The Ising Spin Glass in dimension four; non-universality

P. H. Lundow\textsuperscript{1} and I. A. Campbell\textsuperscript{2}

\textsuperscript{1}Department of Mathematics and Mathematical Statistics, Umeå University, SE-901 87, Sweden
\textsuperscript{2}Laboratoire Charles Coulomb, Université Montpellier II, 34095 Montpellier, France

Extensive simulations are made on Ising Spin Glasses (ISG) with Gaussian and bimodal interaction distributions in dimension four. A detailed explanation is first given of scaling with the ISG scaling variables $\tau = 1 - \beta^2 / \beta_c^2$ and the appropriate scaling expressions, which allow scaling analyses over the entire paramagnetic range of temperatures once the transition temperatures have been established using a combination of finite size scaling and of thermodynamic derivative peak data. Measurements in the thermodynamic limit regime are analysed in order to obtain estimates of the critical exponents and correction terms. The Privman-Fisher ansatz then leads to compact scaling expressions over the complete paramagnetic temperature range and for all sample sizes $L$. Comparisons between the 4d Gaussian and bimodal ISGs show that the critical dimensionless parameters characteristic of a universality class, and the susceptibility and correlation length critical exponents $\gamma$ and $\nu$, depend on the form of the interaction distribution. From these observations it can be deduced that critical exponents are not universal in ISGs, at least in dimension four.

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INTRODUCTION

The universality of critical exponents is an important and remarkably elegant property of standard second order transitions, which has been explored in great detail through the Renormalization Group Theory (RGT). The universality hypothesis states that for all systems within a universality class the critical exponents are rigorously identical and do not depend on the microscopic parameters of the model. However, universality is not strictly universal; there are known "eccentric" models which are exceptions and violate the universality rule in the sense that their critical exponents vary continuously as functions of a control parameter. The most famous example is the eight vertex model solved exactly by Baxter\textsuperscript{1}. The exceptional physical conditions which apply in this case were discussed in detail in Ref. \textsuperscript{2}.

For Ising Spin Glasses (ISGs), the form of the interaction distribution is a microscopic control parameter. It has been assumed tacitly or explicitly that the members of the ISG family of transitions obey standard universality rules, following the generally accepted statement that "Empirically, one finds that all systems in nature belong to one of a comparatively small number of universality classes"\textsuperscript{3}. One should underline the word "empirically"; we know of no formal proof that universality must hold in ISGs. It was found thirty years ago that the \textit{e}-expansion for the critical exponents\textsuperscript{2} in ISGs is not predictive since the first few orders have a non-convergent behavior and higher orders are not known. This can be taken as an indication that a fundamentally different theoretical approach is required for spin glass transitions. Indeed "classical tools of RG analysis are not suitable for spin glasses"\textsuperscript{3,5}, although no explicit predictions have been made so far concerning universality.

ISG transition simulations are much more demand-
glasses the critical exponents depend on the strength of the Dzyaloshinski-Moriya interaction [18].

SCALING VARIABLES AND EXPRESSIONS

We will begin with a detailed discussion of scaling in ferromagnets and in ISGs using the scaling variable \( \tau = 1 - \beta / \beta_c \) in ferromagnets and \( \tau = 1 - \beta^2 / \beta_c^2 \) in ISGs [8], as this approach has been widely misunderstood or simply ignored.

Numerical data on critical transition phenomena are almost always analysed using \( t = 1 - T / T_c \) as the scaling variable, following the standard critical regime prescription of the Renormalization Group Theory (RGT). In particular \( t \) scaling has been used in recent ISG studies [12, 19]. However, with this variable explicit analyses of numerical data are limited to the finite size scaling (FSS) regime within a narrow critical region in temperature; \( t \) diverges at infinite temperature, so when \( t \) is used outside the critical temperature region corrections to scaling inevitably proliferate.

In ferromagnets, for well over fifty years the scaling variable \( \tau \) has been used for the analysis of the high temperature scaling expansion (HTSE) susceptibility; the scaling variable \( \tau \) was used in the original discussion of confluent corrections to scaling in ferromagnets by Wegner [20] which lead to the important ThL susceptibility expression

\[
\chi(\beta) = C_\chi \tau^{-\gamma}[1 + a_\chi \tau^0 + b_\chi \tau + \cdots] \tag{1}
\]

The first correction term is the confluent correction and the second an analytic correction.

In the critical region \( \tau \) and \( t \) become equivalent, but \( \tau \to 1 \) at infinite temperature (where \( \chi(\beta) \to 1 \)) instead of diverging, so the \( \tau \) expression is well controlled over the whole paramagnetic regime. In principle there can be many correction terms in Eq. (1) but in practice to high precision a leading term and one single further effective correction term are generally sufficient for an analysis at the level of precision of numerical data. Thus for the canonical Ising ferromagnet in dimension two, there are five or more further well identified correction terms [8] but in practice with one single weak effective correction term beyond the leading term, the expression Eq. (1) represents the temperature dependence of the ThL susceptibility to high precision from criticality to infinite temperature [21]. The same remark holds for Ising ferromagnets in dimension three [22]. (In ferromagnets the use of \( \tau \) as the scaling variable leads to an “crossover”to a high temperature mean field behavior [24], which is a pure artefact [23].) For ferromagnets with zero temperature ordering \( \tau = 1 - \tanh(\beta) \) is a suitable variable [3].

Following a protocol well-established in ferromagnets [23, 20] one can define a temperature dependent effective exponent

\[
\gamma(\tau) = -\partial \ln \chi(\tau) / \partial \ln \tau \tag{2}
\]

\( \gamma(\tau) \) tends to the critical \( \gamma \) as \( \beta \to \beta_c \), and to \( 2d \beta_c \) as \( \beta \to 0 \) in simple [hyper]-cubic lattices.

For the ThL regime \( \gamma(\tau) \) can be written

\[
\gamma(\tau) = \gamma - [a_\chi \tau^0 + b_\chi \tau^0]/[1 + a_\chi \tau^0 + b_\chi \tau^0] \tag{3}
\]

including the leading order confluent scaling term and a further effective higher order correction term. The exact infinite temperature \( \tau = 1 \) HTSE condition on the fit parameters is

\[
\gamma - [a_\chi + b_\chi]/[1 + a_\chi + b_\chi] = 2d \beta_c^2 \tag{4}
\]

When critical exponent estimates \( \gamma \) and \( \theta \) are also available from FSS, the two sets of values must be consistent if the critical parameters have been identified correctly.

