Notes on Soliton Bound-State Problems in Gauge Theory and String Theory

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Abstract: We review four basic examples where string theory and/or field theory dualities predict the existence of soliton bound-states. These include the existence of threshold bound-states of D0 branes required by IIA/M duality and the closely-related bound-states of instantons in the maximally supersymmetric five dimensional gauge theory. In the IIB theory we discuss \((p,q)\)-strings as bound-states of \(D\) and \(F\) strings, as well as the corresponding bound-states of monopoles and dyons in \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory whose existence was predicted by Sen. In particular we consider the \(\mathcal{L}^2\)-index theory relevant for counting these states. In each case we show that the bulk contribution to the index can be evaluated by relating it to an instanton effect in the corresponding theory with a compact Euclidean time dimension. The boundary contribution to the index can be determined by considering the asymptotic regions of the relevant moduli space.

Keywords: Solitons, D-branes, Supersymmetry.
1. Introduction

Dualities in supersymmetric field theory and string theory, which relate strong and weak coupling, generally provide few predictions which can be tested against existing knowledge. However, theories with extended supersymmetry typically have BPS states which form short representations of the supersymmetry algebra. Consequently, in favourable conditions, the BPS mass spectrum is stable under variations of the coupling constant and the existence of BPS states required by duality can often be tested directly at weak coupling.

Two key examples of this phenomenon arise in Type II string theory on flat ten-dimensional space. The first example arises in the IIA theory. Here, the duality between IIA string theory and M-theory compactified on a circle [1], requires the presence of an infinite tower of BPS states corresponding to Kaluza-Kein modes of the eleven-dimensional metric. These states are realized as threshold bound states of an arbitrary number of D0-branes. The second example occurs in IIB string theory. The IIB theory is believed to have an exact $SL(2, \mathbb{Z})$ duality, known as S-duality, which acts by modular transformations on the complexified IIB coupling $\tau_{\text{IIB}}$. The action of S-duality on the fundamental string leads to a prediction for the existence of an $SL(2, \mathbb{Z})$ multiplet of BPS saturated strings [2]. These states can be thought of as bound-states of $q$ fundamental strings with $k$ D-strings. The $SL(2, \mathbb{Z})$ orbit of the fundamental string contains states corresponding to all coprime pairs $q$ and $k$. When $q$ and $k$ have a common factor $l$, a $(q,k)$-string would be exactly at threshold for decay into $l$, $(q/l,k/l)$ strings. In fact various arguments suggest that there are no normalizable bound-states with non-coprime values of the charges. In the following, we will call these two bound state problems the “IIA” and “IIB” examples, respectively.

Both these examples have close analogs in supersymmetric gauge theories with sixteen supercharges. Most famously, Montonen-Olive duality [3] of $\mathcal{N} = 4$ SUSY Yang-Mills theory with gauge group $SU(2)$ (or $U(2)$), predicts an infinite spectrum of BPS saturated dyons with coprime integral electric and magnetic charges, $q$ and $k$. As in the IIB example, bound-states with non-coprime values of $q$ and $k$, which would be exactly at threshold, do not occur. The connection to IIB string theory is that the $\mathcal{N} = 4$ theory with gauge group $U(N)$ can be realized on the world-volume of $N$ D3-branes in the IIB theory. In this context Montonen-Olive duality of the $U(2)$ gauge theory is mapped onto S-duality of the IIB theory. The $(q,k)$-dyons of the $U(2)$ theory are realized as segments of $(q,k)$-strings stretching between the two D3-branes.

More recently a similarly precise gauge theory analog of the problem of D0-brane bound-states in the IIA theory has emerged. Self-dual configurations of a non-abelian gauge field are most familiar as instantons of finite action in four spacetime dimensions, but they also appear as topologically stable solitons in five dimensional gauge theory. A recent conjecture relates the strong coupling limit of the five dimensional gauge theory with sixteen supercharges to a
theory with chiral $\mathcal{N} = (2, 0)$ supersymmetry in six dimensions [4, 5]. The appearance of the additional sixth dimension is analogous to the appearance of the eleventh direction in M-theory. The Kaluza-Klein modes of the six-dimensional fields correspond to threshold bound-states of Yang-Mills instantons thought of as solitons in five dimensions. Again, the connection to string theory emerges naturally when we realize the gauge theory on the world volume of a D-brane. In this case, the five dimensional theory with gauge group $U(N)$ appears on the world volume of $N$ D4-branes. In this context, Yang-Mills instantons appear as D0-branes which live on the D4-brane world-volume. The required instanton bound-states then correspond directly to the threshold bound-states of D0-branes in the IIA theory.

Hence, the two gauge theory examples in five and four dimensions, respectively, are realized on the world volumes of additional D-branes which act as “spectators” to the basic the IIA/B bound-state problems described above. In the following, we will use this perspective extensively. To avoid confusion we will refer to the five- and four-dimensional gauge theory examples, and their realizations in Type II string theory, as the IIA′ and IIB′ examples, respectively. In all of these examples, duality predicts the existence of BPS saturated bound-states of solitonic extended objects. The natural way to study this phenomenon at weak coupling is to analyse the effective theory which lives on the world-volume of the soliton. After factoring our the center-of-mass (COM) degrees-of-freedom, the required multi-soliton bound-states correspond to normalizable supersymmetric ground states of this world-volume theory. In the IIA/B examples, we have $k$ D-branes in a ten-dimensional spacetime and the world-volume theory is a $U(k)$ supersymmetric gauge theory with sixteen supercharges. In the IIA′/B′ cases, the higher dimensional D-branes introduce matter in the fundamental representation of $U(k)$, breaking half the supersymmetry of the world-volume theory of the lower dimensional D-branes.

For gauge theory solitons, the most familiar formulation of the effective world-volume theory is as a quantum mechanical $\sigma$-model with the soliton moduli space as the target space. This is essentially a supersymmetric version of Manton’s moduli space approximation [6] for soliton dynamics. On the other hand, the fact that the gauge theory solitons of the IIA′/B′ examples can be realized as D-branes means that the corresponding world-volume theories should also have a description as a supersymmetric gauge theory. The connection between the two points-of-view is now well established: for $k$ gauge theory solitons, the standard rules of D-brane calculus yield a description of the world-volume dynamics as a quantum mechanical $U(k)$ gauge theory with eight supercharges. In the IIA′ example, the relevant $D$-term equations reproduce the non-linear constraints of the ADHM construction and the ADHM moduli space of Yang-Mills instantons emerges as the Higgs branch of the world-volume gauge theory [7–9]. In the IIB′ example, the Nahm construction of moduli space of $k$ BPS monopoles emerges in an analogous way [10] as the Higgs branch of an impurity theory [11]. In both cases, on the Higgs branch, the $U(k)$ gauge coupling is irrelevant and the world-volume gauge theory flows

1In the following, we will use the word “soliton” loosely, to describe all of these objects.
in the IR to a non-linear $\sigma$-model with the Higgs branch as the target space. In other words, the conventional moduli space description of the solitons is recovered at low energy. This is consistent with the preservation of exactly eight supercharges, because the moduli spaces of gauge theory monopoles and instantons are non-trivial hyper-Kähler manifolds. In contrast, for the basic IIA/B examples, the preservation of sixteen supercharges requires the moduli space metric to be flat.

In each of the cases described above, the basic problem is to find normalizable supersymmetric ground-states of the world-volume theory which describes the relative motion of two or more solitons. A standard strategy to prove the existence of supersymmetric ground-states is to compute the Witten index \[ I = \text{Tr} (-1)^F e^{-\beta \mathcal{H}}. \] If the index is equal to a positive integer $l$, then the existence of at least $l$ supersymmetric ground-state is guaranteed. Note, however, that the index does not prevent the existence additional vacua which come in bose/fermi pairs. As with other quantum mechanical systems in non-compact spaces the problem of calculating the index is complicated by the fact that $\mathcal{H}$ has a continuous spectrum of scattering states. In cases involving soliton bound-states at threshold, the problem is even more severe as there is no mass gap between the required SUSY ground state and the continuum. These problems necessitate carefully defining the so-called $L^2$-index of $\mathcal{H}$ which correctly counts normalizable ground-states. As we review below, the $L^2$-index $\mathcal{J}_{L^2}$ is the sum of a bulk contribution $\mathcal{J}_{\text{bulk}}$ which is the integral of an index density over the whole configuration space and a “defect” term $\mathcal{J}_{\text{boundary}}$ which comes from an integral over the boundary at infinity [13, 14].

One of the main common features of all the bound-state problems described above is that the index can be evaluated by exploiting the general connection between solitons in $d$-dimensional spacetime and instantons in $d-1$ dimensions. In string theory, this connection is just a special case of a more general phenomena: a finite action instanton can be obtained by wrapping the world-volume of a $p$-brane around a non-contractable $(p+1)$-dimensional cycle in a spacetime of Euclidean signature [15]. The wrapped object can be any BPS brane or BPS bound-state of such branes. Multiple wrappings of the brane world-volume must also be included with appropriate weights. If we compactify the basic IIA example on $\mathbb{R}^9 \times S^1$, then each threshold bound-state of D0-branes has a Euclidean worldline which can wind around the compact direction an arbitrary number of times. After T-duality this yields a series of D-instanton corrections to the $\mathcal{R}^4$ term in the IIB effective action. Via this connection Green and Gutperle [16] showed that the coefficient of the $k$ D-instanton contribution is essentially equal to the bulk contribution to the $L^2$-index for the binding of $k$ D0-branes. However, the $\mathcal{R}^4$ term is also highly constrained by supersymmetry and by the S-duality of the IIB theory [17] and can be determined exactly. The weak coupling expansion of the exact result can then be used to derive the bulk contribution to the index for each value of $k$.

A wide variety of exact results for string theory effective actions can be understood in
this way (for a review see [18]). In particular, a simple set of rules for D-brane instanton calculus have been deduced from the constraints of U-duality [19]. One feature of these rules is that the contributions of multi-D-brane bound states appear on an equal footing with those of single D-branes. In this respect D-brane bound-states are “pointlike”, showing no indications of substructure. The same rules can also be applied to the IIA′/B′ examples where the D-branes have an alternative description in terms of gauge theory solitons and instantons. One of the aims of the present work (and of [20, 21]) is to investigate how these rules emerge from a conventional semiclassical analysis of the corresponding gauge theory. In particular, we will find that gauge theory instanton calculations reproduce characteristically D-braney features such as the contribution of pointlike bound-states and of multiple winding sectors.

In gauge theory, there is already a familiar relation between solitons and instantons. We will start from a classical, static, BPS saturated $k$-soliton configuration with finite mass $kM$ in a $d$-dimensional Lorentzian gauge theory. We then Wick rotate and compactify the resulting Euclidean time dimension on a circle of circumference $\beta$ with SUSY preserving boundary conditions. Now the static soliton becomes an instanton of finite Euclidean action $\beta kM$ in the compactified theory. Because of the BPS property, the resulting instanton will be invariant under half the supersymmetry generators. In a supersymmetric theory with $4N$ supercharges the instanton will then have $2N$ exact fermion zero modes\(^2\) and will contribute to a correlation function with the same number of fermionic insertions. The key point is that each correlation function of the compactified theory can be interpreted as a trace over the BPS sector of the Hilbert space of the original theory [21]:

$$\langle \psi(x_1)\psi(x_2)\ldots\psi(x_{2N}) \rangle = \text{Tr}_{\text{BPS}} \left[ \psi(x_1)\psi(x_2)\ldots\psi(x_{2N})(-1)^F e^{-\beta H} \right].$$

(1.1)

The correlators can therefore be thought of as refinements of the Witten index. Just as in favourable circumstances, the Witten index only gets non-zero contributions from zero energy states, so these correlators only get contributions from BPS states. Similar phenomena in two dimensional QFT have previously been discussed in [22].

A BPS saturated $k$ soliton bound-state of mass $kM$, contributes a term of order $\exp(-\beta kM)$ to the trace on the right-hand side of this equality. In the path integral language, this term can be identified with the contribution of the corresponding instanton in the compactified theory. In the semiclassical limit, we can express the overall coefficient of the contribution as an integral over the soliton moduli space. The main result, which applies equally in the string and gauge theory examples, is that the resulting integral is essentially equal to the bulk contribution to the $L^2$-index of the corresponding bound-state problem in the original $d$-dimensional theory. On the other hand the $2N$-fermion correlator appearing in (1.1) can also be related to a local vertex of the form $\bar{\psi}^{2N}$ in the effective action of the compactified theory. As in the IIA example,

\(^2\)These are the so-called supersymmetric zero modes. If all the vacuum expectation values are turned off an instanton can have additional $2N$ superconformal fermion zero modes.
the low-dimensional terms in this effective action are highly constrained by supersymmetry can often be determined exactly. In all the examples we will consider, obtaining the exact answer requires some additional physical input from duality. Finally, the coefficients of instanton expansion of the exact result then provide a definite prediction for bulk contribution to the $\mathcal{L}^2$-index which counts multi-soliton bound-states.

