Existence of infinite stationary solutions of the $L^2$-subcritical and critical NLSE on compact metric graphs

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October 26, 2017

Abstract
We investigate the existence of stationary solutions for the Nonlinear Schrödinger equation on compact metric graphs. In the $L^2$-subcritical setting, we prove the existence of an infinite number of such solutions, for every value of the mass. In the critical regime, this infinity of solutions is established to exists if and only if the mass is lower or equal to a threshold value. Moreover, the relation between this threshold and the topology of the graph is characterized. The investigation is based on variational techniques and some new versions of Gagliardo-Nirenberg inequalities.

1 Introduction
In this paper we discuss the existence of stationary solutions for the NLS equation on a general compact metric graph $\mathcal{G}$. In particular, we prove existence of critical points for the NLS energy functional

$$E(u, \mathcal{G}) = \frac{1}{2} \|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p} \|u\|^p_{L^p(\mathcal{G})} = \frac{1}{2} \int_\mathcal{G} |u'|^2 dx - \frac{1}{p} \int_\mathcal{G} |u|^p dx \quad (1)$$

with $p \in (2, 6]$, under the mass constraint

$$\|u\|^2_{L^2(\mathcal{G})} = \mu > 0 \quad (2)$$

Such critical points solve, for suitable $\lambda \in \mathbb{R}$, the stationary Schrödinger equation with the focusing pure power nonlinearity

$$u'' + |u|^p-2u = \lambda u \quad (3)$$
on every edge of $\mathcal{G}$, with Kirchhoff conditions (see equation (6)) at the nodes. Throughout all this work, we limit ourselves to deal with real-valued functions.

Our search for critical points of (1) is twofold. On one hand, we investigate the existence of global minimizers of the constrained energy, called ground states. Secondly, we turn our attention to a more general class of critical points, that do not need to be minimizers, usually called bound states.

We analyse both the subcritical regime $p \in (2, 6)$ and critical one $p = 6$, proving that the situation changes significantly.
In the subcritical case, we prove the existence of an infinite number of stationary solutions, with energy increasing to infinity, for every value of the mass and regardless of the topology of $G$. This is stated in the following theorem.

**Theorem 1.1.** Let $G$ be a compact graph and $p \in (2,6)$. Then, for every $\mu > 0$ there exist a ground state of (7) of mass $\mu$, and a sequence of bound states $\{u_k\}_{k \in \mathbb{N}}$ of mass $\mu$, so that:

$$E(u_k, G) \to \infty \quad \text{for } k \to \infty. \quad (4)$$

The critical regime is more subtle. We recall that by a *terminal edge* (i.e. a tip, see Fig. 1) we mean an edge of $G$ such that one of its endpoint is a vertex of degree one. It turns out that both ground state and bound states exist only for masses smaller than a threshold value that depends on whether the graph has at least one terminal edge or not. When $G$ has a terminal edge, the threshold value of the mass is equal to the $L^2$-critical mass on the half-line $\mathbb{R}^+$, $\mu_{\mathbb{R}^+} = \sqrt{3}\pi/4$; in all other cases, it coincides with the $L^2$-critical mass on the whole real line $\mathbb{R}$, $\mu_{\mathbb{R}} = \sqrt{3}\pi/2$ (later on in this section we will briefly recall from where these two quantities arise).

The theorem below establishes our main result at the critical exponent.

**Theorem 1.2.** Let $G$ be a compact graph and $p = 6$. Then:

(i) if $G$ has a terminal edge (Fig. 1), then a ground state of mass $\mu$ exists if and only if $\mu \leq \mu_{\mathbb{R}^+}$. Moreover, for every $\mu < \mu_{\mathbb{R}^+}$ there exists a sequence of bound states $\{u_k\}_{k \in \mathbb{N}}$ of mass $\mu$ and $E(u_k, G) \to \infty$;

(ii) if $G$ has no terminal edge, then ground states of mass $\mu$ exist if and only if $\mu \leq \mu_{\mathbb{R}}$. Moreover, for every $\mu < \mu_{\mathbb{R}}$ there exists a sequence of bound states $\{u_k\}_{k \in \mathbb{N}}$ of mass $\mu$ and $E(u_k, G) \to \infty$.

Roughly speaking, one may interpret our results by saying that the presence of a terminal edge forces $G$ to share a critical behaviour similar to the half-line $\mathbb{R}^+$, while in all other cases compact graphs seem to fake the line $\mathbb{R}$.

