UNBOUNDED TOPOLOGIES AND UO-CONVERGENCE IN LOCALLY SOLID VECTOR LATTICES

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Abstract. Suppose $X$ is a vector lattice and there is a notion of convergence $x_\alpha \sigma \rightarrow x$ in $X$. Then we can speak of an "unbounded" version of this convergence by saying that $x_\alpha \omega u \rightarrow x$ if $|x_\alpha - x| \wedge u \sigma \rightarrow 0$ for every $u \in X_+$. In the literature, the unbounded versions of the norm, order and absolute weak convergence have been studied. Here we create a general theory of unbounded convergence but with a focus on $uo$-convergence and those convergences deriving from locally solid topologies.

1. Preliminaries

The definition of order convergent nets in a vector lattice has been a topic of some disagreement. The two most common definitions are as follows:

Definition 1.1. A net $(x_\alpha)_{\alpha \in A}$ is order convergent to $x$ if there exists a net $(y_\alpha)_{\alpha \in A}$ such that $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all $\alpha \in A$. We will denote this convergence by $x_\alpha \circ 1 \rightarrow x$.

Definition 1.2. A net $(x_\alpha)_{\alpha \in A}$ is order convergent to $x$ if there exists a net $(y_\beta)_{\beta \in B}$ such that $y_\beta \downarrow 0$ and for each $\beta \in B$ there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq y_\beta$ for all $\alpha \geq \alpha_0$. We will denote this convergence by $x_\alpha \circ 0 \rightarrow x$.

Clearly,

(1) $x_\alpha \circ 0 \rightarrow x \Rightarrow x_\alpha \circ 1 \rightarrow x$.
Definition 1.2 is the standard definition used in most papers on $uo$-convergence, and will be used here. Definition 1.1 is also very common and appears in, for example, [2], [3] and [4]. Since we will be studying $uo$-convergence, but our primary reference will be [3], some care will have to be taken. Keeping in mind the discrepancy in the definition of order convergence, the reader is referred to [3] for all undefined terms. Throughout this paper, all vector lattices are assumed Archimedean.

Recall that a net $(x_\alpha)$ in a vector lattice $X$ is **unbounded o-convergent** (or **$uo$-convergent**) to $x \in X$ if $|x_\alpha - x| \wedge u \to 0$ for all $u \in X_+$. We will write $x_\alpha \to uo x$ to denote this convergence. Occasionally, we will need to consider $uo_1$-convergence which is defined analogously but with the notion of order convergence in Definition 1.2 replaced with that of Definition 1.1.

## 2. Basic results on unbounded locally solid topologies

**Definition 2.1.** Suppose that $X$ is a vector lattice and $\tau$ is a (not necessarily Hausdorff) linear topology on $X$. We say that a net $(x_\alpha) \subseteq X$ is **unbounded $\tau$-convergent** to $x \in X$ if $|x_\alpha - x| \wedge u \to 0$ for all $u \in X_+$. We will write $x_\alpha \to u_\tau x$ to denote this convergence. Occasionally, we will need to consider $u_\tau_1$-convergence which is defined analogously but with the notion of order convergence in Definition 1.2 replaced with that of Definition 1.1.

**Definition 2.2.** A (not necessarily Hausdorff) topology $\tau$ on a vector lattice $X$ is said to be **locally solid** if it is linear and has a base at zero consisting of solid sets.

The next theorem justifies the interest in unbounded convergences deriving from locally solid topologies.

**Theorem 2.3.** If $\tau$ is a locally solid topology on a vector lattice $X$ then the unbounded $\tau$-convergence is also a topological convergence on $X$. Moreover, the corresponding topology, $u_\tau$, is locally solid. It is Hausdorff if and only if $\tau$ is.

**Proof.** Since $\tau$ is locally solid it has a base $\{U_i\}_{i \in I}$ at zero consisting of solid neighbourhoods. For each $i \in I$ and $u \in X_+$ define $U_{i,u} := \{x \in X : |x| \wedge u \in U_i\}$. We claim that the collection $\mathcal{N}_0 := \{U_{i,u}\}$ is a base of neighbourhoods of zero for a topology $u_\tau$. Notice that $(x_\alpha)$ unbounded
$\tau$-converges to $x$ iff every set in $N_0$ contains a tail of the net. This means the unbounded $\tau$-convergence is exactly the convergence given by this topology. Notice also that $U_i \subseteq U_{i,u}$ and, since $U_i$ is solid, so is $U_{i,u}$.

We now verify that $N_0$ is a base at zero. Trivially, every set in $N_0$ contains 0. We now show that the intersection of any two sets in $N_0$ contains another set in $N_0$. Take $U_{i,u_1}, U_{j,u_2} \in N_0$. Then $U_{i,u_1} \cap U_{j,u_2} = \{x \in X : |x| \wedge u_1 \in U_i \land |x| \wedge u_2 \in U_j\}$. Since $\{U_i\}$ is a base we can find $k \in I$ such that $U_k \subseteq U_i \cap U_j$. We claim that $U_{k,u_1,u_2} \subseteq U_{i,u_1} \cap U_{j,u_2}$. Indeed, if $x \in U_{k,u_1,u_2}$, then $|x| \wedge (u_1 \lor u_2) \in U_k \subseteq U_i \cap U_j$. Therefore, since $|x| \wedge u_1 \leq |x| \wedge (u_1 \lor u_2) \in U_i \cap U_j \subseteq U_i$ and $U_i$ is solid, we have $x \in U_{i,u_1}$. Similarly, $x \in U_{j,u_2}$.

We know that for every $i$ there exists $j$ such that $U_j + U_j \subseteq U_i$. From this we deduce that for all $i$ and all $u$, if $x, y \in U_{j,u}$ then

$$\text{(2)} \quad |x + y| \wedge u \leq |x| \wedge u + |y| \wedge u \in U_j + U_j \subseteq U_i$$

so that $U_{j,u} + U_{j,u} \subseteq U_{i,u}$.

If $|\lambda| \leq 1$ then $\lambda U_{i,u} \subseteq U_{i,u}$ because $U_{i,u}$ is solid. It follows from $U_i \subseteq U_{i,u}$ that $U_{i,u}$ is absorbing.

To conclude that $u\tau$ is a locally solid linear topology we must show that if $U_{i,u} \in N_0$ and $y \in U_{i,u}$ then $y + V \subseteq U_{i,u}$ for some $V \in N_0$. But $y \in U_{i,u}$ means that $|y| \wedge u \in U_i$ and since $\{U_i\}$ is a base there exists $j$ such that $|y| \wedge u + U_j \subseteq U_i$. We claim that $V = U_{j,u}$ does the trick. Indeed, if $x \in U_{j,u}$ then $|y + x| \wedge u \leq |y| \wedge u + |x| \wedge u \in |y| \wedge u + U_j \subseteq U_i$, and therefore $y + x \in U_{i,u}$.

Suppose further that $\tau$ is Hausdorff; we will verify that $\bigcap N_0 = \{0\}$. Indeed, suppose that $x \in U_{i,u}$ for all $i \in I$ and $u \in X_+$. In particular, $x \in U_{i,|x|}$ which means that $|x| \in U_i$ for all $i \in I$. Since $\tau$ is Hausdorff, $\bigcap U_i = \{0\}$ and we conclude that $x = 0$. 
Finally, if $u\tau$ is Hausdorff then $\tau$ is Hausdorff since $U_i \subseteq U_{i,u}$. □

**Remark 2.4.** If $\tau$ is the norm topology on a Banach lattice $X$, the corresponding $u\tau$-topology is called un-topology; it has been studied in [9], [14] and [15]. It is easy to see that the weak and absolute weak topologies on $X$ generate the same unbounded convergence and, since the absolute weak topology is locally solid, this convergence is topological. It has been denoted $uaw$ and was studied in [17].

From now on, unless explicitly stated otherwise, throughout this paper the minimum assumption is that $X$ is an Archimedean vector lattice and topologies on $X$ are locally solid. The following standard result should be noted. It justifies the name unbounded $\tau$-convergence.

**Proposition 2.5.** If $x_{\alpha} \overset{\tau}{\rightarrow} 0$ then $x_{\alpha} \overset{u\tau}{\rightarrow} 0$. For order bounded nets the convergences agree.

**Remark 2.6.** An important remark is that $uu\tau = u\tau$, so there are no chains of unbounded topologies. To see this note that $x_{\alpha} \overset{uu\tau}{\rightarrow} x$ means that for any $u \in X_+$, $|x_{\alpha} - x| \wedge u \overset{u\tau}{\rightarrow} 0$. Since the net $\langle |x_{\alpha} - x| \wedge u \rangle$ is order bounded, this is the same as $|x_{\alpha} - x| \wedge u \overset{\tau}{\rightarrow} 0$, which means $x_{\alpha} \overset{u\tau}{\rightarrow} x$. In another language, the map $\tau \mapsto u\tau$ from the space of locally solid topologies on $X$ to itself is idempotent. We give a name to the fixed points or, equivalently, the range, of this map:

**Definition 2.7.** A locally solid topology $\tau$ is **unbounded** if $\tau = u\tau$ or, equivalently, if $\tau = u\sigma$ for some locally solid topology $\sigma$.

We next present a few easy corollaries of Theorem 2.3 for use later in the paper.

**Corollary 2.8.** Lattice operations are uniformly continuous with respect to $u\tau$ and $u\tau$-closures of solid sets are solid.

**Proof.** The result follows immediately from Theorem 8.41 and Lemma 8.42 in [2]. □

In the next corollary, the Archimedean property is not assumed. Statement (iii) states that, under very mild topological assumptions, it is satisfied automatically.
Corollary 2.9. Suppose \( \tau \) is both locally solid and Hausdorff then:

(i) The positive cone \( X_+ \) is \( u\tau \)-closed;
(ii) If \( x_\alpha \uparrow \) and \( x_\alpha \xrightarrow{u\tau} x \), then \( x_\alpha \uparrow x \);
(iii) \( X \) is Archimedean;
(iv) Every band in \( X \) is \( u\tau \)-closed.

Proof. The result follows immediately from Theorem 8.43 in [2]. \( \square \)

We next work towards a version of Proposition 3.15 in [11] that is applicable to locally solid topologies. The proposition is recalled here along with a definition.

Proposition 2.10. Let \( X \) be a vector lattice, and \( Y \) a sublattice of \( X \). Then \( Y \) is \( uo \)-closed in \( X \) if and only if it is \( o \)-closed in \( X \).

Definition 2.11. A locally solid topology \( \tau \) on a vector lattice is said to be Lebesgue (or order continuous) whenever \( x_\alpha \xrightarrow{o} 0 \) implies \( x_\alpha \xrightarrow{\tau} 0 \).

Note that the Lebesgue property is independent of the definition of order convergence because, as is easily seen, no matter which definition of order convergence is used, it is equivalent to the property that \( x_\alpha \xrightarrow{\tau} 0 \) whenever \( x_\alpha \downarrow 0 \).

Proposition 2.12. Let \( X \) be a vector lattice, \( \tau \) a Hausdorff Lebesgue topology on \( X \) and \( Y \) a sublattice of \( X \). \( Y \) is \( u\tau \)-closed in \( X \) if and only if it is \( \tau \)-closed in \( X \).

