On stochastic generation of ultrametrics in high-dimensional Euclidean spaces

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Abstract

We present a proof of the theorem which states that a matrix of Euclidean distances on a set of specially distributed random points in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) converges in probability to an ultrametric matrix as \( n \to \infty \). Values of the elements of an ultrametric distance matrix are completely determined by variances of coordinates of random points. Also we preset a probabilistic algorithm for generation of finite ultrametric structures of any topology in high-dimensional Euclidean space. Validity of the algorithm is demonstrated by explicit calculations of distance matrices and ultrametricity indexes for various dimensions \( n \).

Keywords: ultrametrics, ultrametric spaces, ultrametricity index, high-dimensional Euclidean spaces, random distributions, the law of large numbers, disordered systems.

1 Introduction

A set \( M = \{x\} \) is called a metric space if for any pair of elements \( x^a, x^b \in M \) a distance function \( d_{ab} = d(x^a, x^b) \) (called a metric) is defined and satisfies the following four axioms for any triplet of points \( x^a, x^b, x^c \):

\[
d_{ab} \geq 0, \tag{1}
\]
\[
d_{ab} = 0 \iff a = b, \tag{2}
\]
\[
d_{ab} = d_{ba}, \tag{3}
\]
\[
d_{ab} \leq d_{ac} + d_{bc}. \tag{4}
\]

A metric \( d_{ab} \) satisfying the strong triangle inequality

\[
d_{ab} \leq \max\{d_{ac}, d_{bc}\} \tag{5}
\]
is called an ultrametric. A space with an ultrametric is called a space with an ultrametric structure or an ultrametric space.

An arbitrary real valued matrix \( d = \{d_{ab}\} \) is called a \textit{metric matrix} if its elements satisfy the conditions (1)–(4), and it is called an \textit{ultrametric matrix} if its elements satisfy the conditions (1)–(3) and (5).

Classic examples of ultrametric spaces are the field \( Q_p \) of \( p \)-adic numbers [1, 2, 3], and the ring \( Q_m \) of \( m \)-adic numbers (see, for example, [4]).

Ultrametric spaces had long been used in various fields of natural and social sciences to problems of classification and information processing: optimization theory, taxonomy, cluster and factor analysis, and other [5]. During the last 30 years, the mathematical apparatus of ultrametric analysis was developed by V.S.Vladimirov and coworkers. This relatively new scientific field of ultrametric mathematical physics is represented by many books and works devoted to development of the \( p \)-adic analysis, \( p \)-adic mathematical physics and its application to modeling in various fields of physics, biology, computer science, psychology, sociology, and so on (see [1, 2, 3, 6] and references therein).

In fact a lot of physical, biological or socio-economic systems has intrinsic hierarchical structure [5]. Systems with non–explicit hierarchical structure are of considerable interest to researchers. In such systems the hierarchical structure cannot be observed in the original variables, but it becomes observable after transition to some effective (hidden) variables. Typically the number of these effective variables is essentially smaller than the number of degree of freedom of the whole system. There are reasons to believe that hidden ultrametric structures are present in a number of complex systems (i.e. systems with a large number of heterogeneous interacting objects), which include spin glasses, proteins, nucleic acids, etc. Similar ultrametric models arose in the early eighties last century in statistical physics of spin systems with a disorder [7, 5, 8, 9, 10]. Namely it was found that if the system has large number of "internal contradictions" (frustrations) at different scales, equilibrium of the system can be achieved in hierarchically nested regions of phase space, and number of nesting levels increases with decreasing a temperature. In this case there are relations between phase region scales, that satisfy the strong triangle inequality and thus low-temperature spin states are correlated ultrametrically.

Almost immediately after the appearance of ultrametric spin glass models it has been assumed that the conformational state space of protein molecule has ultrametric structure [11, 12]. In this case states are associated with local minima of the potential energy landscape of a protein molecule, and the energy landscape is represented as a hierarchy of nested basins of free energy local minima. Ultrametric models of conformational dynamics of protein molecules have been developed in a series of papers [13, 14, 15, 16, 17, 18, 19].

There is also some evidence that similar ultrametric structures arise in socio-economic systems [20, 21, 22, 23, 24].

In many cases the idea of application of ultrametric models to complex systems such as proteins came from the theory of spin glasses [11, 12] and there are many arguments in favor of that protein should have the ultrametric structure (see [14] for the discussion of energy landscapes and hierarchical disconnectivity graphs). However, the explanation of the origin of ultrametric structures in spin glasses based on the replica method, which is not quite rigorously justified. Thus it is interesting to discuss alternative approaches to ultrametric structures in complex systems.

