An improved explicit bound on $|\zeta(\frac{1}{2} + it)|$

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Abstract
This article proves the bound $|\zeta(\frac{1}{2} + it)| \leq 0.732 t^{\frac{1}{6}} \log t$ for $t \geq 2$, which improves on a result by Cheng and Graham. We also show that $|\zeta(\frac{1}{2} + it)| \leq 0.732 |3.3081 + it|^{\frac{1}{6}} \log |3.3081 + it|$ for all $t$.

1 Introduction

The Riemann zeta-function $\zeta(s)$ is known to satisfy $\zeta(\frac{1}{2} + it) \ll t^{\frac{32}{205} + \epsilon}$ for all $t \gg 1$ [4]. Explicit estimates of the sort

$|\zeta(\frac{1}{2} + it)| \leq At^{\theta_1}(\log t)^{\theta_2}$, \hspace{1em} ($t \geq t_0$)

are difficult to produce since, attempts at small values of $\theta_1$ lead to complicated arguments in the calculation of $A$. Using the approximate functional equation one may show that

$|\zeta(\frac{1}{2} + it)| \leq \frac{4}{(2\pi)^{\frac{1}{2}}} t^{\frac{1}{3}}$, \hspace{1em} ($t \geq 0.2$). \hspace{1em} (1)

Lehman [6, Lem. 2] proved this for $t \geq 128\pi$ — see also [9, Thm 2] and [13, Thm 1] — one may verify that (1) holds in the range $0.2 \leq t < 128\pi$ by direct

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computation. The only other result of which we are aware is due to Cheng and Graham [1], viz.

$$|\zeta(\frac{1}{2} + it)| \leq 3t^{\frac{1}{2}} \log t, \quad (t \geq e).$$  \hspace{1cm} (2)

The upper bound in (2) is smaller than that in (1) when $t \geq 1.4 \times 10^{21}$. This is unfortunate since for some problems one seeks information for $t \geq T_0$, where $T_0$ is at most the height to which the Riemann hypothesis has been verified\(^1\).

In [12, (5.4)] the second author showed that one could combine Theorem 3 of [1] with (1) to show that

$$|\zeta(\frac{1}{2} + it)| \leq 2.38 t^{\frac{1}{2}} \log t, \quad (t \geq e),$$

which is better than the bound in (1) only when $t \geq 10^{10}$. The purpose of this article is to revisit the paper by Cheng and Graham and to prove

**Theorem 1.**

$$|\zeta(\frac{1}{2} + it)| \leq 0.732 t^{\frac{1}{2}} \log t, \quad (t \geq 2).$$

The bound in Theorem 1 improves on that in (1) whenever $t \geq 5.868 \times 10^9$. Three applications are apparent: [8, 11, 12] which respectively relate to explicit estimates for zero-density theorems, bounding $\int_0^T S(t) \, dt$, and bounding $S(t)$, where $\pi S(t)$ is the argument of the zeta-function on the critical line. The estimate for $S(t)$ can be improved immediately to give

**Corollary 1.** If $T \geq e$, then

$$|S(T)| \leq 0.112 \log T + 0.278 \log \log T + 2.359.$$  

*Proof.* Using Theorem 1 one may take $(k_1, k_2, k_3) = (0.631, 1/6, 1)$ in [12, (4.8)]. Instead of choosing $Q_0 = 2$ on page 291 of [12], we choose $Q_0 = 4$. The choice of $\eta = 0.077, r = 2.052$ on the same page establishes Corollary 1. \hfill \Box

The improvement of Theorem 1 over the result in [1] comes from three ideas. First, an explicit form of the ‘standard’ approximate functional equation is used (cf. Lemma 3), in which one needs to estimate sums of the form $\sum_{n \leq Y} n^{it}$, where $Y \asymp t^{\frac{1}{2}}$. Cheng and Graham considered an approximation to $\zeta(\frac{1}{2} + it)$ in which one needs to estimate a longer sum with $Y \asymp t$. Second, trivial estimates are used judiciously to reduce the contribution of lower-order terms. Finally, some minor adjustments are made to some of the results in [1], and more variables are optimised.

