On a theorem by Browder and its application to nonlinear boundary value problems

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Abstract
In a paper from 1960, Felix Browder established a theorem concerning the continuation of the fixed points of a family of continuous functions \( f_t : X \to X \) depending continuously on a parameter \( t \in [0,1] \), where \( X \) is a convex and compact subset of \( \mathbb{R}^n \). Here, the result is presented for a compact mapping \( f : A \times X \to X \) where \( X \) is a convex, closed, and bounded subset of an arbitrary normed space and \( A \) is an arcwise connected topological space. Applications to nonlinear boundary value problems are given; specifically, we shall present new viewpoints of known results, introduce some novel results, and exhibit some open problems.

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1 | INTRODUCTION

A theorem established by Browder in [5] guarantees that if \( f : [0,1] \times X \to X \) is continuous, with \( X \subset \mathbb{R}^n \) is compact and convex, then the set

\[
\mathcal{P}_{[0,1]} := \{(t,x) : f(t,x) = x\} = \bigcup_{t \in [0,1]} \{t\} \times \text{Fix}(f_t)
\]

has a connected component \( C \) that intersects both \( \mathcal{P}_0 := \{0\} \times \text{Fix}(f_0) \) and \( \mathcal{P}_1 := \{1\} \times \text{Fix}(f_1) \). We shall refer to this situation by saying that \( C \) connects \( \mathcal{P}_0 \) and \( \mathcal{P}_1 \). Here, the notation \( \text{Fix}(g) \) stands for the set of fixed points of a mapping \( g \) and, as usual, \( f_t(x) := f(t,x) \). A trivial application
could be the following version of the Poincaré–Miranda theorem: if \( \phi : [0, 1]^2 \to \mathbb{R}^2 \) is continuous and satisfies

\[
\phi_1(0, x) < 0 < \phi_1(1, x) \quad x \in [0, 1]
\]

\[
\phi_2(t, 0) < 0 < \phi_2(t, 1) \quad t \in [0, 1],
\]

then \( \phi \) vanishes at some \( (t, x) \in (0, 1)^2 \). Indeed, it suffices to fix \( M > 0 \) large enough such that the function \( f(t, x) := x - \frac{\phi_2(t, x)}{M} \) satisfies \( f([0, 1]^2) \subset [0, 1] \). Taking \( C \subset \mathcal{P}_{[0, 1]} \) that connects \( \mathcal{P}_0 \) with \( \mathcal{P}_1 \) and noticing that \( \phi|_{\mathcal{P}_0} < 0 < \phi|_{\mathcal{P}_1} \), it is seen that \( \phi_1 \) vanishes at some \( (t, x) \in C \) and, consequently, \( \phi(t, x) = (0, 0) \). This example may give the illusion of an inductive proof of the Brouwer theorem because, if Browder’s result is assumed for \( n \) and \( f = (f_1, f_2) \) is a continuous mapping from \( [0, 1] \times [0, 1]^n \) into itself, then there exists a continuum \( C \) of points \( (t, x) \) such that \( f_2(t, x) = x \) and goes from \( t = 0 \) to \( t = 1 \). Thus, the existence of a fixed point of \( f \) follows, because the continuous mapping \( \Phi(t, x) := t - f_1(t, x) \) changes sign over \( C \). However, as said in [23], the \( n \)-dimensional Browder theorem is already an equivalent form of the \( n + 1 \)-dimensional Brouwer theorem, so the inductive argument fails.

Our applications to boundary value problems will be based on the fact that the result is still valid when \( X \) is (homeomorphic to) a convex, closed, and bounded subset of an arbitrary normed space \( E \), provided that \( \text{Im}(f) \) has compact closure. As a consequence, the following result is deduced in a straightforward manner.

**Theorem 1.1.** Let \( E \) be a normed space and let \( X \subset E \) be bounded, closed, and convex. Consider a continuous mapping \( f : A \times X \to X \), where \( A \) is an arcwise connected topological space. Assume that \( f(K \times X) \) is compact for any compact subset \( K \subset A \). Then for each \( \alpha, \beta \in A \), there exists a connected set

\[
C \subset \mathcal{P}_A := \{ (\sigma, x) \in A \times X : f(\sigma, x) = x \} = \bigcup_{\sigma \in A} \{ \sigma \} \times \text{Fix}(f_\sigma) := \bigcup_{\sigma \in A} \mathcal{P}_\sigma
\]

such that

\[
C \cap \mathcal{P}_\alpha \neq \emptyset, \quad C \cap \mathcal{P}_\beta \neq \emptyset.
\]

Indeed, Theorem 1.1 is deduced from the particular case \( A = [0, 1] \) as follows. For \( \alpha, \beta \in A \), take a continuous curve \( \gamma : [0, 1] \to A \) such that \( \gamma(0) = \alpha, \gamma(1) = \beta \) and define \( f^\gamma : [0, 1] \times X \to X \) by \( f^\gamma(t, x) := f(\gamma(t), x) \). Because \( f^\gamma \) is continuous and \( K := \gamma([0, 1]) \) is compact, the particular case implies the existence of \( C_\gamma \subset \mathcal{P}^\gamma_{[0,1]} \subset [0, 1] \times X \) connecting \( \{0\} \times \text{Fix}(f^\gamma_0) \) and \( \{1\} \times \text{Fix}(f^\gamma_1) \). Thus, the set

\[
C := \{ (\gamma(t), x) : (t, x) \in C_\gamma \} \subset \mathcal{P}_A \subset A \times X
\]

connects \( \mathcal{P}_\alpha \) and \( \mathcal{P}_\beta \).

It is worth mentioning that although \( \gamma \) is a continuous curve, the set \( C \) may not be an arc. Interestingly, it was shown in [15] that if \( E = \mathbb{R}^n \) and \( f \) is smooth, then \( C \) is generically diffeomorphic to \( [0, 1] \): in more precise terms, if one considers the set \( S \) of \( C^2 \) functions \( f : [0, 1] \times X \to X \) endowed with the \( C^1 \) norm, then there exists an open dense set \( S_0 \subset S \) such that, for any \( f \in S_0 \), each connected component \( C \) intersecting \( \mathcal{P}_0 \) is an arc.
The original version of Browder’s theorem has encountered applications in diverse fields, such as the nonlinear complementary theorem in programming theory [8] and game theory [12]. The infinite-dimensional version expressed by Theorem 1.1 with $A = [a, b]$ was employed in [21] to prove the existence of solutions for an elliptic problem at resonance. Here, it is worth mentioning that no proof is given; instead, the author invokes the foundational work of Leray and Schauder [14]. Although it is true that Theorem 1.1 can be obtained by adapting the results in the latter paper (see also [17]), the invariance of the mappings $f_t$ allows a considerably shorter treatment, as shown below.

A more subtle question arises on the following observation. When $A = [0, 1]$, the original result can be equivalently formulated as:

**Theorem 1.2.** Let $E$ be a normed space and let $X \subseteq E$ be bounded, closed, and convex. Consider a continuous mapping $f : [0, 1] \times X \to X$ such that $\text{Im}(f)$ is compact. Then there exists a connected set $C \subseteq \mathcal{F}_{[0,1]}$ such that the projection $\pi : C \to [0, 1]$ defined by $\pi(t, x) := t$ is onto.

