Critical Spectra and Wavefunctions of a One-dimensional Quasiperiodic System

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We numerically study a one dimensional quasiperiodic system obtained from two dimensional electrons on the triangular lattice in a uniform magnetic field aided by the multifractal method. The phase diagram consists of three phases: two metallic phases and one insulating phase separated by critical lines with one bicritical point. Novel transitions between the two metallic phases exist. We examine the spectra and the wavefunctions along the critical lines. Several types of level statistics are obtained. Distributions of the band widths $P_B(w)$ near the origin (in the tail) have a form $P_B(w) \sim w^\beta$ ($P_B(w) \sim e^{-\gamma w}$) ($\beta, \gamma > 0$), while at the bicritical point $P_B(w) \sim w^{-\beta'}$ ($\beta' > 0$). Also distributions of the level spacings follow an inverse power law $P_G(s) \sim s^{-\delta}$ ($\delta > 0$). For the wavefunctions at the centers of spectra, scaling exponents and their distribution in terms of the $\alpha$-$f(\alpha)$-curve are obtained. The results in the vicinity of critical points are consistent with the phase diagram.

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I. INTRODUCTION

A peculiar problem of two-dimensional electrons in a periodic potential with a perpendicular magnetic field has been attracted much attentions since the Hofstadter butterfly. It appears as the spectrum of the underlying one-dimensional system called the Harper model which is deduced from electrons on the square lattice in a uniform magnetic field. It is also essential in physics of the integer quantum Hall effect.

Some of the metal-insulator transitions in one-dimensional quasiperiodic systems have been characterized by multifractal structures of band widths and wavefunctions. See. A one-dimensional tight-binding model is

$$t_{i+1} \psi_{i+1} + t_{i-1} \psi_{i-1} + \epsilon_i \psi_i = E \psi_i, \quad (I.1)$$

where $\psi_i$ denotes the value of the wavefunction at the $i$-th site, $t_i$ and $\epsilon_i$ are the hopping matrix element and the site energy at the $i$-th site respectively, either or both of them can be taken to be quasiperiodic. The Harper model is the case where $t_i = 1$ and $\epsilon_i = \lambda \cos(2\pi \sigma i + \theta)$. When $\sigma$ is an irrational number, it is quasiperiodic. All the eigenstates are extended for $\lambda < 2$ and are localized for $\lambda > 2$ with the metal-insulator transitions at $\lambda = 2\sqrt{2}$. The spectrum has a rich structure (the Hofstadter butterfly). The spectrum as well as the eigenstates becomes multifractal. The total measure of the bands at the critical point $\lambda = 2$ is zero with a fractal dimension less than one. The scaling behaviors of the spectra have been extensively studied by the multifractal analysis. Especially the incommensurate limits of flux per plaquette, such as the inverse of the golden mean $\sigma = \frac{-1 + \sqrt{5}}{2}$ have been extensively studied.

Recently level statistics of some of the quasiperiodic systems have been investigated and turned out to have the behaviors characteristic of criticality. The distributions of the normalized band widths $P_B(w)dw$ have been confirmed that

$$P_B(w) \sim w^\beta \quad (w \to 0), \quad (I.2)$$

and

$$P_B(w) \sim e^{-\gamma w} \quad (w \to \infty), \quad (I.3)$$

These laws have also been confirmed for a variant of the Harper model at criticality. A similar type of statistical law has been confirmed for the Fibonacci model. Remarkably, the form of the distributions of band widths has a similar form as the distributions of the gaps fluctuations observed at the mobility edge of the random systems. Distribution of the energy gaps $P_G(s)$ was also examined. It diverges near the origin and follows an inverse power law.

$$P_G(s) \sim s^{-\delta} \quad (s \to 0), \quad (I.4)$$

For example, $\delta \sim 1.5$ for the critical Harper model($\lambda = 2$).
One of the aims of this paper is to investigate these quantities for the one dimensional quasiperiodic model obtained from two dimensional electrons on the triangular lattice in a uniform magnetic field. This problem was studied by Claro and Wannier, but systematic studies have not been achieved since then. Although the model of two-dimensional electrons on the square lattice with next-nearest hopping, which includes the case of triangular lattice as a special case, were studied previously, statistical techniques such as the multifractal analysis which have been applied for other quasiperiodic systems have not been applied to the triangular lattice model. The same model also appears in the theory of the junction of three wires of Luttinger liquid. These situations motivate us to investigate various aspects of the quasiperiodic system obtained from the triangular lattice model.

The organization of this paper is as follows. In Sec. II, we introduce two dimensional electrons on the triangular lattice in a uniform magnetic field and obtain the one-dimensional quasiperiodic system. We investigate the Aubry and André duality in this model. In Sec. III, we investigate the classical orbits of the model and discuss the phase diagram. In Sec. IV, the distributions of the band widths and the gaps are investigated. In Sec. V, we give a brief review of the general formulation of the multifractal analysis and apply it to the spectra and the wavefunctions. We confirm the phase diagram conjectured in Sec. III Sec. VI is the conclusion.

II. ELECTRONS ON THE TRIANGULAR LATTICE IN A UNIFORM MAGNETIC FIELD

A. Hamiltonian in real space

We consider tight-binding electrons on the triangular lattice in a magnetic field (Fig. 1). We take the lattice spacing to be 1 for simplicity. The Hamiltonian is

\[
H = -t_a \sum_{n,m} c_{n+1,m}^\dagger c_{n,m} \exp(iA_{n+1,m};n,m) - t_a \sum_{n,m} c_{n,m}^\dagger c_{n+1,m} \exp(iA_{n,m};n+1,m) \\
- t_b \sum_{n,m} c_{n,m+1}^\dagger c_{n,m} \exp(iA_{n,m+1};n,m) - t_b \sum_{n,m} c_{n,m}^\dagger c_{n,m+1} \exp(iA_{n,m+1};n,m) \\
- t_c \sum_{n,m} c_{n,m+1}^\dagger c_{n+1,m} \exp(iA_{n+1,m+1};n,m) - t_c \sum_{n,m} c_{n+1,m}^\dagger c_{n,m+1} \exp(iA_{n+1,m+1};n,m+1) \\
\equiv H_a + H_b + H_c
\]  

(II.1)

Here \(t_a, t_b\) and \(t_c\) are the hopping coefficients for each bond, and \(c_{n,m}(c_{n,m}^\dagger)\) is the annihilation (creation) operator at site \((n,m)\) : \(\{c_{n,m}^\dagger, c_{k,l}\} = \delta_{k,n} \delta_{l,m}\). \(A_{n,m;k,l}, k = n \pm 1, l = m \pm 1\) is a gauge field on each bond. We impose \(A_{n,m;k,l} = -A_{k,l;n,m}\) so that \(H\) is hermitian. A uniform magnetic field penetrates each triangle with a flux \(\varphi = \frac{\pi}{2}\).

