Trace-based Deductive Verification

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Abstract. Contracts specifying a procedure’s behavior in terms of pre- and postconditions are essential for scalable software verification, but cannot express any constraints on the events occurring during execution of the procedure. This necessitates to annotate code with intermediate assertions, preventing full specification abstraction. We propose a logic over symbolic traces able to specify recursive procedures in a modular manner that refers to specified programs only in terms of events. We also provide a deduction system based on symbolic execution and induction that we prove to be sound relative to a trace semantics. Our work generalizes contract-based to trace-based deductive verification.

1 Introduction

To make deductive verification scale, modular specification and verification is of essence [14]. In imperative programming languages modularity manifests itself at the granularity of procedure calls. A procedure’s behavior is specified in terms of a contract [27,28]: a pair of first-order pre- and postconditions. During verification each procedure call in the code is replaced with a check that the contract’s precondition holds at this point, the call’s effect is approximated by assuming the postcondition. In consequence, verification of the called code is replaced (or approximated) by first-order constraints. The overall verification effort for a program is proportional to its length, instead of the unfolded (and unbounded) call graph. The contract-based approach works for recursive procedures [19,37]. It requires an induction principle [5,26] and is realized in most state-of-art deductive verification systems, such as [1,22,25].

The fundamental limitation of pre-/postcondition contracts, which in the following are called state-based contracts, is the inability to specify events happening during execution of a procedure. This is not only problematic for specifying concurrent programs, but already an issue for interactive programs or loop specification: It is necessary to annotate code with intermediate assertions, which worsens readability and impedes full abstraction from the implementation during specification. Indeed, contracts written in most specification languages in deductive verification [3,24] cannot be used independently of the specified code.

Lack of the ability to create abstract specifications motivates the design of a logic serving as an abstract, trace-based contract language that is sufficiently expressive to model recursive calls (and loops). The central idea is to represent the control structures embodied in the control flow graph (CFG) of a program. From the CFG view two requirements on a trace-based contract language can
be derived: (i) it must be possible to specify scopes relating to procedure calls and other control structures: to this end we employ events that are semantically connected with the specified code; (ii) one must represent cyclic edges in the CFG in a finite manner: this is achieved by a least fixed-point operator. Both aspects are not new, of course. We draw from ideas first expressed in interval temporal logic [15] and the modal $\mu$-calculus [23], respectively.

However, we use the resulting logic of trace contracts differently from temporal/modal logic: like in Hoare logic [18,19] and dynamic logic [17,33], we admit explicit judgments about a concrete program: if $s$ is a program and $\Phi$ a trace contract, then $s : \Phi$ expresses that any execution of $s$ results in one of the traces characterized by $\Phi$. As mentioned above, we incorporate events into contracts to enable full abstraction from the specified code (cf. [21] who use actions to design an abstract semantics). Consequently, our logic generalizes Hoare/dynamic logic from state-based to trace-based contracts via events (which permit scoping) and recursive specifications. This sounds more complex than it is. Still, to keep the presentation intuitive and manageable, it is crucial to choose (i) a suitable deductive framework and (ii) an adequate semantics to demonstrate soundness.

Regarding the first point, we adapt a symbolic execution calculus for deductive verification in dynamic logic [1]: it reduces a program $s$ via symbolic execution to a sequence of elementary symbolic state updates $\mathcal{U}_s$ (plus path conditions), hence, it reduces judgments of the form $s : \Phi$ to $\mathcal{U}_s : \Phi$. Adding events to this formalism is easy. Dynamic logic is closed under propositional and first-order operators, which enables formulation of a fixed-point induction rule over the syntactic structure of programs. Concerning the second issue, an adequate semantics must be trace-based, i.e. it provides (a) for a given state $\sigma$ and program $s$ the trace of $s$ when started in $\sigma$ and (b) for a given trace formula $\Phi$ the set of all traces it characterizes. Since the deduction rules resemble symbolic execution of a program $s; s'$, where $s$ is the leading statement and $s'$ the remaining code, an adequate semantics is defined locally for each kind of statement $s$ and continuation $s'$. A general local, trace-based semantic framework was suggested in [8,9]—here, we specialize it to sequential programs with recursive procedures.

To summarize, the main contributions of this paper are: (1) A trace-based specification language that permits fully abstract, modular specification of recursive procedures; (2) the extension of a symbolic execution calculus for procedure-modular verification by structural induction and trace abstraction; (3) a soundness proof of the deduction rules based on a local trace semantics.

In Sect. 2 we present the local trace semantics, then define syntax and semantics of trace contracts in Sect. 3. The deductive verification system is given in Sect. 4 (for space reasons, the soundness proofs are in the appendix). Related work is in Sect. 5, conclusion with future work in Sect. 6.

2 Local Trace Semantics of Recursive Procedures

We provide a LAGC-style [8,9] trace semantics for a sequential language with recursive procedures. Its grammar is shown in Fig. 1.
The syntax of an imperative language with recursive procedures is as follows:

\[ P \in \text{Prog} := \overline{M} \{ d \ s \} \]

\[ M \in \text{ProcDecl} := m(x) \{ \text{return } e \} \]

\[ d \in \text{VarDecl} := \epsilon \ | \ x \]

\[ s \in \text{Stmt} := \text{skip} \ | \ x = e \ | \ x = m(e) \ | \ \text{if } e \{ \ s \} \ | \ s ; s \ | \ \text{while } e \{ \ s \} \]

Global lookup table: \[ G = \{ \overline{m(x) \ s} \mid m \in \text{procedures}(P) \} \]

**Fig. 1.** Syntax of an imperative language with recursive procedures

The definition of integer and boolean expressions is standard, they are assumed to be well-typed. Scopes of procedure bodies may only write to local integer variables, initialized to zero, i.e., procedure calls have no side effects. We stress that this is not a fundamental limitation, but to deal with side effects and aliasing [1,32] is orthogonal to the goals of our paper and the restriction greatly simplifies the technical issues.

**Example 1.**

We illustrate the concepts in this paper with the running example shown on the right. The behavior of procedure \( m \) is the identity function for input \( k \), but the result is computed by \( k \) (non tail-)recursive calls.

\[
m(k) \{
\]
 \r; // initialized to 0
if (k \(!=\) 0)
\{  \r = m(k \(-\) 1); \r = \r + 1 ; \}
return \r
\]

**Definition 1 (State, Update).** Let \( \text{Var} \) be a set of program variables and \( \text{Val} \) a set of values, with typical elements \( x \) and \( v \), respectively. A state \( \sigma \in \Sigma \) is a partial mapping \( \sigma : \text{Var} \rightarrow \text{Val} \) from variables to values. The notation \( \sigma[x \rightarrow v] \) expresses the update of state \( \sigma \) at \( x \) with value \( v \) and is defined as \( \sigma[x \rightarrow v](y) = v \) if \( x = y \) and \( \sigma[x \rightarrow v](y) = \sigma(y) \) otherwise.

There is a standard evaluation function \( \text{val}_\sigma \) for expressions, for example, in a state \( \sigma = [x \rightarrow 0, y \rightarrow 1] \) we have \( \text{val}_\sigma(x + y) = \text{val}_\sigma(x) + \text{val}_\sigma(y) = 0 + 1 = 1 \).

**Definition 2 (Context).** A (call) context is a pair \( \text{ctx} = (m, id) \), where \( m \) is a procedure name and \( id \) is a call identifier. Given a call identifier \( id \) we denote with \( \text{res}_id \) the unique name of the return variable associated with the call.

We define events sufficient to characterize the scope and call structure of recursive procedures, but one could define further event types, for example, input events or, in a concurrent setting, suspension events.

**Definition 3 (Event Marker).** Let \( m \) be a procedure name, \( e \) a parameter, \( id \) a call identifier, and \( v \) a return value. Then \( \text{callEv}(m, e, id) \) and \( \text{retEv}(v) \) are event markers associated with a procedure call and a return statement, respectively. We also introduce event markers associated with the start and end of a computation in context \( \text{ctx} \), defined as \( \text{pushEv}(\text{ctx}) \) and \( \text{popEv}(\text{ctx}) \), respectively. We denote with \( \text{ev}(\overline{e}) \) a generic event marker over expressions \( \overline{e} \).

