HALF-LINE NON-SELF-ADJOINT SCHRÖDINGER OPERATORS WITH POLYNOMIAL POTENTIALS: ASYMPTOTICS OF EIGENVALUES

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ABSTRACT. For integers $m \geq 3$, we study the non-self-adjoint eigenvalue problems $-u''(x) + (x^m + P(x))u(x) = Eu(x), \; 0 \leq x < +\infty$, with the boundary conditions $u(+\infty) = 0$ and $\alpha u(0) + \beta u'(0) = 0$ for some $\alpha, \beta \in \mathbb{C}$ with $|\alpha| + |\beta| \neq 0$, where $P(x) = a_1x^{m-1} + a_2x^{m-2} + \cdots + a_{m-1}x$ is a polynomial. We provide asymptotic expansions of the eigenvalue counting function and the eigenvalues $E_n$. Then we apply these to the inverse spectral problem, reconstructing some coefficients of polynomial potentials from asymptotic expansions of the eigenvalues.

Preprint.

1. INTRODUCTION

In this paper, we study non-self-adjoint Schrödinger operators in $L^2([0, +\infty))$, with monic polynomial potentials of degree $m \geq 3$ and provide explicit asymptotic expansions of the eigenvalue counting functions and the eigenvalues $E_n$. Conversely, we reconstruct some coefficients of polynomial potentials from asymptotic expansions of the eigenvalues.

For an integer $m \geq 3$ and $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, we consider the non-self-adjoint eigenvalue problems

\begin{equation}
(H_{P}^{\alpha,\beta} u)(x) := \left[-\frac{d^2}{dx^2} + x^m + P(x)\right] u(x) = Eu(x), \quad 0 \leq x < +\infty,
\end{equation}

for some $E \in \mathbb{C}$, with the boundary condition

\begin{equation}
\alpha u(0) + \beta u'(0) = 0 \quad \text{and} \quad u(+\infty) = 0,
\end{equation}

where $P$ is a polynomial of degree at most $m - 1$ of the form

\[ P(x) = a_1x^{m-1} + a_2x^{m-2} + \cdots + a_{m-1}x, \quad a_j \in \mathbb{C} \quad \text{for} \; 1 \leq j \leq m - 1. \]

If a nonconstant function $u$ satisfies (1.1) with some $E \in \mathbb{C}$ and the boundary condition (1.2), then we call $E$ an eigenvalue of $H_{P}^{\alpha,\beta}$ and $u$ an eigenfunction of $H_{P}^{\alpha,\beta}$ associated with the eigenvalue $E$. Also, the geometric multiplicity of an eigenvalue $E$ is the number of linearly independent eigenfunctions associated with the eigenvalue $E$.

We number the eigenvalues $\{E_n\}_{n \geq n_0}$ in the order of nondecreasing magnitudes, counting their “algebraic multiplicities”, where the integer $n_0$ could depend on the potential and the boundary condition. In Theorem 1.2 we show that for every large $n \in \mathbb{N}$, there exists $E_n$.

Date: February 16, 2005.
satisfying (1.3) below. However, we do not know the number of eigenvalues “near” zero, and this is why we need the number $n_0$.

Throughout this paper, we use $E_n$ to denote the eigenvalues $E_n = E_n(m, P, \alpha, \beta)$ of $H_{P}^{\alpha,\beta}$, without explicitly indicating their dependence on the potential and the boundary condition. Also, we let

$$a := (a_1, a_2, \ldots, a_{m-1}) \in \mathbb{C}^{m-1}$$

be the coefficient vector of $P$.

Before we state our main theorems, we first introduce some known facts by Sibuya [5] about the eigenvalues $E_n$ of $H_{P}^{\alpha,\beta}$.

**Theorem 1.1.** The eigenvalues $E_n$ of $H_{P}^{\alpha,\beta}$ have the following properties.

(I) The set of all eigenvalues is a discrete set in $\mathbb{C}$.

(II) The geometric multiplicity of every eigenvalue is one.

(III) Infinitely many eigenvalues, accumulating at infinity, exist.

This paper contains results on direct and inverse spectral problems. Theorem 1.2 below is the main result, regarding asymptotic expansions of “eigenvalue counting functions”. The other results stated below in the Introduction are deduced from Theorem 1.2.

**Direct spectral problem.** Here, we first introduce the following theorem, regarding asymptotic expansions of a kind of eigenvalue counting functions, where we use multi-index notations with

$$\xi = (\xi_1, \xi_2, \ldots, \xi_{m-1}) \in (\mathbb{N} \cup \{0\})^{m-1}, \quad \text{and} \quad \eta = (1, 2, \ldots, m-1).$$

Also, we use $|\xi| = \xi_1 + \xi_2 + \cdots + \xi_{m-1}$, $\xi! = \xi_1! \xi_2! \cdots \xi_{m-1}!$ and $a^\xi = a_{\xi_1}^1 a_{\xi_2}^2 \cdots a_{\xi_{m-1}}^{m-1}$. Also, $\lfloor x \rfloor$ is the largest integer that is less than or equal to $x \in \mathbb{R}$.

**Theorem 1.2.** For $a \in \mathbb{C}^{m-1}$, the eigenvalues $E_n$ of $H_{P}^{\alpha,\beta}$ satisfy

$$\frac{1}{\pi} \sum_{j=0}^{\left\lfloor \frac{m+2}{2} \right\rfloor} d_j(a) E_n^{\frac{j}{m} + \frac{1}{m} + o(1)} = \begin{cases} n - \frac{1}{4}, & \text{if } \beta = 0, \\ n + \frac{1}{4}, & \text{if } \beta \neq 0, \end{cases}$$

as $n \to +\infty$, where the error term is uniform on any compact set of $a \in \mathbb{C}^{m-1}$ and

$$d_j(a) = \begin{cases} \cos \left( \frac{(j-1)\pi}{m} \right) K_{m,j}(a) & \text{if } 0 \leq j \leq \frac{m+1}{2}, \\ -\frac{\nu(a)}{m} \pi & \text{if } m \text{ is even and } j = \frac{m+2}{2}, \end{cases}$$

where

$$K_{m,0}(a) = K_{m,0,0} = \frac{B_{\frac{1}{2}, 1 + \frac{1}{m}}}{2 \cos \left( \frac{\pi}{m} \right)}, \quad K_{m,j}(a) = \sum_{k=1}^{j} b_{j,k}(a) K_{m,j,k}, \quad 1 \leq j \leq \frac{m+2}{2}.$$
Here $B(\cdot, \cdot)$ is the beta function and

$$K_{m,j,k} = \begin{cases} 
\int_0^\infty \left( \frac{t^{mk-j}}{(m+1)^{k+\frac{1}{2}}} - t^{m-j} \right) \, dt, & \text{if } 1 \leq k \leq j \leq \frac{m+1}{2} \text{ or } k = j = 0, \\
\int_0^\infty \left( \frac{t^{mk-j-1}}{(m+1)^{k+\frac{1}{2}}} - \frac{1}{t+1} \right) \, dt, & \text{if } m \text{ is even and } 1 \leq k \leq j = \frac{m+2}{2},
\end{cases}$$

