Clustering function: a measure of social influence

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Abstract

A commonly used characteristic of statistical dependence of adjacency relations in real networks, the clustering coefficient, evaluates chances that two neighbours of a given vertex are adjacent. An extension is obtained by considering conditional probabilities that two randomly chosen vertices are adjacent given that they have $r$ common neighbours. We denote such probabilities $d(r)$ and call $r \to d(r)$ the clustering function. We compare clustering functions of several networks having non-negligible clustering coefficient. They show similar patterns and surprising regularity. We establish a first order asymptotic (as the number of vertices $n \to +\infty$) of the clustering function of related random intersection graph models admitting nonvanishing clustering coefficient and asymptotic degree distribution having a finite second moment.

key words: clustering coefficient, social network, intersection graph, power law

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1 Introduction

Our study is motivated by the following question: given two vertices of a network, the presence of how many common neighbours would imply with certainty that these two vertices are adjacent. A "softer" question is about the probability that two vertices with (at least) $r$ common neighbours establish a link. The answer is given by the clustering functions (1) and (2).

Let $G = (V, E)$ be a finite graph on vertex set $V$ and with edge set $E$. The number of neighbours of a vertex $v$ is denoted $d(v)$. The number of common neighbours of vertices $v_i$ and $v_j$ is denoted $d(v_i, v_j)$. We are interested in the fraction of adjacent pairs $v_i \sim v_j$ among all pairs $\{v_i, v_j\} \subset V$ having (at least) $r$ common neighbours. Here and below '$\sim$' denotes the adjacency relation of $G$. More formally, let us consider the random pair of distinct vertices $\{v^*_1, v^*_2\}$ drawn from $V$ uniformly at random. Define the clustering functions of $G$

$$r \to d_G(r) := P(v^*_1 \sim v^*_2 | d(v^*_1, v^*_2) = r),$$

$$r \to Cl_G(r) := P(v^*_1 \sim v^*_2 | d(v^*_1, v^*_2) \geq r).$$

In the case of a social network (1), (2) could be interpreted as measures of social influence or pressure exercised by the neighbours on a pair of actors to establish a communication link. We remark that characteristics (1) and (2) are related to the clustering coefficient of $G$. We recall its definition for convenience. Let $(v^*_1, v^*_2, v^*_3)$ be an ordered triple of distinct vertices drawn from $V$ uniformly at random. The conditional probability that $v^*_1$ is adjacent to $v^*_2$, given that $v^*_1$ and $v^*_2$ are both adjacent to $v^*_3$, is called the (global) clustering coefficient ([3], [18], [19], [26]). We denote it $C = C_G = P(v^*_1 \sim v^*_2 | v^*_1 \sim v^*_3, v^*_2 \sim v^*_3)$.

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In this paper we study clustering functions first by considering empirical data and then by a rigorous analysis of related random graph models.

We consider clustering function (1) of real networks admitting positive clustering coefficient: the actor network, where two actors are declared adjacent whenever they have acted in the same film ([29]), and the Facebook network ([1], [14], [27]). We remark that empirical plots show similar pattern and surprising regularity.

Our choice of the random graph model is motivated by an observation of Newman et al. [20] that the clustering property of some social networks (so called affiliation networks) could be explained by the presence of a bipartite graph structure. For example, the bipartite graph, where actors are linked to films, defines the actor network. It seems reasonable that a bipartite graph structure might also be helpful in explaining (at least to some extent) the adjacency relations of Facebook network: two members become adjacent because they share some common interests/attributes.

We secondly consider clustering function (1) of a random intersection graph, where vertices (actors) are prescribed attribute sets independently at random and two vertices are declared adjacent whenever they share at least one common attribute ([17], [15], see also [2], [16]). Random intersection graphs are relatively simple objects and for them rigorous mathematical results can be obtained. We evaluate the probabilities

\[ P(v^*_1 \sim v^*_2 | d(v^*_1, v^*_2) = r), r = 0, 1, 2, \ldots \]

for a random intersection graph in Sect. 3 below.

2 Clustering functions: empirical results

In Figure 1 we plot clustering functions (1) and (2) of three drama actor networks: the English actor network with \( n = 402622 \) actors, \( m = 66127 \) films and the clustering coefficient \( C = 0.32 \) (the clustering coefficients here and below are rounded up to 2 decimal places), the French actor network with \( n = 43204 \) actors, \( m = 5629 \) films and the clustering coefficient \( C = 0.30 \) and the Russian actor network with \( n = 9880 \) actors, \( m = 2459 \) films and \( C = 0.44 \). The data has been obtained from [29]. In Figure 2 we plot clustering function (1) of three networks describing relations between community members at three different universities (data from [27]): the first network has \( n = 17425 \) vertices and the clustering coefficient \( C = 0.16 \) (blue graph); the second network has \( n = 9414 \) vertices and the clustering coefficient \( C = 0.15 \) (green graph); the third network has \( n = 6596 \) vertices and the clustering coefficient \( C = 0.16 \) (red graph).

3 Clustering functions of random intersection graphs

Vertices \( v_1, \ldots, v_n \) of an intersection graph are represented by subsets \( D_1, \ldots, D_n \) of a given ground set \( W = \{w_1, \ldots, w_m\} \). Elements of \( W \) are called attributes or keys. Vertices \( v_i \) and \( v_j \) are declared adjacent if \( D_i \cap D_j \neq \emptyset \). The adjacency relations of such an intersection graph resemble those of some real networks, e.g., the collaboration network, where authors are declared adjacent whenever they have co-authored a paper, or the actor network, where two actors are linked by an edge whenever they have acted in the same film. Random intersection graphs have attracted considerable attention in the recent literature, see, e.g., [4], [5], [7], [10], [13], [22], [21], [28]. They admit a power law degree distribution and tunable clustering. We consider two models of random intersection graphs: the active graph and the inhomogeneous graph.

**Active graph.** In the active random intersection graph \( G_1(n, m, P) \) every vertex \( v_i \in V = \{v_1, \ldots, v_n\} \) selects its attribute set \( D_i \) independently at random ([15], [17]). We assume for simplicity that independent random sets \( D_1, \ldots, D_n \) have the same probability distribution

\[ P(D_i = A) = \binom{m}{|A|}^{-1} P(|A|), \quad \text{for any } A \subset W. \]
Figure 1: Clustering functions for three actor networks: (a) $d(r)$, (b) $Cl(r)$. 
In particular, all attributes have equal probabilities to be selected. Here $P$ is the common probability distribution of the sizes of selected sets $X_i := |D_i|$ (for each $i = 1, \ldots, n$ we have $P(X_i = k) = P(k)$, $k = 0, 1, \ldots, m$). We remark that $X_1, \ldots, X_n$ are independent random variables taking values in $\{0, 1, \ldots, m\}$.

We study the clustering function

$$r \to r \rightarrow \text{cl}(r) = P(v_1^* \sim v_2^*|d(v_1^*, v_2^*) = r) = P(v_1 \sim v_2|d(v_1, v_2) = r) \quad (4)$$

d of a sparse random intersection graph with large number of vertices. We remark, that the second identity of (4) follows from the fact that the probability distribution of $G_{\frac{m}{n}}(n, m, P)$ is invariant under permutation of its vertices. By sparse we mean that the number of edges scales as the number of vertices $n$ as $n \to +\infty$. It is convenient to consider a sequence of random intersection graphs $\{G_{\frac{m}{n}}(n)\}_n$ where $G_{\frac{m}{n}} = G_1(n, m, P)$ and where $m = m_n$ and $P = P_n$ both depend on $n$. We remark that $\{G_{\frac{m}{n}}(n)\}_n$ is a sequence of sparse random graphs whenever the size $X_1$ of the typical random set is of order $(m/n)^{1/2}$ as $m, n \to \infty$ ([6]). Furthermore, assuming that

(i) $X_1 \sqrt{n/m}$ converges in distribution to some random variable $Z$;

(ii) $E(Z) < \infty$ and $E(X_1 \sqrt{n/m})^r$ converges to $E(Z)^r$.

one obtains the asymptotic degree distribution of $\{G_{\frac{m}{n}}(n)\}$

$$\lim_{n \to +\infty} P(d(v_1) = k) = E e^{-Z \frac{E(Z)}{k!}}, \quad \text{for} \quad k = 0, 1, \ldots, \quad (5)$$

see [6, 7, 11, 24]. Here $d(v)$ denotes the degree of a vertex $v$. We remark that a heavy tailed distribution of $Z$ yields a heavy tailed asymptotic degree distribution [5]. Along with the first moment condition (ii) we shall also consider the $r$–th moment condition

(ii-r) $E(Z)^r < \infty$ and $E(X_1 \sqrt{n/m})^r$ converges to $E(Z)^r$.

We denote $z_r = E(Z)^r$ and $\delta_r = E d_\ast^r$ where $d_\ast$ is a random variable with the asymptotic degree distribution $P(d_\ast = k) = E e^{-z_1/Z(z_1 Z)^k/k!}$, $k = 0, 1, \ldots$. We assume below that $E(Z) > 0$, i.e., that the asymptotic degree distribution is non-degenerate. Furthermore, we assume for convenience that the ratio $\beta_n = m/n$ tends to some $\beta \in (0, +\infty]$ as $n \to +\infty$. 

Figure 2: Clustering functions of three university networks.
An important property of the active random intersection graph is that the adjacency relations are statistically dependent events. In particular, the clustering coefficient

$$\alpha = \alpha(G_{(n)}) = P(v_1 \sim v_2 | v_1 \sim v_3, v_2 \sim v_3)$$

of a sparse random intersection graph $G_{(n)}$ is bounded away from zero as $n \to +\infty$ provided that the second moment of the degree distribution is finite and $\beta < \infty$ ([7], [11]). In this case we have (see ([7], [11]))

$$\alpha = \beta^{-1/2} \delta_1^{3/2} (\delta_2 - \delta_1)^{-1} + o(1) \quad \text{as} \quad n \to +\infty. \quad (6)$$

We remark that for $\beta = +\infty$ we have $\alpha = o(1)$. For comparison, the (unconditional) edge probability $p_e := P(v_1 \sim v_2)$ satisfies for any $\beta \in (0, +\infty]$, see, e.g., [7],

$$p_e = \delta_1 n^{-1} + o(n^{-1}) = O(n^{-1}).$$

Theorems 1 and 2 show a first order asymptotics of the conditional probabilities $cl(r)$ as $n \to +\infty$ in the cases where $\beta < \infty$ and $\beta = \infty$, respectively.

**Theorem 1.** Let $m, n \to \infty$. Assume that (i), (ii-2) hold and $EZ > 0$. Suppose that $\beta_n \to \beta \in (0, +\infty)$. Denote $\Lambda = \sqrt{\delta_1/\beta}$. We have

$$cl(r) = \begin{cases} 
    p_e e^{-\Lambda}(1 + o(1)), & r = 0; \\
    \alpha / (\alpha + (1 - \alpha)e^{\Lambda})(1 + o(1)), & r = 1; \\
    1 - o(1), & r \geq 2.
\end{cases} \quad (7)$$

Empirical results of simulated random intersection graphs show that the convergence to the “limiting shape” in [7] is rather slow, see Figures 3 and 4 below.

