The regularizing properties of multistep methods for first kind Volterra integral equations with smooth kernels

Robert Plato*

May 2, 2016

Abstract

We study quadrature methods for solving Volterra integral equations of the first kind with smooth kernels under the presence of noise in the right-hand sides, with the quadrature methods being generated by linear multistep methods. The regularizing properties of an a priori choice of the step size are analyzed, with the smoothness of the involved functions carefully taken into consideration. The balancing principle as an adaptive choice of the step size is also studied. It is considered in a version which sometimes requires less amount of computational work than the standard version of this principle. Numerical results are included.

1 Introduction

In this paper we consider linear Volterra integral equations of the following form,

\[(Au)(x) = \int_a^x k(x, y)u(y) \, dy = f(x) \quad \text{for } a \leq x \leq b,\]

with a sufficiently smooth kernel function \(k : \{ (x, y) \in \mathbb{R}^2 \mid a \leq y \leq x \leq b \} \rightarrow \mathbb{R}\). Moreover, the function \(f : [a, b] \rightarrow \mathbb{R}\) is supposed to be approximately given, and a function \(u : [a, b] \rightarrow \mathbb{R}\) satisfying equation (1) needs to be determined.

In the sequel we suppose that the kernel function does not vanish on the diagonal \(a \leq x = y \leq b\), and without loss of generality we may assume that

\(k(x, x) = 1 \quad \text{for } a \leq x \leq b\)

holds.

Composite quadrature methods for the approximate solution of equation (1) are well-investigated if the right-hand side \(f\) is exactly given, see e.g., Brunner/van der Houwen [3], Brunner [2], Lamm [14], Linz [16] or Hoog/Weiss [4] and the references therein. A special class of composite quadrature methods for the approximate solution of (1) is obtained by using in an appropriate manner multistep methods that usually are used to solve initial value problems for first order ordinary differential equations. That class of methods is considered thoroughly in Wolkenfelt ([25], [26]). A related survey is given in Brunner/van der Houwen [3], and see Holyhead/McKee/Taylor [11], Holyhead/McKee [10] and Taylor [24] for related results. In the present paper, the results and techniques presented in the two papers by Wolkenfelt are modified and extended in order to analyze the regularizing properties of those multistep methods for Volterra integral equations (1) when perturbed right-hand sides are available only. An a priori choice of the step size is considered, followed by the balancing principle as an adaptive choice of the step size. Finally, some numerical illustrations are presented.

*Department of Mathematics, University of Siegen, Walter-Flex-Str. 3, 57068 Siegen, Germany.
2 Numerical integration based on multistep methods

In this section, as a preparation for the numerical solution of Volterra integral equations of the first kind, we introduce linear multistep methods for solving the associated direct problem. For this purpose we consider equidistant nodes

\[ x_s = a + sh, \quad s = 0, 1, \ldots, N, \quad \text{with} \quad h = \frac{b-a}{N}, \]

where \( N \) denotes a positive integer. In a first step we consider – for each \( 1 \leq n \leq N \) – the integral

\[ (I\psi)(x_n) := \int_a^{x_n} \psi(y) \, dy, \]

where \( \psi : [a, x_n] \to \mathbb{R} \) is a given continuous function which may depend on \( n \). In the course of this paper, this integral will be considered for the special case \( \psi(y) = k(x_n, y)u(y), a \leq x \leq x_n \); see Section 2 for the details.

The integral (3) can be computed by solving the elementary ordinary differential equation

\[ \varphi'(y) = \psi(y) \quad \text{for} \quad a \leq y \leq x_n, \quad \varphi(a) = 0, \]

and then obviously \( (I\psi)(x_n) = \varphi(x_n) \). Next we briefly introduce some basic facts about linear multistep methods to solve initial value problems for ordinary differential equations, with a notation that is adapted to the simple situation considered in (4). For a thorough presentation of multistep methods (to solve initial value problems for ordinary differential equations in its general form), see e.g., [21], Hairer / Nørsett / Wanner [7], Henrici [8], or Iserles [12].

2.1 Introduction of multistep methods

A linear \( m \)-step method, with an integer \( m \geq 1 \), is determined by coefficients \( a_j \in \mathbb{R} \) and \( b_j \in \mathbb{R} \) for \( j = 0, 1, \ldots, m \), where \( a_m \neq 0 \) and and \( |a_0| + |b_0| \neq 0 \). When applied to problem (4), this scheme is of the form

\[ \sum_{j=0}^{m} a_j \varphi_{r+j} = h \sum_{j=0}^{m} b_j \psi_{r+j} \quad \text{for} \quad r = 0, 1, \ldots, n-m, \]

where \( n \geq m \), and \( \psi_s = \psi(x_s), s = 0, 1, \ldots, n \) are given, and the step size \( h \) and the nodes \( x_s \) are given by (2). In addition we have \( \varphi_0 = 0 \), and the other starting values \( \varphi_s \approx \varphi(x_s) \) for \( s = 1, 2, \ldots, m-1 \) are determined by some procedure specified below (see Example 2.7). The scheme (5) is used to compute approximations \( \varphi_{r+m} \approx \varphi(x_{r+m}) \) for \( r = 0, 1, \ldots, n-m \).

Example 2.1 (a) We first consider a well-known class of multistep methods of the form (5), depending on three integers \( \tau, \mu \) and \( m \), with \( 1 \leq \tau \leq m \) and \( 0 \leq \mu \leq m \). It is obtained by integrating, for each \( 0 \leq r \leq n-m \), the ordinary differential equation (4) from \( x_{r+m-\tau} \) to \( x_{r+m} \). For the integral of the resulting right-hand side, an interpolatory numerical integration scheme with interpolation nodes \( x_r, x_{r+1}, \ldots, x_{r+m-\mu} \) is applied afterwards. This leads to

\[ \varphi_{r+m} - \varphi_{r+m-\tau} = \int_{x_{r+m-\tau}}^{x_{r+m}} \mathcal{P}_r(y) \, dy, \quad r = 0, 1, \ldots, n-m, \]

where \( \mathcal{P}_r \in \Pi_{m-\mu} \) satisfies \( \mathcal{P}_r(x_s) = \psi_s \) for \( s = r, r+1, \ldots, r+m-\mu \). This means that \( \tau h \) is the length of the interval used for the local integration, and \( m-\mu+1 \) is the number of nodes used for the interpolation of the function \( \psi \). Prominent examples are obtained for \( \mu \in \{0, 1\} \) and \( \tau \in \{1, 2\} \). Next some special cases are considered very briefly. For more details see, e.g., the references given just before the present subsection.
The Adams–Bashfort methods are obtained for \( \tau = 1, \mu = 1 \) and \( m \geq 1 \); for the special case \( m = 1 \) this in fact gives the composite forward rectangular rule. The Adams–Moulton methods are obtained for \( \tau = 1, \mu = 0 \) and \( m \geq 1 \), with the composite trapezoidal rule obtained for the special case \( m = 1 \). The Nyström methods are given by \( \tau = 2, \mu = 1 \) and \( m \geq 2 \). For \( m = 2 \) this gives the repeated midpoint rule. Finally, the Milne–Simpson methods are obtained by \( \tau = 2, \mu = 0 \) and \( m \geq 2 \), with the repeated Simpson’s rule obtained in the case \( m = 2 \). Each of these methods is in fact of the form (5) and leads to a repeated quadrature method for solving (3), with interpolation polynomials \( P \) that, for \( m > \tau \), have nodes outside the local integration interval \([x_{r+m-\tau}, x_{r+m}]\).

(b) Another class of linear multistep methods of the form (5) are BDF methods (backward differentiation formulas), where the left-hand side in (4) is replaced by a finite difference scheme. More precisely, for \( m \) fixed, approximations \( \varphi_{r+m} \approx \varphi(x_{r+m}) \) for \( r = 0, 1, \ldots, n - m \) are given by \( \varphi_{r+m} = P(x_{r+m}) \), where \( P \in \Pi_m \) satisfies \( P(x_s) = \varphi_s \) for \( s = r, r + 1, \ldots, r + m - 1 \) and \( P'(x_{r+m}) = \psi_{r+m} \). For \( m = 1 \) this leads to the composite backward rectangular rule.

### 2.2 Null stability, order of the method

We next recall some basic notation for multistep methods applied to the simple initial value problem (4).

(a) The considered multistep method is called nullstable, if the corresponding first characteristic polynomial

\[
g(\xi) = a_m \xi^m + a_{m-1} \xi^{m-1} + \cdots + a_0
\]

is a simple von Neumann polynomial, i.e.,

(i) \( g(\xi) = 0 \) implies \( |\xi| \leq 1 \),

(ii) \( g(\xi) = 0 \), \( |\xi| = 1 \) implies \( g'(\xi) \neq 0 \).

This means that all roots of the characteristic polynomial \( g \) belong to the closed unit disk, and each root on the unit circle is simple.

(b) We next consider the local truncation error of the considered multistep method. For technical reasons it is introduced here on arbitrary intervals \([c, d]\) which in fact can be \([a, x_n]\) as above, or an interval of fixed length. For a continuous function \( \psi : [c, d] \to \mathbb{R} \) the local truncation error is given by

\[
\eta(\psi, y, h) := \sum_{j=0}^{m} a_j \varphi(y + jh) - \sum_{j=0}^{m} b_j \psi(y + jh), \quad c \leq y \leq d - mh, \quad h > 0,
\]

where \( \varphi : [c, d] \to \mathbb{R} \) satisfies \( \varphi'(y) = \psi(y) \) for \( c \leq y \leq d \) and \( \varphi(c) = 0 \). The multistep method (4) is by definition of (consistency) order \( p \) with an integer \( p \geq 1 \), if on a fixed test interval \([c, d]\) and for each sufficiently smooth function \( \psi : [c, d] \to \mathbb{R} \) and each \( c \leq y < d \), the estimate \( \eta(\psi, y, h) = O(h^{p+1}) \) as \( h \to 0 \) holds. A multistep method is by definition of maximal order \( p_0 \geq 1 \) if it is of order \( p = p_0 \) and not of order \( p = p_0 + 1 \).

**Example 2.2** (a) Each multistep method of the special form (6) is clearly nullstable. The order of this multistep method is at least \( p = m - \mu + 1 \). The maximal order \( p_0 \) may be larger in some cases. For example, for \( \tau = 2, \mu = 0 \) and \( m = 2 \) (the Simpson’s rule from the class of Milne–Simpson methods), the maximal order is \( p_0 = 4 \). For those values of \( \tau \) and \( \mu \), the \( m \)-step methods coincide for \( m = 2 \) and \( m = 3 \) in fact.

(b) The BDF methods are nullstable for \( 1 \leq m \leq 6 \), with respective maximal order \( p_0 = m \).

