A Complex Krein-Rutman Theorem and Its Simple Dynamical Proof

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Abstract. We introduce the notion of rotational strong positivity for complex operators on ordered complex Banach spaces and present a new complex Krein-Rutman Theorem. Our proof is completely self-contained and significantly different from those in the literature for the real Perron-Frobenius Theorem and Krein-Rutman Theorem. It only involves a simple observation on a basic projective property of cones and some preliminary knowledge on linear ordinary differential equations, and therefore reveals a pure dynamical nature of the two theorems mentioned above.

Keywords: Complex operator, cone, positivity, rotational strong positivity, Perron-Frobenius Theorem, Krein-Rutman Theorem.

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1 Introduction

In the pioneering work [26] Perron published his celebrated theorem which asserts that the spectral radius of a nonnegative matrix $A$ is an eigenvalue of $A$ to which there corresponds a nonnegative eigenvector. This result was later extended by Frobenius to nonnegative irreducible matrices [12], which is now known as the Perron-Frobenius Theorem. Since in practice it is often the case that a matrix has nonnegative entries, the theorem has found an astonishing wide range of applications in diverse areas such as the numerical analysis, wireless networks, economy, epidemiology, probability, dynamical systems theory, lower-dimensional topology; see MacCluer [20] for a complete review on this topic. Keener [18] even gave an interesting application of the theorem in the ranking of football teams. The extensive use of the Perron-Frobenius Theorem greatly stimulated mathematicians to provide varied proofs which may help us gain deeper insights into the theorem from different points of view and extend it to more general cases. Many elegant proofs of the theorem were summarized in [20], including the Perron’s original proofs [25] [26] based on a technical use of the Cramer’s rule or the resolvent of matrices, Wielandt’s 1950 proof [32] built on Frobenius’s mini-max idea, Birkhoff’s proof via a clever use of the Jordan canonical form [1], Karlin’s proof using the complex variable theorem about power series with positive coefficients [16]. Some other proofs and generalizations based on geometric methods and fixed-point theorems can be found in [6] [14] [17] [27] [30], etc.

An outstanding extension of the Perron-Frobenius Theorem is the famous Krein-Rutman Theorem (KR Theorem in short) established in 1948 [19] dealing with positive compact linear operators on real infinite-dimensional Banach spaces. The original proof of Krein and Rutman makes a substantial use of the knowledge of spectral theory about compact operators. Since the spectral properties of operators given by the KR Theorem play crucial roles in the investigation
of both linear and nonlinear problems appearing in different areas such as the stability analysis of solutions to nonlinear differential equations, bifurcation theory and topological degree calculation, there appeared a variety of proofs and generalizations of the theorem since 1950’s.

In Ando [2], Bonsall [3] and Schaefer [31] etc. the authors discussed spectral properties of compact or noncompact operators, generalizing the KR Theorem in the frame work of Banach lattices. See also Schaefer [32] for a systematic presentation in this line. Other proofs and generalizations to linear operators use powerful topological and geometric tools such as fixed-point theorems and Hilbert projective metric; see e.g. Du [10], Edmunds et al. [11], Nussbaum [24] and Samuelson [30]. A dynamical systems proof for the KR Theorem was given in Alikakos [1]. The basic idea is to consider the discrete dynamical system of the induced operator on the projective space, and the existence of a positive eigenvector of the original operator reduces to showing that the $\omega$-limit set of the induced operator consists of a single equilibrium. Some new results and ideas on the extension to tangentially positive operators can be found in Kanigowski and Krysiewski [15] and the references cited therein. Chang et al. [8] contains some extensions of the theorem to tensors. There are also nice results in the literature on nonlinear extensions; see Chang [7], Mahadevan [21] and Mallet-Paret et al. [22] etc.

Uniform complex estimates are often needed in the study of Markov additive processes and nonautonomous or random dynamical systems (see e.g. [23, 29]), and spectral gap results as stated in the real KR Theorem for complex operators, if available, can be expected to be useful in deriving such estimates. According to the classical KR Theorem, in an ordered real Banach space the strong positivity of a compact linear operator $A$ guarantees the spectral gap of $A$: the spectral radius $r_\sigma(A)$ is a simple isolated eigenvalue, and the remaining part of the spectrum is contained in a disk centered at zero with radius strictly smaller than $r_\sigma(A)$. Unfortunately in the situation of an ordered complex Banach space with a cone in the usual sense, the fruitful notion of strong positivity widely used for real operators is no longer suitable for complex operators; see Subsection 2.2 below. In [28] Rugh introduced complex cones and associated projective gauges, through which the author obtained, among many other things, a complex KR Theorem for compact and quasi-compact complex operators by generalizing the real Birkhoff cone and its Hilbert metric to complex vector spaces.

On the other hand, in applications cones in the usual terminology seem to be more natural and can be easily constructed. In this work, instead of using generalized complex cones as the one in [28], we introduce the notion of rotational strong positivity for complex operators, based on which we obtain a new complex KR Theorem (Theorem 4.1) and present a completely self-contained pure dynamical proof. As we will see in Section 4, the real KR Theorem is an immediate application of ours. It is worth mentioning that our proof only involves a simple observation on a basic projective property of cones and some preliminary knowledge on linear ODEs, and is significantly different from those in the literature. Although the proof is of a dynamical nature, unlike the one given in [1] for the real KR Theorem, it is so fundamental that even one without any knowledge on the dynamical systems theory and nonlinear analysis can understand it well without any difficulties.

This paper is organized as follows. In Section 2 we make some preliminaries. In particular, we introduce the notion of rotational strong positivity for complex operators and prove a basic projective property for cones. In section 3 we give a complex Perron-Frobenius Theorem and present a completely self-contained dynamical proof for the theorem. In Section 4 we state and prove the complex KR Theorem. A simple proof for the real KR Theorem is also included.
2 Preliminaries

This section is concerned with some preliminaries.

2.1 Basic notions and notations

Let $X$ be a complex Banach space with norm $\|\cdot\|$. Denote $\theta$ the zero element of $X$. Let $M \subset X$. For notational simplicity, denote $\overset{o}{M}$ the interior of $M$. Given $C \subset \mathbb{C}$, we will write

$$CM = \{zx : z \in C, x \in M\},$$

where $\mathbb{C}$ denotes the field of complex numbers.

For $x \in X$, define

$$d(x, M) = \inf_{y \in M} \|x - y\|.$$

If $M$ is a subspace of $X$, it is trivial to check that

$$d(\lambda x, M) = \lambda d(x, M), \quad \forall \lambda \geq 0. \quad (2.1)$$

• Spectrum radius

Let $L(X)$ be the set of bounded linear operators on $X$. Given $A \in L(X)$, denote $\sigma(A)$ and $\sigma_e(A)$ the spectrum and essential spectrum of $A$, respectively; see e.g. [9, 15, 22] for formal definitions. Set

$$r_\sigma := r_\sigma(A) = \sup\{|\mu| : \mu \in \sigma(A)\}, \quad r_e := r_e(A) = \sup\{|\mu| : \mu \in \sigma_e(A)\}.$$

$r_\sigma$ and $r_e$ are called the spectrum radius and essential spectrum radius of $A$, respectively.

• Cone

By a wedge in $X$ we mean a closed subset $P \subset X$ with $P \neq \{\theta\}$ such that $sP \subset P$ for all real numbers $s \geq 0$. A wedge $P$ is said to be proper if $P \cap -P = \{\theta\}$. A wedge $P$ is called a cone if it is convex and proper.

Let $P$ be a cone in $X$. By convexity it is trivial to verify that

$$P + P \subset P.$$

We say that $P$ is total, if $(P - P) = X$. $P$ is called solid (or, inner regular) if it has nonempty interior $\overset{o}{P}$.

2.2 Rotational strong positivity of complex operators

Assume that there has been given a cone $P$ in $X$.

An operator $A \in L(X)$ is called positive (or, $P$-positive when we need to emphasize the role of $P$), if $AP \subset P$.

