Principal pivot transforms: properties and applications

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Abstract

The principal pivot transform (PPT) of a matrix $A$ partitioned relative to an invertible leading principal submatrix is a matrix $B$ such that

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad \text{if and only if} \quad B \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix},$$

where all vectors are partitioned conformally to $A$. The purpose of this paper is to survey the properties and manifestations of PPTs relative to arbitrary principal submatrices, make some new observations, present and possibly motivate further applications of PPTs in matrix theory. We pay special attention to PPTs of matrices whose principal minors are positive.

Key words: pivot transform, principal submatrix, P-matrix, inverse, iterative method

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1 Introduction

Suppose that $A \in M_n(\mathbb{C})$ (the $n$-by-$n$ complex matrices) is partitioned in blocks as
\begin{equation}
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}
\end{equation}
and further suppose that $A_{11}$ is an invertible submatrix. Consider the matrix
\begin{equation}
B = \begin{pmatrix} (A_{11})^{-1} & -(A_{11})^{-1}A_{12} \\ A_{21}(A_{11})^{-1} & A_{22} - A_{21}(A_{11})^{-1}A_{12} \end{pmatrix}.
\end{equation}
The matrices $A$ and $B$ are related as follows: If $x = (x_1^T, x_2^T)^T$ and $y = (y_1^T, y_2^T)^T$ in $\mathbb{C}^n$ are partitioned conformally to $A$, then (see Theorem 3.1)
\begin{equation*}
A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}
\end{equation*}
if and only if
\begin{equation*}
B \begin{pmatrix} y_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_2 \end{pmatrix}.
\end{equation*}
The operation of obtaining $B$ from $A$ has been encountered in several contexts. Tucker [15] considers an equivalence relation among rectangular matrices, which is implicitly determined by a nonsingular (not necessarily principal) submatrix and is defined as follows: two $m$ by $n$ matrices $A$ and $C$ are combinatorially equivalent if there is a one-to-one correspondence between the sets of ordered pairs $\{(x, y) \mid y = Ax\}$ and $\{(u, v) \mid v = Cu\}$, given via a permutation matrix $P$ of order $m + n$ by $(x^T, y^T)^T = P (u^T, v^T)^T$. It is shown in [15] that $C$ is combinatorially equivalent to $A$ in (1.1) if and only if $C$ is, up to independent permutations of its rows and columns, equal to $B$ in (1.2) with the signs of the off-diagonal blocks reversed. The matrix $C$ is referred to as a pivotal transform of $A$. When the equivalence relation is determined by a principal submatrix, Tucker [16] refers to $C$ as a principal pivotal transform of $A$ and asserts that if $A$ has positive principal minors (that is, if $A$ is a P-matrix), then so does every principal pivotal transform of $A$ (see Theorem 5.3).

In the sequel we will adopt the more commonly used term of ‘principal pivot transform’.

Tucker’s motivation for introducing combinatorial equivalence and studying principal pivot transforms is rooted in an effort to generalize Dantzig’s simplex method from ordered to general fields. In turn, the domain-range relation between $A$ and $B$ observed by Tucker is later used by Cottle and Dantzig [5] as an important feature of their “principal pivoting algorithm” for the linear complementarity problem when the coefficient matrix is a real P-matrix. In that algorithm, principal pivot transforms are used to exchange the role of basic and nonbasic variables of the problem and the fact that principal pivot transformations preserve P-matrices is applied effectively. Principal pivot transforms have since found similar uses in the context of mathematical programming (see e.g., Pang [13]).

The relation between $A$ and $B$ above prompted Stewart and Stewart [14] to refer to $B$ as the exchange of $A$ (exc($A$)). The authors use exchanges in order to generate S-orthogonal matrices from hyperbolic Householder transformations, and then apply them to solve the mixed Cholesky updating/downdating problem. In [14] it is also noted that this method of construction of S-orthogonal matrices is a folk result in circuit theory and a reference to Belovitch [1] is made for a special case.
In Johnson and Tsatsomeros [11], a fundamental matrix factorization of the principal pivot transform turns up in a discussion of row-interval nonsingularity and the relation to P-matrices. We review this factorization in Lemma 3.4 and take the opportunity to provide a proof valid for complex matrices of a result claimed in [11] (see Remark 5.4). In a related vein, Elsner and Szulc [8] introduce a generalization of P-matrices to block P-matrices and show that a certain class of block P-matrices is left invariant under principal pivot transformations.

The principal pivot transform also appears under the term *gyration* in Duffin, Hazony, and Morrison [7], and is mentioned in a survey of Schur complements by Cottle [4].

