Decay of correlations and laws of rare events for transitive random maps

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Abstract
We show that a uniformly continuous random perturbation of a transitive map defines an aperiodic Harris chain which also satisfies Doeblin’s condition. As a result, we get exponential decay of correlations for suitable random perturbations of such systems. We also prove that, for transitive maps, the limiting distribution for extreme value laws and hitting/return time statistics is standard exponential. Moreover, we show that the rare event point process converges in distribution to a standard Poisson process.

Keywords: random maps, extreme value laws, hitting/return time statistics, rare event point processes, transitive maps, Harris chains, Doeblin’s condition
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(Some figures may appear in colour only in the online journal)

1. Introduction

Deterministic discrete dynamical systems are often used to model physical phenomena. However, it is more realistic to consider random perturbations of such systems to take into consideration the observational errors. The behaviour of such random systems has been studied thoroughly over the last few decades. We mention, for example [21, 22], for excellent expositions on the subject.

Laws of rare events for chaotic (deterministic) dynamical systems have also been extensively studied over the last few years. By rare events we mean that the probability of the event

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is small. In the literature, these notions were first described as hitting/return times statistics (HTS/RTS). In this setting, rare events correspond to entrances in small regions of the phase space and the goal is to prove distributional limiting laws for the normalised waiting times before hitting/returning to these asymptotically small sets. For a nice review on the subject we mention [37]. More recently, rare events have also been studied through extreme value laws (EVLs). In this setting, rare events correspond to exceedances of a high level and one looks for the distributional limit of the partial maxima of stochastic processes arising from such chaotic systems simply by evaluating an observable function along the orbits of the system. We refer the reader to [15] for a review on this subject. It turns out that these are just two views on the same phenomena: there is a link between these two approaches. This link was already perceivable in the pioneering work of Collet [9], however it has been formally proved in [13, 14], where Freitas et al showed that under general conditions on the observable functions, the existence of HTS/RTS is equivalent to the existence of EVLs. These observable functions achieve a maximum (possibly \(\infty\)) at some chosen point \(\zeta\) in the phase space so that the rare event of an exceedance of a high level occurring corresponds to an entrance in a small ball around \(\zeta\). Moreover, the study of rare events may be enhanced if we enrich the process by considering multiple exceedances (or hits/returns to target sets) that are recorded by rare events point processes (REPP), which count the number of exceedances (or hits/returns) in a certain time frame. Then the aim is to get limits in distribution for such REPP when time is adequately normalised.

Very recently, the connection between EVLs and HTS/RTS for deterministic dynamics was extended to the random case through additive random perturbations in Aytaç et al [6]. There additive random perturbation of expanding and piecewise expanding maps (with finite branches) was studied and a standard exponential law was obtained. Since then several advances have been obtained in this direction (we will mention some of them below). We remark that it was in the random setting that the fundamental theory of extreme value was developed and there are two approaches for studying the recurrence properties of the underlying system. In [26], Marie and Rousseau defined, for the first time, annealed and quenched return times for systems generated by the composition of random maps. On the one hand, in the annealed approach, the realisation is fixed and then integrated over all possible realisations to get the law. In this case the product measure for the skew-product system is studied. On the other hand, to study the quenched approach we take a random realisation, consider sample-stationary measures and get limit laws for almost every realisation.

In [11], Faranda et al studied the additive random perturbation of rational and irrational rotations and proved, using the annealed approach, the existence of extreme value laws for perturbed dynamics, regardless of the intensity of the noise whereas there is no limiting law in deterministic case.

In [33], Rousseau et al got an exponential law for random subshifts of finite type. They showed that for invariant measures with super-polynomial decay of correlations hitting times to dynamically defined cylinders satisfy an exponential distribution. They also got similar results for random expanding maps. Their results were quenched exponential law for hitting times.

In [32], Rousseau studied hitting and return time statistics for observations of dynamical systems and got an annealed exponential law for super-polynomially mixing random dynamical systems. This theory was applied to random expanding maps, random circle maps expanding on average and randomly perturbed dynamical systems.

Again in [34], Rousseau and Todd proved the existence of quenched laws of hitting time statistics for random subshifts of finite type. They showed that it was still possible to get a dichotomy of standard versus non-standard exponential laws for non-periodic and for periodic points respectively, even in the random setting.
One of the main achievements of this paper is the generalisations of the results in [6]. As was pointed out in [6], decay of correlations against all $L^1$ observables was one of the main ingredients in the theory. Hence, the results there were restricted to systems with summable decay of correlations against all $L^1$ observables with some additional conditions on the map. Here, we show that random perturbations of any transitive dynamical system (that is, admitting a forward dense orbit) defines an aperiodic Harris chain which also satisfies Doeblin’s condition (see proposition 4.1) which gives rise to a uniformly ergodic Markov chain. It is well-known that random perturbations are a special case of Markov Chains with suitable transition probabilities. Recently Jost et al in [20] showed that Markov Chains with regular enough transition densities can be represented by continuous random maps or random diffeomorphisms. Then, using the known results for such chains, we conclude that every transitive dynamical system under uniformly continuous random perturbation has exponential decay of correlations against all $L^1$ observables (see theorem A). With this approach, we get laws of rare events for a larger set of dynamics, namely transitive systems, under more general random perturbations.

Our work shows that EVL and HTS/RTS for a large class of randomly perturbed dynamics is an application of the theory of stochastic processes/Markov chains needing very little deterministic dynamical assumptions on the underlying unperturbed system: we only need transitivity and very general random perturbations.

2. Definitions and statement of results

Let $(M, B, \nu, f)$ be a discrete time deterministic dynamical system, where $M$ is a compact connected finite dimensional Riemannian manifold; $\text{dist}(\cdot, \cdot)$ denotes the induced Riemannian distance on $M$ and $\text{Leb}$ a normalised volume form on the $\sigma$-algebra $\mathcal{B}$ of Borel sets of $M$ that we call Lebesgue measure; $f : M \to M$ is a measurable map, and $\nu$ is an $f$-invariant probability measure.

Consider the time series $X_0, X_1, X_2, \ldots$ arising from such a system simply by evaluating a given random variable (r.v.) $\varphi : M \to \mathbb{R} \cup \{-\infty\}$ along the orbits of the system:

$$X_n = \varphi \circ f^n, \quad \text{for each } n \in \mathbb{N}. \quad (2.1)$$

Clearly, $X_0, X_1, \ldots$ defined in this way is not an independent sequence. However, invariance of $\nu$ guarantees that the stochastic process is stationary.

2.1. Random perturbations. Representation of Markov chains

We now consider a random setting constructed from the deterministic system via perturbing the original map. Let $F : M \times X \to M$ be a parameterized family of measurable maps $f_\omega : M \to M, f_\omega(x) := F(x, \omega, \omega \in X, x \in M$, where $(X, d)$ is a compact metric space. We denote the ball of radius $\varepsilon > 0$ around $x \in M$ by $B_\varepsilon(x) := \{ y \in M : \text{dist}(x, y) < \varepsilon \}$ and around $\omega \in X$ by $V_\varepsilon(\omega) := \{ \eta \in X : d(\eta, \omega) < \varepsilon \}$. For a fixed $\omega^* \in X$ which we denote by 0 and some $\varepsilon_0 > 0$, let $\theta = \theta_\varepsilon$ be a Borel probability measure so that $\text{supp}(\theta) \supset V_{\varepsilon_0}(0)$.

We define a random perturbation of $f : M \to M$ by the pair $(f_\omega, \theta)$ which we assume satisfies

$$f_\omega = f \quad \text{and} \quad f^\varepsilon(\text{supp}(\theta)) \supset B_{\rho_0}(fx), \quad \text{Leb} - \text{a.e. } x \in M \quad (2.2)$$

for a constant $\rho_0 > 0$, where we write $f^\varepsilon(\omega) := f_\omega(x)$; and also

$$(f_\omega^\varepsilon)^* \theta = q_* \text{Leb} \quad \text{with} \quad q \leq q_* \leq \bar{q}, \quad \text{Leb} - \text{a.e. on } \text{supp}(f_\omega^\varepsilon \theta) \quad (2.3)$$
for some constants \( q > q > 0 \) and \( \text{Leb}\text{-a.e.} \).

Consider a sequence of i.i.d. random variables (r.v.) \( W_1, W_2, \ldots \) taking values on \( V_1(0) \), where \( \omega^n = 0 \), with common distribution given by \( \theta \). Let \( \Omega = V_1(0)^{\mathbb{N}} \) denote the space of realisations of such processes and \( \theta^\mathbb{N} \) the product measure defined on its Borel subsets. Given a point \( x \in \mathcal{M} \) and the realisation of the stochastic process \( \omega = (\omega_1, \omega_2, \ldots) \in \Omega \), we define the random orbit of \( x \) as \( x_{\omega_1}(x), f_{\omega_2}^2(x), \ldots \) where the evolution of \( x \), up to time \( n \in \mathbb{N} \), is obtained by the concatenation of the respective randomly perturbed maps

\[
  f_{\omega}^n(x) := f_{\omega_n} \circ f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_1}(x),
\]

with \( f_0 \) being the identity map on \( \mathcal{M} \).

