Watson’s theorem and electromagnetism in $K \to \pi\pi$ decay

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Abstract

We consider what constraints unitarity and CPT invariance yield on the strong and electromagnetic phases entering $K \to \pi\pi$ decay. In particular, we show that the relative size of the electromagnetically-induced changes in the $I = 0$ and $I = 2$ phase shifts in the two-pion final state do not depend on the explicit coupling to the $\pi^+\pi^-\gamma$ channel. This demonstrates that Watson’s theorem can be extended to include the presence of electromagnetism. We point out the consequences for the general structure of the $K \to \pi\pi$ decay amplitudes in the presence of isospin violation.

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1. A detailed understanding of the rich phenomenology of $K \to \pi \pi$ decays has remained elusive despite decades of effort. Although progress has been made, the dynamical origin of the $\Delta I = 1/2$ rule, as well as the strength of CP-violating parameter $\text{Re} \left( \epsilon' / \epsilon \right)$, is not yet clear. Another, presumably related, puzzle stems from the apparent violation of Watson’s final-state theorem. Watson’s theorem emerges from unitarity and CPT-invariance, in concert with isospin symmetry, and implies that the strong phase in $K \to \pi \pi$ decay ought to be given by that of $\pi \pi$ scattering. However, the S-wave $\pi \pi$ phase shift difference $\delta_0 - \delta_0$ extracted from the $K \to \pi \pi$ decay modes, using physical masses in the phase-space integrals, is about 57° \cite{1}, whereas its value from $\pi \pi$ scattering data, with the help of chiral perturbation theory and dispersion relations, is about 45° with an uncertainty of roughly 10% \cite{2}. The assumed equality of these quantities is a consequence of isospin symmetry, so that the resolution of the discrepancy has been sought in the computation of isospin-violating effects. Isospin violation can be generated by both strong (up-down quark mass difference) and electromagnetic (virtual photon) interactions, and its effects have been recently studied in great detail, see, e.g., Refs.\cite{3}-[10].

While many interesting results have been obtained and many others are forthcoming, the gap between the phase-shift difference obtained from $K \to \pi \pi$ decay and $\pi \pi$ scattering has thus far eluded a detailed explanation. In the framework of chiral perturbation theory, which is the appropriate theoretical tool in this context, many new low-energy constants appear in the most general Lagrangian of Goldstone bosons coupled to virtual photons and external sources, making certain numerical predictions difficult. It is thus important to explore whether Watson’s theorem can be extended in the presence of isospin violation. In Ref.\cite{8} Watson’s theorem was shown to persist through leading order in the up-down quark mass difference, so that the phase shifts from $K \to \pi \pi$ decay and $\pi \pi$ scattering ought be equal to $O((m_d - m_u)^2)$. Were the phase shifts from $K \to \pi \pi$ decay and $\pi \pi$ scattering equal, the empirical phase-shift discrepancy could nevertheless be resolved, for it could be interpreted in terms of an additional amplitude in $K \to \pi \pi$ decay, of $|\Delta I| = 5/2$ in character \cite{8}. In the isospin-perfect limit, the $K \to \pi \pi$ transition in $O(G_F)$ can be of $|\Delta I| = 1/2$ or $|\Delta I| = 3/2$. In the presence of isospin violation, a $|\Delta I| = 5/2$ transition can be realized from $m_d \neq m_u$ effects in a weak transition or from electromagnetic effects in concert with a $|\Delta I| = 1/2$ weak transition. The empirical enhancement of the $|\Delta I| = 1/2$ weak transition suggests that the latter mechanism is of greater importance. The empirical $|\Delta I| = 5/2$ amplitude required to resolve the phase-shift discrepancy is compatible with that expected from electromagnetic effects \cite{8}, yet it is significantly larger in magnitude than and of opposite sign to that indicated by explicit estimates \cite{1,2}. Moreover, including the estimated phase-shift difference from electromagnetism \cite{1,2} exacerbates this discrepancy. These difficulties prompt the consideration of Watson’s theorem in the presence of electromagnetism, in order to realize what constraints may exist on the strong and electromagnetically induced phase shifts in $K \to \pi \pi$ decay. This is the aim of the present investigation. It has been triggered by the work of Bernstein \cite{11}, who considered isospin violation in near-threshold neutral pion photoproduction from protons, extending the final-state theorem to the situation of three open channels, in that case $\gamma p, \pi^0 p$, and $\pi^+ n$, where $p (n)$ denotes the proton (neutron). In a similar fashion, we consider three open channels for the $K^0$ decays, which are the two–pion final states with total isospin zero and two, denoted as $(\pi \pi)_0$ and $(\pi \pi)_2$, respectively, as well as the inelastic $\pi^+ \pi^- \gamma$ channel, whose inclusion is required to render the electromagnetic corrections to the $K \to (\pi \pi)\gamma$ amplitudes infrared (IR) finite. We will construct a general $4 \times 4$ S-matrix appropriate to this scenario and derive a set of unitarity constraints from it. Our purpose is not a detailed numerical analysis of the various isospin–violating effects, but rather the construction of a theoretical framework which would be helpful in constraining such calculations. Nevertheless, we will be able to derive consequences from the unitarity constraints which thus far have only appeared indirectly in numerical analyses.

