Discontinuity in the specific heat of a weakly interacting Bose gas

Sang-Hoon Kim*

Division of Liberal Arts, Mokpo National Maritime University, Mokpo 530-729, Korea

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We produce the discontinuity in the specific heat of a homogeneous, dilute, and weakly interacting Bose gas in a short-wavelength range with a simple statistical method. The magnitude of the discontinuity at the phase transition temperature is obtained as a function of the density and scattering length of the Bose particles.

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\[ 4\text{He} \] becomes superfluid below 2.18K, so called \( \lambda \) point, at low atmospheric pressure. The specific heat of liquid \( 4\text{He} \) is infinitesimal around 0K, and rises with \( T^{2/3} \) until it reaches a the \( \lambda \) point that shows a dramatic divergence, and then decreases tending asymptotically to a constant classical value. The transition is known as the strongest degeneracy effect of a boson system, but it is still not fully understood. Many physicists have tried to reproduce the discontinuity of the specific since the discovery of the transition, but even until now very little researches has done it successfully.

Kikuchi et al. applied a partition function proposed by Feynman to an Ising model-like 2D cubic lattice and considering nearest-neighbors only, they calculated the specific heat at \( T_c \) numerically \[1\]. Although they produced the discontinuity of the specific heat, their method strongly depends on various atomic bond models and experimental variables. Therefore, this semi-empirical and numerical method could not give any formula between the \( \lambda \) transition and the interaction strength.

The objective of this paper is to reproduce the discontinuity of the specific heat of a homogeneous, dilute, and weakly interacting (HWDI) Bose system analytically. Through the application of a simple statistical method, the relation between the \( \lambda \) transition and the interacting strength will be obtained. The weakly interacting Bose model may prove useful to understand the strongly interacting liquid \( 4\text{He} \). We will write this argument in D-dimensions \((2 < D \leq 3)\) for better applicability.

The dilute is expressed by a dimensionless gas parameter \( \gamma = n(1)a \), where \( n(1) \) is the one-dimensional number density and \( a \) is the s-wave scattering length. In a homogeneous system, the density in D-dimensions is expressed as \( n(D) = n(1)^D \). In a repulsive dilute gas, \( \gamma \) is positive and much less than 1. The \( \gamma \) of liquid \( 4\text{He} \) is known as on the order of one-half, and therefore the HWDI model is different from the real Bose system.

For a momentum-independent interaction \( g(D) \), the mean field contribution to the self-energy is \( n(D)g(D) \). Then, the dispersion relation of the weakly interacting Bose gas can be written as

\[ \varepsilon = \varepsilon_0 + n(D)g(D), \]

where \( \varepsilon_0 = p^2/2m \). The \( g(D) \) is the positive coupling constant in D-dimensions, where \( 2 < D \leq 3 \). With the hard-sphere interaction, it is well-known in 3D as \( g(3) = 4\pi\hbar^2a/m \). The dispersion relation corresponds to a short-wavelength range in the Bogoliubov energy spectrum \[2\]. The exact form of the interaction is \( 2n(D)g(D) \) instead of \( n(D)g(D) \) because the two mean field contributions to the self-energy from Hartree and Fock are equal. However, we will use Eq. \[1\] because the factor of 2 is not a key concept here.

The grand partition function of the momentum-independent potential is given from Eq. \[1\] as

\[
Q(z, n, T) = \prod_p \frac{1}{1 - ze^{-\beta(\varepsilon_0 + ng)}},
\]

where \( \beta = 1/k_BT \). The \( z \) is the fugacity given by \( z = e^{\beta\mu} \) and 1 below \( T_c \). The \( \mu \) is the chemical potential, which is 0 below the \( T_c \) and negative above the \( T_c \). The \( z_e \) is the effective fugacity given by

\[
z_e = ze^{-\beta ng} = e^{-\beta(|\mu| + ng)}. \tag{3}
\]

Note that \( 0 \leq z_e < 1 \). As \( T \to T_c \), then \( z_e \to e^{-\beta n g} = \eta_c \). Note that \( 0 < \eta_c < 1 \), and 1 for the non-interacting Bose system. The effective chemical potential is written in the same way : \( \mu_c = \mu - ng \). It is \(-ng\) instead of zero below \( T_c \). Even for a momentum-independent potential, it shifts the transition temperature.

The equation of the state of the interacting system in D-dimensions is written from Eq. \[2\] as

\[
n(D)\lambda^D = g_D(z_e), \tag{4}
\]

where \( \lambda = \sqrt{2\pi\hbar^2/mk_BT} \) is the thermal wavelength, and \( g_D(z) = \sum_{l=1}^{\infty} z^l/l^s \) is the Bose function. The only difference of the formula from the non-interacting model is the effective fugacity \( z_e \), and the information of the coupling constant \( g(D) \) is in it.
Let $T_0 = 2\pi \hbar^2 n(2)/mk_B = 1$ since $n(D) \hat{\Theta} = n(2)$. It is about 5.92K for liquid $^4$He. In this way from Eq. (4) the relation of the transition temperature between ideal and interacting system is written as

$$T_c(3) = g_\frac{1}{2}(\eta_c) = 1.$$  (5)

The right-hand side is actually $T_c$. In this way the $T_c(3)$ of the interacting systems is obtained self-consistently by numerical method as

$$T_c(3) = g_\frac{1}{2}(\eta_c) - \frac{3}{2}.  \tag{6}$$

The relation between $T_c(3)$ and $T_c^0(3)$ is also written as the power series of the $\gamma$. The leading order in 3D is

$$T_c(3) \simeq T_c^0(3)(1 + c\gamma). \tag{7}$$

We may use the value obtained from the numerical simulation as $c \simeq 1.3 [1, 4]$. We obtain the internal energy in D-dimensions from $U(T) = -(\partial/\partial\beta)_{\mu} \ln Q(z, n, T)$.

At $T < T_c$:

$$\frac{U^-(T)}{Nk_B} = -T^{\frac{3}{2} + 1} g_{\frac{1}{2} + 1}(\eta_c), \tag{8}$$

and at $T > T_c$:

$$\frac{U^+(T)}{Nk_B} = -T^{\frac{3}{2} + 1} g_{\frac{1}{2} + 1}(\eta_c). \tag{9}$$

The internal energy is continous at $T_c$ as $\lim_{T \rightarrow T_c} \Delta U(T) = 0$

The specific heat at constant volume is obtained as $C_v = (\partial/\partial T)_{\mu} U(T)$. Therefore, we have $C_v$ below and above the transition temperature in 3D.

At $T < T_c$:

$$\frac{C^+_v}{Nk_B} = \frac{15}{4} T^\frac{3}{2} g_{\frac{1}{2}}(\eta_c) + 3\gamma T^\frac{1}{2} g_{\frac{1}{2}}(\eta_c). \tag{10}$$

The discontinuity of the specific heat and the interaction strength of the model clearly.

We considered the short-wavelength range only in the dispersion relation, and applied a simple mean field based statistical mechanics to the homogeneous, dilute, and weakly interacting Bose system. In spite of the simplicity of the system, it successfully produced the relation between the discontinuity of the specific heat and the interaction strength of the model clearly.

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