On existence of primitive normal elements of rational form over finite fields of even characteristic

Himangshu Hazarika\textsuperscript{a} and Dhiren Kumar Basnet\textsuperscript{b,*}

\textsuperscript{a,b}Department of Mathematical Sciences, Tezpur University, Assam, India
\textsuperscript{a}diku_95@tezu.ernet.in, \textsuperscript{b}dbasnet@tezu.ernet.in

Key words: Finite field, Primitive element, Free element, Normal basis, Character.

MSC: 12E20, 11T23

Abstract

Let \( q \) be an even prime power and \( m \geq 2 \) an integer. By \( \mathbb{F}_q \), we denote the finite field of order \( q \) and by \( \mathbb{F}_{q^m} \), its extension degree \( m \). In this paper we investigate the existence of primitive normal pair \((\alpha, f(\alpha))\), where \( f(x) = \frac{ax^2+bx+c}{dx+e} \in \mathbb{F}_{q^m}(x) \) with \( a \neq 0, dx + e \neq 0 \) in \( \mathbb{F}_{q^m} \) and establish some sufficient conditions to show that nearly all fields of even characteristic possess such elements. We conclude the paper by providing an explicit small list of genuine exceptional pairs \((q, m)\).

1 Introduction

Given an even prime power \( q \) and an integer \( m \geq 2 \), we denote by \( \mathbb{F}_q \), the finite field of order \( q \) and by \( \mathbb{F}_{q^m} \) its extension field of degree \( m \). A generator of the (cyclic) multiplicative group \( \mathbb{F}_{q^m}^* \) is defined as primitive element. Note that, for any finite field \( \mathbb{F}_q \), there are \( \phi(q - 1) \) primitive elements, where \( \phi \) is the Euler’s phi-function. Further, a basis of the \{\( \alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{m-1}} \)\} of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \) is called a normal basis and such an element is called a normal element or a free element.

The readers are referred to \cite{12} and the references therein for the existence of both primitive and free elements. The simultaneous occurrence of primitive and free elements in \( \mathbb{F}_{q^m} \) is given by the following theorems.

**Theorem 1.1. (Primitive normal basis theorem \cite{5}).** For any prime power \( q \) and positive integer \( m \), the finite field \( \mathbb{F}_{q^m} \) always contains some element which is simultaneously primitive and free.

At first, this result was proved by Lenstra and Schoof in \cite{11}. Later on by using a sieving technique, Cohen and Huczynska \cite{3} provided a computer-free proof which was initially introduced by Cohen \cite{16}.
Theorem 1.2. (Strong primitive normal basis theorem [6]) In the finite field $\mathbb{F}_{q^m}$, there exists some element $\alpha$ such that both $\alpha$ and $\alpha^{-1}$ are primitive normal, with exceptional pairs for $(q, m)$ are $(2, 3), (2, 4), (3, 4), (4, 3)$ and $(5, 4)$.

Tian and Qi were first to provide this result in [13] for $m \geq 32$. Later on Cohen and Huczynska [5, 6] completed the proof up to the above form by using a sieving technique.

The next theorem was given by Kapetanakis in [12] by employing the Sieve technique, which flows to the functions of the quotient form.

Theorem 1.3. [12] For odd prime power $q \geq 23$, an integer $m \geq 17$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(\mathbb{F}_q)$, with the condition that $A$ has exactly two non-zero entries and $q$ is odd, then the quotient of these entries is a square in $\mathbb{F}_{q^m}$. Then there exists some $\alpha \in \mathbb{F}_{q^m}$ such that both $\alpha$ and $\frac{ax+b}{\alpha x+d}$ are simultaneously primitive and free.

The existence of a primitive element $\alpha \in \mathbb{F}_q$ such that $f(\alpha)$ is also primitive for an arbitrary quadratic in $\mathbb{F}_q[x]$ has been completely resolved in [2].

Theorem 1.4 ([2]). For all $q > 211$, there always exists an element $\alpha \in \mathbb{F}_{q^m}$ such that $\alpha$ and $f(\alpha)$ are both primitive, where $f(x) = ax^2 + bx + c$ with $b^2 - 4ac \neq 0$.

It is worth mentioning that, this paper is an extension of Theorem 1.3 for finite fields. We solve the existence question for elements $\alpha$ of $\mathbb{F}_{q^m}$ that both $\alpha$ and $f(\alpha)$ are simultaneously primitive and normal over $\mathbb{F}_q$, where $f(x) = \frac{ax^2+bx+c}{dx+e}$ (with conditions $a \neq 0$ and $dx + e \neq 0$) $\in \mathbb{F}_{q^m}(x)$. Throughout this paper we denote the pair we denote the pair $(q, m)$ as primitive normal pair if the field $\mathbb{F}_{q^m}$ contains such elements.

This work is heavily influenced by the the works of Lenstra and Schoof [11] while character sum plays a very crucial role in our result. We also use Kloosterman sum over finite fields for better approximations. But for more accurate results, we follow the sieving technique provided by Cohen and Huczynska [5, 6] whose techniques have been modified without loss of generality.

In the Section 3, we estimate a lower bound for existence of primitive normal pair. Then in Section 4, by using "the prime sieve technique", we modify the sufficient condition for more efficient results. Finally, in Section 5 we apply the existence conditions on fields of even characteristic for brief study of each and every possible cases and provide an explicit small list of genuine exceptional pairs $(q, m)$.

2 Preliminaries

Under the condition $f \circ \alpha = \sum_{i=1}^{n} a_i \alpha^q$ and $f = \sum_{i=1}^{n} a_i x^i \in \mathbb{F}_q[x]$ for $\alpha \in \mathbb{F}_{q^m}$; the additive group of $\mathbb{F}_{q^m}$ is a $\mathbb{F}_q[x]$-module. The $\mathbb{F}_q$-order of $\alpha \in \mathbb{F}_{q^m}$, is the monic $\mathbb{F}_q$-divisor $g$ of $x^m - 1$ of minimal degree such that $g \circ \alpha = 0$, i.e. the annihilator of $\alpha$ has unique monic generator which we define as $Order$ of $\alpha$ and denote by $Ord(\alpha)$. It is clear that the elements in $\mathbb{F}_{q^m}$ that are free are exactly those of $\mathbb{F}_q$ order $x^m - 1$.

Now the multiplicative order for $\alpha \in \mathbb{F}_{q^m}^*$ is denoted by ord$(\alpha)$ and $\alpha$ is primitive if and only if ord$(\alpha) = q^m - 1$. Furthermore, it follows from the definitions that $q^m - 1$ and $x^m - 1$ can
be freely replaced by their radicals \( q_0 \) and \( f_0 := x^{m_0} - 1 \) respectively, where \( m_0 \) is such that \( m = m_0 p^a \), where \( a \) is a non negative integer and \( \gcd(m_0, p) = 1 \).

Throughout this section we present a couple of functions that characterize primitive and free elements. To represent those functions, the idea of character of finite abelian group is necessary.

**Definition 2.1.** Let \( G \) be a finite abelian group. A character \( \chi \) of \( G \) is a group homomorphism from \( G \) into the group \( S^1 := \{ z \in \mathbb{C} : |z| = 1 \} \). The characters of \( G \) form a group under composition called dual group or character group of \( G \) which is denoted by \( \hat{G} \) and is isomorphic to \( G \). Again the character \( \chi_0 \) defined as \( \chi_0(a) = 1 \) for all \( a \in G \) is denoted for the trivial character of \( G \).

In a finite field \( \mathbb{F}_{q^m} \), additive group \( \mathbb{F}_{q^m} \) and multiplicative group \( \mathbb{F}_{q^m}^* \) are the abelian groups. Throughout this paper we denote the character of additive group \( \mathbb{F}_{q^m} \) as additive character and character of \( \mathbb{F}_{q^m}^* \) as multiplicative character. Multiplicative characters are extended from \( \mathbb{F}_{q^m}^* \) to \( \mathbb{F}_{q^m} \) by the rule \( \chi(0) = \begin{cases} 
0 & \text{if } \chi \neq \chi_0, \\
1 & \text{if } \chi = \chi_0.
\end{cases} \)

Since \( \hat{\mathbb{F}_{q^m}}^* \cong \mathbb{F}_{q^m}^* \), so \( \hat{\mathbb{F}_{q^m}}^* \) is cyclic and for any divisor \( d \) of \( q^m - 1 \), there are exactly \( \phi(d) \) characters of order \( d \) in \( \hat{\mathbb{F}_{q^m}}^* \).

Let \( e \mid q^m - 1 \), then \( \alpha \in \mathbb{F}_{q^m} \) is called \( e \)-free if \( d \mid e \) and \( \alpha = \beta^d \), for some \( \beta \in \mathbb{F}_{q^m} \) implies \( d = 1 \). Furthermore \( \alpha \) is primitive if and only if \( \alpha = \beta^e \), for some \( \beta \in \mathbb{F}_{q^m} \) and \( e \mid q^m - 1 \) implies \( e = 1 \).

For any \( e \mid q^m - 1 \), following Cohen and Huczynska \[5, 6\], we define the character function for the subset of \( e \)-free elements of \( \mathbb{F}_{q^m}^* \) by

\[ \rho_e : \alpha \mapsto \theta(e) \sum_{d \mid e} \left( \frac{\mu(d)}{\phi(d)} \sum_{\chi_d} \chi_d(\alpha) \right), \]

where \( \theta(e) := \frac{\phi(e)}{e} \), \( \mu \) is the Möbius function and \( \chi_d \) stands for any multiplicative character of order \( d \). For any \( e \mid q^m - 1 \), we use “integral” notation due to Cohen and Huczynska \[5, 6\], for weighted sums as follows

\[ \int_{d \mid q^m - 1} \chi_d := \sum_{d \mid q^m - 1} \frac{\mu(d)}{\phi(d)} \sum_{\chi_d} \chi_d \]

Then the character function for the subset of \( e \)-free elements of \( \mathbb{F}_{q^m}^* \) becomes,

\[ \rho_e : \alpha \mapsto \theta(e) \int_{d \mid e} \chi_d(\alpha) \]

Again, for any monic \( \mathbb{F}_q \)-divisor \( g \) of \( x^m - 1 \), a typical additive character \( \psi_g \) of \( \mathbb{F}_q \)-order \( g \) is one such that \( \psi_g \circ g \) is the trivial character of \( \mathbb{F}_{q^m} \) and \( g \) is of minimal degree satisfying this property. Furthermore, there are \( \Phi(g) \) characters \( \psi_g \), where \( \Phi(g) = (\mathbb{F}_q[x]/g\mathbb{F}_q[x])^* \) is the analogue of Euler function over \( \mathbb{F}_q[x] \).
Then the character function for the set of $g$-free elements in $\mathbb{F}_{q^m}$, for any $g|x^m - 1$ is given by

$$
\kappa_g : \alpha \mapsto \Theta(g) \sum_{f|g} \left( \frac{\mu'(f)}{\Phi(f)} \sum \psi_f(\alpha) \right),
$$

where $\Theta(g) := \frac{\Phi(g)}{\phi(g)}$, the sum runs over all additive characters $\psi_f$ of $\mathbb{F}_q$-order $g$ and $\mu'$ is the analogue of the Möbius function which is defined as follows:

$$
\mu'(g) = \begin{cases} 
(-1)^s & \text{if } g \text{ is a product of } s \text{ distinct irreducible monic polynomials} \\
0 & \text{otherwise}
\end{cases}
$$

We use the “integral” notation for weighted sum of additive characters as follows

$$
\int \psi_f := \sum_{f|g} \frac{\mu'(f)}{\Phi(f)} \sum \psi_f
$$

Then the character function for the set of $g$-free elements in $\mathbb{F}_{q^m}$, for any $g|x^m - 1$ is given by

$$
\kappa_g : \alpha \mapsto \Theta(g) \int_{f|g} \psi_f(\alpha)
$$

>From [13], we have the following about the typical additive character. Let $\lambda$ be the canonical additive character of $\mathbb{F}_q$. Thus for $\alpha \in \mathbb{F}_q$, this character is defined as $\lambda(\alpha) = \exp^{2\pi i \text{Tr}(\alpha)/p}$, where $\text{Tr}(\alpha)$ is absolute trace of $\alpha$ over $\mathbb{F}_p$.

