Fast Stochastic Algorithms for SVD and PCA: Convergence Properties and Convexity

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Abstract

We study the convergence properties of the VR-PCA algorithm introduced by [19] for fast computation of leading singular vectors. We prove several new results, including a formal analysis of a block version of the algorithm, and convergence from random initialization. We also make a few observations of independent interest, such as how pre-initializing with just a single exact power iteration can significantly improve the runtime of stochastic methods, and what are the convexity and non-convexity properties of the underlying optimization problem.

1 Introduction

We consider the problem of recovering the top $k$ left singular vectors of a $d \times n$ matrix $X = (x_1, \ldots, x_n)$, where $k \ll d$. This is equivalent to recovering the top $k$ eigenvectors of $XX^\top$, or equivalently, solving the optimization problem

$$
\min_{W \in \mathbb{R}^{d \times k}, \; W^\top W = I} -W^\top \left( \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top \right) W.
$$

(1)

This is one of the most fundamental matrix computation problems, and has numerous uses (such as low-rank matrix approximation and principal component analysis).

For large-scale matrices $X$, where exact eigendecomposition is infeasible, standard deterministic approaches are based on power iterations or variants thereof (e.g. the Lanczos method) [8]. Alternatively, one can exploit the structure of Eq. (1) and apply stochastic iterative algorithms, where in each iteration we update a current $d \times k$ matrix $W$ based on one or more randomly-drawn columns $x_i$ of $X$. Such algorithms have been known for several decades ([14] [17]), and enjoyed renewed interest in recent years, e.g. [2] [4] [3] [10] [6]. Another stochastic approach is based on random projections, e.g. [9] [20].

Unfortunately, each of these algorithms suffer from a different disadvantage: The deterministic algorithms are accurate (runtime logarithmic in the required accuracy $\epsilon$, under an eigengap condition), but require a full pass over the matrix for each iteration, and in the worst-case many such passes would be required (polynomial in the eigengap). On the other hand, each iteration of the stochastic algorithms is cheap, and their number is independent of the size of the matrix, but on the flip side, their noisy stochastic nature means they are not suitable for obtaining a high-accuracy solution (the runtime scales polynomially with $\epsilon$).

Recently, [19] proposed a new practical algorithm, VR-PCA, for solving Eq. (1), which has a “best-of-both-worlds” property: The algorithm is based on cheap stochastic iterations, yet the algorithm’s runtime is logarithmic in the required accuracy $\epsilon$. More precisely, for the case $k = 1$, $x_i$ of bounded norm, and when
there is an eigengap of $\lambda$ between the first and second leading eigenvalues of the covariance matrix $\frac{1}{n}X X^\top$, the required runtime was shown to be on the order of

$$d \left( n + \frac{1}{\lambda^2} \right) \log \left( \frac{1}{\epsilon} \right).$$

(2)

The algorithm is therefore suitable for obtaining high accuracy solutions (the dependence on $\epsilon$ is logarithmic), but essentially at the cost of only $O(\log(1/\epsilon))$ passes over the data. The algorithm is based on a recent variance-reduction technique designed to speed up stochastic algorithms for convex optimization problems ([13]), although the optimization problem in Eq. (1) is inherently non-convex. See Section 3 for a more detailed description of this algorithm, and [19] for more discussions as well as empirical results.

The results and analysis in [19] left several issues open. For example, it is not clear if the quadratic dependence on $1/\lambda$ in Eq. (2) is necessary, since it is worse than the linear (or better) dependence that can be obtained with the deterministic algorithms mentioned earlier, as well as analogous results that can be obtained with similar techniques for convex optimization problems (where $\lambda$ is the strong convexity parameter). Also, the analysis was only shown for the case $k = 1$, whereas often in practice, we may want to recover $k > 1$ singular vectors simultaneously. Although [19] proposed a variant of the algorithm for that case, and studied it empirically, no analysis was provided. Finally, the convergence guarantee assumed that the algorithm is initialized from a point closer to the optimum than what is attained with standard random initialization. Although one can use some other, existing stochastic algorithm to do this “warm-start”, no end-to-end analysis of the algorithm, starting from random initialization, was provided.

In this paper, we study these and related questions, and make the following contributions:

- We propose a variant of VR-PCA to handle the $k > 1$ case, and formally analyze its convergence (Section 3). The extension to $k > 1$ is non-trivial, and requires tracking the evolution of the subspace spanned by the current solution at each iteration.

- In Section 4, we study the convergence of VR-PCA starting from a random initialization. And show that with a slightly smarter initialization – essentially, random initialization followed by a single power iteration – the convergence results can be substantially improved. In fact, a similar initialization scheme should assist in the convergence of other stochastic algorithms for this problem, as long as a single power iteration can be performed.

- In Section 5, we study whether functions similar to Eq. (1) have hidden convexity properties, which would allow applying existing convex optimization tools as-is, and improve the required runtime. For the $k = 1$ case, we show that this is in fact true: Close enough to the optimum, and on a suitably-designed convex set, such a function is indeed $\lambda$-strongly convex. Unfortunately, the distance from the optimum has to be $O(\lambda)$, and this precludes a better runtime in most practical regimes. However, it still indicates that a better runtime and dependence on $\lambda$ should be possible.

2 Some Preliminaries and Notation

We consider a $d \times n$ matrix $X$ composed of $n$ columns $(x_1, \ldots, x_n)$, and let

$$A = \frac{1}{n}X X^\top = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top.$$

Thus, Eq. (1) is equivalent to finding the $k$ leading eigenvectors of $A$. 

We generally use bold-face letters to denote vectors, and capital letters to denote matrices. We let $\text{Tr}(\cdot)$ denote the trace of a matrix, $\| \cdot \|_F$ to denote the Frobenius norm, and $\| \cdot \|_{sp}$ to denote the spectral norm. A symmetric $d \times d$ matrix $B$ is positive semidefinite, if $\inf_{z \in \mathbb{R}^d} z^\top B z \geq 0$. $A$ is positive definite if the inequality is strict. Following standard notation, we write $\| \cdot \|$ to denote the spectral norm.

A twice-differentiable function $F$ on a subset of $\mathbb{R}^d$ is convex, if its Hessian is always positive definite. If it is always positive definite, and $\succ \lambda I$ for some $\lambda > 0$, we say that the function is $\lambda$-strongly convex. If the Hessian is always $\prec s I$ for some $s \geq 0$, then the function is $s$-smooth.

3 The VR-PCA Algorithm and a Block Version

We begin by recalling the algorithm of [19] for the $k = 1$ case (Algorithm 1), and then discuss its generalization for $k > 1$.

**Algorithm 1 VR-PCA: Vector version ($k = 1$)**

1: **Parameters:** Step size $\eta$, epoch length $m$
2: **Input:** Data matrix $X = (x_1, \ldots, x_n)$; Initial unit vector $\mathbf{w}_0$
3: **for** $s = 1, 2, \ldots$ **do**
4: \hspace{1em} $\mathbf{u} = \frac{1}{n} \sum_{i=1}^{n} x_i (x_i^\top \mathbf{w}_{s-1})$
5: \hspace{1em} $\mathbf{w}_0 = \mathbf{w}_{s-1}$
6: \hspace{1em} **for** $t = 1, 2, \ldots, m$ **do**
7: \hspace{2em} Pick $i_t \in \{1, \ldots, n\}$ uniformly at random
8: \hspace{2em} $\mathbf{w}'_t = \mathbf{w}_{t-1} + \eta \left( x_{i_t} (x_{i_t}^\top \mathbf{w}_{t-1} - x_{i_t}^\top \mathbf{w}_{s-1}) + \mathbf{u} \right)$
9: \hspace{2em} $\mathbf{w}_t = \frac{1}{\| \mathbf{w}'_t \|} \mathbf{w}'_t$
10: \hspace{1em} **end for**
11: $\mathbf{w}_s = \mathbf{w}_m$
12: **end for**

The basic idea of the algorithm is to perform stochastic updates using randomly-sampled columns $x_i$ of the matrix, but interlace them with occasional exact power iterations, and use that to gradually reduce the variance of the stochastic updates. Specifically, the algorithm is split into epochs $s = 1, 2, \ldots$, where in each epoch we do a single exact power iteration with respect to the matrix $A$ (by computing $\mathbf{u}$), and then perform $m$ stochastic updates, which can be re-written as

$$\mathbf{w}'_t = (I + \eta A) \mathbf{w}_{t-1} + \eta \left( x_{i_t} x_{i_t}^\top - A \right) (\mathbf{w}_{t-1} - \mathbf{w}_{s-1}) \quad \mathbf{w}_t = \frac{1}{\| \mathbf{w}_t \|} \mathbf{w}_t,$$

The first term is essentially a power iteration (with a finite step size $\eta$), whereas the second term is zero-mean, and with variance dominated by $\| \mathbf{w}_{t-1} - \mathbf{w}_{s-1} \|^2$. As the algorithm progresses, $\mathbf{w}_{t-1}$ and $\mathbf{w}_{s-1}$ both converge toward the same optimal point, hence $\| \mathbf{w}_{t-1} - \mathbf{w}_{s-1} \|^2$ shrinks, eventually leading to an exponential convergence rate.

To handle the $k > 1$ case (where more than one eigenvector should be recovered), one simple technique is deflation, where we recover the leading eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ one-by-one, each time using the $k = 1$ algorithm. However, a disadvantage of this approach is that it requires a positive eigengap between all top $k$ eigenvalues, otherwise the algorithm is not guaranteed to converge. Thus, an algorithm which simultaneously recovers all $k$ leading eigenvectors is preferable.
We will study a block version of Algorithm 1 presented as Algorithm 2. It is mostly a straightforward generalization (similar to how power iterations are generalized to orthogonal iterations), where the \(d\)-dimensional vectors \(w_{t-1}, \tilde{w}_{s-1}, u\) are replaced by \(d \times k\) matrices \(W_{t-1}, \tilde{W}_{s-1}, \tilde{U}\), and normalization is replaced by orthogonalization\(^1\). Indeed, Algorithm 1 is equivalent to Algorithm 2 when \(k = 1\). The main twist in Algorithm 2 is that instead of using \(\tilde{W}_{s-1}, \tilde{U}\) as-is, we perform a unitary transformation (via the \(k \times k\) orthogonal matrix \(B_{t-1}\)) which maximally aligns them with \(W_{t-1}\). Note that \(B_{t-1}\) is a \(k \times k\) matrix, and since \(k\) is assumed to be small, this does not introduce significant computational overhead.

**Algorithm 2 VR-PCA: Block version**

**Parameters:** Rank \(k\), Step size \(\eta\), epoch length \(m\)

**Input:** Data matrix \(X = (x_1, \ldots, x_n)\); Initial \(d \times k\) matrix \(\tilde{W}_0\) with orthonormal columns

for \(s = 1, 2, \ldots\) do

\[
\tilde{U} = \frac{1}{n} \sum_{i=1}^{n} x_i \left( x_i^T \tilde{W}_{s-1} \right)
\]

\(W_0 = \tilde{W}_{s-1}\)

for \(t = 1, 2, \ldots, m\) do

\(B_{t-1} = VU^T\), where \(USV^T\) is an SVD decomposition of \(W_{t-1}^T \tilde{W}_{s-1}\)

\(\triangleright\) Equivalent to \(B_{t-1} = \arg \min_{B^{T}B=I} \| W_{t-1} - \tilde{W}_{s-1} B \|_F^2\)

Pick \(i_t \in \{1, \ldots, n\}\) uniformly at random

\(W_t' = W_{t-1} + \eta \left( x_{i_t} \left( x_{i_t}^T W_{t-1} - x_{i_t}^T \tilde{W}_{s-1} B_{t-1} \right) + \tilde{U} B_{t-1} \right)\)

\(W_t = W_t' \left( W_t'^T W_t' \right)^{-1/2}\)

end for

\(\tilde{W}_s = W_m\)

end for

We now turn to provide a formal analysis of Algorithm 2 which directly generalizes the analysis of Algorithm 1 given in [19]:

**Theorem 1.** Define the \(d \times d\) matrix \(A\) as \(\frac{1}{n} XX^T = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T\), and let \(V_k\) denote the \(d \times k\) matrix composed of the eigenvectors corresponding to the largest \(k\) eigenvalues. Suppose that

- \(\max_i \|x_i\|^2 \leq r\) for some \(r > 0\).
- \(A\) has eigenvalues \(s_1 > s_2 \geq \ldots \geq s_d\), where \(s_k - s_{k+1} = \lambda\) for some \(\lambda > 0\).
- \(k - \|V_k^T \tilde{W}_0\|_F^2 \leq \frac{1}{2}\).

Let \(\delta, \epsilon \in (0, 1)\) be fixed. If we run the algorithm with any epoch length parameter \(m\) and step size \(\eta\), such that

\[
\eta \leq \frac{c\delta^2}{r^2} \lambda , \quad m \geq \frac{c' \log(2/\delta)}{\eta \lambda} , \quad kmn^2 r^2 + rk \sqrt{m n^2 \log(2/\delta)} \leq c''
\]

\(^1\)The normalization \(W_t = W_t' \left( W_t'^T W_t' \right)^{-1/2}\) ensures that \(W_t\) has orthonormal columns. We note that in our analysis, \(\eta\) is chosen sufficiently small so that \(W_t^T W_t'\) is always invertible, hence the operation is well-defined.
Corollary 1. Under the conditions of Theorem 1, there exists an algorithm returning an \( O \) to a high-accuracy approximation of \( V \) probability at least \( 1 \) taking appropriate parameters in an implicit form, but it’s not clear how to implement a similar trick in the block version, where \( k > 1 \).

For any orthogonal \( W \), \( k - \|V_k^T W\|_F^2 \) lies between 0 and \( k \), and equals 0 when the column spaces of \( V_k \) and \( W \) are the same (i.e., when \( W \) spans the \( k \) leading singular vectors). According to the theorem, taking appropriate parameters \( \eta = \Theta(\lambda/(kr)^2) \), and \( m = \Theta((rk/\lambda)^2) \), the algorithm converges with high probability to a high-accuracy approximation of \( V_k \). Moreover, the runtime of each epoch of the algorithm equals \( O(mdk^2 + dnk) \). Overall, we get the following corollary:

**Corollary 1.** Under the conditions of Theorem 1 there exists an algorithm returning \( W_T \) such that \( k - \|V_k^T W_T\|_F^2 \leq \epsilon \) with arbitrary constant accuracy, in runtime \( O \left( dk(n + \frac{r^2k^3}{\lambda^2}) \log(1/\epsilon) \right) \).

This runtime bound is the same as that of \cite{19} for \( k = 1 \).

The proof of Theorem 1 appears in Subsection 6.1 and relies on a careful tracking of the evolution of the potential function \( k - \|V_k^T \tilde{W}_t\|_F^2 \). An important challenge compared to the \( k = 1 \) case is that the matrices \( W_{t-1} \) and \( W_{s-1} \) do not necessarily become closer over time, so the variance-reduction intuition discussed earlier no longer applies. However, the column space of \( W_{t-1} \) and \( \tilde{W}_{s-1} \) do become closer, and this is utilized by introducing the transformation matrix \( B_{t-1} \). We note that although \( B_{t-1} \) appears essential for our analysis, it’s not clear that using it is necessary in practice: In \cite{19}, the suggested block algorithm was Algorithm 2 with \( B_{t-1} = I \), which seemed to work well in experiments. In any case, using this matrix doesn’t affect the overall runtime beyond constants, since the additional runtime of computing and using this matrix (\( O(dk^2) \)) is the same as the other computations performed at each iteration.

