ASYMPTOTIC EXPANSION OF WARLIMONT FUNCTIONS ON WRIGHT SEMIGROUPS

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Abstract. We calculate full asymptotic expansions of prime-independent multiplicative functions on additive arithmetic semigroup that satisfy a strong form of Knopfmacher’s axioms. When applied to the semigroup of unlabeled graphs, our method yields detailed asymptotic information on how graphs decompose into connected components. As a second class of examples, we discuss polynomials in several variables over a finite field.

1. Introduction

Let \( G_n \) be the number of unlabeled graphs with \( n \) vertices and let \( G_n^+ \) be the number of connected unlabeled graphs with \( n \) vertices. Using the fact that the sequences \( \{G_n\} \) and \( \{G_n^+\} \) are related by the identity

\[
\sum_{n=0}^{\infty} G_n x^n = \prod_{m=1}^{\infty} (1 - x^m)^{-G_n^+},
\]

Wright [5] proved that \( G_n \sim G_n^+ \) i.e. almost all graphs are connected. Armed with a full asymptotic expansion for \( G_n \) [6], Wright further improved this result by constructing [7] a sequence \( \{\omega_s\} \) of polynomials such that

\[
G_n^+ = G_n + \sum_{s=1}^{R-1} \omega_s(n) G_{n-s} + O(n^R G_{n-R})
\]

for all positive integers \( R \).

In the context of abstract analytic number theory [4], Knopfmacher [2] observed that (1) is a particular case of an Euler product type of identity that holds for arbitrary additive arithmetical semigroups and the methods of [5] can be used to study the distribution of certain arithmetical functions on additive arithmetical semigroups in which almost all elements are prime. For instance, if \( d_2 \) is the divisor function that to each unlabeled graph \( g \) assigns the number of ways to write \( g \) as a disjoint union of an ordered pair of graphs then

\[
\lim_{n \to \infty} \frac{1}{G_n} \sum d_2(g) = 2 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{G_n} \sum (d_2(g) - 2)^2 = 0,
\]

where both sums are taken over all graphs \( g \) with \( n \) vertices.

The goal of the present paper is to investigate Knopfmacher’s suggestion [2] that restricting to arithmetical semigroups in which the total number of elements is related to the number of prime elements by a formula analogous to (2) might lead to a strengthening of (3). To
illustrate our results with an example, consider again the divisor function $d_2$ on the semigroup of graphs. We prove that for every positive integer $M$, there exists a sequence $\{\tau_s(n)\}$ of polynomials such that

$$
\frac{1}{G_n} \sum(d_2(g) - 2)^M = 2^M \sum_{s=1}^{R-1} \tau_s(n)2^{-sn} + O(n^{2R-1}2^{-Rn})
$$

for every integer $R$. Clearly, (3) can be recovered by setting $M = 1$ and $M = 2$ in (4) and taking the limit as $n \to \infty$. More generally, we show that (4) is a particular case of a formula that holds if $d_2$ is replaced by an arbitrary Warlimont function i.e. a multiplicative prime-independent function whose restriction to power of primes grows in a prescribed way. Even more generally, the semigroup of graphs can be replaced by any Wright semigroup which we define to be an additive arithmetical semigroup subject to a growth condition introduced in [7]. Examples of Wright semigroups include the semigroup of unlabeled graphs with an even number of edges and the semigroup of polynomials in at least two variables over a finite field.

The paper is organized as follows. Section 2 contains the main technical results used in the rest of the paper. We work with triples of sequences related by a generalization of (1) that were introduced in [4]. The main result is Theorem 5 which can be thought of as a generalization of [7], modeled after the way in which [1] generalizes [5]. In Section 3 after introducing the key notions of Wright semigroup and of Warlimont function, provide asymptotic formulas for moments of Warlimont functions in terms of the number of elements of given degree in the underlying (not necessarily Wright) semigroup. In the special case of Wright semigroup, we construct full asymptotic expansions generalizing (1). We illustrate our results in Section 4 by calculating asymptotic expansion of some of the arithmetical functions considered in [2] on three examples of Wright semigroups: the semigroup of all unlabeled graphs, the semigroup of unlabeled graphs with an even number of edges and the semigroup of non-zero polynomials (up to scaling) in at least two variables over a finite field.

Acknowledgments: Part of this work was carried out during the Summer of 2016 at VCU and supported by a UROP Summer Research Fellowship.

2. Warlimont Triples

Definition 1. A Warlimont triple is a triple $\{(T_n), (t_n), (a_n)\}$ of sequences of non-negative real numbers related by the following identity of formal power series

$$
\sum_{n=0}^{\infty} T_n x^n = \prod_{m=1}^{\infty} \left( \sum_{k=0}^{\infty} a_k x^m \right)^{t_m}
$$

and such that

i) $T_0 = a_0 = 1$;

ii) $a_1 > 0$;

iii) $t_m \in \mathbb{Z}$ for all $m$ and $t_m > 0$ for all but finitely many $m$. 

Lemma 2. Let \( \{T_n\}, \{t_n\}, \{a_n\} \) be a Warlimont triple and consider the sequences \( \{v_n\} \), \( \{\beta_n\} \) and \( \{b_n\} \) defined the recursion formulas

\[
\begin{align*}
v_n &= T_n - \sum_{s=1}^{n-1} \frac{s}{n} v_s T_{n-s} \\
\beta_n &= - \sum_{s=0}^{n-1} \beta_s T_{n-s} \\
b_n &= na_n - \sum_{s=1}^{n-1} b_s a_{n-s}
\end{align*}
\]

with initial conditions \( v_1 = T_1, \beta_0 = 1, b_1 = a_1 \). Then for all \( n \)

1) \( v_n = \sum_{d|n} \frac{a_d}{n} b_{n/d} \), where the sum is over all integers \( 1 < d \leq n \) that divide \( n \);
2) \( \beta_n = - \sum_{s=1}^{n} \frac{s}{n} v_s \beta_{n-s} \);
3) for every positive integer \( R \)

\[
\sum_{s=0}^{R-1} \beta_s T_{n-s} = v_n + \frac{1}{n} \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} sv_s T_{n-r-s}.
\]

