Beyond the Heisenberg time: semiclassical treatment of spectral correlations in chaotic systems with spin 1/2

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Abstract
The two-point correlation function of chaotic systems with spin 1/2 is evaluated using periodic orbits. The spectral form factor for all times thus becomes accessible. Equivalence with the predictions of random matrix theory for the Gaussian symplectic ensemble is demonstrated. A duality between the underlying generating functions of the orthogonal and symplectic symmetry classes is semiclassically established.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Energy levels of classically chaotic systems exhibit correlation only slowly subsiding with the energy offset [1]. Around 1980, it became clear, after extensive numerical experiments, that with as few as two degrees of freedom, spectral correlations in the highly excited energy domain have universal properties and obey the same laws as the eigenvalues of the Gaussian random matrix ensembles of the appropriate symmetry class [2–5]. This assertion, known as the Bohigas–Giannoni–Schmitt (BGS) conjecture, took a surprisingly long time to be proven. In most cases, the tool used was the Gutzwiller formula giving the spectral density $\rho(E)$ in a chaotic system as the sum over the classical periodic orbits, each orbit creating a contribution $\sim e^{iS_\gamma/\hbar}$, where $S_\gamma$ is the action of the orbit $\gamma$. Its substitution into the spectral correlation function and its Fourier transform $K(\tau)$, the spectral form factor, lead to double sums over orbit pairs with summands proportional to $e^{i(S_\gamma - S_\gamma')/\hbar}$. Significant contributions can be expected only from pairs with the action difference not large compared with $\hbar$.

The first success in the proof of BGS was connected with the diagonal approximation [6], which takes into account only pairs with $S_\gamma = S_\gamma'$; it explained the fact that at small times, $K(\tau) \approx 2\tau$ (time reversal allowed, orthogonal universality class) or $\tau$ (time reversal forbidden, unitary class). Fifteen years later came the realization that a long periodic orbit with
a small-angle self-crossing dividing it into two pieces has a ‘partner’ orbit with the crossing avoided, but otherwise almost unchanged, up to the sense of traversal of one of the pieces [7]. Contributions of such ‘Sieber–Richter pairs’ sum up the next-to-leading term $-2\tau^2$ in the form factor. Summation over pairs in which the partner consists of pieces of the original orbit reconnected in all thinkable ways restores the small-time form factor to all orders [8, 9].

The form factor experiences a break-up of analyticity at $\tau = 1$, i.e. at the Heisenberg time $T_H = 2\pi \hbar \bar{\rho}$, which reflects the existence of an oscillatory component of the correlation function with the period of the mean level spacing $\Delta = 1/\bar{\rho}$. This component is overlooked in the straightforward semiclassical approach providing correlation functions as asymptotic power series in $1/\varepsilon$; the reason is that semiclassical sums need for their convergence a non-vanishing positive imaginary part of the energy parameters, which leads to suppression of the oscillatory contributions. Early estimates of the oscillatory components are contained in [10, 11]. The systematic approach is based on the formalism of generating functions, i.e. averaged ratios of the spectral determinants. In this approach, the partnership of more than two classical orbits is taken into account and use is made of the semiclassical approximation of the spectral determinant known as the Riemann–Siegel look-alike [12–15]. As a result, complete agreement of the semiclassical correlation functions with random matrix theory (RMT) for spinless systems was demonstrated [16, 17].

Systems with half-integer spin belong to the symplectic universality class whose RMT counterpart is the Gaussian symplectic ensemble. The spin coupling to chaotic translational motion leads to randomization of the spin evolution [18]. The ergodicity of that evolution is instrumental for the evaluation of the relevant periodic orbit expansions such as the diagonal sum for the form factor [19] and the contribution of the Sieber–Richter pairs, which changes its sign in the presence of a half-integer spin [20]. The full expansion of the form factor of systems with symplectic symmetry for times smaller than $T_H$ was obtained for the quantum graphs in [21] and for general dynamical systems in [9].

Here, we close the gap in the proof of BGS for the systems with the half-integer spin by showing the equivalence of their semiclassical correlation function with the RMT predictions including the oscillatory terms; the corresponding form factors coincide for all times. Analytical properties of the correlation function are used to recover the oscillatory term with smaller frequency responsible for the well-known logarithmic singularity of the form factor. The derivation employs the simple duality discovered between the semiclassical four-determinant generating functions of chaotic systems with symplectic and orthogonal symmetries. We talk mostly about the spin-$1/2$ case although the results remain true for other half-integer spins.

2. Complex correlator, form factor and generating function

Our main object of this study is the complex two-point spectral correlation function (complex correlator, for short). This is an analytic function of the complex dimensionless variable $\varepsilon$, which is the double spectral sum:

$$C(\varepsilon) = \frac{\Delta^2}{2\pi^2} \left\{ \sum_{i\neq k} \frac{1}{(E_k - E - i\Delta)} \left( \frac{1}{E_i - E + i\Delta} \right) - \frac{1}{2} \right\}$$  \hspace{1cm} (1)

where $\Delta$ stands for the mean level spacing; $\langle \cdots \rangle$ denotes averaging over an interval of the reference energy $E$, classically small but large compared with $\Delta$. The complex correlator is defined in the half-plane $\text{Im} \; \varepsilon > 0$, where it is analytic and tends to zero, when $|\varepsilon| \to \infty$, and can be continued to the lower half-plane, where (1) would no longer be true. The real part of $C(\varepsilon)$ at the positive real axis coincides with the real level–level correlation function [1],
while its Fourier transform is the spectral form factor $K(\tau)$; the connection between the two functions is given by

$$K(\tau) = \mathcal{F}(C) = \frac{1}{2\pi \tau H} \int_{-\infty}^{\infty} C(\varepsilon) e^{-i2\pi \tau/\hbar} d\varepsilon, \quad \tau > 0, \quad (2)$$

$$C(\varepsilon) = \mathcal{F}^{-1}(K) = 2 \int_{0}^{\infty} e^{i2\tau \varepsilon/\hbar} K(\tau) d\tau, \quad \text{Im} \varepsilon > 0. \quad (3)$$

Here, $\tau_H$ stands 1 for the orthogonal universality class and 2 for the symplectic class; such a choice is equivalent to the replacement $2\varepsilon \rightarrow \varepsilon$ in the symplectic case motivated by the Kramers degeneracy [11].