A limitation of the theorem above is the assumption that the initial point \( \tilde{W}_0 \) is such that \( k - \|V_k^T \tilde{W}_0\|_F^2 \leq \frac{1}{2} \). This is a non-trivial assumption, since if we initialize the algorithm from a random \( d \times O(1) \) orthogonal matrix \( \tilde{W}_0 \), then with overwhelming probability, \( \|V_k^T \tilde{W}_0\|_F^2 = O(1/d) \). However, experimentally the algorithm seems to work well even with random initialization \cite{19}. Moreover, if we are interested in a theoretical guarantee, one simple solution is to warm-start the algorithm with a purely stochastic algorithm for this problem (such as \cite{6,10,4}), with runtime guarantees on getting such a \( \tilde{W}_0 \). The idea is that \( \tilde{W}_0 \) is only required to approximate \( V_k \) up to constant accuracy, so purely stochastic algorithms (which are good in obtaining a low-accuracy solution) are quite suitable. In the next section, we further delve into these issues, and show that in our setting such algorithms in fact can be substantially improved.

### 4 Warm-Start and the Power of a Power Iteration

In this section, we study the runtime required to compute a starting point satisfying the conditions of Theorem 1 starting from a random initialization. Combined with Theorem 1 this gives us an end-to-end analysis of the runtime required to find an \( \epsilon \)-accurate solution, starting from a random point. For simplicity, we will only discuss the case \( k = 1 \), i.e. where our goal is to compute the single leading eigenvector \( v_1 \), although

\[\text{Specifically, we can take } m = c' \log(2/\delta)/\eta \lambda \text{ and } \eta = a \delta^2 / r^2 \lambda, \text{ where } a \text{ is sufficiently small to ensure that the first and second condition in Eq. 3 holds. It can be verified that it’s enough to take } a = \min \left\{ c, \frac{c''}{2 \tau \log(2/\delta)}, \frac{c''}{2 \tau \log(2/\delta)} \right\}.\]

\[\text{[19] showed that it’s possible to further improve the runtime for sparse } X, \text{ replacing } d \text{ by the average column sparsity } d_s. \text{ This is done by maintaining parameters in an implicit form, but it’s not clear how to implement a similar trick in the block version, where } k > 1.\]
our observations can be generalized to $k > 1$. In the $k = 1$ case, Theorem 1 kicks in once we find a vector $w$ satisfying $\langle v_1, w \rangle^2 \geq \frac{1}{2}$.

As mentioned previously, one way to get such a $w$ is to run a purely stochastic algorithm, which computes the leading eigenvector of a covariance matrix $E[xx^\top]$ given a stream of i.i.d. samples $x$. We can easily use such an algorithm in our setting, by sampling columns from our matrix $X = (x_1, \ldots, x_n)$ uniformly at random, and feed to such a stochastic optimization algorithm, guaranteed to approximate the leading eigenvector of $\frac{1}{n} \sum_{i=1}^n x_i x_i^\top$.

To the best of our knowledge, the existing iteration complexity guarantees for such algorithms (assuming the norm constraint $r \leq 1$ for simplicity) scale at least\footnote{For example, this holds for [6], although the bound only guarantees the existence of some iteration which produces the desired output. The guarantee of [4] scale as $d^2/\lambda^2$, and the guarantee of [10] scales as $d/\lambda^3$ in our setting.} as $d/\lambda^2$. Since the runtime of each iteration is $O(d)$, we get an overall runtime of $O((d/\lambda)^2)$.

The dependence on $d$ in the iteration bound stems from the fact that with a random initial unit vector $w_0$, we have $\langle v_1, w_0 \rangle^2 \approx \frac{1}{2}$. Thus, we begin with a vector almost orthogonal to the leading eigenvector $v_1$ (depending on $d$). In a purely stochastic setting, where only noisy information is available, this necessitates conservative updates at first, and in all the analyses we are aware of, the number of iterations appear to necessarily scale at least linearly with $d$.

However, it turns out that in our setting, with a finite matrix $X$, we can perform a smarter initialization: Sample $w$ from the standard Gaussian distribution on $\mathbb{R}^d$, perform a single power iteration w.r.t. the covariance matrix $A = \frac{1}{n} XX^\top$, i.e. $w_0 = A w / \| A w \|$, and initialize from $w_0$. For such a procedure, we have the following simple observation:

**Lemma 1.** For $w_0$ as above, it holds for any $\delta$ that with probability at least $1 - \frac{1}{d} - \delta$,

$$\langle v_1, w_0 \rangle^2 \geq \frac{\delta^2}{12 \log(d) \text{nrank}(A)},$$

where $\text{nrank}(A) = \frac{\| A \|^2_{\text{sp}}}{\| A \|^2_F}$ is the numerical rank of $A$.

The numerical rank (see e.g. [18]) is a relaxation of the standard notion of rank: For any $d \times d$ matrix $A$, $\text{nrank}(A)$ is at most the rank of $A$ (which in turn is at most $d$). However, it will be small even if $A$ is just close to being low-rank. In many if not most machine learning applications, we are interested in matrices which tend to be approximately low-rank, in which case $\text{nrank}(A)$ is much smaller than $d$ or even a constant. Therefore, by a single power iteration, we get an initial point $w_0$ for which $\langle v_1, w_0 \rangle^2$ is on the order of $1/\text{nrank}(A)$, which can be much larger than the $1/d$ given by a random initialization, and is never substantially worse.

**Proof of Lemma 7** Let $s_1 \geq s_2 \geq \ldots \geq s_d \geq 0$ be the $d$ eigenvalues of $A$, with eigenvectors $v_1, \ldots, v_d$. We have

$$\langle v_1, w_0 \rangle^2 = \frac{\langle v_1, A w \rangle^2}{\| A w \|^2} = \frac{\langle s_1 v_1, w \rangle^2}{\left( \sum_{i=1}^d s_i \langle v_i, w \rangle \right)^2} = \frac{s_1^2 \langle v_1, w \rangle^2}{\sum_{i=1}^d s_i^2 \langle v_i, w \rangle^2}.$$

Since $w$ is distributed according to a standard Gaussian distribution, which is rotationally symmetric, we can assume without loss of generality that $v_1, \ldots, v_d$ correspond to the standard basis vectors $e_1, \ldots, e_d$, in which case the above reduces to

$$\frac{s_1^2 w_1^2}{\sum_{i=1}^d s_i^2 w_i^2} \geq \frac{s_1^2}{\sum_{i=1}^d s_i^2 \max_i w_i^2}.$$
where \( w_1, \ldots, w_d \) are independent and scalar random variables with a standard Gaussian distribution.

First, we note that \( s_1^2 \) equals \( \|A\|_{sp}^2 \), the spectral norm of \( A \), whereas \( \sum_{i=1}^d s_i^2 \) equals \( \|A\|_F^2 \), the Frobenius norm of \( A \). Therefore, \( \frac{s_1^2}{\sum_{i=1}^d s_i^2} = \frac{\|A\|_{sp}^2}{\|A\|_F^2} = \frac{1}{\text{rank}(A)} \), and we get overall that

\[
\langle v_1, w_0 \rangle^2 \geq \frac{1}{\text{rank}(A)} \frac{w_1^2}{\max_i w_i^2}.
\] (4)

We consider the random quantity \( \frac{w_1^2}{\max_i w_i^2} \), and independently bound the deviation probability of the numerator and denominator. First, for any \( t \geq 0 \) we have

\[
\Pr(w_1^2 \leq t) = \Pr(w_1 \in [-\sqrt{t}, \sqrt{t}]) = \int_{z=-\sqrt{t}}^{\sqrt{t}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right) \leq \sqrt{\frac{1}{2\pi} * 2\sqrt{t}} = \sqrt{\frac{2}{\pi} t}.
\] (5)

Second, by combining two standard Gaussian concentration results (namely, that if \( W = \max \{|w_1|, \ldots, |w_d|\} \), then \( 0 \leq \mathbb{E}[W] \leq 2\sqrt{2\log(d)} \), and by the Cirelson-Ibragimov-Sudakov inequality, \( \Pr(W - \mathbb{E}[W] > t) \leq \exp(-t^2/2) \)), we get that

\[
\Pr(\max_i |w_i| > 2\sqrt{2\log(d)} + t) \leq \exp(-t^2/2),
\]

and therefore

\[
\Pr(\max_i w_i^2 > (2\sqrt{2\log(d)} + t)^2) \leq \exp(-t/2).
\] (6)

Combining Eq. (5) and Eq. (6), with a union bound, we get that for any \( t_1, t_2 \geq 0 \), it holds with probability at least \( 1 - \sqrt{2\pi t_1} - \exp(-t_2^2/2) \) that

\[
\frac{w_1^2}{\max_i w_i^2} \geq \frac{t_1}{(2\sqrt{2\log(d)} + t_2)^2}.
\]

To slightly simplify this for readability, we take \( t_2 = \sqrt{2\log(d)} \), and substitute \( \delta = \sqrt{\frac{2}{\pi} t_1} \). This implies that with probability at least \( 1 - \delta - 1/d \),

\[
\frac{w_1^2}{\max_i w_i^2} \geq \frac{\pi^2 \delta^2}{18 \log(d)} > \frac{\delta^2}{12 \log(d)}.
\]

Plugging back into Eq. (4), the result follows.

This result can be plugged into the existing analyses of purely stochastic PCA/SVD algorithms, and can often improve the dependence on the \( d \) factor in the iteration complexity bounds to a dependence on the numerical rank of \( A \). We again emphasize that this is applicable in a situation where we can actually perform a power iteration, and not in a purely stochastic setting where we only have access to an i.i.d. data stream (nevertheless, it would be interesting to explore whether this idea can be utilized in such a streaming setting as well).

To give a concrete example of this, we provide a convergence analysis of the VR-PCA algorithm (Algorithm 1), starting from an arbitrary initial point, bounding the total number of stochastic iterations required by the algorithm in order to produce a point satisfying the conditions of Theorem 1 (from which point the analysis of Theorem 1 takes over). Combined with Theorem 1, this analysis also justifies that VR-PCA indeed converges starting from a random initialization.
Theorem 2. Using the notation of Theorem 1 (where \( \lambda \) is the eigengap, \( v_1 \) is the leading eigenvector, and \( r = \max_i \| x_i \|^2 \)), and for any \( \delta \in (0, \frac{1}{2}) \), suppose we run Algorithm 1 with some initial unit-norm vector \( \tilde{w}_0 \) such that
\[
\langle v_1, \tilde{w}_0 \rangle^2 \geq \zeta > 0,
\]
and a step size \( \eta \) satisfying
\[
\eta \leq \frac{c \delta^2 \lambda \zeta^3}{r^2 \log^2 (2/\delta)}
\]
(for some universal constant \( c \)). Then with probability at least \( 1 - \delta \), after
\[
T = \left\lceil \frac{c' \log (2/\delta)}{\eta \lambda \zeta} \right\rceil
\]
stochastic iterations (lines 6–10 in the pseudocode, where \( c' \) is again a universal constant), we get a point \( w_T \) satisfying \( 1 - \langle v_1, w_T \rangle^2 \leq \frac{1}{2} \). Moreover, if \( \eta \) is chosen on the same order as the upper bound in Eq. (7), then
\[
T = \Theta \left( \frac{r^2 \log^3 (2/\delta)}{\delta^2 \lambda^2 \zeta^4} \right).
\]

Note that the analysis does not depend on the choice of the epoch size \( m \), and does not use the special structure of VR-PCA (in fact, the technique we use is applicable to any algorithm which takes stochastic gradient steps to solve this type of problem\(^5\)). The proof of the theorem appears in Section 6.2.

Considering \( \delta, r \) as a constants, we get that the runtime required by VR-PCA to find a point \( w \) such that \( 1 - \langle v_1, w \rangle^2 \leq \frac{1}{2} \) is \( O(d/\lambda^2 \zeta^4) \) where \( \zeta \) is a lower bound on \( \langle v_1, \tilde{w}_0 \rangle^2 \). As discussed earlier, if \( \tilde{w}_0 \) is a result of random initialization followed by a power iteration (requiring \( O(nd) \) time), and the covariance matrix \( A \) has small numerical rank, then \( \zeta = \langle v_1, \tilde{w}_0 \rangle^2 = \tilde{O}(1/\log(d)) \), and the runtime is
\[
O \left( nd + \frac{d}{\lambda^2} \log^4(d) \right) = O \left( d \left( n + \left( \frac{\log^2(d)}{\lambda} \right)^2 \right) \right).
\]

By Corollary 1, the runtime required by VR-PCA from that point to get an \( \epsilon \)-accurate solution is
\[
O \left( d \left( n + \frac{1}{\lambda^2} \right) \log \left( \frac{1}{\epsilon} \right) \right),
\]
so the sum of the two expressions (which is \( d \left( n + \frac{1}{\lambda^2} \right) \) up to log-factors), represents the total runtime required by the algorithm.

Finally, we note that this bound holds under the reasonable assumption that the numeric rank of \( A \) is constant. If this assumption doesn’t hold, \( \zeta \) can be as large as \( d \), and the resulting bound will have a worse polynomial dependence on \( d \). We suspect that this is due to a looseness in the dependence on \( \zeta = \langle v_1, \tilde{w}_0 \rangle^2 \) in Theorem 2 since better dependencies can be obtained, at least for slightly different algorithmic approaches (e.g. \( 4, 10, 6 \)). We leave a sharpening of the bound w.r.t. \( \zeta \) as an open problem.

\(^5\) Although there exist previous analyses of such algorithms in the literature, they unfortunately do not quite apply to our algorithm, for various technical reasons.
5 Convexity and Non-Convexity of the Rayleigh Quotient

As mentioned in the introduction, an intriguing open question is whether the \( d (n + \frac{1}{\lambda^2}) \log (\frac{1}{\epsilon}) \) runtime guarantees from the previous sections can be further improved. Although a linear dependence on \( d, n \) seems unavoidable, this is not the case for the quadratic dependence on \( 1/\lambda \). Indeed, when using deterministic methods such as power iterations or the Lanczos method, the dependence on \( \lambda \) in the runtime is only \( 1/\lambda \) or even \( \sqrt{1/\lambda} \) \(^\[15\]\). In the world of convex optimization from which our algorithmic techniques are derived, the analog of \( \lambda \) is the strong convexity parameter of the function, and again, it is possible to get a dependence of \( 1/\lambda \) or even \( \sqrt{1/\lambda} \) with accelerated schemes (see e.g. \(^\[13, 16, 7\] in the context of the variance-reduction technique we use). Is it possible to get such a dependence for our problem as well?

Another question is whether the non-convex problem that we are tackling (Eq. (1)) is really that non-convex. Clearly, it has a nice structure (since we can solve the problem in polynomial time), but perhaps it actually has hidden convexity properties, at least close enough to the optimal points? We note that Eq. (1) can be “trivially” convexified, by re-casting it as an equivalent semidefinite program \(^\[5\]. However, that would require optimization over \( d \times d \) matrices, leading to poor runtime and memory requirements. The question here is whether we have any convexity with respect to the original optimization problem over “thin” \( d \times k \) matrices.

In fact, the two questions of improved runtime and convexity are closely related: If we can show that the optimization problem is convex in some domain containing an optimal point, then we may be able to use fast stochastic algorithms designed for convex optimization problems, inheriting their good guarantees.

