An implicit relation, relational theoretic approach under $w$-distance and application to nonlinear matrix equations

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Abstract

We propose a new class of implicit relations and an implicit type contractive condition based on it in the relational metric spaces under $w$-distance functional. Further we derive fixed points results based on them. Useful examples illustrate the applicability and effectiveness of the presented results. We apply these results to discuss sufficient conditions ensuring the existence of a unique positive definite solution of the nonlinear matrix equation (NME) of the form $U = Q + \sum_{i=1}^{m} A_i G(U) A_i^*$, where $Q$ is an $n \times n$ Hermitian positive definite matrix, $A_1, A_2, \ldots, A_m$ are $n \times n$ matrices and $G$ is a nonlinear self-mapping of the set of all Hermitian matrices which are continuous in the trace norm. In order to demonstrate the obtained conditions, we consider an example together with convergence and error analysis and visualisation of solutions in a surface plot.

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1 Introductory notes

Several mathematicians have established fixed point findings for contraction type mappings in metric spaces with partial order in recent years. Turinici established some early results in this approach in [21, 22]; it should be noted that their beginning points were amorphous contributions in the area due to Matkowski [10, 11]. Ran and Reurings [16], as well as Nieto and Rodríguez-López [13, 14], have looked at these kinds of findings. Turinici’s findings were expanded upon and refined in articles [13, 14]. Fixed point theorems for nonlinear contraction under symmetric closure of an arbitrary relation were recently developed by Samet and Turinici [18]. Alam and Imdad used an amorphous relation to show a relation-theoretic counterpart of the Banach contraction principle, which combines a number of well-known relevant order-theoretic fixed point theorems. According to Mizoguchi and Takahashi [12], Kada et al. [4] developed the idea of $\omega$-distance on a metric space in 1996 and proved a modified Caristi fixed point theorem, Ekeland’s $\epsilon$-variational principle and the nonconvex minimisation theorem. Rouzkard et al. [17] used the $w$-distance function to find a generalised $(\psi, \phi)$-weakly contractive condition

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for orbital continuous maps. Lakzian et al. [6, 7] established various fixed point theorems for \((\alpha, \psi)\)-contractive mappings and Kannan contraction in the presence of a \(\omega\)-distance. Senapati and Dey [19] recently published a modified version of the \(\omega\)-distance function and demonstrated that Alam and Imdad [1]'s conclusion is false. For more details about the \(\omega\)-distance functional, see [3].

1.1 Motivation
One of the most visually attractive applications of contraction mapping is found in nonlinear matrix equations. The question now is whether the previously described contraction can be improved and generalised. In relational metric spaces with \(w\)-distance functional, we investigate a novel class of implicit relations and implicit type contractions. We establish fixed point findings by justifying the applicability and usefulness of the provided results. Contraction and rational type contraction mappings are included in the proposed implicit contraction. We use these findings to examine the necessary conditions for the existence of a unique positive definite solution to the nonlinear matrix equation (NME) of the form \(U = Q + \sum_{i=1}^{k} A_i^T G(A_i)A_i\), where \(Q\) is an \(n \times n\) Hermitian positive definite matrix, \(A_1, A_2, \ldots, A_n\) are \(n \times n\) matrices and \(G\) is a nonlinear self-mapping of the set of all Hermitian matrices which are continuous in the trace norm. In order to demonstrate the obtained conditions, we consider an example together with convergence and error analysis, and visualisation of solutions in a surface plot.

2 Preliminaries
2.1 Relational metricspaces
Throughout this article, the notations \(\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+\) have their usual meanings, and \(\mathbb{N}^* = \mathbb{N} \cup \{0\}\).

We call \((\mathcal{W}, \mathcal{R})\) a relational set if (i) \(\mathcal{W} \neq \emptyset\) is a set and (ii) \(\mathcal{R}\) is a binary relation on \(\mathcal{W}\).

In addition, if \((\mathcal{W}, d)\) is a metric space, we call \((\mathcal{W}, d, \mathcal{R})\) a relational metric space (RMS, for short).

The following are some standard terms used in the theory of relational sets (see, e.g. [1, 5, 8, 9, 18]).

Let \((\mathcal{W}, \mathcal{R})\) be a relational set, \((\mathcal{W}, d, \mathcal{R})\) be an RMS, and let \(\mathcal{K}\) be a self-mapping on \(\mathcal{W}\). Then:
1. \(v \in \mathcal{W}\) is \(\mathcal{R}\)-related to \(\vartheta \in \mathcal{W}\) if and only if \((v, \vartheta) \in \mathcal{R}\).
2. The set \((\mathcal{W}, \mathcal{R})\) is said to be complete if for all \(v, \vartheta \in \mathcal{W}\), \([v, \vartheta] \in \mathcal{R}\), where \([v, \vartheta] \in \mathcal{R}\) means that either \((v, \vartheta) \in \mathcal{R}\) or \((\vartheta, v) \in \mathcal{R}\).
3. A sequence \((v_n)\) in \(\mathcal{W}\) is said to be \(\mathcal{R}\)-preserving if \((v_n, v_{n+1}) \in \mathcal{R}\), \(\forall n \in \mathbb{N} \cup \{0\}\).
4. \((\mathcal{W}, d, \mathcal{R})\) is said to be \(\mathcal{R}\)-complete if every \(\mathcal{R}\)-preserving Cauchy sequence converges in \(\mathcal{K}\).
5. \(\mathcal{R}\) is said to be \(\mathcal{K}\)-closed if \((v, \vartheta) \in \mathcal{R} \Rightarrow (\mathcal{K}v, \mathcal{K}\vartheta) \in \mathcal{R}\). It is said to be weakly \(\mathcal{K}\)-closed if \((v, \vartheta) \in \mathcal{R} \Rightarrow [\mathcal{K}v, \mathcal{K}\vartheta] \in \mathcal{R}\).
6. \(\mathcal{R}\) is said to be \(d\)-self-closed if for every \(\mathcal{R}\)-preserving sequence with \(v_n \to v\) there is a subsequence \((v_{n_k})\) of \((v_n)\) such that \([v_{n_k}, v] \in \mathcal{R}\) for all \(k \in \mathbb{N} \cup \{0\}\).
7. A subset \(\mathcal{F}\) of \(\mathcal{W}\) is called \(\mathcal{R}\)-directed if for each \(v, \vartheta \in \mathcal{F}\) there exists \(\mu \in \mathcal{W}\) such that \((v, \mu) \in \mathcal{R}\) and \((\vartheta, \mu) \in \mathcal{R}\). It is called \((\mathcal{K}, \mathcal{R})\)-directed if for each \(v, \vartheta \in \mathcal{F}\) there exists \(\mu \in \mathcal{W}\) such that \((v, \mathcal{K}\mu) \in \mathcal{R}\) and \((\vartheta, \mathcal{K}\mu) \in \mathcal{R}\).
Lemma 2.2

Let \( R \) be an arbitrary binary relation and \( W \) a \( R \)-distance space.

**Definition 2.1** ([19]) A function \( \alpha: W \times W \to [0, +\infty) \) is called a \( w \)-distance on \( W \) if it satisfies the following properties:

1. **\((W1)\)** \( \alpha(\nu, \mu) \leq \alpha(\theta, \nu) + \alpha(\nu, \mu) \) for any \( \theta, \nu, \mu \in W \);
2. **\((W2)\)** \( \alpha \) is \( R \)-LSC in its second variable; i.e. if \( \theta \in W \) and \( \nu \mapsto v \in W \) such that \( \nu_n \to \nu \) in \( W \), then \( \liminf_{n \to \infty} \alpha(\theta, v_n) \); then \( \alpha(\theta, v) \)
3. **\((W3)\)** for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \alpha(\mu, \theta) \leq \delta \) and \( \alpha(\omega, \mu) \leq \delta \) imply \( d(\omega, \nu) \leq \epsilon \).

The following lemma is a modified version of Kada et al. [4] due to Senapati and Dey [19].