2. First, we must collect some definitions for the discussion of the $K \to \pi \pi$ amplitudes — we follow the notation and conventions of Ref.\cite{8} and refer the reader to that paper for further details. In the isospin limit $(m_u = m_d, c = 0)$, the decay of a neutral kaon into two pions with isospin $I$ equal to zero or two can be parametrized via\cite{2}

$$
\langle (\pi \pi)_I \mid \mathcal{H}_W \mid K^0 \rangle = A_I \exp(i \delta_I),
$$

$$
\langle (\pi \pi)_I \mid \mathcal{H}_W \mid K^0 \rangle = A_I \exp(i \delta_I),
$$

(1)

where $\mathcal{H}_W$ is the effective weak Hamiltonian for kaon decay. The amplitude $A_I$ is such that $A_I = |A_I| \exp(i \xi_I)$, with $\xi_I$ the weak phase associated with the decay to the final two–pion state of isospin $I$, and $\delta_I$ is the phase shift corresponding to S-wave $\pi \pi$ scattering of isospin $I$. In the isospin-symmetric limit, Bose symmetry requires

\footnote{Here $A_I$ is $i$ times $A_I$ defined in the Ref.\cite{8}.}
the pion pair to have \( I = 0 \) or \( I = 2 \). In that limit, the S–matrix for strong scattering in the \((\pi\pi)_I\) final state is described by a pure phase,
\[
S = \begin{pmatrix}
  e^{2i\delta_0} & 0 \\
  0 & e^{2i\delta_2}
\end{pmatrix}.
\]

Here, we have tacitly assumed that at \( \sqrt{s} = M_{K^0} \) the inelasticities from the opening of the four–pion threshold, \( 2\pi \to 4\pi \), are negligible. This is not only a well–known empirical fact \cite{12, 13}, but it can also be understood in the framework of chiral perturbation theory, for it first occurs in three–loop order (see Ref. \cite{14}, e.g.).

We now turn to the inclusion of isospin-breaking effects. For the moment we neglect the presence of the channel \( K \to \pi^+\pi^-\gamma \), and we consider merely how isospin violation impacts the \( \pi\pi \) subspace. As previously, we introduce the channels \((\pi\pi)_0\) and \((\pi\pi)_2\), where these states are related to the physical basis via
\[
|\pi^+\pi^-\rangle \propto |(\pi\pi)_0\rangle + \frac{1}{\sqrt{2}}|(\pi\pi)_2\rangle,
\]
\[
|\pi^0\pi^0\rangle \propto |(\pi\pi)_0\rangle - \sqrt{2}|(\pi\pi)_2\rangle.
\]

