Continuity on directed spaces

Yuxu Chen, Hui Kou*, Zhenchao Lyu
Department of Mathematics, Sichuan University, Chengdu 610064, China

Abstract

We first investigate two approximation relations on a $T_0$ topological space, the $n$-approximation and the $d$-approximation, which are generalizations of the waybelow relation on a dcpo. Different kinds of continuous spaces are defined by the two approximations and are all shown to be directed spaces. We will show that the continuity of a directed space is very similar to the continuity of a dcpo in many aspects, which indicates that the notion of directed spaces is a suitable topological extension of dcpos.

The main results are: (1) A topological space is continuous iff it is a retract of an algebraic space. (2) A directed space $X$ is core compact iff for any directed space $Y$, $X \times Y = X \otimes Y$, where $X \times Y$ and $X \otimes Y$ are the topological product and categorical product in $D\text{Top}$ of $X$ and $Y$ respectively; (3) A directed space is continuous (resp., algebraic, quasicontinuous, quisialgebraic) iff the lattice of its closed subsets is continuous (resp., algebraic, quasicontinuous, quisialgebraic).

Keywords: directed space, approximation relation, continuous space, c-space, core compact, lattice of open subsets, lattice of closed subsets

1. Introduction

In domain theory, the continuity of a dcpo is defined through the waybelow relation. It was shown that a dcpo is continuous iff the lattice of its Scott open subsets is a completely distributive lattice. More generally, any $T_0$ topological space is a c-space iff the lattice of its open subsets is a completely distributive lattice. A continuous dcpo endowed with the Scott topology can be viewed as a special c-space. The natural question arises whether it is possible to define an approximation relation on $T_0$ topological spaces such that c-spaces are some kind of continuous spaces defined through the approximation, similar to the continuity and the waybelow relation on a dcpo.

Ershov [6] came up with an approximation relation on a topological space defined by $x$ approximates $y$ iff $y \in (\uparrow x)^c$ in 1973. However it is defined through the interior instead of directed subsets like the waybelow relation. In the past several years, some attempts were made to define approximation relations on a $T_0$ space like the waybelow relation on a dcpo. In 2014, Wang and Kou [13] introduced the $n$-approximation relation on $T_0$ topological spaces through convergent nets and then defined $n$-continuous spaces. They showed that a topological space is a $n$-continuous space iff it is a c-space. Thus, the $n$-approximation and $n$-continuity seem appropriate candidate for defining the continuity of a $T_0$ topological space. However, these techniques in domains theory can not be used in $n$-continuous spaces since the $n$-approximation is based on nets rather than directed subsets.

Directed space was introduced by Yu and Kou [14] in 2015, which is a generality of poset endowed with the Scott topologies. The idea is that the priori is a $T_0$ space and its converging directed nets (directed subsets) relative to the specialization order rather than a poset and its existing directed supremums. $D\text{Top}$, the category of directed spaces was shown to be cartesian closed. Directed spaces are similar to dcpos in many aspects and therefore they can be viewed as an extension of dcpos to topological spaces (see [15, 16, 17]). In $T_0$ topological spaces, the notion of directed spaces is equivalent to that of monotone determined spaces defined by Erné in [5].

C-spaces (or $n$-continuous spaces) are all directed spaces. Based on this, Feng and Kou [15] introduced another approximation relation on directed spaces, called $d$-approximation, and defined the quasicontinuous directed spaces in 2017. They showed that quasicontinuous spaces are same to locally hypercompact spaces.

In this paper, we will first investigate the relation between the two approximation relations and the continuity defined by them. It is shown that a $T_0$ topological space is $n$-continuous (resp., $n$-algebraic, $n$-quasicontinuous, $n$-quisialgebraic).
n-quasialgebraic) iff it is a d-continuous (resp., d-algebraic, d-quasicontinuous, d-quasialgebraic) directed space iff it is a c-space (resp., c-space, locally hypercompact space, hypercompactly based space). We call them continuous (resp., algebraic, quasicontinuous, quasialgebraic) space uniformly. In classical domain theory, it is well known that continuous dcpos are just the retracts of algebraic dcpos. It is the same for continuous spaces and algebraic spaces. Here, we give a proof by the structures called topological ideals which are extensions of the rounded ideals of dcpos.

Since all these continuous and quasicontinuous spaces are directed spaces, when investigating spaces with continuity, the scope should be focused on directed spaces. In domain theory, there is a correspondence between and algebraic spaces. Here, we give a proof by the structures called topological ideals which are extensions of well known that continuous dcpos are just the retracts of algebraic dcpos. It is the same for continuous (resp., algebraic, quasicontinuous, quisialgebraic) space uniformly. In classical domain theory, it is correspondences can be extended to directed spaces very smoothly. The key result is that a directed space is \(L\)-space, locally hypercompact space, hypercompactly based space. We call them continuous (resp., algebraic, quasicontinuous, quasialgebraic) iff the lattice of its closed subsets is continuous (resp., algebraic, quasicontinuous, quasialgebraic).

When \(X\) is a dcpo endowed with the Scott topology, it is a special directed space and the concepts of \(d\)-approximation and \(d\)-continuity agree with the waybelow relation and continuity in classical domain theory. The categorical products and exponential objects in the category of dcpos agree with those in \(\mathbf{DTop}\) by viewing dcpos as directed spaces endowed with the Scott topology. Thus, the notion of continuous spaces and directed spaces can be viewed as a natural extension of continuous domains and dcpos to \(T_0\) topological spaces. The aim of defining approximation relations on a \(T_0\) topological space and extend the domain theory to topological space is reached to some extent.

2. Preliminaries

We assume some knowledge of basic domain theory and topology, as in, e.g., \(\mathbb{1}, \mathbb{2}\).

A nonempty set \(P\) endowed with a partial order \(\leq\) is called a poset. For \(A \subseteq P\), we set \(\downarrow A = \{x \in P : \exists a \in A, x \leq a\}\), \(\uparrow A = \{x \in P : \exists a \in P, a \leq x\}\). \(A\) is called a lower or upper set, if \(A = \downarrow A\) or \(A = \uparrow A\) respectively. For any element \(a \in P\), we use \(\downarrow a\) or \(\uparrow a\) instead of \(\downarrow\{a\}\) or \(\uparrow\{a\}\), respectively. A nonempty subset \(D\) of \(P\) is called directed if every nonempty finite subset of \(D\) has an upper bound in \(D\). Particularly, we say that \(P\) is a directed complete poset or a dcpo if every directed subset \(D\) of \(P\) has a supremum (or a least upper bound, denoted by \(\bigvee D\) or \(\bigvee D\)) in \(P\).

Let \(P\) be a poset. We denote \(\sigma(P), v(P)\) and \(A(P)\) to be the Scott topology, the upper topology and the Alexandroff topology respectively. For \(x, y \in P\), we say that \(x\) is way-below \(y\), denoted by \(x \ll_s y\), if for any directed subset \(D\) of \(P\) with \(\bigvee D\) existing, \(y \leq \bigvee D\) implies \(x \leq d\) for some \(d \in D\). \(P\) is called a continuous poset if \(\{a \in P : a \ll_s x\}\) is directed and has \(x\) as its supremum for all \(x \in P\). In Section 3 of this paper, we will define an approximation relation \(\ll\) and a notion of continuity on a \(T_0\) space. Here, we use \(\ll_s\) to denote the above order-theoretic one in domain theory in particular.

Topological spaces will always be supposed to be \(T_0\). For a topological space \(X\), its topology is denoted by \(\mathcal{O}(X)\) or \(\tau\). The partial order \(\subseteq\) defined on \(X\) by \(x \subseteq y \Leftrightarrow x \in \{y\}\) is called the specialization order, where \(\{y\}\) is the closure of \(\{y\}\). From now on, all order-theoretical statements about \(T_0\) spaces, such as upper sets, lower sets, directed sets, and so on, always refer to the specialization order \(\ll\). For any two topological spaces \(X, Y\), we denote \(X^Y\) or \(\mathbf{TOP}(X, Y)\) the set of all continuous maps from \(X\) to \(Y\), endowed with the pointwise order. Denote \([X \to Y]_p\) and \([X \to Y]_I\) to be the topological space equipped with the topology of pointwise convergence and the Isbell topology on \(X^Y\) respectively.

We now introduce the notion of a directed space.

Let \((X, \mathcal{O}(X))\) be a \(T_0\) space. Every directed subset \(D \subseteq X\) can be regarded as a monotone net \(\{d\}_{d \in D}\). Set \(DS(X) = \{D \subseteq X : D\) is directed\} to be the family of all directed subsets of \(X\). For an \(x \in X\), we denote \(D \to x\) to mean that \(x\) is a limit of \(D\), i.e., \(D\) converges to \(x\) with respect to the topology on \(X\). Then the following result is obvious.

**Lemma 2.1.** Let \(X\) be a \(T_0\) space. For any \((D, x) \in DS(X) \times X\), \(D \to x\) if and only if \(D \cap U \neq \emptyset\) for any open neighborhood of \(x\).

Set \(DLim(X) = \{(D, x) \in DS(X) \times X : D \to x\}\) to be the set of all pairs of converging monotone nets and their limits in \(X\). Then \(\{\{y\}, x\} \in DLim(X)\) iff \(x \ll y\) for all \(x, y \in X\).

**Definition 2.2.** Let \(X\) be a \(T_0\) space. A subset \(U \subseteq X\) is called directed-open if for all \((D, x) \in DLim(X), x \in U\) implies \(D \cap U \neq \emptyset\).

Obviously, every open set of \(X\) is directed-open. Set \(d(\mathcal{O}(X)) = \{U \subseteq X : U\) is directed-open\}, then \(\mathcal{O}(X) \subseteq d(\mathcal{O}(X))\).
Theorem 2.3. 14 Let $X$ be a $T_0$ topological space. Then

(1) For all $U \in d(O(X))$, $U = \uparrow U$;
(2) $X$ equipped with $d(O(X))$ is a $T_0$ topological space such that $\subseteq_d = \subseteq$, where $\subseteq_d$ is the specialization order relative to $d(O(X))$.
(3) For a directed subset $D$ of $X$, $D \rightarrow x$ if $D \rightarrow_d x$ for all $x \in X$, where $D \rightarrow_d x$ means that $D$ converges to $x$ with respect to the topology $d(O(X))$.
(4) $d(d(O(X))) = d(O(X))$.