Proof: Using formal term-by-term differentiation it is easy to show that (6) and (7) are equivalent to the formal identities

\[
\log \left( \sum_{n=0}^{\infty} T_n x^n \right) = \sum_{m=1}^{\infty} v_m x^m
\]

and, respectively,

\[
\log \left( \sum_{n=0}^{\infty} a_n x^n \right) = \sum_{s=1}^{\infty} \frac{b_s}{s} x^s.
\]

Taking the formal logarithm of (5) and substituting (9), (10), we obtain

\[
\sum_{m=1}^{\infty} v_m x^m = \sum_{r,s=1}^{\infty} t_r \frac{b_s}{s} x^{rs}
\]

from which 1) easily follows. Since (7) is equivalent to the identity

\[
\left( \sum_{s=0}^{\infty} \beta_s x^s \right) \left( \sum_{n=0}^{\infty} T_n x^n \right) = 1
\]

of formal power series, then taking formal logarithms yields

\[
\log \left( \sum_{s=0}^{\infty} \beta_s x^s \right) = - \sum_{m=1}^{\infty} v_m x^m.
\]
Comparing with (6) and (9) proves 2). It follows from 2) that
\[
R^{-1} \sum_{u=0}^{R-1} \sum_{r=0}^{u} \beta_r ((n-u)v_{n-u}T_{u-r} + (u-r)v_{u-r}T_{n-u}) = v_n - \sum_{u=0}^{R-1} u \beta_u T_{n-u}
\]
and thus
\[
\sum_{s=0}^{R-1} \beta_s T_{n-s} - v_n = \frac{1}{n} \sum_{u=0}^{R-1} \sum_{r=0}^{u} \beta_r \left( \frac{n-r}{R-r} T_{n-r} - (n-u)v_{n-u}T_{u-r} - (u-r)v_{u-r}T_{n-u} \right)
\]
\[
= \frac{1}{n} \sum_{r=0}^{R-1} \beta_r \left( (n-r)T_{n-r} - \sum_{s=0}^{R-r-1} (n-r-s)T_s v_{n-r-s} + \sum_{s=0}^{R-r-1} sv_s T_{n-r-s} \right)
\]
where the last equality is obtained applying (6) to \( T_{n-r} \) for each \( r \in \{0,1,\ldots,R-1\} \).

**Lemma 3.** Let \( \{T_n\}, \{t_n\}, \{a_n\} \) be a Warlimont triple. Then
\[
a_1 \sum_{s=0}^{n/2} T_s t_{n-s} \leq T_n
\]
for all \( n \).

**Proof:** Since
\[
\prod_{m=N+1}^{\infty} \left( \sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} \in 1 + x^{N+1}R[[x]]
\]
for every integer \( N \geq 0 \), then
\[
\sum_{n=0}^{N} T_n x^n = \prod_{m=1}^{N} \left( \sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} + x^{N+1}R[[x]]
\]
and thus
\[
\sum_{n=0}^{N} T_n x^n = \left( \sum_{s=0}^{[N/2]} T_s x^s \right) \prod_{m=[N/2]+1}^{N} \left( \sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} + x^{N+1}R[[x]].
\]
On the other hand by assumption \( t_m \) is a non-negative integer for all \( m \) and thus by the binomial theorem
\[
\left( \sum_{k=0}^{\infty} a_k x^{km} \right)^{t_m} = 1 + a_1 t_m x^m + x^{2m}R[[x]].
\]
Since the sequences \( \{a_k\}, \{t_m\} \) and \( \{T_n\} \) are non-negative, the Lemma follows by substituting (12) into (11) and comparing coefficients.
Lemma 4. Let \((\{T_n\}, \{t_n\}, \{a_n\})\) be a Warlimont triple such that \(\log(a_n) = O(n)\). Then for every non-negative integer \(R\)

\[
|v_n - a_1 t_n| = \begin{cases} 
O(T_{n-R}) & \text{if } T_{n-1} = o(T_n) \\
O(t_{n-R}) & \text{if } t_{n-1} = o(t_n).
\end{cases}
\]

Proof: Assume \(T_{n-1} = o(T_n)\). Since \(\log(a_n) = O(n)\), there exists \(r > 1\) such that \(a_n \leq r^n\) for all \(n\). By induction of the definition of \(\{b_n\}\), we obtain \(|b_n| \leq (3r)^n\) for all \(n\). Moreover, since \(T_{n-1} = o(T_n)\), there exists a constant \(C\) such that \(T_n \leq C(3r)^{-2m}T_{n+m}\) for all \(n, m\). Therefore, using Lemma 2 and Lemma 3

\[
|v_n - a_1 t_n| = \sum_{d/n} d/nT_d(3r)^{n/d} \leq CT_{n-R} \sum_{d/n} (3r)^{-n+2R+2d} = O(T_{n-R}).
\]

The proof for the case \(t_{n-1} = o(t_n)\) is similar and left to the reader.

