F \setminus J} m_\tau$, where $m_\tau = (k_0 - k_\tau)/2 \in \mathbb{N}$. By applying a method of Chai and Faltings [15] Chap.VI, one can prove (see [11] Thm.7.8, Cor.7.9)

Theorem 0.2. Assume that $p$ does not divide $\Delta$. Then

(i) the Galois representation $H^1(Y_{\overline{\mathbb{Q}}}, V_n(\overline{\mathbb{Q}}_p))$ is crystalline at $p$ and its Hodge-Tate weights belong to the set $\{ |p(J)|, J \subset J_F \mid |J| \leq j \}$, and

(ii) the Hodge-Tate weights of $W_f$ are given by the multiset $\{ |p(J)|, J \subset J_F \}$.

For our main arithmetic applications we need to establish a modulo $p$ version of the above theorem. This is achieved under the following additional assumption:

(II) $p - 1 > \sum_{\tau \in J_F} (k_\tau - 1)$.

The integer $\sum_{\tau \in J_F} (k_\tau - 1)$ is equal to the difference $|p(J_F)| - |p(\emptyset)|$ between the largest and the smallest Hodge-Tate weights of the cohomology of the Hilbert modular variety. We use (I) and (II) in order to apply Fontaine-Laffaille’s Theory [17], as well as Faltings’ Comparison Theorem modulo $p$ [14]. By adapting to the case of Hilbert modular varieties some techniques developed by Mokrane, Polo and Tilouine [31, 33] for Siegel modular varieties, such as the construction of an integral Bernstein-Gelfand-Gelfand complex for distribution algebras, we compute the Fontaine-Laffaille weights of $H^\bullet(Y_{\overline{\mathbb{Q}}}, V_n(\kappa))$ (see Thm.5.13).

0.3. Arithmetic results. Consider the $\mathcal{O}$-module of interior cohomology $H^d(Y, V_n(\mathcal{O}))'$, defined as the image of $H^d(Y, V_n(\mathcal{O}))$ in $H^d(Y, V_n(E))$. Let $\mathcal{T} = \mathcal{O}[T_a, a \subset \mathfrak{o}]$ be the full Hecke algebra acting on it, and let $\mathcal{T}' \subset \mathcal{T}$ be the subalgebra generated by the Hecke operators outside a finite set of places containing those dividing $\mathfrak{n} \mathfrak{p}$. Denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{T}$ corresponding to $f$ and $t_\mathfrak{p}$ and put $\mathfrak{m}' = \mathfrak{m} \cap \mathcal{T}'$.

Theorem 0.3. Assume that the conditions (I) and (II) from §0.2 hold.

(i) If (Irр) holds, $d(p - 1) > 5 \sum_{\tau \in J_F} (k_\tau - 1)$ and

$$(\text{MW}) \text{ the middle weight } \left| \frac{|p(J)| + |p(\emptyset)|}{2} \right| = \frac{k_0 - 1}{2} \text{ does not belong to } \{ |p(J)|, J \subset J_F \},$$

then the local component $H^d_{\mathcal{O}_f}(Y, V_n(\mathcal{O}))_{\mathfrak{m}}$ of the boundary cohomology vanishes, and the Poincaré pairing $H^d(Y, V_n(\mathcal{O}))_{\mathfrak{m}}' \times H^d(Y, V_n(\mathcal{O}))_{\mathfrak{m}}' \to \mathcal{O}$ is a perfect duality.

(ii) If (LlInd_s) holds, then $H^\bullet(Y, V_n(\mathcal{O}))_{\mathfrak{m}} = H^d(Y, V_n(\mathcal{O}))_{\mathfrak{m}}$ is a free $\mathcal{O}$-module of finite rank and its Pontryagin dual is isomorphic to $H^d(Y, V_n(E/\mathcal{O}))_{\mathfrak{m}}$.

The proof involves a “local-global” Galois argument. For the (i), we use lemma 4.2(ii) and a theorem of Pink [32] on the étale cohomology of a local system restricted to the boundary of $Y$ (see Thm.4.4). For the (ii), we use lemma 6.5 and the computation of the Fontaine-Laffaille weights of the cohomology of Thm.5.13 (see Thm.6.6).

Let $\Lambda^\ast(Ad^0(f), s)$ be the imprimitive adjoint $L$-function of $f$, completed by its Euler factors at infinity and let $W(f)$ be the complex constant from the functional equation of the standard $L$-function of $f$ (see §4.4). We denote by $\Omega_f^\pm \in \mathbb{C}^\times/\mathcal{O}_F^\times$ any two complementary periods defined by the Eichler-Shimura-Harder isomorphism (see §4.2).
Theorem A (Thm.4.11) Let \( f \) and \( p \) be such that (I), (\text{Irr}_\varpi) and (MW) hold, and \( p - 1 > \max(1, \frac{k}{3}) \sum_{\tau \in J_p} (k_\tau - 1) \). Assume that \( \iota_p(W(f))^{\Lambda(\text{Ad}^0(f)_I)} \in \mathcal{P} \). Then there exists another normalized eigenform \( g \in S_k(n, \psi) \) such that \( f \equiv g \pmod{\mathcal{P}} \), in the sense that \( c(f, a) \equiv c(g, a) \pmod{\mathcal{P}} \) for each ideal \( \mathfrak{a} \subset \mathfrak{o} \).

The proof follows closely the original one given by Hida [21] in the elliptic modular case, and uses Thm.0.3(i) as well as a formula of Shimura relating \( \Lambda^*(\text{Ad}^0(f)_I) \) to the Petersson inner product of \( f \) (see (19)). Let us note that Ghate [18] has obtained a very similar result when the weight \( k \) is parallel. A converse for Thm A is provided by the (ii) of the following

Theorem B (Thm.6.7) Let \( f \) and \( p \) be such that (I), (II) and (\text{LI}_{\text{ind}}_\varpi) hold. Then

(i) \( H^*(Y, V_n(\kappa))[m] = H^d(Y, V_n(\kappa))[m] \) is a \( \kappa \)-vector space of dimension \( 2^d \).
(ii) \( H^*(Y, V_n(\mathcal{O}))[m] = H^d(Y, V_n(\mathcal{O}))[m] \) is free of rank \( 2^d \) over \( T_m \).
(iii) \( T_m \) is Gorenstein.

By [30] it is enough to prove (i), which is a consequence of Thm.0.3(ii) and the \( q \)-expansion principle §1.7.

The last theorem is due, under milder assumptions, to Mazur [30] for \( F = \mathbb{Q} \) and \( k = 2 \), and to Faltings and Jordan [16] for \( F = \mathbb{Q} \). The Gorenstein property is proved by Diamond [8] when \( F \) is quadratic and \( k = (2, 2) \), under the assumptions (I), (II) and (\text{Irr}_\varpi). We expect that Diamond’s approach via intersection cohomology could be generalized in order to prove the Gorenstein property of \( T_m \) under the assumptions (I), (II) and (\text{LI}_{\varpi}) (see lemma 4.2(i) and remark 4.3).

When \( f \) is ordinary at \( p \) (see Def.1.13) we can replace the assumptions (I) and (II) of theorems A and B by the weaker assumptions that \( p \) does not divide \( N_{F/\mathbb{Q}}(\mathfrak{o}) \) and that \( k \) (mod \( p - 1 \)) satisfies (II) (see Cor.6.10). The proof uses Hida’s families of \( p \)-adic ordinary Hilbert modular forms. We prove an exact control theorem for the ordinary part of the cohomology of the Hilbert modular variety, and give a new proof of Hida’s exact control theorem for the ordinary Hecke algebra (see Prop.6.9).

Theorems A and B relate \( \mathcal{O}/\iota_p(W(f))^{\Lambda(\text{Ad}^0(f)_I)} \) to the congruence module associated to the \( \mathcal{O} \)-algebra homomorphism \( T \to \mathcal{O}, T_a \mapsto \iota_p(c(f, a)) \). In a subsequent paper [12] we relate these two with the cardinality of the Selmer group of \( \text{Ad}^0(\rho) \otimes \mathbb{E}/\mathcal{O} \). An interesting question is whether \( \Omega^2 \) are the periods involved in the Bloch-Kato conjecture for \( \text{Ad}^0(f) \) (see the work of Diamond, Flach and Guo [9] for the elliptic modular case).

0.4. Explicit results. By a classical theorem of Dickson, the image in \( \text{PGL}_2(\kappa) \) of an irreducible subgroup of \( \text{GL}_2(\kappa) \) not satisfying (\text{LI}_{\varpi}), is either a dihedral, a tetrahedral, an octahedral or an icosahedral group. In the next proposition we consider the later exceptional cases for the image of \( \varpi \) in \( \text{PGL}_2(\kappa) \).

Denote by \( \mathfrak{o}^\times_2 \) (resp. by \( \mathfrak{o}_{n,1}^\times \)) be the group of totally positive (resp. congruent to 1 modulo \( n \)) units of \( \mathfrak{o} \).
Proposition 0.4. Assume $p$ does not divide $\Delta$ and $p > k_0$.

(i) Assume that $k$ is non-parallel. If for all $J \subset J_F$, there exists an unit $\epsilon \in o_F^x \cap o_{n,1}^x$, such that $p$ does not divide $N_{F/Q}(\prod_{\tau \in J} \tau(\epsilon)^{k_0-m_\tau-1} - \prod_{\tau \in J_p \setminus J} \tau(\epsilon)^{m_\tau}) \neq 0$, then $(\text{Irr}_p)$ holds.

(ii) If $d(p-1) > 5 \sum_{\tau \in J_p} (k_\tau - 1)$, then the image of $p$ in $\text{PGL}_2(\kappa)$ is not a tetrahedral, an octahedral nor an icosahedral group.

(iii) Assume that for all $\tau \in J_F$, $p \neq 2k_\tau - 1$ and the following condition:

(non-CM) for each quadratic extension $K$ of $F$ of discriminant dividing $n$ and splitting all the primes $p$ of $F$ above $p$, one of the following holds:

1. $K$ is CM and there does not exist a Hecke character $\varphi$ of $K$ of conductor dividing $n \Delta_{K/F}^{-1}$ and infinity type $(m_\tau, k_0-1-m_\tau)_{\tau \in J_F}$, such that $\rho \equiv \text{Ind}_F^K \varphi \pmod{\mathcal{P}}$

2. $K$ is not CM and for all extensions $\tau'$ of $\tau \in J_F$ to $K$, there exists a unit $\epsilon \in O_K^x$, $\epsilon - 1 \in n$ such that $p$ does not divide $N_{K/Q}(\prod_{\tau \in J_F} \tau(\epsilon)^{m_\tau} \tau(c(\epsilon))^{k_0-m_\tau-1-1})$.

Then the image of $p$ in $\text{PGL}_2(\kappa)$ is not a dihedral group.

(iv) Assume $(\text{LI}_p)$ and that $k$ is not induced from a weight for a strict subfield $F'$ of $F$. Assume moreover that for all $\tau, \tau' \in J_F$, $p \neq k_\tau + k_{\tau'} - 1$. Then $(\text{LI}_{\text{Ind}_p})$ holds.

By the last proposition, we obtain the following corollary to theorems A and B.

Corollary 0.5. Let $\epsilon$ be any element of $o_+^x \cap o_{n,1}^x$.

(i) Assume $d=2$ and $k=(k_0, k_0-2m_1)$, with $m_1 \neq 0$. If $p \nmid \Delta N_{F/Q}((\epsilon^{m_1-1})(\epsilon^{k_0-m_1-1}))$ and $p-1 > 4(k_0 - m_1 - 1)$, then theorem A holds. If additionally we have the (non-CM) condition then theorem B also holds.

(ii) Assume $d=3$, id $\neq \tau \in J_F$ and $k=(k_0, k_0-2m_1, k_0-2m_2)$, with $0 \leq m_1 \leq m_2 \neq 0$. If $p \nmid \Delta N_{F/Q}((\tau(\epsilon)^{m_1-1} - \epsilon^{-m_2})(\tau(\epsilon)^{m_1} - \epsilon^{-m_2+1-k_0})(\tau(\epsilon)^{m_1+1-k_0} - \epsilon^{-m_2+1-k_0}))$ and $p-1 > \frac{1}{3}(3k_0 - 2m_1 - 2m_2 - 3)$, then theorem A holds. If additionally we have $p \neq 2k_0-1$ and the (non-CM) condition then theorem B also holds.

0.5. Acknowledgements. I would like to thank J. Bellaïche, D. Blasius, G. Chenevier, L. Dieulefait, E. Ghate, M. Kisin, V. Lafforgue, A. Mokrane, E. Urban, J. Wildeshaus and J.-P. Wintenberger for helpful conversations. This article was completed during my visit at UCLA on an invitation by H. Hida. I would like to thank him heartily for his hospitality and for many inspiring discussions. I am grateful to G. Kings and R. Taylor for their interesting comments on an earlier version of this paper that have much improved it. Finally, I would like to thank J. Tilouine who suggested me to study this problem and supported me during the preparation of this article.
1. Hilbert modular forms and varieties.

We define the algebraic groups $D_{/Q} = \text{Res}^F_Q \mathbb{G}_m$, $G_{/Q} = \text{Res}^F_Q \text{GL}_2$ and $G^*_{/Q} = G \times_D \mathbb{G}_m$, where the fiber product is relative to the reduced norm map $\nu : G \to D$. The standard Borel subgroup of $G$, its unipotent radical and its standard maximal torus are denoted by $B$, $U$ and $T$, respectively. We identify $D \times D$ with $T$, by $(u, e) \mapsto \begin{pmatrix} ue & 0 \\ 0 & u^{-1} \end{pmatrix}$.

1.1. Analytic Hilbert modular varieties. Let $D(\mathbb{R})_+$ (resp. $G(\mathbb{R})_+$) be the identity component of $D(\mathbb{R}) = (F \otimes \mathbb{R})^\times$ (resp. of $G(\mathbb{R})$). The group $G(\mathbb{R})_+$ acts by linear fractional transformations on the space $\mathcal{H}_F = \{ z \in F \otimes \mathbb{C} \mid \text{im}(z) \in D(\mathbb{R})_+ \}$. We have $\delta_F \cong \delta^{J_F}$, where $\mathcal{H}_F = \{ z \in \mathbb{C} \mid \text{im}(z) > 0 \}$ is the Poincaré’s upper half-plane (the isomorphism being given by $\xi \otimes z \mapsto (\tau(\xi)z)_{\tau \in J_F}$, for $\xi \in F$, $z \in \mathbb{C}$). We consider the unique group action of $G(\mathbb{R})$ on the space $\mathcal{H}_F$ extending the action of $G(\mathbb{R})_+$ and such that, on each copy of $\mathcal{H}_F$ the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts by $z \mapsto -\overline{z}$. We put $\mathcal{H} = (\sqrt{-i}, ..., \sqrt{-i}) \in \mathcal{H}_F$.

We denote by $\widehat{\mathbb{Z}} = \prod_d \mathbb{Z}_d$ the profinite completion of $\mathbb{Z}$ and we put $\widehat{\mathcal{O}} = \widehat{\mathbb{Z}} \otimes \mathcal{O} = \prod_v \mathcal{O}_v$, where $v$ runs over all the finite places of $F$. Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) be the ring of the adeles (resp. of the finite adeles) of $F$. We consider the following open compact subgroup of $G(\mathbb{A}_f)$:

$$K_1(n) = \left\{ \begin{pmatrix} \frac{a}{c} & \frac{b}{d} \\ c & d \end{pmatrix} \in G(\mathbb{A}) \mid d - 1 \in n, c \in n \right\}.$$ 

The adèlic Hilbert modular variety of level $K_1(n)$ is defined as

$$Y^{an} = Y_1(n)^{an} = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_1(n)K^+_\infty.$$ 

By the Strong Approximation Theorem, the connected components of $Y^{an}$ are indexed by the narrow ideal class group $\text{Cl}^+_F = D(\mathbb{A}) / (D(\mathbb{Q})D(\widehat{\mathbb{Z}})D(\mathbb{R})_+)$ of $F$. For each fractional ideal $\mathfrak{c}$ of $F$, we put $\mathfrak{c}^* = \mathfrak{c}^{-1} \mathfrak{c}^{-1}$. We define the following congruence subgroup of $G(\mathbb{Q})$:

$$\Gamma_1(\mathfrak{c}, n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{Q}) \cap \left( \mathcal{O} \backslash \mathcal{O}_n^{\times} \right) \mid ad - bc \in \mathcal{O}_n^{\times}, d \equiv 1 \mod{n} \right\}.$$ 

Put $M^{an} = M_1(\mathfrak{c}, n)^{an} = \Gamma_1(\mathfrak{c}, n) \backslash \mathcal{H}_F$. Then we have $Y_1(n)^{an} \approx \prod_{i=1}^{h^+} M_1(\mathfrak{c}_i, n)^{an}$, where the ideals $\mathfrak{c}_i$, $1 \leq i \leq h^+$, form a set of representatives of $\text{Cl}^+_F$.

Put $\mathcal{H}_F^* = \mathcal{H}_F \prod \mathbb{P}^1(F)$. The minimal compactification $\tilde{M}^{an}$ of $M^{an}$ is defined as $\tilde{M}^{an} = \Gamma \backslash \mathcal{H}_F^*$. It is an analytic normal projective space whose boundary $\tilde{M}^{an} \backslash M^{an}$ is a finite union of closed points, called the cusps of $M^{an}$.

The same way, by replacing $G$ by $G^*$, we define $\Gamma_1(\mathfrak{c}, n)$, $M_1^{an} = M_1^*(\mathfrak{c}, n)^{an}$ and $M_1^{*an}$.

1.2. Analytic Hilbert modular forms. For the definition of the $\mathbb{C}$-vector space of Hilbert modular forms we follow [24].

**Definition 1.1.** An element $k = \sum_{\tau \in J_F} k_\tau \in \mathbb{Z}[J_F]$ is called a weight. We always assume that the $k_\tau$ have the same parity and are all $\geq 2$. We put $k_0 = \max\{ k_\tau : \tau \in J_F \}$, $n_0 = k_0 - 2$, $t = \sum_{\tau \in J_F} \tau$, $n = \sum_{\tau \in J_F} n_\tau = k - 2t$ and $m = \sum_{\tau \in J_F} m_\tau = (k_0 - k)/2$. 


For \( z \in \mathcal{H}_F, \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) we put \( j_F(\gamma, z) = c \cdot z^d + d \in D(\mathbb{C}) \), where \( z^J = \begin{cases} z_\tau, & \tau \in J, \\ \overline{z}_\tau, & \tau \in J_F \setminus J. \end{cases} \)

**Definition 1.2.** The space \( G_{k,J}(K_1(n)) \) of adèlic Hilbert modular forms of weight \( k \), level \( K_1(n) \) and type \( J \subset J_F \) at infinity, is the \( \mathbb{C} \)-vector space of the functions \( g : G(\mathbb{A}) \to \mathbb{C} \) satisfying the following three conditions:

(i) \( g(axy) = g(x) \) for all \( a \in G(\mathbb{Q}), y \in K_1(n) \) and \( x \in G(\mathbb{A}) \).

(ii) \( g(x\gamma) = \nu(\gamma)^{k+m-j}j_F(\gamma, \underline{z})^{-k}g(x) \), for all \( \gamma \in K_1^\infty \) and \( x \in G(\mathbb{A}) \).

For all \( x \in G(\mathbb{A}_f) \) define \( g_x : \mathcal{H}_F \to \mathbb{C} \), by \( x \mapsto \nu(\gamma)^{t-k-m}j_F(\gamma, \underline{z})^{k}g(x\gamma) \), where \( \gamma \in G(\mathbb{R})^+ \) is such that \( z = \gamma \cdot z_\tau \). By (ii) \( g_x \) does not depend on the particular choice of \( \gamma \).

(iii) \( g_x \) is holomorphic at \( z_\tau \), for \( \tau \in J \), and anti-holomorphic at \( z_\tau \), for \( \tau \in J_F \setminus J \) (when \( F = \mathbb{Q} \) an extra condition of holomorphy at cusps is needed).

The space \( S_{k,J}(K_1(n)) \) of adèlic Hilbert modular cuspforms is the subspace of \( G_{k,J}(K_1(n)) \) consisting of functions satisfying the following additional condition:

(iv) \( \int_{U(\mathbb{Q}) \setminus U(\mathbb{A})} g(ux)du = 0 \), for all \( x \in G(\mathbb{A}) \) and all additive Haar measures \( du \).

The conditions (i) and (ii) of the above definition imply that for all \( g \in G_{k,J}(K_1(n)) \) there exists a Hecke character \( \psi \) of \( F \) of conductor dividing \( \mathfrak{n} \) and of type \( \mathfrak{n} \mathfrak{t}_0 \) at infinity, such that for all \( x \in G(\mathbb{A}) \) and for all \( z \in D(\mathbb{Q})D(\mathbb{Z})D(\mathbb{R}) \), we have \( g(zx) = \psi(z)^{-1}g(x) \).

**Definition 1.3.** Let \( \psi \) be a Hecke character of \( F \) of conductor dividing \( \mathfrak{n} \) and of type \( \mathfrak{n} \mathfrak{t}_0 \) at infinity. The space \( S_{k,J}(\mathfrak{n}, \psi) \) (resp. \( G_{k,J}(\mathfrak{n}, \psi) \)) is defined as the subspace of \( S_{k,J}(K_1(n)) \) (resp. \( G_{k,J}(K_1(n)) \)) of elements \( g \) satisfying \( g(zx) = \psi(z)^{-1}g(x) \), for all \( x \in G(\mathbb{A}) \) and for all \( z \in D(\mathbb{A}) \). When \( J = J_F \) this space is denoted by \( S_k(\mathfrak{n}, \psi) \) (resp. by \( G_k(\mathfrak{n}, \psi) \)).

As the characters of the ideal class group \( \text{Cl}_F = D(\mathbb{A})/D(\mathbb{Q})D(\mathbb{Z})D(\mathbb{R}) \) of \( F \) form a basis of the complex valued functions on this set, we have :

\[
(1) \quad G_{k,J}(K_1(n)) = \bigoplus_{\psi} G_{k,J}(\mathfrak{n}, \psi), \quad S_{k,J}(K_1(n)) = \bigoplus_{\psi} S_{k,J}(\mathfrak{n}, \psi),
\]

where \( \psi \) runs over the Hecke characters of \( F \), of conductor dividing \( \mathfrak{n} \) and infinity type \( \mathfrak{n} \mathfrak{t}_0 \). Let \( \Gamma \) be a congruence subgroup of \( G(\mathbb{Q}) \). We recall the classical definition :

**Definition 1.4.** The space \( G_{k,J}(\Gamma; \mathbb{C}) \) of Hilbert modular forms of weight \( k \), level \( \Gamma \) and type \( J \subset J_F \) at infinity, is the \( \mathbb{C} \)-vector space of the functions \( g : \mathcal{H}_F \to \mathbb{C} \) which are holomorphic at \( z_\tau \), for \( \tau \in J \), and anti-holomorphic at \( z_\tau \), for \( \tau \in J_F \setminus J \), and such that for every \( \gamma \in \Gamma \) we have \( g(\gamma(z)) = \nu(\gamma)^{t-k-m}j_J(\gamma, z)^{-k}g(z) \).

The space \( S_{k,J}(\Gamma; \mathbb{C}) \) of Hilbert modular cuspforms is the subspace of \( G_{k,J}(\Gamma; \mathbb{C}) \), consisting of functions vanishing at all cusps.

Put \( x_i = \left( \begin{array}{cc} \eta_i & 0 \\ 0 & 1 \end{array} \right) \), where \( \eta_i \) is the idèle associated to the ideal \( \mathfrak{c}_i, 1 \leq i \leq \mathfrak{h}^+ \). The map \( g \mapsto (g_{x_i})_{1 \leq i \leq \mathfrak{h}^+} \) (see Def.1.2) induces isomorphisms :

\[
(2) \quad G_{k,J}(K_1(n)) \simeq \bigoplus_{1 \leq i \leq \mathfrak{h}^+} G_{k,J}(\Gamma(\mathfrak{c}_i, n); \mathbb{C}), \quad S_{k,J}(K_1(n)) \simeq \bigoplus_{1 \leq i \leq \mathfrak{h}^+} S_{k,J}(\Gamma(\mathfrak{c}_i, n); \mathbb{C}).
\]

Let \( d\mu(z) = \prod_{\tau \in J_F} y_\tau^{-2}dx_\tau dy_\tau \) be the standard Haar measure on \( \mathcal{H}_F \).
Definition 1.5. The Petersson inner product of two cuspsforms \( g, h \in S_k, j(K_1(n)) \) is given by the formula

\[
(g, h)_n = \sum_{i=1}^{h^+} \int_{\Gamma_1(n) \backslash \mathfrak{H}} g_i(z) h_i(z) y^k d\mu(z),
\]
where \((g_i)_{1 \leq i \leq h^+}\) (resp. \((h_i)_{1 \leq i \leq h^+}\)) is the image of \( g \) (resp. \( h \)) under the isomorphism (2).

1.3. Hilbert-Blumenthal abelian varieties. A sheaf over a scheme \( S \) which is locally free of rank one over \( H \) is denoted by an invertible \( \mathfrak{O}_S \)-bundle on \( S \).

Definition 1.6. A Hilbert-Blumenthal abelian variety (HBAV) over a \( \mathbb{Z} \) is an abelian scheme \( \pi: \mathcal{A} \to S \) of relative dimension \( d \) together with an injection \( \iota: \mathfrak{O} \to \text{End}(\mathcal{A}/S) \), such that \( \omega_{\mathcal{A}/S} := \pi_* \Omega^1_{\mathcal{A}/S} \) is an invertible \( \mathfrak{O} \)-bundle on \( S \).

Let \( c \) be a fractional ideal of \( F \) and \( c_\mathfrak{O} \) be the cone of totally positive elements in \( c \). Given a HBAV \( \mathcal{A}/S \), the functor assigning to a \( S \)-scheme \( X \) the set \( \mathcal{A}(X) \) is representable by another HBAV, denoted by \( \mathcal{A} \otimes \mathfrak{O} \). Then \( \iota \) yields \( c \mapsto \text{Hom}_\mathfrak{O}(\mathcal{A}, \mathcal{A} \otimes \mathfrak{O} c) \). The dual of a HBAV \( \mathcal{A} \) is denoted by \( \mathcal{A}^\vee \).

Definition 1.7. (i) A \( c \)-polarization on a HBAV \( \mathcal{A}/S \) is an \( \mathfrak{O} \)-linear isomorphism \( \lambda: \mathcal{A} \otimes \mathfrak{O} \sim \to \mathcal{A}^\vee \), such that under the induced isomorphism \( \text{Hom}_\mathfrak{O}(\mathcal{A}, \mathcal{A} \otimes \mathfrak{O} c) \cong \text{Hom}_\mathfrak{O}(\mathcal{A}, \mathcal{A}^\vee) \) elements of \( c \) (resp. \( c_\mathfrak{O} \)) correspond exactly to symmetric elements (resp. polarizations).

(ii) A \( c \)-polarization class \( \overline{\lambda} \) is an orbit of \( c \)-polarizations under \( \mathfrak{O}_+^\times \).

Let \( (\mathbb{G}_m \otimes \mathfrak{O}^{-1})[n] \) be the reduced subscheme of \( \mathbb{G}_m \otimes \mathfrak{O}^{-1} \), defined as the intersection of the kernels of multiplications by elements of \( n \). Its Cartier dual is isomorphic to the finite group scheme \( \mathfrak{O}/n \).

Definition 1.8. A \( \mu_n \)-level structure on a HBAV \( \mathcal{A}/S \) is an \( \mathfrak{O} \)-linear closed immersion \( \alpha: (\mathbb{G}_m \otimes \mathfrak{O}^{-1})[n] \to \mathcal{A} \) of group schemes over \( S \).

1.4. Hilbert modular varieties. We consider the contravariant functor \( \mathcal{M}^1 \) (resp. \( \mathcal{M} \)) from the category of \( \mathbb{Z}[1/\mathfrak{O}] \)-schemes to the category of sets, assigning to a scheme \( S \) the set of isomorphism classes of triples \( (\mathcal{A}, \lambda, \alpha) \) (resp. \( (\overline{\mathcal{A}}, \lambda, \alpha) \)) where \( \mathcal{A} \) is a HBAV over \( S \), endowed with a \( c \)-polarization \( \lambda \) (resp. a \( c \)-polarization class \( \overline{\lambda} \)) and a \( \mu_n \)-level structure \( \alpha \). Assume the following condition :

\((\text{NT})\) \( n \) does not divide 2, nor 3, nor \( N_{F/Q}(\mathfrak{O}) \).

Then \( \Gamma_1(c, n) \) is torsion free, and the functor \( \mathcal{M}^1 \) is representable by a quasi-projective, smooth, geometrically connected \( \mathbb{Z}[1/\mathfrak{O}] \)-scheme \( \mathcal{M}^1 = M^1(c, n) \), endowed with an universal HBAV \( \pi: \mathcal{A} \to M^1 \). By definition, the sheaf \( \omega_{\mathcal{A}/M^1} = \pi_* \Omega^1_{\mathcal{A}/M^1} \) is an invertible \( \mathfrak{O} \)-bundle on \( M^1 \). Consider the first de Rham cohomology sheaf \( \mathcal{H}^1_{\text{dR}}(\mathcal{A}/M^1) = R^1 \pi_* \Omega^1_{\mathcal{A}/M^1} \) on \( M^1 \). The Hodge filtration yields an exact sequence :

\[
0 \to \omega_{\mathcal{A}/M^1} \to \mathcal{H}^1_{\text{dR}}(\mathcal{A}/M^1) \to \omega_{\mathcal{A}/M^1}^\vee \otimes \mathfrak{O}^{-1} \to 0.
\]

Therefore \( \mathcal{H}^1_{\text{dR}}(\mathcal{A}/M^1) \) is locally free of rank two over \( \mathfrak{O} \otimes \mathcal{O}_{M^1} \).

The functor \( \mathcal{M} \) admits a coarse moduli space \( M = M^1(c, n) \), which is a quasi-projective, smooth, geometrically connected \( \mathbb{Z}[1/\mathfrak{O}] \)-scheme. The finite group \( \mathfrak{O}_+^\times / \mathfrak{O}_n^\times \) acts properly and
discontinuously on $M^1$ by $[\varepsilon] : (A, \iota, \lambda, \alpha)/S \mapsto (A, \iota, \epsilon \lambda, \alpha)/S$ and the quotient is given by $M$. This group acts also on $\omega_{A/M^1}$ and on $H^1_{\text{dR}}(A/M^1)$ by acting on the de Rham complex $\Omega^1_{A/M^1}$ ($[\varepsilon]$ acts on $\omega_{A/M^1}$ by $\varepsilon^{-1/2}[\varepsilon]^*$).

