SOME PROPERTIES OF POSITIVE SOLUTIONS FOR AN INTEGRAL SYSTEM WITH THE DOUBLE WEIGHTED RIESZ POTENTIALS

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Abstract. In this paper, we study some important properties of positive solutions for a nonlinear integral system. With the help of the method of moving planes in an integral form, we show that under certain integrable conditions, all of positive solutions to this system are radially symmetric and decreasing with respect to the origin. Meanwhile, using the regularity lifting lemma, which was recently introduced by Chen and Li in [1], we obtain the optimal integrable intervals and sharp asymptotic behaviors for such positive solutions, which characterize the closeness of system to some extent.

1. Introduction. In this paper, we consider the following non-linear integral system involving the double weighted Riesz potentials:

\[
\begin{align*}
    u(x) &= \frac{1}{|x|^{\alpha}} \int_{\mathbb{R}^n} \frac{v^p(y)u^q(y)}{|x-y|^{n+\lambda}|y|^{\beta}} dy, \\
v(x) &= \frac{1}{|x|^{\beta}} \int_{\mathbb{R}^n} \frac{u^p(y)v^q(y)}{|x-y|^{n+\lambda}|y|^{\alpha}} dy,
\end{align*}
\]

where $0 < \lambda < n$, $0 < \beta \leq \alpha \leq \beta + \lambda$ with $\lambda \triangleq \lambda + \alpha + \beta < n$ and $p, q \geq 1$ satisfying $p + q = (2n - \lambda)/\lambda$.

Integral equation(s) have attracted lots of attention, since the integral equation(s) theory is an important branch of mathematics and has own’s interest, for
example, the radial symmetry and integrability of solutions, not only are the essential properties to classify all non-negative solution, but also are key ingredients to study the asymptotic behavior of equation(s). On the other hand, due to the fact that under certain integrable conditions integral equation(s) is equivalent to some partial differential equation(s) in \( \mathbb{R}^n \), the integral equation(s) theories also provided a particular skill to investigate the global properties of corresponding differential equation(s). We firstly recall some closely related topics of our investigations as follows.

When \( \alpha = \beta = 0 \) and \( \lambda = n - 2 \), the system (1) can be rewritten as

\[
\begin{align*}
\frac{v(y)}{|x-y|^{n-2}}dy, \\
\frac{u(y)}{|x-y|^{n-2}}dy.
\end{align*}
\]

Under some integrability conditions, (2) is equivalent to the following differential equations

\[
\begin{align*}
-\Delta u(x) &= v^p(x)u^q(x), \\
-\Delta v(x) &= u^p(x)v^q(x),
\end{align*}
\]

which are closely related to the stationary Schrödinger system with critical exponents for the Bose Einstein condensate. C. Li and L. Ma in [8] showed that for \( n \geq 3, 1 \leq p, q \leq (n+2)/(n-2) \) and \( p + q = (n+2)/(n-2) \), any positive solution pair \( (u,v) \) of system (3) in \( L^{2n/(n-2)}(\mathbb{R}^n) \times L^{2n/(n-2)}(\mathbb{R}^n) \) is radially symmetric and unique. Subsequently, Y. Zhao and Y. Lei in [12] considered the following nonlinear system:

\[
\begin{align*}
\frac{u^p(y)i^q(y)}{|x-y|^{\lambda+\beta}}dy, \\
\frac{u^p(y)v^q(y)}{|x-y|^{\lambda+\beta}}dy.
\end{align*}
\]

The authors in [12] showed that as \( \lambda > \beta \geq 0, \lambda + \beta < n \) and \( 0 < p, q \) satisfying \( p + q = (2n - \lambda - \beta)/(\lambda + \beta) \), any positive solutions pair \( (u,v) \) of system (4) in \( L^{2n/(\lambda+\beta)}(\mathbb{R}^n) \times L^{2n/(\lambda+\beta)}(\mathbb{R}^n) \) is symmetric about the origin and monotone decreasing. Additionally, if \( (n-\lambda)(\lambda + \beta) \geq 2n\beta \), then the positive solutions pair \( (u,v) \) of (4) is bounded and satisfies the following asymptotic behaviors:

\[
\lim_{|x| \to \infty} |x|^{\lambda} u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)v^p(y)}{|y|^{\beta}}dy, \quad \lim_{|x| \to \infty} |x|^{\lambda} v(x) = \int_{\mathbb{R}^n} \frac{u^p(y)v^q(y)}{|y|^{\beta}}dy,
\]

which implies that \( u(x) \) and \( v(x) \) in (4) has closed relationship and admits the same order infinitesimal quantity at the infinity. That is to say,

\[
\lim_{|x| \to \infty} \frac{u(x)}{v(x)} = \frac{\int_{\mathbb{R}^n} \frac{u^q(y)v^p(y)}{|y|^{\beta}}dy}{\int_{\mathbb{R}^n} \frac{u^p(y)v^q(y)}{|y|^{\beta}}dy} = C_1 > 0.
\]

Recently, in [11], we studied the following integral system:

\[
\begin{align*}
\frac{v(y)}{|x-y|^{\alpha}}dy, \\
\frac{u(y)}{|x-y|^{\alpha}}dy.
\end{align*}
\]
and showed that if \((u, v) \in L^{n(p+q-1)/(n-\lambda)}(\mathbb{R}^n)\) with \(p+q = (2n - \lambda - 2\beta)/(2\alpha + \lambda)\) is a pair of positive solutions of system (1), then \((u, v)\) is radially symmetric and monotone decreasing about the origin. Meanwhile, we also obtained the sharp integrable interval of system (7) as follows:

\[
u, \ v \in L^r(\mathbb{R}^n), \quad \forall \ \frac{n}{\alpha + \lambda} < r < \frac{n}{\alpha}. \tag{8}\]

Furthermore, we derived the following optimal asymptotic estimates of \((u, v)\) at origin and infinity, respectively,

\[
limit_{|x|\to 0} |x|^\alpha u(x) = \int_{\mathbb{R}^n} \frac{v^q(y)u^p(y)}{|y|^{\beta+\lambda}} dy, \quad \lim_{|x|\to \infty} |x|^\alpha u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)v^p(y)}{|y|^{\beta}} dy; \tag{9}\]

and

\[
limit_{|x|\to 0} |x|^\alpha v(x) = \int_{\mathbb{R}^n} \frac{v^q(y)u^p(y)}{|y|^{\beta+\lambda}} dy, \quad \lim_{|x|\to \infty} |x|^\alpha v(x) = \int_{\mathbb{R}^n} \frac{v^q(y)u^p(y)}{|y|^{\beta}} dy. \tag{10}\]

Another similar integral system is

\[
\begin{align*}
u(x) &= \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v^q(y)}{|x - y|^\lambda |y|^{\beta}} dy, \\
v(x) &= \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{u^p(y)}{|x - y|^\lambda |y|^{\alpha}} dy.
\end{align*} \tag{11}\]

which is closely related to the best constant in the weighted Hardy-Littlewood-Sobolev inequality. The properties of positive solutions for system (11) have been well studied, and we refer the interested readers to [2, 3, 5, 6, 7], among numerous references, for more information.

In this paper, we will prove that the pair of solutions \((u, v)\) in (1) are radially symmetric about the origin and monotone decreasing in \(L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n)\) with \(s = n(p+q-1)/(n-\lambda)\) and \(p+q = (2n - \lambda - \bar{\lambda})/\bar{\lambda}\). Moreover, we will use the regularity lifting lemma to obtain the optimal integrable intervals of the solutions. What’s more, under natural constrain conditions, we will establish the sharp asymptotic estimates of the positive solutions around the origin and near infinity. Our main results can be formulated as follows:

**Theorem 1.1.** Let \((u, v)\) be a pair of positive solutions of (1) and \(p+q = (2n - \lambda)/\bar{\lambda}\). Assume that \((u(x), v(x)) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n)\) for \(s = n(p+q-1)/(n-\lambda)\). Then \(u\) and \(v\) are radially symmetric and monotone decreasing about the origin.

