Electrons, pseudoparticles, and quasiparticles in the
one-dimensional many-electron problem

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We generalize the concept of quasiparticle for one-dimensional (1D) interacting
electronic systems. The \( \uparrow \) and \( \downarrow \) quasiparticles recombine the pseudoparticle colors
\( c \) and \( s \) (charge and spin at zero magnetic field) and are constituted by one many-
pseudoparticle \textit{topological momenton} and one or two pseudoparticles. These excita-
tions cannot be separated. We consider the case of the Hubbard chain. We show that
the low-energy electron – quasiparticle transformation has a singular character which
justifies the perturbative and non-perturbative nature of the quantum problem in
the pseudoparticle and electronic basis, respectively. This follows from the absence
of zero-energy electron – quasiparticle overlap in 1D. The existence of Fermi-surface
quasiparticles both in 1D and three dimensional (3D) many-electron systems sug-
gests there existence in quantum liquids in dimensions \( 1<D<3 \). However, whether
the electron – quasiparticle overlap can vanish in \( D>1 \) or whether it becomes finite
as soon as we leave 1D remains an unsolved question.

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I. INTRODUCTION

The unconventional electronic properties of novel materials such as the superconducting copper oxides and synthetic quasi-unidimensional conductors has attracted much attention to the many-electron problem in spatial dimensions $1 \leq D \leq 3$. A good understanding of both the different and common properties of the 1D and 3D many-electron problems might provide useful indirect information on quantum liquids in dimensions $1 < D < 3$. This is important because the direct study of the many-electron problem in dimensions $1 < D < 3$ is of great complexity. The nature of interacting electronic quantum liquids in dimensions $1 < D < 3$, including the existence or non existence of quasiparticles and Fermi surfaces, remains an open question of crucial importance for the clarification of the microscopic mechanisms behind the unconventional properties of the novel materials.

In 3D the many-electron quantum problem can often be described in terms of a one-particle quantum problem of quasiparticles \[ \sigma \], which interact only weakly. This Fermi liquid of quasiparticles describes successfully the properties of most 3D metals, which are not very sensitive to the presence of electron-electron interactions. There is a one to one correspondence between the $\sigma$ quasiparticles and the $\sigma$ electrons of the original non-interacting problem (with $\sigma = \uparrow, \downarrow$). Moreover, the coherent part of the $\sigma$ one-electron Green function is quite similar to a non-interacting Green function except that the bare $\sigma$ electron spectrum is replaced by the $\sigma$ quasiparticle spectrum and for an electron renormalization factor, $Z_{\sigma}$, smaller than one and such that $0 < Z_{\sigma} < 1$. A central point of Fermi-liquid theory is that quasiparticle - quasihole processes describe exact low-energy and small-momentum Hamiltonian eigenstates and “adding” or “removal” of one quasiparticle connects two exact ground states of the many-electron Hamiltonian.

On the other hand, in 1D many-electron systems \[ 1 \leq D \leq 3 \], such as the hUBbard chain solvable by Bethe ansatz (BA) \[ 1 \leq D \leq 3 \], the $\sigma$ electron renormalization factor, $Z_{\sigma}$, vanishes \[ 1 \leq D \leq 3 \]. Therefore, the many-particle problem is not expected to be described in terms of a one-particle problem of Fermi-liquid quasiparticles. Such non-perturbative electronic
problems are usually called Luttinger liquids \[1\]. In these systems the two-electron vertex function at the Fermi momentum diverges in the limit of vanishing excitation energy \[11\]. In a 3D Fermi liquid this quantity is closely related to the interactions of the quasiparticles \[12,13\]. Its divergence seems to indicate that there are no quasiparticles in 1D interacting electronic systems. A second possibility is that there are quasiparticles in the 1D many-electron problem but without overlap with the electrons in the limit of vanishing excitation energy.

While the different properties of 1D and 3D many-electron problems were the subject of many Luttinger-liquid studies in 1D \[3,4,5\], the characterization of their common properties is also of great interest because the latter are expected to be present in dimensions 1<\(D<3\) as well. One example is the Landau-liquid character common to Fermi liquids and some Luttinger liquids which consists in the generation of the low-energy excitations in terms of different momentum-occupation configurations of anti-commuting quantum objects (quasiparticles or pseudoparticles) whose forward-scattering interactions determine the low-energy properties of the quantum liquid. This generalized Landau-liquid theory was first applied in 1D to contact-interaction soluble problems \[12\] and shortly after also to \(1/r^2\)-interaction integrable models \[13\]. Within this picture the 1D many-electron problem can also be described in terms of weakly interacting “one-particle” objects, the pseudoparticles, which, however, have no one-to-one correspondence with the electrons, as is shown in this paper.

In spite of the absence of the one to one principle in what concerns single pseudoparticles and single electrons, following the studies of Refs. \[12,14,15\] a generalized adiabatic principle for small-momentum pseudoparticle-pseudohole and electron-hole excitations was introduced for 1D many-electron problems in Refs. \[16\]. The pseudoparticles of 1D many-electron systems show other similarities with the quasiparticles of a Fermi liquid, their interactions being determined by \textit{finite} forward-scattering \(f\) functions \[14,15,16\]. At constant values of the electron numbers this description of the quantum problem is very similar to Fermi-liquid theory, except for two main differences: (i) the \(\uparrow\) and \(\downarrow\) quasiparticles are replaced by the \(c\) and \(s\) pseudoparticles \[17,18,19,20,21\], and (ii) the discrete pseudoparticle momentum
(pseudomomentum) is of the usual form $q_j = \frac{2\pi}{N_\alpha} I_j^\alpha$ but the numbers $I_j^\alpha$ (with $\alpha = c, s$) are not always integers. They are integers or half integers depending on whether the number of particles in the system is even or odd. This plays a central role in the present quasiparticle problem. The connection of these perturbative pseudoparticles to the non-perturbative 1D electronic basis remains an open problem. By perturbative we mean here the fact that the two-pseudoparticle $f$ functions and forward-scattering amplitudes are finite \cite{16,18}, in contrast to the two-electron vertex functions.

The low-energy excitations of the Hubbard chain at constant electron numbers and in a finite magnetic field and chemical potential were shown \cite{14,15,16,18,19,20} to be $c$ and $s$ pseudoparticle-pseudohole processes relative to the canonical-ensemble ground state. This determines the $c$ and $s$ low-energy separation \cite{20}, which at zero magnetization leads to the so called charge and spin separation. In this paper we find that in addition to the above pseudoparticle-pseudohole excitations there are also Fermi-surface quasiparticle transitions in the 1D many-electron problem. Moreover, it is the study of such quasiparticle which clarifies the complex and open problem of the low-energy electron – pseudoparticle transformation.

As in 3D Fermi liquids, the quasiparticle excitation is a transition between two exact ground states of the interacting electronic problem differing in the number of electrons by one. When one electron is added to the electronic system the number of these excitations also increases by one. Naturally, its relation to the electron excitation will depend on the overlap between the states associated with this and the quasiparticle excitation and how close we are in energy from the initial interacting ground state. Therefore, in order to define the quasiparticle we need to understand the properties of the actual ground state of the problem as, for instance, is given by its exact solution via the BA. We find that in the 1D Hubbard model adding one $\uparrow$ or $\downarrow$ electron of lowest energy is associated with adding one $\uparrow$ or $\downarrow$ quasiparticle, as in a Fermi liquid. These are many-pseudoparticle objects which recombine the colors $c$ and $s$ giving rise to the spin projections $\uparrow$ and $\downarrow$. We find that the quasiparticle is constituted by individual pseudoparticles and by a many-pseudoparticle ob-
ject of large momentum that we call topological momenton. Importantly, these excitations cannot be separated. Although one quasiparticle is basically one electron, we show that in 1D the quasiparticle – electron transformation is singular because it involves the vanishing one-electron renormalization factor. This also implies a low-energy singular electron - pseudoparticle transformation. This singular character explains why the problem becomes perturbative in the pseudoparticle basis while it is non perturbative in the usual electronic picture.

The singular nature of the low-energy electron - quasiparticle and electron – pseudoparticle transformations reflects the fact that the one-electron density of states vanishes in the 1D electronic problem when the excitation energy $\omega \to 0$. The diagonalization of the many-electron problem is at lowest excitation energy associated with the singular electron – quasiparticle transformation which absorbs the vanishing electron renormalization factor and maps vanishing electronic spectral weight onto finite quasiparticle and pseudoparticle spectral weight. For instance, by absorbing the renormalization factor the electron - quasiparticle transformation renormalizes divergent two-electron vertex functions onto finite two-quasiparticle scattering parameters. These quantities fully determine the finite $f$ functions and scattering amplitudes of the pseudoparticle theory \cite{14,15}. The pseudoparticle $f$ functions and amplitudes determine all the static and low-energy quantities of the 1D many-electron problem and are associated with zero-momentum two-pseudoparticle forward scattering.

The paper is organized as follows: the pseudoparticle operator basis is summarized in Sec. II. In Sec. III we find the quasiparticle operational expressions in the pseudoparticle basis and characterize the corresponding $c$ and $s$ recombination in the $\uparrow$ and $\downarrow$ spin projections. The singular electron – quasiparticle (and electron – pseudoparticle) transformation is studied in Sec. IV. Finally, in Sec. V we present the concluding remarks.
II. THE PERTURBATIVE PSEUDOPARTICLE OPERATOR BASIS

It is useful for the studies presented in this paper to introduce in this section some basic information on the perturbative operator pseudoparticle basis, as it is obtained directly from the BA solution \[18,19,20\]. We consider the Hubbard model in 1D \[8,22,23\] with a finite chemical potential \(\mu\) and in the presence of a magnetic field \(H\) \[16,18,19\]

\[
\hat{H} = -t \sum_{j,\sigma} [c^\dagger_{j,\sigma} c_{j+1,\sigma} + c^\dagger_{j+1,\sigma} c_{j,\sigma}] + U \sum_j [c^\dagger_{j,\uparrow} c_{j,\uparrow} - 1/2] [c^\dagger_{j,\downarrow} c_{j,\downarrow} - 1/2] - \mu \sum_{\sigma} \hat{N}_\sigma - 2\mu_0 H \hat{S}_z, \tag{1}
\]

where \(c^\dagger_{j,\sigma}\) and \(c_{j,\sigma}\) are the creation and annihilation operators, respectively, for electrons at the site \(j\) with spin projection \(\sigma = \uparrow, \downarrow\). In what follows \(k_{F,\sigma} = \pi n_\sigma\) and \(k_F = [k_{F,\uparrow} + k_{F,\downarrow}] / 2 = \pi n / 2\), where \(n_\sigma = N_\sigma / N_a\) and \(n = N / N_a\), and \(N_\sigma\) and \(N_a\) are the number of \(\sigma\) electrons and lattice sites, respectively (\(N = \sum_\sigma N_\sigma\)). We also consider the spin density, \(m = n_\uparrow - n_\downarrow\).

The many-electron problem (1) can be diagonalized using the BA \[7,8\]. We consider all finite values of \(U\), electron densities \(0 < n < 1\), and spin densities \(0 < m < n\). For this parameter space the low-energy physics is dominated by the lowest-weight states (LWS’s) of the spin and eta-spin algebras \[24,25\] of type I \[18,19,21\]. The LWS’s I are described by real BA rapidities, whereas all or some of the BA rapidities which describe the LWS’s II are complex and non-real. Both the LWS’s II and the non-LWS’s out of the BA solution \[24\] have energy gaps relative to each canonical ensemble ground state \[18,19,21\]. Fortunately, the quasiparticle description involves only LWS’s I because these quantum objects are associated with ground-state – ground-state transitions and in the present parameter space all ground states of the model are LWS’s I. On the other hand, the electronic excitation involves transitions to LWS’s I, LWS’s II, and non-LWS’s, but the electron – quasiparticle transformation involves only LWS’s I. Therefore, our results refer mainly to the Hilbert sub space spanned by the LWS’s I and are valid at energy scales smaller than the above gaps. (Note that in simpler 1D quantum problems of symmetry \(U(1)\) the states I span the whole Hilbert space \[26\].)

