Sparse Lerner operators and domination in infinite dimensions

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Abstract

We use the principle of almost orthogonality to give a new and simple proof that a sparse Lerner operator is bounded on a operator-weighted space $L^2_W(\mu)$, where $\mu$ is a locally finite positive measure on $\mathbb{R}^d$ if and only if the weight $W$ satisfies the Muckenhoupt $A_2(\mu)$-condition, restricted to the sparse collection in question. Our method extends to the infinite-dimensional setting, thus allowing for applications to the multi-parameter setting. For the class of Muckenhoupt $A_2$-weights, we obtain bounds in terms of mixed $A_2(\mu)$-$A_\infty(\mu)$-conditions, which is independent of dimension and agrees with the best known bound in the finite-dimensional vectorial setting. As an application, we provide an operator-weighted bound for the (maximal) Bergman projection, where we obtain a new sharp bound in terms of the Békollé-Bonami characteristic. Furthermore, we consider commutators of sparse Lerner operators on operator-valued weighted $L^2$-spaces and some applications to multi-parameters.

1 Introduction

Let $\mu$ be a locally finite positive Borel measure on the Euclidean space $\mathbb{R}^d$. A countable collection $\mathcal{D}$ of cubes in $\mathbb{R}^d$ is said to form a dyadic grid, if the following properties hold:

(i) Every cube $Q \in \mathcal{D}$ has sidelength $2^k$, for some integer $k \in \mathbb{Z}$.

(ii) Each subcollection $\mathcal{D}_k \subset \mathcal{D}$ consisting of cube with sidelengths $2^k$ form a partition of $\mathbb{R}^d$.

(iii) For every pair $Q, Q' \in \mathcal{D}$, we have $Q \cap Q' \in \{\emptyset, Q, Q'\}$.
We shall naturally refer to the elements $Q$ of a dyadic grid $\mathcal{D}$ as dyadic cubes and recall the standard dyadic grid $\mathcal{D}(\mathbb{R}^d)$ given by
\[
\left\{2^n \left([0, 1) + m\right) : m \in \mathbb{Z}^d, n \in \mathbb{Z}\right\}.
\]

We say that a subcollection $\mathcal{S} \subset \mathcal{D}$ is sparse wrt $\mu$, if for any $Q \in \mathcal{D}$:
\[
\sum_{Q' \in \text{Ch}_\mathcal{S}(Q)} \mu(Q') \leq \frac{1}{2} \mu(Q) \tag{1}
\]
where $\text{Ch}_\mathcal{S}(Q)$ denotes the set of maximal (wrt inclusion) cubes in $\mathcal{S}$, which are strictly contained in $Q$. Again, there is nothing particular with the constant $1/2$ and it may be replaced by any fixed $0 < \delta < 1$. For a sparse collection $\mathcal{S}$, we consider the corresponding sparse operator $T^{\mathcal{S}}$ by
\[
T^{\mathcal{S}}(f)(x) = \sum_{Q \in \mathcal{S}} 1_Q(x) \langle f \rangle_{\mu, Q} \tag{2}
\]
where $\langle f \rangle_{\mu, Q} := \frac{1}{\mu(Q)} \int_Q f \, d\mu$ denotes the $\mu$-average of $f$. Sparse operators and its many variations have recently attracted much attention, due to the substantial breakthrough in 2013 where A. Lerner proved his sparse domination theorem, which essentially asserts that general Calderón-Zygmund operators can be pointwise bounded by sparse operators [13]. This result sparked a considerable interest in obtaining sharp bounds for sparse operators on various function spaces. For instance, the sparse domination theorem provided a straightforward proof of the $A_2$-conjecture for general Calderón-Zygmund operators, which was initially solved by T. Hytönen in [11], using rather difficult tools.

Our purpose is to consider sparse operators in a vectorial setting. To this end, we shall denote by $\mathcal{H}$ a separable Hilbert space equipped with the inner-product $(\cdot | \cdot)_{\mathcal{H}}$, and let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$, equipped with the usual operator norm. We say that $W : \mathbb{R}^d \to \mathcal{B}(\mathcal{H})$ is an operator-valued weight, if for every vector $e \in \mathcal{H}$, the function
\[
w_e(x) := (W(x)e|e)_{\mathcal{H}} \tag{3}
\]
is a usual scalar weight. In fact, for the sake of ensuring well-defined Bochner integrals, we shall require that the operator-valued weights $W^{\pm 1}$ are both weakly locally $\mu$-integrable on $\mathbb{R}^d$. That is, for any $u, v \in \mathcal{H}$, the function $x \mapsto (W^{\pm 1}(x)u|v)_{\mathcal{H}}$ is integrable on compacts subsets of $\mathbb{R}^d$ with respect to $\mu$ and satisfies for any dyadic cube $Q \subset \mathbb{R}^d$
\[
\left| \int_Q (W^{\pm 1}(x)u|v)_{\mathcal{H}} \, d\mu(x) \right| \leq C_{\mu, Q} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}
\]
where \( C_{\mu,Q} > 0 \) is a constant possibly depending on \( \mu \) and \( Q \). The bounded linear operators that arise in this way will be denoted by \( \int_Q W^s d\mu \) and they are also automatically invertible, for every dyadic cube \( Q \subset \mathbb{R}^d \). For a subcollection \( S \subseteq \mathcal{D} \), we say that an operator-valued weight \( W : \mathbb{R}^d \to \mathcal{H}(B) \) is said to be a \( A_2^S(\mu) \)-weight, if

\[
[W]^A_2^S(\mu) := \sup_{Q \in S} \left\| (W)^{1/2}_{\mu,Q} (W^{-1})^{1/2}_{\mu,Q} \right\|_{\mathcal{H}(B)} < \infty.
\]  

Note that if \( S = \mathcal{D} \), then we retain the collection of dyadic Muckenhoupt \( A_2 \)-weights wrt \( \mu \), denoted by \( A_2(\mu) \). Another important class of weights for our purposes, is the collection of operator-valued dyadic \( A_\infty \)-weights. An operator-valued weight \( W \) is said to belong to \( A_\infty(\mu) \), if for every \( e \in \mathcal{H} \) the scalar weights \( w_e \) in (3) satisfy the dyadic Fujii-Wilson \( A_\infty(\mu) \)-condition:

\[
[w_e]_{A_\infty(\mu)} := \sup_{e \in \mathcal{H}} \frac{1}{w_e(Q)} \int_Q M^D(1_Q w_e)(x)d\mu(x) < \infty
\]

where \( M^D(f)(x) := \sup_{Q \subseteq \mathcal{D}} 1_Q(x)(f)_{\mu,Q} \) denotes the dyadic maximal function. Due to the scale-invariance of the Fujii-Wilson condition in (5), we conventionally set the dyadic \( A_\infty(\mu) \)-constant to be

\[
[W]_{A_\infty(\mu)} := \sup_{e \in \mathcal{H}} [w_e]_{A_\infty(\mu)} < \infty.
\]  

The dyadic \( A_\infty(\mu) \)-condition is slightly weaker than the dyadic \( A_2(\mu) \)-condition and one can show that \( [W]_{A_\infty(\mu)} \leq C [W]_{A_2(\mu)} \), see for instance [8]. Given an operator-valued weight \( W \), we denote by \( L^2_W := L^2_W(\mu, \mathcal{H}) \) the space of weakly locally \( \mu \)-integrable functions on \( \mathbb{R}^d \), equipped with the norm

\[
\| f \|^2_{L^2_W} := \int_{\mathbb{R}^d} \| W^{1/2} f \|^2_{\mathcal{H}} d\mu = \int_{\mathbb{R}^d} (W f f)_{\mathcal{H}} d\mu < \infty.
\]

Denoting by \( L^\infty_0(\mu) \) the space of complex-valued \( \mu \)-essentially bounded functions with compact support on \( \mathbb{R}^d \), it is not difficult to show that

\[
L^\infty_0(\mu) \otimes \mathcal{H} := \left\{ \sum_{\text{Finite}} f \otimes e : e \in \mathcal{H}, \ f \in L^\infty_0(\mu) \right\}
\]

forms a dense subspace of \( L^2_W \), thus given a linear operator \( T \) well-defined on scalar-valued functions \( L^\infty_0(\mu) \), we denote the canonical \( \mathcal{H} \)-valued extension of \( T \) by \( T \otimes 1_{\mathcal{H}} \), defined on \( L^\infty_0 \otimes \mathcal{H} \) by

\[
(T \otimes 1_{\mathcal{H}}) \left( \sum_{\text{Finite}} f \otimes e \right) := \sum_{\text{Finite}} T(f) \otimes e
\]
where $\mathbb{1}_\mathcal{H}$ is the identity operator on $\mathcal{H}(B)$. If $T : L^\infty_0(\mu) \otimes \mathcal{H} \to L^2_W$ is bounded, then it follows by density that $T \otimes \mathbb{1}_\mathcal{H}$ will have a unique bounded extension to all of $L^2_W$, thus for the sake of abbreviation, we shall denote by $T \otimes \mathbb{1}_\mathcal{H}$ the unique continuous extension.

In the setting of matrix-weights $W$ of dimension $N > 1$ and $\mu$ being the Lebesgue measure on $\mathbb{R}^d$, the following mixed $A_2$-$A_\infty$ bound was proved for general Calderón-Zygmund operators $T$ in [15],

$$\|T \otimes \mathbb{1}_\mathcal{H}\|_{L^2_W \to L^2_W} \leq c_{d,N,T} [W]^{1/2}_{A_2} [W]^{1/2}_{A_\infty} [W^{-1}]^{1/2}_{A_\infty}$$

where $c_{d,N,T} > 0$ is a constant depending on the dimensions and $T$. The authors introduced the technique of so-called convex body domination with sparse operators, extending the of the sparse domination technique in [13] by A. Lerner. In fact, the mixed bound in (6) is a consequence of the bound for sparse operators. Even in the scalar setting, the convex body domination technique gives new results, see e.g. [12]. The proof in [15] of the convex body domination theorem heavily relies on the John-Ellipsoid theorem and equivalence of norms, tools which are both absent tools in the infinite dimensional setting. In fact, one of the results of this paper is that a pointwise domination of Calderón-Zygmund operators by sparse operators is in general not possible in the infinite-dimensional setting. Indeed, it was proved in [4], [5] that the Hilbert transform and dyadic martingale transforms do not in general extend to a bounded linear operator in the operator-valued infinite dimensional setting, even if $W$ is a Muckenhoupt $A_2$-weight. These results relied, among others, on observations by F. Nazarov, S. Treil and A. Volberg in 1997, where they proved that the Carleson embedding theorem fails in the infinite dimensional setting [16]. The first positive result extending weighted boundedness results to an infinite-dimensional, operator-weighted setting was established by A. Aleman and O. Constantin in [1], where they proved that the the family of standard weighted Bergman projections are bounded on $L^2_W$, if and only if the operator-valued weight $W$ satisfies a standard weighted Bekollé-Bonami condition. Sharp bounds for the standard weighted Bergman projection in the scalar-valued setting were proved by M. C. Reguera and the second author in [17], using uniform (as opposed to pointwise) domination by certain sparse operators. More recently, a sparse domination of the Bergman projection on pseudoconvex domains in the matrix-weighted finite dimensional setting was obtained, where the authors in [7] provided a slight improvement of the bound in [1]. Our main purpose is to show that a certain family of sparse operators are bounded in the infinite-dimensional, operator-weighted setting of $L^2_W(\mu, \mathcal{H})$, if and only if $W$ satisfies an appropriate $\mu$-adapted Muckenhoupt $A_2$-condition. In particular, we shall in our setting prove a
similar bound to that of (6), which makes this the best known bound to
date, even in the finite dimensional setting. As an application, we shall pro-
vide a sparse domination result for the Bergman projection in the infinite
dimensional setting of $L^2_W$, which sharpens the results obtained in [1] and
[7].