The problem of the existence of a ground state on the real line $G = \mathbb{R}$ (that can be seen as two half-lines glued together at a single vertex) is nowadays classical (see [16]). In the subcritical regime, for every value of the mass $\mu$, ground states are unique up to translations and change of sign and are the so-called solitons $\phi_\mu$, with strictly negative energy level. When $p = 6$, on the contrary, the infimum of (1) on $\mathbb{R}$ undertakes a sharp transition from 0 to $-\infty$ when the value of $\mu$ exceeds $\mu_{\mathbb{R}}$. A whole family $\{\phi_\lambda\}_{\lambda > 0}$ of solitons realizing $E(\phi_\lambda, \mathbb{R}) = 0$ exists if and only if $\mu = \mu_{\mathbb{R}}$. The same portrait holds when $G = \mathbb{R}^+$, with the threshold value of the mass becoming $\mu_{\mathbb{R}^+}$ and the ground states are given by the half-solitons, i.e. the restriction of $\phi_{2\mu}$ to $\mathbb{R}^+$. 

![Figure 1: A compact graph with a terminal edge.](image-url)
The study of nonlinear dynamics on quantum graphs has recently gained a considerable amount of interest, firstly motivated by a large class of physical applications, that range from optical networks to Bose-Einstein condensates. The evolution equation in this case is the time-dependent NLS

\[ i\partial_t \psi(t, x) = -\psi''(t, x) + g|\psi(t, x)|^{p-2} \psi(t, x) \]

that reduces to (3) when considering an attractive two-body interaction between the elementary components of the condensate (i.e. \( g = -1 \)) and restricting to standing wave solutions \( \psi(t, x) = e^{-i\lambda x} u(x) \) (for a more extended discussion of the physical interpretation see [9]).

Former investigations were initiated in [11, 13, 15], and after that, research has been pushed in several different directions. However, during the past years, the attention has been particularly focused on non-compact graphs with a finite number of nodes and at least a half-line. We point out that in such a physical context one should consider complex-valued functions; however, due to the invariance of eq. (3) under multiplication by a phase factor, one can restrict to the case of real-valued functions.

Evolution of solitary waves ([1]) and existence of ground states ([2, 3, 4]) for star graphs have both been addressed, while, for general graphs with half-lines, the investigations developed in [5, 6] for the subcritical case and in [7] for the critical one revealed that both topological and metric properties of the domain definitely play a key role in allowing or preventing ground states from existing. Particularly, let us just mention that, at the critical exponent, for a certain class of graphs, ground states were proved to exist for a continuum of masses, similarly to what we established for compact graphs in Theorem 1.2.

Problems concerning bound states on graphs with half-lines have recently been addressed in [8] considering minimization of the energy (1) among functions of prescribed mass and fulfilling additional constraints.

Moreover, existence of ground states and bound states on non-compact graphs was investigated also for the NLS equation with concentrated nonlinearity in [23, 22, 21]. Specifically, the general scheme followed in [21] provides the tools we will use in this paper when dealing with bound states (see Section 2).

Finally, we note that analysis of different classes of non-compact graphs has been recently initiated too, for instance in [10], that deals with the ground state existence problem on an infinite, periodic graph (the two-dimensional grid).

Considering compact graphs, the matter still seems to be quite unexplored. Something has been done on specific examples, for instance in [20] and [18] for the cubic NLSE, but the techniques used there cannot be recovered to deal with the general exponent. Our contribution here consists in extending the scope of the analysis to more general graphs and nonlinearity powers. Particularly, we provide here a full topological characterization of the existence of ground states.

Let us briefly recall some standard definitions and facts on compact metric graphs (for more details we refer for instance to [5, 11, 19]). Throughout the paper, a connected metric graph \( \mathcal{G} = (V, E) \) denotes a connected metric space built up by closed line intervals, the edges, glued together with the identification of some of their endpoints, the nodes. The peculiar way in which these identifications are performed defines the topology of the graph \( \mathcal{G} \). Moreover, both multiple edges and self-loops are allowed. Every edge \( e \in E \) is identified with an interval \( I_e = [0, \ell_e] \), and a coordinate \( x_e \) is chosen on \( I_e \) providing orientation.
A metric graph $\mathcal{G}$ is said to be *compact* if and only if it has a finite number of nodes and edges and it has no edge of infinite length.

Functions on a general graph $\mathcal{G}$ can be defined via their restriction to the edges. Indeed, every $u : \mathcal{G} \to \mathbb{R}$ can be interpreted as a family of functions $(u_e)_{e \in E}$, where $u_e : I_e \to \mathbb{R}$ is the restriction of $u$ to the edge represented by $I_e$. Thus, it is straightforward to define the usual functional spaces $L^p(\mathcal{G})$ as

$$L^p(\mathcal{G}) = \{ u : \mathcal{G} \to \mathbb{R} : u_e \in L^p(I_e), \forall e \in E \}$$

and $H^1(\mathcal{G})$ as

$$H^1(\mathcal{G}) = \{ u : \mathcal{G} \to \mathbb{R} \text{ continuous} : u_e \in H^1(I_e), \forall e \in E \}$$

Note that requiring the continuity of $u$ implies that it is continuous at vertices.

