CLASSIFICATION OF GENERALIZED SYMMETRIES OF THE YANG-MILLS FIELDS WITH A SEMI-SIMPLE STRUCTURE GROUP

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Abstract. A complete classification of generalized symmetries of the Yang-Mills equations on Minkowski space with a semi-simple structure group is carried out. It is shown that any generalized symmetry, up to a generalized gauge symmetry, agrees with a first order symmetry on solutions of the Yang-Mills equations. Let $g = g_1 + \cdots + g_n$ be the decomposition of the Lie algebra $g$ of the structure group into simple ideals. First order symmetries for $g$-valued Yang-Mills fields are found to consist of gauge symmetries, conformal symmetries for $g_m$-valued Yang-Mills fields, $1 \leq m \leq n$, and their images under a complex structure of $g_m$.

1. Introduction

Generalized, or Lie-Bäcklund, symmetries of a system of differential equations, roughly speaking, are infinitesimal transformations involving the independent and dependent variables and their derivatives that preserve solutions to the system. Since their introduction by Emmy Noether in her study of the correspondence between transformations preserving the fundamental integral of a variational principle and conservation laws of the associated Euler-Lagrange equations, generalized symmetries have become increasingly important in the geometric analysis of differential equations. Besides the original application to the identification of conservation laws, generalized symmetries also play an important role in the study of infinite dimensional Hamiltonian systems [1] and in various methods, in particular in separation of variables [6], [10], for constructing explicit solutions to differential equations. There also seems to be a close connection between completely integrable equations and generalized symmetries. Bäcklund transformations have been shown to give rise to infinite sequences of generalized symmetries [8] and the existence of infinite number of independent generalized symmetries has, in fact, been proposed as a test for the complete integrability of differential equations [9].

In this paper we carry out a complete analysis of generalized symmetries of the Yang-Mills equations on Minkowski space with a semi-simple structure group. Due to its implications, some of which are discussed below, classification of generalized symmetries and conservation laws of the Yang-Mills fields has been listed by Tsujishita [15] as a significant open problem in the formal geometry of differential equations.

Presently, surprisingly little seem to be known about the existence of higher order symmetries and conservation laws for the Yang-Mills equations. By construction, the Yang-Mills equations admit symmetries arising from the conformal transformations of Minkowski space and from the gauge transformations of the potential. Schwartz [3] has used a computer algebra to verify that for the structure group $SU(2)$, the conformal group together with gauge transformations forms the maximal group of Lie symmetries of the equations. Under the Noether correspondence, the conformal transformations yield 15 independent conservation laws for Yang-Mills fields [4], while, by Noether’s second theorem, the gauge symmetries reflect a differential identity, a divergence identity, satisfied by the Yang-Mills equations.

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A priori, one easily discovers evidence for the existence of hidden symmetries of the Yang-Mills equations. In a recent paper [2], it is found that the simplest case of the Yang-Mills fields, the free electromagnetic field, possesses a family of new generalized symmetries, the representatives of which of order \( r, r \geq 2 \), correspond to Killing spinors of type \((r-2, r+2)\) on Minkowski space. Counterparts of these symmetries of order 2 in a curved spacetime metric are discussed in [7]. Additional evidence is provided by the self-dual Yang-Mills equations and its various completely integrable reductions, which admit an infinite number of independent hidden symmetries. On the other hand, negative evidence for the existence of additional generalized symmetries is provided by [14], in which natural symmetries, i.e., symmetries that transform equivariantly under the Poincaré and gauge groups, are analyzed. It is found that the only natural symmetries are generalized gauge symmetries arising from natural functions. However, the naturality requirement is quite restrictive and excludes, in particular, the conformal symmetries of the Yang-Mills equations.

The foremost application of a symmetry analysis of the Yang-Mills equations is the identification of conservation laws for the equations by Noether’s theorem. However, a full classification of conservation laws is complicated by the degeneracy of the equations, due to which the correspondence between equivalence classes of conservation laws and equivalence classes of characteristics for conservation laws fails to be one-to-one. However, a complete symmetry analysis will still be an integral step in the determination whether the Yang-Mills equations possess any additional non-trivial conservation laws besides those provided by the conformal symmetries. We plan to treat this issue in a future publication.

Vinogradov [16] has argued that a nondegenerate system of nonlinear differential equations involving more than two independent variables do not admit generalized symmetries. Of course, due to the divergence identity satisfied by the Yang-Mills equations, Vinogradov’s arguments do not apply in the present problem. However, the study of the symmetries of degenerate systems may reveal whether under suitable assumptions Vinogradov’s theorem admits a converse of some form.

Spinorial methods are pivotal in our analysis of symmetries of the Yang-Mills fields. Recently, spinor techniques have been employed in the symmetry analysis of the Einstein equations [3], in the analysis of natural symmetries of the Yang-Mills equations [14] and of symmetries and conservation laws of Maxwell’s equations in Minkowski space [1], [2] and in curved background metric [7]. Spinor techniques are also applicable to the problem at hand. The main computational tool is the exact sets of fields of Penrose [12], which can be used as part of the coordinate system for the infinitely prolonged solution manifold determined by the Yang-Mills equations. According to Penrose, the symmetrized covariant derivatives of the Yang-Mills spinor uniquely determine all the unsymmetrized derivatives on the solution manifold, a considerable simplification over the corresponding tensor treatment.

In this paper we will show that the conformal symmetries and their variants provided by the complex structure of a Lie algebra, when it exists, together with the generalized gauge symmetries are, up to equivalence, the only generalized symmetries admitted by the Yang-Mills equations on Minkowski space with a semi-simple structure group. Specifically, we will prove the following Theorem.

**Theorem 1.1.** Let \( \mathfrak{g} \) be a semisimple Lie algebra, and let \( \mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_n \) be the decomposition of \( \mathfrak{g} \) into simple ideals. Let \( Q_i = Q_i(x^1, a_1^\beta, \ldots, a_{k, l_1}^\beta) \) be a generalized symmetry of order \( p \) for \( \mathfrak{g} \)-valued Yang-Mills fields on Minkowski space. Then there is a first order symmetry \( \tilde{Q}_i = \tilde{Q}_i(x^1, a_1^\beta, a_{k, l}^\beta) \) of the Yang-Mills equations and a \( \mathfrak{g} \)-valued function \( \mathcal{X} = \mathcal{X}(x^1, a_1^\beta, \ldots, a_{k, l_1}^\beta) \) so that...
\[ Q_i = \tilde{Q}_i + \nabla_i \mathcal{X} \]

on solutions of the Yang-Mills equations.

Moreover, the function \( \mathcal{X} \) can be chosen so that the symmetry \( \tilde{Q}_i \) is expressible as a sum

\[ \tilde{Q}_i = \tilde{Q}_{1,i} + \cdots + \tilde{Q}_{n,i} \]

of first order symmetries \( \tilde{Q}_{m,i} \) for \( g_m \)-valued Yang-Mills fields, where, in the case \( g_m \) is a real form of a simple complex Lie algebra, the symmetry

\[ \tilde{Q}_{m,i} = \xi^i F_{m,ij} \]

is a conformal symmetry, and, in the case \( g_m \) is the realification of a simple complex Lie algebra with the complex structure \( J_m \), the symmetry

\[ Q_{m,i} = \xi^i F_{m,ij} + \tau^i_m J_m F_{m,ij} \]

is a sum of a conformal symmetry and the image of a conformal symmetry under \( J_m \). In the above expressions \( F_{m,ij} \) denotes the field tensor for \( g_m \)-valued Yang-Mills fields.

The Yang-Mills equations also possess the obvious symmetries consisting of permutations of isomorphic components in the decomposition of the Lie algebra \( g \) into simple ideals. These symmetries, however, are discrete and do not satisfy the defining equations for generalized symmetries, and, as such, do not yield conservation laws under the Noether correspondence.

Our paper is organized as follows. In Section 2 we establish notation and give a summary of some basic properties of generalized symmetries and of semi-simple Lie algebras. We also introduce spinorial methods and derive several technical results needed in the sequel. Section 3 in its entirety is dedicated to the proof of Theorem 1.1. In the Appendix we give the proof of a derivative formula presented in Section 2.

2. Preliminaries

In this Section we establish notation and review some basic definitions and results from the theory of symmetries of differential equations and of semi-simple Lie algebras most relevant for the topic of the paper at hand. We also introduce spinorial methods and present several technical results needed in the proof of Theorem 1.1. For more details, see, for example, [5], [11], [12], [19].

Let \( M \) be Minkowski space with coordinates \( x^i, i = 0, 1, 2, 3 \), and let \( \mathfrak{G} \) be a Lie group with Lie algebra \( g \). Recall that \( \mathfrak{G}, g \) are called simple if \( g \) is not abelian and admits no ideals besides \( \{0\} \) and \( g \), and that \( \mathfrak{G}, g \) are called semi-simple if the Killing form \( \kappa(v, w) = tr(\text{ad} v \circ \text{ad} w) \) of \( g \) is non-degenerate, where \( \text{ad} v(z) = [v, z] \). A semi-simple Lie algebra \( g \) is a direct sum of simple ideals of \( g \). We write \( A = \Lambda^1(M) \otimes g \rightarrow M \) for the bundle of \( g \)-valued Yang-Mills potentials over \( M \). Fix a basis \( \{e_\alpha\} \) for \( g \) and let \( a^\alpha \) denote the components of the Yang-Mills potential. Then, as a coordinate bundle, \( A = \{(x^i, a^\alpha)\} \rightarrow \{(x^i)\} \). We denote the \( p \)th order jet bundle of local section of \( A \) by \( J^p(A) \), \( 0 \leq p \leq \infty \). As a coordinate space, \( J^p(A) \) is given by

\[ J^p(A) = \{(x^i, a^\alpha, a^\alpha_i, a^\alpha_{ij}, a^\alpha_{ij, j_2}, \ldots, a^\alpha_{i_1 \cdot \cdot \cdot i_q})\}, \]

where \( a^\alpha_{i_1 \cdot \cdot \cdot i_q} \) stands for the \( q \)th order derivative variables.

Write \( \partial_i \) for the partial derivative operator \( \partial/\partial x^i \) and define partial derivative operators \( \partial_{a^\alpha i_1 \cdot \cdot \cdot i_p} \) by

\[ \partial_{a^\alpha i_1 \cdot \cdot \cdot i_p} x^k = 0, \quad \text{and} \]

\[ \partial_{a^\alpha i_1 \cdot \cdot \cdot i_p} a^\alpha_{k_1 \cdot \cdot \cdot k_q} = \begin{cases} \delta_k^\alpha \delta_{i_1}^{(j_1)} \cdots \delta_{i_p}^{(j_p)}, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases} \]
A generalized vector field \(X\) on \(A\) is a vector field
\[X = P^i \partial_i + Q^\alpha_i \partial^a_\alpha\],
where the coefficients \(P_i\), \(Q^\alpha_i\) are differential functions on \(A\), that is, functions on some \(J^p(A)\), \(p < \infty\). We call \(p\) the order of \(X\). An evolutionary vector field \(Y\) is a generalized vector field of the form
\[Y = Q^\alpha_i \partial^a_\alpha\].
Here and in what follows we employ the standard Einstein summation convention.

Write \(D_i\) for the total derivative operator
\[D_i = \partial_i + \sum_{p \geq 0} (D_j_1 \cdots D_j_p Q^\alpha_i \partial^a_\alpha) \partial^i_{a_1} \cdots \partial^i_{a_p}\].

The infinite prolongation \(\text{pr}X\) of \(X\) is the unique lift of \(X\) to a vector field on \(J^A\) preserving the contact ideal on \(J^A\). In the coordinates (2.1), \(\text{pr}X\) is given by
\[\text{pr}X = P^i D_i + \sum_{p \geq 0} (D_j_1 \cdots D_j_p Q^\alpha_{ev,i}) \partial^i_{a_1} \cdots \partial^i_{a_p}\],
where the differential functions \(Q^\alpha_{ev,i}\) are the components of the evolutionary form
\[X_{ev} = (Q^\alpha_i - P^j a^\alpha_{a_i,j}) \partial^i_{a}\].

We extend the usual Yang-Mills covariant derivative to \(g\)-valued differential functions \(G^\alpha\) by
\[\nabla_i G^\alpha = D_i G^\alpha + [a_i, G]^\alpha\].
Then due to the Jacobi identity,
\[\nabla_i [G, H] = [\nabla_i G, H] + [G, \nabla_i H]\].

We will use the following result repeatedly in the proof of Theorem 1.1.

**Proposition 2.1.** Let \(G^\alpha\) be a \(g\)-valued and let \(r^\alpha_\beta\) be an \(\text{End}(g)\)-valued differential function. Then
\[\nabla_i (r^\alpha_\beta G^\beta) = (D_i r^\alpha_\beta) G^\beta + r^\alpha_\beta \nabla_i G^\beta - (c^\alpha_\beta \gamma r^\beta_\delta + r^\alpha_\beta c^\beta_\gamma) a^\gamma_i G^\delta\].
In particular, if
\[c^\alpha_\beta \gamma r^\beta_\delta + r^\alpha_\beta c^\beta_\gamma = 0\],
then
\[\nabla_i (r^\alpha_\beta G^\beta) = (D_i r^\alpha_\beta) G^\beta + r^\alpha_\beta \nabla_i G^\beta\].

