On finite approximations of topological algebraic systems

L.Yu. Glebsky, E.I. Gordon, C. Ward Henson*

Abstract

We introduce and discuss a concept of approximation of a topological algebraic system $A$ by finite algebraic systems from a given class $K$. If $A$ is discrete, this concept agrees with the familiar notion of a local embedding of $A$ in a class $K$ of algebraic systems. One characterization of this concept states that $A$ is locally embedded in $K$ iff it is a subsystem of an ultraproduct of systems from $K$. In this paper we obtain a similar characterization of approximability of a locally compact system $A$ by systems from $K$ using the language of nonstandard analysis.

In the signature of $A$ we introduce positive bounded formulas and their approximations; these are similar to those introduced by Henson [14] for Banach space structures (see also [15, 16]). We prove that a positive bounded formula $\varphi$ holds in $A$ if and only if all precise enough approximations of $\varphi$ hold in all precise enough approximations of $A$.

We also prove that a locally compact field cannot be approximated arbitrarily closely by finite (associative) rings (even if the rings are allowed to be non-commutative). Finite approximations of the field $\mathbb{R}$ can be considered as possible computer systems for real arithmetic. Thus, our results show that there do not exist arbitrarily accurate computer arithmetics for the reals that are associative rings.

1 Introduction

The numerical systems implemented in computers for simulation of the field $\mathbb{R}$ are based on representation of reals in floating-point form. These systems are finite algebras with two binary operations $\oplus$ and $\otimes$. The underlying set of any such system $R$ is a finite, symmetric subset of $\mathbb{R}$ ($a \in R$ iff $-a \in R$ for all $a \in \mathbb{R}$) on which the operations $\oplus$ and $\otimes$ are defined as follows. Let $N$ be the maximum of $R$. If $x, y \in R$ and $x + y$ (resp., $x \times y$) $\in [-N, N]$ then $x \oplus y$ (resp., $x \otimes y$) is the element of $R$ nearest to $x + y$ (resp., $x \times y$). Here $+$ and $\times$ are the addition and the multiplication in $\mathbb{R}$. If $x + y$ (resp., $x \times y$) $\notin [-N, N]$ then $x \oplus y$ (resp., $x \otimes y$) is defined more or less arbitrarily. If such overflow happens during a computation, the numerical result might be incorrect; hence it is necessary to take care that the overflow not occur. (In a floating-point system, this is called exponent overflow; see [17] section 4.2.1.)

The elements of a floating-point system $R$ are distributed unevenly in the interval $[-N, N]$; they become especially sparse when one gets close to the endpoints of this interval. This non-uniformity entails a significant loss of accuracy in calculations with large numbers, even if the results of intermediate operations stay within the interval $[-N, N]$. However, there exists an interval $[-M, M] \subseteq [-N, N]$ and a positive $\varepsilon$ such that $R$ is $\varepsilon$-dense in $[-M, M]$ and for every $x, y \in R$ if $x + y$ (resp., $x \times y$) $\in [-M, M]$, then $x \oplus y$, (resp., $x \otimes y$) approximates $x + y$ (resp., $x \times y$) with

*The research of Gordon and Henson for this paper was partially supported by NSF Grant DMS-9970009; Henson’s research was also partially supported by DMS-0100979 and DMS-0140677
an error that does not exceed \( \varepsilon \). By choosing the parameters of the floating-point system correctly, \( M \) can be made arbitrarily large and \( \varepsilon \) arbitrarily small.

According to the main definition in this paper (Definition 1), this means that floating-point systems provide a family of arbitrarily close finite approximations of the field \( \mathbb{R} \) considered as a topological algebra. The algebraic properties of systems based on the floating point representation are discussed in [17], where it is shown that they are neither associative nor distributive.

More generally, we consider in this paper “continuous” expansions of the field of real numbers; these are universal algebras of the form \( \mathcal{R} = (\mathbb{R}, 1, \times, +, f_1, \ldots, f_m) \) where the operations \( f_j \) are continuous. Several interesting questions about the general nature of approximations of such structures arise naturally.

First, is there a general procedure for constructing approximate versions of theorems about continuous expansions of the reals? A strong version of this question is the following: given a proposition \( \varphi \) about such structures, can one construct propositions \( \varphi_{M,\varepsilon} \) such that \( \varphi \) holds for a given continuous expansion \( \mathcal{R} \) of the reals if and only if for all large enough \( M \) and small enough \( \varepsilon \), the proposition \( \varphi_{M,\varepsilon} \) holds for all finite systems \( \mathcal{R} \) approximating \( \mathcal{R} \) on the interval \([-M,M]\) with accuracy bounded by \( \varepsilon \)? In section 4 we do exactly this in an explicit way for positive first order sentences \( \varphi \) in which each quantifier is restricted to a bounded interval of reals (Corollary 2 to Theorem 1). It seems very difficult to do this in a more general setting.

This kind of question may be important for an understanding of the following type of problem. Suppose we use some convergent numerical method for computation of a real function, or a functional, or an operator. The theorem about convergence of this method is a theorem about the field \( \mathbb{R} \) but in our computer-based “applications” of this theorem we use a finite system \( \mathcal{R} \), which only approximates \( \mathbb{R} \). Can we be sure that the result of our computation is approximately correct if we can use large enough numbers and high enough accuracy? The fact that this problem is natural can be demonstrated by the following example (concerning the approximation of \( \sin x \)), which is discussed in [21, section 3.8].

Although the Taylor series for \( \sin x \) converges for all \( x \), the approximate computation of \( \sin x \) for large \( x \) based on its Taylor expansion gives an incorrect answer in a floating-point system. For large \( x \), the first few terms in a partial sum of this series are also very large. Due to the fixed number of digits in the floating-point representation of real numbers, the addition of terms in a partial sum of the series should be done with the terms taken in ascending order, to avoid roundoff error; this is explained in [21, chapter 2]. However, calculation of the \( k^{th} \) term of the Taylor series for \( \sin x \) produces exponent overflow for large \( x \) and \( k \).

A second natural question concerning finite algebraic systems approximating \( \mathbb{R} \) is the following. What properties of continuous expansions of the reals can hold for some finite systems that approximate them arbitrarily closely? For example, let \( \varphi \) be any first order theorem about the field \( \mathbb{R} \); is it true that for any big enough \( M \) and small enough \( \varepsilon \) there exists a finite system \( \mathcal{R} \) approximating \( \mathbb{R} \) on the interval \([-M,M]\) with accuracy \( \varepsilon \) such that \( \varphi \) itself holds for \( \mathcal{R} \)? We mentioned above that the operations \( \oplus \) and \( \otimes \) in numerical systems based on the floating-point representation are neither associative nor distributive. Is it possible to construct finite rings that approximate \( \mathbb{R} \) arbitrarily closely? (Here we answer this question in the negative; see Theorem 1. It is easy to construct approximating systems for \( \mathbb{R} \) that are abelian groups for \( \oplus \); see Example 2 in section 2.)

These problems are discussed in the present paper in a more general setting. We consider a locally compact algebraic system \( \mathcal{A} = (A, \theta) \) of finite signature \( \theta \) with only function symbols (a universal algebra) and give a definition of approximation of this system by a finite system \( \mathcal{A}_f \) on a
compact set $C \subset A$ with accuracy $W$. Here $W$ is an element of the uniformity on $A$ that defines its topology. We call $A_f$ a $(C,W)$-approximation of $A$. For example, if the topology on $A$ is defined by a metric $\rho$, then we may take $W = \{ (x,y) \in A^2 \mid \rho(x,y) < \varepsilon \}$ for some $\varepsilon > 0$. The universal algebra $A$ is said to be approximable by finite algebras from a class $K$ if for any $C$ and $W$ there exists a $(C,W)$-approximation $A_f \in K$. The definition of approximation of a locally compact group by finite groups discussed in [11] is a particular case of this definition. It is known [11] that all locally compact abelian groups are approximable by finite groups but this is false in general for nonabelian groups [12]. There exist groups that are approximable neither by finite groups, nor by finite semigroups, nor even by finite quasigroups [1] [12] [7]. It is proved in [12] that the field $\mathbb{R}$ is not approximable by finite fields; the signature here includes not only the operations of addition and multiplication but also an operation giving the multiplicative inverse of each nonzero element. Based on these results we show here that locally compact fields are not approximable by finite (associative) rings (Theorem 1). That is, it is impossible to implement in a computer a numerical system for arbitrarily accurate simulation of the field of reals that is a finite (associative) ring.

In [2] (see also [6]) finite approximations of locally compact abelian groups are used for a construction of finite dimensional approximations of pseudodifferential operators. In this approach one simultaneously approximates the operators and the group structures associated to them. This allows constructing approximations which have nice properties (e.g., uniform convergence and spectrum convergence). Usually, algebraic and geometric structures connected with operators can be considered as finite dimensional manifolds (e.g., the symmetry groups of operators are often Lie groups). Thus approximations of these structures can be based on approximations of the field $\mathbb{R}$ together with some other continuous functions on $\mathbb{R}$. Approximations of the other locally compact fields can be used in $p$-adic analysis, adelic analysis, etc. This is another reason for investigation of finite approximations of topological algebraic systems.

Nonstandard analysis provides a natural language in which to discuss approximate versions of statements about the reals; here we return to the first general problem discussed in this Introduction. For background on nonstandard analysis see, e.g., the recent books [8], [10], and [19]. A brief introduction adequate for understanding sections 3 and 4 of this paper is contained in [5, Section 4.4].

It is easy to construct approximate versions of first order statements about continuous expansions $\mathcal{R}$ of the field $\mathbb{R}$ using the language of nonstandard analysis, as we describe next. Let $\varphi$ be a first order sentence in the language of $\mathcal{R}$. Prenex rules and the presence of the arithmetic operations $\times,+$ allow us to put $\varphi$ into an equivalent (in $\mathcal{R}$) normal form

$$Q_1x_1 \ldots Q_m x_m [s = t]$$

where each $Q_j$ is either $\forall$ or $\exists$ and $s, t$ are terms. Now let $R$ be any hyperfinite approximation of $\mathcal{R}$ (see Definition 3) whose underlying set is contained in $\mathbb{R}$; the mapping $j: R \rightarrow \mathbb{R}$ is taken to be the inclusion. It is then clear that $\varphi$ holds in $\mathcal{R}$ if and only if the sentence

$$Q_1^{\text{fin}}x_1 \ldots Q_m^{\text{fin}} x_m [s \approx t]$$

holds in $R$; in a quantifier of the form $Q^{\text{fin}} x$ we take $x$ to range over the finite elements of $R$. (See Proposition 1.)

Standard reformulations of such nonstandard approximations can be obtained using Nelson’s algorithm [22, Section 2] for the translation of nonstandard statements into standard language.
Unfortunately, in the general case these standard versions are extremely complicated. (Without using Nelson’s algorithm, we construct (section 4) comprehensible translations for a large class of first-order sentences, the so-called positive bounded sentences that we introduce here.)

Approximate versions of first-order sentences are discussed in this paper for the general case of a locally compact algebra of finite signature. The results obtained for our positive bounded sentences are similar to well-known results about such sentences in the theory of Banach spaces [13] [13] [10] (see also [15]). The problem of constructing (nonstandard or standard) approximate versions of higher order statements about $\mathbb{R}$ is also open and it seems interesting and important. Solving it might lead to a deeper understanding of the interaction between continuous mathematics and its finite computer approximations.

The authors are grateful to the referee for valuable remarks and important suggestions.

2 Approximation of locally compact algebras

Let $A = \langle A, \theta \rangle$ be an algebraic system of finite signature $\theta$ that contains only function symbols. We assume that $A$ is endowed with a locally compact Hausdorff topology and that the function symbols of $\theta$ are interpreted by continuous functions. (We denote these interpretations using the same letters as the corresponding function symbols in $\theta$.)

Let $C \subseteq A$ be a compact set, $\mathcal{U}$ a finite covering of $C$ by relatively compact open sets (an r.c.o. covering), $A_f = \langle A_f, \theta \rangle$ a finite algebra of signature $\theta$ and $j: A_f \to A$ a mapping. The interpretation of a function symbol $g \in \theta$ in $A_f$ is denoted by $g_f$. For $a_1, \ldots, a_n \in A_f$ we denote by $j(\langle a_1, \ldots, a_n \rangle)$ the $n$-tuple $(j(a_1), \ldots, j(a_n))$. We say that $a, b \in C$ are $\mathcal{U}$-close if $\exists U \in \mathcal{U}$ ($a \in U \land b \in U$).

