Lattices with and lattices without spectral gap

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For Fritz Grunewald on his 60th birthday

Abstract

Let \( G = G(k) \) be the \( k \)-rational points of a simple algebraic group \( G \) over a local field \( k \) and let \( \Gamma \) be a lattice in \( G \). We show that the regular representation \( \rho_{\Gamma \backslash G} \) of \( G \) on \( L^2(\Gamma \backslash G) \) has a spectral gap, that is, the restriction of \( \rho_{\Gamma \backslash G} \) to the orthogonal of the constants in \( L^2(\Gamma \backslash G) \) has no almost invariant vectors. On the other hand, we give examples of locally compact simple groups \( G \) and lattices \( \Gamma \) for which \( L^2(\Gamma \backslash G) \) has no spectral gap. This answers in the negative a question asked by Margulis [Marg91, Chapter III, 1.12]. In fact, \( G \) can be taken to be the group of orientation preserving automorphisms of a \( k \)-regular tree for \( k > 2 \).

1 Introduction

Let \( G \) be a locally compact group. Recall that a unitary representation \( \pi \) of \( G \) on a Hilbert space \( \mathcal{H} \) has almost invariant vectors if, for every compact subset \( Q \) of \( G \) and every \( \varepsilon > 0 \), there exists a unit vector \( \xi \in \mathcal{H} \) such that \( \sup_{x \in Q} \| \pi(x)\xi - \xi \| < \varepsilon \). If this holds, we also say that the trivial representation \( 1_G \) is weakly contained in \( \pi \).

Recall that a lattice \( \Gamma \) in \( G \) is a discrete subgroup such that there exists a finite \( G \)-invariant regular Borel measure \( \mu \) on \( \Gamma \backslash G \). Denote by \( \rho_{\Gamma \backslash G} \) the unitary representation of \( G \) given by right translation on the Hilbert space

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\(L^2(\Gamma \backslash G, \mu)\) of the square integrable measurable functions on \(\Gamma \backslash G\). The subspace \(C^1_{\Gamma \backslash G}\) of the constant functions on \(\Gamma \backslash G\) is \(G\)-invariant as well as its orthogonal complement

\[L^2_0(\Gamma \backslash G) = \{ \xi \in L^2(\Gamma \backslash G) : \int_{\Gamma \backslash G} \xi(x) d\mu(x) = 0 \}.\]

Denote by \(\rho^0_{\Gamma \backslash G}\) the restriction of \(\rho_{\Gamma \backslash G}\) to \(L^2_0(\Gamma \backslash G, \mu)\). We say that \(\rho_{\Gamma \backslash G}\) (or \(L^2(\Gamma \backslash G, \mu)\)) has a spectral gap if \(\rho^0_{\Gamma \backslash G}\) has no almost invariant vectors.

(In [Marg91, Chapter III., 1.8], \(\Gamma\) is then called weakly cocompact.) It is well-known that \(L^2(\Gamma \backslash G)\) has a spectral gap when \(\Gamma\) is cocompact in \(G\) (see [Marg91, Chapter III, 1.10]). Margulis (op.cit, 1.12) asks whether this result holds more generally when \(\Gamma\) is a subgroup of finite covolume.

The goal of this note is to prove the following results:

**Theorem 1** Let \(G\) be a simple algebraic group over a local field \(k\) and \(G = G(k)\), the group of \(k\)-rational points in \(G\). Let \(\Gamma\) be a lattice in \(G\). Then the unitary representation \(\rho_{\Gamma \backslash G}\) on \(L^2(\Gamma \backslash G)\) has a spectral gap.

**Theorem 2** For an integer \(k > 2\), let \(X\) be the \(k\)-regular tree and \(G = \text{Aut}(X)\). Then \(G\) contains a lattice \(\Gamma\) for which the unitary representation \(\rho_{\Gamma \backslash G}\) on \(L^2(\Gamma \backslash G)\) has no spectral gap.

So, Theorem 2 answers in the negative Margulis’ question mentioned above.

Theorem 1 is known in case \(k = \mathbb{R}\) ([Bekk98]). It holds, more generally, when \(G\) is a real Lie group ([BeCo08]). Observe also that when \(k - \text{rank}(G) \geq 2\), the group \(G\) has Kazhdan’s Property (T) (see [BHV]) and Theorem 1 is clear in this case. When \(k\) is non-archimedean with characteristic 0, every lattice \(\Gamma\) in \(G(k)\) is uniform (see [Serr, p.84]) and hence the result holds as mentioned above. By way of contrast, \(G\) has many non uniform lattices when the characteristic of \(k\) is non zero (see [Serr] and [Lubo91]). So, in order to prove Theorem 1, it suffices to consider the case where the characteristic of \(k\) is non-zero and where \(k = \text{rank}(G) = 1\).

