NOETHER SYMMETRIES IN EXTENDED GRAVITY QUANTUM COSMOLOGY

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We summarize the use of Noether symmetries in Minisuperspace Quantum Cosmology. In particular, we consider minisuperspace models, showing that the existence of conserved quantities gives selection rules that allow to recover classical behaviors in cosmic evolution according to the so called Hartle criterion. Such a criterion selects correlated regions in the configuration space of dynamical variables whose meaning is related to the emergence of classical observable universes. Some minisuperspace models are worked out starting from Extended Gravity, in particular coming from scalar tensor, $f(R)$ and $f(T)$ theories. Exact cosmological solutions are derived.

We wish to dedicate this paper in loving memory of Mauro Francaviglia. We mourn the loss of a dear friend, a peerless mentor and a traveling fellow in many scientific adventures, discussions and results.

Keywords: Quantum Cosmology; Noether Symmetries; Minisuperspace Models.

1. Introduction

The concept of symmetry has undergone evolution and improvements over the last century: the transition from global continuous symmetries (such as translation and rotation) to local continuous symmetries, and in particular local gauge symmetries, the addition of the internal symmetries to the space-time symmetries, the connection between charge conjugation and time reversal with the discrete symmetry of parity, the utilization of permutation symmetry in Quantum Field Theory and, finally, the use of symmetries in cosmology. Symmetries have become a key tool of theory construction in fundamental physics. For example we recall the role of the Principle of Relativity in Einstein’s construction of Special Relativity and then the role of general covariance as a symmetry principle in the development of General Relativity as well also the role of group theory in Weyl’s approach to Quantum Field Theory or the development of particle physics and, for example, the use of symmetry principles in the construction of the Standard Model, where symmetries
play their role is the framework of Lagrangian field theories, which are the current basis for any approach to fundamental interactions in Physics. This corresponds, to consider a dynamical system with a continuous infinity of degrees of freedom.

In this context, the work of Emmy Noether is of central importance since she provided a general proof that from the continuous symmetries of a given theory, we can derive the conservation laws of that theory. In particular she discovered the relation between symmetries of dynamical systems and their first integrals, i.e., physical quantities which remain constant during the evolution of the system and are related to fundamental physical quantities such as energy, linear momentum, angular momentum and so on \[3,4\].

Also in Quantum Cosmology the Noether symmetries have been of crucial importance, because they provide a subset of the general solution of the Wheeler-De Witt equation where oscillating behaviors with crucial physical meaning can be selected [5]. Specifically, the Hartle criterion can be related to the Noether symmetries of the theory. In fact, in the context of the Minisuperspace Approach, this criterion selects classical trajectories which are solutions of the cosmological field equations [6]. In other words, the presence of Noether Symmetries in Quantum Cosmology allows the emergence of classically observable universes.

In this lecture notes, we will briefly sketch the Minisuperscape Approach in Quantum Cosmology (Sec. 2). Noether symmetries in Lagrangian and Hamiltonian dynamical systems are discussed in Sec. 3. Applications to scalar-tensor, \( f(R) \) and \( f(T) \)-gravity are extensively reported in Sec. 4. In Sec. 5 we draw conclusions on the presented results.

2. The Minisuperspace Approach to Quantum Cosmology

Minisuperspaces are restrictions of superspace, in which the symmetries are fixed a priori on the metric and matter fields. The simplest model of minisuperspace is constructed by choosing homogeneous and isotropic metrics and matter fields. In general, it provides a lapse function \( N = N(t) \) assumed homogeneous, the shift functions \( N^i = 0 \) set to zero and a 4-metric as follow

\[
ds^2 = -N^2(t)dt^2 + h_{ij}(x,t)dx^i dx^j. \tag{1}
\]

where \( h_{ij} \) is a 3-metric homogeneous and it is described by a finite number of functions of \( t, q^\alpha(t) \), where \( \alpha = 0, 1, 2 \cdots (n-1) \) (see [5] and references therein). We can recast the Hilbert-Einstein action as

\[
S[h_{ij}, N, N^i] = \frac{m^2}{16\pi} \int dt d^3x N \sqrt{h} \left[ K_{ij}K^{ij} - K^2 + (3)R - 2\Lambda \right], \tag{2}
\]

and, in general, one gets [4] \[
\text{Here, we have assumed as signature } (-,+,+,+). \text{ The function } f_{\alpha\beta}(q) \text{ is the reduced De Witt metric, and the integration of } t \text{ can range from 0 to 1 by shifting } t \text{ and rescaling the lapse function.} \]
\[ S[q^\alpha(t), N(t)] = \int_0^1 dt N \left[ \frac{1}{2N^2} f_{\alpha\beta}(q) \dot{q}^\alpha \dot{q}^\beta - U(q) \right] \equiv \int L dt . \] (3)

The last equation has the form of a relativistic point particle action where the particles move on an \( n \)-dimensional curved space-time with a self-interaction potential. The variation with respect to \( q^\alpha \), gives the equations of motion

\[ \frac{1}{N} \frac{d}{dt} \left( \frac{\dot{q}^\alpha}{N} \right) + \frac{1}{N^2} \Gamma^\alpha_{\beta\gamma} \dot{q}^\beta \dot{q}^\gamma + f_{\alpha\beta} \frac{\partial U}{\partial q^\beta} = 0 , \] (4)

where \( \Gamma^\alpha_{\beta\gamma} \) are the Christoffel symbols derived from the metric \( f_{\alpha\beta} \). Varying with respect to \( N \), one gets

\[ \frac{1}{2N^2} f_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta + U(q) = 0 , \] (5)

that is a constraint equation.

Eqs(4) and (5) describe geodesic motion in minisuperspace with a forcing term. The general solution of (4), (5) requires \( (2n-1) \) arbitrary parameters to be found.