The equivalence follows immediately from the fact that the only connected subsets of $\mathbb{R}$ are the intervals; however, it is not clear whether or not the latter result may be extended for a given arcwise connected set $A$. Observe, for instance, that if one considers sets $C_{\alpha, \beta}$ as in Theorem 1.1 for all $\alpha, \beta \in A$, then the set

$$C := \bigcup_{\alpha, \beta \in A} C_{\alpha, \beta}$$

may not be connected. As pointed out in [22], an easy extension is obtained when $A$ is a Peano space, namely, when there exists a continuous surjective curve $\gamma : [0, 1] \to A$. These spaces are characterized by the Hahn–Mazurkiewicz theorem and include, in particular, all those sets $A$ that are homeomorphic to a compact convex subset of a normed space. The main result in [22] extends the result to the case in which $A$ is a connected compact Hausdorff space, but not necessarily locally connected. However, the compactness assumption may be dropped in the specific case, frequent in applications, in which $\text{Fix}(f_\alpha)$ is a singleton for some $\alpha \in A$. This is due to the obvious fact that, in such situation, the set

$$C := \bigcup_{\beta \in A} C_{\alpha, \beta}$$

is indeed connected since the intersection of all the sets $C_{\alpha, \beta}$ is nonempty.

**Corollary 1.1.** In the situation of Theorem 1.1, assume that there exists $\alpha \in A$ such that $\text{Fix}(f_\alpha) = \{x\}$ for some $x \in X$. Then there exists a connected set $C \subseteq \mathcal{F}_A$ such that the projection $\pi : C \to A$ defined by $\pi(\beta, x) = \beta$ is onto.

## 2 SHORT PROOF OF THEOREM 1.1 WITH $A = [0, 1]$

By the Dugundji extension Theorem [7], there exists a compact extension of $f$ to the whole space $E$ with range in $X$. Thus, we may assume $f : [0, 1] \times E \to X$. It is clear that the sets $\text{Fix}(f_t)$ remain the same, because all the fixed points of the extended function lie in $X$. Define as before $\mathcal{F} := \mathcal{F}_{[0,1]}$, which is a compact subset of $[0, 1] \times E$ because $\text{Fix}(f_t) \subseteq \text{Im}(f)$ is closed for all $t$. Suppose
the result is false, then by Whyburn’s lemma [26], there exists an open bounded set $U \subset [0, 1] \times E$ containing $F_0$ and disjoint with $F_1$, such that $\partial U \cap F = \emptyset$. The latter property simply says, for all $t$, that $f_t$ has no fixed points $x$ satisfying $(t, x) \in \partial U$. Thus, the homotopy invariance of the Leray–Schauder degree applies, namely, $\deg_{LS}(I - f_1, U_1, 0)$ does not depend on $t \in [0, 1]$, where

$$U_t := \{x : (t, x) \in U\}.$$ 

Next, take $R > 0$ sufficiently large such that $X \cup U_0 \subset B_R(0)$, then $I - \lambda f_0$ does not vanish on $\partial B_R$ for $\lambda \in [0, 1]$. Hence, because $\text{Fix}(f_0) \subset U_0$, the excision property of the degree and the homotopy invariance imply that

$$\deg_{LS}(I - f_0, U_0, 0) = \deg_{LS}(I - f_0, B_R(0), 0) = \deg_{LS}(I, B_R(0), 0) = 1.$$

On the other hand, since $F_1 \cap \mathcal{L} = \emptyset$, it follows that $\text{Fix}(f_1) \cap \mathcal{L} = \emptyset$; thus,

$$\deg_{LS}(I - f_1, U_1, 0) = 0,$$

which contradicts the fact that the degree is constant over the homotopy.

Remark 2.1. The preceding result can be proven by means of Schauder’s theorem only. Indeed, let us give a slightly simplified version of the argument presented in [23]. In the situation of the previous proof, recall that the Whyburn lemma also ensures the existence of disjoint compact sets $K_0$ and $K_1$ containing respectively $F_0$ and $F_1$ and such that $F = K_0 \cup K_1$. Next, define a continuous Urysohn function $\mu : [0, 1] \times X \to [0, 1]$ such that

$$\mu|_{K_j} \equiv j, \quad j = 0, 1$$

and the compact operator $F : [0, 1] \times X \to [0, 1] \times X$ given by

$$F(t, x) = (1 - \mu(t, x), f(t, x)).$$

From Schauder’s theorem, $F$ has a fixed point $(t, x) \in F$, whence $\mu(t, x) = 0$ or $\mu(t, x) = 1$. This implies, respectively, that $t = 1$ or $t = 0$, which contradicts the fact that $F_j \subset K_j$.

3 | APPLICATIONS: FROM KNOWN RESULTS TO OPEN PROBLEMS

In this section, we shall present applications of the Browder theorem to several boundary value problems. In what follows, some new results shall be established, as well as novel viewpoints of known results.

In the first place, we shall consider a nonlocal boundary value problem, for which both the results and the approach are, to the author’s knowledge, new. The second application concerns pendulum-like equations and more general second-order resonant problems. We shall start with very well-known results from [6] and [9], with the aim of showing that the Browder theorem can also be seen as an efficient tool for shortening proofs. In particular, the method of nonwell-ordered upper and lower solutions can be understood as an immediate consequence of the existence of a continuum of fixed points of an appropriate operator. Moreover, we shall unveil a connection between the ideas in the present paper and the celebrated Landesman–Lazer theorem, which seems to be unexplored in the literature. As it shall be pointed out, some new results are deduced
as well, specifically, those related to the range of resonant semilinear operators for systems and its topological properties. Finally, Subsection 3.3 is devoted to a delayed chemostat model previously studied by the author. Here, the goal is to give a more concise approach in terms of the Browder theorem. Furthermore, an open question shall be posed, regarding the possibility of obtaining an alternative proof, based only on a Schauder fixed-point argument.

1. A nonlocal boundary value problem

Let \( \Omega = (-L, L) \subset \mathbb{R} \) be a bounded interval and consider the following problem for some continuous functions \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \):

\[
\begin{aligned}
&\begin{cases}
u''(t) = f(t, \nu(t)) & t \in \Omega \\
\nu(\pm L) = g(\nu(0)).
\end{cases} \\
\end{aligned}
\]  

(1)

Here, the nonlocal boundary condition may be regarded as a nonlinear instance of the three-point boundary condition studied in [24]. For a more general multipoint boundary condition, see, for example, [10].

An easy application of Theorem 1.1 can be implemented as follows. Observe that if \( \nu \) is a solution, then \( \nu|_{\partial \Omega} = g(\nu(0)) := c \) and the function \( v := \nu - c \) is a solution of the problem

\[
\begin{aligned}
v''(t) &= f(t, v(t) + c), \\
v|_{\partial \Omega} &= 0
\end{aligned}
\]  

(2)

satisfying

\[
g(v(0) + c) = c. \]  

(3)