In the real Perron-Frobenius Theorem and KR Theorem, the strong positivity of an operator plays a key role to guarantee the spectral gap property of the operator. However, if one wants to establish parallel results for complex operators in a complex Banach space with a cone in the usual terminology as above, this notion seems to be helpless. This can be seen from the simple observation that if a complex operator $A$ satisfies

$$A(P \setminus \{\theta\}) \subset \overset{o}{P}, \quad (2.2)$$

then no eigenvectors associated with positive real eigenvalues are contained in $P$. Indeed, suppose $A$ satisfies [2.2] and that it has an eigenvector $\xi \in P$ associated with a real eigenvalue $r > 0$. Then $z\xi$ is an eigenvector associated with $r$ for all $z \in S^1$. Because $P \cap -P = \{\theta\}$, we deduce that $-\xi \notin P$. Thus there is a $\nu \in S^1$ such that $w = \nu \xi \in \partial P$, where $\partial P$ denotes the
boundary of $P$. But (2.2) then asserts that $\mathbb{A}w = rw \in \overset{\circ}{P}$. Consequently $w \in \overset{\circ}{P}$, which leads to a contradiction.

To overcome this difficulty, here we introduce the notion of Rotational strong positivity for complex operators.

**Definition 2.1** Assume $P$ is solid. An operator $A \in \mathcal{L}(X)$ is called rotationally strongly positive, if it is positive and

$$
\mathbb{C}(Ax) \cap \overset{\circ}{P} \neq \emptyset, \quad \forall x \in P \setminus \{\theta\}.
$$

(2.3)

**Remark 2.2** One can replace $\mathbb{C}$ in (2.3) with the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This is due to the basic fact that $x \in \overset{\circ}{P}$ if and only if $sx \in \overset{\circ}{P}$ for all positive real numbers $s > 0$.

**Remark 2.3** It is interesting to note that the complexification of a strong positive real operator $A$ (in the usual sense) is rotationally strongly positive. However it does not satisfy (2.2); see Section 4.

### 2.3 Some fundamental properties of cones

Let $P$ be a cone, and let $\mathcal{D} : X = \oplus_{i=1}^{n}X_i$ be a direct sum decomposition of $X$.

**Definition 2.4** We say that $P$ is projectively proper (with respect to $\mathcal{D}$), if

$$(\Pi_i P) \cap (-\Pi_i P) = \{\theta\}, \quad 1 \leq i \leq n,
$$

(2.4)

where $\Pi_i = \Pi_{X_i}$ denotes the projection from $X$ to $X_i$.

Given a direct sum decomposition $\mathcal{D}$, generally $P$ may not be projectively proper (with respect to $\mathcal{D}$). However, we have the following interesting result.

**Proposition 2.5** There exist $1 \leq i_1, \cdots, i_m \leq n$ such that $P_0 := P \cap X_0$ is a cone in $X_0 = \oplus_{k=1}^{m}X_{i_k}$; furthermore, $P_0$ is projectively proper.

**Proof.** We argue by induction.

First, if $n = 1$ then the conclusion trivially holds true. Suppose the conclusion holds true with $n = k \geq 1$. In what follows we show that it remains valid for $n = k + 1$, thus completing the proof of the proposition.

So let $X = \oplus_{i=1}^{k+1}X_i$ be a direct sum decomposition of $X$. If $(\Pi_i P) \cap (-\Pi_i P) = \{\theta\}$ for all $i = 1, 2, \cdots, k + 1$ then we are done. Hence we assume, without loss of generality, that $(\Pi_{k+1} P) \cap (-\Pi_{k+1} P) \neq \{\theta\}$. There is therefore an $x \neq \theta$ such that $x, -x \in \Pi_{k+1} P$. Pick two vectors $y_1, y_2 \in V_1 = \oplus_{i=1}^{k}X_i$ such that

$$
y_1 + x \in P, \quad y_2 - x \in P.
$$

Then

$$
y = y_1 + y_2 = (y_1 + x) + (y_2 - x) \in P.
$$

(2.5)

Note that $y \neq \theta$; otherwise $y_2 = -y_1$, and consequently

$$
y_2 - x = -(y_1 + x) \in P.
$$

This leads to a contradiction. As $y = y_1 + y_2 \in V_1$, by (2.5) we have $P_1 := P \cap V_1 \neq \{\theta\}$. Hence $P_1$ is a cone in $V_1 = \oplus_{i=1}^{k}X_i$. The validity of the conclusion then follows from the induction hypothesis. □
Remark 2.6 Let $P_0$ and $X_0$ be the cone and the subspace of $X$ given in Proposition 2.5, respectively. If $\mathcal{A} \in \mathcal{L}(X)$ is $P$-positive then $\mathcal{A}_0 = \mathcal{A}|_{X_0}$ is $P_0$-positive in $X_0$. Indeed,

$$\mathcal{A}_0 P_0 = \mathcal{A}_0 (P \cap X_0) = \mathcal{A}(P \cap X_0) = (\mathcal{A}P) \cap (\mathcal{A}X_0) \subset P \cap X_0 = P_0.$$ 

The following simple fact will also be frequently used.

Lemma 2.7 Let $X_0$ be a finite-dimensional subspace of $X$ with $P \cap X_0 = \{\theta\}$, and let $x_k \in P$ be a sequence. Suppose $d(x_k, X_0) \to 0$ as $k \to \infty$. Then $x_k \to \theta$ as $k \to \infty$.

Proof. Suppose on the contrary that there is a subsequence of $x_k$, still denoted by $x_k$, such that $\|x_k\| \geq \delta > 0$ for all $k$. Let $y_k = x_k / \|x_k\|$. Then $\|y_k\| = 1$. We observe that $d(y_k, X_0) = d(x_k / \|x_k\|, X_0) = 1 / \|x_k\| d(x_k, X_0) \to 0$ as $k \to \infty$. Thus one can find a bounded sequence $z_k \in X_0$ such that $\|y_k - z_k\| \to 0$. Since $X_0$ is finite-dimensional, up to a subsequence, it can be assumed that $z_k \to z_0 \in X_0$. Then $y_k \to z_0$ as well. Hence $\|z_0\| = 1$. As $y_k \in P$ for all $k$, the closedness of $P$ implies that $z_0 \in P$. Hence $P \cap X_0 \neq \{\theta\}$, a contradiction. □

2.4 Positivity of solutions of linear ODEs

Let $\mathcal{A} \in \mathcal{L}(X)$. Given $\alpha \in \mathbb{R}$, consider the system

$$\dot{x} = \mathcal{A}x + \alpha x, \quad x(0) = x_0. \tag{2.6}$$

The following simple fact is a basic knowledge in the theory of ODEs in Banach spaces; see e.g. Henry [13, pp. 60-61] (Exercises 6-8).

Lemma 2.8 Let $P$ be a cone in $X$. If $\mathcal{A}$ is $P$-positive, then $x_0 \in P \implies x(t) \in P$ for all $t \geq 0$, where $x(t)$ is the solution of (2.6).

Proof. We include a proof for completeness and the reader’s convenience.

The solution $x(t)$ of (2.6) reads as $x(t) = e^{\alpha t} e^{\mathcal{A}t} x_0$, where $e^{\mathcal{A}t}$ is the analytic semigroup generated by $\mathcal{A}$; see [13]. Since $\mathcal{A}$ is a bounded operator, we have

$$e^{\mathcal{A}t} = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{t^k}{k!} \mathcal{A}^k.$$ 

Let $x_0 \in P$. Then $\mathcal{A}^k x_0 \in P$ for all $k \geq 0$. Hence $\sum_{k=0}^{n} \frac{t^k}{k!} \mathcal{A}^k x_0 \in P$ for $t \geq 0$. The closedness of $P$ then implies that $e^{\mathcal{A}t} x_0 \in P$. Therefore $x(t) = e^{\alpha t} e^{\mathcal{A}t} x_0 \in P$. □
2.5 Behavior of liner ODEs in finite-dimensional spaces

In this part we assume $X$ is an $n$-dimensional complex Banach space.

Let $\mathbb{A} \in \mathcal{L}(X)$. If $\xi \in X$ is a generalized eigenvector of $\mathbb{A}$ associated with an eigenvalue $\mu$, then there is a smallest integer $\nu \geq 1$ such that

$$(\mathbb{A} - \mu)^{\nu} \xi \neq \theta, \quad \nu \geq 1, \quad (\mathbb{A} - \mu)^{\nu} \xi = \theta. \quad (2.7)$$

For convenience, we call the number $\nu$ the rank of $\xi$, denoted by $\text{rank}(\xi)$.