In order to classify first order symmetries of the Yang-Mills equations we need to analyze equation (2.4) more closely. Let \(R : g \to g\) be the endomorphism of \(g\) with the matrix \((r^\alpha_\beta)\). Then condition (2.4) simply means that \(R\) commutes with the adjoint representation,
\[R \circ \text{ad} v = \text{ad} v \circ R\] for all \(v \in g\).
If \(g\) is a simple complex Lie algebra, then Schur’s Lemma states that \(R\) must be a scalar multiple of the identity transformation. However, the Yang-Mills fields in this paper are real-valued, and we are thus lead to consider the counterpart of Schur’s Lemma for real Lie algebras.

Let \(h\) be a complex Lie algebra. Then the realification \(h^\mathbb{R}\) of \(h\) is \(h\) regarded as a real Lie algebra. A real form of \(h\) is a subalgebra \(h_o \subset h^\mathbb{R}\) so that the complexification \(h^\mathbb{C}_o\) is isomorphic with \(h\), that is, \(h = h_o + ih_o\).
As is well known, a simple real algebra \( \mathfrak{g} \) is either a real form \( \mathfrak{h}_o \) or the realification \( \mathfrak{h}^R \) of a simple complex Lie algebra \( \mathfrak{h} \). In the latter case the complex structure of \( \mathfrak{h}^R \) is easily seen to commute with the adjoint representation. As we will see, for simple real Lie algebras the identity mapping and the complex structure are essentially the only endomorphisms commuting with the adjoint representation.

Let

\[
(2.7) \quad \mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_n
\]

be the decomposition of a semi-simple Lie algebra \( \mathfrak{g} \) into simple ideals \( \mathfrak{g}_m \). Write \( P_m : \mathfrak{g} \to \mathfrak{g}_m \) for the projection induced by the decomposition \((2.7)\).

**Proposition 2.2.** Let \( \mathfrak{g} \) be a semi-simple Lie algebra as in \((2.7)\), and let \( R \) be an endomorphism of \( \mathfrak{g} \) commuting with the adjoint representation of \( \mathfrak{g} \). Then each \( \mathfrak{g}_m \) is invariant under \( R \). Write \( R_m \) for the restriction of \( R \) to \( \mathfrak{g}_m \) so that \( R = R_1 \circ P_1 + \cdots + R_n \circ P_n \). If \( \mathfrak{g}_m \) is a real form of a simple complex Lie algebra, then \( R_m = a_m \text{id} \) for some \( a_m \in \mathbb{R} \). If \( \mathfrak{g}_m \) is the realification of a simple complex Lie algebra, then \( R_m = a_m \text{id} + b_m J_m \) for some \( a_m, b_m \in \mathbb{R} \), where \( J_m \) is the complex structure of \( \mathfrak{g}_m \).

Conversely, let \( R_m : \mathfrak{g}_m \to \mathfrak{g}_m \) be as above. Then the mapping \( R = R_1 \circ P_1 + \cdots + R_n \circ P_n \) commutes with the adjoint representation of \( \mathfrak{g} \).

**Proof.** It is easy to see that the image and the inverse image of an ideal in \( \mathfrak{g} \) under \( R \) are again ideals of \( \mathfrak{g} \). Thus \( R(\mathfrak{g}_m) \) is an ideal of \( \mathfrak{g} \), which we can assume to be non-trivial. Hence \( R(\mathfrak{g}_m) \) is a direct sum of the members of a subfamily of \( \mathfrak{g}_1, \ldots, \mathfrak{g}_n \). Write \( R(\mathfrak{g}_m) = \mathfrak{g}_{p_1} + \cdots + \mathfrak{g}_{p_q} \), and let \( R_m \) stand for the restriction of \( R \) to \( \mathfrak{g}_m \). If \( R_m^{-1}(\mathfrak{g}_{p_r}) \subset \mathfrak{g}_m \) is non-trivial, then, since \( \mathfrak{g}_m \) is simple, \( R_m^{-1}(\mathfrak{g}_{p_r}) = \mathfrak{g}_m \). Thus \( R_m(\mathfrak{g}_m) = \mathfrak{g}_p \) for some \( p \). If \( m \neq p \), then, by \((2.6)\), we have that

\[
R_m([v, w]) = [v, R_m(w)] = 0 \quad \text{for all } v, w \in \mathfrak{g}_m.
\]

But this contradicts the simplicity of \( \mathfrak{g}_m \) and the non-triviality of \( R_m \). Thus \( R(\mathfrak{g}_m) = \mathfrak{g}_m \).

Next suppose that \( \mathfrak{g}_m \) is the real form \( \mathfrak{h}_{m,o} \) of a complex simple Lie algebra \( \mathfrak{h}_m \). Lift \( R_m \) to the complexification \( \mathfrak{h}_m \) of \( \mathfrak{g}_m \) to obtain an endomorphism \( \mathfrak{R}_m \) of \( \mathfrak{h}_m \) commuting with the adjoint representation of \( \mathfrak{h}_m \). Since \( \mathfrak{h}_m \) is simple, we can use Schur’s lemma to conclude that there is \( a_m \in \mathbb{C} \) such that \( \mathfrak{R}_m = a_m \text{id} \). But \( R_m \) must preserve \( \mathfrak{g}_m \subset \mathfrak{h}_m \), which implies that \( a_m \) is real. Thus \( R_m = a_m \text{id} \), and Schur’s lemma holds in this case.

Suppose in turn that \( \mathfrak{g}_m \) is the realification of a simple complex Lie algebra \( \mathfrak{h}_m \). For simplicity, choose a basis \( e_\alpha \), \( 1 \leq \alpha \leq d_m \), for example a Weyl basis, for \( \mathfrak{h}_m \), in which the structure constants \( c^\alpha_{\gamma} \) are real. Then, in the basis \( e_\alpha, f_\alpha = i e_\alpha, 1 \leq \alpha \leq d_m \), for \( \mathfrak{g}_m \), the bracket relations are

\[
(2.8) \quad [e_\alpha, e_\beta] = c^\alpha_{\beta\gamma} e_\gamma, \quad [e_\alpha, f_\beta] = c^\beta_{\alpha\gamma} f_\gamma, \quad [f_\alpha, f_\beta] = -c^\alpha_{\beta\gamma} e_\gamma.
\]

Let \( t_1, t_2 \) be the subspaces of \( \mathfrak{g}_m^\mathbb{C} \) spanned by the vectors

\[
k_\alpha = \frac{1}{2}(e_\alpha + i f_\alpha), \quad l_\alpha = \frac{1}{2}(e_\alpha - i f_\alpha), \quad \alpha = 1, \ldots, d_m,
\]

respectively. Then by \((2.8)\),

\[
[k_\alpha, k_\beta] = c^\beta_{\alpha\gamma} k_\gamma, \quad [k_\alpha, l_\beta] = 0, \quad [l_\alpha, l_\beta] = c^\beta_{\alpha\gamma} l_\gamma.
\]

Hence \( t_1, t_2 \) are isomorphic to \( \mathfrak{h}_m \), and \( \mathfrak{g}_m^\mathbb{C} \) is the direct sum of the ideals \( t_1, t_2 \).

As above, we show that \( t_1, t_2 \) are invariant under the lift \( R_m' \) of \( R_m \) to \( \mathfrak{g}_m^\mathbb{C} \). Now apply Schur’s Lemma to \( R_m' \) to see that there are \( c_1, c_2 \in \mathbb{C} \) so that

\[
R_m' = c_1 P_{t_1} + c_2 P_{t_2},
\]

where \( P_{t_k} : \mathfrak{g}_m^\mathbb{C} \to t_k \) is the projection induced by the direct sum \( \mathfrak{g}_m^\mathbb{C} = t_1 + t_2 \).
We still need to choose \( c_1, c_2 \) so that \( R'_m \) preserves \( g_m \subset g^C_m \). Write
\[
c_1 = a_1 + ib_1, \quad c_2 = a_2 + ib_2.
\]
Then
\[
R'_m e_\alpha = \frac{1}{2}(a_1 + a_2 + ib_1 + ib_2)e_\alpha + \frac{1}{2}(-b_1 + b_2 + ia_1 - ia_2)f_\alpha,
\]
which is contained in \( g_m \) provided that \( a_2 = a_1 \) and \( b_2 = -b_1 \). With this,
\[
R'_m f_\alpha = b_1 e_\alpha + a_1 f_\alpha.
\]
Thus, when \( g_m \) is the realification of a simple complex Lie algebra, the space of endomorphisms of \( g_m \)
commuting with the adjoint representation is spanned by the identity transformation and the complex structure \( J_m \)
given in the basis (2.8) by
\[
J_m e_\alpha = f_\alpha, \quad J_m f_\alpha = -e_\alpha.
\]
The proof of the converse is now obvious. This completes the proof of the Proposition.

Write
\[
F_{ij}^\alpha = a_{ij}^{\alpha, \beta} a_{i}^{\gamma} a_{j}^{\gamma} + c_{ij}^{\alpha, \beta} a_{i}^{\gamma} a_{j}^{\gamma}
\]
for the components of the Yang-Mills field tensor. The field tensor \( F_{ij}^\alpha \) measures the extent to which
covariant derivatives fail to commute. Specifically,
\[
(2.9) \quad \nabla_i \nabla_j G - \nabla_j \nabla_i G = [F_{ij}, G].
\]
The Yang-Mills equations and the Bianchi identity for \( F_{ij} \) are
\[
(2.10) \quad \nabla^j F_{ij} = 0 \quad \nabla^j \ast F_{ij} = 0.
\]
Here and in what follows, we raise and lower indices using the Minkowski metric \( \eta = \text{diag} (-1, 1, 1, 1) \)
and the symbol \( \ast \) stands for the Hodge duality operator. The Yang-Mills equations determine a submanifold
\( R \subset J^2(A) \), which we call the solution manifold of the equations. The \( r \)-fold, \( r \leq p \), covariant derivatives
\[
\nabla_{i_1} \cdots \nabla_{i_r} \nabla^j F_{i_{r+1} j} = 0
\]
of the field equations, in turn, determine the \( p \)-fold prolonged solution manifold \( R^p \subset J^{p+2}(A) \)
of the equations.

A generalized symmetry of the Yang-Mills equations of order \( p \) is a generalized vector field \( X \)
of order \( p \) satisfying
\[
(2.11) \quad \text{pr}_X (\nabla^j F_{ij}^\alpha) = 0 \quad \text{on} \ R^p.
\]
Note that any total vector field
\[
P = P^i D_i = \text{pr} \left( P^i \partial_i + (P^i a_{i,j}^\alpha) \partial_{a^\alpha} \right),
\]
where \( P^i \) are some differential functions, satisfies the symmetry equations (2.11). Thus, in particular, a
generalized vector field \( X \) is a symmetry if and only if its evolutionary form \( X_{ev} \) is one. Hence we only need
to consider symmetries in evolutionary form. Equation (2.11), when written out for the components of an
evolutionary vector field \( Q = Q_{ij}^\alpha \partial_{a^\alpha} \) of order \( p \), become
\[
(2.12) \quad \nabla^j \nabla_i Q_{ij} - \nabla^i \nabla_j Q_{ij} - [F_{ij}, Q_{ij}] = 0 \quad \text{on} \ R^p.
\]
We call this equation the determining equations for symmetries of the Yang-Mills equations. As is easily verified, the
Yang-Mills equations admit generalized gauge symmetries given in component form by
\[
Q_{ij}^\alpha = \nabla_{ij} \Lambda_{i}^{\alpha},
\]
where $X^\alpha$ is any $g$-valued differential function. Consequently, we will call two generalized symmetries of the Yang-Mills equations in evolutionary form equivalent if their difference agrees with a generalized gauge symmetry on some prolonged solution manifold $R^q$, $q \geq 0$.

By construction, the Yang-Mills equations also admit symmetries arising from the conformal transformations of the underlying Minkowski space. The components of the evolutionary form of the symmetry corresponding to a conformal Killing vector $\xi^\alpha$ is simply given by

$$Q_i^\alpha = \xi^\alpha F^\alpha_{ij}.$$

If the decomposition of the semi-simple Lie algebra $g$ into a direct sum of simple ideals contains factors that are the realifications of simple complex Lie algebras, we can use the result of Proposition 2.2 to see that the Yang-Mills equations possess additional first order symmetries arising from the complex structures of these factors. The construction of the additional symmetries is based on the following result.

**Proposition 2.3.** Let $g$ be a Lie algebra with structure constants $e^\alpha_{\beta\gamma}$ in some basis $e_\alpha$. Suppose that an $\text{End}(g)$-valued vector field $\xi^\alpha_{\beta\gamma}$ on Minkowski space satisfies the equations

$$\partial^i \xi^\alpha_{\beta\gamma} = \eta^i k^\alpha_{\beta}, \quad \xi^\alpha_{\beta\gamma} e_{\gamma\delta} + e^\alpha_{\beta\gamma} \xi^\delta = 0$$

for some functions $k^\alpha_{\beta}$. Then

$$Q_i^\alpha = \xi^\alpha_{\beta\gamma} F^\beta_{ij}$$

are the components of a first order generalized symmetry of the Yang-Mills equations.