Enforcing unitarity and time–reversal invariance, the general S–matrix appropriate to scattering in the two–pion subspace — with zero net charge — contains exactly three parameters. Two parameters characterize \( \pi\pi \) scattering in the isospin-perfect limit, so that the additional parameter must be at least of \( \mathcal{O}(m_d - m_u) \) or \( \mathcal{O}(e) \), as isospin-breaking is generated by both strong and electromagnetic effects. Our neglect of the \( \pi^+\pi^-\gamma \) channel would be appropriate were we to consider strong-interaction isospin-violating effects only. Let us do this and examine the extensions necessary for the treatment of electromagnetism later. Working in analogy to the “bar phase shifts” for \( J = S = 1 \) nucleon–nucleon (NN) scattering in the presence of a tensor force \cite{15}, we parametrize the S–matrix as
\[
S = \begin{pmatrix}
  e^{i\delta_0} & 0 \\
  0 & e^{i\delta_2}
\end{pmatrix}
\begin{pmatrix}
  \cos 2\kappa & i \sin 2\kappa \\
  i \sin 2\kappa & \cos 2\kappa
\end{pmatrix}
\begin{pmatrix}
  e^{i\delta_0} & 0 \\
  0 & e^{i\delta_2}
\end{pmatrix},
\]
where \( \kappa \) is the third parameter which is sensitive to isospin-violating effects. In the absence of isospin violation, \( \kappa = 0 \), and we have \( \delta_I = \delta_f \). G–parity arguments show that strong isospin–violating effects in \( \pi\pi \)–scattering are of \( \mathcal{O}((m_d - m_u)^2) \) \cite{14, 8}.

We parametrize the \( K \to \pi\pi \) amplitudes in the presence of isospin violation via
\[
\langle (\pi\pi)_I | S | K^0 \rangle = iA_I \exp(i\tilde{\delta}_I),
\]
\[
\langle (\pi\pi)_I | S | K^0 \rangle = iA_I^* \exp(i\tilde{\delta}_I),
\]
noting that the \( \tilde{\delta}_I \) are the strong phases of the \( K \to \pi\pi \) amplitude and recalling that \( S = 1 + iT \). Unitarity constrains the explicit relation between the \( \tilde{\delta}_I \) and the \( \delta_I \), note Ref. \[8\]. If the channel-coupling parameter were zero, then \( \delta_f = \tilde{\delta}_f = \delta_f \), and the strong phase in the \( K \to \pi\pi \) decay would be that of \( \pi\pi \) scattering in the isospin-perfect limit. For later use, we introduce the abbreviation
\[
\Delta_I = \tilde{\delta}_I - \delta_I, \quad I = 0, 2,
\]
so that \( \Delta_I = 0 \) for \( \kappa = 0 \).

To end this discussion, we briefly return to the isospin-perfect limit. The \( 3 \times 3 \) S–matrix describing the coupling of the \( K_0 \) to the \( (\pi\pi)_I \) channels takes the form, where \( K_0, (\pi\pi)_0, \) and \( (\pi\pi)_2 \) refer to rows/columns 1,2 and 3, in order:
\[
S = \begin{pmatrix}
  1 & iA_1 e^{i\delta_0} & iA_2 e^{i\delta_2} \\
  iA_1 e^{i\delta_0} & e^{2i\delta_0} & 0 \\
  iA_2 e^{i\delta_2} & 0 & e^{2i\delta_2}
\end{pmatrix},
\]

