REVIEW ARTICLE

Concentration Inequalities for Statistical Inference

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Abstract. This paper gives a review of concentration inequalities which are widely employed in analyzes of mathematical statistics in a wide range of settings, from distribution free to distribution dependent, from sub-Gaussian to sub-exponential, sub-Gamma, and sub-Weibull random variables, and from the mean to the maximum concentration. This review provides results in these settings with some fresh new results. Given the increasing popularity of high dimensional data and inference, results in the context of high-dimensional linear and Poisson regressions are also provided. We aim to illustrate the concentration inequalities with known constants and to improve existing bounds with sharper constants.

Key Words: constants-specified concentration inequalities, sub-Weibull random variables, heavy-tailed distributions, high-dimensional estimation and testing, random matrices.

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In probability theory and statistical inference, researchers often need to bound the probability of a difference between a random quantity from its target, usually the error bound of estimation. Concentration inequalities (CIs) are tools for attaining such bounds, and play important roles in deriving theoretical results for various inferential situations in statistics and probability. The recent developments in high-dimensional (HD) statistical inference, and statistical and machine learning have generated renewed interests in the CIs, as reflected in Koltchinskii (2011), Vershynin (2018) and Wainwright (2019). As the CIs are diverse in their forms and the underlying distributional requirements, and are scattered around in references, there is an increasing need for a review which collects existing results together with some new results from the authors for researchers and graduate students working in statistics and probability. This motivates the writing of this review.

CIs enable us to obtain non-asymptotic results for estimating, constructing confidence intervals, and doing hypothesis testing with a high-probability guarantee. For example,
the first-order optimized condition for HD linear regressions should be held with a high probability to guarantee the well-behavior of the estimator. The concentration inequality for error distributions is to ensure the concentration from first-order optimized conditions to the estimator.

Our review focuses on four types of CIs:

\[ P(Z_n > E(Z_n) + t), \quad P(Z_n < E(Z_n) - t), \quad P(|Z_n - E(Z_n)| > t) \text{ and } \mathbb{E}(\max_{i=1,\cdots,n} |X_i|) \]

where \( Z_n := f(X_1, \cdots, X_n) \) and \( X_1, \cdots, X_n \) are random variables. We present two types of CIs: distribution-free and distribution-dependent. Distribution free CIs are free of distribution assumptions, while the distribution-dependent CIs are based on exponential moment conditions reflecting the tail property for the particular class of distributions. Concentration phenomena for a sum of sub-Weibull random variables will lead to a mixture of two tails: sub-Gaussian for small deviations and sub-Weibull for large deviations from the mean, and it is closely related to Strong Law of Large Numbers, Central Limit Theorem, and Law of the Iterative Logarithm. We provide applications of the CIs to empirical processes and high-dimensional data settings. The latter includes the linear and Poisson regression with a diverging number of covariates. We organize the materials in the forms of Lemmas, Corollaries, Propositions, and Theorems. Lemmas and Corollaries are on existing results usually without proof except for a few fundamental ones. Propositions are also for existing results but with sharper or more precise constants and sometimes come with proofs. Theorems are for new results. This review contains 24 Lemmas, 21 Corollaries, 14 Propositions, and 4 Theorems.

The review is organized as follows. Section 2 outlines distribution-free CIs. CIs for Sub-Gaussian, Sub-exponential, sub-Gamma, and sub-Weibull random variables are given in Section 3, 4, 5, and 6 respectively. Section 7 reports concentration for the maximal of random variables and suprema of empirical processes. Applications for high dimensional linear and Poisson regression are outlined in Section 8. Section 9 discusses extensions to other settings.

## 2 Distribution-free Concentration Bounds

The purpose here is to introduce distribution-free CIs. We first review Markov’s, Chebysh-

eff’s and Chernoff’s tail probability bounds constitute fundamental inequalities for de-

riving most of the concentration bounds; see Chap. 1 of Durrett (2019) or Appendix B in Giraud (2014) for the proofs.

**Lemma 2.1** (Markov’s inequality). Let \( \phi(x):\mathbb{R} \rightarrow \mathbb{R}^+ \) be any non-decreasing positive function. For any real valued r.v. \( X, P(X \geq a) \leq \mathbb{E}[\phi(X)] \frac{1}{\phi(a)}, \forall a \in \mathbb{R}. \)

By letting \( \phi(x) = x^2 \), the following Chebyshev’s inequality is merely an application of Markov’s inequality for \( |X - E(X)| \).
The Chebyshev’s inequality prescribes a polynomial rate of convergence depending on the variance assumption. Another application of Markov’s inequality is the Chernoff’s bound which is sharper by optimizing the upper bounds.

**Lemma 2.3** (Chernoff’s inequality). For a r.v. \( X \) with \( Ee^{tX} < \infty \), \( P(X \geq a) \leq \inf_{t > 0} \{ e^{-ta} Ee^{tX} \} \).

**Proof.** Lemma 2.1 with \( \varphi(x) = e^{tx} \) implies \( P(X \geq a) \leq e^{-ta} Ee^{tX} \) and minimize \( t \) on \( t > 0 \). □

The Jensen’s inequality and its truncated version [Lemma 14.6 in Bühlmann and van de Geer (2011)] are another powerful tool to derive useful inequalities by the convexity.

**Lemma 2.4** (Jensen’s inequality). For any convex function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) and any r.v. \( X \) in \( \mathbb{R}^d \), such that \( \varphi(X) \) is integrable, we have \( E\varphi(X) \leq E[\varphi(E[X])] \).

**Lemma 2.5** (Truncated Jensen’s inequality). Let \( g(\cdot) \) be an increasing function on \([0, \infty)\), which is concave on \([c, \infty)\) for some \( c \geq 0 \). Then \( E_g(|Z|) \leq g[E|Z| + cP(|Z| < c)] \) for r.v. \( Z \).

The Chebyshev’s, Markov’s, Chernoff’s and Jensen’s inequality are also valid for conditional expectations (Chapter 4 of Durrett (2019)). The Chernoff’s bounds typically leads to a tighter bound than Markov’s inequality by optimization via an exponential \( \varphi(x) \) function. A sharper bound for the sum of independent r.vs was attempted in Hoeffding (1963). The following is a slightly sharper bound from Theorem 1.2 in Bosq (1998).

**Corollary 2.1** (Hoeffding’s inequality). Let \( X_1, \ldots, X_n \) be independent r.vs on \( \mathbb{R} \) satisfying bound condition \( a_i \leq X_i \leq b_i \) for \( i = 1, 2, \ldots, n \). Then for \( t, u > 0 \)

(a) **Hoeffding’s lemma:** \( Ee^{u \sum_{i=1}^{n} (X_i - E[X_i])} \leq e^{\frac{u^2}{2} \sum_{i=1}^{n} (b_i - a_i)^2} \) and \( E e^{u|\sum_{i=1}^{n} (X_i - E[X_i])|} \leq 2e^{\frac{u^2}{2} \sum_{i=1}^{n} (b_i - a_i)^2} \);

(b) **Hoeffding’s inequality:** \( P(|\sum_{i=1}^{n} (X_i - E[X_i])| \geq t) \leq 2e^{-2t^2/\sum_{i=1}^{n} (b_i - a_i)^2} \).

Corollary 2.1 has a sharper bound than the Markov’s inequality or Chebyshev’s inequality with the requirement of first or moment condition on \( X \). Hoeffding’s inequality has many applications in statistics as shown in the next example.

**The proof of Hoeffding’s lemma.** Without loss of generality, we assume \( E[X_i] = 0 \). This is from the fact that the concentration inequality is location shift-invariance. Since \( f(x) = e^x \) is convex, for \( u > 0 \), then \( e^{ux} \leq \frac{b_i - x}{b_i - a_i} e^{ua_i} + \frac{x - a_i}{b_i - a_i} e^{ub_i} \), \( a_i \leq x \leq b_i \). Taking expectation, it gives by \( E[X_i] = 0 \)

\[
E e^{uX_i} \leq \frac{b_i}{b_i - a_i} e^{ua_i} - \frac{a_i}{b_i - a_i} e^{ub_i} = \left[ 1 - s e^{u(b_i - a_i)} \right] e^{-su(b_i - a_i)} \triangleq e^{f(r)} ,
\]

where \( r = u(b_i - a_i), s = -a_i/(b_i - a_i) \) and \( f(r) = -sr + \log(1 - s + se^r) \). Hence

\[
f'(r) = -s + \frac{se^r}{1 - s + se^r} , \quad f''(r) = \frac{(1 - s)se^r}{(1 - s + se^r)^2} \leq (1 - s) s \leq \frac{1}{4}
\]
for all $r \geq 0$. Note that $f(0) = f'(0) = 0$. Consider the Taylor’s expansion of $f$, there exists $\xi \in [0,1]$ such that $f(r) = r^2 f''(\xi r)/2 \leq r^2/8 = u^2(b_1 - a_1)^2/8$. Substitute it to (2.1), we get the Hoeffding’s lemma.

The last assertion of Lemma 2.1(a) is by letting $Z = u\sum_{i=1}^n (X_i - EX_i)$, so that

$$Ee^{|Z|} = e^{-Z} \cdot 1(Z \leq 0) + e^Z \cdot 1(Z > 0) \leq 2e^{u^2\sum_{i=1}^n (b_i - a_i)^2}.$$  \hfill (2.2)

The proof of Hoeffding’s inequality. Let $S_n = \sum_{i=1}^n X_i$ and $c_i = a_i - b_i$. For any $t, u > 0$, \[P(S_n - ES_n \geq t) = P(e^{ut(S_n - ES_n)} \geq e^{ut}) \leq \inf_{u > 0} e^{-ut} \prod_{i=1}^n E\{e^{u(X_i - EX_i)}\} \quad [\text{Chernoff’s inequality}]\] \[\text{[Hoeffding’s lemma]} \leq \inf_{u > 0} e^{-ut} \prod_{i=1}^n e^{u^2c_i^2/8} = \inf_{s > 0} e^{-ut + u^2\sum_{i=1}^n c_i^2 / 8} = e^{-2t^2 / \sum_{i=1}^n c_i^2}.\] \hfill (2.3)

The smallest bounded is attained at $u = 4t/\sum_{i=1}^n c_i^2$ and $P(\left|S_n - ES_n\right| \geq t) \leq e^{-2t^2 / \sum_{i=1}^n c_i^2}$ similarly. Hence, the Hoeffding’s inequality is verified \[P(\left|S_n - ES_n\right| \geq t) \leq P(S_n - ES_n \geq t) + P(- \left|S_n - ES_n\right| \geq t) \leq 2e^{-2t^2 / \sum_{i=1}^n c_i^2}.\]

Corollary 2.1 has a sharper bound than the Markov’s inequality or Chebyshev’s inequality with the requirement of first or moment condition on $X$. A second approach for proving Hoeffding’s lemma is given in Lemma 1.8 of Rigollet and Hütter (2019). Hoeffding’s inequality has many applications in statistics as shown in the next example.

**Example 2.1** (Empirical distribution function, EDF). Let $\{X_i\}_{i=1}^n \overset{i.i.d.}{\sim} F(x)$ for a distribution $F$. Let $\mathbb{F}_n(x) := \frac{1}{n} \sum_{i=1}^n I_{\{X_i \leq x\}}(x)$, $x \in \mathbb{R}$ be the empirical distribution. By Hoeffding’s inequality $(a_i - b_i = 1/n)$, $P(\left|\mathbb{F}_n(x) - F(x)\right| > \epsilon) \leq 2e^{-2n\epsilon^2}$, $\forall \epsilon > 0$.

McDiarmid’s inequality (also called bounded difference inequality, see McDiarmid (1989)) is a concentration inequality for a multivariate function of random sequence $\{X_i\}_{i=1}^n$, says $f(X_1, \ldots, X_n)$. As a generalization of Hoeffding’s inequality, it does not require any distribution assumptions about r.v.s and the $f(X_1, \ldots, X_n)$ may be dependent sum of r.v.s. The only requirement is the bounded difference condition by replacing $X_j$ by $X'_j$ meanwhile maintaining the others fixed in $f(X_1, \ldots, X_n)$.

**Lemma 2.6** (McDiarmid’s inequality). Suppose $X_{1'}, \ldots, X_n$ are independent r.v.s all taking values in the set $A$, and assume $f : A^n \rightarrow \mathbb{R}$ satisfies the bounded difference condition

$$\sup_{x_1, \ldots, x_n, x'_n \in A} |f(x_1, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n)| \leq c_k.$$  \hfill (2.4)

Then, $P(\left|f(X_1, \ldots, X_n) - E\{f(X_1, \ldots, X_n)\}\right| \geq t) \leq 2e^{-2t^2 / \sum_{i=1}^n c_i^2}$ $\forall t > 0$.

One method of proof is by the martingale argument, which needs to check the Azuma-Hoeffding’s inequality below, see Section 2.2.2 in Wainwright (2019). Theorem 3.3.14 of Giné and Nickl (2015) gives another proof based on the entropy method.
Lemma 2.7 (Azuma-Hoeffding’s inequality). Let \( \{ X_n \}_{n=1}^{\infty} \) be a sequence of martingale, adapted to an increasing filtration \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \). Suppose \( \{ X_n \}_{n=1}^{\infty} \) satisfies the bounded difference condition \( |X_k - X_{k-1}| < c_k \), a.s. for \( k=1, \ldots, n \). Then, \( P(|X_n - X_0| > t) \leq 2e^{-2t^2/\sum_{k=1}^{n} c_k^2}, \ t \geq 0. \)

Two typical examples with bounded differences function are the concentration for U-statistics (a dependent summation) and the integral error of the kernel density estimation.

Example 2.2 (U-statistics). Let \( \{ X_i \}_{i=1}^{n} \) be independent and identically distributed (IID) r.v.s and \( g: \mathbb{R}^2 \to \mathbb{R} \) be the bounded and symmetric function. Define a U-statistic of order 2 as \( U_n = \frac{1}{\binom{n}{2}} \sum_{i < j} g(X_i, X_j) = f(x_1, \ldots, x_n). \) Its bounded difference condition is

\[
\left| f(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) - f(x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_n) \right| = \frac{1}{\binom{n}{2}} \sum_{j=1, j \neq k}^{n} \left| g(x_k, x_j) - g(x'_k, x_j) \right| \leq 2\frac{\|g\|_\infty}{n(n-1)} \leq \frac{4\|g\|_\infty}{n}.
\]

So we have \( P(|U_n - E U_n| > t) \leq 2e^{-nt^2/8\|g\|_\infty^2}. \)

Example 2.3 (L₁-error in kernel density estimation). Let \( \{ X_i \}_{i=1}^{n} \overset{\text{i.i.d.}}{\sim} F(x) \) with density function \( f(x) \). Define the kernel density estimator by \( \hat{f}_{n,h}(x) = \frac{1}{n} \sum_{i=1}^{n} K \left( \frac{x - X_i}{h} \right) \) where \( K(\cdot) > 0 \) is the kernel function and \( h > 0 \) is a smoothing parameter called the bandwidth. Usually, the kernel function \( K(\cdot) \) is symmetric probability density and \( h > 0 \) with \( h \to 0 \) and \( nh \to \infty \). Define the L₁-error of \( \hat{f}_{n,h}(x) \) by \( Z_n = g(X_1, \ldots, X_n) = \int |\hat{f}_{n,h}(x) - f(x)| \, dx. \) By \( \int K(u) \, du = 1 \), the McDiarmid’s inequality with bound difference condition

\[
\left| g(x_1, \ldots, x_n) - g(x_1, \ldots, x'_n) \right| \leq \frac{1}{h} \int |K \left( \frac{x - x_i}{h} \right) - K \left( \frac{x - x'_i}{h} \right)| \, d \left( \frac{x}{h} \right) \leq \frac{2}{n}
\]

gives \( P(|Z_n - EZ_n| > t) \leq 2e^{-2t^2/n(\hat{F})^2} = 2e^{-nt^2/2}, \) which is free of the bandwidth.

3 Sub-Gaussian Distributions

3.1 Motivations

In probability, there is a well-known inequality for bounding the Gaussian tail. If \( X \sim N(0,1) \), Gordon (1941) obtained for \( x > 0 \)

\[
\left( \frac{1}{x} - \frac{1}{x^2} \right) e^{-x^2/2\sqrt{2\pi}} < \left( \frac{x}{x^2 + 1} \right) e^{-x^2/2\sqrt{2\pi}} \leq P(X \geq x) \leq \frac{1}{x} \cdot \frac{e^{-x^2/2\sqrt{2\pi}}}{\sqrt{2\pi}}, \tag{3.1}
\]

which is called Mills’s inequality, relating to Mills’s ratio (Mills, 1926). The upper bound in (3.1) is mostly used to derive law of the iterated logarithm (Durrett, 1999). However, if \( x \) tends to zero the upper bound goes to \( +\infty \) which makes it meaningless. So the Mill’s inequality are useful only for larger \( x \). We need a better inequality. In fact, the upper
bound in (3.1) can be strengthen as in Lemma B.3 in Giraud (2014): 
\[ P(|X| \geq x) \leq e^{-x^2/2}. \]
We refer it as the **sharper Mill’s inequality**.

In statistics, people want to study a general class of error distributions (beyond Gaussian) whose **moment generating function** (MGF): \( E^{sX} \) have similar Gaussian properties with \( s \) in specific subset of \( \mathbb{R} \). To derive sharper Mill’s inequality, it is natural to define the class of sub-Gaussian r.v. as follows.

**Definition 3.1** (Sub-Gaussian distribution). A r.v. \( X \in \mathbb{R} \) with mean zero is sub-Gaussian with a **variance proxy** \( \sigma^2 \) (denoted \( X \sim \text{subG}(\sigma^2) \)) if its MGF satisfies 
\[ E^{sX} \leq e^{\sigma^2 s^2/2}, \forall s \in \mathbb{R}. \]

With Definition 3.1 and Chernoff’s inequality, we will get the exponential decay of the tail as the alternative definition of sub-Gaussian:
\[ P(X \geq t) \leq \inf_{s>0} e^{-st}E^{sX} \leq \inf_{s>0} e^{-st+\sigma^2 s^2/2} = e^{-t^2/2\sigma^2}. \] (3.2)

Note that \( \text{subG}(\sigma^2) \) denotes a class of distributions rather than a single distribution. Trivially, the Gaussian distribution is a special case of sub-Gaussian. The argument (3.2) is called Cramer-Chernoff method, which is also applied in proving Hoeffding’s lemma.

**Example 3.1** (Normal distributions). Consider the normal r.v. \( X \sim N(\mu, \sigma^2) \). With the MGF of \( X \): 
\[ E^{sX} := e^{\sigma^2 s^2/2}, \forall s \in \mathbb{R}, \] it is sub-Gaussian with the variance proxy \( \sigma^2 = \text{Var}(X) \).

**Example 3.2** (Bounded r.vs). By Hoeffding’s lemma, 
\[ E^{sX} \leq e^{\frac{1}{8}(b-a)^2 s^2} \] for \( s > 0 \) for the centralized bounded variable \( X \in [a,b] \). So \( X \) is essentially sub-Gaussian with variance proxy \( \sigma^2 = \frac{1}{4}(b-a)^2 \). For Bernoulli variable \( X \in \{0,1\} \), we have \( X \sim \text{subG}(\frac{1}{4}) \).

There are at least five equivalent forms for sub-Gaussian as shown in the following.

**Corollary 3.1** (Characterizations of sub-Gaussian). Let \( X \) be a r.v. in \( \mathbb{R} \) with \( EX = 0 \). Then, the following are equivalent for finite positive constants \( \{K_i\}_{i=1}^n \).

1. The MGF of \( X \): 
\[ E^{sX} \leq e^{K_i^2 s^2} \] for all \( s \in \mathbb{R} \);
2. The tail of \( X \): 
\[ P(|X| \geq t) \leq 2e^{-t^2/K_3^2} \] for all \( t \geq 0 \);
3. The moments of \( X \): 
\[ E|X|^k \leq K_3 \sqrt{k} \] for all integer \( k \geq 1 \);
4. The exponential moment of \( X^2 \): 
\[ Ee^{x^2/K_3^2} \leq 2; \]
5. The local MGF of \( X^2 \): 
\[ \text{Exp}(l^2 X^2) \leq \exp(K_3^2 l^2) \] for all \( l \) in a local set \(|l| \leq \frac{1}{K_5} \),

where the \( \{K_i\}_{i=1}^n \) are positive finite constants.
Remark 3.1. The $EX = 0$ is for convenience as the zero mean is used in the proof of Corollary 3.1(1), see Vershynin (2012). The positive integer condition in Corollary 3.1(3) can be relaxed to even integers by the symmetrization technique. By symmetry of $X$, let us consider a negative independent copy $-X'$ which is independent of $X$ and has the same distribution as $X$. If (3) is true and $E(-X') = 0$, from Jensen’s inequality $Ee^{\theta(-X')} \geq e^{\theta E(-X')} = 1$ since $-X'$ has zero mean. So we have by the independence of $X'$ and $X$:

$$Ee^{\theta X} \leq Ee^{\theta X}Ee^{\theta(-X')} = e^{\theta E(-X')} = 1 + \sum_{k=1}^{\infty} \frac{\theta^2 k \cdot 2^k E(|X|)^{2k}}{(2k)!} \leq 1 + \sum_{k=1}^{\infty} \frac{\theta^2 k \cdot 2^k E(|X| + |X'|)^{2k}}{(2k)!}$$

[By (3.7)]

$$\leq 1 + \sum_{k=1}^{\infty} \frac{\theta^2 k \cdot 2^k E(|X|)^{2k}}{(2k)!} = 1 + \sum_{k=1}^{\infty} \frac{(2\theta^2 K^2 \sqrt{2}k)^{2k}}{(2k)!} = e^{\theta^2 k^2} \forall \theta \in \mathbb{R}$$

where the last inequality is due to $(2k)! > k^k \cdot k!$.

3.2 The variance proxy and sub-Gaussian norm

We show that the $\sigma^2$ in Definition 3.1 is indeed the upper bounds of variance of $X$. The $\sigma^2$ not only characterizes the speed of decay in the tail probability (3.2) but also bounds the variance of $n^{-1/2} \sum_{i=1}^{n} X_i$. This is because, by the definition of sub-Gaussian,

$$\frac{\sigma^2 s^2}{2} + o(s^2) = e^{\frac{s^2}{2}} - 1 \geq e^{\sigma^2 X} - 1 = sEX + \frac{s^2}{2}EX^2 + \cdots = \frac{s^2}{2} \cdot \text{Var}X + o(s^2)$$

(3.3)

which implies $\text{Var}X \leq \sigma^2$.

Definition 3.2 (Sub-Gaussian norm). For a sub-Gaussian r.v. $X$, the sub-Gaussian norm of $X$, denoted $\|X\|_{\psi_2}$, is defined by: $\|X\|_{\psi_2} = \inf \{t > 0: Ee^{X^2/t^2} \leq 2\}$.

From Corollary 3.1(4), $\|X\|_{\psi_2}$ is the smallest $K_4$. An alternative definition of the sub-Gaussian norm is $\|X\|_{\psi_2} := \sup_{p \geq 1} P^{-1/2}(E|X|^p)^{1/p}$ (Vershynin, 2012). The definition for sub-Gaussian norm makes Corollary 3.1 easily presented. In fact, if $Ee^{X^2/\|X\|_{\psi_2}^2} \leq 2$, then

$$P(|X| \geq t) = P(e^{-X^2/\|X\|_{\psi_2}^2} \geq e^{-t^2/\|X\|_{\psi_2}^2}) \leq Ee^{X^2/\|X\|_{\psi_2}^2} / e^{t^2/\|X\|_{\psi_2}^2} \leq 2e^{-t^2/\|X\|_{\psi_2}^2}.$$}

(3.4)

Example 3.3 (The sub-Gaussian norm of bounded r.v.s). Consider a r.v. $|X| \leq M < \infty$. Set $Ee^{X^2/t^2} \leq e^{M^2/t^2} \leq 2$ and $t \geq M / \sqrt{\log 2}$, we have $\|X\|_{\psi_2} = M / \sqrt{\log 2}$.

Example 3.4 (The sub-Gaussian norm of Gaussian r.v.s.). For a $N(0, \sigma^2)$ and $t > \sqrt{2} \sigma$, $Ee^{X^2/t^2} = \int e^{x^2/t^2} e^{-x^2/2\sigma^2} dx = \frac{\sigma t}{(t^2 - 2\sigma^2)^{1/2}} \leq 2 \Rightarrow t \geq \sqrt{2} \sigma^2$. By the definition, $\|X\|_{\psi_2} = \sqrt{2} \sigma$.

However, the neat notation for defining sub-Gaussian norm sometime leads to unknown constants in the CIs as shown next.
Corollary 3.2 (Theorem 2.6.3, Vershynin (2018)). Let \( \{X_i\}_{i=1}^n \) be independent mean-zero sub-Gaussian r.vs. Then, \( P\left\{ \frac{1}{n} \sum_{i=1}^n X_i \geq t \right\} \leq 2 \exp\left( -\frac{C t^2}{\sum_{i=1}^n \|X_i\|^2} \right) \), \( \forall \ t \geq 0 \) where \( C \) is an absolute constant.

The unknown constant \( C \) makes the above CIs cannot be used in constructing confidence bands for \( \mu \). To obtain more specific bounds (data dependent bounds as a statistics), we adopt the follow three propositions under sub-Gaussian.

**Proposition 3.1** (Sub-Gaussian properties). Let \( X \sim \text{subG}(\sigma^2) \), then for any \( t > 0 \),

(a) the tail satisfies \( P(\|X\| > t) \leq 2e^{-t^2/2\sigma^2} \);

(b) (a) implies that moments \( \text{E}X^i \leq (2\sigma^2)^{i/2}k! \Gamma\left(\frac{k}{2}\right) \) and \( (\text{E}(X^k))^{1/k} \leq \sigma \frac{1}{\sqrt{k}}, \ k \geq 2 \);

(c) If (a) holds and \( \text{E}X = 0 \), then \( \text{E}e^{sX} \leq e^{4s^2\sigma^2} \) for any \( s > 0 \);

(d) If \( X \sim \text{subG}(\sigma^2) \), then \( |X| \leq 2\sqrt{\log 2} \sigma \); conversely, if \( |X| \leq \sigma \) then \( X \sim \text{subG}(2\sqrt{2}\sigma^2) \).

