Classical and quantum mechanics in the Snyder space

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Abstract. The Snyder model is an example of noncommutative spacetime admitting a
fundamental length scale and invariant under Lorentz transformations. Here, we consider its
nonrelativistic counterpart, i.e. the Snyder model restricted to three-dimensional Euclidean
space. We discuss the classical and the quantum mechanics of a free particle in this framework,
and show that they strongly depend on the sign of a coupling constant $\lambda$, appearing in the
fundamental commutators. If $\lambda$ is negative, momenta are bounded, while for $\lambda > 0$ a minimal
localization length arises. We also give the exact solution of the harmonic oscillator equations
both in the classical and the quantum case, and show that its frequency is energy dependent.

1. Introduction

In 1947, in the attempt to introduce a short distance cutoff in field theory, Snyder proposed a
model of noncommutative spacetime, admitting a fundamental length scale and invariant under
the Lorentz group [1]. This model can be considered as a precursor of both noncommutative
geometries [2] and doubly special relativity [3].

The model of spacetime proposed by Snyder is based on the commutation relations

$$\begin{aligned}
[x_\mu, x_\nu] &= i \lambda J_{\mu\nu}, \\
[p_\mu, p_\nu] &= 0, \\
[x_\mu, p_\nu] &= i (\eta_{\mu\nu} + \lambda p_\mu p_\nu),
\end{aligned}$$

\[(1)\]

where $\lambda$ is a coupling constant, usually assumed to be of the scale of the square of the Planck
length, and the $J_{\mu\nu}$ are the generators of the Lorentz algebra.

Besides its conceptual relevance, the most interesting physical implications of the Snyder
model are a generalization of the uncertainty relations [4], implying a lower bound for the
uncertainty in position, and the discreteness of the spectra of area and volume [5].

The investigation of the Snyder model is usually restricted to the case $\lambda > 0$. However, the
physics strongly depends on the sign of the coupling constant. In particular, the case $\lambda < 0$, that
we call anti-Snyder, is the one relevant for doubly special relativity, since it implies an upper
bound on the mass of free particles. It is therefore interesting to compare the two possibilities.
Following the results of [4], we do this in a 3-dimensional Euclidean version of the Snyder model,
i.e. its nonrelativistic limit. The interest of the nonrelativistic model relies in the fact that, while
the main features of the relativistic model are maintained, one can easily implement quantum
mechanics, whereas the definition of a relativistic quantum mechanics would pose nontrivial
conceptual problems.

As we shall see, if $\lambda > 0$, the momenta can take any real value, but in the quantum theory
a minimal uncertainty in the positions arises. If $\lambda < 0$, instead, the value of the momentum
has an upper bound $1/|\lambda|$, but no minimal uncertainty occurs in the quantum theory. However, in contrast with standard quantum mechanics, states with vanishing position uncertainty have finite momentum uncertainty. Moreover, area and volume are quantized for positive $\lambda$, but not for $\lambda < 0$.

We also briefly discuss the dynamics of the harmonic oscillator. Both in classical and quantum mechanics its motion contains corrections of order $\lambda E$ to the standard case. In particular, its frequency of oscillation is no longer independent from the energy.

2. Classical mechanics of the Snyder model

Classically, the nonrelativistic Snyder model can be realized by postulating a noncanonical symplectic structure, with fundamental Poisson brackets ($i = 1, \ldots, 3$)

$$\{x_i, x_j\} = \lambda J_{ij}, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} + \lambda p_ip_j,$$

where $J_{ij} = x_ip_j - x_jp_i$ are the generators of the group of rotations.

In analogy with its relativistic counterpart [1], the model can be derived from a 4-dimensional momentum space, constraining the momenta to live in a 3-dimensional sphere for $\lambda > 0$ (or a pseudosphere for $\lambda < 0$). The construction can be extended to the full phase space by introducing the 4-dimensional coordinates $X_a$ satisfying canonical Poisson brackets with the momenta $P_a$ [6].