(1.6)

$$b_{j,k}(a) = \left( \frac{1}{k} \right) \sum_{\frac{m+1}{2}} \frac{k! \alpha^k}{\xi^k} \xi \leq j, \quad 1 \leq k \leq \frac{m+2}{2},$$

(1.7)

$$\nu(a) = \begin{cases} 
\sum_{k=1}^{\frac{m+1}{2}} b_{\frac{m+1}{2},k}(a) & \text{if } m \text{ is even}, \\
0 & \text{if } m \text{ is odd}.
\end{cases}$$

One can compute $K_{m,j,k}$ directly (or see [4]):

$$K_{m,j,k} = \begin{cases} 
-\frac{2}{m} & \text{if } j = k = 1, \\
-\frac{2k-1}{m+2-2j} B \left( k - \frac{j-1}{m}, \frac{1}{2} + \frac{j-1}{m} \right) & \text{if } 1 \leq k \leq j \leq \frac{m+1}{2}, j \neq 1, \\
\frac{2}{m} \left( \ln 2 - \frac{1}{1} - \frac{1}{3} - \cdots - \frac{1}{2k-5} - \frac{1}{2k-3} \right) & \text{if } m \text{ is even, } 1 \leq k \leq j = \frac{m+2}{2}.
\end{cases}$$

We obtain (1.3) by investigating the asymptotic expansions of an entire function (the Stokes multiplier) whose zeros are the eigenvalues. In this paper, the “algebraic multiplicity” of an eigenvalue is the multiplicity of the zero of the Stokes multiplier.

Next, we let $N(t), t \in \mathbb{R}$, be the eigenvalue counting function, that is, $N(t)$ is the number of eigenvalues $E$ of $H_P^{a,\beta}$ such that $|E| \leq t$. Then the following theorem on an asymptotic expansion of the eigenvalue counting function is a consequence of Theorem 1.2.

**Theorem 1.3.** Let $a \in \mathbb{C}^{m-1}$ be fixed. Suppose that $\text{Im} \ (K_{m,j}(a)) = 0$ for $1 \leq j \leq \frac{m+2}{2}$. Then $N(t)$ has the asymptotic expansion

$$N(t) = \frac{1}{\pi} \sum_{j=0}^{\lfloor \frac{m+1}{2} \rfloor} \cos \left( \frac{(j-1)\pi}{m} \right) K_{m,j}(a) t^{\frac{j-1}{m}} + O(1), \quad \text{as } t \to +\infty,$$

(1.8)

where the error $O(1)$ is uniform for any compact set of $a \in \mathbb{C}^{m-1}$.

**Proof.** In Corollary 1.5, we show that $|E_n| < |E_{n+1}|$ for all large $n \in \mathbb{N}$.

Suppose that $|E_n| \leq t < |E_{n+1}|$. Then since for $s \in \mathbb{R}$,

$$\left( n + 1 \pm \frac{1}{4} \right)^s = \left( n \pm \frac{1}{4} \right)^s + O \left( n^{s-1} \right), \quad \text{as } n \to \infty,$$

we see from Theorem 1.4 below that $|E_{n+1} - E_n| = O \left( n^{\frac{m-2}{m+2}} \right)$. Thus,

$$E_n^{\frac{\frac{1}{2} - \frac{1}{m}}{m}} = t^{\frac{\frac{1}{2} - \frac{1}{m}}{m}} \left( \frac{1 - t - E_n}{t} \right)^{\frac{\frac{1}{2} - \frac{1}{m}}{m}} = t^{\frac{\frac{1}{2} - \frac{1}{m}}{m}} \left( 1 + O \left( \frac{t - E_n}{t} \right) \right) = t^{\frac{\frac{1}{2} - \frac{1}{m}}{m}} + O(1).$$
Hence, replacing \( E_n^{\frac{1}{2} - \frac{m}{m+2}} \) in (1.3) by \( t^{\frac{1}{2} - \frac{m}{m+2}} + O(1) \) and solving the resulting equation for \( n \) complete the proof. \( \square \)

Next, from (1.3) we get \( E_n \) in terms of \( n \).

**Theorem 1.4.** For each \( a \in \mathbb{C}^{m-1} \), there exist some constants \( e_j(a) \in \mathbb{C} \), \( 2 \leq j \leq \frac{m+2}{2} \), such that

\[
E_n = E_{n,0} + \sum_{j=1}^{\frac{m+2}{2}} e_j(a) E_{n,0}^{1-\frac{j}{m}} + o \left( E_{n,0}^{1-\frac{\frac{m+2}{2}}{m}} \right), \quad \text{as } n \to +\infty,
\]

where the error term is uniform for any compact set of \( a \in \mathbb{C}^{m-1} \) and where

\[
E_{n,0} = \left( \frac{2\sqrt{\pi} \Gamma \left( \frac{3}{2} + \frac{1}{m} \right)}{\Gamma \left( 1 + \frac{1}{m} \right)} \right)^{\frac{2m}{m+2}} \times \begin{cases} 
( n - \frac{1}{4} )^{\frac{2m}{m+2}} , & \text{if } \beta = 0, \\
( n + \frac{1}{4} )^{\frac{2m}{m+2}} , & \text{if } \beta \neq 0,
\end{cases}
\]

and \( e_j(a) \), \( 0 \leq j \leq \frac{m+2}{2} \), are defined recurrently by \( e_0(a) = 1 \) and

\[
e_j(a) = -\frac{2m}{m+2} \left( \frac{d_j(a)}{d_0(a)} + \sum_{|\xi|=k \geq 2} \left( \frac{1}{2} + \frac{1}{m} \right) k! \varepsilon(\xi)^{\xi} \right) + \sum_{r=1}^{j-1} \frac{d_r(a)}{d_0(a)} \sum_{|\xi|=k} \left( \frac{1}{2} + \frac{1-r}{m} \right) k! \varepsilon(\xi)^{\xi},
\]

where \( \varepsilon(\xi) = (\varepsilon_1(a), \varepsilon_2(a), \ldots, \varepsilon_{m-1}(a)) \).

We note, for the first summation in the definition of \( e_j(a) \) above, that \( \xi \cdot \eta = j \) implies \( \xi_\ell = 0 \) whenever \( \ell \geq j \). Also, for the second summation, we point out that \( \xi \cdot \eta = j-r \leq j-1 \) implies \( \xi_\ell = 0 \) whenever \( \ell \geq j \).