Next we consider the local truncation error (9) on variable intervals \([c, d] = [a, x_n]\) and present uniform estimates. As a preparation we introduce for \( p \geq 0 \) and \( L \geq 0 \) the space \( C_L^p[c, d] \) of functions \( u :
\[ [c, d] \rightarrow \mathbb{R} \] that are \( p \)-times differentiable and in addition satisfy \( |u^{(p)}(y_1) - u^{(p)}(y_2)| \leq L|y_1 - y_2| \) for \( y_1, y_2 \in [c, d] \). Occasionally we also use the notation
\[
\tilde{C}^p[c, d] = \bigcup_{L>0} C^p_L[c, d].
\]

**Lemma 2.3** Consider a linear multistep method \( 5 \) of maximal order \( p_0 \geq 1 \) for solving the initial value problem \( 4 \). Then for each Lipschitz constant \( L > 0 \) and each \( 1 \leq p \leq p_0 \), the following estimate for the local truncation error holds:
\[
\eta(\psi, y, h) = O(h^{p+1}) \quad \text{as } h = \frac{b-a}{N} \to 0,
\]
uniformly for \( n, \psi \) and \( y \) satisfying \( m \leq n \leq N, \psi \in C^{p-1}_L[a, x_n], \) and \( a \leq y \leq x_n - mh \).

**Proof.** A Taylor expansion of a function \( f \in C^p_c[c, d] \) on an interval \([c, d]\) gives, for \( c \leq y \) and \( y + h < d \), the following representation for the remainder: \( \bar{R}(f, p, y, h) := f(y + h) - \sum_{s=0}^{p} f^{(s)}(y) \frac{h^s}{s!} = \frac{f(p)(\xi) - f(p)(y)}{p!} h^p \). This means \( R(f, p, y, h) = O(h^{p+1}) \) as \( h > 0, h \to 0 \), uniformly for arbitrary finite intervals \([c, d]\), for \( f \in C^p_c[c, d] \), and \( c \leq y < d \). After these preparations we now consider the special situation in the lemma. From appropriate Taylor expansions for \( \psi, \varphi \), and making use of the consistency equations corresponding to the multistep method \( 5 \) for solving \( 4 \), we finally arrive at \( \eta(\psi, y, h) = \sum_{j=0}^{m} a_j R_j - h \sum_{j=0}^{m} b_j \tilde{R}_j = O(h^{p+1}) \), uniformly for \( n, \psi \) and \( y \) as given in the statement of the lemma, where \( R_j = R(\psi, p, y, jh) \) and \( \tilde{R}_j = R(\psi, p - 1, y, jh) \). \( \square \)

We note that the considered intervals \([a, x_n]\) in Lemma 2.3 depend on \( h \), and we do not require \( x_n \) to be fixed. This causes no problem in \( 10 \), however, since the estimates of the local truncation error are considered uniformly there.

The basic convergence result in multistep method theory is as follows: each nullstable linear multistep method \( 5 \) of order \( p \geq 1 \) is convergent of order \( p \). Details are given in Section 2.4.

### 2.3 Reflected coefficients / polynomials

As a preparation we introduce some more notation. We assume that at least one of the coefficients on the right-hand side of \( 5 \) does not vanish, and we identify the leading nonvanishing coefficient then: let \( 0 \leq \mu \leq m \) such that
\[
b_{m-\mu+1} = \cdots = b_{m-1} = b_m = 0, \quad b_{m-\mu} \neq 0.
\]

In the sequel we make use of a relation between linear difference equations and discrete convolution equations. As a preparation we consider infinite sequences of reflected coefficients \( (\alpha_j)_{j \geq 0} \) and \( (\beta_j)_{j \geq 0} \) of the multistep method under consideration:
\[
\alpha_j = \begin{cases} a_{m-j}, & j \leq m, \\ 0, & j > m, \end{cases} \quad \beta_j = \begin{cases} b_{m-\mu-j}, & j \leq m-\mu, \\ 0, & j > m + \mu. \end{cases}
\]

In addition we introduce sequences \( \alpha_0^{(-1)}, \alpha_1^{(-1)}, \ldots \) and \( \gamma_0, \gamma_1, \ldots \) by the following discrete convolution equations:
\[
\sum_{s=0}^{r} \alpha_{r-s}^{(-1)} = \delta_{0r}, \quad \sum_{s=0}^{r} \alpha_{r-s} \gamma_s = \beta_r, \quad \text{for } r = 0, 1, \ldots,
\]
where \( \delta_{0r} \) denotes the Kronecker symbol, i.e., we have \( \delta_{0r} = 1 \) and \( \delta_{0r} = 0 \) for each \( r \neq 0 \). There is a relation between those discrete convolutions and the products of the associated (formal) power series: for
\[
\alpha(\xi) = \sum_{n=0}^{m} \alpha_n \xi^n, \quad \beta(\xi) = \sum_{n=0}^{m-\mu} \beta_n \xi^n, \quad \gamma(\xi) = \sum_{n=0}^{\infty} \gamma_n \xi^n,
\]
we have
\[ \frac{1}{\alpha(\xi)} = \sum_{n=0}^{\infty} \alpha_n^{(-1)} \xi^n, \quad \alpha(\xi) \gamma(\xi) = \beta(\xi). \]

In addition, there is a relation between products of (formal) power series considered in (14) on one side and the products of associated semicirculant matrices on the other side. This relation will be tacitly used in the sequel. For an introduction to that topic, see, e.g., Henrici [9].

It follows from (13) and standard results for difference equations (see, e.g., Lemma 5.5 on p. 242 in Henrici [8]) that a nullstable multistep method satisfies
\[ \alpha_n^{(-1)} = O(1), \quad \gamma_n = O(1) \quad \text{as} \quad n \to \infty. \quad (15) \]

In the stability analysis to be considered, the coefficients of the inverse power series $1/\beta(\xi)$ and $1/\gamma(\xi)$ also play a significant role. Their behavior will be considered later.

### 2.4 A global error representation

We next present a global error representation in terms of linear combinations of local truncation errors, as well as the starting errors. This representation will be crucial in the subsequent analysis.

**Lemma 2.4** Consider a nullstable linear multistep method (5) for solving the initial value problem (4). Then we have the error representation
\[ \phi_n = (I\psi)(x_n) - \sum_{s=0}^{n-m} \alpha_n^{(-1)} \eta(\psi, x_s, h) + R, \quad |R| \leq \max_{0 \leq r \leq m-1} \left| \varphi_r - \varphi(x_r) \right|, \quad (16) \]

where $n \geq m$, and $\eta(\psi, x_s, h)$ denotes the local truncation error at the node $x_s$, cf. [9]. The constant $C$ in (16) depends on $m$ and the bounds for $(\alpha_n)_{n \geq 0}$ and $(\alpha_n^{(-1)})_{n \geq 0}$ only.

**Proof.** Let $e_r = \varphi_r - \varphi(x_r)$ for $r = 0, 1, \ldots, n$. We have $\sum_{j=0}^{m} a_j e_{r+j} = g_r$ for $r = 0, 1, \ldots, n-m$, where $g_r := -\eta(\psi, x_r, h)$. A reformulation gives $\sum_{i=0}^{m} \alpha_{r-i} e_{i+m} = g_r - \sum_{i=0}^{m-1} \alpha_{r-i} e_{i+m}$ for $r = 0, 1, \ldots, n-m$, which in matrix formulation reads as follows:

\[
\begin{pmatrix}
\alpha_0 & 0 & \cdots & \cdots & \cdots & 0 \\
\alpha_1 & \alpha_0 & \cdots & \cdots & \cdots & \\
\vdots & \vdots & \ddots & \cdots & \cdots & \\
\alpha_m & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \alpha_m \\
0 & \cdots & \cdots & \cdots & \cdots & \alpha_1 \\
0 & \cdots & \cdots & \cdots & \cdots & \alpha_0
\end{pmatrix}
\begin{pmatrix}
e_m \\
\vdots \\
e_{m+1} \\
\vdots \\
e_n
\end{pmatrix}
= \begin{pmatrix}
g_0 + O\left(\max_{0 \leq r \leq m-1} |e_r|\right) \\
g_{m-1} + O\left(\max_{0 \leq r \leq m-1} |e_r|\right) \\
\vdots \\
g_{m} \\
\vdots \\
g_{n-m}
\end{pmatrix}.
\]

The desired result now follows from the fact that the inverse of the semicirculant system matrix is given by
\[
\begin{pmatrix}
\alpha_0^{(-1)} & 0 & \cdots & 0 \\
\alpha_1^{(-1)} & \alpha_0^{(-1)} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n-m}^{(-1)} & \cdots & \cdots & \alpha_1^{(-1)} \\
\alpha_{n-m}^{(-1)} & \cdots & \cdots & \alpha_0^{(-1)}
\end{pmatrix}.
\]

This completes the proof. □
Remark 2.5 It immediately follows from Lemmas 2.3 and 2.4 as well from (15) that, under the conditions stated in those lemmas, we have
\( \varphi_n = (I\psi)(x_n) + O(h^n) \) for \( n \geq m \), provided that the starting errors are \( O(h^p) \). This result, however, does not allow optimal error estimates for the approximate inversion of Volterra integral equations of the first kind to be considered in this paper, so we make use of (16) instead. We note that in the papers by Wolkenfelt ([25], [26]), a global error expansion with an integral representation is used for the inversion process to obtain best possible error estimates. The latter approach, however, requires stronger smoothness assumptions on the involved functions than our approach based on (16) does. △

2.5 Explicit representation of the values \( \varphi_r \)

For the numerical analysis to be considered later on we need to express the values \( \varphi_m, \varphi_{m+1}, \ldots, \varphi_n \) generated by the multistep method (5) in terms of the numbers \( \psi_s \) and the starting values \( \varphi_1, \varphi_2, \ldots, \varphi_{m-1} \) (as indicated, we always choose \( \varphi_0 = 0 \)). To simplify notation somewhat and to adapt our notation to the existing literature on the topic, we shall assume that the starting values are of the form
\[
\varphi_r = h \sum_{s=0}^{m-1} w_{rs} \psi_s, \quad r = 1, 2, \ldots, m - 1,
\]
where \( w_{rs} \in \mathbb{R} \) for \( r = 1, 2, \ldots, m - 1 \) and \( s = 0, 1, \ldots, m - 1 \), are starting weights which are independent of \( h \) and which will be specified below. We note that in (17), each starting value \( \varphi_r \) (\( 1 \leq r \leq m - 2 \)) obviously may depend on future states, in general, which is rather unnatural for a Volterra type problem. Such an approach, however, allows sufficiently good accuracy of those starting values.

As a further preparation for Lemma 2.6 considered below, we introduce weights needed in that lemma:

(a) Consider
\[
w_{ns} = \gamma_{n-m-s} \quad \text{for} \quad m \leq s \leq n - \mu, \quad m + \mu \leq n < \infty,
\]
where the numbers \( \gamma_s \) are given by (13).

(b) For \( n \geq m \) we next consider starting weights \( w_{ns}, 0 \leq s \leq m - 1 \). For \( s \) fixed, they are recursively determined by the following inhomogeneous discrete convolution equation,
\[
\sum_{t=0}^{n} \alpha_{n-t} w_{ts} = \begin{cases} \beta_{n-m-s} & n \geq m + \mu + s, \\ 0 & n < m + \mu + s, \end{cases} \quad n = m, m + 1, \ldots.
\]
The weights introduced in (18), (19) are uniformly bounded in case of a nullstable method, i.e.,
\[
\sup_{m + \mu \leq n \leq \infty, 0 \leq s \leq n - \mu} |w_{ns}| < \infty.
\]
This follows, similarly to (15), (18), from standard results for difference equations.

We are now in a position to represent the multistep method (5) in quadrature form. Note that the numbers \( \psi_0, \psi_1, \ldots, \psi_{n-\mu} \) considered in the following lemma do not necessarily coincide with the values of the previously considered function \( \psi : [a, x_n] \to \mathbb{R} \) at the given nodes.

