Uniformly most reliable three-terminal graph of dense graphs

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Abstract: Suppose that every edge of a graph $G$ survives independently with a fixed probability between 0 and 1. The three-terminal reliability is the connection probability of the fixed three target vertices $r, s$ and $t$ in a three-terminal graph. This research focuses on the uniformly most reliable three-terminal graph of dense graphs with $n$ vertices and $m$ edges in some ranges. First, we give the locally most reliable three-terminal graphs of $n$ and $m$ in a certain range for $p$ close to 0 and for $p$ close to 1. And then, we prove that there is no uniformly most reliable three-terminal graph with certain ranges of $n$ and $m$. Finally, some uniformly most reliable graphs are given for $(\binom{n}{2}) - 2$ ($4 \leq n \leq 6$) and $(\binom{n}{2}) - 1$ ($n \geq 5$). This study of uniformly or locally most reliable three-terminal graph provides helpful guidance for constructing highly reliable network structures involving three key vertices as target vertices.

Keywords: target vertices, $rst$-subgraph, three-terminal graph, locally most reliable graph, uniformly most reliable graph.

1 Introduction

In many applications, the reliability aspect of a network with $n$ vertices and $m$ edges can be modeled as a graph $G$ with the same number of vertices, edges, and interconnections as the network. For all-terminal reliability (connection probability of all vertices of a graph), many studies have been done to determine the existence of a uniformly most reliable (all-terminal) graph for various values of $n$ and $m$ [13, [17], [18], [19]]. However, the research on $k$-terminal reliability (connection probability of a given set of target vertices with size $k$ in a graph) is
mainly about the algorithm of computing the $k$-terminal reliability polynomial \cite{5,6,9,10,12}, and only a few results on the construction of the most reliable $k$-terminal graph.

For large $m$ ($m$ is the number of edges of a graph), it is clear that there is only one graph for a complete graph (that is, $m = \binom{n}{2}$) and a complete graph with one edge removed (that is, $m = \binom{n}{2} - 1$), so they are the uniformly most reliable graphs. In \cite{16}, it is shown that for \((\binom{n}{2}) - \left\lfloor \frac{n}{2} \right\rfloor \leq m \leq (\binom{n}{2}) - 2\) \((\left\lfloor \frac{n}{2} \right\rfloor\) is the largest integer not greater than \(\frac{n}{2}\)), the uniformly most reliable graph is a complete graph with a matching removed (the matching of a graph is a set of edges in a graph that have no common vertices with each other). These results have great significance in network design practices. In fact, in a real network, the design of the network often only needs to ensure the connectivity of $k$ ($2 \leq k < n$) key vertices (target vertices) in the network. Therefore, the construction of the most reliable $k$-terminal graph has high application value. However, by the literature available to the authors of the present research, there is a few research work on the construction of the most reliable $k$-terminal structure. Betrand et al. \cite{2} demonstrated in 2018 that for \((\binom{n}{2}) - \left\lfloor \frac{n-3}{2} \right\rfloor \leq m \leq (\binom{n}{2}) - 2\) \((n \geq 7)\), there is no uniformly most reliable two-terminal graph and for $m = \binom{n}{2} - 1$, the uniformly most reliable two-terminal graph is a complete graph with an edge between non-target vertices removed. Therefore, it is natural to consider the following problem.

**Problem.** For large $m$, is there a uniformly most reliable three-terminal graph? If it exists, how is it constructed? If it does not exist, can we construct the locally most reliable three-terminal graph and how to construct it?

With these questions, we further study the existence of uniformly most reliable three-terminal graphs for large $m$. For three-terminal graphs with $m$ in the range \((\binom{n}{2}) - \left\lfloor \frac{n-3}{2} \right\rfloor \leq m \leq (\binom{n}{2}) - 2\) \((n \geq 7)\), it is proved that there is no uniformly most reliable graph, and the locally most reliable three-terminal graphs are determined, one case is for $p$ close to 0 and the other is for $p$ close to 1. It also determines the uniformly most reliable three-terminal graph with \((\binom{n}{2}) - 2\) \((4 \leq n \leq 6)\) and \((\binom{n}{2}) - 1\) \((n \geq 5)\) edges, respectively.

This present research is organized as follows. In section 2, some related basic definitions and notations are given. In section 3, the locally most reliable three-terminal graphs for $m$ in a certain range are determined and show that there is no uniformly most reliable three-terminal graph for \((\binom{n}{2}) - \left\lfloor \frac{n-3}{2} \right\rfloor \leq m \leq (\binom{n}{2}) - 2\) \((n \geq 7)\) and give the uniformly most reliable graphs for $n = 4, 5, 6$ and $m = \binom{n}{2} - 2$. In section 4, a uniformly most reliable three-terminal graph with $n$ vertices and $m = \binom{n}{2} - 1$ edges is determined. Section 5 summarizes the results of this research.
2 Basic concepts and notations

A graph $G = (V(G), E(G))$ with three specified target vertices $r, s$ and $t$ in $V(G)$ is a three-terminal graph. Using $G_{n,m}$ denotes the set of all simple three-terminal graphs with $n$ vertices and $m$ edges. The connectivity probability of the three specified target vertices $r, s, t$ in graph $G \in G_{n,m}$ when each edge of $G$ survives independently with a fixed probability $p$ is called the three-terminal reliability of $G$, or the three-terminal reliability polynomial of $G$, denote by $R_3(G; p)$. A $v_1v_2 \cdots v_n$-subgraph is a subgraph of $G$ in which vertices $v_1v_2 \cdots v_n$ are connected in the subgraph. The three-terminal reliability polynomial of the graph $G \in G_{n,m}$ can be written as

$$R_3(G; p) = \sum_{i=2}^{m} N_i(G)p^i(1-p)^{m-i},$$

where $N_i(G)$ or simply $N_i$ is the number of $rst$-subgraphs of graph $G$ with $i$ edges.

**Example 1.** Figure 1 shows all types of simple three-terminal graph in $G_{4,4}$ with three target vertices $r, s, t$. Each edge of these graphs survives independently with probability $p$.

By definition, there are

3 $rst$-subgraphs with 2 edges: $\{rs, rt\}$, $\{rs, st\}$, $\{rt, st\}$;

4 $rst$-subgraphs with 3 edges: $\{rs, rt, st\}$, $\{rs, rt, sv_4\}$, $\{rs, st, sv_4\}$, $\{rt, st, sv_4\}$;

1 $rst$-subgraph with 4 edges: $\{rs, rt, st, sv_4\}$.

Thus $N_2(G_1) = 3$, $N_3(G_1) = 4$ and $N_4(G_1) = 1$.

Similarly, by definition, we can calculate $N_i(G_j)$, $2 \leq i, j \leq 4$, are as follows:

$N_2(G_2)$: 1, 3, 1; $N_3(G_3)$: 1, 4, 1; $N_4(G_4)$: 0, 3, 1.