In our setting, the random perturbations we consider satisfy the conditions expressed in the relations (2.2) and (2.3), which can be said to be uniformly continuous random perturbations requiring the small noise to uniformly cover a ball of positive radius around the unperturbed transformation. Weaker assumptions might be sufficient to obtain the same results, but we did not search for the most general possible conditions.

### 2.1.1. Existence of uniformly continuous random perturbations

Families of random maps satisfying (2.2) and (2.3) can be constructed from any \( C^2 \) map on a compact finite \( n \)-dimensional manifold, as showed in [2, example 2], which we present below for completeness. Here we have \( r \geq 0 \) comprising (Hölder) continuous or smooth maps; measurable maps are also allowed.

We start by taking a finite number of coordinate charts \( \{ \psi_i : B(0, 3) \to \mathcal{M}_i \}_{i=1}^l \) such that \( \{ \psi_i(B(0, 3)) \}_{i=1}^l \) is an open cover of \( \mathcal{M} \) and \( \{ \psi_i(B(0, 1)) \}_{i=1}^l \) also (this is a standard construction, see [31, section 1.2]), where \( B(0,a) \) denotes the ball of radius \( a > 0 \) in \( \mathbb{R}^n \). In each of those charts we define \( n \) orthonormal vector fields \( \vec{X}_{ij}, \ldots, \vec{X}_{in} : B(0, 3) \to T_{\psi_i(B(0,3))}\mathcal{M} \) and extend them to the whole of \( \mathcal{M} \) with the help of bump functions. This may be done in such a way that the extensions \( X_{ij} \) are null outside \( \psi_i(B(0,2)) \) and coincide with \( \vec{X}_{ij} \) in \( \psi_i(B(0,1)) \), \( i = 1, \ldots, l; \ j = 1, \ldots, n \). We then see that, since \( \{ \psi_i(B(0, 1)) \}_{i=1}^l \) was an open cover of \( \mathcal{M} \), at every \( x \in \mathcal{M} \) there is some \( 1 \leq i \leq l \) such that \( X_{ij}(x), \ldots, X_{in}(x) \) is an orthonormal basis for \( T_x\mathcal{M} \).

We define the following parameterized family

\[
  F : (\mathbb{R}^n)^l \to C^1(\mathcal{M}, \mathcal{M}), \quad F\left((u_{ij})_{j=1}^l\right)(x) = \Phi\left(f(x), \sum_{i=1}^l \sum_{j=1}^n u_{ij} \cdot X_{ij}, 1\right)
\]

where \( \Phi : T\mathcal{M} \times \mathbb{R} \to \mathcal{M} \) is the geodesic flow associated to the given Riemannian metric.

We now take a small \( \varepsilon_0 > 0 \) and consider the finite dimensional parameterized family of maps \( F : V(0, \varepsilon_0) \to C^1(\mathcal{M}, \mathcal{M}) \), where \( V(0, \varepsilon_0) \) is the \( \varepsilon_0 \)-ball around the origin in \( \mathbb{R}^{n-l} \).

Then by construction of \( F \), every family \( \mathcal{F}_{a,\varepsilon} = \{ F : \|a - t\| < \varepsilon \} \) satisfies conditions (2.2) and (2.3) for some \( \rho_0, q > 0 \), where \( \varepsilon > 0 \) is so that \( \omega^* := a \in V(a, \varepsilon) \subset V(0, \varepsilon_0) \) and we set \( \theta = \text{Leb}\left|V(a, \varepsilon)\right| \).

On parallelizable manifolds the implementation is even simpler since we can perform the previous construction with \( l = 1 \) and obtain so called additive random perturbations, as follows.

### 2.1.2. Additive random perturbations on parallelizable manifolds

If \( \mathcal{M} \) is parallelizable, then \( T\mathcal{M} \) is diffeomorphic to the trivial bundle \( \mathcal{M} \times \mathbb{R}^n \) and we can find \( n \) globally orthonormal (hence nonvanishing) smooth vector fields \( X_1, \ldots, X_n \) on \( \mathcal{M} \). We construct the following family of differentiable maps
\[ F : \mathbb{R}^n \mapsto C^r(\mathcal{M}, \mathcal{M}), \quad F(u_1, \ldots, u_n)(x) = \Phi \left( f(x), \sum_{j=1}^{n} u_j \cdot X_j, 1 \right) \]

where \( \Phi : T \mathcal{M} \times \mathbb{R} \to \mathcal{M} \) is as above. Now for all small enough \( \varepsilon_0 > 0 \) considering \( F_\varepsilon : V(0, \varepsilon_0) \mapsto C^r(\mathcal{M}, \mathcal{M}) \) where \( V(0, \varepsilon_0) \) is the \( \varepsilon_0 \)-ball around the origin in \( \mathbb{R}^n \), the family \( \mathcal{F}_{\varepsilon} \) satisfies conditions (2.2) and (2.3) for some \( \rho_0, \varrho_0 > 0 \).

### 2.1.3. Representation of Markov Chains by random maps.

In our setting the random perturbation is a special case of a Markov Chain with transition probabilities given by

\[ p_\varepsilon(A) = p(A|x) = \theta(\omega \in V_\varepsilon(0) : f_\varepsilon(x) \in A) = \int_A q_\varepsilon \, d\text{Leb}(x) \]

and \( \mu \) is a stationary measure for the Markov Chain with the family \( \{q_\varepsilon\}_{\varepsilon \in \mathcal{M}} \) of transition densities [29].

Recently Jost et al in [20, theorems B and C] showed that Markov Chains with regular enough transition densities can be represented by continuous random maps or random diffeomorphisms.

We state below a result giving sufficient conditions for representation by continuous random maps.

**Theorem 2.1 ([20, proposition 5.1]).** Let \( \{p_\varepsilon\}_{\varepsilon \in \mathcal{M}} \) be a family of probability measures, where each \( p_\varepsilon \) is absolutely continuous with respect to \( \text{Leb} \) and has positive Hölder continuous (for some exponent \( \alpha > 0 \)) probability density \( q_\varepsilon \) and the family \( \{q_\varepsilon\}_{\varepsilon \in \mathcal{M}} \) is pointwise continuous for \( \text{Leb} \)-a.e. \( \varepsilon \in \mathcal{M} \). Then \( \{p_\varepsilon\}_{\varepsilon \in \mathcal{M}} \) can be represented by random continuous maps \( \omega \mapsto f_\varepsilon(\omega) \), that is, there exists a probability measure \( \nu \) on \( C^0(\mathcal{M}, \mathcal{M}) \) so that \( p_\varepsilon(A) = p(A|x) = \nu(g : g(x) \in A) = \int_A q_\varepsilon \, d\text{Leb}(x) \) for every Borel subset \( A \).

### 2.2. Stationary probability measures. Decay of correlations

In this setting the notion of stationary measure replaces the notion of invariant measure by leaving the perturbed map invariant in average over the noise.

**Definition 2.1 (Stationary measure).** We say that the probability measure \( \mu \) on the Borel subsets of \( \mathcal{M} \) is stationary if

\[ \int \int \int \mu(\varphi \circ f_\varepsilon \circ \sigma) \, d\theta(\omega) = \int \varphi \, d\mu \]

for every \( \mu \)-integrable \( \varphi : \mathcal{M} \to \mathbb{R} \).

We can give a deterministic representation of this random setting using the skew product transformation

\[ S : \mathcal{M} \times \Omega \to \mathcal{M} \times \Omega, \quad (x, \omega) \mapsto (f_{\varepsilon_1}, \sigma(\omega)), \]

where \( \sigma : \Omega \to \Omega \) is the one-sided shift \( \sigma(\omega) = \sigma(\omega_1, \omega_2, \ldots) = (\omega_2, \omega_3, \ldots) \). We remark that \( \mu \) is stationary if and only if the product measure \( \mu \otimes \theta^\infty \) is an \( S \)-invariant measure.

Now the process is given by

\[ X_0 = \varphi \circ f_\varepsilon^n, \quad \text{for each } n \in \mathbb{N}, \]

which can also be written as \( X_0 = \varphi \circ \pi \circ S^n \), where \( \pi : \mathcal{M} \times \Omega \to \mathcal{M}, (x, \omega) \mapsto x \) is the natural projection onto the first factor. Note that the stochastic process \( X_0, X_1, \ldots \) is stationary since \( \mu \) is stationary.

Hence, the random evolution of the system is given by a discrete time dynamical system \( (\mathcal{X}, \mathcal{B}, \mathbb{P}, T) \), where \( \mathcal{X} \) is a topological space, \( \mathcal{B} \) is the Borel \( \sigma \)-algebra, \( T : \mathcal{X} \to \mathcal{X} \) is a measurable map and \( \mathbb{P} \) is a \( T \)-invariant probability measure, i.e. \( \mathbb{P}(T^{-1}(B)) = \mathbb{P}(B) \), for all \( B \in \mathcal{B} \).
We set \((\mathcal{X}, \mathcal{B}, \mathbb{P}, T)\) with \(\mathcal{X} = M \times \Omega\) and the product Borel \(\sigma\)-algebra \(\mathcal{B}\) where the product measure \(\mathbb{P} = \mu \times \theta^\mathbb{N}\) is defined. The random dynamics can now be read from the skew product map \(T = S\) since the second factor of \(S\) depends only on the first coordinate of the first factor

\[ p_A = \mathbb{P}\left(\left\{(x, \omega) : (\pi \circ S)(x, \omega) \in A\right\}\right) = \left\{(\pi \circ S)\right\} \mathbb{P} = \left\{(f^x) \theta\right\} \mathbb{P}.\]

In our (random) setting, we will only be interested in Banach spaces of functions that do not depend on \(\omega \in \Omega\). Hence, we assume that \(\phi, \psi\), are functions defined on \(M\) and the correlation between these two observables can be written in a simple form.