**Lemma 2.2** ([19]) Let \( (W, d, R) \) be a relational metric space, and let \( \omega \) be a \( w \)-distance on \( W \). Suppose that \( \{\theta_n\} \) and \( \{v_n\} \) are \( R \)-preserving sequences in \( W \), \( \{\alpha_n\} \) and \( \{\beta_n\} \) are sequences in \( [0, +\infty) \) converging to 0, and let \( \theta, v, \mu \in W \). Then the following assertions hold:

1. If \( \omega(\theta_n, v) \leq \alpha_n \) and \( \omega(\theta_n, \mu) \leq \beta_n \) for all \( n \in \mathbb{N} \), then \( v = \mu \); particularly, if \( \omega(\theta, v) = \omega(\theta, \mu) = 0 \), then \( v = \mu \);
2. If \( \omega(\theta_n, v_n) \leq \alpha_n \) and \( \omega(\theta_n, v) \leq \beta_n \) for all \( n \in \mathbb{N} \), then \( \{v_n\} \) converges to \( v \);
3. If \( \omega(\theta_n, \theta_m) \leq \alpha_n \) for all \( n, m \in \mathbb{N} \) with \( m > n \), then \( \{\theta_n\} \) is a Cauchy sequence;
4. If \( \omega(v, \theta_n) \leq \alpha_n \) for all \( n \in \mathbb{N} \), then \( \{\theta_n\} \) is a Cauchy sequence.
Example 3.2 Let \( q_1, q_2, q_3, q_4, q_5, q_6 \) be a given sequence. We say that for all \( (\Omega, \nu) \in \Omega \), \( \nu \) is nonincreasing in the fifth and sixth variables; \( (\nu_2) \) \( \nu \) is continuous function \( \nu \in (0,1/2) \).

\( (\nu_3) \) For all \( \nu > 0 \), \( \nu(\zeta, \zeta, 0, 0, \zeta, 0) \in (1 - a) \zeta^2 > 0 \).

**Lemma 2.3** ([4, 20]) Let \( \omega \) be a \( w \)-distance on a metric space \( (\mathcal{W}, d) \) and \( \varnothing_n \) be a sequence in \( \mathcal{W} \) such that for each \( \epsilon > 0 \) there exists \( N_\epsilon \in \mathbb{N} \) such that \( m > n > N_\epsilon \) implies \( \omega(\varnothing_n, \varnothing_m) < \epsilon \), i.e., \( \lim_{m,n \to \infty} \omega(\varnothing_n, \varnothing_m) = 0 \). Then \( \{ \varnothing_n \} \) is a Cauchy sequence.

### 3 New class of implicit relations

In this section we introduce a modified version of implicit relations and examples discussed in [2, 15].

Let \( \Omega = \{ H : \mathbb{R}^6 \to \mathbb{R} \text{ is continuous function} \} \) satisfy the following conditions:

\( (H_1) \) \( H \) is nonincreasing in the fifth and sixth variables;

\( (H_2) \) There exists \( h \in [0, 1) \) such that, for all \( \zeta, \xi, \mu \geq 0 \),

\( (H_{2a}) \) \( H(\zeta, \xi, \xi, \zeta, \zeta, \mu) \leq 0 \) implies that \( \zeta \leq h \xi \) and

\( (H_{2b}) \) \( H(\zeta, \xi, \xi, \zeta, \mu, \zeta) \leq 0 \) implies that \( \zeta \leq h \xi \);

\( (H_3) \) \( H(\zeta, \zeta, \zeta, 0, 0, \zeta) > 0 \) and \( H(\zeta, 0, 0, \zeta, 0) > 0 \) for \( \zeta > 0 \).

**Example 3.1** Let \( H(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 + b \frac{q_3^2 + q_4^2}{q_5 + q_6 + 1}, a \in (1/2, 1), b \in (0, 1/2). \)

\( (H_{2a}) \) Then, for \( \zeta, \xi, \mu \geq 0 \), we have

\[
H(\zeta, \xi, \xi, \zeta, \zeta, \mu) = \zeta^2 - a \xi^2 + b \frac{\zeta^2 + \xi^2}{1 + \zeta + \xi + \mu} .
\]

It is evident that \( H(\zeta, \xi, \xi, \zeta, \zeta, \mu) \leq 0 \) implies that \( \zeta \leq a \xi \), \( \xi \leq h \xi \). \( H(\zeta, \xi, 0, 0, \zeta, 0) = (1 - a) \zeta^2 > 0 \).

**Example 3.2** Let \( H(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 + b \frac{q_3^2 + q_4^2}{q_5 + q_6 + 1}, a \in (1/2, 1), b \in (0, 1/2). \)

\( (H_{2a}) \) Then, for \( \zeta, \xi, \mu \geq 0 \), we have

\[
H(\zeta, \xi, \xi, \zeta, \zeta, \mu) = \zeta^2 - a \xi^2 + b \frac{\zeta^2 + \xi^2}{1 + (\zeta + \xi)^2 + \mu} .
\]

It is evident that \( H(\zeta, \xi, \xi, \zeta, \zeta, \mu) \leq 0 \) implies that \( \zeta \leq a \xi \), \( \xi \leq h \xi \). \( H(\zeta, \xi, 0, 0, \zeta, 0) = (1 - a) \zeta^2 > 0 \).

### 4 \( R_w \)-implicit contractive condition under \( w \)-distance

We define an \( R_w \)-implicit contractive mapping in the metric space under \( w \)-distance using the above introduced implicit relation.

**Definition 4.1** Let \( (\mathcal{W}, d, \mathcal{R}) \) be a relational metric space with \( w \)-distance \( w \) and \( K : \mathcal{W} \to \mathcal{W} \) be a given mapping. We say that \( K \) is an \( R_w \)-implicit contractive mapping if there exists a function \( H \in \Omega \) such that

\[
H(\omega(K, \varnothing, \varnothing), \omega(\varnothing, \varnothing), \omega(\varnothing, K, \varnothing), \omega(\varnothing, \varnothing), \omega(\varnothing, K, \varnothing), \omega(\varnothing, K, \varnothing), \omega(\varnothing, K, \varnothing), \omega(\varnothing, K, \varnothing), \omega(\varnothing, K, \varnothing), \omega(\varnothing, K, \varnothing)) \leq 0 \quad (1)
\]

for all \( (\varnothing, \varnothing) \in \mathcal{R} \).
Now, we are equipped to state and prove our first main result as follows.

**Theorem 4.2** Let \((W, d, \mathcal{R})\) be an RMS with \(w\)-distance \(\omega\) and \(\mathcal{K}: W \to W\). Suppose that the following conditions hold:

1. \((A_1)\) \(X(\mathcal{K}, \mathcal{R}) \neq \emptyset\);
2. \((A_2)\) \(\mathcal{R}\) is \(\mathcal{K}\)-closed and \(\mathcal{K}\)-transitive;
3. \((A_3)\) \(W\) is \(\mathcal{K}\)-\(\mathcal{R}\)-complete;
4. \((A_4)\) \(\mathcal{K}\) is a \(\mathcal{R}_w\)-implicit contractive;
5. \((A_5)\) \(\mathcal{K}\) is \(\mathcal{R}\)-continuous.

Then there exists a point \(\omega \in \text{Fix}(\mathcal{K})\). In addition, \(\omega(\vartheta^*, \vartheta^*) = 0\) provided \((\vartheta^*, \vartheta^*) \in \mathcal{R}\) holds.

**Proof** Let \(\theta_0 \in X(\mathcal{K}, \mathcal{R})\) be a point as given in \((A_1)\). If \(\mathcal{K}^n\theta_0 = \mathcal{K}^{n+1}\theta_0\) for some \(n \in \mathbb{N} \cup \{0\}\), then there is nothing to prove. Construct a sequence \(\{\theta_n\}\) of Picard iterates \(\theta_n = \mathcal{K}^n(\theta_0)\) for all \(n \in \mathbb{N}^*\).

Using \((A_1)-(A_2)\), we have that \((\mathcal{K}\theta_0, \mathcal{K}^2\theta_0) \in \mathcal{R}\). Continuing this process inductively, we obtain

\[
(K^n\theta_0, K^{n+1}\theta_0) \in \mathcal{R}
\]

for any \(n \in \mathbb{N} \cup \{0\}\). Hence, \(\{\theta_n\}\) is an \(\mathcal{R}\)-preserving sequence.