The semiclassical evaluation of the complex correlator is based on the generating function $\mathcal{F}$ defined as the averaged ratio of four spectral determinants:

$$Z(\tilde{\varepsilon}) = \left\{ \frac{\det (H - E - i\Delta_0/2) \det (H - E - i\Delta_2/2)}{\det (H - E + i\Delta_0/2) \det (H - E + i\Delta_2/2)} \right\}, \quad (4)$$

where $(\tilde{\varepsilon}) = (\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D)$ and it is assumed that $\text{Im} \varepsilon_{A,C} > 0$, $\text{Im} \varepsilon_{B,D} < 0$. The complex correlator can be obtained from $Z$ as

$$C(\varepsilon) = \lim_{\varepsilon_{A,C} \rightarrow -\varepsilon} \lim_{\varepsilon_{B,D} \rightarrow -\varepsilon} 2 \frac{\partial^2}{\partial \varepsilon_A \partial \varepsilon_B} Z. \quad (5)$$

### 3. Semiclassical generating function for the orthogonal symmetry class

Here, we recapitulate the results for systems with orthogonal symmetry. The semiclassical representation of the generating function follows from the chain of relations [16, 17]

$$\det(H - E) \sim \exp \left[ - \int \int E' \text{Tr}(H - E')^{-1} \right] \sim \exp \left[ -\frac{i\pi E}{\Delta} - \sum \gamma f_0 e^{iS_{\gamma}(E)/\hbar} \right], \quad \text{Im} E > 0. \quad (6)$$

The last step is the Gutzwiller expansion of the exponent into a sum over periodic orbits $\gamma$ with the actions $S_\gamma$ and stability coefficients $f_0$. Expanding all four exponentials, we obtain a sum over quadruplets of ‘pseudo-orbits’ $A, B, C, D$:

$$Z^{(1)}(\tilde{\varepsilon}) = e^{i(\varepsilon_A - \varepsilon_B - \varepsilon_C + \varepsilon_D)} \left[ \sum_{A,B,C,D} F_A F_B F_C F_D (-1)^{\nu_{A,C} + \nu_{B,D}} e^{i\Delta S/\hbar} e^{i(T_{A} + T_{B} + T_{C} + T_{D})/\hbar} \right]. \quad (6)$$

A pseudo-orbit, say $A$, is a set of $\nu_A$ periodic orbits whose actions and periods sum up to $S_A$ and $T_A$, respectively, and whose product of stability coefficients is $F_A$. The difference of actions $\Delta S = S_A - S_B - S_C - S_D$ must be small compared with $\hbar$ for the quadruplets making meaningful contributions. Note the sign factor $(-1)^{\nu_{A,C} + \nu_{B,D}}$ depending on the number of orbits in the pseudo-orbits associated with the numerator of the generating function.

The leading contribution is created by the diagonal quadruplets in which the pseudo-orbit pair $(B, D)$ contains the same periodic orbits as $(A, C)$ and consequently $\Delta S = 0$. The diagonal contributions sum up to

$$Z_{\text{diag}} = e^{i(\varepsilon_A - \varepsilon_B - \varepsilon_C + \varepsilon_D)} \frac{(\varepsilon_A - \varepsilon_D)^2 (\varepsilon_C - \varepsilon_B)^2}{(\varepsilon_A - \varepsilon_B)^2 (\varepsilon_C - \varepsilon_D)^2}.$$
and can be factored out like $Z^{(1)} = Z_{\text{diag}}(1 + Z_{\text{off}})$, where the off-diagonal part $Z_{\text{off}}$ stands for a sum similar to (6) but with all orbits of $(A, C)$ different from those of $(B, D)$. Contributions with a small action mismatch now come from the quadruplets in which the periodic orbits of $(B, D)$ are ‘partners’ of those of $(A, C)$, i.e., consist of practically the same but differently connected pieces. Reconnections occur in ‘encounters’, which are places of close approach of $l \geq 2$ almost parallel stretches of the same or different orbits; the possibility of such a reconnection is a fundamental property of chaotic motion [22]. Using ergodicity summation over all pseudo-orbit quadruplets could be reduced to summation over their topological families (structures). In the end, the expansion $Z_{\text{off}} = \sum_{n=1}^{\infty} Z_n$ was obtained [16, 17]. Here, $Z_n \sim \varepsilon^{-n}$ accumulates contributions of quadruplets with $L - V = n$; $V$ is the number of encounters containing $L = \sum_{i=1}^{V} l_i$ stretches. The explicit expression of $Z_n$ for the orthogonal class is: ('O' = orthogonal):

$$Z_{n,0}(\hat{\varepsilon}) = \frac{(\varepsilon_A - \varepsilon_C)(\varepsilon_B - \varepsilon_D)(-i)^n(n-1)!2^n}{(\varepsilon_A - \varepsilon_D)(\varepsilon_B - \varepsilon_C)(\varepsilon_A - \varepsilon_B)^{n+1}} \left( \frac{1}{\varepsilon_C - \varepsilon_D} + \frac{n}{\varepsilon_A - \varepsilon_B} \right).$$

The true high-energy asymptotics of the generating function is not exhausted by $Z^{(1)}$. Recovery of the missing component was achieved on the basis of the so-called Riemann–Siegel look-alike representation [12, 14, 15] of the spectral determinant, due to the basic quantum mechanical symmetry properties. The additional summand is obtained from $Z^{(1)}(\hat{\varepsilon})$ by the ‘Weyl transposition’ $w(\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D) \equiv (\varepsilon_A, \varepsilon_B, \varepsilon_D, \varepsilon_C)$ of its arguments:

$$Z_0(\hat{\varepsilon}) \sim Z^{(1)}(\hat{\varepsilon}) \equiv Z^{(1)}(\hat{\varepsilon}) + Z^{(2)}(\hat{\varepsilon}),$$

$$Z^{(2)}(\hat{\varepsilon}) = Z^{(1)}(w(\hat{\varepsilon})).$$

As shown in [17], $Z^{(2)}$ can be summed up and then yields the RMT generating function of the orthogonal ensemble $Z_{\text{GOE}}$.

4. Systems with spin 1/2: duality with spinless case

The spin-1/2 evolution must be treated quantum mechanically, while the orbital motion still allows semiclassical description. We assume that the spin is driven by the interaction with the translational motion, while neglecting the back reaction of the spin [18]. The van Vleck propagator of the two-component wave-function then falls into a product of the semiclassical propagator of a spinless particle, and a $2 \times 2$ matrix of the spin evolution. The Gutzwiller formula follows from the van Vleck propagator after going to the energy representation and taking a trace; consequently, contribution of a periodic orbit $\gamma$ in the Gutzwiller expansions is now to be multiplied by $\text{Tr} U_{\gamma}$, where $U_{\gamma}$ is an $SU_2$ matrix describing the change of the spin state after a single traversal of $\gamma$ [19].