**Theorem 2.** Let $m, n \to \infty$. Assume that (i), (ii-2) hold and $EZ > 0$. Suppose that $\beta_n \to +\infty$. We have

$$cl(r) = \begin{cases} 
    p_e (1 + o(1)), & r = 0; \\
    \beta_n^{-1/2}(z_1 / 2)(1 + o(1)) + O(n^{-1}), & r = 1.
\end{cases} \quad (8)$$

In particular, $cl(0) = O(n^{-1})$ and $cl(1) = o(1)$. Furthermore, we have

$$cl(2) = \begin{cases} 
    1 + o(1), & \text{for} \quad \beta_n/n \to 0; \\
    1 + o(1), & \text{for} \quad \beta_n/n \to \beta_* \in (0, +\infty); \\
    o(1), & \text{for} \quad \beta_n/n \to +\infty.
\end{cases} \quad (9)$$

Assuming, in addition, that $\beta_k = o(n)$ for each $k = 1, 2, 3, \ldots$, we obtain

$$cl(r) = 1 + o(1), \quad \text{for} \quad r = 2, 3, \ldots. \quad (10)$$

We conclude from ([7], [8]) that edge dependence measures $cl(1)$ and $\alpha$ are closely related. In particular, we have $cl(1) = 1 - o(1) \iff \alpha = 1 - o(1)$ and $cl(1) = o(1) \iff \alpha = o(1)$. Furthermore, [9] tells us that the characteristic $cl(2)$ is able to distinguish between the cases $\beta_n = o(n)$ and $n = o(\beta_n)$. Finally, [10] tells us that any $cl(r)$, $r = 1, 2, \ldots$ can’t distinguish between sequences $\{\beta_n\}$ and $\{\beta'_n\}$ growing slower that any power of $n$ (take $\beta_n = \ln n$ and $\beta_n' = \ln^2 n$, for example).
Remark 1. It is likely that (9) can be extended to an arbitrary \( r \) as follows

\[
cl(r) = \begin{cases} 
1 + o(1), & \text{for } \frac{\beta_n}{n^{4-2r-1}} \to 0; \\
\frac{c(r, \beta_n^*) + o(1)}{1}, & \text{for } \frac{\beta_n}{n^{4-2r-1}} \to \beta_n^* \in (0, +\infty); \\
o(1), & \text{for } \frac{\beta_n}{n^{4-2r-1}} \to +\infty.
\end{cases}
\]

Here \( c(r, \beta_n^*) = \frac{\beta_n r}{2z_1 - z_1^{-2} z_2 z_2^2 + 1} 
\). We note that numbers \( z_i = \mathbb{E}Z_i \) can be expressed in terms of moments of the asymptotic degree distribution (5).

Proofs of Theorems 1 and 2 are given in Sect. 5.

Fig. 3 illustrates the convergence to the step function shown by Theorem 1. Here we plot clustering function (1) of simulated random intersection graphs \( G_{i} = G(n_i, m_i, P) \), where \( n_i = m_i = 10^2 + i \), \( i = 1, 2, 3 \), and \( P(10) = 1 \).

In Fig. 4 we plot (1) for simulated random intersection graphs \( G_{i} = G(n, m, P_i) \), where \( n = m = 10^4 \) and \( P_i(3^i) = 1 \), \( i = 1, 2, 3 \). Fig. 4 illustrates the influence of the size of random sets.

Inhomogeneous graph. The inhomogeneous random intersection graph \( G_1(n, m, P_1, P_2) \) on the vertex set \( V = \{v_1, \ldots, v_n\} \) is obtained as follows. We first generate independent random variables \( A_1, \ldots, A_n, B_1, \ldots, B_m \) such that each \( A_i \) has the probability distribution \( P_1 \) and each \( B_j \) has the probability distribution \( P_2 \). Then, conditionally on the realized values \( \{A_i, B_j\}_{i,j=1}^{n,m} \), we include the attribute \( w_j \in W \) in the set \( D_i \) with probability \( p_{ij} = \min\{1, A_i B_j (n m)^{-1/2}\} \) independently for each \( i \) and \( j \) (see [2, 8, 9, 23]). Our motivation of studying this random graph model is that its clustering function approximates empirical data remarkably well, see Figure 8 below.

We consider a sequence of inhomogeneous intersection graphs \( \{\tilde{G}_n = G_1(n, m, P_1, P_2)\} \), where \( P_1, P_2 \) remain fixed while \( m = m_n \) and \( n \) tend to infinity. We denote \( a_k = \mathbb{E}A_i^k \) and \( b_k = \mathbb{E}B_1^k \). A simple calculation shows (see Section 5 below) that the edge probability \( p_e = \mathbb{P}(v_1 \sim v_2) \) of \( G_1(n, m, P_1, P_2) \) satisfies

\[
P_e = a_2 b_2 n^{-1} + o(n^{-1}).
\]
Figure 4: Clustering function of random intersection graphs with \( n = m = 10000 \) and \( P_i(3^i) = 1, i = 1, 2, 3 \).

Hence, \( \{\tilde{G}_n\} \) is a sequence of sparse graphs. We remark that this sequence admits a power law asymptotic degree distribution \([8]\).

In Theorem 3 below we show a first order asymptotics of the clustering function \( cI(\cdot) \) in the case where the ratio \( \beta_n = m/n \) has a non-zero finite limit. In addition, we show that \( \tilde{G}_n \) admits a nonvanishing clustering coefficient \( \alpha = \alpha(\tilde{G}_n) = P(v_1 \sim v_2 | v_1 \sim v_3, v_2 \sim v_3) \).

**Theorem 3.** Let \( m, n \to \infty \). Assume that \( 0 < EA_1^2 < \infty \) and \( 0 < EB_1^3 < \infty \). Suppose that \( \beta_n \to \beta \in (0, +\infty) \). Then we have

\[
\alpha = \frac{b_\beta \kappa}{b_\beta \kappa + \sqrt{\beta}} + o(1) \tag{12}
\]

and

\[
cI(r) = \begin{cases} 
\frac{a_1^2 b_2^2 n^{-1}(1 + o(1))}{b_\beta \kappa}, & r = 0; \\
\frac{b_\beta \kappa}{b_\beta \kappa + \sqrt{\beta}}(1 + o(1)), & r = 1; \\
1 - o(1), & r \geq 2.
\end{cases} \tag{13}
\]

Here \( \kappa = a_1 a_2^{-1} b_2^{-2} \) and \( b_\beta^* = EB_1^k e^{-a_1 R_1/\sqrt{\beta}} \).

The proof of Theorem 3 is given in Sect. 5.

**4 Discussion**

The first order asymptotics \([7], [9], [10], [13] \) suggests that the clustering function \( cI(\cdot) \) of a large intersection graph with a square integrable asymptotic degree distribution can be approximated by a step - like function. Furthermore, \( cI(1) \) is closely related to the clustering coefficient.

Simulations in Figures 3 and 4 show that the convergence in \([7] \) can be rather slow and we observe a sigmoid function approximation of the step function. Furthermore, the larger is the
average degree, the more remote is the “step” from the origin and the more gradual is the slope of the clustering function.

Clustering functions of real networks considered in Figures 1 and 3 have even more gradual slope, a phenomena perhaps related to the inhomogeneity of the degree sequence. We remark that the actor network and the Facebook are considered as having power law degree sequences which do not admit a finite (theoretical) second moment, see, e.g., [12], [14]. In order to learn more about the influence of the inhomogeneity of the degree sequence on the slope of the clustering function $r \rightarrow cl(r)$ we select various subnetworks of real networks according to certain regularity conditions satisfied by their degree sequences. We observe that the inhomogeneity (heavy tail) of the degree sequence affects the slope of the clustering function: the heavier the tail the more gradual is the slope of the clustering function. We illustrate these observations in Figures 5 and 6.

Figure 5 plots clustering function (1) of subgraphs of the first university network (see Sect 2.) sampled as follows. $G_1$ is the subgraph that includes all vertices of degree not larger than 50. It has $n_0 = 7165$ vertices. $G_2$ is a subgraph induced by $n_0$ vertices drawn uniformly at random (without replacement) from the vertices of degree not larger than 150. $G_3$ is a subgraph of induced by $n_0$ vertices drawn uniformly at random (without replacement) from the set of all vertices. Now all three graphs have the same number of vertices.

In Figure 6 we plot two subgraphs of the French actor network (data from [29]). The subgraph $G_4$ is induced by the set of marked vertices obtained as follows: we put a mark on each vertex $v$ with probability $d^{-\tau}(v)$ and independently of the other vertices. Choosing $\tau = 0.5$ we obtain a random subgraph denoted $G_4$. In our case the realized number of marked vertices $n_1 = 8871$. $G_5$ is the subgraph of the French actor network induced by $n_1$ vertices drawn uniformly at random (without replacement) from the set of all vertices. Now both subgraphs have the same number of vertices, but the degree sequence of $G_4$ is much more regular than that of $G_5$.

Finally, we examine how well a random intersection graph fits the real data. For this purpose we consider a memoryless actor network obtained as follows. Assume every actor of a given actor graph has forgotten about the titles of movies he or she acted in and only remembers the
number of movies.
We first simulate an instance of the active memoryless graph where each actor chooses films independently and uniformly at random from a given set of $\tilde{m}$ films so that the number of films chosen by each actor is the same as in the true actor graph. In the active memoryless graph all films have equal chances to be selected by any of actors. We remark that in the case where $\tilde{m} = m$, i.e., the number of films in the active memoryless graph is the same as in the real underlying actor network, the expected degree of the memoryless graph does not match the average degree of the real network. We can easily adjust the number of films (of the memoryless graph) so that these degrees match. We denote this number $m'$ and call the active memoryless graph with $\tilde{m} = m'$ adjusted one. In Figure 7 we plot clustering function (1) of two instances of memoryless graphs for comparison with the underlying French actor network: one with the true number of films and another with the adjusted number of films.

We secondly simulate an instance of the inhomogeneous memoryless graph where an actor $v_i$ chooses the film $w_j$ with probability $a_i b_j M^{-1}$ independently for each $i$ and $j$. Here the numbers $a_i, b_j$ are observed characteristics of the underlying actor network: $v_i$ acted in $a_i$ films; $b_j$ actors acted in the film $w_j$. $M = \sum_{1 \leq i \leq n} a_i = \sum_{1 \leq j \leq m} b_j$ is the total number of links of the bipartite graph where actors are linked to films. In Figure 8 we plot clustering function (1) of an instance of the inhomogeneous memoryless graph of the French actor network. Here we observe a remarkable accuracy of the approximation of the real clustering function by that of the memoryless graph. We remark that in comparison with active memoryless graphs of Figure 7, that only use the data $a_1, \ldots, a_n$, the inhomogeneous memoryless graph of Figure 8 uses, in addition, the numbers $b_1, \ldots, b_m$.

We remark that Theorems 1, 2 and 3 establish a first order asymptotics to the clustering function $cl(\cdot)$ of random intersection graphs having a square integrable asymptotic degree distribution. An interesting question were about a power law random intersection graph whose asymptotic degree distribution has infinite second moment: Is there a limiting shape of the clustering function for $n \to +\infty$ in this case? Is there a theoretically valid approximation to the clustering function that explains the gradual slope of $cl(\cdot)$ of observed empirical plots? It would also be
interesting to learn about a higher order asymptotics of the clustering function $\text{cl}(\cdot)$ that refines results of Theorems 1, 2, and could perhaps better explain the empirical data.

5 Proofs

The section is organized as follows: we first we formulate two auxiliary lemmas, then we prove Theorems 1, 2, and 3.