We prove some necessary lemmas in §2. We prove Theorem 1 for large $t$ in §3 and for small $t$ in §4. We conclude with some computational remarks in §5.

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\(^1\)The first author [7] has confirmed that for $0 \leq t \leq 3.06 \times 10^{10}$ all non-trivial zeroes of $\zeta(\sigma + it)$ lie on the critical line.
2 Preparatory Lemmas

It is necessary to record some estimates for exponential sums. Versions of the following lemmas without explicit constants can be found in [10, Thm 5.9 and Lemma 5.10]. Slightly coarser explicit versions can be found in [5, p. 36] and [3, Lemma 2.2].

Lemma 1. Assume that \( f(x) \) is a real-valued function with two continuous derivatives when \( x \in [N+1, N+L] \). If there exist two real numbers \( V < W \) with \( W > 1 \) such that
\[
\frac{1}{W} \leq |f''(x)| \leq \frac{1}{V}
\]
for \( x \in [N+1, N+L] \), then
\[
\left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right| \leq \left( \frac{L-1}{V} + 1 \right) \left( 2\sqrt{\frac{2}{\pi}} W^{1/2} + 2 \right) + 1.
\]

Proof. This is Lemma 3 in [1] with three slight adjustments. First, when applying the mean-value theorem on the first line of page 1268 of [1] one obtains \( k \leq (L-1)/V + 2 \) instead of \( k \leq L/V + 2 \). Second, when estimating the \( 2(k-1) \) intervals trivially, one may note that there are two intervals of length \( W\Delta + 1 \), namely those intervals from \((C_k - \Delta, C_k)\) and \((C_1, C_1 + \Delta)\), whereas there are \( k - 2 \) intervals of length \( 2W\Delta + 1 \). Third, we retain the constant \( 2\sqrt{2/\pi} \) as opposed to (the only slightly larger) \( 8/5 \).

Lemma 2. Let \( f(n) \) be a real-valued function and let \( M \) be a positive integer. Then
\[
\left| \sum_{n=N+1}^{N+L} e^{2\pi i f(n)} \right|^2 \leq \frac{L(L+M-1)}{M} + \frac{2(L+M-1)}{M} \sum_{m=1}^{M-1} \left( 1 - \frac{m}{M} \right) \max_{K \leq L} \left| \sum_{m,K} e^{2\pi i (f(n+m)-f(n))} \right|.
\]

(3)

where
\[
\sum_{m,K} e^{2\pi i (f(n+m)-f(n))}.
\]

Proof. This is Lemma 5 in [1] with \( L+M \) changed to \( L+M-1 \), a substitution that is clearly permitted as per the displayed equation at the bottom of [1] p. 1272. This differs from Lemma 5.10 in [10] in three respects: there is no upper restriction on \( M \), the coefficients are smaller (in [10] both terms in (3) have 4 as their leading coefficients), and the factor \( (1 - m/M) \) is present.

Lemma 3. For \( t \geq 100 \),
\[
|\zeta(\frac{1}{2} + it)| \leq 2 \sum_{n \leq \sqrt{t/\pi}} n^{\frac{1}{2} - it} + \theta(1.53t^{\frac{1}{4}}) + \theta(3.23t^{\frac{1}{4}}).
\]

(4)
Proof. We use Theorem 1 [9], from which it follows that

\[ |\zeta(\frac{1}{2} + it)| \leq 2 \sum_{n \leq \sqrt{\frac{2}{\pi}}} n^{-\frac{1}{2}-it} + \frac{|\Gamma(\frac{1}{2} + it)|}{2\pi} e^{\frac{\pi}{2}t} (2\pi)^{\frac{1}{2}} |g(\tau)| + |R(s)|, \]

where, in Titchmarsh’s expression for \( R(s) \), there appears to be a blemish on the page: the \( 8 \) ought to be \( \frac{8}{3} \), as per equation (4.1) of [9]. By the last line on p. 235 of [9] the function \( g(\tau) \) satisfies

\[ |g(\tau)| \leq (2\pi)^{\frac{1}{4}} t^{-\frac{1}{4}} \left| \frac{\cos 2\pi(x^2 - x - \frac{1}{16})}{\cos 2\pi x} \right|, \]