Definition 1. 1. We say that a set $M \subseteq A$ is a $(C, \mathcal{U})$-grid (equivalently, $M$ is a $\mathcal{U}$-grid for $C$) if for any $c \in C$ there exists an $m \in M$ such that $c$ and $m$ are $\mathcal{U}$-close.

2. We say that $j$ is a $(C, \mathcal{U})$-homomorphism if for any $n$-ary function symbol $g \in \theta$ and for any $\bar{a} \in A_f^n$ such that $j(\bar{a}) \in C^n$ and $g_f(\bar{a}) \in C$, the elements $g(j(\bar{a}))$ and $j(g_f(\bar{a}))$ are $\mathcal{U}$-close.

3. We say that the pair $\langle A_f, j \rangle$ is a $(C, \mathcal{U})$-approximation of $A$ if $j$ is a $(C, \mathcal{U})$-homomorphism and $j(A_f)$ is a $(C, \mathcal{U})$-grid.

4. Let $\mathcal{K}$ be a class of finite algebras of signature $\theta$. We say that the locally compact algebra $A$ is approximable by finite $\mathcal{K}$-algebras if for any compact set $C \subseteq A$ and for any finite r.c.o. covering $\mathcal{U}$ of $C$ there exists a $(C, \mathcal{U})$-approximation $\langle A_f, j \rangle$ of $A$ such that $A_f \in \mathcal{K}$.

Remark 1. If the topology on $A$ is discrete, then condition (4) in Definition 1 is equivalent to the well-known model-theoretic concept of local embedding of an algebraic system $\langle A, \theta \rangle$ in a class $\mathcal{K}$ of algebraic systems of the same signature $\theta$ (see e.g., [20]). The class of discrete groups approximable by finite groups was studied in [25]. It was shown, in particular, that in this case we obtain the same class if we assume that the mapping $j$ is injective. It is not known whether this is true for approximation of topological algebras or even for approximation of discrete algebras other than groups.

Remark 2. Note that if in the item 2 of Definition 1 one has $\text{range}(g) \cap C = \emptyset$ for all $g \in \theta$ or $\text{range}(j) \cap C = \emptyset$, then the mapping $j$ is a $(C, \mathcal{U})$-homomorphism.
Usually we deal with the case of a uniformly locally compact topology on \( A \). This means that the topology on \( A \) is determined by a uniformity \( W \) and there exists \( W \in W \) such that for any \( x \in A \) the set \( W(x) = \{ y \in A \mid \langle x, y \rangle \in W \} \) is relatively compact. For example, all locally compact groups satisfy this condition. For uniformly locally compact algebras of signature \( \theta \), we assume that the interpretations of function symbols are continuous, but not necessary uniformly continuous. For example, \( \mathbb{R} \) is a uniformly locally compact space, but multiplication in \( \mathbb{R} \) is not uniformly continuous. It follows from the general theory of uniform spaces (see, for example, [1]) that the restriction of a continuous function to a compact subset \( C \) is uniformly continuous on \( C \).

For the case of uniformly locally compact algebras Definition 1(4) can be simplified.

We assume now that \( A \) is a uniformly locally compact algebra of signature \( \theta \) and \( W \) is an element of the uniformity \( W \) such that \( \forall x \in A \ [W(x) \text{ is compact}] \). (Here and below the closure of a set \( E \) is denoted by \( \overline{E} \)). Without loss of generality we may assume that \( W \) is symmetric (i.e., \( \langle x, y \rangle \in W \text{ iff } \langle y, x \rangle \in W \) ). The objects \( C, A_f \) and \( j \) satisfy the same assumptions as above. We say that \( a, b \in C \) are \( W \)-close if \( \langle a, b \rangle \in W \).

**Definition 2.**

1. We say that a set \( M \subseteq A \) is a \( (C,W) \)-grid (equivalently, \( M \) is a \( W \)-grid for \( C \)) if for any \( c \in C \) there exists an \( m \in M \) such that \( c \) and \( m \) are \( W \)-close.

2. We say that \( j \) is a \( (C,W) \)-homomorphism if for any \( n \)-ary function symbol \( g \in \theta \) and for any \( \bar{a} \in A^n_f \) such that \( j(\bar{a}) \in C^n \) and \( g(j(\bar{a})) \in C \), the elements \( g(j(\bar{a})) \) and \( j(g(\bar{a})) \) are \( W \)-close.

3. We say that a pair \( \langle A_f, j \rangle \) is a \( (C,W) \)-approximation of \( A \) if \( j \) is a \( (C,W) \)-homomorphism and \( j(A_f) \) is a \( (C,W) \)-grid. If \( A_f \subseteq A \) and \( j: A_f \to A \) is the inclusion map, we say that \( A_f \) is a \( (C,W) \)-approximation of \( A \).

4. Let \( K \) be a class of finite algebras of signature \( \theta \). We say that a uniformly locally compact algebra \( A \) of signature \( \theta \) is approximable by finite \( K \)-algebras if for any compact set \( C \subseteq A \) and for any \( W \in W \) such that \( \forall x \in A \overline{W(x)} \text{ is compact} \), there exists a \( (C,W) \)-approximation \( \langle A_f, j \rangle \) of \( A \) such that \( A_f \in K \).

We omit the simple proofs of the following four propositions.

**Proposition 1.** For every uniformly locally compact algebra \( A \), any compact set \( C \subseteq A \) and any element of the uniformity \( W \in W \) such that \( \forall x \in A \ [\overline{W(x)} \text{ is compact}] \), there exists a finite \( (C,W) \)-approximation of \( A \).

**Proposition 2.** If \( A \) is a compact set, then \( A \) is approximable by finite \( K \)-algebras in the sense of Definition 2 if and only if for any \( W \in W \) there exists a finite \( K \)-algebra that is an \( (A,W) \)-approximation of \( A \).

**Proposition 3.** If \( \langle A_f, j \rangle \) is a \( (C,W) \)-approximation of \( A \) in the sense of Definition 2 and we have a compact set \( C' \subseteq C \) and \( W \supseteq W' \supseteq W \), then the pair \( \langle A_f, j \rangle \) is a \( (C',W') \)-approximation of \( A \).

**Proposition 4.** A uniformly locally compact algebra \( A \) is approximable by finite \( K \)-algebras in the sense of Definition 2 if and only if it is approximable by finite \( K \)-algebras in the sense of Definition 2.

**Remark 3.** Proposition 4 shows that approximability of a uniformly locally compact algebra \( A \) by finite \( K \)-algebras is a topological property; it does not depend on the uniformity on \( A \) but only...
Recall that the floating-point form of a real number is:
\[ \alpha = \pm 10^p \times 0.a_1a_2\ldots, \tag{1} \]
where \( p \in \mathbb{Z} \), and \( a_1a_2\ldots \) is a finite or infinite sequence of digits such that \( 0 \leq a_n \leq 9 \), and \( a_1 \neq 0 \). The integer \( p \) is called the exponent of \( \alpha \), and \( 0.a_1a_2\ldots \), its normalized fraction part or mantissa. Our discussion of floating-point arithmetic mainly follows that of [17], differing only in some inessential technical details.

Fix natural numbers \( P, Q \) and consider the finite set \( A_{PQ} \) of reals of the form (1) such that the exponent \( p \) of \( \alpha \) satisfies \( |p| \leq P \) and the mantissa of \( \alpha \) contains no more than \( Q \) decimal digits. We define binary operations \( \oplus \) and \( \otimes \) on \( A_{PQ} \). In what follows, \( * \) stands for either \( + \) or \( \times \). Let \( \alpha, \beta \in A_{PQ} \) and suppose the normal form of \( \alpha \otimes \beta \) is
\[ \alpha \otimes \beta = \pm 10^r \times 0.c_1c_2\ldots. \tag{2} \]
Note that the mantissa of \( \alpha \otimes \beta \) may contain more than \( Q \) digits. In the following definition, the symbol \( \oplus \) stands for \( \oplus \) or for \( \otimes \), depending on whether \( * \) stands for \( + \) or \( \times \). Then we define
\[ \alpha \oplus \beta = \begin{cases} 
\pm 10^r \times 0.c_1c_2\ldots c_Q & \text{if } |r| \leq P, \\
\pm 10^P \times 0.99\ldots9 & \text{if } r > P, \\
0 & \text{if } r < -P.
\end{cases} \]
If the mantissa of \( \alpha \otimes \beta \) contains fewer than \( Q \) digits we complete it to a \( Q \)-digit mantissa by adding zeros at the right.

Denote by \( A_{PQ} \) the algebra \( \langle A_{PQ}, \oplus, \otimes \rangle \) in which the interpretations of the function symbols \( + \) and \( \times \) are the functions \( \oplus \) and \( \otimes \), respectively. It is easy to see that for any positive \( a \) and \( \varepsilon \) there exist natural numbers \( P \) and \( Q \) such that the algebra \( A_{PQ} \) is an \((a, \varepsilon)\)-approximation of \( \mathbb{R} \). The systems \( A_{PQ} \) are implemented in working computers. What properties of addition and multiplication of the field of reals hold for \( \oplus \) and \( \otimes \)?

It is easy to see that the operations \( \oplus \) and \( \otimes \) are commutative, \( \xi \oplus (-\xi) = 0 \) and \( \xi \otimes 0 = 0 \) for any \( \xi \in A_{PQ} \). Let \( \alpha = \beta = 0.60\ldots06 \) and \( \gamma = 0.60\ldots05 \) (with \( Q \) digits after the decimal point).
Then $\alpha \oplus \beta = \alpha \oplus \gamma$, so the cancellation law fails for $\oplus$. Thus the associative law also fails for $\oplus$. It is easy to construct examples to show that the laws of associativity for $\otimes$ and distributivity for $\oplus, \otimes$ in $A_{PQ}$ also fail. See [17, section 4.2.2] for some other identities of real arithmetic that hold in these floating-point systems.

Example 2 Fix a natural number $M$ and a positive $\varepsilon$. Put $A'_{M,\varepsilon} = \{ k\varepsilon \mid -M \leq k \leq M \}$. Let $N = 2M + 1$. For any $n \in \mathbb{Z}$ we will denote by $n \pmod{N}$ the unique element of the set \{ $k \mid -M \leq k \leq M$ \} that is congruent to $n$ modulo $N$. The operations $\oplus$ and $\otimes$ on $A'_{M,\varepsilon}$ are defined as follows:

\[
\begin{align*}
k\varepsilon \oplus m\varepsilon &= (k + m)(\text{mod } N)\varepsilon, \\
k\varepsilon \otimes m\varepsilon &= |km\varepsilon| \pmod{N}\varepsilon.
\end{align*}
\]

Denote by $A_{M,\varepsilon}'$ the universal algebra in signature $\theta$ with the underlying set $A'_{M,\varepsilon}$ and the interpretation of the function symbols defined by formulas (3) and (4). It is easy to see that $A_{M,\varepsilon}'$ is an $(M\varepsilon,\varepsilon)$-approximation of $\mathbb{R}$. It is obvious that $A_{M,\varepsilon}'$ is an abelian group with respect to $\oplus$ (see (3)). However, one can easily construct examples which show that for any big enough $M$ and small enough $\varepsilon$ the multiplication $\otimes$ satisfies neither the associative law nor the distributive law.