The above varied interest for principal pivot transforms motivates us here to survey and further study their general properties. We will discuss the determinants, the eigenvalues and other basic characteristics of principal pivot transforms relative to arbitrary principal submatrices. The relation and parallelism of the principal pivot transformation to inversion will also be considered, as well as a potential application to iterative techniques for solving linear systems (see sections 3 and 4). We will also discuss matrix classes left invariant under principal pivot transformations, including the aforementioned P-matrices and S-orthogonal matrices (see section 5).

## 2 Notation and preliminaries

Let $n$ be a positive integer and $A \in M_n(\mathbb{C})$. The $i$-th entry of a vector $x$ is denoted by $x(i)$. In the remainder the following notation is also used:

- $\langle n \rangle = \{1, 2, \ldots, n\}$. For any $\alpha \subseteq \langle n \rangle$, the cardinality of $\alpha$ is denoted by $|\alpha|$ and $\alpha^c = \langle n \rangle \setminus \alpha$.

- $A[\alpha, \beta]$ is the submatrix of $A$ whose rows and columns are indexed by $\alpha, \beta \subseteq \langle n \rangle$, respectively; the elements of $\alpha, \beta$ are assumed to be in ascending order. When a row or column index set is empty, the corresponding submatrix is considered vacuous and by convention has determinant equal to 1.

- $A[\alpha] = A[\alpha, \alpha]$, $A(\alpha, \beta) = A[\alpha^c, \beta]$; analogously we define $A[\alpha, \beta]$, $A(\alpha, \beta)$ and $A(\alpha)$.

- $A/A[\alpha]$ is the *Schur complement* of an invertible principal submatrix $A[\alpha]$ in $A$, namely, $A/A[\alpha] = A(\alpha, \alpha) - A(\alpha, \alpha)(A[\alpha, \alpha])^{-1}A(\alpha, \alpha)$. It is well known that $\det(A/A[\alpha]) = \det A / \det(A[\alpha])$.

- $\sigma(A)$ is the spectrum and $\rho(A)$ the spectral radius of $A$.

- $\text{diag}(d_1, \ldots, d_n)$ is the diagonal matrix in $M_n(\mathbb{C})$ with diagonal entries $d_1, \ldots, d_n$.

### Definition 2.1

Given $\alpha \subseteq \langle n \rangle$ and provided that $A[\alpha]$ is invertible, we define the *principal pivot transform* of $A \in M_n(\mathbb{C})$ relative to $\alpha$ as the matrix $\text{ppt}(A, \alpha)$ obtained from $A$ by replacing
By convention, if $\alpha = \emptyset$, then $\text{ppt}(A, \alpha) = A$.

The principal pivot transform is related but distinct from the following block representation of the inverse (obtained by combining formulas in [3] and [18]; see also [10, section 0.7.3]): Given an invertible $A \in M_n(\mathbb{F})$ and $\alpha \subseteq \langle n \rangle$ such that $A[\alpha]$ and $A(\alpha)$ are invertible, $A^{-1}$ is obtained from $A$ by replacing

$$
A[\alpha] \text{ by } (A/A(\alpha))^{-1}, \quad A(\alpha, \alpha) \text{ by } -A[\alpha]^{-1}A[\alpha, \alpha](A/A[\alpha])^{-1},
$$

$$
A(\alpha, \alpha) \text{ by } (A/A[\alpha])^{-1}A(\alpha, \alpha)A[\alpha]^{-1}, \quad \text{ and } A(\alpha) \text{ by } (A/A[\alpha])^{-1}.
$$

In our subsequent discussion, we will also use an easy to verify determinantal formula for $A + D$, where $D = \text{diag}(d_1, \ldots, d_n)$, namely,

$$
\det(A + D) = \sum_{\alpha \subseteq \langle n \rangle \setminus 0} \prod_{i \notin \alpha} d_i \det A[\alpha]. \quad (2.1)
$$

### 3 Basic properties of principal pivot transforms

We begin with a formal statement of the basic domain-range exchange property of $\text{ppt}(A, \alpha)$, and include a proof sketch for the sake of completeness.

**Theorem 3.1** Let $A \in M_n(\mathbb{F})$ and $\alpha \subseteq \langle n \rangle$ so that $A[\alpha]$ is invertible. Given a pair of vectors $x, y \in \mathbb{F}^n$, define $u, v \in \mathbb{F}^n$ by $u[\alpha] = y[\alpha], \ u(\alpha) = x(\alpha), \ v[\alpha] = x[\alpha], \ v(\alpha) = y(\alpha)$. Then $B = \text{ppt}(A, \alpha)$ is the unique matrix with the property that for every such $x, y, y = Ax$ if and only if $Bu = v$. Moreover, $\text{ppt}(B, \alpha) = A$.