Lemma 2.6 Let \( \varphi_1, \varphi_2, \ldots, \varphi_n \) and \( \psi_0, \psi_1, \ldots, \psi_{n-\mu} \) be arbitrary two sequences of real numbers satisfying (17) and the multistep method recurrence (5) with \( n \geq m + \mu \) and \( \varphi_0 = 0 \). Then the following identity holds:
\[
\varphi_n = h \sum_{s=0}^{n-\mu} w_{ns} \psi_s,
\]
where the weights \( w_{ns} \) are given by (18) and (19).
PROOF. It follows by induction that a representation of the form (21) with some weights \( w_{ns} \) exists in general. The special representations of the weights given in (18) and (19) are then obtained by considering in (21) the standard basis of \( \mathbb{R}^{n-\mu+1} \) to represent \( \psi_0, \psi_1, \ldots, \psi_{n-\mu} \). Details are omitted. \( \square \)

A quadrature method (21) generated by a multistep method (5) with starting values as in (17) is called \((\varrho, \sigma)\)-reducible; see, e.g., Brunner/van der Houwen [3], Taylor [24] or Wolkenfelt ([25], [26]).

2.6 A starting quadrature procedure

For multistep methods (5) to solve the initial value problem (4), we next consider, for \( m \geq 2 \), the determination of starting values \( \varphi_1, \varphi_2, \ldots, \varphi_{m-1} \) of the form (17) that have the approximation properties required in Lemma 2.4. A standard procedure is presented in the following example.

Example 2.7 We consider (17) for fixed \( r \in \{1, 2, \ldots, m\} \) with \( m \geq 1 \). The case \( r = m \) is not considered there in fact but this will be needed for the computation of initial approximations to the solution of the Volterra integral equation of the first kind (1) considered below. Note also that in the case \( r = m \) there is a notational conflict with (20), for \( n = m \) there. We will take care of this in every application.

We now consider an interpolatory quadrature method for the integral \( \int_a^x \psi(y)dy \) using interpolation nodes \( x_0, x_1, \ldots, x_{m-1} \). This in fact means that the resulting quadrature scheme \( \varphi_r = h \sum_{s=0}^{m-1} w_{rs} \psi(x_s) \approx \int_a^x \psi(y)dy \) is exact for all polynomials \( \psi \) of degree \( \leq m - 1 \), with quadrature weights that are given by the following nonsingular linear system of equations:

\[
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
0 & 1 & 2 & \cdots & m-1 \\
0 & 1 & 4 & \cdots & (m-1)^2 \\
0 & 1 & 9 & \cdots & (m-1)^3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2^{m-1} & \cdots & (m-1)^{m-1}
\end{pmatrix}
\begin{pmatrix}
w_{r0} \\
w_{r1} \\
w_{r2} \\
\vdots \\
w_{r,m-1}
\end{pmatrix}
=
\begin{pmatrix}
\frac{r}{2} \\
\frac{r^2}{3} \\
\vdots \\
\frac{r^m}{m}
\end{pmatrix}.
\tag{22}
\]

We next study the error of this quadrature scheme, and for this purpose let \( 1 \leq p \leq m \) and \( L > 0 \) be fixed. For functions \( \psi \in C^{p-1}_L[a, x_m] \) and \( \mathcal{P} \in \Pi_m \) with \( \mathcal{P}(x_s) = \psi(x_s) \) for \( s = 0, 1, \ldots, m-1 \), we have

\[
\max\{ |\mathcal{P}(y) - \psi(y)| \mid a \leq y \leq x_m \} = O(h^p)
\tag{23}
\]

uniformly with respect to the considered class of functions \( \psi \). This follows from elementary interpolation theory: for each \( y \notin \{x_0, x_1, \ldots, x_{m-1}\} \) we have \( \mathcal{P}(y) - \psi(y) = \psi[x_0, x_1, \ldots, x_{m-1}, y]w(y) \), where the first factor on the right-hand side denotes a divided difference, and \( w(y) = (y-x_0)\cdots(y-x_{m-1}) \). It follows by induction that \( |\psi[x_0, \ldots, x_{m-1}, y]| \leq \kappa/h^{m-p} \) holds, with the constant \( \kappa = 2^{m-p}L/(m-1)! \). This finally gives (23).

From (23) we immediately obtain \( \varphi_r - \int_a^x \psi(y)dy = O(h^{p+1}) \) for \( r = 1, 2, \ldots, m \) uniformly with respect to the considered class of functions \( \psi \). Note that the assumption \( p \leq m \) made here is no serious restriction; see Remark 3.2 below for details. Note also that the starting weights \( w_{rs} \) given by (22) do not depend on \( h \) and \( n \).

We summarize the results of Lemma 2.4, Lemma 2.6 and Example 2.7.

Corollary 2.8 Consider a nullstable linear multistep method (5), with \( n \geq m \). Let the weights \( \omega_{ns} \) for \( n \geq m \) be given by (18) and (19), with starting weights \( \omega_{ns} \) for \( n \leq m - 1 \) be given as in Example 2.7. Then for each \( 1 \leq p \leq m \) and each Lipschitz constant \( L > 0 \) we have

\[
h \sum_{s=0}^{n-\mu} w_{ns} \psi(x_s) = (I\psi)(x_n) - h \sum_{s=0}^{m-\mu} \alpha_{n-m-s} \eta(\psi, x_s, h) + O(h^{p+1})
\]

uniformly for \( n \) and \( \psi \) satisfying \( m + \mu \leq n \leq N \) and \( \psi \in C^{p-1}_L[a, x_n] \).
3 Linear multistep methods for perturbed first kind Volterra integral equations

3.1 Some preparations

We now return to the first kind Volterra integral equation (1). For the numerical approximation we consider this equation at equidistant nodes.

For each $n = m, m + 1, \ldots, N$, the resulting integral $(I\psi)(x_n) = \int_{x_n}^{x_{n+1}} \psi(y) \, dy$ with $\psi(y) = k(x_n, y)u(y)$ for $a \leq y \leq x_n$ is approximated by the multistep method (5) under consideration.

In the sequel we suppose that the right-hand side of equation (1) is only approximately given, with

$$|f_n^i - f(x_n)| \leq \delta \quad \text{for} \quad n = 1, 2, \ldots, N,$$

(24)

where $\delta > 0$ is a known noise level.

For the main convergence results we impose the following conditions.

**Assumption 3.1** For the Volterra integral equation (1) of the first kind and a given $m$-step method with $m \geq 1$ (see (5)), we introduce the following assumptions and notations.

(a) The considered $m$-step method with $m \geq 1$ is nullstable and has maximal order $1 \leq p_0 \leq m$.

(b) The second characteristic polynomial

$$\sigma(\xi) := b_{m-\mu} \xi^{m-\mu} + b_{m-\mu-1} \xi^{m-\mu-1} + \cdots + b_0,$$

(25)

with $\mu$ as in (1), is a Schur polynomial:

$$\sigma(\xi) = 0 \implies |\xi| < 1 \quad (\xi \in \mathbb{C}),$$

(26)

i.e., all roots of the polynomial $\sigma$ belong to the open unit disk.

(c) There exists a solution $u : [a, b] \to \mathbb{R}$ to the integral equation (1), with $u \in \mathcal{C}^{p-1}[a, b]$ for some $1 \leq p \leq p_0$ (for the definition of the considered function space, see Section 2.2).

(d) For some integer $N_{\text{min}} \geq m$ and $h_{\text{max}} = \frac{b-a}{N_{\text{min}}}$ the kernel function satisfies $k \in \mathcal{C}^{\sigma}(E)$, where $E = \{(x, y) \mid a \leq y \leq x \leq b \text{ or } a \leq y \leq x \leq a + mh_{\text{max}}\}$.

(e) There holds $k(x, x) = 1$ for each $a \leq x \leq b$.

(f) For a given step size $h = \frac{b-a}{N}$ with some integer $N \geq N_{\text{min}}$, let $x_0, x_1, \ldots, x_N$ be uniformly distributed nodes given by (2).

(g) The values of the right-hand side of equation (1) are approximately given by (24).

Next we give some comments on the Schur polynomial property considered in item (b) of Assumption 3.1.

**Remark 3.2** (a) In the stability analysis to be considered, the coefficients of the inverse power series

$$\frac{1}{\beta(\xi)} = \sum_{n=0}^{\infty} \beta_n^{(-1)} \xi^n, \quad \frac{1}{\gamma(\xi)} = \frac{\alpha(\xi)}{\beta(\xi)} = \sum_{n=0}^{\infty} \gamma_n^{(-1)} \xi^n,$$

(27)

of the generating functions $\beta(\xi)$ and $\gamma(\xi)$, respectively (see (14)), play a significant role. The Schur polynomial condition (26) implies that $1/\beta(\xi)$ is analytic in an open set of the complex plane that contains a disk $\{\xi \in \mathbb{C} \mid |\xi| \leq R\}$ for some $R > 1$, and Cauchy’s integral theorem then implies that the coefficients $\beta_n^{(-1)}$ in (27) decay exponentially, i.e.,

$$\beta_n^{(-1)} = \mathcal{O}(\tau^n) \quad \text{as} \quad n \to \infty \text{ for some } 0 < \tau < 1,$$

(28)

with $\tau = 1/R$ in fact. From this $\gamma_n^{(-1)} = \mathcal{O}(\tau^n)$ as $n \to \infty$ follows easily.
(b) It is easy to see that for the \( m \)-step Adams–Bashford methods with \( 1 \leq m \leq 3 \), and the \( m \)-step Nyström method with \( 2 \leq m \leq 3 \) as well, the second characteristic polynomial \( \sigma \) is a Schur polynomial (see condition (26)), respectively. In addition, \( \text{(26)} \) is obviously satisfied by the BDF methods.

(c) The Schur polynomial condition \( \text{(26)} \) is violated for each multistep method of class \( \text{(6)} \) with \( \mu = 0 \) (the implicit case) and with maximal order \( p_0 > m \). More generally, it is an essential observation made by Gladwin/Jeltsch \( [6] \) that the second characteristic polynomial \( \sigma \) is even not a simple von Neumann polynomial in that situation, with the case \( m = \tau = 1 \) (the repeated trapezoidal rule) as an exception. In addition, the associated scheme for solving Volterra integral equations of the first kind introduced below is necessarily divergent then, in general. For the mentioned exception \( m = \tau = 1 \), the associated second characteristic polynomial is obviously a simple von Neumann polynomial but not a Schur polynomial.

As a consequence of the observations made in the beginning of part (c) of this remark, in the special situation \( \mu = 0 \) in the local quadrature approach \( \text{(6)} \), it is no loss of generality to restrict the considerations to \( m \)-step methods of maximal order \( 1 \leq p_0 \leq m \) (see item (a) of Assumption \( \text{3.1} \)).

(d) We note that in Wolkenfelt \( \text{(25), (26)} \), the second characteristic polynomial \( \sigma \) is required to be a von Neumann polynomial, not a Schur polynomial which is the assumption made in the present paper (see condition \( \text{(26)} \)). The latter assumption results in noise amplification terms which in general are smaller than for Neumann polynomials \( \sigma \). Those terms in fact are, up to some factor, of the form \( \delta / h \).