Now let $\psi_0$ be canonical additive character of $\mathbb{F}_{q^m}$; which is simply the lift of $\lambda$ to $\mathbb{F}_{q^m}$, i.e., $\psi_0(\alpha) = \lambda(\text{Tr}(\alpha))$, $\alpha \in \mathbb{F}_{q^m}$. Now for any $\delta \in \mathbb{F}_{q^m}$, let $\psi_\delta$ be the character defined by $\psi_\delta(\alpha) = \psi_0(\delta \alpha)$, $\alpha \in \mathbb{F}_{q^m}$. Define the subset $\Delta_g$ of $\mathbb{F}_{q^m}$ as the set of $\delta$ for which $\psi_\delta$ has $\mathbb{F}_q$-order $g$. So we may also write $\psi_{\delta_g}$ for $\psi_\delta$, where $\delta_g \in \Delta_g$. So with the help of this we can express any typical additive character $\psi_g$ in terms of $\psi_{\delta_g}$ and further we can express this in terms of canonical additive character $\psi_0$.

In the following sections we will encounter various character sums and a lower bound, or at least an estimation, for that the following sums will be necessary. The following results are well established and useful in proving our results in subsequent sections.

**Lemma 2.1.** ([12], theorem 5.4) *(Orthogonality relations)* For any nontrivial character $\chi$ of a finite abelian group $G$ and any nontrivial element $\alpha \in G$, following are orthogonality relations

$$
\sum_{\alpha \in G} \chi(\alpha) = 0 \quad \text{and} \quad \sum_{\chi \in \hat{G}} \chi(\alpha) = 0.
$$

**Lemma 2.2.** ([14], corollary 2.3) Consider any two nontrivial multiplicative characters $\chi_1, \chi_2$ of the finite field $\mathbb{F}_{q^m}$. Again, let $f_1(x)$ and $f_2(x)$ be two monic pairwise co-prime polynomials in $\mathbb{F}_{q^m}[x]$, such that at least one of $f_i(x)$ is of the form $g(x)^{\text{ord}(\chi_i)}$ for $i = 1, 2$; where $g(x) \in \mathbb{F}_{q^m}[x]$ with degree at least 1. Then

$$
\left| \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_1(f_1(\alpha)) \chi_2(f_2(\alpha)) \right| \leq (n_1 + n_2 - 1)q^{m/2},
$$

where $n_1$ and $n_2$ are the degrees of largest square free divisors of $f_1$ and $f_2$, respectively.
Lemma 2.3. ([7], theorem 5.6) Let \( \chi \) and \( \psi \) are two non-trivial multiplicative and additive character for the field \( \mathbb{F}_{q^m} \) respectively. Let \( \mathcal{F}, \mathcal{G} \) be rational functions in \( \mathbb{F}_{q^m}(x) \), where \( \mathcal{F} \neq \beta \mathcal{H}^n \) and \( \mathcal{G} \neq \mathcal{H}^p - \mathcal{H} + \beta \), for any \( \mathcal{H} \in \mathbb{F}_{q^m}(x) \) and any \( \beta \in \mathbb{F}_{q^m} \), and \( n \) is the order of \( \chi \).

Then

\[
\left| \sum_{\alpha \in \mathbb{F}_{q^m} \setminus \mathbb{S}} \chi(\mathcal{F}(\alpha))\psi(\mathcal{G}(\alpha)) \right| \leq [\text{deg}(\mathcal{G}_\infty) + k_0 + k_1 - k_2 - 2]q^{m/2},
\]

where \( \mathbb{S} \) denotes the set of all poles of \( \mathcal{F} \) and \( \mathcal{G} \), \( \mathcal{G}_\infty \) denotes the pole divisor of \( \mathcal{G} \), \( k_0 \) denotes the number of distinct zeroes and poles of \( \mathcal{G} \) in the algebraic closure \( \overline{\mathbb{F}}_{q^m} \) of \( \mathbb{F}_{q^m} \), \( k_1 \) denotes the number of distinct poles of \( \mathcal{G} \) (including infinite pole) and \( k_2 \) denotes the number of finite poles of \( \mathcal{F} \), that are also zeroes or poles of \( \mathcal{G} \).

Lemma 2.4. ([7]) Suppose that \( f_1(x), f_2(x), \ldots, f_s(x) \in \mathbb{F}_{q^m}[x] \) be distinct irreducible polynomials. Let \( \chi_1, \chi_2, \ldots, \chi_s \) are multiplicative characters and \( \psi \) be a non-trivial additive character of \( \mathbb{F}_{q^m} \), then

\[
\left| \sum_{y \in \mathbb{F}_{q^m}, f_i(y) \neq 0} \chi_1(f_1(y))\chi_2(f_2(y)) \ldots \chi_s(f_s(y))\psi(y) \right| \leq kq^{m/2},
\]

where \( k = \sum_{i=1}^{s} \text{deg}(f_i) \).

Definition 2.2. (Kloosterman Sum) For a non-trivial additive character \( \psi \) of the finite field \( \mathbb{F}_{q^m} \), the sum

\[
K(\psi; a, b) := \sum_{\alpha \in \mathbb{F}_{q^m}^*} \psi(a\alpha + b\alpha^{-1}),
\]

where \( a, b \in \mathbb{F}_{q^m} \) is called Kloosterman sum.

Lemma 2.5. ([3], theorem 5.45) If the finite field \( \mathbb{F}_{q^m} \) has a non-trivial additive character \( \psi \) and \( a, b \in \mathbb{F}_{q^m} \) are not both zero, then the Kloosterman sum satisfies

\[
|K(\psi; a, b)| \leq 2q^{m/2}.
\]

3 A lower bound for \( M(e_1, e_2, g_1, g_2) \)

In this section, we try to estimate some results on the element \( \alpha \in \mathbb{F}_{q^m} \) such that both \( \alpha \) and \( f(\alpha) \) are simultaneously primitive normal elements in \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \). We consider \( q \) as even prime power, i.e. \( q = 2^k \), where \( k \) is a positive integer. Take \( e_1, e_2 \) such that \( e_1, e_2|q^m - 1 \) and \( g_1, g_2 \) such that \( g_1, g_2|q^m - 1 \). Considering \( M(e_1, e_2, g_1, g_2) \) to be the number of \( \alpha \in \mathbb{F}_{q^m} \) such that \( \alpha \) is both \( e_1 \)-free and \( g_1 \)-free and \( f(\alpha) \) is \( e_2 \)-free, \( g_2 \)-free; where \( f(x) = \frac{ae^2 + bx + c}{dx + e} \) and \( a, b, c, d, e \in \mathbb{F}_{q^m} \), \( a \neq 0 \) and \( dx + e \neq 0 \).

Throughout this paper, we use the notations \( \omega(n) \) and \( g_d \) to denote number of prime divisors of \( n \) and the number of monic irreducible factors of \( g \) over \( \mathbb{F}_q \) respectively. For calculations we use \( W(n) := 2^{\omega(n)} \) and \( \Omega(g) := 2^{\omega} \).
Theorem 3.1. Let \( f(x) = \frac{ax^2+bx+c}{dx+e} \in \mathbb{F}_q^m(x) \) with \( a \neq 0 \neq dx+e \) and \( f(x) \neq yx, yx^2 \) for any \( y \in \mathbb{F}_q^m \). Suppose \( e_1, e_2 \) divide \( q^m-1 \) and \( g_1, g_2 \) divide \( x^m-1 \). Then

\[
\mathcal{M}(e_1, e_2, g_1, g_2) \geq \theta(e_1)\theta(e_2)\Theta(g_1)\Theta(g_2)q^{m/2} \left( q^{m/2} - 4W(e_1)W(e_2)\Omega(g_1)\Omega(g_2) \right). \tag{3.1}
\]

Hence \( \mathcal{M}(e_1, e_2, g_1, g_2) > 0 \) whenever

\[
q^{m/2} > 4W(e_1)W(e_2)\Omega(g_1)\Omega(g_2). \tag{3.2}
\]

In particular, \( \mathcal{M}(q^m-1, q^m-1, x^m-1, x^m-1) > 0 \) if

\[
q^{m/2} > 4W(q^m-1)^2\Omega(x^m-1)^2, \tag{3.3}
\]

i.e., this is a sufficient condition for a field \( \mathbb{F}_{q^m} \) to have an element \( \alpha \) such that both \( \alpha \) and \( f(\alpha) \) are simultaneously primitive normal.

**Proof.** At first we establish the result for \( d \neq 0 \). From the definition we have,

\[
\mathcal{M}(e_1, e_2, g_1, g_2) = \theta(e_1)\theta(e_2)\Theta(g_1)\Theta(g_2) \int_{d_1|e_1} \int_{d_2|e_2} S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2}), \tag{3.4}
\]

where

\[
S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2}) = \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_{d_1}(\alpha)\chi_{d_2}(f(\alpha))\psi_{h_1}(\alpha)\psi_{h_2}(f(\alpha))
\]

As there exist some \( l_1, l_2 \in \{0, 1, \ldots, q^m-2\} \) such that \( \chi_{l_1}(\alpha) = \chi_{q^m-1}(\alpha^{l_1}) \), for \( i = 1, 2 \) and \( \psi_{h_1}(\alpha) = \psi_{q^m-1}(\beta_i\alpha) \), for some \( \beta_i \in \mathbb{F}_{q^m} \) for \( i = 1, 2 \); so we have the following expression

\[
S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2}) = \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_{q^m-1}(\alpha^{l_1}(f(\alpha))^{l_2})\psi_{q^m-1}(\beta_1\alpha + \beta_2 f(\alpha))
\]

\[
= \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_{q^m-1}(\mathfrak{g}(\alpha))\psi_{q^m-1}(\mathfrak{h}(\alpha)),
\]

where \( \mathfrak{g}(x) = x^{l_1}(ax^2+bx+c)^{l_2} \), and \( \mathfrak{h}(x) = \beta_1 x + \beta_2 (ax^2+bx+c) \), for some \( l_1, l_2 \in \{0, 1, \ldots, q^m-2\} \) and \( \beta_1, \beta_2 \in \mathbb{F}_{q^m} \).

If \( \mathfrak{g} \neq \beta_1 \mathfrak{h}^{q^m-1} \) and \( \mathfrak{h} \neq \beta_2 \mathfrak{h}^{q^m-1} + \mathfrak{g} + \beta \), for any \( \beta \in \mathbb{F}_{q^m} \), then from lemma \( \ref{2.3} \) we have

\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2})| \leq 4q^{m/2}, \tag{3.5}
\]

unless all the four characters are trivial.

Now we consider the following cases. If \( \mathfrak{g} = \beta_1 \mathfrak{h}^{q^m-1} \), for some \( \mathfrak{h} \in \mathbb{F}_{q^m} \) and \( \beta \in \mathbb{F}_{q^m} \), then it can written as \( \mathfrak{h} = \mathfrak{h}_1 \mathfrak{h}_2 \), where \( \mathfrak{h}_1, \mathfrak{h}_2 \) are coprime polynomials over \( \mathbb{F}_{q^m} \).

It follows that \( x^{l_1}(ax^2+bx+c)^{l_2} \mathfrak{h}_2^{q^m-1} = \beta(dx+e)^{l_2} \mathfrak{h}_1^{q^m-1} \), and this implies \( \mathfrak{h}_2^{q^m-1}|(dx+e)^{l_2} \) and hence \( \mathfrak{h}_2 \) is constant.