Figure 2 shows a visualization among all graphs in $G_{4,4}$. Clearly, for all $0 < p < 1$, $R_3(G_1; p) > R_3(G_3; p) > R_3(G_2; p) > R_3(G_4; p)$, so, $G_1$ is the uniformly most reliable graph in $G_{4,4}$.

Figure 1: All simple three-terminal graph in $G_{4,4}$ with three target vertices $r, s, t$. 
Example 2. Figure 3 shows two special simple three-terminal graphs in $G_{8,26}$ with three target vertices $r, s, t$. Each edge of these graphs survives independently with probability $p$.

Directly calculated by Matlab, we give a plot of $R_3(H_1;p) - R_3(H_2;p)$ as shown in Figure 4. Clearly, for $p$ close to 1, $R_3(H_1;p) > R_3(H_2;p)$ and for $p$ close to 0, $R_3(H_1;p) < R_3(H_2;p)$.

In fact, this research later proved that $H_1$ is the locally most reliable graph for $p$ close to 1 and $H_2$ is the locally most reliable graph for $p$ close to 0 in $G_{8,26}$.

Figure 3. Two special three-terminal graphs in $G_{8,26}$ with three target vertices $r, s, t$. The red dotted lines indicate the deleted edges.

Figure 4. A plot of $R_3(H_1;p) - R_3(H_2;p)$. 
In fact, many researches on reliability focuses on determining a uniformly most reliable graph for a given number of vertices $n$ and edges $m$, as shown in Example 1; if there is no uniformly most reliable graph, researchers usually focuses on determining the locally most reliable graph for $p$ close to 0 or 1, as shown in Example 2. So, similar to the definition of uniformly (locally) most reliable graph [1, 4], we give the following definition of uniformly (locally) most reliable three-terminal graph.

Definition 2.1 A graph $G$ is the uniformly most reliable graph in $\mathcal{G}_{n,m}$, if $R_3(G;p) \geq R_3(H;p)$, $H \in \mathcal{G}_{n,m}$ for all $0 \leq p \leq 1$. In particular, if $R_3(G;p) \geq R_3(H;p)$, $H \in \mathcal{G}_{n,m}$ for $p$ close to 0 (for $p$ close to 1), then $G$ is the locally most reliable graph in $\mathcal{G}_{n,m}$ for $p$ close to 0 (for $p$ close to 1).

Here are some notations used in the following. If $G$ is a simple graph, let $N_G(H)$ denote the number of subgraphs as $H$, whose vertices is the subset of non-target vertices in $G$, and $G$ is obtain by deleting the existing edges of $G$ and introducing an edge between all two pairs of non-adjacent vertices in $G$. $G \cup e$ denotes the addition of the edge $e$ to the graph $G$, and $G - e$ denotes the deletion of the edge $e$ from the graph $G$. The union of simple graphs $G$ and $H$, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. In addition, $P_n$ is the path with $n$ vertices, and let $K_n$ denote the complete graph with $n$ vertices, in which there is exactly one edge between each pair of vertices, and $K_{1,n}$ denotes a star with $n + 1$ vertices and $n$ edges.

3 Some locally most reliable three-terminal graphs

In this section, the locally most reliable three-terminal graph for $\left(\begin{array}{c} n \end{array}\right) - (n - 4) \leq m \leq \left(\begin{array}{c} n \end{array}\right) - 2$ for $p$ close to 0 is determined and the locally most reliable three-terminal graph for $\left(\begin{array}{c} n \end{array}\right) - \left\lfloor \frac{n-3}{2} \right\rfloor \leq m \leq \left(\begin{array}{c} n \end{array}\right) - 2$ for $p$ close to 1 is also determined. Then, it is shown that for $n \geq 7$ and $\left(\begin{array}{c} n \end{array}\right) - \left\lfloor \frac{n-3}{2} \right\rfloor \leq m \leq \left(\begin{array}{c} n \end{array}\right) - 2$, there is no uniformly most reliable graph in $\mathcal{G}_{n,m}$, and for $n = 4, 5, 6$ and $m = \left(\begin{array}{c} n \end{array}\right) - 2$, there is a uniformly most reliable graph. To prove these results, we first introduce some related definitions and lemmas.

If the $rst$-subgraph with $i$ edges does not contain any $rst$-subgraph with less than $i$ edges, then it is minimal, otherwise it is non-minimal. A $rst$-cutset is a set of edges whose deletion results in the disconnection of vertices $r, s$ and $t$ in the graph and the number of edges is its size. The edge connectivity of $r, s$ and $t$ is the smallest size of a $rst$-cutset, denoted by $\lambda(rst)$ or simply $\lambda$. 
In general, the calculation of the three-terminal reliability polynomial of a graph is NP-hard \[15,17\]. Therefore, we study the locally most reliable graph by the following lemma.

**Lemma 3.1** Let the three-terminal reliable polynomials of \(G, H \in \mathcal{G}_{n,m}\) be

\[
R_3(G; p) = \sum_{i=2}^{m} N_i(G)p^i(1-p)^{m-i} \quad \text{and} \quad R_3(H; p) = \sum_{i=2}^{m} N_i(H)p^i(1-p)^{m-i}.
\]

Let \(N_i(G) = N_i(H)\) for \(1 \leq i < k\) and for \(l < i \leq m\) and \(k \leq l\), where \(k\) and \(l\) are integers. Then

1. For \(p\) close to 0, \(R_3(G; p) > R_3(H; p)\) if \(N_k(G) > N_k(H)\),
2. For \(p\) close to 1, \(R_3(G; p) > R_3(H; p)\) if \(N_l(G) > N_l(H)\).

**Proof.** Assume that \(G\) and \(H\) satisfy the given conditions, we have

\[
R_3(G; p) - R_3(H; p) = \sum_{i=2}^{m} (N_i(G) - N_i(H))p^i(1-p)^{m-i} = \sum_{i=2}^{k} (N_i(G) - N_i(H))p^i(1-p)^{m-i} \sum_{i=k}^{m} (N_i(G) - N_i(H))p^i(1-p)^{m-i}.
\]

It is clear to see that for \(p\) close to 0, if \(N_k(G) > N_k(H)\), then \(R_3(G; p) - R_3(H; p) > 0\), that is, \(R_3(G; p) > R_3(H; p)\). Similarly, for \(p\) close to 1, if \(N_l(G) > N_l(H)\), then \(R_3(G; p) > R_3(H; p)\).

By Lemma 3.1, we get the following conclusions.

1. If \(G \in \mathcal{G}_{n,m}\) is the locally most reliable graph for \(p\) close to 0, then it must contain the triangle \(rst\) and the value of \(N_3(G)\) is the maximum among all graphs in \(\mathcal{G}_{n,m}\) containing the triangle \(rst\).
2. If \(G \in \mathcal{G}_{n,m}\) is the locally most reliable graph for \(p\) close to 1, then it must have the largest edge connectivity \(\lambda\). Since \(N_i = \binom{m}{i} (m-\lambda+1 \leq i \leq m)\) and \(N_{m-\lambda} = \binom{m}{\lambda}\) – the number of the \(rst\)-cutsets of size \(\lambda\), the number of the \(rst\)-cutsets with size \(\lambda\) of graph \(G\) must be minimized.

