Higher dimensional dyonic black holes

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The paper at hand presents a novel class of dyonic black holes in higher dimensions through a new proposal for the electromagnetic field tensor. The black hole solutions are extracted analytically and their geometrical/physical properties are studied. In addition, the details regarding thermodynamical structure and phase transition behavior of the solutions for 4 different cases are investigated: i) general case, ii) constant electric field, iii) constant magnetic field, and iv) constant electric and magnetic fields. It will be shown that depending on the picture under consideration, the thermodynamical properties are modified. To have better picture regarding the phase transitions, the concept of the extended phase space is employed. It will be shown that in the absence of the electric field, magnetic black holes present van der Waals like phase transition. Furthermore, it will be highlighted that for super magnetized black holes, no phase transition exists.

I. INTRODUCTION

Gravity in higher dimensions has been of interest for several decades. This is due to fact that specific range of the advanced physics theories requires existence of the higher dimensions in order to address different issues in the nature. To name a few, one can point out: I) String, superstring theories and in general M-theory, which have been the most celebrated theories towards unification of the all fundamental forces in nature$^1$. II) Branelows theories which have been employed to address fundamental issues of gravity such as hierarchy problem$^2$. III) Kaluza-Klein compactification which was the pioneering proposal regarding unification of the gravity and electromagnetism$^3,4$. Through all of the mentioned theories, the necessity of existence of the higher dimensionality were highlighted and employed.

In the context of black holes, it was shown that although the laws of black hole mechanic are universal, the properties of black hole are dimension dependent$^5$. Reissner-Nordstrom (RN) black holes in the context of string theory plays an important role in understanding the black hole entropy near extremal limits$^6$, hence and in the context of this theory, Hawking temperature, radiation rate and entropy for these black holes have been studied and it was proposed that quantum evolution of black hole does not lead to information loss$^7$. Hereupon, the study of black holes in higher dimensions has attracted many authors. For example; the generalizations of Schwarzschild and Kerr black holes to arbitrary extra dimensions have been investigated in refs. $^8$ and $^9$, respectively. The existence of black rings and Saturns in higher dimensions have been studied$^{10,11}$. The thermodynamics and stability of higher dimensional Kerr-anti de Sitter black hole has been addressed in ref. $^{12}$.

In ref. $^{13}$, the ultraviolet divergent structures of the matter field in a higher dimensional RN black hole has been studied and the contributions to Bekenstein-Hawking entropy by using the Pauli-Villars regularization method was addressed. Gravitation with superposed Gauss–Bonnet terms and black object solutions in higher dimensions have been obtained in ref. $^{14}$. Uniqueness and non-uniqueness of static (un)charged black holes and black p-branes in higher dimensions have been surveyed$^{15}$. Topology of black holes’ event horizons in higher dimensions has been investigated in ref. $^{16}$. Hawking emission of gravitons and generalization of Hawking’s black hole topology theorem to higher dimensions have been obtained, respectively in refs. $^{17}$ and $^{18}$. Quasinormal modes of Schwarzschild$^{19}$ and Kerr$^{20}$ black holes in higher dimensions have been studied. In addition, the production of higher-dimensional black holes in future colliders becomes a conceivable possibility in scenarios involving large extra dimensions and TeV-scale gravity$^{21}$. In addition, as mathematical objects, black hole spacetimes are among the most important Lorentzian Ricci-flat manifolds in any dimension$^{22}$.

Another motivation for considering higher dimensions is related to the AdS/CFT correspondence which relates the properties of a black hole in d-dimensions with those of a quantum field theory in (d-1)-dimensions$^{23}$. In other words,
we can extract the properties of a complex system in quantum field theory by using the properties of black holes in one higher dimension. Among the different set ups for AdS/CFT studies, the ones including a magnetic field has been of special interests. Historically speaking, Hartnoll and Kovtun in their pioneering work incorporated a background magnetic field and studied low frequency charge transport and Hall conductivity \[25\]. The magnetic field was included by considering a type of black holes known as dyonic black holes. Later, it was shown that large dyonic black holes in anti-de Sitter spacetime are dual to stationary solutions of the equations of relativistic magnetohydrodynamics on the conformal boundary of AdS \[26\]. In addition, the dyonic black holes was employed to induce the effects of external magnetic field on superconductors. It was shown that the size of condensate for the superconductor is magnetic field dependent in a manner which is a reminiscent of the Meissner effect \[27\]. Furthermore, the holographical properties of the dyonic dilatonic black branes including transport coefficients, Hall conductance, DC longitudinal conductivity and response were investigated in ref. \[28\]. So far, a large number of publications was dedicated to study systems including dyonic black holes in different contexts (for a very incomplete list, we refer the reader to refs. \[29–52\]).

In this paper, we introduce a novel approach for constructing electrically-magnetically charged black holes, or simply put dyonic black hole holes, in higher dimension. Our main motivation is to propose a simplified method for constructing higher dimensional dyonic black holes in a manner that magnetic and electric parts of it are stand alone properties. We expand our study to thermodynamical properties of the black holes in order to understand the physical and geometrical properties of the solutions in details. We consider four distinctive cases which correspond to four different scenarios in the context of AdS/CFT correspondence: I) General case: in which no specification is given about electric and magnetic field. II) Constant electric field: which corresponds to immersing the magnetically charged black holes in a finite electric field. III) Constant magnetic field: which corresponds to considering the electrically charged black holes in the presence of finite external magnetic field. IV) Constant electric and magnetic fields: which corresponds to immersing the black holes in a field consisting finite electric and magnetic fields. We will show how the magnetization, electrification and dimensionality of the solutions affect physical properties of the black holes including the behaviors of temperature, enthalpy and heat capacity. In addition, we will explore the possibility of the van der Waals like phase transition for the four mentioned cases and investigate the effects of magnetization, electrification and dimensionality on van der Waals like critical points and phase transition. Among different benefits of our proposal, we can point out two important ones: I) The set up is easy to understand and could be employed without going into trouble of introducing complex system of the equations. II) The set up provides the possibility of including higher dimensional gravities such as Lovelock gravity with simplicity. We intend to provide the possibility of studying the effects of magnetism on holographical systems in higher dimensions (with or without higher dimensional gravity theories). The set up also could be employed to study the effects of non-finite magnetic/electric field on holographical systems.

The structure of the paper is as follows: first, the action and field equations are introduced. The metric function is obtained and, geometrical properties and conditions regarding the existence of black holes are investigated. In section III, thermodynamical quantities of interest in this paper are introduced and obtained. Sections IV-VII are, respectively, dedicated to investigation of four different cases including: General case, constant electric field, constant magnetic field and, constant electric and magnetic fields. The paper is concluded with some closing remarks.

II. BASIC EQUATIONS

The main goal of this paper is construction of the novel dyonic black holes in higher dimensions. Our motivation comes from interesting properties of the dyonic black holes specially in holographical aspects, DC conductivity string theory, etc. Here, we introduce higher dimensional dyonic black holes which could be employed to understand the DC effects in higher dimensions. Furthermore, we are providing the possibility of studying higher dimensional theories of gravity such as Lovelock gravity in the presence of dyonic configuration as well.

Dyonic black holes enjoy existence of magnetic charge as well as electric charge in their structures. It is worthwhile to mention that the set up which is going to be introduced here, provides the possibility of having magnetic field for black holes without introduction of the electric field. For simplicity, we consider Einstein Lagrangian in the presence of the cosmological constant as the gravitational sector of the action. As for the matter field, we simply consider the Maxwell Lagrangian with modified vector potential. Therefore, the $d$-dimensional action will be given by

$$\mathcal{I} = -\frac{1}{16\pi G_d} \int d^d x \sqrt{-g} [\mathcal{R} - 2\Lambda - F_{\mu\nu} F_{\mu\nu}], \quad (1)$$

in which, $\mathcal{R}$ is the Ricci scalar, $\Lambda$ refers to the cosmological constant and $F_{\mu\nu}$ is the electromagnetic field tensor. It is worthwhile to mention the possibility of generalization of this action to include higher dimensional theories of gravity, scalar-tensor field theories and nonlinear electromagnetic field. The magnetic charge, hence the dyonic property lies
within the structure of electromagnetic tensor. The $d$-dimensional metric with topological boundary of $t = cte$ and $r = cte$, is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_k^2,$$

in which $d\Omega_k^2$ is the line element of a $(d - 2)$-dimensional hypersurface with the constant curvature $(d - 2)(d - 3)k$ and volume $V_{d-2}$ with the following explicit form

$$d\Omega_k^2 = \begin{cases} d\theta_1^2 + \sum_{i=2}^{d-2} \prod_{j=1}^{i-1} \sin^2 \theta_i d\theta_i^2 & k = 1 \\ d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sum_{i=3}^{d-2} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 \\ \sum_{i=1}^{d-2} d\phi_i^2 & k = 0 \end{cases}.$$

In order to have consistent field equations with magnetic charge included, we modify the electromagnetic tensor with the following non-zero components

$$F_{tr} = -F_{rt} = \frac{q_E}{r^{d-2}} \quad \text{and} \quad F_{\theta\phi} = -F_{\phi\theta} = \frac{q_M}{r^{d-4}} \Upsilon(\theta)$$

in which $q_E$ and $q_M$ are respectively, electric and magnetic charges, and

$$\Upsilon(\theta) = \begin{cases} \sin \theta & k = 1 \\ \theta & k = 0 \\ \sinh \theta & k = -1 \end{cases}.$$

The $F_{tr}$ is representing the electric part of the electromagnetic field while $F_{\theta\phi}$ is related to the magnetic part. There are several issues that must be pointed out; first of all, the electromagnetic field tensor has been generated by both electric and magnetic charges separately. Therefore, it is possible to cancel out the electric part by setting $q_E = 0$ and have magnetic black hole solutions. Second, we have restricted the magnetic field to one direction which is a common practice for magnetic charges. Finally, except for 4-dimensional case which has constant magnetic field, for large values of the $r$, magnetic field will vanish similar to the electric field which is physically expected. One of the important properties of the proposed electromagnetic tensor is that magnetic field is a stand alone property. In other words, even in the absence of electric charge, one can construct magnetic black holes without resorting to complex field equations.

Using the variational principle, it is a matter of calculation to reach the following field equation

$$e_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu} - \left[2F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{2}g_{\mu\nu}F_{\sigma\rho}F_{\sigma\rho}\right] = 0,$$

which by considering metric and non-zero components of the electromagnetic field tensor, one finds

$$e_{tt} = e_{rr} = (d - 2)(d - 3) [f(r) - k] r^{4d-10} + \left[2\Lambda r + (d - 2) \left(\frac{df(r)}{dr}\right)\right] r^{4d-9} + 2 \left[q_E^2 + q_M^2\right] r^{2d-4} = 0,$$

$$e_{ii} = (d - 3)(d - 4) [f(r) - k] r^{4d-10} + \left[2\Lambda r + 2(d - 3) \left(\frac{df(r)}{dr}\right) + \left(\frac{d^2f(r)}{dr^2}\right)\right] r^{4d-9} - 2 \left[q_E^2 + q_M^2\right] r^{2d-4} = 0.$$

Solving these two equations with respect to metric function, one obtains $f(r)$ as

$$f(r) = k - \frac{m}{r^{d-3}} - \frac{2\Lambda r^2}{(d - 1)(d - 2)} + \frac{2(q_E^2 + q_M^2)}{(d - 2)(d - 3)r^{2d-6}},$$

where $m$ is geometrical mass related to total mass of the black hole. By setting $d = 4$, metric function yields

$$\psi(r) = k - \frac{\Lambda r^2}{3r} - \frac{m}{r} + \frac{q_E^2 + q_M^2}{r^2}.$$
which was previously obtained in ref. [53]. This shows that our proposal includes other types of dyonic black holes as well.