where \( 0 \leq x \leq 1 \). One may plot the quotient of the cosines to see that \( |g(\tau)| \leq (\cos \frac{\pi}{8})(2\pi)^{\frac{1}{4}} t^{-\frac{1}{4}} \). With the version of Stirling’s theorem given in Lemma \( \epsilon \) in [9] we can now bound the second term in (5). Finally, using Titchmarsh’s expression for \( R(s) \), we note that \( R(s)t^{-\frac{1}{2}} \) is decreasing in \( t \) provided that \( t > (5/2)^3 \). A computation of the constants involved proves the lemma. \( \square \)

3 Proof of Theorem 1 for large \( t \).

Write the sum in (4) as

\[ \sum_{n \leq A_0 t^{\frac{3}{4}}} n^{-\frac{1}{2}-it} + \sum_{A_0 t^{\frac{3}{4}} < n < \frac{2}{\pi}} n^{-\frac{1}{2}-it} \]

provided that the interval of summation in the second sum is non-empty, that is, provided that

\[ t_0 > A_0^6 (2\pi)^3. \]

Summing by parts gives

\[ \left| \sum_{n \leq A_0 t^{\frac{3}{4}}} n^{-\frac{1}{2}-it} \right| \leq \sum_{n \leq A_0 t^{\frac{3}{4}}} n^{-\frac{1}{2}} \leq 2A_0^3 t^{\frac{3}{4}} - 1. \]

Now consider

\[ X_j = A_0 k^j t^{\frac{3}{4}}, \quad N_j = [X_j], \]

where \( k > 1 \) is a parameter to be determined later, and \( j = 0, 1, 2, \ldots, J \), where

\[ J \leq \frac{\frac{1}{6} \log t - \log \left\{ A_0 (2\pi)^{\frac{3}{4}} \right\}}{\log k} + 1. \]

It follows that

\[ \sum_{A_0 t^{\frac{3}{4}} < n \leq \sqrt{\frac{2}{\pi}}} n^{-\frac{1}{2}-it} = \sum_{j=1}^{J} \sum_{n=N_{j-1}+1}^{\min\{N_j, \sqrt{\frac{2}{\pi}}\}} n^{-\frac{1}{2}-it}, \]
whence, by partial summation we have

\[ \left| \sum_{A_0 t^+ < n < \sqrt{\pi}} \frac{n^{-\frac{1}{2}-it}}{\sqrt{\pi}} \right| \leq \sum_{j=1}^{J_0} \frac{1}{X_j-1} \max_{L \leq N_j - N_{j-1}} \left| \sum_{n=N_j+1}^{N_j+L} e^{-it \log n} \right|. \]  

(8)

Denote the sum over \( n \) in (8) by \( S_j \). We may estimate \( S_j \) either trivially or using Lemmas 1 and 2. Suppose we use the trivial estimate for \( 1 \leq j \leq J_0 \), whence, for such \( j \)

\[ \max_{L \leq N_j - N_{j-1}} |S_j| \leq L \leq N_j - N_{j-1} \leq (k-1)X_{j-1} + 1. \]

The trivial estimation therefore makes the following contribution to (8)

\[ \sum_{j=1}^{J_0} \frac{L}{X_j-1} \leq k^{\frac{1}{2}}(k-1)A_0^\frac{1}{2} t^+ \left\{ \frac{k^{\frac{1}{2}} t_0^\frac{1}{2} - 1}{k^{\frac{1}{2}} - 1} \right\} + A_0^{-\frac{1}{2}} t_0^{-\frac{1}{2}} \left\{ \frac{1-k^{\frac{1}{2}} t_0}{k^{\frac{1}{2}} - 1} \right\}. \]

Now consider \( j \geq J_0 + 1 \). First apply Lemma 2 to \( S_j \) and thence apply Lemma 1 to the resulting

\[ \sum_{m,K} N_{j+1} \sum_{n=n_j+1} N_{j+1} e^{-it \log (n+m) - \log n}. \]