Theorem 5. Let \((\{T_n\}, \{t_n\}, \{a_n\})\) be a Warlimont triple such that \(\log(a_n) = O(n)\) and let \(R\) be a positive integer. Then the following are equivalent

1) \(T_{n-1} = o(T_n)\) and

\[
\sum_{s=R}^{n-R} T_s T_{n-s} = O(T_{n-R});
\]

2) \(T_{n-1} = o(T_n)\) and

\[
a_1 t_n = \sum_{s=0}^{R-1} \beta_s T_{n-s} + O(T_{n-R});
\]

3) \(t_{n-1} = o(t_n)\) and

\[
T_n = a_1 \sum_{s=0}^{R-1} T_s t_{n-s} + O(t_{n-R});
\]

4) \(t_{n-1} = o(t_n)\) and

\[
\sum_{s=R}^{n-R} t_s t_{n-s} = O(t_{n-R}).
\]

Proof: Assume 1) holds. Using Lemma 3, Lemma 4 and \(T_{n-1} = o(T_n)\), we obtain

\[
|v_n| \leq |v_n - a_1 t_n| + a_1 t_n = O(T_n).
\]
Therefore, exists an integer \( N > R \) and a constant \( C > 0 \) such that \( |v_n| \leq CT_n \leq CT_{n+r} \) for all \( n \geq N \) and for all \( r \in \{0, \ldots, R - 1\} \). Combining this observation with Lemma 2 yields

\[
|v_n - \sum_{s=0}^{R-1} \beta_s T_{n-s}| \leq \sum_{r=0}^{R-1} \beta_r \sum_{s=R-r}^{n-R} \frac{s}{n} |v_s| T_{n-s}
\]

\[
\leq \sum_{r=0}^{R-1} \beta_r \left( \sum_{s=R-r}^{N-1} |v_s| T_{n-r-s} + C \sum_{s=N}^{n-R} T_s T_{n-r-s} \right)
\]

\[
\leq \sum_{r=0}^{R-1} \beta_r \left( \sum_{s=R-r}^{N-1} |v_s| T_{n-r-s} + C^2 \sum_{s=R}^{n-R} T_s T_{n-s} \right)
\]

\[
= O(T_{n-R})
\]

and thus

\[
|a_1 t_n - \sum_{s=0}^{R-1} \beta_s T_{n-s}| \leq |a_1 t_n - v_n| + |v_n - \sum_{s=0}^{R-1} \beta_s T_{n-s}| = O(T_{n-R}).
\]

Hence, 1) implies 2). Assume 2) holds. Then

\[
a_1 \sum_{s=0}^{R-1} T_s t_{n-s} = \sum_{s=0}^{R-1} T_s \sum_{r=0}^{R-1-s} \beta_r T_{n-r-s} + O(T_{n-R})
\]

\[
= \sum_{s=0}^{R-1} \sum_{u=s}^{R-1} T_s \beta_{u-s} T_{n-u} + O(T_{n-R})
\]

\[
= \sum_{u=0}^{R-1} T_{n-u} \sum_{s=0}^{u} T_s \beta_{u-s} + O(T_{n-R})
\]

\[
= T_n + O(T_{n-R})
\]

where the last equality is obtained using the definition of the sequence \( \{\beta_n\} \). In particular, setting \( R = 1 \) we obtain \( a_1 t_n - T_n = O(T_{n-1}) = o(T_n) \) which implies \( a_1 t_n \sim T_n \) and hence \( o(t_{n-1}) = O(T_{n-1}) = o(T_n) = o(t_n) \). Therefore, 2) implies 3). Assume 3) holds. By Lemma 3

\[
a_1^{\left\lceil n/2 \right\rceil} \sum_{s=R}^{n} t_s t_{n-s} \leq a_1 \sum_{s=R}^{\left\lceil n/2 \right\rceil} T_s t_{n-s} \leq T_n - a_1 \sum_{s=0}^{R-1} T_s t_{n-s} = O(t_{n-R}).
\]

This proves 4) since

\[
\sum_{s=R}^{n-R} t_s t_{n-s} = 2 \sum_{s=R}^{\left\lceil n/2 \right\rceil} t_s t_{n-s} + O(t_{n-R}).
\]
Finally, assume 4) holds. Lemma 4 implies \(|v_n - a_1 t_n| = O(t_{n-R}) = o(t_n)\) and thus \(v_n \sim a_1 t_n\).

This implies that there exists an integer \(N \geq R\) and constants \(c, C > 0\) such that

\[0 < ct_n \leq v_n \leq Ct_n\]

for all \(n \geq N\). As a consequence,

\[v_{n-1} = O(t_{n-1}) = o(t_n) = o(v_n)\]

and

\[\sum_{j=N}^{n-R} v_{n-j} v_j \leq C^2 \sum_{j=R}^{n-1} t_{n-j} t_j = O(t_{n-R}) = O(v_{n-R})\].

For each \(n \geq N\), let

\[M_n = \max\{\frac{T_j}{v_j} | N \leq j \leq n\}\].

By Lemma 2 we obtain

\[|T_n - v_n| \leq \sum_{j=1}^{n-1} |v_{n-j}| T_j\]

\[\leq \sum_{j=1}^{N-1} v_{n-j} T_j + M_{n-1} \left(\sum_{j=N}^{n-N} v_{n-j} v_j + \sum_{j=n-N+1}^{n-1} |v_{n-j}| v_j\right)\]

\[= o(v_n)(1 + M_{n-1})\]

where (14) and (15) were used to obtain the last equality. Hence there exists \(N_1 \geq N\) such that for all \(n \geq N_1\)

\[\frac{T_n}{v_n} \leq \frac{3 + M_{n-1}}{2}\]

and thus

\[M_n = \max\left\{M_{n-1}, \frac{T_n}{v_n}\right\} \leq \max\{M_{n-1}, 3\}\].

This shows that the sequence \(\{M_n\}\) is bounded i.e. there exists a constant \(K > 0\) such that \(T_n \leq K v_n\) for all \(n \geq N\). Therefore, using (13) and Lemma 3 we obtain

\[T_{n-1} = O(v_{n-1}) = O(t_{n-1}) = o(t_n) = o(T_n)\].

Moreover (14) yields

\[\sum_{s=R}^{n-R} T_{n-s} T_s \leq 2 \sum_{s=R}^{N} T_s T_{n-s} + K^2 \sum_{s=N}^{n-N} v_{n-s} v_s = O(T_{n-R}) + O(v_{n-R}) = O(T_{n-R})\].

