\[ |V_{ub}| \] and Perturbative QCD Effects in the \( B \rightarrow \pi \) Transition Form Factor

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Abstract: We report on recent improvements for the \( B \rightarrow \pi \) form factor. The updated value of \( |V_{ub}| \) is presented.

1. Motivation. The semileptonic decay \( B \rightarrow \pi \nu_l \) is one of the most important reactions for the determination of the CKM parameter \( V_{ub} \). However, in order to extract \( V_{ub} \) from data one needs an accurate theoretical calculation of the hadronic matrix element

\[
\langle \pi(q)|\bar{u}\gamma_\mu b|B(p+q)\rangle = 2f^+(p^2)q_\mu + (f^+(p^2) + f^-(p^2))p_\mu ,
\]

where \( p+q, q \) and \( p \) denote the \( B \) and \( \pi \) four-momenta and the momentum transfer, respectively, and \( f^\pm \) are two independent form factors. A very reliable approach to calculate \( f^\pm \) in the framework of QCD is provided by the operator product expansion (OPE) on the light-cone \([1,2,3]\) in combination with QCD sum rule techniques. The sum rule for the form factor \( f^+(p^2) \) has been obtained in the leading order (LO) in \( \alpha_s \) in \([4,5]\) taking into account twist 2, 3 and 4 operators. The most important missing elements of these calculations are the perturbative QCD corrections to the correlation function. Here we report on a results of the calculation of the \( O(\alpha_s) \) correction to \( f^+ \) \([13,14]\) which eliminates one of the main uncertainty in the sum rule results.

2. Sum rule. The general idea of the sum rule method is to consider the correlation function of two heavy-light currents,

\[
F_\mu(p,q) = i \int dx e^{ip\cdot x} \langle \pi(q)|T\{\bar{u}(x)\gamma_\mu b(x), m_b\bar{b}(0)i\gamma_5 d(0)\}|0\rangle
\]

which can be calculated in the region \((p+q)^2 < 0 \) and \( p^2 < m_b^2 - O(1\text{GeV}^2) \) using OPE near the light-cone, i.e. at \( x^2 \sim 0 \). In this note we focus on the leading twist 2 contribution. The sum rule for \( f^+ \) in LO in \( \alpha_s \) is given by

\[
f_B f^+(p^2) = \frac{f_\pi m_B^2}{2m_B^2} \int_{m_b^2}^{s_0} \varphi_\pi(u_0) e^{\frac{m_B^2 - s}{M^2}} ds + \text{higher twists} ,
\]

Here \( \varphi_\pi(u) \) is the pion wave function, \( m_b, M_B \) are the masses of the heavy quark and meson, \( M^2 \) is the Borel parameter, \( s_0 \) is the threshold of the continuum, \( f_\pi = 132\text{MeV}, \)

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\[ u_0 = \frac{m_f^2 - p^2}{s - p^2}. \] The calculation has several aspects which are worth pointing out. Firstly, the sum rule is actually derived for the product \( f_B f^+, f_B \) being the \( B \) meson decay constant defined by \( \langle B | i \gamma_5 d | 0 \rangle = m_B^2 f_B/m_b \). The form factor \( f^+ \) itself is then obtained by dividing out \( f_B \) taking the value determined from the corresponding two-point QCD sum rule. In previous estimates, the \( O(\alpha_s) \) correction to \( f_B \) was thereby ignored for consistency because of the lack of the \( O(\alpha_s) \) correction to \( f_B f^+. \) Our calculation now allows to take into account the correction to \( f_B \) which is known to be sizeable. Secondly, knowing the \( O(\alpha_s) \) corrections, also the heavy quark mass entering the sum rule can be properly defined. The calculation for a finite quark mass is new and will have numerous applications.