**Proof.** Consider the permutation matrix $P$ for which

$$
Px = \begin{pmatrix} x[\alpha] \\ x(\alpha) \end{pmatrix} \quad \text{and} \quad PAP^T = \begin{pmatrix} A[\alpha] & A[\alpha, \alpha] \\ A(\alpha, \alpha) & A(\alpha) \end{pmatrix}.
$$

By the construction outlined in Definition 2.1 and on letting $B = \text{ppt}(A, \alpha)$, we have

$$
PBP^T = \begin{pmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A[\alpha, \alpha] \\ A(\alpha, \alpha)A[\alpha]^{-1} & A/A[\alpha] \end{pmatrix}.
$$

Then, with $u$ and $v$ as prescribed, it can be easily verified that $PAP^T(Px) = Py$ if and only if $Pu = PBP^T(Pu)$, or equivalently, $Ax = y$ if and only if $Bu = v$. To show uniqueness, suppose that $B'u = v$ if and only if $Ax = y$. Then $(B - B')u = 0$ for all $u$ such that $u[\alpha] = y[\alpha] = A[\alpha]x[\alpha] + A[\alpha, \alpha]x(\alpha)$ and $u(\alpha) = x(\alpha)$. As $A[\alpha]$ is invertible and $x$ is chosen freely, it follows that $(B - B')u = 0$ for all $u \in \mathbb{F}^n$, that is $B = B'$. To see that $\text{ppt}(B, \alpha) = A$, notice that $\text{ppt}(B, \alpha)x = Ax$ for all $x \in \mathbb{F}^n$. \( \blacksquare \)
It is interesting to note in the next theorem that in certain cases, consecutive principal pivot transforms result into the inverse of a matrix.

**Theorem 3.2** Let $A \in M_n(\mathbb{C})$ and suppose that there exists a partition of $\langle n \rangle$ into subsets $\alpha_i$, $i = 1, 2, \ldots, k$ so that the sequence of matrices

$$A_0 = A, \quad A_i = \text{ppt}(A_{i-1}, \alpha_i), \quad i = 1, 2, \ldots, k$$

is well defined (i.e., the matrices $A_{i-1}[\alpha_i]$ are invertible). Then $A$ is invertible and $A^{-1} = A_k$.

**Proof.** By Theorem 3.1 applied to each of the $A_i$ in sequence, and since the $\alpha_i$ are mutually disjoint and their union is $\langle n \rangle$, we have that $Ax = y$ if and only if $A_ky = x$ for all $x, y \in \mathbb{C}^n$. It follows that $A$ is invertible and by uniqueness of the inverse that $A_k = A^{-1}$.

**Remark 3.3** In [15] it is observed that $A^{-1} \in M_n(\mathbb{C})$ can be found with a sequence of at most $n$ principal pivot transforms (and by interchanging rows or columns if needed). Adopting the definition of a flop as the time required to execute $x = x + t \ast x$, we can compare such a method of inversion of $A \in M_n(\mathbb{R})$ with solving the $n$ linear systems $Ax = e_i$ via the LU factorization of $A$. The latter method of inversion of $A$ entails $n^3/3 + n \cdot n^2/2 = 5n^3/6$ flops. Suppose now that the partition $\alpha_i = \{i\}$, $i = 1, 2, \ldots, n$ of $\langle n \rangle$ yields, as in Theorem 3.2, the inverse of $A$. In the process, it is firstly required to compute $A_1 = \text{ppt}(A, \alpha_1)$, which entails $2(n-1)$ divisions (for the off-diagonal blocks), $(n-1)^2$ multiplications for the computation of the Schur complement (which is a rank one update of $A(\alpha)$), and 1 division for the calculation of $1/a_{11}$. The total is therefore $n^2$ flops for the calculation of $A_1$. Thus to find the inverse by calculating $A_n$, the required flop count is

$$n^2 + (n-1)^2 + \ldots + 2^2 = \frac{n(n+1)(2n+1)}{6} - 1.$$

It follows that there is an economization of $(3n^3 - 3n^2 - n + 6)/6$ flops over inversion via LU factorization that can be realized e.g., when the inverse of a P-matrix is sought (cf. 5.2).

To study further the basic properties of $\text{ppt}(A, \alpha)$, we continue with a useful observation that appears implicitly in the proof of [15, Theorem 4] and in [11].

**Lemma 3.4** Let $A \in M_n(\mathbb{C})$ and $\alpha \subseteq \langle n \rangle$ so that $A[\alpha]$ is invertible. Let $T_1$ be the matrix obtained from the identity by setting the diagonal entries indexed by $\alpha$ equal to 0. Let $T_2 = I - T_1$ and consider the matrices $C_1 = T_2 + T_1A$, $C_2 = T_1 + T_2A$. Then $\text{ppt}(A, \alpha) = C_1C_2^{-1}$.