This concludes the proof that 4) implies 1) and the Theorem is proved.
Remark 6. Let \( \{T_n\}, \{t_n\}, \{a_n\} \) be a Warlimont triple that satisfies the equivalent conditions of Theorem 5 for some \( R > 2 \). Then
\[
\sum_{s=R}^{n-R} T_s T_{n-s} = \sum_{s=R}^{n-R} T_s T_{n-s} + 2T_{R-1}T_{n-R+1} = O(T_{n-R}) + O(T_{n-R+1}) = O(T_{n-R+1})
\]
and thus \( \{T_n\}, \{t_n\}, \{a_n\} \) satisfies the equivalent conditions of Theorem 5 for any positive integer less or equal than \( R \). In particular, \( t_{n-1} = o(t_n) \) and \( T_n \sim a_1 t_n \).

3. Warlimont Functions and Wright Semigroups

Definition 7. An additive arithmetical semigroup is a pair \((G, +, \partial)\) consisting of an abelian semigroup \((G, +)\) with identity and a semigroup homomorphism \( \partial : (G, +) \to (\mathbb{Z}_{\geq 0}, +) \) such that
i) the cardinality \( G_n \) of the preimage \( \partial^{-1}(n) \) is finite for all \( n \);
ii) \( G \) is freely generated by \( G^+ \subseteq G \).
We denote by \( G^+_n \) the cardinality of the set \( \partial^{-1}(n) \cap G^+ \).

Remark 8. Let \((G, +, \partial)\) be an additive arithmetical semigroup. As pointed out in [2,4], \((\{G_n\}, \{G^+_n\}, \{1\})\) is a Warlimont triple.

Definition 9. A Wright semigroup is an additive arithmetical semigroup \((G, +, \partial)\) such that
\[
\log(G_n) = \alpha n^{a+1} + \beta n \log(n) + \gamma n + O(n^b)
\]
for some real numbers \( \alpha, \beta, \gamma, a, b \) such that \( \alpha > 0 \) and \( 0 < b < a \).

Definition 10. Let \( R \) be a positive integer. We say that an additive arithmetical semigroup \((G, +, \partial)\) satisfies axiom \( W_R \) if \( G_{n-1} = o(G_n) \) and
\[
\sum_{s=R}^{n-R} G_s G_{n-s} = O(G_{n-R}).
\]

Remark 11. Let \((G, +, \partial)\) be an additive arithmetical semigroup that satisfies axiom \( W_R \) for some positive integer \( R \). Combining Remark 6 and 8 we conclude that \((G, +, \partial)\) satisfies axiom \( W_{R'} \) for any positive integer \( R' \leq R \). In particular, \( G_n \sim G^+_n \) and \( G^+_n = o(G^+_n) \) i.e. the additive arithmetical semigroup \((G, +, \partial)\) satisfies both axiom \( G_1 \) and axiom \( G_2 \) as defined in [2]. Notice that the combination of Axiom \( G_1 \) and Axiom \( G_2 \) is slightly weaker than axiom \( W_1 \) since \( \sum_{s=1}^{n-1} G_s G_{n-s} = o(G_n) \) does not necessarily imply \( \sum_{s=1}^{n-1} G_s G_{n-s} = O(G_{n-1}) \).

Proposition 12. Every Wright semigroup satisfies axiom \( W_R \) for every positive integer \( R \).

Proof: This is a straightforward consequence of the definitions and Theorem 7 of [7].

Definition 13. Let \((G, +, \partial)\) be an additive arithmetical semigroup. A function \( F : G \to \mathbb{R} \) is multiplicative if \( F(g_1 + g_2) = F(g_1)F(g_2) \) for all \( g_1, g_2 \in G \) coprime. We say that \( F \) is prime-independent if there exists a sequence \( \{F^+_n\} \) such that \( F^+_n = F(np) \) for every \( p \in G^+ \).
and every positive integer \( n \). For every function \( F : G \to \mathbb{R} \), we denote by \( \{ F_n \} \) the sequence defined by setting
\[
F_n = \sum_{d(g)=n} F(g)
\]
for each non-negative integer \( n \). A Warlimont function is a non-negative multiplicative prime-independent function such that \( \log(F_n^+) = O(n) \) and \( F_1^+ > 0 \). The normalization of a Warlimont function \( F \) is the (not necessarily multiplicative) function \( \widetilde{F} : G \to \mathbb{R} \) such that \( \widetilde{F}(g) = F(g)/F_1^+ \) for all \( g \in G \).

**Example 14.** Let \( (G, +, \partial) \) be an additive arithmetical semigroup and let \( F : G \to \mathbb{R} \) be such that \( F(g) = 1 \) for all \( g \in G \). Then \( F \) is a Warlimont function and \( F_n = \widetilde{F}_n = G_n \) for all \( n \).

**Example 15.** Let \( (G, +, \partial) \) be an additive arithmetical semigroup and, for each \( k \geq 2 \), consider the generalized divisor function \( d_k : G \to \mathbb{R} \) that to each \( g \in G \) assigns the number \( d_k(g) \) of \( k \)-tuples \( (g_1, \ldots, g_k) \in G^k \) such that \( g = g_1 + \ldots + g_k \). Then \( d_k \) is multiplicative, prime-independent and \( (d_k)^+_n = \binom{n+k-1}{k-1} \) for each integer \( n \geq 1 \). Therefore, \( d_k \) is Warlimont.

**Example 16.** Let \( (G, +, \partial) \) be an additive arithmetical semigroup and consider the unitary divisor function \( d_\ast : G \to \mathbb{R} \) that to each \( g \in G \) assigns the number \( d_\ast(g) \) of coprime pairs \( (g_1, g_2) \) such that \( g = g_1 + g_2 \). Then \( d_\ast \) is multiplicative, prime-independent and \( (d_\ast)^+_n = 2 \) for each integer \( n \geq 1 \). Therefore, \( d_\ast \) is Warlimont.