**Definition 4 (Trace).** A trace \( \tau \) is defined by the following rules (where \( \epsilon \) denotes the empty trace):

\[
\tau ::= \epsilon \mid \tau \rightarrow t \quad t ::= \sigma \mid \text{ev}(\overline{t})
\]
This definition declares traces as sequences over events and states, but we need to uniquely associate an event $ev(e)$ with a state $\sigma$. This is done by inserting event $ev(e)$ into a trace between two copies of $\sigma$. We define the event trace $ev_{\sigma}(\tau)$ as $ev_{\sigma}(\tau) = \langle \sigma \rangle \cdot ev(\text{val}_{\sigma}(\tau)) \cdot \sigma$. Events do not change a state.

Sequential composition “$r;s$” of statements is semantically modeled as trace composition, where the trace from executing $r$ ends in a state from which the execution trace of $s$ proceeds. Thus the trace of $r$ ends in the same state as where the trace of $s$ begins. This motivates the semantic chop “$\ast\ast$” on traces [15,16,29] that we often use, instead of the standard concatenation operator “$\cdot$”.

**Definition 5 (Semantic Chop on Traces).** Let $\tau_1$, $\tau_2$ be traces, assume $\tau_1$ is non-empty and finite. The semantic chop $\tau_1 \ast\ast \tau_2$ is defined as $\tau_1 \ast\ast \tau_2 = \tau_1 \cdot \tau_2$, where $\tau_1 = \langle \sigma \rangle \cdot ev(\text{val}_{\sigma}(\tau_1)) \cdot \sigma$ and $\tau_2 = \langle \sigma' \rangle \cdot ev(\text{val}_{\sigma'}(\tau_2)) \cdot \sigma'$. When $\sigma \neq \sigma'$ the result is undefined. We overload the semantic chop symbol for sets of traces:

$$T_1 \ast\ast T_2 = \{ \tau_1 \ast\ast \tau_2 \mid \tau_1 \in T_1, \tau_2 \in T_1, \text{last}(\tau_1) = \text{first}(\tau_2) \}.$$  

**Example 2.** Let $\tau_1 = \langle \sigma \rangle \cdot ev(\text{val}_{\sigma}(\tau_1)) \cdot \sigma$, $\tau_2 = \langle \sigma \rangle \cdot ev(\text{val}_{\sigma}(\tau_2)) \cdot \sigma$, then $\tau_1 \ast\ast \tau_2 = \langle \sigma \rangle \cdot ev(\text{val}_{\sigma}(\tau_1)) \cdot \sigma \cdot ev(\text{val}_{\sigma}(\tau_2)) \cdot \sigma$.

Our language is deterministic, so we design local evaluation $\text{val}_{\sigma}(s)$ of a statement $s$ in state $\sigma$ to return a single trace: The result of $\text{val}_{\sigma}(s)$ is of the form $\tau \cdot K(s')$, where $\tau$ is an initial (small-step) trace of $s$ and $K(s')$ contains the remaining, possibly empty, statement $s'$ yet to be evaluated.

**Definition 6 (Continuation Marker).** Let $s$ be a program statement, then $K(s)$ is a continuation marker. The empty continuation is denoted with $K(\emptyset)$ and expresses that nothing remains to be evaluated.

The local evaluation rules defining $\text{val}_{\sigma}(s)$ are in Fig. 2. The rules for call and return emit suitable events, the rule for sequential composition assumes empty leading continuations are discarded, the remaining rules are straightforward.

We define schematic traces that allow us to succinctly characterize sets of traces (not) containing certain events via matching. The notation $\cdots$ represents the set of all non-empty, finite traces without events of type $ev \in Ev$. Symbol $\emptyset \cdots$ is shorthand for $\emptyset \cdots$ and $Ev$ includes all event types in Def. 3. With $\tau_1 \ast\ast \cdots \ast\ast \tau_3$ we denote the set of well defined traces $\tau_1 \ast\ast \cdots \ast\ast \tau_3$ so that $\tau_2 \in Ev$.

**Definition 7 (Last Event, Current Context).** Let $\tau$ be a non-empty trace.

$$\text{lastEv}(\tau) = \begin{cases} ev_{\sigma}(\tau) & \tau \in \ast\ast Ev \cdots Ev \\ \text{NoEvent} & \text{otherwise} \end{cases}$$

\[ \text{Alternatively, one could use state transitions labeled with (possibly empty) events.} \]
\[
\begin{align*}
\text{val}_\sigma(\text{skip}) &= \langle \sigma \rangle \cdot K(\emptyset) \\
\text{val}_\sigma(x = e) &= \langle \sigma \rangle \land \sigma[x \mapsto \text{val}_\sigma(e)] \cdot K(\emptyset) \\
\text{val}_\sigma(\{ s \}) &= \text{val}_\sigma(s) \\
\text{val}_\sigma(\text{return } e) &= \text{retEv}_\sigma(\text{val}_\sigma(e)) \cdot K(\emptyset) \\
\text{val}_\sigma(\text{if } e \{ s \}) &= \begin{cases} 
\langle \sigma \rangle \cdot K(s), & \text{if } \text{val}_\sigma(e) = \text{tt} \\
\langle \sigma \rangle \cdot K(\emptyset) & \text{otherwise}
\end{cases} \\
\text{val}_\sigma(\text{while } e \{ s \}) &= \begin{cases} 
\langle \sigma \rangle \land \sigma[x' \mapsto 0] \cdot K(\{ d \mid s \mid x' \leftarrow x \}) & \text{if } x' \notin \text{dom}(\sigma) \\
\text{callEv}_\sigma(m, \text{val}_\sigma(e), \text{id}) \cdot K(x = \text{res}_\text{id}) & \text{where } \text{res}_\text{id} \notin \text{dom}(\sigma)
\end{cases}
\end{align*}
\]

Fig. 2. Local Program Semantics

\[
\text{currCtx}(\tau) = \begin{cases} 
ctx & \tau \in \cdots \text{pushEv}_{\sigma}(\ctx) \cdots \text{popEv}_{\sigma}(\ctx) \\
\text{main}, \text{nul} & \text{otherwise}
\end{cases}
\]

Local evaluation of a statement \( s \) yields a small step \( \tau \) of \( s \) plus a continuation \( K(s') \). Therefore, traces can be extended by evaluating the continuation and stitching the result to \( \tau \). This is performed by composition rules that operate on a configuration of the form \( \tau, K(s) \). The process terminates when all statements are evaluated, i.e. \( K(s) = K(\emptyset) \). In this case \( \tau \) is the semantics resulting from evaluation of a program. There are three composition rules. The following rule evaluates a statement that has not been directly preceded by a call or return and extends the current trace accordingly.

\[
\frac{\sigma = \text{last}(\tau)}{\tau, K(s) \rightarrow \tau'*s', K(s')}
\]

Procedure calls and returns must be handled differently to model the change of call context and the return of the computed result. Right after a call statement was evaluated, i.e. when \( \tau \) ends with a call event, the call context is switched to the new context \( \ctx \) and the body \( s' \) of the called procedure is inlined:

\[
\frac{\begin{align*}
\ctx &= (m, \text{id}) \\
\tau &\in \cdots \text{callEv}_\sigma(m, v, \text{id})
\end{align*}}{\tau, K(s) \rightarrow \tau'*s'; \text{pushEv}_\sigma(\ctx), K(s')}
\]

Immediately after a return statement the returned value is assigned to the result variable associated with the current context and the context is switched back to the old context \( \ctx \) retrieved from \( \tau \) via matching. Together, rules (2)–(3)
model synchronous semantics of procedure calls.

\[
\text{(RETURN)} \quad \begin{align*}
\text{ctx} &= \text{currCtx}(\tau) = (m, id) \\
\tau, K(s) &\rightarrow \tau \land \sigma[\text{res}_{id} \mapsto v] \Rightarrow \text{popEv}_{\sigma}[\text{res}_{id} \mapsto v](\text{ctx}), K(s)
\end{align*}
\] (3)

Example 3. We evaluate configuration \(\langle \sigma_0 \rangle, K(x = m(1))\) for empty \(\sigma_0\), with \(m\) as defined in Expl. 1. Let \(s = mb[k \leftarrow 1]\), where \(mb\) is the body of \(m\). Applying the progress and call rule we obtain the initial rule sequence:

\[
\langle \sigma_0 \rangle, K(x = m(1)) \rightarrow \text{callEv}_{\sigma_0}(m, 1, 0), K(x = \text{res}_0) \rightarrow \text{callEv}_{\sigma_0}(m, 1, 0) \Rightarrow \text{pushEv}_{\sigma_0}(m, 0), K(s; x = \text{res}_0)
\]

It is straightforward to see that the local evaluation and the composition rules are exhaustive and deterministic, which justifies the following definition:

Definition 8 (Program Trace). Given a program \(s\) and a state \(\sigma\), the trace of \(s\) (with implicit lookup table) is the maximal sequence obtained by repeated application of composition rules, starting from \(\langle \sigma \rangle, K(s)\). If finite, it has the form \(\langle \sigma \rangle, K(s) \rightarrow \cdots \rightarrow \tau, K(\emptyset)\), also written \(\langle \sigma \rangle, K(s) \Rightarrow \tau, K(\emptyset)\).