The contribution of a quadruplet (AC)(BD) in (6) is a product of Gutzwiller amplitudes of all the orbits constituting the quadruplet. Therefore, in the presence of spin, it has to be multiplied by the product of traces of the spin evolution matrices for all its orbits: $\Xi_{\text{AC,BD}} = \prod_{\gamma \in \{\text{AC}\}} \text{Tr} U_{\gamma} \prod_{\gamma' \in \{\text{BD}\}} \text{Tr} U_{\gamma'}$. We recall that the orbits in (AC) and (BD) are constructed of the same $L$ pieces. Therefore, due to the group property of the propagator, $U_{\gamma}$ and $U_{\gamma'}$ can be replaced by products of the matrices $D_i$ describing the spin evolution after traversal of the $i$th piece. Each $D_i$, $i = 1, \ldots, L$, occurs once in (AC) and once in (BD); if the sense of traversal of the $i$th piece is reversed in the partner, the second entry would be $D_i^{-1}$.

The factor $\Xi_{\text{AC,BD}}$ has to be averaged over an interval of the reference energy $E$. Interaction with the chaotic translational motion makes the spin evolution ergodic [18]. Assuming that $D_i$ associated with different non-overlapping pieces are independent quasi-random $SU_2$ matrices,
we can replace averaging by the integration over all $\mathcal{D}$, over the group $SU_2$. The result for the partnership of just two orbits (the only one of interest at sub-Heisenberg times) is well known [20, 21]:

$$\langle \text{Tr } U \text{ Tr } U' \rangle = \frac{(-1)^{L-V}}{2^{L-V}}$$

where $V$ and $L$ are the numbers of encounters and encounter stretches in the orbit pair. We generalize it to the pseudo-orbit quadruplets containing an arbitrary number of orbits, which can change after reconnection in the encounters; see the elementary example in figure 1, where two pieces of the figure-8 orbit become two separate orbits after reconnection.

Calculations in appendix A.2 show that

$$\langle \Xi_{AC, BD} \rangle = \frac{(-1)^{L-V+\nu_B+\nu_D-\nu_A-\nu_C}}{2^{L-V}}$$

(11)

where $V$ and $L$ are now the numbers of encounters and the encounter stretches in the quadruplet $(A, C), (B, D)$. The additional sign factor $(-1)^{V+\nu_B+\nu_D-\nu_A-\nu_C}$ reflects the difference in the number of orbits after the reconnection $(A, C) \rightarrow (B, D)$.

Factor (11) leads to important consequences. Namely, inserting it into the semiclassical expansion (6) of the generating function and combining with $(-1)^{\nu_B+\nu_D-\nu_A-\nu_C}$, we obtain

$$\langle \Xi_{AC, BD} \rangle = \frac{(-1)^{L-V+\nu_B+\nu_D-\nu_A-\nu_C}}{2^{L-V}}$$

(11)

The replacement $A \leftrightarrow C, B \leftrightarrow D$ in the exponent shows that the roles of the numerator and denominator in the generating function are reversed. We recall that a quadruplet with $L-V=n$ contributes to $Z^{(1)}$ in the order $\varepsilon^{-n}$; our rescaling $2\varepsilon \rightarrow \varepsilon$ in the symplectic case absorbs $2^{-V}$. Consequently, all expansion terms of the symplectic off-diagonal sum $Z_{\text{off}}$ are obtained from their orthogonal counterparts (8) by the interchange $A \leftrightarrow C, B \leftrightarrow D$ and the sign change of all arguments:

$$Z^{(1)}_{n,S}(\hat{\varepsilon}) = Z^{(1)}_{n,O}(\varepsilon_A, -\varepsilon_D, -\varepsilon_A, -\varepsilon_B).$$

(12)

The same substitution connects the full periodic orbit expansions of the generating function $Z^{(1)} = Z_{\text{diag}}(1 + \sum_{n=1}^{\infty} Z^{(1)}_n)$, namely

$$Z^{(1)}_S(\hat{\varepsilon}) = Z^{(1)}_O(\varepsilon_A, -\varepsilon_D, -\varepsilon_A, -\varepsilon_B).$$

(13)

We obtain an important and remarkably simple relation (13) between the semiclassical generating functions of systems with or without half-integer spin. (Turning from semiclassics to RMT, we note that numerous identities of that kind between the GOE- and GSE-associated functions are well known under the name of duality relations.) In appendix A.3, we check that a duality relation analogous to (13) does exist between the four-determinant-generating functions of GOE and GSE. A seeming contradiction arises: while the high-energy asymptotic expansion of $Z_{\text{GOE}}$ is identical with its semiclassical counterpart $Z^{(12)}_O$, the analogous expansion of $Z_{\text{GSE}}$ is not. It differs from $Z^{(12)}_S(\hat{\varepsilon}) = Z^{(1)}_S(\hat{\varepsilon}) + Z^{(1)}_S(w(\hat{\varepsilon}))$ by an additional elementary summand proportional to $e^{i(\varepsilon_A - \varepsilon_B)/2}$; see (A.12). The reason is purely mathematical, and the
the missing component of the semiclassical generating function can be recovered by the Borel summation. We prefer to demonstrate the method on the less cumbersome example of the symplectic complex correlator; see section 5.

The averaged spin factor for spins different from 1/2 is given in (A.10). For half-integer spins, the result differs by the replacement of 2 in the denominator of (11) by 2S + 1; however, this is compensated by the changed mean level spacing and Heisenberg time [9]; equation (13) remains in force. For integer spins, the sign of the averaged spin factor is always positive, while 2S + 1 in the denominator is compensated in the way just described; therefore, the generating function is the same as without spin.

5. Borel summation: missing oscillatory component as ‘Stokes’s satellite’

Applying $\partial^2\varepsilon \xi \varepsilon_i \xi_j$ to $Z^{(1)}(\xi) + Z^{(2)}(\xi)$ and going to the limit in (5), we obtain the asymptotic expansion of the complex correlator for both symmetry classes:

$$C(\varepsilon) \sim \sum_{n \geq 2} a_n \varepsilon^n + \varepsilon^{2\varepsilon} \sum_{n \geq 4} b_n \varepsilon^n$$

where the non-oscillatory and oscillatory parts are generated by $Z^{(1)}(\xi)$ and $Z^{(2)}(\xi)$, respectively. The coefficients are easily calculated from (8) and the duality relation; for both symmetries $a_2 = -1$, while for $n > 2$,

$$a_{n,0} = \frac{(n-3)! (n-1)}{2^n}, \quad b_{n,0} = \frac{(n-3)! (n-3)}{2^n},
$$

$$a_{n,S} = a_{n,O} (-1)^n, \quad b_{n,S} = b_{n,O}.$$

The factorial growth of the coefficients signals that the asymptotic series diverge for all $\varepsilon$. Suppose we want to restore the analytic functions behind these series by means of the Borel method [23]. The first stage would be the term-by-term Fourier transform (2) employing

$$\mathcal{F}(\varepsilon^{-n}) = \frac{\tau^{n-1}}{2^n (n-1)!} \theta(\tau), \quad \mathcal{F}(\varepsilon^{2\varepsilon} \varepsilon^{-n}) = \frac{(\tau - 2)^{n-1}}{2^n (n-1)!} \theta(\tau - 2).$$

The resulting series in $\tau$ and $(\tau - 2)$ converge to analytic functions. In the second stage, the inverse Fourier transform produces a closed expression for the complex correlator. This is easily done in the orthogonal case and leads to the form factor $K_{GOE}(\tau)$ and then to the exact $C_{GOE}$; see appendix A.1, equation (A.2).