In this Hilbert sub space the BA solution was shown to refer to an operator algebra which
involves two types of pseudoparticle creation (annihilation) operators $b^\dagger_{q,\alpha} (b_{q,\alpha})$. These obey the usual anti-commuting algebra \[18,19,20\]

\[
\{ b^\dagger_{q,\alpha}, b_{q',\alpha'} \} = \delta_{q,q'} \delta_{\alpha,\alpha'}, \quad \{ b^\dagger_{q,\alpha}, b^\dagger_{q',\alpha'} \} = 0, \quad \{ b_{q,\alpha}, b_{q',\alpha'} \} = 0.
\] (2)

Here $\alpha$ refers to the two pseudoparticle colors $c$ and $s$ \[18,19,20\]. The discrete pseudomomentum values are

\[ q_j = \frac{2\pi I^\alpha_j}{N^\alpha}, \]

(3)

where $I^\alpha_j$ are consecutive integers or half integers. There are $N^\alpha_*$ values of $I^\alpha_j$, i.e. $j = 1, ..., N^\alpha_*$. A LWS I is specified by the distribution of $N^\alpha_*$ occupied values, which we call $\alpha$ pseudoparticles, over the $N^\alpha_*$ available values. There are $N^\alpha_* - N^\alpha_*$ corresponding empty values, which we call $\alpha$ pseudoholes. These are good quantum numbers such that

\[ N^c_* = N^\alpha; \quad N^c = N; \quad N^s_* = N^\uparrow; \quad N^s = N^\downarrow. \] (4)

The numbers $I^c_j$ are integers (or half integers) for $N^s$ even (or odd), and $I^s_j$ are integers (or half integers) for $N^s_*$ odd (or even) \[8\]. All the states I can be generated by acting onto the vacuum $|V\rangle$ (zero-electron density) suitable combinations of pseudoparticle operators \[18,19\]. The ground state

\[ |0; N^\sigma, N_-\sigma\rangle = \prod_{\alpha=c,s} \prod_{q=q_{F\alpha}} \{ q^{(\pm)}_{F\alpha} \} b^\dagger_{q,\alpha} |V\rangle, \]

(5)

and all LWS’s I are Slatter determinants of pseudoparticle levels. In Appendix A we define the pseudo-Fermi points, $q_{F\alpha}^{(\pm)}$, of (5). In that Appendix we also present other quantities of the pseudoparticle representation which are useful for the present study.

In the pseudoparticle basis spanned by the LWS’s I and in normal order relatively to the ground state (5) the Hamiltonian (1) has the following form \[18,20\]

\[ \hat{H} := \sum_{i=1}^{\infty} \hat{H}^{(i)}, \]

(6)

where, to second pseudoparticle scattering order
\[
\hat{H}^{(1)} = \sum_{q, \alpha} \epsilon_{\alpha}(q) : \hat{N}_\alpha(q) : ; \\
\hat{H}^{(2)} = \frac{1}{N_\alpha} \sum_{q, \alpha} \sum_{q', \alpha'} \frac{1}{2} f_{\alpha\alpha'}(q, q') : \hat{N}_\alpha(q) :: \hat{N}_{\alpha'}(q') : .
\]

Here (7) are the Hamiltonian terms which are relevant at low energy \[19\]. Furthermore, at low energy and small momentum the only relevant term is the non-interacting term \(\hat{H}^{(1)}\). Therefore, the c and s pseudoparticles are non-interacting at the small-momentum and low-energy fixed point and the spectrum is described in terms of the bands \(\epsilon_{\alpha}(q)\) (studied in detail in Ref. \[14\]) in a pseudo-Brillouin zone which goes between \(q_c^{(-)} \approx -\pi\) and \(q_c^{(+)}) \approx \pi\) for the c pseudoparticles and \(q_s^{(-)} \approx -k_{F\uparrow}\) and \(q_s^{(+)}) \approx k_{F\uparrow}\) for the s pseudoparticles. In the ground state (5) these are occupied for \(q_{F\alpha}^{(-)} \leq q \leq q_{F\alpha}^{(+)}\), where the pseudo-Fermi points (A1)-(A3) are such that \(q_{F\alpha}^{(+)} \approx \pm 2k_F\) and \(q_{F\alpha}^{(\pm)} \approx \pm k_{F\downarrow}\) (see Appendix A).

At higher energies and (or ) large momenta the pseudoparticles start to interact via zero-momentum transfer forward-scattering processes of the Hamiltonian (6) – (7). As in a Fermi liquid, these are associated with \(f\) functions and Landau parameters \[15,18\], whose expressions we present in Appendix A, where we also present the expressions for simple pseudoparticle-pseudohole operators which are useful for the studies of next sections.

III. THE QUASIPARTICLES AND C AND S RECOMBINATION

In this section we introduce the 1D quasiparticle and express it in the pseudoparticle basis. In Sec. IV we find that this clarifies the low-energy transformation between the electrons and the pseudoparticles. We define the quasiparticle operator as the generator of a ground-state – ground-state transition. The study of ground states of form (5) differing in the number of \(\sigma\) electrons by one reveals that their relative momentum equals precisely the \(U = 0\) Fermi points, \(\pm k_{F\sigma}\). Following our definition, the quasiparticle operator, \(c_{k_{F\sigma},\sigma}^\dagger\), which creates one quasiparticle with spin projection \(\sigma\) and momentum \(k_{F\sigma}\) is such that

\[
c_{k_{F\sigma},\sigma}^\dagger |0; N_{\sigma}, N_{-\sigma}\rangle = |0; N_{\sigma} + 1, N_{\sigma}\rangle .
\]
The quasiparticle operator defines a one-to-one correspondence between the addition of one electron to the system and the creation of one quasiparticle: the electronic excitation, \( c_{kF,\sigma}^\dagger |0; N_\sigma, N_{-\sigma}\rangle \), defined at the Fermi momentum but arbitrary energy, contains a single quasiparticle, as we show in Sec. IV. In that section we will study this excitation as we take the energy to be zero, that is, as we approach the Fermi surface, where the problem is equivalent to Landau’s.

Since we are discussing the problem of addition or removal of one particle the boundary conditions play a crucial role. As discussed in Secs. I and II, the available Hamiltonian eigenstates \( I \) depend on the discrete numbers \( I_j^\alpha \) of Eq. (3) which can be integers of half-integers depending on whether the number of particles in the system is even or odd [the pseudomomentum is given by Eq. (3)]. When we add or remove one electron to or from the many-body system we have to consider the transitions between states with integer and half-integer quantum numbers [or equivalently, between states with an odd (even) and even (odd) number of \( \sigma \) electrons]. The transition between two ground states differing in the number of electrons by one is associated with two different processes: a backflow in the Hilbert space of the pseudoparticles with a shift of all the pseudomomenta by \( \pm \frac{\pi}{N_a} \) [associated with the change from even (odd) to odd (even) number of particles], which we call \textit{topological momenton}, and the creation of one or a pair of pseudoparticles at the pseudo-Fermi points.

According to the integer or half-integer character of the \( I_j^\alpha \) numbers we have four “topological” types of Hilbert sub spaces. Since that character depends on the parities of the electron numbers, we refer these sub spaces by the parities of \( N_\uparrow \) and \( N_\downarrow \), respectively: (a) even, even; (b) even, odd; (c) odd, even; and (d) odd, odd. The ground-state total momentum expression is different for each type of Hilbert sub space in such a way that the relative momentum, \( \Delta P \), of \( U > 0 \) ground states differing in \( N_\sigma \) by one equals the \( U = 0 \) Fermi points, ie \( \Delta P = \pm k_{F\sigma} \). Moreover, we find that the above quasiparticle operator \( \tilde{c}_{kF,\sigma}^\dagger \) involves the generator of one low-energy and large-momentum topological momenton. The \( \alpha \) topological momenton is associated with the backflow of the \( \alpha \) pseudoparticle pseudomomentum band and cannot occur without a second type of excitation associated with
the adding or removal of pseudoparticles. The $\alpha$-topological-momenton generator, $U^{\pm 1}_{\alpha}$, is an unitary operator which controls the topological transformations of the pseudoparticle Hamiltonian (6) – (7). For instance, in the $\Delta P = \pm k_{F\uparrow}$ transitions (a)$\rightarrow$(c) and (b)$\rightarrow$(d) the Hamiltonian (6) – (7) transforms as
\[ H : \rightarrow U^{\pm \Delta N_{\uparrow} s : H : U^{\mp \Delta N_{\uparrow} s} , \] (9)
and in the $\Delta P = \pm k_{F\downarrow}$ transitions (a)$\rightarrow$(b) and (c)$\rightarrow$(d) as
\[ H : \rightarrow U^{\pm \Delta N_{\downarrow} c : H : U^{\mp \Delta N_{\downarrow} c} , \] (10)
where $\Delta N_{\sigma} = \pm 1$ and the expressions of the generator $U^{\pm 1}_{\alpha}$ is obtained below.

In order to arrive to the expressions for the quasiparticle operators and associate topological-momenton generators $U^{\pm 1}_{\alpha}$ we refer again to the ground-state pseudoparticle representation (5). For simplicity, we consider that the initial ground state of form (5) is non degenerate and has zero momentum. Following equations (A1)-(A3) this corresponds to the situation when both $N_{\uparrow}$ and $N_{\downarrow}$ are odd, ie the initial Hilbert sub space is of type (d). However, note that our results are independent of the choice of initial ground state. The pseudoparticle numbers of the initial state are $N_{c} = N_{\uparrow} + N_{\downarrow}$ and $N_{s} = N_{\downarrow}$ and the pseudo-Fermi points $q_{F_{\alpha}}^{(\pm)}$ are given in Eq. (A1).

We express the electronic and pseudoparticle numbers and pseudo-Fermi points of the final states in terms of the corresponding values for the initial state. We consider here the case when the final ground state has numbers $N_{\uparrow}$ and $N_{\downarrow} + 1$ and momentum $k_{F\downarrow}$. The procedures for final states with these numbers and momentum $-k_{F\downarrow}$ or numbers $N_{\uparrow} + 1$ and $N_{\downarrow}$ and momenta $\pm k_{F\uparrow}$ are similar and are omitted here.