Definition 2.4. 14 A topological space $X$ is said to be a directed space if it is $T_0$ and every directed-open set is open; equivalently, $d(O(X)) = O(X)$.

Example 2.6.

(1) The exponential object $X \rightarrow Y$ is an upper set relative to the specialization order of $X \times Y$.

(2) The following two conditions are equivalent to each other:
   - (i) $X$ is a directed space;
   - (ii) For all $U \subseteq X$, $U$ is open iff for any $(D, x) \in DLim(X)$, $x \in U$ implies $U \cap D \neq \emptyset$.
   - (iii) For all $A \subseteq X$, $A$ is closed iff for any directed subset $D \subseteq A$, $D \rightarrow x$ implies $x \in A$ for all $x \in X$.

Directed spaces include many important structures in domain theory.

Theorem 2.7. 14 Directed spaces are extensions of dcpo to topological spaces, where the directed space is defined through

$$D = \{\{x, y\} \mid x \rightarrow Y \}$$

and replacing suprema to limits. It seems to be a successful description of one kind of approximation on a $T_0$ space since it was proved that a space is $n$-continuous iff it is a c-space.

3. Approximating relations and continuity on directed space

As mentioned before, Wang and Kou 13 introduced the notion of the $n$-approximation relation on a $T_0$ topological space, which is similar to the definition of the waybelow relation by replacing directed subsets to nets and replacing suprema to limits. It seems to be a successful description of one kind of approximation on a $T_0$ space since it was proved that a space is $n$-continuous iff it is a c-space.

Directed space are extensions of dcpo to topological spaces, where the directed space is defined through the directed converging nets. Next, Feng and Kou introduced the notion of $d$-approximation relation and $d$-continuity on a directed space. Compared to $n$-approximation, the advantage of $d$-approximation is that the nets are replaced by directed subsets, which seems easier to use and closer to the notion of waybelow relation.
In this section, we will first investigate the relationships between the $n$-approximation and the $d$-approximation on a $T_0$ topological space. It is shown that a $T_0$ topological space is $n$-continuous (resp., $n$-algebraic, $n$-quasicontinuous, $n$-quasialgebraic) iff it is a $d$-continuous (resp., $d$-algebraic, $d$-quasicontinuous, $d$-quasialgebraic) directed space iff it is a $c$-space (resp., $a$-space, locally hypercompact space, hypercompactly based space). These results indicate that the continuity defined by $n$-approximation and $d$-approximation are suitable extension of continuity of a dcpo to topological spaces. Continuous spaces in directed spaces are similar to continuous dcpos in dcpos. Naturally, the relationship between the continuity of a $T_0$ topological space and the lattice of its open subsets are displayed through the line of the two approximation relations.

We first compare the $n$-approximation and the $d$-approximation.

**Definition 3.1.** Let $X$ be a $T_0$ topological space and $x, y \in X$.

1. We say that $x$ $d$-approximates $y$, denoted by $x \ll_d y$, if for any directed subset $D \subseteq X$, $D \rightarrow y$ implies $x \subseteq d$ for some $d \in D$. If $x \ll_d x$, then $x$ is called a $d$-compact element of $X$.
2. We say that $x$ $n$-approximates $y$ if for all net $(y_j)_{j \in J}$ (shortly, $(y_j)_J$) of $X$, $(y_j)_J \rightarrow y$ implies $x \subseteq y_{j_0}$ for some $j_0 \in J$. We call $x$ $n$-compact if it $n$-approximates itself.

We introduce the following notation for $x, y \in X$ and $A \subseteq X$, where $i = d, n$:

- $\downarrow_i x = \{ y \in X : y \ll_i x \}$
- $\uparrow_i x = \{ y \in X : x \ll_i y \}$
- $K_i(X) = \{ x \in X : x \ll_i x \}$

The following is some simple properties of the approximation relation.

**Proposition 3.2.** Let $X$ be a directed space and $i = d, n$. Then

1. $x \ll_i y$ implies $x \subseteq y$,
2. $z \subseteq x, y \subseteq s$ implies $z \ll_i s$.

It is easily seen from the definition of $n$-approximation is stronger than that of $d$-approximation, i.e., $x \ll_n y$ implies $x \ll_d y$. Is the converse true? We give two negative examples, where the first one is endowed with the upper topology and the second one is endowed with the Scott topology and hence it is a directed space.

**Example 3.3.** Let $\mathbb{N}$ be the set of natural numbers.

1. Denote $\mathbb{N}^\uparrow$ the flat domain, i.e., the poset with carrier set $\mathbb{N} \cup \{ \top \}$ and $x \leq y$ iff $y = \top$. It is easily seen that $\mathbb{N}^\uparrow$ is a dcpo. Denote by $\mathbb{N}^\uparrow_1$ the $\mathbb{N}^\uparrow$ adding a bottom element, then it is a complete lattice.
2. Let $\omega = \{ \omega_i : i \in \mathbb{N} \}$ and $A = (\mathbb{N} \times \mathbb{N}) \cup \omega$ endowed with a partial order as follows:
   - (i) $\forall (m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}$, $(m_1, n_1) \leq (m_2, n_2)$ iff $m_1 = m_2$ and $n_1 \leq n_2$.
   - (ii) $\forall i, m, n \in \mathbb{N}$, $(m, n) \leq \omega_i$ iff $m = i$ or $m = 1, n \leq i$.
3. Considering the upper topology on $\mathbb{N}^\uparrow$ (or $\mathbb{N}^\uparrow_1$), we have:
   - (a) $\top \ll_d \top$; $\top \nleq_n \top$;
   - (b) $(\mathbb{N}^\uparrow, v(\mathbb{N}^\uparrow))$ is a locally compact sober space;
   - (c) $(\mathbb{N}^\uparrow_1, v(\mathbb{N}^\uparrow_1))$ is a directed space.
4. Considering the Scott topology on $P$, we have $\forall n \in \mathbb{N}$, $(1, n) \ll_d \omega_1$ & $(1, n) \nleq_n \omega_1$.

From the two examples we know that even in directed spaces, the two approximation relations may be different. We will see from the following statement that when $P$ is a dcpo, the $d$-approximation on $(P, \sigma(P))$ is equal to the waybelow relation on $P$. By Example 3.3(2), the $n$-approximation may be different to the waybelow relation in a dcpo. Thus, the notion of $d$-approximation seems better than $n$-approximation as an extension of waybelow relation.

**Lemma 3.4.** Let $X$ be a $T_0$ topological space and $D$ be a directed subset of $X$.

1. If $D \subseteq \downarrow x$ and $D \rightarrow x$ for $x \in P$, then $x = \sup D$.
2. If $D \rightarrow x$ and $\sup D$ exists, then $x \leq \sup D$.
3. If $D \rightarrow \sup D$, then $D \rightarrow x$ for any $x \leq \sup D$.

**Proof.** (1) Suppose that $D \subseteq \downarrow x$ and $D \rightarrow x$ for $x \in X$. Assume that there exists $y \in X$ such that $D \subseteq \downarrow y$ and $x \neq y$. Then $x \in P \setminus \downarrow y \in O(X)$. Thus $D \cap X \setminus \downarrow y \neq \emptyset$. It is a contradiction. Hence, $x = \sup D$.

(2) Since $D \rightarrow x$, given any open subset $U$ such that $x \in U$, there exists some $d \in D$ such that $d \in U$ and then $\sup D \in U$. Therefore, $x \leq \sup D$.

(3) Obviously. □
By Lemma 3.4, we know that given any dcpo \( P \) and directed subsets \( D \) of \( P \), then \( D \to x \in P \) in \( \sigma(P) \) iff \( x \leq \sup D \). Thus, the \( d \)-approximation is equal to the waybelow relation if we view dcpos to be topological spaces endowed with the Scott topology. We now give the definition of continuous spaces and we will see that for a dcpo \( P \), \( (P, \sigma(P)) \) is a continuous space if and only if \( P \) is a continuous dcpo in the sense of the classical domain theory. Moreover, in any continuous space, the \( d \)-approximation is equal to \( n \)-approximation.

**Definition 3.5.** Let \( X \) be a \( T_0 \) topological space.

(1) \( X \) is called \( d \)-continuous if it is a directed space such that \( \downarrow_{d} x \) is directed and \( \downarrow_{d} x \to x \) for all \( x \in X \).

(2) \( X \) is called \( n \)-continuous if for every \( x \in X \), there exists a net \( (x_j)_J \subseteq \downarrow_{n} x \) such that \( (x_j) \to x \).

**Lemma 3.6.** \[10\] Let \( X \) be a \( d \)-continuous space. Then we have the following statements.

(1) For all \( x, y \in X \), \( x \ll_{d} y \) implies \( x \ll_{d} z \ll_{d} y \) for some \( z \in X \).

(2) \( \uparrow_{d} x = (\uparrow x)^{\circ} \) for all \( x \in X \). Moreover, \( \{\uparrow_{d} x : x \in X\} \) is a basis of the topology of \( X \).

(3) For all \( x, y \in X \), the following are equivalent:
   (i) \( x \ll_{d} y \);
   (ii) \( y \in (\uparrow x)^{\circ} \).
   (iii) For any net \( (x_j)_J \subseteq X \), \( (x_j) \to y \) implies \( x \subseteq x_{j_0} \) for some \( j_0 \in J \).

**Lemma 3.7.** \[13\] For \( T_0 \) space \( X \), the following are equivalent: for all \( x, y \in Y \),

(1) \( x \ll_{n} y \);

(2) \( y \in (\uparrow x)^{\circ} \), where \( (\uparrow x)^{\circ} \) means the interior of \( \uparrow x \);

(3) For all net \( (y_j)_J \to y \) implies \( \{y_j\}_J \) is eventually in \( \uparrow x \).