Recall that when \(k\) is non-archimedean and \(k - \text{rank}(G) = 1\), the group \(G(k)\) acts by automorphisms on the associated Bruhat-Tits tree \(X\) (see [Serr]). This tree is either the \(k\)-regular tree \(X_k\) (in which every vertex has constant degree \(k\)) or is the bi-partite bi-regular tree \(X_{k_0,k_1}\) (where every vertex has either degree \(k_0\) or degree \(k_1\) and where all neighbours of a vertex
of degree $k_i$ have degree $k_{1-i}$). The proof of Theorem 1 will use the special structure of a fundamental domain for the action of $\Gamma$ on $X$ as described in [Lubo91] (see also [Ragh89] and [Baum03]).

Theorems 1 and 2 provide a further illustration of the different behaviour of general tree lattices as compared to lattices in rank one simple Lie groups over local fields; for more on this topic, see [Lubo95].

The proofs of Theorems 1 and 2 will be given in Sections 3 and 4; they rely in a crucial way on Proposition 6 from Section 2, which relates the existence of a spectral gap with expander diagrams. In turn, Proposition 6 is based, much in the spirit of [Broo81], on analogues for diagrams proved in [Mokh03] and [Morg94] of the inequalities of Cheeger and Buser between the isoperimetric constant and the bottom of the spectrum of the Laplace operator on a Riemannian manifold (see Proposition 5). This connection between the combinatorial expanding property and representation theory is by now a very popular theme; see [Lubo94] and the references therein. While most applications in this monograph are from representation theory to combinatorics, we use in the current paper this connection in the opposite direction: the existence or absence of a spectral gap is deduced from the existence of an expanding diagram or of a non-expanding diagram, respectively.

2 Spectral gap and expander diagrams

We first show how the existence of a spectral gap for groups acting on trees is related with the bottom of the spectrum of the Laplacian for an associated diagram.

A graph $X$ consists of a set of vertices $V_X$, a set of oriented edges $EX$, a fix-point free involution $\pi : EX \to EX$, and end point mappings $\partial_i : EX \to V_X$ for $i = 0, 1$ such that $\partial_i(\pi) = \partial_{1-i}(e)$ for all $e \in EX$. Assume that $X$ is locally finite, that is, for every $x \in V_X$, the degree $\deg(x)$ of $x$ is finite, where $\deg(x)$ is the cardinality of the set

$$\partial_0^{-1}(x) = \{ e \in EX : \partial_0(e) = x \}.$$

The group $\text{Aut}(X)$ of automorphisms of the graph $X$ is a locally compact group in the topology of pointwise convergence on $X$, for which the stabilizers of vertices are compact open subgroups.

We will consider infinite graphs called diagrams of finite volume. An edge-indexed graph $(D, i)$ is a graph $D$ equipped with a function $i : ED \to \mathbb{R}^+$. 

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A measure $\mu$ for an edge-indexed graph $(D, i)$ is a function $\mu : V D \cup ED \to \mathbb{R}^+$ with the following properties (see [Mokh03] and [BaLu01, 2.6]):

- $i(e)\mu(\partial_0 e) = \mu(e)$
- $\mu(e) = \mu(\overline{e})$ for all $e \in V D$, and
- $\sum_{x \in V D} \mu(x) < \infty$.

Following [Morg94], we will say that $D = (D, i, \mu)$ is a diagram of finite volume. The in-degree $\text{indeg}(x)$ of a vertex $x \in V D$ is defined by

$$\text{indeg}(x) = \sum_{e \in \partial_0^{-1}(x)} i(e) = \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)}.$$

The diagram $D$ is $k$-regular if $\text{indeg}(x) = k$ for all $x \in V D$.

Let $D = (D, i, \mu)$ be a connected diagram of finite volume. Observe that $\mu$ is determined, up to a multiplicative constant, by the weight function $i$. Indeed, fix $x_0 \in V D$ and set $\Delta(e) = i(e)/i(\overline{e})$ for $e \in ED$. Then

$$\mu(\partial_1 e) = \frac{\mu(\overline{e})}{i(\overline{e})} = \frac{\mu(e)}{i(e)} = \mu(\partial_0 e) \Delta(e)$$

for every $e \in ED$. Hence $\mu(x) = \Delta(e_1)\Delta(e_2)\ldots \Delta(e_n)\mu(x_0)$ for every path $(e_1, e_2, \ldots, e_n)$ from $x_0$ to $x \in V D$.

