Superintegrability and Kontsevich-Hermitian relation

A. Mironov\textsuperscript{a,b,\*} A. Morozov\textsuperscript{d,b,\dag}

\textsuperscript{a} Lebedev Physics Institute, Moscow 119991, Russia
\textsuperscript{b} ITEP, Moscow 117218, Russia
\textsuperscript{c} Institute for Information Transmission Problems, Moscow 127994, Russia
\textsuperscript{d} MIPT, Dolgoprudny, 141701, Russia

Abstract

We analyze the well-known equivalence between the quadratic Kontsevich-Penner and Hermitian matrix models from the point of view of superintegrability relations, i.e. explicit formulas for character averages. This is not that trivial on the Kontsevich side, and seems important for further studies of various deformations of Kontsevich models. In particular, the Brezin-Hikami extension of the above equivalence becomes straightforward.

1 Introduction

According to \cite{1, 2, 11}, the Hermitian matrix model is equivalent to the quadratic Kontsevich-Penner model in the following sense:

\begin{equation}
\int_{N \times N} \exp \left( \sum_k \frac{p_k}{k} \text{Tr} M^k \right) \ e^{-\frac{1}{2} \text{Tr} M^2} dM \sim \int_{n \times n} \det^{-N} (1 - L^{-1} X) \ e^{-\frac{1}{2} \text{tr} X^2} dX
\end{equation}

with

\begin{equation}
p_k = \text{tr} L^{-k}
\end{equation}

and the Gaussian integrals normalized to unity, \( \int e^{-\frac{1}{2} \text{Tr} M^2} dM = \int e^{-\frac{1}{2} \text{tr} X^2} dX = 1 \). Our goal in this note is to make this identity compatible with the superintegrability property \cite{3},

\begin{equation}
\int_{N \times N} \chi_R[M] \ e^{-\frac{1}{2} \text{Tr} M^2} dM = \frac{\chi_R[\delta_{k,2}]}{\chi_R[\delta_{k,1}]} \cdot \chi_R[N]
\end{equation}

which provides explicit expressions for averages of the Schur functions \( \chi_R[M] \) on the both sides. Here the Schur function \cite{4} is a symmetric function of eigenvalues of the matrix \( M \), and it is understood as a function of power sums of the eigenvalues, \( \chi_R[M] := \chi_R[\text{Tr} M^k] \). In the Kontsevich case, these averages were not yet discussed in the literature but are somehow similar to the ones appearing in the cubic Kontsevich \cite{5} (see also \cite{6}) and generalized Kontsevich models \cite{7}, and in the Brezin-Gross-Witten model \cite{8}. For more examples, see \cite{3, 9, 10}.

2 The basic formula

Our consideration is based on a simple generalization of \cite{3} to the Gaussian Hermitian model in the external field with partition function

\begin{equation}
Z = \int_{N \times N} e^{-\frac{1}{2} \text{Tr} M L M L + \sum_k g_k \text{Tr} M^k} dM
\end{equation}
where $M$ is the Hermitian $N \times N$ matrix, $dM$ is the Haar measure, $g_k$ are parameters, and the integral is understood as a formal power series in these parameters $g_k$. The correlators are defined

$$< F(M) >_L := \int_{N \times N} F(M) e^{-\frac{1}{2} \text{Tr} M M L M L} dM$$

(5)

and we normalize the measure in such a way that $< 1 > = 1$.

The average of the Schur functions is easily performed in this model with the help of the Wick theorem following the line of [9, 5], and it gives

$$\langle \chi_R[M] \rangle_L = c_R \cdot \chi_R[L^{-1}]$$

(6)

with

$$c_R = \frac{\chi_R\{\delta_{k,2}\}}{\chi_R\{\delta_{k,1}\}}$$

(7)

This relation (6) is a direct analogue of the result of [5] for the rectangular complex matrix model, which further develops its analogy with the Kontsevich family.