The symplectic case is more interesting. In the first stage, we obtain the form factor as $K = \theta(\tau) [(\tau / 2 - (\tau / 4) \ln |1 - \tau|) + \theta(\tau - 2) [1 - \tau / 2 + (\tau / 4) \ln |1 - \tau|]$. The first summand has a branch cut $(1, +\infty)$; to proceed with Borel, we need to continue $\ln (1 - \tau)$ to all $\tau > 1$. There are three obvious choices: use the logarithm values $\ln |1 - \tau| \pm i\pi$ at the lower or upper lip of the cut, or their average $\ln |1 - \tau|$. Only the last option is admissible since the form factor must be real, in view of reality of the energy eigenvalues [1]; the factor at $\theta(\tau)$ will then be $K^\sim \equiv \tau / 2 - (\tau / 4) \ln |1 - \tau|$. The Fourier transform of $K$ with this choice produces the exact GSE correlator (A.3) whose asymptotic expansion contains an additional oscillatory summand:

$$C_{GSE} \sim \sum_{n=2}^\infty a_{n,S} \varepsilon^n + \varepsilon^{2\varepsilon} \sum_{n=2}^\infty b_{n,S} \varepsilon^n + \frac{\pi}{2 \varepsilon^2} (\varepsilon + i) e^{i\pi}.$$

Incidentally, the latter would be generated by the term $\propto e^{i(\varepsilon S_+ - S_+)}$ of $Z_{GSE}$ (A.12) after application of (5).
It may seem strange that the back-and-forth Fourier transform recovered, free of charge, the missing oscillatory contribution to the correlator. In fact, restoration of an oscillatory term, given an asymptotic power series, is a legitimate mathematical tool described in detail in the book [24]; see [25, 26] for further important developments. The key idea is that the manner in which coefficients of an asymptotic series tend to infinity contains information about the exponentially small terms disregarded in the classical Poincaré approach. Such terms become oscillatory and of crucial importance when the asymptotics is continued to the anti-Stokes lines.

Here is the barest minimum of detail on the method. Consider a diverging asymptotic expansion of an analytic function

$$g(z) \sim \sum_{n=1}^{\infty} \frac{c_n}{z^n}, \quad |z| \to \infty,$$

and assume that in the limit of large $n$, its coefficients tend to

$$c_n \to (n - \beta)!.$$

Then, we have the following.

(a) The real positive semi-axis is the Stokes line at which the power expansion (15) has all its terms positive and, therefore, maximally dominant with respect to an exponentially small additional component $g_{SD}(z)$; the value of $g_{SD}(z)$ changes almost by a jump when the positive semi-axis is crossed.

(b) The subdominant component behaves like

$$g_{SD}(z) \propto e^{-z \frac{\pi}{2\beta-1}}.$$

(c) The imaginary semi-axes are the anti-Stokes lines, where $g_{SD}(z)$ becomes oscillatory and comparable to the power expansion; see figure 2.

(d) Under certain assumptions about the properties of $g(z)$, we have

$$g(z) = \sum_{n=1}^{n^*(z, q)} \frac{c_n}{z^n} + R(z),$$

$$R(z) \approx e^{-z} \frac{\pi}{2\beta-1} \left[ i \text{erf} \left( \frac{\sqrt{2} x}{\sqrt{2\pi}} \right) + i \nu + \eta(z, q) \right].$$

The upper limit of the sum is $n^*(z, q) = \text{Int} \left( |z| + q \right)$, where $q$ is of the order unity and otherwise arbitrary. The error function $\text{erf}(\sigma) = \frac{2}{\sqrt{\pi}} \int_{0}^{\sigma} e^{-t^2} \, dt$ in (17) is close to 1 for $|z|$ large and $\arg z > \delta > 0$, and to $-1$ for $\arg z < -\delta < 0$; on the real axis, it is zero. The almost jump-like change of the subdominant component when the real axis is crossed is the Stokes phenomenon. The small real correction

$$\eta(z, q) = \frac{2}{\sqrt{2\pi} x} \left( \text{Fract} \left( |z| + q \right) + \beta - q - \frac{4}{3} - \frac{\sqrt{2} x}{6x} \right) e^{-y^2/2x}$$

is significantly nonzero only close to the $x$-axis; its dependence on $q$ compensates that of the sum in (17). The constant $\nu$ must be deduced from additional information on the function $g(z)$. In particular, if $g(z)$ is real on the real axis, we must choose $\nu = 0$; the asymptotics of $g(z)$ then contains oscillatory components at both anti-Stokes lines.

Let us apply the method to the non-oscillatory part of the symplectic correlator. The terms of its expansion $a_n \sqrt{\varepsilon^n}$ are all positive at the positive imaginary axis of $\varepsilon$ which is the Stokes line, where the power series is maximally dominant; the anti-Stokes lines are the real semi-axes.
Figure 2. The exponential ‘satellite’ changes by a jump at the Stokes line (the real positive semi-axis, grey/red) and becomes oscillatory at the imaginary axis.

of $\varepsilon$. The above results are applicable with $\varepsilon = iz$, $\beta = 2$. According to its definition (1), the complex correlator must be real for positive imaginary $\varepsilon$. Therefore, we must choose, in (17), $\nu = 0$ such that the oscillatory ‘Stokes satellite’ of the power series must be present on both real semi-axes:

$$R(\varepsilon) \approx e^{i\nu \pi \varepsilon / 2\varepsilon}, \quad \varepsilon \to +\infty,$$

$$R(\varepsilon) \approx -e^{i\varepsilon \pi / 2\varepsilon} = e^{i\varepsilon \pi / 2|\varepsilon|}, \quad \varepsilon \to -\infty.$$  

(19) (20)

This is indeed the leading term in the oscillatory part of the symplectic correlator recovered by the Borel method. It is subdominant in the upper half-plane away from the real axis and experiences the erf-like approximate discontinuity at the positive imaginary axis.