The main goal of these notes is to show how a unified picture of $\mathcal{L}^2$-index theory relevant to each of these $k$ soliton bound-state problems emerges. Strictly speaking, we will make only a little progress towards a first principles calculation of the BPS spectrum. However, numerous consistency checks between seemingly different manifestations of duality in string theory and gauge theory will appear. Before describing the details we list the main results:

1: In each case, the bulk contribution to the $\mathcal{L}^2$-index can be expressed as the partition function of a $U(k)$ matrix model with certain COM degrees-of-freedom factored off. In the IIA'/B' problems the bulk contribution can also be represented as the integral of an Euler density over a smooth moduli space (after a suitable resolution of singularities in the IIA' case).

2: In each case, we find that the bulk contribution can be calculated indirectly by relating it to a corresponding instanton contribution to the Wilsonian effective action of a compactified theory. We review the exact results for instanton contributions which yield predictions for the bulk contributions to the corresponding bound-state problem. In each case the bulk contribution has the interpretation as a sum over wrapped brane world-volumes. The bulk contribution overcounts the $\mathcal{L}^2$-index precisely by including the contribution of sectors of multiple D-brane wrapping.

3: In the gauge theory examples, we can relate the bulk contribution to the $\mathcal{L}^2$-index to the geometric Euler characteristic of the moduli space using classical index theory for a manifold with boundary. We find agreement with recent results on the homology of instanton and monopole moduli spaces due to Nakajima [23] and to Segal and Selby [24], respectively. Our results can be checked very explicitly in the two soliton sector.

4: The defect term is only sensitive to the asymptotic region of the moduli space where it can be calculated without knowing the details of the interactions between solitons. We will review and expand on a heuristic argument due to Yi [14], which is effective in determining the defect term in each case. In each case the total $\mathcal{L}^2$-index is consistent with the predictions of duality discussed above.

In the remainder of this section, we discuss some general features of the problems in question. In §2 we review most of the elements described above in the context of the best under-
stood string theory example: the problem of counting threshold bound-states of D0-branes in IIA string theory on $\mathbb{R}^{9,1}$. The discussion is then extended to the other cases in the remaining sections.

In each of the examples we will consider the world-volume dynamics of $k$ D-branes. Although the dimension of the D-branes in question will vary the common feature is that, in each case, all the spacelike dimensions of the world-volume will be compact. Initially we will work in a spacetime of Lorentzian signature and the single time-like dimension on the world volume will be non-compact. Thus in all cases the effective world-volume dynamics at low energy will be described by a quantum-mechanical $U(k)$ gauge theory with adjoint, and for the IIA'/B' cases fundamental, matter. The eigenvalues of the adjoint-valued scalar fields describe the positions of the $k$ D-branes in their common transverse directions. In the basic IIA/B examples there are no additional branes and the corresponding world-volume theory has sixteen supercharges. These gauge theories have a Coulomb branch of gauge equivalent classical vacua corresponding to the configuration space of the $k$ branes. Since there is fundamental matter for the IIA'/B' examples, there are Coulomb, Higgs and mixed branches arising in the gauge theory examples with eight supercharges.

In all the cases we will consider, the classical flat directions cannot be lifted by quantum corrections because of supersymmetry. In a four-dimensional gauge theory, the corresponding interpretation is that a moduli space of gauge-inequivalent vacua persists at the quantum level. However in quantum mechanics, or in (1+1)-dimensional field theory, this language is not really appropriate as the ground-state wavefunctions spread out over the manifold of classical vacua. Nevertheless it has been established that the distinction between the Higgs and Coulomb phases persists once quantum corrections are taken into account [25–27]. Moreover, just as in higher dimensions, the flat directions are associated with massless modes and we can write down a Wilsonian effective action for these degrees-of-freedom and order the terms according to the number of time derivatives. As in the higher dimensional cases, supersymmetry naturally pairs these time derivatives with the world-volume fermions, order by order in this expansion.

The time derivatives of the scalar fields which parametrize the classical moduli space correspond to the velocities, $v_i$, $i = 1, \ldots, k$, of the $k$ D-branes. The effective action of the $U(k)$ gauge theory then has a non-trivial expansion in powers of these velocities. The first non-trivial term in the velocity expansion has the form $\frac{1}{2} g_{ij} v^i v^j$ where the tensor $g_{ij}$ defines a metric on the manifold of classical vacua $\mathcal{M}_{cl}$. The supersymmetric completion of this term defines a quantum mechanical supersymmetric non-linear $\sigma$ model with target space $\mathcal{M}_{cl}$. In the basic IIA/B cases, the effective theory has sixteen supercharges which forces the metric to be flat. In the IIA'/B' examples, the effective theory has only eight supercharges and a non-trivial hyper-Kähler metric is allowed. In these cases, the off-diagonal terms in the metric describe the leading velocity-dependent interactions between gauge-theory solitons. The order
$v^2$ approximation to the effective world-volume action coincides with Manton’s moduli-space approximation for soliton dynamics [6].

2. The IIA Bound-State Problem

In the IIA case discussed above, the motion of $k$ D0-branes is described by a quantum mechanical $U(k)$ gauge theory with nine adjoint-valued scalar fields and sixteen adjoint-valued fermions. The eigenvalues of the adjoint scalars describe the relative position of the D0-branes in their nine transverse directions. Along these flat directions, non-zero values of the scalar fields break the gauge group down to its Cartan subalgebra and we have a manifold of gauge-in inequivalent classical vacua. This Coulomb branch is just $\text{Sym}_k(\mathbb{R}^9)/(\mathbb{R}^9)^k/S_k$ where the symmetric product in the numerator is simply the configuration space of $k$ identical particles moving in nine dimensional space. The permutation group $S_k$, representing the interchange symmetry of $k$ identical particles, corresponds to the Weyl subgroup of the $U(k)$ gauge group. An overall factor of $\mathbb{R}^9$ in the numerator corresponds to the COM degrees-of-freedom of the D0-brane configuration. After separating out these modes, the relative motion of the branes is described by the corresponding theory with gauge group $SU(k)$. The main problem is to show that this theory has a single normalizable supersymmetric ground-state for each value of $k$.

We begin by investigating the low energy dynamics of the D0-branes in the velocity expansion described above. The first non-trivial terms in the velocity expansion define a quantum mechanical non-linear $\sigma$-model with target space $\text{Sym}_k(\mathbb{R}^9)/(\mathbb{R}^9)^k$. As we have sixteen supercharges, quantum corrections to the classical flat metric on the Coulomb branch are forbidden. As there are no off-diagonal terms in the metric the interactions between D0-branes vanish at order $v^2$. In fact the first non-trivial interactions between D0-branes occur at order $v^4$. The supersymmetric completion of these interactions involves terms with up to eight fermions. Note that the target space has orbifold singularities at the fixed points of the Weyl group where one or more D0-branes coincide. Just as in higher dimensions, a non-abelian subgroup of the gauge group is restored at these points and the low energy description breaks down due to the presence of additional massless fields. In the D0-brane case, the presence of singularities mean that the velocity expansion is only reliable for describing the scattering of D0-branes at large impact parameter. There is no reason to expect this expansion to be adequate for determining the existence or otherwise of D0-brane bound-states. Indeed, as interactions between D0-branes vanish at order $v^2$, it’s obvious that we will not find any bound-states in a naïve “moduli space” approximation. More generally, one expects that the behaviour of the wavefunction near the origin depends on the full non-abelian dynamics.

As mentioned above, the standard approach to determining the presence of supersymmetric
ground-states is to compute the Witten index \( I = \text{Tr} (-1)^F e^{-\beta H} \). The utility of the Witten index comes from the fact that, modulo some caveats to be discussed below, it is invariant under all deformations of the theory which preserve supersymmetry. To avoid the possibility of vacua going to infinity in field space as the parameters are varied, we should restrict our attention to variations which do not alter the asymptotic behaviour of the potential. Such deformations can often be used to reach a regime in parameter space where the index can be computed easily. In particular one may use the \( \beta \)-independence of the index to take the limit \( \beta \to 0 \), in which the quantum mechanical path integral reduces to an ordinary integral. Generally speaking, this procedure is extremely robust whenever the spectrum of the theory is discrete or, at least, has a finite mass gap.

As discussed above, the D0-bound-states in question are exactly at threshold. Correspondingly, the \( SU(k) \) world-volume theory has flat directions which are not lifted by quantum effects and the D0-brane moduli space is non-compact. The main problem introduced by non-compactness is the fact that the theory has a continuous spectrum of scattering states in addition to the discrete bound state spectrum. In particular, even scattering states of non-zero energy can actually contribute to the Witten index. Na"ively states of non-zero energy come in bose-fermi pairs which cancel in the Witten index due to the insertion of \( (-1)^F \) appearing in the trace. However, although supersymmetry demands that the range of the continuous spectrum is the same for bosons and fermions, it does not necessarily require the density of these states to be equal. A difference between the densities of bose and fermi scattering states of non-zero energy can then lead to anomalous \( \beta \) dependence of the index [28]. This contribution of states of non-zero energy can be eliminated by defining the desired index as the \( \beta \to \infty \) limit of \( I(\beta) \). However, we can no longer use the \( \beta \to 0 \) limit to compute the index.

In the present case, the problems related to non-compactness persist even in the \( \beta \to \infty \) limit because there is no mass gap between the required supersymmetric ground-states and the continuum. However, a ground-state corresponding to a D-brane bound-state must have a normalizable wavefunction and this distinguishes it from the continuum of scattering states which are non-normalizable. Hence the trace appearing in the Witten index should be restricted to states in the Hilbert space which have square-integrable wavefunctions. In this case the Witten index coincides with what is known in functional analysis as the “\( L^2 \)-index” of the Hamiltonian. We will now review the approach to computing the \( L^2 \)-index developed by Sethi and Stern [13] and by Yi [14].

A sensible definition of the \( L^2 \)-index requires us to regulate the problem of non-compactness before taking the \( \beta \to \infty \) limit. This can be accomplished by restricting the configuration space to the interior of a ball \( B_R \) of finite radius \( R \). Typically the regulated index will depend on \( \beta \) via the dimensionless parameter \( \kappa = \beta / R \). In the case of \( k \) D0-branes of the IIA theory, the configuration space is the classical Coulomb branch of the \( SU(k) \) gauge theory on the world-
volume, Sym(\(\mathbb{R}^9\))/\(\mathbb{R}^9\) and the ball just the subspace where the distance of any brane from the centre of mass is less than \(R\). The regulator \(R\), should only be taken to infinity after taking the \(\beta \to \infty\) limit. Note that this corresponds to the limit \(\kappa \to \infty\). In contrast, removing the regulator at finite \(\beta\) corresponds to the opposite limit \(\kappa \to 0\). Thus we have

\[
J_{L^2} = \lim_{R \to \infty} \lim_{\beta \to \infty} \int_{B_R} d^p x \text{Tr} \left( -1 \right) F e^{-\beta\kappa}(x, x). \tag{2.1}
\]

Here the notation \((x, x)\) denotes that the Trace is evaluated between position space eigenstates centered at the point \(x\) in \(B_R\).

Sethi and Stern then showed that the resulting index can be written as the sum of bulk and boundary contributions: \(J_{L^2} = J_{\text{bulk}} + J_{\text{boundary}}\) with

\[
J_{\text{bulk}} = \lim_{R \to \infty} \lim_{\beta \to 0} \int_{B_R} d^p x \text{Tr} \left( -1 \right) F e^{-\beta\kappa}, \tag{2.2a}
\]

\[
J_{\text{boundary}} = \lim_{R \to \infty} \lim_{\beta \to 0} \int_{\partial B_R} d^{p-1} x \int_{\beta}^{\infty} d\beta' \text{Tr} e_n(-1)^F Q e^{-\beta\kappa}, \tag{2.2b}
\]

where \(Q\) denotes the supercharge and \(e_n\) is a fermion component in the direction normal to the boundary, \(\partial B_R\), of \(B_R\). For more details see Section 3 of [13].

Following Green and Gutperle [16], the bulk contribution to the index can be calculated by determining a corresponding instanton contribution to the effective action. We now compactify the Euclidean IIA theory down to nine dimensions on a spacelike circle of circumference \(\beta\). The D0-bound-states of the ten-dimensional theory become instantons of finite action after compactification. In particular, the relevant instanton configurations involve D0-brane bound-state with worldlines wrapped around the compact dimension. The BPS configurations are those which involve a single worldline and are labelled by an integer winding number \(l\) as well as the number, \(m\), of constituent D0-branes. When \(\beta \ll \sqrt{\alpha'}\), it is appropriate to perform T-duality to the IIB theory on the dual circle. Under a T-duality transformation of the compact dimension, the wrapped D0-brane worldlines of the IIA theory become D-instantons of the IIB theory. These D-instantons have sixteen unlifted zero modes and contribute at leading semiclassical order to a sixteen fermion vertex which is contained in the supersymmetric completion of the \(\mathcal{R}^4\) term in the IIB effective action. On the one hand, the D-instanton contributions can be related to the bulk contribution to the \(L^2\)-index considered above. On the other hand, the corresponding term in the IIB action can be determined exactly using supersymmetry and the \(SL(2,\mathbb{Z})\) invariance of the IIB theory [17]. Briefly, supersymmetry restricts that the coupling constant dependence to be harmonic and S-duality means that we should therefore solve the Laplace equation on the fundamental domain of \(SL(2,\mathbb{Z})\). As usual for harmonic functions, the desired solution is uniquely determined by specifying the boundary conditions (as well as the absence of singularities). In the present case the boundary conditions are determined by a
tree level calculation at weak coupling. Note that, in this approach, we are assuming S-duality and the final result which will be consistent with the existence of the D0-brane bound-states required by IIA/M duality should simply be regarded as a consistency check on the “web of dualities”.

