Exponential speed-up with a single bit of quantum information:
Testing the quantum butterfly effect

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We present an efficient quantum algorithm to measure the average fidelity decay of a quantum map under perturbation using a single bit of quantum information. Our algorithm scales only as the complexity of the map under investigation, so for those maps admitting an efficient gate decomposition, it provides an exponential speedup over known classical procedures. Fidelity decay is important in the study of complex dynamical systems, where it is conjectured to be a signature of quantum chaos. Our result also illustrates the role of chaos in the process of decoherence.

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“...The flap of a butterfly’s wings in Brazil can set off a tornado in Texas”. This butterfly effect illustrates the canonical feature of chaotic systems: they display extreme sensitivity to their initial conditions. In a chaotic regime, phase space trajectories diverge exponentially in time at a rate governed by the largest Lyapunov exponent of the system. Isolated quantum systems cannot display the butterfly effect, since unitary evolution preserves distances between states. Extensive research over the past two decades has been devoted to examining other manifestations which can be used to distinguish the regular and chaotic regimes of quantum systems. While many signatures of quantum chaos have been proposed, their validity relies mostly on vast accumulations of numerical evidences. Furthermore, obtaining conclusive results using these measures requires manipulating data whose size scales as the dimension (N) of the system’s Hilbert space – that is, exponentially with the number of qubits K required to simulate the system. In this article, we demonstrate a quantum algorithm to evaluate one such signature — the average fidelity decay — with a single bit of quantum information, in a time that scales as poly(K).

Fidelity decay was initially proposed as a signature of chaos by Peres [1], and has since been extensively investigated [2,3]. The closest quantum analogue to the (purely classical) butterfly effect, fidelity decay measures the rate at which identical initial states diverge when subjected to slightly different dynamics. The discrete time evolution of a closed quantum system can be specified by a unitary operator U, where \( \rho(\tau_n) = U^n \rho_0 (U^\dagger)^n \). To examine fidelity decay, we construct a slightly perturbed map \( U_p \), where \( U_p = U P \) with \( P = \exp\{-i\delta V\} \) for some small \( \delta \) and a hermitian matrix V. It is conjectured that the overlap (or fidelity)

\[
F_n(\psi) = \left| \langle \psi | (U^n)^\dagger U_p^n | \psi \rangle \right|^2
\]

(1)

between initially identical states \( \psi \) undergoing slightly different evolutions, \( U \) and \( U_p \), should decay differently (as a function of the discrete time \( n \)) for regular and chaotic dynamics: chaotic dynamics will display exponential fidelity decay, while regular dynamics will produce polynomial fidelity decay. Actual results to date show behavior rather more complex than the preceding simple conjecture. There are various regimes governing the decay rate; fidelity decay depends on the perturbation strength \( \delta \), and also on the degree of correlation between the eigenbasis of \( U \) and the eigenbasis of the perturbation \( P \). Fidelity decay remains a powerful diagnostic of chaotic behavior, but calculating it is computationally hard. Furthermore, because \( F_n(\psi) \) generally shows large fluctuations over time, it is in practice necessary to average \( F_n(\psi) \) over a random set of initial states \( \psi \) to determine its decay rate, thus increasing the numerical burden.

Several classically hard problems can be solved in polynomial (in \( K \)) time on a quantum computer. Since fully controllable and scalable quantum computers are still quite a ways in the future, algorithms which can be performed on a less-ambitious quantum information processor (QIP) are of great interest. A QIP is a quantum device which may fail to satisfy one or more of DiVincenzo’s five criteria, but can nonetheless carry out interesting computations [4]. Of particular interest to us is deterministic quantum computation with a single bit (DQC1) [5], a model of quantum information processing which is believed to be less powerful than universal quantum computation and which is naturally implemented by a high temperature NMR QIP [6]. In this model, universal control over all qubits is still assumed, but state preparation and read-out are limited. The initial state of the \( (K + 1) \)-qubit register is

\[
\left( \gamma |0\rangle |0\rangle + \frac{1 - \gamma}{2} \mathbb{I} \right) \otimes \frac{\mathbb{I}}{2^K},
\]

(2)

i.e., the first qubit (called the probe qubit for reasons which will become clear) is in a pseudo-pure state,
whereas the other $K$ qubits are in the maximally mixed state. Furthermore, the result of the computation is obtained as the noisy expectation value of $\sigma_z$ on the probe qubit. The variance of $\sigma_z$ is determined by i) the polarization $\gamma$ of Eq. 4 (independent of the size of the register) and ii) the inherent noise of the measuring process. Hence, $\langle \sigma_z \rangle$ can be estimated to within arbitrary $\epsilon$ with a probability of error at most $p$ by repeating the computation $O(\log(1/p)/\epsilon^2)$ times. The value of $\gamma$ in high-temperature NMR is independent of the size of the register because only a single qubit needs to be in a pseudo-pure state. The “inherent noise” receives contribution from both electronic noise and statistical fluctuations due to the finite sample size.

