Heterogeneous, Weakly Coupled Map Lattices

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Abstract

A coupled map lattice (CML) consists of an arbitrary number of interacting discrete dynamical systems. The temporal evolution of a CML’s component maps are governed by arbitrary dynamical systems, and different components can be governed by different dynamical systems. Consequently, the study of CMLs is pervasive in a large variety of subjects, and it is important to consider CMLs with heterogeneous components. In this paper, we give an analytical characterization of periodic orbits in heterogeneous weakly coupled map lattices (WCMLs): namely, the period of a periodic orbit of a WCML is the least common multiple of the periods of the isolated maps. Therefore, even for arbitrarily small coupling terms, the dynamics of a CML can be very different from those of its isolated components. For example, it is possible to preserve the period of a WCML for arbitrarily small (but nonzero) coupling strength even when an associated free oscillator is experiencing a period-doubling cascade. Because we characterize periodic orbits both close to and far from saddle-node bifurcations, our work provides an important step for examining the bifurcation structure of heterogeneous CMLs.

1 Introduction

Numerous phenomena in nature — such as human waves in stadiums [1] and flocks of seagulls [2] — result from the interaction of many individual elements,
but they can exhibit fascinating emergent dynamics that cannot arise in individual or even small numbers of components \[3\]. In practice, however, a key assumption in most such studies is that each component is described by the same dynamical system. However, hybrid systems with heterogeneous elements are much more common than homogeneous systems. For example, cars interact with each other, with pedestrians, and with traffic lights. It is therefore important to depart from the usual assumption of homogeneity and examine coupled dynamical systems with heterogeneous components.

The study of coupled map lattices (CMsL) \[4\] is one important way to study cooperative and emergent phenomena that result from interacting systems. They have been used to model systems in numerous fields (ranging from physics and chemistry to sociology to economics) \[5,6\]. In a CML, each component is a discrete dynamical system (i.e., a map). There are a wealth of both theoretical and computational studies of homogeneous CMLs \[4,7–14\], in which the interacting elements are each governed by the same map, and such investigations have yielded insights on a wide variety of phenomena. However, although it is convenient to take a CML’s components as identical, such an assumption is a major simplification that is often not justifiable. For example, a set of interacting cars that treats all cars as the same ignores different types of cars (e.g., their manufacturer, their age, different levels of intoxication among the drivers, etc.), so perhaps a dynamical system that governs the behavior of different cars should have different parameter values or even different functional forms entirely? Unfortunately, because little is known about heterogeneous CMLs \[15,16\], the assumption of homogeneity is an important simplification that allows scholars to apply a bevy of analytical tools. It is thus important to develop tools for the analysis of heterogeneous CMLs. In the present paper, we will do this for periodic orbits, which are one of the most important qualitative features of nonlinear dynamical systems \[17\].

The temporal evolution of a heterogeneous coupled map lattice (CML) with \(p\) components

\[
X_i(n + 1) = f_{R_i}(X_i(n)) + \varepsilon \sum_{h=1, h \neq i}^{p} f_{R_h}(X_h(n)), \quad i \in \{1, \ldots, p\}. \tag{1}
\]

Because we are interested in periodic orbits, we call each component \(X_i\) an “oscillator” (though, in general, one can of course couple components that are not oscillators). Their natural — i.e., uncoupled — dynamics evolve according to the map

\[
X_i(n + 1) = f_{R_i}(X_i(n)), \quad i \in \{1, \ldots, p\}, \tag{2}
\]

where the \(f_{R_i}\) are different functions that depend on a parameter \(R_i\) (where \(i \in \{1, \ldots, p\}\)).

In a homogeneous CML, all oscillators are governed by the same map with the same parameter values. One can think of \(X_i(n)\) as a neuron in a brain, a car on a highway, a person in a stadium, a bird in a flock, etc. If we think of \(X_i(n)\) as a neuron in a brain, then the CML \[1\] amounts to a (grossly) simplified
model of the brain [18]. If each oscillator is the same, then one is assuming (in addition to myriad other assumptions for this example) that all neurons are of the same type and have the same parameter values. We would like to move beyond such assumptions of homogeneity. We will keep the context of a simple model (i.e., a CML), but we will allow our oscillators (e.g., neurons) to be described by different maps and have different parameters.

In the present paper, we examine period orbits in heterogeneous, weakly coupled map lattices (WCMLs). Understanding periodic orbits is interesting by itself and is also crucial for understanding more complicated dynamics (e.g., chaos) [17,19]. The rest of this paper is organized as follows. We will first characterize periodic orbits both far away from and near saddle-node (SN) bifurcations, and we will then draw conclusions from our theorems.