In order to find the Hamiltonian, the canonical momenta have to be defined, that is

\[ p_\alpha = \frac{\partial L}{\partial \dot{q}^\alpha} = f_{\alpha\beta} \frac{\dot{q}^\beta}{N} , \] (6)

and the canonical Hamiltonian is

\[ H_c = p_\alpha \dot{q}^\alpha - L = N \left[ \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) \right] \equiv N \mathcal{H} , \] (7)

where \( f^{\alpha\beta}(q) \) is the inverse metric on minisuperspace. The action in Hamiltonian form is

\[ S = \int_0^1 dt \left[ p_\alpha \dot{q}^\alpha - N \mathcal{H} \right] . \] (8)

A Lagrange multiplier is the lapse function \( N \) and then the Hamiltonian constraint has to be

\[ \mathcal{H}(q^\alpha, p_\alpha) = \frac{1}{2} f^{\alpha\beta} p_\alpha p_\beta + U(q) = 0 . \] (9)

Now, the canonical quantization procedure requires a time-independent wave function \( \Psi(q^\alpha) \) that has to be annihilated by the quantum operator corresponding to the classical constraint (9). This fact gives rise to the Wheeler-De Witt equation,

\[ \hat{\mathcal{H}}(q^\alpha, -i \frac{\partial}{\partial q^\alpha}) \Psi(q^\alpha) = 0 , \] (10)

where \( \Psi(q^\alpha) \) is the so-called Wave Function of the Universe. Since the metric \( f^{\alpha\beta} \) depends on \( q \) there is a factor ordering issue in (10). This may be solved by requiring that the quantization procedure is covariant in minisuperspace, that is unchanged
by field redefinitions of the 3-metric and matter fields, \( q^\alpha \to \tilde{q}^\alpha(q^\alpha) \). This fact restricts the possible operator orderings to
\[
\hat{\mathcal{H}} = -\frac{1}{2} \nabla^2 + \xi \mathcal{R} + U(q),
\]
where \( \nabla^2 \) and \( \mathcal{R} \) are the Laplacian and curvature of the minisuperspace metric \( f_{\alpha\beta} \) and \( \xi \) is an arbitrary constant.

Before concluding this section, an important issue has to be addressed. It is how to interpret the probability measure in Quantum Cosmology. As a final remarks we remember as we can interpret the probability measure. Since the Wheeler-De Witt equation is similar to Klein-Gordon equation we can define a current, that is conserved and satisfies \( \nabla \cdot J = 0 \), in this way
\[
J = i \sqrt{2} (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*),
\]
As in the case of the Klein-Gordon equation (and, in general, of hyperbolic equations), the probability derived from such a conserved current can be affected by negative probabilities. Due to this shortcoming, the correct measure to use should be
\[
dP = |\Psi(q^\alpha)|^2 dV,
\]
where \( dV \) is a volume element of minisuperspace [5,6].

### 3. The Noether Symmetry Approach

As we said before, minisuperspaces are restrictions of the superspace of geometrodynamics. They are finite-dimensional configuration spaces on which point-like Lagrangians can be defined. Cosmological models of physical interest can be defined on such minisuperspaces (e.g. Bianchi models). According to the above discussion, a crucial role is played by the conserved currents that allow to interpret the probability measure and then the physical quantities obtained in Quantum Cosmology. In this context, the search for general methods to achieve conserved quantities and symmetries become relevant. The so-called Noether Symmetry Approach [2], as we will show, can be extremely useful to this purpose.

Let \( \mathcal{L} \) a Lagrangian defined on the tangent space of configurations \( TQ \equiv \{q_i, \dot{q}_i\} \), the vector field \( X \) is
\[
X = \alpha^i(q) \frac{\partial}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial}{\partial \dot{q}^i},
\]
where dot means derivative with respect to \( t \), and
\[
L_X \mathcal{L} = X \mathcal{L} = \alpha^i(q) \frac{\partial \mathcal{L}}{\partial q^i} + \dot{\alpha}^i(q) \frac{\partial \mathcal{L}}{\partial \dot{q}^i},
\]
where is the Lie derivative \( L_X \) of \( \mathcal{L} \). The condition
\[
L_X \mathcal{L} = 0,
\]
implies that the phase flux is conserved along $X$: this means that a constant of motion exists for $\mathcal{L}$ and the Noether theorem holds. In fact, taking into account the Euler-Lagrange equations

$$\frac{d}{dt} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_X \mathcal{L},$$

(18)

If Eq. (16) holds,

$$\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$$

(19)

is a constant of motion. Alternatively, using the Cartan one–form

$$\theta_\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}^i} dq^i$$

(20)

and defining the inner derivative

$$i_X \theta_\mathcal{L} = \langle \theta_\mathcal{L}, X \rangle,$$

(21)

we get, as above,

$$i_X \theta_\mathcal{L} = \Sigma_0,$$

(22)

if condition (16) holds. This representation is useful to identify cyclic variables. Using a point transformation on vector field (14), it is possible to get

$$X = (i_X dQ^k) \frac{\partial}{\partial Q^k} + \left[ \frac{d}{dt} (i_X dQ^k) \right] \frac{\partial}{\partial Q^k}.$$  

(23)

If $X$ is a symmetry also $\tilde{X}$ has this property, then it is always possible to choose a coordinate transformation so that

$$i_X dQ^1 = 1, \quad i_X dQ^i = 0, \quad i \neq 1,$$

(24)

and then

$$\tilde{X} = \frac{\partial}{\partial Q^1}, \quad \frac{\partial \tilde{\mathcal{L}}}{\partial Q^1} = 0.$$  

(25)

It is evident that $Q^1$ is the cyclic coordinate and the dynamics can be reduced [7]. However, the change of coordinates is not unique and a clever choice is always important. Furthermore, it is possible that more symmetries are found. In this case, more than one cyclic variable exists.

A reduction procedure by cyclic coordinates can be implemented in three steps:

bWe indicate the quantities as Lagrangians and vector fields with a tilde if the non–degenerate transformation

$$Q^i = Q^i(q), \quad \tilde{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j,$$

is performed. However the Jacobian determinant $\mathcal{J} = \| \partial Q^i / \partial q^j \|$ has to be non-zero.
we choose a symmetry and obtain new coordinates as above. After this first reduction, we get a new Lagrangian \( \hat{\mathcal{L}} \) with a cyclic coordinate;  
- we search for new symmetries in this new space and apply the reduction technique until it is possible;  
- the process stops if we select a pure kinetic Lagrangian where all coordinates are cyclic.