In this paper we propose a procedure of generation of ultrametric structure in a metric space. In this procedure we do not assume any ultrametric properties for the initial metric space.
It has been observed in several studies (see, for example, [25, 26]), that the effectiveness of clustering algorithms applied to large data sets significantly increases with increasing the dimension of the array. Moreover it has been observed that the distance between randomly distributed points in multidimensional metric spaces shows ultrametric properties with increasing space dimension. In this paper we give the rigorous proof that any finite ultrametric space can be generated by a special random distribution of points in the $n$-dimensional Euclidean metric space taking the limit $n \to \infty$. In this case the ultrametric is completely determined by variances of random point coordinates. We present the algorithm for generating such ultrametric spaces. The validity of the algorithm is numerically demonstrated by calculations of distance matrices and ultrametricity indexes for high dimension spaces.

The paper is organized as follows. In Section 2, we present the formulation of our construction. Also in this section we formulate and prove the theorem that is the main result of this article. In Section 3, we describe the algorithm of a stochastic generation of ultrametrics in high-dimensional Euclidean spaces and check the validity of its by numerical simulations.

2 Stochastic generation of ultrametric matrices

Let us formulate statements which will be used in our construction.

Let $M = \{x^{(a)}\}, a = 1, 2, \ldots, N$ be a finite ultrametric space with an ultrametric $d(x^{(a)}, x^{(b)})$. We say that a space $M$ is an isometric space, and an ultrametric $d(x^{(a)}, x^{(b)})$ is an isometric if for any triplet of points $x^{(a)}, x^{(b)}, x^{(c)}$ in $M$ the following condition holds: $d(x^{(a)}, x^{(b)}) = \max\{d(x^{(a)}, x^{(c)}), d(x^{(b)}, x^{(c)})\}$, i.e. distances between any two non–coinciding points are equal.

The subset $B_r(a) = \{x \in M : d(x, a) \leq r\}$ is said to be the ball of radius $r$ with the center at the point $a$.

We say that an ultrametric space $M$ is homogeneous, if for any fixed value $r$ of ball radius there is exist the number $m(r)$ such that any ball $B_r(a)$ can be represented as a union of $m(r)$ balls of radius $r'$, $r' < r$.

We say that an ultrametric space $M$ is self-similar, if there exist is the number $m$ such that any ball $B_r(a)$ can be represented as a union of $m$ balls of radius $r'$, $r' < r$.

It is obvious that any self-similar ultrametric space is homogeneous.

A finite self-similar ultrametric space is isomorphic to a boundary of a Cayley tree with a finite number of levels. The distance between points at the boundary of the tree is defined as the weighted length of the path in the tree between these points.

Also we need some statements from probability theory (see, for example, [27, 28, 29]). These results will be used to prove the main result formulated in Theorem 7.

Let $\{\Omega, \Sigma, P\}$ be a probability space, where $\{\Omega, \Sigma\}$ is measurable space, $P$ is probability measure. Real random variable $X$ is measurable mapping $X : \Omega \to R$. For any real random variable $X = X(\omega)$ an interegral $\int_A X(\omega) dP(\omega)$, $A \in \Sigma$ can be defined. An expectation and a variance of $X$ are $E[X] = \int_\Omega X(\omega) dP(\omega)$ and $V[X] = E[X^2] - (E[X])^2$ respectively. Let $\Sigma^{(1)} \subset \Sigma$ be a $\sigma$-subalgebra of $\Sigma$, then the conditional expectation $E[X | \Sigma^{(1)}]$ of real random variable $X$ is a random variable $Y$ that $Y$ is $\Sigma^{(1)}$ measurable, and for all $A \in \Sigma^{(1)} \int_A X(\omega) dP(\omega) = \int_A Y(\omega) dP(\omega)$, and the conditional variance $V[X | \Sigma^{(1)}]$ is defined as $V[X | \Sigma^{(1)}] = E[X^2 | \Sigma^{(1)}] - (E[X | \Sigma^{(1)}])^2$.

Theorem 1. (The law of large numbers) Let $X_1, X_2, \ldots$ be a sequence of independent identically
distributed random variables with finite expectations \( \mathbb{E}[X_i] \equiv m_i \) and finite variances \( \mathbb{V}[X_i] \), the variances \( \mathbb{V}[X_i] \) are uniformly bounded, and \( S_n = X_1 + X_2 + \ldots + X_n \). Then \( \frac{S_n}{n} \overset{P}{\to} \frac{\langle S_n \rangle}{n} \) i.e. for any \( \varepsilon > 0 \) one has \( \mathbb{P}\left\{ \left| \frac{S_n}{n} - \frac{\langle S_n \rangle}{n} \right| \geq \varepsilon \right\} \to 0 \) as \( n \to \infty \) (convergence in probability).