#\[Note that the properly symmetrized state \(|\pi^-\pi^+\rangle_{sym} = (|\pi_1^-\pi_2^+\rangle + |\pi_1^+\pi_2^-\rangle)/\sqrt{2} = \sqrt{2}|\pi^+\pi^-\rangle).\]
so that $S_{21} = \langle (\pi \pi)_0 | S | K^0 \rangle$. The amplitude $A_I$ contains a non-trivial weak phase. By CPT invariance, 
$\langle (\pi \pi)_I | T^I | K^0 \rangle = \langle (\pi \pi)_I | T | K^0 \rangle^*$, so that $S_{12} = (K^0 | S | (\pi \pi)_0) = \langle (\pi \pi)_0 | K^0 \rangle$. We work in $O(G_F, e^I)$, though we neglect terms of $O(G_F^2)$ in our parametrization of the $(\pi \pi)_I \leftrightarrow (\pi \pi)_I'$ S-matrix. We do this as our interest is in the constraints which exist on the T-conserving phases associated with an amplitude $A_I$, so that only the unitarity constraints emerging from $(S^I S)_{12} = (S^I S)_{13} = 0$ are of interest. An amplitude $A_I$ is itself of $O(G_F)$, so that $O(G_F)$ contributions to $(\pi \pi)_I \leftrightarrow (\pi \pi)_I'$ scattering play no role in the order of $G_F$ to which we work. Thus it is appropriate to neglect $O(G_F)$ effects in our description of $(\pi \pi)_I \leftrightarrow (\pi \pi)_I'$ scattering; we can parametrize this $2 \times 2$ matrix by a form which is both unitary and T-conserving. The unitarity constraints $(S^I S)_{12} = (S^I S)_{13} = 0$ lead to $\delta_I = \delta_I$ ($I = 0, 2$), so that the strong phases appearing in $K \rightarrow \pi \pi$ decay are exactly those of elastic $\pi \pi$ scattering. This is $Watson's$ $theorem$ in the isospin–perfect world. Equipped with these results, we are now in position to generalize this framework to include the $\pi^+ \pi^- \gamma$ final state as well.

3. We wish to extend Watson’s final–state theorem to include both electromagnetic and strong–interaction isospin–violating effects in $K \rightarrow \pi \pi$ decays. For that, we extend the $3 \times 3$ matrix of Eq. (5) to an appropriate matrix of larger dimension. Before presenting this extension, let us collect and discuss the assumptions of our analysis.

![Diagram](image)

Figure 1: A schematic illustration of how electromagnetism can generate channel-coupling effects. We let a round circle represent a particular final-state channel. The $(\pi \pi)_I \leftrightarrow \pi^+ \pi^- \gamma$ channel coupling starts in $O(e)$, whereas the $(\pi \pi)_0 \leftrightarrow (\pi \pi)_2$ channel coupling, mediated by the $\pi^+ \pi^- \gamma$ channel, starts in $O(e^2)$. A $(\pi \pi)_0 \leftrightarrow (\pi \pi)_2$ channel coupling can also be mediated by photon exchange, so that the presence of an intermediate $\pi^+ \pi^- \gamma$ state is not necessary.

1) We assume that the $\pi^+ \pi^- \gamma$ final state is the $only$ inelastic channel which couples to the two–pion channel of given isospin, thus generating transitions of the type $(\pi \pi)_0 \leftrightarrow (\pi \pi)_2$ as illustrated in Fig. 1. As we have noted, the inelasticities generated by the opening of the four–pion threshold can be safely neglected. Considering the empirical branching ratios of $K_S$ to electromagnetic final states, we note that the branching ratio for $K_S \rightarrow \pi^+ \pi^- \gamma$, presuming photon momenta in excess of 50 MeV/c, is roughly $2 \cdot 10^{-3}$, whereas the next largest measured branching ratio, $K_S \rightarrow \gamma \gamma$, is roughly a factor of 1000 smaller [17]; $K_S \rightarrow 3 \pi$ decay is also possible, but is unimportant in this context, as a $3 \pi \rightarrow 2 \pi$ transition with $J = 0$ violates parity and cannot contribute in $O(G_F)$ to the unitarity relations of interest. Consequently, the $\pi^+ \pi^- \gamma$ final state is the only inelastic channel of interest, and the appropriate extension of Eq. (5) is a $4 \times 4$ matrix.

2) We work in leading order in the Fermi constant, $O(G_F)$, and focus on the unitarity constraints in which the kaon decay amplitudes, $A_I$ for $K \rightarrow (\pi \pi)_I$ decay and $A_s$ for $K \rightarrow \pi^+ \pi^- \gamma$ decay, appear. Thus we may choose the $3 \times 3$ submatrix without kaons, that is, that of the coupled $(\pi \pi)_0$, $(\pi \pi)_2$, and $\pi^+ \pi^- \gamma$ system, which we term the pion–photon system, to be both unitary and T–reversal–invariant. The $3 \times 3$ matrix contains eighteen parameters; nine parameters are constrained by unitarity, and three more are constrained by T invariance, so that the resulting matrix has six non–trivial parameters. Scattering in these channels is driven by strong and electromagnetic effects.

