Minuscule Representations, Invariant Polynomials, and Spectral Covers

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Introduction

Let $G$ be a simple and simply connected complex linear algebraic group, with Lie algebra $\mathfrak{g}$. Let $\rho: G \to \text{Aut} V$ be an irreducible finite-dimensional representation of $G$, and let $\rho_*: \mathfrak{g} \to \text{End} V$ be the induced representation of $\mathfrak{g}$. A goal of this paper is to study $\rho_*$ and $\rho$, and in particular to give normal forms for the action of $\rho_*(X)$ and $\rho(g)$ for regular elements $X$ of $\mathfrak{g}$ or regular elements $g \in G$. Of course, in the case of $\rho_*$, when $X$ is semisimple, the action of $\rho_*(X)$ on $V$ can be diagonalized, and its eigenvalues are given by evaluating the weights of $\rho$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ on an element in $\mathfrak{h}$ conjugate to $X$. If on the other hand $X$ is a principal nilpotent element of $\mathfrak{g}$, then $X$ can be completed to an $\mathfrak{sl}_2$-triple $(X, h_0, X_-)$, where $h_0$ is regular and semisimple. In this case, if $\mathfrak{h}$ is the Cartan subalgebra containing $h_0$, then the eigenvalues for $\rho_*(h_0)$ together with their multiplicities, which are given by evaluating the weights of $\rho$ with respect to $\mathfrak{h}$ on $h_0$, completely determine $V$ as an $\mathfrak{sl}_2$-module and hence determine the action of $\rho_*(X)$ on $V$. A minor modification of these ideas will then describe the action of $\rho_*(X)$ for every regular element $X$.

In this paper, we attempt to give a different algebraic model for the action of $\rho_*(X)$, where $X$ is regular, and to glue these different models together over the set of all regular elements. Related methods also handle the case of $\rho(g)$, where $g$ is a regular element of $G$. While we are only successful in case $\rho$ is minuscule, and partially successful in case $\rho$ is quasiminuscule, it seems likely that the techniques of this paper can be extended to give information about an arbitrary $\rho$. We believe that the techniques and results of this paper are also of independent interest. For example, they have been applied elsewhere to study sections of the adjoint quotient morphism [7].
To explain our results in more detail in the case of \( \rho \), we begin by recalling some results of Kostant on the adjoint quotient of \( g \). Let \( \mathfrak{h} \) be the Lie algebra of a Cartan subgroup \( H \) of \( G \) and let \( W \) be the Weyl group of the pair \((G,H)\). Kostant has shown [11] that the GIT quotient of \( g \) by the adjoint action of \( G \) is isomorphic to \( \mathfrak{h}/W \). Moreover, he has constructed an explicit cross-section \( \Sigma \) of the natural morphism \( g \to \mathfrak{h}/W \), such that the image of \( \Sigma \) is contained in the dense open subset \( g_{\text{reg}} \) of regular elements of \( g \).

Given the Cartan subgroup \( H \), there are the corresponding weights of \( \rho \). Fixing a choice of a set of positive roots for the pair \((G,H)\), we shall let \( \mu \) be the lowest weight of \( \rho \). Let \( W_0 \) be the stabilizer of \( \mu \) in \( W \). Recall that the representation \( \rho \) is minuscule if all of its weights are conjugate under \( W \), so that the set of all weights of \( \rho \) is identified with \( W/W_0 \). In this case, every weight has multiplicity one. The representation \( \rho \) is quasiminuscule if all of its nonzero weights are conjugate under \( W \). In this case, every nonzero weight has multiplicity one.

Let \( S = \text{Sym}^* \mathfrak{h}^* \) be the affine coordinate ring of \( \mathfrak{h} \), so that \( S \) is a polynomial ring. We have the invariant subrings \( S^W \) and \( S^{W_0} \). The inclusions \( S^W \to S^{W_0} \to S \) correspond to the finite morphisms \( \mathfrak{h} \to \mathfrak{h}/W_0 \to \mathfrak{h}/W \). By Chevalley’s theorem [2, p. 107, Thm. 3], \( S^W \) and \( S^{W_0} \) are polynomial algebras, i.e. \( \mathfrak{h}/W \) and \( \mathfrak{h}/W_0 \) are isomorphic to affine space. Hence \( S^{W_0} \) is flat over \( S^W \) and is therefore a free \( S^W \)-module of rank equal to \( \#(W/W_0) \). Note that \( \mu \) is naturally an element of \( S^{W_0} \) and can be completed to a polynomial basis of \( S^{W_0} \). The morphism \( \Sigma: \mathfrak{h}/W \to g \) induces a morphism \( \rho, \Sigma: \mathfrak{h}/W = \text{Spec} S^W \to \text{End} V \), and is thus identified with an element of

\[
(\text{End} V) \otimes_{\mathbb{C}} S^W \cong \text{Hom}_{S^W} (V \otimes_{\mathbb{C}} S^W, V \otimes_{\mathbb{C}} S^W),
\]

which we also denote by \( \rho, \Sigma \). With this said, we have the following:

**Theorem 1.** Let \( \rho: G \to GL(V) \) be an irreducible finite-dimensional representation with lowest weight \( \mu \). Then there is a nonzero linear map \( f: V \to S^{W_0} \), unique up to a nonzero scalar, with the following two properties:

(i) Let \( \hat{f}: V \otimes_{\mathbb{C}} S^W \to S^{W_0} \) be the homomorphism defined by \( f \) via extension of scalars. Then the following diagram commutes:

\[
\begin{array}{ccc}
V \otimes_{\mathbb{C}} S^W & \xrightarrow{\hat{f}} & S^{W_0} \\
\rho, \Sigma \downarrow & & \downarrow \mu \\
V \otimes_{\mathbb{C}} S^W & \xrightarrow{\hat{f}} & S^{W_0},
\end{array}
\]

(ii) If \( g: V \to S^{W_0} \) is any other linear map such that the extension of \( g \) to \( \hat{g}: V \otimes_{\mathbb{C}} S^W \to S^{W_0} \) makes the above diagram commute, then \( g = sf \) for some \( s \in S^{W_0} \).

If \( \rho \) is minuscule, then \( \hat{f} \) is an isomorphism. If \( \rho \) is quasiminuscule, then \( \hat{f} \) is surjective, and \( \text{Ker} \hat{f} = \text{Ker} \rho, \Sigma \).

In fact, we give an explicit description of the map \( f \) in Section 3.

Using very different methods, and based on his results in [9], Victor Ginzburg has sent us the sketch of a proof that \( f \) is surjective for every irreducible representation \( \rho \) (cf. also [10]).

There are two different ways we can interpret Theorem 1. In case \( \rho \) is minuscule, it describes, up to conjugation, the action of every regular element of \( g \) on \( V \), or...
equivalently the action of $\rho_*\Sigma(x)$ for every $x \in \mathfrak{h}/W$, in terms of the algebra structures on $S^W$ and $S^{W_0}$.

**Theorem 2.** Suppose that $\rho$ is minuscule. Let $x \in \mathfrak{h}/W$ and let $\mathfrak{m}_x \subseteq S^W$ be the maximal ideal of $x$, so that the scheme-theoretic fiber over $x$ in $\mathfrak{h}/W_0$ has coordinate ring equal to $S^{W_0}/(\mathfrak{m}_x \cdot S^{W_0})$. Then the map $\tilde{f}$ induces an isomorphism $\tilde{f}_x: V \to S^{W_0}/(\mathfrak{m}_x \cdot S^{W_0})$. Under this isomorphism, multiplication by $\mu$ on the right-hand side becomes the action of $\rho_*(\Sigma(x))$. In particular, for the ideal $\mathfrak{m}_0 \cdot S^{W_0}$ generated by the homogeneous $W$-invariant functions of positive degree, there is an isomorphism $\tilde{f}_0: V \to S^{W_0}/\mathfrak{m}_0 \cdot S^{W_0}$ such that multiplication by $\mu \in S^{W_0}$ on $S^{W_0}/\mathfrak{m}_0 \cdot S^{W_0}$ corresponds to the action $\rho_*(X)$ of a principal nilpotent element $X$ on $V$.

Of course, the ring $S^{W_0}$ is just a polynomial ring and $\mu$ can be chosen to be a coordinate function, so that the action of $\mu$ on $S^{W_0}$ is easy to understand. However, the finite dimensional $\mathbb{C}$-algebras $S^{W_0}/\mathfrak{m}_x \cdot S^{W_0}$ and the action of $\mu$ on them are much more complicated. Thus for example if $x = 0$ the theorem tells us that the nilpotent element $\mu$ of $S^{W_0}/\mathfrak{m}_0 \cdot S^{W_0}$ has Jordan blocks which are the same as those given by the action of a principal nilpotent element $X$ on $V$.

There is a partial analogue of Theorem 2 in the quasiminuscule, non-minuscule case. In this case, the action of $\rho_*\Sigma$ on $V \otimes_S S^W$ corresponds to the action of the matrix

$$
\begin{pmatrix}
0 & * \\
0 & \mu
\end{pmatrix},
$$

where we view $\mu$ as acting in block form on the free $S^W$-module $S^{W_0}$. Here, for any given $x \in \mathfrak{h}/W$, the off-diagonal term $*$ can in principle be determined. We discuss this off-diagonal term in Section 5 for the adjoint representation of a simply laced Lie algebra.

The second interpretation of the surjection $\tilde{f}: V \otimes_S S^W \to S^{W_0}$ in the minuscule or quasiminuscule case is that it describes relations between invariant polynomials for $W$ and $W_0$. In particular, an explicit description of $f$ leads to a set of generators for $S^{W_0}$ as an $S^W$-module. We give examples of such relations in the fourth section.

One basic ingredient in the proof of Theorem 1 is a result concerning the Kostant section $\Sigma$ which (in a slightly different form) was proved by Kostant in [12]. In general, the image of $\Sigma$ is complicated. However, if we make the base change $\mathfrak{h} \to \mathfrak{h}/W$, then, up to the adjoint action of a morphism $\mathfrak{h} \to G$, the structure of $\Sigma$ simplifies considerably:

**Theorem 3.** Let $X$ be a principal nilpotent element of $\mathfrak{g}$ such that $\mathfrak{h}$ is contained in the unique Borel subalgebra containing $X$, and let $\Sigma$ be the composition of $\Sigma$ with the natural projection $\mathfrak{h} \to \mathfrak{h}/W$. Then there exists a morphism $\lambda: \mathfrak{h} \to G$ such that, for all $h \in \mathfrak{h}$,

$$
\text{Ad}(\lambda(h))(\Sigma(h)) = h + X.
$$

In other words, after making the base change $\mathfrak{h} \to \mathfrak{h}/W$, we can conjugate the section $\Sigma$ via the morphism $\lambda$ into the much simpler morphism from $\mathfrak{h}$ to $\mathfrak{g}$ defined by $h \mapsto h + X$, where $X$ is a fixed principal nilpotent in a Borel subalgebra normalized by $\mathfrak{h}$. Using the explicit description of the Kostant section, we are able to give a very explicit construction of the morphism $\lambda$.

Steinberg has defined an analogous cross-section for the quotient for the adjoint action of $G$ on itself, which is a morphism $\overline{\Sigma}: G \to H/W$ [15]. We prove the
following result concerning $\Phi$, which is analogous to Theorem 3 but somewhat weaker:

**Theorem 4.** Let $B$ be a Borel subgroup containing $H$ with unipotent radical $U$, and let $\Phi$ be the composition of $\overline{\Phi}$ with the natural projection $H \to H/W$. Then there exists a morphism $\phi: H \to G$ and a morphism $u: H \to U$ such that, for all $h \in H$, \[
\phi(h)\Phi(h)\phi(h)^{-1} = hu(h).
\]

In contrast to the proof of Theorem 3, the argument does not use any special properties of Steinberg’s construction and would apply to any section of the adjoint quotient morphism. In particular, the morphism $u$ is not given explicitly. It is natural to conjecture that, for an appropriate choice of $\phi$ in the theorem, we can take $u$ to be a constant, in other words that the pullback of the Steinberg section to $H$ is conjugate via a morphism from $H$ to $G$ to the morphism $h \mapsto hu$ for a certain principal unipotent element in a Borel subgroup normalized by $H$. Using Theorem 3, the proofs of Theorem 1 and hence of Theorem 2 go over to the case of the Steinberg section, and to the action of $\rho(g)$ on $V$, for a regular element $g$ in $G$ and a minuscule or quasiminuscule representation $\rho$. In particular, if $S$ is the affine coordinate ring of $H$, then there is a morphism $\tilde{\gamma}: V \otimes_{\mathbb{C}} S^W \to S^W_0$ which intertwines the action of $\rho(\overline{\Phi})$ and multiplication by $\mu$, and $\tilde{\gamma}$ is an isomorphism in case $\rho$ is minuscule and surjective in case $\rho$ is quasiminuscule. It is natural to conjecture that $\tilde{\gamma}$ is always surjective; this does not seem to follow directly from the result of Ginzburg in the Lie algebra case.

Our strategy in the proof of Theorem 1 is to relate the problem to a question concerning holomorphic $G$-bundles on a cuspidal curve. More generally, let $E$ be a Weierstrass cubic curve, i.e. a reduced irreducible curve of arithmetic genus one. Then $E$ is either smooth, isomorphic to a nodal cubic curve in $\mathbb{P}^2$, or isomorphic to a cuspidal cubic curve in $\mathbb{P}^2$. Let $E_{\text{reg}}$ be the Zariski open subset of smooth points. Fixing a base point $p_0 \in E_{\text{reg}}$ identifies $E_{\text{reg}}$ with $\text{Pic}^0 E$ and endows $E_{\text{reg}}$ with the structure of a one-dimensional connected, commutative algebraic group, which is an elliptic curve ($E$ smooth), or is isomorphic to the multiplicative group $\mathbb{G}_m$ (E nodal) or to the additive group $\mathbb{C}$ ($E$ cuspidal). We assume now that $E$ is singular and denote its normalization by $\tilde{E} \cong \mathbb{P}^1$. A $G$-bundle on $E$ which pulls back to the trivial $G$-bundle $\tilde{E} \times G$ on $\tilde{E}$ can be identified (after fixing a trivialization of the pullback) with an element $g \in G$, if $E$ is nodal, or an element $X \in g$, if $E$ is cuspidal. There is a related result for families. In this way, questions about $G$ or $g$ can be translated into questions about $G$-bundles $\xi$ on $E$. In case $E$ is cuspidal, we can use the Kostant section to construct a universal $G$-bundle $\Xi$ over $(\mathfrak{h}/W) \times E$ which pulls back to the trivial bundle over $(\mathfrak{h}/W) \times \tilde{E}$. A similar construction works in the nodal case using the Steinberg section. Although the construction of $\Xi$ seems to depend on a particular choice of a section of the adjoint quotient morphism, we show in [1], using many of the methods of this paper, that all such sections are conjugate in an appropriate sense. If $E$ is smooth, modulo some minor technical difficulties, we can use the parabolic construction of [5]. Although the proof of Theorem 1 only needs the case where $E$ is singular, minor modifications also handle the case of a smooth $E$. This case is interesting for the study of principal $G$-bundles on a smooth elliptic curve $E$ as well as the vector bundles associated to
them by a representation $\rho$ of $G$. In fact, we were led to the results of this paper based on our previous work on $G$-bundles over smooth elliptic curves $E$.

To prove Theorem 1, we shall use the theory of spectral covers associated to semistable vector bundles over Weierstrass cubics. Given an irreducible representation $\rho: G \to \text{Aut } V$ and a $G$-bundle $\xi$ on $E$, we can form the associated vector bundle $\xi \times_G V$. If $\xi$ pulls back to the trivial bundle on $\tilde{E}$, then $\xi \times_G V$ is a semistable vector bundle on $E$ of degree zero. It is well-known that, for every Weierstrass cubic $E$, there is an equivalence between the abelian category of semistable torsion free sheaves on $E$ of degree zero and the abelian category of torsion sheaves on $E$. More precisely, given a semistable torsion free sheaf $V$ on $E$, there is a zero-dimensional subscheme $T_V$ in $E$, the spectral cover, which is an effective Cartier divisor if $V$ is a vector bundle, and a sheaf $Q(V)$ supported on $T_V$ which can be used to recover $V$ from $Q(V)$ and the Poincaré sheaf on $E \times E$. The bundle $V$ is regular if $Q(V)$ is a line bundle over $T_V$. More generally, given a family of semistable vector bundles $V$ over $B \times E$, flat over $B$, the construction yields a relative effective Cartier divisor $T_V$ in $B \times E$ and a sheaf $Q(V)$ over $T_V$, from which we can recover $V$. The bundle $V$ restricts to a bundle which is regular and semistable on every fiber if and only if $Q(V)$ is a line bundle on $T_V$. However, we shall need to consider more general situations, where $Q(V)$ is the pushforward of a line bundle on the normalization of $T_V$.

In case $E$ is cuspidal, we use the Kostant section $\Sigma$ to construct a universal $G$-bundle $\Xi$ over $(\mathfrak{h}/W) \times E$. Then we let $V$ be the vector bundle over $(\mathfrak{h}/W) \times E$ associated to $\Xi$ by the representation $\rho: G \to \text{Aut } V$. Even though $\Xi$ restricts to a regular $G$-bundle $\xi_x$ on every fiber $\{x\} \times E$, the vector bundle $V_x = \xi_x \times_G V$ is not in general regular. The spectral cover $T_V$ and the sheaf $Q(V)$ are also quite complicated. For example, the irreducible components of $T_V$ are indexed by the Weyl orbits of the weights of $\rho$, and each component has multiplicity equal to the multiplicity of the orbit. However, the Weyl orbit of the highest weight (or equivalently of the lowest weight) defines a reduced irreducible component $T'$ of $T_V$. Of course, $\rho$ is minuscule if and only if $T' = T_V$. We explicitly identify the normalization of $T'$ with a finite normal cover of $\mathfrak{h}/W$. The surjection $Q(V) \to Q(V)|_{T'}$ corresponds to a surjection $V \to V'$, where $V'$ is some coherent sheaf on $(\mathfrak{h}/W) \times E$. If $\rho$ is minuscule, then $V = V'$. In the quasiminuscule case, we prove that $V'$ is again a vector bundle. It is natural to conjecture that this is true for an arbitrary irreducible $\rho$. By analyzing how $V'$ fails to be regular in codimension one, we are able to establish Theorem 1 but without explicitly determining $\tilde{f}$. To determine $\tilde{f}$ requires a more detailed analysis of the Kostant section which is given in Section 3. Similar but less explicit constructions using the Steinberg section work in case $E$ is nodal.

The organization of this paper is as follows. In Section 1, we discuss bundles over singular curves and recall the theory of spectral covers. We give a sufficient criterion for a vector bundle to be given by pushing forward a line bundle on the normalization of the spectral cover. In Section 2, we study the spectral covers arising from the vector bundles associated to a universal $G$-bundle via an irreducible representation $\rho$, and in particular identify the normalization of the irreducible component $T'$ associated to the Weyl orbit of the highest weight. These results are then applied to the case of a minuscule or quasiminuscule representation. In Section 3, we prove Theorem 2. We can then easily prove Theorem 1 by translating
the vector bundle results given in Section 2 into an algebraic form. The fourth
section illustrates the main results by studying various explicit examples. In Section
5, we study the quasiminuscule, non-minuscule case in more detail in the simply
laced case, where the corresponding representation is the adjoint representation. It
seems likely that similar methods will also describe the non-simply laced case. We
conclude with some general conjectures concerning arbitrary representations.

It is a pleasure to thank Victor Ginzburg for explaining to us his work [9, 10]
and for calling our attention to the reference [12].

1. Spectral covers of semistable vector bundles over Weierstrass cubics

Throughout this paper, $E$ denotes a Weierstrass cubic, i.e. a reduced irreducible
curve of arithmetic genus one, $E_{\reg}$ is the set of smooth points of $E$, and $p_0 \in E_{\reg}$
is a fixed point used as the origin of the group law on $E_{\reg}$. If $E$ is singular, then
its unique singular point is either a node or a cusp, and we shall refer to $E$ as
either nodal or cuspidal. In this case, we denote the normalization by $\nu: \tilde{E} \to E$.
The point $p_0$ determines an isomorphism $E_{\reg} \to \Pic^0 E$. If $E$ is nodal, $\Pic^0 E$ is
canonically $\mathbb{C}^*$, and if $E$ is cuspidal, the choice of a local coordinate $z$ at $\nu^{-1}(E_{\text{sing}})$
determines an isomorphism of $\Pic^0 E$ with $\mathbb{C}$. Let $\mathcal{P}$ be the Poincaré bundle over
$E \times E$. (For the meaning of this sheaf and various properties of it in case $E$ is
singular, we refer to the proof of Lemma (0.3) in [8].) We shall use the symmetric
form of $\mathcal{P}$: $\mathcal{P} = \mathcal{O}_{E \times E}(\Delta) \otimes \pi_1^* \mathcal{O}_E(-p_0) \otimes \pi_2^* \mathcal{O}_E(-p_0) \otimes H^0(E; K_E)$, where $\Delta \subseteq E \times E$ is the diagonal. In case $E$ is smooth, the first factor is the parameter space
$E = \Pic^0 E$, and the second factor is the original curve. Of course, the symmetry
between the factors means that we can also view the first factor as the curve and the
second as $\Pic^0 E$. In the singular case, for the purposes of this paper, it will suffice
to consider the line bundle $\mathcal{P}|_{E_{\reg} \times E}$. Our goal in this section is to describe basic
results of [13], [8], [16] which describe semistable bundles of degree zero on $E$ in terms
of torsion sheaves on $E$. In [8], we mainly considered the case of bundles whose
restriction to every fiber was regular. Here we shall need a slight generalization
of those results. There are also analogous results relating semistable torsion free
sheaves of degree zero on $E$ to torsion sheaves on $E$, but we shall not need them.

1.1. Bundles over singular curves. Assume in this section that $E$ is singular
and, in case $E$ is cuspidal, that we have fixed once and for all a local coordinate
$z$ on $\tilde{E}$ centered at the inverse image of the singular point, and in particular have
determined an isomorphism $\Pic^0 E \cong \mathbb{C}$. We begin by describing vector bundles
and $G$-bundles over $E$ which become trivial when pulled back to $\tilde{E}$. More generally,
we shall consider parametrized versions. The arguments are standard, and left to
the reader. (See [8] for the case of a single curve.)

Theorem 1.1.1. Suppose that $E$ is cuspidal and that we have fixed the coor-
dinate $z$ on $\tilde{E}$ near the inverse image of the singular point $x$. Let $B$ be a scheme.
Suppose that $\mathcal{V}$ is a vector bundle over $B \times E$ such that $(\text{Id} \times \iota)^* \mathcal{V}$ restricts on every
fiber to the trivial vector bundle of rank $n$. In this case, $\mathcal{W} = \pi_1^* (\text{Id} \times \iota)^* \mathcal{V}$ is a rank
$n$ vector bundle on $B$, and $(\text{Id} \times \iota)^* \mathcal{V} \cong \pi_1^* \mathcal{W}$. Moreover, there is an equivalence
of categories between the category of triples $(\mathcal{V}, \mathcal{W}, \Phi)$, where $\mathcal{V}$ is a rank $n$ vector
bundle on $B \times E$, $\mathcal{W}$ is a rank $n$ vector bundle on $B$, and $\Phi$ is an isomorphism from
$(\text{Id} \times \iota)^* \mathcal{V}$ to $\pi_1^* \mathcal{W}$, and the category of pairs $(\mathcal{W}, \varphi)$, where $\mathcal{W}$ is a rank $n$ vector
bundle on $B$ and $\varphi$ is an element of $\text{End}(W)$. Similar statements hold in the nodal case, where $\varphi \in \text{Aut}(W)$. 

**Remark 1.1.2.** (1) For $E$ cuspidal and $x = \iota^{-1}(E_{\text{sing}})$, the equivalence of categories is defined as follows: given $(W, \varphi)$, $\mathcal{V}$ is the subsheaf of $\pi_1^*W$ consisting of all local sections $s$ such that, identically in $b$,

$$\frac{ds}{dz}(b, x) = \varphi(b)s(b, x).$$

If $E$ is nodal, and $x, y \in \tilde{E}$ are the two preimages of the singular point on $E$, then $\mathcal{V}$ is the subsheaf of $\pi_1^*W$ consisting of all local sections $s$ such that, for all $b$,

$$s(b, y) = \varphi(b)s(b, x).$$

In particular, it follows that if $B$ is normal and $\mathcal{V}$ is a locally free sheaf as in the statement of the theorem, then $\mathcal{V}$ has the Hartogs property: if $X$ is a subset of $B$ of codimension at least 2 and $j: (B - X) \times E \to B \times E$ is the inclusion, then the natural map $\mathcal{V} \to j_*j^*\mathcal{V}$ is an isomorphism. In fact, a similar result is true for every locally free sheaf on $B \times E$, as long as $B$ is normal.