**Theorem 2.** (Slutsky’s theorem, [30, 31]) If \( X_n^{(1)} \overset{P}{\to} X^{(1)} \), \( X_n^{(2)} \overset{P}{\to} X^{(2)} \), \ldots, \( X_n^{(N)} \overset{P}{\to} X^{(N)} \), and \( h(x^{(1)}, x^{(2)}, \ldots, x^{(N)}) \) is a continuous function of \( N \) variables, then \( h\left( X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(N)} \right) \overset{P}{\to} h\left( X^{(1)}, X^{(2)}, \ldots, X^{(N)} \right) \).

**Theorem 3.** Let \( M = \{x\} \) be an ultrametric space with an ultrametric \( d(x,y) \), and let \( f(\lambda) \) be a continuous nonnegative nondecreasing function such that \( f(0) = 0 \). Then \( \delta(x,y) \equiv f(d(x,y)) \) is an ultrametric on \( M \).

**Theorem 4.** Let \( \Omega, \Sigma, \mathbb{P} \) be probability space, and \( x^{(a)} = \left( x_1^{(a)}, x_2^{(a)}, \ldots, x_n^{(a)} \right) \), \( x^{(b)} = \left( x_1^{(b)}, x_2^{(b)}, \ldots, x_n^{(b)} \right) \) be two points in the \( n \)-dimensional space \( R^n \) with independent random coordinates on \( \Omega, \Sigma, \mathbb{P} \) . Suppose coordinates of point \( x^{(a)} \) have the finite expectation \( \mathbb{E}[x_i^{(a)}] = m_a \) and the finite variance \( \mathbb{V}[x_i^{(a)}] = \sigma_a^2 \), and coordinates of the point \( x^{(b)} \) have the finite expectation \( \mathbb{E}[x_i^{(b)}] = m_b \) and the finite variance \( \mathbb{V}[x_i^{(b)}] = \sigma_b^2 \). Then the distance between these points

\[
d_n(x^{(a)}, x^{(b)}) = \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} (x_i^{(a)} - x_i^{(b)})^2}
\]

satisfies the condition

\[
d_n(x^{(a)}, x^{(b)}) \overset{P}{\to} \sqrt{\sigma_a^2 + \sigma_b^2 + (m_a - m_b)^2}
\]

as \( n \to \infty \).

**Proof.** Consider the random variables \( \left( x_i^{(a)} - x_i^{(b)} \right)^2 \) (for all \( i = 1, 2, \ldots, n \)). Their expectations are

\[
\mathbb{E}\left[ \left( x_i^{(a)} - x_i^{(b)} \right)^2 \right] = \mathbb{E}\left[ x_i^{(a)} \right]^2 + \mathbb{E}\left[ x_i^{(a)} \right]^2 - 2\mathbb{E}\left[ x_i^{(a)} \right] \mathbb{E}\left[ x_i^{(a)} \right] = \\
= \left( \sigma_a^2 + m_a^2 \right) + \left( \sigma_b^2 + m_b^2 \right) - 2m_a m_b = \sigma_a^2 + \sigma_b^2 + (m_a - m_b)^2.
\]

By the law of large numbers we obtain that

\[
\frac{\sum_{i=1}^{n} \left( x_i^{(a)} - x_i^{(b)} \right)^2}{n} \overset{P}{\to} \sigma_a^2 + \sigma_b^2 + (m_a - m_b)^2
\]

as \( n \to \infty \). Using Slutsky’s theorem, we get (7). \( \square \)

The following theorem is a direct consequence of Theorem 4.

**Theorem 5.** Let \( \Omega, \Sigma, \mathbb{P} \) be probability space, and \( M_n \equiv \{x^{(a)}\} \ (a = 1, 2, \ldots, N) \) be the set of points \( x^a = (x_1^a, x_2^a, \ldots, x_n^a) \) in \( R^n \) with independent random coordinates on \( \Omega, \Sigma, \mathbb{P} \). Suppose
coordinates of point \(x^{(a)}\) have the finite expectation \(E\left[ x_i^{(a)} \right] = m_i\) and the finite variances \(\text{Var}\left[ x_i^{(a)} \right] = \sigma^2\). Then the metric on \(M_n\)

\[
d_n\left( x^{(a)}, x^{(b)} \right) \equiv \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} \left( x_i^{(a)} - x_i^{(b)} \right)^2}
\]

satisfies the condition

\[
d_n\left( x^{(a)}, x^{(b)} \right) \xrightarrow{P} u_{ab},
\]

as \(n \to \infty\), where

\[
u_{ab} = \begin{cases} \sqrt{2} \sigma, & a \neq b, \\ 0, & a = b. \end{cases}
\]

We also have the following theorem.