In addition, the Schur polynomial assumption on \( \sigma \) allows to use a proof technique which in part is much simpler than the elaborated technique used in \( \text{[25]} \). \( \triangle \)

### 3.2 The numerical scheme

We now consider, under the conditions given in Assumption \( \text{3.1} \), the following scheme for the numerical solution of a Volterra integral equation \( \text{(1)} \):

**Algorithm 3.3** (a) Determine \( m \) initial approximations \( u_s^{\delta} \approx u(x_s) \) for \( s = 0, 1, \ldots, m - 1 \) by solving the following linear system of \( m \) equations,

\[
\begin{align*}
    h & \sum_{s=0}^{m-1} w_{ns} k(x_n, x_s) u_s^{\delta} = f_n^{\delta} , \quad n = 1, 2, \ldots, m ,
\end{align*}
\]  

(29)

where the starting weights \( w_{ns} \) are given by \( \text{(22)} \), with \( r \) replaced by \( n \) there.

(b) Determine then recursively, with \( \mu \) given by \( \text{(11)} \), approximations \( u_{n-\mu}^{\delta} \approx u(x_{n-\mu}) \) for \( n = m + \mu, \ldots, N \) with \( N \geq N_{\min} \) by the following scheme.

For \( n \) fixed and \( u_m^{\delta}, u_{m+1}^{\delta}, \ldots, u_{n-\mu}^{\delta} \) already being computed, the following steps have to be employed to determine \( u_{n-\mu}^{\delta} \):

- Set \( \psi_r^{\delta} = k(x_n, x_s) u_s^{\delta} \) for \( s = 0, 1, \ldots, n - \mu - 1 \),
- set \( \varphi_r^{\delta} = 0 \), and compute (for \( m \geq 2 \)) \( \varphi_r = h \sum_{s=0}^{m-1} w_{rs} \psi_{s}^{\delta} \) for \( r = 1, 2, \ldots, m - 1 \), cf. \( \text{(17)} \), where the starting weights \( w_{rs} \) are given by \( \text{(22)} \),
- compute recursively \( \varphi_{r+m}^{\delta} \) for \( r = 0, 1, \ldots, n - m - 1 \) by using on the interval \([a, x_n]\) the perturbed version of the multistep scheme \( \text{(5)} \):

\[
\sum_{j=0}^{m} a_j \varphi_{r+j}^{\delta} = h \sum_{j=0}^{m-\mu} b_j \psi_{r+j}^{\delta} \quad \text{for} \quad r = 0, 1, \ldots, n - m - 1 ,
\]  

(30)

- set \( \varphi_{m}^{\delta} = f_n^{\delta} \),
- compute \( \psi_{n-\mu}^{\delta} \) by using the identity \( \text{(30)} \) for \( r = n - m \),
- compute \( u_{n-\mu}^{\delta} = \psi_{n-\mu}^{\delta} / k(x_n, x_{n-\mu}) \). \( \triangle \)
Remark 3.4 (a) Note that due to (e) in Assumption 3.1, for $h \ll |k(x_n, x_{n-\mu})| \geq C > 0$ for each $n$, independently of $h$. Thus the numerical procedure considered above can in fact be used for the stable computation of $u_0^{\delta}$.

(b) The scheme (29) results from the quadrature method considered in Example 2.7, applied to the integral $\int_a^x k(x, y) u(y) dy$ for $n = 1, 2, \ldots, m$.

(c) It immediately follows from Lemma 2.6 that the approximations obtained by Algorithm 3.3 satisfy

$$h \sum_{s=0}^{n-\mu} \omega_{ns} k(x_n, x_{s}) u^\delta_{n-s} = f^\delta_n, \quad n = m + \mu, \ldots, N,$$

where the weights $\omega_{ns}$ are given by (18) and (19), respectively. The representation (31) will be used in the proof of the main result, cf. Theorem 3.7. In addition, for multistep methods of the form (6), those weights can also be easily computed in practice, and (31) can then be used for the practical implementation of (30). For an illustration see Example 5.3 below.

(d) The considered numerical scheme in Algorithm 3.3 is quite universal and can be simplified in special cases. For example, for the backward rectangular rule (which is the 1-step BDF method) considered in part (b) of Example 2.1 an implementation of Algorithm 3.3 without the starting procedure considered in (a) is possible. This means, however, that no approximation $u_0^\delta$ will be available then. \(\Delta\)

3.3 Uniqueness, existence and approximation properties of the initial approximations

We now consider uniqueness, existence as well as the approximation properties of the initial approximations $u^\delta_0, u^\delta_1, \ldots, u^\delta_{m-1}$. In a first step we consider in more detail the corresponding linear system of equations (29). This system of equations can be written in the form

$$\begin{pmatrix} w_{10}k(x_1, x_0) & w_{11}k(x_1, x_1) & \cdots & w_{1,m-1}k(x_1, x_{m-1}) \\ w_{20}k(x_2, x_0) & w_{21}k(x_2, x_1) & \cdots & w_{2,m-1}k(x_2, x_{m-1}) \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0}k(x_m, x_0) & w_{m1}k(x_m, x_1) & \cdots & w_{m,m-1}k(x_m, x_{m-1}) \end{pmatrix} \begin{pmatrix} u^\delta_0 \\ u^\delta_1 \\ \vdots \\ u^\delta_{m-1} \end{pmatrix} = \begin{pmatrix} f^\delta_1 \\ f^\delta_2 \\ \vdots \\ f^\delta_m \end{pmatrix}. \quad (32)$$

Note that the matrix $S_h \in \mathbb{R}^{m \times m}$ introduced in (32) depends on the stepsize $h$.

Proposition 3.5 The system matrix $S_h$ in (32) is regular for sufficiently small values of $h$, and $\|S_h^{-1}\|_\infty = O(1)$ as $h \to 0$.

Proof. We first consider the situation $k \equiv 1$. In a first step we observe that (22) applied for $r = 1, 2, \ldots, m$, and a subsequent transposition implies the identity

$$\begin{pmatrix} w_{10} & w_{11} & \cdots & w_{1,m-1} \\ w_{20} & w_{21} & \cdots & w_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{m,m-1} \end{pmatrix} \begin{pmatrix} w_{10} & w_{11} & \cdots & w_{1,m-1} \\ w_{20} & w_{21} & \cdots & w_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{m,m-1} \end{pmatrix} = BD, \quad (33)$$

where $BD$ is

$$\begin{pmatrix} w_{10} & w_{11} & \cdots & w_{1,m-1} \\ w_{20} & w_{21} & \cdots & w_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{m,m-1} \end{pmatrix} \begin{pmatrix} w_{10} & w_{11} & \cdots & w_{1,m-1} \\ w_{20} & w_{21} & \cdots & w_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m0} & w_{m1} & \cdots & w_{m,m-1} \end{pmatrix} = BD,$$
where \( M \in \mathbb{R}^{m \times m} \) denotes the transpose of the system matrix in (22), and
\[
D = \text{diag}(\frac{1}{\tau} : \tau = 1, 2, \ldots, m) \in \mathbb{R}^{m \times m}, \quad B = (n^T)_{n=1, \ldots, m} \in \mathbb{R}^{m \times m}.
\]

The matrices \( D, B \) and \( M \) are regular, and hence the matrix \( T \in \mathbb{R}^{m \times m} \) introduced in (33) is regular.

In the case \( k = 1 \), the latter matrix coincides with the matrix \( S_h \).

We now consider the general case for \( k \). We have \( k(x, x) = 1 \) and \( x_n = a + O(h) \) for \( n = 1, 2, \ldots, m-1 \), and thus \( k(x_n, x_s) = 1 + O(h) \) for \( n = 1, \ldots, m \) and \( s = 0, \ldots, m-1 \). We thus have \( S_h = T + O(h) \) for \( h \to 0 \), and from this the proposition immediately follows. \( \square '\n
Next we consider the approximation properties of the initial approximations.

**Theorem 3.6** Let the conditions of Assumptions [3.1] be satisfied. Then the initial approximations \( u_0^\delta, u_1^\delta, \ldots, u_{m-1}^\delta \), determined by (29) for \( h \) sufficiently small, satisfy
\[
\max_{n=0,1,\ldots,m-1} |u_n^\delta - u(x_n)| = O(h^p + \delta/h) \quad \text{as } (h, \delta) \to 0.
\]

**Proof.** It is clear from (32) and Proposition 3.5 that the initial approximations \( u_0^\delta, u_1^\delta, \ldots, u_{m-1}^\delta \) exist and are uniquely determined for \( h \) sufficiently small. We have
\[
h \sum_{s=0}^{m-1} w_{ns} k(x_n, x_s) e_s^\delta = O(h^{p+1} + \delta) \quad \text{for } n = 1, 2, \ldots, m,
\]
where
\[
e_s^\delta = u_s^\delta - u(x_s), \quad s = 0, 1, \ldots, m-1,
\]
denote the approximation errors. This follows from the considerations in Example 2.7 with the notation \( r = n \) and for \( \psi(y) = k(x_n, y)u(y) \) for \( a \leq y \leq x_m \). A matrix-vector formulation of (34) yields \( hS_h \Delta_h^\delta = O(h^{p+1} + \delta) \) as \( h \to 0 \), with \( \Delta_h^\delta := (e_0^\delta, e_1^\delta, \ldots, e_{m-1}^\delta)^T \in \mathbb{R}^m \), and with the matrix \( S_h \) from (32). According to Proposition 3.5 this matrix \( S_h \) is regular for sufficiently small values of \( h \), and \( \|S_h^{-1}\|_\infty = O(1) \) as \( h \to 0 \). From this the statement of the theorem follows. \( \square '\n
### 3.4 The main result

We next present the main result of this paper which extends the results by Wolkenfelt ([25], [26]) to the case of perturbed right-hand sides.

**Theorem 3.7** Let the conditions of Assumption 3.1 be satisfied, and let the approximations \( u_0^\delta, u_1^\delta, \ldots, u_{N-\mu}^\delta \) be determined by Algorithm 3.3 for \( h \) sufficiently small. Then the following error estimate holds,
\[
\max_{n=0,1,\ldots,N-\mu} |u_n^\delta - u(x_n)| = O(h^p + \delta/h) \quad \text{as } (h, \delta) \to 0.
\]

**Proof.** The initial approximation errors are already covered by Theorem 3.6 so it remains to estimate the error \( u_n^\delta - u(x_n) \) for \( n = m, m+1, \ldots, N - \mu \). For this we may assume \( N \geq m + \mu \), since otherwise nothing is to be done for.

1. In a first step we observe that the following system of error equations holds:
\[
h \sum_{s=m}^{n-\mu} \gamma_{n-\mu-s} k(x_n, x_s) e_s^\delta = r_h(x_n) + O(h^{p+1} + \delta) \quad \text{for } n = m + \mu, \ldots, N,
\]
uniformly in \( n \), where

\[
e^\delta_s = u^\delta_s - u(x_s), \quad s = m, \ldots, N - \mu,
\]

\[
r_h(x_n) = \sum_{s=m}^{n-m} \alpha_{n-m-s} g_h(x_n, x_s), \quad n = m + \mu, \ldots, N.
\]

(37)

Furthermore,

\[
g_h(x, y) := \eta(z \mapsto k(x, z)u(z), y, h), \quad a \leq y \leq x - mh, \quad a < x \leq b,
\]

(38)

denotes the truncation error corresponding to the function \( \psi(y) = k(x, y)u(y), \ a \leq y \leq x \). The error representation (36) follows by considering the difference of the representation (34) on one side and the representations in Corollary 2.8 on the other side. We have taken (34) and \( \sum_{s=0}^{\mu-1} \alpha_{n-m-s} g_h(x_n, x_s) = O(h^{p+1}) \) into consideration here. This allows to start summation in (37) with \( s = \mu \).