(2) The construction is compatible with base change $B' \to B$.

(3) Suppose that $(\text{Id} \times \iota)^*\mathcal{V}$ is trivialized. In this case, there is an equivalence of categories between the category of triples $(\mathcal{V}, V, \Phi)$, where $\mathcal{V}$ is a vector bundle on $B \times E$, $V$ is a finite-dimensional vector space, and $\Phi$ is an isomorphism from $(\text{Id} \times \iota)^*\mathcal{V}$ to $\mathcal{O}_{B \times E} \otimes_C V$, and the category of pairs $(V, \varphi)$, where $V$ is a finite-dimensional vector space and $\varphi: B \to \text{End} V$ is a morphism. A morphism $B \to \text{End} V$ is the same thing as a section of $\mathcal{O}_B \otimes \text{End} V$. In case $B = \text{Spec} R$, this is in turn equivalent to an element of

$$\text{End} V \otimes_C R = \text{End}_R(V \otimes_C R, V \otimes_C R).$$

After choosing a basis of $V$, this can further be identified with an $n \times n$ matrix with coefficients in $R$. Similar remarks hold in the nodal case. For example, in this case, $\varphi$ is a morphism $B \to \text{Aut} V$, and in case $B = \text{Spec} R$, such a morphism is identified with an element of $\text{Aut}_R(V \otimes_C R)$.

(4) The construction is natural under direct image by a finite flat morphism. In other words, suppose that $\nu: B \to B'$ is a finite flat morphism. Thus $\nu_*\mathcal{O}_B$ is a locally free $\mathcal{O}_{B'}$-module. If $\mathcal{V}$ is a vector bundle on $B \times E$, then $(\nu \times \text{Id})_*\mathcal{V}$ is a vector bundle on $B' \times E$. If in addition $(\text{Id} \times \iota)^*\mathcal{V} \cong \pi_1^*W$, then $(\text{Id} \times \iota)^*(\nu \times \text{Id})_*\mathcal{V} = \pi_1^*\nu_*W$. In this case, the global section $\varphi$ of $\text{End}(W)$ induces a global section of $\text{End}(\nu_*W)$, and this corresponds to the bundle $(\nu \times \text{Id})_*\mathcal{V}$ on $B' \times E$.

In case $B$ is a point and $\nu^*\mathcal{V}$ is trivialized, the rank $n$ bundle $\mathcal{V}$ corresponds to an $n \times n$ matrix $A$ (which is arbitrary, in the cuspidal case, and invertible, in the nodal case). For example, in the cuspidal case, if $N$ is the nilpotent matrix defined by $N(e_i) = e_{i-1}$, $i \geq 1$, and $N(e_1) = 0$, where the $e_i$ are the standard basis vectors, the corresponding vector bundle is $I_n$, in the notation of [8]. Here, for each $n > 0$, $I_n$ is the unique indecomposable vector bundle over $E$ of rank $n$ up to isomorphism which has a filtration, all of whose successive quotients are $\mathcal{O}_E$. More generally, if $A = H + N$ is the Jordan decomposition of $A$, where $H$ is semisimple, $N$ is nilpotent, and $[H, N] = 0$, then the action of $H$ decomposes $V$ into a direct sum of eigenspaces $\bigoplus V_i$, with distinct eigenvalues $c_i \in \mathbb{C}$, and each $V_i$ is invariant under $N$. Suppose that the Jordan blocks of $N$ acting on $V_i$ have length $n_{ij}$. Using
the coordinate $z$ to identify $\text{Pic}^0 E$, we can identify $c_i \in \mathbb{C}$ with a line bundle on $E$ of degree zero, which we denote by $\lambda_i$. Then

$$V \cong \bigoplus_i \left( \bigoplus_j I_{n_{ij}} \right) \otimes \lambda_i.$$ 

Of course, a similar statement holds for the nodal case.

An important example in this paper will be the Poincaré bundle $\mathcal{P}|\mathcal{E}_\text{reg} \times E$. It is easy to check the following:

**Proposition 1.1.3.** Suppose that $E$ is cuspidal. The pullback of $\mathcal{P}|\mathcal{E}_\text{reg} \times E$ to $\mathcal{E}_\text{reg} \times \check{E}$ is the trivial line bundle. Given a coordinate $t$ on $\mathcal{E}_\text{reg}$ centered at $p_0$, under the correspondence of Theorem 1.1.1, the Poincaré bundle $\mathcal{P}|\mathcal{E}_\text{reg} \times E$ corresponds to the morphism $\mathbb{C} \to \text{End} \mathbb{C}$ given by multiplication, i.e. corresponds to the function $t$. \hfill \Box

A similar result handles the nodal case, where the coordinate $t$ on $\mathcal{E}_\text{reg} \cong \mathbb{C}^*$ defines an isomorphism of algebraic groups from $\mathcal{E}_\text{reg}$ to $\text{Aut} \mathbb{C}$, again by multiplication.

We shall also use Theorem 1.1.1 in the following situation. Let $\rho: G \to \text{Aut} V$ be a representation of $G$ and let $\rho_\ast: \mathfrak{g} \to \text{End} V$ be the corresponding representation of the Lie algebra. Given $X \in \mathfrak{g}$, the element $\rho_\ast(X)$ defines a vector bundle over $E$. If $X = X_s + X_n$ is the Jordan decomposition of $X$, then $\rho_\ast(X) = \rho_\ast(X_s) + \rho_\ast(X_n)$ is the Jordan decomposition of $\rho_\ast(X)$. Let $\mathfrak{z}_\mathfrak{g}(X)$ be the Lie algebra centralizer of $X$ in $\mathfrak{g}$. The element $X$ is regular if and only if $\dim \mathfrak{z}_\mathfrak{g}(X)$ is equal to the rank of $\mathfrak{g}$ if and only if $X_n$ is a principal nilpotent element in $\mathfrak{z}_\mathfrak{g}(X)$. Now suppose that we are given a morphism $f: B \to \mathfrak{g}$, for example the Kostant section $h/\mathfrak{g} \to \mathfrak{g}$ (which we shall describe later). Then $\rho_\ast \circ f: B \to \text{End} V$ is a morphism, and hence defines a vector bundle $V$ over $B \times E$ which becomes trivial over $B \times \check{E}$. Of course, a similar picture holds in the nodal case.

### 1.2. Spectral covers.

**Definition 1.2.1.** Let $V$ be a semistable bundle of degree zero on $E$. Define $Q(V)$ as the cokernel of the natural global to local map $H^0(\mathcal{E}; V \otimes \mathcal{O}_E(p_0)) \otimes_{\mathcal{O}_E} \mathcal{O}_E \to V \otimes_{\mathcal{O}_E} \mathcal{O}_E(p_0)$. Thus there is an exact sequence

$$0 \to H^0(\mathcal{E}; V \otimes \mathcal{O}_E(p_0)) \otimes_{\mathcal{O}_E} \mathcal{O}_E \xrightarrow{A} V \otimes_{\mathcal{O}_E} \mathcal{O}_E(p_0) \to Q(V) \to 0.$$ 

We shall abbreviate $Q(V) = Q$ when $V$ is clear from the context, and call $Q$ the Fourier-Mukai transform of $V$.

We have the following from [X]:

**Lemma 1.2.2.** Suppose that $V$ is semistable of degree zero and rank $n$. Then $h^0(\mathcal{E}; V \otimes \mathcal{O}_E(p_0)) = n$, $A$ is an injection from $\mathcal{O}_E$ to $V \otimes \mathcal{O}_E(p_0)$, and $Q(V)$ is a torsion sheaf of length $n$. \hfill \Box

For example, if $L = \mathcal{O}_E(q - p_0)$ is a line bundle of degree zero, then $Q(L) = \mathcal{O}_E/m_q$. Let $T_V$ be the zero-dimensional subscheme of $E$ defined by the vanishing of the section $\det A$ of $\bigwedge^n (V \otimes \mathcal{O}_E(p_0)) = \det V \otimes \mathcal{O}_E(np_0)$, in the above notation. By Cramer’s rule, $Q = Q(V)$ is an $\mathcal{O}_{T_V}$-module. Clearly $Q(V_1 \oplus V_2) = Q(V_1) \oplus Q(V_2)$. 
The divisor $T_V$ is additive over exact sequences: given $0 \to V' \to V \to V'' \to 0$, where $V', V, V''$ are semistable of degree zero, we have $T_V = T_{V'} + T_{V''}$ as divisors on $E$.

**Remark 1.2.3.** (1) It follows from the defining exact sequence above that $H^0(E; Q) \cong H^0(E; V \otimes \mathcal{O}_E(p_0)) \otimes H^1(E; \mathcal{O}_E)$ and hence that there is a non-canonical isomorphism $H^0(E; Q) \cong H^0(E; V \otimes \mathcal{O}_E(p_0))$.

(2) Mukai, in [13], defines a torsion sheaf $Q'$ associated to $V$ by:

$$Q' = R^1\pi_1\ast(\pi_2^*V \otimes \mathcal{P}^{-1}).$$

In fact, $Q' = Q \otimes \mathcal{O}_E(p_0)$, which we can see as follows: begin with the exact sequence

$$0 \to \mathcal{P}^{-1} \to \pi_1^*\mathcal{O}_E(p_0) \otimes \pi_2^*\mathcal{O}_E(p_0) \to \mathcal{O}_\Delta \otimes \pi_1^*\mathcal{O}_E(p_0) \otimes \pi_2^*\mathcal{O}_E(p_0) \to 0,$$

and tensor with $\pi_2^*V$. If we apply $R^1\pi_1\ast$, we get an exact sequence

$$0 \to H^0(E; V \otimes \mathcal{O}_E(p_0)) \otimes \mathcal{O}_E(p_0) \to V \otimes \mathcal{O}_E(2p_0) \to R^1\pi_1\ast(\pi_2^*V \otimes \mathcal{P}^{-1}) \to 0.$$

Hence $Q \otimes \mathcal{O}_E(p_0) \cong R^1\pi_1\ast(\pi_2^*V \otimes \mathcal{P}^{-1}) = Q'$.

The support of $V$ is by definition the reduced support of $T_V$, or equivalently of $Q$. If $\text{Supp } V = \{q_1, \ldots, q_a\}$, where the $q_i$ are distinct, then $V = \bigoplus_i V(q_i)$, where each $V(q_i)$ is supported at $q_i$. We shall only be concerned in this paper with bundles $V$ such that the support of $V$ is contained in $E_{\text{reg}}$. We can see the structure of $Q$ concretely in this case: If $\lambda$ is a line bundle of degree zero, let $I_k(\lambda) = I_k \otimes \lambda$. Given a point $q \in E_{\text{reg}}$, we let $\lambda_q = \mathcal{O}_E(q - p)$ be the corresponding line bundle of degree zero. Then it is easy to check the following:

**Lemma 1.2.4.** Suppose that $V \cong \bigoplus_i I_{k_i}(\lambda_{q_i})$. Then $T_V = \sum_i k_i q_i$ and $Q \cong \bigoplus_i Q_i$, where each $Q_i$ is supported at $q_i$ and in fact $Q_i = \mathcal{O}_E/m_{k_i}^\lambda$. Thus $Q$ is a locally free $\mathcal{O}_{E_{\text{reg}}}$-module of rank one if and only if $q_i \neq q_j$ for $i \neq j$, i.e. if and only if $V$ is regular. Finally, $T_V$ depends only on the $S$-equivalence class of $V$. \hfill $\square$

We have the following condition for a semistable vector bundle of degree zero over a singular curve $E$ to have support on $E_{\text{reg}}$:

**Lemma 1.2.5.** Let $E$ be singular. Suppose that $V$ is a semistable vector bundle of rank $n$ and degree zero over $E$. Then the following are equivalent:

(i) $\text{Supp } V \subseteq E_{\text{reg}}$;

(ii) $V$ has a filtration whose successive quotients are line bundles of degree zero;

(iii) The pullback of $E$ to the normalization $\tilde{E}$ is the trivial rank $n$ vector bundle. \hfill $\square$

**Lemma 1.2.6.** Suppose that $E$ is cuspidal. Fixing a local coordinate at the preimage in $\tilde{E}$ of the singular point to identify $E_{\text{reg}}$ with $\mathbb{C}$, let $t$ be the induced coordinate on $E_{\text{reg}}$. Let $V$ be a vector bundle over $E$ whose pullback $\tilde{V}$ to $\tilde{E}$ is trivial. Then, by Theorem [14], a trivialization of $\tilde{V}$ determines an element $A \in \text{End} H^0(\tilde{E}; \tilde{V})$. The conjugacy class of $A$, viewed as an $n \times n$ matrix, is a complete invariant of the isomorphism class of $V$. Finally, $T_V$ is the divisor in $E_{\text{reg}}$ defined by the characteristic polynomial $p_A(t)$ in $E_{\text{reg}}$. A similar statement holds if $E$ is nodal. \hfill $\square$
We return to the case of a general $E$ and $V$. To recover $V$ from $Q$, we have the following result of Mukai \[13\].

**Proposition 1.2.7.** With $Q = Q(V)$ defined as above, there is a canonical isomorphism

$$V \to \pi_2^*(\pi_1^*(Q \otimes O_E(p_0)) \otimes \mathcal{P}).$$

**Proof.** Let $H^0 = H^0(E; V \otimes O_E(p_0))$. Using the exact sequence

$$0 \to H^0 \otimes O_E(p_0) \to V \otimes O_E(2p_0) \to Q \otimes O_E(p_0) \to 0,$$

and the fact that $R^1\pi_2^*(H^0 \otimes \pi_1^*O_E(p_0)) = 0$, we see that it suffices to find a canonical isomorphism

$$\pi_2^*(\pi_1^*(V \otimes O_E(2p_0)) \otimes \mathcal{P})/\pi_2^*(\pi_1^*(H^0 \otimes O_E(p_0)) \otimes \mathcal{P}) \to V.$$

To analyze the quotient, we begin with the exact sequence

$$0 \to O_{E \times E} \to O_{E \times E}(\Delta) \to O_{\Delta} \otimes K_E^{-1} \to 0.$$

Using $\pi_1^*(V \otimes O_E(2p_0)) \otimes \mathcal{P} = \pi_1^*(V \otimes O_E(p_0)) \otimes \pi_2^*O_E(p_0) \otimes O_{E \times E}(\Delta) \otimes H^0(E; K_E)$, we get an exact sequence

$$0 \to \pi_1^*(V \otimes O_E(p_0)) \otimes \pi_2^*O_E(-p_0) \to \pi_1^*(V \otimes O_E(2p_0)) \otimes \mathcal{P} \to i_*(V) \to 0,$$

where $i_*: \Delta \to E \times E$ is the inclusion. Applying $\pi_*$ then gives an exact sequence

$$0 \to H^0 \otimes O_E(-p_0) \to \pi_2^*(\pi_1^*(V \otimes O_E(2p_0)) \otimes \mathcal{P}) \to V \to 0.$$

As for the term $\pi_1^*(H^0 \otimes O_E(p_0)) \otimes \mathcal{P} = H^0 \otimes \pi_2^*O_E(-p_0) \otimes O_{E \times E}(\Delta)$, the fact that $\pi_2^*O_{E \times E} \cong \pi_2^*O_{E \times E}(\Delta)$ via the natural inclusion shows that

$$\pi_2^*(\pi_1^*(H^0 \otimes O_E(p_0)) \otimes \mathcal{P}) \cong H^0 \otimes O_E(-p_0),$$

and it is easy to check that this isomorphism is compatible with the inclusion of $H^0 \otimes O_E(-p_0)$ in $\pi_2^*(\pi_1^*(V \otimes O_E(2p_0)) \otimes \mathcal{P})$ given above. Thus we have identified $\pi_2^*(\pi_1^*(V \otimes O_E(2p_0)) \otimes \mathcal{P})/\pi_2^*(\pi_1^*(H^0 \otimes O_E(p_0)) \otimes \mathcal{P})$ with $V$.

\[\square\]

**Proposition 1.2.8.** Given a torsion sheaf $Q$ on $E$, if we set

$$V(Q) = \pi_2^*(\pi_1^*(Q \otimes O_E(p_0)) \otimes \mathcal{P}),$$

then $V(Q)$ is a semistable vector bundle of degree zero, and there is a canonical isomorphism from the cokernel of the natural map

$$H^0(E; V(Q) \otimes O_E(p_0)) \otimes O_E \to V(Q) \otimes O_E(p_0)$$

to $Q \otimes O_E(p_0)$.

**Proof.** Note that $V(Q) \otimes O_E(p_0) \cong \pi_2^*(\pi_1^*Q \otimes O_{E \times E}(\Delta))$. Applying $\pi_2^*$ to the exact sequence

$$0 \to \pi_1^*Q \otimes O_{E \times E}(\Delta - \pi_2^*p_0) \to \pi_1^*Q \otimes O_{E \times E}(\Delta) \to \pi_1^*Q \otimes \pi_2^*O_E(p_0) \otimes \pi_2^*p_0 \to 0$$

then gives the result.

\[\square\]
In other words, \( V(Q(V)) \cong V \), and \( Q(V(Q)) \cong Q \otimes \mathcal{O}_E(p_0) \cong Q \). Thus we have:

**Corollary 1.2.9.** The functor \( V \mapsto Q(V) \) is an exact functor which defines an equivalence of categories from the abelian category of semistable torsion free sheaves of degree zero over \( E \) to the abelian category of torsion sheaves over \( E \). \( \square \)

**Remark 1.2.10.** Teodorescu \[16\] gives an alternate method for recovering \( V \) from \( Q \). It suffices to recover the extension class of the extension

\[
0 \to H^0 \otimes \mathcal{O}_E \to V \otimes \mathcal{O}_E(p_0) \to Q \to 0,
\]

where \( H^0 \otimes \mathcal{O}_E \cong H^0(E; Q) \otimes K_E \). But \( \operatorname{Ext}^1(E; Q, H^0(E; Q) \otimes K_E) \cong \operatorname{Ext}^1(E; Q, K_E) \otimes H^0(E; Q) \cong H^0(E; Q)^* \otimes H^0(E; Q), \) by Serre duality, and the extension class is \( \text{Id} \in H^0(E; Q)^* \otimes H^0(E; Q) \).

### 1.3. Families of vector bundles.

We turn now to the relative version of the construction. Throughout this section, \( B \) denotes a scheme of finite type over \( \mathbb{C} \) (or analytic space), and \( V \) is a rank \( n \) vector bundle over \( B \times E \). Given \( b \in B \), let \( V_b \) be the bundle on \( E \) induced by the restriction of \( V \) to the slice \( \{ b \} \times E \). We shall always assume that, for all \( b \in B \), \( V_b \) is semistable of degree zero. We can then define \( Q = Q(V) \) by the exact sequence

\[
0 \to \pi_1^* \pi_1* (V \otimes \mathcal{O}_E(p_0)) \to V \otimes \mathcal{O}_E(p_0) \to Q \to 0,
\]

and let \( T_V \) be the Cartier divisor in \( B \times E \) defined by \( \det A \). We call \( T_V \) the spectral cover of \( B \) defined by \( V \). Standard base change arguments show:

**Lemma 1.3.1.** The morphism \( T_V \to B \) is finite and flat of degree \( n \). The coherent sheaf \( Q \) on \( B \times E \) is an \( \mathcal{O}_{T_V} \)-module, flat over \( B \), and the rank of \( Q \) viewed as a vector bundle over \( B \) is also \( n \). Finally, the above construction is compatible with base change: if \( f: B' \to B \) is a morphism of schemes, then the spectral cover associated to \( f^* V \) is \( f^* T_V \) and the corresponding torsion sheaf in \( B' \times E \) is \( f^* Q \). \( \square \)

For example, suppose that \( L \) is a line bundle on \( B \times E \) whose restriction to every slice \( \{ b \} \times E \) has degree zero. Then \( T_L \) is the section of \( B \times \operatorname{Pic}^0 E = B \times E_{\text{reg}} \) corresponding to \( L \), and \( Q(L) = \mathcal{O}_{T_L} \).

Let \( p_1 = \pi_{1,2}: B \times E \times E \to B \times E \), let \( q = \pi_{1,3}: B \times E \times E \to B \times E \), and let \( r = \pi_{2,3}: B \times E \times E \to E \times E \). There is the relative version of Proposition \[16\]:

**Lemma 1.3.2.** There is a natural isomorphism

\[
V \cong q_* (p_1^*(Q \otimes \pi_3^* \mathcal{O}_E(p_0)) \otimes r^* \mathcal{P}) . \quad \square
\]

Similarly, the analogue of Proposition \[16\] holds: If \( Q \) is a sheaf on \( B \times E \), finite and flat over \( B \), and \( V \cong q_* (p_1^*(Q \otimes \pi_3^* \mathcal{O}_E(p_0)) \otimes r^* \mathcal{P}) \), then

\[
Q(V) \cong Q \otimes \pi_3^* \mathcal{O}_E(p_0).
\]

In other words, up isomorphism, \( Q(V) \cong Q \).

We can form the restriction of \( r^* \mathcal{P} \) to \( T_V \times E \), which we shall denote \( \mathcal{P}_{T_V} \).

Let \( M = Q \otimes \pi_3^* \mathcal{O}_E(p_0) \), viewed as an \( \mathcal{O}_{T_V} \)-module, and let \( \nu: T_V \to B \) be the projection. Then the lemma says that

\[
V \cong (\nu \times \text{Id})_*(\pi_1^* M \otimes \mathcal{P}_{T_V}).
\]
There are natural analogues of the above lemmas for flat families $\pi: Z \to B$ of Weierstrass cubics with a section $\sigma$. Given a vector bundle $V$ over $Z$ whose restriction to every fiber of $\pi$ is semistable of degree zero, we define $Q$ and $T_V$ by the exact sequence

$$0 \to \pi^* \pi_*(V \otimes \mathcal{O}_Z(\sigma)) \xrightarrow{\iota} V \otimes \mathcal{O}_Z(\sigma) \to Q \to 0,$$

and $T_V = \det A$, and the obvious analogue of Lemma 1.3.1 holds. Similarly, there is a relative Poincaré sheaf $\mathcal{P} = \mathcal{O}_{Z \times_B Z}(\Delta) \otimes \pi_1^* \mathcal{O}_Z(-\sigma) \otimes \pi_2^* \mathcal{O}_Z(-\sigma) \otimes \pi^* L$ over $Z \times B$, where $L$ is the line bundle on $B$ such that $\omega_{Z/B} = \pi^* L$, where $\omega_{Z/B}$ is the relative dualizing sheaf and $p: Z \times_B Z \to B$ is the projection, and a natural isomorphism

$$V \cong \pi_2^*(\pi_1^*(Q \otimes \mathcal{O}_Z(\sigma)) \otimes \mathcal{P}).$$

We turn now to local properties of $Q$, or equivalently $M$ in the above notation, viewed as an $\mathcal{O}_{T_V}$-module. Applying Lemma 1.2.3 we have:

**Lemma 1.3.3.** Suppose that $\mathcal{V}|\{b\} \times E$ has support in $E_{\text{reg}}$ for every $b \in B$. The $\mathcal{O}_{T_V}$-module $Q$ is a line bundle if and only if the restriction of $V$ to every fiber is regular. \(\square\)

As a consequence of the lemma, suppose that $B$ is reduced and equidimensional and that $\mathcal{V}|\{b\} \times E$ has support in $E_{\text{reg}}$ for every $b \in B$. Then $T$ is generically reduced if and only if it is reduced, if and only if there is an open dense subset $U$ of $B$ such that $\mathcal{V}|\{b\} \times E$ is isomorphic to a direct sum of $n$ distinct line bundles of degree zero for all $b \in U$, and in this case $Q$ is a torsion free rank one $\mathcal{O}_{T_V}$-module. (Here, $Q$ is torsion free if there are no nonzero local sections of $Q$ supported on a subscheme of $T_V$ of strictly smaller dimension.) We shall need the following more precise version:

**Proposition 1.3.4.** Suppose that $B$ is normal and that $T_V$ is reduced and irreducible. Let $\hat{T}_V$ be the normalization of $T_V$, and let $\hat{\nu}: \hat{T}_V \to B$ be the composition $\hat{T}_V \to T_V \xrightarrow{\nu} B$. For each $b \in B$, let $V_b$ be the bundle corresponding to $\mathcal{V}|\{b\} \times E$. Suppose that there is a divisor $D \subseteq B$ and an open dense subset $D' \subseteq D$ such that

(i) For $b \notin D$, $V_b$ is regular and $T_V$ is normal at all points of $(\nu)^{-1}(b)$.