Going back to the point of view interesting in Quantum Cosmology, any symmetry selects a constant conjugate momentum since, by the Euler-Lagrange equations

\[
\frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}_i} = 0 \iff \frac{\partial \hat{\mathcal{L}}}{\partial q_i} = \Sigma_i .
\]  

Viceversa, the existence of a constant conjugate momentum means that a cyclic variable has to exist. In other words, a Noether symmetry exists.

Further remarks on the form of the Lagrangian \( \mathcal{L} \) are necessary at this point. We shall take into account time–independent, non–degenerate Lagrangians \( \mathcal{L} = \mathcal{L}(q^i, \dot{q}^i) \), i.e.

\[
\frac{\partial \mathcal{L}}{\partial t} = 0, \quad \det H_{ij} \equiv \det \left| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0 ,
\]  

where \( H_{ij} \) is the Hessian. As in usual analytic mechanics, \( \mathcal{L} \) can be set in the form

\[
\mathcal{L} = T(q^i, \dot{q}^i) - V(q^i) ,
\]  

where \( T \) is a positive–defined quadratic form in the \( \dot{q}^i \) and \( V(q^i) \) is a potential term. The energy function associated with \( \mathcal{L} \) is

\[
E_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L}(q^i, \dot{q}^i) ,
\]  

and by the Legendre transformations

\[
\mathcal{H} = \pi_j \dot{q}^j - \mathcal{L}(q^i, \dot{q}^i) , \quad \pi_j = \frac{\partial \mathcal{L}}{\partial \dot{q}^j} ,
\]  

we get the Hamiltonian function and the conjugate momenta. Considering again the symmetry, the condition \([\Sigma_j, \mathcal{H}] = 0\) and the vector field \( X \) in Eq.\([14]\) give a homogeneous polynomial of second degree in the velocities plus an inhomogeneous term in the \( q^j \). Due to \([10]\), such a polynomial has to be identically zero and then each coefficient must be independently zero. If \( n \) is the dimension of the configuration space (i.e. the dimension of the minisuperspace), we get \( \{1 + n(n + 1)/2\} \) partial differential equations whose solutions assign the symmetry, as we shall see below. Such a symmetry is over–determined and, if a solution exists, it is expressed in terms of integration constants instead of boundary conditions. In the Hamiltonian formalism, we have

\[
[\Sigma_j, \mathcal{H}] = 0, \quad 1 \leq j \leq m ,
\]
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as it must be for conserved momenta in quantum mechanics and the Hamiltonian has to satisfy the relations

\[ L_\Gamma \mathcal{H} = 0 , \]

in order to obtain a Noether symmetry. The vector \( \Gamma \) is defined by [8]

\[ \Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} . \]

These considerations can be applied to the minisuperspace models of Quantum Cosmology and to the interpretation of the wave function of the universe.

As discussed above, by a straightforward canonical quantization procedure, we have

\[ \pi_j \rightarrow \hat{\pi}_j = -i \partial_j , \]

\[ \mathcal{H} \rightarrow \hat{\mathcal{H}}(q^j, -i \partial_q) . \]

It is well known that the Hamiltonian constraint gives the Wheeler-De Witt equation, so that if \( |\Psi\rangle \) is a state of the system (i.e. the wave function of the universe), dynamics is given by

\[ \mathcal{H}|\Psi\rangle = 0 , \]

where we write the Wheeler-De Witt equation in an operational way. If a Noether symmetry exists, the reduction procedure outlined above can be applied and then, from (26) and (30), we get

\[ \pi_1 = \frac{\partial L}{\partial \dot{Q}_1} = i \chi_1 \theta L = \Sigma_1 , \]

\[ \pi_2 = \frac{\partial L}{\partial \dot{Q}_2} = i \chi_2 \theta L = \Sigma_2 , \]

\[ \ldots \ldots \ldots \ldots , \]

depending on the number of Noether symmetries. After quantization, we get

\[ -i \partial_1 |\Psi\rangle = \Sigma_1 |\Psi\rangle , \]

\[ -i \partial_2 |\Psi\rangle = \Sigma_2 |\Psi\rangle , \]

\[ \ldots \ldots \ldots \ldots , \]

which are nothing else but translations along the \( Q^j \) axis singled out by corresponding symmetry. Eqs. (38) can be immediately integrated and, being \( \Sigma_j \) real constants, we obtain oscillatory behaviors for \( |\Psi\rangle \) in the directions of symmetries, i.e.

\[ |\Psi\rangle = \sum_{j=1}^{m} e^{i\Sigma_j Q^j} |\chi(Q^j)\rangle , \quad m < l \leq n , \]

where \( m \) is the number of symmetries, \( l \) are the directions where symmetries do not exist, \( n \) is the total dimension of minisuperspace. Viceversa, dynamics given by
can be reduced by (38) if and only if it is possible to define constant conjugate momen
ta as in (37), that is oscillatory behaviors of a subset of solutions $|\Psi>$ exist only if Noether symmetry exists for dynamics.

The $m$ symmetries give first integrals of motion and then the possibility to select classical trajectories. In one and two–dimensional minisuperspaces, the existence of a Noether symmetry allows the complete solution of the problem and to get the full semi-classical limit of Quantum Cosmology [9]. In conclusion, we can state that in the semi-classical limit of quantum cosmology, the reduction procedure of dynamics, connected to the existence of Noether symmetries, allows to select a subset of the solution of Wheeler–De Witt equation where oscillatory behaviors are found. This fact, in the framework of the Hartle interpretative criterion of the wave function of the universe, gives conserved momenta and trajectories which can be interpreted as classical cosmological solutions. Vice-versa, if a subset of the solution of Wheeler–De Witt equation has an oscillatory behavior, due to Eq.(16), conserved momenta exist and Noether symmetries are present. In other words, Noether symmetries select classical universes and then are directly related to the validity of the Hartle criterion.