Existence of the black hole solutions depends on satisfaction of specific conditions simultaneously: i) existence of the singularity, and ii) presence of at least one horizon which covers the singularity and is known as event horizon.

The existence of singularity could be determined by divergencies of the curvature scalars. One of the well known curvature scalars is the Kretschmann scalar. It is a matter of calculation to find the Kretschmann for these solutions in the following form

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \left( \frac{d^2 f (r)}{dr^2} \right)^2 + \frac{2 (d - 2)}{r^2} \left( \frac{df (r)}{dr} \right)^2 + \frac{2 (d - 2) (d - 3)}{r^4} (f (r) - k)^2,$$

where

$$\left( \frac{d^2 f (r)}{dr^2} \right) = \frac{4 (2d - 5)}{(d - 2)^2} \left( q_E^2 + q_M^2 \right) - \frac{(d - 2) (d - 3) m}{r^{d-1}} - \frac{4}{(d - 1) (d - 2) \Lambda},$$

$$\left( \frac{df (r)}{dr} \right) = \frac{(d - 3)}{r^{d-2}} m - \frac{4r}{(d - 1) (d - 2) \Lambda} - \frac{4}{(d - 2) r^{2d-8}} \left( q_E^2 + q_M^2 \right).$$

The Kretschmann has the following limit

$$\lim_{r \to 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \to \infty. \quad (10)$$

Eq. (10) confirms the existence of a curvature singularity at \( r = 0 \). By series expansion of this curvature scalar for small values of \( r \), one can find the following relation

$$\lim_{r \to 0} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \propto \left( a_1 q_E^4 + a_2 q_E^2 q_M^2 + a_3 q_M^4 \right) r^{-4(d-2)},$$

in which \( a_i \) are dimension dependent coefficients. It is interesting to note that the singularity is affected by both electric and magnetic fields with same order of magnitude. In the absence of electric field, the dominant term on singular behavior is the magnetic charge. This highlights the contribution of the magnetic part. It is worthwhile to mention that power of the singularity is stronger in higher dimensions and the divergency is reached faster compared to lower dimensions.

Existence of the cosmological constant in the solutions, provides specific complexity which prevents us to extract the root(s) of the metric function, hence event horizon, analytically. In the absence of the cosmological constant \( (\Lambda = 0) \), the roots of metric function are obtained as

$$r (f (r) = 0) = \left[ \frac{m (d - 2) (d - 3) \pm \sqrt{m^2 (d - 2)^2 (d - 3)^2 - 8k (d - 2) (d - 3) (q_E^2 + q_M^2)}}{4 (q_E^2 + q_M^2)} \right]^{\frac{1}{d-3}}, \quad (11)$$

which shows that under certain conditions, two distinct roots for the metric function may exists. The first condition comes from positivity of the expression under square root function. In other words, in order to have real roots for the metric function, the square root function must be positive valued which results into the following condition

$$m^2 \geq \frac{8k (q_E^2 + q_M^2)}{(d - 2) (d - 3)}. \quad (12)$$

This condition has several points which must be highlighted; considering that \( m, q_E \) and \( q_M \) are positive values, the mentioned condition is valid only for spherical case, \( k = 1 \). In other words, for the horizon flat \( (k = 0) \) and hyperbolic horizon \( (k = -1) \), the square root function is always positive valued and mentioned condition is satisfied irrespective of choices for \( m, q_E \) and \( q_M \). If this condition is violated, no real valued root exists for the metric function which indicates that such solutions is a naked singularity. Therefore, it is safe to state that naked singularity only exists for the spherical cases while for horizon flat and hyperbolic horizon, obtained solutions definitely enjoy at least one root,
hence event horizon in their structure. It is interesting to note that for Ricci flat solutions ($k = 0$), the inner horizon goes to $r = 0$, and therefore, the singularity is null.

The coupling between topological factor and the electric and magnetic charges is another important issue. Here, we see that the presence of magnetic charge aﬀected the condition regarding the existence of real valued roots. On the other hand, we see that the presence of magnetic charge has its own effects on the position of root as well. In the absence of the electric charge, the magnetic charge for spherical case upholds the mentioned condition. Returning to obtained root for metric function (11), we see that it is possible to have two real positive valued roots provided that the following inequality holds for negative branch

$$m \left( d - 2 \right) \left( d - 3 \right) - \sqrt{m^2 \left( d - 2 \right)^2 \left( d - 3 \right)^2 - 8k \left( d - 2 \right) \left( d - 3 \right) (q_E^2 + q_M^2)} > 0. \quad (13)$$

Now, considering that for $k = -1,0$, the square root function is positive valued, one can state that for these two cases, the mentioned condition (13) is violated. Therefore, it is safe to conclude that for $k = -1,0$, only one root exists. On the contrary, for spherical horizon, it is possible to have three cases: i) violation of the condition (12) which results in naked singularity. ii) if $m^2 = \frac{8k(q_E^2 + q_M^2)}{(d-2)(d-3)}$, then positive and negative branches of the obtained roots coincide which results in extreme black hole solutions (existence of one root). iii) satisfaction of mentioned conditions (12) and (13) which results in existence of two roots for metric function.

Since it was not possible to obtain the root of metric function in the presence of cosmological constant, we have employed numerical method to plot diagrams (see Fig. 1). Evidently, the existence of root for metric function and its number is a function of the magnetic charge. By suitable choices of this quantity, it is possible to have naked singularity, extreme black holes (one root) and two roots with larger root being event horizon. $q_M$ is linearly related to magnetic charge. Its value determines the power of magnetic field. Considering this fact, Fig. 1 conﬁrms a very important fact: the super magnetized solutions suﬀers the absence of horizon. In other words, the super magnetized solutions are naked singularity.

Our final study in this section is regarding the asymptotic behavior of the solutions. To investigate this, it is suﬃcient to study the behavior of curvature scalar for large $r$. It is a matter of calculation to show that the obtained Kretschmann scalar will have following behavior

$$\lim_{r \to \infty} R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \propto a_4 \Lambda + a_5 \Lambda^2 + O \left( \frac{1}{r^3} \right), \quad (14)$$

in which $a_4$ and $a_5$ are dimension dependent coefficients. Evidently, the dominant term in this limit is $\Lambda$ term which indicates that the asymptotic have AdS/dS behavior depending on the sign of cosmological constant. In the absence of the cosmological constant, the dominant term for asymptotic behavior will be geometrical term in following form

$$\lim_{\Lambda = 0} \frac{r^d}{d-4} R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} = - \frac{a_6}{r^{d-1}} m + \frac{a_7}{r^{2d-4}} (q_E^2 + q_M^2) + O \left( \frac{1}{r^6} \right), \quad (15)$$

which shows that the effects of matter field (electric and magnetic fields) on asymptotic behavior is of secondary importance in the absence of the cosmological constant ( $a_6$ and $a_7$ are dimension dependent coefficients).

In conclusion, we established that these solutions enjoy the presence of a singularity which is located at the origin and depending on choices of diﬀerent parameters, this singularity could be covered by one or two horizons. The role of the magnetic charge on singular behavior and properties of the horizon(s) was pointed out. In addition, the asymptotic behavior was investigated and it was shown that in full form, it is AdS/dS depending on negativity/positivity of the cosmological constant. In the next section, we derive conserved quantities and study the eﬀects of magnetic term and higher dimensions on the thermodynamical behavior of the system.

## III. THERMODYNAMIC PROPERTIES

In the previous section, we established the fact that our solutions could be interpreted as black holes. Having black hole solutions, it is possible to calculate thermodynamic quantities and study their thermodynamical behavior. Here, our main focus is on obtaining thermodynamical properties and employing the first law of thermodynamics to study various properties of these black holes.

The entropy of Einsteinian black holes could be extracted by using the area law, which leads to
FIG. 1: $f(r)$ versus $r$ for $d = 5$, $\Lambda = -1$, $m = 0.2$, $q_E = 0.1$, $k = 1$; $q_M = 0$ (continuous line), $q_M = 0.14$ (dashed line) and $q_M = 0.2$ (dashed-dotted line).

\[
S = \frac{\pi r_+^{d-2}}{4},
\]  

in which $r_+$ is the outer horizon (the largest positive real root of metric function). The total electric charge could be obtained through the use of the Gauss law which leads to

\[
Q_E = \frac{q_E}{4\pi}.
\]  

The same method could be employed to calculate total magnetic charge which is

\[
Q_M = \frac{q_M}{4\pi}.
\]  

It is interesting to note that although the electric and magnetic parts of $F_{\mu\nu}$ are completely different, their conserved charges are of the same nature which are arisen from Gauss’s law. Besides, the total mass of these black holes could be calculated by using the ADM (Arnowitt-Deser-Misner) method which leads to

\[
M = \frac{(d - 2)}{16} m.
\]  

By evaluating the metric function on the horizon, one can also find the geometrical mass. Using obtained entropy \[16\], electric \[17\] and magnetic charges \[18\], one can obtain the following Smarr like formula for these black holes

\[
M (S, Q_E, Q_M) = \frac{(d - 2)}{16} \left( \frac{4S}{\pi} \right)^{\frac{d-2}{d-1}} k - \frac{1}{8(d-1)} \Lambda \left( \frac{4S}{\pi} \right)^{\frac{d-2}{d-1}} + 2\pi^2 \left( \frac{Q_E^2 + Q_M^2}{(d-3)} \right) \left( \frac{4S}{\pi} \right)^{-\frac{d-2}{d-1}}.
\]  

Using the obtained mass together with the first law of black hole thermodynamics

\[
dM = TdS + \Phi_E dQ_E + \Phi_M dQ_M,
\]  

one can extract the electric and magnetic potentials respectively in the following forms
\[ \Phi_E = \frac{\pi q_E}{(d-3) r_+^{d-3}}, \]  
(22) 

\[ \Phi_M = \frac{\pi q_M}{(d-3) r_+^{d-3}}. \]  
(23) 

Recently, there has been a renewed proposal for the cosmological constant; which is taken not as a fixed parameter, 
instead, it is regarded as a thermodynamic variable known as dynamical pressure [54]. The relation between these 
two quantities is given by 

\[ P = -\frac{\Lambda}{8\pi}, \]  
(24) 

which by replacing it in Smarr like formula, one can obtain 

\[ M (S, Q_E, Q_M) = \frac{(d-2)}{16} \left( \frac{4S}{\pi} \right)^{\frac{d-2}{2}} k + \frac{\pi}{(d-1)} P \left( \frac{4S}{\pi} \right)^{\frac{d-2}{2}} + 2\pi^2 \left( \frac{Q_E^2 + Q_M^2}{(d-3)} \right) \left( \frac{4S}{\pi} \right)^{-\frac{d-2}{2}}. \]  
(25) 