Then comparing the degrees of both sides we have \( l_1 + 2l_2 = l_2 + k_1(q^m-1) \), where \( k_1 \) is the degree of \( \mathfrak{h}_1 \) and this gives \( l_1 = 0 \) or \( 1 \) i.e. \( \mathfrak{h}_1(x) = ax + b' \).
When $k_1 = 1$ then $l_1$ must be non-zero, otherwise $l_2 = q^m - 1$, a contradiction. Now,
\[(ax^2 + bx + c)^{l_2} = \beta(dx + c)^{l_2} B(x^{q^m - 1}, x^{q^m - 1 - l_1}), \quad (3.6)\]
where $B(x) = f_1(x)/x \in \mathbb{F}_q[x]$, a constant polynomial. Comparing both sides we have $c = 0$. After putting this in the equation, this is possible only if $gcd(dx + e, ax + b) = x + \frac{c}{d}$ and $q^m - 1 = l_1 + l_2$. In this case $f(x) = \frac{4}{d}x$, which is a contradiction. Hence $k_1 = 0$ and hence $l_1 = l_2 = 0$.

Next, let $\beta_1 = 0$ and $\beta_2 \neq 0$. Then,
\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2})| = \left| \sum_{\alpha \neq \frac{c}{d}} \psi_{x^m-1}(\beta_2(a\alpha^2 + b\alpha + c) \over d\alpha + e) \right|
\]
\[
= \left| \sum_{y \neq 0} \psi_{x^m-1} \left( \frac{\beta_2}{d^2} ay + \left( \frac{\beta_2}{d^2} (c^2 - de + cd^2) y^{-1} \right) \right) \right|
\]
By the lemma 2.5, we have
\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2})| \leq 2q^{m/2} < 4q^{m/2}.
\]
Similarly, if $\beta_1 \neq 0$ and $\beta_2 = 0$, and applying lemma 2.1 we have
\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2})| = \left| \sum_{\alpha \in \mathbb{F}_q^m} \psi_{x^m-1}(\beta_1\alpha) \right| \leq 1 < 4q^{m/2}.
\]
If both $\beta_1$ and $\beta_2$ are non-zero, then we can proceed as follows.
\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2})| = \left| \sum_{\alpha \neq \frac{c}{d}} \psi_{x^m-1} \left( \beta_1\alpha + \frac{\beta_2(a\alpha^2 + b\alpha + c)}{d\alpha + e} \right) \right|
\]
\[
= \left| \sum_{y \neq 0} \psi_{x^m-1} \left( \left( \frac{\beta_1}{d} + \frac{\beta_2 a}{d^2} \right) y + \left( \frac{\beta_2 a e^2}{d^2} - \frac{be}{d} + c \right) y^{-1} + \left( \frac{\beta_2 b}{d} - \frac{\beta_1 e}{d} \right) \right) \right|
\]
Applying lemma 2.5 we have
\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2})| \leq 2q^{m/2} < 4q^{m/2}.
\]
If $\mathfrak{B} := \mathfrak{B}^p - \mathfrak{B} + \beta$ for some $\mathfrak{B} \in \mathbb{F}_{q^m}(x)$ and for some $\beta \in \mathbb{F}_{q^m}$. Then we write $\mathfrak{B} = \frac{H_1}{H_2}$, where $H_1$ and $H_2$ are co-prime polynomials. Continuing this, we have the following.
\[
\frac{\beta_1 x(dx + e) + \beta_2(ax^2 + bx + c)}{dx + e} = \frac{H_1^p - H_1 H_2^{p-1} + \beta H_2^p}{H_2^p}.
\]
Immediately from the restriction on the rational polynomial \( \frac{ax^2+bx+c}{dx+e} \) we get \((dx+e)\) is co-prime to \(\beta_1 x(dx+e)+\beta_2(ax^2+bx+c)\) and hence \(H_2^p\) is co-prime to \(H_1^p_H_2^p+\beta H_2^p\). Then \(dx+e=H_2^p\), which is a contradiction as \(d\neq 0\). It follows that \(\emptyset=0\), i.e. \(\beta_1=\beta_2=0\).

Additionally if at least one of the \(l_1, l_2\) is non-zero, then \(x^{l_1}(ax^2+bx+c)^{l_2}(dx+e)^{q^{m-1}-l_2}\) has at most 4 distinct roots and is not of the form \(\beta S\gamma q^{m-1}\), for \(S\in F_{q^m}\) and for \(\beta\in F_{q^m}\). Then from equation \((3.6)\) we have

\[
S(\chi d_1, \chi d_2, \psi h_1, \psi h_2) = \sum_{\alpha \in F_{q^m}} \chi q^{m-1} (\alpha^{l_1}(a\alpha^2+b\alpha+c)^{l_2}(dx+e)^{q^{m-1}-l_2}).
\]

By lemma \(2.5\) we have the bound as \(|S(\chi d_1, \chi d_2, \psi h_1, \psi h_2)| \leq 2q^{m/2} < 4q^{m/2}.

Now equation \((3.4)\) gives \(M(e_1, e_2, g_2) > 0\) if \(q^{m} > 1 + 4q^{m/2} (W(e_1)W(e_2)\Omega(g_1)\Omega(g_2) - 1)\).

Hence the sufficient condition is

\[
q^{m/2} > 4W(e_1)W(e_2)\Omega(g_1)\Omega(g_2).
\]

(3.7)

Next, we are dealing with the case \(d = 0\). Then \(f(x) = \frac{ax^2+bx+c}{e} = \frac{a}{e}x^2+\frac{b}{e}x+c = a_1x^2+b_1+c_1\) and

\[
M(e_1, e_2, g_1, g_2) = \theta(e_1)\theta(e_2)\Theta(g_1)\Theta(g_2) \int \int S(\chi d_1, \chi d_2, \psi h_1, \psi h_2).
\]

(3.8)

Where

\[
S(\chi d_1, \chi d_2, \psi h_1, \psi h_2) = \sum_{\alpha \in F_{q^m}} \chi d_1(\alpha)\chi d_2(f(\alpha))\psi h_1(\alpha)\psi h_2(f(\alpha))
\]

\[
= \sum_{\alpha \in F_{q^m}} \chi d_1(\alpha)\chi d_2(f(\alpha))\psi h_1(\alpha)\psi h_2'(\alpha)
\]

\[
= \sum_{\alpha \in F_{q^m}} \chi d_1(\alpha)\chi d_2(f(\alpha))\psi h_1'\psi h_2(\alpha)
\]

and \(\psi h_2'(x) = \psi h_2(f(x))\) for all \(x \in F_{q^m}\).

Now, if \((\chi d_1, \chi d_2, \psi h_1, \psi h_2') = \psi h\neq (\chi_0, \chi_0, \psi_0)\), then we consider following cases.

If, \(\psi h_1, \psi h_2 = \psi h\) is non trivial character, then applying lemma \(2.4\) we have

\[
|S(\chi d_1, \chi d_2, \psi h_1, \psi h_2)| = |S(\chi d_1, \chi d_2, \psi h)| \leq 3q^{m/2} < 4q^{m/2}.
\]

If, \(\psi h_1, \psi h_2' = \psi h\) is the trivial character \(\psi h\), then following lemma \(2.3\) we have

\[
|S(\chi d_1, \chi d_2, \psi h_1, \psi h_2)| = |S(\chi d_1, \chi d_2, \psi h)| \leq 2q^{m/2} < 4q^{m/2}.
\]

Again, if \(\chi d_1 = \chi d_2 = \chi 0\) then \(|S(\chi d_1, \chi d_2, \psi h_1, \psi h_2)| = |S(\chi 0, \chi 0, \psi 0)| = 0\). Hence \(|S(\chi d_1, \chi d_2, \psi h_1, \psi h_2)| < 4q^{m/2}\) if \((\chi d_1, \chi d_2, \psi h) \neq (\chi 0, \chi 0, \psi 0)\), where \(\psi h = \psi h_1, \psi h_2'.\) Then from equation \((3.8)\) we have the sufficient condition for \(M(e_1, e_2, g_1, g_2) > 0\) is

\[
q^{m/2} > 4W(e_1)W(e_2)\Omega(g_1)\Omega(g_2).
\]
In particular setting \( e_1 = e_2 = q^m - 1 \) and \( g_1 = g_2 = x^m - 1 \), we have the sufficient condition \( 3.3 \), i.e.,

\[
q^{m/2} > 4W(q^m - 1)^2\Omega(x^m - 1)^2.
\]

We briefly consider the case in which \( c_1 = 0 \) i.e. \( c = 0 \). Then \( f(x) = a_1 x^2 + b_1 x = x(a_1 x + b_1) \), where \( a_1, b_1 \in \mathbb{F}_{q^m} \) with \( b_1 \neq 0 \). This time we have

\[
\mathcal{M}(e_1, e_2, g_1, g_2) = \theta(e_1)\theta(e_2)\Theta(g_1)\Theta(g_2) \int_{d_1 | e_1} \int_{d_2 | e_2} S(\chi_{d_1}, \chi_{d_2}, \psi_1, \psi_2),
\]

where

\[
S(\chi_{d_1}, \chi_{d_2}, \psi_1, \psi_2) = \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_{d_1}(\alpha)\chi_{d_2}(\alpha(a_1 \alpha + b_1))\psi_1(\alpha) = \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_{d_3}(\alpha)\chi_{d_2}(a_1 \alpha + b_1)\psi_1(\alpha).
\]

with \( \chi_{d_3} = \chi_{d_1}\chi_{d_2} \). Now, by lemma 2.4

\[
|S(\chi_{d_1}, \chi_{d_2}, \psi_1, \psi_2)| = \left| \sum_{\alpha \in \mathbb{F}_{q^m}} \chi_{d_3}(\alpha)\chi_{d_2}(a_1 \alpha + b_1)\psi_1(\alpha) \right| \leq 2q^{m/2} < 4q^{m/2}
\]

and (3.3) and (3.1) follow as before.

For the next section in our paper we apply the results on primes dividing \( q^m - 1 \) and irreducible polynomials dividing \( x^m - 1 \) for more specific results. It is worth mentioning that, this was first introduced by Cohen and Huczynska in [5, 6].

### 4 The prime sieve technique

We begin this section by mentioning that the sieving inequality, which was established by Kapetanakis in [10] and we use the inequality by adjusting properly.

**Lemma 4.1. (Sieving Inequality)** Let \( d \) be a divisor of \( q^m - 1 \) and \( p_1, p_2, \ldots, p_n \) are the remaining distinct primes dividing \( q^m - 1 \). Furthermore, let \( g \) be a divisor of \( x^m - 1 \) such that \( g_1, g_2, \ldots, g_k \) are the remaining distinct irreducible polynomials dividing \( x^m - 1 \). Abbreviate \( \mathcal{M}(q^m - 1, q^m - 1, x^m - 1, x^m - 1) \) to \( \mathcal{M} \). Then

\[
\mathcal{M} \geq \sum_{i=1}^{n} \mathcal{M}(p, d, d, g, g) + \sum_{i=1}^{n} \mathcal{M}(d, p, d, g, g) + \sum_{i=1}^{k} \mathcal{M}(d, d, g, g) \tag{4.1}
\]

\[
+ \sum_{i=1}^{k} \mathcal{M}(d, d, g, g) - (2n + 2k - 1)\mathcal{M}(d, d, g, g) \tag{4.2}
\]

**Theorem 4.2.** With all the assumptions in lemma [4.1], define

\[
\vartheta := 1 - 2 \sum_{i=1}^{n} \frac{1}{p_i} - 2 \sum_{i=1}^{k} \frac{1}{q^{\deg(g_i)}}
\]
and
\[ \mathcal{S} := \frac{2n + 2k - 1}{\vartheta} + 2. \]

Suppose \( \vartheta > 0 \). Then a sufficient condition for existence of an element \( \alpha \in \mathbb{F}_q^m \) such that both \( \alpha \) and \( f(\alpha) = \frac{a\alpha^2 + b\alpha + c}{\alpha^2 + c} \) are simultaneously primitive normal over \( \mathbb{F}_q^m \) with \( a \neq 0 \) and \( da + e \neq 0 \) is
\[ q^{m/2} > 4W(d)^2\Omega(g)^2\mathcal{S}. \] (4.3)