**Theorem 6.** Let \(\{\Omega, \Sigma, P\}\) be probability space, \(\Sigma^{(1)} \subset \Sigma\) be a \(\sigma\)-subalgebra of \(\Sigma\). Let \(M_n \equiv \{x^{(a_1a_2)}\}\) (here \(a_1 = 1, 2, \ldots, p_1\), \(a_2 = 1, 2, \ldots, p_2\), and \(a_1a_2\) is two-dimensional index) be the set of \(p_1p_2\) points \(x^{(a_1a_2)} = \left( x_1^{(a_1a_2)}, x_2^{(a_1a_2)}, \ldots, x_n^{(a_1a_2)} \right)\) in \(R^n\) with independent random coordinates on \(\{\Omega, \Sigma, P\}\). Suppose conditional expectations \(E\left[ x_i^{(a_1a_2)} \mid \Sigma^{(1)}\right] \equiv x_i^{(a_1)}\) are identical for all \(a_2\), and \(E\left[ x_i^{(a_1a_2)} \right] = m_i\), \(\text{Var}\left[ x_i^{(a_1)} \right] = \sigma_1^2\), \(E\left[ \text{Var}\left[ (x_i^{(a_1a_2)}) \mid \Sigma^{(1)}\right] \right] = \sigma_2^2\) with real and finite \(m_i, \sigma_1^2, \sigma_2^2\). Then the metric on \(M_n\)

\[
d_n\left( x^{(a_1a_2)}, x^{(b_1b_2)} \right) \equiv \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} \left( x_i^{(a_1a_2)} - x_i^{(b_1b_2)} \right)^2}
\]

satisfies the condition

\[
d_n\left( x^{(a_1a_2)}, x^{(b_1b_2)} \right) \xrightarrow{P} u_{a_1a_2,b_1b_2}
\]

as \(n \to \infty\), where

\[
u_{a_1a_2,b_1b_2} = \begin{cases} \sqrt{2\left( \sigma_1^2 + \sigma_2^2 \right)}, & a_1 \neq b_1, \ a_2 \neq b_2, \\ \sqrt{2}\sigma_1, & a_1 = b_1, \ a_2 \neq b_2, \\ 0, & a_1 = b, \ a_2 = b_2 \end{cases}
\]

is nonisometric ultrametric \(p_1p_2 \times p_1p_2\) matrix.

**Proof.** Consider the random variables \(\left( x_i^{(a_1a_2)} - x_i^{(b_1b_2)} \right)^2\). Their expectations are

\[
E\left[ \left( x_i^{(a_1a_2)} - x_i^{(b_1b_2)} \right)^2 \right] = (1 - \delta_{a_1b_1}\delta_{a_2b_2}) E\left[ \left( x_i^{(a_1a_2)} - x_i^{(b_1b_2)} \right)^2 \mid \Sigma^{(1)}\right] =
\]

\[
= (1 - \delta_{a_1b_1}\delta_{a_2b_2}) E\left[ \left( x_i^{(a_1a_2)} \right)^2 \mid \Sigma^{(1)}\right] +
\]

\[
+ E\left[ \left( x_i^{(b_1b_2)} \right)^2 \mid \Sigma^{(1)}\right] - 2E\left[ x_i^{(a_1a_2)} \mid \Sigma^{(1)}\right] E\left[ x_i^{(b_1b_2)} \mid \Sigma^{(1)}\right] =
\]
\[
(1 - \delta_{a_1b_1} \delta_{a_2b_2}) E \left[ \mathbf{V} \left( \left( x_i^{(a_1a_2)} \right) | \Sigma^{(1)} \right) \right] + \mathbf{V} \left( \left( x_i^{(b_1b_2)} \right) | \Sigma^{(1)} \right) + \\
(1 - \delta_{a_1b_1} \delta_{a_2b_2}) \left( E \left[ \left( x_i^{(a_1a_2)} \right) | \Sigma^{(1)} \right] - E \left[ \left( x_i^{(b_1b_2)} \right) | \Sigma^{(1)} \right] \right)^2 = \\
(1 - \delta_{a_1b_1} \delta_{a_2b_2}) \left( E \left[ \left( x_i^{(a_1a_2)} \right) | \Sigma^{(1)} \right] + E \left[ \left( x_i^{(b_1b_2)} \right) | \Sigma^{(1)} \right] \right) + \\
(1 - \delta_{a_1b_1}) \left( E \left[ \left( x_i^{(a_1a_2)} \right) \right] - E \left[ \left( x_i^{(b_1b_2)} \right) \right] \right)^2 = \\
2 \left( (1 - \delta_{a_1b_1} \delta_{a_2b_2}) \sigma^2 + (1 - \delta_{a_1b_1}) \sigma_1^2 \right),
\]

where \( \delta_{ab} = \begin{cases} 1, & a = b; \\ 0, & a \neq b \end{cases} \) is Kronecker delta. By the law of large numbers, we obtain

\[
\frac{\sum_{i=1}^{n} \left( x_i^{(a_1a_2)} - x_i^{(b_1b_2)} \right)^2}{n} \xrightarrow{P} 2 \left( (1 - \delta_{a_1b_1} \delta_{a_2b_2}) \sigma^2 + (1 - \delta_{a_1b_1}) \sigma_1^2 \right).
\]
as \( n \to \infty \). By Slutsky’s theorem, we get

\[
d_n \left( x^{(a_1a_2)}, x^{(b_1b_2)} \right) \xrightarrow{P} u_{a_1b_2,b_1b_2} = \sqrt{2} \left( (1 - \delta_{a_1b_1} \delta_{a_2b_2}) \sigma^2 + (1 - \delta_{a_1b_1}) \sigma_1^2 \right)^{\frac{1}{2}}.
\]

\[\square\]