Next, we show that

\[
\lim_{n \to \infty} w(K^n\theta_0, K^{n+1}\theta_0) = 0.
\]

Using (1) for \(\vartheta = \theta_{n-1}, \nu = \theta_n\),

\[
\mathcal{H}\left(\omega(K^n\theta_0, K^{n+1}\theta_0), \omega(K^{n-1}\theta_0, K^n\theta_0), \omega(K^{n-1}\theta_0, K^n\theta_0), \omega(K^n\theta_0, K^{n+1}\theta_0), \omega(K^{n+1}\theta_0, K^n\theta_0)\right) \leq 0.
\]

Denoting \(\varrho_n = \omega(K^n\theta_0, K^{n+1}\theta_0)\) for all \(n \in \mathbb{N}^*\) and applying \((\mathcal{H}_1)\) in the fifth variables, we have

\[
\mathcal{H}(\varrho_n, \varrho_{n-1}, \varrho_{n-1}, \varrho_{n-1} + \varrho_n, \omega(K^n\theta_0, K^n\theta_0)) \leq 0. \tag{4}
\]

It follows from \((\mathcal{H}_{2a})\) that there is \(h \in [0, 1)\) such that

\[
\varrho_n \leq h \varrho_{n-1},
\]

that is, the sequence \(\{\varrho_n\}\) is a nonincreasing sequence of real numbers. Therefore there exists \(\zeta\) such that

\[
\lim_{n \to \infty} \varrho_n = \lim_{n \to \infty} \omega(K^n\theta_0, K^{n+1}\theta_0) = \zeta \quad \forall n \in \mathbb{N}.
\]

Applying the limit in (4), by the continuity of \(\mathcal{H}\), we get

\[
\zeta \leq \lim_{n \to \infty} \varrho_n \leq h \zeta,
\]
a contradiction, and therefore \( \zeta = 0 \). Thus we conclude that \( \lim_{n \to \infty} \varrho_n = \lim_{n \to \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^{n+1} \theta_0) = 0 \). Similarly, from (i) \((\mathcal{K} \theta_0, \theta_0) \in \mathcal{R}\), using condition (ii), we get \((\theta_{n+1}, \theta_n) \in \mathcal{R}\) for all \( n \in \mathbb{N}^* \). Using this conclusion and above arguments, it can be shown that

\[
\lim_{n \to \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^n \theta_0) = 0. \tag{5}
\]

Next, we show that \( \{\mathcal{K}^n \theta_0\} \) is a Cauchy sequence in \( \mathcal{W} \). For this we show that

\[
\lim_{m,n \to \infty} \omega(\mathcal{K}^n \theta_0, \mathcal{K}^m \theta_0) = 0. \tag{6}
\]

On the contrary, suppose that condition (6) does not hold. Then we can find \( \delta > 0 \) and increasing sequences \( \{m_k\}_{k=1}^{\infty}, \{n_k\}_{k=1}^{\infty} \) of positive integers with \( m_k > n_k \) such that

\[
\omega(\mathcal{K}^{m_k} \theta_0, \mathcal{K}^{m_k+1} \theta_0) \geq \delta \text{ for all } k \in \{1, 2, 3, \ldots\}. \tag{7}
\]

By (3), there exists \( k_0 \in \mathbb{N} \) such that \( n_k > k_0 \) implies that

\[
\omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{n_k+1} \theta_0) < \delta.
\]

In view of the two last inequalities, we observe that \( m_k \neq n_{k+1} \). We may assume that \( m_k \) is the minimal index such that (7) holds, so that

\[
\omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^r \theta_0) < \delta, \quad \text{for } r \in \{n_{k+1}, n_{k+2}, \ldots, m_k - 1\}.
\]

Now, making use of (7), we get

\[
0 < \delta \leq \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k} \theta_0) \leq \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k-1} \theta_0) + \omega(\mathcal{K}^{m_k-1} \theta_0, \mathcal{K}^{m_k} \theta_0) < \delta + \omega(\mathcal{K}^{m_k-1} \theta_0, \mathcal{K}^{m_k} \theta_0).
\]

Thus,

\[
\lim_{k \to \infty} \omega(\mathcal{K}^{n_k} \theta_0, \mathcal{K}^{m_k} \theta_0) = \delta. \tag{8}
\]

Using the triangle inequality, we have

\[
\omega(\mathcal{K}^{m_k} \theta_0, \mathcal{K}^{m_k} \theta_0) \leq \omega(\mathcal{K}^{m_k} \theta_0, \mathcal{K}^{m_k+1} \theta_0) + \omega(\mathcal{K}^{m_k+1} \theta_0, \mathcal{K}^{m_k} \theta_0)
\leq \omega(\mathcal{K}^{m_k} \theta_0, \mathcal{K}^{m_k+1} \theta_0) + \omega(\mathcal{K}^{m_k+1} \theta_0, \mathcal{K}^{m_k+1} \theta_0)
+ \omega(\mathcal{K}^{m_k+1} \theta_0, \mathcal{K}^{m_k} \theta_0).
\]

Taking the limit on both sides and making use of (3), (5) and (8), we obtain

\[
\lim_{k \to \infty} \omega(\mathcal{K}^{m_k+1} \theta_0, \mathcal{K}^{m_k+1} \theta_0) \geq \delta. \tag{9}
\]
Again, using the triangle inequality, we have
\[ \omega(K^{n_k+1} \theta_0, K^{m_k+1} \theta_0) \leq \omega(K^{m_k+1} \theta_0, K^{m_k} \theta_0) + \omega(K^{m_k} \theta_0, K^{m_k+1} \theta_0) \]
\[ \leq \omega(K^{m_k+1} \theta_0, K^{m_k} \theta_0) + \omega(K^{m_k} \theta_0, K^{m_k-1} \theta_0) + \omega(K^{m_k} \theta_0, K^{m_k+1} \theta_0). \]

Taking the limit on both sides and making use of (3), (5) and (8), we obtain
\[ \lim_{k \to \infty} \omega(K^{n_k+1} \theta_0, K^{m_k+1} \theta_0) \leq \delta. \] (10)

Combining (9) and (10), we have
\[ \lim_{k \to \infty} \omega(K^{n_k+1} \theta_0, K^{m_k+1} \theta_0) = \delta. \] (11)

Now, since \( R \) is transitive and since \( \{ \theta_n \} \) is an \( R \)-preserving sequence, we must have \( (K^n \theta_0, K^m \theta_0) \in R \) for all \( r \in \mathbb{N} \). Therefore, on applying condition (1), we get
\[ \mathcal{H} \left( \omega(K^{m_k+1} \theta_0, K^{m_k} \theta_0), \omega(K^{m_k} \theta_0, K^{m_k+1} \theta_0), \omega(K^{m_k+1} \theta_0, K^{m_k+1} \theta_0) \right) \leq 0. \]

Now applying \((\mathcal{H}_4)\) in the fifth and sixth variables, we have
\[ \mathcal{H} \left( \omega(K^{n_k+1} \theta_0, K^{m_k+1} \theta_0), \omega(K^{m_k} \theta_0, K^{m_k+1} \theta_0), \omega(K^{m_k+1} \theta_0, K^{m_k+1} \theta_0), \omega(K^{m_k+1} \theta_0, K^{m_k+1} \theta_0) \right) \leq 0. \]

Applying the limit and using the continuity of \( \mathcal{H} \), we get
\[ \mathcal{H} \left( \delta, \delta, 0, 0, \delta, \delta \right) \leq 0, \]

a contradiction to \( (\mathcal{H}_3) \). Hence, \( \{K^n \theta_0\} \) must be a Cauchy sequence in \( \mathcal{W} \).

From \( \mathcal{R} \)-completeness of \( \mathcal{W} \), there exists a point \( \vartheta^* \in \mathcal{W} \) such that \( \lim_{n \to \infty} K^n \theta_0 = \vartheta^* \). We shall show that \( \vartheta^* \) is a fixed point of \( K \).

Using the \( \mathcal{R} \)-continuity of \( K \) (due to condition \( A_3 \)), we have \( \lim_{n \to \infty} K^n \theta_0 = K \vartheta^* \).

Owing to the uniqueness of the limit, we obtain \( K \vartheta^* = \vartheta^* \).

Finally, assume that \( \omega(\vartheta^*, \vartheta^*) > 0 \). Then, by (1), we have
\[ \mathcal{H} \left( \omega(K \vartheta^*, K \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, K \vartheta^*), \omega(K \vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, K \vartheta^*) \right) \leq 0 \]

or
\[ \mathcal{H} \left( \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*), \omega(\vartheta^*, \vartheta^*) \right) \leq 0. \]

It follows from \( (\mathcal{H}_{2a}) \) and \( (\mathcal{H}_4) \) (for \( \zeta = \xi = \mu = \vartheta^* \)) that there is \( h \in [0, 1) \) such that
\[ \omega(\vartheta^*, \vartheta^*) \leq h \omega(\vartheta^*, \vartheta^*) \] (12)
a contradiction. Therefore, \( \omega(\theta^*, \varphi^*) = 0 \).

Next, we have the following result.

**Theorem 4.3** The conclusion of Theorem 4.2 remains true if condition (A3) is replaced with the following one:

\( (A'_3) \) For every \( v \in W \) with \( v \neq K\varphi \), \( \inf\{\omega(\varphi, v) + \omega(\varphi, K\varphi) \mid \varphi \in W\} > 0 \).