**Theorem 1.2.** For \(s = n(p+q-1)/(n-\lambda)\), let \((u(x), v(x)) \in L^s(\mathbb{R}^n) \times L^s(\mathbb{R}^n)\) be a pair of positive solutions of (1). Then

\[
\begin{align*}
u(x) &\in L^r(\mathbb{R}^n), \quad \forall \ r \in \left( \frac{n}{\alpha + \lambda}, \frac{n}{\alpha} \right), \\
v(x) &\in L^r(\mathbb{R}^n), \quad \forall \ r \in \left( \frac{n}{\beta + \lambda}, \frac{n}{\beta} \right), \tag{12}\end{align*}
\]

where the above integral intervals of \(u(x)\) and \(v(x)\) are optimal in the following sense,

\[
\|u\|_r = \infty, \quad \text{if} \quad r \notin \left( \frac{n}{\alpha + \lambda}, \frac{n}{\alpha} \right),
\]

\[
\|v\|_r = \infty, \quad \text{if} \quad r \notin \left( \frac{n}{\beta + \lambda}, \frac{n}{\beta} \right). \tag{13}\]

This together with Theorem 1.1, (40) and (44) (see Section 4 below), leads to

\[ \beta p + \alpha q + \lambda + \beta < n, \alpha p + \beta q + \alpha + \lambda < n; \]  
\[ (\lambda + \beta)p + (\lambda + \alpha)q + \beta > n, (\alpha + \lambda)p + (\beta + \lambda)q + \alpha > n. \] (14) (15)

Then

\[ \lim_{|x| \to 0} |x|^\alpha u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)v^p(y)}{|y|^\beta} \, dy < \infty, \]  
\[ \lim_{|x| \to \infty} |x|^\alpha + \lambda u(x) = \int_{\mathbb{R}^n} \frac{u^q(y)v^p(y)}{|y|^\beta} \, dy < \infty; \]  
\[ \lim_{|x| \to 0} |x|^\beta v(x) = \int_{\mathbb{R}^n} \frac{v^q(y)u^p(y)}{|y|^\alpha} \, dy < \infty, \]  
\[ \lim_{|x| \to \infty} |x|^\beta + \lambda v(x) = \int_{\mathbb{R}^n} \frac{v^q(y)u^p(y)}{|y|^\alpha} \, dy < \infty. \] (16) (17) (18) (19)

**Remark 1.** The constrain conditions in Theorem 1.3 is natural. Indeed,

\[ \infty > \int_{\mathbb{R}^n} \left\{ \frac{u^q(y)v^p(y)}{|y|^\beta + \lambda} + \frac{v^q(y)u^p(y)}{|y|^{\alpha + \lambda}} \right\} \, dy \]
\[ \geq \int_{|y| < 1} \left\{ \frac{u^q(y)v^p(y)}{|y|^\beta + \lambda} + \frac{v^q(y)u^p(y)}{|y|^{\alpha + \lambda}} \right\} \, dy. \]

This together with Theorem 1.1, (40) and (44) (see Section 4 below), leads to

\[ \int_{|y| < 1} \frac{u^q(y)v^p(y)}{|y|^\beta + \lambda} \, dy \geq C(\lambda, n, \alpha, \beta) \int_{|y| < 1} |y|^{-(\beta + \lambda) - \alpha q - \beta p} \, dy, \]

and

\[ \int_{|y| < 1} \frac{v^q(y)u^p(y)}{|y|^{\alpha + \lambda}} \, dy \geq C(\lambda, n, \alpha, \beta) \int_{|y| < 1} |y|^{-(\alpha + \lambda) - \alpha p - \beta q} \, dy, \]

which implies the constrains condition (14). Similarly, by (42) and (44), we have

\[ \infty > \int_{\mathbb{R}^n} \left\{ \frac{u^q(y)v^p(y)}{|y|^\beta} + \frac{v^q(y)u^p(y)}{|y|^{\alpha}} \right\} \, dy \]
\[ \geq C(\lambda, n, \alpha, \beta) \int_{|y| > 2} \left\{ |y|^{-(\alpha + \lambda)q - (\beta + \lambda)p - \beta} + |y|^{-(\alpha + \lambda)p - (\beta + \lambda)q - \alpha} \right\} \, dy. \]

This implies (15).

**Remark 2.** Comparing Theorems 1.1 and 1.2 with the results of (11) in [3] and (7) in [11] including the single potential integral system (4), which is a particular form of (7) and was studied in [12], all of positive solutions with certain integrable conditions have the same radial symmetry and decreasing about the origin in critical space which is invariant under some scaling transformation (see [4] for detail). But, the integrable intervals and asymptotic behaviors of solutions for the system (1) are distinct from those of (7) and (11). At first, the structure of system (1) is different from the system (7) and (4). Precisely, the weighted functions of \( u(x) \) and \( v(x) \) in system (7) is exactly the same. Therefore, set \( w(x) = u(x) + v(x) \), the system of (7) and (4) can be transformed into a single integral equation as follows

\[ w(x) = \frac{c(x)}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{w^{p+q}(y)}{|x - y|^\beta} \, dy, \]
where \( c(x) \) is a positive and bounded function. However, system (1) is impossible to rewrite as a single equation. Hence, we have to look for a new way to obtain the optimal interval of \((u, v)\). Here, we will build up a new vectorial-form regularity lifting operator (see (33) and (38) in Section 4 for detail) to obtain the sharp integrability of system (1). Furthermore, this study also finds that the integrability is available to characterize the tightness of system to some extent. Indeed, by (8), (12) and (13), we have learned that the pair of \((u, v)\) in (7) admits the same optimal integral intervals which implies that \( u(x) \) and \( v(x) \) in system (7) is more closed than those of system (1). The construction of (1) is also different from (11), which leads to the thoroughly different integrality. Specifically speaking, by [2], we have learned that for system (11), when \( \alpha, \beta > 0, 0 < \lambda < n \) and \( p, q > 1 \) satisfying
\[
\frac{1}{p+1} + \frac{1}{q+1} = \frac{\lambda}{n}, \quad \frac{1}{p+1} - \frac{\lambda}{n} < \frac{\alpha}{n} < \frac{1}{p+1},
\]
the integrability conditions satisfy that for \((q+1)(\lambda + 2\beta) \geq 2n\),
\[
\frac{1}{r} \in \left( \frac{\max\{\alpha, \beta q + \lambda - n\}}{n}, \frac{\lambda + \alpha}{n} \right), \quad \frac{1}{s} \in \left( \frac{\beta}{n}, \frac{\min\{\lambda + \beta, p(\lambda + \alpha) + \lambda - n\}}{n} \right),
\]
and for \((q+1)(\lambda + 2\beta) < 2n\),
\[
\frac{1}{r} \in \left( \frac{\alpha}{n}, \frac{\min\{\lambda + \alpha, q(\lambda + \beta) + \lambda - n\}}{n} \right), \quad \frac{1}{s} \in \left( \frac{\max\{\beta, p\lambda + \lambda - n\}}{n}, \frac{\beta + \lambda}{n} \right).
\]
However, for the system (1), we know from (12) and (13) that the integrability range of the pair of solutions \((u, v)\) lies in a fixed interval, which is independent of \( p \) and \( q \). This implies that system (1) is more stable than (11).

**Remark 3.** Another significant difference among (1), (7) and (11) is the corresponding asymptotic behavior. Indeed, by (5), (9) and (10), we see that the pair of solutions \((u, v)\) in (4) and (7) has the same decay estimate at the origin as well as at infinity. However, for system (1), by (17) and (18), it is easy to see that \( u \) and \( v \) have different asymptotic behaviors, which implies that the intimacy between \( u \) and \( v \) in (4) and (7) is more closed than those of system (1). On the other hand, system (1) is tighter than the Hardy-Littlewood-Sobolev integral system (11). In fact, for system (11), when \( \alpha + \beta \geq 0 \), the pair of solutions \((u, v)\) have the following asymptotic behaviors at the origin:
\[
u(x) \preceq \begin{cases} A_0 \frac{|x|^{\alpha}}{|x|}, & \text{if } \lambda + (q+1)\beta < n, \\ A_1 \frac{|x|^\beta}{|x|}, & \text{if } \lambda + \alpha(p+1) < n, \\ A_2 \frac{|\ln |x||}{|x|^p}, & \text{if } \lambda + \alpha(p+1) = n, \\ A_3 \frac{|x|^\alpha(p+1)+\beta+\lambda-n}{|x|^\alpha}, & \text{if } \lambda + \alpha(p+1) > n. \end{cases}
\]
At the same time, \((u, v)\) at the infinity admits the following asymptotic behaviors:
\[
u(x) \preceq \begin{cases} B_0 \frac{|x|^{\lambda+\alpha}}{|x|}, & \text{if } \lambda q + \beta(q+1) > n, \\ A_1 \frac{|x|^{\beta}}{|x|}, & \text{if } \lambda q + \beta(q+1) < n, \\ A_2 \frac{|\ln |x||}{|x|^p}, & \text{if } \lambda q + \beta(q+1) = n, \\ A_3 \frac{|x|^\alpha(p+1)+\beta+\lambda-n}{|x|^\alpha}, & \text{if } \lambda q + \beta(q+1) > n. \end{cases}
\]
and
\[
v(x) = \begin{cases} 
\frac{B_1}{|x|^\alpha}, & \text{if } \lambda p + \alpha(p+1) > n, \\
\frac{B_2}{|x|^\alpha \ln |x|}, & \text{if } p + \alpha(p+1) = n, \\
\frac{B_3}{|x|^\alpha (\beta+\|x\|_p+\beta-n)}, & \text{if } p + \alpha(p+1) < n.
\end{cases}
\]

