Gorenstein modules respect to duality pairs over triangular matrix rings

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Abstract

Let $A$, $B$ be two rings and $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with $M$ an $A$-$B$-bimodule. We first construct a semi-complete duality pair $D_T$ of $T$-modules using duality pairs in $A$-Mod and $B$-Mod respectively. Then we characterize when a left $T$-module is Gorenstein $D_T$-projective, Gorenstein $D_T$-injective or Gorenstein $D_T$-flat. These three classes of $T$-modules will induce model structures on $T$-Mod. Finally we show that the homotopy category of each of model structures above admits a recollement relative to corresponding stable categories. Our results give new characterizations to earlier results in this direction.

1. Introduction

Let $A$ and $B$ be two rings. For any bimodule $A M B$, we write $T$ for the upper triangular matrix ring $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$. Such rings play an important role in the study of the representation theory of artin rings and algebras. Some important classes of modules over upper triangular matrix rings have been studied by many authors (e.g., see [16], [15], [30], [31] and [5] and their references). For example, Zhang [31] explicitly described the Gorenstein projective modules over a triangular matrix Artin algebra. Enochs and other authors [5] characterized when a left module over a triangular matrix ring is Gorenstein projective or Gorenstein injective under the “Gorenstein regular” condition. Zhu, Liu and Wang [32, Theorem 3.8] characterized Gorenstein flat modules over a triangular matrix ring $T$. Very recently, Mao [20, Theorem 2.3] further studied Gorenstein flat modules over triangular matrix rings which improves [32, Theorem 3.8].

Duality pairs were introduced by Holm-Jørgensen in [17]. Recall that for a given $R$-module $M$, its character module is defined to be the $R$-module $M^+ = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. A duality pair is essentially a pair of classes $(\mathcal{L}, \mathcal{A})$ such that $L \in \mathcal{L}$ if and only if $L^+ \in \mathcal{A}$, and $\mathcal{A}$ is closed under direct summands and finite direct sums. Recently, Gillespie [12] showed that the entire theory of Gorenstein homological algebra, complete with associated abelian model structures with stable homotopy categories, can be done with respect to a complete duality pair. Assume that $D_R = (\mathcal{L}, \mathcal{A})$ is a complete duality pair. We say that an $R$-module $N$ is $D_R$-Gorenstein projective if $N = Z^0 P$ for some exact $\text{Hom}_R(-, \mathcal{L})$-acyclic complex of projective $R$-modules $P$. That is, both $P$ and $\text{Hom}_R(P, L)$ are exact (acyclic) complexes for all $L \in \mathcal{L}$. Those familiar with Gorenstein homological algebra will guess the definitions of the other concepts, see Definitions 2.3 and 2.4 for precise definitions. When $R$ is a commutative Noetherian ring of finite Krull dimension, then these definitions, applied to the flat-injective duality pair, agree with the usual

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Key words and phrases. duality pair; Ding injective module; triangular matrix ring; recollement.

2010 Mathematics Subject Classification. 16E30, 18E30, 16D90.
definitions of Gorenstein injective, projective and flat modules studied by Enochs and many other authors \([6, 7]\). In fact, the requirement for the duality pair to be complete is too strong, so they defined semi-complete duality pairs in \([13]\) and showed that if we applied to the duality pair \(D_R = (\langle R \mathcal{F} \rangle, \langle I_R \rangle)\), then these definitions agree with the definitions of projectively coresolved Gorenstein flat \([27]\), Ding injective \([4, 10]\) and Gorenstein flat modules \([7]\).

The main goal of this paper is to study Gorenstein homological modules respect to semi-complete duality pairs over triangular matrix rings. Mao \([22]\) constructed a complete duality pair \(D_T\) of \(T\)-modules using duality pairs in \(A\)-Mod and \(B\)-Mod respectively. Based on this result, in Section 3, we further study when \(D_T\) is semi-complete. Then we characterize when a left \(T\)-module is Gorenstein \(D_T\)-projective, Gorenstein \(D_T\)-injective or Gorenstein \(D_T\)-flat (see Theorems 3.4, 3.8 and 3.11). As applications, we investigate when a left \(T\)-module is projectively coresolved Gorenstein flat, Ding injective or Gorenstein flat. In fact, Mao \([23]\) characterized Ding injective modules over a triangular matrix ring \(T\) under the condition “\(T\) is right coherent, \(\_A M\) has finite flat dimension, \(M_B\) is finitely presented and has finite projective or FP-injective dimension”. Our result presents a new characterization of Ding injective modules (see Corollary 3.9). One can compare it with \([23, Theorem 4.4]\).

If we are given two cofibrantly generated model structures \(M_A\) and \(M_B\) on \(A\)-Mod and \(B\)-Mod respectively, we investigated in \([33]\) when there exists a cofibrantly generated model structure \(M_T\) on \(T\)-Mod and a recollement of \(Ho(M_T)\) relative to \(Ho(M_A)\) and \(Ho(M_B)\). Let \(D_R = (\mathcal{L}, \mathcal{A})\) be a semi-complete duality pair. By \([13, Corollary 5.1]\), there are three abelian module structures induced by \(D_R\): the Gorenstein \(D_R\)-projective, \(D_R\)-injective and \(D_R\)-flat model structures. We know that each of these model structures will gives rise to a stable category of modules. In Section 4, using the characterizations in Section 3, we show that the homotopy category of each of model structures above on \(T\)-Mod admits a recollement relative to corresponding homotopy categories (see Theorem 4.5). Finally, we give some applications of our results for projectively coresolved Gorenstein flat, Ding injective and Gorenstein flat model structures. It should be noticed that the recollements of stable categories of Ding injective modules and Gorenstein flat modules over a triangular matrix ring \(T\) have been established in \([28, Theorem 2.10]\) and \([33, Theorem 4.12]\) respectively. Our result Theorem 4.5 gives a new criterion for the existence of these two recollements.

2. Preliminaries

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring \(R\), we write \(R\)-Mod (resp. Mod-\(R\)) for the category of left (resp. right) \(R\)-modules. \(M_R\) (resp. \(R M\)) denotes a right (resp. left) \(R\)-module.

2.1. Duality pairs. \([17, Definition 2.1]\) A duality pair over a ring \(R\) is a pair \((\mathcal{L}, \mathcal{A})\), of classes of \(R\)-modules, satisfying

1. \(L \in \mathcal{L}\) if and only if \(L^+ \in \mathcal{A}\), and
2. \(\mathcal{A}\) is closed under direct summands and finite direct sums.

If \((\mathcal{L}, \mathcal{A})\) is a duality pair, then \(\mathcal{L}\) is closed under pure submodules, pure quotients, and pure extensions.
Definition 2.1. [1, Appendix A] By a symmetric duality pair over $R$ we mean a pair of classes $(\mathcal{L}, \mathcal{A})$ for which both $(\mathcal{L}, \mathcal{A})$ and $(\mathcal{A}, \mathcal{L})$ are duality pairs. A duality pair $(\mathcal{L}, \mathcal{A})$ is called perfect if $\mathcal{L}$ contains the module $R$, and is closed under direct sums and extensions.

As in [13], we call $(\mathcal{L}, \mathcal{A})$ a semi-perfect duality pair if it has all the properties required to be a perfect duality pair except that $\mathcal{L}$ may not be closed under extensions.

Definition 2.2. [13, Definition 2.5] By a semi-complete duality pair $(\mathcal{L}, \mathcal{A})$ we mean that $(\mathcal{L}, \mathcal{A})$ is a symmetric duality pair with $(\mathcal{L}, \mathcal{A})$ being a semi-perfect duality pair. If $(\mathcal{L}, \mathcal{A})$ is indeed perfect, then we call it a complete duality pair.

Several examples of perfect and symmetric duality pairs are given in [2, 12, 17].