It is instructive to investigate what happens if we choose to continue $\log(1 - \tau)$ as $\log(\tau - 1) \pm i\pi$ for $\tau > 1$ in the form factor at the first stage of the Borel summation. The functions obtained by the inverse Fourier transform would then differ from the correct complex correlator by the additional terms

$$\pm i\pi \int_1^\infty \text{d}r \, e^{i\varepsilon \tau} \tau = \pm \pi \frac{\varepsilon}{2\varepsilon^2} (1 + \varepsilon) e^{i\varepsilon \pi / 2\varepsilon} \sim \pm e^{i\varepsilon \pi / 2\varepsilon}.$$  

They would cancel (19) at one of the real semi-axes and double its amplitude at the other one, i.e. exactly what we would obtain if we chose $\nu = \pm 1$ in (17). Therefore, the alternative choices of $\nu$ are equivalent to different continuations of the result of the first Borel stage beyond the branch point in the $\tau$ domain.

Finally, let us convince ourselves that additional oscillatory components do not arise in the orthogonal case. The $\varepsilon^{-1}$-expansion with the coefficients $a_{n,0}$ has its terms all positive on the negative imaginary semi-axis, i.e. in the non-physical half-plane of $\varepsilon$, where (1) is inapplicable and the correlator need not be real. The parameter $\beta = 2$ is the same as in the symplectic
case but we must now set $\varepsilon = -iz$ and the Stokes satellite now behaves like $\sim e^{-i\varepsilon}/\varepsilon$. Let us change the phase of $\varepsilon$ from $-\pi/2$ via 0 to positive values; if the satellite were present with a nonzero amplitude, it would become exponentially large in the physical region $\text{Im} \varepsilon > 0$. This is forbidden, and we must choose thus, in (17), $\nu = -1$, which corresponds to the absence of the exponential term in the sector $-\pi/2 + \delta < \arg \varepsilon < \pi$, in particular, at the real positive semi-axis.

6. Conclusion

We studied the generating function, complex correlator and form factor of systems with spin 1/2. Expanding the generating function into a sum over periodic orbit quadruplets, we showed that in the presence of spin, the terms of the expansion acquire an additional sign factor whose effect is to interchange the role of the numerator and the denominator of the generating function. As a result, the generating functions of the orthogonal and symplectic class turn out to be connected by a simple substitution of their arguments.

The periodic orbit expansion of the generating function supplemented by the Riemann–Siegel look-alike formula for the spectral determinants yields the complex correlator as a combination of two asymptotic series in $\varepsilon^{-1}$, the second one multiplied by $e^{2i\varepsilon}$; they are responsible for the form factor at small times and at times larger than the Heisenberg time. We demonstrate how the Borel summation reveals in the symplectic case one more oscillatory term $\propto e^{i\varepsilon}$ associated with the logarithmic singularity of the form factor; the origin of that term is clarified by the Dingle–Berry method of smart summation of the asymptotic series as a display of the Stokes phenomenon. With the missing oscillatory term restored, the complete equivalence of the correlation functions of chaotic systems with spin 1/2 and the Gaussian symplectic ensemble of RMT is reached.

The inherent ambiguity in restoration of a function from its asymptotic series is solved on the grounds of reality of the energy eigenvalues. The same reasoning is at the heart of the Riemann–Siegel look-alike, such that existence of both oscillatory components of the complex correlator in the symplectic case can be traced to unitarity of the quantum mechanical evolution.

There are several possible further developments of the theory. An obvious generalization would be to consider parametric correlation in systems with spin 1/2 at times comparable with the Heisenberg time [27]. Away from the deep semiclassical limit, system-specific deviations from the universal behavior in systems with half-integer spin can be of physical interest.

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Appendix

A.1. RMT complex correlator of the orthogonal and symplectic case

The complex correlators are conveniently expressed in terms of the functions

$$f_\pm(z) = \int_z^\infty \frac{e^{\pm i(t-z)}}{t} \, dt.$$
These integral representations are applicable whenever the integral converges; for all \( z \), we have
\[
f_{\pm}(z) \equiv e^{i\pi c} \left[ \pm i \frac{\pi}{2} - \text{Ci}(z) \mp i \text{Si}(z) \right], \quad (A.1)
\]
where \( \text{Ci} \) and \( \text{Si} \) are the integral cosine and sine, respectively. The functions \( f_{\pm}(z) \) are analytic in the plane of \( z \) with the cut along the negative real axis.

The complex correlator of the orthogonal case can be represented in terms of \( f_{\pm}(z) \) as
\[
C_{\text{GOE}}(\varepsilon) = 2 \int_0^{\infty} e^{2\varepsilon \tau} K_{\text{GOE}}(\tau) \, d\tau = C^{(1)}_{\text{GOE}} + C^{(2)}_{\text{GOE}},
\]
where
\[
C^{(1)}_{\text{GOE}} = - \frac{1}{2 \varepsilon^2} + \frac{1}{2 \varepsilon^2} (-i \varepsilon + 1) f_+(\varepsilon),
\]
\[
C^{(2)}_{\text{GOE}} = - \frac{\varepsilon}{2 \varepsilon^2} [(-i \varepsilon + 1) f_+(\varepsilon) - 1], \quad (A.2)
\]
while in the symplectic case,
\[
C_{\text{GSE}}(\varepsilon) = 2 \int_0^{\infty} e^{i\varepsilon \tau} K_{\text{GSE}}(\tau) \, d\tau = C^{(1)}_{\text{GSE}} + C^{(2)}_{\text{GSE}} + C^{(3)}_{\text{GSE}},
\]
where
\[
C^{(1)}_{\text{GSE}} = - \frac{1}{2 \varepsilon^2} + \frac{1}{2 \varepsilon^2} (-i \varepsilon + 1) f_-(-\varepsilon),
\]
\[
C^{(2)}_{\text{GSE}} = - \frac{\varepsilon}{2 \varepsilon^2} [(-i \varepsilon + 1) f_+(\varepsilon) - 1],
\]
\[
C^{(3)}_{\text{GSE}} = \frac{\pi}{2 \varepsilon^2} (\varepsilon + i) e^{i\varepsilon}. \quad (A.3)
\]

The components \( C^{(1)} \) in both cases have non-oscillatory asymptotic expansions in powers of \( \varepsilon^{-1} \) at the real positive axis, while \( C^{(2)} \) oscillates like \( e^{2i\varepsilon} \). This follows from the asymptotic representations of \( f_{\pm}(z) \):
\[
f_{\pm}(z) \sim - \sum_{k=0}^{\infty} \frac{k!}{(\pm i \varepsilon)^{k+1}} \equiv \sigma_{\pm}(z), \quad |z| \to \infty, \quad (A.4)
\]
\[
-\pi + \delta < \arg z \leq \pi - 0, \quad (f_+),
\]
\[
-\pi + 0 \leq \arg z < \pi - \delta, \quad (f_-).
\]
The Stokes line, where the power expansion is dominant, is \( z = -i t, \quad t > 0 \), for \( f_+(z) \) and \( z = i t, \quad t > 0 \), for \( f_-(z) \); the subdominant satellite, \( -2\pi i e^{-i\varepsilon} \) for \( \sigma_+(z) \) and \( 2\pi i e^{i\varepsilon} \) for \( \sigma_-(z) \), exists in the quadrant to the left of the respective Stokes line.