In what follows, we will show that such a statement holds for general classes of minisuperspaces and allows to select exact classical solutions, i.e. the presence of Noether symmetries is a selection criterion for classical universes. In the next section, we will shown how they work for Extended Theories of Gravitation, as scalar tensor, $f(R)$ and $f(T)$ gravity.

4. Noether symmetries in cosmology

Let us consider realizations of the above approach in minisuperspace cosmological models derived from Extended Theories of Gravity. As we have seen, the Hartle criterion is directly connected to the presence of Noether symmetries since oscillatory behaviors means correlations among variables [11,12,13,14,15].

Specifically, the approach can be connected to the search for Lagrange multipliers. In fact, imposing Lagrange multipliers allow to modify the dynamics and select the form of effective potentials. By integrating the multipliers, solutions can be obtained.

On the other hand, the Lagrange multipliers are constraints capable of reducing dynamics. Technically they are anholonomic constraints being time-dependent. They give rise to field equations which describe dynamics of the further degrees of freedom coming from Extended Theories of Gravity [11,12,13,14,15]. This fact is extremely relevant to deal with new degrees of freedom under the standard of effective scalar fields [10]. Below, we give minisuperspace examples and obtain exact cosmological solutions. In particular, we show that, by imposing Lagrange multipliers, a given minisuperspace model becomes canonical and Noether symmetries, if exist, can be found out.
4.1. Scalar-Tensor cosmology

A general action for a nonminimally coupled theory of gravity assume the following form

\[ S = \int d^4x \sqrt{-g} \left[ F(\phi)R + \frac{1}{2}g^{\mu\nu}\phi_\mu\phi_\nu - V(\phi) \right], \tag{40} \]

where, \( F(\phi) \) and \( V(\phi) \) are respectively the coupling and the potential of a scalar field.

The cosmological point-like Lagrangian for a Friedman Robertson Walker (FRW) minisuperspace in terms of the scale factor \( a \) is

\[ \mathcal{L} = 6a\dot{a}^2F + 6a^2\dot{\phi}^2 - 6kaF + a^3\left[ \frac{\dot{\phi}^2}{2} - V \right]. \tag{41} \]

The configuration space of such a Lagrangian is \( Q \equiv \{a, \phi\} \), i.e. a bidimensional minisuperspace. A Noether symmetry exists if Eq. (16) holds. In this case, it has to be

\[ X = \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial \phi} + \dot{\alpha} \frac{\partial}{\partial \dot{a}} + \dot{\beta} \frac{\partial}{\partial \dot{\phi}}, \tag{42} \]

where \( \alpha, \beta \) depend on \( a, \phi \). This vector field acts on the \( Q \) minisuperspace. The system of partial differential equation given by (16) is

\[ aF(\phi) \left[ \alpha + 2a\frac{\partial \alpha}{\partial a} \right] + aF'(\phi) \left[ \beta + a\frac{\partial \beta}{\partial a} \right] = 0, \tag{43} \]

\[ 3\alpha + 12F'(\phi)\frac{\partial \alpha}{\partial \phi} + 2a\frac{\partial \beta}{\partial \phi} = 0, \tag{44} \]

\[ a\beta F''(\phi) + \left[ 2\alpha + a\frac{\partial \alpha}{\partial a} + \frac{\partial \beta}{\partial \phi} \right] F'(\phi) + 2\frac{\partial \alpha}{\partial \phi}F(\phi) + a^2\frac{\partial \beta}{\partial a} = 0, \tag{45} \]

\[ [3\alpha V(\phi) + a\beta V'(\phi)]a^2 + 6k[\alpha F(\phi) + a\beta F'(\phi)] = 0. \tag{46} \]

Prime indicates the derivative with respect to \( \phi \). The number of equations is 4 as it has to be, being \( n = 2 \) the \( Q \)-dimension. Several solutions exist for this system. They determine also the form of the model since the system (43)-(46) gives \( \alpha, \beta, F(\phi) \) and \( V(\phi) \). For example, if the spatial curvature is \( k = 0 \), a solution is

\[ \alpha = -\frac{2}{3}p(s)\beta_0a^{s+1}\phi^{m(s)-1}, \quad \beta = \beta_0a^s\phi^{m(s)}, \tag{47} \]

\[ F(\phi) = D(s)\phi^2, \quad V(\phi) = \lambda\phi^{2p(s)}, \tag{48} \]