These actions are defined over the integer field $F(\varepsilon^{1/2}, \varepsilon \in \mathcal{O}^\times_\mathfrak{p})$.

Let $\mathfrak{o}'$ be the integer ring of $F(\varepsilon^{1/2}, \epsilon \in \mathcal{O}^\times_\mathfrak{p})$. For every $\mathbb{Z}[\frac{1}{\mathfrak{m}}]$-scheme $X$ we put

$$X' = X \times \text{Spec}(\mathfrak{o}'[\frac{1}{\mathfrak{m}}]).$$

The sheaf of $\mathfrak{o}^\times_\mathfrak{m}/\mathfrak{o}^\times_{n,1}$- invariants of $\omega_{A/M^1}$ (resp. of $H^1_{\text{dR}}(A/M^1)$) is locally free of rank one (resp. two) over $\mathfrak{o} \otimes \mathcal{O}_{M'}$, and is denoted by $\omega$ (resp. $H^1_{\text{dR}}$).

We put $Y = Y_1(n) = \prod_{i=1}^{h^+} M_1(c_i, n)$ and $Y^1 = Y_1(n) = \prod_{i=1}^{h^+} M_1(c_i, n)$, where the ideals $c_i$, $1 \leq i \leq h^+$, form a set of representatives of $\text{Cl}_1^1$.

1.5. Geometric Hilbert modular forms. Under the action of $\mathfrak{o}$, the invertible $\mathfrak{o}$-bundle $\omega$ on $M'$ decomposes as a direct sum of line bundles $\omega_\tau$, $\tau \in J_F$. For every $k = \sum_{r} k_r \tau \in \mathbb{Z}[J_F]$ we define the line bundle $\omega^k = \otimes \omega_\tau^{k_\tau}$ on $M'$.

One should be careful to observe, that the global section of $\omega^k$ on $M'^{an}$ are given by the cocycle $\gamma \mapsto \nu(\gamma)^{-k/2} j(\gamma, z)^k$, meanwhile we are interested in finding a geometric interpretation of the cocycle $\gamma \mapsto \nu(\gamma)^{t-k-m} j(\gamma, z)^k$, used in Def.1.4.

The universal polarization class $\mathfrak{X}$ endows $H^1_{\text{dR}}$, with a perfect symplectic $\mathfrak{o}$-linear pairing. Consider the invertible $\mathfrak{o}$-bundle $\mathfrak{X} := \wedge^2 \mathfrak{o} \otimes \mathcal{O}_{M'} H^1_{\text{dR}}$ on $M'$. Note that $(k + m - t) - \frac{k}{2} = \frac{n}{2} t$.

**Definition 1.9.** Let $R$ be an $\mathfrak{o}'[\frac{1}{\mathfrak{m}}]$-algebra. A Hilbert modular forms of weight $k$, level $\Gamma$ and coefficients in $R$, is a global section of $\omega^k \otimes \mathfrak{X}^{-n/2}$ over $M \times \text{Spec}(\mathbb{Z}[\frac{1}{\mathfrak{m}}]) \text{Spec}(R)$. We denote by $G_k(\Gamma; R) = H^0(M \times \text{Spec}(\mathbb{Z}[\frac{1}{\mathfrak{m}}])) \text{Spec}(R), \omega^k \otimes \mathfrak{X}^{-n/2}$ the $R$-module of these Hilbert modular forms.

1.6. Toroidal compactifications. The toroidal compactifications of the moduli space of $\mathfrak{o}$-polarized HBAV with principal level structure have been constructed by Rapoport [34]. Several modifications need to be made in order to treat the case of $\mu_\mathfrak{m}$-level structure. These are described in [10]Thm.7.2.

Let $\Sigma$ be a smooth $\Gamma_1^1(\mathfrak{c}, n)$-admissible collection of fans (see [10]Def.7.1). Then, there exists an open immersion of $M^1$ into a proper and smooth $\mathbb{Z}[\frac{1}{\mathfrak{m}}]$-scheme $\mathcal{M} = \mathcal{M}_\Sigma$, called the toroidal compactification of $M^1$ with respect to $\Sigma$. The universal HBAV $\pi : \mathcal{A} \to M^1$ extends uniquely to a semi-abelian scheme $\pi : \mathfrak{G} \to \mathcal{M}^\dagger$. The group scheme $\mathfrak{G}$ is endowed with an action of $\mathfrak{o}$ and its restriction to $\mathcal{M}^\dagger \backslash M^1$ is a torus. Moreover, the sheaf $\omega_{\mathfrak{G}/\mathcal{M}^\dagger}$ of $\mathfrak{G}$-invariants sections of $\pi_* \Omega^1_{\mathfrak{G}/\mathcal{M}^\dagger}$ is an invertible $\mathfrak{o}$-bundle on $\mathcal{M}^\dagger$, extending $\omega_{A/M^1}$.

The scheme $\mathcal{M} \backslash M^1$ is a divisor with normal crossings, and the formal completion of $\mathcal{M}^\dagger$ along this divisor can be completely determined in terms of $\Sigma$ (see [10]Thm.7.2). For the sake of simplicity, we will only describe the completion of $\mathcal{M}^\dagger$ along the connected component of $\mathcal{M}^\dagger \backslash M^1$ corresponding to the standard cusp at $\infty$. Let $\Sigma^\infty \in \Sigma$ be the fan corresponding to the cusp at $\infty$. It is a complete, smooth fan of $\mathfrak{c}_\mathfrak{m} \cup \{0\}$, stable by the action of $\mathfrak{o}_\mathfrak{m}^\times$, and containing a finite number of cones modulo this action. Put
\[ R_\infty = \mathbb{Z}[q^\xi; \xi \in \mathcal{C}] \text{ and } S_\infty = \text{Spec}(R_\infty) = \mathbb{G}_m \otimes \mathcal{C}^*. \] Associated to the fan \( \Sigma_\infty \), there is a toroidal embedding \( S_\infty \hookrightarrow S_{2\infty} \) (it is obtained by gluing the affine toric embeddings \( S_\infty \hookrightarrow S_{2\infty,\sigma} = \text{Spec}(\mathbb{Z}[q^\xi; \xi \in \mathcal{C} \cap \sigma]) \) for \( \sigma \in \Sigma_\infty \)). Let \( S_{2\infty}' \) be the formal completion of \( S_{2\infty} \) along \( S_{2\infty} \setminus S_\infty \). By construction, the formal completion of \( \overline{M}' \) along the connected component of \( \overline{M}' \setminus M_1 \) corresponding to the standard cusp at \( \infty \), is isomorphic to \( S_{2\infty}' / \mathfrak{a}_{n,1}^{\times 2} \).

Assume that \( \Sigma \) is \( \Gamma(\mathfrak{c},\mathfrak{n}) \)-admissible (for the cusp at \( \infty \), it means that \( \Sigma_\infty \) is stable under the action of \( \mathfrak{a}_{n,1}^{\times} \)). Then the finite group \( \mathfrak{a}_{n,1}^{\times} / \mathfrak{a}_{n,1}^{\times 2} \) acts properly and discontinuously on \( \overline{M}' \), and the quotient \( \overline{M} = M_\Sigma \) is a proper and smooth \( \mathbb{Z}[1/\mathfrak{a}] \)-scheme, containing \( M \) is an open subscheme. Again by construction, the formal completion of \( \overline{M} \) along the connected component of \( \overline{M} \setminus M_1 \) corresponding to the standard cusp at \( \infty \), is isomorphic to \( S_{2\infty}' / \mathfrak{a}_{n,1}^{\times 2} \).

The invertible \( \mathfrak{a} \)-bundle \( \omega_{\mathfrak{a}/\overline{M}'} \) on \( \overline{M}' \) descends to an invertible \( \mathfrak{a} \)-bundle on \( \overline{M}' \), extending \( \omega \). We still denote this extension by \( \omega \). For each \( k \in \mathbb{Z}[J_F] \) this gives us an extension of \( \omega^k \) to a line bundle on \( \overline{M}' \), still denoted by \( \omega^k \).

1.7. \textit{q-expansion and Koecher Principles.} The Koecher Principle states (see [10]Thm.8.3)

\[ H^0(M \times \text{Spec}(R), \omega^k \otimes \nu^{-n_0t/2}) = H^0(\overline{M} \times \text{Spec}(R), \omega^k \otimes \nu^{-n_0t/2}) \]

For simplicity, we will only describe the \( q \)-expansion at the standard (unramified) cusp at \( \infty \). For every \( \sigma \in \Sigma_\infty \), and every \( \mathfrak{a}'[1/\mathfrak{a}] \)-algebra \( R \), the pull-back of \( \omega \) to \( S_{2\infty}' \times \text{Spec}(R) \) is canonically isomorphic to \( \mathfrak{a} \otimes \mathcal{O}_{S_{2\infty}'} \otimes R \). Thus

\[ H^0(S_{2\infty}' \times \text{Spec}(R) / \mathfrak{a}_{n,1}^{\times}, \omega^k \otimes \nu^{-n_0t/2}) = \left\{ \frac{\sum_{\xi \in \mathfrak{a}_{n,1}^{\times}} a_\xi q^{\xi}}{a_{\xi} \in R, a_{\nu} = u^{k+\xi} \text{ for } \nu(u, \epsilon) \in \mathfrak{a}_{n,1}^{\times}} \right\} \]

By the above construction, to each \( g \in G_k(\Gamma; R) \), we can associate an element \( g_\infty = \sum_{\xi \in \mathfrak{a}_{n,1}^{\times}} a_\xi q^{\xi} \), called the \( q \)-expansion of \( g \) at the cusp at \( \infty \). The element \( a_0(g) \in R \) is the value of \( g \) at the cusp at \( \infty \).

**Proposition 1.10.** Let \( R \) be a \( \mathfrak{a}'[1/\mathfrak{a}] \)-algebra.

(i) (\( q \)-expansion Principle) \( G_k(\Gamma; R) \rightarrow \mathbb{R}[g, g \in \mathfrak{a}_{n,1}^{\times} \cup \{0\} ] \), \( g \mapsto g_\infty \) is injective.

(ii) If there exists \( g \in G_k(\Gamma; R) \), such that \( a_0(g) \neq 0 \), then \( g^{k+m-t} - 1 \) is a zero-divisor in \( R \), for all \( \epsilon \in \mathfrak{a}_{n,1}^{\times} \).

1.8. \textit{The minimal compactification.} There exist a projective, normal \( \mathbb{Z}[1/\mathfrak{a}] \)-scheme \( M_1^* \), containing \( M \) as an open dense subscheme and such that the scheme \( M_1^* \setminus M_1 \) is finite and étale over \( \mathbb{Z}[1/\mathfrak{a}] \). Moreover, for each toroidal compactification \( \overline{M}' \) of \( M_1 \) there is a natural surjection \( \overline{M}' \rightarrow M_1^* \), inducing the identity map on \( M_1 \). The scheme \( M_1^* \) is called the minimal compactification of \( M_1 \). The action of \( \mathfrak{a}_{n,1}^{\times} / \mathfrak{a}_{n,1}^{\times 2} \) on \( M_1 \) extends to an action on \( M_1^* \), and the minimal compactification \( M^* \) of \( M \) is defined as the quotient for this action. In general \( M_1^* \rightarrow M^* \) is not étale.
We summarize the above discussion in the following commutative diagram:

\[
\begin{array}{ccc}
\mathfrak{G} & \xrightarrow{\pi} & M^1 \\
\downarrow & & \downarrow \pi \\
A & \xrightarrow{\pi} & M^1
\end{array}
\]

\[
\begin{array}{ccc}
& & M^1 \rightarrow M \\
& M^1 & \rightarrow M^* \\
\end{array}
\]

1.9. **Toroidal compactifications of the Kuga-Sato varieties.** Let \(s\) be a positive integer. Let \(\pi_s : \mathcal{A}^s \to M^1\) be the \(s\)-fold fiber product of \(\pi : A \to M^1\), and \((\pi)_s : \mathfrak{G}^s \to M^1\) be the \(s\)-fold fiber product of \(\pi : \mathfrak{G} \to M^1\).

Let \(\Sigma\) be a \((\mathfrak{c} \oplus \mathfrak{c}) \times \Gamma_1(c,n)\)-admissible, polarized, equidimensional, smooth collection of fans, above the \(\Gamma_1(c,n)\)-admissible collection of fans \(\Sigma\) of §1.6. Using Faltings-Chai's method [15], the main result of [11] Sect.6 is the following: there exists an open immersion of a \(\mathcal{A}^s\) into a projective smooth \(\mathbb{Z}[\Delta]\)-scheme \(\mathcal{A}^s = \mathcal{A}^s_{\Sigma}\), and a proper, semi-stable homomorphism \(\pi_s : \mathcal{A}^s \to M^1\) extending \(\pi_s : \mathcal{A}^s \to M^1\), and such that \(\mathcal{A}^s / \mathcal{A}^s\) is a relative normal crossing divisor above \(M^1 / M^1\). Moreover, \(\mathcal{A}^s\) contains \(\mathfrak{G}^s\) as an open dense subscheme and \(\mathfrak{G}^s\) acts on \(\mathcal{A}^s\) extending the translation action of \(\mathcal{A}^s\) on itself.

The sheaf \(H^1_{\text{log-dr}}(\mathcal{A}/M^1) = R^1\pi_!\Omega^*_\mathcal{A}/M^1(d\log \infty)\) is independent of the particular choice of \(\Sigma\) above \(\Sigma\) and is endowed with a filtration:

\[
0 \to \omega_{\mathfrak{G}/M^1} \to H^1_{\text{log-dr}}(\mathcal{A}/M^1) \to \omega^\vee_{\mathfrak{G}/M^1} \otimes \mathcal{O}^{-1} \to 0.
\]

It descends to a sheaf \(H^1_{\text{log-dr}}\) on \(M\) which fits in the following exact sequence:

\[
0 \to \omega \to H^1_{\text{log-dr}} \to \omega^\vee \otimes \mathcal{O}^{-1} \to 0.
\]

1.10. **Hecke operators on modular forms.** Let \(\mathbb{Z}[K_1(n) \backslash G(\mathbb{A}_f) / K_1(n)]\) be the free abelian group with basis the double cosets of \(K_1(n)\) in \(G(\mathbb{A}_f)\). It is endowed with algebra structure, where the product of two basis elements is given by:

\[
[K_1(n)xK_1(n)] \cdot [K_1(n)yK_1(n)] = \sum_i [K_1(n)x_iyK_1(n)],
\]

where \([K_1(n)xK_1(n)] = \prod_i K_1(n)x_i\). For \(g \in S_{k,j}(K_1(n))\) we put:

\[
g([K_1(n)xK_1(n)](\cdot)) = \sum_i g(x_i^{-1}).
\]

This defines an action of the algebra \(\mathbb{Z}[K_1(n) \backslash G(\mathbb{A}_f) / K_1(n)]\) on \(S_{k,j}(K_1(n))\) (resp. on \(G_{k,j}(K_1(n))\)). Unfortunately, this algebra is not commutative when \(n \neq \mathfrak{o}\). We will now define a commutative subalgebra. Consider the semi-group:

\[
\Delta(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{A}_f) \cap M_2(\mathfrak{o}) \mid d_v \in \mathfrak{o}_v^\times, \ c_v \in \mathfrak{n}_v, \text{ for all } v \text{ dividing } n \right\}.
\]

The abstract Hecke algebra of level \(K_1(n)\), is defined as \(\mathbb{Z}[K_1(n) \backslash \Delta(n) / K_1(n)]\) endowed with the convolution product (4). This algebra has the following explicit description.

For each ideal \(\mathfrak{a} \subset \mathfrak{o}\) we define the Hecke operator \(T_{\mathfrak{a}}\) as the finite sum of double cosets \([K_1(n)xK_1(n)]\) contained in the set \(\{ x \in \Delta(n) | \nu(x) \mathfrak{o} = \mathfrak{a} \}\). In the same way, for a prime
to \( n \) ideal \( \mathfrak{a} \subset \mathfrak{o} \), we define the Hecke operator \( S_{\mathfrak{a}} \) by the double coset for \( K_1(n) \) containing the scalar matrix of the idèle attached to the ideal \( \mathfrak{a} \).

For each finite place \( v \) of \( F \), we have \( T_v = K_1(n) \left( \begin{array}{cc} \varpi_v & 0 \\ 0 & 1 \end{array} \right) K_1(n) \), and for each \( v \) not dividing \( n \) we have \( S_v = K_1(n) \left( \begin{array}{cc} \varpi_v & 0 \\ 0 & \varpi_v \end{array} \right) K_1(n) \), where \( \varpi_v \) is an uniformizer of \( F_v \).

Then, the abstract Hecke algebra of level \( K_1(n) \) is isomorphic to the polynomial algebra in the variables \( T_v \), where \( v \) runs over the prime ideals of \( F \), and the variables \( S_v \), where \( v \) runs over the prime ideals of \( F \) not dividing \( n \). The action of Hecke algebra obviously preserves the decomposition (1) and moreover, \( S_v \) acts on \( S_{k,j}(n, \psi) \) as the scalar \( \psi(v) \).

Let \( \mathbb{T}(\mathbb{C}) = \mathbb{T}_k(n, \psi; \mathbb{C}) \) be the subalgebra of \( \text{End}_\mathbb{C}(S_{k,j}(n, \psi)) \) generate by the operators \( S_v \) for \( v \nmid n \) and \( T_v \) for all \( v \) (we will see in \( \S 1.11 \) that \( \mathbb{T}(\mathbb{C}) \) does not depend on \( J \)).

The algebra \( \mathbb{T}(\mathbb{C}) \) is commutative, but not semi-simple in general. Nevertheless, for \( v \nmid n \) the operators \( S_v \) and \( T_v \) are normal with respect to the Petersson inner product (see Def.1.5). Denote by \( \mathbb{T}(\mathbb{C}) \) the subalgebra of \( \mathbb{T}(\mathbb{C}) \) generated by the Hecke operators outside a finite set of places containing those dividing \( n \). The algebra \( \mathbb{T}(\mathbb{C}) \) is semi-simple, that is to say \( S_{k,j}(n, \psi) \) has a basis made of eigenvectors for \( \mathbb{T}(\mathbb{C}) \).

We will now describe the relation between Fourier coefficients and eigenvalues for the Hecke operators. By (2) we can associate to \( g \in S_k(K_1(n)) \) a family of classical cusp forms \( g_i \in S_k(\Gamma_1(\mathfrak{c}_i, n); \mathbb{C}) \), where \( \mathfrak{c}_i \) are representatives of the narrow ideal class group \( \mathbb{C}l_F^\times \).

Each form \( g_i \) is determined by its \( q \)-expansion at the cusp \( \infty \) of \( M_1(\mathfrak{c}_i, n) \mathbb{Z} \). For each fractional ideal \( \mathfrak{a} = \mathfrak{c}_i \mathfrak{t} \), with \( \mathfrak{t} \in F_+^\times \), we put \( c(g, \mathfrak{a}) = \mathfrak{t}^m a_\mathfrak{c}_i(g_i) \). By \( \S 1.7 \) for each \( \mathfrak{c}_i \in F_+^\times \), we have \( a_\mathfrak{c}_i = e^{k+m-t}a_\mathfrak{c}_i \) and therefore the definition of \( c(g, \mathfrak{a}) \) does not depend on the choice of \( \mathfrak{c}_i \) (nor on the particular choice of the ideals \( \mathfrak{c}_i \); see [20] IV.4.2.9.).

**Definition 1.11.** We say that \( g \in S_k(n, \psi) \) is an eigenform, if it is an eigenvector for \( \mathbb{T}(\mathbb{C}) \).

We say that an eigenform \( g \) is normalized if \( c(g, \mathfrak{o}) = 1 \).

**Lemma 1.12.** ([24] Prop.4.1, [20] (4.64)) If \( g \in S_k(n, \psi) \) is a normalized eigenform, then the eigenvalue of \( T_{\mathfrak{a}} \) on \( g \) is equal to the Fourier coefficient \( c(g, \mathfrak{a}) \).

A consequence of this lemma and the \( q \)-expansion Principle (see \( \S 1.7 \)), is the Weak Multiplicity One Theorem stating that two normalized eigenforms having the same eigenvalues are equal.

**Ordinary modular forms.** When the weight \( k \) is non-parallel, the definition of the Hecke operators should be slightly modified. We put \( T_{0,v} = \varpi_v^m T_v \) and \( S_{0,v} = \varpi_v^{-2m} S_v \) (see [24] Sect.3 ; in the applications our base ring will be the \( p \)-adic ring \( \mathcal{O} \) which satisfies the assumptions of this reference).

The advantage of the Hecke operators \( T_{0,v} \) and \( S_{0,v} \) is that they preserve in an optimal way the \( \mathcal{O} \)-integral structures on the space of Hilbert modular forms and on the cohomology of the Hilbert modular variety.

**Definition 1.13.** A Hilbert modular eigenform is ordinary at \( p \) if, for all primes \( p \) of \( F \) dividing \( p \), the image by \( \iota_p \) of its \( T_{0,p} \)-eigenvalue is a \( p \)-adic unit.
Primitive modular forms. For each $n_1$ dividing $n$ and divisible by the conductor of $\psi$, and for all $n_2$ dividing $n_1^{-1}$ we consider the linear map

$$S_k(n_1, \psi) \rightarrow S_k(n, \psi), \ g \mapsto g|_{n_2},$$

where $g|_{n_2}$ is determined by the relation $c(a, g|_{n_2}) = c(a n_2^{-1}, g)$.

We define the subspace $S^\text{old}_k(n, \psi)$ of $S_k(n, \psi)$ as the subspace generated by the images of all these linear maps. This space is preserved by the Hecke operators outside $n$. We define the space $S^\text{new}_k(n, \psi)$ of the primitive modular forms as the orthogonal of $S^\text{old}_k(n, \psi)$ in $S_k(n, \psi)$ with respect to the Petersson inner product (see Def.1.5). Because the Hecke operators outside $n$ are normal for the Petersson inner product, the direct sum decomposition $S_k(n, \psi) = S^\text{new}_k(n, \psi) \oplus S^\text{old}_k(n, \psi)$ is preserved by $T'(\mathbb{C})$. The Strong Multiplicity One Theorem, due to Miyake in the Hilbert modular case, asserts that if $f \in S^\text{new}_k(n, \psi)$ is an eigenform $T'(\mathbb{C})$, then it is an eigenform for $T(\mathbb{C})$.

A normalized primitive eigenform is called a \textit{newform}.

The pairing $T(\mathbb{C}) \times S_k(n, \psi) \rightarrow \mathbb{C}, (T, g) \mapsto c(g|_T, \sigma)$ is a perfect duality (see [24] Thm.5.2).

1.11. External and Weyl group conjugates. For an element $\sigma \in \text{Aut}(\mathbb{C})$ we define the \textit{external conjugate} of $g \in S_k(K_1(n))$, as the unique element $g^\sigma \in S_k(K_1(n))$ satisfying $c(g^\sigma, a) = c(g, a)^\sigma$, for each ideal $a$ of $\mathfrak{o}$.

We identify $\{\pm 1\}^{|J_P|}$ with the Weyl group $K_\infty/K_\infty^+$ of $G$, by sending $\epsilon_J = (-1_J, 1_{J \setminus J})$ to $c_J K_\infty^+$, where for all $\tau \in J_P$, $\det(\epsilon_{J, \tau}) < 0$ if and only if $\tau \in J$. The length of $\epsilon_J$ is $|J|$.

We have an action of the Weyl group on the space of Hilbert modular forms. More precisely, $\epsilon_J$ acts as the double class $[K_1(n) c_J K_1(n)]$, and maps bijectively $S_k(K_1(n))$ onto $S_{k, J \setminus J}(K_1(n))$. The action of $\epsilon_J$ commutes with the action of the Hecke operators. For an element $g \in S_k(K_1(n))$ we put $g_J = \epsilon_{J \setminus J} \cdot g$.

1.12. Eichler-Shimura-Harder isomorphism. Let $R$ be an $\mathcal{O}$-algebra and $V_n(R)$ be the polynomial ring over $R$ in the variables $(X_\tau, Y_\tau)_{\tau \in J_P}$ which are homogeneous of degree $n_\tau$ in $(X_\tau, Y_\tau)$. We have a pairing (perfect if $n_0$ is invertible in $R$)

$$(5) \quad \langle \ , \ , \rangle : V_n(R) \times V_n(R) \rightarrow R, \ \text{given by}$$

$$\sum_{0 \leq j \leq n} a_j X^{n-j} Y^j, \sum_{0 \leq j \leq n} b_j X^{n-j} Y^j = \sum_{0 \leq j \leq n} (-1)^j a_j b_{n-j} (n_j)^j, \text{where} \ (n_j) = \prod_{\tau \in J_P} (n_{\tau,j}),$$

The $R$-module $V_n(R)$ realizes the algebraic representation $V_n = \bigotimes_\tau (\text{Sym}^{n_\tau} \otimes \text{det}^{m_\tau})$ of $G(R)$. We endow $V_n(R)$ with an action of $(M_2(\mathcal{O}) \cap \text{GL}_2(E))^{|J_P|}$ given by

$$\gamma. P((X_\tau, Y_\tau)_{\tau \in J_P}) = \nu(\gamma)^m P((\det(\gamma)\gamma^{-1})^j (X_\tau, Y_\tau)_{\tau \in J_P}).$$

Let $\mathbb{V}_n(R)$ be the sheaf of continuous (thus locally constant) sections of

$$G(\mathbb{Q}) \setminus G(\mathbb{A}) \times V_n(R)/K_1(n)K_\infty^+ \rightarrow G(\mathbb{Q}) \setminus G(\mathbb{A})/K_1(n)K_\infty^+ = Y^{an},$$

where $y \in K_1(n)K_\infty^+$ acts on $V_n(R)$ via its $p$-part $y_p$.

For each $y \in \Delta(n)$ the map $[y] : G(\mathbb{A}) \times V_n(R) \rightarrow G(\mathbb{A}) \times V_n(R), (x, v) \mapsto (xy, y_p v)$ is a homomorphism of sheaves. This induces an action of the Hecke operator $[K_1(n) y K_1(n)]$ on $H^d(Y^{an}, \mathbb{V}_n(R))$ preserving the cuspidal cohomology $H^d_{\text{cusp}}(Y^{an}, \mathbb{V}_n(R))$. 

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The action of \( \epsilon_J \) on \((M_{\text{an}}, \mathcal{V}_{\text{an}})\) given by \( \epsilon_J \cdot ((z_J, z_{J^c}), v) = ((\overline{z}_J, z_{J^c}), v) \), induces an action of the Weyl group on \( H^d(Y_{\text{an}}, \mathcal{V}_{\text{an}}) \) commuting with the Hecke action.

By Harder [19] we know that, if \( n \neq 0 \), then \( H^d_c(Y_{\text{an}}, \mathcal{V}_n(\mathbb{C})) = H^d_c(Y_{\text{an}}, \mathcal{V}_n(\mathbb{C})) \).

By (5) we have a Poincaré pairing \( \langle \cdot, \cdot \rangle : H^d_c(Y_{\text{an}}, \mathcal{V}_n(R)) \times H^d_c(Y_{\text{an}}, \mathcal{V}_n(R)) \to R \).

Let \( \eta \) be the idèle corresponding to the ideal \( n \) and let \( \iota = \left( \begin{array}{cc} 0 & 1 \\ -\eta & 0 \end{array} \right) \) be the Atkin-Lehner involution. By putting \([x, y] = \langle x, iy \rangle\) we obtain a new pairing

\[
\delta : \bigoplus_{\psi} S_{k,J}(n, \psi) \cong H^d_c(Y_{\text{an}}, \mathcal{V}_n(\mathbb{C})),
\]

where \( \psi \) runs over the Hecke characters of conductor dividing \( n \) and type \(-n_0t\) at infinity. This isomorphism is equivariant for the actions of the Hecke algebra and the Weyl group.

For each \( J \subset J_F \) let \( \hat{\epsilon}_J : \{ \pm 1 \}^{J_F} \to \{ \pm 1 \} \) be the unique character of the Weyl group sending \( \epsilon_\tau = (-1_\tau, 1^\tau) \) to 1, if \( \tau \in J \), and to -1 if \( \tau \in J_{F^c} \setminus J \). The restriction of the Eichler-Shimura-Harder isomorphism (7) to \( S_{k,J}(n, \psi) \), followed by the projection on the \((\psi, \hat{\epsilon}_J)\)-part yields a Hecke equivariant isomorphism

\[
\delta_J : S_{k,J}(n, \psi) \cong H^d_c(Y, \mathcal{V}_n(\mathbb{C}))[\psi, \hat{\epsilon}_J].
\]

Moreover, after twisting by the complex conjugation \( c \) on the coefficients, we still have a direct sum decomposition:

\[
H^d(M_{\text{an}}, \mathcal{V}_n(\mathbb{C})) = \bigoplus_{J \subset J_F} H^d(M_{\text{an}}, \mathcal{V}_n(\mathbb{C}))[\epsilon_J \otimes c].
\]

This decomposition is finer than the usual Hodge decomposition, whose graded are given by \((0 \leq a \leq d)\):

\[
\text{gr}^a H^d(M_{\text{an}}, \mathcal{V}_n(\mathbb{C})) = \bigoplus_{J \subset J_F, |J| = a} H^d(M_{\text{an}}, \mathcal{V}_n(\mathbb{C}))[\epsilon_J \otimes c].
\]

The transcendental decomposition (9) has an algebraic interpretation, via the so-called BGG complex, that we will describe in the next section.