Example 3 Consider approximation of the locally compact field $\mathbb{Q}_p$ of $p$-adic numbers. Recall that any $p$-adic number $\alpha \neq 0$ can be uniquely represented in the form

\[ \alpha = \sum_{\nu=n}^{\infty} a_\nu p^\nu, \]

where $n \in \mathbb{Z}$ and for all $\nu \geq n$ in $\mathbb{Z}$ one has $0 \leq a_\nu < p$; moreover, the representation is normalized by taking $a_n \neq 0$. The $p$-adic norm of $\alpha$ is then given by the formula

\[ |\alpha|_p = p^{-n}, \]

The set $\mathbb{Z}_p = \{ \alpha \mid |\alpha|_p \leq 1 \}$ is a compact subring of $\mathbb{Q}_p$, the ring of $p$-adic integers. For any $m \in \mathbb{Z}$ consider the compact additive subgroup $p^{-m}\mathbb{Z}_p = \{ \alpha \mid |\alpha|_p \leq p^m \}$. The sequence $\{p^{-m}\mathbb{Z}_p \mid m \in \mathbb{Z} \}$ is a monotone sequence of compact sets that covers $\mathbb{Q}_p$. Hence, it is enough to consider only the $(p^{-m}\mathbb{Z}_p, p^{-n})$-approximations of $\mathbb{Q}_p$ for all $m, n \in \mathbb{N}$.

For any $n > 0$, the set $p^n\mathbb{Z}_p$ is an ideal in $\mathbb{Z}_p$ and its quotient ring $K_n$ is equal to $\mathbb{Z}/p^n\mathbb{Z}$. We represent an element of this ring by its positive residue modulo $p^n$, so $K_n = \{ 0, 1, \ldots, p^n - 1 \}$. We have $K_n \subset \mathbb{Z}_p$ as sets. However, the ring operations in these sets are distinct. Indeed, addition $+$ and multiplication $\times$ of natural numbers in $\mathbb{Z}_p$ are the same as in $\mathbb{N}$, while addition $\oplus$ and multiplication $\otimes$ in $K_n$ are equal to addition and multiplication modulo $p^n$. For $\alpha = \sum_{\nu=0}^{\infty} a_\nu p^\nu \in \mathbb{Z}_p$ denote by $\alpha_n$ the number $\sum_{\nu=0}^{n-1} a_\nu p^\nu \in K_n$. Then $|\alpha - \alpha_n|_p \leq p^{-n}$. Hence, $K_n$ is a $p^{-n}$-grid for $\mathbb{Z}_p$.

It is easy to see that

\[ |\alpha + \beta - (\alpha_n \oplus \beta_n)|_p, \ |\alpha \times \beta - (\alpha_n \otimes \beta_n)|_p \leq p^{-n}. \]

Thus, the inclusion map of $K_n$ into $\mathbb{Z}_p$ is a $p^{-n}$-homomorphism. Hence, the ring $K_n$ is a $(\mathbb{Z}_p, p^{-n})$-approximation of $\mathbb{Z}_p$. It follows that the compact ring $\mathbb{Z}_p$ is approximable by finite commutative associative rings. (See Proposition [2])

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To construct a \((p^{-m}\mathbb{Z}_p, p^{-n})\)-approximation \((0 < m < n)\) of \(\mathbb{Q}_p\) consider the set \(H_{m,n} \subset p^{-m}\mathbb{Z}_p\) of all numbers of the form \(\sum_{\nu=-m}^{n-1} a_{\nu} p^{\nu}\). Obviously, \(H_{m,n}\) is a \(p^{-n}\)-grid for \(p^{-m}\mathbb{Z}_p\). We define operations \(\oplus\) and \(\otimes\) such that the inclusion map of \(H_{m,n}\) into \(p^{-m}\mathbb{Z}_p\) is a \((p^{-m}\mathbb{Z}_p, p^{-n})\)-homomorphism.

Note that \(\alpha \in H_{m,n}\) iff \(p^m\alpha \in K_{m+n}\). For any \(\alpha, \beta \in H_{m,n}\) put

\[\alpha \oplus \beta = p^{-m}(p^m\alpha \oplus p^m\beta),\]

where \(\oplus\) is the addition in \(K_{m+n}\).

The definition of \(\otimes\) is more complicated. Let

\[p^m\alpha \times p^m\beta = c_0 + c_1 p + \cdots + c_{m-1}p^{m-1} + c_mp^m + \cdots + c_{2m+2n-2}p^{2m+2n-2}.\]

Put

\[\alpha \otimes \beta = c_mp^{-m} + \cdots + c_{2m+n-1}p^{n-1}.\]

It is easy to see that for all \(\alpha, \beta \in H_{m,n}\)

\[|\alpha + \beta - \alpha \otimes \beta|_p < p^{-n};\]

and if \(\alpha \times \beta \in p^{-m}\mathbb{Z}_p\), then

\[|\alpha \times \beta - \alpha \otimes \beta|_p < p^{-n}.\]

Note that \(\alpha \times \beta \in p^{-m}\mathbb{Z}_p\) iff \(c_k = 0\) for \(k < m\).

This proves that the inclusion map of \(H_{m,n}\) in \(\mathbb{Q}_p\) is a \((p^{-m}\mathbb{Z}_p, p^{-n})\)-homomorphism.

Obviously, \((H_{m,n}, \oplus)\) is an abelian group isomorphic to the additive group of \(K_{m+n}\).

It is easy to see that for any integer \(c\) such that \(0 \leq c < p\) one has \(\frac{1}{p^m} \otimes \frac{1}{p^m} = 0\). Thus

\[\frac{1}{p^m} \otimes \frac{1}{p^m} = 0,\]

while \((\frac{1}{p^m} \otimes \frac{1}{p^m}) \otimes \frac{1}{p^m} = \frac{1}{p^{2m}}\). This shows that the distributive law fails for \(\oplus\) and \(\otimes\).

Since \(0 \otimes p = 0\) and \(\frac{1}{p} \otimes p = 1\), we have \((\frac{1}{p^m} \otimes \frac{1}{p^m}) \otimes p = 0\), while \(\frac{1}{p^m} \otimes (\frac{1}{p} \otimes p) = \frac{1}{p^{2m}}\). This shows that the associative law fails for \(\otimes\).

In all these examples, the finite algebras that approximate the locally compact fields fail to be rings. Indeed, this is inevitable, as the following theorem shows:

**Theorem 1.** No infinite locally compact field can be approximated by finite (associative) rings.

**Proof.** Let \(K\) be a locally compact field, \(K_+\) the additive group of \(K\), and \(K^\times\) the multiplicative group of \(K\). In this proof we denote the multiplication in \(K\) by \(\cdot\). This multiplication is a continuous action of \(K^\times\) on \(K_+\). It is obvious that this action does not preserve the Haar measure on \(K_+\).

Recall that a locally compact group is said to be unimodular if the left and right Haar measures coincide.

It is well known [13] that if a unimodular group \(G\) acts continuously on a unimodular locally compact group \(H\) by automorphisms and this action does not preserve the Haar measure on \(H\), then the semidirect product of \(G\) and \(H\) is non-unimodular.

---

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Thus, the semidirect product $K_+ \ltimes K^\times$ is a non-unimodular group. This semidirect product is isomorphic to the matrix group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a \in K^\times, b \in K \right\}.$$

Let us assume that $K$ is approximable by finite associative rings and prove under this assumption that $G$ is approximable by finite semigroups.

The group $G$ is homeomorphic to $K^\times \times K$ as a topological space. Put

$$W_\varepsilon = \{ \langle (a,b),(a',b') \rangle \mid a,a' \in K^\times, b,b' \in K, \max\{|a-a'|_K,|b-b'|_K\} < \varepsilon \},$$

where $|\cdot|_K$ is the norm in $K$.

We have to show that for any compact sets $A \subset K^\times$ and $B \subset K$ there exists a $(A \times B,W_\varepsilon)$-approximation $\langle S,j \rangle$ such that $S$ is a semigroup.

Let $D = A \cup B \cup (A \cdot B)$. Then $D$ is a compact subset of $K$ (here $A \cdot B = \{ a \cdot b \mid a \in A, b \in B \}$). For any positive $\delta$ denote by $U_\delta$ the set $\{ (a,b) \in K \mid |a-b|_K < \delta \}$ and put $C = \overline{U_{\varepsilon/2}(D)}$, where $U_{\varepsilon/2}(D) = \bigcup_{d \in D} U_{\varepsilon/2}(d)$. Since $D$ is a compact set and any open ball in $K$ is relatively compact, we have that $C$ is a compact set also.

According to our assumption, there exists a finite associative ring $\langle F,\oplus,\odot \rangle$ and a map $j\colon F \to K$ such that the pair $\langle F,j \rangle$ is a $(C,U_{\varepsilon/2})$-approximation of the field $K$. Our group $G$ is equal to $K^\times \times K$ as a set. The multiplication in $G$ is given by the formula

$$\langle a,b \rangle \cdot \langle c,d \rangle = \langle a \cdot c, a \cdot d + b \rangle.$$

Consider the finite set $S = F^* \times F$ and the multiplication in $S$ given by the formula

$$\langle s,p \rangle \odot \langle q,r \rangle = \langle s \odot q, s \odot r \oplus p \rangle.$$ 

Since $F$ is an associative ring, it is easy to see that $S$ is a semigroup.

Define the map $i\colon S \to G$ by the formula

$$i(\langle a,b \rangle) = \langle j(a),j(b) \rangle.$$ 

Since $j(F)$ is an $U_{\varepsilon/2}$-grid for $C$ and thus, for $A$ and $B$, it is obvious that $i(S)$ is a $W_{\varepsilon/2}$-grid for $A \times B$.

Let $i(\langle s,p \rangle),i(\langle q,r \rangle),i(\langle s,p \rangle \odot \langle q,r \rangle) \in A \times B$. By the definition of $i$, we have $j(s),j(q),j(s \odot q) \in A$. Hence,

$$|j(s) \cdot j(q) - j(s \odot q)|_K < \frac{\varepsilon}{2}.$$ 

Since $j(p),j(r) \in B$, we have $j(s) \cdot j(r) \in A \cdot B \subset D \subset C$. Thus, $|j(s) \cdot j(r) - j(s \odot r)|_K < \frac{\varepsilon}{2}$ and $j(s \odot r) \in C$. Hence,

$$|j(s) \cdot j(r) + j(p) - (j(s \odot r) + j(p))|_K < \frac{\varepsilon}{2},$$

and

$$|j(s \odot r) + j(p) - j(s \odot r \oplus p)|_K < \frac{\varepsilon}{2}. $$
Thus,
\[ |j(s) \cdot j(r) + j(p) - j((s \circ r) \oplus p)|_K < \varepsilon. \]

This shows that \( i \) is an \((A \times B, W_\varepsilon)\)-homomorphism.

Thus, the non-unimodular group \( G \) is approximable by finite semigroups.

By Theorem 4 of [7], if a locally compact group is approximable by finite semigroups, then it is approximable by finite groups.

By Corollary 1 of Theorem 1 of [7], if a locally compact group \( G \) is approximable by finite groups (indeed, even if only by finite quasigroups), then \( G \) is unimodular. This contradiction completes the proof. \( \Box \)

3 Nonstandard characterization of approximability

We first recall some well-known notions and results from nonstandard analysis. (See the books [8] [10] [19] or the brief introduction in [5] Section 4.4] for necessary background.) In this section, \( #(M) \) denotes the cardinality of the set \( M \).

Let \( U \) be a nonstandard universe and \( \kappa \) an infinite cardinal. Recall that \( U \) is \( \kappa^+ \)-saturated if for any family \( F \) of internal sets in \( U \) such that \( #(F) \leq \kappa \) and \( F \) satisfies the finite intersection property one has \( \bigcap F \neq \emptyset \).

Let \( \mathcal{A} = \langle A, \theta \rangle \) be as in the previous section, a uniformly locally compact algebra of finite signature \( \theta \). We again assume that \( \theta \) contains only function symbols and that they are interpreted by continuous functions, which we denote by the same letters as the respective function symbols.

Let \( \lambda \) be the least infinite cardinal greater than the weight of the topology on \( A \) and the weight of the uniformity on \( A \). (The weight of a topology on \( A \) is the minimal cardinality of a base of this topology and the weight of a uniformity on \( A \) is the minimal cardinality of a base of this uniformity).

**Proposition 5.** There exists a family \( C_\lambda \) of compact subsets of \( A \) with the following properties:

- \( #(C_\lambda) \leq \lambda \);
- \( C_\lambda \) is closed under finite unions;
- for any \( C \in C_\lambda \) the interior \( C^o \) of \( C \) is nonempty and \( \bigcup \{C^o \mid C \in C_\lambda\} = A \).