We take the Landau gauge

\[
A_{n+1,m;n,m} = 0, A_{n,m+1;n,m} = 2\pi \phi \quad \text{and} \quad A_{n+1,m+1;n,m+1} = 2\pi \phi(n + \frac{1}{2}).
\]

(II.2)

thus \(\sum_{\text{triangle}} A_{n,m;k,l} = \frac{\pi}{2}\). A state \(|\Psi\rangle\) is written

\[
|\Psi\rangle = \sum_{n,m} \Psi_{n,m} c_{n,m}^\dagger |0\rangle.
\]

(II.3)

The Schrödinger equation \(H|\Psi\rangle = E|\Psi\rangle\) is

\[
- t_a (\Psi_{n-1,m} + \Psi_{n+1,m}) - t_b (e^{2\pi i \phi n} \Psi_{n,m-1} + e^{-2\pi i \phi n} \Psi_{n,m+1}) \\
- t_c (e^{-2\pi i \phi (n+\frac{1}{2})} \Psi_{n-1,m+1} + e^{2\pi i \phi (n+\frac{1}{2})} \Psi_{n+1,m}) = E \Psi_{n,m}.
\]

(II.4)

We take the form of the wavefunction \(\Psi_{n,m} = e^{ik_y m} \psi_n\), then the Schrödinger equation becomes

\[
- (t_a + t_c e^{-2\pi i \phi (n+\frac{1}{2}) + ik_y}) \psi_{n-1} - (t_a + t_c e^{2\pi i \phi (n+\frac{1}{2}) - ik_y}) \psi_{n+1} - 2t_b \cos(2\pi \phi n + k_y) \psi_n = E \psi_n.
\]

(II.5)

When \(\phi = \frac{p}{q}\) (\(p\) and \(q\) are coprime integers), (II.5) is periodic with period \(q\). The Bloch theorem tells that one can put \(\Psi_n = \exp(ik_x n) \psi_n\) where \(\psi_n\) satisfies \(\psi_n = \psi_{n+q}\), which implies that \(\Psi_{n+q} = e^{ik_x q} \Psi_n\). Thus, if we introduce a row
vector \( \Psi = (\Psi_1, \Psi_2, \cdots, \Psi_{q-1}, \Psi_q)^t \) (\( t \) means the transpose of a matrix) and \( a_n(k_y) = t_a + t_c \exp(2\pi i q(n + \frac{1}{2}) - ik_y) \) and \( b_n(k_y) = 2t_b \cos(2\pi \frac{n}{q} + k_y) \), [II.6] is reduced to an eigenvalue problem of a finite size matrix

\[
H_q(k_x, k_y) = \begin{pmatrix}
  b_1(k_y) & a_1(k_y) & 0 & \cdots & 0 & e^{ik_ya_0(k_y)} \\
  a_1^*(k_y) & b_2(k_y) & a_2(k_y) & 0 & \cdots & 0 \\
  0 & a_2^*(k_y) & b_3(k_y) & a_3(k_y) & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & a_{q-3}(k_y)^* & b_{q-2}(k_y) & a_{q-2}(k_y) \\
  e^{-ik_ya_0(k_y)} & 0 & \cdots & 0 & a_{q-2}(k_y)^* & b_{q-1}(k_y) \\
  & & & & & a_{q-1}(k_y)^* & b_q(k_y)
\end{pmatrix}
\]

(II.6)

Also, in terms of \( \psi_n \), [II.5] becomes

\[
-e^{-ik_y}(t_a + t_c e^{-2\pi i \phi(n + \frac{1}{2}) + ik_y}) \psi_{n-1} - e^{ik_y}(t_a + t_c e^{2\pi i \phi(n + \frac{1}{2}) - ik_y}) \psi_{n+1} - 2t_b \cos(2\pi \phi n + k_y) \psi_n = E \psi_n.
\]

(II.7)

At \( t_c = 0 \), this is reduced to the Harper equation. We define \( \lambda \equiv \frac{2t_a}{t_c} \) and \( \mu \equiv \frac{t_a}{t_c} \) then [II.5] becomes

\[
-e^{-ik_y}(t_a + t_c e^{-2\pi i \phi(n + \frac{1}{2}) + ik_y}) \psi_{n-1} - e^{ik_y}(t_a + t_c e^{2\pi i \phi(n + \frac{1}{2}) - ik_y}) \psi_{n+1} - 2t_b \cos(2\pi \phi n + k_y) \psi_n = E \psi_n.
\]

(II.8)

B. Hamiltonian in momentum space

We denote \( k = (k_x, k_y) \). The electron annihilation operator \( c(k) \) in momentum space is

\[
c_{n,m} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y \exp[ik_x n + ik_y m] c(k).
\]

(II.9)

The commutation relation for \( c(k), c^\dagger(k') \) is

\[
\{c(k), c^\dagger(k')\} = (2\pi)^2 \delta_2(k_x - k'_x) \delta_2(k_y - k'_y),
\]

(II.10)

where \( \delta_2(k) = \sum_{n \in \mathbb{Z}} \delta(k + 2\pi n) \). In terms of \( c(k), c^\dagger(k) \), the tight-binding Hamiltonian [II.1] is

\[
H = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y H(k),
\]

(II.11)
with
\[
H(k) = -2t_c \cos k_x c^\dagger(k) c(k) \\
-(t_b e^{-i k_y} + t_c e^{-i k_x + i k_y - i \pi \phi}) c^\dagger(k_x + 2 \pi \phi, k_y) c(k_x, k_y) \\
-(t_b e^{i k_y} + t_c e^{i k_x - i k_y - i \pi \phi}) c^\dagger(k_x - 2 \pi \phi, k_y) c(k_x, k_y).
\]

(II.12)

When \( \phi = p/q \), since \( k_x \) couples only to \( k_x \pm 2 \pi \phi \), we write \( k_x \) as \( k_x^0 + 2 \pi \phi j \) with \( j \in \mathbb{Z} \). Here \( k_x^0 \) is in the magnetic Brillouin zone
\[
-\frac{\pi}{q} < k_x^0 < \frac{\pi}{q}.
\]

(II.13)

The Hamiltonian \( H(k) \) acts on the Hilbert space spanned by
\[
|\Psi\rangle = \sum_{j} \tilde{\psi}_j c^\dagger(k_x^0 + 2 \pi \phi j, k_y)|0\rangle,
\]

(II.14)

with \( \tilde{\psi}_{j+q} = \tilde{\psi}_j \). The Schrödinger equation \( H|\Psi\rangle = E|\Psi\rangle \) is
\[
-(t_b e^{-i k_y} + t_c e^{-i k_x^0 + i k_y - 2 \pi \phi (j - \frac{1}{2})}) \tilde{\psi}_{j-1} - (t_b e^{i k_y} + t_c e^{i k_x^0 - i k_y + 2 \pi \phi (j + \frac{1}{2})}) \tilde{\psi}_{j+1} - 2t_a \cos(2 \pi \phi j + k_x^0) \tilde{\psi}_j = E \tilde{\psi}_j.
\]