**Example 17.** Let \( (G, +, \partial) \) be an additive arithmetical semigroup and consider the prime divisor function \( B : G \to \mathbb{R} \) such that \( B(k_1p_1 + k_2p_2 + \ldots + k_rp_r) = k_1k_2\cdots k_r \) for any \( p_1, \ldots, p_r \in G \) primes and \( k_1, \ldots, k_r \) positive integers. Then \( B \) is multiplicative, prime-independent and \( B^+_n = n \) for each integer \( n \geq 1 \). Therefore, \( B \) is Warlimont.

**Remark 18.** Let \( F \) be a Warlimont function on an additive arithmetical semigroup \( (G, +, \partial) \). Then \( F^m : G \to \mathbb{R} \) such that \( F^m(g) = (F(g))^m \) is again a Warlimont function for every integer \( m \geq 1 \) since
\[
\log((F^m)^+_n) = m \log(F^+_n) = O(n).
\]
Moreover, \( \widetilde{F}^m = (\widetilde{F})^m \).

**Remark 19.** Let \( F \) be a Warlimont function on an additive arithmetical semigroup \( (G, +, \partial) \). Then, as observed in [4], \( \{ F_n \}, \{ G_n \}, \{ F^+_n \} \) is a Warlimont triple.

**Theorem 20.** Let \( (G, +, \partial) \) be an additive arithmetical semigroup that satisfies axiom \( W_1 \) and let \( F \) be a Warlimont function on \( G \). Then for every positive integer \( M \) there exist constants \( \xi_1, \ldots, \xi_{R-1} \) such that
\[
\sum_{d(g)=n} (\widetilde{F}(g)-1)^M = \sum_{s=1}^{R-1} \xi_s G_{n-s} + O(G_{n-R}).
\]
Proof: \((\{G_n\}, \{G^*_n\}, \{1\}) \text{ and } (\{F_n\}, \{G^*_n\}, \{F^*_n\})\) are both Warlimont triple by Remark 19 and Example 14. Since \(G_n\) satisfies axiom \(W_R\), it follows from Theorem 5 applied to the Warlimont triple \((\{G_n\}, \{G^*_n\}, \{1\})\) that 

\[
G^+_{n-1} = o(G^*_n),
\]

Moreover, if \(\{\beta_n\}\) is the sequence defined recursively by setting \(\beta_0 = 1\) and

\[
\beta_n = - \sum_{s=0}^{n-1} \beta_s G_{n-s}
\]

for every positive integer \(n\), then

\[
G^+_{n-s} = \sum_{r=0}^{R-1} \beta_r G_{n-s-r} + O(G^+_{n-s-R}).
\]

for all \(s \geq 0\). In particular, we can apply Theorem 5 to the Warlimont triple \((\{F_n\}, \{G^*_n\}, \{F^*_n\})\) and obtain

\[
F_n = F^+_1 \sum_{s=0}^{R-1} F_s G^+_{n-s} + O(G^+_n).
\]

Since by definition \(G^+_n \leq G_n\) for all \(n\) and \(G_{n-s-R} = o(G^+_n)\) for all \(s > 0\), then substituting (20) into (21) yields

\[
\tilde{F}_n = \sum_{s=0}^{R-1} \left( \sum_{r=0}^{s} \beta_r F_{s-r} \right) G_{n-s} + O(G_{n-R}).
\]

Using the binomial theorem and Remark 18 we obtain

\[
\sum_{\partial(g) = n} (\tilde{F}(g) - 1)^M = (-1)^M \sum_{\partial(g) = n} \sum_{m=0}^{M} (-1)^m \binom{M}{m} \tilde{F}^m(g)
\]

\[
= (-1)^M \sum_{m=0}^{M} (-1)^m \binom{M}{m} (\tilde{F}^m)_n.
\]

Applying (22) to the Warlimont function \(F^m\) and substituting into (24) (after an obvious rearrangement) yields

\[
\sum_{\partial(g) = n} (\tilde{F}(g) - 1)^M = \sum_{s=0}^{R-1} \xi_s G_{n-s} + O(G_{n-R}).
\]
with

\begin{align}
\xi_s &= (-1)^M \sum_{m=0}^{M} (-1)^m \binom{M}{m} \sum_{r=0}^{s} \beta_r (F^m)_{s-r} \\
&= (-1)^M \sum_{m=1}^{M} (-1)^m \binom{M}{m} \sum_{r=0}^{s} \beta_r (F^m)_{s-r}
\end{align}

for all \( s \in \{0, \ldots, R-1\} \) where the second equality follows from (19) and Example 14. This implies (17) since (by combining Remark 18 and Remark 19) \((F^m)_0 = 1\) for all \( m \) and thus

\[ \xi_0 = (-1)^M \sum_{m=0}^{M} (-1)^m \binom{M}{m} \beta_0 (F^m)_0 = 0. \]

**Definition 21.** Let \( F \) be a Warlimont function on an additive arithmetical semigroup \((G, +, \partial)\) and let \( M \) be a positive integer. We define the **normalized \( M \)-th moments of \( F \)** to be the functions \( \mu_{F,M} : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R} \) defined by

\[ \mu_{F,M}(n) = \frac{1}{G_n} \sum_{\partial(g)=n} (\bar{F}(g) - 1)^M \]

for all \( n \geq 0 \).

**Remark 22.** Let \( F \) be a Warlimont function on an additive arithmetical semigroup \((G, +, \partial)\). The average value of \( F \) on \( \partial^{-1}(n) \) is given by

\[ \frac{F_n}{G_n} = F_1^+ (1 + \mu_{F,1}(n)). \]

The higher normalized moments can be thought of as capturing the deviation of \( F \) from \( F_1^+ \). For instance, if \( \mu_{F,1}(n) = o(1) \), then

\[ \frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = (F_1^+)^2 \mu_{F,2}(n) \]

can be thought of as an asymptotic measure of the variance of \( F \) on \( \partial^{-1}(n) \).