(2) We next consider a matrix-vector formulation of (36). As a preparation we introduce the notation

\[
N_1 := N - m - \mu + 1
\]

(39)

and consider the system matrix \( A_h \in \mathbb{R}^{N_1 \times N_1} \) given by

\[
A_h = \begin{pmatrix}
\gamma_0 k_{m+\mu,0} & 0 & \cdots & 0 \\
\gamma_1 k_{m+\mu+1,m} & \gamma_0 k_{m+\mu+1,m+1} & \cdots & 0 \\
\vdots & \gamma_1 k_{m+\mu+2,m+1} & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\gamma_{N-m-\mu} k_{Nm} & \cdots & \cdots & \gamma_1 k_{N,N-\mu} & \gamma_0 k_{N,N-\mu}
\end{pmatrix},
\]

(40)

with the notation

\[
k_{ns} = k(x_n, x_s) \quad \text{for} \ m \leq s \leq n - \mu, \quad m + \mu \leq n \leq N.
\]

In addition we consider the vectors

\[
\Delta^\delta_h = (e^\delta_s)_{s=m,\ldots,N-\mu}, \quad R_h = (r_h(x_n))_{n=m+\mu,\ldots,N}.
\]

(41)

Using these notations, the linear system of equations (36) obviously takes the form

\[
h A_h \Delta^\delta_h = R_h + G^\delta_h, \quad \text{with some} \ G^\delta_h \in \mathbb{R}^{N_1}, \ |G^\delta_h|_\infty = O(h^{p+1} + \delta),
\]

(42)

where \( |\cdot|_\infty \) denotes the maximum norm on \( \mathbb{R}^{N_1} \).

(3) For a further treatment of the identity (42), let the matrices \( W_h \in \mathbb{R}^{N_1 \times N_1} \) and its inverse \( W_h^{-1} \in \mathbb{R}^{N_1 \times N_1} \) be given by

\[
W_h = \begin{pmatrix}
\gamma_0 & 0 & \cdots & 0 \\
\gamma_1 & \gamma_0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
\gamma_{N-m-\mu} & \cdots & \gamma_1 & \gamma_0
\end{pmatrix}, \quad W_h^{-1} = \begin{pmatrix}
\gamma_0^{(-1)} & 0 & \cdots & 0 \\
\gamma_1^{(-1)} & \gamma_0^{(-1)} & \cdots & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots & \vdots \\
\gamma_{N-m-\mu}^{(-1)} & \cdots & \gamma_1^{(-1)} & \gamma_0^{(-1)}
\end{pmatrix}.
\]

(43)
We next show that
\[ \|W_h^{-1}\|_\infty = \mathcal{O}(1), \quad \|A_h^{-1}W_h\|_\infty = \mathcal{O}(1), \quad \|A_h^{-1}\|_\infty = \mathcal{O}(1) \quad \text{as } h \to 0, \]
where \( \|\cdot\|_\infty \) denotes the matrix norm induced by the maximum vector norm on \( \mathbb{R}^{N_1 \times N_1} \). In fact, the estimate \( \|W_h^{-1}\|_\infty = \mathcal{O}(1) \) as \( h \to 0 \) follows immediately from the exponential decay of the coefficients of the inverse of the generating function \( \gamma(\xi) \), cf. part (a) of Remark 3.2. For the proof of the second statement in (44), below it will be shown that the matrix \( W_h^{-1}A_h \) can be written in the form
\[ W_h^{-1}A_h = I_h + L_h, \]
where \( I_h \in \mathbb{R}^{N_1 \times N_1} \) denotes the identity matrix, and \( L_h = (\ell_{nj}(h)) \in \mathbb{R}^{N_1 \times N_1} \) denotes some lower triangular matrix which satisfies \( \max_{0 \leq j < n \leq N - m - \mu} |\ell_{nj}(h)| = \mathcal{O}(h) \) as \( h \to 0 \). The representation (45) shows that the matrix \( W_h^{-1}A_h \) is nonsingular for \( h \) small enough, and the discrete version of Gronwall’s inequality then yields \( \|A_h^{-1}W_h\|_\infty = \mathcal{O}(1) \) as \( h \to 0 \). The third estimate in (44) follows immediately from the other two estimates considered in (44).

In the sequel it will be shown that the representation (45) is valid, and for this purpose we consider the lower triangular matrix
\[ W_h^{-1}A_h = (b_{nj}) \in \mathbb{R}^{N_1 \times N_1} \]
in more detail. In fact, we have for \( 0 \leq j < n \leq N - m - \mu \)
\[ b_{nj} = \sum_{\ell=0}^{n-j} \gamma_{n-j-\ell}^{(-1)} \gamma_\ell k(x_{m+m+\ell}, x_{m+j}) = \sum_{\ell=0}^{n-j} \gamma_{n-j-\ell}^{(-1)} \gamma_\ell k(x_{m+m+\ell+j}, x_{m+j}) = 0 \]
\[ = k(x_{m+m+n}, x_{m+j}) \sum_{\ell=0}^{n-j} \gamma_{n-j-\ell}^{(-1)} \gamma_\ell \]
\[ + \sum_{\ell=0}^{n-j-1} \left[ \gamma_{n-j-\ell}^{(-1)} \gamma_\ell \left( k(x_{m+m+\ell+j}, x_{m+j}) - k(x_{m+m+n}, x_{m+j}) \right) \right]. \]

Thus we have
\[ |b_{nj}| = \mathcal{O} \left( h \sum_{\ell=0}^{n-j-1} |\gamma_{n-j-\ell}^{(-1)}| |\gamma_\ell| (n-j-\ell) \right) = \mathcal{O}(h) \quad \text{for } 0 \leq j < n \leq N - m - \mu \] (47)

uniformly with respect to \( j \) and \( n \), where \((*)\) follows immediately from (15) and the end of part (a) of Remark 3.2. Moreover we have
\[ b_{nn} = \gamma_0^{(-1)} k(x_{n+m+m}, x_{n+m}) \gamma_0 = 1 + \mathcal{O}(h) \quad \text{for } n = 0, 1, \ldots, N - m - \mu, \] (48)
which follows from the identities \( \gamma_0^{(-1)} = 1/\gamma_0 \) and \( k(x, x) \equiv 1 \), cf. (6) in Assumption 3.1. The statements (47) and (48) show that the lower triangular matrix \( W_h^{-1}A_h \) in fact can be written as in (45). (4) We still have to take a closer look at the vector \( R_h \in \mathbb{R}^{N_1} \) considered in (41). It can be written as follows,
\[ R_h = B_h \xi_h, \] (49)
where \( B_h \in \mathbb{R}^{N_1 \times N_1} \) is the following matrix,

\[
\begin{pmatrix}
\alpha_0^{(-1)} g_h(x_{m+\mu}, x_\mu) & 0 & \cdots & 0 \\
\alpha_1^{(-1)} g_h(x_{m+\mu+1}, x_\mu) & \alpha_0^{(-1)} g_h(x_{m+\mu+1}, x_{\mu+1}) & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \ddots & \ddots \\
\alpha_{N-m-\mu}^{(-1)} g_h(x_N, x_\mu) & \cdots & \alpha_1^{(-1)} g_h(x_N, x_{N-m-1}) & \alpha_0^{(-1)} g_h(x_N, x_{N-m})
\end{pmatrix}
\]

and \( \mathcal{E}_h = (1, 1, \ldots, 1) \in \mathbb{R}^{N_1} \). The representations \((42)\) and \((49)\) give \( h A_h \Delta_h^4 = B_h \mathcal{E}_h + G_h^3 \), and due to \((44)\) it remains to show that

\[
\|W_h^{-1} B_h\|_\infty = \mathcal{O}(h^{p+1})\quad \text{as } h \to 0
\]

holds. For this purpose we introduce the notation

\[
U_h = \begin{pmatrix}
\alpha_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\alpha_m & \cdots & 0 & 0 \\
0 & \cdots & \alpha_m & \alpha_0
\end{pmatrix}, \quad U_h^{-1} = \begin{pmatrix}
\alpha_0^{(-1)} & 0 & \cdots & 0 \\
\alpha_1^{(-1)} & \alpha_0^{(-1)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{N-m-\mu}^{(-1)} & \cdots & \alpha_1^{(-1)} & \alpha_0^{(-1)}
\end{pmatrix} \quad \in \mathbb{R}^{N_1 \times N_1},
\]

\[
V_h = \begin{pmatrix}
\beta_0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
\beta_{m-\mu} & \cdots & 0 & 0 \\
0 & \cdots & \beta_{m-\mu} & \beta_0
\end{pmatrix}, \quad V_h^{-1} = \begin{pmatrix}
\beta_0^{(-1)} & 0 & \cdots & 0 \\
\beta_1^{(-1)} & \beta_0^{(-1)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{N-m-\mu}^{(-1)} & \cdots & \beta_1^{(-1)} & \beta_0^{(-1)}
\end{pmatrix} \quad \in \mathbb{R}^{N_1 \times N_1},
\]

and observe that

\[
W_h = V_h U_h^{-1}, \quad W_h^{-1} = V_h^{-1} U_h, \quad (51)
\]

holds. From the fact that the second characteristic polynomial (see \((26)\)) is a Schur polynomial it follows

\[
\|V_h^{-1}\|_\infty = \mathcal{O}(1)\quad \text{as } h \to 0.
\]

(52)

In the sequel we consider the lower triangular matrix \( U_h B_h \) in more detail. It can be written as follows,

\[
U_h B_h = M_h + C_h \quad \text{with the diagonal matrix } M_h = \text{diag}(g_h(x_{m+n}, x_n) : n = \mu, \mu+1, \ldots, N-m),
\]

with \( \|M_h\|_\infty = \mathcal{O}(h^{p+1}) \) as \( h \to 0 \). In addition, \( C_h = (c_{ij}(h)) \in \mathbb{R}^{N_1 \times N_1} \) denotes some strictly lower triangular matrix with max\( 0 \leq j < n \leq N-m-\mu \mid c_{ij}(h) \mid = \mathcal{O}(h^{p+2}) \). See the third part of this proof for similar results with respect to the matrix \( W_h^{-1} A_h \). Here we additionally use the mean value theorem with respect to the first variable of \( g \) and the fact that the local truncation error \( g \) defined in \((38)\) satisfies

\[
\frac{\partial}{\partial x} g_h(x, y) = \eta(z \mapsto \frac{\partial}{\partial x} k(x, z) u(z), y, h) = \mathcal{O}(h^{p+1})
\]
uniformly for \( a \leq y \leq x - mh \) and \( a < x \leq b \).

This in particular means \( \| U_h B_h \|_{\infty} = O(h^{p+1}) \) as \( h \to 0 \), and this together with (51) and (52) implies (50).

The statement of the theorem now follows easily from the error representation (36) and its matrix version (41), (42), from the stability estimates in (44), and from the considerations in part (4) of this proof. \( \Box \)

**Remark 3.8** The stability analysis presented in the third part of the proof of Theorem 3.7 uses techniques similar to those used in Eggermont [5]; see also Lubich [18] as well as [22] and [23].