The form factors obtained from the complex correlators by the transformation (2) are as follows.

• GOE:
\[
K(\tau) = 2\tau - \tau \ln(1 + 2\tau), \quad \tau < 1,
\]
\[
K(\tau) = 2 - \tau \ln \frac{2\tau + 1}{2\tau - 1}, \quad \tau > 1.
\]

• GSE:
\[
K(\tau) = \frac{\tau}{2} - \frac{\tau}{4} \ln |1 - \tau|, \quad \tau < 2,
\]
\[
K(\tau) = 1, \quad \tau > 2.
\]
A.2. Spin factor for quadruplets

Contribution of the quadruplet \((AC)(BD)\) in the periodic orbit expansion of the generating function \((4)\) contains, in the symplectic case, the spin factor \(\mathcal{Z}_{AC,BD} \equiv \prod_{\gamma \in (AC)} \text{Tr} \ U_{\gamma} \prod_{\gamma' \in (BD)} \text{Tr} \ U_{\gamma'}\), which is the product of traces of the periodic orbits composing the quadruplet. Non-vanishing contributions to the sum are created by the quadruplets such that the orbits of \((BD)\) are constructed from pieces of the orbits in \((AC)\) connected in different order and possibly traversed with a different sense. We assume ergodicity of the spin motion and independence of the spin evolution along different orbit pieces. Averaging is then done step by step by integration over the evolution matrices \(D_{\gamma}\) associated with the orbit pieces; the integration domain is the group \(SU_2\). Possible outcomes of a single integration are summed up by the relations \([21]\)

\[
\int d\mathcal{D} \text{Tr} (ABBD) = -\frac{1}{2} \text{Tr}(AB^{-1}),
\]

\[
\int d\mathcal{D} \text{Tr} (ABBD^{-1}) = \frac{1}{2} \text{Tr}A \text{Tr} B,
\]

\[
\int d\mathcal{D} \text{Tr} (AD) \text{Tr} (BD) = \frac{1}{2} \text{Tr}(AB^{-1}).
\]

Here, \(A\) and \(B\) are any fixed \(SU_2\) matrices. Applying these rules, e.g., to the structure with \(V = 1, L = 2\) shown in figure 1, we obtain the result

\[
\langle \mathcal{Z}_{AC,BD} \rangle = \int d\mathcal{D}_1 d\mathcal{D}_2 \text{Tr}(D_1 D_2) \text{Tr}(D_1) \text{Tr}(D_2) = \frac{1}{2},
\]

which is opposite in sign compared with the Sieber–Richter pair \([20]\).

We shall find the average of \(\mathcal{Z}_{AC,BD}\) for all structures extending the inductive method of Bolte and Harrison from pairs of orbits to the pseudo-orbits quadruplets. We recall the essence of the method. Consider an orbit pair \(\gamma, \gamma'\) differing in \(V\) encounters with \(L\) encounter stretches and assume that averaging of the spin factor produces the factor \(C_{\gamma\gamma'} = (-1)^{L-V}/2^{L-V}\). Introduce one more orbit \(\gamma''\) differing from \(\gamma'\) by an additional 2-encounter such that the pair \(\gamma, \gamma''\) contains \(V'' = V + 1\) active encounters with \(L'' = L + 2\) stretches. It is then shown using the recurrence relations that \(C_{\gamma\gamma''} = -C_{\gamma\gamma'}/2 = (-1)^{L''-V''}/2^{L''-V''}\). Starting from a pair without active encounters \(L = V = 0\) when the formula is true, and adding 2-encounters one by one we obtain that the result is true for an arbitrary number of 2-encounters. Finally, reconnection in any 1-encounter with \(l > 2\) can be reduced to \(l - 1\) successive reconnections in 2-encounters, with the factor \((-1)^{L-V}/2^{L-V}\) correct in all steps. Indeed, reconnection in an \(l\)-encounter can be described by a permutation of \(l\) elements; however, any permutation can be represented as a chain of transpositions of just two elements.

Let us apply this method to the pseudo-orbit quadruplets. Let \(\Gamma = (A,C)\) be the initial pseudo-orbit pair, and \(\Gamma' = (B,D)\) be its partner differing from \(\Gamma\) in \(V\) active encounters with \(L\) stretches. Let \(\Gamma''\) be a pseudo-orbit pair differing from \(\Gamma'\) by reconnection in a single additional 2-encounter whose stretches belong to some orbit \(\gamma\) in \(\Gamma\) and to \(\gamma'\) in \(\Gamma'\), where \(\gamma, \gamma'\) may differ in an arbitrary number of other encounters. The only new case to be considered is that of a 2-encounter with almost parallel stretches such that \(\gamma'\) breaks up after reconnection into two orbits \(\gamma_1''\) and \(\gamma_2''\) belonging to \(\Gamma''\); see the example in figure 1.

The averaged spin factors for the quadruplets \(\Gamma\Gamma'\) and \(\Gamma\Gamma''\) can be written as multiple integrals over \(SU_2\) with the Haar measure:

\[
C_{\Gamma\Gamma'} = \int d(\cdots) \int d(\gamma)d(\gamma'),
\]

\[
C_{\Gamma\Gamma''} = \int d(\cdots) \int d(\gamma)d(\gamma_1'')(\gamma_2''),
\]

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Here, \( d(\gamma) \) and \( d(\gamma') \) denote \( SU_2 \) integration over matrices associated with pieces of \( \gamma \) and \( \gamma' \), respectively, while \( d(\gamma'')d(\gamma'')' \) indicate integration over matrices associated with pieces of the disconnected orbit pair in \( \Gamma'' \). Integration \( d(\cdots) \) is over the remaining variables, the same as for \( \Gamma'' \) and \( \Gamma'''' \). In the way of induction, let us assume that

\[
C_{\Gamma''''} = \frac{(-1)^{L-V+v_\gamma-v_{\gamma'}}}{2^{L-V}},
\]

(A.8)

where \( v_\gamma = v_A + v_c \) and \( v_{\gamma'} = v_B + v_D \) are the number of periodic orbits in \( \Gamma \) and \( \Gamma' \), respectively, and prove that the analogous formula will be true for \( C_{\Gamma''''} \).

