Triple Derivations and Triple Homomorphisms of Perfect Lie Superalgebras

Jia Zhou, Liangyun Chen, Yao Ma

1 Information and Computational Science, Jilin Agriculture University, Changchun 130118, China
2 School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

Abstract

In this paper, we study triple derivations and triple homomorphisms of perfect Lie superalgebras over a commutative ring $R$. It is proved that, if the base ring contains $\frac{1}{2}$, $L$ is a perfect Lie superalgebra with zero center, then every triple derivation of $L$ is a derivation, and every triple derivation of the derivation algebra $\text{Der}(L)$ is an inner derivation. Let $L, L'$ be Lie superalgebras over a commutative ring $R$, the notion of triple homomorphism from $L$ to $L'$ is introduced. We proved that, under certain assumptions, homomorphisms, anti-homomorphisms, and sums of homomorphisms and anti-homomorphisms are all triple homomorphisms.

Keywords: Perfect Lie superalgebras; Triple derivations; Triple homomorphisms; Enveloping Lie superalgebras.

1 Introduction

In studies to derivations of associative algebras ([2], [4], [12]) appear naturally different sorts of triple derivations such as associative triple derivations, Jordan triple derivations and Lie triple derivations. Lie triple derivations are interesting not only to studies of associative rings and associative algebras, but also to studies such as that of Lie groups [9]...
and operator algebras ([8], [11], [13]). Triple derivation of Lie algebra is apparently a generalization of derivation, and is an analogy of triple derivation of associative algebra and of Jordan algebra. It was first introduced independently [9] by Müller where it was called prederivation. It can be easily checked that, for any Lie algebra, every derivation is a Lie triple derivation, but the converse does not always hold [14].

The relations of homomorphisms, anti-homomorphisms, Jordan homomorphisms, Lie homomorphisms, and Lie triple homomorphisms became attractive questions. Bresar gave a characterization of Lie triple isomorphisms associated to certain associative algebras [1]. Jacobson and Rickart gave some conditions such that every Jordan homomorphism of a ring is either a homomorphism or an anti-homomorphism [10]. Similar problems arose in the study of operator algebras ([6], [7]). An analogous result was proved for more general perfect Lie algebras [15].

Lie superalgebras are the natural generalization of Lie algebras, and have important applications both in mathematics and physics. Lie superalgebras are also interesting from a purely mathematical point of view. The aim of this article is to generalize some results in [14] and [15] to the triple derivations and triple homomorphisms of Lie superalgebras. Throughout the following sections, $L$ always denote a Lie superalgebra over a commutative ring $R$ with 1. A Lie superalgebra $L$ is called perfect if the derived subalgebra $[L, L] = L$. The center of $L$ is denoted by $Z(L)$. $L$ is called centerless if $Z(L) = 0$. For a subset $S$ of $L$, denote by $C_L(S)$ the centralizer of $S$ in $L$, the enveloping Lie superalgebra of $S$ is by definition the Lie subalgebra of $L$ generated by $S$. A Lie superalgebra is called indecomposable if it cannot be written as a direct sum of two nontrivial ideals. $\text{Der}(L)$ is the derivation algebra of $L$.

Some definitions needed in this paper are as follows.

Definition 1.1. [5] $L = L_0 \oplus L_1$ is a $\mathbb{Z}_2$-graded algebra over a commutative ring $R$ with 1, we call $L$ a Lie superalgebra if the multiplication $[,]$ satisfies the following identities:

1. $[x, y] = -(1)^{|x||y|}[y, x]$; (graded skew-symmetry)
2. $[x, [y, z]] = [[x, y], z] + (1)^{|x||y|}[y, [x, z]]$; (graded Jacobi identity)

where $x, y, z \in \text{hg}(L)$, $\text{hg}(L)$ denotes the set of all $\mathbb{Z}_2$-homogeneous elements of $L$. If $|x|$ occurs in some expression in this paper, we always regard $x$ as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as the $\mathbb{Z}_2$-degree of $x$.