Furthermore, we introduce the space

$$H^1_\mu(\mathcal{G}) = \{ u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu \}$$

that embodies the mass constraint (2).

The paper is organized as follows. In Section 2 we recall some standard notions from the Calculus of Variation we need for our purposes. In Section 3 we prove a general compactness property that holds for every compact graphs and we completely deal with the subcritical regime, proving Theorem 1.1. In Section 4 we derive new technical tools playing a key role in the critical case, and we use them in Section 5 to provide the proof of Theorem 1.2.

### 2 Variational framework

As anticipated in the Introduction, our aim is to prove the existence of stationary solutions for the nonlinear Schrödinger equation with a pure power nonlinearity, defined as follows.

**Definition 2.1.** Let $\mathcal{G} = (V, E)$ be a connected metric graph. A function $u \in H^1_\mu(\mathcal{G})$ is said to be a *stationary solution of mass $\mu$ for the NLS equation on $\mathcal{G}$ with Kirchhoff conditions at the nodes* if:

1. there exists $\lambda \in \mathbb{R}$ such that, for every $e \in E$
   $$u_e'' + |u_e|^{p-2}u_e = \lambda u_e,$$
   (5)
2. for every vertex $v \in V$,
   $$\sum_{e > v} \frac{d}{d x_e}(u_e(v)) = 0$$
   (6)

The symbol $e > v$ means that the sum above is extended to all edges $e$ incident at $v$, and $\frac{d}{d x_e}(u_e(v))$ can be either $u_e'(0)$ or $-u_e'(\ell_e)$, depending on the fact that $e$ reaches $v$ in $x_e = 0$ or $x_e = \ell_e$.

Solutions of the stationary NLS equation as in Definition 2.1 are usually called *bound states*, and a simple variational argument (see [5]) shows that they can be found as constrained critical points of the *energy functional* $E : H^1(\mathcal{G}) \to \mathbb{R}$ defined as

$$E(u, \mathcal{G}) := \frac{1}{2}\|u''\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|_{L^p(\mathcal{G})}^p = \frac{1}{2} \int_{\mathcal{G}} |u''|^2 dx - \frac{1}{p} \int_{\mathcal{G}} |u|^p dx$$

(7)
with the mass condition
\[ \|u\|_{L^2(G)} = \mu. \]

Among all bound states, of particular interest are those critical points that globally minimize the NLS energy \( (7) \), the so-called ground states. We introduce the shorthand notation
\[ \mathcal{E}(\mu) := \inf_{u \in H^1_\mu(G)} E(u, G), \quad \mu \geq 0 \]  \( (8) \)

In order to investigate the existence of bound states that are not in general global minimizers, we need some known results from the Calculus of Variations. Let us thus recall the following definition.

**Definition 2.2.** Let \( c \in \mathbb{R} \). A sequence \( \{u_n\}_{n \in \mathbb{N}} \subset H^1_\mu(G) \) is called a Palais-Smale sequence for \( E \) at level \( c \) if, as \( n \to \infty \),

(i) \( E(u_n, G) \to c \),

(ii) \( \|E'(u_n, G)\|_{T_{u_n} H^1_\mu(G)} \to 0 \)

One says that \( E \) satisfies the Palais-Smale condition at level \( c \) (denoted by \( (PS)_c \)) if every Palais-Smale sequence at level \( c \) admits a subsequence strongly convergent in \( H^1_\mu(G) \). \( E \) is said to satisfy the Palais-Smale condition \( (PS) \) if \((PS)_c \) holds for every \( c \) admitting a Palais-Smale sequence.

The following theorem, that is a unified version of some results presented in Section 10.2 of \[12\], states that the Palais-Smale condition ensures the existence of bound states.

**Theorem 2.1.** Let \( \mu > 0 \) and \( E \in C^1(H^1_\mu(G), \mathbb{R}) \) as in \( (7) \). Suppose that the Palais-Smale condition \( (PS) \) holds. Then there exists a sequence of bound states \( \{u_k\}_{k \in \mathbb{N}} \subset H^1_\mu(G) \), and
\[ E(u_k, G) \to +\infty \quad \text{for } k \to \infty \] \( (9) \)

**Remark 2.1.** In general, the results of \[12\] summarized in the above theorem only imply that
\[ E(u_k, G) \to \sup_{u \in H^1_\mu(G)} E(u, G). \]
However, it is immediate to see that, for every metric graph \( G \)
\[ \sup_{u \in H^1_\mu(G)} E(u, G) = +\infty \]
thus justifying our statement of Theorem 2.1.