(ii) Given $b \in D'$ and $q \in \text{Supp} V_b$, exactly one of the following holds:

(a) $T_V$ is normal at $(b, q)$ and $V_b(q)$ is regular;

(b) $q \in E_{\text{reg}}$, there exists a neighborhood of $(b, q)$ in $T_V$ of the form $U_1 \cup U_2$, where each $U_i$ is smooth, $(b, q) \in U_1 \cap U_2$, $U_1$ meets $U_2$ transversally, and $V_b(q) \cong \mathcal{O}_E(q - p_0) \oplus \mathcal{O}_E(q - p_0)$.

Then there exists a rank one reflexive sheaf $M$ on $\hat{T}_V$ such that

$$\mathcal{V} \cong (\hat{\nu} \times \text{Id})_*(\pi_1^* M \otimes \mathcal{P}_{\hat{T}_V}),$$

where $\mathcal{P}_{\hat{T}_V}$ is the pullback of $\mathcal{P}_{T_V}$ to $\hat{T}_V \times E$. If $\hat{T}_V$ is smooth, then $M$ is a line bundle.

**Proof.** If $b \notin D$, then $T_V$ is normal along $(\nu)^{-1}(b)$, so that $\hat{T}_V = T_V$ there, and $Q$ is locally free of rank one in a neighborhood of $(\nu)^{-1}(b)$. Now suppose that $b \in D'$. If $T_V$ is normal at $(b, q)$, then again $Q$ is locally free of rank one in a neighborhood of $(b, q)$. Suppose that $T_V$ is not normal at $(b, q)$. We first find locally a rank two subbundle of $V$ corresponding to the two branches of $T_V$. There is a small analytic
Stein neighborhood $U$ of $b$ in $B$ such that $(\nu)^{-1}(U) = U' \cup U''$, where $(b, q) \in U'$, $U'$ is connected, and $U' \cap U'' = \emptyset$. By hypothesis, if $U$ is sufficiently small, we can write $U' = U_1 \cup U_2$, where $U_1$ and $U_2$ are smooth and meet transversally along some component $D_0$ of $(\nu)^{-1}(D)$. Since $\nu$ is finite and flat and the local degree of $\nu$ at $(b, q)$ is 2, $\nu|_{U_i}$ is an isomorphism of $U_i$ onto $U$. Corresponding to $(\nu)^{-1}(U) = U' \cup U''$, we have $\nu|_{U \times E} = \nu' \oplus \nu''$, where the rank of $\nu'$ is 2. The restriction of $P_{TV}$ to $U_i$ defines line bundles $L_i$ on $U_i \times E$ and hence on $U \times E$. Here $D_0$ is the preimage in $U_i$ of the set of points $u \in U$ where $L_1\{u\} \times E \cong L_2\{u\} \times E$. Since $(\nu') \otimes L_1^{-1}\{t\} \times E$ has a nonzero section for each $t \in U$, and has exactly one if $t \notin D$, it is easy to see that there is a nonzero map $L_1 \to \nu'$. There is an effective divisor $Y$ on $U \times E$ such that the map $L_1 \to \nu'$ extends to a map $L_1 \otimes O_{U \times E}(Y) \to \nu'$ which does not vanish in codimension one. The quotient is thus of the form $L'_2 \otimes I_Z$, where $Z$ is a codimension two subscheme of $U \times E$ (or empty). Clearly $O_{U \times E}(Y)$ restricts to the trivial line bundle on every fiber $\{u\} \times E$ and hence is pulled back from a line bundle on $U$. After shrinking $U$, we may assume that this line bundle is trivial. There is a homomorphism from $L_2$ to $L'_2$ which is an isomorphism away from $D$. Thus as before $L'_2 \cong L_2 \otimes O_{U \times E}(Y')$ and as before we may assume that the line bundle $O_{U \times E}(Y')$ is trivial. By semistability, the scheme $Z$ is the preimage of a subset $Z'$ of $U$, which is either empty or of codimension 2. Thus in the complement of $Z'$, there is an exact sequence

$$0 \to L_1 \to \nu' \to L_2 \to 0.$$  

For $u \in U - Z'$, $\nu'|\{u\} \times E$ is a direct sum of two line bundles of degree zero. It follows that the extension

$$0 \to L_1\{u\} \times E \to \nu'|\{u\} \times E \to L_2\{u\} \times E \to 0$$  

is split for every $u \in U - Z'$. For each $u \in U - Z'$, let $U_u$ be a Stein neighborhood of $u$. Now

$$\text{Ext}^1(U_u \times E; L_2, L_1) = H^1(U_u \times E; (L_2)^{-1} \otimes L_1)) = H^0(U_u; R^1\pi_{2*}((L_2)^{-1} \otimes L_1)).$$

By the above discussion about $\nu'|\{u\} \times E$, if $s$ is the corresponding global section of $R^1\pi_{2*}((L_2)^{-1} \otimes L_1)$, then the image of $s$ is zero in the fiber of $R^1\pi_{2*}((L_2)^{-1} \otimes L_1)$ over $t$ for every $t \in U_u$. To see that $s$ is actually zero, it suffices to prove that $R^1\pi_{2*}((L_2)^{-1} \otimes L_1)$ is supported along $D_0 = U_1 \cap U_2$ and that its length is one there. Each line bundle $L_i$ defines a morphism $g_i$ from $U$ to $\text{Pic}^0 E \cong E$, and the morphism $U_i \to TV \subseteq U \times E$ is given by $(\text{Id}, g_i)$. Let $g = g_1 - g_2$, so that $D_0 = g^{-1}(p_0)$. Then $(L_2)^{-1} \otimes L_1$ is the pullback to $U \times E$ via $(g \times \text{Id})^* of \mathcal{P}$. The assumption that $U_1$ and $U_2$ meet transversally is the assumption that $g$ is smooth at the origin and hence that $D_0 = g^*(p_0)$ as reduced divisors. By flat base change, $R^1\pi_{2*}((L_2)^{-1} \otimes L_1) = R^1\pi_{2*}(g \times \text{Id})^*\mathcal{P} = g^* R^1\pi_{1*}\mathcal{P}$.

It is a standard result that $R^1\pi_{1*}\mathcal{P}$ is supported at $p_0$ and has length one there. It follows that $R^1\pi_{2*}((L_2)^{-1} \otimes L_1)$ is supported along $D_0$ and has length one there. Hence $s = 0$.

Thus $\nu'|U_u \cong (L_1 \oplus L_2)|U_u$. It follows that, at least over the complement of the set $Z'$ of codimension at least two, $Q(\nu'|)(U - Z') \times E$, which is a direct summand of $Q(\nu')(U - Z') \times E$, is $Q(L_1) \oplus Q(L_2)(U - Z') \times E \cong O_{U_1} \oplus O_{U_2}((U - Z') \times E$. Hence, in the complement of the codimension two subset $(\nu)^{-1}(Z')$ of $TV$, $Q$ is a line bundle $M_0$ over the normalization $\tilde{T}_V$ of $TV$. Since $\tilde{T}_V$ is normal, $M_0$ extends uniquely to
a rank one reflexive sheaf \( M \) on \( \hat{T}_V \). We have constructed \( M \) and an isomorphism from \( \mathcal{V} \) to \( (\hat{\nu} \times \text{Id})_* (\pi_1^* M \otimes \mathcal{P}_{\hat{T}_V}) \) over \((B - Z') \times E\), where now \( Z' \) is some global subscheme of \( B \) of codimension at least two. Since the codimension of \( Z' \) in \( B \) is at least two, if \( j: (B - Z') \times E \to B \times E \) is the inclusion, then it follows from (1) of Remark \ref{rem:extension} that \( \mathcal{V} \cong j_* j^* \mathcal{V} \). A similar result holds for \( (\hat{\nu} \times \text{Id})_* (\pi_1^* M \otimes \mathcal{P}_{\hat{T}_V}) \), since \( M \) is reflexive. Thus the isomorphism from \( \mathcal{V} \) to \( (\hat{\nu} \times \text{Id})_* (\pi_1^* M \otimes \mathcal{P}_{\hat{T}_V}) \) over \((B - Z') \times E\) extends to give an isomorphism from \( \mathcal{V} \) to \( (\hat{\nu} \times \text{Id})_* (\pi_1^* M \otimes \mathcal{P}_{\hat{T}_V}) \). The final statement follows since every rank one reflexive sheaf on a smooth scheme is a line bundle. \( \Box \)

### 1.4. Cohomology and extensions

For applications to the study of quasiminscular, non-minuscule representations, we shall need a general result concerning extensions of a family of semistable bundles by the trivial bundle. We begin with the case of a single bundle over \( E \):

**Proposition 1.4.1.** Let \( t \) be a local coordinate centered at \( p_0 \in E \). Let \( V \) be a torsion free semistable sheaf on \( E \) of degree zero and let \( Q = Q(V) \). Then

\[
\begin{align*}
(i) \quad & H^0(E; V) \cong \text{Ker}\{ xt: Q \to Q \}; \\
(ii) \quad & H^1(E; V) \cong \text{Coker}\{ xt: Q \to Q \} = Q/tQ.
\end{align*}
\]

**Proof.** To prove (i), let \( \mathcal{C}_{p_0} \) be the torsion module corresponding to \( \mathcal{O}_E \), i.e. \( \mathcal{C}_{p_0} = \mathcal{O}_E / \mathfrak{m}_{p_0} = \mathcal{O}_{E,p_0} / t\mathcal{O}_{E,p_0} \). Then, by the equivalence of categories given by the Fourier-Mukai correspondence,

\[
H^0(E; V) = \text{Hom}(\mathcal{O}_E, V) \cong \text{Hom}(\mathcal{C}_{p_0}, Q) = \text{Ker}\{ xt: Q \to Q \}.
\]

To see (ii), the group \( H^1(E; V) \) classifies the set of isomorphism classes of extensions \( 0 \to V \to E \to \mathcal{O}_E \to 0 \), and via the Fourier-Mukai correspondence this set is the same as the set of isomorphism classes of extensions \( 0 \to Q \to \mathcal{Q} \to \mathcal{C}_{p_0} \to 0 \), which is classified by \( \text{Ext}^1_{\mathcal{O}_E}(\mathcal{C}_{p_0}, Q) \). But \( \text{Ext}^1_{\mathcal{O}_E}(\mathcal{C}_{p_0}, Q) \cong Q/tQ \) via the resolution \( 0 \to \mathcal{O}_{E,p_0} \xrightarrow{xt} \mathcal{O}_{E,p_0} \to \mathcal{C}_{p_0} \to 0 \).

A minor extension of the argument to the relative case gives:

**Corollary 1.4.2.** Let \( B \) be a scheme, \( \mathcal{V} \to B \times E \) a flat family of torsion free semistable sheaves of degree zero and \( Q = Q(V) \) the corresponding sheaf on \( B \times E \). Then, as sheaves on \( B \),

\[
\begin{align*}
(i) \quad & R^0 \pi_1^* \mathcal{V} \cong \text{Ker}\{ xt: Q \to Q \}; \\
(ii) \quad & R^1 \pi_1^* \mathcal{V} \cong \text{Coker}\{ xt: Q \to Q \} = Q/tQ.
\end{align*}
\]

As an application, we have:

**Corollary 1.4.3.** Let \( B \) be a scheme, and let \( \mathcal{V} \to B \times E \) a family of torsion free semistable vector bundles of degree zero. Let \( \mathcal{W} \) be a vector bundle on \( B \). Then the relative extension sheaf \( \text{Ext}^1_{\mathcal{O}_B}(\mathcal{V}, \pi_1^* \mathcal{W}) \cong R^1 \pi_1^*(\mathcal{V}) \otimes \mathcal{W} \) is isomorphic to \((Q'/tQ') \otimes \mathcal{W} \), where \( Q' = Q(\mathcal{V}) \). \( \Box \)
2. Covers of the moduli space of \(G\)-bundles

Throughout the remainder of this paper, the rank of \(G\) will be denoted by \(r\), \(H\) is a maximal torus in \(G\), \(\mathfrak{h} = \text{Lie} \: H\), and \(\Lambda \subseteq \mathfrak{h}\) is the coroot lattice. The algebraic group \(E_{\text{reg}} \otimes \mathbb{Z} \Lambda\), is abstractly isomorphic to the \(r\)-fold Cartesian product \((E_{\text{reg}})^r\). If \(E\) is cuspidal, then \(E_{\text{reg}} \otimes \mathbb{Z} \Lambda \cong \mathfrak{h}\), and if \(E\) is nodal then \(E_{\text{reg}} \otimes \mathbb{Z} \Lambda \cong H\). The Weyl group \(W = W(G, H)\) acts on \(\Lambda\). We let \(M = (E_{\text{reg}} \otimes \mathbb{Z} \Lambda)/W\), and let \(\tilde{\nu}: E_{\text{reg}} \otimes \mathbb{Z} \Lambda \to M\) be the natural projection. For \(E\) smooth, \(M\) is the moduli space of semistable holomorphic \(G\)-bundles on \(E\), modulo S-equivalence. For \(E\) nodal, \(\tilde{M} = H/W\) can be identified with the adjoint quotient of \(G\) by the conjugation action of \(G\), where quotient is to be taken in the GIT sense, i.e. again up to S-equivalence. By \([15\, 56]\), \(H/W\) is an affine space \(\mathbb{A}^r\). Likewise, if \(E\) is cuspidal, then \(\tilde{M} = \mathfrak{h}/W\) is the (GIT) quotient of \(\mathfrak{g}\) via the adjoint action of \(G\). By Chevalley’s theorem, \(\mathfrak{h}/W\) is again an affine space \(\mathbb{A}^r\). (In case \(E\) is nodal or \(E\) is cuspidal and \(G\) is not of type \(E_8\), the parabolic construction gives a compactification of \(\tilde{M}\) \([6]\), which however will not concern us here.)

2.1. Definition of certain covers. Fix a primitive weight \(\mu\) for the pair \((G, H)\), i.e. a surjective homomorphism \(\Lambda \to \mathbb{Z}\) (minor modifications handle the case where \(\mu\) is not primitive). Let \(\Lambda_0 = \text{Ker} \: \mu\) and let \(W_0 = \text{Stab}_{W}(\mu) \subseteq W\). Set \(\tilde{T}_0 = (E_{\text{reg}} \otimes \mathbb{Z} \Lambda)/W_0\). Thus there is a finite morphism \(\tilde{\nu}: \tilde{T}_0 \to M\). Since \(W_0\) acts on \(\Lambda_0\), it also acts on \(E_{\text{reg}} \otimes \mathbb{Z} \Lambda_0\). We let \(M_0 = (E_{\text{reg}} \otimes \Lambda_0)/W_0\). The homomorphism \(\mu\) defines a \(W_0\)-invariant morphism \(E_{\text{reg}} \otimes \mathbb{Z} \Lambda \to E_{\text{reg}}\) and hence induces a morphism \(r: \tilde{T}_0 \to E_{\text{reg}}\).

**Lemma 2.1.1.** The space \(\tilde{T}_0\) is an étale fiber bundle over \(E_{\text{reg}}\) with fibers isomorphic to \(M_0\).

**Proof.** There exists a primitive \(v \in \Lambda\) which is \(W_0\)-invariant and whose image under \(\mu\) is \(d > 0\). Define the surjection \(\Lambda \oplus \mathbb{Z} \to \mathbb{Z}\) by \((\lambda, k) \mapsto \mu(\lambda) - kd\). Clearly, the kernel is exactly \(\Lambda_0 \oplus \mathbb{Z}\), included via the standard inclusion on \(\Lambda_0 \oplus \{0\}\) and where \((0, 1) \mapsto (v, 1)\). Tensoring with \(E_{\text{reg}}\) gives a \(W_0\)-equivariant isomorphism \(E_{\text{reg}} \otimes \mathbb{Z} (\Lambda_0 \oplus \mathbb{Z}) = (E_{\text{reg}} \otimes \mathbb{Z} \Lambda_0) \times E_{\text{reg}} \to (E_{\text{reg}} \otimes \mathbb{Z} \Lambda) \times E_{\text{reg}}\), where in the second factor the map \(E_{\text{reg}} \to E_{\text{reg}}\) is multiplication by \(d\). Taking the quotient by \(W_0\) gives an isomorphism \(\tilde{T}_0 \times_{E_{\text{reg}}} E_{\text{reg}} \cong M_0 \times E_{\text{reg}}\) as claimed. □

There is the following standard fact about groups generated by reflections:

**Lemma 2.1.2.** The group \(W_0\) is generated by the reflections in \(W\) which fix \(\mu\), and hence by the reflections in the roots dual to the coroots lying in \(\text{Ker} \: \mu = \Lambda_0\). □

In the main application, \(\mu\) will be a minuscule or quasiminuscule weight, and in particular will be a fundamental weight \(\varpi_\alpha\) for some simple root \(\alpha\) unless \(G\) is of type \(A_n\) and \(\mu\) is a root. Clearly, if \(\mu = \varpi_\alpha\) for some \(\alpha\), then \(\Lambda_0\) is spanned by the simple coroots \(\beta\) for \(\beta \neq \alpha\). More generally, if \(\Lambda_0\) is spanned by the coroots it contains, then \(\Lambda_0\) is the coroot lattice for a subroot system \(R_0\) of \(R\) of rank \(n - 1\) and \(W_0\) contains the Weyl group of \(R_0\). In fact, it follows from Lemma 2.1.2 that \(W_0\) is exactly the Weyl group of \(R_0\).

Next we turn to the possible singularities of the moduli space and their relation to the singularities of \(\tilde{T}_0\). Let \(M_{\text{sing}}\) be the singular locus of \(M\), and let \(M_{\text{reg}} = \)
\( \mathcal{M} - \mathcal{M}_{\text{sing}} \). Note that \( \mathcal{M}_{\text{reg}} = \mathcal{M} \) if \( E \) is singular, or if \( E \) is smooth and \( G \) is of type \( A_n \) or \( C_n \), and in no other cases.

**Lemma 2.1.3.** Suppose that \( E \) is smooth. In the above notation, \((\hat{T}_0)_{\text{sing}} \subseteq \hat{\nu}^{-1}(\mathcal{M}_{\text{sing}})\).

**Proof.** Let \( K \) be the compact form of \( G \). Viewing a point \( e \in E \otimes_\mathbb{Z} \Lambda \) as corresponding to a pair of elements \((x, y) \in T \times T\), where \( T \) is the maximal torus of \( K \), let \( K_e \) be the centralizer of \( \{x, y\} \) in \( K \). By [1] Theorem 6.12, the image of \( e \) is a singular point of \( \mathcal{M} \) if and only if \( \pi_0(K_e) \neq \{1\} \). Let \( R_e \) be the set of roots \( \alpha \) such that \( e \) is in the kernel of the homomorphism \( E \otimes_\mathbb{Z} \Lambda \to E \) defined by \( \alpha \), and let \( W(R_e) \) be the subgroup of \( W \) generated by reflections in the elements of \( R_e \). A formal argument (cf. [1] Lemma 7.1.4) shows that there is an exact sequence

\[ \{1\} \to W(R_e) \to \text{Stab}_W(e) \to \pi_0(K_e) \to \{1\}. \]

Then the image of \( e \) in \( \mathcal{M} \) is a singular point of \( \mathcal{M} \) if and only if \( \text{Stab}_W(e) \neq W(R_e) \).

Suppose that \( e \in E \otimes_\mathbb{Z} \Lambda \) lies over a smooth point of \( \mathcal{M} \). Then \( \text{Stab}_W(e) = W(R_e) \). Thus \( \text{Stab}_W(e) \) is the stabilizer in \( W(R_e) \) of \( \mu \), and hence by Lemma 2.1.3, it is generated by the reflections in \( W(R_e) \) which fix \( \mu \). These generators fix \( e \in E \otimes_\mathbb{Z} \Lambda \) and act as reflections on the tangent space to \( E \otimes_\mathbb{Z} \Lambda \) at \( e \). Hence by Chevalley’s theorem \( \hat{T}_0 \) is smooth in a neighborhood of the image of \( e \).

\[ \square \]

### 2.2. Existence of universal conformal bundles.

We now discuss the construction of universal \( G \)-bundles and universal vector bundles over \( \mathcal{M} \times E \). Let \( E \) be a smooth elliptic curve. Recall the following from [5]: for each \( G \), there is

- An integer \( n > 0 \) and an element \( c \in Z(G) \) of order \( n \);
- A \( G \)-bundle \( \Xi_0 \) over \( \mathbb{A}^{r+1} \times E \) and an action of \( \mathbb{C}^* \) on \( \Xi_0 \), lifting a linear action on \( \mathbb{A}^{r+1} \),

with the following properties:

- \( \Xi_0/\{x\} \times E \) is semistable for all \( x \neq 0 \);
- the induced \( \mathbb{C}^* \)-invariant morphism \( \mathbb{A}^{r+1} - \{0\} \to \mathcal{M} \) induces an isomorphism from \( \mathbb{A}^{r+1} - \{0\}/\mathbb{C}^* \) to \( \mathcal{M} \).

Moreover, the primitive \( n \)th roots of unity act trivially on \( \mathbb{A}^{r+1} \), the quotient \( \mathbb{C}^*/(\mathbb{Z}/n\mathbb{Z}) \) acts effectively, and the action of \( \mathbb{Z}/n\mathbb{Z} \) on \( \Xi_0 \) is via \( \langle c \rangle \). It follows that, away from the singularities of \( \mathcal{M} \), i.e. away from the non-free \( \mathbb{C}^* \)-orbits, there is an induced \( G/\langle c \rangle \) bundle.

In practice, we shall modify this construction slightly. Let \( \hat{G} = G \times_{\mathbb{Z}/n\mathbb{Z}} \mathbb{C}^* \), where \( 1 \in \mathbb{Z}/n\mathbb{Z} \) acts via \( c \in G \) and as a primitive \( n \)th root of unity \( \zeta \) in \( \mathbb{C}^* \). Given \( \zeta \), write \( \zeta = \exp(2\pi \sqrt{-1}a/n) \), where \( \gcd(a, n) = 1 \). Of course, we could always fix \( a = 1 \) if desired. There is a \( \mathbb{C}^* \)-linearization of the trivial \( \mathbb{C}^* \)-bundle over \( \mathbb{A}^{r+1} \), covering the given action on the base, as follows: \( t(\lambda, x) = (t^{-a} \lambda, t \cdot x) \). Taking the product bundle \( \mathbb{C}^* \times \Xi_0 \) with its product linearization, we see that there is an induced \( \hat{G} \)-bundle \( \hat{\Xi}_0 \) over \( \mathcal{M}_{\text{reg}} \times E \).

**Lemma 2.2.1.** For each \( x \in \mathcal{M}_{\text{reg}} \), the \( \hat{G} \)-bundle \( \hat{\Xi}_0/\{x\} \times E \) has a canonical lift to a \( G \)-bundle \( \xi_x \), which is the regular \( G \)-bundle corresponding to the point \( x \).
Proof. Begin with the central extension
\[ \{1\} \to G \to \hat{G} \to \mathbb{C}^* \to \{1\}. \]
By construction, the bundle \( \hat{\xi}_x = \hat{\Xi}_a \{x\} \times E \) lifts to a \( G \)-bundle. Let \( \det \hat{\xi}_x \) be the \( \mathbb{C}^* \)-bundle induced from the quotient map \( \hat{G} \to \mathbb{C}^* \). The inclusion \( \mathbb{C}^* \subseteq \hat{G} \) defines a surjective homomorphism \( \text{Aut} \hat{\xi}_x \to \text{Aut} \det \hat{\xi}_x = \mathbb{C}^* \), and hence the lift of \( \hat{\xi}_x \) is unique. \( \Box \)

For \( E \) nodal or cuspidal, we could use slight variations of this construction (with special care in case \( E \) is cuspidal and \( G = E_8 \)). In fact, the moduli space \( \mathcal{M} \) is isomorphic to an affine subspace \( \mathbb{A}^r \) in \( \mathbb{A}^{r+1} - \{0\} \) as we show in \[8\], and hence there is a universal \( G \)-bundle \( \Xi \) over \( \mathcal{M} \times E \). On the other hand, the Steinberg and Kostant sections already give us universal \( G \)-bundles as described in \[8\]. As we shall show in \[7\], all of these sections are conjugate in the appropriate sense, so that the choice does not really matter. However, in the next section, for the cuspidal case, we shall use the explicit form of the Kostant section.