Conversely, if \( \nu \) is a solution of (2) for some \( c \in \mathbb{R} \) such that (3) is fulfilled, then \( \nu := \nu + c \) solves the original problem (1). Thus, we may consider the space \( E := \{ v \in C(\overline{\Omega}) : v|_{\partial \Omega} = 0 \} \) and the operator \( T : \mathbb{R} \times E \to E \) given by \( T(c, v) := v \), where \( v \) is the unique element of \( E \) that solves the linear equation \( v''(t) = f(t, v(t) + c) \). Recalling the standard estimate

\[
\|v\|_{\infty} \leq k\|v''\|_{\infty} \quad v \in E,
\]

it is immediately seen that \( T \) is compact. In order to verify the assumptions of Theorem 1.1, we need to find a closed convex set \( X \subset E \) such that \( T_c(X) \subset X \) for all \( c \). Assume for simplicity that \( f \) is bounded, then

\[
\|T(c, w)\|_{\infty} \leq k\|f(\cdot, w + c)\|_{\infty} \leq k\|f\|_{\infty} := R,
\]

and we conclude that \( X := B_R(0) \) is \( T_c \)-invariant for all \( c \). From Theorem 1.1, for each \( a < b \), there exists a subset

\[
C \subset \bigcup_{a \leq c \leq b} \{ c \} \times \text{Fix}(T_c)
\]

connecting \( \{a\} \times \text{Fix}(T_a) \) with \( \{b\} \times \text{Fix}(T_b) \). Furthermore, the function

\[
\Phi(c, w) := c - g(w(0) + c)
\]

is continuous; thus, in order to solve the original problem, it suffices to find \( a, b \in \mathbb{R} \) such that \( \Phi \) takes different signs when restricted to \( \{a\} \times \text{Fix}(T_a) \) and \( \{b\} \times \text{Fix}(T_b) \). A trivial occurrence of
such situation is when $g$ is sublinear or, more generally, when

$$\limsup_{|u|\to\infty} \left| \frac{g(u)}{u} \right| < 1. \quad (4)$$

It might be argued that there is no need to invoke Browder’s theorem here, because the proof is readily deduced from a direct fixed-point argument, by defining instead the compact operator $\tilde{T} : C(\overline{\Omega}) \to C(\overline{\Omega})$ by $\tilde{T}(v) := u$, the unique solution of the Dirichlet problem

$$u''(t) = f(t, v(t)), \quad u(\pm L) = g(v(0)).$$

Indeed, since there exist $A < 1$ and $B > 0$ such that $|g(v)| \leq A|v| + B$, the fact that

$$\|Tv - g(v(0))\|_{\infty} \leq R,$$

implies

$$\|Tv\|_{\infty} \leq A|v(0)| + B + R \leq A\|v\|_{\infty} + B + R.$$ 

Thus, taking $M := \frac{B+R}{1-A}$, it is verified that $T(BM(0)) \subset BM(0)$ and Schauder’s theorem applies.

The situation is different when $g$ is superlinear or, more generally, when

$$\liminf_{|u|\to\infty} \frac{g(u)}{u} > 1. \quad (5)$$

Here, a nontrivial-bounded invariant region for $\tilde{T}$ may not exist. This is comparable with the existence of fixed points for expansive operators: for instance, the only invariant region for the real function $f(x) := 2x$ is the set composed by the (unique) fixed point of $f$. Although it is still possible to deduce the existence of solutions by computing the Leray–Schauder degree of $I - T$, the previous setting in terms of Theorem 1.1 allows to reduce the proof to a single line, since it is clear that $\Phi(c, v) < 0 < \Phi(-c, v)$ for all $v \in B_R(0)$ and $c \gg 0$. A more general nonasymptotic assumption reads

$$g(a + r) \leq a, \quad g(b + r) \geq b \quad (6)$$

for some $a, b \in \mathbb{R}$ and $|r| \leq R$, which clearly implies $\Phi|_{R_a} \leq 0 \leq \Phi|_{R_b}$. For example, if $g(u) = u + \sin(\sqrt{u})$, then the problem has infinitely many solutions for arbitrary bounded $f$, although $\lim_{|u|\to\infty} \frac{g(u)}{u} = 1$.

**Remark 3.1.** When $g$ satisfies (4), the assumption that $f$ is bounded may be relaxed: for example, it suffices to assume that $f$ is sublinear or, more specifically, that

$$|f(t, u)| \leq \varepsilon|u| + C$$

for some $C > 0$ and $\varepsilon > 0$ small enough.

Indeed, fix $\varepsilon > 0$ such that $k\varepsilon < 1 - \limsup_{|u|\to\infty} \left| \frac{g(u)}{u} \right|$ and observe that, if $u$ is a solution of the problem with $u(\pm L) = c$, then

$$\|u\|_{\infty} \leq k(\varepsilon\|u\|_{\infty} + C) + |c|.$$
As before, set $A, B > 0$ with $k \varepsilon + A < 1$ such that $|g(v)| \leq A|v| + B$, then, because $g(u(0)) = c$,

$$\|u\|_{\infty} \leq k(\varepsilon\|u\|_{\infty} + C) + A|u(0)| + B,$$

whence

$$\|u\|_{\infty} \leq \frac{kC + B}{1 - k\varepsilon - A} : = R. \quad \text{(1)}$$

Since the latter constant does not depend on $f$, the proof follows from a truncation argument.

The same conclusion holds if $g$ satisfies (5), although the proof is more tricky. We may proceed as follows: in the first place, there exists a constant (still denoted $k$) such that if $u$ is a solution, then

$$\|u - u(0)\|_{\infty} \leq k\|u''\|_{\infty} \leq k\varepsilon\|u\|_{\infty} + kC.$$

This, in turn, implies

$$\|u\|_{\infty} \leq k\varepsilon\|u\|_{\infty} + kC + |u(0)|. \quad \text{(2)}$$

Next, fix $A > 1$ and $B > 0$ such that $|g(u)| \geq A|u| - B$, then

$$A|u(0)| \leq |g(u(0))| + B = |u(\pm L)| + B \leq \|u\|_{\infty} + B,$$

that is,

$$\left(1 - k\varepsilon - \frac{1}{A}\right)\|u\|_{\infty} \leq \frac{B}{A} + C. \quad \text{(3)}$$

In other words, it suffices to take $\varepsilon$ such that $(1 - k\varepsilon)\liminf_{|u| \to \infty} \frac{g(u)}{u} > 1.$

A less elementary matter arises when dealing with systems rather than a scalar equation. Observe, indeed, that if $f : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is bounded, then the previous machinery can be arranged exactly in the same way as before and, by the “strong” version of the Browder theorem, for an arbitrary closed ball $B \subset \mathbb{R}^N$, there exists a connected set

$$C \subset \bigcup_{c \in B} \{c\} \times \text{Fix}(T_c)$$

whose projection to $\overline{B}$ is onto. In spite of that, it is not clear if the $N$-dimensional analog of the preceding function $\Phi$ vanishes at some $(c, u)$ because, in principle, $\Phi(C)$ may not be simply connected. Solutions are obtained as fixed points of the previous operator $\tilde{T}$, both in the sublinear and the superlinear cases, namely,

$$\limsup_{|u| \to \infty} \frac{|g(u)|}{|u|} < 1$$

and

$$\liminf_{|u| \to \infty} \frac{\langle g(u), u \rangle}{|u|^2} > 1,$$
respectively, where $| \cdot |$ stands now for the norm of $\mathbb{R}^N$. More generally, let us observe that the nonasymptotic condition (6) has an $N$-dimensional (strict) analog for systems.

**Proposition 3.1.** In the previous setting, assume there exists $D \subset \mathbb{R}^N$ open such that:

(a) $g(r + c) \neq c$ for every $c \in \partial D$ and $|r| \leq R$,
(b) $\deg(I - g, D, 0) \neq 0$.

Then system (1) admits at least one solution $u$ with $u|_{\partial \Omega} \in D$.

**Proof.** Define the operator $\mathcal{T}_\lambda : \overline{D \times B_R(0)} \rightarrow \mathbb{R}^N \times \overline{B_R(0)}$ given by

$$\mathcal{T}_\lambda(c, \nu) := (g(\nu(0) + c), \lambda T(c, \nu)).$$

If $\mathcal{T}_1$ has a fixed point on the boundary, then there is nothing to prove. Otherwise, suppose that $\mathcal{T}_\lambda(c, \nu) = (c, \nu)$ for some $(c, \nu) \in \partial(D \times B_R(0))$ and $0 \leq \lambda < 1$, then $c \in D$ and $R = \|\nu\|_\infty = \lambda\|T(c, \nu)\|_\infty < R$, a contradiction. This implies

$$\deg(I - \mathcal{T}_1, D \times B_R(0), 0) = \deg(I - \mathcal{T}_0, D \times B_R(0), 0) = \deg(I - g, D, 0),$$

and the result follows. \hfill \square

Besides the preceding degree argument, it would be interesting to analyze if a proof is possible by means of Theorem 1.1 only.