**Proof.** The proofs of (a)-(c) are in Lemma 1.4 and Lemma 1.5 in Rigollet and Hütter (2019)). For (d), note that

\[
\text{E}\exp(s^2 X^2) = 1 + \sum_{k=1}^{\infty} \frac{s^{2k} \text{E}X^{2k}}{k!} \leq 1 + \sum_{k=1}^{\infty} \frac{2s^{2k}(2\sigma^2)^{k/2}k!}{k!} = 1 + 4s^2\sigma^2 \sum_{k=0}^{\infty} \left(\frac{2s^2\sigma^2}{k!}\right)^k \leq 1 + 8s^2\sigma^2 \leq e^{8s^2\sigma^2}.
\]

By (3.5), set \( \text{E}e^{s_0^2 X^2} \leq e^{8s_0^2\sigma^2} \leq 2 \) for some \( s_0 \). Then \( |s_0| \leq \frac{\sqrt{\log 2}}{2\sqrt{\sigma}} \leq \frac{1}{2\sigma} \). Put \( |s| = \frac{\sqrt{\log 2}}{2\sqrt{\sigma}} \), and the sub-Gaussian norm gives \( \text{E}X^2 + \frac{2\sqrt{\log 2}}{\sqrt{2\sigma}} \leq 2 \Rightarrow |X|_{\text{Q}} \leq 2\sqrt{\log 2} \sigma \). Conversely, if \( |X|_{\text{Q}} = \sigma \) then the (3.4) gives \( P(\|X\| > t) \leq 2e^{-t^2/2\sigma^2} \). Proposition 3.1(c) concludes \( X \sim \text{subG}(2\sqrt{2}\sigma^2) \). \( \square \)

We next introduce the sub-Gaussian CIs for the non-random weighted sum of exponential family (EF) r.vs with compact parameter space, adapted from Lemma 6.1 in Rigollet (2012) with more specific constants.

**Proposition 3.2** (Concentration for weighted E-F summation). Let \( \{Y_i\}_{i=1}^n \) be a sequence of r.vs, its density

\[
f(y;\theta) = c(y) \exp\{y\theta - b(\theta)\}
\]

with \( \text{E}Y_i = b(\theta) \) and \( \text{Var}Y_i = \tilde{b}(\theta) \). We assume

- (E.1): Uniformly bounded variances condition: there exist a compact set \( \Omega \) and some constant \( C_b \) such that \( \sup_{\theta \in \Omega} \tilde{b}(\theta) \leq C_b \) for all \( i \).
Let \( w := (w_1, \ldots, w_n)^T \in \mathbb{R}^n \) be a non-random vector and define \( S_n^w := \sum_{i=1}^n w_i Y_i \). Then

(a) Closed under addition: \( S_n^w - E S_n^w \sim \text{subG}(C_b^2 \| w \|_2^2) \) and \( P\{ |S_n^w - E S_n^w| > t \} \leq 2e^{-t^2/(2C_b^2 \| w \|_2^2)} \);

(b) Let \( C_n := S_n^w - E S_n^w \) and \( \Gamma(t) := \int_0^\infty x^{t-1} e^{-x} \, dx \) be the Gamma function. For all integer \( k \geq 1 \), we have moments bound: \( E|C_n|^k \leq k(2C_b)^{k/2} \Gamma(k/2) \| w \|_2^k \).

(c) The MGF of centralized \( |C_n|^2 \): \( Ee^{s^2|C_n|^2 - E|C_n|^2} \leq e^{c^2(8\sqrt{2}C_b \| w \|_2^2)^2} e^{1/8c^2 \| w \|_2^2} \), for integer \( k \geq 1 \). 

Proof. Based on the MGF and uniformly bounded variances condition, the proof of (a)-(b) can be found in Lemma 6.1 of Rigollet (2012). We give the proof of (c) since we update the constant, and (d) is similar for the sub-Gaussian case.

The proof will resort to Proposition 3.2(b). Then, under \( |4sC_b^2 \| w \|_2^2| < 1 \), we have

\[
Ee^{s^2|C_n|^2 - E|C_n|^2} = 1 + 2 \sum_{k=2}^\infty (4sC_b^2 \| w \|_2^2)^k = 1 + \frac{2(4sC_b^2 \| w \|_2^2)^2}{1 - 4s^2C_b^2 \| w \|_2^2} \leq e^{c^2(8\sqrt{2}C_b \| w \|_2^2)^2/2}.
\]

(d) It follows by defining \( C_b^2 := \max_{1 \leq i \leq n} \sigma_i^2 > 0 \) as the common variance proxy for \( \{ Y_i \}_i \). For \( i = 1, 2, \cdots, n \), we have: \( Ee^{s^2(Y_i - EY_i)} \leq e^{c^2s^2\sigma_i^2/2}, \forall s \in \mathbb{R} \).

Proposition 3.2(a) yields the following results (The first result is in Corollary 1.7 of Rigollet and Hütter (2019)). The second sub-Gaussian CI below specifies the unknown constant in Theorem 2.6.2 of Vershynin (2018).

Proposition 3.3. Let \( \{ X_i \}_i \) be \( n \) independent \text{subG}(\sigma_i^2). Define \( \sigma^2 = \max_{1 \leq i \leq n} \sigma_i^2 \).
\[
P(\left| \sum_{i=1}^{n} w_i X_i \right| > t) \leq 2e^{-t^2/(2\varpi_2^2)} \text{ and } P(\left| \sum_{i=1}^{n} w_i X_i \right| > t) \leq 2e^{-t^2/(8\Sigma_{i=1}^{n} \|w_i X_i\|_{\varpi_2}^2)}, \quad \forall \ t \geq 0
\]
for any non-random vector \( w := (w_1, \ldots, w_n)^T \).

**Proof.** To see the second CI, just use the Proposition 3.1(d) and the first result above, by noticing that if \( \|X\|_{\varpi_2} \leq \infty \) then \( X \sim \text{subG}(2\sqrt{2}\|X\|_{\varpi_2}^2) \).

### 3.3 Randomly weighted sum of independent sub-Gaussian variables

In this part, we outline the sub-Gaussian type CIs for the randomly weighted sum of exponential family r.v.s: \( S_n^W := \sum_{i=1}^{n} W_i Y_i \) where \( \{W_i\}_{i=1}^{n} \) are called the multipliers (or random weights) which is independent from \( \{Y_i\}_{i=1}^{n} \). The normalized sum \( \frac{1}{\sqrt{n}}(S_n^W - ES_n^W) \) is also called multiplier empirical processes, and it serves for the multiplier Bootstrap inference where the multipliers \( \{W_i\} \) are r.v.s independent from \( \{Y_i\}_{i=1}^{n} \), see Chapter 2.9 of van der Vaart and Wellner (1996). To get sub-Gaussian concentration, some regularity conditions for the parameter space are required.

- (E.2): Let \( W := (W_1, \ldots, W_n)^T \in \mathbb{R}^n \) be a random vector with some bounded components, i.e. \( |W_i| \leq w_i < \infty \) for a non-random vector \( w := (w_1, \ldots, w_n)^T \in \mathbb{R}^n \).

**Theorem 3.1** (Concentration inequalities for randomly weighted sum). Let \( \{Y_i\}_{i=1}^{n} \) belong to the canonical exponential family (3.6), and let \( \{W_i\}_{i=1}^{n} \) be independent of \( \{Y_i\}_{i=1}^{n} \). Define the randomly weighted sum \( S_n^W := \sum_{i=1}^{n} W_i Y_i \), then under (E.1) and (E.2)

\[
P(\left| S_n^W - ES_n^W \right| \geq t) \leq 2e^{-t^2/(2\varpi_2^2)}.
\]

**Proof.** Let \( Y_i = b(\theta_i) + Z_i \), where \( \{Z_i\}_{i=1}^{n} \) are centralized and independent E-F r.v.s. From \( EY_i = b(\theta_i) \) and the identity (3.8) for a dominating measure \( \mu(\cdot) \)

\[
\int dF_{Y_i}(y) = 1 \Leftrightarrow \int c(y)e^{y \theta_i} \mu(dy) = e^{b(\theta_i)}.
\]

Let \( E_{|W}(\cdot) := E(\cdot | W) \) and \( s \) be in \((-\delta, \delta)\) (a neighbourhood of zero). The conditional MGF of \( W_i Y_i \) given \( W \) is

\[
E_{|W}[e^{sW_i Y_i}] = \int e^{sW_i Y_i} dF_{W_i Y_i}(y) = \int e^{sW_i Y_i} dF_{Y_i}(y) \quad \text{by } \{W_i\}_{i=1}^{n} \perp \{Y_i\}_{i=1}^{n}
\]

\[
= \int c(y)e^{y b(\theta_i)} e^{W_i Y_i} \mu(dy) \overset{(3.8)}{=} e^{b(\theta_i + s W_i) - b(\theta_i)}.
\]

It can be easily derived from (E.2) and Taylor’s expansion,

\[
E_{|W}[e^{s(W_i Y_i - E_{|W}(W_i Y_i))}] = e^{b(\theta_i + s W_i) - b(\theta_i) - b(\theta_i) W_i \theta_i} \overset{\exists \eta_i \in [b(\theta_i), b(\theta_i) + s W_i]}{=} e^{\frac{s^2 W_i^2}{2} b(\theta_i)} \leq e^{\frac{s^2 \varpi_2^2}{2}}.
\]
By the conditional independence for \( \{W_i, Z_i | W\} \) and (3.9), it follows that when \( s \in (-\delta, \delta) \)
\[
E_{|W|^s} \left[ \sum_{i=1}^n [W_i Z_i - E_{|W|}(W_i Z_i)] \right] = \prod_{i=1}^n E_{|W|^s} [W_i Z_i - E_{|W|}(W_i Z_i)] \leq \prod_{i=1}^n e^{-s(\lambda - \frac{1}{2})} \leq 2e^{-s(\lambda - \frac{1}{2})}, 
\]  
(3.10)
where the last inequality is from \( \{|W_i| \leq w_i\} \) for a non-random vector \( w := (w_1, \ldots, w_n)^T \).

By the conditional Markov’s inequality and symmetry of \( Z_i \), we have, for \( s \in (-\delta, \delta) \)
\[
P(\sum_{i=1}^n |W_i Z_i - E_{|W|}(W_i Z_i)| \geq t | W) \leq \inf_{s > 0} e^{-st} E_{|W|^s} e^{( \lambda - \frac{1}{2} )} |W|^s e^{-st} E_{|W|^s} e^{( \lambda - \frac{1}{2} )} |W|^s 
\leq 2e^{-st} \frac{2s^2|w|^2}{2s^2} = 2e^{-st} \frac{s^2|w|^2}{2s^2}.
\]  
(3.11)
where the last equality is minimized by setting \( s = t/2|w|^2 \).

Note that Lemma 6.1 in Rigollet (2012) is about the concentration for the non-random weighted sum of exponential family r.v.s. The assumption of compact parameter space for exponential family is vital for obtaining the sub-Gaussian type concentration. If we do not impose condition (E.2) and the assumption that \( \{W_i\}_{i=1}^n \) and \( \{Y_i\}_{i=1}^n \) are dependent, a counterexample for sub-Gaussian concentration is \( W_i = Y_i \). Thus, \( S_n^W \) is a quadratic form, and \( S_n^W - E S_n^W \) is sub-exponential by Lemma 4.2 below. If \( \{W_i\}_{i=1}^n \) are dependent but \( \{Y_i\}_{i=1}^n \) are still bounded, another counterexample is \( W_i = \text{sign}(Y_i) \). Therefore, \( S_n^W = \sum_{i=1}^n |Y_i| \) is not zero-mean, and the concentration of \( \sum_{i=1}^n |Y_i| \) fails.

### 3.4 Concentration for Lipschitz functions of random vectors

In the analyses of high-dimensional statistics by empirical processes, researches often resort to the CIs of Lipschitz functions for either bounded or strongly log-concave random vectors (Wainwright, 2019).

**Lemma 3.1** (Theorem 2.26, Wainwright (2019)). Let \( N \sim N(0, I_p) \). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be \( L \)-Lipschitz with respect to (w.r.t.) the Euclidean norm: \( |f(a) - f(b)| \leq L \|a - b\|_2 \) for any \( a, b \in \mathbb{R}^n \). Then, \( P(|f(N) - E f(N)| \geq t) \leq 2e^{-t^2/(2L^2)}, \forall t > 0 \).

A non-negative function \( f(x) : \mathbb{R}^n \to \mathbb{R} \) is log-concave if
\[
\log f(\lambda x + (1-\lambda) y) \geq \lambda \log f(x) + (1-\lambda) \log f(y), \forall \lambda \in [0,1] \text{ and } x, y \in \mathbb{R}^n.
\]  
(3.12)
A function \( \psi(x) : \mathbb{R}^n \to \mathbb{R} \) is \( \gamma \)-strongly concave if there is some \( \gamma > 0 \) s.t.
\[
\lambda \psi(x) + (1-\lambda) \psi(y) - \psi(\lambda x + (1-\lambda) y) \leq \frac{\gamma}{2} \lambda (1-\lambda) \|x - y\|_2^2, \forall \lambda \in [0,1] \text{ and } x, y \in \mathbb{R}^n,
\]
A continuous probability density \( f(x) \) and the corresponding r.v. are log-concave (or strongly log-concave) if \( f(x) \) is a log-concave function (or strongly log-concave function), see Saumard and Wellner (2014) for a review of the log-concavity in statistics.
Lemma 3.2 (Theorem 3.16, Wainwright (2019)). Let P be any $\gamma-$strongly log-concave distribution on $\mathbb{R}^n$ with parameter $\gamma > 0$. Then for any function $f: \mathbb{R}^n \to \mathbb{R}$ that is L-Lipschitz w.r.t. the Euclidean norm: $P[f(X) - Ef(X) \geq t] \leq e^{-\frac{4t^2}{L^2}}$ for $X \sim P$ and $t \geq 0$.

The standard Gaussian random vector is 1—strongly log-concave distributed. However, Lemma 3.1 has the sharper constant $2L^2$ than the Gaussian case of Lemma 3.2 with constant $4L^2$. Beyond Gaussian and strongly log-concave, it is possible to establish concentration for distributions involving bounded r.v.s. A function $f(x): \mathbb{R}^n \to \mathbb{R}$ is said to be separately convex if, the univariate function $y_k \mapsto f(x_1, x_2, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)$ for each index $k \in \{1, 2, \ldots, n\}$, is convex for each fixed vector $(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$.

Lemma 3.3 (Theorem 3.4, Wainwright (2019)). Let $\{X_i\}_{i=1}^n$ be independent r.v.s, each supported on the interval $[a, b]$. Let $f: \mathbb{R}^n \to \mathbb{R}$ be separately convex, and L-Lipschitz w.r.t. the Euclidean norm. Then, $P[f(X) - Ef(X) \geq t] \leq e^{-t^2/4L^2(b-a)^2}$ for $X \sim P$ and $t \geq 0$.

Example 3.5 (Order Statistics). From Lemma 3.2 and Lemma 3.3, suppose that $\{X_i\}_{i=1}^n$ are independent r.v.s which are $\gamma-$strongly log-concave distributed satisfying $P[f(X) - Ef(X) \geq t] \leq e^{-\frac{4t^2}{L^2}}$ for any function $f: \mathbb{R}^n \to \mathbb{R}$ that is L-Lipschitz w.r.t. the Euclidean norm. Let $X_{(k)}$ be the k-th order statistic of $X_1, \ldots, X_n$, it can be shown that

$$P(|X_{(k)} - EX_{(k)}| > \delta) \leq 2e^{-\delta^2/2}$$

by checking $|X_{(k)} - Y_{(k)}| \leq \|X - Y\|_2$, i.e. $\delta = 1$.

Indeed, we have $X_{(k)} - Y_{(k)} \leq |X_{(k)} - Y_{(k)}| \leq \|X - Y\|_2$ for some $l \in \{1, 2, \ldots, n\}$. More results of the tail bounds for order statistics of IID r.v.s are reported in Boucheron (2012).

## 4 Sub-exponential distributions

### 4.1 Characterizations

The requirement in definition of sub-Gaussian r.v. $\text{E}e^{sX} \leq e^{\frac{s^2}{2}}$, $\forall s \in \mathbb{R}$, is too strong. We consider the MGF of exponential distributions:

Example 4.1 (MGF of exponential distributions). Consider the exponential r.v. $X \sim \text{Exp}(\mu)$ with $EX = \mu > 0$. The MGF of $X - \mu$ satisfies

$$\text{E}e^{s(X-\mu)} = \frac{e^{-s\mu}}{1-s\mu} = (e^{-s\mu}/(1-s\mu))^{-1/2} \leq e^{2(s\mu/2)^2} \leq e^{2(2\mu)^2/2} < e^{\mu^2/2}, \forall |s| < (2\mu)^{-1}$$

(4.1)

where the second last inequality is by $e^{-t}/\sqrt{1-2t} \leq e^{2t^2}$ for $|t| < 1/4$.

In (4.1), the MGF of the exponential r.v. is divergent on $s = 1/\mu$ and it cannot be bounded by a Gaussian MGF of $s$ in $\mathbb{R}$, and the exponential MGF is bounded by Gaussian MGF for $|s| \leq \frac{1}{2\mu}$ via inequality (4.1). Motivated by Example 4.1, the first definition of sub-exponential distribution (4.2) below is exactly the locally sub-Gaussian property.
Definition 4.1 (Sub-exponential distributions). A r.v. \( X \in \mathbb{R} \) with \( EX = 0 \) is sub-exponential with parameter \( \lambda \) (denoted \( X \sim \text{subE}(\lambda) \)) if its MGF satisfies
\[
E e^{sX} \leq e^{\frac{2s^2 \lambda}{\bar{\lambda}}} \quad \text{for all } |s| < 1/\bar{\lambda}.
\] (4.2)

In Wainwright (2019), sub-exponential distributions are generally defined by two positive parameters \((\lambda, \alpha)\) (denoted \( X \sim \text{subE}(\lambda, \alpha) \)): \( E e^{sX} \leq e^{\frac{2s^2 \lambda}{\bar{\lambda}} + \frac{|s|^\alpha \lambda}{\bar{\lambda}}} \) for all \( |s| < 1/\bar{\lambda} \).

The \( \lambda^2 \) in (4.2) is treated as a variance proxy and \( \alpha \) is seen as a locally sub-Gaussian factor, see Remark 4.1 below. Specifically, \( \text{subE}(\lambda) = \text{subE}(\lambda, \lambda) \). Sub-Gaussian r.v.s are sub-exponential by definition, but not vice versa. In Corollary 3.1, one equivalence of sub-Gaussian r.v.s is that the survival function is bounded by the Gaussian-like survival function up to a constant. Similarly, the sub-exponential r.v. has a characterization that the survival function is bounded by that of an exponential distribution. Similar to sub-Gaussian characterizations, there are at least six equivalent forms for sub-exponential distributions which are useful for checking the sub-exponential distribution.

**Corollary 4.1** (Characterizations of sub-exponential). Let \( X \) be a r.v. in \( \mathbb{R} \) with \( EX = 0 \). Then the following properties are equivalent, where \( \{K_i\}_{i=1}^6 \) are positive constants.

1. The tails of \( X \) satisfy \( P\{|X| \geq t\} \leq 2e^{-t/K_1} \) for all \( t \geq 0 \);
2. The MGF of \( X \) satisfies \( E e^{sX} \leq e^{K_2s^2} \) for all \( |s| \leq 1/K_2 \);
3. The moments of \( X \) satisfy \( (E|X|^k)^{1/k} \leq K_3k \) for integer \( k \geq 1 \);
4. The MGF of \( |X| \) satisfies \( E e^{s|X|} \leq e^{K_4s} \) for all \( 0 \leq s \leq 1/K_4 \);
5. The MGF of \( |X| \) is bounded at some point: \( E e^{s|X|/K_5} \leq 2 \);
6. Bounded MGF of \( X \) in a compact set: \( E e^{sX} < \infty, \forall |t| < 1/K_6 \).

The zero mean is only used in the proof of (2) of Corollary 4.1. The equivalence among (1)–(5) is proved in Vershynin (2018) and that between (5) and (6) can be found in Lemma 5 of Petrov (1975). The (6) is the called Cramer’s condition which signifies that: All r.v.s are sub-exponential if their MGF exist in a neighborhood of zero. Pistone and Wynn (1999) names the property (6) as the exponentially integrable r.v.

Example 4.2 (Moment of exponential distributions). The \( P(X - \mu \geq t) = e^{-(t+\mu)/\mu} \leq 2e^{-t/\mu} \) implies \( K_1 = \mu \) in Corollary 4.1. Continue to Example 4.1, the (4.1) implies \( X - \mu \sim \text{subE}(2\mu) \) and \( X - \mu \sim \text{subE}(\mu, 2\mu) \). So \( K_2 = \mu/\sqrt{2} \) and \( K_6 = 2\mu \) in Corollary 4.1. Next, we evaluate the moment of \( X \) for any \( p \geq 1 \), \( E|X|^p = \int_0^\infty x^p \cdot \frac{e^{-x}}{x^{\mu-1}x} \cdot \mu e^{-x} \cdot x \cdot \frac{x^{\mu-1}x}{\mu} = \mu^p \int_0^\infty y^p e^{-y} dy = \Gamma(p+1)\mu^p \).

By \( \Gamma(p+1) \leq p^p \) for \( p \geq 1 \), it gives: \( E|X|^p \leq \Gamma(p+1) \mu^p \leq p\mu \), which shows that
Thus $K_4=2eK_3$. That $Ee^{λ|X|}≤e^{K_3λ}$ for $0<λ≤1/(2eK_2)$ in (4.3) implies $E|X|/(2eK_3)<e^{1/2}<2$. Hence $K_5=2eK_3$.

**Example 4.3** (Geometric distributions). The geometric distribution $X \sim \text{Geo}(q)$ for r.v. $X$ is defined by: $P(X=k) = (1-q)^{k-1}q$, $q \in (0,1)$, $k = 1,2,\ldots$. The mean and the variance of Geo$(q)$ are $(1-q)/q$ and $(1-q)/q^2$ respectively. Apply Lemma 4.3 in Hillar and Wibisono (2013), we get $(E|X|^k)^{1/k} < -2k/\log(1-q)$. It follows from the Minkowski’s inequality and Jensen’s inequality $(E|Z|^k)^{1/k} ≤ E|Z|^k$ for integer $k ≥ 1$ that

$$(E|X-EX|^k)^{1/k} ≤ (E|X|^k)^{1/k} + |EX| ≤ 2(E|X|^k)^{1/k} ≤ -4k/\log(1-q)$$

and Corollary 4.1(3) suggests the centralized Geo$(q)$ is sub-exponential with $K_3 = -4/\log(1-q)$.

**Example 4.4** (Discrete Laplace r.v.). A r.v. $X \sim \text{DL}(q)$ obeys the discrete Laplace distribution if $f_q(k) = P(X=k) = \frac{1-q}{1-q-q^k}$, $k \in \mathbb{Z} = \{0,±1,±2,\ldots\}$ with parameter $q \in (0,1)$. The discrete Laplace r.v. is the difference of two IID Geo$(q)$). The geometric distribution is sub-exponential, thus Corollary 4.2(a) implies that the discrete Laplace is also sub-exponential distributed. In differential privacy of network models, the noises are assumed following the discrete Laplace distribution, see Fan et al. (2020) and references therein.

The next result shows that a sum of independent sub-exponential r.v.s has two tails with difference convergence rate, which is slightly different from Hoeffding’s inequality. Deviating from the mean, it tells us that the tail of the sum of sub-exponential r.v.s behaves like a combination of a Gaussian tail and a exponential tail.

**Corollary 4.2** (Concentration for weighted sub-exponential sums). Let $\{X_i\}_{i=1}^n$ be independent $\{\text{subE}(\lambda_i)\}_{i=1}^n$ distributed with zero mean. Define $\lambda = \max_{1 ≤ i ≤ n} \lambda_i > 0$ and the non-random vector $w := (w_1,\ldots,w_n)^T \in \mathbb{R}^n$ with $w = \max_{1 ≤ i ≤ n} |w_i| > 0$, we have

(a) Closed under addition: $\sum_{i=1}^n w_iX_i \sim \text{subE}(\|w\|_2\lambda)$;

(b) $P(|\sum_{i=1}^n w_iX_i| ≥ t) ≤ 2e^{-\frac{t^2}{2\|w\|_2^2\lambda^2}} = \begin{cases} 2e^{-t^2/2\|w\|_2^2\lambda^2}, & 0 ≤ t ≤ \|w\|_2^2\lambda/\|w\|_2 \\ 2e^{-t/2\|w\|_2\lambda}, & t > \|w\|_2^2\lambda/\|w\|_2 \end{cases}$. 

(c) Let $\{X_i\}_{i=1}^n$ be independent zero-mean $\{\text{subE}(\lambda_i,\alpha_i)\}_{i=1}^n$ distributed. Define $\alpha := \max_{1 ≤ i ≤ n} \alpha_i > 0$, $\|\lambda\|_2 := (\sum_{i=1}^n \lambda_i^2)^{1/2}$ and $\bar{\lambda} := (\frac{1}{n} \sum_{i=1}^n \lambda_i^2)^{1/2}$. Then $\sum_{i=1}^n X_i \sim \text{subE}(\|\lambda\|_2,\alpha)$ and

$P\left(\left|\frac{1}{n} \sum_{i=1}^n X_i\right| ≥ t\right) ≤ 2e^{-\frac{t^2}{2\|\lambda\|_2^2\alpha}} = \begin{cases} 2e^{-\frac{t^2}{2\alpha}}, & 0 ≤ t ≤ \frac{\bar{\lambda}^2}{\alpha} \\ 2e^{-\frac{t}{\alpha}}, & t > \frac{\bar{\lambda}^2}{\alpha} \end{cases}$, $\forall t ≥ 0$. (4.5)
Remark 4.1. The $(\frac{\mu^2}{n^2} \wedge \frac{\lambda^2}{n})$ in (4.5) reveals that the smaller $\alpha$ (locally sub-Gaussian factor) leads to sharper sub-exponential concentration. The sub-exponential concentration tends to the sub-Gaussian concentration with variance proxy $\lambda^2$ when $\alpha \to 0$, which coincides with the locally sub-Gaussian definition for sub-exponential distribution in Definition 4.1.