Let $D = (D, i, \mu)$ be a diagram of finite volume. An inner product is defined for functions on $V D$ by

$$\langle f, g \rangle = \sum_{x \in V D} f(x) \overline{g(x)} \mu(x).$$

The Laplace operator $\Delta$ on functions $f$ on $V D$ is defined by

$$\Delta f(x) = f(x) - \frac{1}{\text{indeg}(x)} \sum_{e \in \partial_0^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)).$$

The operator $\Delta$ is a self-adjoint positive operator on $L^2(V D)$. Let

$$L^2_0(V D) = \{ f \in L^2(V D) : \langle f, 1_{V D} \rangle = 0 \}.$$
and set
\[
\lambda(D) = \inf_f \langle \Delta f, f \rangle,
\]
where \(f\) runs over the unit sphere in \(L^2_0(VD)\). Observe that
\[
\lambda(D) = \inf \{ \lambda : \lambda \in \sigma(\Delta) \setminus \{0\} \},
\]
where \(\sigma(\Delta)\) is the spectrum of \(\Delta\).

Let now \(X\) be a locally finite tree, and let \(G\) be a closed subgroup of \(\text{Aut}(X)\). Assume that \(G\) acts with finitely many orbits on \(X\). Let \(\Gamma\) be a discrete subgroup of \(G\) acting without inversion on \(X\). Then the quotient graph \(\Gamma \setminus X\) is well-defined. Since \(\Gamma\) is discrete, for every vertex \(x\) and every edge \(e\), the stabilizers \(\Gamma_x\) and \(\Gamma_e\) are finite. Moreover, \(\Gamma\) is a lattice in \(G\) if and only if \(\Gamma\) is a lattice in \(\text{Aut}(X)\) and this happens if and only if
\[
\sum_{x \in D} \frac{1}{|\Gamma_x|} < \infty,
\]
where \(D\) is a fundamental domain of \(\Gamma\) in \(X\) (see [Serr]). The quotient graph \(\Gamma \setminus X \cong D\) is endowed with the structure of an edge-indexed graph given by the weight function \(i: ED \to \mathbb{R}^+\) where \(i(e)\) is the index of \(\Gamma_e\) in \(\Gamma_x\) for \(x = \partial_0(e)\). A measure \(\mu: VD \cup ED \to \mathbb{R}^+\) is defined by
\[
\mu(x) = \frac{1}{|\Gamma_x|} \quad \text{and} \quad \mu(e) = \frac{1}{|\Gamma_e|}
\]
for \(x \in VD\) and \(e \in ED\). Observe that \(\mu(VD) = \sum_{x \in D} 1/|\Gamma_x| < \infty\). So, \(D = (D, i, \mu)\) is a diagram of finite volume.

Let \(G\) be a group acting on a tree \(X\). As in [BuMo00, 0.2], we say that the action of \(G\) on \(X\) is locally \(\infty\)-transitive if, for every \(x \in VX\) and every \(n \geq 1\), the stabilizer \(G_x\) of \(x\) acts transitively on the sphere \(\{y \in X : d(x, y) = n\}\).

**Proposition 3** Let \(X\) be either the \(k\)-regular tree \(X_k\) or the bi-partite bi-regular tree \(X_{k_0,k_1}\) for \(k \geq 3\) or \(k_0 \geq 3\) and \(k_1 \geq 3\). Let \(G\) be a closed subgroup of \(\text{Aut}(X)\). Assume that the following conditions are both satisfied:

- \(G\) acts transitively on \(VX\) in the case \(X = X_k\) and \(G\) acts transitively on the set of vertices of degree \(k_0\) as well as on the set of vertices of degree \(k_1\) in the case \(X = X_{k_0,k_1}\);
the action of $G$ on $X$ is locally $\infty$-transitive.

Let $\Gamma$ be a lattice in $G$ and let $D = \Gamma \setminus X$ be the corresponding diagram of finite volume. The following properties are equivalent:

(i) the unitary representation $\rho_{\Gamma \setminus G}$ on $L^2(\Gamma \setminus G)$ has a spectral gap;

(ii) $\lambda(D) > 0$.

For the proof of this proposition, we will need a few general facts. Let $G$ be a second countable locally compact group and $U$ a compact subgroup of $G$. Let $C_c(U \setminus G/U)$ be the space of continuous functions $f : G \to \mathbb{C}$ which have compact support and which are constant on the double cosets $UgU$ for $g \in G$.

Fix a left Haar measure $\mu$ on $G$. Recall that $L^1(G, \mu)$ is a Banach algebra under the convolution product, the $L^1$-norm and the involution $f^*(g) = \overline{f(g^{-1})}$; observe that $C_c(U \setminus G/U)$ is a *-subalgebra of $L^1(G, \mu)$. Let $\pi$ be a (strongly continuous) unitary representation of $G$ on a Hilbert space $\mathcal{H}$. A continuous *-representation of $L^1(G)$, still denoted by $\pi$, is defined on $\mathcal{H}$ by

$$
\pi(f)\xi = \int_G f(x)\pi(x)\xi d\mu(x), \quad f \in L^1(G), \quad \xi \in \mathcal{H}.
$$

Assume that the closed subspace $\mathcal{H}^U$ of $U$-invariant vectors in $\mathcal{H}$ is non-zero. Then $\pi(f)\mathcal{H}^U \subset \mathcal{H}^U$ for all $f \in C_c(U \setminus G/U)$. In this way, a continuous *-representation $\pi_U$ of $C_c(U \setminus G/U)$ is defined on $\mathcal{H}^U$.