**Proof.** A key step is to write (4.1) in the equivalent form
\[
\mathcal{M} \geq \sum_{i=1}^{n} \left( \mathcal{M}(p_i d, d, g, g) - \left(1 - \frac{1}{p_i}\right) \mathcal{M}(d, d, g, g) + \sum_{i=1}^{n} \left( \mathcal{M}(d, d p_i, g, g) - \left(1 - \frac{1}{p_i}\right) \mathcal{M}(d, d, g, g) \right) \right) + \sum_{i=1}^{k} \left( \mathcal{M}(d, d, g, g) - \left(1 - \frac{1}{q^{\deg(g_i)}}\right) \mathcal{M}(d, d, g, g) \right) + \vartheta \mathcal{M}(d, d, g, g). \] (4.4)

On the right side of (4.4), since \( \vartheta > 0 \), we can bound the last term below using (3.1). Thus
\[ \vartheta \mathcal{M}(d, d, g, g) \geq \vartheta \theta^2(d) \Theta^2(g) q^m \left( q^m - 4W^2(d) \Omega^2(g) \right). \] (4.5)

Moreover, since \( \theta(p_i d) = \theta(p_i) \theta(d) = \left(1 - \frac{1}{p_i}\right) \) and \( \Theta(g, g) = \Theta(g_i) \Theta(g) = \left(1 - \frac{1}{q^{\deg(g_i)}}\right) \) we have from (4.4),
\[
\mathcal{M}(p_i d, d, g, g) - \left(1 - \frac{1}{p_i}\right) \mathcal{M}(d, d, g, g) = \left(1 - \frac{1}{p_i}\right) \theta^2 \Theta^2 \int_{d_1 | d} \int_{d_2 | d} \int_{h_1 | g} \int_{h_2 | g} S(\chi_{p_i d_1}, \chi_{d_2}, \psi_{h_1}, \psi_{h_2}).
\]

and
\[
\mathcal{M}(d, d, g, g) - \left(1 - \frac{1}{q^{\deg(g_i)}}\right) \mathcal{M}(d, d, g, g) = \left(1 - \frac{1}{q^{\deg(g_i)}}\right) \theta^2 \Theta^2 \int_{d_1 | d} \int_{d_2 | d} \int_{h_1 | g} \int_{h_2 | g} S(\chi_{d_1}, \chi_{d_2}, \psi_{g_i h_1}, \psi_{h_2}).
\]

Hence, as for (3.1),
\[
\left| \mathcal{M}(p_i d, d, g, g) - \left(1 - \frac{1}{p_i}\right) \mathcal{M}(d, d, g, g) \right| \leq 4 \left(1 - \frac{1}{p_i}\right) \theta^2(d) \Theta^2(g) \left(W(p_i d) - W(p_i)\right) W(d)
= 4 \left(1 - \frac{1}{p_i}\right) \theta^2(d) W^2(d). \] (4.6)

\[
\left| \mathcal{M}(d, d, g, g) - \left(1 - \frac{1}{q^{\deg(g_i)}}\right) \mathcal{M}(d, d, g, g) \right| \leq 4 \left(1 - \frac{1}{p_i}\right) \theta^2(d) \Theta^2(g) \left(\Omega(g, g) - \Omega(g_i)\right) \Omega(g)
= 4 \left(1 - \frac{1}{q^{\deg(g_i)}}\right) \Theta^2(g) \Omega^2(g). \] (4.7)
Similarly,
\[
|\mathcal{M}(d,d,g,g) - \left(1 - \frac{1}{p_i}\right)\mathcal{M}(d,p_id,g,g)| \leq 4 \left(1 - \frac{1}{p_i}\right)\theta^2(d)\Theta^2(g)W^2(d) \tag{4.8}
\]
and
\[
|\mathcal{M}(d,d,g,g,g) - \left(1 - \frac{1}{q^{\deg(g_i)}}\right)\mathcal{M}(d,d,g,g)| \leq 4\theta^2(d)\left(1 - \frac{1}{q^{\deg(g_i)}}\right)\Theta^2(g)\Omega^2(g). \tag{4.9}
\]
Inserting (4.5), (4.6), (4.8) and (4.9) in (4.4) and cancelling the common factor \(\theta^2(d)\Theta^2(g)\), we obtain (4.3) as a condition for \(\mathcal{M}\) to be positive (since \(\vartheta\) is positive). This completes the proof.

We conclude our paper by discussing all the possible cases for fields of characteristic 2.

5 Some estimations for fields of even characteristic

The prime purpose of this section is to analyse the conditions (3.3) and (4.3) for the existence of elements of desired properties in fields of even characteristic. Towards that, we express the pairs \((q,m)\) with the desired properties with extending and developing the techniques employed in [13], [14] and [7] by the functions presented earlier, leading us to character sums. We have already defined such pairs \((q,m)\) as primitive normal pair.

Also, it is worth mentioning that due to the complexity of the character sums and its fragile behaviour on fields of different orders, it is necessary to distinguish few cases depending on the order of prime sub-field. From now on we suppose \(q = 2^k\), where \(k\) is a positive integer.

>From now on we use the concept of radical of \(m\) i.e. \(m'\) and radical of \(x^m - 1\) which is \(x^{m'} - 1\). Where \(m'\) is such that \(m = 2^km'\), where \(\gcd(2,m') = 1\) and \(k\) is a non-negative integer. In fact, when \(m' = 1\), trivially \(k\) is positive. For further computation, we need some additional results.

Furthermore, we consider the two cases

- \(m'|q - 1\)
- \(m' \nmid q - 1\)

We recall the fact that, in this case \(x^{m'} - 1\) splits at most into a product of \(m'\) linear factors over \(\mathbb{F}_q\). The following result is inspired from the lemma 6.1, given by Cohen in [4].

**Lemma 5.1.** For \(q = 2^k\), where \(k \geq 1\), let \(d = q^m - 1\) and let \(g|x^m - 1\) with \(g_1, g_2, \ldots, g_r\) be the remaining distinct irreducible polynomials dividing \(x^m - 1\). Furthermore, let us write \(\vartheta := 1 - \sum_{i=1}^{r} \frac{1}{q^{\deg(g_i)}}\) and \(\mathfrak{S} := \frac{r-1}{\vartheta} + 2\), with \(\vartheta > 0\). Let \(m = m'2^k\), where \(k\) is a non-negative integer and \(\gcd(m', 2) = 1\). If \(m'|q - 1\), then

\[
\mathfrak{S} = \frac{q^2 - 3q + aq + 2}{aq - q + 1}
\]

where \(m' = \frac{q-1}{a}\). In particular, \(\mathfrak{S} < q^2\).
In order to apply our results, we also need the following lemma 6.2 by Cohen in [4]. We use this result in next case and all the subsequent cases unless other lemmas are stated.

**Lemma 5.2.** For any odd positive integer \( n \), \( W(n) < 6.46n^{1/5} \), where \( W \) has same meaning as stated earlier.

>From Theorem [4,3] it is clear that some concepts regarding the factorization of \( x^m - 1 \) can be used in order to effectively use the results of the previous section. Such as if \( m'|q - 1 \), then \( x^{m'} - 1 \) splits into \( m' \) distinct linear polynomials. Throughout this section we use prime sieve technique result to establish the rest.

**Proposition 5.3.** For \( f(x) \in \mathbb{F}_{q^n}(x) \), such that \( f(x) = x \) or \( f(x) = x^2 \), we have \( \mathcal{M}(q^m - 1, q^m - 1, x^m - 1, x^m - 1) > 0 \).

**Proof.** The proof follows lemma 4.1 of [11]. Since \( q \) is of even characteristic, \( q^m - 1 \) is odd, hence both \( \alpha \) and \( f(\alpha) \) are simultaneously primitive. In similar way since \( m' \) is odd hence \( \alpha \) and \( f(\alpha) \) are simultaneously normal.

**Proposition 5.4.** Suppose \( m' \) is such that \( m'|q - 1 \), then for all the pairs \( (q, m) \), \( \mathcal{M} > 0 \) i.e. all the pairs \( (q, m) \) are primitive normal pairs except \( (q, m) \) is one of the pairs \( (2, 2), (2, 4), (2, 8), (2, 16), (4, 2), (4, 3), (4, 4), (4, 6), (4, 8), (4, 12), (8, 2), (8, 4), (8, 7), (8, 8), (8, 14), (16, 2), (16, 3), (16, 4), (16, 5), (16, 6), (16, 15), (32, 2), (64, 2), (64, 4), (128, 2), (256, 2), (512, 2), (1024, 2), (4096, 2).

**Proof.** Taking \( g = 1 \) in inequality [4,3] and applying Lemma [5,2] we have the sufficient condition

\[
q^{\frac{m}{10}} > 167 q^2
\]

Then for \( m' = q - 1 \), the inequality transforms to

\[
q^{\frac{q-1}{10} - 2} > 167,
\]

which holds for \( q \geq 64 \).

Next, we consider \( q = 32 \) and \( m = m' = q - 1 = 31 \). Then, by factorising, \( \omega(27^{26} - 1) = 12 \) and the pair \( (q, m) = (32, 31) \) satisfies the condition [1.3]. Hence \( \mathbb{F}_{2^{32}} \) contains an element \( \alpha \) such that both \( \alpha \) and \( f(\alpha) \) are simultaneously primitive normal with given conditions.

In order to reduce our calculations, we now consider the range \( 19 \leq m' < \frac{q-1}{3} \), for \( q \geq 64 \). Then, by Lemma 5.1 we have \( S < \frac{4}{2} \). Hence the inequality [4.3] is satisfied if \( q^{\frac{m'-1}{10}-1} > 83.5 \) and this holds for \( m' \geq 19 \).

When \( m' = \frac{q-1}{3} \), then \( S \leq q \) and then the inequality is \( q^{\frac{m'-1}{10}-1} > 167 \) and this holds for \( m' \geq 19 \). Since \( m' = 19 \neq \frac{q-1}{3} \) for any \( q = 2^k \) hence we leave this case.

Next, we investigate all cases with \( m' < 19 \). In the next part, we set \( d = q^m - 1, g = 1 \) unless mentioned otherwise.

- **Case 1:** \( m' = 1 \)

Then \( m = 2^j \). Initially we take \( j \geq 2 \). To check the condition we take \( g = X + 1 \). In that case \( \vartheta = 1 \) and \( S = 1 \). Then the inequality becomes

\[
q^{\frac{2^j}{10}} > 334.
\]
Taking $q = 2$, we have that the condition holds for $j \geq 7$. Again for $g = 4$, $j \geq 6$; for $8 \leq q \leq 32$, $j \geq 5$; for $64 \leq q \leq 2^{10}$, $j \geq 4$; for $2^{11} \leq q \leq 2^{20}$, $j \geq 3$ and for $q \geq 2^{21}$ the condition holds for $j \geq 2$. So we calculate the rest of the pairs $(q, m)$ by calculating $\omega = \omega(q^{m/2} - 1)$ i.e., the number of distinct prime divisors of $q^{m/2} - 1$. Hence it suffices to check that $q^{m/2} > 4 \times W(q^{m/2} - 1)^2 \times 2^s$, where $W(q^{m/2} - 1) = 2^x$ and there are pairs $(2, 4), (2, 8), (2, 16), (4, 4), (4, 8), (8, 8), (16, 4), (32, 4), (64, 4), (128, 4), (512, 4)\) which don’t satisfy the condition. Again taking $g = 1$ and appropriate value of $d$ we apply the sieve condition (4.3) to verify $(128, 4), (512, 4)$ as primitive normal pairs and declare others are exceptional pairs.

Now we discuss the case when $m = 2$. Then any pair $(q, 2)$ is primitive normal pair if and only if it is a primitive pair, i.e. there exists $\alpha \in \mathbb{F}_q$ such that both $\alpha$ and $f(\alpha)$ are simultaneously primitive element of $\mathbb{F}_q$. For all $q$ such that $q^2 - 1$ is a Mersenne prime (the primes which are of the form $2^j - 1$ for some positive integer $j$ are called "Mersenne primes") except $(2, 2)$, all the elements of $\mathbb{F}_q$ are primitive except the identity and hence pairs $(q, 2)$ are primitive normal pairs. $(2, 2)$ does not fit into this category as $\mathbb{F}_q \cong \mathbb{Z}_{2^{q-2}}$ and primitive elements of $\mathbb{F}_q$ satisfy $x^2 + x + 1$, i.e. $f(\alpha)$ is not primitive when $\alpha$ is primitive.