Proof. (a) By definition of sub-exponential r.vs, $\text{E} e^{iw_iX_i} \leq e^{\frac{2}{\alpha}w_i^2\lambda_i^2/2} \forall |s| \leq \frac{1}{|w_i|\lambda_i}$, $i = 1, 2, \cdots, n$, and it implies $\text{E} e^{iw_iX_i} \leq e^{\frac{2}{\alpha}w_i^2\lambda_i^2/2}$, $|s| \leq \frac{1}{\sqrt{\alpha}}$ for all $i$. By the independence among $\{X_i\}_{i=1}^n$,

$$\text{E}\exp\{s \sum_{i=1}^n w_i X_i\} = \prod_{i=1}^n \text{E} e^{iw_iX_i} \leq \exp\{s^2 \sum_{i=1}^n w_i^2\lambda_i^2/2\} \leq e^{s^2\|w\|^2\lambda^2/2}, |s| \leq \frac{1}{\sqrt{\alpha}}.$$ 

(b) The proof can be found in Theorem 1.13 of Rigollet and Hütter (2019).

(c). The proof is similar to (b), see page 29 of Wainwright (2019).

Corollary 4.2(b) is due to Petrov, and it is also called Petrov’s exponential inequalities, see Lin and Bai (2011). Although Corollary 4.2(b,c) are non-asymptotically valid for any number of summands. Nevertheless, it also has asymptotical merit, which implies: Strong Law of Large Numbers (SLNN), Central Limit Theorem (CLT), and Law of the Iterated Logarithm (LIL) for sub-exponential sums, as discussed below.

(1) **SLNN.** Let $w_i = 1/n$. Consider the sample mean $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$ for IID $\{\text{subE}(\lambda_i)\}_{i=1}^n$ data $\{X_i\}_{i=1}^n$ with population mean $\mu$, and we can use Corollary 4.2(b) to prove that $\bar{X}_n \overset{a.s.}{\to} \mu$. We verify the Borel-Cantelli lemma by observing that $\sum_{n=1}^\infty P(\bar{X}_n - \mu > \epsilon) \leq \sum_{n=1}^\infty 2e^{-\frac{n\epsilon^2}{2\lambda^2/\epsilon}} < \infty$, which shows the strong convergence: $\bar{X}_n \overset{a.s.}{\to} \mu$. Corollary 4.2(b) also implies the rate of convergence for sample mean for all $n$ with high probability. It is easy to see that the sample mean $\bar{X}_n$ has the non-asymptotic error bounds by

$$|\bar{X}_n - \mu| \leq \sqrt{\frac{2\lambda^2 t}{n}} \sqrt{\frac{2\lambda t}{n}} = \begin{cases} \sqrt{\frac{2\lambda^2 t}{n}}, & n \geq 2t \text{ (slow global rate)} \\ \sqrt{\frac{2\lambda t}{n}}, & n < 2t \text{ (fast local rate)} \end{cases}$$

$\forall t \geq 0$ with the probability at least $1 - 2e^{-t}$.

(2) **CLT.** In order to study the convergence rate of CLT, we standardize the sum by letting $w_i = 1/\sqrt{n}$ and apply Corollary 4.2(b) to

$$P(|\sqrt{n}\bar{X}_n| \geq t) \leq 2\exp\left\{-\frac{1}{2} \left(\frac{\mu^2}{\lambda^2} \wedge \frac{t^2}{\lambda^2/\sqrt{n}}\right)\right\} \begin{cases} 2e^{-ct^2/\lambda^2}, & t \leq \lambda \sqrt{n}; \\ 2e^{-t^2/\lambda^2/\sqrt{n}}, & t > \lambda \sqrt{n}. \end{cases}$$

The above deviation inequality is powerful as it indicates the phase transition about the tail behavior of $\sqrt{n}\bar{X}_n$:

**Small Deviation Regime.** In the regime $t \leq \lambda \sqrt{n}$, we have a sub-Gaussian tail bound with variance proxy $\lambda^2$ as if the sum had the normal distribution with a constant variance. Note that the domain $t \leq \lambda \sqrt{n}$ widens as $n$ increases and then the central limit theorem becomes more powerful.
Large Deviation Regime. In the regime \( t \geq \lambda \sqrt{n} \), the sum has a heavier. The sub-exponential tail bound is affected from the extreme variable among \( \{ \text{subE}(\lambda_i) \}_{i=1}^n \) with parameter \( \lambda / \sqrt{n} \).

(3) LIL. Let \( w_i = 1/n \) and \( t = \frac{R \sqrt{\log \log n}}{\sqrt{n}} \leq \|w\|^2_2 \lambda / w = \lambda \) for some positive constant \( R \).

Corollary 4.2(b) claims

\[
P(|X_n| \geq \frac{R \sqrt{\log \log n}}{\sqrt{n}}) \leq 2e^{-t^2/2\|w\|^2_2} = 2\exp\left\{ -\frac{n}{2\lambda^2} \frac{R^2 \log \log n}{n} \right\} = 2\exp\{\log(\log n)^{-R^2/2\lambda^2}\} = 2/(\log n)^{R^2/2\lambda^2}.
\]

Therefore, with probability \( 1 - 2/(\log n)^{R^2/2\lambda^2} \), \( |X_n| \leq \frac{R \sqrt{\log \log n}}{\sqrt{n}} \). Although some researchers claims that LIL is useless, we clarify that there are still some meaningful applications of LIL, see Jamieson et al. (2014) and Yang et al. (2019) for the statistical and machine learning applications of the LIL.

4.2 Sub-exponential norm

Recall the Corollary 4.1(5): The absolute value of sub-exponential r.v. \( |X| \) has a bound MGF at point \( K_5^{-1} : \phi_{|X|}(K_5^{-1}) := \text{E} e^{X/K_5} \leq 2 \). Similar to the definition of sub-Gaussian norm, we define the sub-exponential norm.

**Definition 4.2** (sub-exponential norm). The sub-exponential norm of \( X \) is defined as

\[
\|X\|_{\psi_1} = \inf \{ t > 0 : \text{E} \exp(|X|/t) \leq 2 \}.
\]

An alternative definition of the sub-exponential norm is \( \|X\|_{\psi_1} := \sup_{p \geq 1} 2^{-1/p} (\text{E} |X|^p)^{1/p} \) as in Vershynin (2012). The sub-exponential r.v. \( X \) satisfies the equivalent properties in Corollary 4.1 (Characterizations of sub-exponential). Next, we present a useful lemma below which is to determine the sub-exponential parameter in the Definition 4.1 by its MGF if we adopt Definition 4.2 by the sub-exponential norm.

**Proposition 4.1** (Properties of sub-exponential norm). If \( \text{E} \exp(|X|/\|X\|_{\psi_1}) \leq 2 \), then

(a) Tail bounds \( P(|X| > t) \leq 2e^{-t^2/\|X\|_{\psi_1}} \) for all \( t \geq 0 \);

(b) Moment bounds \( \text{E}|X|^k \leq 2\|X\|_{\psi_1}^k k! \) for all integer \( k \geq 1 \);

(c) If \( EX = 0 \), the MGF bounds \( \text{E} e^{sX} \leq e^{(2\|X\|_{\psi_1}^2 s^2)} \) for all \( |s| < 1/(2\|X\|_{\psi_1}) \), i.e. \( X \sim \text{subE}(2\|X\|_{\psi_1}) \).

**Proof.** (a). To verify (a), using exponential Markov’s inequality, we have \( P(|X| \geq t) = P(e^{s|X|/\|X\|_{\psi_1}} \geq e^{s/t\|X\|_{\psi_1}}) \leq e^{-s/t\|X\|_{\psi_1}} \text{E} e^{s|X|/\|X\|_{\psi_1}} \leq 2e^{-s/t\|X\|_{\psi_1}} \) where the last inequality stems from the definition of the sub-exponential norm.
(b). Similar to the proof of Theorem 5.1 (b), we get from (a)
\[ \mathbb{E}|X|^k = \int_0^\infty P(|X| \geq t)kt^{k-1}dt \leq 2k\int_0^\infty e^{-t/|X|_{\phi_1}}tk^{k-1}dt \]

[let \( s = t/|X|_{\phi_1} \)]
\[ = 2k\int_0^\infty e^{-s}(s||X|_{\phi_1})^{k-1}|X|_{\phi_1}ds = 2||X|_{\phi_1}^k\Gamma(k-1) = 2||X|_{\phi_1}t. \]

(c). Applying Taylor’s expansion to MGF, we have
\[ \text{E}^{\text{exp}}(c). \]
\[ \text{Applying T aylor’s expansion to MGF , we have } \]
\[ \text{Example 4.6} \]
\[ \text{Proof. Note that } \]
\[ \text{Application 4.1.} \]
\[ \text{Lemma 4.1.} \]
\[ \text{Concentration for r .v . with sub-exponential sum) \]
\[ \text{asymptotic confident interval for sub-exponential sample me an.} \]
\[ \text{includes an un-specific constant, which makes it is inefficacious when constructing non-} \]
\[ \text{asymptotic confident interval for sub-exponential sample mean.} \]

**Proposition 4.2** (Concentration for r.v. with sub-exponential sum). Let \( \{X_i\}_{i=1}^n \) be zero mean independent sub-exponential distributed with \( |X_i|_{\phi_1} \leq \infty \). Then for every \( t \geq 0 \),
\[ P(|\sum_{i=1}^n X_i| \geq t) \leq 2\exp\{-\frac{1}{4}(\frac{t^2}{\max_{1 \leq i \leq n} \|X_i\|_{\phi_1}})\} \]

**Proof.** If \( \text{E}^{\text{exp}}(|X|/\|X\|_{\phi_1}) \leq 2, \) then \( X \sim \text{subE}(2\|X\|_{\phi_1}) \) by using Lemma 4.1(c). The result follows by employing Corollary 4.2(b). \( \blacksquare \)

Götze et al. (2019) mentions an explicitly calculation the sub-exponential norm with example of Poisson distributions. Therefore, it is convenient to apply Proposition 4.2 to get the concentration of sub-exponential summation.

**Lemma 4.1.** If \( \|X\|_{\phi_1} \) exists, then \( |X|_{\phi_1} = 1/\phi_{|X|}^{-1}(2) \) for the MGF \( \phi_{X}(t) := \mathbb{E}e^{tX} \).

**Proof.** Note that \( \|\cdot\|_{\phi_1} \) is the smallest \( t \) such that \( \mathbb{E}e^{tX} = \phi_{|X|}(t-1) \leq 2 \), so \( t \geq 1/\phi_{|X|}^{-1}(2) \). By the definition of \( \|\cdot\|_{\phi_1} \) again, we have \( \|X\|_{\phi_1} = 1/\phi_{|X|}^{-1}(2) \). \( \blacksquare \)

**Example 4.5** (The sub-exponential norm of bounded r.v.). Consider a r.v. \( |X| \leq M < \infty \). Set \( \mathbb{E}e^{tX/2} \leq e^{M/t} \leq 2 \) and \( t \geq M/\log 2 \). By the definition of \( \|X\|_{\phi_1} \), we have \( \|X\|_{\phi_1} = M/\log 2 \).

**Example 4.6** (The sub-exponential norm of Poisson r.v.). Poisson r.v. \( X \) has the probability mass function \( \text{P}(X = k) = \frac{\lambda^ke^{-\lambda}}{k!}(k = 1,2,\cdots,n; \lambda > 0) \). We denote it as \( X \sim \text{Poisson}(\lambda) \). The MGF of the Poisson(\( \lambda \)) is \( \phi_{X}(t) := e^{\lambda(e^t-1)} \). We have \( \|X\|_{\phi_1} = [\log(\log(2)\lambda^{-1}+1)]^{-1} \), and the triangle inequality shows \( \|X-EX\|_{\phi_1} \leq \|X\|_{\phi_1} + \|EX\|_{\phi_1} = \|X\|_{\phi_1} + \frac{\lambda}{\log 2} \leq [\log(\log(2)\lambda^{-1}+1)]^{-1} + \frac{\lambda}{\log 2} \propto \lambda \), where we use inequality \( \|EX\|_{\phi_1} = \frac{EX}{\log 2} \) by Example 4.5.

Corollary 4.2 is useful in the next subsection for the concentration for quadratic forms.
4.3 Concentration for quadratic forms

All concentration results in the above sections are about the mean. The inference for the variance and covariance in high-dimensional models is an important problem, see \( \text{?} \). It is connected with squares of r.v.s. The sample variance is a quadratic form (with shift term) of the data. The data are often postulated as sub-Gaussian. For the square of a sub-Gaussian r.v., it is natural to ask what is the behavior of the tail (or the exponential moment). The answer is sub-exponential by using (5) in Corollary 3.1.

A simple example that the quadratic form of Gaussian is Chi-squared distributed, and the Chi-square distribution of 2 degrees of freedom is exponentially distributed with mean 2. Let us look at the Chi-squared concentration below:

**Example 4.7** (Chi-squared r.v.s). If \( \{X_i\}_{i=1}^n \overset{\text{i.i.d.}}{\sim} N(0,1) \), then we say \( Y_n := \sum_{i=1}^n X_i \) follows \( \chi^2 \)-distribution with \( n \)-degree of freedom, denoted as \( Y_n \sim \chi^2(n) \). The density function is \( f(y) = \Gamma^{-1}(\frac{n}{2}) \frac{1}{2^{n/2}} y^{n/2-1} e^{-\frac{y}{2}} \cdot 1(y > 0) \). As \( s < 1/2 \), the MGF of \( X_i^2 - 1 \) is

\[
E e^{s(X_i^2 - 1)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{s(x^2-1)} e^{-x^2/2} dx = \frac{\sqrt{e^s}}{\sqrt{1-2s}} \leq e^{(2s)^2/2}, \quad \text{for all } |s| < \frac{1}{4}
\]

where the second last inequality is due to \( \frac{e^{-t}}{\sqrt{1-2t}} \leq e^{2t^2} \) for \( |t| < 1/4 \). Then \( X_i^2 \sim \text{subE}(2,4) \). Applying Corollary 4.2(c), we have \( Y_n \sim \text{subE}(2\sqrt{n},4) \), therefore \( P(|\frac{Y_n-n}{n}| \geq t) \leq 2e^{-\frac{t^2}{2(1+t)}} \).

Similar sub-exponential results also hold for independent sum of square of sub-Gaussian r.v.s. The following two lemmas in Page 31 of Vershynin (2018) confirm this simple example to the general situation.

**Lemma 4.2** (Square and product of sub-Gaussian are sub-exponential). (a). A r.v. \( X \) is sub-Gaussian if and only if \( X^2 \) is sub-exponential. Moreover, \( \|X^2\|_{\psi_1} = \|X\|_{\psi_2}^2 \). (b). Let \( X \) and \( Y \) be sub-Gaussian r.v.s. Then \( XY \) is sub-exponential and \( \|XY\|_{\psi_1} \leq \|X\|_{\psi_2} \|Y\|_{\psi_2} \).

For Lemma 4.2(a), it is follows from \( \|X^2\|_{\psi_1} = \|X\|_{\psi_2}^2 \) and Lemma 4.1 that Corollary 4.2 coincides 3.3 as \( \max_{1 \leq i \leq n} \|X_i\|_{\psi_1} \rightarrow 0 \), i.e. the sub-exponential r.v. degenerates to the sub-Gaussian r.v. the next corollary gives the accurately sub-exponential parameter for the square of sub-Gaussian r.v. in Definition 4.1, and it improves the constant in Lemma 1.12 of Rigollet and Hütter (2019) (from \( \text{subE}(16\sigma^2) \) to \( \text{subE}(8\sqrt{2}\sigma^2) \)).

**Proposition 4.3.** Let \( X \sim \text{subG}^2(\sigma^2) \), then \( Z := X^2 - EX^2 \sim \text{subE}(8\sqrt{2}\sigma^2) \) or \( \sim \text{subE}(8\sqrt{2}\sigma^2, 8\sigma^2) \).

**Proof.** The proof is immediately from Proposition 3.2(c) by letting \( w := (1,0,\cdots,0)^T \).

In below, we deal with a sharper Hanson-Wright inequality in Bellec (2019) recently. The Hanson-Wright inequality is a general concentration result for quadratic forms of sub-Gaussian r.v.s, which was first studied in Hanson and Wright (1971). Let \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) be a real matrix and the \( \xi = (\xi_1, \cdots, \xi_n)^T \) be a centered random vector with independent components. Define the Frobenius norm \( \|A\|_F := \sqrt{\text{tr}(A^T A)} = \sqrt{\sum_{i,j} A_{ij}^2} \) and the
Corollary 4.3 (Gaussian chaos of order 2). Let $\xi_1, \ldots, \xi_n$ be zero-mean Gaussian with $E\xi_i^2 = \sigma_i^2$. Define $D_\sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$, then for any $x > 0$

$$P(\xi^T A\xi - E[\xi^T A\xi] > t) \leq e^{-cK^2F(A\xi)^2/F(x^2)}$$

(4.7)

The similar concentration phenomenon is also available for sub-Gaussian r.v.s. which is named as the Hanson-Wright inequality, and Rudelson and Vershynin (2013) gives a modern proof by the so-called decoupling argument attributed to Bourgain (1996).

Corollary 4.4 (Hanson-Wright inequality, Rudelson and Vershynin (2013)). Let $n \geq 1$ and $\xi := (\xi_1, \ldots, \xi_n)^T$ be a independent zero-mean sub-Gaussian r.v with $\max_{i=1, \ldots, n} ||\xi_i||_{\psi_2} \leq K$ for some real number $K > 0$. Let $A$ be any $n \times n$ real matrix. Then there exists an absolute constant $c > 0$ such that the following holds for all $t > 0$,

$$P(\xi^T A\xi - E[\xi^T A\xi] > t) \leq e^{-cK^2F(A\xi)^2/F(x^2)}$$

(4.8)

Furthermore, for any $x > 0$, $P(\xi^T A\xi - E[\xi^T A\xi] \leq cK^2(\|A\xi\|_2 + \|A\xi\|_F \sqrt{x}) \geq 1 - e^{-x}$.

Intuitively, the term $K^2\|A\xi\|_F \sqrt{x}$ is seen as the “variance term”. When $A$ is diagonal-free (i.e. the $A$ matrix has zeros down its diagonal: $a_{ii} = 0$), the r.v. $\xi^T A\xi$ is zero-mean. Pollard (2015) shortens the proof without unknown constant.

Corollary 4.5 (Diagonal-free Hanson-Wright inequality). Let $\zeta_1, \ldots, \zeta_n$ be independent, centered sub-Gaussian r.v with $\max_{i=1, \ldots, n} ||\zeta_i||_{\psi_2} \leq K$. Let $A$ be an $n \times n$ matrix of real numbers with $a_{ii} = 0$ for each $i$. Then $P(\xi^T A\xi \geq t) \leq e^{-cK^2F(A\xi)^2/F(x^2)}$ for $t > 0$.

Under assumptions on the moments of $\zeta_1, \ldots, \zeta_n$ (do not need sub-Gaussian assumption), the next theorem provides a concentration inequality for quadratic forms of independent r.v.s satisfying Bernstein’s moment condition (discussed in the next subsection).

Corollary 4.6 (Quadratic forms concentration with moment conditions). Assume that the r.v. $\xi = (\zeta_1, \ldots, \zeta_n)^T$ satisfies the condition on independent variables $\zeta_1^2, \ldots, \zeta_n^2$: $E|\xi_i|^{2p} \leq \frac{1}{p} p! \sigma_i^{2p} \kappa^{2p-2}$ for some $\kappa > 0$. Let $A$ be any $n \times n$ real matrix. Then for all $t > 0$,

$$P(\xi^T A\xi - E[\xi^T A\xi] > t) \leq e^{-cK^2F(A\xi)^2/F(x^2)}$$

(4.9)

where $D_\sigma := \text{diag}(\sigma_1, \ldots, \sigma_n)$. Furthermore, with probability greater than $1 - e^{-x}$,

$$\xi^T A\xi - E[\xi^T A\xi] \leq 256\kappa^2\|A\|_2^2 \sqrt{x} + 8\sqrt{3\kappa} \|A\|_F \sqrt{x}, \forall x \geq 0.$$  

(4.10)
The right hand side of (4.9) is exactly \( \exp(-\frac{t^2}{192c^2||A||_F^4}) \) if \( t \) is small, and while the \( \exp(-c\frac{t^2}{n||X||_F^2}) \) in right hand side of the Hanson-Wright inequality (4.8) has an unspecified absolute constant \( c > 0 \). We finish this subsection with an exponential probability tail inequality for quadratic forms of a sub-Gaussian random vector, see Hsu et al. (2012).

**Corollary 4.7** (Tail inequality for quadratic forms of sub-Gaussian vectors). Let \( \Sigma = AA^T \) for \( p \times n \) matrix \( A \). Consider a sub-Gaussian random vector \( \xi = (\xi_1, \ldots, \xi_n)^T \) with independent components for \( \mu = E\xi \) and \( \sigma^2 \) s.t. \( Ee^{a^T\xi - \mu} \leq e^{\frac{||a||^2}{2a^T\mu}} \), \( \forall \ a \in \mathbb{R}^n \). Then,

\[
P\{|A\xi|^2 > \sigma^2[\text{tr}(\Sigma) + 2\text{tr}(\Sigma^2t)^{1/2} + 2||\Sigma||_2t] + \text{tr}(\Sigma\mu\mu^T)(1 + 2\frac{||\Sigma||_2^2t}{\text{tr}(\Sigma^2t)})\} \leq e^{-t}, \ t > 0.
\]

An application of Corollary 4.7 for the prediction error (8.5) in regressions with sub-Gaussian noise and divergence number of non-random covariates is given in Section 8.1.

## 5 Sub-Gamma distributions and Bernstein’s inequality

### 5.1 Sub-Gamma distributions

Comparing to the classical Chebyshev’s inequality, Bernstein-type inequalities have more precise concentration, it originally is an extension of the Hoeffding’s inequality with bounded assumption [see Bernstein (1924), Bennett (1962)]. As mentioned by Pollard (2015), the proof of Hoeffding’s inequality with endpoints of the interval \([a, b]\) in Lemma 2.1 (with \( n = 1 \)) crudely depends on the variance bound:

\[
\text{Var}X = E(X - EX)^2 \leq E(X - \frac{b-a}{2})^2 \leq \frac{(b-a)^2}{4} \quad \text{if} \ a \leq X \leq b \quad (5.1)
\]

without any other variance information.

The following tail bound for the sum \( S_n := \sum_{i=1}^n X_i \) needs extra variance information.

**Corollary 5.1** (Bernstein’s inequality with the bounded condition). Let \( X_1, \ldots, X_n \) be centralized independent variables such that \( |X_i| \leq M \) a.s. for all \( i \). Then, \( \forall \ t > 0 \)

\[
P(|S_n| \geq t) \leq 2e^{-\frac{t^2}{12\sum_{i=1}^n \text{Var}X_i + Mt^3}}, P\{|S_n| \geq (2t\sum_{i=1}^n \text{Var}X_i)^{1/2} + \frac{M}{2}\} \leq 2e^{-t}.
\]

The next example illustrates a sharp confidence interval for sample mean if we know that the variance is sufficient small.

**Example 5.1** (Non-asymptotic confidence intervals). Let \( \{X_i\}_{i=1}^n \overset{i.i.d.}{\sim} X \) with the support \([-c, c]\) and the mean \( \mu \). Hoeffding’s and Bernstein’s inequalities show for \( \bar{X} := n^{-1}\sum_{i=1}^n X_i \)

\[
P(|\bar{X} - \mu| \leq \sqrt{\frac{2c^2\log(2/\delta)}{n}}) \geq 1 - \delta \quad \text{Hoeffding};
\]

\[
P(|\bar{X} - \mu| \leq \frac{c}{\sqrt{n}}\log(2/\delta) + \sqrt{\frac{2\text{Var}X\log(2/\delta)}{n}}) \geq 1 - \delta \quad \text{Bernstein}.
\]
For large $n$, the Bernstein’s confidence interval is substantially shorter if $X_i$ has relatively small variance, i.e. $\text{Var}X \ll c^2$ (the factor $\sqrt{\frac{\log(2/\delta)}{n}}$ is a dominated term). The Hoeffding’s confidence is shorter as $\text{Var}X = c^2$ (This extreme case attains the upper bound $\text{Var}X \leq c^2$ in (5.1) due to $b - a = 2c$). But, for the case $\text{Var}X < c^2$, if $n$ is sufficient small s.t. $\frac{n}{2}\log(2/\delta) \geq \left( c - \sqrt{\text{Var}X} \right) \frac{2\log(2/\delta)}{n}$, i.e. we need restrictions $n \leq \frac{1}{18} \left( \frac{c}{c - \sqrt{\text{Var}X}} \right)^2 \log(\frac{2}{\delta}) \geq 1$ to ensure Hoeffding’s confidence interval is more accurate when $\delta \leq 2\exp\{ -\frac{1}{18}(c - \sqrt{\text{Var}X})^2 \}$. 

To prove Corollary 5.1, we need get the sharp bounds of the MGF of the single variable and then do aggregation for the summation. By the Taylor expansion, we have

$$ Ee^{sX_i} = 1 + \sum_{k=2}^{\infty} s^k \frac{X_i}{k!} \leq 1 + \sum_{k=2}^{\infty} s^k \frac{M^{k-2}\text{Var}X_i}{k!} \leq 1 + s^2\text{Var}X_i \sum_{k=2}^{\infty} \left( \frac{|s|M}{3} \right)^{k-2}, \quad 1 \leq i \leq n. $$

Applying the inequality $k!/2 \geq 3^{k-2}$ for any $k \geq 2$, it implies

$$ Ee^{sX_i} \leq 1 + \frac{s^2\text{Var}X_i}{2} \sum_{k=2}^{\infty} \left( \frac{|s|M}{3} \right)^{k-2} = 1 + s^2\text{Var}X_i/2 \frac{1 - |s|M/3}{1 - |s|M/3} \leq \exp\left( s^2\text{Var}X_i/2 \frac{1 - |s|M/3}{1 - |s|M/3} \right). $$

The upper bounds of MGF essentially have the same form in comparison with Gamma distribution below whose MGF is bounded by (5.3) in following example.

**Example 5.2** (Gamma r.v.s). The Gamma distribution with density $f(x) = \frac{x^{a-1}e^{-x/b}}{\Gamma(a)b^a}$, $x \geq 0$ is denote as $\Gamma(a, b)$. We have $EX = ab$ and $\text{Var}X = ab^2$ for $X \sim \Gamma(a, b)$. The Page28 of Boucheron et al. (2013) illustrates that the log-MGF of a centered $\Gamma(a, b)$ is bounded by

$$ \log(Ee^{s(X - EX)}) = a(-\log(1 - sb) - sb) \leq \frac{s^2}{2} \frac{ab^2}{1 - bs}, \quad \forall \ 0 < s < b^{-1}. \quad (5.3) $$

Like the sub-Gaussian, Boucheron et al. (2013) defines the sub-Gamma r.v. based on the right tail and left tail with variance factor $\nu$ and scale factor $b$.