**Proposition 4** With the previous notation, let $f \in C_c(U \setminus G/U)$ be a function with the following properties: $f(x) \geq 0$ for all $x \in G$, $\int_G f d\mu = 1$, and the subgroup generated by the support of $f$ is dense in $G$. The following conditions are equivalent:

(i) the trivial representation $1_G$ is weakly contained in $\pi$;

(ii) $1$ belongs to the spectrum of the operator $\pi_U(f)$.

**Proof** Assume that $1_G$ is weakly contained in $\pi$. There exists a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that

$$
\lim_n \|\pi(x)\xi_n - \xi_n\| = 0,
$$
uniformly over compact subsets of $G$. Let

$$\eta_n = \int_U \pi(u)\xi_n du,$$

where $du$ denotes the normalized Haar measure on $U$. It is easily checked that $\eta_n \in \mathcal{H}^U$ and that

$$\lim_n \|\pi(f)\eta_n - \eta_n\| = 0.$$

Since

$$\|\eta_n - \xi_n\| \leq \int_U \|\pi(u)\xi_n - \xi_n\| du,$$

we have $\|\eta_n\| \geq 1/2$ for sufficiently large $n$. This shows that 1 belongs to the spectrum of the operator $\pi_U(f)$.

For the converse, assume that 1 belongs to the spectrum of $\pi_U(f)$. Hence, 1 belongs to the spectrum of $\pi(f)$, since $\pi_U(f)$ is the restriction of $\pi(f)$ to the invariant subspace $\mathcal{H}^U$. As the subgroup generated by the support of $f$ is dense in $G$, this implies that $1_G$ is weakly contained in $\pi$ (see [BHV, Proposition G.4.2]).

**Proof of Proposition 3** We give the proof only in the case where $X$ is the bi-regular tree $X_{k_0,k_1}$. The case where $X$ is the regular tree $X_k$ is similar and even simpler.

Let $X_0$ and $X_1$ be the subsets of $X$ consisting of the vertices of degree $k_0$ and $k_1$, respectively. Fix two points $x_0 \in X_0$ and $x_1 \in X_1$ with $d(x_0, x_1) = 1$. So, $X_0$ is the set of vertices $x$ for which $d(x_0, x)$ is even and $X_1$ is the set of vertices $x$ for which $d(x_0, x)$ is odd. Let $U_0$ and $U_1$ be the stabilizers of $x_0$ and $x_1$ in $G$. Since $G$ acts transitively on $X_0$ and on $X_1$, we have $G/U_0 \cong X_0$ and $G/U_1 \cong X_1$.

We can view the normed $*$-algebra $C_c(U_0 \setminus G/U_0)$ as a space of finitely supported functions on $X_0$. Since $U_0$ acts transitively on every sphere around $x_0$, it is well-known that the pair $(G, U_0)$ is a Gelfand pair, that is, the algebra $C_c(U_0 \setminus G/U_0)$ is commutative (see for instance [BLRW09, Lemma 2.1]). Observe that $C_c(U_0 \setminus G/U_0)$ is the linear span of the characteristic functions $\delta_n(x_0)$ (lifted to $G$) of spheres of even radius $n$ around $x_0$. Moreover, $C_c(U_0 \setminus G/U_0)$ is generated by $\delta_2(x_0)$; indeed, this follows from the formulas (see
Let $f_0 = \frac{1}{\|\delta_2^{(0)}\|_1} \delta_2^{(0)}$. We claim that $f_0$ has all the properties listed in Proposition 4.

Indeed, $f_0$ is a non-negative and $U_0$-bi-invariant function on $G$ with $\int_G f_0(x)dx = 1$. Moreover, let $H$ be the closure of the subgroup generated by the support of $f_0$. Assume, by contradiction, that $H \neq G$. Then there exists a function in $C_c(U_0 \backslash G/U_0)$ whose support is disjoint from $H$. This is a contradiction, as the algebra $C_c(U_0 \backslash G/U_0)$ is generated by $f_0$. This shows that $H = G$.