Let $x(t)$ be the solution the system

$$\dot{x} = A x, \quad x(0) = x_0. \quad (2.8)$$

**Lemma 2.9** For any $x_0 \neq \theta$, there exist $\alpha \in \mathbb{R}$ and integer $\nu \geq 0$ such that

$$\lim_{t \to +\infty} ||t^{-\nu}e^{-\alpha t}x(t) - \Gamma(t)|| = 0 \quad (2.9)$$

for some function $\Gamma$ given by

$$\Gamma(t) = \sum_{i=1}^{k} c_i e^{i \beta_{m_i} t} w_{m_i}, \quad c_i \neq 0, \quad 1 \leq m_i \leq n, \quad (2.10)$$

where $w_{m_i}$ $(1 \leq i \leq k)$ are linearly independent eigenvectors of $\mathbb{A}$, and each $w_{m_i}$ corresponds to an eigenvalue $\mu_{m_i} = \alpha_m + i \beta_{m_i}$, with $\alpha_m = \alpha$.

**Remark 2.10** The function $\Gamma$ given above is a solution of the system

$$\dot{\Gamma} = (\mathbb{A} - \alpha) \Gamma. \quad (2.11)$$

Furthermore, $\Gamma(t) \neq 0$ for all $t \in \mathbb{R}$.

Indeed, each number $i \beta_{m_i}$ in (2.10) is an eigenvalue of $\mathbb{A} - \alpha$ with eigenvector $w_{m_i}$. Thus $\Gamma$ solves (2.11). The second conclusion follows from the linear independence of $w_{m_i}$ $(1 \leq i \leq k)$ and the fact that $c_i \neq 0$.

**Proof of Lemma 2.9.** By the basic knowledge in the theory of linear algebra (see any textbook in this line), $X$ has a unique primary direct sum decomposition $\mathcal{D} : X = \oplus_{i=1}^{m} X_i$ induced by $\mathbb{A}$, that is, each $X_i$ is a cyclic subspace of $\mathbb{A}$ corresponding to an eigenvalue $\mu_i = \alpha_i + i \beta_i \in \sigma(\mathbb{A})$. Pick a generalized eigenvector $\xi_i$ of $\mathbb{A}$ such that $X_i$ is spanned by $\{\xi_i, (\mathbb{A} - \mu_i) \xi_i, \cdots, (\mathbb{A} - \mu_i)^{\nu_i} \xi_i\}$ (such a vector is available), where $\nu_i := \text{rank}(\xi_i) = \text{dim}(X_i)$. Set

$$\xi_{ij} = (\mathbb{A} - \mu_i)^{j} \xi_i, \quad j = 0, 1, \cdots, \nu_i - 1.$$

Then $\xi_{ij}$ is a generalized eigenvector of $\mathbb{A}$ with $\nu_{ij} := \text{rank}(\xi_{ij}) = \nu_i - j$. Let

$$w_i = (\mathbb{A} - \mu_i)^{\nu_i - 1} \xi_{ij} = (\mathbb{A} - \mu_i)^{\nu_i - 1} \xi_i.$$

$w_i$ is an eigenvector of $\mathbb{A}$ associated with $\mu_i$. As $w_i \in X_i$, evidently $w_1, w_2, \cdots, w_m$ are linearly independent.

By simple computations we can easily verify that

$$x(t) = e^{\mu_i t} \left( I + \frac{t}{1!} (\mathbb{A} - \mu_i) + \cdots + \frac{t^{\nu_i - 1}}{(\nu_i - 1)!} (\mathbb{A} - \mu_i)^{\nu_i - 1} \right) \xi_{ij}$$
is the solution of the system $\dot{x} = Ax$ with $x(0) = \xi_{ij}$ (one can also consult any text book on the theory of linear ODEs). Clearly

$$\lim_{t \to +\infty} ||t^{-\nu_{ij} + 1 \nu} x(t) - \gamma(t)|| = 0,$$

where

$$\gamma(t) = \frac{1}{(\nu_{ij} - 1)!} e^{i\beta t} \left( (\mathbb{A} - \mu_i)^{\nu_{ij} - 1} \xi_{ij} \right) = \frac{1}{(\nu_{ij} - 1)!} e^{i\beta t} w_i.$$  \tag{2.12}

Now assume that $x_0 \in X_i$, $x_0 \neq \theta$. We may write $x_0 = \sum_{j=k}^{\nu_{ij} - 1} a_j \xi_{ij}$, where $k \geq 0$ is such that $a_k \neq 0$. Then the solution $x(t)$ of (2.8) with $x(0) = x_0$ reads

$$x(t) = \sum_{j=k}^{\nu_{ij} - 1} a_j e^{\mu_i t} \left( I + \frac{t}{\nu_{ij} - 1} (\mathbb{A} - \mu_i) + \sum_{j=k+1}^{\nu_{ij} - 1} \frac{t^{\nu_{ij} - 1}}{(\nu_{ij} - 1)!} (\mathbb{A} - \mu_i)^{\nu_{ij} - 1} \right) \xi_{ij},$$

Noting that $\nu_{ik} > \nu_{ij}$ for all $j > k$, by (2.12) it can be easily seen that

$$\lim_{t \to +\infty} ||t^{-\nu_{ik} + 1} e^{-\alpha_{ij} t} x(t) - \gamma(t)|| = 0,$$

where $\gamma(t) = \frac{a_k}{(\nu_{ik} - 1)!} e^{i\beta t} w_i.$ \tag{2.13}

We are now ready to complete the proof of the lemma. Let $x_0 \in X$, $x_0 \neq \theta$. Write

$$x_0 = \sum_{i=1}^{l} a_{m_i} x_{m_i}, \quad x_{m_i} \in X_{m_i},$$

where $1 \leq m_i = m \leq n$, and $a_{m_1}, \cdots, a_{m_l} \in \mathbb{C}$, $a_{m_i} \neq 0$. Then $l \geq 1$. The solution $x(t)$ of (2.8) is the sum of $x_i(t)$ (1 \leq i \leq l), where $x_i(t)$ is the solution of the system $\dot{x} = Ax$ with $x(0) = a_{m_i} x_{m_i}$. By (2.13) we deduce that there is a number $\eta_i \geq 0$ such that

$$\lim_{t \to +\infty} ||t^{-\eta_i} e^{-\alpha_{m_i} t} x_i(t) - \gamma_i(t)|| = 0,$$ \tag{2.14}

where $\gamma_i(t) = c_i e^{i\beta_{m_i} t} w_{m_i}$, and $c_i \neq 0$.

For the sake of clarity, we may assume

$$\alpha_{m_1} = \alpha_{m_2} = \cdots = \alpha_{m_s} > \alpha_{m_{s+1}} \geq \cdots \geq \alpha_{m_l},$$

for some $1 \leq s \leq l$. For $\eta_1, \eta_2, \cdots, \eta_s$, it can be also assumed that

$$\eta_1 = \eta_2 = \cdots = \eta_k > \eta_{k+1} \geq \cdots \geq \eta_s,$$

where $k$ is a number with $1 \leq k \leq s$. Let us write $\alpha_{m_1} = \alpha$, and $\eta_1 = \nu$. Then by (2.14) it is trivial to see that

$$\lim_{t \to +\infty} ||t^{-\nu} e^{-\alpha t} x(t) - \Gamma(t)|| = 0,$$ \tag{2.15}

where $\Gamma(t) = \sum_{i=1}^{k} c_i e^{i\beta_{m_i} t} w_{m_i}. \square$

## 3 A Complex Perron-Frobenius Theorem

In this section we give a complex Perron-Frobenius Theorem and present a completely self-contained pure dynamical proof of the theorem.

Let $X$ be an $n$-dimensional complex Banach space, and suppose there has been given a cone $P$ in $X$. Let $A \in \mathcal{L}(X)$ be a positive operator.
**Theorem 3.1** If $P$ is total then the spectral radius $r_\sigma$ is an eigenvalue of $A$ with an eigenvector $v \in P$. If we further assume $P$ is solid and that $A$ is rotationally strongly positive, then the following assertions hold.

1. $|\mu| < r_\sigma$ for any $\mu \in \sigma(A) \setminus \{r_\sigma\}$.
2. $P$ contains no generalized eigenvectors of any other eigenvalue $\mu \neq r_\sigma$.
3. The algebraic and the geometric multiplicities of $r_\sigma$ coincide. Moreover,

$$\mathbb{C}\xi \cap \hat{P} \neq \emptyset$$

for any eigenvector $\xi$ corresponding to $r_\sigma$.