2.2. Gorenstein modules relative to a duality pair. Throughout this subsection we let $D_R = (\mathcal{L}, \mathcal{A})$ denote a fixed semi-complete duality pair over $R$.

Definition 2.3. Given an $R$-module $N$, a chain complex $I$ of injective $R$-modules is called $N$-acyclic if $\text{Hom}_R(N, I)$ is exact. In a similar way, given a class $\mathcal{N}$ of $R$-modules, $I$ will be called $\mathcal{N}$-acyclic if it is $N$-acyclic for all $N \in \mathcal{N}$. On the other hand, if $P$ is a chain complex of projective (or even flat) $R$-modules, we call it $\mathcal{N}^\mathcal{G}$-acyclic if $N \otimes_R P$ is exact; and similarly we define $\mathcal{N}^\mathcal{G}$-acyclic complexes of projective (or even flat) $R$-modules for a class $\mathcal{N}$.

Definition 2.4. An $R$-module $N$ is called

1. Gorenstein $(\mathcal{L}, \mathcal{A})$-projective (or Gorenstein $D_R$-projective) if $N = Z^0P$ for some exact $\mathcal{A}^\mathcal{G}$-acyclic complex of projectives $P$. Let $\mathcal{GP}_{D_R}$ denote the class of all Gorenstein $D_R$-projective modules.

2. $(\mathcal{L}, \mathcal{A})$-Gorenstein projective (or $D_R$-Gorenstein projective) if $N = Z^0P$ for some exact complex of projectives $P$ which remains exact after applying $\text{Hom}_R(\cdot, L)$ for any $L \in \mathcal{L}$. Let $\mathcal{GP}^{D_R}$ denote the class of all $D_R$-Gorenstein projective modules.

3. An $R$-module $N$ is called Gorenstein $(\mathcal{L}, \mathcal{A})$-injective (or Gorenstein $D_R$-injective) if $N = Z^0I$ for some exact $\mathcal{A}$-acyclic complex of injectives $I$. Let $\mathcal{GI}_{D_R}$ denote the class of all Gorenstein $D_R$-injective modules.

4. An $R$-module $N$ is called Gorenstein $(\mathcal{L}, \mathcal{A})$-flat (or Gorenstein $D_R$-flat) if $N = Z^0F$ for some exact $\mathcal{A}^\mathcal{G}$-acyclic complex of flat modules $F$. Let $\mathcal{GF}_{D_R}$ denote the class of all Gorenstein $D_R$-flat modules.

Note that a Gorenstein $D_R$-projective module $N$ is always Gorenstein $D_R$-flat. If $D_R$ is a symmetric duality pair, then $\mathcal{GP}_{D_R} = \mathcal{GP}^{D_R}$ by [1, Theorem A6].

2.3. Triangular matrix rings. Let $A$, $B$ be two rings and $T = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ with $M$ an $A$-$B$-bimodule. Next, we recall the description of left $T$-modules via column vectors. Let $X_1 \in A$-Mod and $X_2 \in B$-Mod, and let $\phi^X : M \otimes_B X_2 \rightarrow X_1$ be a homomorphism of left $A$-modules.

The left $T$-module structure on $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ is defined by the following identity

$$\begin{pmatrix} a & m \\ 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + \phi^X(m \otimes x_2) \\ bx_2 \end{pmatrix},$$
where \(a \in A, \ b \in B, \ m \in M, \ x_i \in X_i\) for \(i = 1, \ 2\). According to [14, Theorem 1.5], \(T\)-\text{Mod} is equivalent to the category whose objects are triples \(X = (X_1, X_2, \phi_X)\), where \(X_1 \in A\)-\text{Mod}, \(X_2 \in B\)-\text{Mod} and \(\phi_X : M \otimes_B X_2 \rightarrow X_1\) is an \(A\)-homomorphism, and whose morphisms between two objects \(X = (X_1, X_2, \phi_X)\) and \(Y = (Y_1, Y_2, \phi_Y)\) are pairs \((f_1, f_2)\) such that \(f_1 \in \text{Hom}_A(X_1, Y_1), \ f_2 \in \text{Hom}_B(X_2, Y_2)\), satisfying that the diagram

\[
\begin{array}{ccc}
M \otimes_B X_2 & \xrightarrow{f_2} & M \otimes_B Y_2 \\
\phi_X & \downarrow & \phi_Y \\
X_1 & \xrightarrow{f_1} & Y_1
\end{array}
\]

is commutative. In the rest of the paper we identify \(T\)-\text{Mod} with this category and, whenever there is no possible confusion, we omit the homomorphism \(\phi\). Consequently, throughout the paper, a left \(T\)-module is a pair \((X_1, X_2, \phi_X)\). Given such a module \(X\), we denote by \(\tilde{\phi}^X\) the morphism from \(X_2\) to \(\text{Hom}_A(M, X_1)\) given by \(\tilde{\phi}^X(x)(m) = \phi^X(m \otimes x)\) for each \(x \in X_2, \ m \in M\).

Note that a sequence of \(T\)-modules

\[
0 \rightarrow \left( \frac{M_1}{M_2} \right) \rightarrow \left( \frac{M_2}{M_2} \right) \rightarrow \left( \frac{M_2'}{M_2''} \right) \rightarrow 0
\]

is exact if and only if both sequences \((0 \rightarrow M_1' \rightarrow M_1 \rightarrow M_1'' \rightarrow 0)\) of \(A\)-modules and \((0 \rightarrow M_2' \rightarrow M_2 \rightarrow M_2'' \rightarrow 0)\) of \(B\)-modules are exact.

Recall that each right \(T\)-module is identified with a triple \((X, Y)_{\varphi}\), where \(X \in A\)-\text{Mod}, \(Y \in B\)-\text{Mod} and \(\varphi : X \otimes A \rightarrow Y\) is a right \(B\)-homomorphism, and a right \(T\)-map is identified with a pair \((f_1, f_2) : (X_1, Y_1)_{\varphi_1} \rightarrow (X_2, Y_2)_{\varphi_2}\), where \(f_1 : X_1 \rightarrow X_2\) is an \(A\)-map and \(f_2 : Y_1 \rightarrow Y_2\) a \(B\)-map, such that \(f_2 \varphi_1 = \varphi_2(f_1 \otimes 1)\). For a right \(T\)-module \((X, Y)_{\varphi}\), we denote \(\tilde{\varphi} : X \rightarrow \text{Hom}_B(M, Y)\) the involution map given by \(\tilde{\varphi}(x)(m) = \varphi(x \otimes m)\) for each \(x \in X, \ m \in M\).

Let \(\left( \frac{X_1}{X_2} \right)_{\phi_X}\) be a left \(T\)-module and \((W_1, W_2)_{\phi_w}\) be a right \(T\)-module. By [19, Proposition 3.6.1], there is an isomorphism of abelian groups

\[
(W_1, W_2)_{\phi_w} \otimes_T \left( \frac{X_1}{X_2} \right)_{\phi_X} = (W_1 \otimes_A X_1 \otimes W_2 \otimes_B X_2) / H,
\]

where \(H\) is generated by all elements of the form \(\phi_w(w_1 \otimes x_2) - w_1 \otimes \phi^X(m \otimes x_2)\) with \(w_1 \in W_1, x_2 \in X_2, m \in M\).