Proof. If \( Y \) is \( u\tau \)-closed in \( X \) it is clearly \( \tau \)-closed in \( X \). Suppose now that \( Y \) is \( \tau \)-closed in \( X \) and let \( (y_\alpha) \) be a net in \( Y \) that \( u\tau \)-converges in \( X \) to some \( x \in X \). Since lattice operations are \( u\tau \)-continuous we have that \( y_\alpha \xrightarrow{u\tau} x_\pm \) in \( X \). Thus, WLOG, we may assume that \( (y_\alpha) \subseteq Y_+ \) and \( x \in X_+ \). Observe that for every \( z \in X_+ \),

\[
|y_\alpha \land z - x \land z| \leq |y_\alpha - x| \land z \xrightarrow{\tau} 0. \tag{3}
\]

In particular, for any \( y \in Y_+ \), \( y_\alpha \land y \xrightarrow{\tau} x \land y \). Since \( Y \) is \( \tau \)-closed, \( x \land y \in Y \) for any \( y \in Y_+ \). On the other hand, given any \( 0 \leq z \in Y^d \), we have \( y_\alpha \land z = 0 \) for all \( \alpha \) so that \( x \land z = 0 \) by (3) and the assumption that \( \tau \) is Hausdorff. Therefore, \( x \in Y^{dd} \), which is the band generated
by $Y$ in $X$. It follows that there is a net $(z_{\beta})$ in the ideal generated by $Y$ such that $0 \leq z_{\beta} \uparrow x$ in $X$. Furthermore, for every $\beta$ there exists $w_{\beta} \in Y$ such that $z_{\beta} \leq w_{\beta}$. Then $x \geq w_{\beta} \land x \geq z_{\beta} \land x = z_{\beta} \uparrow x$ in $X$, and so $w_{\beta} \land x \overset{\tau}{\rightarrow} x$ in $X$. By order continuity, $w_{\beta} \land x \overset{\tau}{\rightarrow} x$ in $X$. Since $w_{\beta} \land x \in Y$ and $Y$ is $\tau$-closed, we get $x \in Y$. \hfill \Box$

Proposition 2.12 will be used to prove a much deeper statement: see Theorem 5.10.

We next proceed to generalize many results from the aforementioned papers on $un$-convergence.

**Definition 2.13.** A locally solid topology $\tau$ on a vector lattice is said to be $uo$-**Lebesgue** (or **unbounded order continuous**) whenever $x_{\alpha} \overset{uo}{\rightarrow} 0$ implies $x_{\alpha} \overset{\tau}{\rightarrow} 0$.

It is clear that the $uo$-Lebesgue property implies the Lebesgue property but not conversely:

**Example 2.14.** The norm topology of $c_0$ is order continuous but not unbounded order continuous.

**Proposition 2.15.** If $\tau$ is Lebesgue then $u\tau$ is $uo$-Lebesgue. In particular, $u\tau$ is Lebesgue.

*Proof.* Suppose $x_{\alpha} \overset{uo}{\rightarrow} x$, i.e., $\forall u \in X_+, |x_{\alpha} - x| \land u \overset{uo}{\rightarrow} 0$. The Lebesgue property implies that $|x_{\alpha} - x| \land u \overset{\tau}{\rightarrow} 0$ so that $x_{\alpha} \overset{u\tau}{\rightarrow} x$. \hfill \Box

For a partial converse see Corollary 4.5.

There is much more to say about the $uo$-Lebesgue property and, in fact, a whole section on it. We continue now with more easy observations.

Next we present Lemmas 2.1 and 2.2 of [15] which carry over with minor modification. These lemmas correspond to lemmas 8 and 9 in [17].

**Lemma 2.16.** Let $(X, \tau)$ be a locally solid vector lattice, $u \in X_+$ and $U$ a solid neighbourhood of zero for $\tau$. Then $U_u := \{ x \in X : |x| \land u \in U \}$ is either contained in $[-u, u]$ or contains a non-trivial ideal.
Proof. Modify Lemma 2.1 in [15].

Lemma 2.17. If $U_u$ is contained in $[-u, u]$ then $u$ is a strong unit.

Proof. Modify Lemma 2.2 in [15].

Next we present a trivialized version of Theorem 2.3 in [15].

Proposition 2.18. Let $(X, \tau)$ be a locally solid vector lattice and suppose that $\tau$ has a neighbourhood $U$ of zero containing no non-trivial ideal. If there is a $u\tau$-neighbourhood contained in $U$ then $X$ has a strong unit.

Proof. Let $\{U_i\}$ be a solid base at zero for $\tau$ and suppose there exists $i$ and $u > 0$ s.t. $U_{i,u} \subseteq U$. We conclude that $U_{i,u}$ contains no non-trivial ideal and, therefore, $u$ is a strong unit.

In particular, if $\tau$ is unbounded and $X$ does not admit a strong unit then every neighbourhood of zero for $\tau$ contains a non-trivial ideal.

Definition 2.19. A subset $A$ of a locally solid vector lattice $(X, \tau)$ is $\tau$-almost order bounded if for every solid $\tau$-neighbourhood $U$ of zero there exists $u \in X_+$ with $A \subseteq [-u, u] + U$.

It is easily seen that for solid $U$, $x \in [-u, u] + U$ is equivalent to $(|x| - u) \in U$. The proof is the same as the norm case. This leads to a generalization of Lemma 2.9 in [9].

Proposition 2.20. If $x_\alpha \overset{ur}{\rightarrow} x$ and $(x_\alpha)$ is $\tau$-almost order bounded then $x_\alpha \overset{\tau}{\rightarrow} x$.

Proof. Easy modification that is left to the reader.

In a similar vein, the following can easily be proved; just follow the proof of Proposition 3.7 in [12] or notice it is an immediate corollary of Proposition 2.20.

Proposition 2.21. Let $(X, \tau)$ be an order continuous locally solid vector lattice. If $(x_\alpha)$ is $\tau$-almost order bounded and $u\circ$-converges to $x$, then $(x_\alpha)$ $\tau$-converges to $x$.

One direction of [9] Theorem 4.4 can also be generalized. The proof is, again, easy and left to the reader.
Proposition 2.22. Let \((x_n)\) be a sequence in \((X, \tau)\) and assume \(\tau\) is Lebesgue. If every subsequence of \((x_n)\) has a further subsequence which is \(uo\)-null then \((x_n)\) is \(u\tau\)-null.

Recall that a net \((x_\alpha)\) in a vector lattice \(X\) is \(uo\)-Cauchy if the net \((x_\alpha - x_\alpha')_{(\alpha, \alpha')}\) \(uo\)-converges to zero. \(X\) is \(uo\)-complete if every \(uo\)-Cauchy net is \(uo\)-convergent. A study of \(uo\)-complete spaces was undertaken in [6]. A weaker property involving norm boundedness was introduced in [11]. Here is a generalization of both definitions to locally solid vector lattices.

Definition 2.23. A locally solid vector lattice \((X, \tau)\) is \textit{boundedly \(uo\)-complete} (respectively, \textit{sequentially boundedly \(uo\)-complete}) if every \(\tau\)-bounded \(uo\)-Cauchy net (respectively, sequence) is \(uo\)-convergent.

Proposition 2.24. Let \((X, \tau)\) be a locally solid vector lattice. If \((X, \tau)\) is boundedly \(uo\)-complete then it is order complete. If \((X, \tau)\) is sequentially boundedly \(uo\)-complete then it is \(\sigma\)-order complete.

\[\text{Proof.}\] By [3] Theorem 2.19, the order bounded subsets of \(X\) are \(\tau\)-bounded. The rest of the argument is exactly as given in [11] Proposition 5.7. \(\square\)

Notice that a vector lattice \(X\) is \(uo\)-complete if and only if \(X\) equipped with the trivial topology (which is locally solid) is boundedly \(uo\)-complete. Thus, this is a more general concept than both \(uo\)-complete vector lattices and boundedly \(uo\)-complete Banach lattices. Notice also that the order completeness assumption in [6] Proposition 2.8 may now be dropped.

One should compare the definition of boundedly \(uo\)-complete spaces with the Levi property.\(^1\) They are very similar. A locally solid vector lattice \((X, \tau)\) is said to satisfy the \textbf{Levi property} if every increasing \(\tau\)-bounded net of \(X_+\) has a supremum in \(X\). Notice that the Levi property is not, however, equivalent to every \(\tau\)-bounded order Cauchy net being order convergent. This can easily be seen by letting \(\tau\) be

\[^1\text{For Banach lattices, this property is referred to as monotonically complete in [10].}\]
the trivial topology on $\mathbb{R}$. The Levi and Fatou properties together are enough to ensure that a space is boundedly $uo$-complete. The formal statement is Theorem 8.11.

**Remark 2.25.** It should be noted that if $\tau$ is Hausdorff then every $\tau$-convergent $uo$-Cauchy net $uo$-converges to its $\tau$-limit. This follows since lattice operations are $\tau$-continuous and the positive cone is $\tau$-closed: see [3] Theorem 2.21. The following is a slight generalization of Proposition 4.2 in [12]. The proof is similar but is provided for convenience of the reader.

**Proposition 2.26.** Suppose that $\tau$ is a complete Hausdorff Lebesgue topology on a vector lattice $X$. If $(x_\alpha)$ is a $\tau$-almost order bounded $uo$-Cauchy net in $X$ then $(x_\alpha)$ converges $uo$ and $\tau$ to the same limit.

**Proof.** Suppose $(x_\alpha)$ is $\tau$-almost order bounded and $uo$-Cauchy. Then the net $(x_\alpha - x_{\alpha'})$ is $\tau$-almost order bounded and is $uo$-convergent to zero. By Proposition 2.21, $(x_\alpha - x_{\alpha'})$ is $\tau$-null. It follows that $(x_\alpha)$ is $\tau$-Cauchy and thus $\tau$-convergent to some $x \in X$ since $\tau$ is complete. By Remark 2.25, $(x_\alpha)$ $uo$-converges to $x$. \qed

3. Products, Quotients and Sublattices

3.1. **Products.** Let $\{(X_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a family of locally solid vector lattices and let $X = \prod X_\alpha$ be the Cartesian product, ordered componentwise, and equipped with the product topology $\prod \tau_\alpha$. It is known that $X$ has the structure of a locally solid vector lattice. See [3] pages 8 and 56 for details.

**Theorem 3.1.** Let $\{(X_\alpha, \tau_\alpha)\}$ be a family of locally solid vector lattices. Then $(\prod X_\alpha, u \prod \tau_\alpha) = (\prod X_\alpha, \prod u \tau_\alpha)$.

**Proof.** Let $\{U_\alpha^i\}_{i \in I_\alpha}$ be a solid base for $(X_\alpha, \tau_\alpha)$ at zero. We know the following:

$\{U_{i,u}^\alpha\}_{i \in I_\alpha, u \in X_\alpha^+}$ is a solid base for $(X_\alpha, u \tau_\alpha)$ at zero where $U_{i,u}^\alpha = \{x \in X_\alpha : |x| \wedge u \in U_i^\alpha\}$. 
A solid base of \((\prod X_\alpha, \prod u_\tau)\) at zero consists of sets of the form \(\prod U_\alpha\) where \(U_\alpha = X_\alpha\) for all but finitely many \(\alpha\) and if \(U_\alpha \neq X_\alpha\) for some \(\alpha\) then \(U_\alpha = U_{i_\alpha}^\alpha\) for some \(i \in I_\alpha\) and \(u \in X_{\alpha+}\).

A solid base for \((\prod X_\alpha, \prod \tau_\alpha)\) at zero consists of sets of the form \(\prod V_\alpha\) where \(V_\alpha = X_\alpha\) for all but finitely many \(\alpha\) and if \(V_\alpha \neq X_\alpha\) for some \(\alpha\) then \(V_\alpha = U_{i_\alpha}^\alpha\) for some \(i \in I_\alpha\) and \(u \in X_{\alpha+}\). Therefore, a solid base for \((\prod X_\alpha, u\prod \tau_\alpha)\) at zero consists of sets of the form \((\prod V_\alpha)^w\) where \(w = (w_\alpha) \in (\prod X_\alpha)^+ = \prod X_{\alpha+}\) and \((\prod V_\alpha)^w = \{x = (x_\alpha) \in \prod X_\alpha : |(x_\alpha)\wedge (w_\alpha) = (|x_\alpha|\wedge w_\alpha) \in \prod V_\alpha\}\). Here we used the fact that lattice operations are componentwise.