Definition 1.2. An endomorphism of $R$-module $D$ of $L$ is called a triple derivation of $L$, if $\forall x, y, z \in L$, $D$ satisfies

$$D([[x, y], z]) = [[D(x), y], z] + (1)^{|D||x|}[x, D(y)], z] + (1)^{|D||x||y|}[x, y], D(z)]$$

Denote by $T\text{Der}(L)$ the set of all triple derivations of $L$. It is not difficult to show that $T\text{Der}(L)$ is a Lie superalgebra under the usual bracket of endomorphisms of $R$-module (See Lemma 2.1).
Definition 1.3. Let $L$, $L'$ be two Lie superalgebras over $R$. An $R$-linear mapping $f : L \to L'$ is called:

(i) a homomorphism from $L$ to $L'$ if it satisfies $f([x, y]) = [f(x), f(y)]$, $\forall x, y \in L$.

(ii) an anti-homomorphism if it satisfies $f([x, y]) = (-1)^{|x||y|}[f(y), f(x)]$, $\forall x, y \in hg(L)$.

(iii) a triple homomorphism if it satisfies $f([x, [y, z]]) = [f(x), [f(y), f(z)]]$, $\forall x, y, z \in L$.

Definition 1.4. Let $L$, $L'$ be Lie superalgebras. A mapping $g : L \to L'$ is called a direct sum of $g_1$ and $g_2$, if $g = g_1 + g_2$ and there exist ideals $I_1, I_2$ of the enveloping Lie superalgebra of $g(L)$ such that $I_1 \cap I_2 = 0$, and $g_1(L) \subseteq I_1$, $g_2(L) \subseteq I_2$.

The main results of this article are the following two theorems.

Theorem 1.1 Let $L$ be a Lie superalgebra over commutative ring $R$. If $\frac{1}{2} \in R$, $L$ is perfect and has zero center, then we have that:

(1) $T Der(L) = Der(L)$;

(2) $T Der(Der(L)) = ad(Der(L))$.

Theorem 1.2 Suppose that $R$ is a commutative ring with 1, and 2 is invertible in $R$. Let $L$ and $L'$ be Lie superalgebras over $R$, $f$ a triple homomorphism from $L$ to $L'$, and $M$ the enveloping Lie superalgebra of $f(L)$. Assume the following statements:

(1) $L$ is perfect;

(2) $M$ is centerless and can be decomposed into a direct sum of indecomposable ideals.

Then $f$ is either a homomorphism or an anti-homomorphism or a direct sum of a homomorphism and an anti-homomorphism.

2 Triple Derivations of Perfect Lie Superalgebras

We proceed to prove Theorem 1.1 by the follow lemmas.

Lemma 2.1. For any Lie superalgebra $L$, $T Der(L)$ is closed under the usual Lie bracket.

Proof. Let $D_1, D_2 \in T Der(L), x_1, x_2 \in hg(L), x_3 \in L$. By the definition of triple
derivation, we have

\[ D_1 D_2 ([|x_1, x_2], x_3]) \]
\[ = D_1 ([D_2 (x_1), x_2], x_3) + (-1)^{|D_2| |x_1|} [D_1 (x_1), D_2 (x_2), x_3] + (-1)^{|D_2| (|x_1| + |x_2|)} [x_1, D_2 (x_2), x_3] \]
\[ = [D_1 D_2 (x_1), x_2], x_3] + (-1)^{|D_1| |x_1|} [D_2 (x_1), D_1 (x_2), x_3] + (-1)^{|D_1| (|x_1| + |x_2|)} [x_1, D_1 D_2 (x_2), x_3] \]
\[ + (-1)^{|D_2| |x_1|} [D_1 (x_1), D_2 (x_2), x_3] + (-1)^{|D_2| (|x_1| + |x_2|)} [D_1 (x_1), x_2, D_2 (x_2)] + (-1)^{|D_2| (|x_1| + |x_2|)} [D_1 (x_1), x_2, D_2 (x_2)] \]
\[ + (-1)^{|D_2| |x_1|} [D_1 (x_1), D_2 (x_2), x_3] + (-1)^{|D_2| (|x_1| + |x_2|)} [D_1 (x_1), x_2, D_2 (x_2)] \]