2. Hodge-Tate weights of the Hilbert modular varieties.

The aim of this section is to determine the Hodge-Tate weights of the \( p \)-adic étale cohomology of the Hilbert modular variety \( H^\bullet(M_{\text{an}}, \mathcal{V}_n(\overline{\mathbb{Q}}_p)) \), as well as those of the \( p \)-adic Galois representation associated to a Hilbert modular form. In all this section we assume

(I) \( p \) does not divide \( \Delta = N_{E/\mathbb{Q}}(n \mathfrak{d}) \).
The proof relies on Faltings’ Comparison Theorem [14] relating the étale cohomology of $M$ with coefficients in the local system $V_n(\overline{\mathbb{Q}_p})$ to the de Rham logarithmic cohomology of the corresponding vector bundle $\mathcal{V}_n$ over a smooth toroidal compactification $\overline{M}$ of $M$. The Hodge-Tate weights are given by the jumps of the Hodge filtration of the associated de Rham complex. These are computed, following [15], using the so-called Bernstein-Gelfand-Gelfand complex (BGG complex).

Instead of using Faltings’ Comparison Theorem, one can use Tsuji’s result for the étale cohomology with constant coefficients of the Kuga-Sato variety $A$ above the fine moduli space $M^1$ associated to $M$; see [11] Sect.6 for the construction of toroidal compactifications of $A^e$.

For each subset $J$ of $J_F$ we put $p(J) = \sum_{\tau \in J} (k_0 - m_\tau - 1) \tau + \sum_{\tau \in J_F \setminus J} m_\tau \tau \in \mathbb{Z}[J_F]$ and for each $a = \sum_{\tau \in J_F} a_\tau \tau \in \mathbb{Z}[J_F]$ we put $|a| = \sum_{\tau \in J_F} a_\tau \in \mathbb{Z}$.

2.1. Motivic weight of the cohomology. Consider the smooth sheaf $R^1 \pi_+ \overline{\mathbb{Q}_p}$ on $M^1$, where $\pi : A \to M^1$ is the universal HBAV. It corresponds to a representation of the fundamental group of $M^1$ in $G(\overline{\mathbb{Q}_p})$. By composing this representation with the algebraic representation $V_n$ of $G$ of highest weight $n$ (see §1.12), we obtain a smooth sheaf on $M^1$ (thus on $Y^1$). It descends to a smooth sheaf on $Y$ denoted $V_n(\overline{\mathbb{Q}_p})$.

Let $W_f = \bigcap_{a \in \mathbb{Q}} \ker(T_a - c(f, a))$ be the subspace of $H^d(Y_{\overline{\mathbb{Q}_p}}, V_n(\overline{\mathbb{Q}_p}))$ corresponding to the Hilbert modular newform $f \in S_k(n, \psi)$. Put $s = \sum_{\tau} (n_\tau + 2m_\tau) = dns_0$.

Proposition 2.1. $W_f$ is pure of weight $d + s$, that is to say for all prime $l \nmid p\Delta$ the eigenvalues of the geometric Frobenius $\text{Frob}_l$ at $l$ are Weil numbers of absolute value $l^{d+s}$.

Proof: As $f$ is cuspidal $W_f \subset H^d(Y_{\overline{\mathbb{Q}_p}}, V_n(\overline{\mathbb{Q}_p}))$. We recall that $Y_{\overline{\mathbb{Q}_p}}$ is a disjoint union of its connected components $M_{\overline{\mathbb{Q}_p}} = M_1(\mathbb{C}, n, \overline{\mathbb{Q}_p})$ where the $\mathbb{C}$’s form a set of representatives of $\text{Cl}_F^+$. Let $\mathfrak{c}$ be one of the $\mathbb{C}$’s and $M^1 = M_1^1(\mathbb{C}, n)$. For $\ast = \emptyset, c$ we have

$$H^0(\mathfrak{a}_+^2 \mathfrak{a}_{n,1}^2, H^d(M_{\overline{\mathbb{Q}_p}}^1, V_n(\overline{\mathbb{Q}_p}))) = H^d(M_{\overline{\mathbb{Q}_p}} V_n(\overline{\mathbb{Q}_p})),$$

and therefore, it is enough to prove that $H^d(M_{\overline{\mathbb{Q}_p}}^1, V_n(\overline{\mathbb{Q}_p}))$ is pure of weight $d + s$. We use Deligne’s method [4]. Let $\pi : A \to M^1$ be the universal abelian variety (see §1.4). The sheaf $V_n(\overline{\mathbb{Q}_p})$ corresponds to the representation $\bigotimes_{\tau \in J_F} \text{Sym}^{m_\tau} \otimes \text{det}^{m_\tau}$ of the group $G^a$ and can therefore be cut out by algebraic correspondences in $(R^1 \pi_+ \overline{\mathbb{Q}_p})_{\otimes \ast}$. Let $\pi_\ast : A^\ast \to M^1$ be the Kuga-Sato variety. By the Kunmeth’s formula we have

$$H^d(M_{\overline{\mathbb{Q}_p}}^1, R^1 \pi_+ \overline{\mathbb{Q}_p}) \subset H^d(M_{\overline{\mathbb{Q}_p}}^1, R^s \pi_{ss} \overline{\mathbb{Q}_p}) \subset H_1^{d+s} (M_{\overline{\mathbb{Q}_p}}^1, \overline{\mathbb{Q}_p}) \subset H_1^{d+s} (A^e_{\overline{\mathbb{Q}_p}}),$$

where the middle inclusion comes from the degeneration of the of the Leray spectral sequence $E_2^{i,j} = H^i_\ast (M_{\overline{\mathbb{Q}_p}}^1, R^j \pi_{ss} \overline{\mathbb{Q}_p}) \Rightarrow H_\ast^{i+j} (A^e_{\overline{\mathbb{Q}_p}})$ for $\ast = \emptyset, c$ (see [4]). The proposition is then a consequence of the Weil conjectures for the eigenvalues of the Frobenius, proved by Deligne [5].
2.2. The Bernstein-Gelfand-Gelfand complex over $\mathbb{Q}$. In this and the next sections we give, following Faltings [13], an algebraic construction of the transcendental decomposition of the Betti cohomology described in (9).

In this section all the objects are defined over a characteristic zero field splitting $G$.

Let $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$ and $\mathfrak{u}$ denote the Lie algebras of $G$, $B$, $T$ and $U$, respectively. Consider the canonical splitting $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{u}^-$. Let $U(\mathfrak{g})$, $U(\mathfrak{b})$ be the enveloping algebras of $\mathfrak{g}$ and $\mathfrak{b}$, respectively.

The aim of this section is to write down a resolution of $V_n$ of the type:

$$0 \leftarrow V_n \leftarrow U(\mathfrak{g}) \otimes U(\mathfrak{b}) K^\bullet_n,$$

where the $K^j_n$ are finite dimensional semi-simple $\mathfrak{b}$-modules, with explicit simple components.

We start by the case $n = 0$. If we put $K^j_0 = \wedge^j (\mathfrak{g}/\mathfrak{b})$ we obtain the so-called bar-resolution of $V_0$. Note that $\wedge^j (\mathfrak{g}/\mathfrak{b})$ is a $\mathfrak{b}$-module with trivial action of $\mathfrak{u}$, therefore $K^j_0 = \oplus \omega_\mu$, where $\mu$ runs over the weights of $B$ that are sum of $j$ distinct negative roots.

By tensoring this resolution with $V_n$ we obtain the Koszul’s complex:

$$(10) \quad 0 \leftarrow V_n \leftarrow U(\mathfrak{g}) \otimes U(\mathfrak{b}) (\wedge^j (\mathfrak{g}/\mathfrak{b}) \otimes V_n|_{\mathfrak{b}}),$$

which is a resolution of $V_n$ by $\mathfrak{b}$-modules $\wedge^j (\mathfrak{g}/\mathfrak{b}) \otimes V_n|_{\mathfrak{b}}$, not semi-simple in general.

The BGG complex that we are going to define is a direct factor of the Koszul’s complex, cut by the action of the center $U(\mathfrak{g})^G$ of $U(\mathfrak{g})$.

Denote by $\chi_n$ the character of $U(\mathfrak{g})^G$ corresponding to the weight $n$. It is a classical result that

Lemma 2.2. $\chi_n = \chi_\mu$, if and only if there exists $J \subset J_F$ such that $\mu = \epsilon_j(n + t) - t$.

By taking the $\chi_n$-part of the bar resolution (10) of $V_n$ we obtain the complex:

$$(11) \quad 0 \leftarrow V_n \leftarrow U(\mathfrak{g}) \otimes U(\mathfrak{b}) K^\bullet_n,$$

with $K^j_n = \bigoplus_{J \subset J_F, |J| = i} \omega_{\epsilon_j(n + t) - t}$.

which is still a resolution of $V_n$, as it is a direct factor of a resolution. We call this resolution the BGG complex.

2.3. Hodge-Tate decomposition of $H^\bullet(M \otimes \mathbb{Q}_p, V_n(\mathbb{Q}_p))$. In this paragraph we summarize the results of [11] Sect.7. The algebraic groups $G$, $B$, $T$ and $D$ of §1 have models over $\mathbb{Z}$, denoted by the same letters. For every scheme $X$, we put $X' = X \times \text{Spec}(\phi'[\mathbb{Z}])$.

By §1.9, we can extend the vector bundles $\omega$ and $\mathcal{H}_{dR}^1$ to $\overline{M}$. Only the construction depends on a choice of a toroidal compactification $\overline{\pi} : \overline{A} \to \overline{M}$ of $\pi : A \to M^1$.

The sheaf $\mathcal{M}_D = \text{Isom}_A \circ \mathcal{O}_{\overline{M}}(\omega, \mathfrak{a} \circ \mathcal{O}_{\overline{M}})$ is a $D'$-torsor over $\overline{M}'$ (for the Zariski topology). We have a functor $\mathcal{F}_D$ from the category of algebraic representations of $D'$ to the category of vector bundles on $\overline{M}'$ which are direct sum of invertible bundles. To an algebraic representation $W$ of $D'$, $\mathcal{F}_D$ associates the fiber product $\overline{W} := \mathcal{M}_D^{D'} \times W$. 

The sheaf $\mathcal{M}_B = \text{Isom}_{\text{fil}}^\otimes \mathcal{O}_{\overline{M}}(\mathcal{H}_{\log-dR}^1, (\mathcal{O} \otimes \mathcal{O}_{\overline{M}}')^2)$ is a $B'$-torsor over $\overline{M}'$. We have a functor $\mathcal{F}_B$ from the category of algebraic representations of $B'$ to the category of filtered vector bundles on $\overline{M}'$ whose graded are sums of invertible bundles. To an algebraic representation $V$ of $B'$, $\mathcal{F}_B$ associates the fiber product $\overline{V} := \mathcal{M}_B \times V$.

A representation of $G$ (resp. $T$) can be considered as a representation of $B$ by restriction (resp. by making $U$ act trivially). Thus, we may define the filtered vector bundle $\overline{V}_n$ on $\overline{M}'$ associated to the algebraic representation $V_n$ of $G$, and the invertible bundle $\overline{W}_{n,n_0}$ on $\overline{M}'$ associated to the algebraic representation of $T = D \times D$, given by $(u, \epsilon) \mapsto u^n \epsilon^m$.

The sheaf $\mathcal{M}_G = \text{Isom}_{\text{fil}}^\otimes \mathcal{O}_{\overline{M}}(\mathcal{H}_{\log-dR}^1, (\mathcal{O} \otimes \mathcal{O}_{\overline{M}}')^2)$ is a $G'$-torsor over $\overline{M}'$. We have a functor $\mathcal{F}_G$ from the category of algebraic representations of $G'$ to the category of flat vector bundles on $\overline{M}'$ (that is vector bundles endowed with an integrable quasi-nilpotent logarithmic connection). To any algebraic representation $V$ of $B'$, $\mathcal{F}_G$ associates the fiber product $\overline{V}^G := \mathcal{M}_G \times V$. For $j \in \mathbb{N}$, we put $H^j_{\log-dR}(\overline{M}', \overline{V}) = R^j\phi_* (\overline{V} \otimes \mathcal{O}_{\overline{M}}'(d\log \infty))$, where $\phi : \overline{M}' \to \text{Spec}(\mathcal{O}(\overline{\mathbb{F}}))$ denotes the structural homomorphism.

By the Faltings’ Comparison Theorem [14], the $G_{\mathbb{Q}_p}$-representation $H^*(\mathcal{M}_1^1, \mathcal{V}_n(\overline{\mathbb{Q}}_p))$ is crystalline, hence de Rham, and we have a canonical isomorphism

$$H^*(\mathcal{M}_1^1, \mathcal{V}_n(\overline{\mathbb{Q}}_p)) \otimes B_{\text{dR}} \cong H^*_{\log-dR}(\overline{M}/\mathbb{Q}_p, \overline{V}_n) \otimes B_{\text{dR}}.$$ 

By [11] Sect.7, the Hodge to de Rham spectral sequence

$$E_r^{i,j} = H^{i+j}(\overline{M}/\mathbb{Q}_p, \text{gr}^i(\overline{V} \otimes \Omega_{\overline{M}}^j(d\log \infty))) \Rightarrow H^{i+j}_{\log-dR}(\overline{M}/\mathbb{Q}_p, \overline{V}_n),$$

degenerates at $E_1$ (the filtration is the tensor product of the two Hodge filtrations). In order to compute the jumps of the resulting filtration we introduce the BGG complex:

$$\mathcal{K}_n^i = \bigoplus_{J \in \mathcal{J}_F, |J| = i} \overline{W}_{J, (n+t) - t, n_0}.$$

The fact that $\mathcal{K}_n^i$ is a complex follows from (11) and from the following isomorphism (see [15] Prop.VI.5.1)

$$\text{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes U(\mathfrak{b}) W_1), U(\mathfrak{g}) \otimes U(\mathfrak{b}) W_2) \to \text{Diff}(\overline{W}_2, \overline{W}_1).$$

Define a filtration on $\mathcal{K}_n^i$ by $\text{Fil}^j \mathcal{K}_n^i = \bigoplus_{J \in \mathcal{J}_F, |p(J)| \geq j} \overline{W}_{J, (n+t) - t, n_0}$.

As the image of the Koszul’s complex (10) by the contravariant functor $W \mapsto \overline{W}$ is equal to the de Rham complex, and as the BGG complex is a direct (filtered) factor of the Koszul’s complex, we have:

**Theorem 2.3.** ([11] Thm.7.8) (i) There is a quasi-isomorphism of filtered complexes

$$\mathcal{K}_n^i \hookrightarrow \overline{V}_n \otimes \Omega_{\overline{M}}^j(d\log \infty).$$

(ii) The spectral sequence given by the Hodge filtration

$$E_1^{i,j} = \bigoplus_{J \in \mathcal{J}_F, |J| = i} H^{i+j-|J|}(\overline{M}/\mathbb{Q}_p, \overline{W}_{J, (n+t) - t, n_0}) \Rightarrow H^{i+j}_{\log-dR}(\overline{M}/\mathbb{Q}_p, \overline{V}_n).$$
degenerates at $E_1$.

(iii) For all $j \leq d$, the Hodge-Tate weights of the $p$-adic representation $\mathbb{H}^j(M_{\mathbb{Q}_p}^1, V_n(\overline{\mathbb{Q}_p}))$ belong to the set $\{p(j)\}$, $|J| \leq j$.

2.4. Hecke operators on the cohomology. We describe the standard Hecke operator $T_a$ as a correspondence on $Y^1$. We are indebted to M. Kisin for pointing out us that the usual definition of Hecke operators on $Y$ extends to $Y^1$ (see [29]§1.9-1.11). Note that the corresponding Hecke action on analytic modular forms for $G^*$ (see §1.10) is not easy to write down, because the double class for the Hecke operator $T_v$ does not belong to $G^*(\mathbb{A}_f)$, unless $v$ is inert in $F$.

Recall that $Y^1_1(n) = \bigsqcup_{i=1}^{h_1} M^1_a(c_i, n)$, where $c_1, \ldots, c_{h_1}$ are a set of representatives of $\text{Cl}_{F_1}^+$. Assume that $c_i a$ and $c_j$ have the same class in $\text{Cl}_{F_1}^+$. Then, consider the contravariant functor $M^1_\mathbb{A}$ from the category of $\mathbb{Z}([1])$-schemes to the category of sets, assigning to a scheme $\mathcal{S}$ the set of isomorphism classes of quintuples $(A, \lambda, C, \beta)$ where $(A, \lambda, \alpha)/S$ is a $c_i$-polarized HBAV with $\mu_\alpha$-level structure, $C$ is a closed subscheme of $A[a]$ which is $\sigma$-stable, disjoint from $\alpha(G_m \otimes \sigma^{-1})$ and locally isomorphic to the constant group scheme $\sigma/a$ over $S$, and $\beta$ is an $\alpha(a)\sigma$-orbit of isomorphisms $(c_i a, (c_i a)^-) \sim (c_j, c_j+)$. We have a projection $M^1_\mathbb{A} \rightarrow M^1_\mathbb{A}$, $(A, \lambda, C, \beta) \mapsto (A, \lambda, \alpha)$ which is relatively representable by $\pi_1 : M^1_\mathbb{A}(c_i, n) \rightarrow M^1_\mathbb{A}(c_i, n)$. We have also a projection $\pi_2 : M^1_\mathbb{A}(c_i, n) \rightarrow M^1_\mathbb{A}(c_j, n)$, coming from $(A, \lambda, \alpha, C, \beta) \mapsto (A/C, \lambda', \alpha')$, where $\alpha'$ is the composed map of $\alpha$ and $A \rightarrow A/C$, and $\lambda'$ is a $c_j$-polarization of $A/C$ (defined via $\lambda$ and $\beta$).

Put $Y^1_a = \bigsqcup_{i=1}^{h_1} M^1_\mathbb{A}(c_i, n)$. As $c_i \mapsto c_j \simeq c_i a$ is a permutation of $\text{Cl}_{F_1}^+$, we get two finite projections $\pi_1, \pi_2 : Y^1_a \rightarrow Y^1$.

$$
\begin{array}{ccc}
A & \rightarrow & A_a \\
\uparrow \pi & & \downarrow \pi_a \\
Y^1 & \rightarrow & Y^1_a \\
\pi_1 & \rightarrow & \pi_2 \\
Y^1 & \rightarrow & Y^1_a
\end{array}
$$

From this diagram we obtain $\pi_2^* \mathbb{H}^1_{dR} \rightarrow \pi_1^* \mathbb{H}^1_{dR}$. Therefore, for every algebraic representation $V$ of $G$, we have $\pi_2^* V^V \rightarrow \pi_1^* V^V$. By composing this morphisms by $\pi_1$, and taking the trace, we obtain $V^V \rightarrow \pi_1^* \pi_2^* V^V \rightarrow \pi_1^* \pi_1^* V^V \rightarrow \pi_1^* V^V$, and thus we obtain an action of $T_a$ on $\mathbb{H}^*(Y^1, V^V)$.

The same way from the above diagram we obtain $\pi_2^* \omega \rightarrow \pi_1^* \omega$ and $\pi_2^* \nu \rightarrow \pi_1^* \nu$. Therefore, for each algebraic representation $W$ of $T$, we get $\pi_2^* W \rightarrow \pi_1^* W$. In order to define the good action of $T_a$ on on Hilbert modular forms, we should modify slightly the last arrow: we decompose $W$ as $(W_{\omega^2}, \omega^{2t})$ and we define $\pi_{2*}(W_{\omega^2}, \omega^{2t}) \rightarrow \pi_{1*}(W_{\omega^2}, \omega^{2t})$ as above and $\pi_{2*}(\omega^2, \omega^{2t}) \rightarrow \pi_{1*}(\omega^2, \omega^{2t})$ as in [29]§1.11 (via the Kodaira–Spencer isomorphism $\Omega^1_{Y_1} \simeq \omega^2 \otimes \sigma \epsilon^{-1}$). Thus we obtain $W \rightarrow \pi_{1*} \pi_2^* W \rightarrow \pi_{1*} \pi_1^* W \rightarrow W$, and an action of $T_a$ on $\mathbb{H}^*(Y^1, W)$.

In particular, we obtain an action of $T_a$ on the space of geometric Hilbert modular forms for $G^*$, $\mathbb{H}^0(Y^1, \omega_k \otimes \omega^{n \ell/2})$. As it has been observed in [29]1.11.8, this action is given by the projection

$$
\frac{1}{\mathcal{O} \mathcal{O}_{F_1}} \sum_{t \in \mathcal{O} \mathcal{O}_{F_1}} [t] : \mathbb{H}^0(Y^1, \omega_k \otimes \omega^{n \ell/2}) \to \mathbb{H}^0(Y, \omega_k \otimes \omega^{n \ell/2}),
$$

followed by the usual Hecke operator on the space of Hilbert modular forms (see §1.10).
2.5. **Hodge-Tate weights of \( \otimes \text{Ind}_F^Q \rho \) in the crystalline case.** We first recall the notion of induced representation. Let \( V_0 \) be vector space over a field \( L \), and let \( \rho_0 : G_F \to \text{GL}(V_0) \) be a linear representation. The induced representation \( \text{Ind}_F^Q \rho_0 \) of \( \rho_0 \) from \( F \) to \( Q \) is by definition the \( L \)-vector space

\[
\text{Hom}_{g_F}(G_Q, V_0) := \{ \phi : G_Q \to V_0 \mid \forall h \in G_F, \ g \in G_Q, \ \phi_0(gh) = \rho_0(h^{-1})(\phi_0(g)) \},
\]

where \( g \in G_Q \) acts on \( \phi_0 \in \text{Hom}_{g_F}(G_Q, V_0) \) by \( g \cdot \phi_0(\cdot) = \phi_0(g^{-1} \cdot) \).

For any fixed decomposition \( G_Q = \prod_{\tau \in J_F} \tilde{T} G_F \), the map \( \phi \mapsto (\phi(\tilde{\tau}))_\tau \) gives an isomorphism between \( \text{Hom}_{g_F}(G_Q, V_0) \) and the direct sum \( \bigoplus V_\tau \) (where each \( V_\tau \) is isomorphic to \( V_0 \)). Via this identification, the action of \( G_Q \) on \( \bigoplus V_\tau \) is given by:

\[
(\text{Ind}_F^Q \rho_0)(g)((v_\tau)_\tau) = (\rho_0(\tilde{\tau}^{-1} g \tilde{\tau}_g)(v_\tau_g))_\tau,
\]

where \( g^{-1} \tilde{\tau} \in \tilde{\tau}_g G_F \). In fact \( (\rho_0(\tilde{\tau}))_\tau \) gives an isomorphism between \( \text{Hom}_{g_F}(G_Q, V_0) \) and the direct sum \( \bigoplus V_\tau \to \bigoplus V_\tau \) (where each \( V_\tau \) is isomorphic to \( V_0 \)). Via this identification, the action of \( G_Q \) on \( \bigoplus V_\tau \) is given by:

\[
(\otimes \text{Ind}_F^Q \rho_0)(g)(\otimes v_\tau) = \otimes_{\tau} \rho_0(\tilde{\tau}^{-1} g \tilde{\tau}_g)(v_\tau_g).
\]

**Remark 2.4.** For each \( g \in G_Q \) the map \( \tau \mapsto \tau_g \) is a permutation of \( J_F \), and it is trivial if and only if \( g \in G_F \). Therefore, for each \( g \in G_F \), we have \( (\otimes \text{Ind}_F^Q \rho_0)(g) = \otimes_{\tau} \rho_0(\tilde{\tau}^{-1} g \tilde{\tau}) \).

Moreover for all \( g, g' \in G_Q \) we have \( (\tau_g)_{g'} = \tau_{gg'} \).

**Definition 2.5.** The internal conjugate \( g_{\tau} \) of \( g \) by \( \tau \in J_F \), is defined as the unique element \( g_{\tau} \in S_{k^\tau,F}(\tau(n), \psi_\tau)/\tau(F) \) satisfying \( c(g_{\tau}, a) = c(g, \tau(a)) \), for each ideal \( a \) of \( a \), where \( k^\tau = \sum_{\tau'} k_\tau \tau' \) and \( \psi_\tau(a) = \psi(\tau(a)) \).

If \( \rho = \rho_{f,p} \) by the previous remark we have \( (\otimes \text{Ind}_F^Q \rho)(g) = \otimes_{\tau} \rho_{f,\tau}(g) \), for each \( g \in G_F \).

Brylinski and Labesse [3] have shown (see [40] for this formulation)

**Theorem 2.6.** (Brylinski-Labesse) The restrictions to \( G_F \) of the two \( G_Q \)-representations \( W_f \) and \( \otimes \text{Ind}_F^Q \rho \) have the same characteristic polynomial.

**Corollary 2.7.** ([11] Cor.7.9) (i) The spectral sequence given by the Hodge filtration

\[
E_1^{ij} = \bigoplus_{J \subset J_F, |p(J)| = i} H^{i+j-|J|}(\mathcal{M}/\mathcal{G}_p, \mathcal{W}_g(j(n+\bar{t})-t,n_0)) \Rightarrow H^{i+j}_{\log-dR}(\mathcal{M}/\mathcal{G}_p, \mathcal{V}_n)
\]

degenerates at \( E_1 \) and is Hecke equivariant.

(ii) The Hodge-Tate weights of \( W_f \) are the integers \( |p(J)|, J \subset J_F \), counted with multiplicity.

**Proof:** (i) By taking the invariants of the Hodge filtration of \( \mathcal{V}_n \otimes \Omega_{\mathcal{M}/\mathcal{M}}^\bullet(d\log \infty) \) by the Galois group of the etale covering \( \mathcal{M}^\ell \to \mathcal{M} \), we obtain a filtration of the complex \( \mathcal{V}_n \otimes \Omega_{\mathcal{M}/\mathcal{M}}^\bullet(d\log \infty) \) on \( \mathcal{M}^\ell \), still called the Hodge filtration. The same way, we define the BGG complex over \( \mathcal{M}^\ell \) by taking the invariants of the BGG complex over \( \mathcal{M}^\ell \). The associated
spectral sequence is given by the invariants of the spectral sequence of Thm.2.3(ii). We have now to see that it is Hecke equivariant.

The Hecke operator $T_a$ extends to $\overline{Y}^T$. One way to define it is to take the schematic closure of $T_a \subset Y^1 \times Y^1$ in $\overline{Y}^T \times \overline{Y}^T$. Another way is to take a toroidal compactification $\overline{Y}'$ of $Y^1$ over the toroidal compactification $\overline{Y}^T$ of $Y^1$. In both cases we obtain an action of $T_a$ on $H^*(\overline{Y}^T,\overline{\mathcal{W}})$ and on $H^*(\overline{Y}^T,\overline{\nabla})$. Although it is not clear in general that these extended Hecke operators commute. Nevertheless, they commute on the right hand side of Thm.2.3(ii), because by Faltings’ Comparison Theorem this side is independent of the toroidal compactification. Since the spectral sequence of Thm.2.3(ii) degenerates at $E_1$, they should also commute on the left hand side.

(ii) We have $\overline{W}_{\varepsilon J(n+t)-t,n_0} = \varphi^{-\varepsilon J(n+t)+t} \otimes \varphi^J$. It follows from Thm.2.3 (as in [15] Thm.5.5 and [31] Sect.2.3) that the jumps of the Hodge filtration are among $\{p(J)|, J \subset J_F\}$. Moreover $gr^{[J]} H^d(\overline{M}/\overline{Q}_p, \nabla_n \otimes \Omega^*_M(dlog \infty)) = H^d-\{J\}(\overline{M}/\overline{Q}_p, \varphi^{-\varepsilon J(n+t)+t} \otimes \varphi^J)$.

It is enough to see that the $\overline{Q}_p$-vector space $H^d-\{J\}(\overline{Y}/\overline{Q}_p, \varphi^{-\varepsilon J(n+t)+t} \otimes \varphi^J)/\mathbb{C}$ is of dimension 1, for all $J \subset J_F$.

Because of the existence of a BGG complex over $\overline{Q}$ giving by base change the BGG complexes over $\overline{Q}_p$ and $\mathbb{C}$, we have an Hecke-equivariant isomorphism

$$H^d-\{J\}(\overline{Y}/\overline{Q}_p, \varphi^{-\varepsilon J(n+t)+t} \otimes \varphi^J) \otimes \overline{Q}_p \cong H^d(Y^\text{an}, \nabla_n(\mathbb{C})[\varepsilon J \otimes \mathbb{C}].$$

Finally, the $f$-part $H^d(Y^\text{an}, \nabla_n(\mathbb{C})[\varepsilon J \otimes \mathbb{C}, f]$ is equal to $H^d(Y^\text{an}, \nabla_n(\mathbb{C})[\varepsilon J \otimes \mathbb{C}, f]$ and is therefore one dimensional by (8), for all $J \subset J_F$. \qed

Remark 2.8. 1) We proved that $W_f$ is pure of weight $d(k_0-1)$. The set of its Hodge-Tate weights is stable by the symmetry $h \mapsto d(k_0-1)-h$, corresponding $[p(J)| \mapsto [p(J\setminus J)].$ This symmetry is induced by the Poincaré duality $W_f \times W_f \rightarrow \overline{Q}_p(-d(k_0-1))$.

2) If $F$ is a real quadratic field and $\tau$ denotes the non-trivial embedding of $F$, then the Hodge-Tate weights of $W_f$ are given by $m, k_0-m, 1, k_0+m, 1, 2k_0-3m-2$.

2.6. Hodge-Tate weights of $\rho$ in the crystalline case. The embedding $\iota_p : \overline{Q} \hookrightarrow \overline{Q}_p$ allows us to identify $J_F$ with $\text{Hom}_{\overline{Q}_p}^{\text{alg}}(F, \overline{Q}_p)$. For each prime $p$ of $F$ dividing $p$, we put $J_{F,p} = \text{Hom}_{\overline{Q}_p-\text{alg}}(F_p, \overline{Q}_p)$. Thus we get a partition $J_F = \bigsqcup_p J_{F,p}$. Let $D_p$ (resp. $I_p$) be a decomposition (resp. inertia) subgroup of $G_F$ at $p$.

The following result is due to Wiles if $k$ is parallel, and to Hida in the general case.

Theorem 2.9. (Wiles [43], Hida [23]) Assume that $f$ is ordinary at $p$ (see Def.1.13).

Then $\rho|_{I_p}$ is reducible and :

$$(\text{ORD}) \quad \rho|\iota_p \sim \begin{pmatrix} \varepsilon_p & \ast \\ 0 & \delta_p \end{pmatrix},$$

where $\delta_p$ (resp. $\varepsilon_p$) is obtained by composing the class field map $I_p \rightarrow O_p^\times$ with the map $\mathfrak{o}_p^\times \rightarrow \overline{Q}_p^\times, \ x \mapsto \prod_{\tau \in J_{F,p}} \tau(x)^{-m} \ast \prod_{\tau \in J_{F,p}} \tau(x)^{-(k_0-m-1)}.$

Breuil [2] has shown that if $p > k_0$ and $p$ does not divide $\Delta$, then $\rho$ is crystalline at each prime $p$ of $F$ dividing $p$, with Hodge-Tate weights between 0 and $k_0-1$. 