Lemma 1. (See, e.g., [25]) Let $S = I_1 + I_2 + \cdots + I_n$ be the sum of independent random indicators with probabilities $P(I_i = 1) = p_i$. Let $\Lambda$ be Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions $P_S$ of $P_\Lambda$ of $S$ and $\Lambda$

$$\sup_{A \subset \{0,1,2,\ldots\}} |P(S \in A) - P(\Lambda \in A)| \leq \sum_i p_i^2. \quad (14)$$

Lemma 2. ([7]) Given integers $1 \leq s \leq d_1 \leq d_2 \leq m$, let $D_1, D_2$ be independent random subsets of the set $W = \{1,\ldots,m\}$ such that $D_1$ (respectively $D_2$) is uniformly distributed in the class of subsets of $W$ of size $d_1$ (respectively $d_2$). The probabilities $p' := P(|D_1 \cap D_2| = s)$ and $p'' := P(|D_1 \cap D_2| \geq s)$ satisfy

$$\left(1 - \frac{(d_1 - s)(d_2 - s)}{m + 1 - d_1}\right) p^*_{d_1,d_2,s} \leq p' \leq p'' \leq p^*_{d_1,d_2,s}. \quad (15)$$

Here we denote $p^*_{d_1,d_2,s} = \binom{d_1}{s} \binom{d_2}{s} \binom{m}{s}^{-1}$.

5.1. Active graph. By $X_i = X_{ni}$ we denote the size of the set $D_i$ in $G(n)$. Furthermore, we write $Z_i = Z_{ni} = \beta_n^{-1/2} X_{ni}$ and put $Z_{01} := Z$. We denote $\bar{z}_k = E Z_{n1}^k$ and introduce the function

$$t \to \varphi(t) = \sup_{n \geq 0} E Z_{n1}^2 1_{\{Z_{n1} \geq t\}}. \quad (16)$$
We remark that conditions (i), (ii-2) imply $\varphi(t) = o(1)$ as $t \to +\infty$ (see, e.g., [7]) and $z_{2} := \sup_{n \geq 0} EZ_{n1}^{2} < \infty$. By $\mathbf{P}$ and $\mathbf{E}$ we denote the conditional probability and expectation given $X_{1}, X_{2}$. By $\mathbf{P}'$ and $\mathbf{E}'$ we denote the conditional probability and expectation given $D_{1}, D_{2}$. We introduce events $A = \{v_{1} \sim v_{2}\}$, $A_{i} = \{|D_{1} \cap D_{2}| = i\}$ and probabilities $p_{i}(r) = \mathbf{P}(A_{i} \cap \{d_{12} = r\})$.

By $f_{r}(\lambda) = e^{-\lambda} \lambda^{r}/r!$ we denote the Poisson probability.

Proof of Theorems 1 and 2. We have

\[ cl(r) = \mathbf{P}(A|d_{12} = r) = \frac{\mathbf{P}(A \cap \{d_{12} = r\})}{\mathbf{P}(d_{12} = r)}. \quad (17) \]

In order to evaluate the numerator we write $A = \bigcup_{i \geq 1} A_{i}$ and apply the total probability formula

\[ \mathbf{P}(A \cap \{d_{12} = r\}) = \sum_{i \geq 1} p_{i}(r) = \sum_{1 \leq i \leq k} p_{i}(r) + R_{k}(r). \quad (18) \]

Here $R_{k}(r) = \sum_{i>k} p_{i}(r) \leq \mathbf{P}(|D_{1} \cap D_{2}| \geq k + 1)$. Similarly we expand the denominator of (17)

\[ \mathbf{P}(d_{12} = r) = \sum_{i \geq 0} p_{i}(r) = \sum_{0 \leq i \leq k} p_{i}(r) + R_{k}(r). \quad (19) \]

In order to prove Theorem 1 we choose $k = 1$ in (18), (19) and invoke the asymptotic expressions of $p_{i}(r)$ and the upper bound for $\mathbf{P}(|D_{1} \cap D_{2}| \geq k + 1)$ shown in Lemma 3. Then, observing that as $n \to +\infty$ we have $\bar{z}_{k} = z_{k} + o(1)$, for $k = 1, 2$, and $\alpha = \beta^{-1/2} z_{1}/z_{2} + o(1)$ (see (6)), we obtain (7).

Theorem 2 is obtained in the same way, but now we choose $k = 2$. \qed

Figure 8: The French actor network and a simulated inhomogeneous memoryless network.
Given a sequence of random variables \( \{Y_n\} \) and \( r = 0, 1, \ldots \) we write \( Y_n \prec O_r \) to denote the fact that \( \mathbb{E}|Y_n| = O(n^{-r}) \), for \( r = 0, 1 \), and \( \mathbb{E}|Y_n| = O(n^{-2}\beta_n^{-1/2}) + o(n^{-2}) \), for \( r \geq 2 \).

**Lemma 3.** Assume that \( \beta_n \to \beta \in (0, +\infty] \). Suppose that (i), (ii-2) hold. Denote \( \Lambda_1 = \beta_n^{-1/2}z_1 \) and \( \Lambda_2 = z_2 - \beta_n^{-1/2}z_1 \). We have as \( n \to +\infty \)

\[
\begin{align*}
p_0(0) &= 1 - o(1), \quad p_0(r) = o(n^{-2}), \quad r \geq 3, \quad (20) \\
p_0(r) &= n^{-r}(r!-1)\Lambda_nz_1 + o(n^{-r}), \quad r = 1, 2, \quad (21) \\
p_1(r) &= n^{-1}z_1^2f_r(\Lambda_1) + o_r(r), \quad r \geq 0, \quad (22) \\
p_2(r) &= 2^{-1}n^{-2}f_r(2\Lambda_1)^2 + O_2, \quad r \geq 0. \quad (23)
\end{align*}
\]

Furthermore, we have

\[
\mathbb{P}(|D_1 \cap D_2| \geq 3) = o(n^{-2}) \quad \text{and} \quad \mathbb{P}(|D_1 \cap D_2| \geq k) = O(n^{-k}), \quad k = 1, 2. \quad (24)
\]

**Proof of Lemma 3.** Before the proof we introduce some notation and collect auxiliary inequalities. Then we give an outline of the proof. Afterwards we prove (20), (21) and (22), (23).

By \( c_\ast \) we denote a generic positive constant. By \( \mathbb{I}_B \) we denote the indicator of an event \( B \) and write \( \mathbb{I}_B = 1 - \mathbb{I}_B \). In the proof we use several indicators

\[
\begin{align*}
\mathbb{I} &= \mathbb{I}_{\{X_1+X_2<2\min\beta_n^{1/2}\}} , \quad \mathbb{I}_j = \mathbb{I}_{\{X_j<0.5\varepsilon_n\beta_n^{1/2}\}}, \quad \mathbb{I}_{\ast j} = \mathbb{I}_{\{X_j<\beta_n^{1/2}\varepsilon\}} , \\
\mathbb{I}_{\ast j} &= \mathbb{I}_{\{X_j<\varepsilon m\}}, \quad \mathbb{I}_j = \mathbb{I}_{\{X_j<0.5\varepsilon_m\beta_n^{1/2}\}}, \quad \mathbb{I}_{\ast j} = \mathbb{I}_{\{X_j<0.5\varepsilon_m\}}.
\end{align*}
\]

Some of them depend on \( \varepsilon > 0 \), value of which will be clear from the context. We denote

\[
\tilde{q}_0 = n^{-2}\Lambda_2Z_1Z_2, \quad \tilde{q}_1 = n^{-1}\Lambda_1, \quad \tilde{q}_2 = 2n^{-1}\Lambda_1,
\]

and, for \( i = 1, 2 \) we write

\[
\lambda_i = n\tilde{q}_i, \quad \lambda_i = (n-2)\tilde{q}_i, \quad q_i = \tilde{P}(v_1 \sim v_3, v_2 \sim v_3)\mathbb{I}_{A_i}, \quad \kappa_i = (X_1)(X_2)i/(i!(m)_i).
\]

We note that (15) implies

\[
\kappa_i(1 - X_1X_2/(m - X_1)) \leq \tilde{P}(A_i) \leq \kappa_i. \quad (26)
\]

In particular, we have

\[
\tilde{P}(A_i) \leq \tilde{P}(A_i)(\mathbb{I}_1\mathbb{I}_2 + \mathbb{I}_1 + \mathbb{I}_2) \leq \kappa_i\mathbb{I}_1\mathbb{I}_2 + \mathbb{I}_1 + \mathbb{I}_2 \leq n^{-1/2} + \mathbb{I}_1 + \mathbb{I}_2. \quad (27)
\]

We will use the following properties of the function \( \lambda \to f_r(\lambda) \). For \( r = 0, 1, \ldots \), it follows from the mean value theorem \( f_r(t) - f_r(s) = f'_r(\xi)(t - s) \), where \( 0 < s \leq \xi \leq t \), combined with inequalities \( |f'_r(\xi)| \leq 1 \) and \( |f'_r(\xi)| \leq \xi \) that

\[
|f_r(s) - f_r(t)| \leq |s - t| \quad \text{and} \quad |f_{2+r}(s) - f_{2+r}(t)| \leq (s + t)|s - t|. \quad (28)
\]

Now we outline the proof. In order to evaluate \( p_i(r) \) we write

\[
p_i(r) = \mathbb{E}(\mathbb{E}(\mathbb{I}_{A_i}\mathbb{I}_{(d_{12}=r)})) = \mathbb{E}(\mathbb{I}_{A_i}\mathbb{E}(\mathbb{I}_{(d_{12}=r)})) = \mathbb{E}(\mathbb{I}_{A_i}\tilde{P}(d_{12} = r)) \quad (29)
\]

and observe that, given \( D_1, D_2 \) satisfying \( |D_1 \cap D_2| = i \), the random variable

\[
d_{12} = \sum_{\beta \leq j \leq n} \mathbb{I}_{\{v_{1\sim v_j}\}} \mathbb{I}_{\{v_2 \sim v_j\}}
\]

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has binomial distribution $\text{Bin}(n - 2, q_i)$. We first approximate $\tilde{P}'(d_{12} = r)$ in \eqref{eq:30} by the Poisson probability $f_r(\lambda_i)$. Then, we approximate $\lambda_i$ by $\hat{\lambda}_i$, and $f_r(\lambda_i)$ by $f_r(\hat{\lambda}_i)$. We obtain

$$p_i(r) = \mathbb{E}(\mathbb{I}_{A_i} f_r(\hat{\lambda}_i)) + \mathbb{E}(\mathbb{I}_{\tilde{A}_i} \Delta_{r,i}) = \mathbb{E}(\tilde{P}(A_i) f_r(\hat{\lambda}_i)) + \mathbb{E}(\mathbb{I}_{\tilde{A}_i} \Delta_{r,i}),$$

where, for $|D_1 \cap D_2| = i$, we denote

$$\Delta_{r,i} := \tilde{P}'(d_{12} = r) - f_r(\hat{\lambda}_i) = \Delta_{r,i}' + \Delta_{r,i}''.$$  \hspace{1cm} (31)

$$\Delta_{r,i}' = \tilde{P}'(d_{12} = r) - f_r(\lambda_i), \quad \Delta_{r,i}'' = f_r(\lambda_i) - f_r(\hat{\lambda}_i).$$

Next we show that the remainder term $\mathbb{E}(\mathbb{I}_{\tilde{A}_i} \Delta_{r,i})$ of \eqref{eq:30} is negligible. For this purpose we estimate using LeCam’s lemma (see Lemma \ref{lem:LeCam}).

$$|\Delta_{r,i}| \leq n q_i^2,$$ \hspace{1cm} (32)

and estimate $\Delta_{r,i}''$ combining \eqref{eq:28} with the approximations $q_i \approx \tilde{q}_i$. We briefly explain these approximations. Let $\{w_1, \ldots, w_i\}$ denote the intersection $D_1 \cap D_2$ provided it is non empty. Denote $n_j = |D_3 \cap D_j|, \ j = 1, 2$. We split

$$q_0 = q_{01} + q_{02}, \quad q_1 = q_{11} + q_{12}, \quad q_2 = q_{21} + q_{22} + q_{23} + q_{24},$$

where

$$q_{01} = \tilde{P}'(n_1 = 1, n_2 = 1) \mathbb{I}_{A_0}, \quad q_{02} = \tilde{P}'(n_1 + n_2 \geq 3, n_1 \geq 1, n_2 \geq 1) \mathbb{I}_{A_0},$$

$$q_{11} = \tilde{P}'(w_1^* \in D_3) \mathbb{I}_{A_1}, \quad q_{12} = \tilde{P}'(w_1^* \notin D_3, n_1 \geq 1, n_2 \geq 1) \mathbb{I}_{A_1},$$

$$q_{21} = \tilde{P}'(w_1^* \in D_3, w_2^* \notin D_3) \mathbb{I}_{A_2}, \quad q_{22} = \tilde{P}'(w_1^* \notin D_3, w_2^* \notin D_3) \mathbb{I}_{A_2},$$

$$q_{23} = \tilde{P}'(w_1^*, w_2^* \in D_3) \mathbb{I}_{A_2}, \quad q_{24} = \tilde{P}'(w_1^*, w_2^* \notin D_3, n_1 \geq 1, n_2 \geq 1) \mathbb{I}_{A_2}.$$

and approximate $q_0 \approx q_{01} \approx \tilde{q}_0 \mathbb{I}_{A_0}, \ q_1 \approx q_{11} = \tilde{q}_1 \mathbb{I}_{A_1}$ and $q_2 \approx q_{21} + q_{22} \approx \tilde{q}_2 \mathbb{I}_{A_2}$.