and

\[ D_2 D_1 ([|x_1, x_2], x_3]) \]
\[ = D_2 ([D_1 (x_1), x_2], x_3) + (-1)^{|D_1| |x_1|} [D_2 (x_1), D_1 (x_2), x_3] + (-1)^{|D_1| (|x_1| + |x_2|)} [x_1, D_1 (x_2), x_3] \]
\[ = [D_2 D_1 (x_1), x_2], x_3] + (-1)^{|D_1| |x_1|} [D_2 (x_1), D_2 (x_2), x_3] + (-1)^{|D_1| (|x_1| + |x_2|)} [D_1 (x_1), D_2 (x_2), x_3] \]
\[ + (-1)^{|D_2| (|x_1| + |x_2|)} [D_2 (x_1), x_2, D_2 (x_3)] + (-1)^{|D_2| |x_1|} [D_2 (x_1), x_2, D_2 (x_3)] + (-1)^{|D_2| (|x_1| + |x_2|)} [D_2 (x_1), x_2, D_1 (x_3)] \]
\[ + (-1)^{|D_2| |x_1|} [D_2 (x_1), x_2, D_2 (x_3)] \]

Then from easy computation we have

\[ [D_1, D_2] ([|x_1, x_2], x_3)] \]
\[ = D_1 D_2 ([|x_1, x_2], x_3) - (-1)^{|D_1| |D_2|} D_2 D_1 ([|x_1, x_2], x_3)] \]
\[ = [D_1 D_2 (x_1), x_2], x_3] + (-1)^{|x_1| (|D_1| + |D_2|)} [x_1, [D_1, D_2] (x_2)], x_3 \]
\[ + (-1)^{|D_1| + |D_2|} (|x_1| + |x_2|) [x_1, D_2 (x_2), D_1 (x_3)] + (-1)^{|D_1| |x_1| + |x_2|} [D_2 (x_1), x_2, D_1 (x_3)] \]
\[ + (-1)^{|D_1| |x_1|} [D_2 (x_1), x_2, D_1 (x_3)] \]

Hence, \([D_1, D_2] \in TDer(L)\). The lemma is proved. \(\square\)

Clearly, \(ad(L)\), \(Der(L)\) are all subalgebras of \(TDer(L)\). Since \(L\) is perfect, every element \(x \in h\) \((L)\) can be written as a finite sum of Lie brackets, i.e., there exists a finite index set \(I\) such that \(x = \sum_{i \in I} [x_{i_1}, x_{i_2}], x_{i_1}, x_{i_2} \in \mathcal{L}\) in this article, we always put \(\sum\) in place of \(\sum_{i \in I, x_{i_1}, x_{i_2} \in \mathcal{L}}\) for convenience.

Moreover, we have the following lemma.

**Lemma 2.2.** If \(L\) is perfect, then \(ad(L)\) is an ideal of Lie superalgebra \(TDer(L)\).

**Proof.** Let \(D \in TDer(L), x \in h\) \((L)\). \(\forall z \in L\), we have

\[ [D, adx](z) = Dadx(z) - (-1)^{|D||x|} adx(D(z)) \]
\[= D[x, z] - (-1)^{|D||x|}[x, D(z)]\]
\[= D\left(\sum [x_{i1}, x_{i2}], z\right) - (-1)^{|D||x|}\left(\sum [x_{i1}, x_{i2}], D(z)\right)\]
\[= \sum D([x_{i1}, x_{i2}], z) - \sum (-1)^{|D||x|}[[x_{i1}, x_{i2}], D(z)]\]
\[= \sum[[D(x_{i1}), x_{i2}], z] + \sum (-1)^{|D||x_{i1}|}[[x_{i1}, D(x_{i2})], z]\]
\[+ \sum (-1)^{|D||x|}[[x_{i1}, x_{i2}], D(z)] - \sum (-1)^{|D||x|}[[x_{i1}, x_{i2}], D(z)]\]
\[= ad\left(\sum[D(x_{i1}), x_{i2}] + \sum (-1)^{|D||x_{i1}|}[x_{i1}, D(x_{i2})]\right)(z).\]

By the arbitrariness of \(z\), \([D, adx]\) is an inner derivation. Hence, \(ad(L)\) is an ideal of \(TDer(L)\). The lemma holds. \(\square\)

**Lemma 2.3.** If \(L\) is a perfect Lie superalgebra with zero center, then there exists a \(R\)-module homomorphism \(\delta : TDer(L) \to \text{End}(L), \delta(D) = \delta_D\) such that \(\forall x \in L, D \in TDer(L), \text{one has } [D, adx] = ad\delta_D(x)\).