The theorem follows easily from this. Consider a set \(\prod U_\alpha\) and assume \(U_\alpha = X_\alpha\) except for the indices \(\alpha_1, \ldots, \alpha_n\) where \(U_{i_\alpha}^\alpha_j = U_{i_{\alpha_j}^\alpha}^\alpha_j\) for \(j \in \{1, \ldots, n\}\), \(i_j \in I_{\alpha_j}\) and \(u_j \in X_{\alpha_j+}\). Then \(\prod U_\alpha = (\prod V_\alpha)^w\) where \(w_{\alpha_j} = u_j\) for \(j = 1, \ldots, n\) and \(w_\alpha = 0\) otherwise and \(V_{i_\alpha}^\alpha_j = U_{i_\alpha}^\alpha_j\) for \(j = 1, \ldots, n\) and \(V_\alpha = X_\alpha\) otherwise.

Conversely, consider a set of the form \((\prod V_\alpha)^w\) and assume \(V_\alpha = X_\alpha\) except for the indices \(\alpha_1, \ldots, \alpha_n\) in which case \(V_{i_\alpha}^\alpha_j = U_{i_\alpha}^\alpha_j\) for \(j \in \{1, \ldots, n\}\) and \(i_j \in I_{\alpha_j}\). Then \((\prod V_\alpha)^w = \prod U_\alpha\) where \(U_{i_\alpha}^\alpha_j = U_{i_\alpha}^\alpha_j\) for \(j \in \{1, \ldots, n\}\) and \(U_\alpha = X_\alpha\) otherwise.

Since each base is contained in the other, the topologies agree. \(\Box\)

### 3.2. Quotients.

For notational purposes, we now recall Theorem 2.24 in [3]:

**Theorem 3.2.** Let \((X, \tau)\) be a locally solid vector lattice and let \(Q\) be a lattice homomorphism from \(X\) onto a vector lattice \(Y\). Then \(Y\) equipped with the quotient topology \(\tau_Q\) is a locally solid vector lattice. Moreover, if \(\{U_i\}_{i \in I}\) is a solid base at zero for \(\tau\) then \(\{Q(U_i) : i \in I\}\) is a solid base at zero for \(\tau_Q\).

**Remark 3.3.** A particular case of the theorem tells us that when \(I\) is an ideal of \(X\), \((X/I, \tau/I)\) is a locally solid vector lattice. It is possible
that $X/I$ fails to be Archimedean, but we are only interested in the case when it is.

We next investigate how $u\tau_Q$ and $(u\tau)_Q$ relate and prove that one inclusion holds in general.

**Lemma 3.4.** With the notation as in Theorem 3.2, $u\tau_Q \subseteq (u\tau)_Q$ as topologies on $Y$.

**Proof.** We begin by listing the relevant bases:

- Solid base of $\tau$ at zero: $\{U_i\}_{i \in I}$;
- Solid base of $u\tau$ at zero: $\{U_{i,u}\}_{i \in I, u \in X_+}$ where $U_{i,u} := \{x \in X : |x| \wedge u \in U_i\}$;
- Solid base of $(u\tau)_Q$ at zero: $\{Q(U_{i,u})\}_{i \in I, u \in X_+}$ where $Q(U_{i,u}) := \{Qx : x \in U_{i,u}\}$;
- Solid base of $u\tau_Q$ at zero: $\{Q(U_i)_z\}_{i \in I, z \in Y_+}$ where $Q(U_i)_z := \{y \in Y : |y| \wedge z \in Q(U_i)\}$.

Let $Q(U_i)_z$ with $i \in I$ and $z \in Y_+$ be a fixed solid base neighbourhood of zero in the $u\tau_Q$ topology. We must find a $j \in I$ and $u \in X_+$ such that $Q(U_j) \subseteq Q(U_i)_z$. Take $j = i$ and $u$ any vector such that $Q(u) = z$. $u$ exists because $Q$ is surjective and, WLOG, $u$ is positive since, if not, replace $u$ with $|u|$ and notice that $Q(|u|) = |Q(u)| = |z| = z$. The claim is that $Q(U_{i,u}) \subseteq Q(U_i)_z$.

Let $x \in Q(U_{i,u})$; then $x = Qw$ where $|w| \wedge u \in U_i$. We wish to show $x \in Q(U_i)_z$ so we need only verify that $|x| \wedge z \in Q(U_i)$. This follows since

$$|x| \wedge z = |Qw| \wedge Qu = Q(|w| \wedge u) \in Q(U_i)$$

and that completes the proof. \qed

3.3. **Sublattices.** Let $Y$ be a sublattice of a locally solid vector lattice $(X, \tau)$. The reader should convince themselves that $Y$, equipped with the subspace topology, $\tau|_Y$, is a locally solid vector lattice in its own right. It would be natural to now compare $u\tau|_Y$ and $(u\tau)|_Y$, but this was already implicitly done in [15]. In general, $u\tau|_Y \nsubseteq (u\tau)|_Y$, even if $Y$ is an ideal. If $(y_\alpha)$ is a net in $Y$ we will write $y_\alpha \overset{u\tau}{\to} 0$ in $Y$ to
mean $y_\alpha \to 0$ in $(Y, u\tau|_Y)$. We now look for conditions that make all convergences agree.

**Lemma 3.5.** Let $Y$ be a sublattice of a locally solid vector lattice $(X, \tau)$ and $(y_\alpha)$ a net in $Y$ such that $y_\alpha \overset{u\tau}{\to} 0$ in $Y$. Each of the following conditions implies that $y_\alpha \overset{u\tau}{\to} 0$ in $X$.

1. $Y$ is majorizing in $X$;
2. $Y$ is $\tau$-dense in $X$;
3. $Y$ is a projection band in $X$.

**Proof.** WLOG, $y_\alpha \geq 0$ for every $\alpha$. (i) gives no trouble. To prove (ii), take $u \in X_+$ and fix solid $\tau$-neighbourhoods $U$ and $V$ of zero (in $X$) with $V + V \subseteq U$. Since $Y$ is dense in $X$ we can find a $v \in Y$ with $v - u \in V$. WLOG, $v \in Y_+$ since $V$ is solid and $|v - u| = |v| - |u| \leq |v - u| \in V$. By assumption, $y_\alpha \wedge v \overset{\tau}{\to} 0$ so we can find $\alpha_0$ such that $y_\alpha \wedge v \in V$ whenever $\alpha \geq \alpha_0$. It follows from $u \leq v + |u - v|$ that $y_\alpha \wedge u \leq y_\alpha \wedge v + |u - v|$. This implies that $y_\alpha \wedge u \in U$ for all $\alpha \geq \alpha_0$ since

$$0 \leq y_\alpha \wedge u \leq y_\alpha \wedge v + |u - v| \in V + V \subseteq U$$

where, again, we used that $U$ and $V$ are solid. This means that $y_\alpha \wedge u \overset{\tau}{\to} 0$. Hence, $y_\alpha \overset{u\tau}{\to} 0$ in $X$.

To prove (iii), let $u \in X_+$. Then $u = v + w$ for some positive $v \in Y$ and $w \in Y_d$. It follows from $y_\alpha \perp w$ that $y_\alpha \wedge u = y_\alpha \wedge v \overset{\tau}{\to} 0$. \hfill \Box

Let $X$ be a vector lattice and $\tau$ a locally solid topology on $X$. It is known that $\tau$ can be extended to a locally solid topology $\tau^\delta$ on $X^\delta$, the order completion of $X$. See Exercise 8 on page 73 of [3] for details. Since $X$ is majorizing in $X^\delta$, Lemma 3.5 gives the following.

**Corollary 3.6.** If $(X, \tau)$ is a locally solid vector lattice and $(x_\alpha)$ is a net in $X$ then $x_\alpha \overset{u\tau}{\to} 0$ in $X$ if and only if $x_\alpha \overset{u\tau^\delta}{\to} 0$ in $X^\delta$.

4. **Pre-Lebesgue property and disjoint sequences**

Recall the following definition from page 75 of [3]:
**Definition 4.1.** Let \((X, \tau)\) be a locally solid vector lattice. We say that \((X, \tau)\) satisfies the **pre-Lebesgue property** (or that \(\tau\) is a **pre-Lebesgue topology**), if \(0 \leq x_n \uparrow \leq x\) in \(X\) implies that \((x_n)\) is a \(\tau\)-Cauchy sequence.

Recall that Theorem 3.23 of [3] states that in an Archimedean locally solid vector lattice the Lebesgue property implies the pre-Lebesgue property. It is also known that in a topologically complete Hausdorff locally solid vector lattice that the Lebesgue property is equivalent to the pre-Lebesgue property and these spaces are always order complete. This is Theorem 3.24 of [3]. The next theorem tells us exactly when disjoint sequences are \(u\tau\)-null. Parts (i)-(iv) are Theorem 3.22 of [3], (v) and (vi) are new.

**Theorem 4.2.** For a locally solid vector lattice \((X, \tau)\) TFAE:

1. \((X, \tau)\) satisfies the pre-Lebesgue property;
2. If \(0 \leq x_\alpha \uparrow \leq x\) holds in \(X\), then \((x_\alpha)\) is a \(\tau\)-Cauchy net of \(X\);
3. Every order bounded disjoint sequence of \(X\) is \(\tau\)-convergent to zero;
4. Every order bounded \(k\)-disjoint sequence of \(X\) is \(\tau\)-convergent to zero;
5. Every disjoint sequence in \(X\) is \(u\tau\)-convergent to zero;
6. Every disjoint net in \(X\) is \(u\tau\)-convergent to zero.\(^2\)

**Proof.** (iii)\(\Rightarrow\)(v): Suppose \((x_n)\) is a disjoint sequence. For every \(u \in X_+, (|x_n| \land u)\) is order bounded and disjoint, so is \(\tau\)-convergent to zero. This proves \(x_n \xrightarrow{u\tau} 0\).

(v) \(\Rightarrow\) (iii): Let \((x_n) \subseteq [−u, u]\) be a disjoint order bounded sequence. By (v), \(x_n \xrightarrow{u\tau} 0\) so, in particular, \(|x_n| = |x_n| \land u \xrightarrow{\tau} 0\). This proves \((x_n)\) is \(\tau\)-null.

Next we prove (v)\(\Leftrightarrow\)(vi). Clearly (vi)\(\Rightarrow\)(v). Assume (v) holds and suppose there exists a disjoint net \((x_\alpha)\) which is not \(u\tau\)-null. Let \(\{U_i\}\)
be a solid base of neighbourhoods of zero for $\tau$ and $\{U_{i,u}\}$ the solid base for $u\tau$ described in Theorem 2.3. Since $(x_\alpha)$ is not $u\tau$-null there exists $U_{i,u}$ such that for every $\alpha$ there exists $\beta > \alpha$ with $x_\beta \notin U_{i,u}$. Inductively, we find an increasing sequence $(\alpha_k)$ of indices such that $x_{\alpha_k} \notin U_{i,u}$. Hence the sequence $(x_{\alpha_k})$ is disjoint but not $u\tau$-null. 

Since, for a Banach lattice, the norm topology is complete, the pre-Lebesgue property agrees with the Lebesgue property. This theorem can therefore be thought of as a generalization of Proposition 3.5 in [15]. Theorem 4.2 has the following corollaries:

**Corollary 4.3.** $\tau$ has the pre-Lebesgue property if and only if $u\tau$ does.

*Proof.* $\tau$ and $u\tau$-convergences agree on order bounded sequences. Apply (iii). 

**Corollary 4.4.** If $\tau$ is pre-Lebesgue and unbounded then every disjoint sequence of $X$ is $\tau$-convergent to 0.

We also get a partial converse to Proposition 2.15.

**Corollary 4.5.** Suppose $\tau$ is a complete and Hausdorff locally solid topology. $\tau$ is Lebesgue if and only if $u\tau$ is.

*Proof.* Proposition 2.15 says that $u\tau$ inherits the Lebesgue property from $\tau$. Suppose $u\tau$ is Lebesgue. Since $\tau$ is Hausdorff, $X$ is Archimedean (Corollary 2.9) and, therefore, $u\tau$ satisfies the pre-Lebesgue property. By Corollary 4.3, we conclude that $\tau$ is pre-Lebesgue and hence Lebesgue because $\tau$ is complete and Hausdorff. 