**Proof** Let \( W_\lambda \) be a base of the uniformity on \( A \) of cardinality less or equal to \( \lambda \). Without loss of generality we assume that \( W_\lambda \) consists of elements \( W \) such that for all \( x \in A \) the set \( W(x) \) is open and relatively compact. Let \( D \) be a dense subset of \( A \) such that \( #(D) \leq \lambda \) (to obtain such \( D \) pick an element from each set in a base of the topology on \( A \) of least cardinality). Take \( C_\lambda \) to be the family of all finite unions of the sets \( W(d), \) where \( W \in W_\lambda \) and \( d \in D \). Then \( C_\lambda \) satisfies the conditions of the proposition. \( \Box \)

Throughout this section we deal with an arbitrary but fixed \( \lambda^+ \)-saturated nonstandard universe \( \mathbb{U} \), with a fixed family \( C_\lambda \) satisfying Proposition 5 and with a base \( W_\lambda \) of the uniformity \( W \) such that \( #(W_\lambda) \leq \lambda \).

The nonstandard extension \( \ast \mathcal{A} \) of \( \mathcal{A} \) is the algebraic system \( \langle \ast \mathcal{A}, \theta \rangle \), where any function symbol \( f \in \theta \) is interpreted in \( \ast \mathcal{A} \) by the nonstandard extension \( \ast f \) of the operation \( f \) in \( \mathcal{A} \). In what follows
we omit the symbol * in notations of *A-operations; i.e., we denote the interpretations of a function symbol \( f \in \theta \) in \( A \) and in *\( A \) by the same letter \( f \).

For \( \alpha, \beta \in *A \) we write \( \alpha \approx \beta \) if \( \forall W \in W \langle \alpha, \beta \rangle \in *W \). In this case we say that \( \alpha \) and \( \beta \) are infinitesimally close. Obviously, \( \alpha \approx \beta \) iff \( \langle \alpha, \beta \rangle \in *W \) holds for all \( W \in W_\lambda \). An element \( \alpha \) of *\( A \) is called nearstandard if \( \alpha \approx a \) holds for some \( a \in A \). Since \( W \) is a Hausdorff uniformity, if such an \( a \in A \) exists, then it is unique. In this case \( a \) is called the standard part or the shadow of \( \alpha \) and it is denoted by \( \circ \alpha \).

An element \( \alpha \) of *\( A \) is called nearstandard if \( \alpha \approx a \) holds for some \( a \in A \). Since \( W \) is a Hausdorff uniformity, if such an \( a \in A \) exists, then it is unique. In this case \( a \) is called the standard part or the shadow of \( \alpha \) and it is denoted by \( \circ \alpha \).

Denote the family of all compact subsets of \( A \) by \( C \).

For \( B \subseteq *A \), we denote by ns(\( B \)) the set of all nearstandard elements of \( B \). It is well-known that a set \( C \subseteq A \) is compact iff \( \text{ns}(^*C) = C \). Thus, since \( A \) is a locally compact space, we have

\[
\text{ns}(^*A) = \bigcup_{C \in C} ^*C = \bigcup_{C \in C_\lambda} ^*C
\]

Let \( f \in \theta \) and \( f: A^n \to A \) for some standard natural number \( n \). Since \( f \) is a continuous function, it is well-known that for any \( \bar{\alpha}, \bar{\beta} \in (\text{ns}(^*A))^n \) one has

\[
\bar{\alpha} \approx \bar{\beta} \implies f(\bar{\alpha}) \approx f(\bar{\beta})
\]

Note that implication (10) holds for arbitrary \( \bar{\alpha}, \bar{\beta} \in (^*A)^n \) iff the function \( f \) is uniformly continuous.

The statements (9) and (10) and some of their obvious modifications can be found in any of the books concerning nonstandard analysis that were mentioned above.

Statement (10) implies the following:

**Proposition 6.** 1. The external set \( \text{ns}(^*A) \) is closed under \( \theta \)-operations; i.e., \( \langle \text{ns}(^*A), \theta \rangle \) is a subalgebra of *\( A \). We denote this subalgebra by \( \text{ns}(^*A) \).

2. The mapping \( \text{st}: \text{ns}(^*A) \to A \) defined by the formula \( \text{st}(\alpha) = \circ \alpha \) is a surjective homomorphism of algebras such that

\[
\text{st}(\alpha) = \text{st}(\beta) \iff \alpha \approx \beta.
\]

Thus the equivalence relation \( \approx \) restricted to \( \text{ns}(^*A) \) is a congruence relation on \( \text{ns}(^*A) \) and the algebra \( \text{ns}(^*A)/\approx \) is isomorphic to \( A \).

Let \( A_h \) be a hyperfinite algebra of signature \( \theta \); i.e \( A_h = \langle A_h, \theta \rangle \), where \( A_h \) is a hyperfinite set and every function symbol \( f \in \theta \) is interpreted by an internal function, which is denoted by \( f_h \).

**Definition 3.** Let \( A_h \) be a hyperfinite algebra of signature \( \theta \) and let \( j: A_h \to *A \) be an internal mapping satisfying the following conditions:

1. \( \forall a \in A \exists b \in A_h j(b) \approx a \);
2. if \( f \in \theta \) is an \( n \)-ary function symbol in \( \theta \), \( \bar{b} \in A^n_h \) and \( j(\bar{b}) \in (\text{ns}(^*A))^n \), then \( j(f_h(\bar{b})) \approx f(j(\bar{b})) \).

Then we say that the pair \( \langle A_h, j \rangle \) is a hyperfinite approximation of the algebra \( A \).

Assume that \( \langle A_h, j \rangle \) is a hyperfinite approximation of \( A \).

Put \( (A_h)_b = j^{-1}(\text{ns}(^*A)) \). Elements of the set \((A_h)_b\) are said to be feasible.

For \( b_1, b_2 \in A_h \) put \( b_1 \sim b_2 \) if \( j(b_1) \approx j(b_2) \).

We call \( \sim \) the indiscernibility relation. If \( b_1 \sim b_2 \) we say that the elements \( b_1 \) and \( b_2 \) are indiscernible.

Definition 2 and Proposition 3 imply immediately the following:

**Proposition 7.** 1. The external set \((A_h)_b\) is closed under \( \theta \)-operations; i.e., \( \langle (A_h)_b, \theta \rangle \) is a subalgebra of \( A_h \). We denote this subalgebra by \( (A_h)_b \).

2. The mapping \( st \circ j = \iota : (A_h)_b \to A \) is a surjective homomorphism of algebras such that for any \( b_1, b_2 \in (A_h)_b \) one has
   \[ \iota(b_1) = \iota(b_2) \iff b_1 \sim b_2. \]

Thus the indiscernibility relation \( \sim \) restricted to \((A_h)_b\) is a congruence relation on \((A_h)_b\) and the algebra \((A_h)_b/\sim\) is isomorphic to \( A \).

Put \( M = C_\lambda \times W_\lambda \) and consider the partial order \( \leq \) on \( M \) such that if \( m_i = \langle C_i, W_i \rangle \in M \), \( i = 1, 2 \), then
   \[ m_1 \leq m_2 \iff C_1 \supseteq C_2 \land W_1 \subseteq W_2. \]

For \( m \in M \) denote the set \( \{m' \in M \mid m' \leq m\} \) by \( M_m \).

Let \( \langle C, W \rangle \in \text{ns}(^*M) \). By the transfer principle, the internal set \( C \) is \text{ns}(^*M)\)-compact. We recall the meaning of this notion. Let \( T \) be the topology on \( A \). For any statement \( P \), the \( \text{ns}(^*M) \)-version of \( P \) is obtained by restricting all quantifiers to internal sets. Any standard set \( S \) involved in \( P \) should be replaced by its nonstandard extension \( ^*S \). Thus, an internal set \( C \subseteq A \) is \( \text{ns}(^*M) \)-compact if for any internal family \( U \subseteq \text{ns}(^*T) \) such that \( C \subseteq \bigcup U \), there exists a hyperfinite subfamily \( V \subseteq U \) such that \( C \subseteq \bigcup V \).

Again let \( A_h \) be a hyperfinite algebra of signature \( \theta \) and let \( j : A_h \to ^*A \) be an internal mapping. If \( \langle C, W \rangle \in ^*M \), then we say that \( \langle A_h, j \rangle \) is a \( (C, W) \)-approximation of \( ^*A \) if \( \langle A_h, j \rangle \) and \( ^*A \) satisfy Definition 2 and Proposition 3.

We say that a pair \( \langle C, W \rangle \in \text{ns}(^*M) \) is infinitesimal if for any \( \langle D, V \rangle \in M \) one has \( \langle C, W \rangle \leq \langle ^*D, ^*V \rangle \). Obviously, if \( \langle C, W \rangle \in \text{ns}(^*M) \) is infinitesimal then \( C \supset ns(^*A) \).

The following lemma is an immediate consequence of our assumption that the nonstandard universe \( U \) is \( \lambda^+ \)-saturated.

**Lemma 1.** Let \( N \) be an internal subset of \( ^*M \). The following statements hold.

1. If for every \( \langle D, V \rangle \in M \) one has \( ^*D, ^*V \in N \), then there exists an infinitesimal element \( \langle C, W \rangle \in N \);  

2. if \( N \) contains all infinitesimal elements of \( ^*M \), then there exists an \( m = \langle D, V \rangle \in M \) such that \( ^*M_m \subseteq N \).
Lemma 2. A pair $\langle A_h, j \rangle$ is a hyperfinite approximation of the algebra $A$ iff $\langle A_h, j \rangle$ is a $(C, W)$-approximation of $*A$ for some infinitesimal $\langle C, W \rangle \in *M$.

Proof ($\Leftarrow$) Let $a \in A$. Since $j(A_h)$ is a $(C, W)$-grid, there exists $b \in A_h$ such that $\langle a, j(b) \rangle \in W$. Thus, for any $V \in W$ one has $\langle a, j(b) \rangle \in *V$; i.e., $j(b) \approx a$. Let $f \in \theta$ be an $n$-ary function symbol and $f_h$ its interpretation in $A_h$; take $\bar{a} \in (A_h)_b$; i.e., $j(\bar{a}) \in \text{ns}(*A)^n \subset C^n$. Since $f$ is a continuous function, we have $f(j(\bar{a})) \in \text{ns}(*A)$. Hence $f(j(\bar{a})) \in C$. By Definition 2 and the transfer principle, $\langle j(f_h(\bar{a})), f(j(\bar{a})) \rangle \in W$. Thus, $j(f_h(\bar{a})) \approx f(j(\bar{a}))$. So $\langle A_h, j \rangle$ is a hyperfinite approximation of $A$.

($\Rightarrow$) Let $\langle A_h, j \rangle$ be a hyperfinite approximation of $A$. Obviously, for any $\langle D, V \rangle \in M$ the pair $\langle A_h, j \rangle$ is a $(\langle D, V \rangle)-approximation of $*A$. By Lemma 1, there exists an infinitesimal $\langle C, W \rangle$ such that $\langle A_h, j \rangle$ is a $(C, W)$-approximation of $*A$.

Theorem 2. A uniformly locally compact universal algebra $A$ of finite signature $\theta$ is approximable by finite algebras from a class $\mathcal{K}$ iff there exist a hyperfinite algebra $A_h = \langle A_h, \theta \rangle \in *\mathcal{K}$ and an internal mapping $j: A_h \to *A$ such that the pair $\langle A_h, j \rangle$ is a hyperfinite approximation of $A$.

Proof ($\Rightarrow$) Let $A$ be approximable by finite $\mathcal{K}$-algebras and let $\langle C_0, W_0 \rangle$ be an infinitesimal element of $*M$. By the transfer principle, there exists a hyperfinite algebra $A_h \in *\mathcal{K}$ and an internal mapping $j: A_h \to A$ such that the pair $\langle A_h, j \rangle$ is $(C_0, W_0)$-approximation of $*A$. By Lemma 2, $\langle A_h, j \rangle$ is a hyperfinite approximation of $A$.