To discuss these questions, we will focus on the \( k = 1 \) case for simplicity (i.e., our goal is to find a leading eigenvector of the matrix \( A = \frac{1}{n} X X^\top = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top \)), and study potential convexity properties of the negative Rayleigh quotient,

\[
F_A(w) = -\frac{w^\top A w}{\|w\|^2} = \frac{1}{n} \sum_{i=1}^{n} \left( -\frac{(w, x_i)^2}{\|w\|^2} \right).
\]

Note that for \( k = 1 \), this function coincides with Eq. (1) on the unit Euclidean sphere, and with the same optimal points, but has the nice property of being defined on the entire Euclidean space (thus, at least its domain is convex).

At a first glance, such functions \( F_A \) appear to potentially be convex at some bounded distance from an optimum, as illustrated for instance in the case where \( A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) (see Figure 1). Unfortunately, it turns out that the figure is misleading, and in fact the function is not convex almost everywhere:

**Theorem 3.** For the matrix \( A \) above, the Hessian of \( F_A \) is not positive semidefinite for all but a measure-zero set.

**Proof.** The leading eigenvector of \( A \) is \( v_1 = (1, 0) \), and \( F_A(w) = -\frac{w_1^2}{w_1^2 + w_2^2} \). The Hessian of this function at some \( w \) equals

\[
\begin{pmatrix}
\frac{2}{(w_1^2 + w_2^2)^3} & \frac{w_2^2(3w_1^2 - w_2^2)}{w_1^2 + w_2^2} & -2w_1w_2(w_1^2 - w_2^2) \\
-2w_1w_2(w_1^2 - w_2^2) & \frac{w_1^2(3w_2^2 - w_1^2)}{w_1^2 + w_2^2} & w_1^2(w_1^2 - 3w_2^2)
\end{pmatrix}
\].
The function \((w_1, w_2) \mapsto -\frac{w_2^2}{w_1^2 + w_2^2}\), corresponding to \(F_A(w)\) where \(A = (1 \ 0 ; 0 \ 0)\). It is invariant to re-scaling of \(w\), and attains a minimum at \((a, 0)\) for any \(a \neq 0\).

The determinant of this \(2 \times 2\) matrix equals

\[
\frac{4}{(w_1^2 + w_2^2)^6} \left( w_1^2 w_2^2 (3w_1^2 - w_2^2)(w_1^2 - 3w_2^2) - 4w_1^2 w_2^2 (w_1^2 - w_2^2)^2 \right)
\]

\[
= \frac{4w_1^2 w_2^2}{(w_1^2 + w_2^2)^6} \left( (3w_1^2 - w_2^2)(w_1^2 - 3w_2^2) - 4(w_1^2 - w_2^2)^2 \right)
\]

\[
= \frac{4w_1^2 w_2^2}{(w_1^2 + w_2^2)^6} \left( -(w_1^2 + w_2^2)^2 \right)
\]

which is always non-positive, and strictly negative for \(w\) for which \(w_1 w_2 \neq 0\) (which holds for all but a measure-zero set of \(\mathbb{R}^d\)). Since the determinant of a positive semidefinite matrix is always non-negative, this implies that the Hessian isn’t positive semidefinite for any such \(w\).

The theorem implies that we indeed cannot use convex optimization tools as-is on the function \(F_A\), even if we’re close to an optimum. However, the non-convexity was shown for \(F_A\) as a function over the entire Euclidean space, so the result does not preclude the possibility of having convexity on a more constrained, lower-dimensional set. In fact, this is what we are going to do next: We will show that if we are given some point \(w_0\) close enough to an optimum, then we can explicitly construct a simple convex set, such that

- The set includes an optimal point of \(F_A\).
- The function \(F_A\) is \(O(1)\)-smooth and \(\lambda\)-strongly convex in that set.
This means that we can potentially use a two-stage approach: First, we use some existing algorithm (such as VR-PCA) to find \( w_0 \), and then switch to a convex optimization algorithm designed to handle functions with a finite sum structure (such as \( F_A \)). Since the runtime of such algorithms scale better than VR-PCA, in terms of the dependence on \( \lambda \), we can hope for an overall runtime improvement.

Unfortunately, this has a catch: To make it work, we need to have \( w_0 \) very close to the optimum – in fact, we require \( \|v_1 - w_0\| \leq O(\lambda) \), and we show (in Theorem 5) that such a dependence on the eigengap \( \lambda \) cannot be avoided (perhaps up to a small polynomial factor). The issue is that the runtime to get such a \( w_0 \), using stochastic-based approaches we are aware of, would scale at least quadratically with \( 1/\lambda \), but getting dependence better than quadratic was our problem to begin with. For example, the runtime guarantee using VR-PCA to get such a point \( w_0 \) (even if we start from a good point as specified in Theorem 1) is on the order of

\[
d\left(n + \frac{1}{\lambda^2}\right) \log\left(\frac{1}{\lambda}\right),
\]

whereas the best known guarantees on getting an \( \epsilon \)-optimal solution for \( \lambda \)-strongly convex and smooth functions (see [1]) is on the order of

\[
d\left(n + \sqrt{n/\lambda}\right) \log\left(\frac{1}{\epsilon}\right).
\]

Therefore, the total runtime we can hope for would be on the order of

\[
d\left(n + \frac{1}{\lambda^2}\right) \log\left(\frac{1}{\lambda}\right) + \left(n + \sqrt{n/\lambda}\right) \log\left(\frac{1}{\epsilon}\right).
\] (8) 

In comparison, the runtime guarantee of using just VR-PCA to get an \( \epsilon \)-accurate solution is on the order of

\[
d\left(n + \frac{1}{\lambda^2}\right) \log\left(\frac{1}{\epsilon}\right).
\] (9) 

Unfortunately, Eq. (9) is the same as Eq. (8) up to log-factors, and the difference is not significant unless the required accuracy \( \epsilon \) is extremely small (exponentially small in \( n, 1/\lambda \)). Therefore, our construction is mostly of theoretical interest. However, it still shows that asymptotically, as \( \epsilon \to 0 \), it is indeed possible to have runtime scaling better than Eq. (9). This might hint that designing practical algorithms, with better runtime guarantees for our problem, may indeed be possible.

To explain our construction, we need to consider two convex sets: Given a unit vector \( w_0 \), define the hyperplane tangent to \( w_0 \),

\[
H_{w_0} = \{w : \langle w, w_0 \rangle = 1\}
\]
as well as a Euclidean ball of radius \( r \) centered at \( w_0 \):

\[
B_{w_0}(r) = \{w : \|w - w_0\| \leq r\}
\]

The convex set we use, given such a \( w_0 \), is simply the intersection of the two, \( H_{w_0} \cap B_{w_0}(r) \), where \( r \) is a sufficiently small number (see Figure 2).

The following theorem shows that if \( w_0 \) is \( O(\lambda) \)-close to an optimal point (a leading eigenvector \( v_1 \) of \( A \)), and we choose the radius of \( B_{w_0}(r) \) appropriately, then \( H_{w_0} \cap B_{w_0}(r) \) contains an optimal point, and the function \( F_A \) is indeed \( \lambda \)-strongly convex and smooth on that set. For simplicity, we will assume that \( A \) is scaled to have spectral norm of 1, but the result can be easily generalized.
Theorem 4. For any positive semidefinite $A$ with spectral norm 1, eigengap $\lambda$ and a leading eigenvector $v_1$, and any unit vector $w_0$ such that $\|w_0 - v_1\| \leq \frac{\lambda}{44}$, the function $F_A(w)$ is 20-smooth and $\lambda$-strongly convex on the convex set $H_{w_0} \cap B_{w_0}(\frac{\lambda}{22})$, which contains a global optimum of $F_A$.

The proof of the theorem appears in Subsection 6.3. Finally, we show below that a polynomial dependence on the eigengap $\lambda$ is unavoidable, in the sense that the convexity property is lost if $w_0$ is significantly further away from $v_1$.

Theorem 5. For any $\lambda, \epsilon \in (0, \frac{1}{2})$, there exists a positive semidefinite matrix $A$ with spectral norm 1, eigengap $\lambda$, and leading eigenvector $v_1$, as well as a unit vector $w_0$ for which $\|v_1 - w_0\| \leq \sqrt{2(1 + \epsilon)}\lambda$, such that $F_A$ is not convex in any neighborhood of $w_0$ on $H_{w_0}$.

Proof. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which $v_1 = (1, 0, 0)$, and take

$$w_0 = (\sqrt{1 - p^2}, 0, p),$$

where $p = \sqrt{(1 + \epsilon)\lambda}$ (which ensures $\|v_1 - w_0\|^2 = 2p^2 = 2(1 + \epsilon)\lambda$). Consider the ray $\{(\sqrt{1 - p^2}, t, p) : t \geq 0\}$, and note that it starts from $w_0$ and lies in $H_{w_0}$. The function $F_A$ along that ray (considering it as a function of $t$) is of the form

$$-\frac{(1 - p^2) + (1 - \lambda)t^2}{(1 - p^2) + t^2 + p^2} = -\frac{1 - p^2 + (1 - \lambda)t^2}{1 + t^2}.$$

The second derivative with respect to $t$ equals

$$-2\frac{(3t^2 - 1)(\lambda - p^2)}{(t^2 + 1)^3} = 2\frac{(3t^2 - 1)\epsilon\lambda}{(t^2 + 1)^3},$$

where we plugged in the definition of $p$. This is a negative quantity for any $t < \frac{1}{\sqrt{3}}$. Therefore, the function $F_A$ is strictly concave (and not convex) along the ray we have defined and close enough to $w_0$, and therefore isn’t convex in any neighborhood of $w_0$ on $H_{w_0}$.

6 Proofs

6.1 Proof of Theorem 1

Although the proof structure generally mimics the proof of Theorem 1 in [19] for the $k = 1$ special case, it is more intricate and requires several new technical tools. To streamline the presentation of the proof, we begin with proving a series of auxiliary lemmas in Subsection 6.1.1 and then move to the main proof in Subsection 6.1. The main proof itself is divided into several steps, each constituting one or more lemmas.

Throughout the proof, we use the well-known facts that for all matrices $B, C, D$ of suitable dimensions,

$$\text{Tr}(B + C) = \text{Tr}(B) + \text{Tr}(C), \text{Tr}(BC) = \text{Tr}(CB), \text{Tr}(BCD) = \text{Tr}(DBC), \text{Tr}(B^\top B) = \|B\|^2_F.$$

Moreover, since Tr is a linear operation, $E[\text{Tr}(B)] = E[\text{Tr}(B)]$ for a random matrix $B$. 12
6.1.1 Auxiliary Lemmas

**Lemma 2.** For any $B, C, D \succeq 0$, it holds that $\text{Tr}(BC) \geq \text{Tr}(B(C - D))$ and $\text{Tr}(BC) \geq \text{Tr}((B - D)C)$.

*Proof.* It is enough to prove that for any positive semidefinite matrices $E, G$, it holds that $\text{Tr}(EG) \geq 0$. The lemma follows by taking either $E = B, G = D$ (in which case, $\text{Tr}(BC) = \text{Tr}(B(C - D)) + \text{Tr}(BD) \geq \text{Tr}(B(C - D))$), or $E = D, G = C$ (in which case, $\text{Tr}(BC) = \text{Tr}((B - D)C) + \text{Tr}(DC) \geq \text{Tr}((B - D)C)$).

Any positive semidefinite matrix $M$ can be written as the product $M^{1/2}M^{1/2}$ for some symmetric matrix $M^{1/2}$ (known as the matrix square root of $M$). Therefore,

$$\text{Tr}(EG) = \text{Tr}(E^{1/2}E^{1/2}G^{1/2}G^{1/2}) = \text{Tr}(G^{1/2}E^{1/2}E^{1/2}G^{1/2})$$

$$= \text{Tr}((E^{1/2}G^{1/2})^\top(E^{1/2}G^{1/2})) = \|E^{1/2}G^{1/2}\|_F^2 \geq 0.$$ 

\[\square\]

**Lemma 3.** If $B \succeq 0$ and $C > 0$, then

$$\text{Tr}(BC^{-1}) \geq \text{Tr}(B(2I - C)),$$

where $I$ is the identity matrix.

*Proof.* We begin by proving the one-dimensional case, where $B, C$ are scalars $b \geq 0, c > 0$. The inequality then becomes $bc^{-1} \geq b(2 - c)$, which is equivalent to $1 \geq c(2 - c)$, or upon rearranging, $(c - 1)^2 \geq 0$, which trivially holds.

Turning to the general case, we note that by Lemma 2 it is enough to prove that $C^{-1} - (2I - C) \succeq 0$. To prove this, we make a couple of observations. The positive definite matrix $C$ (like any positive definite matrix) has a singular value decomposition which can be written as $USU^\top$, where $U$ is an orthogonal matrix, and $S$ is a diagonal matrix with positive entries. Its inverse is $US^{-1}U^\top$, and $2I - C = 2I - USU^\top = U(2I - S)U^\top$. Therefore,

$$C^{-1} - (2I - C) = US^{-1}U^\top - U(2I - S)U^\top = U(S^{-1} - (2I - S))U^\top.$$ 

To show this matrix is positive semidefinite, it is enough to show that each diagonal entry of $S^{-1} - (2I - S)$ is non-negative. But this reduces to the one-dimensional result we already proved, when $b = 1$ and $c > 0$ is any diagonal entry in $S$. Therefore, $C^{-1} - (2I - C) \succeq 0$, from which the result follows. \[\square\]

**Lemma 4.** For any matrices $B, C$,

$$\text{Tr}(BC) \leq \|B\|_F\|C\|_F$$

and

$$\|BC\|_F \leq \|B\|_{sp}\|C\|_F.$$ 

*Proof.* The first inequality is immediate from Cauchy-Shwartz. As to the second inequality, letting $c_i$ denote the $i$-th column of $C$, and $\| \cdot \|_2$ the Euclidean norm for vectors,

$$\|BC\|_F = \sqrt{\sum_i \|Be_i\|_2^2} \leq \sqrt{\sum_i (\|B\|_{sp}\|c_i\|_2)^2} = \|B\|_{sp}\sqrt{\sum_i \|c_i\|_2^2} = \|B\|_{sp}\|C\|_F.$$

\[\square\]
Lemma 5. Let $B_1, B_2, Z_1, Z_2$ be $k \times k$ square matrices, where $B_1, B_2$ are fixed and $Z_1, Z_2$ are stochastic and zero-mean (i.e. their expectation is the all-zeros matrix). Furthermore, suppose that for some fixed $\alpha, \gamma, \delta > 0$, it holds with probability 1 that

- For all $\nu \in [0, 1]$, $B_2 + \nu Z_2 \succeq \delta I$.
- $\max\{\|Z_1\|_F, \|Z_2\|_F\} \leq \alpha$.
- $\|B_1 + \eta Z_1\|_{sp} \leq \gamma$.

Then

$\mathbb{E}\left[\text{Tr}\left((B_1 + Z_1)(B_2 + Z_2)^{-1}\right)\right] \geq \text{Tr}(B_1 B_2^{-1}) - \frac{\alpha^2(1 + \gamma/\delta)}{\delta^2}$.