4.1. **$\sigma$-Lebesgue topologies.** Recall that a locally solid topology $\tau$ is $\sigma$-**Lebesgue** if $x_n \downarrow 0 \Rightarrow x_n \xrightarrow{\tau} 0$ or, equivalently, $x_n \overset{\omega}{\rightarrow} 0 \Rightarrow x_n \xrightarrow{\tau} 0$. Example 3.25 in [3] shows that the $\sigma$-Lebesgue property does not imply the pre-Lebesgue property.

By Theorem 4.2(v) it may be tempting to conclude that if $\tau$ is $\sigma$-Lebesgue then $\tau$ is pre-Lebesgue (since disjoint sequences are $uo$-null). This is not the case, however, as the example above illustrates. This allows us to prove the following:
Theorem 4.6. By [11] Corollary 3.6 every disjoint sequence in a vector lattice $X$ is $uo$-null. It is not the case that every disjoint sequence in $X$ is $uo_1$-null.

Proof. Let $(X, \tau)$ be as in Example 3.25 of [3] and assume every disjoint sequence is $uo_1$-null. Then, since $\tau$ is $\sigma$-Lebesgue, every disjoint sequence is $u\tau$-null. By Theorem 4.2(v) $\tau$ has the pre-Lebesgue property, a contradiction. \qed

5. The $uo$-Lebesgue property and universal completions

Throughout this section $X$ is a vector lattice, all topologies are assumed locally solid, and all unmarked references are to [3]. We first deal with the deep connection between universal completions, unbounded topologies and $uo$-convergence. Recall Theorem 7.54:

Theorem 5.1. For a vector lattice $X$ we have the following:

(i) $X$ can admit at most one Hausdorff Lebesgue topology that extends to its universal completion as a locally solid topology;

(ii) $X$ admits a Hausdorff Lebesgue topology if and only if $X^u$ does.

We can now add an eighth and nineth equivalence to Theorem 7.51.

Theorem 5.2. For a Hausdorff locally solid vector lattice $(X, \tau)$ with the Lebesgue property the following statements are equivalent.

(i) $\tau$ extends to a Lebesgue topology on $X^u$;

(ii) $\tau$ extends to a locally solid topology on $X^u$;

(iii) $\tau$ is coarser than any Hausdorff $\sigma$-Fatou topology on $X$;

(iv) Every dominable subset of $X_+$ is $\tau$-bounded;

(v) Every disjoint sequence of $X_+$ is $\tau$-convergent to zero;

(vi) Every disjoint sequence of $X_+$ is $\tau$-bounded;

(vii) The topological completion $\hat{X}$ of $(X, \tau)$ is Riesz isomorphic to $X^u$, that is, $\hat{X}$ is the universal completion of $X$;

(viii) Every disjoint net of $X_+$ is $\tau$-convergent to zero;

(ix) $\tau$ is unbounded.

Proof. The proof that (v)\Leftrightarrow(viii) is the same technique as in Theorem 4.2.
We now prove that (v)⇔(ix). Assume $\tau$ is unbounded and is a Hausdorff Lebesgue topology. Since $\tau$ is Hausdorff, $X$ is Archimedean. Since $X$ is Archimedean and $\tau$ is Lebesgue, $\tau$ is pre-Lebesgue. Now apply Corollary 4.4.

Now assume (v) holds so that every disjoint sequence of $X_+$ is $\tau$-convergent to zero. Since $\tau$ is Hausdorff and Lebesgue, so is $u\tau$. Since $\tau$-convergence implies $u\tau$-convergence, every disjoint positive sequence is $u\tau$-convergent to zero so that, by (ii), $u\tau$ extends to a locally solid topology on $X^u$. We conclude that $\tau$ and $u\tau$ are both Hausdorff Lebesgue topologies that extend to $X^u$ as locally solid topologies. By Theorem 5.1, $\tau = u\tau$. □

**Remark 5.3.** Theorem 5.2(vii) also tells us when unbounded Hausdorff Lebesgue topologies are topologically complete. Compare this with [15] Proposition 6.2 and [3] Theorem 7.47. It can also be deduced that if $\tau$ is a topologically complete unbounded Hausdorff Lebesgue topology then it is the only Hausdorff Fatou topology on $X$. See Theorem 7.53.

By Exercise 5 on page 72 of [3], if an unbounded Hausdorff Lebesgue topology $\tau$ is extended to a Lebesgue topology $\tau^u$ on $X^u$, then $\tau^u$ is also Hausdorff. By Theorem 7.53 it is the only Hausdorff Lebesgue (even Fatou) topology $X^u$ can admit. It must therefore be unbounded. By the uniqueness of Hausdorff Lebesgue topologies on $X^u$ we deduce uniqueness of unbounded Hausdorff Lebesgue topologies on $X$ since these types of topologies always extend to $X^u$.

We summarize in a theorem:

**Theorem 5.4.** Let $X$ be a vector lattice. We have the following:

(i) $X$ admits at most one unbounded Hausdorff Lebesgue topology. (It admits an unbounded Hausdorff Lebesgue topology if and only if it admits a Hausdorff Lebesgue topology);

(ii) Let $\tau$ be a Hausdorff Lebesgue topology on $X$. $\tau$ is unbounded if and only if $\tau$ extends to a locally solid topology on $X^u$. In this
case, the extension of $\tau$ to $X^u$ can be chosen to be Hausdorff, Lebesgue and unbounded.

**Example 5.5.** $X$ be an order continuous Banach lattice. Both the norm and $un$ topologies are Hausdorff and Lebesgue. Since these topologies generally differ, it is clear that a space can admit more than one Hausdorff Lebesgue topology. Notice, however, that when $X$ is order continuous, $un$ is the same as $uaw$. The deep reason for this is that $un$ and $uaw$ are two unbounded Hausdorff Lebesgue topologies, so, by the theory just presented, they must coincide.

Recall that every Lebesgue topology is Fatou; this is Lemma 4.2 of [3]. Also, if $\tau$ is a Hausdorff Fatou topology on a universally complete vector lattice $X$ then $(X, \tau)$ is $\tau$-complete. This is Theorem 7.50 of [3] and can also be deduced from previous facts presented in this paper.

**Corollary 5.6.** Suppose $(X, \tau)$ is Hausdorff and Lebesgue. Then $u\tau$ extends to an unbounded Hausdorff Lebesgue topology $(u\tau)^u$ on $X^u$ and $(X^u, (u\tau)^u)$ is topologically complete.

**Example 5.7.** Recall by Theorem 6.4 of [3] that if $X$ is a vector lattice and $A$ an ideal of $X^\sim$ then the absolute weak topology $|\sigma|(A, X)$ is a Hausdorff Lebesgue topology on $A$. This means that the topology $u|\sigma|(A, X)$ is the unique unbounded Hausdorff Lebesgue topology on $A$. In particular, if $X$ is a Banach lattice then $u|\sigma|(X^*, X)$ is the unique unbounded Hausdorff Lebesgue topology on $X^*$ so, if $X^*$ is (norm) order continuous, then $un = uaw = u|\sigma|(X^*, X)$ on $X^*$.

We next characterize the $uo$-Lebesgue property:

**Theorem 5.8.** Let $(X, \tau)$ be a Hausdorff locally solid vector lattice. TFAE:

(i) $\tau$ is $uo$-Lebesgue, i.e., $x_\alpha \xrightarrow{uo} 0 \Rightarrow x_\alpha \xrightarrow{\tau} 0$;

(ii) $\tau$ is Lebesgue and unbounded.

**Proof.** Recall that the Lebesgue property is independent of the definition of order convergence.
(ii)⇒(i) is known since if $\tau$ is Lebesgue then $u\tau$ is $uo$-Lebesgue and, therefore, since $\tau = u\tau$, $\tau$ is $uo$-Lebesgue.

Assume now that $\tau$ is $uo$-Lebesgue. Note first that this trivially implies that $\tau$ is Lebesgue. Assume now that $(x_n)$ is a disjoint sequence in $X_+$. By known results, $(x_n)$ is $uo$-null. Since $\tau$ is $uo$-Lebesgue $(x_n)$ is $\tau$-null. Therefore, $\tau$ satisfies condition (v) of Theorem 5.2. It therefore also satisfies (ix) which means $\tau$ is unbounded. □

**Question 5.9.** Does the above theorem remain valid if $uo$ is replaced by $uo_1$? In other words, is the $uo$-Lebesgue property independent of the definition of order convergence?

There are two natural ways to incorporate unboundedness into the literature on locally solid vector lattices. The first is to take some property relating order convergence to topology and then make the additional assumption that the topology is unbounded. The other is to take said property and replace order convergence with $uo$-convergence. For the Lebesgue property, these approaches are equivalent: $\tau$ is unbounded and Lebesgue iff it is $uo$-Lebesgue (with the overlying assumption $\tau$ is Hausdorff). Later on we will study the Fatou property and see that these approaches differ.

We can now strengthen Proposition 2.12. Compare this with Theorem 4.20 and 4.22 in [3]. The latter theorem says that all Hausdorff Lebesgue topologies induce the same topology on order bounded subsets. It will now be shown that, furthermore, all Hausdorff Lebesgue topologies have the same topologically closed sublattices.

**Theorem 5.10.** Let $\tau$ and $\sigma$ be Hausdorff Lebesgue topologies on a vector lattice $X$ and let $Y$ be a sublattice of $X$. Then $Y$ is $\tau$-closed in $X$ if and only if it $\sigma$-closed in $X$.

**Proof.** By Proposition 2.12, $Y$ is $\tau$-closed in $X$ if and only if it is $u\tau$-closed in $X$ and it is $\sigma$-closed in $X$ if and only if it is $u\sigma$-closed in $X$. By Theorem 5.4, $u\tau = u\sigma$, so $Y$ is $\tau$-closed in $X$ if and only if it $\sigma$-closed in $X$. □
Next we present a partial answer to the question of whether unbounding and passing to the order completion is the same as passing to the order completion and then unbounding. First, a proposition:

**Proposition 5.11.** Let $X$ be a vector lattice and $\tau$ a locally solid topology on $X$ with the $\text{uo}$-Lebesgue property. If $Y$ is a regular sublattice of $X$ then $\tau$ induces a $\text{uo}$-Lebesgue topology on $Y$.

**Proof.** The reader should convince themselves that the subspace topology defines a locally solid topology on $Y$; we will prove that $\tau|_Y$ is $\text{uo}$-Lebesgue. Suppose $(y_\alpha)$ is a net in $Y$ and $y_\alpha \xrightarrow{\text{uo}} y$ in $Y$ for some $y \in Y$. Since $Y$ is regular, $y_\alpha \xrightarrow{\text{uo}} y$ in $X$, so that $y_\alpha \xrightarrow{\tau} y$ as $\tau$ is $\text{uo}$-Lebesgue. This is equivalent to $y_\alpha \xrightarrow{\tau|_Y} y$. □

Example 7.5 shows that the assumption that $Y$ is regular cannot be dropped.

Let $\sigma$ be a Hausdorff Lebesgue topology on a vector lattice $X$. By Theorem 4.12 in [3] there is a unique Hausdorff Lebesgue topology $\sigma^\delta$ on $X^\delta$ that extends $\sigma$. We have the following:

**Lemma 5.12.** For any Hausdorff Lebesgue topology $\tau$ on a vector lattice $X$, $u\tau^\delta = (u\tau)^\delta = (u\tau)^u|_{X^\delta}$.

**Proof.** As stated, $\tau$ extends uniquely to a Hausdorff Lebesgue topology $\tau^\delta$ on $X^\delta$. $u\tau^\delta$ is thus a Hausdorff $\text{uo}$-Lebesgue topology on $X^\delta$.

Alternatively, since $\tau$ is a Hausdorff Lebesgue topology, $u\tau$ is a Hausdorff $\text{uo}$-Lebesgue topology. This topology extends uniquely to a Hausdorff Lebesgue topology $(u\tau)^\delta$ on $X^\delta$. It suffices to prove that $(u\tau)^\delta$ is $\text{uo}$-Lebesgue since then, by uniqueness of such topologies, it must equal $u\tau^\delta$.