Definition 9 (Program Semantics). The semantics of a program \(s\) is only defined for terminating programs as \([s]_{\tau} = \tau'\) if \(\tau, K(s) \Rightarrow \tau'\).

From this definition and the local semantics of sequential statements it is easy to prove Proposition 1 in the appendix by straightforward induction.

Example 4 (Continuing Example 3). Fig. 3 visualizes the semantics \([x = m(1)]_{\langle \sigma_0 \rangle}\) with the intermediate states \(\sigma_1 = \sigma_0[\tau' \mapsto 0], \sigma_2 = \sigma_1[\tau'' \mapsto 0], \sigma_3 = \sigma_2[\text{res}_1 \mapsto 0], \sigma_4 = \sigma_3[\tau' \mapsto 0], \sigma_5 = \sigma_4[\tau' \mapsto 1],\) and \(\sigma_6 = \sigma_5[\text{res}_0 \mapsto 1]\).

We only consider traces that are adequate, i.e. consistent with local evaluation and composition rules: context switches can only occur in event pairs \(\text{callEv-popEv}\) and \(\text{retEv-popEv}\) consistently with the current context, and the freshness of call identifiers has to be preserved. We define \(\text{Traces}\) as the set of all adequate traces. For details, see Appendix A.2.

3 A Logic for Trace Contracts

We present a logic for specifying properties over finite program traces. The logic is a temporal \(\mu\)-calculus [35] with two binary temporal operators, corresponding to concatenation and chop over sets of traces, respectively. We consider the syntax fragment without negation, which guarantees that fixed-point formulas indeed denote fixed points of the corresponding semantic transformers, and with least fixed-point recursion only. Then we show this logic to be suitable for expressing relevant finite-trace properties of recursive programs.
3.1 Syntax

The formulas of our logic are built from a set $\text{LVar}$ of first-order ("logical") variables and a set $\text{RecVar}$ of recursion variables. The syntax of the logic is defined by the following grammar:

$$\Phi ::= [P] | X(t) | Ev | \Phi \land \Phi | \Phi \lor \Phi | \Phi \cdot \Phi | \Phi^* \Phi | (\mu X(t).\Phi)(\overline{t})$$

where $P$ ranges over first-order predicates, $X$ over recursion variables, $\overline{y}$ over tuples of first-order variables, $\overline{t}$ over tuples of terms over first-order variables, and where in the last clause the arities of $\overline{y}$ and $\overline{t}$ agree.

Events $Ev$ have the form $\text{startEv}(m, e, i)$ or $\text{finishEv}(m, e, i)$. There are no push or pop events, but the finish events record the context. As stated in the introduction, we aim at procedure-modular specification and verification. Therefore, it is not necessary to record all context switches in a global trace, as long as corresponding calls and returns can be uniquely identified.

Remark 1. Events $Ev$ in the logic are syntactic representations of events in the local semantics (Sect. 2), thus different entities. This permits to tailor events to the desired degree of abstraction of specifications and the deduction system. For example, if one is only interested in the interface behavior of a program, but not in internal state changes, one might choose abstract events of the form $\text{callEv}(m)/\text{retEv}(m)$, where call identifiers and arguments are dropped.
**Example 5.** To illustrate our logic, we introduce a formula template (or pattern) that we make extensive use of later. Let \( m \) be a procedure name, and let \( \text{NoEv}(m) \) be a predicate that is true of a state if the latter is not an event that involves \( m \). Consider the recursive formula:

\[
\Psi_m = \mu X. (\text{NoEv}(m) \lor \text{NoEv}(m) \cdot X)
\]

It is true for any trace that is either a singleton trace which is not an event involving \( m \) (base case), or else for a trace that starts with a singleton trace that is not an event involving \( m \) and continues as a trace satisfying \( \Psi_m \) (induction case). Equivalently, \( \Psi_m \) holds for finite traces not containing any event involving \( m \).

With help of \( \Psi_m \), we define the binary logic operator \( \Phi_1 \cdot \cdot \cdot \Phi_2 \) as shorthand for \( \Phi_1 \cdot \cdot \cdot \Psi_m \cdot \cdot \cdot \Phi_2 \), expressing the trace property of satisfying \( \Phi_1 \) initially, satisfying \( \Phi_2 \) in the end, and not containing any event involving \( m \) in between. This is the logic equivalent of the semantic operator \( \cdot \cdot \cdot \) on traces introduced earlier.

### 3.2 Semantics

To define the semantics of our logic, we need a first-order variable assignment \( \beta : \text{LVar} \rightarrow D \) and a recursion variable assignment \( \rho : \text{RecVar} \rightarrow (D \rightarrow 2^\text{Traces}) \) that assigns each recursion variable to a set of traces. The (finite-trace) semantics \([\Phi]_{\beta,\rho}\) of formulas as a set of traces is inductively defined in Fig. 4.

By a *trace formula* we mean a formula of our logic that is closed with respect to both first-order and recursion variables. Since the semantics of a trace formula does not depend on any variable assignments, we sometimes use \( [\Phi] \) to denote \([\Phi]_{\beta,\rho}\) for arbitrary \( \beta \) and \( \rho \). We also permit a slight extension, where we allow trace formulas to be syntactically closed with respect to first-order operators, as long as fixed-point formulas occur only in positive positions.
3.3 Specifying Procedure Contracts

In general, a (finite-trace) procedure contract should capture the sequences of states and events that are allowed to occur as a result of a call to that procedure. Since the procedure may recursively call other procedures, such contracts need to be stated recursively. The base case(s) of a recursive contract should specify the traces that do not involve any procedure calls. The induction case(s), on the other hand, should specify the remaining traces, and use recursion variables, properly applied to arguments, at the points where calls to procedures are made.

While it can be cumbersome to propose a general pattern for procedure contracts, we illustrate the idea on a particular class of contracts that generalize traditional, Hoare-style state-based contracts to trace-based ones. As usual, we use state predicates over logical variables to formulate pre- and postconditions that express the intended relationship between the values of the program variables upon procedure call and return, but in addition, we capture the structure and arguments of recursive procedure calls. We propose the following pattern, where $m$ is the name of the specified procedure, $n$ the value of its (sole) formal parameter with which it is called, and $i$ the call identifier:

$$H_m = \mu X_m(n, i). \left( \left[ \text{pre}_{m}^{\text{base}} \right] \circledast \text{startEv}(m, n, i) \overset{m}{\circledast} \text{finishEv}(m, f_m(n), i) \circledast \left[ \text{res}_i \equiv f_m(n) \right] \right) \lor \left[ \text{pre}_{m}^{\text{step}} \right] \circledast \text{startEv}(m, n, i) \overset{m}{\circledast} X_m(\text{step}^{-1}(n), \#(i)) \overset{m}{\circledast} \text{finishEv}(m, f_m(n), i) \circledast \left[ \text{res}_i \equiv f_m(n) \right]$$

Both base and inductive case use state predicates $\text{pre}_{m}^{\text{base/step}}$ to establish the precondition. Both cases use the state predicate $\text{res}_i \equiv f_m(n)$ to specify the return value $\text{res}_i$ of $m$ as a function $f_m$ depending on the value $n$ of the formal parameter. The inductive case also specifies a recursive call to $m$. The first argument is the inverse of the recursive step function, for example, $\text{step}^{-1} = n - 1$ when $\text{step} = n + 1$. The symbol $\#$ ensures that the call context identifier is fresh.

**Example 6.** We illustrate the use of pattern $H_m$ to provide a contract for the procedure $m$ from Example 1. To specify that the returned value equals the value of the parameter, we should define $\text{pre}_{m}^{\text{base}} = n \equiv 0$, $f_m$ as the identity function, $\text{pre}_{m}^{\text{step}} = n > 0$, and $\text{step}(n) = n + 1$:

$$\mu X_m(n, i). \left( \left[ n \equiv 0 \right] \circledast \text{startEv}(m, 0, i) \overset{m}{\circledast} \text{finishEv}(m, 0, i) \circledast \left[ \text{res}_i \equiv 0 \right] \right) \lor \left[ n > 0 \right] \circledast \text{startEv}(m, n, i) \overset{m}{\circledast} X_m(n - 1, \#(i)) \overset{m}{\circledast} \text{finishEv}(m, n, i) \circledast \left[ \text{res}_i \equiv n \right]$$

The traditional big-step semantics of state-based contracts can be expressed simply as $H_m \subseteq [\text{pre}_{m}^{\text{base}} \lor \text{pre}_{m}^{\text{step}}] \circ \left[ \text{res}_i \equiv f_m(n) \right]$.