2 Theoretical Results

In this section, we examine periodic orbits both far from and near SN bifurcations. Such orbits exhibit different dynamics [20,21], so it is important to distinguish between these two cases. From now on, we will assume that each \( f_{R_i} \) is a \( C^1 \) unimodal function with a critical point \( C \) at \( R_i \).

2.1 Periodic Orbits Far From a Saddle-Node Bifurcation.

Lemma 1

The temporal evolution of the CML

\[
X_i(n + 1) = f_{R_i}(X_i(n)) + \varepsilon \sum_{h=1 \atop h \neq i}^{p} f_{R_h}(X_h(n)), \quad i \in \{1, \ldots, p\},
\]

(3)

with initial condition \( X_i(n) = x_{i,1} + \varepsilon A_{i,1} \), is given by

\[
X_i(n+t) = x_{i,t+1} + \left[ \prod_{k=1}^{i} \frac{\partial f_{R_k}}{\partial x}(x_{i,k}) A_{i,1} + \sum_{l=2}^{i} \left( \prod_{k=l}^{i} \frac{\partial f_{R_k}}{\partial x}(x_{i,k}) \sum_{h=1 \atop h \neq i}^{p} x_{h,l} \right) + \sum_{h=1 \atop h \neq i}^{p} x_{h,t+1} \right] \varepsilon + O(\varepsilon^2), \quad t \geq 2,
\]

(4)

where \( |\varepsilon| \ll 1 \) and \( \{x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,q_i}\} \) is an orbit of period \( q_i \) for the function \( f_{R_i} \) (and \( i \in \{1, \ldots, p\} \)).
Proof of Lemma 1.

We proceed by induction. For \( t = 1 \), we have

\[
X_i(n + 1) = f_{R_i}(x_{i,1} + \varepsilon A_{i,1} + O(\varepsilon^2)) + \varepsilon \sum_{h \neq i}^{p} f_{R_h}(x_{h,1} + \varepsilon A_{h,1} + O(\varepsilon^2))
\]

\[
= x_{i,2} + \varepsilon \left( \frac{\partial f_{R_i}}{\partial x}(x_{i,1})A_{i,1} + \sum_{h = 1}^{p} x_{h,2} \right) + O(\varepsilon^2). \tag{5}
\]

Using (5) then yields

\[
X_i(n + 2) = f_{R_i} \left( x_{i,2} + \varepsilon \left( \frac{\partial f_{R_i}}{\partial x}(x_{i,1})A_{i,1} + \sum_{h = 1}^{p} x_{h,2} \right) \right) + \varepsilon \sum_{h = 1}^{p} f_{R_h}(x_{h,2} + O(\varepsilon))
\]

\[
= x_{i,3} + \varepsilon \left( \frac{\partial f_{R_i}}{\partial x}(x_{i,2}) \left( \frac{\partial f_{R_i}}{\partial x}(x_{i,1})A_{i,1} + \sum_{h = 1}^{p} x_{h,2} \right) + \sum_{h = 1}^{p} x_{h,3} \right) + O(\varepsilon^2). \tag{6}
\]

We assume the induction hypothesis

\[
X_i(n + t) = x_{i,t+1} + \left[ \prod_{k = 1}^{t} \frac{\partial f_{R_i}}{\partial x}(x_{i,k})A_{i,1} + \sum_{l = 2}^{t} \left( \prod_{k = l}^{t} \frac{\partial f_{R_i}}{\partial x}(x_{i,k}) \sum_{h = 1}^{p} x_{h,l} \right) + \sum_{h = 1}^{p} x_{h,t+1} \right] \varepsilon + O(\varepsilon^2). \tag{7}
\]
We then obtain

\[ X_i(n+1) = f_{R_i}(X_i(n)) + \varepsilon \sum_{h=1}^{p} f_{R_h}(X_h(n)) \]

\[ = f_{R_i}(x_{i,t+1} + \varepsilon \sum_{l=2}^{t} \left( \prod_{k=l}^{t} \frac{\partial f_{R_i}}{\partial x}(x_{i,k}) \sum_{h=1}^{p} x_{h,l} \right) + \sum_{h=1}^{p} x_{h,t+1} ) + \varepsilon + O(\varepsilon^2) \]

\[ + \varepsilon \sum_{h=1}^{p} f_{R_h}(x_{h,t+1} + O(\varepsilon)) \]

\[ = f_{R_i}(x_{i,t+1}) + \varepsilon \left( \prod_{k=1}^{t} \frac{\partial f_{R_i}}{\partial x}(x_{i,k}) A_{i,1} + \sum_{h=1}^{p} x_{h,1} \right) + \sum_{h=1}^{p} x_{h,t+1} \]

\[ + \varepsilon \sum_{h=1}^{p} \left( x_{h,t+2} + O(\varepsilon^2) \right) \]

\[ = x_{i,t+1} + \left( \prod_{k=1}^{t+1} \frac{\partial f_{R_i}}{\partial x}(x_{i,k}) A_{i,1} + \sum_{h=1}^{p} x_{h,1} \right) + \sum_{h=1}^{p} x_{h,t+1} \]

\[ + \varepsilon + O(\varepsilon^2), \quad \text{(8)} \]

which completes the proof. \( \square \)