**Definition 5.1** (Sub-Gamma r.v.). Motivated by the MGF bounds in (5.3), a centralized r.v. $X$ is *sub-Gamma* with variance factor $\nu > 0$ and scale parameter $c > 0$ (denoted by $X \sim \text{sub}\Gamma(\nu, c)$) if

$$ \log(Ee^{\nu X}) \leq \frac{s^2}{2} \frac{\nu}{1 - c|s|}, \quad \forall \ 0 < |s| < c^{-1}. \quad (5.4) $$

If the restriction $0 < |s| < b^{-1}$ is replaced by one side conditions $0 < s < b^{-1}$ (or $0 < -s < b^{-1}$), we call it *sub-Gamma on the right tail* (or *sub-Gamma on the left tail*), denoted as sub$\Gamma_+(\nu, c)$ (or sub$\Gamma_-(\nu, c)$). In Example 5.2, the Gamma r.v. $X \sim \text{sub}\Gamma_+(ab^2, b)$. The (5.4) is called *two-sided Bernstein’s condition*.

**Example 5.3** (Sub-exponential r.v.s). The sub-exponential distribution with positive support implies the sub-Gamma condition: $\log(Ee^{\nu X}) \leq \frac{s^2\lambda^2}{2} \leq \frac{s^2\lambda^2}{2(1 - \lambda|s|)}$, $\forall \ |s| < \frac{1}{\lambda}$. This shows that $X \sim \text{sub}\text{E}(\lambda)$ implies $X \sim \text{sub}\Gamma(\lambda^2, \lambda)$. 
The sub-Gamma condition (5.4) leads to the useful tail bounds and moment bounds.

**Lemma 5.1** (Sub-gamma properties, Boucheron et al. (2013)). If \( X \sim \text{sub}\Gamma(v, c) \), then

\[
P(\{|X| > t\}) \leq 2e^{-\frac{v}{2}\frac{\log t}{4}} \leq 2e^{-\frac{\sqrt{2}t}{v+ct}}, \tag{5.5}
\]

where \( h(u) = 1 + u - \sqrt{1 + 2|u|} \). Moreover, we have \( P(\{|X| > \sqrt{2}vt\}) \leq e^{-t} \).

The tail bound in Lemma 5.1 verifies that, the sub-Gamma variable has sub-Gaussian tail behavior with parameter \( v \) for suitably small \( t \), and it has exponential tail behavior for larger \( t \). The proof is originated from Bennett (1962).

**Proof.** By Chernoff’s inequality, \( P(X - EX \geq t) \leq \inf_{s \geq 0} e^{-st} E e^{s(X-EX)} \). It remains to bound

\[
\inf_{c^{-1} \geq s > 0} \log(e^{-st}Ee^{s(X-EX)}) \leq \inf_{c^{-1} \geq s > 0} \left( \frac{u^2}{2} \frac{p}{1-ct} - ut \right) = -\frac{u}{c} h(u) \leq -\frac{\sqrt{2}t}{v+ct},
\]

where the last inequality is from \( h(u) = 1 + u - \sqrt{1 + 2|u|} \geq \frac{\sqrt{2}t}{1+t} \). So we conclude (5.5). \( \square \)

**Proposition 5.1** (Concentration for sub-Gamma sum). Let \( \{X_i\}_{i=1}^n \) be independent \( \{\text{sub}\Gamma(v_i, c_i)\}_{i=1}^n \) distributed with zero mean. Define \( c = \max_{1 \leq i \leq n} c_i \), we have

(a) Closed under addition: \( S_n := \sum_{i=1}^n X_i \sim \text{sub}\Gamma(\sum_{i=1}^n v_i, c) \);

(b) For every \( t \geq 0 \): \( P(|S_n| \geq t) \leq 2\exp \left( -\frac{\sqrt{2}n}{4(c+\sum_{i=1}^n v_i)} \right) \) and \( P(|S_n| > (2t\sum_{i=1}^n v_i)^{1/2} + ct) \leq 2e^{-t} \);

(c) If \( X \sim \text{sub}\Gamma(v, c) \), the moments bounds satisfy for any integer \( k \geq 1 \):

\[
EX^k \leq 2k^{2k-2}[2(\sqrt{2}v)^k \Gamma(k) + c(\sqrt{2}v)^k \Gamma(\frac{k+1}{2}) + 3c^k \Gamma(k)].
\]

(d) If \( X \sim \text{sub}\Gamma(v, c) \), the even moments bounds satisfy \( EX^{2k} \leq k!(8v)^k + (2k)!(4c)^k \), \( k \geq 1 \).

(e) If \( P\{X > (2tv)^{1/2} + ct\} \leq e^{-t} \), then \( X \sim \text{sub}\Gamma(32(\nu + 2\nu^2), 8c) \).

**Proof.** (a) By definition of \( \{\text{sub}\Gamma(v_i, c_i)\}_{i=1}^n \), we have \( \log(E e^{X_i}) \leq \frac{\nu}{2(1-c_i|s|)} \), \( 0 < |s| < c^{-1} \), from which and the independence among \( \{X_i\}_{i=1}^n \), thus

\[
\log(E e^{S_n}) \leq \frac{\nu}{2} \sum_{i=1}^n \frac{v_i}{1-c_i|s|} \leq \frac{\nu}{2} \sum_{i=1}^n v_i \quad \forall 0 < |s| < c^{-1}.
\]

(b) Employing Proposition 5.1, we immediately obtain (b) due to (a).

(c) Applying the integration form of the expectation formula, it yields

\[
EX^k \leq E|X|^k = k \int_0^\infty x^{k-1} P\{|X| > x\} dx = k \int_0^\infty x^{k-1} P\{|X| > \sqrt{2}vt + ct\} \left( \frac{\sqrt{2}v}{2\sqrt{t}} + c \right) dt
\]

\[
\leq 2k \int_0^\infty \left( \frac{\sqrt{2}vt + ct}{2t} \right)^{k-1} \left( \frac{\sqrt{2}vt + 2ct}{2t} \right) e^{-t} dt = k \int_0^\infty \left( \frac{\sqrt{2}vt + ct}{2t} \right)^k + ct \left( \frac{\sqrt{2}vt + ct}{2t} \right)^{k-1} e^{-t} dt.
\]
should be noted that (5.6) can be replaced by

\[
E X^k \leq k \int_0^\infty \left\{ 2^{k-1} \left[ \left( \sqrt{2t} \right)^k + (ct)^k \right] + c t 2^{k-2} \left[ (\sqrt{2t})^{k-1} + (ct)^{k-1} \right] \right\} \frac{e^{-t}}{t} dt
\]

\[
= k 2^{k-2} \int_0^\infty \left\{ 2 (\sqrt{2t})^{k(2-1)} + c (\sqrt{2t})^{k-1} t^{(k+1)/(2-1)} + 3c^k k^{-1} \right\} e^{-t} dt
\]

\[
= k 2^{k-2} \left\{ 2 (\sqrt{2})^k \Gamma\left( \frac{k}{2} \right) + c (\sqrt{2})^{k-1} \Gamma\left( \frac{k-1}{2} \right) + 3c^k (k-1)! \right\}.
\]

(d,e) The proofs are in Theorem 2.3 of Boucheron et al. (2013).

Having obtained Proposition 5.1(b), from the upper bound in (5.2), we finish the proof of Proposition 5.1 by treating \( X_i \sim \text{sub}\Gamma(\text{Var}X_i/2, M/3) \) for \( i = 1, 2, \ldots, n \).

5.2 Bernstein’s growth of moments condition

In some settings, one cannot assume the random variables being bounded. Bernstein’s inequality for the sum of independent r.v.s allows us to estimate the tail probability by a weaker version of an exponential condition on the growth of the \( k \)-moment without the boundedness.

**Corollary 5.2** (Bernstein’s inequality with the growth of moment condition). If the centred independent r.v.s \( X_1, \ldots, X_n \) satisfy the growth of moments condition

\[
E |X_i|^k \leq 2^{-1} v_i^2 \kappa_i^k k!, \quad (i = 1, 2, \ldots, n), \text{ for all } k \geq 2
\]

(5.6) where \( \{\kappa_i\}_{i=1}^n, \{v_i\}_{i=1}^n \) are constants independent of \( k \). Let \( v_i^2 = \sum_{j=1}^n v_j^2 \) (the fluctuation of sums) and \( \kappa = \max_{1 \leq i \leq n} \kappa_i \). Then, we have \( X_i \sim \text{sub}_{\Gamma}(v_i, \kappa_i) \) and for \( t > 0 \)

\[
P(\{|S_n| \geq t\}) \leq 2e^{-\frac{t^2}{v_i^2(2-\kappa_i)|\kappa_i|}}, \quad P(\{|S_n| \geq \sqrt{2v_i^2 t + \kappa t}\}) \leq 2e^{-1}.
\]

(5.7)

**Proof.** Given that \( \kappa_i |s| < 1 \) for all \( i \), (5.6) implies that \( X_i \sim \text{sub}_{\Gamma}(v_i, \kappa_i) \) for \( 1 \leq i \leq n \)

\[
E e^{sX_i} \leq 1 + \frac{e^s v_i^2}{2|1 - |\kappa_i||} \leq e^{s v_i^2/(2 - 2\kappa_i |s|)}.
\]

The independence among \( \{X_i\}_{i=1}^n \) and Proposition 5.1(a,b) implies (5.7). □

The (5.6) is also called Bernstein’s moment condition. Corollary 5.2 slightly extends Lemma 2.2.11 in van der Vaart and Wellner (1996) for the case \( \kappa_i = \kappa \) (a fixed number). It should be noted that (5.6) can be replaced by \( \frac{1}{n} \sum_{i=1}^n E |X_i|^k \leq \frac{1}{2} v^2 \kappa^{k-2} k!, k = 3, 4, \ldots, \forall i \), where the \( v^2 \) is a variance-depending constant such that \( \frac{1}{n} \sum_{i=1}^n E |X_i|^2 \leq v^2 \). Then (5.7) still holds with \( v_n^2 = n v \), see Theorem 2.10 in Boucheron et al. (2013).

**Example 5.4** (Normal r.v.). Applying the relation between MGF and moment, the \( k \)-th moment of \( X \sim N(\mu, \sigma^2) \) is

\[
E |X|^{2k-1} = 0; \quad E |X|^{2k} = \sigma^{2k} (2k-1)(2k-3) \cdots 3 \cdot 1 \leq 2^{-1} (2\sigma^2) \sigma^{2k-2} (2k)!,
\]

which satisfies (5.6) with \( v^2 = 2\sigma^2, \kappa = \sigma^2 \).
5.3 Concentration of exponential family without compact space

Theory and statistical applications of natural exponential family (3.6) have attracted renewed attention in the past years (Lehmann and Romano, 2006). In Lasso penalized generalized linear models (GLMs), the results of oracle inequalities lie on CIs of a quantity that can be represent as Karush-Kuhn-Tucker conditions (see (8.23)) related to the centralized exponential family empirical process: \( \sum_{i=1}^{n} w_i(Y_i - \text{E}_i Y_i) \) for no-random weights \( \{w_i\}_{i=1}^{n} \) depending on the fixed design. Kakade et al. (2010) has studied the sub-exponential growth of the cumulants of an exponential family distribution and studied oracle inequalities of Lasso regularized GLMs, but the constant in their result is not specific.

In this part, we obtain central moments bounds with a specific constant, which gives the Bernstein’s inequality for the general exponential family, and the proof is based on the Cauchy formula of higher-order derivatives for complex function [Corollary 4.3 in Stein and Shakarchi (2010)].

Lemma 5.2 (Cauchy’s derivative inequalities). If \( f \) is analytic in an open set that contains the closure of a disk \( D \) centered at \( z_0 \) of radius \( 0 < r < \infty \), then \( |f^{(n)}(z_0)| \leq \frac{n!}{r^n} \sup_{|z - z_0| = r} |f(z)| \).

Zhang et al. (2014) adopts a similar approach for recovering the probability mass function (p.m.f.) from the characteristic function.

It is well-known that exponential families on the natural parameter space, \( \Theta \subset \{ \theta \in \mathbb{R}^k : e^\theta(b(y)) := \int c(y)e^{xb}\mu(dy) < \infty \} \) have finite analytic (standardized) moments and cumulants, see Lemma 3.3 in Kakade et al. (2010). The natural parameter space of an exponential family is convex, see Lehmann and Romano (2006). A nice property in Lehmann’s measure-theoretical statistical inference book is that:

Lemma 5.3 (Analytic property of MGF in the exponential family). The MGF \( m_{\theta}(s) := E_{\theta} e^{sY_i} \) on \( s \in C \) of exponential family r.vs indexed by \( \theta \), is analytic on \( \Theta \) (see Theorem 2.7.1 in Lehmann and Romano (2006) or Theorem 2 in Pistone and Wynn (1999)).

First, let us check that the following lemma which is deduced by Cauchy’s inequalities for the Taylor’s series coefficients of a complex analytic function.

Proposition 5.2. The \( s \rightarrow \tilde{m}_{\theta}(s) := E_{\theta} e^{s[Y_i - b(\theta)]} \) is analytic on the natural parameter space \( \Theta \) with radius \( r(\Theta) \), and the k-th absolute central moment of \( \{w_iY_i\}_{i=1}^{n} \) is bounded by

\[
E_{\theta} |w_i(Y_i - \text{E}_i Y_i)|^k \leq \frac{n!}{2(kwC_{\theta})^2} \sup_{|z| \leq r} |f(z)|^k,
\]

where \( \{w_i\}_{i=1}^{n} \) are non-random with \( w := \max_{1 \leq i \leq n} |w_i| > 0 \), and \( C_{\theta} := \inf_{0 < r \leq r(\Theta)} \frac{1}{r^k} E_{\theta} |e^{s(X_i - b(\theta))}| \).

Proof. Let \( s \in \mathbb{C} := \{b_i: b \in \mathbb{R}\} \) be a given complex number on imaginary axis.

\[
\tilde{m}_{\theta}(s) = E_{\theta} \left(e^{s[Y_i - b(\theta)]} 1\{Y_i \geq b(\theta)\} \right) + E_{\theta} \left(e^{s[b(\theta) - Y_i]} 1\{Y_i < b(\theta)\} \right)
\]

\[
= \int x \geq b(\theta) e^{c(x)} e^{s(x + \theta)} e^{-b(\theta)} \mu(dx) + \int x < b(\theta) e^{c(y)} e^{s(x - \theta)} e^{-b(\theta)} \mu(dx)
\]

\[
= e^{-b(\theta)} \left[ \int x \geq b(\theta) e^{c(x)} e^{s(x + \theta)} \mu(dx) + \int x < b(\theta) e^{c(x)} e^{s(x - \theta)} \mu(dx) \right].
\]

(5.8)
The natural parameter space implies \( \int c(x)e^{\theta_0} \mu(dx) \) is finite and analytic for \( \theta_0 \in \Theta \), so
\[
\int 1 \{ x \geq b(\theta_1) \} c(x)e^{\theta_0} \mu(dx) \quad \text{and} \quad \int 1 \{ x < b(\theta_1) \} c(y)e^{\theta_0} \mu(dx)
\]
are finite and analytic for \( s \in \Theta \). By Lemma 5.3, \( \bar{m}_{\theta_1}(s) \) in (5.8) is analytic on
\[
D_{\theta_1} := \{ s \in \mathbb{C} : \text{Re}(\theta_1 + s) \in \text{Int}(\Theta) \ \text{and} \ \text{Re}(\theta_1 - s) \in \text{Int}(\Theta) \}.
\]
by using analytic continuation [i.e. the \( \bar{m}_{\theta_1}(s) \) has an analytic continuation from \( \bar{m}_{\theta_1}(s) \) on \( s \in D_{\theta_1} \) to \( \bar{m}_{\theta_1}(s) \) on \( s \in \mathbb{C} \), see Corollary 4.9 in Stein and Shakarchi (2010)].

Since \( 0 + \theta_1 = \theta_1 \in D_{\theta_1} \subset \text{Int}(\Theta) \), \( \bar{m}_{\theta_1}(s) \) is analytic at the point 0 and hence the function is also analytic in a neighborhood of 0. By the analyticity of the functions \( \{ \bar{m}_{\theta_1}(s) \}_{\theta_1 \in \Theta} \) on \( s \in \text{Int}(\Theta) \), and Cauchy’s derivative inequality with \( z_0 = 0 \), we have
\[
E_{\theta_1} |Y_1 - b(\theta_1)|^k = \bar{m}_{\theta_1}^{(k)}(0) \leq k! \sup_{|s| = r} \left| E_{\theta_1} e^{r|Y_1 - b(\theta_1)|} \right|, \quad 0 < r \leq r(\Theta).
\]

Let \( s = r(\cos \omega + \sin \omega), \omega \in [0,2\pi] \). Then,
\[
E_{\theta_1} e^{r|Y_1 - b(\theta_1)|} = E_{\theta_1} e^{r|\cos \omega + \sin \omega|} e^{r|Y_1 - b(\theta_1)|} = E_{\theta_1} [e^{r|\cos \omega|} e^{r|Y_1 - b(\theta_1)|} e^{r|\sin \omega|}] = E_{\theta_1} [e^{r|\cos \omega|} e^{r|Y_1 - b(\theta_1)|} e^{r|\sin \omega|}].
\]

Hence, (5.9) gives
\[
k! \sup_{|s| = r} \left| E_{\theta_1} e^{s|Y_1 - b(\theta_1)|} \right| \leq k! \sup_{\omega \in [0,2\pi]} \left| E_{\theta_1} e^{r|Y_1 - b(\theta_1)|} \right| \leq k! \frac{E_{\theta_1} e^{r|Y_1 - b(\theta_1)|}}{r^k},
\]

(Due to \( E_{\theta_1} e^{r|Y_1 - b(\theta_1)|} \geq 1 \)),
\[
\left( \frac{[E_{\theta_1} e^{r|Y_1 - b(\theta_1)|}]^{1/k}}{r} \right)^k \leq k! \left( \frac{[E_{\theta_1} e^{r|Y_1 - b(\theta_1)|}]^{1/k}}{r} \right)^k.
\]

From (5.9), it shows that by take infimum over \( 0 < r \leq r(\Theta) \),
\[
E_{\theta_1} |Y_1 - b(\theta_1)|^k \leq k! \inf_{0 < r \leq r(\Theta)} \left( r^{-1}[E_{\theta_1} e^{r|Y_1 - b(\theta_1)|}] \right)^k \leq k! C_{\theta_1}^k = \left( \frac{4}{\sqrt{2 \pi}} \right)^2 C_{\theta_1}^{k-2}
\]
where \( C_{\theta_1} := \inf_{0 < r \leq r(\Theta)} \frac{r^{-1}[E_{\theta_1} e^{r|Y_1 - b(\theta_1)|}]}{k!} \leq \frac{1}{2} k! \left( \sqrt{2 \pi C_{\theta_1}} \right)^2 C_{\theta_1}^{k-2} = \frac{1}{2} k! \left( \frac{w}{\sqrt{2 \pi C_{\theta_1}}} \right)^2 C_{\theta_1}^{k-2}, \quad k = 2,3,\ldots.
\]

Therefore, \( w_i X_i \sim \text{subf}(\sqrt{2 \pi C_{\theta_1}}, w C_{\theta_1}) \) by Proposition 5.2 and we can apply the Bernstein’s inequality with the growth of moments condition to get the following concentration of exponential family on a natural parameter space.

**Theorem 5.1 (Concentration of exponential family).** Let \( \{ Y_i \}_{i=1}^n \) be a sequence of independent r.v.s with their densities \( \{ f(y;\theta_i) \}_{i=1}^n \) belong to canonical exponential family (3.6) on the natural parameter space \( \theta_i \in \Theta \). Given non-random weights \( \{ w_i \}_{i=1}^n \) with \( w =\max_{1 \leq i \leq n} w_i > 0 \), then
\[
P(\sum_{i=1}^n w_i (Y_i - E Y_i) \geq t) \leq 2 \exp \left( -\frac{t^2}{4 w^2 \sum_{i=1}^n C_{\theta_i} + 2 w \max_{1 \leq i \leq n} C_{\theta_i} t} \right),
\]
(5.10)
Theorem 5.1 has no compact space assumption. If we impose the compact space assumption (E.1) in Proposition 3.2, it leads to the sub-Gaussian concentration as presented in Proposition 5.1. The constant $C_{\beta_i}$ in Theorem 5.1 is hard to determine in general exponential family with infinite support. However, if the exponential family is Poisson, the $C_{\beta_i}$ can be obtained as explicit form.

**Theorem 5.2** (Concentration for weighted Poisson summation). Let $\{Y_i\}_{i=1}^n$ be independent Poisson$(\lambda_i)$ r.v.s with $\lambda_i > 0$ for all $i$. For non-random weights $\{w_i\}_{i=1}^n$ with $w = \max_{1 \leq i \leq n} |w_i| > 0$, put $S^n_w := \sum_{i=1}^n w_i (Y_i - EY_i)$, then for all $x > 0$

$$P(\|S^n_w\| \geq t) \leq 2 \exp(-\frac{t^2/2}{w^2 \sum_{i=1}^n \lambda_i + wt/3}), \quad P(\|S^n_w\| > w(2t \sum_{i=1}^n \lambda_i)^{1/2} + t/3) \leq e^{-t}. \quad (5.11)$$

**Proof.** We evaluate the log-MGF of centered Poisson r.v.s

$$\log E e^{s Y_i} = -s \lambda_i + \log E e^{s Y_i} = -\lambda_i s + \log e^{\lambda_i (e^{sw_i} - sw_i - 1)} = \lambda_i (e^{sw_i} - sw_i - 1).$$

Note that, for $s$ in a small neighbourhood of zero,

$$\lambda_i (e^{sw_i} - sw_i - 1) = \lambda_i \sum_{k=2}^{\infty} \frac{(sw_i)^k}{k!} \leq \lambda_i \sum_{k=2}^{\infty} \frac{|sw_i|^k}{k!} = \lambda_i s^2 w^2 \sum_{k=2}^{\infty} \frac{w_k}{k(k-1)\cdots 3} \leq \lambda_i s^2 w^2 \sum_{k=2}^{\infty} \frac{|sw_i|^k}{k^2} = \frac{s^2}{2} \frac{w^2 \lambda_i}{1 - w/3} \quad (5.12)$$

for $|s| \leq 3/w$, which implies $w_i (Y_i - EY_i) \sim \text{sub}\Gamma(w^2 \lambda_i, w/3)$. By Proposition 5.1(a), we have $S^n_w \sim \text{sub}\Gamma(w^2 \sum_{i=1}^n \lambda_i, w/3)$. Then applying Proposition 5.1(b), we get (5.11).

Before ending this section, we show a result for checking Bernstein’s moment condition by the moment recurrence condition of log-concave distributions.

**Definition 5.2** (Moment recurrence condition). A r.v. $Z$ is called moment bounded with parameter $L > 0$ if it has recurrence condition

$$E|Z|^p \leq p L \cdot E|Z|^{p-1} \quad \text{for any integer } p \geq 1.$$

By the recursion relation, Definition 5.2 implies that any moment bounded r.v. $Z$ satisfies $E|Z|^p \leq p L \cdot E|Z|^{p-1}$. Hence, the tails of its moment bounded r.v.s. decay as the Bernstein’s growth of moment condition. So the constant $C_{\beta_i}$ in Theorem 5.1 is relatively easy to find. Lemmas 7.2, 7.3, 7.6 and 7.7 in Schudy and Sviridenko (2012) showed that any log-concave continuous distribution (see Section 3.4) and log-concave discrete distribution $X$ with density $f$ is moment bounded with parameter $L \propto E|X|$.

**Example 5.5** (Log-concave continuous distributions, Bagnoli and Bergstrom (2005)). Many continuous distributions, such as normal distribution, exponential distribution, uniform distribution over any convex set, logistic distribution, extreme value distribution, chi-square distribution, chi distribution, hyperbolic secant distribution, Laplace distribution, Weibull distribution (the shape parameter $\theta \geq 1$), Gamma distribution (the shape parameter $a \geq 1$) and Beta distribution (both shape parameters are $\geq 1$) have log-concave continuous densities.
Analogous to the definition of log-concave continuous function in (3.12), we define log-concave sequence for the p.m.f. of discrete distributions, which are also have Bernstein-type concentrations.

**Definition 5.3 (Log-concave discrete distributions).** A sequence $\{p_i\}_{i \in \mathbb{Z}}$ (or $\{p_i\}_{i \in \mathbb{N}}$) is said to be log-concave if
\[
p_{i+1}^2 \geq p_ip_{i+2} \text{ for all } i \in \mathbb{Z} \text{ (or } i \in \mathbb{N}).
\]

An integer-valued r.v. $X$ is log-concave if its probability mass function (p.m.f.) $p_i := P(X = i)$ is log-concave sequence.

**Example 5.6 (Log-concave discrete distributions).** Bernoulli and binomial distributions, Poisson distribution, geometric distribution, and negative binomial distribution (with number of success $> 1$) and hypergeometric distribution have log-concave integer-valued p.m.f., see Johnson et al. (2005).

## 6 Sub-Weibull distributions

### 6.1 Sub-Weibull r.v.s and $\psi_\theta$-norm

A r.v. is heavy-tailed if its distribution function fails to be bounded by a decreasing exponential function (Foss et al., 2011). We first give a simple example of the heavy-tailed distributions arisen by multiplying sub-Gaussian r.v.s. The proof is motivated by Lemmas 2.7.7 of Vershynin (2018).

**Lemma 6.1 (The product of sub-Gaussians).** Suppose $\{X^{(m)}\}_{m=1}^d$ are sub-Gaussian (may be dependent). Then $\prod_{m=1}^d X^{(m)}$ is sub-exponential and $\|\prod_{m=1}^d X^{(m)}\|^2_\psi \leq \prod_{m=1}^d \|X^{(m)}\|^2_\psi$.