The above final state belongs the Hilbert sub space (c). Our goal is to find the quasi-particle operator $\tilde{c}^{\dagger}_{k_{F\downarrow}}$ such that
\[ |0; N_{\uparrow}, N_{\downarrow} + 1 \rangle = \tilde{c}^{\dagger}_{k_{F\downarrow}} |0; N_{\uparrow}, N_{\downarrow} \rangle . \] (11)

Taking into account the changes in the pseudoparticle quantum numbers associated with this (d)$\rightarrow$(c) transition we can write the final state as follows
\[ |0; N_{\uparrow}, N_{\downarrow} + 1 \rangle = \prod_{q=q_{p,c}^{(-)} - \frac{\pi}{N_{\alpha}}}^{q_{p,c}^{(+)}} \prod_{q=q_{p,s}^{(-)} - \frac{\pi}{N_{\alpha}}}^{q_{p,s}^{(+)}} b_{q,c}^{\dagger} b_{q,s}^{\dagger} |V \rangle, \]

which can be rewritten as

\[ |0; N_{\uparrow}, N_{\downarrow} + 1 \rangle = b_{q,c}^{\dagger} b_{q,s}^{\dagger} \prod_{q=q_{p,c}^{(-)} - \frac{\pi}{N_{\alpha}}}^{q_{p,c}^{(+)}} \prod_{q=q_{p,s}^{(-)} - \frac{\pi}{N_{\alpha}}}^{q_{p,s}^{(+)}} b_{q,c}^{\dagger} |V \rangle, \]

and further, as

\[ |0; N_{\uparrow}, N_{\downarrow} + 1 \rangle = b_{q,c}^{\dagger} b_{q,s}^{\dagger} \prod_{q=q_{p,c}^{(-)} - \frac{\pi}{N_{\alpha}}}^{q_{p,c}^{(+)}} \prod_{q=q_{p,s}^{(-)} - \frac{\pi}{N_{\alpha}}}^{q_{p,s}^{(+)}} b_{q,c}^{\dagger} |0; N_{\uparrow}, N_{\downarrow} \rangle, \]

where \( U_{c}^{+1} \) is the generator of expression (10). Both this operator and the operator \( U_{s}^{+1} \) of Eq. (9) obey the relation

\[ U_{\alpha}^{\pm 1} b_{q,\alpha}^{\dagger} U_{\alpha}^{-1} = b_{q,\alpha}^{\dagger} \frac{\pi}{N_{\alpha}}, \]

The pseudoparticle vacuum remains invariant under the application of \( U_{\alpha}^{\pm 1} \)

\[ U_{\alpha}^{\pm 1} |V \rangle = |V \rangle. \]

(The \( s \)-topological-momenton generator, \( U_{s}^{+1} \), appears if we consider the corresponding expressions for the up-spin electron.) Note that the \( \alpha \) topological momenton is an excitation which only changes the integer or half-integer character of the corresponding pseudoparticle quantum numbers \( I_{j}^{\alpha} \). In Appendix B we derive the following expression for the generator \( U_{\alpha}^{\pm 1} \)

\[ U_{\alpha}^{\pm 1} = U_{\alpha} \left( \pm \frac{\pi}{N_{\alpha}} \right), \]

where

\[ U_{\alpha} (\delta q) = \exp \{-i \delta q G_{\alpha} \}, \]

and

\[ G_{\alpha} = -i \sum_{q} \left[ \frac{\partial}{\partial q} b_{q,\alpha}^{\dagger} \right] b_{q,\alpha}, \]
is the Hermitian generator of the $\mp \frac{\pi}{N_{\alpha}}$ topological $\alpha$ pseudomomentum translation. The operator $U_{\alpha}^{\pm 1}$ has the following discrete representation

$$U_{\alpha}^{\pm 1} = \exp\left\{ \sum_{q} b_{q_{\pm},\alpha}^{\dagger} b_{q,\alpha} \right\}. \quad (20)$$

When acting on the initial ground state of form (5) the operator $U_{\alpha}^{\pm 1}$ produces a vanishing-energy $\alpha$ topological momenton of large momentum, $k = \mp N_{\alpha} \frac{\pi}{N_{\alpha}} \simeq q_{F\alpha}^{(\pm)}$. As referred above, the topological momenton is always combined with adding or removal of pseudoparticles.

In the two following equations we change notation and use $q_{F\alpha}^{(\pm)}$ to refer the pseudo-Fermi points of the final state (otherwise our reference state is the initial state). Comparing equations (11) and (14) it follows that

$$\tilde{c}_{\pm k_{F\uparrow}}^{\dagger} = b_{q_{F\uparrow}^{(\pm)},c}^{\dagger} b_{q_{F\uparrow}^{(\pm)},s}^{\dagger} U_{c}^{\pm 1}, \quad (21)$$

and a similar procedure for the up-spin electron leads to

$$\tilde{c}_{\pm k_{F\uparrow}}^{\dagger} = b_{q_{F\uparrow}^{(\pm)},c}^{\dagger} U_{s}^{\pm 1}. \quad (22)$$

According to these equations the $\sigma$ quasiparticles are constituted by one topological momenton and one or two pseudoparticles. The topological momenton cannot be separated from the pseudoparticle excitation, ie both these excitations are confined inside the quasiparticle. Moreover, since the generators (17) – (20) have a many-pseudoparticle character, following Eqs. (21) – (22) the quasiparticle is a many-pseudoparticle object. Note also that both the $\downarrow$ and $\uparrow$ quasiparticles (21) and (22), respectively, are constituted by $c$ and $s$ excitations. Therefore, the $\sigma$ quasiparticle is a quantum object which recombines the pseudoparticle colors $c$ and $s$ (charge and spin in the limit $m \to 0$) giving rise to spin projection $\uparrow$ or $\downarrow$. It has “Fermi surface” at $\pm k_{F\sigma}$.

However, two-quasiparticle objects can be of two-pseudoparticle character because the product of the two corresponding many-pseudoparticle operators is such that $U_{\alpha}^{+1} U_{\alpha}^{-1} = \mathbb{1}$, as for the triplet pair $\tilde{c}_{+k_{F\uparrow}}^{\dagger} \tilde{c}_{-k_{F\uparrow}}^{\dagger} = b_{q_{F\uparrow}^{(\pm)},c}^{\dagger} b_{q_{F\uparrow}^{(\pm)},c}^{\dagger}$. Such triplet quasiparticle pair is constituted only by individual pseudoparticles because it involves the mutual annihilation
of the two topological momentons of generators $U^{+1}_\alpha$ and $U^{-1}_\alpha$. Therefore, relations (21) and (22) which connect quasiparticles and pseudoparticles have some similarities with the Jordan-Wigner transformation.

Finally, we emphasize that the Hamiltonian-eigenstate generators of Eqs. (26) and (27) of Ref. [19] are not general and refer to finite densities of added and removed electrons, respectively, corresponding to even electron numbers. The corresponding general generator expressions will be studied elsewhere and involve the topological-momenton generators (17)–(20).

IV. THE ELECTRON - QUASIPARTICLE TRANSFORMATION

In this section we study the relation of the 1D quasiparticle introduced in Sec. III to the electron. This study brings about the question of the low-excitation-energy relation between the electronic operators $c_{k,\sigma}^\dagger$ in momentum space at $k = \pm k_{F\sigma}$ and the pseudoparticle operators $b_{q,\alpha}^\dagger$ at the pseudo-Fermi points.

The quasiparticle operator, $\tilde{c}_{k_{F\sigma}}^\dagger$, which creates one quasiparticle with spin projection $\sigma$ and momentum $k_{F\sigma}$, is defined by Eq. (8). In the pseudoparticle basis the $\sigma$ quasiparticle operator has the form (21) or (22). However, since we do not know the relation between the electron and the pseudoparticles, Eqs. (21) and (22) do not provide direct information on the electron content of the $\sigma$ quasiparticle. Equation (8) tells us that the quasiparticle operator defines a one-to-one correspondence between the addition of one electron to the system and the creation of one quasiparticle, exactly as we expect from the Landau theory in 3D: the electronic excitation, $c_{k_{F\sigma}}^\dagger|0; N^\uparrow = N_c - N_s, N^\downarrow = N_s\rangle$, defined at the Fermi momentum but arbitrary energy, contains a single $\sigma$ quasiparticle, as we show below. When we add or remove one electron from the many-body system this includes the transition to the suitable final ground state as well as transitions to excited states. The former transition is nothing but the quasiparticle excitation of Sec. III.

Although our final results refer to momenta $k = \pm k_{F\sigma}$, in the following analysis we
consider for simplicity only the momentum $k = k_{F\sigma}$. In order to relate the quasiparticle operators $\tilde{c}^\dagger_{k_{F\sigma}\sigma}$ to the electronic operators $c^\dagger_{k_{F\sigma}\sigma}$ we start by defining the Hilbert sub space where the low-energy $\omega$ projection of the state

$$c^\dagger_{k_{F\sigma}\sigma}|0; N_{\sigma}, N_{-\sigma}\rangle,$$

is contained. Notice that the electron excitation (23) is not an eigenstate of the interacting problem: when acting onto the initial ground state $|0; i\rangle \equiv |0; N_{\sigma}, N_{-\sigma}\rangle$ the electronic operator $c^\dagger_{k_{F\sigma}\sigma}$ can be written as

$$c^\dagger_{k_{F\sigma}\sigma} = \left[ \langle 0; f | c^\dagger_{k_{F\sigma}\sigma} | 0; i \rangle + \hat{R} \right] \tilde{c}^\dagger_{k_{F\sigma}\sigma},$$

where

$$\hat{R} = \sum_\gamma \langle \gamma; k = 0 | c^\dagger_{k_{F\sigma}\sigma} | 0; i \rangle \hat{A}_\gamma,$$

and

$$| \gamma; k = 0 \rangle = \hat{A}_\gamma \tilde{c}^\dagger_{k_{F\sigma}\sigma} | 0; i \rangle = \hat{A}_\gamma | 0; f \rangle.$$

Here $|0; f\rangle \equiv |0; N_{\sigma}+1, N_{-\sigma}\rangle$ denotes the final ground state, $\gamma$ represents the set of quantum numbers needed to specify each Hamiltonian eigenstate present in the excitation (23), and $\hat{A}_\gamma$ is the corresponding generator. The first term of the rhs of Eq. (24) refers to the ground state - ground state transition and the operator $\hat{R}$ generates $k = 0$ transitions from $|0, f\rangle$ to states I, states II, and non LWS’s. Therefore, the electron excitation (23) contains the quantum superposition of both the suitable final ground state $|0; f\rangle$, of excited states I relative to that state which result from multiple pseudoparticle-pseudohole processes associated with transitions to states I, and of LWS’s II and non-LWS’s. All these states have the same electron numbers as the final ground state. The transitions to LWS’s II and to non-LWS’s require a minimal finite energy which equals their gap relative to the final ground state. The set of all these Hamiltonian eigenstates spans the Hilbert sub space where the electronic operators $c^\dagger_{k_{F\sigma}\sigma}$ (24) projects the initial ground state.
In order to show that the ground-state – ground-state leading order term of (24) controls the low-energy physics, we study the low-energy sector of the above Hilbert subspace. This is spanned by low-energy states I. In the case of these states the generator $\hat{A}_\gamma$ of Eq. (26) reads

$$\hat{A}_\gamma \equiv \hat{A}_{\{N_{ph}^{\alpha,\iota}\}, l} = \prod_{\alpha = c, s} \hat{L}_{N_{ph}^{\alpha}}^{\alpha}(l),$$

(27)

where the operator $\hat{L}_{N_{ph}^{\alpha}}^{\alpha}(l)$ is given in Eq. (56) of Ref. [19] and produces a number $N_{ph}^{\alpha,\iota}$ of $\alpha, \iota$ pseudoparticle-pseudohole processes onto the final ground state. Here $\iota = sgn(q)1 = \pm 1$ defines the right ($\iota = 1$) and left ($\iota = -1$) pseudoparticle movers, $\{N_{ph}^{\alpha,\iota}\}$ is a short notation for

$${\{N_{ph}^{\alpha,\iota}\} \equiv N_{ph}^{c, +1}, N_{ph}^{c, -1}, N_{ph}^{s, +1}, N_{ph}^{s, -1}}$$

(28)

and $l$ is a quantum number which distinguishes different pseudoparticle-pseudohole distributions characterized by the same values for the numbers (28). In the case of the lowest-energy states I the above set of quantum numbers $\gamma$ is thus given by $\gamma \equiv \{N_{ph}^{\alpha,\iota}\}, l$. (We have introduced the argument ($l$) in the operator $L_{-N_{ph}^{\alpha}}^{\alpha}(l)$ which for the same value of the $N_{ph}^{\alpha}$ number defines different $\alpha$ pseudoparticle - pseudohole configurations associated with different choices of the pseudomomenta in the summation of expression (56) of Ref. [19].) In the particular case of the lowest-energy states expression (26) reads

$$|\{N_{ph}^{\alpha,\iota}\}, l; k = 0\rangle = \hat{A}_{\{N_{ph}^{\alpha,\iota}\}, l} c_{K_{\sigma, \sigma}}^{\dagger} |0; i\rangle = \hat{A}_{\{N_{ph}^{\alpha,\iota}\}, l} |0; f\rangle.$$  

(29)

The full electron – quasiparticle transformation (24) involves other Hamiltonian eigenstates which are irrelevant for the quasiparticle problem studied in the present paper. Therefore, we omit here the study of the general generators $\hat{A}_\gamma$ of Eq. (26).