We can see from \[3, 7\] that \( x \ll_{n} y \) iff \( x \) approximates \( y \) in the sense of Ershov. Thus, the notion of \( n \)-approximation seems closer to \( c \)-spaces. For the concepts of continuity in domain theory, the notion of a \( c \)-space is another famous generalization.

**Definition 3.8.** A topological space \( X \) is called a \( c \)-space, if it is \( T_0 \) and every point \( y \) has a neighborhood basis of sets of the form \( \uparrow x \).

\( c \)-spaces had been studied by Erné \[3, 4\] and by Ershov \[7, 8\] under the name of an \( \alpha \)-space. Adding the notions of \( d \)-continuity and \( n \)-continuity we introduce above, there are three generalizations of continuity in domain theory. What is the relationship among them? Combining the results in \[13\] and \[10\], we have the following statement.

**Theorem 3.9.** For a \( T_0 \) topological space \( X \), the following are equivalent:

(1) \( X \) is a \( d \)-continuous directed space;

(2) \( X \) is a directed space and for each \( x \in X \), there is a directed subset \( D \subseteq \downarrow_{d} x \) such that \( D \to x \).

(3) \( X \) is \( n \)-continuous;

(4) for every \( x \in X \), \( \downarrow_{n} x \) is directed and converges to \( x \);

(5) for every \( x \in X \), there exists a directed subset \( D \subseteq \downarrow_{n} x \) such that \( D \to x \);

(6) \( X \) is a \( c \)-space;

(7) \( \mathcal{O}(X) \) is a completely distributive lattice.

**Proof.** (3) \( \iff \) (4) \( \iff \) (6) \( \iff \) (7) was proved in \[13\]. (6) \( \iff \) (1) \( \iff \) (2) was proved in \[10\]. (4) \( \Rightarrow \) (5) and (5) \( \Rightarrow \) (3) are obvious. \( \square \)

**Remark 3.10.** When defining a \( d \)-continuous space, we require it to be a directed space. Otherwise, it may not be a \( c \)-space. Example \[3, 7\] (1) is an example. \( n \)-continuous spaces and \( c \)-spaces are all directed spaces. Therefore, it would better restrict the spaces in directed spaces when discussing continuity.

From now on, we give a uniform name to replace the three notions of continuity.

**Definition 3.11.** A topological space is called continuous if one of the seven equivalent conditions in Theorem 3.9 holds.

From Lemma \[3, 6\] and Lemma \[3, 7\], the \( d \)-approximation and \( n \)-approximation coincide when \( X \) is a continuous space. For convenience, both \( d \)-approximation and \( n \)-approximation in a continuous space are replaced by \( \ll \) uniformly in the sense of Definition 3.11. From now on, we do not use the subscripts \( d, n \).

**Remark 3.12.** (1) For not leading to confusion, a continuous dcpo or poset always means the order-theoretic one in domain theory as defined in Section 2.
(2) Every continuous poset is exactly a continuous space endowed with the Scott topology.
(3) Every $T_0$ Alexandroff space is a continuous space. So continuity in our sense is a proper extension of the one in Domain theory.

We next give the definition of an algebraic space.

**Definition 3.13.** Let $X$ be a topological space. $X$ is called $d$-algebraic if it is a directed space such that \( \downarrow x \cap K_d(X) \) is directed and converges to $x$ for all $x \in X$. $X$ is called $n$-algebraic if \( \downarrow x \cap K_n(X) \) is directed and converges to $x$ for all $x \in X$.

Obviously, all $d$-algebraic spaces and $n$-algebraic spaces are continuous spaces. By Theorem 3.6 and Theorem 3.7, we know that $d$-compact elements coincide with $n$-compact elements for continuous spaces. Thus, the notion of a $d$-algebraic space is equivalent to a $n$-algebraic space. We call them algebraic spaces uniformly.

Similarly as Theorem 3.9, we have the following result.

**Theorem 3.14.** For a $T_0$ topological space $X$, the following conditions are equivalent:

1. $X$ is algebraic;
2. $X$ is a B-space, i.e., $\forall x \in U \in \mathcal{O}(X), \exists y \in U, x \in \uparrow y = \uparrow y.$
3. $\mathcal{O}(X)$ is an algebraic distributively complete lattice.

The notion of a B-space is introduced by Erné [3] and by Ershov [8] under the name of a $\varphi$-space. When $X$ is a poset with the Scott topology, the above notion of an algebraic space coincides with the order theoretical one in domain theory.

In classical domain theory, it is shown that continuous domains are the retracts of algebraic domains. It can be extended to continuous spaces as well. We will construct a canonical algebraic space for each $T_0$ space and show that a $T_0$ space is continuous if and only if it is a continuous retract of an algebraic space.

**Definition 3.15.** Let $X$ be a $T_0$ space. A directed subset $D \subseteq X$ is called an ideal net if there exists $x \in X$ such that:

1. $D \to x,$
2. $\forall d \in D, d \subseteq x.$

Obviously, if $X$ is a poset endowed with the Scott topology, then every directed subset with a existing supremum is an ideal net.

**Lemma 3.16.** If $D \subseteq X$ is an ideal net of a $T_0$ space $X$, then $\bigvee D$ exists and $D \to \bigvee D$.

**Proof.** By definition, there exists some $x \in X$ such that $\bigvee D \leq x$ and $D \to x$, then we have $D \to \bigvee D$. \( \square \)

**Definition 3.17.** Let $X$ be a $T_0$ space. A subset $A$ of $X$ is called a topological ideal if there is an ideal net $D \subseteq X$ such that $A = \downarrow D$.

So a topological ideal $A$ is a directed lower subset such that it has a supremum (denoted by $\bigvee A$) and converges to its supremum. Particularly, we set $I_T(X)$ to be the set of all topological ideals of a $T_0$ space $X$.

Obviously, we have \( \{ \downarrow x : x \in X\} \subseteq I_T(X) \). Set

$$\Omega(I_T(X)) = \{ U \subseteq I_T(X) : A \in U \Leftrightarrow \exists x \in X, \downarrow x \in U \land \downarrow x \subseteq A \}.$$  

Particularly, for any $x \in X$, we set $U_x = \{ A \in I_T(X) : x \in A \}$, then $U_x \in \Omega(I_T(X))$.

**Lemma 3.18.** Let $X$ be a $T_0$ space. Then \( (I_T(X), \Omega(I_T(X))) \) is an algebraic space such that:

1. Its specialization order is equal to the set-theoretical inclusion;
2. $K(I_T(X)) = \{ \downarrow x : x \in X \}$.

**Proof.** Obviously, $\emptyset, I_T(X) \in \Omega(I_T(X))$ and $\Omega(I_T(X))$ is closed under unions. For $U, V \in \Omega(I_T(X))$ and $A \in U \cap V$, there are $x, y \in X$ such that $\downarrow x \in U$, $\downarrow y \in V$ and $x, y \in A$. Since $A$ is directed, there exists $a \in A$ with $x, y \subseteq a$. Thus $\downarrow a \in U \cap V$ and $\downarrow a \subseteq A$, i.e., $U \cap V \in \Omega(I_T(X))$. Hence, $\Omega(I_T(X))$ is a topology. If $A \not\subseteq B$ for $A, B \in I_T(X)$, then there is $a \in A$ together with $a \not\in B$; thus $A \in U_a$ and $B \not\subseteq U_a$. If $A \not\subseteq B$ for $A, B \in I_T(X)$ and $A \in U \in \Omega(I_T(X))$, then there exists some $\downarrow x$ such that $\downarrow x \subseteq A, \downarrow x \in U$; we have $\downarrow x \subseteq B$ and thus $B \in U$.

Hence, $(I_T(X), \Omega(I_T(X)))$ is $T_0$ and its specialization order is equal to the set-theoretical inclusion.

Let $x \in X, B$ be a directed subset of $I_T(X)$ and $B \to \downarrow x$. Since $U_x$ is open, there must exists some $B \subseteq B$ such that $B \in U_x$, i.e., $\downarrow x \subseteq B$. Hence, we have that $\downarrow x$ is a compact element of $(I_T(X), \Omega(I_T(X)))$. For any $A \in I_T(X)$, $A = \{ \downarrow x : x \in A \}$ is directed. For $U \in \Omega(I_T(X))$ with $A \in U$, there exists $a \in A$ such that
Thus, \( A \) converges to \( A \) in \( \mathcal{I}_r(X) \). It follows that \( \downarrow x : x \in X \) is a \( \ll \)-base of \( \mathcal{I}_r(X) \). Therefore, \( (\mathcal{I}_r(X), \Omega(\mathcal{I}_r(X))) \) is an algebraic space and \( K(\mathcal{I}_r(X)) = \{ \downarrow x : x \in X \} \).

By definition, every topological ideal is directed and converges to its supremum. Hence, there exists a surjective map from \( \mathcal{I}_r(X) \) onto \( X \) as follows:

\[
\bigvee: \mathcal{I}_r(X) \to X, \quad \forall A \in \mathcal{I}_r(X), \ A \mapsto \bigvee A.
\]

**Proposition 3.19.** For a \( T_0 \) space \( X \), \( \bigvee: \mathcal{I}_r(X) \to X \) is a continuous surjective.

**Proof.** Obviously, \( \bigvee: \mathcal{I}_r(X) \to X \) is surjective. For any \( U \in \mathcal{O}(X) \) and for \( A \in \mathcal{V}^{-1}(U) \), we have \( \bigvee A \subset U \).

Note that since \( A \) is a topological ideal, we have that \( A = \downarrow A \) and \( A \to \bigvee A \) in \( X \). There exists \( a \in A \) such that \( a \notin U \). Thus, \( \downarrow a \in \mathcal{V}^{-1}(U) \) and \( \downarrow a \subset A \). Hence, \( \mathcal{V}^{-1}(U) \in \Omega(\mathcal{I}_r(X)) \), i.e., \( \bigvee: \mathcal{I}_r(X) \to X \) is continuous. \( \square \)

Hence, every \( T_0 \) space is a continuous image of some algebraic space. Particularly, when \( X \) is a dcpo endowed with the Scott topology \( \sigma(X) \), \( \mathcal{I}_r(X) \) is exactly the ideal completion of \( X \) (see [3]).