**Corollary 23.** Let \( F \) be a Warlimont function on an additive arithmetical semigroup \((G, +, \partial)\) that satisfies axiom \( W_1 \). Then

\[ \lim_{n \to \infty} \frac{F_n}{G_n} = F_1^+ \]

and

\[ \lim_{n \to \infty} \frac{1}{G_n} \sum_{\partial(g)=n} (F(g) - F_1^+)^2 = 0. \]
Proof: Combining Remark 22 and Theorem 20 (with $R = M = 1$), we obtain
\[ \frac{F_n}{G_n} = F_1^+ (1 + \mu_{F,1}(n)) = F_1^+ + o(1). \]

Similarly, \[ \frac{1}{G_n} \sum_{\partial(g) = n} (F(g) - F_1^+)^2 = (F_1^+)^2 \left( \xi_1 \frac{G_{n-1}}{G_n} + O \left( \frac{G_{n-1}}{G_n} \right) \right) = o(1). \]

Remark 24. A slightly stronger (see Remark 11) version of Corollary 23 is proved in [2] for particular choices of $F$. A sharper result is given in [4] where it is shown that the assumption $G_{n-1} = O(G_n)$ (which is part of axiom $W_1$) is unnecessary.

Theorem 25. Let $F$ be a Warlimont function on a Wright semigroup $(G, +, \partial)$ with $\alpha, a, b$ as in Definition 9 and let $q = e^{\alpha(a+1)}$.

1) For every positive integer $M$ there exists a sequence $\{\lambda_s\}$ of functions $\lambda_s : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that $\log(\lambda_s(n)) = O(n^{a-1} + n^b)$ and
\begin{equation}
\mu_{F,M}(n) = \sum_{s=1}^{R-1} \lambda_s(n) q^{-sn^a} + O\left( \lambda_R(n) q^{-Rn^a} \right)
\end{equation}
for every integer $R > 0$.

2) Assume further that there exists constants $0 \leq d_2 \leq d_1$ and sequence $\{\psi_s\}$ of polynomials such that $\deg(\psi_s) \leq d_1 s - d_2$ for all $s \geq 1$ and
\begin{equation}
\frac{G_{n-1}}{G_n} = \sum_{s=1}^{R-1} \psi_s(n) q^{-ns} + O(n^{d_1 R - d_2} q^{-Rn})
\end{equation}
for every integer $R > 0$. Then there exists a sequence $\{\tau_s\}$ of polynomials such that $\deg(\tau_s) \leq d_1 s - d_2$ and
\begin{equation}
\mu_{F,M}(n) = \sum_{s=1}^{R-1} \tau_s(n) q^{-sn^a} + O\left(n^{d_1 R - d_2} q^{-Rn}\right)
\end{equation}
for every integer $R > 0$.

Proof: Let $\xi_s$ be defined by (26) for all $s \geq 1$. By Proposition 12 and Theorem 20
\begin{equation}
\mu_{F,M}(n) = \sum_{s=1}^{R-1} \xi_s \frac{G_{n-1-s}}{G_n} + O\left( \frac{G_{n-R}}{G_n} \right)
\end{equation}
for every integer $R > 0$. Since
\[ \log \left( \frac{G_{n-s}}{G_n} \right) = \alpha ((n-s)^{a+1} - n^{a+1}) + O(n^b) = -\alpha (a+1) sn^a + O(n^{a-1} + n^b), \]
then in order to prove 1) it suffices to choose \( \lambda_s \) such that
\[
\lambda_s(n) = \xi_s q^{s^n} \frac{G_{n-s}}{G_n}
\]
for all \( n \geq s \geq 1 \). Using (28) repeatedly and induction on \( t \), we obtain
\[
\frac{G_{n-t}}{G_n} = \frac{G_{n-1}}{G_n} \cdots \frac{G_{n-t}}{G_{n-t+1}} = \sum_{s=t}^{R-1} \nu_{s,t}(n) q^{-sn} + O(n^{DR} q^{-Rn})
\]
where
\[
\nu_{s,t}(n) = \sum_{i_1 + \cdots + i_t = s} \psi_{i_1}(n) \psi_{i_2}(n-1) \cdots \psi_{i_t}(n-t+1) q^{i_2+2i_3+\cdots+(t-1)i_t}.
\]
is a polynomial in \( n \) of degree at most \( d_1 s - d_2 t \) for all \( 1 \leq t \leq s \). Substituting (31) in into (30) yields
\[
\mu_{F,M}(n) = \sum_{t=1}^{R-1} \xi_t \sum_{s=t}^{R-1} \nu_{s,t}(n) q^{-sn} + O(n^{DR} q^{-nR})
\]
\[
\quad = \sum_{s=1}^{R-1} \left( \sum_{t=1}^{s} \xi_t \nu_{s,t}(n) \right) q^{-sn} + O(n^{DR} q^{-nR}),
\]
which proves 2) upon setting
\[
\tau_s(n) = \sum_{t=1}^{s} \xi_t \nu_{s,t}(n)
\]
for all \( s, n \).

**Remark 26.** Comparison of (28) and (16) shows that the assumptions of 2) in Theorem 25 require in particular that (16) holds with \( a = 1 \).