In the sequel, for step sizes \( h = h(\delta) = \frac{b-a}{N} \), with \( N = N(\delta) \), with a slight abuse of notation we write \( h \sim \delta^\beta \) as \( \delta \to 0 \), if there exist real constants \( c_2 \geq c_1 > 0 \) such that \( c_1 h \leq \delta^\beta \leq c_2 h \) holds for \( \delta \to 0 \). As an immediate consequence of Theorem 3.7 we obtain the following main result of this paper.

**Corollary 3.9** Let Assumption 3.1 be satisfied. For \( h = h(\delta) \sim \delta^{1/(p+1)} \) we have

\[
\max_{n=0,1,\ldots,N-\mu} |u_{\delta,n} - u(x_n)| = O(\delta^{p/(p+1)}) \quad \text{as} \quad \delta \to 0,
\]

where the approximations \( u_{\delta,0}, u_{\delta,1}, \ldots, u_{\delta,N-\mu} \) are determined by Algorithm 3.3.

We conclude this section with some remarks.

**Remark 3.10** (a) Assumption 3.1 and Corollary 3.9 imply that the order of the method should be chosen as large as possible to allow best possible estimates for a wide range of smoothness degrees of solutions. Note that, for \( m \) fixed, both the computational complexity and the number of function evaluations for the implementation of Algorithm 3.3 are \( O(N^2) \) as \( N \to \infty \). Thus the number of steps \( m \) in the considered multistep method has no impact here.

(b) For results on the regularization properties of the composite midpoint rule, see e.g., Apartsin [11] or Kaltenbacher [13]. For other special regularization methods for the approximate solution of Volterra integral equations of the first kind with smooth kernels and perturbed right-hand sides, see e.g., Lamm [14].

## 4 The balancing principle

### 4.1 Preparations

The a priori choice of the step size \( h \) considered in Corollary 3.9 requires knowledge of the smoothness of the exact solution \( u : [a, b] \to \mathbb{R} \). The balancing principle as an a posteriori strategy for choosing \( h \) has no such requirement and thus seems to be an interesting alternative. Its implementation, however, requires a determination of the coefficient of the error propagation term \( \delta/h \) that appears in the basic error estimate (35). This is the subject of the following proposition.

**Proposition 4.1** Under the conditions of Assumption 3.1 we have

\[
\max_{n=0,1,\ldots,N-\mu} |u_{\delta,n} - u(x_n)| \leq C_1 h^p + C_2 \frac{\delta}{h} \quad \text{for} \quad 0 < h \leq \overline{h},
\]

where \( C_1 \) and \( C_2 \) denote some constants chosen independently of \( h \), and \( \overline{h} \) is chosen sufficiently small.
The constant $C_2$ may be chosen as follows:

$$C_2 = \max \left\{ C_{2a}, C_{2b} \left( 1 + C_{2a} \|k\|_\infty \right) \right\},$$

where

$$C_{2a} = (1 + L)\|T^{-1}\|_\infty, \quad C_{2b} = (1 + \mu L) \left( \sum_{s=0}^{\infty} \|\gamma_s^{(-1)}\| \right) \exp \left( (1 + \mu L)C_1 T (b - a) \right),$$

with $C_3 = \left\{ \sup_{r \geq 0} |\gamma_r| \right\} \sum_{s=1}^{\infty} |\gamma_s^{(-1)}| s,$

where the notation $\|k\|_\infty = \max_{(x,y) \in E} |k(x,y)|$ is used, and $L \geq 0$ denotes a Lipschitz constant of the kernel $k$ with respect to the first variable. In addition, for the definition of the sequence $(\gamma_s^{(-1)})$ and the matrix $\Gamma$, see $\Gamma_{22}$ and $\Gamma_{23}$, respectively.

Moreover, $h$ in $\Gamma_{32}$ can be chosen as follows, $h = \min \left\{ \frac{1}{m(1+L)\|c\|_\infty}, h_{\text{max}} \right\}$, where $h_{\text{max}}$ is taken from Assumption $3.1$. In the special case $k \equiv 1$, the estimate $\Gamma_{33}$ holds with $h_{\text{max}} = h_{\text{max}}$.

PROOF. Let $e^\delta = u^\delta - u(x_s)$ for $s = 0, \ldots, N - \mu$. We first consider the starting error. A closer look at the proof of Theorem 3.6 shows that

$$\max_{s=0, \ldots, m-1} |e^\delta_s| \leq \|S_h^{-1}\|_\infty (C_4 h^p + \frac{\delta}{h})$$

for $h > 0$, (54)

where $S_h$ denotes the system matrix considered in (32) and (34), and $h$ is chosen so small (details are given below) such that the inverse matrix of $S_h$ exists. In addition, $C_4$ denotes some constant that may be chosen independently of $h$. So we need to estimate $\|S_h^{-1}\|_\infty$ which is done below. First we consider the error of the present multistep scheme. A closer look at the reasoning of (56) shows that

$$h \sum_{s=m}^{n-\mu} \gamma_{n-\mu-s} (x_n, x_s) e^\delta_s = f^{\delta}_n - f(x_n) + r_h(x_n) - h \sum_{s=0}^{m-1} w_{ns} k(x_n, x_s) e^\delta_s + O(h^{p+1})$$

holds uniformly for $n = m + \mu, \ldots, N$, where $\gamma_0, \gamma_1, \ldots$ are given by (13). Representation (42) in the proof of Theorem 3.6 thus can be written as

$$h A_h \Delta^\delta = R_h + G_{h,1} + G_{h,2}$$

with some $G_{h,1} \in \mathbb{R}^N$, $\|G_{h,1}\|_\infty = O(h^{p+1})$, (55)

and some vector $G_{h,2} \in \mathbb{R}^N$ with

$$\|G_{h,2}\|_\infty \leq \delta + h \|k\|_\infty \left\{ \max_{m+\mu \leq n \leq N} \sum_{s=0}^{m-1} |w_{ns}| \right\} \max_{0 \leq s \leq m-1} |e^\delta_s|.$$

(56)

So in view of (54)–(56) we need to provide upper bounds for $\|S_h^{-1}\|_\infty$ and $\|A_h^{-1}\|_\infty$. For this purpose let $L \geq 0$ denote a Lipschitz constant of the kernel $k$ with respect to the first variable, i.e.,

$$|k(x_1, y) - k(x_2, y)| \leq L|x_1 - x_2| \quad \text{for} \quad (x_1, y), (x_2, y) \in E,$$

where the set $E$ is introduced in Assumption 3.1. Then the matrix $S_h$, $h \leq h_{\text{max}}$, can be written in the form $S_h = T + F_h$, where the perturbation matrix $F_h \in \mathbb{R}^{m \times m}$ satisfies $\|F_h\|_\infty \leq \|T\|_\infty m h$. It then follows from standard perturbation results for matrices that

$$\|S_h^{-1}\|_\infty \leq (1 + L)\|T^{-1}\|_\infty = C_{2a}$$

for $0 < h \leq \frac{1}{m(1+L)\|c\|_\infty}$.

(57)

where $\|c\|_\infty = \|T\|_\infty \|T^{-1}\|_\infty$. and the upper bound for $h$ in (57) can be ignored if $L = 0$.

For the estimation of $\|A_h^{-1}\|_\infty$ we have to take a closer look at part (3) of the proof of Theorem 3.7. We obviously have $\|W_h^{-1}\|_\infty \leq \sum_{s=0}^{\infty} |\gamma_s^{(-1)}|$ for $h > 0$, and we next estimate the entries of $W_h^{-1} A_h =
In the sequel we consider the following sequence of geometrically increasing step sizes:

\[ h = s \geq s \]  

where \( s \) is chosen sufficiently small such an \( h \) exists. Proceeding from (61) yields \( |b_{nj}| \leq C_3 Lh \) for \( j < n \), where the constant \( C_3 \) is chosen as in the statement of the proposition. An application of the discrete version of Gronwall’s lemma now results in

\[
\| A_h^{-1} \|_\infty \leq \sum_{s=0}^\infty \| (W_h^{-1} A_h)^{-1} \|_\infty \| W_h^{-1} \|_\infty \\
\leq (1 + \mu L) \sum_{s=0}^\infty | \eta_s^{\gamma-1} | \exp((1 + \mu L)(b - a) = C_{2b} \quad \text{for } 0 < h \leq \frac{1}{1 + \mu L},
\]

where the considered upper bound for \( h \) can be ignored if \( \mu = 0 \) or \( L = 0 \) holds. Note also that this upper bound for \( h \) is not smaller than the upper bound for \( h \) given in (57), which justifies the definition of \( \bar{h} \) given in the proposition. This completes the proof. \( \Box \)

### 4.2 Implementation of the balancing principle

In the sequel we assume that the conditions of Assumption 3.1 are satisfied. It is convenient to introduce new notation for the set of nodes and for the approximations generated by the considered multistep method to indicate dependence on the step size \( h \):

\[
\Delta(h) = \{ x_n = a + nh \mid n = 0, 1, \ldots, N - \mu \}, \quad \text{where } h = \frac{b-a}{N}, \quad N \geq N_{\text{min}}, \\
u^\delta(h) : \Delta(h) \to \mathbb{R}, \quad x_n \mapsto u_n^\delta.
\]

(58)

In the sequel we consider the following sequence of geometrically increasing step sizes:

\[
h_s = \frac{b-a}{N_s}, \quad N_s = \prod_{\Sigma} s \quad \text{for } s = 0, 1, \ldots, \Sigma,
\]

(59)

where \( \Sigma = \pi(\delta) \geq 0 \) and \( N = \prod_{\Sigma} s \geq 1 \) are some integers that may depend on \( \delta \), and \( \kappa \geq 1 \) is some fixed integer. The set of those step sizes will be denoted by \( \Sigma \), i.e.,

\[
\Sigma = \{ h_0 < h_1 < \cdots < h_{\Sigma} \}.
\]

Note that due to the special form of the step sizes we have

\[
\Delta(h_{\Sigma}) \subset \Delta(h_{\Sigma-1}) \subset \cdots \subset \Delta(h_0).
\]

In the sequel we assume that \( \Sigma \geq 0 \) and \( N \geq 1 \) are chosen so that the step sizes \( h_0 \) and \( h_{\Sigma} \) are respectively sufficiently small and sufficiently large. More precisely, we assume the following:

\[
h_0 \leq c_s \delta_s^{1/2}, \quad c_{ss} \delta_s^{1/(\nu_0 + 1)} \leq h_{\Sigma} \leq \bar{h} \quad (0 < \delta \leq \delta_0),
\]

(60)

where \( c_s, c_{ss} \) and \( \delta_0 > 0 \) denote some constants, and \( \bar{h} \) is chosen as in Proposition 4.1. In addition, \( c_{ss} \) is chosen sufficiently small such an \( h_{\Sigma} \) satisfying (60) exists.