This consideration modifies the role of the mass from internal energy to enthalpy and the first law of black hole 
thermodynamics (21) will be modified into 

\[ dH = TdS + \Phi_E dQ_E + \Phi_M dQ_M + VdP. \]  
(26) 

Using this relation, it is a matter of calculation to extract the corresponding volume 

\[ V = \left( \frac{\partial H (S, Q_E, Q_M, P)}{\partial P} \right)_{S, Q_M, Q_E} = \left( \frac{\partial M (S, Q_E, Q_M, P)}{\partial P} \right)_{S, Q_M, Q_E} = \frac{\pi r_+^{d-1}}{(d-1)}, \]  
(27) 

which is geometrically expected. The temperature of black hole is generally obtained through the concept of surface 
gravity which is given by 

\[ \kappa = \sqrt{-\frac{1}{2} \left( \nabla_\mu \chi^\nu \right) \left( \nabla_\mu \chi^\nu \right)}, \]  
(28) 

where \( \chi^\nu \) is a Killing vector. Considering the fact that our solutions are static, the Killing vector will be \( \chi = \partial_t \), and 
therefore, temperature is calculated as 

\[ T = \frac{\kappa}{2\pi} = \frac{1}{4\pi} \frac{df(r)}{dr}, \]  
(29) 

By replacing the cosmological constant with its correspondence pressure in temperature [21], it is possible to extract 
an equation of state. Using the equation of state and checking the existence of its inflection point 

\[ \left( \frac{\partial P}{\partial r_+} \right)_T = \left( \frac{\partial^2 P}{\partial r_+^2} \right)_T = 0, \]  
(30) 

one is able to extract the critical points. Furthermore, it is possible to obtain the free energy by using the following 
relation 

\[ F = H - TS - \Phi_E Q_E. \]  
(31) 

The last thermodynamic quantity of interest is the heat capacity which could be used to determine thermal stability 
and possible phase transition of the black holes. This quantity is given by
\[ C = T \left( \frac{\partial S}{\partial M} \right)_{Q,M,Q_E,P} \]  

(32)

In next sections, we will investigate thermodynamical behavior of the solutions for the following ensembles; i) general case, ii) constant electric field, iii) constant magnetic field, and iv) constant electric and magnetic fields.

Unfortunately, it was not possible to extract all the properties for mentioned cases in arbitrary dimension, analytically. Therefore, we consider 5-dimensional black holes (as a prototype higher dimensional solutions) as a case study to understand the effects of higher dimensions on properties of solutions.

IV. GENERAL CASE

Here, we consider unspecified electric and magnetic fields and study the general behavior. Using obtained mass (30), it is a matter of calculation to show that mass/enthalpy for this case is given by

\[ M = H = \frac{(d - 2) r_+^{d-3}}{16} k^2 - \frac{\pi r_+^{d-1}}{16} P + \frac{q_+^{3-d}}{8(d - 3)} (q_E^2 + q_M^2). \]  

(33)

As it was pointed out, it is not possible to obtain roots of the mass/enthalpy for arbitrary \( d \), analytically. Therefore, we consider 5-dimensional solution as a case study. So, in 5-dimensions, the root is given by

\[ r_+(M(d = 5) = 0) = \frac{\sqrt{A_1^{2/3} + k^2 - kA_1^{1/3}}}{2\sqrt{\pi PA_1^{1/6}}}, \]  

(34)

in which \( A_1 = 4\pi P\sqrt{q_E^2 + q_M^2} \sqrt{4\pi^2 P^2 (q_E^2 + q_M^2) + k^2 - 8\pi^2 P^2 (q_E^2 + q_M^2)} - k^3. \)

In addition, the high energy limit and asymptotical behavior of the mass are given by

\[ \lim_{r_+ \to 0} M = \frac{1}{16r_+^2} (q_E^2 + q_M^2) + \frac{3}{16} r_+^2 k + O \left( \frac{1}{r_+^2} \right), \]  

(35)

\[ \lim_{r_+ \to \infty} M = \frac{\pi}{4} r_+^4 P + \frac{3}{16} r_+^2 k + O \left( \frac{1}{r_+} \right). \]  

(36)

The high energy limit of the mass (enthalpy) for this case is governed by magnetic and electric charge terms. This indicates that in the absence of the electric field, the dominant term in the high energy limit of the mass is magnetic charge of the solutions. The effects of electric and magnetic charges are of the same order. On the other hand, the asymptotic behavior is governed by the pressure term which is essentially the cosmological constant term. In general, one can state that for small black holes, the effects of matter field, hence the magnetic and electric charges become dominant over other quantities contributing to the mass, whereas for large black holes, the significant effect comes from the pressure term. This difference in the limiting behavior could be employed to determine the size of black hole, although there are other matters that should be considered before making a final statement. But, the important subject is that here, the contribution of the magnetic charge becomes significant as we study the high energy limit. Considering that for both limits, the dominant terms are positive valued with power of the horizon radius presented for different terms, one expects a minimum for the mass. Plotting mass versus \( r_+ \) confirms this statement (see left panel of Fig. 8). The minimum takes place where the topological term, \( k \), becomes dominant which is the case for medium black holes. This indicates that the smallest mass (enthalpy) available for these black holes belongs to medium black holes.

Using Eq. (29) or \( \left( \frac{\partial H}{\partial S} \right)_{Q,M,Q_E,P} \), it is possible to extract the temperature as

\[ T = \frac{1}{2\pi} \left[ \frac{(d - 3)}{2r_+} k^2 - \frac{8\pi r_+}{(d - 2)} P + \frac{q_+^2}{(d - 2) r_+^{d-4}} \right]. \]  

(37)

The root of this quantity could not be calculated for \( d \)-dimensional case, whereas for the 5-dimensional case it is obtained as
in which \( A_2 = 8\pi P\sqrt{q_E^2 + q_M^2} \sqrt{16\pi^2 P^2 (q_E^2 + q_M^2) + k^3 + 32\pi^2 P^2 (q_E^2 + q_M^2) - k^3} \). The high energy limit and asymptotic behavior of the temperature are given by

\[
\lim_{r_+ \to 0} T = -\frac{1}{6\pi r_+} (q_E^2 + q_M^2) + \frac{1}{2\pi r_+} k + O(r_+),
\]

\[
\lim_{r_+ \to \infty} T = \frac{4}{3} r_+ P + \frac{1}{2\pi r_+} k + O\left(\frac{1}{r_+^2}\right).
\]

Here too, similar to enthalpy, the dominant term in high energy limit includes electric and magnetic charges but contrary to enthalpy case, its sign is negative. On the other hand, the asymptotic behavior of the temperature is governed by the pressure term whereas for medium black holes, the topological factor determines the behavior of temperature. Since the high energy limit has negative sign and asymptotic behavior is positive, it is expected that there is at least one real valued positive root for temperature which could be observed in plotted diagrams (see middle panel of Fig. 2). Studying the diagrams for temperature confirms that depending on the choices of magnetic charge, temperature: i) could be an increasing function of horizon radius with one root. ii) could have one extremum and one root. iii) could have one root with one minimum and one maximum which are located after the root. Existence of the extremum for temperature confirms the existence of divergency for the heat capacity. In other words, extrema are where the heat capacity acquires divergencies, and therefore, they are phase transition points. Here, we see that modification in magnetic charge results in existence/absence of extremum for the temperature. In addition, since before root, the temperature is negative and solutions are not physical, one can conclude that modification in magnetic charge changes the valid range for existence of physical black holes. The root of temperature is an increasing function of the magnetic charge.

It is a matter of calculation to obtain the equation of state for this case as

\[
P = \frac{1}{2\pi} \left[ \frac{\pi (d-2)}{2r_+} T - \frac{(d-2)(d-3)}{8r_+^2} k + \frac{q_E^2 + q_M^2}{4r_+^{2d-4}} \right].
\]

Using the concept of inflection point \(^{[30]}\), one can obtain the following equation for calculating the critical horizon radius

\[
\frac{(q_E^2 + q_M^2) (4d-10)}{r_+^{2d-6}} - (d-3) k = 0,
\]

in which the critical horizon radius, temperature and pressure are, respectively, obtained as

\[
r_c = \left[ \frac{(q_E^2 + q_M^2) (4d-10)}{(d-3) k} \right]^\frac{1}{2d-5},
\]

\[
T_c = \frac{1}{\pi} \left( \frac{k}{2d-5} \right)^\frac{2d-5}{2d-6} (d-3)^\frac{4d-11}{2d-6} (2q_E^2 + 2q_M^2)^{-\frac{1}{2d-5}},
\]

\[
P_c = \frac{k^{\frac{2d-7}{2d-5}}}{16\pi (q_E^2 + q_M^2)^{1/(d-3)}} \left[ 2^{\frac{2d-7}{d-5}} (2d-5)^{\frac{2-d}{2d-7}} \left[ (d-2)(d-3)^{\frac{2d-5}{d-5}} + \frac{(d-3)^{\frac{2d-5}{d-5}}}{4} \right] 
- \left( \frac{d-3}{2d-5} \right)^{\frac{1}{(d-3)}} \frac{d}{2^{1/d-3}} - 2^{\frac{d-7}{2d-5}} \right].
\]
Evidently, the critical horizon radius is an increasing function of the magnetic charge while critical temperature and pressure are decreasing function of it. This indicates that for magnetized black holes, phase transition takes place in larger horizon radius for smaller pressure and temperature. Therefore, one can conclude that for super magnetized black holes (large magnetic charge), van der Waals like phase transition takes place in very small values of temperature and pressure but for large volume. This enables one to understand the effects of magnetization on van der Waals like behavior of the black holes. In order to confirm the existence of van der Waals like behavior, we have plotted the following diagrams for the pressure (see Fig. 3). Evidently, by variation of the magnetic charge, the pressure can acquire i) one extremum which indicates existence of critical behavior taking place at one point. ii) two extrema which show the existence of phase transition over range (rather than a single point) iii) without any extremum which is interpreted as the absence of critical behavior.