Corollary 2.10. Assume \( p > k_0 \) and that \( p \) does not divide \( \Delta \). Then for each prime \( p \) of \( F \) dividing \( p \), \( \rho_{|D_p} \) is crystalline with Hodge-Tate weights the \( 2[F_p : \mathbb{Q}_p] \) integers \( (m_\tau, k_0 - m_\tau - 1)_{\tau \in J_{F,p}} \).

Proof: Assume first that \( n \neq 0 \). Let \( K \) be a CM quadratic extension of \( F \), in which each prime \( p \) of \( F \) (dividing \( p \)) splits as \( p = \mathfrak{p}\mathfrak{p}' \). Blasius and Rogawski [1] have then constructed a pure motive over \( K \) with coefficients in \( E \), of Hodge weights \( (m_\tau, k_0 - m_\tau - 1)_{\tau \in J_F} \) and whose \( p \)-adic realization is isomorphic to the restriction of \( \rho \) to \( \mathcal{G}_K \). This shows that \( \rho_{|D_p} \) is de Rham, and even crystalline for \( p \) big enough.

By Faltings’ Comparison Theorem the Hodge weights of this motif correspond naturally (via \( \iota_p : \mathbb{Q} \to \overline{\mathbb{Q}}_p \)) to the Hodge-Tate weights of its \( \mathfrak{p} \)-adic realization, which are the same as the Hodge-Tate weights of \( \rho \). This proves the corollary for \( n \neq 0 \).

If \( n = 0 \) (or more generally if \( k \) is parallel) we can complete the proof using the following

Lemma 2.11. Let \( a \) and \( b \) be two positive integers and let \((a_\tau)_{\tau \in J_F} \) (resp. \((b_\tau)_{\tau \in J_F} \)) be integers satisfying \( 0 \leq 2a_\tau < a \) (resp. \( 0 \leq 2b_\tau < b \)). Assume that the following two sets are equal (with multiplicities)
\[
\left\{ \sum_{\tau \in J} a_\tau + \sum_{\tau \in J_F \setminus J}(a - a_\tau), J \subset J_F \right\} = \left\{ \sum_{\tau \in J} b_\tau + \sum_{\tau \in J_F \setminus J}(b - b_\tau), J \subset J_F \right\}.
\]

Then \( a = b \) and we have equality (with multiplicities) \((a_\tau, \tau \in J_F) = (b_\tau, \tau \in J_F)\).

Using this lemma together with Thm.2.6 and Cor.2.7(ii) we obtain the Hodge-Tate weights of \( \rho \) at the primes \( p \) dividing \( p \), up to permutation. In particular, we know exactly the Hodge-Tate weights of \( \rho \) when \( k \) is parallel. \( \square \)

2.7. Fontaine-Laffaille weights of \( \mathfrak{p} \) in the crystalline case. Our aim is to find the weights of \( \mathfrak{p}_{|I_p} \) for \( p \) dividing \( p \). If \( f \) is ordinary at \( p \) we know by Thm.2.9, that \( \rho_{|D_p} \) is reducible and by a simple reduction modulo \( \mathcal{P} \) we obtain the weights of \( \mathfrak{p}_{|I_p} \).

Proposition 2.12. Assume \( p > k_0 \) and that \( p \) does not divide \( \Delta \). Then \( \mathfrak{p} \) is crystalline at each \( p \) dividing \( p \) with Fontaine-Laffaille weights \((m_\tau, k_0 - m_\tau - 1)_{\tau \in J_{F,p}} \).

Proof: It is a consequence of the theory of Fontaine and Laffaille [17], and of the computation of Hodge-Tate weights of \( \rho_{|D_p} \) in §2.6.

Consider a Galois stable lattice \( \mathcal{O}^2 \) in the crystalline representation \( \rho \), as well as the sub-lattice \( \mathcal{P}^2 \). The representation \( \mathfrak{p} \) is equal to the quotient of these two lattices. It is crystalline, as a sub-quotient of a crystalline representation. Its weights are determined by the associated filtered Fontaine-Laffaille module. Since the Fontaine-Laffaille’s functor is exact, it is a quotient of the Fontaine-Laffaille’s filtered modules associated to the two lattices. By compatibility of the filtrations on these two lattices, and by the condition \( p > k_0 \), the graded of the quotient have the right dimension. See the Appendix for more details. \( \square \)

Corollary 2.13. Let \( p \) be a prime of \( F \) above \( p \). Then
\[
\mathfrak{p}_{|I_p} \sim \begin{pmatrix} \varepsilon_p & \ast \\ 0 & \delta_p \end{pmatrix},
\]
where \( \varepsilon_p, \delta_p : I_p \to \overline{\mathfrak{p}}^\times \) are two tame characters of level \( |J_{F,p}| \), whose product equals the \((1-\kappa_0)\)th power of the modulo \( p \) cyclotomic character and whose sum has Fontaine-Laffaille weights \((m_\tau, k_0 - m_\tau - 1)_{\tau \in J_{F,p}} \).
3. Study of the images of $\overline{\rho}$ and $\text{Ind}_F^Q \overline{\rho}$.

In all this section we assume that $p > k_0$ and that $p$ does not divide $6\Delta$.

Let $\omega : \mathbb{G}_Q \to \mathbb{F}_p^\times$ be the modulo $p$ cyclotomic character and let $\text{pr} : \text{GL}_2(\kappa) \to \text{PGL}_2(\kappa)$ be the canonical projection.

3.1. Lifting of characters and irreducibility criterion for $\overline{\rho}$.

Proposition 3.1. (i) For all but finitely many primes $p$ ($\text{Irr}_p^{\overline{\rho}}$) holds, that is $\overline{\rho} = \overline{\rho}_{f,p}$ is absolutely irreducible.

(ii) Assume that $k$ is non-parallel. If for all $J \subset J_F$ there exists $\epsilon \in \mathfrak{o}_+^\times$, $\epsilon - 1 \in \mathfrak{n}$ such that $p$ does not divide the non zero integer $N_{F/\mathbb{Q}}(\epsilon^{p(J)} - 1)$, then ($\text{Irr}_p^{\overline{\rho}}$) holds.

Remark 3.2. Assume that $k = k_0t$ is parallel and that for all $\varnothing \subsetneq J \subsetneq J_F$, there exists $\epsilon \in \mathfrak{o}_+^\times$, $\epsilon - 1 \in \mathfrak{n}$ such that $p$ does not divide the constant term of an Eisenstein series of weight $k$ and level dividing $\mathfrak{n}$, which is the numerator of the value at $1 - k_0$ of the $L$-function of a finite order Hecke character of $F$ of conductor dividing $\mathfrak{n}$ (see [16]§3.2 for the case $F = \mathbb{Q}$).

Proof : As $\overline{\rho}$ is totally odd, if it is irreducible, then it is absolutely irreducible. Assume that $\overline{\rho}$ is reducible : $\overline{\rho}^{\mathfrak{a}_+} = \varphi_{\text{gal}} \oplus \varphi'_{\text{gal}}$. The characters $\varphi_{\text{gal}}, \varphi'_{\text{gal}} : \mathbb{G}_F \to \kappa^\times$ are unramified outside $\mathfrak{n}p$ and $\varphi_{\text{gal}}(\varphi'_{\text{gal}} = \text{det}(\overline{\rho}) = \overline{\varphi}_{\text{gal}}^{-1}$ (recall that $\psi$ is a Hecke character of infinity type $-n_0t$). Denote by $\mathfrak{a}_{\kappa_1}^\times$ the subgroup of $\mathfrak{a}_n^\times$ of elements $\equiv 1 \pmod{\mathfrak{n}}$. Then $\mathfrak{a}_{\kappa_1}^\times$ is a product of its $p$-part $\prod_{\mathfrak{p} | \mathfrak{p}} \mathfrak{a}_p^\times$ and its part outside $p$, denoted by $\mathfrak{a}_{\kappa_1}^\times(p)$.

By the global class field theory, the Galois group of the maximal $n$-ramified (resp. $n \mathfrak{p}$-ramified) abelian extension of $F$ is isomorphic to $\text{Cl}^+_{F,n} = \mathbb{A}_F^\times / F^\times \mathfrak{a}_{\kappa_1}^\times D(\mathbb{R})^\times$ (resp. $\text{Cl}^+_{F,n \mathfrak{p}} := \lim_{\leftarrow} \text{Cl}^+_{F,n \mathfrak{p}} = \mathbb{A}_F^\times / F^\times \mathfrak{a}_{\kappa_1}^\times D(\mathbb{R})^\times$). We choose the convention in which an uniformizer corresponds to a geometric Frobenius. We have the following exact sequence

$$1 \to \left( \prod_{\mathfrak{p} | p} \mathfrak{a}_p^\times \right)/\left\{ \epsilon \in \mathfrak{o}_+^\times | \epsilon - 1 \in \mathfrak{n} \right\} \to \text{Cl}^+_{F,n \mathfrak{p}} \to \text{Cl}^+_{F,n} \to 1.$$  

(13)

By Cor.2.13, for each $\mathfrak{p} | p$, $\varphi_{\text{gal}} \oplus \varphi'_{\text{gal}}$ is crystalline at $\mathfrak{p}$ of weights $(m_\tau, k_0 - m_\tau - 1)_{\tau \in J_F,p}$.

By (13) for every $\epsilon \in \mathfrak{o}_+^\times$, $\epsilon - 1 \in \mathfrak{n}$ we have the following equality in $\kappa$ :

$$1 = \varphi_{\text{gal}}(\epsilon) = \prod_{\mathfrak{p} | p} \varphi_{\text{gal},p}(\epsilon) = \prod_{\mathfrak{p} | p} \prod_{\tau \in J_F,p} \tau(\epsilon)^{m_\tau} \text{ or } (k_0 - m_\tau - 1) = \epsilon^{p(J)},$$

for some subset $J \subset J_F$. Because of the assumption $p > k_0$, if $k$ is non-parallel, then $\epsilon^{p(J)} \neq 1$ for all $J \subset J_F$. Thus we obtain (ii) and (i) when $k$ is non-parallel.

Assume now that $k = k_0t$ is parallel and that for all $\varnothing \subsetneq J \subsetneq J_F$, there exists $\epsilon \in \mathfrak{o}_+^\times$, $\epsilon - 1 \in \mathfrak{n}$ such that $p$ does not divide the non zero integer $N_{F/\mathbb{Q}}(\epsilon^{p(J)} - 1)$. The same arguments as above show that the restriction to $\prod_{\mathfrak{p} | p} \mathfrak{a}_p^\times$ of the character $\varphi_{\text{gal}}$ (resp. $\varphi'_{\text{gal}}$) $\text{Cl}^+_{F,n \mathfrak{p}} \to \kappa^\times$ is trivial (resp. given by the $(1 - k_0)$-th power of the norm). By the following lemma (applied to $P = \text{Cl}^+_{F,n \mathfrak{p}}$ and $Q = (\prod_{\mathfrak{p} | p} \mathfrak{a}_p^\times)/\left\{ \epsilon \in \mathfrak{o}_+^\times | \epsilon - 1 \in \mathfrak{n} \right\}$) there
exists an unique character $\tilde{\varphi}_{\text{gal}}$ (resp. $\tilde{\varphi}'_{\text{gal}}$) : $\text{Cl}^+_{F, n, p^\infty} \to O^\times$ lifting $\varphi_{\text{gal}}$ (resp. $\varphi'_{\text{gal}}$) and whose restriction to $\prod_{p \mid n} \mathfrak{p}_p^\times$ is also trivial (resp. given by the $(1 - k_0)$-th power of the norm).

**Lemma 3.3.** Let $P$ be an abelian group and $Q$ be a subgroup, such that the factor group $P/Q$ is finite. Let $\varphi : P \to \kappa^\times$ and $\varphi' : Q \to O^\times$ be two characters such that $\varphi \mid Q = \varphi' \mod p$. Then, there exists an unique character $\tilde{\varphi}_P : P \to O^\times$, whose restriction to $Q$ is $\tilde{\varphi}_Q$ and such that $\tilde{\varphi}_P \mod p = \varphi_P$.

For $x \in \mathbb{A}_F^\times$, we put $\varphi(x) := \tilde{\varphi}_{\text{gal}}(x)$ and $\varphi'(x) := \tilde{\varphi}'_{\text{gal}}(x)x_p^{-k}x_\infty^k$. Then $\varphi$ (resp. $\varphi'$) is a Hecke character of $F$, of conductor dividing $n$ and infinity type $0$ (resp. $(1 - k_0)t$). It is crucial to observe that there are only finitely many such $\varphi$ and $\varphi'$.

Assume now that for infinitely many primes $p$, $\mathfrak{p}$ is reducible. Then there exist Hecke characters $\varphi$ and $\varphi'$ as above, such that for infinitely many primes $p$ we have $\mathfrak{p}^{s, \text{ss}} \equiv \varphi_{\text{gal}} \oplus \varphi'_{\text{gal}} \mod \mathfrak{p}$. Hence for each prime $v$ of $F$ not dividing $n$ we have $c(f, v) \equiv \varphi(\tau_v) + \varphi'(\tau_v)$ (mod $\mathfrak{p}$) for infinitely many $\mathfrak{p}$’s and hence $c(f, v) = \varphi(\tau_v) + \varphi'(\tau_v)$. By the Cebotarev Density Theorem we obtain $\rho^{s, \text{ss}} = \varphi \oplus \varphi'$. This contradicts the absolute irreducibility of $\rho$ (see [41]).

### 3.2. The exceptional case

The aim of this paragraph is to find a bound for the primes $p$ such that $\text{pr}(\overline{\mathfrak{p}_{f,p}(G_F)})$ is isomorphic to one of the groups $A_4$, $S_4$ or $A_5$. We will only use the fact that the elements of these groups are of order at most $5$.

Assume that $\text{pr}(\overline{\mathfrak{p}_{f,p}(G_F)}) \cong A_4$, $S_4$ or $A_5$. By Cor. 2.13 there exist $\epsilon_\tau \in \{\pm 1\}$, $\tau \in J_F$, such that for all $p \mid p$ and for any generator $x$ of $\mathbb{F}_{p^h}^\times$, where $h = |J_{F,p}|$, the element

$$\prod_{\tau \in \text{Gal}(\mathbb{F}_{p^h}/\mathbb{F}_p)} \tau(x)^{\epsilon_\tau(k_\tau - 1)} \in \mathbb{F}_{p^h}^\times$$

belongs to $\text{pr}(\overline{\mathfrak{p}(I_p)})$ and is therefore of order at most $5$ (if (ORD) holds we may assume that $\epsilon_\tau = 1$ for all $\tau$). Denote by $\tau_1, \ldots, \tau_n$ the elements of $J_{F,p}$. Then

$$\epsilon_{\tau_1}(k_{\tau_1} - 1) + \epsilon_{\tau_2} p(k_{\tau_2} - 1) + \ldots + \epsilon_{\tau_n} p^{h-1}(k_{\tau_n} - 1) \in \mathbb{Z}/(p^h - 1)$$

is of order $\leq 5$, hence $5((k_{\tau_1} - 1) + p(k_{\tau_2} - 1) + \ldots + p^{h-1}(k_{\tau_n} - 1)) \geq p^h - 1$.

If we replace the generator $x$ by $x^p, x^{p^2}, \ldots, x^{p^{h-1}}$ and then sum these inequalities we find

$$5 \sum_{\tau \in J_{F,p}} (k_\tau - 1) \geq |J_{F,p}|(p - 1).$$

We conclude that $\text{pr}(\overline{\mathfrak{p}(G_F)})$ cannot be isomorphic to $A_4$, $S_4$ or $A_5$ if

$$d(p - 1) > 5 \sum_{\tau \in J_F} (k_\tau - 1).$$

Note that this assumption follows from (II) if $d \geq 5$.

### 3.3. The dihedral case

In this paragraph we study the case when $\text{pr}(\overline{\mathfrak{p}_{f,p}(G_F)})$ is isomorphic to the dihedral group $D_{2n}$, where $n \geq 3$ is an integer prime to $p$. Let $C_n$ be the cyclic subgroup of order $n$ of $D_{2n}$. Since $\text{pr}^{-1}(C_n)$ is a commutative group containing only semi-simple elements ($p$ does not divide $n$), it is diagonalizable. Since $\text{pr}^{-1}(D_{2n}\setminus C_n)$ is contained in the normalizer of $\text{pr}^{-1}(C_n)$, it is contained in the set of anti-diagonal matrices.
Let $\varepsilon : D_{2n} \to \{\pm 1\}$ be the signature map and let $K$ be the fixed field of $\ker(\varepsilon \circ \text{pr} \circ \bar{\rho})$. The extension $K/F$ is quadratic and unramified outside $\mathfrak{p}$. 

Let $c$ be the non-trivial element of the Galois group $\text{Gal}(K/F)$. As $\bar{\mathfrak{p}}$ is absolutely irreducible, but $\bar{\mathfrak{p}}|\mathcal{O}_K$ is not, there exists a character $\varphi_{\text{gal}} : \mathcal{G}_K \to \kappa^\times$ distinct from its Galois conjugate $\mathfrak{c}_{\text{gal}}$ and such that $\bar{\rho}|\mathcal{G}_K = \varphi_{\text{gal}} \oplus \mathfrak{c}_{\text{gal}}$.

**Lemma 3.4.** Let $\mathfrak{p}$ be a prime of $F$ dividing $p$. Assume $p \neq 2k_\tau - 1$, for $\tau \in J_{F,p}$. Then

(i) the field $K$ is unramified at $\mathfrak{p}$,

(ii) the prime $\mathfrak{p}$ splits in $K$ as $\mathfrak{p}\mathfrak{p}^c$, and $\varphi_{\text{gal}}$ is crystalline at $\mathfrak{p}$ (resp. $\mathfrak{p}^c$) of weights $(p_\tau)_{\tau \in J_{F,p}}$ (resp. $(q_\tau)_{\tau \in J_{F,p}}$), where $\{p_\tau, q_\tau\} = \{m_\tau, k_0 - m_\tau - 1\}$ for each $\tau \in J_{F,p}$.

**Proof:** (i) Otherwise $\bar{\rho}(I_\mathfrak{p})$ would contain at least one anti-diagonal matrix and the basis vectors would not be eigen for $(\varphi_{\mathfrak{p}})$, but the group $\bar{\rho}(I_\mathfrak{p})$ has a common eigenvector. Hence, the elements of $\text{pr}(\bar{\rho}(I_\mathfrak{p}))$ would be of order $\leq 2$. Using the computations of §3.2 and $p > k_0$, we find that for all $\tau \in J_{F,p}$ we have $2(k_\tau - 1) = p - 1$. Contradiction.

(ii) By Cor.2.13, $\varphi_{\text{gal}} \oplus \mathfrak{c}_{\text{gal}}$ is crystalline at $\mathfrak{p}$ (resp. $\mathfrak{p}^c$) of weights $(m_\tau, k_0 - m_\tau - 1)_{\tau \in J_{F,p}}$.

Let $\mathcal{D}$ be the integer ring of $K$, and $\hat{\mathcal{D}}$ its profinite completion. Denote by $\hat{\mathcal{D}}_{n,1}^\times$ the subgroup of $\hat{\mathcal{D}}^\times$ of elements $\equiv 1 \pmod{n}$. Then $\hat{\mathcal{D}}_{n,1}^\times$ is a product of its $p$-part $\prod_{\mathfrak{p} \nmid p} \mathcal{D}_{\mathfrak{p}}^\times$ and its part outside $p$, denoted by $\hat{\mathcal{D}}_{n,1}^\times(p)$.

By the global class field theory, the Galois group of the maximal $n$-ramified (resp. $np^\infty$-ramified) abelian extension of $K$ is isomorphic to $\text{Cl}_{K,n} := \mathbb{A}_K^\times / K^\times \hat{\mathcal{D}}_{n,1}^\times K_\infty^\times$ (resp. to $\text{Cl}_{K,np^\infty} := \mathbb{A}_K^\times / K^\times \hat{\mathcal{D}}_{n,1}^{\times(p)} K_\infty^\times$). We have the following exact sequence:

$$1 \to (\prod_{\mathfrak{p} \nmid p} \mathcal{D}_{\mathfrak{p}}^\times)/\{\epsilon \in \mathcal{D}^\times | \epsilon - 1 \in n\} \to \text{Cl}_{K,np^\infty} \to \text{Cl}_{K,n} \to 1$$

**Proposition 3.5.** (i) Assume that for all $\tau \in J_F$, $p \neq 2k_\tau - 1$ and that $\text{pr}(\bar{\rho}(G_F))$ is dihedral. Let $K/F$ be the quadratic extension defined above. Then

- either $K$ is CM and there exists a Hecke character $\varphi$ of $K$ of conductor of norm dividing $n \Delta_{K/F}^{-1}$ and infinity type $(m_\tau, k_0 - 1 - m_\tau)_{\tau \in J_F}$ such that $\rho \equiv \text{Ind}_F^K \varphi \pmod{\mathcal{P}}$,

- either $K$ is not CM and we can choose places $\mathfrak{p}$ of $K$ above each $\tau \in J_F$ such that for all $\epsilon \in \mathcal{D}^\times$, $\epsilon - 1 \in n$ the prime $p$ divides $N_{K/Q}(\prod_{\tau \in J_F} \mathfrak{p}(\tau)^{m_\tau - 1}(c(\epsilon))^{k_0 - m_\tau - 1})$.

(ii) Assume that $f$ is not a theta series. Then for all but finitely many primes $p$ the group $\text{pr}(\bar{\rho}(G_F))$ is not dihedral.

**Remark 3.6.** (i) The primes $p$ for which the congruence $\rho \equiv \text{Ind}_F^K \varphi \pmod{\mathcal{P}}$ may occur should be controlled by the special value of the $L$-function associated to the CM character $\varphi/\varphi^c$ (in the elliptic case it is proved by Hida [22] and Ribet [36]; see also Thms A and B).

(ii) We would like to thank E. Ghate for having pointed out the possible existence of dihedral primes for non CM fields $K$. It would be interesting to explore the converse
Hence the congruence \( \varphi_{\text{gal}} : \text{Cl}_{K, np} \to \kappa^\times \) whose restriction to \( \prod \mathcal{P}_p \kappa^\times \) is given by the reduction modulo \( p \) of an algebraic character \( x \mapsto x^k \), where \( k = \sum_{\tau \in J_F} m_{\tau} \tau + (k_0 - m_{\tau} - 1) \tau \circ c \), for some choice of places \( \tau \) of \( K \) above \( \tau \in J_F \).

We observe that the character \( x \mapsto x^k \) is trivial on \( \mathfrak{o}_K^\times \), whereas it is only trivial modulo \( p \) on \( \{ \epsilon \in \mathcal{D}^\times | \epsilon - 1 \in \mathfrak{n} \} \). The case when \( K \) is not CM follows immediately.

Assume now that \( K \) is CM. In this case \( \{ \epsilon \in \mathcal{D}^\times | \epsilon - 1 \in \mathfrak{n} \} \) is a finite index subgroup of \( \{ \epsilon \in \mathcal{D}^\times | \epsilon - 1 \in \mathfrak{n} \} \). Since \( \ker(\mathcal{D}^\times \to \kappa^\times) \) does not contain elements of finite order, the above character is trivial on \( \{ \epsilon \in \mathcal{D}^\times | \epsilon - 1 \in \mathfrak{n} \} \).

By the lemma 3.3 (applied to \( P = \text{Cl}_{K, np}^\infty \) and \( Q = (\prod \mathcal{P}_p \mathcal{D}_p^\times)/\{ \epsilon \in \mathcal{D}^\times | \epsilon - 1 \in \mathfrak{n} \} \)) there exists a lift \( \tilde{\varphi}_{\text{gal}} : \text{Cl}_{K, np}^\infty \to \mathcal{O}_p^\times \) whose restriction to \( \mathcal{P}_p \mathcal{D}_p^\times \) is given by \( x \mapsto x^k \).

We put \( \varphi(x) := \tilde{\varphi}_{\text{gal}}(x)x_p^{-k}x_\infty^k \). Then \( \varphi \) is a Hecke character of \( K \) as desired.

(ii) There are finitely many fields \( K \) as above. For those \( K \) that are not CM it is enough to choose \( \epsilon \in \mathcal{D}^\times \), \( \epsilon - 1 \in \mathfrak{n} \) of infinite order in \( \mathcal{D}^\times/\kappa^\times \).

For each of the CM fields \( K \) that are only finitely many characters \( \varphi \) as above. Therefore, if \( \text{pr}(\mathcal{T}(\mathcal{G}_F)) \) is dihedral for infinitely many primes \( p \), then there would exist \( K \) and \( \varphi \) as above, such that the congruence \( \rho \equiv \text{Ind}_K^F \varphi \mod \mathcal{P} \) happens for infinitely many \( \mathcal{P} \)'s. Hence \( f \) would be equal to the theta series associated to \( \varphi \).

3.4. The image of \( \mathcal{T} \) is “large”.

**Theorem 3.7.** (Dickson) (i) An irreducible subgroup of \( \text{PSL}_2(\kappa) \) of order divisible by \( p \) is conjugated inside \( \text{PGL}_2(\kappa) \) to \( \text{PSL}_2(\mathbb{F}_q) \) or to \( \text{PGL}_2(\mathbb{F}_q) \), for some power \( q \) of \( p \).

(ii) An irreducible subgroup of \( \text{PSL}_2(\kappa) \) of order prime to \( p \) is either dihedral, either isomorphic to one of the groups \( A_4, S_4 \) or \( A_5 \).

As an application of this theorem, Prop.3.1, Prop.3.5 and §3.2 we obtain the following

**Proposition 3.8.** Assume that \( f \in S_k(\mathfrak{n}, \psi) \) is a newform, which is not a theta series. Then for all but finitely many primes \( p \), the image of the \( p \)-adic representation \( \mathcal{T} \) associated to \( f \) is large, in the following sense:

\((\text{LI}_\mathcal{T}) \) there exists a power \( q \) of \( p \) such that \( \text{SL}_2(\mathbb{F}_q) \subset \text{im}(\mathcal{T}) \subset \kappa^\times \text{GL}_2(\mathbb{F}_q) \).

Let \( \widehat{F} \) be the compositum of \( F \) and of the subfield of \( \overline{\mathbb{Q}} \) fixed by the Galois group \( \ker(\overline{\psi}^n) \). The extension \( \widehat{F}/F \) is Galois and unramified at \( p \), because \( \widehat{F} \) is unramified at \( p \) and \( \psi \) is of conductor prime to \( p \). Therefore \( \mathcal{G}_{\widehat{F}} \) is a normal subgroup of \( \mathcal{G}_F \) containing the inertia subgroups \( I_p \), for all \( p \) dividing \( p \).

We put \( \mathcal{D} = \det(\mathcal{T}(\mathcal{G}_{\widehat{F}})) = (\mathbb{F}_q^n)^{1-k_0} \).

**Proposition 3.9.** Assume \((\text{LI}_\mathcal{T})\). Then there exists a power \( q \) of \( p \) such that,

- either \( \mathcal{T}(\mathcal{G}_{\widehat{F}}) = \text{GL}_2(\mathbb{F}_q) \mathcal{D} := \{ x \in \text{GL}_2(\mathbb{F}_q) \mid \det(x) \in \mathcal{D} \} \),
- either \( \mathcal{T}(\mathcal{G}_{\widehat{F}}) = (\mathbb{F}_q^n)^{1-k_0} \text{GL}_2(\mathbb{F}_q) \mathcal{D} := \{ x \in \mathbb{F}_q^n \text{GL}_2(\mathbb{F}_q) \mid \det(x) \in \mathcal{D} \} \).
Proof: We first show that $\operatorname{pr}(\overline{\rho}(G_F))$ is still irreducible of order divisible by $p$. By \((\text{LI}_\overline{\rho})\) the group $\operatorname{pr}(\overline{\rho}(G_F))$ is isomorphic to $\PSL_2(F_q)$ or $\PGL_2(F_q)$. The group $\operatorname{pr}(\overline{\rho}(G_F))$ is a non-trivial normal subgroup of $\operatorname{pr}(\overline{\imath}(\overline{\rho}))$ (because it contains $\operatorname{pr}(\overline{\rho}(I_p))$ and $p > k_0$; see Cor.2.13). As $\PSL_2(F_q)$ is a simple group of index $2$ in the group $\PGL_2(F_q)$, we deduce

$$\PSL_2(F_q) \subset \operatorname{pr}(\overline{\rho}(G_F)) \subset \operatorname{pr}(\overline{\rho}(G_F)) \subset \PGL_2(F_q).$$

**Lemma 3.10.** Let $H$ be a group of center $Z$ and let $\operatorname{pr} : H \to H/Z$ the canonical projection. Let $P$ and $Q$ be two subgroups of $H$ such that $\operatorname{pr}(P) \supset \operatorname{pr}(Q)$. Assume moreover that $Q$ does not have non-trivial abelian quotients. Then $P \supset Q$.

It follows from this lemma that $\overline{\rho}(G_F) \supset \SL_2(F_q)$, hence

$$(\kappa^\times \GL_2(F_q))^D \supset \overline{\rho}(G_F) \supset \GL_2(F_q)^D.$$

Since $[(\kappa^\times \GL_2(F_q))^D : \GL_2(F_q)^D] \leq 2$ we’re done.

Let $y \in F_{q^2} \setminus F_q$ be such that $y^2 \in F_q$. Then $(F_{q^2}^\times \GL_2(F_q))^D = \GL_2(F_q)^D \cap (y \GL_2(F_q))^D$ and hence $\operatorname{tr}((F_{q^2}^\times \GL_2(F_q))^D) = F_q \cup y F_q$. Therefore, the $F_{p^r}$-algebra generated by the traces of the elements of $(F_{q^2}^\times \GL_2(F_q))^D$ is $F_{q^2}$, while $\operatorname{pr}((F_{q^2}^\times \GL_2(F_q))^D) \subset \PGL_2(F_q)$. This reflects the existence of a congruence with a form having inner twists.

### 3.5. The image of $\Ind^{\overline{\rho}}_{\overline{\rho}}$ is “large”.

We assume in this paragraph that \((\text{LI}_\overline{\rho})\) holds.