\( ^{c} \)Using physical units \( 8\pi G = c = \hbar = 1 \), the standard Einstein coupling is recovered for \( F(\phi) = -\frac{1}{2} \).
where
\[
D(s) = \frac{(2s + 3)^2}{48(s + 1)(s + 2)},
\]
\[
p(s) = \frac{3(s + 1)}{2s + 3},
\]
\[
m(s) = \frac{2s^2 + 6s + 3}{2s + 3},
\]
and \(s, \lambda\) are free parameters. The change of variables \([24]\) gives
\[
w = \sigma_0 a^3 \phi^{2p(s)}, \quad z = \frac{3}{\beta_0 \chi(s)} a^{-s} \phi^{1-m(s)},
\]
where \(\sigma_0\) is an integration constant and
\[
\chi(s) = -\frac{6s}{2s + 3}.
\]
Lagrangian \([41]\) becomes, for \(k = 0\),
\[
\mathcal{L} = \gamma(s) w^{s/3} \dot{w} - \lambda w,
\]
where \(z\) is cyclic and
\[
\gamma(s) = \frac{2s + 3}{12 \sigma_0^2 (s + 2)(s + 1)}.
\]
The conjugate momenta are
\[
\pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = \gamma(s) w^{s/3} \dot{w}, \quad \pi_w = \frac{\partial \mathcal{L}}{\partial \dot{w}} = \gamma(s) w^{s/3} \dot{z},
\]
and the Hamiltonian is
\[
\hat{\mathcal{H}} = \frac{\pi_z \pi_w}{\gamma(s) w^{s/3}} + \lambda w.
\]
The Noether symmetry is given by
\[
\pi_z = \Sigma_0.
\]
Quantizing Eqs. \([58]\), we get
\[
\pi \rightarrow -i \partial_z, \quad \pi_w \rightarrow -i \partial_w,
\]
and then the Wheeler-De Witt equation
\[
[(i \partial_z)(i \partial_w) + \hat{\lambda} w^{1+s/3}]|\Psi> = 0,
\]
where \(\hat{\lambda} = \gamma(s) \lambda\). The quantum version of constraint \([58]\) is
\[
- i \partial_z |\Psi> = \Sigma_0 |\Psi>.
\]
so that dynamics results reduced. A straightforward integration of Eqs. (60) and (61) gives
\[ |\Psi> = |\Omega(w)\rangle > |\chi(z)\rangle > \propto e^{i\Sigma_0 z} e^{-i\tilde{\lambda} w^{2/s^3}}, \] (62)
which is an oscillating wave function and the Hartle criterion is recovered. In the semi–classical limit, we have two first integrals of motion: \(\Sigma_0\) (i.e. the equation for \(\pi_z\)) and \(E_{\mathcal{L}} = 0\), i.e. the Hamiltonian (57) which becomes the equation for \(\pi_w\). Classical trajectories in the configuration space \(\mathcal{Q} \equiv \{w, z\}\) are immediately recovered
\[ w(t) = [k_1 t + k_2]^{3/(s+3)}, \]
\[ z(t) = [k_1 t + k_2]^{(s+6)/(s+3)} + z_0, \] (64)
then, going back to \(\mathcal{Q} \equiv \{a, \phi\}\), we get the classical cosmological behaviour
\[ a(t) = a_0 (t - t_0)^{(l(s)}), \]
\[ \phi(t) = \phi_0 (t - t_0)^{(q(s)}), \] (66)
where
\[ l(s) = \frac{2s^2 + 9s + 6}{s(s + 3)}, \quad q(s) = -\frac{2s + 3}{s}, \] (67)
which means that Hartle criterion selects classical universes. Depending on the value of \(s\), we get Friedman, power–law, or pole–like behaviors. The considerations on the oscillatory regime of the wave function of the universe and the recovering of classical behaviors are exactly the same.

4.2. \(f(R)\) cosmology

Similar results can be obtained also for higher–order gravity minisuperspaces. In particular, let us consider fourth–order gravity given by the action
\[ S = \int d^4x \sqrt{-g} f(R), \] (68)
where \(f(R)\) is a generic function of Ricci scalar [14]. Rewriting the action to a point-like FriedmanRobertsonWalker one, we obtain
\[ S = \int dt \mathcal{L}(a, \dot{a}; R, \dot{R}), \] (69)
where dot means derivative with respect to the cosmic time. The scale factor \(a\) and the Ricci scalar \(R\) are the canonical variables. This position could seem arbitrary since \(R\) depends on \(a, \dot{a}, \ddot{a}\), but it is generally used in canonical quantization [19,21,20]. The definition of \(R\) in terms of \(a, \dot{a}, \ddot{a}\) introduces a constraint which eliminates second and higher order derivatives in action (69), and yields to a system of second order differential equations in \(\{a, R\}\). Action (69) can be written as
\[ S = 2\pi^2 \int dt \left\{ a^3 f(R) - \lambda \left[ R + 6 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \right] \right\}, \] (70)
where the Lagrange multiplier $\lambda$ is derived by varying with respect to $R$. It is
\begin{equation}
\lambda = a^3 f'(R),
\end{equation}
(71)
Here prime means derivative with respect to $R$. To recover the analogy with previous scalar–tensor models, let us introduce the auxiliary field
\begin{equation}
p = f'(R),
\end{equation}
(72)
so that the Lagrangian in (70) becomes
\begin{equation}
L = 6a\dot{a}^2 p + 6a^2 \dot{a} \dot{p} - 6kap - a^3 W(p),
\end{equation}
(73)
which is of the same form of (41) a part the kinetic term. This is an Helmhotz–like Lagrangian \cite{22} and $a, p$ are independent fields. The potential $W(p)$ is defined as
\begin{equation}
W(p) = h(p)p - r(p),
\end{equation}
(74)
where
\begin{equation}
r(p) = \int f'(R)dR = \int pdR = f(R),
\end{equation}
(75)
such that $h = (f')^{-1}$ is the inverse function of $f'$. The configuration space is now $Q = \{a, p\}$ and $p$ has the same role of the above $\phi$. Condition (10) is now realized by the vector field
\begin{equation}
X = \alpha(a, p) \frac{\partial}{\partial a} + \beta(a, p) \frac{\partial}{\partial p} + \dot{a} \frac{\partial}{\partial a} + \dot{p} \frac{\partial}{\partial p},
\end{equation}
(76)
and explicitly it gives the system
\begin{equation}
a \dot{p} \left[ \alpha + 2a \frac{\partial \alpha}{\partial a} \right] + a \left[ \beta + a \frac{\partial \beta}{\partial a} \right] = 0,
\end{equation}
(77)
\begin{equation}
a^2 \frac{\partial \alpha}{\partial p} = 0,
\end{equation}
(78)
\begin{equation}
2\alpha + a^2 \frac{\partial \alpha}{\partial a} + 2\alpha \frac{\partial \alpha}{\partial p} + a \frac{\partial \alpha}{\partial p} = 0,
\end{equation}
(79)
\begin{equation}
6k[\alpha p + \beta a] + a^2[3\alpha W + a\beta \frac{\partial W}{\partial p}] = 0.
\end{equation}
(80)
The solution of this system, i.e. the existence of a Noether symmetry, gives $\alpha$, $\beta$ and $W(p)$. It is satisfied for
\begin{equation}
\alpha = \alpha(a), \quad \beta(a, p) = \beta_0 a^s p,
\end{equation}
(81)
where $s$ is a parameter and $\beta_0$ is an integration constant. In particular,
\begin{equation}
s = 0 \longrightarrow \alpha(a) = -\frac{\beta_0}{3} a, \quad \beta(p) = \beta_0 p,
\end{equation}
\begin{equation}
W(p) = W_0 p, \quad k = 0,
\end{equation}
(82)
\[ s = -2 \rightarrow \alpha(a) = -\frac{\beta_0}{a}, \quad \beta(a, p) = \frac{\beta_0 p}{a^2}, \]

\[ W(p) = W_1 p^3, \quad \forall \ k, \quad (83) \]

where \( W_0 \) and \( W_1 \) are constants. Let us discuss separately the solutions \( (82) \) and \( (83) \).

