Lattice bosons in a quasi-disordered environment: The effects of next-nearest-neighbor hopping on localization and Bose-Einstein condensation

R. Ramakumar\textsuperscript{1}, A. N. Das\textsuperscript{2}, and S. Sil\textsuperscript{3}

\textsuperscript{1}Department of Physics and Astrophysics, University of Delhi, Delhi-110007, India
\textsuperscript{2}Saha Institute of Nuclear Physics, 1/AF Bidhannagar, Kolkata-700064, India
\textsuperscript{3}Department of Physics, Visva Bharati, Santiniketan-731235, India

(19 August 2013)

We present a theoretical study of the effects of the next-nearest-neighbor (NNN) hopping ($t_2$) on the properties of non-interacting bosons in optical lattices in the presence of an Aubry-Andrè quasi-disorder. First we investigate, employing exact diagonalization, the effects of $t_2$ on the localization properties of a single boson. The localization is monitored using an entanglement measure as well as with inverse participation ratio. We find that the sign of $t_2$ has a significant influence on the localization effects. We also provide analytical results in support of the trends found in the localization behavior. Further, we extend these results including the effects of a harmonic potential which obtains in experiments. Next, we study the effects of $t_2$ on Bose-Einstein condensation. We find that, a positive $t_2$ strongly enhances the low temperature thermal depletion of the condensate while a negative $t_2$ reduces it. It is also found that, for a fixed temperature, increasing the quasi-disorder strength reduces the condensate fraction in the extended regime while enhancing it in the localized regime. We also investigate the effects of boundary conditions and that of the phase of the AA potential on the condensate. These are found to have significant effects on the condensate fraction in the localization transition region.

PACS numbers: 03.75.Hh, 03.75.Lm, 37.10.Jk, 67.85.Hj, 72.15.Rn

I. INTRODUCTION

Cold atoms in harmonic traps and optical lattices continue to be an important controllable system for investigations into various properties of condensed matter. A case in point is the direct observation of Anderson localization\textsuperscript{1} of matter waves by several experimental groups\textsuperscript{2–4} in recent years. That the Anderson localization is a strongly dimension dependant phenomenon was recognized early on\textsuperscript{5}. An infinitesimal random disorder localizes all the single particle states in two and lower dimensions\textsuperscript{5,6} (see also the note in Ref. 7). This also rules out the development of mobility edges in two and lower dimensions. However, if the disorder distribution is deterministic, it is possible to have extended states if the quasi-disorder strength ($\lambda$) is below a critical value ($\lambda_c$), as is found in the one-dimensional Aubry-Andrè (AA) model\textsuperscript{8}. Recently, localization properties of non-interacting bosons loaded into a one-dimensional optical lattice with an AA quasi-disorder potential were experimentally investigated by the LENS group\textsuperscript{9}. Detailed theoretical studies of the AA model have been carried out both in the past\textsuperscript{10–26} and in the recent times\textsuperscript{27–37} (for recent reviews see Refs. 38–40). In the AA model, where the hopping is restricted to between nearest neighbors (NN), all the single particle states remain extended for $\lambda$ below $\lambda_c$ and become localized above it, which implies, again, the absence of mobility edges. Following a line of research into the effects of longer range hopping on the localization properties\textsuperscript{41–45}, recently it was discovered that the hopping beyond the NN can lead to the development of mobility edges in what may be called extended AA models\textsuperscript{46–49}. Considering the fact that the AA model has already been realized in experiments and that the experimental investigations of various aspects of the localization phenomena using cold atoms is ongoing\textsuperscript{50–53}, further theoretical studies of the beyond-NN-effects on the localization and the Bose condensation in extended AA models are certainly of current interest. The purpose of this paper is to present some interesting results obtained in such an investigation.