Now suppose that \( \rho: G \to GL(V) \) is an irreducible representation. Then \( \rho(e) = \zeta' \text{Id} \), where \( \zeta' \) is a root of unity whose order divides \( n \). Thus there is \( b \in \mathbb{Z} \) such that \( \zeta^b = \zeta' \). The map \( \chi_b(t) = t^b \text{Id} \) is a homomorphism from \( \mathbb{C}^* \) to \( GL(V) \), and \( \rho \otimes \chi_b \) factors to give a representation \( \rho_b: G \to GL(V) \). There is then the associated bundle \( V_{a,b} = \hat{\Xi}_a \times_G V \) over \( \mathcal{M}_{reg} \times E \), where \( \hat{G} \) acts on \( V \) via \( \rho_b \). It is easy to see that the isomorphism type of the vector bundle \( V_{a,b} \) only depends on the product \( ab \). For each \( x \in \mathcal{M}_{reg}, V_{a,b} \{x\} \times E = \xi_x \times_G V = V_x \), and hence does not depend on \( a, b \). Since the image of \( \rho \) is contained in \( SL(V) \), the bundle \( V_x \) has trivial determinant. Thus \( \det V_{a,b} \) is pulled back from the factor \( \mathcal{M}_{reg} \). We can take the associated spectral cover \( T_{V_{a,b}} \) in the sense of \( \S 1 \), and it is also independent of \( a, b \). Finally, it is not hard to see that \( Q(V_{a,b}) \) is independent of the choice of \( (a,b) \), up to twisting by a line bundle on \( T_{V_{a,b}} \). Fix some pair \((a,b)\), and set \( V = V_{a,b} \).

The fiber of \( T_V \) over \( x \) only depends on the \( S \)-equivalence class of \( \xi_x \). Thus we have:

Lemma 2.2.2. Let \( \mu_1, \ldots, \mu_n \) be the weights of \( V \), counted with multiplicity. Suppose that \( \xi_x \) is \( S \)-equivalent to the split bundle corresponding to the point \( e \in E \otimes \mathbb{Z} \Lambda \). Then the fiber over \( x \) of \( T_V \) is the divisor \( \sum_{i=1}^n \mu_i(e) \).

Corollary 2.2.3. Suppose that \( V \) is an irreducible representation, with lowest weight \( \mu \). Let \( T_0 = (E \otimes \mathbb{Z} \Lambda)/W_0 \) be as in \( \S 2.1 \). Then there is a component \( T_0 \) of \( T_V \), which has multiplicity one in \( T_V \), and a finite birational morphism \( \varphi: \hat{T}_0 \to T_0 \) covering the identity on \( \mathcal{M} \), where the composite map \( \hat{T}_0 \to T_0 \to \mathcal{M} \times E_{reg} \) is \((\hat{\nu}, r)\). Thus \( \hat{T}_0 \) is identified with the normalization of a component \( T_0 \) of \( T_V \).

Proof. The reduction of the spectral cover \( T_V \subseteq \mathcal{M} \times E \) is the set of all points \((x,p)\) such that there exists a lift of \( x \) to \( e \in E_{reg} \otimes \mathbb{Z} \Lambda \) and a weight \( \mu_i \) such that \( \mu_i(e) = p \). Define the map \( \hat{\varphi}: E_{reg} \otimes \mathbb{Z} \Lambda \to T_V \) by \( \hat{\varphi}(e) = (\hat{\nu}(e), \mu(e)) \), where \( \hat{\nu}: E_{reg} \otimes \mathbb{Z} \Lambda \to \mathcal{M} \) is the quotient morphism. Clearly, \( \hat{\varphi} \) descends to a morphism \( \varphi: \hat{T}_0 \to T_V \), covering the projection to \( \mathcal{M} \), whose image is set-theoretically a component \( T_0 \) of \( T_V \). Since all weights in \( W \cdot \mu \) have multiplicity one, \( T_0 \) is generically reduced and hence is reduced. It will suffice to show that \( \varphi \) has degree one. Since both \( \hat{T}_0 \) and \( T_0 \) map to \( \mathcal{M} \) with degree \( n \), this is clear. \( \Box \)
2.3. Some lemmas on weights. In this section, our goal is to analyze the morphism $T_0 \to T_0$ of Corollary 2.3 more carefully. To do so, we shall have to analyze the weights in the $W$-orbit of $\mu$. Given two weights $\mu_1, \mu_2$, define the divisor $D(\mu_1 - \mu_2) \subseteq \mathcal{E}_{\text{reg}} \otimes \mathbb{Z} \Lambda$ via

$$D(\mu_1 - \mu_2) = \{e \in \mathcal{E}_{\text{reg}} \otimes \mathbb{Z} \Lambda : \mu_1(e) = \mu_2(e)\}.$$ 

More generally, for any weight $\lambda$, let $D(\lambda) = \text{Ker}\{\lambda : \mathcal{E}_{\text{reg}} \otimes \mathbb{Z} \Lambda \to \mathcal{E}_{\text{reg}}\}$.

**Proposition 2.3.1.** Let $V$ be an irreducible representation of $G$ with lowest weight $\mu$. Suppose that $\mu_1, \mu_2 \in W : \mu$ are distinct weights of $V$ in the $W$-orbit of $\mu$. Then exactly one of the following holds:

(i) $\mu_1 - \mu_2 = k\alpha$ for some root $\alpha$ and $k \in \mathbb{Z}$. In this case, $\mu_2$ is the unique element of $W : \mu$ such that $\mu_1 - \mu_2$ is a nonzero real multiple of $\alpha$.

(ii) There is an open dense subset of $D(\mu_1 - \mu_2)$ on which no root vanishes.

**Proof.** First suppose that $\mu_1 - \mu_2 = k\alpha$ for some root $\alpha$. Since the difference of two weights of an irreducible representation is a sum of roots, and the roots are primitive elements in the root lattice, $k \in \mathbb{Z}$. The affine line $\mu_1 + t\alpha$ meets the orbit $W : \mu$ at least at the two points $\mu_1, \mu_2$. Since every element of $W : \mu$ has the same length (with respect to any Weyl invariant inner product) and an affine line can meet a sphere in at most two points, we have proved (i).

Conversely, suppose that $\mu_1 - \mu_2$ is not a multiple of any root. Then, for every root $\alpha$, $	ext{Ker}(\mu_1 - \mu_2)$ and $	ext{Ker} \alpha$ are distinct hyperplanes in $\mathfrak{h}$. It follows that, for every root $\alpha$, $D(\alpha)$ meets $D(\mu_1 - \mu_2)$ in a proper subset. Since there are only finitely many roots, (ii) follows. \hfill $\square$

Let us analyze these two possibilities separately. Before we do so, let us introduce the following notation:

**Definition 2.3.2.** For $e \in \mathcal{E}_{\text{reg}} \otimes \mathbb{Z} \Lambda$, let $\xi_e$ be the regular $G$-bundle whose $S$-equivalence class is the image of $e$ in $\mathcal{M}$. Let $\mathcal{V}_e$ be the vector bundle $\xi_e \times_G V$.

**Lemma 2.3.3.** In the above notation, suppose that $\mu_1 - \mu_2$ is not a multiple of a root. Then there is an open dense subset $U$ of $D(\mu_1 - \mu_2)$ such that, for all $e \in U$, the following hold:

(i) $\text{Stab}_W(e) = \{1\}$.

(ii) The multiplicity of every point of the divisor $\sum_{w \in W/W_\alpha}(w\mu)(e)$ is at most two.

(iii) If $\xi'$ is the $H$-bundle defined by $e$, then $\xi_e = \xi' \times_H G$, and the vector bundle $\mathcal{V}_e$ defined in Definition 2.3.2 is a direct sum of line bundles.

**Proof.** An element $w \in W$ fixes a codimension one subset of $\mathcal{E}_{\text{reg}} \otimes \mathbb{Z} \Lambda$ if and only if $w$ fixes a codimension one subset of $\mathfrak{h}$ if and only if $w = r_\alpha$ is the reflection in a root $\alpha$. In this case, since $r_\alpha(e) = e - \alpha(e)\alpha^\vee$ and $\alpha^\vee$ is primitive, the fixed set of $r_\alpha$ is $D(\alpha)$. Since by hypotheses $D(\alpha) \neq D(\mu_1 - \mu_2)$, (i) follows for an open dense subset of $D(\mu_1 - \mu_2)$.

To prove (ii), first suppose that $\mu', \mu'' \in W : \mu$ and that $\mu' - \mu''$ vanishes along a component of $D(\mu_1 - \mu_2)$. Then, in $\mathfrak{h}$, $\text{Ker}(\mu' - \mu'') = \text{Ker}(\mu_1 - \mu_2)$. Hence $\mu' - \mu''$ is a multiple of $\mu_1 - \mu_2$. Thus, if $\mu', \mu'', \mu'''$ are three distinct elements of $W : \mu$ and $\mu', \mu'', \mu'''$ all agree along a component of $D(\mu_1 - \mu_2)$, then $\mu' - \mu''$ and $\mu'' - \mu'''$
are both multiples of $\mu_1 - \mu_2$, and so $\mu', \mu'', \mu'''$ are all contained in an affine line in $\mathfrak{h}\mathbb{R}$. But this is impossible since, as $\mu', \mu'', \mu'''$ are all conjugate via $W$, they all have the same length, and a sphere meets an affine line in at most two points.

To see (iii), it is clear that $\xi_e$ is regular since the regular bundle corresponding to $e$ fails to be split if and only if a root vanishes on $e$ (by [4, 6.2], in case $E$ is smooth, and by the direct descriptions of §1.1 in case $E$ is singular). The vector bundle $\nu_e$ is thus a direct sum of line bundles.

The following handles the case where $\mu_1 - \mu_2$ is a multiple of a root:

**Lemma 2.3.4**. In the above notation, suppose that $\mu_1 - \mu_2 = k\alpha$ for some root $\alpha$ and some $k \in \mathbb{Z}$, $k \neq 0$. Then there is an open dense subset $U$ of $D(\mu_1 - \mu_2) = D(k\alpha)$ such that, for all $e \in U$, the following hold:

(i) If $\mu', \mu'' \in W \cdot \mu$ and $\mu'(e) = \mu''(e)$, then $\mu' - \mu'' = \ell\alpha$ for some $\ell \in \mathbb{Z}$.

(ii) The multiplicity of every point of the divisor $\sum_{w \in W/W_\alpha} (w\mu)(e)$ is at most two.

(iii) $\text{Stab}_W(e) = \{1, r_\alpha\}$ if $e \in D(\alpha) \subseteq D(k\alpha)$ and $\text{Stab}_W(e) = \{1\}$ if $e \in D(k\alpha) \setminus D(\alpha)$. Moreover, $r_\alpha(\mu') = \mu''$ for each pair of distinct weights $\mu', \mu'' \in W \cdot \mu$ such that $\mu'(e) = \mu''(e)$.

(iv) If $e \in D(k\alpha) \setminus D(\alpha)$, then $\xi_e$ is split, and $\nu_e$ is a direct sum of line bundles.

(v) Suppose that $e \in D(\alpha)$, and let $G_\alpha$ be the connected subgroup of $G$ whose Lie algebra is $\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{h}$. Then $G_\alpha$ is isomorphic to $\text{SL}_2 \times H'$ or to $\text{SL}_2 \times_{\mathbb{Z}/2\mathbb{Z}} H'$, where $H'$ is the connected subtorus of $H$ whose Lie algebra is $\text{Ker} \alpha$. If $\xi_e$ is the corresponding regular bundle, then $\xi_e = \xi' \otimes (\text{SL}_2 \times H') G$, where $\xi' = \xi_\xi \otimes \xi_2$ is the bundle over $\text{SL}_2 \times H'$ such that $\xi_1$ is the principal $\text{SL}_2$-bundle corresponding to $I_2$ and $\xi_2$ is the $H'$-bundle such that $\xi' \otimes H' G$ is the split bundle whose image in $\mathcal{M}$ is $e$.

**Proof.** If $\mu', \mu'' \in W \cdot \mu$ and $\mu' - \mu''$ vanishes on a nonempty open subset of $D(\mu_1 - \mu_2)$, then $\mu' - \mu'' = \ell(\mu_1 - \mu_2) = \ell\alpha$ for some real number $\ell$. By the proof of Proposition 2.3.1, $\ell \in \mathbb{Z}$, proving (i). Part (ii) and the first part of (iii) follow as in the proof of the previous lemma. The second part follows from the fact [3, p. 124, Prop. 3(i)] that $\mu' - j\alpha$ is a weight whose length is the same as that of $\mu'$ if and only if $j = \mu'(\alpha')$, and in this case by definition $\mu' - j\alpha = r_\alpha(\mu')$. Parts (iv) and (v) follow from the explicit description of the regular representative $\xi_e$ given in [4] in the smooth case, and by the discussion of §1.1 in the singular case.

We can now describe the morphism $\varphi: \tilde{T}_0 \to T_0 \subseteq T_V$ in more detail. Recall that $\varphi$ is the morphism induced from $\tilde{\varphi}: E_{\text{reg}} \otimes_{\mathbb{Z}} \Lambda \to T_0 \subseteq \mathcal{M} \times E$ defined by $\tilde{\varphi}(e) = (\nu(e), \mu(e))$, where $\nu: E_{\text{reg}} \otimes_{\mathbb{Z}} \Lambda \to \mathcal{M}$ is the quotient map.

**Lemma 2.3.5.** In the above notation, suppose that $e \in E_{\text{reg}} \otimes_{\mathbb{Z}} \Lambda$, and let $y \in \tilde{T}_0$ be the image of $e$.

(i) If, for all $\mu' \in W \cdot \mu$ such that $\mu' \neq \mu$, $e \notin D(\mu - \mu')$, then $\varphi^{-1}(\varphi(y)) = \{y\}$, and the morphism $\varphi: \tilde{T}_0 \to T_0$ is an isomorphism in a neighborhood of $y$.

(ii) If $\mu' \in W \cdot \mu$, $\mu' \neq \mu$, and $e$ is a generic point of $D(\mu - \mu')$, where either $\mu - \mu'$ is not a multiple of a root, or $\mu - \mu' = k\alpha$ for some root $\alpha$ and $e \in D(k\alpha) \setminus D(\alpha)$, then there is an open neighborhood $U$ in $T_0$ of $\varphi(y)$ such
that \( \varphi^{-1}(U) = U_1 \cup U_2 \), \( \varphi|_{U_1} \) is an isomorphism onto a smooth divisor in \( \mathcal{M} \times E \), and \( \varphi(U_1) \) meets \( \varphi(U_2) \) transversally.

(iii) If \( \mu' \in W \cdot \mu \), \( \mu' \neq \mu \), and \( e \) is a generic point of \( D(\mu - \mu') \), where \( \mu - \mu' = k\alpha \) for some root \( \alpha \) and \( e \in D(\alpha) \), then \( \varphi: \hat{T}_0 \to T_0 \) is an isomorphism in a neighborhood of \( y \).

Proof. Suppose that \( e \notin D(\mu - \mu') \) for every \( \mu' \in W \cdot \mu \) such that \( \mu' \neq \mu \). Given \( y' \in \hat{T}_0 \) which is the image of \( e' \in E_{\text{reg}} \otimes \Lambda \), suppose that \( \varphi(y) = \varphi(y') \). Then \( e' = w(e) \) for some \( w \in W \) and \( \mu(e') = \mu(we) = \mu(e) \). It follows that \( w\mu = \mu \), so that \( w \in W_0 \), i.e. \( e \) and \( e' \) have the same image \( y \in \hat{T}_0 \). Thus \( \varphi^{-1}(\varphi(y)) = \{y\} \). If \( w \in \text{Stab}_W(e) \), then \( (we)(e) = \mu(e) \), and so \( w\mu = \mu \), i.e. \( w \in W_0 \). Thus \( \text{Stab}_W(e) = \text{Stab}_{W_0}(e) \), so that the morphism \( \hat{\nu}: \hat{T}_0 \to \mathcal{M} \) is a local diffeomorphism near \( y \). Since \( \hat{\nu} = \pi_1 \circ \varphi \), the morphism \( \varphi \) must also be a local diffeomorphism onto its image near \( y \). This proves (i).

Now suppose that the hypotheses of (ii) hold and let \( e' \) and \( y' \) be as above. By (ii) of Lemma 2.3.3 or (ii) of Lemma 2.3.4, if \( e \) is generic, then \( \mu' \) is the unique element of \( W \cdot \mu \) such that \( \mu'(e) = \mu(e) \). Suppose that \( \mu' = w\mu \). If \( \varphi(y') = \varphi(y) \), then necessarily \( e' = w^{-1}(e) \), and so \( \varphi^{-1}(\varphi(y')) = \{y, y'\} \). Moreover, since \( \text{Stab}_W(e) = \{1\} \), \( y \neq y' \) and the map \( \hat{T}_0 \to \mathcal{M} \) is a local diffeomorphism onto its image at both \( y \) and \( y' \). Hence the same is true for \( \varphi \). Since \( w\mu = \mu' \neq \mu \), it is easy to see that the two divisors meet transversally at \( \varphi(y) \). This proves (ii).

Finally suppose that we are in Case (iii). In this case \( \text{Stab}_W(e) = \{1, r_\alpha\} \) and \( \mu' = r_\alpha(\mu) \). Thus \( \text{Stab}_{W_0}(e) = \{1\} \). It follows as in the previous case that, if \( \varphi(y') = \varphi(y) \), then \( e' = r_\alpha(e) = e \). Thus \( \varphi^{-1}(\varphi(y)) = \{y\} \). For \( e \) generic, we can identify the tangent space to \( \hat{T}_0 \) at \( y \) with \( \mathfrak{h} \) and the kernel of the differential of the map from \( \hat{T}_0 \) to \( \mathcal{M} \) can be identified with \( \mathbb{C} \cdot \alpha^\vee \subseteq \mathfrak{h} \). Since \( \mu(\alpha^\vee) \neq 0 \), it follows that the differential of \( \varphi \) at \( y \) is injective. This proves (iii).

2.4. The case of a minuscule or quasinominuscule representation. We now apply the above results in case the representation \( \rho \) is minuscule or quasinominuscule. Given \( \rho \), for each pair of integers \((a, b)\), we have constructed a vector bundle \( V_{a,b} \) in §2.2, and the spectral cover \( T_{V_{a,b}} \) is independent of \((a, b)\). As before, we fix some pair \((a, b)\) and set \( V = V_{a,b} \).

**Theorem 2.4.1.** Let \( E \) be singular, and suppose that \( \rho \) is minuscule. Then there exists a line bundle \( M \) on \( T_0 \) such that \( V = (\hat{\nu} \times \text{Id})_*(\pi_1^* M \otimes (r \times \text{Id})^* \mathcal{P}) \). A similar result holds if \( E \) is smooth, provided that we replace \( \hat{\nu}: \hat{T}_0 \to \mathcal{M} \) by \( \hat{\nu}: (T_0)_{\text{reg}} \to \mathcal{M}_{\text{reg}} \).

**Proof.** In the minuscule case, \( T_0 = T_V \). Note that the pullback of the Poincaré line bundle on \( T_V \times E \) to \( \hat{T}_0 \) is simply \((r \times \text{Id})^* \mathcal{P}\). Thus, it suffices to show that the normalization morphism \( \varphi: \hat{T}_0 \to T_V \) and the bundle \( V \) satisfy the hypotheses of Proposition 1.3.3. Let \( D_0 \) be the union in \( \mathcal{M} \) of the images of the divisors \( D(\mu - \mu') \), and let \( D'_0 \) be the dense open subset of \( D_0 \) implicitly defined by Lemmas 2.3.3 and 2.3.4. Note that, since \( \rho \) is minuscule, if the difference \( \mu - \mu' \) of two weights is a multiple of a root \( \alpha \), then in fact this multiple is \( \pm 1 \), so that Case (iv) of Lemma 1.3.3 does not arise.
If \( x \in \mathcal{M} \) does not lie in \( D_0 \), then \( T_V \) is smooth over \( x \) and \( V_x \) is regular. If \( x \in D_0 \) lies under a point of \( D(\mu - \mu') \) satisfying the hypotheses of Lemma \[ \ref{lem:smoothness} \], then \( V_x \) satisfies (ii)(a) (with \( V_x(q) \) of rank one) or (ii)(b) of Proposition \[ \ref{prop:regularity} \]. Otherwise, we are in case (v) of Lemma \[ \ref{lem:smoothness} \] and \( x \in \mathcal{M} \) lies under a generic point \( e \in D(\alpha) \) for some root \( \alpha \). In this case, there is a connected subgroup \( G_0 \) of \( G \) isomorphic to \( SL_2 \times H' \) or to \( SL_2 \times Z_{2/2} H' \), the structure group of \( \xi \) reduces to \( G_0 \), and lifts to \( SL_2 \times H' \), and on the \( SL_2 \) factor it is the principal bundle corresponding to \( I_2 \). Under the induced action of \( SL_2 \) on \( V, V \) decomposes into \( \bigoplus_i W_i \oplus \bigoplus_j U_j \), where the \( W_i \) are irreducible, nontrivial representations of \( SL_2 \), and the \( U_i \) are the trivial representation. Since \( \rho \) is minuscule, \( \dim W_i = 2 \) by e.g. \[ \ref{ref:2} \] p. 128, Prop. 7]. Thus, for the action of \( SL_2 \times H' \) on \( V, V \) decomposes into \( \bigoplus_i (W_i \otimes \chi_i) \oplus \bigoplus_j (U_j \otimes \chi_j) \), where the \( \chi_i \) and \( \chi_j \) are characters of \( H' \). It then follows that the vector bundle \( V_x \) associated to \( \xi \) is of the form \( \bigoplus_i (J_2 \otimes \lambda_i) \oplus \bigoplus_j \lambda_j \). Here the \( \lambda_i \) in the first summand are the line bundles of degree zero corresponding to \( \mu'(e) \) for \( \mu' \in W \cdot \mu \) such that \( \mu' - \mu'' = \pm \alpha \) for some \( \mu'' \in W \cdot \mu \), and the \( \lambda_i \) in the second summand are the line bundles of degree zero corresponding to \( \mu'(e) \) for \( \mu' \in W \cdot \mu \) such that \( \mu'(e) \neq \mu''(e) \) if \( \mu'' \in W \cdot \mu, \mu'' \neq \mu' \). In particular, the line bundles corresponding to the various summands are all distinct. Thus, the vector bundle associated to \( \xi \) is regular. In this case, \( \varphi \) fulfills the hypotheses of (ii)(a) of Proposition \[ \ref{prop:regularity} \], completing the proof of the theorem. \[ \square \]

**Remark 2.4.2.** It is an interesting problem to identify the line bundle \( M \) of Theorem \[ \ref{thm:spectral_cover} \] in case \( E \) is smooth, where it will depend on the choice of \((a, b)\). For groups of type \( A_n \) or \( C_n \), the answer is essentially contained in \[ \ref{ref:2} \] Corollary 3.4].

Now we turn to quasiminuscule representations.

**Theorem 2.4.3.** Let \( E \) be singular, and suppose that \( \rho \) is quasiminuscule but not minuscule. Define the sheaf \( \nabla \) by the exact sequence

\[
0 \to \pi_1^* \pi_1^* \mathcal{V} \to \mathcal{V} \to \nabla \to 0.
\]

Then \( \nabla \) is a vector bundle, and its associated spectral cover is \( T_0 \), the image of \( \tilde{T}_0 \). Moreover, there exists a line bundle \( M \) on \( \tilde{T}_0 \) such that

\[
\nabla = (\hat{\nu} \times \text{Id})_* (\pi_1^* M \otimes (r \times \text{Id})^* \mathcal{P}).
\]

A similar result holds if \( E \) is smooth, provided that we replace \( \hat{\nu}: \tilde{T}_0 \to \mathcal{M} \) by \( \hat{\nu}: (\tilde{T}_0)_{\text{reg}} \to \mathcal{M}_{\text{reg}} \).