3 The main relation

There are many proofs of formula (1), see, for instance, [11]. Here we are going to derive (1) using the basic formula (6). To this end, we make a change of variables $M \rightarrow L^{-1} M$ in the integral (4) so that the averages in (6) are evaluated with the Gaussian weight $e^{-\frac{1}{2} \text{Tr} M^2}$, but the Schur functions become depending on $L^{-1} M$:

$$\langle \chi_R[L^{-1}M] \rangle = c_R \cdot \chi_R[L^{-1}]$$

(8)

where $< \ldots >$ is understood as the average with the Gaussian measure (3), i.e. without the external matrix.

At the l.h.s. of (1), it is sufficient to apply the Cauchy formula [12]

$$\exp \left( \sum_k \frac{p_k p'_k}{k} \right) = \sum_R \chi_R\{p\} \chi_R\{p'\}$$

(9)

where the sum goes over all Young diagrams $R$, in order to get

$$\int_{N \times N} \exp \left( \sum_k \frac{p_k p'_k}{k} M^k \right) e^{-\frac{1}{2} \text{Tr} M^2} dM = \sum_R \chi_R\{p\} \int_{N \times N} \chi_R[M] e^{-\frac{1}{2} \text{Tr} M^2} dM$$

(10)

Applying the result of [3] (formula (3) above), we get for the l.h.s. of (1)

$$\int_{N \times N} \exp \left( \sum_k \frac{p_k p'_k}{k} M^k \right) e^{-\frac{1}{2} \text{Tr} M^2} dM = \sum_R \frac{\chi_R\{p\}\chi_R\{N\}\chi_R\{\delta_{k,2}\}}{\chi_R\{\delta_{k,1}\}}$$

(11)

Now consider the r.h.s. of (1). In [1, 2], the determinant in this formula entered in a positive degree (see, e.g., [11 Eqs.(I.1),(I.17)]), which gave rise to a simple definition of the integral, since it was just the Gaussian average of a polynomial. It was for the price of various imaginary units and of the minus sign in [2]. Here we consider the integral at the r.h.s. of (1) as a power series at large $L$:

$$\int_{n \times n} \det^{-N}(1 - L^{-1} X) e^{-\frac{1}{2} \text{tr} X^2} dX = \int_{n \times n} \exp \left( N \sum_k \frac{1}{k} \text{tr} (L^{-1} X)^k \right) e^{-\frac{1}{2} \text{tr} X^2} dX$$

Applying once again the Cauchy formula, we obtain

$$\int_{n \times n} \det^{-N}(1 - L^{-1} X) e^{-\frac{1}{2} \text{tr} X^2} dX = \sum_R \chi_R\{N\} \langle \chi_R[L^{-1} X] \rangle$$
Using now (8), we finally come to
\[
\int_{n \times n} \det^{-N}(1 - L^{-1}X) e^{-\frac{1}{2} \text{tr} X^2} dX = \sum_R \frac{\chi_R[L^{-1}] \chi_R[N] \chi_R[\delta_{k,2}]}{\chi_R[\delta_{k,1}]} \int_{n \times n} \exp \left( \sum_k \frac{p_k}{k} \text{Tr} M^k \right) e^{-\frac{1}{2} \text{Tr} M^2} dM
\]
(12)

An example. As a small illustration of how this works, in the first approximation to (12), we have
\[
\int N\text{tr}(L^{-1}XL^{-1}X) + N^2(\text{tr} L^{-1}X)^2 e^{-\frac{1}{2} \text{tr} X^2} dX = \frac{N^2p_2 + Np_1^2}{2}
\]
where we used the relation
\[
\langle X_{ij}X_{kl} \rangle = \frac{\delta_{jk}\delta_{il}}{n^2} \langle \text{tr} X^2 \rangle = \delta_{jk}\delta_{il}
\]
for the Gaussian correlators. At the same time,
\[
\sum_{R: |R|=2} \frac{\chi_R[\text{tr} L^{-k}] \chi_R[N] \chi_R[\delta_{k,2}]}{\chi_R[\delta_{k,1}]} = \chi_{[2]}[\text{tr} L^{-k}] \chi_{[2]}[N] - \chi_{[1,1]}[\text{tr} L^{-k}] \chi_{[1,1]}[N] = \\
= \frac{N(N + 1)(p_2 + p_1^2) - N(N - 1)(-p_2 + p_1)}{4} = \frac{N^2p_2 + Np_1^2}{2}
\]