The momentum expression (relative to the final ground state) of Hamiltonian eigenstates with generators of the general form (27) is [19]

$$k = \frac{2\pi}{N_a} \sum_{\alpha, \iota} l N_{ph}^{\alpha,\iota},$$

(30)
Since our states \( |\{N_{ph}\}, l; k = 0\rangle \) have zero momentum relative to the final ground state they have restrictions in the choice of the numbers (28). For these states these numbers are such that

\[
\sum_{\alpha,\ell} l N_{ph}^{\alpha,\ell} = 0, \tag{31}
\]

which implies that

\[
\sum_{\alpha} N_{ph}^{\alpha,1} = \sum_{\alpha} N_{ph}^{\alpha,-1} = \sum_{\alpha} N_{ph}^{\alpha,\ell} . \tag{32}
\]

Since

\[
N_{ph}^{\alpha,\ell} = 1, 2, 3, \ldots, \tag{33}
\]

it follows from Eqs. (31) – (33) that

\[
\sum_{\alpha,\ell} N_{ph}^{\alpha,\ell} = 2, 4, 6, 8, \ldots, \tag{34}
\]

is always an even positive integer.

The vanishing chemical-potential excitation energy,

\[
\omega_0^\sigma = \mu(N_{\sigma} + 1, N_{-\sigma}) - \mu(N_{\sigma}, N_{-\sigma}) , \tag{35}
\]

can be evaluated by use of the Hamiltonian (6) – (7) and is given by

\[
\omega_0^\uparrow = \frac{\pi}{2N_a} \left[ v_c + F_{cc}^0 + v_s + F_{ss}^0 - 2F_{cs}^1 + v_c + F_{cc}^0 \right] , \tag{36}
\]

and

\[
\omega_0^\downarrow = \frac{\pi}{2N_a} \left[ v_s + F_{ss}^1 + v_c + F_{cc}^0 + v_s + F_{ss}^0 + 2F_{cs}^0 \right] , \tag{37}
\]

for up and down spin, respectively, and involves the pseudoparticle velocities (A6) and Landau parameters (A8). Since we measure the chemical potential from its value at the canonical ensemble of the reference initial ground state, i.e., we consider \( \mu(N_{\sigma}, N_{-\sigma}) = 0 \), \( \omega_0^\sigma \) measures also the ground-state excitation energy \( \omega_0^\sigma = E_0(N_{\sigma} + 1, N_{-\sigma}) - E_0(N_{\sigma}, N_{-\sigma}) \).
The excitation energies \( \omega(\{N_{ph}^{\alpha,i}\}) \) of the states \( |\{N_{ph}^{\alpha,i}\}, l; k = 0 \rangle \) (relative to the initial ground state) involve the energy \( \omega_0 \) and are \( l \) independent. They are given by

\[
\omega(\{N_{ph}^{\alpha,i}\}) = \omega_0 + \frac{2\pi}{N_a} \sum_{\alpha,i} v_{\alpha,i} N_{ph}^{\alpha,i}.
\] (38)

We denote by \( N_{\{N_{ph}^{\alpha,i}\}} \) the number of these states which obey the condition-equations (31), (32), and (34) and have the same values for the numbers (28).

In order to study the main corrections to the (quasiparticle) ground-state – ground-state transition it is useful to consider the simplest case when \( \sum_{\alpha,i} N_{ph}^{\alpha,i} = 2 \). In this case we have \( N_{\{N_{ph}^{\alpha,i}\}} = 1 \) and, therefore, we can omit the index \( l \). There are four of such Hamiltonian eigenstates. Using the notation of the right-hand side (rhs) of Eq. (28) these states are

\[
|1, 1, 0, 0; k = 0 \rangle = \prod_{\alpha = \pm 1} \hat{\rho}_{\alpha,i}(\alpha) \hat{c}_{kF,\sigma}^\dagger |0; i \rangle,
\] (39)

\[
|0, 0, 1, 1; k = 0 \rangle = \prod_{\alpha = \pm 1} \hat{\rho}_{\alpha,i}(\alpha) \hat{\tilde{c}}_{kF,\sigma}^\dagger |0; i \rangle,
\] (40)

\[
|1, 0, 0, 1; k = 0 \rangle = \hat{\rho}_{c,+1}(\frac{2\pi}{N_a}) \hat{\tilde{c}}_{kF,\sigma}^\dagger |0; i \rangle,
\] (41)

\[
|0, 1, 1, 0; k = 0 \rangle = \hat{\rho}_{c,-1}(\frac{2\pi}{N_a}) \hat{\tilde{c}}_{kF,\sigma}^\dagger |0; i \rangle,
\] (42)

where \( \hat{\rho}_{\alpha,i}(k) \) is the fluctuation operator of Eq. (A12). This was studied in some detail in Ref. [20].

From equations (26), (27), and (29) we can rewrite expression (24) as

\[
c_{kF,\sigma}^\dagger = \langle 0; f | c_{kF,\sigma}^\dagger |0; i \rangle \left[ 1 + \sum_{\{N_{ph}^{\alpha,i}\}, l} \frac{\langle \{N_{ph}^{\alpha,i}\}, l; k = 0 | c_{kF,\sigma}^\dagger |0; i \rangle}{\langle 0; f | c_{kF,\sigma}^\dagger |0; i \rangle} \prod_{\alpha = c,s} \hat{L}_{\alpha N_{ph}^{\alpha,i}}(l) \right] \hat{c}_{kF,\sigma}^\dagger
\]

\[
+ \sum_{\gamma'} \langle \gamma'; k = 0 | c_{kF,\sigma}^\dagger |0; i \rangle \hat{A}_{\gamma'} \hat{\tilde{c}}_{kF,\sigma}^\dagger,
\] (43)

where \( \gamma' \) refers to the Hamiltonian eigenstates of form (26) whose generator \( \hat{A}_{\gamma'} \) are not of the particular form (27).
In Appendix C we evaluate the matrix elements of expression (43) corresponding to transitions to the final ground state and excited states of form (29). Following Ref. [19], these states refer to the conformal-field-theory [22,23] critical point. They are such that the ratio \( N_{ph}^{\alpha,i} / N_a \) vanishes in the thermodynamic limit, \( N_a \to 0 \). Therefore, in that limit the positive excitation energies \( \omega(\{N_{ph}^{\alpha,i}\}) \) of Eq. (38) are vanishing small. The results of that Appendix lead to

\[
\langle 0; f | e_{kF,\sigma}^\dagger | 0; i \rangle = \sqrt{Z_\sigma},
\]

where, as in a Fermi liquid [27], the one-electron renormalization factor

\[
Z_\sigma = \lim_{\omega \to 0} Z_\sigma(\omega),
\]

is closed related to the \( \sigma \) self energy \( \Sigma(\pm kF, \omega) \). Here the function \( Z_\sigma(\omega) \) is given by the small-\( \omega \) leading-order term of

\[
|\varsigma_\sigma| |1 - \frac{\partial \text{Re} \Sigma(\pm kF, \omega)}{\partial \omega}|^{-1},
\]

where

\[
\varsigma_\uparrow = -2 + \sum_\alpha \frac{1}{2} [(\xi_{1,\alpha}^1 - \xi_{1,\alpha}^1)^2 + (\xi_{0,\alpha}^0)^2],
\]

and

\[
\varsigma_\downarrow = -2 + \sum_\alpha \frac{1}{2} [(\xi_{1,\alpha}^1)^2 + (\xi_{0,\alpha}^0 + \xi_{0,\alpha}^0)^2],
\]

are \( U, n, \) and \( m \) dependent exponents which for \( U > 0 \) are negative and such that \(-1 < \varsigma_\sigma < -1/2\). In equations (47) and (48) \( \xi_{\alpha\alpha'}^j \) are the parameters (A7). From equations (46), (C11), and (C15) we find

\[
Z_\sigma(\omega) = a_0^\sigma \omega^{1+\varsigma_\sigma},
\]

where \( a_0^\sigma \) is a real and positive constant such that

\[
\lim_{U \to 0} a_0^\sigma = 1.
\]
Equation (49) confirms that the renormalization factor (45) vanishes, as expected for a 1D many-electron problem [10]. It follows from Eq. (44) that in the present 1D model the electron renormalization factor can be identified with a single matrix element [10,28]. We emphasize that in a Fermi liquid $ς = -1$ and Eq. (46) recovers the usual Fermi-liquid relation. In the different three limits $U \to 0$, $m \to 0$, and $m \to n$ the exponents $ς_\uparrow$ and $ς_\downarrow$ are equal and given by $-1$, $-2 + \frac{1}{2} \left| \frac{8}{2} + \frac{1}{ξ_0} \right|^2$, and $-\frac{1}{2} - η_0|1 - \frac{3n}{2}|$, respectively. Here the $m \to 0$ parameter $ξ_0$ changes from $ξ_0 = \sqrt{2}$ at $U = 0$ to $ξ_0 = 1$ as $U \to \infty$ and $η_0 = \left( \frac{2}{π} \right) \tan^{-1} \left( \frac{4t \sin(πn)}{U} \right)$.

The evaluation in Appendix C for the matrix elements of the rhs of expression (43) refers to the thermodynamic limit and follows the study of the small-$ω$ dependencies of the one-electron Green function $G_\sigma(±k_{Fσ}, ω)$ and self energy $Σ_σ(±k_{Fσ}, ω)$. This leads to $ω$ dependent quantities [as (46) and (49) and the function $F_σ^{α,ι}(ω)$ of Eq. (51) below] whose $ω \to 0$ limits provide the expressions for these matrix elements. Although these matrix elements vanish, it is physically important to consider the associate $ω$-dependent functions. These are matrix-element expressions only in the limit $ω \to 0$, yet at small finite values of $ω$ they provide relevant information on the electron - quasiparticle overlap at low energy $ω$. In addition to expression (44), in Appendix C we find the following expression which is valid only for matrix elements involving the excited states of form (29) referring to the conformal-field-theory critical point

$$\langle \{N^α_{ph}\}, l; k | c^+_i k_{Fσ,σ} | 0; i \rangle = \lim_{ω \to 0} F_σ^{α,ι}(ω) = 0, \quad F_σ^{α,ι}(ω) = e^{iχ_σ(\{N^α_{ph}\}, l)} \sqrt{\frac{a^σ(\{N^α_{ph}\}, l)}{a_0^σ}} \sqrt{Z_σ(ω) \omega^{\sum_α N^α_{ph}}}. \quad (51)$$

Here $χ_σ(\{N^α_{ph}\}, l)$ and $a^σ(\{N^α_{ph}\}, l)$ are real numbers and the function $Z_σ(ω)$ was defined above. Notice that the function $F_σ^{α,ι}(ω)$ vanishes with different powers of $ω$ for different sets of $N^α_{ph}$ numbers. This is because these powers reflect directly the order of the pseudoparticle-pseudohole generator relative to the final ground state of the corresponding state I.

Although the renormalization factor (45) and matrix elements (51) vanish, Eqs. (49) and (51) provide relevant information in what concerns the ratios of the different matrix ele-
ments which can either diverge or vanish. Moreover, in the evaluation of some \(\omega\)-dependent quantities we can use for the matrix elements (51) the function \(F_{\sigma}^{\alpha,\iota}(\omega)\) and assume that \(\omega\) is vanishing small, which leads to correct results. This procedure is similar to replacing the renormalization factor (45) by the function (49). While the renormalization factor is zero because in the limit of vanishing excitation energy there is no overlap between the electron and the quasiparticle, the function (49) is associated with the small electron - quasiparticle overlap which occurs at low excitation energy \(\omega\).

