LAGRANGIAN NON-SQUEEZING AND A GEOMETRIC INEQUALITY

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Abstract. We prove that if the unit codisc bundle of a closed Riemannian manifold embeds symplectically into a symplectic cylinder of radius one then the length of the shortest non-trivial closed geodesic is at most half the area of the unit disc.

1. Introduction

Consider a metric $g$ on the unit circle $S^1 = \partial D$. After a reparametrization by arc length $g = r^2 g_0$ becomes a positive multiple of the standard metric on $S^1$. If the unit codisc bundle of $(S^1, g)$ embeds into the unit disc $D$ such that the area and the orientation are preserved then

$$2 \text{length}_g(S^1) \leq \pi.$$ 

This is equivalent to $r \leq \frac{1}{4}$. In other words

$$\inf(g) \leq \frac{\pi}{2},$$

where $\inf(g)$ denotes the length of the shortest non-trivial closed geodesic. In this article we prove a symplectic generalization of this inequality. For any closed Riemannian manifold $(L, g)$ such that the unit codisc bundle $D^*(g)L$ has a symplectic embedding into the cylinder $Z = D \times \mathbb{R}^{2n-2}$ which is provided with the standard split symplectic form $\inf(g) \leq \frac{\pi}{2}$ holds true. In fact, if the dimension is $\geq 4$ we only require that the symplectic embedding exists in a neighbourhood of the unit cosphere bundle $S^*(g)L$, see Corollary 3.3.

The proof of this result is based on the link capacity $\ell$ that we introduce in Theorem 3.1. For a given subset $U$ the capacity $\ell$ measures the largest minimal total action of null-homologous Reeb links on a contact type hypersurface in $U$ that is diffeomorphic to a unit cotangent bundle. This is a variant of the capacity introduced in [10, 9], cf. [19] and fits into a larger class of so called embedding capacities, see [4].

The first embedding capacity appeared in the unpublished work [5], cf. [4]. In [5] Cieliebak and Mohnke defined a Lagrangian embedding capacity for 2-connected symplectic manifolds $(V, \omega)$

$$c_L(V, \omega) := \sup \{ \inf(L) \mid L \subset (V, \omega) \},$$

where $\inf(L)$ denotes the least positive symplectic area of a smooth disc in $V$ with boundary on $L$. Here the supremum runs over all Lagrangian tori in $(V, \omega)$. The

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values on the unit symplectic cylinder $Z$, unit ball $B$, and unit polydisc $P$ are

\[ c_L(Z) = \pi, \quad c_L(B) = \frac{\pi}{n}, \quad \text{and} \quad c_L(P) = \pi. \]

This capacity gives an alternative proof of the Ekeland-Hofer non-squeezing theorem \[7, \text{Corollary 3}\]. It states that if the polydisc

\[ P(r_1, \ldots, r_n) := D_{r_1} \times \ldots \times D_{r_n} \]

with radii $0 < r_1 \leq \ldots \leq r_n$ embeds symplectically into the ball $B_R$ of radius $R$ then $\sqrt{n} r_1 \leq R$. Generalizing to arbitrary Lagrangian submanifolds Swoboda and Ziltener \[15, 16\] obtained a symplectic capacity $a_L \geq c_L$ for 2-connected symplectic manifolds $(V, \omega)$ via

\[ a_L(V, \omega) := \sup \left\{ \inf(L) \mid L \subset (V, \omega) \text{ closed Lagrangian submanifold} \right\} \]

such that

\[ a_L(Z) = \pi, \quad a_L(B) \geq \frac{\pi}{2}. \]

The precise value on the unit ball is not known. In fact Swoboda and Ziltener assign a capacity to a large class of coisotropic submanifolds for every possible codimension and prove non-squeezing results for so-called small sets, see \[15, 16\]. The link capacity (see Theorem 3.1) can be seen as another example in this direction.

In Corollary 2.3 and 2.6 we give further non-squeezing results for Lagrangian submanifolds. For that we use Chekanov’s elementary tori, see \[2\], Damian’s proof of the Audin conjecture in the monotone case, see \[6\], and Lagrangian embedding capacities that we introduce in Section 2.