Here, we use the notation \(w(x) \leq C/|x|^t\) to denote that \(\lim_{x \to 0} |x|^t w(x) = C\) for a function \(w(x)\), a real number \(t\) and a non-zero real number \(C\).

**Remark 4.** It is easy to see that when \(\alpha = \beta\), the system (7) and (1) have the same integral intervals and asymptotic behaviors at the origin as well as at infinity. However, the solutions space \(L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)\) with \(s = n(p+q-1)/(n-\lambda) = 2n/\lambda\) in our theorem is different from \(L^s(\mathbb{R}^n) \times L^r(\mathbb{R}^n)\) with \(s = n(p+q-1)/(n-\lambda) = 2n/(2n+\lambda)\) in [11] since the index \(p+q = (2n-\lambda)/\lambda = (2n-\alpha-\beta-\lambda)/(\alpha+\beta+\lambda)\) in this paper is different from \(p+q = (2n-\lambda-2\beta)/(\lambda+2\alpha)\) in [11]. This indicates that our results are new and interesting.

The rest of this paper is organized as follows. In Section 2, we will recall some technical lemmas, and the proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4, respectively. Finally, we will build up the sharp asymptotic estimates of (1) in Section 5.

Throughout this paper, we always use the letter \(C\) to denote positive constants that may vary at each occurrence but are independent of the essential variables.

2. Preliminary. In this section, we recall some standard ingredients, which will be used in the proofs of our main results. Firstly, we introduce the following Regularity Lifting Lemma, which is used extensively to the variant integral systems.

**Lemma 2.1** (Regularity Lifting Lemma, see [1]). Let \(X\) and \(Y\) be both Banach spaces with norms \(\|\cdot\|_X\) and \(\|\cdot\|_Y\), respectively. The subspace of \(X\) and \(Y\), \(Z = X \cap Y\) is endowed with a new norm by
\[
\|\cdot\|_Z = \sqrt[p]{\|\cdot\|_X^p + \|\cdot\|_Y^p}, \quad p \in [1, \infty).
\]

Suppose that \(\mathcal{T}\) is a contraction map from Banach space \(X\) into itself and from Banach space \(Y\) into itself. If \(f \in X\) and there exists a function \(g \in Z = X \cap Y\) such that \(f = \mathcal{T}f + g\), then \(f\) also belongs to \(Z\).

**Lemma 2.2** (Weighted Hardy-Littlewood-Sobolev inequality, see [9]). Let \(1 < r, s < \infty, 0 < \lambda < \alpha, \alpha + \beta \geq 0, 1/r + 1/s + (\lambda + \alpha + \beta)/n = 2\) and \(1 - 1/r - \lambda/n \leq \alpha/n < 1 - 1/r\). Then
\[
\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x|^{-\alpha} f(x) g(y) |x-y|^{-\lambda} |y|^{-\beta} dx dy \right| \leq C(\lambda, \alpha, \beta, n) \|f\|_{L^r(\mathbb{R}^n)} \|g\|_{L^s(\mathbb{R}^n)}
\]
for all \(f \in L^r(\mathbb{R}^n), g \in L^s(\mathbb{R}^n)\). The weighted Hardy-Littlewood-Sobolev inequality can also be written in the following equivalent form:
\[
\|R_{\lambda, \alpha, \beta}(g)\|_{L^p(\mathbb{R}^n)} \leq C(n, \alpha, \lambda, \beta, p) \|g\|_{L^{\frac{n}{\alpha+\lambda-n}}(\mathbb{R}^n)},
\]
where \(p\) satisfies \(\alpha/n < 1/p < \min \{\lambda/n, (\alpha+\lambda)/n\}\) and \(R_{\lambda, \alpha, \beta}(g)(x)\) is given by
\[
R_{\lambda, \alpha, \beta}(g)(x) = \frac{1}{|x|^{\alpha}} \int_{\mathbb{R}^n} \frac{g(y)}{|x-y\lambda| |y|^{\beta}} dy.
\]
3. **Proof of Theorem 1.1.** This section is devoted to proving Theorem 1.1. We will use the method of moving plane in integral forms recently introduced by Chen et al. in [1] to prove the radial symmetry and monotonicity of positive solutions of system (1).

**Proof.** For a given real number \( \mu \in \mathbb{R} \), define

\[
\Sigma_\mu \triangleq \{ x = (x_1, ..., x_n) \in \mathbb{R}^n : x_1 \geq \mu \},
\]

and let \( x^\mu \triangleq (2\mu - x_1, x_2, ..., x_n) \), \( u_\mu(x) \triangleq u(x^\mu) \) and \( v_\mu(x) \triangleq v(x^\mu) \).

For any pair of solutions \((u, v)\) of system (1), it is easy to check that

\[
u_\mu(x) - u(x) = \int_{\Sigma_\mu} \left\{ - \left( \frac{1}{|x|^{p}} \frac{1}{|x|^{\alpha} |x - y|^{\lambda} |y|^{\beta} - \frac{1}{|x|^{\beta} |x - y|^{\lambda} |y|^{\alpha}} \right) (v_\mu^p v_\mu^q(y) - u_\mu^p u_\mu^q(y)) + \left( \frac{1}{|x|^{p}} \frac{1}{|x|^{\beta} |x - y|^{\lambda} |y|^{\alpha}} - \frac{1}{|x|^{\alpha} |x - y|^{\lambda} |y|^{\beta}} \right) \right\} dy
\]

\( \triangleq A_1(x) + A_2(x) + A_3(x) + A_4(x), \) (20)

and

\[
v_\mu(x) - v(x) = \int_{\Sigma_\mu} \left\{ - \left( \frac{1}{|x|^{p}} \frac{1}{|x|^{\alpha} |x - y|^{\lambda} |y|^{\beta} - \frac{1}{|x|^{\beta} |x - y|^{\lambda} |y|^{\alpha}} \right) (u_\mu^p v_\mu^q(y) - u_\mu^p v_\mu^q(y)) + \left( \frac{1}{|x|^{p}} \frac{1}{|x|^{\beta} |x - y|^{\lambda} |y|^{\alpha}} - \frac{1}{|x|^{\alpha} |x - y|^{\lambda} |y|^{\beta}} \right) u_\mu^p v_\mu^q(y)
\]

\( \triangleq B_1(x) + B_2(x) + B_3(x) + B_4(x). \) (21)

The proof of Theorem 1.1 consists of two steps. In step 1, we will compare the values of \( u(x) \) with \( u_\mu(x) \) and \( v(x) \) with \( v_\mu(x) \) on \( \Sigma_\mu \), respectively, and show that for sufficiently negative \( \mu < 0 \), there holds

\[
u_\mu(x) - u_\mu(x) \quad \text{and} \quad v(x) - v_\mu(x), \quad \forall x \in \Sigma_\mu - \{0\}. \] (22)