Proof of \eqref{eq:27}, \eqref{eq:21}. In order to prove \eqref{eq:27}, \eqref{eq:21} we show that

$$\mathbb{E}(\mathbb{I}_{A_0} \Delta_{r,0} | \beta) = o(n^{-r \wedge 2}), \quad r \geq 0$$ \hspace{1cm} (34)

$$\mathbb{E}(\mathbb{I}_{A_0} f_r(\hat{\lambda}_0)) = (r!)^{-1} E \tilde{\lambda}_r^0 + o(n^{-r}), \quad r = 0, 1, 2,$$ \hspace{1cm} (35)

$$\mathbb{E}(\mathbb{I}_{A_0} f_r(\lambda_0)) = o(n^{-2}), \quad r \geq 3.$$ \hspace{1cm} (36)

We firstly prove \eqref{eq:34}. In the case where $\beta < \infty$ we find $n_0 > 0$ such that $\beta < 2\beta_n$ for $n \geq n_0$. In the case where $\beta = +\infty$ we find $n_0$ such that $\beta_n > 1$ for $n \geq n_0$. In order to prove \eqref{eq:34} we show that for any $0 < \varepsilon < \min\{0.5\beta_n^{1/2}, 0.1\}$ and $n \geq n_0$ we have

$$\mathbb{E}(\mathbb{I}_{A_0} | \Delta_{r,0}) \leq c_n n^{-3} + c_n n^{-r \wedge 2} R_1(\varepsilon) + c_n n^{-2} \varepsilon^{-4} R_2(\varepsilon),$$ \hspace{1cm} (37)

$$R_1(\varepsilon) := \varphi(\varepsilon^{-1}) + \varepsilon + m^{-1} + n^{-1}, \quad R_2(\varepsilon) := \varphi(0.5\varepsilon^2 n)(1 + \varepsilon^{-4} n^{-2}).$$

We remark that \eqref{eq:37} combined with the relation $\lim_{t \to +\infty} \varphi(t) = 0$ implies \eqref{eq:34}. Let us prove \eqref{eq:37}. Given $\varepsilon$, we write $\Delta_{r,0} = \Delta_{r,0}^A + \Delta_{r,0}^\beta$ and show that

$$\mathbb{E}(\mathbb{I}_{A_0} | \Delta_{r,0}) \leq \mathbb{E} \leq c_n n^{-2} \varepsilon^{-4} R_2(\varepsilon),$$ \hspace{1cm} (38)

$$\mathbb{E}(\mathbb{I}_{A_0} | \Delta_{r,0}) \leq c_n n^{-3} + c_n n^{-r \wedge 2} R_1(\varepsilon).$$ \hspace{1cm} (39)

The first inequality of \eqref{eq:38} is obvious. In order to prove the second one we combine the inequalities

$$\varepsilon^4 n^2 \mathbb{E} \leq \beta_n^{-1} \mathbb{E}(X_1 + X_2)^2 \mathbb{E} \leq 2\beta_n^{-1} \mathbb{E}(X_1^2 + X_2^2) \mathbb{E} = 4\beta_n^{-1} \mathbb{E} X_1^2 \mathbb{E},$$

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which follow from Markov’s inequality, with the inequalities
\[ \beta^{-1}_n \mathbb{E}X^2_1 \mathbb{I} \leq \beta^{-1}_n \mathbb{E}X^2_1 (\mathbb{I}_1 + \mathbb{I}_2) \leq \beta^{-1}_n \mathbb{E}X^2_1 (\mathbb{I}_1 + 4\varepsilon^{-4}n^{-2}\beta^{-1}_n X^2_2 \mathbb{I}_2) \leq c_n R_2(\varepsilon). \]

Here we applied the inequality \( \mathbb{I} \leq \mathbb{I}_1 + \mathbb{I}_2 \) and then Markov’s inequality.

In order to prove (39) we write \( \Delta_{r,0} \mathbb{I} = \Delta'_{r,0} \mathbb{I} + \Delta''_{r,0} \mathbb{I} \), see (31), and invoke the inequalities
\[ \mathbb{E}\mathbb{I}_{\mathbb{D}_0}|\Delta'_{r,0} \mathbb{I}| \leq c_n n^{-3} \quad \text{and} \quad \mathbb{E}\mathbb{I}_{\mathbb{D}_0}|\Delta''_{r,0} \mathbb{I}| \leq c_n n^{-r^2} R_1(\varepsilon). \]  

The first inequality of (40) follows from (25), (32) and inequalities \( q_0^2 \leq 2q_0^2 + 2(q_0 - \tilde{q}_0)^2 \), and
\[ \mathbb{I}_\mathbb{D}_0|q_0 - \tilde{q}_0| \leq c_n n^{-m-1}X_1X_2R_1(\varepsilon). \]  

The second inequality of (40) follows from (28) and (41).

We complete the proof of (34) by showing (41). To this aim we prove that for \( D_1, D_2 \) satisfying \( |D_1 \cap D_2| = 0 \) the following inequalities hold true
\[ (1 - 3\varepsilon)\tilde{q}_0 \mathbb{I} - \varphi(\varepsilon^{-1}) \frac{X_1X_2}{nm} \mathbb{I} \leq q_0 \mathbb{I} \leq (1 + 2\varepsilon)\tilde{q}_0 \mathbb{I}, \]
\[ q_0 \mathbb{I} \leq 2n^{-m-1}X_1X_2(\varphi(\varepsilon^{-1}) + 2\varepsilon z_2). \]  

Let us prove (42). We write
\[ q_0 = \tilde{E}q_0^*, \quad \text{where} \quad q_0^* = \tilde{P}'(n_1 = 1, n_2 = 1|X_3) = \tau_1 \tau_2, \]
\[ \tau_1 = \tilde{P}'(n_1 = 1|X_3), \quad \tau_2 = \tilde{P}'(n_2 = 1|n_1 = 1, X_3), \]  

and apply (15) to probabilities \( \tau_1 \) and \( \tau_2 \). We obtain
\[ \frac{X_1X_2(X_3)^2}{m^2}(1 - \theta_1)(1 - \theta_2) \leq \tau_1 \tau_2 \leq \frac{X_1X_2(X_3)^2}{m^2} \theta_3. \]

Here \( \theta_1 = \frac{(X_1-1)(X_2-1)}{m-X_1+1}, \quad \theta_2 = \frac{(X_2-1)(X_2-2)}{m-X_2+1} \) and \( \theta_3 = \frac{m}{m-X_2} \). Next, we observe that, by our choice of \( \varepsilon \), we have \( \varepsilon \leq \beta^{1/2}_n \) for \( n \geq n_0 \). In particular, the inequality \( X_1 + X_2 \leq \varepsilon^2 n \beta^{1/2}_n \) implies \( X_1 + X_2 \leq \varepsilon m \). Assuming, in addition, that \( X_3 \leq \varepsilon^{-1} \beta^{1/2}_n \), we obtain \( X_1X_3 \leq \varepsilon m, \) for \( i = 1, 2 \).

These inequalities imply \( \theta_i \leq \varepsilon/(1 - \varepsilon), \) \( i = 1, 2, \) and \( \theta_3 \leq 1 + 2\varepsilon \). Note that \( \varepsilon < 0.1 \). Hence, we have
\[ (1 - \theta_1)(1 - \theta_2) \geq 1 - \theta_1 - \theta_2 \geq 1 - 3\varepsilon. \]

Now, we write
\[ \frac{X_1X_2(X_3)^2}{m^2}(1 - 3\varepsilon)\mathbb{I}_{X_3} \mathbb{I} \leq \mathbb{I}_{X_3} \mathbb{I}_{\tau_1 \tau_2} \leq \mathbb{I}_{\tau_1 \tau_2} \leq \frac{X_1X_2(X_3)^2}{m^2}(1 + 2\varepsilon) \mathbb{I} \]
and, using identities \( \mathbb{I}_{q_0} = \tilde{E}\tilde{q}_0^* = \tilde{E}\tilde{E}^\dagger q_0^* = \tilde{E}\tilde{E}^\dagger \mathbb{I}_{\tau_1 \tau_2} \), we obtain
\[ \mathbb{I}_{q_0} \leq (1 + 2\varepsilon) \frac{X_1X_2}{m^2} \tilde{E}'(X_3) \mathbb{I} \leq (1 + 2\varepsilon) \tilde{q}_0, \]
\[ \mathbb{I}_{q_0} \geq (1 - 3\varepsilon) \frac{X_1X_2}{m^2} \tilde{E}'(\mathbb{I}_{X_3}) \mathbb{I} \geq (1 - 3\varepsilon) \tilde{q}_0 - \mathbb{I}\varphi(\varepsilon^{-1}) \frac{X_1X_2}{nm}. \]

In the last step we used the inequalities \( \beta^{-1}_n \tilde{E}'(X_3) \mathbb{I} \leq \beta^{-1}_n \tilde{E}'X_3^2(1 - \mathbb{I}_{X_3}) \leq \varphi(\varepsilon^{-1}). \) Now we prove (43). To this aim we write
\[ q_0 \leq q_0 + q_4, \quad q_0 := \tilde{P}'(n_1 \geq 2, n_2 \geq 1) \mathbb{I}_{\mathbb{D}_0}, \quad q_4 := \tilde{P}'(n_1 \geq 1, n_2 \geq 2) \mathbb{I}_{\mathbb{D}_0} \]
and show that for $D_1, D_2$ satisfying $|D_1 \cap D_2| = 0$ the following inequalities hold true

$$q_0 \leq n^{-1}m^{-1}X_1X_2(4\varepsilon\bar{z}_2 + 2\varphi(\varepsilon^{-1})).$$  \hspace{1cm} (46)