**Proof** Following the proof of Theorem 4.2, we observe that the sequence \( \{K^n\theta_0\} \) is a Cauchy sequence, and so there exists a point \( \varphi^* \) in \( W \) such that \( \lim_{n \to \infty} K^n\theta_0 = \varphi^* \). Since \( \lim_{m,n \to \infty} \omega(K^n\theta_0, K^m\theta_0) = 0 \) for each \( \epsilon > 0 \), there exists \( N_\epsilon \in \mathbb{N} \) such that \( n > N_\epsilon \) implies \( \omega(K^n\theta_0, K^m\theta_0) < \epsilon \). Hence, \( \lim_{n \to \infty} \omega(K^n\theta_0, \varphi^*) = 0 \), which contradicts our assumption. Therefore, \( K\varphi^* = \varphi^* \).

The last conclusion is derived as in the proof of Theorem 4.2. \( \square \)

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 4.2 and 4.3.

To obtain uniqueness, we add the following additional conditions for the functions \( H \in \Omega \) and call them by \( \Omega' \): for all \( \zeta, \xi > 0 \),

\( (H_4) \) \( H(\xi, \zeta, 0, 0, \xi, \zeta) \leq 0 \) implies that there exists \( \psi \in [0, 1) \) such that \( \zeta \leq \psi(\xi) \).

**Theorem 4.4** In addition to the hypotheses of Theorem 4.2 (or Theorem 4.3), if any of the following conditions is fulfilled:

(I) For all \( \varphi, \nu \in W \), there exists \( z \in W \) such that

\[
\{ (z, Kz), (z, \varphi), (z, \nu) \} \subseteq R;
\]

(13)

(II) The set \( K(W) \) is \( R \)-directed;

(III) \( R_{|K(W)} \) is complete;

(IV) \( \Psi(\varphi, \nu, \text{Fix}(K), R^r) \) is nonempty for each \( \varphi, \nu \in \text{Fix}(K) \), then \( K \) has a unique fixed point.

**Proof** In view of Theorem 4.2 (or Theorem 4.3), \( \text{Fix}(T) \neq \emptyset \) and \( w(\varphi, \varphi) = 0 \) for each \( \varphi \in \text{Fix}(T) \).
• Assumption (I). Suppose that there exist distinct fixed points \( \vartheta \) and \( \nu \) of \( K \). We will consider the following two cases.

Case (A): \( \vartheta \) and \( \nu \) are \( R \)-comparable. Then \( K^n \vartheta = \vartheta \) and \( K^n \nu = \nu \) are comparable for \( n = 0, 1, \ldots \). Therefore, using condition (1),

\[
H \left( \omega(K^n \vartheta, K^{n-1} \vartheta), \omega(K^n \nu, K^{n-1} \nu), \omega(K^n \vartheta, K^n \nu), \omega(K^{n-1} \vartheta, K^{n-1} \nu) \right) \leq 0
\]

or

\[
H \left( \omega(\vartheta, \nu), 0, 0, \omega(\vartheta, \nu), \omega(\vartheta, \nu) \right) \leq 0
\]

which contradict \((H_3)\), then \( w(\vartheta, \nu) = 0 \). Since \( w(\vartheta, \vartheta) = 0 \), by using Lemma 2.2, we have \( \vartheta = \nu \); i.e. the fixed point of \( K \) is unique.

Case (B): By assumption (I), there exists \( z \in W \), satisfying condition (13). Due to the \( K \)-closedness of \( R \), we get

\[
(K^{n-1} z, \vartheta) \in R, \quad (K^{n-1} z, \nu) \in R,
\]

and, using (1), it follows that

\[
H \left( \omega(K^n z, \nu), \omega(K^{n-1} z, \nu), \omega(K^{n-1} z, K^n z), \omega(v, K^n z), \omega(K^{n-1} z, K^n z), \omega(K^n z, \nu) \right) \leq 0. \tag{14}
\]

Using \((z, Kz) \in R\), similarly as in the proof of Theorem 4.2, it can be shown that \( w(K^{n-1} z, K^n z) \to 0 \) as \( n \to \infty \). Therefore, for \( n \) sufficiently large, from (14), we have

\[
H \left( \omega(K^n z, \nu), \omega(K^{n-1} z, \nu), 0, 0, \omega(K^{n-1} z, K^n z), \omega(K^n z, \nu) \right) \leq 0. \tag{15}
\]

It follows from \((H_4)\) that there is \( h \in [0, 1) \) such that

\[
\omega(K^n z, \nu) \leq h \omega(K^{n-1} z, \nu).
\]

It follows that the sequence \( \{w(K^n z, \nu)\} \) is nonincreasing. As earlier, we have

\[
\lim_{n \to \infty} w(K^n z, \nu) = 0.
\]

Also, since \((z, \vartheta) \in R\), proceeding as earlier, we can prove that

\[
\lim_{n \to \infty} w(K^n z, \vartheta) = 0,
\]

and by using Lemma 2.2 we infer that \( \nu = \vartheta \); i.e. the fixed point of \( K \) is unique.

• Assumption (II). For any two fixed points \( \vartheta, \nu \) of \( K \), there must be an element \( z \in K(W) \) such that

\[
(z, \vartheta) \in R \quad \text{and} \quad (z, \nu) \in R.
\]
As \( \mathcal{R} \) is \( \mathcal{K} \)-closed, so for all \( n \in \mathbb{N} \cup \{0\} \),

\[
(\mathcal{K}^n z, \vartheta) \in \mathcal{R} \quad \text{and} \quad (\mathcal{K}^n z, \nu) \in \mathcal{R}.
\]

In the line of proof of Case(B) (I), we obtain \( \vartheta = \nu \), i.e. \( \mathcal{K} \) has a unique fixed point.

- **Assumption (III).** Suppose that \( \vartheta^*, \nu^* \) are two fixed points of \( \mathcal{K} \). Then we must have \( (\vartheta^*, \nu^*) \in \mathcal{R} \) or \( (\nu^*, \vartheta^*) \in \mathcal{R} \). For \( (\vartheta^*, \nu^*) \in \mathcal{R} \), we obtain

\[
\mathcal{H} \left( [\omega(\mathcal{K} \vartheta^*, \mathcal{K} \nu^*), \omega(\vartheta^*, \nu^*), \omega(\vartheta^*, \mathcal{K} \vartheta^*)], \omega(\nu^*, \mathcal{K} \nu^*), \omega(\vartheta^*, \mathcal{K} \nu^*), \omega(\mathcal{K} \vartheta^*, \nu^*) \right) \leq 0,
\]

that is,

\[
\mathcal{H}(\omega(\vartheta^*, \nu^*), \omega(\vartheta^*, \nu^*), 0, 0, \omega(\vartheta^*, \nu^*), \omega(\vartheta^*, \nu^*)) \leq 0,
\]

which contradicts \( (H_4) \), then \( w(\vartheta^*, \nu^*) = 0 \). Since \( w(\vartheta^*, \vartheta^*) = 0 \), by using Lemma 2.2, we have \( \vartheta^* = \nu^* \); i.e. the fixed point of \( \mathcal{K} \) is unique.

In a similar way, if \( (\nu^*, \vartheta^*) \in \mathcal{R} \), we have \( \vartheta^* = \nu^* \).

- **Assumption (IV).** Suppose that \( \vartheta^*, \nu^* \) are two fixed points of \( \mathcal{K} \). Let \( \{z_0, z_1, \ldots, z_k\} \) be an \( \mathcal{R}^2 \)-path in Fix\( (\mathcal{K}) \) connecting \( \vartheta^* \) and \( \nu^* \). As in Case (I, A), it must be \( z_{i-1} = z_i \) for each \( i = 1, 2, \ldots, k \), and it follows that \( \vartheta^* = \nu^* \).

By choosing \( \mathcal{H} \in \Omega \) from Example 3.1–Example 3.2, we have the following consequences.

**Corollary 4.5** Let all the conditions of Theorem 4.2–Theorem 4.3 be satisfied, except that the assumption of orbitally \( \omega \)-implicit contractive mapping for \( \mathcal{H} \in \Omega \) is replaced with either of the forms

(I)

\[
\left[ \omega(\mathcal{K} \vartheta, \mathcal{K} \nu) \right]^2 \leq a \left[ \omega(\vartheta, \nu) \right]^2 - b \left[ \omega(\vartheta, \mathcal{K} \vartheta) \right]^2 + \left[ \omega(\nu, \mathcal{K} \nu) \right]^2 \left/ \left[ 1 + \omega(\vartheta, \mathcal{K} \nu) + \omega(\mathcal{K} \vartheta, \nu) \right] \right., \quad (16)
\]

where \( 1/2 < a < 1, 0 < b < 1/2 \), or

(II)

\[
\left[ \omega(\mathcal{K} \vartheta, \mathcal{K} \nu) \right]^2 \leq a \left[ \omega(\vartheta, \nu) \right]^2 - b \left[ \omega(\vartheta, \mathcal{K} \vartheta) \right]^2 + \left[ \omega(\nu, \mathcal{K} \nu) \right]^2 \left/ \left[ 1 + \omega(\vartheta, \mathcal{K} \nu) + \omega(\mathcal{K} \vartheta, \nu) \right] \right., \quad (17)
\]

where \( 1/2 < a < 1, 0 < b < 1/2 \).