In Ising ferromagnets the analogous ThL expression for the second moment correlation length \( \xi(\tau) \) is [16]

\[
\xi(\tau) = C_\xi \beta^{1/2} \tau^{-\nu[1 + a_\xi \tau^0 + \cdots]} \tag{5}
\]

The factor \( \beta^{1/2} \) arises because the generic infinite temperature limit behavior is \( \xi(\tau)/\beta^{1/2} \to 1 \). The temperature dependent effective exponent is then

\[
\nu(\tau) = -\partial \ln(\xi(\tau)/\beta^{1/2}) / \partial \ln \tau \tag{6}
\]

so with a two correction term relation:

\[
\nu(\tau) = \nu - [a_\xi \theta \tau^0 + b_\xi y \tau^0]/[1 + a_\xi \tau^0 + b_\xi \tau^0] \quad (7)
\]

\( \nu(\tau) \) tends to the critical \( \nu \) as \( \beta \to \beta_c \), and to \( d \beta_c \) as \( \beta \to 0 \) for ferromagnets on simple [hyper]-cubic lattices.

In ISGs because the effective interaction energy parameter is \( \langle J^2 \rangle \) not \( \langle J \rangle \), the appropriate inverse “temperature”parameter is \( \beta^2 \) not \( \beta \), so the appropriate scaling variable is \( \tau = 1 - (\beta^2 / \beta_c^2) \), (or \( w = 1 - (\tanh(\beta) / \tanh(\beta_c))^2 \) for the bimodal ISG case). This ISG \( \tau \) (or \( w \)) has been used for many years for the analysis of the ThL ISG susceptibility [16, 29–31]. Then with this \( \tau \) the ThL susceptibility relations Eqs. (1), (2), (3) are formally the same as in the ferromagnet.

\[
\chi(\tau) = C_\chi \tau^{-\gamma}[1 + a_\chi \tau^0 + \cdots] \tag{8}
\]

and \( \gamma(\tau) \) tends to the critical \( \gamma \) as \( \beta^2 \to \beta_c^2 \). The exact high temperature limit from HTSE is \( \gamma(\tau) \to 2d \beta_c^2 \) as \( \beta^2 \to 0 \) in simple [hyper]-cubic lattices.

The appropriate ISG correlation length expression is

\[
\xi(\tau) = C_\xi \beta \tau^{-\nu[1 + a_\xi \tau^0 + \cdots]} \tag{9}
\]

The factor \( \beta \) arises from the generic form of the ISG \( \xi(\beta) \) high temperature series [16]. The temperature dependent ISG effective exponent is

\[
\nu(\tau) = -\partial \ln(\xi(\beta)/\beta) / \partial \ln \tau \tag{10}
\]
The exact $\beta = 0$ limit in ISGs in simple hyper-cubic lattices of dimension $d$ is $\nu(\beta) = (d - K/3)2^x$ where $K$ is the kurtosis of the interaction distribution.

FSS analyses rely mainly on the size dependence of the critical behavior of observables $U(\beta_c, L)$ and their derivatives $[\partial U(\beta_c, L)]/\partial \beta$. For the dimensionless parameters such as the cumulant $U_2(\beta, L) = (m(\beta, L)^4)/(m(\beta, L)^2)^2$ or alternatively the Binder parameter $g(\beta, L) = (3 - U_4)/2$, and the correlation length ratio $\xi(\beta, L)/L$, the form of the critical size dependence

\[ U(\beta_c, L) = U_{\beta_c}L^{1/\nu}[1 + a_u L^{-\omega} + \cdots] \]  \hspace{1cm} (11)

and the critical derivative expression

\[ \left[ \frac{\partial U(\beta, L)}{\partial \beta} \right]_{\beta_c} = K_U L^{1/\nu}[1 + a_u L^{-\omega} + \cdots] \]  \hspace{1cm} (12)

can be retained unaltered with $\tau$ scaling for very small $1 - \beta/\beta_c$ both for ferromagnets and for ISGs.

It has been pointed out on general grounds \[11, 12\] that the logarithmic derivative of the susceptibility has the form

\[ \frac{\partial \chi(\beta, L) / \partial \beta}{\chi(\beta, L)} = K_1 L^{1/\nu}[1 + a_\chi L^{-\omega} + \cdots] + K_1 \]  \hspace{1cm} (13)

No evaluation was proposed for the constant term $K_1$ in Ref. \[11, 12\]. From the leading order $\tau$ scaling finite size expression for $\chi(\beta, L) \[16\]$ it is easy to show that $K_1 = -(2 - \eta)/2\beta_c$ in a ferromagnet \[22\]. In an ISG the constant term in $\partial \chi(\beta^2, L) / \partial \beta^2 / \chi(\beta^2, L)$ is $K_1 = -(2 - \eta)/2\beta_c$. As pointed out in Ref. \[11\], for many years the published estimates of the exponent $\nu$ in ISGs were wrong by factors of the order of 2 because a constant $K_1$ term was not included in the FSS susceptibility analyses.

The ThL regime is limited for each $L$ by a condition for which a rule of thumb is $L/\xi(\beta) > \sim 6$, and an extrapolation to criticality must complete the overall fit to all the ThL data in order to estimate critical exponents. The accuracy of this extrapolation depends on a figure of merit, the minimum value of $\tau$ for which the ThL condition still holds for samples of size $L$. This figure of merit in ISGs can be taken to be

\[ \tau_{\text{min}} \sim (L/\beta_c)^{-1/\nu} \]  \hspace{1cm} (14)

In dimension 4 with $\beta_c \sim 0.5$ and $\nu \sim 1$ the condition implies $\tau_{\text{min}} \sim 0.25$ if the largest size used is $L_m = 12$. This $\tau_{\text{min}}$ corresponds to $\beta_{\text{min}} \sim 0.9\beta_c$, so that a temperature fairly close to the critical temperature. (It should be underlined that with the appropriate parameters for ISGs in dimension 3, $\beta_c \sim 1, \nu \sim 2.5$, to reach $\tau_{\text{min}} \sim 0.25$ would require sample sizes to $L_m \sim 200$, well beyond the maximum sizes which have been studied numerically so far in 3d ISGs.) When fitting to obtain the extrapolation, no a priori assumption is made as to the value of the dominant scaling correction exponent, which can be that of the confluent correction (as in the 3d ferromagnet \[22\]), of an analytic correction (as in the 2d ferromagnet \[21\]), or of a high order effective correction if the prefactors of the low order terms happen to be very weak. The exponent values and the prefactor $a_\chi$ for the leading term come out of the fit. It can be noted that an exactly equivalent procedure is followed in the traditional fits to FSS data, where extrapolations are made from finite $L$ to infinite size. The strength of the correction prefactor $a_\chi$ is an important parameter, which is rarely quoted explicitly in publications on numerical work on ISGs.