Choosing projective coordinates $p_i$ on a 3-sphere of radius $\beta = \sqrt{\lambda}$,

$$p_i = \frac{P_i}{\beta P_4} = \frac{P_i}{\sqrt{1 - \beta^2 P_i^2}},$$

with $P_i^2 < 1/\beta^2$ and defining 3-dimensional position coordinates $x_i = \sqrt{1 - \beta^2 P_i^2} X_i$, that transform covariantly with respect to the $p_i$, one obtains the Poisson brackets of the Snyder model, namely

$$\{x_i, x_j\} = \beta^2 J_{ij}, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} + \beta^2 p_ip_j.$$

The momentum components $p_i$ so defined can take any real values. Of course, the transformation relating the coordinates $X_i$, $P_i$ with $x_i$, $p_i$ is not canonical.

Similarly, the anti-Snyder model, with $\lambda = -\beta^2 < 0$, can be obtained by embedding a pseudosphere in 4-dimensional momentum space with Minkowskian signature. Choosing again projective coordinates

$$p_i = \frac{P_i}{\beta P_4} = \frac{P_i}{\sqrt{1 + \beta^2 P_i^2}},$$

and defining $x_i = \sqrt{1 + \beta^2 P_i^2} X_i$, one obtains the Poisson brackets

$$\{x_i, x_j\} = -\beta^2 J_{ij}, \quad \{p_i, p_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} - \beta^2 p_ip_j.$$

In this case, the momenta are bounded by the relation $P_i^2 < 1/\beta^2$.

The model is invariant under the Poincaré group. While the phase space coordinates transform as ordinary vectors under the action of the generators of rotations $J_{ij} = x_ip_j - x_jp_i =$ $X_iP_j - X_jP_i,$

$$\{J_{ij}, x_k\} = \delta_{ik}x_j - \delta_{ij}x_k, \quad \{J_{ij}, p_k\} = \delta_{ik}p_j - \delta_{ij}p_k,$$

the translation symmetry must be realized nonlinearly as in doubly special relativity. The most natural choice is to identify the translation generators with the momenta $p_i$, leading, for a translation of infinitesimal parameter $a_i$, to the relation

$$\delta x_i = a_j \{x_i, p_j\} = a_i + \lambda a_j p_j p_i, \quad \delta p_i = 0.$$
3. The classical harmonic oscillator
The classical motion in Snyder space can be deduced from the Hamilton equations, using the deformed Poisson structure. A nontrivial example of dynamics is that of a one-dimensional harmonic oscillator, with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m \omega_0^2 x^2}{2}. \quad (9)$$

For unit mass, the Hamilton equations read

$$\dot{x} = (1 + \lambda p^2) p, \quad \dot{p} = -\omega_0^2 (1 + \lambda p^2) x. \quad (10)$$

To solve these equations for $\lambda = \beta^2 > 0$, it is convenient to define a new variable $\bar{p} = \arctan \beta p$. From (10), one obtains

$$\frac{d^2 \bar{p}}{dt^2} = -\omega_0^2 \sin \bar{p} \cos^3 \bar{p}, \quad (11)$$

which admits the first integral

$$\frac{1}{2} \left( \frac{d\bar{p}}{dt} \right)^2 + \frac{\omega_0^2}{2} \tan^2 \bar{p} = \text{const} = \omega_0^2 \beta^2 E, \quad (12)$$

where the integration constant $E$ is the total energy of the oscillator. These equations are identical to those of a classical particle moving in the effective potential $V = \omega_0^2 \tan^2 \bar{p} / 2$.

Integrating the previous equation, one obtains

$$p = \frac{\sqrt{2E} \sin \omega t}{\sqrt{1 + 2\beta^2 E \cos^2 \omega t}}, \quad x = \frac{\sqrt{2E(1 + 2\beta^2 E)} \cos \omega t}{\omega_0 \sqrt{1 + 2\beta^2 E \cos^2 \omega t}}, \quad (13)$$

with $\omega = \sqrt{1 + 2\beta^2 E \omega_0}$. Hence, the motion of the harmonic oscillator differs from the classical one. The solution is still periodic, but its frequency $\omega$ depends on the energy. Also the amplitude acquires corrections of order $\beta^2 E$ and is no longer sinusoidal.