**Definition 2.2 (Annealed decay of correlations).** Let \(C_1, C_2\) denote Banach spaces of real valued measurable functions defined on \(M\). We denote the annealed correlation of non-zero functions \(\phi \in C_1\) and \(\psi \in C_2\) w.r.t. the measure \(\mu \times \theta^\mathbb{N}\) as

\[
\text{Cor}_{\mu \times \theta^n}(\phi, \psi, n) := \frac{1}{\|\phi\|_{C_1} \|\psi\|_{C_2}} \left| \int \left( \int \psi \circ f^n \, d\theta^\mathbb{N} \right) \phi \, d\mu - \int \phi \, d\mu \int \psi \, d\mu \right|.
\]

We say that we have **annealed decay of correlations**, w.r.t. the measure \(\mu \times \theta^\mathbb{N}\), for observables in \(C_1\) against observables in \(C_2\) if, for every \(\phi \in C_1\) and every \(\psi \in C_2\), then it holds that

\[
\text{Cor}_{\mu \times \theta^n}(\phi, \psi, n) \xrightarrow{n \to \infty} 0.
\]

We say that we have **annealed decay of correlations against \(L^1\) observables** whenever we have decay of correlations, with respect to the measure \(\mu \times \theta^\mathbb{N}\), for observables in \(C_1\) against observables in \(C_2\) and \(C_2 = L^1(\text{Leb})\) is the space of Leb-integrable functions on \(M\) and \(\|\psi\|_{C_2} = \|\psi\| = \int |\psi| \, d\text{Leb}\). Note that when \(\mu\) is absolutely continuous with respect to \(\text{Leb}\) and the respective Radon-Nikodym derivative is bounded above and below by positive constants, then \(L^1(\text{Leb}) = L^1(\mu)\).

### 2.3. Regularity conditions on the observable function and the measure

We assume that the r.v. \(\varphi : M \to \mathbb{R} \cup \{\pm \infty\}\) achieves a global maximum at \(\zeta \in M\) (we allow \(\varphi(\zeta) = +\infty\)). We also assume that \(\varphi\) and \(\mathbb{P}\) are sufficiently regular so that:

- **(R1)** for \(u\) sufficiently close to \(u_F := \varphi(\zeta)\), the event

\[
U(u) = \{X_0 > u\} = \{x \in M : \varphi(x) > u\}
\]

coincides with a topological ball centred at \(\zeta\). Moreover, the quantity \(\mathbb{P}(U(u))\), as a function of \(u\), varies continuously on a neighbourhood of \(u_F\).

In what follows, an **exceedance** of the level \(u \in \mathbb{R}\) at time \(j \in \mathbb{N}\) means that the event \(\{X_j > u\}\) occurs. We denote by \(F\) the distribution function (d.f.) of \(X_0\), i.e. \(F(x) = \mathbb{P}(X_0 \leq x)\). Given any d.f. \(F\), let \(\tilde{F} = 1 - F\), the so-called **tail distribution**, and \(u_F\) denote the right endpoint of the d.f. \(F\), i.e. \(u_F = \sup\{x : F(x) < 1\}\).

### 2.4. Extreme value laws

Given the dynamically defined time series \(X_0, X_1, \ldots\) we want to study its extremal behaviour. Hence we define a new sequence of random variables \(M_k, M_2, \ldots\) as the partial maximum of the first \(n\) random variables, i.e.
\[ M_n = \max\{X_0, \ldots, X_{n-1}\}. \quad (2.8) \]

**Definition 2.3.** We say that we have an extreme value law (EVL) for \( M_n \) if there is a non-degenerate d.f. \( H : \mathbb{R} \to [0, 1] \) with \( H(0) = 0 \); and if for every \( \tau > 0 \), there exists a sequence of levels \( u_n = u_n(\tau), n = 1, 2, \ldots \), such that

\[ n \mathbb{P}(X_0 > u_n) \to \tau, \quad \text{as } n \to \infty \quad (2.9) \]

and for which the following holds

\[ \mathbb{P}(M_n \leq u_n) \to H(\tau), \quad \text{as } n \to \infty. \quad (2.10) \]

For every sequence \((u_n)_{n \in \mathbb{N}}\) satisfying (2.9) we define:

\[ U_n := \{X_0 > u_n\}. \quad (2.11) \]

The normalising sequences \( u_n \) come from the i.i.d. case. Namely, if \( X_0, X_1, X_2, \ldots \) are independent and identically distributed, then it is clear that \( \mathbb{P}(M_n \leq u) = (F(u))^n \) where \( F \) is the d.f. of \( X_0 \). Hence, condition (2.9) implies that

\[ \mathbb{P}(M_n \leq u_n) = (1 - \mathbb{P}(X_0 > u_n))^n \sim \left(1 - \frac{\tau}{n}\right)^n \to e^{-\tau}, \quad \text{as } n \to \infty. \quad (2.12) \]

Moreover, the reciprocal is also true; see [23, theorem 1.5.1] for more details. Note that in this case \( H(\tau) = 1 - e^{-\tau} \) is the standard exponential d.f.

When \( X_0, X_1, X_2, \ldots \) are not independent, the standard exponential law still applies under some conditions on the dependence structure. These conditions are as follows.

**Condition \((D_2(u_n))\).** We say that \( D_2(u_n) \) holds for the sequence \( X_0, X_1, \ldots \) if for all \( \ell, t, n \) \( \mathbb{P}(X_0 > u_n) \cap \max\{X_0, \ldots, X_{t+\ell-1}\} \leq u_n) = \mathbb{P}(X_0 > u_n)\mathbb{P}(M_t \leq u_n) \leq \gamma(n, t) \),

where \( \gamma(n, t) \) is decreasing in \( t \) for each \( n \) and \( n\gamma(n, t) \to 0 \) when \( n \to \infty \) for some sequence \( t_n = o(n) \).

Now, let \((k_n)_{n \in \mathbb{N}}\) be a sequence of integers such that

\[ k_n \to \infty \quad \text{and} \quad k_d n = o(n). \quad (2.13) \]

**Condition \((D'(u_n))\).** We say that \( D'(u_n) \) holds for the sequence \( X_0, X_1, X_2, \ldots \) if there exists a sequence \((k_n)_{n \in \mathbb{N}}\) satisfying (2.13) and such that

\[ \lim_{n \to \infty} n \sum_{j=1}^{[n/k_n]} \mathbb{P}(X_0 > u_n, X_j > u_n) = 0. \quad (2.14) \]

By [12, theorem 1], if conditions \( D_2(u_n) \) and \( D'(u_n) \) hold for \( X_0, X_1, \ldots \), then there exists an EVL for \( M_n \) and \( H(\tau) = 1 - e^{-\tau} \). Besides, as it can be seen in [12, section 2], condition \( D_2(u_n) \) follows immediately if \( X_0, X_1, \ldots \) is given by (2.1) and the system has sufficiently fast decay of correlations.

In this paper, we extend the results for the random case from Aytaç et al [6] to transitive maps and for more general random perturbations. The class of maps that can be perturbed includes interval maps with unbounded derivatives, as in the Lorenz map for finite branch case; interval maps with infinitely many branches, and also maps in higher dimensional manifolds. The random perturbation setting is not restricted to additive noise as in [6]. Moreover, we also consider Markov Chains under conditions that guarantee their representation by random maps
satisfying conditions (2.2) and (2.3), which imply an almost uniform distribution of perturbed images on a neighbourhood of the original value of the unperturbed map.

First of all, we show that, for transitive systems, we have decay of correlations against $L^1$ observables for the type of random perturbations we consider here.

**Theorem A.** Every measurable map $f : \mathcal{M} \to \mathcal{M}$ which is Leb-a.e. continuous admitting $x_0 \in \mathcal{M}$ so that $\{f^n x_0 : n \geq 1\}$ is both a dense subset of $\mathcal{M}$ and a set of continuity points of $f$ is such that any random perturbation of $f$ satisfying (2.2) and (2.3) has exponential decay of correlations against $L^1$ observables.

We remark that the assumptions on the underlying unperturbed dynamics are very weak and the conclusion in theorem A is rather strong.

The assumptions on the random perturbation in theorem 2.1 ensure that $\rho_B f x, \rho(x)$, that is, the perturbed images cover a full neighbourhood of the image of the original map $f$, and $\rho_B f(x, \rho(x)) \geq g(x)$, i.e. the distribution of images in this neighbourhood is essentially uniform, for some Leb-a.e. continuous map $\rho : \mathcal{M} \to \mathbb{R}^+$ and continuous functions $\rho : \mathcal{M} \to \mathbb{R}^+$. Hence we can state the following version of theorem A in the language of Markov chains.