While it has been known for some time that the dynamics of some quantized chaotic systems can be efficiently (i.e., in poly($K$) time) simulated on quantum computers, it was shown only recently that this ability can also be used to efficiently evaluate certain proposed signatures of quantum chaos. In Ref. 10, an efficient quantum circuit is constructed to evaluate the coarse grained local density of states (LDOS) — the average profile of the eigenstates of $\hat{U}$ over the eigenbasis of $U_p$ — which is believed to be a valid indicator of chaos and is formally related to fidelity decay via Fourier transform [12]. In Ref. 14, an efficient procedure to estimate the fidelity decay using the standard model of quantum computation is presented. Finally, in Ref. 11, a DQC1 circuit is presented to estimate the form factors $t_n = |\langle \nu^n | U_p \rangle|^2$ of a unitary map $U$ which, under the random matrix conjecture (see [12] and refs. therein), is a good signature of quantum chaos. The proposed algorithm offers only a quadratic speedup, but since entanglement is very limited in DQC1 [10, 11], this result raises doubts about the common belief that massive entanglement is responsible for quantum computational speed-up [10].

Drawing upon all this previous work, we will now construct an efficient DQC1 algorithm to evaluate the average fidelity decay associated with any pair of unitary operators $U$ and $U_p$, provided they can be implemented efficiently, e.g. as those of Refs. 12. We begin by proving a crucial identity required to implement the efficient algorithm.

Let $f(\psi)$ be a complex-valued function on the space of pure state of a $N$-dimensional quantum system. We denote its average by $\langle f(\psi) \rangle = \int f(\psi) d\psi$, where $d\psi$ is the uniform measure induced by the Haar measure, such that $\int d\psi = 1$. For sake of compactness let $\langle \psi | A | \psi \rangle = \langle A \rangle$. Theorem: Let $A, B, C, \ldots$ be $\ell$ linear operators on a $N$-dimensional Hilbert space. Then

$$\langle A \rangle = \frac{\text{Tr} \left\{ (A \otimes B \otimes C \ldots) F_S^{(\ell)} \right\}}{N^{2 + \ell - 1}}$$

where $F_S^{(\ell)}$ is the projector on the symmetric subspace of $\ell$ systems, see Ref. 14 for details on $F_S^{(\ell)}$.

Proof: First, note that

$$\langle A \rangle = \frac{\text{Tr} \left\{ |\psi(\psi)^{\otimes \ell} (A \otimes B \otimes C \ldots) \right\}}{N^{2 + \ell - 1}}.$$ 

Therefore, the average over the pure states $\psi$ yields,

$$\text{Tr} \left\{ |\psi(\psi)^{\otimes \ell} (A \otimes B \otimes C \ldots) \right\}.$$ 

Since $|\psi(\psi)^{\otimes \ell}$ annihilates any state which is antisymmetric under interchange of two of the $\ell$ systems, and is by construction symmetric under such interchange, it must be proportional to the projector $P_S^{(\ell)}$ onto the symmetric subspace. To establish the theorem it is sufficient to find the proportionality factor $\lambda$ between these two quantities. Letting $A = B = C = \ldots = I$, we get $1 = \text{Tr} \left\{ (\psi(\psi)^{\otimes \ell} \right\} = \lambda \text{Tr} \{ P_S^{(\ell)} \} = \lambda (N^{2 + \ell - 1})$ (see Ref. 14), which completes the proof.

A useful corollary to this Theorem for any specific $\ell$ can be obtained by expanding $P_S^{(\ell)}$ in Eq. 3. In the case $\ell = 2$, it reads

$$\langle A \rangle = \sum_{ijmn} 2A_{ij}B_{mn}(P_S^{(2)})_{ijmn}$$

$$= \sum_{ijmn} A_{ij}B_{mn} \left( \delta_{ij} \delta_{mn} + \delta_{in} \delta_{mj} \right)$$

$$= \frac{\text{Tr} \{ A \} \text{Tr} \{ B \} + \text{Tr} \{ AB \}}{N^{2 + \ell - 1}}.$$ 

Similar expressions can be derived for $\ell > 2$, which involves the properly normalized sum of all combinations of traces of products and products of traces.