We state the following generalization of Theorem 6 which can be proved similarly.

**Theorem 7.** Let \( \{ \Omega, \Sigma, P \} \) be a probability space, \( \Sigma^{(n)} \) be an increasing sequence of \( \sigma \)-subalgebras \( \Sigma^{(1)} \subset \Sigma^{(2)} \subset \ldots \subset \Sigma^{(N)} = \Sigma \). Let \( M_n \equiv \{ x^{(a_1a_2 \ldots a_N)} \} \), \( (a_1 = 1, 2, \ldots, p_1, a_2 = 1, 2, \ldots, p_2, \ldots, a_N = 1, 2, \ldots, p_N) \), and \( a_1a_2 \ldots a_N \) is \( N \)-dimensional index be the sets of \( p_1p_2 \ldots p_N \) points \( x^{(a_1a_2 \ldots a_N)} = \left( x_1^{(a_1a_2 \ldots a_N)}, x_2^{(a_1a_2 \ldots a_N)}, \ldots, x_N^{(a_1a_2 \ldots a_N)} \right) \) in \( \mathbb{R}^n \) with independent random coordinates. Suppose conditional expectations \( E \left[ x_i^{(a_1a_2 \ldots a_N)} | \Sigma^{(k)} \right] \equiv x_i^{(a_1a_2 \ldots a_k)} \) \((k = 1, 2, \ldots, N - 1) \) are identical for all \( a_{k+1}, a_{k+2}, \ldots, a_N \). Suppose \( E \left[ x_i^{(a_1a_2 \ldots a_N)} \right] = m_i, \)

\[
E \left[ x_i^{(a_1a_2 a_k)} \right] = \sigma_i^2,
\]

\( E \left[ \mathbf{V} \left[ x_i^{(a_1a_2 a_k)} | \Sigma^{(k)} \right] \right] = \sigma_{k+1}^2 (k = 1, \ldots, N - 1) \) with real and finite \( m_i, \sigma_i^2, \sigma_{k+1}^2 \). Then the metric on \( M_n \)

\[
d_n \left( x^{(a_1a_2 \ldots a_N)}, x^{(b_1b_2 \ldots b_N)} \right) \equiv \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^{n} \left( x_i^{(a_1a_2 \ldots a_N)} - x_i^{(b_1b_2 \ldots b_N)} \right)^2}
\]

has the property

\[
d_n \left( x^{(a_1a_2 \ldots a_N)}, x^{(b_1b_2 \ldots b_N)} \right) \xrightarrow{P} u_{a_1a_2 \ldots a_N,b_1b_2 \ldots b_N}
\]
as \( n \to \infty \), where

\[
u_{a_1a_2 \ldots a_N,b_1b_2 \ldots b_N} = \sqrt{2} \left( (1 - \delta_{a_1b_1} \delta_{a_2b_2} \ldots \delta_{a_Nb_N}) \sigma_N^2 + \right.
\]

\[
\left. (1 - \delta_{a_1b_1} \delta_{a_2b_2} \ldots \delta_{a_Nb_N}) \sigma_1^2 \right).
\]
$$+ (1 - \delta_{a_1b_1}\delta_{a_2b_2} \cdots \delta_{a_Nb_{N-1}}) \sigma_{N-1}^2 + \cdots + (1 - \delta_{a_1b_1}) \sigma_1^2 \right)^{\frac{1}{2}}.$$  (11)

is nonisometric ultrametric $p_1p_2 \cdots p_N \times p_1p_2 \cdots p_N$ matrix.

The proof Theorem 7 is analogous to the proof of Theorem 6 and is based on the identity

$$\mathbb{E} \left[ \left( x_i^{(a_1a_2 \cdots a_k)} - x_i^{(b_1b_2 \cdots b_k)} \right)^2 \right] =$$

$$= (1 - \delta_{a_1b_1}\delta_{a_2b_2} \cdots \delta_{a_kb_k}) \mathbb{E} \left[ V \left[ x_i^{(a_1a_2 \cdots a_k)} \mid \Sigma^{(k-1)} \right] + V \left[ x_i^{(b_1b_2 \cdots b_k)} \mid \Sigma^{(k-1)} \right] \right] +$$

$$+ (1 - \delta_{a_1b_1}\delta_{a_2b_2} \cdots \delta_{a_{k-1}b_{k-1}}) \mathbb{E} \left[ \left( x_i^{(a_1a_2 \cdots a_{k-1})} - x_i^{(b_1b_2 \cdots b_{k-1})} \right)^2 \right].$$  for $k = 1, \ldots, N$.