Since $u\tau$ is Hausdorff and $\text{uo}$-Lebesgue, it also extends to a Hausdorff $\text{uo}$-Lebesgue topology $(u\tau)^u$ on $X^u$. By page 187 of [3], the universal completion of $X$ and $X^\delta$ coincide so we can restrict $(u\tau)^u$ to $X^\delta$. This gives a Hausdorff $\text{uo}$-Lebesgue topology, $(u\tau)^u|_{X^\delta}$, on $X^\delta$ that extends
By uniqueness of Hausdorff Lebesgue extensions to $X^\delta$, $(u\tau)^u|_{X^\delta} = (u\tau)^\delta$ and so $(u\tau)^\delta$ is $uo$-Lebesgue. □

In particular, if $\tau$ is also unbounded, so that $\tau$ is a Hausdorff $uo$-Lebesgue topology, then $u\tau^\delta = \tau^\delta$. This means we can also include $uo$-Lebesgue topologies in [3] Theorem 4.12:

**Corollary 5.13.** If $\tau$ is a Hausdorff $uo$-Lebesgue topology on a vector lattice $X$ then there exists a unique $uo$-Lebesgue topology $\tau^\delta$ on the order completion $X^\delta$ of $X$ that induces $\tau$ on $X$.

### 6. Minimal topologies

In this section we will see that $uo$-convergence “knows” exactly which topologies are minimal. Much work has been done on minimal topologies and, unfortunately, the section in [3] is out-of-date both in terminology and sharpness of results. The primary reference for this section will be [7]. First we fix our definitions; they are inconsistent with [3].

**Definition 6.1.** A Hausdorff locally solid topology $\tau$ on a vector lattice $X$ is said to be **minimal** if it follows from $\tau_1 \subseteq \tau$ and $\tau_1$ a Hausdorff locally solid topology that $\tau_1 = \tau$.

**Definition 6.2.** A Hausdorff locally solid topology $\tau$ on a vector lattice $X$ is said to be **least** or, to be consistent with [7], **smallest**, if $\tau$ is coarser than any other Hausdorff locally solid topology $\sigma$ on $X$, i.e., $\tau \subseteq \sigma$.

A crucial result, not present in [3], is Proposition 6.1 of [7]:

**Proposition 6.3.** A minimal topology is a Lebesgue topology.

This allows us to prove the following elegant result:

**Theorem 6.4.** Let $\tau$ be a Hausdorff locally solid topology on a vector lattice $X$. TFAE:

(i) $\tau$ is $uo$-Lebesgue;
(ii) $\tau$ is Lebesgue and unbounded;
(iii) $\tau$ is minimal.
Proof. We have already shown (i)⇔(ii). We show (ii)⇔(iii).

Suppose $\tau$ is minimal. By the last proposition, it is Lebesgue. It is also unbounded since $u\tau$ is a Hausdorff locally solid topology and $u\tau \subseteq \tau$. Minimality forces $\tau = u\tau$.

Conversely, suppose that $\tau$ is Hausdorff, Lebesgue and unbounded and that $\sigma \subseteq \tau$ is a Hausdorff locally solid topology. It is clear that $\sigma$ is then Lebesgue and hence $\sigma$-Fatou. By Theorem 5.2(iii), $\tau$ is coarser than $\sigma$. Therefore, $\tau = \sigma$ and $\tau$ is minimal.

In particular, we can deduce the (already known) fact that minimal topologies, if they exist, are unique. They exist if and only if $X$ admits a Hausdorff Lebesgue topology. More importantly, the whole paper [7] is now at our disposal. We learn that if $X$ has a weak unit then a Hausdorff $uo$-Lebesgue topology is metrizable if and only if $X$ has the countable sup property. This is Proposition 3.4 in [7] and is consistent with Theorem 3.2 in [15] since if $X$ is an order continuous Banach lattice then every principal band has the countable sup property (see [6] Theorem 2.10) and $X = B$, where $e$ is the weak unit. We also get many generalizations of results on unbounded topologies in Banach lattices. The following is proved in [3].

**Theorem 6.5.** If $X$ is an order continuous Banach lattice then $X$ has a least topology.

The least topology on an order continuous Banach lattice is, simply, $un$. This proves that $un$ is “special”, but also that it has been studied before. Since $u|\sigma|(X^*, X)$ is Hausdorff and $uo$-Lebesgue, it also has a minimality property.

**Lemma 6.6.** $u|\sigma|(X^*, X)$ is the (unique) minimal topology on the dual of a Banach lattice.

For more examples of spaces that admit minimal and least topologies the reader is referred to [3] and [7].
It is interesting to compare our approach to obtaining minimal topologies to \cite{7}'s. We work from the “top-down”. Explicitly, we take any Hausdorff Lebesgue topology, project it via the map \( \tau \mapsto u\tau \), and arrive at the minimal topology. \cite{7}'s approach is “bottom-up”, similar to \cite{14}, and will have corollaries in our section on unbounded convergence witnessed by ideals. \cite{7} starts with an order dense sublattice (WLOG ideal) equipped with a Hausdorff Lebesgue topology and “lifts it” to a minimal topology.

7. Local convexity and dual spaces of \( u\sigma \)-Lebesgue topologies

We now make some remarks about \( u\tau \)-continuous functionals and, surprisingly, generalize many results in \cite{15} whose proofs heavily rely on AL-representation theory and the norm.

First recall that by \cite{3} Theorem 2.22, if \( \sigma \) is a locally solid topology on \( X \) then \((X,\sigma)^* \subseteq X^\sim\) as an ideal. \((X,\sigma)^* \) is, therefore, an order complete vector lattice in its own right. Here \((X,\sigma)^* \) stands for the topological dual and \( X^\sim \) for the order dual.

**Proposition 7.1.** \((X,u\tau)^* \subseteq (X,\tau)^* \) as an ideal.

**Proof.** It is easy to see that the set of all \( u\tau \)-continuous functionals in \((X,\tau)^* \) is a linear subspace. Suppose that \( \varphi \) in \((X,\tau)^* \) is \( u\tau \)-continuous; we will show that \( |\varphi| \) is also \( u\tau \)-continuous. Fix \( \varepsilon > 0 \) and let \( \{U_i\}_{i \in I} \) be a solid base for \( \tau \) at zero. By \( u\tau \)-continuity, one can find an \( i \in I \) and \( u > 0 \) such that \( |\varphi(x)| < \varepsilon \) whenever \( x \in U_{i,u} \). Fix \( x \in U_{i,u} \). Since \( U_{i,u} \) is solid, \( |y| \leq |x| \) implies \( y \in U_{i,u} \) and, therefore, \( |\varphi(y)| < \varepsilon \). By the Riesz-Kantorovich formula, we get that

\[
|\varphi|(x) \leq |\varphi|(|x|) = \sup\{|\varphi(y)| : |y| \leq |x|\} \leq \varepsilon.
\]

It follows that \( |\varphi| \) is \( u\tau \)-continuous and, therefore, the set of all \( u\tau \)-continuous functionals in \((X,\tau)^* \) forms a sublattice. It is straightforward to see that if \( \varphi \in (X,\tau)_+^* \) is \( u\tau \)-continuous and \( 0 \leq \psi \leq \varphi \) then \( \psi \) is also \( u\tau \)-continuous and, thus, the set of all \( u\tau \)-continuous functionals in \((X,\tau)^* \) is an ideal. \( \square \)
We next need some definitions. Our definition of discrete element is slightly different than [7] since we require them to be positive and non-zero. It is consistent with [3].

**Definition 7.2.** Let $X$ be a vector lattice. $x > 0$ in $X$ is called a **discrete element** or **atom** if the ideal generated by $x$ equals the linear span of $x$.

**Definition 7.3.** A vector lattice $X$ is **discrete** or **atomic** if there is a complete disjoint system $\{x_i\}$ consisting of discrete elements in $X_+$, i.e., $x_i \land x_j = 0$ if $i \neq j$ and $x \in X$, $x \land x_i = 0$ for all $i$ implies $x = 0$.

By [3] Theorem 1.78, $X$ is atomic if and only if $X$ is lattice isomorphic to an order dense sublattice of some vector lattice of the form $\mathbb{R}^A$.

By page 291 of [7], a pre-$L_0$ space is the same as a vector lattice that admits a Hausdorff Lebesgue topology. Proposition 3.5 of that paper can be reworded and becomes a generalization of Theorem 5.2 in [15]:

**Lemma 7.4.** Let $\tau$ be a Hausdorff $uo$-Lebesgue topology on a vector lattice $X$. $\tau$ is locally convex if and only if $X$ is atomic. Moreover, if $X$ is atomic (and Archimedean) then a Hausdorff $uo$-Lebesgue topology exists, it is least, and it is the topology of pointwise convergence.

**Proof.** The first part is just a re-wording of Proposition 3.5 in [7]. The moreover part follows from Theorem 7.70 in [3] (remember, minimal in [3] means least). \qed

Consider Remark 4.15 in [15]. It is noted that $\ell_\infty$ is atomic yet un-convergence is not the same as pointwise convergence. Lemma 7.4 tells us that $\ell_\infty$ does admit a least topology that coincides with the pointwise convergence. Since $\ell_\infty$ is a dual Banach lattice, $u|\sigma|(\ell_\infty, \ell_1)$ is defined and must be the least topology on $\ell_\infty$. This is an example where $X^*$ is not an order continuous Banach lattice but $u|\sigma|(X^*, X)$ is still a least topology.

**Example 7.5.** Example 4.11 in [3] shows that the hypothesis that $Y$ is regular in Proposition 5.11 cannot be dropped. Let $X = [0, 1)$ and let $\tau$ be the Hausdorff Lebesgue topology of pointwise convergence.
on $X$. Since $X$ is atomic, Lemma 7.4 tells us that $\tau$ is $uo$-Lebesgue. Let $Y = C[0,1]$. $Y$ is a sublattice of $X$ (but is not regular) and the restriction of $\tau$ to $Y$ is not even Fatou.

Theorem 7.71 in [3] is a perfectly reasonable generalization of [15] Corollary 5.4(ii) since the $un$-topology in order continuous Banach lattices is least. What we want, however, is to replace the least topology assumption in Theorem 7.71 with the assumption that the topology is minimal. The reason being that this is a paper on $uo$-convergence and $uo$ can detect if a topology is minimal but not necessarily if it is least. To prove Corollary 5.4 in [15] the authors go through the theory of dense band decompositions. A similar theory of $\tau$-dense band decompositions can be developed, but there is an easier proof of this result utilizing the recent paper [10].

**Proposition 7.6.** Let $\tau$ be a $uo$-Lebesgue topology on a vector lattice $X$. If $0 \neq \varphi \in (X,\tau)^*$ then $\varphi$ is a linear combination of the coordinate functionals of finitely many atoms.

**Proof.** Suppose $0 \neq \varphi \in (X,\tau)^*$. Since $\tau$ is $uo$-Lebesgue and $\varphi$ is $\tau$-continuous, $\varphi(x_\alpha) \to 0$ whenever $x_\alpha \xrightarrow{uo} 0$. The conclusion now follows from [10] Proposition 2.2. \qed

**Question 7.7.** Does Proposition 5.1 in [15] admit a generalization? In other words, does a non-atomic Hausdorff $uo$-Lebesgue topology admit no non-trivial convex neighbourhoods of zero?

8. **Fatou topologies**

**Definition 8.1.** A locally solid vector lattice $(X,\tau)$ is said to satisfy the **Fatou property** if $\tau$ has a base at zero consisting of solid and order closed sets.

The Fatou property is independent of the definition of order convergence. This can be easily deduced via an argument similar to that of Lemma 1.15 in [3]. It is trivial to verify that if $\tau$ has the Fatou property then so does $u\tau$. There is also a natural $uo$ version of this property:
Definition 8.2. A locally solid vector lattice \((X, \tau)\) is said to satisfy the \textit{uo-Fatou property} if \(\tau\) has a base at zero consisting of solid and \(uo\)-closed sets.

Surprisingly, these two concepts coincide.