We only prove (46) for $j = 4$ (both cases $j = 3, 4$ are identical). Observing that probabilities $p_{k*} := \tilde{P}'(n_1 \geq 1, n_2 \geq k|X_3)$ satisfy the inequality $p_{k*} \leq p_{1*}$, we write

$$q_{04} = \tilde{E}'p_{2*} = \tilde{E}'p_{2*}(\mathbb{I}_{k+2}) \leq \tilde{E}'p_{2*}(\mathbb{I}_{k+3}) + \tilde{E}'p_{1*}(\mathbb{I}_{k+3}).$$  \hspace{1cm} (47)

Next, we split

$$p_{k*} = \tau_{k*}\tau_*,$$

and apply (15) to the probabilities $\tau_*$ and $\tau_{k*}$. We have

$$\tau_* \leq X_1X_3 \frac{m}{m - X_1}, \quad \tau_{1*} \leq \frac{X_2(X_3 - 1)}{m - X_1} \leq \frac{X_2X_3}{m - X_1} \frac{\theta_3}{\theta_3}, \quad \tau_{2*} \leq \frac{\theta_3^2}{(m - X_1)^2} \leq \frac{\theta_3^2}{m^2} \theta_3^2.$$  

We recall that $\theta_3 = m/(m - X_1)$ satisfies $\theta_3 \leq (1 + 2\varepsilon)\leq 2\varepsilon$. Collecting these inequalities in (47) we obtain (46):

$$q_{04} \leq 4\varepsilon X_1X_2 \frac{m}{m - X_1} \frac{\theta_3^2}{m^2} \tilde{E}'X_3^3\mathbb{I}_{k+3} + 2\varepsilon\frac{X_1X_2}{m} \tilde{E}'X_3^2\mathbb{I}_{k+3} \leq 4\varepsilon X_1X_2 \frac{m}{m - X_1} \frac{\theta_3^2}{m^2} \varepsilon + 2\varphi(\varepsilon^{-1}) \frac{X_1X_2}{m} \theta_3^2.$$  

In the last step we used identity $m^{-1}\tilde{E}'X_3^3\mathbb{I}_{k+3} = n^{-1}\varphi(\varepsilon^{-1})$ and inequalities

$$X_1X_2 \frac{\theta_3^2}{m^2} \leq \varepsilon X_1X_2 \frac{\theta_3^2}{m^2}.$$  

We secondly prove (35). Denote

$$R_{01} = f_r(\tilde{\lambda}_0) - (r!)^{-1}\tilde{\lambda}_0, \quad R_{02} = (\tilde{P}(A_0) - 1)(r!)^{-1}\tilde{\lambda}_0.$$

We observe that $1 - e^{-\tilde{\lambda}_0} \leq \tilde{\lambda}_0$ implies $|R_{01}| \leq \tilde{\lambda}_0^{r+1}$. Furthermore, from the inequality

$$1 - \tilde{P}(A_0) = \tilde{P}(D_1 \cap D_2 \neq \emptyset) \leq X_1X_2 m^{-1},$$

see (15), we obtain $|R_{02}| \leq \tilde{\lambda}_0^r X_1X_2 m^{-1}$. We remark that, for $r = 0, 1$, relation (35) follows from the bounds $E\tilde{\lambda}_0^{r+1} = O(n^{-r-1})$ and $E\tilde{\lambda}_0^r X_1X_2 m^{-1} = O(n^{-r-1})$. Indeed, we have

$$E\|A_0 f_r(\tilde{\lambda}_0) - E(r!)^{-1}\tilde{\lambda}_0 = E I_{A_0} R_{01} + E(I_{A_0} - 1)(r!)^{-1}\tilde{\lambda}_0 = E I_{A_0} R_{01} + E R_{02} = O(n^{-r-1}).$$

In the case where $r = 2$ we invoke the truncation argument. Denote

$$R_{03} = f_2(\tilde{\lambda}_0)(1 - I_{I_2}) \quad R_{04} = (2!)^{-1}\tilde{\lambda}_0^{I_1}(1 - I_{I_2}).$$

We observe that inequalities

$$I_{I_2} \leq 1 \leq I_{I_2} + I_{I_2}$$

imply, for $j = 3, 4$,

$$E|I_{I_2}| \leq E\tilde{\lambda}_0^j(I_{I_2} + I_{I_2}) \leq c_n n^{-2}\varphi(n^{1/4}) = o(n^{-2}).$$

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Finally, we obtain \(35\) from the identities
\[
\begin{align*}
\mathbb{E}[A_0 f_2(\hat{\lambda}_0) - \mathbb{E}(2!)^{-1} \hat{\lambda}^2_0 = \mathbb{E}[A_0 I_1 I_2 f_2(\hat{\lambda}_0) - \mathbb{E}_1 I_2(2!)^{-1} \hat{\lambda}^2_0 + \mathbb{E}[A_0 R_{03} - \mathbb{E} R_{04},
\mathbb{E}[A_0 I_1 I_2 f_2(\hat{\lambda}_0) - \mathbb{E}_1 I_2(2!)^{-1} \hat{\lambda}^2_0 = \mathbb{E}[A_0 I_1 I_2 R_{01} + \mathbb{E}_1 I_2 R_{02}
\end{align*}
\]
combined with bounds \(52\) and
\[
\mathbb{E}_1 I_2(|R_{01}| + |R_{02}|) \leq \mathbb{E}_1 I_2(\hat{\lambda}^3_0 + \hat{\lambda}^2_0 X_1 X_2 m^{-1}) \leq c_n n^{-5/2}.
\]
Let us prove \(36\). We write
\[
\mathbb{E}[A_0 f_r(\hat{\lambda}_0) \leq \mathbb{E}[f_r(\hat{\lambda}_0) I_1 I_2 + I_1 + I_2)
\]
and apply the inequalities \(f_r(t) \leq t^j f_r-j(t) \leq t^j, 0 \leq j \leq r\). For \(r \geq 3\) we obtain
\[
\begin{align*}
\mathbb{E}[f_r(\hat{\lambda}_0) I_1 I_2 & \leq \mathbb{E}[\hat{\lambda}^3_0 I_1 I_2 \leq c_n n^{-1/2} \mathbb{E}\hat{\lambda}^2_0 = O(n^{-5/2})
\mathbb{E}[f_r(\hat{\lambda}_0) (I_1 + I_2) & \leq \mathbb{E}[\hat{\lambda}^2_0 (I_1 + I_2) \leq c_n n^{-2} \varphi(n^{1/4}) = o(n^{-2})
\end{align*}
\]
Proof of \(22\), \(23\). We remark that \(22\), \(23\) follows from \(30\) and the bounds, for \(i = 1, 2\),
\[
\begin{align*}
\mathbb{I}[A_1 \Delta_{r,i} \prec O_r,
\mathbb{P}(A_i) - x_r) f_r(\hat{\lambda}_i) \prec O_{r\cap i}.
\end{align*}
\]
We first prove \(53\). For this purpose we combine identities
\[
\Delta r,i = \Delta_{r,i} I_{s1} + \Delta_{r,i} I_{s1} = \Delta\prime r,i I_{s1} + \Delta\prime r,i I_{s1}
\]
with the bounds, which are shown below,
\[
\begin{align*}
\mathbb{I}[A_1 \Delta_{r,i} I_{s1} \prec O_r,
\mathbb{I}[A_1 \Delta\prime r,i I_{s1} \prec O_r,
\mathbb{I}[A_1 \Delta r,i I_{s1}] \prec O_2
\end{align*}
\]
We remark that the third bound of \(55\) is an easy consequence of Markov’s inequality,
\[
\mathbb{E}[A_1 \Delta r,i I_{s1} \leq \mathbb{E}[I_{s1} \leq 4(nm)^{-1} \varphi(0.5\sqrt{nm}) = o(n^{-2})
\]
Now we prove the first and second bound of \(55\) in the case where \(i = 1\). In the proof we use the simple identity \(q_{11} = q_1\) and inequality
\[
q_{12} I_{14} \leq 2n^{-1} m^{-1} \varepsilon_2 X_1 X_2
\]
which hold whenever conditions of event \(A_1\) are satisfied. We note that \(56\) follows from identities
\[
\begin{align*}
q_{12} &= \mathbf{P}(w_1^{\prime} \notin D_3, n_1 \geq 1, n_2 \geq 1|X_3) = \mathbf{P}(w_1^{\prime} \notin D_3|X_3)\mathbf{P}(n_1 \geq 1|w_1^{\prime} \notin D_3, X_3),
\tau_1^{\prime} &= \mathbf{P}(n_1 \geq 1|w_1^{\prime} \notin D_3, X_3),
\tau_2^{\prime} &= \mathbf{P}(n_2 \geq 1|n_1 \geq 1, w_1^{\prime} \notin D_3, X_3)
\end{align*}
\]
and inequalities, see \(15\),
\[
\tau_1^{\prime} \leq (m - 1)^{-1} (X_1 - 1) X_3, \quad \tau_2^{\prime} \leq (m - X_1)^{-1} (X_2 - 1) (X_3 - 1).
\]
Let us prove the first bound of \(55\). Combining \(32\) with inequality \(q_{12}^{2} \leq 2q_{11}^{2} + 2q_{12}^{2}\) we write \(|\Delta_{r,1}^{\prime}| \leq n q_{12}^{2} \leq 2n q_{11}^{2} + 2n q_{12}^{2}\). Hence, we obtain
\[
\mathbb{E}[A_1 |\Delta_{r,1}^{\prime} I_{s1} \leq \mathbb{E}[A_1) 2n q_{11}^{2} + \mathbb{E}[A_1) 2n q_{12}^{2} I_{s1}.
\]
Furthermore, invoking inequality $\mathbf{E}nq^2_1\mathbf{P}(A_1) \leq c_* n^{-2} \beta_n^{-1}$, which follows from (26), and bound $\mathbf{E}nq^2_1I_4 = O(n^{-3})$, which follows from (56), we obtain the first bound of (55). Let us prove the second bound of (55). In the proof we use the inequalities

$$|\lambda_1 - \tilde{\lambda}_1| \leq 2\tilde{q}_1 + nq_{12}, \quad \lambda_1 + \tilde{\lambda}_1 \leq 2n\tilde{q}_1 + nq_{12}. \quad (57)$$

For $r = 0, 1$ we apply (28) and obtain

$$\mathbb{E}I_{A_1} |\Delta''_{1}|_{I_{1}} \leq \mathbb{E}I_{A_1} |\lambda_1 - \tilde{\lambda}_1|_{I_{1}} \leq 2\tilde{q}_1 \mathbb{E}I_{A_1} + n\mathbb{E}I_{A_1} q_{12}I_{1} \quad (58)$$

Then we invoke the bounds $\mathbb{E}I_{A_1} = \mathbb{E}\mathbf{P}(A_1) = O(n^{-1})$, see (26), and

$$n\mathbb{E}I_{A_1} q_{12} I_{1} \leq 2\tilde{z}_m^{-1} \mathbb{E}I_{A_1} X_1 X_2 = 2\tilde{z}_m^{-1} \mathbf{E}I_{A_1} X_1 X_2 = O(n^{-2}), \quad (59)$$

see (26), (56). Clearly, (58), (59) imply the second bound of (55). For $r \geq 2$ we derive the second bound of (55) from inequalities, see (28), (57).