($\Leftarrow$) Let $\langle A_h, j \rangle \in *\mathcal{K}$ be a hyperfinite approximation of $A$. By Lemma 2, $\langle A_h, j \rangle \in *\mathcal{K}$ is a $(C_0, W_0)$-approximation for some infinitesimal $\langle C_0, W_0 \rangle \in *M$. Then by Proposition 3 and the transfer principle, the pair $\langle A_h, j \rangle$ is a $(\langle C_0, W_0 \rangle, *\mathcal{K})$-approximation of $*A$ for all $\langle C, W \rangle \in M$. By the transfer principle (used in the opposite direction), for every $\langle C, W \rangle \in M$ there exists a finite $(C, W)$-approximation of $A$ that belongs to $\mathcal{K}$.

Corollary 1. For every uniformly locally compact algebra $A$, there exists a hyperfinite approximation of $A$.

Proof Take $\mathcal{K}$ to be the class of all finite algebras of signature $\theta$ and apply Theorem 2 and Proposition 4.

Remark 4. It follows from Definition 4 and Proposition 4 that Theorem 2 holds if we take our nonstandard universe only to be $\nu^+$-saturated, where $\nu$ is the weight of the topology on $A$.

The topology on $(A_h)_b/\sim$ induced by its isomorphism to $A$ can be defined in terms of the triple $\langle A_h, (A_h)_b, \sim \rangle$. We will now do this in a more general setting.

Recall that an external subset of a $\lambda^+$-saturated universe is called a $\sigma$-set (respectively, a $\pi$-set) if it can be represented by a union (respectively, an intersection) of a family of internal sets of cardinality $\leq \lambda$. Obviously $(A_h)_b$ is a $\sigma$-subset of $A$, while $\sim$ is a $\pi$-set contained in $A^2$.

The above considerations provide motivation for the following:

Definition 4. We say that a triple $\tau = \langle T, T_b, \rho \rangle$ is an abstract nonstandard topological triple if $T$ is an internal set, $T_b \subseteq T$ is a $\sigma$-subset and $\rho$ is a $\pi$-equivalence relation on $T$ such that for every $\alpha \in T_b$ the set $\rho(\alpha) = \{ \beta \in T \mid \langle \alpha, \beta \rangle \in \rho \}$ is contained in $T_b$. We call $T_b$ the set of abstractly feasible elements and $\rho$ the abstract indiscernibility relation. If $T$ is hyperfinite, we call $\tau$ a hyperfinite topological triple.
We now introduce a topology on the quotient set \( \hat{T} = T_b/\rho \). For \( \alpha \in T \) denote by \( \alpha^\rho \) the \( \rho \)-equivalence class of \( \alpha \).

Let \( F \subseteq T \). Put \( i(F) = \{ \alpha \in F \mid \alpha^\rho \subseteq F \} \). Denote by \( \mathcal{I} \) the family of all internal subsets of \( T_b \). Let \( \mathcal{T}_\tau \) be the topology on \( \hat{T} \) obtained by taking the family \( \{ i(F) \mid \alpha \in i(F), F \in \mathcal{I} \} \) to be a base of neighborhoods of the point \( \alpha^\rho \), for each \( \alpha \in T_b \). Here \( i(F)^\rho = \{ \beta^\rho \mid \beta^\rho \subseteq F \} \).

The construction of the topological space \((T_b, \mathcal{T}_\tau)\) is a generalization of the well-known construction of the nonstandard hull. This generalization was introduced in [9] for the case of hyperfinite abelian groups (see also [11]).

**Theorem 3.**

1. The weight of the topology \( \mathcal{T}_\tau \) is \( \leq \lambda \).

2. The topological space \((\hat{T}, \mathcal{T}_\tau)\) is locally compact iff for every internal set \( F \subseteq T_b \) and for every internal set \( G \) such that \( \rho \subseteq G \subseteq T \times T \) there exists a set \( K \subseteq F \) of standard finite cardinality that satisfies the following condition:

\[
F \subseteq \bigcap_{\alpha \in K} G(\alpha).
\]

3. If \( \varphi : T^n \to T \) is an internal \( n \)-ary operation on \( T \) for some standard \( n \), and we assume that the set \( T_b \) of feasible elements of \( T \) is closed under \( \varphi \) and \( \varphi \upharpoonright T_b \) is stable under the indiscernibility relation \( \rho \); i.e.,

\[
\forall \bar{a}, \bar{a}' \in T^n_b \quad (\bar{a} \rho \bar{a}') \implies \varphi(\bar{a}) \rho \varphi(\bar{a}'),
\]

then the induced \( n \)-ary operation \( \varphi^\# \) on \( \hat{T} \) (i.e., \( \varphi^\# \) is such that for every \( \bar{a} \in T^n_b \) one has \( \varphi^\#(\bar{a}^\rho) = \varphi(\bar{a})^\rho \)) is continuous in the topology \( \mathcal{T}_\tau \).

4. Let \( \langle A_h, j \rangle \) be a hyperfinite approximation of a uniformly locally compact algebra \( \mathcal{A} \), let \( \langle A_h \rangle_b \) and \( \sim \) be as defined in Proposition [4] let \( \tau = \langle A_h, (A_h)_b, \sim \rangle \), and \( \hat{A}_h = (A_h)_b/\sim \). Then the isomorphism of algebras \( \hat{A}_h \) and \( \mathcal{A} \) induced by the homomorphism \( \iota : (A_h)_b \to \mathcal{A} \) of Proposition 7 is an isomorphism of topological algebras with respect to the topology \( \mathcal{T}_\tau \) on \( \hat{A}_h \).

A proof of this theorem for the case of locally compact abelian groups is contained in [9] and in [11]. It can be transferred without any changes to the general case.

Let \( A_h = \langle A_h, \theta \rangle \) be an internal algebra, let \( \langle A_h \rangle_b = \langle (A_h)_b, \theta \rangle \) and let \( \rho \) be a \( \pi \)-equivalence relation on \( A_h \). We say that the triple \( \tau = \langle A_h, (A_h)_b, \rho \rangle \) is a nonstandard topological \( \theta \)-triple, if \( \rho \upharpoonright (A_h)_b \) is a congruence relation on \( (A_h)_b \) (i.e., (11) holds for all operations \( \varphi \) from \( \theta \)) and \( \langle A_h, (A_h)_b, \rho \rangle \) is an abstract nonstandard topological triple.

Theorem [4](2) shows that if \( \tau = \langle A_h, (A_h)_b, \rho \rangle \) is a nonstandard topological \( \theta \)-triple, then the quotient algebra \( \hat{A}_h = (A_h)_b/\rho \) is a topological algebra with respect to the topology \( \mathcal{T}_\tau \).

We say that a topological algebra \( \mathcal{A} = \langle A, \theta \rangle \) is abstractly approximable by finite algebras from a class \( \mathcal{K} \), if there exists a hyperfinite topological \( \theta \)-triple \( \tau = \langle A_h, (A_h)_b, \rho \rangle \) such that \( A_h \in \mathcal{K} \) and \( \mathcal{A} \) is topologically isomorphic to \( \hat{A}_h \).

Theorem 3 together with Proposition 7 show that if \( \mathcal{A} \) is approximable by finite \( \mathcal{K} \)-algebras, then it is abstractly approximable by finite \( \mathcal{K} \)-algebras.

The following question is open. Is it true that any locally compact algebra \( \mathcal{A} \) that is abstractly approximable by finite \( \mathcal{K} \)-algebras is approximable by finite \( \mathcal{K} \)-algebras in the sense of Definition 11?
It is easy to see that Theorems 1 and 4 of [7] stay true if we replace approximability (of groups by finite quasigroups and finite semigroups) by abstract approximability (of groups by finite quasigroups and finite semigroups).

This implies the following proposition, which strengthens Theorem 1.

**Proposition 8.** No infinite locally compact field is abstractly approximable by finite (associative) rings.

An interesting discussion about the relations between real analysis and discrete analysis is contained in [27]. The main idea of that paper is expressed as follows: “Continuous analysis and geometry are just degenerate approximations to the discrete world . . . . While discrete analysis is conceptually simpler . . . than continuous analysis, technically it is usually much more difficult. Granted, real geometry and analysis were necessary simplifications to enable humans to make progress in science and mathematics . . . ”.

The discussion in this section shows how the idea that continuous mathematics is an approximation of the discrete could be formalized. We may assume that we deal only with finite sets, but some of these sets are so big that they contain some only vaguely defined subclasses, which do not satisfy all the properties of sets. For example, the induction principle fails for these subclasses. For example, recall the well-known paradox of the pile of sand, due to Eubulides, IV century B.C.: one grain of sand is not a pile, and if \( n \) grains of sand do not form a pile, then \( n + 1 \) grains also do not form a pile; so, how can we get a pile of sand? According to our approach, hyperfinite sets of infinite cardinality simulate such large sets and external subsets simulate their vaguely defined subclasses. This follows from the obviously true statement: “A hyperfinite set has a standard cardinality iff all its subsets are internal”. Under this approach the set of all grains of sand is hyperfinite and a pile of sand is an external subset of this set\(^1\).

According to Proposition 7 and Corollary 1, for every locally compact algebra \( \mathcal{A} \) there exists a hyperfinite algebra \( \mathcal{A}_h \), an external subalgebra \( (\mathcal{A}_h)_b \) and an equivalence relation \( \sim \) such that \( \mathcal{A} \) is isomorphic to \( (\mathcal{A}_h)_b / \sim \). So \( (\mathcal{A}_h)_b \) can be viewed as a subclass of feasible elements and \( \sim \) as an indiscernibility relation.

Proposition 8 together with the results on non-approximability of Lie groups from [1] explain, in some sense, why continuous analysis is simpler than discrete analysis. The discrete algebraic structures that are used in science need not have algebraic properties as good as those possessed by the corresponding continuous structures.

We complete this section with a formulation of the concept of approximability in terms of ultraproducts.

If an algebra \( \mathcal{A} \) is discrete (see Remark 1), then Definition 2 of the concept of approximation of \( \mathcal{A} \) by finite \( \mathcal{K} \)-algebras can be reformulated in the following way.

**Proposition 9.** A discrete algebra \( \mathcal{A} \) of a finite signature \( \theta \) is approximable by finite \( \mathcal{K} \)-algebras iff for any finite subset \( C \subset A \) there exists a finite algebra \( \mathcal{A}_C \in \mathcal{K} \) and a map \( j: \mathcal{A}_C \to A \) such that

1. \( C \subset j(\mathcal{A}_C) \).

\(^1\)P.Vopenka [26] suggested an axiomatic set theory for the formalization of this idea. The main defect of his approach is its opposition to classical mathematics. Another axiomatization of hyperfinite sets was suggested in [3], where classical mathematics was interpreted in the framework of hyperfinite sets.
2. For any $n$-ary function symbol $f \in \theta$ and for any $\bar{a} \in A^n$, such that $j(\bar{a}) \in C$ and $f(j(\bar{a})) \in C$, one has

$$ j(f_C(\bar{a})) = f(j(\bar{a})), $$

where $f_C$ is the interpretation of $f$ in $A_C$.

In $[25]$ are presented some examples of locally compact groups $G$ such that $G$ is approximable by finite groups as a discrete group but $G$ is not approximable by finite groups as a topological group.

The following proposition is contained in $[20]$. (A proof can also be found in $[1]$.)

**Proposition 10.** A discrete algebra $A$ is approximable by finite $\mathcal{K}$-algebras iff $A$ is isomorphic to a subalgebra of an ultrapower of finite $\mathcal{K}$-algebras.

Theorem 2 together with Proposition 7 can be considered as a generalization of Proposition 10 to the setting of approximation of topological algebras.

Indeed, if our nonstandard universe is a $\lambda^+$-saturated ultrapower of a standard universe, then any hyperfinite algebra $A_h \in \mathcal{K}$ is isomorphic to an ultraproduct of finite $\mathcal{K}$-algebras. Internal subsets of $A_h$ correspond to subsets of this ultraproduct that are ultraproducts themselves. Unions (respectively, intersections) of at most $\lambda$ many internal subsets are called $\sigma$-sets (respectively, $\pi$-sets). Combining Theorem 2 and Proposition 7 with these remarks we obtain the following:

**Proposition 11.** If a uniformly locally compact algebra $A$ of signature $\theta$ is approximable by finite algebras from a class $\mathcal{K}$, then $A$ is isomorphic to a quotient algebra of a $\sigma$-subalgebra $B_\sigma$ of some $\lambda^+$-saturated ultrapower $B$ of finite $\mathcal{K}$-algebras with respect to some $\pi$-equivalence relation $\rho$ on $B$, such that $\rho \mid B_\sigma$ is a congruence relation. (Here $\lambda$ is the weight of the topology on $A$).