In the anti-Snyder case, $\lambda = -\beta^2 < 0$, the solution can be obtained in an analogous way, defining the variable $\bar{p} = \arctanh \beta p$. Proceeding as before, one obtains an effective potential $V = \omega_0^2 \tanh^2 \bar{p} / 2$, and the solution has the same form as (13) with $\beta^2 \rightarrow -\beta^2$. Moreover, from the condition $p^2 < 1/\beta^2$, it follows that $E < 1/2\beta^2$, which is also the asymptotic value of the potential. The other properties of the solutions are analogous to those found in the Snyder case.

4. Quantum mechanics of the Snyder model
As it is well known, when passing from classical to quantum mechanics, the Poisson brackets go to commutators, and the deformation of the latter implies a modification of the Heisenberg uncertainty relations.

For Snyder space, the commutation relations between the position operators $\hat{x}_i$ and the momentum operators $\hat{p}_i$ read

$$[\hat{x}_i, \hat{x}_j] = i\hbar \beta^2 \tilde{J}_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar (\delta_{ij} + \beta^2 \hat{p}_i \hat{p}_j). \quad (14)$$

In the simple case of one dimension, the uncertainty relations following from (14) are

$$\Delta x \Delta p \geq \frac{1}{2} |\langle \hat{x}, \hat{p} \rangle| = \frac{\hbar}{2} [1 + \beta^2 (\Delta p)^2 + \beta^2 (\Delta \hat{p})^2]. \quad (15)$$
where $\langle \rangle$ denotes expectation values. These generalized uncertainty relations have been thoroughly studied in [7], where it was shown that they imply the existence of a minimal position uncertainty, given by

$$\Delta x_M = \hbar \beta \sqrt{1 + \beta^2 \langle \hat{p} \rangle^2}.$$  

(16)

Its minimum value is obtained when $\langle \hat{p} \rangle = 0$, as $\Delta x_0 = \hbar \beta$, in which case $\Delta p = 1/\beta$.

Exploiting (3), one can define the position and momentum operators $\hat{x}$ and $\hat{p}$ acting on functions defined on a momentum space parametrized by $P$, that satisfy the commutation relations (14), as

$$\hat{p} \psi(P) = P \psi(P), \quad \hat{x} \psi(P) = i\hbar \sqrt{1 - \beta^2 P^2} \frac{\partial \psi(P)}{\partial P}.$$  

(17)

The range of allowed values of $P$ is bounded by $P^2 < 1/\beta^2$, but the spectrum of the momentum operator $\hat{p}$ is unbounded. In order for the operators $\hat{x}$ and $\hat{p}$ to be symmetric the scalar product must be defined as

$$\langle \psi, \phi \rangle = \int_{-1/\beta}^{1/\beta} dP \sqrt{1 - \beta^2 P^2} \psi^*(P) \phi(P),$$  

(18)

and the wave functions must satisfy the boundary conditions $\phi(1/\beta) = \phi(-1/\beta)$.

Of special interest is the study of the spectrum of the position operator. Its eigenfunctions are determined by the equation

$$i\hbar \sqrt{1 - \beta^2 P^2} \frac{\partial \phi_x}{\partial P} = x \phi_x.$$  

(19)

However, the solutions of this equation are not physical states, since they have diverging expectation value of the energy. This is a consequence of the modified uncertainty relations that do not allow the existence of states with a definite value of the position.

As proposed in [7], a more relevant basis for the position operator can be obtained by considering states of maximal localization $\phi_x$, i.e. states having minimal uncertainty $\Delta x_0$ around the position $x$. These states satisfy the equation [4]

$$\left( i\hbar \sqrt{1 - \beta^2 P^2} \frac{\partial}{\partial P} - x + \frac{\hbar \beta^2 P}{\sqrt{1 - \beta^2 P^2}} \right) \phi_x = 0,$$  

(20)

which has solution

$$\phi_x = C \sqrt{1 - \beta^2 P^2} e^{-\frac{\beta}{\hbar} \arcsin \beta P}.$$  

(21)

These states have finite energy, but nonvanishing $\Delta x$.