Choose \( M = [k^j \theta] + 1 \), for some \( \theta \) to be determined later, subject to the restriction that \( M \geq 2 \). We need to determine \( V \) and \( W \) in Lemma 1. We have

\[ f(x) = -\frac{t}{2\pi} \{ \log(x + m) - \log x \}, \quad |f''(x)| = \frac{tm}{2\pi} \left( \frac{m + 2x}{x^2} \right)^2. \]

Since \( (m + 2x)/(x(x + m))^2 \) is decreasing in both \( x \) and \( m \) we take \( m = 0, x = A_0 k^{j-1} t^+ \), and \( m = M - 1 \leq k^j \theta, x = A_0 k^j t^+ \) to find that \( 1/W \leq |f''(x)| \leq 1/V \), where

\[ V = \pi A_0^{\frac{3}{2}} k^{3j}/k^{3m}, \quad W = \pi k^{3j} A_0^3 / m \left( 1 + \frac{\theta}{A_0 t^+_0} \right)^2. \]

In order to apply Lemma 1 it remains only to note that

\[ L \leq (k-1)X_{j-1} + 1 \leq (k-1)A_0 k^{j-1} t^+ + 1. \]  

(9)

One may now apply Lemma 1 to find that

\[ \left| \sum_{m,K} \right| \leq A_1 t^+ m^\frac{3}{2} k^{-\frac{1}{2}} j + A_2 t^+ m k^{-2j} + A_3 m^{-\frac{1}{2}} k^j + 3, \]

\[ \text{One of the advantages of using Lemma 1 over Lemma 3 in 1 is that, according to 1, } \]

\[ \text{L - 1 generates only one term.} \]
where
\[ A_1 = 2\sqrt{2}(k-1)k^2Y_0, \quad A_2 = 2(k-1)k^2, \quad A_3 = 2\sqrt{2}A_0^2Y_0, \quad Y_0 = 1+\frac{\theta}{A_0t_0^2}. \]

The displayed formulae on page 1277 of [1] show that
\[ \sum_{1 \leq m \leq M-1} \left( 1 - \frac{m}{M} \right) m^{\frac{2}{3}} \leq \frac{4}{15} M^{\frac{5}{3}}, \quad \sum_{1 \leq m \leq M-1} \left( 1 - \frac{m}{M} \right) m^{-\frac{1}{2}} \leq \frac{4}{3} M^{\frac{4}{3}}. \]

Applying this gives
\[ \frac{1}{M} \sum_{m=1}^{M-1} \left( 1 - \frac{m}{M} \right) | \sum_{m,K} | \leq \frac{4}{15} A_1 t_0^\frac{2}{3} M^\frac{5}{3} k^{-\frac{1}{3}} + \frac{1}{6} A_2 t_0^\frac{2}{3} M k^{-2} + \frac{4}{3} A_3 M^{-\frac{1}{3}} k \frac{2}{3} j + \frac{3}{2}. \]

Return now to Lemma 2
\[ |S_j|^2 \leq \frac{L(L+M-1)}{M} + 2(L+M-1) \left\{ \frac{1}{M} \sum_{m=1}^{M-1} \left( 1 - \frac{m}{M} \right) | \sum_{m,K} | \right\}. \]

For \( \alpha > 0 \), \( (L+M-1)M^\alpha \) is an increasing function of \( M \); \( (L+M-1)/M \) is decreasing. We use an upper bound for the numerator and a lower bound for the denominator in \( (L+M-1)/M^{1/2} \). With \( M = |k^j\theta| + 1 \) we have,
\[ |S_j|^2 \leq B_1 k^j t_0^\frac{1}{2} + B_2 k t_0^\frac{2}{3} + B_3 k^2 t_0^\frac{2}{3} + B_4 k^3 t_0^\frac{2}{3}, \]