Obviously, if we introduced in the rhs of Eq. (43) zero for the matrix elements (44) and (51) we would lose all information on the associate low-energy singular electron - quasiparticle transformation (described by Eq. (58) below). The vanishing of the matrix elements (44) and (51) just reflects the fact that the one-electron density of states vanishes in the 1D many-electron problem when the excitation energy \(\omega \to 0\). This justifies the lack of electron - quasiparticle overlap in the limit of zero excitation energy. However, the diagonalization of that problem absorbs the renormalization factor (45) and maps vanishing electronic spectral weight onto finite quasiparticle and pseudoparticle spectral weight. This process can only be suitable described if we keep either \(1/N_a\) corrections in the case of the large finite system or small virtual \(\omega\) corrections in the case of the infinite system. (The analysis of Appendix C has considered the thermodynamic limit and, therefore, we consider in this section the case of the infinite system.)

In spite of the vanishing of the matrix elements (44) and (51), following the above discussion we introduce Eqs. (44) and (51) in Eq. (43) with the result

\[
\begin{align*}
c_{\pm k_{F\sigma},\sigma}^\dagger &= \lim_{\omega \to 0} \sqrt{Z_{\sigma}(\omega)} \left[ 1 + \sum_{\{N_{ph}^{\alpha,\iota}\},l} e^{i\chi_{\sigma}\{N_{ph}^{\alpha,\iota}\},l} \sqrt{\frac{a_{\sigma}\{\{N_{ph}^{\alpha,\iota}\},l\}}{a_{\sigma}^0}} \omega \sum_{\alpha,s} N_{ph}^{\alpha,s} \prod_{\alpha=s} \hat{L}_{\alpha N_{ph}^{\alpha}(l)} \right] c_{\pm k_{F\sigma},\sigma}^\dagger \\
+ \sum_{\gamma} \langle \gamma' : k = 0 | c_{\pm k_{F\sigma},\sigma}^\dagger | 0; i \rangle \hat{A}_{\gamma'} c_{\pm k_{F\sigma},\sigma}.
\end{align*}
\]

(52)

(Note that the expression is the same for momenta \(k = k_{F\sigma}\) and \(k = -k_{F\sigma}\).)

Let us confirm the key role played by the “bare” quasiparticle ground-state – ground-state transition in the low-energy physics. Since the \(k = 0\) higher-energy LWS’s I and
finite-energy LWS’s II and non-LWS’s represented in Eq. (52) by \(|\gamma'; k = 0\rangle\) are irrelevant for the low-energy physics, we focus our attention on the lowest-energy states of form (29).

Let us look at the leading-order terms of the first term of the rhs of Eq. (52). These correspond to the ground-state – ground-state transition and to the first-order pseudoparticle-pseudohole corrections. These corrections are determined by the excited states (39) – (42). The use of Eqs. (34) and (39) – (42) allows us rewriting the leading-order terms as

$$\lim_{\omega \to 0} \sqrt{Z_{\sigma}(\omega)} \left[ 1 + \omega^2 \sum_{\alpha,\alpha',\lambda} C_{\alpha,\alpha'}^{\sigma}(l) \rho_{\alpha \lambda}(\frac{2\pi}{N_a}) \rho_{\alpha' \lambda}(-\frac{2\pi}{N_a}) + \mathcal{O}(\omega^4) \right] \hat{c}_{\pm kF,\sigma}^\dagger, \quad (53)$$

where \(C_{\alpha,\alpha'}^{\sigma}\) are complex constants such that

$$C_{c,c}^{1} = C_{c,c}^{-1} = e^{i\chi_{\sigma}(1,1,0,0)} \sqrt{\frac{a_{\sigma}^\sigma(1,1,0,0)}{a_{0}^\sigma}}, \quad (54)$$

$$C_{s,s}^{1} = C_{s,s}^{-1} = e^{i\chi_{\sigma}(0,0,1,1)} \sqrt{\frac{a_{\sigma}^\sigma(0,0,1,1)}{a_{0}^\sigma}}, \quad (55)$$

$$C_{c,s}^{1} = C_{s,c}^{-1} = e^{i\chi_{\sigma}(1,0,0,1)} \sqrt{\frac{a_{\sigma}^\sigma(1,0,0,1)}{a_{0}^\sigma}}, \quad (56)$$

$$C_{c,s}^{-1} = C_{s,c}^{1} = e^{i\chi_{\sigma}(0,1,1,0)} \sqrt{\frac{a_{\sigma}^\sigma(0,1,1,0)}{a_{0}^\sigma}}, \quad (57)$$

and \(\hat{\rho}_{\alpha,\lambda}(k) = \sum_{q} b_{q+k,\alpha,\lambda}^\dagger b_{q,\alpha,\lambda}\) is a first-order pseudoparticle-pseudohole operator. The real constants \(a_{\sigma}\) and \(\chi_{\sigma}\) in the rhs of Eqs. (54) – (57) are particular cases of the corresponding constants of the general expression (51). Note that the \(l\) independence of the states (39)–(42) allowed the omission of the index \(l\) in the quantities of the rhs of Eqs. (54) – (57) and that we used the notation (28) for the argument of the corresponding \(l\)-independent \(a_{\sigma}(\{N_{\rho h}^{\alpha,\lambda}\})\) constants and \(\chi_{\sigma}(\{N_{\rho h}^{\alpha,\lambda}\})\) phases.

The higher-order contributions to expression (53) are associated with low-energy excited Hamiltonian eigenstates I orthogonal both to the initial and final ground states and whose matrix-element amplitudes are given by Eq. (51). The corresponding functions \(F_{\sigma}^{\alpha,\lambda}(\omega)\) vanish as \(\lim_{\omega \to 0} \omega \sum_{j} \frac{1}{\omega^{1+\alpha_{\rho,\lambda}+\lambda_{\rho,\lambda}}/2} (with \ 2j \ the \ number \ of \ pseudoparticle-pseudohole \ processes \ relative}
to the final ground state and \( j = 1, 2, \ldots \)). Therefore, the leading-order term of \( (52) - (53) \) and the exponent \( \varsigma_{\sigma} \) \((47) - (48)\) fully control the low-energy overlap between the \( \pm k_{F\sigma} \) quasiparticles and electrons and determines the expressions of all \( k = \pm k_{F\sigma} \) one-electron low-energy quantities. That leading-order term refers to the ground-state – ground-state transition which dominates the electron – quasiparticle transformation \((24)\). This transition corresponds to the “bare” quasiparticle of Eq. (8). We follow the same steps as Fermi liquid theory and consider the low-energy non-canonical and non-complete transformation one derives from the full expression \((53)\) by only taking the corresponding leading-order term which leads to

\[
\hat{c}^\dagger_{\pm k_{F\sigma}, \sigma} = \frac{c^\dagger_{\pm k_{F\sigma}, \sigma}}{\sqrt{Z_{\sigma}}}. \quad (58)
\]

This relation refers to a singular transformation. Combining Eqs. \((21) - (22)\) and \((58)\) provides the low-energy expression for the electron in the pseudoparticle basis. The singular nature of the transformation \((58)\) which maps the vanishing-renormalization-factor electron onto the one-renormalization-factor quasiparticle, explains the perturbative character of the pseudoparticle-operator basis \([18, 19, 20]\).

If we replace in Eq. \((58)\) the renormalization factor \( Z_{\sigma} \) by \( Z_{\sigma}(\omega) \) or omit \( \lim_{\omega \to 0} \) from the rhs of Eqs. \((52)\) and \((53)\) and in both cases consider \( \omega \) being very small leads to effective expressions which contain information on the low-excitation-energy electron – quasiparticle overlap. Since these expressions correspond to the infinite system, the small \( \omega \) finite contributions contain the same information as the \( \frac{1}{N_a} \) corrections of the corresponding large but finite system at \( \omega = 0 \).

It is the perturbative character of the pseudoparticle basis that determines the form of expansion \((53)\) which except for the non-classical exponent in the \( \sqrt{Z_{\sigma}(\omega)} \propto \omega^{\frac{1+\varsigma_{\sigma}}{2}} \) factor [absorbed by the electron - quasiparticle transformation \((58)\)] includes only classical exponents, as in a Fermi liquid \([27]\). At low energy the BA solution performs the singular transformation \((58)\) which absorbs the one-electron renormalization factor \((45)\) and maps vanishing electronic spectral weight onto finite quasiparticle and pseudoparticle spectral weight. By
that process the transformation (58) renormalizes divergent two-electron scattering vertex functions onto finite two-quasiparticle scattering quantities. These quantities are related to the finite $f$ functions [15] of form given by Eq. (A4) and amplitudes of scattering [16] of the pseudoparticle theory.

It was shown in Refs. [15,16,19] that these $f$ functions and amplitudes of scattering determine all static and low-energy quantities of the 1D many-electron problem, as we discuss below and in Appendices A and D. The $f$ functions and amplitudes are associated with zero-momentum two-pseudoparticle forward scattering. These scattering processes interchange no momentum and no energy, only giving rise to two-pseudoparticle phase shifts. The corresponding pseudoparticles control all the low-energy physics. In the limit of vanishing energy the pseudoparticle spectral weight leads to finite values for the static quantities, yet it corresponds to vanishing one-electron spectral weight.

To diagonalize the problem at lowest energy is equivalent to perform the electron-quasiparticle transformation (58): it maps divergent irreducible (two-momenta) charge and spin vertices onto finite quasiparticle parameters by absorbing $Z_\sigma$. In a diagramatic picture this amounts by multiplying each of these vertices appearing in the diagrams by $Z_\sigma$ and each one-electron Green function (propagator) by $\frac{1}{Z_\sigma}$. This procedure is equivalent to renormalize the electron quantities onto corresponding quasiparticle quantities, as in a Fermi liquid. However, in the present case the renormalization factor is zero.