**Theorem B.** Let $(p_\lambda)_{\lambda \in \mathcal{M}}$ be a continuous family of probability measures such that $p_\lambda = q_\lambda \text{Leb}$, where $q_\lambda$ is a positive Hölder continuous (for some exponent $\alpha > 0$) probability density $q_\lambda$ varying continuously with $x \in \mathcal{M}$ with respect to the $C^0$-topology. Assume that there are $\rho > 0, \varrho \geq q > 0$ and a full Leb-measure subset $Y$ so that the map $f : \mathcal{M} \to \mathcal{M}$ is continuous on $Y$ satisfying $\rho_B f x, \rho(x)$ and $q \leq q_\lambda \leq \varrho$ for $x \in Y$ and admitting a point $x_0 \in \mathcal{M}$ so that $\{f^n x_0 : n \geq 0\}$ is both dense in $\mathcal{M}$ and contained in $Y$.

Then the Markov Chain defined by $(p_\lambda)_{\lambda \in \mathcal{M}}$ has a unique stationary measure $\mu$ with exponential decay of correlations against $L^1$ observables.

Using this general result on decay of correlations for random maps/Markov Chains we show that under suitable random perturbation of the original transitive system, we get a standard exponential distribution for the extreme values as well as the hitting time statistics for any point $\zeta \in \mathcal{M}$.

**Theorem C.** Let $(\mathcal{M} \times \Omega, (B, \mu \times \theta^\mathcal{M}, S)$ be a dynamical system where $\mathcal{M}$ is a finite dimensional compact Riemannian manifold and $f : \mathcal{M} \to \mathcal{M}$ is a map which is continuous on the full Leb-measure subset $Y$ admitting a point $x_0 \in \mathcal{M}$ so that $\{f^n x_0 : n \geq 0\}$ is both dense in $\mathcal{M}$ and contained in $Y$. Assume that $f$ is randomly perturbed by a random maps scheme satisfying (2.2) and (2.3) or by a Markov Chain given by a family $(p_\lambda)_{\lambda \in \mathcal{M}}$ of transition probabilities satisfying the conditions of theorem B.

For any point $\zeta \in \mathcal{M}$, consider that $X_0, X_1, \ldots$ is defined as in (2.6), let $u_n$ be such that (2.9) holds and assume that $U_n$ is defined as in (2.11).

Then the stochastic process $X_0, X_1, \ldots$ satisfies $D^2(u_n)$ and $D^1(u_n)$, which implies that we have an EVL for $M_n$ such that $H(\tau) = e^{-\tau}$.

### 2.5. Hitting/return time statistics

Next we consider the second approach in the statistical study of rare events. In the deterministic setting the definition of first hitting/return time (function) is given as follows.

Given a set $A \in \mathcal{B}$ we define a function that we refer to as first hitting time function to $A$ and denote by $r_A : \mathcal{X} \to \mathbb{N} \cup \{+\infty\}$ where

\[ r_A(\mathcal{X}) = \inf\{n \geq 0 : x \in A\} \]
The restriction of \( r_A \) to \( A \) is called the first return time function to \( A \). We define the first return time to \( A \), which we denote by \( R(A) \), as the minimum of the return time function to \( A \), i.e.

\[
R(A) = \min_{x \in A} r_A(x).
\]

In the random setting, one has to make a choice regarding the type of definition for the first hitting/return times (functions). Essentially, there are two possibilities. The quenched perspective which consists of fixing a realisation \( \omega \in \Omega \) and defining the objects in the same way as in the deterministic case. The annealed perspective consists of defining the same objects by averaging over all possible realisations \( \omega \). (We refer to [26] for more details on both perspectives.) In [6], the quenched perspective was used to define hitting/return times (functions) as it facilitates the connection between EVL and hitting/return time statistics in the random setting. Here, we follow the same setting.

For some \( \omega \in \Omega \) fixed, some \( \mathcal{M} \in \mathcal{X} \) and \( \mathcal{M} \subset A \) measurable, we define the first random hitting time

\[
r^A_{\omega}(x) := \min \{ j \in \mathbb{N} : f^j_{\omega}(x) \in A \}
\]

and the first random return from \( A \) to \( A \) as

\[
R^A(A) = \min \{ r^A_{\omega}(x) : x \in A \}.
\]

**Definition 2.4.** Given a sequence of measurable subsets of \( \mathcal{X} : (V_n)_{n \in \mathbb{N}} \), such as \( \mathbb{P}(V_n) \to 0 \), the system has (random) Hitting Time Statistics (HTS) \( G \) for \( (V_n)_{n \in \mathbb{N}} \) if for all \( t \geq 0 \)

\[
\mathbb{P} \left( r_{V_n} \leq \frac{t}{\mathbb{P}(V_n)} \right) \to G(t) \quad \text{as} \quad n \to \infty,
\]

(2.15)

and the system has (random) Return Time Statistics (RTS) \( \tilde{G} \) for \( (V_n)_{n \in \mathbb{N}} \) if for all \( t \geq 0 \)

\[
\mathbb{P} \left( r_{V_n} \leq \frac{t}{\mathbb{P}(V_n)} \right) \to \tilde{G}(t) \quad \text{as} \quad n \to \infty.
\]

(2.16)

We observe that in the random setting, \( \mathcal{X} = \mathcal{M} \times \Omega \), \( \mathbb{P} = \mu \times \theta^\mathbb{N} \), \( T = \mathcal{S} \) as defined in (2.5), \( V_n = V_n^* \times \Omega \), where \( V_n^* \subset \mathcal{M} \) and \( \mu(V_n^*) \to 0 \) as \( n \to \infty \).

We also observe that

\[
\mathbb{P} \left( r_{V_n} \leq \frac{t}{\mathbb{P}(V_n)} \right) = \mu \times \theta^\mathbb{N} \left( r^A_{\omega}(x) \leq \frac{t}{\mu(V_n^*)} \right).
\]

The normalising sequences to obtain HTS/RTS, are motivated by Kac’s Lemma. It asserts that the expected value of \( r_A \) w.r.t. \( \mathbb{P}_A \) is \( \int_A r_A \, d\mathbb{P}_A = 1/\mathbb{P}(A) \). So, the appropriate normalising factor in the study of the fluctuations of \( r_A \) on \( A \) is \( 1/\mathbb{P}(A) \).

The relation between the existence of HTS and that of RTS is given by the Main theorem in [19]. It states that a system has HTS \( G \) if and only if it has RTS \( \tilde{G} \) and

\[
G(t) = \int_0^t (1 - \tilde{G}(s)) \, ds.
\]

(2.17)

So, the existence of exponential HTS is equivalent to the existence of exponential RTS.
In [13], the link between HTS/RTS (for balls) and EVLs of stochastic processes given by (2.1) was established for invariant measures \( \nu \) absolutely continuous w.r.t. Leb. Essentially, it was proved that if such time series have an EVL \( H \) then the system has HTS \( H \) for balls ‘centred’ at \( \zeta \) and vice versa. (Recall that having HTS \( H \) is equivalent to saying that the system has RTS \( \tilde{H} \), where \( H \) and \( \tilde{H} \) are related by (2.17)). This was based on the elementary observation that for stochastic processes given by (2.1) we have:

\[
\{(x_u) \} \Rightarrow f_{\text{uniform}}.
\]

(2.18)

This connection was exploited to prove EVLs using tools from HTS/RTS and the other way around. In [14], the authors carried the connection further to include more general measures, which, in particular, allowed the coauthors to obtain the connection in the random setting in [6]. For that, it was sufficient to use the skew product map to look at the random setting as a deterministic system and to take the observable \( f \) defined as in (2.6) with \( f_{\text{uniform}} \) as in [14, equation (4.1)]. Namely,

\[
\varphi : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}, \quad x \mapsto g(B_{\text{first}}, \zeta(\zeta))
\]

(2.19)

where \( \zeta \) is a chosen point in the phase space \( \mathcal{M} \) and the function \( g : [0, +\infty) \to \mathbb{R} \cup \{+\infty\} \) is such that 0 is a global maximum (+\infty is allowed), \( g \) is a strictly decreasing bijection in a neighbourhood of 0 and has one of the three types coming from the classical extreme value theory; see [14]. Then [14, theorems 1 and 2] guarantee that if we have an EVL, in the sense that \( (2.10) \) holds for some d.f. \( H \), then we have HTS for sequences \( \{M_n\} \subset \mathcal{V} \) where \( M_n = B_\delta \times \Omega \) and \( \delta_n \to 0 \) as \( n \to \infty \), with \( G = H \) and vice versa.

Using the connection between EVLs and HTS/RTS provided by [6], we immediately get from theorem C

Corollary D. Under the same hypothesis of theorem C we have exponential HTS/RTS for balls around \( \zeta \), in the sense that \( (2.15) \) and \( (2.16) \) hold with \( G(t) = \tilde{G}(t) = 1 - e^{-t} \) and \( V_n = B_\delta(\zeta) \times \Omega \), where \( \delta_n \to 0 \) as \( n \to \infty \).