In the anti-Snyder case, the commutation relations are

$$[\hat{x}_i, \hat{x}_j] = -i\hbar \beta^2 \hat{J}_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar (\delta_{ij} - \beta^2 \hat{p}_i \hat{p}_j),$$  

(22)

and in one dimension, the uncertainty relations read

$$\Delta x \Delta p \geq \frac{1}{2} \frac{1}{\hbar} \left| \langle \Delta \hat{x}, \hat{p} \rangle \right| = \frac{1}{2} \left| 1 - \beta^2 (\Delta p)^2 - \beta^2 \langle \hat{p} \rangle^2 \right|.$$  

(23)

In this case no minimal position uncertainty arises. However, contrary to standard quantum mechanics, the states with vanishing position uncertainty have finite momentum uncertainty, given by $\Delta p = 1/\beta$. 


In analogy with the previous case, the position and momentum operators \( \hat{p} \) and \( \hat{x} \) can be realized on functions defined on a momentum space parametrized by \( P \), as

\[
\hat{p} \psi(P) = p \psi(P) = \frac{P}{\sqrt{1 + \beta^2 P^2}} \psi(P), \quad \hat{x} \psi(P) = i\hbar \sqrt{1 + \beta^2 P^2} \frac{\partial \psi(P)}{\partial P}.
\]  

(24)

From the definition follows that the spectrum of the momentum is bounded, \( p^2 < 1/\beta^2 \). The scalar product for which the operators (24) are symmetric is given by

\[
(\psi, \phi) = \int_{-\infty}^{\infty} \frac{dP}{\sqrt{1 + \beta^2 P^2}} \psi^*(P) \phi(P).
\]  

(25)

The position eigenfunctions \( \psi_x \) are determined by the equation

\[
i\hbar \sqrt{1 + \beta^2 P^2} \frac{\partial \psi_x}{\partial P} = x \psi_x,
\]  

whose solution is

\[
\psi_x = C \exp \left[ -\frac{ix}{\hbar \beta \arcsinh \beta P} \right].
\]  

(27)

The functions (27) are in this case physical eigenstates, with vanishing \( \Delta x \), and finite expectation value of the energy.

5. The quantum harmonic oscillator

Let us consider the one-dimensional harmonic oscillator, with Hamiltonian

\[
H = \frac{\hat{p}^2}{2m} + m\omega_0^2 \hat{x}^2.
\]  

(28)

For unit mass, the Schrödinger equation for the Snyder oscillator is, in the representation (17),

\[
\frac{d^2 \psi}{dP^2} - \frac{\beta^2 P}{1 - \beta^2 P^2} \frac{d \psi}{dP} - \frac{1}{\hbar^2 \omega_0^2} \left[ \frac{P^2}{(1 - \beta^2 P^2)^2} - \frac{2E}{1 - \beta^2 P^2} \right] \psi = 0,
\]  

(29)

with \( P^2 < 1/\beta^2 \). In terms of a variable \( \bar{P} = \arcsin \beta P \), this becomes the standard Schrödinger equation for a potential \( V = \tan^2 \bar{P}/2\omega_0^2 \), that coincides with classical potential.

The explicit solution of eq. (29) can be found defining the variable \( z = (1 + \beta P)/2 \), in terms of which the equation can be written in the standard hypergeometric form

\[
\frac{d^2 \psi}{dz^2} + \frac{z - \frac{1}{2}}{z(z - 1)} \frac{d \psi}{dz} - \left[ \frac{\mu(z - \frac{1}{2})^2}{z^2(z - 1)^2} + \frac{\epsilon}{z(z - 1)} \right] \psi = 0,
\]  

(30)

with \( \mu = 1/\hbar^2 \omega_0^2 \beta^4 \), \( \epsilon = 2E/\hbar^2 \omega_0^2 \beta^2 \). The solution reads

\[
\psi = \text{const} \times (1 - \beta^2 P^2)^{1(1 + \sqrt{1 + 4\mu})/4} F \left( a, b, c; \frac{1 + \beta P}{2} \right),
\]  