The necessity in Proposition 10 is a special case of Proposition 11. Indeed, it is easy to see that if the topology on $A$ is discrete, then the equivalence relation $\rho$ is the relation of equality. (See Proposition 7)

4 Positive bounded formulas and finite approximations

In this section we consider first order statements true of a locally compact algebra $A$, and we investigate approximate versions of those statements that hold in finite approximations of $A$. We start with approximations of statements that are formulated in the language of nonstandard analysis.

Let $L_\theta$ be the set of all first order formulas in the signature $\theta$ and let $\varphi(x_1, \ldots, x_n) \in L_\theta$. Denote by $\varphi_\sim$ the formula obtained from $\varphi$ by the replacement of each atomic subformula $t_1 = t_2$ by the formula $t_1 \sim t_2$; here $t_1$ and $t_2$ are terms in the signature $\theta$.

Let $(A_h, j)$ be a hyperfinite approximation of $A$. Then the formula $\varphi_\sim$ has an obvious interpretation in the algebra $(A_h)_b$ of feasible elements of $A_h$ (cf. Proposition 7). Every term $t(x_1, \ldots, x_n)$ of signature $\theta$ is interpreted by a function $t_b$ on $(A_h)_b^n$, obtained by substitution of the function $f_b$ for any function symbol $f$ involved in $t$ (we denote the restriction of $f_h$ to $(A_h)_b$ by $f_b$). Then for any $a_1, \ldots, a_n \in (A_h)_b$ one has $(A_h)_b \models t_1(a_1, \ldots, a_n) \sim t_2(a_1, \ldots, a_n)$ iff the elements $(t_1)_b(a_1, \ldots, a_n)$ and $(t_2)_b(a_1, \ldots, a_n)$ are indiscernible. The following proposition is an immediate corollary of Proposition 7.
Proposition 12. If \((A_h,j)\) is a hyperfinite approximation of \(A\) then for any formula \(\varphi(x_1,\ldots,x_n) \in L_\theta\) and any \(a_1,\ldots,a_n \in (A_h)_b\) one has
\[
(A_h)_b \models \varphi(a_1,\ldots,a_n) \iff A \models \varphi(\circ j(a_1),\ldots,\circ j(a_n)).
\]

Remark 5. The same proposition is true also for any abstract hyperfinite approximation of \(A\) if we replace \(A\) by \(\widehat{A}_h\) and \(\circ j(a_i)\) by the canonical image of \(a_i\) in \(\widehat{A}_h\).

From the point of view of computer numerical systems discussed in the Introduction, Proposition 2 has the following interpretation. In the setting of nonstandard analysis, we can consider an idealized computer that has a hyperfinite memory. Then the numerical system \(\mathbb{R}_h\) for simulating the field of reals that is implemented in this computer is a hyperfinite algebra in the signature \(\sigma = \langle +, \times \rangle\) and \(\mathbb{R}_h\) is a hyperfinite approximation of \(\mathbb{R}\). So Proposition 12 provides a lot of information about \(\mathbb{R}_h\).

Suppose \(N = \max\{\{a| \alpha \in \mathbb{R}_h\}\). Then the elements of \((\mathbb{R}_h)_b\) can be considered as elements that are far enough from the end points of the interval \([-N,N]\) so that exponent overflow never occurs in computations involving them. It is very natural that the property of being “far enough from the end points of the interval \([-N,N]\)” is an external property: if a natural number \(n\) is “far enough from the end points” then obviously the same is true for \(n+1\). Thus the induction principle fails for this property. Proposition 12 shows that the first order properties of \(\mathbb{R}\) hold approximately for the computer implementation of \(\mathbb{R}\), as long as we only consider elements that are far enough from the end points of the interval \([-N,N]\). This fact seems to be very clear for those who use computers for numerical computation. The language of nonstandard analysis makes it possible to formulate a rigorous mathematical theorem that expresses this phenomenon.

Example 4 Consider the algebra \(A_{PQ}\) discussed in Example 1 of section 2. It is easy to see that if \(P,Q \in \mathbb{N}\setminus\mathbb{N}\), then \(A_{PQ}\) is a hyperfinite approximation of \(\mathbb{R}\) (here \(j\) is the inclusion map). Consider a formula \(\varphi(x,y)\) of the signature \(\sigma = \langle +, \times \rangle\). Let \(\mathbb{R} \models \forall x \exists y \varphi(x,y)\). Put \(\psi(x) = \exists y \varphi(x,y)\), \(\eta(x) = \exists y_1, y_2 (y_1 \neq y_2 \land \varphi(x,y_1) \land \varphi(x,y_2))\), where \(\exists y \varphi(x,y)\) means that there exists a unique \(y\) such that \(\varphi(x,y)\). Assume that for every rational number \(\alpha\) one has
\[
\mathbb{R} \models \psi(\alpha) \quad (12)
\]
Thus, \(\mathbb{R} \models \psi(\alpha)\) holds for every \(\alpha \in A_{PQ}\). Let us assume also that there exists an irrational \(\alpha\) such that
\[
\mathbb{R} \models \eta(\alpha) \quad (13)
\]
Consider the following question. Given an arbitrary \(\alpha \in \mathbb{R}\), how can we determine whether \(\alpha\) satisfies (12) or (13) using only our computer? The qualitative answer to this question is the following. If \(\alpha\) satisfies (12), then for all precise enough approximations \(a_1\) and \(a_2\) of \(\alpha\), any \(b_1\) and \(b_2\) such that \(\varphi(a_1,b_1)\) and \(\varphi(a_2,b_2)\) are true with a high accuracy must be very close to each other. If \(\alpha\) satisfies (13), then there exist two arbitrarily precise approximations \(a_1\) and \(a_2\) of \(\alpha\) and two significantly distinct \(b_1\) and \(b_2\) such that \(\varphi(a_1,b_1)\) and \(f(a_2,b_2)\) are approximately true.

A rigorous mathematical statement that reflects this qualitative answer follows from Proposition 12. Indeed, it is easy to see that
\[
(A_{PQ})_b \models \forall x_1,x_2 (\psi_<(x_1) \land \psi_<(x_2) \land x_1 \sim x_2 \rightarrow \forall y_1,y_2 (\varphi_<(x_1,y_1) \land \varphi_<(x_2,y_2) \rightarrow y_1 \sim y_2)) \quad (14)
\]
and
\[(A_{PQ})_b \models \forall x(\eta_\sim(x) \to \exists x_1, x_2, y_1, y_2(\varphi_\sim(x_1, y_1) \land \varphi_\sim(x_2, y_2) \land x_1 \sim x \land x_2 \sim x \land \neg(y_1, \sim y_2))) \quad (15)\]

Let us illustrate this discussion by a very simple numerical example. Consider the following system
\[
\begin{align*}
x + ay &= b \\
ax + by &= 2
\end{align*}
\] (16)

This system has
1. a unique solution, if \(a^2 \neq b\),
2. no solutions if \(a^2 = b \neq \sqrt[4]{4}\),
3. infinitely many solutions if \(a^2 = b = \sqrt[4]{4}\).

In the last case the general solution is given by the formula
\[x + \sqrt[4]{2} \cdot y = \sqrt[4]{4} \quad (17)\]

Performing numerical calculations on a computer, we deal only with rational numbers. Thus, the third case cannot occur in computer calculations.

Taking the 5-digit approximations to \(\sqrt[4]{2}\) and \(\sqrt[4]{4}\) for \(a\) and \(b\) and solving the system (16) on a computer we obtain the solution \(x' = 0.74552, y' = 0.6682\), which satisfies (17) with accuracy \(10^{-5}\).

Taking the 10-digit approximations to \(\sqrt[4]{2}\) and \(\sqrt[4]{4}\) for \(a\) and \(b\) and solving the system (16) we obtain the solution \(x' = -0.9979450387, y' = 2.051990552\), which satisfies (17) with accuracy \(10^{-10}\). We see that these two approximate solutions of the system (16) are significantly distinct (compare with (15)).

In the language of nonstandard analysis it is only possible to formulate mathematical theorems that give us some qualitative picture of the connection between continuous problems and their computer simulations. To obtain specific estimates it is necessary (but not sufficient) to formulate a standard version of Proposition 12.

In the language of classical mathematics, we can only consider approximate properties of reals that hold eventually when the memory of computers increases to infinity and the accuracy becomes more and more precise. We will see that only a restricted result can be obtained in this way.

We say that a formula \(\varphi \in L_\theta\) is positive if it can be built up from atomic formulas using only conjunctions, disjunctions and quantifiers. The main result of this section concerns positive formulas in prenex form
\[Q_1y_1 \ldots Q_my_m \psi(x_1, \ldots, x_n, y_1, \ldots, y_m), \quad (18)\]
where the \(Q_i\) are quantifiers and \(\psi\) is a disjunction of conjunctions of atomic formulas.

An arbitrary (not necessary positive) formula \(\varphi\) is equivalent to a formula in the form (18), where \(\psi\) is a disjunction of conjunctions of atomic subformulas of \(\psi\) and negations of atomic subformulas of \(\psi\). Let \(\gamma_1, \ldots, \gamma_k\) be the list of all atomic formulas and their negations involved in \(\psi\). For any \(1 \leq i \leq k\) fix \(W_i \in W\) and denote by \(\gamma_i[W_i]\) the formula \((t_1, t_2) \in W_i\) if \(\gamma_i = t_1 = t_2\) and the formula \((t_1, t_2) \notin W_i\) if \(\gamma_i = \neg(t_1 = t_2)\). Here \(t_1\) and \(t_2\) are terms in the signature \(\theta\).

Define the interpretations of the formula \((t_1, t_2) \in W\) in \(A\) and in an arbitrary \((C', W')\)-approximation \((A_f, j_f)\) of \(A\), where \((C', W') \in M\), as follows:
Let \( \tau_1, \tau_2 \in A \) be interpretations of the terms \( t_1 \) and \( t_2 \) in \( A \). Then \( A \models \langle t_1, t_2 \rangle \in W \) iff \( \langle \tau_1, \tau_2 \rangle \in W \).

If \( \xi_1 \) and \( \xi_2 \) are interpretations of the terms \( t_1 \) and \( t_2 \) in \( A_f \), then \( A_f \models \langle t_1, t_2 \rangle \in W \) iff \( \langle j_f(\xi_1), j_f(\xi_2) \rangle \in W \).

Denote by \( \varphi[W_1, \ldots, W_k] \) the formula that is obtained from \( \varphi \) by replacement of each \( \gamma_i \) by \( \gamma_i[W_i] \) respectively. The formula \( \varphi[W_1, \ldots, W_k] \) is called an approximation of \( \varphi \). Obviously, if \( \varphi \) is positive, then for any \( W_1, \ldots, W_k \in W \) one has \( \varphi \Rightarrow \varphi[W_1, \ldots, W_k] \) (for both interpretations). This is not true in general if \( \varphi \) is non-positive. Similarly, if \( W_i' \subseteq W_i \), \( i = 1, \ldots, k \), then \( \varphi[W_i', \ldots, W_k'] \Rightarrow \varphi[W_1, \ldots, W_k] \) for a positive \( \varphi \) but not generally if \( \varphi \) is non-positive. For positive formulas we say in this case that \( \varphi[W_1, \ldots, W_k] \) is a finer approximation than \( \varphi[W_1, \ldots, W_k] \). Obviously, for any approximation \( \varphi[W_1, \ldots, W_k] \) of a positive formula \( \varphi \) there exists a finer approximation \( \varphi[W_1', \ldots, W_k'] \) such that \( W_1' = \cdots = W_k' = W \) (it is enough to put \( W = W_1 \cap \cdots \cap W_k \)). In this case we write \( \varphi[W] \) instead of \( \varphi[W_1, \ldots, W_k] \). In what follows we deal only with approximations of the form \( \varphi[W] \) of a positive formula \( \varphi \).