3. QCD correction. The correlator can be written as a convolution of a hard amplitude \( T(p^2, (p + q)^2, u) \) calculable within perturbation theory, with the pion wave function \( \varphi_\pi(u) \) containing the long-distance effects:

\[ F(p^2, (p + q)^2) = -f_\pi \int_0^1 \! du \varphi_\pi(u) T(p^2, (p + q)^2, u). \quad (4) \]

The evolution of the light-cone wave function \( \varphi_\pi(u) \) is controlled by the Brodsky-Lepage equation [2]

\[ d\varphi_\pi(u, \mu)/d \ln \mu = \int_0^1 \! d\omega V(u, \omega) \varphi_\pi(\omega, \mu) \quad (5) \]

The first step is to calculate the \( O(\alpha_s) \) correction to the hard amplitude \( T \). The calculation is performed in general covariant gauge in order to have a possibility to check the gauge invariance of the result. Both the ultraviolet (UV) and infrared divergences are regularized by dimensional regularization and renormalized in the \( \overline{MS} \) scheme with totally anticommuting \( \gamma_5 \). This choice is motivated by the fact that the same scheme is used in the calculation of the NLO evolution kernel of the wave function \( \varphi_\pi(u) \) [6]. After UV renormalization, IR factorization and reexpressing of the \( \overline{MS} \) mass by the pole mass, we have obtained

\[
T(r_1, r_2, u, \mu) = \frac{1}{\rho - 1} + \frac{\alpha_s(\mu) C_F}{4\pi} \left\{ \frac{1}{\rho - 1} \left( -4 + 3 \ln \frac{m_b^2}{\mu^2} \right) + \frac{2}{\rho - 1} \left[ 2G(\rho) - G(r_1) - G(r_2) \right] \right. \\
+ \frac{2}{(r_1 - r_2)^2} \left( \frac{1 - r_2}{u} G(\rho) - G(r_1) \right) + \frac{1 - r_1}{1 - u} \left( G(\rho) - G(r_2) \right) \right. \\
+ \frac{\rho(1 - \rho) \ln(1 - \rho)}{\rho^2} + \frac{2}{\rho - 1} \left( (1 - r_2) \ln(1 - r_2) \right) - \frac{2}{\rho - 1} \\
- \frac{2}{(1 - u)(r_1 - r_2)} \left( \frac{(1 - \rho) \ln(1 - \rho)}{\rho} - \frac{(1 - r_2) \ln(1 - r_2)}{r_2} \right) \right\}. \quad (6)
\]

We used convenient dimensionless variables \( r_1 = p^2/m_b^2 \) and \( r_2 = (p + q)^2/m_b^2 \) and

\[
\Delta = \frac{2}{4 - d} - \gamma_E + \ln(4\pi), \quad \rho = r_1 + u(r_2 - r_1), \quad (7)
\]

\[
G(\rho) = \text{Li}_2(\rho) + \ln^2(1 - \rho) + \ln(1 - \rho) \left( \ln \frac{m_b^2}{\mu^2} + 1 \right),
\]

2
\[ \text{Li}_2(x) = - \int_0^x \frac{dt}{t} \ln(1-t) \] being the Spence function. The UV renormalization scale and the factorization scale of the collinear (COL) divergences are taken to be equal and denoted by \( \mu \). As an additional check on the origin of the various divergent terms we have performed additional explicit calculations. In particular, we have used mass regularization by giving the light quarks a small but finite mass, and momentum regularization keeping the light quarks off mass shell.

4. **Numerical results.** The next step is to determine the decay constant \( f_B \) and the pion wave function \( \varphi_\pi(u, \mu) \) in NLO. For that purpose we have analyzed the two-point sum rule for \( f_B \) obtained from the renormalization-group-invariant correlation function \( m_B^2(0) \mid T \{ J_5^+(x)J_5(0) \} \mid 0 \) in \( O(\alpha_s) \) \([7]\). For the running coupling constant we use the two-loop expression with \( N_f = 4 \) and \( \Lambda^{(4)} = 234 \text{ MeV} \) \([8]\) corresponding to \( \alpha_s(M_Z) = 0.112 \). For \( \mu^2 \) we take the value \( \mu_B^2 = m_B^2 - m_b^2 \). corresponding to the average virtuality of the correlation function. With this choice the following correlated results are extracted from the two-point sum rule:

\[
 f_B = 180 \pm 30 \quad \text{MeV} \quad m_b^* = 4.7 \mp 0.1 \quad \text{GeV}, \quad s_0 = 35 \pm 2 \quad \text{GeV}^2. \quad (8)
\]

In the following, we adopt the central values in the above intervals. Note that without \( O(\alpha_s) \) correction one obtains \( f_B = 140 \pm 30 \text{ MeV} \). The remaining parameters entering the sum rules are directly measured: \( m_B = 5.279 \text{ GeV} \) and \( f_\pi = 132 \text{ MeV} \).