This also holds true for more involved four-momenta divergent two-electron vertices at the Fermi points. In this case the electron-quasiparticle transformation multiplies each of these vertices by a factor $Z_\sigma Z_{\sigma'}$, the factors $Z_\sigma$ and $Z_{\sigma'}$ corresponding to the pair of $\sigma$ and $\sigma'$ interacting electrons. The obtained finite parameters control all static quantities. Performing the transformation (58) is equivalent to sum all vertex contributions and we find that this transformation is unique, ie it maps the divergent Fermi-surface vertices on the same finite quantities independently on the way one chooses to approach the low energy limit. This cannot be detected by looking only at logarithmic divergences of some diagrams [3,5]. Such non-universal contributions either cancel or are renormalized to zero by the electron-
quasiparticle transformation. We have extracted all our results from the exact BA solution which takes into account all relevant contributions. We can choose the energy variables in such a way that there is only one ω dependence. We find that the relevant vertex function divergences are controlled by the electron - quasiparticle overlap, the vertices reading

\[
\Gamma_{\sigma\sigma'}(k_{F\sigma}, k_{F\sigma'}; \omega) = \frac{1}{Z_{\sigma}(\omega)Z_{\sigma'}(\omega)} \left\{ \sum_{\iota' = \pm 1} (\iota')^{1/2} \left[ v_{\rho} + (\delta_{\sigma,\sigma'} - \delta_{\sigma,-\sigma'})v_{\sigma_z} - \delta_{\sigma,\sigma'}v_{F,\sigma} \right] \right\},
\]

(59)

where the expressions for the charge \(v_{\rho}\) and spin \(v_{\sigma_z}\) velocities are given in Appendix D. The divergent character of the function (59) follows exclusively from the \(1/Z_{\sigma}(\omega)Z_{\sigma'}(\omega)\) factor, with \(Z_{\sigma}(\omega)\) given by (49). The transformation (58) maps the divergent vertices onto the ω-independent finite quantity \(Z_{\sigma}(\omega)Z_{\sigma'}(\omega)\Gamma_{\sigma\sigma'}(k_{F\sigma}, k_{F\sigma'}; \omega)\). The low-energy physics is determined by the following \(v_{F,\sigma}\)-independent Fermi-surface two-quasiparticle parameters

\[
L_{\iota\sigma,\sigma'} = \lim_{\omega \to 0} \left[ \delta_{\sigma,\sigma'}v_{F,\sigma} + Z_{\sigma}(\omega)Z_{\sigma'}(\omega)\Gamma_{\sigma\sigma'}(k_{F\sigma}, k_{F\sigma'}; \omega) \right].
\]

(60)

From the point of view of the electron - quasiparticle transformation the divergent vertices (59) originate the finite quasiparticle parameters (60) which define the above charge and spin velocities. These are given by the following simple combinations of the parameters (60)

\[
v_{\rho} = \frac{1}{4} \sum_{\iota' = \pm 1} (\iota')^{1/2} \left[ L_{\alpha,\sigma}^{\iota'} + L_{\alpha,-\sigma}^{\iota'} \right],
\]

\[
v_{\sigma_z} = \frac{1}{4} \sum_{\iota' = \pm 1} (\iota')^{1/2} \left[ L_{\alpha,\sigma}^{\iota'} - L_{\alpha,-\sigma}^{\iota'} \right].
\]

(61)

As shown in Appendix D, the parameters \(L_{\iota\sigma,\sigma'}\) can be expressed in terms of the pseudoparticle group velocities (A6) and Landau parameters (A8) as follows

\[
L_{\iota\sigma,\sigma'}^{\pm 1} = 2 \left[ \left( \frac{v_{F,ss}}{L^0} \right) \pm \left( v_{F,cc} + \frac{v_{F,cs}}{2} \right) \right],
\]

\[
L_{\pm 1}^{\iota \sigma,\sigma' - \sigma} = -4 \left[ \left( \frac{v_{F,cc} + F_{cs}^0}{L^0} \right) \pm \left( v_{F,ss} + F_{cs}^0 - F_{cs}^1 \right) \right],
\]

(62)

where \(L^0 = (v_{F,cc} + F_{cs}^0)(v_{F,ss} + F_{cs}^0) - (F_{cs}^0)^2\). Combining equations (61) and (62) we find the expressions of the Table for the charge and spin velocities. These velocities were already known through the BA solution and determine the expressions for all static quantities \[20\]. Equations (62) clarify their origin which is the singular electron - quasiparticle transformation.
(58). It renders a non-perturbative electronic problem into a perturbative pseudoparticle problem. In Appendix D we show how the finite two-pseudoparticle forward-scattering functions and amplitudes which determine the static quantities are directly related to the two-quasiparticle finite parameters (60) through the velocities (61). This study confirms that it is the singular electron - quasiparticle transformation (58) which justifies the finite character of the $f_{\alpha \alpha'}(q, q')$ functions (A4) and the associate perturbative origin of the pseudoparticle Hamiltonian (6) – (7).

In order to further confirm that the electron - quasiparticle transformation (58) and associate electron - quasiparticle overlap function (49) control the whole low-energy physics we close this section by considering the one-electron spectral function. The spectral function was studied numerically and for $U \rightarrow \infty$ in Refs. [29] and [30], respectively. The leading-order term of the real-part expression for the $\sigma$ Green function at $k = \pm k_{F\sigma}$ and small excitation energy $\omega$ (C10)-(C11) is given by, Re$G_{\sigma}(\pm k_{F\sigma}, \omega) = a_0^\sigma \omega^\sigma$. From Kramers-Kronig relations we find Im$G_{\sigma}(\pm k_{F\sigma}, \omega) = -i\pi a_0^\sigma (1 + \varsigma_\sigma)\omega^\sigma$ for the corresponding imaginary part. Based on these results we arrive to the following expression for the low-energy spectral function at $k = \pm k_{F\sigma}$

$$A_{\sigma}(\pm k_{F\sigma}, \omega) = 2\pi a_0^\sigma (1 + \varsigma_\sigma)\omega^\sigma = 2\pi \left( \frac{\partial Z_\sigma(\omega)}{\partial \omega} \right).$$

This result is a generalization of the $U \rightarrow \infty$ expression of Ref. [30]. It is valid for all parameter space where both the velocities $v_c$ and $v_s$ (A6) are finite. (This excludes half filling $n = 1$, maximum spin density $m = n$, and $U = \infty$ when $m \neq 0$.) The use of Kramers-Kronig relations also restricts the validity of expression (63) to the energy continuum limit. On the other hand, we can show that (63) is consistent with the general expression

$$A_{\sigma}(\pm k_{F\sigma}, \omega) = \sum_{\{N_{ph}^{\alpha+}\}, l} \sum_{\{N_{ph}^{\alpha-}\}, l} |\langle \{N_{ph}^{\alpha+}\}, l; k = 0|c_{k_{F\sigma, \sigma}}^\dagger|0; i \rangle|^2 2\pi \delta(\omega - \omega(\{N_{ph}^{\alpha-}\}))$ + \sum_{\gamma'} |\langle \gamma'; k = 0|c_{k_{F\sigma, \sigma}}^\dagger|0; i \rangle|^2 2\pi \delta(\omega - \omega_{\gamma'}) ;$$

whose summations refer to the same states as the summations of expressions (43) and (52). The restriction of the validity of expression (63) to the energy continuum limit requires the
consistency to hold true only for the spectral weight of (64) associated with the quasiparticle ground-state – ground-state transition. This corresponds to the first δ peak of the rhs of Eq. (64). Combining equations (44) and (64) and considering that in the present limit of vanishing ω replacing the renormalization factor (45) by the electron - quasiparticle overlap function (49) leads to the correct result (as we confirm below) we arrive to

$$A_σ(±k_{Fσ}, ω) = a_0^σω^{1+ς}2πδ(ω − ω_0^σ) = Z_σ(ω)2πδ(ω − ω_0^σ).$$  \hspace{1cm} (65)$$

Let us then show that the Kramers-Kronig continuum expression (63) is an approximation consistent with the Dirac-delta function representation (65). This consistency just requires that in the continuum energy domain from ω = 0 to the ground-state – ground-state transition energy ω = ω_0^σ (see Eq. (35)) the functions (63) and (65) contain the same amount of spectral weight. We find that both the $A_σ(±k_{Fσ}, ω)$ representations (63) and (65) lead to

$$\int_0^{ω_0^σ} A_σ(±k_{Fσ}, ω) = 2πa_0^σ[ω_0^σ]^{ς+1},$$  \hspace{1cm} (66)$$

which confirms they contain the same spectral weight. The representation (63) reveals that the spectral function diverges at ±k_{Fσ} and small ω as a Luttinger-liquid power law. However, both the small-ω density of states and the integral (66) vanish in the limit of vanishing excitation energy.

Using the method of Ref. [17] we have also studied the spectral function $A_σ(k, ω)$ for all values of k and vanishing positive ω. We find that $A_σ(k, ω)$ [and the Green function $ReG_σ(k, ω)$] vanishes when ω → 0 for all momentum values except at the non-interacting Fermi-points $k = ±k_{Fσ}$ where it diverges as the power law (63). This divergence is fully controlled by the quasiparticle ground-state - ground-state transition. The transitions to the excited states (29) give only vanishing contributions to the spectral function. This further confirms the dominant role of the bare quasiparticle ground-state - ground-state transition and of the associate electron - quasiparticle transformation (58) which control the low-energy physics.

It follows from the above behavior of the spectral function at small ω that for ω → 0 the density of states,
\[ D_\sigma(\omega) = \sum_k A_\sigma(k, \omega), \] (67)

results, exclusively, from contributions of the peaks centered at \( k = \pm k_{F\sigma} \) and is such that
\[ D_\sigma(\omega) \propto \omega A_\sigma(\pm k_{F\sigma}, \omega) \] [11]. On the one hand, it is known from the zero-magnetic field
studies of Refs. [30,31] that the density of states goes at small \( \omega \) as
\[ D_\sigma(\omega) \propto \omega^{\nu_\sigma}, \] (68)

where \( \nu_\sigma \) is the exponent of the equal-time momentum distribution expression,
\[ N_\sigma(k) \propto |k \mp k_{F\sigma}|^{\nu_\sigma}, \] (69)

[23,32]. (The exponent \( \nu_\sigma \) is defined by Eq. (5.10) of Ref. [23] for the particular case of the
\( \sigma \) Green function.) On the other hand, we find that the exponents (47) – (48) and \( \nu_\sigma \) are
such that
\[ \varsigma_\sigma = \nu_\sigma - 1, \] (70)
in agreement with the above analysis. However, this simple relation does not imply that
the equal-time expressions [23,32] provide full information on the small-energy instabilities.
For instance, in addition to the momentum values \( k = \pm k_{F\sigma} \) and in contrast to the spectral
function, \( N_\sigma(k) \) shows singularities at \( k = \pm [k_{F\sigma} + 2k_{F_{-\sigma}}] \) [32]. Therefore, only the direct
low-energy study reveals all the true instabilities of the quantum liquid.

Note that in some Luttinger liquids the momentum distribution is also given by \( N(k) \propto |k \mp k_F|^{\nu} \) but with \( \nu > 1 \) [3,33,34]. We find that in these systems the spectral function
\( A(\pm k_F, \omega) \propto \omega^{\nu-1} \) does not diverge.

V. CONCLUDING REMARKS

One of the goals of this paper was, in spite of the differences between the Luttinger-
liquid Hubbard chain and 3D Fermi liquids, detecting common features in these two limiting
problems which we expect to be present in electronic quantum liquids in spatial dimensions
1 <D< 3. As in 3D Fermi liquids, we find that there are Fermi-surface quasiparticles in the Hubbard chain which connect ground states differing in the number of electrons by one and whose low-energy overlap with electrons determines the $\omega \to 0$ divergences. In spite of the vanishing electron density of states and renormalization factor, the spectral function vanishes at all momenta values except at the Fermi surface where it diverges (as a Luttinger-liquid power law).

While low-energy excitations are described by $c$ and $s$ pseudoparticle-pseudohole excitations which determine the $c$ and $s$ separation [20], the quasiparticles describe ground-state–ground-state transitions and recombine $c$ and $s$ (charge and spin in the zero-magnetization limit), being labelled by the spin projection $\sigma$. They are constituted by one topological momenton and one or two pseudoparticles which cannot be separated and are confined inside the quasiparticle. Moreover, there is a close relation between the quasiparticle contents and the Hamiltonian symmetry in the different sectors of parameter space. This can be shown if we consider pseudoholes instead of pseudoparticles [11] and we extend the present quasiparticle study to the whole parameter space of the Hubbard chain.

Importantly, we have written the low-energy electron at the Fermi surface in the pseudoparticle basis. The vanishing of the electron renormalization factor implies a singular character for the low-energy electron–quasiparticle and electron–pseudoparticle transformations. This singular process extracts from vanishing electron spectral weight quasiparticles of spectral-weight factor one. The BA diagonalization of the 1D many-electron problem is at lowest excitation energy equivalent to perform such singular electron–quasiparticle transformation. This absorbs the vanishing one-electron renormalization factor giving rise to the finite two-pseudoparticle forward-scattering $f$ functions and amplitudes which control the expressions for all static quantities [13,16,18]. It is this transformation which justifies the perturbative character of the many-electron Hamiltonian in the pseudoparticle basis [18].