**Theorem 1**

When the hypotheses of Lemma 1 are satisfied, the CML (3) has a periodic solution of period \( m = \text{lcm}(q_1, q_2, \ldots, q_p) \), where “lcm” stands for the least common multiple of the ensuing arguments. This periodic solution is

\[ X_i(n+j) = x_{i,j} + \varepsilon A_{i,j} + O(\varepsilon^2), \quad i \in \{1, \ldots, p\}, \quad j \in \mathbb{N}, \quad \text{(9)} \]
where

\[
A_{i,j} = \frac{\sum_{k=j+1}^{j+m-1} \left( \prod_{t=k}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,t}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,j+m}}{1 - \prod_{k=j}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,k})}
\]

satisfies the periodicity condition

\[
A_{i,j+m} = A_{i,j}, \quad x_{i,j+q_i} = x_{i,j}.
\]

**Proof of Theorem 1.**

A periodic solution

\[
X_i(n+j) = x_{i,j} + \epsilon A_{i,j} + O(\epsilon^2), \quad i \in \{1, \ldots, p\}, \quad j \in \mathbb{N} \tag{10}
\]

exists when the system

\[
\begin{align*}
X_i(n) &= x_{i,1} + \epsilon A_{i,1} \\
X_i(n + 1) &= x_{i,2} + \epsilon A_{i,2} \\
&\vdots \\
X_i(n + m - 1) &= x_{i,m} + \epsilon A_{i,m} \\
X_i(n + m) &= x_{i,1} + \epsilon A_{i,1} 
\end{align*}
\]

has an associated matrix with full rank. Because the system (11) is linear, its associated matrix has full rank if and only if it has a solution (i.e., if the \( A_{i,j} \) exist).

To find these solutions, we substitute \( X_i(n) = x_{i,1} + \epsilon A_{i,1} + O(\epsilon^2) \) into (3) to obtain

\[
X_i(n+1) = x_{i,2} + \epsilon \left( \frac{\partial f_R}{\partial x}(x_{i,1}) A_{i,1} + \sum_{h=1}^{p} x_{h,2} \right) + O(\epsilon^2) = x_{i,2} + \epsilon A_{i,2} + O(\epsilon^2). \tag{12}
\]

By using Lemma 1 and making \( m \) iterations (for \( m \geq 2 \)), we obtain

\[
X_i(n + m) = x_{i,1} + \left[ \prod_{k=1}^{m} \frac{\partial f_R}{\partial x}(x_{i,k}) A_{i,1} + \sum_{k=2}^{m} \left( \prod_{t=k}^{m} \frac{\partial f_R}{\partial x}(x_{i,t}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,m+1} \right] \epsilon + O(\epsilon^2)
\]

\[
= x_{i,1} + \epsilon A_{i,1} + O(\epsilon^2), \tag{13}
\]
where we note that we can cancel $x_{i,1}$ from both sides of the equation. To first order in $\varepsilon$, equation (13) yields

$$A_{i,1} = \frac{\sum_{k=2}^{m} \left( \prod_{t=k}^{m} \frac{\partial f_R}{\partial x}(x_{i,t}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,1}}{1 - \prod_{k=1}^{m} \frac{\partial f_R}{\partial x}(x_{i,k})}. \quad (14)$$

We then obtain expressions for $A_{1j}$ via recursion by using (5) and (14). This, in turn, yields expressions for $A_{i,j}$ by symmetry because the 1-st oscillator plays the same role as the $j$-th oscillator. Consequently,

$$A_{i,j} = \frac{\sum_{k=j+1}^{j+m-1} \left( \prod_{t=k}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,t}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,j+m}}{1 - \prod_{k=j}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,k})}. \quad (15)$$

Additionally, observe that

$$1 - \prod_{k=j}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,k}) \in O(1)$$

because orbits of free oscillators are far from SN orbits. (We will discuss SN orbits below.)