### 3 The algorithm of the stochastic generation of ultrametric structures in high-dimensional spaces and the numerical simulation

In this section, we describe the procedure for constructing an ultrametric space which is induced by the special random distribution of points in the space $\mathbb{R}^n$ for high $n$. We restrict ourselves to finite homogeneous ultrametric spaces. Such spaces are equivalent to the boundary of some $N$-level hierarchical tree with fixed numbers of branching $p_k$ for every $k$-th level. For $p_1 = p_2 = \cdots = p_N = p$ the homogeneous ultrametric space is self-similar, and the corresponding hierarchical tree is the $N$-level Cayley tree, with the number of branching equal to $p$. However, following our procedure, one can construct a non–homogeneous finite ultrametric space isomorphic to the boundary of $N$-level hierarchical tree with arbitrary number of branches at each node. The procedure of building such an ultrametric space in accordance with Theorem 7 can be described as follows. We generate $p_1$ independent random points $x^{(a_1)} = \left( x_1^{(a_1)}, x_2^{(a_1)}, \ldots, x_n^{(a_1)} \right)$ ($a_1 = 1, 2, \ldots, p_1$) in the $n$-dimensional space $\mathbb{R}^n$ with the normal distribution $\mathcal{N} (0, \sigma_1)$ for each coordinate. Next we generate $p_1p_2$ independent random points $x^{(a_1a_2)} = \left( x_1^{(a_1a_2)}, x_2^{(a_1a_2)}, \ldots, x_n^{(a_1a_2)} \right)$ ($a_1 = 1, 2, \ldots, p_1$, $a_2 = 1, 2, \ldots, p_1p_2$) in $\mathbb{R}^n$ with normal distribution $\mathcal{N} (x_i^{(a_1)}, \sigma_2)$ for $i$-th coordinate. Next we generate $p_1p_2p_3$ independent random points $\left( x_1^{(a_1a_2a_3)}, x_2^{(a_1a_2a_3)}, \ldots, x_n^{(a_1a_2a_3)} \right)$ ($a_1 = 1, 2, \ldots, p_1$, $a_2 = 1, 2, \ldots, p_1p_2$, $a_3 = 1, 2, \ldots, p_1p_2p_3$) in $\mathbb{R}^n$ with the normal distribution $\mathcal{N} (x_i^{(a_1a_2)}, \sigma_3)$ for $i$-th coordinate and so on. We repeat this procedure for the generation of random points $N$ times. On the last step we generate $p_1p_2 \cdots p_N$ independent random points $x^{(a_1a_2 \cdots a_N)} = \left( x_1^{(a_1a_2 \cdots a_N)}, x_2^{(a_1a_2 \cdots a_N)}, \ldots, x_n^{(a_1a_2 \cdots a_N)} \right)$ ($a_1 = 1, 2, \ldots, p_1$, $a_2 = 1, 2, \ldots, p_1p_2$, $a_3 = 1, 2, \ldots, p_1p_2p_3$, $a_4 = 1, 2, \ldots, p_1p_2p_3p_4$, $\ldots, a_N = 1, 2, \ldots, p_1p_2 \cdots p_N$) in $\mathbb{R}^n$ with the normal distribution $\mathcal{N} (x_i^{(a_1a_2 \cdots a_{N-1})}, \sigma_N)$ for $i$-th coordinate. The set of points $M_n^{(N)} = \{ x^{(a_1a_2 \cdots a_N)} \}$ forms the metric space with the metric (10). The metric (10) will converge in probability for $n \to \infty$ to some ultrametric in the sense of Theorem 7.
We introduce some new definitions.

Let $M$ be the finite metric space with elements $x^{(a)} \in R^n, a = 1, 2, \ldots, N$ and let $d (x^{(a)}, x^{(b)})$ be the metric on $M$. For for any triplet of points (or for any triangle) $x^a, x^b, x^c$ in $M$ we define two functions:

$$I(a, b, c) = 1 - \frac{\text{mid} \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\}}{\max \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\}},$$

and

$$J(a, b, c) = 1 - \frac{\min \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\}}{\max \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\}},$$

where

$$\text{mid} \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\} = \frac{d(x^{(a)}, x^{(b)}) + d(x^{(b)}, x^{(c)}) + d(x^{(c)}, x^{(a)}) - \max \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\} - \min \{d(x^{(a)}, x^{(b)}), d(x^{(b)}, x^{(c)}), d(x^{(c)}, x^{(a)})\}}{3},$$

and we call this the middle length of the of the triangle $\{x^{(a)}, x^{(b)}, x^{(c)}\}$. We say that $I(a, b, c)$ is the ultrametricity index and $I(a, b, c)$ is the isometricity index of the triangle $\{x^{(a)}, x^{(b)}, x^{(c)}\}$.