In this paper, we study the effects of the NNN hopping on some properties of bosons in optical lattices with AA quasi-disorder without and with confining harmonic potentials. In the first part of this paper, we investigate the effects of $t_2$ on the localization properties of a single boson employing the exact diagonalization method. We consider a single boson moving with NN and NNN hoppings in a one-dimensional (1d) optical lattice with AA quasi-disorder. We will show that the sign of $t_2$ plays an important role on the localization. The numerical results obtained are supplemented with analytical results for the energy dependence and the $t_2$ dependence of the critical disorder strength required for the localization transition. We also compare some of the results obtained using periodic boundary conditions to those obtained employing open boundary conditions\textsuperscript{54} after including a harmonic potential, which is usually present in the experiments. In the second part of this paper, we consider a many boson system and study the effects of $t_2$ on the Bose-Einstein condensation. The condensate fraction is found to show a significant dependence on $t_2$ and the
quasi-disorder strength. We also show that the phase of the AA potential has a significant effect on the condensate fraction. Since we are dealing with a quasi-disordered system, it is appropriate for us to place this work in the larger field of the studies on the effects of random disorder on Bose-Einstein condensates. There have been extensive studies on the random disorder effects on interacting continuum bosons\textsuperscript{55–64}. These investigations found that the condensate fraction decreases with increasing disorder strength. Further, recent studies on non-interacting lattice bosons\textsuperscript{65,66} reached the conclusion that the Bose condensation temperature slightly enhances with disorder for large filling whereas it reduces for small filling. Extensive studies have also been conducted on interacting disordered lattice bosons employing the Bose-Hubbard model and its variants\textsuperscript{67–73}. On the experimental front, it has been reported that the random disorder reduces the condensate fraction of lattice bosons in a harmonic trap\textsuperscript{74}. The rest of this paper is organized as follows. The numerical and the analytical studies on the single particle localization properties are presented in Sec. II. The studies on effect of $t_2$ on the Bose-Einstein condensation and the effects of the different phases of the AA potential are presented in Sec. III. The conclusions are given in Sec. IV.

II. THE EFFECTS OF THE NNN HOPPING ON THE LOCALIZATION PROPERTIES

In the first part of this section we study a lattice boson in the AA disorder potential. The Hamiltonian of this system is:

$$H = \sum_{i,\delta} \left( t_1 c_i^\dagger c_{i+\delta} + t_2 c_i^\dagger c_{i+2\delta} \right) + \lambda \sum_i \cos(2\pi q_i) c_i^\dagger c_i, \quad (1)$$

where $i$ is a site index in a one-dimensional optical lattice with a lattice constant $a$, $t_1$ and $t_2$ are the NN and the NNN hopping matrix elements, $c_i^\dagger$ is a creation operator for a boson at site $i$, $\delta$ is the locator of a NN site, $\lambda$ is the strength of the AA potential, and $q = (\sqrt{3}+1)/2$ is the incommensurability parameter. Here $t_1$, $t_2$ and $\lambda$ have energy units. All the energies in this paper are measured in units of $|t_1|$. We note here that Biddle \emph{et. al.}\textsuperscript{48,49} have shown that, for positive $t_1$ and $t_2$, the $\lambda_c$ decreases with $t_2$ for low energy states while it increases for high energy states. In our studies we have considered both negative and positive $t_2$ values. In this paper, the results presented for localization studies are for positive $t_1$ unless stated otherwise. We numerically diagonalize $H$ to obtain its eigen energies and eigen functions. All our results presented in this paper are obtained considering a lattice of 233 sites. We monitor the localization properties by calculating the Shannon entropy which is a measure of the quantum entanglement\textsuperscript{75}. The Shannon entropy for the ground state is given by

$$S = -\sum_i p_i \log_2 p_i, \quad (2)$$

where,

$$p_i = |\langle i | \psi_G \rangle|^2 = |a_i|^2 \quad (3)$$

in which the $a_i$ is the $i$th site amplitude of the ground state wave-function

$$|\psi_G\rangle = \sum_i a_i |i\rangle. \quad (4)$$

We also supplement these results with calculations of the Inverse Participation Ratio (IPR). The IPR is defined by

