Renormalization Group Equations in integro-differential form describing the evolution of cascades or resumming logarithmic scaling violations have been known in quantum field theory for a long time. These equations have been traditionally solved by turning to Mellin moments, since in this space they become algebraic. x-space solutions are less known, but special asymptotic expansions exist which allow a fast numerical implementation of these equations. We illustrate how the equations can be solved using recursion relations in the next-to-leading order approximation.

1. Introduction

Evolution equations describing the high energy behaviour of scattering amplitudes carry significant information on the factorization/renormalization scale dependence of such amplitudes, and allow to link the behaviour of processes at a given energy scale to collisions taking place at another (usually much higher) scale.

In QCD, the accepted theory of the strong interactions, confinement forbids the detection of the fundamental states of the theory, such as quarks and gluons. However, asymptotic freedom allows to separate the perturbative dynamics at short distances from the non-perturbative one, due to confinement, through factorization theorems. The functions evolved by these equations are called parton distributions. We will very briefly introduce them below, and we will exploit the density matrix formulation of quantum mechanics as an analogy to illustrate the matter.
2. Nonlocal correlators, Wigner functions and all that

The mathematical construct which is the closest to a parton distribution function (p.d.f.) $q(x, Q)$ is a Wigner function. The analogy is, of course, limited.

We recall that Wigner’s description of quantum mechanics via quasi-probabilities of phase space $(x, p)$

$$f(x, p) = \frac{1}{(2\pi)^2} \int dy \psi^\dagger(x - \frac{\hbar}{2} y) \psi(x + \frac{\hbar}{2} y)$$

is fully equivalent to Schrodinger’s formulation $^3$.

Differently from Wigner functions, in a p.d.f. a variable $x$, called “Bjorken x”, takes now the role of the momentum “p”. Also, parton distributions are correlation functions of a special type, being defined just on the light-cone. In this sense they are not generic nonlocal correlators. This limitation, due to the special nature of high energy collisions in asymptotically free theories, sets the boundary of validity of the parton model approach to QCD.

In QCD one starts by introducing, via arguments based on unitarity, the hadronic tensor, which is the key construct describing the collision

$$W^{\mu\nu} = \int \frac{d^4x}{(2\pi)^4} e^{iq \cdot x} \langle P_A S_A; P_B S_B | [J^\mu(0), J^\nu(x)] | P_A S_A; P_B S_B \rangle,$$  \hspace{1cm} (2)

with $P_A$ and $P_B$ being the momenta of the colliding hadrons and $S_A$ and $S_B$ their covariant spins. The $J$’s are electromagnetic currents.

The distribution functions that emerge -at leading order- from this factorized picture are correlation functions of non-local operators in configuration space. They are the quark-quark and the quark-antiquark correlators.

Their expression simplifies in the axial gauge, in which the gauge link is removed by the gauge condition. For instance, the quark-quark correlator takes the form

$$\langle \Phi_{a/A} \rangle_{\alpha\beta}^{\mu\nu} (P, S, k) = \int \frac{d^4z}{(2\pi)^4} e^{ik \cdot z} \langle \overline{\psi}^a_\beta (0) \psi^a_\alpha (z) | PS \rangle.$$  \hspace{1cm} (3)

In (3) we have included the quark flavour index $a$ and an index $A$ for the hadron, as usual. Fields are not time ordered since they can be described by the good light cone components of $\psi$ and by $A_T$, a transverse component of the gauge field $A_{\mu}$, as discussed in $^7$. The non-perturbative information in a collision is carried by matrix elements of this type.

Further considerations allow to show that the leading contributions to (3) come from the light-cone region. The leading expansion of the quark-
quark correlator then is of the form

\[
\int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle PS|\bar{\psi}(0)\psi(\lambda n)|PS\rangle = \frac{1}{2} (\not\!p f_1(x) + \lambda \gamma_5 \not\!p g_1(x) + \gamma_5 S_T \not\!p h_1(x)),
\]

where we have used all the four vectors at our disposal (spin S, momentum P of the hadron) and introduced invariant amplitudes (parton distributions) \( f_1, g_1, h_1 \), now expressed in terms of a scaling variable \( x \) (Bjorken x). \( n^\mu \) is a light-cone four-vector (\( n^2 = 0 \)), approximately perpendicular to the hadron momentum.