Consider figure A1, where the orbits \( \gamma \) and \( \gamma' \) and the orbit pair \( \gamma'' \) are depicted. In \( \gamma \) and \( \gamma' \), we see a parallel crossing; however, it is not switched between \( \gamma \) and \( \gamma' \), i.e. it is inactive and not counted in \( V = V_{\Gamma''''} \). On the other hand, it is activated in the pair \( \gamma \gamma'' \) such that

\[
V_{\Gamma''''} = V + 1, \quad L_{\Gamma''''} = L + 2, \quad V_{\gamma''} = V_{\gamma''} + 1.
\]

In figure A1, the orbit pieces adjacent to the encounter and incorporating parts of its stretches are denoted \( a, b, c, d \); they are assumed to coincide in \( \gamma \) and \( \gamma' \), i.e. do not contain any additional active encounters. The associated matrices \( D_a \), etc involved in the \( SU_2 \) integration will be denoted by the same letters for compactness, \( a \equiv D_a \), etc. (We hope that \( d \) as the integration variable will not be confused with \( d \) as the differential!) The two links attached to the selected 2-encounter are denoted \( L_1 \) and \( L_2 \) in \( \gamma \) and \( L'_1 \) and \( L'_2 \) in \( \gamma' \). Unlike \( a, b, c, d \), the links \( L_i \) and \( L'_i \) need not coincide; indeed, \( L_i \) can contain any amount of encounters active in \( \Gamma'''' \) such that \( L'_i \) can contain pieces of all orbits of the original quasi-orbit pair \( \Gamma \) other than \( \gamma \).

The \( SU_2 \) integrals can be written, with \( dL_1 \) being shorthand for \( dD_{L_1} \), etc, as

\[
C_{\Gamma''''} = \int d(\cdots)dL_1 dL_2 dL'_1 dL'_2 \int da \, db \, dc \, dd \, \text{Tr}(L_1 abL_2 cd) \text{Tr}(L'_1 abL'_2 cd);
\]

\[
C_{\Gamma''''} = \int d(\cdots)dL_1 dL_2 dL'_1 dL'_2 \int da \, db \, dc \, dd \, \text{Tr}(L_1 abL_2 cd) \text{Tr}(L'_1 abL'_2 cd) \text{Tr}(L_1 cd) \text{Tr}(L'_2 cd).
\]

Let us transform the integrals over \( a, b, c, d \). In \( C_{\Gamma''''} \), we can take the matrices \( x = ab \) and \( y = cd \) as the new integration variables writing

\[
\int da \, db \, dc \, dd \, \text{Tr}(L_1 abL_2 cd) \text{Tr}(L'_1 abL'_2 cd)
\]

\[
= \int dx \, dy \, \text{Tr}(L_1 xL_2 y) \text{Tr}(L'_1 xL'_2 y)
\]

\[
= \frac{1}{2} \int dy \, \text{Tr}[L_2 yL_1 (L_1)^{-1} y^{-1} (L_2)^{-1}]
\]

\[
= \frac{1}{4} \text{Tr}[L_1 (L_1)^{-1}] \text{Tr}[L_2 (L_2)^{-1}].
\]
The cyclic invariance of the trace and relations (A.7) and (A.6) were used.

Now, let us carry out similar transformations of the integral in $C_{\Gamma'\Gamma}$, introducing consecutively new integration variables $x = ad$, $y = cb$, $z = cd$:

$$\int da \, db \, dc \, dd \, Tr \left( L_{1} ab L_{2} cd \right) Tr \left( L_{1}' ad \right) Tr \left( L_{2}' cb \right)$$

$$= \int dx \, db \, dc \, dd \, Tr \left( L_{1} x d^{-1} b L_{2} cd \right) Tr \left( L_{1}' x \right) Tr \left( L_{2}' cb \right)$$

$$= \frac{1}{2} \int db \, dc \, dd \, Tr \left[ d^{-1} b L_{2} c d L_{1}' \right] Tr \left( L_{2}' cb \right)$$

$$= \frac{1}{2} \int dy \, dc \, dd \, Tr \left[ d^{-1} c^{-1} y L_{2} c d L_{1}' \right] Tr \left( L_{2}' y \right)$$

$$= \frac{1}{4} \int dc \, dd \, Tr \left[ L_{2} c d L_{1}' \left( L_{1}' \right)^{-1} d^{-1} c^{-1} \left( L_{2}' \right)^{-1} \right]$$

$$= \frac{1}{4} \int dc \, dd \, Tr \left[ L_{1} \left( L_{1}' \right)^{-1} z \left( L_{2}' \right)^{-1} L_{2} z^{-1} \right]$$

$$= \frac{1}{8} Tr \left[ L_{1} \left( L_{1}' \right)^{-1} \right] Tr \left[ L_{2} \left( L_{2}' \right)^{-1} \right].$$

Comparing with (A.9), we see that

$$C_{\Gamma'\Gamma} = \frac{1}{2} C_{\Gamma'\Gamma},$$

which agrees with (A.8) with $L \to L'' = L + 2$, $V \to V'' = V + 1$, $v_{\Gamma'} \to v_{\Gamma''} = v_{\Gamma'} + 1$; therefore, if (A.8) is true for the quadruplet $\Gamma'\Gamma$, it will also hold for $\Gamma''\Gamma$.

Evidently, the reversed process, when two orbits of $\Gamma'$ merge into a single orbit in $\Gamma''$ after reconnection in a 2-encounter, also agrees with (A.8); reconnection in an $l$-encounter with $l > 2$ is reducible to a sequence of reconnections in 2-encounter. By repeated activation of the encounters resulting in joining and disjoining the orbits, we can construct any pseudo-orbit quadruplet out of an orbit pair for which (A.8) is known to be correct; hence, by induction, it is true for all quadruplets.

For the spin different from 1/2, the Bolte–Harrison recurrence relations (A.5)–(A.7) differ by the replacement of 2 in the denominator by $2S + 1$; for integer spins, the sign in (A.5) is plus. Otherwise, the reasoning remains unchanged with result (A.8) replaced by

$$C_{\Gamma'\Gamma} = \frac{(-1)^{L-V+v_{\Gamma'}-v_{\Gamma}}}{(2S + 1)^{L-V}}, \quad \text{half-integer } S,$$

$$C_{\Gamma'\Gamma} = \frac{1}{(2S + 1)^{L-V}}, \quad \text{integer } S.$$

(A.10)