### 3. Gorenstein modules respect to duality pairs over triangular matrix rings

For two classes \(C\) and \(\mathcal{F}\) of modules, we set the following classes of \(T\)-modules:

\[
\mathcal{W}_C^F = \{ N = \left( \frac{X_1}{X_2} \right)_{\phi_N} \ | \ N_1 \in C, \ N_2 \in \mathcal{F} \};
\]

\[
\mathfrak{W}_C^F = \{ X = \left( \frac{X_1}{X_2} \right)_{\phi_X} \ | \ \phi^X \text{ is monomorphic, Coker}\phi^X \in C, \ X_2 \in \mathcal{F} \};
\]

\[
\mathfrak{Y}_C^F = \{ Y = \left( \frac{Y_1}{Y_2} \right)_{\phi_Y} \ | \ \phi^Y \text{ is epimorphic, } Y_1 \in C, \ \text{Ker}\phi^Y \in \mathcal{F} \}.
\]

There are similar symbols \(\mathfrak{B}_{C, D}, \mathfrak{X}_{C, D}\) for the case of right \(T\)-modules.

L. Mao [22] studied symmetric or perfect duality pairs over formal triangular matrix rings.
**Theorem 3.4.** Assume Setup 3.3. Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be a class of left (resp. right) $A$-modules and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be a class of left (resp. right) $B$-modules. Suppose that $M_B$ is finitely presented and $\text{Tor}^B_1(M, \mathcal{F}_1) = 0$. Then $\langle \mathcal{B}^{\mathcal{C}_1}_{\mathcal{F}_1}, \mathcal{I}_{\mathcal{C}_2, \mathcal{F}_2} \rangle$ is a complete duality pair if and only if $(\mathcal{C}_1, \mathcal{C}_2)$ and $(\mathcal{F}_1, \mathcal{F}_2)$ are complete duality pairs.

By the proof of Lemma 3.1, one can get the following result.

**Proposition 3.2.** Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be a class of left (resp. right) $A$-modules and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be a class of left (resp. right) $B$-modules. Suppose that $M_B$ is finitely presented. Then $\langle \mathcal{B}^{\mathcal{C}_1}_{\mathcal{F}_1}, \mathcal{I}_{\mathcal{C}_2, \mathcal{F}_2} \rangle$ is a semi-complete duality pair if and only if $(\mathcal{C}_1, \mathcal{C}_2)$ and $(\mathcal{F}_1, \mathcal{F}_2)$ are semi-complete duality pairs.

Let $\mathcal{X}$ be a class of left $R$-modules closed under direct summands and finite direct sums. Given a natural number $n$ and a left $R$-module $N$, we shall say that $N$ has finite $\mathcal{X}$-projective (resp., $\mathcal{X}$-injective) dimension less than or equal to $n$ if there exists an exact sequence

$$0 \to X_n \to \cdots \to X_1 \to X_0 \to N \to 0 \quad (0 \to N \to X_0 \to X_1 \to \cdots \to X_n \to 0)$$

such that $X_i$ belongs to $\mathcal{X}$ for $i = 0, 1, \cdots, n$.

**Setup 3.3.** Let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) be a class of left (resp. right) $A$-modules and $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) be a class of left (resp. right) $B$-modules. In this section, we always assume that

1. $\mathcal{D}_A = (\mathcal{C}_1, \mathcal{C}_2)$ and $\mathcal{D}_B = (\mathcal{F}_1, \mathcal{F}_2)$ are semi-complete duality pairs;
2. $M_B$ is finitely presented and has finite $\mathcal{F}_2$-injective dimension.

In the rest of this paper, we denote by $\mathcal{D}_T = (\mathcal{B}^{\mathcal{C}_1}_{\mathcal{F}_1}, \mathcal{I}_{\mathcal{C}_2, \mathcal{F}_2})$ the semi-complete duality pair in $T\text{-Mod}$ induced by semi-complete duality pairs $\mathcal{D}_A$ and $\mathcal{D}_B$.

### 3.1. Gorenstein $\mathcal{D}_T$-projective modules.

**Theorem 3.4.** Assume Setup 3.3 and let $X = \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right)_{\phi^X}$ be a left $T$-module. Suppose that $\text{Hom}_B(M, D)$ has finite $\mathcal{C}_2$-injective dimension for each $D \in \mathcal{F}_2$. Then the following conditions are equivalent:

1. $X$ is a Gorenstein $\mathcal{D}_T$-projective left $T$-module.
2. $X_2$ is a Gorenstein $\mathcal{D}_B$-projective left $B$-module, $\text{Coker}\phi^X$ is a Gorenstein $\mathcal{D}_A$-projective left $A$-module, and $\phi^X$ is a monomorphism.

**Proof.** By Proposition 3.2, we know that $\mathcal{D}_T$ is a semi-complete duality pair if and only if $\mathcal{D}_A$ and $\mathcal{D}_B$ are semi-complete duality pairs.

1. $(1) \Rightarrow (2)$: If $X$ is a Gorenstein $\mathcal{D}_T$-projective left $T$-module, then there is an exact $\mathcal{I}_{\mathcal{C}_2, \mathcal{F}_2}^\otimes$-acyclic complex of projective left $T$-modules

$$P : \cdots \to \left( \begin{array}{c} P_1^{i+1} \\ P_2^{i+1} \end{array} \right)_{\phi^{i+1}} \xrightarrow{\partial^{i+1}} \left( \begin{array}{c} P_1^i \\ P_2^i \end{array} \right)_{\phi^i} \xrightarrow{\partial^i} \left( \begin{array}{c} P_1^0 \\ P_2^0 \end{array} \right)_{\phi^0} \xrightarrow{\phi^1} \cdots$$

with $X = \text{Ker}(\partial^1_2)$. By [16, Theorem 3.1], we get the exact sequence

$$P_2 : \cdots \to P_2^{i-1} \xrightarrow{\partial^{i-1}_2} P_2^i \xrightarrow{\partial^i_2} P_2^1 \to \cdots$$
of projective left $B$-modules with $X_2 = \text{Ker}(\partial_2^0)$. Let $D \in \mathcal{F}_2$. Then there is an exact sequence of right $T$-modules

$$0 \to (0, D) \to (\text{Hom}_B(M, D), D) \to (\text{Hom}_B(M, D), 0) \to 0,$$

which induces the exact sequence of complexes

$$0 \to (0, D) \otimes_T P \to (\text{Hom}_B(M, D), D) \otimes_T P \to (\text{Hom}_B(M, D), 0) \otimes_T P \to 0.$$

Since $(\text{Hom}_B(M, D), D)$ is belonging to $\mathcal{T}_{C_2, F_2}$, the complex $(\text{Hom}_B(M, D), D) \otimes_T P$ is exact. By assumption, $\text{Hom}_B(M, D)$ has finite $C_2$-injective dimension, so one can check that $(\text{Hom}_B(M, D), 0)$ has finite $\mathcal{T}_{C_2, F_2}$-injective dimension and the complex $(\text{Hom}_B(M, D), 0) \otimes_T F$ is exact. It follows that $D \otimes_B P_2 = (0, D) \otimes_T P$ is exact. Whence $X_2$ is a Gorenstein $\mathcal{D}_B$-projective left $B$-module.

Let $i_1 : X_1 \to P_1^0$ and $i_2 : X_2 \to P_2^0$ be the inclusions. Consider the following commutative diagram in $A$-Mod:

\[
\begin{array}{ccc}
M \otimes_B X_2 \ar[d]_{\phi^X} & \ar[l]_{1_M \otimes_B I^2} M \otimes_B P_2^0 \ar[d]_{\phi^{P_0}} & \\
X_1 & \ar[l]_{i_1} P_1^0 & \\
\end{array}
\]

By assumption that $M_B$ has finite $\mathcal{F}_2$-injective dimension, one can check that $M \otimes_B P_2$ is exact. Thus $1_M \otimes_B i_2$ is a monomorphism. Also $\phi^{P_0}$ is a monomorphism by [16, Theorem 3.1]. So $\phi^X$ is a monomorphism by the commutative diagram above.

For any $i \in \mathbb{Z}$, there exists $\partial_i^1 : P_i^1/\text{Im}(\phi^i) \to P_{i-1}^1/\text{Im}(\phi^{i+1})$ such that the following diagram with exact rows is commutative.