2 Main results and outline

For a sparse family $\mathcal{S}$ of dyadic cubes on $\mathbb{R}^d$, we shall consider the family of
sparse Lerner operators $\{T^S_\kappa\}_\kappa$ defined by

\[
T^S_\kappa f(x) = \sum_{Q \in \mathcal{S}} \int_Q \kappa_Q(x, y) f(y) \mu(y)
\]

where $\kappa_Q$ are complex-valued functions supported on $Q \times Q$ with $\|\kappa_Q\|_{L^\infty(\mu \times \mu)} \leq \mu(Q)^{-1}$. We shall see that the family of sparse Lerner operators naturally
appear as convex bodies of sparse operators in (2) (see [15], Lemma 2.7). We
now state our first main result.

**Theorem 2.1.** The family of sparse operators $\{T^S_\kappa\}_\kappa$ defined in (7) extend
to bounded linear operators on $L^2_W(\mu, \mathcal{H})$ if and only if $W$ belongs to $\mathcal{A}_2^S$.
Moreover, there exists an absolute constant $C > 0$, such that

\[
\frac{1}{C} [W]_{\mathcal{A}_2^S(\mu)}^{1/2} \leq \sup_{\kappa} \|T^S_\kappa \otimes 1_{\mathcal{H}}\|_{L^2_W \to L^2_W} \leq C [W]_{\mathcal{A}_2^S(\mu)}^{3/2} \tag{8}
\]

where the supremum is taken over all sequences $\{\kappa_Q\}_{Q \in \mathcal{S}}$ defined in the previous paragraph.

If we assume that the locally finite positive Borel measure $\mu$ on $\mathbb{R}^d$ satisfies
the doubling condition:

\[
K_\mu := \sup_{Q \in \mathbb{R}^d \text{ cube}} \frac{\mu(2Q)}{\mu(Q)} < \infty \tag{9}
\]

where $2Q$ denotes the dilate of a cube $Q$ in $\mathbb{R}^d$ by a factor 2, then we actually
obtain the following mixed-bound identical to (6), for operator-valued dyadic
Muckenhoupt weights.

**Corollary 2.2.** If $\mu$ satisfies the doubling condition in (9) and $W$ is a dyadic
$\mathcal{A}_2(\mu)$-weight, then we have the following improved mixed-bound:

\[
\sup_{\kappa, S} \|T^S_\kappa \otimes 1_{\mathcal{H}}\|_{L^2_W \to L^2_W} \leq C_\mu [W]_{\mathcal{A}_2(\mu)}^{1/2} [W]_{\mathcal{A}_\infty(\mu)}^{1/2} [W^{-1}]_{\mathcal{A}_\infty(\mu)}^{1/2} \tag{10}
\]

for some constant $C_\mu > 0$, only depending on $\mu$. Note that we also take the
supremum over all sparse collections $\mathcal{S} \subset \mathcal{D}$ of dyadic cubes in (1).
Some comments are now in order. The proof of Theorem 2.1 relies on a stopping time argument, which allows us to decompose any sparse Lerner operator into a sum of simpler operators, so that the principle of almost orthogonality by M. Cotlar and E. Stein is applicable. For this reason and in contrast to previous dimensionally dependent proofs, our method extends to infinite dimensions. This in turn allows for certain applications to the multi-parameter setting. The proof of Corollary 2.2 is similar to Theorem 2.1, but the improvement hinges on a sharp reverse Hölder inequality on homogenous type spaces (see Theorem 1.1, [10]), which requires the doubling condition on the measure \( \mu \).

We stress the fact that even though the operator-weighted Hilbert transform is generally unbounded in an infinite-dimensional setting, regardless if the operator-valued Muckenhoupt \( A_2 \)-condition holds, we show here that sparse Lerner operators are bounded in this setting. This implies in particular that a pointwise domination of the Hilbert transform by sparse Lerner operators is not possible in the infinite-dimensional setting. A striking result by A. Aleman and O. Constantin in [1] shows that the Bergman projection is bounded on \( L^2_W \) if and only if the operator-valued weight \( W \) is a Bécollé-Bonami weight. In their approach, they relate the norms of analytic functions on weighted Bergman spaces to weighted norms of their derivatives. As an application of Theorem 2.1, we shall provide a convex body domination result for the Bergman projection. We shall consider the family of Bergman projections \( P^+_\gamma \) with \( \gamma > -1 \), defined on the upper half-plane \( \mathbb{C}_+ := \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) via

\[
P^\gamma_\gamma(f)(z) = \int_{\mathbb{C}_+} \frac{f(\xi)}{|z - \xi|^{2+\gamma}} dA_\gamma(\xi) \quad z \in \mathbb{C}_+, \tag{11}
\]

where \( dA_\gamma(\xi) = \text{Im}(\xi)^\gamma d\lambda(\xi) \). These are projections onto the subspace of analytic functions in \( L^2(dA_\gamma, \mathbb{C}) \). However, analyticity will play no role in our considerations, hence we shall also consider family of maximal Bergman projections

\[
P^+\gamma(f)(z) = \int_{\mathbb{C}_+} \frac{f(\xi)}{|z - \xi|^{2+\gamma}} dA_\gamma(\xi) \quad z \in \mathbb{C}_+. \tag{12}
\]

In this setting, the relevant class of operator-valued weights \( W : \mathbb{C}_+ \rightarrow B(H) \) will be the so-called Bécollé-Bonami weights \( B_2(\gamma) \) on \( \mathbb{C}_+ \), defined by

\[
[W]_{B_2(\gamma)} := \sup_{J \subset \mathbb{R}} \left\| \langle f \rangle_{Q^J_{J}}, \gamma (W^{-1})_{Q^J_{J}}, \gamma \right\|_{H(B)} < \infty
\]

where \( \langle f \rangle_{Q^J_{J}}, \gamma := \frac{1}{A_\gamma(Q^J_{J})} \int_{Q^J_{J}} f dA_\gamma \) and \( Q^J_{J} := \{ z \in \mathbb{C}_+ : \text{Re}(z) \in J, \text{Im}(z) \in (0, |J|) \} \) denotes the Carleson square associated to the interval \( J \subset \mathbb{R} \). It was
proved in [1] that both the operators $P^{(+)}_\gamma$ are bounded on $L^2_W(dA_\gamma,\mathcal{H})$ if and only if the operator-valued weight $W$ belongs to $B_2(\gamma)$, and there exists a constant $C_\gamma > 0$, only depending on $\gamma$, such that

$$\frac{1}{C_\gamma} [W]^{1/2}_{B_2(\gamma)} \leq \|P^{(+)}_\gamma \otimes 1_{\mathcal{H}}\|_{L^2_W \to L^2_W} \leq C_\gamma [W]^{5/2}_{B_2(\gamma)}.$$ 

The upper bound was later improved in [7] by reducing the exponent of $5/2$ to 2, in the finite dimensional setting of matrix-weights and in the setting on pseudoconvex domains. The content of our next result is to provide in the setting of operator-valued weights, an upper bound, which sharpens both of these results for the family of (maximal) Bergman projections introduced above.

**Theorem 2.3.** For every $\gamma > -1$, the (maximal) Bergman projection $P^{(+)}_\gamma$ is bounded on $L^2_W(dA_\gamma,\mathcal{H})$ with

$$\|P^{(+)}_\gamma \otimes 1_{\mathcal{H}}\|_{L^2_W \to L^2_W} \leq C_\gamma [W]^{3/2}_{B_2(\gamma)}.$$ 

As indicated, the proof of Theorem 2.3 relies on a convex body domination result (see Proposition 5.2), which we defer to section 5, together with our main result in Theorem 2.1. We remark that the general nature of Theorem 2.1 does not rely on reverse Hölder properties of the matrix weights $W$, which is an absent tool in the setting of Béckollé-Bonami weights, hence a significant obstacle. With these perspectives in mind, it should not come as a surprise if Theorem 2.3 extends to even more general settings, with regards to the domain. A linear bound in terms of the $A_2$ characteristic for sparse Lerner operators or in terms of Béckollé-Bonami condition for the (maximal) Bergman projection would certainly improve our main results, but this question (notoriously known as the $A_2$-conjecture) remains open even in the finite dimensional setting and is expected to be very difficult.

This manuscript is organized as follows. In the preliminary section 3, we have collected the preparatory work for our main results, Theorem 2.1, Corollary 2.2 and Theorem 2.3. It is divided into subsections, consisting of decompositions of sparse collections into stopping times, averaging operators and sharp estimates for scalar weights. Section 4 is devoted to the proof of Theorem 2.1 and Corollary 2.2, while section 5 contains the proof of Theorem 2.3. In our final section, we provide some applications to boundedness results for commutators of sparse Lerner operators and their iterated versions in the multi-parameter setting.
3 Preliminary results

3.1 The sparseness condition Here we shall briefly discuss a more conventional notion of sparseness and justify our seemingly particular choice in (1). Given a number $0 < \delta < 1$, we say that a subcollection $F \subset D$ is weakly $\delta$-sparse wrt $\mu$, if for every $Q \in F$ there exists Borel sets $E_Q \subset Q$ with the properties that $\mu(E_Q) \geq \delta \mu(Q)$, and such that the collection $\{E_Q\}_{Q \in F}$ is pairwise disjoint. Evidently, every sparse collection is weakly $1/2$-sparse with $E_Q := Q \setminus \bigcup_{Q' \in \text{Ch}_S(Q)} Q'$. Conversely, if $F$ is a weakly $\delta$-sparse collection, then

$$\sum_{Q' \in \text{Ch}_F(Q)} \mu(Q') \leq \frac{1}{\delta} \left( \sum_{Q' \in \text{Ch}_F(Q) \cup \{Q\}} \mu(E_Q') - \mu(Q) \right) \leq \left( \frac{1}{\delta} - 1 \right) \mu(Q).$$

However, the constant $(1/\delta - 1)$ may still exceed 1, hence to remedy this, we pick an integer $m \geq 2$ with $(1/\delta - 1)/m \leq 1/2$. Adapting the techniques from Lemma 6.6 in [14], we can decompose any weakly $\delta$-sparse collection $F$ into a union of $m \geq 2$ disjoint sparse subcollections $S_1, \ldots, S_m$ in the sense of (1). Consequently, any weakly $\delta$-sparse Lerner operator can be written as a sum of $m$ sparse Lerner operators, thus our main results continue to hold for weakly sparse Lerner operators, at the cost of $\delta$-dependent constants. For our purposes and for the sake of convenience, we shall restrict our attention to sparse collections $S$ in the sense of (1).