By Prop.3.9 there exists a power $q$ of $p$ such that $\operatorname{pr}(\overline{\rho}(G_{\overline{F}})) = \PSL_2(F_q)$ or $\PGL_2(F_q)$.

Consider the representation $\operatorname{pr}(\Ind^{Q}_{\overline{\rho}} \overline{\rho}) : \overline{G}_{\overline{F}} \to \PGL_2(F_q)^{J_F}$.

Any automorphism of the simple group $\PSL_2(F_q)$ is the composition of a conjugation by an element of $\PGL_2(F_q)$ and of a Galois automorphism of $F_q$. By a lemma of Serre (see [35]), there exist a partition $J_F = \bigsqcup_{i \in I} J^i_F$ and for each $i \in I$, and for each $\tau \in J^i_F$ an element $\sigma_{i, \tau} \in \Gal(F_q/F_p)$ such that

$$\operatorname{pr}(\phi(\SL_2(F_q)^J)) \subset \operatorname{pr}(\Ind^{Q}_{\overline{\rho}} \overline{\rho}(G_{\overline{F}})) \subset \operatorname{pr}(\phi(\GL_2(F_q)^J)),$$

where $\phi = ((\phi^i)_{i \in I} : \GL_2(F_q)^J \hookrightarrow \GL_2(F_q)^{J_F}$ is given by $\phi^i(M_i) = (M_{i, r, \tau}^{\sigma_{i, \tau}})_{r \in J^i_F}$.

Keeping these notations, we introduce the following assumption on the image of $\Ind^{Q}_{\overline{\rho}} \overline{\rho}$

\begin{enumerate}
\item [(LI$_{\Ind^{\overline{\rho}}}$)] the condition \((\text{LI}_\overline{\rho})\) holds and $\forall \ i \in I, \ \forall \ \tau, \tau' \in J^i_F \ (\tau \neq \tau' \Rightarrow \sigma_{i, \tau} \neq \sigma_{i, \tau'})$.
\end{enumerate}

We now introduce a genericity assumption on the weight $k$.

**Definition 3.11.** We say that the weight $k \in \mathbb{Z}[J_F]$ is non-induced, if there does not exist a strict subfield $F'$ of $F$ and a weight $k' \in \mathbb{Z}[J_{F'}]$ such that for each $\tau \in J_F$, $k_\tau = k'_\tau|_{J_{F'}}$.

**Remark 3.12.** Define $\overline{k} = \sum_{\tau \in J_F} k_\tau \overline{\tau} \in \mathbb{Z}[J_{\overline{F}}]$ by putting $k_{\overline{\tau}} = k_{\overline{\tau}|_{J_{\overline{F}}}}$, for all $\overline{\tau} \in J_{\overline{F}}$. The group $G_Q$ acts on $\mathbb{Z}[J_{\overline{F}}]$ by $\overline{k} = \sum_{\tau \in J_{\overline{F}}} k_{\overline{\tau}} \overline{\tau} \mapsto \overline{k'} = \sum_{\overline{\tau} \in J_{\overline{F}}} k_{\overline{\tau}} \overline{\tau}$. It is easy to see that $k \in \mathbb{Z}[J_{\overline{F}}]$ is non-induced, if and only if $\{ \overline{\tau}' \in G_Q \mid \overline{k} = \overline{k}' \}$ equals $G_F$.

**Proposition 3.13.** Assume that \((\text{LI}_\overline{\rho})\) hold and $k$ is non-induced. Assume moreover that for all $\tau \neq \tau' \in J_F, \ p \neq k_\tau + k_{\tau'} - 1$. Then \((\text{LI}_{\Ind^{\overline{\rho}}})\) hold.
Proof: Let $\tilde{\tau}_1, \tilde{\tau}_2 \in G_Q$ be such that for all $g \in G_{\bar{F}}$ we have $\text{pr}(\bar{p}(\tilde{\tau}_1^{-1}g\tilde{\tau}_1)) = \text{pr}(\bar{p}(\tilde{\tau}_2^{-1}g\tilde{\tau}_2))$. We have to prove that $\tilde{\tau}_1^{-1}\tilde{\tau}_2 \in G_F$. We put $\bar{p}_i(g) = \bar{p}(\tilde{\tau}_i^{-1}g\tilde{\tau}_i)\ (i = 1, 2)$.

Let $\mathfrak{P}$ be a prime ideal of $\bar{F}$ above a prime ideal $\mathfrak{p}$ of $F$ dividing $p$. Denote by $h'_i$ (resp. $h_i$) the residual degree of $\mathfrak{P}^{\tilde{\tau}_i}$ (resp. of $\mathfrak{p}^{\tau}$), $i = 1, 2$. By Cor.2.13 we have $\bar{p}_i|_{\mathfrak{p}^{s,s}} = \epsilon_i \oplus \delta_i$, where $\epsilon_i$ (resp. $\delta_i$) $I_{\mathfrak{P}} \to I_{\mathfrak{P}^{\tilde{\tau}_i}} \to \mathbb{F}_{p^{h'_i}}^\times \to \mathbb{F}_{p^{h_i}}^\times \to \kappa^\times$ is obtained by composing the conjugation by $\tilde{\tau}_i$, the projection on the tame inertia, the norm map, and the character $x \mapsto \prod_{\tau \in J_{F,\mathfrak{p}^{\tilde{\tau}_i}}} \tau(x)^{p_{\tau}}$ (resp. $x \mapsto \prod_{\tau \in J_{F,\mathfrak{p}^{\tilde{\tau}_i}}} \tau(x)^{q_{\tau}}$), where $\{p_{\tau}, q_{\tau}\} = \{m_{\tau}, k_0 - 1 - m_{\tau}\}$. In the case (ORD) we can even assume that for all $\tau \in J_F$, we have $p_{\tau} = m_{\tau}$ and $q_{\tau} = k_0 - 1 - m_{\tau}$.

Note that $\epsilon_1\delta_1 = \epsilon_2\delta_2 = \omega^{1-k_0}$. Since $I_{\mathfrak{P}} \subset G_{\bar{F}}$ and $\text{pr} \circ \overline{\mathfrak{p}} = \text{pr} \circ \overline{\mathfrak{p}}$ on $G_{\bar{F}}$, we may assume that $\epsilon_1\delta_1 = \epsilon_2\delta_2$. By varying $\mathfrak{P}$ we deduce that for all $\tilde{\tau} \in J_{\bar{F}}$, $\tilde{k}_{\tilde{\tau}} = \tilde{k}_{\tilde{\tau}_1^{-1}\tilde{\tau}_2}$ (here we use $p > k_0$ and the assumption $p \neq k_0 + k_\tau - 1$). As $k$ is non-induced, it follows from the remark 3.12 that $\tilde{\tau}_1^{-1}\tilde{\tau}_2 \in G_F$. \hfill \Box

The following corollary generalizes a result of Ribet [35] on the image of a Galois representation associated to a family of classical modular forms, to the case of the family of internal conjugates of a Hilbert modular form.

Corollary 3.14. Assume that (LI$\mathfrak{P}$) hold and $k$ is non-induced. Assume moreover that $p > 2k_0$ is totally split in $F$. Then,

$$(\text{GL}_2(\mathbb{F}_q)^{J_F})^D \subset \text{Ind}_{K^{(\mathfrak{p})}}^Q(\mathfrak{P}(G_F)) \subset (\mathfrak{P}(G_F)^{J_F})^D,$$ where $D = \mathbb{F}_p^{(1-k_0)}$.

Put $H(\mathbb{F}_q) = \left( \prod_{i \in I} \text{GL}_2(\mathbb{F}_q) \right)^D = \left\{ (M_i)_{i \in I} \mid \prod_{i \in I} \text{GL}_2(\mathbb{F}_q) \mid \exists \delta \in D, \forall i, \det(M_i) = \delta \right\}$.

Lemma 3.15. Assume $p > 2k_0$. Then,

(i) for all $p$ dividing $p$, $\mathfrak{P}(I_p)$ is contained (possibly after conjugation by an element of $\text{GL}_2(\mathbb{F}_q)$), either in the Borel’s subgroup of $\text{GL}_2(\mathbb{F}_q)$, either in the non-split torus of $\text{GL}_2(\mathbb{F}_q)$. The second case cannot occur if $f$ is ordinary at $p$.

(ii) $\text{Ind}_{K^{(\mathfrak{p})}}^Q(\mathfrak{P}(I_p)) \subset \phi(H(\mathbb{F}_q))$.

Proof: (i) Put $h = |J_{F,p}|$. By Cor.2.13 we have $\mathfrak{P}(I_p)|_{\mathfrak{p}^{s,s}} = \epsilon_p \oplus \delta_p$, where $\epsilon_p$ (resp. $\delta_p$) $I_p \to \kappa^\times$ are obtained by composing the tame inertia map $I_p \to \mathbb{F}_{p^h}^\times$ and the character $\epsilon : x \mapsto \prod_{\tau \in J_{F,p}} \tau(x)^{p_{\tau}}$ (resp. $\delta : x \mapsto \prod_{\tau \in J_{F,p}} \tau(x)^{q_{\tau}}$), where $\{p_{\tau}, q_{\tau}\} = \{m_{\tau}, k_0 - 1 - m_{\tau}\}$.

Let $x_h$ be a generator of $\mathbb{F}_q^\times$. As the traces of the elements of $\mathfrak{P}(G_F)$ are in $F_q \prod y \mathbb{F}_q$, we have $(\epsilon(x_h) + \delta(x_h))^2 \in \mathbb{F}_q$ and therefore $\epsilon(x_h)^2 + \delta(x_h)^2 \in \mathbb{F}_q$.

If $\epsilon(x_h)^2, \delta(x_h)^2 \in \mathbb{F}_q^\times$, then it is easy to see that $\epsilon(x_h), \delta(x_h) \in \mathbb{F}_q^\times$ (we use $p > k_0$ and $p \neq 2$, $k_\tau - 1$). Therefore $I_p$ fixes a $\mathbb{F}_q$-rational line, and $\mathfrak{P}(I_p)$ is contained in a Borel subgroup of $\text{GL}_2(\mathbb{F}_q)$.

Otherwise $\epsilon(x_h)^2$ and $\delta(x_h)^2$ are conjugated by the non-trivial element of $\text{Gal}(\mathbb{F}_q^2 / \mathbb{F}_q)$, hence $\epsilon(x_h)^2 = \delta(x_h)^2q_i$. Since $p > 2k_0$, we have $\epsilon(x_h) = \delta(x_h)^q$ and so $\epsilon(x_h) + \delta(x_h)^q \in \mathbb{F}_q^\times$. Hence $\text{tr}(\mathfrak{P}(I_p)) \subset \mathbb{F}_q$, and therefore $\mathfrak{P}(I_p) \subset \text{GL}_2(\mathbb{F}_q)$. In this case $\mathfrak{P}(I_p)$ is contained in a
Corollary 3.18. □

Assume that Proposition 3.17.  

\( p > 2k_0 \). □

Lemma 3.16. \( \phi(H(\mathbb{F}_q)) \subset \text{Ind}^G_{\hat{F}} \mathfrak{m}(G_{\hat{F}}) \),

Proof: We have seen in the beginning of this paragraph that \( \text{pr}(\phi(\text{SL}_2(\mathbb{F}_q)^f)) \subset \text{pr}(\text{Ind}^G_{\hat{F}} \mathfrak{m}(G_{\hat{F}})) \). By lemma 3.10, we deduce that \( \phi(\text{PSL}_2(\mathbb{F}_q)^f) \subset \text{Ind}^G_{\hat{F}} \mathfrak{m}(G_{\hat{F}}) \).

As \( \phi(H(\mathbb{F}_q)) = \phi(\text{SL}_2(\mathbb{F}_q)^f) \text{Ind}^G_{\hat{F}} \mathfrak{m}(I_p) \), we’re done. □

Proposition 3.17. Assume that \( f \) is not a theta series and that \( (\text{LI}_{\text{Ind}_p}) \) does not hold for infinitely many primes \( p \). Then, there exists \( \tau \in J_F, \tau \neq \text{id} \) and a finite order Hecke character \( \varepsilon \) of \( \hat{F} \) of conductor dividing \( N_{F/\mathbb{Q}}(n) \), such that for all prime \( v \mid N_{F/\mathbb{Q}}(n) \) which splits completely in \( \hat{F} \), we have \( c(f, \tau, v) = \varepsilon(v)c(f, v) \).

Proof: Since \( f \) is not a theta series, we know that \( (\text{LI}_p) \) hold for all but finitely many primes \( p \). Take such a \( p \), and assume that \( (\text{LI}_{\text{Ind}_p}) \) does not hold. Then there exist \( \tilde{\tau}_1, \tilde{\tau}_2 \in G_{\hat{Q}} \) such that \( \tau := \tilde{\tau}_2^{-1}\tilde{\tau}_1 |_{F} \neq \text{id} \) and for all \( g \in G_{\hat{F}} \), \( \text{pr}(\tilde{\tau}_2^{-1}g\tilde{\tau}_1) = \text{pr}(\tilde{\tau}_2^{-1}g\tilde{\tau}_2) \).

As \( (\text{LI}_p) \) hold and \( G_{\hat{F}} \) is a normal subgroup of \( G_{\hat{F}} \), the above relation hold for every \( g \in G_{\hat{F}} \). Therefore, there exist a character \( \varepsilon_{\text{gal}} : G_{\hat{F}} \rightarrow \kappa^x \), such that for all \( g \in G_{\hat{F}} \), \( \tilde{\psi}_{f, \tau}(g) = \varepsilon_{\text{gal}}(g)\tilde{\psi}_{f}(g) \). Assume that \( p > 2k_0 \). Then, the same argument as in the proof of Prop.3.13 shows that \( \varepsilon_{\text{gal}} \) is unramified at primes dividing \( p \). By lemma 3.3 \( \varepsilon_{\text{gal}} \) can then be lifted to a finite order Hecke character \( \varepsilon \) of \( \hat{F} \) of conductor dividing \( N_{F/\mathbb{Q}}(n) \). Because of the determinant relation \( \frac{1}{\psi_{\tau}} = \varepsilon^2_{\text{gal}}/\psi \), there finitely many such \( \varepsilon \)'s.

For every prime \( v \mid np \) which splits completely in \( \hat{F} \), we have \( c(f, \tau, v) \equiv \varepsilon(v)c(f, v) \) (mod \( P \)). If \( (\text{LI}_{\text{Ind}_p}) \) fails for infinitely many \( P \)'s, then the congruence above become an equality. □

Corollary 3.18. Assume that \( F \) is a Galois field of odd degree and the central character of \( \psi \) of \( f \) is trivial \( (F = \hat{F}) \). Assume moreover that \( f \) is not a theta series and that \( (\text{LI}_{\text{Ind}_p}) \) does not hold for infinitely many primes \( p \). Then, there exist a subfield \( F'' \subsetneq F \) and a Hilbert modular form \( f' \) on \( F'' \), such that the base change of \( f' \) to \( F \) is a twist of \( f \) by a quadratic character of conductor dividing \( N_{F/\mathbb{Q}}(n) \).

Proof: As in the proof of Prop.3.17 there exist a quadratic character \( \varepsilon \) of \( F \) of conductor dividing \( N_{F/\mathbb{Q}}(n) \) and \( \text{id} \neq \tau \in \text{Gal}(F/\mathbb{Q}) \) such that we have \( \rho_{f, \tau} = \varepsilon_{\text{gal}} \otimes \rho \). Let \( F' \subset F \) (resp. \( F' \supset F \)) be the fixed field of \( \tau \) (resp. of \( \text{ker}(\varepsilon_{\tau}) \)). By assumption \( F/F' \) is a cyclic extension of odd degree \( h \). Let \( F'' = \prod_{i=1}^{h} F_i \). Then we have \( \text{Gal}(F''/F') = \{(u_1, \ldots, u_h) \in \{\pm 1\}^h \mid \prod_{i=1}^{h} u_i = 1\} \times \{\tau^i \mid 0 \leq i \leq h-1\} \),
where $\tau$ acts on $(u_1, \ldots, u_h)$ by cyclic permutation. When $h = 3$ the group $\text{Gal}(F''/F')$ is isomorphic to $A_4$.

The representation $\rho|_{G_{F''}}$ is invariant by $\text{Gal}(F''/F')$, but Langlands’ Cyclic Descend does not apply directly, because the order of $\text{Gal}(F''/F')$ is even. Consider the quadratic character $\delta = \varepsilon \cdot \varepsilon_2 \cdot \ldots \cdot \varepsilon_{h-1}$. Then the $G_F$-representation $\delta_{\text{gal}}\rho$ is invariant by $\text{Gal}(F/F')$, so extends to a representation of $G_{F'}$. By applying Langlands’ Cyclic Descend to $\delta \otimes f$ we obtain $f'$ as desired. \qed

4. Boundary cohomology and congruence criterion.

We recall that $f \in S_k(n, \psi)$ is a Hilbert modular newform.

**Definition 4.1.** We say that a normalized eigenform $g \in S_k(n, \psi)$ is congruent to $f$ modulo $\mathcal{P}$, if their respective eigenvalues for the Hecke operators (that is their Fourier coefficients) are congruent modulo $\mathcal{P}$.

We say that a prime $\mathcal{P}$ is a congruence prime for $f$, if there exists a normalized eigenform $g \in S_k(n, \psi)$ distinct from $f$ and congruent to $f$ modulo $\mathcal{P}$.

One expects that, as in the elliptic modular case (carried out by Hida [21, 22] and Ribet [36]), the congruence primes for $f$ are controlled by the special value at 1 of the adjoint $L$-function of $f$. Such results have been obtained by Ghate [18] when $k$ is parallel.

Following [21], [18] and using a vanishing result of the boundary cohomology, we obtain a new result in this direction (see Thm.4.11 and Thm.6.7(ii)).

4.1. Vanishing of certain local components of the boundary cohomology. We introduce the following condition:

(MW) the middle weight $\frac{|p(J_F)|+|p(\mathcal{P})|}{2} = \frac{d(k_0-1)}{2}$ does not belong to $\{ |p(J)|, J \subset J_F \}$.

This condition is automatically satisfied when the motivic weight $d(k_0-1)$ is odd, or when $d = 2$ and $k$ is non-parallel.

**Lemma 4.2.** Let $\rho_0$ be a representation of $G_F$ on a finite dimensional $\kappa$-vector space $W$. Assume that for every $g \in G_F$, the characteristic polynomial of $(\otimes \text{Ind}_F^Q \mathcal{P})(g)$ annihilates $\rho_0(g).

(i) If (I), (II) and (LLT) hold, then for all $h \in \mathbb{Z}$, the weights $h$ and $d(k_0-1) - h$ occur with the same multiplicity in each $G_F$-irreducible subquotient of $\rho_0$.

(ii) If (I), (Irr$\mathcal{P}$) and (MW) hold, and $p - 1 > \max(1, \frac{5}{6}) \sum_{\tau \in J_F} (k_{\tau} - 1)$, then each $G_F$-irreducible subquotient of $\rho_0$ contains at least two different weights for the action of the tame inertia at $p$.

**Proof:** We may assume that $\rho_0$ is irreducible.

(i) By the lemmas 3.15(ii) and 3.16 we have $\text{Ind}_F^Q \mathcal{P}(I_p) \subset \phi(H(\mathbb{F}_q)) \subset \text{Ind}_F^Q \mathcal{P}(G_F)$. Let $T'$ be the torus of $H(\mathbb{F}_q)$ containing the image of the tame inertia, and $N'$ be the normalizer of $T'$ in $H(\mathbb{F}_q)$. The image by $\phi$ of $N'/T' \cong \{ \pm 1 \}^I$ is the subgroup of the Weyl group $N/T = \{ \pm 1 \}^{J_F}$ of $G$ containing the elements which are constant on the partition $J_F = \coprod_{i \in I} J^F_i$. In particular, the longest Weyl element $\epsilon_{J_F}$ belongs to the image of $N'/T'$.

Let $x \in W$ be an eigenvector for the action of $T'$. By the annihilation condition, there exists a subset $J_x \subset J_F$, such that $I_p$ acts on $x$ by the weight $|p(J_x)|$. 

Let \( g_{J_F} \in \mathcal{G}_F \) be such that \( \text{Ind}_F^Q \rho(g_{J_F}) = \epsilon_{J_F} \mod T' \). Then \( \rho_0(g_{J_F})(x) \) is of weight \( |p(J_x \Delta J_F)| = d(k_0 - 1) - |p(J_x)| \). Therefore, for each \( h \in \mathbb{Z} \), \( \rho_0(g_{J_F}) \) gives a bijection between the eigenspaces for the tame inertia of weight \( h \) and \( d(k_0 - 1) - h \).

(ii) If \((\mathbf{LL})\) hold, then the statement follows from (i) and \((\mathbf{MW})\). Otherwise, by Prop.3.8 the image \( \text{pr}(\rho(\mathcal{G}_F)) \) is dihedral. Since \( \tilde{F} \) is totally real, \( \text{pr}(\rho(\mathcal{G}_F)) \) is also dihedral (see §3.3).

Denote by \( N \) the normalizer of the standard torus \( T \) in \( G \). Put \( N' = \text{Ind}_F^Q \rho(\mathcal{G}_F) \subset N(\kappa) \) and \( T' = N' \cap T(\kappa) \). Then \( N'/T' \) is a subgroup of the Weyl group \( \{ \pm 1 \}^{J_F} = N/T \) of \( G \).

As we have seen in §3.3, the representation \( \text{Ind}_F^Q \rho \) is tamely ramified at \( p \) and the image of the inertia group \( I_p \) is contained in \( T' \).

Let \( x \in W \) be an eigenvector for the action of \( T' \). By the annihilation condition, there exists a subset \( J_x \subset J_F \), such that \( I_p \) acts on \( x \) by the weight \( |p(J_x)| \). For every element \( \epsilon_j \in N'/T' \), \( J \subset J_F \), let \( g_j \in \mathcal{G}_F \) be such that \( \text{Ind}_F^Q \rho(g_j) = \epsilon_j \mod T' \). Then \( \rho_0(g_j)(x) \) is of weight \( |p(J_x \Delta J)| \). It remains to show that the \( |p(J_x \Delta J)| \) are not all equal when \( \epsilon_j \) runs over the elements of \( N'/T' \). Note that, for all \( \tau \in J_F \), the \( \tau \)-projection \( N'/T' \to \{ \pm 1 \} \) is a surjective homomorphism (because the group \( \text{pr}(\rho(\mathcal{G}_F)) \) is also dihedral). Therefore, we have:

\[
\sum_{\epsilon_j \in N'/T'} |p(J_x \Delta J)| = |N'/T'| \frac{d(k_0 - 1)}{2}.
\]

The statement now follows from the \((\mathbf{MW})\) assumption. \( \square \)

**Remark 4.3.** The (i) in the previous lemma is a generalization, from the quadratic to the arbitrary degree case, of the key lemma in [8]. This lemma is false in general under the only assumptions \((\mathbf{I})\), \((\mathbf{II})\) and \((\mathbf{Irr}_p)\), when the degree is \( \geq 3 \). In fact, consider the following construction in the cubic case: let \( L \) be a Galois extension of \( \mathbb{Q} \) of group \( A_4 \), such that the cubic subfield \( F \) fixed by the Klein group is totally real; let \( K \) be a quadratic extension of \( F \) in \( L \), and consider a theta series \( f \) of weight \( (2,2,2) \) attached to a Hecke character of \( K \); then the tensor induced representation \( \otimes \text{Ind}_F^Q \rho \) has two irreducible four dimensional subquotients of Hodge-Tate weights \( (0,2,2,2) \) and \( (1,1,1,3) \).

Let \( T' \subset T \) be the subalgebra generated by the Hecke operators outside a finite set of places containing those dividing \( np \).

**Theorem 4.4.** Assume that \((\mathbf{I})\), \((\mathbf{Irr}_p)\) and \((\mathbf{MW})\) hold, and \( p - 1 > \max(1, \frac{5}{2}) \sum_{\tau \in J_F} (k_\tau - 1) \).

Denote by \( \mathfrak{m} \) the maximal ideal of \( T \) corresponding to \( f \) and \( p \) and put \( \mathfrak{m}' = \mathfrak{m} \cap T' \). Then

(i) the \( \mathfrak{m}' \)-torsion of the boundary cohomology \( H^*(Y, \mathcal{V}_n(\mathcal{O}))[\mathfrak{m}'] \) vanishes,

(ii) the Poincaré pairing \( H^2(Y, \mathcal{V}_n(\mathcal{O}))[\mathfrak{m}'] \times H^2(Y, \mathcal{V}_n(\mathcal{O}))[\mathfrak{m}'] \to \mathcal{O} \) is a perfect duality of free \( \mathcal{O} \)-modules of finite rank,

(iii) \( H^*(Y, \mathcal{V}_n(\mathcal{O}))[\mathfrak{m}'] = H^*(Y, \mathcal{V}_n(\mathcal{O}))[\mathfrak{m}'] = H^*(Y, \mathcal{V}_n(\mathcal{O}))[\mathfrak{m}'] \).

**Proof:** (i) Consider the minimal compactification \( Y_\mathbf{Q}^\dagger \overset{\tilde{j}}{\rightarrow} Y_\mathbf{Q}^\dagger \overset{i}{\leftarrow} \partial Y_\mathbf{Q}^\dagger \). The Hecke correspondences extend to \( Y_\mathbf{Q}^\dagger \). By the Betti-étale comparison isomorphism, we identify (in a Hecke-equivariant way) the following two long exact cohomology sequences:
Consider the $G_Q$-module $W^*_j = H^*(\partial Y^{*}_Q, i^* Rj_* \mathbb{V}_n(\kappa))$. We have to show that $W^*_j[m'] = 0$.

By the Cebotarev Density Theorem and the congruence relations at totally split primes of $F$, we can apply lemma 4.2 to $W^*_j[m']$. Therefore each $G_Q$-irreducible subquotient of $W^*_j[m']$ has at least two different weights for the action of the tame inertia at $p$. So it is enough to show that each $G_Q$-irreducible subquotient of $W^*_j$ is pure (=contains a single weight for the action of the tame inertia at $p$). Because $\partial M^*_Q$ is zero dimensional, the spectral sequence $H^*(\partial Y^{*}_Q, i^* Rj_* \mathbb{V}_n(\kappa)) \Rightarrow H^*(\partial Y^{*}_Q, i^* Rj_* \mathbb{V}_n(\kappa))$ shows that $W^*_j = H^0(\partial Y^{*}_Q, i^* Rj_* \mathbb{V}_n(\kappa))$.

As $H^0(\partial Y^{*}_Q, i^* Rj_* \mathbb{V}_n(\kappa))$ is a subquotient of $H^0(\partial Y^{*}_Q, i^* Rj_* \mathbb{V}_n(\kappa))$ it is enough to show that each $G_Q$-irreducible subquotient of this last is pure.

This will be done using a result of Pink[32]. We had to replace $Y$ by $Y^1$, because the group $G$ does not satisfy the conditions of this reference, while $G^*$ satisfies them.

Consider the decomposition $T = D_I \times D_h$, according to $\left( \begin{array}{cc} u \epsilon & 0 \\ 0 & u^{-1} \end{array} \right) = \left( \begin{array}{cc} u & 0 \\ 0 & u^{-1} \end{array} \right) \left( \begin{array}{cc} \epsilon & 0 \\ 0 & 1 \end{array} \right)$. Put $\Gamma^1 = \Gamma^1(c, n)$. By [32]Thm.5.3.1, the restriction of the étale sheaf $i^* Rj_* \mathbb{V}_n(\mathbb{F}_p)$ to a cusp $C = \gamma \infty$ of $Y^{1,+}_Q$, is the image by the functor of Pink of the $\gamma^{-1} \Gamma^1(c, n) \cap B/\gamma^{-1} \Gamma^1(c, n) \cap D_I U$-module

$$\bigoplus_{a+b=r} H^a(\gamma^{-1} \Gamma^1(c, n) \cap D_I, \mathbb{V}_n(\mathbb{F}_p))).$$

Under the assumption (II), a modulo $p$ version of a theorem of Kostant (see [33]) gives an isomorphism of $T$-module $H^0(\gamma^{-1} \Gamma^1(c, n) \cap U, \mathbb{V}_n(\mathbb{F}_p)) = \bigoplus_{[J]=b} W_{\epsilon J(n+t) - t}$. By decomposing $W_{\epsilon J(n+t) - t} = W_{\epsilon J(n+t) - t,l} \otimes W_{\epsilon J(n+t) - t,h}$ according to $T = D_I \times D_h$, we get

$$H^0(\gamma^{-1} \Gamma^1(c, n) \cap D_I, \mathbb{V}_n(\mathbb{F}_p)) = \bigoplus_{[J]=b} H^0(\gamma^{-1} \Gamma^1(c, n) \cap D_I, W_{\epsilon J(n+t) - t,l}) \otimes W_{\epsilon J(n+t) - t,h},$$

where Galois acts only on the second factors of the right hand side.

Therefore $H^0(\partial M^*_Q, i^* Rj_* \mathbb{V}_n(\mathbb{F}_p))$ is a direct sum of subspaces $H^0(C, W_{\epsilon J(n+t) - t,h}(\mathbb{F}_p))$, $[J] \leq r$, each containing a single Fontaine-Laffaille weight, namely the weight $|p(J)|$.

(ii) As the Poincaré duality is perfect over $E$, it is enough to show that the $m'$-localization of natural map $H^d(O)/H^d(O) \rightarrow H^d(E)/H^d(E)$ is injective. For this, it is sufficient to show that $H^d(O)_m' := H^d(\partial M, \mathbb{V}_n(O))_{m'}$ is torsion free, which is a consequence of the vanishing of $H^d(\partial M, \mathbb{V}_n(O))_{m'}$. By (i) and Nakayama’s lemma $H^d(\partial M, \mathbb{V}_n(O))_{m'} = 0$ and moreover we have a surjection $H^d(\partial M, \mathbb{V}_n(O))_{m'} \rightarrow H^d(\partial M, \mathbb{V}_n(O))_{m'}[\overline{\omega}]$, where $\overline{\omega}$ is an uniformizer of $O$.

(iii) The vanishing of $H^d(\partial M, \mathbb{V}_n(O))_{m'}$ gives the vanishing of $H^d(\partial M, \mathbb{V}_n(O))_{m'} = 0$. 