$$\mathbb{I}_{A_1} |\Delta''_{r}|_{I_{1}} \leq \mathbb{I}_{A_1} |\lambda_1 - \tilde{\lambda}_1|_{I_{1}} (\lambda_1 + \tilde{\lambda}_1) \leq \mathbb{I}_{A_1} (4n\tilde{q}_1 + 3n^2\tilde{q}_1 + n^2\tilde{q}_1)$$

combined with relations

$$\mathbb{E}I_{A_1} n\tilde{q}^2_1 = \mathbf{E}nq^2_1\mathbf{P}(A_1) = O(n^{-2}\beta_n^{-1}), \quad (60)$$

$$\mathbb{E}I_{A_1} n^2\tilde{q}_1 I_{1} I_{2} = n^2\tilde{q}_1 \mathbb{E}I_{A_1} q_{12} I_{1} = O(n^{-2}\beta_n^{-1/2}), \quad (61)$$

$$\mathbb{E}I_{A_1} n^2\tilde{q}_1 I_{1} I_{2} \leq 4\tilde{z}_m^{-2} \mathbb{E}I_{A_1} X_1^2 X_2^2 \leq c_* n^{-1/2} + c_* n^{-2} \varphi(n^{-1/4}). \quad (62)$$

Here (60) follows from (26), (61) follows from (59). The first inequality of (62) follows from (56). To show the second inequality of (62) we invoke (27) and write

$$\mathbb{E}I_{A_1} X_1^2 X_2^2 = \mathbf{E}I_{A_1} X_1^2 X_2^2 \leq \mathbf{E}(n^{-1/2} + I_1 + I_2) X_1^2 X_2^2 \leq c_* n^{-1/2} + c_* n^{-2} \varphi(n^{-1/4}).$$

Now we establish the first two bounds of (55) for $i = 2$. In the proof we use the relations

$$q_{24} I_{1} \leq c_* (nm)^{-1} X_1 X_2, \quad (63)$$

$$q_{23} = \mathbf{E}(X_3) (m)^{-1} \leq c_* n^{-2}, \quad (64)$$

$$q_{21} = q_{22} = 2^{-1} \tilde{q}_2 + R_s, \quad (65)$$

where $|R_s| \leq c_* n^{-2}(\beta_n^{-3/2} + \beta_n^{-1})$. Here (63) is obtained in the same way as (56) above, and (64) follows from (15). Furthermore, the first identity of (65) is obvious and second one is obtained from the identities

$$q_{22} = \mathbf{E}^\prime P_2(w_2^* \in D_3 | w_1^* \notin D_3, X_3) \mathbf{P}^\prime(w_1^* \notin D_3 | X_3) = \mathbf{E}^\prime \frac{X_3}{m-1} \left( 1 - \frac{X_3}{m} \right) = \frac{\mathbf{E}X_3}{m} + R_s.$$

To prove the first bound of (55) for $i = 2$ we write, see (32),

$$\mathbb{I}_{A_2} |\Delta'_{r}|_{A_2} \leq \mathbb{I}_{A_2} nq^2_{2} \leq 2n\mathbb{I}_{A_2} (q^2_{21} + q^2_{22} + q^2_{23} + q^2_{24})$$

and invoke the bounds, which follow from (63), (64), (65).

$$\mathbb{E}q^2_{23} \leq c_* n^{-4}, \quad \mathbb{E}q^2_{24} \leq c_* n^{-4}, \quad \mathbb{E}q^2_{21} \leq c_* n^{-4} \beta_n^{-1}.$$

In the last step we used inequalities $\mathbb{E}I_{A_2} = \mathbf{E}I_{A_2} \leq c_* \mathbf{E}I_{22} \leq c_* n^{-2}$, see (26).
The second bound of (55) for $i = 2$ follows from the relations shown below

$$|\Delta''_{\nu}| \leq |\lambda_2 - \tilde{\lambda}_2| \leq n(q_{23} + q_{24} + 2|R|) + 2\tilde{q}_2.$$  \hspace{1cm} (66)

$$\mathbb{E}\pi_{A_2}(q_{23} + 2|R| + 2\tilde{q}_2) = (q_{23} + 2|R| + 2\tilde{q}_2)\mathbb{E}\pi_{A_2} \leq c_s(n^{-4} + n^{-3}\beta_{n-1/2})$$,

$$\mathbb{E}\pi_{A_2}q_{24}I_1 \leq c_s(nm)^{-1}\mathbb{E}\pi_{A_2}X_1X_2 = c_s(nm)^{-1}\mathbb{E}P(A_2)X_1X_2 = o(n^{-3}).$$  \hspace{1cm} (68)

Here the first inequality of (66) follows from (28), and the second inequality follows from (65) and the identity

$$\lambda_2 - \tilde{\lambda}_2 = (n - 2)(q_{23} + q_{24} + (q_{21} + q_{22} - \tilde{q}_2)) - 2\tilde{q}_2.$$

Furthermore, (67) follows from (64), (65) and inequality $\mathbb{E}\pi_{A_2} \leq c_s n^{-2}$. Finally, the first inequality of (68) follows from (63), and in the the last step of (68) we use the inequality

$$P(A_2) \leq \epsilon_n^{1/2}(n^{1/2} + \bar{I}_1 + \bar{I}_2),$$

which follows from (27).

Now we prove (64). Since $f_r(\tilde{\lambda}_1) \leq 1$ it suffices to show that $\kappa_i - \tilde{P}(A_i) \prec O_{r\sqrt{i}}$. For $i = 1$ we write, see (26),

$$\kappa_1 \geq \tilde{P}(A_1) \geq P(A_1)I_{a_1} \geq \kappa_1\left(1 - \frac{X_1X_2}{m - X_1}\right) I_{a_1} = \kappa_1 - R_{11} - R_{12},$$  \hspace{1cm} (69)

where

$$R_{11} = \kappa_1I_{a_1} \leq 2\kappa_1\frac{X_1}{m}, \quad R_{12} = \kappa_1\frac{X_1X_2}{m - X_1}I_{a_1} \leq 2\kappa_1\frac{X_1X_2}{m}.$$  \hspace{1cm} (70)

Hence, we have $E|\kappa_1 - \tilde{P}(A_1)| \leq E|R_{11}| + E|R_{12}| = O(n^{-2})$.

For $i = 2$ we proceed as follows. Given $0 < \varepsilon < 1$ we write, see (26),

$$\kappa_2 \geq \tilde{P}(A_2) \geq P(A_2)\|I_{a_1}\|I_{a_2} \geq \kappa_2\left(1 - \frac{X_1X_2}{m - X_1}\right) I_{a_1}I_{a_2} = \kappa_2 - R_{12} - R_{22},$$  \hspace{1cm} (71)

where

$$R_{12} = \kappa_2(1 - \|I_{a_1}\|\|I_{a_2}\|) \leq \kappa_2(\bar{I}_{a_1} + \bar{I}_{a_2}), \quad R_{22} = \kappa_2\frac{X_1X_2}{m - X_1}I_{a_1}I_{a_2} \leq \kappa_2\frac{\beta_n\varepsilon^{-2}}{m - \beta_n\varepsilon^{-1}}.$$  \hspace{1cm} (72)

We let $\varepsilon = \varepsilon_n \rightarrow 0$ slowly enough to get

$$\frac{\beta_n\varepsilon^{-2}}{m - \beta_n\varepsilon^{-1}} = o(1).$$

Then we obtain $E|\kappa_2 - \tilde{P}(A_2)| \leq E|R_{21}| + E|R_{22}| = o(n^{-2})$.

Proof of (24). We write, for short, $\tilde{p}_k := \tilde{P}(|D_1 \cap D_2| \geq k)$. We have, see (15),

$$\tilde{p}_k \leq \kappa_k \leq Z_k^k Z_k^k n^{-k}.$$  \hspace{1cm} (73)

Taking the expected values in (73) we obtain (24) for $k = 1, 2$. For $k = 3$ we apply (27) and write

$$\tilde{p}_3 \leq \tilde{p}_3^{2/3} \tilde{p}_3^{1/3} \leq \kappa_3^{2/3}(n^{-3/2} + \bar{I}_1 + \bar{I}_2)^{1/3} \leq \kappa_3^{2/3}(n^{-1/2} + \bar{I}_1 + \bar{I}_2).$$

Hence, we have $P(|D_1 \cap D_2| \geq 3) \leq E\kappa_3^{2/3}(n^{-1/2} + \bar{I}_1 + \bar{I}_2) = o(n^{-2})$. \hfill $\square$
5.2. Inhomogeneous graph. Before the proof of Theorem 3 we introduce some notation and show (11). By $P^*$ and $E^*$ we denote the conditional probability and expectation given $A_1, A_2, B_m$. Given $i, j, l \in [m]$ and $k, s, t \in [n]$, denote

\[
I_{ki} = \mathbb{I}_{\{A_k B_i \leq \sqrt{nm}\}}, \quad \hat{I}_{ki} = \mathbb{I}_{\{w_i \in D_k\}},
\]

\[
\mathcal{H}_i = \{w_i \in D_1 \cap D_2\}, \quad \mathcal{H}_{ijk} = \{w_i \in D_1 \cap D_k, w_j \in D_2 \cap D_k\},
\]

\[
\mathcal{H}^*_i = \{w_i = D_1 \cap D_2\}, \quad \mathcal{H}_{ij} = \{w_i, w_j \in D_1 \cap D_2\},
\]

\[
\mathcal{L}_i = \{w_i \in D_1 \cap D_2 \cap D_3\}, \quad \mathcal{L}_{ijl} = \{w_i \in D_1 \cap D_2, w_j \in D_1 \cap D_3, w_l \in D_2 \cap D_3\},
\]

\[
\mathcal{U}_e = \{v_1 \sim v_2, v_s \sim v_1, v_s \sim v_2\}, \quad \mathcal{U}_{st} = \mathcal{U}_e \cap \mathcal{U}_s,
\]

\[
\mathcal{H}^* = \cup_{i \in [m]} \mathcal{H}^*_i, \quad \mathcal{H}^{**} = \cup_{\{i,j\} \subset [m]} \mathcal{H}_{ij}, \quad \mathcal{H}^{***} = \cup_{k=3}^n \cup_{\{i,j\} \subset [m-1]} \mathcal{H}_{ijk} \cup \mathcal{H}_{jik},
\]

\[
\mathcal{L} = \cup_{i \in [m]} \mathcal{L}_i, \quad \mathcal{L}^* = \cup_{\{i,j,l\} \subset [m]} \mathcal{L}_{ijl}, \quad \mathcal{L}^{**} = \cup_{1 \leq i,j \leq m, i \neq j} \mathcal{H}_{ij3}.
\]

Introduce the random variable $S = \sum_{3 \leq k \leq n} \mathbb{I}_{km}$ and probability $p_r = P(\mathcal{H}_r^* \cap \{S = r\})$. Let us prove (11). It follows from identity (77) and the bounds \[\text{Inclusion-exclusion, that}
\]

\[
\sum_{i \in [m]} P(\mathcal{H}_i) - \sum_{\{i,j\} \subset [m]} P(\mathcal{H}_i \cap \mathcal{H}_j) \leq P(v_1 \sim v_2) \leq \sum_{i \in [m]} P(\mathcal{H}_i).
\]

We derive (11) from these inequalities using relations

\[
P(\mathcal{H}_i) = (nm)^{-1}(a_1^2 b_2 + o(1)), \quad P(\mathcal{H}_i \cap \mathcal{H}_j) \leq E p_{1i} p_{2i} p_{1j} p_{2j} \leq (nm)^{-2} a_3^2 b_5^2.
\]

To show the first relation we apply the inequality $I_{1i} I_{2i} \geq 1 - \hat{I}_{1i} - \hat{I}_{2i}$ and write

\[
\frac{A_1 A_2 B^2_i}{nm} \geq p_{1i} p_{2i} \geq p_{1i} p_{2i} I_{1i} I_{2i} = \frac{A_1 A_2 B^2_i}{nm} (I_{1i} I_{2i} - 1 - \hat{I}_{1i} - \hat{I}_{2i}).
\]

Then we take the expected values in (75), use the identity $P(\mathcal{H}_i) = E p_{1i} p_{2i}$ and the bound $E A_1 A_2 B^2_i (I_{1i} + I_{2i}) = o(1)$.