In addition to the final state, we can also specify behavior related to intermediate states. The contract in Example 6 can be extended to relate the result of the internal recursive call to the final state by replacing the trace after the
recursive call $X_m(n - 1, \#(i))$ with

$$**[\text{res}_{\#(i)} = n - 1] *^m \text{finishEv}(m, n, i) **[\text{res}_i \equiv \text{res}_{\#(i)} + 1]$$

Trace formulas (by design) completely abstract away from a specified program, to which they are only connected via events. The set $Ev$ of events can be extended if needed. It is also conceivable to expose parts of the program state in traces: Let predicate $\text{expose}(l, e, i)$ be true in all states $\sigma$, such that local variable $l$ in the call identified by $i$ has value $\beta(e)$. Then one can, for example, insert $"**[\text{expose}(r, n - 1, i)]"$ right after the recursive call to $X_m$ above to express that $r$ has the value $n - 1$ immediately after the call returns.

4 A Calculus for Deductive Verification

To verify whether a program fulfills its trace contract, we design a symbolic execution calculus that reduces programs to a sequence of syntactic state updates containing recursive calls. These are matched against a trace contract formula with a novel abstraction rule.

4.1 Updates

Updates [1] can be seen as explicit substitutions recording state changes. An elementary update assigns an expression $e$ to variable $v$, denoted by $\{v := e\}$. We also admit event updates $\{\text{Ev}(\bar{e})\}$ to record that event $\text{Ev}$ with parameters $\bar{e}$ occurred (here, $\text{Ev}$ is $\text{startEv}$ or $\text{finishEv}$, but this is generalizable). A composite update $U$ is a (possibly empty) sequence of elementary/event updates, see the grammar on top of Fig. 5. Updates $U$ precede statements $s$ with the meaning that in $Us$ statement $s$ is evaluated under the state changes embodied by $U$.

Example 7. An expression like $"\{\text{startEv}(m, 0, i)\}\{r := 0\} \text{return } r"$ results after partial symbolic execution of a procedure $m'$, where only the return statement is left to be executed in a state where $r$ is assigned value 0.

Local Evaluation. We extend the local semantic evaluation rules to programs with leading updates. Fig. 5 contains semantic rules for expressions of the form $uUs$, where $u$ is either an elementary update or an event update. If $U$ is empty then after the evaluation of $u$ the rules from Sect. 2 apply. The rules are similar to those for statements. The first rule is similar to the rule for sequential composition. Three rules evaluate elementary updates: (a) the case corresponding to assignments; (b) if a result variable $\text{res}_i$ is assigned, then the update is simply ignored: it is redundant, because its evaluation always follows $\text{finishEv}(m, e, i)$; (c) the semantics of procedure calls invokes the program semantics. The last two rules evaluate event updates occurring in the deduction rules. They represent the start and end of the execution of $m$, respectively, and generate suitable events.

Example 8. One step of local evaluation of Example 7 in a state $\sigma$ yields the expression $"\text{callEv}_\sigma(m', 0, i) **\text{pushEv}_\sigma((m', i)) \cdot K(\{r := 0\} \text{return } r)\"$. 
\[ U ::= \epsilon \mid \{ v := e \} U \mid \{ \text{Ev}(\tau) \} U \quad (\epsilon \text{ empty sequence of updates}) \]

\[ \text{val}_s(uU) = \tau \cdot K(Us), \text{ if } \text{val}_s(u) = \tau \cdot K(\emptyset) \]

\[ \text{val}_s(\{ v := e \}) = \text{val}_s(v = e), \text{ with } v \neq \text{res}, \quad \text{val}_s(\{ \text{res} := e \}) = (\sigma) \cdot K(\emptyset) \]

\[ \text{val}_s(\{ v := m(e) \}) = [v = m(e)](\sigma) \cdot K(\emptyset), \text{ when } [v = m(e)](\sigma) \text{ is defined} \]

\[ \text{val}_s(\{ \text{startEv}(m, e, i) \}) = \text{callEv}_s(m, e, i) \cdot \text{pushEv}_s((m, i)) \cdot K(\emptyset) \]

\[ \text{val}_s(\{ \text{finishEv}(m, e, i) \}) = \text{retEv}_s(\text{val}_s(e)) \cdot \text{popEv}_s[\text{res} := \text{val}_s(e)]((m, i)) \cdot K(\emptyset) \]

\[
\text{Fig. 5.} \quad \text{Update syntax and local evaluation of statements with leading updates}
\]

**Updates over Expressions.** Expressions \( e \) are evaluated in the current program state, represented by a preceding composite update \( U \). To evaluate \( e \) under \( U \), written \( U(e) \), we apply updates from inner- to outermost. Applying an elementary update \( \{ v := e' \} \) to an expression \( e \) corresponds to syntactic substitution.

Events have no effect on the value of expressions. We obtain the following rules:

\[ U'u(e) = U'(u(e)) \quad \{ v := e' \}(e) = e[v/e'] \quad \{ \text{Ev}(\_\_\_\_) \}(e) = \epsilon(e) = e \]

The trace composition is performed by applying rule (1) \( \text{(PROGRESS)} \), overloaded to support the occurrence of updates. Therefore, the semantics is defined in a similar manner as in Def. 9:

**Definition 10 (Semantics of Programs with Updates).** We define the semantics of a terminating program \((\text{undefined else})\) with leading updates as

\[ [Us]_\tau = \tau', \text{ if } \tau, K(Us) \xrightarrow{\tau} \tau', K(\emptyset), \text{ with } s \text{ possibly empty}. \]

When symbolically executing the return statement we need to generate a \( \text{finishEv} \) whose context matches the current context of the leading update. We retrieve the call context from the leading update with a helper function:

**Definition 11 (Current Context for Updates).**

\[
\text{currCtx}(U) = \begin{cases} 
(m, i) & U = U'\{\text{startEv}(m, \_\_, i)\} \\
\text{currCtx}(U') & U = U'\{v := e\} \\
\text{currCtx}(U') & U = U'\{\text{startEv}(m, \_\_, i)\}U''\{\text{finishEv}(m, \_\_, i)\} \\
(\text{main}, \text{null}) & \text{otherwise}
\end{cases}
\]

**Example 9.** The context in which the return statement of Example 7 executes is: \( \text{currCtx}(\{\text{startEv}(m', 0, i)\}\{\tau := 0\}) = \text{currCtx}(\{\text{startEv}(m', 0, i)\}) = (m', i) \).

### 4.2 Calculus for Straight-line Programs

First we present a calculus for programs that do neither contain loops nor procedure calls. Our deductive proof system is a Gentzen-style sequent calculus,
where partial symbolic execution of a program is represented by $U_s$, with $U$ the executed part and $s$ the remaining part. If we judge $U_s$ to conform to a trace specification $\Phi$ we write $U_s : \Phi$, where the judgment $U_s : \Phi$ is evaluated in a current state $\sigma$, formally:

**Definition 12 (Judgment, Assertion, Sequent).** A judgment has the shape $U_s : \Phi$, where $U$ is an update, $s$ a program statement, and $\Phi$ a trace formula. An assertion is either a closed first-order predicate $P$ or a judgment. A sequent has the shape $\Gamma \vdash U_s : \Phi$, where $\Gamma$ is a set of assertions.

**Definition 13 (Semantics of Sequents).** Let $\sigma$ be a state. A first-order predicate $P$ is true in $\sigma$ if $\sigma \models P$, as usual in first-order logic. A judgment $U_s : \Phi$ is true in $\sigma$, denoted $\sigma \models U_s : \Phi$, when $[U_s]_\sigma$ is undefined or $[U_s]_\sigma \in [\Phi]$. A sequent $\Gamma \vdash U_s : \Phi$ is true in $\sigma$, denoted $\sigma \models \Gamma \vdash U_s : \Phi$, if one of the assertions in $\Gamma$ is not true in $\sigma$ or $\sigma \models U_s : \Phi$. A sequent is valid if it is true in all states $\sigma$.