4.2. Definition of periods. By taking the subspace $\bigcap_{a \in \mathcal{O}} \ker(T_a - c(f, a))$ of (8) we obtain

$$\delta_J : \mathbb{C}f_J \xrightarrow{\sim} H^d(\mathbb{V}_n(\mathbb{C}))[\overline{\omega}, f].$$
Fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ compatible with $i_p$. We recall that $H_d^1(Y^{an}, V_n(\mathcal{O}))'$ denotes the image of the natural map $H_d^2(Y^{an}, V_n(\mathcal{O})) \to H_d^4(Y^{an}, V_n(\mathbb{C}))$. As $\mathcal{O}$ is principal, the $\mathcal{O}$-module $L_{f,J} := H_d^1(Y^{an}, V_n(\mathcal{O}))(\mathcal{O}, f)$ is free (of rank 1). We fix a basis $\eta(f, J)$ of $L_{f,J}$.

**Definition 4.5.** For each $J \subset J_F$ we define the period $\Omega(f, J) = \frac{\delta_J(f)}{\eta(f, J)} \in \mathbb{C}^* / \mathcal{O}^*$. We fix $J_0 \subset J_F$ and put $\Omega_f^+ = \Omega(f, J_0)$ and $\Omega_f^- = \Omega(f, J_F \setminus J_0)$.

**Remark 4.6.** The periods $\Omega_f^\pm$ differ from the ones originally introduced by Hida in [21]. The Hida’s periods put together all the external conjugates of $f$. Our slightly different definition is motivated by the congruence criterion that we want to show (Thm.4.11). As we can prove the perfectness of the twisted Poincaré pairing only for certain local components of the middle degree cohomology $H_d^1(Y^{an}, V_n(\mathcal{O}'))'$, and that in general $f$ and his external conjugates do not belong to the same local component, we have to separate them in the period’s definition.

### 4.3. Computation of a discriminant.

The aim of this paragraph is the computation of the discriminant $\text{disc}(L_f)$ of the $\mathcal{O}$-lattice $L_f := H_d^1(Y^{an}, V_n(\mathcal{O}))(\mathcal{O}, f) = \oplus_{J \subset J_F} L_{f,J}$, with respect twisted Poincaré pairing $[\ , \ ]$. We follow [18] Sect.6.

We have $\text{disc}(L_f) = \det((\langle \eta(f, J), \eta(f, J') \rangle)_{J,J' \subset J_F})$.

By [18] (41), for every $J \subset J_F$ and $x,y \in H_d^1(Y^{an}, V_n(\mathbb{C}))$ we have $[\epsilon_x \cdot x, y] = -[x, \epsilon_y \cdot y]$. The embedding $\mathcal{O} \hookrightarrow \mathbb{C}$ that we have fixed gives an embedding $\mathbb{Z}_p : F \hookrightarrow \mathbb{C}$. We have

$$\text{disc}(L_f) = \prod_{\tau_0 \in J \subset J_F} \begin{vmatrix} 0 & \langle \eta(f, J), \eta(f, f_J J) \rangle \\ \langle \eta(f, J), \eta(f, J) \rangle & 0 \end{vmatrix} = \prod_{\tau_0 \in J \subset J_F} -\left( \frac{[\delta_J(f), \delta_{J/J}(f)]}{\Omega(f, J)\Omega(f, J_{J/J})} \right)^2$$

We have $[\delta_J(f), \delta_{J/J}(f)] = 2^d \langle \epsilon_{J/J}, \delta(f) \rangle = 2^d W(f) \langle \epsilon_{J/J}, \delta(f) \rangle = 2^d W(f)(f, f_n)$, where $f^c$ denotes the complex conjugate of $f$, $i$ is the Atkin-Lehner involution and $W(f)$ is the complex constant of the functional equation of the standard $L$-function of $f$. Thus we get the following equality in $E^* / \mathcal{O}^*$:

$$\text{disc}(L_{f,J_0} \oplus L_{f,J_{J/F} \setminus J_0}) = \left( \frac{W(f)(f, f_n)}{\Omega_f^+ \Omega_f^-} \right)^2.$$  

### 4.4. Shimura’s formula for $L(\text{Ad}^0(f), 1)$.

For each prime ideal $v$ of $F$ we define $\alpha_v$ and $\beta_v$ by the equations

$$\alpha_v + \beta_v = c(f, v), \quad \alpha_v \beta_v = \begin{cases} \psi(v) N_{F/Q}(v) & , \text{if } v \nmid n, \\ 0 & , \text{if } v \mid n. \end{cases}$$

The naive adjoint $L$-function of $f$ is then given by the Euler product:

$$L^0(\text{Ad}^0(f), s) = \prod_{v \mid n} \left[ (1 - \alpha_v \beta_v^{-1} N_{F/Q}(v)^{-s})(1 - N_{F/Q}(v)^{-s})(1 - \beta_v \alpha_v^{-1} N_{F/Q}(v)^{-s}) \right]^{-1}.$$  

We denote by $f^c$ the (external) complex conjugate of $f$. We introduce a twisted version of the $L$-function associated to the tensor product $f \otimes f^c$ (see [38]):

$$D(f, f^c, s) = \prod_v \left( 1 - \alpha_v \beta_v \alpha_v \beta_v N_{F/Q}(v)^{-2s} \right) (1 - \alpha \beta N_{F/Q}(v)^{-s})^{-1}.$$
\[
(1-\alpha_v^*v N_{F/Q}(v)^{-s})^{-1} \left( 1-\beta_v^*v N_{F/Q}(v)^{-s} \right)^{-1} \left( 1-\gamma_v^*v N_{F/Q}(v)^{-s} \right)^{-1}
\]
as well as a naive version \(D^0(f, f^c, s)\) obtained by removing the factors for \(v|n\).

Using that for all \(v \not|n\), \(c_f(v) = \bar{c}_v(v) c(f, v)\), a direct computation, shows that

\[
(18) \quad \zeta_F^0(2s) D^0(f, f^c, s+k_0-1) = \zeta_F^0(s) L^0(\text{Ad}^0(f), s).
\]

The advantage to switch to \(D(f, f^c, s)\) is that we have the following formula, proved by Shimura (see [26] lemma 7.2, for a proof that the \(D(f, f^c, s)\) defined above equals the one studied by Shimura):

**Theorem 4.7.** (Shimura [38] (2.31), Prop. 4.13) Let \(f \in S_k(n, \psi)\) be a newform. Then

\[
\text{Res}_{s=1} D(f, f^c, s+k_0-1) = 2^{d-1}(4\pi)^{|\mathbf{c}|} \prod_{\tau \in J_F} \Gamma(k_\tau)^{-1} R_F[\sigma_F^2 : \omega^2](f, f),
\]

\[
\langle f, f \rangle = \mu(\Gamma \backslash \mathbf{H}_F)^{-1}(f, f)_n, \quad \text{and} \quad \mu(\Gamma \backslash \mathbf{H}_F) = \frac{2 N_{F/Q}(\psi)^{3/2} \zeta_F^0(2) N_{F/Q}(n)}{\pi^d|\sigma_F^2| \prod_{v|n} (1+N_{F/Q}(v))^{-1}}.
\]

We deduce the formula:

\[
(18) \quad \zeta_F^0(2) \text{Res}_{s=1} D(f, f^c, s+k_0-1) = \frac{2^{d-1}(4\pi)^{|\mathbf{c}|} \pi^d \text{Res}_{s=1} \zeta_F^0(s)}{2 \Delta h_F \prod_{\tau \in J_F} \Gamma(k_\tau)} (f, f)_n.
\]

We define the imprimitive adjoint \(L\)-function \(L^*(\text{Ad}^0(f), s)\) by completing the naive adjoint \(L\)-function \(L^0(\text{Ad}^0(f), s)\), defined in (16), in order to have the relation

\[
L^*(\text{Ad}^0(f), s) D^0(f, f^c, s+k_0-1) = L^0(\text{Ad}^0(f), s) D(f, f^c, s+k_0-1).
\]

An explicit computation of [26] (7.7) gives \(L^*(\text{Ad}^0(f), s) = L^0(\text{Ad}^0(f), s) \prod_{v|n} L^*(\text{Ad}^0(f), s),\)

where for \(v|n\)

\[
L^*(\text{Ad}^0(f), s) = \begin{cases} 1-\alpha_{v|n}^{-s} & \text{if } f \text{ is principal series and minimal at } v, \\ 1-\beta_{v|n}^{-s} & \text{if } f \text{ is special and minimal at } v, \\ 1 & \text{otherwise}. \end{cases}
\]

Following Deligne [6] we associate to \(L^*(\text{Ad}^0(f), s)\) the Euler factor at infinity

\[
\Gamma(\text{Ad}^0(f), s) = \prod_{\tau \in J_F} \pi^{-(s+1)/2} \Gamma((s+1)/2)(2\pi)^{s+k_\tau} \Gamma(s+k_\tau-1).
\]

Finally from (17) and (18) we obtain

\[
(19) \quad \Lambda^*(\text{Ad}^0(f), 1) := \Gamma(\text{Ad}^0(f), s) L^*(\text{Ad}^0(f), s) = \frac{2^{d-1} \pi^d}{\Delta h_F} (f, f)_n.
\]

**Remark 4.8.** Consider the adjoint \(L\)-function \(L(\text{Ad}^0(\rho), s)\) of the three dimensional \(G_F\)-representation \(\text{Ad}^0(\rho)\) (on trace zero matrices). By the compatibility between local and global Langlands correspondence \(L(\text{Ad}^0(\rho), s)\) equals the adjoint \(L\)-function \(L(\text{Ad}^0(f), s)\) associated to the automorphic representation attached to \(f\). Nevertheless \(L(\text{Ad}^0(f), s)\) may differ from \(L^*(\text{Ad}^0(f), s)\) at some places \(v\) dividing \(n\) (see [26] (7.3c)).
4.5. Construction of congruences.

**Lemma 4.9.** Let $V_1$ and $V_2$ be two finite dimensional $E$-vector spaces and let $L$ be a $O$-lattice in $V = V_1 \oplus V_2$. For $j = 1, 2$, put $L_j = L \cap V_j$ and denote $L^j$ the projection of $L$ in $V_j$ following the above direct sum decomposition. Then:

(i) $L_j \subset L^j$ are two lattices of $V_j$, and $L_j$ is a direct factor in $L$.

(ii) we have isomorphisms of finite $O$-modules:

\[
L^1 / L \overset{\sim}{\rightarrow} L/L_1 \oplus L_2 \overset{\sim}{\rightarrow} L^2 / L_2
\]

This finite $O$-module is called the congruence module, and is denoted by $C_0(L; V_1, V_2)$.

The following proposition follows from Deligne-Serre (Lemma 6.11 of [7]) and will be used to construct congruences:

**Proposition 4.10.** Keep the notations of the lemma 4.9. Let $T$ be a commutative $O$-algebra consisting of endomorphisms of $V$, preserving the lattice $L$ and the direct sum decomposition $V_1 \oplus V_2$. Denote, for $j = 1, 2$, by $T_j$ the image of $T$ in $\text{End}(V_j)$.

Assume that $C_0(L; V_1, V_2)$ is non zero: $\{P\} = \text{Ass}(C_0(L; V_1, V_2)) = \text{Supp}(C_0(L; V_1, V_2))$.

Let $m_1$ be maximal ideal $T_1$, of residue field $\kappa_1$, such that $L^1 / L_1 \otimes T_1, \kappa_1$ is non zero, and denote by $\bar{\theta_1} : T_1 \rightarrow \kappa_1$ the corresponding character.

Then there exists a discrete valuation ring $O'$ of maximal ideal $P'$ (with $P' \cap O = P$), of residue field $\kappa' \supset \kappa_1$ and whose fraction field $E'$ is a finite extension of $E$, and there exists a character $\theta_2 : T_2 \rightarrow O'$ such that for each $T \in T$, $\bar{\theta_1}(T) \equiv \theta_2(T) \pmod{P'}$.

**Proof:** Denote by $\pi_j$ the projection of $T$ onto $T_j$, $j = 1, 2$. Then $m = \pi_1^{-1}(m_1)$ is a maximal ideal of $T$ of residue field $\kappa_1$. Put $m_2 = \pi_2(m)$. As the isomorphism (20) of the lemma 4.9 is $T$-equivariant, we get

\[
(L^1 / L_1) \otimes T_1, (T_1 / m_1) \cong (L / (L_1 \oplus L_2)) \otimes_T (T / m) \cong (L^2 / L_2) \otimes_T (T_2 / m_2)
\]

By assumption $(L^1 / L_1) \otimes T_1 (T_1 / m_1)$ is non zero. Therefore $m_2$ is a maximal ideal of $T_2$ of residue field $\kappa_2$ and the corresponding character $\bar{\theta_2} : T_2 \rightarrow \kappa_1$ fits in the following commutative diagram:

\[
\begin{array}{ccc}
T & \xrightarrow{\theta_3} & \kappa_1 \\
\downarrow & & \downarrow \\
T_2 & \xrightarrow{\theta_2} & \kappa_1
\end{array}
\]

As $T_2$ is a (finite) flat $O$-algebra, there exist a prime ideal $P_2$, contained in $m_2$ and such that $P_2 \cap O = 0$. The reduction modulo $P_2$ gives a character $\theta_2$ of $T_2$ as in the statement.

**Theorem 4.11.** (Theorem A) Let $f$ and $p$ be such that $(I)$, $(\text{Irr}_{p})$ and $(\text{MW})$ hold, and $p - 1 > \text{max}(1, \frac{1}{d}) \sum_{\tau \in J_F} (k_\tau - 1)$. If $t_p(W(f)\Lambda^*(\text{Ad}^0(f), 1)) \in P$, then $P$ is a congruence prime for $f$.

**Proof:** We put $L = H^1_t((Y^an, V_n(O)))_{\text{m}}, \pm \epsilon_{j_0, \psi} \in V = H^1_t((Y^an, V_n(E)))_{\text{m}}, \pm \epsilon_{j_0, \psi}$. We define $V_1 = H^1_t((Y^an, V_n(E)))_{\pm \epsilon_{j_0, \psi}}$ (with the notations of §4.2 we have $L_1 = L \cap V_1 = L_{f, j_0} \oplus L_{f, j_0, f}$).
5. Fontaine-Laffaille weights of the Hilbert modular varieties.

In this section all the objects are over $\mathcal{O}$. The aim is to establish a modulo $p$ version of Thm.2.3 under the assumptions that $p$ does not divide $\Delta$ and $p - 1 > |n| + d$.

5.1. The BGG complex over $\mathcal{O}$.

Koszul’s complex. The Koszul’s complex of the trivial $G$-module $\mathcal{O}$ is given by

$$\ldots \rightarrow U_{\mathcal{O}}(g) \otimes \land^2 \mathcal{O} g \rightarrow U_{\mathcal{O}}(g) \otimes g \rightarrow U_{\mathcal{O}}(g) \rightarrow \mathcal{O} \rightarrow 0$$

As $g = b \oplus u^\perp$, the $\mathcal{O}[b]$-module $g/b$ is a direct factor in $g$ and we have a homomorphism of $B$-modules $U_{\mathcal{O}}(g) \otimes \land^i \mathcal{O} g \rightarrow U_{\mathcal{O}}(g) \otimes_{U_{\mathcal{O}}(b)} \land^i (g/b)$. Thus, we deduce another complex

$$U_{\mathcal{O}}(g) \otimes_{U_{\mathcal{O}}(b)} \land^i (g/b) \rightarrow \mathcal{O} \rightarrow 0,$$

denoted by $S^i_{\mathcal{O}}(g, b)$.

More generally, for a free $\mathcal{O}$-module $V$ endowed with an action of $U_{\mathcal{O}}(g)$, we consider the complex $S^i_{\mathcal{O}}(g, b) \otimes V$, endowed with the diagonal action of $U_{\mathcal{O}}(g)$.

For every $U_{\mathcal{O}}(b)$-module $W$, free over $\mathcal{O}$, we have a canonical isomorphism of $U_{\mathcal{O}}(g)$-modules

$$(U_{\mathcal{O}}(g) \otimes_{U_{\mathcal{O}}(b)} W) \otimes V \cong U_{\mathcal{O}}(g) \otimes_{U_{\mathcal{O}}(b)} (W \otimes V|_b),$$

(21)

So we get another complex

$$U_{\mathcal{O}}(g) \otimes_{U_{\mathcal{O}}(b)} (\land^i (g/b) \otimes V|_b) \rightarrow V \rightarrow 0,$$

denoted $S^i_{\mathcal{O}}(g, b, V)$. In the case where $V = V_n$ we denote it $S^i_{\mathcal{O}}(g, b, n)$.

Verma modules. For each weight $\mu \in \mathbb{Z}[J_F]$, we define a $U_{\mathcal{O}}(g)$-module $V_{\mathcal{O}}(\mu) := U_{\mathcal{O}}(g) \otimes_{U_{\mathcal{O}}(b)} W_\mu(\mathcal{O})$, called the Verma module of weight $\mu$.

Lemma 5.1. Let $W$ be a $B$-module, free of finite rank over $\mathcal{O}$, whose weights are smaller than $(p - 1)t$. Then, there exists a filtration of $B$-modules $0 = W_0 \subset W_1 \subset \ldots \subset W_r = W$ such that for each $1 \leq i \leq r$, $W_i/W_i+1 \cong W_\mu_i(\mathcal{O})$, for some $\mu_i \in \mathbb{Z}[J_F]$. Moreover the $W_\mu_i(\mathcal{O})$, $1 \leq i \leq r$, are the irreducible factors of the $T$-module $W$.

In particular, if $U$ acts trivially on $W$, then $W \cong \oplus_{i=1}^r W_\mu_i(\mathcal{O})$.

Proof: Let $\mu_1$ be a maximal weight of $W$ (for the partial order given by the positive roots of $G$) and let $v \in W$ be a $\mathcal{O}$-primitive vector of weight $\mu_1$. Let $W'$ be the $U_{\mathcal{O}}(b)$-submodule generated by $v$. Then $W' \cong W_{\mu_1}(\mathcal{O})$ and $W' \otimes \kappa$ is irreducible, because $\mu_1$ is
smaller than \((p-1)t\) (and \(W'\) is free of rank 1). As \(W\) is free over \(O\) we have an exact sequence of \(B\)-modules

\[
0 \to \text{Tor}^O_0(W/W', \kappa) \to W' \otimes \kappa \to W \otimes \kappa.
\]

As \(W' \otimes \kappa\) is irreducible and \(v\) is primitive, the last arrow is injective. Therefore

\[
\text{Tor}^O_1(W/W', \kappa) = 0,
\]

that is \(W/W'\) is free over \(O\). The lemma follows then by induction. \(\square\)

**Lemma 5.2.** The module \(S^\bullet_O(g, b, n)\) has a finite filtration by \(U_O(g)\)-submodules whose graded pieces are of the form \(V_O(\mu), \mu \in \Omega^n(n)\), where \(\Omega^n(n)\) denoted the set of weights of the \(t\)-module \(\wedge^i_O(g/b) \otimes V_n(O)|_b\).

**Proof:** Since \(p-1 > |n| + d\) the previous lemma applies to \(\wedge^i_O(g/b) \otimes V_n(O)|_b\). This gives a filtration \(0 = W_0 \subset W_1 \subset ... \subset W_r = \wedge^i_O(g/b) \otimes V_n(O)|_b\) whose graded pieces are \(W_n(O), \mu \in \Omega^n(n)\). As \(U_O(g)\) is \(U_O(b)\)-free, the functor \(U_O(g) \otimes U_O(b) \bullet\) is exact. \(\square\)

**Central characters.** Let \(U_O(g) \to U_O(t)\) be the projection coming from the Poincaré-Birkoff-Witt decomposition \(U_O(g) = U_O(t) \oplus (u^- U_O(g) + U_O(g) u)\). We take its restriction to the invariants for the adjoint action \(\theta : U_O(g)^G \to U_O(t)\). Note that \(U_{\overline{T}_p}(t)\) identifies itself with the algebra of regular functions on \(\text{Hom}_O(t, \overline{F}_p) \cong \overline{F}_p[J_F]\) (a Laurent polynomial algebra). The Weyl group \(\{\pm 1\}^J_F\) of \(G\) acts on it by \((\epsilon_J \cdot P)(\mu) = P(\epsilon_J(\mu + t) - t)\). The following result is a analogous to a theorem of Harish-Chandra:

**Theorem 5.3.** (Jantzen [28]) \(\theta_{\overline{T}_p}\) induces an algebra isomorphism \(U_{\overline{T}_p}(g)^G \to U_{\overline{T}_p}(t)^{\{\pm 1\}^J_F}\).

For every \(\mu \in \mathbb{Z}[J_F]\) and every \(O\)-algebra \(R\), we denote by \(d_{\mu, R} : t_R \to R\) the corresponding character and by \(\chi_{\mu, R} = d_{\mu, R} \circ \theta_R\) the composed map \(U_R(g)^G \to U_R(t) \to R\). This definition is compatible with the \(O\)-algebra homomorphisms.

If \(V\) is a \(U_R(g)\)-module generated by a vector \(v\) of weight \(\mu\), that is annihilated by \(u\), then \(U_R(g)^G\) acts over \(V\) by \(\chi_{\mu, R}\). Put \(\chi_{\mu, p} = \chi_{\mu, O}\) and \(\overline{\chi}_{\mu, p} = \chi_{\mu, \overline{T}_p}\).

**Corollary 5.4.** Let \(\mu \in \mathbb{Z}[J_F]\). If \(\overline{\chi}_{n, p} = \overline{\chi}_{\mu, p}\), then there exists \(J \subset J_F\) such that \(\mu - (\epsilon_J(n+t) - t) \in p\mathbb{Z}[J_F]\). In particular, if \(\mu\) is smaller than \((p-1)t\), then we have \(\mu = \epsilon_J(n+t) - t\).

**Proposition 5.5.** Let \(\mu \in \Omega^n(n)\) (see lemma 5.2). Then \(\overline{\chi}_{n, p} = \overline{\chi}_{\mu, p}\), if and only if there exists a subset \(J \subset J_F\) containing \(i\) elements and such that \(\mu = \epsilon_J(n+t) - t\).

**Proof:** By the corollary, it remains to show that for \(J \subset J_F\), we have \(\epsilon_J(n+t) - t \in \Omega^n(n)\), if and only if \(|J| = i\). By the lemma 5.2, we have to show that \(W_{\epsilon_J(n+t)-t}(E)\) occurs in \(\wedge^i_E(g/b) \otimes V_n(E)|_t\) (with multiplicity one) if and only if \(|J| = i\). The weight of \(\wedge^i_E(g/b) \otimes V_n(E)|_t\) are of the form \(\epsilon_{J'}(n+t) - t + \nu\), where \(J' \subset J_F\) is a subset containing \(i\) elements and \(\nu\) is a weight of \(V_n(E)\). Therefore \(\epsilon_J(n+t) - t = \epsilon_{J'}(n+t) - t + \nu\) and so \(n = \epsilon_J(\nu) + \epsilon_J(\epsilon_{J'}(t)) - t\). Because \(n\) is a maximal weight of \(V_n(E)\), we deduce that \(J = J'\).

**Decomposition with respect to the central characters.** By the lemma 5.2 \(S^\bullet_O(g, b, n)\) admits a finite filtration by \(U_O(g)\)-submodules, whose graded are of the form \(V_O(\mu), \mu \in \Omega^n(n)\). Therefore \(S^\bullet_O(g, b, n)\) is annihilated by a power of the ideal \(I := \prod_{\mu \in \Omega^n(n)} \ker(\chi_{\mu, p})\) of the commutative ring \(U_O(g)^G\). As we will see at the end of this section as a consequence
of Prop.5.5, $S_{\mathcal{O}}^\bullet(g,b,n)$ is annihilated by $I$ itself. We have the following commutative algebra lemma:

**Lemma 5.6.** Let $P_1, ..., P_r$ be ideals of the commutative ring $R$, such that $P_1, ..., P_r = 0$ and for all $i \neq j$, $P_i + P_j = R$. Then each $R$-module $W$ admits a direct sum decomposition $W = \oplus_{1 \leq i \leq r} W^{P_i}$, with $W^{P_i} = \{ m \in W | I_i = 0 \}$.

Consider the maximal ideals $pR + \ker(\chi_{\mu,p}) = \ker(\chi_{\mu,p})$ of $R$, where $\mu \in \Omega^\bullet(n)$. Let $\chi_1 = \chi_{n,p}, \chi_2, ..., \chi_r$ be the set of distinct characters among $\chi_{\mu,p}, \mu \in \Omega^\bullet(n)$. Put $P_i = \prod_{\chi_{\mu,p} \neq \chi_i} \ker(\chi_{\mu,p})$. By the above lemma we get a decomposition

$$S_{\mathcal{O}}^\bullet(g,b,n) = \oplus_{i=1}^r S_{\mathcal{O}}^\bullet(g,b,n)^{P_i}$$

which is a direct sum, because the differentials are $U_{\mathcal{O}}(g)$-equivariant. Moreover, $V_{\mathcal{O}}(\mu)_{\chi_{\mu,p}} = V_{\mathcal{O}}(\mu)$, if $\chi_{\mu,p} = \chi_{n,p}$, and $V_{\mathcal{O}}(\mu)_{\chi_{\mu,p}} = 0$, otherwise. From here and from Prop.5.5 we get:

**Theorem 5.7.** The complex $S_{\mathcal{O}}^\bullet(g,b,n)_{\chi_{\mu,p}}$ is a direct factor in $S_{\mathcal{O}}^\bullet(g,b,n)$ and we have $S_{\mathcal{O}}^\bullet(g,b,n)_{\chi_{n,p}} = V_n(\mathcal{O})$. For each $i \geq 1$, $S_{\mathcal{O}}^i(g,b,n)_{\chi_{n,p}}$ has a filtration whose graded are given by the $V_{\mathcal{O}}(\varepsilon J(n+t) - t)$ where $J \subset J_F$, $|J| = i$ (with multiplicity one).

### 5.2. The BGG complex for distributions algebras

Let $U_{\mathcal{O}}(G)$ be the distribution $\mathcal{O}$-algebra over $G$. For each $G$-module $V$, free over $\mathcal{O}$, we define the complex

$$0 \leftarrow V \leftarrow U_{\mathcal{O}}(G) \otimes_{U_{\mathcal{O}}(B)} (\Lambda^\bullet(\mathfrak{g}/b) \otimes V|_b),$$

and denote it by $S_{\mathcal{O}}^\bullet(G,B,V)$. In the case where $V = V_n(\mathcal{O})$ we denote this complex by $S_{\mathcal{O}}^\bullet(G,B,n)$.

**Remark 5.8.** The complex $S_{\mathcal{O}}^\bullet(G,B,V)$ is not exact. It will become exact after applying the Grothendieck linearization functor to the associated complex of vector bundles over the Hilbert modular variety.

For all $\mu \in \mathbb{Z}[J_F]$, we define the Verma module $\mathcal{V}(\mu) = U_{\mathcal{O}}(G) \otimes_{U_{\mathcal{O}}(B)} W_\mu(\mathcal{O})$ (see §5.1). We recall that, since $p - 1 > |n| + d$, $\Omega^\bullet(n)$ is the set of $\mu \in \mathbb{Z}[J_F]$ such that $W_\mu(\mathcal{O})$ is a irreducible subquotient of $\otimes_i^\bullet(\mathfrak{g}/b) \otimes V_n(\mathcal{O})|_b$. The lemma 5.2 translates as:

**Lemma 5.9.** The modules $S_{\mathcal{O}}^\bullet(G,B,n)$ has a finite filtration by $U_{\mathcal{O}}(G)$-submodules whose successive quotients are given by $\mathcal{V}_\mathcal{O}(\mu)$, with $\mu \in \Omega^\bullet(n)$.

As $U_{\mathcal{O}}(g) \subset U_{\mathcal{O}}(G) \subset U_{E}(g)$, the center $U_{\mathcal{O}}(g)^G$ of $U_{\mathcal{O}}(g)$ is contained in the center of $U_{\mathcal{O}}(G)$. Consider the central characters $\chi_{\mu,p} = \chi_{\mu,\mathcal{O}}$ and $\chi_{\mu,p} = \chi_{\mu,\mathcal{O}}$ (see §5.1).

If $W$ is a $U_{\mathcal{O}}(G)$-module generated by a vector $v$ of weight $\mu$, that is annihilated by $u$, then $U_{\mathcal{O}}(g)^G$ acts on $W$ by the character $\chi_{\mu,p}$. Put $I = \prod_{\mu \in \Omega^\bullet(n)} \ker(\chi_{\mu,p})$. By the last lemma the finite $\mathcal{O}$-module $S_{\mathcal{O}}^\bullet(G,B,n)$ is an $R := U_{\mathcal{O}}(g)^G/I$-module. Let $\chi_1 = \chi_{n,p}, \chi_2, ..., \chi_r$ be the distinct algebra homomorphisms from $R$ in $\mathcal{O}_p$. For $1 \leq j \leq r$, we put

$$S_{\mathcal{O}}^\bullet(G,B,n)_{\chi_j} = \left\{ x \in S_{\mathcal{O}}^\bullet(G,B,n) \left| \prod_{\mu \in \Omega^\bullet(n) \chi_{\mu,p} = \chi_j} \ker(\chi_{\mu,p}) \right| x = 0 \right\}.$$
The same way as in Thm.5.7 we obtain a decomposition
\begin{equation}
S^*_O(G, B, n) = \oplus_{j=1}^r S^*_O(G, B, n)_{X_j}.
\end{equation}

The main theorem of this section states then:

**Theorem 5.10.** \(S_i^*(G, B, n)_{X_{n,p}} \cong \bigoplus_{J \subseteq J_F, |J| = i} \mathcal{V}_\mathcal{O}(\epsilon_j(n + t) - t).\)

**Proof:** We start with the case \(n = 0\).

As \(u\) is abelian, \(U\) acts trivially on \(\wedge^i (g/b)\) and therefore, by the lemma 5.2, we have
\begin{equation}
\wedge^i (g/b) \cong \bigoplus_{J \subseteq J_F, |J| = i} W_{\epsilon_j(t) - t}(\mathcal{O}).
\end{equation}

As \(U(\mathcal{O})\) is free over \(U(\mathcal{O})(B)\) we obtain:
\begin{equation}
S_i^*(G, B, 0) = S_i^*(G, B, 0)_{X_{0,p}} \cong \bigoplus_{J \subseteq J_F, |J| = i} \mathcal{V}_\mathcal{O}(\epsilon_j(t) - t).
\end{equation}

If \(n \geq 0\), we deduce from the \(n = 0\) case a decomposition
\begin{equation}
S_i^*(G, B, n) \cong \bigoplus_{J \subseteq J_F, |J| = i} U(\mathcal{O})(G) \otimes U(\mathcal{O})(B) \left( W_{\epsilon_j(t) - t}(\mathcal{O}) \otimes V_n(\mathcal{O}) \right).
\end{equation}

Using (23) the theorem is a consequence of the following lemma, whose proof is a direct application of the proof of Prop.5.5.