**Proof.** We split the proof of the theorem into several lemmas below.

### 3.1 $r_\sigma$ is an eigenvalue with an eigenvector $v \in P$

**Lemma 3.2** $A$ has at least one real eigenvalue.

**Proof.** Let $\mathcal{D} : \mathbb{X} = \oplus_{i=1}^m X_i$ be the decomposition given in the proof of Lemma 2.9. By virtue of Proposition 2.5 it can be assumed that $P$ is projectively proper with respect to $\mathcal{D}$, namely,

$$\bigcap_{i=1}^m P_i = \emptyset$$

where $P_i = \Pi \cap (-\Pi_i P) = \{0\}$, $1 \leq i \leq m$, \hspace{1cm} (3.1)

where $\Pi_i = \Pi \cap X_i$, is the projection from $\mathbb{X}$ to $X_i$.

Pick an $x_0 \in P$, $x_0 \neq \theta$, and consider the solution $x(t)$ of the system $\dot{x} = Ax$ with initial value $x(0) = x_0$. Thanks to Lemma 2.9 and Remark 2.10 there exist $\alpha \in \mathbb{R}$ and $\nu \geq 0$ such that

$$\lim_{t \to +\infty} \|t^{-\nu}e^{-\alpha t}x(t) - \gamma(t)\| = 0,$$

where $\gamma(t) = \sum_{i=1}^k c_i w_m e^{i\beta_m t}$ ($1 \leq m \leq n$, and $c_i \in \mathbb{C}$, $c_i \neq 0$), and $w_m$ is an eigenvector associated with some eigenvalue $\mu_m = |m| + i\beta_m$ with $\alpha_m = \alpha$ for each $1 \leq i \leq k$. Furthermore, $\gamma(t) \neq 0$ for all $t \in \mathbb{R}$.

Since $A$ is positive, we have $x(t) \in P$ for all $t \geq 0$. Consequently

$$y(t) = t^{-\nu}e^{-\alpha t}x(t) \in P, \quad t \geq 0.$$ \hspace{1cm} (3.3)

Therefore $Qy(t) \in QP := P_1$ for all $t \geq 0$, where $Q = \Pi m_1$. We show that

$$\gamma_1(t) = c_1 w_m e^{i\beta_m t} \in P_1$$

for all $t \in \mathbb{R}$. Indeed, since $\gamma_1$ is a periodic function, for any $t$ fixed we have

$$\|Qy(t + kT) - \gamma_1(t)\| = \|Qy(t + kT) - \gamma_1(t + kT)\|$$

$$= \|Qy(t + kT) - Q\gamma(t + kT)\|$$

$$\leq \|y(t + kT) - \gamma(t + kT)\| \to 0 \text{ (by (3.2))}$$

as $k \to \infty$, where $T > 0$ is a period of $\gamma_1$. As $P_1$ is closed and $Qy(t + kT) \in P_1$ for all $k$, one immediately concludes that $\gamma_1(t) \in P_1$.

Now we argue by contradiction and suppose that $A$ has no real eigenvalues. Then $\beta_m \neq 0$. Thus one can find a number $s > 0$ such that $e^{i\beta_m s} = -1$. Therefore

$$\gamma_1(s) = c_1 w_m e^{i\beta_m s} = -c_1 w_m = -\gamma_1(0).$$

As $\gamma_1(t) \in P_1$ for all $t \in \mathbb{R}$, we deduce that $\pm \gamma_1(0) \in P_1$, which contradicts (3.1). \hspace{1cm} $\square$
Lemma 3.3 Assume $P$ is total. Then $r_\sigma \in \sigma(\mathbb{A})$ with an eigenvector $w \in P$.

Proof. We may assume $r_\sigma = 1$.

Step 1. Take a $\delta > 0$ sufficiently small so that all the eigenvalues $\mu$ of $\mathbb{A}$ with $|\mu| > 1 - 2\delta$ are contained in $S^1 = \{z : |z| = 1\}$; see Fig. 3.1. Let

$$
\sigma_0 = \{z \in \sigma(\mathbb{A}) : |z| \leq 1 - 2\delta\}, \quad \sigma_1 = \{z \in \sigma(\mathbb{A}) : |z| = 1\}.
$$

$\sigma_0$ and $\sigma_1$ form a spectral decomposition of $\sigma(\mathbb{A})$. Let $X = X_0 \oplus X_1$ be the corresponding direct sum decomposition of $X$, where $X_0$ and $X_1$ are invariant subspaces of $\mathbb{A}$. (Such a decomposition can be easily obtained by using the primary one used in the proof of Lemma 2.9.) We show that

$$
P \cap X_1 \neq \{\theta\}. \quad (3.4)
$$

Figure 3.1: Distribution of eigenvalues

Pick a real number $\kappa > 1$ such that $\kappa \sigma_0 \subset \{z \in \mathbb{C} : |z| \leq 1 - \delta\}$. Set $\hat{\mathbb{A}} = \kappa \mathbb{A}$. Then $\sigma(\hat{\mathbb{A}}|_{X_0}) = \kappa \sigma_0 \subset \{z \in \mathbb{C} : |z| \leq 1 - \delta\}$, and

$$
\sigma(\hat{\mathbb{A}}|_{X_1}) = \kappa \sigma_1 \subset \{z \in \mathbb{C} : |z| = \kappa\}.
$$

Hence we deduce that

$$
\|\hat{\mathbb{A}}^k x\| \to 0 \ (x \in X_0), \quad \text{and} \quad \|\hat{\mathbb{A}}^k x\| \to \infty \ (x \in X_1)
$$

as $k \to \infty$. We claim that $P \not\subseteq X_0$. Indeed, if $P \subset X_0$ then $(P - P) \subset X_0 \neq X$, which contradicts the assumption that $P$ is total. Pick an $x \in P \setminus X_0$. Let $x = x_0 + x_1$, where $x_i \in X_i$. Clearly $x_1 \neq \theta$. We observe that

$$
\lim_{k \to \infty} d(\hat{\mathbb{A}}^k x, X_1) = \lim_{k \to \infty} d(\hat{\mathbb{A}}^k x_0 + \hat{\mathbb{A}}^k x_1, X_1) = 0. \quad (3.5)
$$

Now if $P \cap X_1 = \{\theta\}$ then by Lemma 2.7 we find that $\hat{\mathbb{A}}^k x \to \theta$. Consequently $\hat{\mathbb{A}}^k x_1 \to \theta$. This leads to a contradiction and verifies the validity of (3.4).

By (3.4) $P_1 := P \cap X_1$ is a cone in $X_1$. Since $\mathbb{A}X_1 \subset X_1$, we clearly have $\mathbb{A}P_1 \subset P_1$. Set $Y = (P_1 - P_1)$. $Y$ is an $\mathbb{A}$-invariant subspace of $X$ with $P_1$ being a total cone in $Y$.