**Proof.** The proof amounts to showing that $Q_i^\alpha$ in (2.13) satisfy the determining equations (2.12) for a symmetry. This is a standard computation based on equation (2.5) and on elementary properties of conformal Killing vectors and will therefore be omitted. □

Let $g = g_1 + \cdots + g_n$ be the decomposition of a semi-simple Lie algebra into simple ideals. Order the ideals so that for some $0 \leq p \leq n + 1$, the ideals $g_m$, $m < p$, are real forms of simple complex Lie algebras and the ideals $g_m$, $m \geq p$, are realifications of simple complex Lie algebras with complex structures $J_m$. Let $P_m : g \rightarrow g_m$ be the projection induced by the above decomposition. Write $F^\alpha_{m,ij}$ for the Yang-Mills field tensor for $g_m$-valued fields.

Let $\xi^\alpha$, $\tau^\gamma$ be conformal Killing vectors and let $Q_m[\xi]$, $1 \leq m \leq n$, $Q_{J,m}[\tau]$, $p \leq m \leq n$, be evolutionary vector fields on $J^1(A)$ with components $Q^\alpha_{m,i}[\xi]$, $Q^\alpha_{m,i}[\tau]$ determined by

$$P_l Q_{m,i}[\xi] = \delta_{lm} \xi^j F^\alpha_{m,ij}, \quad P_l Q_{J,m,i}[\tau] = \delta_{lm} \tau^j J_m F^\alpha_{m,ij}, \quad 1 \leq l \leq n.$$

**Corollary 2.4.** Let $\xi^\alpha$, $\tau^\gamma$ be conformal Killing vectors. Then the vector fields $Q_m[\xi]$, $1 \leq m \leq n$, $Q_{J,m}[\tau]$, $p \leq m \leq n$, are first order generalized symmetries of the Yang-Mills equations for $g$-valued fields.

**Proof.** The proof of the Corollary is an immediate consequence of Propositions 2.2 and 2.3. □
Spinorial methods play a crucial role in our analysis of symmetries of the Yang-Mills fields. Given a tensorial object \(T_{i_1\ldots i_p}\), we write \(T_{i_1'i_1\ldots i_p'}\) for its spinor representative, where

\[
T_{i_1'i_1\ldots i_p'} = \sigma_{i_1'i_1}^{i_1} \cdots \sigma_{i_p'i_p}^{i_p} T_{i_1\ldots i_p}.
\]

Here, apart from a constant factor, the matrices \(\sigma_{i_j}^{i_j}\) are the identity matrix and the Pauli spin matrices. We use bar to denote complex conjugation and we raise and lower spinor indices using the spin metric \(\epsilon_{ij} = \epsilon_{[ij]}, \epsilon_{01} = 1\), and its complex conjugate \(\epsilon_{IJ}, \epsilon_{I', J'}\). For more details, see [12]. Accordingly, we write

\[
\partial_I = \sigma_I^j \partial_j, \quad D_{I'} = \sigma_{I'}^j D_j, \quad \nabla_I = \sigma_I^i \nabla_i
\]

for the spinor representatives of the partial, the total, and the covariant derivative operators. Note that in spinor form equation (2.9) becomes

\[
\nabla_I \nabla_{I'} G - \nabla_{I'} \nabla_I G = \epsilon_{I'J'} [\Phi_I, G] + \epsilon_{IJ} [\Phi_{I'}, G].
\]

Let \(\Phi^\alpha_{IJ; K_1K_2\ldots K_pK_{p'}} = \Phi^\alpha_{(IJ); K_1K_2\ldots K_pK_{p'}}\), \(p \geq 0\), be the spinorial variables determined by the equations

\[
\sigma_{I'}^j \sigma_{JJ'}^k \sigma_{K_1K_2'\ldots K_pK_{p'}} \nabla_{k_1} \cdot \cdots \nabla_{k_p} F_{I}^\alpha = \epsilon_{I'J'} \Phi^\alpha_{IJ; K_1K_2\ldots K_pK_{p'}} + \epsilon_{IJ} \Phi^\alpha_{IJ; K_1K_2\ldots K_pK_{p'}}.
\]

Then, in spinor form, the Yang-Mills equations (2.10) reduce to

\[
\nabla_{I'} \Phi_{IJ} = 0,
\]

while the determining equations for symmetries (2.12) become

\[
\nabla_{I'} Q_{JJ'} - \nabla_{JJ'} Q_{I'I'} - [\Phi_I', Q_{I'I'}] - [\Phi_{I'}, Q_{I'I'}] = 0 \quad \text{on } \mathcal{R}^p.
\]

Next define symmetrized variables \(a^\alpha_{i_1i_2\ldots i_{p+1}}\), \(\Phi^\alpha_{IJK_1K_2\ldots K_pK_{p'}}\), \(p \geq 0\), by

\[
a^\alpha_{i_1i_2\ldots i_{p+1}} = a^\alpha_{(i_1i_2\ldots i_{p+1})}, \quad \Phi^\alpha_{IJK_1K_2\ldots K_pK_{p'}} = \Phi^\alpha_{(IJK_1K_2\ldots K_p)}
\]

where round brackets indicate symmetrization in the enclosed indices.

In order to avoid excessive proliferation of indices we will streamline our notation by employing multi-indices of integers to designate groups of indices in which an object is symmetric. In the case of the space-time indices, we denote multi-indices by boldface lower case letters, and in the case of spinorial indices, by boldface capital letters. Hence, for example, \(i_p = (i_1, i_2, \ldots, i_p)\), where each \(i_j\) is either 0, 1, 2, or 3, and \(K_p' = (K_1', K_2', \ldots, K_p')\), where each \(K_j'\) is either 0 or 1. We also combine multi-indices by the rule \(i_{p+1} = i_p + 1\). Accordingly, we write

\[
a^\alpha_{i_p} = a^\alpha_{i_1i_2\ldots i_p}, \quad \Phi^\alpha_{K_{p+2}K_{p+1}K_{p+1}\ldots K_{p+1}} = \Phi^\alpha_{K_{p+2}K_{p+1}K_{p+1}\ldots K_{p+1}}
\]

We will, moreover, collectively designate the variables \(a^{[p]}\), \(\Phi^{[p]}\) by \(\partial^p a\), \(\partial^p \Phi\), and we let \(a^{[p]}\), \(\Phi^{[p]}\) stand for the variables \(\partial^q a\), \(\partial^q \Phi\), \(0 \leq q \leq p\). Note that the variables \(\partial^p \Phi\) are of order \(p + 1\).

The proof of the following result appears in [14].

**Proposition 2.5.** The variables

\[
x^j, \quad a^{[p+2]}, \quad \Phi^{[p+1]},
\]

form a coordinate system on the \(p\)-fold prolonged solution manifold \(\mathcal{R}^p\), \(p \geq 0\).
Thus, in particular, any symmetry of the Yang-Mills equations of order \( p \geq 1 \) is equivalent to one depending on the variables \( x^i, a^{[p]}, \Phi^{[p-1]} \) only.

The following result is pivotal in our analysis of symmetries of the Yang-Mills equations. The proof of the first part of the Proposition is based on standard index manipulations and will be omitted. The second part, however, relies on a lengthy computation and we will therefore defer its proof to the Appendix.

**Proposition 2.6.** (i) The derivative \( D_{i_{p+1}} a^\alpha_{i_{p+1}} \) of the symmetrized variable \( a^\alpha_{i_{p+1}} \) satisfies the equation

\[
D_{i_{p+1}} a^\alpha_{i_{p+1}} = a^\alpha_{i_{p+1}} - \frac{1}{p+1} \nabla (i_1 \cdots \nabla_{i_{p+1}} a^\alpha_{i_{p+1}} + b^\alpha_{i_{p+1}}),
\]

where the functions \( b^\alpha_{i_{p+1}} \) are of order \( p - 1 \).

(ii) When restricted to the solution manifold \( \mathcal{R}^p \), the covariant derivative \( \nabla_{K_{p+3}} K^\alpha_{p+2} \) of the symmetrized variable \( \Phi^\alpha_{K_{p+2}} \), \( p \geq 1 \), satisfies the equation

\[
\nabla_{K_{p+3}} K^\alpha_{p+2} = \Phi^\alpha_{K_{p+2}} + \frac{p^2 + p - 2}{2(p+1)} \nabla (i_1 \cdots \nabla_{i_{p+1}} (K^\alpha_{K_{p+2}}) + \frac{3(1 - \delta_{p1})}{p+3} \nabla_{K_{p+3}} [\Phi^\alpha_{K_{p+2}}] + \frac{p(p+2)}{2(p+3)} \nabla_{K_{p+3}} [\Phi^\alpha_{K_{p+2}}] + \frac{(p+1)(p+2)}{p+3} \nabla_{K_{p+3}} [\Phi^\alpha_{K_{p+2}}] + \frac{p(p-1)}{p+1} \nabla_{K_{p+3}} [\Phi^\alpha_{K_{p+2}}])
\]

where \( \Psi^\alpha_{K_{p+3}} = \Psi^\alpha_{K_{p+2}} \) only depend on the variables \( \Phi^{[p-2]} \). Moreover, on \( \mathcal{R}^{[p+1]} \), we have that

\[
\nabla^2 \Phi ^\alpha_{K_{p+2}} = 2(p+2 - \delta_{p0}) \nabla [\Phi_S (K_{p+2}) + \Phi^S K^\alpha_{K_{p+2}}] + 2p \nabla [\Phi^S K^\alpha_{K_{p+2}}] + \Psi^\alpha_{K_{p+2}},
\]

where \( \Psi^\alpha_{K_{p+2}} = \Psi^\alpha_{K_{p+2}} \) only depend on the variables \( \Phi^{[p-1]} \).

Next write

\[
u^\alpha = \Re \Phi^\alpha K^\alpha_{K_{p+2}}, \quad \nu^\alpha = \Im \Phi^\alpha K^\alpha_{K_{p+2}}
\]

for the real and imaginary parts of the symmetrized variables \( \Phi^\alpha K^\alpha_{K_{p+2}} \). Suppose that a differential function \( G \) only depends on the variables \( x^i, a^{[p]}, u^{[p-1]}, v^{[p-1]} \). We can always assume that this is the case with a symmetry of the Yang-Mills equations of order \( p \). We write

\[
\partial_{a^\alpha} G, \quad \partial_{a^\alpha} K_{q+2} G, \quad \partial_{a^\alpha} K_{q+2} G,
\]

for the weighted partial derivatives of \( G \) with respect to the variables \( a^\alpha, u^\alpha K^\alpha_{K_{q+2}}, v^\alpha K^\alpha_{K_{q+2}} \). Define

\[
\partial_{a^\alpha} K_{q+2} = \frac{1}{2} (\partial_{a^\alpha} K_{q+2} - i \partial_{a^\alpha} K_{q+2}), \quad \partial_{a^\alpha} K_{q+2} = \partial_{a^\alpha} K_{q+2}
\]
where \( \iota \) is the imaginary unit. Thus, in particular,
\[
\frac{\partial_{i}}{\partial_{j}} q_{\beta} = \begin{cases} 
\delta_{j}^{i} \delta_{j_{1}}^{i_{1}} \cdots \delta_{j_{q}}^{i_{q}}, & \text{if } q = r, \\
0, & \text{if } q \neq r;
\end{cases}
\]
\[(2.21) \quad \partial_{\Phi}^{3} \frac{Q_{\alpha}}{Q_{\beta}} = \left. \frac{\partial_{\Phi}}{\partial_{\alpha}} \right|^{3} \frac{Q_{\alpha}}{Q_{\beta}} = \left. \frac{\partial_{\Phi}}{\partial_{\alpha}} \right|^{3} \frac{Q_{\alpha}}{Q_{\beta}} = 0.
\]

### 3. Generalized symmetries of the Yang-Mills equations

In this Section we prove Theorem 1.1 by completely classifying generalized symmetries of the Yang-Mills equations. Let
\[
\bar{Q}_{i}^{\alpha} = \bar{Q}_{i}^{\alpha}(x^{j}, a^{[p]}, \Phi^{[p-1]}),
\]
be the components of a generalized symmetry \( \bar{Q} \) of order \( p \) of the equations, which, without loss of generality, we assume to be a function of \( x^{j} \) and the symmetrized variables \( a^{[p]}, \Phi^{[p-1]} \) only.

We start by analyzing the highest order terms in the symmetrized variables \( \partial^{a} \alpha \) in the determining equations for \( \bar{Q} \) to show that \( \bar{Q} \) is equivalent to a symmetry that does not depend on \( \partial^{a}, q \geq 1 \). The proof of the following Proposition is a straightforward computation and will be omitted.