**Proof.** Without loss of generality, we can assume that $\alpha = \langle k \rangle$ (otherwise we can apply our argument to a permutation similarity of $A$). Observe then that

$$C_1 = \begin{pmatrix} I & 0 \\ A(\alpha, \alpha) & A(\alpha) \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} A[\alpha] & A[\alpha, \alpha] \\ 0 & I \end{pmatrix}.$$
and thus
\[ C_1C_2^{-1} = \begin{pmatrix} I & 0 \\ A(\alpha) & 0 \end{pmatrix} \begin{pmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A(\alpha, \alpha) \\ 0 & I \end{pmatrix} = \begin{pmatrix} A[\alpha]^{-1} & -A[\alpha]^{-1}A(\alpha, \alpha) \\ A(\alpha)A[\alpha]^{-1} & A/A[\alpha] \end{pmatrix} = \text{ppt}(A, \alpha). \]

**Definition 3.5** Referring to the matrices of Lemma 3.4, we call \( \text{ppt}(A, \alpha) = C_1C_2^{-1} \) the basic factorization of \( \text{ppt}(A, \alpha) \).

In connection with a remark added in proof in [15], we have the following result that sheds more light on the combinatorial relationship between a matrix and its principal pivot transforms.

**Theorem 3.6** Let \( A \in M_n(\mathbb{F}) \) and \( \alpha \subseteq \langle n \rangle \) so that \( A[\alpha] \) is invertible. Let \( B = \text{ppt}(A, \alpha) \) and \( I \) be the identity in \( M_n(\mathbb{F}) \). Then there exists a permutation matrix \( P \in M_{2n}(\mathbb{F}) \) such that
\[
(-B \quad I) \quad P \begin{pmatrix} I \\ A \end{pmatrix} = 0.
\]

Moreover, if \( T_1 \) and \( T_2 \) are as in Lemma 3.4, then
\[
P = \begin{pmatrix} T_1 & T_2 \\ T_2 & T_1 \end{pmatrix}.
\]

Conversely, if \( B = \text{ppt}(A, \alpha) \) for some \( \alpha \subseteq \langle n \rangle \), then (3.1) holds for an appropriately defined permutation matrix \( P \in M_{2n}(\mathbb{F}) \).

**Proof.** In the notation of Lemma 3.4 we have that \( B = \text{ppt}(A, \alpha) \) if and only if
\[
(-B \quad I) \begin{pmatrix} C_2 \\ C_1 \end{pmatrix} = 0.
\]

The claims of the theorem follow by substituting \( C_1 = T_2 + T_1A \) and \( C_2 = T_1 + T_2A \). That \( P \) as above is a permutation matrix follows from the fact that \( T_1 + T_2 = I \).

**Example 3.7** To illustrate the definitions and observations so far, let \( \alpha = \{1, 3\} \) so that
\[
A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 8 & 1 \end{pmatrix} \quad \text{and} \quad B = \text{ppt}(A, \alpha) = \begin{pmatrix} -1 & -6 & 1 \\ -1 & -5 & 1 \\ 2 & 4 & -1 \end{pmatrix}.
\]

Notice the exchange taking place relative to the index set \( \alpha \) in the equations
\[
A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 11 \end{pmatrix} \quad \text{and} \quad B \begin{pmatrix} 4 \\ 1 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.
\]
The basic factorization of $B$ is $C_1C_2^{-1}$, where
\[
C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 8 & 1 \end{pmatrix}.
\]
Also if $\beta = \{2\}$, then
\[
\text{ppt} (B, \beta) = \begin{pmatrix} .2 & 1.2 & -.2 \\ -.2 & -.2 & .2 \\ 1.2 & -.8 & -.2 \end{pmatrix} = A^{-1}.
\]

**Theorem 3.8** Let $A \in M_n(\mathbb{F})$ and $\alpha \subseteq \langle n \rangle$ so that $A[\alpha]$ is invertible. Then

(i) $\det(\text{ppt} (A, \alpha)) = \frac{\det A(\alpha)}{\det A[\alpha]}$, and

(ii) if in addition $A(\alpha)$ is invertible, $\text{ppt} (A, \alpha)^{-1} = \text{ppt} (A, \alpha^c)$.

**Proof.** Let $C_1C_2^{-1}$ be the basic factorization of $\text{ppt} (A, \alpha)$. The conclusions follow, respectively, from Lemma 3.4 and by directly verifying that $\text{ppt} (A, \alpha)^{-1} = C_2C_1^{-1} = \text{ppt} (A, \alpha^c)$.