The definition of p.d.f.’s in (4) involves also an underlying physical scale \( Q (Q >> \Lambda_{QCD}, \text{with } \Lambda_{QCD} \text{ being the scale of confinement}), \) not apparent from that equation and characterizing the energy scale at which these matrix elements, summarized by (4), are defined. Truly: \( f_1 = f_1(x, Q^2), \) \( h_1 = h_1(x, Q^2) \) and so on.

The role of the various renormalization group equations is to describe the perturbative change in these functions as the scale \( Q \) is raised (lowered). Each equation involves kernels \( (P(x)) \) of various types, of well known form, and asymptotic expansions of the solutions exist (see for instance 4 and the implementation given in 5). Here, however, we will illustrate an alternative method to solve these equations which is computationally very efficient.

3. The equations

Parton distributions, though fully identified by their operatorial form, are not currently calculable from first principles. In fact, the theory that they describe is a (nonlinear) gauge theory characterized by a large QCD coupling constant \( \alpha \) at the scale at which they are usually introduced. On the other hand, the equations describing their evolution under the renormalization group are identified using perturbation theory

\[
Q^2 \frac{d}{dQ^2} q_i^{(-)}(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} P_{(-)}(x, \alpha(Q^2)) \otimes q_i^{(-)}(x, Q^2)
\]

\[
Q^2 \frac{d}{dQ^2} \chi_i(x, Q^2) = \frac{\alpha(Q^2)}{2\pi} P_{(-)}(x, \alpha(Q^2)) \otimes \chi_i(x, Q^2),
\]

having defined

\[
f \otimes g = \int_x^1 \frac{dy}{y} f\left(\frac{x}{y}\right) g(y)
\]

with

\[
\chi_i(x, Q^2) = q_i^{(+)}(x, Q^2) - \frac{1}{n_F} q_i^{(+)}(x, Q^2)
\]
for the non-singlet distributions and

\[ Q^2 \frac{d}{dQ^2} \left( \frac{q^+(x, Q^2)}{G(x, Q^2)} \right) = \frac{\alpha}{2\pi} \left( \frac{P_{qq}(x, Q^2)}{P_{qq}(x, Q^2)} \right) \frac{q^+(x, Q^2)}{G(x, Q^2)} \]  

(8)
in the singlet sector. Here, \( G(x, Q^2) \) is the gluon density, while \( q(x, Q^2) \) is the quark density. \( \alpha \) is the QCD coupling constant.

Similar RG equations can be derived for the photon structure function \( q_\gamma(x, Q^2) \), now with an inhomogeneous term included (\( \alpha_{em} \) is the QED coupling constant)

\[ Q^2 \frac{d}{dQ^2} q_\gamma^-(x, Q^2) = \frac{\alpha_{em}}{2\pi} \left( K^{(0)} + \frac{\alpha}{2\pi} K^{(1)} \right) \]
\[ + \frac{\alpha}{2\pi} \left( P^{(0)} + \frac{\alpha}{2\pi} P^{(1)} \right) \otimes q_\gamma(x, Q^2). \]  

(9)

Let’s start from the latter equation. The ansatz for the solution of (9) is chosen of the form

\[ q_\gamma(x, Q^2) = \frac{\alpha_{em}}{2\pi} \left( \frac{4\pi}{\beta_0} \sum_{n=0}^\infty \frac{A_n(x)}{n!} \ln^n \left( \frac{\alpha}{\alpha_0} \right) + \sum_{n=0}^\infty \frac{B_n(x)}{n!} \ln^n \left( \frac{\alpha}{\alpha_0} \right) \right) \]  

(10)
and the recursion relations for the functions \( A_n(x) \) and \( B_n(x) \) appearing in the expansion are obtained by comparing terms of the same order in \( a^k \log^n(\alpha/\alpha_0) \), with \( k=0,1 \) and \( n = 0, 1, ... \). We use a running QCD coupling at the desired perturbative order

\[ \frac{d\alpha}{d\log(Q^2)} = \beta(\alpha) = -\frac{\beta_0}{4\pi} \alpha^2 - \frac{\beta_1}{16\pi^2} \alpha^3 \]  

(11)
and \( \alpha_0 \equiv \alpha(Q_0) \), with \( Q_0 \) being the initial scale at which the evolution starts. In (11) \( \beta_0 \) and \( \beta_1 \) are the first two coefficients of the QCD beta function.