In step 2, we will continuously move the plane \( x_1 = \mu \) along \( x_1 \) direction from near negative infinity to the right as long as (20) holds. By moving this plane in this way, we finally show that the plane will stop at the origin. Now we turn our attention to step 1.
Step 1: Noting that as \( y \in \Sigma_{\mu} \), it is easy to verify that \(|y|^\alpha > |y|\) and 
\( A_2(x) , A_3(x), A_4(x), B_2(x), B_3(x), B_4(x) \leq 0 \) for \( x \in \Sigma_{\mu} \). Therefore
\[
\begin{align*}
u_{\mu}(x) - u(x) & \leq \int_{\Sigma_{\mu}} \left( \frac{1}{|x|^\alpha} \frac{1}{|x|} - \frac{1}{|x|^\alpha} \frac{1}{|x|} \right) \frac{1}{|y|^\beta} (v_{\mu} q^p - v^q p^q)(y)dy \\
& \leq \int_{\Sigma_{\mu}} \left( \frac{1}{|x|^\alpha} \frac{1}{|x|} - \frac{1}{|x|^\alpha} \frac{1}{|x|} \right) \frac{1}{|y|^\beta} (v_{\mu} q^p - v^q p^q)(y)dy \\
& + \int_{\Sigma_{\mu}} \left( \frac{1}{|x|^\alpha} \frac{1}{|x|} - \frac{1}{|x|^\alpha} \frac{1}{|x|} \right) \frac{1}{|y|^\beta} (v_{\mu} q^p - v^q p^q)(y)dy \\
& \triangleq A_{11}(x) + A_{12}(x),
\end{align*}
\]
and
\[
\begin{align*}
u_{\mu}(x) - v(x) & \leq \int_{\Sigma_{\mu}} \left( \frac{1}{|x|^\alpha} \frac{1}{|x|} - \frac{1}{|x|^\alpha} \frac{1}{|x|} \right) \frac{1}{|y|^\beta} (v_{\mu} q^p - v^q p^q)(y)dy \\
& \leq \int_{\Sigma_{\mu}} \left( \frac{1}{|x|^\alpha} \frac{1}{|x|} - \frac{1}{|x|^\alpha} \frac{1}{|x|} \right) \frac{1}{|y|^\beta} (v_{\mu} q^p - v^q p^q)(y)dy \\
& + \int_{\Sigma_{\mu}} \left( \frac{1}{|x|^\alpha} \frac{1}{|x|} - \frac{1}{|x|^\alpha} \frac{1}{|x|} \right) \frac{1}{|y|^\beta} (v_{\mu} q^p - v^q p^q)(y)dy \\
& \triangleq B_{11}(x) + B_{12}(x),
\end{align*}
\]
where \( \Sigma_{\mu} = \{ x \in \Sigma_{\mu} : \ u(x) < u_{\mu}(x) \} \) and \( \Sigma_{\mu} = \{ x \in \Sigma_{\mu} : \ v(x) < v_{\mu}(x) \} \).

Since \( p + q = (2n - \lambda)/\lambda \) and \( p = \alpha \leq \lambda + \beta \), then
\[
s \triangleq \frac{n(p + q - 1)}{n - \alpha} \in \left( \frac{n}{\alpha + \lambda}, \frac{n}{\alpha} \right).
\]

By Lemma 2.2, we conclude that
\[
\begin{align*}
\| A_{11} \|_{L^*(\Sigma_{\mu})} & \leq C(n, p, q) \| v_{\mu} q^p - u^q \| \| u_{\mu} - u \|_{L^{\frac{n}{\alpha + \lambda - \alpha}}(\Sigma_{\mu})} \\
& \leq C(n, p, q) \| v_{\mu} q^p - u^q \| \| u_{\mu} - u \|_{L^*(\Sigma_{\mu})} \\
& \leq C(n, p, q) \| v_{\mu} q^p \|_{L^*(\Sigma_{\mu})} \| u_{\mu} \|_{L^{\frac{n}{\alpha}}(\Sigma_{\mu})} \| u_{\mu} - u \|_{L^*(\Sigma_{\mu})},
\end{align*}
\]
and
\[
\begin{align*}
\| A_{12} \|_{L^*(\Sigma_{\mu})} & \leq C(n, p, q) \| u^q \|_{L^*(\Sigma_{\mu})} \| v_{\mu} q^p - u^q \|_{L^*(\Sigma_{\mu})} \\
& \leq C(n, p, q) \| u \|_{L^*(\Sigma_{\mu})} \| v_{\mu} q^p \|_{L^*(\Sigma_{\mu})} \| u_{\mu} - u \|_{L^*(\Sigma_{\mu})}.
\end{align*}
\]

This together with (25) implies that
\[
\| u_{\mu} - u \|_{L^*(\Sigma_{\mu})} \leq \| A_{11} \|_{L^*(\Sigma_{\mu})} + \| A_{12} \|_{L^*(\Sigma_{\mu})} \\
\leq C(n, p, q) \left\{ \| v_{\mu} q^p \|_{L^*(\Sigma_{\mu})} \| u_{\mu} \|_{L^{\frac{n}{\alpha}}(\Sigma_{\mu})} \| u_{\mu} - u \|_{L^*(\Sigma_{\mu})} \\
+ \| u \|_{L^*(\Sigma_{\mu})} \| v_{\mu} q^p \|_{L^*(\Sigma_{\mu})} \| u_{\mu} - u \|_{L^*(\Sigma_{\mu})} \right\}.
\]

Since \( u , v \in L^*(\mathbb{R}^n) \), we can choose \( N > 0 \) large enough, such that for any \( \mu \leq -N < 0 \),
\[
C(n, p, q) \| v_{\mu} q^p \|_{L^*(\Sigma_{\mu})} \| u_{\mu} \|_{L^{\frac{n}{\alpha}}(\Sigma_{\mu})} \leq 1/4 \text{ and } C(n, p, q) \| u \|_{L^*(\Sigma_{\mu})} \| v_{\mu} q^p \|_{L^*(\Sigma_{\mu})} \leq 1/4,
\]
which combining with (27) implies that
\[ \|u_{\mu} - u\|_{L^r(\Sigma^u_{\mu})} \leq \frac{1}{4}\|u_{\mu} - u\|_{L^r(\Sigma^u_{\mu})} + \frac{1}{4}\|v_{\mu} - v\|_{L^r(\Sigma^v_{\mu})}. \] (28)

Similarly, since \( n(p + q - 1)/(n - \lambda) \in (n/(\beta + \lambda), n/\beta) \) and \( u, v \in L^s(\mathbb{R}^n) \), we have
\[ \|v_{\mu} - v\|_{L^r(\Sigma^v_{\mu})} \leq \frac{1}{4}\|u_{\mu} - u\|_{L^r(\Sigma^u_{\mu})} + \frac{1}{4}\|v_{\mu} - v\|_{L^r(\Sigma^v_{\mu})}. \] (29)

This together with (28) leads to that
\[ \|u_{\mu} - u\|_{L^r(\Sigma^u_{\mu})} + \|v_{\mu} - v\|_{L^r(\Sigma^v_{\mu})} = 0. \]

Therefore \( \Sigma^u_{\mu} \) and \( \Sigma^v_{\mu} \) must be two zero-measure sets, which completes the assertion of Step 1.

**Step 2:** We will continuously move the plane \( x_1 = \mu \) to the right as long as (22) holds. Indeed, suppose that at \( x_1 = \mu^0 < 0 \), we have, for any \( x \in \Sigma^0_{\mu} \)
\[ u(x) \geq u_{\mu^0}(x) \text{ and } v(x) \geq v_{\mu^0}(x), \text{ but } u(x) \neq u_{\mu^0}(x) \text{ or } v(x) \neq v_{\mu^0}(x). \]

Next, we will show that the plane can be moved further to the right. Precisely, there exists an \( \epsilon \) depending on \( n, \alpha, \lambda, \beta, \mu \) and the solution \((u(x), v(x))\) itself such that
\[ u(x) \geq u_{\mu}(x) \text{ and } v(x) \geq v_{\mu}(x), \quad \forall x \in \Sigma_{\mu} - \{0\}, \quad \mu \in [\mu_0^0, \mu^0 + \epsilon). \] (30)

Under the assumption that \( v(x) \neq v_{\mu^0}(x) \) on \( \Sigma^0_{\mu^0} \), by (20), (21) and the non-negativity of \( u, v \), we have \( u(x) > u_{\mu^0}(x) \) in the interior of \( \Sigma^0_{\mu^0} \).