From the existence of Fermi-surface quasiparticles both in the 1D and 3D limits, our results suggest their existence for quantum liquids in dimensions $1<D<3$. However, the effect of increasing dimensionality on the electron–quasiparticle overlap remains an unsolved
problem. The present 1D results do not provide information on whether that overlap can vanish for $D>1$ or whether it always becomes finite as soon as we leave 1D.

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APPENDIX A: SOME USEFUL QUANTITIES OF THE PSEUDOPARTICLE REPRESENTATION

In this Appendix we present some quantities of the pseudoparticle picture which are useful for the present study. We start by defining the pseudo-Fermi points and limits of the pseudo-Brillouin zones. When $N_\alpha$ (see Eq. (4)) is odd (even) and the numbers $I_{j\alpha}$ of Eq. (3) are integers (half integers) the pseudo-Fermi points are symmetric and given by

$$q^{(+)}_{F\alpha} = -q^{(-)}_{F\alpha} = \frac{\pi}{N_\alpha} [N_\alpha - 1].$$

(A1)

On the other hand, when $N_\alpha$ is odd (even) and $I_{j\alpha}$ are half integers (integers) we have that

$$q^{(+)}_{F\alpha} = \frac{\pi}{N_\alpha} N_\alpha, \quad -q^{(-)}_{F\alpha} = \frac{\pi}{N_\alpha} [N_\alpha - 2],$$

(A2)

or

$$q^{(+)}_{F\alpha} = \frac{\pi}{N_\alpha} [N_\alpha - 2], \quad -q^{(-)}_{F\alpha} = \frac{\pi}{N_\alpha} N_\alpha.$$  

(A3)

Similar expressions are obtained for the pseudo-Brioullin zones limits $q^{(\pm)}_\alpha$ if we replace in Eqs. (A1)-(A3) $N_\alpha$ by the numbers $N_\alpha^*$ of Eq. (4).

The $f$ functions were studied in Ref. [15] and read

$$f_{\alpha\alpha'}(q, q') = 2\pi v_\alpha(q)\Phi_{\alpha\alpha'}(q, q') + 2\pi v_{\alpha'}(q')\Phi_{\alpha'\alpha}(q, q') + \sum_{j=\pm 1} \sum_{\alpha''=c,s} 2\pi v_{\alpha''}\Phi_{\alpha''\alpha}(jq_{F\alpha''}, q)\Phi_{\alpha''\alpha'}(jq_{F\alpha''}, q'),$$

(A4)

where the pseudoparticle group velocities are given by

$$v_\alpha(q) = \frac{d\epsilon_\alpha(q)}{dq},$$

(A5)

and

$$v_\alpha = \pm v_\alpha(q^{(\pm)}_{F\alpha}),$$

(A6)

are the pseudo-Fermi points group velocities. In expression (A4) $\Phi_{\alpha\alpha'}(q, q')$ measures the phase shift of the $\alpha'$ pseudoparticle of pseudomomentum $q'$ due to the forward-scattering
collision with the $\alpha$ pseudoparticle of pseudomomentum $q$. These phase shifts determine the pseudoparticle interactions and are defined in Ref. [15]. They control the low-energy physics. For instance, the related parameters

$$\xi^j_{\alpha\alpha'} = \delta_{\alpha\alpha'} + \Phi_{\alpha\alpha'}(q^{(+)}_{F\alpha}, q^{(+)}_{F\alpha'}) + (-1)^j \Phi_{\alpha\alpha'}(q^{(+)}_{F\alpha}, q^{(-)}_{F\alpha'}) , \quad j = 0, 1 ,$$  \hspace{1cm} (A7)

play a determining role at the critical point. ($\xi^1_{\alpha\alpha'}$ are the entries of the transpose of the dressed-charge matrix [22].) The values at the pseudo-Fermi points of the $f$ functions (A4) include the parameters (A7) and define the Landau parameters,

$$F^j_{\alpha\alpha'} = \frac{1}{2\pi} \sum_{\iota=\pm 1} (\iota)^j f_{\alpha\alpha'}(q^{(\mp)}_{F\alpha}, q^{(\mp)}_{F\alpha'}) , \quad j = 0, 1 .$$  \hspace{1cm} (A8)

These are also studied in Ref. [15]. The parameters $\delta_{\alpha,\alpha'} v_{\alpha} + F^j_{\alpha\alpha'}$ appear in the expressions of the low-energy quantities.

We close this Appendix by introducing pseudoparticle-pseudohole operators which will appear in Sec. IV. Although the expressions in the pseudoparticle basis of one-electron operators remains an unsolved problem, in Ref. [20] the electronic fluctuation operators

$$\hat{\rho}(k) = \sum_{k'} c^\dagger_{k'+k\sigma} c_{k'\sigma} ,$$  \hspace{1cm} (A9)

were expressed in terms of the pseudoparticle fluctuation operators

$$\hat{\rho}_\alpha(k) = \sum_q b^\dagger_{q+k\alpha} b_{q\alpha} .$$  \hspace{1cm} (A10)

This study has revealed that $\iota = \text{sgn}(k)1 = \pm 1$ electronic operators are made out of $\iota = \text{sgn}(q)1 = \pm 1$ pseudoparticle operators only, $\iota$ defining the right ($\iota = 1$) and left ($\iota = -1$) movers.

Often it is convenient measuring the electronic momentum $k$ and pseudomomentum $q$ from the $U = 0$ Fermi points $k_{F\sigma}^{(\pm)} = \pm \pi n_{\sigma}$ and pseudo-Fermi points $q_{F\alpha}^{(\pm)}$, respectively. This adds the index $\iota$ to the electronic and pseudoparticle operators. The new momentum $\tilde{k}$ and pseudomomentum $\tilde{q}$ are such that

$$\tilde{k} = k - k_{F\sigma}^{(\pm)} , \quad \tilde{q} = q - q_{F\alpha}^{(\pm)} ,$$  \hspace{1cm} (A11)
respectively, for \( \nu = \pm 1 \). For instance,

\[
\hat{\rho}_{\sigma,\nu}(k) = \sum_{k} c_{k+\nu}^\dagger c_{k\sigma}, \quad \hat{\rho}_{\alpha,\nu}(k) = \sum_{\bar{q}} b_{\bar{q}+\nu}^\dagger b_{\bar{q}\alpha},
\]  

(A12)
APPENDIX B: THE TOPOLOGICAL-MOMENTON GENERATOR

In this Appendix we evaluate the expression for the topological-momenton generator (17) – (19). In order to derive the expression for \( U_c^{\pm 1} \) we consider the Fourier transform of the pseudoparticle operator \( b_{q,c}^\dagger \) which reads

\[
\beta_{x,c}^\dagger = \frac{1}{\sqrt{N_a}} \sum_{q_{c}^{(+)}} e^{-i q x} b_{q,c}^\dagger.
\]  

(B1)

From Eq. (15) we arrive to

\[
U_c^{+1} \beta_{x,c}^\dagger U_c^{-1} = \frac{1}{\sqrt{N_a}} \sum_{q_{c}^{(-)}} e^{-i q x} b_{q,c}^\dagger.
\]  

(B2)

By performing a \( \frac{\pi}{N_a} \) pseudomomentum translation we find

\[
U_c^{+1} \beta_{x,c}^\dagger U_c^{-1} = e^{i \frac{\pi}{N_a} x} \beta_{x,c}^\dagger.
\]  

(B3)

and it follows that

\[
U_c^{\pm 1} = \exp\left\{ \pm i \frac{\pi}{N_a} \sum_y y \beta_{y,c}^\dagger \beta_{y,c} \right\}.
\]  

(B4)

By inverse-Fourier transforming expression (B4) we find expression (17) – (19) for this unitary operator, which can be shown to also hold true for \( U_s^{\pm 1} \).
APPENDIX C: ONE-ELECTRON MATRIX ELEMENTS

In this Appendix we derive the expressions for the matrix elements (44) and (51).

At energy scales smaller than the gaps for the LWS’s II and non-LWS’s referred in this paper and in Refs. [18,19,20] the expression of the \( \sigma \) one-electron Green function \( G_\sigma(k_{F\sigma}, \omega) \) is fully defined in the two Hilbert sub spaces spanned by the final ground state \( |0; f\rangle \) and associate \( k = 0 \) excited states \( |\{N_{\text{ph}}^{\alpha,\iota}\}, l; k = 0\rangle \) of form (29) belonging the \( N_\sigma + 1 \) sector and by a corresponding set of states belonging the \( N_\sigma - 1 \) sector, respectively. Since \( |0; f\rangle \) corresponds to zero values for all four numbers (28) in this Appendix we use the notation \( |0; f\rangle \equiv |\{N_{\text{ph}}^{\alpha,\iota} = 0\}, l; k = 0\rangle \). This allows a more compact notation for the state summations. The use of a Lehmann representation leads to

\[
G_\sigma(k_{F\sigma}, \omega) = G^{(N_\sigma+1)}_{\sigma}(k_{F\sigma}, \omega) + G^{(N_\sigma-1)}_{\sigma}(k_{F\sigma}, \omega),
\]

(C1)

where

\[
G^{(N_\sigma+1)}_{\sigma}(k_{F\sigma}, \omega) = \sum_{\{N_{\text{ph}}^{\alpha,\iota}\}, l} \frac{|\langle \{N_{\text{ph}}^{\alpha,\iota}\}, l; k = 0 | c_{k_{F\sigma,\sigma}}^\dagger |0; i\rangle|^2}{\omega - \omega(\{N_{\text{ph}}^{\alpha,\iota}\}) + i\xi},
\]

(C2)

has divergences for \( \omega > 0 \) and \( G^{(N_\sigma-1)}_{\sigma}(k_{F\sigma}, \omega) \) has divergences for \( \omega < 0 \). We emphasize that in the \( \{N_{\text{ph}}^{\alpha,\iota}\} \) summation of the rhs of Eq. (C2), \( N_{\text{ph}}^{\alpha,\iota} = 0 \) for all four numbers refers to the final ground state, as we mentioned above. Below we consider positive but vanishing values of \( \omega \) and, therefore, we need only to consider the function (C2). We note that at the conformal-field critical point \([22,23]\) the states which contribute to (C2) are such that the ratio \( N_{\text{ph}}^{\alpha,\iota}/N_a \) vanishes in the thermodynamic limit, \( N_a \to 0 \) \([19]\). Therefore, in that limit the positive excitation energies \( \omega(\{N_{\text{ph}}^{\alpha,\iota}\}) \) of Eq. (C2), which are of the form (38), are vanishing small. Replacing the full Green function by (C2) (by considering positive values of \( \omega \) only) we find

\[
\lim_{N_a \to \infty} \text{Re}G_\sigma(k_{F\sigma}, \omega) = \sum_{\{N_{\text{ph}}^{\alpha,\iota}\}} \left[ \frac{\sum_l |\langle \{N_{\text{ph}}^{\alpha,\iota}\}, l; k = 0 | c_{k_{F\sigma,\sigma}}^\dagger |0; i\rangle|^2}{\omega} \right].
\]

(C3)

We emphasize that considering the limit (C3) implies that all the corresponding expressions for the \( \omega \) dependent quantities we obtain in the following are only valid in the limit of
vanishing positive energy $\omega$. Although many of these quantities are zero in that limit, their $\omega$ dependence has physical meaning because different quantities vanish as $\omega \to 0$ in different ways, as we discuss in Sec. IV. Therefore, our results allow the classification of the relative importance of the different quantities.

In order to solve the present problem we have to combine a suitable generator pseudoparticle analysis [19] with conformal-field theory [22,23]. Let us derive an alternative expression for the Green function (C3). Comparison of both expressions leads to relevant information. This confirms the importance of the pseudoparticle operator basis [18,19,20] which allows an operator description of the conformal-field results for BA solvable many-electron problems [22,23].