**Proof.** By the definition of sub-Gaussian norm, $E e^{\|X^{(m)}\|^2_\psi} \leq 2$, $m = 1, 2, \ldots, d$. Applying the elementary inequality $\prod_{m=1}^d a_m \leq \frac{1}{d} \sum_{m=1}^d a_m^d$, we get by Jensen’s inequality
\[
\prod_{m=1}^d \|X^{(m)}\|^2_\psi \leq \frac{1}{d} \sum_{m=1}^d E \|X^{(m)}\|^2_\psi \leq 2.
\]

The proof is finished by the definition of the sub-exponential norm.

In probability, Weibull r.v.s are generated from the power of the exponential r.v.s.

**Example 6.1 (Weibull r.v.s).** The Weibull r.v. $X \in \mathbb{R}^+$ is defined by its survival function
\[
P(X \geq x) = e^{-bx^\theta} \quad (x \geq 0)
\]
for the scale parameter $b > 0$ and the shape parameter $\theta > 0$.

Sub-Weibull distribution is characterized by the right tail of the Weibull distribution and is a generalization of both sub-Gaussian and sub-exponential distributions.
Definition 6.1 (Sub-Weibull distributions). A r.v. $X$ satisfying $P(|X| \geq x) \leq a e^{-b x^\theta}$ for given $a,b,\theta > 0$, is called a sub-Weibull r.v. with tail parameter $\theta$ (denoted by $X \sim \text{subW}(\theta)$).

A subW($\theta$)'s tail is no heavier than that of a Weibull r.v. with tail parameter $\theta$. It is emphasized that $X \sim \text{subW}(\theta)$ r.v.s with $\theta < 1$ belongs to heavy-tailed r.v.s. Recently, the Weibull-like tail condition is also studied in high-dimensional statistics and random matrix theory [see Tao and Vu (2013), Kuchibhotla and Chakrabortty (2018) and Wong et al. (2020)]. Götze et al. (2019) names subW($\theta$) as $\theta$-sub-exponential r.v. There are 4 equivalent conditions to reveal the sub-Weibull tail condition which is useful in applications.

Corollary 6.1 (Characterizations of sub-Weibull condition). Let $X$ be a r.v.. Then the following properties are equivalent.

1. The tails of $X$ satisfy $P(|X| \geq x) \leq e^{-(x/K_3)^\theta}$, for all $x \geq 0$.
2. The moments of $X$ satisfy $\|X\|_k := (E|X|^k)^{1/k} \leq K_2 k^{1/\theta}$ for all $k \geq 1$;
3. The MGF of $|X|^{1/\theta}$ satisfies $E e^{\lambda^{1/\theta} |X|^{1/\theta}} \leq e^{\lambda^{1/\theta} K_3^{1/\theta}}$ for $|\lambda| \leq 1$;
4. The MGF of $|X|^{1/\theta}$ is bounded at some point: $E e^{X/K_4^{1/\theta}} \leq 2$.

The proof can be founded in Wong et al. (2020), Vladimirova et al. (2020) by mimicking the proof of Proposition 2.5.2 in Vershynin (2018). It follows from Corollary 6.1 (4) that $X$ is sub-Weibull with tail parameter $\theta$ if and only if $|X|^{1/\theta}$ is sub-exponential.

Let $\theta_1$ and $\theta_2$ ($0 < \theta_1 \leq \theta_2$) be two sub-Weibull tail parameters. Corollary 6.1 implies subW($\theta_1$) $\subset$ subW($\theta_2$). The following sub-Weibull and Orlicz-type norms play crucial roles in deriving tail probability and maximal inequality for sub-Weibull r.v.s. without the zero-mean assumption.

Definition 6.2 (Sub-Weibull norm or $\psi_\theta$-norm). Let $\psi_\theta(x) := e^{x^\theta} - 1$. The sub-Weibull norm of $X$ for any $\theta > 0$ is defined as $\|X\|_{\psi_\theta} := \inf\{C \in (0,\infty) : E|g(X)/C^{\theta}| \leq 2\}$.

A r.v. with finite $\psi_\theta$-norm is said to be sub-Weibull. Sub-Weibull norm is a special case of the Orlicz norm (Wellner, 2017).

Definition 6.3 (Orlicz Norms). Let $g: [0,\infty) \rightarrow [0,\infty]$ be a non-decreasing convex function with $g(0) = 0$. The “$g$-Orlicz norm” of a r.v. $X$ is $\|X\|_g := \inf\{\eta > 0 : E[g(|X|/\eta)] \leq 1\}$.

Let $g(x) = e^{x^\theta} - 1$ and $E[g(|X|/\eta)] \leq 1$ implies $E[\exp(|X|^{\theta}/\eta^\theta)] \leq 2$, which is the definition of sub-Weibull norm. Similar to sub-exponential, Zajkowski (2019), Wong et al. (2020), Vladimirova et al. (2020) attained the following.

Corollary 6.2 (Properties of sub-Weibull norm). If $E e^{X/\|X\|_{\psi_\theta}^\theta} \leq 2$, then (a). $P(|X| > t) \leq 2e^{-(t/\|X\|_{\psi_\theta}^\theta)}$ for all $t \geq 0$; (b). Moment bounds: $E|X|^k \leq 2 \|X\|_{\psi_\theta}^k \Gamma(k/\theta + 1)$.
6.2 Concentrations for sub-Weibull summation

The Chernoff inequality tricks in the derivation of Corollary 4.2 for sub-exponential concentration is not valid for sub-Weibull distributions, since the exponential moment equivalent conditions of sub-Weibull are on the absolute value $|X|$. However, Bernstein’s moment condition is the exponential moment of the absolute value. An alternative method is given by Kuchibhotla and Chakrabortty (2018), who defines the so-called Generalized Bernstein-Orlicz (GBO) norm. A promising development is that the following GBO norm helps us derive tail behaviors for sub-Weibull r.v.s.

**Definition 6.4** (Generalized Bernstein-Orlicz Norm). Fixed $\alpha > 0$ and $L \geq 0$, define a function $\Psi_{\theta,L}(\cdot)$ with its inverse function $\Psi_{\theta,L}^{-1}(t) := \sqrt{\log(t+1)+L\log(t+1)^{1/\theta}} \quad \forall \, t \geq 0$. The Generalized Bernstein-Orlicz (GBO) norm of a r.v. $X$ is then given by:

$$
\|X\|_{\Psi_{\theta,L}} := \inf \{ \eta > 0 : \mathbb{E}[\Psi_{\theta,L}(|X|/\eta)] \leq 1 \}.
$$

The monotone function $\Psi_{\theta,L}(\cdot)$ is motivated by the classical Bernstein’s inequality for sub-exponential r.v.s. Like the sub-Weibull norm properties Corollary 6.2(a), the following proposition in Kuchibhotla and Chakrabortty (2018) allows us to get the concentration inequality for r.v.s with the finite GBO norm.

**Corollary 6.3** (Concentration inequality based on GBO norm). For any r.v. $X$ with $\|X\|_{\Psi_{\theta,L}} < \infty$, we have $P(|X| \geq \|X\|_{\Psi_{\theta,L}} \sqrt{T+Lt^{1/\theta}}) \leq 2e^{-t}$, for all $t \geq 0$.

With the upper bounds of GBO norm and Corollary 6.3, it is easy to derive the concentration inequality for a single sub-Weibull r.v. or even the sum of independent sub-Weibull r.v.s. Theorem 3.1 in Kuchibhotla and Chakrabortty (2018) obtains an upper bound for the GBO norm of the summation.

**Corollary 6.4** (Concentration for sub-Weibull summation). If $\{X_i\}_{i=1}^n$ are independent centralized r.v.s such that $\|X_i\|_{q_i} < \infty$ for all $1 \leq i \leq n$ and some $\theta > 0$, then for any weight vector $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $\sum_{i=1}^n w_i X_i \|\Psi_{\theta,L}(\cdot)\|_{\Psi_{\theta,L}} \leq 2eC(\theta) \|b\|_2$ and

$$
P(\sum_{i=1}^n w_i X_i \geq \|b\|_2 \sqrt{T+Ln(\theta)\frac{1}{\theta}}) \leq 2e^{-t} \quad (6.2)
$$

where $b = (\|X_1\|_{q_1}, \ldots, \|X_n\|_{q_n})^T \in \mathbb{R}^n$, $L_n(\theta) := \frac{4^{1/\theta}}{\sqrt{\|b\|_2}} \times \left\{ \begin{array}{ll} \|b\|_{\infty} & , \text{if } \theta < 1, \\ 4e \|b\|_2 / C(\theta) & , \text{if } \theta \geq 1 \end{array} \right.$

and $C(\theta) := \max \{ \sqrt{2} \cdot 2^{1/\theta} \times \left\{ \begin{array}{ll} \sqrt{2}e^3(2\pi)^{1/4}e^{1/24}(e^{2/\theta} / \theta)^{1/\theta} & , \text{if } \theta < 1, \\ 4e+2(\log 2)^{1/\theta}, & , \text{if } \theta \geq 1. \end{array} \right.$

The upper bound of sub-Weibull norm for summation provided by Corollary 6.4 depends on $\|X_i\|_{q_i}$ and the $w$. Zhang and Wei (2020) gives a sharper version of Corollary 6.4. The $\theta = 1$ is the phrase transition point, and reflect the fact that Weibull r.v.s are log-convex for $\theta \leq 1$ and log-concave for $\theta \geq 1$. At last, we mention a generalized Hanson-Wright inequality for sub-Weibull r.v.s in Götzte et al. (2019).
Corollary 6.5 (Concentration for the quadratic form of sub-Weibull r.v.s.). Let \( q \in \mathbb{N}, A = (a_{ij}) \) be a symmetric \( n \times n \) matrix and let \( \{X_i\}_{i=1}^{n} \) be a set of independent, centered r.v.s. with \( \|X_i\|_{2/4} \leq M \) and \( \text{EX}_i^2 = a_{ii}^2 \). For any \( t > 0 \)
\[
P(\left| \sum_{i,j} a_{ij} X_i X_j - \sum_{i=1}^{n} a_{ii}^2 \right| \geq t) \leq 2 \exp\left(-\eta(A,q,t/M^2) / C \right)
\]
where \( \eta(A,q,t) := \min \left( \frac{t}{\|A\|_{4/2}}, \frac{t}{\|A\|_{op}}, \left( \frac{\max_{i=1,...,n} \| a_{ii} \|_2}{\| A \|_{\infty}} \right)^{1/2}, \left( \frac{\| A \|_{\infty}}{\| A \|_{op}} \right)^{1/4} \right) \) and \( C \) is a constant.

7 Concentration for Extremes

The CIs presented so far only concern with linear combinations of independent r.v.s or Lipschitz function of random vectors. In many statistics applications, we have to control the maximum of the Lipschitz function of random vectors. In many statistics applications, we have to control the maximum of the r.v.s when deriving the error bounds, while these r.v.s may be arbitrarily dependent. The development of this section is based on advanced proof skills. So we present the proofs even for existing results, which are applications of CIs in a probability aspect.

7.1 Maximal inequalities

This section presents the maximal inequalities for r.v.s. \( \{X_i\}_{i=1}^{n} \) which may not be independent. In the theory of empirical process, it is of interest to bound \( \text{E} \max_{1 \leq i \leq n} |X_i| \) [Section 2.2, van der Vaart and Wellner (1996)]. If \( \{X_i\}_{i=1}^{n} \) are arbitrary sequence of real-valued r.v.s. and have finite \( r \)-th moments \((r \geq 1)\), Aven (1985) gives a crude upper bounds for \( \text{E} \max_{1 \leq i \leq n} |X_i| \) by Jensen’s inequality

\[
\text{E}\left\{ \max_{1 \leq i \leq n} |X_i|^r \right\}^{1/r} \leq \left\{ \text{E} \max_{1 \leq i \leq n} |X_i|^r \right\}^{1/r} \leq \left\{ \sum_{i=1}^{n} \text{E}|X_i|^r \right\}^{1/r} \leq n^{1/r} \max_{1 \leq i \leq n} (\text{E}|X_i|^r)^{1/r} \tag{7.1}
\]

Page314 of van der Vaart (1998) mentions a sharper version of (7.1) without the proof. In below, we introduce the proof by the truncation technique.

Corollary 7.1 (Sharper maximal inequality). Let \( \{X_i\}_{i=1}^{n} \) be identically distributed but not necessarily independent and assume \( \text{E}|X_i|^p < \infty \), \( (p \geq 1) \). Then, \( \text{E} \max_{1 \leq i \leq n} |X_i| = o(n^{1/p}) \).

Proof. Let \( M_n := \max_{1 \leq i \leq n} |X_i| \). For any \( e > 0 \), we truncate \( M_n \) by \( en^{1/p} \),
\[
\text{EM}_n = \int_{0}^{en^{1/p}} t \text{P}(M_n > t) dt + \int_{en^{1/p}}^{\infty} t \text{P}(M_n > t) dt \leq \int_{0}^{en^{1/p}} t dt + \int_{en^{1/p}}^{\infty} \text{nP}(|X_i| > t) dt
\]
\[
= en^{1/p} + n^{1/p} \int_{en^{1/p}}^{\infty} n^{(p-1)/p} t P(|X_i| > t) dt \leq en^{1/p} + \frac{n^{1/p}}{e^{p-1}} \int_{en^{1/p}}^{\infty} t^{p-1} P(|X_i| > t) dt.
\]
Thus, by dividing \( n^{1/p} \) we have \( \frac{\text{EM}_n}{n^{1/p}} \leq e + \frac{1}{e^{p-1}} \int_{en^{1/p}}^{\infty} t^{p-1} P(|X_i| > t) dt = e + o(1) \), where we adopt the fact \( \int_{en^{1/p}}^{\infty} t^{p-1} P(|X_i| > t) dt = o(1) \) from moment condition: \( \text{E}|X_i|^p < \infty \). Finally, it implies that \( \limsup_{n \to \infty} \frac{\text{EM}_n}{n^{1/p}} \leq e \), which gives \( \text{EM}_n = o(n^{1/p}) \) by letting \( e \to 0 \). \( \square \)
Corollary 7.1 reveals that $\max_{1 \leq i \leq n} |X_i|$ diverges at rate slower than $n^{1/r}$ under the $r$-th moment condition. If we have arbitrary finite $r$-th moment conditions (such as Gaussian distribution), it means that the divergence rate of maxima is slower than any polynomial rate $n^{1/r}$. This suggests that the rate may be logarithmic. With the sub-Gaussian assumptions, Rigollet and Hütter (2019) presents the logarithmic divergence rate below and the proof is based on controlling the expectation of the supremum of variables, from the argument in Pisier (1983).

**Corollary 7.2** (Sub-Gaussian maximal inequality). Let \( \{X_i\}_{i=1}^n \) be r.v.s (without independence assumption) such that \( X_i \sim \text{subG}(\sigma^2) \). Then,

(a) \( \mathbb{E}[\max_{1 \leq i \leq n} X_i] \leq \sigma \sqrt{2 \log n} \) and \( \mathbb{E}[\max_{1 \leq i \leq n} |X_i|] \leq \sigma \sqrt{2 \log(2n)} \).

(b) \( P(\max_{1 \leq i \leq n} X_i > t) \leq ne^{-\frac{t^2}{2\sigma^2}} \) and \( P(\max_{1 \leq i \leq n} |X_i| > t) \leq 2ne^{-\frac{t^2}{2\sigma^2}} \).

**Proof.** (a) By the property of maximum, sub-Gaussian MGF and Jensen’s inequality,

\[
\mathbb{E}_{X_i \sim \text{subG}(\sigma^2)} \max_{1 \leq i \leq n} X_i = \inf_{s > 0} s \mathbb{E}^{\max_{1 \leq i \leq n} X_i} \leq \inf_{s > 0} s \log\mathbb{E}^{\max_{1 \leq i \leq n} X_i} \leq \inf_{s > 0} s \log \sum_{i=1}^n \mathbb{E}^{X_i} \leq \inf_{s > 0} s \log \sum_{i=1}^n e^{\frac{s^2}{2\sigma^2}} = \inf_{s > 0} s \left( \log n + \frac{\sigma^2 s^2}{2} \right) = \sigma \sqrt{2 \log n} \text{ [Setting } s = \sqrt{\frac{2 \log n}{\sigma^2} \text{ as the optimal bound].}
\]

Let \( Y_{2i-1} = X_i \) and \( Y_{2i} = -X_i (1 \leq i \leq n) \). It gives \( \max_{1 \leq i \leq n} |X_i| = \max_{1 \leq i \leq n} X_i \max_{1 \leq i \leq n} (X_i - X_i) = \max_{1 \leq i \leq n} X_i \max_{1 \leq i \leq 2n} Y_i \). The previous result for sample size $2n$ finishes the proof of the second part.

(b) By Chernoff inequality and the sub-Gaussian MGF, we have \( P(\max_{1 \leq i \leq n} X_i > t) \leq \inf_{s > 0} \mathbb{E}^{\max_{1 \leq i \leq n} X_i} \mathbb{E}^{\max_{1 \leq i \leq n} \sigma X_i} \leq \inf_{s > 0} \mathbb{E}^{\max_{1 \leq i \leq n} \sigma X_i} \mathbb{E}^{\max_{1 \leq i \leq n} \sigma X_i} \leq \inf_{s > 0} ne^{-st:} \sum_{i=1}^n \mathbb{E}^{X_i} \leq \inf_{s > 0} ne^{-st:} \sum_{i=1}^n \mathbb{E}^{X_i} = ne^{-\frac{t^2}{2\sigma^2}}. \) For the second part, note that \( P(\max_{1 \leq i \leq n} |X_i| > t) = P(\max_{1 \leq i \leq n} X_i > t, \max_{1 \leq i \leq n} X_i > t, \max_{1 \leq i \leq n} X_i > t) \leq 2P(\max_{1 \leq i \leq n} X_i > t). \)

By a similar proof, Corollary 7.2 can be extended to other r.v.s, such as sub-Gamma r.v.s and r.v.s characterized by sub-Weibull norm (or Orlicz norm) as presented before, see Corollary 2.6 of Boucheron et al. (2013) for the case of sub\(\Gamma_+(\nu,c)\).

**Corollary 7.3** (Concentration for maximum of sub-Gamma r.v.s). Let \( \{X_i\}_{i=1}^n \) be independent zero-mean \( \{\text{sub}\Gamma(\nu,c_i)\}_{i=1}^n \). Then

\[
\mathbb{E}(\max_{i=1,...,n} |X_i|) \leq \sqrt{2\nu \log(2n)} + c\log(2n)
\]

for \( \max_{i=1,...,n} u_i =: u \) and \( \max_{i=1,...,n} c_i =: c \).

In below, based on the sub-Weibull norm condition, a fundamental result due to Pisier (1983) is given for obtaining the divergence rate of the maxima of sub-Weibull r.v.s.

**Corollary 7.4** (Maximal inequality for sub-Weibull r.v.s). For $\theta > 0$, consider the sub-Weibull norm $\|X\|_{\psi_{\theta}} := \inf_{c \in (0,\infty)} \{ E|x|^\theta/c^\theta \leq 2 \}$ for $\psi_{\theta}(x) = x^\theta - 1$. For any r.v. \( \{X_i\}_{i=1}^n \),

\[
\mathbb{E}(\max_{1 \leq i \leq n} |X_i|) \leq \psi_{\theta}^{-1}(n) \max_{1 \leq i \leq n} \|X_i\|_{\psi_{\theta}} = (\log(1+n))^{1/\theta} \max_{1 \leq i \leq n} \|X_i\|_{\psi_{\theta}}
\] (7.2)
If the function \( \psi_\theta(x) \) is replaced by any non-decreasing convex function \( g(x) \) with \( g(0)=0 \) in the definition of Orlicz norm: \( \|X\|_g:=\inf\{\eta>0: E[g(|X|/\eta)]\leq 1\} \), then
\[
E(\max_{1\leq i\leq n} |X_i|) \leq g^{-1}(n) \max_{1\leq i\leq n} \|X_i\|_g \quad \text{for finite} \max_{1\leq i\leq n} \|X_i\|_g.
\]

**Proof.** From Jensen’s inequality, for \( C \in (0,\infty) \) and \( \psi_\theta(x)=e^{x\theta} - 1 \) we get
\[
\psi_\theta\left[ E(\max_{1\leq i\leq n} |X_i|/C) \right] \leq E[\max_{1\leq i\leq n} \psi_\theta(|X_i|/C)] \leq \sum_{i=1}^n E\psi_\theta(|X_i|/C) \leq n
\]
where the last inequality is by the definition of sub-Weibull norm: \( E\psi_\theta(|X_i|/C) \leq 1 \).

Let \( C := \max_{1\leq i\leq n} \|X_i\|_{\psi_\theta} \). Applying the non-decreasing property of \( \psi_\theta(x) \) (so does its inverse \( \psi^{-1}_\theta(x) \)), the (7.3) implies \( E(\max_{1\leq i\leq n} |X_i|/C) \leq \psi^{-1}_\theta(n) \) by operating the map \( \psi^{-1}_\theta \), and so we have (7.2). The derivation of Orlicz norm case is the same. \( \square \)

By Hoeffding’s lemma, the following results on the maxima of the sum of independent r.v.s, useful for bounding empirical processes, is from Lemma 14.14 in Bühlmann and van de Geer (2011).

**Corollary 7.5** (Maximal inequality for the sum of independent bounded r.v.s). Let \( X_1, \ldots, X_n \) be independent r.v.s on \( \mathcal{X} \) and \( f_1, \ldots, f_n \) be real-valued functions on \( \mathcal{X} \) which satisfy \( Ef_j(X_i) = 0 \), \( |f_j(X_i)| \leq a_{ij} \) for all \( j=1, \ldots, p \) and all \( i=1, \ldots, n \). Then
\[
E(\max_{1\leq i\leq p} \sum_{i=1}^n f_j(X_i)) \leq [2\log(2p)]^{1/2} \max_{1\leq i\leq p} \left( \sum_{j=1}^n a_{ij}^2 \right)^{1/2}.
\]

If Hoeffding’s lemma for moment is replaced by Bernstein’s moment conditions, then the maximal inequality for the sum of independent bounded r.v.s in Corollary 7.5 can be extended to Bernstein’s moment conditions. We give a modified version of Corollary 14.1 in Bühlmann and van de Geer (2011) based on truncated Jensen’s inequality.

**Proposition 7.1** (Maximal inequality with Bernstein’s moment conditions). If \( \{X_{ij}\}_{j=1}^p \) are read-valued independent variables across \( i=1, \ldots, n \), assume \( E X_{ij} = 0 \) and Bernstein’s moment conditions: \( \frac{1}{n} \sum_{i=1}^n E|X_{ij}|^k \leq \frac{1}{2} \alpha^2 \kappa^{-2} k! \), \( k=2,3, \ldots \), \( \forall j \). Then,
\[
E(\max_{1\leq j\leq p} \frac{1}{n} \sum_{i=1}^n |X_{ij}|^m) \leq \left[ \frac{x\log(2p)}{n} + (\alpha^2 + 1) \sqrt{\frac{\log(2p)}{n}} \right]^m, \quad 1 \leq m \leq 1+\log p \text{ and } p \geq 2.
\]

**Proof.** Let \( M_n,m := \max_{1\leq j\leq p} \frac{1}{n} \sum_{i=1}^n |X_{ij}|^m \). First, we show for any r.v. \( X \) and all \( m \geq 1 \),
\[
E|X|^m \leq \log^m(E|X| - 1 + e^{m-1}). \quad (7.4)
\]
The function \( g(x) = \log^m(x+1), x \geq 0 \) is concave for all \( x \geq e^{m-1} - 1 \). By the truncated Jensen’s inequality in Lemma 2.5 with \( Z := e^{|X|} - 1, c = e^{m-1} - 1 \), we have
\[ E |X|^m = E \log^m (e^{X} - 1 + 1) \leq \log^m [E (e^{X} - 1 + 1) + (e^m - 1)] = \log^m [E (e^{X} - 1 + e^m - 1)]. \]

Then for all \( L, m > 0, \)
\[
\left( \frac{L}{n} \right)^m E_{n,m} \leq \log^m \left[ E \max \left( \frac{\sum X_{ij}}{L} \right) - 1 + e^m - 1 \right] \leq \log^m \left( \sum_{j=1}^{p} E \left( e^{\frac{\sum X_{ij}}{L}} - 1 \right) + e^m - 1 \right). \tag{7.5}
\]

Therefore, it is sufficient to bound \( E e^{\frac{\sum X_{ij}}{L}} \) uniformly in \( j. \)

Second. To bound the MGF in (7.5), then we show that for any real-valued r.v. \( X, \)
\[ E e^X \leq E e^{X - 1 - E|X|}, \text{ with } EX = 0. \tag{7.6} \]

Indeed, for any \( c > 0, \) we have \( e^{X-c} - 1 \leq \frac{e^{|X|}-1}{1+c} \leq \frac{e^{|X|}-1+|X|-c}{1+c}. \) Let \( c = \text{E}\exp[|X|] - 1 - \text{E}|X|. \) Note that \( EX = 0, \) so \( E e^{X-c} - 1 \leq E e^{X} - \text{E}|X| - c = 0. \) Hence \( \log (E e^X) \leq c. \)

Using Taylor’s expansion, the (7.6) and \( e^x \leq e^x + e^{-x} \) give
\[ E \left( \frac{\sum X_{ij}}{L} \right) - 1 \leq E e^{\sum X_{ij}/L} + E e^{-\sum X_{ij}/L} - 1 \leq 2 e^{\sum_{m=1}^{\infty} (e^{X_{ij}/L} - 1)^m} \leq 2 e^{\sum_{m=2}^{\infty} (e^{X_{ij}/L})^{m-2} - 1} = 2 e^{2 e^{2(\kappa/|X|)}} - 1. \tag{7.7} \]

Combining (7.7) and (7.5), we obtain for \( L > \kappa = L - \sqrt{n/2 \log (p+e^{-m-1})}, \)
\[
\text{E}_{n,m} \leq \left( \frac{L}{n} \right)^m \log^m \left( \frac{2 e^{2 e^{2(\kappa/|X|)}} - 1 + e^m - 1}{2 e^{2 e^{2(\kappa/|X|)}} - 1} \right) \leq \left( \frac{L}{n} \right)^m \log^m \left( \frac{p+e^{m-1} e^{2 e^{2(\kappa/|X|)}}}{2 e^{2 e^{2(\kappa/|X|)}} - 1} \right) \leq \left( \frac{L}{n} \right)^m \log^m \left( \frac{p+e^{m-1}}{2 e^{2 e^{2(\kappa/|X|)}} - 1} \right) \leq \left( \frac{L}{n} \right)^m \log^m \left( \frac{p+e^{m-1}}{2 e^{2 e^{2(\kappa/|X|)}} - 1} \right).
\]

where the second and last inequality is by \( \frac{2 e^{2 e^{2(\kappa/|X|)}} - 1}{2 e^{2 e^{2(\kappa/|X|)}} - 1} \) and \( e^m - 1 \leq p. \)

\section{7.2 Concentration for suprema of empirical processes}

Let \( \{ X_{ij} \}_{j=1}^{n} \) be a random sample from a measure \( \mathbb{P} \) on a measurable space \( (\mathcal{X}, \mathcal{A}). \)

The empirical distribution \( \mathbb{P}_n := n^{-1} \sum_{i=1}^{n} \delta_{X_i}, \) where \( \delta_{X_i} \) is the probability mass of 1 at \( x. \)

Given a measurable function \( f : \mathcal{X} \rightarrow \mathbb{R}, \) let \( \mathbb{P}_n f := \frac{1}{n} \sum_{i=1}^{n} f (X_i) \) be the expectation of \( f \) under the empirical measure \( \mathbb{P}_n, \) and \( \mathbb{P} f := \int f d\mathbb{P} \) be the expectation under \( \mathbb{P}. \) The \( \mathbb{P}_n f \) is called the empirical process index by \( n. \)

The study of the empirical processes begins with the uniform limit law of EDF in Example 2.1. The Glivenko-Cantelli theorem extends the LLN for EDF and gives uniform convergence: \( \| \mathbb{P}_n - F \|_{\infty} = \sup_{t \in \mathbb{R}} \left| \mathbb{P}_n (t) - F (t) \right| \xrightarrow{a.s} 0. \) Moreover, a stronger result than Example 2.1 is the Dvoretzky-Kiefer-Wolfowitz inequality (DKW)(Dvoretzky et al., 1956)
\[ P(\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| > \varepsilon) \leq 2e^{-2n\varepsilon^2} \quad \forall \varepsilon > 0. \]

The DKW inequality is a uniform version of Hoeffding’s inequality. It also strengthens the Glivenko-Cantelli theorem since DKW implies Glivenko-Cantelli: \( \|F_n - F\|_{\infty} \xrightarrow{a.s.} 0 \) by Borel-Cantelli lemma: \( X_n \xrightarrow{a.s.} 0 \Leftrightarrow \sum_{n=1}^{\infty} P(|X_n| \geq \varepsilon) < \infty \) for any \( \varepsilon > 0 \). Dvoretzky et al. (1956) proves the inequality with an unspecified multiplicative constant \( C \) in the exponent of the upper bounds. Massart (1996) attained the sharper constant \( C = 2 \).