3.2 The principle of almost orthogonality via stopping times In this section, we shall decompose any sparse collection of dyadic cubes into union of stopping times, by identifying collections of dyadic cubes with collections of vertices of graphs. These notions are deeply inspired by ideas from [14], which we refer the reader to further details on these matters. With this decomposition at hand, we shall see that any sparse operator can be written as a sum of sparse operators, for which the principle of almost orthogonality by Cotlar and Stein can be utilized.

Given a sparse collection $S$, we view the cubes of $S \subset D$ as a set of vertices of a graph $\Gamma_S$ by declaring that two distinct cubes $Q, Q' \in S$ are joined by a graph edge if either $Q \subset Q'$ or $Q' \subset Q$ and there is no intermediate cube $Q'' \in S$, which lies strictly between $Q$ and $Q'$. We say that any two cubes $Q, Q' \in S$ are connected if there is a path of graph edges between them. With this at hand, we can define $d_S(Q', Q)$ to be the minimal number of graph edges from $Q'$ to $Q$ (if two cubes $Q', Q \in S$ are not connected, we set $d_S(Q', Q) = \infty$ by default). Connectedness of cubes in $S$ induces an equivalence relation on $\Gamma_S$, hence $\Gamma_S$ decomposes into a collection of at most
finitely many connected subgraphs. Each connected subgraph of $\Gamma_S$ can be viewed as a branching tree, with the natural motions of either moving upwards to larger cubes within the subgraph or moving downwards to smaller cubes within the subgraph. Now pick exactly one vertex in each connected subgraph of $\Gamma_S$ and denote the collection of the cubes corresponding to these vertices by $J^0 \subset S$. Recursively, we may define the stopping times

$$J^{n+1} = \bigcup_{Q \in J^n} \text{Ch}_S(Q) \quad J^{-(n+1)} = \bigcup_{Q \in J^{-n}} \text{Pr}_S(Q) \quad n \geq 0.$$  

Here $\text{Ch}_S(Q)$ denotes the maximal cubes in $S$ which are strictly contained in $Q$ and $\text{Pr}_S(Q)$ denotes the minimal cube in $S$, which strictly includes $Q$. For positive integers $n$, $J^n$ corresponds to moving down $n$ generations in all the connected subgraphs of $S$, from every cube in $J^0$, while for negative integers $n$, it corresponds to moving up $n$ generations in all the connected subgraphs of $S$, from every cube in $J^0$. In a similar way, we define the $n$-generation stopping time relative to an arbitrary cube $Q \in S$ by

$$J^n(Q) := \{Q' \in S : d_S(Q',Q) = n, Q' \nsubseteq Q\} \quad n \geq 0.$$  

With these constructions at hand, we obtain a collection of families $\{J^n\}_{n \in \mathbb{Z}}$, satisfying the following properties:

(i) $J^n$ is a disjoint collection of cubes in $S$, for all $n \in \mathbb{Z}$.

(ii) $\bigcup_{n \in \mathbb{Z}} J^n = S$.

(iii) For every $Q \in S$, the collection $\{J^n(Q)\}_{n=0}^\infty$ forms a decaying stopping time family. That is, for every $Q \in S$, we have

$$\sum_{Q' \in J^n(Q)} \mu(Q') \leq 2^{-n} \mu(Q) \quad n \geq 0. \quad (13)$$

These properties are all immediate consequences of the constructions of $J^n$, while the third property incorporates an iteration of the sparseness condition of $S$. Decomposing the sparse family $S$ in this way, we may express any sparse operator as

$$T^S_\kappa = \sum_{n \in \mathbb{Z}} T_n$$

where each term consist of sums of averaging operators restricted to a disjoint family $J^n$, given by
\[ T_n f(x) := \sum_{Q \in J^n} \int_Q \kappa_Q(x, y) f(y) d\mu(y) \quad n \in \mathbb{Z}. \quad (14) \]

It turns out that the decaying stopping time property in (13) makes the family of operators \( \{ T_n \}_{n \in \mathbb{Z}} \) in (14) ”almost orthogonal”. In order to make the notion of almost orthogonality more precise, we will need the following tailor-made version of the Cotlar-Stein lemma.

**Lemma 3.1 ((Cotlar-Stein lemma)).** Let \( \{ T_n \}_{n \in \mathbb{Z}} \) be a sequence of bounded linear operators on a Hilbert space \( \mathcal{H} \) and suppose there are sequences of positive real numbers \( \{ \alpha(n) \}_{n \in \mathbb{Z}}, \{ \beta(n) \}_{n \in \mathbb{Z}} \), with the properties

\[
\| T_n^* T_m \|_{\mathcal{H}(B)} \leq \alpha(n - m), \\
\| T_n T_m^* \|_{\mathcal{H}(B)} \leq \beta(n - m). \quad (15)
\]

for all \( m, n \in \mathbb{Z} \). Furthermore, assume that

\[
A := \sum_{n \in \mathbb{Z}} \sqrt{\alpha(n)} < \infty, \quad B := \sum_{n \in \mathbb{Z}} \sqrt{\beta(n)} < \infty.
\]

Then the operator \( \sum_n T_n \) converges unconditionally and enjoys the bound

\[
\left\| \sum_{n \in \mathbb{Z}} T_n \right\|_{\mathcal{H}(B)} \leq 2\sqrt{AB}.
\]

The standard proof of Lemma 8.5.1, in [6], can be easily be adapted to prove this version of the lemma, thus we omit the proof.

### 3.3 Averaging operators on \( L^2_W \)

Now in order to satisfy the hypothesis of the lemma Lemma 3.1, we necessarily need to establish boundedness of the \( T_n \)'s, uniformly in \( n \in \mathbb{Z} \), which accounts for the diagonal case \( m = n \) in the hypothesis (15). Since \( J^n \) is a disjoint collection, it suffices to find a uniform bound for the individual terms of \( T_n \). This task is captured by the following proposition.

**Proposition 3.2.** Let \( Q \subset \mathbb{R}^d \) be an arbitrary cube and and \( \kappa_Q \) be a complex-valued function supported on \( Q \times Q \) and satisfying \( \| \kappa_Q \|_{L^\infty(\mu Q)} \leq \mu(Q)^{-1} \). Then the family of averaging operators

\[
A_{\kappa_Q, \mu}(f)(x) := \int_Q \kappa_Q(x, y) f(y) d\mu(y)
\]

with \( \kappa_Q \) as above satisfies the following bound

\[
\sup_\kappa \left\| A_{\kappa_Q, \mu} \right\|_{L^2_W \rightarrow L^2_W} \leq \left\langle W \right\rangle_{\mu Q}^{1/2} \left\langle W^{-1} \right\rangle_{\mu_Q}^{1/2} \left\| \mathcal{H}(B) \right\|.
\]
In order to prove Proposition 3.2, we shall need a couple of simple lemmas.

**Lemma 3.3.** Let \( \kappa_Q \) be an arbitrary complex-valued function supported on \( Q \times Q \) with \( \| \kappa_Q \|_{L^\infty(\mu \otimes \mu)} \leq \mu(Q)^{-1} \). Then there exists sequences of complex-valued functions \( \{ \psi_Q^N \}, \{ \varphi_Q^N \} \), where both \( \psi_Q^N, \varphi_Q^N \) are supported on \( Q \) with \( \| \psi_Q^N \|_{L^\infty(\mu)} \leq 1 \), such that the tensor product \( \psi_Q^N \otimes \varphi_Q^N \to \mu(Q) \kappa_Q \) converges in the weak-star topology of \( L^\infty(\mathbb{R}^d \times \mathbb{R}^d, \mu \otimes \mu) \), as \( N \to \infty \).

**Proof.** By means of taking \( \kappa_Q \) convolved with a continuous and compactly supported approximate to the identity, it follows from the dominated convergence theorem that \( \kappa_Q \) can be approximated by a sequence of continuous functions \( \{ \kappa_Q^N \} \), all supported within say \( 2Q \times 2Q \). However, according to the Stone-Weierstrass Theorem, any such continuous function \( \kappa_Q^N \) can at its turn be uniformly approximated by tensor products \( \psi_Q^N \otimes \varphi_Q^N \), where \( \psi_Q^N, \varphi_Q^N \) are continuous and supported on \( 2Q \). Thus the sequence of functions \( 1_Q \psi_Q^N \otimes 1_Q \varphi_Q^N \) has the required properties. \( \square \)

**Lemma 3.4.** Let \( A_N(f)(x) := \psi_Q^N(x) \langle \varphi_Q^N f \rangle_{\mu,Q} \) be a sequence of averaging operators with integral kernel \( \frac{1}{\mu(Q)} \left( \psi_Q^N \otimes \varphi_Q^N \right)_\mu \), where \( \psi_Q^N, \varphi_Q^N \) are defined as in Lemma 3.3. Then for any \( f \in L^\infty_\mu(\mathcal{H}) \), we have

\[
\| A_{\kappa_Q,\mu}(f) \|_{L^2_\mathcal{H}} \leq \liminf_{N \to \infty} \| A_N(f) \|_{L^2_\mathcal{H}}.
\]

**Proof.** Note that it suffices to prove that the sequence of operators \( \{ M_{W^{1/2}} A_{\kappa_Q,M_{W^{1/2}}} \} \) on the non-weighted space \( L^2 := L^2(\mu, \mathcal{H}) \) (think \( W = \mathbb{1}_\mathcal{H} \)) converges in the weak operator topology to \( M_{W^{1/2}} A_{\kappa_Q,\mu} M_{W^{-1/2}} \), where \( M_{W^{1/2}} \) denotes the multiplication operator by \( W^{1/2} \). The proof then follows from the fact that the \( L^2(\mathcal{H}, \mu) \)-norm is weakly lower semicontinuous. To this end, note that if \( f, g \in L^\infty_\mu(\mathcal{H}) \), then since \( W^\pm \) are weights, it follows by the Cauchy-Schwarz inequality that the complex-valued function

\[
(x, y) \mapsto \left( W^{-1/2}(y)f(y)|W^{1/2}(x)g(x) \right)_\mathcal{H}
\]

is locally \( (\mu \otimes \mu) \)-integrable on \( \mathbb{R}^d \times \mathbb{R}^d \). Consequently, by Lemma 3.3 we have that

\[
\left( (M_{W^{1/2}} A_{\kappa_Q,M_{W^{-1/2}}})f \right) \langle g \rangle_{L^2} = \int \int \frac{1}{\mu(Q)} \psi_Q^N(x) \varphi_Q^N(y) \left( W^{-1/2}(y)f(y)|W^{1/2}(x)g(x) \right)_\mathcal{H} d\mu(y) d\mu(x) \to \int \int \kappa_Q(x, y) \left( W^{-1/2}(y)f(y)|W^{1/2}(x)g(x) \right)_\mathcal{H} d\mu(y) d\mu(x) = \left( (M_{W^{1/2}} A_{\kappa_Q,\mu} M_{W^{-1/2}})f \right) \langle g \rangle_{L^2} ; \text{ as } N \to \infty.
\]