Let
\[ \tilde{\Sigma}^u_{\mu^0} = \{x \in \Sigma^0_{\mu^0} : u(x) \leq u_{\mu^0}(x)\} \text{ and } \tilde{\Sigma}^v_{\mu^0} = \{x \in \Sigma^0_{\mu^0} : v(x) \leq v_{\mu^0}(x)\}. \]

From the analysis mentioned above, it is easy to check that the \( \tilde{\Sigma}^u_{\mu^0} \) is a zero measure set in \( \mathbb{R}^n \). Similarly, we also have \( m(\tilde{\Sigma}^v_{\mu^0}) = 0 \). This together with the integrability conditions \( u, v \in L^s(\mathbb{R}^n) \) ensures that one can choose \( \epsilon \) small enough such that for all \( \mu \in [\mu^0, \mu^0 + \epsilon) \)
\[ C(n, p, q)\|v_{\mu} - v\|_{L^r(\Sigma^v_{\mu})} \leq \frac{1}{4}, \quad C(n, p, q)\|u_{\mu} - u\|_{L^r(\Sigma^u_{\mu})} \leq \frac{1}{4}. \]

So (28) and (29) hold. Thus we also have
\[ \|u_{\mu} - u\|_{L^r(\Sigma^u_{\mu})} = 0 \quad \text{and} \quad \|v_{\mu} - v\|_{L^r(\Sigma^v_{\mu})} = 0, \] (31)

which implies that the measures of \( \tilde{\Sigma}^u_{\mu} \) and \( \tilde{\Sigma}^v_{\mu} \) must be zero. This verifies (30).

Finally, we show that the plane can’t stop before hitting the origin. On the contrary, if the plane stops at \( x_1 = \mu^0 < 0 \), then \( u(x) \) and \( v(x) \) must be symmetric about the plane \( x_1 = \mu^0 \), i.e.,
\[ u(x) = u_{\mu^0}(x) \text{ and } v(x) = v_{\mu^0}(x), \quad \forall x \in \Sigma^0_{\mu^0}. \] (32)
On the other hand, noting that $|x^\mu| > |x|$ for any $x \in \Sigma_{\mu,0}$, we have

$$u(x) - u_\mu(x) = \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} v^p(y) u_\mu(y) \frac{1}{|y|^\beta} dy - \frac{1}{|x^\mu|^\alpha} \int_{\mathbb{R}^n} v^p(y) u_\mu(y) \frac{1}{|y|^\beta} dy$$

$$\geq \int_{\mathbb{R}^n} \left[ \frac{1}{|x|^\alpha} - \frac{1}{|x^\mu|^\alpha} \right] \left( \frac{v^p(y) u_\mu(y)}{|y|^\beta} - \frac{v^p(y) u_\mu(\mu)}{|y|^\beta} \right) dy$$

$$= 0,$$ which obviously contradicts with (32). Since the direction is arbitrary, we derive that $u$ and $v$ are radially symmetric and decreasing about the origin. This completes the proof of Theorem 1.1.

4. **Proof of Theorem 1.2.** In this section, we will apply the regularity lift Lemma 2.1 to obtain the optimal integrable intervals of the solutions of system (1). For convenience, we firstly introduce some necessary notations. For given $w(x)$ and every fixed real number $A > 0$, set

$$w_A(x) = \begin{cases} w(x), & \text{if } w(x) \geq A \text{ or } |x| \geq A; \\ 0, & \text{otherwise.} \end{cases}$$

And for simplicity, we also denote $\| \cdot \|_{L^p(\mathbb{R}^n)}$ by $\| \cdot \|_p$.

**Proof.** Let $(\tau, r)$ be a pair of positive constants such that $(\tau, r) \in (n/(\alpha + \lambda), n/\alpha) \times (n/(\beta + \lambda), n/\beta)$. For $(f, g) \in L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n)$ with the normal norm $\|f\|_r + \|g\|_r$, define the operator $\mathcal{F}_A(f, g)$ by

$$\mathcal{F}_A(f, g)(x) = (T_{A,1}(f)(x), M_{A,1}(g)(x)) + (T_{A,2}(f)(x), M_{A,2}(g)(x)), \quad (33)$$

where $T_{A,1}(f)(x), \ T_{A,2}(f)(x), \ M_{A,1}(g)(x)$, and $M_{A,2}(g)(x)$ is denoted, respectively, by

$$T_{A,1}(f)(x) \triangleq \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{v^p u_A^{-1}(y) f(y)}{|x-y|^\lambda} \frac{1}{|y|^\beta} dy,$$

$$T_{A,2}(f)(x) \triangleq \frac{1}{|x|^\alpha} \int_{\mathbb{R}^n} \frac{u_A^{-1}(y-v_A) f(y)}{|x-y|^\lambda} \frac{1}{|y|^\beta} dy,$$

$$M_{A,1}(g)(x) \triangleq \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{v^p u_A(y) g(y)}{|x-y|^\lambda} \frac{1}{|y|^{\alpha}} dy,$$

$$M_{A,2}(g)(x) \triangleq \frac{1}{|x|^\beta} \int_{\mathbb{R}^n} \frac{v^p (y-u_A) g(y)}{|x-y|^\lambda} \frac{1}{|y|^\alpha} dy.$$

By Lemma 2.2 and Hölder’s inequality, we deduce that for $n/(\alpha + \lambda) < \tau < n/\alpha$,

$$\|T_{A,1} f\|_r \leq C(n, \alpha, \beta, \lambda) \|v^p u_A^{-1} f\|_{\frac{\tau}{n/(\alpha + \lambda)}} \leq C(n, \alpha, \beta, \lambda) \|v_A\|_{\frac{\tau}{n/(\beta + \lambda)}} \|u_A\|_{\frac{\tau}{n/(\alpha + \lambda)}} \|f\|_r,$$  \quad (34)
\[ |T_{A,2}f|_r \leq C(n, \alpha, \beta, \lambda) \| v - v_A \|_{\frac{p}{n(p+q-1) - \lambda}} \| u_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} \| f \|_r, \]  

(35)

and for \( n/(\beta + \lambda) < r < n/\beta, \)

\[ \| M_{A,1}g \|_r \leq C(n, \alpha, \beta, \lambda) \| v_{\frac{q-1}{A}} \|_{\frac{p}{n(p+q-1) - \lambda}} \leq C(n, \alpha, \beta, \lambda) \| v_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} \| u_A \|_{\frac{p}{n(p+q-1) - \lambda}} \| g \|_r, \]  

(36)

\[ \| M_{A,2}r \| \leq C(n, \alpha, \beta, \lambda) \| u - u_A \|_{\frac{p}{n(p+q-1) - \lambda}} \| v_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} \| g \|_r. \]  

(37)

Here \( C(n, \alpha, \beta, \lambda) \) is the sharp constant of weighted Hardy-Littlewood-Sobolev inequality.

Write

\[ F(x) \triangleq \frac{1}{|x|^{\alpha}} \int_{\mathbb{R}^n} \frac{(u - u_A)^q(y)(v - v_A)^p(y)}{|x - y|^{\lambda}} \frac{1}{|y|^{\beta}} \, dy \]

\[ + \frac{1}{|x|^{\alpha}} \int_{\mathbb{R}^n} \frac{v_A^q(y)(u - u_A)^q(y)}{|x - y|^{\lambda}} \frac{1}{|y|^{\beta}} \, dy \]

\[ \triangleq F_1(x) + F_2(x), \]

and

\[ G(x) \triangleq \frac{1}{|x|^{\beta}} \int_{\mathbb{R}^n} \frac{(u - u_A)^q(y)(v - v_A)^q(y)}{|x - y|^{\lambda}} \frac{1}{|y|^{\alpha}} \, dy \]

\[ + \frac{1}{|x|^{\beta}} \int_{\mathbb{R}^n} \frac{u_A^q(y)(v - v_A)^q(y)}{|x - y|^{\lambda}} \frac{1}{|y|^{\alpha}} \, dy \]

\[ \triangleq G_1(x) + G_2(x). \]

Then \((u(x), v(x))\) is a pair of solutions of the following equation

\[ (u(x), v(x)) = \mathcal{T}_A(u(x), v(x)) + (F(x), G(x)). \]  

(38)