The Privman-Fisher ansatz Eq. (15) was originally presented \[17\] in terms of scaling near criticality, with $\tau$ as the scaling variable. With the $\tau$ scaling expressions the ansatz can be applied successfully over the entire paramagnetic temperature range. Once explicit expressions $\chi(\tau, \infty)$ and $\xi(\tau, \infty)$ have been obtained by fits to the ThL regime with estimated values for $\gamma, \nu, \theta, a_\chi, a_\xi$ (and possibly higher order terms if necessary), Privman-Fisher ansatz \[17\] scaling plots can be made for all $L$ and all the paramagnetic data from $\beta_c$ to $\beta = 0$:

\[ \frac{\chi(\tau, L)}{\chi(\tau, \infty)} = \frac{F[L/\xi(\tau, \infty)] + a(\omega, \chi) L^{-\omega} G_\chi[L/\xi(\tau, \infty)]}{F[L/\xi(\tau, \infty)] + a(\omega, \xi) L^{-\omega} G_\xi[L/\xi(\tau, \infty)]} \]  \hspace{1cm} (15)

and

\[ \frac{\xi(\tau, L)}{\xi(\tau, \infty)} = \frac{F[L/\xi(\tau, \infty)] + a(\omega, \xi) L^{-\omega} G_\xi[L/\xi(\tau, \infty)]}{F[L/\xi(\tau, \infty)] + a(\omega, \xi) L^{-\omega} G_\xi[L/\xi(\tau, \infty)]} \]  \hspace{1cm} (16)

Scaling, with $\omega = \theta/\nu$, should be ”perfect” for all $L$ and over the whole paramagnetic temperature range including the critical FSS regime if the critical parameter estimates have been chosen correctly. This overall scaling can be considered to provide an ultimate validation of the coherence of ThL and FSS fits.

The Privman-Fisher procedure with $\tau$ scaling has been followed successfully in the case of the simple cubic Ising ferromagnet \[22\], where as a further ansatz, a simple explicit form for the principal Privman-Fisher scaling function was proposed:

\[ F_\chi(x) = [1 - \exp(-bx(2 - \eta)/a)]a \]  \hspace{1cm} (17)

with $x = L/\xi(\beta, \infty)$, and $a, b$ fit parameters. This extremely compact form automatically fulfills the limit conditions for large and small $x$. The parameters $\eta, a$ and $b$ should be characteristics of a universality class.

For the $\xi(\beta, L)$ scaling plot the fit ansatz becomes even simpler:

\[ F_\xi(x) = [1 - \exp(-bx^{1/4})]a \]  \hspace{1cm} (18)

The same approach will be applied below to the ISGs in dimension four.

It can be noted that the ThL data analyses show no evidence for the presence of an analytic correction term.
proportional to $\tau$, which if it exists should become dominant as criticality is approached if the confluent $\theta$ is greater than 1. In a ferromagnet the leading analytic term is due to the field dependence of the analytic part of the free energy. We know of no \textit{a priori} estimate of the strength of such terms in the ISG context, but it is plausible that they are intrinsically weak because the field is only present at higher order. HTSE analyses, particularly the $M1$ and $M2$ techniques, are sensitive to the presence of analytic terms. In the HTSE measurements of Klein et al.\cite{30} an explicit test was made for an analytic correction in the 4d bimodal ISG. No evidence was found for such a term. In all the more extensive HTSE analyses of Ref.\cite{31} the leading ThL correction term effective exponent $\theta$ was always significantly greater than 1.

It can be noted that for dimensional reasons, FSS corrections $L^{-\omega_1}$ have exponents related to the equivalent ThL exponents through $\omega_1 = \theta_1/\nu$. So for ISGs in dimension 3 with $\nu \sim 2.5$\cite{12,13}, the leading analytic correction term would have an FSS exponent $\omega_a = 1/\nu \sim 0.4$. The leading correction term estimated from extensive 3d bimodal ISG numerical data analysis\cite{12,13} has an exponent, $\omega \sim 1.1$, implying a dominant confluent ThL correction with exponent $\theta \sim 2.75$. There is no mention in these publications of any analytic FSS term with an exponent $\omega_a \sim 0.4$.

We conclude empirically that quite generally in ISGs the ThL analytic correction terms proportional to $\tau$ are small or negligible, presumably due to vanishingly weak prefactors.

Working with the $t$ scaling variable for ISGs as in\cite{12,19} means that information coming from temperatures well above criticality is lost. Comments which have been published such as “The difference between the $[\tau$ scaling] expressions and the standard expressions is only in the corrections to scaling,”\cite{11} or “[The $\tau$ scaling] approach might partly take into account the scaling corrections...”\cite{12} are incorrect and follow from a misunderstanding. They refer to the initial lowest order form of $\tau$ scaling\cite{10} where corrections to scaling were explicitly left out of the analysis. Full expressions have been used in the analysis of ferromagnets\cite{21,22} and are used here for ISGs. It is helpful to note that because of exact infinite temperature closure conditions on $\chi(\tau), \xi(\tau)/\beta, \gamma(\tau)$ and $\nu(\tau)$ from HTSE, a potentially infinite set of high temperature corrections can be grouped together into a single effective correction.