Lemma 8.3. Let \(A \subseteq X\) be a solid subset of a vector lattice \(X\). \(A\) is \(o\)-closed if and only if it is \(uo\)-closed.

Proof. If \(A\) is \(uo\)-closed then it is clearly \(o\)-closed.

Suppose \(A\) is \(o\)-closed, \((x_\alpha) \subseteq A\) and \(x_\alpha \xrightarrow{uo} x\). We must prove \(x \in A\). By the same computation as in Lemma 3.6 of [12],

\[
\|x_\alpha \wedge |x| - |x| \wedge |x|\| \leq \|x_\alpha - |x| \wedge |x|\| \xrightarrow{o} 0.
\]

Thus \(|x_\alpha \wedge |x|\) \xrightarrow{o} \(|x|\). Since \(A\) is solid, \((|x_\alpha| \wedge |x|) \subseteq A\), and since \(A\) is \(o\)-closed we conclude that \(|x| \in A\). Finally, using the solidity of \(A\) again, we conclude that \(x \in A\).

\[
\square
\]

A similar proof to Lemma 8.3 gives the following. Compare with [9] Lemma 2.8.

Lemma 8.4. If \(x_\alpha \xrightarrow{u\tau} x\) then \(|x_\alpha| \wedge |x| \xrightarrow{\tau} |x|\). In particular, \(\tau\) and \(u\tau\) have the same closed solid sets.

This leads to the following elegant result:

Theorem 8.5. Let \(\tau\) and \(\sigma\) be Hausdorff Lebesgue topologies on a vector lattice \(X\) and let \(A\) be a solid subset of \(X\). Then \(A\) is \(\tau\)-closed if and only if it is \(\sigma\)-closed.

Proof. Suppose \(A\) is \(\tau\)-closed. By Lemma 8.4, \(A\) is \(u\tau\)-closed. Since \(X\) can admit only one unbounded Hausdorff Lebesgue topology, \(u\sigma = u\tau\) and, therefore, \(A\) is \(u\sigma\)-closed. Since \(u\sigma \subseteq \sigma\), \(A\) is \(\sigma\)-closed. \(\square\)

We can also strengthen Lemma 3.6 in [12]. For properties and terminology involving Riesz seminorms, the reader is referred to [3].

Lemma 8.6. Let \(X\) be a vector lattice and suppose \(\rho\) is a Riesz seminorm on \(X\) satisfying the Fatou property. Then \(x_\alpha \xrightarrow{uo} x \Rightarrow \rho(x) \leq \liminf \rho(x_\alpha)\).
Proof. First we prove the statement for order convergence. Assume \( x_\alpha \overset{\omega}{\to} x \) and pick a dominating net \( y_\beta \downarrow 0 \). Fix \( \beta \) and find \( \alpha_0 \) such that
\[
| x_\alpha - x | \leq y_\beta \text{ for all } \alpha \geq \alpha_0.
\]
Since
\[
(|x| - y_\beta)^+ = |x| - |x| \wedge y_\beta \leq |x| - |x| \wedge |x_\alpha - x| = (|x| - |x_\alpha - x|)^+ \leq |x_\alpha|
\]
we conclude that \( \rho((|x| - y_\beta)^+) \leq \rho(x_\alpha) \). Since this holds for all \( \alpha \geq \alpha_0 \) we can conclude that \( \rho((|x| - y_\beta)^+) \leq \liminf \rho(x_\alpha) \). Since \( \rho \) is Fatou and \( 0 \leq (|x| - y_\beta)^+ \uparrow |x| \) we conclude that \( \rho((|x| - y_\beta)^+) \uparrow \rho(x) \) and so \( \rho(x) \leq \liminf \rho(x_\alpha). \)

Now assume that \( x_\alpha \overset{uo}{\to} x \). Then \( |x_\alpha| \wedge |x| \overset{\omega}{\to} |x| \). Using the above result and properties of Riesz seminorms, \( \rho(x) = \rho(|x|) \leq \liminf \rho(|x_\alpha| \wedge |x|) \leq \liminf \rho(|x_\alpha|) = \liminf \rho(x_\alpha). \)

Recall, again, that every Lebesgue topology is Fatou.

**Example 8.7.** Let \( X = L_2[0,1] \) and \( \tau = \| \cdot \|_2 \). Clearly, \( X \) is an order continuous Banach lattice and hence Fatou, and the norm topology is not unbounded. Therefore, unlike for the Lebesgue property, uo-Fatou is not the same as Fatou and unbounded.

**Example 8.8.** Here is an example where \( \tau \) is Hausdorff, Fatou and unbounded but not Lebesgue. Consider the norm topology on \( C[0,1] \). Since \( C[0,1] \) has a strong unit, the norm topology is unbounded. By [3] Example 4.3 the norm topology is Fatou but not Lebesgue.

We get the following proposition immediately from Theorem 4.8 in [3]:

**Proposition 8.9.** For a Hausdorff locally solid vector lattice \((X, \tau)\) TFAE:

(i) \( \tau \) is uo-Lebesgue;

(ii) \( \tau \) is Fatou, pre-Lebesgue and unbounded.

We next investigate how unbounded Fatou topologies lift to the order completion. Theorem 4.12 of [3] asserts that if \( \sigma \) is a Fatou topology on an (Archimedean) vector lattice \( X \) then \( \sigma \) extends uniquely to a Fatou topology \( \sigma^\delta \) on \( X^\delta \). We will use this notation in the following theorem.
Theorem 8.10. Let $X$ be a vector lattice and $\tau$ a Fatou topology on $X$. $u\tau^\delta = (u\tau)^\delta$.

Proof. Since $\tau$ is Fatou, $\tau$ extends uniquely to a Fatou topology $\tau^\delta$ on $X^\delta$. Clearly, $u\tau^\delta$ is still Fatou. Suppose $(x_\alpha)$ is a net in $X$ and $x \in X$. $x_\alpha \xrightarrow{u\tau^\delta} x$ means $\forall u \in X^\delta_+: |x_\alpha - x| \wedge u \xrightarrow{\tau^\delta} 0$. Since $X$ is majorizing in $X^\delta$, this is the same as $\forall u \in X_+, |x_\alpha - x| \wedge u \xrightarrow{\tau^\delta} 0$ and since $(|x_\alpha - x| \wedge u)$ is now a net in $X$, this is the same as $\forall u \in X_+, |x_\alpha - x| \wedge u \xrightarrow{\tau^\delta} 0$, which is equivalent to $x_\alpha \xrightarrow{u\tau} x$.

Since $\tau$ is Fatou, so is $u\tau$. Therefore, $u\tau$ extends uniquely to a Fatou topology $(u\tau)^\delta$ on $X^\delta$. Suppose $(x_\alpha)$ is a net in $X$ and $x \in X$. Then $x_\alpha \xrightarrow{(u\tau)^\delta} x$ is the same as $x_\alpha \xrightarrow{u\tau} x$.

Thus, $(u\tau)^\delta$ and $u\tau^\delta$ are two Fatou topologies on $X^\delta$ that agree with the Fatou topology $u\tau$ when restricted to $X$. By uniqueness of extension $u\tau^\delta = (u\tau)^\delta$. □

Recall that a Banach lattice is a KB-space iff the norm topology is Lebesgue and Levi. Here is a partial generalization of Theorem 4.7 in [12].

Theorem 8.11. If $(X, \tau)$ is Fatou and Levi then it is boundedly $uo$-complete.\(^3\)

Proof. Suppose that $(x_\alpha)$ is $uo$-Cauchy and $\tau$-bounded. From $|x^\pm - y^\pm| \leq |x - y|$, we see that $(x^\pm_\alpha)$ are both $uo$-Cauchy. The nets $(x^\pm_\alpha)$ are also $\tau$-bounded since $X$ is locally solid. Thus, WLOG, $x_\alpha \geq 0 \forall \alpha$.

First assume that $X$ has a weak unit $e > 0$. Fix $k \in \mathbb{N}$. Note that $|x_\alpha \wedge ke - x_\alpha' \wedge ke| \leq |x_\alpha - x_\alpha'| \wedge ke$. Hence, $(x_\alpha \wedge ke)$ is order Cauchy. By page 112 of [3], $X$ is order complete so $(x_\alpha \wedge ke)$ converges in order to some $y_k \in X$ by Proposition 2.3 of [11]. We next show that $\{y_k\}$ is $\tau$-bounded. Since $\tau$ is Fatou, it has a base at zero consisting of solid order closed sets. Let $U$ be a solid order closed $\tau$-neighbourhood of zero.

\(^3\)Comparing with Definition 4.35 of [3], Theorem 8.11 simply states that every Nakano space is boundedly $uo$-complete.
Since \( \{x_\alpha\} \) is \( \tau \)-bounded, there exists \( \lambda > 0 \) such that \( \{x_\alpha\} \subseteq \lambda U \). Since \( U \) is solid and \( 0 \leq x_\alpha \wedge ke \leq x_\alpha \in \lambda U \), we conclude that \( x_\alpha \wedge ke \in \lambda U \) for every \( k \) and \( \alpha \). Since \( U \) is order closed and \( x_\alpha \wedge ke \xrightarrow{o^\ast} y_k, y_k \in \lambda U \). This proves that \( \{y_k\} \subseteq \lambda U \) and, therefore, \( \{y_k\} \) is \( \tau \)-bounded.

Since the positive cone is \( o \)-closed we conclude that \( y_k \uparrow \). Thus, since \( (y_k) \) is increasing, positive and \( \tau \)-bounded, the Levi property yields that \( y_k \uparrow y \) for some \( y \in X \).

It remains to prove that \( x_\alpha \xrightarrow{\omega} y \) or, equivalently, \( |x_\alpha - y| \wedge e \xrightarrow{\omega} 0 \) since \( e \) is a weak unit. Put \( x_{\alpha,\alpha'} = \sup_{\beta \geq \alpha, \beta' \geq \alpha'} |x_\beta - x_{\beta'}| \wedge e \). By assumption, \( x_{\alpha,\alpha'} \downarrow 0 \). Now for any \( k \geq 1 \), we have

\[
(9) \quad |x_\beta \wedge ke - x_{\beta'} \wedge ke| \wedge e \leq |x_\beta - x_{\beta'}| \wedge e \leq x_{\alpha,\alpha'} \forall \beta \geq \alpha, \beta' \geq \alpha'.
\]

Taking the order limit in \( \beta' \), using that the positive cone is \( o \)-closed and continuity of lattice operations with respect to order convergence, we conclude that \( \forall \beta \geq \alpha \),

\[
(10) \quad |x_\beta \wedge ke - y_k| \wedge e \leq x_{\alpha,\alpha'} \forall k \geq 1.
\]

Since \( e \) is a weak unit, for any \( x \in X_+ \), \( x \wedge ke \uparrow x \). Taking order limits in \( k \) yields,

\[
(11) \quad |x_\beta - y| \wedge e \leq x_{\alpha,\alpha'} \forall \beta \geq \alpha,
\]

from which it follows that \( |x_\alpha - y| \wedge e \xrightarrow{\omega} 0 \).