**Proof.** From Lemma 2.2, if \(L\) is perfect and \(L\) has zero center, \(D \in TDer(L)\), we can define a module endomorphism \(\delta_D\) on \(L\) such that for \(\forall x = \sum[x_{i1}, x_{i2}] \in hg(L),\)
\[\delta_D(x) = \sum([D(x_{i1}), x_{i2}] + (-1)^{|D||x_{i1}|}[x_{i1}, D(x_{i2})]).\]
In fact, the definition is independent of the form of expression of \(x\). For proving it, let
\[\alpha = \sum([D(x_{i1}), x_{i2}] + (-1)^{|D||x_{i1}|}[x_{i1}, D(x_{i2})]).\]
If there exists another finite index set \(J\) and \(y_{j_i} \in L\) such that \(x\) can be expressed in the form \(x = \sum_{j \in J} \sum_{j_1, j_2} y_{j_1} y_{j_2}\), we also put \(\sum\) in place of \(\sum_{|y_{j_1}|+|y_{j_2}|=|x|}\) when necessary, let
\[\beta = \sum([D(y_{j1}), y_{j2}] + (-1)^{|D||y_{j1}|}[y_{j1}, D(y_{j2})]).\]
Since \(D \in TDer(L), \forall z \in L,\) we have
\[[\alpha, z] = \sum([[D(x_{i1}), x_{i2}], z] + (-1)^{|D||x_{i1}|}[[x_{i1}, D(x_{i2})], z])\]
\[= \sum([[D(x_{i1}, x_{i2})], z] - (-1)^{|D||x|}[[x_{i1}, x_{i2}], D(z)])\]
\[= D([x, z]) - (-1)^{|D||x|}[x, D(z)]\]
\[= \sum(D([[y_{j1}, y_{j2}], z]) - (-1)^{|D||x|}[[y_{j1}, y_{j2}], D(z)])\]
\[= \sum([[D(y_{j1}), y_{j2}], z] + (-1)^{|D||y_{j1}|}[[y_{j1}, D(y_{j2})], z]) = [\beta, z].\]
Hence, \([\alpha - \beta, z] = 0\). This means that \(\alpha - \beta \in Z(L)\). Since \(Z(L) = 0, \alpha = \beta.\) Hence, \(\delta_D\) is well-defined. The rests of the lemma follow from the proof of Lemma 2.2. \(\square\)
Using the mapping $\delta_D$ and the proof of Lemma 2.2, we have the following lemmas.

**Lemma 2.4.** If $L$ is a perfect Lie superalgebra with zero center, then for $\forall D \in T\text{Der}(L), \delta_D \in \text{Der}(L)$.

**Proof.** Suppose $D \in T\text{Der}(L), x \in h\text{g}(L), y \in L$. Then $[D, \text{ad}([x, y])] = \text{ad}\delta_D([x, y])$.

In other hand,

$$[D, \text{ad}([x, y])] = [D, [\text{ad}x, \text{ad}y]]$$

$$= [[D, \text{ad}x], \text{ad}y] + (-1)^{|D||x|}[\text{ad}x, [D, \text{ad}y]]$$

$$= [\text{ad}\delta_D(x), \text{ad}y] + (-1)^{|D||x|}[\text{ad}x, \text{ad}\delta_D(y)]$$

$$= \text{ad}([\delta_D(x), y] + (-1)^{|D||x|}[x, \delta_D(y)]).$$

Hence, $\text{ad}\delta_D([x, y]) = \text{ad}([\delta_D(x), y] + (-1)^{|D||x|}[x, \delta_D(y)])$. Since $Z(L) = 0$, $\delta_D([x, y] = [\delta_D(x), y] + (-1)^{|D||x|}[x, \delta_D(y)])$. By the arbitrariness of $x, y, \delta_D \in \text{Der}(L)$. \hfill $\square$

**Lemma 2.5.** If the base ring $R$ contains $\frac{1}{2}, L$ is perfect, then the centralizer of $\text{ad}(L)$ in $T\text{Der}(L)$ is trivial, i.e., $C_{T\text{Der}(L)}(\text{ad}(L)) = 0$. In particular, the center of $T\text{Der}(L)$ is zero.