\( \square \)
Proof of Proposition 3.2. According to Lemma 3.4, it actually suffices to find the asserted uniform bound for the family of averaging operators

$$f \mapsto \frac{1}{\mu(Q)} \langle \psi_Q \otimes \varphi_Q \rangle_{\mu} (f) := \psi_Q(\varphi_Q f)_{\mu,Q}$$

where $\psi_Q, \varphi_Q$ are arbitrary pairs of complex-valued functions supported on $Q$ with $\|\psi_Q\|_{L^\infty(\mu)} \leq 1$. To this end, fix an arbitrary $f \in L^\infty_0(\mu) \otimes \mathcal{H}$, so that $\|\psi_Q(\varphi_Q f)_{\mu,Q}\|_{L^2_\mathcal{W}} < \infty$, and note that by Cauchy-Schwarz inequality, we can write

$$\|\psi_Q(\varphi_Q f)_{\mu,Q}\|_{L^2_\mathcal{W}}^2 = \int_Q \langle W^{-1/2}(\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q} \rangle_{\mathcal{H}}^2 d\mu \leq \left( \int_Q \|W^{-1/2}(\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q}\|_{\mathcal{H}}^2 d\mu \right)^{1/2} \|\varphi_Q f\|_{L^2_\mathcal{W}}^2.$$  \hspace{1cm} (16)

We now estimate the integral on the right hand side of (16), according to

$$\int_Q \|W^{-1/2}(\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q}\|_{\mathcal{H}}^2 d\mu = \mu(Q) \left\|W^{-1/2}(\psi_Q^2W)_{\mu,Q} \left( (\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q} \right) \right\|_{\mathcal{H}}^2 \leq \left\|W^{-1/2}(\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q}\right\|_{\mathcal{H}}^2.$$

Going back to the expression in (16) and cancelling the common factors, we obtain

$$\|\psi_Q(\varphi_Q f)_{\mu,Q}\|_{L^2_\mathcal{W}} \leq \left\|W^{-1/2}(\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q}\right\|_{\mathcal{H}}.$$  \hspace{1cm} (16)

Note that by the $C^*$-identity, we can write

$$\left\|W^{-1/2}(\psi_Q^2W)_{\mu,Q} \varphi_Q f_{\mu,Q}\right\|_{\mathcal{H}}^2 = \sup_{\|e\|_{\mathcal{H}} = 1} \left( \|\psi_Q^2W\rangle_{\mu,Q} (W^{-1/2})_{\mu,Q} e \langle W^{-1/2} \mu,Q e \right)_{\mathcal{H}}.$$

Expanding $\|\psi_Q^2W\rangle_{\mathcal{H}}$ and estimating the positive function $\|\psi_Q\|_{L^\infty(\mu)} \leq 1$, the $C^*$-identity yields

$$\|\psi_Q^2W\rangle_{\mathcal{H}}^2 \leq \|W^{-1/2}\rangle_{\mathcal{H}} \langle W^{-1/2} \mu,Q \|_{\mathcal{H}}^2.$$
Consequently, we deduce that

$$\sup_{\psi_Q, \varphi_Q} \left\| \frac{1}{\mu(Q)} (\psi \otimes \varphi_Q) \right\|_{L^2_W \rightarrow L^2_W} \leq \left\| (W)^{1/2}_{\mu,Q} (W^{-1})^{1/2}_{\mu,Q} \right\|_{\mathcal{H}(B)}$$

where the supremum is taken over all pairs $\psi_Q, \varphi_Q$ as above. However, according to Lemma 3.4, this is enough to complete the proof.

**Remark 1.** We remark that if $\psi_Q, \varphi_Q$ are both equal to the indicator function $1_Q$, then we actually have the following norm equality for the averaging operator

$$\left\| \frac{1}{\mu(Q)} (1_Q \otimes 1_Q) \right\|_{L^2_W \rightarrow L^2_W} = \left\| (W)^{1/2}_{\mu,Q} (W^{-1})^{1/2}_{\mu,Q} \right\|_{\mathcal{H}(B)}.$$

Indeed, one can show that the norm-equality is attained testing functions of the form $f = 1_Q W^{-1} e$, with $e \in \mathcal{H}$.

We also remark that we have not been able to find a straightforward proof of Proposition 3.2 by directly estimating the operator norm of $A_{\kappa_Q}$ for a general kernel $\kappa_Q$. Thus, the point of using Lemma 3.4 is that the operator norm of averaging operators with separable kernels $\psi \otimes \varphi_Q$ is easier to compute.

### 3.4 Sharp estimates for scalar-valued weights

In this section, we include a couple of auxiliary lemmas about scalar-valued weights, which will be of crucial in the proof of Theorem 2.1. The following result is essentially borrowed from Lemma 4.3 in [15] and will allow us to reduce estimates of operator-valued weights to scalar-weights. For the reader’s convenience and for the sake of reassuring that the lemma carries over to our general setting, we include the short proof.

**Lemma 3.5.** Let $W : \mathbb{R}^d \rightarrow \mathcal{H}(B)$ be an $\mathcal{A}_2^S(\mu)$-weight and $e \in \mathcal{H}$ a non-zero vector. Then $w_e(x) := (W(x)e|e)_\mathcal{H}$ is a scalar-valued $\mathcal{A}_2^S(\mu)$-weight and satisfies

$$[w_e]_{\mathcal{A}_2^S(\mu)} \leq [W]_{\mathcal{A}_2^S(\mu)}.$$

**Proof.** Pick an arbitrary scalar-valued function $\psi \in L^2_{w_e}$ and set $f_e = \psi \otimes e \in L^2_W$. Fix an arbitrary cube $Q \in \mathcal{S}$ and notice that the averaging operator $f \mapsto 1_Q(f)_{\mu,Q}$ evaluated at $f_e$ can be expressed as

$$\left\| 1_Q(f_e)_{\mu,Q} \right\|_{L^2_W}^2 = \int_Q |\psi|_{\mu,Q}^2 (W(x)e|e)_\mathcal{H} \, d\mu(x) = \left\| 1_Q(\psi)_{\mu,Q} \right\|_{L^2_{w_e}}^2.$$
As a consequence of Remark 1 on norms of the averaging operator on weighted spaces, we immediately get

$$\left| \langle w_e \rangle_{\mu,Q}^{1/2} (w_e)^{-1/2} \right|^2 = \sup_{\|\psi\|_{L_w^2} = 1} \left\| 1_Q f_e \right\|_{L_w^2}^2 \leq \sup_{\|f\|_{L_w^2} = 1} \left\| 1_Q f \right\|_{L_w^2}^2 = \left\| (W^{-1})_{\mu,Q} \right\|_{H^2(B)}.$$

The proof now follows by taking supremum over $Q \in S$. □

Reducing inequalities operator-weighted inequalities to that of scalar weights, allows for applications of sharp estimates for scalar weights. The following lemma is essentially a quantitative version of the portion preserving property of scalar-valued $A^S_2(\mu)$-weights.

**Lemma 3.6.** Let $w$ be a scalar-valued $A^S_2(\mu)$-weight and $0 < \delta < 1$. Then for every $Q \in S$ and $S \subset Q$ with $\mu(S) \leq \delta \mu(Q)$, we have that

$$\int_S w d\mu \leq \left( 1 - \frac{(1 - \delta)^2}{[w]_{A^S_2(\mu)}} \right) \int_Q w d\mu.$$

**Proof.** Set $E_S := Q \setminus S$ and notice that $(1 - \delta) \mu(Q) \leq \mu(E_S)$. With this at hand, we estimate according to

$$\int_Q w d\mu \leq \frac{[w]_{A^S_2(\mu)}(\mu)}{(1 - \delta)^2} \int_{E_S} w^{-1} d\mu \leq \frac{[w]_{A^S_2(\mu)}(\mu)}{(1 - \delta)^2} \int_{E_S} w^{-1} d\mu \leq \int_{E_S} w d\mu.$$

The proof readily follows by writing $\int_{E_S} w d\mu = \int_Q w d\mu - \int_S w d\mu$ and rearranging in the previous inequality. □

In the context of dyadic Muckenhoupt weights, we can actually obtain a sharper version of the portion preserving property, which does not utilize the full strength of the dyadic $A_2$-condition and instead relies on the weaker notion of dyadic $A_\infty$-weights. In the context of $A_\infty(\mu)$, we shall need to assume that $\mu$ is a doubling measure with constant $K_\mu$, previously defined in (9). We state and prove a tailor-made version of this principle in the following context.

**Lemma 3.7.** Let $w$ be a scalar-valued dyadic $A_\infty(\mu)$-weight and let $0 < \delta < 2^{-16K_\mu^2[w]_{A_\infty(\mu)}}$. Then for any $Q \in D$ and Borel set $S \subset Q$ with $\mu(S) \leq \delta \mu(Q)$, there exists $0 < \eta < \frac{1}{2}$, such that $w(S) \leq \eta w(2Q)$. In fact, we can take $\eta = 2K_\mu^2 \delta^2 / 6$, with $\varepsilon = \frac{1}{6K_\mu^2[w]_{A_\infty(\mu)}}$. 14
Proof. This proof relies on a sharp version of the reverse Hölder inequality (see [10], Theorem 1.1), adapted to our setting. For instance, it asserts that for $0 < \varepsilon \leq \frac{1}{6K^2_{\mu}[w]_{A^\infty(\mu)}}$, one has

$$\left\langle w^{1+\varepsilon} \right\rangle^{1/(1+\varepsilon)}_{\mu,Q} \leq 2K^2_{\mu}(w)_{\mu,2Q}$$

for all $Q \in \mathcal{D}$. Now, let $S \subset Q$ with $\mu(S) \leq \delta \mu(Q)$. By Hölder’s inequality and the sharp version the of the reverse Hölder inequality, we get

$$\int_S wd\mu \leq \mu(S)^{1/(1+\varepsilon)} \mu(Q)^{1/(1+\varepsilon)} \left\langle w^{1+\varepsilon} \right\rangle^{1/(1+\varepsilon)}_{\mu,Q} \leq 2K^2_{\mu}\delta^{1/(1+\varepsilon)} \int_Q wd\mu \leq 2K^2_{\mu}\delta^{1/2} \int Q wd\mu.$$

It is straightforward to check that $2K^2_{\mu}\delta^{1/2} \leq \frac{1}{2}$, whenever $0 < \delta < 2^{-16K^2_{\mu}[w]_{A^\infty(\mu)}}$. \hfill $\Box$

4 Proof of main results

4.1 The lower bound

Proof of the lower bound of Theorem 2.1. Given a sparse collection $S \subset \mathcal{D}$, we let $\sigma : S \rightarrow \mathbb{N}$ an injective function and consider the orthogonal system on $L^2((0,1),\mathbb{R})$ given by the Rademacher functions $\mathcal{R} := \{\rho_n(t) := \text{sgn}(\sin(2^{n+1}\pi t)) : n \in \mathbb{N}\}$. With this at hand, we define the sparse Lerner operators

$$T^S_{\kappa}\rho f(x) = \sum_{Q \in S} \int_Q \rho_{\sigma(Q)}(t)\kappa_Q(x,y)f(y)d\mu(y). \quad (17)$$