**Proposition 3.20.** Let \( X \) be a dcpo endowed with the Scott topology \( \sigma(X) \). Then \( \mathcal{I}_r(X) = \text{Id}(X) \), where \( \text{Id}(X) \) is the set of all ideal of \( X \) endowed with the set-theoretical inclusion.

**Proof.** Given any topological ideal \( D \), it is an ideal by definition. Thus, \( D \in \text{Id}(X) \). Conversely, suppose that \( A \) is an ideal of \( X \). Since \( X \) is a dcpo endowed with the Scott topology and \( A \) is a directed subset, we know that \( A \to \bigvee A \). Hence, \( A \) is a topological ideal. \( \square \)

The following is the definition of adjunctions between posets.

**Definition 3.21.** Let \( f: P \to Q \) and \( g: Q \to P \) be a pair of monotone maps between posets \( P \) and \( Q \). We say that \((f, g)\) is a pair of adjunctions provided the following holds:

\[
\forall x \in P, y \in Q, \ f(x) \leq y \Leftrightarrow x \leq g(y).
\]

In this case, \( f \) is called the lower adjoint of \( g \) and \( g \) is called the upper adjoint of \( f \).

**Theorem 3.22.** Let \( X \) be a \( T_0 \) space. Then following conditions are equivalent to each other:

1. \( X \) is a continuous space;
2. \( \bigvee: \mathcal{I}_r(X) \to X \) has a continuous lower adjoint;
3. \( X \) is a continuous retract of some algebraic space.

**Proof.** (1) \( \Rightarrow \) (2): Let \( X \) be a continuous space. Then \( \downarrow x \) is directed and converges to \( x \) for any \( x \in X \) by Theorem. Note that since \( \downarrow x = \downarrow(\downarrow x) \), we have that \( \downarrow x \) is a topological ideal. Define a mapping \( \downarrow: X \to \mathcal{I}_r(X) \) as follows:

\[
\forall x \in X, \ \downarrow(x) = \downarrow x.
\]

Then \( \downarrow: X \to \mathcal{I}_r(X) \) is well defined and monotone. For any \( a \in X \) and for \( x \in \mathcal{V}^{-1}(U_a) \), we have \( a \leq x \).

Thus, there is some \( y \in X \) such that \( a \ll y \ll x \). Set \( U = \uparrow y \). Then \( U \) is an open neighborhood of \( x \) with \( U \subset \mathcal{V}^{-1}(U_a) \). Hence, \( \downarrow(X) \) is an open subset of \( X \). Note that since \( \{U_a: a \in X\} \) is a base of \( \Omega(\mathcal{I}_r(X)) \), we have that \( \downarrow(X) \to \mathcal{I}_r(X) \) is a continuous map. For any \( x \in X \) and \( A \in \mathcal{I}_r(X) \), \( x \leq \bigvee A \) if and only if \( \downarrow x \subset A \). Therefore, \( \downarrow: X \to \mathcal{I}_r(X) \) is the lower adjoint of \( \bigvee: \mathcal{I}_r(X) \to X \) by definition.

(2) \( \Rightarrow \) (3): Let \( f: X \to \mathcal{I}_r(X) \) be a continuous lower adjoint of \( \bigvee: \mathcal{I}_r(X) \to X \). Then \( \bigvee f = \text{id}_X \) for that \( \bigvee \) is surjective. Hence, \( X \) is a continuous retract of the algebraic space \( \mathcal{I}_r(X) \).

(3) \( \Rightarrow \) (1): Let \( Y \) be an algebraic space such that \( X \) is a continuous retract of \( Y \). Then there exist two continuous maps \((f, g): Y \to X \) and \((g, f): X \to Y \) with \( f \circ g = \text{id}_X \). For any \( x \in X \) and \( k \in K(Y) \), if \( k \leq g(x) \), then \( f(k) \leq f(g(x)) = x \). Let \( (x_j)_{j \in J} \) be a net of \( X \) with \( x_j \to x \). Since \( g \) is continuous, we have \( g(x_j) \to g(x) \).

As \( k \) is compact and \( k \leq g(x) \), there exists \( j_0 \in J \) such that \( k \leq g(x_{j_0}) \). Hence, \( f(k) \leq f(g(x_{j_0})) = x_{j_0} \), i.e.,

\[
f(k) \ll x \text{ by Definition.}
\]

Note that since \( Y \) is an algebraic space, \( \downarrow g(x) \cap K(Y) \) is directed and converges to \( g(x) \).

Thus, \( \{f(z) : z \in \downarrow g(x) \cap K(Y) \} \subseteq \downarrow x \) and \( (f(z))_{z \in \downarrow g(x) \cap K(Y)} \to f(g(x)) = x \). Therefore, \( X \) is a continuous space. \( \square \)

4. **Continuity and the lattice of open subsets**

Similarly with continuous spaces and algebraic spaces in the previous section, we introduce the concepts of quasicontinuous spaces and quasialgebraic spaces. We will show that \( n \)-quasicontinuous spaces are equal to \( d \)-quasicontinuous spaces as well and they are equal to locally hypercompact space. Finally, we summarize the correspondences between the continuity of a directed space and the lattice of its open subsets in this section.

Given any set \( X \), we use \( F \subseteq_f X \) to denote that \( F \) is a finite subset of \( X \). Given any two finite subsets \( G, H \subseteq_f X \), we define \( G \leq H \) iff \( \uparrow H \subseteq \uparrow G \). A family of finite sets is said to be directed if given \( F_1, F_2 \) in the family, there exists a \( F \) in the family such that \( F_1, F_2 \leq F \).
Definition 4.1. Let $X$ be a directed space, $y \in X$ and $G, H \subseteq_f X$. We say that $G$ $d$-approximates $H$, denoted by $G \ll_d H$, if for every directed subset $D \subseteq X$, $D \rightarrow y$ for some $y \in H$ implies $D \cap ↑G \neq \emptyset$. We write $G \ll_d x$ for $F \ll_d \{x\}$. $G$ is said to be compact if $G \ll_d G$.

Let $X$ be a $T_0$ topological space and $D(M)$ be a directed family of finite subsets of $X$. Then $F_M = \{A \subseteq X : \exists M \in D(M), ↑M \subseteq A\}$ is a filter under the inclusion order. We say that $D(M)$ converges to $x$ if $F_M$ converges to $x$, i.e., for any open neighbourhood $U$ of $x$, there exists some $A \in D(M)$ such that $A \subseteq U$.

Definition 4.2. A topological space $X$ is called $d$-quasicontinuous if it is a directed space such that for any $x \in X$, the family $\text{fin}_d(x) = \{F : F$ is finite, $F \ll_d x\}$ is a directed family and converges to $x$.

Obviously, every continuous space is $d$-quasicontinuous. When $P$ is a poset endowed with the Scott topology $\sigma(P)$, $P$ is $d$-quasicontinuous if and only if $P$ is quasicontinuous in the sense of the classical domain theory. Given any quasicontinuous space $X$ and $F \subseteq_f X$, we write $\sqcup_d F = \{x \in X : F \ll_d x\}$.

Lemma 4.3. Let $X$ be a $d$-quasicontinuous space. Then we have the following.

1. Given any $H \subseteq_f X$ and $y \in X$, $H \ll_d y$ implies $H \ll_d F \ll_d y$ for some finite subset $F \subseteq_f X$.
2. Given any $F \subseteq_f X$, $\sqcup_d F = (\sqcup F)^\circ$. Moreover, $\{\sqcup_d F : F \subseteq_f X\}$ is a basis of the topology of $X$.
3. Given any $G, H \subseteq_f X$, the following are equivalent:
   
   1. $G \ll_d H$;
   2. $H \subseteq (\sqcup G)^\circ$;
   3. For any net $(x_j)_J \subseteq X$, $(x_j) \rightarrow y, y \in H$ implies $\sqcup G \cap (x_j)_J \neq \emptyset$.

Proof. (1) and (2) were proved in [15] and the proof is analogous to that for quasicontinuous domains. We only prove (3).

(i)⇒(ii) follows directly from (2) and the fact that $G \ll_d H$ if $G \ll_d x$ for every $x \in H$. (ii)⇒(iii) and (iii)⇒(i) are obvious from the definitions of convergence and $d$-approximation. □

Now we define $n$-quasicontinuous spaces.

Definition 4.4. Let $X$ be a $T_0$ space and $G, H \subseteq_f X$. We say that $G$ $n$-approximates $H$, denoted by $G \ll_n H$, if for every net $(x_j)_J$ of $X$, $(x_j)_J \rightarrow y$ for some $y \in H$ implies $(x_j)_J \cap ↑G \neq \emptyset$. We write $G \ll_n x$ for $G \ll_n \{x\}$. $G$ is said to be $n$-compact if $G \ll_n G$.

Theorem 4.5. For any $T_0$ space $X$, the following are equivalent: for all $x, y \in Y$,

1. $G \ll_n H$,
2. $H \subseteq (\sqcup G)^\circ$, where $(\sqcup G)^\circ$ means the interior of $\sqcup G$;
3. For every net $(y_j)_J$, $(y_j) \rightarrow y$ and $y \in H$ implies that $\{y_j\}_J$ is eventually in $\sqcup G$.

Proof. (2) ⇒ (3) and (3) ⇒ (1) are obvious. We only show (1) ⇒ (2). Assume that $y \in H, y \notin (\sqcup G)^\circ$. Then, for any open set $U \subseteq X$ with $y \in U$, there exists a $y_U \in U$ such that $y_U \notin \sqcup G$. Set $J = \{U \in \mathcal{O}(X) : y \in U\}$ ordered as: $\forall U, V \in J$, $U \subseteq V \Leftrightarrow V \subseteq U$.