The ultrametricity index of the metric space $M$ is the number $U(M)$ defined as the average ultrametricity index of all possible triangles in $M$:

$$U(M) = \frac{3!(N-3)!}{N!} \sum_{a=1}^{N} \sum_{b=a+1}^{N} \sum_{c=b+1}^{N} I(a, b, c).$$

The isometricity index of the metric space $M$ is the number $E(M)$ defined as the average isometricity index of all possible triangles in $M$:

$$E(M) = \frac{3!(N-3)!}{N!} \sum_{a=1}^{N} \sum_{b=a+1}^{N} \sum_{c=b+1}^{N} I(a, b, c).$$

Note that $0 \leq U(M) \leq \frac{1}{2}, 0 \leq E(M) \leq 1$ and we have $U(M) = 0$ if $M$ is ultrametric and $E(M) = 0$ if $M$ is isometric. It is possible to claim that for metric space $M^{(N)}_n$ generated by the procedure described in the previous section

$$U(M^{(N)}_n) \xrightarrow{p} 0$$
as $n \to \infty$.

Let $d^{(p_1, p_2, \ldots, p_N)}(N)$ be the metric matrix generated numerically according to the above procedure, where $N$ is the number of procedure steps (the level number of the ultrametric tree), $p_i, i = 1, 2, \ldots, N$ is the numbers of points at the $i$-th step of the procedure. Below we present the metric matrix $d^{(p_1, p_2)}(2)$ and $d^{(p_1, p_2, p_3)}(3)$. The values of the variances are $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$. The dimension $n$ of the space $R^n$ is chosen to be 10^4.
with the matrix (9) for \( \sigma \) respectively up to the third digit in the values of its elements.

\[
\begin{bmatrix}
0 & 1.4055 & 1.4199 & 2.0049 & 1.9981 & 1.9989 & 1.9968 & 1.9958 & 2.0010 \\
1.4055 & 0 & 1.4116 & 2.0141 & 1.9973 & 1.9984 & 2.0133 & 2.0173 & 2.0211 \\
1.4199 & 1.4116 & 0 & 2.0148 & 2.0082 & 2.0022 & 2.0240 & 2.0245 & 2.0275 \\
2.0049 & 2.0141 & 2.0148 & 0 & 1.4166 & 1.4163 & 1.9921 & 1.9770 & 1.9958 \\
1.9981 & 1.9973 & 2.0082 & 1.4166 & 0 & 1.4137 & 1.9955 & 1.9794 & 1.9921 \\
1.9989 & 1.9984 & 2.0022 & 1.4163 & 1.4137 & 0 & 1.9894 & 1.9592 & 1.9809 \\
1.9968 & 2.0133 & 2.0240 & 1.9921 & 1.9955 & 1.9894 & 0 & 1.4124 & 1.4019 \\
1.9958 & 2.0173 & 2.0245 & 1.9770 & 1.9794 & 1.9592 & 1.4124 & 0 & 1.4027 \\
2.0010 & 2.0211 & 2.0275 & 1.9958 & 1.9921 & 1.9809 & 1.4019 & 1.4027 & 0 
\end{bmatrix}
\]

\[
d^{(3,3)}(2) = \begin{bmatrix}
0 & 1.4168 & 2.0359 & 2.0355 & 2.4578 & 2.4540 & 2.4434 & 2.4260 \\
1.4168 & 0 & 2.0113 & 2.0119 & 2.4398 & 2.4402 & 2.4433 & 2.4147 \\
2.0359 & 2.0113 & 0 & 1.4238 & 2.4573 & 2.4580 & 2.4466 & 2.4422 \\
2.0355 & 2.0119 & 1.4238 & 0 & 2.4572 & 2.4541 & 2.4472 & 2.4467 \\
2.4578 & 2.4398 & 2.4573 & 2.4572 & 0 & 1.4294 & 2.0108 & 2.4572 \\
2.4540 & 2.4402 & 2.4580 & 2.4541 & 1.4294 & 0 & 2.0146 & 2.0033 \\
2.4434 & 2.4433 & 2.4466 & 2.4472 & 2.0108 & 2.0146 & 0 & 1.4142 \\
2.4260 & 2.4147 & 2.4422 & 2.4467 & 2.0005 & 2.0033 & 1.4142 & 0 
\end{bmatrix}
\]