**Proof of Theorem 3.** In order to prove (13) we write $cl(r) = p^*_r/(p^*_r + p^*_{r'})$, where

\[
p^*(r) = P(v_1 \sim v_2, d_{12} = r), \quad p^*_r = P(v_1 \sim v_2, d_{12} = r),
\]

and invoke relations

\[
p^*_r = E f_l(r) a_1^2 B_m n^{-1} + o(n^{-1}), \quad r = 0, 1, \ldots, \quad (76)
\]

\[
\bar{p}_0 = 1 - O(n^{-1}), \quad \bar{p}_r = O(n^{-2}), \quad r \geq 2. \quad (77)
\]

\[
\bar{p}^*_1 = n^{-1} a_1^2 a_2 b_5^2 + o(n^{-1}). \quad (78)
\]

Here we denote $\Lambda = a_1 B_m \beta^{-1/2}_n$.

Let us prove (77). For $r = 0$ we write

\[
P(v_1 \sim v_2, d_{12} = 0) = P(v_1 \sim v_2) - P(v_1 \sim v_2, d_{12} \geq 1)
\]

and invoke the bounds

\[
1 - P(v_1 \sim v_2) = O(n^{-1}), \quad P(v_1 \sim v_2, d_{12} \geq 1) = O(n^{-1}).
\]
The first bound follows from \([11]\). In order to show the second bound we note that the event \(\{v_1 \not\sim v_2, d_{12} \geq 1\}\) implies that there exist \(i, j \in [m]\), \(i \neq j\) and \(3 \leq k \leq n\) such that \(I_{i1} I_{k2} I_{kj} = 1\). Hence, by Markov’s inequality,

\[
P(\{v_1 \not\sim v_2, d_{12} \geq 1\}) \leq E \sum_{3 \leq k \leq n} \sum_{i, j \in [m], i \neq j} I_{i1} I_{k2} I_{kj} = \sum_{3 \leq k \leq n} \sum_{i, j \in [m], i \neq j} EP_{i1} P_{k2} P_{kj}.
\]

By the inequality \(EP_{i1} P_{k2} P_{kj} \leq (nm)^{-2} E A_1 A_2 B_1^2 B_2^2\), the right hand side sum is \(O(n^{-1})\). For \(r \geq 2\) we write \(p^*_r \leq \bar{p}\), where \(\bar{p} = P(\{v_1 \not\sim v_2, d_{12} \geq 2\})\), and invoke the bound \(\bar{p} = O(n^{-2})\). Let us prove this bound. Given \(3 \leq s < t \leq n\) introduce events

\[U_{1, st} = \{\exists i \neq j \text{ such that } w_i \in D_1 \cap D_s \cap D_t \text{ and } w_j \in D_2 \cap D_s \cap D_t\};
\]

\[U_{2, st} = \{\exists i_1 \neq i_2 \neq j \text{ such that } w_{i_1} \in D_1 \cap D_s, w_{i_2} \in D_1 \cap D_t \text{ and } w_j \in D_2 \cap D_s \cap D_t\};
\]

\[U_{3, st} = \{\exists i \neq j_1 \neq j_2 \text{ such that } w_i \in D_1 \cap D_s \cap D_t \text{ and } w_j \in D_2 \cap D_s \cap D_t\};
\]

\[U_{4, st} = \{\exists i_1 \neq i_2 \neq j_1 \neq j_2 \text{ such that } w_{i_1} \in D_1 \cap D_s, w_{i_2} \in D_1 \cap D_t \text{ and } w_{j_1} \in D_2 \cap D_s, w_{j_2} \in D_2 \cap D_t\}
\]

and observe that \(U_{st} = \cup_{k \in [4]} U_{k, st}\). Next, using the identity \(\{v_1 \not\sim v_2, d_{12} \geq 2\} = \cup_{s < t} U_{st}\) we obtain

\[
\bar{p} = P(\cup_{s < t} U_{st}) \leq \sum_{s < t} P(U_{st}) = \binom{n-2}{2} \sum_{k \in [4]} P(U_{k, st}) = O(n^{-2}).
\] (79)

In the last step we invoke the bounds that follow by Markov’s inequality

\[
P(U_{1, st}) \leq E \sum_{i, j \in [m], i \neq j} I_{i1} I_{i2} I_{j1} I_{j2} I_{ij} \leq \frac{a_1^2 a_2^2 b_3^3}{n^3 m},
\]

\[
P(U_{3, st}) = P(U_{2, st}) \leq E \sum_{i_1, i_2, j \in [m], i_1 \neq i_2 \neq j} I_{i_11} I_{i_12} I_{i_21} I_{i_22} I_{j1} I_{j2} I_{ij} \leq \frac{a_1^2 a_2^2 b_3^3}{n^{7/2} m^{1/2}},
\]

\[
P(U_{4, st}) \leq \sum_{i_1, i_2, j_1, j_2 \in [m], i_1 \neq i_2 \neq j_1 \neq j_2} I_{i_11} I_{i_12} I_{i_21} I_{i_22} I_{j_11} I_{j_12} I_{j_21} I_{j_22} I_{ij} \leq \frac{a_1 b_3^3}{n^4}.
\]

Proof of (77) is complete.

Let us prove (78). We have, see (79),

\[
p^*_r = P(\{v_1 \not\sim v_2, d_{12} \geq 1\}) - P(\{v_1 \not\sim v_2, d_{12} \geq 2\}) = P(\{v_1 \not\sim v_2, d_{12} \geq 1\}) - O(n^{-2}).
\]

Furthermore, from the identity \(\{v_1 \not\sim v_2, d_{12} \geq 1\} = \cup_{3 \leq s \leq n} U_s\) we obtain, by inclusion-exclusion,

\[
0 \leq \sum_{3 \leq s \leq n} P(U_s) - P(\{v_1 \not\sim v_2, d_{12} \geq 1\}) \leq \sum_{3 \leq s < t \leq n} P(U_{st}) = O(n^{-2}).
\]

In the last step we used (79). It remains to evaluate the sum \(\sum_{3 \leq s \leq n} P(U_s) = (n - 2)P(U_3)\).

We observe that \(U_3 = \cup_{(i, j) \in M} H_{i,j,3}\), where \(M\) is the set of vectors \((i, j) \in [m]^2\) satisfying \(i \neq j\). We write, by inclusion-exclusion, \(S_1 - S_2 \leq P(U_3) \leq S_1\), where

\[
S_1 = \sum_{(i, j) \in M} P(H_{i,j,3}), \quad S_2 = \sum_{(i, j), (k, l) \in M, (i, j) \neq (k, l)} P(H_{i,j,3} \cap H_{k,l,3}),
\]

and complete the proof of (78) by showing that

\[
S_1 = n^{-2}a_1^2 a_2 b_3^3(1 + o(1)), \quad S_2 = o(n^{-2}).
\] (80)
The first relation of (80) follows from the identity
\[ P(\mathcal{H}_{i,j,3}) = E_{p_1}p_{2j}p_{3k}p_{3j} = (nm)^{-2}a_1^2a_2b_2^2(1 + o(1)), \]
which is obtained using the same truncation argument as in (75) above. The second bound of (80) follows from the inequalities that hold for any \( \varepsilon > 0 \)
\[ P(\mathcal{H}_{i,j,3} \cap \mathcal{H}_{k,l,3}) \leq c_\varepsilon^{-2}n^{-2}m^{-4} + o(n^{-3}m^{-3}), \] (81)
\[ P(\mathcal{H}_{i,j,3} \cap \mathcal{H}_{k,l,3}) \leq c_\varepsilon^{-2}n^{-2}m^{-3} + o((nm)^{-5/2}). \] (82)

In order to show (81) we write \( 1 = I + \bar{I} \), where \( I = I_{\{A_1 \leq \varepsilon n\}} \) and invoke the inequalities
\[ p_{3k}p_{3l} \leq p_{3k}p_{3l}I + \bar{I} \leq \varepsilon n^2B_kB_l nm + \bar{I} \]
in the identity \( P(\mathcal{H}_{i,j,3} \cap \mathcal{H}_{k,l,3}) = E_{p_1}p_{2j}p_{3k}p_{3j}p_{1k}p_{2l}p_{3k}p_{3l}. \) Here we also use the bound \( EA_{3\bar{I}} = o(1) \). To show (82) we invoke the inequality
\[ p_{3k} \leq p_{3k}I + \bar{I} \leq \varepsilon nB_k \sqrt{nm} + \bar{I} \]
in the identity \( P(\mathcal{H}_{i,j,3} \cap \mathcal{H}_{k,l,3}) = E_{p_1}p_{2j}p_{3k}p_{3j}p_{1k}p_{3l}. \) Proof of (78) is complete.

Now we prove (76). Firstly, from relations \( \mathcal{H}^* \subset \{v_1 \sim v_2\} \subset \mathcal{H}^* \cup \mathcal{H}^{**} \) we derive inequalities
\[ 0 \leq P(v_1 \sim v_2, d_{12} = r) - P(\mathcal{H}^* \cap \{d_{12} = r\}) \leq P(\mathcal{H}^{**}). \] (83)
Here \( P(\mathcal{H}^{**}) \leq \binom{m}{2} P(\mathcal{H}_{ij}) = O(n^{-2}) \), since \( P(\mathcal{H}_{ij}) \leq E_{p_1}p_{1j}p_{2j}p_{2j} \leq a_2^2b_2^2(nm)^{-2}. \) Secondly, we write, by symmetry,
\[ P(\mathcal{H}^* \cap \{d_{12} = r\}) = \sum_{j \in [m]} P(\mathcal{H}_j^* \cap \{d_{12} = r\}) = mP(\mathcal{H}_m^* \cap \{d_{12} = r\}) \] (84)
and approximate \( P(\mathcal{H}_m^* \cap \{d_{12} = r\}) \) by \( \tilde{p}_r \). We remark that relations \( \mathcal{H}_m^* \cap \{d_{12} = r\} \subset \mathcal{H}_m^* \cap \{d_{12} = r\} \subset \mathcal{H}_m^* \cap \{\{S = r\} \cup \mathcal{H}^{**} \} \)
implies inequalities
\[ 0 \leq P(\mathcal{H}_m^* \cap \{d_{12} = r\}) - \tilde{p}_r \leq P(\mathcal{H}_m^* \cap \mathcal{H}^{**}) \] (85)
and observe that the probability
\[ P(\mathcal{H}_m^* \cap \mathcal{H}^{**}) \leq (n-2)(m-1)(m-2)P(\mathcal{H}_m^* \cap \mathcal{H}_{1j,3}) = O(m^{-1}n^{-2}) \] (86)
because
\[ P(\mathcal{H}_m^* \cap \mathcal{H}_{1j,3}) = E_{p_1p_2p_3p_{ki}p_{kj}}P_{p_1p_{2j}} \leq a_2^3b_2^3(nm)^{-3}. \]
It follows from (83), (84), (85), (86) that
\[ P(v_1 \sim v_2, d_{12} = r) = m\tilde{p}_r + O(n^{-2}). \]
We complete the proof of (76) by showing that
\[ \tilde{p}_r = Ef_{A}(r)A_1A_2B_m^2(mn)^{-1} + o(n^{-2}). \] (87)
Let us show (87). Using LeCam’s inequality, see (14), we write
\[ |P^*(S = r) - f_{\Lambda_0}(r)| \leq \Delta, \quad \Lambda_0 := \sum_{3 \leq k \leq n} p_{km}^r, \quad \Delta := \sum_{3 \leq k \leq n} (p_{km}^r)^2. \] (88)