For the remaining pairs we use the sufficient condition $q^{1/5} > 668$, which holds for $q \geq 2^{47}$. Now the remaining pairs we use sieve condition (4.3) to test the existence of the property. When $d = q^2 - 1$ and $g = x + 1$, the condition holds for all $q = 2^k$, where $k = 13, 17$ and $k \geq 19$. Again choosing appropriate $d$ as in table 1, we conclude that among the above pairs; $(2^{11}, 2), (2^{14}, 2), (2^{15}, 2), (2^{16}, 2), (2^{18}, 2)$ are primitive normal pairs and rest are possible exceptions.

Finally we declare that the following pairs are possible exceptional pairs.

$$(2, 2), (2, 4), (2, 8), (2, 16), (4, 2), (4, 4), (4, 8), (8, 2), (8, 4), (8, 8), (16, 2), (16, 4), (32, 2), (64, 2), (64, 4), (128, 2), (256, 2), (512, 2), (1024, 2), (4096, 2)$$

- **Case 2: $m' = 3$**

  In this case, $m$ is of the form $m = 3 \cdot 2^j$, where $j$ is a positive integer and $q = 2^{2k}$ for some $k \geq 1$. For $q = 4$, take $g = x^{3m} - 1$ so that $\mathcal{S} = 1$ and the sufficient condition is $4^{\frac{3m}{2}} > 167 \times (3^3)^2$, which holds for $j \geq 5$. Hence the pairs under the above condition are primitive normal pairs except the pairs $(4, 3), (4, 6), (4, 12), (4, 24), (4, 48)$. From table 1, by sieving in condition (4.3) we conclude that $(4, 24), (4, 48)$ are primitive normal pairs and hence possible exceptional pairs are $(4, 3), (4, 6), (4, 12)$.

  Then we take $g = 1$ and then calculate the following. For $q = 16$, $\mathcal{S} \leq 22$ and the sufficient condition is $q^{\frac{32}{144}} > 3672.8$, which holds for $j \geq 4$.

  For $q = 64, 256$, $\mathcal{S} < 3.3$ and $m'|q - 1$, the condition holds for $j \geq 3$. Again for $1024 \leq q \leq 2^{16}$, $\mathcal{S} < 7.029$ and we need to check $q^{\frac{32}{144}} > 1251.95$, which holds when $j \geq 2$. For $2^{18} \leq q \leq 2^{34}$, $\mathcal{S} < 7.0001$ and the condition holds for $j \geq 1$; and for $q \geq 2^{35}$ such that $m'|q - 1$ the condition holds for $j \geq 0$. 

13
We calculate the remaining pairs by taking \( g = x^3 - 1 \) and using \( W(q^m - 1), \Omega(x^3 - 1) \). So the condition is \( q^{m/2} > 4 \times W(q^m - 1)^2 \times (2^3)^2 \). Then all but the pairs \((16,3), (16,6), (64,3), (64,6), (256,3), (1024,3), (2^{12},3), (2^{16},3), (2^{20},3)\) fail to satisfy. Now we choose compatible values of \( g \) and \( d \) to declare \((16,12), (16,24), (64,3), (256,3), (1024,3), (2^{12},3), (2^{16},3), (2^{20},3)\) as primitive normal pairs as shown in table 1.

Then we have the following pairs as possible exceptional pairs \((16,3)\).

\[
\begin{align*}
(4,3), & (4,6), (4,12), (16,3), (16,6)
\end{align*}
\]

From now on take \( m = m'2^j \) with \( j \geq 0 \).

- **Case 3: \( m' = 5 \)**

Here \( m = 5.2^j \), with non-negative integer \( j \). As there are 5 distinct factors of \( x^{m'} - 1 \), so by calculation we have \( \vartheta > 0 \) if \( q \geq 16 \). Then \( \mathcal{S} < 26 \) for \( q = 16 \) and the sufficient condition is \( q^{5/2^j} > 4340.09 \) which holds for \( j \geq 3 \).

For \( q = 256 \), \( \mathcal{S} \leq 11.7627 \), and sufficient condition is \( q^{5/2^j} > 1963 \). This holds when \( j \geq 2 \).

Again, \( 4096 \leq q \leq 2^{20} \) and \( m'|q - 1 \), the condition is \( q^{5/2^j} > 1843.57 \) and holds for \( j \geq 1 \). When \( q \geq 2^{21} \) and \( m'|q - 1 \) the condition holds for \( j \geq 0 \).

Taking \( g = x^5 - 1 \), we check the remaining pairs for the inequality \( q^{m/2} > 4 \times 2^{2\vartheta} \times (2^5)^2 \) and have the following as possible exceptional pairs \((16,5), (16,10), (256,5), (2^{12},5)\). Then we choose proper \( d \) and \( g \), and verify condition \((4,3)\) and have \((16,10), (256,5), (2^{12},5)\) are primitive normal pairs. Then the following pair is a possible exceptional pair.

\[
(16,5)
\]

- **Case 4: \( m' = 7 \)**

Here \( m = 7.2^j \), with non-negative integer \( j \). Let \( g = x^{m'} - 1 \) for \( q = 8 \), then \( \vartheta = 1 \) and \( \mathcal{S} = 1 \). Then the sufficient condition is \( q^{7/2^j} > 2736128 \) which holds for \( j \geq 4 \).

For \( q = 64 \), take \( g = 1 \) and \( \mathcal{S} \leq 18.64 \), then sufficient condition \( q^{7/2^j} > 3112.8 \) holds for \( j \geq 2 \). Again, \( q = 512, 2^{12}, 2^{15} \), \( \mathcal{S} < 15.3655 \) and the condition holds for \( j \geq 1 \). For \( q \geq 2^{16} \), whenever \( m'|q - 1 \) the condition holds for \( j \geq 0 \).

Taking \( g = x^7 - 1 \), we check the remaining pairs for the inequality \( q^{m/2} > 4 \times 2^{2\vartheta} \times (2^7)^2 \). After calculation, we conclude that all the pairs are primitive normal pairs except the following pairs .

\[
(8,7), (8,14)
\]

- **Case 5: \( m' = 9 \)**

Here \( m = 9.2^j \), with non-negative integer \( j \). As \( m' = 9 \), there are 9 distinct factors of \( x^{m'} - 1 \). When \( g = 1 \) we have \( \vartheta > 0 \) if \( q \geq 32 \). Then \( \mathcal{S} < 38.2667 \) for \( q = 64 \) and the sufficient condition is \( q^{9/2^j} > 6387.72 \) which holds for \( j \geq 2 \).
For \( q = 2^{12} \), sufficient condition holds for \( j \geq 1 \). When \( q \geq 2^{13} \) and \( m'|q - 1 \) the condition holds for \( j \geq 0 \).

Taking \( g = x^9 - 1 \), we check the remaining pairs for the inequality \( q^{m/2} > 4 \times 2^{2\omega} \times (2^9)^2 \) and have the pair \((64, 9)\), which does not satisfy the inequality. After calculating with compatible values of \( d \) and \( g \) as in table 1, we conclude that \((64, 9)\) is also a primitive normal pair.

- **Case 6: \( m' = 11 \)**

  Here \( m = 11.2^j \), with non-negative integer \( j \). As there are 11 distinct factors of \( x^{m'} - 1 \), so by calculation for \( g = 1 \) we have \( \vartheta > 0 \) if \( q \geq 32 \).

  For \( q = 2^{10} \), \( \mathfrak{S} < 27.3585 \) and sufficient condition \( q^{\frac{11m/2}{4}} > 4566.86 \) holds for \( j \geq 1 \). When \( q \geq 2^{11} \) and \( m'|q - 1 \) the condition holds for \( j \geq 0 \).

  Taking \( g = x^{11} - 1 \), we check the remaining pairs for the inequality \( q^{m/2} > 4 \times 2^{2\omega} \times (2^{11})^2 \), then we have 3 possible exceptional pairs. By calculating with compatible values of \( d \) and \( g = x^{m'} - 1 \), we have all the pairs are primitive normal pairs.

- **Case 7: \( m' = 13 \)**

  Here \( m = 13.2^j \), with non-negative integer \( j \). As there are 13 distinct factors of \( x^{m'} - 1 \), so by calculation we have \( \vartheta > 0 \) if \( q \geq 32 \). For \( q \geq 64 \) and \( m'|q - 1 \), take \( g = 1 \) then \( \mathfrak{S} < 44.1053 \) and the sufficient condition holds for \( j \geq 0 \). We conclude that all the pairs are primitive normal pairs.

- **Case 8: \( m' = 15 \)**

  Here \( m = 15.2^j \), with non-negative integer \( j \). As \( m' = 15 \), there are 15 distinct factors of \( g = x^{m'} - 1 \). For \( q = 16 \), the sufficient condition for existence of primitive normal element is \( q^{\frac{15m/2}{10}} > 167 \times (2^{15})^2 \). This condition holds for \( j \geq 3 \).

  For \( q = 256 \), and \( g = 1 \) we have \( \mathfrak{S} < 56.882 \) and the sufficient condition \( q^{\frac{15m/2}{16}} > 9446.06 \) holds for \( j \geq 1 \). When \( q > 256 \) and \( m'|q - 1 \) the condition holds for \( j \geq 0 \).

  Taking \( g = x^{15} - 1 \), we check the remaining pairs for the inequality \( q^{m/2} > 4 \times 2^{2\omega} \times (2^{15})^2 \), then we have \((16, 15), (16, 30), (256, 15)\) as possible exceptional pairs. By calculating with compatible values of \( d, g \) in prime sieve condition \( [13] \), we have \((16, 30), (256, 15)\) as primitive normal pairs. Hence we declare the following as an exceptional pair.

  \[ (16, 15) \]

- **Case 9: \( m' = 17 \)**

  Here \( m = 17.2^j \), with non-negative integer \( j \). As there are 17 distinct factors of \( x^{m'} - 1 \), so by calculation for \( g = 1 \), we have \( \vartheta > 0 \) if \( q \geq 64 \).