3 Compactness, Gagliardo-Nirenberg inequalities and the sub-critical regime

As one may expect, dealing with compact graphs reflects in a gain of compactness in \( H^1_\mu(G) \). The following proposition makes this fact more precise, as it establishes that, for Palais-Smale sequences, weak limits turn into strong ones.
Proposition 3.1. Let $G$ be a compact graph and $\{u_n\}_{n \in \mathbb{N}} \subset H^1_\mu(G)$ a Palais-Smale sequence for (7) bounded in $H^1_\mu(G)$. Then there exists $u \in H^1_\mu(G)$ such that, up to subsequences, $u_n \to u$ strongly in $H^1(G)$.

Proof. Since $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^1_\mu(G)$, then $u_n \rightharpoonup u$ weakly in $H^1(G)$, for some $u \in H^1(G)$. By Sobolev compact embeddings, this implies (possibly passing to a subsequence)

$$u_n \to u \text{ strongly in } L^p(G), \quad p \geq 1$$

so that $u \in H^1_\mu(G)$.

For every $u \in H^1_\mu(G)$, we define the quantity

$$\lambda = \lambda(u) := -\frac{1}{\mu} E'(u)u = \frac{1}{\mu} \left( \int_G |u|^p dx - \int_G |u'|^2 dx \right)$$

and the linear functional $J(u) : H^1(G) \to \mathbb{R}$

$$J(u)v := \int_G u'v' dx - \int_G |u|^{p-2}uv dx + \lambda \int_G uv dx$$

As proved in [21] (Section 2), since $\{u_n\}_{n \in \mathbb{N}}$ is a bounded Palais-Smale sequence, condition (ii) in Definition 2.2 can be conveniently rewritten as $J(u_n) \to 0$ in $H^{-1}(G)$. Moreover, if $\lambda_n := \lambda(u_n)$ is as in (11), then $\{\lambda_n\}_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}$, and (up to subsequences)

$$\lambda_n \to \bar{\lambda}$$

for some $\lambda \in \mathbb{R}$.

Let us now define the operator $A(u) : H^1(G) \to \mathbb{R}$

$$A(u)v := \int_G u'v' dx + \bar{\lambda} \int_G uv$$

and note that $A(u)(u_n - u) \to 0$ by weak convergence of $u_n$ to $u$.

Then we have:

$$o(1) = (J(u_n) - A(u))(u_n - u)$$

$$= \int_G |u'_n - u'|^2 dx + \int_G |u_n|^{p-2}u_n(u_n - u) dx + \lambda_n \int_G u_n(u_n - u) dx$$

$$- \bar{\lambda} \int_G u(u_n - u) dx$$

$$= \int_G |u'_n - u'|^2 dx + o(1)$$

by strong convergence in $L^p(G)$ of $u_n$ to $u$ and by convergence of $\lambda_n$ to $\bar{\lambda}$, and this concludes the proof. \[\square\]

One of the key tools in the study of the NLS energy functional (7) is the Gagliardo-Nirenberg inequality

$$\|u\|_{L^p(G)}^p \leq K_G \|u\|_{L^2(G)}^{\frac{p}{2}+1} \|u\|_{H^1(G)}^{\frac{p}{2}-1}$$

(14)
holding for every \( p \geq 2, \ u \in H^1(\mathcal{G}) \) and any compact graph \( \mathcal{G} \). Here \( K_\mathcal{G} \) denotes the optimal constant.

Combining inequality (14) and Proposition 3.1 we are able to deal with the subcritical regime, proving Theorem 1.1. On the other hand, it turns out that the standard compact Gagliardo-Nirenberg inequality is not enough to manage the critical case. Theorem 1.2 thus requires modified versions of Gagliardo-Nirenberg inequalities that will be derived in the next section.

Proof of Theorem 1.1: Plugging (14) into (7) and using \( p \leq 6 \), we have

\[
E(u, \mathcal{G}) \geq \frac{1}{2} \|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p} K_\mathcal{G} \mu \frac{p+4}{2} \|u\|_{H^1(\mathcal{G})}^{\frac{p}{2}-1} \geq \frac{1}{2} \|u'\|^2_{L^2(\mathcal{G})} \left( 1 - C_1 \mu \frac{p+4}{2} \|u'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-3} \right) - C_2 \mu \frac{p}{2}
\]

(15)

where \( C_1, C_2 > 0 \) are proper constants, and so it follows

\[
E_\mathcal{G}(\mu) > -\infty
\]

(16)

for every \( \mu > 0 \).