**Proof.** Suppose that the multiplicity of the trivial weight in \( \rho \) is \( s \). Let \( e \in E_{\text{reg}} \otimes \mathbb{Z} \Lambda \) and let \( \xi_e \) be the regular vector bundle whose \( S \)-equivalence class corresponds to the image of \( e \) in \( \mathcal{M} \). We first claim that \( H^0(E; \xi_e \otimes GV) = s \) for every \( e \). Of course, if \( G \) is simply laced, then \( \rho \) is the adjoint representation, \( s = r \), and the claim is essentially the definition of a regular bundle. In case \( G \) is not simply laced, the nonzero weights are the short roots. Let \( e \in E_{\text{reg}} \otimes \mathbb{Z} \Lambda \) and let \( R_e \) be the set of roots which are trivial on \( e \). Let \( X \) be a principal nilpotent element for the reductive Lie algebra \( \mathfrak{g}_e \) which is generated by \( \mathfrak{h} \) and the root spaces \( \mathfrak{g}^\alpha, \alpha \in R_e \). By the recipe for constructing \( \xi_e \) given in \[ \ref{ref:4} \] in case \( E \) is smooth, and by the discussion in §1.1 in case \( E \) is singular, \( H^0(E; \xi_e \otimes GV) \) is given by the kernel of \( \rho_e(X) \) acting on \( V_e \), where \( V_e \) is the sum of the weight spaces (including those for the trivial weight).
which annihilate $e$. The trivial weight contributes a subspace of $V_c$ of dimension $s$ and the remaining weight spaces annihilating $e$ are one-dimensional subspaces of $V_c$ corresponding to the short roots in $R_c$. We may assume that we have chosen $X$ so that there is an $h \in h \subseteq \mathfrak{g}(e)$ such that $(h, X)$ completes to an $\mathfrak{s}\mathfrak{l}_2$-triple in $\mathfrak{g}(e)$. Thus $(\rho_\star(h_c), \rho_\star(X))$ completes to some $\mathfrak{s}\mathfrak{l}_2$-triple in End $V$. Since $X$ is principal in $\mathfrak{g}(e)$, for all $\alpha \in R_\alpha$, $\alpha(h_c)$ is an even nonzero integer. Thus the kernel of $\rho_\star(h_c)$ is the weight zero subspace of $V$ and so $\dim \ker\{\rho_\star(h_c) : V_c \to V_c\} = s$. Since all of the eigenvalues of $\rho_\star(h_c)$ are even, it follows from the classification of representations of $\mathfrak{s}\mathfrak{l}_2$ that $\dim \ker\{\rho_\star(X) : V_c \to V_c\} = s$ as well.

Since $h^0(E; \xi \times_G V)$ is constant, the direct image $\pi_1^* V$ is locally free of rank $s$ and the induced map $\pi_1^* V/\mathfrak{m}_x \pi_1^* V \to H^0(E; V_x)$ is an isomorphism for every $x \in M$. By semistability, the map $H^0(E; V_x) \otimes \mathcal{O}_E \to V_x$ is injective for every $x \in M$. It follows that the induced homomorphism $\pi_1^* \pi_1^* V \to V$ is injective on every fiber, and so the cokernel $\nabla$ is a subbundle as claimed. Clearly the spectral cover corresponding to $\pi_1^* \pi_1^* V$ is just the trivial section of $\mathcal{M} \times E$ with multiplicity $s$, and hence $T_\nabla = T_0$.

As in the proof of Theorem 2.5.1, we take $D_0$ to be the union in $\mathcal{M}$ of the images of the divisors $D(\mu - \mu')$, and $D'_0$ to be the dense open subset of $D_0$ implicitly defined by Lemmas 2.5.2 and 2.5.3. To complete the proof, we must analyze the corresponding bundles $\nabla_x$, where either $x \notin D_0$ or $x \in D'_0$. The arguments are very similar to those given in the proof of Theorem 2.5.1 and we shall be brief. The only essentially new case is when a weight, i.e. a short root $\alpha$, vanishes. In this case, for a generic $x \in D(\alpha)$, there is a unique factor of $V_x$ isomorphic to $I_2$, and the remaining summands of $V_x$ are of the form $I_2 \otimes \lambda_i$ or $\lambda_j$, where the $\lambda_i, \lambda_j$ are pairwise distinct nontrivial line bundles, together with $s - 1$ copies of $\mathcal{O}_E$. In this case, $\nabla_x$ has exactly the same summands, except that the $I_2$ factor is replaced by an $I_2$ and the remaining $s - 1$ copies of $\mathcal{O}_E$ are no longer present. Hence $\nabla_x$ is regular.

2.5. Some remarks on representations. Given a regular $G$-bundle $\xi$ and an irreducible representation $\rho : G \to GL(V)$ of $G$, when is the associated vector bundle $\xi \times_G V$ regular? In this section, we state some results along these lines. The proofs are straightforward and will not be given.

Proposition 2.5.1. Let $\rho : G \to GL(V)$ be an irreducible representation. Then the following are equivalent:

(i) For every smooth elliptic curve $E$ and every regular $G$-bundle $\xi$ over $E$, the associated vector bundle $\xi \times_G V$ is a regular vector bundle.

(ii) For every regular element $g \in G$, $\rho(g)$ is a regular element of $GL(V)$.

(iii) The multiplicity of every weight of $\rho$ is one, and for every pair $\mu_1, \mu_2$ of distinct weights of $\rho$, $\mu_1 - \mu_2$ is a root.

(iv) Either $G = SL_n$ and $\rho$ is the standard representation or its dual, or $G = Sp(2n)$ and $\rho$ is the standard representation.

A weaker but related question is the following: let $\xi$ be the regular $G$-bundle $S$-equivalent to the trivial $G$-bundle. When is $\xi \times_G V$ regular? The answer is as follows:

Proposition 2.5.2. Let $\rho : G \to GL(V)$ be an irreducible representation. Then the following are equivalent:
(i) For every smooth elliptic curve $E$ and every regular $G$-bundle $\xi$ over $E$ $S$-equivalent to the trivial bundle, the associated vector bundle $\xi \times_G V$ is a regular vector bundle.

(ii) For every regular element $X \in \mathfrak{g}$, $\rho_\ast(X)$ is a regular element of $\text{End} V$.

(iii) If $X$ is a principal nilpotent element in $\mathfrak{g}$, then $\rho_\ast(X)$ is a regular element of $\text{End} V$.

(iv) The multiplicity of every weight of $\rho$ is one, and for every pair $\mu_1, \mu_2$ of weights of $\rho$, $\mu_1 - \mu_2$ is a multiple of a root.

(v) The group $G$ and the representation $\rho$ are as in (iv) of Proposition 2.5.1. If $G = \text{SL}_2$ and $\rho$ is any irreducible representation, or $G$ is of type $B_n$ or $G_2$ and $\rho$ is the unique quasiminuscule, non-minuscule representation of $G$. □

3. Kostant and Steinberg sections and the proof of the main theorem

3.1. Conjugation into the slice. Every $\text{ad} G$-invariant function on $\mathfrak{g}$ may be identified with a $W$-invariant function on $\mathfrak{h}$. In this way, the adjoint quotient of $\mathfrak{g}$ is identified with $\mathfrak{h}/W$. Let $\psi: \mathfrak{g} \rightarrow \mathfrak{h}/W$ be the induced morphism, and let $\mathfrak{g}_{\text{reg}}$ be the open dense subset of $\mathfrak{g}$ consisting of regular elements. In [11], Kostant constructed a section of $\psi$ whose image lies in $\mathfrak{g}_{\text{reg}}$ as follows. Let $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{h}$, with nilpotent radical $\mathfrak{u}$. Thus, there is a unique set of positive roots $R^+$ for $(\mathfrak{g}, \mathfrak{h})$ such that $\mathfrak{u} = \bigoplus_{\alpha \in R^+} \mathfrak{g}\alpha$. Fix a principal $\mathfrak{sl}_2$-triple $(X, h_0, Y)$ adapted to $\mathfrak{b}$. By definition, this means that $X \in \mathfrak{u}$ is a principal nilpotent element of $\mathfrak{g}$, $[h_0, X] = 2X$, $[h_0, Y] = -2Y$, and $[X, Y] = h_0 \in \mathfrak{h}$. It follows automatically that $Y \in \mathfrak{u}$, $\mathfrak{u} = \bigoplus_{\alpha \in R^+} \mathfrak{g}\alpha$. Kostant has proved the following [11] Theorem 0.10:

**Theorem 3.1.1.** Let $Z$ be a vector subspace of $\mathfrak{g}$ which is complementary to $\text{Im}(\text{ad} X)$ and is invariant under $\text{ad} h_0$. Then the affine space $Z + X$ is contained in $\mathfrak{g}_{\text{reg}}$ and the morphism $Z + X \rightarrow \mathfrak{h}/W$ induced by the projection $\mathfrak{g} \rightarrow \mathfrak{h}/W$ is an isomorphism. □

We call the inverse morphism $\Sigma_Z: \mathfrak{h}/W \rightarrow Z + X \subseteq \mathfrak{g}_{\text{reg}}$ the Kostant section defined by $Z$. Denote the composition $\mathfrak{h} \rightarrow \mathfrak{h}/W \rightarrow Z + X \rightarrow \mathfrak{g}_{\text{reg}} \subseteq \mathfrak{g}$ by $\Sigma_Z: \mathfrak{h} \rightarrow \mathfrak{g}$. Clearly, $\Sigma_Z$ is a $W$-invariant morphism from $\mathfrak{h}$ to $\mathfrak{g}$.

We would like to compare the $\Sigma_Z$ for different choices of $Z$, and would also like to compare $\Sigma_Z$ with the inclusion $i: \mathfrak{h} \rightarrow \mathfrak{g}$. Of course, $\Sigma_Z$ can never be conjugate to the inclusion, not even pointwise, because not all elements of $\mathfrak{h}$ are regular in $\mathfrak{g}$. According to the next result, for every choice of $Z$, if we replace the inclusion morphism $i$ by $i + X$, then the resulting morphism is indeed conjugate to $\Sigma_Z$ via a morphism $\mathfrak{h} \rightarrow G$:

**Theorem 3.1.2.** Let $(X, h_0, Y)$ be a principal $\mathfrak{sl}_2$-triple adapted to $\mathfrak{b}$ and let $Z$ and $Z'$ be linear complements to $\text{Im}(\text{ad} X)$ invariant under $\text{ad} h_0$. Let $\Sigma_Z$ and $\Sigma_{Z'}$ be the Kostant sections defined by $Z$ and $Z'$ respectively. Let $i + X: \mathfrak{h} \rightarrow \mathfrak{g}$ be the map $h \mapsto h + X$. Then:

(i) $\Sigma_Z$ and $\Sigma_{Z'}$ are conjugate under the adjoint action of $G$ on $\mathfrak{g}$, i.e. there exists a morphism $\psi: \mathfrak{h}/W \rightarrow G$ such that $\text{Ad}(\psi) \circ \Sigma_Z = \Sigma_{Z'}$.

(ii) The map $i + X$ is an embedding of $\mathfrak{h}$ into $\mathfrak{g}_{\text{reg}}$. 

(iii) For each $h \in \mathfrak{h}$, $p(h + X) = p(h)$, so that $h + X$ and $h$ are identified in the adjoint quotient. More precisely, there is a one-parameter subgroup $\beta: \mathbb{C}^* \to G$ such that, for all $h \in \mathfrak{h}$,

$$\lim_{t \to 0} \text{Ad} \beta(t)(h + X) = h.$$ 

(iv) There exists a morphism $\Psi: \mathfrak{h} \to \mathfrak{u}_-$ such that, for all $h \in H$,

$$\text{Ad}(\exp(\Psi(h)))(h + X) = \Sigma_Z(h).$$

(v) The morphism $\Psi$ is the unique morphism from $\mathfrak{h}$ to $\mathfrak{u}_-$ with the property that, for all $h \in \mathfrak{h}$,

$$\text{Ad}(\exp(\Psi(h)))(h + X) \in Z + X.$$

By (i), all Kostant sections are equivalent under the adjoint action of $G$ on $\mathfrak{g}$, a result which is subsumed in the main result of [7]. By (iv), after making the base change $\mathfrak{h} \to \mathfrak{h}/W$, we can conjugate any Kostant section until its image lies in a Borel subalgebra. This is of course not possible for the Kostant section itself, since otherwise we would be able to find a section to the morphism $\mathfrak{h} \to \mathfrak{h}/W$.

We will deduce Theorem 3.1.2 from a more general result, proved in §3.2.

3.2. More general \(\mathfrak{sl}_2\)-triples. For the moment, let $(X, h_0, Y)$ denote an arbitrary \(\mathfrak{sl}_2\)-triple, not necessarily principal or adapted to $\mathfrak{b}$. The endomorphism $\text{ad} h_0$ of $\mathfrak{g}$ is semisimple and its eigenvalues are integers. Let $\mathfrak{g}_\ell$ be the $\ell$-eigenspace of $\text{ad} h_0$. Thus $\mathfrak{g}$ is graded and the bracket is compatible with the grading. Moreover, $X \in \mathfrak{g}_2$. Let $\mathfrak{g}_-$ be the nilpotent subalgebra $\bigoplus_{\ell \leq 0} \mathfrak{g}_\ell$, let $\mathfrak{g}_-$ be the corresponding connected unipotent subgroup of $G$, and let $\mathfrak{g}_{\leq 0} = \bigoplus_{\ell \leq 0} \mathfrak{g}_\ell$. By the degree $d(x)$ of an element $x \in \mathfrak{g}$, we mean the largest value of $\ell$ such that the component of $x$ in $\mathfrak{g}_\ell$ is nonzero. By the classification of finite-dimensional representations of $\mathfrak{sl}_2$, $\text{ad} X : \mathfrak{g}_\ell \to \mathfrak{g}_{\ell + 2}$ is injective if $\ell < 0$ and surjective if $\ell \geq -1$.

**Proposition 3.2.1.** Let $(X, h_0, Y)$ be an \(\mathfrak{sl}_2\)-triple. Let $Z \subseteq \mathfrak{g}_{\leq 0}$ be a complementary linear subspace to the image of $\text{ad} X$ invariant under $h_0$. Let $\tau \in \mathfrak{g}_{\leq 0}$. Then there is a unique $\psi \in \mathfrak{g}_-$ such that $\text{Ad}(\exp(\psi))(\tau + X) \in Z + X$.

**Proof.** Write $\tau = \tau_0 + \text{ad} X(\tau_1)$, where $\tau_0 \in Z$ and we may assume that $\tau_1 \in \mathfrak{g}_-$, in fact that $d(\tau_1) \leq -2$. The proof of existence of $\psi$ is by induction on $k = d(\tau_1) \leq -2$. If $k$ is sufficiently negative, then $\tau_1 = 0$ and we can take $\psi = 0$. Suppose that we know the result for all $\tau$ such that $d(\tau_1) \leq k - 1$, and let $\tau$ be an element such that $d(\tau_1) = k$. Let $\psi' = \tau_1$. Then

$$\text{Ad}(\exp(\psi'))(\tau + X) = \exp(\text{ad} \psi')(\tau + X)$$

$$= \tau + X - \text{ad} X(\tau_1) + \sum_{n=1}^{\infty} \frac{1}{n!} \text{ad}(\tau_1)^n(\tau) + \sum_{n=2}^{\infty} \frac{1}{n!} \text{ad}(\tau_1)^n(X).$$

We can write this as $\tau_0 + X + \alpha$, where $d(\alpha) \leq d(\tau_1)$. Let $\alpha = \tau'_0 + \text{ad} X(\tau_2)$, where $\tau'_0 \in Z$ and $\tau_2 \in \mathfrak{g}_-$. Since $Z$ is $h_0$-invariant, $d(\tau') \leq d(\alpha)$ and thus $d(\text{ad} X(\tau_2)) \leq d(\alpha)$. We can thus assume that $d(\tau_2) \leq d(\alpha) - 2 \leq k - 2$. Thus by induction there exists a $\psi_1$ such that $\text{Ad}(\exp(\psi_1))(\tau'_0 + \alpha + X) \in Z + X$. Since $\exp(\psi_1)$ and $\exp(\psi')$ lie in the unipotent group $G_-$, $\psi = \log(\exp(\psi_1) \exp(\psi'))$ is defined and clearly $\text{Ad}(\exp(\psi))(\tau + X) \in Z + X$. This completes the inductive step and proves the existence of $\psi$. 
To see the uniqueness, suppose that \( \text{Ad}(\exp(\psi_i))(\tau + X) \in Z + X \) for \( i = 1, 2 \). Writing \( \log(\exp(\psi_1)\exp(-\psi_2)) = \psi \), we see that it suffices to prove: if there exists \( \zeta \in Z \) and \( \psi \in \mathfrak{g}_- \) such that \( \text{Ad}(\exp(\psi))(\zeta + X) \in Z + X \), then \( \psi = 0 \). If \( \psi \neq 0 \), write \( \psi = \psi_k + \psi' \), where \( \psi_k \) is homogeneous of degree \( k < 0 \) and \( d(\psi') < k \). As before, we write

\[
\text{Ad}(\exp(\psi))(\zeta + X) = \zeta + X + [\psi_k, X] + \gamma,
\]

where \( d(\gamma) < d([\psi_k, X]) = k + 2 \). Since \( \zeta \in Z \) and \( Z \) is a vector subspace of \( \mathfrak{g} \), it follows that \([\psi_k, X] + \gamma \in Z \). Since \( Z \) is invariant under the action of \( h_0 \), each homogeneous component of \([\psi_k, X] + \gamma \) lies in \( Z \). But as \( d(\gamma) < d([\psi_k, X]) \), it follows that \([\psi_k, X] \in Z \). Since \( Z \) is a complement to the image of \( \text{ad} \), \([\psi_k, X] = 0 \), and hence \( \psi_k = 0 \) since \( \psi_k \in \mathfrak{g}_k \) and \( k < 0 \). This is a contradiction. Hence \( \psi = 0 \).

**Remark 3.2.2.** Suppose that \( \mathfrak{g}_{-1} = 0 \), which is the case if \((X, h_0, Y)\) is principal. Then, for all \( g \in G_- \), \( \text{Ad}(g)(X) = X \in \mathfrak{g}_{\leq 0} \). It follows that, for all \( g \in G_- \) and \( z \in \mathfrak{g}_- \), \( \text{Ad}(g)(z + X) = X \in \mathfrak{g}_{\leq 0} \). Thus

\[
(g, z) \mapsto \text{Ad}(g)(z + X) - X
\]
defines a morphism

\[
G_- \times (Z + X) \to \mathfrak{g}_{\leq 0}.
\]

It follows from Proposition 3.2.1 that this morphism defines an isomorphism from \( G_- \times (Z + X) \) to \( \mathfrak{g}_{\leq 0} \). In this essentially equivalent form, Proposition 3.2.1 is due to Kostant [12, Theorem 1.2].

**3.3. A parametrized version.** We continue to assume that \((X, h_0, Y)\) is an arbitrary \( \mathfrak{sl}_2 \)-triple. Let \( R \) be a finitely generated \( \mathbb{C} \)-algebra. The exponential map \( \exp: \mathfrak{g}_- \to G \) extends to a map \( \mathfrak{g}_- \otimes \mathbb{C} R \to G(R) \), where \( G(R) \) denotes the group of \( R \)-valued points of \( G \), in other words the set of morphisms \( \text{Spec} \, R \to G \). Viewing \( \mathfrak{g}_- \otimes \mathbb{C} R \) as the set of morphisms \( \text{Spec} \, R \to \mathfrak{g}_- \), this map is the composition \( \psi \mapsto \exp \circ \psi: \text{Spec} \, R \to G \). The group \( G(R) \) acts on \( \mathfrak{g} \otimes \mathbb{C} R \) via \( \text{Ad} \). We still have the formula \( \text{Ad}(\exp(\psi)) = \exp(\text{ad} \, \psi) \). From this, it is clear that the proof of Proposition 3.2.1 also proves:

**Proposition 3.3.1.** Let \((X, h_0, Y)\) be an \( \mathfrak{sl}_2 \)-triple. Let \( Z \subseteq \mathfrak{g}_{\leq 0} \) be a complementary linear subspace to the image of \( \text{ad} \, X \) invariant under \( h_0 \). Let \( \tau \in \mathfrak{g}_{\leq 0} \otimes \mathbb{C} R \). Then there is a unique \( \psi \in \mathfrak{g}_- \otimes \mathbb{C} R \) such that \( \text{Ad}(\exp(\psi))(\tau + X) \in Z \otimes \mathbb{C} R + X \subseteq \mathfrak{g} \otimes \mathbb{C} R \).

**Remark 3.3.2.** An examination of the inductive argument shows the following: suppose that \( R = \mathbb{C}[s_1, \ldots, s_n] \) is the coordinate ring of affine \( n \)-space with the usual grading, and that \( \tau \in \mathfrak{g}_0 \otimes \mathbb{C} R_1 \), so that the components of \( \tau \) are linear polynomials. Then \( \psi = \sum_{k=1}^N \psi_{-k} \), where \( \psi_{-k} \in \mathfrak{g}_{-2k} \otimes \mathbb{C} R_{k+1} \) and so the components of \( \psi_{-k} \) are homogeneous polynomials of degree \( k + 1 \).

We will also need the following lemma to prove Part (i) of Theorem 3.1.2.

**Lemma 3.3.3.** Let \( Z, Z' \subseteq \mathfrak{g}_- \) be two linear subspaces, both complementary to \( \text{Im} \, (\text{ad} \, X) \) and invariant under \( h_0 \). Then there exists a morphism \( \gamma: Z \to \mathfrak{g}_- \) with \( \gamma(0) = 0 \) such that the morphism

\[
z \mapsto \text{Ad}(\exp(\gamma(z)))(z + X) - X
\]
is an isomorphism from \( Z \) to \( Z' \).
Proposition 3.3.1 that there is a unique morphism \( \Psi: \sum \to \kappa \) such that \( \Psi = t \) for all \( z \in Z \). Clearly \( \gamma(0) = 0 \) by uniqueness. Define the morphism \( \alpha: Z \to Z' \) by \( \alpha(z) = \text{Ad}(\exp(\gamma(z)))(z + X) - X \). Symmetrically, there is a unique morphism \( \gamma': Z' \to g_- \) such that, for all \( z' \in Z' \), \( \text{Ad}(\exp(\gamma(z')))(z' + X) \in Z + X \). If we set \( \beta(z') = \text{Ad}(\exp(\gamma(z')))(z' + X) - X \), then \( \beta \) is a morphism from \( Z' \) to \( Z \). By the uniqueness statements, it follows that

\[
\text{Ad}(\exp(\gamma'(\alpha(z)))) \circ \text{Ad}(\exp(\gamma(z)))(z + X) = z + X
\]

for all \( z \in Z \), and hence that \( \beta \circ \alpha = \text{Id} \). Symmetrically, \( \alpha \circ \beta = \text{Id} \) as well, so that \( \alpha \) and \( \beta \) are inverse to one another. \( \square \)

3.4. Proof of Theorem 3.1.2 \( \) We now prove Theorem 3.1.2. Let \((X, h_0, Y)\) be a principal \( s_0 \)-triple adapted to \( \mathfrak{b} \). Since \( h_0 \) is regular, \( g_0 = \mathfrak{h} \) and \( g_- = u_- \). Let \( Z \) and \( Z' \) be complementary linear subspaces to \( \text{Im}(\text{ad} X) \) and invariant under \( h_0 \). Both \( Z \) and \( Z' \) lie in \( g_- \). By Lemma 3.3.3 there is a morphism \( \gamma: Z \to g_- \) such that \( z + X \mapsto \text{Ad}(\exp(\gamma(z)))(z + X) \) is an isomorphism \( I: Z + X \to Z' + X \). Since \( I \) is given by the adjoint action of \( G \) on \( g \), it follows that \( I \) is compatible with the projections to the adjoint quotient \( \mathfrak{h}/W \). In particular, \( \text{Ad}(\exp(\gamma)) \circ \sum h = \sum h' \), proving Part (i) of Theorem 3.1.2.