  When \( q \geq 64 \), the sufficient condition is \( q^{\frac{17m/2}{16}} > 12085.5 \) which holds for \( j \geq 0 \) whenever \( m'|q - 1 \). Hence we have all the pairs of this case as primitive normal pairs.
For each of the individual pairs \((g, m)\) listed above that does not satisfy the sufficient condition based on lemma 5.2, we can test them more precisely by means of the sufficient condition (1.3) after factorising completely \(x^m - 1\) and \(q^m - 1\) and making a choice of polynomial divisor \(g\) of \(x^m - 1\) and factor \(d\) of \(q^m - 1\). In practice, the best choice is to choose \(p_1, \ldots, p_n\) and sometimes, the “largest” irreducible factors \(g_1, \ldots, g_k\) of \(x^m - 1\) to ensure that \(\psi\) is positive (and not too small). Here the multiplicative aspect of the sieve is more significant. Table 1 summarises the pairs in which the test yielded some positive conclusion.

| \((q, m)\) | \(d\) | \(n\) | \(g\) | \(k\) | \(\mathcal{G}\) | \(q^m/2\) | \(4W(d)^2\Omega^2(g)\Lambda\) |
|------------|-------|------|------|-----|-------------|--------|------------------|
| (128,4)    | 3     | 5    | \(x + 1\) | 0   | 21.9523     | 16384  | 1404.95          |
| (512,4)    | 15    | 6    | \(x + 1\) | 0   | 32.9531     | 262144 | 8435.99          |
| (2\(^{11}\), 2) | 3 | 3 | \(x + 1\) | 0   | 7.6329      | 2048   | 488.506          |
| (2\(^{14}\), 2) | 3 | 5 | \(x + 1\) | 0   | 21.9523     | 16384  | 1404.95          |
| (2\(^{15}\), 2) | 3 | 5 | \(x + 1\) | 0   | 22.0596     | 32768  | 1411.81          |
| (2\(^{16}\), 2) | 3 | 4 | \(x + 1\) | 0   | 16.7511     | 65536  | 1072.07          |
| (2\(^{18}\), 2) | 15 | 6 | \(x + 1\) | 0   | 32.9531     | 262144 | 8435.99          |
| (4,24)      | 15    | 7    | \(x^3 - 1\) | 0   | 34.2484     | 1.6777 \times 10^7 | 140283 |
| (4,48)      | 15    | 10   | \(x^3 - 1\) | 0   | 50.3795     | 2.81475 \times 10^{14} | 206354 |
| (16,12)     | 15    | 7    | \(x^3 - 1\) | 0   | 34.2484     | 1.6777 \times 10^7 | 140283 |
| (16,24)     | 15    | 10   | \(x^3 - 1\) | 0   | 50.3795     | 2.81475 \times 10^{14} | 206354 |
| (64,3)      | 3     | 3    | \(x + 1\) | 2   | 9.3369      | 512    | 309.39           |
| (256,3)     | 15    | 4    | \(x + 1\) | 2   | 28.2612     | 4096   | 452.179          |
| (1024,3)    | 3     | 5    | \(x^3 - 1\) | 0   | 22.0596     | 32768  | 22589            |
| (2\(^{12}\), 3) | 15 | 6 | \(x^3 - 1\) | 0   | 32.9531     | 262144 | 134976           |
| (2\(^{16}\), 3) | 15 | 7 | \(x^3 - 1\) | 0   | 34.2484     | 1.67772 \times 10^7 | 140281 |
| (2\(^{20}\), 3) | 15 | 9 | \(x^3 - 1\) | 0   | 82.2883     | 1.07374 \times 10^9 | 337053 |
| (16,10)     | 3     | 6    | \(x^3 - 1\) | 0   | 60.7588     | 1.04858 \times 10^9 | 995472 |
| (256,5)     | 3     | 6    | \(x^3 - 1\) | 0   | 60.7588     | 1.04858 \times 10^9 | 995472 |
| (2\(^{12}\), 5) | 15 | 9 | \(x^3 - 1\) | 0   | 87.8157     | 1.07374 \times 10^9 | 5.75509 \times 10^6 |
| (8,28)      | 15    | 10   | \(x^3 - 1\) | 0   | 49.0678     | 4.39805 \times 10^8 | 5.14313 \times 10^7 |
| (8,56)      | 15    | 15   | \(x^3 - 1\) | 0   | 106.643     | 1.93428 \times 10^{25} | 1.11828 \times 10^9 |
| (64,9)      | 3     | 5    | \(x^3 - 1\) | 0   | 17.4747     | 1.34218 \times 10^8 | 7.32942 \times 10^7 |
| (16,30)     | 15    | 13   | \(x^3 - 1\) | 0   | 293.517     | 1.15292 \times 10^{18} | 2.01703 \times 10^{13} |
| (256,15)    | 15    | 13   | \(x^3 - 1\) | 0   | 293.517     | 1.15292 \times 10^{18} | 2.01703 \times 10^{13} |

Table 1

From the above table and calculation, we conclude that the following pairs are the final possible exceptional pairs, where \(m'|q - 1\) and \((g, m)\) does not satisfy the sufficient condition.

\[
(2, 2), (2, 4), (2, 8), (2, 16), (4, 2), (4, 3), (4, 4), (4, 6), (4, 8), (4, 12), (8, 2), (8, 4), (8, 7), (8, 8), (8, 14), (16, 2), (16, 3), (16, 4), (16, 5), (16, 6), (16, 15), (32, 2), (64, 2), (64, 4), (128, 2), (256, 2), (512, 2), (1024, 2), (4096, 2)
\]
For our next proposition, we need the following results.
Let \( u \) be the order of \( q \) mod \( m' \). Then \( x^{m'} - 1 \) is a product of irreducible polynomial factors of degree less than or equal to \( u \) in \( \mathbb{F}_q[x] \). In particular, \( u \geq 2 \) if \( m' \nmid q - 1 \).

Let \( M \) be the number of distinct irreducible polynomials of \( x^{m'} - 1 \) over \( \mathbb{F}_q \) of degree less than \( u \).

Let \( \sigma(q, m) \) denotes the ratio \( \sigma(q, m) := \frac{M}{m'} \).

where \( m\sigma(q, m) = m'\sigma(q, m') \).

From proposition 5.3 in [5], we deduce the following bounds.

**Lemma 5.5.** Suppose \( q = 2^k \). Then the following hold.

- \( \sigma(2, 3) = \frac{1}{3} \); \( \sigma(2, 5) = \frac{1}{5} \); \( \sigma(2, 9) = \frac{2}{5} \); \( \sigma(2, 21) = \frac{4}{21} \) otherwise \( \sigma(2, m) \leq \frac{1}{6} \).
- \( \sigma(4, 9) = \frac{1}{3} \); \( \sigma(4, 45) = \frac{11}{45} \) otherwise \( \sigma(4, m) \leq \frac{1}{5} \).
- \( \sigma(8, 3) = \sigma(8, 21) = \frac{1}{3} \) otherwise \( \sigma(8, m) \leq \frac{1}{5} \).
- If \( q \geq 16 \), then \( \sigma(q, m) \leq \frac{1}{3} \).

Now, to discuss the conditions, we need the lemma 7.2 from [4].

**Lemma 5.6.** Assume that \( q = 2^k \) and \( m \) is a positive integer such that \( m' \nmid q - 1 \). Let \( \sigma \) denotes the order of \( q \) mod \( m' \). Let \( g \) be the product of the irreducible factors of \( x^{m'} - 1 \) of degree less than \( \sigma \). Then, in the notation of Lemma 5.1, we have \( \mathcal{S} \leq m' \).

We need few more conditions, which we can derive from the lemma 4.2 of [8].

**Lemma 5.7.** For any \( n, \alpha \in \mathbb{N} \), \( W(n) \leq b_{\alpha,n}n^{1/\alpha} \), where \( b_{\alpha,n} = \frac{2^\alpha}{(p_1 p_2 \ldots p_s)^{1/\alpha}} \) and \( p_1, p_2, \ldots, p_s \) are primes \( \leq 2^\alpha \) that divide \( n \). \( W \) has same meaning as mentioned earlier.

>From these we derive the next lemma.

**Lemma 5.8.** For \( n \in \mathbb{N} \) and

(i) \( \alpha = 6 \), \( W(n) < 37.4683n^{1/6} \),

(ii) \( \alpha = 8 \), \( W(n) < 4514.7n^{1/8} \),

(iii) \( \alpha = 14 \), \( W(n) < (5.09811 \times 10^{67})n^{1/14} \),

where \( W \) has same meaning as mentioned earlier.

**Proposition 5.9.** Let \( q = 2 \), and \( m' \nmid q - 1 \), then there exists an element \( \alpha \in \mathbb{F}_q \) such that \( \alpha \), \( f(\alpha) \) such are simultaneously primitive normal elements over \( \mathbb{F}_q \) i.e., \( (q, m) \) are primitive normal pairs except for a few pairs viz. \( (2, 3), (2, 5), (2, 6), (2, 7), (2, 9), (2, 10), (2, 11), (2, 12), (2, 14), (2, 15), (2, 18), (2, 21), (2, 24), (2, 30) \).
Proof. At first let \( m' = 3 \), then \( x' - 1 \) can be factorised into one linear and one quadratic factor. Then the condition becomes \( 2^{m/10} > 2672 \), which holds for \( m \geq 114 \). Next let \( m = 96 \), then \( \omega = 12 \) and the condition is \( q^{m/2} > 2^{2\omega+6} \) and the condition holds. But the remaining pairs \( (2, 3), (2, 6), (2, 12), (2, 24), (2, 48) \) doesn’t satisfy the above condition. We perform further research on these pairs by taking compatible \( d \) and \( g \) in sieve condition \( 1.3 \) as in table 2 and conclude that \( (2, 48) \) is primitive normal pair and \( (2, 3), (2, 6), (2, 12), (2, 24) \) are exceptional ones.

Again, if \( m' = 5 \), then \( x' - 1 \) can be factorised into one linear and one fourth degree polynomial. Then the condition becomes \( 2^{m/10} > 2672 \), which holds for \( m \geq 114 \). Then for the remaining pairs condition is \( q^{m/2} > 2^{2\omega+6} \) and by calculating \( \omega(q^m - 1) = \omega \), we have the following exceptional pairs \( (2, 5), (2, 10), (2, 20), (2, 40) \). Again from table 2 we can conclude that the only possible exceptional pairs are \( (2, 5), (2, 10), (2, 20) \).

For \( m' = 9 \), \( x' - 1 \) is a product of one linear, one quadratic and one sextic polynomial and the condition is \( 2^{m/10} > 10688 \) and the condition holds for \( m \geq 134 \). For the remaining pairs we use the condition \( q^{m/2} > 2^{2\omega+8} \) and by calculating the value of \( \omega \), we have the following exceptional pairs \( (2, 9), (2, 18), (2, 36) \). From table 2, we can conclude that \( (2, 36) \) is a primitive normal pair and hence final possible exceptional pairs are \( (2, 9), (2, 18) \).

Now, for \( m' = 21 \), \( x' - 1 \) is a product of one linear, one quadratic, two cubic, two distinct sextic polynomials. Then the condition is \( 2^{m/10} > 684032 \) and the condition holds for \( m \geq 194 \). For the remaining pairs we use the condition \( q^{m/2} > 2^{2\omega+14} \) and then calculating the value of \( \omega(q^m - 1) = \omega \), we have the following exceptional pairs \( (2, 21), (2, 42) \), from which we can declare the pair \( (2, 42) \) as primitive normal pair from table 2. Hence possible exceptional pair is \( (2, 21) \).

For the remaining pairs i.e. \( q = 2, m' \not\equiv 1 \mod 2 \) and \( m' \not\equiv 3, 5, 9, 21 \), we consider two cases viz. (i) \( m \) is odd and (ii) \( m \) is even.

Case (i): \( m \) is odd. We apply the lemma \( 5.6 \) to obtain the condition \( q^{m/2} > 4 \times 2^{2\omega} \times 2^{2\sigma(q, m)} \times m \). Then by lemma \( 5.5 \) and lemma \( 5.8 \) the condition transforms to \( 2^{m/42} > 1.03991 \times 10^{136} \cdot m \), which holds for \( m \geq 19577 \). Let \( m \leq 19576 \), then \( \omega \leq 1620 \), and apply these on condition \( 2^{m/6} > m^{2\omega+2} \), we conclude that the condition holds for \( m \geq 19538 \). Maintaining the flow we have the condition holds for \( m \geq 19333 \).

For the remaining pairs we calculate the exact value of \( \omega \) and able to detect 37 pairs where \( m = 7, 11, 13, 15, 17, 19, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 51, 53, 55, 57, 59, 65, 67, 69, 71, 73, 75, 77, 79, 81, 135, 165 and 225 \); which don’t satisfy the condition. Again for \( d = q^{m} - 1 \) and \( g = x^{m'} - 1 \), applying the prime sieve we are able to declare 20 of them as primitive normal pair. Then by choosing compatible \( d \) and \( g \) (as shown in table 2) we are able to determine another 13 pairs \( (2, 17), (2, 19), (2, 23), (2, 25), (2, 27), (2, 29), (2, 31), (2, 33), (2, 35), (2, 39), (2, 45), (2, 51) \) as primitive normal pairs. Hence, we conclude that following are the possible exceptional pairs \( (2, 7), (2, 11), (2, 13), (2, 15) \).

Case (ii): \( m \) is even. Once again we shall break this discussion into two parts.