2.6. Rare event point processes

If we consider multiple exceedances we are lead to point processes of rare events counting the number of exceedances in a certain time frame. For every \( A \subset \mathbb{R} \) we define

\[
\mathcal{N}_u(A) := \sum_{i \in A/\mathbb{N}\delta} 1_{[x_i > u]}.
\]

In the particular case where \( A = I = [a, b] \) we simply write \( \mathcal{N}_{u,a} := \mathcal{N}_u(I(a, b)) \). Observe that \( \mathcal{N}_{u,a} \) counts the number of exceedances amongst the first \( n \) observations of the process \( X_0, X_1, \ldots, X_n \) or, in other words, the number of entrances in \( U(u) \) up to time \( n \). Also, note that

\[
\{\mathcal{N}_{u,a} = 0\} = \{M_n \leq u\}.
\]

(2.20)

In order to define a point process that captures the essence of an EVL and HTS through (2.20), we need to re-scale time using the factor \( v := 1/\mathbb{E}(X > u) \) given by Kac’s theorem. However, before we give the definition, we need some formalism.

Let \( \mathcal{G} \) denote the semi-ring of subsets of \( \mathbb{R}_0^+ \) whose elements are intervals of the type \( [a, b] \), for \( a, b \in \mathbb{R}_0^+ \). Let \( \mathcal{R} \) denote the ring generated by \( \mathcal{G} \). Recall that for every \( J \in \mathcal{R} \) there are \( k \in \mathbb{N} \) and \( k \) intervals \( I_1, \ldots, I_k \in \mathcal{G} \) such that \( J = \cup_{i=1}^k I_i \). In order to fix notation, let \( a_j, b_j \in \mathbb{R}_0^+ \) be such that \( I_j = [a_j, b_j] \in \mathcal{G} \). For \( I = [a, b] \in \mathcal{G} \) and \( \alpha \in \mathbb{R} \), we denote
\[ \alpha I := [\alpha a, \alpha b] \text{ and } I + \alpha := [a + \alpha, b + \alpha]. \]

Similarly, for \( J \in \mathcal{R} \) define \( \alpha J := \alpha I_1 \cup \cdots \cup \alpha I_k \) and \( J + \alpha := (I_1 + \alpha) \cup \cdots \cup (I_k + \alpha) \).

**Definition 2.5.** We define the rare event point process (REPP) by counting the number of exceedances (or hits to \( U(u_n) \)) during the (re-scaled) time period \( v_nJ \in \mathcal{R}, \) where \( J \in \mathcal{R} \). To be more precise, for every \( J \in \mathcal{R} \), set

\[
N_n(J) := \mathcal{F}_{\alpha J}(v_n J) = \sum_{j \in \mathcal{F}(v_n J)} \mathbf{1}_{\{|X_j| > u_n\}}.
\]

When \( D'(u_n) \) holds then, since there is no clustering, due to a criterion proposed by Kallenberg [21, theorem 4.7] which applies only to simple point processes (without multiple events), we can adjust condition \( D_2(u_n) \) to this scenario of multiple exceedances in order to prove that the REPP converges in distribution to a standard Poisson process. We denote this adapted condition by:

**Condition \( D_3(u_n) \).** Let \( A \in \mathcal{R} \) and \( t \in \mathbb{N} \). We say that \( D_3(u_n) \) holds for the sequence \( X_0, X_1, \ldots \) if

\[
\mathbb{P}((X_0 > u_n) \cap \{\mathcal{F} (A + t) = 0\}) - \mathbb{P}((X_0 > u_n)\mathcal{F}(\mathcal{F} (A) = 0)) \leq \gamma(n, t),
\]

where \( \gamma(n, t) \) is nonincreasing in \( t \) for each \( n \) and \( n\gamma(n, t_n) \to 0 \) as \( n \to \infty \) for some sequence \( t_n = o(n) \), which means that \( t_n/n \to 0 \) as \( n \to \infty \).

Condition \( D_3(u_n) \) follows, as easily as \( D_2(u_n) \), from sufficiently fast decay of correlations.

In [13, theorem 5] a strengthening of [12, theorem 1] is proved, which essentially says that, under \( D_2(u_n) \) and \( D'(u_n) \), the REPP \( N_n \) defined in (2.21) converges in distribution to a standard Poisson process.

Since, under the same assumptions of theorem C, condition \( D_3(u_n) \) holds trivially, then applying [13, theorem 5] we obtain

**Corollary E.** Under the same hypothesis of theorem C, the stochastic process \( X_0, X_1, \ldots \) satisfies \( D_3(u_n) \) and \( D'(u_n) \), which implies that the REPP \( N_n \) defined in (2.21) is such that \( N_n \overset{d}{\to} N \), as \( n \to \infty \), where \( N \) denotes a Poisson Process with intensity 1.

### 2.7. Organization of the work

We present some examples of applications focusing on specific families for which our results directly improve recent advances, in section 3.

In section 4 we explain how our setting fits into a Markov Chain satisfying Harris and Doeblin conditions, enabling us to obtain exponential decay of correlations from already known results. Then, in section 4.3, we prove the main results on extreme value laws, hitting/return time statistics and rare event point processes.

### 3. Applications

Here we give some examples for which we can apply our results. The assumptions are rather weak: any transitive map (that is, admitting a subset of points with dense forward orbit) of a compact manifold can be randomly perturbed in our setting.

In what follows we focus on specific families for which our results directly improve recent advances.
3.1. Lorenz-like maps

Lorenz maps are the one-dimensional maps associated to the geometric Lorenz models, which were constructed as an attempt to understand the numerically observed behaviour of the Lorenz attractor introduced by Lorenz in [24]. The Lorenz equations

\[ \dot{x} = a(y - x), \quad \dot{y} = (r - z)x - y, \quad \dot{z} = xy - bz, \]  

(3.1)

with the parameters \( a = 10, r = 28/3 \) and \( b = 8/3 \) were intended as an extremely simplified model for thermal fluid convection, in order to understand the atmospheric circulation. Numerical simulations for an open neighbourhood of these values of the parameters pointed to the existence of a strange attractor, but this non-linear system of differential equations poses both numerical and analytical challenges to its understanding. Ten years after the introduction of this system, the so-called geometric Lorenz models were constructed as an attempt to rigorously understand the phenomena observed by Lorenz. They were proposed by Afraimovich et al in [1] and Guckenheimer, Williams in [16], independently. These models are three-dimensional flows for which it is possible to prove the existence of a strange attractor with regular solutions accumulating a singular (or an equilibrium) point. Moreover, this attractor is sensitive to initial conditions and can not be destroyed by small perturbations of the original flow, that is to say it is robust. Finally, Tucker [38, 39] proved the existence and robustness of the Lorenz attractor and, as a consequence of the method of his proof, showed that these models do describe the behaviour of (3.1). For more information on the history of the subject and the construction of the geometric models, we refer the reader to Araujo et al [4, 40] and references therein.

Basically, the study of the geometric Lorenz flows is done through the reduction to a Poincaré first return map to a global singular two-dimensional cross-section, which is then further reduced to the study of a one-dimensional transformation. This one-dimensional transformation is obtained by quotienting the return map over an invariant contracting foliation by curves which partition the cross-section. The map \( f \) satisfies (see figure 1)

1. \( f \) is discontinuous at \( x = 0 \) with \( \lim_{x \to 0^+} f(x) = +1 \) and \( \lim_{x \to 0^-} f(x) = -1 \);
2. \( f \) is differentiable on \([-1/2, 0) \cup (0, 1/2], f'(x) > \sqrt{2} \) and \( \lim_{x \to 0^+} \frac{f'(x)}{x} = \alpha_\# \), where \( 0 < \beta < 1 \);
3. \( f \) is topologically exact and thus transitive.

In [17], Gupta et al established exponential limiting laws for the extremal study of Lorenz-like maps in the deterministic setting. Later, in [11], Faranda et al gave numerical results for additive random perturbations of a family of Lorenz maps, pointing to the convergence of extreme values to the classical EVL distributions for increasing values of the noise. Here, we give an analytic solution for an arbitrary noise level as a result of theorems A, C and corollaries D, E.

**Corollary F.** Let \( f : S^1 \to S^1 \) be a map satisfying conditions (1)–(3) listed above, which is randomly perturbed as in (2.2) with noise distribution given by (2.3). For any point \( \zeta \in \mathcal{M} \), consider that \( X_0, X_1, \ldots \) is defined as in (2.6) and let \( u_\# \) be such that (2.9) holds. Then the stochastic process \( X_0, X_1, \ldots \) satisfies \( D_2(u_\#), D_3(u_\#) \) and \( D_4(u_\#) \), which implies that we have an EVL for \( M_n \) such that \( H(\tau) = e^{-\tau} \) and we have exponential HTS/RTS for balls around \( \zeta \). Moreover, the REPP \( N_n \) defined in (2.21) is such that \( N_n \xrightarrow{d} N \), as \( n \to \infty \), where \( N \) denotes a Poisson Process with intensity 1.
3.2. Countable branch case

We can also apply our results to full branch Markov maps with countable number of branches, like the Gauss map or maps in the setting of Rychlik’s theorem [35], as studied in [6, section 3.2.1] in the deterministic setting, since these classes of maps are transitive. Thus our results about EVL, HTS/RTS and REPP also hold for this class of systems.