The map \( C \to \mathfrak{h} \) defined by \( a \mapsto ax0 \) exponentiates to a one-parameter subgroup \( \beta: C^* \to H \) such that \( \text{Ad} \beta(t)(X) = t^2 X \). Thus, for all \( h \in H \), \( \text{Ad} \beta(t)(h + X) = h + t^2 X \), and hence \( \lim_{t \to 0} \text{Ad} \beta(t)(h + X) = h \). This proves Part (iii) of Theorem 3.1.2.

Viewing the inclusion \( i: \mathfrak{h} \to g \) as a morphism from \( \mathfrak{h} \) to \( g_0 \), it follows from Proposition 3.3.1 that there is a unique morphism \( \Psi: \mathfrak{h} \to g_- \) such that

\[
\text{Ad}(\Psi(h))(h + X) \in Z + X
\]

for all \( h \in \mathfrak{h} \). This proves Part (v) of Theorem 3.1.2. Note that, by Remark 3.3.2, \( \Psi = \sum_{k=1}^{N} \Psi_k \), where \( \Psi_k \in g_- \otimes C R \) is a homogeneous polynomial of degree \( k + 1 \).

Now define \( \Sigma_Z: \mathfrak{h} \to Z + X \) by

\[
\Sigma_Z(h) = \text{Ad exp}(\Psi(h))(h + X).
\]

Lemma 3.4.1. The morphism \( \Sigma_Z \) is invariant under \( W \) and the induced morphism from \( \mathfrak{h}/W \) to \( Z + X \) is the Kostant section. In other words, \( \Sigma_Z = \Sigma_Z \).

Proof. Clearly \( \Sigma_Z(h) \) is conjugate to \( h + X \), and hence by Part (iii) of Theorem 3.1.2 maps to the same point of \( \mathfrak{h}/W \) as \( h \). Since there is a unique such point in \( Z + X \) with this property, namely \( \Sigma_Z(h) \), it follows that \( \Sigma_Z = \Sigma_Z \). \( \square \)

We have thus proved (iv) of Theorem 3.1.2. Since the image of the Kostant section lies in \( g_{\text{reg}} \), it follows that the image of \( i + X \) has this property as well, proving (ii) of Theorem 3.1.2. This completes the proof of the theorem. \( \square \)
3.5. Consequences for invariant polynomials. Let $\rho: G \to \text{Aut } V$ be an irreducible representation and let $\rho_*: \mathfrak{g} \to \text{End } V$ be the induced representation of Lie algebras. We can decompose $V$ into a direct sum of weight spaces $V_\mu$. Let $\mu$ be the lowest weight of $\rho$, with corresponding weight space $V_\mu$, and let $\pi_\mu: V \to V_\mu$ be the induced projection, i.e. $\pi_\mu$ is the identity on $V_\mu$ and is zero on $V_\nu, \nu \neq \mu$. As in §2, let $W_0$ be the stabilizer in $W$ of $\mu$.

The endomorphism $\rho_*(h_0)$ of $V$ is semisimple and its eigenvalues are integral. Thus $V = \bigoplus k V_k$, where $\rho_*(h_0)$ acts on $V_k$ with eigenvalue $k$, and each $V_k$ is a sum of weight spaces. Clearly $\rho_*(X)(V_k) \subseteq V_{k+2}$. The eigenspace $V_k$ corresponding to the minimal value of $k$ is one dimensional, and in fact is exactly $V_\mu$.

Let $S = \text{Sym}^* \mathfrak{h}^*$ be the affine coordinate ring of $\mathfrak{h}$. The groups $W$ and $W_0$ act on $S$, and we have the inclusions $S^W \subseteq S^{W_0} \subseteq S$, corresponding to the finite surjective morphisms $\mathfrak{h} \to \mathfrak{h}/W_0 \to \mathfrak{h}/W$. Note that the linear function $\mu$ is naturally an element of $S$ and in fact lies in $S^{W_0}$. Of course, $W$ acts on $\mathfrak{h}$ as a group generated by reflections. But, by Lemma 2.1.2, $W_0$ also acts on $\mathfrak{h}$ as a group generated by reflections. Thus, by Chevalley’s theorem both $S^W$ and $S^{W_0}$ are polynomial algebras. We wish to compare these polynomial algebras.

**Lemma 3.5.1.** The ring $S$ is a faithfully flat extension of $S^{W_0}$. Likewise, $S^{W_0}$ is a faithfully flat extension of $S^W$. In particular, as an $S^W$-module, $S^{W_0}$ is free of rank $\#(W/W_0)$.

**Proof.** This is an immediate consequence of the fact that $S$, $S^{W_0}$, and $S^W$ are regular and that the morphisms in question are finite and surjective.

For the rest of this section we fix a linear complement $Z \subseteq \mathfrak{g}$ to $\text{Im}(\text{ad } X)$, invariant under $h_0$, and we let $\Sigma = \Sigma_Z$ and $\Psi$ be as given in Theorem 3.1.2.

**Lemma 3.5.2.** Let $\lambda = \exp(-\Psi): \mathfrak{h} \to U_-$, so that $\rho(\lambda)$ is a morphism from $\mathfrak{h}$ to $\text{Aut } V$. Then, for all $h \in \mathfrak{h}$, we have the following equality in $\text{End } V$:

$$\rho(\lambda(h)) \circ \rho_*(\Sigma(h)) = \rho_*(h + X) \circ \rho(\lambda(h)).$$

**Proof.** This is immediate from Theorem 3.1.2.

A morphism $\mathfrak{h} \to \text{End } V$ is the same thing as an element of

$$\text{(End}_C V) \otimes_C S \cong \text{End}_S(V \otimes_C S).$$

Thus, after choosing a basis of $V$, we can identify the morphism $\rho(\lambda)$ from $\mathfrak{h}$ to $V$ with an $n \times n$ matrix with coefficients in $S$. The decomposition of $V$ into the direct sum of its weight spaces means that we can write $\rho(\lambda)$ in block form as a sum of elements $\rho(\lambda)_{\nu_1, \nu_2} \in \text{Hom}(V_{\nu_1}, V_{\nu_2})$, where $\nu_1, \nu_2$ are weights of $\rho$. It follows from Remark 3.3.2 that the entries of $\rho(\lambda)_{\nu_1, \nu_2}$ are homogeneous polynomials of the appropriate degree. Similarly, the morphism $\Sigma: \mathfrak{h}/W \to \mathfrak{g}$ induces a morphism $\rho_*\Sigma: \mathfrak{h}/W = \text{Spec } S^W \to \text{End } V$, and is thus identified with an element of $\text{End } V \otimes_C S^W$, which we continue to denote by $\rho_*\Sigma$. The morphism $\rho_*\Sigma$ can likewise be viewed as an element of $\text{End } V \otimes_C S$. Since $\Sigma$ is the pullback of $\Sigma$, it follows that $\rho_*\Sigma$ is the image of $\rho_*\Sigma$ under the inclusion $\text{End } V \otimes_C S^W \subseteq \text{End } V \otimes_C S$.

**Proposition 3.5.3.** Define the $S$-submodule $M$ of $\text{Hom}_S(V \otimes_C S, S)$ by

$$M = \{ \phi \in \text{Hom}_S(V \otimes_C S, S) : \phi \circ \rho_*(\Sigma) = \mu \cdot \phi \}.$$
Then $M$ is a free $S$-module of rank one, with a generator given by $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$ (under any identification of $V_\mu$ with $\mathbb{C}$).

**Proof.** Since $\rho_\ast(\Sigma)$ and $\rho_\ast(i + X)$ are conjugate in $\text{Aut}_S(V \otimes \mathbb{C} S)$ (via $\rho(\lambda)$), it will suffice to prove the same statement with $\rho_\ast(\Sigma)$ replaced by $\rho_\ast(i + X)$ and $M$ replaced by

$$M' = \{ \phi \in \text{Hom}_S(V \otimes \mathbb{C} S, S) : \phi \circ \rho_\ast(i + X) = \mu \cdot \phi \}.$$

In this case, $\phi \in M'$ if and only if $\phi \circ \rho(\lambda) \in M$.

Fix a basis $\{v_i\}$ of $V$ such that, for every $i$, $1 \leq i \leq n$, $v_i \in V_{k_i}$. We may assume that, for $i \leq j$, $k_i \leq k_j$. In particular, $v_1 \in V_\mu$. In this basis, the element $\rho_\ast(i + X)$ is a lower triangular matrix whose diagonal entries are the weights of $V$, viewed as elements of $S$. Clearly, then, $M'$ is identified with the set of all $\phi: V \otimes \mathbb{C} S \to S$ which vanish on all weight spaces except for $V_\mu$. This space is identified with $V_\mu^\ast \otimes \mathbb{C} S$, where $V_\mu^\ast \subseteq V^\ast$ is the inclusion dual to the surjection $\pi_\mu: V \to V_\mu$. Thus, $M'$ is a free rank one $S$-module with basis $\pi_\mu \otimes \text{Id}$. It follows that $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$ is a basis for $M$.

Let $W$ act on $S$ in the usual way and trivially on $V$. There are induced actions of $W$ on $V \otimes \mathbb{C} S$ and on $\text{Hom}_S(V \otimes \mathbb{C} S, S)$, via $w \cdot \phi(v) = w \cdot \phi(w^{-1}v)$.

**Lemma 3.5.4.** The stabilizer $W_0$ of $\mu$ in $W$ acts trivially on $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$.

**Proof.** By the previous proposition, $\tilde{f} = (\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$ is an eigenvector for the action of $\rho_\ast(\Sigma)$ on $\text{Hom}_S(V \otimes \mathbb{C} S, S)$, with eigenvalue $\mu$, and the corresponding eigenspace is free of rank one as an $S$-module. Since $\rho_\ast(\Sigma)$ is $W$-invariant, $w \cdot \tilde{f}$ is an eigenvector for $\rho_\ast(\Sigma)$ with eigenvalue $w \cdot \mu$. In particular, if $w \in W_0$, then $w \cdot \tilde{f} = c \tilde{f}$ for some $c \in S$. Since $\lambda \in G_-$, $\rho(\lambda)|V_\mu = \text{Id}$. Thus, $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)|V_\mu \otimes \mathbb{C} S$ is the identity. On the other hand, given $v \in V_\mu \otimes \mathbb{C} S$ and $w \in W_0$, clearly $w^{-1}v \in V_\mu \otimes \mathbb{C} S$ as well, so that $(w \cdot \tilde{f})(v) = w \cdot w^{-1}v = v$. It follows that $w \cdot \tilde{f}|V_\mu \otimes \mathbb{C} S = \text{Id}$, so that $c = 1$. Hence $\tilde{f}$ is $W_0$-invariant.

**Corollary 3.5.5.** Let $M^{W_0}$ be the submodule of $M$ fixed by $W_0$, so that

$$M^{W_0} = \{ \phi \in \text{Hom}_{S^{W_0}}(V \otimes \mathbb{C} S^{W_0}, S^{W_0}) : \phi \circ \rho_\ast(\Sigma) = \mu \cdot \phi \}.$$

Then $M^{W_0}$ is a free rank one $S^{W_0}$-module, generated by $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$.

**Proof.** We have a natural inclusion $M^{W_0} \otimes_{S^{W_0}} S \to M$. The element $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$ lies in $M^{W_0}$, and so this inclusion is an isomorphism. Thus, $M^{W_0} \otimes_{S^{W_0}} S$ is free of rank one. Since $S$ is faithfully flat over $S^{W_0}$, it follows that $M^{W_0}$ is a free rank one $S^{W_0}$-module and that $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$ is a generator. By Lemma 3.5.4, the map $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda): V \otimes \mathbb{C} S \to S$ is invariant under $W_0$ and hence induces a map

$$(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda): V \otimes \mathbb{C} S^{W_0} \to S^{W_0}.$$

Let $f: V \to S^{W_0}$ be the restriction of $(\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)$ to $V \otimes \{1\} \subset V \otimes \mathbb{C} S^{W_0}$. Of course, the map $f$ depends on $\lambda$, which depends on the choice of the slice $Z$. 
Corollary 3.5.6. Let $m_0 \subseteq S^W$ be the ideal generated by all $W$-invariant polynomials whose constant term is zero. Then the map

$$\overline{f}_0 : V \to S^{W_0}/m_0 \cdot S^{W_0}$$

obtained by composing $f$ with the natural quotient mapping $S^{W_0} \to S^{W_0}/m_0 \cdot S^{W_0}$ is independent of the choice of slice $Z$.

**Proof.** If $Z'$ is another slice for $[X, g]$, then by Lemma 3.3.3 there is a map $\gamma : Z \to u_-$ with $\gamma(0) = 0$ such that

$$\text{Ad}(\exp(\gamma(z)))(z + X) \in Z' + X$$

for all $z \in Z$. Let $\lambda = \exp(-\Psi)$ and $\lambda' = \exp(-\Psi')$, where $\Psi, \Psi'$ are the maps given by Theorem 3.2.1 for the slices $Z$ and $Z'$, respectively. Then

$$\text{Ad}(\exp(-\gamma(\Sigma_Z(h) - X))) \cdot (\lambda')^{-1} \cdot \text{Ad}(\exp(\gamma(\Sigma_Z(h) - X))) \cdot \lambda : Z + X \to Z + X.$$  

By uniqueness, it follows that

$$\exp(\gamma(\Sigma_Z(h) - X)) \cdot \lambda = \lambda' \cdot \exp(\gamma(\Sigma_Z(h) - X)).$$

Now $\exp(\gamma(\Sigma_Z(h) - X))$ is the exponential of a $W$-invariant map from $\mathfrak{h}$ to $u_-$, and thus $\exp(\gamma(\Sigma_Z(h) - X)) \equiv 1 \mod m_0$. Hence, $\lambda(h) \equiv \lambda'(h) \mod m_0 \cdot S^{W_0}$. \qed

A Kostant section can be viewed as parametrizing a universal family of regular, semistable $G$-bundles on the cuspidal Weierstrass cubic $E$ whose support is contained in $E_{\text{reg}}$. We now apply Theorem 2.4.1 and Theorem 2.4.3 to the family of vector bundles on $E$ obtained from this universal family by a minuscule or quasiminuscule representation $\rho$.

**Theorem 3.5.7.** Let $\rho : G \to GL(V)$ be an irreducible representation with lowest weight $\mu$ and let

$$f = (\pi_\mu \otimes \text{Id}) \circ \rho(\lambda) : V \to S^{W_0}$$

be the element given in Corollary 3.5.6. Then by extension of scalars $f$ induces a homomorphism $\hat{f} : V \otimes_C S^W \to S^{W_0}$ making the following diagram commute:

$$
\begin{array}{ccc}
V \otimes_C S^W & \xrightarrow{f} & S^{W_0} \\
\rho_* \Sigma \downarrow & & \mu \downarrow \\
V \otimes_C S^W & \xrightarrow{\hat{f}} & S^{W_0}
\end{array}
$$

If $\rho$ is minuscule, then $\hat{f}$ is an isomorphism. If $\rho$ is quasiminuscule, then $\hat{f}$ is surjective.

**Proof.** By Corollary 3.5.6 the $S^{W_0}$-module map $f \otimes \text{Id} : V \otimes_C S^{W_0} \to S^{W_0}$ makes the analogous diagram commute, where $S^W$ is replaced by $S^{W_0}$ and $\rho_* \Sigma$ by $\rho_* \Sigma$, and the set of all such maps $V \otimes_C S^{W_0} \to S^{W_0}$ with this property is a free $S^{W_0}$-module with $f \otimes \text{Id}$ as a generator. Let $\hat{f} = f \otimes \text{Id} | V \otimes_C S^W$. Since $\rho_* (\Sigma)$ is $S^W$-invariant we see that the diagram as defined in the theorem is commutative. It remains to show that $\hat{f}$ is surjective in the quasiminuscule case and an isomorphism in the minuscule case.

Let $E$ be a cuspidal curve of arithmetic genus one. We consider the $G$-bundle over $(\mathfrak{h}/W) \times E$ which is trivial on $(\mathfrak{h}/W) \times \overline{E}$ and which is given by the Kostant
section \( \Sigma: \mathfrak{h}/W \to \mathfrak{g}_{\text{reg}} \subseteq \mathfrak{g} \) under the correspondence of Theorem 1.1.1. Let \( \mathcal{V} \to (\mathfrak{h}/W) \times E \) be the vector bundle induced from this \( G \)-bundle by \( \rho \). This vector bundle is also trivialized on \((\mathfrak{h}/W) \times \tilde{E} \) and is given by the map \( \rho_{\ast} \Sigma: \mathfrak{h}/W \to \text{End}(\mathcal{V}) \).

Applying Theorem 2.5.1 and Theorem 2.4.3 and using the fact that every line bundle over \( \mathfrak{h}/W_0 \) is trivial since \( \mathfrak{h}/W_0 \) is isomorphic to an affine space, we see that there is a surjection from \( \mathcal{V} \) to the vector bundle

\[
(\tilde{\nu} \times \text{Id})_{\ast}(r \times \text{Id})_{\ast} \mathcal{P},
\]

where \( \tilde{\nu}: \mathfrak{h}/W_0 \to \mathfrak{h}/W \) is the covering projection and \( r: \mathfrak{h}/W_0 \to \mathbb{C} \subseteq E \) is the map induced by the \( W_0 \)-invariant weight \( \mu: E_{\text{reg}} \otimes \Lambda \to E_{\text{reg}} \). By Proposition 1.1.2, \( \mathcal{P} \to E_{\text{reg}} \times E \) pulls back to the trivial line bundle on \( E_{\text{reg}} \times \tilde{E} \) and hence is given by a map \( E_{\text{reg}} \to \text{End}(\mathbb{C}) = \mathbb{C} \). This map is the fixed identification \( E_{\text{reg}} \cong \mathbb{C} \). Thus, when we use this identification to produce an isomorphism of the coordinate ring of \( E_{\text{reg}} \) with \( \mathbb{C}[t] \), \( \mathcal{P} \) becomes the element \( t \in \mathbb{C}[t] \). Hence, \( (r \times \text{Id})_{\ast} \mathcal{P} \) is identified with the element \( r_{\ast}(t) \) which is the element \( \mu \in S^{W_0} \). In this way we identify \( (r \times \text{Id})_{\ast} \mathcal{P} \) with the element \( \mu \in S^{W_0} \) viewed as an \( S^{W_0} \)-valued endomorphism of \( \mathbb{C} \). Applying \((\tilde{\nu} \times \text{Id})_{\ast}\), produces the \( S^W \)-module \( S^{W_0} \) with the \( S^W \)-linear endomorphism given by multiplication by \( \mu \).

Theorem 2.4.3 now implies that there is a surjective \( S^W \)-linear homomorphism \( f': V \otimes_{\mathbb{C}} S^W \to S^{W_0} \) making the following diagram commute:

\[
\begin{array}{ccc}
V \otimes S^W & \longrightarrow & S^{W_0} \\
\rho_{\ast} \Sigma \downarrow & & \downarrow \mu \\
V \otimes S^W & \longrightarrow & S^{W_0}.
\end{array}
\]

Its extension to a map \( \tilde{f}' : V \otimes_{\mathbb{C}} S^{W_0} \to S^{W_0} \) is an element of the free module \( M^{W_0} \) and hence \( \tilde{f}' = s(f \otimes \text{Id}) \) for some \( s \in S^{W_0} \). Since \( \tilde{f}' \) is surjective, \( s \in S^{W_0} \) is invertible. Restricting to \( V \otimes_{\mathbb{C}} S^W \), we see that \( f' = sf \) and, since \( f' \) is surjective, so is \( \tilde{f} \). Finally, if \( \rho \) is minuscule, then \( f' \) is an isomorphism and hence \( \tilde{f} \) is an isomorphism as well.

**Corollary 3.5.8.** Suppose that \( \rho \) is minuscule. Let \( x \in \mathfrak{h}/W \) and let \( m_x \subseteq S^W \) be the maximal ideal of \( x \). Then the scheme-theoretic fiber over \( x \) in \( \mathfrak{h}/W_0 \) has coordinate ring \( S^{W_0}/m_x \cdot S^{W_0} \). The map \( \tilde{f} \) induces an isomorphism \( \mathcal{F}_x : V \to S^{W_0}/m_x \cdot S^{W_0} \). Under this isomorphism, multiplication by \( \mu \) on the right-hand side becomes the action of \( \rho_{\ast} \Sigma(x) \). In particular, under the isomorphism \( \mathcal{F}_0 : V \to S^{W_0}/m_0 \cdot S^{W_0} \) given in Corollary 3.5.6, multiplication by \( \mu \in S^{W_0} \) on \( S^{W_0}/m_0 \cdot S^{W_0} \) corresponds to the action \( \rho_{\ast}(X) \) of the principal nilpotent element \( X \) on \( V \).

In particular, the corollary describes the action of every regular element of \( \mathfrak{g} \) on \( V \) up to conjugation.

Using his deep results on perverse sheaves on the loop group of the Langlands dual of \( \mathfrak{g} \), Ginzburg has proved the following generalization of Theorem 3.5.7.

**Theorem 3.5.9.** Let \( \rho \) be an arbitrary irreducible representation. The map \( \tilde{f} : V \otimes_{\mathbb{C}} S^W \to S^{W_0} \) is always surjective.
It is natural to ask if there is a more elementary and direct proof of Theorem 3.5.9. The image of \( f \) is the \( S^W \)-submodule of \( S^{W_0} \) generated by \((\pi_\mu \otimes \text{Id}) \circ \rho(\lambda)_{\nu,\mu}\), where \( \rho(\lambda)_{\nu,\mu} : V_\nu \otimes_C S \to V_\mu \otimes_C S \) is the linear map induced by \( \rho(\lambda) \). It follows the image of \( f \) is generated by homogeneous elements, when \( S^W \) and \( S^{W_0} \) are given their natural gradings as subrings of \( S \). By the homogeneous Nakayama lemma, to prove Theorem 3.5.9 it would suffice to prove that the induced homomorphism \( \tilde{f}_0 : V \to S^{W_0}/m_0 \cdot S^{W_0} \) is surjective.

3.6. The Steinberg section. Let \( \overline{\Phi} : H/W \to G_{\text{reg}} \) be a section of Steinberg type. By this we mean that \( \overline{\Phi} \) is any morphism from \( H/W \) to \( G_{\text{reg}} \), the set of regular elements of \( G \), which is a section of the adjoint quotient morphism \( G \to H/W \). We begin by proving a weaker analogue of Theorem 3.1.2 for \( G \).

**Theorem 3.6.1.** Let \( \overline{\Phi} : H/W \to G_{\text{reg}} \) be the Steinberg section, and let \( \Phi : H \to G_{\text{reg}} \) be the composition \( H \to H/W \to G_{\text{reg}} \). Then there exists a Borel subgroup \( B \) of \( G \) containing \( H \), with unipotent radical \( U \), and a morphism \( \phi : H \to G \) such that, for all \( h \in H \), \( \phi(h) \Phi(h) \phi(h)^{-1} \in B \), and moreover \( \phi(h) \Phi(h) \phi(h)^{-1} = hu(h) \) for some morphism \( u : H \to U \).

**Proof.** Suppose that we can show that there exists some Borel subgroup \( B \) and a morphism \( \phi : H \to G \) such that, for all \( h \in H \), \( \phi(h) \Phi(h) \phi(h)^{-1} \in B \). Then after a further conjugation we may assume that \( B \) contains \( H \). In this case, we can write \( \phi(h) \Phi(h) \phi(h)^{-1} = h'u(h) \) for some morphism \( u : H \to U \), where \( h \) and \( h' \) lie in \( H \) and have the same image in \( H/W \). It follows that there is a \( w \in W \) such that, for an open dense set of \( h \in H \), \( h' = w(h) \). Hence we can assume that \( h' = w(h) \) for all \( h \). After further conjugating \( B \) by a representative for \( w \) in the normalizer of \( H \) in \( G \), we can then assume that \( \phi(h) \Phi(h) \phi(h)^{-1} = hu(h) \) for some morphism \( u : H \to U \) as desired.