**Rules.** The rules of our calculus for the symbolic execution of straight-line programs are shown in Fig. 6. For space reasons, we omit the standard rules for $\text{Assign} \quad \Gamma \vdash U\{v := e\} s : \Phi$

$\quad \Gamma \vdash U v = e; s : \Phi$

$\text{Prestate} \quad \Gamma; P \vdash Q \quad \Gamma; P \vdash U_s : \Phi$

$\quad \Gamma \vdash U \{Q\} * \Phi$

$\text{Cond} \quad \Gamma; U(e) \vdash U_s; s' : \Phi \quad \Gamma; U(e) \vdash U_s : \Phi$

$\quad \Gamma \vdash U \text{ if } e \{s\}; s' : \Phi$

$\text{Return} \quad \text{currCtx}(U) = (m, i) \quad \Gamma \vdash U \{\text{finishEv}(m, e, i)\} \text{res}_i = e : \Phi$

$\quad \Gamma \vdash U \text{ return } e : \Phi$

$\text{VarDecl} \quad \Gamma \vdash U\{v' := 0\}\{s[v'/v]\} : \Phi \quad v' \text{ fresh for } s, \Gamma, \Phi$

$\quad \Gamma \vdash U\{v; s\} : \Phi$

Fig. 6. Sequent rules for straight-line programs

Gentzen-style first-order sequent calculi. In the following we present a number of specific rules, starting with the rule for unfolding fixed-point formulas:

**Unfold**

$\Gamma \vdash U_s : \Phi[(\mu X(\overline{y}). \Phi)/X, \overline{y}/\overline{y}]

\quad \Gamma \vdash U_s : (\mu X(\overline{y}). \Phi)(\overline{y})$

**Example 10.** We show the first step of the symbolic execution of the straight-line program “if (k!=0) { r=k−1; r=r+1; } return r” with premise $\Gamma \equiv k > 0$ and under a similar event update $U = \{\text{startEv}(m, k, i')\}$ as in Example 7. The first statement is a conditional, so rule $\text{Cond}$ is applied. Observe that $U(k!=0)$ evaluates to $k != 0$ and is subsumed by $k > 0$. We show only the left premise as the right premise is immediately closed due to $k > 0$. The final sequent is the result of applying rule $\text{Assign}$ twice, followed by an application of the $\text{Return}$ rule. An unabbreviated version is in appendix B.
Specifying and verifying trace contracts for each procedure allows us to verify procedure calls in a modular way. First we show how to specify a procedure \( m \) with a contract \( C_m \), then we present the rule ProcedureContract that is used to prove \( C_m \) in our calculus. Let

\[
C_m = \forall n, i. (\text{pre}_m(n) \rightarrow m(n) : \Phi_m(n, i) \Rightarrow [\text{res}_i = f_m(n)]) ,
\]

where \( f_m(n) \) is the result of \( m \) given input \( n \).\(^4\) Eq. (4) can be seen as the generalization of \( \forall n. (\{\text{pre}_m(n)\} m(n)\{\text{res} = f_m(n)\}) \), a state-based contract, where \( n \) is a first-order parameter. When proving correctness of trace contracts \( C_m \) of recursive procedures \( m \) [1,20,37], one establishes that \( \Phi_m \) is a specification invariant for the implementation of \( m \). One proves that the inlined body of \( m \) respects \( C_m \) and when symbolic execution arrives at a recursive call to \( m \), one can assume that \( m \) holds already. This yields partial correctness. We generalize this approach to traces in the following rule for recursive self-calls:

\[
\text{ProcedureContract} \quad \frac{\Gamma; \text{pre}_m(n'), C_m \vdash \text{inline}(m, n', i') : \Phi_m(n', i') \quad i', n' \text{ fresh}}{\Gamma \vdash C_m}
\]

The rule expresses that for any parameter \( n' \) and any recursion depth \( i' \), the trace specification \( \Phi_m(n', i') \) is an invariant for the inlined procedure body, where \( C_m \) can be assumed in recursive calls. To represent inlining succinctly, we use

\[
\text{inline}(m, e, i) = \{\text{startEv}(m, e, i)\} e' = e; mb[e'/p] ,
\]

where \( p \) is the procedure parameter, and \( e' \) is fresh.

In general, there might be more than one procedure call, so the assumption in (ProcedureCall) should actually be \( \bigwedge_{m \in \text{procedures}(p)} C_m \).

**Example 11.** Symbolic execution for procedure \( m \) from Example 1 with contract \( C_m = \forall n, i. (m(n) : \Phi_m(n, i) \Rightarrow [\text{res}_i = n]) \), with \( \mathcal{U} = \{\text{startEv}(m, n', i')\} \):

\[
\begin{align*}
\text{(here symbolic execution of procedure body starts)} \\
n' \geq 0, C_m \vdash & \mathcal{U}\{k' := n'\}\{r' := 0\} s[k/k', r/r'] : \Phi_m(n', i') \\
\vdots \\
\text{ProcedureContract} \\
n' \geq 0, & C_m \vdash \text{inline}(m, n', i') : \Phi_m(n', i') \\
\end{align*}
\]

\(^4\) For technical reasons, the formalization of contract \( C_m \) uses a procedure call outside of an assignment. Its local trace semantics \( \text{val}_m(e) = \text{callEv}_m(m, \text{val}_m(e), \text{id}) \cdot K(\emptyset) \) is the same as for procedure calls with assignment, except for the empty continuation.
4.4 Procedure Calls

As Example 11 shows, after applying rule (ProcedureContract), a procedure body can be fully symbolically executed, where recursive calls in assignments are simply handled by the (Assign) rule. The semantics of elementary updates with a call on the right ensures that this is sound provided that we make the assumption that a procedure call has no side effects. In this way, we can retrofit standard, state-based verification into our trace-based framework. Complete symbolic execution of a procedure body with a recursive call yields a judgment of the form

\[
U_1 \{ v := m(e) \} U_2 : \Phi_1 \bowtie \Phi_m(e, k) \bowtie \Phi_2.
\]

The updates are here followed by the “empty” program. For the trace specification the above shape is also justified, because typically we have \( \Phi_m = \mu X(y).(\Phi_1(y) \lor \cdots \lor \Phi_n(y)) \). This specification one strengthens into \( \mu X(y).\Phi_1(y) \), where \( \Phi \) is the specification case corresponding to the current symbolic execution path. At this point, and after symbolic execution finished, we can use pattern (6) and assumption \( C_m \) in the following trace abstraction rule:

\[
\text{TrAbs} \quad \Gamma \vdash U_1 : \Phi_1 \quad \Gamma \vdash U_1(\text{pre}_m(e)) \quad C_m \vdash \{ v := f_m(U_1(e)) \} U_2 : \Phi_2 \quad \Gamma, C_m \vdash U_1 \{ v := m(e) \} U_2 : \Phi_1 \bowtie \Phi_m(e, k) \bowtie \Phi_2
\]

where \( f_m(\cdot) \) is the function computed by \( m \). The trace abstraction rule decomposes the conclusion into three premises. The first premise verifies that the trace represented by \( U_1 \) conforms to \( \Phi_1 \). The second premise guarantees that the precondition of contract \( C_m \) is satisfied. Contract \( C_m \) is now implicitly used to guarantee that the trace of update \( \{ v := m(e) \} \) conforms to \( \Phi_m(e, k) \), up to assignment of the procedure’s result to program variable \( v \). The task of the third premise is then to ensure that the trace consisting of the latter assignment followed by update \( U_2 \) conforms to \( \Phi_2 \).

Example 12. Trace abstraction is applied in the continuation of Example 11, after the recursive call in the body was symbolically executed and moved to an update. At this point the goal sequent has the form

\[
n' > 0, C_m \vdash \{ U_1 \} \{ r' := m(n' - 1) \} U_2 : \Phi_1 \bowtie \Phi_m(n' - 1, #(i')) \bowtie \Phi_2
\]

with \( U_1 \equiv \{ \text{startEv}(m, n', i') \} \), \( U_2 \equiv \{ r' := r' + 1 \} \{ \text{finishEv}(m, n', i') \} \{ \text{res}_{1'} := r' \} \), \( \Phi_1 \equiv [n' > 0] \bowtie \text{startEv}(m, n', i') \bowtie \Phi_m(n' - 1, #(i')) \bowtie X_{n'}(n' - 1, #(i')) \), and \( \Phi_2 \equiv \bowtie \text{finishEv}(m, n', i') \bowtie \text{res}_{1'} \bowtie n' \). Applying the trace abstraction rule results

\footnote{We stress that the absence of side effects is for ease of presentation and not a fundamental limitation of our approach. How to model side effects in symbolic execution is well-known [1].}
in the following three sequents:

\[
\begin{align*}
n' > 0 & \vdash \{\text{startEv}(m, n', i')\} : [n' > 0] \ast \text{startEv}(m, n', i') \vdash m \\
n' > 0 & \vdash n' > 0 \\
\text{res}_{1'} &= n' - 1 \vdash \{r' := \text{res}_{1'}\} \{r' := r' + 1\} \{\text{finishEv}(m, r', 0)\} \{\text{res}_{1'} := r'\} : \\
& \vdash \text{finishEv}(m, n', i') \cdot [\text{res}_{1'} = n']
\end{align*}
\]

The first subgoal is provable with rule (Pprestat) and a simplification rule, the second is trivial, the third requires simplification rules found in Appendix B.