Now, we demonstrate the locally most reliable graph for three-terminal graphs. We first introduce two important graphs for this section, as follows.

Let \(n \geq 7\) and \(2 \leq l \leq n-4\) be positive integers. Using \(A_{n,l}\) denotes the three-terminal graph on \(n\) vertices and \(\binom{n}{2} - l\) edges with vertex set \(V(A_{n,l}) = \{r = v_1, s = v_2, t = v_3, v_4, \ldots, v_n\}\) and edge set \(E(A_{n,l}) = \{v_i v_j | 1 \leq i < j \leq n\} - \{v_4 v_{i+3} | 2 \leq i \leq l+1\}\).
Let \( n \geq 7 \) and \( 2 \leq l \leq \left\lfloor \frac{n-3}{2} \right\rfloor \) be positive integers. Using \( A'_{n,l} \) denotes the three-terminal graph on \( n \) vertices and \((\binom{n}{2}) - l\) edges with vertex set \( V(A'_{n,l}) = \{r = v_1, s = v_2, t = v_3, v_4, \ldots, v_n\} \) and edge set \( E(A'_{n,l}) = \{v_i v_j | 1 \leq i < j \leq n\} - \{v_2 v_2+i | 2 \leq i \leq l + 1\} \).

Figure 5 depicts these two three-terminal graphs with 11 vertices and 51 edges.

![Figure 5: Graph \( A_{11,51} \) (left) and Graph \( A'_{11,51} \) (right)](image)

The red dotted lines indicate the deleted edges.

**Lemma 3.2** \cite{2} Let \( n \geq 1 \) and \( 0 \leq m \leq n - 1 \) be positive integers.

If \( m \neq 3 \), then the unique simple graph on \( n \) vertices and \( m \) edges with the maximum number of \( P_3 \) is \( K_{1,m} \cup \overline{K_{n-m-1}} \).

If \( m = 3 \), there are two simple graphs with the maximum number of \( P_3 \): \( K_3 \cup \overline{K_{n-3}} \) and \( K_{1,3} \cup \overline{K_{n-4}} \).

We give the following Theorem 3.1.

**Theorem 3.1** Let \( n \geq 7 \), \( 2 \leq l \leq n - 4 \) and \( m = \binom{n}{2} - l \) be positive integers. Then

1. If \( l = 3 \), then the graph \( A^{*}_{n,3} = A_{n,3} \cup \{v_4 v_7\} - \{v_5 v_6\} \) is the unique locally most reliable graph in \( G_{n,m} \) for \( p \) close to 0,

2. If \( l \neq 3 \), then the graph \( A_{n,l} \) is the unique locally most reliable graph in \( G_{n,m} \) for \( p \) close to 0.

**Proof.** Suppose that \( n, l \) and \( m \) satisfy the conditions and \( G \) is the locally most reliable graph in \( G_{n,m} \) for \( p \) close to 0 and the vertex set is \( V(G) = \{r = v_1, s = v_2, t = v_3, v_4, \ldots, v_n\} \). Then by Lemma 3.1, \( G \) must contain the triangle \( rst \) and the value of \( N_3 \) must reach the maximum among all graphs in \( G_{n,m} \) containing the triangle \( rst \).
It is no hard to see that \( N_3 = a + b + c + d \), where \( a \) is the number of set \( \{rs, rt, st\} \); \( b \) is the number of sets \( \{rs, st, v_iv_j\}, \{rs, rt, v_iv_j\} \) and \( \{rt, st, v_iv_j\} \) \((1 \leq i \leq n, 4 \leq j \leq n)\), \( c \) is the number of sets \( \{v_iv_j, tv_iv_j \in \{1, 2\}\}; \{v_iv_j, sv_iv_j, rt\} \in \{1, 3\}\}, \{rv_iv_j, sv_iv_j, st\} \in \{2, 3\}\) \((4 \leq i \leq n)\) and \( d \) is the number of set \( \{v_ir, v_is, v_it\} \) \((4 \leq i \leq n)\).

Clearly, for all graphs in \( G_{n,m} \) containing the triangle \( rst \), \( a = 3 \) and \( b = 3(m - 3) \) are constants and \( N_3 \) take the maximum value if and only if \( c \) and \( d \) attains the maximum value. Note that if \( d \) takes the maximum value \( n - 3 \), then the value of \( c \) also reaches its maximum, that is, \( E(G) \) contains the edges \( v_ir, v_is, v_it \) for all \( 4 \leq i \leq n \).

Now, consider the remaining \( \binom{n}{2} - l - 3n - 6 \) edges between non-target vertices in \( G \) that have not been described. Since \( G \) is a dense graph, it is often easier to consider the position of the \( l \) edges deleted between non-target vertices.

By Lemma 3.1, we need to continue to calculate the coefficients \( N_i = b_i + c_i \), where \( b_i \) and \( c_i \) are the number of minimal \( rst \)-subgraphs and non-minimal \( rst \)-subgraphs with \( i \) edges, respectively. We now calculate \( N_4 = b_4 + c_4 \).

Clearly, the value of \( b_4 \) is the sum of the numbers of sets \( \{sv_i, v_it, sv_j, v_jr\}, \{sv_i, v_iv_j, v_jt, sr\} \) and \( \{sv_i, v_iv_j, v_jt, v_jr\} \) and the non-minimal \( rst \)-subgraph with 4 edges includes two parts: the smallest \( rst \)-subgraph is the minimal \( rst \)-subgraph with 2 edges and the smallest \( rst \)-subgraph is the minimal \( rst \)-subgraph with 3 edges. By calculation, \( b_4 = 6^{(n-3)} + 12(m - 3n + 6) + 6(m - 3n - 6) \) and \( c_4 = 3^{(m-3)} + (m - 3) + (n - 3)(m - 3) + 6(n - 3)(m - 6) \) are constants. So, \( N_4 \) is a constant. Then we calculate \( N_5 = b_5 + c_5 \).