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 \ar[r] & M \otimes_B P_2^{-1} \ar[d]_{1 \otimes \partial_1^{-1}} \ar[r]^{\phi^{-1}} & P_1^{-1} \ar[r]^{\phi_1^{-1}} & P_1^{-1}/\text{Im}(\phi^{-1}) \ar[r] & 0 \\
0 \ar[r] & M \otimes_B P_2^0 \ar[d]_{1 \otimes \partial_0^0} \ar[r]^{\phi_0} & P_1^0 \ar[r]^{\phi_1} & P_1^0/\text{Im}(\phi^0) \ar[r] & 0 \\
0 \ar[r] & M \otimes_B P_2^1 \ar[r]^{\phi_1} & P_1^1 \ar[r] & P_1^1/\text{Im}(\phi^1) \ar[r] & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
\end{array}
\]

Since the first column and the second column are exact, we get the exact sequence of projective left $A$-modules

\[
\overline{\text{P}_1} : \cdots \to P_1^{-1}/\text{Im}(\phi^{-1}) \ar[r]^{\overline{\partial_1^{-1}}} & P_1^0/\text{Im}(\phi^0) \ar[r]^{\overline{\partial_0^0}} & P_1^1/\text{Im}(\phi^1) \to \cdots
\]

with $X_1/\text{Im}(\phi^X) \cong \text{Ker}(\overline{\partial_1^1})$. 

Let \( G \in \mathcal{C}_2 \). Then each exact sequence of left \( A \)-modules

\[
0 \longrightarrow M \otimes_B P^i_2 \xrightarrow{\phi^i} P^i_1 \longrightarrow \text{Im}(\phi^i) \longrightarrow 0
\]

induces the exact sequence

\[
G \otimes_A M \otimes_B P^i_2 \xrightarrow{1 \otimes \phi^i} G \otimes_A P^i_1 \longrightarrow G \otimes_A (\text{Im}(\phi^i)) \longrightarrow 0
\]

So we have

\[
G \otimes_A (\text{Im}(\phi^i)) \cong G \otimes_A P^i_1/\text{Im}(1 \otimes \phi^i) \cong (G, 0) \otimes_T \left( \frac{P^i_1}{\phi^i} \right).
\]

Since \((G, 0) \in \mathcal{F}_{2, 2}, G \otimes_A P^1_1 \cong (G, 0) \otimes_T P \) is exact. So \( \text{Coker}\phi^X \) is a Gorenstein \( \mathcal{D}_A \)-projective left \( A \)-module.

(2) \( \Rightarrow \) (1) Since \( \phi^X \) is a monomorphism, there exists an exact sequence in \( T\text{-Mod} \)

\[
0 \rightarrow \left( \frac{M \otimes_B X^2}{X_2} \right)_{\text{id}} \rightarrow \left( \frac{X_1}{X_2} \right)_{\phi^X} \rightarrow \left( \text{coker}\phi^X \right)_0 \rightarrow 0.
\]

By the dual of [13, Lemma 3.5], one can check that the class of Gorenstein \( \mathcal{D}_T \)-projective left \( T \)-module is closed under extensions. So we only need to verify that \( \left( \frac{M \otimes_B X^2}{X_2} \right) \) and \( \left( \text{coker}\phi^X \right)_0 \)

are Gorenstein \( \mathcal{D}_T \)-projective.

We first prove that \( \left( \frac{M \otimes_B X^2}{X_2} \right) \) is a Gorenstein \( \mathcal{D}_T \)-projective module. In fact, there is an exact \( \mathcal{F}_2 \)-acyclic complex

\[
U : \cdots \rightarrow Q^{-1} \xrightarrow{\partial^{-1}} Q^0 \xrightarrow{\partial^0} Q^1 \rightarrow \cdots
\]

of projective left \( B \)-modules with \( X_2 = \text{Ker}(\partial^0) \). Since \( M_B \) has finite \( \mathcal{F}_2 \)-injective dimension, one can check that \( M \otimes_B U \) is exact. So we get the exact sequence of projective left \( T \)-modules

\[
V : \cdots \rightarrow \left( \frac{M \otimes_B Q^{-1}}{Q^{-1}} \right) \left( \xrightarrow{1 \otimes \partial^{-1}} \frac{M \otimes_B Q^0}{Q^0} \right) \left( \xrightarrow{1 \otimes \partial^0} \frac{M \otimes_B Q^1}{Q^1} \right) \rightarrow \cdots
\]

with \( \left( \frac{M \otimes_B X^2}{X_2} \right) \cong \text{Ker}(1 \otimes \partial^0) \). For any right \( T \)-module \( (H_1, H_2) \in \mathcal{F}_{2, 2}, \) there exists an exact sequence in \( \text{Mod-}T \)

\[
0 \rightarrow (0, H_2) \rightarrow (H_1, H_2) \rightarrow (H_1, 0) \rightarrow 0.
\]

Since each \( \left( \frac{M \otimes_B P^i}{P^i} \right) \) is a projective left \( T \)-module, we get the exact sequence

\[
0 \rightarrow (0, H_2) \otimes_T \left( \frac{M \otimes_B P^i}{P^i} \right) \rightarrow (H_1, H_2) \otimes_T \left( \frac{M \otimes_B P^i}{P^i} \right) \rightarrow (H_1, 0) \otimes_T \left( \frac{M \otimes_B P^i}{P^i} \right) \rightarrow 0.
\]

Note that \( (H_1, 0) \otimes_T \left( \frac{M \otimes_B P^i}{P^i} \right) \cong (H_1 \otimes_A M \otimes_B P^i)/(H_1 \otimes_A M \otimes_B P^i) = 0 \). Thus \( (H_1, H_2) \otimes_T \left( \frac{M \otimes_B P^i}{P^i} \right) \cong (0, H_2) \otimes_T \left( \frac{M \otimes_B P^i}{P^i} \right) \). So \( (H_1, H_2) \otimes_T V \cong (0, H_2) \otimes_T V \cong H_2 \otimes_B U \) is exact since \( H_2 \in \mathcal{F}_2 \). Hence \( \left( \frac{M \otimes_B X^2}{X_2} \right) \) is a Gorenstein \( \mathcal{D}_T \)-projective left \( T \)-module.

Next we prove that \( \text{coker}\phi^X \) is a Gorenstein \( \mathcal{D}_T \)-projective left \( T \)-module. There is an exact \( \mathcal{F}_2 \)-acyclic sequence

\[
L : \cdots \rightarrow L^{-1} \xrightarrow{d^{-1}} L^0 \xrightarrow{d^0} L^1 \rightarrow \cdots
\]
of projective left $A$-modules with $\operatorname{coker}\phi^X = X_1/\operatorname{Im}(\phi^X) = \operatorname{Ker}(d^0)$. Then we get the exact sequence of projective left $T$-modules

\[
\begin{array}{ccccccc}
L_0 & : & \cdots & \rightarrow & L_{-1}^{-1} & (d_0^{-1}) & \rightarrow & L_0^0 & (d_0^0) & \rightarrow & L_1^1 & \rightarrow & \cdots
\end{array}
\]

such that $(\operatorname{Coker}\phi^X) = \operatorname{Ker}(d^0)$.