When \( P \) is real (i.e., \( a \in \mathbb{R}^{m-1} \)) and \( x^m + P(x) \) is increasing and convex downwards on \([0, +\infty)\), Titchmarsh [6, Chap. 7] showed that

\[
N(t) = \frac{1}{\pi} \int_{t^{-\infty}}^{x_0} \sqrt{t-x^m - P(x)} \, dx + O(1),
\]

where \( x_0 = x_0(t) > 0 \) such that \( t = x_0^m + P(x_0) \), provided that \( \alpha = 0 \) or \( \beta/\alpha \) real. Then from (1.10) one could get (1.8) and hence (1.3).

Voros [7] (cf. [8]) studied (1.1) with arbitrary real polynomials \( P \) under Dirichlet (\( \beta = 0 \)) and Neumann (\( \alpha = 0 \)) boundary conditions at \( x = 0 \), and computed \( d_0(a) \) and \( d_1(a) \) explicitly.

Fedoryuk [1, §3.3] considered (1.1) with complex polynomial potentials and showed the existence of asymptotic expansions of the eigenvalues to all orders. Also, he computed \( E_{n,0} \) explicitly. However, to the best of my knowledge Theorem 1.2 in this generality does not appear in the literature to the date.

Regarding monotonicity of modulus of \( E_n \) for all large \( n \in \mathbb{N} \).

**Corollary 1.5.** For each \( a \in \mathbb{C}^{m-1} \) there exists \( M > 0 \) such that \( |E_n| < |E_{n+1}| \) if \( n \geq M \).
**Proof.** This is a consequence of Theorem 1.4. Or one can see that proof of Theorem 3 in [3] can be easily adapted for this case. □

**Inverse spectral problem.** Here, we introduce results on inverse spectral problems, but first the following corollary is an easy consequence of Theorems 1.2 and 1.4, regarding how the coefficients of the asymptotic expansions depend on $a \in \mathbb{C}^{m-1}$.

**Corollary 1.6.** Let $1 \leq j \leq \frac{m+2}{2}$ be a fixed integer. Then

(i) $d_j(a)$ and $e_j(a)$ are polynomials in $a_1, a_2, \ldots, a_{j-1}, a_j$. In particular, $d_j(a)$ and $e_j(a)$ are nonconstant linear functions of $a_j$.

(ii) $d_j(a)$ and $e_j(a)$ do not depend on $a_{j+1}, a_{j+2}, \ldots, a_{m-1}$.

**Proof.** Statements on $d_j(a)$ are direct consequences of the definition of $d_j(a)$ in Theorem 1.2. One can use statements on $d_j(a)$ and induction on $j$ to prove statements on $e_j(a)$. □

Next, one can reconstruct some coefficients of the polynomial potential from the asymptotic expansion of the eigenvalues.

**Theorem 1.7.** Let $1 \leq j \leq \frac{m+1}{2}$ be a fixed integer. Then the asymptotic expansions of the eigenvalues $E_n$ of $H_{\alpha,\beta}^P$ of type (1.9) with an error term $o\left(n^{\frac{2(m-2j)}{m+2}}\right)$ uniquely and explicitly determine $a_k$ for all $1 \leq k \leq j$.

**Proof.** From the asymptotic expansion of the eigenvalues, one gets $e_k(a)$ as an explicit polynomial in $a_1, a_2, \ldots, a_k$ for every $1 \leq k \leq j$. Then since $e_k(a)$ is a nonconstant linear function of $a_k$ and since $e_k(a)$ does not depend on $a_\ell$, $\ell > k$, all $a_1, a_2, \ldots, a_j$ can be found uniquely and explicitly. □

When $m$ is even, $j = \frac{m+2}{2}$ is allowed in Corollary 1.6 while it is not allowed in Theorem 1.7. This is due to the fact that our method in this paper does not determine the number $n_0$ in $\{E_n\}_{n \geq n_0}$.

2. **Properties of the solutions**

In this section, we introduce work of Hille [2] and Sibuya [5] about properties of the solutions of (1.1).

We first set

$$\lambda = -E$$

and extend (1.1) to the complex plane so that if $u$ is a solution of (1.1) then

$$-u''(z) + |z|^m + P(z) + \lambda |u(z) = 0, \quad z \in \mathbb{C}. \quad (2.1)$$

It is known that solutions of (2.1) have rather simple asymptotic behavior near infinity in the complex plane [2, §7.4]. We will describe this simple asymptotic behavior of the solutions near infinity by using the following definition.
Figure 1. The Stokes sectors for $m = 3$. The dashed rays represent $\arg z = \pm \frac{\pi}{5}, \pm \frac{3\pi}{5}, \pi$.

**Definition.** The Stokes sectors $S_k$ of the equation (2.1) are

$$S_k = \left\{ z \in \mathbb{C} : \left| \arg(z) - \frac{2k\pi}{m+2} \right| < \frac{\pi}{m+2} \right\}$$

for $k \in \mathbb{Z}$.

See Figure 1.

Hille [2, §7.4] showed that every nonconstant solution of (2.1) either decays to zero or blows up exponentially, in each Stokes sector $S_k$.

**Lemma 2.1** ([2, §7.4]).

(i) For each $k \in \mathbb{Z}$, every solution $u$ of (2.1) is asymptotic to

$$z^{-\frac{m}{2}} \exp \left[ \pm \int z^{m} + P(z) + \lambda \frac{3}{2} dz \right]$$

as $z \to \infty$ in every closed subsector of $S_k$.

(ii) If a nonconstant solution $u$ of (2.1) decays in $S_k$, it must blow up in $S_{k-1} \cup S_{k+1}$.

However, when $u$ blows up in $S_k$, $u$ need not be decaying in $S_{k-1}$ or in $S_{k+1}$.

Lemma 2.1 (i) implies that if $u$ decays along one ray in $S_k$, then it decays along all rays in $S_k$. Also, if $u$ blows up along one ray in $S_k$, then it blows up along all rays in $S_k$.

We will use

$$\omega = \exp \left[ \frac{2\pi i}{m+2} \right]$$

and we define

$$b_j(a) = \sum_{k=1}^{j} b_{j,k}(a), \quad 1 \leq j \leq \frac{m+2}{2}.$$

We further define $r_m = -\frac{m}{4}$ if $m$ is odd, and $r_m = -\frac{m}{4} - \frac{b_{m+1}}{2}$ if $m$ is even.

Now we are ready to introduce some results of Sibuya [5] that is the main ingredient of the proof of Theorem 1.2.
Theorem 2.2. Equation (2.1), with \( a \in \mathbb{C}^{m-1} \), admits a solution \( f(z, a, \lambda) \) with the following properties.

(i) \( f(z, a, \lambda) \) is an entire function of \( z, a \) and \( \lambda \).