Clearly, the value of \( b_5 \) is the sum of the numbers of sets \( \{sv_i, v_iv_j, v_jr, rv_kv_k, v_kt\}, \{sv_i, v_iv_j, v_jk, v_kr, v_kt\} \) and \( \{rv_i, sv_i, v_iv_j, v_jv_k, v_kt\} \) \((4 \leq i, j, k \leq n, i \neq j \neq k)\) and the non-minimal \( rst \)-subgraph with 5 edges includes three parts: the smallest \( rst \)-subgraph is the minimal \( rst \)-subgraph with 2 edges, the smallest \( rst \)-subgraph is the minimal \( rst \)-subgraph with 3 edges and the smallest \( rst \)-subgraph is the minimal \( rst \)-subgraph with 4 edges. By calculation, \( c_5 = 3^{(m-3)} + \binom{m-3}{2} + 7(n - 3)(m - 6) + 3(n - 3)(m - 6) - 12^{(n-3)} + 18(m - 3n + 6)(m - 10) + 6(m - 3n + 6) + 6(m - 9)^{(n-3)} \) is a constant and

\[
b_5 = 12(m - 3n + 6)(n - 5) + 24 \sum_{i=4}^{n} \frac{d(v_i) - 3}{2}
\]
\[
= 12 \sum_{i=4}^{n} (d(v_i) - 3)^2 + 12(m - 3n + 6)(n - 7)
\]
\[
= 24 \sum_{i=4}^{n} \frac{d_i^2}{2} + 24l + 12(n - 3)(n - 4)^2 - 48(n - 4)l + 12(m - 3n + 6)(n - 7),
\]

where \( d(v_i) \) is the degree of the vertex \( v_i \), which is the number of edges associated with vertex.
$v_i$ and $d_i$ is the number of edges deleted on the non-target vertex $v_i$ ($4 \leq i \leq n$).

Note that the value of $\sum_{i=4}^{n} \binom{d_i}{2}$ is the number of subgraphs as $P_3$, whose vertices is the subset of non-target vertices in $G$.

(1) By Lemma 3.2, if $l = 3$, then the number of $P_3$ in a simple graph with $n - 3$ vertices and $l$ edges reaches the maximum if the graph is either $K_3 \cup \overline{K_{n-6}}$ or $K_{1,3} \cup \overline{K_{n-7}}$. So, we get that $N_5(A_{n,3}) = N_5(A_{n,3}^*)$ attains the maximum among the graphs whose $N_i$ ($1 \leq i \leq 4$) satisfy the above calculations, that is, $G$ is either $A_{n,3}$ or $A_{n,3}^*$. So, we need to calculate the value of $N_6(A_{n,3}) - N_6(A_{n,3}^*)$.

Similar to the analysis and solution process of $N_5$, it can be calculated,

$$N_6(A_{n,3}) - N_6(A_{n,3}^*) = 30[N_{A_{n,3}}(P_4) - N_{A_{n,3}^*}(P_4)] + 6[N_{A_{n,3}}(K_1,3) - N_{A_{n,3}^*}(K_1,3)]$$

$$- 66[N_{A_{n,3}}(K_3) - N_{A_{n,3}^*}(K_3)].$$

By calculation, we can get $N_{A_{n,3}}(P_4) = N_{A_{n,3}^*}(P_4)$, $N_{A_{n,3}}(K_1,3) - N_{A_{n,3}^*}(K_1,3) = -1$ and $N_{A_{n,3}}(K_3) - N_{A_{n,3}^*}(K_3) = 1$. So, $N_6(A_{n,3}) - N_6(A_{n,3}^*) = -72 < 0$.

Therefore, if $l = 3$, then the graph $A_{n,l}^*$ is the graph $G$ which is the unique locally most reliable graph in $G_{n,m}$ for $p$ close to 0.

(2) By Lemma 3.2, if $l \neq 3$, then the number of $P_3$ in a simple graph with $n - 3$ vertices and $l$ edges is maximized only if the graph is $K_{1,l} \cup \overline{K_{n-4-l}}$. So, we get that $N_5(A_{n,l})$ attains the maximum among the graphs whose $N_i$ ($1 \leq i \leq 4$) satisfy the above calculations.

Therefore, if $l \neq 3$, then the graph $A_{n,l}$ is the graph $G$ which is the unique locally most reliable graph in $G_{n,m}$ for $p$ close to 0.

Now, we will show that for $\binom{n}{2} - \left\lfloor \frac{n-3}{2} \right\rfloor \leq m \leq \binom{n}{2} - 2$ ($n \geq 7$), $A_{n,l}^*$ is the unique locally most reliable graph in $G_{n,m}$ for $p$ close to 1, as shown in Theorem 3.2. To prove this, we need to give some lemmas first.

**Lemma 3.3** For positive integers $n \geq 7$ and $2 \leq l \leq \left\lfloor \frac{n-3}{2} \right\rfloor$ and every graph $G \in G_{n,m}$ with $m = \binom{n}{2} - l$ edges, let $C$ be a minimal rst-cutset and the component containing $r$ in $G - C$ with $k + 1$ vertices ($0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor$) and if $s$ is not in the component containing $r$ in $G - C$, the component containing $s$ has $k'$ ($1 \leq k' \leq n - k - 1$) vertices. Then

$$n - 1 - l + k(n - k - 2) \leq |C| \leq n - 1 + k(n - k - 2) + k'(n - k - k' - 1)$$

**Proof.** Suppose that $G \in G_{n,m}$ and $C$ satisfy the given hypotheses. Then when all the components obtained by $G - C$ are both complete graphs, the number of edges in the graph $G - C$ is the maximum. There are two cases of components obtained by the graph $G - C$:
Case 1. Obtain two components: one containing \( r \) (or \( rs \) or \( rt \)), and the other containing \( st \) (or \( t \) or \( s \)).

Case 2. Obtain three components containing \( r, s, t \) respectively.

In Case 1, the two components contain \( k+1 \) and \( n-k-1 \) vertices, respectively. Thus, the number of edges in \( G-C \) is \( \binom{k+1}{2} + \binom{n-k-1}{2} \). In Case 2, since the component containing \( s \) has \( k' \) vertices with \( 1 \leq k' \leq n-k-1 \), the number of edges in \( G-C \) is \( \binom{k+1}{2} + \binom{k'}{2} + \binom{n-k-1-k'}{2} \).

By calculation,

\[
\left( \binom{k+1}{2} + \binom{n-k-1}{2} \right) - \left( \binom{k+1}{2} + \binom{k'}{2} + \binom{n-k-1-k'}{2} \right) = k'[(n-k-1)-k'] \geq 0. \ (1 \leq k' \leq n-k-1)
\]

Thus, the maximum number of the edges in \( G-C \) is \( \binom{k+1}{2} + \binom{n-k-1}{2} \). Then

\[
|C| \geq \left( \binom{n}{2} - l - \left( \binom{k+1}{2} + \binom{n-k-1}{2} \right) \right) = n-1-l+k(n-k-2).
\]

On the other hand, the minimum number of the edges in \( G-C \) is \( \binom{k+1}{2} + \binom{k'}{2} + \binom{n-k-1-k'}{2} - l \).

Thus, we have

\[
|C| \leq \left( \binom{n}{2} - l - \left( \binom{k+1}{2} + \binom{k'}{2} + \binom{n-k-1-k'}{2} - l \right) \right) = \left( \binom{n}{2} - \left( \binom{k+1}{2} + \binom{k'}{2} + \binom{n-k-1-k'}{2} \right) \right) = n-1+k(n-k-2)+k'(n-k-k'-1).
\]

The proof is thus complete.