For the wave function \( \varphi_\pi \) we adopt the ansatz suggested in \([9]\):

\[
 \varphi_\pi(u, \mu_0) = \Psi_0(u) + a_2(\mu_0)\Psi_2(u) + a_4(\mu_0)\Psi_4(u), \quad (9)
\]

where \( \Psi_n(u) = 6u(1-u)C_{n/2}^0(2u-1) \). The coefficients \( a_2(\mu_0) = 2/3 \) and \( a_4(\mu_0) = 0.43 \) at the scale \( \mu_0 = 500 \text{ MeV} \) have been extracted \([9]\) from a two-point QCD sum rule for the moments of \( \varphi_\pi(u) \) \([1]\).

Now we are ready to perform a numerical analysis of the sum rule. In Fig. 1, the product \( f_B f^+(0) \) is plotted as a function of the Borel parameter \( M^2 \). The \( O(\alpha_s) \) correction turns out to be large, between 30% and 35%, and stable under variation of \( M^2 \). Fig. 2 shows the momentum dependence of the form factor \( f^+(p^2) \) in the region \( 0 < p^2 < 15 \div 17 \text{ GeV}^2 \) for \( M^2 = 10 \text{ GeV}^2 \), where the sum rule is expected to be valid. Note the almost complete cancellation of the NLO correction in \( f^+ \). Finally, it is interesting to compare the \( \mu \) dependence in LO and NLO (Fig.3). The very mild \( \mu \)-dependence in LO results from the evolution of the wave function. In NLO, the \( \mu \)-dependence is stronger than in LO but similar to the \( \mu \)-dependence of \( f_B \). As a result, the residual scale dependence of \( f^+ \) is again mild.

5. **Application.** The above results refer to the leading twist 2 approximation. The thorough numerical analysis of the NLO sum rules have been performed in \([10]\) taking into account LO twist 3 and 4 contributions. Here we give preliminary numbers. The final result for the form factor \( f^+(r) = \frac{p^2}{m_B^2} \) can be approximate by the function \([10]\) (see also \([14,15]\))

\[
f^+(r) = \frac{f^+(0)}{1 - ar + br^2}.
\]

with \( a = 1.5, b = 0.52 \) and

\[
f^+(0) = 0.27 \pm 0.02 \pm 0.02.
\]
The first uncertainty is connected with the unknown perturbative corrections to the twist-2 \( O(\alpha_s^2) \) and twist-3 \( O(\alpha_s) \) contributions and the second one is connected with the wave functions. This value is 10\% lower than the \( LO \) estimate \( f^+(0) = 0.30 \) obtained in [4,11].

Integrating over momentum one obtains the decay width [10]

\[
\Gamma(B^0 \to \pi^- e^+ \nu_e) = (7.5 \pm 2)|V_{ub}|^2 \text{ps}^{-1}.
\]

And finally, using the current CLEO number for the \( Br(B^0 \to \pi l \nu) = (1.8 \pm 0.4) \cdot 10^{-4} \) [12] and the world average of the \( B^0 \) lifetime \( \tau_{B^0} = (1.56 \pm 0.06) \) ps [8] one obtains

\[
|V_{ub}|^{B \to \pi} = 0.0039 \pm 0.0005_{\text{exp}} \pm 0.0005_{\text{th}}.
\]

where we indicate the theoretical and experimental uncertainty.

6. **Outlook.** Here we stress that the light cone sum rule gives a reliable estimation for the \( f^+(p^2) \) form factor. The present accuracy of the result is estimated to be 15–20\% and can be improved up to 10\% by including the unknown perturbative \( O(\alpha_s^2) \) correction to twist-2 and the \( O(\alpha_s) \) correction to the twist-3 contributions and by more accurate extraction of the pion wave functions from the data.

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Figure 1: Light-cone sum rule estimate for $f_B f^+(0)$ in leading twist 2 approximation as a function of the Borel parameter $M^2$: NLO (solid) in comparison to LO (dashed).

Figure 2: Momentum dependence of the form factor $f^+(p^2)$ in leading twist 2 approximation: LO (dashed) in comparison to NLO (solid).
Figure 3: Scale dependence of the light-cone sum rule estimate of $f_{Bf^+(0)}$ in leading twist 2 approximation: NLO (solid) in comparison to LO (dotted).