**THERMODYNAMIC DERIVATIVE ANALYSIS**

The thermodynamic derivative analysis can be an efficient method for analyzing data in a ferromagnet or an ISG. We will cite as an example the case of the susceptibility in a ferromagnet; other observables can replace the susceptibility. A plot is made of $y = \partial \beta/\partial \ln \chi(\beta, L)$ against $x = \beta$. This plot is purely a presentation of measured data and does not require $\beta_c$ or any other parameter as input. The part of the data which for each $L$ in the ThL regime can be fitted by an expression:

$$\frac{\partial \beta}{\ln \chi(\beta)} = \frac{(\beta - \beta_c) (1 + a_\chi \omega^\theta)}{\gamma + (\gamma - \theta) a_\chi \omega^\theta}$$

(19)

A further correction term can be readily included if needed. The intercept $y = 0$ of the fit curve occurs at the critical point where $x = \beta_c$, and the initial slope starting at the intercept is $\partial y/\partial x = -1/\gamma$. The fit parameters are $\beta_c, \gamma, \theta$ and $a_\chi$. The fit must obey the condition that at $x = 0$, $y \equiv 1/2d$, so if $\beta_c$ is known and corrections beyond the leading one are negligible, there are just two free fit parameters.

Near criticality in a ferromagnet for many observables $U$ the heights of the peaks of the thermodynamic derivatives $\partial U(\beta, L)/\partial \beta$ scale for large $L$ as\cite{25,28}

$$[\partial U(\beta, L)/\partial \beta]_{\text{max}} \propto L^{1/\nu} \left(1 + a_\chi \omega^\theta \right)$$

(20)

Without needing a value of $\beta_c$ as input, the large $L$ limit peak height against $L$ plot provides $1/\nu$ directly. The temperature location of the derivative peak $\beta_{\text{max}}(L)$ also scales as $\beta_c - \beta_{\text{max}}(L) \propto L^{-1/\nu} \left(1 + b_\chi \omega^\theta \right)$ and the observables used for $U(\beta, L)$\cite{27} can be for instance the Binder cumulant $g(\beta, L)$, the logarithm of the finite size susceptibility $\ln \chi(\beta, L)$, or the logarithm of the absolute value of the magnetisation $\ln \langle |m| \rangle(\beta, L)$. Each of these data sets can give independent estimates of $\nu$ and $\beta_c$ without any initial knowledge of either parameter. For a ferromagnet, Ferrenberg and Landau\cite{27} find this form of analysis significantly more accurate than the standard Binder cumulant crossing approach.

We note that both the minimum of the inverse derivative $[\partial \beta/\partial U(\beta, L)]_{\text{min}}$ and the temperature location difference $[\beta_c - \beta_{\text{min}}(L)]$ are proportional to $L^{-1/\nu}[1 + a_\chi \omega^\theta + \cdots]$ (with different factors $a$). Hence for instance in the $y$ against $x$ plot above the set of minima of $[\partial \beta/\partial \ln \chi(\beta, L)]_{\text{min}}$ plotted against $\beta_{\text{min}}(L)$ must tend linearly towards an intercept $[\partial \beta/\partial \ln \chi(\beta, L)]_{\text{min}} = 0$ at $\beta_{\text{min}} \equiv \beta_c$ for large $L$. Other plots of the same type should all extrapolate consistently to the true $\beta_c$. The confluent correction only appears as a small $L$ modification to the straight line. Provided that the peaks for the chosen observable fall reasonably close to $\beta_c$, these data are in principle much simpler to analyse than those from the Binder crossing technique where one must estimate simultaneously $\beta_c, \nu$, and the strength $a_\chi$ and the exponent $\omega$ of the correction term.

In the ISG context exactly the same methodology can be used as in the ferromagnet; Fig. 4 shows as an example the 4d bimodal ISG susceptibility logarithmic derivative data and the fits. As far as we are aware this analysis has not been used previously in ISGs.
example, for $L = 12$ this means about 200000 sweep-exchange steps.

After equilibration, at least 200000 measurements were made for each sample for all sizes, taking place after every sweep-exchange step. Data were registered for the energy $E(\beta, L)$, the correlation length $\xi(\beta, L)$, for the spin overlap moments $\langle |q| \rangle$, $\langle q^2 \rangle$, $\langle |q|^3 \rangle$, $\langle q^4 \rangle$ and the corresponding link overlap $q_f$ moments. In addition the correlations $(E(\beta, L), U(\beta, L))$ between the energy and observables $U(\beta, L)$ were also registered so that thermodynamic derivatives could be evaluated using the relation $\partial U(\beta, L)/\partial \beta = \langle U(\beta, L), E(\beta, L) \rangle - \langle U(\beta, L) \rangle \langle E(\beta, L) \rangle$ where $E(\beta, L)$ is the energy [21]. Bootstrap analyses of the errors in the derivatives as well as in the observables $U(\beta, L)$ themselves were carried out.

For the present analysis we have first observed the FSS behavior of various dimensionless parameters, not only the familiar Binder cumulant and the correlation length ratio $\xi(\beta, L)/L$ but also other observables showing critical behavior. One alternative dimensionless parameter

$$W = \frac{1}{\pi - 2} \left( \pi \frac{\langle |m| \rangle^2}{\langle m^2 \rangle} - 2 \right)$$

(22)

was introduced in the Ising ferromagnet context in [32]. In the ISGs $m$ can be replaced by $q$ so

$$W_q(\beta, L) = \frac{1}{\pi - 2} \left( \pi \frac{\langle |q| \rangle^2}{\langle q^2 \rangle} - 2 \right)$$

(23)

with $\langle \cdot \rangle$ denoting the sample average of the time averages. In the same spirit we will also make use of the dimensionless parameter

$$h(\beta, L) = \frac{1}{\sqrt{\pi} \sqrt{8}} \left( \sqrt{\pi} \frac{\langle |q|^3 \rangle}{\langle q^2 \rangle^{3/2}} - \sqrt{8} \right)$$

(24)

having properties similar to $g$ and $W_q$.

We have also registered the non-self averaging parameter $U_{22}$, the kurtosis of the spin overlap distribution, and the moments of the absolute spin overlap distribution, together with the variance and kurtosis of the link overlap distribution [33, 34]. Only a fraction of these data are reported here.