**Lemma 3.4** For positive integers \( n \geq 7 \) and \( 2 \leq l \leq \lfloor \frac{n-3}{2} \rfloor \) and every graph \( G \in G_{n,m} \) with \( m = \binom{n}{2} - l \) edges, the smallest \( rst \)-cutset of \( G \) contains all edges incident with one of the target vertices \( r, s, t \) and the next smallest minimal \( rst \)-cutset of \( G \) will obtain an order 2 components containing either \( r \) or \( s \) or \( t \) when it is removed.

**Proof.** By Lemma 3.3, if \( C \) is a minimal \( rst \)-cutset of \( G \), then the component containing either \( r \) or \( s \) or \( t \) in \( G-C \) has \( k+1 \ (0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor) \) vertices. Without loss of generality, let \( C \) be a minimal \( rst \)-cutset of \( G \) and the component containing \( r \) in \( G-C \) has \( k+1 \ (0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor) \) vertices.
By Lemma 3.3, \( n - 1 - l + k(n - k - 2) \leq |C| \leq n - 1 + k(n - k - 2) + k'(n - k' - 1) \), \( 0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor \). It is easy to see that \( k(n - k - 2) \) increases as \( 0 \leq k \leq \lfloor \frac{n-3}{2} \rfloor \) increases, so, we have

If \( k = 0 \) (that is, \( r \) is the unique vertex in the component that contains \( r \) in \( G - C \)), then \( |C| \leq n - 1 + k'(n - k' - 1) \).

If \( k \geq 1 \), then

\[
|C| \geq n - 1 - l + k(n - k - 2) \\
\geq n - 1 - \frac{n - 3}{2} + 1(n - 1 - 2) \\
\geq 2n - 4 - \frac{n - 3}{2} \\
= \frac{3n - 5}{2}.
\]

Since for \( k = 0 \), \( |C| \leq n - 1 \) when \( k' = 0 \) and \( n - 1 < \frac{3n-5}{2} \) \( (n \geq 7) \), so, when \( k = 0 \), the smallest \( rst \)-cutset of \( G \) can be obtained, that is, the smallest \( rst \)-cutset of \( G \) contains all the edges incident with either \( r \) or \( s \) or \( t \).

If \( k = 1 \) (that is, there exist \( r \) and one other vertex in the component that contains \( r \) in \( G - C \)), then \( |C| \leq 2n - 4 + k'(n - k' - 1) \).

If \( k \geq 2 \), then

\[
|C| \geq n - 1 - l + k(n - k - 2) \\
\geq n - 1 - \frac{n - 3}{2} + 2(n - 2 - 2) \\
\geq 3n - 9 - \frac{n - 3}{2} \\
= \frac{5n - 15}{2}.
\]

Since for \( k = 1 \), \( |C| \leq 2n - 4 \) when \( k' = 0 \) and \( 2n - 4 \leq \frac{5n-15}{2} \) \( (n > 7) \) and no hard to get that the size of the next smallest minimal \( rst \)-cutset of \( G \) for \( k = 1 \) is smaller than for \( k \geq 2 \) \( (n = 7) \), so, when \( k = 1 \), the next smallest minimal \( rst \)-cutset of \( G \) can be obtained, that is, the next smallest minimal \( rst \)-cutset of \( G \) will obtain an order 2 components containing either \( r \) or \( s \) or \( t \) when it is removed.

The proof is now complete.
**Theorem 3.2** Let \( n \geq 7, \ 2 \leq l \leq \lfloor \frac{n-3}{2} \rfloor \) and \( m = \binom{n}{2} - l \) be positive integers. Then \( A_{n,l}' \) is the unique locally most reliable graph in \( G_{n,m} \) for \( p \) close to 1.

**Proof.** Assume that \( n, l \) and \( m \) satisfy the given hypotheses. Let \( G \in G_{n,m} \) be the unique most reliable graph for \( p \) close to 1. Then by Lemma 3.1, \( G \) must have the largest edge connectivity \( \lambda \), that is, the size of the smallest \( rst \)-cutset of \( G \) must be as large as possible. By Lemma 3.4, we get that for \( G, \lambda = d(r) = d(s) = d(t) = n - 1 \). There are many graphs satisfying this condition, so, the size of the next smallest minimal \( rst \)-cutset of \( G \) also must be as large as possible with \( \lambda = n - 1 \).

By Lemma 3.4, the next smallest minimal \( rst \)-cutset leaves \( r \) and one other vertex \( v \) in a component, whose size is \( n - 3 + d(v) \). Thus, for each \( v \in V(G) - \{r\}, d(v) \geq n - 2 \).

Therefore, \( A_{n,l}' \) is the unique locally most reliable graph in \( G_{n,m} \) for \( p \) close to 1.

As a straightforward consequence of Theorems 3.1 and 3.2, we obtain the following Theorem 3.3.

**Theorem 3.3** Let \( n \geq 7, \ 2 \leq l \leq \lfloor \frac{n-3}{2} \rfloor \) be positive integers. If \( m = \binom{n}{2} - l \), then there is no uniformly most reliable three-terminal graph in \( G_{n,m} \).

**Remark 3.1** For \( n = 4 \) and \( m = \binom{4}{2} - 2 = 4 \), there is a uniformly most reliable three-terminal graph in \( G_{4,4} \) (see Example 1). For \( n = 5 \) or \( 6 \), \( m = \binom{5}{2} - 2 \), there is also a uniformly most reliable three-terminal graph in \( G_{n,m} \) (see Appendix A).

### 4 A uniformly most reliable three-terminal graph

For the three-terminal graph with \( m = \binom{n}{2} \) edges, there is only one graph, thus, it is easy to see that it is the uniformly most reliable graph in \( G_{n,\binom{n}{2}} \). In this section, we determine a uniformly most reliable graph in \( G_{n,m} \) with \( m = \binom{n}{2} - 1 \) edges. First, we introduce three graphs used in the following Theorem.

Clearly, when we remove one edge, there are only three distinct cases: the edge between target vertices; the edge between a target vertex and a non-target vertex; the edge between non-target vertices. Let \( n \geq 5 \) and \( m = \binom{n}{2} - 1 \) be positive integers.

1. Using \( X_n \) denotes the three-terminal graph on \( n \) vertices and \( m \) edges with vertex set \( V(X_n) = \{r = x_1, s = x_2, t = x_3, x_4, \ldots, x_n\} \) and edge set \( E(X_n) = \{x_ix_j | 1 \leq i < j \leq n\} \) - \{rs\}. 

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(2) Using $Y_n$ denotes the three-terminal graph on $n$ vertices and $m$ edges with vertex set $V(Y_n) = \{r = y_1, s = y_2, t = y_3, y_4, \ldots, y_n\}$ and edge set $E(Y_n) = \{y_iy_j|1 \leq i < j \leq n\} - \{ry_4\}.$

(3) Using $Z_n$ denotes the three-terminal graph on $n$ vertices and $m$ edges with vertex set $V(Z_n) = \{r = z_1, s = z_2, t = z_3, z_4, \ldots, z_n\}$ and edge set $E(Z_n) = \{z_iz_j|1 \leq i < j \leq n\} - \{z_4z_5\}.$

Now, we can give a uniformly most reliable graph in $G_{n,m}$ for $n \geq 5$ and $m = \binom{n}{2} - 1$, as shown in Theorem 4.1.