Now suppose there exists a constant $C > 0$, possibly depending on $W$, such that

$$\sup_{\kappa} \left\| (T^S_{\kappa} \otimes I_{L^2_W}) f \right\|_{L^2_W} \leq C \left\| f \right\|_{L^2_W}$$

for all $f \in L^2_W$. Since elements in $\mathcal{R}$ are unimodular, the family of operators $T^S_{\kappa,\mathcal{R}}$ are still within the family of sparse Lerner operators in (7), thus their operator-norms on $L^2_W$ are also uniformly bounded by $C$. Now using the orthogonality assumption of $\mathcal{R}$, we get that

$$\sum_{Q \in S} \left\| A_{\kappa,Q,\mu}(f) \right\|_{L^2_W}^2 = \int_0^1 \left\| (T^S_{\kappa,\mathcal{R}} \otimes I_{L^2_W}) f \right\|_{L^2_W}^2 dt \leq C^2 \left\| f \right\|_{L^2_W}^2$$

for all $f \in L^2_W$. In particular, this means that for any $Q \in \mathcal{S}$, we have

$$\sup_{\kappa} \left\| A_{\kappa,Q,\mu}(f) \right\|_{L^2_W} \leq C \left\| f \right\|_{L^2_W},$$
where the supremum is taken over all $\kappa_Q$ supported on $Q \times Q$ and belonging to the unit-ball of $L^\infty(\mu \otimes \mu)$. Considering a specific class of averaging operators and the observation in Remark 1, we get

$$\| (W)^{1/2}_{\mu,Q} (W^{-1})^{1/2}_{\mu,Q} \|_{H(B)} = \sup_{\|f\|_{L^2_W} = 1} \left\| \frac{1}{\mu(Q)} (1_Q \otimes 1_Q) \mu(f) \right\|_{L^1_W} \leq C.$$ 

Taking supremum over $Q \in \mathcal{S}$, we finally conclude that

$$[W]^{1/2}_{\mathcal{A}^2(\mu)} \leq \sup_{\kappa} \left\| (T^S_{\kappa} \otimes 1_{\mathcal{H}}) \right\|_{L^2_W \to L^2_W}.$$ 

We remark that if a single sparse Lerner operator is bounded on $L^2_W$, then we have only managed to obtain the lower bound of its operator norm in terms of $[W]^{1/3}_{\mathcal{A}^2(\mu)}$, which is slightly weaker. We are not certain whether this can be improved.

4.2 The upper bound In order to establish the upper bound of Theorem 2.1, we need to verify that the sequence of linear operators $\{T_n\}_{n \in \mathbb{Z}}$ in (14) satisfy the prerequisites of the Lemma 3.1. For the sake of abbreviation, we shall in many instances simply denote by $T_n$ the unique canonical extensions to $L^2_W$. We shall also slightly abuse the notation and write $T_n$ instead of $T_n \otimes 1_{\mathcal{H}}$.

Proof of the upper bound of Theorem 2.1. Fix an arbitrary sequence of bounded complex-valued functions $\{\kappa_Q\}$, where each $\kappa_Q$ is supported on $Q \times Q$ and satisfy $\|\kappa_Q\|_{L^\infty(\mu \otimes \mu)} \leq \mu(Q)^{-1}$. Recall that the decomposition in subsection 3.2 allows us to express an arbitrary sparse Lerner operator as

$$T^S_{\kappa} = \sum_{n \in \mathbb{Z}} T_n$$

where

$$T_n f(x) := \sum_{Q \in \mathcal{J}^n} \int_Q \kappa_Q(x,y) f(y) d\mu(y).$$

We shall now verify that the hypothesis of Lemma 3.1 are satisfied. Note that since each $\mathcal{J}^n$ is collection of disjoint dyadic cubes, we can apply Proposition 3.2 to get

$$\|T_n f\|_{L^2_W}^2 = \sum_{Q \in \mathcal{J}^n} \| A_{\kappa_Q,\mu}(f) \|_{L^1_W}^2 \leq [W]_{\mathcal{A}^2(\mu)} \sum_{Q \in \mathcal{J}^n} \|1_Q f\|_{L^1_W}^2 \leq [W]_{\mathcal{A}^2(\mu)} \|f\|_{L^2_W}^2.$$ 

(18)
This yields the following uniform bound for all the diagonal terms

\[ \|T_n^* T_n\|_{L_W^2 \to L_W^2} = \|T_n T_n^*\|_{L_W^2 \to L_W^2} = \|T_n\|_{L_W^2 \to L_W^2} \leq [W]_{A^*_\mathcal{H}}. \]  

(19)

It now remains to establish bounds for the non-diagonal terms. To this end, we may without loss of generality assume that \( n > m \), since by the \( C^* \)-identity, we can write

\[ \|T_n^* T_m\|_{L_W^2 \to L_W^2} = \|T_m^* T_n\|_{L_W^2 \to L_W^2}. \]

Now observe that the boundedness of \( T_n : L_W^2 \to L_W^2 \) is equivalent to the boundedness of the following composition of operators, all considered in the non-weighted \( \mathcal{H} \)-valued \( L^2 \)-space (think \( L_W^2 \) with \( W \) being the identity operator \( 1_{\mathcal{H}} \) on \( \mathcal{H} \)), namely

\[ L_n := M_{W^{1/2}} T_n M_{W^{-1/2}} : L^2 \to L^2 \]

where \( M_{W^{\pm 1/2}} \) denotes the usual multiplication operator with \( W^{\pm 1/2} \). Now we can easily compute the adjoint of the operator \( L_n \) on \( L^2 \), which is given by

\[ L_n^* = M_{W^{-1/2}} T_n^* M_{W^{1/2}} = M_{W^{-1/2}} \left( \sum_{Q \in \mathcal{F}^n} A_{\kappa Q, \mu}^* \right) M_{W^{1/2}} \]

where the adjoint of the averaging operator on \( L^2 \) takes the form

\[ A_{\kappa Q, \mu}^*(f)(x) = \int_Q \kappa_Q(y, x) f(y) d\mu(y). \]

By the \( C^* \)-identity, we actually seek a bound for the operator norm of

\[ \|T_n^* T_m\|_{L_W^2 \to L_W^2}^2 = \|L_n^* L_m\|_{L^2 \to L^2}^2. \]

(20)

Using the fact that \( \mathcal{F}^n \) is a disjoint collection we can explicitly compute the kernel expression of the positive linear operator \( L_n L_m^* = M_{W^{1/2}} T_n M_{W^{-1/2}} T_m^* M_{W^{1/2}} \), given by

\[ L_n L_m^* = \sum_{Q \in \mathcal{F}^n} M_{W^{1/2}} A_{\kappa Q, \mu} M_{W^{-1/2}} \left( M_{W^{1/2}} A_{\kappa Q, \mu} M_{W^{-1/2}} \right)^* . \]

As a consequence, we can write

\[ \|L_n^* L_m f\|_{L^2}^2 = (L_n L_n^* L_m f, L_m f)_{L^2} = \sum_{Q \in \mathcal{F}^n} \|(M_{W^{1/2}} A_{\kappa Q, \mu} M_{W^{-1/2}})^* L_m f\|_{L^2}^2. \]

According to Proposition 3.2, we have that the family of averaging operators (hence also their adjoints) satisfy \( M_{W^{1/2}} A_{\kappa Q, \mu} M_{W^{-1/2}} : L^2 \to L^2 \) with operator
norm uniformly bounded by \([W]_{A^2_N}(\mu)\). Since the averaging operators are localized, we get
\[
\|L_n^*L_m f\|_{L^2}^2 \leq [W]_{A^2_N}(\mu) \sum_{Q \in \mathcal{J}^n} \|1_Q L_m f\|_{L^2}^2. \tag{21}
\]

It now remains to estimate the sum on the right hand side of (21), and here the fact that norms are localized to 1 for \(Q \in \mathcal{J}^n\) will play a crucial role.

To this end, using the fact that \(n > m\) together with \(\mathcal{J}^m\) being disjoint and changing the order of summation, it is not difficult to check that we can express
\[
\sum_{Q \in \mathcal{J}^n} \|1_Q L_m f\|_{L^2}^2 = \sum_{R \in \mathcal{J}^m} \sum_{Q \in \mathcal{J}^{(n-m)}(R)} \|1_Q (M_{W^{1/2}} A_{\kappa,R,\mu} M_{W^{-1/2}}) f\|_{L^2}^2. \tag{22}
\]

Now fix an arbitrary \(R \in \mathcal{J}^m\) and note that by applying Proposition 3.2 to each averaging operator \(A_{\kappa,R,\mu}\) and using Fatou’s lemma, we can write
\[
\sum_{Q \in \mathcal{J}^{(n-m)}(R)} \|1_Q (M_{W^{1/2}} A_{\kappa,R,\mu} M_{W^{-1/2}}) f\|_{L^2}^2 \leq \liminf_{N \to \infty} \sum_{Q \in \mathcal{J}^{(n-m)}(R)} \|1_Q (M_{W^{1/2}} A^N M_{W^{-1/2}}) f\|_{L^2}^2
\]
\[
= \liminf_{N \to \infty} \sum_{Q \in \mathcal{J}^{(n-m)}(R)} \int_Q \|\psi_R^N\|_{L^2}^2 \left(W (\varphi_R^N W^{-1/2} f)_{\mu,R} (\varphi_R^N W^{-1/2} f)_{\mu,R}\right)_N d\mu
\]
\[
\leq \limsup_{N \to \infty} \sum_{Q \in \mathcal{J}^{(n-m)}(R)} \int_Q \left(W (\varphi_R^N W^{-1/2} f)_{\mu,R} (\varphi_R^N W^{-1/2} f)_{\mu,R}\right)_N d\mu. \tag{23}
\]

Recall that we used the fact that \(\|\psi_R^N\|_{L^2(\mu)} \leq 1\). For any fixed \(N > 1\), we set \(\epsilon_N = (\varphi_R^N W^{-1/2} f)_{\mu,R}\) and consider the scalar-weight \(w_{\epsilon_N}(x) = (W(x) \epsilon_N)_{\mu,R}\). Lemma 3.5 asserts that \(w_{\epsilon_N}\) is a scalar-valued \(A^2_N\)-weight with \([w_{\epsilon_N}]_{A^2_N}(\mu) \leq [W]_{A^2_N}(\mu)\) (independent \(\epsilon_N \in \mathcal{H}\)). According to the decaying stopping time property in (13) we have
\[
\sum_{Q \in \mathcal{J}^{(n-m)}(R)} \mu(Q) \leq \left(\frac{1}{2}\right)^{(n-m)} \mu(R).
\]