To find the morphism \( \phi \), we first claim that there exists a morphism \( \overline{\phi} \) from \( H \) to the space \( B \cong G/B \) of Borel subgroups of \( G \), such that, for all \( h \in H \), \( h \in \overline{\phi}(h) \). To see this, fix a Borel subgroup \( B \) and let \( I \subseteq G \times (G/B) \) be the incidence variety:

\[
I = \{(g,xB) : x^{-1}gx \in B\} = \{(g,xB) : g \in xBx^{-1}\}.
\]

Identifying the set of all Borel subgroups of \( G \) with \( G/B \), \( I \) is the set of pairs consisting of an element \( g \) of \( G \) and a Borel subgroup containing \( g \). Clearly \( I \) is a closed subvariety of \( G \times (G/B) \). The morphism \((g,xB) \mapsto (x,x^{-1}gx)\) defines an isomorphism from \( I \) to \( G \times_B B \), where \( B \) operates on \( G \) by right multiplication and on itself by conjugation, with inverse \((g,b) \mapsto (gb^{-1},gB)\) (cf. [14] 4.3]. Let \( I_{\text{reg}} \) be the inverse image of \( G_{\text{reg}} \) under the projection \( \pi_1 : I \to G \). There is a morphism \( \theta : I \to H \) defined by the composition

\[
I \cong G \times_B B \to G \times_B H \cong (G/B) \times H \to H,
\]

where the morphism \( G \times_B B \to G \times_B H \) is induced by the homomorphism \( B \to H \), and the morphism \((G/B) \times H \to H \) is projection onto the second factor. It is straightforward to verify that the following diagram is commutative:

\[
\begin{array}{ccc}
I & \xrightarrow{\pi_1} & G \\
\downarrow^\theta & & \downarrow \\
H & \longrightarrow & H/W.
\end{array}
\]
By a theorem of Grothendieck \[14\, §4.4\], this diagram identifies \( I \to H \) with a simultaneous resolution of the morphism \( G \to H/W \), in the terminology of \[14\].

Since the adjoint quotient morphism \( G \to H/W \) is smooth along \( G \reg \), the above diagram identifies \( I \reg \) with the fiber product \( H \times_{H/W} G \reg \). Using the morphisms \( \Phi: H/W \to G \) and \( \pi_1: I \to G \), we can take the fiber product \( (H/W) \times_G I \). By the above remarks (all products below are fiber products),

\[
(H/W) \times_G I = (H/W) \times_{G \reg} I \reg \cong (H/W) \times_{H/W} G \reg \cong H.
\]

The isomorphism \( H \cong (H/W) \times_G I \) together with the projection \( (H/W) \times_G I \to I \) identify an element \( h \) of \( H \) with a pair \((x, B_h)\), where \( x \in H/W \) is the image of \( h \) and \( B_h \) is a Borel subgroup containing \( \Phi(x) = \Phi(h) \). Here \( B_h \) is the image of \( h \) under the morphism \( \overline{\phi}: H \to G/B \), which is the composition

\[
\overline{\phi}: H \to G/B
\]

and we identify \( G/B \) with the variety of all Borel subgroups of \( G \).

Now suppose that we can lift the morphism \( \overline{\phi}: H \to G/B \) to a morphism \( \overline{\phi}: H \to G \). It then follows that, for all \( h \in H \), \( \phi(h)\Phi(h)\phi(h)^{-1} \in B \) as claimed.

To see that such a lift is possible, let \( B \) be the sheaf of morphisms from \( H \) to \( B \), and similarly for \( G \) and \( G/B \). Then the following is an exact sequence of sheaves of sets in the \( \acute{e}tale \) topology:

\[
\{1\} \to B \to G \to G/B \to \{1\}.
\]

In particular, we have the following long exact sequence (of pointed sets):

\[
H^0(H; G) \to H^0(G; G/B) \to H^1(H; B).
\]

On the other hand, the solvable group \( B \) has a filtration by normal subgroups, such that the successive quotients are either \( G_m \) or \( G_a \). Since \( H \) is a torus, all line bundles on it are trivial, and since it is affine, all higher coherent sheaf cohomology vanishes. Thus \( H^1(H; B) \) is trivial. It follows that every morphism from \( H \) to \( G/B \) lifts to a morphism from \( H \) to \( G \), as claimed. This concludes the proof. \( \square \)

The method of proof above also proves a weak form of Theorem 3.1.2. However, it does not give the homogeneity statements.

Arguments very similar to those used in the proof of Theorem 3.5.7 then show the following:

**Theorem 3.6.2.** Let \( S \) be the affine coordinate ring of \( H \). Let \( \rho: G \to GL(V) \) be an irreducible representation with lowest weight \( \mu \) and let

\[
g = (\pi_\mu \otimes \text{Id}) \circ \rho(\phi)^{-1}: V \to S^{W_0}.
\]

Then by extension of scalars \( g \) induces a homomorphism \( \tilde{g}: V \otimes_C S^W \to S^{W_0} \) making the following diagram commute:

\[
\begin{array}{ccc}
V \otimes_C S^W & \xrightarrow{\tilde{g}} & S^{W_0} \\
\rho_*, \Phi & \downarrow & \mu_* \\
V \otimes_C S^W & \xrightarrow{\tilde{g}} & S^{W_0}.
\end{array}
\]

If \( \rho \) is minuscule, then \( \tilde{g} \) is an isomorphism. If \( \rho \) is quasiminuscule, then \( \tilde{g} \) is surjective. \( \square \)
In case $\rho$ is minuscule, there is also a corollary to the above theorem, along the lines of Corollary 3.5.8, describing the action of $\Phi(x)$ on $V$ in terms of multiplication by $\mu$ on $S^W_0/m_\rho \cdot S^W$, which we leave to the reader to formulate.

Finally, it is natural to make the following conjecture, which does not seem to follow from the Lie algebra analogue:

**Conjecture 3.6.3.** In the above notation, the homomorphism $\hat{g}$ is always surjective.

### 4. Examples

**The case of $A_{n-1}$.** Let $\mathfrak{h}$ be the Cartan subalgebra of $SL_n$, so that $\mathfrak{h} \subset \mathbb{C}^n$ is identified with

$$\mathfrak{h} = \{(x_1, \ldots, x_n) | \sum_i x_i = 0\}.$$  

The coordinate ring of $\mathfrak{h}$ is $S = \mathbb{C}[x_1, \ldots, x_n]/(\sum_i x_i)$. Let $\sigma_2, \ldots, \sigma_n$ be the elementary symmetric functions in $x_1, \ldots, x_n$ (note that $\sigma_1 = 0$ on $\mathfrak{h}$). Let $W = \mathfrak{S}_n$ be the symmetric group on $n$ letters and let $W_0 = \mathfrak{S}_{n-1}$, which is embedded in $W$ as the stabilizer of $n$. Thus $W$ acts on $S$ and $S^W = \mathbb{C}[\sigma_2, \ldots, \sigma_n]$. Let $\sigma_i(k)$ be the $i$th elementary symmetric function in the variables $x_1, \ldots, x_k$. Thus for example $\sigma_i(n) = \sigma_i$. Note that $x_n = -\sigma_1(n-1)$. More generally, we have the relation

$$\sigma_i(n-1) + x_n \sigma_{i-1}(n-1) = \sigma_i. \quad (*)$$

Clearly

$$S^W_0 = \mathbb{C}[x_n, \sigma_1(n-1), \ldots, \sigma_{n-1}(n-1)]/(x_n + \sigma_1(n-1) = 0)$$

$$= \mathbb{C}[\sigma_1(n-1), \sigma_2(n-1), \ldots, \sigma_{n-1}(n-1)].$$

By $(*)$, it follows that $S^W_0 = S^W[x_n]$. Since $x_n$ satisfies the monic equation $p(x) = \sum_{i=0}^n (-1)^i \sigma_i x^{n-i} = 0$, $S^W_0 = S^W \cdot 1 \oplus \cdots \oplus S^W \cdot x_n$. Again by using $(*)$, we can write $x_n$ as $\sigma_i(n-1)$ plus terms involving $x_j^n$ for $j < i$ as well as elements of $S^W$. Thus

$$S^W_0 = S^W \cdot 1 \oplus S^W \cdot \sigma_1(n-1) \oplus \cdots \oplus S^W \cdot \sigma_{n-1}(n-1)$$

as well, so that $1, \sigma_1(n-1), \ldots, \sigma_{n-1}(n-1)$ is a basis for $S^W_0$ as an $S^W$-module.

Let $V$ be the defining $n$-dimensional representation of $SL_n$. The vector space $V$ has the standard basis $e_1, \ldots, e_n$. Define the $n \times n$ matrix $\Sigma$ by

$$\Sigma = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^{n-1} \sigma_n & (-1)^{n-2} \sigma_{n-1} & (-1)^{n-3} \sigma_{n-2} & (-1)^{n-4} \sigma_{n-3} & \cdots & -\sigma_2 & 0
\end{pmatrix}.$$  

As is well-known, $\Sigma$ is a Kostant section for $\mathfrak{sl}_n$. 

For each \( k, 1 \leq k \leq n \), let \( m_i(k) \) be the sum of all monomials of degree \( i \) in the variables \( x_1, \ldots, x_k \). Define the column vector \( c_k \) by

\[
c_k = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ m_1(k) \\ \vdots \\ m_{n-k}(k) \end{pmatrix},
\]

and let \( A = (c_1 \cdots c_n) \).

It is not difficult to show that

\[
A^{-1} \Sigma A = \begin{pmatrix} x_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x_3 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & x_n \end{pmatrix}
\]

and that \( A^{-1} = \)

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
-\sigma_1(1) & 1 & 0 & \cdots & 0 & 0 \\
\sigma_2(1) & -\sigma_1(2) & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
(-1)^{n-1} \sigma_{n-1}(n-1) & (-1)^{n-2} \sigma_{n-2}(n-1) & (-1)^{n-3} \sigma_{n-3}(n-1) & \cdots & -\sigma_1(n-1) & 1
\end{pmatrix}.
\]

Thus, \( A^{-1} \in \text{End}(V) \otimes S \) is the matrix realizing the endomorphism \( \rho(\lambda) \) of Theorem 1 for the standard \( n \)-dimensional representation \( \rho \) of \( SL_n \). The lowest weight of this representation is \( x_n \) and hence \( \pi_\mu = x_n \). Thus, \( (\pi_\mu \otimes \text{Id}) \circ \rho(\lambda) \) is the row matrix given by the last row of \( A^{-1} \). Note that the last row of \( A^{-1} \) is invariant under \( W_0 = S_{n-1} \) and is in fact up to sign the basis for \( S^{W_0} \) over \( S^W \) described above.

We now consider the minuscule representation \( \wedge^k V \). The lowest weight \( \mu \) for this representation is \( e_{n+\cdots+e_{n-k+1}} \) and hence the map \( (\pi_\mu \otimes \text{Id}) \circ \rho(\lambda) \) is the map given by the \( k \times k \) minors of \( A^{-1} \) which involve the last \( k \) rows. The stabilizer of \( \mu \) is \( S_{n-k} \times S_k \), where the first factor acts on the first \( n-k \) variables and the second on the last \( k \) variables. It is clear from the explicit description of the matrix for \( A^{-1} \) that each of the last \( k \) rows is invariant under the group \( S_{n-k} \) and hence the determinants we are considering are also invariant under this group. The theorem implies that these determinants are also invariant under \( S_k \) acting on the last \( k \) variables. In fact, this is easy to check directly.

Let \( \tau_j(k) \) be the \( j \)th elementary symmetric function in the last \( k \) variables. We have the relations

\[
\sigma_\ell = \sum_{i=0}^\ell \sigma_i(n-k) \tau_{\ell-i}(k), \ell = 1, \ldots, n.
\]

The ring \( S^{W_0} \) is the quotient of the polynomial ring \( \mathbb{C}[\sigma_i(n-k), \tau_j(k)] \) modulo the above relations. The theorem implies that the \( k \times k \)-minors of the \( k \times n \) matrix
obtained from $A^{-1}$ by only considering the last $k$ rows are a basis for $S^{W_0}$ over $S^W$.

To complete the picture for $SL_n$, we consider the adjoint representation. In this case $\mu = e_n - e_1$ and the group $W_0$ is a copy of $\mathfrak{S}_{n-2}$, embedded in $\mathfrak{S}_n$ as the stabilizer of $1$ and $n$. It is straightforward to check in this case that the image of $V \otimes C^S$ in $S^{W_0}$ is the $S^W$-submodule of $S^{W_0}$ generated by $\sigma_{n-i}(n-1)x_{i-1}^1, 1 \leq i, j \leq n$, and that this submodule is all of $S^{W_0}$.

The case of $C_n$. The unique minuscule representation of $Sp(2n, \mathbb{C})$ is the defining $2n$-dimensional representation. Its weights are $\pm x_i, 1 \leq i \leq n$, with lowest weight $-x_1$. The ring $S^W$ is a polynomial algebra $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$, where $\alpha_i$ is the $i^{\text{th}}$ elementary symmetric function on $x_1^2, \ldots, x_n^2$. If $W_0$ is the stabilizer of the lowest weight $-x_1$, then $S^{W_0}$ is a polynomial algebra with generators $x_1, \beta_1, \ldots, \beta_{n-1}$ where $\beta_i$ is the $i^{\text{th}}$ elementary symmetric function on $x_1^2, \ldots, x_n^2$. As in the case of $A_{n-1}$, there is a set of relations

$$x_1^2 \beta_{i-1} + \beta_i = \alpha_i, i = 1, \ldots, n.$$ 

Note that $x_1$ satisfies a monic polynomial over $S^W$ of degree $2n$. It follows that $S^{W_0} = S^W[x_1] = S^W \cdot 1 \oplus S^W \cdot x_1 \oplus \cdots \oplus S^W \cdot x_1^{2n-1}$. In particular, $-x_1$ acts on $S^{W_0}/m_0 \cdot S^{W_0}$ with a Jordan block of length $2n$, corresponding to the fact that the image of a principal nilpotent element of $\mathfrak{sp}(2n)$ under the standard representation is a principal nilpotent matrix. In terms of bundles, this is equivalent to the statement that the bundle $I_{2n}$ has a nondegenerate symplectic form.

The orthogonal representation of $Spin(2n)$. Choose coordinates $x_1, \ldots, x_n$ on the Cartan subalgebra of $\mathfrak{spin}(2n)$ so that the coroot lattice is the even integral lattice. The orthogonal representation of $Spin(2n)$ has weights $\pm x_1, \ldots, \pm x_n$, and $-x_1$ is the lowest weight. Let $W_0$ be the subgroup of the Weyl group $W$ stabilizing $-x_1$. It is the Weyl group for $Spin(2n-2)$ in the variables $x_2, \ldots, x_n$. As before, we let $\alpha_i$ be the $i^{\text{th}}$ elementary symmetric function on $x_1^2, \ldots, x_n^2$ and $\beta_i$ the $i^{\text{th}}$ elementary symmetric function on $x_1^2, \ldots, x_n^2$. The ring $S^W$ is the ring generated by $\alpha_1, \ldots, \alpha_n$ and the Pfaffian $\text{Pfaff}_n = x_1 \cdots x_n$ subject to the relation $\text{Pfaff}_n^2 = \alpha_n$. Thus $S^W$ is a polynomial algebra on $\alpha_i, 1 \leq i < n$, and $\text{Pfaff}_n$. The algebra $S^{W_0}$ is the algebra generated by $x_1, \beta_1, 1 \leq i \leq n-1$, and $\text{Pfaff}_{n-1}$, subject to the relation $\text{Pfaff}_{n-1}^2 = \beta_{n-1}$, and thus $S^{W_0}$ is a polynomial algebra generated by $x_1, \beta_i, 1 \leq i < n-1$, and $\text{Pfaff}_{n-1}$. The ring $S^{W_0}/m_0 \cdot S^{W_0}$ is the quotient of $S^{W_0}$ by the relations

$$x_1^2 \beta_i + \beta_{i+1} = 0;$$
$$x_1 \text{Pfaff}_{n-1} = 0.$$ 

Hence $S^{W_0}/m_0 \cdot S^{W_0}$ is generated by $x_1$ and $\text{Pfaff}_{n-1}$ subject to the relations

$$x_1^{2n-2} = \text{Pfaff}_{n-1}^2;$$
$$x_1 \text{Pfaff}_{n-1} = 0.$$ 

It follows that

$$x_1^{2n-1} = x_1 \text{Pfaff}_{n-1}^2 = 0.$$ 

A $\mathbb{C}$-basis for $S^{W_0}/m_0 \cdot S^{W_0}$ is $1, x_1, \ldots, x_1^{2n-2}, \text{Pfaff}_{n-1}$. As a module over $\mathbb{C}[x_1]$, $S^{W_0}/m_0 \cdot S^{W_0} \cong \mathbb{C}[x_1]/(x_1^{2n-2}) \oplus \mathbb{C}[x_1]/(x_1)$. This reflects the fact, which is easy to check directly, that a principal nilpotent element of $\mathfrak{spin}(2n)$ acts on $V$ with
two Jordan blocks, one of dimension $2n - 1$ and one of dimension 1. This implies that the orthogonal bundle of rank $2n$ which is regular as a Spin$(2n)$-bundle and $S$-equivalent to the trivial bundle is $I_{2n-1} \oplus O_E$. In particular, it is not regular as a vector bundle.

The case of $E_6$. Direct computation shows that the action of the principal nilpotent element $X$ in the Lie algebra of type of $E_6$ on the 27-dimensional minuscule representation $V$ has three Jordan blocks of dimensions 17, 9, and 1. The bottom of the Jordan block of rank 17 is the lowest weight space $V_\mu$. Thus, Theorem 4.1 implies that multiplication by $\mu$ in $S^{W_0}/m_0 \cdot S^{W_0}$ is nilpotent of order 17. One can show that $S^{W_0} = S^W[\mu, a, b]$, where $\mu, a, b$ are homogeneous in degrees 1, 4, 8 respectively. Note that $S^W$ is a graded polynomial algebra with generators in degrees $2, 5, 6, 8, 9, 12$. On the other hand, since $W_0$ is the Weyl group of $D_5$, and its action on $\mathfrak{h}$ decomposes as a copy of the standard representation of $W(D_5)$ plus a trivial factor, it follows that $S^{W_0}$ has generators in degrees 1, 2, 4, 5, 6, 8. Given these degrees and the fact that $S^{W_0}$ has rank 27 over $S^W$, it is easy to check directly that $S^{W_0}$ is generated as an $S^W$-algebra by elements in degrees 1, 4, 8.

5. Quasiminuscule representations

Let $\rho$ be a quasiminuscule, nonminuscule representation, and let $V$ be the corresponding vector bundle over $\mathcal{M} \times E$. In this section, we describe the extension class corresponding to the extension

$$(*) \quad 0 \to \pi_1^* \pi_4^* V \to V \to V \to 0$$

of Theorem 5.1.1 in the case where $G$ is simply laced and hence $\rho$ is the adjoint representation. It seems likely that similar geometric methods can also describe the quasiminuscule, nonminuscule representation in the non-simply laced case. We shall just write out the case of $g$, i.e. where $E$ is cuspidal, although minor modifications handle the nodal and smooth cases. Recall the notation of §2.1: we have the finite cover $\hat{\nu}: \hat{T}_0 \to \mathcal{M}$, as well as the normalization map $\varphi: T_0 \to T_0 \subseteq \mathcal{M} \times E_{\text{reg}}$, where $T_0$ is the image of $(\hat{\nu}, r): \hat{T}_0 \to \mathcal{M} \times E_{\text{reg}}$. Fixing a coordinate $t$ for $E$ at $p_0$, the pullback $r^* t$ in the coordinate ring of $\hat{T}_0$, i.e. as an element of $S^{W_0}$, is just the weight $\mu$. In the above notation, $\mathcal{V} = (\hat{\nu} \times \text{Id})_*(r \times \text{Id})^*(\mathcal{P})$.

5.1. The extension group. We begin by identifying the module of extensions for exact sequences whose first and third terms are above.

**Proposition 5.1.1.** Let $W$ be a vector bundle over $\mathcal{M}$ and let $M$ be the projective and hence free $S^W$-module corresponding to $W$. Then $\text{Ext}_E^1(\mathcal{V}, \pi_1^* W)$, which is a module over $S^W$, is isomorphic to the $S^W$-module $(S^{W_0}/\mu S^{W_0}) \otimes_{S^W} M$.

**Proof.** By Corollary 1.3.3 the module of such extensions is isomorphic to $(Q^\vee/tQ^\vee) \otimes_{S^W} M$, where $Q^\vee$ is the Fourier-Mukai transform of $(\mathcal{V})^\vee$. Thus it suffices to show that $Q^\vee \cong S^{W_0}$. By Theorem 5.1.3 $\mathcal{V}$ is the direct image under $\hat{\nu} \times \text{Id}$ of the pullback of a line bundle on $\hat{T}_0$, necessarily trivial, tensored with $(r \times \text{Id})^*(\mathcal{P})$. Applying duality for the finite flat morphism $\hat{\nu} \times \text{Id}$, and using the fact that the relative dualizing sheaf $\omega_{\hat{T}_0/\mathcal{M}}$ is a line bundle on $\hat{T}_0$, and hence is trivial as well, it follows that $(\mathcal{V})^\vee$ is the direct image under $\hat{\nu} \times \text{Id}$ of $(r \times \text{Id})^*(\mathcal{P})^{-1}$. Note
that reflection in the root $\mu$ defines an element of $W$ of order two normalizing $W_0$, and hence an involution $\tau$ of $\tilde{T}_0$ such that $\tilde{\nu} \circ \tau = \tilde{\nu}$ and $r \circ \tau = -\text{Id} \circ r$. Thus
\[
(\tilde{\nu} \times \text{Id})_*(r \times \text{Id})^*P^{-1} = (\tilde{\nu} \times \text{Id})_*(\tau \times \text{Id})^*(r \times \text{Id})^*P^{-1} = (\tilde{\nu} \times \text{Id})_*(r \times \text{Id})^*(-\text{Id} \times \text{Id})^*P^{-1}.
\]
On the other hand, from the symmetric definition of $P$, it follows that
\[
(-\text{Id} \times \text{Id})^*P^{-1} \cong P.
\]
Thus $(\overline{V})^\vee \cong (\tilde{\nu} \times \text{Id})_*(r \times \text{Id})^*P \cong \overline{V}$, and in particular $Q^\vee \cong S^{W_0}$.

Let $D = T_0 \cap (M \times \{p_0\})$ and let $\hat{D}$ be the pullback of $D$ to $\tilde{T}_0$. Since the affine coordinate ring of $\hat{D}$ is equal to $S^{W_0}/\mu S^{W_0}$, we can rephrase the proposition as follows:

**Corollary 5.1.2.** In the above notation, the $S^W$-module $\text{Ext}^1_{M \times E}(\overline{V}, \pi^*W)$ is isomorphic to the $S^W$-module of global sections of $\tilde{\nu}^*W|D$.

Clearly, $\hat{D} = r^{-1}(0)$, and hence $\hat{D}$ is isomorphic to $M_0$, in the terminology of §2.1. In particular, $\hat{D}$ is smooth. It follows from Lemma 2.3.4 that the induced morphism $\hat{D} \to D$ is birational. We shall say more about the structure of $\hat{D} \to D$ below.

### 5.2. Identification of the extension class.

Assume now that $\rho$ is the adjoint representation. The nonzero weights of $\rho$ are the roots of $\mathfrak{g}$, and $\mathcal{V} = \text{ad} \Xi$. First, we identify the vector bundle $\mathcal{W} = \pi_1^*\mathcal{V}$:

**Proposition 5.2.1.** The vector bundle $\pi_1^*\mathcal{V}$ is isomorphic to $\Omega^1_M$.

**Proof.** The bundle $\Xi$ is a universal $G$-bundle over $M \times E$. It follows that the Kodaira-Spencer homomorphism $T_M \to R^1\pi_{1*}\text{ad} \Xi$ is an isomorphism. (Of course, this is just a restatement of the result of Kostant [11] that the adjoint quotient morphism is smooth at a regular element.) Using relative duality, the isomorphism $(\text{ad} \Xi)^\vee \cong \text{ad} \Xi$ given by the Killing form, and the fact that the relative dualizing sheaf $\omega_{M \times E/M}$ is trivial, it follows that $(R^1\pi_{1*}\text{ad} \Xi)^\vee \cong R^0\pi_{1*}\text{ad} \Xi$. Thus $\Omega^1_M \cong \pi_{1*}\text{ad} \Xi = \pi_{1*}\mathcal{V}$, as claimed.