3) Recalling the form of the bar phase shifts in NN scattering [14], we write the $3 \times 3$ S–matrix of the pion–photon system in an analogous form, $A \cdot B \cdot \mathcal{A}$, where $\mathcal{A}$ is a diagonal matrix parametrized in terms of the phase shifts of $\pi \pi$ and $\pi^+ \pi^- \gamma$ scattering, namely $\delta_I$ and $\delta_\gamma$, so that

$$
\mathcal{A} = \begin{pmatrix}
    e^{i\delta_1} & 0 & 0 \\
    0 & e^{i\delta_0} & 0 \\
    0 & 0 & e^{i\delta_2}
\end{pmatrix},
$$

#6 The cut on the photon momentum is required; the $\pi^+ \pi^- \gamma$ final state generated by bremsstrahlung from a charged pion is infrared divergent.
cf. Eq. (4), and \( B \) is a unitary, T-invariant \( 3 \times 3 \) matrix containing three parameters. Our form of \( B \) is inspired by the form of the Kobayashi-Maskawa parametrization of the Cabibbo-Kobayashi-Maskawa matrix [18]. The latter, however, contains 4 parameters, but one can readily define one of the parameters in terms of the others to yield a T–invariant matrix. More precisely, we introduce two angles \( \Theta_{1,2} \) and one phase \( \delta \). This assignment will be discussed below. The matrix \( B \) is chosen to be:

\[
B = \begin{pmatrix}
\cos \Theta & i \sin \Theta \\
-1 \sin \Theta & \cos \Theta
\end{pmatrix},
\]

where we adopt the conventional abbreviations \( s_i = \sin \Theta_i \) and \( c_i = \cos \Theta_i \).

4) The form of \( B \) in Eq. (4) is compatible with the hierarchy of channel couplings. The transitions \( (\pi^+ \pi^-) \leftrightarrow (\pi^0 \pi^0) \) start at \( O(e) \), whereas the couplings between the two–pion final states \( (\pi^0 \pi^0) \leftrightarrow (\pi^0 \pi^0) \) are of \( O(e^2) \). It is seemly, then, that \( \Theta_1 \) appears as the \( (\pi^0 \pi^-) \leftrightarrow (\pi^0 \pi^0) \) elements and as \( \Theta_2 \) in the \( (\pi^0 \pi^0) \leftrightarrow (\pi^0 \pi^0) \) elements. Furthermore, at \( O(m_\pi - m_\eta)^2 \), there are no transitions between \( (\pi^0 \pi^-) \) and \( (\pi^0 \pi^0) \), whereas there are transitions between \( (\pi^0 \pi^-) \) and \( (\pi^0 \pi^-) \). Thus the introduction of the phase \( \delta \) is convenient as this can describe the presence of \( m_\eta \neq m_\pi \) effects as distinct from the electromagnetic isospin–violating effects characterized by \( \Theta_1 \). Of course \( \delta \) can contain electromagnetic contributions as well. Note that taking \( \Theta_1 = 0, \delta = 0 \) sets all the channel couplings to zero. Moreover, including the parameter \( \delta \) as a phase means that in the limit in which only \( O(\delta, \Theta_1) \) terms are kept, all the off-diagonal terms are imaginary. The remaining parameter, \( \Theta_2 \), can be thought of as characterizing the difference of the inelasticity parameters in the \( I = 0 \) and \( I = 2 \) channels due to the presence of the third channel.

5) We work with IR-finite amplitudes throughout. For a detailed discussion of the extraction of the IR-finite parts from the full amplitudes, we refer the reader to Ref. [6]. Here, it suffices to say that the potentially troublesome contributions can be factored and are thus not relevant to our discussion.