Finally, Green and Gutperle obtained the explicit prediction,

\[ J_{\text{bulk}} = \sum_{d|k} \frac{1}{d^2}. \] (2.3)

As the bulk piece corresponds to the \( \beta \to 0 \) limit of the Witten index, an explicit formula may be obtained by reducing the partition function of the quantum mechanical gauge theory to zero dimensions. For \( k \) D0-branes, this yields an ordinary integral over traceless \( k \times k \) matrices modulo \( SU(k) \) gauge transformations. Initially this integral was evaluated explicitly for the case of two D0-branes in [13, 14], yielding results in agreement with (2.3). Subsequently the integral was considered for all values of \( k \), by Moore, Nekrasov and Shatashvilli [29], again confirming (2.3) (see also [30]).

The formula (2.3) has a nice interpretation in terms of the original IIA theory on a circle. The term corresponding to the divisor \( d \), corresponds to the contribution of a bound-state of \( l = d \) D0-branes with world-line wrapped \( m = k/d \) times round the circle. This suggests (correctly) that the first term in the sum is the contribution of the desired bound-state of \( k \) D0-branes wrapped only once around the compact dimension. What remains to be explained is how the boundary contribution subtracts out the spurious additional terms as well as the origin of the weighting factor \( 1/d^2 \) for each term in the sum.

The boundary contribution is hard to analyse in a precise way. The complexity of the problem is due to the fact that the boundary contains “clustering regions” where the separation between one or more subset of the \( k \) D0-branes remains small. This problem is absent for \( k = 2 \), and first appears in the \( k = 3 \) case where the boundary contains a region where two D0-branes are far from the third one but remain close to each other. Despite this problem, there is a heuristic argument due to Yi [14] (and subsequently developed by Green and Gutperle [16]) which is effective in determining the boundary contribution which we will now review. In the process we will expand the original argument and place it in a more general setting which will apply to the other bound-state problems we wish to consider.

The basic argument is tantamount to the usual assumption in quantum mechanics (and quantum field theory) that we can define asymptotic scattering states. Essentially, this means that we can split the Hamiltonian of the system as: \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \). Here \( \mathcal{H}_0 \) is an appropriate free Hamiltonian. The asymptotic particle states are eigenstates of \( \mathcal{H}_0 \) and \( \mathcal{H}_{\text{int}} \) accounts for the interactions between these states which are assumed to have finite range. This is reasonable
in the case of D0-branes as the large distance interactions between them are supressed by four powers of the velocity. The velocity of states of fixed momentum is very small in the weak coupling limit because their masses scale with an inverse power of the coupling. In the first instance the asymptotic particle states will be the D0-branes themselves, however we will subsequently allow for the fact that bound-states of D0-branes can also appear as asymptotic states using an inductive argument.

The idea is that the boundary conribution to the index comes from the asymptotic region where the interactions between the D0-branes (and any other asymptotic states) can be neglected. It can therefore be calculated with respect to free Hamiltonian, $\mathcal{H}_0$. Thus, we have

$$I^{(0)}_\text{boundary} = \lim_{R \to \infty} \lim_{\beta \to 0} \int_{\partial B_R} d^{p-1}x \int_{\beta}^{\infty} d\beta' \text{Tr} e^{\beta} Q e^{-\beta' \mathcal{H}_0} .$$

(2.4)

As the free Hamiltonian certainly has no bound-states we have, $I^{(0)}_\mathcal{L}^2 = I^{(0)}_\text{bulk} + I^{(0)}_\text{boundary} = 0$. Consequently we have, $I^{(0)}_\text{boundary} = -I^{(0)}_\text{bulk}$ where,

$$I^{(0)}_\text{bulk} = \lim_{R \to \infty} \lim_{\beta \to 0} \int_{B_R} d^p x \text{Tr} (-1)^F e^{-\beta \mathcal{H}_0} .$$

(2.5)

Neglecting the interactions D0-branes are simply massive identical non-relativistic particles moving in $\mathbb{R}^9$. Generalizing slightly, we will evaluate (2.5) in the case where, $\mathcal{H}_0$ is the supersymmetric Hamiltonian of $k$ identical free massive particles moving on $\mathbb{R}^m$ in the center of mass frame. The bosonic degrees of freedom are therefore real variables $X^i_a$ where $a = 1, 2, \ldots, b$ is a vector index of the $SO(b)$ R-symmetry group and $i = 1, \ldots, k$ is an additional index labelling the $k$ particles. This system is supersymmetrized by including $k$ fermions corresponding to the fermionic zero modes of D0-branes. These Grassmann variables are denoted, $\Psi^i_\alpha$, each with $f$ real components where $\alpha = 1, 2, \ldots, f$ is an appropriate spinor index of the R-symmetry group. In the case of D0-branes the relevant values of $b$ and $f$ are 9 and 16, respectively, but we will meet other cases below.

The relevant Hamiltonian is the sum of free Hamiltonians for Bosons and Fermions: $\mathcal{H}^0 = \mathcal{H}_X^0 + \mathcal{H}_\Psi^0$. However, we still need to account for the interchange symmetry of the identical particles. Specifically we must mod out by all permutations $\pi \in S_k$ which act on the bose and fermi degrees of freedom via the $k \times k$ representation matrix $M_\pi$:

$$\pi: \quad X^a_i \to (M_\pi)^a_j X^a_j , \quad \Psi^\alpha_i \to (M_\pi)^\alpha_j \Psi^\alpha_j .$$

(2.6)

To mod out the interchange symmetry we must insert a projector onto $S_k$ invariant states inside a trace over the full Hilbert space.

$$I^0_\text{bulk} = \lim_{R \to \infty} \lim_{\beta \to 0} \int_{\mathbb{R}^9k} d^9k x \text{Tr} (-1)^F \mathcal{P} e^{-\beta (\mathcal{H}_X^0 + \mathcal{H}_\Psi^0)} .$$

(2.7)
with
\[ P = \frac{1}{k!} \sum_{\pi \in S_k} M_\pi, \]  \tag{2.8}

where \( M_\pi \) acts on the bose and fermi variables as in (2.6). This sum over permutations has a very simple physical interpretation in terms of particle worldlines. Recall that an arbitrary permutation of \( k \) objects admits a decomposition into a product of mutually commuting cyclic permutations of different lengths, \( l_1, l_2, \ldots, l_r \) with \( \sum_{j=1}^r l_j = k \). A cyclic permutation of length one is trivial and corresponds to the worldline of a single particle brane winding around the compact dimension. The identity element of \( S_k \) corresponds to a product of cycles each of length one: \( l_j = 1 \) for \( j = 1, \ldots, k \). This corresponds to \( k \) particles each with a worldline which wraps the compact dimension once. In contrast a cyclic permutation of length \( l > 1 \) corresponds to a single particle with worldline wrapped around the compact dimension \( l \) times. A generic element of \( S_k \) specified by integers \( l_j \) with \( \sum_{j=1}^r l_j = k \) above corresponds to a configuration with \( r \) particles where the worldline of the \( j \)th particle wraps the compact dimension \( l_j \) times.

The traces (2.7) are elementary to evaluate via a path integral representation in the \( \beta \to 0 \) limit and yield,
\[ I_{\text{bulk}}^0 = \frac{1}{k!} \sum_{\pi \in S_k} \left( \det \left[ 1 - M_\pi \right] \right)^{\frac{1}{2} f - b}. \]  \tag{2.9}

As explained in [16], the only non-vanishing contribution to the sum comes from the \((k - 1)\)! cyclic permutations of length \( k \), for which \( \det \left[ 1 - M_\pi \right] = k \). Thus the states with worldline wrapped \( k \) times around the compact dimension contribute to the partition function. The natural interpretation is that multi-particle states have additional exact fermion zero modes beyond those associated with the COM degrees-of-freedom which we have already modded out. We obtain, \( I_{\text{bulk}}^0 = k^{\frac{1}{2} f - b - 1} \). In the D0-brane case we have \( \frac{1}{2} f - b = 8 - 9 = -1 \) and thus we find that \( I_{\text{boundary}} = -I_{\text{bulk}}^0 = -k^{-2} \).

We have so far accounted only for the contributions of the D0-branes themselves. However, if bound-states of D0-branes exist it is natural to expect that these also appear as asymptotic states. In the semiclassical language, this corresponds to the asymptotic regions of the moduli space in which one has several “clusters” of D0-branes with a large distance between clusters. Obviously, we must beware of making a circular argument here as the presence of bound-states is precisely what we are trying to demonstrate! However, bearing this in mind, we may make the following inductive argument. Let us assume that a single threshold bound-state of \( k \) D0-branes is present in the spectrum for all \( k < k' \). We will now analyse the boundary contribution in the case \( k = k' \). By assumption, the possible asymptotic states correspond to all possible sets of D0-brane bound-states with total D0 brane number \( k = k' \). As before we treat the constituents as non-interacting particles and look for the non-zero contributions to the bulk
index. By the same reasoning used above, the only non-zero contributions come from cyclic permutations of all the particles. Such a permutation only occurs when all the particles present are identical. Thus the only non-zero contributions come from sectors with \( d > 1 \) identical bound-states, each with D0-number \( k'/d \). Thus, for \( k = k' \), we have,

\[
J_{\text{boundary}} = -J_{\text{bulk}}^{(0)} = 1 - \sum_{d|k} \frac{1}{d^2}.
\]  

(2.10)

As anticipated, the boundary term has precisely the effect of subtracting out the higher winding number sectors. Finally, using (2.3) we obtain \( J_{L^2} = J_{\text{bulk}} + J_{\text{boundary}} = +1 \). This value of the \( L^2 \)-index is consistent with the existence of exactly one bound-state at threshold for \( k \) D0-branes when \( k = k' \). As the result holds trivially for \( k = 1 \), we may now extend it to all values of \( k \) by induction.

3. The IIB' Bound-State Problem

In this section we will review the problem of finding BPS saturated monopole-dyon bound-states required by Montonen-Olive duality in \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory with gauge group \( SU(2) \) in four dimensions. Before discussing the relevant application of index theory, we review the standard formulation of the problem due to Sen [31].

In the moduli space approximation, the spectrum of BPS states with magnetic charge \( k \) can be determined by considering the spectrum of supersymmetric quantum mechanics on the classical moduli space of \( k \) BPS monopoles \([32, 33]\), \( \mathcal{M}_k \). This space has the isometric decomposition,

\[
\mathcal{M}_k = \mathbb{R}^3 \times \frac{S^1 \times \widehat{\mathcal{M}}_k}{\mathbb{Z}_k},
\]

(3.1)

where \( \mathbb{R}^3 \) is parametrized by the COM coordinates of the monopoles \( \vec{X} \) and \( S^1 \), parametrized by \( \theta \in [0, 2\pi] \), corresponds global charge rotations of the configuration. The relative degrees of freedom of the \( k > 1 \) monopoles are coordinates on a smooth, hyper-Kähler manifold, \( \widehat{\mathcal{M}}_k \), of real dimension \( d = 4(k - 1) \). The hyper-Kähler metric on \( \widehat{\mathcal{M}}_k \) is only known explicitly in the \( k = 2 \) case. The \( \mathbb{Z}_k \) reflects the invariance of a multi-monopole configuration under a \( 2\pi \) charge angle of each constituent monopole.

A bound-state in supersymmetric quantum mechanics is a state whose wavefunction is non-normalizable as a function of its COM coordinates. For example, a bosonic state of definite COM momentum \( \vec{P} \), has wavefunction which goes like \( e^{i\vec{P} \cdot \vec{X}} \) just like that of a particle without substructure. On the other hand, the part of the wavefunction which depends on the relative
coordinates must be normalizable. Furthermore, the quantum mechanics of the COM degrees-of-freedom saturate the BPS bound in the sector of charge \( k \) for each value of \( p \), which means that the relative component of the bound-state wavefunction does not contribute to the mass of the state. Hence the key question for determining the bound-state spectrum is the existence of normalizable zero energy states in supersymmetric quantum mechanics on \( \widehat{M}_k \). If we introduce real coordinates \( Y_q \) with \( q = 1, 2, \ldots, d = 4(k - 1) \) on \( \widehat{M}_k \) the effective action takes the form,

\[
S^{(k)} = \frac{1}{2} \int dt \left[ g_{pq} d_r Y^p d_s Y^q + g_{pq} i \bar{\alpha}^p \gamma^0 D_r \alpha^q + \frac{j}{12} R^{pqrs}(\bar{\alpha}^p \alpha^r)(\bar{\alpha}^q \alpha^s) \right], \tag{3.2}
\]

where \( \alpha_q \) are two-component fermionic superpartners to the coordinates \( Y_q \) with \( \bar{\alpha} = \alpha \gamma^0 \) and \( \gamma^0 = \sigma^2 \). Here, \( D_r \alpha^q = d_r \alpha^q + d_r Y^r \Gamma^p_{rs} \alpha^s \) and \( R^{pqrs} \) are the covariant derivative and Riemann curvature tensor formed from the hyper Kähler metric metric \( g_{pq} \).