The recursion relations are in leading order given by

\[ A_1(x) = A_0(x) - K^{(0)}(x) - \frac{2}{\beta_0} P^{(0)}(x) \otimes A_0(x) \]
\[ A_{m+1}(x) = A_m(x) - \frac{2}{\beta_0} P^{(0)}(x) \otimes A_m(x) \quad m = 1, 2, 3, ... \]  

(12)
and

\[ B_1(x) = -\frac{\beta_1}{\beta_0^2} (A_{m+1}(x) - A_m(x)) - 2 \frac{K^{(1)}(x)}{\beta_0} - \frac{2}{\beta_0} P^{(0)}(x) \otimes B_m(x) - \frac{4}{\beta_0^2} P^{(1)}(x) \otimes A^{(0)}(x) \]
\[ B_{m+1}(x) = \frac{-\beta_1}{\beta_0^2} (A_{m+1}(x) - A_m(x)) - \frac{2}{\beta_0} P^{(0)}(x) \otimes B_m(x) - \frac{4}{\beta_0^2} P^{(1)}(x) \otimes A^{(0)}(x) \]

(13)

at order \( \alpha^2 \). The initial condition is easily shown to be of the form

\[ \frac{\alpha_{em}}{2\pi} \left( \frac{4\pi}{\beta_0\alpha_0} A_0(x) + B_0(x) \right) = q_0(x), \quad (14) \]

with \( q_0(x) \) defining the initial functional choice for the parton distribution at the lowest scale. It is possible to show \(^2\) that the freedom in choosing the original values for \( A_0(x) \) and \( B_0(x) \) is not relevant at the order at which we are working \( (\alpha^2) \), as far as (14) is satisfied.

In the case of other equations, such as eqs. (5), we get the recursion relations

\[ A_{n+1}(x) = -\frac{2}{\beta_0} P^{(0)} \otimes A_n(x) \quad (15) \]

and

\[ B_{n+1}(x) = -B_n(x) - \left( \frac{\beta_1}{4\beta_0} A_{n+1}(x) \right) - \frac{1}{4\pi\beta_0} P^{(1)} \otimes A_n(x) - \frac{2}{\beta_0} P^{(0)} \otimes B_n(x) \]

\[ = -B_n(x) + \left( \frac{\beta_1}{2\beta_0^2} P^{(0)} \otimes A_n(x) \right) \]

\[-\frac{1}{4\pi\beta_0} P^{(1)} \otimes A_n(x) - \frac{2}{\beta_0} P^{(0)} \otimes B_n(x), \quad (16) \]

which are solved with the initial condition \( B_0(x) = 0 \). The initial conditions for the coefficients \( A_0(x) \) and \( B_0(x) \) are specified in a slightly different way from the photon case, with \( q(x,Q_0^2) \) now identified as the leading order ansatz for the initial distribution

\[ A_0(x) = \delta(1-x) \otimes q(x,Q_0^2) \equiv q_0(x) \quad (17) \]

which also requires \( B_0(x) = 0 \), since we have to satisfy the boundary condition

\[ A_0(x) + \alpha_0 B_0(x) = q_0(x). \quad (18) \]

Again, other boundary choices are possible for \( A_0(x) \) and \( B_0(x) \) as far as (18) is fulfilled.
4. Distributional Singularities and Finite Elements

Once the recursion are given, it remains to be seen how to actually implement the method. In practice it is not so easy, but it is easier than in other methods. The codes can range from 1,000 lines for QCD to several thousands lines for supersymmetry. The advantage of these implementations is that the codes can run in few minutes (2-3), compared to the much slower codes obtained before. The reason of such improvement is related to the use of analytical expressions (in a finite element discretization of the integrals) which drastically reduce the computational time required to actually perform the recursive integrations. There are some important points, however, to keep into account. The kernels $P(x)$ are defined, in fact, in a distributional sense and are plagued with artificial numerical singularities the most critical ones being tied to plus (“+”) distributions and defined as

$$
\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dy \frac{f(y) - f(1)}{1-y}.
$$

(19)

A simple trick to eliminate this singular behaviour is to use the identity

$$
\frac{1}{(1-x)_+} \otimes f(x) \equiv \int_0^1 dy \frac{yf(y) - xf(x)}{y-x} + f(x) \log(1-x)
$$

(20)

and proceed with a finite element discretization of the resulting integral.