(II.15)

In (II.10) and (II.14), the terms \( H_a \) and \( H_b \) are diagonal respectively. We can also diagonalize \( H_c \) which is proportional to \( t_c \) by changing the gauge. For example, we take the gauge
\[
A_{n+1,m;n,m} = 2 \pi \phi (n + \frac{1}{2}), \quad A_{n,m+1;n,m} = 2 \pi \phi n, \quad \text{and} \quad A_{n,m+1;n+1,m} = 0.
\]

(II.16)

The gauge transformation which transforms from (II.12) to (II.16) is given by
\[
c_{n,m} \rightarrow c_{n,m} \exp(iff_n), \\
c_{n,m}^\dagger \rightarrow c_{n,m}^\dagger \exp(-iff_n), \\
A_{n,m;n',m'} \rightarrow A_{n,m;n',m'} + f_n m - f_n', m', \\
f_n = \phi(n - 1).
\]

(II.17)

In this gauge, the Hamiltonian in momentum space is
\[
H = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} dk_x \int_{-\pi}^{\pi} dk_y H'(k),
\]

(II.18)

with
\[
H'(k) = -2t_c \cos(k_x - k_y) c^\dagger(k) c(k) \\
-(t_b e^{-i k_y} + t_a e^{-i k_x + i \pi \phi}) c^\dagger(k_x + 2 \pi \phi, k_y) c(k_x, k_y) \\
-(t_b e^{i k_y} + t_a e^{-i k_x + i \pi \phi}) c^\dagger(k_x - 2 \pi \phi, k_y) c(k_x, k_y)
\]

(II.19)

From (II.19), we get the Schrödinger equation :
\[
-(t_b e^{i k_y} + t_a e^{-i k_x^0 - 2 \pi \phi (\ell - \frac{1}{2})}) \tilde{\psi}_{\ell-1} - (t_b e^{-i k_y} + t_a e^{i k_x^0 + 2 \pi \phi (\ell + \frac{1}{2})}) \tilde{\psi}_{\ell+1} - 2t_c \cos(2 \pi \phi \ell + k_x^0 - k_y) \tilde{\psi}_\ell = E \tilde{\psi}_\ell.
\]

(II.20)

Apparently, if one exchange \( k_x^0 \) by \( k_x^0 - k_y \) and \( t_a \) by \( t_c \) in (II.15), we get (II.20). This is due to the symmetry of the triangular lattice.

C. Duality

At \( t_c = 0 \), it is known that (II.17) has the duality of Aubry and André who showed the existence of a transition between localized and extended states of \( \psi_j \) when \( \phi \) is an irrational number. When \( \lambda > 2 \), the states are all localized, and when \( \lambda < 2 \), the states are all extended. At \( \lambda = 2 \), all the states are critical.
In the present case, take $\phi = \frac{p}{q}$ and write

$$\psi_n = \sum_{l=0}^{q-1} e^{2\pi i \phi nl} f_l.$$  \hspace{1cm} (II.21)

and substitute it into (II.17), then

$$-(t_b e^{ik_x} + t_c e^{ik_x-ik_y+2\pi \phi(l-\frac{1}{2})}) f_{l-1} - (t_b e^{-ik_x} + t_c e^{-ik_x+ik_y-2\pi \phi(l+\frac{1}{2})}) f_{l+1} - 2t_a \cos(2\pi \phi l + k_x) f_l = Ef_l.$$  \hspace{1cm} (II.22)

When $t_c = 0$, (II.22) becomes (II.15) by substituting $k_x \rightarrow k^0_x$ and $k_y \rightarrow -k_y$ and $\lambda \rightarrow \frac{2}{3}$. This is just the Aubry-André duality when we take the incommensurate limit of $\phi$. However, when $t_c \neq 0$, (II.22) and (II.15) are not transformed by (II.21) due to the term proportional to $t_c$.

Because of the symmetry of the triangular lattice, we can consider duality involving $t_c$ by putting $t_a$ or $t_b$ to be zero. Let us consider small $t_a$ limit. In the limit, (II.20) has the Aubry-André duality for exchanging $t_b$ and $t_c$. This implies that there is a duality between $\lambda$ and $\mu$ for small $t_a$ limit. It relates a state at $(\lambda, \mu)$ to the one at $(2\mu, \frac{2}{3})$ by the transformation (II.21). Thus the phase diagram in $(\lambda, \mu)$ should have a localization transitions on the line $\lambda = \mu$ for small $t_a$ limit i.e. large $\lambda$ and $\mu$.

D. Characteristic Polynomial

When $\phi = p/q$ is rational, (II.15) is reduced to the eigenvalue problem of the matrix (II.6). The eigenvalues are determined by the zeroes of the characteristic polynomial

$$P(E) = \det(E - H_q(k_x, k_y)).$$  \hspace{1cm} (II.23)

which has been studied previously. In Ref. it was shown that $P(E)$ can be written in terms of Chebyshev polynomial of order $q$. In the present case, the characteristic polynomial takes a simple form as follows:

$$P(E) = P_0(E) - Q(k_x, k_y)$$

$$Q(k_x, k_y) = (-1)^q (t_a^q \cos qk_x + t_b^q \cos qk_y) + (-1)^q t_a^q \cos q(k_x - k_y),$$

where $P_0(E)$ is independent of $k_x$ and $k_y$. The energy bands are determined by the zeroes of the polynomial $P(E)$ as we vary $k_x$ and $k_y$. Especially, the edges of energy bands are determined by the minimum and the maximum of the function $Q(k_x, k_y)$. When $k_y = 0$, they are given by $k_x = 0, \pi$.

In Ref. the total band width $W$ of the triangular lattice is estimated when $t_b > t_a > t_c$ as

$$W \sim (t_b - t_a) g \left( \frac{t_b - t_a}{q} \right),$$  \hspace{1cm} (II.26)

where the scaling function $g(x)$ behaves as when its argument is small. Since $t_c$ does not enter the argument, the scaling of the total band width of the triangular lattice is the same as the square lattice. This suggests that the universality class of the scaling property of the spectral measure of the triangular lattice is the same as that of the square lattice.

III. CLASSICAL ORBITS AND PHASE DIAGRAM

A. Classical Orbits

The Hamiltonian (II.1) consists of three terms $H_x$, $H_y$, and $H_z$ which are noncommutative each other. They are diagonalized in different bases as in (II.13), (II.15) and (II.20). The “classical” Hamiltonian is thus

$$H_{\text{classical}} = 2t_a \cos k_x + 2t_b \cos k_y + 2t_c \cos(k_x - k_y).$$  \hspace{1cm} (III.1)

In a magnetic field, $k_y$ is canonically conjugate to $k_x$ and vice versa. Thus to analyze classical orbits, we replace $k_y$ by $x$ and $k_x$ by $y$. Setting $t_a = 1$, we plot the contour of

$$H_{\text{classical}} = \cos y + \frac{\lambda}{2} \cos x + \mu \cos(y - x)$$  \hspace{1cm} (III.2)
FIG. 2: Contour plots of the classical orbits for \((\lambda, \mu) = (1.0, 0.5), (1.0, 1.0)\) and \((1.0, 0.5)\).

