The Study of Two-dimensional Polytropic Stars

Sanchari De and Somenath Chakrabarty†
Department of Physics, Visva-Bharati, Santiniketan-731235, India
†somenath.chakrabarty@visva-bharati.ac.in

PACS:73.20.-r, 71.10.Ca, 21.65.+f, 13.75.Cs

Abstract

In this article we have studied the structure of hypothetical two-dimensional polytropic stars. Considering some academic interest, we have developed a formalism to investigate some of the gross properties of such stellar objects. However, we strongly believe that the formalism developed here may be prescribed as class problem for post-graduate level students in physics or a post-graduate dissertation project work in physics.

1 Introduction

Since the temperature inside the main-sequence stars are quite high, the thermal energy of the hydrogen ions are large enough to overcome the inter-ionic Coulomb barrier. As a result the conversion of hydrogen to helium is continuously going on inside such stars. For the stars of mass very close to the sun, the thermonuclear reactions are taking place through $p - p$-chain reactions. Whereas for the massive main-sequence stars, the hydrogen to helium conversion process is called the CNO-cycle. To study the structure and also some of the gross properties of main-sequence stars the solution of the well known Lane-Emden equation \[1\] is used. This equation is essentially obtained from the hydrostatic equilibrium condition inside the star. For the physically acceptable values of various parameters of the star, in particular the mass and the radius of the star, the numerical solution of Lane-Emden equation is used. To obtain the Lane-Emden equation from hydrostatic equilibrium condition, the pressure and the density of stellar matter are connected by some equation of state, called polytropic equation of state and the general form of this equation is given by

\[ P = K \rho^n \]  

where $P$ and $\rho$ are respectively the pressure and the density of stellar matter, $K$ is a constant which depends on the nature of stellar matter and the constant $\Gamma$ is called the polytropic index. In stellar astrophysics, instead of $\Gamma$, for the sake of convenience it is expressed as

\[ \Gamma = 1 + \frac{1}{n} \]  

and $n$ is then treated as the polytropic index. In the usual main-sequence stellar model $n = 3/2$ for extreme non-relativistic case and $n = 3$ for the ultra-relativistic situation.

In this article we have developed a formalism to study the structure and some of the gross properties of two-dimensional hypothetical main-sequence stars. To the best of our knowledge such studies have not been reported in the past. The work is essentially an extension of standard polytropic model for main-sequence stars, discussed in many standard text book on astrophysics \[1, 2\] However, we have already mentioned that the formalism may be useful for the post-graduate physics students.

2 Basic Formalism

For the two-dimensional stellar objects, we redefine the pressure as the force per unit length. Then it is very easy to show that the hydro-static equilibrium condition is given by

\[ dP + g(r)\rho(r)dr = 0 \]  

Now in two-dimension, the gravitational force per unit mass may be expressed in the form

\[ g(r) = \frac{G}{r} \int_0^r 2\pi \rho(r') r' dr' \]  

Combining these two equations (eqn.(3) and eqn.(4)), we can rewrite the hydro-static equilibrium condition in the following form

\[ \frac{r}{\rho} \frac{dP}{dr} + G \int_0^r 2\pi \rho r' dr' = 0 \]  

Now differentiating throughout by \( r \), we have after rearranging some of the terms

\[ \frac{1}{r} \frac{d}{dr} \left( \frac{r}{\rho(r)} \frac{dP}{dr} \right) + 2\pi G \rho = 0 \]  

Now using the polytropic equation of state (eqn.(1)) and replacing the radial coordinate \( r \) by the usual scaled radial parameter \( x \), defined by \( r = ax \), where \( a \) is an unknown constant, we have

\[ \left( \frac{n + 1}{a^2 nx} \right) \frac{d}{dx} \left( x^\frac{1}{n+1} \frac{d\rho}{dx} \right) + 2\pi G \rho = 0 \]  