(ii) \( f(z, a, \lambda) \) and \( f'(z, a, \lambda) = \frac{\partial}{\partial z} f(z, a, \lambda) \) admit the following asymptotic expansions. Let \( \varepsilon > 0 \). Then

\[
\begin{align*}
    f(z, a, \lambda) &= z^{r_m} (1 + O(z^{-1/2})) \exp \left[ -F(z, a, \lambda) \right], \\
    f'(z, a, \lambda) &= -z^{r_m+\frac{m}{2}} (1 + O(z^{-1/2})) \exp \left[ -F(z, a, \lambda) \right],
\end{align*}
\]

as \( z \) tends to infinity in the sector \( |\arg z| \leq \frac{3\pi}{m+2} - \varepsilon \), uniformly on each compact set of \((a, \lambda)\)-values. Here

\[
F(z, a, \lambda) = \frac{2}{m+2} z^{\frac{m}{2}+1} + \sum_{1 \leq j < \frac{m}{2}+1} \frac{2}{m+2-2j} b_j(a) z^{\frac{1}{2}(m+2-2j)}.
\]

(iii) For each fixed \( a \in \mathbb{C}^{m-1} \) and \( \delta > 0 \), \( f \) and \( f' \) also admit the asymptotic expansions,

\[
\begin{align*}
    f(0, a, \lambda) &= [1 + o(1)] \lambda^{-1/4} \exp \left[ L(a, \lambda) \right], \\
    f'(0, a, \lambda) &= - [1 + o(1)] \lambda^{1/4} \exp \left[ L(a, \lambda) \right],
\end{align*}
\]

as \( \lambda \to \infty \) in the sector \( |\arg(\lambda)| \leq \pi - \delta \), uniformly on each compact set of \( a \in \mathbb{C}^{m-1} \), where

\[
L(a, \lambda) = \begin{cases} 
    \int_0^{+\infty} \left( \sqrt{t^m + P(t)} + \lambda - t^{\frac{m}{2}} - \sum_{j=1}^{m+1} b_j(a) t^{\frac{m}{2}-j} \right) dt & \text{if } m \text{ is odd}, \\
    \int_0^{+\infty} \left( \sqrt{t^m + P(t)} + \lambda - t^{\frac{m}{2}} - \sum_{j=1}^{m} b_j(a) t^{\frac{m}{2}-j} - \frac{b_{m+1}(a)}{t+1} \right) dt & \text{if } m \text{ is even}.
\end{cases}
\]

(iv) The entire functions \( \lambda \mapsto f(0, a, \lambda) \) and \( \lambda \mapsto f'(0, a, \lambda) \) have orders \( \frac{1}{2} + \frac{1}{m} \).

Proof. In Sibuya’s book [5], see Theorem 6.1 for a proof of (i) and (ii), and Theorem 19.1 for a proof of (iii). Moreover, (iv) is a consequence of (iii) along with Theorem 20.1 in [5]. Note that properties (i), (ii), and (iii) are summarized on pages 112–113 of Sibuya [5]. \( \square \)

Remarks. (I) Uniformity of the error term in Theorem 1.2 is essentially due to uniformity of error terms in \( (2.3) \) and \( (2.4) \).

(II) In this paper we will deal with numbers like \( (\omega^\nu \lambda)^s \) for some \( s \in \mathbb{R} \), and \( \nu \in \mathbb{C} \). As usual, we will use

\[
\omega^\nu = \exp \left[ \nu \frac{2\pi i}{m+2} \right]
\]

and if \( \arg(\lambda) \) is specified, then

\[
\arg((\omega^\nu \lambda)^s) = s [\arg(\omega^\nu) + \arg(\lambda)] = s \left[ \Re(\nu) \frac{2\pi}{m+2} + \arg(\lambda) \right], \quad s \in \mathbb{R}.
\]

If \( s \not\in \mathbb{Z} \) then the branch of \( \lambda^s \) is chosen to be the negative real axis.

In [3], the following asymptotic expansion of \( L(a, \cdot) \) is proved.
Lemma 2.3. Let \( m \geq 3 \) and \( a \in \mathbb{C}^{m-1} \) be fixed. Then

\[
L(a, \lambda) = \begin{cases} 
\sum_{j=0}^{\frac{m+1}{3}} K_{m,j}(a) \lambda^{\frac{j+1}{m}} + O \left( |\lambda|^{-\frac{3}{2m}} \right) \text{ if } m \text{ is odd}, \\
\sum_{j=0}^{\frac{m+1}{2}} K_{m,j}(a) \lambda^{\frac{j+1}{m}} - \frac{b_{m+1}(a)}{m} \ln(\lambda) + O \left( |\lambda|^{-\frac{1}{m}} \right) \text{ if } m \text{ is even},
\end{cases}
\]

as \( \lambda \to \infty \) in the sector \( |\arg(\lambda)| \leq \pi - \delta \), uniformly on each compact set of \( a \in \mathbb{C}^{m-1} \).

\[\text{Proof.} \text{ See [3] for a proof.} \]}

Sibuya [5] introduced solutions of (2.1) that decays in \( S_k, k \in \mathbb{Z} \). Before we introduce this, we let

\[(2.5) \quad G^\ell(a) := (\omega^{-\ell} a_1, \omega^{-2\ell} a_2, \ldots, \omega^{-(m-1)\ell} a_{m-1}) \quad \text{for} \quad \ell \in \mathbb{Z}.
\]

Then we have the following lemma, regarding properties of \( G^\ell(\cdot) \).

Lemma 2.4. For \( a \in \mathbb{C}^{m-1} \) fixed, and \( \ell_1, \ell_2, \ell \in \mathbb{Z} \), \( G^{\ell_1}(G^{\ell_2}(a)) = G^{\ell_1+\ell_2}(a) \), and

\[
b_{j,k}(G^\ell(a)) = \omega^{-\ell} b_{j,k}(a), \quad \ell \in \mathbb{Z}.
\]

Next, recall that the function \( f(z, a, \lambda) \) in Theorem 2.2 solves (2.1) and decays to zero exponentially as \( z \to \infty \) in \( S_0 \), and blows up in \( S_{-1} \cup S_1 \). One can check that the function

\[(2.6) \quad f_k(z, a, \lambda) := f(\omega^{-k} z, G^k(a), \omega^{2k} \lambda), \quad k \in \mathbb{Z},
\]

which is obtained by scaling \( f(z, G^k(a), \omega^{2k} \lambda) \) in the \( z \)-variable, also solves (2.1). It is clear that \( f_0(z, a, \lambda) = f(z, a, \lambda) \), and that \( f_k(z, a, \lambda) \) decays in \( S_k \) and blows up in \( S_{k-1} \cup S_{k+1} \) since \( f(z, G^k(a), \omega^{2k} \lambda) \) decays in \( S_0 \). Since no nonconstant solution decays in two consecutive Stokes sectors (see Lemma 2.1 (ii)), \( f_0 \) and \( f_{-1} \) are linearly independent and hence any solution of (2.1) can be expressed as a linear combination of these two. Especially, there exist some coefficients \( C(a, \lambda) \) and \( \tilde{C}(a, \lambda) \) such that

\[
f_1(z, a, \lambda) = C(a, \lambda) f_0(z, a, \lambda) + \tilde{C}(a, \lambda) f_{-1}(z, a, \lambda).
\]