Loops. For space reasons, we do not provide rules for dealing with loops. Conceptually, it is well-known that contracts can be used to specify loops. A systematic overview and comparison between invariant-based and contract-based loop specification is in [10]. A suitable adaptation of our contract rules to the case of loops will be the topic of future work.

4.5 Soundness

**Definition 14 (Soundness)**. A rule of the calculus is sound if the validity of the conclusion follows from the validity of the premises. A calculus is sound if it can prove only valid statements.

**Theorem 1 (Calculus soundness)**. The presented sequent calculus is sound.

**Proof**. Direct consequence of the local soundness of each rule. Proof sketches for the soundness of the rules are given in Appendix A.

5 Related Work

We specify symbolic traces, so it is not surprising to find related work in extensions of LTL model checking and program synthesis. CaReT logic [2] can specify the call structure of programs which is modeled as pushdown automata. It has abstract versions of the temporal next and until operators that jump over balanced calls. The call and event structure is fixed. In [12] Systems of Procedural Automata model procedure calls with context-free rules for atomic actions and procedure calls. The goal is to learn automata from observed traces. Temporal Stream Logic [11] features uninterpreted function terms and updates in addition to standard LTL operators, aiming at program synthesis of Büchi stream automata and FPGA programs. In each case the setup is finite or propositional, not first-order. Matching and fixed-point specifications are not possible.

Process logic [16] and interval temporal logic [15] feature the chop operator, which was taken up by Nakata & Uustalu [29], who used infinite symbolic traces to characterize non-terminating loops. These were extended to a rich dynamic logic [6] and equipped with events and a local trace semantics [9].

Cyclic proof systems to prove inductive claims, including contracts of recursive procedures, date back to Hoare’s axiomatization of recursive procedures
[19]. Gurov & Westman [13] provide an abstract framework based on denotational semantics to formally justify (cyclic) procedure-modular verification, while Brotherston & Simpson [5] investigate the expressive power of sequent calculi for cyclic and infinite arguments that are often used as a basis in deductive verification. Recursive predicate specifications are standard, for example, in separation logic [31], but not used to specify program traces. Several papers [7,34] present a first-order $\mu$-calculus, but do not feature explicit programs or events.

Constructive logical frameworks [4,30] feature abstract specifications in terms of typed higher-order logic formulas, but they target functional languages and feature neither events nor traces.

6 Conclusion and Future Work

In this paper we established the fundamental theory of trace-based contracts, generalizing specification and deductive verification with state-based contracts. The ingredients are (i) an expressive fixed-point logic to characterize complex event structures over recursive procedures; (ii) a uniform, local trace-event semantics for programs, state updates, and the trace logic; (iii) a sound symbolic execution calculus with rules to prove trace contracts and programs with recursive procedures. Programs and trace contracts communicate semantically via a configurable set of events. This permits fully abstract specification of programs and a highly flexible approach to specify concepts like user input, concurrency, etc. Indeed, to harvest these opportunities arising from an expressive, trace-based specification and verification approach, will be the topic of follow-up papers, where we will look at concurrent programs, loops, and complex case studies.

Further interesting questions concern the precise expressivity of our trace logic, in particular, which events are required for completeness.

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A Proofs

A.1 Trace Adequacy

We only consider traces consistent with local evaluation and composition rules: at most one variable update occurs between two consecutive states and the events in a trace properly record call events, context switches, and return events.

**Definition 15 (Trace Adequacy).** We say \( \tau = \tau_1 * * \tau_2 \) is an adequate trace if either \( \tau_1 = \tau_2 = \sigma[x \mapsto v] \) or \( \tau_1 \) is adequate and exactly one of the following holds:

1. \( \tau_2 = \sigma[x \mapsto v] \)
2. \( \tau_2 = \text{callEv}_\sigma((m, \_), \text{id}) \) \( \tau_1 \in \text{pushEv}_\sigma((m, \_), \text{id}) \) \( \text{callEv}_\sigma((m, \_), \text{id}) \) and \( \text{lastEv}(\tau_1) \notin \{ \text{callEv}, \text{retEv} \} \)
3. \( \tau_2 = \text{retEv}_\sigma((m, \_), \text{id}) \) \( \text{lastEv}(\tau_1) \notin \{ \text{callEv}, \text{retEv} \} \)
4. \( \tau_2 = \text{pushEv}_\sigma((m, \_), \text{id}) \) \( \tau_1 \in \text{callEv}_\sigma((m, \_), \text{id}) \) \( \text{retEv}_\sigma((m, \_), \text{id}) \)
5. \( \tau_2 = \text{popEv}_\sigma((m, \_), \text{id}) \) \( \tau_1 \in \text{callEv}_\sigma((m, \_), \text{id}) \) \( \text{currCtx}(\tau_1) = (m, \_), \text{id} \).

Traces is the set of all adequate traces.

The second clause expresses that call identifiers are unique as well as one part of the requirement that call/return events are followed immediately by push/pop events. That requirement is ensured together with the remaining clauses.

The events callEv and retEv are always followed by pushEv and popEv respectively. Therefore they can be the last event of a trace only if they occur at the end of the trace. This result is formalized in the following lemma.

**Lemma 1.** If \( \langle \sigma \rangle, K(s) \xrightarrow{n} \tau, K(s') \) and callEv = lastEv(\( \tau \)) or retEv = lastEv(\( \tau \)) then \( \tau \) is adequate.

If \( \tau, K(s) \xrightarrow{n} \tau', K(s') \) in exactly \( n \) steps then we write \( \tau, K(s) \xrightarrow{n} \tau', K(s') \).

**Lemma 2.** Given a program \( s \), if \( \langle \sigma \rangle, K(s) \xrightarrow{n} \tau, K(s') \) then \( \tau \) is adequate.

**Proof.** We prove the lemma by induction on \( n \), i.e. on the number of applications of composition rules.

**Base case (n=1).** For non trivial programs, \( s \) has the form “\( x \_d \_r \)”, i.e. at least a variable \( x \) is declared. Otherwise, assignment and procedures calls cannot occur. Therefore if \( n = 1 \) then the rule applied is the progress rule.

\( \langle \sigma \rangle, K(x \_d \_r) \rightarrow \sigma[x \mapsto 0], K(x \_d \_r) \)

where \( \langle \sigma \rangle \cap \sigma[x \mapsto 0] \) satisfies condition (1) of Def. 15.

**Inductive Step.** Let’s assume as IH that \( \langle \sigma \rangle, K(s) \xrightarrow{n} \tau, K(s') \) with \( \tau \) adequate and \( s' \neq \emptyset \). Let also \( \sigma' = \text{last}(\tau) \). There are three main cases:

1. \( \tau = \tau' * * \text{callEv}_\sigma((m, \_), \_i) \): condition (4) is satisfied since applying call rule we have \( \tau, K(s') \rightarrow \tau' * * \text{callEv}_\sigma((m, \_), \_i) * * \text{pushEv}_\sigma((m, \_), \_i), K(s'') \)
– $\tau = \tau' \ast \ast \text{retEv}_{\sigma'}(\_)$: condition (5) is satisfied since applying \textit{return rule} we have

$$\tau, K(s') \rightarrow \tau' \ast \ast \text{retEv}_{\sigma'}(\_), \ast \ast \text{popEv}_{\sigma'}((m, id)), K(s'')$$

where $\text{currCtx}(\tau') = (m, id)$ by definition.

– $\tau \neq \tau' \ast \ast \text{callEv}_{\sigma'}$ and $\tau \neq \tau' \ast \ast \text{retEv}_{\sigma'}$: the \textit{progress rule} applies and we have three cases

- $\langle \sigma \rangle, K(s) \xrightarrow{n} \tau, K(x := m(\tau); r)$: a \text{callEv} with a fresh call identifier is generated. Since, by Lemma 1 we have that $\text{last}(\tau) \notin \{\text{callEv}, \text{retEv}\}$ condition (2) is satisfied.