Proof. Define the function

$f(\nu) = \text{Tr}\left((B_1 + \nu Z_1)(B_2 + \nu Z_2)^{-1}\right), \quad \nu \in [0, 1].$

Since $B_2 + \nu Z_2$ is positive definite, it is always invertible, hence $f(\nu)$ is indeed well-defined. Moreover, it can be differentiated with respect to $\nu$, and we have

$f'(\nu) = \text{Tr}\left(Z_1(B_2 + \nu Z_2)^{-1} - (B_1 + \nu Z_1)(B_2 + \nu Z_2)^{-1}Z_2(B_2 + \nu Z_2)^{-1}\right).$

Again differentiating with respect to $\nu$, we have

$f''(\nu) = \text{Tr}\left(-2Z_1(B_2 + \nu Z_2)^{-1}Z_2(B_2 + \nu Z_2)^{-1}
+ 2(B_1 + \nu Z_1)(B_2 + \nu Z_2)^{-1}Z_2(B_2 + \nu Z_2)^{-1}Z_2(B_2 + \nu Z_2)^{-1}\right)
= 2 \text{Tr}\left(-Z_1 + (B_1 + \nu Z_1)(B_2 + \nu Z_2)^{-1}Z_2\right)(B_2 + \nu Z_2)^{-1}Z_2(B_2 + \nu Z_2)^{-1}.$

Using Lemma 4 and the triangle inequality, this is at most

$2\| - Z_1 + (B_1 + \nu Z_1)(B_2 + \nu Z_2)^{-1}Z_2\|_F\|(B_2 + \nu Z_2)^{-1}Z_2(B_2 + \nu Z_2)^{-1}\|_F
\leq 2 \left(\|Z_1\|_F + \|(B_1 + \nu Z_1)(B_2 + \nu Z_2)^{-1}Z_2\|_F\right)\|(B_2 + \nu Z_2)^{-1}\|_{sp}\|Z_2\|_F.
\leq 2 \left(\|Z_1\|_F + \|B_1 + \nu Z_1\|_{sp}\|(B_2 + \nu Z_2)^{-1}\|_{sp}\|Z_2\|_F\right)\|(B_2 + \nu Z_2)^{-1}\|_{sp}\|Z_2\|_F.
\leq 2 \left(\alpha + \gamma/\delta\right)\|\eta\|_{sp} \alpha = 2 \frac{\alpha^2(1 + \gamma/\delta)}{\delta^2}.$

Applying a Taylor expansion to $f(\cdot)$ around $\nu = 0$, with a Lagrangian remainder term, and substituting the values for $f'(\nu), f''(\nu)$, we can lower bound $f(1)$ as follows:

$f(1) \geq f(0) + f'(0) \ast (1 - 0) - \frac{1}{2} \max_{\nu} |f''(\nu)| \ast (1 - 0)^2
= \text{Tr}\left(B_1 B_2^{-1}\right) + \text{Tr}\left(Z_1 B_2^{-1} - B_1 B_2^{-1}Z_2 B_2^{-1}\right) - \frac{\alpha^2(1 + \gamma/\delta)}{\delta^2}.$

Taking expectation over $Z_1, Z_2$, and recalling they are zero-mean, we get that

$\mathbb{E}[f(1)] \geq \text{Tr}\left(B_1 B_2^{-1}\right) - \frac{\alpha^2(1 + \gamma/\delta)}{\delta^2}.$

Since $\mathbb{E}[f(1)] = \mathbb{E}\left[\text{Tr}\left((B_1 + Z_1)(B_2 + Z_2)^{-1}\right)\right]$, the result in the lemma follows. □
Lemma 6. Let $U_1, \ldots, U_k$ and $R_1, R_2$ be positive semidefinite matrices, such that $R_2 - R_1 \succeq 0$, and define the function

$$f(x_1 \ldots x_k) = \text{Tr} \left( \left( \sum_{i=1}^{k} x_i U_i + R_1 \right) \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} \right).$$

over all $(x_1 \ldots x_k) \in [\alpha, \beta]^d$ for some $\beta \geq \alpha \geq 0$. Then $\min_{(x_1 \ldots x_k) \in [\alpha, \beta]^d} f(x) = f(\alpha, \ldots, \alpha)$.

Proof. Taking a partial derivative of $f$ with respect to some $x_j$, we have

$$\frac{\partial}{\partial x_j} f(x)$$

$$= \text{Tr} \left( U_j \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} - \left( \sum_{i=1}^{k} x_i U_i + R_1 \right) \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} U_j \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} \right)$$

$$= \text{Tr} \left( \left( I - \left( \sum_{i=1}^{k} U_i + R_1 \right) \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} \right) U_j \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} \right)$$

$$= \text{Tr} \left( \left( \sum_{i=1}^{k} x_i U_i + R_2 \right) - \left( \sum_{i=1}^{k} x_i U_i + R_1 \right) \right) \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} U_j \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1}$$

$$= \text{Tr} \left( (R_2 - R_1) \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1} U_j \left( \sum_{i=1}^{k} x_i U_i + R_2 \right)^{-1}.$$  

By the lemma’s assumptions, each matrix in the product above is positive semidefinite, hence the product is positive semidefinite, and the trace is non-negative. Therefore, $\frac{\partial}{\partial x_j} f(x) \geq 0$, which implies that the function is minimized when each $x_j$ takes its smallest possible value, i.e. $\alpha$. \qed

Lemma 7. Let $B$ be a $k \times k$ matrix with minimal singular value $\delta$. Then

$$1 - \frac{\|B^T B\|_F^2}{\|B\|_F^2} \geq \max \left\{ 1 - \frac{\|B\|_F^2}{\|B\|_F^2}, \frac{\delta^2}{k} \left( k - \frac{\|B\|_F^2}{\|B\|_F^2} \right) \right\}.$$

Proof. We have

$$1 - \frac{\|B^T B\|_F^2}{\|B\|_F^2} \geq 1 - \frac{\|B\|_F^2}{\|B\|_F^2} = 1 - \frac{\|B\|_F^2}{\|B\|_F^2},$$

so it remains to prove $1 - \frac{\|B^T B\|_F^2}{\|B\|_F^2} \geq \frac{\delta^2}{k} \left( k - \frac{\|B\|_F^2}{\|B\|_F^2} \right)$. Let $\sigma_1, \ldots, \sigma_k$ denote the vector of singular values of $B$. The singular values of $B^T B$ are $\sigma_1^2, \ldots, \sigma_k^2$, and the Frobenius norm of a matrix equals the Euclidean norm of its vector of singular values. Therefore, the lemma is equivalent to requiring

$$1 - \frac{\sum_{i=1}^{k} \sigma_i^4}{\sum_{i=1}^{k} \sigma_i^2} \geq \frac{\delta^2}{k} \left( k - \sum_{i=1}^{k} \sigma_i^2 \right),$$

assuming $\sigma_i \in [\delta, 1]$ for all $i$. This holds since

$$1 - \frac{\sum_{i=1}^{k} \sigma_i^4}{\sum_{i=1}^{k} \sigma_i^2} = \frac{\sum_{i} \sigma_i^2 - \sum_{i} \sigma_i^4}{\sum_{i} \sigma_i^2} = \frac{\sum_{i} \sigma_i^2 (1 - \sigma_i^2)}{\sum_{i} \sigma_i^2} \geq \frac{\delta^2 \sum_{i} (1 - \sigma_i^2)}{k} = \frac{\delta^2}{k} \left( k - \sum_{i} \sigma_i^2 \right).$$  

\qed
Lemma 8. For any $d \times k$ matrices $C, D$ with orthonormal columns, let

$$D_C = \arg \min_{DB : (DB)^\top (DB) = I} \|C - DB\|^2_F$$

be the nearest orthonormal-columns matrix to $C$ in the column space of $D$ (where $B$ is a $k \times k$ matrix). Then the matrix $B$ minimizing the above equals $B = VU^\top$, where $C^\top D = USV^\top$ is the SVD decomposition of $C^\top D$, and it holds that

$$\|C - D_C\|^2_F \leq 2(k - \|C^\top D\|^2_F).$$

Proof. Since $D$ has orthonormal columns, we have $D^\top D = I$, so the definition of $B$ is equivalent to

$$B = \arg \min_{B : B^\top B = I} \|C - DB\|^2_F.$$

This is the orthogonal Procrustes problem (see e.g. [8]), and the solution is easily shown to be $B = VU^\top$ where $USV^\top$ is the SVD decomposition of $C^\top D$. In this case, and using the fact that $\|C\|^2_F = \|D\|^2_F = k$ (as $C, D$ have orthonormal columns), we have that $\|C - D_C\|^2_F$ equals

$$\|C - DB\|^2_F = \|C\|^2_F + \|D\|^2_F - 2\text{Tr}(C^\top DB) = 2\left(k - \text{Tr}(USV^\top (VU^\top))\right) = 2\left(k - \text{Tr}(USU^\top)\right).$$

Since the trace function is similarity-invariant, this equals $2k - \text{Tr}(S)$. Let $s_1, \ldots, s_k$ be the diagonal elements of $S$, and note that they can be at most 1 (since they are the singular values of $C^\top D$, and both $C$ and $D$ have orthonormal columns). Recalling that the Frobenius norm equals the Euclidean norm of the singular values, we can therefore upper bound the above as follows:

$$2\left(k - \text{Tr}(USU^\top)\right) = 2\left(k - \sum_{i=1}^{k} s_i\right) \leq 2\left(k - \sum_{i=1}^{k} s_i^2\right) = 2\left(k - \|C^\top D\|^2_F\right).$$

\hfill \Box

Lemma 9. Let $W_t, W'_t$ be as defined in Algorithm 2 where we assume $\eta < \frac{1}{3}$. Then for any $d \times k$ matrix $V_k$ with orthonormal columns, it holds that

$$\left|\|V_k^\top W_t\|^2_F - \|V_k^\top W_{t-1}\|^2_F\right| \leq \frac{12k\eta}{1 - 3\eta}.$$

Proof. Letting $s_t, s_{t-1}$ denote the vectors of singular values of $V_k^\top W_t$ and $V_k^\top W_{t-1}$, and noting that they are both in $[0, 1]^k$ (as $V_k, W_{t-1}, W_t$ all have orthonormal columns), the left hand side of the inequality in the lemma statement equals

$$\|s_t\|^2 - \|s_{t-1}\|^2 = (\|s_t\|_2 + \|s_{t-1}\|_2) \|s_t - s_{t-1}\|_2 \leq 2\sqrt{k}\|s_t - s_{t-1}\|_2 \leq 2k\|s_t - s_{t-1}\|_\infty,$$

where $\| \cdot \|_\infty$ is the infinity norm. By Weyl’s matrix perturbation theorem\footnote{Using its version for singular values, which implies that the singular values of matrices $B$ and $B + E$ are different by at most $\|E\|_sp$.}, this is upper bounded by

$$2k\|V_k^\top W_t - V_k^\top W_{t-1}\|_{sp} \leq 2k\|V_k\|_{sp}\|W_t - W_{t-1}\|_{sp} \leq 2k\|W_t - W_{t-1}\|_{sp}. \quad (10)$$
Recalling the relationship between \(W_t\) and \(W_{t-1}\) from Algorithm\(\textsuperscript{2}\), we have that
\[
W'_t = W_{t-1} + \eta N,
\]
where
\[
\|N\|_{sp} \leq \|x_i x_i^\top W_{t-1}\|_{sp} + \|x_i x_i^\top \tilde{W}_{s-1} B_{t-1}\|_{sp} + \|\frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top \tilde{W}_{s-1} B_{t-1}\|_{sp} \leq 3,
\]
as \(W_{t-1}, \tilde{W}_{s-1}, B_{t-1}\) all have orthonormal columns, and \(x_i, x_i^\top\) and \(\frac{1}{n} \sum_{i=1}^{n} x_i x_i^\top\) have spectral norm at most 1. Therefore, \(W'_t\) equals \(W_{t-1}\), up to a matrix perturbation of spectral norm at most \(3\eta\). Again by Weyl’s theorem, this implies that the \(k\) non-zero singular values of the \(d \times k\) matrix \(W'_t\) are different from those of \(W_{t-1}\) (which has orthonormal columns) by at most \(3\eta\), and hence all lie in \([1 - 3\eta, 1 + 3\eta]\). As a result, the singular values of \((W'_t W'_t)^{-1/2}\) all lie in \([\frac{1}{1+3\eta}, \frac{1}{1-3\eta}]\). Collecting these observations, we have
\[
\|W_t - W_{t-1}\|_{sp} = \|(W_{t-1} + \eta N) (W'_{t-1} W'_{t-1})^{-1/2} - W_{t-1}\|_{sp}
\leq \|W_{t-1} (W'_t W'_t)^{-1/2} - I\|_{sp} + \|N\|_{sp} \|W'_t W'_t\|_{sp}^{-1/2}
\leq \frac{3\eta}{1 - 3\eta} + \frac{3\eta}{1 - 3\eta} = \frac{6\eta}{1 - 3\eta}.
\]
Plugging back to Eq. (10), the result follows. \(\Box\)

6.1.2 Main Proof

To simplify the technical derivations, note that the algorithm remains the same if we divide each \(x_i\) by \(\sqrt{r}\), and multiply \(\eta\) by \(r\). Since \(\max_i \|x_i\|^2 \leq r\), this corresponds to running the algorithm with step-size \(\eta r\) rather than \(\eta\), on a re-scaled dataset of points with squared norm at most 1, and with an eigengap of \(\lambda/r\) instead of \(\lambda\). Therefore, we can simply analyze the algorithm assuming that \(\max_i \|x_i\|^2 \leq 1\), and in the end plug in \(\lambda/r\) instead of \(\lambda\), and \(\eta r\) instead of \(\eta\), to get a result which holds for data with squared norm at most \(r\).

Part I: Establishing a Stochastic Recurrence Relation

We begin by focusing on a single iteration \(t\) of the algorithm, and analyze how \(\|V_k^T W_t\|_F^2\) (which measures the similarity between the column spaces of \(V_k\) and \(W_t\)) evolves during that iteration. The key result we need is Lemma\(\textsuperscript{10}\) below, which is specialized for our algorithm in Lemma\(\textsuperscript{11}\).

Lemma 10. Let \(A\) be a \(d \times d\) symmetric matrix with all eigenvalues \(s_1 \geq s_2 \geq \ldots \geq s_d\) in \([0, 1]\), and suppose that \(s_k - s_{k+1} \geq \lambda\) for some \(\lambda > 0\).

Let \(N\) be a \(d \times k\) zero-mean random matrix such that \(\|N\|_F \leq \sigma_N^F\) and \(\|N\|_{sp} \leq \sigma_N^{sp}\) with probability 1, and define
\[
r_N = 46 (\sigma_N^F)^2 \left(1 + \frac{8}{3} \left(\frac{1}{4}\sigma_N^{sp} + 2\right)^2\right).
\]

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Let $W$ be a $d \times k$ matrix with orthonormal columns, and define

$$W' = (I + \eta A)W + \eta N, \quad W'' = (W'^\top W'^\top)^{-1/2},$$

for some $\eta \in \left[0, \frac{1}{\max\{1, \sigma_k^2\}}\right]$.