Here \( p_{km}^r = E^*H_{km} = E^*p_{km} \leq a_1 B_m(nm)^{-1/2} \). In particular, we have \( \Delta \leq a_1^2 B_m^2 m^{-1} \). This inequality and \( (88) \) imply
\[ P_r = \mathbb{E}P^*(S = r)I_{H_m^*} = \mathbb{E}f_{\Lambda_0}(r)I_{H_m^*} + R_1, \]
\[ |R_1| \leq \mathbb{E}\Delta I_{H_m^*} = \mathbb{E}\Delta p_{1m}p_{2m} \leq \mathbb{E}\Delta p_{1m}(A_2 n^{-1/2} + I_{\{B_m \geq \sqrt{m}\}}) = o(n^{-2}). \] (89)

Here we used inequalities
\[ p_{2m} \leq p_{2m}(I_{\{B_m \leq \sqrt{m}\}} + I_{\{B_m \geq \sqrt{m}\}}) \leq A_2 n^{-1/2} + I_{\{B_m \geq \sqrt{m}\}}, \]
\[ \mathbb{E}\Delta p_{1m}A_2 n^{-1/2} \leq n^{-1} m^{-3/2} a_1^2 \mathbb{E}A_1 A_2 B_m^2 = O(n^{-5/2}), \]
\[ \mathbb{E}\Delta p_{1m}I_{\{B_m \geq \sqrt{m}\}} \leq n^{-1} m^{-3/2} a_1^2 \mathbb{E}A_1 A_2 B_m^2 I_{\{B_m \geq \sqrt{m}\}} = o(n^{-2}). \]

Let us now evaluate the term \( \mathbb{E}f_{\Lambda_0}(r)I_{H_m^*} \) of \( (89) \). From relations
\[ H_m^* \subset H_m \subset H_m^* \cup \tilde{H}, \quad \tilde{H} := \cup_{i \in [m-1]} H_{im} \]
we obtain inequalities \( 0 \leq I_{H_m^*} - I_{H_m^*} \leq I_{\tilde{H}} \) which yield the approximation
\[ \mathbb{E}f_{\Lambda_0}(r)I_{H_m^*} = \mathbb{E}f_{\Lambda_0}(r)I_{H_m} + R_2, \]
\[ |R_2| \leq \sum_{i \in [m-1]} P(H_{im}) = \sum_{i \in [m-1]} \mathbb{E}p_{1i}p_{2i}p_{1m}p_{2m} \leq n^{-2} m^{-1} a_1^2 b_2^2. \] (90)

Furthermore, we have
\[ \mathbb{E}f_{\Lambda_0}(r)I_{H_m} = \mathbb{E}f_{\Lambda_0}(r)p_{1m}p_{2m} = \mathbb{E}f_{\Lambda_0}(r)A_1 A_2 B_m^2 (mn)^{-1} + o(n^{-2}). \] (91)

In the last step we replaced \( p_{1m}p_{2m} \) by \( A_1 A_2 B_m^2 (mn)^{-1} \) as in \( (75) \) above.

Now we are going to replace \( f_{\Lambda_0}(r) \) by \( f_{\lambda}(r) \). For this purpose we combine the mean value theorem and the inequality \( |\frac{\partial}{\partial \alpha} f_{\lambda}(r)| \leq 1 \). We obtain
\[ |f_{\Lambda}(r) - f_{\Lambda_0}(r)| \leq |\Lambda - \Lambda_0|. \] (92)

Furthermore, we write \( \Lambda_0 = (n - 2) \mathbb{E}p_{3m} \) and \( \Lambda = n \mathbb{E}(A_3 B_m^3 \sqrt{mn}) \) and estimate
\[ |\Lambda - \Lambda_0| \leq (n - 2) |\mathbb{E}(A_3 B_m^3 \sqrt{mn}) - p_{3m}| + 2 \frac{a_1 B_m^3}{\sqrt{mn}} \leq (n - 2) |\mathbb{E}(A_3 B_m^3 \sqrt{mn}) - p_{3m}| + 2 \frac{a_1 B_m^3}{\sqrt{mn}} \]
\[ \leq (n - 2) |\mathbb{E}(A_3 B_m^3 \sqrt{mn}) - p_{3m}| + 2 \frac{a_1 B_m^3}{\sqrt{mn}}. \]

The latter inequalities and \( (92) \) yield
\[ \mathbb{E}f_{\Lambda_0}(r)A_1 A_2 B_m^2 (mn)^{-1} = \mathbb{E}f_{\Lambda}(r)A_1 A_2 B_m^2 (mn)^{-1} + o(n^{-2}), \] (93)

since \( \mathbb{E}A_1 A_2 A_3 B_m^3 I_{3m} = o(1) \). Finally, \( (89), (90), (91) \) and \( (93) \) imply \( (87) \). Proof of \( (76) \) is complete.

Let us prove \( (92) \). To this aim we write \( \alpha = P(B)/P(D) \), where \( D \) denotes the event \( \{v_1 \sim v_3, v_2 \sim v_3\} \) and \( B = D \cap \{v_1 \sim v_2\} \), and show that
\[ P(B) = \kappa_1 + o(n^{-2}), \quad P(D) = \kappa_1 + \kappa_2 + o(n^{-2}). \] (94)
Here \( \kappa_1 := a_1^3 b_3 n^{-3/2} m^{-1/2} \) and \( \kappa_2 := a_2^3 b_2 n^{-2} \). To show the first relation of (94) we observe that event \( L \) implies \( B \) and event \( B \) implies \( L \cup L^* \). In particular, we have \( 0 \leq P(B) - P(L) \leq P(L^*) \). Here

\[
P(L^*) = \left( \frac{m}{3} \right) P(L_{123}) \leq \left( \frac{m}{3} \right) a_2^3 b_2^3 (nm)^{-3} = O(n^{-3}).
\]

Hence, \( P(B) = P(L) + O(n^{-3}) \). Next we approximate \( P(L) \) using inclusion-exclusion

\[
\sum_{s \in [m]} P(L_s) - \sum_{\{s, t\} \subset [m]} P(L_s \cap L_t) \leq P(L) \leq \sum_{s \in [m]} P(L_s)
\]

and obtain \( P(L) = \kappa_1 + o(n^{-2}) \). Here we invoked the bound

\[
\sum_{\{s, t\} \subset [m]} P(L_s \cap L_t) = \left( \frac{m}{2} \right) P(L_1 \cap L_2) \leq \left( \frac{m}{2} \right) a_2^3 b_2^2 (nm)^{-3} = O(n^{-4})
\]

and approximated, see (73),

\[
\sum_{s \in [m]} P(L_s) = mP(L_s) = mE_{p_{1s}p_{2s}p_{3s}} = m(a_1^3 b_3 (nm)^{-3/2} + o(n^{-3})) = \kappa_1 + o(n^{-2}).
\]

Let us prove the second relation of (94). We observe that \( D = L \cup L^{**} \) and approximate

\[
P(D) \approx P(L) + P(L^{**}) \approx mP(L_1) + m(m-1)P(H_{123}) = \kappa_1 + \kappa_2 + o(n^{-2}).
\]

Our rigorous proof is a bit more involved since we operate under minimal moment conditions. Introduce event \( A^* = \{ A_3 < n^{-1/4} \} \) and its indicator function \( I_{A^*} \). We derive upper and lower bounds for \( P(D) \) from the inequalities

\[
P(L \cap A^*) + P(L^{**} \cap A^*) - P(L \cap L^{**} \cap A^*) \leq P(D \cap A^*) \leq P(D) \leq P(L) + P(L^{**}).
\]

By the union bound, the right hand side is bounded from above by

\[
mP(L_1) + m(m-1)P(H_{123}) = \kappa_1 + \kappa_2 + o(n^{-2}).
\]

Next we show a matching lower bound for \( P(D) \). Proceeding as in (95) we write

\[
P(L \cap A^*) = mP(L_1 \cap A^*) + O(n^{-4}),
\]

where

\[
P(L_1 \cap A^*) = E_{p_{11}p_{21}p_{31}} I_{A^*} = E_{p_{11}p_{21}p_{31}} + o(n^{-3}) = a_1^3 b_3 (nm)^{-3/2} + o(n^{-3}).
\]

Hence, we have \( P(L \cap A^*) = \kappa_1 + o(n^{-2}) \). It remains to show that

\[
P(L^{**} \cap A^*) \geq \kappa_2 + o(n^{-2}), \quad P(L \cap L^{**} \cap A^*) = o(n^{-2}). \tag{96}
\]

Let us prove the first inequality of (96). We write, by inclusion-exclusion,

\[
P(L^{**} \cap A^*) \geq S_3 - S_4, \quad S_3 := \sum_s P(H_{st3} \cap A^*), \quad S_4 := \sum_{st} P(H_{st3} \cap H_{xy3} \cap A^*).
\]

Here and below \( \sum_s \) denotes the sum over all vectors \( (s, t) \) with \( s \neq t \), \( 1 \leq s, t \leq m \). By \( \sum_{st} \) we denote the sum over unordered pairs of distinct vectors \( \{(s, t), (x, y)\} \) with \( s \neq t \), \( x \neq y \) and \( 1 \leq s, t, x, y \leq m \). Next, we calculate

\[
S_3 = m(m-1)P(H_{st3} \cap A^*) = m(m-1)(a_1^3 a_2 b_2^2 (nm)^{-2} + o((nm)^{-2})) = \kappa_2 + o(n^{-2})
\]
and estimate

\[ S_4 = m(m - 1)((m - 2)R_3 + \binom{m-2}{2} R_4) = o(n^{-2}). \]

Here

\[ R_3 = \mathbb{P}(H_{st3} \cap \mathcal{H}_{xy3} \cap \mathcal{A}^*) \leq (nm)^{-3} E A_1 A_2^2 A_3 B_s B_t B_y^2 = O(n^{-2.75}m^{-3}). \]
\[ R_4 = \mathbb{P}(H_{st3} \cap \mathcal{H}_{xy3} \cap \mathcal{A}^*) \leq (nm)^{-4} E (A_1 A_2)^2 A_4^4 (B_s B_t B_x B_y)^2 = O(n^{-3.5}m^{-4}). \]

In the last step we used inequalities \( A_3^{44} A^* < A_2^2 n^{1/2} \) and \( A_3^{33} A^* < A_2^2 n^{1/4} \).

It remains to prove the second bound of (96). We apply the union bound

\[ \mathbb{P}(L \cap L^{**} \cap \mathcal{A}^*) \leq \sum \mathbb{P}(L \cap H_{st3} \cap \mathcal{A}^*) \leq \sum (r_s + r_t + \sum_{u \in [m]\backslash \{s,t\}} r'_u) \]

where

\[ r_s = \mathbb{P}(L_s \cap H_{st3} \cap \mathcal{A}^*), \quad r_t = \mathbb{P}(L_t \cap H_{st3} \cap \mathcal{A}^*), \quad r'_u = \mathbb{P}(L_u \cap H_{st3} \cap \mathcal{A}^*) \]

satisfy \( r_s = r_t \) and estimate

\[ r_s \leq (nm)^{-5/2} E A_1 A_2^2 A_3 B_s B_t^2 = O((nm)^{-5/2}), \]
\[ r'_u \leq (nm)^{-7/2} E A_1^2 A_2^2 A_3 B_s B_t^2 B^3 A^* = O((nm)^{-7/2} n^{1/4}). \]

\[ \square \]

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