2. Measuring the area

We are interested in special capacities $a$ on the standard symplectic vector space $\mathbb{R}^{2n}$, which are (1) monotone on subsets of $\mathbb{R}^{2n}$, i.e. $a(U_1) \leq a(U_2)$ provided $U_1 \subset U_2$, (2) invariant under global symplectomorphisms of $\mathbb{R}^{2n}$, (3) conformal in the sense that $a(rU) = r^2 a(U)$ for all $U \subset \mathbb{R}^{2n}$ and $r \in \mathbb{R}$, and (4) satisfy

\[ a(Z) < \infty, \quad a(B) > 0, \]

see \[12, \text{p. 172}\]. The aim is to measure the minimal symplectic area $\inf(L)$ of closed Lagrangian submanifolds $L$ in $\mathbb{R}^{2n}$ among all smooth discs attached to $L$. In other words we consider the Liouville class $\lambda_L = [\lambda]_{TL}$ for any primitive $\lambda$ of $dx \wedge dy$. The image of $H_1(L, \mathbb{Z})$ under $\lambda_L$ generates a subgroup $\Lambda_L$ of $\mathbb{R}$. If this group is discrete we call $L$ rational and $\inf(L)$ is the positive generator of $\Lambda_L$; otherwise $\inf(L)$ is zero, see \[13\].

For our first version of a special capacity we consider for real numbers

\[ 0 < r_1 \leq \ldots \leq r_n \]

the elementary Lagrangian tori

\[ T(r_1, \ldots, r_n) := \partial D_{r_1} \times \ldots \times \partial D_{r_n} \]

in $\mathbb{R}^{2n}$. We call two closed Lagrangian submanifolds of $\mathbb{R}^{2n}$ symplectomorphic if there exists a global symplectomorphism of $\mathbb{R}^{2n}$ which maps one to the other. It follows from \[2, \text{Theorem A}\] that the first radius

\[ r_1 = r_1(L) \]
of a Lagrangian torus \( L \) symplectomorphic to \( T(r_1, \ldots, r_n) \) is an invariant under global symplectomorphisms.

**Theorem 2.1.** For subsets \( U \) in \( \mathbb{R}^{2n} \) the quantity
\[
a_c(U) := \sup \{ \pi \left( r_1(L) \right)^2 | L \subset U \},
\]
where the supremum is taken over all Lagrangian tori \( L \) symplectomorphic to an elementary torus, defines a special capacity in \( \mathbb{R}^{2n} \) such that
\[
a_c(Z) = \pi, \quad a_c(B) = \frac{\pi}{n}.
\]

**Proof.** We only have to verify the normalization axiom. For the lower bounds consider the tori \( T_1 \) and \( T_1/\sqrt{n} \) that have minimal symplectic action \( \pi \) and \( \pi/n \), resp. To obtain upper bounds consider a Lagrangian torus \( L \) in \( \mathbb{R}^{2n} \) which is symplectomorphic to the elementary torus \( T(r_1, \ldots, r_n) \). For \( r_1 = r_1(L) \) the values of the first and \( n \)-th Ekeland-Hofer capacity of \( L \) are
\[
c_{EH}^1(L) = \pi r_1^2, \quad c_{EH}^n(L) = n \pi r_1^2,
\]
see [2, Theorem 2.1]. The claim follows now from
\[
c_{EH}^1(Z) = \pi = c_{EH}^n(B)
\]
and the monotonicity property of the Ekeland-Hofer capacities, see [7]. \( \square \)

**Remark 2.2.** Because \( c_{EH}^1 \) takes the value \( \pi r_1^2 \) on the polydisc \( P(r_1, \ldots, r_n) \) the proof shows that \( a_c(P(r_1, \ldots, r_n)) = \pi r_1^2 \).

A direct consequence of the theorem is that the torus \( T(r_1, \ldots, r_n) \) admits a global symplectic embedding into the symplectic cylinder \( Z_R \) of radius \( R \) if and only if \( r_1 \leq R \). This non-squeezing result follows alternatively from the stronger [3, Main Theorem], which gives an upper bound on the area of a non-constant holomorphic disc (for example for the standard complex structure) attached to \( T(r_1, \ldots, r_n) \) by its displacement energy. The rational case was observed by Sikorav in [14]. Note that Sikorav's theorem implies the general case by approximating irrational radii by rational numbers.

**Corollary 2.3.** If the torus \( T(r_1, \ldots, r_n) \) admits a global symplectic embedding into the ball \( B_R \) of radius \( R \) then \( \sqrt{n} r_1 \leq R \).

**Remark 2.4.** This follows alternatively with the Cieliebak-Mohnke capacity, see [5], via an approximation by rational Lagrangian tori.