4. **Examples**

4.1. **Graphs.** Let \((G, +)\) be the semigroup of (simple, unlabeled) graphs with semigroup operation \(+\) given by disjoint union. If \( \partial \) is the map that to each graph \( g \) assigns the cardinality of its set of vertices, then \((G, +, \partial)\) is an additive arithmetical semigroup and \( g \in G^+ \) if and only if the graph \( g \) is connected. As proved in [6], there exists a sequence \( \{\varphi_s\} \) of polynomials such that \( \varphi_s \) has degree \( 2s \) for every \( s \) and
\[
G_n = \frac{2^\left(\frac{3}{2}\right)}{n!} \left( \sum_{s=0}^{R-1} \varphi_s(n) 2^{-sn} + O(n^{2R} 2^{-Rn}) \right)
\]
for every positive integer $R$. The polynomials $\varphi_s$ can be calculated explicitly, the first few being

\[
\begin{align*}
\varphi_0(n) &= 1; \\
\varphi_1(n) &= 2n^2 - 2n; \\
\varphi_2(n) &= 8n^4 - \frac{128}{3}n^3 + 72n^2 - \frac{112}{3}n; \\
\varphi_3(n) &= \frac{256}{3}n^6 - \frac{3712}{3}n^5 + \frac{20672}{3}n^4 - \frac{54272}{3}n^3 + 21952n^2 - 9600n.
\end{align*}
\]

In particular,

\[
\log(G_n) = \log(\sqrt{2})n^2 - n \log(n) + (1 - \log(\sqrt{2}))n + O(n^b)
\]

for any $b > 0$ and thus $(G, +, \partial)$ is a Wright semigroup. Moreover, using (34) and expanding the denominator as a geometric series we obtain

\[
\begin{align*}
\frac{G_{n-1}}{G_n} &= 2n2^{-n} \left( \sum_{s=0}^{R-1} 2^s \varphi_s(n - 1)2^{-sn} \right) \sum_{r=0}^{R-1} \left( - \sum_{s=1}^{R-1} \varphi_s(n)2^{-sn} \right)^r + O(n^{2R+1}2^{-(R+1)s}) \\
&= \sum_{s=1}^{R-1} \psi_s(n)2^{-sn} + O(n^{2R-1}2^{-Rn}),
\end{align*}
\]

where the $\psi_s$ are polynomials of degree $\deg(\psi_s) = 2s - 1$ which can be explicitly calculated in terms of the polynomials $\varphi_s$ in (34). For instance

\[
\begin{align*}
\psi_1(n) &= 2n; \\
\psi_2(n) &= 4n^3 - 20n^2 + 16n; \\
\psi_3(n) &= 40n^5 - 464n^4 + 1768n^3 - 2624n^2 + 1280n; \\
\psi_4(n) &= \frac{3248}{3}n^7 - 24176n^6 + \frac{630608}{3}n^5 - 908496n^4 + \frac{6137792}{3}n^3 - 2250240n^2 + 925696n.
\end{align*}
\]

Substitution into (32) yields $\nu_{s,1}(n) = \psi_s(n)$ for all $s$ and

\[
\begin{align*}
\nu_{2,2}(n) &= 8n^2 - 8n; \\
\nu_{3,2}(n) &= 48n^4 - 352n^3 + 688n^2 - 384n; \\
\nu_{3,3}(n) &= 64n^3 - 192n^2 + 128n; \\
\nu_{4,2}(n) &= 864n^6 - 13472n^5 + 77216n^4 - 203488n^3 + 245376n^2 - 106496n; \\
\nu_{4,3}(n) &= 896n^5 - 9728n^4 + 35200n^3 - 50944n^2 + 24576n; \\
\nu_{4,4}(n) &= 1024n^4 - 6144n^3 + 11264n^2 - 6144n.
\end{align*}
\]

Inspection of graphs with up to four vertices shows that $G_1 = 1$, $G_2 = 2$, $G_3 = 4$ and $G_4 = 11$. Substitution into (19) yields $\beta_1 = \beta_2 = \beta_3 = -1$ and $\beta_4 = -4$. 
Example 27. Consider the Warlimont function $d_2$ from Example 15. When specialized to the semigroup of graphs, $d_2$ counts the number of ways of writing a given graph as the disjoint union of two graphs. The order is taken into account, so that if $g_1$ is not isomorphic to $g_2$, then $g = g_1 + g_2$ and $g = g_2 + g_1$ count as two distinct decompositions. Moreover, decompositions in which one of the components is the empty graph are allowed. Combining Remark 22 and Theorem 25 we obtain (4). In particular, setting $M = 1$ yields a full asymptotic expansion for the average of $d_2$ of the form

$$
\frac{1}{G_n} \sum_{\partial(g)=n} d_2(g) = 2 + 2 \sum_{s=1}^{R-1} \tau_s(n) 2^{-sn} + O(n^{2R-1}2^{-Rn}) ,
$$

valid for every positive integer $R$ where the $\tau_s(n)$ are polynomials of degree $2s - 1$. For instance, direct inspection of graphs with up to four vertices yields $(d_2)_1 = 2$, $(d_2)_2 = 5$, $(d_2)_3 = 12$ and $(d_2)_4 = 34$. Substituting into (26) and then into (33) we obtain

- $\tau_1(n) = 2n$;
- $\tau_2(n) = 4n^3 - 4n^2$;
- $\tau_3(n) = 40n^5 - 368n^4 + 1320n^3 - 2016n^2 + 1024n$;
- $\tau_4(n) = \frac{3248}{3} n^7 - 22448n^6 + \frac{560528}{3} n^5 - 781712n^4 + \frac{5136512}{3} n^3 - 1839360n^2 + 743424n$.

4.2. Graphs with an even number of edges. Let $(G, +)$ be the semigroup of (simple, unlabeled) graphs with an even number of edges with semigroup operation $+$ given by disjoint union. If $\partial$ is the map that to each graph $g$ assigns the cardinality of its set of vertices, then $(G, +, \partial)$ is an additive arithmetical semigroup. $G^+$ consists of graphs $g$ with an even number of edges that cannot be written as the disjoint union of two nonempty graphs with an even number of edges. While $G$ is a subsemigroup of the semigroup of all unlabeled graphs, not all graphs in $G^+$ are connected. For instance, while $2K_1$ is not connected it is nevertheless prime in the semigroup of graphs with even edges. As pointed out in [1], there exists a sequence $\{\varphi_s\}$ of polynomials such that $\varphi_s$ has degree $2s$ for every $s$ and

$$
G_n = \frac{2^{(s)}}{2n!} \left( \sum_{s=0}^{R-1} \varphi_s(n) 2^{-sn} + O(n^{2R-1}2^{-Rn}) \right)
$$