We first claim that \( \mathcal{T}_A \) is a contraction map from \( L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n) \) into itself, provided

\[ (\tau, r) \in \left( \frac{n}{\alpha + \lambda}, \frac{n}{\alpha} \right) \times \left( \frac{n}{\beta + \lambda}, \frac{n}{\lambda} \right). \]

Indeed, noting that \((u, v) \in L^{\frac{n(p+q-1)}{n-\lambda}}(\mathbb{R}^n) \times L^{\frac{n(p+q-1)}{n-\lambda}}(\mathbb{R}^n)\), there exists a positive number \( A \) large enough such that

\[ S_A \triangleq C(n, \alpha, \beta, \lambda) \left\{ \| v_A \|_{\frac{p}{n(p+q-1) - \lambda}} \| u_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} + \| v_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} \| u - u_A \|_{\frac{p}{n(p+q-1) - \lambda}} + \| v - v_A \|_{\frac{p}{n(p+q-1) - \lambda}} \| u_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} + \| v_A \|_{\frac{q-1}{n(p+q-1) - \lambda}} \| u_A \|_{\frac{p}{n(p+q-1) - \lambda}} \right\} \leq \frac{1}{2}. \]

This together implies that

\[ \| \mathcal{T}_A(f, g) \|_{r \times r} \leq \| T_{A,1}f \|_r + \| T_{A,2}f \|_r + \| M_{A,1}g \|_r + \| M_{A,2}g \|_r \]

\[ \leq S_A \left( \| f \|_r + \| g \|_r \right) \leq \frac{1}{2} \left( \| f \|_r + \| g \|_r \right). \]

which together with the linearity of \( \mathcal{T}_A \) implies that the claim holds. At the same time, noting that \( u - u_A \) and \( v - v_A \) are bounded functions with supported set, as
\((\tau, t) \in (n/(\alpha + \lambda), n/\alpha) \times (n/(\beta + \lambda), n/\beta)\), we conclude that
\[
\|F\|_r + \|G\|_r \leq \|F_1\|_r + \|F_2\|_r + \|G_1\|_r + \|G_2\|_r
\leq \|(u - u_A)^q (v - v_A)^p\|_{\frac{n}{\beta + n - \lambda}}
+ \|v_A\|_{\frac{n}{\beta + n - \lambda}}\|u - u_A\|_{\frac{n - 1}{n - \lambda}} \|u - u_A\|_r
+ \|(u - u_A)^p (v - v_A)^q\|_{\frac{n}{\beta + n - \lambda}}
+ \|u_A\|_{\frac{n}{\beta + n - \lambda}}\|v - v_A\|_{\frac{n - 1}{n - \lambda}} \|v - v_A\|_r.
\]

Now, we are in the position to apply the Lemma 2.2 to obtain the sharp integrability of system (1). Note that
\[
s = \frac{n(p + q - 1)}{(n - \lambda)} \in \left(\frac{n}{\alpha + \lambda}, \frac{n}{\alpha}\right) \cap \left(\frac{n}{\beta + n}, \frac{n}{\beta}\right).
\]
Combining with \(u, v \in L^s(\mathbb{R}^n)\) and taking \(\mathcal{X} = L^r(\mathbb{R}^n) \times L^r(\mathbb{R}^n)\) for any \((\tau, t) \in (n/(\alpha + \lambda), n/\alpha) \times (n/(\beta + \lambda), n/\beta)\), \(\mathcal{Y} = L^{\frac{n}{\alpha + \lambda}}(\mathbb{R}^n) \times L^{\frac{n}{\beta + n}}(\mathbb{R}^n)\) and \(\mathcal{Z} = \mathcal{X} \cap \mathcal{Y}\) in Lemma 2.1, we derive that
\[
u \in L^r(\mathbb{R}^n), \quad s \in \left(\frac{n}{\alpha + \lambda}, \frac{n}{\alpha}\right); \quad v \in L^r(\mathbb{R}^n), \quad r \in \left(\frac{n}{\beta + n}, \frac{n}{\beta}\right).
\]

Next, we turn to show that above integrable interval of \((u, v)\) is optimal. Observe that \(p + q = (2n - \lambda)/\lambda\) and \(\alpha < \beta + \lambda\), we have
\[
0 < \frac{\lambda(p + q) - n}{n(p + q - 1)} = \frac{\lambda}{2n} < 1,
\]
and
\[
\frac{(\lambda + \beta)n(p + q - 1)}{\lambda(p + q) - n} - n = n\left\{\frac{(\lambda + \beta)(p + q - 1) - [\lambda(p + q) - n]}{\lambda(p + q) - n}\right\}
= \frac{n(n - \lambda)(\lambda + \beta - \alpha)}{\lambda(p + q) - n} \lambda > 0.
\]
This combining with Hölder’s inequality leads to
\[
\int_{|y| > 1} \frac{v(y)^p u(y)^q}{|y|^\lambda \beta^2} dy \leq \left\|v\right\|_{\frac{n}{\alpha + \lambda}}\left\|u\right\|_{\frac{n}{\beta + n}} \times \left\{\int_{|y| > 1} |y|^{-(\lambda + \beta) n(p + q - 1)/(\alpha + \lambda)p} dy\right\} \frac{\lambda(p + q) - n}{\lambda(p + q - 1)} < \infty.
\]
Therefore, as \(|x| < 1\) and \(|y| > 2\), we conclude that
\[
u(x) \geq \frac{1}{|x|^\alpha} \int_{|y| \geq 2} \frac{v(y)^p u(y)^q}{|x - y|^\lambda |y|^\beta} dy \geq \frac{1}{|x|^\alpha} \int_{|y| \geq 2} \frac{v(y)^p u(y)^q}{(|x| + |y|)^\lambda |y|^\beta} dy
\geq C(\lambda) \frac{1}{|x|^\alpha} \int_{|y| \geq 2} \frac{v(y)^p u(y)^q}{|y|^\lambda |y|^{\lambda + \beta}} dy \geq C(\lambda) |x|^{-\alpha},
\]
which implies that
\[
\|u\|_s^s \geq \int_{|x| < 1} |x|^{-\alpha s} dx = \infty, \quad \text{as } s \geq \frac{n}{\alpha}.
\]
Now, we consider the other case. Noting that for $\beta + \lambda > \alpha \geq \beta$,
\[
 n - \frac{n \beta (p + q - 1)}{\lambda(p + q) - n} = \frac{n(\lambda - 2\beta)}{\lambda} = \frac{n(\alpha + \lambda - \beta)}{\lambda} > 0,
\]
and
\[
\int_{|y| \leq 1} \frac{v(y)^p u(y)^q}{|y|^\lambda + \beta} dy \leq \|v\|^p_{n(p+q-1)} \times \|u\|^q_{n(p+q-1)}
\]
\[
\times \left( \int_{|y| \leq 1} |y|^{-\beta n(p+q-n)} dy \right)^{\frac{\lambda(p+q-n)}{n(p+q-n)}} < \infty.
\]
Then, as $|x| > 2$ and $|y| < 1$, we have
\[
u(x) \geq \frac{1}{|x|^\alpha} \int_{|y| \leq 1} \frac{v(y)u(y)}{|x|+|y|} \frac{1}{|y|^\beta} dy
\]
\[
\geq \frac{1}{|x|^\alpha} \int_{|y| \leq 1} \frac{v(y)u(y)}{|y|^\beta} dy \geq C(\lambda, n, p, q)|x|^{-\alpha - \lambda},
\]
Hence, we have
\[
\|u\|_\tau \geq \int_{|y| \geq 2} |x|^{-\alpha - \lambda} dx = \infty, \quad \text{as } \tau \leq \frac{n}{\alpha + \lambda}.
\]
Similarly, with the same arguments, it is easy to check that
\[
v(x) \geq C(\lambda, \beta, \alpha, p, q) |x|^{-\beta}, \quad \forall |x| < 1;
\]
\[
v(x) \geq C(\lambda, \beta, \alpha, p, q) |x|^{-\beta - \lambda}, \quad \forall |x| \geq 2,
\]
and
\[
\|v\|_\tau \geq \int_{|x| < 1} |x|^{-\beta} dx = \infty, \quad \text{as } \tau \geq \frac{n}{\beta};
\]
\[
\|v\|_\tau \geq \int_{|x| > 2} |x|^{-(\beta + \lambda)} dx = \infty, \quad \text{as } \tau \leq \frac{n}{\beta + \lambda}.
\]
This completes the proof of Theorem 1.2. \(\square\)