A second special capacity on \( \mathbb{R}^{2n} \) can be constructed as follows: Consider closed connected monotone Lagrangian submanifolds \( L \subset \mathbb{R}^{2n} \) which admit a metric of non-positive sectional curvature (and are therefore aspherical by the Hadamard-Cartan Theorem). Notice that \( L \) is allowed to be non-orientable so that for example in dimension 4 the curvature condition is not a restriction, see [11, 0.4.A2].

**Theorem 2.5.** For subsets \( U \) in \( \mathbb{R}^{2n} \) the quantity
\[
a_m(U) := \sup \{ \inf(L) | L \subset U \},
\]
where the supremum is taken over all closed connected monotone Lagrangian submanifolds \( L \subset \mathbb{R}^{2n} \) which admit a metric of non-positive sectional curvature, defines a special capacity in \( \mathbb{R}^{2n} \) such that
\[
a_m(Z) = \pi, \quad a_m(B) = \frac{\pi}{n}.
\]
Proof. We only have to show that $a_m(Z) = \pi$ and $a_m(B) = \pi/n$. The tori $T_1$ and $T_{1/\sqrt{n}}$ yield lower bounds. Uniform upper bounds are obtained as follows: Consider a Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ as above. Because $L$ is monotone the Liouville class $\lambda_L$ and the Maslov class $\mu_L$ are related by
\[
\lambda_L = \eta \mu_L
\]
for some $\eta > 0$. By Damian’s proof of the Audin conjecture we find a closed curve $\gamma$ on $L$ such that $\mu_L(\gamma) \leq 2$ (equality if and only if $L$ is orientable), see [6, Theorem 1.5.(a)]. This gives
\[
\inf(L) \leq \lambda_L(\gamma) \leq 2 \eta.
\]
Moreover, by Bates [1, Theorem 3], the $k$-th Ekeland-Hofer capacity satisfies
\[
2k\eta \leq c_{EH}^k(L),
\]
where the curvature condition is used, so that
\[
\inf(L) \leq \frac{c_{EH}^k(L)}{k}.
\]
The claim follows now from the properties of the first and $n$-th Ekeland-Hofer capacity. $\square$

Corollary 2.6. Let $L \subset B_R$ be a closed connected Lagrangian submanifold. Then
\[
\inf(L) \leq \frac{\pi}{n} R^2
\]
provided $L$ is monotone and admits a metric of non-positive sectional curvature.

Remark 2.7. The case without the monotonicity assumption follows from the Cieliebak-Mohnke capacity [5].

3. Measuring the length

For irrational Lagrangian submanifolds the symplectic area can be arbitrary small, thus do not give a sensible invariant. But the length of closed unit speed geodesics on $L$ for certain Riemannian metrics is an alternative way to measure the size of Lagrangian submanifolds $L$ symplectically. [10, 9] construct the so-called orbit capacity
\[
c(V, \omega) = \sup \{ \inf_\beta(\alpha) \mid \exists \text{ contact type embedding } (M, \alpha) \hookrightarrow (V, \omega) \}
\]
for all symplectic manifolds $(V, \omega)$ with dimension $\geq 4$. Here $\inf_\beta(\alpha)$ is the infimum of the total action of null-homologous Reeb links on the closed contact manifold $(M, \alpha)$. As the arguments in [9] show closed Reeb orbits that constitute a null-homologous one-component Reeb link can be assumed to be contractible. The supremum is taken over all embeddings $j : M \rightarrow V$ such that near $j(M)$ there is a Liouville vector field $Y$ for $\omega$ satisfying $\alpha = j^*(i_Y \omega)$. Notice that this is equivalent to $d\alpha = j^*\omega$ for the contact form $\alpha$, see [12, p. 119].

If one restricts in the definition of the orbit capacity $c$ to manifolds $M$ that are diffeomorphic to the unit cosphere bundle $S^*Q$ of closed Riemannian manifolds $Q$ one obtains the link capacity $\ell$.

Theorem 3.1. For symplectic manifolds $(V, \omega)$ with dimension $\geq 4$ the quantity $\ell(V, \omega)$ is an intrinsic capacity such that
\[
\ell(Z) = \pi, \quad \ell(B) \geq \frac{\pi}{n}.
\]
Proof. Because of $\ell \leq c$ we only need to compute the values on the ball and the cylinder. Identify the cotangent bundle of the unit circle $S^1 = \partial D$ with $(\mathbb{R} \times S^1, sdt)$. Consider polar coordinates on $\mathbb{C}$ such that the radial Liouville primitive of the standard symplectic form equals $\frac{1}{2}r^2 d\theta$. For $a > 0$ we define a symplectic embedding

$$\varphi_a(s, t) = \sqrt{a + 2s} e^{it}$$

of $\{s > -\frac{a}{2}\}$ into $\mathbb{C}$. The image of the $b$-codisc bundle

$$D_b^* S^1 = (-b, b) \times S^1,$$

$b \in (0, \frac{a}{2})$, is the annulus

$$A(a, b) = A_{\sqrt{a^2 - 2b}, \sqrt{a - 2b}} = D_{\sqrt{a^2 - 2b}} \setminus D_{\sqrt{a - 2b}}.$$  