Using obtained thermodynamical quantities, it is possible to calculate the free energy as

\[ F = \frac{q_d^d - 1}{16} k - \frac{\pi r_+^{d-1}}{(d-1)(d-2)} P + \frac{(2d - 5) r_+^{d-3}}{8(d-2)(d-3)} q_M^2 + \frac{2}{16\pi(d-2)(d-3)} \frac{q^2}{d^2}. \]  

(46)

Considering that \( q_E \) term is horizon radius independent, for high energy limit, this term becomes dominant, whereas, for asymptotic behavior, the dominant term is pressure term. Finally, the heat capacity of this case is obtained as

\[ C = \frac{(d - 2)(d - 3) \pi r_+^{3d-2k} + 16\pi^2 r_+^{3d} P - 2\pi r_+^{d+4} (q_E^2 + q_M^2)}{64\pi r_+^{d+4} P - 4 (d - 2)(d - 3) r_+^d k + 8 (2d - 5)(q_E^2 + q_M^2) r_+^d} (d - 2). \]  

(47)

Divergencies of the heat capacity could not be extracted analytically for arbitrary dimension, while for the 5-dimensional case, divergence occurs at

\[ r_+ (C(d = 5) \rightarrow \infty) = \frac{\sqrt{A_3^{2/3} + k^2 - k A_3^{1/3}}}{\sqrt{8\pi PA_3^{1/6}}}. \]  

(48)

where \( A_3 = 8\sqrt{\pi} P \sqrt{q_E^2 + q_M^2} \sqrt{80\pi^2 P^2 (q_E^2 + q_M^2) - k^2} - 160\pi^2 P^2 (q_E^2 + q_M^2) + k^3. \)

In addition, based on series expansion, we find that high energy limit and asymptotic behavior of the heat capacity are given by

\[ \lim_{r_+ \to 0} C = -\frac{3\pi}{20} r_+^3 + \frac{9}{25} q_E^2 + q_M^2 r_+^3 + O(r_+^3), \]  

(49)

\[ \lim_{r_+ \to \infty} C = \frac{3\pi}{4} r_+^3 + \frac{9}{16} k r_+ + O\left(\frac{1}{r_+^3}\right). \]  

(50)

The effect of the magnetic charge on the obtained divergency for the heat capacity is evident. This shows that phase transition points, root and stability conditions of the black holes are modified due to the magnetic charge. If certain conditions are satisfied, the heat capacity could acquire two divergencies (see right panel of Fig. 2). Between the divergencies, solutions suffer from thermal instability, since the heat capacity is negative valued. But comparing this case with plotted diagrams for the pressure, one can see that extrema of pressure coincide with divergencies of the heat capacity. Therefore, the region between divergencies is where physical black holes are absent. As a result, one can draw the following conclusions regarding thermal stability of the solutions: i) in the absence of divergencies, stable black holes exist after root of the heat capacity which is the same as root of the temperature. ii) in the case of one divergency, there is a phase transition between medium stable black holes and larger stable ones taking place at the divergency. Before root of the heat capacity (temperature), solutions are unstable and non-physical (due to negative temperature). iii) in case of two divergencies for the heat capacity, the phase transition takes place over a region between small stable black holes and large stable ones. Once more, we emphasize that between two divergencies, due to thermodynamical concepts, no physical black holes exist.

Interestingly, the high energy limit of heat capacity and asymptotic behavior of it depends only on horizon radius with some factors. In other words, the high energy limit and asymptotic behavior of the heat capacity are depending neither on electric and magnetic charges nor on pressure. It seems that for the heat capacity, these two limits are only affected by the gravitational part of the action, since there is no trace of matter field and cosmological constant generalization present in them. It is worthwhile to mention that the second dominant term in high energy limit of the heat capacity includes electric and magnetic charges which highlights the contribution of magnetic charge.
FIG. 2: $H$ (left panel), $T$ (middle panel) and $C$ (right panel) versus $r_+$ for $d = 5$, $k = 1$, $P = 0.1$ and $q_E = 0.1$; $q_M = 0$ (continuous line), $q_M = 0.34155$ (dashed line) and $q_M = 0.4$ (dashed-dotted line).

FIG. 3: $P$ versus $r_+$ for $d = 5$, $k = 1$, $T = 0.2855$ and $q_E = 0.1$; $q_M = 0$ (continuous line), $q_M = 0.34155$ (dashed line) and $q_M = 0.4$ (dashed-dotted line).

V. CONSTANT ELECTRIC FIELD

In this section, we work in an ensemble where the temporal component of the electromagnetic tensor, $A_t$, is constant everywhere. Therefore, we can replace the electric charge, $q_E$, by its value at the horizon with following relation

$$q_E = \frac{(d - 3) r_+^{d-3} \Phi_E}{\pi}.$$  \hspace{1cm} (51)

So, the mass/enthalpy of this case is given by

$$M = \frac{(d - 2) r_+^{d-3}}{16} k + \frac{\pi r_+^{d-1}}{d - 1} P + \frac{r_+^{3-d}}{8 (d - 3)} q_M^2 + \frac{(d - 3) r_+^{d-3}}{8 \pi^2} \Phi_E^2,$$  \hspace{1cm} (52)

in which, its roots could not be obtained analytically, except for the 5-dimensional case which are
two roots and presence of a minimum originates from contributions of the magnetic charge. Specially topological factor, the enthalpy can have no root, one root, or two roots. It is notable that existence of magnetic charge. In this case, the enthalpy acquires a minimum. Depending on choices of different parameters of magnetic charge, enthalpy is an increasing function of horizon radius. The situation is modified in the presence of black holes the magnetization has a significant role on the behavior of enthalpy. That being said, one can see that due to the presence of magnetic charge (none constant magnetic field), the enthalpy for vanishing the horizon radius could only occur for black holes with hyperbolic horizon. This specific behavior is rooted in the assumption of electric field leads to the high energy limit of enthalpy becomes modified at a significant level. The enthalpy is an increasing function of the magnetic charge. Taking a closer look at the high energy limit and asymptotic behavior, one can see that for medium range of horizon radius dominant terms of enthalpy are both electric potential and topological term. Interestingly, by choosing it is given by

\[ r_+(M(d = 5) = 0) = \frac{1}{6} \sqrt{\frac{\left(2\pi^2 k + \frac{4\pi^2}{A_4^{1/3}} \right)^2 + 3A_4^{1/3} - 12\Phi_E^2 - 9\pi^2 k}{\pi^3 P}}, \]  

(53)

where \( A_4 = -A_5 + 12\sqrt{3}\pi^4 Pq_M\sqrt{A_5} \) and \( A_5 = 216\pi^8 P^2q_M^2 + 27\pi^6 k^3 + 108\pi^4 k^2\Phi_E^2 + 144\pi^2 k\Phi_E^4 + 64\Phi_E^6 \). Also the high energy limit and asymptotic behavior are given by

\[ \lim_{r_+ \to 0} M = \frac{1}{16r_+^2} q_M^2 + \left( \frac{3}{16} + \frac{1}{4\pi^2 \Phi_E^2} \right) r_+^2 + O \left( \frac{1}{r_+^4} \right), \]  

(54)

\[ \lim_{r_+ \to \infty} M = \frac{\pi}{4} r_+^4 + P + \left( \frac{3}{16} + \frac{1}{4\pi^2 \Phi_E^2} \right) r_+^2 + O \left( \frac{1}{r_+^4} \right), \]  

(55)

The high energy limit of mass/enthalpy of this case is governed by the magnetic charge. This means that for small black holes the magnetization has a significant role on the behavior of enthalpy. That being said, one can see that due to the presence of magnetic charge (none constant magnetic field), the enthalpy for vanishing the horizon radius diverges. Comparing this case with previous one, it can be stated that the presence of magnetic charge with constant electric field leads to the high energy limit of enthalpy becomes modified at a significant level. The enthalpy is an increasing function of the magnetic charge. Taking a closer look at the high energy limit and asymptotic behavior, one can see that for medium range of horizon radius dominant terms of enthalpy are both electric potential and topological term. Interestingly, by choosing

\[ \Phi_E = \pi \sqrt{-\frac{3k}{4}}, \]  

(56)

for \( k = -1 \), it is possible to cancel out all the effects of electric field on enthalpy. It is clear that such a situation could only occur for black holes with hyperbolic horizon. This specific behavior is rooted in the assumption of electric field being constant. In order to have a better picture regarding thermodynamic behavior of the enthalpy for this case, we have plotted qa diagram (see left panel of Fig. 4). Evidently, for constant electric field and in the absence of magnetic charge, enthalpy is an increasing function of horizon radius. The situation is modified in the presence of magnetic charge. In this case, the enthalpy acquires a minimum. Depending on choices of different parameters (specially topological factor), the enthalpy can have no root, one root, or two roots. It is notable that existence of two roots and presence of a minimum originates from contributions of the magnetic charge.

The temperature for this case is obtained as

\[ T = \frac{1}{2\pi} \left[ \frac{(d - 3)}{2r_+} k + \frac{8\pi r_+}{(d - 2)} P - \frac{q_M^2}{(d - 2)} r_+^{2d - 5} - \frac{(d - 3)^2}{\pi^2 (d - 2) r_+} \Phi_E^2 \right]. \]  

(57)

The root of the temperature could not be obtained analytically for general d-dimensional case, but in 5- dimensions, it is given by

\[ r(T(d = 5) = 0) = \frac{1}{12} \sqrt{\frac{\left(54 \left(\frac{\pi^2 k - 4\Phi_E^2}{A_6^{1/3}} \right)^2 + 6A_6^{1/3} + 24\Phi_E^2 - 18\pi^3 k \right)}{\pi^3 P}}, \]  

(58)

where \( A_6 = A_7 + 24\sqrt{3}\pi^4 Pq_M\sqrt{A_8} \), in which \( A_7 \) and \( A_8 \) are

\[ A_7 = 864\pi^8 P^2 q_M^2 - 27\pi^6 k^3 + 108\pi^4 k^2\Phi_E^2 - 144\pi^2 k\Phi_E^4 + 64\Phi_E^6, \]  

(59)

\[ A_8 = 432\pi^8 P^2 q_M^2 - 27\pi^6 k^3 + 108\pi^4 k^2\Phi_E^2 - 144\pi^2 k\Phi_E^4 + 64\Phi_E^6. \]  

(60)
For high energy limit and asymptotic behavior we have

\[
\lim_{r_+ \to 0} T = -\frac{1}{6\pi r_+^5} q_M^2 + \left(\frac{1}{2} k - \frac{2}{3\pi^3} \Phi^2_E\right) \frac{1}{r_+} + O\left(\frac{1}{r_+}\right),
\]

(61)

\[
\lim_{r_+ \to \infty} T = \frac{4}{3} + P + \left(\frac{1}{2\pi} k - \frac{2}{3\pi^3} \Phi^2_E\right) \frac{1}{r_+} + O\left(\frac{1}{r_+}\right).
\]

(62)

The dominant term in the high energy limit is the magnetic term with negative sign. For medium black holes, topological and electric potential terms are governing the behavior of temperature. Finally, the asymptotic behavior is governed by pressure, hence cosmological term. It is worthwhile to mention that for \( \Phi = \pi \sqrt{\frac{q}{M}} \), the effects of topological and electric potential cancel each other. This could only take place for spherical black holes \((k = 1)\). Since the high energy limit of the temperature is negative valued and asymptotic behavior is positive, one can confirm that there exists at least one root for the temperature. To show this, we have plotted it in middle panel of Fig. 4. In the absence of magnetic charge, temperature has a minimum. By taking a non-zero value for the magnetic charge, one can find the following behavior for temperature: There exists a critical magnetic charge, \( q_M - \text{critical} \) at which the temperature acquires one extremum. For magnetic charges less than this critical magnetic charge, \( q_M < q_M - \text{critical} \), the temperature has two extrema: one minimum and one maximum. On the other hand, for \( q_M > q_M - \text{critical} \), no extremum is available for the temperature. Remembering that extrema are where the heat capacity diverges, one can state that the presence of magnetic charge enables the possibility of existence of van der Waals like behavior for these black holes, provided by mentioned cases. In the absence of magnetic charge, temperature indeed has a minimum, but the type of phase transition for these two cases are different: while one is van der Waals like phase transition the other one (absence of magnetic charge) does not enjoy this type of phase transition. This highlights the importance of the contribution of the magnetic charge. It is worthwhile to mention that there exists a root for temperature which is an increasing function of the magnetic charge.