where
\[
A_4 = \frac{(k-1)^2 A_0^2}{k^2 \theta} \left( 1 + \frac{1}{(k-1)A_0 k^{j_0} t_0^\frac{1}{2}} \right) \left( 1 + \frac{\theta k}{(k-1)A_0 t_0^\frac{1}{2}} \right), \\
A_5 = \frac{2(k-1)A_0}{k} \left\{ 1 + \frac{1}{(k-1)k A_0 k^{j_0} t_0^\frac{1}{2}} + \frac{\theta k}{(k-1)A_0 t_0^\frac{1}{2}} \right\}, \\
A_6 = \frac{4}{15} A_1 \theta^\frac{2}{3} \left\{ 1 + \frac{1}{k^{j_0+1} \theta} \right\} ^\frac{1}{2}, \quad A_7 = \frac{A_2 \theta}{6} \left\{ 1 + \frac{1}{k^{j_0+1} \theta} \right\}, \\
A_8 = \frac{4A_3}{3\theta^{\frac{2}{3}}}, \quad B_1 = A_4 + A_5 A_6, \quad B_2 = A_5 A_7, \quad B_3 = \frac{3}{2} A_5, \quad B_4 = A_5 A_8. \]

Note that choosing \( J_0 \) large, that is, estimating many terms trivially, keeps most of the parenthetical terms in (10) close to unity.

Using the inequality \( \sqrt{(x+y+\cdots)} \leq \sqrt{x} + \sqrt{y} + \cdots \) we have
\[
\sum_{j=J_0+1}^{1} \frac{1}{X_{j-1}^\frac{1}{2}} |S_j| \leq \frac{k^{\frac{1}{2}}}{A_0^\frac{1}{2}} \left\{ (\sqrt{B_1 t_0^\frac{1}{2}} + \sqrt{B_3}) \sum_{j=J_0+1}^{1} 1 \right. \\
+ \sqrt{B_2 t_0^\frac{2}{3}} \sum_{j=J_0+1}^{1} k^{-\frac{2}{3} j} + \sqrt{B_4} \sum_{j=J_0+1}^{1} k^{\frac{2}{3} j} \right\}. \]
Since
\[
\sum_{j=j_0+1}^{j} k^{-\frac{j}{2}} = k^{-\frac{1}{2}}(j_0+1) \left\{ \frac{1 - k^{-\frac{1}{2}(J-J_0)}}{1 - k^{-\frac{1}{2}}} \right\},
\]
this gives
\[
\sum_{j=j_0+1}^{j} \frac{1}{X_j^\frac{1}{2}} |S_j| \leq \left( \frac{k}{A_0} \right)^{\frac{1}{2}} \left\{ C_1 t^{\frac{1}{2}} \log t + C_2 t^{\frac{1}{2}} + C_3 t^{\frac{1}{2}} + C_4 \log t + C_5 \right\},
\]
where
\[
C_1 = \frac{\sqrt{B_1}}{6 \log k}, \quad C_2 = \sqrt{B_1} \left( 1 - J_0 - \frac{\log \left( A_0 (2\pi)^{\frac{1}{2}} \right)}{\log k} \right) + \sqrt{B_2} k^{-\frac{1}{2}(J_0+1)}
\]
\[
C_3 = \frac{\sqrt{B_3} k}{A_0^\frac{1}{2} (2\pi)^{\frac{1}{4}} (k^{\frac{1}{2}} - 1)} - \frac{\sqrt{B_2} A_0^\frac{1}{2} (2\pi)^{\frac{1}{4}}}{k (1 - k^{-\frac{1}{2}})}, \quad C_4 = \frac{\sqrt{B_3}}{6 \log k}
\]
\[
C_5 = \frac{\sqrt{B_3}}{1 - J_0} - \frac{\log A_0 (2\pi)^{\frac{1}{2}}}{\log k} - \frac{\sqrt{B_3} k^{\frac{1}{2}(J_0+1)}}{k^{\frac{1}{2}} - 1}.
\]
This means that
\[
|\zeta(\frac{1}{2} + it)| \leq D_1 t^{\frac{1}{2}} \log t + D_2 t^{\frac{1}{2}} + D_3 t^{\frac{1}{2}} + D_4 \log t + D_5,
\]
where
\[
D_1 = 2C_1 \left( \frac{k}{A_0} \right)^{\frac{1}{2}}, \quad D_2 = 2 \left\{ 2A_0^{\frac{1}{2}} + C_2 \left( \frac{k}{A_0} \right)^{\frac{1}{2}} + (k - 1)A_0^{\frac{1}{2}} \left( \frac{k^{\frac{1}{2}} - J_0}{k^{\frac{1}{2}} - 1} \right) \right\},
\]
\[
D_3 = 2C_3 \left( \frac{k}{A_0} \right)^{\frac{1}{2}}, \quad D_4 = 2C_4 \left( \frac{k}{A_0} \right)^{\frac{1}{2}},
\]
\[
D_5 = 2 \left\{ C_5 \left( \frac{k}{A_0} \right)^{\frac{1}{2}} - 1 + A_0^{\frac{1}{2}} t_0^{-\frac{1}{2}} \left( \frac{1 - k^{-\frac{1}{2}J_0}}{1 - k^{-\frac{1}{2}}} \right) + \frac{0.64}{t_0^\frac{1}{2}} + \frac{0.33}{t_0^\frac{1}{2}} \right\}.
\]
Choosing
\[
k = 1.16, \quad \theta = 7.5, \quad A_0 = 3.37, \quad J_0 = 0, \quad T_0 = 5.861 \times 10^9
\]
means that \( |\zeta(\frac{1}{2} + it)| \leq 0.732 t^{\frac{1}{2}} \log t \) for \( t \geq 5.861 \times 10^9 \), \( \mathbb{1} \) is satisfied, and that \( M \geq 2 \). We now turn our attention to \( t < 5.861 \times 10^9 \).