Then Fix\( (\mathcal{K}) \) is a singleton.

**5 Illustrations**

In this part, we show how important our results are when compared to metric distances.

**Example 5.1** (Inspired by [19]) Consider the set \( \mathcal{W} = [0, \frac{3}{2}] \) with the usual metric \( d \). Define a \( w \)-distance \( \omega: \mathcal{W} \times \mathcal{W} \to [0, \infty) \) by \( \omega(\vartheta, \nu) = \nu \) for all \( \vartheta, \nu \in \mathcal{W} \) and the binary relation \( \mathcal{R} \) by

\[
(\vartheta, \nu) \in \mathcal{R} \iff \nu \leq 1 \quad \text{or} \quad \vartheta \leq 1.
\]
Consider the self-mapping $K$ on $W$ given by

$$K(\theta) = \begin{cases} 
\frac{\theta}{4}, & 0 \leq \theta \leq \frac{2}{3}; \\
1 - \theta, & \frac{2}{3} < \theta < 1; \\
\frac{2}{3}, & \theta = 1; \\
\theta - \frac{1}{2}, & \theta > 1.
\end{cases}$$

It is clear that $W$ is $K - R$ is complete, $X(K, R) \neq \emptyset$ and $R$ is $K$-closed.

Considering Example 3.2, we can define $H \in \Omega$ as

$$H(q_1, q_2, q_3, q_4, q_5, q_6) = q_1^2 - aq_2^2 - b\frac{q_2^2 + q_4^2}{1 + q_5^2 + q_6^2},$$

where $1/2 < a < 1, 0 < b < 1/2$. Then (1) reduces to

$$\left[\omega(K, \partial, K, v)\right]^2 \leq a\left[\omega(\partial, v)\right]^2 - b\frac{[\omega(\partial, v, K, v)]^2 + [\omega(v, K, v)]^2}{1 + [\omega(\partial, v, K, v)]^2 + [\omega(K, \partial, v)]^2}, \quad (18)$$

where $1/2 < a < 1, 0 < b < 1/2$.

We show that $K$ satisfies (18). We divide the proof into three parts for $\theta \leq 1$. Similarly, we can show for $\nu \leq 1$.

**Case 1:** Let $\nu = 0$, then $v \in [0, \frac{3}{2}]$ such that $(\theta, v) \in R$. Then

- Let $0 \leq v \leq \frac{1}{2}$. Then $\omega(K, \partial, K, v) = v/3$, $\omega(\partial, v) = v$, $\omega(\partial, K, \partial) = 0$, $\omega(v, K, v) = v/3$, $\omega(\partial, v, K, v) = v/3$, $\omega(\partial, \partial, v) = v$. Therefore, condition (18) reduces to

$$\frac{v^2}{9} \leq av^2 - b\frac{v^2}{2v^2 + 9}.$$

- Let $\frac{1}{2} < v < 1$, then $K\nu \in (0, \frac{1}{2})$. Then $\omega(K, \partial, K, v) = 1 - v$, $\omega(\partial, v) = v$, $\omega(\partial, K, \partial) = 0$, $\omega(v, K, v) = 1 - v$, $\omega(\partial, \partial, v) = v$. Therefore, condition (18) reduces to

$$(1 - v)^2 \leq av^2 - b \frac{(1 - v)^2}{2 + 2v^2 - 2v}.$$ 

- Let $v = 1$. Then $\omega(K, 0, K, 1) = \frac{3}{4}$, $\omega(\partial, v) = 1$, $\omega(\partial, K, \partial) = 0$, $\omega(v, K, v) = 3/4$, $\omega(\partial, v, K, v) = 3/4$, $\omega(\partial, \partial, v) = 1$. Therefore, condition (18) reduces to

$$\frac{9}{16} \leq a - b \frac{9}{41}.$$ 

- Let $v > 1$. Then we have $\omega(K, \partial, K, v) = v - \frac{1}{2}$, $\omega(\partial, v) = v$, $\omega(\partial, K, \partial) = 0$, $\omega(v, K, v) = v - 1/2$, $\omega(\partial, v, K, v) = v - 1/2$, $\omega(K, \partial, v) = v$. Therefore, condition (18) reduces to

$$\left(v - \frac{1}{2}\right)^2 \leq av^2 - b \frac{(v - 1/2)^2}{2v^2 - v + 5/4}.$$

**Case 2:** Let $\theta \in [0, 3/2]$ and $v = 0$. Then $\omega(K, \partial, K, v) = 0$, and condition (18) is trivially true.

**Case 3:** Let $\theta \neq 0$. Then $v \leq 1$. Therefore, we have:
\[ \begin{align*}
\text{Let } & 0 \leq \nu \leq \frac{2}{3}, \quad \omega(K\vartheta, K\nu) = \nu/3, \quad \omega(\vartheta, \nu) = \nu, \quad \omega(\vartheta, K\vartheta) = K\vartheta, \quad \omega(\nu, K\nu) = \nu/3, \\
& \omega(\vartheta, K\nu) = v/3, \quad \omega(K\vartheta, \nu) = v. \quad \text{Therefore, condition (18) reduces to} \\
& \frac{\nu^2}{9} \leq a\nu^2 - b \left( \frac{(K\vartheta)^2 + \nu^2}{1 + \nu^2/9 + \nu^2} \right). \\
\end{align*} \]

\[ \begin{align*}
\text{Let } & \frac{2}{3} < \nu < 1, \quad \omega(K\vartheta, K\nu) = 1 - \nu, \quad \omega(\vartheta, \nu) = \nu, \quad \omega(\vartheta, K\vartheta) = K\vartheta, \quad \omega(\nu, K\nu) = 1 - \nu, \\
& \omega(\vartheta, K\nu) = 1 - \nu, \quad \omega(K\vartheta, \nu) = \nu. \quad \text{Therefore, condition (18) reduces to} \\
& (1 - \nu)^2 \leq a\nu^2 - b \left( \frac{(K\vartheta)^2 + (1 - \nu)^2}{2 + 2\nu^2 - 2\nu} \right). \\
\end{align*} \]

\[ \begin{align*}
\text{Let } & \nu = 1, \quad \omega(K\vartheta, K\nu) = \frac{3}{4}, \quad \omega(\vartheta, \nu) = 1, \quad \omega(\vartheta, K\vartheta) = K\vartheta, \quad \omega(1, K\nu) = 3/4, \quad \omega(\vartheta, K\nu) = 3/4, \\
& \omega(K\vartheta, \nu) = 1. \quad \text{Therefore, condition (18) reduces to} \\
& \frac{9}{16} \leq a - b \left( \frac{(K\vartheta)^2 + 9}{41/16} \right). \\
\end{align*} \]

\[ \begin{align*}
\text{Let } & \vartheta \leq 1 \text{ and } \nu > 1. \quad \text{Then} \quad \omega(K\vartheta, K\nu) = \nu - 1/2, \quad \omega(\vartheta, \nu) = \nu, \quad \omega(\vartheta, K\vartheta) = K\vartheta, \\
& \omega(\nu, K\nu) = \nu - 1/2, \quad \omega(\vartheta, K\nu) = \nu - 1/2, \quad \omega(K\vartheta, \nu) = \nu. \quad \text{Therefore, condition (18) reduces to} \\
& \left( \nu - \frac{1}{2} \right)^2 \leq a\nu^2 - b \left( \frac{(K\vartheta)^2 + (\nu - 1/2)^2}{2\nu^2 - \nu + 5/4} \right). \\
\end{align*} \]

It can be easily checked that the above cases hold true for \( a = 9/10, \ b = 1/10. \)
Thus \( K \) is an \( R_\omega \)-implicit contractive mapping.