Then $J$ is directed. Easily, one sees that $(y_U)_{U \in J}$ is a net converging to $y$. Since $G \ll_n y$, there exists $U_0 \in J$ such that $x \subseteq y_{U_0}$ for some $x \in G$ by definition. It is a contradiction for $y_{U_0} \notin \sqcup G$. Hence, $y \in (\sqcup G)^\circ$. □

From Lemma [1.3] and Theorem 4.5, the notion of $d$-approximation and $n$-approximation between finite subsets coincide when $X$ is a $d$-quasicontinuous directed space. Since the $n$-approximation does not depend on directed space, a notion of $n$-quasicontinuity for a $T_0$ space can be defined as follows.

Definition 4.6. A $T_0$ topological space $X$ is called $n$-quasicontinuous if for each $x \in X$, there is a directed family $D(x) \subseteq \text{fin}_n(x)$ such that $D(x) \rightarrow x$.

Definition 4.7. A $T_0$ topological space $X$ is called locally hypercompact, if for any open subsets $U$ and $x \in U$, there exists some $F \subseteq_f X$ such that $x \subseteq \sqcup F \subseteq U$. $X$ is called locally hypercompact if for any open subsets $U$ and $x \in U$, there exists some $F \subseteq_f X$ such that $x \subseteq \sqcup F = \sqcup F \subseteq U$.

Theorem 4.8. For a $T_0$ topological space $X$, the following are equivalent:

1. $X$ is a $d$-quasicontinuous directed space;
2. $X$ is a directed space and for each $x \in X$, there is a directed family $D(x) \subseteq \text{fin}_d(x)$ such that $D(x) \rightarrow x$.
3. $X$ is $n$-quasicontinuous;
4. for every $x \in X$, $\text{fin}_n(x) = \{F : F$ is finite, $F \ll_n x\}$ is a directed family and converges to $x$;
5. $X$ is a locally hypercompact space;
(6) $\mathcal{O}(X)$ is hypercontinuous.

(7) $X^S$ is a quasiadicnous dcpo.

Proof. (4) $\Rightarrow$ (3) is obvious. (1) $\Rightarrow$ (2) $\iff$ (5) $\iff$ (6) $\iff$ (7) was proved in [15].

(2) $\Rightarrow$ (3): by (2) $\Rightarrow$ (1), we know $X$ is $d$-quasicontinuous. By Lemma [13] and [14] for any $G \subseteq_f X$ and $x \in X$, $G \in \text{fin}_n(x)$ iff $G \in \text{fin}_n(x)$. Thus $D(x) \subseteq \text{fin}_n(x)$ and $D(x) \to x$, i.e., $X$ is $n$-quasicontinuous.

(3) $\Rightarrow$ (4): Suppose that $D(x) \subseteq \text{fin}_n(x)$ and $D(x) \to x$. Given any $G, H \in \text{fin}_n(x)$, then $x \in (\uparrow G) \cap (\uparrow H)$. By $D(x) \to x$, we have that there exists some $F \in D(x)$ such that $F \subseteq (\uparrow G) \cap (\uparrow H)$. Then $F \leq_d G, H$ by Theorem [13] and thus $F \leq G, H$, i.e., $\text{fin}_n(x)$ is directed. $\text{fin}_n(x) \to x$ follows from $D(x) \to x$ and $D(x) \subseteq \text{fin}_n(x)$.

(4) $\Rightarrow$ (5): Given any open subsets $U \subseteq X$ and $x \in U$, since $\text{fin}_n(x) \to x$, there exists some $F \in \text{fin}_n(x)$ such that $F \subseteq U$. By Theorem [13] $F \in \text{fin}_n(x)$ implies $x \in (\uparrow F)^0$. Therefore, $x \in (\uparrow F)^0 \subseteq \uparrow F \subseteq U$. $\square$

We give a uniform name to replace these notions of quasicontinuity.

Definition 4.9. A $T_0$ topological space is called quasicontinuous if one of the seven equivalent conditions in Theorem [13] holds.

The $\prec$ relation on the lattice of open subsets of a topological space can be characterized as follows.

Lemma 4.10. Let $X$ be a topological space and $U, V \in \mathcal{O}(X)$. $U \prec V$ iff $U \subseteq \uparrow F \subseteq V$ for some $F \subseteq_f X$.

Let $\text{compfin}_n(X) = \{F : F$ is finite, $F \leq_d F\}$ for $i = d, n$. A directed space $X$ is called $d$-quasicontinuous if for any $x \in X$, $\text{compfin}_d(X) \cap \text{fin}_d(x)$ is a directed family and converges to $x$. A $T_0$ topological space $X$ is called $n$-quasicontinuous if for any $x \in X$, there exists a directed family $D(x) \subseteq \text{compfin}_n(X) \cap \text{fin}_n(x)$ converging to $x$. When $X$ is quasicontinuous, we have that $\text{compfin}_n(X) = \text{compfin}_d(X)$, $\text{fin}_n(x) = \text{fin}_d(x)$ for any $x \in X$. Similarly, we get the following statement.

Theorem 4.11. For a $T_0$ topological space $X$, the following are equivalent:

1. $X$ is a $d$-quasicontinuous directed space;
2. $X$ is a directed space and for each $x \in X$, there is a directed family $D(x) \subseteq \text{compfin}_d(X) \cap \text{fin}_d(x)$ such that $D(x) \to x$.
3. $X$ is $n$-quasicontinuous;
4. for each $x \in X$, $\text{compfin}_n(X) \cap \text{fin}_n(x)$ is a directed family and converges to $x$;
5. $X$ is a hyperquasicontinuously compact space;
6. $\mathcal{O}(X)$ is hyperalgebraic;
7. $X^S$ is a quasiadicnous dcpo.

Proof. (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are obvious.

(2) $\Rightarrow$ (1): Obviously, $X$ is quasicontinuous. Assume that $G, H \in \text{compfin}_d(X) \cap \text{fin}_d(x)$. Since $D(x) \to x$, there must exist some $G', H' \in D(x)$ such that $G \leq G', H \leq H'$. Then there exists some $F \in \text{compfin}_d(X) \cap \text{fin}_d(x)$ with $G, H \leq F$ by that $D(x)$ is directed, that is, $\text{compfin}_d(X) \cap \text{fin}_d(x)$ is a directed family.

(3) $\Rightarrow$ (4): Obviously, $X$ is quasicontinuous. Thus $\text{compfin}_n(X) \cap \text{fin}_n(x) = \text{compfin}_d(X) \cap \text{fin}_d(x)$. Then we know it is a directed family by (2) $\Rightarrow$ (1). And we have (1) $\iff$ (4).

(2) $\Rightarrow$ (5): Assume that $x \in U \in \mathcal{O}(X)$ and $D(x) \to x$. Then there exists some $F \in D(x)$ such that $F \subseteq U$. Since $F \in \text{compfin}_d(X) \cap \text{fin}_d(x)$, we have $F \subseteq (\uparrow F)^0$ and $x \in (\uparrow F)^0$ by Lemma [13]. Therefore, $x \in \uparrow F = (\uparrow F)^0 \subseteq U$, i.e., $X$ is locally $\uparrow$compact.

(5) $\Rightarrow$ (2): Since $X$ is locally $\uparrow$compact, $X$ is locally hyperquasicontinuous and then it is a quasicontinuous directed space. Given any $x \in X$, let $D(x) = \{F : F \subseteq_f X, \uparrow F = (\uparrow F)^0, x \in \uparrow F\}$. By definition, $D(x) \to x$. We show that $D(x)$ is a directed family. For any $G, H \in D(x)$, we have $x \in \uparrow G \cap \uparrow H \in \mathcal{O}(X)$. Then there exists some $F \in D(x)$ such that $x \in \uparrow F \subseteq \uparrow G \cap \uparrow H$ by definition. Then $G, H \leq F$, that is, $D(x)$ is a directed family. Since $X$ is quasicontinuous, $x \in \uparrow F = (\uparrow F)^0$ implies $x \leq_d F \leq_d F$. Thus, $D(x) \subseteq \text{compfin}_d(X) \cap \text{fin}_d(x)$ and $D(x) \to x$.

(5) $\Rightarrow$ (6): For any $U \in \mathcal{O}(X)$ and $U \neq \emptyset$, we have $U = \cup \{\uparrow F : F \subseteq_f U, (\uparrow F)^0 = \uparrow F\}$. By Lemma [13] for any $F \subseteq U$ such that $(\uparrow F)^0 = \uparrow F$, we have $\uparrow F \leq_d \uparrow F$ and $\uparrow U$. Thus, $\mathcal{O}(X)$ is hyperalgebraic.

(6) $\Rightarrow$ (5): Let $x \in U \in \mathcal{O}(X)$. Then there exists some $V \in \mathcal{O}(X)$ such that $x \in V \preceq V \subseteq U$. By Lemma [13] we have $V \subseteq \uparrow F \subseteq V$ for some $F \subseteq_f X$. Thus, $\uparrow F = V$ and $(\uparrow F)^0 = \uparrow F$. So, $X$ is locally $\uparrow$compact.

(6) $\Rightarrow$ (7): By $O(X^S) = O(X)$, we have that $O(X^S)$ is hyperalgebraic and then $X^S$ is $d$-quasicontinuous. Since very sober space is a monotone convergence space, $X^S$ must be a dcpo endowed with the Scott topology, that is, $X^S$ is a quasiadicnous dcpo.

(7) $\Rightarrow$ (6): Since $X^S$ is quasiadicnous, then $O(X^S)$ is hyperalgebraic and $O(X) = O(X^S)$ is hyperalgebraic. $\square$
Definition 4.12. A $T_0$ topological space is called quasialgebraic iff one of the seven equivalent conditions in Theorem 4.11 holds.

Obviously, every quasialgebraic space is a quasicontinuous space. In [15], the notion of meet continuous was extended to directed spaces and it was shown that a directed space is continuous iff it is quasialgebraic and meet continuous.

Definition 4.13. [15] A directed space $X$ is called meet continuous if for any $x \in X$ and directed subsets $D \subseteq X$, $D \to x$ implies $x \in \downarrow D \cap \downarrow x$.

Theorem 4.14. [15] For any directed space $X$, the following statements hold.