**Remark 3.2.** The problem treated in the present section may be considered as an Ordinary Differential Equations (ODE) analog of the more general problem studied in [13]. Here, not only the boundary condition but also the equation may contain nonlocal terms. An example of a nonlocal ODE with local boundary conditions is introduced in [25], arising on a two ion electrodiffusion model. In [2], a two-dimensional shooting technique is developed in order to prove the existence of solutions; however, it is not clear whether or not the result can be obtained by means of a Browder-type argument only.

2. Range of some semilinear operators at resonance

In a celebrated paper from 1980, A. Castro [6] posed the following question: for which $\omega$-periodic forcing terms $p(t)$ the pendulum equation

$$u''(t) + \sin u(t) = p(t)$$

admits an $\omega$-periodic solution? The problem is called resonant, because the kernel of the associated linear operator $L u := u''$ over the space of $\omega$-periodic solutions is nontrivial. Together with the fact that the nonlinear term $g(u) := \sin u$ is bounded, this implies that the problem has no $\omega$-periodic solutions for some choices of $p$; indeed, taking average at both sides of the equation yields the necessary condition $|\overline{p}| \leq 1$. However, this condition is not sufficient, so the answer to the previous question is far from being obvious. Using variational methods, it is easy to prove that the equation admits at least one $\omega$-periodic solution when $\overline{p} = 0$, because the ($\omega$-periodic) associated functional achieves a minimum. In view of this, it proves convenient to write $p = p_0 + s$, where $p_0$ has zero average; thus, the problem of describing the range of the semilinear operator $Su := u'' + \sin u$ is reduced to find, for each
$p_0$, the set $I(p_0)$ of values $s \in [-1, 1]$ such that an $\omega$-periodic solution of the equation exists. If for simplicity, we consider $S : D \subset C_\omega \to C_\omega$, where $C_\omega$ denotes the space of $\omega$-periodic continuous functions and $D := C_\omega \cap C^2$, then the results in [6] for $\omega \leq 2\pi$ imply that, for each $p_0 \in \overline{C}_\omega = \{ \varphi \in C_\omega : \overline{\varphi} = 0 \}$, the set $I(p_0)$ is a compact interval containing the origin, whose endpoints depend continuously on $p_0$. Thus, identifying $C_\omega$ with $\overline{C}_\omega \times \mathbb{R}$, the range of $S$ can be expressed as

$$\text{Im}(S) = \bigcup_{p_0 \in \overline{C}_\omega} \{p_0\} \times I(p_0).$$

The proofs in [6] rely on the variational structure of the problem; a few years later, Fournier and Mawhin [9] extended the results for arbitrary periods and assuming that the equation contains a friction term $au'$, so the problem is nonvariational when $a \neq 0$. Here, we shall analyze the method used in [9] through the lens of the Browder theorem. It is worthy to mention that the setting applies for more general problems such as

$$u''(t) + au'(t) + g(u(t)) = p_0(t) + s$$

with $g$ continuous and bounded, which has been treated, for example, in [11]. With this in mind, let us observe that the periodic problem for (7) can be regarded as nonlocal, in the following sense. Because $p_0$ is $\omega$-periodic, if $u$ is a solution of (7), then $u$ is $\omega$-periodic if and only if

$$u(0) = u(\omega), \quad u'(0) = u'(\omega)$$

or, since $p_0 = 0$, if and only if

$$u(0) = u(\omega), \quad g(u) = s.$$ 

As it may be recalled, this procedure is the essence of the Lyapunov–Schmidt decomposition. Again, a compact operator is defined by solving a homogeneous Dirichlet problem; namely, for $(c, w) \in \mathbb{R} \times C_\omega$, let $v := T(c, w)$ be defined as the unique solution of the linear problem

$$v''(t) + av'(t) = p_0(t) - g(c + w(t)) + \overline{g(c + w)}, \quad v(0) = v(\omega) = 0.$$ 

Upon integration, it follows that $v \in C_\omega$, which implies $\text{Im}(T) \subset C_\omega$. Hence, $v$ is a fixed point of $T_c$ if and only if $u := c + v$ is an $\omega$-periodic solution of the problem for $s = g(u)$. Furthermore, using again standard estimates, we obtain

$$\|v\|_{\infty} \leq k \|v'' + av'\|_{\infty} \leq R$$

for some constant $R$, that is, $T(\mathbb{R} \times B_R(0)) \subset B_R(0)$. Thus, if we define as before

$$F_{\mathbb{R}} = \bigcup_{c \in \mathbb{R}} \{c\} \times \text{Fix}(T_c),$$

then the set $I(p_0)$ coincides with $I(F_{\mathbb{R}})$, where the continuous mapping $I : \mathbb{R} \times C_\omega \to \mathbb{R}$ given by $I(c, w) := g(c + w)$. It is clear that the set $I(p_0)$ is contained in the interval $[-\|g\|_{\infty}, \|g\|_{\infty}]$, although its compactness is not guaranteed unless we assume extra conditions on $g$. One of these possible extra conditions is that, as in the pendulum equation, $g$ is $\sigma$-periodic for some $\sigma > 0$, which allows to restrict the mapping $I$ to the compact set $F_{[0, \sigma]}$. The same happens when a priori
bounds exist: if one knows that all the possible $\omega$-periodic solutions of the problem are bounded by a constant $M$ depending only on $\|p_0\|_{\infty}$, then $I$ can be restricted to the set $F_{[-M,M]}$.

A typical situation in which compactness fails is the Landesman–Lazer case, that is, when the limits

$$g_{\pm} := \lim_{u \to \pm\infty} g(u)$$

exist and are distinct. If furthermore $g(u) \neq g_{\pm}$ for all $u \in \mathbb{R}$, then it is well known that $I(p_0)$ is the (open) interval of all those values lying strictly between $g_-$ and $g_+$. Perhaps, we do not need a new proof of this but, again, Browder’s theorem reduces it to an easy exercise. To fix ideas, suppose that $g_- < g_+$ and define as before a mapping $T(c,w)$ from solving the linear problem (8) and, for arbitrary $s \in (g_-,g_+)$, define the continuous function $\Phi(c,w) := s - g(w+c)$. Take $M$ sufficiently large such that

$$g(u) > s > g(-u) \quad u \geq M,$$

then, for $w \in B_{g}(0)$, we deduce, for $c := R + M$, that $w(t) + c \geq M$ and $w - c \leq -M$. Thus, if $C \subset F_{[-c,c]}$ connects $F_{-c}$ and $F_c$, the mapping $\Phi$ changes sign over $C$ and, consequently, vanishes at some point. Conversely, when $s \geq g_+$ or $s \leq g_-$, the nonexistence of solutions is deduced simply by integrating the equation.