The asymptotic expression of the Green function in $x$ and $t$ space is given by the summation of many terms of form (3.13) of Ref. [22] with dimensions of the fields suitable to that function. For small energy the Green function in $k$ and $\omega$ space is obtained by the summation of the Fourier transforms of these terms, which are of the form given by Eq. (5.2) of Ref. [23]. However, the results of Refs. [22,23] do not provide the expression at $k = k_{F\sigma}$ and small positive $\omega$. In this case the above summation is equivalent to a summation in the final ground state and excited states of form (29) obeying to Eqs. (31), (32), and (34) which correspond to different values for the dimensions of the fields.

We emphasize that expression (5.7) of Ref. [23] is not valid in our case. Let us use the notation $k_0 = k_{F\sigma}$ (as in Eqs. (5.6) and (5.7) of Ref. [23]). While we consider $(k - k_0) = (k - k_{F\sigma}) = 0$ expression (5.7) of Ref. [23] is only valid when $(k - k_0) = (k - k_{F\sigma})$ is small but finite. We have solved the following general integral

$$\tilde{g}(k_0, \omega) = \int_0^\infty dt e^{i\omega t} F(t), \quad (C4)$$

where

$$F(t) = \int_{-\infty}^\infty dx \prod_{\alpha,\tau} \frac{1}{(x + i\nu_{\alpha} t)^{2\Delta_{\alpha}}}, \quad (C5)$$

with the result
\[ \tilde{g}(k_0, \omega) \propto \omega^{\left[ \sum_{\alpha, \iota} 2\Delta^\iota_\alpha - 2 \right]} . \]  \hspace{1cm} (C6)

Comparing our expression (C6) with expression (5.7) of Ref. [23] we confirm these expressions are different.

In the present case of the final ground state and excited states of form (29) obeying Eqs. (31), (32), and (34) we find that the dimensions of the fields are such that

\[ \sum_{\alpha, \iota} 2\Delta^\iota_\alpha = 2 + \varsigma_\sigma + 2 \sum_{\alpha, \iota} N_{ph}^\alpha, \iota, \] \hspace{1cm} (C7)

with \( \varsigma_\sigma \) being the exponents (47) and (48). Therefore, equation (C6) can be rewritten as

\[ \tilde{g}(k_0, \omega) \propto \omega^{\varsigma_\sigma + 2 \sum_{\alpha, \iota} N_{ph}^\alpha} . \] \hspace{1cm} (C8)

Summing the terms of form (C8) corresponding to different states leads to an alternative expression for the function (C3) with the result

\[ \lim_{N_a \to \infty} \text{Re} G_{\sigma}(k_{F\sigma}, \omega) = \sum_{\{N_{ph}^\alpha, \iota\}} \left[ a^\sigma(\{N_{ph}^\alpha, \iota\}) \omega^{\varsigma_\sigma + 2} \sum_{\alpha, \iota} N_{ph}^\alpha \right] / \omega , \] \hspace{1cm} (C9)

or from Eq. (34),

\[ \lim_{N_a \to \infty} \text{Re} G_{\sigma}(k_{F\sigma}, \omega) = \sum_{j=0,1,2,...} \left[ a^\sigma_j \omega^{\varsigma_\sigma + 1 + 4j} \right] / \omega , \] \hspace{1cm} (C10)

where \( a^\sigma(\{N_{ph}^\alpha, \iota\}) \) and \( a^\sigma_j \) are complex constants. From equation (C10) we find

\[ \text{Re} \Sigma_\sigma(k_{F\sigma}, \omega) = \omega - \frac{1}{\text{Re} G_{\sigma}(k_{F\sigma}, \omega)} = \omega [1 - \frac{\omega^{-1-\varsigma_\sigma}}{a^\sigma_0 + \sum_{j=1}^{\infty} a^\sigma_j \omega^{4j}}] ; \] \hspace{1cm} (C11)

While the function \( \text{Re} G_{\sigma}(k_{F\sigma}, \omega) \) (C9)-(C10) diverges as \( \omega \to 0 \), following the form of the self energy (C11) the one-electron renormalization factor (45) vanishes and there is no overlap between the quasiparticle and the electron, in contrast to a Fermi liquid. (In equation (C11) \( \varsigma_\sigma \to -1 \) and \( a^\sigma_0 \to 1 \) when \( U \to 0 \).)

Comparision of the terms of expressions (C3) and (C9) with the same \( \{N_{ph}^\alpha, \iota\} \) values, which refer to contributions from the same set of \( N_{ph}^\alpha, \iota \) Hamiltonian eigenstates \( |\{N_{ph}^\alpha\}, l; k = 0 \rangle \) and refer to the limit \( \omega \to 0 \), leads to
\[ \sum_l |\langle \{ N_{ph}^{\alpha,i} \}, l; k = 0 | c_{kF,s,\sigma}^\dagger |0; i \rangle|^2 = \lim_{\omega \to 0} a^\sigma(\{ N_{ph}^{\alpha,i} \}) \omega^{\sigma+1+2} \sum_{\alpha,i} N_{ph}^{\alpha,i} = 0. \] (C12)

Note that the functions of the rhs of Eq. (C12) corresponding to different matrix elements go to zero with different exponents.

On the other hand, as for the corresponding excitation energies (38), the dependence of functions associated with the amplitudes \( |\langle \{ N_{ph}^{\alpha,i} \}, l; k = 0 | c_{kF,s,\sigma}^\dagger |0; i \rangle| \) on the vanishing energy \( \omega \) is \( l \) independent. Therefore, we find

\[ |\langle \{ N_{ph}^{\alpha,i} \}, l; k = 0 | c_{kF,s,\sigma}^\dagger |0; i \rangle|^2 = \lim_{\omega \to 0} a^\sigma(\{ N_{ph}^{\alpha,i} \}, l) \omega^{\sigma+1+2} \sum_{\alpha,i} N_{ph}^{\alpha,i} = 0, \] (C13)

where the constants \( a^\sigma(\{ N_{ph}^{\alpha,i} \}) \) are \( l \) dependent and obey the normalization condition

\[ a^\sigma(\{ N_{ph}^{\alpha,i} \}) = \sum_l a^\sigma(\{ N_{ph}^{\alpha,i} \}, l). \] (C14)

It follows that the matrix elements of Eq. (C12) have the form given in Eq. (51).

Moreover, following our notation for the final ground state when the four \( N_{ph}^{\alpha,i} \) vanish Eq. (C13) leads to

\[ |\langle 0; f | c_{kF,s,\sigma}^\dagger |0; i \rangle|^2 = \lim_{\omega \to 0} a_0^\sigma \omega^{\sigma+1} = \lim_{\omega \to 0} Z_\sigma(\omega) = Z_\sigma = 0, \] (C15)

where \( a_0^\sigma = a^\sigma(\{ N_{ph}^{\alpha,i} = 0 \}, l) \) is a positive real constant and \( Z_\sigma(\omega) \) is the function (49).

Following equation (C11) the function \( Z_\sigma(\omega) \) is given by the leading-order term of expression (46). Since \( a_0^\sigma \) is real and positive expression (44) follows from Eq. (C15).
APPENDIX D: DIVERGENT TWO-ELECTRON AND FINITE TWO-PSEUDOPARTICLE QUANTITIES

In this Appendix we confirm that the finite two-quasiparticle functions (60) of form (62) which are generated from the divergent two-electron vertex functions (59) by the singular electron - quasiparticle transformation (58) control the charge and spin static quantities of the 1D many-electron problem.

On the one hand, the parameters $v^i_\rho$ and $v^i_{\sigma_z}$ of Eq. (59) can be shown to be fully determined by the two-quasiparticle functions (60). By inverting relations (60) with the vertices given by Eq. (59) expressions (61) follow. Physically, the singular electron - quasiparticle transformation (58) maps the divergent two-electron functions onto the finite parameters (60) and (61).

On the other hand, the “velocities” (61) play a relevant role in the charge and spin conservation laws and are simple combinations of the zero-momentum two-pseudoparticle forward-scattering $f$ functions and amplitudes introduced in Refs. [15] and [16], respectively. Here we follow Ref. [20] and use the general parameter $\vartheta$ which refers to $\vartheta = \rho$ for charge and $\vartheta = \sigma_z$ for spin. The interesting quantity associated with the equation of motion for the operator $\hat{\varrho}^{(\pm)}(k, t)$ defined in Ref. [20] is the following ratio

$$\frac{i\partial_k \hat{\varrho}^{(\pm)}(k, t)}{k} |_{k=0} = \left[ \frac{\varrho^{(\pm)}(k, t)}{k} : \hat{\mathcal{H}} : \right] |_{k=0} = v^{\mp 1}_\vartheta \hat{\varrho}^{(\mp)}(0, t),$$

(D1)

where the functions $v^{\pm 1}_\vartheta$ (61) are closely related to two-pseudoparticle forward-scattering quantities as follows

$$v^{\pm 1}_\vartheta = \frac{1}{\sum_{\alpha, \alpha'} k_{\vartheta \alpha} k_{\vartheta \alpha'} \left( v_{\alpha} \delta_{\alpha, \alpha'} - \frac{[A^1_{\alpha \alpha'} + A^{-1}_{\alpha \alpha'}]}{2\pi} \right)}$$

(D2)

and

$$v^{-1}_\vartheta = \sum_{\alpha, \alpha'} k_{\vartheta \alpha} k_{\vartheta \alpha'} \left( v_{\alpha} \delta_{\alpha, \alpha'} + \frac{[f^1_{\alpha \alpha'} - f^{-1}_{\alpha \alpha'}]}{2\pi} \right)$$
\[
= \sum_{\alpha} v_{\alpha} \left( \sum_{\alpha'} k_{\vartheta \alpha \alpha'} \xi_{\alpha \alpha'}^1 \right)^2.
\] (D3)

Here \( k_{\vartheta \alpha} \) are integers given by \( k_{\rho c} = k_{\sigma z c} = 1, k_{\rho s} = 0, \) and \( k_{\sigma z s} = -2, \) and the parameters \( \xi_{\alpha \alpha'}^j \) are defined in Eq. (A7). In the rhs of Eqs. (D2) and (D3) \( v_{\alpha} \) are the \( \alpha \) pseudoparticle group velocities (A6), the \( f \) functions are given in Eq. (A4) and \( A_{\alpha \alpha'}^1 = A_{\alpha \alpha'}(q_{F \alpha}^{(\pm)}, q_{F \alpha'}^{(\pm)}) \) and \( A_{\alpha \alpha'}^{-1} = A_{\alpha \alpha'}(q_{F \alpha}^{(\pm)}, q_{F \alpha'}^{(\mp)}) \), where \( A_{\alpha \alpha'}(q, q') \) are the scattering amplitudes given by Eqs. (83) – (85) of Ref. [16].

The use of relations (61) and of Eqs. (A5), (A6), (A8), (D2), and (D3) shows that the parameters (60) and corresponding charge and spin velocities \( v_{\vartheta}^{\pm 1} \) can also be expressed in terms of the pseudoparticle group velocities (A6) and Landau parameters (A8). These expressions are given in Eq. (62) and in the Table.

The charge and spin velocities control all static quantities of the many-electron system. They determine, for example, the charge and spin susceptibilities,

\[
K'_{\vartheta} = \frac{1}{\pi v_{\vartheta}^{+1}},
\] (D4)

and the coherent part of the charge and spin conductivity spectrum, \( v_{\vartheta}^{-1} \delta(\omega) \), respectively [15,16,20].
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Table I - Alternative expressions of the parameters $v^\ell_\rho$ (D1)-(D4) and $v^\ell_{\sigma_z}$ (D2)-(D5) in terms of the pseudoparticle velocities $v_\alpha$ (A6) and Landau parameters $F^\alpha_{\alpha\alpha'}$ (A8), where $L^0 = (v_c + F^0_{cc})(v_s + F^0_{ss}) - (F^0_{cs})^2$.