Let us now check that the Palais-Smale condition (PS) holds, for every value of the mass \( \mu \). Fix \( \mu > 0 \) and suppose \( \{u_n\}_{n \in \mathbb{N}} \subset H^1_\mu(\mathcal{G}) \) is a Palais-Smale sequence at level \( c \). Then, by (15):

\[
c + o(1) = E(u_n, \mathcal{G}) \geq \frac{1}{2} \|u_n'\|^2_{L^2(\mathcal{G})} \left( 1 - C_1 \mu \frac{p+4}{2} \|u_n'\|_{L^2(\mathcal{G})}^{\frac{p}{2}-3} \right) - C_2 \mu \frac{p}{2}
\]

Thus \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathcal{G}) \). Hence, by Proposition 3.1 \( u_n \to u \) strongly in \( H^1(\mathcal{G}) \cap L^p(\mathcal{G}) \), for some \( u \in H^1_\mu(\mathcal{G}) \). Since this is true for every \( c \), (PS) is proved.

Since it always exists a minimizing sequence that is also a Palais-Smale sequence at level \( c = E_\mathcal{G}(\mu) \) (that is finite due to (16)), ground states exist for every \( \mu > 0 \). Moreover, Theorem 2.1 applies and a sequence of bound states \( \{u_k\}_{k \in \mathbb{N}} \) exists for every value of the mass and

\[
E(u_k, \mathcal{G}) \to \infty \quad \text{for} \ k \to \infty.
\]

\[\square\]

4 Modified Gagliardo-Nirenberg inequalities

In order to deal with the critical case \( p = 6 \), we state here a modified version of Gagliardo-Nirenberg inequalities that holds for general compact graph. The importance of the inequality in this form is that the \( H^1 \)–norm of the function \( u \) appearing in (11) is replaced by the \( L^2 \)-norm of the first derivative \( u' \), as in the non-compact case (see for instance [6]). Moreover, the constant involved here instead of \( K_\mathcal{G} \) directly relates the mass value \( \mu \) to \( \mu_{\mathbb{R}^+} \) or \( \mu_{\mathbb{R}} \), depending on the possible presence of a terminal edge.

A similar modified Gagliardo-Nirenberg inequality was derived for the first time in Lemma 4.4 in [7] for general non-compact graphs, so, to some extent, our proof is an adaptation of the one given in [7].

First, let us recall the standard Gagliardo-Nirenberg inequality with \( p = 6 \) on \( \mathbb{R} \) and \( \mathbb{R}^+ 

\[
\|u\|^6_{L^6(\mathcal{G})} \leq K \|u\|^4_{L^2(\mathcal{G})} \|u'\|^2_{L^2(\mathcal{G})}
\]

(17)
where \( K = \frac{3}{\mu_R} \), \( \frac{3}{\mu_{R^+}} \) denotes the optimal constant on \( \mathbb{R} \) and \( \mathbb{R}^+ \) respectively.

**Proposition 4.1.** Assume \( G \) is a compact graph with at least one terminal edge. Fix \( \mu \in (0, \mu_{R^+}] \) and let \( u \in H^1(\mu) \). Then there exists a number \( \theta = \theta(u) \in [0, \mu] \) such that

\[
\|u\|^6_{L^6(G)} \leq 3 \left( \frac{\mu - \theta}{\mu_{R^+}} \right)^2 \|u'\|^2_{L^2(G)} + C \theta^{1/2}
\]  

(18)

where \( C > 0 \) is a constant that depends only on \( G \).

**Proof:** If \( u \) is constant on \( G \), then the result is immediate, so let us consider a non-constant function \( u \). Replacing \( u \) with \( |u| \), we may assume \( u > 0 \).

Let \( \ell := |G| \) be the total length of \( G \). Considering the decreasing rearrangement \( u^* \) of \( u \) on \([0, \ell)\), it is well-known (see e.g. [5]) that \( u^* \in H^1(0, \ell) \), \( u^* \) is non-increasing on \([0, \ell)\) and

\[
\|u^*\|_{L^6(0, \ell)} = \|u\|_{L^6(G)} \\
\|(u^*)'\|_{L^2(0, \ell)} \leq \|u'\|_{L^2(G)}
\]  

(19)

Rephrasing Step 2 of the proof of Lemma 4.4. in [7], one can construct a function \( v \in H^1(\mathbb{R}^+) \) so that, for some \( \theta = \theta(u) \in [0, \mu] \):

(i) \( v(0) = u^*(0) \);

(ii) \( \int_0^\infty |v|^2 \, dx = \int_0^\ell |u^*|^2 \, dx - \theta = \mu - \theta \);

(iii) \( \int_0^\infty |v'|^2 \, dx \leq \int_0^\ell |(u^*)'|^2 \, dx + C \theta^{1/2} \);

(iv) \( \int_0^\infty |v|^6 \, dx \geq \int_0^\ell |u^*|^6 \, dx - C \theta \);

with the constant \( C > 0 \) depending only on \( G \).