The equation of state for the pressure is obtained in the following form

\[
P = \frac{1}{2\pi} \left[ \frac{\pi (d - 2)}{2r_+} T - \frac{(d - 2) (d - 3)}{8r_+^2} k + \frac{q_M^2}{4r_+^{2d - 4}} + \frac{(d - 3)^2}{4\pi^2 r_+^2} \Phi^2_E \right] .
\]

(63)

Using the properties of the inflection point, one can derive the following relation governing the critical horizon radius

\[
\frac{(d - 2)(4d - 10)}{r_+^{2d - 6}} \pi^2 q_M^2 - (d - 3) \left[ k (d - 2) \pi^2 - 2 (d - 3) \Phi^2_E \right] = 0,
\]

(64)

which leads to

\[
r_c = \left[ \frac{(d - 3) \left[ k (d - 2) \pi^2 - 2 (d - 3) \Phi^2_E \right]}{2 (d - 2) (2d - 5) \pi^2 q_M^2} \right]^{\frac{1}{d - 6}} .
\]

(65)

It is a matter of calculation to obtain the critical temperature and pressure as

\[
T_c = (2d - 4) \pi^{\frac{2d}{d - 6}} \left[ (2d - 5) q_M^2 \right]^{\frac{1}{d - 6}} \pi^{\frac{4d - 2d}{d - 6}} \left[ (d - 2) (d - 3) \pi^2 k - 2 (d - 3)^2 \Phi^2_E \right]^{\frac{5 - 2d}{2d - 6}}
\]

\[
\pi^4 (d - 2)^2 (d - 3)^2 k^2 - 4 (d - 2) (d - 3)^3 \pi^2 \Phi^2_E k - 4\pi^4 (d - 2)^2 (2d - 5) q_M^4 + 4 (d - 3)^4 \Phi^4_E ) ,
\]

(66)
The van der Waals like phase transition takes place over this region. For a critical magnetic charge, the behavior of pressure diagrams is modified. There exists a root and maximum which appears after root. Therefore, for this case, there is a region of negative pressure before root. After such root, there exists a maximum which marks the existence of a phase transition which is a decreasing function of the horizon radius with no extremum, hence critical horizon radius. But here, the electric potential is in the numerator of the obtained critical horizon radius with negative sign. The positive real valued critical horizon radius only exits for the spherical case. On the other hand, critical temperature and pressure are decreasing functions of the magnetic charge and electric potential. Here, the obtained critical temperature and pressure show a significant modification compared to the previous case. This indicates that consideration of the electric field being constant resulted in different classes and electric potential. Here, the obtained critical temperature and pressure show a significant modification compared to the previous case. This indicates that consideration of the electric field being constant resulted in different classes of black holes which have different thermodynamical properties, therefore critical structure. We should point it out that in the absence of electric field, the black holes will have critical behavior which indicates that magnetic black holes have also van der Waals like behavior in their structure considering the set up proposed in this paper. In order to have a better picture regarding the effects of magnetic charge on this case, we have plotted Fig. 5. Evidently, in this case, pressure has a root and maximum which appears after root. Therefore, for this case, there is a region of negative pressure before root. After such root, there exists a maximum which marks the existence of a phase transition which is a decreasing function of the horizon radius with no extremum, hence critical temperature and pressure. In the presence of magnetic charge, the behavior of pressure diagrams is modified. There exists a critical magnetic charge, \( q_{M\text{-critical}} \) in which for \( q_{M} < q_{M\text{-critical}} \), pressure has one minimum and one maximum. The van der Waals like phase transition takes place over this region. For \( q_{M} = q_{M\text{-critical}} \), the pressure acquires an extremum which is a critical point. In other words, this is the case in which phase transition at a single point. For \( q_{M\text{-critical}} < q_{M} \), pressure will be a decreasing function of the horizon radius without any extremum, hence critical point.

It is possible to obtain the free energy of this case in following form

\[
P_{c} = \left( \frac{\Phi_{E}^{2}}{4 (d-3)^{2} + \pi^{2} k} \right)^{2/d} \left( \frac{q_{E}^{2} + q_{M}^{2}}{2 \pi^{d-1} (d-2)^{d-2} (2d-5)^{d-2}} \right)^{1/(d-3)}
+ \left[ (d-2) (2d-5) \right]^{1/(d-3)} \left( \frac{2^{d-4}}{3^{d-4}} \frac{\pi^{d-4}}{\Phi_{E}^{2}} \left( \frac{q_{E}^{2} + q_{M}^{2}}{2 \pi^{d-1}} \right)^{1/(d-3)} \left( \frac{q_{M}^{2}}{2 \pi^{d-1}} \right)^{1/(d-3)} \right) \left( \frac{2^{d-4}}{3^{d-4}} \frac{\pi^{d-4}}{\Phi_{E}^{2}} \left( \frac{q_{E}^{2} + q_{M}^{2}}{2 \pi^{d-1}} \right)^{1/(d-3)} \left( \frac{q_{M}^{2}}{2 \pi^{d-1}} \right)^{1/(d-3)} \right) \right]
- \frac{1}{16} \left( \frac{2^{d-4} (d-2)^{d-2} \pi^{d-4}}{\Phi_{E}^{2} (d-2) + \pi^{2} k (d-2)} \right)^{1/(d-3)}
\]

in which \( \Gamma = 52 \left( \frac{13 d^{2} - 28 d + 20}{13} \right) \left( \frac{\pi^{d-1} q_{M}^{4d-10}}{d-3} \right)^{1/(d-3)} - 4 k \Phi_{E}^{2} (d-2) (d-3)^{3} \left( \frac{q_{M}^{2}}{\pi^{d-1}} \right)^{1/(d-3)}
+ \left( \frac{4 (d-3)^{4} \Phi_{E}^{2}}{\pi^{d-4}} + \pi^{d-4} k^{2} (d^{4} + 37 d^{2} - 60 d + 36) \right) \frac{4q_{M}^{10}}{q_{M}^{4}} \right) \).

Evidently, here, the critical horizon radius is a decreasing function of the electric potential and magnetic charge while it is an increasing function of the topological parameter (if taken as a continuous variable). In previous case, the critical horizon radius was a decreasing function of the electric charge but the presence of the electric charge was in the denominator of the critical horizon radius. But here, the electric potential is in the numerator of the obtained critical horizon radius with negative sign. The positive real valued critical horizon radius only exits for the spherical case. On the other hand, critical temperature and pressure are decreasing functions of the magnetic charge and electric potential. Here, the obtained critical temperature and pressure show a significant modification compared to the previous case. This indicates that consideration of the electric field being constant resulted in different classes of black holes which have different thermodynamical properties, therefore critical structure. We should point it out that in the absence of electric field, the black holes will have critical behavior which indicates that magnetic black holes have also van der Waals like behavior in their structure considering the set up proposed in this paper. In order to have a better picture regarding the effects of magnetic charge on this case, we have plotted Fig. 5. Evidently, in the absence of magnetic charge, no van der Waals like behavior is present for the pressure vs horizon radius. In this case, pressure has a root and maximum which appears after root. Therefore, for this case, there is a region of negative pressure before root. After such root, there exists a maximum which marks the existence of a phase transition which is not van der Waals like. Later, in studying the heat capacity diagrams, we will see what type of phase transition these diagrams represent. In the presence of magnetic charge, the behavior of pressure diagrams is modified. There exists a critical magnetic charge, \( q_{M\text{-critical}} \) in which for \( q_{M} < q_{M\text{-critical}} \), pressure has one minimum and one maximum. The van der Waals like phase transition takes place over this region. For \( q_{M} = q_{M\text{-critical}} \), the pressure acquires an extremum which is a critical point. In other words, this is the case in which phase transition at a single point. For \( q_{M\text{-critical}} < q_{M} \), pressure will be a decreasing function of the horizon radius without any extremum, hence critical point.

It is possible to obtain the free energy of this case in following form

\[
F = \frac{r_{+}^{d-3} k - \frac{\pi_{+}^{d-1}}{(d-1) (d-2)} P + \frac{(2d-5) r_{+}^{d-3}}{8 (d-2) (d-3)} q_{E}^{2} + \frac{2 (2d-5) \pi - (d-2)}{16 \pi^{3} (d-2)} (d-3) r_{+}^{d-3} \Phi_{E}^{2}}{16 (d-3) r_{+}^{d-3} \Phi_{E}^{2}}
\]

The final subject of interest in this section is the heat capacity. It is a matter of calculation to obtain the heat capacity as

\[
C = \frac{(d-2) (d-3) \pi_{+}^{3d-2} k + 16 \pi_{+}^{3d} P - 2 \pi_{+}^{d+4} q_{E}^{2} - 2 (d-3)^{2} \pi_{+}^{3d-3} \Phi_{E}^{2}}{64 \pi_{+}^{2d+1} P + 4 \pi^{2} P (d-2) (d-3) r_{+}^{2d} k + 8 \pi^{2} (2d-5) q_{E}^{2} + 8 (d-3)^{2} r_{+}^{2d} \Phi_{E}^{2}} (d-2),
\]

in which, for the 5-dimensional case, one can obtain the corresponding divergencies as
Therefore, we replace the magnetic charge, \( q \), where \( A \) are, respectively, given by behavior and critical points for magnetically charged black holes. In this case, the stability conditions, hence of magnetic charge (for fixed values of other parameters). Once again, we point it out that in the absence of electric field, it is possible to obtain critical behavior for the magnetic black holes. In this case, the stability conditions, regions of stability and absence of critical behavior depend on the value of magnetic charge is in the denominator of this term. Interestingly, for both cases of the asymptotical behavior and high energy limit, the presence of electric field is observed in the second leading order term in the numerator of both cases. The gravitational part of solutions since the dominant terms for both cases depend only on the horizon radius. The effects of magnetic charge could be seen in the second leading order term of the high energy limit in which the magnetic charge is in the denominator of this term. Interestingly, for both cases of the asymptotical behavior and high energy limit, the presence of electric field is observed in the second leading order term in the numerator of both cases. This confirms that the effects of magnetic charge on the thermal stability of medium black holes become significant. It should be highlighted that in the presence of constant electric field, the high energy limit and asymptotic behavior of heat capacity are highly modified as compared to other thermodynamical quantities. Once more, this highlights the differences between these black holes and previous ones.

\[
\lim_{r_+ \to 0} C = \frac{3\pi^3}{20} + \frac{3}{25} \frac{3k\pi^2 - 4\Phi E^2}{\pi q_M^3} r_+^7 + O\left(\frac{1}{r_+^2}\right),
\]

\[
\lim_{r_+ \to \infty} C = \frac{3\pi^3}{4} + \frac{3}{16} \frac{3k\pi^2 - 4\Phi E^2}{\pi^2 P} r_+ + O\left(\frac{1}{r_+}\right).
\]

In order to understand more details regarding the types of phase transitions of this case, we have plotted the necessary diagrams (see right panel of Fig. 4). First of all, in the absence of magnetic charge, it is seen that only one divergency for the heat capacity exits. The sign of the heat capacity changes from negative to positive as we cross such divergence point. Therefore, we have a phase transition between smaller unstable black holes to larger stable ones. In the presence of magnetic charge, the stability conditions, regions of stability and absence of critical behavior depend on the value of magnetic charge, heat capacity could enjoy: i) one root and two divergencies in which stable black holes exist only between the root and the smaller divergency, and also after larger divergency. ii) one root and one divergency around which the sign of heat capacity is positive. In this case, the phase transition is between two stable phases of smaller and larger black holes. iii) one root in which after it, the heat capacity is positive valued and solutions are thermally stable. Here, we see that consideration of the constant electric field resulted in the presence of peculiarities in thermodynamical behavior of the black holes. In the presence of magnetic charge, the stability conditions, regions of stability and absence of critical behavior depend on the value of magnetic charge (for fixed values of other parameters). Once again, we point it out that in the absence of electric field, it is possible to obtain critical behavior for the magnetic black holes. In this case, the stability conditions, hence place of the divergencies of heat capacity, are different but the end results show the possibility of van der Waals like behavior and critical points for magnetically charged black holes.