1. $X$ is continuous iff $X$ is quasicontinuous and meet continuous.
2. $X$ is meet continuous iff $C(X)$ is a frame.

By the above discussion, we conclude the relationships between the continuity of a directed space and the lattice of its open subsets, which will be used when investigate the relationships between the continuity of a directed space and the continuity of the lattice of its closed subsets.

Theorem 4.15. Let $X$ be a directed space.

1. $X$ is continuous (algebraic) iff $\mathcal{O}(X)$ is a completely distributive lattice (completely distributive algebraic lattice).
2. $X$ is quasicontinuous iff the lattice $\mathcal{O}(X)$ is a hypercontinuous lattice.
3. $X$ is quasialgebraic iff the lattice $\mathcal{O}(X)$ is an algebraic lattice iff $\mathcal{O}(X)$ is a hyperalgebraic lattice.

5. Core compactness of directed spaces

In [12], Lawson gave some equivalent conditions for a $T_0$ topological space $X$ to be quasicontinuous. One of the key equivalent conditions is that for any $T_0$ topological space $X$, $X \times Y = X \otimes Y$, where $X \otimes Y$ is the tensor product of $X$ and $Y$. In [13], the equivalent conditions for a $T_0$ topological space to be core compact also be given. Given any poset $P$, $\Sigma P$ denotes the topological space $(P, \sigma(P))$. Then it was proved that for a depo $P$, $P$ is core compact iff for any depo $Q$, $\Sigma(P \times Q) = \Sigma P \times \Sigma Q$.

Given any topological space $X$, when $\mathcal{O}(X)$ is hypercontinuous, $X$ is a quasicontinuous space and hence it is a directed space. When $\mathcal{O}(X)$ is a continuous lattice, i.e., when $X$ is core compact, $X$ may not be a directed space. All nontrivial compact $T_2$ spaces are the examples and Example 3.3.1 is locally compact sober space, which is neither a directed space nor a $T_1$ space. In this section, we investigate the core compactness of a directed space. We will first show that for any two directed spaces $X$ and $Y$, their tensor product $X \otimes Y$ is same to their categorical product $X \times Y$ in $D\text{Top}$. Then, we show that a directed space is core compact iff for any directed spaces $X$, $X \times Y = X \otimes Y$. Finally, through the similar line of [12], we give more equivalent conditions for a directed space to be core compact.

Definition 5.1. For any two topological spaces $X, Y$, the tensor product $X \otimes Y$ of $X$ and $Y$ has the same carrier set of $X \times Y$. A set $W$ is open in $X \otimes Y$ if for every $(x, y) \in W$, there are open sets $U$ of $X$ and $V$ of $Y$ such that $(x, y) \in \{x\} \times V \subseteq W$ and $(x, y) \in U \times \{y\} \subseteq W$.

Given any two topological spaces $X, Y$, a map from a topological space $(X \times Y, \tau)$ to $Z$ is called separately continuous if it is continuous at each argument, i.e., for any $(x_0, y_0) \in X \times Y$, the maps $f_{x_0} : Y \to Z$ and $f_{y_0} : X \to Z$ are continuous, where $f_{x_0}(y) = f(x_0, y)$, $f_{y_0}(x) = f(x, y_0)$.

Here, we give some equivalent characterizations of tensor product.

Lemma 5.2. For any two topological spaces $X$ and $Y$ and a topological space $(X \times Y, \tau)$. The following statement are equivalent.

1. $(X \times Y, \tau)$ is equal to the tensor product.
2. A set $W$ is open in $\tau$ iff for all $(x, y) \in X \times Y$, the slices $W_x = \{y \in Y : (x, y) \in W\}$ and $W^y = \{x \in X : (x, y) \in W\}$ are open in $Y$ and $X$ respectively.
3. $\tau$ is the topology $\mathcal{E}(X \times Y)$ determined by

$$\mathcal{E} = \{\{(x, y)\}_{1, (x, y)} : \{x\}_{1 \to x} \cup \{(x, y)\}_{1, (x, y)} : \{y\}_{1 \to y}\}.$$

4. $\tau$ is the coarsest topology such that all separately continuous functions from the product are continuous.
Proof. (1) \(\Rightarrow\) (2): Given any \(x \in X\) and \(y \in W_z\), since \((x, y) \in W\), there exists a \(V_y\) such that \((x, y) \in \{x\} \times V_y \subseteq W\). Thus, \(y \in V_y \subseteq W_x\). Then \(W_z \subseteq \bigcup_{y \in W_x} V_y \subseteq W_x\) and \(W_x\) is open in \(Y\). Similarly, \(W^y\) is open in \(X\) for any \(y \in Y\).

(2) \(\Rightarrow\) (3): Let \((x, y) \in W \in \tau\). For any \(\{x_i\}_I \rightarrow x\) in \(X\), since \(W^y\) is open in \(X\) and \(x \in W^y\), \(\{x_i\}_I\) is eventually in \(W^y\). Thus \(\{\{x_i, y\}\}_I \rightarrow (x, y)\) \(\subseteq W\). Therefore, \(\{(x_i, y)\}_I \rightarrow (x, y)\) relative to \(\tau\). Similarly, \(\{(x_i, y)\}_I \rightarrow (x, y)\) for any \(\{y_i\}_I \rightarrow y\). Thus, \(\tau \subseteq \mathcal{E}(X \times Y)\). Conversely, assume that \(W \in \mathcal{E}(X \times Y)\). We show that \(W \in \tau\). Suppose that \(W^y\) is not open. Then there exists some \(y \in W^y\) such that for any open subset \(V\) of \(X\) with \(x \in V\), there exists a \(x_V\) with \(x_V \in V\) and \(x_V \notin W^y\). Let \(V\) range over all open set containing \(x\) in \(X\). Now, \(\{x_V\} \rightarrow x\) and \(\{x_V, y\} \rightarrow (x, y)\) relative to \(\mathcal{E}(X \times Y)\). We have \(\{x_V, y\} \in W\) for some \(V\), i.e., \(x_V \notin W^y\), a contradiction. Thus, \(W^y\) is open. Similarly, \(W_x\) is open and then \(W \in \tau\).

(3) \(\Rightarrow\) (4): Let \(Z\) be any topological space. Since \(\tau\) is determined by \(\mathcal{E}\), a map \(f : (X \times Y, \tau) \rightarrow Z\) is continuous if it preserves all the limits in the class if it is separately continuous. Now, we show \(\tau\) is the coarsest one. Assume that \(\delta\) is a topology on \(X \times Y\) such that all separately continuous map are continuous. Consider the identity map \(id : (X \times Y, \delta) \rightarrow (X \times Y, \tau)\). It is separately continuous and it is continuous. Thus, \(\tau\) is coarser than \(\delta\).

(4) \(\Rightarrow\) (1): Let \(id : (X \times Y, \tau) \rightarrow X \otimes Y\) be the identity map. For any \(y \in Y\), let \(id_y : X \rightarrow X \otimes Y\) be the map such that \(id_y(x) = id(x, y) = (x, y)\). Let \(W\) be any open subset of \(X \otimes Y\), then \(f_{\tau}^{-1}(W) = W^y\) and thus it is open. We get that \(id_y\) is continuous. Similarly, for any \(x \in X\), \(id_x\) is continuous. Therefore, \(id : (X \times Y, \tau) \rightarrow X \otimes Y\) is separately continuous and hence continuous. Then \(\tau\) is finer than the tensor product topology.

Now, it is enough to show that for the tensor product topology, if a map is separately continuous, then it is continuous. Assume that \(f : X \otimes Y \rightarrow Z\) is separately continuous. Let \(U\) be any open subset of \(Z\) and \((x_0, y_0) \in f^{-1}(U)\). Since \(f\) is separately continuous, the set \(A = \{x \in X : f(x, y_0) \in U\}\) is open in \(X\) and we have \((x_0, y_0) \in A \times \{y_0\} \subseteq f^{-1}(U)\). Similarly, the set \(B = \{y \in Y : f(x_0, y) \in U\}\) is open in \(Y\) and \((x_0, y_0) \in \{x_0\} \times B \subseteq f^{-1}(U)\). Thus, \(f^{-1}(U)\) is open in \(X \otimes Y\), i.e., \(f\) is continuous. \(\square\)

**Corollary 5.3.** Let \(X, Y\) be any two topological spaces.

(1) The topology of tensor product is the coarsest topology on \(X \times Y\) such that separately continuous maps are the same to continuous maps.

(2) A map \(f : X \times Y \rightarrow Z\) is separately continuous iff \(f\) is continuous from \(X \otimes Y\) to \(Z\).

**Lemma 5.4.** For any two directed spaces \(X\) and \(Y\), \(X \otimes Y = X \times Y\).

**Proof.** Assume that \(W\) is open in \(X \otimes Y\), we show that \(W\) is open in \(X \times Y\). Given any \((x, y) \in W\) we show that the slice \(W^y\) is open in \(X\). For any directed subset \(\{x_i\}_I\) of \(X\) with \(\{x_i\}_I \rightarrow x\) \(\in W^y\) in \(X\), since \(\{(x_i, y)\}_I \rightarrow (x, y)\) \(\in X \otimes Y\), we have \(\{(x_i, y)\}_I\) is finally in \(W\). Then \(\{x_i\}_I\) is finally in \(W^y\). So, \(W^y\) is open in \(X\). Similarly, \(W_x\) is open in \(Y\). Therefore, \(W\) is open in \(X \otimes Y\).

Conversely, assume that \(W\) is open in \(X \otimes Y\). Given any directed subset \(\{x_i\}_I \rightarrow x\) in \(X\), then for \(D = \{(x_i, y)\}_I\) we have \(D \rightarrow (x, y)\) \(\in W\) in \(X \times Y\). Since \(W^y\) is open in \(X\) and \(X\) is a directed space, then there exists some \(i_0 \in I\) such that \(x_{i_0} \in W^y\). Then \((x_{i_0}, y_0) \in W \cap D\). Similarly, for any directed subsets \(D = \{(x_i, y_i)\}_I \rightarrow (x, y)\) \(\in W\), there exists some \(y_i\) such that \((x_i, y_i) \in W \cap D\). Therefore, \(W\) is open in \(X \otimes Y\) by Lemma 2.3. \(\square\)

**Corollary 5.5.** Let \(X, Y, Z\) be directed spaces. A map \(f : X \otimes Y \rightarrow Z\) is continuous if it is separately continuous.