We can now give the generalized \( 4 \times 4 \) matrix, where the \( R^0, \pi^+ \pi^- \), \( (\pi^0 \pi^0) \), and \( (\pi^0 \pi^0) \) channels are associated with rows/columns 1, 2, 3, and 4, respectively. This assignment is prompted by the foregoing remarks. We thus have

\[
S = \begin{pmatrix}
1 & iA_+ e^{i\delta_7} & iA_0 e^{i\delta_0} & iA_2 e^{i\delta_2} \\
iA_+ e^{i\delta_7} & c_1 e^{2i\delta_7} & iA_0 e^{i(\delta_0 + \delta_7)} & iA_2 e^{i(\delta_0 + \delta_2)} \\
iA_0 e^{i\delta_0} & iA_0 e^{i(\delta_0 + \delta_7)} & c_1 c_2 e^{i(\delta_0 + \delta_7)} & c_2 (c_1 - e^{i\delta_7}) e^{i(\delta_0 + \delta_2)} \\
iA_2 e^{i\delta_2} & iA_2 e^{i(\delta_0 + \delta_7)} & c_2 s_2 (c_1 - e^{i\delta_7}) e^{i(\delta_0 + \delta_2)} & c_2 s_2 (c_1 + e^{i\delta_7} + s_2 e^{i\delta_7} + e^{2i\delta_2})
\end{pmatrix},
\]

(10)

Based on this matrix, we can now derive the consequences of the extension of Watson’s theorem including electromagnetism and strong-interaction isospin violation. To the best of our knowledge, this form of the unitarity constraints has not appeared previously in the literature.

Before working out the unitarity constraints realized from this \( 4 \times 4 \) matrix, we briefly discuss the relation of its \( 2 \times 2 \) submatrix for \( \pi \pi \) scattering to existing parametrizations in the literature. Were the inelastic channel not present, three parameters would suffice in characterizing it. We have five parameters, whereas the following parametrization for the \( \pi \pi \) transition matrix in the isospin basis is proposed in Ref. [18]:

\[
\sqrt{1 - 4 M_\eta^2 / M_K^2} T_{iso} = \begin{pmatrix}
\frac{1}{21} (\eta_2 e^{i\delta_0} - 1) & a e^{i(\delta_0 + \delta_2 + \Delta)} \\
1/24 (\eta_2 e^{i\delta_0} + 1) & a e^{i(\delta_0 + \delta_2 + \Delta)}
\end{pmatrix}.
\]

(11)

Only T invariance is imposed, and the “bar” denotes the IR finite amplitudes. The parameter \( a \) controls the isospin mixing, and the opening of possible new channels, yielding a violation of unitarity in the \( (\pi^0 \pi^0) \) sector, is parametrized in terms of the inelasticity parameters \( \eta_0 \) and \( \eta_2 \). The assumption of having only one additional open channel \( (\pi^+ \pi^-) \) leads to a correlation between \( \eta_0 \) and \( \eta_2 \), as seen in Eq. (10). We should also note that in the limit \( \Theta_1 = 0 \), our \( 2 \times 2 \) submatrix for the \( \pi \pi \) system is characterized by four parameters, so that one of the parameters is redundant, as the resulting submatrix is both unitary and T–reversal–invariant.
resulting parameters of the 2×2 matrix and those of Eq. (6) can be determined, but are not transparent.

4. We proceed to derive the explicit form of the \textit{unitarity constraints} from Eq. (\ref{eq:unitarity}). Specifically, \((S^\dagger S)_{21} = (S^\dagger S)_{31} = (S^\dagger S)_{41} = 0\) yields