The states we are looking for have a simple geometric interpretation which was first described by Witten [12]. As usual the wavefunctions of purely bosonic states just correspond to ordinary functions or zero forms on \( \widehat{M}_k \). The corresponding wavefunctions for states of definite fermion number \( F = r \) can be represented as differential forms of degree \( r \) on this manifold. Hence the Hilbert space of the quantum mechanics (3.2) is essentially the de Rham complex \( \Lambda^* = \oplus^d_{r=0} \Lambda^r \) of \( \widehat{M}_k \). It will also be useful to divide the Hilbert space into sectors of definite charge \( p \) under the Z\(_k\) appearing in the denominator of (3.1), thus e.g. \( \Lambda^{(k)}_{r,p} \) is the space of \( C^\infty \) \( r \)-forms, \( \omega_r \) on \( \widehat{M}_k \) which transform as \( \omega \rightarrow e^{2\pi ip/k} \omega \) under the generator of \( Z_k \). As \( Z_k \) acts non-trivially on the \( S^1 \) factor of the full moduli space \( M_k \), the \( Z_k \) charge of states in quantum mechanics on the relative moduli space \( \widehat{M}_k \) is correlated with the electric charge of the resulting BPS state. In fact a form with with \( Z_k \) charge \( p \) corresponds to a state with electric charge \( q = -p \mod k \).

By standard arguments the Hamiltonian of the quantum mechanical system described above is precisely the Laplacian \( \Delta \) acting on forms. And zero energy states, being zero modes of the Laplacian correspond to harmonic forms. The existence of the \( (q, k) \) dyons required by Montonen-Olive duality then requires the existence of a unique, normalizable, harmonic form \( \omega_r \) of \( \mu \) on \( \widehat{M}_k \) only when \( \langle p, k \rangle = 1 \).\(^4\) As the harmonic form must be unique it must be invariant under Hodge duality which maps \( r \) forms to \( d - r \) forms, preserving the \( Z_k \) charge and normalizability. This means that the resulting forms must be middle dimensional, ie they must be forms of degree \( r = d/2 = 2k - 2 \). They must also be (anti-)self-dual. As the hyper-Kähler metric on \( \widehat{M}_2 \) has been determined explicitly by Atiyah and Hitchin, the existence of the required normalizable, self-dual, harmonic two-form on this manifold can be established by a direct calculation [31]. The metric is unknown for \( k > 2 \) and the statement that the corresponding harmonic forms exist is known as Sen’s conjecture.

\(^3\)All notation is as in [34].

\(^4\)Here and in the following \( \langle m, n \rangle \) denotes the highest common divisor of two integers \( m \) and \( n \).
As described in the introduction, the problem of finding the \((q,k)\) dyons is directly related to that of finding BPS saturated \((q,k)\) strings in the IIB theory. In particular we can consider a configuration where \(k\) D-strings are stretched between two parallel D3-branes. As in all our examples, the world-volume dynamics of the D-strings are described by a \(U(k)\) gauge theory. The gauge theory is formulated on a finite spacelike interval between the two D3-branes and the effective theory is therefore quantum mechanical at low energy. The resulting theory includes adjoint scalar fields whose eigenvalues describe the positions of the \(k\) D-strings along the D3-world-volume. Dirichlet boundary conditions for the D-string prevent motion in the transverse directions and leave only eight unbroken supercharges. The world-volume theory also has multiplets in the fundamental representation of \(U(k)\) which are localised at the endpoints of the interval. Electric charges can be realized by including non-zero electric flux on the two dimensional world sheet.

The \(D\)-term equations of the world-volume theory are ordinary differential equations on the interval known as Nahm’s equations \([10]\). The boundary conditions at the endpoints of the interval constitute the appropriate Nahm data for the construction of exact \(k\)-monopole solutions in \(SU(2)\) gauge theory. The classical vacuum moduli space of the theory is obtained by solving the \(D\)-term differential equations and modding out by the \(U(k)\) gauge symmetry. This is a Higgs branch in the sense that the fields which acquire VEVs are adjoint hypermultiplets in the language appropriate for a theory with eight supercharges. This procedure yields the classical moduli space of \(k\) BPS monopoles, denoted as \(\mathcal{M}_k\) above, as an infinite dimensional hyper-Kähler quotient. At low-energy the effective theory flows to a quantum mechanical non-linear supersymmetric \(\sigma\)-model with target \(\mathcal{M}_k\). In other words, at low velocities we recover the standard moduli-space description of BPS monopoles in the \(\mathcal{N} = 4\) theory described above.

We will now briefly describe the elements necessary to discuss the \(L^2\)-index theory relevant to the Sen bound state problem. As above we must consider the \(L^2\)-index of the Hamiltonian of the \(SU(k)\) supersymmetric gauge theory which describes the relative motion of the D-strings. The new feature is that the gauge theory has a non-singular effective low-energy description in terms of SUSY quantum mechanics on \(\mathfrak{M}_k\) described by the Lagrangian \((3.2)\). Standard reasoning suggests that it is sufficient to determine the presence or otherwise of the required ground-states in the low energy theory. As before we expect that the \(L^2\)-index will include bulk and boundary contributions.

\[
\gamma_{L^2}^{(p,k)} = \gamma_{\text{bulk}}^{(p,k)} + \gamma_{\text{boundary}}^{(p,k)} \quad (3.3)
\]

As in the D0-brane example the bulk piece can be extracted from the effective action of the four dimensional theory compactified on a spacelike circle. As the \(k\)-monopole solution is Bogomolnyi saturated it yields an instanton with eight exact fermion zero modes. The corresponding eight fermion term in the effective action of the \(\mathcal{N} = 4\) theory on \(\mathbb{R}^3 \times S^1\) was
recently evaluated in [20, 21], building on previous work in the three-dimensional theory [35]. The elements which enter this calculation are very similar to those entering in the determination of the exact $\mathcal{R}^4$ in the IIB effective action described in the previous section. The requirements of supersymmetry force the eight fermion term to obey Laplace’s equation on the classical Coulomb branch. Again the uniqueness properties of harmonic functions can be exploited. In this case, the only additional assumption required is the existence of a $\text{Spin}(8)_{\mathcal{R}}$ invariant superconformal fixed point in three-dimensions. This is in turn a direct consequence of IIA/M duality [36]. The final result is, $\mathcal{I}_{p,k}^{\text{bulk}} = +1$ for all values of $p$ and $k$. Note that as in the IIA example the bulk contribution to the index overcounts by precisely the sectors where multiple winding can occur. In otherwords, if $\langle p, k \rangle = l > 1$ then we have an additional instanton contribution which comes from the worldsheet of a $(p/l, k/l)$-string stretched between the two D3-branes and wrapped $l$ times around the compact Euclidean time dimension. As in the IIA example, we will use Yi’s argument to show that the boundary contribution precisely subtracts out the contributions of the multiple winding sectors.

We will consider the possible asymptotic states in the sector of the theory with magnetic charge $k = k'$ and electric charge $q = -p \mod k$. As before we will assume that the Sen’s conjecture holds for all $k < k'$. The simplest case is that electric charge divisible by $k'$, $q = k's$. In this case we have $k'$ dyons of electric charge $s$. For the purposes of evaluating the boundary contribution we will treat these as free massive non-relativistic particles moving in three-dimensional space. As before the boundary contribution to the index of the free system is minus the bulk contribution: $\mathcal{I}_{p,k}^{\text{boundary}} = -\mathcal{I}_{p,k}^{\text{bulk}}$. The later contribution can be evaluated using (2.10). As we have particles moving in three-dimensional space now with eight superpartners the relevant values of $b$ and $f$ are 3 and 8 respectively, which yields $\mathcal{I}_{p,k}^{\text{boundary}} = -\mathcal{I}_{p,k}^{\text{bulk}} = -1$.

Now suppose we have non-zero electric charge $q = -p \mod k$ and that $k$ and $q$ have lowest common divisor $l$. If $l$ is one then there are no asymptotic states in this sector and $\mathcal{I}_{p,k}^{\text{boundary}} = 0$ If $l > 1$, we get an asymptotic state of $l$ identical dyons of charges $(k/l, q/l)$. Again we will treat these as $l$ identical particles moving in three dimensional space with eight fermionic superpartners. This yields, $\mathcal{I}_{p,k}^{\text{boundary}} = -1$. Hence finally we obtain,

$$\mathcal{I}_{\mathcal{L}^2}^{(p,k)} = \mathcal{I}_{\text{bulk}}^{(p,k)} + \mathcal{I}_{\text{boundary}}^{(p,k)} = \begin{cases} +1 & \langle p, k \rangle = 1 \\ 0 & \langle p, k \rangle > 1 \end{cases},$$

in accordance with Sen’s conjecture.

There is also an interesting relation between the above results and the topological properties of the moduli space of BPS monpoles. To see this we consider the $\mathcal{L}^2$-index of supersymmetric quantum mechanics on the moduli space $\hat{\mathcal{M}}_k$ without implementing the $\mathbb{Z}_k$ quotient. The relevant $\mathcal{L}^2$ Witten index counts all normalizable harmonic forms on $\hat{\mathcal{M}}_k$ regardless of $\mathbb{Z}_k$ charge and is equal to $\sum_{p=0}^{k-1} \mathcal{I}_{\mathcal{L}^2}^{(p,k)}$. We now consider the path integral formula for this index. This is
just a Euclidean path integral with action obtained from (3.2) by Wick rotation evaluated with periodic boundary conditions on the coordinates and their superpartners. Taking the $\beta \to 0$ limit we obtain the formula [37],

$$
\sum_{p=0}^{k-1} g^{(p,k)}_{\text{bulk}} = \frac{1}{(8\pi)^{d/2}(d/2)!} \int_{\hat{\mathcal{M}}_k} \varepsilon^{p_1 p_2 \ldots p_d} \varepsilon^{q_1 q_2 \ldots q_d} R_{p_1 p_2 q_1 q_2} \ldots R_{p_{d-1} p_d q_{d-1} q_d} .
$$

(3.5)

Thus the bulk contribution to the $L^2$-index can be represented as an integral of a $d = 4(k - 1)$ form density over the moduli space $\hat{\mathcal{M}}_k$. Our previous results imply that this quantity is equal to $k$. In the case $k = 2$, where the explicit metric on the moduli space was given by Atiyah and Hitchin one may evaluate this integral explicitly and confirm that it equals two [38]. To explain the significance of this result (for all $k$) we review some elementary facts about classical index theory which is distinct from the $L^2$-index theory described above in non-compact cases.

Classical index theory relates the zero mode spectrum of an elliptic differential operator to the topological invariants of the manifold on which it is defined. In the following we will be interested in the index theorem for the Laplacian $\Delta$ acting on differential forms. The theorem takes it easiest to state in the simple (but unrealistic) case of a compact, smooth, manifold without boundary, $\mathcal{M}$, of real dimension $d$. In this case the Laplacian will have a discrete spectrum and all its eigenfunction are normalizable. The classical index of $\Delta$ is identical to the $L^2$-index in this simple case.

We denote the restriction of the Laplacian on $\mathcal{M}$ to the forms of degree $r$, as $\Delta_r$. The number of linearly independent harmonic forms of degree $r$ is therefore $b_r = \dim \ker \Delta_r$. The integers $b_r$, for $0 \leq r \leq d$ are known as the Betti numbers of $\mathcal{M}$. By Hodges theorem the Betti number $b_r$ is also equal to the dimension of, $H^r(\mathcal{M})$, the $r^{th}$ cohomology group of $\mathcal{M}$, and are therefore purely topological (in otherwords they do not depend of the choice of metric on $\mathcal{M}$). The index, $\chi$ of the Laplacian $\Delta$ is defined as the alternating sum of the Betti numbers,

$$
\chi = \sum_{r=0}^{d} (-1)^r b_r .
$$

Thus $\chi$ is just the Euler characteristic of $\mathcal{M}$ and in this well behaved case we have

$$
\text{Tr}(-1)^F e^{-\beta \mathcal{E}} = \text{Ind} \Delta = \chi .
$$

(3.6)

The corresponding index theorem for the Laplacian is the Gauss-Bonnet-Chern (GBC) theorem,

$$
\chi = \text{Ind} \Delta = \int_{\mathcal{M}} e (T^*\mathcal{M}) ,
$$

(3.7)

where the Euler density $e (T^*\mathcal{M})$ is defined by

$$
e (T^*\mathcal{M}) = \frac{1}{(8\pi)^{d/2}(d/2)!} \varepsilon^{p_1 p_2 \ldots p_d} \varepsilon^{q_1 q_2 \ldots q_d} R_{p_1 p_2 q_1 q_2} \ldots R_{p_{d-1} p_d q_{d-1} q_d} .
$$

(3.8)
This is precisely the integral appearing on the RHS of (3.5). In fact, in the compact case, one may prove the Gauss-Bonnet theorem directly from $\beta \to 0$ limit of the path integral [37].