Let $\pi$ be the unitary representation of $G$ on $L^2_0(\Gamma \backslash X)$ defined by right translations. Observe that the space of $\pi(U_0)$-invariant vectors is $L^2_0(\Gamma \backslash X_0)$, so, we have a $*$-representation $\pi_{U_0}$ of $C_c(U_0 \backslash G/U_0)$ on $L^2(\Gamma \backslash X_0, \mu)$, where $\mu$ is the measure on the diagram $D = \Gamma \backslash X$, as defined above.

Similar facts are also true for the algebra $C_c(U_1 \backslash G/U_1)$: this is a commutative normed $*$-algebra, it is generated by the characteristic function $\delta_2^{(1)}$ of the sphere of radius 2 around $x_1$, and the representation $\pi$ of $G$ on $L^2_0(\Gamma \backslash G)$ induces a $*$-representation $\pi_{U_1}$ of $C_c(U_1 \backslash G/U_1)$ on $L^2_0(\Gamma \backslash X_1, \mu)$. Likewise, the function $f_1 = \frac{1}{\|\delta_2^{(1)}\|_1} \delta_2^{(1)}$ has all the properties listed in Proposition 4.

Let $A_X$ be the adjacency operator defined on $\ell^2(X)$ by

$$A_X f(x) = \frac{1}{\deg(x)} \sum_{e \in \partial^{-1}(x)} f(\partial_1(e)), \quad f \in \ell^2(X).$$

Since $A_X$ commutes with automorphisms of $X$, it induces an operator $A_D$ on $L^2(VD, \mu)$ given by

$$A_D f(x) = \frac{1}{\indeg(x)} \sum_{e \in \partial^{-1}(x)} \frac{\mu(e)}{\mu(x)} f(\partial_1(e)), \quad f \in L^2(VD, \mu),$$

where $D$ is the diagram obtained from the quotient graph $\Gamma \backslash X$. So, $\Delta = I - A_D$, where $\Delta$ is the Laplace operator on $D$. 

Let $B_D$ denote the restriction of $A_D$ to the space $L^2_0(VD, \mu)$. It follows that $\lambda(\Delta) > 0$ if and only if 1 does not belong to the spectrum of $B_D$.

Proposition 3 will be proved, once we have shown the following

**Claim:** 1 belongs to the spectrum of $B_D$ if and only if $1 \in \pi U_0(g_0)$.

For this, we consider the squares of the operators $A_X$ and $A_D$ and compute

$$A^2_X f(x) = \frac{1}{k_0 k_1} \deg(x) f(x) + \frac{1}{k_0 k_1} \sum_{d(x,y) = 2} f(y), \quad f \in \ell^2(X).$$

The subspaces $\ell^2(X_0)$ and $\ell^2(X_1)$ of $\ell^2(X)$ are invariant under $A^2_X$ and the restrictions of $A^2_X$ to $\ell^2(X_0)$ and $\ell^2(X_1)$ are given by right convolution with the functions

$$g_0 = \frac{1}{k_0 k_1} \delta_e + (1 - \frac{1}{k_0 k_1}) f_0$$
$$g_1 = \frac{1}{k_0 k_1} \delta_e + (1 - \frac{1}{k_0 k_1}) f_1,$$

where $\delta_e$ is the Dirac function at the group unit $e$ of $G$.

It follows that the restrictions of $B^2_D$ to the subspaces $L^2_0(\Gamma \backslash X_0, \mu)$ and $L^2_0(\Gamma \backslash X_1, \mu)$ coincide with the operators $\pi U_0(g_0)$ and $\pi U_1(g_1)$, respectively.

For $i = 0, 1$, the spectrum $\sigma(\pi U_i(g_i))$ of $\pi U_i(g_i)$ is the set

$$\sigma(\pi U_i(g_i)) = \left\{ \frac{1}{k_0 k_1} + (1 - \frac{1}{k_0 k_1}) \lambda : \lambda \in \sigma(\pi U_i(f_i)) \right\}.$$

Thus, 1 belongs to the spectrum of $\pi U_0(f_i)$ if and only if 1 belongs to the spectrum of $\pi U_0(g_i)$.

To prove the claim above, assume that 1 belongs to the spectrum of $B_D$. Then 1 belongs to the spectrum of $B^2_D$. Hence 1 belongs to the spectrum of either $\pi U_0(g_0)$ or $\pi U_1(g_1)$ and therefore 1 belongs to the spectrum of either $\pi U_0(f_0)$ or $\pi U_1(f_1)$. It follows from Proposition 4 that $1 \in \sigma(\pi U_0(g_i))$.