**Proof.** Let $D \in C_{T\text{Der}(L)}(\text{ad}(L))$. Then for $\forall x \in L, [D, \text{ad}x] = 0$. Hence, for $\forall x, y \in h\text{g}(L), D([x, y]) = (-1)^{|D||x|}[x, D(y)] = [D, \text{ad}x](y) = 0$. Thus, $D([x, y]) = [D(x), y] = (-1)^{|D||x|}[x, D(y)]$.

For $x_1, x_2, x_3 \in h\text{g}(L)$, we always have that

$$D([x_1, x_2], x_3) = (-1)^{|D|(|x_1|+|x_2|)}[[x_1, x_2], D(x_3)]$$

$$= (-1)^{|D||x_1|}[[x_1, x_2], D(x_3)] = [[D(x_1), x_2], x_3].$$

Therefore,

$$D([x_1, x_2], x_3)$$

$$=[[D(x_1), x_2], x_3] + (-1)^{|D||x_1|}[[x_1, x_2], D(x_3)] + (-1)^{|D|(|x_1|+|x_2|)}[[x_1, x_2], D(x_3)]$$

$$=3D([[x_1, x_2], x_3]).$$

Hence, $2D([[x_1, x_2], x_3]) = 0$. Because $\frac{1}{2} \in R, D([[x_1, x_2], x_3]) = 0$. Since $L$ is perfect, every element of $L$ can be expressed as the linear combination of elements of the form $[[x_1, x_2], x_3]$, we have that $D = 0$. This completes the proof. \hfill $\square$

The next lemma is well known.

**Lemma 2.6.** [5] For all Lie superalgebra $L$, if $x \in L, D \in \text{Der}(L)$, then $[D, \text{ad}x] = \text{ad}(D(x))$.

Now we can prove the first conclusion of the theorem.
Lemma 2.7. If the base ring $R$ contains $\frac{1}{2}$, $L$ is perfect and has trivial center, then $T\text{Der}(L) = \text{Der}(L)$.

Proof. Suppose $x \in L$, $D \in T\text{Der}(L)$. By Lemma 2.3, $[D, \text{ad}x] = \text{ad}\delta_D(x)$. By Lemma 2.4 and Lemma 2.6, $\text{ad}\delta_D(x) = [\delta_D, \text{ad}x]$. Hence, $D - \delta_D \in C_{T\text{Der}(L)}(\text{ad}(L))$. By Lemma 2.5, $D - \delta_D = 0$, i.e., $D = \delta_D \in \text{Der}(L)$. Hence, $T\text{Der}(L) \subseteq \text{Der}(L)$. The lemma follows from Lemma 2.4.

The remainder of the chapter aims to prove the second conclusion of theorem 1.1.

Lemma 2.8. If $L$ is a perfect Lie superalgebra, $D \in T\text{Der}(T\text{Der}(L))$, then $D(\text{ad}(L)) \subseteq \text{ad}(L)$.

Proof. Since $L$ is perfect, we have

$$D(\text{ad}x) = \sum D(\text{ad}[[x_{i1}, x_{i2}], x_{i3}])$$

$$= \sum D([\text{ad}x_{i1}, \text{ad}x_{i2}], \text{ad}x_{i3})$$

$$= \sum ([D(\text{ad}x_{i1}), \text{ad}x_{i2}], \text{ad}x_{i3} + (-1)^{|D||x_{i1}|}[\text{ad}x_{i1}, D(\text{ad}x_{i2})], \text{ad}x_{i3}$$

$$+ (-1)^{|D||x_{i1}|}[\text{ad}x_{i1}, \text{ad}x_{i2}], D(\text{ad}x_{i3})).$$

Hence, $D(\text{ad}x) \subseteq \text{ad}(L)$. The lemma holds thanks to Lemma 2.2.

Lemma 2.9. If the base ring $R$ contains $\frac{1}{2}$, $L$ is a perfect Lie superalgebra with zero center. $D \in T\text{Der}(T\text{Der}(L))$. If $D(\text{ad}(L)) = 0$, then $D = 0$.