FIG. 3: Contour plots of the classical orbits for \((\lambda, \mu) = (2.0, 1.0)\) and \((3.0, 0.5)\).
in Fig. 2 for \((\lambda, \mu) = (1.0, 0.5), (1.0, 1.0)\) and \((1.0, 1.5)\), and in Fig. 3 for \((\lambda, \mu) = (2.0, 1.0)\) and \((3.0, 1.5)\). In Fig. 2 we see that all the contours for \((\lambda, \mu) = (1.0, 0.5)\) and \((1.0, 1.0)\) are extended in the \(x\)-direction and localized in the \(y\)-direction, while, for \((\lambda, \mu) = (1.0, 1.5)\), there is a separatrix which is extended in both directions. We also see in Fig. 3 that the contours for \((\lambda, \mu) = (3.0, 0.5)\) are extended in \(x\)-direction and localized in \(y\)-direction, while there is a separatrix for \((\lambda, \mu) = (2.0, 1.0)\).

**B. Phase Diagram for irrational \(\phi\)**

From behaviors of the classical orbits shown in the previous section, we may deduce the phase diagram of the equation

\[
-\left[1 + \mu e^{-2\pi i \phi (n-\frac{1}{\tau}) + ik_y}\right] \Psi_{n-1} - \left[1 + \mu e^{-2\pi i \phi (n+\frac{1}{\tau} - ik_y)}\right] \Psi_{n+1} - \lambda \cos (2\pi \phi n + k_y) \Psi_n = E \Psi_n. \tag{III.3}
\]

for irrational limit of \(\phi = \frac{p}{q}\). The phase diagram is shown in Fig. 4. One intriguing aspect is the transitions between phase I and phase III which are the transitions between metals. Indeed a transition in the quantum case is not characterized by an appearance of a separatrix at a certain energy. For the Harper model, it is known that metal-insulator transitions occur for whole energies at \(\lambda = 2\). This is generalized to the triangular lattice model we consider.

As an example of an incommensurate limit, in the sections hereafter, we perform numerical scaling analysis for the energy spectra and the critical wavefunctions when \(\phi = \frac{2}{q}\) approaches the inverse of the golden mean \(\frac{1}{\tau} = \frac{\sqrt{5} - 1}{2}\). A standard sequence which corresponds to the continued fraction expansion of \(\frac{1}{\tau}\) is the Fibonacci series \(F_n\), which is defined by \(F_0 = F_1 = 1\), \(F_n = F_{n-1} + F_{n-2}\). \(F_n\) behaves \(\sim \tau^n\) for large \(n\). By taking \(p = F_{n-1}, q = F_n, \phi = \frac{p}{q}\) approaches \(\frac{1}{\tau}\). \(F_n\) is called a Fibonacci number and \(n\) is referred to Fibonacci index.

To take this incommensurate limit of (III.3), the off-diagonal terms in (III.3) need a special care. Namely, when \(\mu = 1\), these terms can be zero if the exponential becomes \(-1\). For the sequence above, this actually happens when \(q = F_n\) with \(n = 3\ell + 1\) for some integer \(\ell\). In that case, the energy spectrum has no dependence on \(k_x\), and the dispersion relation is flat.

**IV. LEVEL STATISTICS**

Consider (III.3) when \(k_y = 0\). On the critical lines, the spectral measure and the wavefunctions are expected to show characteristic behaviors of criticality. See Fig. 4. In order to obtain the distributions of the band widths, the
\[ \lambda = 2.0, \mu = 0.4 \]

q \times q matrices (II.6) are diagonalized. The normalizations are

\[ \int_0^{\infty} P_B(w)dw = 1 \]

\[ \langle w \rangle = \int_0^{\infty} wP_B(w)dw = 1. \] (IV.1)

Similarly the distributions of the gaps \( P_G(s) \) are obtained and normalized by

\[ \int_0^{\infty} P_G(s)ds = 1 \]

\[ \langle s \rangle = \int_0^{\infty} sP_G(s)ds = 1. \] (IV.2)

As we discussed in Sec. III, the edges of the energy bands are found at \( k_x = 0, \pi \) and \( k_y = 0, \pi \). Thus, to study the measure of the spectrum of (III.3), it is sufficient to study those points in the Brillouin zone. When \( \phi = \frac{p}{q} \) is a rational number, the problem is reduced to the eigenvalue problem of the finite size matrix (II.6). Furthermore, when \( q \) is odd and \( k_x = 0, \pi \) and \( k_y = 0, \pi \), the matrix (II.6) reduces to a tridiagonal form by the symmetric and antisymmetric eigenstates.

For \( \mu = 1 \) with \( q = F_n \) and \( n = 3\ell + 1 \), as we noted above, the hopping term becomes zero at a bond and all the band has zero width. Thus we study only the case of \( q = F_n \) with \( n = 3\ell \) when \( \mu = 1 \).

### A. Distributions of Band Widths

Consider the distributions of the band widths along the line \( \lambda = 2 \) with \( \mu = 0.2, 0.4, 0.6 \) and 0.8. In Fig. 5 \( P_B(w) \) at \( (\lambda, \mu) = (2.0, 0.4) \) for \( q = F_n \) with \( n = 25, 27 \) and 28 are plotted. It shows convergence to a limit, indicating the existence of a limit of the distributions of the gaps for the incommensurate flux \( \varphi \). For \( 0 \leq \mu < 1 \) the distributions depend on \( \mu \). The semi-log plots of \( P_B(w) \) is shown in Fig. 6. One sees the linear behaviors for large \( w \), implying an asymptotic form

\[ P_B(w) \sim e^{-\gamma w}, \quad \text{as} \quad w \to \infty, \] (IV.3)

where \( \gamma > 0 \). The optimized values of \( \gamma \) are shown in Table II for several \( \mu \)’s. The inset of Fig. 6 shows \( P_B(w) \) near the origin which indicates that the distributions of the band widths \( P_B(w) \) are zero at the origin with a power law decay. To characterize this behavior, we make an ansatz

\[ P_B(w) \sim w^\beta, \quad \text{as} \quad w \to 0. \] (IV.4)
where $\beta > 0$. The optimized values of $\beta$ are shown in Table I. One sees that $\beta$ becomes smaller as approaching to $\mu = 1$.

Next, we investigate $P_B(w)$ on the other lines $\mu = 1$ and $\lambda = 2\mu$. In Fig. 7, the semi-log and the log-log plots of $P_B(w)$ are shown for $(\lambda, \mu) = (1.0, 1.0)$. We find similar type of behaviors (IV.3) and (IV.4) for the $\lambda = 2$ line. We also investigate the critical line $\lambda = 2\mu$ and find similar type of behaviors. We collect the values of $\beta, \gamma$ in Table II.