Conversely, suppose that $1 \in \sigma(\pi U_0(g_i))$. Then, again by Proposition 4, 1 belongs to the spectrum of $\pi U_0(f_0)$ and $\pi U_1(f_1)$. Hence, 1 belongs to the spectrum of $\pi U_0(g_0)$ and $\pi U_1(g_1)$. We claim that 1 belongs to the spectrum of $B_D$. }

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Indeed, assume by contradiction that 1 does not belong to the spectrum of $B_D$, that is, $B_D - I$ has a bounded inverse on $L^2_0(VD, \mu)$. Since 1 belongs to the spectrum of the self-adjoint operator $\pi_{U_0}(g_0)$, there exists a sequence of unit vectors $\xi_n^{(0)}$ in $L^2_0(\Gamma \setminus X_0, \mu)$ with

$$\lim_n \|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = 0.$$ 

As the restriction of $B_D^2$ to $L^2_0(\Gamma \setminus X_0, \mu)$ coincides with $\pi_{U_0}(g_0)$, we have

$$\|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = \|B_D^2 - I\|\xi_n^{(0)}\|$$

$$= \|(B_D - I)(B_D + I)\xi_n^{(0)}\|$$

$$\geq \frac{1}{\|(B_D - I)^{-1}\|}\|(B_D + I)\xi_n^{(0)}\|$$

So, $\lim_n \|B_D\xi_n^{(0)} + \xi_n^{(0)}\| = 0$. On the other hand, observe that $B_D$ maps $L^2_0(\Gamma \setminus X_0, \mu)$ to the subspace $L^2(\Gamma \setminus X_1, \mu)$ and that these subspaces are orthogonal to each other. Hence,

$$\|B_D\xi_n^{(0)} + \xi_n^{(0)}\|^2 = \|B_D\xi_n^{(0)}\|^2 + \|\xi_n^{(0)}\|^2$$

This is a contradiction since $\|\xi_n^{(0)}\| = 1$ for all $n$. The proof of Proposition 3 is now complete.

Next, we rephrase Proposition 3 in terms of expander diagrams. Let $(D, i, w)$ be a diagram with finite volume. For a subset $S$ of $VD$, set

$$E(S, S^c) = \{e \in ED : \partial_0(e) \in S, \partial_1(e) \notin S\}.$$ 

We say that $D$ is an expander diagram if there exists $\varepsilon > 0$ such that

$$\frac{\mu(E(S, S^c))}{\mu(S)} \geq \varepsilon$$

for all $S \subset VD$ with $\mu(S) \leq \mu(D)/2$. The motivation for this definition comes from expander graphs (see [Lubo94]).

We quote from [Mokh03] and [Morg94] the following result which is standard in the case of finite graphs.

**Proposition 5** ([Mokh03], [Morg94]) Let $(D, i, w)$ be a diagram with finite volume. Assume that $\sup_{e \in ED} i(\overline{e})/i(e) < \infty$ and that $\sup_{x \in VD} \text{indeg}(x) < \infty$ The following conditions are equivalent:

1. $D$ is an expander diagram.
2. $\lim_n \|\pi_{U_0}(g_0)\xi_n^{(0)} - \xi_n^{(0)}\| = 0$.
3. $\lim_n \|B_D\xi_n^{(0)} + \xi_n^{(0)}\| = 0$.

The proof of this proposition is analogous to the proof of Proposition 3.
As an immediate consequence of Propositions 3 and 5, we obtain the following result which relates the existence of a spectral gap to an expanding property of the corresponding diagram.

**Proposition 6** Let $X$ be either the $k$-regular tree $X_k$ or the bi-partite bi-regular tree $X_{k_0,k_1}$ for $k \geq 3$ or $k_0 \geq 3$ and $k_1 \geq 3$. Let $G$ be a closed subgroup of $\text{Aut}(X)$ satisfying both conditions from Proposition 3. Let $\Gamma$ be a lattice in $G$ and let $D = \Gamma \setminus X$ be the corresponding diagram of finite volume. The following properties are equivalent.

(i) The unitary representation $\rho_{\Gamma \setminus G}$ on $L^2(\Gamma \setminus G)$ has a spectral gap;

(ii) $D$ is an expander diagram.

3 Proof of Theorem 1

Let $G = G(k)$ be the $k$-rational points of a simple algebraic group $G$ over a local field $k$ and let $\Gamma$ be a lattice in $G$. As explained in the Introduction, we may assume that $k$ is non-archimedean and that $k - \text{rank}(G) = 1$. By the Bruhat-Tits theory, $G$ acts on a regular or bi-partite bi-regular tree $X$ with one or two orbits. Moreover, the action of $G$ on $X$ is locally $\infty$-transitive (see [Chou94, p.33]).

Passing to the subgroup $G^+$ of index at most two consisting of orientation preserving automorphisms, we can assume that $G$ acts without inversion. Indeed, assume that $L^2(\Gamma \cap G^+) \setminus G^+$ has a spectral gap. If $\Gamma$ is contained in $G^+$, then $L^2(\Gamma \setminus G)$ has a spectral gap since $G^+$ has finite index (see [BeCo08, Proposition 6]). If $\Gamma$ is not contained in $G^+$, then $\Gamma \cap G^+ \setminus G^+$ may be identified as a $G^+$-space with $G \setminus G^+ = G \setminus G$. Hence, $1_{G^+}$ is not weakly contained in the $G^+$-representation defined on $L^2_0(\Gamma \setminus G)$.