Now, by Gagliardo-Nirenberg inequality [17] on \( \mathbb{R}^+ \),

\[
\|v\|^6_{L^6(\mathbb{R}^+)} \leq \frac{3}{\mu_{R^+}} \|v\|^4_{L^2(\mathbb{R}^+)} \|v'\|^2_{L^2(\mathbb{R}^+)} = 3 \left( \frac{\mu - \theta}{\mu_{R^+}} \right)^2 \|v'\|^2_{L^2(\mathbb{R}^+)}.
\]  

(20)

Using properties (iii)-(iv) of \( v \), we can then get back to \( u^* \), and then to \( u \). Indeed, by (iii) and (19) we have

\[
\|v\|^6_{L^6(\mathbb{R}^+)} \leq \|u\|^6_{L^6(G)} - C \theta
\]  

(21)

while (iv) and (19) again give

\[
\|v'\|^2_{L^2(\mathbb{R}^+)} \leq \|u'\|^2_{L^2(G)} + C \theta^{1/2}
\]  

(22)

Finally, plugging (21) and (22) into (20) and properly changing \( C \), we get (18) and the proof is complete. □

The following proposition is the analogue of the previous one if there is no tip in \( G \), where \( \mu_\mathbb{R} \) takes the place of \( \mu_{R^+} \).
Proposition 4.2. Assume $G$ is a compact graph with no terminal edge. Fix $\mu \in (0, \mu_{\mathbb{R}}]$ and let $u \in H^1_{\mu}(G)$. Then there exists a number $\theta = \theta(u) \in [0, \mu]$ such that

$$
\|u\|^6_{L^\infty(G)} \leq 3 \left( \frac{\mu - \theta}{\mu_{\mathbb{R}}} \right)^2 \|u'\|^2_{L^2(G)} + C\theta^{1/2}
$$

where $C > 0$ is a constant that depends only on $G$.

Proof. The line of the proof is similar to the one of Proposition 4.1. The main difference is that it is possible to make use of the standard Gagliardo-Nirenberg inequality on $\mathbb{R}$ instead of $\mathbb{R}^+$. Again, we can consider $u$ not everywhere constant on $G$, $u > 0$ on $G$. Let us denote by $2\gamma$ the length of the shortest loop in $G$. Suppose that $u$ realizes its maximum at some point $x_0$, and let $\Sigma$ be the shortest path in $G$ going from $x_0$ to a point in which $u$ attains its minimum (it exists since $G$ is connected). Since there is no terminal point in $G$, $\Sigma$ can be extended for an additional length $\gamma$ from $x_0$ in the direction opposite to the minimum. We call $\Gamma$ the path starting at $x_0$ of length $\gamma$ obtained this way (see Figure 2). Since $G$ is connected and $u$ is continuous, it follows that $u|_{\Gamma}$ reaches all the values in the range of $u$ at least once. Hence, rearranging monotonically $u|_{\Gamma}$ on $[0, \gamma]$ and $u|_{G\setminus\Gamma}$ on $(\gamma - \ell, 0]$ and gluing them together at the origin, we get a function $\varphi \in H^1(\gamma - \ell, \gamma)$ such that:

(a) $\varphi(0) = \|\varphi\|_{L^\infty(\gamma - \ell, \gamma)} = \|u\|_{L^\infty(G)}$ and $\varphi(\gamma - \ell) = m$;

(b) $\varphi$ is monotonically increasing on $[\gamma - \ell, 0]$ and monotonically decreasing on $[0, \gamma]$;

(c) $\int_{\gamma - \ell}^\gamma \varphi^2 dx = \mu$;

(d) $\int_{\gamma - \ell}^\gamma |\varphi|^6 dx = \int_G |u|^6 dx$, while $\int_{\gamma - \ell}^\gamma |\varphi'|^2 dx \leq \int_G |u'|^2 dx$.

Now, starting from $\varphi$, we construct a function $w \in H^1(\mathbb{R})$ and $\theta \in [0, \mu]$ satisfying:

1. $\int_{\mathbb{R}} |w|^2 dx = \mu - \theta$;

2. $\int_{\mathbb{R}} |w'|^2 dx \leq \int_{\gamma - \ell}^{\gamma - \ell} |\varphi'|^2 dx + C\theta^{1/2}$;

3. $\int_{\mathbb{R}} |w|^6 dx \geq \int_{\gamma - \ell}^{\gamma - \ell} |\varphi|^6 dx - C\theta$
where \( C > 0 \) depends once again only on \( G \). Indeed, applying Step 2 of the proof of Lemma 4.4 in [7] independently to \( \varphi_{[\gamma-\ell,0]} \) and \( \varphi_{[0,\gamma]} \), we get two functions \( v_1, v_2 \in H^1(\mathbb{R}^+) \) satisfying properties (i)-(iv) with some non negative \( \theta_1, \theta_2 \) so that \( \theta_1 + \theta_2 \in [0, \mu] \). Gluing together \( v_1(x) \) and \( v_2(-x) \) at \( x = 0 \) and setting \( \theta := \theta_1 + \theta_2 \) we get \( w \) as above. Here \( \theta = \theta(u) \) depends by construction on the original function \( u \).