3.3. Smooth interval maps

In [7, 8] Benedicks and Carleson proved the existence of a positive Lebesgue measure subset of parameters $P \subset [1, 2]$ of the family of quadratic maps $x \mapsto f_a(x) = a - x^2$ for which there exists an absolutely continuous $f_a$-invariant probability measure (acim) $\mu_a$ and $f_a$ is topologically mixing on the support $I_a = [f_a(0), f_a^2(0)]$ of $\mu_a$ for $a \in P$. In particular, these maps admit a dense forward orbit on the interval $I_a$.

In [28], Freitas and Freitas studied the extremal behaviour of Benedicks-Carleson (BC) maps in deterministic case and they got a standard exponential extreme value law. Later, in [11, section 4.3], the authors numerically studied the additive random perturbations of quadratic maps, which includes BC maps, and concluded that under suitable normalisations one should get standard exponential laws as well.

In fact, Lyubich [25] shows that almost all quadratic maps either admit an ergodic absolutely continuous invariant probability measure, where we can apply our results, or there exists a periodic sink whose basin covers the interval except a zero Lebesgue measure subset. This is typical of unimodal families [5]. In particular no requirements on decay of correlations are needed and so our results can also be applied to multimodal maps exhibiting an absolutely continuous invariant probability measure.

Applying our results, we can get the standard exponential limiting laws under random perturbations analytically. Moreover, we get results for HTS/RTS and REPP, as expected from the previous numerical studies.

3.4. Infinite modal maps

In [3, 30] certain parametrized families of one-dimensional maps with infinitely many critical points were analyzed from the measure-theoretical point of view; see figure 1.

It was proved that such families admit absolutely continuous invariant probability measures and are topologically mixing (in particular topologically transitive and so admit a dense orbit) for a positive Lebesgue measure subset of parameters. Moreover, the densities of these
measures vary continuously with the parameter and each measure exhibits exponential rate of mixing for Hölder observables.

We can apply our results to each map in these families obtaining EVL, HTS/RTS and REPP in the setting of the main results stated.

3.5. Piecewise expanding maps in higher dimensions

Again, we can apply all our main results on EVL, HTS/RTS and REPP to piecewise expanding maps in higher dimensions in the setting of [36], as studied in [6] in the deterministic setting.

4. Markov chains and decay against $L^1$ for random perturbations

In this section we show that random perturbations, as defined in section 2, of a transitive dynamical system define a Harris chain which also satisfies Doeblin’s condition, from which we get fast decay of correlations against all $L^1$ observables as a consequence of exponentially fast convergence to the equilibrium or stationary distribution.

We have already seen in section 2.1.3 that our random perturbations are a particular case of a Markov Chain with transition probabilities given by (2.4), with transition densities $(q_n)_{n \in \mathcal{M}}$ and stationary measure $\mathcal{P} = \mu$.

4.1. Harris and Doeblin conditions

Roughly speaking, a Harris chain is a Markov chain that returns to a particular part of the state space an unbounded number of times with positive probability. For the precise definition we follow Durrett [10].

We denote the $n$-th step transition probability by

$$p^n_x(A) = \int_0^\infty l_x(f^n_x) \, d\theta^n(\omega) = \int_A q^n_x \, d\text{Leb,} \quad n \geq 1.$$

Definition 4.1 (Harris chain). A Markov chain $\Phi_n$ is a Harris chain if one can find measurable sets $A, B \in \mathcal{B}$, a function $g$ and a constant $\xi > 0$ with $g(x, y) \geq \xi$ for $x \in A, y \in B$ and a probability measure $m$ concentrated on $B$ such that

(i) $p_x((x : \tau_B(x) < \infty)) > 0$ for Leb-a.e. $z$, where $\tau_B(x) = \inf\{n \geq 0 : p^n_x > 0\}$;
(ii) $x \in A, C \subset B \implies p_x(C) \geq \int_C g(x, y) \, dm(y)$.

A probability measure $\pi$ on $\mathcal{M}$ with $\int p_A \, \pi(dx) = \pi(A)$, for all $A \in \mathcal{B}$, is a stationary distribution for a Markov chain. As proved in [18], there exists a unique stationary distribution for Harris chains. In our setting $\pi = \mathcal{P} = \mu \times \theta^{\mathbb{N}}$.

Definition 4.2 (Aperiodicity). A Markov chain is aperiodic if there is no partition $\mathcal{M} = \bigsqcup_{i=1}^{\ell} \mathcal{M}_i$ (where $\bigsqcup$ represents a disjoint union) for some $\ell \geq 2$ which satisfies

$p_x \mathcal{M}_{i+1} = 1, \quad \forall x \in \mathcal{M}_i, i = 1, \ldots, \ell - 1$ and $p_x \mathcal{M}_1 = 1, \quad \forall x \in \mathcal{M}_\ell$.

Aperiodicity ensures ergodicity of the Markov Chain: the state space admits no decomposition into strictly smaller sets which are invariant under finitely many iterates of the process.
Definition 4.3 (Doeblin’s condition). There are $0 < \gamma < 1$, $\delta > 0$, an integer $k \geq 1$ and a probability measure $m$ so that $m(A) > \gamma \Rightarrow \rho^k_A \geq \delta$ for any measurable set $A$ and $m$-a.e. $x$.

This condition together with the previous one ensures uniform ergodicity of the chain, that is, any initial probability distribution in the state space converges to the unique stationary measure exponentially fast.

Next result gives a sufficient condition to obtain an aperiodic Harris chain satisfying Doeblin’s condition via random perturbation of a dynamical system.

Proposition 4.1. Let $f$ be a map randomly perturbed according to (2.2) with noise distribution given by (2.3) and such that there exists a full Leb-measure subset $Y$ and a point $x_0 \in \mathcal{M}$ satisfying $\{f^n x_0 : n \geq 0\}$ is both dense in $\mathcal{M}$ and a subset of $Y$. Then the random perturbation defines an aperiodic Harris chain which satisfies Doeblin’s condition.

Proof. For the Harris conditions we need to find the sets $A$, $B$ as in definition 4.1. Fix any positive Lebesgue measure Borel set $A$. For condition (i) of definition 4.1, we show that $\theta^\delta(\mathcal{M} : \exists n \geq 1$ s.t. $f^n(z) \in A) > 0$ for Leb a.e. $z \in \mathcal{M}$.

By assumption on $f$, there exists a point $x_0 \in Y$ so that $[f^n x_0 : n \geq 1] = \mathcal{M}$. Hence we can find $N \in \mathbb{N}$ so that $\{f^n x_0 : 0 \leq n \leq N\}$ is $\frac{\delta}{\kappa^k}$-dense and $\{f^n x_0 : k \leq n \leq N + k\}$ is also $\frac{\delta}{k}$-dense for all $k = 1, \ldots, N$.

Let $w \in A$ be a Lebesgue density point of $A$. Then, for Leb-a.e. $z \in \mathcal{M}$, we can find $x_1 \in B_{\rho_0}(fz) \subset f^k(\text{supp}(\theta))$ so that $x_1 = f^k x_0$ for some $0 \leq k \leq N$; and there exists $k \leq n \leq N + k$ such that $f^k x_1 \in B_{\rho_0}(w)$. In particular, $\text{Leb}(B_{\rho_0}(f^n x_1) \cap A) > 0$.

Then $\text{dist}(w,f^n x_1) < \frac{\delta}{\kappa^k}$ and $\text{dist}(fz,x_1) < \rho_0$. Since $x_1 \in Y$ we have that $q_{f^n x_1} \geq q$ in a $\rho_0$-neighborhood of $f^{k+1} x_1$ for $k = 0, \ldots, n$. Consequently, by definition of $(q_\cdot)_{x \in \mathcal{M}}$, for all small enough $\delta > 0$

\[
\int (f^{n+1})^\delta(B(w)) = \int d\theta^\delta(\mu) 1_{B(w)} \circ f^{n+1}(z) = \int d\zeta_1 \cdots d\zeta_N q(z_1) q(z_2) q(z_3) \cdots q(z_n) 1_{B(w)}(z_n) \geq q^\delta \cdot \text{Leb}(B(w)) > 0
\]

In particular, we get $\int (f^{n+1})^\delta(A) > 0$ by the choice of $w$. This also shows that

\[
\text{Leb}(A) > 0 \Rightarrow \tau_A(x) \leq 2N, \quad \text{Leb - a.e. } x \in \mathcal{M}
\]

and proves the following stronger statement than item (i) of definition 4.1

\[
\text{Leb}(A) > 0 \Rightarrow \text{Leb - a.e. } z \in \mathcal{M} \exists 0 \leq n(z) \leq 2N : p_{z}^{n+1}(A) > \frac{\delta^2}{4} \text{Leb}(A \cap B(w))
\]

for every Lebesgue density point $w$ of $A$ and every small enough $\delta > 0$.

Remark 4.1. The function $n = n(z)$ is locally constant for Leb-a.e. $z$ since $x_1 \in B_{\rho_0}(fz)$ for all $\zeta$ in a neighborhood of $z$, by the continuity of $f$ on $Y$.