Let $X$ be the Bruhat-Tits tree associated to $G$. It is shown in [Lubo91, Theorem 6.1] (see also [Baum03]) that $\Gamma$ has fundamental domain $D$ in $X$ of the following form: there exists a finite set $F \subset D$ such that $D \setminus F$ is a union of finitely many disjoint rays $r_1, \ldots, r_s$. (Recall that a ray in $X$ is an infinite
path beginning at some vertex and without backtracking.) Moreover, for every ray \( r_j = \{x^j_0, x^j_1, x^j_2, \ldots \} \) in \( D \setminus F \), the stabilizer \( \Gamma_{x^j_i} \) of \( x^j_i \) is contained in the stabilizer \( \Gamma_{x^j_{i+1}} \) of \( x^j_{i+1} \) for all \( i \).

To prove Theorem 1, we apply Proposition 6. So, we have to prove that \( D \) is an expander diagramm.

Choose \( i \in \{0, 1, \ldots \} \) such that, with \( D_1 = F \cup \bigcup_{j=1}^s \{x^j_0, \ldots, x^j_i\} \), we have \( \mu(D_1) > 1/2 \).

Let \( S \) be a subset of \( D \) with \( \mu(S) \leq \mu(D)/2 \). Then \( D_1 \not\subseteq S \). Two cases can occur.

• First case: \( S \cap D_1 = \emptyset \). Thus, \( S \) is contained in \( \bigcup_{j=1}^s \{x^j_{i+1}, x^j_{i+2}, \ldots \} \).

Fix \( j \in \{1, \ldots, s\} \). Let \( i(j) \in \{0, 1, \ldots \} \) be minimal with the property that \( x^j_{i(j)+1} \in S \). Then \( e_j := (x^j_{i(j)+1}, x^j_{i(j)}) \in E(S, S^c) \). Observe that \( |\Gamma_{x^j_{i(j)+1}}| = \deg(x^j_i)|\Gamma_{x^j_i}| \) for all \( l \geq 0 \). Let \( k \) be the minimal degree for vertices in \( X \) (so, \( k = \min\{k_0, k_1\} \) if \( X = X_{k_0, k_1} \)). Then \( \mu(x^j_{i+1}) \leq \mu(x^j_i)/k \) for all \( l \) and

\[
\mu(e_j) = \frac{1}{|\Gamma_{e_j}|} \geq \frac{k}{|\Gamma_{x^j_{i(j)}}|} = k\mu(x^j_{i(j)}).
\]
Therefore, we have
\[
\frac{\mu(E(S, S^c))}{\mu(S)} \geq \frac{\sum_{j=1}^{s} \mu(e_j)}{\sum_{j=1}^{s} \mu(x_{i(j)}^{j})} \geq k \frac{\sum_{j=1}^{s} \mu(x_{i(j)}^{j})}{\sum_{j=1}^{\infty} \mu(x_{i(j)}^{j})} \geq k \frac{\sum_{j=1}^{s} \mu(x_{i(j)}^{j})}{\sum_{j=1}^{\infty} \mu(x_{i(j)}^{j})} \geq \frac{k}{1-k^{-1}} \sum_{j=1}^{\infty} \mu(x_{i(j)}^{j}) = k - 1.
\]

● *Second case:* \(S \cap D_1 \neq \emptyset\). Then there exist \(x \in S \cap D_1\) and \(y \in D_1 \setminus S\). Since \(D_1\) is a connected subgraph, there exists a path \((e_1, e_2, \ldots, e_n)\) in \(ED_1\) from \(x\) to \(y\). Let \(l \in \{1, \ldots, n\}\) be minimal with the property \(\partial_0(e_l) \in S\) and \(\partial_1(e_l) \notin S\). Then \(e_l \in E(S, S^c)\). Hence, with \(C = \min\{\mu(e) : e \in ED_1\}\ > 0\), we have
\[
\frac{\mu(E(S, S^c))}{\mu(S)} \geq \frac{C}{\mu(D)}.
\]

This completes the proof of Theorem 1. ■

4 Proof of Theorem 2

Let \((D, i, \mu)\) be a \(k\)-regular diagram. By the “inverse Bass–Serre theory” of groups acting on trees, there exists a lattice \(\Gamma\) in \(G = \text{Aut}(X_k)\) for which \(D = \Gamma \setminus X_k\). Indeed, we can find a finite grouping of \((D, i)\), that is, a graph of finite groups \(D = (D, \mathcal{D})\) such that \(i(e)\) is the index of \(D_e\) in \(D_{\partial e}\) for all \(e \in ED\). Fix an origin \(x_0\). Let \(\Gamma = \pi_1(D, x_0)\) be the fundamental group of \((D, x_0)\). The universal covering of \((D, x_0)\) is the \(k\)-regular tree \(X_k\) and the diagram \(D\) can identified with the diagram associated to \(\Gamma \setminus X_k\). For all the see (2.5), (2.6) and (4.13) in [BaLu01].