### 4.2.1. The case \( s = 0 \)

The induced change of variables \( \mathcal{Q} \equiv \{a, p\} \rightarrow \tilde{\mathcal{Q}} \equiv \{w, z\} \) can be

\[ w(a, p) = a^3 p, \quad z(p) = \ln p. \quad (84) \]

The Lagrangian \( (73) \) becomes

\[ \tilde{\mathcal{L}}(w, \dot{w}, \dot{z}) = \dot{z} \dot{w} - 2 w \dot{z}^2 + \frac{\dot{w}^2}{w} - 3 W_0 w. \quad (85) \]

and, obviously, \( z \) is the cyclic variable. The conjugate momenta are

\[ \pi_z = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} = \dot{w} - 4 \dot{z} = \Sigma_0, \quad (86) \]

\[ \pi_w = \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{w}} = \dot{z} + 2 \frac{\dot{w}}{w}. \quad (87) \]

and the Hamiltonian is

\[ \mathcal{H}(w, \pi_w, \pi_z) = \pi_w \pi_z - \frac{\pi_z^2}{w} + 2 w \pi_w^2 + 6 W_0 w. \quad (88) \]

By canonical quantization, reduced dynamics is given by

\[ \left[ \dot{\pi}_z^2 - 2 w^2 \partial_w^2 - w \partial_w \partial_z + 6 W_0 w^2 \right] |\Psi > = 0, \quad (89) \]

\[ -i \partial_z |\Psi > = \Sigma_0 |\Psi >. \quad (90) \]

However, we have done simple factor ordering considerations in the Wheeler-De Witt equation \( (89) \). Immediately, the wave function has an oscillatory factor, being

\[ |\Psi > \sim e^{i\Sigma_0 z} |\chi(w) >. \quad (91) \]

The function \( |\chi > \) satisfies the Bessel differential equation

\[ \left[ w^2 \partial_w^2 + i \frac{\Sigma_0}{2} w \partial_w + \left( \frac{\Sigma_0^2}{2} - 3 W_0 w^2 \right) \right] \chi(w) = 0, \quad (92) \]

whose solutions are linear combinations of Bessel functions \( Z_{\nu}(w) \)

\[ \chi(w) = w^{1/2 - i\Sigma_0/4} Z_{\nu}(\lambda w), \quad (93) \]

where

\[ \nu = \pm \frac{1}{4} \sqrt{4 - 9 \Sigma_0^2 - i 4 \Sigma_0}, \quad \lambda = \pm 9 \sqrt{\frac{W_0}{2}}. \quad (94) \]
The oscillatory regime for this component depends on the reality of $\nu$ and $\lambda$. The wave function of the universe, from Noether symmetry (82) is then
\[ \Psi(z, w) \sim e^{i\Sigma_0[z-(1/4)\ln w]} w^{1/2} Z_{\nu}(\lambda w). \] (95)

For large $w$, the Bessel functions have an exponential behavior, so that the wave function (95) can be written as
\[ \Psi \sim e^{i[\Sigma_0 z-(\Sigma_0/4)\ln w \pm \lambda w]}. \] (96)

Due to the oscillatory behaviour of $\Psi$, Hartle’s criterion is immediately recovered. By identifying the exponential factor of (96) with $S_0$, we can recover the conserved momenta $\pi_z, \pi_w$ and select classical trajectories. Going back to the old variables, we get the cosmological solutions
\[ a(t) = a_0 e^{(\lambda/6)t} \exp\left\{ -\frac{z_1}{3} e^{-(2\lambda/3)t} \right\}, \] (97)
\[ p(t) = p_0 e^{(\lambda/6)t} \exp\{ z_1 e^{-(2\lambda/3)t} \}, \] (98)
where $a_0, p_0$ and $z_1$ are integration constants. It is clear that $\lambda$ plays the role of a cosmological constant and inflationary behavior is asymptotically recovered.

4.2.2. The case $s = -2$

The new variables adapted to the foliation for the solution (83) are now
\[ w(a, p) = ap, \quad z(a) = a^2. \] (99)

and Lagrangian (83) assumes the form
\[ \hat{L}(w, \dot{w}, \dot{z}) = 3\dot{z}w - 6kw - W_1 w^3, \] (100)

The conjugate momenta are
\[ \pi_z = \frac{\partial \hat{L}}{\partial \dot{z}} = 3\dot{w} = \Sigma_1, \] (101)
\[ \pi_w = \frac{\partial \hat{L}}{\partial \dot{w}} = 3\dot{z}. \] (102)

The Hamiltonian is given by
\[ \mathcal{H}(w, \pi_w, \pi_z) = \frac{1}{3} \pi_z \pi_w + 6kw + W_1 w^3. \] (103)

Going over the same steps as above, the wave function of the universe is given by
\[ \Psi(z, w) \sim e^{i[\Sigma_1 z + 9kw^2 + (3W_1/4)w^4]}, \] (104)

and the classical cosmological solutions are
\[ a(t) = \pm \sqrt{h(t)}, \quad p(t) = \pm \frac{c_1 + (\Sigma_1/3) t}{\sqrt{h(t)}}, \] (105)
where
\[ h(t) = \left( \frac{W_1 \Sigma_1}{36} \right) t^4 + \left( \frac{W_1 w_1 \Sigma_1}{6} \right) t^3 + \left( k \Sigma_1 + \frac{W_1 w_1^2 \Sigma_1}{2} \right) t^2 + w_1 (6k + W_1 w_1^2) t + z_2. \]

\[ w_1, z_1 \text{ and } z_2 \text{ are integration constants. Immediately we see that, for large } t \]
\[ a(t) \sim t^2, \quad p(t) \sim \frac{1}{t}. \]

which is a power-law inflationary behavior. An extensive discussion of Noether symmetries in \( f(R) \) gravity is in [23,25].