First one is for those \( m \) such that \( 4 \mid m \). Then by lemma \( 5.8 \) \( W(q^m - 1) < 37.4683 \cdot q^{m/6} \) and for \( 4 \mid m \), \( \sigma(q, m) \leq m/24 \). Then to show \( \mathfrak{N}(q^m - 1, q^m - 1, x^m - 1, x^m - 1) > 0 \) it is sufficient to show \( 2^{m/12} > 5615.49 \cdot m \), which holds for \( m \geq 144 \). Then we calculate the exact value of \( \omega \) and check the condition \( 2^{m/12} > 2^{2\omega+2} \), for \( m \leq 143 \) and identify the pairs \( (2, 28), (2, 44), (2, 52), (2, 56), (2, 60) \) which don’t satisfy the condition. But from table 2, we can conclude that all of them are primitive normal pairs. Hence in this particular case all pairs \( (q, m) \) are primitive normal pairs.

For the second part i.e. for the case \( 2 \mid m \) but \( 4 \nmid m \), we use the lemma \( 5.8 \) \( W(q^m - 1) < 4514.7 \cdot q^{m/8} \) and in this case \( \sigma(q, m) \leq m/12 \). For these, the sufficient condition for existence of
primitive normal pair is $2^{m/12} > 8.153 \times 10^7$, which holds for $m \geq 420$. For the remaining pairs we use the prime sieve condition (4.3) for $d = q^m - 1$ and $g = x^{m'} - 1$ and identify the pairs $(2, 14), (2, 22), (2, 30), (2, 70)$ which fail to satisfy the condition. Again observing the condition (4.3) for appropriate values of $d, g$ we are able to clarify the pairs $(2, 22), (2, 70)$ as primitive normal pairs and the calculations are listed in table 2. Hence the only possible exceptional pairs are $(2, 14), (2, 30)$.

After all the observations for the case $q = 2$ and $m \nmid q - 1$, we conclude that the following pairs are only possible exceptional pairs.

$$(2, 3), (2, 5), (2, 6), (2, 7), (2, 9), (2, 10), (2, 11), (2, 12), (2, 13), (2, 14), (2, 15), (2, 18), (2, 20), (2, 21), (2, 24), (2, 30)$$

Following lemma is derived from lemma 5.7

**Lemma 5.10.** For $n \in \mathbb{N}$, $W(n) < 1.10992 \times 10^9 n^{1/10}$ and $W(n) < 4.24455 \times 10^{14} n^{1/11}$.

**Proposition 5.11.** For $q = 4$ and $m \nmid q - 1$, all the pairs $(q, m)$ are primitive normal pairs, except for the possible genuine exceptional pairs $(4, 5), (4, 7), (4, 9), (4, 10)$.

**Proof.** We shall start this discussion with the case $m' = 45$. In this case $x^{m'}$ is a product of 3 linear, 6 quadratic, 2 cubic and 4 sextic factors. Let $g$ be the product of linear factors, then $\vartheta = 0.5927$ and $\mathbb{G} = 20.56$. After this, the sufficient condition becomes $4^{m/10} > 167 \times (2^3)^2 \times 20.56$, which holds for $m \geq 90$. When $m = 45$, then $\omega = \omega(4^m - 1) = 11$ and the pair $(4, 45)$ satisfies the condition $4^{m/2} > 2^{2\omega+8} \times 20.56$. Hence $(4, 45)$ is also a primitive normal pair.

Now we are heading towards the next case, which is $m' = 9$. Then $x^{m'} - 1$ is a product of 3 linear and 2 cubic factors. Now we take $g$ as the product of three linear factors, then $\vartheta = 0.9375$ and $\mathbb{G} = 5.5$. These yield the condition $4^{m/10} > 167 \times (2^3)^2 \times 5.5$ and this holds for $m \geq 144$.

For the remaining pairs we verify the sufficient condition $4^{m/2} > 2^{2\omega+8} \times 5.5$ by calculating the exact value of $\omega$. After this we can conclude that the pairs $(4, 36), (4, 72)$ satisfy it and they are primitive normal pairs. From the table 2, we conclude that $(4, 18)$ is also a primitive normal pair, thus the only possible exceptional pair is $(4, 9)$.

Next we have the case $q = 4, m' \nmid q - 1$ and $m' \neq 9, 45$. At first we consider when $m$ is even. In this case $\sigma(q, m) \leq m/10$ and by lemma 5.10 $W(q^m - 1) < 1.10992 \times 10^9 q^{m/10}$. Hence the sufficient condition for existence of primitive normal pair is $4^{m/5} > 4.83296 \times 10^{18} m$, which holds for $m \geq 174$. For the remaining pair we use condition $2^{2\omega+2} m$ by calculating $\omega = \omega(4^m - 1)$. Among the rest of the pairs , $(4, 10), (4, 14), (4, 20), (4, 22), (4, 28), (4, 30)$ don’t satisfy the condition. Again for the appropriate values of $d$ and $g$, $(4, 14), (4, 20), (4, 22), (4, 28), (4, 30)$ satisfy the sieve condition as given in table 2. Hence the only possible exceptional pair is $(4, 10)$.

Now, we consider the case when $m$ is odd. Here $\sigma(q, m) = 1/5$ and from lemma 5.10 we have $W(q^m - 1) < 4.24455 \times 10^{14} q^{m/11}$. Then the sufficient condition is $4^{m/11} > 7.20647 \times 10^{29} m$, which holds for $m \geq 597$. Then we use the condition $4^{m/11} > 2^{2\omega+2} m$ to test the remaining pairs by calculating $\omega = \omega(q^m - 1)$. Then $(4, 5), (4, 7), (4, 11), (4, 13), (4, 15), (4, 25), (4, 27), (4, 29), (4, 33), (4, 35), (4, 39)$ are the pairs which don’t satisfy the condition. Now we take $d = q^m - 1$ and $g = x^m - 1$ in the prime sieve condition (4.3) and detect $(4, 27), (4, 29), (4, 33)$ and $(4, 39)$ as primitive normal pairs. Again, by choosing compatible values of $d$ and $g$ in condition (4.3) (as shown in table 2) we conclude that all of the remaining pairs are primitive normal pairs.
After the above investigation for the case \( q = 4 \) and \( m' \nmid q - 1 \), we conclude that except the following possible pairs, all the pairs \((q, m)\) are primitive normal pairs.

\[
(4, 5), (4, 7), (4, 9), (4, 10)
\]

\[\Box\]

**Proposition 5.12.** Let \( q = 8 \) and \( m' \nmid q - 1 \), then all the pairs \((q, m)\) are primitive normal pairs, unless \((q, m)\) is one of the pairs \((8, 3), (8, 5)\) and \((8, 7)\).

*Proof.* We begin our discussion with \( m' = 3 \), then \( x^{m'} - 1 \) is a product of linear and quadratic polynomial. Let \( g \) be the linear polynomial, then \( \vartheta = 0.96875 \) and \( \mathcal{S} < 3.04 \). Then sufficient condition for the existence of primitive normal pair is \( 8^{m/10} > 167 \times 2^2 \times 3.04 \) and this holds for all \( m \geq 48 \). For the remaining pairs we use the condition \( 8^{m/2} > 2^{2\omega+4} \times 3.04 \) by calculating the value of \( \omega \). Then the pairs \((8, 3), (8, 6), (8, 12)\) are the ones which fail to satisfy the inequality. Then choosing appropriate values of \( d \) and \( g \) in condition \( (4.3) \) as shown in table 2, we conclude that \((8, 3)\) is the only possible exceptional pair.

For the next stage we choose \( m' = 21 \) and \( x^{m'} - 1 \) is product of one linear, one quadratic, two cubic and two sextic polynomials. We choose \( g \) as product of the linear and the quadratic factor. Then \( \vartheta = 0.992172 \) and \( \mathcal{S} < 9.06 \) which yields the sufficient condition as \( 8^{m/10} > 167 \times (2^4)^2 \times 9.06 \), which holds for \( m \geq 84 \). Then the condition \( 8^{m/2} > 2^{2\omega+10} \times 9.06 \) comes into play to detect the primitive normal pairs by taking the value of \( \omega \). From this we declare that the remaining pairs \((8, 21), (8, 42)\) are also primitive normal pairs.

Now, we are heading for the final stage i.e. \( q = 8, m' \nmid q - 1 \) and \( m' \neq 3, 21 \). Form the lemma 5.5 and lemma 5.7 we have \( \sigma(q, m) \leq 1/5 \) and \( W(q^m - 1) < 37.4683q^{m/6} \). Then for the existence of primitive normal pairs, sufficient condition is \( 8^{m/30} > 5616m \), which holds for \( m \geq 202 \).

For the remaining pairs, we use the condition \( 8^{11m/30} > 2^{2\omega+12}m \) by determining the value of \( \omega \), this holds for \( m \geq 164 \). Next we take \( m \leq 163 \) and then \( \omega \leq 72 \). For these the condition holds for \( m \geq 140 \). Now repeating the above process we get the condition holds for \( m \geq 92 \) and among the remaining pairs \((8, 5), (8, 9), (8, 10), (8, 11), (8, 15), (8, 20)\) are the ones which fail to satisfy the condition. Then choosing appropriate value of \( l \) and \( g = x^{m'} - 1 \) in the condition \( (4.3) \) we are able to declare all but the pair \((8, 5)\) as primitive normal pairs.

After the above investigation for the case \( q = 8 \) and \( m' \nmid q - 1 \), we conclude that except the following possible pairs, all the pairs \((q, m)\) are primitive normal pairs.

\[
(8, 3), (8, 5), (8, 6)
\]

\[\Box\]

**Proposition 5.13.** Let \( q \geq 16 \) and \( m' \nmid q - 1 \), then all the pairs \((q, m)\) are primitive normal pairs, unless \((q, m)\) is \((32, 3)\).

*Proof.* We shall break the discussion into 4 cases (I–IV). For all the cases we suppose, after Lemma 5.5 that \( \vartheta(q, m) \leq \frac{1}{3} \). In this situation let \( g \) be the product of irreducible polynomials dividing \( x^m - 1 \) of degree less than \( u \).

**Case I:** \( q = 16 \); For this case we apply the lemma 5.8 i.e. \( W(q^m - 1) < 4514.7q^{m/8} \). Then to show \( \mathcal{M}(q^m - 1, q^m - 1, x^m - 1, x^m - 1) > 0 \) it is sufficient to show that \( 16^{m/12} > 8.15265 \times 10^7m \), which holds for \( m \geq 110 \). We use the condition \( 16^{m/2} > 2^{2\omega+2}m \) to test the remaining pairs by...
plotting value of $\omega$ and conclude that the pairs $(16, 7), (16, 9), (16, 11), (16, 13), (16, 14), (16, 18), (16, 21)$ fail to satisfy the condition. Further, we can choose compatible $l$ and $g = x^{m'} - 1$ in condition (4.3) to conclude all of them except $(16, 7)$ are primitive normal pairs. Finally from table 2, we have $(16, 7)$ is also a primitive normal pair.

**Case II:** $q = 32$; From lemma 5.8 we have $W(q^m - 1) < 37.4683q^{m/2}$ and proceeding as above with the sufficient condition $32^{m/30} > 1403.87m$, which is true for all $m \geq 103$. For rest of the pairs we use the condition $32^{1m/30} > 2^{2\omega+2}m$, which proves that all the pairs $(q, m)$ are primitive normal pairs unless $(q, m)$ is one of the pairs $(32, 3), (32, 5), (32, 6), (32, 9), (32, 10), (32, 12)$. Furthermore, applying the prime sieve condition (4.3) for compatible $l$ and $g = x^{m'} - 1$, we confirm that all of them are primitive normal pairs except $(32, 3)$.

**Case III:** $q = 64$; Using lemma 5.8 we have $W(q^m - 1) < 37.4683q^{m/2}$ and for $\mathfrak{M}(q^m - 1, q^m - 1, x^m - 1, x^m - 1) > 0$ the sufficient condition is $64^{m/18} > 1403.87m$, which is true for all $m \geq 49$. We use the condition $64^{7m/18} > 2^{2\omega+2}m$, to investigate the existence of the property in rest of the pairs and all the pairs $(q, m)$ are primitive normal pairs unless $(q, m)$ is one of the pairs $(64, 5), (64, 10)$. Later applying the prime sieve condition (4.3) for compatible $l$ and $g = x^{m'} - 1$, we confirm that all of them are primitive normal pairs.