Note that invertibility of $\text{ppt} (A, \alpha)$ does not necessarily imply invertibility of $A$ (Meenakshi [12]). A simple counterexample is provided by
\[
A = \begin{pmatrix} 1 & .2 \\ 1 & .2 \end{pmatrix} \quad \text{and} \quad \text{ppt} (A, \{1\}) = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}.
\]

4 Eigenvalues of principal pivot transforms

We continue with what to our knowledge are new observations on the eigenvalues of principal pivot transforms.

**Theorem 4.1** Let $A \in M_n(\mathbb{F})$ and $\alpha \subseteq \langle n \rangle$ so that $A[\alpha]$ is invertible. Let $C_1C_2^{-1}$ be the basic factorization of $\text{ppt} (A, \alpha)$. Then the following are equivalent:

(i) $\lambda \in \sigma(\text{ppt} (A, \alpha))$

(ii) $\lambda$ is a finite eigenvalue of the matrix pencil $C_1 - \lambda C_2$.

When, in addition, $\lambda \neq 0$, then the following condition is also equivalent to (i) and (ii):

(iii) $A - D$ is singular, where $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i = \lambda^{-1}$ if $i \in \alpha$ and $d_i = \lambda$ otherwise.
Proof. The equivalence of (i) and (ii) follows from Lemma 3.4 and the fact that \( \lambda \) is a finite eigenvalue of the matrix pencil \( C_1 - \lambda C_2 \) if and only if \( \lambda \) is an eigenvalue of \( C_1 C_2^{-1} \). For the equivalence of (ii) and (iii) when \( \lambda \neq 0 \), observe that up to a permutation similarity of \( A \),

\[
C_1 - \lambda C_2 = \begin{pmatrix}
I|\alpha| - \lambda A|\alpha| & -\lambda A(\alpha, \alpha) \\
A(\alpha, \alpha) & A(\alpha) - \lambda I(\alpha)
\end{pmatrix},
\]

where \( I \) is the identity matrix in \( M_n(\mathbb{F}) \). Thus, multiplying the leading \(|\alpha|\) rows of \( C_1 - \lambda C_2 \) by \(-\lambda^{-1}\), we obtain that (ii) holds if and only if

\[
A - \begin{pmatrix}
\lambda^{-1} I|\alpha| & 0 \\
0 & \lambda I(\alpha)
\end{pmatrix}
\]

is singular.

It is worth noting the parallelism in viewing a principal pivot transform as ‘partial inversion’ with the fact that its nonzero eigenvalues are the zeros of \( \det(A - D) \) as in (iii) of the above theorem. A more precise account of \( \det(A - D) \) as a function of \( \lambda \) and of its relation to the spectrum of the principal pivot transform is given next. Note that unless \( \alpha = \emptyset \), \( \det(A - D) \) is not a polynomial in \( \lambda \).

Proposition 4.2 Let \( A \in M_n(\mathbb{F}) \) and \( \alpha \subseteq \langle n \rangle \) such that \( A|\alpha| \) and \( A(\alpha) \) are invertible. Let \( \lambda \) be an indeterminate, and let \( D = \text{diag}(d_1, \ldots, d_n) \) with \( d_i = \lambda^{-1} \) if \( i \in \alpha \) and \( d_i = \lambda \) otherwise. Then

\[
g(\lambda) = (-1)^{|\alpha^c|}\lambda^{|\alpha|} \frac{\det(A - D)}{\det A|\alpha|}
\]

is the characteristic polynomial of \( \text{ppt} (A, \alpha) \). Moreover, the coefficients of \( g(\lambda) \) can be expressed as real linear combinations of the principal minors of \( A \).

Proof. Since \( A|\alpha| \) and \( A(\alpha) \) are invertible, we respectively have that \( B = \text{ppt} (A, \alpha) \) is well defined and, by Theorem 3.8, nonsingular. It then follows from Theorem 4.1 (iii) that \( \lambda \) is an eigenvalue of \( B \) if and only if \( \det(A - D) = 0 \), where \( D \) is as described above. Since \( D \) is diagonal, by (2.1) we obtain

\[
\det(A - D) = \sum_{\beta \subseteq \langle n \rangle} (-1)^{|\beta^c|} \prod_{i \notin \beta} d_i \det A|\beta|.
\]  \( (4.2) \)

Since \( \beta^c = (\beta^c \cap \alpha^c) \cup (\beta^c \cap \alpha) \) and \( (\beta^c \cap \alpha^c) \cap (\beta^c \cap \alpha) = \emptyset \), we have

\[
\prod_{i \notin \beta} d_i = \sum_{\beta \subseteq \langle n \rangle} \lambda^{|\beta^c \cap \alpha^c| - |\beta^c \cap \alpha|}.
\]  \( (4.3) \)