As for the high energy limit and asymptotic behavior of heat capacity, evidently, these two limits are governed by the gravitational part of solutions since the dominant terms for both cases depend only on the horizon radius. The effects of magnetic charge could be seen in the second leading order term of the high energy limit in which the magnetic charge is in the denominator of this term. Interestingly, for both cases of the asymptotical behavior and high energy limit, the presence of electric field is observed in the second leading order term in the numerator of both cases. This confirms that the effects of magnetic charge on the thermal stability of medium black holes become significant. It should be highlighted that in the presence of constant electric field, the high energy limit and asymptotic behavior of heat capacity were highly modified as compared to other thermodynamical quantities. Once more, this highlights the differences between these black holes and previous ones.

**VI. CONSTANT MAGNETIC FIELD**

In this ensemble, we assume that the non-zero spatial component of the electromagnetic tensor, \( A_\theta \), is constant. Therefore, we replace the magnetic charge, \( q_M \), with the following relation

\[
q_M = \frac{(d - 3)r_d^{-3}\Phi M}{\pi},
\]
FIG. 4: $H$ (left panel), $T$ (middle panel) and $C$ (right panel) versus $r_+$ for $d = 5$, $k = 1$, $P = 0.1$ and $\Phi_E = 0.1$; $q_M = 0$ (continuous line), $q_M = 0.2$ (dashed line), $q_M = 0.3553$ (dashed-dotted line) and $q_M = 0.4$ (dotted line).

FIG. 5: $P$ versus $r_+$ for $d = 5$, $k = 1$, $T = 0.286$ and $\Phi_E = 0.1$; $q_M = 0$ (continuous line), $q_M = 0.2$ (dashed line), $q_M = 0.3553$ (dashed-dotted line) and $q_M = 0.4$ (dotted line).

in thermodynamical quantities. It is easy to show that the mass(enthalpy) of this case is given by

$$M = \frac{(d-2)r_+^{d-3}}{16} - k + \frac{\pi r_+^{d-1}}{d-1}P + \frac{r_+^{3-d}}{8(d-3)q_E^2} + \frac{(d-3)r_+^{d-3}}{8\pi^2} \Phi_E^2.$$

Such as before, since the roots of $M$ could not be extracted analytically for general $d$, we only present the roots for 5-dimensional case

$$r_+(M(d = 5) = 0) = \frac{1}{6} \sqrt{\frac{27(\pi^2 k - \frac{4\Phi_E^2}{3})^2}{A_{12}^{1/3}} + 3A_{12}^{1/3} - 12\Phi_E^2 - 9\pi^2 k}}.$$

where $A_{12} = A_{13} + 12\sqrt{3}\pi^4 P q_E \sqrt{A_{14}}$, in which $A_{13}$ and $A_{14}$ are in the following forms.
\[ A_{13} = -216\pi^8 P^2 q_E^2 - 27\pi^6 k^3 - 108\pi^4 k^2 \Phi_M^2 - 144\pi^2 k \Phi_M^4 - 64\Phi_M^6, \]  
(78)

\[ A_{14} = 108\pi^8 P^2 q_E^2 + 27\pi^6 k^3 + 108\pi^4 k^2 \Phi_M^2 + 144\pi^2 k \Phi_M^4 + 64\Phi_M^6. \]  
(79)

The high energy limit of the mass/enthalpy and its asymptotic behavior are, respectively, as follows:

\[
\lim_{r_+ \to 0} M = \frac{1}{16\pi^2} q_E^2 + \left( \frac{3}{16} k + \frac{1}{4\pi^2} \Phi_M^2 \right) r_+^2 + O \left( r_+^4 \right), \quad (80)
\]

\[
\lim_{r_+ \to \infty} M = \frac{\pi}{4} r_+^4 P + \left( \frac{3}{16} k + \frac{1}{4\pi^2} \Phi_M^2 \right) r_+^2 + O \left( \frac{1}{r_+^2} \right). \quad (81)
\]

For the high energy limit, contrary to the previous cases, the dominant term includes a coupling between electric charge and horizon radius. Here, the effects of the magnetic field could be detected in the second dominant term of high energy limit. Interestingly, for

\[ k = -\frac{4\Phi_M^2}{3\pi^2}, \quad (82) \]

the effects of topological structure and magnetic potential on mass/enthalpy cancel out. It is evident that such a case, only happens for black holes with hyperbolic horizon. By studying the asymptotic behavior, one can observe that the first dominant term includes the pressure while the second dominant term in this limit is the same as second dominant term in the high energy limit. This property enables us to categorize the mass/enthalpy behavior of black holes into three categories: i) small black holes in which the general behavior significantly depends on the electric charge. ii) medium black holes which are governed by the magnetic potential and topological structure of black holes. iii) large black holes which are mainly described by the pressure, hence the cosmological constant. The existence of root for the mass/enthalpy depends on the following condition

\[ k < -\frac{4\Phi_M^2}{3\pi^2}. \quad (83) \]

If this condition is satisfied, mass/enthalpy of the black holes could acquire root. Such case could only happen for black holes with hyperbolic horizon. In order to have a better picture regarding the behavior of the mass/enthalpy, we have plotted them in left panel of Fig. 6. The mass/enthalpy for this case has a minimum which is an increasing function of the magnetic potential.

The temperature for this case is calculated as

\[ T = \frac{1}{2\pi} \left[ \frac{(d - 3)}{2r_+} k + \frac{8\pi r_+}{(d - 2)} P - \frac{q_E^2}{(d - 2) r_+^2} - \frac{(d - 3)^2}{\pi^2 (d - 2) r_+} \Phi_M^2 \right]. \quad (84) \]

Such as before, it is not possible to extract the roots of temperature analytically for the general \( d \)-dimensional case, but in 5-dimension, we have the following root

\[ r(T(d = 5) = 0) = \frac{1}{12} \left[ \frac{54 \left( \frac{\pi^2 k - 4\Phi_M^2}{A_{15}^{1/3}} \right)^2}{A_{15}^{1/3}} + 6A_{15}^{1/3} + 24\Phi_M^2 - 18\pi^2 k \right], \quad (85) \]

where \( A_{15} = A_{16} + 24\sqrt{3\pi^4} P q_E \sqrt{A_{17}}, \) in which \( A_{16} \) and \( A_{17} \) are

\[ A_{16} = 864\pi^8 P^2 q_E^2 - 27\pi^6 k^3 + 108\pi^4 k^2 \Phi_M^2 - 144\pi^2 k \Phi_M^4 + 64\Phi_M^6, \quad (86) \]

\[ A_{17} = 432\pi^8 P^2 q_E^2 - 27\pi^6 k^3 + 108\pi^4 k^2 \Phi_M^2 - 144\pi^2 k \Phi_M^4 + 64\Phi_M^6. \quad (87) \]
The high energy limit and asymptotic behavior of the temperature, for this case could be written as

$$\lim_{r_+ \to 0} T = \frac{1}{6\pi r_+^2} q_E^2 + \left( \frac{1}{2\pi} k - \frac{2}{3\pi^3} \Phi_M^2 \right) \frac{1}{r_+} + O(r_+), \quad (88)$$

$$\lim_{r_+ \to \infty} T = \frac{4}{3} r_+ P + \left( \frac{1}{2\pi} k - \frac{2}{3\pi^3} \Phi_M^2 \right) \frac{1}{r_+} + O\left(\frac{1}{r_+^2}\right). \quad (89)$$

Here too, similar to the previous case, the dominant term of high energy limit is governed by the electric charge. But contrary to mass/enthalpy, the sign of the dominant term for high energy limit is negative. The second dominant term includes both topological factor and magnetic potential. It is possible to cancel out the effects of topological factor and magnetic potential by following adjustment

$$k = \frac{4\Phi_M^2}{3\pi^2}, \quad (90)$$

which takes place only for spherical black holes. Considering that the dominant term in the asymptotic behavior is positive valued, one can conclude that at least one root is available for the temperature of these black holes. In order to have a better picture regarding the effects of magnetic potential on the behavior of temperature, we have plotted the corresponding diagrams in the middle panel of Fig. 6. Evidently, the place of root and existence of extremum for the temperature and its number are functions of the magnetic potential. The root of temperature is an increasing function of the magnetic potential while the number of extrema is a decreasing function of it. There exists a critical magnetic potential, $\Phi_{M-critical}$, which for magnetic potentials less than it, temperature has two extrema. Beyond this magnetic potential, temperature will be an increasing function of the horizon radius with one root and without extremum. Remembering that extremum in temperature is where the heat capacity acquires divergency (thermal phase transition), one can conclude that by increasing the magnetic potential to certain values, the black hole will not have any thermal phase transition. Therefore, for fixing other quantities and varying the magnetic potential, one can state that for super magnetized black holes, no phase transition exists. It is worthwhile to mention that the phase transition observed in temperature versus horizon radius diagrams is van der Waals like. The nature of the phase transition will be discussed when we turn to the heat capacity.