Let $(H_1, H_2) \in \mathfrak{C}_2 \mathfrak{F}_2$. Then there exists an exact sequence

\[
0 \rightarrow \operatorname{Ker}(\overline{\varphi_H}) \rightarrow H_1 \rightarrow \operatorname{Hom}_B(M, H_2) \rightarrow 0
\]

with $H_2 \in \mathfrak{F}_2$ and $\operatorname{Ker}(\overline{\varphi_H}) \in \mathfrak{C}_2$. Since $L$ is a complex consisting of projective modules, we have a short exact sequence of complexes

\[
0 \rightarrow \operatorname{Ker}(\overline{\varphi_H}) \otimes_A L \rightarrow H_1 \otimes_A L \rightarrow \operatorname{Hom}_B(M, H_2) \otimes_A L \rightarrow 0.
\]

By hypotheses, $\operatorname{Hom}_B(M, H_2)$ has finite $\mathfrak{C}_2$-injective dimension. It follows that the complexes $\operatorname{Ker}(\overline{\varphi_H}) \otimes_A L$ and $\operatorname{Hom}_B(M, H_2) \otimes_A L$ are exact. Therefore, the complex $(H_1, H_2) \otimes_T \left( \frac{L}{L} \right) \cong H_1 \otimes_A L$ is exact, so $(\operatorname{Coker}\phi^X)_{0}$ is a Gorenstein $\mathfrak{D}_T$-projective $T$-module.

By [1, Theorem A6], for any ring $R$ and a symmetric duality pair $\mathfrak{D}_R$, the class of Gorenstein $\mathfrak{D}_R$-projective modules and the class of $\mathfrak{D}_R$-Gorenstein projective modules are coincide. We have the following characterization.

**Corollary 3.5.** Assume Setup 3.3 and let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \phi^X$ be a left $T$-module. Suppose that $\operatorname{Hom}_B(M, D)$ has finite $\mathfrak{C}_2$-injective dimension for each $D \in \mathfrak{F}_2$. Then the following conditions are equivalent:

1. $X$ is a $\mathfrak{D}_T$-Gorenstein projective left $T$-module.
2. $X_2$ is a $\mathfrak{D}_B$-Gorenstein projective left $B$-module, $\operatorname{Coker}\phi^X$ is a $\mathfrak{D}_A$-Gorenstein projective left $A$-module, and $\phi^X$ is a monomorphism.

Let $R$ be any ring. The class of all flat left $R$-modules and injective right $R$-modules is denoted by $\mathfrak{I}_R$ and $\mathfrak{R} \mathfrak{F}$, respectively. Using results from [25] it is shown very succinctly in [3, Lemmas 5.5-5.7], that we have a semi-complete duality pair $(\langle (R \mathfrak{F}), \langle \mathfrak{I}_R \rangle \rangle)$ where $(\langle R \mathfrak{F} \rangle)$ is the definable class (meaning it is closed under products, direct limits, and pure submodules) generated by the class of all flat left $R$-modules and $(\langle \mathfrak{I}_R \rangle)$ is the definable class generated by the class of all injective right $R$-modules.

Recently, in order to prove that Gorenstein flat modules are always closed under extensions over any ring, Šaroch and Štovíček [27] introduced the notion of PGF-modules. Recall that a projectively coresolved Gorenstein flat module, or a PGF-module for short, is a syzygy module in an acyclic complex

\[
\cdots \rightarrow P_{-1} \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \cdots
\]

consisting of projective modules which remains exact after tensoring by an arbitrary injective left $R$-module. Denote by $\mathcal{PGF}_R$ the class of PGF-left $R$-modules. Let $\mathfrak{D}_A = (\langle A \mathfrak{F} \rangle, \langle \mathfrak{I}_A \rangle)$ and $\mathfrak{D}_B = (\langle B \mathfrak{F} \rangle, \langle \mathfrak{I}_B \rangle)$, respectively. By [13, Corollary 2.12] or [27, Theorem 3.4], an $R$-module is PGF if and only if it is a Gorenstein $(\langle R \mathfrak{F} \rangle, \langle \mathfrak{I}_R \rangle)$-projective module. Then we have the following result.
Corollary 3.6. Let $X = \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right)_{\tilde{\phi}^X}$ be a left $T$-module. Suppose that $M_B$ has finite injective dimension, $A \mathcal{M}$ has finite flat dimension. Then the following conditions are equivalent:

1. $X$ is a PGF left $T$-module.
2. $X_2$ is a PGF left $B$-module, $\text{Coker} \phi^X$ is a PGF left $A$-module, and $\phi^X$ is a monomorphism.
Moreover, if $M_B$ is finitely presented, then the above conditions are also equivalent to

3. $X$ is a Gorenstein $(\mathfrak{B}_{(n,F)}^{(A,F)}, \mathfrak{T}_{(I_A), (I_B)})$-projective left $T$-module.

Proof. The equivalence of (1) and (2) follows by [29, Theorem 2.8]. If a module $N$ is finitely presented as a right $B$-module and flat as a left $A$-module, then by [20, Lemma 2.2], one can check that the functor $\text{Hom}_B(N, -)$ preserves injectives, products, colimits and pure embedding. Therefore, we obtain that $\text{Hom}_B(N, (I_B)) \subseteq (I_A)$. It follows that if $A \mathcal{M}$ has finite flat dimension, then the $(I_A)$-injective dimension of $\text{Hom}_B(M, E)$ is finite for each module $E \in (I_B)$. Moreover, since $M_B$ has finite injective dimension, $M_B$ has finite $(I_B)$-injective dimension. Therefore, all the assumptions of Setup 3.3 and Theorem 3.4 are satisfied. Thus the equivalence of (2) and (3) follows.

Corollary 3.7. Suppose that $M_B$ is finitely presented and has finite injective dimension, $A \mathcal{M}$ has finite flat dimension. Then the duality pairs $(\mathfrak{B}_{(n,F)}^{(A,F)}, \mathfrak{T}_{(I_A), (I_B)})$ and $(\langle T, F \rangle, \langle I_T \rangle)$ are coincide.

Proof. By [13, Corollary 2.12], a $T$-module is PGF if and only if it is Gorenstein-$\langle (T, F), (I_T) \rangle$ projective. Therefore, Corollary 3.6 tells us that the class of Gorenstein $(\mathfrak{B}_{(n,F)}^{(A,F)}, \mathfrak{T}_{(I_A), (I_B)})$-projective modules and the class of Gorenstein-$\langle (T, F), (I_T) \rangle$ projective modules are the same. Consider the PGF module $T$. It is a syzygy module in complex $\cdots \to 0 \to T \to T \to 0 \to \cdots$ which remains exact after applying functor $X \otimes_T -$ and $Y \otimes_T -$ for any $X \in \mathfrak{T}_{(I_A), (I_B)}$ and $Y \in (I_T)$. This implies that $\mathfrak{T}_{(I_A), (I_B)} = (I_T)$. Thus these two duality pairs are the same.

3.2. Gorenstein $\mathcal{D}_T$-injective modules.

Theorem 3.8. Assume Setup 3.3 and let $X = \left( X_1, X_2 \right)_{\tilde{\phi}^X}$ be a right $T$-module. Suppose that $\text{Hom}_B(M, D)$ has finite $C_2$-injective dimension for each $D \in \mathcal{F}_2$. Then the following conditions are equivalent:

1. $X$ is a Gorenstein $\mathcal{D}_T$-injective right $T$-module.
2. $X_2$ is a Gorenstein $\mathcal{D}_A$-injective right $B$-module, $\text{Ker} \left( \tilde{\phi}^X \right)$ is a Gorenstein $\mathcal{D}_B$-injective right $A$-module, and $\tilde{\phi}^X$ is an epimorphism.

Proof. By [15, Proposition 5.1], a right $T$-module $X = \left( X_1, X_2 \right)_{\tilde{\phi}^X}$ is injective if and only if $X_2$ is an injective right $B$-module, $\text{Ker} \left( \tilde{\phi}^X \right)$ is an injective right $A$-module, and $\tilde{\phi}^X$ is an epimorphism. Then the proof is dual to that of Theorem 3.4.