We next move on to the general case. Let \( \{y_\gamma : \gamma \in \Gamma\} \) be a maximal collection of positive pairwise disjoint non-zero elements of \( X \). Let \( \Delta \) be the collection of all finite subsets of \( \Gamma \) directed by inclusion. For each \( \delta = \{\gamma_1, \ldots, \gamma_n\} \in \Delta \), let \( B_\delta \) be the band generated by \( \{y_{\gamma_i}\}_{i=1}^n \) in \( X \). By [3] Theorem 4.10, \( (B_\delta, \tau|_{B_\delta}) \) is Fatou. We claim that \( (B_\delta, \tau|_{B_\delta}) \) is also Levi. Indeed, let \( (x_\alpha) \) be an increasing \( \tau|_{B_\delta} \)-bounded net in \( B_{\delta,+} \). Then \( (x_\alpha) \) viewed as a net in \( X_+ \) is increasing and \( \tau \)-bounded and therefore \( x_\alpha \uparrow x \) for some \( x \in X \). Since \( B_\delta \) is a band, \( x \in B_\delta \) and \( x_\alpha \uparrow x \) in \( B_\delta \).
Next notice that \( y_\delta = \sum_{i=1}^{n} y_{\gamma_i} \) is a weak unit in \( B_\delta \). Let \( P_\delta \) be the band projection onto \( B_\delta \). \( P_\delta \) exists because, as noted, if \( X \) admits a Levi topology then it is order complete and order completeness implies the projection property. See [3] Theorem 1.60. We next observe that \( P_\delta x \uparrow x \) for all \( x \in X_+ \). Indeed, \( (P_\delta x) \) is positive, increasing and order bounded since \( P_\delta \) is a band projection. Since \( X \) is order complete, \( P_\delta x \uparrow y \) for some \( y \in X \). Then for each \( \gamma \in \Gamma \), \( |x - y| \land y_\gamma = o\text{-}\lim ((I - P_\delta)x \land y_\gamma) = 0 \) where \( o\text{-}\lim \) denotes the order limit in \( \delta \).

By Lemma 3.3 in [12], \( (P_\delta x_{\alpha}) \) is uo-Cauchy in \( B_\delta \). It is also \( \tau|B_\delta \)-bounded. By the preceding case, there exists \( 0 \leq z_\delta \in B_\delta \) such that \( P_\delta x_{\alpha} \xrightarrow{uo} z_\delta \) in \( B_\delta \) and hence in \( X \) since bands are regular. Lemma 3.1 of [12] tells us that the positive cone is uo-closed and thus that the net \( (z_\delta) \) is positive and increasing. We claim that \( (z_\delta) \) is order convergent to some \( x \in X \). By the Levi property, it suffices to show that \( \{z_\delta\} \) is \( \tau \)-bounded. Let \( U \) be a solid order closed \( \tau \)-neighbourhood of zero. Since \( \{x_{\alpha}\} \) is \( \tau \)-bounded there exists \( \lambda > 0 \) with \( \{x_{\alpha}\} \subseteq \lambda U \). Since \( U \) is solid \( P_\delta x_{\alpha} \in \lambda U \) for all \( \delta \) and \( \alpha \). Since solid order closed sets are uo-closed by Lemma 8.3, \( z_\delta \in \lambda U \) for all \( \delta \). Therefore \( \{z_\delta\} \) is \( \tau \)-bounded and we conclude that \( z_\delta \uparrow x \) for some \( x \in X \).

It remains to show that \( x_{\alpha} \xrightarrow{uo} x \). Pick any \( y \in X_+ \). Let \( P_y \) be the band projection onto \( B_y \). A similar argument shows that \( P_y x_{\alpha} \) uo-converges to some \( 0 \leq y_0 \in B_y \) in \( X \). We have

\[
(12) \quad |x_{\alpha} - y_0| \land y = P_y (|x_{\alpha} - y_0| \land y) = |P_y x_{\alpha} - y_0| \land y \xrightarrow{o} 0 \text{ in } X. 
\]

Thus for any \( \delta \), \( |P_\delta x_{\alpha} - P_\delta y_0| \land y \leq |x_{\alpha} - y_0| \land y \xrightarrow{o} 0 \text{ in } X. \) Since \( P_\delta x_{\alpha} \xrightarrow{uo} z_\delta \text{ in } X, |P_\delta x_{\alpha} - z_\delta| \land y \xrightarrow{o} 0 \text{ in } X. \) This implies that

\[
(13) \quad |z_\delta - P_\delta y_0| \land y = 0. 
\]

Since \( P_\delta y_0 \uparrow y_0 \) and \( z_\delta \uparrow x \), taking order limit in \( \delta \) in (13) yields \( 0 = |x - y_0| \land y = |P_y x - y_0| \land y. \) Since \( y \) is a weak unit in \( B_y \), it follows that
\[ y_0 = P_y x. \] Therefore, \( |x_\alpha - x| \wedge \gamma = |P_y x_\alpha - P_y x| \wedge \gamma = |P_y x_\alpha - y_0| \wedge \gamma \xrightarrow{\alpha} 0 \)
in \( X \) by (12). This proves that \( x_\alpha \xrightarrow{u_0} x \) in \( X \).

9. **Unbounded convergence witnessed by ideals**

In this section we see which results in [14] move to the general setting.

To decide whether \( x_\alpha \xrightarrow{u_\tau} x \) one has to check if \( |x_\alpha - x| \wedge u \xrightarrow{\tau} 0 \) for every “test” vector \( u \in X_+ \). A natural question is, why do we take our test vectors from \( X_+ \)? In this section we study unbounded convergence against a smaller test set.

**Definition 9.1.** Let \((X, \tau)\) be a locally solid vector lattice and \( A \subseteq X \). We say a net \((x_\alpha)\) **unbounded \( \tau \)-converges to \( x \) with respect to \( A \)** if \( |x_\alpha - x| \wedge |a| \xrightarrow{\tau} 0 \) for all \( a \in A \).

Since \( I(A) \), the ideal generated by \( A \), can be defined via
\[
I(A) = \{ x \in X : |x| \leq \sum_{i=1}^{n} \lambda_i |a_i|, \ n \in \mathbb{N}, \ \lambda_1, \ldots, \lambda_n \in \mathbb{R}_+, \ a_1, \ldots, a_n \in A \},
\]
there is no loss in generality in assuming \( A \) is an ideal of \( X \). We make this assumption throughout. When \( A \) is an ideal we only have to test against \( a \in A_+ \) so, to be clear, in this section \( A \) always denotes an ideal and we always test against \( a \in A_+ \). Now that we have fixed \( A \) to be an ideal, \( \tau \) need only be defined on \( A \) for the unbounded \( \tau \)-convergence with respect to \( A \) to be well-defined on \( X \). This is the motivation for the paper [14] and will be assumed throughout. With these insights noted, we replace Definition 9.1 with the following.

**Definition 9.2.** Let \( X \) be a vector lattice, \( A \subseteq X \) an ideal and \( \tau \) a locally solid topology on \( A \). Let \((x_\alpha)\) be a net in \( X \) and \( x \in X \). We say that \((x_\alpha)\) **unbounded \( \tau \)-converges to \( x \) with respect to \( A \)** if \( |x_\alpha - x| \wedge |a| \xrightarrow{\tau} 0 \) for all \( a \in A_+ \).

**Proposition 9.3.** If \((A, \tau)\) is a locally solid vector lattice sitting as an ideal in a vector lattice \( X \) then the unbounded \( \tau \)-convergence with respect to \( A \) is a topological convergence on \( X \). Moreover, the corresponding topology, \( u_{A_+} \), is locally solid.
Proof. There are no issues arising from a minor modification of the proof of Theorem 2.3. \qed

Notice the change in notation from [14]. One may think of $u$ and $u_A$ as maps from the space of locally solid topologies on $X$ to itself. In particular, $u_A u \tau$ should make sense and equal $u_A(u(\tau))$.\footnote{Here, of course, when $\tau$ is a locally solid topology on $X$ and $A$ an ideal of $X$, by $u_A \tau$ we mean $u_A \tau |_A$.} This is why this notation is chosen. Also, it is evident that $u = u_X$ so this subject is more general than the previous sections of the paper.

It is clear that $u_A \tau$ agrees with the topology $u \tau$ on $A$. It is also clear that if $A$ and $B$ are ideals of a vector lattice $X$ with $A \subseteq B$ and $\tau$ defined on $B$ that $u_A \tau \subseteq u_B \tau$. Since $u_A \tau$ is locally solid, Proposition 1.1 in [14] comes for free. We now present the analog of Proposition 1.4:

**Proposition 9.4.** Let $(A, \tau)$ be a locally solid vector lattice sitting as an ideal in a vector lattice $X$. If $\tau$ is not Hausdorff then neither is $u_A \tau$. If $\tau$ is Hausdorff then $u_A \tau$ is Hausdorff if and only if $A$ is order dense in $X$.

**Proof.** Routine modification of the proof of Proposition 1.4 in [14]. \qed

We now move on to the analog of [14] Proposition 2.2.

**Proposition 9.5.** Let $(B, \tau)$ be a locally solid vector lattice sitting as an ideal in a vector lattice $X$. Suppose $A \subseteq B$ is a $\tau$-dense ideal of $B$. Then the topologies $u_A \tau$ and $u_B \tau$ on $X$ agree.

**Proof.** It suffices to show that $x_\alpha \xrightarrow{u_A \tau} 0$ iff $x_\alpha \xrightarrow{u_B \tau} 0$ for every net $(x_\alpha)$ in $X_+$. It is clear that $x_\alpha \xrightarrow{u_B \tau} 0$ implies $x_\alpha \xrightarrow{u_A \tau} 0$. To prove the converse, suppose that $x_\alpha \xrightarrow{u_A \tau} 0$. Fix $b \in B_+$, a solid base neighbourhood $V$ of zero for $\tau$, and a solid base neighbourhood $U$ of zero for $\tau$ with $U + U \subseteq V$. By density, there exists $a \in A$ such that $a \in b + U$. WLOG $a \in A_+$ because, by solidity, $|a| - |b| \leq |a - b| \in U$ implies $|a| \in b + U$.\footnote{Here, of course, when $\tau$ is a locally solid topology on $X$ and $A$ an ideal of $X$, by $u_A \tau$ we mean $u_A \tau |_A$.}
By assumption, \( x_\alpha \wedge a \xrightarrow{\tau} 0 \). This implies that there exists \( \alpha_0 \) such that \( x_\alpha \wedge a \in U \) whenever \( \alpha \geq \alpha_0 \). It follows by solidity that

\[
(15) \quad x_\alpha \wedge b = x_\alpha \wedge (b - a + a) \leq x_\alpha \wedge |b - a| + x_\alpha \wedge a \in U + U \subseteq V,
\]

so that \( x_\alpha \xrightarrow{u_{B\tau}} 0 \). □

[7] Proposition 3.2 gives us a very nice generalization of Theorem 2.6 in [14]. The proof is given in that paper. This next theorem is just a re-wording into our language:

**Theorem 9.6.** Let \( X \) be a vector lattice and \( Y_1, Y_2 \subseteq X \) order dense ideals of \( X \). Suppose \( Y_i \) admits a Hausdorff Lebesgue topology \( \tau_i \) for \( i = 1, 2 \). Then the topologies \( u_{Y_1}\tau_1 \) and \( u_{Y_2}\tau_2 \) agree on \( X \). Moreover, this topology is the minimal topology on \( X \) so is Hausdorff and \( uo\)-Lebesgue (in particular, unbounded).

We next generalize Corollary 4.6 of [15].

**Lemma 9.7.** Suppose \( Y \) is a sublattice of a vector lattice \( X \). If \( \tau \) is a Hausdorff Lebesgue topology on \( X \) then \( u\tau|_Y = (u\tau)|_Y \).

**Proof.** It is clear that \( u\tau|_Y \subseteq (u\tau)|_Y \).

Suppose \( (y_\alpha) \) is a net in \( Y \) and \( y_\alpha \xrightarrow{u\tau|_Y} 0 \). Since \( Y \) is majorizing in \( I(Y) \), the ideal generated by \( Y \) in \( X \), \( y_\alpha \xrightarrow{u_{I(Y)}\tau} 0 \). By Theorem 1.36 of [4], \( I(Y) \oplus I(Y)^d \) is an order dense ideal in \( X \). Let \( v \in (I(Y) \oplus I(Y)^d)_+ \). Then \( v = a + b \) where \( a \in I(Y) \) and \( b \in I(Y)^d \). Notice \( |y_\alpha| \wedge v \leq |y_\alpha| \wedge |a| + |y_\alpha| \wedge |b| = |y_\alpha| \wedge |a| \xrightarrow{\tau} 0 \). This proves that \( y_\alpha \xrightarrow{u_{I(Y) \oplus I(Y)^d} \tau} 0 \). We conclude that \( (u_{I(Y) \oplus I(Y)^d} \tau)|_Y \subseteq u\tau|_Y \). Since the other inclusion is obvious, \( (u_{I(Y) \oplus I(Y)^d} \tau)|_Y = u\tau|_Y \).