$$IPR = \frac{\sum_i p_i^2}{(\sum_i p_i)^2}. \quad (5)$$

In Fig. 1, we have shown the variations of the S and the IPR of the ground state as a function of the quasi-disorder strength for various values of $t_2$. The transition from the extended to the localized state is signalled by a large drop in $S$ and a large rise in IPR. In the absence of the NNN hopping, the transition occurs at a critical disorder strength $\lambda_c \approx 2$. It is seen that $\lambda_c$ changes as $t_2$ is introduced and the sign of $t_2$ has a significant effect on the localization transition. The $\lambda_c$ for the ground state decreases with $|t_2|$ for $t_2 > 0$ and increases with $|t_2|$ for $t_2$ negative. In fig. 1 we have also presented the results for $t_1 = -1.0$ and $t_2 = -0.2$. We have seen that the results do not depend on the sign of $t_1$\textsuperscript{76}. Next we have studied the energy dependence of the $\lambda_c$. We generalize the definitions of S and IPR to excited states by replacing $|\psi_G\rangle$ with corresponding excited state wave functions in Eq. (3). We have numerically obtained the
\( \lambda_c \) for all the 233 eigen states from the calculations of the \( dS/d\lambda \) as well as the \( d(\text{IPR})/d\lambda \) as a function of \( \lambda \) as shown in Fig. 2 for three states. For each eigen state, the value of \( \lambda \) for which \( dS/d\lambda \) is minimum and \( d(\text{IPR})/d\lambda \) is maximum is taken as the \( \lambda_c \) of that state and the corresponding eigen energy is \( E_c \). We note here that \( dS/d\lambda \) is a better indicator of the localization transition compared to \( d(\text{IPR})/d\lambda \). In Fig. 3, we have shown the variation of the \( \lambda_c \) with \( E_c \) for \( t_2 = -0.1 \). The \( \lambda_c \) is seen to decrease approximately linearly with increasing energy. It may be noted that for positive \( t_2 \) this trend is reverse\(^{49} \).

![Figure 1](image1.png)

**FIG. 1.** Entanglement (solid) and the IPR (dashes) as a function of the AA quasi-disorder potential strength (\( \lambda \)) for a closed chain with 233 lattice sites for different values of \( t_2 \). From left to right, the curves are for \( t_2 = 0.2, 0.15, 0.1, 0.05, 0, -0.05, -0.1, -0.15, \) and -0.2, respectively. The solid circles represent the results for \( t_1 = -1.0 \) and \( t_2 = -0.2 \).

![Figure 2](image2.png)

**FIG. 2.** The variation of \( dS/D\lambda \) and \( d(\text{IPR})/d\lambda \) with \( \lambda \) for the the ground state (solid), 39th excited state (dash), and the 78th excited state (dot) of a boson in a closed chain of 233 sites.

The numerical results obtained for negative \( t_2 \) can be understood from an extension of the analytical calculations given in Ref. 48. We start with the following Hamiltonian for an infinite system:

\[
\hat{H} = \sum_{n,n'} t_{nn'} c_{n'}^\dagger c_n + \lambda \sum_n \cos(2\pi qn) c_{n'}^\dagger c_n, \tag{6}
\]
where

\[ t_{nn'} = e^{i\pi t e^{-(p+i\pi)|n-n'|}} \]  

(7)

\( n \) and \( n' \) are the site indices and \( s = 0 \) or \( 1 \). Here, the hopping strength decreases exponentially with increasing distance and our choice (Eq. 7) makes \( t_1 \) positive whereas \( t_2 \) is positive or negative depending on \( s = 0 \) or \( 1 \). The relevance of this model in the context of ultracold atoms in optical lattices has been discussed in Ref. 48. The eigenvalue equation is then

\[ (E + e^{is\pi t} - \lambda \cos(2\pi q n)) u_n = e^{is\pi \sum_{n'} t e^{-(p+i\pi)|n-n'|} u_{n'}}, \]  