We then see that

\[(2.7) \quad C(a, \lambda) = \frac{W_{-1,1}(a, \lambda)}{W_{-1,0}(a, \lambda)} \quad \text{and} \quad \tilde{C}(a, \lambda) = \frac{W_{1,0}(a, \lambda)}{W_{-1,0}(a, \lambda)},
\]

where \( W_{j,\ell} = f_j f'_\ell - f'_j f_\ell \) is the Wronskian of \( f_j \) and \( f_\ell \). Since both \( f_j, f_\ell \) are solutions of the same linear equation (2.1), we know that the Wronskians are constant functions of \( z \). Also, \( f_k \) and \( f_{k+1} \) are linearly independent, and hence \( W_{k,k+1} \neq 0 \) for all \( k \in \mathbb{Z} \).

Moreover, we have the following lemma that is useful later on.

Lemma 2.5. Suppose \( k, j \in \mathbb{Z} \). Then

\[(2.8) \quad W_{k+1,j+1}(a, \lambda) = \omega^{-1} W_{k,j}(G(a), \omega^2 \lambda),
\]
and $W_{0,1}(a, \lambda) = 2\omega^{\mu(a)}$, where

$$
\mu(a) = \begin{cases} 
\frac{m}{4} - b\frac{m}{2} + 1(a) & \text{if } m \text{ is odd}, \\
\frac{m}{4} & \text{if } m \text{ is even}.
\end{cases}
$$

Proof. See Sibuya [5, pages 116-118].

Thus, by Lemma 2.5,

$$
\tilde{C}(a, \lambda) = W_{1,0}(a, \lambda) = - \frac{2\omega^{\mu(a)}}{2\omega^{\mu(G^{-1}(a))}} = -\omega^{-1-2\nu(a)},
$$

where $\nu(a) = \frac{m}{4} - \mu(a)$, that is,

$$
(2.9) \quad \nu(a) = \begin{cases} 
0 & \text{if } m \text{ is odd}, \\
b\frac{m}{2} + 1(a) & \text{if } m \text{ is even}.
\end{cases}
$$

3. Asymptotics of $f(0, a, \lambda)$ and $f'(0, a, \lambda)$

The asymptotics of $f(0, a, \lambda)$ and $f'(0, a, \lambda)$ as $\lambda \to \infty$ in the sector $|\arg(\lambda)| \leq \pi - \delta$ are given by (2.3) and (2.4), respectively. In this section, we provide the asymptotics of $f(0, a, \lambda)$ and $f'(0, a, \lambda)$ as $\lambda \to \infty$ in a sector near the negative real axis.

In [3], we showed the following asymptotic expansion of $W_{-1,1}(a, \lambda)$ as $\lambda \to \infty$ in a sector near the negative real axis.

**Theorem 3.1.** Let $m \geq 3$, $a \in \mathbb{C}^{m-1}$ and $0 < \delta < \frac{\pi}{m+2}$ be fixed. Then

$$
W_{-1,1}(a, \lambda) = [2i + o(1)] \exp \left[ L(G^{-1}(a), \omega^{-2}\lambda) + L(G(a), \omega^{-m}\lambda) \right],
$$

as $\lambda \to \infty$ along the rays in the sector

$$
(3.2) \quad \pi - \frac{4\pi}{m+2} + \delta \leq \arg(\lambda) \leq \pi + \frac{4\pi}{m+2} - \delta,
$$

where the error term is uniform on any compact set of $a \in \mathbb{C}^{m-1}$.

Proof. See [3, Theorem 12] for a proof.

We will use this in the next theorem, regarding asymptotics of $f(0, a, \lambda)$ and $f'(0, a, \lambda)$ near the negative real axis.

**Theorem 3.2.** Let $a \in \mathbb{C}^{m-1}$ be fixed. Then

$$
\begin{align*}
f(0, a, \lambda) &= \left( \frac{i}{2} \omega^{-\nu(a)} + o(1) \right) \lambda^{-\frac{i}{4}} \exp \left[ -L(G^{-1}(a), \omega^{-2}\lambda) \right] \\
&\quad + \left( \frac{1}{2} \omega^{-3\nu(a)} + o(1) \right) \lambda^{-\frac{i}{4}} \exp \left[ -L(G(a), \omega^{-m}\lambda) \right], \\
f'(0, a, \lambda) &= \left( \frac{i}{2} \omega^{-\nu(a)} + o(1) \right) \lambda^{\frac{i}{4}} \exp \left[ -L(G^{-1}(a), \omega^{-2}\lambda) \right] \\
&\quad - \left( \frac{1}{2} \omega^{-3\nu(a)} + o(1) \right) \lambda^{\frac{i}{4}} \exp \left[ -L(G(a), \omega^{-m}\lambda) \right], \quad \text{as } \lambda \to \infty \text{ in } (3.2),
\end{align*}
$$
where the error terms are uniform on any compact set of \( a \in \mathbb{C}^{m-1} \) and where \( \arg \left( \lambda^{\frac{1}{4}} \right) = \pm \frac{1}{4} \arg(\lambda) \) in the sector (3.2).

Proof. From (2.6) and (2.7), and Lemma 2.5, we have

\[
f(z, a, \lambda) = f_0(z, a, \lambda) = \frac{1}{C(a, \lambda)} \left[ f_1(z, a, \lambda) - \tilde{C}(a, \lambda)f_{-1}(z, a, \lambda) \right]
\]

(3.5)

\[
= 2\omega^{1+\mu(G^{-1}(a))} W_{-1,1}(a, \lambda) \left[ f(\omega^{-1} z, G(a), \omega^2 \lambda) + \omega^{-1-2\nu(a)} f(\omega z, G^{-1}(a), \omega^{-2} \lambda) \right].
\]