To obtain item (ii) of definition 4.1, we take $B = B_{\rho_0/2}(f(x))$, for any given fixed $x \in A$. Then, for any $C \subseteq B$, using properties (2.3) we get
and so taking \( \xi = q \cdot \text{Leb}(B) \), \( m(C) := \frac{\text{Leb}(C \cap B)}{\text{Leb}(B)} \) for any Borel set \( C \subseteq B \) and \( g(x, y) := q_\xi(y) \) we are done.

Aperiodicity is a consequence of properties (2.3) together with the existence of a dense unperturbed orbit. Let us assume that there is a partition of \( \mathcal{M} \) as in definition 4.2. We can assume without loss of generality that \( \text{Leb}(\mathcal{M}_i) > 0 \) for all \( i = 1, \ldots, \ell \). Let \( \tilde{\mathcal{M}}_j \) denote the subset of Lebesgue density points of \( \mathcal{M}_i \) so that \( \text{Leb}(\mathcal{M}_i \setminus \tilde{\mathcal{M}}_i) = 0 \) and \( i = 1, \ldots, \ell \) and thus \( \mathcal{M} = \bigcup_{i=1}^{\ell} \tilde{\mathcal{M}}_i \), \( \text{Leb} \) mod 0. In particular we obtain \( \mathcal{M} = \bigcup_{i=1}^{\ell} \overset{\sim}{\mathcal{M}}_i \) and this cannot be a disjoint union, for otherwise \( \overset{\sim}{\mathcal{M}}_i = \mathcal{M} \setminus \overset{\sim}{\mathcal{M}}_i \) and \( \overset{\sim}{\mathcal{M}}_i \) would be open and closed, contradicting the connectedness of \( \mathcal{M} \) because \( \ell \geq 2 \).

Hence there exists \( x \in \overset{\sim}{\mathcal{M}}_i \cap \overset{\sim}{\mathcal{M}}_j \) for some \( i \neq j \) and we can find \( y = f^m x_0 \in \mathcal{M} \) for some \( n \geq 1 \) such that \( B_{p_\eta/\ell}(f^m) \ni x \) (recall that \( x_0 \) has dense positive orbit). Then we obtain

\[
[(f^m), \theta](\overset{\sim}{\mathcal{M}}_j) \ni \int_{B_{p_\eta/\ell}(f^m) \cap \overset{\sim}{\mathcal{M}}_j} q_\xi \text{d}\text{Leb} \geq q \cdot \text{Leb}\left(B_{p_\eta}(f^m) \cap \overset{\sim}{\mathcal{M}}_j\right) > 0
\]

since \( B_{p_\eta/\ell}(f^m) \cap \overset{\sim}{\mathcal{M}}_j \neq \emptyset \), by definition of Lebesgue density point. Analogously we get \( [(f^m), \theta](\overset{\sim}{\mathcal{M}}_j) > 0 \). Therefore we have found \( y \in \mathcal{M} \) such that \( 0 < p_\eta \mathcal{M}_i < 1 \) and \( 0 < p_\eta \mathcal{M}_j < 1 \). This shows that a partition of \( \mathcal{M} \) as in definition 4.2 cannot exist.

To obtain Doeblin’s condition we use properties (2.3). Let \( \gamma \in (0, 1) \) be such that \( \text{Leb}(B(z, \rho_0)) \gg \gamma \) for all \( x \in \mathcal{M} \), which exists by compactness of \( \mathcal{M} \). Let \( B \) be a Borel subset of \( \mathcal{M} \) such that \( \text{Leb}(B) \geq 1 - \gamma/2 \). Then for any Borel subset \( A \)

\[
\text{Leb}(A) > \gamma \implies p_\eta B = \int_B q_\eta \text{d}\text{Leb} \geq q \text{Leb}(B \cap B(f^m, \rho)) \geq \frac{\gamma}{2} \quad \text{Leb} - \text{a.e.} \ x \in \mathcal{M}.
\]

Hence letting \( \delta = q \gamma/2 \) we have obtained Doeblin’s condition for \( k = 1 \).

4.2. Strictly positive stationary density

We note that the absolute continuity assumption on \( f^* \theta \ll \text{Leb} \) for \( \text{Leb} \)-a.e. \( x \in \mathcal{M} \) ensures that the unique stationary probability measure is given by a distribution, that is, \( p = \mu \times \theta^\mathbb{N} \) where \( \mu = h \text{Leb} \) with \( h \geq 0 \), \( h \in L^1(\text{Leb}) \). Indeed, by definition of stationary measure, for \( \phi \in L^1(\mu) \)

\[
\mu(\phi) = \int \text{d}\mu(x) \int \text{d}\theta(\omega) \phi \circ f^m(x) = \int \text{d}\mu(x) \int \text{d}(f^m)^* \phi = \int \text{d}\mu(x) \int \text{d}\text{Leb} \phi \cdot q_\xi
\]

and if \( \phi = 1_A \) with \( A \in \mathcal{B} \) so that \( \text{Leb}(A) = 0 \), we get \( \mu(A) = \int \text{d}\mu(x) \int_A \text{d}\text{Leb} q_\xi = 0 \), showing that \( \mu \ll \text{Leb} \) and that \( h = \frac{d\mu}{d\text{Leb}} \) is as claimed.

This density is strictly positive as a consequence of (4.1) and (4.2) together with remark 4.1. Indeed, because \( \mu \) is stationary we get, fixing \( z \in Y \) and a Borel subset \( A \) such that \( \text{Leb}(A) > 0 \), the existence of \( 0 \leq n \leq 2N \) satisfying

\[
\mu(A^\mathbb{N}) = \int \text{d}(\mu \times \theta^\mathbb{N})(x, \omega) 1_{A^\mathbb{N}} \circ f^m(x) = \int \text{d}\mu(x) \text{d}(f^m)^0(x, \theta^\mathbb{N}) 1_{A^\mathbb{N}} \geq q^{n-1} \mu(B_\rho(z)) \text{Leb}(A^\mathbb{N})
\]
where \( A^w := A \cap B_\varepsilon(w) \), for every Lebesgue density point \( w \) of \( A \) and every small enough \( \delta, \delta > 0 \). Note that \( \delta \) depends on \( w \) and \( A \) and \( \delta \) depends on \( z \), but \( n = n(z) \) is uniformly bounded from above. Since \( d\mu = h \ d\text{Leb} \) we obtain for all small enough \( \delta > 0 \)

\[
\frac{1}{\text{Leb}(B_\varepsilon(w))} \int_h 1_{A \cap B_\varepsilon(w)} \ d\text{Leb} \geq g^{n-1} \mu(B_\varepsilon(z)) \frac{\text{Leb}(A \cap B_\varepsilon(w))}{\text{Leb}(B_\varepsilon(w))}
\]

and by the Lebesgue differentiation theorem and the choice of \( w \) we obtain

\[
h(w) \geq \mu(B_\varepsilon(z)) \cdot \inf \{ q^k : 0 \leq k \leq 2N \} =: h
\]

Hence we conclude \( h \geq h > 0 \), Leb-a.e..

4.3. Deduction of the main results

Now we use the previous observations to complete the proofs of the main results.

**Proof of theorems A and B.** Since the chain is an aperiodic Harris chain which also satisfies Doeblin’s condition (see proposition 4.1), we can use the equivalence of items (ii) and (iv) in [27, theorem 16.0.2] and conclude uniform ergodicity: there exist \( \lambda > 1 \) and \( C < \infty \) such that for Leb-a.e. \( x \in \mathcal{M} \) and each \( n \geq 1 \)

\[
\| p^n_x - \mu \| \leq C \lambda^{-n}, \tag{4.3}
\]

where \( \| \cdot \| \) stands for the total variation norm, i.e. \( \| p^n_x - \mu \| = \sup_{A \in \mathcal{B}} | p^n_x A - \mu(A) | \). Since all probability measures involved here have densities, (4.3) is equivalent to

\[
\frac{1}{2} \| q^n_x - h \|_{L^1(\text{Leb})} = \frac{1}{2} \int |q^n_x - h| \ d\text{Leb} \leq C \lambda^{-n}. \tag{4.4}
\]

If \( \psi \in L^\infty(\text{Leb}) = L^\infty(\mu) \) and \( \phi \in L^1(\mu) \), then

\[
\int \psi \circ f^n(x) \cdot \phi(x) \ d\mathbb{P}(x, \omega) - \mu(\psi) \mu(\phi) = \int d\mu(x) \phi(x) \int \mathbb{P}(f^n(x), \omega)^{\theta^n} \psi - \int d\mu(\phi) \cdot \int d\mu(\psi)
\]

\[
= \int d\mu(x) \phi(x) \left( \int d\text{Leb} \psi \cdot (q^n_x - h) \right) \quad \text{(using (4.4))}
\]

\[
\leq \int d\mu \phi \| \psi \|_{L^\infty(\mu)} \| \phi \|_{L^1(\mu)} 2C \lambda^{-n} \leq 2C \| \psi \|_{L^\infty(\mu)} \| \phi \|_{L^1(\mu)} \lambda^{-n} \tag{4.5}
\]

concluding the proof of annealed decay of correlations against \( L^1 \) observables. \( \square \)

**Proof of theorem C and corollary E.** As already explained in section 2, it is enough to show that (4.5) implies all conditions \( D_2(u_n), D_3(u_n) \) and \( D'(u_n) \).