Since \( I(Y) \oplus I(Y)^d \) is order dense in \( X \), \( u_{I(Y) \oplus I(Y)^d} \tau \) is a Hausdorff locally solid topology on \( X \). Clearly, \( u_{I(Y) \oplus I(Y)^d} \tau \subseteq u\tau \) so, since \( u\tau \) is a Hausdorff \( uo\)-Lebesgue topology and hence minimal, \( u_{I(Y) \oplus I(Y)^d} \tau = u\tau \). This proves the claim. □

The next proposition is an analogue of [9] Lemma 2.11.
Proposition 9.8. Let \( \tau \) be defined on an ideal \( A \subseteq X \) and suppose there exists \( e \in A_+ \) such that \( \overline{I_e}^\tau = A \). If \( (x_\alpha) \) and \( x \) are in \( X \) then \( x_\alpha \xrightarrow{uA\tau} x \) if and only if \( |x_\alpha - x| \wedge e \xrightarrow{\tau} 0 \) in \( A \).

Proof. The forward implication is trivial. Suppose \( |x_\alpha - x| \wedge e \xrightarrow{\tau} 0 \) in \( A \). Then \( |x_\alpha - x| \wedge a \xrightarrow{\tau} 0 \) for every positive \( a \) in \( I_e \). Since \( \overline{I_e}^\tau = A \), Proposition 9.5 tells us that \( u_{I_e} \tau = u_A \tau \). From this we get that \( x_\alpha \xrightarrow{uA\tau} x \Leftrightarrow x_\alpha \xrightarrow{u_{I_e} \tau} x \Leftrightarrow |x_\alpha - x| \wedge e \xrightarrow{\tau} 0 \). \( \square \)

Next we present an easy generalization of Corollary 3.2 in [14].

Corollary 9.9. Suppose \((A, \tau)\) is a metrizable locally solid vector lattice sitting as an ideal in a vector lattice \( X \). Suppose that \( e \in A_+ \) satisfies \( \overline{I_e}^\tau = A \). If \( x_\alpha \xrightarrow{uA\tau} 0 \) in \( X \) then there exists \( \alpha_1 < \alpha_2 < \ldots \) such that \( x_\alpha \xrightarrow{uA\tau} 0 \).

The analogue of [14] Theorem 6.1 is:

Corollary 9.10. Let \( X \) be an order complete vector lattice and \( \tau \) a locally solid topology on \( X \). \( u_X \tau \) is defined on \( X^u \) and is Hausdorff if and only if \( \tau \) is.

Remarks 6.5 and 6.6 in [14] have little to do with the norm and more to do with lattice theory. There are obvious generalizations to the setting of this paper. Next we present a more general version of [14] Theorem 6.7. Using the machinery we have built, the proof is very simple.

Theorem 9.11. Let \( \tau \) be a Hausdorff order continuous topology on an order complete vector lattice \( X \). Viewing \( X \) as an order dense ideal of \( X^u \), \( u_X \tau \) is the unique Hausdorff order continuous topology on \( X^u \). It is unbounded and topologically complete.

Proof. Since \( X \) is order complete, \( X \) sits as an order dense ideal of \( X^u \). Combine Theorem 9.6 with Theorem 5.6 and [3] Theorem 7.53. \( \square \)

Note that we can replace the order completeness assumption with \( \tau \)-completeness because in a topologically complete Hausdorff vector lattice, order continuity implies order completeness.

Finally, we look for analogues of Propositions 9.1 and 9.2 in [14]. Compare them with Theorem 4.2, Corollary 4.3 and Corollary 4.5.
Proposition 9.12. Let $(A, \tau)$ be a locally solid vector lattice sitting as an ideal in a vector lattice $X$. TFAE:

(i) $(A, \tau)$ satisfies the pre-Lebesgue property;
(ii) Every disjoint sequence in $X$ is $u_{A\tau}$-null;
(iii) Every disjoint net in $X$ is $u_{A\tau}$-null;
(iv) $(X, u_{A\tau})$ satisfies the pre-Lebesgue property.

Proof. To prove that (i)$\Rightarrow$(ii) let $(x_n)$ be a disjoint sequence in $X$. Then for every $a \in A_+$, $|x_n| \wedge a$ is an order bounded disjoint sequence in $A$ and hence $\tau$-converges to zero by Theorem 4.2. This proves $x_n \xrightarrow{\tau} 0$. An argument already presented proves (ii)$\Leftrightarrow$(iii). (ii)$\Rightarrow$(iv) is obvious.

(iv)$\Rightarrow$(i): Suppose $u_{A\tau}$ is a pre-Lebesgue topology on $X$. We first show that $(u_{A\tau})|_A$ is a pre-Lebesgue topology on $A$. We again use Theorem 4.2. Let $(a_n)$ be a disjoint order bounded sequence in $A$. Then $(a_n)$ is also a disjoint order bounded sequence in $X$ and hence $a_n \xrightarrow{u_{A\tau}} 0$. Thus $(u_{A\tau})|_A$ satisfies (iii) of Theorem 4.2 and we conclude that $(u_{A\tau})|_A$ is pre-Lebesgue. Next notice that $(A, (u_{A\tau})|_A) = (A, u\tau)$, so $u\tau$ has the pre-Lebesgue property. Finally, apply Corollary 4.3. $\square$

Proposition 9.13. Let $(A, \tau)$ be a locally solid vector lattice sitting as an ideal in a vector lattice $X$ and suppose $\tau$ is complete and Hausdorff. TFAE:

(i) $\tau$ is order continuous.
(ii) $u\tau$ is order continuous.
(iii) $u\tau$ is unbounded order continuous.
(iv) $u_{A\tau}$ is unbounded order continuous.
(v) $u_{A\tau}$ is order continuous.

Moreover, if $A$ is order dense in $X$ then $u_{A\tau}$ is unbounded.

Proof. The equivalence of (i), (ii) and (iii) has already been proven.

(i)$\Rightarrow$(iv): Suppose $x_\alpha \xrightarrow{u_\tau} 0$ in $X$ where $(x_\alpha)$ is a net in $X$. Fix $a \in A_+$. Then $|x_\alpha| \wedge a \xrightarrow{u_\tau} 0$ in $X$ and hence in $A$ since $A$ is an ideal. Since the net $(|x_\alpha| \wedge a)$ is order bounded in $A$, this is equivalent to $|x_\alpha| \wedge a \xrightarrow{\tau} 0$ in $A$. Since $\tau$ is order continuous this means $|x_\alpha| \wedge a \xrightarrow{\tau} 0$. We conclude
that \( x_\alpha \xrightarrow{u_A\tau} 0 \) and, therefore, \( u_A\tau \) is unbounded order continuous.  

(iv) \( \Rightarrow \) (v) is trivial.

(v) \( \Rightarrow \) (ii): Suppose \( u_A\tau \) is order continuous and let \((a_\alpha)\) be a net in \( A \) with \( a_\alpha \xrightarrow{uo} 0 \) in \( A \). To show that \( u\tau \) is order continuous, it suffices to show that \( |a_\alpha| \land a \xrightarrow{\tau} 0 \) for every \( a \in A_+ \). But \( a_\alpha \xrightarrow{uo} 0 \) in \( A \) implies that for all \( a \in A_+ \), \( |a_\alpha| \land a \xrightarrow{o} 0 \) in \( A \) and hence in \( X \) since \( A \) is an ideal of \( X \). Since \( u_A\tau \) is order continuous we conclude that \( |a_\alpha| \land a \xrightarrow{u_A\tau} 0 \) and, since this net is order bounded in \( A \), \( |a_\alpha| \land a \xrightarrow{\tau} 0 \).

If \( A \) is order dense then \( u_A\tau \) is Hausdorff. Combine condition (iv) with Theorem 5.8.

\[ \square \]

10. Comments as to when \( uo\)-convergence is locally solid

Since \( uo\)-convergence is modelled off of convergence a.e. - a non-topological convergence - we cannot expect \( uo\)-convergence to be topological in general. In this brief section we will investigate necessary and sufficient conditions for \( uo\)-convergence to be locally solid. In [8] it is shown that order convergence is not topological in infinite dimensional vector lattices.

If the \( uo\)-convergence is that of a locally solid topology on \( X \) then it is, tautologically, \( uo\)-Lebesgue. Since \( uo\)-limits are unique, it is also Hausdorff. By Theorem 6.4 it is minimal. In particular, a necessary condition for \( uo\)-convergence to be locally solid is that \( X \) admits a minimal topology. We immediately deduce that \( uo\)-convergence in \( C[0,1] \) is not locally solid since \( C[0,1] \) admits no Hausdorff locally solid Lebesgue topology.

By Proposition 1 of [8], \( uo\)-convergence in atomic vector lattices agrees with pointwise convergence and is, therefore, locally solid. By Lemma 7.4, pointwise convergence is the least topology on \( X \). This allows us to give an instructive alternative proof of Proposition 4.1 in [14].
Let $X$ be an order complete atomic Banach lattice represented as an order dense ideal in $\mathbb{R}^A$. Since pointwise convergence is the least topology on $\mathbb{R}^A$, un-convergent nets in $\mathbb{R}^A$ are pointwise convergent. Pointwise convergent nets in $\mathbb{R}^A$ are un-convergent iff $X$ is order continuous by Theorem 6.4 and Proposition 9.13.

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REFERENCES

[1] Y.A. Abramovich and G. Sirotkin, On Order Convergence of Nets, *Positivity*, Vol. 9, No.3, 287-292, 2005.
[2] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis: A hitchhiker’s guide*, 3rd edition, Springer, Berlin, 2006.
[3] C.D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces with Applications to Economics*, 2nd ed., AMS, Providence, RI, 2003.
[4] C.D. Aliprantis and O. Burkinshaw, *Positive Operators*, 2nd edition, Springer 2006.
[5] A. Basile and T. Traynor, Monotonely Cauchy locally-solid topologies, *Order* 7, 407-416, 1991.
[6] Z. Chen and H. Li, Some Loose Ends on Unbounded Order Convergence, *Positivity*, to appear. arXiv:1609.09707v2 [math.FA]
[7] J.J. Conradie, The Coarsest Hausdorff Lebesgue Topology, *Quaestiones Math.* 28, 287-304, 2005.
[8] Y.A. Dabboorasad, E.Y. Emelyanov and M.A.A. Marabeh, Order Convergence in Infinite-dimensional Vector Lattices is Not Topological, preprint. arXiv:1705.09883 [math.FA].
[9] Y. Deng, M. O’Brien and V.G. Troitsky, Unbounded norm convergence in Banach lattices, *Positivity*, to appear. doi:10.1007/s11117-016-0446-9.
[10] N. Gao, D. Leung and F. Xanthos, Duality for unbounded order convergence and applications, preprint. arXiv:1705.06143 [math.FA]
[11] N. Gao, V.G. Troitsky and F. Xanthos, Uo-convergence and its applications to Cesàro Means in Banach Lattices, *Israel J. Math.*, to appear. arXiv:1509.07914 [math.FA].
[12] N. Gao and F. Xanthos, Unbounded order convergence and application to martingales without probability, *J. Math. Anal. Appl.*, 415 (2014), 931-947.
[13] H. Jarchow, *Locally Convex Spaces*, Teubner, Stuttgart, 1981.
[14] M. Kandić, H. Li and V.G. Troitsky. Unbounded Norm Topology Beyond Normed Lattices, preprint. arXiv:1703.10654 [math.FA].
[15] M. Kandić, M. Marabeh and V.G. Troitsky, Unbounded Norm Topology in Banach Lattices, *J. Math. Anal. Appl.*, 451 (2017), no. 1, 259-279.

[16] A.C. Zaanen, *Riesz Spaces II*, North-Holland Publishing Co., Amsterdam-New York, 1983. MR 86b:46001

[17] O. Zabeti, Unbounded absolute weak convergence in Banach lattices, preprint. arXiv:1608.02151 [math.FA].

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