The relative moduli space of $k$-BPS monopoles is not compact and so the GBC theorem stated above does not apply. To regulate the problem one must replace the manifold $\hat{M}_k$ by a compact manifold $\hat{M}_k^{\text{cpt}}$ with boundary $\partial \hat{M}_k^{\text{cpt}}$ of finite volume $V$. For a review of classical index theory on a manifold with boundary see [39]. This could be done, for example, defining $\hat{M}_k^{\text{cpt}}$ as the submanifold of $\hat{M}_k$ where the distance between any pair of monopoles is less than or equal to a fixed length $R$. Although this sounds similar to the $L^2$-index introduced above there is an important difference: in this standard approach we also include finite-volume boundary conditions which render the spectrum of $\Delta$ discrete. As all states are normalizable in finite volume, the resulting index typically overcounts the number of normalizable groundstates by including states whose normalization diverges in the limit of infinite volume. The modified version of the Gauss-Bonnet theorem which holds under these conditions reads [39]

$$\chi = \text{Ind} \Delta = \int_{\hat{M}_k^{\text{cpt}}} e(T^*\hat{M}_k^{\text{cpt}}) + \int_{\partial \hat{M}_k^{\text{cpt}}} Q. \quad (3.9)$$

The second term on the right-hand side is a surface term which involves the integral of the second fundamental form $Q$ over the boundary $\partial \hat{M}_k^{\text{cpt}}$. The geometrical Euler characteristic $\chi$ is defined as the alternating sum over the Betti numbers just as in the compact case. This will be an integer which is independent of the volume $V$. In contrast, the bulk and boundary contributions may depend on $V$ and will not be integral in general. However, the individual terms will each be finite in the $V \to \infty$ limit and we denote these limiting values $\bar{\chi}$ and $\delta \chi$ respectively and the theorem then states $\chi = \bar{\chi} + \delta \chi$. In the case of two BPS monopoles one may use the explicit metric on the moduli space to show that $\delta \chi = 0$ and $\chi = \bar{\chi} = 2$. Our result (3.5) shows that $\bar{\chi} = k$ for all values of $k$ [34]. Note that the bulk contribution to the Euler characteristic is the same as the bulk contribution to the corresponding $L^2$-index while the boundary contributions to these two quantities are different.

The homology of the moduli-space $\hat{M}_k$ has been determined by Segal and Selby, using the description of this manifold as a space of rational maps. On a non-compact manifold homology is dual to cohomology with compact support. This cohomology with complex coefficients, $H^*(\hat{M}_k)$, is divided into different sectors according to the action of the discrete symmetry group, $\mathbb{Z}_k$. Let $H^*(\hat{M}_k)_p$ denote the cohomology with compact support restricted to the sector of forms with $\mathbb{Z}_k$ charge $p$. Then the result of Ref. [24] is that $H^*(\hat{M}_k)_p$ has complex dimension one whenever $r = 2k - 2(p, k)$ and dimension zero otherwise. The Euler characteristic of $\hat{M}_k$ can be deduced directly from the results of Segal and Selby described in the previous section. As the non-vanishing de Rham cohomology groups are of even dimension, the Euler characteristic
is simply obtained by counting the total number of solutions of the condition $r = 2k - 2\langle k, p \rangle$:

$$\chi(\hat{M}_k) = \sum_{r=0}^{4(k-1)} \sum_{p=0}^{k-1} \delta_{r,2k-\langle k, p \rangle} = k.$$  

(3.10)

This agrees with our expectations as long as $\delta \chi = 0$. As mentioned above, one may check this explicitly in the case $k = 2$.

4. The IIA' Bound-State Problem

In this section we will consider the $\mathcal{L}^2$-index theory relevant for determining the existence of bound-states of Yang-Mills instantons thought of as solitons in five-dimensional gauge theory.

We begin by discussing the Type IIA theory on flat $\mathbb{R}^{9,1}$ with $N$ D4-branes. The theory on the D4-world-volume is a five dimensional $U(N)$ gauge theory with sixteen supercharges. Dimensional reduction of this theory in one direction yields $\mathcal{N} = 4$ SUSY Yang-Mills theory in four dimensions. As described above, the IIA theory also contains D0-branes which are believed to form bound-states at threshold. The D0-branes appear as Bogomol’nyi saturated solitons on the five-dimensional world-volume of the D4-branes. These static field configurations which have finite energy in five dimensions correspond to instantons of finite Euclidean action in four dimensions.

The theory on the D0-brane world volume is a quantum mechanical gauge theory with eight supercharges. As in the absence of D4-branes, this theory contains the same degrees of freedom as any $U(k)$ gauge theory with sixteen supercharges (e.g. the $\mathcal{N} = 4$ theory in four dimensions). In particular, as in the basic IIA example, the D0-world-volume theory contains nine scalar fields in the adjoint representation of $U(k)$. The classical vacuum manifold includes a Coulomb branch where the adjoint scalars have non-zero VEVs breaking the gauge group down to its Cartan subalgebra. The eigenvalues of these fields describe the positions of the $k$ D0-branes in their nine transverse directions. However, the presence of D4-branes leads to additional hypermultiplets on the D0-world-volume which transform in the $(k, N)$ representation of $U(k) \times U(N)$. These multiplets break half the supersymmetry of the $N = 0$ case and the corresponding scalars parametrize a new branch of the vacuum moduli space (the Higgs branch) on which the gauge group is broken completely. The resulting soliton configurations on the D4-brane correspond to $k$ finite size instantons of gauge group $U(N)$. The Higgs branch of the D0-brane theory coincides with the moduli space of instantons in four dimensions. Unbroken supersymmetry and the Atiyah-Singer index theorem dictates that this is a hyper-Kähler manifold of real dimension $4kN$. As usual in a theory with eight supercharges, we obtain the Higgs branch by imposing the $D$-term equations and dividing out by $U(k)$ gauge
transformations. In the present case, this standard procedure coincides with the hyper-Kähler quotient construction of the corresponding instanton moduli space $M_{k,N}$, which is also known as the ADHM construction [40]. In more detail, it is convenient to specify the field content of the theory in the language of $d = 4$ superfields. First of all, there is a vector multiplet of $\mathcal{N} = 2$ supersymmetry, decomposing as a vector multiplet $V$ of $\mathcal{N} = 1$ SUSY and an adjoint chiral multiplet $\Phi$. On top of this, there is an adjoint hypermultiplet, consisting of chiral multiplets $X$ and $\tilde{X}$, and $N$ fundamental hypermultiplets, consisting of $N$ chiral multiplets $Q$ and $\tilde{Q}$. The scalar components of $\Phi$, which we denote with the same symbol, along with the 3 scalars that appear after dimensional reduction of the gauge field, represent the positions of the D0-branes transverse to the D4-branes. The adjoint hypermultiplet $(X, \tilde{X})$ specifies the positions of the D0-branes within the D4-branes. On the Higgs branch, the D0-branes lie within the D4-branes, i.e. $\Phi = 0$, and the $D$-flatness conditions are

\begin{align}
Q\tilde{Q} + [X, \tilde{X}] &= 0 \, , \\
QQ^\dagger - \tilde{Q}^\dagger\tilde{Q} + [X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] &= 0 \, ,
\end{align}

respectively. These are the ADHM equations which, modulo the $U(k)$ gauge symmetry, specify the moduli space $M_{k,N}$. The variables $Q$ and $\tilde{Q}$ encode the instanton sizes and orientation in the $U(N)$ gauge group, while, as stated above, $(X, \tilde{X})$ specify the positions of the instantons in the D4-branes. The $U(k)$ gauge theory also has Coulomb, as well as mixed, branches which classically are connected to the Higgs branch at points where an instanton shrinks to zero size, i.e. where some component of $Q$ and $\tilde{Q}$ goes to zero and a $U(1) \subset U(k)$ does not act freely. At these points, the D0-brane can move off the D4-brane into the bulk and one moves out along another branch of the theory.

The terms in the Lagrangian of the theory can be grouped into three:

$$L = g_1^{-2}L_{V,\Phi} + L_{X,\tilde{X}} + L_{Q,\tilde{Q}} \, ,$$

where $L_{V,\Phi}$ is the Lagrangian for the vector multiplet, while $L_{X,\tilde{X}}$ and $L_{Q,\tilde{Q}}$ are the Lagrangians describing the hypermultiplets. The vector multiplet involves the dimensionful coupling constant $g_1^2 \sim (\alpha')^{-3/2}$. Naïvely, at energy scales much less than $g_1^{2/3}$, the first term in (4.2) is irrelevant. Without a kinetic term, the vector multiplet becomes non-dynamical and can be integrated out. The 3 auxiliary fields in $V$ impose the ADHM constraints (4.1a) and (4.1b), while the gauge field enforces $U(k)$ invariance. In the infra-red, therefore, the gauge theory flows to an effective theory which corresponds to the supersymmetric quantum mechanics of a non-relativistic particle moving geodesically on the ADHM moduli space $M_{k,N}$. This effective description is a supersymmetric version of Manton’s moduli space approximation for soliton dynamics. As in the case of BPS monopoles the metric is in curved space and hence there are

\footnote{We choose the $U(N)$ flavour symmetry, which corresponds to the gauge symmetry on the D4-branes, to act as $Q \rightarrow QU^\dagger$ and $\tilde{Q} \rightarrow U\tilde{Q}$.}
non-trivial interactions between solitons at order $v^2$ in the velocity expansion. However, an important difference with this case is the fact that the metric has orbifold singularities at the points where the classical Higgs phase meets the mixed/Coulomb phases, i.e. where, as mentioned above, instantons shrink to zero size. In this respect the problem has common features to the basic problem of D0-binding in the IIA theory. In particular, the moduli space approximation breaks down near the singularities due to the presence of new light degrees of freedom and therefore may not be reliable for determining the existence of bound-states. The story of the phases structure of these theories, both in quantum mechanics, as here, and in $1+1$ dimensions, where one is concerned with the D1/D5 system, is a very interesting one (see [25–27] and references therein). For our purposes the singularities in the Higgs branch can be resolved in a standard way [41] by introducing non-commutivity in the five-dimensional gauge theory, where the instantons appear as BPS solitons. In terms of the string theory picture this corresponds to introducing a background anti-symmetric tensor field. In terms of the $U(k)$ gauge theory on the D0-brane world volume, the non-commutivity parameters $(\zeta_R, \zeta_C)$, corresponds to Fayet-Illiopoulos terms in the center of the gauge group. This coupling lifts the Coulomb branch of the world-volume theory and resolves the orbifold singularities of the Higgs branch, yielding a smooth hyper-Kähler manifold $M_{k,1}$ described by the deformed ADHM equations:

$$Q\tilde{Q} + [X, \tilde{X}] = \zeta_C 1_{[k] \times [k]},$$

$$QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} + [X, X^\dagger] + [\tilde{X}, \tilde{X}^\dagger] = \zeta_R 1_{[k] \times [k]}.$$  \hspace{1cm} (4.3a)

(4.3b)

One of the most interesting features of the non-commutative gauge theory on the D4-branes is that the theory admits non-singular instantons even in the abelian case $N = 1$. In fact, the ADHM construction of the moduli space of $U(1)$ instantons as a hyper-Kähler quotient is well defined even when the non-commutivity parameter is zero. The resulting moduli space is simply a symmetric product $\mathcal{M}_{k,1} = \text{Sym}_k(\mathbb{R}^4)$ reflecting the fact that abelian instantons, or D0-branes sitting on a single D4-brane, correspond to identical pointlike objects on $\mathbb{R}^4$. In the simplest non-trivial case of two such objects we have $\mathcal{M}_{2,1} = \mathbb{R}^4 \times \mathbb{R}^4/\mathbb{Z}_2$, where the $\mathbb{R}^4$ factor describes the COM degrees-of-freedom and the orbifold $\mathbb{R}^4/\mathbb{Z}_2$ describes the relative positions of the two identical objects. As the metric is flat, the instantons do not interact at order $v^2$ and we certainly cannot hope to find bound-states in a naïve moduli space approximation. However, when the FI couplings are turned on, the orbifold singularities are resolved and we have a smooth manifold of the form $\mathbb{R}^4 \times \tilde{\mathcal{M}}_{k,1}$ where the manifold of relative positions $\tilde{\mathcal{M}}_{k,1}$ has non-zero curvature. In the $k = 2$ case the orbifold $\mathbb{R}^4/\mathbb{Z}_2$ is replaced by the Eguchi-Hanson manifold, as has been shown explicitly in [42].