(8)

where \( u_n \) is the amplitude of the wave function at site \( n \) and \( E \) the energy eigenvalue. Now defining

\[ \cosh(p_o) = e^{is\pi \left( \frac{E + e^{is\pi t}}{\lambda} \right)} \]  

(9)

which immediately gives

\[ \sinh(p_o) = e^{is\pi \frac{\omega}{\lambda}} \]  

(10)

with

\[ \omega = e^{is\pi \sqrt{(E + e^{is\pi t})^2 - \lambda^2}}, \]  

(11)

the eigenvalue equation becomes

\[ \omega T_n(p_o) u_n = e^{is\pi \sum_{n'} t e^{-(p+i\pi)|n-n'|} u_{n'}}, \]  

(12)

where

\[ T_n(p_o) = \frac{\cosh(p_o) - e^{is\pi \cos(2\pi q n)}}{\sinh(p_o)}. \]  

(13)

The dual to the preceding eigenvalue equation is obtained by multiplying both sides of it with \( \exp(i2\pi mnq) \) and summing over \( m \), an integer. The dual is obtained as

\[ \omega T_n(p) \tilde{u}_n = e^{is\pi \sum_{n'} t e^{-(p+i\pi)|n-n'|} \tilde{u}_{n'}}, \]  

(14)

where

\[ \tilde{u}_m = \sum_{n} T_n(p_o) e^{i2\pi mnq} u_n. \]  

(15)

The Eq. (12) and Eq. (14) become self dual for \( p = p_o \). At the self dual point, following Eq. (9), one gets

\[ \cosh(p) = e^{is\pi \left( \frac{E + e^{is\pi t}}{\lambda} \right)}. \]  

(16)

Noting that \( t_1 = te^{-p} \) and \( t_2 = e^{is\pi t} e^{-2p} \), one obtains from the preceding equation an equation for \( \lambda_c \) as

\[ \lambda_c = \frac{2t_1 + 2e^{is\pi t} E e^{-p}}{1 + e^{-2p}} = \frac{2t_1 + 2E (t_2/t_1)}{1 + (t_2/t_1)^2}. \]  

(17)

When \( t_2 = 0 \), the critical disorder strength \( \lambda_c \) is energy independent and is equal to \( 2t_1 \), a result obtained for the original AA model. When \( t_2 \neq 0 \) and energy \( (E) \) is fixed, the change in \( \lambda_c \) is proportional to \( t_2 \) provided \( t_2/t_1 \) is small. For a fixed value of \( t_2 \), \( \lambda_c \) increases or decreases linearly with the energy eigenvalue \( E \) depending on \( t_2 \) being positive or negative. In Fig. 3, the solid line is obtained by using Eq. (17) for \( t_2/t_1 = -0.1 \). The numerical results are in reasonable agreement with the analytical results. Note that the numerical results are obtained considering
NN and NNN hoppings only while the analytical results are derived considering long range hopping which decays exponentially with distance. In Fig. 4, we plot the $t_2$ dependence of $\lambda_c$ for the ground state. We find that an increase in $|t_2|$ for negative $t_2$ increases the $\lambda_c$ while increasing positive $t_2$ reduces $\lambda_c$ and that it decreases almost linearly with increasing $t_2$.

From now on, keeping the cold atom experiments in mind, we consider the effects of a harmonic confining potential. The Hamiltonian of the system is then

$$\hat{H} = H + \sum_i k_i r_i^2 c_i^\dagger c_i,$$

(18)