Now regrouping all cubes in \(Q \in \mathcal{J}^{(n-m)}(R)\) with common predecessors in \(\mathcal{J}^{(n-m-1)}(R)\) and successively applying Lemma 3.6 in each step as we move towards the top cube \(R\), we obtain
\[
\sum_{Q \in \mathcal{J}^{(n-m)}(R)} \int_Q w_{\epsilon_N} d\mu \leq \left(1 - \frac{1}{4[W]_{A^2_N}(\mu)}\right)^{(n-m)} \int_R w_{\epsilon_N} d\mu. \tag{24}
\]
Recall that $\varepsilon_N = (\varphi_N^R W^{-1/2} f)_{\mu,R}$ and consider the integral kernels $\kappa^N_R = \mu(R)^{-1}(1_R \otimes \varphi_N^R)$. Going back to the estimate in (23) and applying Proposition 3.2, we obtain the following upper bound, independent of both $N > 1$ and $R \in \mathcal{J}^m$:

\[
\sum_{Q \in \mathcal{J}^{(n-m)}(R)} \left\| 1_Q (M_{W^{1/2}} A_{\kappa^N_R, \mu} M_{W^{-1/2}}) f \right\|_{L^2}^2 \\
\leq \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)} \limsup_{N \to \infty} \int_R w_{e_N} d\mu \\
= \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)} \limsup_{N \to \infty} \left\| 1_R (\varphi_N^N W^{-1/2} f)_{\mu,R} \right\|_{L^2_W}^2 \\
= \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)} \limsup_{N \to \infty} \left\| (M_{W^{1/2}} A_{\kappa^N_R, \mu} M_{W^{-1/2}}) f \right\|_{L^2}^2 \\
\leq \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)} [W]_{A^2_2(\mu)} \| 1_R f \|_{L^2}^2 .
\]

Using these observations together with (21) and (22), we deduce that

\[
\| L^*_n L_m f \|_{L^2}^2 \leq \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)} [W]^2_{A^2_2(\mu)} \sum_{R \in \mathcal{J}^m} \| 1_R f \|_{L^2}^2 \\
\leq \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)} [W]^2_{A^2_2(\mu)} \| f \|_{L^2}^2 .
\]

Consequently, we have thus established

\[
\| T^*_n T_m \|_{L^2_W \to L^2_W} = \| L^*_n L_m \|_{L^2 \to L^2} \leq \left( 1 - \frac{1}{4 [W]_{A^2_2(\mu)}} \right)^{(n-m)/2} [W]_{A^2_2(\mu)} .
\]

In an identical manner, we can estimate the operator norms

\[
\| T_n T^*_m \|_{L^2_W \to L^2_W} = \| L_n L^*_m \|_{L^2 \to L^2} .
\]

Indeed, this is done by running through the same argument as before, with the exception of the dual weight $W^{-1}$ playing the previous role of $W$. Since the $A^2_2(\mu)$-condition is symmetric, that is $[W^{-1}]_{A^2_2(\mu)} = [W]_{A^2_2(\mu)}$; the proof principally remains unchanged. We then analogously get
Together with the lower bound in subsection 4.1, the proof of Theorem 2
is complete.


\[ \| T_n T_m^* \|_{L^2_W \to L^2_W} \leq \left( 1 - \frac{1}{4 [W]_{A_2^\infty(\mu)}} \right)^{(n-m)/2} [W]_{A_2^\infty(\mu)}. \]

The hypothesis of Lemma 3.1 are now fulfilled, hence we conclude that
\( \Sigma_n T_n \) converges unconditionally to \( T_n^S \) and satisfies the upper bound:

\[ \left\| \sum_{n \in \mathbb{Z}} T_n \right\|_{L^2_W \to L^2_W} \leq 2 \sum_{n \in \mathbb{Z}} [W]_{A_2^\infty(\mu)}^{1/2} \left( 1 - \frac{1}{4 [W]_{A_2^\infty(\mu)}} \right)^{|n|/4} \leq \frac{4 [W]_{A_2^\infty(\mu)}^{1/2}}{1 - \left( 1 - \frac{1}{4 [W]_{A_2^\infty(\mu)}} \right)^{1/4}}. \]

Using the simple inequality \( \frac{1}{1-t^4} = \frac{(1+t^4)(1+t^2)}{1-t^2} \leq \frac{4}{1-t} \), for \( 0 < t < 1 \), and the fact that the upper bound is independent of \( \kappa \), we finally arrive that

\[ \sup_{\kappa} \| T_n^S \otimes 1_R \|_{L^2_W \to L^2_W} \leq 64 [W]_{A_2^\infty(\mu)}^{3/2}. \]

Together with the lower bound in subsection 4.1, the proof of Theorem 2.1
is now complete.

\[ \square \]

4.3 Dyadic Muckenhoupt \( A_2(\mu) \)-weights \( W \)

We now turn to the proof of Corollary 2.2.

Proof of Corollary 2.2. This time, we assume that \( \mu \) is a doubling measure
with constant \( K_\mu \) and that \( W \) is a dyadic Muckenhoupt \( A_2(\mu) \)-weight. The
proof follows verbatim that of Theorem 2.1 and only need to substitute the estimate in (24) by slightly sharper inequality, which is now available in this setting. To this end, we shall continue from the (23). Again, fix \( R \in \mathcal{J}^m \),
let \( \hat{R} \) denote the unique child of \( R \) in \( S \), which contains \( \cup_{Q \in \mathcal{J}^{(n-m)}(R)} Q \). For
an arbitrary fixed \( N > 1 \), we set \( e_N = (\varphi_{\hat{R}}^N W^{-1/2} f)_{\mu,R} \) and consider the scalar
weight \( w_{e_N} = (W e_N | e_N)_R \). Recall the decaying stopping time property in
(13) gives

\[ \mu \left( \bigcup_{Q \in \mathcal{J}^{(n-m)}(R)} Q \right) = \sum_{Q \in \mathcal{J}^{(n-m)}(R)} \mu(Q) \leq 2^{-(n-m)} \mu(R). \]

According to the definition of the dyadic operator-valued \( A_\infty(\mu) \)-condition
in (5), the sequence of weights \( w_{e_N} \) are scalar \( A_\infty(\mu) \)-weights with constant
less than \( [W]_{A_\infty(\mu)} \) and independent of \( e_N \). Consequently, if \( (n-m) > \)
16K^{12}_{\mu}[W]_{A_{\infty}(\mu)} then we may apply Lemma 3.7 to each scalar-weight \(w_{\varepsilon_{N}}\) and to the sets \(\cup Q_{\varepsilon}J^{(\varepsilon_{m})}Q\) and \(\tilde{R}_{\varepsilon}\), which gives

\[
\int_{\cup Q_{\varepsilon}J^{\varepsilon_{m}}} w_{\varepsilon_{N}}d\mu = \sum_{Q_{\varepsilon}J^{\varepsilon_{m}}(R)} \int_{Q} w_{\varepsilon_{N}}d\mu \leq \eta_{\varepsilon}(n-m) \int_{2\tilde{R}} w_{\varepsilon_{N}}d\mu \leq \eta_{\varepsilon}(n-m) \int_{R} w_{\varepsilon_{N}}d\mu
\]

where \(\eta_{\varepsilon}(n-m) = 4K^{-2\varepsilon_{m}}(n-m)\varepsilon_{+}\) with \(\varepsilon_{+} = 1/(12K^{2}_{\mu}[W]_{A_{\infty}(\mu)})\). If \(n-m \leq 16K^{12}_{\mu}[W]_{A_{\infty}(\mu)}\), then we instead use the trivial estimate

\[
\sum_{Q_{\varepsilon}J^{(\varepsilon_{m})}(R)} \int_{Q} w_{\varepsilon_{N}}d\mu \leq \int_{R} w_{\varepsilon_{N}}d\mu.
\]

These are the desired and sought after substitutions for the estimate in (24), hence following the identical line of reasoning as in the remain part of proof in Theorem 2.1, and recalling that \(n > m\), we obtain

\[
\|T_{n}^{*}T_{m}\|_{L_{W}^{2} \rightarrow L_{W}^{2}} \leq \alpha(n-m)
\]

where

\[
\alpha(n-m) := \begin{cases} 
[W]_{A_{2}(\mu)} & |n-m| \leq 16K^{12}_{\mu}[W]_{A_{\infty}(\mu)} \\
[W]_{A_{2}(\mu)} \eta_{-}(|n-m|)^{1/2} & |n-m| > 16K^{12}_{\mu}[W]_{A_{\infty}(\mu)}.
\end{cases}
\]

In an identical manner, we can estimate the norm \(\|T_{n}^{*}T_{m}\|_{L_{W}^{2} \rightarrow L_{W}^{2}}\). Again, we repeat the same procedure as before, with the exception of substituting \(W\) with its dual weight \(W^{-1}\) and using the symmetry that \(W \in A_{2}(\mu)\) iff \(W^{-1} \in A_{2}(\mu)\). In this case, we apply Lemma 3.7 to \(\sigma_{\varepsilon} = (W^{-1}e_{H})\) which gives rise to the parameters \(\varepsilon_{-} = 1/(12K^{2}_{\mu}[W^{-1}]_{A_{\infty}(\mu)})\) and \(\eta_{-}(n-m) = 4K^{2\varepsilon_{m}}(n-m)\varepsilon_{-}\). At the end, we analogously arrive at

\[
\|T_{n}^{*}T_{m}\|_{L_{W}^{2} \rightarrow L_{W}^{2}} \leq \beta(n-m)
\]

where

\[
\beta(n-m) := \begin{cases} 
[W]_{A_{2}(\mu)} & |n-m| \leq 16K^{12}_{\mu}[W^{-1}]_{A_{\infty}(\mu)} \\
[W]_{A_{2}(\mu)} \eta_{-}(|n-m|)^{1/2} & |n-m| > 16K^{12}_{\mu}[W^{-1}]_{A_{\infty}(\mu)}.
\end{cases}
\]

Lemma 3.1 on the principle of almost orthogonality now applies and we deduce

\[
\left\| \sum_{n \in \mathbb{Z}} T_{n} \right\|_{L_{W}^{2} \rightarrow L_{W}^{2}} \leq 2 \left( \sum_{n \in \mathbb{Z}} \sqrt{\alpha(n)} \cdot \sum_{n \in \mathbb{Z}} \sqrt{\beta(n)} \right)^{1/2}. \tag{25}
\]
It now remains to estimate the sums on the right hand side of (25). We write

\[
\sum_{n=1}^{\infty} \sqrt{\alpha(n)} = [W]_{A_2(\mu)}^{1/2} \left( \sum_{|n| \leq 16^K_\mu [W]_{A_\infty(\mu)}} 1 + \sum_{|n| > 16^K_\mu [W]_{A_\infty(\mu)}} \eta_+ (|n|)^{1/4} \right) = S_1 + S_2.
\]

The first sum is trivially bounded by

\[
S_1 \leq 32K^{12}_\mu [W]_{A_2(\mu)}^{1/2} [W]_{A_\infty(\mu)}.
\]

Recalling that \( \eta_+ (|n|) = 4K^{2\mu}|n|^{-\mu} \) with \( \varepsilon_+ = 1/(12K^{12}_\mu [W]_{A_\infty(\mu)}) \), the second sum is a geometric series and can be estimated according to

\[
S_2 \leq 2[W]_{A_2(\mu)}^{1/2} \sum_{n=3}^{\infty} \eta_+ (n)^{1/4} = \frac{8K^{2}_\mu [W]_{A_2(\mu)}^{1/2}}{1 - 2^{-\varepsilon_+/4}}.
\]