**Proposition 3.1.** Let \( T_{\alpha_{i}}^{(k_{1}\cdots k_{p})} \) be some constants satisfying the equations
\[
T_{\alpha_{i}}^{(k_{1}\cdots k_{p})} = T_{\alpha_{j}}^{(k_{1}\cdots k_{p})} a^{\alpha_{k_{1}\cdots k_{p+1}}}. \]

Then there are constants \( S_{\alpha}^{(k_{1}\cdots k_{p-1})} \) so that
\[
T_{\alpha_{i}}^{(k_{1}\cdots k_{p})} = \delta_{i}^{(k_{1}\cdots k_{p})} S_{\alpha}^{(k_{1}\cdots k_{p-1})}.
\]

**Lemma 3.2.** Let
\[
\tilde{Q}_{i}^{\alpha} = \tilde{Q}_{i}^{\alpha}(x^{j}, a^{[p]}, \Phi^{[p-1]}), \quad p \geq 0,
\]
be the components of a symmetry \( \tilde{Q} \) of the Yang-Mills equations of order \( p \). Then \( \tilde{Q} \) is equivalent to a symmetry \( Q \) with components \( Q_{i}^{\alpha} \) of the form
\[(3.1) \quad Q_{i}^{\alpha} = r_{i}^{\alpha}(x^{j}, a^{[p]}, \Phi^{[p-1]}).\]

**Proof.** Substitute \( \tilde{Q}_{i}^{\alpha} \) in the determining equations \((2.14)\) and collect the coefficients of the terms \( \partial^{a} \alpha \). On account of Proposition 2.6, this yields the equation
\[
(\partial_{a}^{k_{p+1}} \tilde{Q}_{i}^{\alpha}) a_{k_{p+1}}^{\beta} j - (\partial_{a}^{k_{p+1}} \tilde{Q}_{i}^{\alpha}) a_{k_{p+1}j}^{\beta} = 0.
\]

Thus, by Proposition 3.1, there are smooth differential functions \( q_{\beta}^{a_{k_{p}}} = q_{\beta}^{a_{k_{p}}} (x^{j}, a^{[p]}, \Phi^{[p-1]}) \) such that
\[(3.2) \quad \partial_{a}^{k_{p+1}} \tilde{Q}_{i}^{\alpha} = \delta_{i}^{(k_{p+1})} a_{k_{p+1}}^{\beta} q_{\beta}^{a_{k_{p}}}.\]

After differentiation, equation \((3.2)\) yields
\[(3.3) \quad \delta_{i}^{(k_{p+1})} \partial_{a}^{k_{p}} q_{\beta}^{a_{k_{p}}} = \delta_{i}^{(k_{p+1})} \partial_{a}^{k_{p+1}} q_{\beta}^{a_{k_{p}}}.\]

Multiply \((3.3)\) by \( X_{k_{p+1}}, Y_{k_{p+1}} \) to see that
\[
(\partial_{a}^{k_{p+1}} q_{\beta}^{a_{k_{p}}} - \partial_{a}^{k_{p+1}} q_{\beta}^{a_{k_{p}}}) X_{k_{p+1}} Y_{k_{p+1}} = (\partial_{a}^{k_{p+1}} q_{\beta}^{a_{k_{p}}}) X_{k_{p+1}} Y_{k_{p+1}} Y_{u},
\]
which implies that $\partial_{a_1}^{p+1} q_{\alpha, k_{p}} X_{k_{p}} Y_{i_{p+1}} = 0$ whenever $X_i, Y_i$ are linearly independent, and thus, by continuity,

$$\partial_{a_1}^{p+1} q_{\alpha, k_{p}} = 0.$$  

Hence $q_{\alpha, k_{p}}^{(\beta)} = q_{\beta}^{(\alpha, k_{p})}(x^j, a^{[p-1]}, \Phi^{[p-1]})$, and consequently, by (3.3), we have that

(3.4)  

$$\tilde{Q}_{i}^{\alpha} = q_{\beta}^{(\alpha, k_{p})}(x^j, a^{[p-1]}, \Phi^{[p-1]}) d_{\beta}^{\gamma} a_{i_{p+1}} + t_{i}^{p}(x^j, a^{[p-1]}, \Phi^{[p-1]}),$$

for some functions $t_{i}^{p}(x^j, a^{[p-1]}, \Phi^{[p-1]})$.

If $p = 0$, the above arguments show that (3.1) holds. Suppose that $p \geq 1$. Next we collect terms in the determining equations for $\tilde{Q}_{i}^{\alpha}$ involving products of $\partial^{a} a, \partial^{p+1} a$. Since $p \geq 1$, we see from Proposition 2.4 that these only arise from the derivatives of $Q_{i}^{\alpha}$ with respect to the variables $\partial^{p+1} a, \partial^{a} a$.

After some manipulations we obtain the equation

(3.5)

$$(\partial_{a_1}^{p} q_{\alpha, k_{p}}^{(\beta)} - \partial_{a_2}^{k_{p}} q_{\gamma}^{(\alpha, 1_{p})}) (a_{\beta}^{\gamma} a_{i_{p+1}} + a_{\gamma}^{\beta} a_{k_{p}, j}) = 0.$$  

Now apply the operators $A_{V, X}^{a}$ and $A_{W, Y}^{p+1}$, where

$$A_{V, X}^{a} = v^{a} X_{i} \partial_{a}^{k_{p}}$$

to the above equation to see that

$$(\partial_{a_1}^{p} q_{\alpha, k_{p}}^{(\beta)} - \partial_{a_2}^{k_{p}} q_{\gamma}^{(\alpha, 1_{p})}) (v^{a} w^{\gamma} X_{k_{p}} Y_{i_{p+1}} + v^{\gamma} w^{a} X_{k_{p}} Y_{k_{p}, j}) = 0.$$  

This implies that

(3.6)  

$$(\partial_{a_1}^{p} q_{\alpha, k_{p}}^{(\beta)} - \partial_{a_2}^{k_{p}} q_{\gamma}^{(\alpha, 1_{p})}) X_{k_{p}} Y_{i_{p+1}} = 0,$$

whenever $X_i, Y_i$ are linearly independent and $X_j Y_j \neq 0$. By continuity, equation (3.6) holds for all $X_i, Y_i$, and hence, the functions $q_{\alpha, k_{p}}^{(\beta)}$ satisfy the integrability conditions

(3.7)  

$$\partial_{a_1}^{p} q_{\alpha, k_{p}}^{(\beta)} = \partial_{a_2}^{k_{p}} q_{\gamma}^{(\alpha, 1_{p})}.$$  

Consequently, if we define a $\g$-valued differential function $X^{\alpha}$ by

$$X^{\alpha}(x^j, a^{[p-1]}, \Phi^{[p-1]}) = \int_{0}^{1} q_{\alpha, k_{p}}^{(\beta)}(x^j, a^{[p-2]}, t \partial^{p+1} a, \Phi^{[p-1]}) a_{k_{p}}^{\beta} dt,$$

then, by (3.4), (3.7), the difference $\tilde{Q}_{i}^{\alpha} - \nabla_{i} X^{\alpha}$, when restricted to $\tilde{R}^{[p-1]}$, only involves the variables $\partial^{\beta} a$ up to order $p - 1$. Thus $\tilde{Q}$ is equivalent to a symmetry $\tilde{Q}$ with components

$$\tilde{Q}_{i}^{\alpha} = \tilde{Q}_{i}^{\alpha}(x^j, a^{[p-1]}, \Phi^{[p]}).$$

Now one can inductively repeat the above argument to conclude that $\tilde{Q}$ is equivalent to a symmetry $\tilde{Q}$ with components $Q_{i}^{\alpha}$ as in (3.1).  

Next suppose that we have a symmetry $\tilde{Q}$ with components $\tilde{Q}_{i}^{\alpha}$, given by

(3.8)  

$$\tilde{Q}_{i}^{\alpha} = \tilde{Q}_{i}^{(\alpha, 1_{p})}(x^j, \Phi^{[p]}) + \tilde{Q}_{i}^{(\alpha, k_{p})}(x^j, \Phi^{[p]}) a_{k_{p}}^{\beta}, \quad p \geq 1.$$  

In the following Lemma we analyze terms in the determining equations for $\tilde{Q}$ involving the variables $\partial^{p+1} \Phi, \partial^{p+2} \Phi$. We use the symbol c.c. to denote the complex conjugates of the terms preceding it in an expression.
Lemma 3.3. Let \( \tilde{Q} \) be a symmetry of the Yang-Mills equations with components \( \tilde{Q}_{IJ}^\alpha \) as in (3.8). Then \( \tilde{Q} \) is equivalent to a symmetry \( Q_{IJ}^\alpha \) given by

\[
Q_{IJ}^\alpha = s^p_{\beta} K_{IJ}^{p+1} (x^j) \Phi_\beta^J K_{p+1} + t^p_{\beta} K_{IJ}^{p+2} (x^j) \Phi_\beta^J K_{p+2} +
\]

(3.9)

\[
u_\beta K_{IJ}^{p+1} (x^j) \Phi_\beta^J K_{p+1} + c.c. + q_\beta (x^j, \Phi^{[p-2]}) a_\beta^I + v_\beta (x^j, \Phi^{[p-1]}),
\]

where \( s^p_{\beta} K_{p+1} \), \( t^p_{\beta} K_{p+2} \) are Killing spinors and satisfy the equations

(3.10)

\[c_\beta \gamma K_{p+1} + c_\beta \gamma K_{p+1} c_\gamma = 0, \quad c_\beta \gamma K_{p+2} + c_\beta \gamma K_{p+2} c_\gamma = 0,
\]

\[w_\beta K_{p+2} \]

are given by

(3.11)

\[w_\beta K_{p+2} = \left\{ \begin{array}{l}
\frac{p-2}{p+2} q_\beta (K_{p+2} [\alpha] K_{p+1}^\rho) \Phi^J K_{p+1}^\rho + 
\frac{p+2}{p+4} \partial_{\rho} (K_{p+2} [\alpha] K_{p+1}^\rho)
\end{array} \right.
\]

and where

(3.12)

\[\partial_\phi (K_{p+1}^\rho [\alpha] K_{p+2}^\rho) = 0, \quad \partial_{\rho} (K_{p+2}^\rho [\alpha] K_{p+1}^\rho) = 0.
\]

Proof. We substitute \( \tilde{Q}_{IJ}^\alpha \) in (3.8) into the determining equations (2.10) and use Proposition 2.6 to conclude that in the resulting equations terms involving the variables \( \partial^{p+2} \Phi \) yield the expression

\[
(\partial_\phi K_{p+2}^\rho [\alpha] K_{p+1}^\rho) \Phi^J K_{p+1}^\rho + (\partial_\phi K_{p+2}^\rho [\alpha] K_{p+1}^\rho) \Phi^J K_{p+1}^\rho +
\]

(3.13)

which must vanish. Hence we have that

Next we collect terms quadratic in the variables \( \partial^{p+1} \Phi \), \( \partial^{p+1} \Phi \) in the determining equations for \( \tilde{Q} \). By (3.13) these only arise from the term \( \tilde{Q}_{IJ}^\alpha \) in \( \tilde{Q}_{IJ}^\alpha \). Note that the derivative \( \nabla^J \nabla \tilde{Q}_{IJ}^\alpha \) yields the following quadratic terms of order \( p+2 \),

\[((\partial_\phi K_{p+2}^\rho [\alpha] K_{p+1}^\rho) \Phi^J K_{p+1}^\rho + (\partial_\phi K_{p+2}^\rho [\alpha] K_{p+1}^\rho) \Phi^J K_{p+1}^\rho +
\]

(3.14)

all of which vanish due to (3.13). Thus the terms quadratic in \( \partial^{p+1} \Phi \), \( \partial^{p+1} \Phi \) in the determining equations for \( \tilde{Q} \) yield the equation

(3.15)

from which it follows that

(3.16)

...
that is, the functions $\tilde{Q}^{\alpha}_{II'}$ are linear in the highest order field variables. Hence by (3.13), (3.14), the components $\tilde{Q}^{\alpha}_{II'}$ reduce to

$$
\tilde{Q}^{\alpha}_{II'} = s_{\beta}^{\alpha}K_{p+1}^{\beta} (x^j, \Phi^{[p-1]}_\beta) \Phi^{\beta K'_{p+2}}_{II} + t_{\beta}^{\alpha}K^{\beta}_{p+1} (x^j, \Phi^{[p-1]}_\beta) \Phi^{\beta K'_{p+2}}_{II'} +
$$

(3.15)

where $s_{\beta}^{\alpha}K_{p+1}$, $t_{\beta}^{\alpha}K_{p+2}$, $\tilde{v}_{\alpha}^{\beta}$ are symmetric in their spinorial indices.

We next analyze terms linear in the variables $\partial^{p+1} \Phi$ in the determining equations for $\tilde{Q}$. Write (3.13) as

$$
\tilde{Q}^{\alpha}_{II'} = r_{\beta}^{\alpha} (I^{p} K_{p+2}) = 0.
$$

so that

(3.16)

For example, we use (2.3) to compute

$$
\nabla J^J \nabla I^J ' (\alpha K_{p+2}^{\beta} J^{\beta K'_{p+2}}_{II}) = r_{\beta}^{\alpha} (I^{p} K_{p+2}) \nabla J^J \nabla I^J ' (\beta K^{\beta K'_{p+2}}_{II}) +
$$

$$
(3.17)
$$

(3.18)

where $\overline{\Upsilon}^{\alpha}_{II}$, $\overline{\Upsilon}^{\alpha}_{II'}$ are of order $p + 1$.