\begin{align}
A_\gamma &= A_\gamma c_1 e^{2i(\delta_\gamma + \delta_\gamma)} - A_0 i s_1 c_2 e^{i(\delta_0 - \delta_0 + \delta_\gamma - \delta_\gamma)} - A_2 i s_1 s_2 e^{i(\delta_2 - \delta_2 + \delta_\gamma - \delta_\gamma)} ,
A_0 &= -A_\gamma is_1 c_2 e^{i(\delta_0 - \delta_0 + \delta_\gamma - \delta_\gamma)} + A_0 (c_1 c_2 + s_2 e^{-i\delta}) e^{2i(\delta_0 - \delta_0)} + A_2 c_2 s_2 (c_1 - e^{-i\delta}) e^{i(\delta_0 - \delta_0 + \delta_2 - \delta_2)} ,
A_2 &= -A_\gamma is_1 s_2 e^{i(\delta_2 - \delta_2 + \delta_\gamma - \delta_\gamma)} + A_0 c_2 s_2 (c_1 - e^{-i\delta}) e^{i(\delta_0 - \delta_0 + \delta_2 - \delta_2)} + A_2 (c_1 s_2 + s_2 e^{-i\delta}) e^{2i(\delta_2 - \delta_2)} .
\end{align}

For \(\Theta_1 = \delta = 0\) we recover \(\delta_0 = \delta_0, \delta_2 = \delta_2,\) and \(\delta_\gamma = \delta_\gamma\) as we would expect. Recalling Eq. (6) and defining \(\Delta_2 = \delta_\gamma - \delta_\gamma\), we proceed by eliminating \(A_2, \exp(i\Delta_2)\) from Eqs. (\ref{eq:12} \ref{eq:14}) to recover

\begin{equation}
A_0 s_2 \left(2i \sin \Delta_0 + (e^{-i\delta} - 1) e^{i\Delta_0}\right) - A_2 c_2 \left(2i \sin \Delta_2 + (e^{-i\delta} - 1) e^{i\Delta_2}\right) = 0 .
\end{equation}

Remarkably these formulae do not depend on \(\Theta_1\), and we can factor \(e^{-i\delta/2}\) to obtain

\begin{equation}
\frac{A_2}{A_0} = \tan \Theta_1 \left(\frac{\sin(\Delta_0 - \delta/2)}{\sin(\Delta_2 - \delta/2)}\right) .
\end{equation}

We find, as discussed previously, that \(\Delta_2 \gg \Delta_0\) since the \(I = 2\) phase shift is enhanced by a factor of \(A_0/A_2 \sim 22\)\footnote{\cite{ref1}}. Moreover, no terms in \(\Theta_1\) appear, so that the explicit coupling to the \(\pi^+\pi^-\gamma\) channel is irrelevant to \(\Delta_2\). Interestingly, \(\Theta_1\) does not enter either Eq. (\ref{eq:13}) or Eq. (\ref{eq:14}). This was also observed in Ref. \cite{ref1}, but arose from the results of a numerical analysis, whereas here it emerges as a consequence of unitarity. Let us make one more comment about Eq. (\ref{eq:14}) before proceeding. The right-hand side is explicitly real, whereas the left-hand side is not, as the amplitudes \(A_I\) carry weak phase information. Requiring the imaginary part of the left-hand side of Eq. (\ref{eq:14}) to be zero yields the constraint \(\text{Im} A_2/\text{Re} A_2 - \text{Im} A_0/\text{Re} A_0 = 0\), apparently suggesting that the CP-violating parameter \(\text{Re} (\epsilon'/\epsilon)\) is zero as a consequence of unitarity. However, this is not the case: the amplitudes in \(A_I\) contain isospin violation as well, and Eq. (\ref{eq:14}) becomes indefinite in the isospin-perfect limit. If we proceed to examine the relationship between \(A_I\) and the amplitudes appearing in the general parametrization of Ref. \cite{ref1} of the isospin-breaking effects in the \(K \rightarrow \pi\pi\) amplitudes, we find that \(\text{Re} A_2\) and \(\text{Im} A_2\) are distinguished by an additional, isospin-violating, CP-conserving function. Physically this implies that our parametrization of the \(S_{11}\) and \(S_{41}\) matrix elements is not sufficiently general, that in the presence of isospin violation, there is an additional, path-dependent CP-conserving piece. For our purposes we can neglect this additional contribution, though it could well impact the Standard Model prediction of \(\text{Re} (\epsilon'/\epsilon)\), and we thus proceed to neglect weak phases throughout. Thus we interpret \(A_\gamma\) and \(A_I\) as \(\text{Re} A_\gamma\) and as \(\text{Re} A_I\), respectively.