Step 2. We continue our argument with the subspace $Y$, the cone $P_1$ in $Y$, and the operator $\mathbb{B} := \mathbb{A}|_Y$ in place of $X$, $P$ and $\mathbb{A}$, respectively.
It is clear that $\mathcal{B}$ is $P_1$-positive in $Y$, and $\sigma(\mathcal{B}) \subset S^1$. Set
\[ \alpha = \max\{\Re \mu : \mu \in \sigma(\mathcal{B})\}. \]
The spectrum $\sigma(\mathcal{B})$ has a decomposition:
\[ \sigma(\mathcal{B}) = \{ \mu \in \sigma(\mathcal{B}) : \Re \mu = \alpha \} \cup \{ \mu \in \sigma(\mathcal{B}) : \Re \mu < \alpha \} := \sigma_1 \cup \sigma_2. \]
Let $Y = Y_1 \oplus Y_2$ be the corresponding direct sum decomposition. We claim that
\[ P_1 \cap Y_1 \neq \{ \theta \}. \quad (3.6) \]
To see this, pick an $\varepsilon > 0$ sufficiently small so that
\[ \tilde{\sigma}_2 := \sigma_2 - \alpha + \varepsilon \subset \{ z \in \mathbb{C} : \Re z < 0 \}. \]
(Such an $\varepsilon$ is available because $\sigma_2 - \alpha \subset \{ z \in \mathbb{C} : \Re z < 0 \}$ and contains only a finite number of elements.) Note that $\tilde{\sigma}_1 := \sigma_1 - \alpha + \varepsilon \subset \{ z \in \mathbb{C} : \Re z \geq 0 \}$, and
\[ \sigma(\mathcal{B} - \alpha + \varepsilon) = \tilde{\sigma}_1 \cup \tilde{\sigma}_2. \quad (3.7) \]
Consider the initial value problem
\[ \dot{x} = (\mathcal{B} - \alpha + \varepsilon)x, \quad x(0) = x_0. \quad (3.8) \]
Denote $x(t)$ the solution of $\mathcal{B} - \alpha + \varepsilon$ as $t \to +\infty$, if $x_0 \in Y_i$ then $x(t) \in Y_i$ for all $t \geq 0$. By (3.7) we deduce that
\[ \lim_{t \to +\infty} \|x(t)\| = \infty \quad (\forall x_0 \in Y_1 \setminus \{ \theta \}), \quad \lim_{t \to +\infty} \|x(t)\| = 0 \quad (\forall x_0 \in Y_2). \quad (3.9) \]
In what follows we argue by contradiction and suppose $P_1 \cap Y_1 = \{ \theta \}$. Let $x_0 \in P_1$ be given arbitrary, $x_0 \neq \theta$. Write
\[ x_0 = u_1 + u_2, \quad u_i \in Y_i, \; i = 1, 2. \]
We first check that $u_1 = \theta$. Indeed, let $x_i(t)$ be the solution of the system in $\mathcal{B} - \alpha + \varepsilon$ with initial value $x_i(0) = u_i$. Then $x(t) = x_1(t) + x_2(t)$. Noticing that $x_2(t) \to \theta$ as $t \to +\infty$, we have
\[ \lim_{t \to +\infty} d(x(t), Y_1) = 0. \quad (3.10) \]
On the other hand, because $\mathcal{B}$ is $P_1$-positive, by Lemma 2.8 we deduce that $x(t) \in P_1$ for all $t \geq 0$. As $P_1 \cap Y_1 = \{ \theta \}$, by Lemma 2.7 and (3.10) we deduce that $x(t) \to \theta$ as $t \to +\infty$. It then follows that $x_1(t) \to \theta$ as $t \to +\infty$. Thus by (3.9) one necessarily has $u_1 = \theta$.

Now by what we have just proved above one has $P_1 \subset Y_2$. Therefore $P_1 - P_1 \subset Y_2$. Consequently $Y = (P_1 - P_1) \subset Y_2$. This contradiction finishes the proof of our claim in (3.6).

**Step 3.** Now we set $\mathcal{C} = P_1 \cap Y_1$. Then $\mathcal{C}$ is a cone in $Y_1$. Applying Lemma 3.2 to the restriction $\mathcal{B}_1 := \mathcal{B}|Y_1$ of $\mathcal{B}$ on $Y_1$, one immediately deduces that $\sigma_1$ contains a real eigenvalue of $\mathcal{A}$ in $S^1$. Further by the definition of $\sigma_1$ we see that either $\alpha = 1$ or $\alpha = -1$; moreover, $\alpha$ is an eigenvalue of $\mathcal{A}$ with $\sigma_1 = \{ \alpha \}$.

We first show that there is an eigenvector $w \in \mathcal{C} \subset P$ associated with $\alpha$. Take a $\xi \in \mathcal{C}$, $\xi \neq \theta$. Since $\sigma(\mathcal{B}_1) = \sigma_1 = \{ \alpha \}$ consists of exactly one eigenvalue, $\xi$ is necessarily a generalized eigenvector of $\mathcal{B}_1$ with $\text{rank}(\xi) := \nu \geq 1$. Consider the solution $y(t)$ of $\dot{y} = \mathcal{B}_1 y$ in $Y_1$ with initial value $y(0) = \xi$. We have
\[ y(t) = e^{\alpha t} \left( I + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots + \frac{(\nu - 1)!}{(\nu - 1)!} (\mathcal{B}_1 - \alpha)^{\nu - 1} \right) \xi. \]
Clearly
\[ z(t) := (\nu - 1)! t^{-\nu + 1} e^{-\alpha t} y(t) \longrightarrow (B_1 - \alpha)^{\nu - 1} \xi := w \]
as \( t \to +\infty \). Note that \( w \) is an eigenvector of \( B_1 \) associated with \( \alpha \).

On the other hand, since \( \xi \in \mathcal{C} \), by Lemma 2.8 we deduce that \( z(t) \in \mathcal{C} \) for all \( t \geq 0 \). Consequently \( w \in \mathcal{C} \).

Finally we check that \( \alpha \neq -1 \), and therefore \( \alpha = 1 \), which completes the proof of the lemma. Suppose the contrary. Then one would have \( B_1 w = -w \). Hence \( \pm w \in \mathcal{C} \subset P \), a contradiction. \( \square \)

### 3.2 Proof of assertions (1) – (3)

We now prove assertions (1)-(3) in Theorem 3.1. The argument below is also self-contained and of a simple pure dynamical nature.

Let \( P \) be a solid cone, and assume that \( \mathcal{A} \) is rotationally strongly positive.

**Lemma 3.4** \( P \) contains no generalized eigenvectors associated with any other eigenvalue \( \mu \neq r_\sigma \).

**Proof.** For simplicity, we write \( r_\sigma = r \). We argue by contradiction and suppose there is a generalized eigenvector \( \zeta \in P \) associated with an eigenvalue \( \mu \neq r \).

Step 1. We first check that such an eigenvalue \( \mu \) is necessarily a positive real number. Furthermore, \( \hat{P} \) contains an eigenvector \( v \) associated with \( \mu \).

Indeed, let \( \mu = \alpha + i\beta \). Then the solution \( x(t) \) of the system \( \dot{x} = \mathcal{A} x \) with \( x(0) = \zeta \) reads

\[
x(t) = e^{(\alpha + i\beta)t} \left( I + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^{\nu - 1}}{(\nu - 1)!} (\mathcal{A} - \mu)^{\nu - 1} \right) \zeta,
\]

where \( \nu = \text{rank}(\zeta) \). We observe that

\[
\|(\nu - 1)! t^{-(\nu - 1)} e^{-\alpha t} x(t) - \gamma(t)\| \to 0 \quad \text{as} \quad t \to +\infty,
\]

where \( \gamma(t) = e^{i\beta t} \eta \), and \( \eta = (\mathcal{A} - \mu)^{\nu - 1} \zeta \) is an eigenvector associated with \( \mu \). Since \( x(t) \in P \) (by Lemma 2.8), as in the proof of Lemma 3.2 it can be easily shown that \( \gamma(t) \in P \) for all \( t \in \mathbb{R} \). Hence \( \eta = \gamma(0) \in P \).

We claim that \( \beta = 0 \), hence \( \mu = \alpha \in \mathbb{R} \). Suppose the contrary. Then one could find a \( \tau > 0 \) such that \( e^{i\beta \tau} = -1 \). Thus \( -\eta = e^{i\beta \tau} \eta \in P \), which yields a contradiction.

Now we check that \( \mu > 0 \). First, by the rotational strong positivity of \( \mathcal{A} \) that \( z\mathcal{A} \eta = z\mu \eta \in \hat{P} \) for some \( z \in \mathcal{C} \). Hence \( \mu \neq 0 \). On the other hand, if \( \mu < 0 \) then since \( \eta \in P \) is an eigenvector of \( \mu \), we have

\[
-\eta = \frac{\mu}{|\mu|} \eta = \frac{1}{|\mu|} \mathcal{A} \eta \in P,
\]

which leads to a contradiction. Therefore \( \mu > 0 \).

It is obvious that \( \mu < r \). Thus \( 0 < \mu < r \).

As \( (\mathcal{C} \mathcal{A} \eta) \cap \hat{P} \neq \emptyset \), we can pick an \( z_0 \in \mathcal{C} \) such that \( z_0 \mathcal{A} \eta \in \hat{P} \). Hence

\[
\mu(z_0 \eta) = z_0 \mathcal{A} \eta = \mathcal{A}(z_0 \eta) \in \hat{P}.
\]

Consequently \( v := z_0 \eta \in \hat{P} \) and is an eigenvector associated with \( \mu \).
Step 2. By Lemma 3.3 \( A \) has an eigenvector \( \xi \in P \) associated with the eigenvalue \( r = r_\sigma \). As above one can find a number \( z_1 \in \mathbb{C} \) such that \( u := z_1 \xi \in \hat{P} \) is an eigenvector associated with \( r \).