Recall that a left $R$-module $X$ is $FP$-injective or absolutely pure if $\text{Ext}_R^1(N, X) = 0$ for every finitely presented left $R$-module $N$. Let $\mathcal{F}_R$ be the class of all absolutely pure modules. As in [10], an right $R$-module $M$ Ding injective if there exists an exact complex of injectives

$$
\cdots \to I_1 \to I_0 \to I^0 \to I^1 \to \cdots
$$
with \( M = \text{Ker}(I^0 \rightarrow I^1) \) and which remains exact after applying \( \text{Hom}_R(E, -) \) for any absolutely pure module \( E \). Denote by \( \mathcal{D}_R \) the class of all Ding injective right \( R \)-modules. Let \( \mathcal{D}_A = (\langle A \mathcal{F} \rangle, \langle I_A \rangle) \) and \( \mathcal{D}_B = (\langle B \mathcal{F} \rangle, \langle I_B \rangle) \), respectively. By \([13, Proposition 2.11]\), an \( R \)-module is Ding injective if and only if it is a Gorenstein \((\langle R \mathcal{F} \rangle, \langle I_R \rangle)\)-injective module. Then we have the following result. One can compare it with \([23, Theorem 4.4]\).

**Corollary 3.9.** Let \( X = (X_1, X_2) \) be a right \( T \)-module. Suppose that \( M_B \) is finitely presented and has finite FP-injective dimension, \( _AM \) has finite flat dimension. Then the following conditions are equivalent:

1. \( X \) is a Gorenstein \((\mathfrak{B}^{(A \mathcal{F})}, \langle I_A \rangle, \langle I_B \rangle)\)-injective right \( T \)-module.
2. \( X_2 \) is a Ding injective right \( B \)-module, \( \text{Ker}(\overline{\varphi X}) \) is a Ding injective right \( A \)-module, and \( \overline{\varphi X} \) is an epimorphism.

Moreover, if \( M_B \) has finite injective dimension, then the above conditions are also equivalent to

3. \( X \) is a Ding injective right \( T \)-module.

**Proof.** By the proof of Corollary 4.6, we know that if \( _AM \) has finite flat dimension, then the \( \langle I_A \rangle \)-injective dimension of \( \text{Hom}_B(M, E) \) is finite for each module \( E \in \langle I_B \rangle \). Moreover, since \( M_B \) has finite FP-injective dimension and \( \mathcal{F}I_B \subseteq \langle I_B \rangle \), \( M_B \) has finite \( \langle I_B \rangle \)-injective dimension. Then all the assumptions of Setup 3.3 and Theorem 3.8 are satisfied. Thus the equivalence of (1) and (2) follows. Moreover, if \( M_B \) has finite injective dimension, then the equivalence of (2) and (3) follows by Corollary 3.7. \( \square \)

### 3.3. Gorenstein \( \mathcal{D}_T \)-flat modules.

**Lemma 3.10.** \([13, Proposition 4.6 or Corollary 5.3]\) If \( \mathcal{D}_R = (\mathcal{L}, \mathcal{A}) \) is a semi-complete duality pair, then the class of all Gorenstein \( \mathcal{D}_R \)-flat modules is closed under extensions.

**Theorem 3.11.** Assume Setup 3.3 and let \( X = (X_1, X_2) \) be a left \( T \)-module. Suppose that \( \text{Hom}_B(M, D) \) has finite \( C_2 \)-injective dimension for each \( D \in \mathcal{F}_2 \). Then the following conditions are equivalent:

1. \( X \) is a Gorenstein \( \mathcal{D}_T \)-flat left \( T \)-module.
2. \( X_2 \) is a Gorenstein \( \mathcal{D}_B \)-flat left \( B \)-module, \( \text{Coker} \phi^X \) is a Gorenstein \( \mathcal{D}_A \)-flat left \( A \)-module, and \( \phi^X \) is a monomorphism.

**Proof.** From Lemma 3.10, we know that the class of Gorenstein \( \mathcal{D}_T \)-flat left \( T \)-module is closed under extensions. By \([8, Proposition 1.14]\), we know that \( \left( \begin{array}{c} F_1 \\ F_2 \end{array} \right) \phi^F \) is flat if and only if \( F_2 \) is flat in \( B \text{-Mod} \), \( \text{coker} \phi^F \) is flat in \( A \text{-Mod} \) and \( \phi^F \) is monomorphic. Then the proof follows by an argument similar to that in Theorem 3.4. \( \square \)

Recall that a module \( M \) is called **Gorenstein flat** \([7]\) if there exists an exact sequence

\[
\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots
\]
of flat modules such that $M \cong \ker(F_0 \to F_{-1})$ and $I \otimes -$ leaves the sequence exact whenever $I$ is an injective right module. Let $\mathcal{D}_A = (\langle A_0 \rangle, \langle I_A \rangle)$ and $\mathcal{D}_B = (\langle B_0 \rangle, \langle I_B \rangle)$, respectively. By [13, Proposition 2.11], an $R$-module is Gorenstein flat if and only if it is a Gorenstein $(\langle R_0 \rangle, \langle I_R \rangle)$-flat module. Then we have the following result. One can compare it with [20, Theorem 2.8]

**Corollary 3.12.** Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \otimes \phi_X$ be a left $T$-module. Suppose that $M_B$ has finite injective dimension, $A^M$ has finite flat dimension. Then the following conditions are equivalent:

1. $X$ is a Gorenstein flat left $T$-module.
2. $X_2$ is a Gorenstein flat left $B$-module, $\text{Coker} \phi^X$ is a Gorenstein flat left $A$-module, and $\phi^X$ is a monomorphism.

Moreover, if $M_B$ is finitely presented, then the above conditions are also equivalent to

3. $X$ is a Gorenstein $(\mathcal{B}_R, \langle \Sigma_I \rangle)$-projective left $T$-module.

**Proof.** The equivalence of (1) and (2) follows by the proof of [20, Lemma 2.2] and the fact that Gorenstein flat modules are closed under extensions over any ring. Moreover, if $M_B$ is finitely presented, by an argument similar to Theorem 3.4, we see that all the assumptions of Setup 3.3 and Theorem 3.11 are satisfied. Thus the result follows. \qed

### 4. Recollements of stable categories relative to triangular matrix rings

Let $\mathcal{D}_R = (\mathcal{L}, \mathcal{A})$ denote a semi-complete duality pair over a ring $R$. There are following three model structures induced by Gorenstein modules respect to $\mathcal{D}_R$. A nice introduction to the basic idea of a model category can be found in [18].

**Lemma 4.1.** [13, Corollary 5.1] The following abelian model structures are induced by $\mathcal{D}_R = (\mathcal{L}, \mathcal{A})$.

1. The **Gorenstein $\mathcal{D}_R$-projective model structure** exists on $R$-Mod. It is a cofibrantly generated projective abelian model structure whose cofibrant objects are the Gorenstein $\mathcal{D}_R$-projective left $R$-modules.

2. The **Gorenstein $\mathcal{D}_R$-injective model structure** exists on $\text{Mod-}R$. It is a cofibrantly generated injective abelian model structure whose fibrant objects are the Gorenstein $\mathcal{D}_R$-injective right $R$-modules.

3. The **Gorenstein $\mathcal{D}_R$-flat model structure** exists on $R$-Mod. It is a cofibrantly generated abelian model structure whose cofibrant objects (resp. trivially cofibrant objects) are the Gorenstein $\mathcal{D}_R$-flat left modules (resp. flat left modules). Moreover, the trivial objects in this model structure coincide with those in the Gorenstein $\mathcal{D}_R$-projective model structure.