Applying Gagliardo-Nirenberg (17) with \( K = \frac{3}{\mu_R} \) to \( w \) and combining with 2.-3. and the properties of \( \varphi \), we have

\[
\|u\|_{L^6(G)}^6 - C\theta \leq \|w\|_{L^6(\mathbb{R})}^6 \leq 3 \left( \frac{\mu - \theta}{\mu_R} \right)^2 \|w'\|_{L^2(\mathbb{R})}^2 \leq 3 \left( \frac{\mu - \theta}{\mu_R} \right)^2 \|w'\|_{L^2(G)}^2 + C\theta^{1/2}
\]

Rearranging terms and modifying \( C \) if necessary conclude the proof. \( \square \)

Remark 4.1. Proposition 4.2 states a stronger result than Proposition 4.1, in the sense that it holds for a wider interval of masses, since \( \mu_R > \mu_R^+ \). This difference relies on the fact that, if \( G \) has no terminal edges, then every function on the graph shares at least two pre-images for almost every value it achieves. On the contrary, the presence of a terminal edge allows to exhibit functions on \( G \) with only one pre-image for an interval of values. Such a difference is strictly related to the properties of rearrangements and it is where it enters the field through our proofs. For a detailed overview on the connection with rearrangements we refer to [6].

5 Proof of the main result

We present here the proof of Theorem 1.2, that takes advantage of the modified Gagliardo-Nirenberg inequalities derived before.

Proof of Theorems 1.2. Let us begin by assuming that \( G \) is a compact graph with at least one terminal edge.

We first deal with the case \( \mu > \mu_R^+ \). Let \( v \in H^1_\mu(\mathbb{R}^+) \) so that \( \text{supp}(v) = [0,1] \) and \( E(v, \mathbb{R}^+) < 0 \) (it surely exists since \( E \) is unbounded from below on \( \mathbb{R}^+ \) when \( \mu > \mu_R^+ \) [10]). Defining, for \( \lambda > 0 \),

\[
v_\lambda(x) := \sqrt{\lambda}v(\lambda x), \quad x \in \mathbb{R}^+
\]

we get a family of functions so that:

\[
\int_{\mathbb{R}^+} |v_\lambda|^2 dx = \mu
\]

\[
\text{supp}(v_\lambda) = \left[0, \frac{1}{\sqrt{\lambda}}\right]
\]

\[
E(v_\lambda, \mathbb{R}^+) = \lambda^2 E(v, \mathbb{R}^+)
\]

Thus, when \( \lambda \) is large enough, the support of \( v_\lambda \) can be contained by any edge of \( G \). Therefore, suppose \( e \) is an edge of \( G \) and choose \( \lambda > 0 \) so that \( \text{supp}(v_\lambda) \subseteq I_e \). Then, defining \( w_\lambda \in H^1_\mu(G) \) as follows

\[
w_\lambda(x) = \begin{cases} v_\lambda(x) & \text{if } x \in I_e \\ 0 & \text{elsewhere on } G \end{cases}
\]

one gets \( E(w_\lambda, G) \to -\infty \) as \( \lambda \to \infty \), thus

\[
\mathcal{E}_G(\mu) = -\infty
\]
for every $\mu > \mu_R^+$, and ground states do not exist.

On the contrary, in order to ensure existence of ground states and bound states when $\mu \leq \mu_R^+$, it is sufficient to check the validity of the Palais-Smale condition.

Let us first consider $\mu < \mu_R^+$. Since $\theta \leq \mu$, for every $u \in H_{\mu}^1(G)$, plugging inequality (18) in the energy functional (7) one gets rid of $\theta$,

$$E(u, G) \geq \frac{1}{2} \|u'\|^2_{L^2(G)} \left(1 - \frac{\mu^2}{\mu_R^+}ight) - C\mu^{1/2}$$

thus implying

$$\mathcal{E}_G(\mu) > -\infty.$$  (27)

Let $\{u_n\}_{n \in \mathbb{N}} \subset H_{\mu}^1(G)$ be a Palais-Smale sequence at level $c \in \mathbb{R}$. Then, by $E(u_n) \to c$,

$$c + o(1) = E(u_n, G) \geq \frac{1}{2} \|u_n\|^2 \left(1 - \frac{\mu^2}{\mu_R^+}ight) - C\mu^{1/2}$$

and $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $H^1(G)$. Then, by Proposition 3.1, $u_n \to u$ strongly, for some $u \in H_{\mu}^1(G)$. Since this holds for every $c$, (PS) is proved.