**Periodic Orbits Near Saddle-Node Bifurcations.**

A period-$m$ SN orbit is a periodic orbit that consists of $m$ SN points. Each of these $m$ SN points is a fixed points of $f_R^m$ at which $f_R^m$ undergoes an SN bifurcation. Period-$m$ SN orbits play an important role in a map’s bifurcation structure because they occur at the beginning of periodic windows in bifurcation diagrams. Studying them is thus an important step towards examining bifurcation structure in maps.

If $f_R$ has a period-$m$ SN orbit, then

$$\frac{\partial f_R^m}{\partial x}(x_{i,k}) = 1 = \prod_{k=j}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,k}).$$

It then follows that

$$1 - \prod_{k=j}^{j+m-1} \frac{\partial f_R}{\partial x}(x_{i,k}) = 0,$$
Orbits that are near an SN orbit thus satisfy
\[
1 - \prod_{k=j}^{j+m-1} \frac{\partial f_{R_i}(x_{i,k})}{\partial x} \in O(\varepsilon),
\]
so Theorem 1 does not hold. We address this issue as follows. As we show in Fig. 1, there is a point \(x_{i,k}\) of the SN orbit that is close to the critical point \(C\) of \(f_{r_i}\). As one considers SN orbits of larger period, the point \(x_{i,k}\) becomes closer to \(C\). Because \(\frac{\partial f_{r_i}(C)}{\partial x} = 0\), it follows that
\[
\left| \frac{\partial f_{r_i}(x_{i,k})}{\partial x} \right| \ll \varepsilon. \quad (16)
\]
The expression (16) is true for orbits close to SN orbits — that is, for parameter values \(R_i = r_i + \varepsilon\) with \(|\varepsilon| \ll 1\).

**Lemma 2**
The temporal evolution of the CML
\[
X_i(n + 1) = f_{R_i}(X_i(n)) + \varepsilon \sum_{\substack{h=1 \atop h \neq i}}^{p} f_{R_h}(X_h(n)), \quad i \in \{1, \ldots, p\}, \quad (17)
\]
with initial conditions \(X_i(n) = x_{i,1} + \varepsilon A_{i,1}\) satisfies the following equations.

1. For \(i \in \{1, \ldots, s\}\), we have
\[
X_i(n + t) = x_{i,t+1} + \left( \frac{\partial f_{r_i}(x_{i,t})}{\partial r} \sum_{k=1}^{t-1} \frac{\partial f_{r_i}(x_{i,k})}{\partial r} \prod_{l=k+1}^{t} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \right) + \sum_{\substack{h=1 \atop h \neq i}}^{p} f_{r_h}(x_{h,t+1}) \varepsilon + O(\varepsilon^2). \quad (18)
\]

2. For \(i \in \{s + 1, \ldots, p\}\), we have
\[
X_i(n + t) = x_{i,t+1} + \left[ \prod_{k=1}^{t} \frac{\partial f_{r_i}(x_{i,k})}{\partial x} A_{i,1} + \sum_{k=2}^{t} \left( \prod_{l=k}^{t} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \sum_{\substack{h=1 \atop h \neq i}}^{p} x_{h,k} \right) + \sum_{\substack{h=1 \atop h \neq i}}^{p} x_{h,t+1} \right] \varepsilon + O(\varepsilon^2). \quad (19)
\]
In equations (18,19), \(\{x_{i,1}, x_{i,2}, x_{i,3}, \ldots, x_{i,q_i}\}\) is an orbit of period \(q_i\) for the function \(f_{R_i}\) (for \(i \in \{1, \ldots, p\}\)) such that \(R_i = r_i + \varepsilon\) (with \(|\varepsilon| \ll 1\)) if \(i \in \{1, \ldots, s\}\) and \(R_i\) is far from \(r_i\) if \(i \in \{s + 1, \ldots, p\}\). For \(R_i = r_i\), the function \(f_{r_i}\) has a period-\(q_i\) SN orbit.
Proof of Lemma 2.

Remark: The function $f_r$ has a period-$q_i$ SN orbit for $i \in \{s+1, \ldots, p\}$, so in this case we implicitly make the assumption that $f_r$ satisfies the necessary conditions to have the associated bifurcations.

To prove Lemma 2, we proceed by induction. We substitute $X_i(n) = x_{i,1} + \varepsilon A_{i,1} + O(\varepsilon^2)$ into (17), and we note that we need to consider $i \in \{1, \ldots, s\}$ and $i \in \{s+1, \ldots, p\}$ separately.