Now using the techniques followed in the conventional three-dimensional case, we put

\[ \rho = \rho_c \theta^n \]  

where \( \rho_c \) is the central density and \( \theta \) is some dimensionless variable. Then the above equation (eqn.(7)) can be rewritten in the following form

\[ \frac{1}{x} \frac{d}{dx} \left( x \frac{d\theta}{dx} \right) = -\theta^n \]  

with

\[ a = \left[ \frac{(n + 1)K \rho_c^{\frac{1}{n+1}}}{2\pi G} \right]^\frac{1}{2} \]  

Eqn.(9) is the two-dimensional version of Lane-Emden equation. Whereas the conventional three dimensional form is given by

\[ \frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d\theta}{dx} \right) = -\theta^n \]  

Of course with different expression for the constant \( a \). To solve the Lane-emden equation (eqn.(9)), which is a second order differential equation, we need two initial conditions or one initial and one boundary condition. At the centre, \( \rho = \rho_c \), therefore \( \theta = 1 \), which is the maximum value of \( \theta \), therefore \( d\theta/dx = 0 \) at the center. The surface of the star is given by \( \rho = 0 \), which gives \( \theta = 0 \), the corresponding radial coordinate will be the radius of the star. We have noticed that only for \( n = 0 \) and \( n = 1 \), the Lane-Emden equation in two-dimension can be solved analytically. For \( n = 0 \), eqn.(9) becomes

\[ \frac{1}{x} \frac{d}{dx} \left( x \frac{d\theta}{dx} \right) = -1 \]  

The solution of this equation is given by

\[ \theta = 1 - \frac{x^2}{4} \]  

Since \( a = \infty \) for \( n = 0 \), the radius of the star also becomes infinitely large. The above solution is therefore not physically acceptable (this is also true for the three dimensional case). Now for \( n = 1 \), the Lane-Emden equation can in two-dimension be expressed in the form

\[ \frac{d^2 \theta}{dx^2} + \frac{1}{x} \frac{d\theta}{dx} + \theta = 0 \]  

\[ \frac{d^2 \theta}{dx^2} + \frac{1}{x} \frac{d\theta}{dx} + \theta = 0 \]
which is the well known Bessel differential equation of order zero. The solution is given by $\theta(x) = AJ_0(x)$, where $A$ is a constant. Since at $x = 0$, $\theta = 1$ and $J_0(0) = 1$, the constant $A = 1$. Now for $n = 1$, the radius of the object is found to be independent of central density $\rho_c$ ($a$ becomes independent of $\rho_c$). Therefore the analytical solution of Lane-Emden equation with $n = 1$ is also unphysical (this is also true for three dimensional situation). The Lane-Emden equation therefore solved numerically for general $n$-values, with the initial conditions mentioned above.

The mass of the star is obtained from the integral

$$M = \int_0^R 2\pi r dr \rho(r)$$

(15)

Expressing $\rho$ in terms of the variable $\theta$, $r$ in terms of $x$ in the Lane-Emden equation (eqn.(9)), we finally have

$$M = -2\pi a^2 \rho_c x_s \frac{d\theta}{dx} \bigg|_{x=x_s}$$

(16)

where $x_s$ is the surface value of the scaled radius parameter. The surface term of the derivative is obtained from the numerical solution of Lane-Emden equation. The actual radius of the star is then given by

$$R = sx_s$$

(17)

Now to obtain mass-radius relation for such hypothetical stellar object, we consider two different scenarios for the internal structure of the object. We first assume that both the kinetic pressure and the thermal energy density of the stellar matter are coming from the baryonic part only. Then following any standard text book on statistical mechanics (see for example [3]), it can very easily be shown that for the non-relativistic situation, the energy density and kinetic pressure of the matter are given by

$$\epsilon = \frac{g_B}{4\pi m_p \hbar^2} \int_0^\infty p^3 f(p) dp$$

(18)

and

$$P = \frac{g_B}{4\pi m_p \hbar^2} \int_0^\infty p^3 f(p) dp$$

(19)