We consider the following a posteriori choice of the step size \( h = h(\delta) \):

\[
h(\delta) = \max H^\delta, \quad \text{where } H^\delta := \{ h_\ast \in \Sigma : \text{for } h, h' \in \Sigma \text{ with } h < h' \leq h_\ast \text{ we have}
\]

\[
\max_{y \in \Delta(h') |u^\delta(y, h') - u^\delta(y, h)| \leq \beta h^\delta \}
\]

(61)

where \( \beta > 2C_2 \) holds, with \( C_2 \) chosen as in Proposition 4.1. Note that by definition we have \( h_0 = \min \Sigma \in H^\delta \) so that \( H^\delta \neq \emptyset \), and thus \( h(\delta) \) in (61) is well-defined. The adaptive choice of the step size given by (61) is in fact a balancing principle. For a general introduction to this class of a posteriori parameter choice strategies see, e.g., Lepskii [15], Mathé [19], Pereverzev/Schock [20], or Lu/Pereverzev [17].
Remark 4.2 The strategy (61) is in fact a nonstandard balancing principle. We recall that the classical balancing principle chooses, in our framework, the maximum from the set \( H^\delta := \{ h' \in \Sigma : |u^\delta(y, h') - u^\delta(y, h)| \leq \beta \frac{\delta}{h} \text{ for } y \in \Delta(h'), \ h \in \Sigma, \ h < h' \} \). The latter maximum may be larger than \( h(\delta) \) introduced in (61), in general. In turns out, however, that the step size \( h(\delta) \) is sufficiently large to get similar estimates as for the standard balancing principle; see the following theorem for details.

The nonstandard version (61) of the balancing principle is considered for computational reasons: it may require less computational amount than the standard version. In fact, a possible strategy to determine \( h(\delta) \) is to verify for \( s = 1, 2, \ldots \) whether \( h_s \in H^\delta \) is satisfied, and this procedure stops if \( h_s \notin H^\delta \) holds for the first time, or if \( s = \infty \). In the former case we have \( h(\delta) = h_{s-1} \), and then there is no need to consider the step sizes \( h_{s+1}, h_{s+2}, \ldots, h_\infty \).

We have the following convergence result:

**Theorem 4.3** Let Assumption 3.1 be satisfied, and let \( u^\delta(., h) \) and \( h(\delta) \) be given by (58) and (61), respectively. Then the following estimates hold,

\[
\max_{y \in \Delta(h(\delta))} |u^\delta(y, h(\delta)) - u(y)| = O(\delta^{p/(p+1)}) \quad \text{as } \delta \to 0,
\]

\[
h(\delta) \geq C\delta^{1/(p+1)},
\]

where \( C > 0 \) denotes some constant which is independent of \( \delta \).

**Proof.** The proof is a compilation of techniques used, e.g., in Lu/Pereverzev [17], and we thus give a sketch of a proof only. A basic ingredient in the following analysis is provided by the following estimate, which follows from Proposition 4.1 and (61):

\[
\max_{y \in \Delta(h(\delta))} |u^\delta(y, h(\delta)) - u(y)| \leq \max_{y \in \Delta(h(\delta))} |u^\delta(y, h(\delta)) - u^\delta(y, h)| + \max_{y \in \Delta(h)} |u^\delta(y, h) - u(y)|
\]

\[
\leq C_1 h^p + (\beta + C_2) \frac{\delta}{h} \quad \text{for each } h \in \Sigma, \ h \leq h(\delta).
\]

It now remains to determine some \( h \in \Sigma \) with \( h \leq h(\delta) \) and \( h \sim \delta^{1/(p+1)} \); the estimates (62)–(63) then easily follow from (64). For this purpose we consider the set

\[
M^\delta := \{ h \in \Sigma : h^{p+1} \leq C_3 \delta \},
\]

where \( C_3 > 0 \) is chosen so small such that \( 2(C_1 C_3 + C_2) \leq \beta \) holds, with \( C_1 \) and \( C_2 \) being chosen as in Proposition 4.1. That choice of \( C_3 \) guarantees

\[
M^\delta \subset H^\delta
\]

which is shown in the sequel. For this purpose let \( h_*, h' \in \Sigma \) with \( h < h' \leq h_* \). We then have

\[
\max_{y \in \Delta(h')} |u^\delta(y, h') - u^\delta(y, h)| \leq \max_{y \in \Delta(h')} |u^\delta(y, h') - u(y)| + \max_{y \in \Delta(h)} |u^\delta(y, h) - u(y)|
\]

\[
\leq C_1 h^p + C_2 \frac{\delta}{h'} + C_1 h^p + C_2 \frac{\delta}{h} \leq 2(C_1 C_3 + C_2) \frac{\delta}{h},
\]

where \( h, h' \in M^\delta \) is taken into account. This shows \( h_* \in H^\delta \) and completes the proof of the relation \( M^\delta \subset H^\delta \).

We are now in a position to verify (62)–(63), and for this we consider two situations. In the case \( M^\delta \neq \emptyset \) we define \( h^+(\delta) = \max M^\delta \) and obtain

\[
h^+(\delta) \leq h(\delta), \quad h^+(\delta) \sim \delta^{1/(p+1)},
\]

where we assume that \( \delta \leq \delta_0 \) holds. The first statement in (65) follows immediately from \( M^\delta \subset H^\delta \) and the definition of \( h(\delta) \), see (61). The second statement in (65) follows in the case \( h^+(\delta) = \max \Sigma \).
(which is \(h^r\) in fact) from the second estimate in (60), and in the case \(h^+ (\delta) < \max \Sigma\) it follows from \(2^* h^+ (\delta) \in \Sigma \setminus M^\delta\). Estimate (63) is an immediate consequence of (65), and estimate (62) then follows easily from estimate (64), applied with \(h = h^+ (\delta)\).

On the other hand, \(M^\delta = \emptyset\) means \(h_0 \notin M^\delta\), and the first estimate in (60) then implies \(\min \Sigma \sim \delta^{1/\nu + 1}\) for \(0 < \delta \leq \delta_0\). This shows (63), and estimate (62) follows easily from (64), applied with \(h = \min \Sigma\). \(\square\)

5 Numerical experiments

As an illustration of the main results considered in Corollary 3.9 and Theorem 4.3 we next present the results of numerical experiments for four Volterra integral equations of the first kind with smooth kernels of the form (1), treated by different kind of multistep methods, respectively.

Here are two comments on the first three numerical tests, where a priori choices of the step size are considered in fact:

- Numerical experiments on the interval \([a, b] = [0, 1]\) are employed for step sizes \(h = 1/2^\nu\) for \(\nu = 5, 6, \ldots, 12\), with the exception of the order 4 BDF method. In the latter method, the influence of rounding errors becomes clearly visible for \(\nu \geq 10\).

- For each considered step size \(h\) and each considered multistep method with maximal order \(p_0\), we consider (1) with some function \(u \in C^0 [0, 1]\), and the noise level \(\delta = h^{1/(p_0 + 1)}\) is considered.

In all numerical experiments, the perturbations are of the form \(f^\delta_n = f(x_n) + \Delta f_n\) with uniformly distributed random values \(\Delta f_n\), with \(|\Delta f_n| \leq \delta\).

Example 5.1 First we consider the repeated midpoint rule which in fact coincides with the 2-step Nyström method (see Example 2.1). In the formulation (5), this quadrature method reads as follows, \(\varphi_{r+2} - \varphi_r = 2h\psi_{r+1}\) for \(r = 0, 1, \ldots, n - 2\). This method is applied to the following linear Volterra integral equation of the first kind,

\[
\int_0^x \cos (x - y) u(y) \, dy = \sin x =: f(x) \quad \text{for} \ 0 \leq x \leq 1,
\]

with exact solution \(u(y) = 1\) for \(0 \leq y \leq 1\). The conditions of Assumption 3.1 are satisfied with \(m = p_0 = p = 2\). The numerical results are shown in Table 1. There, \(\|f\|_\infty\) denotes the maximum norm of the function \(f\). All numerical experiments are employed using the program system OCTAVE (http://www.octave.org).

| \(N\) | \(\delta\) | \(100 \cdot \|f\|_\infty\) | max \(|u^*_n - u(x_n)|\) | max \(|u^*_n - u(x_n)| / \delta^{2/3}\) |
|-------|----------|----------------|----------------|-------------------------------|
| 32    | 3.1 \cdot 10^{-9} | 3.70 \cdot 10^{-5} | 1.05 \cdot 10^{-5} | 1.07 |
| 64    | 3.8 \cdot 10^{-6} | 4.58 \cdot 10^{-4} | 3.09 \cdot 10^{-4} | 1.27 |
| 128   | 4.8 \cdot 10^{-7} | 5.70 \cdot 10^{-5} | 6.56 \cdot 10^{-5} | 1.08 |
| 256   | 6.0 \cdot 10^{-8} | 7.10 \cdot 10^{-6} | 1.69 \cdot 10^{-5} | 1.11 |
| 512   | 7.5 \cdot 10^{-9} | 8.87 \cdot 10^{-7} | 7.25 \cdot 10^{-6} | 1.90 |
| 1024  | 9.3 \cdot 10^{-10} | 1.11 \cdot 10^{-7} | 1.09 \cdot 10^{-6} | 1.14 |
| 2048  | 1.2 \cdot 10^{-10} | 1.38 \cdot 10^{-8} | 2.71 \cdot 10^{-7} | 1.14 |
| 4096  | 1.5 \cdot 10^{-11} | 1.73 \cdot 10^{-9} | 6.71 \cdot 10^{-8} | 1.13 |

Table 1: Numerical results of the repeated midpoint rule applied to equation (66)

Example 5.2 Next we present some numerical results for the order 4 BDF method which in the formulation (5) reads as follows, \(\frac{1}{12} (25 \varphi_{r+4} - 48 \varphi_{r+3} + 36 \varphi_{r+2} - 16 \varphi_{r+1} + 3 \varphi_r) = h\psi_{r+4}\) for \(r = 0, 1, \ldots, n - 2\). In the formulation (5), this quadrature method reads as follows, \(\varphi_{r+2} - \varphi_r = 2h\psi_{r+1}\) for \(r = 0, 1, \ldots, n - 2\). This method is applied to the following linear Volterra integral equation of the first kind,
This method is applied to the example considered above. The results are shown in Table 3.

Next we present the results of numerical experiments with the second order Adams–Bashfort method.

Example 5.3 Next we present the results of numerical experiments with the second order Adams–Bashfort method \( \varphi_{r+2} - \varphi_{r+1} = \frac{4}{3}(3\varphi_{r+1} - \psi_r) \) for \( r = 0, 1, \ldots, n - 2 \). The quadrature scheme formulation of this method, see (31), is \( \varphi_n = \frac{h}{3}(3\varphi_{n-1} + 2\varphi_{n-2} + \cdots + 2\varphi_2 + 3\varphi_1) \), where the latter identity follows from the fact that \( w_{10} = w_{11} = \frac{1}{2} \), see (22).

This method is applied to the following test problem:

\[
\int_0^x (1 + x - y)u(y) \, dy = x - 1 + e^{-x} \quad \text{for} \quad 0 \leq x \leq 1, \quad \Rightarrow: f(x)
\]

with exact solution \( u(y) = ye^{-y} \) for \( 0 \leq y \leq 1 \). The conditions of Assumption 3.1 are satisfied with \( m = p_0 = p = 4 \). Step sizes, noise levels, initial approximations and starting values are chosen similar to the example considered above. The results are shown in Table 3.