(31)

with

\[
a = \frac{1}{2} (1 + \sqrt{1 + 4\mu}) - \sqrt{\mu + \epsilon}, \quad b = \frac{1}{2} (1 + \sqrt{1 + 4\mu}) + \sqrt{\mu + \epsilon}, \quad c = 1 + \frac{1}{2} \sqrt{1 + 4\mu}.
\]
Requiring the vanishing of $\psi$ at $P = \pm 1/\beta$, one obtains the condition $a = -n$ or $b = -n$. In both cases,
\[ E = \hbar\omega_0 \left[ \left( n + \frac{1}{2} \right) \left( \sqrt{1 + \frac{\hbar^2 \omega_0^2 \beta^4}{4} + \frac{\hbar\omega_0 \beta^2}{2}} \right) + \frac{\hbar\omega_0 \beta^2}{2} n^2 \right]. \] (32)

Hence corrections of order $\hbar\omega_0 \beta^2$ occur in the spectrum of the harmonic oscillator, and the relation between the energy spectrum and $\omega_0$ is no longer linear.

The same calculation can be performed for the anti-Snyder oscillator. The Schrödinger equation reads in this case
\[ \frac{d^2\psi}{dP^2} + \frac{\beta^2 P}{1 + \beta^2 P^2} \frac{d\psi}{dP} - \frac{1}{\hbar^2 \omega_0^2} \left[ \frac{P^2}{(1 + \beta^2 P^2)^2} + \frac{2E}{1 + \beta^2 P^2} \right] \psi = 0. \] (33)

Again one may define a new variable $\bar{P} = \operatorname{arcsinh} \beta P$, in terms of which (33) becomes the standard Schrödinger equation for a potential identical to that obtained for the classical motion, $V = \tanh^2 \bar{P}/2\omega_0^2$, and therefore bound states are possible for $0 \leq E < \beta^2/2$. The latter inequality is also a consequence of the bound on the momentum, $p^2 < 1/\beta^2$.

The Schrödinger equation can be solved in the same way as for the Snyder case, and its solution is simply the analytic continuation of (29) for $\beta^2 \to -\beta^2$. In particular, the energy spectrum gives
\[ E = \hbar\omega_0 \left[ \left( n + \frac{1}{2} \right) \left( \sqrt{1 + \frac{\hbar^2 \omega_0^2 \beta^4}{4} - \frac{\hbar\omega_0 \beta^2}{2}} \right) - \frac{\hbar\omega_0 \beta^2}{2} n^2 \right]. \] (34)

Its properties are analogous to those holding in the Snyder case. However, an important difference arises: the energy (34) becomes negative for large $n$. In order to preserve the bound $E \geq 0$, one must impose that $n \leq (1 - \hbar\omega_0 \beta^2/2 + \sqrt{1 + \hbar\omega_0 \beta^2/4})/\hbar\omega_0 \beta^2$, and hence only a finite number of energy levels are present.

6. Further developments
A further difference between Snyder and anti-Snyder models is that the Snyder model gives rise to the quantization of areas, as was first shown in ref. [5]. This follows from the fact that the operators $x_1$, $x_2$ and $J_{12}$, from which the area of a surface in the 1-2 plane can be defined, generate an $so(3)$ algebra with discrete spectrum. In the anti-Snyder case, the corresponding algebra is instead the noncompact $so(2,1)$ and the spectrum is not quantized [6].

It is also possible to extend the Snyder model to a curved background, following the proposal of [8]. This is allowed if the curvature of position and of momentum space have the same sign. The commutation relations in this case
\[ [x_\mu, x_\nu] = i\hbar \beta^2 J_{\mu\nu}, \quad [p_\mu, p_\nu] = i\hbar \alpha^2 J_{\mu\nu}, \]
\[ [x_\mu, p_\nu] = i\hbar [\eta_{\mu\nu} + \alpha^2 x_\mu x_\nu + \beta^2 p_\mu p_\nu + 2\alpha \beta p_\mu x_\nu], \]
with $\alpha^2$ the curvature of spacetime, and the can be realized through a linear but nonsymplectic transformation of the phase space variables of the Snyder model [9].

Also the extension of the present investigations to the relativistic theory is of special interest. This gives rise to interesting conceptual problems, in particular in the quantum mechanical case, where a field theory should be defined. The dependence of the oscillator frequency on the energy may have important implications for the quantum fields.
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