Recall that $p \geq 1$. Apply the operator $B_{w,X,Y}$, where

$$
(3.19)
$$

to the determining equations for $\tilde{Q}$. With the help of equations (2.3), (2.13), (2.21), (3.17), (3.18) we conclude that in the resulting equations the terms not involving the variables $a_{II'}^{\alpha}$ yield the equation

$$
(3.20)
$$

where we have factored out $w^\gamma$. First multiply equation (3.20) by $X^I$ and sum over $I$ and then multiply equation (3.20) by $Y^{I'}$ and sum over $I'$ to get the equations

$$
D_{(J')^{\alpha} (\beta I^{p+2})} = 0, \quad D_{(J')^{\alpha} (\beta I^{p+2})} = 0.
$$
Thus
\[ D^{(j)}_{s\alpha}K_{p+1} = 0, \quad D^{(j)}_{s\alpha}K_{p+3} = 0. \]

It is easy to see that equations (3.21) imply that the coefficients \( s_{\beta}^{\alpha}K_{p+1}^{p+1} = t_{\beta}^{\alpha}K_{p+1}^{p+1} = t_{\beta}^{\alpha}K_{p+3}^{p+3} \) are, in fact, functions of \( x^j \) only, and hence that they are Killing spinors of type \((p, p)\) and \((p + 3, p - 1)\), respectively.

By virtue of (3.21) equations (3.20) simplify to
\[ D_{J\alpha}(u_{\gamma}^{\alpha}K_{p+1}^{p+1} - w_{\gamma}^{\alpha}K_{p+1}^{p+1} + \partial_{\gamma}^{\alpha}K_{p+1}^{p+1})X_{K_{p+1}^{p+1}}Y_{K_{p+1}^{p+1}} = 0, \]
where \( w_{\beta}^{\alpha}K_{p+2} \) is as in (3.11). The above equations in turn imply that
\[ \partial_{\gamma}^{\alpha}K_{p+1}^{p+1} - w_{\gamma}^{\alpha}K_{p+1}^{p+1} + \partial_{\gamma}^{\alpha}K_{p+1}^{p+1} = 0. \]

Let
\[ \mathcal{X}^{\alpha} = \int_0^1 u_{\beta}^{\alpha}K_{p+1}^{p+1}(x^j, \Phi^{[p-2]}, t\partial^{p-1}\Phi)\Phi^{\beta}K_{p+1}^{p+1} dt + c.c. \]
By (3.15), (3.23), we can subtract the generalized gauge symmetry with components \( \nabla_{II}^{\alpha} \mathcal{X}^{\alpha} \) from \( Q \) to get an equivalent symmetry \( Q^{a}_{II} \) given by
\[ Q^{a}_{II} = s_{\beta}^{\alpha}K_{p+1}^{p+1} \Phi^{\beta}K_{p+1}^{p+1} + t_{\beta}^{\alpha}K_{p+2}^{p+2} \Phi^{\beta}K_{p+2}^{p+2} + c.c. + q_{\beta}^{a}(x^j, \Phi^{[p-1]}), \]

Substitute \( Q \) in the determining equations and apply the operator \( B^{p+1}_{w,x} \) to the resulting expression. Steps analogous to those leading to (3.22) now yield the equation
\[ (u_{\gamma}^{\alpha}K_{p+1}^{p+1} - \partial_{\gamma}^{\alpha}K_{p+1}^{p+1})X_{K_{p+1}^{p+1}}Y_{K_{p+1}^{p+1}} = 0. \]
This together with its complex conjugate equation imply that
\[ \partial_{\gamma}^{\alpha}(K_{p+1}^{p+1}v^{\alpha}K_{p+2}^{p+2}) = \Phi^{\beta}K_{p+1}^{p+1}v^{\alpha}K_{p+2}^{p+2} + \Phi^{\beta}K_{p+1}^{p+1}. \]

Hence if we write
\[ v^{a}_{II} = v^{a}_{II} + u_{\beta}^{\alpha}K_{p+1}^{p+1} \Phi^{\beta}K_{p+1}^{p+1} + t_{\beta}^{\alpha}K_{p+2}^{p+2} \Phi^{\beta}K_{p+2}^{p+2} + \]
we have that
\[ Q^{a}_{II} = s_{\beta}^{\alpha}K_{p+1}^{p+1}(x^j)\Phi^{\beta}K_{p+1}^{p+1} + t_{\beta}^{\alpha}K_{p+2}^{p+2} \Phi^{\beta}K_{p+2}^{p+2} + \]
\[ u_{\beta}^{\alpha}K_{p+1}^{p+1} \Phi^{\beta}K_{p+2}^{p+2} + c.c. + q_{\beta}^{a}(x^j, \Phi^{[p-1]}), \]
where
\[ \partial_{\gamma}^{\alpha}(K_{p+1}^{p+1}v^{\alpha}K_{p+2}^{p+2}) = 0, \quad \partial_{\gamma}^{\alpha}(K_{p+1}^{p+1}v^{\alpha}K_{p+2}^{p+2}) = 0. \]
Thus, in particular, equations (3.12) hold.

We still need to show that the coefficient functions \( s_{\beta}^{\alpha}K_{p+1}^{p+1}, t_{\beta}^{\alpha}K_{p+3}^{p+3} \) satisfy equation (3.10) and that \( q_{\beta}^{a} \) does not depend on the variables \( \partial^{p-1}\Phi \). For this we first use equations (3.17), (3.18)
together with (3.16) to conclude that terms involving the variable \( a^{\gamma j} j' \) in the equations obtained by applying the operator \( B^{p+1}_{w1} X_{Y} \) to the determining equations for \( Q \) yield the equation

\[
(\partial_{\alpha} K^p_{I_p+1} q^\alpha - (c^\beta_{\delta} r^\beta_{J J'}) K^p_{I_p+2} X_{Y} - r^\alpha_{J J'} K^p_{I_p+2} c^\beta_{\gamma \delta} X_{Y}) - 2r^\alpha_{J J'} K^p_{I_p+2} c^\beta_{\gamma \delta} X_{Y} - 2i^\alpha_{J J'} K^p_{I_p+2} c^\beta_{\gamma \delta} X_{Y} Y^J K_I = 0.
\]

(3.25)

First multiply (3.25) by \( X^I \) and sum over \( I \) and then multiply (3.25) by \( Y^{J'} \) and sum over \( I' \) to derive the equations

\[
e^\alpha_{\beta} r^\beta_{I I'} K_I^p + r^\alpha_{I I'} K_I^p c_\gamma^\delta = 0, \quad e^\alpha_{\beta} r^\beta_{I I'} (J') K_{I'}^{p+2} + r^\alpha_{I I'} (J') K_{I'}^{p+2} c_\gamma^\delta = 0,
\]

that is,

\[
e^\alpha_{\beta} r^\beta_{I I'} K_I^p + s^\alpha_{I I'} K_{I+1}^{p+1} c_\gamma^\delta = 0, \quad e^\alpha_{\beta} r^\beta_{I I'} (J') K_{I'}^{p+2} + t^\alpha_{I I'} (J') K_{I'}^{p+2} c_\gamma^\delta = 0.
\]

(3.26)

Hence (3.10) holds. Recall that by (3.24), the contraction \( r^\alpha_{\beta} K_{I+2}^{p+2} K_{I+1}^{p+2} \) vanishes. Thus with (3.26), equation (3.25) and its complex conjugate equation furthermore imply that

\[
q^\alpha_{\beta} = q^\alpha_{\beta} (x^I, \Phi[p-2]).
\]

This concludes the proof of Lemma 3.3.

Next let \( \tilde{Q} \) be a symmetry with components \( \tilde{Q}_{I I'} \) of the form

\[
\tilde{Q}_{I I'} = s^\alpha_{\beta} K_{I+1}^{p+1} (x^I) \Phi^\beta K_{I+1}^p + r^\alpha_{\beta} K_{I+2}^{p+2} (x^I) \Phi^\beta K_{I+2}^p + u^\alpha_{\beta} K_{I+2}^{p+3} (x^I) \Phi^\beta K_{I+3}^p + v^\alpha_{\beta} (x^I, \Phi[p-1]), \quad p \geq 1,
\]

(3.27)

where the functions \( s^\alpha_{\beta} K_{I+1}^{p+1}, r^\alpha_{\beta} K_{I+2}^{p+2}, u^\alpha_{\beta} K_{I+2}^{p+3}, v^\alpha_{\beta} \), satisfy (3.10), (3.11), (3.12). We now analyze terms involving \( \partial^\alpha \Phi \) in the determining equations for \( \tilde{Q} \).

**Lemma 3.4.** Let \( \tilde{Q} \) be a symmetry of the Yang-Mills fields with components \( \tilde{Q}^a_{I I'} \) as in (3.27). Then \( \tilde{Q} \) is equivalent to a symmetry \( Q^{a}_{I I'} \) given by

\[
Q^a_{I I'} = s^\alpha_{\beta} K_{I+1}^{p+1} (x^I) \Phi^\beta K_{I+1}^p + r^\alpha_{\beta} K_{I+2}^{p+2} (x^I) \Phi^\beta K_{I+2}^p + u^\alpha_{\beta} K_{I+2}^{p+3} (x^I) \Phi^\beta K_{I+3}^p + v^\alpha_{\beta} (x^I, \Phi[p-1]), \quad p \geq 1,
\]

(3.28)

where the functions \( s^\alpha_{\beta} K_{I+1}^{p+1}, r^\alpha_{\beta} K_{I+2}^{p+2}, u^\alpha_{\beta} K_{I+2}^{p+3} \) are symmetric in their spinorial indices.

**Proof.** Define \( r^\alpha_{\beta} K_{I+3}^{p+3} \) by

\[
r^\alpha_{\beta} K_{I+3}^{p+3} = s^\alpha_{\beta} K_{I+1}^{p+1} \Phi_{I+1}^\beta + r^\alpha_{\beta} K_{I+2}^{p+2} \Phi_{I+2}^\beta + u^\alpha_{\beta} K_{I+2}^{p+3} \Phi_{I+3}^\beta.
\]

By the proof of Lemma 3.3 conditions (3.10), (3.11), (3.12) guarantee that the highest order terms in the determining equations for \( \tilde{Q} \) are \( \partial^\alpha \Phi \). Note that due to condition (3.10), we can apply
equation (2.3) in Proposition 2.1 in computing covariant derivatives of the term $\gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2}$. Thus, when substituted in the determining equations (2.16), it yields the expression

$$
(\partial I_p \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} - \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} + \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} ) \nabla I_p \Phi K_{p}^{'} K_{p}^{+2} +
$$

(3.29)

$$
(\partial J_p \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} - \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} + \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} ) \nabla I_p \Phi K_{p}^{'} K_{p}^{+2} +
$$

Now by (2.19), it is clear that (3.29) gives rise to terms of order $p + 1$ that are either linear expressions in $\partial^p \Phi$, $\partial^p \Phi'$ or in the products of either $\Phi$, $\Phi'$ with $\partial^p \Phi$ or $\partial^p \Phi'$ with coefficients that are functions of $x^\gamma$ only. Thus, since $p \geq 1$, the only terms in the determining equations for $Q_p$ involving quadratic expressions in $\partial^p \Phi$, $\partial^p \Phi'$ arise from the function $v^\gamma_{J_p}$. Hence, steps identical to those leading to equation (3.14) in the proof of Lemma 3.3 show that $v^\gamma_{J_p}$ must be linear in the variables $\partial^{-1} \Phi$ with coefficients that are functions of the variables $x^\gamma$, $\Phi[\gamma]$. In particular, if $p = 1$, it follows that equation (3.28) holds with $\tilde{v}^\gamma_{J_p} = 0$.