In view of Proposition 6, Theorem 2 will be proved once we present examples of \(k\)-regular diagrams with finite volume which are not expanders.
An example of such a diagram appears in [Mokh03, Example 3.4]. For the convenience of the reader, we review the construction.

Fix $k \geq 3$ and let $q = k - 1$. For every integer $n \geq 1$, let $D_n$ be the finite graph with $2n + 1$ vertices:

\[
\circ\ x^{(n)}_1 - \circ\ x^{(n)}_2 - \cdots - \circ\ x^{(n)}_{2n} - \circ\ x^{(n)}_{2n+1}
\]

Let $D$ be the following infinite ray:

\[
\circ - \circ - D_1 - \circ - \circ - D_2 - \circ - \circ - \cdots - \circ - \circ - D_n - \circ - \circ - \cdots
\]

We first define a weight function $i_n$ on $ED_n$ as follows:

- $i_n(e) = 1$ if $e = (x^{(n)}_1, x^{(n)}_2)$ or $e = (x^{(n)}_2, x^{(n)}_1)$
- $i_n(e) = q$ if $e = (x^{(n)}_m, x^{(n)}_{m+1})$ for $m$ even
- $i_n(e) = 1$ if $e = (x^{(n)}_m, x^{(n)}_{m+1})$ for $m$ odd
- $i_n(e) = q$ if $e = (x^{(n)}_{m+1}, x^{(n)}_m)$ for $m$ even
- $i_n(e) = 1$ if $e = (x^{(n)}_{m+1}, x^{(n)}_m)$ for $m$ odd.

Observe that $i_n(e)/i_n(\tau) = 1$ for all $e \in ED_n$. Define now a weight function $i$ on $ED$ as follows:

- $i(e) = q + 1$ if $e = (x_0, x_1)$
- $i(e) = q$ if $e = (x_1, x_0)$
- $i(e) = 1$ if $e = (x_m, x_{m+1})$ for $m \geq 1$
- $i(e) = q$ if $e = (x_{m+1}, x_m)$ for $m \geq 1$
- $i(e) = i_n(e)$ if $e \in ED_n$.

One readily checks that, for every vertex $x \in D$,

\[
\sum_{e \in \partial^{-1}_n(x)} i(e) = q + 1 = k,
\]

that is, $(D, i)$ is $k$-regular. The measure $\mu : VD \rightarrow \mathbb{R}^+$ corresponding to $i$ (see the remark at the beginning of Section 2) is given by
• \( \mu(x_0) = 1/(q + 1) \)
• \( \mu(x_{2m-2}) = 1/q^{m-1} \) for \( m \geq 2 \)
• \( \mu(x_{2m-1}) = 1/q^m \) for \( m \geq 1 \)
• \( \mu(x) = 1/q^n \) if \( x \in D_n \).

One checks that, if we define \( \mu(e) = i(e)\mu(\partial_0 e) \) for all \( e \in ED \), we have \( \mu(\overline{e}) = \mu(e) \). Moreover,

\[
\mu(D_n) = (2n + 1) \frac{1}{q^n}
\]

and hence

\[
\mu(D) \leq \frac{1}{q + 1} + 2 \sum_{n \geq 0} \frac{1}{q^n} + \sum_{n \geq 1} \mu(D_n) < \infty.
\]

We have also

\[
E(D_n, D_n^c) = \{(x_{2n-1}, x_{2n-2}), (x_{2n}, x_{2n+1})\},
\]

so that

\[
\mu(E(D_n, D_n^c)) = q \frac{1}{q^n} + \frac{1}{q^n} = \frac{q + 1}{q^n}.
\]

Hence

\[
\frac{\mu(E(D_n, D_n^c))}{\mu(D_n)} = \frac{\frac{q+1}{q^n}}{(2n+1)\frac{1}{q^n}} = \frac{q + 1}{2n + 1}
\]

and

\[
\lim_n \frac{\mu(E(D_n, D_n^c))}{\mu(D_n)} = 0.
\]

Observe that, since \( \lim_n \mu(D_n) = 0 \), we have \( \mu(D_n) \leq \mu(D)/2 \) for sufficiently large \( n \). This completes the proof of Theorem 2.\( \blacksquare \).

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