If \( B \subseteq A \) and \( Q \) is either \( \forall \) or \( \exists \) then \( Q_B x \ldots \) is interpreted in \( A \) by \( \forall x(x \in B \Rightarrow \ldots) \) or \( \exists x(x \in B \land \ldots) \) and in a finite \((C, W)\)-approximation \((A_f, j_f)\) of \( A \) by \( \forall x(x \in j_f^{-1}(B) \Rightarrow \ldots) \) or \( \exists x(x \in j_f^{-1}(B) \land \ldots) \).

Quantifiers of the form \( Q_B \) are called bounded quantifiers. If all quantifiers in a formula \( \varphi \) are bounded then we say that \( \varphi \) is bounded.

Let \( c = (C_1, \ldots, C_m) \) be an \( m \)-tuple of subsets of \( A \) and let \( \varphi \) be a positive prenex formula as in (18). Then \( \varphi[c] \) is the formula

\[
Q_{1C_1} y_1 \cdots Q_{mC_m} y_m \psi. \tag{19}
\]

A formula of the form (19) is said to be a positive bounded formula.

In what follows we consider only positive bounded formulas \( \varphi[c] \) that satisfy the following condition:

for any \( i \leq m \) such that \( Q_i = \forall \) (respectively, \( Q_i = \exists \)) the set \( C_i \) is a relatively compact open (respectively, compact) set.

In this case we say that an \( m \)-tuple \( c \) of subsets of \( A \) is \( \varphi \)-regular.

**Example 5** Consider the signature \( \sigma' \) obtained from the signature \( \sigma = (\ominus, \otimes) \) by adding a constant for each real number. Let \( \varphi \) be a formula of the form (18) in the signature \( \sigma' \) and \( c \) be a \( \varphi \)-regular \( m \)-tuple that consists only of open and closed intervals. Since the relation \( x \leq y \) is expressed by the positive formula \( \exists z(y = x + z^2) \) and the universal quantifiers are restricted to open intervals, while the existential quantifiers are restricted to closed intervals, it is easy to see that \( \varphi[c] \) is equivalent to a positive formula of the signature \( \sigma' \).

For two \( \varphi \)-regular \( m \)-tuples \( c \) and \( c' \) we say that \( c \ll c' \) if for any \( i \leq m \) the following property holds:

if \( Q_i = \forall \) then \( C_i \subseteq C_{i'} \) and if \( Q_i = \exists \) then \( C_i \subseteq \text{int}(C_{i'}) \). Here \( \overline{B} \) is the closure of \( B \) and \( \text{int}(B) \) is the interior of \( B \).

If \( c \ll c' \) and \( W \in W \) then the formula \( \varphi[c'][W] \) is called a strong approximation of \( \varphi[c] \). The following lemma is obvious.

**Lemma 3.** Let \( \varphi(x_1, \ldots, x_n) \) be a positive formula of \( L_\omega \) of the form (18), \( c_1 \ll c_2 \), be \( \varphi \)-regular \( m \)-tuples of subsets of \( A \), let \( W_2 \subseteq W_1 \) be elements of the uniformity \( W \) and \( \langle A_f, j_f \rangle \) be a \((C, W)\)-approximation of \( A \) for some \( \langle C, W \rangle \in M \). Then
1. \( \forall a_1, \ldots, a_n \in A \ (A \models \varphi[c](a_1, \ldots, a_n) \implies A \models \varphi[c](a_1, \ldots, a_n)) \);

2. \( \forall a_1, \ldots, a_n \in A_f \ (A_f \models \varphi[c](W_2)(a_1, \ldots, a_n) \implies A_f \models \varphi[c](W_1)(a_1, \ldots, a_n)) \);

3. \( \forall a_1, \ldots, a_n \in A_f \ (A_f \models \varphi[c](W_2)(a_1, \ldots, a_n) \implies A_f \models \varphi[c](W_2)(a_1, \ldots, a_n)) \).

This notion of approximation for positive bounded formulas is similar to the one introduced in [14][16][15] for structures based on Banach spaces.

The following theorem is the main result of this section.

**Theorem 4.** Let \( \varphi[c](x_1, \ldots, x_n) \) be a positive bounded formula and \( a_1, \ldots, a_n \in A \). Then \( A \models \varphi[c](a_1, \ldots, a_n) \) iff for any strong approximation \( \varphi[c'][W'] \) of \( \varphi[c] \) there exists a pair \( \langle C_0, W_0 \rangle \in M \) such that the following conditions hold:

1. \( \bigcup_{i=1}^{n} W_0(a_i) \subseteq C_0 \);

2. for any \( \langle C, W \rangle \leq \langle C_0, W_0 \rangle \), for any \( (C, W) \)-approximation \( \langle A_f, j_f \rangle \) of \( A \) and for any \( b_1, \ldots, b_n \in A_f \) such that \( (a_i, j(b_i)) \in W_0, \ i = 1, \ldots, n \), one has \( A_f \models \varphi[c'][W'](b_1, \ldots, b_n) \).

If for some property \( P \) there exists a \( \langle C_0, W_0 \rangle \in M \) such that \( P \) holds for all \( (C, W) \)-approximations of \( A \) such that \( \langle C, W \rangle \leq \langle C_0, W_0 \rangle \), then we say that \( P \) holds for all precise enough approximations of \( A \).

**Corollary 2.** A positive bounded sentence \( \varphi[c] \) holds in \( A \) iff all of its strong approximations \( \varphi[c'][W] \) hold in all precise enough approximations of \( A \).

From the point of view of numerical systems implemented in computers this corollary means that approximate versions of positive bounded theorems about the reals hold for numerical computer systems that simulate the field of reals in powerful enough computers.

Before we start to prove Theorem 4 consider the following three examples. In these examples we deal with the algebra \( \langle \mathbb{R}; 1, +, \times \rangle \) and its \( (a, \varepsilon) \)-approximations \( \langle A_f, j_f \rangle \) (see Example 1) such that \( j_f \) is the inclusion map. According to Definition 2(3), in this case we say that \( A_f \) is an \( (a, \varepsilon) \)-approximation of \( \mathbb{R} \).

**Example 6** Fix any positive \( d > 1 \). Then the following positive bounded formula holds for the field \( \mathbb{R} \):

\[
\forall_D \exists y (xy = 1),
\]

where \( D = \{ x \in \mathbb{R} \mid d^{-1} < |x| < d \} \).

It is easy to see that for any strong approximation of this formula there exists a finer strong approximation of the following form:

\[
\forall_C \exists_B (|xy - 1| < \delta),
\]

where \( C = \{ x \in \mathbb{R} \mid c^{-1} < |x| < c \} \), \( B = \{ x \in \mathbb{R} \mid b^{-1} \leq |x| \leq b \} \), \( 1 < c < d < b \) and \( \delta > 0 \).

We have to show that there exist \( a_0, \varepsilon_0 \) such that for any \( a > a_0, \ \varepsilon < \varepsilon_0 \), formula (20) holds for any finite \( (a, \varepsilon) \)-approximation \( A_f \) of \( \mathbb{R} \). Fix any \( x \) such that \( c^{-1} < |x| < c \) and let \( y = x^{-1}, \ b^{-1} < |y| < b \). Take \( \xi, \eta \in A_f \) such that \( |x - \xi| < \varepsilon \) and \( |y - \eta| < \varepsilon \). The \( a \) and \( \varepsilon \) have to satisfy the following conditions: \( \xi, \eta, \xi \times \eta \in [-a, a], \ |\xi \otimes \eta - 1| < \delta \), where \( \otimes \) is the multiplication in \( A_f \).
By the definition of \((a, \varepsilon)\)-approximation, it is easy to see that the following \(a_0\) and \(\varepsilon_0\) satisfy the required conditions:

\[
\varepsilon_0 = \sqrt{\left(\frac{2b+1}{2}\right)^2 + \delta} - \frac{2b+1}{2}, \quad a_0 = \max\{b + \varepsilon_0, (2b + \varepsilon_0)\varepsilon_0 + 1\}.
\]

**Example 7** Let \(\varphi(x, y)\) be the positive formula \(\exists z(x + z^2 = y)\), which defines the relation \(\leq\) in \(\mathbb{R}\).

Consider a bounded version of this formula \(\varphi[b](x, y) = \exists_{|z| \leq b}(x + z^2 = y)\), which defines the relation \(x \leq y \leq x + b^2\). A strong approximation of this formula is of the form \(\varphi[c][\alpha](x, y) = \exists_{|z| \leq c}(|x + z^2 - y| < \alpha)\) for some \(\alpha > 0\) and \(0 < b < c\). Let \(x_0, y_0 \in [-d, d]\) and \(\mathbb{R} \models \varphi[b](x_0, y_0)\). Put \(a_0 = (c + \alpha)^2 + d + \alpha + 1\) and \(\varepsilon_0 = \max\{c - b, \frac{\alpha}{5 + 2a_0}\}\). Then it is easy to see that \(A_f \models \varphi[c][\alpha](\xi, \eta)\), where \(A_f\) is a \((a, \varepsilon)\)-approximation of \(\mathbb{R}\), \(a > a_0, \varepsilon < \varepsilon_0, \xi, \eta \in A_f\) and \(|x_0 - \xi| < \varepsilon_0, |y_0 - \eta| < \varepsilon_0\). If \(x_0 > y_0\) and \(\alpha < \frac{1}{2}(x_0 - y_0)\), we may take \(\varepsilon_0\) such that \(x_0 - \varepsilon_0 > y_0 + \alpha\). Thus, the formula \(\varphi[c][\alpha](\xi, \eta)\) fails in \(A_f\) for any \((a, \varepsilon)\)-approximation \(A_f\) of \(\mathbb{R}\) and for any \(\xi, \eta \in A_f\) such that \(|x_0 - \xi| < \varepsilon_0, |y_0 - \eta| < \varepsilon_0\). A similar consideration holds for \(y_0 > x_0 + b^2\).

**Example 8** The relation \(<\) can also be defined by a positive formula. Indeed:

\[
x < y \iff \exists z((y - x)z^2 = 1) = \varphi(x, y)
\]

A bounded version of this formula \(\varphi[b](x, y) = \exists_{|z| \leq b}((y - x)z^2 = 1)\) defines the relation \(y > x + \frac{1}{y}\).

A strong approximation of this formula is of the form \(\varphi[c][\alpha](x, y) = \exists_{|z| \leq c}(|(y - x)z^2 - 1| < \alpha)\) for some \(\alpha > 0\) and \(0 < b < c\). It is easy to see that for \(\alpha < 1\), for small enough \(\varepsilon\), big enough \(a\) and for any \((a, \varepsilon)\)-approximation \(A_f\) of \(\mathbb{R}\) if \(\xi, \eta \in [-a, a]\) then \(A_f \models \varphi[c][\alpha](\xi, \eta)\) \(\iff\) \(\eta > \xi + \frac{1 - 2\varepsilon}{2}\).

**Remark 6.** It is easy to see that the relation \(u < v\) between normalized floating-point numbers \(u, v\), introduced in [17] page 200], is a special case of the approximation \(\varphi[c][\alpha]\) in Example 8.

**Remark 7.** By a classical result of Tarski [24] any formula in the signature \(\langle+, \times\rangle\) is equivalent in the first order theory of the ordered field of real numbers (\(\text{Th}(\mathbb{R})\)) to a quantifier free formula in the signature \(\langle 1, +, \times, \leq \rangle\). Therefore, the examples considered above show that any formula of the language of rings is equivalent in \(\text{Th}(\mathbb{R})\) to a positive formula and thus has its approximate versions.

Let the topological space \(A\) be totally disconnected; i.e., the clopen sets form a base of its topology. Consider a positive bounded formula \(\varphi[c]\) with an \(m\)-tuple \(c\) that consists of clopen sets. In this case we say that \(c\) is clopen. Since for a clopen set \(V\) one has \(\overline{V} \subseteq V\) and \(V \subseteq \text{int}(V)\), then for a clopen \(m\)-tuple \(c\) one has \(c \subseteq c\). Thus if \(W \in \mathcal{W}\), then \(\varphi[c][W]\) is a strong approximation of \(\varphi[c]\). So the formulation of Theorem 11 can be simplified for this case.