These behaviors of $P_B(w)$ on these lines are consistent with the behavior of $P_B(w)$ in other quasiperiodic model\textsuperscript{12,13,14} thus gives a support for the phase diagram of Fig. 4.

### B. Distributions of Gaps

The distribution of the gaps at $(\lambda, \mu) = (2.0, 0.0)$ has been known to follow an inverse power law\textsuperscript{17} which diverges at the origin

$$P_G(s) \sim s^{-\delta}$$  

with $\delta \sim 1.5$. In Fig. 8, the distributions of the gaps for $(\lambda, \mu) = (2.0, 0.4), (1.0, 1.0)$ and $(2.0, 1.0)$ are shown. It is clear that $P_G(s)$ shows a power law of the inverse. The estimated value of $\delta$ is $\sim 1.5$ for these cases. We also investigate other points on the lines $\lambda = 2, \mu = 1$ and $\lambda = 2\mu$ and find a similar behavior with $\delta \sim 1.5$ within statistical error.
This behavior of $P_G(s)$ shows that the spectra are singular continuous on these lines, gives a further support for the phase diagram Fig. 4. Also the value $\delta \sim 1.5$ seems to be a characteristic quantity for this model.

C. Bicritical Point

We also investigate the point $(\lambda, \mu) = (2.0, 1.0)$. The result is shown in Fig. 9. In sharp contrast to other points on the critical lines, it shows the inverse power law

$P_B(w) \sim w^{-\beta'}$, \hspace{1cm} (IV.6)

($\beta' > 0$) for whole the range. We estimate the exponent of the law as $\beta' \sim 1.4$. This implies that the spectrum at this point is a qualitively different fractal-like set. On the other hand, the gap distribution is shown in Fig. 10. It is an inverse power law

$P_G(s) \sim s^{-\delta}$, \hspace{1cm} (IV.7)

($\delta > 0$) with the exponent $\delta \sim 1.5$, analogous to the ones found for other critical points. Thus the band width distribution gives a finer characterization of the spectra than the gap distribution.

V. MULTIFRACTAL ANALYSIS

We apply the method of multifractal analysis to the spectra and the critical wavefunctions. In Ref. 11, the entropy function was introduced which reformulates the theory along the way that standard statistical mechanics is formulated. We use it in our analysis.
A. Review of Multifractal Analysis

We consider quantities $l_i$ and their probability measure $p_i$ of a fractal-like set. Though we shall only consider the cases where $l_i$ or $p_i$ is a constant, a general formulation is reviewed for convenience. It is natural to consider

| $\lambda$ | $\mu$ | $\beta$ | $\gamma$ | $\delta$ |
|-----------|-------|---------|---------|---------|
| 2.0       | 0.0   | 2.5     | 1.4     | 1.5     |
| 2.0       | 0.2   | 2.5     | 1.3     | 1.5     |
| 2.0       | 0.4   | 2.5     | 1.2     | 1.5     |
| 2.0       | 0.6   | 2.3     | 1.2     | 1.5     |
| 2.0       | 0.8   | 2.1     | 0.9     | 1.5     |
| 2.5       | 1.25  | 2.1     | 0.9     | 1.5     |
| 3.0       | 1.5   | 2.3     | 1.1     | 1.5     |
| 4.0       | 2.0   | 2.4     | 1.2     | 1.5     |
| 0.0       | 1.0   | 2.6     | 1.6     | 1.5     |
| 0.5       | 1.0   | 2.6     | 1.5     | 1.5     |
| 1.0       | 1.0   | 2.4     | 1.3     | 1.5     |
| 1.5       | 1.0   | 2.2     | 1.0     | 1.5     |

TABLE I: Estimated exponents on the critical lines. For definitions of $\beta$, $\gamma$, and $\delta$, see Eqs. (IV.3), (IV.4), and (IV.5) respectively.
distributions of logarithm of $l_i$

$$\epsilon_i = -\frac{\ln l_i}{n}, \text{ i.e. } l_i = \exp(-n\epsilon).$$  \tag{V.1}

As $n$ becomes large, $l_i$ approaches zero, but $\epsilon_i$ takes a finite nonzero value for critical points. We introduce a scale index $\alpha_i$ as the exponent of $p_i$ measured by $l_i$ as

$$p_i = l_i^{\alpha_i}, \quad \alpha_i = -\frac{1}{\epsilon_i n} \ln p_i.$$  \tag{V.2}

We write the number of $l_i$ whose scale index lies between $\epsilon$ and $\epsilon + d\epsilon$, $\alpha_i$ and $\alpha_i + d\alpha$ as $\Omega(\epsilon, \alpha)$

$$\Omega(\epsilon, \alpha) = \exp[Q(\epsilon, \alpha)],$$  \tag{V.3}

where $Q(\epsilon, \alpha)$ can be seen as a kind of entropy function.

Following\textsuperscript{10,11}, we consider the generalized partition function

$$\Gamma(q, \beta) = \sum_i p_i^q l_i^\beta$$  \tag{V.4}

$$= \sum_i \exp[-n\epsilon_i (\alpha_i q + \beta)].$$  \tag{V.5}

The generalized free energy is

$$G(q, \beta) = \frac{1}{n} \ln \Gamma(q, \beta),$$  \tag{V.6}

Using $Q(\epsilon, \alpha)$, \textsuperscript{V.6} is written

$$\Gamma(q, \beta) = \int d\epsilon \int d\alpha \exp[nQ(\epsilon, \alpha) - (\alpha q + \beta)\epsilon]].$$  \tag{V.7}

For large $n$, the maximum of the exponent dominates the integral and gives

$$G(q, \beta) = Q(\langle \epsilon \rangle, \langle \alpha \rangle) - (\langle \alpha \rangle q + \beta)\langle \epsilon \rangle,$$  \tag{V.8}

where $\langle \epsilon \rangle$ and $\langle \alpha \rangle$ give the maximum of $Q(\epsilon, \alpha) - (\alpha q + \beta)\epsilon$, so we have

$$\frac{\partial Q(\epsilon, \alpha)}{\partial \epsilon}\bigg|_{\epsilon = \langle \epsilon \rangle, \alpha = \langle \alpha \rangle} = (\alpha) + \beta.$$  \tag{V.9}

and

$$\frac{\partial Q(\epsilon, \alpha)}{\partial \alpha} = \langle \epsilon \rangle q.$$  \tag{V.10}

Thus $G(q, \beta)$ is obtained from $Q(\epsilon, \alpha)$ using \textsuperscript{V.8} and \textsuperscript{V.9}. From \textsuperscript{V.10}, the maximum of $Q(\epsilon, \alpha)$ with respect to $\alpha$ occurs when $q = 0$. On the other hand, once $G(q, \beta)$ is calculated, $\langle \epsilon \rangle, \langle \alpha \rangle$ and $Q(\langle \epsilon \rangle, \langle \alpha \rangle)$ are given by

$$\langle \epsilon \rangle = -\frac{\partial}{\partial \beta}G(q, \beta), \quad \langle \alpha \rangle \langle \epsilon \rangle = -\frac{\partial}{\partial q}G(q, \beta),$$  \tag{V.11}

and

$$Q(\langle \epsilon \rangle, \langle \alpha \rangle) = G(q, \beta) - q \frac{\partial G(q, \beta)}{\partial q} - \beta \frac{\partial G(q, \beta)}{\partial \beta}.$$  \tag{V.12}

Since $\langle \epsilon \rangle$ and $\langle \alpha \rangle$ are functions of $q$ and $\beta$, different regions with scaling indices $\epsilon$ and $\alpha$ are explored by changing the values of the parameters $q$ and $\beta$. Thus $Q(\langle \epsilon \rangle, \langle \alpha \rangle)$ is implicitly a function of $q$ and $\beta$.