### A.3. Generating functions of RMT: GOE–GSE duality

The generating function of GOE found by Zirnbauer [28] has a Weyl symmetric form:

$$Z_{\text{GOE}}(\hat{\epsilon}) = F_{\text{GOE}}(\hat{\epsilon}) + F_{\text{GOE}}(w(\hat{\epsilon})),$$

where $\hat{\epsilon} = (\epsilon_{A} \epsilon_{C} \epsilon_{B} \epsilon_{D})$; $w$ interchanges $C$ and $D$. Denoting $ad = \epsilon_{A} - \epsilon_{D}$, etc, and assuming $\text{Im} \, ab > 0$, we can write

$$F_{\text{GOE}}(\hat{\epsilon}) = e^{\frac{1}{2} (\overline{a}b - \overline{a}b)} \frac{ad \, cb}{ab \, cd} \left[ 1 + \frac{1}{2} \frac{\overline{a}c bd}{cd} + \frac{1}{cd} \, f_{+} \left( \frac{\overline{a}b}{2} \right) \right]$$

with $f_{+}$ defined above; see (A.1).
The generating function of GSE is given in [28] as

\[ Z_{\text{GSE}}(\varepsilon) = Z_{\text{GUE}}(\varepsilon) + \frac{ac}{ab \cdot cd} \cdot \frac{\bar{a} \bar{d}}{\bar{b} \cd} 1 G_1 \left( \frac{ab}{2} \right) G_0 \left( \frac{cd}{2} \right) \]

\[ G_1(z) = i \int_1^\infty e^{qz} dq = -\frac{e^{iz}}{z^2} (i + z), \]
\[ G_0(x) = \int_1^1 \frac{\sin qx}{q} dq = \pi + \int_1^\infty \frac{e^{-iqx}}{iq} dq - \int_1^\infty \frac{e^{iqx}}{iq} dq \]
\[ = (\pi - i e^{-i\varepsilon} f_+^+(x)) + i e^{i\varepsilon} f_+^+(x) = G_0^{(1)} + G_0^{(2)}. \]

Here, \( Z_{\text{GUE}}(\varepsilon) \) is the generating function of the unitary ensemble; in the last line, \( x = \sqrt{\varepsilon} / \sqrt{2} \) is assumed real positive.

The GSE generating function can be transformed into a sum of two components connected by the Weyl substitution \( w \). First, let us write \( Z_{\text{GSE}}(\varepsilon) = F^{(1)} + F^{(2)} \) with

\[ F^{(1)} = e^{i \frac{\pi}{2} - \varepsilon} \frac{ad}{ab \cdot cd} \frac{\bar{a} \bar{d}}{\bar{b} \cd} \left[ 1 + \frac{1}{2} \frac{ac}{ab \cdot cd} (2 - iab) (\frac{2 im}{ab}) \right] \]
\[ + i \pi e^{i \frac{\pi}{2} - \varepsilon} \frac{ad}{ab \cdot cd} (2 - iab) \left( \frac{ab}{2} \right) f_+^+ \left( \frac{cd}{2} \right) \]

and

\[ F^{(2)} = e^{i \frac{\pi}{2} - \varepsilon} \frac{ac}{ab \cdot cd} \left[ 1 + \frac{1}{2} \frac{ad}{ab \cdot cd} (2 - iab) (\frac{2 im}{ab}) \right] \]

The part \( F^{(1)} \) proportional to \( e^{i \frac{\pi}{2} - \varepsilon} \) generates the non-oscillatory component of the symplectic correlator, while the part proportional to \( e^{i \frac{\pi}{2} - \varepsilon} \) generates the correlator term proportional to \( \propto e^{i\varepsilon} \).

The part \( F^{(2)} \propto e^{i \frac{\pi}{2} - \varepsilon} \) is responsible for the component of the correlator \( \propto e^{2i\varepsilon} \).

Let us define the function \( f_+^+(x) \) for the real positive \( x \), i.e. at the branch cut, as the average of the values of the analytic function \( f_+^+(z) \) at the lips of the cut:

\[ f_+^+(x) = \frac{f_+^+(x + i0) + f_+^+(x - i0)}{2}, \quad x > 0. \]

Taking into account that

\[ f_+^+(x + i0) = f_+^+(x), \]
\[ f_+^+(x - i0) = -f_+^+(x - i0) = -2\pi i e^{i\varepsilon}, \]

we have \( G_0(x) = -if_+^+(x) e^{-i\varepsilon} + if_+^+(x) e^{i\varepsilon} \); we have concealed \( \pi \) in the first summand. Now, considering that the substitution \( w \) changes the sign of \( \sqrt{cd} \), we obtain

\[ Z_{\text{GSE}}(\varepsilon) = F_{\text{GSE}}(\varepsilon) + F_{\text{GSE}}(w(\varepsilon)) \quad (A.11) \]

with

\[ F_{\text{GSE}}(\varepsilon) = F^{(1)} = e^{i \frac{\pi}{2} - \varepsilon} \frac{ad}{ab \cdot cd} \left[ 1 + \frac{1}{2} \frac{ac}{ab \cdot cd} (2 - iab) (\frac{2 im}{ab}) \right] \]

The duality relation now holds:

\[ F_{\text{GSE}}(\varepsilon_A, \varepsilon_B, \varepsilon_C, \varepsilon_D) = F_{\text{GSE}}(-\varepsilon_C, -\varepsilon_D, -\varepsilon_A, -\varepsilon_B), \]
\[ \text{Re}(\varepsilon_A - \varepsilon_B) > 0, \quad \text{Im}(\varepsilon_A - \varepsilon_B) = +0, \]
\[ \varepsilon_C - \varepsilon_D > 0. \]
It is instructive to compare the RMT function (A.11) with the semiclassical results. Taking (8) into account and using the semiclassical duality, we have the following equivalence in the high-energy limit:

\[ F_{\text{GSE}}(\hat{\varepsilon}) \sim Z^{(1)}_S(\hat{\varepsilon}) + e^{i(\varepsilon_A - \varepsilon_B)/2} \frac{cd}{2ab} \frac{ac}{cd} \frac{ba}{cd} (2i + ab) \frac{1}{ab^2} \pi, \]

\[ F_{\text{GSE}}(w(\hat{\varepsilon})) \sim Z^{(2)}_S(\hat{\varepsilon}) \equiv Z^{(1)}_S(w(\hat{\varepsilon})). \]

We stress that the asymptotics of \( F_{\text{GSE}}(w(\hat{\varepsilon})) \) is not simply the Weyl-transposed asymptotics of \( F_{\text{GSE}}(\hat{\varepsilon}) \). The reason is the Stokes phenomenon; the Weyl operation changes the sign of \( cd \), and whereas the asymptotics of \( f_+(cd/2) \) is purely power like, \( f_+(-cd/2) \) contains an additional exponential summand, in accordance with

\[ f_+(x) \sim \sigma_+(x), \quad f_+(-x) \sim i\pi e^{i\pi} + \sigma_+(-x), \]

\[ x \rightarrow +\infty, \]

where \( \sigma_+ \) is the asymptotic power series defined in (A.4).

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