Also notice that \( |\alpha| \geq |\beta^c \cap \alpha| \geq |\beta^c \cap \alpha^c| - |\beta^c \cap \alpha^c| \), i.e.,

\[
|\beta^c \cap \alpha^c| - |\beta^c \cap \alpha| \geq -|\alpha|,
\]  \( (4.4) \)

and that

\[
|\beta^c \cap \alpha^c| - |\beta^c \cap \alpha| \leq |\beta^c \cap \alpha^c| \leq |\alpha^c|.
\]  \( (4.5) \)
Equalities hold in \((4.4)\) and \((4.5)\) if and only if \(\beta^c = \alpha\) and \(\beta = \alpha\), respectively. Thus, multiplying the equation in \((4.2)\) by \(\lambda^{\alpha}\) and using \((4.3)-(4.5)\), we obtain that \(\lambda\) is an eigenvalue of \(B\) if and only if \(\lambda\) is a (nonzero) root of the polynomial

\[
\lambda^{\alpha} \det(A - D) = \sum_{\beta \subseteq \langle n \rangle} (-1)^{|\beta^c|} \lambda^{\alpha + |\beta^c\cap\alpha^c| - |\beta^c\cap\alpha|} \det A[\beta]. \tag{4.6}
\]

The term of highest degree in \((4.6)\) appears when \(\beta = \alpha\) and equals \((-1)^{|\alpha^c|}\lambda^n \det A[\alpha]\). The constant term in \((4.6)\) appears when \(\beta = \alpha^c\) and equals \((-1)^{|\alpha|}\det A(\alpha)\). Thus, by Theorem \(3.8\) (i), \(g(\lambda)\) as in the statement of the theorem is indeed the characteristic polynomial of \(B\) and its coefficients are real linear combinations of the principal minors of \(A\) as seen by \((4.3)\).

Note that under the assumptions (and as a consequence) of the above proposition, if \(A \in M_n(\mathbb{F})\) has real principal minors, then the spectrum of ppt \((A, \alpha)\) is closed under complex conjugation.

**Corollary 4.3** Let \(A \in M_n(\mathbb{F})\) and \(\alpha \subseteq \langle n \rangle\) so that \(A[\alpha]\) is invertible. Then \(1 \in \sigma(\text{ppt}(A, \alpha))\) (resp., \(-1 \in \sigma(\text{ppt}(A, \alpha))\)) if and only if \(1 \in \sigma(A)\) (resp., \(-1 \in \sigma(A)\)). Also ppt \((A, \alpha)\) is singular if and only if \(A(\alpha)\) is singular.

**Proof.** The results on the \pm1\ eigenvalues follow from Proposition \(4.2\). The singularity condition for ppt \((A, \alpha)\) follows either from Theorem \(3.8\) (i) or from Theorem \(4.1\) (ii).

We continue with an application to iterative techniques for solving a linear system \(Ax = b\), where \(A \in M_n(\mathbb{F})\) is invertible. Such iterative techniques are obtained by expressing the unique solution \(x\) as a fixed point of a matrix equation \(x = Tx + c\) for an appropriate matrix \(T\). In fact, based on a splitting of \(A\) into \(A = M - N\) and assuming that \(M\) is invertible, we take \(T = M^{-1}N\) and \(c = M^{-1}b\). Then the sequence \(\{x_k\}_{k=0}^\infty\) generated by \(x_k = Tx_{k-1} + c\) for arbitrary \(x_0\) converges to the solution \(x\) if and only if \(\rho(T) < 1\) (see e.g., Varga \([17]\) ). The Jacobi method is obtained when \(M = \text{diag}(a_{11}, \ldots, a_{nn})\) and \(N = M - A\). In many instances, certain splittings lead to divergent sequences. This may be overcome by considering a principal pivot transform \(\hat{T}\) of \(T\) and an equation \(x = \hat{T}x + d\) equivalent to \(x = Tx + c\), as suggested by the following result and illustrated by the subsequent example.

**Proposition 4.4** Let \(T \in M_n(\mathbb{F})\) and \(x, c \in \mathbb{F}^n\). Let \(\alpha \subseteq \langle n \rangle\) so that \(T[\alpha]\) is invertible. Consider the vector \(u\) defined by

\[
u(i) = \begin{cases} c(i) & \text{if } i \in \alpha \\ 0 & \text{otherwise}. \end{cases}
\]

Then \(x = Tx + c\) if and only if \(x = \hat{T}x + d\), where \(d = c - (I + \hat{T})u\).