- $\langle \sigma \rangle, K(s) \xrightarrow{n} \tau, K(\text{return } e; r)$: a \text{retEv} is generated. Since, by Lemma 1 we have that $\text{last}(\tau) \notin \{\text{callEv}, \text{retEv}\}$ condition (2) is satisfied.

- Otherwise we have $\tau, K(s') \rightarrow \tau', K(s'')$) where either $\tau' = \tau$ is adequate by IH, or $\tau' = \tau \lhd \sigma'[x \mapsto v]$, that satisfies condition (1).

Therefore, given

$$\langle \sigma \rangle, K(s) \xrightarrow{n} \tau, K(s')$$

with $\tau$ adequate we have

$$\langle \sigma \rangle, K(s) \xrightarrow{n+1} \tau', K(s'')$$

where $\tau'$ is adequate.

\begin{theorem}[Adequacy of Program Semantics] The program semantics $\llbracket s \rrbracket_{\langle \sigma \rangle}$ is an adequate trace for any state $\sigma$ and terminating program $s$.
\end{theorem}

\begin{proof}
From Lemma 2 it follows that given a program $s$ if $\langle \sigma \rangle, K(s) \xrightarrow{*} \tau, K(\emptyset)$ then $\tau$ is adequate. In other words $\llbracket s \rrbracket_{\langle \sigma \rangle}$ is adequate.
\end{proof}

\section*{A.2 Soundness of the Calculus}

\begin{proposition}
$\llbracket r; s \rrbracket_{\tau} = \tau' \ast \ast \llbracket s \rrbracket_{\tau'}$, with $\tau' = \llbracket r \rrbracket_{\tau}$.
\end{proposition}

\begin{proposition}
$\llbracket U s \rrbracket_{\tau} = \llbracket U \rrbracket_{\tau} \ast \ast \llbracket s \rrbracket_{\llbracket U \rrbracket_{\tau}}$, and $\llbracket U \udash \llbracket U \udash' \rrbracket_{\tau} = \llbracket U \rrbracket_{\tau} \ast \ast \llbracket U' \rrbracket_{\llbracket U \udash \rrbracket_{\tau}}$, and $\llbracket \{ v := m(e) \} \rrbracket_{\tau} = \llbracket \{ v := m(e) \} \rrbracket_{\text{last}(\tau)}$.
\end{proposition}

\begin{theorem}[Soundness] The presented sequent calculus is sound.
\end{theorem}

\begin{proof}
The result is a direct consequence of the (local) soundness of each rule, which we show here for selected rules. Recall that a rule is sound if its conclusion is a valid sequent whenever all its premises are. For some rules we also show \textit{reversibility} (also called \textit{backward soundness}), which means that whenever the conclusion of the rule is a valid sequent, the rule can be applied backwards in a way so that all premises are valid.

\end{proof}
Rule Assign. We shall prove soundness and reversibility of the rule. Let \( \sigma \) be a state. We have, with \( \tau = [U]_{(\sigma)} \):

\[
\begin{align*}
s & \models U\{v := e\}s : \Phi \\
\iff & [U\{v := e\}s]_{(\sigma)} \in [\Phi] \\
\iff & \tau^{*\star}[\{v := e\}]_{\tau}^{*\star}[s]_{\tau} \in [\Phi] \quad \text{(Def. 13)} \\
\iff & \tau^{*\star}[v := e]^{*\star}[s]_{\tau} \in [\Phi] \quad \text{(Def. val}_{\tau}\{\{v := e\}\}) \\
\iff & [Uv = e]_{\tau}s_{(\sigma)} \in [\Phi] \quad \text{(Prop. 1.2)} \\
\iff & \sigma \models Uv = e; s : \Phi \quad \text{(Def. 13)} 
\end{align*}
\]

We therefore have that the sequent \( \Gamma \vdash \{v := e\}s : \Phi \) is valid if and only if the sequent \( \Gamma \vdash v = e; s : \Phi \) is valid.

Rule Cond. We shall prove soundness and reversibility of the rule. Let \( \sigma \) be a state. We have, with \( \tau = [U]_{(\sigma)} \):

\[
\begin{align*}
s & \models U(e) \Rightarrow \sigma \models Us; s' : \Phi \\
& \land \sigma \models U(\ell e) \Rightarrow \sigma \models Us' : \Phi \\
\iff & \sigma \models U(e) \Rightarrow [Us; s']_{(\sigma)} \in [\Phi] \\
& \land \sigma \models U(\ell e) \Rightarrow [Us']_{(\sigma)} \in [\Phi] \\
\iff & \sigma \models U(e) \Rightarrow \tau^{*\star}[s; s']_{\tau} \in [\Phi] \\
& \land \sigma \models U(\ell e) \Rightarrow \tau^{*\star}[s']_{\tau} \in [\Phi] \\
\iff & \text{val}_{\text{last}(\tau)}(e) = \text{tt} \Rightarrow \tau^{*\star}[s; s']_{\tau} \in [\Phi] \\
& \land \text{val}_{\text{last}(\tau)}(e) = \text{ff} \Rightarrow \tau^{*\star}[s']_{\tau} \in [\Phi] \quad \text{(Def. } \sigma \models U(e)\}) \\
\iff & \text{U if } e \{s\}; s'_{(\sigma)} \in [\Phi] \quad \text{(Prop. 2)} \\
\iff & \sigma \models U\text{ if } e \{s\}; s' : \Phi \quad \text{(Def. 13)} 
\end{align*}
\]

Explanation: Given Fig. 2, Def. 9 we have

\[
\begin{align*}
\tau, K(\text{if } e \{s\}; s') & \rightarrow_{\tau} \tau, K(s; s') \quad \text{if } \text{val}_{\text{last}(\tau)}(e) = \text{tt}, \text{ and} \\
\tau, K(\text{if } e \{s\}; s') & \rightarrow_{\tau} \tau, K(s') \quad \text{if } \text{val}_{\text{last}(\tau)}(e) = \text{ff}
\end{align*}
\]

Therefore

\[
\begin{align*}
[[\text{if } e \{s\}; s']_{\tau} & = [s; s']_{\tau} \quad \text{if } \text{val}_{\text{last}(\tau)}(e) = \text{tt}, \text{ and} \\
[[\text{if } e \{s\}; s']_{\tau} & = [s']_{\tau} \quad \text{if } \text{val}_{\text{last}(\tau)}(e) = \text{ff}
\end{align*}
\]

We therefore have that the sequent \( \Gamma \vdash U\text{ if } (e) s; s' : \Phi \) is valid if and only if the sequents \( \Gamma, U(e) \vdash Us; s' : \Phi \) and \( \Gamma, U(\ell e) \vdash Us' : \Phi \) are valid.

Rule Unfold. We shall prove soundness and reversibility of the rule. By Tarski’s fixed-point theorem for complete lattices [36], the semantics \( [(\mu X(\tau).\Phi)/(\tau)]_{\beta, \rho} \) of a fixed-point predicate \( \mu X(\tau).\Phi \) is indeed a fixed point of the trace predicate transformer \( \lambda F.\Delta \{\Phi\}_{\beta[\tau \rightarrow \tau], \rho[X \rightarrow F]} \). We therefore have the following fixed-point unfolding equivalence:

\[
(\mu X(\tau).\Phi)(\tau) \equiv \Phi[(\mu X(\tau).\Phi)/X, \tau/\tau]
\]
where $\Phi_1 \equiv \Phi_2$ is defined to hold whenever $[\Phi_1]_{\beta,\rho} = [\Phi_2]_{\beta,\rho}$ for all $\beta$ and $\rho$. The soundness and reversibility of the rule are a direct consequence of this equivalence.

**Rule ProcedureContract.** Because the details are somewhat technical, soundness is only be sketched here. We follow the approach taken in [37] to prove the soundness of a similar rule, but in the context of Hoare logic. The essence of the approach is to find a suitable notion of validity of sequents that allows to capture an inductive argument on the recursive depth of procedure calls. In [37], this is achieved by augmenting the notion of sequent validity with an explicit parameter $n$ of that depth, in turn relying on a modified version of the operational semantics that is also parameterised on $n$ as a bound on the maximal recursion depth when going from an initial state to a final one. Here, we follow the same approach, and apply it to traces.