Analyses with the traditional technique of estimating crossing point temperatures $[\beta_{\text{cross}}, L]$, defined through $U(\beta_{\text{cross}}, L) = U(\beta_{\text{cross}}, 2L)$, have disadvantages. The statistical errors in both sizes $L$ and $2L$ contribute to the uncertainty of the crossing temperature; the scaling correction to the smaller size $L$ dominates and is combined with the numerical difficulty in equilibrating at the larger size $2L$. Instead we interpolate using the data points for each $U(\beta, L)$ so as to obtain sets of data $U(\beta_f, L)$ for a few fixed $\beta_f$ in the critical region, after making a first rough estimate of $\beta_c$. (It is important to span the range of temperatures on both sides of the true $\beta_c$.) We then make a global fit with the standard FSS expression, valid

**FIG. 1:** (Color online) The 4d bimodal derivative $\partial^2 \beta^2 / \partial \ln \chi(\beta, L)$ as a function of $\beta^2$. (Throughout the paper, the symbol code for sizes is: $L = 14$ navy heptagons, $L = 13$ brown hexagons, $L = 12$ purple squares, $L = 11$ brown hexagons, $L = 10$ pink circles, $L = 9$ orange xx, $L = 8$ black triangles, $L = 7$ olive inverted triangles, $L = 6$ red diamonds, $L = 5$ green left triangles, $L = 4$ blue right triangles. For display purposes not all sizes are shown in each Figure). Green curve : HTSE data points. Blue curve : ThL regime data fit. Open squares : minima locations for each $L$.

**ISING SPIN GLASS SIMULATIONS**

The Hamiltonian is as usual

$$\mathcal{H} = - \sum_{ij} J_{ij} S_i S_j$$

(21)

with the near neighbor symmetric distributions normalized to $\langle J_{ij}^2 \rangle = 1$. The Ising spins live on simple hyper-periodic cubic lattices with periodic boundary conditions. We have studied the bimodal $(\pm J)$, Gaussian, and Laplacian distributions in 4d. Here we will discuss the Gaussian and bimodal ISGs and will compare with published measurements on 4d ISGs.

The simulations were carried out using the exchange Monte Carlo method for equilibration using so called multi-spin coding, on 4096 individual samples at each size. An exchange was attempted after every sweep with a success rate of at least 30%. At least 40 temperatures were used forming a geometric progression reaching down to $\beta_{\text{max}} = 0.55$ for $4d$ and $\beta_{\text{max}} = 0.58$ for $4d$. This ensures that our data also come from the low-temperature region which is useful during the parameter fits. Near the critical temperature the $\beta$ step length was at most 0.003. The various systems were deemed to have reached equilibrium when the sample average susceptibility for the lowest temperature showed no trend between runs.
in the critical region if there is a single dominant scaling correction term:

$$U(\beta, L) = AL^{-\omega} + B(\beta - \beta_c)L^{1/\nu}$$  \hspace{1cm} (25)

The fit uses all the FSS region data, and gives output estimates, with error bars, for $\beta_c$ and the critical exponents. The parameters of Eq. (25), and their error bars at the 95% level, were found by using Mathematica’s built-in routines for nonlinear model fitting. The data points for $U$ were obtained at some 20 fixed $\beta_f$ near $\beta_c$ (0.98 < $\beta_f$/$\beta_c$ < 1.02), using cubic spline interpolation. As a reality-check we also tried skipping the interpolation, thus using the actual $\beta$-values of the measurements (these differ between the various lattice sizes). The outputs were however very similar to the ones based on interpolation. Thus the six parameters are based on some 200-300 data points from different $\beta$ and $L$. The quality of the fit is checked by looking at not only the adjusted R-square index, which is always extremely close to 1 in our fits. A better test is to see whether the distribution of deviations between data and fitted model, so called standardized residuals, can be considered zero by using Mathematica’s built-in zero-location tests, using the sign-test or T-test. The fitted models reported here pass these tests at the 5% level in all cases except one which is adressed in the text.

We also use inverse derivative plots and inverse derivative peak data to obtain independent estimates of $\nu$ and $\beta_c$. With estimated values of $\beta_c$ and the derivative data we make up plots of $\gamma(\tau)$ and $\nu(\tau)$, and we fit the ThL data to estimate effective values of the critical parameters $\gamma$, $\nu$, $\theta$ and the coefficients $C_\chi$, $C_\xi$, $a_\chi$ and $a_\xi$. The pure FSS estimates should be consistent with these values.

Finally, we make up Privman-Fisher scaling plots which include all the measured data, from the ThL regime, the FSS regime, and the intermediate temperature regime, using the fit parameter estimates. These plots provide an overall validation of the physical parameter estimates.

THE GAUSSIAN ISG IN DIMENSION 4

We will now address the question of ISGs in dimension four, starting with the Gaussian interaction distribution. Simulation measurements up to $L = 12$ were published on the 4d Gaussian ISG, together with a 4d bimodal ISG with diluted interactions (65% of the interactions being set to $J = 0$). The critical temperature for the 4d Gaussian ISG was estimated from Binder parameter and correlation length ratio measurements to be $T_c = 1.805(10)$ so $\beta_c = 0.554(3)$, in full agreement with earlier simulation estimates 0.555(3) \cite{37, 38} and consistent with a high temperature series expansion (HTSE) estimate $\beta_c^2 = 0.314(4)$, so $\beta_c = 0.560(4)$ \cite{31}. The simulation analyses led to essentially identical exponents for the two systems: $\nu = 1.02(2)$ and $\eta = -0.275(25)$ and so through scaling rules $\gamma = 2.32(8)$ \cite{14}. The HTSE critical exponent estimates were $\gamma = 2.31(1)$ and $\theta \sim 1.35(3)$. The two systems of Ref. \cite{14} happen to show almost negligible corrections to scaling for the Binder parameter, rendering the $\beta_c$ and critical parameter estimates particularly reliable.

We have repeated the measurements of the Binder parameter $g(\beta, L)$ and the correlation length ratio $\xi(\beta, L)/L$, and have measured the dimensionless parameter $W_q(\beta, L)$, Eq. (25), in the critical region. Plots of $g(\beta, L)$, $h(\beta, L)$, $W_q(\beta, L)$ and $\xi(\beta, L)/L$ for chosen fixed $\beta$ as functions of $L^{-\omega}$ are shown in Figs. 2, 3, 4 and 5, together with the fits as described above. Due to very small corrections to scaling for Gaussian interactions ($A$ in Tab. I) we had to fix the $\omega$-parameter or the fitting process would not converge properly. Here it was chosen to be $\omega = 2$, but using 1.8 or 2.2 gave very similar results for the other parameters, well within their error bars. We should mention that the data-model residual distribution did not pass the zero-location test at the 5%-level for $\xi/L$. Increasing $\omega$ to 2.5 corrects this but does not change the picture much. We retain the $\omega = 2$ and note that the data points agree very well with the model and we see no cause for alarm here.