In some statistical applications, given an estimator \( \hat{\theta} \), and \( f_\theta(X_i) \) is a function of \( X_i \) and \( \hat{\theta} \). We want to study its asymptotic properties for sums of \( f_\theta(X_i) \) that changes with both \( n \) and \( \hat{\theta} \),

\[ \frac{1}{n} \sum_{i=1}^{n} [f_\theta(X_i) - E f_\theta(X_i)], \text{ (a dependent sum)}. \]

A possible route to attain results is via the suprema of the empirical process for all possible the “true” parameter \( \theta_0 \) on a set \( K \):

\[ \frac{1}{n} \sum_{i=1}^{n} [f_\theta(X_i) - E f_\theta(X_i)] \leq \sup_{\theta \in K} \frac{1}{n} \sum_{i=1}^{n} [f_\theta(X_i) - E f_\theta(X_i)] = \sup_{\theta \in K} |P_n f_\theta - P f_\theta|. \]

Fortunately, the summation in the sup enjoys independence. So, the study of convergence rate suprema of empirical processes is important if we consider a functional class \( \mathcal{F} \) instead of the set \( K \) such that: \( \sup_{f \in \mathcal{F}} |P_n f - Pf| = \sup_{\theta \in K} |P_n f_\theta - P f_\theta| \).

Let \( (\mathcal{F}, \| \cdot \|) \) be a normed space of real functions \( f : \mathcal{X} \rightarrow \mathbb{R} \) on \( \mathcal{X} \). Define the \( L_r(Q) \)-norm by \( \|f\|_{L_r(Q)} = \left( \int |f|^r dQ \right)^{1/r} \). For a probability measure \( Q, L_r(Q) \)-spaces is endowed by the \( L_r(Q) \)-norm. Given two functions \( l(\cdot) \) and \( u(\cdot) \), the bracket \( [l, u] \) is the set of all functions \( f \in \mathcal{F} \) with \( l(x) \leq f(x) \leq u(x) \), for all \( x \in \mathcal{X} \). An \( \varepsilon \)-bracket is a bracket \( [l, u] \) with \( \|l - u\|_{L_r(Q)} < \varepsilon \). One condition to get the convergence of \( \sup_{f \in \mathcal{F}} |P_n f - Pf| \) is the finite bracketing number condition with \( L_1(P) \)-norm in Theorem 19.4 of van der Vaart (1998).

**Lemma 7.1** (Glivenko-Cantelli class). The bracketing number \( N_1(\varepsilon, \mathcal{F}, L_r(Q)) \) is minimum number of \( \varepsilon \)-brackets needed to cover \( \mathcal{F} \), i.e.

\[ N_1(\varepsilon, \mathcal{F}, L_r(Q)) = \inf \{ n : \exists l_1, u_1, \ldots, l_n, u_n \text{ s.t. } \cup_{i=1}^{n} [l_i, u_i] = \mathcal{F} \text{ and } \|l_i - u_i\|_{L_r(Q)} < \varepsilon \}. \]

Every class \( \mathcal{F} \) of measurable functions s.t. \( N_1(\varepsilon, \mathcal{F}, L_1(P)) < \infty \) for every \( \varepsilon > 0 \) is \( P \)-Glivenko-Cantelli, i.e. \( \sup_{f \in \mathcal{F}} |P_n f - Pf| \xrightarrow{a.s.} 0 \).

**Example 7.1** (Empirical process with indicator functions). Let \( \mathcal{F} \) be the collection of all indicator functions of the form \( f_i = 1_{(-\infty,t]} \), with \( t \) ranging over \( \mathbb{R} \). Then, \( \mathcal{F} \) is \( P \)-Glivenko-Cantelli, see Example 19.4 of van der Vaart (1998).

**Example 7.2** (Weighted empirical process with dependent weights). Suppose we observe a sequence of IID observations \( \{(X_i, Y_i)\}_{i=1}^{\infty} \) drawn from a random pair \( (X, Y) \). Given some weighted functions \( W(\cdot) \) and a bounded estimator \( \hat{t} \in (0, \tau] \), we want to study the stochastic convergence of dependent weighted empirical processes.
\[
\frac{1}{n} \sum_{i=1}^{n} 1(Y_i \geq t) W(X_i) - \mu(t; W) \leq \sup_{0 \leq t \leq T} \frac{1}{n} \sum_{i=1}^{n} 1(Y_i \geq t) W(X_i) - \mu(t; W)
\]
where \( \mu(t; W) = E_{X,Y} \{1(Y \geq t) W(X)\} < \infty \) and \( W(X) \leq U_f < \infty \) and \( \tau < \infty \).

Consider the class of functions indexed by \( t \),
\[
\mathcal{F} = \{1(y \geq t) W(x)/U_f : t \in [0, \tau], y \in \mathbb{R}, W(x) \leq U_f \}.
\]
It is crucial to evaluate the bracketing number of \( \mathcal{F} \). Given \( \epsilon \in (0,1) \), let \( t_i \) be the \( i \)-th \( \lceil 1/\epsilon \rceil \) quantile of \( Y \), thus \( P(Y \leq t_i) = i \epsilon, i = 1, \ldots, \lceil 1/\epsilon \rceil - 1 \). Furthermore, take \( t_0 = 0 \) and \( t_{\lceil 1/\epsilon \rceil} = +\infty \).

For \( i = 1, \ldots, \lfloor 1/\epsilon \rfloor \), define brackets \( [L_i, U_i] \) with
\[
L_i(x,y) = 1(y \geq t_i) \frac{W(x)}{U_f}, \quad U_i(x,y) = 1(y > t_{i-1}) \frac{W(x)}{U_f}
\]
such that \( L_i(x,y) \leq 1(y \geq t) e^{\Delta(x)/U_f} \leq U_i(x,y) \) as \( t_{i-1} < t \leq t_i \). The Jensen’s inequality gives
\[
E[U_i - L_i] \leq E[\frac{W(X)}{U_f} \{1(Y \geq t_i) - 1(Y > t_{i-1})\}] < \|P(t_{i-1} < Y \leq t_i)\| = \epsilon.
\]
Therefore, \( N_{s}([\epsilon, \mathcal{F}, L_2(P)]) < 1/\epsilon < \infty \) for every \( \epsilon > 0 \). So the class \( \mathcal{F} \) is \( P \)-Glivenko-Cantelli.

To extend DKW inequality to general empirical processes with any bounded function \( f \), the following tail bounds inequality is obtained by Talarand (1994) which characterizes the convergence rate of suprema of empirical processes in Lemma 7.1.

**Lemma 7.2** (Sharper bounds for suprema of empirical processes). Consider a probability space \((\Omega, \Sigma, P)\), and consider \( n \) IID r.v.s \( \{X_i\}_{i=1}^{n} \) valued in \( \Omega \), of law \( P \). Let \( \mathcal{F} \) be a class of measurable functions \( f: \mathcal{X} \rightarrow [0,1] \) that satisfies bracketing number conditions with \( L_2(P) \)-norm: \( N_{s}([\epsilon, \mathcal{F}, L_2(P)]) \leq (K/\epsilon)^V \), for every \( 0 < \epsilon < K \). Then, for every \( t > 0 \)
\[
P(\sqrt{n} \sup_{f \in \mathcal{F}} |P_{n}f - Pf| \geq t) \leq \left( \frac{D(K)}{\sqrt{V}} \right)^V e^{-2t^2}
\]
with a constant \( D(K) \) that depends on \( K \) only.

Kong and Nan (2014) derives the rate of convergence for the Lasso regularized high-dimensional Cox models by using sharper concentration inequality for the suprema of empirical processes in Example 7.2 related to the negative log partial likelihood function. The explicit constant \( D(K) \) can be found in Zhang (2006), who studies the tail bounds for the supremums of the empirical process \( \{n^{-1/2} \sum_{i=1}^{n} [f(X_i) - Ef(X_i)]\} \), where \( \{X_i\} \) is a sequence of (non-i.i.d. unbounded) independent random vectors with values in a general measurable space \((\mathcal{X}, A)\), and \( f \) is a measurable real function on \((\mathcal{X}, A)\). In Example 7.2,
\[
\{E[U_i - L_i]^2\}^{1/2} \leq \{E[\frac{W(X_i)}{U_m} \{1(Y \geq t_i) - 1(Y > t_{i-1})\}]^2\}^{1/2} \leq |P(t_{i-1} < Y \leq t_i)|^{1/2} = \sqrt{\epsilon},
\]
which implies \( N_{s}([\sqrt{\epsilon}, \mathcal{F}, L_2(P)]) \leq 1/\epsilon \leq 2/\epsilon^2 \) for every \( \epsilon > 0 \). Hence, \( N_{s}([\epsilon, \mathcal{F}, L_2(P)]) \leq 2/\epsilon^2 \). By applying Lemma 7.2 with \( V = 2, K = \sqrt{2} \), we have
\[
P(\sup_{0 \leq t \leq T} \frac{1}{U_f \sqrt{n}} \sum_{i=1}^{n} [1(Y_i \geq t) W(X_i) - \mu(t; W)] \geq t) \leq \frac{D^2(2\sqrt{2})}{2} t^2 e^{-2t^2}.
\]
Let $t = R \sqrt{\log \log n}$ for a constant $R > 0$. We obtain $\frac{D^2(\sqrt{T})}{2} e^{-2t^2} = \frac{D^2(\sqrt{T})}{2} (R^2 \log \log n) e^{\log(\log n) - 2t^2} = \frac{D^2(2) R^2 \log \log n}{2(\log n)^2}$. Then it derives the non-asymptotic version of the LIL

$$P\left( \sup_{0 \leq t \leq T} \left| \frac{1}{n} \sum_{i=1}^{n} 1(Y_i \geq t) W(X_i) - \mu(t; W) \right| \leq \frac{R \sqrt{\log \log n}}{\sqrt{n}} \right) \geq 1 - \frac{D^2(\sqrt{T}) R^2 \log \log n}{2(\log n)^2}.$$  

More result of the LIL in empirical processes are presented in Page 268 of van der Vaart (1998). The last two results are the symmetrization theorem and the contraction theorem, which are fundamental tools to get sharper bounds for suprema of empirical processes.

**Lemma 7.3** (Symmetrization theorem). Let $\{X_i\}_{i=1}^{n}$ independent r.v.s with values in some space $\mathcal{X}$ and $\mathcal{F}$ a class of measurable real-valued functions on $\mathcal{X}$. Let $\{\epsilon_i\}_{i=1}^{n}$ be a Rademacher sequence, independent of $\{X_i\}_{i=1}^{n}$ and $f \in \mathcal{F}$. If $E|f(X_i)| < \infty \ \forall \ i$, then

$$E\{\sup_{f \in \mathcal{F}} \Phi(\sum_{i=1}^{n} f(X_i)) \} \leq E\{\sup_{f \in \mathcal{F}} \Phi(2 \sum_{i=1}^{n} \epsilon_i f(X_i)) \}$$

for every nondecreasing, convex $\Phi : \mathbb{R} \mapsto \mathbb{R}$ and class of measurable functions $\mathcal{F}$.

**Lemma 7.4** (Contraction theorem). Let $x_1, \ldots, x_n$ be the non-random elements of $\mathcal{X}$ and $\epsilon_1, \ldots, \epsilon_n$ be Rademacher sequence. Consider $c$-Lipschitz functions $g_i$, i.e. $|g_i(s) - g_i(t)| \leq c|s - t|, \forall s, t \in \mathbb{R}$. Then for any function $f$ and $h$ in $\mathcal{F}$, we have

$$E_{\epsilon} \{ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} \epsilon_i \left( f(x_i) - g_i(h(x_i)) \right) \right| \} \leq 2c E_{\epsilon} \{ \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} \epsilon_i \left( f(x_i) - h(x_i) \right) \right| \}.$$  

A gentle introduction to the theory of suprema of empirical processes and it statistical applications are nicely presented in Sen (2018).

8 Concentration for High-dimensional Statistics

With the emergence of high-dimensional (HD) data such as the gene expression data, there are renewed interests on the CIs. One aspect of the HD data is such that the number of variables $p$ can be comparable to or even great than the sample size $n$. This section provide results in three commonly encountered settings: increasing-dimensional ($p_n = o(n) < n$), large-dimensional ($p_n = O(n)$) and high-dimensional setting ($p_n \gg n$, $p_n = e^{o(n)}$).

8.1 Linear models with diverging number of covariates

Suppose that we have an $n$-dimensional random vector $Y$ which contains $n$ responses $\{Y_i\}_{i=1}^{n}$ to $p$ covariates $X_i = (x_{i1}, \cdots, x_{ip})^T$, respectively. The $n$ copies of $X_i$ as row vectors make a $n \times p$ design matrix $X = (X_1, \cdots, X_n)^T$. The conditional expectation $E[Y_i | X_i]$ is linearly related to a coefficient vector $\beta^* = (\beta_{11}^*, \cdots, \beta_{p1}^*)^T$ such that

$$Y = X \beta^* + \epsilon$$  

(8.1)
where \( \{\varepsilon_i\}_{i=1}^n \) in the error vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T \) are IID with zero mean and finite variance \( \sigma^2 \). The \( \beta^* \) needs to be estimated.

This subsection only considers the case that \( p \) is increasing but \( p < n \). The ordinary least square (OLS) estimator is

\[
\hat{\beta}_{LS} = \arg\min_{\beta \in \mathbb{R}^p} \| Y - X\beta \|_2^2.
\]  

(8.2)

Assume rank \( (X) = p \), which is not hard to meet since \( p < n \), \( \hat{\beta}_{LS} = (X^TX)^{-1}X^TY \) is the unique solution of the (8.2). The following result for the OLS estimator is well known.

**Lemma 8.1.** Under the assumptions on the linear models and the rank of \( X \) is \( p \), then

(i) Let \( A \) be a \( p \times n \) matrix, then \( E\|A\varepsilon\|_2^2 = E(\varepsilon^T A^T A \varepsilon) = \sigma^2 \text{tr}(A^T A) \).

(ii) (The curse of dimensionality) The mean square error and the average in-sample \( \ell_2 \) risk of the OLS estimator are \( E\| \hat{\beta}_{LS} - \beta^* \|_2^2 = \text{tr}((X^TX)^{-1}) \sigma^2 \) and \( \frac{1}{n} E\| X(\hat{\beta}_{LS} - \beta^*) \|_2^2 = \frac{\sigma^2}{n} \).

**Remark 8.1.** As \( p, n \to \infty \) with \( p < n \), part (ii) implies that the OLS estimator may had poor performance if \( p/n \to c > 0 \). The average in-sample \( \ell_2 \)-risk tends to zero if \( p_n = o(n) \).

Put \( \tilde{\beta} := \hat{\beta}_{LS} \). Let \( \{\lambda_i(X^TX)\}_{i=1}^p \) be the eigenvalue values of \( X^TX \). Markov’s inequality and Lemma 8.1 with \( A = (X^TX)^{-1}X^T \) implies

\[
P\{ \| \tilde{\beta} - \beta^* \|_2 > t \} \leq \frac{\sigma^2 \text{tr}(X^TX)^{-1}}{t^2} = \frac{\sigma^2}{t^2} \sum_{i=1}^p \frac{1}{\lambda_i(X^TX)} \leq \frac{\sigma^2}{t^2} \frac{p}{\lambda_{\min}(X^TX)} =: \delta_n,
\]

(8.3)

which implies that, with probability greater than \( 1 - \delta_n \),

\[
\| \tilde{\beta} - \beta^* \|_2 \leq \sigma \sqrt{\frac{p}{n} \delta_n \lambda_{\min}(\frac{1}{n}X^TX)}^{1/2}.
\]

(8.4)

Assume that \( p := p_n = o(n^t) \) as \( n \to \infty \), \( p_n < n \). We specific two groups of regularity conditions and the value of \( r \) such that the \( \ell_2 \) consistency \( (\| \tilde{\beta} - \beta^* \|_2 \overset{p}{\to} 0) \) is true.

(1) By Lemma 8.1, if \( \frac{1}{n}X^TX \) is uniformly positive definite (\( \exists c > 0 \) s.t. \( \frac{1}{n}X^TX \succ cI_p \)) then

\[
\frac{p\sigma^2}{n} = \frac{1}{n} E\|X(\tilde{\beta} - \beta^*)\|_2^2 = E[(\tilde{\beta} - \beta^*)^T \frac{1}{n}X^TX(\tilde{\beta} - \beta^*)] \geq c E\| \tilde{\beta} - \beta^* \|_2^2.
\]

If \( p = o(n) \), then \( E\| \tilde{\beta} - \beta^* \|_2 = o(1) \) which implies \( \| \tilde{\beta} - \beta^* \|_2 = o_p(1) \).

(2) From (8.4), if \( p = o(\lambda_{\min}(X^TX)) \), we have \( \| \tilde{\beta} - \beta^* \|_2 = o_p(1) \). In this case, if we consider: “\( \frac{1}{n}X^TX \) is positive definite” in (1), it also leads to \( p = o(\lambda_{\min}(X^TX)) = o(n) \).

In (8.1) with fixed design, suppose that the \( \varepsilon_1, \ldots, \varepsilon_n \) are sub-Gaussian zero-mean noise for which there exists a \( \sigma > 0 \) such that \( \mathbb{E}\varepsilon_i^{2r} \leq \sigma^2 \varepsilon_i^{2r} \), \( \forall \varepsilon_1, \ldots, \varepsilon_n \in \mathbb{R} \). Suppose that the Gram matrix \( S_n := \frac{1}{n}X^TX \) is invertible. The excess in-sample prediction error \( R(\tilde{\beta}) \) is the difference between the expected squared error for \( X^TX \tilde{\beta} \) and for \( X^TX \beta^* \):

\[
R(\tilde{\beta}) := \frac{1}{n} \left[ E\left\{ \sum_{i=1}^n (X_i^T \beta - Y_i)^2 \right\} - E\left\{ \sum_{i=1}^n (X_i^T \beta^* - Y_i)^2 \right\} \right]_{\beta = \tilde{\beta}}
\]

\[
= \frac{1}{n} \| X(\tilde{\beta} - \beta^*) \|_2^2 + \frac{1}{n} E\left\{ \sum_{i=1}^n (X_i^T \beta - X_i^T \beta^*) \cdot \varepsilon_i \right\}_{\beta = \tilde{\beta}} = \frac{1}{n} \| X(X^TX)^{-1}X^T \varepsilon \|_2^2.
\]

(8.5)
which is a quadratic form of sub-Gaussian vector.

By Corollary 4.7 with \( A := X(X^T X)^{-1} X^T / \sqrt{n}, \xi := \epsilon, \mu = 0 \) and \( \Sigma := A^T A = X(X^T X)^{-1} X^T / n, \)

\[
\text{tr}(\Sigma) = \text{tr}((X^T X)^{-1} X^T X) / n = p / n, \quad \text{tr}(\Sigma^2) = p / n^2, \quad ||\Sigma||_2 = 1 / n.
\]

where last identity is due to \( X(X^T X)^{-1} X^T \) being a projection matrix. Thus \( P[R(\hat{\beta}) > \sigma^2/(p+2\sqrt{p+2})]\) \( \leq e^{-t} \), i.e. with probability \( 1 - e^{-t} \),

\[
R(\hat{\beta}) \leq \sigma^2/(p+2\sqrt{p+2})
\]

For Gaussian noise, \( ER(\hat{\beta}) = \sigma^2 p / n \) in Lemma 8.1, so \( P\{ R(\hat{\beta}) - ER(\hat{\beta}) \leq \sigma^2/(2\sqrt{p+2}) \} \geq 1 - e^{-t} \).

### 8.2 Non-asymptotic Bai-Yin theorem for random matrix

Let \( A \) be a \( p \times p \) Hermitian matrix with real eigenvalues: \( \lambda_{\max} := \lambda_1 \geq \cdots \geq \lambda_p =: \lambda_{\min} \). The empirical spectral distribution (ESD) of \( A \) is

\[
F_A(x) = \frac{1}{p} \sum_{j=1}^p 1(\lambda_j \leq x),
\]

which resembles the empirical distribution of a random sample. Let \( \{A_n\}_{n \geq 1} \) be a sequence of \( p \times p \) Hermitian random matrices indexed by the sample size \( n \), and \( F_{A_n} \) be the ESD of \( A_n \). A major interest in random matrix theory is to investigate the convergence of \( F_{A_n} \) as a sequence of distributions to a limit \( F \). In multivariate statistics, it is of interest to study the sample covariance matrix \( S_n := \frac{1}{p} XX^T \) where the double array \( X = \{ X_{ij} : i = 1, \ldots, p; j = 1, \ldots, n \} \) contains zero-mean IID r.v.s \( X_{ij} \) with variance \( \sigma^2 \). Suppose that the dimensions \( n \) and \( p \) grow to infinity while \( p / n \) converges to a constant in \([0,1]\). Marčenko and Pastur (1967) gives the limit behavior of the ESD of \( S_n \). Bai and Yin (1993) obtained a strong version of the Marčenko-Pastur law.

**Corollary 8.1 (Bai-Yin theorem).** Let \( X \) be an \( n \times p \) random matrix whose entries are independent copies of a r.v. with zero mean, unit variance, and finite fourth moment \( (E|X|_4^4 < \infty) \). As \( n \to \infty, p \to \infty, p / n \to y \in (0,1) \), then

\[
\lim_{n \to \infty} \lambda_{\min}(S_n) = \sigma^2 (1 - \sqrt{y})^2, \quad \lim_{n \to \infty} \lambda_{\max}(S_n) = \sigma^2 (1 + \sqrt{y})^2.
\]

Note that \( \lambda_i(S_n) = \lambda_i(X / \sqrt{n}) \) for all \( i \), Bai-Yin’s law asserts that if \( \sigma^2 = 1 \): \( \lambda_{\min}(X / \sqrt{n}) = 1 - \sqrt{p/n + o(\sqrt{p/n})} \), \( \lambda_{\max}(X / \sqrt{n}) = 1 + \sqrt{p/n + o(\sqrt{p/n})} \) a.s..

**Theorem 4.6.1 in Vershynin (2018) studies the non-asymptotic upper and lower bounds of the extreme eigenvalues of sample covariance with independent sub-exponential entries, but the bounds contained un-specific constants. We give a constant-specified version in the following.** The \( n \)-dimensional unit Euclidean sphere \( S^{n-1} := \{ x \in \mathbb{R}^n : ||x||_2 = 1 \} \). We say that a random vector \( X \) in \( \mathbb{R}^n \) is sub-Gaussian (sub-exponential) if the one-dimensional marginals \( \langle X, x \rangle \) are sub-Gaussian (sub-exponential) r.v.s for all \( x \in \mathbb{R}^n \). The sub-Gaussian (sub-exponential) norm of \( X \) is defined as \( ||X||_{\psi_2} := \sup_{x \in S^{n-1}} ||\langle X, x \rangle||_{\psi_2} \) ( \( ||X||_{\psi_1} := \sup_{x \in S^{n-1}} ||\langle X, x \rangle||_{\psi_1} \)).
Proposition 8.1 (Constants-specified non-asymptotic Bai-Yin theorem). Let $X$ be an $n \times p$ matrix whose rows $X_j$ are independent sub-Gaussian random vectors in $\mathbb{R}^p$ with $\text{Var}(X_j) = I_p$. Define $Z_j := \langle X_j, x \rangle$, $\forall x \in S^{n-1}$. Further assume that $\{Z_i^2 - 1\}_{i=1}^n$ are subE($\theta$), then

$$P\left\{ \left\| n^{-1}X^T X - I_p \right\| \leq 2c\theta \max(\delta, \delta^2) \right\} \geq 1 - 2e^{-c\delta^2}, \quad t \geq 0$$

(8.6)

where $\delta = 2c(\sqrt{p/n + t}/\sqrt{n})$ with $t = c\theta \max(\delta, \delta^2)$ and $c \geq 2n \log 9/p$. Moreover,

$$P \{ 1 - t^2 \leq \lambda_{\text{min}}(S_n) \leq \lambda_{\text{max}}(S_n) \leq 1 + t^2 \} \geq 1 - 2e^{-c\delta^2}.$$  

(8.7)

Proposition 8.1 does not require $p/n \to y \in (0, 1)$ as in Corollary 8.1.