Now we have two eigenvectors \( u, v \in \hat{P} \) of \( A \) corresponding to different eigenvalues \( r \) and \( \mu \), respectively. Denote \( \pi \) the real plane spanned by \( u \) and \( v \),

\[ \pi = \{ au + bv : a, b \in \mathbb{R} \}. \]

Then \( \pi \) is a 2-dimensional real space isomorphic to \( \mathbb{R}^2 \). Set \( P_\pi = P \cap \pi \). \( P_\pi \) is a cone in \( \pi \). One trivially checks that \( u, v \in \overset{\circ}{P}_\pi \), the interior of \( P_\pi \) in \( \pi \).

The two half-lines \( l_1 = \{ su : s \geq 0 \} \) and \( l_2 = \{ sv : s \geq 0 \} \) split \( P_\pi \) into three subcones \( C_1, C_2 \) and \( C_3 \) with nonempty disjoint interiors in \( \pi \) (see Fig. 3.2) such that

\[ u \in C_1, \quad v \in C_3, \quad \text{and} \quad C_1 \cap C_3 = \{ \theta \}. \]

![Figure 3.2: \( P_\pi = C_1 \cup C_2 \cup C_3 \)](image-url)

Evidently \( \mathbb{A}_\pi \subset \pi \). Let \( A = \mathbb{A}_\pi \) be the restriction of \( \mathbb{A} \) on \( \pi \). Then \( A \) is a real linear operator on \( \pi \). Pick a real number \( \lambda \) with \( \mu < \lambda < r \) and consider the planar system

\[ \dot{y} = (A - \lambda)y, \quad y(0) = y_0. \quad (3.11) \]

The operator \( A - \lambda \) has two eigenvalues \( \mu_1 = r - \lambda > 0 \) and \( \mu_2 = \mu - \lambda < 0 \) corresponding to the eigenvectors \( u \) and \( v \), respectively. Hence both the half-lines \( l_1 \) and \( l_2 \) are invariant under the system. Therefore any solution of (3.11) in \( P_\pi \) can not cross the half-lines \( l_1 \) and \( l_2 \). The same argument as in the proof of Lemma 2.8 applies to show that \( P_\pi \) is invariant under the system (3.11). Thus one concludes that all the subcones \( C_1, C_2 \) and \( C_3 \) are invariant under (3.11).

Take a \( y_0 \in \overset{\circ}{C}_3 \). Let \( y_0 = au + bv \). Then \( a \neq 0 \neq b \). The solution of (3.11) reads

\[ y(t) = ae^{(r - \lambda)t}u + be^{(\mu - \lambda)t}v. \]

Since \( be^{(\mu - \lambda)t}v \to 0 \) and \( |ae^{(r - \lambda)t}u| \to \infty \) as \( t \to +\infty \), we see that

\[ \lim_{t \to +\infty} d(y(t), l_1) = 0; \quad (3.12) \]

furthermore, \( |y(t)| \to \infty \) as \( t \to +\infty \).

On the other hand, \( y(t) \in C_3 \) for all \( t \geq 0 \). Because \( l_1 \cap C_3 = \{ \theta \} \), by (3.12) it follows that \( y(t) \to \theta \) as \( t \to +\infty \), which leads to a contradiction. \( \Box \)

**Lemma 3.5** \( |\mu| < r_\sigma \) for all \( \mu \in \sigma(\mathbb{A}) \setminus \{ r_\sigma \} \).
Proof. It can be assumed that $r_\sigma = 1$; otherwise one can argue with $r_\sigma^{-1}A$ in place of $A$.

We argue by contradiction and suppose the contrary. Then $A$ has an eigenvalue $\mu \neq 1$ with $|\mu| = 1$. Let $\eta$ be an eigenvector associated with $\mu$. Clearly $z\eta$ is also an eigenvector associated with $\mu$ for all $z \in S^1$. Set

$$C = S^1\eta = \{z\eta : z \in S^1\}.$$  

By virtue of Lemma 3.4 we deduce that $C \cap P = \emptyset$. Hence by the compactness of $C$ we have

$$\inf_{u \in C} d(u,P) = \delta > 0. \quad (3.13)$$

We infer from the proof of Lemma 3.4 that $A$ has an eigenvector $\xi \in \hat{P}$ associated with the eigenvalue $r_\sigma = 1$. Set

$$M(t) = \{z_1\xi + tz_2\eta : z_1, z_2 \in S^1\}.$$  

By (3.13) it is trivial to see that $(t^{-1}M(t)) \cap P = \emptyset$ for $t > 0$ sufficiently large, and consequently $M(t) \cap P = \emptyset$. On the other hand, because $\xi \in \hat{P}$, if $t > 0$ is sufficiently small then $\xi + tz_2\eta \in \hat{P}$ for all $z_2 \in S^1$, hence $M(t) \cap P \neq \emptyset$. Define

$$\tau = \inf\{t > 0 : M(t) \cap P = \emptyset\}.$$  

Then $\tau > 0$. It can be easily seen that $M(\tau) \cap P \neq \emptyset$.

We claim that

$$M(\tau) \cap P \subset \partial P.$$  

Indeed, if there is a $v \in M(\tau)$ such that $v \in \hat{P}$, then one can find an $\varepsilon > 0$ such that $M(t) \cap P \neq \emptyset$ for all $t \in [\tau, \tau + \varepsilon]$. This contradicts the definition of $\tau$ and proves our claim.

Pick a $v \in M(\tau) \cap P$. By the rotational strong positivity of $A$ there is a $z' \in S^1$ such that $z'A\upsilon \in \hat{P}$. Let $v = z_1\xi + \tau z_2\eta$ for some $z_1, z_2 \in S^1$. We observe that

$$z'A\upsilon = z'\upsilon(z_1\xi + \tau z_2\eta) = (z'z_1)\xi + \tau(z'z_2\upsilon)\eta.$$  

Since $|z'z_1| = |z'z_2\upsilon| = 1$, by definition we have $z'A\upsilon \in M(\tau)$. Hence $z'A\upsilon \in \partial P$, which yields a contradiction. \Box

To complete the proof of Theorem 3.1 there remains to prove the following result.

**Lemma 3.6** The algebraic and the geometric multiplicities of the eigenvalue $r_\sigma$ coincide. Moreover, $(C\xi) \cap \hat{P} \neq \{\theta\}$ for any eigenvector $\xi$ associated with $r_\sigma$.

Proof. We first show that if $\xi$ is an eigenvector of $A$ associated with $r_\sigma$, then $(C\xi) \cap P \neq \{\theta\}$. Suppose the contrary. Then $C \cap P = \emptyset$, where $C = \{z\xi : z \in S^1\}$. Repeating the same argument as in the proof of Lemma 3.5 one immediately obtains a contradiction.

Now we prove that $A$ has no generalized eigenvectors $w$ with rank($w$) $\geq 2$ associated with $r_\sigma := r$. Suppose on the contrary that $A$ has such a generalized eigenvector $w$. Let $m := \text{rank}(w)$ ($m \geq 2$). Then $u = (A - r)^{m-2}w$ is a generalized eigenvector with rank($u$) $= 2$. Hence $u_1 = (A - r)u$ is an eigenvector of $A$ associated with $r$. By what we have just proved above and the rotational strong positivity of $A$, one can therefore find a $z_1 \in S^1$ such that $v = z_1u_1 \in \hat{P}$. $v$ is an eigenvector of $A$ associated with $r$.

Since $v \in \hat{P}$, if $c \in \mathbb{C}$ is sufficiently small then $cu + v \in P$. Let $x(t)$ be the solution of the system $\dot{x} = Ax$ with $x(0) = cu + v$. Then $x(t) \in P$ for all $t \geq 0$. On the other hand,

$$x(t) = ce^{rt}(u + t(A - r)u) + e^{rt}v = ce^{rt}(u + tu_1) + e^{rt}v,$$  

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from which it can be easily seen that
\[
\lim_{t \to +\infty} t^{-1} e^{-rt} x(t) = cu_1 = cz_1^{-1}v.
\]
Because \( t^{-1} e^{-rt} x(t) \in P \) for all \( t \geq 0 \), we see that \( cz_1^{-1}v \in P \). Taking \( c = -\varepsilon z_1 \) with \( \varepsilon > 0 \) sufficiently small such that \( cu + v \in P \), one concludes that \( -\varepsilon v \in P \). Hence \( -v \in P \), which leads to a contradiction. \( \square \)

The proof of Theorem 3.1 is complete. \( \square \)

4 Complex Krein-Rutman Theorem

In this section we state and prove our complex KR Theorem by reducing the original infinite-dimensional problem into a finite-dimensional one. As an immediate application, we will also recover the real KR Theorem.