**Proof.** Let \(T, \hat{T}, x, c, u\) and \(d\) as prescribed. Observe that by Theorem \(3.1\), \(Tx = x - c\) is equivalent to \(\hat{T}(x - u) = x - (c - u)\), which in turn is equivalent to \(x = \hat{T}x - \hat{T}u + (c - u)\), that is, \(x = \hat{T}x + d\).
Example 4.5 Consider the matrix $A$ and the corresponding Jacobi iteration matrix $T$ given by

$$
A = \begin{pmatrix}
1 & -3/2 & -1/4 \\
-3/2 & 1 & -5/2 \\
-1/2 & -1/2 & 1
\end{pmatrix}
\quad \text{and} \quad
T = \begin{pmatrix}
0 & 3/2 & 1/4 \\
3/2 & 0 & 5/2 \\
1/2 & 1/2 & 0
\end{pmatrix}.
$$

We find that $\sigma(T) = \{2.1419, -0.6419, -1.5\}$. That is, as $\rho(T) > 1$, the Jacobi iteration $x_k = T x_{k-1} + c$ fails to converge to the solution of a system $Ax = b$. However, if we consider

$$
\hat{T} = \text{ppt}(T, \{1, 2\}) = \begin{pmatrix}
0 & 2/3 & -5/3 \\
2/3 & 0 & -1/6 \\
1/3 & 1/3 & -11/12
\end{pmatrix},
$$

then $\sigma(\hat{T}) = \{-1/4, 0, 2/3\}$ and thus $\rho(\hat{T}) = 2/3 < 1$. It follows that the iteration $x_k = \hat{T} x_{k-1} + d$ with $d$ as in Proposition 4.4, converges to the solution of $Ax = b$. In passing we mention that $T$ above satisfies the assumptions of the Stein-Rosenberg theorem in [17] and hence the Gauss-Seidel iteration for $A$ also fails to converge to the solution of the system.

5 Principal pivot transforms of special matrices

One of the main matrix classes discussed in association with principal pivot transforms is the class of P-matrices, that is, matrices in $M_n(\mathbb{C})$ all of whose principal minors are positive. Tucker [16] asserts that principal pivot transformations preserve the class of P-matrices. In the case of real P-matrices, a simple proof of this assertion can indeed be based on Theorem 3.1 and on the following characteristic property of real P-matrices (see Fiedler [9, Theorem 5.22]): $A \in M_n(\mathbb{R})$ is a P-matrix if and only if for every nonzero $x \in \mathbb{R}^n$, $x$ and $Ax$ have at least one pair of corresponding entries whose product is positive. Here we present a proof of the assertion in [16] for the general case of complex P-matrices, based on the following well known result.

Lemma 5.1 Let $A \in M_n(\mathbb{C})$ be a P-matrix and $\alpha \subseteq \langle n \rangle$. Then $A/A[\alpha]$ is a P-matrix.

Proof. Assuming that $A$ is a P-matrix and by considering the block representation of $A^{-1}$ mentioned in section 2, it is enough to show that $A^{-1}$ is also a P-matrix. Indeed, since $A[\alpha]$ is invertible for all $\alpha \subseteq \langle n \rangle$, each principal submatrix of $A^{-1}$ is of the form $(A/A[\alpha])^{-1}$ for some $\alpha \subseteq \langle n \rangle$ and thus its determinant is $\det A[\alpha]/\det A > 0$.

Theorem 5.2 Let $A \in M_n(\mathbb{C})$ be a P-matrix and $\alpha \subseteq \langle n \rangle$. Then ppt $(A, \alpha)$ is a P-matrix.

Proof. Let $A$ be a P-matrix and consider first the case where $\alpha$ is a singleton; without loss of generality assume that $\alpha = \{1\}$. Let $B = \text{ppt}(A, \alpha) = (b_{ij})$. By definition, the principal submatrices of $B$ that do not include entries from the first row of $B$ coincide with the principal submatrices of $A/A[\alpha]$ and thus, by Lemma 5.1, have positive determinants. The principal submatrices of $B$ that include entries from the first row of $B$ are equal to the corresponding
principal submatrices of the matrix $B'$ obtained from $B$ using $b_{11} = (A[\alpha])^{-1} > 0$ as the pivot and eliminating the nonzero entries below it. Notice that

$$B' = \begin{pmatrix} 1 & 0 \\ -A(\alpha, \alpha) & I \end{pmatrix} \begin{pmatrix} b_{11} & -b_{11}A[\alpha, \alpha] \\ A(\alpha, \alpha)b_{11} & A(\alpha, \alpha) \end{pmatrix} = \begin{pmatrix} b_{11} & -b_{11}A[\alpha, \alpha] \\ 0 & A(\alpha) \end{pmatrix}.$$ 

That is, $B'$ is itself a P-matrix, as it is block upper triangular with the diagonal blocks being P-matrices. It follows that all the principal minors of $B$ are positive and thus $B$ is a P-matrix.