Note that the function \( t \mapsto \frac{t}{1 - 2^{-t}} \) is bounded on \([0, 1]\), hence we can find a numerical constant \( c > 0 \) (\( c = 7 \) will do), such that

\[
S_2 \leq 8K^{2}_\mu [W]_{A_2(\mu)}^{1/2} [W]_{A_\infty(\mu)}.
\]

Here \( c_\mu > 0 \) is a constant only depending \( \mu \), which we shall allow to change from line to line. Adding up the estimates of \( S_1 \) and \( S_2 \) yields

\[
\sum_{n \in \mathbb{Z}} \sqrt{\alpha(n)} \leq c_\mu [W]_{A_2(\mu)}^{1/2} [W]_{A_\infty(\mu)}.
\]

An identical argument also shows that

\[
\sum_{n \in \mathbb{Z}} \sqrt{\beta(n)} \leq c_\mu [W]_{A_2(\mu)}^{1/2} [W^{-1}]_{A_\infty(\mu)}.
\]

Going back to (25) with these estimates at hand, we ultimately arrive at

\[
\sup_{\psi, \varphi, \sigma} \left\| T^S_{\psi, \varphi} \otimes I_M \right\|_{L_{\mu}^1 \to L_{\nu}^1} \leq c_\mu [W]_{A_2(\mu)}^{1/2} [W]_{A_\infty(\mu)}^{1/2} [W^{-1}]_{A_\infty(\mu)}^{1/2}.
\]

This completes the proof of Corollary 2.2. \( \square \)

5 Weighted bounds for the Bergman projection

In this section, we shall prove Theorem 2.3 based on a convex body domination by sparse operators. To this end, we shall need to introduce the following systems of dyadic grids on \( \mathbb{R} \):

\[
\mathcal{D}^\omega(\mathbb{R}) := \{2^j([0,1] + m + (-1)^j \omega) : m \in \mathbb{Z}, j \in \mathbb{Z} \}
\]
for \( \omega \in \{0, 1/3\} \). These systems have previously appeared in many different works on sparse dominations, for instance, see [13], [9] and references therein. Note that \( D^0(\mathbb{R}) \) is just the standard grid on \( \mathbb{R} \), while \( D^{1/3}(\mathbb{R}) \) is a shifted and alternating grid, but when both are combined, they have the following useful property.

**Lemma 5.1** ([Lemma 3.1, [17]]). For any interval \( I \subset \mathbb{R} \), there exists a dyadic interval \( J \in D^\omega(\mathbb{R}) \) for some \( \omega \in \{0, 1/3\} \), such that \( I \subset J \) and \( |J| \leq 8|I| \).

Using these dyadic grids, we shall consider the corresponding collections of Carleson squares

\[
Q^\omega := \{ Q_J := J \times (0, |J|) : J \in D^\omega(\mathbb{R}) \}
\]

with \( \omega \in \{0, 1/3\} \), which are easily seen to be sparse collection of dyadic cubes on \( \mathbb{C}_+ \). For a fixed \( f \in L_0^\infty(\mathbb{R}) \otimes \mathcal{H} \), we introduce the convex body averages

\[
\langle f \rangle_{\gamma, Q_J} := \left\{ \frac{1}{A_{\gamma}(Q_J)} \int_{Q_J} \varphi f dA_{\gamma} : \varphi : Q_J \to \mathbb{C}, \|\varphi\|_\infty \leq 1 \right\}.
\]

The convex body averages are symmetric, convex and compacts subsets of \( \mathcal{H} \), where the compactness follows from weak-compactness of the unit-ball of \( \mathcal{H} \) together with the weak-star closure of the unit-ball of \( L^\infty(\mathbb{R}) \). With this at hand, we can define the set-valued sparse operators

\[
L_\gamma f(z) = \sum_{\omega \in \{0, 1/3\}} \sum_{J \in D^\omega} 1_{Q_J}(z) \langle f \rangle_{Q_J, \gamma}
\]

regarded as Minkowski sums of convex body averages. It follows from arguments identical to Lemma 2.5 in [15], that the corresponding Minkowski sum in (26) is for each \( f \in L_0^\infty(\mathbb{R}) \otimes \mathcal{H} \) and a.e. \( z \in \mathbb{C}_+ \), is a bounded symmetric convex set of \( \mathcal{H} \). The main reason for introducing these operators is show that there exists a dilation constant \( C_\gamma > 0 \), such that for each \( z \in \mathbb{C}_+ \) and \( f \in L_0^\infty(\mathbb{R}) \otimes \mathcal{H} \), the family of (maximal) Bergman projections \( P_\gamma^{(+)} \) with \( \gamma > -1 \), belongs to the \( C_\gamma \)-dilation of the set (26). The content of our next result is the following convex body domination, using techniques inspired from [17].

**Proposition 5.2.** There exists a constant \( C_\gamma > 0 \), only depending on \( \gamma > -1 \) such that for any \( f \in L_0^\infty(\mathbb{R}) \otimes \mathcal{H} \) and \( z \in \mathbb{C}_+ \), we have

\[
P_\gamma^{(+)}(f)(z) \in C_\gamma L_\gamma(f)(z).
\]
Proof. It suffices to prove the claim for the maximal Bergman projections $P_γ^+$. To this end, fix an arbitrary $z ∈ ℂ_+$ and $f ∈ L_0^∞(dA_γ) ⊗ ℋ$. Note that we can write

$$P_γ^+ f(z) = \sum_{k=−∞}^{∞} \int_{2^k |z−ξ| < 2^{k+1}} \frac{f(ξ)}{|z−ξ|^{2+γ}} dA_γ(ξ).$$

It thus suffices to find a constant $C_γ > 0$, such that each term satisfies

$$\int_{2^k |z−ξ| < 2^{k+1}} \frac{f(ξ)}{|z−ξ|^{2+γ}} dA_γ(ξ) ∈ C_γ ∥f∥_{Q,J_κ,γ}$$

for some $J_κ ∈ D^∞(ℝ)$ with $ω ∈ \{0,1/3\}$ and the collection $\{J_κ\}_{κ ∈ ℤ}$ has finite multiplicity. To this end, fix an integer $κ$ and pick an arbitrary $ξ ∈ ℂ_+$ satisfying $2^κ ≤ |z−ξ| < 2^{κ+1}$. If $\text{Re}(z) ≤ \text{Re}(ξ)$, then $ξ ∈ Q_I(z)$ where $I(z) := [\text{Re}(z), \text{Re}(z)+2^{κ+1})$, and if $\text{Re}(z) > \text{Re}(ξ)$ then we instead pick $I(z) := [\text{Re}(z)−2^{κ+1}, \text{Re}(z))$ for which $ξ ∈ Q_I(z)$. According to Lemma 5.1, there exists an interval $J_κ ∈ D^∞$, for some $ω ∈ \{0,1/3\}$, such that $I(z) ∈ J_κ$ and $|J_κ| ≤ 8|I(z)|$. From this it follows that

$$\frac{1}{|z−ξ|^{2+γ}} ≤ 1_{Q,J_κ} 2^{−κ(2+γ)} ≤ 2^{2+γ} 8^{2+γ} 1_{Q,J_κ} |J_κ|^{2+γ}.$$

This establishes (27) with $C_γ = 2^{2+γ} 8^{2+γ}$, thus it suffices to prove that each interval $J_κ$ in (27) appears at most finitely many times. However, note that by construction each interval $J_κ$ contains an interval of length $2^{κ+1}$ and is itself of length no more than $2^{κ+1}$, thus for any pair of integers $κ, m ∈ ℤ$ with $|m−κ| > 4$, the intervals $J_κ, J_m$ must necessarily be distinct. Consequently the collection of intervals $\{J_κ\}_{κ ∈ ℤ}$ have at most multiplicity 4, which completes the proof of this proposition.

Proposition 5.2 tells us that in order to find operator-weighted bounds for the (maximal) Bergman projections $P_γ^{(+)}$, it suffices to bound the set-valued sparse operators in (26). Fortunately, the following result allows to understand these set-valued operators as sparse Lerner operators in (7). The result in question is independent of dimension and we shall for convenience phrase it according to our setting.

**Lemma 5.3** (Lemma 2.7, [15]). Let $f ∈ L_0^∞(dA_γ) ⊗ ℋ$ and let $g(z) ∈ ∥f∥_{γ,Q_J}$ for a.e $z ∈ Q_J$. Then there exists a Borel measurable function $K_{Q_J} : Q_J × Q_J → ℂ$ with $∥K_{Q_J}∥_{L_∞} ≤ |J|^{−(2+γ)}$ such that

$$g(z) = \int_{Q_J} K_{Q_J}(z,ξ) f(ξ) dA_γ(ξ)$$

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Now according to Proposition 5.2 and Lemma 5.3, it thus suffices to find a uniform bound for sparse Lerner operators in (7). However, this is precisely the content of our main result, hence the proof of Theorem 2.3 is an immediate consequence of Theorem 2.1.

6 Applications to commutators in the multi-parameter setting

In this section, we shall use our main results to show new boundedness results on commutators of sparse operators with operator-valued functions. This in turn can be applied to prove boundedness results for iterated commutators in the bi-parameter setting.

For a locally weakly integrable function $B : \mathbb{R}^d \rightarrow \mathcal{H}(B)$ and a collection of dyadic cubes $\mathcal{S}$, we define the strong operator BMO-norm relative to $\mathcal{S}$ by

$$\|B\|_{SBMO_\mathcal{S}}^2 := \sup_{Q \in \mathcal{S}} \sup_{|e| \leq 1} \frac{1}{|Q|} \int_Q \|B(x)e - \langle Be\rangle_Q\|^2 dx + \sup_{Q \in \mathcal{S}} \sup_{|e| \leq 1} \frac{1}{|Q|} \int_Q \|B^*(x)e - \langle B^*e\rangle_Q\|^2.$$

We shall denote the space of all such functions with finite norm by $SBMO_\mathcal{S}$. In case that $\mathcal{S}$ is the collection of all dyadic cubes in $\mathbb{R}^d$, we simply write $SBMO^D$. For any Banach space $X$, we write $BMO(\mathbb{R}^d, X)$ for the so-called norm-BMO space consisting of all locally Bochner integrable functions

$$f : \mathbb{R}^d \rightarrow X, \quad \sup_{Q \in \mathcal{D}, 	ext{cube}} \frac{1}{|Q|} \int_Q \|f(x) - \langle f\rangle_Q\|_X^2 dx < \infty,$$

and we write $BMO^D(\mathbb{R}^d, X)$ in case the supremum is only taken over dyadic cubes $\mathcal{D}$. It is well-known that the John-Nirenberg Theorem holds in this context, so the $L^2(X)$ norm can be replaced by any $L^p(X)$ norm for $1 < p < \infty$. It is also well-known that $BMO^D(\mathcal{H}(B))$ is strictly contained in $SBMO^D$. Here is the main result of this section.