**Case IV:** $q \geq 128$; Lemma 5.8 yields $W(q^m - 1) < 37.4683q^{m/2}$ and for $\mathfrak{M}(q^m - 1, q^m - 1, x^m - 1, x^m - 1) > 0$ it is sufficient to show $q^{m/6} > 1403.87 \times 2^{m/3}m$, which is true for all $q \geq 128$ and $m \geq 18$. We use the condition $q^{m/2} > 2^{2\omega+2+2m/3}m$, to test the existence of the property in rest of the pairs and all the pairs $(q, m)$ are primitive normal pairs except $(128, 3)$. Then from the table 2, we confirm that all of them are primitive normal pairs.

Finally, we conclude that for $q \geq 128$ and $m' \nmid q - 1$, all the pairs $(q, m)$ are primitive normal pairs with genuine exceptional pair,

$$(32, 3).$$

Finally, we explain all the pairs in this table which we eliminate through further calculation by means of condition (4.3) for suitable $d$ and $g$. 

| $(q,m)$  | $d$ | $n$ | $g$ | $k$ | $\Lambda$ | $q^{m/2}$ | $3W(d)^2\Omega(g)\Lambda$ |
|-----------|-----|-----|-----|-----|-----------|-----------|-----------------|
| $(2,48)$  | 105 | 6   | $x + 1$ | 1   | 70.8428   | 1.67772   | 72543           |
| $(2,40)$  | 3   | 6   | 1    | 2   | 82.1256   | 1.04858   | 21031.1        |
| $(2,36)$  | 15  | 6   | $x^3 - 1$ | 0  | 32.9531   | 262144    | 134976         |
| $(2,42)$  | 3   | 5   | $x^{21} - 1$ | 0 | 15.9379   | 2.09751   | 16320.4        |
| $(2,17)$  | $q^m - 1$ | 0   | $x + 1$ | 2   | 5.04762   | 362.039   | 323.048        |
| $(2,19)$  | $q^m - 1$ | 0   | $x + 1$ | 1   | 3.00001   | 724.077   | 192.001        |
| $(2,23)$  | 47   | 1   | $x + 1$ | 2   | 7.00984   | 2896.31   | 448.63         |
| $(2,25)$  | 31   | 2   | $x + 1$ | 2   | 10.0408   | 5792.62   | 642.611        |
| $(2,27)$  | 7   | 2   | $x + 1$ | 3   | 22.3926   | 11585.2   | 1433.13        |
| $(2,29)$  | 233  | 2   | $x^{23} - 1$ | 0 | 5.00834   | 23170.5   | 1282.14        |
| $(2,31)$  | $q^m - 1$ | 0   | $x + 1$ | 6   | 19.6      | 46341     | 1254.4         |
| $(2,33)$  | 7   | 2   | $x + 1$ | 4   | 35.7918   | 92681.9   | 2290.68        |
| $(2,35)$  | 31   | 3   | $x + 1$ | 5   | 47.4422   | 185364    | 3036.3         |
| $(2,39)$  | 7   | 3   | $(x + 1)(x^2 + x + 1)$ | 3 | 13.3057   | 741455    | 3406.26        |
\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
(q,m) & d & n & g & k & \Lambda & q^{n/2} & 3W(d)^2\Omega(g)\Lambda \\
\hline
(2,45) & 7 & 5 & (x+1)(x^2+x+1) & 6 & 32.9687 & 5.93164 \times 10^6 & 8439.99 \\
(2,51) & 7 & 4 & x^{51} - 1 & 0 & 9.14684 & 4.74531 \times 10^7 & 9.59166 \times 10^6 \\
(2,28) & 3 & 5 & x+1 & 2 & 66.6522 & 16384 & 4265.74 \\
(2,44) & 3 & 6 & x^{11} - 1 & 0 & 24.8377 & 4.1943 \times 10^6 & 6358.45 \\
(2,52) & 3 & 6 & x^{13} - 1 & 0 & 22.0983 & 6.71089 \times 10^6 & 5657.16 \\
(2,56) & 15 & 6 & x^r - 1 & 0 & 16.9888 & 2.68435 \times 10^8 & 17395.6 \\
(2,60) & 15 & 9 & x^{15} - 1 & 0 & 82.2883 & 1.07374 \times 10^9 & 5.39285 \times 10^6 \\
(2,22) & 3 & 3 & x^{11} - 1 & 0 & 7.6329 & 2048 & 1954.02 \\
(2,70) & 3 & 8 & x^{35} - 1 & 0 & 24.8631 & 3.43597 \times 10^{10} & 1.62943 \times 10^9 \\
(4,18) & 15 & 6 & (x+1)(x^2+x+1) & 1 & 42.1079 & 262144 & 42.1079 \\
(4,14) & 3 & 5 & x+1 & 2 & 35.4555 & 16384 & 2269.15 \\
(4,20) & 3 & 6 & x^5 - 1 & 0 & 60.7588 & 1.04858 \times 10^6 & 15554.3 \\
(4,22) & 3 & 6 & x^{11} - 1 & 0 & 24.8377 & 4.1943 \times 10^6 & 6358.45 \\
(4,28) & 3 & 7 & x^7 - 1 & 0 & 40.9888 & 2.68435 \times 10^8 & 41972.5 \\
(4,30) & 15 & 9 & x^{15} - 1 & 0 & 82.2883 & 1.07374 \times 10^9 & 5.39285 \times 10^6 \\
(4,11) & 3 & 3 & x^{11} - 1 & 0 & 7.6329 & 2048 & 1954.02 \\
(4,13) & 3 & 2 & x^{13} - 1 & 0 & 5.00293 & 8192 & 1280.75 \\
(4,15) & 3 & 5 & (x+1)(x^2+x+1) & 3 & 44.2638 & 32768 & 11332.7 \\
(4,25) & 3 & 6 & x^{25} - 1 & 0 & 16.8495 & 3.35544 \times 10^7 & 17253.9 \\
(4,35) & 33 & 7 & x+1 & 5 & 31.9641 & 3.43596 \times 10^{10} & 2045.7 \\
(8,6) & 3 & 3 & x+1 & 1 & 14.7186 & 512 & 235.498 \\
(8,12) & 15 & 6 & x^3 - 1 & 0 & 32.9531 & 262144 & 33744 \\
(16,7) & 3 & 5 & x+1 & 2 & 30.8825 & 16384 & 1976.48 \\
(128,3) & 7 & 2 & x^3 - 1 & 0 & 5.06649 & 1448.15 & 1297.02 \\
\hline
\end{array}
\]

Table 2

As immediate consequence of above results, we have our final proposition.

**Proposition 5.14.** Let \(\mathbb{F}_{q^m}\) be a finite field of even characteristic and \(m \nmid q - 1\). Then there exists a primitive normal element \(\alpha\) in \(\mathbb{F}_{q^m}\) such that both \(\alpha\) and \(f(\alpha)\) are simultaneously primitive normal in \(\mathbb{F}_{q^m}\) over \(\mathbb{F}_q\), where \(f(x) = ax^2 + bx + c\), with \(a,b,c,d,e \in \mathbb{F}_{q^m}\), \(a \neq 0\), and \(dx + e \neq 0\) unless \((q,m)\) is one of the following pairs.

\[
\begin{align*}
(2,3), & \quad (2,5), \quad (2,6), \quad (2,7), \quad (2,9), \quad (2,10), \quad (2,11), \quad (2,12), \\
(2,13), & \quad (2,14), \quad (2,15), \quad (2,18), \quad (2,20), \quad (2,21), \quad (2,24), \quad (2,30), \\
(4,5), & \quad (4,7), \quad (4,9), \quad (4,10), \quad (8,3), \quad (8,5), \quad (8,6), \quad (32,3)
\end{align*}
\]

**Acknowledgement**

This work was funded by Council of Scientific and Industrial Research, New Delhi, Government of India’s research grant no. 09/796(0099)/2019-EMR-I.

**References**

[1] Anju and R.K.Sharma, Existence of some special primitive normal elements over finite fields, *Finite Fields Appl.* 46 (2017) 280-303.
[2] A.R. Booker, S.D. Cohen, N. Sutherland and T. Trudgian, Primitive values of quadratic polynomials in a finite field, *Math. Comp.* 88 (318) (2019) 1903-1912.

[3] L.Carlitz, Primitive roots in a finite fields, *Trans. Amer. Math. Soc.* 73(3) (1952) 314-318.

[4] S.D.Cohen, Pair of primitive elements in fields of even order, *Finite Fields Appl.* 28 (2014) 22-42.

[5] S.D.Cohen and S.Huczynska, The primitive normal basis theorem – without a computer, *J. Lond. Math. Soc.* 67(1) (2003) 41-56.

[6] S.D.Cohen and S.Huczynska, The strong primitive normal basis theorem, *Acta. Arith.* 143(4) (2010) 299-332.

[7] L.Fu and D.Q.Wan, A class of incomplete character sums, *Q.J.Math.Soc* 43, (1968) 21-39.

[8] T.Garefalakis and G.Kapetanakis, On the existence of primitive completely normal bases of finite fields, *J. Pure Appl. Algebra* 223(3) (2018) 909-921.

[9] G.Kapetanakis, An extension of the (strong) primitive normal basis theorem, *Appl. Algebra Eng. Commun. Comput.*, 25 (2013) 311-337.

[10] G. Kapetanakis, Normal bases and primitive elements over finite fields, *Finite Fields Appl.* 26(2014) 123-143.

[11] H.W.Lenstra,Jr. and R.J.Schoof, Primitive Normal Bases for Finite Fields, *Math. Comp.* 48 (1987) 217-231.

[12] R. Lidl and H. Niederreiter, *Finite Fields* 2nd edn. (Cambridge University Press, Cambridge, 1997).

[13] T. Tian and W.F. Qi, Primitive normal elements and its inverse in finite fields, *Acta. Math. Sinica*(Chin. Ser.) 49(3) (2006) 657-668.

[14] D. Wan, Generators and irreducible polynomials over finite fields, *Math. Comp.* 66(219) (1997) 1195-1212.

[15] P.P. Wang, X.W. Cao and R.Q. Feng, On the existence of some specific elements in finite fields of characteristic 2, *Finite Fields Appl.* 18(4) (2012) 800-8013.

[16] S.D. Cohen, Kloosterman sums and primitive elements in Galois fields, *Acta Arithmetica* XCIV(2) (2000) 173-201.

[17] F.N. Castro, C.J. Moreno, Mixed exponential sums over finite fields, *Proc. Am. Math. Soc.* 128(9) (2000) 2529-2537.

[18] S.D. Cohen, Consecutive primitive roots in a finite field, *Proc. Am. Math. Soc.* 93(2) (1985) 189-197.

[19] H. Davenport, Bases for finite fields, *J. Lond. Math. Soc.* 43 (1968) 21-39.
[20] L. Carlitz, Some problems involving primitive roots in a finite filed, *Proc. Natl. Acad. Sci. USA*, 38(4) (1952) 314-318.

[21] W.S. Chou, S.D. Cohen, Primitive elements with zero traces, *Finite Fields Appl.*, 7 (2001) 125-141.

[22] G.James and M.Liebeck, *Representations and Characters of Groups*, 2nd edn. (Cambridge University Press, Cambridge, 2001).

[23] L.B. He, W.B. Han,. Research on primitive elements in the form $\alpha + \alpha^{-1}$ over $\mathbb{F}_q$, *J. Inf. Eng. Univ.*, 4(2) (2003) 97-98.

[24] G.B. Agnew, R.C. Mullin, I.M. Onyszchuk, and S.A. Vanstone. An implementation for a fast public key cryptosystem, *J. Cryptol.*, 3 (1991) 63-79.