According to the recollement constructed by [26] and [31], we have the following recollement of abelian categories:

\[
\begin{array}{ccc}
A:\text{-Mod} & \xrightarrow{i^*} & T:\text{-Mod} \\
\downarrow{i_*} & & \downarrow{j_*} \\
B:\text{-Mod} & \xrightarrow{j^*} & \\
\end{array}
\] (4.1)
where \( i^* \) is given by \( (\chi)^\phi \mapsto \text{coker}\phi; \ i_* \) is given by \( X \mapsto (\frac{X}{0}); \ i^! \) is given by \( (\chi)^\phi \mapsto X; \ j_! \) is given by \( Y \mapsto (\phi): \ j^* \) is given by \( (\chi)^\phi \mapsto Y; \ j_* \) is given by \( Y \mapsto (0) \).

If we are given two cofibrantly generated model structures \( \mathcal{M}(A) = (\mathcal{A}, \mathcal{W}, \mathcal{A}) \) and \( \mathcal{M}(B) = (\mathcal{B}, \mathcal{W}, \mathcal{B}) \) on \( \text{A-Mod} \) and \( \text{B-Mod} \) respectively, we investigate in [33] when there exists a cofibrantly generated model structure \( \mathcal{M}(T) \) on \( T-\text{Mod} \) and a recollement of \( \text{Ho}(\mathcal{M}(T)) \) relative to \( \text{Ho}(\mathcal{M}(A)) \) and \( \text{Ho}(\mathcal{M}(B)) \). We defined in [33] that a bimodule \( \mathcal{M}_B \) is perfect relative to \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \), if \( \mathcal{M}_B \) exists a cofibrantly generated abelian model structure on \( \mathcal{M}(T) \) on \( T-\text{Mod} \) and a recollement of \( \text{Ho}(\mathcal{M}(T)) \) relative to \( \text{Ho}(\mathcal{M}(A)) \) and \( \text{Ho}(\mathcal{M}(B)) \). Theorem 5.6], [45], we know that the homotopy category \( \text{Ho}(\mathcal{M}(A)) \) is triangle equivalent to the stable category \( \mathcal{M}_B \) is perfect relative to \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \), of projective left \( \mathcal{A} \) and \( \mathcal{B} \)-modules. The Hovey triple corresponding to the Gorenstein \( \mathcal{A} \)-projective right \( \mathcal{A} \)-modules, respectively. Let \( \mathcal{M}(A) = (\mathcal{A}, \mathcal{W}, \mathcal{A}) \) and \( \mathcal{M}(B) = (\mathcal{B}, \mathcal{W}, \mathcal{B}) \) be two cofibrantly generated model structures on \( \text{A-Mod} \) and \( \text{B-Mod} \) respectively. If \( \text{Tor}^B(M, X) = 0 \) for any \( X \in \mathcal{A} \) and \( M \) is perfect relative to \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \), then \( \mathcal{M}(T) = (\mathcal{M}_A, \mathcal{W}, \mathcal{M}_A) \) is a cofibrantly generated abelian model structure on \( T-\text{Mod} \) and we have a recollement as shown below

\[
\begin{array}{ccc}
\text{Ho}(\mathcal{M}(A)) & \xleftarrow{\mathcal{L} i^*} & \text{Ho}(\mathcal{M}(T)) \\
\mathcal{L} i_* & \cong & \mathcal{R} i^* \\
\mathcal{R} i^! & \cong & \mathcal{L} j_!
\end{array}
\]

where \( \mathcal{L} i^* \), \( \mathcal{L} i_* \), \( \mathcal{L} j^* \), \( \mathcal{R} i^* \), \( \mathcal{R} j_* \), \( \mathcal{R} i^! \) and \( \mathcal{R} j_* \) are the total derived functors of those in (4.1).

Let \( \mathcal{M}_A' = (\mathcal{A}, \mathcal{W}, \mathcal{A}_L) \) and \( \mathcal{M}_B' = (\mathcal{B}, \mathcal{W}, \mathcal{B}) \) be two cofibrantly generated model structures on \( \text{A-Mod} \) and \( \text{B-Mod} \) respectively. We define that a bimodule \( \mathcal{M}_B \) is coperfect relative to \( \mathcal{M}_A' \) and \( \mathcal{M}_B' \), if \( \mathcal{M}_B = (\mathcal{U}_{\mathcal{A}_L, \mathcal{B}} \cap \mathcal{F}_{\mathcal{W}_A \cap \mathcal{L}_A, \mathcal{W}_B} \cap \mathcal{L}_B = \mathcal{U}_{\mathcal{A}_L, \mathcal{W}_A \cap \mathcal{W}_B} \cap \mathcal{F}_{\mathcal{L}_A, \mathcal{L}_B} \) is a cofibrantly generated abelian model structure on \( T-\text{Mod} \) and we have a recollement as shown below

\[
\begin{array}{ccc}
\text{Ho}(\mathcal{M}_A') & \xrightarrow{\mathcal{L} i^*} & \text{Ho}(\mathcal{M}_T) \\
\mathcal{L} i_* & \cong & \mathcal{R} i^* \\
\mathcal{R} i^! & \cong & \mathcal{L} j_!
\end{array}
\]

where \( \mathcal{L} i^* \), \( \mathcal{L} i_* \), \( \mathcal{L} j^* \), \( \mathcal{R} i^* \), \( \mathcal{R} j_* \), \( \mathcal{R} i^! \) and \( \mathcal{R} j_* \) are the total derived functors of those in (4.1).

Let \( \mathcal{G} \mathcal{P}_{D_T} \) and \( \mathcal{G} \mathcal{L}_{D_T} \) denote the class of all Gorenstein \( D_T \)-projective left \( T \)-modules and Gorenstein \( D_T \)-injective right \( T \)-modules, respectively. Let \( R \) be a ring. The Hovey triple corresponding to the Gorenstein \( D_R \)-projective model structure is \( \mathcal{M}_R = (\mathcal{G} \mathcal{P}_{D_R}, \mathcal{R} \mathcal{W}, \mathcal{R} \text{-Mod}) \). By [11, Section 4.2], we know that the homotopy category \( \text{Ho}(\mathcal{M}_R) \) is a triangulated category and it is triangle equivalent to the stable category \( \mathcal{G} \mathcal{P}_{D_R} := \mathcal{G} \mathcal{P}_{D_R}/\mathcal{R} \mathcal{P} \), where \( \mathcal{R} \mathcal{P} \) is the class of projective left \( R \)-modules. The Hovey triple corresponding to the Gorenstein \( D_R \)-injective
model structure is $\mathcal{M}'_R = (\text{Mod}-R, \mathcal{V}_R, \mathcal{G}\mathcal{T}_{D_R})$. By [11, Section 4.2], we know that the homotopy category $\text{Ho}(\mathcal{M}'_R)$ is triangle equivalent to the stable category $\mathcal{G}\mathcal{T}_{D_R} := \mathcal{G}\mathcal{P}_{D_R}/I_R$.

**Lemma 4.4.** (1) Let $M$ be a right $B$-module with finite $F_2$-injective dimension and $G$ be a Gorenstein $D_B$-projective left module. Then $\text{Tor}^B_i(M, G) = 0$ for all $i > 0$.

(2) Let $M$ be a right $B$-module with finite $F_2$-injective dimension and $E$ be a Gorenstein $D_B$-injective right module. Then $\text{Ext}^B_i(M, E) = 0$ for all $i > 0$.