So we examine asymptotics of \( f(\omega^{-1} z, G(a), \omega^2 \lambda) + \omega^{-1-2\nu(a)} f(\omega z, G^{-1}(a), \omega^{-2} \lambda) \) and its derivative at \( z = 0 \). Using (2.3) and the fact that \( f(0, G(a), \omega^2 \lambda) = f(0, G(a), \omega^{-m} \lambda) \), we have

\[
f(0, G(a), \omega^2 \lambda) + \omega^{-1-2\nu(a)} f(0, G^{-1}(a), \omega^{-2} \lambda) = f(0, G(a), \omega^{-m} \lambda) + \omega^{-1-2\nu(a)} f(0, G^{-1}(a), \omega^{-2} \lambda)
\]

\[
= (1 + o(1)) (\omega^{-m} \lambda)^{-\frac{1}{4}} \exp[L(G(a), \omega^{-m} \lambda)]
\]

\[
+ \omega^{-1-2\nu(a)} (1 + o(1)) (\omega^{-2} \lambda)^{-\frac{1}{4}} \exp[L(G^{-1}(a), \omega^{-2} \lambda)]
\]

\[
= (\omega^\frac{m}{2} + o(1)) \lambda^{-\frac{1}{4}} \exp[L(G(a), \omega^{-m} \lambda)] + (\omega^{-1-2\nu(a)} + o(1)) \lambda^{-\frac{1}{4}} \exp[L(G^{-1}(a), \omega^{-2} \lambda)],
\]

as \( \lambda \to \infty \) in the sector (3.2). Then (3.3) is obtained from (3.1) and (3.5).

Next, we differentiate (3.5) with respect to \( z \) and evaluate the resulting equation at \( z = 0 \) to get

\[
f'(0, a, \lambda) = 2\omega^{1+\mu(G^{-1}(a))} W_{-1,1}(a, \lambda) \left[ \omega^{-1} f'(0, G(a), \omega^{-m} \lambda) + \omega^{-2\nu(a)} f'(0, G^{-1}(a), \omega^{-2} \lambda) \right].
\]

(3.6)

Using (2.4), we have

\[
\omega^{-1} f'(0, G(a), \omega^{-m} \lambda) + \omega^{-2\nu(a)} f'(0, G^{-1}(a), \omega^{-2} \lambda) = \left(i \omega^{-\frac{1}{2}} + o(1)\right) \lambda^{\frac{1}{4}} \exp[L(G(a), \omega^{-m} \lambda)] - \left(\omega^{-1-2\nu(a)} + o(1)\right) \lambda^{\frac{1}{4}} \exp[L(G^{-1}(a), \omega^{-2} \lambda)],
\]

as \( \lambda \to \infty \) in the sector (3.2). Then this along with (3.1) and (3.6) yields (3.4).

Finally, the uniformity of the error terms in (3.5) and (3.6) is due to the uniformity of the error terms in (2.3), (2.4), and (3.1). \( \square \)

4. Proof of Theorem 1.2

In this section we prove Theorem 1.2.

Proof of Theorem 1.2 for Dirichlet boundary condition at \( x = 0 \). From (3.3)

\[
2\omega^3 a \lambda^{\frac{1}{4}} \exp[L(G^{-1}(a), \omega^{-2} \lambda) + o(1)] f(0, a, \lambda)
\]

\[
= \exp[L(G^{-1}(a), \omega^{-2} \lambda) - L(G(a), \omega^{-m} \lambda) + o(1)] + i\omega^{2\nu(a)}, \quad \text{as } \lambda \to \infty \text{ in } (3.2).
\]
Since
\[ L(G^{-1}(a), \omega^{-2} \lambda) - L(G(a), \omega^{-m} \lambda) \]
\[ = K_m (\omega^{-2} \lambda)^{\frac{1}{2} + \frac{1}{m}} (1 + o(1)) - K_m (\omega^{-m} \lambda)^{\frac{1}{2} + \frac{1}{m}} (1 + o(1)) \]
\[ = K_m \left( \exp \left[ -\frac{2\pi}{m} i \right] - \exp [-\pi i] \right) \lambda^{\frac{1}{2} + \frac{1}{m}} (1 + o(1)) \]
\[ = K_m \left( 1 + \exp \left[ -\frac{2\pi}{m} i \right] \right) \lambda^{\frac{1}{2} + \frac{1}{m}} (1 + o(1)), \]
and since \( \arg \left( 1 + \exp \left[ -\frac{2\pi}{m} i \right] \right) = -\frac{\pi}{m} \), we have
\[ \arg \left( L(G^{-1}(a), \omega^{-2} \lambda) - L(G(a), \omega^{-m} \lambda) \right) = -\frac{\pi}{m} + \frac{m + 2}{2m} \arg(\lambda) + o(1). \]
Thus, if \( \pi - \frac{4\pi}{m+2} + \delta \leq \arg(\lambda) \leq \pi + \frac{4\pi}{m+2} - \delta \) and \(|\lambda|\) is large, we have
\[ (4.1) \quad \frac{\pi}{2} - \frac{2\pi}{m} + o(1) \leq \arg \left( L(G^{-1}(a), \omega^{-2} \lambda) - L(G(a), \omega^{-m} \lambda) \right) \leq \frac{\pi}{2} + \frac{2\pi}{m} + o(1). \]
So \( \lambda \mapsto L(G^{-1}(a), \omega^{-2} \lambda) - L(G(a), \omega^{-m} \lambda) \) maps the sector (3.2) near infinity onto a region containing \( |\arg(\lambda) - \frac{\pi}{2}| \leq \epsilon_1 \) and \( |\lambda| \geq M_0 \) for some positive real numbers \( \epsilon, M_0 \). Hence, there exists a sequence of the numbers \( \lambda_n \) in (3.2) such that
\[ (4.2) \quad L(G^{-1}(a), \omega^{-2} \lambda_n) - L(G(a), \omega^{-m} \lambda_n) + o(1) \underset{n \to +\infty}{\to} 2n\pi i + 2\nu(a) \frac{2\pi i}{m+2} - \frac{\pi}{2} i, \]
for all large \( n \in \mathbb{N} \) so that \( f(0, a, \lambda_n) = 0 \). Next, by (1.5) and Lemma 2.3
\[ L(G^{-1}(a), \omega^{-2} \lambda_n) - L(G(a), \omega^{-m} \lambda_n) \]
\[ = \sum_{j=0}^{[\frac{m}{2}]+1} \left( K_{m,j}(G^{-1}(a))(\omega^{-2} \lambda_n)^{\frac{1}{2} + \frac{1}{m}} - K_{m,j}(G(a))(\omega^{-m} \lambda_n)^{\frac{1}{2} + \frac{1}{m}} \right) \]
\[ = \frac{\nu(G^{-1}(a))}{m} \ln(\omega^{-2} \lambda_n) + \frac{\nu(G(a))}{m} \ln(\omega^{-m} \lambda_n) + o(1) \]
\[ = \sum_{j=0}^{[\frac{m}{2}]+1} \left( \omega^j \omega^{-2(\frac{1}{2} + \frac{1}{m})} - \omega^{-j} \omega^{-m(\frac{1}{2} + \frac{1}{m})} \right) K_{m,j}(\lambda_n^{\frac{1}{2} + \frac{1}{m}}) \]
\[ - \frac{\nu(a)}{m} \frac{4\pi}{m+2} i + \frac{\nu(a)}{m} \frac{2m\pi}{m+2} i + o(1) \]
\[ = 2i \sum_{j=0}^{[\frac{m}{2}]+1} \sin \left( \frac{(m - 2 + 2j)\pi}{2m} \right) K_{m,j}(\lambda_n^{\frac{1}{2} + \frac{1}{m}}) + \nu(a) \frac{(2m - 4)\pi i}{m+2} i + o(1). \]
So this and (4.2) yield
\[ 2i \sum_{j=0}^{[\frac{m}{2}]+1} \sin \left( \frac{(m - 2 + 2j)\pi}{2m} \right) K_{m,j}(\lambda_n^{\frac{1}{2} + \frac{1}{m}}) - \frac{2\nu(a)}{m} \pi i + o(1) \underset{n \to +\infty}{\to} \left( 2n - \frac{1}{2} \right) \pi i. \]
Finally, we use \( \sin(\pi/2 + \theta) = \cos(\theta) \) and \( E_n = -\lambda_n \) to complete the proof. \( \square \)