The limit of $G(q, \beta)$ for large $n$, may be obtained by

$$G(q, \beta_c(q)) = 0,$$  \tag{V.13}
and $\beta_c(q)$ can be regarded as a set of generalized dimensions. The scaling index $\langle \varepsilon \rangle_c$ which corresponds to $\beta_c(q)$ could be considered as being a representative for a particular value of $q$.

From (V.8), (V.9), and (V.13), we see that $Q(\langle \varepsilon \rangle_c, \langle \alpha \rangle_c)$ at the critical point satisfies the relation

$$Q(\langle \varepsilon \rangle_c, \langle \alpha \rangle_c) = f(\langle \alpha \rangle_c) \langle \varepsilon \rangle_c,$$  

(V.14)

where $f(\langle \alpha \rangle_c)$ is given by

$$f(\langle \alpha \rangle_c) = \frac{\partial Q(\langle \varepsilon \rangle_c, \langle \alpha \rangle_c)}{\partial \varepsilon} \bigg|_{\varepsilon = \langle \varepsilon \rangle_c}.$$  

(V.15)

By substituting (V.15) into (V.9) and (V.10), we obtain

$$f(\langle \alpha \rangle_c) = \langle \alpha \rangle_c q + \beta_c(q),$$  

(V.16)

and

$$\frac{df(\langle \alpha \rangle_c)}{d\alpha} \bigg|_{\alpha = \langle \alpha \rangle_c} = q,$$  

(V.17)

respectively. And (V.16) and (V.17) give

$$\langle \alpha \rangle_c = -\frac{d\beta_c(q)}{dq}.$$  

(V.18)

Thus once $\beta_c(q)$ is known by solving (V.13), $\langle \alpha \rangle_c$ and $f(\langle \alpha \rangle_c)$ are obtained from (V.16) and (V.18). In terms of $f(\alpha)$, the density function of $\varepsilon$ and $\alpha$ $\Omega(\varepsilon, \alpha)$ is written, using (V.3) and (V.14) as

$$\Omega(\langle \varepsilon \rangle_c, \langle \alpha \rangle_c) = \exp(n(\langle \varepsilon \rangle_c f(\langle \alpha \rangle_c)) = \langle l \rangle_c \exp(f(\langle \alpha \rangle_c),$$  

(V.19)

where $\langle l \rangle_c = \exp(-n(\langle \varepsilon \rangle_c)$ is a representative length, and $\langle \varepsilon \rangle_c$ and $\langle \alpha \rangle_c$ are functions of $q$ [see (V.11) and (V.18)]. $f(\alpha)$ can be considered to be a set of generalized dimensions. In numerical approach, we calculate $f(\alpha)$ for a given fractal-like object for a finite $n$ and extrapolate it to the limit $n \to \infty$. From $G(q, \beta)$ (V.6), $G_n(q, \beta)$ at finite $n$ should behave as $G_n(q, \beta) \sim G(q, \beta) + O(\frac{1}{n})$. Thus we should extrapolate $G_n(q, \beta)$ as a function of $\frac{1}{n}$ and estimate the limit for $n \to \infty$.

We denote the support of $f(\alpha)$ by $[\alpha_{\min}, \alpha_{\max}]$ and the value of $\alpha$ which gives the maximum of $f(\alpha)$ by $\alpha_0$.

### B. Spectrum

In this section, scaling properties of energy spectra are analyzed by the multifractal analysis.

#### 1. Multifractal Analysis of Spectrum

We apply the general formulation to characterization of the energy spectra. Take the band widths as variables $l_i$, and let $p_i$ be constants

$$p_i = \frac{1}{F_n} \sim \frac{1}{\tau^n},$$  

(V.20)

where $\tau$ is the golden mean. As seen from (V.6),

$$G(q, \beta) = -q \ln \tau + F(\beta), \quad F(\beta) = G(0, \beta) = \frac{1}{n} \ln \Gamma(0, \beta).$$  

(V.21)

For each $q$, the critical value $\beta$ is determined and vice versa. From (V.11) and (V.21), the scaling index $\langle \varepsilon \rangle$ is written

$$\langle \varepsilon \rangle = -\frac{\partial}{\partial \beta} G(q, \beta) = -\frac{\partial}{\partial \beta} F(\beta).$$  

(V.22)
Thus $\langle \varepsilon \rangle$ depends only on $\beta$. From (V.11) and (V.21), $\langle \alpha \rangle$ is related to $\langle \varepsilon \rangle$ by

$$\langle \alpha \rangle = \ln \tau / \langle \varepsilon \rangle,$$

and $Q(\varepsilon, \alpha)$ is nonzero only for $\alpha$ satisfying (V.23), thus depends only on $\varepsilon$. We may put $Q(\varepsilon, \alpha)$ as $S(\varepsilon)$, and from (V.12) and (V.14) we get

$$S(\varepsilon) = F(\beta) + \varepsilon \beta = \varepsilon f(\alpha).$$

Then $f(\alpha)$ is calculated from the formula

$$f(\alpha) = \frac{S(\varepsilon)}{\varepsilon}.$$  (V.25)

2. Numerical Results

We apply the method above to the spectra of our model. In Fig. 11, we show $\alpha$-$f(\alpha)$ curve for the spectrum at $(\lambda, \mu) = (1.0, 1.0)$. The estimated values of $\alpha_{\text{min}}$, $\alpha_{\text{max}}$ and $\alpha_0$ are 0.421, 0.547 and 0.495 respectively. These values coincide with the corresponding values of the Harper model. We also investigated other points on lines $\lambda = 2, \mu = 1, \lambda = 2 \mu$ in Fig. 4. Except for $(\lambda, \mu) = (2.0, 1.0)$, it turns out that the estimated values of $\alpha_{\text{min}}$, $\alpha_{\text{max}}$ and $\alpha_0$ are the same as those of the Harper model. This implies that the universality class for these lines is the same as the Harper model. This is consistent with the scaling of the total bandwidth (II.26) for $\lambda = 2, \mu < 1$. On the other hand, $\alpha$-$f(\alpha)$ curve for the spectrum at $(\lambda, \mu) = (2.0, 1.0)$ has a different shape as shown in Fig. 12. We see that $\alpha_{\text{min}} = 0.381$, $\alpha_{\text{max}} = 0.755$ and $\alpha_0 = 0.498$. Thus the universality of this point is different from the Harper model.

