Gluon condensate from the Polyakov loop

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We estimate the temperature dependence of the gluon condensate from the Polyakov loop effective potential. It is presented how this analytic approach provides a simple picture for the electric gluon condensate around the deconfinement temperature, showing that it drops to zero in a temperature range which is in good agreement with different pure gauge lattice results.

PACS numbers: 67.85.De, 14.70.Dj, 11.15.Ha

I. INTRODUCTION

Although Quantum Chromodynamics (QCD) is a first principle theory of hadron interactions, it has the drawback of being a theory where the low energy regime is not available using standard perturbative methods. One characteristic feature of QCD is the presence of nontrivial gluon and chiral condensates in the system due to the strong interaction. Usually, for these situations, it is necessary to use effective models of QCD to describe the low energy physics.

Conservation of energy and momentum is a consequence of translation invariance, and holds whether or not the theory is scale invariant. Scale invariance implies that the trace of the energy-momentum tensor is zero. This trace condition is generally broken by quantum corrections on account of scale anomalies [1], and relates the trace of the energy-momentum tensor to the expectation value of the squared gluon field strength, the gluon condensate, through the renormalization group beta function.

The extension of the relationship of the gluon condensate and the trace anomaly was studied by Leutwyler [2]. As it is well known, the energy momentum tensor at finite temperature can be separated into the zero temperature part and the finite temperature contribution. The zero temperature part contains all the ultraviolet infinities which determine the anomaly. On the other side, the finite temperature part is connected to the thermodynamic contribution to the energy density.

In this letter we estimate the temperature dependence of the gluon condensate through the effective energy density of a pure gauge theory, i.e. disregarding the quark formation belonging to the gauge group center. Then, as is well known, the energy momentum tensor at finite temperature can be separated into the zero temperature part and the finite temperature contribution. The zero temperature part contains all the ultraviolet infinities which determine the anomaly. On the other side, the finite temperature part is connected to the thermodynamic contribution to the energy density.

In this letter we estimate the temperature dependence of the gluon condensate through the effective energy density of a pure gauge theory, i.e. disregarding the quark contribution. We work under the hypothesis that the effective potential of self interacting gluons is given by the Polyakov loop potential.

This work is organized as follows. In Sec. II and Sec. III we introduce the general formalism, including analytical results for the temperature dependence of the QCD gluon condensate. In Sec. IV we present our analysis and results, and in Sec. V we summarize our results and conclusions.

II. POLYAKOV LOOP

In the pure Yang-Mills theory the center symmetry plays a crucial part in the description of the confinement-deconfinement phase transition. The order parameter for the latter is the trace of the Polyakov line [3, 4]

$$\Phi(x) = \frac{1}{N_c} \text{Tr} \, \mathcal{P} \exp \left( i \int_0^{1/T} dx \, A_4 \right). \quad (1)$$

The Polyakov line is not invariant under gauge transformations belonging to the gauge group center. Then, if $\Phi = 0$ the $Z(N)$ symmetry is manifest, this situation describes confinement. If for any reason $\Phi \neq 0$, the symmetry must have been broken, this corresponds to the deconfined phase. Therefore, the order parameter $\Phi$ is zero in the confined phase below the critical temperature, and assumes a non-zero value in the deconfined phase above this critical temperature.

For our explicit calculations we shall use the freedom to rotate the $A_4$ field to a diagonal form and consider it to be static. In this gauge, $\Phi(x)$ is a diagonal matrix $\exp(i A_4/T)$, with $A_4$ a constant background field $A_4 = i A_0 = i g \delta_{\alpha0} A_\alpha^a \lambda^a / 2$, where $A_\alpha^a$ are the SU(3) color gauge fields. Then the traced Polyakov loop (PL) is given by $\Phi = \text{Tr} \, \exp(i A_4/T)$. We will work in the so-called Polyakov gauge, in which the matrix $\Phi = \phi_3 \lambda_3 + \phi_8 \lambda_8$ is diagonal [5]. Owing to the charge conjugation properties of the QCD Lagrangian [6], the mean field traced Polyakov loop field is expected to be a real quantity.
Assuming that $\phi_3$ and $\phi_8$ are real-valued, $\phi_8$ has to be zero [7], and therefore
\[ \Phi = \frac{1}{3}(1 + 2 \cos(\phi_3/T)) \]. (2)