Next, we prove Theorem 1.2 for the case when \( \beta \neq 0 \) in (1.2).
Proof of Theorem 1.2 for other boundary conditions. Using (3.3) and (3.4), one gets
\[\alpha f(0, a, \lambda) + \beta f'(0, a, \lambda)\]
\[= \left\{ \frac{\alpha}{2\lambda^4} (i \omega^{-\nu(a)} + o(1)) + \frac{\beta \lambda^4}{2} (i \omega^{-\nu(a)} + o(1)) \right\} \exp \left[ -L(G^{-1}(a), \omega^{-2}\lambda) \right] + \left\{ \frac{\alpha}{2\lambda^4} (\omega^{-3\nu(a)} + o(1)) - \frac{\beta \lambda^4}{2} (\omega^{-3\nu(a)} + o(1)) \right\} \exp \left[ -L(G(a), \omega^{-m}\lambda) \right]\]
(4.3)
\[= \frac{\beta \lambda^4}{2} \left\{ (i \omega^{-\nu(a)} + o(1)) \exp \left[ -L(G^{-1}(a), \omega^{-2}\lambda) \right] + (\omega^{-3\nu(a)} + o(1)) \exp \left[ -L(G(a), \omega^{-m}\lambda) \right] \right\},\]
as \(\lambda \to \infty\) in the sector (3.2), where the error terms are uniform on any compact set of \(a \in \mathbb{C}^{-1}\) and where \(\arg\left(\lambda^4 \right) = \pm \frac{\pi}{4} \arg(\lambda)\) in the sector (3.2). Since \(\beta \neq 0\),
\[\frac{2}{\beta \lambda^4} \exp \left[ L(G^{-1}(a), \omega^{-2}\lambda) + o(1) \right] [\alpha f(0, a, \lambda) + \beta f'(0, a, \lambda)] = i \omega^{-\nu(a)} - \omega^{-3\nu(a)} \exp \left[ L(G^{-1}(a), \omega^{-2}\lambda) - L(G(a), \omega^{-m}\lambda) + o(1) \right],\]
as \(\lambda \to \infty\) in the sector (3.2).
Thus, like in the proof of Theorem 1.2 for \(\beta = 0\), there exists a sequence of \(\lambda_n\) such that
\[L(G^{-1}(a), \omega^{-2}\lambda_n) - L(G(a), \omega^{-m}\lambda_n) + o(1) = \left(2n + \frac{1}{2}\right) \pi i + 2\nu(a) \frac{2\pi i}{m + 2},\]
for all large \(n \in \mathbb{N}\) so that \(\alpha f(0, a, \lambda_n) + \beta f'(0, a, \lambda_n) = 0\). Here we have \(\left(2n + \frac{1}{2}\right) \pi i\) in the place of \(\left(2n - \frac{1}{2}\right) \pi i\) in (4.2). So one can complete the proof by following the methods in the proof for \(\beta = 0\) case. \(\square\)

5. PROOF OF THEOREM 1.4
We will prove existence of \(e_j(a)\) by induction on \(j\). In doing so we will recurrently find \(e_j(a)\).
From (1.3) we have
\[\sum_{j=0}^{\left\lfloor \frac{m}{2} + 1 \right\rfloor} \frac{d_j(a)}{d_0(a)} E_n^{j + \frac{1}{2} m} + o(1) = \left\{ \begin{array}{ll}
\left( \frac{\frac{1}{2} \pi}{d_0(a)} \right)^{\frac{2n}{m+2}}, & \text{if } \beta = 0, \\
\left( \frac{\left( \frac{1}{2} \pi \right)^2}{d_0(a)} \right)^{\frac{2n}{m+2}}, & \text{if } \beta \neq 0.
\end{array} \right.\]
(5.1)
We then introduce the decomposition
\[E_n = E_{n,0} + E_{n,1},\]
where
\[E_{n,0} = \left\{ \begin{array}{ll}
\left( \frac{\frac{1}{2} \pi}{d_0(a)} \right)^{\frac{2n}{m+2}}, & \text{if } \beta = 0, \\
\left( \frac{\left( \frac{1}{2} \pi \right)^2}{d_0(a)} \right)^{\frac{2n}{m+2}}, & \text{if } \beta \neq 0.
\end{array} \right.\]
and
\[\frac{E_{n,1}}{E_{n,0}} = o(1).\]
So we have

\[ E_{n,0}^{\frac{1}{2}+\frac{1}{m}} = E_{n,0}^{\frac{1}{2}+\frac{1}{m}} \left( 1 + \frac{E_{n,1}}{E_{n,0}} \right)^{\frac{1}{2}+\frac{1}{m}} + \sum_{j=1}^{\left[ \frac{m+1}{2} \right]} \frac{d_j(a)}{d_0(a)} E_{n,0}^{\frac{1}{2}+\frac{1}{m}} \left( 1 + \frac{E_{n,1}}{E_{n,0}} \right)^{\frac{1}{2}+\frac{1}{m}} + o(1) \]

Thus,

\[ 0 = \left( \frac{1}{2} + \frac{1}{m} \right) \frac{E_{n,1}}{E_{n,0}} + \sum_{k=2}^{\infty} \left( \frac{1}{2} + \frac{1}{m} \right) \left( \frac{E_{n,1}}{E_{n,0}} \right) \]

and hence

\[ \left( \frac{1}{2} + \frac{1}{m} \right) \frac{E_{n,1}}{E_{n,0}} + \sum_{k=2}^{\infty} \left( \frac{1}{2} + \frac{1}{m} \right) \left( \frac{E_{n,1}}{E_{n,0}} \right) \]