Using the temperature, it is a matter of calculation to obtain the following relation for the pressure

$$P = \frac{1}{2\pi} \left[ \frac{\pi (d-2)}{2r_+} T - \frac{(d-2) (d-3) k}{8r_+^2} \right] + \frac{q_E^2}{4r_+^{2d-4}} - \frac{(d-3) q_E^2}{4\pi^2 r_+^2} \Phi_M^2. \quad (91)$$

The following relation can be obtained for the critical horizon radius

$$\frac{(d-2) (4d-10)}{r_+^{2d-6}} \pi^2 q_E^2 - (d-3) [k (d-2) \pi^2 - 2 (d-3) \Phi_M^2] = 0, \quad (92)$$

which yields the following result

$$r_c = \left[ \frac{(d-3) [k (d-2) \pi^2 - 2 (d-3) \Phi_M^2]}{2 (d-2) (2d-5) \pi^2 q_E^2} \right]^{\frac{1}{d-6}}. \quad (93)$$

Using this critical horizon radius, one can extract critical temperature and pressure in the following forms

$$T_c = (2d-4) \pi^2 r_c^{d-6} \left[ (2d-5) q_E^2 \right]^{\frac{d-3}{d-6}} \pi \frac{10-3d}{\pi^2} \left[ (d-3) (d-2) \pi^2 k - 2 (d-3)^2 \Phi_M^2 \right]^{\frac{5-2d}{d-6}}$$

$$\left[ \pi^4 (d-2)^2 (d-3)^2 k^2 - 4 (d-2) (d-3)^3 \pi^2 \Phi_M^2 k - 4 \pi^4 (d-2)^2 (2d-5) q_E^2 + 4 (d-3)^4 \Phi_M^4 \right], \quad (94)$$
\[ P_e = \frac{\Phi_M^2 (6 - 2d) + (d - 2) \pi^2 k}{4(d - 3)^{d-3}} \left\{ \frac{2d-4}{q_E} \left( q_E^2 + \frac{q_M^2}{2} \right) \left[ 2\pi^{d-1} (d - 2)^{d-2} (2d - 5)^{d-2} \right] \right\}^{1/(d-3)} + [(d - 2)(2d - 5)]^{1/(d-3)} \]
\[ \frac{1}{16} \left( \frac{2\pi^5 d \pi^2 k}{\Phi_M^2 (6 - 2d) + \pi^2 k (d - 2)} \right)^{1/(d-3)} \]

in which

\[ \Gamma' = 52 \left( \frac{13d^2 - 28d + 20}{13} \right) \left( \frac{4 \pi^{d-1}}{q_E^{d-10}} \right)^{1/(d-3)} - 4k \Phi_M^2 (d - 2)(d - 3)^3 \left( \frac{q_E^2}{\pi^{d-6}} \right)^{1/(d-3)} + \left( \frac{4(d - 3)^4 \Phi_M^2}{\pi^{d-10}} + \pi^{d-3} k^2 (d^4 + 37d^2 - 60d + 36) \right) \frac{d^{d-10}}{q_E^{d-10}} \]

The plotted diagram for the pressure (Fig. 7) confirms that depending on choices for the magnetic potential, the solutions would have a van der Waals like phase transition. Interestingly, the critical horizon radius obtained for this case has fundamental differences comparing to the previous case in which electric field was constant. Here, the presence of magnetic potential is present in the numerator of obtained critical horizon radius while the electric charge is in the denominator. This is opposite to what was observed for critical horizon radius in case of constant electric field. It is worthwhile to mention that critical horizon radius is a decreasing function of magnetic potential and electric charge, whereas the critical temperature and pressure are increasing functions of them.

It is possible to obtain the free energy of this case as

\[ F = \frac{r_+^{d-3}}{16} - \frac{\pi \pi^{d-4}}{(d - 1)(d - 2)} P + \frac{2(d - 5)(d - 3) r_+^{d-3}}{8\pi^2 (d - 2)} \Phi_M^2 + \frac{2}{16\pi (d - 2)(d - 3)} r_+^{3-d} q_E^2. \]

Our final remark in this section concerns the thermal stability in the context of canonical ensemble, hence heat capacity. It is a matter of calculation to obtain the heat capacity as

\[ C = \frac{(d - 2)(d - 3) \pi^3 \pi^{3d-4k} + 16\pi^4 \pi^{d+4} P - 2\pi^3 \pi^{4d+4} q_E^2 - 2(d - 3)^2 \pi \pi^{3d-2} \Phi_M^2}{64\pi^3 \pi^{2d+2} P - 4\pi^2 (d - 2)(d - 3) r_+^{d} k - 8\pi^2 (2d - 5) q_E^2 r_+^{d} + 8(d - 3)^2 \pi^{d} \Phi_M^2} (d - 2). \]

The divergencies of heat capacity for d-dimensional case could not be obtained analytically, but for the 5-dimensional case, the divergencies are given by

\[ r(C(d = 5) \to \infty) = \frac{1}{12} \sqrt{\left( \frac{54(\pi k - 4\pi^3 \Phi_M^2)}{A_{18}^{1/3} \Gamma} \right)^2 + 6(\Phi_M^2 + 18\pi^2 k) \pi^3 P}, \]

where \( A_{18} = A_{19} + 24\sqrt{15} \pi^4 P q_E \sqrt{A_{20}}, \) in which \( A_{19} \) and \( A_{20} \) are in the following forms

\[ A_{19} = -4320\pi^8 P^2 q_E^2 + 27\pi^6 k^3 - 108\pi^4 k^2 \Phi_M^2 + 144\pi^2 k \Phi_M^4 - 64\Phi_M^6. \]

\[ A_{20} = 2160\pi^8 P^2 q_E^2 - 27\pi^6 k^3 + 108\pi^4 k^2 \Phi_M^2 - 144\pi^2 k \Phi_M^4 + 64\Phi_M^6. \]

It is possible to obtain the high energy limit and asymptotic behavior of the heat capacity for this case in the following forms
First of all, the dominant terms in the high energy limit and asymptotic behavior depend only on the horizon radius which confirms that these two terms are governed by the gravitational sector of the action. On the other hand, the second dominant term of high energy limit depends on topological factor, magnetic potential and electric charge. This term is an increasing function of the topological term while it is a decreasing function of the magnetic potential and electric charge. The presence of electric charge could be seen in the denominator of the second dominant term. As for the asymptotical behavior, the effects of pressure could be observed in the second dominant term of it. The pressure is in denominator of this term. In order to understand the behavior of the heat capacity in more details, one may refer to right panel of Fig. 6. In the case of $\Phi_M = \Phi_M - \text{critical}$, the heat capacity enjoys the presence of a root and divergency which is located after the root. Around the divergency, the sign of heat capacity is positive. Therefore, here we have a phase transition between two stable black holes. On the other hand, for $\Phi_M < \Phi_M - \text{critical}$, the heat capacity enjoys a root and two divergencies. The divergencies are located after the root, and between them, the heat capacity is negative. In other words, between the root and smaller divergency, and after larger divergency, the heat capacity is positive. This indicates that over a specific region (between divergencies), black holes suffer from instability. But comparing this case with pressure or temperature diagrams, one can see that no-physical black holes exists for this region. Therefore, there is a phase transition over a region (between the divergencies) for black holes in this case. By increasing the magnetic potential beyond $\Phi_M - \text{critical}$, the divergencies in the heat capacity are eliminated and black holes will have no thermal phase transition in their structure. It is worthwhile to mention that before root, both temperature and heat capacity are negative valued which indicates that solutions in this region are not physical ones (according to classical concepts of thermodynamics of black holes). This case takes place for all mentioned cases of the magnetic potential.

VII. CONSTANT MAGNETIC AND ELECTRIC FIELDS

In this section, we assume that the temporal and spatial components of the electromagnetic tensor, $A_t$ and $A_\theta$, are constant. So, we replace the electric and magnetic charges, $q_E$ and $q_M$, with the following relations

$$q_M = \frac{(d - 3) r^d - 3}{\pi} \Phi_M,$$

$$q_E = \frac{(d - 3) r^{d - 3} \Phi_E}{\pi}.$$
FIG. 7: $P$ versus $r_+$ for $d = 5$, $k = 1$, $T = 0.187$ and $q_E = 0.1$; $\Phi_M = 0$ (continuous line), $\Phi_M = 1.7$ (dashed line), $\Phi_M = 2.056$ (dashed-dotted line) and $\Phi_M = 3$ (dotted line).

\[
M = \frac{(d - 2)r_+^{d-3}}{16}k + \frac{\pi r_+^{d-1}}{d - 1}P + \frac{(d - 3)r_+^{d-3}}{8\pi^2} (\Phi_M^2 + \Phi_E^2). \tag{104}
\]

for which, the roots are obtained, analytically, as

\[
r_+(M = 0) = \left(4\pi^2 \sqrt{-\frac{P}{(d - 1)\pi^2(d - 2)k + 2(d - 3)(\Phi_M^2 + \Phi_E^2)}}\right)^{-1}. \tag{105}
\]

Evidently, in order to have a root for the mass/enthalpy, the following inequality should be satisfied

\[
k < -\frac{2(d - 3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d - 2)}. \tag{106}
\]

This relation could be satisfied for black holes with hyperbolic horizon. In other words, the mass/enthalpy enjoys the absence of root in the cases of black holes with flat or spherical horizons. It is worthwhile to mention that the root of mass is an increasing function of electric and magnetic potentials while it is a decreasing function of pressure. The high energy and asymptotic behavior of the mass/enthalpy for 5-dimensional case are given by

\[
\lim_{r_+ \to 0} M = \left(\frac{3}{16}k + \frac{1}{4\pi^2} [\Phi_M^2 + \Phi_E^2]\right)r_+^2 + \frac{\pi}{4}r_+^4 P, \tag{107}
\]

\[
\lim_{r_+ \to \infty} M = \frac{\pi}{4}r_+^4 P + \left(\frac{3}{16}k + \frac{1}{4\pi^2} [\Phi_M^2 + \Phi_E^2]\right)r_+^2. \tag{108}
\]

Here, the dominant term in high energy limit includes electric and magnetic potential with topological factors. Comparing this with previous cases, one can notice that for vanishing horizon radius, mass also vanishes which is opposite to previous cases. Such a behavior is rooted in consideration of the constant electric and magnetic fields. This is, in the absence of horizon radius, the mass/enthalpy does diverge. Instead it tends to a constant (zero) (see left panel of Fig. 8). On the other hand, the asymptotic behavior is governed by the pressure term. This confirms that highest modification for considering constant electric and magnetic fields could be observed for small black holes only. Careful examination of these limits also confirms one more result: with such consideration, one can see that black holes are separated into only two groups of small and large black holes. In other words, the class of the medium black holes for the case under consideration is eliminated. It is worth to mention that for the choice of
The spherical one, as long as 2 holes would have root in their temperature. It is worth mentioning that for radius. The high energy limit and asymptotic behavior of the solutions for 5-dimensional black holes are given by and the behavior of temperature is only determined by the pressure which will be only an increasing function of horizon radius. The effects of electric and magnetic potentials, and topological term of the black holes on the temperature are canceled out and the mass/enthalpy of the black holes, irrespective of their size, is determined by the pressure.

The absence of divergent temperature for temperature for temperature for

The mentioned condition is automatically satisfied for black holes with flat and hyperbolic horizons, whereas for the spherical one, as long as \(2 (\Phi_M^2 + \Phi_E^2) \geq \pi^2\), the mentioned condition would be satisfied, hence spherical black holes would have root in their temperature. It is worth mentioning that for

the effects of electric and magnetic potentials, and topological term of the black holes on the temperature are canceled and the behavior of temperature is only determined by the pressure which will be only an increasing function of horizon radius. The high energy limit and asymptotic behavior of the solutions for 5-dimensional black holes are given by

\[
\lim_{r_+ \to 0} T = \left( \frac{1}{2\pi} k - \frac{2}{3\pi^3} [\Phi_M^2 + \Phi_E^2] \right) \frac{1}{r_+} + \frac{4}{3} Pr_+, \tag{114}
\]

\[
\lim_{r_+ \to \infty} T = \frac{4}{3} Pr_+ + \left( \frac{1}{2\pi} k - \frac{2}{3\pi^3} [\Phi_M^2 + \Phi_E^2] \right) \frac{1}{r_+}. \tag{115}
\]

The high energy limit of temperature, similar to mass, is governed by electric and magnetic potentials, and topological factor. If, \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)} = k \), then the temperature of black holes for vanishing horizon radius, vanishes too. In other words, in this case, for vanishing horizon radius, temperature will have finite (zero) value. On the other hand, if \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)} = k \) is not satisfied, the temperature for \( r_+ \to 0 \) will diverge. The possibility of zero temperature for \( r_+ \to 0 \) resulted from the constant electric and magnetic fields. In the absence of electric field, this property is still valid. The absence of divergent temperature for \( r_+ \to 0 \) is observed for specific black holes. Here, we see that by modification in the action and with special consideration for matter field, this behavior happens. In order to have a better picture regrading the behavior of temperature, we have plotted a diagram (see middle panel of Fig. 8). Evidently, the behavior of the temperature could be divided into three classes: i) if \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)} > k \), the temperature is an increasing function of the horizon radius with one root. Before the root, temperature is negative valued and solutions are not physical. ii) if \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)} = k \), the effects of the electric and magnetic potentials, and topological factor are canceled out and temperature will be positive valued everywhere and only an increasing
function of the horizon radius. Finally for the case \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)^2} < k \), the temperature will be positive valued with one minimum. The minimum is a decreasing function of the magnetic and electric potentials. Since the presence of extremum in temperature indicates the existence of divergency (phase transition) in the heat capacity, one can conclude that only for the third case (for \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)^2} < k \)) solutions could undergo phase transition like behavior.