C. Wavefunctions

We investigate scaling properties of the wavefunctions by the multifractal analysis. We concentrate on the eigenfunctions at the centers of the spectra. This enables us to confirm the phase diagram.

1. Multifractal Analysis of Wavefunctions

We apply the general formulation to characterize the wavefunctions. Take squares modulus of the wavefunctions to be variables $p_i$, while take $l_i$ to be constants

$$l_i = l = \frac{1}{F_n} \sim \frac{1}{\tau^n}, \quad \varepsilon = -\frac{1}{n} \ln l \sim \ln \tau.$$  (V.26)
Thus $\varepsilon$ is a constant in this case. From (V.5) and (V.6), one has
\[
G(q, \beta) = -\beta \varepsilon + G(q, 0).
\] (V.27)

Using (V.11) and (V.12), we obtain the generalized entropy
\[
Q(\varepsilon, \alpha) = G(q, 0) - q \varepsilon \langle \alpha \rangle, \quad \langle \alpha \rangle = -\frac{1}{\varepsilon} \frac{\partial G(q, 0)}{\partial q}.
\] (V.28)

Since $Q(\varepsilon, \alpha)$ is nonzero only for $\langle \alpha \rangle$, we write it as $S'(\alpha)$. From (V.14), we have
\[
f(\alpha) = \frac{S'(\alpha)}{\varepsilon}.
\] (V.29)

We calculate $f(\alpha)$ for finite Fibonacci index $n$ by this formula and extrapolate them to $n \to \infty$. Actually only a part of $f(\alpha)$ is required to distinguish localized, extended and critical states. For a localized state, $f(\alpha)$ has a point support and takes nonzero value only at $\alpha_{\text{min}} = 0$ and $\alpha_{\text{max}} = \infty$ and $f(\alpha_{\text{min}}) = 0$, $f(\alpha_{\text{max}}) = 1$. For an extended state, it has $\alpha_{\text{min}} = \alpha_{\text{max}} = 1$ and $f(\alpha_{\text{min}}) = f(\alpha_{\text{max}}) = 1$. For a critical state, $f(\alpha)$ may have a finite interval $[\alpha_{\text{min}}, \alpha_{\text{max}}]$ as a support and $f(\alpha)$ takes various values. We shall use this method to distinguish states near critical points.

2. Numerical Results

We numerically obtain the wavefunctions at the centers of the spectra for odd $q$ on the $\lambda = 2$, $\mu = 1$ and $\lambda = 2\mu$ lines in the phase diagram Fig.4. For $\mu = 1$, we investigate $q = F_n$ with $n = 3\ell$ as well as $3\ell + 1$. Although the dispersion relations are flat when $n = 3\ell + 1$, we find that the wavefunctions still show a characteristic behavior of a critical state.

In Fig.13 the square moduli of the wavefunctions at the centers of spectra for $(\lambda, \mu) = (1.0, 1.1), (1.0, 1.0)$ and $(1.0, 0.9)$ are displayed for $n = 21$ and $F_n = 17711$. From these figures, we see that the wavefunctions are extended for $(1.0, 1.1)$ and for $(1.0, 0.9)$, and critical for $(1.0, 1.0)$ which is in accord with the phase diagram Fig.4. In Fig.14 the square modulus of the wavefunctions at the band center for $(\lambda, \mu) = (2.0, 1.0), (2.0, 1.1), (1.9, 0.9)$ and $(2.1, 1.0)$ i.e. in the vicinity of the bicritical point of Fig.4 are displayed for $n = 21$ and $F_n = 17711$ ($n = 18$ and $F_n = 4181$ for $(2.1, 1.0)$). It is rather clear that the wavefunction is extended for $(2.0, 1.1)$ and $(1.9, 0.9)$, localized for $(1.0, 0.9)$ and critical for $(2.0, 1.0)$. To draw convincing conclusions, however, it is necessary to study the scaling properties by multifractal analysis.

We plot $\alpha_{\text{min}}$ for $(\lambda, \mu) = (1.0, 1.1), (1.0, 1.0)$ and $(1.0, 0.9)$ in Fig.15. For $(\lambda, \mu) = (1.0, 0.9)$ and $(1.0, 1.1)$, it is clearly seen that $\alpha_{\text{min}}$ extrapolates to 1 for $n \to \infty$. On the other hand, $\alpha_{\text{min}}$ is extrapolated to 0.358 for $(\lambda, \mu) = (1.0, 1.0)$. This value of $\alpha_{\text{min}}$ is actually the same as the one found in the Harper model within statistical error. As shown in Fig.10 $f(\alpha_{\text{min}})$ extrapolates to 1 for $(\lambda, \mu) = (1.0, 1.1)$ and $(1.0, 0.9)$, and 0 for $(1.0, 1.0)$. The behaviors of $\alpha_{\text{min}}$ and $f(\alpha_{\text{min}})$ in Fig.15-16 indicate that the state is extended for $(\lambda, \mu) = (1.0, 1.1)$ and $(1.0, 0.9)$,

![Fig. 12: $\alpha-f(\alpha)$ curves of the spectrum for $(\lambda, \mu) = (2.0, 1.0)$.](image-url)
and critical for \((1,0,0)\). This confirms a part of the phase diagram in Fig. 14, especially the metal-metal transitions at \(\mu = 1.0\).

In Fig. 17 \(\alpha_{\text{min}}\)’s are shown for the states near the bicritical point \((\lambda, \mu) = (2.0, 1.0), (2.0, 1.1), (1.9, 0.9)\) and \((2.1, 1.0)\). Also Fig. 18 shows \(f(\alpha_{\text{min}})\)’s for them. For \((2.0, 1.1)\) and \((1.9, 0.9)\), both \(\alpha_{\text{min}}\) and \(f(\alpha_{\text{min}})\) are extrapolated to 1, telling that the state is extended. For \((2.1, 1.0)\), both \(\alpha_{\text{min}}\) and \(f(\alpha_{\text{min}})\) are extrapolated to 0, which means that the state is localized. At \((2.0, 1.0)\), the convergence of \(\alpha_{\text{min}}\) seems slow but the plots show a tendency to converge to a finite value near 0.47. Similarly \(f(\alpha_{\text{min}})\) converges to zero. Thus we conclude that the states are critical at \((2,0,1,0)\) and it is the bicritical point of metal-insulator and metal-metal transitions. See Fig. 14.