Also, we cannot ensure that $0 \in I(p_0)$: this may be false even for $g(u) = \sin u$, provided that $a \neq 0$. This was shown in the early work [19], see, for example, [16] for a complete survey of the problem. Due to Schauder’s theorem, the mapping $T_c$ has fixed points for all $c$; thus, $I(p_0)$ is nonempty although, in general, it cannot be proven that it contains more than one point. This is one of the most famous open problems concerning the pendulum equation: decide whether or not there exists $p_0$ such that $I(p_0)$ is a singleton. Such a situation is called degenerate and it was demonstrated that, generically, it does not occur, that is, except possibly for a meager subset of $\tilde{C}_\omega$. Observe, incidentally, that if $I(p_0) = \{s\}$, then all the solutions of the problem

$$u''(t) + au'(t) + \sin(u(t)) = p_0(t) + \sin(u), \quad u(0) = u(\omega) = c$$

for arbitrary $c$ satisfy $\sin(u) = s$, so we can deduce the existence of a continuum $C$ of such solutions $u_c \in C_\omega$ with $c \in \mathbb{R}$. Differently from what occurs in the general abstract case, here $C$ is indeed a curve, because, as shown by Ortega and Tarallo in [20], $u_c$ is unique and the map $c \mapsto u_c$ is continuous. Amazingly, this can be seen, once more, as a consequence of Theorem 1.1. To this end, following the ideas in [20], let us first prove that different solutions do not intersect at any point. Suppose that $u$ and $v$ are solutions such that $\eta(t_0) > \eta(t_0) := \eta$ at some $t_0$. Without loss of generality, we may assume $t_0 = 0$ and take, for $a < \eta - 2R$, a subset $C \subset F_{[a,\eta]}$ that connects $F_a$ and $F_\eta$. Let $\hat{C} \subset C_\omega$ be the projection of $C$, then $\hat{C} \neq \hat{C}$ and observe that if $\hat{C}$ intersects some $w \in \hat{C}$ at some $t$, then the uniqueness of the initial value problem implies $u'(t) \neq w'(t)$. In turn, this implies that the set

$$A_v := \{w \in \hat{C} : w \text{ intersects } v\}$$

is open and closed; in other words, $v$ does not intersect any element of $\hat{C}$. Now suppose $u \notin \hat{C}$ and observe that, as before, the set $A_u$ is open and closed in $\hat{C}$. Moreover, $u$ intersects some element of $\hat{C}$ at $t = 0$, so $A_u = \hat{C}$. But, on the other hand, there exists $w \in \hat{C}$ such that $w(0) = a$, which
satisfies, for all $t$,

$$w(t) < a + R < \eta - R < u(t),$$

a contradiction. Thus, $u \in \hat{\mathcal{C}}$ and, consequently, does not intersect $v$. The continuity of the map $c \mapsto u_c$ follows now straightforwardly.

In contrast with the problem of degeneracy, proving that $I(p_0)$ is always an interval is easy and, again, relies on the existence of a continuum. Indeed, it suffices to prove that if $s_2 < s_1$ are two elements of $I(p_0)$, then $s \in I(p_0)$ for any $s \in (s_2, s_1)$. Take $(c_j, u_j) \in P$ preimages of $s_j$ for $j = 1, 2$, then

$$u''_1(t) + au'_1(t) + g(u_1(t)) > p_0(t) + s > u''_2(t) + au'_2(t) + g(u_2(t)),$$

that is, $u_1$ and $u_2$ are, respectively, a lower and an upper solution of the problem. The issue here is that $u_1$ and $u_2$ are not necessarily well ordered, although the existence of a solution is still verified because $g$ is bounded. This is a well-known fact, whose proof becomes almost effortless with the help of Theorem 1.1. To this end, fix a constant $M$ such that

$$M > \max\{\|u_1\|_{\infty}, \|u_2\|_{\infty}\} + R.$$

Since any solution $u$ satisfies $\|u - u(0)\|_{\infty} \leq R$, it is deduced, for $u(0) = \pm M$, that $\max\{\|u_1\|_{\infty}, \|u_2\|_{\infty}\} < M - |u(t) - u(0)|$ for all $t$, that is

$$u_1(t), u_2(t) < u(t) \quad \text{if } u(0) = M,$$

$$u_1(t), u_2(t) > u(t) \quad \text{if } u(0) = -M.$$

Next, consider a set $\mathcal{C} \subset \mathcal{P}$ connecting $\mathcal{P}_{-M}$ and $\mathcal{P}_M$. If $I$ takes the value $s$ at some point in $\mathcal{C}$, then we are done; otherwise, $I - s$ has constant sign, say, for example, $I|_{\mathcal{C}} > s$. Then, we may take an element $(-M, u) \in \mathcal{C}$ and observe that, because $g(u) = I(-M, u) > s$, the function $u$ is a lower solution of the problem for $s$ and, consequently, $(u, u_2)$ is a well-ordered couple of a lower and an upper solution. If we assume, instead, that $I|_{\mathcal{C}} < s$, then taking $(M, v) \in \mathcal{C}$ produces the well-ordered couple $(u_1, v)$.

Next, let us consider the problem of establishing the continuity of the mapping $p_0 \mapsto I(p_0)$. As mentioned, for the pendulum equation, this was proven in [6] for the variational case and later extended to the nonvariational case by means of topological arguments (see [16]). However, the subject was not tackled in [11] for arbitrary bounded $g$, although the proof can be also performed in a very simple and direct way as follows. Assume that $p_n \in C_\omega$ has zero average for all $n$ and $p_n \to p_0$ uniformly. In order to prove that $I(p_n) \to I(p_0)$, we shall proceed in two steps. First, let $b_n := \sup I(p_n)$ and consider an arbitrary subsequence, still denoted as $\{b_n\}$, converging to some value $b$. Fix $s < b$, then there exist $\bar{b}_n \in I(p_n)$ such that $s < \bar{b}_n - \varepsilon$ for some $\varepsilon > 0$. Taking $u_n \in C_\omega$ solutions for $p_n + \bar{b}_n$, we obtain

$$u''_n(t) + au'_n(t) + g(u_n(t)) = p_0(t) + s + p_n(t) - p_0(t) + \bar{b}_n - s > p_0(t) + s$$

for $n \gg 0$, that is, $u_n$ is a lower solution of the problem for $p_0 + s$. In the same way, if we take an arbitrary subsequence of $a_n := \inf I(p_n)$ converging to some $a < s$, then a lower solution for $p_0 + s$ is obtained and, as before, we conclude that a solution exists when $a < s < b$. Because the
subsequences are arbitrary, this already proves that

\[(\lim\inf_{n\to\infty} a_n, \lim\sup_{n\to\infty} b_n) \subset I(p_0).\]

In the second place, suppose that \(s \in I(p_0)\) satisfies \(s > b\) and fix \(\varepsilon > 0\) such that \(s > b_n + 2\varepsilon\) for all \(n\). Let \(u \in C_\omega\) be a solution for \(p_0 + s\), then it follows as before that \(u\) is a lower solution for the problem corresponding to \(p := p_n + b_n + \varepsilon\), provided that \(n \gg 0\). On the other hand, setting \(\bar{b}_n \in I(p_n)\) and a solution \(u_n \in C_\omega\) for \(p_n + \bar{b}_n\), then it is clear that \(u_n\) is an upper solution for \(p\). This implies that \(b_n + \varepsilon \in I(p_n)\), a contradiction. An analogous argument holds if \(s < a\) and, again, due to the fact that the subsequences are arbitrary, we conclude that

\[
\bar{I}(p_0) \subset [\lim\sup_{n\to\infty} a_n, \lim\inf_{n\to\infty} b_n].
\]

By combining the two steps, it is deduced that

\[(\lim\inf_{n\to\infty} a_n, \lim\sup_{n\to\infty} b_n) \subset [\lim\sup_{n\to\infty} a_n, \lim\inf_{n\to\infty} b_n];\]

summarizing, the sequences \(\{a_n\}\) and \(\{b_n\}\) converge, respectively, to the lower and upper endpoints of \(I(p_0)\).