We thus obtain consistent estimates, Tab. I, $\beta_c = 0.5555(2)$, $\nu = 1.02(2)$ (taking the average values) together with the infinite size limit dimensionless critical parameter values $g(\beta_c, \infty) = 0.474(1)$, $h(\beta_c, \infty) = 0.3847(11)$, $W_q(\beta_c, \infty) = 0.241(1)$ and $\xi/L(\beta_c, L \rightarrow \infty) = 0.448(1)$. The estimates from the Binder cumulant are in full agreement with those of Ref. \cite{14} where $g(\beta_c) = 0.470(5)$ (no value for $\xi/L(\beta_c, L \rightarrow \infty)$ or $h(\beta_c, \infty)$, was cited explicitly; $h(\beta, L)$ and $W_q(\beta, L)$ were not measured).

The present $\beta_c$ value is rather more accurate mainly because of a more closely spaced temperature grid and
better statistics at higher $L$. At criticality $\chi(\beta_c, L) \propto L^{2-\nu}$. Using this $\beta_c$ value a FSS log-log plot of $\chi(\beta_c, L)/L^2$ against $L$ gives a straight line of slope $-\eta = 0.307(10)$, consistent with the estimate $-\eta = 0.275(25)$ of [14]. From the scaling rule $\gamma = (2-\eta)\nu$ we obtain a FSS estimate $\gamma = 2.35(1)$.

As an example of the use of thermodynamic derivative peak height and location measurements we exhibit the results for the peak maximum $H_{\text{max}}(L)$ and the corresponding inverse temperature location $\beta_m(L)$ for the derivative $\partial W_q(\beta, L)/\partial \beta$. From the scaling rule Eq. (19) a log-log plot of $H_{\text{max}}(L)$ against $L$ has a limiting slope of $1/\nu$, with no need of an estimate for $\beta_c$. Fig. 6 shows $1/\nu = 1.00(1)$. A plot of $\beta_m(L)$ against $1/H_{\text{max}}(L)$ shows a limiting straight line extrapolating to $\beta_m(\infty) = 0.554(1)$ at $1/H_{\text{max}}(L) = 0$, an independent estimate for $\beta_c$.

The temperature dependent derivatives $\gamma(\tau, L)$ and $\nu(\tau, L)$, Eqs. 2, 13, assuming $\beta_c = 0.555$ are shown in Figs. 7 and 8. The ThL regime correlation length exponent $\nu(\tau)$ has a weak temperature variation, with an exponent $\theta \sim 1.8$ compatible with the HTSE value $\theta \sim 1.35$ [31]. The overall fit to the $\nu(\tau, L)$ data in the ThL regime (this regime can be recognised by the condition $\nu(\tau, L)$ is independent of $L$) or from the figure of merit Eq. (14) gives a ThL temperature dependence

$$\nu(\tau) = 1.015 - (0.052)(1.8^{1.8}/[1 + 0.052^{1.8}])$$

i.e. $\nu = 1.015(10), \theta \sim 1.8, a_\xi = 0.052(5)$ or

$$\xi(\beta) = 0.95\tau^{-1.015}[1 + 0.052^{1.8}]$$

The $\gamma(\tau)$ curve evaluated directly from the HTSE se-
ries \cite{31} is exact at high and moderate temperatures once $\beta_c$ is known and is fully consistent with the ThL $\gamma(\tau, L)$ simulation data in the appropriate $\tau$ range. The simulation and HTSE data taken together show that for the leading confluent correction term $\tau(\theta)$ the prefactor is tiny, $a_\chi \sim 0$, leaving only a weak high order effective correction term, so
\[
\gamma(\tau) = 2.37 + (0.012)(8)\tau^8/(1 - 0.012\tau^8)
\]  
(28)
i.e. $\gamma = 2.37(2), \theta_{eff} \sim 8(2), a_\chi = -0.012(2)$ or
\[
\chi(\beta) = 1.012\tau^{-2.37}[1 - 0.012\tau^8]
\]  
(29).

From the scaling rule $\gamma = (2 - \eta)\nu$ and the estimates for the critical $\gamma$ and $\nu, \eta = -0.33(4)$.

These critical exponents : $\nu = 1.015(10), \gamma = 2.37(2), \eta = -0.33(4)$ estimated from the ThL data are thus in full agreement with the FSS estimates above, and with the FSS estimations of Ref. \cite{14} : $\nu = 1.02(2), \gamma = 2.32(8), \eta = -0.275(25)$. The high effective correction exponent observed in $\chi(\tau)$ is also consistent with the high effective FSS correction exponent $\omega \sim 2.5$ reported in Ref. \cite{14}. There is thus good overall consistency. The critical temperature and exponent values (except for $\omega$) are particularly reliable in this case because of the accidental weakness of the correction to scaling terms.

With the ThL $\chi(\beta, \infty)$ and $\xi(\beta, \infty)$ expressions in hand we make up the Privman-Fisher susceptibility plot $\chi(\beta, L)/\chi(\beta, \infty) = F(L/\xi(\beta, \infty))$, Fig. 6 The whole data set for the entire paramagnetic temperature region and all $L$ (so covering the FSS regime, the ThL regime, and the intermediate regime) shows an excellent scaling plot, which can be fitted by the ansatz Eq. (17) with the fit parameters $\eta = -0.33, a = 1.88, b = 0.41$. The Privman-Fisher correlation length plot with large $L$ fit parameters $a = 0.80$ and $b = 0.35$ in Eq. (18) is shown in Fig. 11. There is a small but visible finite size scaling correction which we have not attempted to analyse.

We have no data for the diluted bimodal ISG of Ref. \cite{14} but in view of the fact that the ThL data show even weaker corrections to scaling than for the Gaussian, the exponent and dimensionless parameter estimates $\nu = 1.025(15), \gamma = 2.33(6), \eta = -0.275(25), g(\beta_c) = 0.472(2)$ are certainly very reliable also.