**Theorem 6.1.** Let $B : \mathbb{R}^d \rightarrow \mathcal{B}(\mathcal{H})$ be a locally strongly integrable function, let $\mathcal{S}$ be a sparse collection of dyadic cubes in $\mathbb{R}^d$ and $T^S_\kappa$ be the corresponding sparse Lerner operators in (7). Then the family of commutators $[T^S_\kappa, B]$ given by

$$[T^S_\kappa, B]f = T^S_\kappa Bf - BT^S_\kappa f$$

(28)
for $\mathcal{H}$-valued functions $f$ with finite Haar expansion, extends a bounded linear operator on $L^2(\mathbb{R}^d, \mathcal{H})$, if and only if $B \in SBMO_S$. In this case,

$$
\sup_{\kappa} \|[T^S_\kappa, B]\|_{L^2(\mathbb{R}^d, \mathcal{H}) \to L^2(\mathbb{R}^d, \mathcal{H})} \approx \|B\|_{SBMO_S}.
$$

In particular, if $B \in SBMO^D$, then any commutator with a sparse Lerner operator as in (28) is bounded on $L^2(\mathbb{R}^d, \mathcal{H})$.

Before proving this theorem, we shall first establish a corollary about iterated commutators in the biparameter setting, which requires one more definition. Let $b : \mathbb{R}^d \times \mathbb{R}^s \to \mathbb{C}$ be a locally integrable function and let $\mathcal{S}$ be a collection of dyadic cubes in $\mathbb{R}^d$. We define

$$
\|b\|_{rect, \mathcal{S}}^2 := \sup_{Q \in \mathcal{S}, R \in \mathbb{R}^s} \frac{1}{|Q||R|} \int_Q \int_R |b(x,y) - \langle b \rangle_Q(y) - \langle b \rangle_R(s) + \langle b \rangle_{Q \times R}|^2 dydx.
$$

In case that $\mathcal{S}$ is the collection of all dyadic cubes in $\mathbb{R}^d$, we want to write $\|b\|_{rect, d}$. Note that this is not quite the usual dyadic rectangular $BMO$ norm (see e.g. [2]), but a mixture of a dyadic and a non-dyadic rectangular $BMO$ norm.

**Corollary 6.2.** Let $b : \mathbb{R}^d \times \mathbb{R}^s \to \mathbb{C}$ be a locally integrable function, let $(T^S_\kappa)^{(1)}$ be a sparse Lerner operator for a sparse family $\mathcal{S}$ in $\mathbb{R}^d$ as in (7), and $T^{(2)}$ be a Calderón-Zygmund operator on $\mathbb{R}^s$. Suppose that $\|b\|_{rect, \mathcal{S}} < \infty$. Then the iterated commutator $[(T^S_\kappa)^{(1)}, [T^{(2)}, b]]$, given by

$$
[(T^S_\kappa)^{(1)}, [T^{(2)}, b]]f = \frac{1}{\|b\|_{rect, \mathcal{S}}} \left(T^S_\kappa)^{(1)} T^{(2)} b - (T^S_\kappa)^{(1)} T^{(2)} - T^{(2)} (T^S_\kappa)^{(1)} f + b (T^S_\kappa)^{(1)} T^{(2)} f\right)
$$

for suitable functions $f$ on $\mathbb{R}^d \times \mathbb{R}^s$, extends to a bounded linear operator on $L^2(\mathbb{R}^d \times \mathbb{R}^s)$.

We start with the proof of Corollary 6.2.

**Proof of Corollary 6.2.** Let us assume for the moment that $b$ is bounded and contained in $L^2(\mathbb{R}^d \times \mathbb{R}^s)$. Let $\mathcal{H} = L^2(\mathbb{R}^s)$. We define

$$
B : \mathbb{R}^d \to \mathcal{B}(\mathcal{H}), \quad B(x) = [T^{(2)}, b(x, \cdot)].
$$

Since $b$ is bounded and $T^{(2)}$ is a Calderón-Zygmund operator, $B(x) \in \mathcal{H}(B)$ for each $x \in \mathbb{R}^d$. Let $\epsilon \in L^2(\mathbb{R}^s)$ and $Q \in \mathcal{S}$, then
\[
\frac{1}{|Q|} \int_Q \|B(x)e - (Be)_Q\|^2 dx
= \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} \left| \left( [T^{(2)}, b(x, \cdot)]e \right)(y) - \left( [T^{(2)}, b(x, \cdot)]e \right)_Q \right|^2 dydx
= \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} \left| \left( [T^{(2)}, b(x, \cdot)]e \right)(x, y) - \left( [T^{(2)}, b(x, \cdot)]e \right)_Q \right|^2 dydx
= \frac{1}{|Q|} \int_{\mathbb{R}^n} \left\| \left( [T^{(2)}, b(x, \cdot)]e \right)(\cdot, y) \right\|^2_{L^2(Q)} dy.
\]

Note that the function \( \tilde{b} : \mathbb{R}^n \to L^2(\mathbb{R}^d) \), defined by \( \tilde{b}(y) = b(\cdot, y) - b_Q(y) \), belongs to \( BMO(L^2(Q)) \) with norm less or equal to \( |Q|^{1/2} \| b \|_{BMO_{rect,S}} \), since for any cube \( R \subset \mathbb{R}^n \), we have

\[
\frac{1}{|R|} \int_R \| \tilde{b}(y) - \{ \tilde{b} \}_R \|^2_{L^2(Q)} dy
= \frac{1}{|R|} \int_R \int_Q \left| b(x, y) - b_Q(y) - b_R(x) + b_Q \right|^2 dx dy \leq |Q| \| b \|_{BMO_{rect,S}}^2.
\]

Noting that the Coifman-Rochberg-Weiss Theorem for commutators \cite{3} holds even in the case of Hilbert-space valued functions, we hence find that

\[
\frac{1}{|Q|} \int_Q \|B(x)e - (Be)_Q\|^2 dx \leq \frac{1}{|Q|} \left\| \tilde{b} \right\|_{BMO(L^2(Q))}^2 \| e \|_{L^2(\mathbb{R}^d)}^2 \leq \| b \|_{BMO_{rect,S}}^2.
\]

Hence \( B \in SBMO_S \). Using Theorem 6.1, we find that

\[
\left\| \left[ (T^{(2)}_{\kappa} \right), [T^{(2)}, b] \right\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d)} = \left\| \left[ (T^{(2)}_{\kappa} \right), B \right\|_{L^2(\mathbb{R}^d, L^2(\mathbb{R}^d)) \to L^2(\mathbb{R}^d, L^2(\mathbb{R}^d))} \leq \| B \|_{SBMO_S} \leq \| b \|_{BMO_{rect,S}}.
\]

The case for a general \( b \in BMO_{rect,S} \) follows now by a standard approximation argument.

We are now ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** We follow a calculation from \cite{4} in a slightly more general setting. For a function \( B \) and a sparse family \( S \) as in the statement
of the Theorem, we define the operator-valued weight $W_B : \mathbb{R}^d \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by

$$W_B(x) = V_B^*(x)V_B(x) = \begin{pmatrix} \mathbb{1}_\mathcal{H} & 0 \\ B^*(x) & \mathbb{1}_\mathcal{H} \end{pmatrix} \begin{pmatrix} \mathbb{1}_\mathcal{H} & B(x) \\ 0 & \mathbb{1}_\mathcal{H} \end{pmatrix}.$$  

We claim that there exists a numerical constant $c > 0$, such that

$$\frac{1}{c} [W_B]_{A_2} \leq \|B\|_{S_{BMO}}^2 + 1 \leq [W_B]_{A_2}.$$  

(29)

Let $\rho(A)$ denote the spectral radius of an element $A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and recall that $\rho(A) = \|A\|$ for positive operators $A$. Note that

$$W_B^{-1}(x) = V_B^{-1}(x)(V_B(x))^{-1} = \begin{pmatrix} \mathbb{1}_\mathcal{H} & -B(x) \\ 0 & \mathbb{1}_\mathcal{H} \end{pmatrix} \begin{pmatrix} \mathbb{1}_\mathcal{H} & 0 \\ -B^*(x) & \mathbb{1}_\mathcal{H} \end{pmatrix}.$$  

For any cube $Q \subset \mathbb{R}^d$, we may compute

$$\rho\left( \frac{1}{|Q|^2} \int_Q \int_Q \begin{pmatrix} \mathbb{1}_\mathcal{H} & B(x) \\ B^*(x) & \mathbb{1}_\mathcal{H} + B^*(x)B(x) \end{pmatrix} \begin{pmatrix} \mathbb{1}_\mathcal{H} + B(y)B^*(y) & -B(y) \\ -B^*(y) & \mathbb{1}_\mathcal{H} \end{pmatrix} dxdy \right) =$$

$$\rho\left( \frac{1}{|Q|^2} \int_Q \int_Q \begin{pmatrix} \mathbb{1}_\mathcal{H} + B(y)B^*(y) - B(x)B^*(y) \\ * & \mathbb{1}_\mathcal{H} + B^*(x)B(x) - B^*(x)B(y) \end{pmatrix} dxdy \right) =$$

$$\rho\left( \frac{1}{|Q|^2} \int_Q \int_Q \begin{pmatrix} \mathbb{1}_\mathcal{H} + (BB^*)_Q - (B)_Q(B^*)_Q \\ 0 \\ B_1 + (B^*)_Q - (B)_Q(B_*)_Q \end{pmatrix} dxdy \right) =$$

$$= \rho\left( \frac{1}{|Q|^2} \int_Q \int_Q \begin{pmatrix} (BB^*)_Q - (B)_Q(B^*)_Q \\ * \end{pmatrix} dxdy \right) = \max\{\|1 + (BB^*)_Q - (B)_Q(B^*)_Q\|_{\mathcal{H}(\mathcal{H})}, \|1 + (B^*)_Q - (B)_Q(B)_Q\|_{\mathcal{H}(\mathcal{H})}\}.$$  

In the previous paragraphs, we denoted the lower non-diagonal element of the matrix by $*$, due to its lack of relevance when computing spectral radius of a lower triangular matrix. Now noting that

$$(BB^*)_Q - (B)_Q(B^*)_Q, e, e)_{\mathcal{H}} = \frac{1}{|Q|} \int_Q \|(B^*)_Q - (B)_Q\|_{\mathcal{H}(\mathcal{H})} dxdy$$

and

$$(B^*)_Q - (B)_Q(B)_Q, e, e)_{\mathcal{H}} = \frac{1}{|Q|} \int_Q \|(B)_Q - (B)_Q\|_{\mathcal{H}(\mathcal{H})} dxdy.$$  

This proves the claim in (29). On the other hand, we also have

$$\|T^S_\kappa \|_{L^2_{W_B} \to L^2_{W_B}} = \|V_B T^S_\kappa V_B^{-1}\|_{L^2(\mathcal{H} \oplus \mathcal{H}) \to L^2(\mathcal{H} \oplus \mathcal{H})}$$

$$= \left\| \begin{pmatrix} T^S_\kappa & \mathbb{1} \\ 0 & T^S_\kappa \end{pmatrix} \right\|_{L^2(\mathcal{H} \oplus \mathcal{H}) \to L^2(\mathcal{H} \oplus \mathcal{H})}.$$  

According to Theorem 2.1 and (29), this is enough to conclude the proof of this Theorem.
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