Proof. For $\forall d \in T\text{Der}(L)$, $x \in \text{hg}(L)$, since $L$ is perfect, $x = \sum [x_{i1}, x_{i2}]$. We have that

$$[\text{ad}x, D(d)] = [\sum [\text{ad}x_{i1}, \text{ad}x_{i2}], D(d)]$$

$$= \sum ((-1)^{|D||x|} D([[\text{ad}x_{i1}, \text{ad}x_{i2}], d]) - (-1)^{|D||x|}[D(\text{ad}x_{i1}), \text{ad}x_{i2}], d]$$

$$- (-1)^{|D||x|} (-1)^{|D||x|} [[\text{ad}x_{i1}, D(\text{ad}x_{i2})], d).$$

By Lemma 2.2, $[\text{ad}x, d] \in \text{ad}(L)$, so $D([\text{ad}x, d]) = 0$. Hence, $[\text{ad}x, D(d)] = 0$. Therefore, $D(d) \in C_{T\text{Der}(L)}(\text{ad}(L))$. By Lemma 2.6, $D(d) = 0$. Hence, $D = 0$. The lemma holds.

Lemma 2.10. Let $L$ is a Lie superalgebra over commutative ring $R$. Suppose that $\frac{1}{2} \in R$, $L$ is perfect and has zero center. If $D \in T\text{Der}(\text{Der}(L))$, then there exists $d \in \text{Der}(L)$ such that for $\forall x \in L$, $D(\text{ad}x) = \text{ad}(d(x))$.

Proof. For $\forall d \in T\text{Der}(\text{Der}(L))$, $x \in L$, by Lemma 2.8, $D(\text{ad}x) \subseteq \text{ad}(L)$. Let $y \in L$ and $D(\text{ad}x) = \text{ad}y$. Since the center $Z(L)$ is trivial, such $y$ is unique. Clearly, the map $d : x \rightarrow y$ is a $R$-module endomorphism of $L$. Let $x_1, x_2 \in \text{hg}(L), x_3 \in L$. We have

$$\text{ad}(d([[x_1, x_2], x_3])) = D(\text{ad}([[x_1, x_2], x_3]))$$

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That is to say, $$d_{\mathcal{Z}}$$ since L is a morphism from F to L of characteristic 2. Thus, by Lemma 2 and 3.1.

There exists an L-derivation $$\forall x \in L$$ with $$\delta(x) = \sum_{i \in I} \frac{1}{|x|} [g(x), f(x)]$$.

Since $$Z(L) = 0$$,

$$d([x_1, x_2, x_3]) = [d(x_1), x_2, x_3] + (-1)^{|x_1|} [x_1, d(x_2), x_3] + (-1)^{|x_1|+|x_2|} [x_1, x_2, d(x_3)].$$

That is to say, $$d \in TDer(L)$$. By Lemma 2.7, $$d \in Der(L)$$.

**Proof of Theorem 1.1.** By Lemma 2.7, it remains only to prove the second assertion. By Lemma 2.10, for $$\forall D \in TDer(Der(L))$$, $$x \in L$$, there exists $$d \in Der(L)$$ such that for $$\forall x \in L$$, $$D(adx) = ad(d(x))$$. Thanks to Lemma 2.6, $$ad(d(x)) = [d, adx]$$. Hence,

$$D(adx) = ad(d(x)) = [d, adx] = ad(d)(adx).$$

Thus,

$$(D – ad(d))(adx) = 0.$$

By Lemma 2.9, $$D = ad(d)$$. Therefore, $$TDer(Der(L)) = ad(Der(L))$$. The theorem holds.

**Remark 2.11.** [11] The condition $$\frac{1}{2}$$ is necessary. For example, if the base ring is field $$F$$ of characteristic 2 and L is not abelian, then the identity map is a triple derivation but not a derivation.

3 Triple Homomorphisms of Perfect Lie Superalgebras

Throughout this section, L and $$L'$$ are Lie superalgebras over $$R$$, f is a triple homomorphism from L to $$L'$$, and M is the enveloping Lie superalgebra of f(L). We always assume that L is perfect and that M is centerless and can be decomposed into a direct sum of indecomposable ideals. We proceed to prove the theorem by a series of lemmas.