4 Brezin-Hikami identity

Relation (8) allows one to write a more symmetric version of (12), which appears to be the Brezin-Hikami generalization [13,14] of the Chekhov-Makeenko relation [1]. To emphasize the symmetry, we first write it in a slightly different notation:
\[
\int_{N_1 \times N_1} \text{Det}^{-1}(1 - L_{1}^{-1}M_1 \otimes L_{2}^{-1}) \cdot e^{-\frac{1}{2} \text{Tr}_1 M_1^2} dM_1 = \int_{N_2 \times N_2} \text{Det}^{-1}(1 - L_{1}^{-1} \otimes L_{2}^{-1}M_2) \cdot e^{-\frac{1}{2} \text{Tr}_2 M_2^2} dM_2
\]
(13)

In this identity, there are determinants of the tensor product, and we write 1 instead of $Id \otimes Id$ in order to simplify the notation. By the Cauchy formula, the inverse determinant at the l.h.s. is equal to
\[
\text{Det}^{-1}(1 - L_{1}^{-1}M_1 \otimes L_{2}^{-1}) = \exp \left( \sum_k \frac{1}{k} \text{Tr}_1(L_{1}^{-1}M_1)^k \text{Tr}_2L_{2}^{-k} \right) = \sum_R \chi_R \{ \text{Tr}_1(L_{1}^{-1}M_1)^k \} \chi_R \{ \text{Tr}_2L_{2}^{-k} \}
\]
(14)
and the l.h.s. itself becomes
\[
\int_{N_1 \times N_1} \text{Det}^{-1}(1 - L_{1}^{-1}M_1 \otimes L_{2}^{-1}) e^{-\frac{1}{2} \text{Tr}_1 M_1^2} dM_1 = \sum_R \left( \chi_R \{ \text{Tr}_1(L_{1}^{-1}M_1)^k \} \right)_{N_1 \times N_1} \cdot \chi_R \{ \text{Tr}_2L_{2}^{-k} \} = \sum_R c_R \cdot \chi_R \{ \text{Tr}_1L_{1}^{-k} \} \chi_R \{ \text{Tr}_2L_{2}^{-k} \}
\]
(15)
Clearly, the r.h.s. of (6) is just the same because of the symmetry 1 $\leftrightarrow$ 2, and this proves the relation (13).

An example. It is again instructive to look at the first approximation to (13). At the l.h.s., we have a clearly symmetric expression:
\[
\int_{N_1 \times N_1} \frac{\text{Tr}_1(L_{1}^{-1}M_1L_{1}^{-1}M_1) \cdot \text{Tr}_2L_{2}^{-2} + (\text{Tr}_1L_{1}^{-1}M_1)^2 \cdot (\text{Tr}_2L_{2}^{-1})^2}{2} e^{-\frac{1}{2} \text{Tr}_1 M_1^2} dX = \\
= \frac{(\text{Tr}_1L_{1}^{-1})^2 \cdot \text{Tr}_2L_{2}^{-2} + \text{Tr}_1L_{1}^{-2} \cdot (\text{Tr}_2L_{2}^{-1})^2}{2}
\]
where we used
\[
\langle M_{ij}M_{kl} \rangle = \frac{\delta_{jk}\delta_{il}}{N^2} \langle \text{Tr} M^2 \rangle = \delta_{jk}\delta_{il}
\]
for the Gaussian correlators.
5 Conclusion

In this paper, we exploited a new relation \( (6) \) for the Gaussian averages, and demonstrated that it stands behind the Chekhov-Makeenko identity \( (1) \) between the Hermitian and quadratic Kontsevich-Penner models, and behind its further Brezin-Hikami extension.