Proof. Step 1. We introduce a counting measure for measuring the complexity of a set in some space. The covering number $\mathcal{N}(K, \epsilon)$ is the smallest number of closed balls centered at $K$ with radii $\epsilon$ whose union covers $K$. For some $\epsilon \in [0, 1]$, a subset $\mathcal{N}_\epsilon \subseteq \mathbb{R}$ is an $\epsilon$-net for $S^{n-1}$ if for all $x \in S^{n-1}$, there is a $y \in \mathcal{N}_\epsilon$ such that $\|x - y\| < \epsilon$. We use the following results in Lemma 5.2 and 5.4 of Vershynin (2012).

Lemma 8.2 (Covering numbers of the sphere). $\mathcal{N}(S^{n-1}, \epsilon) \leq (1 + \frac{1}{2})^n$ for every $\epsilon > 0$.

Lemma 8.3 (Computing the spectral norm on a net). Let $B$ be an $p \times p$ matrix. Then

$$\|B\| := \max_{\|x\|_2 = 1} \|Bx\|_2 = \sup_{x \in S^{p-1}} |\langle Bx, x \rangle| \leq (1 - 2\epsilon)^{-1} \sup_{x \in \mathcal{N}_\epsilon} |\langle Bx, x \rangle|.$$ 

Lemma 8.3 shows that $\|\frac{1}{n}X^T X - I_p\| \leq 2\max_{x \in \mathcal{N}_{1/4}} |\frac{1}{n}\|Xx\|_2^2 - 1|$. Indeed, note that $\langle \frac{1}{n}X^T X x, x \rangle - 1 = \frac{1}{n}\|Xx\|_2^2 - 1$. By setting $\epsilon = 1/4$ in Lemma 8.3, we get

$$\|n^{-1}X^T X - I_p\| \leq (1 - 2\epsilon)^{-1} \sup_{x \in \mathcal{N}_{1/4}} |\langle n^{-1}X^T X x, x \rangle| = 2\max_{x \in \mathcal{N}_{1/4}} |\|n^{-1}Xx\|_2^2 - 1|.$$  

(8.8)

By (8.8), we have

$$P\{ \|n^{-1}X^T X - I_p\| \geq 2t \} \leq P\{ 2 \|Xx\|_2^2 - 1 \geq 2t \} \leq \sum_{x \in \mathcal{N}_{1/4}} P\{ |\|n^{-1}Xx\|_2^2 - 1| \geq t \}\leq \mathcal{N}(S^{n-1}, 1/4) P\{ |\|n^{-1}Xx\|_2^2 - 1| \geq t \}, \forall x \in \mathcal{N}_{1/4},$$

(8.9)

where the last inequality follows Lemma 8.2 with $\epsilon = 1/4$.

Step 2. It is sufficient to bound $P\{ \frac{1}{n^n} \|Xx\|_2^2 - 1 \geq t \}$. Let $Z_i := |\langle X_i, x \rangle|$, $\forall x \in S^{n-1}$. Observe that $\|Xx\|_2^2 = \sum_{i=1}^{n} |\langle X_i, x \rangle|^2 = \sum_{i=1}^{n} Z_i^2$. Apply the sub-exponential concentration inequality in Corollary 4.2, $P\{ |\|n^{-1}Xx\|_2^2 - 1| \geq t \} = P\{ |\sum_{i=1}^{n} (Z_i^2 - 1) | \geq t \} \leq 2e^{-\frac{t^2}{2(\bar{B}^2)}}$. Specially, let $t = c\theta \max(\delta, \delta^2) = c\theta [\theta I_{\{\delta \leq 1\}} + \delta^2 I_{\{\delta > 1\}}]$ with $\delta := 2c(p/n + t/\sqrt{n})$. From (8.9),

$$P\{ \|n^{-1}X^T X - I_p\| \geq 2t \} \leq 9^n P\{ |\|n^{-1}Xx\|_2^2 - 1| \geq c\theta \max(\delta, \delta^2) \} \leq 2 \cdot 9^n e^{-\frac{t^2}{2}} \leq 2 \cdot 9^n e^{-c(p+t^2)/2}$$

(8.10)
where the last inequality is obtained by using the inequality \((a+b)^2 \geq a^2 + b^2\) for \(a, b \geq 0\). For \(c \geq n \log 9/p\), \(2.9^n e^{-c(p+1)} \leq 2e^{-ct^2}\), which proves (8.6).

**Step 3.** To show (8.7), the \(\max_{|x|_2=1} \left\| \frac{1}{\sqrt{n}} X x \right\|_2^2 - 1 = \max_{|x|_2=1} \left\| \left( \frac{1}{\sqrt{n}} X^T X - I_p \right) x \right\|_2^2 = \left\| \frac{1}{\sqrt{n}} X^T X - I_p \right\|_2^2 \leq t^2\) implies that \(1 - t^2 \leq \lambda_{\max}(S_n) \leq 1 + t^2\). Similarly, for \(\lambda_{\min}(S_n)\),

\[
\min_{|x|_2=1} \left\| \frac{1}{\sqrt{n}} X x \right\|_2^2 - 1 = \min_{|x|_2=1} \left\| \left( \frac{1}{\sqrt{n}} X^T X - I_p \right) x \right\|_2^2 \leq \max_{|x|_2=1} \left\| \left( \frac{1}{\sqrt{n}} X^T X - I_p \right) x \right\|_2^2 \leq t^2.
\]

So \(\lambda_{\min}(S_n) \in [1 - t^2, 1 + t^2]\) and \(\left\{ \|X^T X - I_p]\|^2 \leq t^2\right\} \subset \{1 - t^2 \leq \lambda_{\min}(S_n) \leq \lambda_{\max}(S_n) \leq 1 + t^2\}\). Then \(P\{1 - t^2 \leq \lambda_{\min}(S_n) \leq \lambda_{\max}(S_n) \leq 1 + t^2\} \geq P\{\left\| \frac{1}{\sqrt{n}} X^T X - I_p \right\|_2^2 \leq t^2\} \geq 1 - 2e^{-ct^2}\). \(\square\)

### 8.3 Oracle inequalities for penalized linear models

This section introduces the proofs of the error bounds from the perspective of Lasso penalized linear models with the \(\ell_2\)-loss function. When \(p > n\), the OLS estimator is no longer available as \(\frac{1}{n} \sum_{i=1}^n X_i X_i^T\) is of invertible. A common way for obtaining a plausible estimator for the true parameter \(\beta^*\) is by adding penalized function to the square loss function. For \(0 < q \leq \infty\), we write \(\|\beta\|_q := (\sum_{j=1}^p |\beta_j|^q)^{1/q}\) as the \(\ell_q\)-norm for \(\beta \in \mathbb{R}^p\). If \(q = \infty\), \(\|\beta\|_\infty := \max_{j=1,...,p} |\beta_j|; \) if \(q = 0\), \(\|\beta\|_0 := \sum_{j=1}^p 1(\beta_j \neq 0)\). There are two types statistical guarantees of \(\hat{\beta}\) as mentioned in Bartlett et al. (2012).

1. **Persistence**: \(\hat{\beta}\) performs well on a new sample \(X^* \overset{d}{=} X\) (equal in distribution), i.e.
\[
E\{X^T(\hat{\beta} - \beta^*)|X^*\} \to 0.
\]

2. \(\ell_q\)-**consistency** (\(q \geq 1\)): \(\hat{\beta}\) approximates \(\beta^*\), i.e. with high probability \(\|\hat{\beta} - \beta^*\|_q \to 0\).

The persistence and \(\ell_1\)-consistency are respectively obtained by error bounds:

\[
\|\hat{\beta} - \beta^*\|_1 \leq O_p(s\lambda_n), \quad E\{|X^T(\hat{\beta} - \beta^*)|^2|X^*\} \leq O_p(s\lambda_n^2), \text{ (say oracle inequalities)}
\]

where \(\lambda_n \to 0\) is a tuning parameter and \(s := \|\beta^*\|_0\). In the following, we focus on the \(\ell_1\) estimation and prediction consistencies for the penalized linear models. Let \(\lambda > 0\) be a tuning parameter, the **Lasso estimator** (Tibshirani, 1996) for Model (8.1) is

\[
\hat{\beta}_L = \underset{\beta \in \mathbb{R}^p}{\text{arg min}} \{\|Y - X\beta\|_2^2/n + \lambda \|\beta\|_1\}. \quad (8.10)
\]

By sub-derivative techniques in convex optimizations, the Karush-Kuhn-Tucker (KKT) conditions of Lasso optimization function is

\[
\begin{cases}
2[\dot{X}^T(Y - X\hat{\beta}_L)]_j/n = -\lambda \text{sign}(\hat{\beta}_L)_j \text{ if } \hat{\beta}_L \neq 0, \\
2|[X^T(Y - X\hat{\beta}_L)]_j|/n \leq \lambda \text{ if } \hat{\beta}_L_j = 0.
\end{cases}
\]

which implies \(\|\frac{1}{\sqrt{n}} X^T(Y - X\hat{\beta})\|_\infty \leq \frac{\lambda}{2}\). Another approach to get the Lasso-like sparse estimator is attained by Dantzig selector (DS)

\[
\hat{\beta}_{DS} = \underset{\beta \in \mathbb{R}^p}{\text{arg min}} \{\|\beta\|_1 : \|X^T(Y - X\beta)\|_\infty/n \leq \lambda/2\}. \quad (8.11)
\]
see Candès and Tao (2007). Lasso and DS are capable of producing sparse estimates with only a few (hence sparse) nonzero coefficients among the \( p \) coefficients of the covariates. The idea of Lasso and DS was presented in a geophysics literature (Levy and Fullagar, 1981). By (8.11), we get \( \|\hat{\beta}_{DS}\|_1 \leq \|\hat{\beta}_L\|_1 \), which signifies that the DS may be more sparse than the Lasso.

It is well-known that \( \Sigma := \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T \) is singular when \( p > n \). To obtain oracle inequalities for the Lasso estimator with the minimax optimal rate (Ye and Zhang, 2010), the restricted eigenvalues proposed in Bickel et al. (2009) is usually needed. Let \( S(\beta^*):= \{j: \beta_j^* \neq 0, \beta^* = (\beta_1^*, \ldots, \beta_p^*)^T\} \) and \( S := |S(\beta^*)| \). For any vector \( b \in \mathbb{R}^p \) and any index set \( H \subset \{1, 2, \ldots, p\} \), define the sub-vector indexed by \( H \) as \( b_H = (\ldots, b_j, \ldots)^T \in \mathbb{R}^p \) with \( b_j = b \) if \( j \in H \) and \( b_j = 0 \) if \( j \notin H \). Define the conic set for a sparse \( \beta^* \) with support \( S(\beta^*) \):

\[
C(\eta, S(\beta^*)) = \{b \in \mathbb{R}^p: \|b_{S(\beta^*)^c}\|_1 \leq \eta \|b_{S(\beta^*)}\|_1\}, \quad \eta > 0.
\]

(8.12)

Denote the restricted eigenvalue condition (RE) as \( RE(\eta, S(\beta^*), \Sigma) = \inf_{0 \neq b \in C(\eta, S(\beta^*))} \frac{\langle b^T \Sigma b \rangle^{1/2}}{\|b\|_2} > 0 \) for any \( p \times p \) matrix \( \Sigma \). In the following, we present a modified version of Theorem 7.2 in Bickel et al. (2009) from Lemma 2.5 of Li and Jia (2017) beyond Gaussian noise.

**Proposition 8.2** (The rate of convergence of the Lasso). Suppose that \( X \) is the fixed design matrix and the error sequence \( \{\varepsilon_i\}_{i=1}^{n} \overset{i.i.d.}{\sim} N(0, \sigma^2) \) or \( \{\varepsilon_i/\sigma\}_{i=1}^{n} \overset{i.i.d.}{\sim} 2 -$strongly log-concave distribution satisfying Lemma 3.2. Let \( \{X_{(j)}\}_{j=1}^{n} \) be column vectors of \( X \). We assume that \( \frac{1}{n} X_{(j)}^T X_{(j)} = 1 \). If \( \lambda = A \sigma \sqrt{\log p/n} \) satisfies the KKT condition for \( \beta^* \),

\[
\|X^T (Y - X \beta^*)/n\|_\infty \leq \lambda/2.
\]

(8.13)

1. Then the estimated error \( u := \hat{\beta}_L - \beta^* \) satisfies \( \|u_{S(\beta^*)^c}\|_1 \leq 3\|u_{S(\beta^*)}\|_1 \), i.e. \( u \in C(3, S(\beta^*)) \).
2. Suppose that \( X \) satisfies the RE condition \( \gamma := RE(3, S(\beta^*), \frac{1}{n} \sum_{i=1}^{n} X_i X_i^T) > 0 \). We have non-asymptotic oracle inequalities with probability greater than \( 1 - 2p^{-1/2} \):

\[
(a) . \|\hat{\beta}_L - \beta^*\|_1 \leq \frac{3A \sigma }{ \gamma^2 } n^{1/2} \sqrt{\frac{\log p}{n}}; \quad (b) . \|\hat{\beta}_L - \beta^*\|_2^2 \leq \frac{9A \sigma^2 }{ \gamma^2 } n \log p / n;
\]

(8.14)

\[
(c) . \frac{1}{n} \|X(\hat{\beta}_L - \beta^*)\|_2^2 \leq \frac{9A \sigma^2 }{ \gamma } n \log p / n, \quad A > 2 \sqrt{2}.
\]

(8.15)

**Proof.** The proof consists 3 steps. **Step1:** By the Lasso optimization (8.10),

\[
(2n)^{-1} \|Y - X \hat{\beta}_L\|_2 + \lambda \|\hat{\beta}_L\|_1 \leq (2n)^{-1} \|Y - X \beta^*\|_2 + \lambda \|\beta^*\|_1.
\]

(8.16)

From \( \frac{\|X - X \hat{\beta}_L\|_2^2}{2n} = \frac{1}{2n} \|X \beta^* - X \hat{\beta}_L\|_2^2 = \frac{1}{2n} \|X \beta^* - X \hat{\beta}_L\|_2^2 + \frac{\|\varepsilon\|_2^2}{2n} - \frac{1}{2} \varepsilon^T X (\hat{\beta}_L - \beta^*) \) and \( \frac{1}{2n} \|Y - X \beta^*\|_2^2 = \frac{\|\varepsilon\|_2^2}{2n} \), thus \( \frac{1}{2n} \|X \beta^* - X \hat{\beta}_L\|_2^2 + \frac{\|\varepsilon\|_2^2}{2n} - \frac{1}{2} \varepsilon^T X (\hat{\beta}_L - \beta^*) + \lambda \|\hat{\beta}_L\|_1 \leq \frac{\|\varepsilon\|_2^2}{2n} + \lambda \|\beta^*\|_1 \). Then,

\[
(2n)^{-1} \|X(\hat{\beta}_L - \beta^*)\|_2^2 + \lambda \|\hat{\beta}_L\|_1 \leq n^{-1} \varepsilon^T X (\hat{\beta}_L - \beta^*) + \lambda \|\beta^*\|_1.
\]

(8.17)
The (8.17) is usually called the basic inequality in the proof of Lasso oracle inequalities. The first term in the left side of inequality (8.17) is the empirical prediction error, while on the right side, \( \frac{1}{n}e^TX(\hat{\beta} - \beta^*) \) is random and \( \lambda ||\beta^*||_1 \) is still fixed and unknown. For \( \frac{1}{n}e^TX(\hat{\beta} - \beta^*) \), if we can get a sharper upper bound and it approaching 0 as \( n \to \infty \), then we can achieve a sharper oracle inequality in below. By (8.13),

\[
\frac{\|X(\hat{\beta} - \beta^*)\|_2}{2n} + \lambda \|\hat{\beta}\|_1 \leq \frac{1}{n}e^TX_{\infty}\|\hat{\beta} - \beta^*\|_1 + \lambda \|\beta^*\|_1 \leq \frac{\lambda}{2}||\hat{\beta} - \beta^*||_1 + \lambda ||\beta^*||_1. \tag{8.18}
\]

Let \( S := S(\beta^*) \) and notes that \( ||\hat{\beta}_S||_1 = ||\beta_S^* + (\hat{\beta}_S - \beta_S^*)||_1 \geq ||\beta_S^*||_1 - ||\hat{\beta}_S - \beta_S^*||_1 \), then

\[
||\hat{\beta}||_1 = ||\hat{\beta}_S^*||_1 + ||\hat{\beta}_S||_1 \geq ||\beta_S^*||_1 - ||\hat{\beta}_S - \beta_S^*||_1 + ||\hat{\beta}_S^*||_1. \tag{8.19}
\]

From (8.18), we get \( ||u_S^*||_1 \leq 3||u_S||_1 \) be checking

\[
0 \leq (2n)^{-1}||X(\hat{\beta} - \beta^*)||_2^2 \leq \lambda \|\hat{\beta} - \beta^*\|_1/2 + \lambda \|\beta^*\|_1 - \lambda \|\hat{\beta}\|_1
\]

\[
\leq \frac{\lambda}{2} \{ (||\hat{\beta}_S - \beta_S^*||_1 + ||\hat{\beta}_S^*||_1) + \lambda ||\beta_S^*||_1 - \lambda \{ ||\beta_S^*||_1 - ||\hat{\beta}_S - \beta_S^*||_1 + ||\hat{\beta}_S^*||_1 \} \} \quad \text{[By (8.19)]}
\]

\[
= \frac{3\lambda}{2} ||\hat{\beta}_S - \beta_S^*||_1 - \frac{\lambda}{2} ||\hat{\beta}_S^*||_1 =: \frac{3\lambda}{2} ||u_S||_1 - \frac{\lambda}{2} ||u_S^*||_1. \tag{8.20}
\]

**Step2:** The Gaussian error vector \( \epsilon \) enables us to get the Gaussian concentration around its mean, we can shows that (8.13) occurs with a high probability. So next we need to check the Lipschitz condition in Lemma 3.1. Use Lemma 3.1, it implies that

\[
P(n^{-1}|X_{(j)}^T(Y - X\beta^*)| \geq t) \leq 2pe^{-nt^2/2\sigma^2}, \forall j. \tag{8.21}
\]

The Lipschitz condition depends on the design matrix \( X \). The \( ||X_{(j)}||_2^2 = X_{(j)}^TX_{(j)} = n \) is a presupposition due to (8.21). The different types of CIs require different assumptions on the design matrix (the random design is allowed if we adopt empirical process theory). In Lemma 3.1, put \( f(a) := \frac{1}{n}X_{(j)}^T(\sigma a - X\beta^*) \). Then, Cauchy’s inequality implies

\[
f(a) - f(b) \leq \frac{\sigma}{\sqrt{n}}||X_{(j)}^T(b-a)||_2 \leq \frac{\sigma}{\sqrt{n}}||X_{(j)}||_2 \cdot ||b-a||_2 = \frac{\sigma}{\sqrt{n}}||b-a||_2 \forall j.
\]

Hence, \( f(a) \) is \( \sigma/\sqrt{n} \)-Lipschitz. Recall \( \lambda = \Lambda\sigma\sqrt{\log p}/n \). So (8.21) implies

\[
P(||\frac{1}{n}X^T(Y - X\beta^*)||_\infty \geq \frac{t}{2}) \leq \sum_{j=1}^n P(\frac{1}{n}X_{(j)}^T(Y - X\beta^*) \geq \frac{1}{2}\Lambda\sigma\sqrt{\frac{\log p}{n}}) \leq 2p^{-1-t^2/2\sigma^2}.
\]

By Lemma 3.3, (8.21) is also held for \( \{\epsilon_i/\sigma\}_{i=1}^n \sim 2\text{-strongly log-concave distribution.}

**Step3:** Next we can start the proof based on cone set condition (8.12). Since the \( X \) satisfies RE condition \( \gamma := \text{RE}(3,S, \frac{1}{n}\sum_{i=1}^n X_iX_i^T) > 0 \), by (8.12) we have

\[
\gamma ||u||_2^2 \leq \frac{1}{n}||Xu||_2^2 \leq \lambda(3||u_S||_1 - ||u_S^*||_1) \leq 3\lambda ||u_S||_1 \leq 3\lambda \sqrt{S}||u_S||_2 \leq 3\lambda \sqrt{S}||u||_2.
\]
where the second last inequality is by Cauchy’s inequality. Therefore,
\[
\|\hat{\beta}_L - \beta^*\|_2^2 = \| \mathbf{u} \|_2^2 \leq \frac{9\lambda^2_s}{\gamma^2} - \frac{9A^2e^2\log p}{\gamma^2}n, \quad \|X(\hat{\beta}_L - \beta^*)\|_2^2 = \|X\hat{u}\|_2^2 \leq \frac{9\lambda^2_s}{\gamma} - \frac{9A^2e^2\log p}{\gamma}n.
\]
So \(\|\hat{\beta}_L - \beta^*\|_1 = \| \mathbf{u} \|_1 \leq \sqrt{s}\| \mathbf{u} \|_2 \leq \frac{3\lambda s}{\gamma} \sqrt{\log p}n\) by Cauchy’s inequality.

According to (8.4), the OLS with diverging number of covariates has the convergence rate \(O(\sqrt{p/n})\) under the minimal eigenvalue condition \(\lambda_{\min}(X^T X) = O(n)\). In contrast, due to the sparse restriction and the RE condition in Corollary 8.2, the factor \(\sqrt{\log p}\) is much more smaller that the factor \(\sqrt{p}\) in the convergence rate (8.4). Under the RE condition, Corollary 8.2 reveals that Lasso is \(\ell_2\)-consistent if \(\frac{s\log p}{n} \to 0\), and \(s\sqrt{\log p/n} \to 0\) guarantees \(\ell_1\)-consistency. Theorem 7.1 in Bickel et al. (2009) also gives oracle inequalities (8.14) and (8.15) for the DS estimator (8.11).

### 8.4 High-dimensional Poisson regressions with random design

The Poisson regression (McCullagh and Nelder, 1983) is a model for nonnegative integers response variables, i.e. \(Y_i \sim \text{Poisson}(\lambda_i)\), where \(\log(\lambda_i) = X_i^T \beta\) for \(i = 1, \ldots, n\). We presume that the \(\{X_i\}_{i=1}^n\) are IID r.v.s on some space \(\mathcal{X}\), and we observe \(n\) copies of \(\{(Y_i, X_i)\}_{i=1}^n \sim (Y, X) \in \mathbb{R} \times \mathbb{R}^p\). The average negative log-likelihood of Poisson regressions is \(\ell_n(\beta) := -\frac{1}{n} \sum_{i=1}^n [Y_i X_i^T \beta - e^{X_i^T \beta}]\) and the Lasso penalized likelihood is for the estimated parameter. But, the necessary and sufficient condition for the Lasso estimates (8.22) is

\[
\hat{\beta} := \hat{\beta}(\lambda) = \arg\min_{\beta \in \mathbb{R}^p} \{ \ell_n(\beta) + \lambda \| \beta \|_1 \} \text{ with a turning parameter } \lambda > 0.
\]  

Lemma 4.2 in Bühlmann and van de Geer (2011) showed the first-order conditions for the optimization in (8.22).

**Lemma 8.4** (Necessary and sufficient condition). Let \(j \in \{1, 2, \ldots, p\}\) and \(\lambda > 0\). Then, a necessary and sufficient condition for the Lasso estimates (8.22) is

\[
\left\{ \begin{array}{ll}
\frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - e^{X_i^T \hat{\beta}}) = -\lambda \text{sign}(\hat{\beta}_j) & \text{if } \hat{\beta}_j \neq 0, \\
\frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - e^{X_i^T \hat{\beta}}) \leq \lambda & \text{if } \hat{\beta}_j = 0.
\end{array} \right.
\]  

Let \(l(Y, X, \beta) = -Y \sum X_i^T \beta + e^{X_i^T \beta}\) be the Poisson loss function. The true coefficient \(\beta^*\) is the minimizer of the expected Poisson loss, i.e.

\[
\beta^* = \arg\min_{\beta \in \mathbb{R}^p} \mathbb{E}[l(Y, X, \beta)].
\]

The KKT condition of the \(\ell_1\)-penalized likelihood is for the estimated parameter. But, here we use the true parameter version of the KKT conditions: \(\frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - EY_i) \leq \lambda, \; j = 1, \ldots, p\) by replacing \(e^{X_i^T \hat{\beta}}\) by \(EY_i = e^{X_i^T \hat{\beta}^*}\) to approximate the estimated version (8.23). To motivate the next two propositions concerning high-probability events, let us consider the following notations and the decomposition of empirical process.
The Poisson loss \( l(\beta, \mathbf{X}, Y) = l_1(\beta, \mathbf{X}, Y) + l_2(\beta, \mathbf{X}) \) is decomposed into two parts where 
\( l_1(\beta) := l_1(\beta, \mathbf{X}, Y) := -Y \mathbf{X}^T \beta \) and 
\( l_2(\beta) := l_2(\beta, \mathbf{X}) := e^{\mathbf{X}^T \beta} \) is free of response. Let \( \mathbb{P}(l(\beta)) = \mathbb{(\mathbb{P}_n - \mathbb{P})} l(\beta) \) be the expected loss. We are interested in the centralized empirical loss \( (\mathbb{P}_n - \mathbb{P}) l(\beta) \) 
representing fluctuations between the expected and empirical losses. Note that

\[
(\mathbb{P}_n - \mathbb{P}) l(\beta) = (\mathbb{P}_n - \mathbb{P}) l_1(\beta) + (\mathbb{P}_n - \mathbb{P}) l_2(\beta),
\]

which is crucial in attaining the convergence rate of \( \| \hat{\beta} - \beta^* \|_1 \). Motivated from rate of convergence theorem [Theorem 3.2.5 of van der Vaart and Wellner (1996)] for M-estimation with functional parameter in some metric space, we study the upper bounds (or the rate) for the first and second part of the difference of the centralized empirical process between \( \beta^* \) and \( \hat{\beta} \): 
\( (\mathbb{P}_n - \mathbb{P})(l_m(\beta^*) - l_m(\hat{\beta})) \), for \( m = 1, 2 \).