4.1 Complex Krein-Rutman Theorem

Let \( X \) be a complex Banach space, and let \( P \) be a total cone in \( X \). Our complex KR Theorem reads as follows.

**Theorem 4.1** Assume \( A \in L(X) \) is positive, and that \( r_e < r_\sigma \). Then \( r_\sigma \) is an eigenvalue of \( A \) with an eigenvector \( \xi \in P \). If we further assume \( P \) is solid and that \( A \) is rotationally strongly positive, then the following assertions hold.

1. \( |\mu| < r_\sigma \) for all \( \mu \in \sigma(A) \setminus \{r_\sigma\} \).
2. \( P \) contains no generalized eigenvectors of any other eigenvalue \( \mu \neq r_\sigma \) with \( |\mu| > r_e \).
3. The algebraic and the geometric multiplicities of \( r_\sigma \) coincide; furthermore, \( (\mathbb{C}\xi) \cap \hat{P} \neq \emptyset \) for any eigenvector \( \xi \) associated with \( r_\sigma \).

**Proof.** By the basic Knowledge in the spectral theory of linear operators, for any \( 0 < \varepsilon < r_\sigma - r_e \), the region \( \{ z \in \mathbb{C} : r_e + \varepsilon < |z| \leq r_\sigma \} \) contains only a finite number of elements in \( \sigma(\hat{A}) \). One can therefore pick a \( \delta > 0 \) sufficiently small such that
\[
\sigma(\hat{A}) \cap \{ z \in \mathbb{C} : r_e + \varepsilon < |z| < r_e + \varepsilon + \delta \} = \emptyset;
\]
see Fig. 4.1 (a). Let
\[
\sigma_0 = \sigma(\hat{A}) \cap \{ z \in \mathbb{C} : |z| \leq r_e + \varepsilon \}, \quad \sigma_1 = \sigma(\hat{A}) \cap \{ z \in \mathbb{C} : |z| \geq r_e + \varepsilon + \delta \}.
\]
Then \( \sigma(\hat{A}) = \sigma_0 \cup \sigma_1 \) is a spectral decomposition of \( \sigma(\hat{A}) \). Let \( X = X_0 \oplus X_1 \) be the corresponding direct sum decomposition of \( X \). We show that \( P \cap X_1 \neq \{0\} \).

Pick a number \( \mu \) with \( 1/(r_e + \varepsilon + \delta) < \mu < 1/(r_e + \varepsilon) \). Then
\[
r_0 := \mu(r_e + \varepsilon) < 1, \quad \text{and} \quad r_1 := \mu(r_e + \varepsilon + \delta) > 1. \tag{4.1}
\]
Consider the operator \( \hat{A} = \mu A \). \( \hat{A} \) has a spectral decomposition \( \sigma(\hat{A}) = (\mu \sigma_0) \cup (\mu \sigma_1) \), and the corresponding decomposition of \( X \) remains the same: \( X = X_0 \oplus X_1 \). As
\[
\mu \sigma_0 \subset \{ z \in \mathbb{C} : |z| \leq r_0 \}, \quad \mu \sigma_1 \subset \{ z \in \mathbb{C} : |z| \geq r_1 \},
\]
by \((4.1)\) we deduce that
\[
\|\tilde{A}^k u\| \to 0 \quad (u \in X_0), \quad \text{and} \quad \|\tilde{A}^k u\| \to \infty \quad (u \in X_1)
\]
as \(k \to \infty\). Using some similar argument as in the proof of Lemma 3.3, one can easily check that \(P \cap X_1 \neq \{\theta\}\).

Now we argue by cases.

Case 1). \(P\) is total and \(A\) is positive. In this case we can choose an \(\varepsilon > 0\) such that \(r_\varepsilon + \varepsilon\) is close to \(r_\sigma\) so that \(\sigma_1\) is contained in the circle \(S_{r_\sigma} = \{z \in \mathbb{C} : |z| = r_\sigma\}\); see Fig. 4.1 (b). Applying Theorem 3.1 to the restriction of \(A\) on \(X_1\) with \(P_1 = P \cap X_1\) in place of \(P\), one immediately concludes that \(r_\sigma \in \sigma_1\) and possesses an eigenvector in \(P_1 \subset P\).

Case 2). \(P\) is solid and that \(A\) is rotationally strongly positive. We first verify that \(P_1 = P \cap X_1\) is a solid cone in \(X_1\). Pick a \(\xi \in P_1 \setminus \{\theta\}\). Then there is a \(z \in \mathbb{C}\) such that \(u = zA\xi \in \overset{\circ}{P}\). Take a small neighborhood \(N\) of \(u\) with \(N \subset P\). Since \(u = zA\xi \in X_1\) (by the invariance property of \(X_1\)), \(N_1 := N \cap X_1\) is a neighborhood of \(u\) in \(X_1\). Observing that
\[
N_1 = N \cap X_1 = (N \cap P) \cap X_1 = N \cap (P \cap X_1) \subset P \cap X_1 = P_1,
\]
we deduce that \(u \in P_1\), where the interior of \(P_1\) is taken in \(X_1\). Hence \(P_1\) is solid in \(X_1\).

The above argument can be also used to check the rotational strong \(P_1\)-positivity of \(A_1\) in \(X_1\).

Applying Theorem 3.1 to \(A_1\) on \(X_1\) with \(P_1\) in place of \(P\), we deduce that assertions (1) and (3) in Theorem 4.1 hold. Moreover, \(P\) contains no generalized eigenvectors of any other eigenvalue \(\mu \in \sigma_1 \setminus \{r_\sigma\}\). Since \(\varepsilon\) can be chosen arbitrarily small, we finally conclude that assertion (2) holds true for any eigenvalue \(\mu \neq r_\sigma\) of \(A\) with \(|\mu| > r_\varepsilon\).

The proof of Theorem 4.1 is complete. \(\square\)

### 4.2 Real Krein-Rutman Theorem

As a simple application of Theorem 4.1 we finally prove a generalized real KR Theorem given in Nussbaum [24, Corollary 2.2] and Zhang [34, Theorem 1.3].
Let $X$ be a real Banach space. A closed subset $P \subset X$ with $P \neq \{\theta\}$ is called a \textit{wedge} in $X$ if $sP \subset P$ for all $s \geq 0$. A wedge $P$ is said to be \textit{proper} if $P \cap -P = \{\theta\}$. A wedge $P$ is called a \textit{cone} if it is convex and proper.

Let there be given a cone $P$ in $X$. We say that $P$ is \textit{total}, if $(P - P) = X$. We say that $P$ is \textit{solid}, if it has nonempty interior $\overset{\circ}{P}$.

A bounded linear operator $A \in \mathcal{L}(X)$ is called \textit{positive}, if $AP \subset P$. $A$ is called \textit{strongly positive}, if $A(P \setminus \{\theta\}) \subset \overset{\circ}{P}$.

**Theorem 4.2** Let $A \in \mathcal{L}(X)$ be a positive operator. Assume $P$ is total, and that $r_e < r_\sigma := r$. Then $r \in \sigma(A)$ and has an eigenvector $\xi \in P$. If we further assume $P$ is solid and that $A$ is strongly positive, then

1. $r$ is simple with an eigenvector $\xi \in P$;
2. any other eigenvalue is strictly smaller than $r$ in modulus; and
3. $P$ contains no eigenvectors associated with any other eigenvalue $\mu \neq r$ with $|\mu| > r_e(A)$.

**Proof.** Let $X = X + iX$ be the complexification of $X$, and let $\mathbb{P} = P + iP$. Then $\mathbb{P}$ is a cone in $\mathbb{X}$. It is trivial to see that if $P$ is total (solid) then $\mathbb{P}$ is total (solid) as well. The complexification of $A$ on $\mathbb{X}$, denote by $\hat{A}$, is defined as

$$\hat{A}u = Ax + iAy, \quad \forall u = x + iy \in \mathbb{X}.$$ 

Assume $A$ is positive. Then $\hat{A}$ is $\mathbb{P}$-positive. Thus if we assume $P$ is total and that $r_e < r$, by virtue of Theorem 4.1 one concludes that $r \in \sigma(A)$; furthermore, there is an associated eigenvector $\zeta \in \mathbb{P}$. Write $\zeta = u + iv$. Then $u, v \in P$. As $\hat{A}\zeta = r\zeta$, we find that both $u$ and $v$ are eigenvectors of $A$ associated with $r$.