Next, consider the case $\alpha = \{i_1, \ldots, i_k\} \subseteq \langle n \rangle$ with $k \geq 1$. By the proof completed so far, the sequence of matrices

$$A_0 = A, \quad A_j = \text{ppt}(A_{j-1}, \{i_j\}), \quad j = 1, 2, \ldots, k$$

is well defined and comprises P-matrices. Moreover, from the uniqueness of $B = \text{ppt}(A, \alpha)$ shown in Theorem 3.1, it follows that $A_k = \text{ppt}(A, \alpha) = B$ and thus $B$ is a P-matrix.

The next theorem summarizes our discussion of principal pivot transforms of P-matrices and follows readily from the above result.

**Theorem 5.3** Let $A \in M_n(\mathbb{F})$. Then the following are equivalent:

(i) $A$ is a P-matrix.

(ii) there exists $\alpha \subseteq \langle n \rangle$ such that $\text{ppt}(A, \alpha)$ is a P-matrix.

(iii) for all $\alpha \subseteq \langle n \rangle$, $\text{ppt}(A, \alpha)$ is a P-matrix.

**Remark 5.4** Theorem 5.3 is stated in similar terms in [11, Theorem 4.4]. However, the proof provided in [11], unless modified, is valid only when $A$ is a real matrix.

We continue with a few words on some other matrix classes that are invariant under principal pivot transformations. One such class is the $S$-matrices or semipositive matrices, consisting of matrices $A \in M_n(\mathbb{R})$ such that $Ax > 0$ for some $x > 0$ (inequalities here are entrywise.) Clearly, by Theorem 3.1, a principal pivot transform of an S-matrix is an S-matrix.

Next, recall that $A \in M_n(\mathbb{R})$ is called a Z-matrix if its off-diagonal entries are all nonpositive. Of course, principal pivot transformations do not, in general, preserve Z-matrices. In particular, they do not preserve M-matrices (i.e., Z-matrices that are also P-matrices; see Berman and Plemmons [2]). However, principal pivot transformations do preserve a class that generalizes M-matrices, which is introduced in [13]. The matrix $A \in M_n(\mathbb{R})$ is called a hidden Z-matrix provided there exist Z-matrices $X, Y$ such that

$$AX = Y \quad \text{and} \quad r^TX + s^TY > 0$$

for some vectors $r, s \geq 0$. As it is shown in [13], principal pivot transformations preserve the intersection of the classes of hidden Z-matrices and P-matrices. For example, the principal pivot transform of an M-matrix is a hidden Z-matrix and a P-matrix.
We now return to the S-orthogonal matrices mentioned in the introduction. The matrix \( Q \in M_n(\mathbb{R}) \) is called S-orthogonal if there exists a signature matrix \( S \in M_n(\mathbb{R}) \) (that is, a diagonal matrix \( S \) whose diagonal entries are \( \pm 1 \)) such that \( Q^T S Q = S \). When \( S = I \), then an S-orthogonal matrix is simply an orthogonal matrix. In [14] it is formally shown that S-orthogonal matrices can be constructed for any prescribed signature matrix \( S \) in the following way. Suppose that \( S = \text{diag}(s_1, \ldots, s_n) \) and that \( s_i = 1 \) for all \( i \in \alpha \subseteq \langle n \rangle \) and \( s_i = -1 \) for all \( i \in \alpha^c \). Let \( R \in M_n(\mathbb{R}) \) be an orthogonal matrix such that \( R[\alpha] \) is invertible. Then \( Q = \text{ppt}(R, \alpha) \) exists and is S-orthogonal.

As is the case with Schur complements, the notion of a principal pivot transform can be extended to the case of non-invertible principal submatrices by considering generalized inverses. Some work in this direction is presented in [12], where it also shown that under certain assumptions, the principal pivot transform of an EP-matrix is an EP-matrix. (Recall that \( A \in M_n(\mathbb{F}) \) is an EP-matrix if \( \text{Nul}(A) = \text{Nul}(A^*) \).)

6 Some questions

We conclude with a couple of questions about principal pivot transforms, hoping to motivate their further theoretical development and to promote their applicability.

It has been shown in Coxson [3] that the important problem of testing for P-matrices is co-NP-complete. In view of Theorem 5.3, we are led to ask: Is there a computationally advantageous utilization of principal pivot transforms to check whether a given matrix is a P-matrix or not?

As we saw in section 4, principal pivot transformations in certain instances can map the eigenvalues to desired regions, e.g., the open unit disk. When and how can we choose \( \alpha \) so that the eigenvalues of \( \text{ppt}(A, \alpha) \) lie in given regions of the complex plane?

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