It is worth recalling, in pendulum-like equations, that the periodicity of \(g\) also implies, when \(s\) is an interior point of \(I(p_0)\), the existence of a second solution which is geometrically different, in the sense that it does not differ with the first one by an integer multiple of the period. This is a consequence of the so-called three solutions theorem; we sketch a proof for the sake of completeness. Assume that \(g(t + \sigma) \equiv g(t)\) and set solutions \(u_1, u_2\) corresponding to some \(s_2 < s_1\). Fix \(s \in (s_2, s_1)\), then there exists \(k \in \mathbb{Z}\) such that

\[u_1(t) < u_2(t) + k\sigma, \quad u_1(t) + \sigma \not< u_2(t) + k\sigma,\]

and the excision property of the Leray–Schauder degree shall be applied to the sets

\[\Omega_1 := \{u \in C_\omega : u_1(t) < u_2(t) + k\sigma\},\]

\[\Omega_2 := \{u \in C_\omega : u_1(t) + \sigma < u_2(t) + (k + 1)\sigma\},\]

both contained in

\[\Omega := \{u \in C_\omega : u_1(t) < u_2(t) + (k + 1)\sigma\}.\]

To this end, as usual, we may define \(K : C_\omega \to C_\omega\) given by \(Kw := v\), the unique \(\omega\)-periodic solution of the (nonresonant) problem

\[v''(t) + av'(t) - v(t) = p_0(t) + s - g(w(t)) - w(t).\]

A standard result in the theory of strict well-ordered upper and lower solutions ensures that the degree of \(I - K\) over \(\Omega_1, \Omega_2,\) and \(\Omega\) is equal to 1; hence, there exist solutions \(w_j \in \Omega_j\) and \(w \in \Omega \setminus (\Omega_1 \cup \Omega_2)\) and, clearly, at least two of the three solutions \(w, w_1,\) and \(w_2\) are geometrically distinct.

As in our first example, we may also make some considerations about the nonscalar case. The construction can be done exactly in the same way as before, and the existence of a nonempty
bounded set $I(p_0) = \text{Im}(I) \subset \mathbb{R}^N$ follows. Using the mean value theorem for vector integrals, it is readily verified that $I(p_0)$ is contained in the convex hull of $\text{Im}(g)$; moreover, $I(p_0)$ is compact if, for example, $g$ satisfies some periodicity assumption, for example,

$$g(u + \sigma_j e_j) \equiv g(u),$$

(9)

where $\sigma_j > 0$ and $\{e_1, \ldots, e_N\}$ is a basis of $\mathbb{R}^N$. But an issue occurs when one tries to prove the connectedness, because the method of well-ordered upper and lower solutions for systems usually requires stronger conditions, for example, $\alpha_j \leq \beta_j$ and

$$\alpha_j''(t) + a \alpha_j'(t) + g_j(\dot{\alpha}(t)) \geq p(t) \geq \beta_j''(t) + a \beta_j'(t) + g_j(\dot{\beta}(t))$$

for $j = 1, \ldots, N$, where $\dot{\alpha}_j = \alpha_j$, $\dot{\beta}_j = \beta_j$ and $\alpha_k \leq \hat{\alpha}_k, \hat{\beta}_k \leq \beta_k$ for all $k$. These conditions might be of some help when dealing with weakly coupled systems but, in general, the problem of determining the shape of $I(p_0)$, or even if it is connected, is open.

An alternative approach can be employed if one assumes the same assumption imposed in [6] for the scalar case, namely, the relaxed monotonicity condition

$$\langle g(u) - g(v), u - v \rangle \left| u - v \right|^2 < \left( \frac{2\pi}{\omega} \right)^2 \forall u \neq v.$$  

(10)

This case was already analyzed in [1] for the variational case, that is, with $a = 0$ and $g = \nabla G$. However, the following extension was not previously treated in the literature.

**Proposition 3.2.** In the previous situation, assume that (10) holds. Then $I(p_0)$ is arcwise connected.

**Proof.** Given $x \in \mathbb{R}^N$, let us prove the following claim: there exists a unique $\omega$-periodic function $w_x$ such that $w_x'' + aw_x' + g(x + w_x) - p_0$ is constant and $w_x = 0$.

**Existence:** We may consider now the operator

$$T^x(c, w) := (x - T(c, v), T(c, v)),$$

and prove as before that the degree of $I - T^x$ over a large ball of $\mathbb{R}^N \times C_\omega$ is equal to 1. This implies the existence of $v$ such that $v = T(c, v)$ and $\overline{v} + c = x$; thus, $w_x := c + v - x$ fulfills all the requirements.

**Uniqueness:** Suppose that $w_1 \neq w_2$ are solutions, then $w := w_1 - w_2$ verifies

$$w''(t) + aw'(t) + g(x + w_1(t)) - g(x + w_2(t)) = C$$

for some $C \in \mathbb{R}^N$. Multiplying by $w$ and using that $\overline{w} = 0 = w(\omega) - w(0)$, we get

$$\int_0^\omega \left| w'(t) \right|^2 dt = \int_0^\omega \langle g(x + w_1(t)) - g(x + w_2(t)), w(t) \rangle dt$$

$$< \left( \frac{2\pi}{\omega} \right)^2 \int_0^\omega \left| w(t) \right|^2 dt,$$

which contradicts Wirtinger’s inequality.

Furthermore, observe that the mapping $x \mapsto w_x$ is continuous. Indeed, assume $x_n \rightarrow x$ and consider the respective solutions $w_n := w_{x_n}$ and $w := w_x$. If $w_n \rightarrow w$, then passing to a
subsequence, we may assume that $w_n$ converges to some $z \neq w$ for the $C^1$ norm and $w''_n$ converges weakly in $L^2$. Multiplying the equation by test functions, it is readily seen that $z = w$, a contradiction.

We conclude that the set $I(p_0)$ is characterized as the range of the continuous function of $\mathbb{R}^N$ defined by

$$x \mapsto g(x + w_x)$$

and so completes the proof.

Regarding the continuity of $I$ with respect to $p_0$, it is observed that, as far as we do not have any information about the shape of the set $I(p_0)$, we need to use some notion of distance between bounded sets. The obvious choice is the Hausdorff metric, namely, given $K_1, K_2$ compact subsets of $\mathbb{R}^N$, the distance given by

$$d_H(K_1, K_2) := \max \left\{ \sup_{y \in K_1} d(y, K_2), \sup_{y \in K_2} d(y, K_1) \right\}.$$  

We shall prove that the mapping $I$ is continuous when the nonlinearity is periodic or of Landesman–Lazer type and leave the general situation for future research. Again, we recall that, up to the author’s knowledge, only the variational case under condition (9) was previously treated in the referred work [1].

**Proposition 3.3.** Assume that (10) holds and the sequence $\{p_n\} \subset \tilde{C}_\omega$ converges to $p_0$ uniformly. Assume, furthermore, that either $g$ satisfies (9) or has radial limits

$$g_v := \lim_{r \to +\infty} g(rv)$$

uniformly for $v$ in the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Then $\overline{I(p_n)} \to \overline{I(p_0)}$ for the Hausdorff metric.