Note that \( K \) is not \( R \)-continuous at \( \vartheta = 1 \) and \( \vartheta = \frac{2}{3}. \)

Finally, we will show that condition \((A_5')\) of Theorem 4.3 holds true. If \( \nu > 0, \) we have \( \nu \neq K\nu \) so that

\[ \inf \{ \omega(\vartheta, \nu) + \omega(\vartheta, K\vartheta) : \vartheta \in W \} = \inf \{ \nu + K\vartheta : \vartheta \in W \} > 0. \]

Thus, all the conditions of Theorem 4.3 are satisfied and \( \text{Fix}(K) = \{0\}. \)

Remark 5.2 In the above example, it is noted that (18) is not true in a metric space, that is,

\[ |K\vartheta - K\nu|^2 \leq a|\vartheta - \nu|^2 - b \left( \frac{|\vartheta - K\vartheta|^2 + |v - K\nu|^2}{1 + |\vartheta - K\vartheta|^2 + |K\vartheta - \nu|^2} \right). \quad \text{(19)} \]

For instance, if \( \vartheta = 1, \nu = 3/4, \) (19) implies that

\[ \frac{1}{4} \leq \frac{a}{16} - \frac{b}{5} = \frac{1}{16} - \frac{b}{1 + \frac{1}{16} + \frac{1}{4} + \frac{1}{16} + 0}, \]

which is not true for any \( 1/2 < a < 1, \ 0 < b < 1/2. \)
Example 5.3 Consider the set \( W = \{0, 1, 2\} \) with the usual metric \( d \), and order relation \( R \) is defined as \( (\vartheta, v) \in R \Leftrightarrow \vartheta \leq v \). Define a \( w \)-distance \( \omega: W \times W \rightarrow [0, \infty) \) by \( \omega(\vartheta, v) = v \) for all \( \vartheta, v \in W \). Define a mapping \( K: W \rightarrow W \) by

\[
K = \begin{pmatrix}
0 & 1 & 2 \\
0 & 0 & 1
\end{pmatrix}.
\]

It is clear that \( W \) is \( K - R \) is complete, \( X(K, R) \neq \emptyset \) and \( R \) is \( K \)-closed.

Considering Example 3.1, we can define \( H \in \Omega \) as

\[
H(q_1, q_2, q_3, q_4, q_5, q_6) = q_2^1 - aq_2^2 + b \frac{q_2^3 + q_2^4}{q_5 + q_6 + 1},
\]

where \( 1/2 < a < 1, 0 < b < 1/2 \). Then (1) reduces to

\[
[\omega(K\vartheta, K\nu)]^2 \leq a[\omega(\vartheta, \nu)]^2 - b \frac{[\omega(\vartheta, K\vartheta)]^2 + [\omega(\nu, K\nu)]^2}{1 + \omega(\vartheta, K\nu) + \omega(K\vartheta, \nu)},
\]

where \( 1/2 < a < 1, 0 < b < 1/2, \vartheta \leq \nu \).

It is easy to check that \( K \) satisfies (20). Thus, by Theorem 4.2, \( \text{Fix}(K) = \emptyset \).

It is noted that (20) is not true in metric distance \( d \). For instance, if \( \vartheta = 1, \nu = 2 \), (20) implies that

\[
1 \leq a - \frac{b}{4} = a - b \frac{0 + 1}{1 + 0 + 2},
\]

which is not true for any \( 1/2 < a < 1, 0 < b < 1/2 \).

Example 5.4 Consider the set \( W = [0, 1] \) with the usual metric \( d \). Define a \( w \)-distance \( \omega: W \times W \rightarrow [0, \infty) \) by \( \omega(\vartheta, v) = \vartheta + v \) for all \( \vartheta, v \in W \) and the binary relation \( R \) by

\[
R = \left\{ (0, 0), (0, 1), \left( \frac{1}{4}, 1 \right), \left( \frac{1}{4}, 0 \right), \left( 0, \frac{1}{4} \right), \left( \frac{1}{4}, \frac{1}{4} \right) \right\}.
\]

Consider the self-mapping \( K \) on \( W \) given by

\[
K(\vartheta) = \begin{cases} 
0, & 0 \leq \vartheta \leq \frac{1}{4} \\
\frac{1}{4}, & \frac{1}{4} < \vartheta \leq 1.
\end{cases}
\]

It is clear that \( W \) is \( K - R \) is complete, \( X(K, R) \neq \emptyset \) and \( R \) is \( K \)-closed.

Considering Example 3.2, we can define \( H \in \Omega \) as

\[
H(q_1, q_2, q_3, q_4, q_5, q_6) = q_2^1 - aq_2^2 + b \frac{q_2^3 + q_2^4}{q_5 + q_6 + 1},
\]

where \( 1/2 < a < 1, 0 < b < 1/2 \). Then (1) reduces to

\[
[\omega(K\vartheta, K\nu)]^2 \leq a[\omega(\vartheta, \nu)]^2 - b \frac{[\omega(\vartheta, K\vartheta)]^2 + [\omega(\nu, K\nu)]^2}{1 + [\omega(\vartheta, K\nu)]^2 + [\omega(K\vartheta, \nu)]^2},
\]

where \( 1/2 < a < 1, 0 < b < 1/2 \).
We show that $K$ satisfies (21).

Case 1: If $(\vartheta, v) \in \{0, 0\}, (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4})$, then $\omega(K \vartheta, K v) = 0$, and condition (21) is trivially satisfied.

Case 2: If $(\vartheta, v) \in \{(0, 1), (\frac{1}{2}, 1)\}$.
- Let $(\vartheta, v) = (0, 1)$. Then $\omega(K \vartheta, K v) = 1/4$, $\omega(\vartheta, K \vartheta) = 1$, $\omega(v, K v) = 5/4$, $\omega(\vartheta, K v) = 1/4$, $\omega(K \vartheta, v) = 1$. Therefore, condition (21) reduces to

$$
\frac{1}{16} \leq a - \frac{25c}{33}.
$$

- Let $(\vartheta, v) = \left(\frac{1}{2}, 1\right)$. Then $\omega(K \vartheta, K v) = 1/4$, $\omega(\vartheta, K \vartheta) = 5/4$, $\omega(\vartheta, K \vartheta) = 1/4$, $\omega(v, K v) = 5/4$, $\omega(\vartheta, K v) = 1/2$, $\omega(K \vartheta, v) = 1$. Therefore, condition (21) reduces to

$$
\frac{1}{16} \leq \frac{5a}{4} - \frac{13b}{18}.
$$

It can be easily checked that the above cases hold true for $a = 3/4, b = 1/10$. Thus $K$ is an $R_w$-implicit contractive mapping.

Finally, we will show that condition $(A_{5}')$ of Theorem 4.3 holds true. If $v > 0$, we have $v \neq K v$ so that

$$
\inf \left\{ \omega(\vartheta, v) + \omega(\vartheta, K \vartheta) : \vartheta \in W \right\} = \inf \{2v + \vartheta + K \vartheta : \vartheta \in W \} > 0.
$$

Thus, all the conditions of Theorem 4.3 are satisfied and $\text{Fix}(K) = \{0\}$.

Example 5.5 Consider the set $W = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ with $d(\vartheta, v) = |\vartheta - v|$, where $(\vartheta, v) \in R \Leftrightarrow \vartheta < v$. Define a $w$-distance $\omega : W \times W \to [0, +\infty)$ by $\omega(\vartheta, v) = v$ for all $\vartheta, v \in W$.

Consider the self-mapping $K$ on $W$ given by

$$
K(\vartheta) = \begin{cases} 
\frac{1}{2}, & \text{if } \vartheta = 0 \\
\frac{1}{2^{n+1}}, & \text{if } \vartheta = \frac{1}{2^n}, n \in \mathbb{N} \cup \{0\}.
\end{cases}
$$

Considering Example 3.1, (1) reduces to

$$
\left[\omega(K \vartheta, K v)\right]^2 \leq a \left[\omega(\vartheta, v)\right]^2 - b \left[\omega(\vartheta, K \vartheta)\right]^2 + \left[\omega(\vartheta, K v)\right]^2 - \frac{\omega(\vartheta, K \vartheta) + \omega(\vartheta, K v)}{1 + \omega(\vartheta, K \vartheta) + \omega(\vartheta, K v)},
$$

where $1/2 < a < 1, 0 < b < 1/2, \vartheta < v$.

Consider two cases:
- If $\vartheta = 0$ and $v = 1/2^n, n \in \mathbb{N}$, then (22) reduces to

$$
\frac{1}{2^{2n+2}} \leq a \frac{1}{2^{2n}} - b \frac{\frac{1}{2^{2n+1}}}{1 + \frac{1}{2^{2n+1}} + \frac{1}{2^n}}
$$

and is fulfilled for $1/2 < a < 1, 0 < b < 1/2$.
- Let $\vartheta, v \in \{1/2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Consider $\vartheta = \frac{1}{2^m}$ and $v = \frac{1}{2^m}(m > n)$. Then inequality (22) has the form

$$
\frac{1}{2^{2m+2}} \leq a \frac{1}{2^{2m}} - b \frac{\frac{1}{2^{2m+1}} + \frac{1}{2^{2m+2}}}{1 + \frac{1}{2^{2m+1}} + \frac{1}{2^n}}.
$$
which is obviously true for $1/2 < a < 1$, $0 < b < 1/2$.