**Lemma 3.1.** There exists an R-linear mapping $$\delta_f : L \to L'$$ such that for $$\forall x \in hg(L)$$ with $$x = \sum_{|x_{i_1}| + |x_{i_2}| = |x|} [x_{i_1}, x_{i_2}](x_{i_1}, x_{i_2} \in L)$$, $$\delta_f = \sum_{|x_{i_1}| + |x_{i_2}| = |x|} [f(x_{i_1}), f(x_{i_2})].$$
**Proof.** It is sufficient to prove that $\sum [f(x_1), f(x_2)]$ is independent of the expression of $x$. Suppose that $x = \sum [x_{1i}, x_{2i}] = \sum [y_{j1}, y_{j2}]$.

Let
\[
\alpha = \sum [f(x_{1i}), f(x_{2i})], \quad \beta = \sum [f(y_{j1}), f(y_{j2})].
\]

For $\forall z \in L$, we have that
\[
[f(z), \alpha - \beta] = [f(z), \sum [f(x_{1i}), f(x_{2i})] - \sum [f(y_{j1}), f(y_{j2})]] = \sum [f(z), [f(x_{1i}), f(x_{2i})]] - \sum [f(z), [f(y_{j1}), f(y_{j2})]] = f([z, \sum [x_{1i}, x_{2i}]] - [z, \sum [y_{j1}, y_{j2}]]) = f([z, x] - [z, x]) = 0.
\]

It follows $\alpha - \beta \in Z(M)$ and thus $\alpha = \beta$ since $M$ is centerless. This completes the proof.

**Lemma 3.2.** Let $\delta_f$ be the mapping in Lemma 3.1. Then, for $\forall x \in L$, we have that $fad_x = ad_{\delta_f(x)} f$.

**Proof.** Let $x = \sum [x_{1i}, x_{2i}] \in L$. For $\forall z \in L$, we have
\[
fad_x(z) = f([x, z]) = \sum f([x_{1i}, x_{2i}], z)] = \sum [f(x_{1i}), f(x_{2i})] f(z) = [\sum [f(x_{1i}), f(x_{2i})], f(z)] = [\delta_f(x), f(z)] = ad_{\delta_f(x)} f(z).
\]

Thus $fad_x = ad_{\delta_f(x)} f$, and the lemma holds.

**Lemma 3.3.** The mapping $\delta_f$ is a homomorphism of Lie superalgebras.

**Proof.** For $\forall x, y \in hg(L), z \in L$, it follows from Lemmas 3.1 and 3.2 that
\[
[f(x), f(y)] = [\delta_f(x), f(y)], \quad f(z) = [\delta_f(x), f(z)] + (-1)^{|x||y|}[\delta_f(y), f(z)], \quad f(z) = [f(x), f(y)] + (-1)^{|x||y|}[\delta_f(y), f(x)],
\]

The last equality is due to the Jacobi identity. Since $M$ is the enveloping Lie superalgebra of $f(L)$ and $z$ is arbitrary in $L$, $\delta_f([x, y]) = [\delta_f(x), \delta_f(y)] \in Z(M)$. Hence, $\delta_f([x, y]) = [\delta_f(x), \delta_f(y)]$ since $M$ is centerless. Therefore, the lemma follows from the arbitrariness of $x, y \in L$.\qed
Lemma 3.4. Denote $M^+ = \text{Im}(f + \delta_f)$, $M^- = \text{Im}(f - \delta_f)$. Then, $M^+$ and $M^-$ are both ideals of $M$.

Proof. Similar with [15].

Lemma 3.5. $[M^+, M^-] = 0$.