Condition \( D_3(u_n) \) is designed to follow easily from fast decay of correlations. In fact, recalling (2.11), if we choose \( \phi = 1_{U_0} \) and \( \psi = \int_1 \phi(x, \varphi_{B_{z_1}(x), \ldots, \varphi_{B_{z_{n-1}}(x)}(x) \in u_n}) \ d\theta^{n-1}(\omega) \), then we can take \( \gamma(n, t) = \gamma(t) = C^t \lambda^{-t} \) in condition \( D_3(u_n) \) for some \( C^t > 0, \lambda > 1 \) and \( t_n = o(n) \) coming from (4.5). A very similar reasoning applies to get condition \( D_3(u_n) \) by choosing the same \( \phi \) but \( \psi = \int_{l_2} \ d\theta^{n}(\omega) \); where \( A \in \mathcal{R} \) and \( Z = Z(\omega) = \bigcap_{l \in \mathbb{A} \cap \mathbb{N}} \{ x : f^l(x) \leq u_n \} \); see the proof of [6, theorem D and corollary F].
Now we show that $D'(u_n)$ holds automatically in our random setting. By assumption (R1) we have that, for $n$ sufficiently large, $U_n$ is topologically a ball around a chosen point $\zeta$ and $n\mu(U_n) \to \tau$ by (2.9). Moreover, we have from (2.3) that

$$
(f^i_x \theta)U_n = \int q_x \cdot 1_{U_n} \, d\text{Leb} \leq \frac{\tilde{q}}{B} \cdot \text{Leb}(U_n), \quad \text{Leb} - \text{a.e.} \, x \in \mathcal{M}.
$$

Consequently, we can estimate

$$
\mathbb{P}((x, \omega) \in \Omega : x \in U_n \quad \text{and} \quad f^i_x(x) \in U_n) = \int_{U_n} \mathcal{d}\mu(x) 1_{U_n}(x) \int \mathcal{d}\theta \mathcal{H}(\omega) 1_{U_n} \circ f^i_x(x) \\
\leq \mu(U_n) \int \mathcal{d}\mu(x) \int \mathcal{d}\theta \mathcal{H}(\omega) \mathcal{H}(f^i_x \circ f^{i-1}_x)(U_n) \\
= \leq \mu(U_n) \cdot \frac{\tilde{q}}{B} \cdot \text{Leb}(U_n) \leq \frac{\tilde{q}}{B} \cdot n\mu(U_n)^2
$$

since $\mu \geq \frac{\tilde{q}}{B} \text{Leb}$ by section 4.2. Hence we get

$$
\begin{align*}
\sum_{k=1}^{n} \mathbb{P}(X_0 > u_n, X_1 > u_n) &= \sum_{k=1}^{n} \mathbb{P}((x, \omega) \in \Omega : x \in U_n \quad \text{and} \quad f^i_x(x) \in U_n) \\
&\leq \frac{n \tilde{q}}{B} \mu(U_n)^2 \leq \frac{\tilde{q}}{B} \cdot (n\mu(U_n))^2 \cdot \frac{1}{k_n} \to 0 \quad \text{as} \, \frac{k_n}{n} \to \infty
\end{align*}
$$

since $k_n \to \infty$ by definition (2.13). This completes the proof of condition $D'(u_n)$. \hfill \Box

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References

[1] Afraimovich V S, Bykov V V and Shil’nikov L P 1977 On the appearence and structure of the Lorenz attractor Dokl. Acad. Sci. USSR 234 336–9
[2] Araújo V 2000 Attractors and time averages for random maps Inst. Henri Poincaré 17 307–69
[3] Araújo V and Pacifico M J 2009 Physical measures for infinite-modal maps Fundam. Math. 203 211–62
[4] Araújo V and Pacifico M J 2010 Three-dimensional flows Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics (Results in Mathematics and Related Areas. 3rd Series vol 53) (New York: Springer) (with a foreword by Marcelo Viana)
[5] Avila A and Moreira C G 2005 Statistical properties of unimodal maps: physical measures, periodic orbits and pathological laminations Publ. Math. Inst. Hautes Études Sci. 101 1–67
[6] Aytaç H, Freitas J M and Vaienti S 2015 Laws of rare events for deterministic and random dynamical systems Trans. Am. Math. Soc. 367 8229–78
[7] Benedicks M and Carleson L 1985 On iterations of $1 - ax^2$ on $(-1, 1)$ Ann. Math. 122 1–25
[8] Benedicks M and Carleson L 1991 The dynamics of the Hénon map Ann. Math. 133 73–169
[9] Collet P 2001 Statistics of closest return for some non-uniformly hyperbolic systems Ergod. Theor. Dynam. Syst. 21 401–20
[10] Durrett R 2010 Probability: Theory and Examples (Cambridge Series in Statistical and Probabilistic Mathematics) (Cambridge: Cambridge University Press)

[11] Faranda D, Freitas J M, Lucarini V, Turchetti G and Vaijenti S 2013 Extreme value statistics for dynamical systems with noise Nonlinearity 26 2597–622

[12] Freitas A C M and Freitas J M 2008 On the link between dependence and independence in extreme value theory for dynamical systems Stat. Probab. Lett. 78 1088–93

[13] Freitas A C M, Freitas J M and Todd M 2010 Hitting time statistics and extreme value theory Probab. Theory Relat. Fields 147 675–710

[14] Freitas A C M, Freitas J M and Todd M 2011 Extreme value laws in dynamical systems for non-smooth observations J. Stat. Phys. 142 108–26

[15] Freitas J M 2013 Extremal behaviour of chaotic dynamics Dyn. Syst. 28 302–32

[16] Guckenheimer J and Williams R F 1979 Structural stability of Lorenz attractors Publ. Math. IHES 50 59–72

[17] Gupta C, Holland M and Nicol M 2011 Extreme value theory and return time statistics for dispersing billiard maps and flows, Lozi maps and Lorenz-like maps Ergod. Theor. Dynam. Syst. 31 1363–90

[18] Harris T E 1956 The existence of stationary measures for certain Markov processes Proc. Third Berkeley Symp. on Mathematical Statistics and Probability, 1954–1955 vol II (Berkeley, CA: University of California Press) pp 113–24

[19] Haydn N, Lacroix Y and Vaienti S 2005 Hitting and return times in ergodic dynamical systems Ann. Probab. 33 2043–50

[20] Jost J, Kell M and Rodrigues C S 2015 Representation of Markov chains by random maps: existence and regularity conditions. Calculus Variations PDE 54 2637–55

[21] Kifer Y 1986 Ergodic theory of random transformations Progress in Probability and Statistics vol 10 (Boston, MA: Birkhäuser)

[22] Kifer Y and Liu P-D 2006 Chapter 5: Random dynamics Handbook of Dynamical Systems (Handbook of Dynamical Systems vol 1) ed B Hasselblatt and A Katok (Amsterdam: Elsevier) pp 379–499

[23] Leadbetter M R, Lindgren G and Rootzén H 1988 Extremes and Related Properties of Random Sequences and Processes (Springer Series in Statistics) (New York: Springer)

[24] Lorenz E N 1963 Deterministic nonperiodic flow J. Atmosph. Sci. 20 130–41

[25] Lyubich M 2002 Almost every real quadratic map is either regular or stochastic Ann. Math. 156 1–78

[26] Marie P and Rousseau J 2011 Recurrence for random dynamical systems Discrete Contin. Dyn. Syst. 30 1–16

[27] Meyn S and Tweedie R L 2009 Markov Chains andStochastic Stability 2nd edn (Cambridge: Cambridge University Press) (with a prologue by Peter W Glynn)

[28] Moreira Freitas A C and Freitas J M 2008 Extreme values for Benedicks–Carleson quadratic maps Ergod. Theor. Dynam. Syst. 28 1117–33

[29] Ohno T 1983 Asymptotic behaviors of dynamical systems with random parameters Publ. RIMS Kyoto Univ. 19 83–98

[30] Pacifico M J, Rovella A and Viana M 1998 Infinite-modal maps with global chaotic behavior Ann. Math. 148 441–84

[31] Palis J and de Melo W 1982 Geometric Theory of Dynamical Systems (New York: Springer)

[32] Rousseau J 2014 Hitting time statistics for observations of dynamical systems Nonlinearity 27 2377

[33] Rousseau J, Saussol B and Varandas P 2014 Exponential law for random subshifts of finite type Stoch. Process. Appl. 124 3260–76

[34] Rousseau J and Todd M 2015 Hitting times and periodicity in random dynamics J. Stat. Phys. 161 131–50

[35] Rychlik M 1983 Bounded variation and invariant measures Stud. Math. 76 69–80

[36] Saussol B 2000 Absolutely continuous invariant measures for multi-dimensional expanding maps Israel J. Math. 116 223–48

[37] Saussol B 2009 An introduction to quantitative poincaré recurrence in dynamical systems Rev. Math. Phys. 21 949–79

[38] Tucker W 1999 The Lorenz attractor exists C. R. Acad. Sci., Paris I 328 1197–202

[39] Tucker W 2002 A rigorous ODE solver and Smale’s 14th problem Found. Comput. Math. 2 53–117

[40] Viana M 2000 What’s new on Lorenz strange attractors Math. Intelligencer 22 6–19