where $k$, which has an energy unit, is the strength of the harmonic confining potential. In the presence of harmonic potential, we use open boundary conditions and measure the position coordinate of a lattice site from the center of the harmonic trap. In such cases the effect of second nearest-neighbor hopping on the entanglement is shown in Fig. 5. In this figure one finds a small region of $\lambda$ where the entanglement ($S$) increases with $\lambda$, in contrast to the usual behavior of suppression of $S$ with increasing $\lambda$. Then $S$ reaches a peak and falls abruptly with increasing $\lambda$ signifying localization of the wave function. The small enhancement of $S$ preceding the localization transition results from a competition between the AA potential trying to move the boson away from the center of the trap and the confining potential trying to bring it to the center of the trap, as explained in our earlier paper (Ref. 37). We note here that the disorder induced enhancement of $S$ becomes prominent and the peak becomes sharper with increasing positive $t_2$, whereas the peak becomes broader for negative values of $t_2$. In Fig. 6 we have plotted the $\lambda_c$ for the ground state as a function of $t_2$ for different values of the strength ($k$) of the harmonic potential. $\lambda_c$ decreases with increasing $t_2$ and the curves are almost linear as similar to that for a closed chain. For a fixed $t_2$ the $\lambda_c$ increases with increasing $k$. The physical origin of this effect is in the harmonic potential opposing the disorder trying to localize the boson away from the trap center.

FIG. 3. Plot of $\lambda_c$ vs $E$ for $t_2 = -0.1$. The plus signs represent numerically determined $\lambda_c$ and energy eigen values for different eigen states for a closed chain of 233 lattice sites and the solid line is the approximate analytical result.
FIG. 4. The variation of the $\lambda_c$ with $t_2$ for a closed chain of 233 lattice sites.

FIG. 5. Entanglement ($S$) as a function of the AA quasi-disorder potential strength ($\lambda$) for an open chain with 233 lattice sites in a harmonic trap of $k = 0.00005$ for different values of $t_2$. From left to right, the curves are for $t_2 = 0.08, 0.05, 0, -0.05, -0.08$, respectively. The AA potential is placed symmetrically about the center of the harmonic trap with phase factor $\phi = 0$ (see Eq. 22). The solid circles represent the results for $t_1 = -1.0$ and $t_2 = 0.05$. 
III. THE EFFECTS OF NNN HOPPING ON THE BOSE-EINSTEIN CONDENSATION

In this section we first study the effect of $t_2$ on the Bose condensate fraction of a collection of bosons in a finite one-dimensional periodic optical lattice with AA disorder in a harmonic confining potential. The system Hamiltonian is

$$\tilde{H} = \hat{H} - \mu \sum_i c_i^\dagger c_i,$$

where $\mu$ is the chemical potential. In terms of the single particle energy levels ($E_i$) obtained by numerically diagonalizing the Hamiltonian given in Eq. (18), the boson number equation is

$$N = \sum_{i=0}^{m} N(E_i),$$

where $E_0$ and $E_m$ are the lowest and the highest energy levels, and

$$N(E_i) = \frac{1}{e^\beta (E_i - \mu) - 1}$$

in which $\beta = 1/k_B T$ with $k_B$ the Boltzmann constant and $T$ the temperature. We first determine the chemical potential and then the boson populations in various energy levels using the boson number equation.

The temperature dependence of the condensate fraction for various values of the quasi-disorder strength for fixed $t_2$ values are shown in Fig. 7. In the low temperature regime, increasing $\lambda$ is seen to suppress the condensate fraction. Beyond this temperature range, increasing $\lambda$ leads to a reduction of the $N_0/N$ until $\lambda = \lambda_c$ and then to an enhancement for $\lambda > \lambda_c$. The effect of $t_2$ on the condensate fraction for fixed value of the quasi-disorder strength is shown in Fig. 8. It is clear that a negative $t_2$ enhances the $N_0/N$ while a positive $t_2$ reduces it. It is also seen that the effect of a positive $t_2$ is more prominent in comparison with a negative $t_2$. 
FIG. 7. The variation of the condensate fraction \( \frac{N_0}{N} \) with temperature for 10000 bosons in an open chain with 233 sites in a harmonic trap with \( k = 0.00001 \) for different strengths of the AA potential and different values of \( t_2 \). Top panel: \( \lambda = 0 \) (dots), 2.0 (short dashes), 3.05 (solid), 3.5 (long dash), and 4.0 (dash-dot). Middle panel: \( \lambda = 0 \) (dots), 1.5 (short dash), 2.06 (solid), 2.2 (long dash), and 2.5 (dash dot). Bottom panel: 0 (dots), 0.8 (short dash), 0.94 (solid), 1.2 (long dash), and 1.5 (dash dot). The solid line is for \( \lambda = \lambda_c \). The AA potential is placed symmetrically about the center of the harmonic trap with phase factor \( \phi = 0 \).
FIG. 8. The variation of the condensate fraction with temperature for 10000 bosons in an open chain with 233 sites in a harmonic trap with $k = 0.00001$ for $\lambda = 0.5$ and different values of $t_2$. From top to bottom, the curves are for $t_2 = -0.2, -0.1, 0, 0.1$, and 0.2, respectively. The AA potential is placed symmetrically about the center of the harmonic trap with phase factor $\phi = 0$.