1. For $i \in \{1, \ldots, s\}$, we have

\[
X_i(n+1) = f_r(x_{i,1}) + \varepsilon \frac{\partial f_r}{\partial x}(x_{i,1}) + \varepsilon \sum_{h=1}^{p} x_{h,2} + O(\varepsilon^2)
\]

\[
= x_{i,2} + \varepsilon \left( \frac{\partial f_r}{\partial r}(x_{i,1}) + \sum_{h=1, h \neq i}^{p} x_{h,2} \right) + O(\varepsilon^2),
\]

where we have used equation (16). In the last step, we have neglected the terms that contain $\frac{\partial f_r}{\partial x}(x_{i,1})$ because they are of order $O(\varepsilon^2)$.

2. For $i \in \{s+1, \ldots, p\}$, we have

\[
X_i(n+1) = x_{i,2} + \varepsilon \left( \frac{\partial f_r}{\partial r}(x_{i,1}) A_{i,1} + \sum_{h=1, h \neq i}^{p} x_{h,2} \right) + O(\varepsilon^2)
\]

With (17), equations (20) and (21) yield the following equations.

1. For $i \in \{1, \ldots, s\}$, we have

\[
X_i(n+2) = f_{r,+\varepsilon} \left( x_{i,2} + \varepsilon \left( \frac{\partial f_r}{\partial r}(x_{i,1}) + \sum_{h=1, h \neq i}^{p} x_{h,2} \right) \right) + \varepsilon \sum_{h=1, h \neq i}^{p} f_{R,h}(x_{h,2}) + O(\varepsilon^2)
\]

\[
= x_{i,3} + \left( \frac{\partial f_r}{\partial r}(x_{i,2}) + \frac{\partial f_r}{\partial x}(x_{i,2}) \frac{\partial f_r}{\partial r}(x_{i,1}) \right) \varepsilon + \frac{\partial f_r}{\partial x}(x_{i,2}) \sum_{h=1, h \neq i}^{p} (x_{h,2}) \varepsilon + \sum_{h=1, h \neq i}^{p} (x_{h,3}) \varepsilon + O(\varepsilon^2).
\]
2. For $i \in \{s+1, \ldots, p\}$, we have

\[
X_i(n+2) = f_{r_i} \left( x_{i,2} + \varepsilon \left( \frac{\partial f_{r_i}}{\partial r}(x_{i,1})A_{i,1} + \sum_{h=1}^{p} x_{h,2} \right) \right) + \varepsilon \sum_{h=1}^{p} f_{R_h}(x_{h,2}) + O(\varepsilon^2)
\]

\[
= x_{i,3} + \left( \frac{\partial f_{r_i}}{\partial x}(x_{i,2}) \left( \frac{\partial f_{r_i}}{\partial x}(x_{i,1})A_{i,1} + \sum_{h=1}^{p} x_{h,2} \right) + \sum_{h=1}^{p} x_{h,3} \right) \varepsilon + O(\varepsilon^2).
\]

(23)

When using the induction hypothesis, we need to distinguish the two cases.

1. For $i \in \{1, \ldots, s\}$, we write the induction hypothesis as

\[
X_i(n+t) = x_{i,t+1} + \left( \frac{\partial f_{r_i}}{\partial r}(x_{i,t}) + \sum_{k=1}^{t-1} \frac{\partial f_{r_i}}{\partial r}(x_{i,k}) \prod_{l=k+1}^{t} \frac{\partial f_{r_i}}{\partial x}(x_{i,l}) \right)
\]

\[
+ \sum_{k=2}^{t} \left( \prod_{l=k}^{t} \frac{\partial f_{r_i}}{\partial x}(x_{i,l}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,t+1} \right) \varepsilon + O(\varepsilon^2),
\]

(24)
which implies that

\[ X_i(n + t + 1) = f_{R_i}(X_i(n + t)) + \varepsilon \sum_{h=1 \atop h \neq i}^p f_{R_h}(X_h(n + t)) \]

\[ = f_{R_i}\left( x_{i,t+1} + \left( \frac{\partial f_{r_i}}{\partial x}(x_{i,t}) + \sum_{k=1}^{t-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,k}) \prod_{l=k+1}^{t} \frac{\partial f_{r_i}}{\partial x}(x_{i,l}) \right) + \sum_{h=1 \atop h \neq i}^p \frac{\partial f_{r_i}(x_{i,t+1})}{\partial x} \sum_{h=1 \atop h \neq i}^p x_{h,t+1} \right) \varepsilon \] 

\[ + \sum_{k=2}^{t} \left( \prod_{l=k}^{t} \frac{\partial f_{r_i}}{\partial x}(x_{i,l}) \sum_{h=1 \atop h \neq i}^p x_{h,k} \right) \varepsilon \] 