We will examine the type of this phase transition in the context of heat capacity later.

Using the obtained temperature, it is possible to have pressure in the following form

\[
P = \frac{1}{2\pi} \left[ \frac{\pi(d-2)}{2r_+} T - \frac{(d-2)(d-3)}{8r_+^2} k + \frac{(d-3)^2}{4\pi^2r_+^2} (\Phi_M^2 + \Phi_E^2) \right].
\]

(116)

Now, using the properties of inflection point, one can find the following relation governing critical point

\[
(d-3) \left[ k (d-2) \pi^2 - 2 (d-3) (\Phi_M^2 + \Phi_E^2) \right] = 0.
\]

(117)

Evidently, this relation does not explicitly depend on the horizon radius. Remembering that horizon radius is directly related to volume of the black hole, one can conclude the absence of critical volume. This indicates that although the black holes with constant magnetic and electric fields could have critical behavior, but this critical behavior is not van der Waals like. In plotted diagrams for the pressure (see Fig. 9), one can see that for the cases \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)^2} \geq k \), the pressure will be a decreasing function of horizon radius without any root or extremum. On the other hand, for \( \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)^2} < k \), pressure will acquire a root and a maximum which is located after the root. The root and maximum are decreasing functions of the magnetic potential. Before the root, pressure is negative valued and at the maximum, a phase transition will take place.

It is a matter of calculation to obtain the free energy of this case in following form

\[
F = \frac{r_+^{d-3}}{16} k - \frac{\pi r_+^{d-1}}{(d-1)(d-2)} P + \frac{(2d-5)(d-3)}{8\pi^2(d-2)} \Phi_M^2 + \frac{2(2d-5)\pi - (d-2)}{16\pi^3(d-2)} (d-3) r_+^{d-3} \Phi_E^2,
\]

(118)

which has the following root

\[
r_+ (F = 0) = \left( 4\pi^2 \sqrt{\frac{P}{(d-1)\pi^3(d-2)k + 2\pi(d-3)(2d-5)\Phi_M^2 + (d-3)(2\pi(2d-5) - d + 2)\Phi_E^2}} \right)^{-1}.
\]

(119)

The existence of root for free energy depends on the following condition

\[
\pi^3(d-2)k + 2\pi(d-3)(2d-5)\Phi_M^2 + (d-3)(2\pi(2d-5) - d + 2)\Phi_E^2 > 0,
\]

(120)

The root of free energy for this case is an increasing function of topological factor, electric and magnetic potentials, whereas, it is a decreasing function of the pressure.

Finally, the heat capacity for this case is obtained as

\[
C = \frac{(d-2)(d-3)\pi^2k + 16\pi^3r_+^2P - 2(d-3)^2(\Phi_M^2 + \Phi_E^2)}{64\pi^3r_+^2P - 4\pi^2(d-2)(d-3)k + 8(d-3)^2(\Phi_M^2 + \Phi_E^2)} \pi(d-2)r_+^{d-2}.
\]

(121)

The divergent point of the heat capacity is obtained in the following form

\[
r_+(C \rightarrow \infty) = \frac{1}{4P\pi^2} \sqrt{(d-3)\pi^2(d-2)k - 2(d-3)(\Phi_M^2 + \Phi_E^2)].
\]

(122)

Considering that pressure, electric and magnetic potentials are positive valued, the existence of divergency, hence phase transition, depends on the satisfaction of the following condition.
FIG. 8: \( M \) (left panel), \( T \) (middle panel) and \( C \) (right panel) versus \( r_+ \) for \( d = 5 \), \( P = 0.1 \) and \( \Phi_E = 0.1 \); \( k = 1 \), \( \Phi_M = 0 \) (continuous line), \( \Phi_M = 2 \) (dashed line), \( \Phi_M = 3 \) (dashed-dotted line) and \( \Phi_M = 4 \) (dotted line). The bold line: \( k = \frac{2(d - 3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d - 2)} \) and \( \Phi_M = 4 \).

\[
k > \frac{2(d - 3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d - 2)}.
\] (123)

Comparing this condition for the presence of divergency in heat capacity with the one for existence of root for temperature, one can see that the following statement holds: i) in the case where the condition for having root for the temperature holds, the condition for having divergency in heat capacity is violated. Therefore, the temperature and heat capacity have the same root beyond which both temperature and heat capacity are positive valued and solutions are physical and thermally stable. ii) in the case where the condition for having divergency in heat capacity is satisfied, the condition for temperature having root is violated. Therefore, solutions will have positive temperature everywhere. But before divergency, the heat capacity is negative while after that, it will be positive. Therefore, for this case, there is a phase transition between smaller unstable black holes with larger stable ones. iii) finally, for the case \( k = \frac{2(d - 3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d - 2)} \), no divergencies and roots are available for both heat capacity and temperature. In this case, black holes are everywhere physical and thermally stable. It is worthwhile to mention that in this case, the effects of topological structure of the black holes are canceled by the effects of the electric and magnetic potentials. It should be pointed out that such a case only takes place for spherical black holes. In order to show the mentioned behaviors for heat capacity, see the right panel of Fig. 8. Evidently, the root of temperature (heat capacity) is an increasing function of the magnetic potential while the divergency of the heat capacity is a decreasing function of it.

The high energy limit and asymptotic behavior of the heat capacity for 5-dimensional case are in the following forms

\[
\lim_{r_+ \to 0} C = \frac{3\pi}{4} r_+^3 - \frac{12P\pi^4}{3k\pi^2 - 4(\Phi_M^2 + \Phi_E^2)} r_+^5 + O\left(\frac{1}{r_+}\right),
\] (124)

\[
\lim_{r_+ \to \infty} C = \frac{3\pi}{4} r_+^3 + \frac{3k\pi^2 - 4(\Phi_M^2 + \Phi_E^2)}{16\pi^2 P} r_+ + O\left(\frac{1}{r_+}\right).
\] (125)

In this case, similar to previous cases, the dominant terms for high energy limit and asymptotic behavior of the heat capacity only include horizon radius which originated purely from gravitational part of the action. On the other hand, the second leading term in the high energy limit includes all the quantities of the black hole: the pressure is present in the numerator of this term while the topological factor, electric and magnetic potentials are in numerator. The second leading term is an increasing function of the topological factor and pressure while it is a decreasing function of the electric and magnetic potentials. In other words, since we regard constant electric and magnetic fields, one expects to keep their effects asymptotically.
FIG. 9: $P$ versus $r_+$ for $d = 5$, $T = 0.1$ and $\Phi_E = 0.1$; $k = 1$, $\Phi_M = 0$ (continuous line), $\Phi_M = 2$ (dashed line), $\Phi_M = 3$ (dashed-dotted line) and $\Phi_M = 4$ (dotted line). The bold line: $k = \frac{2(d-3)(\Phi_M^2 + \Phi_E^2)}{\pi^2(d-2)}$ and $\Phi_M = 4$.

VIII. CONCLUSION

The paper at hand investigated the higher dimensional electric-magnetic black holes through a novel proposal. The novel proposal employed for constructing electromagnetic tensors, resulted into components which have valid physical properties and for Maxwell invariant, they were distinguishable. Interestingly, the obtained metric function for this case has electric and magnetic components of the same order of magnitude with same factors. The event horizon and geometrical properties of the solutions were investigated and it was shown that the number and location of root of metric function depend on magnetic charge.

Next, thermodynamical properties of the solutions were investigated in detail for different cases. In order to enrich the study, the concept of extended phase space (equivalency between pressure and negative cosmological constant) was employed as well. The investigation was done in the context of four distinctive cases:

I) General case, in which it was shown that high energy limit of different thermodynamical quantities except for heat capacity are governed by both the magnetic and electric charges. The existence of the van der Waals like behavior was reported for this case and dependency of phase transition points on the magnetic charge was pointed out. The presence of electric and magnetic charges were observed in denominator of the second dominant term of the high energy limit.

II) Constant electric field in which comparing to previous case, the high energy limit of the thermodynamical quantities, except heat capacity, was determined by magnetic charge. The presence of electric potential was evident in the second dominant term in the numerator, even for heat capacity. Interestingly, it was possible to trace out the effects of topological structure of the black holes by suitable choices of the electric field. It was shown also, that in the absence of magnetic charge, no van der Waals like phase transition is present, although there exists a phase transition. The type of phase transition was different for this case.

III) Constant magnetic field. For this case, the dominant term of high energy limit of different thermodynamic quantities, except heat capacity, included only electric charge while the effects of magnetic potential were observed at the second leading term. Contrary to the previous case, here, it was possible to cancel out the effects of topological parameter of black holes via a specific condition involving the magnetic potential. The existence of van der Waals behavior and dependency on number of the divergencies in heat capacity and stability condition on magnetic potential were highlighted. It was shown that for super-magnetized black holes (large magnetic field), thermal structure of the black holes are modified on level of the absence of van der Waals like behavior and phase transition. In other words, for super magnetized black holes, heat capacity is a smooth function of the horizon radius.

IV) Constant magnetic and electric fields, simultaneously. In this particular case, there was no van der Waals like behavior available for black holes under any circumstances. Although the heat capacity enjoyed a divergency, hence critical behavior, in this case for specific values of the magnetic potential, the type of phase transition was different from van der Waals like phase transition. It was also shown that for this case, it is possible to remove the effects of topological parameter of the black holes alongside with electric and magnetic potentials for specific choices of these
parameters. In that case, both the high energy limit and asymptotic behavior of black holes are governed only by pressure, hence cosmological constant.

One of the main goals here was to provide the possibility of investigating dyonic (electric-magnetic) black holes in higher dimensions. The novel proposal here provided such a possibility. The next step would be understanding the holographical aspects of this proposal in higher dimensional solutions and see how higher dimensionality would modify the ferromagnetism/diamagnetism phase transitions. In addition, higher curvature gravities that are dimension-dependent could be coupled with this theory, as well, which help understanding the effects of these higher derivative gravity theories on the ferromagnetism/diamagnetism phase transitions and holographical principles. Furthermore, the investigation of superconductivity in the presence of this magnetic field is yet another interesting subject which we leave for a future work.

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