Table 2: Numerical results of the 4th order BDF method applied to equation (67)

| \( N \) | \( \delta \) | \( 100 \cdot \delta / ||f||_\infty \) | \( \max_n |u_n^\delta - u(x_n)| \) | \( \max_n |u_n^\delta - u(x_n)| / \delta^{2/3} \) |
|-------|---------|-------------------------------|----------------|---------------------------------|
| 32    | \( 3.0 \times 10^{-8} \) | \( 6.48 \times 10^{-6} \) | \( 7.14 \times 10^{-6} \) | \( 7.48 \) |
| 64    | \( 9.3 \times 10^{-10} \) | \( 2.03 \times 10^{-7} \) | \( 4.85 \times 10^{-7} \) | \( 8.14 \) |
| 128   | \( 2.9 \times 10^{-11} \) | \( 6.33 \times 10^{-9} \) | \( 2.85 \times 10^{-8} \) | \( 7.65 \) |
| 256   | \( 9.1 \times 10^{-13} \) | \( 1.98 \times 10^{-10} \) | \( 2.11 \times 10^{-9} \) | \( 9.07 \) |
| 512   | \( 2.8 \times 10^{-14} \) | \( 6.18 \times 10^{-12} \) | \( 1.28 \times 10^{-10} \) | \( 8.83 \) |
| 1024  | \( 8.9 \times 10^{-16} \) | \( 1.93 \times 10^{-13} \) | \( 2.32 \times 10^{-11} \) | \( 25.50 \) |

Table 3: Numerical results of the 2nd order Adams–Bashfort method applied to equation (68)

| \( N \) | \( \delta \) | \( 100 \cdot \delta / ||f||_\infty \) | \( \max_n |u_n^\delta - u(x_n)| \) | \( \max_n |u_n^\delta - u(x_n)| / \delta^{2/3} \) |
|-------|---------|-------------------------------|----------------|---------------------------------|
| 32    | \( 3.1 \times 10^{-9} \) | \( 8.76 \times 10^{-3} \) | \( 1.93 \times 10^{-3} \) | \( 1.98 \) |
| 64    | \( 3.8 \times 10^{-6} \) | \( 1.07 \times 10^{-3} \) | \( 5.21 \times 10^{-4} \) | \( 2.13 \) |
| 128   | \( 4.8 \times 10^{-7} \) | \( 1.31 \times 10^{-4} \) | \( 1.29 \times 10^{-4} \) | \( 2.11 \) |
| 256   | \( 6.0 \times 10^{-8} \) | \( 1.63 \times 10^{-5} \) | \( 3.84 \times 10^{-5} \) | \( 2.52 \) |
| 512   | \( 7.5 \times 10^{-9} \) | \( 2.03 \times 10^{-6} \) | \( 8.99 \times 10^{-6} \) | \( 2.36 \) |
| 1024  | \( 9.3 \times 10^{-10} \) | \( 2.54 \times 10^{-7} \) | \( 2.36 \times 10^{-6} \) | \( 2.47 \) |
| 2048  | \( 1.2 \times 10^{-10} \) | \( 3.17 \times 10^{-8} \) | \( 5.95 \times 10^{-7} \) | \( 2.50 \) |
| 4096  | \( 1.5 \times 10^{-11} \) | \( 3.96 \times 10^{-9} \) | \( 1.60 \times 10^{-7} \) | \( 2.68 \) |

Note that the relative errors in the right-hand side presented in the third column (of all three tables in fact) are rather small, respectively.
Example 5.4 Here we consider again the second order Adams–Bashforth method, see Example 5.3, this time applied to the problem of numerical differentiation:

\[
\int_0^1 u(y) \, dy = f(x) \quad \text{for} \quad 0 \leq x \leq 1, \quad \text{with} \quad u(y) = \begin{cases} 
2y, & 0 \leq y \leq \frac{1}{2}, \\
2(1-y), & \frac{1}{2} < y \leq 1,
\end{cases}
\]

which means \( u \in C^0[0,1] \) in fact. We consider the balancing principle, and for this we need to take a closer look at Proposition 4.1. Elementary computations show that results of the numerical experiments are shown in Table 4.

which in fact results from an application of the discrete Gronwall inequality in the proof of Theorem 3.7. This shows that estimate (53) holds with \( C = \frac{6}{7} \) and \( \sum_{n=0}^{\infty} |g^{0.1}| = \frac{4}{7} \). This shows that (53) holds with \( C = \frac{19}{10} \), and thus we may choose \( \beta = 13.0 \) in (61).

For each considered noise level \( \delta \), the integers \( s \) and \( N \) are chosen such that \( h_0 \) is the largest step size \( \leq \delta^{1/2} \), and \( h_\ast \) is the smallest step size satisfying \( \geq \delta^{1/3} \) (see (60)). We choose \( \kappa = 1 \) in (59). The results of the numerical experiments are shown in Table 4.

| \( \delta \)   | \( 100 \cdot \delta/\|f\|_\infty \) | \( N(\delta) \) | \( h(\delta)/\delta^{1/3} \) | \( \max_n |e_n^0| \) | \( \max_n |e_n^0|/\delta^{1/2} \) |
|---------------|-------------------------------------|----------------|-----------------|-----------------|-----------------|
| \( 1.0 \cdot 10^{-6} \) | \( 2.00 \cdot 10^{-3} \) | 92 | 3.44 | 1.45 \cdot 10^{-2} | 4.60 |
| \( 2.5 \cdot 10^{-6} \) | \( 5.00 \cdot 10^{-4} \) | 146 | 4.33 | 9.10 \cdot 10^{-3} | 5.76 |
| \( 6.2 \cdot 10^{-7} \) | \( 1.25 \cdot 10^{-4} \) | 232 | 5.45 | 5.77 \cdot 10^{-3} | 7.30 |
| \( 1.6 \cdot 10^{-7} \) | \( 3.13 \cdot 10^{-5} \) | 740 | 3.42 | 1.80 \cdot 10^{-3} | 4.55 |
| \( 3.9 \cdot 10^{-8} \) | \( 7.81 \cdot 10^{-6} \) | 1176 | 4.30 | 1.15 \cdot 10^{-3} | 5.83 |

Table 4: Numerical results of the 2nd order Adams–Bashforth method, applied to equation (69)

6 Conclusions

In the present paper we consider the regularization of linear first-kind Volterra integral equations with smooth kernels and perturbed given right-hand sides. As regularization scheme we consider quadrature methods that are generated by linear multistep methods for solving ODEs, with an appropriate starting procedure. The regularizing properties of an a priori choice of the step size as well as the balancing principle as an adaptive choice of the step size are analyzed, with a variant of the balancing principle which sometimes requires less amount of computational work than the standard version of this principle.

In the case of exact data, the considered scheme is similar to that in Wolkenfelt ([25], [26]). However, our analysis is different from that in those two papers and allows less smoothness of the involved functions in fact. All used smoothness assumptions in the present paper are of the form \( \tilde{C}^{p} \) instead of \( C^{p} \) which enlarge the classes of admissible functions further.

It turns out that an application of the balancing principle for the choice of the step size is possible, but for general kernels \( k \) the coefficient of the error propagation term \( \delta/h \) turns out to be rather large which in fact results from an application of the discrete Gronwall inequality in the proof of Theorem 3.7.

References

[1] A. S. Apartsin. The numerical solution of Volterra integral equations of the first kind (in Russian). Technical report, Preprint No. 1, Sib. Energ. Inst., Sib. Otd. Akad. Nauk SSSR, Irkutsk, 1981.

[2] H. Brunner. Collocation Methods for Volterra Integral and Related Functional Differential Equations. Cambridge University Press, Cambridge, 2004.

[3] H. Brunner and P. J. van der Houwen. The Numerical Solution of Volterra Equations. Elsevier, Amsterdam, 1986.

[4] F. de Hoog and R. S. Anderssen. On the solution of Volterra integral equations of the first kind. Numer. Math., 21:22–32, 1973.
[5] P. P. B. Eggermont. A new analysis of the trapezoidal-discretization method for the numerical solution of Abel-type integral equations. *J. Integral Equations*, 3:317–332, 1981.

[6] C. J. Gladwin and R. Jeltsch. Stability of quadrature rule methods for Volterra integro-differential equations. *BIT*, 14:144–151, 1974.

[7] E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I, Nonstiff Problems*. Springer, Berlin, 2 edition, 2008.

[8] P. Henrici. *Discrete Variable Methods in Ordinary Differential Equations*. Wiley, New York, 1962.

[9] P. Henrici. *Applied and Computational Complex Analysis, Vol. I*. Wiley, New York, 1974.

[10] P. A. W. Holyhead and S. McKee. Stability and convergence of multistep methods for solving linear Volterra integral equations of the first kind. *SIAM J. Math. Anal.*, 13(2):269–292, 1976.

[11] P. A. W. Holyhead, S. McKee, and P. J. Taylor. Multistep methods for solving linear Volterra integral equations of the first kind. *SIAM J. Math. Anal.*, 12(5):698–711, 1975.

[12] A. Iserles. *A First Course in the Numerical Analysis of Differential Equations*. Cambridge University Press, Cambridge, 2nd edition, 2008.

[13] B. Kaltenbacher. A convergence analysis of the midpoint rule for first kind Volterra integral equations with noisy data. *J. Integral Equations*, 22:313–339, 2010.

[14] P. Lamm. A survey of regularization methods for first-kind Volterra equations. In D. Colton, H. W. Engl, A. K. Louis, J. R. McLaughlin, and W. Rundell, editors, *Surveys on Solution Methods for Inverse Problems*, pages 53–82, Vienna, New York, 2000. Springer.

[15] O. V. Lepskiĭ. On a problem of adaptive estimation of gaussian white noise. *Theory Probab. Appl.*, 36:454–466, 1990.

[16] P. Linz. *Analytical and Numerical Methods for Volterra Equations*. SIAM, Philadelphia, 1 edition, 1985.

[17] S. Lu and S. V. Pereverzev. *Regularization Theory for Ill-posed Problems*. de Gruyter, Berlin, 2013.

[18] Ch. Lubich. Fractional linear multistep methods for Abel-Volterra integral equations of the first kind. *IMA J. Numer. Anal.*, 7:97–106, 1987.

[19] P. Mathé. The Lepskiĭ principle revisited. *Inverse Problems*, 22:L11–L15, 2006.

[20] S. V. Pereverzev and E. Schock. On the adaptive selection of the parameter in regularization of the ill-posed problems. *SIAM J. Numer. Anal.*, 43(5):2060–2076, 2005.

[21] R. Plato. *Concise Numerical Mathematics*. AMS, Providence, Rhode Island, 2003.

[22] R. Plato. Fractional multistep methods for weakly singular Volterra equations of the first kind with noisy data. *Numer. Funct. Anal. Optimization*, 26(2):249–269, 2005.

[23] R. Plato. The regularizing properties of the composite trapezoidal method for weakly singular Volterra integral equations of the first kind. *Adv. Comput. Math.*, 36(2):331–351, 2012.

[24] P. J. Taylor. The solution of Volterra integral equations of the first kind using inverted differentiation formulae. *BIT*, 16:416–425, 1976.

[25] P. H. M. Wolkenfelt. Linear multistep methods and the constructions of quadrature formulae for Volterra integral equations and integro-differential equations. Technical report, Report NW 76/79, Mathematical Centrum, Amsterdam, 1979.
[26] P. H. M. Wolkenfelt. Reducible quadrature methods for Volterra integral equations of the first kind. *BIT*, 21:232–241, 1981.