Now assume that $p \geq 2$. Write

$$
\gamma_p^{\alpha}(L_{\beta}^{K_{p}^{+2}}) (x^j) \Phi K_{p}^{'} K_{p}^{+2} + \gamma_p^{\alpha}(L_{\beta}^{K_{p}^{+2}}) (x^j) \Phi K_{p}^{'} K_{p}^{+2} = \gamma_p^{\alpha}(L_{\beta}^{K_{p}^{+2}}) (x^j) \Phi K_{p}^{'} K_{p}^{+2}
$$

so that $\gamma_p^{\alpha}(L_{\beta}^{K_{p}^{+2}}) (x^j) \Phi K_{p}^{'} K_{p}^{+2} = \gamma_p^{\alpha}(L_{\beta}^{K_{p}^{+2}}) (x^j) \Phi K_{p}^{'} K_{p}^{+2}$. By virtue of Proposition 2.6 we have that

$$
\nabla_{K_{p}^{+2}} K_{p}^{+2} \Phi K_{p}^{+2} = \Phi K_{p}^{+2} + G_{\alpha} K_{p}^{+2} K_{p}^{+2} L_{p}^{+2} + H_{\beta} K_{p}^{+2} K_{p}^{+2} L_{p}^{+2}
$$

where $G_{\alpha} K_{p}^{+2} K_{p}^{+2} L_{p}^{+2}$ and $H_{\beta} K_{p}^{+2} K_{p}^{+2} L_{p}^{+2}$ are constant coefficients in the field variables $\Phi_{J_p}^{\alpha}$, $\Phi_{J_p}^{\alpha}$, and $\Phi_{K_{p}^{+2}}^{\alpha}$ when restricted to the solution manifold $R^p$, are functions of the variables $\Phi[p-1]$ only. Now apply the operator $B^\gamma_{\nu,\chi,\gamma} X_{\gamma,\chi}$ to the determining equations for $Q$ and factor out $v^\gamma$ to see that on $R^p$,

$$
(\partial J_p \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} - \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} + \gamma_p^{\alpha} L_{\beta}^{K_{p}^{+2}} \Phi K_{p}^{'} K_{p}^{+2} ) \nabla I_p \Phi K_{p}^{'} K_{p}^{+2} +
$$

(3.30)
the terms on lines 4 and 5 in (3.30). Specifically, these terms yield the equation
\[ \alpha^s c(p_J J') X_{1} Y_{p+1} + (\partial_{\Phi} \kappa_{p-2}^{\beta} \gamma J' L'_{p-1}^{l}) \Phi_{\beta} L'_{p+1}^{l} = 0. \]

where the terms \( S_{\alpha} \delta \gamma J' L'_{p+2}^{l} \kappa_{p+2} \) are determined by the equation
\[ \partial_{\Phi} \gamma J' L'_{p+2}^{l} = \frac{\partial_{\Phi} \gamma J' L'_{p+2}^{l}}{\partial_{\Phi} \delta \gamma J' L'_{p+2}^{l}}. \]

Our next goal is to show that equation (3.30) forces the symmetrizations \( \bar{\gamma} \) to be functions of \( x^j \) only. Suppose on the contrary that \( \bar{\gamma} \) depend on the variables \( \Phi[q] \) for some \( 0 \leq q \leq p-2 \). Multiply (3.30) by \( X_j \) and sum over \( I \). In the resulting equations the only terms involving the variables \( \beta \) arise from the total derivative term \( D J' \bar{\gamma}^{l} \kappa_{p+1} \). Hence upon an application of the operator \( B_{w,l}^{+1} \) we get the equation
\[ (\partial_{\Phi} \gamma_{j+1} L_{q+2}^{p+1} \kappa_{p+1}^{l} X_{j+1} Y_{p+1}^{l} - \Phi_{p+1}^{l} Z_{j}^{p+1} W_{q+2}^{l} = 0, \]
from which it follows that
\[ \partial_{\Phi} \gamma_{j+1} L_{q+2}^{p+1} \kappa_{p+1}^{l} = 0. \]

One can similarly show that
\[ \partial_{\Phi} \gamma_{j+1} L_{q+2}^{p+1} \kappa_{p+1}^{l} = 0, \]
which contradicts our assumption. Hence we have that
\[ \bar{\gamma} \frac{\partial_{\Phi} \kappa_{p+1}^{l} \kappa_{p+1}^{l}}{\partial_{\Phi} \delta \kappa_{p+1}^{l}} = \frac{\partial_{\Phi} \gamma_{j+1} L_{q+2}^{p+1} \kappa_{p+1}^{l} \kappa_{p+1}^{l}}{\partial_{\Phi} \delta \kappa_{p+1}^{l}} = 0, \]
which yields the result.

Since \( p \geq 2 \), the only terms in the resulting equations involving the variables \( \beta \) arise from the terms on lines 4 and 5 in (3.30). Specifically, these terms yield the equation
\[ \left( \partial_{\Phi} \gamma_{p+2}^{l} \gamma_{p+1}^{l} L_{p+2}^{l} \Phi_{p+1}^{l} \right) + \left( \partial_{\Phi} \gamma_{p+2}^{l} \gamma_{p+1}^{l} L_{p+2}^{l} \Phi_{p+1}^{l} \right) = 0, \]

where \( \bar{\gamma} \), \( \bar{\gamma} \), \( \bar{\gamma} \), \( \bar{\gamma} \) are symmetric in their spinorial indices. Now substitute (3.31) into (3.30).
which implies that
\[ \partial_{\Phi^\gamma} L^p_{\rho-2} \hat{u}_{\beta}^\gamma K_{\rho-2}^p = \partial_{\Phi^\gamma} K_{\rho-2}^p \hat{u}_{\beta}^\gamma L^p_{\rho-2}, \quad \bar{\hat{u}}_{\gamma}^\gamma L^p_{\rho-2} \hat{u}_{\beta}^\gamma K_{\rho-2}^p = \partial_{\Phi^\gamma} K_{\rho-2}^p \bar{\hat{u}}_{\beta}^\gamma L^p_{\rho-2}. \]

Thus we can subtract a gauge symmetry from \( \bar{\hat{Q}} \) to obtain a symmetry of the required form (3.28). This concludes the proof of Lemma 3.4.

We now come to the crucial step in the proof of Theorem 1.3, which consists of analyzing terms in the determining equations for a symmetry \( Q \) in (3.28) involving products of the variables \( \Phi \) and \( \partial^\beta \Phi \).

**Lemma 3.5.** Let \( Q \) be a symmetry of the Yang-Mills equations with components \( Q_{IL}^\alpha \) as in (3.28) with \( p \geq 1 \). Then
\[ s^\alpha_{\beta K_{p+1}} = 0, \quad t^\alpha_{\beta K_{p+1}} = 0. \]

**Proof.** Write \( \mathcal{E}_{II}^\alpha (\mathcal{E}_{II}^\beta) \) for the expression obtained by substituting differential functions \( \mathcal{E}_{II}^\beta \) on the left-hand side of the determining equations (2.16). Our goal is to analyze the equations
\[ (3.32) \]
\[ B^\alpha_{\nu, Z} B^\beta_{\nu, X, Y} \mathcal{E}_{II}^\alpha (Q_{II}^\beta) = 0, \]
where the operators \( B^\alpha_{\nu, Z}, B^\beta_{\nu, X, Y} \) are as in (3.19).

First note that the functions \( z_{II}^\alpha \), when substituted in the determining equations, yield the expression
\[ (3.33) \]
\[ B^\alpha_{\nu, Z} B^\beta_{\nu, X, Y} \mathcal{E}_{II}^\alpha (z_{II}^\beta) = (\partial_{\Phi^\gamma} K_{p-2}^\gamma \hat{u}_{\beta}^\gamma L^p_{\rho-2}) Z_{LM} X^I_{IKp} Y^J_{IKp} Y^J_{IKp} = 0. \]

One similarly shows that
\[ (3.34) \]
\[ B^\alpha_{\nu, Z} B^\beta_{\nu, X, Y} \mathcal{E}_{II}^\alpha (q_{II}^\beta) = (\partial_{\Phi^\gamma} K_{p-2}^\gamma \hat{u}_{\beta}^\gamma L^p_{\rho-2}) Z_{LM} X^I_{IKp} Y^J_{IKp} = 0. \]

Next note that the terms
\[ s^\alpha_{\beta K_{p+1}} = 0, \quad t^\alpha_{\beta K_{p+1}} = 0 \]
and their complex conjugate terms in (3.28), when substituted in the determining equations (2.16), only yield terms of order \( p + 1 \) that are linear in the variables \( \partial^\beta \Phi \) with coefficients that are functions of \( x^I \). Consequently, for \( \mathcal{E}_{II}^\alpha (T_{II}^\beta) \) any of the terms
\[ (3.35) \]
\[ B^\alpha_{\nu, Z} B^\beta_{\nu, X, Y} \mathcal{E}_{II}^\alpha (T_{II}^\beta) = 0. \]
Note that
\[
B_{w,Z}^\alpha B_{v,X,Y}^\alpha \varepsilon_{II'}(s_\beta^\alpha K_{p+1}^\beta \phi^\gamma K_p^{J K_{p+1}}) =
\]
(3.36)
\[
B_{w,Z}^\alpha B_{v,X,Y}^\alpha (s_\gamma^\beta_{J'K_p^\beta} \nabla^{JJ'} \nabla_{II'} \phi^\beta_{J'K_p^\beta} - s_\alpha^\beta_{J'K_p^\beta} \nabla_{JJ'} \phi_{J'K_p^\beta} [\Phi^J_{II'}, \Phi^\gamma_{J'K_{p+1}}]) =
\]
and
\[
B_{w,Z}^\alpha B_{v,X,Y}^\alpha \varepsilon_{II'}(t_\gamma^\alpha_{JK_{p+1}^\beta} \phi^\beta K_p^{J'K_{p+1}}) =
\]
(3.37)
\[
B_{w,Z}^\alpha B_{v,X,Y}^\alpha (t_\beta^\gamma_{JK_{p+1}^\beta} \nabla^{JJ'} \nabla_{II'} \phi^\beta_{JK_{p+1}^\beta} - t_\alpha^\gamma_{JK_{p+1}^\beta} \nabla_{JJ'} \phi_{JK_{p+1}^\beta} [\Phi^J_{II'}, \Phi^\gamma_{JK_{p+1}^\beta}]).
\]
We use equation (2.13) to compute
\[
s_\beta^\alpha_{JK_{p+1}^\beta} \nabla^{JJ'} \nabla_{II'} \phi_{JK_{p+1}^\beta} = s_\beta^\alpha_{JK_{p+1}^\beta} \nabla_{II'} \phi_{JK_{p+1}^\beta} +
\]
(3.38)
\[
s_\beta^\alpha_{JK_{p+1}^\beta} [\Phi^J_{II'}, \Phi^\gamma_{JK_{p+1}^\beta}] = s_\beta^\alpha_{JK_{p+1}^\beta} [\Phi^J_{II'}, \Phi^\gamma_{JK_{p+1}^\beta}].
\]
But by equation (2.19),
\[
s_\beta^\alpha_{JK_{p+1}^\beta} \nabla^{JJ'} \nabla_{II'} \phi_{JK_{p+1}^\beta} =
\]
(3.39)
\[
-\frac{s_\beta^\alpha_{JK_{p+1}^\beta} \nabla_{II'} (p(p+5))}{2(p+2)} \left( \phi_{JK_{p+1}^\beta}^{\gamma J'K_{p+1}^\beta} \right) + \Psi_{2,II'} =
\]
\[
-\frac{s_\beta^\alpha_{JK_{p+1}^\beta} (p(p+5))}{2(p+2)} \left( \phi_{JK_{p+1}^\beta}^{\gamma J'K_{p+1}^\beta} \right) -
\]
\[
\Psi_{2,II'} =
\]
where \(\Psi_{2,II'}\), \(\Psi_{3,II'}\) involve variables of order at most \(p\). Moreover, by equation (2.20), we have that
\[
s_\beta^\alpha_{JK_{p+1}^\beta} \nabla^{JJ'} \nabla_{JJ'} \phi_{JK_{p+1}^\beta} = 2(p+2) s_\beta^\alpha_{JK_{p+1}^\beta} [\Phi^J_{II'}, \Phi^\gamma_{JK_{p+1}^\beta}]
\]
(3.40)
\[
2p s_\beta^\alpha_{JK_{p+1}^\beta} [\Phi^J_{II'}, \Phi^\gamma_{JK_{p+1}^\beta}] + \Psi_{4,II'}
\]
where again \(\Psi_{4,II'}\) is of order \(p\). It follows from equations (3.36), (3.38), (3.39), (3.40) that
\[
B_{w,Z}^\alpha B_{v,X,Y}^\alpha \varepsilon_{II'}(s_\gamma^\beta_{J'K_p^\beta} \phi^\gamma K_p^{J'K_{p+1}}) =
\]
(3.41)
\[-2(p+1) s_\gamma^\beta_{J'K_p^\beta} Z_{SK_{p+1}^\gamma} Y_{K_p^{J'K_{p+1}}} [w, v]].
\]
Next we compute

\[ t_0^{K_{p+2}} \nabla^J \nabla^I \Psi^J K_{p+1}^{-1} = t_0^{JK_{p+2}} \nabla^I \nabla^J \Psi^J K_{p+1}^{-1} + \]

\[ t_0^{JK_{p+2}} [\Phi_{IJ}, K_{p+1}^{-1}] + t_0^{IJK_{p+2}} [\nabla^J K_{p+1}^{-1}]\beta. \]

(3.42)