**Lemma 5.11.** \((U(\mathcal{O})(G) \otimes U(\mathcal{O})(B) \left( W_{\epsilon_j(t) - t}(\mathcal{O}) \otimes V_n(\mathcal{O}) \right))_{X_{n,p}} \cong \mathcal{V}_\mathcal{O}(\epsilon_j(n + t) - t)\).

5.3. **BGG complex for crystals.** Our reference is the section 4 of [31]. For every integer \(r \geq 0\) we put \(S_r = \text{Spec}(\mathbb{Z}/p^{r+1})\). For a \(\mathbb{Z}[\frac{1}{2}]\)-scheme \(X\), we put \(X_r = X \times S_r\).

We have an equivalence of categories between the category of crystals over \((\mathbb{X}_0/S_r)_{crys}\) and the category of \(\mathcal{O}_{\mathbb{X}_r}\)-modules \(\mathcal{M}\) which are locally free and endowed with integrable, quasi-unipotent connection with logarithmic poles \(\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes \mathcal{O}_{\mathbb{X}_r} \Omega^1_{\mathbb{X}_r/S_r} (d\log(\infty_X))\).

We have a functor \(L\), called the linearization functor, from the category of sheaves of \(\mathcal{O}_{\mathbb{X}_r}\)-modules to the category of crystals on \((\mathbb{X}_0/S_r)_{crys}\).

By the log-crystalline Poincaré lemma, we have a resolution:
\begin{equation}
0 \rightarrow \mathcal{M} \rightarrow L(\mathcal{M} \otimes \mathcal{O}_{\mathbb{X}_r} \Omega^*_{\mathbb{X}_r/S_r} (d\log(\infty))).
\end{equation}

Let \(W_1\) and \(W_2\) be two \(B\)-modules with weights smaller than \((p-1)t\). Put \(\overline{W}_i = F_B(W_i), i = 1, 2\) (see §2.3). By [31]§5.2.4 we have a homomorphism
\begin{equation}
\text{Hom}_{U(\mathcal{O})(G)}(U(\mathcal{O})(G) \otimes U(\mathcal{O})(B) W_1, U(\mathcal{O})(G) \otimes U(\mathcal{O})(B) W_2) \rightarrow \text{Diff. Op.}(\overline{W}_{2,r}, \overline{W}_{1,r}),
\end{equation}
which becomes an isomorphism after tensoring with \(E\) (see (12)).

We apply now the above construction to the toroidal compactification over of the Hilbert modular variety \(\overline{M}\) and the vector bundle \(V_n\). For every \(r \geq 0\) we have an injective homomorphism of complexes of vector bundles over \(\overline{M}_r\)
\begin{equation}
K^*_r := \bigoplus_{J \subseteq J_F} W_{\epsilon_j(n + t) - t} \rightarrow \overline{V}_n \otimes \mathcal{O}_{\overline{M}_r} \Omega^*_{\overline{M}_r/S_r} (d\log(\infty)).
\end{equation}
Proposition 5.12. The map (28) is a strict injective homomorphism of filtered complexes.

By the last proposition $L(K^*_n)$ is a direct factor in $L(Y_n \otimes_{M_r} \Omega^\cdot \Omega_{M_r/S_r} \,(d\log \infty))$, which is exact by the Poincaré’s crystalline lemma. Therefore $L(K^*_n)$ is also exact. As the functor $L$ is exact, we deduce filtered isomorphisms $H^j_{\text{log-dR}}(M_r/S_r, Y_n) \cong H^j(M_r/S_r, K^*_n)$.

Recall that $p$ does not divide $\Delta$ and $p - 1 > |n| + d$. Under this assumptions we have

Theorem 5.13. The spectral sequence given by the Hodge filtration

$$E_{i,j}^1 = \bigoplus_{J \subset J_F, |p(J)| = i} H^{i+j-|J|}(M_r, W_{\epsilon_J(n+t)-t,n_0}) \Rightarrow H^{i+j}_{\text{log-dR}}(M_r, Y_n)$$

degenerates at $E_1$:

$$(29) \quad \text{gr}^i H^r_{\text{log-dR}}(M_r, Y_n) = \bigoplus_{J \subset J_F, |J| \leq r, |p(J)| = i} H^{r-|J|}(M_r, W_{\epsilon_J(n+t)-t})$$

Proof: The proof is formally the same as the one of Thm.2.3(ii), once we have Prop.5.12. The degeneration of the spectral sequence follows from a result of Illusie [27] Prop.4.13. applied to the semi-stable morphism $\pi : \overline{M}^T \to \overline{M}^T$ of smooth $\mathbb{Z}_p$-schemes. □

Remark 5.14. (i) It follows from the same arguments as in Cor.2.7(i), that the above decomposition is Hecke equivariant, except for the $T_p$ operators, when $p$ divides $p$. When $p$ is totally split in $F$, we could use Wedhorn’s results [42] to write $T_p$ as a sum of correspondences and try adapt to this case the method of [16]. Unfortunately, this approach is not available when $p$ is not totally split in $F$.

In the proof of Thm.6.7, we will use different method to prove the $T_p$-equivariance of the above decomposition after a localization outside $p$.

(ii) The commutativity of the Hecke operators outside $p$ follows from the degeneration at $E_1$ as in the proof of Cor.2.7(i). The last graded piece $H^0(Y, W_{\epsilon_J(n+t)-t,n_0})$ of the filtration is independent of the toroidal compactification by the Koecher principle (3).

6. Integral cohomology over certain local components of the Hecke algebra.

6.1. The key lemma. Let $q = p^r$ and denote by $\sigma_1, \ldots, \sigma_r$ the elements of Gal$(\mathbb{F}_q / \mathbb{F}_p)$.

Theorem 6.1. (Brauer-Nesbitt, Steinberg [39]) The group SL$_2(\mathbb{F}_q)$ has exactly $q$ irreducible representations on finite dimensional $\mathbb{F}_q$-vector spaces, namely the $\otimes^r_{j=1}(\text{Sym}^{a_j})^{\sigma_j}$, for $0 \leq a_j \leq p - 1$.

Corollary 6.2. For every finite set $I$, the group $\prod_{i \in I} \text{SL}_2(\mathbb{F}_q)$ has exactly $q^{|I|}$ irreducible representations on finite dimensional $\mathbb{F}_q$-vector spaces, namely the $\otimes_{i \in I}(\otimes_{j=1}^r(\text{Sym}^{a_{i,j}})^{\sigma_i})$, for $0 \leq a_{i,j} \leq p - 1$.

In [30] Mazur states the following:

Lemma 6.3. Let $\Phi$ be a group and $\rho_0$ be a representation of $\Phi$ on a finite dimensional $\mathbb{F}_q$-vector space $W$. Let $\rho : \Phi \to \text{GL}_2(\mathbb{F}_q)$ be an absolutely irreducible representation such that for all $g \in \Phi$, the characteristic polynomial of $\rho(g)$ annihilates $\rho_0(g)$. Then, $\rho_0^{\Phi} = \rho \oplus \ldots \oplus \rho$ and in particular $\rho \subset \rho_0$. 
The corresponding statement for another group than $GL_2$ is false in general. Here is an example for $GL_3$: take $\rho = \text{Sym}^2 : GL_2(\mathbb{F}_q) \to GL_3(\mathbb{F}_q)$ and $\rho_0 = \text{det} : GL_2(\mathbb{F}_q) \to GL_1(\mathbb{F}_q)$. Nevertheless, as we have a generalization for the special group:

$$H(\mathbb{F}_q) = \left( \prod_{i \in I} GL_2(\mathbb{F}_q) \right)^D := \left\{ (M_i)_{i \in I} \in \prod_{i \in I} GL_2(\mathbb{F}_q) \mid \exists \delta \in D, \ \forall i \in I, \ \text{det}(M_i) = \delta \right\}$$

and the particular representation

$$\rho_1 = \bigotimes_{i \in I, \tau \in J_F} \text{St}_{i,\tau}^{\sigma_i,\tau} : H(\mathbb{F}_q) \to GL_2(\mathbb{F}_q), \ (M_i)_{i \in I} \mapsto \bigotimes_{i \in I, \tau \in J_F} M_i^{\sigma_i,\tau},$$

where $(J_F)_{i \in I}$ is a partition of $J_F$ and for all $i \in I$, $(\sigma_i,\tau)_{\tau \in J_F}$ are two by two distinct elements of $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ (St = $\text{Sym}^1$ is the standard two-dimensional representation of $GL_2$).

**Lemma 6.4.** Let $\rho_0$ be a representation of $H(\mathbb{F}_q)$ on a finite dimensional $\mathbb{F}_q$-vector space $W$, such that for all $g \in H(\mathbb{F}_q)$ the characteristic polynomial of $\rho_1(g)$ annihilates $\rho_0(g)$. Then $\rho_0^{\text{ss}} = \rho_1 \oplus \cdots \oplus \rho_1$ (each irreducible subquotient of $\rho_0$ is isomorphic to $\rho_1$).

**Proof:** We can assume that $\rho_0$ is absolutely irreducible. Consider the exact sequence $1 \to H_1(\mathbb{F}_q) = \prod_{i \in I} \text{SL}_2(\mathbb{F}_q) \to H(\mathbb{F}_q) \to D \to 1$. By Cor.6.2, we know that each irreducible subquotient of $\rho_0|_{H_1(\mathbb{F}_q)}$ is of the form $\otimes_{i \in I} \left( \otimes_{j=1}^{r_i} (\text{Sym}^{a_i,j}_i)^{\sigma_j} \right)$, with $0 \leq a_i,j \leq p - 1$.

The subspace corresponding to the highest weight of the representation $\rho_0|_{H_1(\mathbb{F}_q)}$ is preserved by the standard torus of $H(\mathbb{F}_q)$, and therefore contains an eigenvector $x$ for the action of this torus. Because $\rho_0$ is irreducible, it is generated by $x$, and therefore $\rho_0$ isomorphic to a twist of $\otimes_{i \in I} \left( \otimes_{j=1}^{r_i} (\text{Sym}^{a_i,j}_i)^{\sigma_j} \right)$ by some power of the character $\nu$ (in particular, $\rho_0|_{H_1(\mathbb{F}_q)}$ is also irreducible).

As the characteristic polynomial of $\rho_1$ annihilates $\rho_0$, the set of the weights of $\rho_0$ is a subset of the set of the weights of $\rho_1$, and therefore $\rho_0 = \rho_1$. \hfill \square

In §3.5 we proved under the assumption $(LI_{\text{Ind}_{\mathbb{F}^*}})$, that $\text{Ind}_{\mathbb{F}^*}^{\mathbb{F}}(\mathcal{G}_{\mathbb{F}})$ contains the image of the map $\phi = (\phi_i)_{i \in I} : H(\mathbb{F}_q) \to GL_2(\mathbb{F}_q)^{J_F}$.

Denote by $\tilde{F}^\nu$ the fixed field of $\mathcal{F}^{-1}(\phi(H(\mathbb{F}_q)))$.

**Lemma 6.5.** (Key Lemma) Let $\rho_0$ be a representation of $\mathcal{G}_{\mathbb{F}^*}$ on a finite dimensional $\kappa$-vector space $W$. Assume $(LI_{\text{Ind}_{\mathbb{F}^*}})$ and assume that, for every $g \in \mathcal{G}_{\mathbb{F}^*}$, the characteristic polynomial of $(\otimes \text{Ind}_{\mathbb{F}^*}^{\mathbb{F}}(\mathcal{P})) (g)$ annihilates $\rho_0(g)$. Then each $\mathcal{G}_{\mathbb{F}^*}$-irreducible subquotient of $\rho_0$ is isomorphic to $\otimes \text{Ind}_{\mathbb{F}^*}^{\mathbb{F}}(\mathcal{P})$.

**Proof:** It is enough to treat the case where $\rho_0$ is irreducible. The idea is show that the action of $\mathcal{G}_{\mathbb{F}^*}$ on $W$ is through the algebraic group $H(\mathbb{F}_q)$, and then use the lemma 6.4.

Put $\mathcal{P} = (\text{Ind}_{\mathbb{F}^*}^{\mathbb{F}}(\mathcal{P}))|_{\mathcal{G}_{\mathbb{F}^*}}$. Because of the annihilating assumption, the group $\rho_0(\text{ker}(\mathcal{P}))$ is unipotent $p$-group and therefore $W^{\text{ker}(\mathcal{P})}$ is non-zero. Moreover the subspace $W^{\text{ker}(\mathcal{P})}$ is preserved by $\mathcal{G}_{\mathbb{F}^*}$. Because $W$ is irreducible, we get $W^{\text{ker}(\mathcal{P})} = W$ and therefore the action of $\mathcal{G}_{\mathbb{F}^*}$ on $W$ is through $H(\mathbb{F}_q)$. Thus we get a homomorphism $\rho_0^{\text{ss}}$ fitting in the following...
Let \( \mathcal{T}' \subset \mathcal{T} \) be the subalgebra generated by the Hecke operators outside a finite set of places containing those dividing \( np \). Put \( \mathfrak{m}' = \mathfrak{m} \cap \mathcal{T}' \).

**Theorem 6.6.** Assume \( f \) and \( p \) satisfy (I), (II) and (LI_{\text{Ind}}). Then

(i) \( H^r(Y, \mathcal{V}_n(\kappa))_{\mathfrak{m}'} = H^d(Y, \mathcal{V}_n(\kappa))_{\mathfrak{m}'} \),  
(ii) \( H^r(Y, \mathcal{V}_n(\mathcal{O}))_{\mathfrak{m}'} = H^d(Y, \mathcal{V}_n(\mathcal{O}))_{\mathfrak{m}'} \) is a free \( \mathcal{O} \)-module of finite rank and the \( \mathcal{O} \)-module \( H^r(Y, \mathcal{V}_n(E/\mathcal{O}))_{\mathfrak{m}'} = H^d(Y, \mathcal{V}_n(E/\mathcal{O}))_{\mathfrak{m}'} \) is divisible of finite corank.

(iii) \( H^d(Y, \mathcal{V}_n(\mathcal{O}))_{\mathfrak{m}'} \times H^d(Y, \mathcal{V}_n(E/\mathcal{O}))_{\mathfrak{m}'} \to \mathcal{O} \) is a perfect Pontryagin pairing.

**Proof:** (i) By Faltings’ Comparison Theorem [14] and Thm.5.13(i) the integer \( |p(J)| \) is not a Fontaine-Laffaille weight of \( H^r(\kappa) \), when \( r < d \). Wedhorn [42] has established the congruence relations for all totally split primes of \( F \). By the Cebotarev Density Theorem the assumptions of the key lemma 6.5 are fulfilled. We deduce that \( H^r(\kappa)_{\mathfrak{m}'} = 0 \), and therefore \( H^r(\kappa)_{\mathfrak{m}'} = 0 \) by Nakayama’s lemma. The case \( n > d \) follows by Poincaré duality.

(ii)(iii) By the long exact cohomology sequence

\[
\ldots \to H^{r-1}(\kappa) \to H^r(\mathcal{O}) \to H^r(\kappa) \to \ldots,
\]

and by the vanishing of \( H^r(\kappa)_{\mathfrak{m}'} \), for \( r \neq d \), we deduce that (for \( r \neq d \)) the multiplication by an uniformizer \( \varpi \) is a surjective endomorphism of \( H^r(\mathcal{O})_{\mathfrak{m}'} \), so this last vanishes.

The same way, by the long exact sequence

\[
\ldots \to H^r(\varpi^{-1} \mathcal{O} / \mathcal{O}) \to H^r(E/\mathcal{O}) \to H^r(E/\mathcal{O}) \to H^{r+1}(\varpi^{-1} \mathcal{O} / \mathcal{O}) \to \ldots,
\]

we deduce a surjection \( H^r(\kappa)_{\mathfrak{m}'} \to H^r(E/\mathcal{O})_{\mathfrak{m}'}[\varpi] \), when \( r \neq d \). Since \( H^r(E/\mathcal{O})_{\mathfrak{m}'} \) is a torsion \( \mathcal{O} \)-module, it vanishes (for \( r \neq d \)).

The localization at \( \mathfrak{m}' \) of the long exact sequence of \( \mathcal{O} \)-modules:

\[
\ldots \to H^{r-1}(E/\mathcal{O}) \to H^r(\mathcal{O}) \to H^r(E) \to H^r(E/\mathcal{O}) \to \ldots,
\]

is concentrated at the three terms of degree \( r = d \). From this we deduce the freeness. \( \square \)
6.3. On the Gorensteiness of Hecke the algebra.

**Theorem 6.7.** (Theorem B) Let $f$ and $p$ be such that (I), (II) and (LI\text{Ind}_\mathcal{P}) hold. Then

(i) $H^\bullet(Y, \mathcal{V}_n(\kappa))[m] = H^d(Y, \mathcal{V}_n(\kappa))[m]$ is a $\kappa$-vector space of dimension $2^d$.

(ii) $H^\bullet(Y, \mathcal{V}_n(\mathcal{O}))_m = H^d(Y, \mathcal{V}_n(\mathcal{O}))_m$ is free of rank $2^d$ over $T_m$.

(iii) $T_m$ is Gorenstein.

**Proof:** In this proof we put $W = H^d(Y_{\overline{\mathbb{Q}}}, \mathcal{V}_n(\kappa))_m$. By using an auxiliary level structure as in [8], we can assume that the condition (\textbf{NT}) of §1.4 is fulfilled.

(i) As in the proof of Thm.6.6(i), by lemma 6.5 we have an isomorphism of $\mathcal{G}_\mathcal{F}_\kappa$-modules

$$W[m]^{s.s.} = (\otimes \text{Ind}_\mathcal{F}^\mathcal{G}_\kappa \mathcal{P})^\oplus.$$ 

It is crucial to observe that $\mathcal{I}_p \subset \mathcal{G}_\mathcal{F}_\kappa$. By Thm.2.6 we have $r \geq 1$. In order to show that $r = 1$ we consider the restriction of these representations to $\mathcal{I}_p$. The multiplicity of the maximal Fontaine-Laffaille weight $|p(J_\mathcal{F})|$ in the right hand side is $r$, by Thm.2.6, Cor.2.7(ii) and Fontaine-Laffaille’s theory.

On the other hand, the multiplicity of $|p(J_\mathcal{F})|$ in the left hand side is equal, by Thm.5.13, to the dimension of $H^0(\mathcal{Y} \otimes \kappa, \mathcal{V}_{\kappa}^{\text{m}\text{-loc}})|m|$. In fact, by the remark 5.14 it is sufficient to check the $T_\mathcal{P}$-equivariance of the $\mathfrak{m}$-localization of the projective limit over $r$ of (29). By Thm.6.6(ii) it is question of checking the $T_\mathcal{P}$-equivariance of an isomorphism of free $\mathcal{O}$-modules. Therefore, it is enough to be checked after extending the scalars to $\mathbb{C}$. Then the Strong Multiplicity One Theorem applies (recall that $p$ is prime to the level $n$). We owe this idea to Diamond ([8] proof of Prop.1).

We will now see that $\dim_k H^0(\mathcal{Y} \otimes \kappa, \mathcal{V}_{\kappa}^{\text{m}\text{-loc}})|m| = 1$. We have $\mathcal{V}_{\kappa}^{\text{m}\text{-loc}} = \mathbb{A}^k \otimes \mathbb{P}^n(t/2)$. So we are led to show that two normalized Hilbert modular forms of weight $k$, level $n$ and coefficients in $\kappa = T_m/m$ having the same eigenvalues for all the Hecke operators are equal. One should be careful to observe that the Hecke operators permute the connected components $M_1(\mathfrak{c}, n)$ of the Shimura variety $Y = Y_1(n)$ (here the ideal $\mathfrak{c}$ runs over a set of representatives of $\mathcal{C}^+_\mathcal{F}$). We use then the Hecke relations between Fourier coefficients and eigenvalues for the Hecke operators and the $q$-expansion principle (see 1.7) at the $\infty$-cusp of each connected component $M_1(\mathfrak{c}, n)$.

Even if we do not know the degeneration at $E_1$ of the Hodge to de Rham spectral sequence, we obtain by the same arguments that $r \leq 1$ (instead of $r = 1$), because we have always $H^0(\mathcal{Y} \otimes \kappa, \mathcal{V}_{\mathfrak{c}, \mathcal{P}}^{\text{m}\text{-loc}})|m| \supset \text{gr}[p(J_\mathcal{F})] W[m]$. But $H^\bullet(Y, \mathcal{V}_n(\mathcal{O}))_m$ is non zero as $H^\bullet(Y, \mathcal{V}_n(\mathcal{O}))_m \otimes \mathbb{Q}$ is free of rank $2^d$ over $T_m \otimes \mathbb{Q}$, and therefore $r = 1$.

(ii)(iii) Mazur’s argument in the elliptic modular case remains valid. By the theorem A, the twisted Poincaré pairing (6) on $H^d(Y, \mathcal{V}_n(\mathcal{O}))_m = H^d_c(Y, \mathcal{V}_n(\mathcal{O}))_m$ is a perfect duality of $T_m$-modules, so it would be enough to show (ii).

Again using the perfectness of the twisted Poincaré pairing $W \times W \to \kappa$ we obtain $W \cong \text{Hom}_{T_m}(W, \kappa)$, and so $W \otimes_{T_m} \kappa = W/ \mathfrak{m} W \cong \text{Hom}(W[m], \kappa)$, and therefore

$$\dim_\kappa(W \otimes_{T_m} \kappa) = \dim_\kappa(W[m]),$$

which equals $2^d$, by (i). Then (ii) follows from the following
Lemma 6.8. Let \( T \) a torsion free local \( \mathcal{O} \)-algebra \((T \hookrightarrow T \otimes \mathcal{O})\) of maximal ideal \( m \) and residue field \( \kappa = T / m \).

Let \( M \) be a finitely generated \( T \)-module such that \( M \otimes \mathcal{O} E \) is free of rank \( r \) over \( T \otimes \mathcal{O} E \). If \( M \otimes T \kappa \) is a \( \kappa \)-vector space of dimension \( \leq r \), then \( M \) is free of rank \( r \) over \( T \).

**Proof:** Since \( M \otimes T k \) is of dimension \( \leq r \), the Nakayama’s lemma gives a surjective homomorphism of \( T \)-modules \( T^r \to M \). Denote by \( I \) its kernel. We have an exact sequence of \( \mathcal{O} \)-modules

\[
0 \to I \to T^r \to M \to 0.
\]

By tensoring it by \( \otimes \mathcal{O} E \) (or equivalently by \( \otimes \mathcal{T} (T \otimes \mathcal{O} E) \)) we obtain another exact sequence

\[
0 \to I \otimes \mathcal{O} E \to (T \otimes \mathcal{O} E)^r \to M \otimes \mathcal{O} E \to 0.
\]

By comparing the dimensions over \( E \) we get \( I \otimes \mathcal{O} E = 0 \). Since \( I \) is torsion free, \( I = 0 \). □

6.4. **An application to \( p \)-adic ordinary families.** For \( r \geq 1 \), consider the following open compact subgroups of \( G(\mathbb{A}_f) \)

\[
K_0(p^r) = \left\{ u \in K_1(n) | u \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p^r} \right\},
\]

\[
K_1(p^r) = \left\{ u \in K_1(n) | u \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p^r} \right\}.
\]

Let \( Y_0(p^r) \) (resp. \( Y_1(p^r) \)) be the Hilbert modular variety of level \( K_0(p^r) \) (resp. \( K_1(p^r) \)).

The cohomology group \( H^*Y_1(p^r), \mathbb{V}_n(E/\mathcal{O})^* \) has a natural action of \( K_0(p^r)/K_1(p^r) \cong (\mathfrak{o}/p^r)^* \times (\mathfrak{o}/p^r)^* \) (we denote by * the Pontryagin dual). Therefore the group \( T(Z_p)/\mathfrak{o}^\times \) acts on the inductive limit \( H^*Y_1(p^r), \mathbb{V}_n(E/\mathcal{O})^* := \lim H^*Y_1(p^r), \mathbb{V}_n(E/\mathcal{O})^* \).

By Hida’s stabilization lemma, the ordinary part of \( H^*Y_1(p^r), \mathbb{V}_n(E/\mathcal{O})^* \) (that is the part where the Hecke operators \( T_{p, \mathfrak{p}} \) of Def.1.13 are invertible for all \( \mathfrak{p} \) dividing \( p \)) is independent on \( n \). We denote it by \( H^*_\text{ord} := H^*_\text{ord}(Y_1(p^\infty), E/\mathcal{O})^* \).

By the above discussion \( H^*_\text{ord} \) has a structure of a \( \Lambda := \mathcal{O}[[T(Z_p)/\mathfrak{o}^\times]] \)-module. It is of finite type, by a theorem of Hida.

We also define the \( p \)-adic ordinary Hecke \( \Lambda \)-algebra \( T^\infty_{k, \text{ord}} := \lim T_{k, \text{ord}}(Y_1(p^r)) \). As \( T^\infty_{k, \text{ord}} \) is independent of \( k \), we denote it by \( T^\infty_\text{ord} \). Then \( H^*_\text{ord} \) is a \( T^\infty_\text{ord} \)-module.

An arithmetic character of \( T(Z_p)/\mathfrak{o}^\times \) is by definition a character whose restriction to an open subgroup is given by an algebraic character. It is immediate that such a character is a product of an algebraic character and a finite order character. An algebraic character of \( T(Z_p) \cong D(Z_p) \times D(Z_p) \) trivial on \( \mathfrak{o}^\times \) is necessarily of the form \( (u, \epsilon) \mapsto u^n \epsilon^{-m} \), where \( m, n \in \mathbb{Z}[J_F] \) and \( m + 2n \in \mathbb{Z} t \). Hence, the general form of an arithmetic character \( \psi \) of \( T(Z_p)/\mathfrak{o}^\times \) is \( (u, \epsilon) \mapsto u^n \epsilon^{-m} \psi_1(u) \psi_2(\epsilon) \), where \( \psi_1, \psi_2 \) are finite order characters. Every such \( \psi \) induces an \( \mathcal{O} \)-algebra homomorphism \( \Lambda \to \mathcal{O} \), whose kernel is denoted by \( P_{\psi} \).

Let \( m \) be a maximal ordinary ideal of \( \mathcal{T} = \mathbb{T}_k(n) \) and \( m_\infty \) be a maximal ideal of \( T^\infty_\text{ord} \) above \( m \). We denote by \( T^\infty_{m_\infty} \) (resp. \( H^*_\text{m_\infty} \)) the localization of \( T^\infty_\text{ord} \) (resp. of \( H^*_\text{ord} \)) at \( m_\infty \).
Proposition 6.9. Let \( m \) be such that (I), (II) and (LI \( \text{Ind} \rho \)) hold. Then

(i) \( H^d_{m,\infty} \) is free of finite rank over \( \Lambda \) and we have exact control:

\[
H^d_{m,\infty} / P_{\psi} H^d_{m,\infty} \simeq H^*(Y_{11}(p^r), V_{\psi}(E/\mathcal{O}))^*_{m,}\]

(ii) \( H^d_{m,\infty} \) is free of rank \( 2^d \) over \( T^\infty_{m,\infty} \), and

(iii) Hida’s control theorem for the Hecke algebra holds, that is \( T^\infty_{m,\infty} \) is a free \( \Lambda \)-algebra of finite rank and for every \( \psi \) we have \( T^\infty_{m,\infty} / P_{\psi} T^\infty_{m,\infty} \simeq T_{\psi}(Y_{11}(p^r))_{m,}\).

Proof: (i) The proof is very similar to the one of [31]Thm.9. It uses that a \( \Lambda \)-module is free, if it is free of constant rank over \( \mathcal{O} \) for infinitely many specializations. In our case, it is enough to specialize at the weight of the form \( k + (p - 1)k' \), and use the exact control criterion and Thm.6.6. We omit the details. Note that (i) follows from (ii) and (iii).

(ii) Consider \( \Lambda \rightarrow T^\infty_{m,\infty} \rightarrow \text{End}_{\mathcal{O}}(H^d_{m,\infty}) \). The specialization at \( \psi = \psi_k \) gives

\[
\mathcal{O} \rightarrow T^\infty_{m,\infty} / P_k T^\infty_{m,\infty} \rightarrow \text{End}_{\mathcal{O}}(H^d_{m,\infty} / P_k H^d_{m,\infty}).
\]

By Thm.6.6 we have \( H^d(Y_0(p), V_n(E/\mathcal{O}))^*_{m} \simeq H^d(Y, V_n(\mathcal{O}))^*_{m} \) and an exact control: \( H^d_{m,\infty} / P_k H^d_{m,\infty} \simeq H^d(Y, V_n(\mathcal{O}))^*_{m} \).

From here and from Thm.B we obtain that \( H^d_{m,\infty} \otimes T^\infty_{m,\infty} (T^\infty_{m,\infty} / P_k T^\infty_{m,\infty}) \simeq H^d_{m,\infty} \otimes \Lambda / P_k \)

is free of rank \( 2^d \) over \( T_m \). Hence \( H^d_{m,\infty} \otimes T^\infty_{m,\infty} \kappa \) is free of rank \( 2^d \) over \( T_m \otimes T^\infty_{m,\infty} / P_k T^\infty_{m,\infty} \kappa = \kappa \).

Then lemma 6.8 applies to the \( T^\infty_{m,\infty} \)-module \( H^d_{m,\infty} \) which is finitely generated over the local algebra \( \Lambda \).