THE BIMODAL ISG IN DIMENSION 4

For the 4d bimodal ISG, from early simulation measurements up to $L = 10$ a critical temperature $\beta_c = 0.493(7)$ (i.e. $\beta_c^2 = 0.243(7)$) was estimated \cite{40} using the Binder parameter crossing point criterion. However, finite size corrections to scaling were not allowed for. The exponent estimates were $\nu = 1.0(1)$ and $\eta = -0.30(5)$. From extensive domain wall free energy measurements to

\begin{figure}[h]
\centering
\includegraphics[width=0.5\linewidth]{fig6}
\caption{(Color online) The 4d Gaussian $\partial W_q(\beta, L)/\partial \beta$ peak maxima heights as a function of $L$. Straight line : fit with slope $1/\nu = 1.00(1)$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\linewidth]{fig7}
\caption{(Color online) The 4d Gaussian effective $\gamma(\tau) = \partial \chi(\beta, L)/\partial \tau$ for $\beta_c = 0.5555$. Symbol code as in Fig. 4 Green curve : HTSE data points. Blue curve : ThL regime data fit.}
\end{figure}

$L = 10$ Hukushima gave an estimate $\beta_c = 0.50(1)$ (i.e. $\beta_c^2 = 0.25(1)$) \cite{41}. Inspection of the raw data \cite{42} shows strong finite size corrections; extrapolation to larger $L$ leads to an infinite size limit $\beta_c$ definitely greater than 0.50. The HTSE critical temperature and exponent estimates are \cite{31} $\beta_c^2 = 0.26(2)$ or $\beta_c = 0.51(2), \gamma = 2.5(3)$, and a leading confluent correction exponent $\theta \sim 1.5$.

From bimodal 4d simulations with impressive numbers of samples up to $L = 16$ to a maximum inverse temperature $\beta = 0.5025$ Baños et al. \cite{19} give estimates $\omega = 1.04(10), \nu = 1.068(7), \eta = -0.320(13)$ (so $\gamma = 2.48(1)$}
by the scaling rules) and \( \beta_c = 0.5023(6) \), all from a FSS analysis with \( t \) as scaling variable. We can note that the inverse temperatures for the \( U_4(\beta, L) \), \( U_2(\beta, 2L) \) and \( \xi/L(\beta, L) \), \( \xi/L(\beta, 2L) \) crossing points should scale linearly with \( 1/L^{\omega+1/\nu} \), Eq. 30 of [19]. However, it can be seen that in the crossing point plot, Fig. 9 of Ref. [19], with the authors’ preferred values of \( \omega \) and \( \nu \) the scaling against \( 1/L^{\omega+1/\nu} \) is far from being linear. Baños et al. obtain fits by discarding the lower \( L \) points or by introducing \textit{ad hoc} higher order terms. An alternative explanation could be that despite the precautions taken complete equilibration has not quite been achieved for the largest \( \beta \) measurements at the largest size \( L = 16 \) (where equilibration is the most difficult), so the \( L = 16 \) to \( L = 8 \) crossing points should be dropped. In this case the remaining data points appear more consistent and indicate a rather larger critical inverse temperature \( \beta_c \) and a rather larger \( \omega + 1/\nu \). Our independent data reported below are compatible with these modified values.

We have made measurements in the critical region of the standard finite size Binder cumulant \( g(\beta, L) \), the dimensionless parameters \( h(\beta, L) \), \( W_q(\beta, L) \), and the normalized correlation length \( \xi/L(\beta, L) \). The present data are for all \( L \) from 3 to 12 and span the estimated inverse critical temperature \( \beta_c \). From fits to the \( g(\beta, L) \), \( h(\beta, L) \), \( W_q(\beta, L) \) and \( \xi/L(\beta_c, L) \) data we obtain estimates for the critical temperature \( \beta_c \) and the exponent \( \nu \) together with the dimensionless critical values \( g(\beta_c, \infty) \), \( h(\beta_c, \infty) \), \( W_q(\beta_c, \infty) \) and \( \xi/L(\beta_c, \infty) \). As explained above rather than following the traditional crossing point analysis, the raw dimensionless parameter data are fitted directly to Eq. 25 at fixed temperatures in the critical region, Figs. 11, 12, 13 and 14. Consistent independent fits could be made to the \( g(\beta, L) \), \( h(\beta, L) \), \( W_q(\beta, L) \) and \( \xi/L(\beta, L) \) results for the data from \( L = 4 \) to \( L = 12 \). Due to higher order corrections we have eliminated \( L = 3 \) from the fitting process. The fit values with error bars are given in Tab. III. However, the \( \xi/\beta, L \) data have stronger corrections and so are less informative. As for the Gaussian data we have chosen a fixed \( \omega = 1.3 \) when determining the other parameters. Using \( \omega = 1.2 \) or \( \omega = 1.4 \) changes the fitted parameter only slightly but not outside any margin of error we can hope to aspire to.
Estimates for the important critical values of the dimensionless parameters for 4d bimodal ISG are not quoted explicitly in Ref. [19], but strict limits $U_4(\beta_c) < 2.00$ i.e. $g(\beta_c, \infty) > 0.50$, and $\xi/L(\beta_c) > 0.45$ can be read off the relevant figures.

From the present bimodal analysis, $g(\beta_c, \infty) = 0.526(6)$ (or $U_4(\beta_c, \infty) = 1.948(12)$), $h(\beta_c, \infty) = 0.4412(27)$, $[\xi(\beta, L)/L](L \to \infty) = 0.478(3)$ and $W_q(\beta_c, \infty) = 0.279(3)$. The $U_4$ and $[\xi(\beta, L)/L](L \to \infty)$ estimates appear fully compatible with the raw data of Ref. [19].

These binomial ISG critical values can be compared with the Gaussian ISG estimates above: $g(\beta_c, \infty) = 0.474(1)$ (or $U_4 = 2.052(2)$), $h(\beta_c, \infty) = 0.3847(11)$,
\[ W_k(\beta_c, \infty) = 0.241(1) \text{ and } \xi/L(\beta, L \to \infty) = 0.448(2), \]
and with the values quoted for the 4d Gaussian and diluted bimodal models in Ref. [14] : \( g(\beta_c, \infty) = 0.470(5) \) (or \( U_4(\beta_c) = 2.060(10) \)) and \( g(\beta_c, \infty) = 0.472(2) \) (or \( U_4(\beta_c) = 2.056(4) \)) respectively; together with \( \xi/L(\beta_c) \sim 0.44 \) for both systems by inspection of the figures of Ref. [14]. The differences between the bimodal ISG values and those of the Gaussian and diluted bimodal ISGs are not identical; as these critical limit values are characteristics of a universality class, even before discussing the critical exponents the system-to-system differences already clearly demonstrate that the 4d bimodal ISG is not in the same universality class as the other two 4d ISGs.