### 4.3. \( f(T) \) cosmology

In analogy to the \( f(R) \) gravity, a new sort of Extended Gravity, the so-called \( f(T) \) theory, has been recently proposed. It is a generalized and extended version of the teleparallel gravity originally proposed by Einstein [27,28,29]. Teleparallelism uses as dynamical object a vierbein field \( e_i(x^\mu), i = 0, 1, 2, 3 \), which is an orthonormal basis for the tangent space at each point \( x^\mu \) of the manifold: \( \eta_{ij} = \eta_{ij} = \text{diag}(-1, 1, 1, 1) \). Each vector \( e_i \) can be described by its components \( e_i^\mu, \mu = 0, 1, 2, 3 \) in a coordinate basis; i.e. \( e_i = e_i^\mu \partial_\mu \). Notice that latin indexes refer to the tangent space, while greek indexes label coordinates on the manifold. The metric tensor is obtained from the dual vierbein as \( g_{\mu\nu}(x) = \eta_{ij} e_i^\mu(x) e_j^\nu(x) \). Furthermore, the Weitzenböck connection is used, whose non-null torsion is
\[ T^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\nu\mu} - \tilde{\Gamma}^\lambda_{\mu\nu} = e_i^\lambda (\partial_\mu e_i^\nu - \partial_\nu e_i^\mu). \]

rather than the Levi-Civita connection which is used in General Relativity.

This tensor encompasses all the information about the gravitational field. The Lagrangian is built by the torsion \[ T^\lambda_{\mu\nu} \] and its dynamical equations for the vierbein imply the Einstein equations for the metric. The teleparallel Lagrangian is
\[ T = S_{\rho}^{\mu\nu} T^\rho_{\mu\nu}, \]
where
\[ S_{\rho}^{\mu\nu} = \frac{1}{2} (K^\mu_{\rho} + \delta^\mu_{\rho} T^\theta_{\mu\theta} - \delta^\nu_{\rho} T^\theta_{\mu\theta}) \]
and \( K^\mu_{\rho} \) is the contorsion tensor
\[ K^\mu_{\rho} = -\frac{1}{2} (T^\mu_{\rho} - T^\nu_{\rho} - T^\rho_{\mu\nu}), \]
which equals the difference between Weitzenböck and Levi-Civita connections. Thus the general action assume the following form
\[ S = \int d^4 x f(T), \]
where $e = \det(e^i_\mu) = \sqrt{-g}$. If matter couples to the metric in the standard form then the variation of the action with respect to the vierbein leads to the equations

\[ e^{-1} \partial_\mu (e S_i^{\mu \nu}) f'(T) - e^\lambda T^\mu_{\lambda \mu} S_\rho^{\nu \mu} f'(T) + S_i^{\mu \nu} \partial_\mu (T) f''(T) + \frac{1}{4} e^i_\mu f(T) = e^\rho T^\rho_{\nu}, \]

(113)

where a prime denotes differentiation with respect to $T$, $S_i^{\mu \nu} = e^i_\rho S_\rho^{\mu \nu}$ and $T_{\mu \nu}$ is the matter energy-momentum tensor. In order to derive the cosmological equations in a FRW metric, we need to infer, as above, a point-like Lagrangian from the action(112). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. Then we can write

\[ e^i_\mu = \text{diag}(1, a(t), a(t), a(t)), \]

(114)

where $a(t)$ is the cosmological scale factor, and the action is

\[ S = \int \mathcal{L}(a, \dot{a}, T, \dot{T}) dt \]

(115)

considering $a$ and $T$ as canonical variables, whereas $Q = \{a, T\}$ is the configuration space, and $TQ = \{a, \dot{a}, T, \dot{T}\}$ is the related tangent bundle on which $\mathcal{L}$ is defined. As above, one can use the method of Lagrange multipliers to set $T$ as a constraint of the dynamics. Selecting the suitable Lagrange multiplier and integrating by parts, the Lagrangian $\mathcal{L}$ becomes canonical and then we have

\[ S = 2 \pi^2 \int dt \ a^3 \left[ f(T) - \lambda \left( T + 6 \frac{\dot{a}^2}{a^2} \right) - \frac{\rho_{m0}}{a^3} \right], \]

(116)

where $\lambda$ is a Lagrange multiplier. The variation with respect to $T$, gives

\[ \lambda = f'(T). \]

(117)

Therefore, the action (116) can be rewritten as

\[ S = 2 \pi^2 \int dt \ a^3 \left[ f(T) - f'(T) \left( T + 6 \frac{\dot{a}^2}{a^2} \right) - \frac{\rho_{m0}}{a^3} \right], \]

(118)

and then the point-like Lagrangian reads

\[ \mathcal{L}(a, \dot{a}, T, \dot{T}) = a^3 \left[ f(T) - f'(T)T \right] - 6 f'(T) a \dot{a}^2 - \rho_{m0}. \]

(119)

Substituting Eq. (119) into the Euler-Lagrange equation, we obtain

\[ a^3 f''(T) \left( T + 6 \frac{\dot{a}^2}{a^2} \right) = 0, \]

(120)

\[ f(T) - f'(T)T + 2 f_T H^2 + 4 \left[ f'(T) \frac{\ddot{a}}{a} + H f''(T) \dot{T} \right] = 0. \]

(121)
If $f''(T) \neq 0$, from Eq. (120), it is easy to find that

$$T = -6 \left( \frac{\dot{a}}{a} \right)^2 = -6H^2. \quad (122)$$

This is the Euler constraint for dynamics. Substituting Eq. (122) into Eq. (121) and using $\ddot{a}/a = H^2 + \dot{H}$, we get

$$48H^2 f''(T)\dot{H} - 4f'(T) \left( 3H^2 + \dot{H} \right) - f(T) = 0. \quad (123)$$

By a Legendre transformation on Lagrangian (119), we obtain the Hamiltonian constraint

$$H(a, \dot{a}, T, \dot{T}) = a^3 \left[ -6f'(T) \frac{\dot{a}^2}{a^2} - f(T)T + \frac{\rho m_0}{a^3} \right]. \quad (124)$$