For instance, both of the above cases are true for $a = 99/100$, $b = 2/100$.

It can be noted that, for $0 \neq K0$, we have

$$\inf \left\{ \omega \left( \frac{1}{2^a}, 0 \right) + \omega \left( \frac{1}{2^b}, \frac{1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} = \inf \left\{ \frac{1}{2^{n+1}} : n \in \mathbb{N} \right\} = 0.$$

Thus, all the conditions of Theorem 4.3 are satisfied except (A5). Clearly, $\mathcal{K}$ has no fixed points in $\mathcal{W}$.

**Remark 5.6** It is observed that for $0 \neq K0$ in the metric space

$$\inf \left\{ d \left( \frac{1}{2^a}, 0 \right) + d \left( \frac{1}{2^b}, \frac{1}{2^{n+1}} \right) : n \in \mathbb{N} \right\} = \inf \left\{ \frac{1}{2^a} + \frac{1}{2^{n+1}} : n \in \mathbb{N} \right\} = 0.$$

That is, condition (A5) of Theorem 4.3 is also not satisfied in the metric space.

### 6 Application to nonlinear matrix equations

Let $\mathcal{H}(n)$ (resp. $\mathcal{K}(n)$, $\mathcal{P}(n)$) denote the set of all $n \times n$ Hermitian (resp. positive semi-definite, positive definite) matrices over $\mathbb{C}$ and $\mathcal{M}(n)$ be the set of all $n \times n$ matrices over $\mathbb{C}$. Denote by $s(\mathcal{U})$ any singular value of a matrix $\mathcal{U}$, and its trace norm by $s'(\mathcal{U}) = \| \mathcal{U} \|$. For $\mathcal{C}, \mathcal{D} \in \mathcal{H}(n), \mathcal{C} \succeq \mathcal{D}$ (resp. $\mathcal{C} \succ \mathcal{D}$) will mean that the matrix $\mathcal{C} - \mathcal{D}$ is positive semi-definite (resp. positive definite).

**Theorem 6.1** Consider the equation

$$\mathcal{U} = \mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i, \quad (23)$$

where $\mathcal{Q} \in \mathcal{P}(n)$, $\mathcal{A}_i \in \mathcal{M}(n)$, $i = 1, \ldots, k$, and the operator $\mathcal{G} : \mathcal{P}(n) \to \mathcal{P}(n)$ is continuous in the trace norm. Let, for some $M, N_1 \in \mathbb{R}$ and for any $\mathcal{U} \in \mathcal{P}(n)$ with $\| \mathcal{U} \| \leq M$, $s(\mathcal{G}(\mathcal{U})) \leq N_1$ hold for all singular values of $\mathcal{G}(\mathcal{U})$. Assume that:

1. $\| \mathcal{Q} \| \leq M - NN_1 n$, where $\sum_{i=1}^{k} \| \mathcal{A}_i^* \| \| \mathcal{A}_i \| = N$;
2. For any $\mathcal{Q} \in \mathcal{P}(n)$ with $\| \mathcal{Q} \| \leq M$, $\sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{A}_i \succeq \mathcal{O}$ holds;
3. There exist $1/2 < a < 1$, $0 < b < 1/2$ such that, for any $s(\mathcal{Q})$,

$$2NN_1 + 2s(\mathcal{Q}) \leq \left[ \Upsilon(\mathcal{U}, \mathcal{V}) \right]^{1/2} \quad (24)$$

holds for all $\mathcal{U}, \mathcal{V} \in \mathcal{P}(n)$ with $\| \mathcal{U} \|, \| \mathcal{V} \| \leq M, \mathcal{U} \preceq \mathcal{V}$ and

$$\sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i \neq \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i, \text{ where}$$

$$\Upsilon(\mathcal{U}, \mathcal{V}) = a \left( |s'(\mathcal{U})| + |s'(\mathcal{V})| \right)^2 - \beta \left( |s'(\mathcal{U})| + |s'(\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)| \right)^2 + (|s'(\mathcal{V})| + |s'(\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)|)^2 \left[ 1 + |s'(\mathcal{U})| + |s'(\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{U}) \mathcal{A}_i)| + |s'(\mathcal{Q} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{G}(\mathcal{V}) \mathcal{A}_i)| + |s'(\mathcal{V})| \right]^{-1} \quad (25)$$
Then NME (23) has a unique solution \( \hat{U} \in \mathcal{P}(n) \) with \( \| \hat{U} \| \leq M \). Further, the solution can be obtained as the limit of the iterative sequence \( \{ U_n \} \), where for \( j \geq 0 \),

\[
U_{j+1} = Q + \sum_{i=1}^{k} A_i^* G(U_j) A_i
\]

(26)

and \( U_0 \) is an arbitrary element of \( \mathcal{P}(n) \) satisfying \( \| U_0 \| \leq M \).

**Proof** Denote by \( \Lambda := \{ U \in \mathcal{P}(n) : \| U \| \leq M \} \) a closed subset of \( \mathcal{P}(n) \). According to (II), any solution of (23) in \( \Lambda \) has to be positive definite. We have, for any \( U \in \Lambda \),

\[
\left\| Q + \sum_{i=1}^{k} A_i^* G(U) A_i \right\| \leq \| Q \| + \sum_{i=1}^{k} \| A_i \| \| G(U) \| = \| Q \| + N \| G(U) \|.
\]

(27)

Since all singular values of \( U \) satisfy \( s(G(U)) \leq N_1 \), it follows that \( \| G(U) \| \leq N_1 n \). Thus, (27) implies

\[
\left\| Q + \sum_{i=1}^{k} A_i^* G(U) A_i \right\| \leq \| Q \| + NN_1 n \leq M - NN_1 n + NN_1 n = M.
\]

Define now an operator \( \mathcal{K} : \Lambda \to \Lambda \) by

\[
\mathcal{K}(U) = Q + \sum_{i=1}^{k} A_i^* G(U) A_i
\]

for all \( U \in \Lambda \).

Also, define a binary relation

\[
\mathcal{R} = \{ (U, V) \in \Lambda \times \Lambda : UV \preceq U \text{ or } UV \preceq V \}.
\]

It is clear that finding positive definite solution(s) of equation (23) is equivalent to finding fixed point(s) of \( \mathcal{K} \). Notice that \( \mathcal{K} \) is well defined, \( \mathcal{R} \)-continuous and \( \mathcal{R} \) is \( \mathcal{K} \)-closed. Since

\[
\sum_{i=1}^{k} A_i^* G(Q) A_i \succ 0
\]

for some \( Q \in \Lambda \), we have \( (Q, \mathcal{K}(Q)) \in \mathcal{R} \) or \( (\mathcal{K}(Q), Q) \in \mathcal{R} \), and hence \( \Lambda(\mathcal{K}; \mathcal{R}) \neq \emptyset \).

Now, for \( (U, V) \in \mathcal{R} \), we have

\[
\| \mathcal{K}(U) \| + \| \mathcal{K}(V) \| = \| Q + \sum_{i=1}^{k} A_i^* G(U) A_i \| + \| Q + \sum_{i=1}^{k} A_i^* G(V) A_i \|
\]

\[
\leq 2 \| Q \| + \| \sum_{i=1}^{k} A_i^* G(U) A_i \| + \| \sum_{i=1}^{k} A_i^* G(V) A_i \|
\]
Thus, for any $\mathcal{U}, \mathcal{V} \in \Lambda$ with $\mathcal{U} \preceq \mathcal{V}$, we have

$$
\| \mathcal{K}(\mathcal{U}) \| + \| \mathcal{K}(\mathcal{V}) \| \leq 2\| \mathcal{Q} \| + 2NN_1n. \tag{28}
$$

For some fixed $\mathcal{U}, \mathcal{V} \in \Lambda$ with $\mathcal{U}, \mathcal{V} \in \mathcal{R}$, from (24) and (25), we have

$$
\left[2NN_1 + 2s(\mathcal{Q})\right]^2 \leq a\left(\| s'(\mathcal{U}) \| + \| s'(\mathcal{V}) \| \right)^2 - b\left(\| \mathcal{Q} + \sum_{i=1}^k A_i^* G(\mathcal{U}, A_i) \| + \| \mathcal{Q} + \sum_{i=1}^k A_i^* G(\mathcal{V}, A_i) \| \right)^2 \\
= a\left(\| \mathcal{U} \| + \| \mathcal{V} \| \right)^2 - b\left(\| \mathcal{U} \| + \| \mathcal{Q} + \sum_{i=1}^k A_i^* G(\mathcal{U}, A_i) \| + \| \mathcal{Q} + \sum_{i=1}^k A_i^* G(\mathcal{V}, A_i) \| \right)^2 \\
\frac{1}{1 + \| \mathcal{U} \| + \| \mathcal{Q} + \sum_{i=1}^k A_i^* G(\mathcal{U}, A_i) \| + \| \mathcal{Q} + \sum_{i=1}^k A_i^* G(\mathcal{V}, A_i) \| + \| \mathcal{V} \|}.
$$