**Proof.** Observe, in the first place, that the same argument employed before shows that the mapping $x \mapsto w_x$ also depends continuously on $p_0$, that is: if $\{x_n, p_n\} \to (x, p_0)$, then the corresponding sequence $\{w^n_{x_n}\}$ converges to $w_x$. Let $s \in I(p_0)$, that is, $s = g(x + w_x)$ for some $x$. By dominated convergence, $g(x + w^n_{x_n}) \to s$ which, in turn, implies that $d(s, I(p_n)) \to 0$. On the other hand, suppose that $d(s_n, I(p_n)) \not\to 0$ for some sequence $s_n = g(x_n + w^n_{x_n})$, then passing to a subsequence, we may assume $s_n \to s$, with $s \notin I(p_0)$ and $x_n \to L$, with $L = x \in \mathbb{R}^N$ or $L = \infty$. In the first case, it follows as before that $x_n + w^n_{x_n} \to x + w_x$ for some $x$; hence, $s \in I(p_0)$, a contradiction. This already covers the case in which (9) is satisfied. Thus, we may assume that $g$ have uniform radial limits and $x_n \to \infty$. Taking a subsequence, we may also assume that $\frac{x_n}{|x_n|}$ converges to some $v \in S^{N-1}$. Consequently, both sequences $\frac{x_n + w^n_{x_n}}{|x_n + w^n_{x_n}|}$ and $\frac{x_n + w_{x_n}}{|x_n + w_{x_n}|}$ converge uniformly to $v$. This implies that $s_n \to g_v \in \overline{I(p_0)}$, a contradiction. \qed

Exactly the same conclusions can be obtained for the elliptic analog of the preceding problem, that is,

$$\Delta u(x) + \langle a, \nabla u(x) \rangle + g(u(x)) = p(x) \quad x \in \Omega,$$
with the nonlocal condition
\[ u|_{\partial\Omega} = c, \quad \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS = 0, \]
where \( c \) is an undetermined constant and \( \nu \) denotes the outer normal or, equivalently
\[ u|_{\partial\Omega} = c, \quad \int_{\Omega} g(u(x)) \, dx = \int_{\Omega} p(x) \, dx. \]

3. **A delayed chemostat model revisited.** In [3], the following model was considered

\[
\begin{align*}
s'(t) &= D(t)[s^0(t) - s(t)] - \mu(s(t)) \frac{s(t)}{\gamma} \quad x'(t) = x(t)[\mu(s(t-\tau)) - D(t)],
\end{align*}
\]
where \( D, s^0 > 0 \) are continuous \( \omega \)-periodic functions, \( \gamma > 0 \) is a constant, and the mapping \( \mu : [0, +\infty) \to [0, +\infty) \) is locally Lipschitz and strictly increasing, with \( \mu(0) = 0 \). It is assumed that \( \tau > 0 \) is a fixed delay and, regarding the existence of \( \omega \)-periodic solutions, we may also assume without loss of generality that \( \tau < \omega \). From the standard theorem of existence and uniqueness, for arbitrary initial values \( \varphi \in C[-\tau, 0] \) with \( \varphi \geq 0 \) and \( x_0 \geq 0 \), there exists a unique solution defined on an interval \([ -\tau, \delta) \) for some \( \delta \in (0, +\infty) \). In particular, when \( x_0 > 0 \), it is seen that \( x(t) > 0 \) for all \( t \) and, if \( s \) vanishes at some \( t_0 \in [0, \delta) \), then \( s'(t_0) > 0 \). This implies that \( s(t) > 0 \) for \( t > 0 \). Observe, moreover, that if \( x_0 = 0 \), then \( x \equiv 0 \) and \( s \) coincides with the unique solution of the problem

\[ v'(t) = D(t)(s^0(t) - v(t)) \quad (11) \]

satisfying
\[ v(0) = \varphi(0). \]

It is readily verified that (11) has a unique \( \omega \)-periodic solution \( v^* \), which is positive, and we may call \( \varphi^* := v^*|_{-\tau, 0} \). On the other hand, for \( x_0 > 0 \) and \( \varphi \in X \), where
\[ X := \{ \varphi \in C[-\tau, 0] : 0 \leq \varphi \leq \varphi^* \}, \]
the corresponding solution \( s \) satisfies \( s(t) < v^*(t) \) for all \( t > 0 \). Indeed, it is noticed that \( w := v^* - s \) satisfies
\[ w'(t) > -D(t)w(t), \quad t > 0 \]
and, upon integration, we obtain
\[ w(t) > e^{-\int_0^t D(s) \, ds} w(0) \geq 0. \]

As a consequence, all the trajectories \((s, x)\) with initial values \( \varphi \in X \) and \( x_0 \in [0, +\infty) \) are globally defined. Thus, we may consider the Poincaré operator
\[ P(\varphi, x_0) := (s_\omega, x(\omega)), \]
where \((s, x)\) is the solution corresponding to the initial values \((\varphi, x_0)\) and \(s_\omega \in C[-\tau, 0]\) is given as usual by \(s_\omega(t) := s(\omega + t)\). It follows that \(P : X \times [0, +\infty) \to X \times [0, +\infty)\) is well defined, and a direct application of the Arzelà–Ascoli theorem shows that \(P\) is compact. In order to find a fixed point of \(P\), the previous setting suggests considering the application \(f : [0, +\infty) \times X \to X\) given by

\[
f(x_0, \varphi) := P^1(\varphi, x_0),
\]

where \(P^1\) denotes the first coordinate of \(P\). Observe, in this case, that \(\text{Fix}(f_0) = \{\varphi^*\}\); thus, Corollary 1.1 implies the existence of a connected subset \(C \subset F_{[0, +\infty)}\) whose projection to \([0, +\infty)\) is onto. It is seen, anyway, that the weaker conclusion of Theorem 1.1 is already enough to deduce the existence of \(\omega\)-periodic solutions of the system, provided that an appropriate condition is fulfilled. To this end, let us begin by noticing that, if \((s, x)\) is a positive \(\omega\)-periodic solution, then integration of the first equation yields

\[
D_s < Ds_0 = Dv^*,
\]

where, for an arbitrary continuous function, we define its average \(\overline{w}\) by \(\overline{w} := \frac{1}{\omega} \int_0^\omega w(t) \, dt\). The previous inequality proves that \(s(t) < v^*(t)\) for some \(t\) and, as shown before, this implies \(s(t) < v^*(t)\) for all \(t\). In other words, when searching for \(\omega\)-periodic solutions, we may restrict ourselves to look for initial values \((\varphi, x_0) \in X \times (0, +\infty)\). Furthermore, since \(\frac{x'}{x}\) is also \(\omega\)-periodic, it follows that

\[
\int_0^\omega [\mu(s(t - \tau)) - D(t)] \, dt = 0,
\]

that is, using the fact that \(\mu\) is strictly increasing,

\[
\int_0^\omega D(t) \, dt < \int_0^\omega \mu(v^*(t - \tau)) \, dt = \int_0^\omega \mu(v^*(t)) \, dt
\]

or, equivalently,

\[
\overline{D} < \overline{\mu(v^*)}.
\]

As we shall see, the necessary condition (12) is also sufficient for the existence of a positive \(\omega\)-periodic solution. This was already proven in [3], where the strategy consisted in transforming the original system into a one-parameter family of integrodifferential equations. The main goal of the present section is to emphasize the fact that the result can be retrieved as a direct consequence of the Browder theorem in a simple and concise way. In order to demonstrate this assertion, let us consider the continuous mapping \(\Phi : [0, +\infty) \times X \to \mathbb{R}\) defined by

\[
\Phi(x_0, \varphi) := \overline{\mu(s)} - \overline{D},
\]

where \(s\) is the first coordinate of the trajectory corresponding to the initial value \((\varphi, x_0)\). Condition (12) ensures that \(\Phi(0, \varphi^*) > 0\); moreover, when \(x_0 > 0\), the inequality \(x'(t) > -D(t)x(t)\) implies, as before,

\[
x(t) > e^{-\int_0^\omega D(r) \, dr} x_0 = e^{-\omega \overline{D}} x_0 \quad t \in [0, \omega].
\]
Hence, setting \( w := v^* - s \geq 0 \), it follows that

\[
w'(t) > -D(t)w(t) + cx_0 \mu(s(t)),
\]

where the constant \( c := e^{-\omega \over \gamma} \) is independent of \( x_0 \) and \( \phi \). This means, in turn,

\[
v^*(t) > w(t) > w(0)e^{-\int_0^t D(r)dr} + cx_0 \int_0^t e^{-\int_r^t D(r)dr} \mu(s(\xi)) d\xi,
\]

and, in particular,

\[
v^*(\omega) > cx_0 e^{-\omega \over \gamma} \int_0^\omega \mu(s(\xi)) d\xi.
\]