All the results presented in the presence of the harmonic trap are obtained by placing the AA potential symmetric about the center of the trap. Further to study the effects of varying the phase factor of the AA potential on the condensate fraction we consider the potential at a site $i$ with a phase factor $\phi$ as

$$V(i) = \lambda \cos[2\pi i + \phi],$$

where the coordinate of the site $i$ is measured from the center of the trap. For $\phi = 0$ the potential is symmetric about the center and has its highest value there. This is an unique situation where the AA potential and the harmonic trap compete with each other in the localization transition regime as mentioned previously. In the top panel of Fig. 9, we have compared the $\lambda$ dependence of the low temperature $N_0/N$ for $\phi = 0$ with two finite values of $\phi$. The phase factor is found to have a strong influence. The drop in the condensate fraction to 0.5 in the localization transition region occurs only for $\phi = 0$. To understand the $\lambda$ dependence of $N_0/N$, we looked at the energy difference between the ground state and the first excited state ($\Delta E$), as shown in the lower panel of the Fig. 9. The physical origin of the variations seen is in the disorder induced changes in the lower energy levels of the boson. For $\phi = 0$, the drop to 0.5 occurs because the ground state and the first excited state are equally populated since $\Delta E$ is extremely small for $\lambda \leq \lambda_c$. For non zero values of $\phi$, the $\Delta E$ shows a minimum at $\lambda_c$, and then it becomes large enough so that almost all the bosons are in the ground state for all $\lambda$ except at $\lambda_c$ where $N_0/N$ shows a dip. In the upper panel of Fig. 10, we have shown the dependences of $N_0/N$ in the entire of range (0 to $\pi$) of $\phi$ for two temperatures for a fixed AA disorder strength. The changes in $N_0/N$ increases with increasing temperature. The variation of $N_0/N$ closely tracks the changes in $\Delta E$ given in the lower panel of the Fig. 10.

IV. CONCLUSIONS

In this paper we investigated the effects of next nearest-neighbor (NNN) hopping ($t_2$) on the properties of non-interacting bosons in optical lattices in the presence of an Aubry-André quasi-disorder. In closed chains, for the ground state, a negative $t_2$ enhances the critical disorder strength ($\lambda_c$) for the ground state required for the localization transition while a positive $t_2$ reduces. For high energy states ($E > 0$), the trend is opposite. These results obtained numerically were complemented with analytical calculations. We further extended these studies of the single particle localization including the effects of a harmonic confining potential with open boundary conditions which usually obtains in cold atom experiments. The harmonic potential was found to increase the $\lambda_c$. It was also found that while negative $t_2$ enhances the entanglement ($S$), the effect of a positive $t_2$ is to reduce it. Further, a positive $t_2$ enhances the disorder induced enhancement of the $S$ when the harmonic trap competes with the AA potential in the localization
transition region. Next we considered a many boson system and studied the effects of \( t_2 \) on the Bose condensation. It is found that the thermal depletion of the condensate is enhanced by a negative \( t_2 \) while it is reduced for a positive \( t_2 \). Finally, we investigated the effects of the phase of the AA potential on the condensate fraction (\( N_0/N \)) to find that it has a very strong effect on the \( N_0/N \) for \( \lambda \geq \lambda_c \).