Considering the total energy $H = 0$ \[32,23,31\] and using Eq. (122), we get

$$12H^2 f'(T) + f(T) = \frac{\rho m_0}{a^3}, \quad (125)$$

In summary, the point-like Lagrangian (119) yields all the equations of motion \[33\]. To obtain the Noether symmetries, we substitute Eq. (119) into Eq. (15) imposing $L_{X\mathcal{L}} = 0$ and using the relations $\dot{a} = (\partial\alpha/\partial a) \dot{a} + (\partial\alpha/\partial T) \dot{T}$, $\beta = (\partial\beta/\partial a) \dot{a} + (\partial\beta/\partial T) \dot{T}$, we obtain

$$3\alpha a^2 \left[ f(T) - f'(T)T \right] - \beta a^3 f''(T)T +$$

$$-6\dot{a}^2 \left[ \alpha f'(T) + \beta af''(T) + 2af'(T) \frac{\partial\alpha}{\partial a} \right] - 12a\dot{a}\dot{T} \frac{\partial\alpha}{\partial T} = 0. \quad (126)$$

As mentioned above, requiring the coefficients of $\dot{a}^2$, $\dot{T}^2$ and $\dot{a}\dot{T}$ in Eq. (126) to be zero, we find that

$$\frac{\partial\alpha}{\partial T} = 0, \quad (127)$$

$$\alpha f'(T) + \beta af''(T) + 2af'(T) \frac{\partial\alpha}{\partial a} = 0, \quad (128)$$

$$3\alpha a^2 \left( f(T) - f'(T)T \right) - \beta a^3 f''(T)T = 0. \quad (129)$$

In particular, the constraint (129) is the \textit{Noether condition} \[23,33\]. The corresponding constant of motion (\textit{Noether charge}), reads

$$Q_0 = -12\alpha f'(T)a\dot{a} = \text{const.} \quad (130)$$

A solution of Eqs. (127), (128) and (129) exists if explicit forms of $\alpha$ and $\beta$ are found. In this case, as above, a symmetry exists. Obviously, from Eq. (127), it is easy to see that $\alpha$ is independent of $T$, and hence it is a function of $a$ only, \textit{i.e.}, $\alpha = \alpha(a)$. On the other hand, from Eq. (129), we have

$$\beta af''(T)T = 3\alpha \left( f(T) - f'(T)T \right). \quad (131)$$
Multiplying by $T$ Eq. \((128)\), and then substituting Eq. \((131)\) into it, we obtain 

$$f'(T)T \left(2a \frac{d\alpha}{da} - 2\alpha\right) + 3\alpha f(T) = 0. \tag{132}$$

One can perform a separation of variables and recast Eq. \((132)\) as 

$$1 - a \frac{d\alpha}{\alpha da} = \frac{3}{2f'(T)T}. \tag{133}$$

Since its left-hand side is a function of $a$ only and its right-hand side is a function of $T$ only, they must be equal to a constant in order to ensure that Eq. \((133)\) holds. For convenience, we let this constant be $3/(2n)$, and then Eq. \((133)\) can be separated into two ordinary differential equations, i.e., 

$$nf(T) = f'(T)T, \quad 1 - a \frac{d\alpha}{\alpha da} = \frac{3}{2n}. \tag{134}$$

It is easy to find the solutions of these two ordinary differential equations, namely 

$$f(T) = \mu_0 T^n, \quad \alpha(a) = \alpha_0 a^{1-3/(2n)}, \tag{135}$$

where $\mu_0$ and $\alpha_0$ are integral constants. Obviously, $f(T)$ and $\alpha(a)$ are both power-law functions. Substituting Eqs. \((134)\) and \((135)\) into Eq. \((131)\), we find that 

$$\beta(a, T) = -\frac{3\alpha_0}{n} a^{-3/(2n)} T. \tag{136}$$

Therefore a Noether symmetry exists. Finally, we find out the exact solution $a(t)$ for this family of $f(T)$. Substituting Eqs. \((134)\), \((135)\) and \((122)\) into Eq. \((130)\), we obtain an ordinary differential equation for $a(t)$, namely 

$$a^c_1 \dot{a} = c_2, \tag{137}$$

where 

$$c_1 = \frac{3}{2n} - 1, \quad c_2 = \left[\frac{Q_0}{-12\alpha_0 \mu_0 n (-6)^{n-1}}\right]^{1/(2n-1)}. \tag{138}$$

It is easy to find that the general solution for Eq. \((137)\) is 

$$a(t) = -(1 + c_1)(c_3 - c_2 t)^{1/(1+c_1)} = (-1)^{1+2n/3} \cdot \frac{3}{2n} (c_2 t - c_3)^{2n/3}, \tag{139}$$

where $c_3$ is another integration constant. Obviously, in the late time $|c_2 t| \gg |c_3|$ and the universe experiences a power-law expansion. Requiring $a(t = 0) = 0$, it is easy to see that the integration constant $c_3$ is zero. Finally, we have a behavior of the form 

$$a(t) \sim t^{2n/3}, \tag{140}$$

which is clearly a Friedman behavior. Note that the condition $n > 0$ is required to ensure that the universe is expanding.
5. Conclusions

In this lecture notes, we have discussed the Noether Symmetry Approach for Minisuperspace Quantum Cosmology. The method allows to identify conserved quantities that select peaked behaviors in the solutions of the Wheeler-De Witt equation. Peaked behaviors mean correlations among variables and then the possibility to obtain classical universes according to the interpretative Hartle criterion. Specifically, such a criterion states that classical observable universes are solutions of dynamics as soon as correlations among physical variables are identified. Here, we search for Noether symmetries that allow to reduce dynamics coming from minisuperspaces and then find out exact solutions.

The method has been worked out for several examples of Extended Theories of Gravity, namely scalar-tensor gravity, $f(R)$-gravity and $f(T)$-gravity. The common feature of such dynamical systems is that, in any case, specific Lagrange multipliers, related to symmetries, can be found out. Such multipliers allow to reduce dynamics and then exact cosmological solutions can be easily found.

Here we have worked out some very didactic examples but more physically interesting systems can be taken into account (see for example [23,24,25,26]).

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