\[ = f_{r_i}(x_{i,t+1}) + \left( \frac{\partial f_{r_i}}{\partial x}(x_{i,t+1}) + \left[ \sum_{h=1 \atop h \neq i}^p \frac{\partial f_{r_i}(x_{i,t+1})}{\partial x} \sum_{h=1 \atop h \neq i}^p x_{h,t+1} \right] \right) \varepsilon \] 

\[ + \varepsilon \sum_{h=1 \atop h \neq i}^p x_{h,t+2} + O(\varepsilon^2) \]

\[ = x_{i,t+2} + \left( \frac{\partial f_{r_i}}{\partial x}(x_{i,t+1}) + \sum_{k=1}^{t} \frac{\partial f_{r_i}(x_{i,k})}{\partial x} \prod_{l=k+1}^{t+1} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+2} \varepsilon + O(\varepsilon^2). \] 

(25)

2. For \( i \in \{ s + 1, \ldots, p \} \), we write the induction hypothesis as

\[ X_i(n+t) = x_{i,t+1} + \left( \prod_{k=1}^{t} \frac{\partial f_{r_i}}{\partial x}(x_{i,k}) A_{i,1} + \sum_{k=2}^{t} \left( \prod_{l=k}^{t} \frac{\partial f_{r_i}}{\partial x}(x_{i,l}) \sum_{h=1 \atop h \neq i}^p x_{h,k} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+1} \right) \varepsilon + O(\varepsilon^2), \] 

(26)
which implies that

\[
X_i(n + t + 1) = f_{R_i}(X_i(n + t)) + \varepsilon \sum_{h=1 \atop h \neq i}^p f_{R_h}(X_h(n + t)) \\
= f_{r_i}(x_{i,t+1}) + \left[ \prod_{k=1}^t \frac{\partial f_{r_i}(x_{i,k})}{\partial x}(A_{i,1}) + \sum_{k=2}^t \left( \prod_{l=k}^t \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \sum_{h=1 \atop h \neq i}^p x_{h,l} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+1} \right] \varepsilon + O(\varepsilon^2) \\
+ \varepsilon \sum_{h=1 \atop h \neq i}^p f_{R_h}(x_{h,t+1} + O(\varepsilon)) \\
= f_{r_i}(x_{i,t+1}) + \varepsilon \frac{\partial f_{r_i}(x_{i,t+1})}{\partial x} \left( \prod_{k=1}^t \frac{\partial f_{r_i}(x_{i,k})}{\partial x}(A_{i,1}) + \sum_{k=2}^t \left( \prod_{l=k}^t \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \sum_{h=1 \atop h \neq i}^p x_{h,l} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+1} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+2} + O(\varepsilon^2) \\
= f_{r_i}(x_{i,t+1}) + \varepsilon \left( \prod_{k=1}^{t+1} \frac{\partial f_{r_i}(x_{i,k})}{\partial x}(A_{i,1}) + \sum_{k=2}^{t+1} \left( \prod_{l=k}^{t+1} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \sum_{h=1 \atop h \neq i}^p x_{h,l} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+2} \right) \varepsilon + O(\varepsilon^2) \\
+ \sum_{h=1 \atop h \neq i}^p x_{h,t+2} + O(\varepsilon^2) \\
= x_{i,t+2} + \left[ \prod_{k=1}^{t+1} \frac{\partial f_{r_i}(x_{i,k})}{\partial x}(A_{i,1}) + \sum_{k=2}^{t+1} \left( \prod_{l=k}^{t+1} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \sum_{h=1 \atop h \neq i}^p x_{h,l} \right) + \sum_{h=1 \atop h \neq i}^p x_{h,t+2} \right] \varepsilon + O(\varepsilon^2) .
\]

(27)

\[\square\]

**Theorem 2**

When the hypotheses of Lemma 2 are satisfied, the CML (17) has a periodic solution of period \( m = \text{lcm}(q_1, q_2, \ldots, q_p) \). This solution is

\[
X_i(n + j) = x_{i,j} + \varepsilon A_{i,j} + O(\varepsilon^2) , \quad j \in \mathbb{N} .
\]

(28)

The quantities \( A_{i,j} \) satisfy the following formulas.
1. For $i \in \{1, \ldots, s\}$, we have
\[
A_{i,j} = \frac{\partial f_r}{\partial r}(x_{i,j+m-1}) + \sum_{k=j}^{j+m-2} \frac{\partial f_r}{\partial r}(x_{i,k}) \prod_{l=k+1}^{j+m-1} \frac{\partial f_r}{\partial x}(x_{i,l}) \]
\[
+ \sum_{k=j+1}^{j+m-1} \left( \prod_{l=k}^{j+m-2} \frac{\partial f_r}{\partial x}(x_{i,l}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,j+m}, \quad j \in \{1, \ldots, m\}.
\]
(29)

2. For $i \in \{s+1, \ldots, p\}$, we have
\[
A_{i,j} = \frac{\sum_{k=j+1}^{j+m-1} \left( \prod_{l=k}^{j+m-2} \frac{\partial f_r}{\partial x}(x_{i,l}) \sum_{h=1}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,j+m}}{1 - \prod_{k=j}^{j+m-1} \frac{\partial f_r}{\partial x}(x_{i,k})}, \quad j \in \{1, \ldots, m\}.
\]
(30)

To ensure periodicity, $A_{i,j+m} = A_{i,j}$ and $x_{i,j+q} = x_{i,j}$.