But by equation (2.10),

\[ t_0^{JK_{p+2}} \nabla^I \nabla^J \Phi^J K_{p+1}^{-1} = \]

\[ \frac{p^2 + p - 2}{2p} t_0^{JK_{p+2}} \nabla^I [\Phi_{JK_{p+2}}, K_{p+1}^{-1}] + \Psi_5^{K_{p+1}} = \]

\[ \frac{p^2 + p - 2}{2p} t_0^{JK_{p+2}} [\Phi_{JK_{p+2}}, K_{p+1}^{-1}] + \Psi_5^{K_{p+1}}, \]

where \( \Psi_5^{K_{p+1}} \) involve variables of order at most \( p \). Moreover, by equation (3.20), we have that

\[ t_0^{JK_{p+2}} \nabla^I \nabla^J \Phi^J K_{p+1}^{-1} = t_0^{JK_{p+2}} (2(p + 2) [\Phi_{SP_{K+2}}, K_{p+1}^{-1}]^2 + \]

\[ 2p [\Phi_{SP_{K+2}}, K_{p+1}^{-1}]^2 (K_{p+1}^{-1}) + \Psi_5^{K_{p+1}}, \]

where again \( \Psi_5^{K_{p+1}} \) is of order \( p \). It follows from equations (3.37), (3.42), (3.43), (3.44) that

\[ B_0^{w,z} B_{v,x}^p \nabla^\alpha_{II} (\nabla_{IK_{p+1}^{-1}} K_{p+1}^{-1}) = \]

\[ \left( \frac{p^2 + p - 2}{2p} t_0^{JK_{p+2}} Z_{K_{p+3}K_{p+2}} X_I + 2t_0^{JK_{p+2}} Z_{K_{p+3}K_{p+2}} X_{K_{p+2}} - \right) \]

\[ 2(p + 2) t_0^{JK_{p+2}} Z_{SK_{p+2}X} X_{K_{p+2}} Y_{K_{p+2}^{-1}} [w, v]^{\beta}. \]

One can easily verify that

\[ B_0^{w,z} B_{v,x}^p \nabla^\alpha_{II} (\nabla_{IK_{p+1}^{-1}} K_{p+1}^{-1}) = 0, \]

\[ B_0^{w,z} B_{v,x}^p \nabla^\alpha_{II} (\nabla_{IK_{p+1}^{-1}} K_{p+1}^{-1}) = 0. \]

Finally, by equations (3.33), (3.34), (3.35), (3.41), (3.42), (3.43), (3.44), we see that

\[ X^I B_0^{w,z} B_{v,x}^p \nabla^\alpha_{II} (Q_{JJK}) = \]

\[ 2(p + 3) t_0^{JK_{p+2}} Z_{K_{p+3}K_{p+2}} X_{K_{p+2}} Y_{K_{p+2}^{-1}} Z_{S} X_{S} [w, v]^{\beta} = 0, \]

\[ Y^I B_0^{w,z} B_{v,x}^p \nabla^\alpha_{II} (Q_{JJK}) = \]

\[ -2(p + 1) t_0^{JK_{p+2}} Z_{K_{p+3}K_{p+2}} X_{K_{p+2}} Y_{K_{p+2}^{-1}} Z_{S} X_{S} [w, v]^{\beta} = 0, \]

from which it immediately follows that

\[ s_0^{K_{p+1}} = 0, \quad t_0^{K_{p+3}} = 0. \]
as required.

**Proposition 3.6.** Let $\mathfrak{g}$ be a semi-simple Lie algebra with structure constants $c^\gamma_{\beta\gamma}$ in a basis $\{e_\alpha\}$. Suppose that $\varepsilon_\beta^\gamma = z_\beta^\gamma(x^i)$ are smooth functions satisfying
\[
\varepsilon_\beta^\gamma \varepsilon_\delta^\gamma + z_\beta^\gamma c^\gamma_\beta = 0.
\]
Then there is a $\mathfrak{g}$-valued smooth function $w^\alpha = w^\alpha(x^i)$ such that
\[
\varepsilon_\beta^\gamma = c^\gamma_\beta, w^\gamma.
\]
**Proof.** Let $Z = Z(x^j)$ be the $\text{End}(\mathfrak{g})$-valued function defined by
\[
Z(x^j)e_\alpha = z_\alpha^\beta(x^j)e_\beta.
\]
Then the assumptions imply that
\[
Z(x^j)[v_1, v_2] = [Z(x^j)v_1, v_2] + [v_1, Z(x^j)v_2],
\]
that is, $Z(x^j)$ is a derivation of $\mathfrak{g}$ for every $x^j$. Since $\mathfrak{g}$ is semi-simple, any derivation is inner, that is, there is a $\mathfrak{g}$-valued function $w^\alpha = w^\alpha(x^i)$ such that
\[
Z(x^j)v = [v, w(x^j)] \quad \text{for all } v \in \mathfrak{g}.
\]
This implies that
\[
w^\alpha(x^j) = -\kappa_\alpha^\beta c_\beta^\gamma z_\gamma^\gamma(x^i),
\]
where $\kappa_\alpha^\beta$ is the inverse of the Killing form $\kappa_\alpha^\beta = \kappa_\alpha^\beta, c_\beta^\gamma$. Thus $w^\alpha(x^j)$ is smooth.

**Proof of Theorem 1.1.** By applying Lemmas 3.2, ..., 3.3 repeatedly, we see that any generalized symmetry of the Yang-Mills equations is equivalent to a first order symmetry $\tilde{Q}$ with components $\tilde{Q}^\alpha_{I'J}$ of the form
\[
\tilde{Q}^\alpha_{I'J} = q_\alpha^\beta(x^j)a^\beta_{II} + s^\alpha_{I'J}(x^j)\Phi^\beta_{IJK} + \Phi^\beta_{I'J'}(x^j) + v^\alpha_{I'J}(x^j).
\]
On $\mathcal{R}$ we have that
\[
D_i(q_\beta^\alpha a^\beta_j) - D_j(q_\beta^\alpha a^\beta_i) = (\partial_i q_\beta^\alpha)a^\beta_j - (\partial_j q_\beta^\alpha)a^\beta_i + q_\beta^\gamma(F^\gamma_{ij} - c^\gamma_{\gamma\delta} a^\gamma_i a^\gamma_j),
\]
and
\[
D^j(D_i(q_\beta^\alpha a^\beta_j) - D_j(q_\beta^\alpha a^\beta_i)) = (\partial^j \partial_i q_\beta^\alpha)a^\beta_j - (\partial^j \partial_j q_\beta^\alpha)a^\beta_i +
\]
\[
(\partial_i q_\beta^\alpha)a^\beta_j - \partial^j q_\beta^\alpha(a^\gamma_j - \frac{3}{2} F^\gamma_{ij} + \frac{3}{2} c^\gamma_{\gamma\delta} a^\gamma_i a^\gamma_j) -
\]
\[
q_\beta^\gamma c^\gamma_\delta (a^\gamma j F^\gamma_{ij} + (a^\gamma_j - \frac{1}{2} F^\gamma_{ij} + \frac{1}{2} c^\gamma_{\gamma\delta} a^\gamma_i a^\gamma_j) a^\delta j + a^\gamma_i a^\delta j).
\]
Consequently, the only terms involving the variables $\partial^j \Phi$ in the determining equations for $\tilde{Q}$ arise from the term $s^\alpha_{I'J}(x^j)\Phi^\beta_{IJK}$ and its complex conjugate. Hence we can repeat the computations leading to equations (3.21), (3.26) in the proof of Lemma 3.3 to see that the $s^\alpha_{I'J}$. are Killing spinors satisfying
\[
s^\alpha_{I'J} \Phi^\beta_{IJK} + c^\gamma_\beta s^\beta_{I'J} = 0.
\]
Our next goal is to show that $s^\alpha_{I'J}$ must be real. For this, we analyze terms quadratic in the variables $\Phi, \overline{\Phi}$ in the determining equations for $\tilde{Q}$. These only arise from the second order covariant derivatives of $\Phi, \overline{\Phi}$ and from the bracket term in (2.16). Note that
\[
\nabla_{I'J} \Phi_{JK} = \Phi_{I'J'K} \quad \text{on } \mathcal{R}.
\]
We use the above equation and \((2.14)\) to compute
\[
(3.52) \quad s_{\beta,j}^\alpha K_j^l \nabla_j \nabla_I^j \Phi_{J,K} = -s_{\beta,j}^\alpha K_I^l \Phi_{J,K} + s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l - s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l
\]
\[
(3.53) \quad s_{\beta,j}^\alpha K_j^l \nabla_j \nabla_{J,K}^j \Phi_{J,K} = -2s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l
\]
on \(\mathcal{R}^1\). Moreover, on account of \((3.51)\), we have that
\[
(3.54) \quad c_{\beta,j}^\gamma K_j^l \nabla_j \nabla_{J,K}^j \Phi_{J,K}^l = s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l + s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l
\]
It follows from \((3.52)\), \((3.53)\), \((3.54)\) and their complex conjugate equations that the determining equations for \(Q\) yield the terms
\[
2s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l - 2s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l
\]
that is, \(s_{\beta,j}^\alpha K_I^l\) is real.

Next by \((3.51)\), for fixed \(K, K'\), the endomorphisms \(S_{K,K'}^I\) of \(G\) defined by \(S_{K,K'}^I(e_a) = s_{\alpha\beta}^\gamma K_J^l \Phi_{J,K}^l\) commute with the adjoint representation of \(G\). Write \(g = g_1 + \cdots + g_n\), where \(g_m \subset g\) is a simple ideal. By Proposition \(2.2\), \(S_{K,K'}^I\) leaves each ideal \(g_m\) invariant and the restriction \(S_{m,K'}^I\) of \(S_{K,K'}^I\) to \(g_m\) is either a multiple of an identity mapping or a linear combination of the identity mapping and the almost complex structure \(J_m\) of \(g_m\). It follows that the terms \(s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l = s_{\beta,j}^\alpha K_I^l \Phi_{J,K}^l\) in \(Q_{\mathcal{R}^1}\) agree with the components of a sum of the symmetries \(Q_m[e], Q_{J,m}[\gamma]\) defined in \((2.14)\). Thus we can subtract this sum from \(Q\) to get a symmetry \(Q\) with components
\[
Q_a^\alpha = q_a^\alpha(x^j) + v_a^\alpha(x^j).
\]
Our next goal is to show that \(Q\) must be a gauge symmetry. For this collect terms in the determining equations for \(Q\) involving the product \(a \partial^1 a\). With the help of \((3.50)\) we get the equation
\[-q_{\beta}^\alpha q_{\delta}^\gamma a_\gamma^\alpha a_\delta^\beta + q_{\beta}^\alpha q_{\delta}^\gamma a_\gamma^\alpha a_\delta^\beta - q_{\beta}^\alpha q_{\delta}^\gamma a_\gamma^\alpha a_\delta^\beta + q_{\beta}^\alpha q_{\delta}^\gamma a_\gamma^\alpha a_\delta^\beta = 0,
\]
which simplifies to
\[
(q_{\beta}^\alpha q_{\delta}^\gamma + q_{\delta}^\alpha q_{\beta}^\gamma - q_{\beta}^\alpha q_{\delta}^\gamma)(a_\gamma^\alpha a_\delta^\beta - a_\gamma^\beta a_\delta^\alpha) = 0.
\]
Hence we have
\[
q_{\beta}^\alpha q_{\delta}^\gamma + q_{\delta}^\alpha q_{\beta}^\gamma - q_{\beta}^\alpha q_{\delta}^\gamma = 0,
\]
which, by Proposition \(3.6\) implies that
\[
q_{\beta}^\alpha(x^j) = c_{\beta}^\alpha w^\gamma(x^j),
\]
for some \(w^\gamma = w^\gamma(x^j)\). Next collect terms involving \(\partial^1 a\) in the determining equations for \(Q\).
This yields the equation
\[
c_{\beta}^\gamma(\partial_j w^\gamma - v_j^\gamma)a_\beta^j - c_{\beta}^\gamma(\partial_j w^\gamma - v_j^\gamma)a_\beta^j = 0,
\]
from which it follows that
\[
v_j^\gamma = \partial_j w^\gamma.
\]
By \((3.54)\), \((3.56)\), \((3.57)\), \(Q\) is a gauge symmetry, as required.
Appendix A.

In this Section we prove the second part of Proposition 2.6. We start by collecting together a few identities needed in the course of the proof.

The commutation formula (2.15) implies that

\[ \nabla_{I'} \nabla_{J'} G = \frac{1}{2} \varepsilon_{I'J'} [\Phi_{IJ}, G], \quad \nabla_{[I} \nabla_{J']} G = \frac{1}{2} \varepsilon_{IJ} [\Phi^{I'J'}, G], \]

and

\[ \nabla_{K'}(I \nabla_{J'}) G = [\Phi_{IJ}, G], \quad \nabla_{K'}(\nabla_{J'} G) = [\Phi^{I'J'}, G]. \]

Write

\[ \nabla^2 = \nabla_{AA'}\nabla^{AA'}, \quad \Delta_{I'I'} = \nabla^{I'}_{I} \Phi_{IJ} \]

for the wave operator and the Yang-Mills equations in spinor form.

**Proposition A.1.** Let \( G \) be a \( g \)-valued differential function and let \( \nabla^2 \) be the wave operator. Then, on solutions of the Yang-Mills equations,

\[ \nabla^2 \Phi_{IJ} = 2[\Phi_{IK}, \Phi^K_{J}], \]

\[ \nabla^2 \nabla_{I'I'} G = 2[\Phi_{IJ}, \nabla_{J'} G] + 2[\Phi^{I'J'}, \nabla_{J'} G]. \]

**Proof.** We use (A.2) to compute

\[ \nabla_{K'}(I \nabla_{J'}) G = [\Phi_{IJ}, G], \quad \nabla_{K'}(\nabla_{J'} G) = [\Phi^{I'J'}, G]. \]

from which (i) follows.