The effective gauge field self-interactions are given by the Polyakov loop potential $U[A(x)]$. At finite temperature $T$, it is usual to take for this potential a functional form based on properties of pure gauge QCD. One possible Ansatz is that based on the logarithmic expression of the Haar measure associated with the SU(3) color group integration. The corresponding potential is given by [7]
\[ \frac{U_{\log}(\Phi, T)}{T^4} = -\frac{1}{2} a(T) \Phi^2 + b(T) \log \left(1 - 6 \Phi^2 + 8 \Phi^3 - 3 \Phi^4\right), \] (3)

where
\[ a(T) = a_0 + a_1 \left(\frac{T_0}{T}\right) + a_2 \left(\frac{T_0}{T}\right)^2 + a_3 \left(\frac{T_0}{T}\right)^3, \]
\[ b(T) = b_3 \left(\frac{T_0}{T}\right)^3. \]

The parameters can be fitted to pure gauge lattice QCD data so as to properly reproduce the corresponding equation of state and Polyakov loop behavior. This leads to [7]
\[ a_0 = 3.51, \quad a_1 = -2.47, \quad a_2 = 15.2, \quad b_3 = -1.75. \]

The values of $a_i$ and $b_i$ are constrained by the condition of reaching the Stefan-Boltzmann limit at $T \to \infty$ and by imposing the presence of a first order phase transition at $T_0$. In absence of dynamical quarks, from lattice calculations one expects a deconfinement temperature $T_0 = 270$ MeV. However, in the presence of light dynamical quarks this temperature scale should be adequately reduced to about 200 MeV, with an uncertainty of about 30 MeV [8].

Another widely used potential is that given by a polynomial function based on a Ginzburg-Landau Ansatz [9, 10]:
\[ \frac{U_{\text{poly}}(\Phi, T)}{T^4} = -\frac{b_2(T)}{2} \Phi^2 - \frac{b_3}{3} \Phi^3 + \frac{b_4}{4} \Phi^4, \] (4)

where
\[ b_2(T) = b_0 + a_1 \left(\frac{T_0}{T}\right) + a_2 \left(\frac{T_0}{T}\right)^2 + a_3 \left(\frac{T_0}{T}\right)^3, \]
with
\[ a_0 = 6.75, \quad a_1 = -1.95, \quad a_2 = 2.625, \quad a_3 = -7.44, \]
\[ b_3 = 0.75, \quad b_4 = 7.5. \]

Here the reference temperature $T_0$ plays the same role as in the logarithmic potential of Eq. (3). Once again, the parameters can be fitted to pure gauge lattice QCD results so as to reproduce the corresponding equation of state and Polyakov loop behavior.

In addition, other considered form is the PL potential proposed by Fukushima [11, 12], which includes both a logarithmic piece and a quadratic term with a coefficient that falls exponentially with the temperature:
\[ U_{\text{fuku}}(\Phi, T) = -b T \left[54 \exp(-a/T) \Phi^2 + \log \left(1 - 6 \Phi^2 + 8 \Phi^3 - 3 \Phi^4\right)\right]. \] (5)

Values of dimensionful parameters $a$ and $b$ are given in Ref. [12].

### III. GLUON CONDENSATE

We start by considering the pure gauge QCD Lagrangian
\[ \mathcal{L} = -\frac{1}{4g_s^2} G^a_{\mu\nu} G^{a \mu\nu}, \] (6)

where
\[ G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - f^{abc}_{\mu\nu} A^b_\mu A^c_\nu \] (7)
is the gauge invariant gluon field strength tensor, where $A^a_\mu$ are the color gauge fields and $f^{abc}_{\mu\nu}$ the totally antisymmetric structure constants of SU(3). The strong coupling constant $g_s$ was absorbed in the gluon field, and it can be restored replacing $A_\mu$ by $g_s A_\mu$, with $g_s^2 = 4\pi \alpha_s$.