Once the singularities of the moduli space have been resolved, the IR limit of the $U(k)$ gauge theory which describes the relative dynamics of the solitons is simply a supersymmetric $\sigma$-model with target space $\tilde{\mathcal{M}}_{k,1}$ of precisely the same form (3.2) as occurs in the BPS monopole case. In other words, the moduli space approximation is now perfectly well defined and should
yield sensible answers. As in the monopole case, the required bound-states precisely correspond to normalizable harmonic forms of middle dimension on $\mathcal{M}^{(\zeta)}_{k,1}$. In the case $k = 2$, the fact that the Eguchi-Hanson manifold there is a single such form in agreement with the prediction of the existence of single threshold bound state of two abelian instantons [42]. We will now discuss the index theory responsible for establishing the existence of required threshold bound states for all values of $k$. As before the $L^2$-index is the sum of bulk and boundary pieces $J_{L^2} = J_{\text{bulk}} + J_{\text{boundary}}$

The main points are as follows:

1: The bulk contribution has two equivalent representations. Firstly, by dimensionally reducing the partition function of the world-volume gauge theory, it can be represented as the partition function of a $U(k)$ matrix model with a single flavour of matter in the fundamental representation. Secondly, by dimensionally reducing the low energy effective theory, it is equal to the Gauss-Bonnet integral of the Euler density over the smooth hyper-Kähler manifold $\mathcal{M}^{(\zeta)}_{k,1}$. The equality of these two representations as ordinary integrals can be demonstrated by using cohomological field theory techniques applied to the reductions of supersymmetric gauge theories to matrix integrals, along the lines described in [29]. Actually, the relevant point can already be seen in the D1/D5 system [43]. In that case, the kinetic term of the vector multiplet is a $D$-term and hence $Q$-exact, where $Q$ is a nilpotent combination of the supersymmetries [44]. This fact descends to the matrix integral, and allows one to argue à la Moore et al [29], that the resulting integral is independent of the gauge coupling $g_0$. In particular, we can take the strong coupling limit $g_0 \to \infty$, as long as the resulting integral is well defined. In the presence of non-commutativity, the FI terms ensures that the resulting integral is well defined, since it is precisely the integral of the Euler density over the smooth manifold $\mathcal{M}^{(\zeta)}_{k,1}$. In a similar way, one can argue that with finite $g_0$, one can take the FI couplings to zero, since in zero dimensions it is also $Q$-exact, without altering the value of the integral. In other words, both non-commutativity and $\alpha'$ effects, through $g_0$, regulate the singular behaviour of the partition function, a fact that was used in [45].

2: As in the other examples we have considered the bulk contribution to the $L^2$-index can be related to an instanton effect in a suitably compactified theory. In the present case, the relevant effects are the D-instanton contributions to the higher-derivative terms in the effective action of a single D3-brane of the IIB theory. Last year Green and Gutperle determined the order-$\alpha'^4$ term in the derivative expansion of the world-volume action. In terms of the bosonic fields on the D3 world-volume, it is given by [46]

$$S_{D3}^{(4)} \text{ eff} = \alpha' \pi^2 \frac{1}{12} \int d^4 x \sqrt{|g^E|} h(\tau, \bar{\tau}) \left( (\partial^2 \varphi)^4 + \tau_2 (\partial^2 \varphi)^2 \partial F^+ \partial F^- + \tau_2^2 (\partial F^+)^2 (\partial F^-)^2 \right), \quad (4.4)$$

where $F_{mn}^\pm$ denote the (anti)-self-dual gauge field-strength, $\phi_{AB}$ are the six real scalars of the $\mathcal{N} = 4$ SYM theory on the D3 world-volume, and $g^E$ is the Einstein metric. The function $h(\tau, \bar{\tau})$ was determined by considering the $SL(2, \mathbb{Z})$ invariant completion of a known tree level
open string contribution

\[ S_{D3}^{(4)} \text{ Born} = \frac{\pi^3 \alpha'^4}{12} \int d^4 x \sqrt{g(E)} \tau_2 \left( (\partial^2 \varphi)^4 + \tau_2 (\partial^2 \varphi)^2 \partial F^+ \partial F^- + \tau_2^2 (\partial F^+)^2 (\partial F^-)^2 \right). \]  

(4.5)

In order for the full nonperturbative effective action to be modular invariant Green and Gutperle [46] proposed to replace the overall factor of \( \tau \) in (4.5) by a modular invariant function, \( h(\tau, \bar{\tau}) \),

\[ h(\tau, \bar{\tau}) = \ln |\tau_2 \eta(\tau)|^4, \]

(4.6)

where \( \eta \) is the Dedekind function. The function \( h \) has the weak coupling expansion,

\[ h(\tau, \bar{\tau}) = \left( -\frac{\pi}{3} \tau_2 + \ln \tau_2 - 2 \sum_{k=1}^{\infty} \sum_{d|k} \frac{1}{d} \left( e^{2\pi i k \tau} + e^{-2\pi i k \bar{\tau}} \right) \right), \]

(4.7)

which contains the Born contribution from (4.5), a one-loop term and an infinite set of multi-D-instanton and multi-D-anti-instanton corrections. In the Appendix we will derive an expression for the generating functional \( Z_{k,1}[\Phi] \) of the instanton-induced contributions to the scattering amplitudes on the D3-brane. This generating functional will contain an integral over the resolved centered instanton moduli space as an overall coefficient. This integral is precisely the bulk contribution to the \( L^2 \)-index on \( \hat{M}_{k,1}^{(C)} \). By comparing the 4-point amplitudes obtained from \( Z_{k,1}[\Phi] \) to those derived from \( S_{D3}^{(4)} \text{ eff} \), we find that the result of Green and Gutperle (4.7), along with the \( Q \)-exactness argument described in 1 above, imply

\[ I_{\text{bulk}}(k, 1) = \sum_{d|k} \frac{1}{d}. \]

(4.8)

As before, the terms in the sum can be interpreted in the IIA theory on \( \mathbb{R}^9 \times S^1 \) as coming from the worldline of a threshold bound-state of \( k/d \) D0-branes wrapped \( d \) times on \( S^1 \).

3: As previously we can apply the Yi argument to determine the boundary contribution. The details are identical to those given for the basic IIA bound-state problem in §2. The only difference is that the D0’s move in four dimensions and have eight single component fermions as superpartners: thus we have \( b = 4 \) and \( f = 8 \). We can now use (2.9), and the discussion following it, to deduce

\[ I_{\text{boundary}}(k, 1) = 1 - \sum_{d|k} \frac{1}{d}. \]

(4.9)

Hence, as expected, the boundary term subtracts of the contributions of sectors of multiple winding to give results consistent with the existence of a single threshold bound-state for each value of \( k \).

4: As in §3, the bulk contribution to the \( L^2 \)-index is also equal to the bulk contribution Euler characteristic \( \chi_k \) of \( \hat{M}_{k,1}^{(C)} \) in the Gauss-Bonnet theorem. Thus we have \( \chi_k = \bar{\chi}_k + \delta \chi_k \)
with $\bar{\chi}_k = \sum_{d|k} d^{-1}$. Interestingly the full Euler characteristic has recently been determined by Nakajima using equivariant Morse theory [23]. His result is $\chi_k = \sigma(k)$ where $\sigma(k)$ is the number of partitions of $k$. Thus we have a new (and non-zero) prediction for the boundary contribution to the Gauss-Bonnet theorem on $\hat{\mathcal{M}}^{(c)}_{k,1}$:

$$\delta \chi_k = \sigma(k) - \sum_{d|k} \frac{1}{d}$$  \hspace{1cm} (4.10)

This can be checked explicitly for the $k = 2$ case. As explained in [39], the Eguchi-Hanson manifold has Euler characteristic $\chi_2 = 2$. This agrees with Nakajima [23] as $\sigma(2) = 2$. The bulk contribution to the Gauss-Bonnet theorem was evaluated in [39] using the explicit metric to obtain $\bar{\chi}_2 = \frac{4}{3}$ in agreement with points 2 and 4 above. Finally the evaluation of the boundary term as an explicit integral of the second fundamental form over the $S^3/\mathbb{Z}_2$ boundary at infinity of the Eguchi-Hanson manifold gives [39] $\delta \chi_2 = \frac{1}{2}$ in agreement with (4.10).

It would be interesting to generalize this analysis to the case of non-abelian instantons, $N > 1$. Some results for the corresponding theory with one compactified dimension were presented in [47], indicating the existence of $N$ threshold bound-states for arbitrary $N$. Applying our index theory approach to the non-compact theory is complicated by the fact that that, for $N > 1$, instantons have a genuine size modulus. This means that the notion of a clustering region is more complicated since even though instantons can be spatially a long way apart they can still overlap by being large. Nevertheless, although we have not calculated the bulk contribution to the $L^2$-index for general $N$ and $k$, we have calculated some special cases which may offer some clues [43]. Firstly, the bulk contribution to the $L^2$-index on $\hat{\mathcal{M}}^{(c)}_{1,N}$ is

$$I_{\text{bulk}}(1, N) = \frac{2^{1-2N} (2N)!}{N!(N-1)!}.$$  \hspace{1cm} (4.11)

The case with $N = 2$ is somewhat special, since the unresolved (centered) moduli space is $\hat{\mathcal{M}}_{1,2} = \mathbb{R}^4/\mathbb{Z}_2$, which is the same as $\hat{\mathcal{M}}_{2,1}$. The interpretation, however, is different: the radial parameter is the scale size of the instanton and the $S^3$ solid angle gives the $SU(2)$ orientation of the instanton (the $\mathbb{Z}_2$ orbifolding corresponds to the center of the gauge group). As in the abelian case of two instantons, the resolved space $\hat{\mathcal{M}}^{(c)}_{1,2}$ is also the Eguchi-Hanson manifold and so $I_{\text{bulk}} = \frac{3}{2}$, which agrees with (4.11) for $N = 2$. In this case we also know that the boundary contribution to the $L^2$-index is $-\frac{1}{2}$, since it is the same as for the $k = 2$ and $N = 1$ case. Hence, $I_{L^2}(1, 2) = 1$. For $N > 2$, the situation is not so simple because $\mathcal{M}^{(c)}_{1,N}$ does not have a clustering region where it looks like $\text{Sym}_N(\mathbb{R}^4)/\mathbb{R}^4$, and we cannot, at least in any obvious way, apply the Yi argument to these situations.

The situation for $k > 1$ and $N > 1$ is even more complicated. In this case the only data that we have is the bulk contribution to the $L^2$-index of $\hat{\mathcal{M}}^{(c)}_{k,N}$ in the limit where $k$ is fixed and
$N$ is large [43]:

$$I_{\text{bulk}}(k, N) \overset{N \to \infty}{=} 2^{3-2k} \pi^{6k-13/2} \sqrt{N} k^{3/2} \sum_{d|k} \frac{1}{d^2}. \quad (4.12)$$

5. The IIB Bound-State Problem

We now return to the IIB theory in ten flat dimensions. The problem is to prove the existence of a unique bound-state of $k$ D-strings and $q$ fundamental strings whenever $k$ and $q$ are coprime and not otherwise. The worldvolume theory is a two dimensional $U(k)$ gauge theory with sixteen supercharges analysed by Witten in [48]. He started by wrapping the spatial dimension on a circle so that we have quantum mechanics as in the other examples. Fundamental string charge corresponds to a constant electric flux, which is the same as a charge at spatial infinity in this world sheet theory. The aim as before is to prove the existence of appropriate normalizable SUSY ground states of the corresponding $SU(k)$ gauge theory which describes the relative degrees of freedom of the D-strings. The fundamental string charge then shows up as charge under the $\mathbb{Z}_k$ center of $SU(k)$ placed at infinity. The coprime case is easy because the non-zero $\mathbb{Z}_k$ flux produces a mass gap in the $SU(k)$ theory. The existence of the required ground-state can then be confirmed reliably by a suitable perturbation [48].

The non-coprime case is much harder because it involves bound-states at threshold. However the relevant $\mathcal{L}^2$-index theory can be studied using the methods discussed in the preceding sections. In the following, we will only present a brief sketch of how we believe this works. As in the other examples, the bulk contribution to the index is obtained by considering the appropriate instanton contributions for the IIB theory on $\mathbb{R}^8 \times T^2$. The instantons in question are just the $(q, k)$ strings wrapped on $T^2$. These have sixteen zero modes and contribute to the $\mathcal{R}^4$ term in the IIB action of this compactified theory. The exact $\mathcal{R}^4$ term was given by Kiritsis and Pioline, extending the approach of Green and Sethi to the compactified theory. The $(q, k)$ contribution is given in Eqn (3.46) of [49] as,

$$F_{q, k} = -8\pi \text{ Re log } \left[ \prod_{n=1}^{\infty} (1 - \exp(2\pi i n T_{q, k})) \right], \quad (5.1)$$

where the action $T_{q, k}$ is the action of a single wrapped $(q, k)$-string. If we take $T^2$ as a rectangular torus with sides $R_1$ and $R_2$ and turn off the $B$ fields, then the action, $T_{q, k}$, is the simply product of the world sheet area, $R_1 R_2$, and string tension $|q + k\tau|/\alpha'$. The product over $n$ inside the logarithm yields a sum over multiply-wrapped string world sheets for each type of string.