**FIG. 9.** Top panel: The variation of the condensate fraction with \( \lambda \) for 10000 bosons in an open open chain with 233 sites in a harmonic trap with \( k = 0.00001 \), \( t_2 = 0.1 \) and \( k_B T = 0.2 \). The results are for: \( \phi = 0 \) (solid line), 0.05 (long dashes), 0.08 (short dash) and 0.1 (dots). The bottom panel shows the corresponding variation of \( \Delta E (= E_2 - E_1) \) with \( \lambda \).

**FIG. 10.** The variation of the condensate fraction (top panel) and \( \Delta E \) (bottom panel) for 10000 bosons in an open chain with 233 sites in a harmonic trap with \( k = 0.00001 \) as a function of the phase factor (\( \phi \)) for \( t_2 = 0.1 \) and \( \lambda = 2.0 \). The results in the top panel are for: \( k_B T = 0.4 \) (open circles with a guiding dotted line) and 2.0 (solid line).
ACKNOWLEDGMENTS

RRK thanks Professor M. K. Sanyal, Director, SINP and Professor S. N. Karmakar, Head, TCMP Division, SINP for hospitality at SINP.

1. P. W. Anderson, Phys. Rev. 109, 1492 (1958).
2. J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, and A. Aspect, Nature 453, 891 (2008).
3. S. S. Kondov, W. R. McGehee, J. J. Zirbel, and B. DeMarco, Science 334, 66 (2011).
4. F. Jendrzejewski, A. Bernard, K. Müller, P. Cheinet, V. Josse, M. Piraud, L. Pezzé, L. Sanchez-Palencia, A. Aspect, and P. Bouyer, Nature Phys. 8, 398 (2012). See also M. Piraud, L. Pezzé, L. Sanchez-Palencia, Europhys. Lett. 99 (2012) 50003.
5. N. F. Mott and W. D. Twose, Adv. Phys. 10, 107 (1961).
6. E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).
7. For a study of the trapping potential induced changes on Anderson localization, see L. Pezzé and L. Sanchez-Palencia, Phys. Rev. Lett. 106, 040601 (2011).
8. S. Aubry and G. Andrie, Ann. Isr. Phys. Soc. 3, 133 (1980).
9. G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature 453, 895 (2008).
10. J. B. Sokoloff, Phys. Rev. B 23, 6422 (1981).
11. M. Ya Azbel, Phys. Rev. Lett. 43, 1954 (1982).
12. I. M. Suslov, Zh. Éksp. Teor. Fiz. 83, 1079 (1982) [Sov. Phys. JETP 56, 612 (1982)].
13. C. M. Soukoulis and E. N. Economou, Phys. Rev. Lett. 48, 1043 (1982).
14. K. S. Dy and T. C. Ma, J. Phys. C 15, 6971 (1982).
15. D. J. Thouless, Phys. Rev. Lett. 28, 4272 (1983).
16. D. J. Thouless and Q. Niu, J. Phys. A 16, 1911 (1983).
17. M. Kohmoto, Phys. Rev. Lett. 51, 1198 (1983).
18. M. Kohmoto, L. P. Kadanoff, and C. Tang, Phys. Rev. Lett. 50, 1870 (1983).
19. D. Weire and J. P. Kermode, Phys. Status Solidi B 118, K163 (1983).
20. S. Ostlund and R. Pandit, Phys. Rev. B 29, 1394 (1984).
21. C. Wiecko and E. Roman, Phys. Rev. B 30, 1603 (1984).
22. M. Wilkinson, Proc. R. Soc. A 391, 305 (1984).
23. K. A. Chao, R. Riklund, and G. Wahlström, J. Phys. A 18, L403 (1985).
24. A. D. Zlatesis, C. M. Soukoulis, and E. N. Economou, Phys. Rev. B 33, 4936 (1986).