5. Proof of Theorem 1.3.

\textit{Proof.} To show Theorem 1.3, firstly, we conclude that the following improper integrals are convergent:
\[
D_0 = \int_{\mathbb{R}^n} \frac{v^\tau(y)u^p(y)}{|y|^\beta + \lambda} dy, \quad D_\infty = \int_{\mathbb{R}^n} \frac{u^\tau(y)v^p(y)}{|y|^\beta} dy;
\]
and
\[
E_0 = \int_{\mathbb{R}^n} \frac{v^\tau(y)u^p(y)}{|y|^\alpha + \lambda} dy, \quad E_\infty = \int_{\mathbb{R}^n} \frac{u^\tau(y)v^p(y)}{|y|^\alpha} dy.
\]
Since $\beta p + \alpha q + \lambda + \beta < n$, then there exists $\varepsilon > 0$ small enough such that
\[
\varepsilon < \min \left\{ \frac{n - (\beta p + \alpha q + \lambda + \beta)}{4n}, \frac{\lambda q}{4n}, \frac{\lambda p}{4n} \right\}.
\]
This together with Hölder’s inequality and Theorem 1.2 implies that
\[
\int_{|y|<1} \frac{u^q(y)v^p(y)}{|y|^{\beta+\lambda}} \, dy \leq \int_{|y|<1} \frac{u^q(y)v^p(y)}{|y|^{\beta+\lambda}} \, dy
\]
\[
\leq C(n, \lambda, \beta, \alpha) \left( \int_{|y|<1} u^{q_{\tau_1}}(y)dy \right)^{1/\tau_1} \left( \int_{|y|<1} v^{p_{\tau_1}}(y)dy \right)^{1/\tau_2}
\]
\[
\times \left( \int_{|y|<1} |y|^{-(\beta+\lambda)s_1} \, dy \right)^{1/s_1},
\]
(47)

where \(1/\tau_1 = (\alpha q)/n + \varepsilon\), \(1/r_1 = (\beta p)/n + \varepsilon\) and \(1/s_1 = (\beta + \lambda)/n + \varepsilon\) with
\[
\frac{1}{\tau_1} + \frac{1}{r_1} + \frac{1}{s_1} \leq 1.
\]

Similarly, when \(\alpha p + \beta q + \alpha + \lambda < n\), taking \(\tau_2, r_2, s_2\) as follows
\[
1/\tau_2 = (\beta q)/n + \varepsilon, 1/r_2 = (\alpha p)/n + \varepsilon, 1/s_2 = (\alpha + \lambda)/n + \varepsilon,
\]
where \(\varepsilon\) is small enough and satisfies
\[
\varepsilon < \min \left\{ \frac{n - (\alpha p + \beta q + \lambda + \alpha)}{4n}, \frac{\lambda q}{4n}, \frac{\lambda p}{4n} \right\}.
\]

Then \(1/\tau_2 + 1/r_2 + 1/s_2 \leq 1\) and by Hölder’s inequality,
\[
\int_{|y|<1} \frac{v^q(y)u^p(y)}{|y|^{\alpha}} \, dy \leq \int_{|y|<1} \frac{v^q(y)u^p(y)}{|y|^{\alpha+\lambda}} \, dy
\]
\[
\leq C(n, \lambda, \beta, \alpha) \left( \int_{|y|<1} v^{q_{\tau_2}}(y)dy \right)^{1/\tau_2} \left( \int_{|y|<1} u^{p_{\tau_2}}(y)dy \right)^{1/r_2}
\]
\[
\times \left( \int_{|y|<1} |y|^{-(\alpha+\lambda)s_2} \, dy \right)^{1/s_2}.
\]
(48)

Now we consider the other case \(|y| \geq 1\). Since \(\beta p + \alpha q + \lambda + \beta < n\) and \((\beta + \lambda)p + (\alpha + \lambda)q + \beta > n\), then there exists a real number \(k > 1\) such that
\[
0 < \frac{1}{n} \left( \frac{\lambda(p+q)}{k} + \beta p + \alpha q + \beta - n \right) < \frac{\beta}{n}.
\]

Let
\[
\frac{1}{\tau_3} = \frac{\alpha q}{n} + \frac{\lambda q}{k n}, \quad \frac{1}{t_3} = \frac{\beta p}{n} + \frac{\lambda p}{k n}, \quad \frac{1}{s_3} = 1 - \frac{\lambda(p+q)}{n k} - \frac{\beta p}{n} - \frac{\alpha q}{n},
\]

Then
\[
\frac{n}{\alpha + \lambda} < \tau_3q < \frac{n}{\alpha}, \quad \frac{n}{\lambda + \beta} < p t_3 < \frac{n}{\beta}, \quad s_3 \beta > n,
\]
respectively, as follows.

By Lebesgue’s dominated convergence theorem, we derive that
\[
\int_{|y|>1} \frac{u^q(y)v^p(y)}{|y|^{\beta+\lambda}} dy \leq \int_{|y|>1} \frac{u^q(y)v^p(y)}{|y|^\beta} dy
\]
\[
\leq C(n, \lambda, \beta, \alpha) \left( \int_{|y|>1} u^{q_{\tau_3}}(y) dy \right)^{1/\tau_3} \left( \int_{|y|>1} v^{p_{\tau_3}}(y) dy \right)^{1/\tau_3}
\times \left( \int_{|y|>1} |y|^{-\beta s_3} dy \right)^{1/s_3}.
\]

Similarly, when \( \alpha p + \beta q + \alpha + \lambda < n \) and \( (\alpha + \lambda)p + (\beta + \lambda)q + \alpha + \lambda > n \), there exists a positive real number \( k > 1 \) such that
\[
0 < \frac{1}{n} \left( \frac{\lambda(p+q)}{k} + \beta q + \alpha p - \alpha n \right) < \frac{\alpha}{n}.
\]

Taking \( \tau_4, t_4, s_4 \) as follows
\[
\frac{1}{\tau_4} = \frac{\alpha p}{n} + \frac{\lambda p}{n k}, \quad \frac{1}{t_4} = \frac{\beta q}{n} + \frac{\lambda p}{n k}, \quad \frac{1}{s_4} = 1 - \frac{1}{n} \left( \frac{\lambda(p+q)}{k} + \beta q + \alpha p \right).
\]

Then we derive that
\[
\int_{|y|>1} \frac{v^q(y)u^p(y)}{|y|^{\alpha + \lambda}} dy \leq \int_{|y|>1} \frac{v^q(y)u^p(y)}{|y|^{\alpha}} dy
\]
\[
\leq C(n, \lambda, \beta, \alpha) \left( \int_{|y|>1} v^{q_{t_4}}(y) dy \right)^{1/t_4} \left( \int_{|y|>1} u^{p_{t_4}}(y) dy \right)^{1/t_4}
\times \left( \int_{|y|>1} |y|^{-\alpha s_4} dy \right)^{1/s_4}.
\]

Thus (45) and (46) immediately follow from (47)–(50).