We recall some equivalent conditions for a \(T_0\) topological space and a dcpo to be core compact.

**Proposition 5.6.** If \(X\) is a space such that \(\mathcal{O}(X)\) is a continuous lattice and \(Y\) is an injective space, then \([X \rightarrow Y]_I\) is injective. In particular, the Isbell topology on \(TOP(X, Y)\) is the Scott topology.

**Theorem 5.7.** Let \(X\) be a \(T_0\) space. Then the following statements are equivalent.

(1) \(\mathcal{O}(X)\) is a continuous lattice.

(2) The set \(\{(U, x) \in \mathcal{O}(X) \times X : x \in U\}\) is open in \(\Sigma \mathcal{O}(X) \times X\).

**Proposition 5.8.** Let \(P\) be a dcpo. Then the following conditions are equivalent.

(1) \(\sigma(P)\) is a continuous lattice.

(2) \(\Sigma P \times \Sigma Q = \Sigma (P \times Q)\) for any dcpo \(Q\).

**Theorem 5.9.** Let \(X\) be a directed space. Then the following conditions are equivalent.

(1) \(X\) is core compact.

(2) \(X \otimes Y = X \times Y\) for any directed space \(Y\).
(3) $\Sigma\mathcal{O}(X) \times X = \Sigma\mathcal{O}(X) \otimes X$.

**Proof.** Assume that $\Sigma\mathcal{O}(X) \times X = \Sigma\mathcal{O}(X) \otimes X$. Since $\mathcal{O}(X)$ is a complete lattice, $\Sigma\mathcal{O}(X)$ is a directed space. To show that $X$ is core compact, we need only to show that $E = \{(U, x) \in \mathcal{O}(X) \times X : x \in U\}$ is open in $\Sigma\mathcal{O}(X) \times X$ by Lemma 5.7. Then it is equivalent to showing that $E$ is open in $\Sigma\mathcal{O}(X) \otimes X$. Assume that a directed set $\{(U_i, x_i)\}_{i \in I}$ converges to $(U, x) \in E$ in $\Sigma\mathcal{O}(X) \times X$. Then $\{x_i\}_{i \in I} \rightarrow x$, $\{U_i\}_{i \in I} \rightarrow U$. Therefore, $\exists i_0 \in I$, such that $x_{i_0} \in U_{i_1}$. Let $i_0, i_1 \leq i_2$, we have $(U_{i_2}, x_{i_2}) \in E$, proved.

Conversely, suppose that $X$ is core compact. We show that $X \otimes Y = X \times Y$ for any directed space $Y$. Since $X \otimes Y$ is finer than $X \times Y$, we need only to show that every open subset $U$ of $X \otimes Y$ is open in $X \times Y$. Given $(x_0, y_0) \in U$, let $V = \{(x, y) : (x, y_0) \in U\}$. It is easily seen that $V$ is an open subset of $X$. Since $X$ is core compact, there exists a family of open subsets $\{V_n : n \in N\}$ such that

$x_0 \in V_0 \sqsubset \cdots \sqsubset V_{n+1} \sqsubset V_n \sqsubset \cdots \sqsubset V_2 \sqsubset V_1 \sqsubset V.

Let $W = \bigcup_{n \leq 0} \{y \in Y : V_n \times \{y\} \subseteq U\}$. Obviously, $y_0 \in W$. Since $V_n \subseteq V_0$ for any $n \geq 1$, then $V_0 \times W \subseteq U$. We need only to show that $W$ is an open subset of $Y$. Given any directed subset $D \rightarrow y \in W$ in $Y$, there exists some $n$ such that $V_n \times \{y\} \subseteq U$. For any $x \in V_n$, $\{(x, d) : d \in D \rightarrow (x, y) \in U \in X \otimes Y\}$. Thus, there is some $d_x \in D$ such that $(x, d_x) \in U$. Then there exists an open neighborhood $V_{y,n}$ of $x$ such that $V_{y,n} \times \{d_x\} \subseteq U$. Since $V_{n+1} \sqsubset V_n \subseteq \bigcup_{x \in D} V_{x}$, we have that $V_{n+1} \subseteq \bigcup_{x \in D} V_{x}$, for some finite subset $B = \{x_1, \ldots, x_n\}$ of $V_{y,n}$. Since $B$ is finite, there exists some $d_0 \in D$ such that $d_{x_i} \leq d_0$. Then $V_{n+1} \subseteq \{x \in (x, d_0) \in U\}$ and thus $V_{n+1} \times \{d_0\} \subseteq U$, i.e., $d_0 \in W$. Therefore, $W$ is open in $Y$, proved. □

Now, we give more equivalent conditions for a directed space to be core compact. We denote $\Sigma \mathcal{O}$ to be the Sierpinski space, i.e., the poset $\{0, 1\}$ endowed with the Scott topology.

**Lemma 5.10.** The topology of pointwise convergence on $[X \rightarrow \Sigma \mathcal{O}]_{\mathcal{P}}$ is the upper topology, which corresponds to the upper topology on $\mathcal{O}(X)$.

Notice that given any two directed spaces, there are three kinds of topologies on $TOP(X, Y)$. $[X \rightarrow Y]$ denotes the exponential objects in $\textbf{DTop}$; $[X \rightarrow Y]_{\mathcal{P}}$ denotes the space endowed with the topology of pointwise convergence; $[X \rightarrow Y]_{\mathcal{L}}$ denote the space endowed with the Isbell topology.

**Theorem 5.11.** Let $X$ be a directed space. The following conditions are equivalent.

1. $[X \rightarrow Y]$ is injective for all injective $T_0$ spaces $Y$.
2. $X \rightarrow \Sigma \mathcal{O}$ is injective.
3. $\mathcal{O}(X)$ is continuous.
4. $\{(U, x) : x \in U\}$ is open in $\Sigma\mathcal{O}(X) \times X$.
5. The evaluation map $ev : [X \rightarrow \Sigma \mathcal{O}] \times X \rightarrow \Sigma \mathcal{O}$ is continuous.
6. For all directed spaces $Y$, $X \otimes Y = X \otimes Y = X \times Y$.
7. For all directed spaces $Y, Z$, if a map $f : X \times Y \rightarrow Z$ is separately continuous, then it is jointly continuous.
8. For any directed space $Y$, the evaluation map $[X \rightarrow Y] \times X \rightarrow Y$ is continuous.
9. The natural map $[Z \times X \rightarrow Y] \rightarrow [Z \rightarrow [X \rightarrow Y]]$ is onto (and a homeomorphism) for all directed spaces $Y$ and $Z$.

**Proof.** (1) $\Rightarrow$ (2) Obviously.

(2) $\Leftrightarrow$ (3): Since $[X \rightarrow \Sigma \mathcal{O}] = D([X \rightarrow \Sigma \mathcal{O}]_{\mathcal{P}}) = \Sigma\mathcal{O}(X)$ and an injective space is a continuous lattice endowed with the Scott topology, then $\mathcal{O}(X)$ is continuous iff $[X \rightarrow \Sigma \mathcal{O}]$ is injective.

(3) $\Rightarrow$ (1): Assume that $Y$ is an injective space. By Proposition 5.9, $[X \rightarrow Y] = \Sigma(TOP(X, Y))$ is an injective space. Thus, $TOP(X, Y)$ is a continuous lattice. Since the topology of pointwise convergence is coarser than the Isbell topology, $[X \rightarrow Y]_{\mathcal{P}}$ is coarser than $\Sigma(TOP(X, Y))$ and then $[X \rightarrow Y] = D([X \rightarrow Y]_{\mathcal{P}} = \Sigma(TOP(X, Y)) = [X \rightarrow Y]_{\mathcal{L}}$.

(3) $\Leftrightarrow$ (4): By Theorem 5.7.1.

(4) $\Leftrightarrow$ (5): It is a direct conclusion by the fact that $[X \rightarrow \Sigma \mathcal{O}] = \Sigma\mathcal{O}(X)$.

(3) $\Leftrightarrow$ (6) $\Leftrightarrow$ (10): By Theorem 5.7 and Lemma 5.3.

(6) $\Rightarrow$ (7): $f$ is separately continuous is equivalent to that $f$ is continuous from $X \otimes Y$ to $Z$, thus $f$ is jointly continuous.

(7) $\Rightarrow$ (8) $\Rightarrow$ (5): Since $ev : [X \rightarrow Y] \otimes X \rightarrow Y$ is separately continuous, then $ev$ is continuous from $[X \rightarrow Y] \times X$ to $Y$. (8) $\Rightarrow$ (5) is obvious.

(8) $\Rightarrow$ (9): Since the natural map $[Z \times X \rightarrow Y] \rightarrow [Z \rightarrow [X \rightarrow Y]]$ is a homeomorphism for all directed spaces $Y$ and $Z$ (see 14), we need only to show (6). This has been proved.

(9) $\Rightarrow$ (8): Let $Z = [X \rightarrow Y]$. Then the inverse of identity map $id : [[X \rightarrow Y] \times [X \rightarrow Y]]$ is $ev : [[X \rightarrow Y] \times X \rightarrow Y]$. □
6. The lattice of closed subsets of directed spaces

In [18], Chen and Kou give a uniform proof for the correspondence between the continuity of a dcpo and the lattice of its Scott closed subsets. Dcpos can be viewed as special directed spaces. By Theorem 5.9 we can extend the result to directed spaces very smoothly.

Proposition 6.1. [18] Let $L$ be a continuous lattice, if $L$ satisfies the condition that $v(L^{op}) = \sigma(L^{op})$, then $(L^{op}, \sigma(L^{op}))$ is a sober and locally compact space.