To arrive at our algorithm, it is sufficient to write the average fidelity as $F_n(\psi) = \langle \langle U^n \rangle_U^n \rangle \langle \langle U^n \rangle_U^n \rangle_\psi$, and apply the identity from Eq. 4 to obtain

$$F_n(\psi) = \langle \langle U^n \rangle_U^n \rangle_\psi^2 + N.$$ 

The specific form of our theorem with $\ell = 2$, unitary $A$, and $B = A^\dagger$ was discovered by M., P., and R. Horodecki [12], but our proof simplifies the presentation. An efficient DQC1 algorithm to evaluate the trace of any unitary operator [here, $(U^n)_U^n$], provided that it admits an efficient gate decomposition, was presented in Ref. 11. If the perturbed map takes the form $U_p = UP$ for some unitary operator $P$ (e.g., $P = \exp \{-i\delta V\}$ as above), the circuit can be further simplified into the one illustrated on Fig. 4.

We now analyze the complexity of our algorithm. We assume that $U$ and $U_p$ admit $\epsilon$-accurate gate decompositions whose sizes grow as $L(K, \epsilon) \in \text{poly}(K, 1/\epsilon)$. This implies that the controlled version of these gates also scale as $L(K, \epsilon)$. We see from Eq. 4 that the variance of $F_n(\psi)$ is at most twice the variance of $\text{Tr} \{ (U^n)_{U_p}^\dagger \}/N$. Therefore, the overall algorithm — estimating $F_n(\psi)$, to within $\epsilon$, with error probability at most
FIG. 1: Quantum circuit evaluating the average fidelity $F_n(\psi)$ between the perturbed and unperturbed maps $U$ and $U_p = UP$. The gates $R_k$ are $\pi/2$ rotation in the Bloch sphere around axis $x = k$ or $y$. When $k$ is set to $x$, we get the real part of $\text{Tr}(U^n U_p^n)/N$ while $k = y$ yields the imaginary part. The unitary operator $P$ is applied conditionally: when the probe qubit is in state $|1\rangle$, the unitary $P$ is applied to the lower register while no transformation is performed when the state of the probe qubit is $|0\rangle$.

$p$— requires resources growing as $L(K, c)n \log(1/p)/\epsilon^2$, so it is efficient. (The range in $n$ over which the decay is studied should be independent of the system’s size.) This algorithm thus provides an exponential speed-up over all known classical procedures and uses a single bit of quantum information. Furthermore, it eliminates any cost of averaging the fidelity over a random set of initial states, as this averaging is done directly.

In order to implement certain unitary maps on $K$ qubits efficiently, it is necessary to introduce a number $K_a$ of ancillary qubits (a “quantum work-pad”) in the fiducial state $|\psi_0\rangle$. Ancillary qubits in pseudo-pure states can be used in the DQC1 setting. As a first step of the computation, part of the polarization of the probe qubit of Eq. 2 can be transferred to ancillas initially in maximally mixed states. Thus, as long as the size $K_a$ of the work-pad is at most poly-logarithmic in $K$, the algorithm remains efficient.

Perhaps the most surprising feature of the quantum algorithm as it is presented in Fig. 4 is that the probe never gets entangled with the system throughout the computation. To show this, consider a generalized version of the circuit of Fig. 4 where the $P$’s and the $U$’s are free to differ at each iteration, i.e. at step $j$, we apply $P_j$ conditionally on the probe qubit, followed by $U_j$. This generalization is necessary since the controlled $P$ gate will in general be decomposed as a sequence of elementary controlled and regular gates [18]. Initially, the probe qubit is in state $|\alpha\rangle = \alpha|0\rangle + |1\rangle$. After $k$ steps, the state of the QIP is

$$
\rho_k = \frac{1}{N} \left( |\alpha|^2 |0\rangle \langle 0| \otimes I + \alpha^* \beta |1\rangle \langle 1| \otimes S^\dagger + \alpha \beta |1\rangle \langle 0| \otimes S + |\beta|^2 |1\rangle \langle 1| \otimes I \right)
$$

where $S = U_k P_k \ldots U_2 P_2 U_1 P_1 U_1^\dagger \ldots U_2^\dagger$. Decomposing this state in the eigenbasis of the unitary matrix $S|\phi_j\rangle = e^{i\gamma_j}|\phi_j\rangle$, we get

$$
\rho_k = \frac{1}{N} \sum_j |\alpha_j\rangle \langle \alpha_j| \otimes |\phi_j\rangle \langle \phi_j|
$$

where $|\alpha_j\rangle = \alpha|0\rangle + \beta e^{i\gamma_j}|1\rangle$; the state is separable. Its separability supports the point of view that the power of quantum computing derives not from the special features of quantum states — such as entanglement — but rather from fundamentally quantum operations [17].