For real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ with $0 < b_j < \frac{a_j}{2}$ for $j = 1, \ldots, n$ the embedding

$$\varphi_{a_1} \times \ldots \times \varphi_{a_n}$$

maps $D_{b_1}^* S^1 \times \ldots \times D_{b_n}^* S^1$ onto the polyannulus $A(a_1, b_1) \times \ldots \times A(a_n, b_n)$ symplectically.

In order to compute the quantity $\ell$ we consider the flat torus $T^n = \mathbb{R}^n/2\pi \mathbb{Z}^n$. The $b$-cosphere bundle $S_b^k T^n$ is provided with the canonical contact form. Each closed geodesic induces a null-homologous Reeb link with two components corresponding to the opposing orientations of the geodesic. Hence, the smallest total action $\inf \ell$ equals $4\pi b$, see [3 Section 1.5]. Because the $b$-codisc bundle $D_b^* T^n$ is contained in $(D_b^* S^1)^n$ the images of the $b'$-cosphere bundles under $(\varphi_a)^n$ for $b' < b$ are hypersurfaces of contact type. Taking the limits $a \downarrow \frac{1}{2}$ and $b \uparrow \frac{1}{4}$, resp., $a \downarrow \frac{1}{2n}$ and $b \uparrow \frac{1}{4n}$ proves the claim. □

Remark 3.2. For $\varepsilon > 0$ sufficiently small the disc of radius $2\sqrt{b} - \varepsilon$ embeds into the square $(-b, b) \times (0, 2\pi)$ preserving the orientation and the area, cf. [12 p. 171]. Composing this with $\varphi_{a_1} \times \ldots \times \varphi_{a_n}$ appropriately yields an symplectic embedding of the polydisc

$$P(2\sqrt{b_1} - \varepsilon, \ldots, 2\sqrt{b_n} - \varepsilon)$$

into the polyannulus

$$A(a_1, b_1) \times \ldots \times A(a_n, b_n) \subset P(\sqrt{a_1^2 + b_1}, \ldots, \sqrt{a_n + b_n}).$$

Therefore, one shows that $\ell(P(r_1, \ldots, r_n)) = \pi r_1^2$ as in the proof above. Moreover, if we consider the metric on $T^n$ induced from $\mathbb{R}^n$ we see together with the remark after [3] Theorem 4.5 that

$$\ell(D_b^* T^n) = 4\pi b_1 = \ell(D_{b_1}^* S^1 \times \ldots \times D_{b_n}^* S^1),$$

where we assume that $0 < b_1 \leq \ldots \leq b_n$.

Consider a closed monotone Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ that admits a metric $g$ of non-positive sectional curvature. By Weinstein’s neighbourhood theorem [14] there exists $r > 0$ such that the $r$-codisc bundle of $L$ embeds symplectically. We denote its image by $U_r \subset \mathbb{R}^{2n}$. Then [1] Theorem 2.1 implies that for all $k \in \mathbb{N}$

$$\inf(L) + r \inf(g) \leq c_k^{EH}(U_r),$$

where $\inf(g)$ denotes the length of the shortest non-trivial closed geodesic of $(L, g)$. In particular, if $U_r \subset Z_R$ then $r \inf(g) \leq \pi R^2$. This observation generalizes to the following non-squeezing result.
Corollary 3.3. Let \((Q, g)\) be a closed Riemannian manifold such that all closed geodesics are not contractible. If a neighbourhood of the \(r\)-cosphere bundle in \(T^*Q\) embeds into \(Z_R\) symplectically then

\[ 2r \inf(g) \leq \pi R^2. \]

Proof. The claim is an application of the capacity \(\ell\). Notice that \(2r \inf(g) \leq \inf(\alpha_r)\) if computed with respect to the restriction \(\alpha_r\) of the canonical Liouville form to \(T^*_r(g)Q\). \(\Box\)

For a continuative discussion the reader is referred to [18, Section 3.6].

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