Thus, one concludes \( E_{n,1} = E_{n,2} + E_{n,3} \), where

\[ E_{n,2} = -\frac{2m}{m + 2d_0(a)} E_{n,0}^{-\frac{1}{m}} \quad \text{and} \quad E_{n,3} = o \left( E_{n,0}^{-\frac{1}{m}} \right) \]

Hence, from (5.2) and (5.3) we have

\[ \left( \frac{1}{2} + \frac{1}{m} \right) (E_{n,2} + E_{n,3}) + \sum_{k=2}^{\infty} \left( \frac{1}{2} + \frac{1}{m} \right) (E_{n,2} + E_{n,3})^k \]

Thus provides the induction basis.
Next, suppose that \( \frac{E_{n,1}}{E_{n,0}} = E_{n,2} + E_{n,4} + \cdots + E_{n,2s} + E_{n,2s+1} \), where \( E_{n,2s+1} = o \left( \frac{1}{E_{n,0}^{\frac{1}{m}}} \right) \) and \( E_{n,2t} = e_t(a)E_{n,0}^{-\frac{1}{m}} \), \( 1 \leq t \leq s < \frac{m+2}{2} \) for some \( e_t(a) \in \mathbb{C} \). Then from (5.2)
\[
\sum_{k=1}^{\infty} \left( \frac{\frac{1}{2} + \frac{1}{m}}{k} \right) (E_{n,2} + \cdots + E_{n,2s} + E_{n,2s+1})^k
\]
\[
+ \sum_{j=1}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \frac{d_j(a)}{d_0(a)} E_{n,0}^{-\frac{1}{m}} \sum_{k=1}^{\infty} \left( \frac{\frac{1}{2} + \frac{1-j}{m}}{k} \right) (E_{n,2} + \cdots + E_{n,2s} + E_{n,2s+1})^k + o \left( E_{n,0}^{\frac{1}{m}} \right)
\]
\[
= - \sum_{j=1}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \frac{d_j(a)}{d_0(a)} E_{n,0}^{-\frac{1}{m}}.
\]

Hence,
\[
\sum_{k=1}^{\infty} \left( \frac{\frac{1}{2} + \frac{1}{m}}{k} \right) (E_{n,2} + \cdots + E_{n,2s})^k
\]
\[
+ \sum_{j=1}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \frac{d_j(a)}{d_0(a)} E_{n,0}^{-\frac{1}{m}} \sum_{k=1}^{\infty} \left( \frac{\frac{1}{2} + \frac{1-j}{m}}{k} \right) (E_{n,2} + \cdots + E_{n,2s})^k
\]
\[
= - \sum_{j=1}^{\left\lfloor \frac{m+1}{2} \right\rfloor} \frac{d_j(a)}{d_0(a)} E_{n,0}^{-\frac{1}{m}} - \left( \frac{\frac{1}{2} + \frac{1}{m}}{1} \right) E_{n,2s+1} + o \left( E_{n,0}^{\frac{1}{m}} \right) + o \left( E_{n,0}^{\frac{1}{2} \cdot \frac{1}{m}} \right).
\]

Next, for \( 1 \leq k \geq s + 1 \)
\[
(E_{n,2} + \cdots + E_{n,2s} + E_{n,2s+1})^k
\]
\[
= \left( e_1(a) E_{n,0}^{\frac{1}{m}} + e_2(a) E_{n,0}^{\frac{1}{m}} + \cdots + e_s(a) E_{n,0}^{\frac{1}{m}} \right)^k
\]
\[
= \sum_{k_1=0}^{k} \binom{k}{k_1} \left( e_1(a) E_{n,0}^{\frac{1}{m}} + e_2(a) E_{n,0}^{\frac{1}{m}} + \cdots + e_s(a) E_{n,0}^{\frac{1}{m}} \right)^{k-k_1} o \left( E_{n,0}^{\frac{k-k_1}{m}} \right)
\]
\[
= \left( e_1(a) E_{n,0}^{\frac{1}{m}} + e_2(a) E_{n,0}^{\frac{1}{m}} + \cdots + e_s(a) E_{n,0}^{\frac{1}{m}} \right)^k + o \left( E_{n,0}^{\frac{s+k-k_1}{m}} \right)
\]
\[
= \sum_{i_1 \geq 0, \ldots, i_t \geq 0} \frac{k!}{i_1! \cdots i_t!} e_{j_1}(a)^{i_1} e_{j_2}(a)^{i_2} \cdots e_{j_t}(a)^{i_t} E_{n,0}^{\frac{j_1 i_1 + \cdots + j_t i_t}{m}} + o \left( E_{n,0}^{\frac{s+k-k_1}{m}} \right).
\]

Also, if \( k > s + 1 \) then \( (E_{n,2} + \cdots + E_{n,2s} + E_{n,2s+1})^k = o \left( E_{n,0}^{\frac{s+k-k_1}{m}} \right) \).

Then in (5.5) comparing coefficients of \( E_{n,0}^{-\frac{1}{m}} \), \( 1 \leq j \leq s \), we have
\[
\frac{d_j(a)}{d_0(a)} = \sum_{\substack{|\xi|=k \xi \cdot \eta = j}} \left( \frac{\frac{1}{2} + \frac{1}{m}}{k} \right) k! \epsilon(a)^{\xi} + \sum_{r=1}^{j-1} \frac{d_r(a)}{d_0(a)} \sum_{\substack{|\xi|=k \xi \cdot \eta = j-r}} \left( \frac{\frac{1}{2} + \frac{1-r}{m}}{k} \right) k! \epsilon(a)^{\xi},
\]
where \( \eta = (1, 2, \ldots, m - 1) \). Moreover, if \( \frac{s+1}{m} \leq \frac{1}{2} + \frac{1}{m} \) (i.e., \( s + 1 \leq \frac{m+2}{2} \)) then there exists some constant \( e_{s+1}(a) \in \mathbb{C} \) such that

\[
E_{n,2s+1} = e_{s+1}(a)E_n^{\frac{s+1}{m}} + o\left(E_n^{-\frac{s+1}{m}}\right).
\]

Now we let \( E_{n,2s+2} = e_{s+1}(a)E_n^{-\frac{s+1}{m}} \) and \( E_{n,2s+3} = o\left(E_n^{-\frac{s+1}{m}}\right) \).

If \( s + 1 > \frac{m+2}{2} \) then \( E_n^{\frac{s+1}{m}} \) could be smaller than the error term \( o\left(E_n^{-\frac{1}{2}} - \frac{1}{m}\right) \) in (5.5), and hence we cannot deduce existence of \( e_{s+1}(a) \) like we do in (5.7). This completes induction step and hence proof of Theorem 1.4.

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