There is another description of $\pi_{1*}\mathcal{V} \cong \Omega^1_M$ as follows:

**Lemma 5.2.2.** The sheaf $\pi_{1*}\mathcal{V} \cong \Omega^1_M$ corresponds to the $S^W$-module $(S \otimes_{\mathbb{C}} \mathfrak{h})^W$, where $W$ acts on both factors of the tensor product by the natural actions.

**Proof.** The sheaf $\Omega^1_M$ is the coherent sheaf associated to the $S^W$-module $\Omega^1_{S^W/\mathbb{C}}$. By a theorem of Solomon (see for example [2] p. 135, ex. 3]), $\Omega^1_{S^W/\mathbb{C}} = (\Omega^1_{S/\mathbb{C}})^W$. But $\Omega^1_{S/\mathbb{C}} \cong S \otimes_{\mathbb{C}} \mathfrak{h}^*$, and the invariant bilinear form defines a $W$-equivariant isomorphism from $\mathfrak{h}^*$ to $\mathfrak{h}$. Thus $\Omega^1_{S^W/\mathbb{C}} \cong (S \otimes_{\mathbb{C}} \mathfrak{h})^W$. \square
The extension class corresponding to \( V \) is thus a global section of \( \nu^*\Omega^1_M|\hat{D} \). Our goal now will be to describe this section. Viewing \( D \) as a divisor on \( M \times \{p_0\} \cong M \), there is the conormal morphism \( I_D/I^2_D \to \Omega^1_M|D \). The image of this morphism is the conormal sheaf. At a smooth point of \( D \), this subsheaf is a line subbundle which is the dual of the normal bundle to \( D \). Since \( M \) is an affine space, \( \text{Pic} M \) is trivial, and hence there is an \( \bar{s} \in S^{w_0} \), unique up to a nonzero scalar, such that \( I_D = (\bar{s}) \). In fact, we can take \( \bar{s} = \prod_{w \in W_{\nu}} w \mu \). Thus \( \bar{d}s \) defines a section of \( \Omega^1_M|D \). There is then the induced section \( \nu^*d\bar{s} \) of \( \nu^*\Omega^1_M|\hat{D} \). Since \( \hat{D} \cong M_0 \), which is again an affine space, there exists an indivisible section \( s \) of \( \nu^*\Omega^1_M|\hat{D} \), unique up to a constant, and an element \( a \in \Gamma(\hat{D}, \mathcal{O}_D) \cong S^{w_0}/\mu S^{w_0} \) such that \( \nu^*d\bar{s} = as \). Concretely, choosing a basis of the trivial vector bundle \( \Omega^1_M \) and hence of \( \nu^*\Omega^1_M|\hat{D} \cong \mathcal{O}_D \), we can write \( s = (f_1, \ldots, f_r) \), where the \( f_i \) have no common factor, and \( as = (a_f f_1, \ldots, a_f f_r) \) for functions \( f_i \) on \( \hat{D} \).

**Theorem 5.2.3.** The above section \( s \) defines the extension class corresponding to \( V \). Moreover, \( s \) can be completed to a basis for \( \nu^*\Omega^1_M|\hat{D} \). In other words, the sheaf \( (\nu^*\Omega^1_M|\hat{D})/\mathcal{O}_D \cdot s \) is a locally free sheaf of \( \mathcal{O}_D \)-modules.

**Proof.** We begin by verifying that, at a generic point \( x \) of \( D \), the corresponding extension class lies in the image of the conormal line \( N^{-1}_{D/M,x} \) in \( \Omega^1_{M,x} \). For such an \( x \in D \), we may assume that \( D \) and \( T_0 \) are smooth at \( x \). Let \( \xi \) be the \( G \)-bundle over \( E \) corresponding to \( x \in M \). By the proof of Theorem 2.4.1 and 2.4.3, \( \xi \) is described as follows: let \( H' \) be the subtorus of \( H \) given by \( \text{Ker} \mu \) and let \( SL_2 \) be the subgroup of \( G \) corresponding to the Lie algebra spanned by \( \mathfrak{g}^\mu, \mathfrak{g}^{-\mu} \), and \( \mu^\vee \). Then there is the natural embedding \( SL_2 \times_{\mathbb{Z}/2\mathbb{Z}} H \to G \). Let \( \xi_1 \) be the \( SL_2 \)-bundle corresponding to \( I_2 \). Then there exists a generic \( H' \)-bundle \( \xi_2 \) such that, if \( \xi' \) is the bundle on \( SL_2 \times_{\mathbb{Z}/2\mathbb{Z}} H' \) induced by \( \xi_1 \boxtimes \xi_2 \) on \( SL_2 \times H' \), then \( \xi \) is the induced \( G \)-bundle. Note that \( \text{ad}_{SL_2} \xi_1 = I_3 \) and that \( \text{ad}_{H'} \xi_2 = \mathcal{O}_E^{-1} \). There is the induced injection \( \text{ad}_{SL_2} \xi_1 \oplus \text{ad}_{H'} \xi_2 \to \text{ad}_G \xi \). The arguments of Theorem 2.4.3 imply that the corresponding homomorphism on \( H^1 \) is an isomorphism. In particular,

\[
T_{M,x} \cong H^1(E; \text{ad}_G \xi) = H^1(E; \text{ad}_{SL_2} \xi_1) \oplus H^1(E; \text{ad}_{H'} \xi_2).
\]

Clearly, \( H^1(E; \text{ad}_{H'} \xi_2) \) is the image of the tangent space \( T_{D,x} \) to \( D \) at \( x \), and hence the normal bundle sequence is split at \( x \). Another way to give the same splitting on the normal bundle sequence is as follows. The Killing form induces a quadratic form on \( H^1(E; \text{ad} \xi) \), which is easily seen degenerate with radical equal to \( H^1(E; \text{ad}_{SL_2} \xi_1) \), and this splits the sequence. Dualizing, there is a direct sum decomposition

\[
\Omega^1_{M,x} \cong H^0(E; \text{ad} \xi) = H^0(E; \text{ad}_{SL_2} \xi_1) \oplus H^0(E; \text{ad}_{H'} \xi_2).
\]

Under the identification of \( \Omega^1_{D,x} \) with \( H^0(E; \text{ad}_{H'} \xi_2) \), the above is a splitting of the conormal sequence such that the natural morphism \( \Omega^1_{M,x} \to \Omega^1_{D,x} \) is the projection to \( H^0(E; \text{ad}_{H'} \xi_2) \).

The extension \((\ast)\) restricted to the slice \( \{x\} \times E \) gives an extension

\[
0 \to H^0(E; \text{ad} \xi) \otimes \mathcal{O}_E \to \text{ad} \xi \to \overline{V} \to 0,
\]

and hence defines an extension class \( \varepsilon \in H^1(E; \overline{V}^\vee) \otimes H^0(E; \text{ad} \xi) \). By construction, the part of \( \overline{V} \) supported at \( p_0 \in E \) is isomorphic to \( I_2 \) and so \( H^1(E; \overline{V}^\vee) \cong \mathbb{C} \). Thus
up to a scalar we can view $\varepsilon$ as an element of $H^0(E; \text{ad} \xi)$. We claim that, if $V'$ is
defined via the commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^0(E; \text{ad} \xi) \otimes_{\mathcal{O}_E} \mathcal{O}_E & \longrightarrow & \text{ad} \xi & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow & & \downarrow & & = & & \downarrow \\
0 & \longrightarrow & H^0(E; \text{ad} H, \xi_2) \otimes_{\mathcal{O}_E} \mathcal{O}_E & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & 0
\end{array}
$$

then the corresponding extension $V'$ is split. There are many ways to see this.

For example, it follows from the explicit knowledge of $\text{ad} \xi$ given in the proof of
Theorem 5.2.3. Another way is to use that fact that orthogonal projection from $\mathfrak{g}$ to
$\mathfrak{h}'$ is equivariant with respect to the action of the subgroup $SL_2 \times \mathbb{Z}/2\mathbb{Z}H'$. Thus there
is an induced homomorphism of vector bundles from $\text{ad} \xi$ to $H^0(E; \text{ad} H', \xi_2) \otimes_{\mathcal{O}_E} \mathcal{O}_E$.
In turn, this homomorphism induces a surjection from $V'$ to $H^0(E; \text{ad} H, \xi_2) \otimes_{\mathcal{O}_E} \mathcal{O}_E$
which splits the extension. In any case, we see that $\varepsilon \in H^0(E; \text{ad} \xi) \cong \Omega^1_{M,x}$ lies
in the kernel of the projection $\Omega^1_{M,x} \rightarrow \Omega^1_{D,x}$, and hence $\varepsilon$ lies in the image of the
conormal line $N_{D/M,x}^{-1}$. Thus, at a generic point of $D$, the extension class $\varepsilon$ is equal
to a nonzero multiple of $\nu^* \tilde{d}s$ at $x$ and hence is a nonzero multiple of $s$ at $x$.

Suppose that $s'$ is the section of $\nu^* \Omega^1_M \mid \tilde{D}$ corresponding to the extension $(\ast)$.
The above shows that there exists a regular function $c$ on $\tilde{D}$ such that $s' = cs$.
Fixing a basis for $\Omega^1_{M}$, we can write $s' = cs = (cf_1, \ldots, cf_r)$. To complete the
proof of Theorem 5.2.3, it suffices to show the following: for all $y \in \tilde{D}$, there
exists an $i$ such that $cf_i \notin \mathfrak{m}_y$. This will show that $c$ is a nowhere vanishing
function on $\tilde{D}$ and hence a constant, and that $s$ is a nowhere vanishing section
of $\nu^* \Omega^1_M \mid \tilde{D}$. To see this, let $x \in M$ lie under $y$, let $\xi$ be the corresponding $G$-
bundle, and let $0 \rightarrow \mathcal{O}_E^r \rightarrow V \rightarrow \mathcal{V} \rightarrow 0$ be the corresponding extension, where
we use the given basis of $\Omega^1_M$ to trivialize the subbundle $H^0(E; V) \otimes \mathcal{O}_E$ of $V$.
The fact about $V$ that we shall need is that $h^0(E; V) = r$ and hence that the
coboundary map $\delta: H^0(E; \mathcal{V}) \rightarrow H^1(E; \mathcal{O}_E^r)$ is injective. Write $\delta = (\delta_1, \ldots, \delta_r)$, where $\delta_i \in \text{Ext}^1(\mathcal{V}, \mathcal{O}_E) \cong \text{Hom}(H^0(E; \mathcal{V}), H^1(E; \mathcal{O}_E))$. By Proposition 5.1.1
we have identified $Q = Q(V)$ with $\tilde{R} = S^W_0/\mathfrak{m}_y S^W_0$ and $\text{Ext}^1_{\mathcal{O}_E}(\mathcal{V}, \mathcal{O}_E)$ with
$(\tilde{R}/\mu \tilde{R})^r$. Under this identification, the $i$th component $\delta_i$ of $\delta$ corresponds to the
$i$th component of $s'$, namely $cf_i$. Thus, if $cf_i \in \mathfrak{m}_y$ for every $i$, then $cf_i$ lies in the
subspace $\mathfrak{m}_y/\mu \tilde{R}$ of $\tilde{R}/\mu \tilde{R}$, which is a proper subspace since $y \in \tilde{D}$, i.e. $\mathfrak{m}_y$ is
not the unit ideal. This would imply that all of the $\delta_i$ lie in a proper subspace
of $\text{Hom}(H^0(E; \mathcal{V}), H^1(E; \mathcal{O}_E))$, and hence that they have a common kernel. This
contradicts the injectivity of $\delta$. Thus $cf_i \notin \mathfrak{m}_y$ for some $i$, proving that both $c$ and
$f_i$ are units at $y$.

We can say more about the function $a$ such that $\nu^* \tilde{d}s = as$ in the discussion
preceding Theorem 5.2.3. To determine $a$ up to a nonzero constant, it suffices
to describe the divisor of zeroes of $a$ in $\tilde{D}$. We begin by describing the divisor $D$ in
more detail. Let $y \in \tilde{D}$, so that by definition $y$ is the image of a point $e \in E_{eg} \otimes \Lambda$
such that $\mu(e) = 0$. If $\pm \mu$ are the only roots which vanish at $e$, then by Lemma 5.3.8
$\tilde{T}_0 \rightarrow \tilde{T}_0$ is a local isomorphism near $y$, and hence $\tilde{D} \rightarrow D$ is also an isomorphism.
Thus we may assume that there exists a root $\mu' \neq \pm \mu$ such that $\mu'(e) = 0$ as well.
The possibilities are as follows (we leave the calculations to the reader):
Lemma 5.2.4. Let $e \in E_{\text{reg}} \otimes \Lambda$ lie over $y \in \hat{D}$. Suppose that there exists a root $\mu' \neq \pm \mu$ such that $\mu'(e) = 0$, and that $e$ is a generic point of $D(\mu) \cap D(\mu')$.

(i) If $\mu$ and $\mu'$ are orthogonal, then $\varphi^{-1}(\varphi(y)) = \{y, y'\}$ consists of two distinct points, both $\hat{T}_0$ and $D$ have two local branches at $\varphi(y)$, meeting transversally, and $\varphi: \hat{T}_0 \rightarrow T_0$ is a local diffeomorphism at $y$ onto one of the two branches of $T_0$ passing through $\varphi(y)$. There exist local coordinates $x_1, \ldots, x_{r-2}, s_1, t_2$ for $\hat{T}_0$ at $y$ and local coordinates $x_1, \ldots, x_{r-2}, t_1, t_2$ for $\mathcal{M}$ at $\check{\nu}(y)$ such that

$$\check{\nu}(x, s_1, t_2) = (x, t_1, t_2),$$

where $t_1 = s_1^2$. A local form for the equation of $D$ is $t_1 t_2 = 0$. Finally, $\mu = r^* t = s_1$.

(ii) If $\mu$ and $\mu'$ are not orthogonal, then they span a root system of type $A_2$. The preimage $\varphi^{-1}(\varphi(y)) = \{y\}$. There exist local coordinates $x_1, \ldots, x_{r-2}, s_1, s_2$ for $\hat{T}_0$ at $y$ and local coordinates $x_1, \ldots, x_{r-2}, \sigma_2, \sigma_3$ for $\mathcal{M}$ at $\check{\nu}(y)$ such that

$$\check{\nu}(x, s_1, s_2) = (x, \sigma_2, \sigma_3),$$

where $\sigma_2 = -(s_1^2 + s_2^2 + s_1 s_2)$ and $\sigma_3 = -(s_1^2 s_2 + s_1 s_2^2)$. A local form for the equation of $D$ is $4\sigma_2^2 + 27\sigma_3^2 = 0$. Finally, $\mu = r^* t = s_1 - s_2$. □

Corollary 5.2.5. Let $\hat{D}_{1,1}$ the the hypersurface in $\hat{D}$ which is the image of the set of $e \in E_{\text{reg}} \otimes \Lambda$ such that there exists a root $\mu' \neq \pm \mu$, orthogonal to $\mu$, with $\mu'(e) = 0$. Let $\hat{D}_2$ the the hypersurface in $\hat{D}$ which is the image of the set of $e \in E_{\text{reg}} \otimes \Lambda$ such that there exists a root $\mu' \neq \pm \mu$, not orthogonal to $\mu$ with $\mu'(e) = 0$. Then the function $a$ vanishes to order 1 along every component of $\hat{D}_{1,1}$ and to order 3 along every component of $\hat{D}_2$. □

6. Further conjectures

In this final section, we speculate on how some of the previous results and conjectures might be generalized to an arbitrary irreducible representation $\rho$. For simplicity, we shall only discuss the case of $\mathfrak{g}$, where these questions might be accessible to the methods of Ginzburg. The general idea is to find a filtration of the corresponding vector bundle $\mathcal{V}$, or equivalently of the module $V \otimes_{\mathbb{C}} S^W$, so that the associated graded module is free and action of $\rho_{\hat{\Sigma}}$ on the associated graded can be described explicitly. For simplicity, we shall shift our point of view and work with highest weights instead of lowest weights in what follows.

6.1. Notation. Given an irreducible representation $\rho: G \rightarrow GL(V)$, let $\lambda$ be a weight of $\rho$ and let $m_\lambda$ be the multiplicity of $\lambda$, so that $V = \bigoplus_\lambda V_\lambda$, where $H$ acts on $V_\lambda$ by the character corresponding to $\lambda$, and $\dim V_\lambda = m_\lambda$. Recall that, if $\lambda$ is a weight, then so is $w \lambda$ for every $w \in W$. In particular, each $W$-orbit of weights contains a unique dominant weight. We define a partial ordering on the set of all dominant weights as follows: If $\lambda_1$ and $\lambda_2$ are two dominant weights, then $\lambda_1 \geq \lambda_2$ if the difference $\lambda_1 - \lambda_2$ is a rational linear combination of simple roots with nonnegative coefficients, and $\lambda_1 > \lambda_2$ to mean $\lambda_1 \geq \lambda_2$ and $\lambda_1 \neq \lambda_2$. 
6.2. Statement of the conjecture. For each dominant weight \( \lambda \) of \( \rho \), define the operator \( P_\lambda \in \text{End}(V \otimes_C S^W) \) by
\[
P_\lambda = \prod_{\lambda' \in W \cdot \lambda} (\rho_\ast \Sigma - \lambda' \cdot \text{Id}).
\]
Note that \( P_\lambda \) does in fact lie in \( \text{End}(V \otimes_C S^W) \), although the individual factors only lie in \( \text{End}(V \otimes_C S) \), and that \( P_\lambda \) commutes with \( \rho_\ast \Sigma \). Moreover, \( \text{Ker} \, P_\lambda \) is a saturated submodule of \( V \otimes_C S^W \) which corresponds to the component of the spectral cover whose reduction is given by the Weyl orbit of \( \lambda \). It seems natural, however, that the ordering of the weights must somehow be taken into account. To do so, make the following construction: For \( \lambda \) dominant, define
\[
Q_\lambda = \prod_{\lambda' \leq \lambda} P_\lambda,
\]
where the product is over all dominant weights \( \lambda' \leq \lambda \), and define and
\[
Q_\lambda^0 = \prod_{\lambda' < \lambda} P_\lambda.
\]
Then \( F_\lambda = \text{Ker} \, Q_\lambda \subseteq V \otimes_C S^W \) is invariant under \( \rho_\ast \Sigma \), as is \( F_\lambda^0 = \text{Ker} \, Q_\lambda^0 \), and \( F_\lambda^0 \subseteq F_\lambda \). Note that, since the dominant weights of \( \rho \) might only be partially ordered, \( \{F_\lambda\} \) need not be a filtration of \( V \otimes_C S^W \).

**Conjecture 6.2.1.** Let \( \lambda \) be a dominant weight. Let \( W_\lambda \) be the stabilizer in \( W \) of \( \lambda \) and \( m_\lambda \) the multiplicity of \( \lambda \) in \( \rho \). Then, in the above notation, there is an isomorphism of \( S^W \)-modules
\[
\hat{h} : F_\lambda / F_\lambda^0 \cong (S^W_\lambda)^{m_\lambda},
\]
and \( \hat{h} \circ \rho_\ast \Sigma = \lambda h \).

In case \( \lambda \) is the highest weight of \( \rho \), the conjecture is Ginzburg’s result. At the other extreme, if \( \lambda = 0 \), the conjecture is contained in the following:

**Proposition 6.2.2.** \( \text{Ker} \, \rho_\ast \Sigma \) is a free submodule of \( V \otimes_C S^W \) of rank \( m_0 \).

**Proof.** If \( 0 \) is not a weight of \( \rho \), then \( \text{Ker} \, \rho_\ast \Sigma(x) = 0 \) for each \( x \) such that \( \Sigma(x) \) is semisimple. Thus \( \text{Ker} \, \rho_\ast \Sigma = 0 \), and hence it is free of rank \( m_0 = 0 \). Thus we may assume that \( 0 \) is a weight of \( \rho \). In this case, the proof uses the following result of Kostant \((\text{I})\) (5.1.1): Suppose that \( m_0 \), the multiplicity of the weight \( 0 \) in \( V \), is positive. Let \( X \) be a regular element of \( \mathfrak{g} \), let \( \mathfrak{g}^X \) denote the centralizer of \( X \), and define
\[
V^{\mathfrak{g}^X} = \{ v \in V : \rho_\ast(Y)(v) = 0 \text{ for all } Y \in \mathfrak{g}^X \}.
\]
Then \( \dim V^{\mathfrak{g}^X} = m_0 \), and in particular it is independent of the choice of \( X \).

For each \( x \in \mathfrak{h} / W \), we just write \( \mathfrak{g}^x \) for \( \mathfrak{g}^{\Sigma(x)} \). Then \( \bigcup_{x \in \mathfrak{h} / W} \mathfrak{g}^x \) is an \( r \)-dimensional subbundle of \( \mathfrak{g} \) of the trivial bundle \( (\mathfrak{h} / W) \times \mathfrak{g} \). Since \( S^W \) is a polynomial algebra, the vector bundle \( \mathfrak{g}^x \) corresponds to a free \( S^W \)-submodule of \( \mathfrak{g} \otimes_C S^W \), with basis \( X_1, \ldots, X_r \) (although the main point is rather that the submodule is projective). Let \( M = \bigcap_{x \in \mathfrak{h} / W} \text{Ker} \, \rho_\ast(X_i) \subseteq V \otimes_C S^W \). Since \( \dim V^{\mathfrak{g}^x} \) is independent of \( x \), it follows that \( M \) is a projective, and hence free, submodule of \( V \otimes_C S^W \), clearly saturated. As \( \Sigma(x) \in \mathfrak{g}^x \) for every \( x \), \( M \subseteq \text{Ker} \, \rho_\ast \Sigma \). Finally, if \( x \) is the image of a regular semisimple element, then it is easy to see that \( V^{\mathfrak{g}^x} = \text{Ker} \, \Sigma(x) \),
and since $M$ is saturated, we must have equality everywhere. Thus $\text{Ker } \rho_*\Sigma = M$, and in particular $\text{Ker } \rho_*\Sigma$ is free.

References

[1] A. Borel, R. Friedman, and J.W. Morgan, \textit{Almost commuting elements in compact Lie groups}, math.GR/9907007.
[2] N. Bourbaki, \textit{Groupes et Algèbres de Lie}, Chap. 4, 5, et 6, Masson, Paris, 1981.
[3] N. Bourbaki, \textit{Groupes et Algèbres de Lie}, Chap. 7 et 8, Masson, Paris, 1990.
[4] R. Friedman and J.W. Morgan, \textit{Holomorphic principal bundles over elliptic curves}, math.AG/9811130.
[5] R. Friedman and J.W. Morgan, \textit{Holomorphic principal bundles over elliptic curves II: The parabolic construction}, math.AG/0006174.
[6] R. Friedman and J.W. Morgan, \textit{Holomorphic principal bundles over elliptic curves III: Singular curves and fibrations}, math.AG/0108104.
[7] R. Friedman and J.W. Morgan, \textit{Automorphism sheaves, spectral covers, and the Kostant and Steinberg sections}, math.AG/0209033.
[8] R. Friedman, J.W. Morgan, and E. Witten, \textit{Vector bundles over elliptic fibrations}, J. Algebraic Geometry \textbf{8} (1999), 279–401.
[9] V. Ginzburg, \textit{Perverse sheaves on a loop group and Langlands’ duality}, math.AG/9511007.
[10] V. Ginzburg, \textit{Loop Grassmannian cohomology, the principal nilpotent and Kostant theorem}, math.AG/9803141.
[11] B. Kostant, \textit{Lie group representations on polynomial rings}, Am. J. Math. \textbf{85} (1963), 327–404.
[12] B. Kostant, \textit{On Whittaker vectors and representation theory}, Invent. Math. \textbf{48} (1978), 101–184.
[13] S. Mukai, \textit{Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves}, Nagoya Math. J. \textbf{81} (1981), 101–116.
[14] P. Slodowy, \textit{Simple Singularities and Simple Algebraic Groups}, Lecture Notes in Mathematics \textbf{815}, Berlin New York, Springer, 1980.
[15] R. Steinberg, \textit{Regular elements of semisimple algebraic groups}, Publ. Math. Inst. Hautes Études Sci. \textbf{25} (1965), 49–80.
[16] T. Teodorescu, \textit{Semistable Torsion-Free Sheaves over Curves of Arithmetic Genus One}, Columbia University thesis (1999).

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