\[
d^{(2,2,2)}(3) = \begin{bmatrix}
0 & 1.4168 & 2.0359 & 2.0355 & 2.4578 & 2.4540 & 2.4434 & 2.4260 \\
1.4168 & 0 & 2.0113 & 2.0119 & 2.4398 & 2.4402 & 2.4433 & 2.4147 \\
2.0359 & 2.0113 & 0 & 1.4238 & 2.4573 & 2.4580 & 2.4466 & 2.4422 \\
2.0355 & 2.0119 & 1.4238 & 0 & 2.4572 & 2.4541 & 2.4472 & 2.4467 \\
2.4578 & 2.4398 & 2.4573 & 2.4572 & 0 & 1.4294 & 2.0108 & 2.4572 \\
2.4540 & 2.4402 & 2.4580 & 2.4541 & 1.4294 & 0 & 2.0146 & 2.0033 \\
2.4434 & 2.4433 & 2.4466 & 2.4472 & 2.0108 & 2.0146 & 0 & 1.4142 \\
2.4260 & 2.4147 & 2.4422 & 2.4467 & 2.0005 & 2.0033 & 1.4142 & 0 
\end{bmatrix}
\]

In full agreement with Theorem 6 and Theorem 7 the matrix \( d^{(3,3)}(2) \) and \( d^{(2,2,2)}(3) \) coincides with the matrix (9) for \( \sigma_1 = \sigma_2 = 1 \)

\[
u_{a_1a_2b_1b_2} = \sqrt{2} \left( (1 - \delta_{a_1b_1}\delta_{a_2b_2}) + (1 - \delta_{a_1b_1})^{\frac{1}{2}} \right)
\]

and (11) for \( N = 3 \) and \( \sigma_1 = \sigma_2 = \sigma_3 = 1 \)

\[
u_{a_1a_2a_3b_1b_2b_3} = \sqrt{2} \left( (1 - \delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}) + (1 - \delta_{a_1b_1}\delta_{a_2b_2}) + (1 - \delta_{a_1b_1})^{\frac{1}{2}} \right)
\]

respectively up to the third digit in the values of its elements.

Below we present a plots of the ultrametricity index \( U(M) \) and isometricity index \( E(M) \) from the space dimension \( n \) for the three step procedure of stochastic generation of ultrametric space (Fig. 14). The values of parameters are \( p_1 = p_2 = p_3 = 3, \sigma_1 = \sigma_2 = \sigma_3 = 1 \). As one can see from Fig. 14 even at dimensions \( n \geq 15 \) the metric \( 27 \times 27 \) matrix can be satisfactorily considered as an ultrametric matrix.
Figure 1: plot of the ultrametricity index $U(M)$ (a) and plot of the isometricity index $E(M)$ (b) from the space dimension $n$ for three step procedure of stochastic generation of ultrametric space in $\mathbb{R}^n$. 
4 Conclusion

In this paper a probabilistic mechanism for generating an ultrametrics in Euclidean metric spaces of high dimension is described. It is based on a special probability distribution of statistically independent random points. It is proved that the Euclidean metric on a set of independent random points in $\mathbb{R}^n$ with a special distribution convergence in probability to ultrametric as $n \to \infty$. Ultrametric distance matrix is completely determined by variances of distributions of coordinates of points. The probabilistic algorithm for the generation of finite ultrametric structures of any topology in high-dimensional Euclidean spaces is described. The validity of the algorithm is demonstrated by the explicit calculations of distance matrices and ultrametricity indexes for different dimensions.

The main result of this paper is Theorem 7. Note that this theorem like the previous ones since Theorem 4 can be generalized. For example, there is no need to require that the expectations of conditional variances $\mathbb{E} \left[ \mathcal{V} \left( x_{i}^{(a_1a_2\cdots a_N)} | \Sigma^{(k-1)} \right) \right] = \sigma_{k,i}^2$ of random coordinates of points $x^{(a_1a_2\cdots a_k)}$ are independent on $i$. It is sufficient to require that the limit $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} (\sigma_{k,i})^2$ is finite. Furthermore one can consider the case when the coordinates of random points are correlated in some way. We left the generalization of these theorems for future.

We emphasize that the problem discussed in the present paper is not about a mechanism of stochastic generation of hidden ultrametric structures being realized in real high-dimensional complex systems, such as disordered ferromagnets, spin glasses, etc. We have described only a possible scenario of appearance of ultrametric for such systems. But even if there is an implementation of a similar scenario, it is difficult to track it because of limitation of analytical methods of modeling and research of these systems. However it seems reasonable that this mechanism could be realized in complex disordered systems consisting of a large number of elements interacting in a random way. Therefore the study of toy models of disordered systems with a special type of random interactions allowing an accurate analytical study for the stochastic generation of hidden ultrametric structures in the line with the described mechanism is of particular interest.

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