Now, we turn to the proofs of (17) and (18). For fixed \( R > 0 \), define \( \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \), respectively, as follows.

\[
\mathcal{D}_1(x) = \left( \int_{B_R(0)} \left( \frac{|x|}{|x-y|} \right)^\lambda \frac{v^p(y)u^q(y)}{|y|^\beta} dy, \right. \quad (51)
\]

\[
\mathcal{D}_2(x) = \left( \int_{\mathbb{R}^n \setminus B_R(0) \setminus B_{R/2}(x)} \left( \frac{|x|}{|x-y|} \right)^\lambda \frac{v^p(y)u^q(y)}{|y|^\beta} dy, \right. \quad (52)
\]

and

\[
\mathcal{D}_3(x) = \left( \int_{B_{R/2}(x) \setminus B_{R/4}(x)} \left( \frac{|x|}{|x-y|} \right)^\lambda \frac{v^p(y)u^q(y)}{|y|^\beta} dy. \right. \quad (53)
\]

Noting that as \( |x| \geq 2R \) and \( |y| < R \), by (45), we conclude that
\[
\frac{v^p(y)u^q(y)}{|y|^\beta} \left( \frac{|x|}{|x-y|} \right)^\lambda \leq C(\lambda) \frac{v^p(y)u^q(y)}{|y|^\beta} \in L^1(\mathbb{R}^n).
\]

By Lebesgue’s dominated convergence theorem, we derive that
\[
\lim_{|x| \to \infty} \int_{B_R(0)} \left\{ \left( \frac{|x|}{|x-y|} \right)^\lambda - 1 \right\} \frac{v^p(y)u^q(y)}{|y|^\beta} dy = 0,
\]

\[
\int_{|y|>1} \frac{v^q(y)u^p(y)}{|y|^{\alpha + \lambda}} dy \leq \int_{|y|>1} \frac{v^q(y)u^p(y)}{|y|^{\alpha}} dy
\]
\[
\leq C(n, \lambda, \beta, \alpha) \left( \int_{|y|>1} v^{q_{\tau_4}}(y) dy \right)^{1/\tau_4} \left( \int_{|y|>1} u^{p_{\tau_4}}(y) dy \right)^{1/\tau_4}
\times \left( \int_{|y|>1} |y|^{-\alpha s_4} dy \right)^{1/s_4}.
\]
and
\[
\lim_{{R \to \infty}} \lim_{{|x| \to \infty}} \int_{{B_R(0)}} \left( \frac{|x|}{|x-y|} \right)^{\lambda} \frac{v^p(y)u^q(y)}{|y|^\beta} dy = \int_{{\mathbb{R}^n}} \frac{v^p(y)u^q(y)}{|y|^\beta} dy = D_\infty. \quad (54)
\]

Therefore, to obtain the second equality of (45), it suffices to show that \(\lim_{{|x| \to \infty}} \mathcal{D}_2(x) = 0\) and \(\lim_{{|x| \to \infty}} \mathcal{D}_3(x) = 0\). Firstly, we turn to \(\mathcal{D}_2(x)\). Observe that \(|x| > 2R\) and \(y \in \mathbb{R}^n \setminus B_R(0) \setminus B_{|x|/2}(x)\), together with (45) we have
\[
|\mathcal{D}_2(x)| = \left| \int_{{\mathbb{R}^n \setminus B_R(0) \setminus B_{|x|/2}(x)}} \left( \frac{|x|}{|x-y|} \right)^{\lambda} \frac{v^p(y)u^q(y)}{|y|^\beta} dy \right| \leq C(\lambda) \int_{{\mathbb{R}^n \setminus B_R(0)}} \frac{v^p(y)u^q(y)}{|y|^\beta} dy \to 0, \quad R \to \infty.
\]

Next, we turn our attention to \(\mathcal{D}_3(x)\). By Theorems 1.1 and 1.2, we have
\[
u^s(|x|) \int_{B_{|x|/2}(x)} dx \leq \int_{B_{|x|/2}(x)} \nu^s(x) dx \leq \|\nu\|^s_s, \quad \forall \ s \in \left( \frac{n}{\alpha + \lambda}, \frac{n}{\beta} \right), \quad (56)
\]
and
\[
u^t(|x|) \int_{B_{|x|/2}(x)} dx \leq \int_{B_{|x|/2}(x)} \nu^t(x) dx \leq \|\nu\|^t_t, \quad \forall \ t \in \left( \frac{n}{\beta + \lambda}, \frac{n}{\beta} \right). \quad (57)
\]
This together with \(y \in B_{|x|/2}(x)\) and Theorem 1.1 implies that \(|x|/2 < |y| < 2|x|\) and
\[
\mathcal{D}_3(x) = \int_{B_{|x|/2}(x)} \left( \frac{|x|}{|x-y|} \right)^{\lambda} \frac{v^p(y)u^q(y)}{|y|^\beta} dy \leq C(\lambda, \beta, \alpha) u^q\left( \frac{|x|}{2} \right) v^p\left( \frac{|x|}{2} \right) |x|^{\lambda - \beta} \int_{B_{|x|/2}(x)} |x-y|^{-\lambda} dy \leq C(\lambda, \beta, \alpha, n) \left\{ \left( \frac{|x|}{2} \right)^n \nu^s\left( \frac{|x|}{2} \right) \right\} \left\{ \left( \frac{|x|}{2} \right)^n \nu^t\left( \frac{|x|}{2} \right) \right\} |x|^{n - \beta - \frac{m}{q} - \frac{n}{p}}.
\]
By (56), (57) and (15), take \(1/s = (\alpha + \lambda)/n - \varepsilon\) and \(1/r = (\beta + \lambda)/n - \varepsilon\), we have
\[
\lim_{{|x| \to \infty}} |\mathcal{D}_3(x)| \leq C(\lambda, \alpha, \beta) \lim_{{|x| \to \infty}} \left( |x|^{n - \beta - q(\alpha + \lambda) - p(\beta + \lambda) - (nq + np)\varepsilon} \right) = 0, \quad (58)
\]
provided
\[
0 < \varepsilon < \frac{\beta + (\beta + \lambda)p + q(\alpha + \lambda) - n}{2(np + nq)}.
\]
Therefore, this together with (54), (55) and (58) completes the proof of the second equality of (17).

Now we turn to the first equality in (17). For simplification, we introduce some notations as follows.
\[
\mathcal{G}_1(x) \triangleq \int_{{\mathbb{R}^n \setminus B_R(0)}} \frac{v^p(y)u^q(y)}{|x-y|^{\lambda}|y|^\beta} dy,
\]
and
\[
\mathcal{G}_2(x) \triangleq \int_{{B_R(0)}} \frac{v^p(y)u^q(y)}{|x-y|^{\lambda}|y|^\beta} dy.
\]
Noting that $|x| < R/2$ and $|y| > R$, we have

$$\left| \frac{1}{|x-y|^\lambda} - \frac{1}{|y|^\lambda} \right| \leq C(\lambda) \frac{1}{|y|^\lambda},$$

and

$$\left| \int \mathbb{R}^n \setminus B_R(0) v^p(y)u^q(y) \frac{1}{|y|^\lambda} \left( \frac{1}{|x-y|^\lambda} - \frac{1}{|y|^\lambda} \right) dy \right| \leq C(\lambda) \int \mathbb{R}^n \frac{v^p(y)u^q(y)}{|y|^\lambda + \lambda} dy.$$

This together with (45) and Lebesgue’s dominated convergence implies that

$$\lim_{R \to 0} \lim_{|x| \to 0} G_1(x) = \int \mathbb{R}^n \frac{v^p(y)u^q(y)}{|y|^\lambda + \lambda} dy. \tag{59}$$

Observe that $\alpha + \beta + \alpha + \lambda < n$. Then, there exists $\varepsilon > 0$ small enough such that

$$0 < \varepsilon < \min \left\{ \frac{-\beta p}{8n}, \frac{\alpha q}{8n}, \frac{\lambda s}{8n} \right\}.$$

Take the parameters $t, \tau, s, d$ such that

$$\frac{1}{t} = \frac{\beta p}{n} + \varepsilon, \quad \frac{1}{\tau} = \frac{\alpha q}{n} + \varepsilon,$$

$$\frac{1}{s} = \frac{\beta}{n} + \varepsilon, \quad \frac{1}{d} = \frac{\lambda}{n} + \varepsilon.$$

Then

$$\frac{1}{t} + \frac{1}{\tau} + \frac{1}{s} + \frac{1}{d} \leq 1.$$

By Hölder’s inequality, as $|x| < R$, we have

$$G_2(x) = \int \mathbb{R}^n \frac{v^p(y)u^q(y)}{|x-y|^\lambda|y|^\beta} dy$$

$$\leq \left( \int \mathbb{R}^n v^p(y)dy \right)^{1/t} \left( \int \mathbb{R}^n u^q(y) \right)^{1/\tau}$$

$$\times \left( \int \mathbb{R}^n |y|^{-\beta} dy \right)^{1/s} \left( \int \mathbb{R}^n \frac{|x-y|^{-\lambda} dy}{|y|^\lambda} \right)^{1/d} \tag{60}$$

$$\leq C(\lambda, \beta, n, p, q) R^{n(1-1/t-1/\tau-1/s-1/d)} R^{n-\beta} R^{n-\lambda},$$

which tends to zero as $R \to 0$. This combining with (59) implies that the first equation in (17). Similarly, we can conclude (18) and complete the proof of Theorem 1.3. \qed

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