If $V_k = [v_1, v_2, \ldots, v_k]$ is the $d \times k$ matrix of $A$’s first $k$ eigenvectors, then the following holds:

- $\mathbb{E} \left[1 - \|V_k^\top W''\|_F^2\right] \leq \left(1 - \frac{4}{5} \eta \lambda \|V_k^\top W\|_F^2\right) \left(1 - \|V_k^\top W\|_F^2\right) + \eta^2 r_N$

- If $\|V_k^\top W\|_F^2 \geq k - \frac{1}{2}$, then

$$\mathbb{E}_N \left[k - \|V_k^\top W''\|_F^2\right] \leq \left(k - \|V_k^\top W\|_F^2\right) \left(1 - \frac{1}{10} \eta \lambda\right) + \eta^2 r_N.$$

**Proof.** Using the fact that $\text{Tr}(BCD) = \text{Tr}(CDB)$ for any matrices $B, C, D$, we have

$$\mathbb{E} \left[\|V_k^\top W''\|_F^2\right] = \mathbb{E} \left[\text{Tr} \left(W''^\top V_k^\top V_k^\top W''\right)\right]
= \mathbb{E} \left[\text{Tr} \left((W'^\top W')^{-1/2} W'^\top V_k^\top V_k^\top W' (W'^\top W')^{-1/2}\right)\right]
= \mathbb{E} \left[\text{Tr} \left((W'^\top V_k^\top V_k^\top W' (W'^\top W')^{-1}\right)\right].$$

By definition of $W'$, we have

$$W'^\top V_k^\top V_k^\top W' = ((I + \eta A)W + \eta N)^\top V_k^\top ((I + \eta A)W + \eta N)
= B_1 + Z_1,$$

where we define

$$B_1 = W^\top (I + \eta A)V_k^\top V_k^\top (I + \eta A)W + \eta^2 N^\top V_k^\top V_k^\top N
Z_1 = \eta N^\top V_k^\top V_k^\top (I + \eta A)W + \eta W^\top (I + \eta A)V_k^\top V_k^\top N.$$

Also, we have

$$W'^\top W' = ((I + \eta A)W + \eta N)^\top ((I + \eta A)W + \eta N)
= B_2 + Z_2,$$

where

$$B_2 = W^\top (I + \eta A)(I + \eta A)W + \eta^2 N^\top N
Z_2 = \eta N^\top (I + \eta A)W + \eta W^\top (I + \eta A)N.$$

With these definitions, we can rewrite Eq. (11) as $\mathbb{E} \left[\text{Tr}((B_1 + Z_1)(B_2 + Z_2)^{-1})\right]$. We now wish to remove $Z_1, Z_2$, by applying Lemma 5. To do so, we check the lemma’s conditions:

- $Z_1, Z_2$ are zero mean: This holds since they are linear in $N$, and $N$ is assumed to be zero-mean.
• $B_2 + \nu Z_2 \succeq \frac{3}{2}I$ for all $\nu \in [0, 1]$: Recalling the definition of $B_2, Z_2$, and the facts that $A \succeq 0, N^\top N \succeq 0$ (by construction), and $W^\top W = I$, we have that $B_2 \succeq I$. Moreover, the spectral norm of $Z_2$ is at most

$$2\eta\|N^\top (I + \eta A)W\|_{sp} \leq 2\eta\|N\|_{sp}\|I + \eta A\|_{sp}\|W\|_{sp} \leq 2\eta\sigma^s_N (1 + \eta) \leq 2\eta\sigma^F_N (1 + \eta),$$

which by the assumption on $\eta$ is at most $2\frac{1}{4} (1 + \frac{1}{4}) = \frac{5}{8}$. This implies that the smallest singular value of $B_2 + \nu Z_2$ is at least $1 - \nu(5/8) \geq 3/8$.

• $\max\{\|Z_1\|_F, \|Z_2\|_F\} \leq \frac{5}{2}\eta\sigma^F_N$: By definition of $Z_1, Z_2$, and using Lemma 4, the Frobenius norm of these two matrices is at most

$$2\eta\|N\|_F\|(I + \eta A)\|_{sp}\|W\|_{sp} \leq 2\eta\sigma^F_N (1 + \eta),$$

which by the assumption on $\eta$ is at most $2\eta\sigma^F_N (1 + \frac{1}{4}) = \frac{5}{2}\eta\sigma^F_N$.

• $\|B_1 + \eta Z_1\|_{sp} \leq \left(\frac{1}{3}\sigma^s_N + 2\right)^2$: Using the definition of $B_1, Z_1$ and the assumption $\eta \leq \frac{1}{4}$,

$$\|B_1 + \eta Z_1\|_{sp} \leq \|B_1\|_{sp} + \eta\|Z_1\|_{sp} \leq (1 + \eta)^2 + \eta^2(\sigma^s_N)^2 + 2\eta\sigma^s_N (1 + \eta) \leq \left(\frac{5}{4}\right)^2 + \frac{1}{16}(\sigma^s_N)^2 + \frac{5}{8}\sigma^s_N \leq \left(\frac{1}{4}\sigma^s_N + 2\right)^2.$$

Applying Lemma 5 and plugging back to Eq. (11), we get

$$\mathbb{E} \left[\|V_k^\top W\|_F^2\right] \geq \mathbb{E} \left[\text{Tr}((B_1 + Z_1)(B_2 + Z_2)^{-1})\right] \geq \text{Tr} \left(B_1B_2^{-1} - \frac{400}{9}(\eta\sigma^F_N)^2 \left(1 + \frac{8}{3}\left(\frac{1}{4}\sigma^s_N + 2\right)^2\right)\right).$$

(12)

We now turn to lower bound $\text{Tr} \left(B_1B_2^{-1}\right)$, by first re-writing $B_1, B_2$ in a different form. For $i = 1, \ldots, d$, let

$$U_i = W^\top v_i v_i^\top W,$$

where $v_i$ is the eigenvector of $A$ corresponding to the eigenvalue $s_i$. Note that each $U_i$ is positive semidefinite, and $\sum_{i=1}^d U_i = W^\top W = I$. We have

$$B_1 = W^\top (I + \eta A)V_k V_k^\top (I + \eta A)W + \eta^2 N^\top V_k V_k^\top N = W^\top ((I + \eta A)V_k) ((I + \eta A)V_k)^\top W + \eta^2 N^\top V_k V_k^\top N \leq \sum_{i=1}^k (1 + \eta s_i)^2 W^\top v_i v_i^\top W + \eta^2 N^\top V_k V_k^\top N \leq \sum_{i=1}^k (1 + \eta s_i)^2 U_i + \eta^2 N^\top V_k V_k^\top N.$$  

(13)
Similarly,
\[ B_2 = W^\top (I + \eta A)(I + \eta A)W + \eta^2 N^\top N \]
\[ = \sum_{i=1}^d (1 + \eta s_i)^2 W^\top v_i v_i^\top W + \eta^2 N^\top N \]
\[ = \sum_{i=1}^d (1 + \eta s_i)^2 U_i + \eta^2 N^\top N. \] (14)

Plugging Eq. (13) and Eq. (14) back into Eq. (12), we get
\[ E \left[ \|V_k^T W''\|_F^2 \right] \geq \text{Tr} \left( \left( \sum_{i=1}^k (1 + \eta s_i)^2 U_i + \eta^2 N^\top V_k V_k^\top N \right) \left( \sum_{i=1}^d (1 + \eta s_i)^2 U_i + \eta^2 N^\top N \right)^{-1} \right) \]
\[ - \frac{400}{9} (\eta \sigma_F^N)^2 \left( 1 + \frac{8}{3} \left( \frac{1}{4} \sigma^p N + 2 \right)^2 \right). \] (15)

Recalling that \( s_1 \geq s_2 \geq \ldots \geq s_k \) and letting \( \alpha = (1 + \eta s_k)^2, \beta = (1 + \eta s_1)^2 \), the trace term can be lower bounded by
\[ \min_{x_1, \ldots, x_k \in [\alpha, \beta]} \text{Tr} \left( \left( \sum_{i=1}^k x_i U_i + \eta^2 N^\top V_k V_k^\top N \right) \left( \sum_{i=1}^k x_i U_i + \sum_{i=k+1}^d (1 + \eta s_i)^2 U_i + \eta^2 N^\top N \right)^{-1} \right). \]

Applying Lemma 6 (noting that as required by the lemma, \( \sum_{i=k+1}^d (1 + \eta s_i)^2 U_i + \eta^2 N^\top (I - V_k V_k^\top) N \geq 0 \)), we can lower bound the above by
\[ \text{Tr} \left( \left( 1 + \eta s_k \right)^2 \sum_{i=1}^k U_i + \eta^2 N^\top V_k V_k^\top N \right) \left( 1 + \eta s_k \right)^2 \sum_{i=1}^k U_i + \sum_{i=k+1}^d (1 + \eta s_i)^2 U_i + \eta^2 N^\top N \right)^{-1} \].

Using Lemma 2 this can be lower bounded by
\[ \text{Tr} \left( \left( 1 + \eta s_k \right)^2 \sum_{i=1}^k U_i \right) \left( 1 + \eta s_k \right)^2 \sum_{i=1}^k U_i + \sum_{i=k+1}^d (1 + \eta s_i)^2 U_i + \eta^2 N^\top N \right)^{-1} \]
\[ = \text{Tr} \left( \sum_{i=1}^k U_i \right) \left( \sum_{i=1}^k U_i + \sum_{i=k+1}^d \left( 1 + \frac{\eta s_i}{1 + \eta s_k} \right)^2 U_i + \left( \frac{\eta}{1 + \eta s_k} \right)^2 N^\top N \right)^{-1} \]

Applying Lemma 3 this is at least
\[ \text{Tr} \left( \sum_{i=1}^k U_i \right) \left( 2I - \sum_{i=1}^k U_i - \sum_{i=k+1}^d \left( 1 + \frac{\eta s_i}{1 + \eta s_k} \right)^2 U_i - \left( \frac{\eta}{1 + \eta s_k} \right)^2 N^\top N \right). \]

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Recalling that \( I = \sum_{i=1}^{d} U_i = \sum_{i=1}^{k} U_i + \sum_{i=k+1}^{d} U_i \), this can be simplified to

\[
\text{Tr} \left( \left( \sum_{i=1}^{k} U_i \right) \left( \sum_{i=1}^{k} U_i + \sum_{i=k+1}^{d} \left( 2 - \frac{1 + \eta s_i}{1 + \eta s_k} \right)^2 U_i - \left( \frac{\eta}{1 + \eta s_k} \right)^2 N^\top N \right) \right). \tag{16}
\]

Since \( U_i \succeq 0 \), then using Lemma 3, we can lower bound the expression above by shrinking each of the \( 2 - \left( \frac{1 + \eta s_i}{1 + \eta s_k} \right)^2 \) terms. In particular, since \( s_i \leq s_k - \lambda \) for each \( i \geq k + 1 \),

\[
2 - \left( \frac{1 + \eta s_i}{1 + \eta s_k} \right)^2 \geq 2 - \frac{1 + \eta (s_k - \lambda)}{1 + \eta s_k} = 1 + \frac{\eta \lambda}{1 + \eta s_k},
\]

which by the assumption that \( \eta \leq 1/4 \) and \( s_k \leq s_1 \leq 1 \), is at least \( 1 + \frac{\eta \lambda}{5} \). Plugging this into Eq. (16), and recalling that \( \sum_{i=1}^{d} U_i = I \), we get the lower bound

\[
\text{Tr} \left( \left( \sum_{i=1}^{k} U_i \right) \left( \sum_{i=1}^{k} U_i + \sum_{i=k+1}^{d} \left( 1 + \frac{4}{5} \eta \lambda \right) \left( I - \sum_{i=1}^{k} U_i \right) - \left( \frac{\eta}{1 + \eta s_k} \right)^2 N^\top N \right) \right)\]

Again using Lemma 2 this is at least

\[
\text{Tr} \left( \left( \sum_{i=1}^{k} U_i \right) \left( I + \frac{4}{5} \eta \lambda \left( I - \sum_{i=1}^{k} U_i \right) \right) - \left( \frac{\eta}{1 + \eta s_k} \right)^2 \text{Tr} \left( \sum_{i=1}^{k} U_i \right) N^\top N \right)
\]

Recall that this is a lower bound on the trace term in Eq. (15). Plugging it back and slightly simplifying, we get

\[
\mathbb{E} \left[ \| V_k^W W' \|_F^2 \right] \geq \text{Tr} \left( \left( \sum_{i=1}^{k} U_i \right) \left( I + \frac{4}{5} \eta \lambda \left( I - \sum_{i=1}^{k} U_i \right) \right) \right) - \eta^2 r_N,
\]

where

\[
r_N = 46 \left( \sigma_N \right)^2 \left( 1 + \frac{8}{3} \left( \frac{1}{4} \sigma_N + 2 \right)^2 \right).
\]
The trace term above can be re-written (using the definition of $U_i$ and the fact that $\text{Tr}(B^TB) = \|B\|_F^2$) as
\[
\text{Tr} \left( \left( W^T \sum_{i=1}^k v_i v_i^T W \right) \left( I + \frac{4}{5} \eta \lambda \left( I - W^T \sum_{i=1}^k v_i v_i^T W \right) \right) \right)
\]
\[= \left( 1 + \frac{4}{5} \eta \lambda \right) \text{Tr} \left( W^T V_k V_k^T W \right) - \frac{4}{5} \eta \lambda \left( W^T V_k V_k^T W \right) \left( W^T V_k V_k^T W \right)
\]
\[= \left( 1 + \frac{4}{5} \eta \lambda \right) \| V_k^T W \|_F^2 - \frac{4}{5} \eta \lambda \| W^T V_k V_k^T W \|_F^2
\]
\[= \| V_k^T W \|_F^2 \left( 1 + \frac{4}{5} \eta \lambda \left( 1 - \frac{\| W^T V_k V_k^T W \|_F^2}{\| V_k^T W \|_F^2} \right) \right).
\]
Applying Lemma 7 and letting $\delta$ denote the minimal singular value of $V_k^T W$, this is lower bounded by
\[
\| V_k^T W \|_F^2 \left( 1 + \frac{4}{5} \eta \lambda \max \left\{ 1 - \| V_k^T W \|_F^2 , \frac{\delta^2}{k} \left( k - \| V_k^T W \|_F^2 \right) \right\} \right).
\]
Overall, we get that
\[
\mathbb{E} \left[ \| V_k^T W'' \|_F^2 \right] \geq \| V_k^T W \|_F^2 \left( 1 + \frac{4}{5} \eta \lambda \left( 1 - \| V_k^T W \|_F^2 \right) \right) - \eta^2 N. \tag{17}
\]
We now consider two options:

- **Taking the first argument of the max term in Eq. (17), we get**
  \[
  \mathbb{E} \left[ \| V_k^T W'' \|_F^2 \right] \geq \| V_k^T W \|_F^2 \left( 1 + \frac{4}{5} \eta \lambda \left( 1 - \| V_k^T W \|_F^2 \right) \right) - \eta^2 N.
  \]
  Subtracting 1 from both sides and simplifying, we get
  \[
  \mathbb{E} \left[ 1 - \| V_k^T W'' \|_F^2 \right] \leq \left( 1 - \frac{4}{5} \eta \lambda \| V_k^T W \|_F^2 \right) \left( 1 - \| V_k^T W \|_F^2 \right) + \eta^2 N.
  \]