The relevant coefficients in the weak coupling expansion of (5.1) can be identified with the R-R partition function of the $U(K)$ theory on a Euclidean $T^2$. This has actually been
calculated independently by Kostov and Vanhove in [50]. In our notation, equation (33) of [50] reads,

\[ Z_K \sim \sum_{d|K} \frac{1}{d} \sum_{Q = -\infty}^{+\infty} \exp \left( -\frac{Q^2}{2R_1 R_2 K} \right) . \]  

(5.2)

The weak coupling limit of (5.1) agrees with (5.2) if we identify \( K = kd \) and \( Q = qd \) with \( d = n \). To isolate the bulk contribution, we send \( R_1 \) and \( R_2 \) to zero and obtain,

\[ I_{\text{bulk}} = \sum_{d|k} \frac{1}{d} . \]  

(5.3)

As in the preceding examples, the over counting of the index corresponds to the sectors with multiply wrapped branes and the compensating boundary contribution can be evaluated using Yi’s argument as in the preceding sections. In the present case we can apply the argument by considering the winding modes of the \((p, q)\) strings on \( \mathbb{R}^8 \times S^1 \) as massive particles in eight non-compact spatial dimension. Now we may apply (2.9) with the values are \( b = 8 \) and \( f = 16 \), which yields \( \delta = 8 - 8 = 0 \). Thus we have a boundary contribution of \(-1/d\) from each clustering sector with \( d > 1 \) identical particles. The resulting index is unity when \( k \) and \( q \) are coprime and zero otherwise, as expected.

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**Appendix A: Bulk contribution to the \( L^2 \)-index on \( \hat{\mathcal{M}}_{k,1}^{(\zeta)} \)**

In this Appendix we consider the generating functional \( Z_{k,1}[\Phi] \) of the \( k\)-D-instanton contributions to the scattering amplitudes on a single D3-brane. The expression for \( Z_{k,1}[\Phi] \) will contain an integral over the centered resolved \( k\)-instanton moduli space \( \hat{\mathcal{M}}_{k,1}^{(\zeta)} \). Matching the 4-point amplitudes obtained from \( Z_{k,1}[\Phi] \) with the predictions of (4.4), (4.7) we will derive the expression (4.8) for the bulk contribution to the \( L^2 \)-index on \( \hat{\mathcal{M}}_{k,1}^{(\zeta)} \).

In order to determine the generating functional \( Z_{k,1}[\Phi] \), we will first consider the partition function of the \( k\)D(-1)/ND3 brane system\(^6\) on the \( k\)D(-1) world-volume [51],

\[ Z_{k,N} = \int d\mu_{k,N} e^{-S_{k,N}} . \]  

(A.1)

Here the instanton integration measure \( d\mu_{k,N} \) and action \( S_{k,N} \) are over the instanton collective coordinates, and the D-3 brane degrees of freedom are turned off. Next, we will turn on

\(^6\)We are ultimately interested in the case of a single D3-brane, but our formalism applies to all values of \( N \). In due course we will set \( N = 1 \) to simplyfy final expressions.
the fields living on the world-volume of the D3-branes. From the point of view of the zero-dimnsional instanton world-volume, the D3-branes are infinitely heavy and their world-volume fields are represented by static sources $\Phi$ coupled to instanton collective coordinates in the action $S_{k,N}(\Phi)$. The partition function (A.1) in the presence of the D3-sources $\Phi$,

$$Z_{k,N}[\Phi] = \int d\mu_{k,N} e^{-S_{k,N}(\Phi)}, \quad (A.2)$$

defines the generating functional of the D-instanton-induced amplitudes on the D3-branes in the semi-classical approximation.

The $kDp/ND(p+4)$ brane system can live in the maximal dimension $p = 5$ which corresponds to the 6-dimensional gauge theory on the world-volume of the D5-branes. Then the cases $5 \geq p \geq -1$ follow by dimensional reduction. In Section 4 we specified the ADHM collective coordinates in the language of $d = 4$ superfields. This corresponds to choosing $p = 3$ as the starting point of dimensional reduction to $p = -1$. For practical calculations it is more convenient to start with the maximal case $p = 5$ and follow conventions of Ref. [51]. The field content of the $kD5/ND9$ system is described by the $(1,1)$ vector multiplet and two bi-fundamental hypermultiplets in the 6-dimensional world-volume of $kD5$ branes. The component fields are introduced in the same way as in [51] and are listed in the Tables 1-2.

The relation between the $d = 4$ language of Section 4 and the $d = 6$ language is as follows:

$$\chi^{1\ldots4} \equiv V, \quad \chi^5 \pm i\chi^6 \equiv \Phi^{(f)}, \quad a'^{\alpha\dot{\alpha}} \equiv \left( X^\dagger \tilde{X} \right), \quad w^{\dot{\alpha}} \equiv \left( Q^\dagger \tilde{Q} \right), \quad \bar{w}^{\dot{\alpha}} \equiv \left( Q \tilde{Q}^\dagger \right). \quad (A.3)$$

The $\mathcal{N} = 1$ superfields on the right hand sides of equations above are identified with their bosonic components.
Table 2: Bi-fundamental hypermultiplets in $d = 6$.

The D-instanton integration measure in the $\mathcal{N} = 4$ supersymmetric $U(N)$ gauge theory has the following form [51]:

$$ Z_{k,N} = \frac{g_4^4}{\text{Vol} U(k)} \int d^{4k^2} a' d^{8k^2} \mathcal{M}' d^{8k^2} \lambda d^{8k^2} D d^{2kN} w d^{2kN} \bar{w} d^{4kN} \mu d^{4kN} \bar{\mu} \exp[-S_{k,N}] \tag{A.4} $$

where $S_{k,N} = g_0^{-2} S_G + S_K + S_D$ and

$$ S_G = \text{tr}_k \left( -[\chi_a, \chi_b]^2 + \sqrt{2} i \pi \lambda_{\dot{\alpha}} A[\chi_{AB}, \lambda_{\dot{B}}] + 2 D^c D^c \right), \tag{A.5a} $$

$$ S_K = -\text{tr}_k \left( [\chi_a, a_n']^2 + \chi_a \bar{w}_a w \dot{w} \chi_a + \sqrt{2} i \pi \mathcal{M}^A [\chi_{AB}, \mathcal{M}^B] + 2 \sqrt{2} i \pi \bar{\mu}_a A[\chi_{AB}, \mu^B] \right), \tag{A.5b} $$

$$ S_D = \pi \text{tr}_k \left( [a_n', \mathcal{M}^A] \lambda_{\dot{A}} + \bar{\mu}_a A(w \dot{w}) \lambda_{\dot{A}} + \bar{\mu}_u B \lambda_{\dot{B}} + \pi^{-1} D^c(\tau^c)_{\dot{A}} (\bar{w}^\dot{A} w + \bar{a}_a A a_n') \right). \tag{A.5c} $$

We use the same conventions$^7$ as in [51]. For future convenience we also introduce the centre of mass coordinates of the $k$-instanton,

$$ a^{CM}_n = k^{-1} \text{tr}_k a_n', \quad \mathcal{M}^{ssA}_a = k^{-1} \text{tr}_k (\mathcal{M}^A_a) \tag{A.6} $$

These simply correspond to translations along the D3-branes in superspace of the multi-instanton configuration as a whole.

$^7$The index assignment is $i, j, \ldots = 1, \ldots, k$ and $u, v, \ldots = 1, \ldots, N$, together with $A, B, \ldots = 1, \ldots, 4$ and $a, b, \ldots = 1, \ldots, 6$ and $c, \ldots = 1, \ldots, 3$. An overall normalization of the measure is suppressed in (A.4). It can be determined by comparison with the ADHM multi-instanton measure in field theory. Properly normalized centered partition function $\hat{Z}_{k,N}$ is discussed in the Appendix of [43].
gauge theory one must take the limit $\alpha' \to 0$. In this limit $g_0^2 \to \infty$ equations of motion for $D^c$ are precisely the non-linear ADHM constraints (4.1a), (4.1b). Similarly equations of motion for $\lambda$ are the fermionic ADHM constraints. Integration over $D^c$ and $\lambda_A^\alpha$ yields $\delta$-functions which impose the constraints.

For D-instanton applications in string theory we must keep the $\alpha'$ corrections in the D-instanton measure (A.4). In this case, the integrals over $D^c$ and $\lambda_\dot{\alpha}$ do not lead to the ADHM constraints and their fermionic analogues; instead of being imposed via delta functions, the arguments of the ADHM constraints appear as Gaussian factors:

\[
\int d^3D e^{-\text{tr}_k(2g_0^{-2}D^cD_c+iD^c\mathcal{B}^c)} = (g_0^2\pi/2)^{3k^2/2}e^{-\frac{1}{8g_0^2}\text{tr}_k\mathcal{B}^2}.
\]

(A.7)

This provides a natural string theory regularization of the singularities of the instanton moduli space. We also note that the $g_0^{-2} \propto \alpha'^2$ terms in the action of (A.4) lift the superconformal fermion zero modes. When $\alpha'$ is set to zero in (A.4) and the $U(N)$ gauge group is unbroken, the superconformal fermion zero modes are exact. (This is only relevant for $N > 1$.) In the latter situation to obtain non-zero answers one has to saturate the supersymmetric as well as the superconformal fermion zero modes by the field insertions in the pre-exponent as in [51], or otherwise.

We now need to include on the right hand side of (A.4) the interactions with the D3-brane world-volume fields transforming in the adjoint representation of $U(N)$: the $d = 4$ gauge fields $A_m(x)$, the six scalars $\varphi_{AB}(x)$, and the gauginos $\Lambda_\alpha^A(x), \bar{\Lambda}_\dot{\alpha}^A(x)$. Note that so far we have ignored these fields, the bifundamental hypermultiplets $w$ and $\bar{w}$ represent merely the strings stretched between the D-instantons and the D3-branes. The gauge interactions between the D9-gauge fields, $A_\mu = (\varphi_{AB}, A_m)$, and the D5-matter fields, $a'_m, w_\alpha$ and $\bar{w}_{\dot{\alpha}}$, occur in covariant derivatives. These interactions are introduced by an extension of the gauge sector in $d = 6$ matter actions $S_K$ and $S_D$

\[
(\chi_{AB})_{ij} \delta^{uv} \to (\chi_{AB})_{ij} \delta^{uv} - \varphi_{AB}^u(a^CM) \delta_{ij},
\]

\[
(D^c)_{ij} \delta^{uv} \to (D^c)_{ij} \delta^{uv} - F_{mn}^{+uv}(a^CM) \eta_{mn}^c \delta_{ij},
\]

(A.8)

and keeping the gauge action $S_G$ unchanged. Note that $S_K$ and $S_D$ are defined on the D-instanton world-volume and do not depend on the D3 world-volume coordinates $x_m$. Thus, the D3 world-volume fields in (A.8) are taken at the specific point $x_m \equiv a^CM_m$, the centre of mass location of the instanton (A.6) along the D3 brane. This is consistent with the earlier observation that from the instanton world-volume point of view, the D3 fields are infinitely heavy and should appear as static sources.\(^8\)

\(^8\)We further note that for the general $N > 1$ case the separations between the D3-branes are automatically incorporated in (A.8) by turning on the VEVs of the scalar fields, $\langle \varphi^a \rangle_{uv} = \text{diag}_{uv}(\langle \varphi^a \rangle_1, \cdots, \langle \varphi^a \rangle_N)$. When the separations are non-vanishing for all $u \neq v$ the $U(N)$ gauge group is spontaneously broken to $U(1)^N$ and we are on the Coulomb branch of the ND3 system.
However, the interaction term generated by the substitution \( (A.8) \) is not the full story. With \( (A.8) \) we have introduced an explicit dependence on the centre of mass instanton coordinates \( a_{CM}^m \), but not on their fermionic superpartners \( M^{ss} \). With respect to the (super)symmetry transformations broken by D\((-1)\) branes, but not by D3-branes the total action of the D\((-1)/D3\) system \( S_{k,N}^{tot} \) should transform as follows:

\[
\begin{align*}
  a_m^{CM} & \rightarrow a_m^{CM} + z_m : \quad S_{k,N}^{tot} (a_m^{CM}, M^{ss}, \ldots) \rightarrow S_{k,N}^{tot} (a_m^{CM} + z_m, M^{ss}, \ldots), & (A.9a) \\
  M^{ssA}_\alpha \rightarrow M^{ssA}_\alpha + \eta^{\alpha} : \quad S_{k,N}^{tot} (a_m^{CM}, M^{ss}, \ldots) \rightarrow S_{k,N}^{tot} (a_m^{CM}, M^{ss} + \eta, \ldots). & (A.9b)
\end{align*}
\]

The factor \( \int d^4a^{CM} d^8M^{ss} \) in the generating functional \( (A.2) \) will ensure that when the D\((-1)\) branes are integrated out, the symmetries broken by instantons on the D3 branes will be restored.