**Theorem 4.1** Let $n \geq 5$ and $m = \binom{n}{2} - 1$ be positive integers. Then $Z_n$ is the unique uniformly most reliable graph in $G_{n,m}.$

**Proof.** To prove this theorem, we will prove that there are more $rst$-subgraphs with $i$ ($2 \leq i \leq \binom{n}{2} - 1$) edges in $Z_n$ than in $X_n$ and $Y_n$.

We complete this proof by construct two injective maps $f_X$ and $f_Y$, from the $rst$-subgraphs with $i$ edges in $X_n$ and $Y_n$ to the $rst$-subgraphs with $i$ edges in $Z_n$, respectively.

**Construct the map $f_X$:**

Let $S$ be a $rst$-subgraph with $i$ edges in $X_n$, where $2 \leq i \leq \binom{n}{2} - 1$.

Case 1. If $S$ does not contain the edge $x_4x_5$, then $f_X(S) = \{z_iz_j|x_ix_j \in S\}$.

The image is a $rst$-subgraph of $Z_n$ with the same number of edges as $S$. And this image does not contain the edge $rs$.

Case 2. Assume that $S$ contains the edge $x_4x_5$.

Case 2.1. If $S - \{x_4x_5\}$ is still a $rst$-subgraph, then $f_X(S) = \{z_iz_j|x_ix_j \in S\} \cup \{rs\} - \{z_4z_5\}$.

The image is a $rst$-subgraph of $Z_n$ with the same number of edges as $S$. Since this image contains the edge $rs$, it is distinct from Case 1. And $f_X(S) - \{rs\}$ is still a $rst$-subgraph.

Case 2.2. If $S - \{x_4x_5\}$ is not a $rst$-subgraph, but a $rt$-subgraph or a $st$-subgraph, then $f_X(S) = \{z_iz_j|x_ix_j \in S\} \cup \{rs\} - \{z_4z_5\}$.

The image is a $rst$-subgraph of $Z_n$ with the same number of edges as $S$. Since the image contains $rs$ and $f_X(S) - \{rs\}$ is not a $rst$-subgraph, it is distinct from the above cases. It is clear to see that in $f_X(S)$, it contains either the edge $st$ and an edge $rz_i$ for some $4 \leq i \leq n$ or the edge $rt$ and an edge $sz_j$ for some $4 \leq j \leq n$ or an edge $rz_i$ and an edge $sz_j$ for some $4 \leq i, j \leq n$. 13
Case 2.3. Assume that $S - \{x_4x_5\}$ is neither a $rst$-subgraph, nor a $rt$-subgraph, nor a $st$-subgraph.

It is easy to see that for this case, all $rst$-subgraph and all $rt$-subgraph and all $st$-subgraph in $S$ contains the edge $x_4x_5$. Thus, the image of the map defined by the above cases is not a $rst$-subgraph of $Z_n$. Let $S'$ be a minimal $rst$-subgraph in $S$. Then $S'$ consists of a minimal $rsx_4$-subgraph, the edge $x_4x_5$ and a minimal $x_5t$-subgraph.

Case 2.3.1. If $S'$ consists of an edge $sx_j$, a minimal $x_4x_j$-subgraph, the edge $rx_4$, the edge $x_4x_5$ and a minimal $x_5t$-subgraph, then $f_X(S) = \{z_i|z_j \in S\} \cup \{z_5z_j|sx_j \in S\} \cup \{rs\} - \{z_4z_5\}$.

According to the condition of $S$, $S$ does not have both edges $x_5x_j$ and $sx_j$, otherwise, an edge $x_5x_j$, an edge $sx_j$ and a $x_5t$-subgraph will get a $st$-subgraph that does not contain the edge $x_4x_5$. Therefore, $f_X(S)$ has the same size as $S$. In $f_X(S)$, we have a $rst$-subgraph of $Z_n$ which consists of an edge $z_5z_j$, a $x_5z_j$-subgraph, the edge $rz_4$, a $z_5t$-subgraph and the edge $rs$. Thus, the image of the map defined by the above cases is not a $rst$-subgraph and it does not contain any edge $sz_j$ $(3 \leq j \leq n)$, it is distinct from the above cases. In $f_X(S)$, it contains the edge $rz_4$.

Case 2.3.2. If $S'$ consists of an edge $rx_j$, a minimal $sx_4x_j$-subgraph, the edge $x_4x_5$ and a minimal $x_5t$-subgraph, then $f_X(S) = \{z_i|z_j \in S\} \cup \{z_5z_j|rx_j \in S\} \cup \{rs\} - \{z_4z_5\}$.

Similarly, $S$ does not have both edges $x_5x_j$ and $rx_j$. Therefore, $f_X(S)$ has the same size as $S$. In $f_X(S)$, we have a $rst$-subgraph of $Z_n$ which consists of an edge $z_5z_j$, a $z_4z_j$-subgraph, a $z_5t$-subgraph and the edge $rs$. Since $f_X(S)$ contains the edge $rz_4$ and $f_X(S) - \{rs\}$ is not a $rst$-subgraph and it contains an edge $sz_j$ for some $4 \leq j \leq n$ and does not contain any edge $z_i$ $(3 \leq i \leq n)$, it is distinct from the above cases.

Case 2.3.3. If $S'$ consists of the edge $rx_4$, the edge $sx_4$, the edge $x_4x_5$ and a minimal $x_5t$-subgraph, then $f_X(S) = \{z_i|z_j \in S\} \cup \{rz_5|sx_4 \in S\} \cup \{rs\} - \{z_4z_5\}$.

It is easy to see that $f_X(S)$ has the same size as $S$. In $f_X(S)$, we have a $rst$-subgraph of $Z_n$ consists of the edge $rz_4$, the edge $rz_5$, a $z_5t$-subgraph and the edge $rs$. Since the image has no inverse image in the above mappings, it is distinct from the above cases.

Therefore, all of these mappings are different. Since the map $f_X(S)$ defined on each of these cases of $Z_n$ as disjoint images, the map is injective.

Because there are at least as many $rst$-subgraphs with $i$ edges in $Z_n$ as in $X_n$ for $2 \leq i \leq \binom{9}{2} - 1$, $Z_n$ is more reliable than $X_n$ for all $p$ $(0 \leq p \leq 1)$.

Construct the map $f_Y$:
Let $S$ be a $rst$-subgraph with $i$ edges in $Y_n$, where $2 \leq i \leq \binom{n}{2} - 1$.