Following a decomposition sugested in Ref. [13], we have
\[ Z = \int [dA] \exp \left\{ \sum f_{cf} \mathcal{L}_{cf} \right\} \approx \int [dA] \exp \left\{ -\frac{1}{16\pi\alpha_s(T)} \sum G^a_{\mu\nu} G^{a \mu\nu} \right\}. \] (8)

The thermal fluctuation of the gluon condensate can be formally obtained by varying the grand canonical potential $U$, defined as
\[ U = -T \ln Z, \] (9)

with respect to the inverse of the coupling constant $\alpha_s$, this is
\[ \langle G^a_{\mu\nu} G^{a \mu\nu} \rangle_T = 16\pi \frac{\partial U}{\partial \alpha_s}. \] (10)

It has been argued that the gluon condensate can be separated in an electric and magnetic part. The later, as function of the temperature remains almost constant at its zero temperature value. While the electric contribution rapidly drops to zero near above the deconfinement
temperature. In consequence, the Polyakov loop effective potential, as an estimate for the gluons self interaction, should be related to the electric contribution of the gluon condensate. Accordingly, we can estimate the temperature dependence of the QCD gluon condensate \( \langle G^2 \rangle \) through the thermodynamical potential \( U(\Phi, T) \), assuming that all the extra energy comes from the Polyakov loop effective potential.

Restoring the coupling constant, we have

\[
\langle \alpha_s G_{\mu\nu} G^{\mu\nu} \rangle \equiv \langle G^2 \rangle = \langle G^2 \rangle_0 + \langle G^2 \rangle_{T,E}, \tag{11}
\]

where the value of the gluon condensate at zero temperature \( \langle G^2 \rangle_0 = 0.037 \text{ GeV}^4 \) was taken from Ref. [14], and the thermal electric fluctuations are given by

\[
\langle G^2 \rangle_{T,E} = -\frac{4}{\pi} \alpha_s(T)^2 \left( \frac{\partial \alpha_s(T)}{\partial T} \right)^{-1} \frac{\partial U(\Phi,T)}{\partial T}. \tag{12}
\]

For the behavior of the strong coupling constant we choose two different parametrizations \( \alpha_s^{(1,2)}(T) \) from Ref. [15]. Here, we had assume that the temperature is the energy scale of the system, a rather valid assumption in a temperature range around \( T_0 \). These coupling constants obtained through the ghost-gluon vertex, as a truncation prescription fitted in a pure gauge theory, are

\[
\alpha_s^{(1)}(T) = \frac{1}{1.16 + T^2/\Lambda_1^2} \left[ 3.49 \left( 1.16 - 0.07 \left( \frac{T^2}{\Lambda_1^2} \right)^{1/2} \right) + \left( \frac{T^2}{\Lambda_1^2} + 2 \right) \frac{T^4}{\Lambda_1^4} \alpha_s^{(1)} \right], \tag{13}
\]

where \( \Lambda_1 = 0.856 \text{ GeV} \), with

\[
\alpha_s^{(1)} = \frac{4\pi}{11 \ln(T^2/\Lambda_1^2)} \left[ 1 - \frac{102 \ln(T^2/\Lambda_2^2)}{11^2 T^2/\Lambda_1^4} \right].
\]

and

\[
\alpha_s^{(2)}(T) = \frac{1}{1.15 + T^2/\Lambda_2^2} \left[ 15 \times 2.6 + \frac{4\pi}{11} \left( \ln(T^2/\Lambda_2^2) - \frac{1}{T^2/\Lambda_2^4 - 1} \right) \right], \tag{14}
\]

with \( \Lambda_2 = 0.33 \text{ GeV} \).

IV. RESULTS

We present here our analytic results for the thermal fluctuations of the QCD gluon condensate defined in the previous section, considering different Polyakov loop effective potentials and parametrizations for the temperature dependence of the strong coupling constant.

In Fig. 1 we plot for the logarithmic, polynomial and Fukushima PL effective potentials, defined in Eqs. (3), (4) and (5) in solid, dashed and dotted line, respectively, as functions of the reduced temperature \( T/T_0 \) the finite temperature part of the electric gluon condensate \( \langle G^2 \rangle_{T,E} \), Eq. (12). We set for the QCD strong coupling constant the temperature dependence of Eq. (13). It is worth mentioning that an equivalent behavior was found for the parametrization of Eq. (14).

As we are neglecting the quark contribution, we adopt \( T_0 = 270 \text{ MeV} \). Since, this value corresponds to the deconfinement transition temperature of pure gauge QCD obtained from lattice calculations [8].