Our algorithm also illustrates why chaotic environments are expected to produce decoherence more rapidly than integrable ones [21]. Consider the probe qubit of Fig. 4 as a quantum system interacting with a complex environment consisting of $K$ two-level systems. After a “time” $n$, the state of the system is given by tracing out the $K$ environmental qubits from Eq. 4. The diagonal elements of the reduced density matrix $|\alpha|^2$ and $|\beta|^2$ are left intact while the off-diagonal elements $\alpha^* \beta$ and $\alpha \beta^*$ are decreased by a factor $|\text{Tr}(S)|$ which is roughly equal to $\sqrt{F_n(\psi)}$. Thus, for an environment with chaotic dynamics, the system will decohere at an exponential rate, whereas the rate of decoherence should be slower for non-chaotic environments. This analogy also provides a very simple example of decoherence without entanglement [21].

On the circuit of Fig. 4 only the perturbation gates $P$ are conditioned on the state of the probe qubit. This suggests a dual interpretation of the algorithm as quantum circuit and quantum probe. On the one hand, $U$ could be a known unitary transformation which is being simulated on the lower $K$-qubit register over which we have universal control. Then, the gate $U$ would simply be decomposed as a sequence of elementary gates as prescribed in Refs. [4] for example. On the other hand, the lower register could be a real quantum system undergoing its natural evolution $U$ which might not even be known. Then, the probe qubit should really be regarded as a probe which is initialized in a quantum superposition, used to conditionally kick the system, and finally measured to extract information about the system under study. In this case, it is not necessary to have universal control over the lower register (the quantum system), we must simply be able to apply a conditional small unitary transformation to it.

Finally, Eq. 5 provides a useful numerical tool that can be used to compute the exact average fidelity instead of estimating it by averaging over a finite random sample of initial states. In Ref. [4], fidelity decay was illustrated on the quantum kicked top map $U_{QKT} = \exp(-i\pi J_{ij}/2) \exp(-ik J^2_{ij})$ acting on the $N = 2j + 1$ dimensional Hilbert space of angular momentum operator $J$. The chosen perturbation operator was $P = \prod_{k=1}^K \exp(-i\delta J_k^2/2)$, a collective rotation of all $K$ qubits of the QIP by an angle $\delta$. The decay rate (governed by the Fermi golden rule in this regime) was evaluated to $\Gamma = 2.50\delta^2$ for this perturbation [4]. $F_n(\psi)$ was estimated in both chaotic ($k = 12$) and regular ($k = 1$) regimes of the kicked top by averaging over 50 initial states. We reproduce these results on Fig. 4 and compare
them with the exact average Eq. \ref{exact_avg} and theoretical prediction $e^{-\Gamma_n}$. The random sample is in good agreement with the exact average except that the former shows fluctuations. Furthermore, the decay in the chaotic regime is in excellent agreement with the Fermi golden rule. While the decay is slower in the regular regime, it is not clear from these results that it is not exponential.

We have presented an efficient quantum algorithm which computes the average fidelity decay of a quantum map under perturbation using a single bit of quantum information. The quantum circuit for this algorithm establishes a link between decoherence by a chaotic environment and fidelity decay. Using a special case of our theorem, we have numerically evaluated the exact average fidelity decay for the quantum kicked top, and found good agreement with previous estimations using random samples. Although we have mainly motivated our algorithm for the study of quantum chaos, we believe that it has many other applications such as characterizing noisy quantum channels and computing correlation functions for many-body systems. We have also shown that our algorithm can be viewed as a special experiment where a quantum probe is initialized in a superposition and used to conditionally kick the system under study. This type of quantum information science byproduct might open the horizon to new types of experimental measurements where a small QIP is used to extract information from the quantum system under study. Finally, the effective speed-up despite the limited presence of entanglement — in particular its complete absence between the quantum probe and the mixed register — is a step forward in our understanding of the origin of quantum-computational speed-up.

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\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fidelity_decay.png}
\caption{Fidelity decay $F_n(\psi)$ averaged over 50 initial computational basis states for $U_{QKT}$ in a regular regime ($k = 1$, squares) and chaotic regime ($k = 12$, circles). The dashed lines represent the exact average Eq. \ref{exact_avg} and the full line shows the exponential decay at the Fermi golden rule rate $\Gamma$.}
\end{figure}