Fixing \( \beta_c = 0.506 \), a log-log plot of \( \chi(\beta_c, L)/L^2 \) against \( L \) can be fitted using \( \chi(\beta_c, L)/L^2 = L^{-\eta} \) with \( \eta = -0.43(2) \) and a weak correction term. The FSS estimate for \( \gamma \) is then \( \gamma = (2 - \eta)\nu = 2.67(5) \).

We show in Fig. 15 and Fig. 16 \( \gamma(\tau) \) and \( \nu(\tau) \) plots for 4d bimodal ISG using as the inverse critical temperature \( \beta_c = 0.506 \) as estimated from the FSS analysis. Satisfactory fits can be obtained with \( \nu = 1.14(2) \), \( \theta \sim 1.8 \), and \( \gamma = 2.76(3) \) with factors \( a_\xi = 0.10 \), \( a_\chi = 0.69 \) and a weak high order correction term for \( \gamma(\tau) \), \( \eta \sim 8 \), \( b_\chi \sim -0.01 \). There is reasonable consistency between the FSS exponent estimates and the ThL estimates. The estimated critical exponents \( \gamma \) and \( \nu \) for the 4d bimodal ISG are considerably higher than the values estimated above for the 4d Gaussian, \( \gamma = 2.37(1) \) and \( \nu = 1.015(10) \), and from \( \gamma = 2.33(6) \) and \( \nu = 1.025(15) \) and for the diluted bimodal ISGs [14].

In Ref. [19], the FSS estimate of the bimodal ISG critical temperature is \( \beta_c = 0.5023(6) \). Fixing \( \beta_c \) at this value, a ThL \( \gamma(\tau) \) fit leads to an estimate \( \gamma = 2.60(3) \), still considerably above the Gaussian value. However, the fit requires a correction exponent \( \theta \sim 2.0 \) corresponding to \( \omega \sim 1.8 \), which is considerably higher than the Ref. [19] value \( \omega = 1.04(10) \).

Finally Privman-Fisher ansatz plots can be made up for all the data. Assuming \( \beta_c = 0.506 \) and the fit parameters from the \( \gamma(\tau) \) and \( \nu(\tau) \) plots, the Privman-Fisher scalings are shown in Figs. 17 and 18. It can be seen that the \( \chi \) scaling is excellent on the scale of the figure over the entire range of paramagnetic temperatures and of sizes. The scaling curve can be fitted accurately by the explicit form used to fit the simple cubic Ising ferromagnet data, Eq. (17) with \( \eta = -0.40 \), \( a = 1.95 \), and \( b = 0.43 \) for the \( \chi \) scaling plot. For the \( \xi \) scaling plot a good large-\( L \) fit is obtained using the ansatz Eq. (18) with \( a = 0.74 \), and \( b = 0.35 \) with indications of finite size corrections which we have not attempted to analyse. It can be noted that both for \( \chi \) and \( \xi \) the values of the fit parameters (which should be characteristic of a universality class) obtained for the bimodal and Gaussian ISGs are not identical.

The present critical exponents \( \gamma \) for both the bimodal and Gaussian ISGs are consistent with but are more accurate than the HTSE estimates, principally because the uncertainty in \( \beta_c^2 \) is considerably reduced thanks to the FSS simulation analysis. The 4d bimodal \( \gamma \) and \( \nu \) exponent estimates can be compared to the values found above for the 4d Gaussian, and with the published estimates Gaussian and diluted bimodal systems [14]. The critical exponents of the 4d bimodal ISG are quite distinct from those of the 4d Gaussian and diluted bimodal ISGs.
FIG. 17: (Color online) The 4d bimodal Privman-Fisher $\chi$ plot, $\chi(\tau, L)/\chi(\tau, \infty)$ against $L/\xi(\tau, \infty)$. Symbol code as in Fig. 1. Curve: fit (see text)

FIG. 18: (Color online) The 4d bimodal Privman-Fisher $\xi$ plot, $\xi(\tau, L)/\xi(\tau, \infty)$ against $L/\xi(\tau, \infty)$. Symbol code as in Fig. 1. Curve: fit (see text)

CONCLUSIONS

We have first spelt out in detail the procedure used for scaling over the whole paramagnetic temperature range with the appropriate ISG scaling variable, $\tau = 1 - (\beta/\beta_c)^2$, and scaling expressions which include the correction terms.

Simulations on the 4d Gaussian and bimodal ISGs up to size $L = 12$ and $L = 14$ respectively are first analysed in the critical temperature range to obtain estimates for the critical inverse temperatures $\beta_c$, together with FSS estimates for the critical values for dimensionless parameters $g(\beta_c, \infty)$, $h(\beta_c, \infty)$, $W_q(\beta_c, \infty)$ and $[\xi/L(\beta_c, L \to \infty)]$, and for the susceptibility and correlation length critical exponents $\gamma$ and $\nu$. The critical temperatures $\beta_c$ are in full agreement with, but are considerably more precise than, the estimates from high temperature scaling expansions alone [31]. They also improve on previous numerical estimates [14, 19]. Using the FSS estimates for $\beta_c$, data for the temperature dependent effective exponents $\gamma(\tau)$ and $\nu(\tau)$ in the thermodynamic limit (ThL) regime were fitted to obtain complementary critical exponent $\gamma$, $\nu$ and $\theta$ estimates together with the strengths of the correction terms. There is good agreement between the FSS and ThL estimates. Overall Privman-Fisher scalings with the estimated critical parameters, covering the whole paramagnetic temperature range and all sizes $L$, validate the analysis.

The critical values of the dimensionless parameters and the critical exponents are characteristic of a universality class. For each of these observables the values for the 4d bimodal ISG are quite different from those of the 4d Gaussian ISG (or from those of the 4d diluted bimodal ISGs [14]), showing that these systems lie in different universality classes. It can be concluded that for ISG transitions in dimension four at least and probably more generally [35, 36], the critical parameters depend on the form of the interaction distribution, so the standard RGT universality rules do not apply.

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