Case 1. If $S$ does not contain the edge $y_4y_5$, then $f_Y(S) = \{z_iz_j | y_iy_j \in S\}$.

The image is a $rst$-subgraph of $Z_n$ with the same number of edges as $S$. And this image does not contain the edge $rz_4$.

Case 2. Assume that $S$ contains the edge $y_4y_5$.

Case 2.1. If $S - \{y_4y_5\}$ is still a $rst$-subgraph, then $f_Y(S) = \{z_iz_j | y_iy_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$.

The image is a $rst$-subgraph of $Z_n$ with the same number of edges as $S$. Since the image contains the edge $rz_4$, it is distinct from Case 1. And $f_Y(S) - \{rz_4\}$ is still a $rst$-subgraph.

Case 2.2. If $S - \{y_4y_5\}$ is not a $rst$-subgraph, but a $rsy_5$-subgraph and a $y_4t$-subgraph, or a $ry_5$-subgraph and a $sty_4$-subgraph, or a $rtys$-subgraph and a $sy_4$-subgraph, then $f_Y(S) = \{z_iz_j | y_iy_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$.

The image is a $rst$-subgraph of $Z_n$ with the same number of edges as $S$. Since the image contains $rz_4$ and $f_Y(S) - \{rz_4\}$ is not a $rst$-subgraph, it is distinct from the above cases. Since $S$ contains an edge $ry_j$ for some $2 \leq j \leq n$ and $n \neq 4$, the image also contains an edge $rz_j$ for some $2 \leq j \leq n$ and $j \neq 4$.

Case 2.3. Assume that $S - \{y_4y_5\}$ does not satisfy all of the following four cases: a $rst$-subgraph; a $rsy_5$-subgraph and a $y_4t$-subgraph; a $ry_5$-subgraph and a $sty_4$-subgraph; a $rtys$-subgraph and a $sy_4$-subgraph.

It is easy to see that for this case, all $rst$-subgraph in $S$ contains the edge $y_4y_5$. And $S - \{y_4y_5\}$ is either a $rsy_4$-subgraph and a $y_4t$-subgraph, or a $rtys$-subgraph and a $sy_4$-subgraph, or a $y_4t$-subgraph and a $sty_4$-subgraph. Therefore, the image of the map defined for the above cases is not a $rst$-subgraph of $Z_n$. Let $S'$ be a minimal $rst$-subgraph in $S$.

Case 2.3.1. If $S'$ consists of an edge $ry_j$, a minimal $sy_4y_j$-subgraph, the edge $y_4y_5$ and a minimal $y_4t$-subgraph, then $f_Y(S) = \{z_iz_j | y_iy_j \in S\} \cup \{z_5z_j | ry_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$.

According to the condition of $S$, $S$ does not have both edges $y_5y_j$ and $ry_j$. Because if $S$ contains both edges $y_5y_j$ and $ry_j$, then an edge $y_5y_j$, an edge $ry_j$, a $sy_4y_j$-subgraph and a $y_5t$-subgraph will get a $rst$-subgraph that does not contain the edge $y_4y_5$. Therefore, $f_Y(S)$ has the same size as $S$. In $f_Y(S)$, we have a $rst$-subgraph of $Z_n$ which consists of an edge $z_5z_j$, a $sz_4z_j$-subgraph, a $szt$-subgraph and the edge $rz_4$. Since $f_Y(S)$ contains the edge $rz_4$ and $f_Y(S) - \{rz_4\}$ is not a $rst$-subgraph and it does not contain any edge $rz_j (2 \leq j \leq n, j \neq 4)$, it is distinct from the above cases. In $f_Y(S)$, any $rs$-subgraph does not contain $z_5$.

Case 2.3.2. If $S'$ consists of an edge $ry_j$, a minimal $ty_4y_j$-subgraph, the edge $y_4y_5$ and a
minimal $sy_5$-subgraph, then $f_Y(S) = \{z_iz_j|yi,j \in S\} \cup \{z_5z_j|ry_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$.

Similarly, $S$ does not have both edges $y_5y_j$ and $ty_j$. Therefore, $f_Y(S)$ has the same size as $S$. In $f_Y(S)$, we have a $rst$-subgraph of $Z_n$ which consists of an edge $z_5z_j$, a $tz_4z_j$-subgraph, a $sz_5$-subgraph and the edge $rz_4$. Since $f_Y(S)$ contains the edge $rz_4$ and $f_Y(S) - \{rz_4\}$ is not a $rst$-subgraph and it does not contain any edge $rz_j$ ($2 \leq j \leq n, j \neq 4$) and all $rs$-subgraph in $f_Y(S)$ contain $z_5$, it is distinct from the above cases. In $f_Y(S)$, any $rt$-subgraph does not contain $z_5$.

Case 2.3.3. If $S'$ consists of an edge $ry_j$, a minimal $y_4y_j$-subgraph, the edge $y_4y_5$ and a minimal $styz_5$-subgraph, then $f_Y(S) = \{z_iz_j|yi,j \in S\} \cup \{z_5z_j|ry_j \in S\} \cup \{rz_4\} - \{z_4z_5\}$.

Similarly, $S$ does not have both edges $y_5y_j$ and $ry_j$. Therefore, $f_Y(S)$ has the same size as $S$. In $f_Y(S)$, we have a $rst$-subgraph of $Z_n$ which consists of an edge $z_5z_j$, a $z_4z_j$-subgraph, a $styz_5$-subgraph and the edge $rz_4$. Since $f_Y(S)$ contains the edge $rz_4$ and $f_Y(S) - \{rz_4\}$ is not a $rst$-subgraph and it does not contain any edge $rz_j$ ($2 \leq j \leq n, j \neq 4$) and all $rs$-subgraphs and $rt$-subgraphs in $f_Y(S)$ contain $z_5$, it is distinct from the above cases.

Therefore, all of these mappings are different. Since the map $f_Y(S)$ defined on each of these cases of $Z_n$ as disjoint images, the map is injective.

Because there are at least as many $rst$-subgraphs with $i$ edges in $Z_n$ as in $Y_n$ for $2 \leq i \leq \binom{n}{2} - 1$, $Z_n$ is more reliable than $Y_n$ for all $p$ ($0 \leq p \leq 1$).

From the above argument, we conclude that the graph $Z_n$ is the unique most reliable graph in $G_{n,m}$ for all $p$ ($0 \leq p \leq 1$).

5 Conclusion

This research focuses on determining the existence of the uniformly most reliable graph for three-terminal graphs with number of edges in a given large range. If there is a uniformly most reliable graph, the uniformly most reliable graph is given; if there is no uniformly most reliable graph, the locally most reliable graphs are given. Based on the results of this research, the following conclusions can be drawn.

- When the number of vertices is $n = 4$ or $5$ or $6$ and the number of edges is $m = \binom{n}{2} - 2$, the uniformly most reliable graph is determined with comparisons in Example 1 and Appendix A.