After the substitution \( (A.8) \), the action satisfies \( (A.9a) \), but not \( (A.9b) \). An elegant way to satisfy \( (A.9b) \) is to upgrade the bosonic fields on the right hand sides of \( (A.8) \) to the corresponding components of the \( N = 4 \) on-shell superfields \( W_{AB}(\theta, \bar{\theta}) \) [52]. Using supersymmetric covariant derivatives one can also define \( W_{B}^\alpha = (3/2)D^A \theta^\alpha W_{AB} \) and \( W_{mn} = D^A \bar{\sigma}_{mn} D^B W_{AB} \). The dependence on \( M^{ss} \) is introduced as follows [46]:

\[
\Phi_{mn} = W_{mn}|_{\theta, \bar{\theta} = M^{ss}}, \quad \Phi_A^\alpha = W_{A\alpha}^\dagger|_{\theta, \bar{\theta} = M^{ss}}, \quad \Phi_{AB} = W_{AB}|_{\theta, \bar{\theta} = M^{ss}}. \tag{A.10}
\]

By construction, these superfields are invariant under the supersymmetry transformations generated by shifts of \( M^{ss} \rightarrow M^{ss} + \eta \),

\[
\delta_\eta \Phi = \eta^A_{\alpha} \frac{\partial}{\partial M^{ss\alpha}_A} \Phi. \tag{A.11}
\]

Thus, the total interaction with the ND3-brane sources is determined by the substitution

\[
\begin{align*}
  (\chi_{AB})_{ij} \delta^{uv} & \rightarrow (\chi_{AB})_{ij} \delta^{uv} - \Phi_{AB}^\alpha (a^{CM}_m, M^{ss}) \delta_{ij}, \\
  (\lambda_A^\dagger)_{ij} \delta^{uv} & \rightarrow (\lambda_A^\dagger)_{ij} \delta^{uv} - \Phi_A^{uv\dagger} (a^{CM}_m, M^{ss}) \delta_{ij}, \tag{A.12} \\
  (D^c)_{ij} \delta^{uv} & \rightarrow (D^c)_{ij} \delta^{uv} - \Phi_{mn}^{uv} (a^{CM}_m, M^{ss}) \delta_{mn}^c \delta_{ij},
\end{align*}
\]

into the instanton matter-field actions \( S_K \) and \( S_D \). This inclusion of the D3-brane sources from now on will be denoted as \( S_{K}(\Phi) \) and \( S_{D}(\Phi) \). Equation \( (A.12) \) was originally derived by Green and Gutperle [46] from slightly different considerations. We will need an explicit realization of only one of these superfields which we copy from [46]

\[
\Phi_{mn} (a^{CM}_n, \bar{M}) = F(m_n^+ + i M^A \sigma_{[m} \partial_{n]} \Lambda_A + 4 M^B \sigma_{[m} \partial_{n]} \Lambda M \partial_{n]} \partial_{p} \varphi_{AB} + 2 \epsilon_{ABCD} M^B \sigma_{p[m} \sigma_{n]} \partial_{n]} \partial_{p} \Lambda + \epsilon_{ABCD} M^B \sigma_{p[m} \sigma^{kl} \Lambda M \partial_{n]} \partial_{p} F_{kl}^- . \tag{A.13}
\]

The fact that an on-shell superfield is used in the above analysis is not an obstacle for deriving instanton-induced amplitudes (which are also on-shell quantities) on the 3-branes.
The $k$D(-1)/$ND3$ partition function on the $k$D(-1) world-volume and in the presence of the D3-brane sources is given by

$$Z_{k,N}(\Phi) = \frac{g^4_4}{\text{Vol} U(k)} \int d^{6k^2} \chi d^{8k^2} \lambda d^{3k^2} D d^{4k} a' d^{8k^2} \mathcal{M}' d^{2kN} w d^{2kN} \bar{w} d^{4kN} \mu d^{4kN} \bar{\mu} \quad (A.14)$$

$$\times \exp -S_{k,N}(\Phi) ,$$

where

$$S_{k,N}(\Phi) = g_0^{-2} S_G + S_K(\Phi) + S_D(\Phi) .$$

It is worthwhile to note that the bosonic zero modes associated with 4-translations $a^{CM}$ and the supersymmetric fermionic zero modes $\mathcal{M}^{ss}$ are both lifted in the action (A.15) via explicit insertions of the superfields $\Phi(a^{CM}, \mathcal{M}^{ss})$. There remain no unlifted bosonic or fermionic zero modes in the partition function $Z_{k,N}(\Phi)$.

The generating functional (A.14) can be simplified for the case of a single D3-brane. Thus from now on we set $N = 1$. The simplification comes about by noticing that the superfields in the shifts (A.12) of $S_K(\langle \varphi \rangle)$ and $S_D$ are now constants (not $[N] \times [N]$ matrices). These constant shifts can be undone by the opposite constant shifts in the integration variables $\chi, \lambda$ and $D$ in the partition function and in the previously unshifted action $S_G$. Furthermore, since $\chi$ and $\lambda$ variables appear in $S_G$ only in the commutators, the shifts of $\chi$ and $\lambda$ cancel and only the shift of $D$ contributes:

$$g_0^{-2} S_G \to g_0^{-2} \text{tr}_k \left( -[\chi_a, \chi_b] + \sqrt{2i\pi} \lambda_\alpha \lambda_\beta \chi_A^B, \chi_B^\alpha + 2(D^c + \Phi_{mn}\bar{\eta}_{mn})^2 \right) . \quad (A.16)$$

Consider now the terms in the total action involving the $D$-files:

$$2g_0^{-2}(D^c + \Phi_{mn}\bar{\eta}_{mn})^2 + iD^c(\tau^c)\bar{\alpha}(\bar{w}^\alpha w^\beta + \bar{a}^\alpha a^\beta) \right) =$$

$$2g_0^{-2}(\Phi_{mn}\bar{\eta}_{mn})^2 + 2g_0^{-2}(D^c)^2 + i(D^c(\tau^c)\bar{\alpha}(\bar{w}^\alpha w^\beta - i4g_0^{-2}\Phi_{mn}\bar{\eta}_{mn} + \bar{a}^\alpha a^\beta)) . \quad (A.17)$$

Using this rearrangement we can finally express the generating functional in a simple form:

$$Z_{k,1}(\Phi) = g_4^4 \int d^4 a^{CM} d^8 \mathcal{M}^{ss} \exp[-2g_0^{-2}(\Phi_{mn}(a^{CM}, \mathcal{M}^{ss})\bar{\eta}_{mn})^2] \times J_{k,1} , \quad (A.18)$$

where $J_{k,1}$ denotes an integral over the centered $k$-instanton moduli space,

$$J_{k,1} = \frac{1}{\text{Vol} U(k)} \int d^{6k^2} \chi d^{8k^2} \lambda d^{3k^2} D d^{4k^2-1} a' d^{8k^2-1} \mathcal{M}' d^{2k} w d^{2k} \bar{w} d^{4k} \mu d^{4k} \bar{\mu} \quad (A.19)$$

$$\exp[-g_0^{-2} S_G - S_K - S_D(\zeta)] .$$

Here $S_G$ and $S_K$ are given by, correspondingly (A.5a) and (A.5b), and contain no D3-sources. $S_D(\zeta)$ is given by (A.5c) with the $D$-term shifted by $\zeta \equiv i4g_0^{-2}\Phi_{mn}\bar{\eta}_{mn}$,

$$S_D(\zeta) = i\pi \text{tr}_k ([a'_{\alpha\dot{\alpha}}, \mathcal{M}^{\alpha\dot{\alpha}}] \lambda^\dot{\alpha} + \bar{\mu}^A w^\dot{\alpha} \lambda^\dot{\alpha} + \bar{w}_\dot{\alpha} \mu^A \lambda^\dot{\alpha} + \pi^{-1} D^c(\tau^c)\bar{\alpha}(w^\dot{\alpha} w^\beta - \zeta + \bar{a}^\alpha a^\beta) ) . \quad (A.20)$$
Generally speaking, \( J_{k,1} \) is a certain function of two parameters: \( \zeta \) and \( g_0 \), expressed as an integral (A.19) over the centered \( k \)-instanton moduli space \( \hat{\mathfrak{M}}_{k,1}^{(\zeta)} \). This integral is the partition function of the \( U(k) \) instanton matrix model with a single flavour of matter in the fundamental representation; \( g_0 \) is the \( U(k) \) gauge coupling, and \( \zeta \) is the abelian FI parameter. Both \( g_0 \) and \( \zeta \) provide a resolution of the singularities on \( \hat{\mathfrak{M}}_{k,1} \). For \( g_0 < \infty \) and arbitrary \( \zeta \) the singularities of \( \hat{\mathfrak{M}}_{k,1} \) are absent due to a string-theory resolution and the integral is well-defined. Alternatively, for \( \zeta \neq 0 \) and arbitrary \( g_0 \) the singularities of \( \hat{\mathfrak{M}}_{k,1} \) are again absent due to a FI resolution\(^9\), and the integral \( J_{k,1} \) is well-defined again. The Q-exactness argument of Section 4 implies that \( J_{k,1} \) does not depend on \( g_0 \) and does not depend on \( \zeta \) and is precisely equal to the bulk contribution to the \( \mathcal{L}^2 \)-index on the resolved centered instanton moduli space \( \hat{\mathfrak{M}}_{k,1}^{(\zeta)} \).

The expression (A.18) is the generating functional for the instanton-induced scattering amplitudes. In particular, the 4-point amplitudes \( A_{2\rightarrow 2} \) generated by (A.18) correspond precisely to those derived from the effective action (4.4)\(^!\) A simple way to see this is to single out specific components of the on-shell superfield \( \Phi_{mn}(a, \mathcal{M}) \). For example, it is easy to show that the \( (\partial^2 \varphi)^2 \) term in the effective action (4.4) follows from the \( 4\mathcal{M}^B \sigma_{[m} \mathcal{M}^A \partial_{n]} \partial_p \varphi_{AB} \) component in the expansion of \( \Phi_{mn}(a, \mathcal{M}) \) in Eq. (A.13). The 4-point scalar amplitude is obtained from the generating functional (A.18) via the functional differentiation with respect to the Fourier components of the scalar fields \( \tilde{\varphi}^a(k_i) \), and multiplications by the polarizations \( \zeta_i^a \),

\[
A_{2\rightarrow 2}^{\text{scalar}}(k_1, k_2, k_3, k_4) = \frac{1}{4!} \prod_{i=1}^{4} \left( \zeta_i^a \frac{\delta}{\delta \tilde{\varphi}^a(k_i)} \right) Z_{k,1}[\Phi] ,
\]

and with the substitution of the selected superfield component

\[
\Phi_{mn}(a, \mathcal{M}) = 4\mathcal{M}^B \sigma_{[m} \mathcal{M}^A \partial_{n]} \partial_p \varphi_{AB}(a) .
\]

For a non-zero answer we need to saturate precisely 8 supersymmetric fermion zero modes \( \mathcal{M} \), and the argument in the exponent of (A.18) needs to be brought down 8 times. Schematically we have

\[
\prod_{i=1}^{4} \left( \zeta_i \frac{\delta}{\delta \tilde{\varphi}(k_i)} \right) \int d^4a \int d^8\mathcal{M} \frac{1}{4!} (\mathcal{M} \mathcal{M} \partial \partial \varphi(a))^4 =
\]

\[
\prod_{i=1}^{4} \left( \zeta_i \frac{\delta}{\delta \tilde{\varphi}(k_i)} \right) \int d^4a \int d^8\mathcal{M} \frac{1}{4!} \left( \mathcal{M} \mathcal{M} \int d^4pe^{-ip\alpha_m} \alpha_m pp \tilde{\varphi}(p) \right)^4 =
\]

\[
(2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 + k_4) (s^2 + t^2 + u^2) t_{a_1}^{a_1} t_{a_2}^{a_2} t_{a_3}^{a_3} t_{a_4}^{a_4} \zeta_1 \zeta_2 \zeta_3 \zeta_4 k_1 k_2 k_3 k_4 ,
\]

\(^9\)As in Section 4 the \( \zeta \)-resolved moduli space describes instantons in gauge theory on a spacetime with non-commuting coordinates [41]. Note, however, that we started with the ordinary commutative theory on the 3-brane and derived the \( \zeta \)-resolution from the string-theory resolution via the inclusion of the D3-brane sources: when \( \alpha' \) is set to zero, \( \zeta \propto g_0^{-2} \Phi \propto \alpha'^2 \Phi \) is zero as well.
where the factor of \((2\pi)^4 \delta^{(4)}(k_1 + k_2 + k_3 + k_4)\) comes from \(\int d^4a\), and the remaining factor of \((s^2 + t^2 + u^2) t_8^{a_1 c_1 \ldots a_4 c_4} \zeta^{a_1} k_1^{c_1} \ldots \zeta^{a_4} k_4^{c_4}\) is the result of integrations over \(\int d^8M\) and of a proper care over the index contractions in (A.23). An eight-rank tensor \(t_8\) is an appropriately symmetrized sum of products of four Kroneckers [46].

Finally, the \(k\)-instanton contribution to the 4-point scalar amplitude on the 3-brane derived from the generating functional (A.18) reads

\[
A^\text{scalar}_{2\to 2} = \text{const} \, \alpha' \sum d_k \frac{1}{d} \int d^4x \sqrt{g(E)} (\partial^2 \varphi)^4,
\]

where \(\text{const}\) denotes some numerical factor independent of \(k\), and we have used \(g_4^4/g_5^4 \propto \alpha'\). This amplitude has precisely the same kinematic form as dictated by the Green-Gutperle effective action (4.4)

\[
S^{(4)}_{\text{scalar}} = \text{const} \, \alpha' \sum d_k \frac{1}{d} \int d^4x \sqrt{g(E)} (\partial^2 \varphi)^4,
\]

hence we conclude that \(I_{\text{bulk}}(k, 1) = \sum d_k \, d^{-1}\).

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