We can also eliminate \(A_2\) from Eqs. (\ref{eq:12} \ref{eq:13}) to yield

\begin{equation}
A_\gamma c_2 \left[\cos(\Delta_\gamma + \delta/2) - c_1 \cos(\Delta_\gamma - \delta/2)\right] = A_0 s_1 \sin(\Delta_0 - \delta/2) ,
\end{equation}

and, similarly, \(A_0\) from Eqs. (\ref{eq:12} \ref{eq:14}),

\begin{equation}
A_\gamma s_2 \left[\cos(\Delta_\gamma + \delta/2) - c_1 \cos(\Delta_\gamma - \delta/2)\right] = A_2 s_1 \sin(\Delta_2 - \delta/2) .
\end{equation}

It is worth noting that in Eqs. (\ref{eq:12} \ref{eq:18}) the parameter \(\Theta_1\) does explicitly appear, controlling the relation between \(A_\gamma\) and \(A_I\). Combining these latter two equations yields Eq. (\ref{eq:16}), as it should. Since the \(A_I\) and \(A_\gamma\) are complex, our remarks concerning the most general parametrization of \(A_I\) apply to the \(A_\gamma\) amplitude as well.

We also point out that \(\Delta_\gamma\) itself is only non-zero in \(\mathcal{O}(e^3)\), \(\delta_\gamma\) being given by the \(I = 1, L = 1\) \(\pi\pi\) phase shift, so that additional expressions are possible. We refrain from reporting these. Nevertheless, we have extended Watson’s theorem to include the presence of electromagnetism, for the parameters of \(\pi\pi\) and \(\pi^+\pi^-\gamma\) scattering suffice to relate the electromagnetically generated phases in \(K \rightarrow \pi\pi\).
5. In this letter, we have considered the unitarity constraints on the strong and electromagnetically induced phases in $K \to \pi\pi$ decays. Assuming that the $(\pi^+\pi^-\gamma)$ final state is the only inelastic channel, and working in $O(G_F)$, we have derived the constraints between the decay amplitudes $A_I$ for $K \to (\pi\pi)_I$ decay (for isospin $I = 0, 2$) and $A_\gamma$ for $K \to \pi^+\pi^-\gamma$. The corresponding S–matrix is characterized by three mixing parameters; we choose two angles $\Theta_{1,2}$ and one phase $\delta$. The angle $\Theta_1$ is chiefly responsible for the channel couplings $(\pi^+\pi^-\gamma) \leftrightarrow (\pi\pi)_I$, whereas $\delta$ describes the strong isospin violation effects due to the light quark mass difference, as well as any “direct” electromagnetic coupling between the $(\pi\pi)_0 \leftrightarrow (\pi\pi)_2$ states. From the general $4 \times 4$ S-matrix for $K^0$ decays into these three final states, cf. Eq.(10), one can derive a set of unitarity constraints. Most remarkably, it can be shown that the explicit coupling of the $(\pi^+\pi^-\gamma)$–channel is irrelevant to the difference between the T-conserving $\pi\pi$ phases measured in $\pi\pi$ scattering and extracted from $K \to \pi\pi$ (and $K \to \pi^+\pi^-\gamma$) decays. We have also argued that in the presence of isospin violation, the parametrization for the complex-valued decay amplitudes $A_I$ ought be modified; we illuminate the sources of isospin breaking which give rise to the general parametrization of Ref. [8]. As a next step, it will be interesting to work out the numerical consequences of these constraints on the determination of the $|\Delta I| = 5/2$ amplitude in $K \to \pi\pi$ decay, as well as the impact on the Standard Model prediction for the CP-violating parameter $\text{Re}(\epsilon'/\epsilon)$.

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#7Recall that we work with IR finite amplitudes throughout.