- **Suppose that $\| V_k^T W \|_F^2 \geq k - \frac{1}{2}$. Taking the second argument of the max term in Eq. (17), we get**
  \[
  \mathbb{E} \left[ \| V_k^T W'' \|_F^2 \right] \geq \| V_k^T W \|_F^2 \left( 1 + \frac{4 \eta \lambda \delta^2}{5k} \left( k - \| V_k^T W \|_F^2 \right) \right) - \eta^2 N.
  \]
  Subtracting both sides from $k$, we get
  \[
  \mathbb{E} \left[ k - \| V_k^T W'' \|_F^2 \right] \leq \left( k - \| V_k^T W \|_F^2 \right) - \frac{4 \eta \lambda \delta^2}{5k} \| V_k^T W \|_F^2 \left( k - \| V_k^T W \|_F^2 \right) + \eta^2 N
  \]
  \[= \left( k - \| V_k^T W \|_F^2 \right) \left( 1 - \frac{4 \eta \lambda \delta^2}{5k} \| V_k^T W \|_F^2 \right) + \eta^2 N
  \]
  \[\leq \left( k - \| V_k^T W \|_F^2 \right) \left( 1 - \frac{4 \eta \lambda \delta^2}{5k} \left( k - \frac{1}{2} \right) \right) + \eta^2 N.
\]
Since \( k \geq 1 \), we can lower bound the \( (k - \frac{1}{2}) \) term by \( \frac{k}{2} \). Moreover, the condition \( k - \|V_k^TW\|^2_F \leq \frac{1}{2} \) implies that the singular values \( \sigma_1, \ldots, \sigma_k \) of \( V_k^TW \) satisfy \( k - \sum_{i=1}^{k} \sigma_i^2 \leq \frac{1}{2} \). But each \( \sigma_i \) is in \([0,1]\) (as \( V_k, W \) have orthonormal columns), so no \( \sigma_i \) can be less than \( \frac{1}{2} \). This implies that \( \delta \geq \frac{1}{2} \). Plugging the lower bounds \( k - \frac{1}{2} \geq \frac{k}{2} \) and \( \delta \geq \frac{1}{2} \) into the above, we get

\[
\mathbb{E} \left[ k - \|V_k^TW\|^2_F \right] \leq \left( k - \|V_k^TW\|^2_F \right) \left( 1 - \frac{1}{10} \delta \right) + \eta^2 r_N.
\]

Lemma 11. Let \( A, W_t \) be as defined in Algorithm 2 and suppose that \( \eta \in \left[ 0, \frac{1}{23\sqrt{k}} \right] \). Then the following holds for some positive numerical constants \( c_1, c_2, c_3 \):

- \( \mathbb{E} \left[ 1 - \|V_k^TW''\|^2_F \right] \leq (1 - c_1 \eta \|V_k^TW\|^2_F) \left( 1 - \|V_k^TW\|^2_F \right) + c_2 k^2 \eta \)
- If \( \|V_k^TW_i\|^2_F \geq k - \frac{1}{2} \), then
  \[
  \mathbb{E} \left[ k - \|V_k^TW_{i+1}\|^2_F \right] \leq \left( k - \|V_k^TW_i\|^2_F \right) \left( 1 - c_1 \eta (\lambda - c_2 \eta) \right) + c_3 \eta^2 (k - \|V_k^TW_i\|^2_F).
  \]

In the above, the expectation is over the random draw of the index \( i_t \), conditioned on \( W_t \) and \( \tilde{W}_{s-1} \).

Proof. To apply Lemma 10 we need to compute upper bounds \( \sigma_N^F \) and \( \sigma_N^{sp} \) on the Frobenius and spectral norms of \( N \), which in our case equals \((x_i^t x_i^T - A)(W_t - \tilde{W}_{s-1}B_t)\). Since \( \|A\|_{sp}, \|x_i^t x_i^T\|_{sp} \leq 1 \), and \( W_t, \tilde{W}_{s-1}, B_t \) have orthonormal columns, the spectral norm of \( N \) is at most

\[
\|x_i^t x_i^T - A\|_{sp} \leq \left( \|x_i^t x_i^T\|_{sp} + \|A\|_{sp} \right) \left( \|W_t\|_{sp} + \|\tilde{W}_{s-1}\|_{sp} \|B_t\|_{sp} \right) \leq 4,
\]

so we may take \( \sigma_N^{sp} = 4 \). As to the Frobenius norm, using Lemma 4 and a similar calculation, we have

\[
\|N\|^2_F \leq 4 \|W_t - \tilde{W}_{s-1}B_t\|^2_F.
\]

To upper bound this, define

\[
W_{W_t} = \arg \min_{V_k B : (V_k B)^T (V_k B) = I} \|W_t - V_k B\|^2_F
\]

to be the nearest orthonormal-columns matrix to \( W_t \) in the column space of \( V_k \), and

\[
\tilde{W}_V = \arg \min_{W_{s-1} B : (W_{s-1} B)^T (W_{s-1} B) = I} \|V_{W_t} - \tilde{W}_{s-1} B\|^2_F
\]

to be the nearest orthonormal-columns matrix to \( V_{W_t} \) in the column space of \( \tilde{W}_{s-1} \). Also, recall that by definition,

\[
\tilde{W}_{s-1} B_t = \arg \min_{W_{s-1} B : (W_{s-1} B)^T (W_{s-1} B) = I} \|W_t - \tilde{W}_{s-1} B\|^2_F
\]

is the nearest orthonormal-columns matrix to \( W_t \) in the column space of \( \tilde{W}_{s-1} \). Therefore, we must have

\[
\|W_t - \tilde{W}_{s-1} B_t\|^2_F \leq \|W_t - \tilde{W}_V\|^2_F.
\]

Using this and Lemma 9 we have

\[
\|W_t - \tilde{W}_{s-1} B_t\|^2_F \leq \|W_t - \tilde{W}_V\|^2_F
\]

\[
= \|(W_t - V_{W_t}) - (\tilde{W}_V - V_{W_t})\|^2_F
\]

\[
\leq 2\|W_t - V_{W_t}\|^2_F + 2\|\tilde{W}_V - V_{W_t}\|^2_F
\]

\[
= 4 \left( k - \|V_k^TW_t\|^2_F \right) + 4 \left( k - \|V_{W_t}\|_{sp} \|W_t\|^2_F \right).
\]
By definition of $V_{W_t}$, we have $V_{W_t} = V_k B$ where $B^T B = B^T V_k^T V_k B = (V_k B)^T (V_k B) = I$. Therefore $B$ is an orthogonal $k \times k$ matrix, and $\|V_{W_t}^T \tilde{W}_{s-1}\|_F^2 = \|B^T V_k^T \tilde{W}_{s-1}\|_F^2 = \|V_k^T \tilde{W}_{s-1}\|_F^2$, so the above equals $4(k - \|V_k^T \tilde{W}_t\|_F^2) + 4(k - \|V_k^T \tilde{W}_{s-1}\|_F^2)$. Overall, we get that the squared Frobenius norm of $N$ can be upper bounded by

$$(\sigma_N^N)^2 = 16 \left(k - \|V_k^T \tilde{W}_t\|_F^2 + (k - \|V_k^T \tilde{W}_{s-1}\|_F^2)\right).$$

Plugging $\sigma_N^N$ and $(\sigma_N^N)^2$ into the $r_N$ as defined in Lemma 10 and picking any $\eta \in \left[0, \frac{1}{44 + 2k}\right]$ (which satisfies the condition in Lemma 10 that $\eta \in \left[0, \frac{1}{4 \max\{1, \sigma_N^N\}\lambda}\right]$, since $4 \max\{1, \sigma_N^N\} \leq 4 \max\{1, \sqrt{16 + 2k}\} < 23\sqrt{k}$), we get

$$r_N = 736 \left(k - \|V_k^T \tilde{W}_t\|_F^2 + (k - \|V_k^T \tilde{W}_{s-1}\|_F^2)\right) \left(1 + \frac{8}{3} \left(\frac{1}{4} + 2\right)^2\right) \leq 18400 \left(k - \|V_k^T \tilde{W}_t\|_F^2 + (k - \|V_k^T \tilde{W}_{s-1}\|_F^2)\right).$$

This implies that $r_N \leq 36800k$ always, which by application of Lemma 10 gives the first part of our lemma. As to the second part, assuming $\|V_k^T \tilde{W}_t\|_F^2 \geq k - \frac{1}{2}$ and applying Lemma 10 we get that

$$\mathbb{E}\left[k - \|V_k^T W_t\|_F^2\right] \leq \left(k - \|V_k^T \tilde{W}_t\|_F^2\right) \left(1 - \frac{1}{10} \eta \lambda\right) + 18400 \eta^2 \left(k - \|V_k^T \tilde{W}_t\|_F^2 + (k - \|V_k^T \tilde{W}_{s-1}\|_F^2)\right)
= \left(k - \|V_k^T \tilde{W}_t\|_F^2\right) \left(1 - \eta \left(\frac{1}{10} \lambda - 18400\eta\right)\right) + 18400 \eta^2 (k - \|V_k^T \tilde{W}_{s-1}\|_F^2).
$$

This corresponds to the lemma statement. \hfill \square

**Part II: Solving the Recurrence Relation for a Single Epoch**

Since we focus on a single epoch, we drop the subscript from $\tilde{W}_{s-1}$ and denote it simply as $\tilde{W}$. Suppose that $\eta = \alpha \lambda$, where $\alpha$ is a sufficiently small constant to be chosen later. Also, let

$$b_t = k - \|V_k^T \tilde{W}_t\|_F^2 \quad \text{and} \quad \tilde{b} = k - \|V_k^T \tilde{W}\|_F^2.$$

Then Lemma 11 tells us that if $\alpha$ is a sufficiently small constant, $b_t \leq \frac{1}{2}$, then

$$\mathbb{E}\left[b_{t+1}|W_t\right] \leq \left(1 - c_2 \alpha \lambda^2\right) b_t + c' \alpha^2 \lambda^2 \tilde{b}$$

for some numerical constants $c, c'$.

**Lemma 12.** Let $B$ be the event that $b_t \leq \frac{1}{2}$ for all $t = 0, 1, 2, \ldots, m$. Then for certain positive numerical constants $c_1, c_2, c_3$, if $\alpha \leq c_1$, then

$$\mathbb{E}[b_{m+1}|B] \leq \left(1 - c_2 \alpha \lambda^2\right)^m + c_3 \alpha \tilde{b},$$

where the expectation is over the randomness in the current epoch.
Proof. Recall that $b_t$ is a deterministic function of the random variable $W_t$, which depends in turn on $W_{t-1}$ and the random instance chosen at round $t$. We assume that $W_0$ (and hence $\tilde{b}$) are fixed, and consider how $b_t$ evolves as a function of $t$. Using Eq. (18), we have

$$\mathbb{E}[W_{t+1}|W_t, B] = \mathbb{E} \left[ b_{t+1}|W_t, b_{t+1} \leq \frac{1}{2} \right] \leq \mathbb{E}[b_{t+1}|W_t] \leq (1 - ca\lambda^2) b_t + c'\alpha^2 \lambda^2 \tilde{b}.$$  

Note that the first equality holds, since conditioned on $W_t$, $b_{t+1}$ is independent of $b_1, \ldots, b_t$, so the event $B$ is equivalent to just requiring $b_{t+1} \leq 1/2$.

Taking expectation over $W_t$ (conditioned on $B$), we get that

$$\mathbb{E}[b_{t+1}|B] \leq \mathbb{E} \left[ (1 - ca\lambda^2) b_t + c'\alpha^2 \lambda^2 \tilde{b} | B \right] = (1 - ca\lambda^2) \mathbb{E}[b_t|B] + c'\alpha^2 \lambda^2 \tilde{b}.$$  

Unwinding the recursion, and using that $b_0 = \tilde{b}$, we therefore get that

$$\mathbb{E}[b_m|B] \leq (1 - ca\lambda^2)^m \tilde{b} + c'\alpha^2 \lambda^2 \tilde{b} \sum_{i=0}^{m-1} (1 - ca\lambda^2)^i \leq (1 - ca\lambda^2)^m \tilde{b} + c'\alpha^2 \lambda^2 \tilde{b} \frac{1}{ca\lambda^2} = \left( (1 - ca\lambda^2)^m + \frac{c'}{c} \right) \tilde{b}.$$  

as required.  

We now turn to prove that the event $B$ assumed in Lemma 12 indeed holds with high probability:

**Lemma 13.** The following holds for certain positive numerical constants $c_1, c_2, c_3$: If $\alpha \leq c_1$, then for any $\beta \in (0, 1)$ and $m$, if

$$\tilde{b} + c_2 km\alpha^2 \lambda^2 + c_3 k \sqrt{m\alpha^2 \lambda^2 \log(1/\beta)} \leq \frac{1}{2},$$  

(19)

then it holds with probability at least $1 - \beta$ that

$$b_t \leq \tilde{b} + c_2 km\alpha^2 \lambda^2 + c_3 k \sqrt{m\alpha^2 \lambda^2 \log(1/\beta)} \leq \frac{1}{2}$$  

for all $t = 0, 1, 2, \ldots, m$.

Proof. To prove the lemma, we analyze the stochastic process $b_0(= \tilde{b}), b_1, b_2, \ldots, b_m$, and use a concentration of measure argument. First, we collect the following facts:

- $\tilde{b} = b_0 \leq \frac{1}{2}$: This directly follows from the assumption stated in the lemma.
- As long as $b_t \leq \frac{1}{2}$, $\mathbb{E}[b_{t+1}|W_t] \leq b_t + c_2\alpha^2 \lambda^2 \tilde{b}$ for some constant $c_2$: Supposing $\alpha$ is sufficiently small, then by Eq. (18),

$$\mathbb{E}[b_{t+1}|W_t] \leq (1 - ca\lambda^2) b_t + c'\alpha^2 \lambda^2 \tilde{b} \leq b_t + c'\alpha^2 \lambda^2 \tilde{b}.$$  

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• $|b_{t+1} - b_t|$ is bounded by $c'_k k\alpha \lambda$ for some constant $c'_k$: Applying Lemma 9 and assuming that $\alpha$ is at most some sufficiently small constant $c_1$ (e.g. $\alpha \leq \frac{1}{12}$, so $\eta = \alpha \lambda \leq \frac{1}{12}$),

\[
|b_{t+1} - b_t| = \left| \| V_k^T W_{t+1} \|_F^2 - \| V_k^T W_t \|_F^2 \right| \leq \frac{12k\eta}{1-3\eta} \leq \frac{12k\alpha \lambda}{3/4} = 16k\alpha \lambda.
\]

Armed with these facts, and using the maximal version of the Hoeffding-Azuma inequality [11], it follows that with probability at least $1 - \beta$, it holds simultaneously for all $t = 1, \ldots, m$ (and for $t = 0$ by assumption) that

\[
\int b_t \leq \tilde{b} + c_2 k m \alpha^2 \lambda^2 + c_3 k \sqrt{m \alpha^2 \lambda^2 \log(1/\beta)}
\]

for some constants $c_2, c_3$, as long as the expression above is less than $\frac{1}{2}$. If the expression is indeed less than $\frac{1}{2}$, then we get that $b_t \leq \frac{1}{2}$ for all $t$. Upper bounding $\tilde{b}$ by $k$ and slightly simplifying, we get the statement in the lemma.

Combining Lemma 12 and Lemma 13 and using Markov’s inequality, we get the following corollary:

**Lemma 14.** Let confidence parameters $\beta, \gamma \in (0, 1)$ be fixed. Suppose that $m, \alpha$ are chosen such that $\alpha \leq c_1$ and

\[
\tilde{b} + c_2 k m \alpha^2 \lambda^2 + c_3 k \sqrt{m \alpha^2 \lambda^2 \log(1/\beta)} \leq \frac{1}{2},
\]

where $c_1, c_2, c_3$ are certain positive numerical constants. Then with probability at least $1 - (\beta + \gamma)$, it holds that

\[
b_m \leq \frac{1}{\gamma} \left( (1 - c_0 \alpha \lambda^2)^m + c' \alpha \right) \tilde{b}.
\]

for some positive numerical constants $c, c'$.