- Under the conditions of $n \geq 7$ and $\binom{n}{2} - (n - 4) \leq m \leq \binom{n}{2} - 2$, the locally most reliable graph in $G_{n,m}$ for $p$ close to $0$ is determined with proofs and for $\binom{n}{2} - \lfloor \frac{n-3}{2} \rfloor \leq m \leq \binom{n}{2} - 2$, the locally most reliable graph in $G_{n,m}$ for $p$ close to $1$ is determined with proofs.
the locally most reliable graph in $\mathcal{G}_{n,m}$ for $p$ close to 1 is also determined with proofs. Then it shows that there is no uniformly most reliable three-terminal graph for $(n\choose 2) - \lfloor \frac{n-3}{2} \rfloor \leq m \leq (n\choose 2) - 2$. It is worth considering whether there is a uniformly most reliable graph in the class of three-terminal graphs which delete more edges.

• With a complex proof, the uniformly most reliable graph in $\mathcal{G}_{n,(n\choose 2)-1}$ is determined, which is a graph with $(n\choose 2)$ edges that removes an edge between non-target vertices. This conclusion is significant in comparison with the conclusion given by Bertrand et al. [2], which states that the uniformly most reliable graph with $(n\choose 2) - 1$ edges for two-terminal graphs is also a graph with $(n\choose 2)$ edges that removes an edge between non-target vertices. By these comparison, for $m = (n\choose 2) - 1$, it is most probably that the uniformly most reliable graph for $k$-terminal graphs is a graph with $(n\choose 2)$ edges that removes an edge between non-target vertices.

The results of the research provide guiding significance for characterizing and determining the uniformly most reliable graphs or the locally most reliable graphs of general $k$-terminal networks. In fact, the results of the research can be useful for designing highly reliable networks with three key vertices (target vertices).

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Appendix A  Reliability polynomials for three-terminal graphs
with \( n < 7 \) vertices and \( m = \binom{n}{2} - 2 \) edges

All three-terminal graphs with 5 vertices and 8 edges

Calculated by Matlab, the reliable polynomials of these graphs are:

\[
R_3(G_1; p) = 3p^2(1-p)^6 + 25p^3(1-p)^5 + 60p^4(1-p)^4 + 55p^5(1-p)^3 + 28p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_2; p) = 3p^2(1-p)^6 + 23p^3(1-p)^5 + 57p^4(1-p)^4 + 54p^5(1-p)^3 + 28p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_3; p) = 3p^2(1-p)^6 + 20p^3(1-p)^5 + 51p^4(1-p)^4 + 50p^5(1-p)^3 + 27p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_4; p) = 3p^2(1-p)^6 + 20p^3(1-p)^5 + 56p^4(1-p)^4 + 54p^5(1-p)^3 + 28p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_5; p) = p^2(1-p)^6 + 16p^3(1-p)^5 + 55p^4(1-p)^4 + 54p^5(1-p)^3 + 28p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_6; p) = p^2(1-p)^6 + 12p^3(1-p)^5 + 46p^4(1-p)^4 + 49p^5(1-p)^3 + 27p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_7; p) = p^2(1-p)^6 + 13p^3(1-p)^5 + 51p^4(1-p)^4 + 53p^5(1-p)^3 + 28p^6(1-p)^2 + 8p^7(1-p) + p^8;
\]
\[
R_3(G_8; p) = 6p^3(1-p)^5 + 42p^4(1-p)^4 + 48p^5(1-p)^3 + 27p^6(1-p)^2 + 8p^7(1-p) + p^8.
\]

It is clear to see that, \( G_1 \) is the uniformly most reliable graph in \( G_{5,8} \).

All three-terminal graphs with 6 vertices and 13 edges

Calculated by Matlab, the reliable polynomials of these graphs are:
Figure 7: All simple three-terminal graph in $G_{6,13}$ with three target vertices $r, s, t$.

$$R_3(G_1; p) = 3p^2(1-p)^{11} + 52p^3(1-p)^{10} + 337p^4(1-p)^9 + 1017p^5(1-p)^8 + 1605p^6(1-p)^7 + 1689p^7(1-p)^6 + 1284p^8(1-p)^5 + 715p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_2; p) = 3p^2(1-p)^{11} + 47p^3(1-p)^{10} + 304p^4(1-p)^9 + 955p^5(1-p)^8 + 1550p^6(1-p)^7 + 1661p^7(1-p)^6 + 1276p^8(1-p)^5 + 714p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_3; p) = 3p^2(1-p)^{11} + 47p^3(1-p)^{10} + 297p^4(1-p)^9 + 953p^5(1-p)^8 + 1552p^6(1-p)^7 + 1662p^7(1-p)^6 + 1276p^8(1-p)^5 + 714p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_4; p) = 3p^2(1-p)^{11} + 45p^3(1-p)^{10} + 283p^4(1-p)^9 + 907p^5(1-p)^8 + 1501p^6(1-p)^7 + 1634p^7(1-p)^6 + 1268p^8(1-p)^5 + 713p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_5; p) = 3p^2(1-p)^{11} + 42p^3(1-p)^{10} + 259p^4(1-p)^9 + 849p^5(1-p)^8 + 1428p^6(1-p)^7 + 1577p^7(1-p)^6 + 1240p^8(1-p)^5 + 705p^9(1-p)^4 + 285p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_6; p) = 3p^2(1-p)^{11} + 42p^3(1-p)^{10} + 264p^4(1-p)^9 + 889p^5(1-p)^8 + 1494p^6(1-p)^7 + 1633p^7(1-p)^6 + 1268p^8(1-p)^5 + 713p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_7; p) = p^2(1-p)^{11} + 26p^3(1-p)^{10} + 227p^4(1-p)^9 + 863p^5(1-p)^8 + 1486p^6(1-p)^7 + 1632p^7(1-p)^6 + 1268p^8(1-p)^5 + 713p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_8; p) = p^2(1-p)^{11} + 22p^3(1-p)^{10} + 188p^4(1-p)^9 + 761p^5(1-p)^8 + 1365p^6(1-p)^7 + 1548p^7(1-p)^6 + 1232p^8(1-p)^5 + 704p^9(1-p)^4 + 285p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_9; p) = p^2(1-p)^{11} + 23p^3(1-p)^{10} + 199p^4(1-p)^9 + 805p^5(1-p)^8 + 1432p^6(1-p)^7 + 1604p^7(1-p)^6 + 1260p^8(1-p)^5 + 712p^9(1-p)^4 + 286p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13};$$

$$R_3(G_{10}; p) = 9p^3(1-p)^{10} + 132p^4(1-p)^9 + 687p^5(1-p)^8 + 1308p^6(1-p)^7 + 1520p^7(1-p)^6 + 1224p^8(1-p)^5 + 703p^9(1-p)^4 + 285p^{10}(1-p)^3 + 78p^{11}(1-p)^2 + 13p^{12}(1-p) + p^{13}. $$

It is clear to see that, $G_1$ is the uniformly most reliable graph in $G_{6,13}$. 

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