Thus, \( \mu(s) < k \) for some constant \( k \) independent of \( x_0 \) and \( \phi \), which yields \( \Phi(x_0, \phi) = \mu(s) - D < 0 \) for any \( \phi \in X \), provided that \( x_0 \) is sufficiently large. By Theorem 1.1, for such \( x_0 \), there exists \( C_{0,x_0} \subset F_{[0, +\infty)} \) connecting \( F_0 = \{(0, v^*)\} \) with \( F_{x_0} = \{x_0\} \times \text{Fix}(f_{x_0}) \). The continuous map \( \Phi \) changes sign over \( C_{0,x_0} \), so it vanishes at some point \((\tilde{x}_0, \tilde{\phi}) \in C_{0,x_0} \) with \( \tilde{x}_0 > 0 \). Because \( \tilde{\phi} \in \text{Fix}(f_{\tilde{x}_0}) \), it follows that

\[
\ln \tilde{x}(\omega) - \ln \tilde{x}_0 = \int_0^\omega [\mu(\tilde{s}(t - \tau)) - D(t)] dt = \int_0^\omega [\mu(s(t)) - D(t)] dt = \omega \Phi(\tilde{x}_0, \tilde{\phi}) = 0,
\]

that is, \( x(\omega) = \tilde{x}(0) \). In other words, \((\tilde{\phi}, \tilde{x}_0)\) is a fixed point of the Poincaré map and an easy computation shows that the corresponding solution \((\tilde{s}, \tilde{x})\) is \( \omega \)-periodic.

A natural question concerning the previous proof is: can it be carried out by means of Schauder’s theorem only? Of course, this is possible using the trick described in Remark 2.1, but this does not mean that the Schauder theorem can be directly applied to the Poincaré map. In our particular situation, we need to observe, on the one hand, that we need to avoid the so-called trivial solution, namely, \((v^*, 0)\), so the domain of \( P \) should consider only values of \( x_0 \) larger than a positive constant. But, on the other hand, although the reasoning for \( x_0 \gg 0 \) may start in a similar way, the conclusion that \( \mu(s) < k \) for arbitrary \( \phi \in X \) does not necessarily imply \( x(\omega) < x_0 \) for \( \phi \notin \text{Fix}(f_{x_0}) \), because \( \int_0^\omega \mu(\phi(t)) dt \) might be larger than \( \omega D \). This means that the region \( X \times [\varepsilon, x_0] \) may not be invariant for the Poincaré operator. A sharper argument allows to conjecture that it is possible to find an invariant set of the form

\[
\{(\phi, x) : 0 \leq \phi \leq r(x) v^*, \varepsilon \leq x \leq x_0\}
\]

for some appropriate function \( r \leq 1 \), where \( r \) is initially equal to 1 and then decays in such a way that \( \int_0^\omega \mu(r(x) \phi(t)) dt < \omega D \) for \( x \gg 0 \). However, the choice of \( r \) is not obvious and shall studied in a forthcoming paper.

To conclude, it is worthy mentioning that the previous method does not seem to be applicable to different models, such as the one studied in [4], namely,

\[
\begin{cases}
s'(t) = D(t)[s^0(t) - s(t)] - \mu(s(t)) \frac{x(t)}{\gamma} \\
x'(t) = e^{-d(t)} \mu(s(t - \tau)) x(t - \tau) - D(t)x(t),
\end{cases}
\]

The continuous map \( \Phi \) changes sign over \( C_{0,x_0} \), so it vanishes at some point \((\tilde{x}_0, \tilde{\phi}) \in C_{0,x_0} \) with \( \tilde{x}_0 > 0 \). Because \( \tilde{\phi} \in \text{Fix}(f_{\tilde{x}_0}) \), it follows that

\[
\ln \tilde{x}(\omega) - \ln \tilde{x}_0 = \int_0^\omega [\mu(\tilde{s}(t - \tau)) - D(t)] dt = \int_0^\omega [\mu(s(t)) - D(t)] dt = \omega \Phi(\tilde{x}_0, \tilde{\phi}) = 0,
\]

that is, \( x(\omega) = \tilde{x}(0) \). In other words, \((\tilde{\phi}, \tilde{x}_0)\) is a fixed point of the Poincaré map and an easy computation shows that the corresponding solution \((\tilde{s}, \tilde{x})\) is \( \omega \)-periodic.

A natural question concerning the previous proof is: can it be carried out by means of Schauder’s theorem only? Of course, this is possible using the trick described in Remark 2.1, but this does not mean that the Schauder theorem can be directly applied to the Poincaré map. In our particular situation, we need to observe, on the one hand, that we need to avoid the so-called trivial solution, namely, \((v^*, 0)\), so the domain of \( P \) should consider only values of \( x_0 \) larger than a positive constant. But, on the other hand, although the reasoning for \( x_0 \gg 0 \) may start in a similar way, the conclusion that \( \mu(s) < k \) for arbitrary \( \phi \in X \) does not necessarily imply \( x(\omega) < x_0 \) for \( \phi \notin \text{Fix}(f_{x_0}) \), because \( \int_0^\omega \mu(\phi(t)) dt \) might be larger than \( \omega D \). This means that the region \( X \times [\varepsilon, x_0] \) may not be invariant for the Poincaré operator. A sharper argument allows to conjecture that it is possible to find an invariant set of the form

\[
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To conclude, it is worthy mentioning that the previous method does not seem to be applicable to different models, such as the one studied in [4], namely,
where \( d(t) = \int_{t-\tau}^{t} D(s) \, ds \) is also \( \omega \)-periodic. Indeed, here the delay appears in both unknown variables; thus, although the Poincaré operator can be still defined, its two coordinates lie on an infinite-dimensional space, so the Theorem 1.1 does not suffice as a tool for proving the existence of solutions.

4 CONCLUSIONS

An infinite-dimensional version of a theorem by Browder was introduced. The focus of the paper was put on applications to nonlinear boundary value problems. With this in mind, new viewpoints of known results were presented, as well as some novel results and open problems.

It is important to observe that since the Browder theorem provides only a *continuum* of fixed points, it is not clear how to obtain information about the topological degree of the involved maps when dealing with a system instead of a scalar equation. This is the spirit of the open problem proposed in Subsection 3.3. In the same direction, the application of Theorem 1.1 in Subsection 3.3 sheds some light on the difficulties that may arise when dealing with other models that involve a parameter (namely, the initial condition for one of the unknown variables) in an infinite-dimensional space. Regarding Subsection 3.3, the problem of determining the shape of \( I(\rho_0) \) in the \( N \)-dimensional case looks like a new and challenging open question. The matter of nondegeneracy for the pendulum equation is still intriguing and, hopefully, the fact that some of the arguments in [20] can be established in terms of Theorem 1.1 may give a new insight to the subject.

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