Now we assume $P$ is solid and that $A$ is strongly positive. We check that $\hat{A}$ is rotationally strongly $\mathbb{P}$-positive in $\mathbb{X}$. Indeed, let $w = x + iy \in \mathbb{P} \setminus \{\theta\}$. If $x \neq \theta \neq y$, then $Ax, Ay \in \overset{\circ}{P}$. Consequently $\hat{A}w \in \overset{\circ}{P}$. Thus we may assume, say, that $x = \theta$. In such a case we necessarily have $y \neq \theta$. Hence $Ay \in \overset{\circ}{P}$. Taking $z = 1 - i$, we see that

$$z\hat{A}w = (1 - i)(iAy) = Ay + iAy \in \overset{\circ}{P},$$

which proves what we desired.

Applying Theorem 4.1 to $\hat{A}$ we immediately deduce that $r$ is an eigenvalue of $A$ with an eigenvector $\xi$; furthermore, assertions (2) and (3) hold true. By strong positivity of $A$ one can easily see that $\xi \in \overset{\circ}{P}$.

To complete the proof of the theorem, there remains to check that $r$ is both algebraically and geometrically simple. For this purpose, by virtue of assertion (3) in Theorem 4.1 it suffices to show that if $\eta \in \mathbb{X}$ is another eigenvector of $A$ associated with $r$, then $\eta = a\xi$ for some $a \in \mathbb{R}$.

Suppose the contrary. Then $\xi$ and $\eta$ are linearly independent. Let $P = \mathcal{Z}\{\xi, \eta\}$ be the plane spanned by $\xi$ and $\eta$. Then $\pi \cap \partial P \neq \{\theta\}$. Take a $v \in \pi \cap \partial P$, $v \neq \theta$. Clearly $v$ is an eigenvector associated with $r$. On the other hand, by the strong positivity of $A$ we have $Av \in \overset{\circ}{P}$. Hence $v = r^{-1}Av \in \overset{\circ}{P}$, a contradiction. $\square$

**Remark 4.3** We have seen that in the real Krein-Rutman Theorem, if $P$ is solid and $A$ is a strongly positive bounded operator with $r_e < r_\sigma$ then $r_\sigma$ is a simple eigenvalue, which fact is a consequence of the strong positivity of $A$. The situation seems to be somehow different in the complex case, and the simplicity of $r_\sigma$ in Theorem 4.2 remains unknown. Here we would like to propose this problem as an open question.
Open Question. Let $X$ be a complex Banach space, and $P$ be a solid cone in $X$. Let $A$ be a rotationally strongly positive bounded linear operator on $X$ with $r_e < r_\sigma$. Is the geometric multiplicity of $r_\sigma$ equals one? Or equivalently, are any two eigenvectors of $A$ associated with $r_\sigma$ linearly dependent?

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References

[1] N.D. Alikakos and G.A. Fusco, A dynamical systems proof of the Krein-Rutman Theorem and an extension of the Perron Theorem, Proc. Roy. Soc. Edinburgh Ser. A 117(3-4) (1991) 209-214.
[2] T. Ando, Positive linear operators in semi-ordered linear spaces, J. Fac. Science, Hokkaido University Ser. I XIII (1957) 214-228.
[3] F.F. Bonsall, Endomorphisms of partially ordered vector spaces, J. London Math. Society 30 (1955) 133-144.
[4] G. Birkhoff, Linear transformations with invariant cones, Amer. Math. Monthly 74 (3) (1967) 274-276.
[5] G. Birkhoff, Extensions of Jentzschs theorem, Trans. Amer. Math. Soc. 85(1) (1957) 219-227.
[6] A. Borobia, U.R. Trías, A geometric proof of the Perron-Frobenius theorem, Rev. Mat. Univ. Complut. Madrid 5(1) (1992) 57-63.
[7] K.C. Chang, Nonlinear extensions of the Perron-Frobenius theorem and the Krein-Rutman theorem, J. Fixed Point Theory Appl. 15(2) (2014) 433-457.
[8] K.C. Chang, K. Pearson and T. Zhang, Primitivity, the convergence of the NQZ method and the largest eigenvalue for nonnegative tensors, SIAM J. Matrix Anal. Appl. 32(3) (2011) 806-819.
[9] K. Deimling, Nonlinear Functional Analysis, Springer-Verlag, Berlin, 1985.
[10] Y. Du, Order structure and topological methods in nonlinear partial differential equations, World Scientific Publishing Co. Pte. Ltd., Hackensack NJ, 2006.
[11] D.E. Edmunds, A.J.B. Potter and C.A. Stuart, Non-compact positive operators, Proc. Roy. Soc. Edinburgh Ser. A 328 (1972) 67-81.
[12] F.G. Frobenius, Über Matrizen aus nicht negativen Elementen, S.-B. Preuss. Akad. Wiss. (1908 and 1912) 456-477.
[13] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lect. Notes in Math. 840, Springer Verlag, Berlin New York, 1981.
[14] R. Jentzsch, Über integralgleichungen mit positivem kern, Crelles J. 141 (1912) 235-244.
[15] A. Kanigowski and W. Kryszewski, Perron-Frobenius and Krein-Rutman theorems for tangentially positive operators, Cent. Eur. J. Math. 10(6) (2012) 2240-2263.
[16] S. Karlin, Positive operators, J. Math. Mech. 8(6) (1959) 907-937.
[17] E. Kohlberg and J.W. Pratt, The Contraction mapping approach to the Perron-Frobenius theory: why Hilbert’s metric?, Math. Oper. Res. 7(2) (1982) 198-210.

[18] James P. Keener, The Perron-Frobenius Theorem and the ranking of football teams, SIAM Rev. 35(1) (1993) 80-93.

[19] M.G. Krein and M.A. Rutman, Linear operators leaving invariant a cone in a Banach space, Uspekhi Mat. Nauk 3(1) (1948) 3-95.

[20] C.R. MacCluer, The many proofs and applications of Perron’s theorem, SIAM Review 42(3) (2000) 487-498.

[21] R. Mahadevan, A note on a non-linear Krein-Rutman theorem. Nonlinear Anal. 67 (2007) 3084-3090.

[22] J. Mallet-Paret and R.D. Nussbaum, Generalizing the KreinC.Rutman theorem, measures of noncompactness and the fixed point index, J. Fixed Point Theory Appl. 7(1) (2010) 103-143.

[23] P. Ney and E. Numellin, Markov Additive Processes, Ann. Probab. 15(2) (1987) 561-592.

[24] R. Nussbaum, Eigenvectors of nonlinear positive operators and the linear Krein-Rutman theorem, In: Fixed Point Theory, Lecture Notes in Math. 886, Springer, Berlin, 1981, 309-331.

[25] O. Perron, Grundlagen für eine theorie des Jacobischen kettenbruchalgorithmus, Math. Ann. 64(1) (1907) 1-76.

[26] O. Perron, Zur theorie der matrices, Math. Ann. 64(2) (1907) 248-263.

[27] N.J. Pullman, A geometric approach to the theory of nonnegative matrices, Linear Algebra Appl. 4(4) (1971) 297-312.

[28] H.H. Rugh, Cones and gauges in complex spaces: Spectral gaps and complex Perron-Frobenius theory, Ann. Math. 171(3) (2010) 1707-1752.

[29] H.H. Rugh, Coupled maps and analytic function spaces, Ann. Sci. Éc. Norm. Sup. 35(4) (2002) 489-535.

[30] H. Samelson, On the Perron-Frobenius theorem, Michigan Math. J. 4(1) (1957) 57-59.

[31] H.H. Schaefer, Some properties of positive linear operators, Pacific J. Math. 10(3) (1960) 1009-1019.

[32] H.H. Schaefer, Banach Lattices and Positive Operators, Springer-Verlag, Berlin Heidelberg New York 1974

[33] H. Wielandt, Unzerlegbare, nicht negative Matrizen, Math. Z. 52(1) (1950) 642C648.

[34] L. Zhang, A generalized Krein-Rutman Theorem, preprint.