The above relation holds for every singular value of $\mathcal{Q}$, so adding up, we obtain

$$
(2NN_1n + 2\| \mathcal{Q} \|)^2 \leq a\left(\| \mathcal{U} \| + \| \mathcal{V} \| \right)^2 - b\left(\| \mathcal{U} \| + \| \mathcal{Q} \| + \| \mathcal{K}(\mathcal{U}) \| \right)^2 \\
\frac{1}{1 + \| \mathcal{U} \| + \| \mathcal{Q} \| + \| \mathcal{K}(\mathcal{U}) \| + \| \mathcal{V} \|}.
$$

Therefore, from (28) we get

$$
\left(\| \mathcal{K}(\mathcal{U}) \| + \| \mathcal{K}(\mathcal{V}) \| \right)^2 \leq a\left(\| \mathcal{U} \| + \| \mathcal{V} \| \right)^2 - b\left(\| \mathcal{U} \| + \| \mathcal{Q} \| + \| \mathcal{K}(\mathcal{U}) \| \right)^2 \\
\frac{1}{1 + \| \mathcal{U} \| + \| \mathcal{Q} \| + \| \mathcal{K}(\mathcal{U}) \| + \| \mathcal{V} \|}.
$$

Let $\omega: \Lambda \times \Lambda \to \mathbb{R}$, be defined by

$$
\omega(\mathcal{U}, \mathcal{V}) = \| \mathcal{U} \| + \| \mathcal{V} \| \quad \text{for all } \mathcal{U}, \mathcal{V} \in \Lambda.
$$

Then $(\Lambda, \| \cdot \|, \omega)$ is a complete relational metric space with the above $\omega$-distance. It follows from (29) that

$$
\left[\omega(\mathcal{K}(\mathcal{U}), \mathcal{K}(\mathcal{V}))\right]^2 \leq a[\omega(\mathcal{U}, \mathcal{V})]^2 - b\left[\omega(\mathcal{U}, \mathcal{K}(\mathcal{U}))^2 + [\omega(\mathcal{V}, \mathcal{K}(\mathcal{V}))]^2 \\
\frac{1}{1 + \omega(\mathcal{U}, \mathcal{K}(\mathcal{U}))) + \omega(\mathcal{K}(\mathcal{U}), \mathcal{V})}.
$$

(30)
Taking $\mathcal{H} \in \Omega$ as Example 3.1, (30) can be rewritten as
\[
\mathcal{H} \left( \omega(\mathcal{K}(U), \mathcal{K}(V)), \omega(U, V), \omega(\mathcal{K}(U), \mathcal{K}(V)) \right) \leq 0.
\]

Now, all the hypotheses of Theorem 4.2 are satisfied, and therefore there exists $\hat{X} \in \mathcal{P}(n)$ such that $\mathcal{K}(\hat{X}) = \hat{X}$. Hence, matrix equation $\text{(23)}$ has a solution in $\mathcal{P}(n)$. Furthermore, due to the existence of the least upper bound and the greatest lower bound for each pair $U, V \in \mathcal{K}(\mathcal{P}(n))$, we have $\mathcal{Y} \neq \emptyset$ for all $U, V \in \mathcal{K}(\mathcal{P}(n))$. Hence, using Theorem 4.4, $\mathcal{K}$ has a unique fixed point, and hence we conclude that matrix equation $\text{(23)}$ has a unique solution in $\mathcal{P}(n)$. □

7 Numerical analysis

Example 7.1 Consider NME $\text{(23)}$ for $a = 0.75$, $b = 0.1$ with $\mathcal{G}(X) = X^{1/4}$, i.e.
\[
X = Q + A_1^* X^{1/4} A_1 + A_2^* X^{1/4} A_2 + A_3^* X^{1/4} A_3,
\]
where
\[
A_1 = \begin{bmatrix}
0.0446 & 0.0600 & 0.1186 \\
0.0483 & 0.0293 & 0.1212 \\
0.1243 & 0.0490 & 0.0232
\end{bmatrix},
Q = \begin{bmatrix}
9.2231 & 0.1203 & 0.1978 \\
0.1203 & 9.1168 & 0.1645 \\
0.1978 & 0.1645 & 9.2731
\end{bmatrix}.
\]
\[
A_2 = \begin{bmatrix}
0.0246 & 0.0319 & 0.0363 \\
0.0517 & 0.0345 & 0.0405 \\
0.0539 & 0.0526 & 0.0350
\end{bmatrix},
A_3 = \begin{bmatrix}
0.1739 & 0.2720 & 0.0854 \\
0.1455 & 0.0759 & 0.1644 \\
0.3036 & 0.1138 & 0.1771
\end{bmatrix}.
\]
\[
A_3 = \begin{bmatrix}
0.2230 & 0.1200 & 0.1978 \\
0.1200 & 0.1079 & 0.1638 \\
0.1978 & 0.1638 & 0.2730
\end{bmatrix},
\mathcal{Y} = \begin{bmatrix}
9.0001 & 0.0003 & 0.0001 \\
0.0003 & 9.0089 & 0.0007 \\
0.0001 & 0.0007 & 9.0001
\end{bmatrix}.
\]

The conditions of Theorem 31 can be checked numerically by considering different particular values of matrices involved. For instance, it can be tested (and verified to be true) for
\[
X = \begin{bmatrix}
0.2230 & 0.1200 & 0.1978 \\
0.1200 & 0.1079 & 0.1638 \\
0.1978 & 0.1638 & 0.2730
\end{bmatrix},
\mathcal{Y} = \begin{bmatrix}
9.0001 & 0.0003 & 0.0001 \\
0.0003 & 9.0089 & 0.0007 \\
0.0001 & 0.0007 & 9.0001
\end{bmatrix}.
\]

To see the behaviour of convergence of the sequence $\{U_n\}$ defined in $\text{(26)}$, we take up three different initial values:
\[
U_0 = \begin{bmatrix}
0.0220 & 0.0048 & 0.0016 \\
0.0048 & 0.0320 & 0.0040 \\
0.0016 & 0.0040 & 0.0215
\end{bmatrix},
V_0 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

with $\|U_0\| = 0.0755$, $\|V_0\| = 1$, $W_0 = \begin{bmatrix}
29.5737 & 0.9425 & 1.1894 \\
0.9425 & 28.9075 & 0.8807 \\
1.1894 & 0.8807 & 29.3474
\end{bmatrix}$ with $\|W_0\| = 87.8286$.

After 10 successive iterations, the approximations of the unique PDS of NME $\text{(23)}$ are the following:
\[
\hat{U} \approx U_{10} = \begin{bmatrix}
9.5246 & 0.3142 & 0.3965 \\
0.3142 & 9.3025 & 0.2936 \\
0.3965 & 0.2936 & 9.4491
\end{bmatrix}\text{ with Error } = 7.0958 \times 10^{-16};
\]
\[
V_{10} = \begin{bmatrix}
9.0000 & 0.0003 & 0.0001 \\
0.0003 & 9.0089 & 0.0007 \\
0.0001 & 0.0007 & 9.0001
\end{bmatrix}\text{ with Error } = 7.0958 \times 10^{-16};
\]
\[
W_{10} = \begin{bmatrix}
29.5737 & 0.9425 & 1.1894 \\
0.9425 & 28.9075 & 0.8807 \\
1.1894 & 0.8807 & 29.3474
\end{bmatrix}\text{ with Error } = 7.0958 \times 10^{-16}.
\]
It can be also verified that the elements of each sequence are order-preserving. The convergence behaviour (see Fig. 1) and the solution surface plot (see Fig. 2) are shown graphically.

\[
\hat{V} \approx V_{10} = \begin{bmatrix} 9.5246 & 0.3142 & 0.3965 \\ 0.3142 & 9.3025 & 0.2936 \\ 0.3965 & 0.2936 & 9.4491 \end{bmatrix} \quad \text{with Error } = 3.7139 \times 10^{-16};
\]

\[
\hat{W} \approx W_{10} = \begin{bmatrix} 9.5246 & 0.3142 & 0.3965 \\ 0.3142 & 9.3025 & 0.2936 \\ 0.3965 & 0.2936 & 9.4491 \end{bmatrix} \quad \text{with Error } = 2.2887 \times 10^{-16}.
\]
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