**Corollary 3.** Let \(A\) be a totally disconnected algebra, \(\varphi[c](x_1, \ldots, x_n)\) be a positive bounded formula (19) with a clopen \(m\)-tuple \(c\), and \(a_1, \ldots, a_n \in A\). Then \(A \models \varphi[c](a_1, \ldots, a_n)\) iff for any \(W' \in \mathcal{W}\) there exists a pair \((C_0, W_0) \in M\) such that the following conditions hold:

1) \(\bigcup_{i=1}^{n} W_0(a_i) \subseteq C_0\);

2) for any \(\langle C, W \rangle \leq \langle C_0, W_0 \rangle\), for any \((C, W)\)-approximation \(\langle A_f, j_f \rangle\) of \(A\), and for any \(b_1, \ldots, b_n \in A_f\) such that \(\langle a_i, j(b_i) \rangle \in W_0\), \(i = 1, \ldots, n\), one has \(A' \models \varphi[c][W'](b_1, \ldots, b_n)\).
Now we turn to the proof of Theorem 4. First we consider an equivalent nonstandard statement.

Let \((A_h, j)\) be a hyperfinite approximation of \(A\) in the sense of Definition \(\text{K}\). Then a strong approximation \(\varphi[c][W]\) of a positive formula \(\varphi\) in the form (18) has an obvious interpretation in \(A_h\): a quantifier \(Q_C x \ldots\) is interpreted as on page 19 and a formula \((t_1, t_2) \in W\) is interpreted by \((j(t_1), j(t_2)) \in W\). Obviously, the statements (2) and (3) of Lemma \(\text{K}\) hold for hyperfinite approximations of \(A\).

**Lemma 4.** For any \(\beta_1, \ldots, \beta_n \in (A_h)_h\) one has
\[
A_h \models \varphi[c]_\sim(\beta_1, \ldots, \beta_n) \iff \forall W \in W_\lambda \ A_h \models \varphi[c][W](\beta_1, \ldots, \beta_n).
\]

**Proof** Obviously, \(A_h \models \varphi[c]_\sim(\beta_1, \ldots, \beta_n) \iff \forall W \in W_\lambda \ A_h \models \varphi[c][W](\beta_1, \ldots, \beta_n).\) So we have to prove only the converse implication. Consider first the case of a quantifier free formula, i.e., the case when \(\varphi = \psi\) in the form (18). We have \(\psi = P_1 \lor \cdots \lor P_r\), where each \(P_i\) is a conjunction of atomic formulas. Assume that \(\forall W \in W_\lambda\) one has \(\psi[W]\). If \(\psi\) is false then for each \(i \leq r\) there exists \(W_i \in W_\lambda\) such that \(P_i[W_i]\) is false. Take \(W \in W_\lambda\) such that \(W \subseteq \bigcap_{i=1}^r W_i\). Then by Lemma \(\text{K}(2)\) for any \(i \leq r\) the formula \(P_i[W]\) is false. Thus, the formula \(\psi[W]\) is false.

We have to prove now that
\[
\forall W \in W_\lambda Q_{1C_1}y_1 \ldots Q_{mC_m}y_m \psi[W] \implies Q_{1C_1}y_1 \ldots Q_{mC_m}y_m \forall W \in W_\lambda \psi[W]
\]

To prove this implication, it is enough to prove that for any positive bounded formula \(\tau(x)\) and any compact set \(C\) one has
\[
\forall W \in W_\lambda \exists Cx\tau[W](x) \implies \exists Cx\forall W \in W_\lambda \tau[W](x).
\] (21)

Assume that the left hand side of this implication holds. Put \(B(W) = \{x \mid j(x) \in ^*C, \tau[W](x)\}\). Then \(B(W) \neq \emptyset\). Since for any \(W_1, \ldots, W_s \in W_\lambda\) there exists \(W \in W_\lambda\) such that \(W \subseteq \bigcap_{i=1}^s W_i\), using Lemma \(\text{K}(2)\), we obtain that the family \(\{B(W) \mid W \in W_\lambda\}\) has the finite intersection property. Thus, by saturation, we obtain that the right hand side of the implication (21) holds. \(\square\)

**Lemma 5.** Let \(\varphi(x_1, \ldots, x_n)\) be a positive formula in \(L_\phi\) of the form (18), \(c = \langle C_1, \ldots, C_m\rangle\) a \(\varphi\)-regular \(m\)-tuple of subsets of \(A\), \(a_1, \ldots, a_n \in A\), and \((A_h, j)\) a hyperfinite approximation of \(A\). Then \(A \models \varphi[c](a_1, \ldots, a_n)\) iff for any \(\varphi\)-regular \(c' = \langle C'_1, \ldots, C'_m\rangle\) such that \(c' \gg c\) and for any \(\alpha_1, \ldots, \alpha_n \in A_h\) such that \(j(\alpha_i) \approx a_i, \ i = 1, \ldots, n\) one has \(A_h \models \varphi[c'\sim(\alpha_1, \ldots, \alpha_n)]\).

**Proof** We prove this lemma by induction on \(m\). For \(m = 0\) it follows from Proposition \(\text{I}(2)\). Assume that it is proved for \(m-1\). Denote by \(\tau(y_1, x_1, \ldots, x_n)\) the formula \(Q_{2y_2}Q_{m}y_m \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\), by \(c_-\) the \((m-1)\)-tuple \(\langle C_2, \ldots, C_m\rangle\), by \(c'_-\) the \((m-1)\)-tuple \(\langle C'_2, \ldots, C'_m\rangle\), so that \(\varphi = Q_{1}y_1 \tau, \ \varphi[c] = Q_{1C_1}y_1 \tau[c_-], \ \varphi[c'] = Q_{1C'_1}y_1 \tau[c'_-]\). Consider two cases.

a). \(Q_1 = \exists\). In this case \(C_1, C'_1\) are compact sets and \(C_1 \subseteq \text{int}(C'_1)\).

\(\Rightarrow\) Let \(A \models \varphi[c](a_1, \ldots, a_n)\). Then there exists \(b \in C_1\) such that \(A \models \tau[c_-](b, a_1, \ldots, a_n)\). Let \(\beta \in A_h\) be such that \(j(\beta) \approx b\). Then, by the induction assumption, \(A_h \models \varphi[c'\sim(\beta, a_1, \ldots, a_n)]\). Since \(b \in C_1 \subseteq \text{int}(C'_1) \subseteq \text{int}(C'_1)\), \(j(\beta) \approx b\) and \(\text{int}(C'_1)\) is an open set, we have \(j(\beta) \in \text{int}(C'_1)\). This proves that \(A_h \models \varphi[c'\sim(\alpha_1, \ldots, \alpha_n)]\).

\(\Leftarrow\) Obviously \(C_1 = \bigcap_{W \in W_\lambda} W(C_1)\). Fix any \(V, W \in W_\lambda\). By the assumption of the lemma, for any \(\tau\)-regular \((m-1)\)-tuple \(c'_- \gg c_-\) one has \(A_h \models \exists_{W(C_1)} y_1 \tau[c'_-\sim(\alpha_1, \ldots, \alpha_n)]\). Thus,
by Lemma 4, we have $B(W, V, c_-) = \{ \beta \in j^{-1}(\overline{W(C_1)}) \mid A_h \models \tau[c_-, [V](\beta, \alpha_1, \ldots, \alpha_n)] \neq \emptyset \}$. 

Let $\Xi$ be the set of all $\tau$-regular $(m-1)$-tuples $c_-$. Then it is easy to see that there exists a cofinal subset $\Xi_\lambda$ (i.e., $\forall d \in \Xi \exists d_i \in \Xi_\lambda (d_i \ll d)$) of cardinality $\lambda$. By Lemma 3(3), if for some $b, \alpha_1, \ldots, \alpha_n \in (A_h)_b$ for all $d \in \Xi_\lambda$ one has $A_h \models \tau[d][V]$, then the same holds for all $d \in \Xi$. It is easy to see also that, similar to $W_\lambda$, the family $\Xi_\lambda$ has the following property: for any $c^{(1)}, \ldots, c^{(s)} \in \Xi_\lambda$ there exists a $c_- \in \Xi_\lambda$ such that $c_- \ll c^{(1)}, \ldots, c_- \ll c^{(s)}$. All this shows that the family $\{B(W, V, c_-) \mid V, W \in W_\lambda, c_- \in \Xi_\lambda \}$ has the finite intersection property and thus, by saturation, has nonempty intersection. By our construction and Lemma 4 any element $\beta$ in this intersection has the following properties: $j(\beta) \in \mathcal{C}_1$ and $A_h \models \tau[c_-](\beta, \alpha_1, \ldots, \alpha_n)$ for any $c_- \in \Xi$. By the induction assumption this implies that $A \models \tau[c_-](\beta, \alpha_1, \ldots, \alpha_n)$. Since $C$ is a compact set, we obtain that $\vdash j(\beta) \in C$. This proves a).

b) $Q_1 = \forall$. In this case $C_1$ and $C_1'$ are relatively compact open sets and $C_1 \subseteq C_1'$. Let $A \models \varphi[c](a_1, \ldots, a_n)$ and $A_h \models \tau[c_-][\varphi][W][x_1, \ldots, x_n]$. Consider the internal set $N$ of all pairs $\langle C_0, W_0 \rangle \in \mathcal{M}_\lambda$ such that for all $\langle C, W \rangle \leq \langle C_0, W_0 \rangle$, for any $\mathcal{W}$-approximation $\langle A_h, j \rangle$ of $\mathcal{A}$ and for any $\alpha_1, \ldots, \alpha_n \in A_h$ satisfying the condition $\langle a_i, j(\alpha_i) \rangle \in W_0$ one has $A_h \models \varphi[c][W](\alpha_1, \ldots, \alpha_n)$. Lemmas 4 and 2 imply that $N$ contains all infinitesimal pairs $\langle C_0, W_0 \rangle \in \mathcal{M}_\lambda$. By Lemma 4(2), there exists $\langle C_0, W_0 \rangle \in \mathcal{M}_\lambda$ such that $\langle C, W \rangle \in N$. By the transfer principle, this completes the proof.

The following corollary of Theorem 4 shows that the approximation of continuous functions by polynomials on closed intervals holds for all precise enough approximations of the field $\mathbb{R}$ (cf. the example concerning $\sin x$ which was discussed in the Introduction).

**Corollary 4.** Let $\mathcal{A} = \langle \mathbb{R}, \sigma \rangle$ be such that $\sigma$ contains the symbols $+$ and $\times$ and a unary function symbol $g$. Suppose that the continuous function $g$ is approximable on an interval $[-d, d]$ by a polynomial $b_n x^n + \cdots + b_1 x + b_0$ with accuracy $\delta$. Then for any $0 < d' < d$ and $\delta' > \delta$ there exist $a_0, \varepsilon_0 > 0$ such that any $(a, \varepsilon)$-approximation $\langle A_f, j_f \rangle$ of $\mathcal{A}$ with $a > a_0$ and $\varepsilon < \varepsilon_0$ has the following property: for any $\beta_0, \ldots, \beta_n, \xi \in A_f$, if $|j(\beta_i) - b_i| < \varepsilon_0$ for all $i = 0, \ldots, n$ and $j(\xi) \in [-d', d']$, then $|j_f(g_f(\xi)) - j_f(\beta_n \xi^n + \cdots + \beta_1 \xi + \beta_0)| < \delta'$. (Here $g_f$ is the interpretation of the symbol $g$ in $A_f$.)
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Universidad Autonoma de San Luis Potosi, Mexico (Glebsky)
Eastern Illinois University, USA (Gordon)
University of Illinois at Urbana-Champaign, USA (Henson)

1991 *Mathematics Subject Classification*. Primary 26E35, 03H05; Secondary 28E05, 42A38
Instituto de Investigacion en Communicacion Optica
Universidad Autonoma de San Luis Potosi, Mexico
AvKarakorum 1470
Lomas 4ta Session
San Luis Potosi SLP 7820
Mexico
e-mail:glebsky@cactus.iico.uaslp.mx

Department of Mathematics and Computer Science
Eastern Illinois University
600 Lincoln Avenue
Charleston, IL 61920-3099
USA
e-mail: cfyig@eiu.edu

Department of Mathematics
University of Illinois at Urbana-Champaign
1409 West Green Street
Urbana, IL 61801
USA
www: http://www.math.uiuc.edu/~henson