Proof. We need only to prove that $\sigma(C(X)) \subseteq v(C(X))$. At first, for each $n \in \mathbb{N}$, we define a map $s_n : \mathcal{D}(\prod X) \to (C(X), \sigma(C(X)))$ as follows:

$$\forall (x_1, x_2, \ldots, x_n) \in \prod X, s_n(x_1, x_2, \ldots, x_n) = \bigcup_{k=1}^n \downarrow x_k.$$  

We show that $s_n$ is continuous, i.e., $s_n$ preserves $D \to x$ for every $(D, x) \in DLim(\prod X)$. Let $\{(x_{1i}, x_{2i}, \ldots, x_{ni}) : i \in I\}$ be a directed subset of $\prod X$ converging to $(x_1, x_2, \ldots, x_n)$ in $\prod X$. Then for each $1 \leq k \leq n$, $\{(x_{ki}) : i \in I\}$ converges to $x_k$ by the definition of topological product. We have

$$s_n((x_1, x_2, \ldots, x_n)) = \bigcup_{k=1}^n \downarrow x_k \subseteq \bigcup_{k=1}^n \bigcup_{i \in I} \downarrow x_{ki} = \bigcup_{i \in I} \bigcup_{k=1}^n \downarrow x_{ki} = \bigcup_{i \in I} s_n(x_{1i}, x_{2i}, \ldots, x_{ni}).$$

Thus $s_n$ is a continuous map from $\mathcal{D}(\prod X)$ into $(C(X), \sigma(C(X)))$.

Next, let $U$ be a Scott open subset of $C(X)$ and $A \in \mathcal{U}$. Without loss of generality, we assume $A \neq \emptyset$. Note that since $A = \bigcup \{\downarrow F : F \subseteq_f A\}$ and $\{\downarrow F : F \subseteq_f A\}$ is a directed family in $C(X)$, there exists a non-empty finite subset $F$ of $A$ such that $\downarrow F \in \mathcal{U}$. Let $F = \{x_1, x_2, \ldots, x_n\}$, then $s_n(x_1, x_2, \ldots, x_n) \subseteq \downarrow F \in \mathcal{U}$. It follows that $s_n^{-1}(U)$ is a directed family in $\prod X$, there exists a family of open subsets $U_k$, $k = 1, 2, \ldots, n$, of $C(X)$ such that $U_1 \times U_2 \times \cdots \times U_n$ is open in $\mathcal{D}(\prod X)$ and

$$(x_1, x_2, \ldots, x_n) \in U_1 \times U_2 \times \cdots \times U_n \subseteq s_n^{-1}(U).$$

Since $x_k \in A$ for $1 \leq k \leq n$, we have $A \subseteq \bigcup_{k=1}^n U_k = \{B \in C(X) : B \cap U_k \neq \emptyset\}$. It follows that $A \subseteq \bigcup_{k=1}^n U_k = v(C(X))$. For any $B \in \bigcup_{k=1}^n U_k$, there exists $y_k \in B \cap U_k$ for $1 \leq k \leq n$. Since $(y_1, y_2, \ldots, y_n) \in s_n^{-1}(U)$, we have

$$\bigcup_{k=1}^n y_k \in U.$$  

It follows that $B \in \mathcal{U}$, i.e., $A \subseteq \bigcup_{k=1}^n U_k \subseteq \mathcal{U}$. □

Proposition 6.3. Let $X$ be a directed space. If $X$ is core compact, then $\sigma(C(X)) = v(C(X))$. Moreover, $(C(X), \sigma(C(X)))$ is sober and locally compact.

Proof. Let $X$ be a core compact directed space. We have that for every $n \in \mathbb{N}$, $\mathcal{D}(\prod X) = \prod X$. By Proposition 6.2 $\sigma(C(X)) = v(C(X))$. Letting $L = \mathcal{O}(X)$, then $L$ is a continuous lattice and $C(X) = L^{op}$. We have that $(C(X), \sigma(C(X)))$ is sober and locally compact by Proposition 5.1. □

In [18], an adjunction between $\sigma(P)$ and $\sigma(C(P))$ serves as a useful tool in studying the relation between $P$ and $C(P)$. It can be extended to directed spaces as well.

Proposition 6.4. Let $X$ be a directed space.

1. $\eta : X \to (C(X), \sigma(C(X))), \forall x \in X, \eta(x) = \downarrow x$. Then $\eta$ is continuous.
2. A map $f : \mathcal{O}(X) \to \sigma(C(X))$ is defined by $f(U) = \bigcup U = \{A \in C(X) : A \cap U \neq \emptyset\}$. Then $f$ preserves arbitrary supers.
3. Let the map $\eta^{-1} : \sigma(C(X)) \to \mathcal{O}(X)$ be defined by $\eta^{-1}(U) = \{x \in X : \downarrow x \subseteq U\}$. Then $\eta^{-1} \circ f = 1_{\sigma(C(X))}$, $f \circ \eta^{-1} \leq 1_{\sigma(C(X))}$, which implies that $(\eta^{-1}, f)$ is an adjunction.

Proof. (1) $\eta$ is the special case of $s_n$ for $n = 1$ in Proposition 6.2. Thus it is continuous.

(2) Let $\{U_i : i \in I\}$ be any subset of $\sigma(C(X))$. Then $f(\bigcup_{i \in I} U_i) = \{A \in C(X) : A \cap \bigcup_{i \in I} U_i \neq \emptyset\} = \{A \in C(X) : \exists i \in I, A \cap U_i \neq \emptyset\} = \bigcup_{i \in I} f(U_i)$

(3) Since $\eta$ is continuous, $\eta^{-1}$ preserves arbitrary supers and finite infs. Given any $U \in \mathcal{O}(X)$, $x \in \eta^{-1}(f(U)) \iff \eta(x) \in f(U) \iff x \cap U \neq \emptyset \iff x \in U$, hence $\eta^{-1} \circ f = 1_{\sigma(C(X))}$. For any $U \in \sigma(C(X))$, $A \in f \circ \eta^{-1}(U) \iff A \cap \eta^{-1}(U) \neq \emptyset \Rightarrow A \in U$, i.e., $f \circ \eta^{-1} \leq 1_{\sigma(C(X))}$. □
**Theorem 6.5.** Let $X$ be a directed space, then the following statements are equivalent:

1. $X$ is core-compact.
2. $(C(X), σ(C(X)))$ is core compact.
3. $(C(X), σ(C(X)))$ is sober and locally compact
4. $(C(X), σ(C(X)))$ is sober and locally compact with $σ(C(X)) = υ(C(X))$.

**Proof.** It is analogous to the proof of Theorem 3.13 in [18]. □

**Theorem 6.6.** Let $X$ be a directed space,

1. $X$ is continuous iff the lattice $C(X)$ is a continuous lattice.
2. $X$ is quasicontinuous iff the lattice $C(X)$ is a quasicontinuous lattice.
3. $X$ is algebraic iff the lattice $C(X)$ is an algebraic lattice.
4. $X$ is quasialgebraic iff the lattice $C(X)$ is a quasialgebraic lattice.

**Proof.** It is analogous to the proof of Theorem 3.17 in [18]. □

**References**

[1] S. Abramsky and A. Jung, Domain theory, in: Handbook of Logic in Computer Science, Vol. III, OUP, 1994 pp. 1-168.
[2] A. Jung, "Cartesian Closed Categories of Domains." CWI Tracts (1988).
[3] M. Ernè, Scott convergence and Scott topology in partially ordered sets II, In: B. Banaschewski, R. E. Hoffmann (eds.) Continuous Lattices, Lecture Notes in Mathematics, vol. 871, pp. 61-96, Springer, Berlin, 1981.
[4] M. Ernè, The ABC of order and topology, In: H. Herrlich, H. E. Porst (des.) Category Theory at Work, pp. 57-83, Heldermann, Berlin, 1991.
[5] M. Ernè, Infinite distributive laws versus local connectedness and compactness properties, Topology and its Applications, 2009, 156(12): 2054-2069.
[6] Yu. L. Ershov, The theory of A-spaces, Algebra & Logic 12.4(1973):208-232.
[7] Yu. L. Ershov, Theory of domains and nearby, In: D. Björner, M. Broy, I.V. Pottosin (eds) Formal Methods in Programming and Their Applications, Lecture Notes in Computer Science, vol. 735, pp. 1-7, Springer, Berlin, 1993.
[8] Yu. L. Ershov, The bounded complete hull of an α-space, Theoretic Computer Science, 175(1997), 3-13.
[9] G. Gierz, K.H.Holmman, K.Keimel, J.D.Lawson, M. Mislove and D. Scott, Continuous Lattices and Domains, CUP, 2003.
[10] A. Jung, The classification of continuous domains, in: Proceedings of 5th Annual Symposium on Logic in Computer Science, 1990, pp. 35-40.
[11] K. Keimel, Topological cone: functional analysis in $T_0$-setting, Semigroup Forum, (2008)77, 109-142.
[12] J.D. Lawson, T0-spaces and pointwise convergence, Topology and Its Applications, 1985, 21(1):73-76.
[13] W. Wang, H. Kou, Approximation structures on $T_0$ topological spaces(Chinese), Journal of Sichuan University(Natural Science Edition), 2014, 51(04): 681-683.
[14] Y. Yu, H. Kou, Directed spaces defined through $T_0$ topological order(Chinese), Journal of Sichuan University (Natural Science Edition), 2015, 52(2): 217-222.
[15] H. Feng, H. Kou, Quasicontinuity and meet-continuity of $T_0$ spaces(Chinese), Journal of Sichuan University (Natural Science Edition), 2017, 54(05): 905-910.
[16] M.J. Che, H. Kou, A cartesian closed full subcategory of c-spaces(Chinese), Journal of Sichuan Normal University(Natural Science), 2020, 43(06):756-762.
[17] X.L. Xie, H. Kou, The lower powerspace of directed spaces(Chinese), Journal of Sichuan University (Natural Science Edition), 2020, 57(2):7.
[18] Y. Chen, H. Kou, Z.C. Lyu, Two topologies on the lattice of Scott closed subsets, Topology and its Applications, 2022.