![Figure 1: Electric part of the gluon condensate for the PL potentials of Eqs. (3), (4) and (5) in solid black, dashed red and dotted blue line, respectively, as functions of the reduced temperature \( T/T_0 \) for \( \alpha_s^{(1)}(T) \).](image)

The mean field traced Polyakov loop \( \Phi \) is zero for lower temperatures than the deconfinement temperature transition \( T_0 \), this implies that the effective potential vanishes for those temperatures, therefore the gluon condensate remains constant at its zero temperature value \( \langle G^2 \rangle_0 \) up to \( T/T_0 = 1 \). For higher temperatures, the gluon condensate should decreases monotonically since it is related to the free energy density. Hence, from Fig. 1, we see that the polynomial effective potential has the best thermal behavior. Consequently, we will use this potential for the estimations of the QCD gluon condensate.

To compare our results, in Fig. 2, we plot Eq. (11) with \( T_0 = 270 \text{ MeV} \) for the polynomial effective PL potential against lattice calculations for a pure gauge theory. The grey circles were taken from Ref. [18] and correspond to a quenched lattice gauge theory. There, the authors show that only the electric part of the gluon condensate has a temperature dependence. While the blue line was obtained from Ref. [17], here the analysis was done for a SU(3) lattice gauge theory using data from Monte Carlo simulations of the interaction measure. The qualitative structure of their results is the same as ours, in both cases the gluon condensate drops rapidly around \( T_0 \) and
vanishes at higher temperatures.

![Figure 2: Normalized electric gluon condensate as a function of the reduced temperature for the polynomial potential and the coupling constant $\alpha_s^{(1)}(T)$ ($\alpha_s^{(2)}(T)$) in solid black (dashed red) line. Grey circles and blue line correspond to lattice results from Ref. [18] and [17], respectively.](image)

Through finite energy sum rules [19], with inputs from lattice QCD [20] or Nambu–Jona-Lasinio models [21] it is possible to obtain from the dimension 4 term in the operator product expansion, the temperature dependence of the gluon condensate via the thermal behavior of the continuum threshold. These estimations also shown that the QCD gluon condensate remains constant up to near the transition temperature and then decreases monotonically. Nevertheless, this formalism provides predictions less accurate than our simple analytic approach.

Finally, as proof of consistency, we can check the parameterizations for the strong coupling constant, Eqs. (13) and (14).

Previous integration of Eq. (11) and using lattice results from Ref. [18] for the electric gluon condensate, it is possible to obtain an approximate temperature dependence for $\alpha_s$ as a function of the PL effective potential,

$$\alpha_s(T) = \frac{\langle G^2 \rangle - \langle G^2 \rangle_0}{\frac{2}{3} U(\Phi, T) + C},$$  \hspace{1cm} (15)

where $C$ is the integration constant, that could be fixed by normalization.

In Fig. 3 we plot the two parameterizations $\alpha_s^{(1)}$ and $\alpha_s^{(2)}$ in solid black and dashed red line, respectively, against the approximate strong coupling constant of Eq. (15) for the polynomial potential (grey circles), normalized by its zero temperature value as functions of the reduced temperature $T/T_0$. We can see that both parametrizations are in good agreement with the results from Eq. (15) in the temperature range of interest.

![Figure 3: Strong coupling constant $\alpha_s^{(1)}$ ($\alpha_s^{(2)}$) in solid black (dashed red) line as functions of $T/T_0$ and estimations from Eq. (15) in grey circles obtained from Ref. [18].](image)

V. SUMMARY AND CONCLUSIONS

In this work we have presented an analytic formalism without free parameters for the thermal fluctuations of the QCD gluon condensate. It was seen that the Polyakov loop effective potential it is related to the electric part of the gluon condensate.

We have analyzed the temperature dependence of the gluon condensate for the three most used PL potentials in the literature, and proposed two parametrizations for the temperature dependence of the strong coupling constant.

We showed that the polynomial effective potential has the best thermal behavior, while the qualitative dependence for both of the strong coupling parametrizations is equivalent.

Finally, we conclude saying that the predictions obtained within this approach are in better agreement with estimates from lattice QCD than other more complex effective theories, and provides an accurate simple picture for the description of the electric gluon condensate around the deconfinement critical temperature.

ACKNOWLEDGEMENTS

This work has been partially funded by CONICET under Grant No. PIP 449; by the National University of La Plata, Project No. X718; by FAPESP grant 2016/01343-7 and by FONDECYT under grants No.1150471, No 1150547 and No 1170107.

The authors would like to thank M. Loewe for useful discussions.
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