for every positive integer $R$, where the polynomials $\varphi_s(n)$ coincide with those of Section 4.1. In particular, $(G, +, \partial)$ is a Wright semigroup and

$$
\frac{G_{n-1}}{G_n} = \sum_{s=1}^{R-1} \psi_s(n) 2^{-sn} + O(n^{2R-1}2^{-Rn})
$$

where the polynomials $\psi_s(n)$ coincide with those calculated in Section 4.1. Inspection of graphs with up to four vertices shows that $G_1 = G_2 = 1$, $G_3 = 2$ and $G_4 = 6$. Substitution into (19) yields $\beta_1 = -1$, $\beta_2 = 0$, $\beta_3 = -1$ and $\beta_4 = -3$. 
Example 28. Consider the Warlimont function $d_*$ from Example 16. Combining Remark 22 and Theorem 25 we obtain a full asymptotic expansion for the second moment of $d_*$ about 2 is

$$
\frac{1}{G_n} \sum_{\partial(g) = n} (d_*(g) - 2)^2 = 4 \sum_{s=1}^{R-1} \tau_s(n) 2^{-sn} + O \left( n^{2R-1} 2^{-Rn} \right)
$$

for every positive integer $R$, where the $\tau_s(n)$ are polynomials of degree $2s - 1$. To calculate these explicitly for small values of $s$, we first observe that (by direct calculation $(d_*)_1 = 2$, $(d_*)_2 = 2$, $(d_*)_3 = 4$, $(d_*)_4 = 14$ as well as $(d^2_*)_1 = 4$, $(d^2_*)_2 = 4$, $(d^2_*)_3 = 8$, $(d^2_*)_4 = 36$. Substitution into (26) (upon setting $M = 2$) and then into (33) yields

$$
\begin{align*}
\tau_1(n) &= 2n ; \\
\tau_2(n) &= 4n^3 - 20n^2 + 16n ; \\
\tau_3(n) &= 40n^5 - 464n^4 + 1832n^3 - 2816n^2 + 1408n ; \\
\tau_4(n) &= \frac{3248}{3} n^7 - 24176n^6 + \frac{633296}{3} n^5 - 906960n^4 + \frac{6040640}{3} n^3 - 2177280n^2 + 882688n.
\end{align*}
$$

4.3. Polynomials over a finite field. Consider the field $\mathbb{F}_q$ with $q$ elements and let $G$ be the set of non-zero polynomials in $\mathbb{F}_q[x_1, \ldots, x_k]$ modulo the equivalence relation $f \sim g$ if and only if $f = \lambda g$ for some $\lambda \in \mathbb{F}_q$. $G$ has a natural structure of additive semigroup with semigroup operation $+$ given by multiplication of polynomials. If $\partial$ is the semigroup homomorphism that to each polynomial $f \in G$ assigns its total degree, then $(G, +, \partial)$ is an additive arithmetical semigroup and $G^+$ is the set of equivalent classes of irreducible polynomials in $\mathbb{F}_q[x_1, \ldots, x_k]$. Since

$$
G_n = \frac{q \binom{n+k}{k} - q \binom{n-1+k}{k}}{q - 1}
$$

for every $n$, then

$$
\log(G_n) = \log(q) \frac{n^k}{k!} + O(n^{k-1})
$$

for every $k \geq 2$. On the other hand if $k = 1$, then $\log(G_n) = \log(q)n$ for every $n$. Hence $(G, \cdot, \partial)$ is a Wright semigroup if and only if $k \geq 2$. If $k = 2$ then for every positive integer $R$

$$
\frac{G_{n-1}}{G_n} = q^{-n-1} \frac{1 - q^{-n}}{1 - q^{-n-1}} = \sum_{s=1}^{R-1} \psi_s(n) q^{-sn} + O(q^{-Rn})
$$

where, $\psi_1(n) = q^{-1}$ and $\psi_s(n) = q^{-s}(1 - q)$ for all $s \geq 2$. By Theorem 25 each $\mu_{F,M}$ admits an asymptotic expansion as a power series in $q^{-n}$ with constant coefficients. For instance,
substitution into (32) yields

\[ \begin{align*}
\nu_{2,1}(n) &= q^{-2} - q^{-1}; \\
\nu_{2,2}(n) &= q^{-1}; \\
\nu_{3,1}(n) &= q^{-3} - q^{-2}; \\
\nu_{3,2}(n) &= q^{-2} - 1; \\
\nu_{3,3}(n) &= 1.
\end{align*} \]

**Example 29.** Let us further specialize to the case where \( G \) is the semigroup of non-zero polynomial in two variables over the field with two elements. By Theorem 25, there exist constants \( \tau_s \) such that for every positive integer \( R \) the average of the Warlimont function \( B \) (as defined in Example 17) on polynomials of degree \( n \) is

\[ \frac{B_n}{G_n} = 1 + \sum_{s=1}^{R-1} \tau_s 2^{-sn} + O(2^{-Rn}). \]

Since \( B_1 = 6, B_2 = 62 \) and \( B_3 = 1002 \), substituting (35) into (19) and then into (26) shows that in particular \( \tau_1 = 0, \tau_2 = 3 \) and \( \tau_3 = \frac{3}{2} \).

**Example 30.** If \( k > 2 \), then by Remark 26 the second part of Theorem 25 Nevertheless, the asymptotic behavior of Warlimont functions can be described using (27) as follows. Consider for instance the Warlimont function \( B \) of Example 17 on the semigroup of polynomials in 3 variables with coefficients in \( \mathbb{F}_q \). Since

\[ G_1 = -\beta_1 = (B^m)_1 \]

for all \( m \), substitution in (26) yields \( \xi_1 = 0 \) and thus

\[ \frac{1}{G_n} \sum_{\partial(g) = n} (B(g) - 1)^M = O \left( \frac{G_{n-2}}{G_n} \right) = O \left( q^{n^2 - 2n} \right) \]

for all \( M \).

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