**Proof of Theorem 2.**

The periodic solution
\[
X_i(n+j) = x_{i,j} + \varepsilon A_{i,j} + O(\varepsilon^2), \quad i \in \{1, \ldots, p\}, \quad j \in \mathbb{N}
\]
exist when the system
\[
X_i(n) = x_{i,1} + \varepsilon A_{i,1}
\]
\[
X_i(n+1) = x_{i,2} + \varepsilon A_{i,2}
\]
\[
\vdots
\]
\[
X_i(n+m-1) = x_{i,m} + \varepsilon A_{i,m}
\]
\[
X_i(n+m) = x_{i,1} + \varepsilon A_{i,1}
\]
(32)
(33)

has an associated matrix with full rank. Because the system (32) is linear, its associated matrix has full rank if and only if it has a solution (i.e., if the $A_{i,j}$ exist).

We now determine the values of the coefficients $A_{i,j}$. Using Lemma 2 and applying $m$ iterations (for $m \geq 2$) yields the following equations.
We also obtain the coefficients $A_{i,j}$ for $j \in \{2, \ldots, m\}$ by symmetry because the 1-st oscillator plays the same role as the $j$-th oscillator. Therefore,

1. For $i \in \{1, \ldots, s\}$, we have

$$A_{i,j} = \frac{\partial f_{r_i}(x_{i,j+m-1})}{\partial r} + \sum_{k=j}^{j+m-2} \frac{\partial f_{r_i}(x_{i,k})}{\partial r} \prod_{l=k+1}^{j+m-1} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} + \sum_{k=j+1}^{j+m-1} \left( \prod_{l=k}^{j+m-1} \frac{\partial f_{r_i}(x_{i,l})}{\partial x} \sum_{b=1}^{p} x_{h,k} \right) + \sum_{b=1}^{p} x_{h,j+m} .$$

(38)
2. For \( i \in \{s + 1, \ldots, p\} \), we have

\[
A_{i,j} = \sum_{k=j+1}^{j+m-1} \left( \prod_{l=k}^{j+m-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,l}) \sum_{h=k}^{p} x_{h,k} \right) + \sum_{h=1}^{p} x_{h,j+m} \left(1 - \prod_{k=j}^{j+m-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,k})\right).
\]

Observe that the formula for \( A_{i,j} \) for \( i \in \{1, \ldots, s\} \) in equation (38) does not contain the term

\[
1 - \prod_{k=j}^{j+m-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,k})
\]

in the denominator. Otherwise, \( A_{i,j} \) would be of size \( O(\varepsilon) \) and the expansion that we used to prove the theorem would not be valid. By contrast, the formula for \( A_{i,j} \) for \( i \in \{s + 1, \ldots, p\} \) in (39) includes the term

\[
1 - \prod_{k=j}^{j+m-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,k})
\]

in the denominator because the oscillators are far from SN bifurcations for \( i \in \{s + 1, \ldots, p\} \). Therefore,

\[
1 - \prod_{k=j}^{j+m-1} \frac{\partial f_{r_i}}{\partial x}(x_{i,k}) \in O(1),
\]

and it follows that \( A_{i,j} \) also has size \( O(1) \).

### 3 Discussion and Conclusions

In summary, we have proven theorems to describe periodic orbits both near and far from saddle-node orbits in weakly coupled map lattices. An important implication of our results is that WCMLs of oscillators need not behave approximately like their associated free-oscillator counterparts. In particular, they can have periodic-orbit solutions with completely different periods even for arbitrarily small coupling strengths \( \varepsilon \neq 0 \). Additionally, both discrete and continuous weakly coupled systems can exhibit phenomena (e.g., phase separation because of additive noise \([22]\)) that do not arise in strongly coupled systems, and one can even use weak coupling plus noise to fully synchronize nonidentical oscillators \([23]\).