respectively. Where $g_B$ is the spin degeneracy of baryons, $f(p)$ is the Fermi distribution and $m_p$ is the baryon mass. Therefore in this scenario, the polytropic form of equation of state will be of the same type as we have written for the non-relativistic case, of course with different $K$. If we now compare the equation of state with the standard form of polytropic equation, the index $n$ will be $\infty$. As a consequence, mass of the star $M \propto \rho_0^{1/n}$ becomes independent of central density and the radius of the star $R \propto \rho_0^{1/(1-n)2n}$ will become $R \propto \rho_0^{-1/2}$. Therefore the mass of the star becomes independent of central density. If we consider the matter as a mixture of proton and electron, then for the non-relativistic case, the baryon mass has to be replaced by $m_B = \mu m_p$, where $\mu$ is a number, called mean molecular weight and is 0.5 for the case of fully ionized hydrogen gas. Of course the ultra-relativistic expressions will not change, only $g_B$ will be 4 instead of 2.

Next we consider a white dwarf like star in two dimension. The degeneracy pressure of the electron gas makes the star stable against gravitational collapse. On the other hand the mass of
the object is coming from the massive baryonic part, which are assumed to be in static condition, therefore does not contribute in pressure. For such a stellar object, the mass density is given by

$$\rho = n_e m_p = \frac{m_p}{2\pi\hbar^2}P_{F_e},$$  \hspace{1cm} (22)

where $n_e$ is the electron surface density, given by

$$n_e = \frac{p_{F_e}}{2\pi\hbar^2},$$  \hspace{1cm} (23)

$m_p$ is the baryon mass and $p_{F_e}$ is the electron Fermi momentum. Following [3], the degeneracy pressure for electron gas in the non-relativistic and relativistic scenarios are given by

$$P = \frac{1}{8\pi\hbar^2 m_e}p_{F_e}^4$$  \hspace{1cm} (24)

and

$$P = \frac{c}{6\pi\hbar^2}p_{F_e}^3$$  \hspace{1cm} (25)

respectively. Here we have taken the spin degeneracy for electron $g_e = 2$. Hence we can write down the polytropic equation of states in the non-relativistic and ultra-relativistic cases in the form

$$P = K\rho^n \quad \text{and} \quad P = K'\rho^{3/2}$$  \hspace{1cm} (26)

Hence $n = 1$ for the non-relativistic case and $n = 2$ for the ultra-relativistic situation. Then it can very easily be shown that for non-relativistic scenario, the mass of the star $M \propto \rho_c$ and the radius becomes independent of central density, which is physically unacceptable. On the other hand for $n = 2$ case, $M \propto \rho_c^{1/2}$ and the radius $R \propto \rho_c^{-1/4}$. Hence the mass-radius relation is $MR^2 = \text{constant}$.

Whereas for the conventional white dwarf scenario, $n = 3/2$ for the non-relativistic case and $n = 3$ for the ultra-relativistic case are used. If we consider these values for two-dimensional case, the mass-radius relations for non-relativistic and ultra-relativistic scenarios are given by $MR^4 = \text{constant}$ and $MR = \text{constant}$ respectively.

3 Conclusion

In this work we have solved an hypothetical problem associated with two-dimensional polytropic stars. We strongly believe that the derivations will be useful for Master of Science level students. This problem can also be suggested as dissertation project work for the students.

References

[1] S.L. Shapiro and S.A. Teukolsky, Black Holes, White Dwarfs and Neutron Stars, John Wiley and Sons, New York, (1983).

[2] H.Q. Huang and K.N, Yu, Stellar Astrophysics, Springer, (1998).

[3] L.D. Landau and E.M Lifshitz, Statistical Physics, Part-I, Butterworth/Heinemann, Oxford, (1998).

[4] M. Abramowitz and I.A Stegun (ed.), Handbook of Mathematical Functions, Dovar Publication, NY, (1972) pp. 355.