**Part III: Analyzing the Entire Algorithm’s Run**

Given the analysis in Lemma 14 for a single epoch, we are now ready to prove our theorem. Let

\[
\tilde{b}_s = k - \| V_k^T \tilde{W}_s \|_F^2.
\]

By assumption, at the beginning of the first epoch, we have $\tilde{b}_0 = k - \| V_k^T \tilde{W}_0 \|_F^2 \leq \frac{1}{2}$. Therefore, by Lemma 14 for any $\beta, \gamma \in (0, \frac{1}{2})$, if we pick any

\[
\alpha \leq \min \left\{ c_1, \frac{1}{2c_0 \gamma^2} \right\} \quad \text{and} \quad m \geq \frac{3 \log(1/\gamma)}{c_0 \alpha \lambda^2} \quad \text{such that} \quad \frac{1}{2} + c_2 k m \alpha^2 \lambda^2 + c_3 k \sqrt{m \alpha^2 \lambda^2 \log(1/\beta)} \leq \frac{1}{2},
\]

then we get with probability at least $1 - (\beta + \gamma)$ that

\[
b_m \leq \frac{1}{\gamma} \left( (1 - c_0 \alpha \lambda^2)^m + \frac{1}{2} \gamma^2 \right) \tilde{b}_0
\]

Using the inequality $(1 - (1/x))^{ax} \leq \exp(-a)$, which holds for any $x > 1$ and any $a$, and taking $x = 1/(c_0 \alpha \lambda^2)$ and $a = 3 \log(1/\gamma)$, we can upper bound the above by

\[
\frac{1}{\gamma} \left( \exp \left( -3 \log \left( \frac{1}{\gamma} \right) \right) + \frac{1}{2} \gamma^2 \right) \tilde{b}_0
\]

\[
= \frac{1}{\gamma} \left( \gamma^3 + \frac{1}{2} \gamma^2 \right) \tilde{b}_0 \leq \gamma \tilde{b}_0.
\]

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Since \( b_m \) equals the starting point \( \tilde{b}_1 \) for the next epoch, we get that \( \tilde{b}_1 \leq \gamma \tilde{b}_0 \leq \gamma^{\frac{1}{2}} \). Again applying Lemma 14 and performing the same calculation we have that with probability at least \( 1 - (\beta + \gamma) \) over the next epoch, \( \tilde{b}_2 \leq \gamma \tilde{b}_1 \leq \gamma^{\frac{1}{2}} \tilde{b}_0 \). Repeatedly applying Lemma 14 and using a union bound, we get that after \( T \) epochs, with probability at least \( 1 - T(\beta + \gamma) \),
\[
k - \| V_k^T \tilde{W}_T \| F^2 = \tilde{b}_T \leq \gamma T \tilde{b}_0 < \gamma^T.
\]
Therefore, for any desired accuracy parameter \( \epsilon \), we simply need to use \( T = \left\lceil \frac{\log(1/\epsilon)}{\log(1/\gamma)} \right\rceil \) epochs, and get \( k - \| V_k^T \tilde{w}_s \| F^2 \leq \epsilon \) with probability at least \( 1 - T(\beta + \gamma) = 1 - \left\lceil \frac{\log(1/\epsilon)}{\log(1/\gamma)} \right\rceil (\beta + \gamma) \).

Using a confidence parameter \( \delta \), we pick \( \beta = \gamma = \frac{\epsilon}{2} \), which ensures that the accuracy bound above holds with probability at least
\[
1 - \left\lceil \frac{\log(1/\epsilon)}{\log(2/\delta)} \right\rceil \delta \geq 1 - \left\lceil \frac{\log(1/\epsilon)}{\log(2)} \right\rceil \delta = 1 - \left\lceil \log_2 \left( \frac{1}{\epsilon} \right) \right\rceil \delta.
\]
Substituting this choice of \( \beta, \gamma \) into Eq. (20), and recalling that the step size \( \eta \) equals \( \alpha \lambda \), we get that \( k - \| V_k^T \tilde{w}_T \| F^2 \leq \epsilon \) with probability at least \( 1 - \lceil \log_2(1/\epsilon) \rceil \delta \), provided that
\[
\eta \leq c \delta^2 \lambda , \quad m \geq \frac{c'}{\eta \lambda} \log(2/\delta), \quad km\eta^2 + k\sqrt{m\eta^2 \log(2/\delta)} \leq \epsilon''
\]
for suitable positive constants \( c, c', \epsilon'' \).

To get the theorem statement, recall that the analysis we performed pertains to data whose squared norm is bounded by 1. By the reduction discussed at the beginning of the proof, we can apply it to data with squared norm at most \( r \), by replacing \( \lambda \) with \( \lambda/r \), and \( \eta \) with \( \eta r \), leading to the condition
\[
\eta \leq \frac{c \delta^2}{r^2} \lambda , \quad m \geq \frac{c'}{\eta \lambda} \log(2/\delta), \quad km\eta^2 + r k\sqrt{m\eta^2 \log(2/\delta)} \leq \epsilon''
\]
and establishing the theorem.

### 6.2 Proof of Theorem 2

The proof relies mainly on the techniques and lemmas of Section 6.1 used to prove Theorem 1. As done in Section 6.1, we will assume without loss of generality that \( r = \max_i \| x_i \|^2 \) is at most 1, and then transform the bound to a bound for general \( r \) (see the discussion at the beginning of Subsection 6.1.2).

First, we extract the following result, which is essentially the first part of Lemma 11 (for \( k = 1 \)):

**Lemma 15.** Let \( A, w_t \) be as defined in Algorithm 1 and suppose that \( \eta \in \left[ 0, \frac{1}{2} \right] \). Then
\[
E_{w_t} \left[ 1 - \langle v_1, w_{t+1} \rangle^2 \right| w_t, \tilde{w}_{s-1} \leq \left( 1 - c\eta \lambda \langle v_1, w_t \rangle \right) \left( 1 - \langle v_1, w_t \rangle \right) + c' \eta^2,
\]
for some positive numerical constants \( c, c' \).

Note that this bound holds regardless of what is \( \tilde{w}_{s-1} \), and in particular holds across different epochs of Algorithm 1. Therefore, it is enough to show that starting from some initial point \( w_0 \), after sufficiently many stochastic updates as specified in line 6-10 of the algorithm (or in terms of the analysis, sufficiently many applications of Lemma 15), we end up with a point \( w_T \) for which \( 1 - \langle v_1, w_T \rangle \leq \frac{1}{2} \), as required.
Note that to simplify the notation, we will use here a single running index $w_0, w_1, w_2, \ldots, w_T$ (whereas in the algorithm we restarted the indexing after every epoch).

The proof is based on martingale arguments, quite similar to the ones in Subsection 6.1.2 but with slight changes. First, we let

$$b_t = 1 - \langle v_1, w_t \rangle^2$$

to simplify notation. We note that $b_0 = 1 - \langle v_1, w_0 \rangle^2$ is assumed fixed, whereas $b_1, b_2, \ldots$ are random variables based on the sampling process. Lemma 11 tells us that if $\eta$ is sufficiently small, and $b_t \leq 1 - \xi$ for some $\xi \in (0, 1)$, then

$$E[b_{t+1} | b_t] \leq (1 - c\eta \lambda \xi) b_t + c' \eta^2. \quad (21)$$

for some numerical constants $c, c'$.

**Lemma 16.** Let $B$ be the event that $b_t \leq 1 - \xi$ for all $t = 0, 1, \ldots, T$. Then for certain positive numerical constants $c_1, c_2, c_3$, if $\eta \leq c_1 \lambda$, then

$$E[b_T | B] \leq \left(1 - c_2 \eta \lambda \xi\right)^T + c_3 \eta \lambda \xi.$$

**Proof.** Using Eq. (21), we have for any $b_t$ satisfying event $B$ that

$$E[b_{t+1} | b_t, B] = E[b_{t+1} | b_t, b_{t+1} \leq 1 - \xi] \leq E[b_{t+1} | b_t] \leq (1 - c\eta \lambda \xi) b_t + c' \eta^2.$$

Taking expectation over $b_t$ (conditioned on $B$), we get that

$$E[b_{t+1} | B] \leq E \left[(1 - c\eta \lambda \xi) b_t + c' \eta^2 | B\right]$$

$$= (1 - c\eta \lambda \xi) E[b_t | B] + c' \eta^2.$$

Unwinding the recursion, we get

$$E[b_T | B] \leq (1 - c\eta \lambda \xi)^T b_0 + c' \eta^2 \sum_{i=0}^{T-1} (1 - c\eta \lambda \xi)^i$$

$$\leq (1 - c\eta \lambda \xi)^T + c' \eta^2 \sum_{i=0}^{\infty} (1 - c\eta \lambda \xi)^i$$

$$= (1 - c\eta \lambda \xi)^T + c' \eta^2 \frac{1}{c\eta \lambda \xi} \leq (1 - c\eta \lambda \xi)^T + \frac{c'}{c} \frac{\eta}{\lambda \xi}.$$  

We now turn to prove that the event $B$ assumed in Lemma 12 indeed holds with high probability:

**Lemma 17.** The following holds for certain positive numerical constants $c_1, c_2, c_3$: If $\eta \leq c_1 \lambda$, then for any $\beta \in (0, 1)$, if

$$b_0 + c_2 T \eta^2 + c_3 \sqrt{T} \eta^2 \log(1/\beta) \leq 1 - \xi, \quad (22)$$

then it holds with probability at least $1 - \beta$ that

$$b_t \leq b_0 + c_2 T \eta^2 + c_3 \sqrt{T} \eta^2 \log(1/\beta) \leq 1 - \xi$$

for all $t = 0, 1, \ldots, T$.  

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Proof. To prove the lemma, we analyze the stochastic process $b_1, b_2, \ldots, b_T$, and use a concentration of measure argument. First, we collect the following facts:

- $b_0 \leq 1 - \xi$: This directly follows from the assumption stated in the lemma.
- $\mathbb{E}[b_{t+1}|b_t] \leq b_t + c'\eta^2$ for some constant $c'$: By Eq. (21),
  $$\mathbb{E}[b_{t+1}|W_t] \leq (1 - c\eta\lambda\xi)b_t + c'\eta^2 \leq b_t + c'\eta^2.$$

- $|b_{t+1} - b_t|$ is bounded by $c\eta$ for some constant $c$: Applying Lemma 9 for the case $k = 1$, and assuming $\eta \leq 1/12$,
  $$|b_{t+1} - b_t| = |\langle v_1, w_{t+1} \rangle - \langle v, w_t \rangle| \leq \frac{12\eta}{1 - 3\eta} \leq \frac{12\eta}{3/4} = 16\eta.$$

Armed with these facts, and using the maximal version of the Hoeffding-Azuma inequality [11], it follows that with probability at least $1 - \beta$, it holds simultaneously for all $t = 0, 1, \ldots, T$ that

$$b_t \leq b_0 + c_2 T\eta^2 + c_3 \sqrt{T\eta^2 \log(1/\beta)}$$

for some constants $c_2, c_3$. If the expression is indeed less than $1 - \xi$, then we get that $b_t \leq 1 - \xi$ for all $t$, from which the lemma follows.

Combining Lemma 16 and Lemma 17, and using Markov’s inequality, we get the following corollary:

**Lemma 18.** Let confidence parameters $\beta, \gamma \in (0, 1)$ be fixed. Then for some positive numerical constants $c_1, c_2, c_3, c, c'$, if $\eta \leq c_1 \lambda$ and

$$b_0 + c_2 T\eta^2 + c_3 \sqrt{T\eta^2 \log(1/\beta)} \leq 1 - \xi,$$

then with probability at least $1 - (\beta + \gamma)$, it holds that

$$b_T \leq \frac{1}{\gamma} \left( (1 - c\eta\lambda\xi)^T + c'\eta^2 \lambda\xi \right).$$

We are now ready to prove our theorem. By Lemma 18 for any $\beta, \gamma \in (0, 1/2)$ and any

$$\eta \leq \min \left\{ c_1, \frac{1}{2c}\gamma^2 \right\} \lambda\xi \quad \text{and} \quad T \geq \frac{3\log(1/\gamma)}{c\eta\lambda\xi}$$

such that $b_0 + c_2 T\eta^2 + c_3 \sqrt{T\eta^2 \log(1/\beta)} \leq 1 - \xi$, (23)

we get with probability at least $1 - (\beta + \gamma)$ that

$$b_T \leq \frac{1}{\gamma} \left( (1 - c\eta\lambda\xi)^{3\log(1/\gamma)/c\eta\lambda\xi} + \frac{1}{2} \gamma^2 \right).$$

Using the inequality $(1 - (1/x))^{ax} \leq \exp(-a)$, which holds for any $x > 1$ and any $a$, and taking $x = 1/(c\eta\lambda\xi)$ and $a = 3 \log(1/\gamma)$, we can upper bound the above by

$$\frac{1}{\gamma} \left( \exp\left(-3\log\left(\frac{1}{\gamma}\right)\right) + \frac{1}{2} \gamma^2 \right) = \frac{1}{\gamma} \left( \gamma^3 + \frac{1}{2} \gamma^2 \right),$$

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and since we assume $\gamma < \frac{1}{2}$, this is at most $\frac{1}{2}$. Overall, we got that with probability at least $1 - \beta - \gamma$, $b_T \leq \frac{1}{2}$, and therefore $1 - \langle v_1, w_T \rangle^2 \leq \frac{1}{2}$ as required.

It remains to show that the parameter choices in Eq. (23) can indeed be satisfied. First, we fix $\xi \leq \frac{1}{2} \zeta$ (where we recall that $0 < \zeta \leq \langle v_1, w_0 \rangle^2$), which trivially ensures that $b_0 = 1 - \langle v_1, w_0 \rangle^2$ is at most $1 - 2\xi$. Moreover, suppose we pick $\beta = \gamma$ in $(0, \exp(-1))$, and $\eta, T$ so that

$$\eta \leq \frac{c_* \gamma^2 \lambda \zeta^3}{\log^2(1/\gamma)}$$

where $c_*, c'_*$ are sufficiently small constants so that the bounds on $\eta, T$ in Eq. (23) are satisfied. This implies that the third bound in Eq. (23) is also satisfied, since by plugging in the values / bounds of $T$ and $\eta$, and using the assumptions $\gamma = \beta \leq \exp(-1)$ and $\xi \leq 1$, we have

$$b_0 + c_2 T \eta^2 + c_3 \sqrt{T \eta^2 \log(1/\gamma)}$$

$$\leq 1 - 2\xi + c_2 - \frac{3 \log(1/\gamma) \eta}{c'_* \lambda \zeta} + c_3 \sqrt{\frac{3 \log(1/\gamma) \eta \log(1/\gamma)}{c'_* \lambda \zeta}}$$

$$\leq 1 - 2\xi + c_2 - \frac{3 c_* \gamma^2 \lambda \zeta^2}{c'_* \log(1/\gamma)} + c_3 \sqrt{\frac{3 c_* \gamma^2 \lambda \zeta^2}{c'_* \lambda \zeta}}$$

$$\leq 1 - 2\xi + \left(\frac{3 c_* c_1 \delta \lambda \zeta^3}{c'_* \lambda \zeta} + c_3 \sqrt{\frac{3 c_*}{c'_* \lambda \zeta}}\right) \xi,$$

which is less than $1 - \xi$ if we pick $c_*$ sufficiently small compared to $c'_*$.