**Rule TrAbs.** We prove soundness of the rule.

\[
\frac{\Gamma \vdash U_1 : \Phi_1 \quad \Gamma \vdash U_1 (\text{pre}_m(e)) \quad C_m \vdash \{ v := f_m(U_1(e)) \} U_2 : \Phi_2}{\Gamma, C_m \vdash U_1 \{ v := m(e) \} U_2 : \Phi_1 * \Phi_m(e,k) ** \Phi_2}
\]

where $f_m(\cdot)$ is the function computed by $m$.

Assuming the premisses (I)–(III) (from left to right) are valid, we have to show that the conclusion is valid. This means that for all states $\sigma$

\[
\sigma \models (\Gamma \land C_m) \rightarrow U_1 \{ v := m(e) \} U_2 : \Phi_1 * \Phi_m(e,k) ** \Phi_2
\]

holds. We consider only the non-trivial case where $\sigma \models \Gamma \land C_m$ holds. This means we have to prove that

\[
[U_1 \{ v := m(e) \} U_2]_{(\sigma)} \in [\Phi_1 * \Phi_m(e,k) ** \Phi_2]
\]

We can decompose the left side as follows:

\[
[U_1 \{ v := m(e) \} U_2]_{(\sigma)} = \tau' \cdot [v = m(e)]_{\tau'} \cdot \tau'', \text{ with } [U_1]_{(\sigma)} = \tau \text{ and } [U_2]_{\tau'} = \tau''
\]

Validity of premise (I) ensures already that $\tau \in [\Phi_1]_{(\sigma)}$.

For the middle part, we observe that

\[
[v = m(e)]_{\tau} = \frac{\text{callEv}_{\text{last}(\tau)}(m,e,k) * \text{pushEv}_{\text{last}(\tau)}((m,i)) * \text{mb}[p \rightarrow e]; v = \text{res}_k]}{\tau'}
\]

with $\tau' = [m(e); \tau]$.

By assumption $\sigma \models C_m$, i.e.,

\[
\sigma \models \forall n, i. (\text{pre}_m(n) \rightarrow m(n) : \Phi_m(n,i) * \text{res}_i \models \text{f}_m(n))
\]
and hence,
\[ \sigma \models \text{pre}_m(e_1) \rightarrow m(e_1) : \Phi_m(e_1, k) \ast \ast [\text{res}_k \doteq f_m(e_1)] \]
with \( \text{val}_\sigma(e_1) = \text{val}_\sigma(U_1 e) \) and \( e_1 \) fresh rigid constant symbol and \( i \) instantiated with \( k \). Validity of premise (II) asserts that
\[ \sigma \models U_1 \text{pre}_m(e) \equiv \sigma \models \text{pre}_m(U_1 e) \equiv \sigma \models \text{pre}_m(e_1) . \]

Consequently (modus ponens),
\[ \sigma \models m(e_1) : \Phi_m(e_1, k) \ast \ast [\text{res}_k \doteq f_m(e_1)] \]
\[ \equiv [m(e_1)]_{\sigma} \in [\Phi_m(e_1, k) \ast \ast [\text{res}_k \doteq f_m(e_1)]]_{\sigma} \]
\[ \Rightarrow [m(e_1)]_{\sigma} \in [\Phi_m(e_1, k)]_{\sigma} \]

Because there are no side effects from procedure calls on the state, we have that if for any two states \( \sigma, \sigma' \)
\[ \text{val}_\sigma(e_1) = \text{val}_{\sigma'}(e) \quad \text{and} \quad \text{val}_\sigma(k) = \text{val}_{\sigma'}(k) \]
then
\[ [m(e)]_{\sigma} = [m(e)]_{\sigma'} \quad \text{and} \quad [\Phi_m(e_1, k)]_{\sigma} = [\Phi_m(e_1, k)]_{\sigma'} . \]

Thus we can deduce:
\[ [m(e_1)]_{\sigma} = [m(e)]_{\text{last}(\tau)} = [m(e)]_{\tau} = \hat{\tau} , \quad \text{and further,} \quad \hat{\tau} \in [\Phi_m(e_1, k)]_{\tau} . \]

In summary, we have now that \( \tau \cdot \hat{\tau} \in [\Phi_1 \ast \ast \Phi_m(e_1, k)]_{\sigma} \). We have not yet considered the whole trace of the procedure call update. Remember:
\[ [v = m(e)]_{\tau} = [m(e); v = \text{res}_k]_{\tau} = [m(e); \text{res}_k]_{\tau} \ast \ast [v = \text{res}_k]_{\hat{\tau}} \]

It remains to show that
\[ [v = \text{res}_k]_{\tau} \ast \ast [U_2]_{\hat{\tau}} \in [\Phi_2]_{\hat{\tau}} \]

This is a direct consequent of premise (III), the only critical point being the equality of the value of variable \( v \). This follows from the fact that in the conclusion \( v \) has the value computed by the procedure when called with parameters \( U_1(e) \) which is the same value to which \( f_m(U_1(e)) \) evaluates by definition of \( f_m \).

\[ \Box \]

B Extended Examples

**Example 13 (Extended Version of Example 10).** Here is the symbolic execution of the straight-line program “s = if (k!=0) {r=k-1; r=r+1;} return r” with premise \( \Gamma := k > 0 \). For well-formedness, since the program ends with a return statement it has to be preceded by a leading update \( \{ \text{startEv}(m, k, i') \} \mathcal{U} \). We assume \( \mathcal{U} \) to be empty.
(symbolic execution finished)

| Rule               | Premise                                                                 | Conclusion          |
|--------------------|-------------------------------------------------------------------------|---------------------|
| Assign             | $k > 0 \vdash U_1 \{r := k - 1\} \{r := r + 1\} \{\text{finishEv}(m, r, i')\} \{\text{res}_{i'} := r\} : \Phi$ | $k > 0 \vdash U_1 \{r := k - 1\} \{r := r + 1\} \{\text{finishEv}(m, r, i')\} \{\text{res}_{i'} := r\} : \Phi$ |
| Return             | $k > 0 \vdash U_1 \{r := k - 1\} \{r := r + 1\} \{\text{finishEv}(m, r, i')\} \{\text{res}_{i'} := r\} : \Phi$ | $k > 0 \vdash U_1 \{r := k - 1\} \{r := r + 1\} \{\text{finishEv}(m, r, i')\} \{\text{res}_{i'} := r\} : \Phi$ |

**Example 14 (Extended Version of Example 11).** Symbolic execution for procedure $m$ from Example 1 with contract $C_m = \forall n, i. (m(n) : \Phi_m(n, i) \ast \text{res}_i = n)$, with $U = \{\text{startEv}(m, n, i')\}$:

```
(symbolic execution of procedure body starts)
```

| Rule               | Premise                                                                 | Conclusion          |
|--------------------|-------------------------------------------------------------------------|---------------------|
| VarDecl            | $n' \geq 0, C_m \vdash U\{k' := n'\}\{r' := 0\} s[k/k', r/r'] : \Phi_m(n', i')$ | $n' \geq 0, C_m \vdash U\{k' := n'\}\{r; s[k/k']\} : \Phi_m(n', i')$ |
| Assign             | $n' \geq 0, C_m \vdash U\{k' := n'\}\{r; s[k/k']\} : \Phi_m(n', i')$ | $n' \geq 0, C_m \vdash inline(m, n', i') : \Phi_m(n', i')$ |

**Example 15 (Extended Version of Example 12).** We provide some of the needed update simplification rules:

| Rule               | Premise                                                                 | Conclusion          |
|--------------------|-------------------------------------------------------------------------|---------------------|
| elimUpdate_1       | $\Gamma \vdash U : \Phi, \Delta \quad \Gamma \vdash U\{x := e\} : \Phi.e, \Delta$ | $\Gamma \vdash U\{x := e\} : \Phi.e, \Delta$ |
| elimUpdate_2       | $\Gamma \vdash U : \Phi, \Delta \quad \Gamma \vdash U\{\text{finishEv}(m, e, i)\} : \Phi \ast \text{finishEv}(m', e', i'), \Delta$ | $\Gamma \vdash U\{\text{finishEv}(m, e, i)\} : \Phi \ast \text{finishEv}(m', e', i'), \Delta$ |
| subsumeUpdates_1   | $\Gamma \vdash U_1 : \Phi, \Delta \quad \Gamma \vdash U_1U_2 : \Phi \ast U_2, \Delta$ | $\Gamma \vdash U_1U_2 : \Phi \ast U_2, \Delta$ if $U_2$ does not contain any update involving $m$ |