(iii) As \( H^d_{m,\infty} \) is a free \( \Lambda \)-module, it admits a direct sum decomposition with respect to the Weyl group action on the Betti cohomology:

\[
H^d_{m,\infty} = \bigoplus_{J \subset J_F} H^d_{m,\infty}[\widetilde{e}_J].
\]

Every \( H^d_{m,\infty}[\widetilde{e}_J] \) is free of rank 1 over \( T^\infty_{m,\infty} \) and free over \( \Lambda \) Therefore \( T^\infty_{m,\infty} \) is free over \( \Lambda \) and exact control holds. \( \square \)

Corollary 6.10. Let \( f \in S_{k+(p-1)k'}(Y_0(p^r)) \) be a newform and \( p \) be a prime not dividing \( N_{F/\mathbb{Q}}(0) \), such that \( p - 1 > \sum (k_\tau - 1) \) and (LI \( \text{Ind} \rho \)) holds. Then theorems A and B hold.
LIST OF SYMBOLS

\begin{align*}
A & \quad \text{HBAV, Def.1.6} & K_1(n), K_1^\dagger(n) & \quad \Sect{1.1} & Y, Y^1 & \quad \text{HMV \Sect{1.4}} \\
A' & \quad \text{dual HBAV} & K_n^\ast & \quad \Sect{2.2} & \alpha & \quad \text{\(\mu_n\)-level structure, Def.1.8} \\
A & \quad \text{universal HBAV \Sect{1.4}} & K_n^\bullet & \quad \Sect{2.3} & \beta & \quad \Sect{2.4} \\
a & \quad \text{ideal of \(\mathfrak{o}\)} & L & \quad \Sect{1.3} & \gamma & \quad \text{element of \(G(\mathbb{R})\)} \\
B & \quad \text{standard Borel of \(G\)} & m = (k_0 t - k) / 2 & \quad \text{Def.1.1} & \Gamma_1(c, n) & \quad \Sect{1.1} \\
b & \quad \text{Lie algebra of \(B\)} & M, M^1 & \quad \text{connected HMV \Sect{1.4}} & \Gamma_1(c, n) & \quad \Sect{1.1} \\
c, c_+ & \quad \Sect{1.3} & M' & \quad \Sect{1.4} & \delta, \delta_J & \quad \Sect{1.12} \\
c(f, a) & \quad \Sect{1.10} & \mathcal{M}, \mathcal{M}^\dagger & \quad \Sect{1.6} & \Delta & \quad \text{\(N_{F/Q}(n \mathfrak{d})\)} \\
\Cl_F & \quad \text{class group \Sect{1.2}} & M^*, M_1^* & \quad \Sect{1.8} & \delta_p, \epsilon_p & \quad \text{tame characters of \(I_p\)} \\
\Cl_F^+ & \quad \text{narrow class group \Sect{1.1}} & \mathfrak{m} & \quad \text{maximal ideal of \(\mathbb{T}\) \Sect0.3} & \varepsilon & \quad \text{quadratic character} \\
d & \quad \text{degree of \(F\)} & \mathfrak{m}' & \quad \text{maximal ideal of \(\mathbb{T}'\) \Sect{0.2}} & \epsilon & \quad \text{unit of \(F\)} \\
D & \quad \text{Res}_{\mathbb{Q}}^F \mathbb{G}_m & n & \quad \text{weight of \(G\) \Sect{1.11}} & \epsilon_J & \quad \Sect{1.11} \\
D_p & \quad \text{decomposition group \Sect{2.6}} & N & \quad \text{normalizer of \(T\) \Sect{1.2}} & \epsilon_J & \quad \Sect{1.12} \\
\mathfrak{d} & \quad \text{different of \(F\)} & \mathfrak{n} & \quad \text{level ideal \(\eta\)} & \eta & \quad \text{idèle} \\
\mathcal{D} & \quad \Sect{3.4} & \mathcal{O} & \quad \text{integer ring of \(E\)} & \iota & \quad \text{Def.1.6} \\
E & \quad \text{large \(p\)-adic field} & \mathfrak{o} & \quad \text{integer ring of \(F\)} & \iota_p & \quad \text{embedding of \(\mathbb{Q}\) in \(\mathbb{Q}_p\)} \\
f & \quad \text{Hilbert modular newform} & \mathfrak{o}' & \quad \Sect{1.4} & \epsilon & \quad \text{unit of \(F\)} \\
F & \quad \text{totally real number field} & \mathfrak{o}^+_\mathbb{T}, \mathfrak{o}^\dagger_{n_1} & \quad \Sect{0.4} & \kappa & \quad \text{residue field of \(\mathcal{O}\)} \\
F & \quad \text{Galois closure of \(F\)} & p & \quad \text{prime number} & \lambda & \quad \text{\(c\)-polarization, Def.1.7} \\
F' & \quad \text{before lemma 6.5} & p(J) & \quad \Sect{2.2} & \lambda & \quad \text{\(c\)-polarization class, Def.1.7} \\
\mathcal{F} & \quad \Sect{3.4} & \mathcal{P} & \quad \text{maximal ideal of \(\mathcal{O}\)} & \lambda & \quad \text{\(c\)-polarization class, Def.1.7} \\
\mathcal{F}_F, \mathcal{F}_B, \mathcal{F}_G & \quad \text{functors \Sect{2.3}} & \mathfrak{p} & \quad \text{prime of \(F\) dividing \(p\)} & \mu & \quad \Sect{1.5} \\
g & \quad \text{Hilbert modular form} & q & \quad \text{power of \(p\)} & \mu & \quad \Sect{1.5} \\
G & \quad \text{Res}_{\mathbb{Q}}^F \mathbb{G}_2 & R & \quad \text{ring} & \xi & \quad \Sect{1.4} \\
g & \quad \text{Lie algebra of \(G\)} & s & \quad \Sect{2.1} & \varpi_v & \quad \text{uniformizer of \(F_v\)} \\
G^* & \quad \text{\(G \times_D \mathbb{G}_m\)} & S & \quad \text{base scheme} & \pi & \quad \Sect{1.6} \\
g_t & \quad \text{Def.1.1} & S_t(n, \psi) & \quad \text{Def.1.3} & \rho = \rho_{f, p} & \quad \text{\(p\)-adic repr. \Sect{0.1}} \\
g_{r_t} & \quad \text{internal conjugate \Sect{2.5}} & S_n, T_n & \quad \text{Hecke operators \Sect{1.10}} & \rho_1 & \quad \Sect{6.1} \\
g_{t_1} & \quad \text{Galois group of \(L\)} & t = \sum_{\tau \in \mathcal{I}_p} \tau & \quad \text{Def.1.1} & \sigma & \quad \text{cone \Sect{1.6}} \\
\mathfrak{G} & \quad \Sect{1.6} & T & \quad \text{standard torus of \(G\)} & \sigma & \quad \text{fan \Sect{1.6}} \\
H & \quad \text{above lemma 3.15} & t & \quad \text{Lie algebra of \(T\)} & \Sigma & \quad \text{fan \Sect{1.6}} \\
\mathfrak{H}_F, \mathfrak{H} & \quad \Sect{1.1} & \mathfrak{T} & \quad \text{Hecke algebra \Sect{0.3}} & \sigma_{\mathfrak{t}, \tau} & \quad \Sect{3.5} \\
h^+ & \quad \Sect{1.1} & \mathfrak{T}' & \quad \text{reduced Hecke algebra} & \tau & \quad \text{infinite place of \(F\)} \\
\mathfrak{H}_F & \quad \Sect{1.4} & U & \quad \text{standard unipotent of \(G\)} & \phi & \quad \Sect{3.5} \\
I_p & \quad \text{inertia group at \(p\) \Sect{2.6}} & U(b), U(g) & \quad \Sect{2.2} & \varphi & \quad \text{Hecke character} \\
J & \quad \text{subset of \(J_F\)} & u & \quad \text{Lie algebra of \(U\)} & \chi_n & \quad \Sect{2.2} \\
J_F & \quad \text{set of infinite places of \(F\)} & v & \quad \text{finite place of \(F\)} & \psi & \quad \text{Hecke character, Def.1.3} \\
J_F, \mathfrak{J} & \quad \Sect{3.5} & V, V_n & \quad G\text{-modules \Sect{1.12}} & \Omega_f^\dagger & \quad \text{periods, Def.4.5} \\
J_F, \mathfrak{J}_p & \quad \Sect{2.6} & \mathbb{V}_n & \quad \text{local system \Sect{1.12}} & \omega & \quad \text{mod \(p\) cycl. char.} \\
k = n + 2t & \quad \text{Def.1.1} & \mathbb{V}, \mathbb{W} & \quad \Sect{2.1} & \omega^k & \quad \Sect{1.4} \\
k_0, n_0 & \quad \text{Def.1.1} & W, W_\mu & \quad \text{\(B\)-modules \Sect{2.1}} \\
K & \quad \text{CM field} & \mathbb{W}_f & \quad \Sect{2.1} & (, )_n & \quad \Sect{1.4} \\
K_{\infty}, K_{\infty}^+ & \quad \Sect1.1
\end{align*}
Appendix

The notations in this section are independent from the rest of the text.

Let \( K \) be a local field of characteristic zero with a perfect residue field \( \kappa \) of characteristic \( p \). Let \( W \) be the ring of Witt vectors of \( \kappa \) and \( K_0 \) be the fraction field of \( W \). We denote by \( \sigma \) the Frobenius of \( \kappa \), \( W \) and \( K_0 \). Let \( E \) be a finite extension of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}}_p \).

A.1. \( p \)-adic representations. A \( p \)-adic representation of \( \mathcal{G}_K \) is a finite dimensional \( \mathbb{Q}_p \)-vector space, endowed with continuous action of \( \mathcal{G}_K \). The \( p \)-adic representations form an abelian category, denoted \( \text{Rep}(\mathcal{G}_K) \). We denote by \( \text{Rep}_E(\mathcal{G}_K) \) the subcategory of \( \text{Rep}(\mathcal{G}_K) \) consisting of \( E \)-linear representations.

There are several interesting subcategories of \( \text{Rep}(\mathcal{G}_K) \), as the one of Hodge-Tate representations \( \text{Rep}_{HT}(\mathcal{G}_K) \), the one of de Rham representations \( \text{Rep}_{dR}(\mathcal{G}_K) \), the one of semistables representations \( \text{Rep}_{st}(\mathcal{G}_K) \), and the one of crystalline representations \( \text{Rep}_{crys}(\mathcal{G}_K) \)

\[
\text{Rep}(\mathcal{G}_K) \supset \text{Rep}_{HT}(\mathcal{G}_K) \supset \text{Rep}_{dR}(\mathcal{G}_K) \supset \text{Rep}_{st}(\mathcal{G}_K) \supset \text{Rep}_{crys}(\mathcal{G}_K).
\]

Let \( \text{Rep}_{\overline{\mathbb{Q}}_p}(\mathcal{G}_K) \) be the category of continuous representations of \( \mathcal{G}_K \) on finite dimensional \( \overline{\mathbb{Q}}_p \)-vector spaces. All these representations are obtained by scalar extension from \( \text{Rep}_E(\mathcal{G}_K) \) (for some \( E \)). By an abuse of language will still call them \( p \)-adic representations.

A.2. Hodge-Tate weights. Denote by \( C \) the \( p \)-adic completion of the algebraic closure \( \overline{K} \) of \( K \). The \( \mathcal{G}_K \)-action on \( \overline{K} \) extends by continuity to an action on \( C \). Put \( \mathcal{B}_{HT} = \oplus_i C(i) \), where the \( \mathcal{G}_K \)-action on \( C(i) \) is twisted by the \( i \)-th power of the cyclotomic character.

Let \( V \in \text{Rep}(\mathcal{G}_K) \). Then, by definition, \( V \in \text{Rep}_{HT}(\mathcal{G}_K) \), if and only, if \( \dim_K (V \otimes_{\mathbb{Q}_p} B_{HT})^\mathcal{G}_K = \dim_{\mathbb{Q}_p} V \). For \( V \in \text{Rep}_{HT}(\mathcal{G}_K) \), we say that \( i \) is a Hodge-Tate weight of \( V \), if \( V_i := (V \otimes_{\mathbb{Q}_p} C(i))^\mathcal{G}_K \neq 0 \) and we call \( h^i \) its multiplicity. We have a equality of \( \mathcal{G}_K \)-modules \( V \otimes_{\mathbb{Q}_p} C = \bigoplus_i V_i \otimes_K C(-i) \).

If \( V \in \text{Rep}_E(\mathcal{G}_K) \), then for all \( i \in \mathbb{Z} \), \( V_i = (V \otimes_{\mathbb{Q}_p} C(i))^\mathcal{G}_K \) is a \( E \otimes_{\mathbb{Q}_p} K \)-module in a natural way. It is not free in general. By decomposing the \( \mathbb{Q}_p \)-algebra \( E \otimes_{\mathbb{Q}_p} K \) as a product of fields \( \prod_j L_j \) (endowed with injections \( \sigma : E \to L_j, \tau : K \to L_j \)), we obtain:

\[
(V \otimes_{\mathbb{Q}_p} C(i))^\mathcal{G}_K \otimes_{E \otimes_K} L_j = (V \otimes_{E} C(i))^\mathcal{G}_{L_j}
\]

There is another way to index the Hodge-Tate weights that is more appropriate to the modular case. Consider the functor \( \text{Rep}_E(\mathcal{G}_K) \to \text{Rep}_{\overline{\mathbb{Q}}_p}(\mathcal{G}_K) \) sending \( V \) to \( V_{\overline{\mathbb{Q}}_p} := V \otimes_{E} \overline{\mathbb{Q}}_p \).

**Definition A.1.** For all \( \tau : K \to \overline{\mathbb{Q}}_p \) we put \( h_{\tau,i} = \dim_{\overline{\mathbb{Q}}_p} (V_{\overline{\mathbb{Q}}_p} \otimes_{\tau,K} C(i))^\mathcal{G}_K \). The integer \( h_{\tau,i} \) is called the multiplicity of \( i \) as Hodge-Tate weight of \( V \). For all \( \tau \), we have \( \sum_{i \in \mathbb{Z}} h_{\tau,i} = \dim_E V \).

**Example A.2.** Assume that \( E = \mathbb{Q}_p \). Then \( V_i = (V \otimes_{\mathbb{Q}_p} C(i))^\mathcal{G}_K \) is a \( K \)-vector space and the \( i \)-th Hodge-Tate number is given by \( h^i = \dim_K (V_i), i \in \mathbb{Z} \).

If we change the action of \( \mathcal{G}_K \) on \( C \) by an automorphism \( g \to g^{-1} \), with \( \tau \in \mathcal{G}_{\overline{\mathbb{Q}}_p} \), then we send \( V_i \) onto \( V_i^\tau \) by \( v \otimes a \mapsto v \otimes \tau(a) \) that does not change \( h^i \) (since the cyclotomic character is invariant by \( g \to g^{-1} \)).
A.3. Crystalline representations and filtered modules. The category $\text{Rep}_{\text{crys}}(G_K)$ of crystalline representations is the $p$-adic analogue of the unramified $l$-adic representations.

**Definition A.3.** (i) A filtered $\phi$-module over $K$ is a $K_0$-vector space $D$ of finite dimension, endowed with a $\sigma$-linear bijective map $\phi : D \to D$ and a filtration $\text{Fil}^iD$ of $D_K = D \otimes_K K$ which is decreasing $(\text{Fil}^iD \supset \text{Fil}^{i+1}D)$, exhaustive $(\cup \text{Fil}^iD = D_K)$ and separated $(\cap \text{Fil}^iD = 0)$. We denote by $\text{MF}_K$ the additive category of filtered $\phi$-module over $K$.

(ii) A filtered $\phi$-module $D$ over $K$ is called weakly admissible, if it contains a $W$-lattice $M$, such that $\sum p^{-i}\phi(\text{Fil}^iD \cap M) = M$. Such a lattice is called strongly divisible (or adapted to $D$). The weakly admissible filtered $\phi$-module over $K$ form a full subcategory of $\text{MF}_K$, denoted by $\text{MF}_fK$.

**Remark A.4.** To an object $D \in \text{MF}_K$ one can associate Newton and Hodge polygons and the notion of be weakly admissible can be expressed in terms of inequalities between these two polygons.

Fontaine’s theory gives an equivalence of categories $D_{\text{crys}}$ between $\text{Rep}_{\text{crys}}(G_K)$ and a certain full subcategory of admissible objects $\text{MF}_aK$ of $\text{MF}_K$. The Hodge-Tate weights of $V \in \text{Rep}_{\text{crys}}(G_K)$ are given by the jumps of the filtration on $D_{\text{crys}}(V)$.

It is known that admissible implies weakly admissible, and recently Colmez and Fontaine proved the converse, in the more general semi-stable case. When $K$ is an unramified extension of $\mathbb{Q}_p$ and the length of the filtration is $\leq p-1$, this has been established earlier by Fontaine and Laffaille [17].

A.4. Crystalline representations modulo $p$. In the sequel, we assume $K$ to be unramified ($K = K_0$). Fontaine and Laffaille [17] have introduced

**Definition A.5.** (i) A filtered $F$-module over $W$ is defined by the following data

- a $W$-module $M$,  
- a filtration $\text{Fil}^iM$ by $W$-submodules which is decreasing exhaustive and separated,  
- $\sigma$-linear maps $\varphi_i : \text{Fil}^iM \to M$ satisfying $\varphi_i|_{\text{Fil}^{i+1}M} = p\varphi_{i+1}$.

We denote by $\text{MF}_W$ the $\mathbb{Z}_p$-linear additive category of filtered $F$-modules over $W$.

(ii) We define two full abelian subcategories $\text{MF}_{W,tf} \subset \text{MF}_{W,tf} \subset \text{MF}_W$, by $M \in \text{MF}_W$

\[
M \in \text{MF}_{W,tf} \iff \begin{cases} 
M \text{ is of finite type over } W, \\
\text{Fil}^iM \text{ are direct factors in } M, \\
\sum \varphi_i(\text{Fil}^iM) = M.
\end{cases}
\]

\[
M \in \text{MF}_{W,tf} \iff \begin{cases} 
M \text{ is of finite length over } W, \\
\sum \varphi_i(\text{Fil}^iM) = M.
\end{cases}
\]

**Remark A.6.** Let $M$ be a strongly divisible lattice for $D \in \text{MF}_fK$, and put $\text{Fil}^iM = \text{Fil}^iD \cap M$ and $\varphi_i = \phi/p^i$. Then we have $M \in \text{MF}_{W,tf}$. If moreover $M$ is a free $W$-module of finite type, then we have $M/p^rM \in \text{MF}_{W,tf}$, for all $r \in \mathbb{N}$. 
A.5. **The tame inertia.** We choose to uniformize the Local Class Field Theory isomorphism by sending a uniformizer to the geometric Frobenius.

The tame inertia $I^t_K$ is the quotient of the inertia group $I_K$ by its maximal pro-$p$ subgroup (called the wild inertia). There is a canonical isomorphism $I^t_K \cong \lim_{\leftarrow} \mathbb{F}_p^\times$ (see [37]). A tame character of level $h \in \mathbb{N}$ is a character of $I^t_K$ factoring through $\mathbb{F}_p^\times$.

By the Local Class Field Theory, the inertia group of the maximal abelian extension of $K$ is isomorphic to $W^\times$. Thus we get a homomorphism $I_K \to W^\times \to \kappa^\times$ equal to the tame character $I_K \to I^t_K \to \kappa^\times$ of level $[\kappa : \mathbb{F}_p]$ (see [37]).

A.6. **Fontaine-Laffaille’s theory and a theorem of Wintenberger.** In [17] Fontaine and Laffaille have introduced a contravariant functor

$$V_{FL} : MF_{W,lf} \to \text{Rep}_{\mathbb{Z}_p}(\mathcal{G}_K),$$

such that

- the restriction to $MF_{W,lf}^{<p^{-1}}$ is exact and fully faithful,
- if $M \in MF_{W,lf}^{<p^{-1}}$, then $\text{length}_W M = \text{length}_{\mathbb{Z}_p} V_{FL}(M)$,
- if $M \in MF_{W,lf}^{<p^{-1}}$ is free, then $V_{FL}(M)$ is free and $\text{rank}_W M = \text{rank}_{\mathbb{Z}_p} V_{FL}(M)$.

Let $X$ be the abelian group of periodic map $\xi : \mathbb{Z} \to \mathbb{Z}$. By a result of Wintenberger [44] we can decompose a filtered $F$-module $M$ of finite type over $W$ as a sum of isotypic components indexed by $X$, $M = \bigoplus_{\xi \in X} M_\xi$.

A.7. **Hodge-Tate weights and Fontaine-Laffaille weights.** The aim of this paragraph is to explain how the theory of Fontaine and Laffaille relates the Hodge-Tate weights of a crystalline representation to the weights of the tame inertia acting on the semi-simplification of its reduction modulo $p$. This formulation is due to Wintenberger [44].

Let $V$ be a $p$-adic crystalline representation of Hodge-Tate weights between 0 and $p-1$ ($V \in \text{Rep}_{\text{crys}}^{<p^{-1}}(\mathcal{G}_K)$). We can associate to it $D = D_{\text{crys}}(V) \in MF_{K}$. The multiplicity of $i \in \mathbb{Z}$ as a Hodge-Tate weight of $V$ is equal to $h^i = \dim_K(\text{Fil}^i D) - \dim_K(\text{Fil}^{i+1} D)$.

Let $M \in MF_{W,lf}^{<p^{-1}}$ be a $W$-lattice adapted to $D$. By a theorem of Wintenberger we have two natural decompositions of $M$:

$$M = \bigoplus_{i \in \mathbb{Z}} M_i, \quad M = \bigoplus_{\xi \in X} M_\xi.$$

Let's define $D_i = M_i \otimes_W K_0$ and $D_\xi = M_\xi \otimes_W K_0$. Then we have $h^i = \dim D_i$ and

$$D_i = \bigoplus_{\xi \in X, \xi(0) = i} D_\xi.$$

For each $r \in \mathbb{N}$, we have $M/p^r M \in MF_{W,lf}^{<p^{-1}}$. Put $L_r = V_{FL}(M/p^r M)$ and $L = \lim_{\leftarrow} L_r$.

Then $L$ is a lattice of $V$ and by construction we have $L/pL = L_1$.

Moreover $M/pM = \bigoplus_{\xi \in X} M_\xi/pM_\xi$. But $M_\xi/pM_\xi$ is a sum of copies of a simple object of $MF_{W,lf}^{<p^{-1}}$ and $V_{FL}(M_\xi/pM_\xi)$ is equal to the sum of the same number of copies of the tame character $\theta(\xi)$. We deduce that the tame characters occurring in $L/pL$ correspond exactly to the $\xi$ occurring in $M$. By the theorem of Brauer-Nesbitt the semi-simplification of $L/pL$ does not depend of the particular choice of a lattice.
Theorem A.7. (Fontaine-Laffaille) Assume $V \in \text{Rep}_{\text{crys}}^{<p-1}(G_K)$ has Hodge-Tate weights $i_1, \ldots, i_r$ (with multiplicities). Let $L$ be a stable lattice of $V$ and consider the tame inertia action on $(L/pL)$. Take a decomposition $(L/pL)^{ss} = \bigoplus L_j$, such that the tame inertia acts on $L_j$ by a certain tame character of level $h_j = \dim_{\mathbb{F}_p} L_j$ and weights $i_{1j}, \ldots, i_{rj}$ ($r = \sum h_j$). Then the multisets $\{i_1, \ldots, i_r\}$ and $\{i_{1k} | 1 \leq k \leq h_j\}$ are equal.

The result remains valid for $V \in \text{Rep}_{\text{crys}, E}^{<p-1}(G_K)$ (see Def.A.1).

References
[1] D. Blasius and J. Rogawski, Motives for Hilbert Modular Forms, Invent. Math., 114 (1993), pp. 55–87.
[2] C. Breuil, Une remarque sur les repr´esentations locales $p$-adiques et les congruences entre formes modulaires de Hilbert, Bull. Soc. Math. France, 127 (1999), pp. 459–472.
[3] J.-L. Brylinski and J.-P. Labesse, Cohomologie d’intersection et fonctions $L$ de certaines vari´et´es de Shimura, Ann. Sci. ´Ec. Norm. Sup., 17 (1984), pp. 361–412.
[4] P. Deligne, Formes modulaires et repr´esentations $l$-adiques, in S´eminaire Bourbaki, expos´e 355, vol. 179 of LNM, Springer-Verlag, 1971.
[5] ———, La conjecture de Weil. I, Publ. Math. IHES, 43 (1974), pp. 273–307.
[6] ———, Valeurs de fonctions $L$ et p´eriodes d’int´egrales, in Proceedings of Symposia of Pure Mathematics, vol. 33, 1979, pp. 313–346.
[7] P. Deligne and J.-P. Serre, Formes modulaires de poids 1, Ann. Sci. ´Ec. Norm. Sup., 7 (1974), pp. 507–530.
[8] F. Diamond, On the Hecke action on the cohomology of Hilbert-Blumenthal surfaces, Contemporary Math., 210 (1998), pp. 71–83.
[9] F. Diamond, M. Flach, and L. Guo, Adjoint motives of modular forms and the Tamagawa number conjecture. preprint.
[10] M. Dimitrov, Compactifications arithm´etiques des vari´et´es de Hilbert et formes modulaires de Hilbert pour $\Gamma_1(c,n)$, in Geometric Aspects of Dwork Theory, A. Adolphson, F. Balda ssarri, P. Berthelot, N. Katz, and F. Loeser, eds., Walter de Gruyter, 2004, pp. 525-551.
[11] M. Dimitrov and J. Tilouine, Vari´et´es et formes modulaires de Hilbert arithm´etiques pour $\Gamma_1(c,n)$, in Geometric Aspects of Dwork Theory, A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, and F. Loeser, eds., Walter de Gruyter, 2004, pp. 553-610.
[12] M. Dimitrov, On Iwara’s lemma for Hilbert modular varieties. preprint.
[13] G. Faltings, On the cohomology of locally symmetric hermitian spaces, in S´eminaire d’algèbre, vol. 1029 of Lecture Notes, Springer, 1983, pp. 349–366.
[14] ———, Crystalline cohomology and $p$-adic Galois representations, in Algebraic analysis, geometry, and number theory, J. H. U. Press, ed., 1989, pp. 25–80.
[15] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, Springer-Verlag, 1990.
[16] G. Faltings and B. Jordan, Crystalline cohomology and $GL(2, \mathbb{Q})$, Israel J. Math., 90 (1995), pp. 1–66.
[17] J.-M. Fontaine and G. Laffaille, Construction de repr´esentations $p$-adiques, Ann. Sci. ENS, 15 (1982), pp. 547–608.
[18] E. Ghate, Adjoint $L$-values and primes of congruence for Hilbert modular forms, Compositio Math., 132 (2002), pp. 243–281.
[19] G. Harder, Eisenstein cohomology of arithmetic groups. The case $GL_2$, Invent. Math., 89 (1987), pp. 37–118.
[20] H. Hida, $p$-adic automorphic forms on Shimura varieties, Springer, 2004.
[21] ———, Congruences of cusp forms and special values of their zeta functions, Invent. Math., 63 (1981), pp. 225–261.
[22] ———, On congruence divisors of cusp forms as factors of the special values their zeta functions, Invent. Math., 64 (1981), pp. 221–262.
[23] Nearly ordinary Hecke algebras and Galois representations of several variables, in Algebraic analysis, geometry and number theory, Proceedings of the JAMI Inaugural Conference, 1988, pp. 115–134.

[24] On p-adic Hecke algebras for $GL_2$ over totally real fields, Ann. Math., 128 (1988), pp. 295–384.

[25] On the critical values of $L$-functions of $GL_2$ and $GL_2 \times GL_2$, Duke Math. J., 74 (1994), pp. 431–529.

[26] H. Hida and J. Tilouine, Anti-cyclotomic Katz $p$-adic $L$-functions and congruence modules, Ann. Sci. ENS, 26 (1993), pp. 189–259.

[27] L. Illusie, Réduction semi-stable et décomposition de complexes de de Rham à coefficients., Duke Math. J., 60 (1990), pp. 139–185.

[28] J. Jantzen, Representations of algebraic groups, Academic Press, 1987.

[29] M. Kisin and K. Lai, Overconvergent Hilbert modular forms. preprint.

[30] B. Mazur, Modular curves and the Eisenstein ideal, Inst. Hautes Études Sci. Publ. Math., 47 (1977), pp. 33–186.

[31] A. Mokrane and J. Tilouine, Cohomology of Siegel varieties with $p$-adic integral coefficients and applications, in Cohomology of Siegel Varieties, Astérisque, 280 (2002), pp. 1–95.

[32] R. Pink, On l-adic sheaves on Shimura varieties and their higher images in the Baily-Borel compactification, Math. Ann., 292 (1992), pp. 197–240.

[33] P. Polo and J. Tilouine, Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups over $Z(p)$ for representations with $p$-small weights, in Cohomology of Siegel Varieties, Astérisque, 280 (2002), pp. 97–135.

[34] M. Rapoport, Compactification de l’espace de modules de Hilbert-Blumenthal, Compositio Math., 36 (1978), pp. 255–335.

[35] K. Ribet, On l-adic representations attached to modular forms, Invent. Math., 28 (1975), pp. 245–275.

[36] Mod $p$ Hecke operators and congruences between modular forms, Invent. Math., 71 (1983), pp. 193–205.

[37] J.-P. Serre, Propriétés galoisienes des points d’ordre fini des courbes elliptiques, Invent. Math., 15 (1972), pp. 259–331.

[38] G. Shimura, The special values of the zeta functions associated with Hilbert modular forms, Duke Math. J., 45 (1978), pp. 637–679.

[39] R. Steinberg, Representations of algebraic groups, Nagoya Math. J., 22 (1963), pp. 33–56.

[40] R. Taylor, On Galois representations associated to Hilbert modular forms, Invent. Math., 98 (1989), pp. 265–280.

[41] On Galois representations associated to Hilbert modular forms II, in Elliptic curves, modular forms and Fermat’s last theorem (Hong Kong, 1993), J. Coates and S.-T. Yau, eds., International Press, 1997, pp. 185–191.

[42] T. Wedhorn, Congruence relations on some shimura varieties, J. Reine Angew. Math., 524 (2000), pp. 43–71.

[43] A. Wiles, On ordinary $\lambda$-adic representations associated to modular forms, Invent. Math., 94 (1988), pp. 529–573.

[44] J.-P. Wintenberger, Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux, Ann. of Math., 119 (1984), pp. 511–548.

[45] H. Yoshida, On the zeta functions of Shimura varieties and periods of Hilbert modular forms, Duke Math. J., 74 (1994), pp. 121–191.

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