To summarize, we get that for any $\gamma \in (0, \exp(-1))$, by picking $\eta$ as in Eq. (24), we have that after $T$ iterations (where $T$ is specified in Eq. (24)), with probability at least $1 - 2\gamma$, we get $w_T$ such that $1 - \langle v_1, w_T \rangle \leq \frac{1}{2}$. Substituting $\delta = 2\gamma$ and $\zeta = 2\xi$, we get that if $\langle v_1, \tilde{w}_0 \rangle^2 \geq \zeta > 0$,

and $\eta$ satisfies

$$\eta \leq \frac{c_1 \delta^2 \lambda \zeta^3}{\log^2(2/\delta)}$$

(for some universal constant $c_1$), then with probability at least $1 - \delta$, after

$$T = \left\lceil \frac{c_2 \log(2/\delta)}{\eta \lambda \zeta} \right\rceil$$

stochastic iterations, we get a satisfactory point $w_T$.

As discussed at the beginning of the proof, this analysis is valid assuming $r = \max_i \|x_i\|^2 \leq 1$. By the reduction discussed at the beginning of Subsection 6.1.2, we can get an analysis for any $r$ by substituting $\lambda \rightarrow \lambda/r$ and $\eta \rightarrow \eta r$. This means that we should pick $\eta$ satisfying

$$\eta r \leq \frac{c_1 \delta^2 (\lambda/r) \zeta^3}{\log^2(2/\delta)} \Rightarrow \eta \leq \frac{c_1 \delta^2 \lambda \zeta^3}{r^2 \log^2(2/\delta)}$$

and getting the required point after

$$T = \left\lceil \frac{c_2 \log(2/\delta)}{(\eta r)(\lambda/r) \zeta} \right\rceil = \left\lceil \frac{c_2 \log(2/\delta)}{\eta \lambda \zeta} \right\rceil$$

iterations.
6.3 Proof of Theorem

For simplicity of notation, we drop the $A$ subscript from $F_A$, and refer simply to $F$.

We first prove the following two auxiliary lemmas:

**Lemma 19.** If $A$ is a symmetric matrix, then the gradient of the function $F(w) = -\frac{w^T A w}{\|w\|^2}$ at some $w$ equals

$$-\frac{2}{\|w\|^2} (F(w) I + A) w,$$

and its Hessian equals

$$\frac{1}{\|w\|^2} \left( \left( I - \frac{4}{\|w\|^2} w w^T \right) (F(w) I + A) \right),$$

where $B^\perp = B + B^\top$ (i.e., a matrix $B$ plus its transpose).

**Proof.** By the product and chain rules (using the fact that $\frac{1}{\|w\|^2}$ is a composition of $w \mapsto \|w\|^2$ and $z \mapsto \frac{1}{z}$), the gradient of $F(w) = -\frac{1}{\|w\|^2} (w^T A w)$ equals

$$w \frac{2}{\|w\|^4} (w^T A w) - (A w) \frac{2}{\|w\|^2}, \tag{25}$$

giving the gradient bound in the lemma statement after a few simplifications.

Differentiating the vector-valued Eq. (25) with respect to $w$ (using the product and chain rules, and the fact that $\frac{1}{\|w\|^2}$ is a composition of $w \mapsto \|w\|^2$, $z \mapsto z^2$, and $z \mapsto \frac{1}{z}$), we get that the Hessian of $F$ equals

$$I \frac{2}{\|w\|^4} (w^T A w) + w \left( -\frac{2}{\|w\|^4} + 2 \frac{2}{\|w\|^2} \right)^T \left( w^T A w \right) + w \frac{2}{\|w\|^4} \left( 2 A w \right)^T$$

$$- A \frac{2}{\|w\|^2} - (A w) \left( -\frac{2}{\|w\|^4} + 2 w \right)^T$$

$$= - \frac{2 F(w)}{\|w\|^2} I + \frac{8 F(w)}{\|w\|^4} w w^T + \frac{4}{\|w\|^4} w w^T A - \frac{2}{\|w\|^2} A + \frac{4}{\|w\|^4} A w w^T$$

$$= - \frac{1}{\|w\|^2} \left( 2 F(w) I - \frac{8 F(w)}{\|w\|^2} w w^T - \frac{4}{\|w\|^2} w w^T A + 2 A - \frac{4}{\|w\|^2} A w w^T \right),$$

which can be verified to equal the expression in the lemma statement (using the fact that $A$, $w w^T$ and $I$ are all symmetric matrices, hence equal their transpose). \hfill $\square$

**Lemma 20.** Let $w_0, v_1$ be two unit vectors such that $\|w_0 - v_1\| \leq \epsilon < \frac{1}{2}$ (which implies $\langle w_0, v_1 \rangle > 0$). Let $v_1'$ be the intersection of the ray $\{a v_1 : a \geq 0\}$ with the hyperplane $H_{w_0} = \{w : \langle w, w_0 \rangle = 1\}$. Then $\|v_1' - w_0\| \leq \frac{3}{4} \epsilon$.

**Proof.** See Figure 2 in the main text for a graphical illustration.

Letting $v_1' = a v_1$, $a$ must satisfy $\langle v_1, w_0 \rangle = 1$. Since $v_1, w_0$ are unit vectors, this implies

$$a = \frac{1}{\langle v_1, w_0 \rangle} = \frac{2}{2 - \|v_1 - w_0\|^2},$$

Thus $\|v_1' - w_0\| = \|a v_1 - w_0\| = \|2 v_1 - 2 w_0\| = \frac{3}{2} \epsilon$. \hfill $\square$
and since \( \|v_1 - w_0\| \leq \epsilon \), this means that
\[
a \in \left[1, \frac{2}{2 - \epsilon^2}\right].
\]

Therefore,
\[
\|v'_1 - w_0\| \leq \|v_1 - w_0\| + \|v'_1 - v_1\| \leq \epsilon + \|av_1 - v_1\| \leq \epsilon + |a - 1| \leq \epsilon + \frac{2}{2 - \epsilon^2} - 1 = \epsilon + \frac{\epsilon^2}{2 + \epsilon^2},
\]
and since \( \epsilon < \frac{1}{2} \), this is at most \( \frac{5}{4} \epsilon \).

We now turn to prove the theorem. Let \( \nabla^2(w) \) denote the Hessian at some point \( w \). To show smoothness and strong convexity as stated in the theorem, it is enough to fix some unit \( w_0 \) which is \( \epsilon \)-close to the leading eigenvector \( v_1 \) (where \( \epsilon \) is assumed to be sufficiently small), and show that for any point \( w \) on \( H_{w_0} \) which is \( O(\epsilon) \) close to \( w_0 \), and any direction \( g \) along \( H_{w_0} \) (i.e. any unit \( g \) such that \( \langle g, w_0 \rangle = 0 \)), it holds that \( g^\top \nabla^2(w)g \in [\lambda, 20] \). This implies that the second derivative in an \( O(\epsilon) \) neighborhood of \( w_0 \) on \( H_{w_0} \) is always in \([\lambda, 20] \), hence the function is both \( \lambda \)-strongly convex in that neighborhood.

More formally, letting \( \epsilon \in (0, 1) \) be a small parameter to be chosen later, consider any \( w_0 \) such that
\[
\|w_0\| = 1 \text{, } \|w_0 - v_1\| \leq \epsilon,
\]
any \( w \) such that
\[
\langle w - w_0, w_0 \rangle = 0 \text{, } \|w - w_0\| \leq 2\epsilon,
\]
and any \( g \) such that
\[
\|g\| = 1 \text{, } \langle g, w_0 \rangle = 0.
\]

Our goal is to show that for an appropriate \( \epsilon \), we have \( g^\top \nabla^2(w)g \in [\lambda, 20] \). Moreover, by Lemma 20, the neighborhood set \( H_{w_0} \cap B_{w_0}(2\epsilon) \) would also contain a point \( av_1 \) for some \( a \), which is a global optimum of \( F \) due to its scale-invariance. This would establish the theorem.

The easier part is to show the upper bound on \( g^\top \nabla^2(w)g \). Since \( g \) is a unit vector, it is enough to bound the spectral norm of \( \nabla^2(w) \), which equals

\[
\left\| \frac{1}{\|w\|^2} \left( \left( I - \frac{4}{\|w\|^2}ww^\top \right) \left(F(w)I + A\right) \right) \right\|_sp \leq \frac{2}{\|w\|^2} \left\| \left( I - \frac{4}{\|w\|^2}ww^\top \right) \left(F(w)I + A\right) \right\|_sp \leq \frac{2}{\|w\|^2} \left\| \left( I - \frac{4}{\|w\|^2}ww^\top \right) \right\|_sp \|F(w)I + A\|_sp \leq \frac{2}{\|w\|^2} \left( \|I\|_sp + \frac{4}{\|w\|^2} \left\|ww^\top \right\|_sp \right) \left(\|F(w)I\|_sp + \|A\|_sp \right).
\]

Since the spectral norm of \( A \) is 1, and \( \|w\|^2 \geq 1 \) (as \( w \) lies on a hyperplane \( H_{w_0} \) tangent to a unit vector \( w_0 \)), it is easy to verify that this is at most \( 2(1 + 4)(1 + 1) = 20 \) as required.

We now turn to lower bound \( g^\top \nabla^2(w)g \), which by Lemma 19 equals
\[
- \frac{1}{\|w\|^2} g^\top \left( \left( I - \frac{4}{\|w\|^2}ww^\top \right) \left(F(w)I + A\right) \right) \downarrow g.
\]
Since $g^\top B^\perp g = g^\top B g + g^\top B^\top g = 2g^\top B g$, the above equals

$$-\frac{2}{\|w\|^2}g^\top \left( I - \frac{4}{\|w\|^2} w w^\top \right) \left( F(w)I + A \right) g. \quad (26)$$

Using the fact that $w = w_0 + (w - w_0)$, and $\langle g, w_0 \rangle = 0$, we get that $\langle g, w \rangle = \langle g, w - w_0 \rangle$. Moreover, since $A$ is positive semidefinite and has spectral norm of 1, $F(w) = -\frac{w^\top A w}{\|w\|^2} \in [-1, 0]$. Expanding Eq. (26) and plugging these in, we get

$$-\frac{2}{\|w\|^2} \left( F(w)g^\top \left( I - \frac{4}{\|w\|^2} w w^\top \right) g + g^\top \left( I - \frac{4}{\|w\|^2} w w^\top \right) Ag \right)$$

$$= \frac{2}{\|w\|^2} \left( -F(w)\|g\|^2 + \frac{4F(w)}{\|w\|^2} \langle g, w - w_0 \rangle^2 - g^\top Ag + \frac{4}{\|w\|^2} \langle g, w - w_0 \rangle w^\top Ag \right)$$

$$\geq \frac{2}{\|w\|^2} \left( -F(w)\|g\|^2 - \frac{4}{\|w\|^2} \|g\|^2 \|w - w_0\|^2 - g^\top Ag - \frac{4}{\|w\|^2} \|g\| \|w - w_0\| \|A\|_{sp} \|g\| \right).$$

Since $\|g\| = 1$, $\|A\|_{sp} = 1$, $\|w - w_0\| \leq 2\epsilon$, and $\|w\|^2 = \|w_0\|^2 + \|w - w_0\|^2$ is between 1 and $1 + 4\epsilon^2$, this is at least

$$\frac{2}{\|w\|^2} \left( -(F(w)) - 16\epsilon^2 - g^\top Ag - 8\epsilon \sqrt{1 + 4\epsilon^2} \right) = \frac{2}{\|w\|^2} \left( -F(w) - g^\top Ag - 8\epsilon \left( 2\epsilon + \sqrt{1 + 4\epsilon^2} \right) \right). \quad (27)$$

Let us now analyze $-F(w)$ and $g^\top Ag$ more carefully. The idea will be to show that since we are close to the optimum, $-F(w)$ is very close to 1, and $g$ (which is orthogonal to the near-optimal $w_0$) is such that $g^\top Ag$ is strictly smaller than 1. This would give us a positive lower bound on Eq. (27).

- By the triangle inequality and the assumptions $\|w_0 - v_1\| \leq \epsilon$, $\|w - w_0\| \leq 2\epsilon$, we have $\|w - v_1\| \leq 3\epsilon$. Also, we claim that $F(\cdot)$ is 4-Lipschitz outside the unit Euclidean ball (since the gradient of $F$ at any point with norm $\geq 1$ according to Lemma 19 has norm at most 4). Therefore, $|F(w) + 1| = |F(w) - F(v_1)| \leq 4\|w - v_1\| \leq 12\epsilon$, so overall,

$$F(w) \leq -1 + 12\epsilon. \quad (28)$$

- Since $\langle w_0, g \rangle = 0$, and $\|w_0 - v_1\| \leq \epsilon$, it follows that

$$|\langle v_1, g \rangle| \leq |\langle v_1 - w_0, g \rangle| + |\langle w_0, g \rangle| \leq \|v_1 - w_0\| \|g\| + 0 \leq \epsilon.$$

Letting $v_1, \ldots, v_d$ and $1 = s_1 > s_2 \geq \ldots \geq s_d \geq 0$ be the eigenvectors and eigenvalues of $A$ in decreasing order (and recalling that $s_2 \leq s_1 - \lambda = 1 - \lambda$ for some eigengap $\lambda > 0$), we get

$$g^\top Ag = \sum_{i=1}^{d} s_i \langle v_i, g \rangle^2 \leq \langle v_1, g \rangle^2 + (1 - \lambda) \sum_{i=1}^{d} \langle v_i, g \rangle^2$$

$$= \langle v_1, g \rangle^2 + (1 - \lambda)(1 - \langle v_1, g \rangle^2) = \lambda \langle v_1, g \rangle^2 + (1 - \lambda)$$

$$\leq \lambda \epsilon^2 + (1 - \lambda) = 1 - (1 - \epsilon^2) \lambda. \quad (29)$$
Plugging Eq. (28) and Eq. (29) back into Eq. (27), we get a lower bound of
\[
\frac{2}{\|w\|^2} \left( 1 - 12\epsilon - (1 - (1 - \epsilon^2)\lambda) - 8\epsilon \left( 2\epsilon + \sqrt{1 + 4\epsilon^2} \right) \right) \\
\frac{2}{\|w\|^2} \left( (1 - \epsilon^2)\lambda - 8\epsilon \left( 1.5 + 2\epsilon + \sqrt{1 + 4\epsilon^2} \right) \right) \\
\frac{2}{\|w\|^2} \left( 1 - \epsilon^2 - \frac{8\epsilon \left( 1.5 + 2\epsilon + \sqrt{1 + 4\epsilon^2} \right)}{\lambda} \right) \lambda.
\]
Using the fact that \(\sqrt{1 + z^2} \leq 1 + z\), this can be loosely lower bounded by
\[
\frac{2}{\|w\|^2} \left( 1 - \epsilon - \frac{8\epsilon (2.5 + 4\epsilon)}{\lambda} \right) \lambda.
\]
Recalling that \(\|w\|^2 = \|w_0\|^2 + \|w - w_0\|^2\) is at most \(1 + 4\epsilon^2\), and picking \(\epsilon\) sufficiently small compared to \(\lambda\), (say \(\epsilon = \lambda/44\)), we get that the above is at least \(\lambda\), which implies the required strong convexity condition. To summarize, by picking \(\epsilon = \lambda/44\), we have shown that the function \(F(w)\) is \(\lambda\)-strongly convex and 20-smooth in a neighborhood of size \(2\epsilon = \frac{\lambda}{44}\) around \(w_0\) on the hyperplane \(H_{w_0}\), provided that \(\|w_0 - v_1\| \leq \epsilon = \frac{\lambda}{44}\). By Lemma 20, we are guaranteed that this neighborhood contains \(v_1\) up to some rescaling (which is immaterial for our scale-invariant function \(F\)), hence by optimizing \(F\) in that neighborhood, we will get a globally optimal solution.

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