25. G.-L. Ingold, A. Wobst, C. Aulbach, and P. Hanggi, Eur. Phys. J. B 30, 175 (2002).
26. C. Aulbach, A. Wobst, G.-L. Ingold, P. Hänggi, and I. Varga, New J. Phys. 6, 70 (2004).
27. G. Roux, T. Barthel, I. P. McCulloch, C. Kollath, U. Schollwöck, and T. Giamarchi, Phys. Rev. A 78, 023628 (2008).
28. X. Deng, R. Citro, A. Minguzzi, and E. Orignac, Phys. Rev. A 78, 013625 (2008).
29. M. Modugno, New J. Phys. 11, 033023 (2009).
30. M. Larcher, F. Dalfovo, and M. Modugno, Phys. Rev. A 80, 053606 (2009).
31. S. K. Adhikari and L. Salasnich, Phys. Rev. A 80, 023606 (2009).
32. X. Deng, R. Citro, E. Orignac, and A. Minguzzi, Eur. Phys. J. B 68, 435 (2009).
33. J. C. C. Cestari, A. Foerster, and M. A. Cusmáo, Phys. Rev. A 82, 063634 (2010).
34. For an investigation into the similarities and the differences between the localization in the Anderson model and the AA model, see M. Albert and P. Leboeuf, Phys. Rev. A 81, 013614 (2010). See also M. Piraud, A. Aspect, and L. Sanchez-Palencia, Phys. Rev. A 85, 063611 (2012).
35. M. Larcher, M. Modugno, and F. Dalfovo, Phys. Rev. A 83, 013624 (2011).
36. J. C. C. Cestari, A. Foerster, M. A. Gusmáo, and M. Continento, Phys. Rev. A 84, 055601 (2011).
37. R. Ramakumar and A. N. Das, Physica A 392, 4271 (2013).
38. G. Modugno, Rep. Prog. Phys. 73, 102401 (2010).
39. L. Sanchez-Palencia and M. Lewenstein, Nature Phys. 6, 87 (2010).
40. B. Shapiro, J. Phys. A 45, 143001 (2012).
41. R. Riklund, Y. Liu, G. Wahlström, and Z. Zhao-bo, J. Phys. C 19, L705 (1986).
42. A. Rodríguez, V. A. Malyshev, G. Sierra, M. A. Martín-Delgado, J. Rodríguez-Laguna, and F. Domínguez-Adame, Phys. Rev. Lett. 90, 027404 (2003).
43. S.-J. Xiong and G.-P. Zhang, Phys. Rev. B 68, 174201 (2003).
In the numerical calculations, the closed chain results are obtained by taking $t_{N_s,1} = t_{1,N_s} = t_1$ and $t_{N_s-1,1} = t_{1,N_s-1} = t_{1,N_s-2} = t_{2,N_s} = t_2$, while for the open chains these matrix elements are taken as zero, where $N_s$ is the number of sites in the lattice.

There have been detailed investigations on the effects of disorder on lattice bosons within the framework of the Bose-Hubbard model since the late eighties (Ref. 68). Since our focus is not interacting bosons in this paper, we request the reader to consult the recent papers (Refs. 69–73), and the references contained therein.

In presence of nearest and next nearest neighbor hopping the Schrödinger equation reduces to $(E - \epsilon_n)u_n = t_1(u_{n+1} + u_{n-1}) + t_2(u_{n+2} + u_{n-2})$. If we transform $u_n$ to $e^{i\pi x}u'_n$, then $u'_n$ satisfies similar equation of motion with negative $t_1$. Consequently, the eigenvalues remain unchanged if the sign of $t_1$ is changed. Moreover, $|u_n|^2 = |u'_n|^2$ makes the localization behaviour of the system unaltered with the change of the sign of $t_1$. 