Let us now consider the case $\mu = \mu_R^+$. We denote here by $\theta_u$ the constant appearing in (18) to stress its dependence on $u$. Even though it is no longer possible to ignore the role of $\theta$ as in (26), we are still able to deal with functions $u$ that realize strictly negative energy. Indeed, if $E(u, G) = -\alpha < 0$, then by (18) one has

$$\frac{1}{2} \|u'\|^2_{L^2(G)} \left(1 - \frac{\theta_u^2}{\mu_R^+}ight) - C\theta_u^{1/2} \leq E(u, G) = -\alpha < 0$$

showing that $\theta_u$ is bounded away from 0.

On one hand, this ensures again that

$$\mathcal{E}_G(\mu_{\mathbb{R}^+}) > -\infty$$  (30)

On the other hand, it implies that the Palais-Smale condition can be recovered at least at negative levels $c < 0$, thanks to the lower bound provided by inequality in (29). Therefore, if $\{u_n\}_{n \in \mathbb{N}} \subset H_{\mu_R^+}^1(G)$ is a Palais-Smale sequence at $c < 0$, then $u_n \to u$ strongly in $H_{\mu_R^+}^1(G)$.

Remember that it is not restrictive to consider minimizing sequences that are also Palais-Smale sequences at level $c = \mathcal{E}_G(\mu_R^+)$, and $c > -\infty$ for every $\mu \leq \mu_R^+$ by (27)–(30).

Moreover, the constant function $\varsigma \in H_{\mu_R^+}^1(G)$ on $G$

$$\varsigma := \sqrt{\frac{\mu}{\ell}}$$

ensures that, on every compact graph:

$$\mathcal{E}_G(\mu_{\mathbb{R}^+}) \leq E(\varsigma, G) = -\frac{\mu_{\mathbb{R}^+}^3}{6\ell^2} < 0$$

Since the Palais-Smale condition always holds at negative levels for $\mu \leq \mu_{\mathbb{R}^+}$, if $\{u_n\}_{n \in \mathbb{N}} \subset H_{\mu}^1(G)$ is a minimizing Palais-Smale sequence, then it is compact in $H_{\mu}^1(G)$, and ground states exist for every value of the mass less or equal to $\mu_{\mathbb{R}^+}$. This proves the first part of (i) in
Theorem 1.2. Moreover, since (PS) was proved to hold whenever \( \mu \) is strictly less than \( \mu_{\mathbb{R}^+} \), then Theorem 2.1 applies in this case, and the second part of (i) in Theorem 1.2 follows.

Statement (ii) follows the same argument, using the proper modified Gagliardo-Nirenberg (23) instead of (18) whenever needed.

\[ \square \]

Remark 5.1. Theorem 1.2 does not apply when \( \mu = \mu_{\mathbb{R}^+} \) and \( G \) has a terminal edge or \( \mu = \mu_{\mathbb{R}} \) and \( G \) has no terminal edge. The problem in this situation is that we do not know if the Palais-Smale condition holds at every level. Indeed, at the threshold mass, it is still true (thanks to the methods we used) that this condition holds at negative levels \( c < 0 \). Unfortunately, taking for instance sequences of functions with compact support approximating different half-solitons (resp. solitons), it is easy to see that it fails at level \( c = 0 \) when \( \mu = \mu_{\mathbb{R}^+} \) (resp. \( \mu = \mu_{\mathbb{R}} \)) on a graph with a terminal edge (resp. with no terminal edge). Moreover, when \( c > 0 \), we are not able to establish whether Palais-Smale sequences are compact or may not converge.

Remark 5.2. Let \( \ell := |G| \) denote the total length of \( G \). Using standard methods in the theory of stability developed in [17], it is possible to prove that, both in the subcritical and in the critical domain, there exists a threshold value of the mass, say \( \mu^* = \mu^*(G, p) \), such that the constant function \( \varsigma \in H^1(G) \) as in (31) is a local minimum for (7) on the constrained manifold \( \|u\|_{L^2(G)}^2 = \mu \), whenever \( \mu < \mu^* \). Even though partial results in highly specific cases have been developed in [20] and [18], understanding when \( \varsigma \) is actually a ground state of the energy on a general compact graph seems to lay a quite involved question.

Acknowledgements

The author is grateful to Enrico Serra for all enlightening suggestions he provided during the preparation of this work and to Riccardo Adami for constant support.

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