In addition to these "practical" applications, one can also wonder about the theoretical meaning of our results. The superintegrability property \( (3) \) is a direct corollary of \( (6) \) at \( L = 1 \), with the obvious substitution of the \( n \times n \) matrix \( X \) by the \( N \times N \) matrix \( M \). This can make \( (6) \) a reasonable enhancement of \( (3) \). However, the inverse claim, i.e. whether \( (6) \) is directly implied by \( (3) \), remains unclear. A possible approach to this problem can require supplementing \( (3) \) with some version of the Wick theorem which can be used as a consistency relation for \( (3) \) and a source of some stronger statements like \( (6) \). Alternatively, one can just postulate \( (6) \) in addition to \( (3) \), but this requires a study of possible mutual restrictions on the two definitions. A separate interesting question is the relation between \( (3) \), \( (6) \) and factorization property of single-trace Harer-Zagier functions \[15, 16\]. All these issues remain for the future work.

Acknowledgements

This work was supported by the Russian Science Foundation (Grant No.20-12-00195).

References

[1] L. Chekhov, Y. Makeenko, Phys. Lett. B278 (1992) 271, hep-th/9202006
S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, Nucl.Phys. B397 (1993) 339-378, hep-th/9203043
[2] A. Morozov, Phys.Usr.(UFN) 37 (1994) 1; hep-th/9502091, hep-th/0502010
A. Mironov, Int.J.Mod.Phys. A9 (1994) 4355; Phys.Part.Nucl. 33 (2002) 537; hep-th/9409190
[3] H. Itoyama, A. Mironov, A. Morozov, JHEP 1706 (2017) 115, arXiv:1704.08648
JHEP 1912 (2019) 127, arXiv:1909.06021
A.Mironov, A.Morozov, Phys.Lett. B 802 (2020) 135237, arXiv:1910.03261
A.Mironov, A.Morozov, Phys.Lett. B 771 (2017) 503-507, arXiv:1705.11056
JHEP 1808 (2018) 163, arXiv:1807.02409
[4] I.G. Macdonald, Symmetric functions and Hall polynomials, Second Edition, Oxford University Press, 1995
[5] A.Mironov, A.Morozov, arXiv:2011.12917
A.Mironov, A.Morozov, S.Natanzon, A.Orlov, arXiv:2012.09847
[6] P. Di Francesco, C. Itzykson, J. B. Zuber, Commun. Math. Phys. 151 (1993) 193, hep-th/9206090
[7] A.Mironov, A.Morozov, arXiv:2101.08759
[8] A. Alexandrov, arXiv:2012.07753
[9] A.Mironov, A.Morozov, Phys.Lett. B774 (2017) 210-216, arXiv:1706.03667
[10] C. Cordova, B. Heidenreich, A. Popolitov, S. Shakirov, Commun. Math. Phys. 361 (2018) 1235, arXiv:1611.03142
[11] A.Alexandrov, A.Mironov and A.Morozov, JHEP 12 (2009) 053, arXiv:0906.3305
[12] W. Fulton, Young tableaux: with applications to representation theory and geometry, London Mathematical Society, 1997
For a review adapted for our purposes, see: A. Morozov, Eur.Phys.J. C79 (2019) 76, arXiv:1812.03853
[13] E. Brezin, S. Hikami, Comm. Math. Phys. 283 (2008) 507, arXiv:0708.2210
J. Phys. A. 40 (2007) 13545, arXiv:0704.2044
JHEP 10 (2007) 096, arXiv:0709.3378
JHEP 04 (2009) 110, arXiv:0810.1085
[14] E. Brezin, S. Hikami, JHEP 1007 (2010) 067, arXiv:1005.4730
[15] J. Harer, D. Zagier, Invent.Math. 85 (1986) 457-485
C. Itzykson, J.-B. Zuber, Comm.Math.Phys. 134 (1990) 197-208
S.K. Lando, A.K. Zvonkin, Embedded graphs, Max-Plank-Institut für Mathematik, Preprint 2001 (63)
[16] A. Morozov, S. Shakirov, JHEP 0904 (2009) 064, arXiv:0902.2627