From Theorems 1 and 2, we know that periodic solutions of the WCMLs have period \( m = \text{lcm}(q_1, q_2, \ldots, q_p) \), where \( q_1, \ldots, q_p \) are the periods of the free
oscillators. One can adjust the periods $q_1 \ldots q_p$ such that $m = \text{lcm}(q_1, q_2, \ldots, q_p)$ remains constant. For example, if $q_1 = 3$ and $q_2 = 4$, then $m = \text{lcm}(3, 4) = 12$. If the first oscillator undergoes a period-doubling cascade, then its period is $3, 3 \times 2, 3 \times 2^2,$ and so on. However, the period $m$ of the WCMLs is $m = \text{lcm}(3, 4) = \text{lcm}(3 \times 2, 4) = \cdots = \text{lcm}(3 \times 2^k, 4) = 12$ (for integers $k > 0$), so it does not change even after an arbitrary number of period-doubling bifurcations. That is, for arbitrarily small $\varepsilon \neq 0$, the CML period remains the same even amidst a period-doubling cascade.

Let’s illustrate this phenomenon with a simple computation. Consider the CML

$$
X(n + 1) = f(X(n)) + \varepsilon g(Y(n)),
$$

$$
Y(n + 1) = g(Y(n)) + \varepsilon f(X(n)),
$$

(40)

where $f(x) = r_1 x (1 - x)$ and $g(y) = \cos(r_2 y)$. Additionally, suppose that $r_1 = 3.83$ and $r_2 = 1.9$. When $\varepsilon = 0$ (i.e., when there is no coupling), free oscillator $X(n)$ has a period-3 orbit and the free oscillator $Y(n)$ has a period-4 orbit. However, when $\varepsilon = 0.001$, both $X(n)$ and $Y(n)$ have a periodic $m = \text{lcm}(3, 4) = 12$ (see Fig.2), as expected according to our previous results. As we shown in Table 1, the free oscillator $X(n)$ undergoes period-doubling bifurcations, but the CML still has period-12 orbits for $\varepsilon = 0.001$. (We did our computations using Fortran code and double precision.)

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Figure Legends
The temporal evolution of the $i$-th oscillator of the CML (1) is given by $X_i(n+1) = f_{R_i}(X_i(n))$. A period-$m$ SN orbit for this oscillator consists of $m$ SN points of the map $f_{R_i}^m$ for the parameter value $R_i = r_i$. The map $f_{R_i}^m$ has $m$ SN bifurcations when $R_i = r_i$. In this figure, we show the three SN points (which belong to a period-3 SN orbit) of $f_{r_i}^3$ when $f_{r_i}$ is the logistic map. The SN point $x_{i,k}$ is close to the critical point $C$ of the function $f_{r_i}$. As one considers SN orbits of larger period, the point $x_{i,k}$ becomes closer to $C$. Because $\frac{\partial f_{r_i}(C)}{\partial x} = 0$, it follows that $\frac{\partial f_{r_i}(x_{i,k})}{\partial x}$ is very small. The magnitude $|\frac{\partial f_{r_i}(x_{i,k})}{\partial x}|$ is smaller for SN orbits of larger period.
Figure 2. Temporal evolution of CML in (40) for $r_1 = 3.83$ and $r_2 = 1.9$. (a,b) When $\varepsilon = 0$ (i.e., there is no coupling), (a) oscillator $X_0(n)$ has period 3, and (b) oscillator $Y_0(n)$ has period 4. When $\varepsilon = 0.001$ (i.e., weak coupling), oscillators $X_\varepsilon(n)$ and $Y_\varepsilon(n)$ both have period $\text{lcm}(3, 4) = 12$. In panels (c) and (d), respectively, we plot $X_\varepsilon(n) - X_0(n)$ and $Y_\varepsilon(n) - Y_0(n)$ (i.e., the solution for the coupled case minus the solution for the $\varepsilon = 0$ case) to better observe the period-12 orbit.
Tables

Table 1. Period of CML given by (40) for \( r_2 = 1.9 \) and \( \varepsilon = 0.001 \). The parameter \( r_1 \) indicates when the logistic map, which is satisfied by the free oscillator \( X(n) \), exhibits supercycles of various periods during a period-doubling cascade in the window of period-3 orbits in the bifurcation diagram. Although the period of \( X(n) \) changes, the period of the CML remains constant.

| \( r_1 \)                  | Period of \( X(n) \) | Period of the CML       |
|---------------------------|----------------------|-------------------------|
| 3.8318740552831556        | 3                    | lcm(3, 4) = 12          |
| 3.844656879219443297      | \( 3 \times 2 \)     | lcm(3 \times 2, 4) = 12 |
| 3.84834465689184598       | \( 3 \times 2^2 \)   | lcm(3 \times 2^2, 4) = 12 |