We briefly recall the numerical strategy employed in this analysis. We define $\bar{P}(x) \equiv xP(x)$ and $\bar{A}(x) \equiv xA(x)$. We also define the convolution product

$$
J(x) \equiv \int_x^1 \frac{dy}{y} \left( \frac{x}{y} \right) P \left( \frac{x}{y} \right) \bar{A}(y).
$$

(21)

The integration interval in $y$ at any fixed $x$-value is partitioned in an array of increasing points ordered from left to right $(x_0, x_1, x_2, ..., x_n, x_{n+1})$ with $x_0 \equiv x$ and $x_{n+1} \equiv 1$ being the upper edge of the integration region. One constructs a rescaled array $(x, x/x_n, ..., x/x_2, x/x_1, 1)$. We define $s_i \equiv x/x_i$, and $s_{n+1} = x < s_n < s_{n-1} < ... s_1 < s_0 = 1$. We get

$$
J(x) = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} \frac{dy}{y} \left( \frac{x}{y} \right) P \left( \frac{x}{y} \right) \bar{A}(y)
$$

(22)

At this point we introduce the linear interpolation

$$
\bar{A}(y) = \left( 1 - \frac{y-x_i}{x_{i+1} - x_i} \right) \bar{A}(x_i) + \frac{y-x_i}{x_{i+1} - x_i} \bar{A}(x_{i+1})
$$

(23)

and perform the integration on each subinterval with a change of variable $y- > x/y$ and replace the integral $J(x)$ with its discrete approximation
\[ J_N(x) = \tilde{A}(x_0) \frac{1}{1 - s_1} \int_{s_1}^{1} \frac{dy}{y} P(y)(y - s_1) \]

\[ + \sum_{i=1}^{N} \tilde{A}(x_i) \frac{s_i}{s_i - s_{i+1}} \int_{s_{i+1}}^{s_i} \frac{dy}{y} P(y)(y - s_{i+1}) \]

\[ - \sum_{i=1}^{N} \tilde{A}(x_i) \frac{s_i}{s_{i-1} - s_i} \int_{s_i}^{s_{i-1}} \frac{dy}{y} P(y)(y - s_{i-1}). \]

Introducing the coefficients \( W(x, x) \) and \( W(x_i, x) \), the integral is cast in the form

\[ J_N(x) = W(x, x)\tilde{A}(x) + \sum_{i=1}^{n} W(x_i, x)\tilde{A}(x_i) \]  

(25)

where

\[ W(x, x) = \frac{1}{1 - s_1} \int_{s_1}^{1} \frac{dy}{y} (y - s_1)P(y), \]

\[ W(x_i, x) = \frac{s_i}{s_i - s_{i+1}} \int_{s_{i+1}}^{s_i} \frac{dy}{y} (y - s_{i+1})P(y) \]

\[ - \frac{s_i}{s_{i-1} - s_i} \int_{s_i}^{s_{i-1}} \frac{dy}{y} (y - s_{i-1})P(y). \]

The results of the integration of the recursion relations can be given in analytical form, with obvious care in handling the “+” distributions. We have solved by this method all the leading twist evolution equations of QCD to higher order. Application of the method to supersymmetry has also been illustrated 6.

5. Applications

As a simple illustration of the method we have included two figures which illustrate the variation in shape of the evolved functions with a varying final evolution scale. Scaling violations are usually quite small, however they are very important both to uncover important new physics and for precision studies. Good algorithms are always welcomed.

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Figure 1. Plot of the supersymmetric evolution of the up and down quarks for a final scale of 100 GeV in a specific model.

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Figure 2. Density of gluons, obtained by the recursion method for the QCD (reg.) and Supersymmetric QCD (SQCD) evolution. To vary here is the mass of the supersymmetric partner, the gluino, defining a matching scale in the QCD/SQCD evolution.