**Proposition 8.3** (Convergence rate of \( (\mathbb{P}_n - \mathbb{P})(l_1(\beta^*) - l_1(\hat{\beta})) \)). Suppose that 
\[
\sup_{1 \leq i \leq \infty} \| X_i \|_{\infty} \leq L < \infty \text{ a.s. and } \| \beta^* \|_1 \leq B.
\]

In the event of \( \mathcal{A} := \bigcap_{j=1}^p \{ \frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - EY_i) \leq \frac{1}{4} \} \), we have 
\[
(\mathbb{P}_n - \mathbb{P})(l_1(\beta^*) - l_1(\hat{\beta})) \leq \frac{\lambda}{4} \| \hat{\beta} - \beta^* \|_1.
\]

If \( \lambda \geq \max \{ \frac{16A^2 \log(2p)}{3n}, 8AL e^{L/2} \sqrt{\frac{\log(2p)}{n}} \} \) with \( A > 1 \), we have \( P(\mathcal{A}) \geq 1 - (2p)^{-A^2} \).

**Proof.** Note that, on the event \( \mathcal{A} \)

\[
(\mathbb{P}_n - \mathbb{P})(l_1(\beta^*) - l_1(\hat{\beta})) = \frac{-1}{n} \sum_{i=1}^n (Y_i - EY_i) \mathbf{X}_i^T (\beta^* - \hat{\beta}) = \frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - EY_i)
\]

\[
\leq \sum_{j=1}^p |\hat{\beta}_j - \beta^*_j| \leq \frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - EY_i) \leq \frac{\lambda}{4} \| \hat{\beta} - \beta^* \|_1.
\]

Next, we show that \( \mathcal{A} \) is a high probability event if the \( \lambda \) is well chosen. For \( j = 1, ..., p \) and \( i=1,...,n, P(\mathcal{A}^c) = \sum_{j=1}^p P(\{ |\frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - EY_i) | > \frac{1}{4} \}) \). Given \( \mathbf{X} \), \( \{ S_{nj}(Y, X) := \frac{1}{n} \sum_{i=1}^n X_{ij}(Y_i - EY_i) \} \) are conditional independent for each \( j = 1, ..., p \). Thus Corollary 5.2 with \( w_i = X_{ij}/n \) gives 

\[
P(\{ S_{nj}(Y, X) \geq t \mid \mathbf{X} \}) \leq 2 \exp \left( -\frac{nt^2/2}{\frac{1}{n} \sum_{i=1}^n e^{X_{ij}^T \beta} \max_{1 \leq i \leq n} |X_{ij}|^2 + \max_{1 \leq i \leq n} \frac{|X_{ij}|^2}{3} \} \right) \leq 2(e^{\frac{t^2}{2}} \vee e^{\frac{3}{2} t})
\]

where the last inequality is from \( e^{-\frac{t^2}{2}} \leq e^{\frac{t^2}{4}} \vee e^{\frac{3}{2} t} \) for any positive numbers \( a, b \) and \( c \).

Let \( t = \frac{1}{4} \). Assumptions (8.26) and (8.28) give for \( j = 1, ..., p \).
\[ P\left( \frac{1}{n} \sum_{i=1}^{n} X_{ij} (Y_{i} - EY_{i}) \geq \frac{1}{4} \right) = \text{EP}\left( \frac{1}{n} \sum_{i=1}^{n} X_{ij} (Y_{i} - EY_{i}) \geq \frac{1}{4} \big| X \right) \leq 2\max\left\{ e^{\frac{-M}{4n}}, e^{\frac{-3n}{4}} \right\}, \]

which implies that \( P(A^c) \leq 2p\max\{ e^{\frac{-M}{4n}}, e^{\frac{-3n}{4}} \} \).

Finally, if \( \lambda \geq \max\{ \frac{16A^2 \log(2p)}{3n}, 8AL \varepsilon^{L_B/2} / \sqrt{\log(2p) / n} \} \) (\( A > 1 \), so \( P(A^c) \leq (2p)^{1-A^2} \).

Next, we provide a crucial lemma to bound \((\mathbb{P}_n - \mathbb{P})(l_2(\beta^*) - l_2(\beta))\). Let \( v_n(\beta, \beta^*) := (\mathbb{P}_n - \mathbb{P})(l_2(\beta^*) - l_2(\beta))\) and the normalized empirical process indexed by \( \beta \). Denote the \( l_1 \)-ball by \( S_M(\beta^*) := \{ \beta \in \mathbb{R}^p : \| \beta - \beta^* \|_1 \leq M < \infty \} \), we define the local stochastic Lipschitz constant:

\[ Z_M(\beta^*) := \sup_{\beta \in S_M(\beta^*)} |v_n(\beta, \beta^*)| \] and a random event \( \mathcal{B} := \{ Z_M(\beta^*) \leq \lambda_1 / 4 \} \).

It is easy to see \( |v_n(\beta, \beta^*)| \leq \sup_{\beta \in S_M(\beta^*)} |v_n(\beta, \beta^*)| \leq \frac{\lambda_1}{4} \), which gives \( |(\mathbb{P}_n - \mathbb{P})(l_2(\beta) - l_2(\beta^*))| \leq \frac{\lambda_1}{4} \| \beta - \beta^* \|_1 \), provided that \( \beta \in S_M(\beta^*) \). Then we have follow result.

**Proposition 8.4** (Convergence rate of \((\mathbb{P}_n - \mathbb{P})(l_2(\beta^*) - l_2(\beta))\). Assume that there exists a large constant \( M \) such that \( \beta \) is in the \( l_1 \)-ball \( S_M(\beta^*) \). Under assumption (8.26), we have

\[ P(Z_M(\beta^*) \geq 5AL \varepsilon^{L_B} \sqrt{\frac{\log 2p}{n}}) \leq (2p)^{-A^2}. \quad (8.29) \]

If \( \lambda \geq 20AL \varepsilon^{L_B} \sqrt{\frac{\log 2p}{n}} \), we get \( P\left\{ |(\mathbb{P}_n - \mathbb{P})(l_2(\beta) - l_2(\beta^*))| \leq \frac{\lambda_1}{4} (\| \beta - \beta^* \|_1) \right\} \geq 1 - (2p)^{-A^2} \).

**Proof.** In the first step, we apply following McDiarmid's inequality to \( Z_M(\beta^*) \) by showing that \( Z_M(\beta^*) \) is fluctuated of no more than \( \frac{2\varepsilon^{L_B}}{n} \). Let us check it. Put \( \mathbb{P}_n := \frac{1}{n} \sum_{i=1}^{n} 1_{x_i, y_i} \) and \( \mathbb{P}_n' := \frac{1}{n} \sum_{i=1, i \neq j}^{n} X_i, Y_i + 1_{X_i, Y_i}' \) where \( (X_i, Y_i) \) is the independent copy of \( (X_i, Y_i) \).

Let \( X_i^T \hat{\beta}, (X_i^T \hat{\beta})' \) be an intermediate point between \( X_i^T \beta, (X_i^T \beta)' \) and \( X_i^T \beta^*, (X_i^T \beta^*)' \) from the Taylor’s expansion of function \( F(x) := e^x \). It deduces

\[
\begin{aligned}
&\sup_{\beta \in S_M} \left| \frac{(\mathbb{P}_n - \mathbb{P})(l_2(\beta^*) - l_2(\beta))}{\| \beta^* - \beta \|_1} - \sup_{\beta \in S_M} \left| \frac{(\mathbb{P}_n' - \mathbb{P})(l_2(\beta^*) - l_2(\beta))}{\| \beta^* - \beta \|_1} \right| \right|
\leq \sup_{\beta \in S_M} \frac{n}{\| \beta^* - \beta \|_1} \left| l_2(\beta^*, X_i) - l_2(\beta, X_i) - l_2(\beta^*, X_i') + l_2(\beta, X_i') \right|
\leq \sup_{\beta \in S_M} \frac{1}{n} e^{X_i^T \beta^*} \frac{|X_i^T \beta^* - X_i^T \beta|}{\| \beta^* - \beta \|_1} + \sup_{\beta \in S_M} \frac{1}{n} e^{X_i^T \beta} \frac{|X_i^T \beta^* - X_i^T \beta|}{\| \beta^* - \beta \|_1} \leq \sup_{\beta \in S_M} \frac{2L \varepsilon^{L_B}|X_i^T \beta^* - X_i^T \beta|}{n} \frac{2L \varepsilon^{L_B}}{n} \frac{\| \beta^* - \beta \|_1}{\| \beta^* - \beta \|_1} = \frac{2L \varepsilon^{L_B}}{n} \frac{\| \beta^* - \beta \|_1}{\| \beta^* - \beta \|_1}. 
\end{aligned}
\]

Apply McDiarmid’s inequality to \( Z_M(\beta^*) \), we have \( P(Z_M(\beta^*) - EZ_M(\beta^*) \geq \lambda) \leq e^{-\frac{\lambda^2}{2L \varepsilon^{L_B}}} \). Let \((2p)^{-A^2} = \exp\left\{ -\frac{n^2}{2L \varepsilon^{L_B}} \right\}, \) we get \( \lambda \geq AL \varepsilon^{L_B} \sqrt{\log(2p) / n} \) for \( A > 0 \), therefore

\[ P(Z_M(\beta^*) - EZ_M(\beta^*) \geq \lambda) \leq (2p)^{-A^2}. \quad (8.30) \]
for Lasso penalized Poisson regression, we consider the following regularity conditions: 

- (H.1): The covariate $X$ is almost surely bounded $\|X\|_\infty \leq L$ a.s. for $L > 0$;
- (H.2): There exists a constant $B > 0$ such that $\|\beta^*\|_1 \leq B$;
- (H.3): (Stablility Condition) For $\Sigma := E(XX^T)$, there exist a $k \in (0, 1)$ such that

$$
\delta^T \Sigma \delta \geq k \sum_{j \in S} \delta_j^2 \text{ for any } \delta \in C(c_0, S) := \{ \delta \in \mathbb{R}^p : \sum_{j \in S} |\delta_j| \leq c_0 \sum_{j \in S} |\delta_j| \}.
$$

The next step is to estimate the sharper upper bounds of $EZ_M(\beta^*)$ by Lemma 7.3 with $\Phi(t) = |t|$ and Lemma 7.4. Note that $\{P_n - P\} \{l_2(\beta^*) - l_2(\beta)\} = P_n \{l_2(\beta^*) - l_2(\beta)\} - E\{l_2(\beta^*) - l_2(\beta)\}$, by symmetrization theorem, the expected terms is canceled. To see contraction phenomenon, for $Z_M(\beta^*) = \sup_{\beta \in S_M} \left\{ \frac{1}{n} \left| \sum_{i=1}^n (e^{X_i^T \beta^*} - e^{X_i^T \beta}) - nE[l_2(\beta^*) - l_2(\beta)] \right| \right\}$, it is required to check the Lipschitz property of $g_i$ in Lemma 7.4 with $\mathcal{F} = \mathbb{R}^p$. Let $f(x_i) = x_i^T \beta^*/\|\beta^* - \beta\|_1$, $h(x_i) = x_i^T \beta^*/\|\beta^* - \beta\|_1$ and $g_i(t) = \frac{e^{x_i^T \beta^*} - e^{x_i^T \beta}}{n\|\beta^* - \beta\|_1} (|t| \leq LB/\|\beta^* - \beta\|_1)$. Then the function $g_i(t)$ here is $\frac{e^{x_i^T \beta^*}}{n\|\beta^* - \beta\|_1}$-Lipschitz. In fact 

$$
|g_i(s) - g_i(t)| = \frac{4e^{x_i^T \beta^*}}{n\|\beta^* - \beta\|_1} \cdot |s - t| \leq \frac{4e^{x_i^T \beta^*}}{n\|\beta^* - \beta\|_1} |s - t|,
$$

where $\tilde{t} \in [-LB/\|\beta^* - \beta\|_1, LB/\|\beta^* - \beta\|_1]$ is an intermediate point between $t$ and $s$ given by applying Lagrange mean value theorem.

The symmetrization theorem and the contraction theorem imply

$$
EZ_M(\beta^*) \leq \frac{4e^{LB}}{n} E(\sup_{\beta \in S_M} \left| \sum_{i=1}^n \epsilon_i X_i^T (\beta^* - \beta) \right|) \leq \frac{4e^{LB}}{n} E(\sup_{\beta \in S_M} \max_{1 \leq j \leq p} \left| \sum_{i=1}^n \epsilon_i X_{ij} \right| \|\beta - \beta^*\|_1).
$$

From Corollary 7.5, with $E(\epsilon_i X_{ij} | X) = 0$ we get $\frac{4e^{LB}}{n} E(\max_{1 \leq j \leq p} \left| \sum_{i=1}^n \epsilon_i X_{ij} \right| | X) \leq \frac{4e^{LB}}{n} \sqrt{2 \log 2p}$. Thus, for $A \geq 1$,

$$
EZ_M(\beta^*) \leq 4e^{LB} L \sqrt{\frac{2 \log 2p}{n}} \leq 4A e^{LB} L \sqrt{\frac{2 \log 2p}{n}}. \quad (8.31)
$$

With $\lambda \geq A e^{LB} L \sqrt{\frac{2 \log 2p}{n}}$ and (8.31), we conclude from (8.30) that $P(Z_M(\beta^*) \geq 5A e^{LB} \sqrt{\frac{\log 2p}{n}}) \leq P(Z_M(\beta^*) \geq \lambda + E Z_M(\beta^*)) \leq (2p)^{-A^2}$. Finally, we complete the proof of Proposition 8.4 by letting $\frac{\lambda}{4} \geq 5A e^{LB} L \sqrt{\frac{2 \log 2p}{n}}$ and setting $\beta = \hat{\beta} \in Z_M(\beta^*)$. 

Let $S := S(\beta^*)$ for $\beta^*$ defined in (8.24) and $s := |S|$. To obtain sharp oracle inequalities for Lasso penalized Poisson regression, we consider the following regularity conditions:

- (H.1): The covariate $X$ is almost surely bounded $\|X\|_\infty \leq L$ a.s. for $L > 0$;
- (H.2): There exists a constant $B > 0$ such that $\|\beta^*\|_1 \leq B$;
- (H.3): (Stablility Condition) For $\Sigma := E(XX^T)$, there exist a $k \in (0, 1)$ such that

$$
\delta^T \Sigma \delta \geq k \sum_{j \in S} \delta_j^2 \text{ for any } \delta \in C(c_0, S) := \{ \delta \in \mathbb{R}^p : \sum_{j \in S} |\delta_j| \leq c_0 \sum_{j \in S} |\delta_j| \}.
$$


The Stabil Condition (H.3) is denoted as \( S(c_0, S, k, \Sigma) \) which is a similar version of the RE condition in the Lasso linear models proposed in Bunea (2008). Due to the random variance, Poisson regression is more complex than the linear model with the constant variance assumption. Thus, (H.1) and (H.2) are stronger than those assumed for the linear models. Based on the high probability event \( A \) and \( \overline{B} \), we have the oracle inequalities for estimation and prediction for Lasso estimator \( \hat{\beta} \) in (8.22) for the Poisson regressions.

**Theorem 8.1.** Assume conditions (H.1) – (H.3) hold. Let \( \lambda \) be chosen such that
\[
\lambda \geq \max\{\frac{16A^2L\log(2p)}{3n}, 8ALe^{L/2} \sqrt{\frac{\log(2p)}{n}}, 20ALe^{L/2} \sqrt{\frac{2\log2p}{n}}\} \text{ for } A > \sqrt{2}. \tag{8.32}
\]

Suppose that we have a new covariate vector \( X^* \) (as the test data) which is an independent copy of \( X \) (as the training data), and \( E^* \) represents the expectation w.r.t. \( X^* \) only, then
\[
P(E^*|X^*(\hat{\beta} - \beta^*))^2 \leq \frac{12e^{10LB}}{k}s\lambda^2 \text{ and } P(\|\hat{\beta} - \beta^*\|_1 \leq \frac{4e^{5LB}}{k}s\lambda) \geq 1 - (2p)^{1-A^2} - (2p)^{-A^2/2}.
\]

The Theorem 8.1 lead to the persistence and \( \ell_1 \)-consistency if \( \max\{s\lambda, s\lambda^2\} \to 0 \).

**Proof.** The proof consists of three steps.

**Step1: Check \( \hat{\beta} - \beta^* \in C(3, S) \).** From the definition of the Lasso estimates \( \hat{\beta} \) (see (8.22)),
\[
P_nl(\hat{\beta}) + \lambda\|\hat{\beta}\|_1 \leq P_nl(\beta^*) + \lambda\|\beta^*\|_1. \tag{8.33}
\]

By adding \( P(l(\hat{\beta}) - l(\beta^*)) + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 \) to both sides of (8.33), we have
\[
P(l(\hat{\beta}) - l(\beta^*)) + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 \leq (P_n - P)(l(\beta^*) - l(\hat{\beta})) + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 + \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1),
\]
which leads
\[
P(l(\hat{\beta}) - l(\beta^*)) + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 \leq (P_n - P)(l(\beta^*) - l(\hat{\beta})) + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 + \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1)
\leq \lambda\|\hat{\beta} - \beta^*\|_1 + \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1). \tag{8.34}
\]

By the definition of \( \beta^* \), \( P(l(\hat{\beta}) - l(\beta^*)) \geq 0 \). The above inequality and the fact: \( |\hat{\beta}_j - \beta^*_j| + |\beta^*_j| - |\hat{\beta}_j| = 0 \) for \( j \notin S \) and \( |\hat{\beta}_j| - |\beta^*_j| \leq |\hat{\beta}_j - \beta^*_j| \) for \( j \in S \) lead to
\[
\lambda\|\hat{\beta} - \beta^*\|_1/2 \leq \lambda\|\hat{\beta} - \beta^*\|_1 + \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1) \leq 2\lambda\|\hat{\beta} - \beta^*\|_1. \tag{8.35}
\]

Thus, \( \frac{1}{2}\|\hat{\beta} - \beta^*\|_1 \leq 1.5\lambda\|\hat{\beta} - \beta^*\|_1 \) and then \( \hat{\beta} - \beta^* \in C(3, S) \).

**Step2: Choosing the order of tuning parameter under high probability events.** Since \( P(l(\hat{\beta}) - l(\beta^*)) \geq 0 \), (8.34) implies
\[
\lambda\|\hat{\beta} - \beta^*\|_1/2 \leq \lambda\|\hat{\beta} - \beta^*\|_1 + \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1)
\leq \lambda\|\hat{\beta}\|_1 + \lambda\|\beta^*\|_1 + \lambda(\|\beta^*\|_1 - \|\hat{\beta}\|_1) = 2\lambda\|\beta^*\|_1. \tag{8.36}
\]

Thus (H.2) implies \( \|\hat{\beta} - \beta^*\|_1 \leq 4B \). After having showed Propositions 8.3 and 8.4, we need the result on the high probability of the event \( A \cap B \), whose proof is skipped.
Proposition 8.5. Under the event $A \cap B$ with (H.1)-(H.3), we have $\hat{\beta} \in S_{4B}(\beta^*)$. And if $\lambda$ are chosen as (8.32), then $P(A \cap B) \geq 1 - (2p)^{1/2} - (2p)^{1/2}/2$.

Step3: Error bounds from Stabil Condition. As $X^*$ is an independent copy of $X$,

$$
P\{l(\hat{\beta}) - l(\beta^*)\} = E^*[E\{l(\hat{\beta}) - l(\beta^*)|X^*\}] = E^*[E[-YX^T(\beta - \beta^*) + e^{X^T\beta} - e^{X^T\beta^*}|X^*]|_{\beta = \beta^*} = E^*[E[-Y|X^*|X^T(\beta - \beta^*) + (e^{X^T\beta} - e^{X^T\beta^*})|X^*]|_{\beta = \beta^*} (E^*[|Y|X^*] = e^{X^T\beta^*}) = E^*[-e^{X^T\beta^*} + e^{X^T\beta^*} + 2^{-1}e^{X^T\beta^*}|X^T(\beta - \beta^*)|_2^2]|_{\beta = \beta^*} = 2^{-1}E^*[e^{X^T\beta^*}|X^T(\beta - \beta^*)|_2^2]|_{\beta = \beta^*}
$$

where $X^T\hat{\beta} = (1-t)X^T\beta^* + tX^T\beta$ is an intermediate point of $X^T$ and $X^T\beta$ with $t \in [0,1]$.

Note that $\|\beta^*\|_1 \leq B$ by (H.1) and $\|\hat{\beta} - \beta^*\|_1 \leq 4B$, (H.2) yields

$$
|X^T\hat{\beta}| \leq t|X^T\hat{\beta} - X^T\beta^*| + |X^T\beta^*| \leq \|X^*\|_\infty \cdot \|\hat{\beta} - \beta^*\|_1 + |X^T\beta^*| \leq 4LB + LB = 5LB,
$$

which implies for $c := e^{-5LB}/2$

$$
P\{l(\hat{\beta}) - l(\beta^*)\} \geq \inf_{\|\beta^*\|=1} E^*\{e^{X^T\beta^*}|X^T(\beta - \beta^*)|^2\}|_{\beta = \beta^*} = cE^*[X^T(\beta - \beta^*)]^2. \quad (8.37)
$$

As $E^*[X^*X^T] = \Sigma, E^*[X^*(\hat{\beta} - \beta^*)]^2 = (\hat{\beta} - \beta^*)\Sigma(\hat{\beta} - \beta^*)$.

Having checked the cone condition $C(3,S)$, we apply the Stabil Condition

$$
c(\hat{\beta} - \beta^*)\Sigma(\hat{\beta} - \beta^*) \geq c\|\beta - \beta^*\|_S. \quad (8.38)
$$

From (8.34), (8.35) and (8.37), we get

$$
cE^*[X^*(\hat{\beta} - \beta^*)]^2 + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 \leq P(l(\hat{\beta}) - l(\beta^*)) + \frac{\lambda}{2}\|\hat{\beta} - \beta^*\|_1 \leq 2\lambda\|\beta - \beta^*\|_S|_1, \quad (8.39)
$$

which gives $c\|\beta - \beta^*\|_S|_1^2 + \frac{\lambda}{2}\|\beta - \beta^*\|_1 \leq 2\lambda\|\beta - \beta^*\|_S|_1$ by plugging (8.38) into (8.39). Then, employing Cauchy’s inequality, we have

$$
2c\|\beta - \beta^*\|_S|_1^2 + \lambda\|\beta - \beta^*\|_1 \leq 4\lambda(s\cdot\|\beta - \beta^*\|_S|_1^2)^{1/2} \leq 4t\lambda^2s^{1/2} \|\beta - \beta^*\|_S|_1^2, \quad (8.40)
$$

where the last inequality is from the elementary inequality $2xy \leq tx^2 + y^2/1$ for all $t > 0$. Let us set $t = (2ck)^{-1}$ in (8.40), thus $\|\beta - \beta^*\|_1 \leq 4t\lambda s = \frac{2\lambda}{ck} = \frac{4\lambda}{k}s\lambda$.

To derive the oracle inequality of prediction error, from (8.39), we obtain

$$
cE^*[X^*(\hat{\beta} - \beta^*)]^2 \leq 1.5\lambda\|\beta - \beta^*\|_S|_1 \leq 1.5\lambda\|\beta - \beta^*\|_1
$$

which implies $E^*[X^*(\hat{\beta} - \beta^*)]^2 \leq 1.5\lambda\|\beta - \beta^*\|_1/c \leq \frac{3\lambda^2}{ck} = \frac{12\lambda}{k}s\lambda^2$, where the last inequality is from $\|\beta - \beta^*\|_1 \leq \frac{4\lambda}{k}s\lambda$. \quad \Box
9 Extensions

The review has been focused on the sum of independent r.v.s in the Euclidean space. However, independence structure may not be suitable for some applications, for instance, econometrics, survival analysis, and graphical models. At the same time, the Euclidean valued r.v.s may not be appropriate for functional data and image data. In the following we point out results in settings not covered to broaden this review.

By CIs for the martingales, oracle inequalities have been proposed for Lasso penalized Cox models, see Huang et al. (2013). Some statistical models, such as the Ising model involving Markov’s chains. Miasojedow and Rejchel (2018) applies Hoeffding’s inequality for Markov’s chains to deal with this difficulty, see Fan et al. (2020) for a review. In time series analysis, Xie and Xiao (2018) studies the square-root Lasso method for HD linear models with $\alpha, \rho, \phi$-mixing or $m$-dependent errors. The Hoeffding’s and Bernstein’s CIs for weakly dependent summations can be found in Bosq (1998). Via sub-Weibull concentrations under $\beta$-mixing, non-asymptotic inequalities for estimation errors, and the prediction errors are obtained by Wong et al. (2020) for the Lasso-regularized sparse VAR model with sub-Weibull innovations. U-Statistic is another dependent sum, and Example 2.2 provides a concentration results by McDiarmid’s inequality. Borovskikh (1996) introduces the concentration for the Banach-valued U-statistics.

In non-parametric regressions, the corresponding score functions may be r.v.s in Banach (or Hilbert) space; see the monographs Ledoux and Talagrand (1991), Yurinsky (1995) for introductions. Exponential tail bounds for Banach- or Hilbert-valued r.v.s are indispensable for deriving sharp oracle inequalities of the error bounds, see Zhang (2005), Lei and Zhang (2020). Recently, Banach-valued CIs are applied to conceive non-asymptotic hypothesis testing for non-parametric regressions, see Yang et al. (2020). To extend the empirical covariance matrix from finite to infinite dimension, the sample covariance operator is treated as a random element in Banach spaces. The concentrations of empirical covariance operator also have been raised attention in kernel principal components analysis, and functional data analysis, see Rosasco et al. (2010), Bunea and Xiao (2015).

Testing hypotheses on the regression coefficients is a necessity in measuring the effects of covariates on the certain response variables. Scientists are interested in testing the significance of a large number of covariates simultaneously. From this backgrounds, Zhong and Chen (2011) proposes simultaneous tests for coefficients in HD linear models under the “large $p$, small $n$” situations by U-statistics motivated by Chen and Qin (2010). However, their HD tests are asymptotical without a non-asymptotic guarantee. Motivated by Arlot et al. (2010), Zhu and Bradic (2018) invents a new methodology for testing the linearity hypothesis in HD linear models, and the test they proposed does not impose any restriction of model sparsity. Based on the concentration of Lipschitz functions of Gaussian distributions or strongly log-concave distribution, Zhu (2018) develops a new concentration-based test in HD regressions. Recently, Wang et al. (2020) studies non-asymptotical two-sample testing using Projected Wasserstein Distance, via McDiarmid’s inequality.
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