Next Fig. 19 and Fig. 20 show the scaling of \(\alpha_{\text{min}}\) and \(f(\alpha_{\text{min}})\) respectively near \((\lambda, \mu) = (3.0, 1.5)\). The extrapolated values of \(\alpha_{\text{min}}\) and \(f(\alpha_{\text{min}})\) are consistent with the phases diagram Fig. 14. We investigate \(\alpha_{\text{min}}\) and \(f(\alpha_{\text{min}})\) for other points on the critical lines and the results are consistent with the \((\lambda, \mu)\)-phase diagram Fig. 14.

Let us next turn to whole \(\alpha-f(\alpha)\) curve. In Fig. 21 the \(\alpha-f(\alpha)\) curve at \((\lambda, \mu) = (2.0, 0.4)\) is shown. Within statistical error, \(\alpha_{\text{min}}\) is 0.358. This value of \(\alpha_{\text{min}}\) holds on the critical lines except in the vicinity of the bicritical point, where \(\alpha_{\text{min}}\) slightly changes about 0.05. In Fig. 21 the value of \(\alpha_{0}\) which gives the maximum of \(f(\alpha)\) is observed to be 1.31 which is also the same as the Harper model\(^2\). It is the same for the other points on the critical lines \(\lambda = 2.0\) and \(\lambda = 2\mu\) within statistical error, except in the vicinity of the bicritical point where \(\alpha_{0}\) slightly changes within 0.05. We may interpret these slight changes of \(\alpha_{\text{min}}\) and \(\alpha_{0}\) in the vicinity of the bicritical point as an effect of slow convergence there (see below). For \(f(\alpha)\) for \(\alpha > \alpha_{0}\), the convergence of \(f(\alpha)\) is not good, especially for \(\alpha_{\text{max}}\).

For the critical line \(\mu = 1\), \(\alpha_{0}\) splits for \(n = 3\ell\) and \(n = 3\ell+1\). In Fig. 22 plots of \(\alpha_{0}\) and \(\alpha_{\text{min}}\) for \((\lambda, \mu) = (1.0, 1.0)\) are shown. It is seen that \(\alpha_{0}\) goes to different values for \(n = 3\ell\) and \(n = 3\ell+1\). Similar behavior of \(\alpha_{0}\) is observed for other points on the \(\mu = 1.0\) critical line. This implies that the universality class is different for these two series. Also the \(\alpha-f(\alpha)\) curves for \(n = 3\ell\) and \(3\ell+1\) are shown in Fig. 23. The estimated values of \(\alpha_{\text{min}}\) and \(\alpha_{0}\) for \(n = 3\ell\) and \(3\ell+1\) with \(\mu = 1\) are shown in Table I.

For the bicritical point \((\lambda, \mu) = (2.0, 1.0)\), the split of \(3\ell\) and \(3\ell+1\) is not so obvious in the numerical data. For example, Fig. 17 shows \(\alpha_{\text{min}}\) for \(n = 12, 13, \ldots\) and they don’t clearly split into two series. Also the convergence is not as good as those for other points. The numeric for \((\lambda, \mu) = (2.0, 1.0)\) in Table I is obtained based on both series.
VI. CONCLUSIONS

We study two dimensional electrons on the triangular lattice in a uniform magnetic field and the one dimensional quasiperiodic system obtained from it. We conjectured a phase diagram of the one dimensional model as in Fig. 14. As a typical example, we investigated the incommensurate limit of the golden mean via the level statistics, namely the distributions of the band widths and the gaps, and scaling properties of spectra and wavefunctions on the conjectured critical lines. For level statistics, we find the characteristic behaviors similar to the ones previously found for other quasiperiodic models. We also obtain $\alpha - f(\alpha)$ curve for spectra and the wavefunctions at the centers of the spectra. As for the spectra, $\alpha - f(\alpha)$ curve is the same as one in the Harper model on the critical lines except for the bicritical point. For the wavefunctions, we find that $\alpha_{\min}$ is the same as the Harper model except near the bicritical point. For

FIG. 14: Plots of the wavefunctions at the centers of the spectra for (a) $(\lambda, \mu) = (2.0, 1.0)$, (b) $(2.0, 1.1)$, (c) $(1.9, 0.9)$ and (d) $(2.1, 1.0)$. Here $g = 17711$ for (a), (b) and (c), and $g = 4181$ for (d).

FIG. 15: Plots of $\alpha_{\min}$ vs. $1/n$ near $(\lambda, \mu) = (1.0, 1.0)$ with $n = 12, 13, 15, 16, 17, 18, 19, 21$ and 22.
the line $\lambda = 2$, $\alpha$-$f(\alpha)$ curves are the same as the Harper model. For $\mu = 1$, the Fibonacci sequence $F_n$ splits into two classes $n = 3\ell$ and $n = 3\ell + 1$ according to the appearance of zero of the hopping terms. The dispersion relation is flat for $n = 3\ell + 1$. Also the $\alpha$-$f(\alpha)$ curve of the wavefunction is different for $n = 3\ell$ and $n = 3\ell + 1$. At the bicritical point where the triangular lattice symmetry retains, both level statistics and multifractal analysis show qualitively different behaviors from those of other critical points.
FIG. 19: Plots of $\alpha_{\text{min}}$ against $\frac{1}{n}$ near $(\lambda, \mu) = (3.0, 1.5)$.

FIG. 20: Plots of $f(\alpha_{\text{min}})$ against $\frac{1}{n}$ near $(\lambda, \mu) = (3.0, 1.5)$.

FIG. 21: $\alpha$-$f(\alpha)$ curve for $(\lambda, \mu) = (2.0, 0.4)$. 
FIG. 22: The upper two series are plots of $\alpha_0$ and the lower two series are plots of $\alpha_{\text{min}} (\lambda, \mu) = (1.0, 1.0)$.

FIG. 23: $\alpha$-$f(\alpha)$ curve at $(\lambda, \mu) = (1.0, 1.0)$, (a) for index=12,15,18 and 21 and (b) for index=13,16,19 and 22.

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| index | $\lambda$ | $\mu$ | $\alpha_{\text{min}}$ | $\alpha_0$ |
|-------|----------|-------|----------------------|----------|
| $3\ell$ | 0.0      | 1.0   | 0.358                | 1.82     |
|       | 0.5      | 1.0   | 0.358                | 1.57     |
|       | 1.0      | 1.0   | 0.357                | 1.55     |
|       | 1.5      | 1.0   | 0.357                | 1.54     |
| $3\ell + 1$ | 0.0 | 1.0   | 0.358                | 1.65     |
|       | 0.5      | 1.0   | 0.358                | 1.77     |
|       | 1.0      | 1.0   | 0.358                | 1.77     |
|       | 1.5      | 1.0   | 0.358                | 1.73     |
|       | 2.0      | 1.0   | 0.47                 | 1.4      |

TABLE II: $\alpha_{\text{min}}$ and $\alpha_0$ on the critical line $\mu = 1.0$. 

\[ \alpha = 1.0, \mu = 1.0 \]
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