**Proof.** We just prove (1) since (2) follows by a similar way. Denote by $\text{fid}(M)$ the $F_2$-injective dimension of $M_B$. We shall induction on the $F_2$-injective dimension of $M_B$. If $\text{fid}(M) = 0$ the result is trivial. Let $n > 0$ and assume that the result is true for any right $B$-module with $F_2$-injective dimension equal to $n$. Moreover, suppose that $\text{fid}(M) = n + 1$. Then there exists a short exact sequence of right $B$-modules $0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow 0$ with $I_0$ belong to $F_2$ and $\text{fid}(I_1) = n$. From this exact sequence, we obtain the following exact sequence:

$$\text{Tor}^B_{i+1}(I_1, G) \rightarrow \text{Tor}^B_i(M, G) \rightarrow \text{Tor}^B_i(I_0, G)$$

where $G$ is Gorenstein $D_B$-projective. By induction hypothesis, we have $\text{Tor}^B_{i+1}(I_1, G) = \text{Tor}^B_i(I_0, G) = 0$ for all $i > 0$. Hence $\text{Tor}^B_i(M, G) = 0$ for all $i > 0$ and every Gorenstein $D_B$-projective module $G$ if $M$ has finite $F_2$-injective dimension. □

Let $R$ be any ring. Let $\mathcal{G}\mathcal{F}_{D_T}$ denotes the class of all Gorenstein $D_T$-flat modules. Denote by $R\mathcal{C}T := R\mathcal{F}^{-1}$ the class of cotorsion left $R$-modules. The Hovey triple corresponding to the Gorenstein $D_R$-flat model structure is $\mathcal{N}_R = (\mathcal{G}\mathcal{F}_{D_R}, R\mathcal{E}, R\mathcal{C}T)$. According to [9, Proposition 3.1], we see that $\mathcal{G}\mathcal{F}_{D_R} \cap \mathcal{G}\mathcal{F}_{D_R}^{-1} = R\mathcal{F} \cap R\mathcal{C}T$. Its homotopy category $\text{Ho}(\mathcal{N}_A)$ is triangle equivalent to the stable category $\mathcal{G}\mathcal{F}_{D_R} \cap R\mathcal{C}T := (\mathcal{G}\mathcal{F}_{D_R} \cap R\mathcal{C}T)/(R\mathcal{F} \cap R\mathcal{C}T)$.

**Theorem 4.5.** Assume Setup 3.3 and suppose that $\text{Hom}_B(M, D)$ has finite $C_2$-injective dimension for each $D \in \mathcal{F}_2$. Then we have recollements

(1) $\frac{\mathcal{G}\mathcal{P}_{D_A}}{\mathcal{L}^i_1} \frac{\mathcal{L}^i_2}{\mathcal{L}^j_1} \frac{\mathcal{G}\mathcal{P}_{D_T}}{\mathcal{L}^j_2} \frac{\mathcal{G}\mathcal{P}_{D_B}}{\mathcal{L}^i_1}$

(2) $\frac{\mathcal{G}\mathcal{I}_{D_A}}{\mathcal{L}^i_1} \frac{\mathcal{L}^i_2}{\mathcal{L}^j_1} \frac{\mathcal{G}\mathcal{I}_{D_T}}{\mathcal{L}^j_2} \frac{\mathcal{G}\mathcal{I}_{D_B}}{\mathcal{L}^i_1}$

(3) $\frac{\mathcal{G}\mathcal{F}_{D_A} \cap A\mathcal{C}T}{\mathcal{L}^i_1} \frac{\mathcal{G}\mathcal{F}_{D_T} \cap T\mathcal{C}T}{\mathcal{L}^j_1} \frac{\mathcal{G}\mathcal{F}_{D_B} \cap B\mathcal{C}T}{\mathcal{L}^i_1}$

**Proof.** From Lemma 4.1, there are abelian model structures $\mathcal{M}(A) = (\mathcal{G}\mathcal{P}_{D_A}, \mathcal{A}\mathcal{W}, \text{A-Mod})$ and $\mathcal{M}(B) = (\mathcal{G}\mathcal{P}_{D_B}, \mathcal{B}\mathcal{W}, \text{B-Mod})$ on $\text{A-Mod}$ and $\text{B-Mod}$, respectively. By Lemma 4.4 we have
\[ \text{Tor}_B^1(M, E) = 0 \text{ for any } E \in \mathcal{GP}_D. \]

By Theorem 3.4 and the fact that the abelian module structure \( \mathcal{M}(R) = (\mathcal{GP}_D, R^W, R{-}\text{Mod}) \) is projective for any ring, we have
\[ \mathcal{O}_{\mathcal{GP}_D A} \cap \mathcal{U}_{\mathcal{GP}_D B} = \mathcal{GP}_D T \cap \mathcal{T}^P = \mathcal{O}_{\mathcal{GP}_D B} \cap \mathcal{U}_{\mathcal{GP}_D B}{-}\text{Mod}. \]

It follows that the bimodule \( M \) is perfect relative to \( \mathcal{M}(A) \) and \( \mathcal{M}(B) \). Thus the first recollement follow from Lemma 4.2. On the other hand, by Lemma 4.1, there are abelian model structures \( \mathcal{M}'_A = (\text{Mod}-A, \mathcal{V}_A, \mathcal{GI}_D A) \) and \( \mathcal{M}'_B = (\text{Mod}-B, \mathcal{V}_B, \mathcal{GI}_D B) \) on \( \text{Mod}-A \) and \( \text{Mod}-B \), respectively. By Lemma 4.4 we have \( \text{Ext}_A^i(M, E) = 0 \) for any \( E \in \mathcal{GI}_D A \). By Theorem 3.8 and the fact that the abelian module structure \( \mathcal{M}'_R = (\text{Mod}-R, \mathcal{V}_R, \mathcal{GI}_D R) \) is injective for any ring, we have
\[ \mathcal{U}_{\text{Mod}-A, \text{Mod}-B} \cap \mathcal{I}_{A, \text{Mod}-A} = \mathcal{I}_T = \mathcal{V}_T \cap \mathcal{GI}_D T = \mathcal{U}_{\mathcal{V}_A, \mathcal{V}_B} \cap \mathcal{I}_{\mathcal{GI}_D A, \mathcal{GI}_D B}. \]

It follows that the bimodule \( M \) is coperfect relative to \( \mathcal{M}'_A \) and \( \mathcal{M}'_B \). Thus the first recollement follow from Lemma 4.4. Finally, by Lemma 4.1, there are abelian model structures \( \mathcal{N}(A) = (\mathcal{GF}_D A, A^E, \mathcal{CT}_A) \) and \( \mathcal{N}(B) = (\mathcal{GF}_D B, B^E, \mathcal{CT}_B) \) on \( \text{A{-}Mod} \) and \( \text{B{-}Mod} \), respectively. By the same proof of Lemma 4.4, we see that \( \text{Tor}_B^1(M, E) = 0 \) for any \( E \in \mathcal{GF}_D B \). By Theorem 3.11, we have
\[ \mathcal{B}_{\mathcal{GF}_D A} \cap \mathcal{U}_{A^E, A^E} = \mathcal{GF}_D T \cap \mathcal{GF}_D T = \mathcal{B}_{B^E} \cap \mathcal{CT}_B. \]

It follows that the bimodule \( M \) is perfect relative to \( \mathcal{N}(A) \) and \( \mathcal{N}(B) \). Thus the second recollement follow from Lemma 4.2.

Finally, we give some applications of Theorem 4.5 for PGF, Ding injective and Gorenstein flat model structures, respectively. One can compare it with [28, Theorem 2.10] and [33, Theorem 4.12].

**Corollary 4.6.** Suppose that \( M_B \) is finitely presented and has finite injective dimension, \( _A M \) has finite flat dimension. Then we have recollements

1. \[ \mathcal{PGF}_A \xrightarrow{L_1} \mathcal{PGF}_T \xrightarrow{L_2} \mathcal{PGF}_B ; \]
2. \[ \mathcal{DL}_A \xrightarrow{L_1} \mathcal{DL}_T \xrightarrow{L_2} \mathcal{DL}_B ; \]
3. \[ \mathcal{GF}_A \cap _A \mathcal{CT} \xrightarrow{L_1} \mathcal{GF}_T \cap _T \mathcal{CT} \xrightarrow{L_2} \mathcal{GF}_B \cap _B \mathcal{CT} . \]

**Proof.** It follows by Theorem 4.5, Corollaries 3.6, 3.9 and 3.12. \( \square \)
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