Next we use (2.15) to compute

\[ \nabla_{I'I'} \nabla^2 G = \nabla_{I'I'} \nabla^{J'I'} G + \frac{1}{2} \varepsilon^{J'I'} [\Phi^I_{J'I'} G] + \frac{1}{2} \varepsilon^I_{J'I'} [\Phi_{J'I'} G]. \]

Hence

\[ \nabla_{J'J''} \nabla_{I'I'} G = \nabla_{J'J''} \nabla^{J'I'} G - \nabla_{J'J''} [\Phi^I_{J'I'} G] - \nabla_{J'J''} [\Phi_{J'I'} G] = \]

\[ \nabla_{J'J''} \nabla_{I'I'} G - [\Phi^I_{J'I'} G] - [\Phi_{J'I'} G] = \]

But by (2.15),

\[ \nabla_{J'J''} \nabla_{I'I'} G = \nabla_{I'I'} \nabla_{J'J''} G + [\Phi_{IJ}, \nabla_{J'J''} G] + [\Phi_{I'I'}, \nabla_{J'J''} G]. \]

Now (ii) follows from equations (A.5), (A.6). \( \square \)
Proof of Proposition 2.6 (ii). We first prove (2.20). By (A.4),

\[ \nabla^2 \Phi_{K_{p+2}}^{K_p} = \nabla^2 \nabla(K_1^{K_p} \ldots \nabla(K_2^{K_p} \ldots \nabla(K_p^{K_p}) \Phi_{K_{p+1}K_{p+2}}) = \]

\[ \nabla(K_1^{K_p} \nabla(K_2^{K_p} \ldots \nabla(K_p^{K_p}) \Phi_{K_{p+1}K_{p+2}}) + 2[\Phi_{S(K_{p+2}, \Phi_{K_{p+1}} ^{K_p})} + 2[\Phi_{S^p, \Phi_{K_{p+2}} ^{K_p}}] + \nabla{\hat{W}_1} = \ldots = \]

\[ \nabla(K_1^{K_p} \ldots \nabla(K_p^{K_p}) \nabla(K_p^{K_p}) \Phi_{K_{p+1}K_{p+2}}) + 2[\Phi_{S(K_{p+2}, \Phi_{K_{p+1}} ^{K_p})} + 2[\Phi_{S^p, \Phi_{K_{p+2}} ^{K_p}}] + \nabla{\hat{W}_2} = \]

\[ 2(p + 2 - \delta_{0p})[\Phi_{S(K_{p+2}, \Phi_{K_{p+1}} ^{K_p})} + 2[\Phi_{S^p, \Phi_{K_{p+2}} ^{K_p}}] + \nabla{\hat{W}_3}, \]

where \( \hat{W}_1, \hat{W}_2, \hat{W}_3 \), when restricted to \( R_p \), depend on the variables \( \Phi_{[p-1]} \). This proves (2.20).

In order to prove (2.19) we first reduce the derivative \( \nabla(K_{p+1}^{K_p} \Phi_{K_{p+3}K_{p+2}}) \) into symmetric components to derive the expression

\[ \nabla(K_{p+1}^{K_p} \Phi_{K_{p+3}K_{p+2}}) = \Phi_{K_{p+3}K_{p+2}} + \frac{p}{p+1} K_{p+1}^{K_p} \nabla S(K_{p+3} \Phi_{K_{p+2}}) \]

\[ \frac{p + 2}{p + 3} \epsilon^{K_{p+3}(K_{p+2}) \nabla S(K_{p+3} \Phi_{K_{p+2}})^{S(K_{p+3} \Phi_{K_{p+2}})}} \]

\[ \frac{p(p + 2)}{(p + 1)(p + 3)} \epsilon^{K_{p+1}(K_{p+2}) \nabla S(K_{p+3} \Phi_{K_{p+2}})^{S(K_{p+3} \Phi_{K_{p+2}})}} \]

We simplify the terms \( \nabla S(K_{p+3} \Phi_{K_{p+2}})^{K_{p+1}S(K_{p+3} \Phi_{K_{p+2}})}, \nabla S(K_{p+3} \Phi_{K_{p+2}})^{S(K_{p+3} \Phi_{K_{p+2}})}, \nabla S(K_{p+3} \Phi_{K_{p+2}})^{S(K_{p+3} \Phi_{K_{p+2}})} \) in (A.8) separately. First, we have that

\[ \nabla S(K_{p+3} \Phi_{K_{p+2}})^{K_{p+1}S(K_{p+3} \Phi_{K_{p+2}})} = \nabla S(K_{p+3} \nabla S(K_{p+3} \Phi_{K_{p+2}})^{K_{p+1}S(K_{p+3} \Phi_{K_{p+2}})}) = \]

\[ \frac{1}{p} \sum_{s=1}^{p} \nabla S(K_{p+3} \nabla S(K_{p+3} \Phi_{K_{p+2}})^{K_{p+1}S(K_{p+3} \Phi_{K_{p+2}})}) \]

Write

\[ A_s = \nabla S(K_{p+3} \nabla S(K_{p+3} \Phi_{K_{p+2}})^{K_{p+1}S(K_{p+3} \Phi_{K_{p+2}})}) \]

so that

\[ \nabla S(K_{p+3} \Phi_{K_{p+2}})^{K_{p+1}S(K_{p+3} \Phi_{K_{p+2}})} = \frac{1}{p} \sum_{s=1}^{p} A_s. \]
Then, by (A.1),

\[ A_s = \nabla S'(K_{p+3}) \nabla (K'_1) \cdots \nabla K'_{p+1} |S'| \nabla K'_{-2} \cdots \nabla K'_{p+1} \phi_{K_{p+1} K_{p+2}} + \]

\( (A.10) \)

\[ \nabla S'(K_{p+3}) \nabla (K'_1) \cdots \nabla K'_{p+1} |S'| \nabla K'_{-2} \cdots \nabla K'_{p+1} \phi_{K_{p+1} K_{p+2}} \]

Consequently, equations (A.9), (A.11), (A.12) imply that

\[ A_{s-1} + \nabla (K'_1) \cdots \nabla K'_{p+1} |S'| \nabla K'_{-2} \cdots \nabla K'_{p+1} \phi_{K_{p+1} K_{p+2}} \]

\( (A.13) \)

\[ A_{s-1} + (1 - \delta_{sp}) \phi_{(K_{p+3} K_{p+2}, K_{p+1})} + \hat{A}_{o, s}, \]

where \( \hat{A}_{o, s} \) is a function of the variables \( \phi^{[p-2]} \). Now a repeated application of (A.10) yields the equation

\[ A_s = A_1 + (s - 1 - \delta_{sp}) \phi_{(K_{p+3} K_{p+2}, K_{p+1})} + \hat{A}_s, \]

where \( \hat{A}_s \) is a function of the variables \( \phi^{[p-2]} \). By equation (A.2),

\[ A_1 = \phi_{(K_{p+3} K_{p+2}, K_{p+1})}. \]

Consequently, equations (A.9), (A.11), (A.12) imply that

\[ \nabla S'(K_{p+3}) \phi_{K_{p+2}} = \frac{(p - 1)(p + 2)}{2p} \phi_{(K_{p+3} K_{p+2}, K_{p+1})} + \hat{A}, \]

where \( \hat{A} \) is a function of the variables \( \phi^{[p-2]} \).

Next note that

\[ \nabla S(K'_{p+1}) \phi_{K_{p+1} S} = \frac{2}{p + 2} \nabla S(K'_{p+1}) \nabla (K'_1) \cdots \nabla K'_{p} |S| \nabla K_{-2} \cdots \nabla K'_{p} \phi_{K_{p+1} K_{p+1}} + \]

\( (A.14) \)

\[ \frac{1}{p + 2} \sum_{s=1}^{p} \nabla S(K'_{p+1}) \nabla (K'_1) \cdots \nabla K'_{p} |S| \nabla K_{-2} \cdots \nabla K'_{p} \phi_{K_{p+1} K_{p+1}}. \]

Write

\[ B_s = \nabla S(K'_{p+1}) \nabla (K'_1) \cdots \nabla K'_{p} |S| \nabla K_{-2} \cdots \nabla K'_{p} \phi_{K_{p+1} K_{p+1}}. \]

Note that

\[ B_1 = \phi_{(K'_{p+1}, K_{p+1})}, \quad \text{if } p = 1. \]

When \( p \geq 2 \), we use (A.1) to compute

\[ B_s = B_{s-1} - \nabla (K'_1) \cdots \nabla K'_{p-2} [\phi_{K'_{p+1} K_{p+1}}, \nabla K'_{p-1} \phi_{K_{p+1} K_{p+1}}] = \]

\[ B_{s-1} - \phi_{(K'_{p+1} K_{p+1})} \phi_{K_{p+1} K_{p+1}} - \delta_{sp} \phi_{(K'_{p+1} K_{p+1})} + \hat{B}_{o, s}, \]

where \( \hat{B}_{o, s} \) is a function of the variables \( \phi^{[p-2]} \). Hence

\[ B_s = B_1 - (s - 1) \phi_{(K'_{p+1} K_{p+1})} \phi_{K_{p+1} K_{p+1}} - \delta_{sp} \phi_{(K'_{p+1} K_{p+1})} + \hat{B}_s, \]

where \( \hat{B}_s \) is a function of the variables \( \phi^{[p-2]} \).

By virtue of the identity

\[ \nabla K'_{p+1} \phi_{K_{p+1} K_{p+1} S} = \nabla K'_{p+1} \phi_{K_{p+1} K_{p} + \epsilon S K_p \Delta K'_{p+1}} \]
we have that
\[\nabla^S(K'_{p+1}K'_1 \cdots K'_p) \Phi_{K_{p+1}S} = B_p + \nabla^{(K'_1 \cdots K'_p)} \Delta_{K_{p+1}}.\]

Clearly
\[B_1 = -\frac{p(p + 5)}{2(p + 2)}(K'_{p+1}K'_p, \Phi_{K_{p+1}}) - \left(\frac{3(1 - \delta_p)}{p + 2}\right) \Phi_{K_{p+1}} + \hat{\mathcal{B}},\]

where \(\hat{\mathcal{B}}\), when restricted to \(\mathcal{R}^{p-1}\), is a function of the variables \(\Phi^{[p-2]}\).

Next note that
\[\nabla^S(K'_{p-1}S')_{K_{p+1}} = \frac{1}{p} \sum_{s=1}^{p} \nabla^S \nabla^{(K'_1 \cdots K'_{s-1}) \cdots K'_p} \Phi_{K_{p+1}S}.\]

Write
\[C_s = \nabla^S \nabla^{(K'_1 \cdots K'_{s-1}) \cdots K'_p} \Phi_{K_{p+1}S} + \hat{\mathcal{C}}_s,\]

Suppose that \(p \geq 2\). Then
\[C_s = C_{s-1} + \nabla^S[K'_1 \cdots K'_{s-1} \cdots K'_p] \Phi_{K_{p+1}S} + \hat{\mathcal{C}}_s,\]

where \(\hat{\mathcal{C}}_s\), when restricted to \(\mathcal{R}^{p-1}\), are functions of the variables \(\Phi^{[p-2]}\). It follows that for \(p \geq 1\),
\[\nabla^S \nabla^S \Phi_{K_{p+1}S} = \nabla^S \nabla^{(K'_1 \cdots K'_{s-1}) \cdots K'_p} \Phi_{K_{p+1}S} + \hat{\mathcal{C}},\]

where again \(\hat{\mathcal{C}}\), when restricted to \(\mathcal{R}^{p-2}\), are functions of the variables \(\Phi^{[p-2]}\). We use the identity
\[\nabla^S \nabla^{(A \nabla B)_{S'}} G = \nabla^S(A \nabla B)_{S'} G + \frac{1}{2} e^{AB} \nabla^2 G\]
and equations (2.21), (A.2) to compute
\[
\nabla_S S'_{\nu} (K_1, K_2, \ldots, K_p) \Phi_{p+1} =
\]
\[
\frac{p+1}{p+2} [\Phi_{S(K_{p+1}), \Phi_{K_p}}] - \frac{p+3}{2(p+2)} \nabla^2 \Phi_{K_{p+1}} =
\]
\[
-\frac{(p+1)^2}{p} - 2\delta_p \left[ \Phi_{S(K_{p+1}), \Phi_{K_p}} \right] -
\]
\[
\frac{(p-1)(p+3)}{p+2} \left( \Phi_{S(K_{p+1}), \Phi_{K_p}} \right) \]
where \( \hat{D} \), when restricted to \( \mathbb{R}^{p-1} \), depends on the variables \( \Phi[p-2] \).

Finally, by (A.8), (A.13), (A.20), (A.21), (A.22), we see that (2.19) holds.

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