Proof. Take $x, y, z \in hg(L)$, we have that

\[
[[f(x) + \delta_f(x), f(y) - \delta_f(y)], f(z)] = [[f(x), f(y)], f(z)] - [[f(x), \delta_f(y)], f(z)]
\]

\[
+ [[\delta_f(x), f(y)], f(z)] - [[\delta_f(x), \delta_f(y)], f(z)]
\]

\[
= f([[x, y], z]) + (-1)^{|x||y|}[ad_{\delta_f(y)}f(x), f(z)] + [ad_{\delta_f(x)}f(y), f(z)]
\]

\[
- [\delta_f(x), [\delta_f(y), f(z)]] + (-1)^{|x||y|}[\delta_f(y), ad_{\delta_f(x)}f(z)]
\]

\[
= f([[x, y], z]) + (-1)^{|x||y|}[f([y, x]), f(z)] + [f(x, y), f(z)]
\]

\[
- [\delta_f(x), f([y, z])] + (-1)^{|x||y|}(-1)^{|y||z|}[f([x, z]), \delta_f(y)]
\]

\[
= f([[x, y], z]) - [\delta_f(x), f([y, z])] - (-1)^{|y||z|}[f([x, z]), \delta_f(y)]
\]

\[
= f([[x, y], z]) - f([x, [y, z]]) + (-1)^{|y||z|}(-1)^{|y||z|}[f([y, [x, z]])]
\]

\[
= f([[x, y], z]) - [x, [y, z]] + (-1)^{|x||y|}[y, [x, z]] = 0.
\]

Therefore, $[f(x) + \delta_f(x), f(y) - \delta_f(y)] \in Z(M)$. Since $Z(M) = 0$, we have that $[f(x) + \delta_f(x), f(y) - \delta_f(y)] = 0$. The lemma follows.

Lemma 3.6. $M^+ \cap M^- = 0$.

Proof. Similar with [15].

Lemma 3.7. If $M$ can not be decomposed into a direct sum of two nontrivial ideals. Then $f$ is either a homomorphism or an anti-homomorphism of Lie superalgebras.

Proof. For $\forall x \in L$, let $m^+ = \frac{1}{2}(f(x) + \delta_f(x))$, $m^- = \frac{1}{2}(f(x) - \delta_f(x))$. Then $m^+ \in M^+$, $m^- \in M^-$ and $f(x) = m^+ + m^-$. Hence, $f(L) \subseteq M^+ + M^-$. Therefore, $M \subseteq M^+ + M^-$. By Lemma 3.6, $M = M^+ \oplus M^-$. Since $M$ cannot be decomposed into direct sum of two nontrivial ideals, either $M^+$ or $M^-$ must be trivial. If $M^+$ is trivial i.e. $(f + \delta_f)([x, y]) = 0$, then $f([x, y]) = -\delta_f([x, y]) = -[f(x), f(y)] = (-1)^{|x||y|}[f(y), f(x)]$. So $f$ is an anti-homomorphism. If $M^-$ is trivial i.e. $(f - \delta_f)([x, y]) = 0$, then $f([x, y]) = \delta_f([x, y]) = [f(x), f(y)]$. So $f$ is a homomorphism. Hence the lemma follows.
Proof of Theorem 1.2. By Lemma 3.7, it remains to prove the theorem in case $M$ is decomposable. By the assumptions, $M$ can be written as the sum

$$M = M_1 \oplus M_2 \oplus ... \oplus M_s,$$

where each $M_i$ is an indecomposable ideal of $M$. Since $M$ is centerless, each $M_i$ is also centerless (See Lemma 3.1 in [3]). Let $p_i$ be the projection of $M$ into $M_i$. Then, $f = \sum_{i=1}^s p_i f$ and $p_i f$ is a triple homomorphisms from $L$ to $M_i$, and $M_i$ is the enveloping Lie superalgebra of $p_i f(L)$ for $i = 1, 2, ..., s$. Since each $M_i$ is indecomposable, by Lemma 3.7, $p_i f$ is either a homomorphism or an anti-homomorphism from $L$ to $M_i$. Let $P = \{ i \mid p_i f \text{ is a homomorphism} \}$, $Q$ is the complementary set of $P$ in the set $\{1, 2, ..., s\}$. Let $M_1 = \sum_{i \in P} M_i$, $M_2 = \sum_{i \in Q} M_i$. Let $f_1 = \sum_{i \in P} p_i f$, $f_1 = \sum_{i \in Q} p_i f$. It can be checked by direct verification that $M = M_1 \oplus M_2$, $[M_1, M_2] = 0$. $f = f_1 + f_2$, $f_1$ is a homomorphism and $f_2$ is an anti-homomorphism of Lie superalgebras. The theorem is proved.

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