4 Proof of Theorem \( \mathbb{1} \) for small \( t \)

Lemma 4. For \( t \in [2, 5.861 \times 10^9] \) we have
\[
|\zeta(\frac{1}{2} + it)| < 0.732 t^{\frac{1}{2}} \log t.
\]
Proof. The trivial bound (1) is tighter than our new bound at $t = 5.861 \times 10^9$ and remains so for $t$ all the way down to $t = 226.7088 \ldots$. We checked the range $[2, 230]$ rigorously by computer as follows.

We implemented an interval arithmetic version of the Euler–MacLaurin summation formula that, given an interval $I$, returns an interval that includes $|\zeta(1/2 + it)|$ for all $t \in I$. We divided the line segment $[2, 230]$ into pieces of length $1/1024$ and for each piece, checked that $|\zeta(1/2 + it)|$ did not exceed our bound. Specifically, if we are considering $I = [a, a + 1/1024]$ and we know that for $t \in I$ that $|\zeta(1/2 + it)| \in [x, y]$, then we check $y < 0.732a^{\frac{1}{6}} \log a$. No counter examples exist for $t \in [2, 230]$ and this establishes the lemma.

\[ \text{Lemma 5.} \quad \text{For } t \text{ real and } Q \geq 3.3081 \text{ we have} \]
\[ |\zeta(1/2 + it)| < 0.732|Q + it|^{\frac{1}{6}} \log |Q + it|. \]

Proof. For $|t| \geq 2$ we use Lemma 4. For $t \in (-2, 2)$ we know that $|\zeta(1/2 + it)|$ attains a maximum at $t = 0$ so we determine a $Q$ such that
\[ |\zeta(1/2)| < Q^{\frac{1}{6}} \log Q \]
and we are done.

5 Conclusion

Since an Euler–MacLaurin computation of $\zeta(1/2 + it)$ becomes inefficient as $t$ increases, we also implemented an interval version of the Riemann–Siegel formula (R-S) for $t \geq 200$. Above this height we have explicit error bounds due to Gabcke [2]. The only nuance is that the main sum of R-S runs from 1 to $\lfloor \sqrt{t}/2\pi \rfloor$ and we must be careful not to compute with intervals $I = [a, b]$ such that $\lfloor \sqrt{a}/2\pi \rfloor \neq \lfloor \sqrt{b}/2\pi \rfloor$. We get around this by using Euler–MacLaurin for such intervals.

So armed, we can continue to compute $|\zeta(1/2 + it)|$ for $t \in [a, b]$ and each time we come across an interval where (possibly) $|\zeta(1/2 + i[a, b])|$ sets a new record $[x, y]$, we store $a$ and $y$. Running through the data files produced, it is a trivial matter to find an $A$ such that $|\zeta(1/2 + it)| < Ait^{\frac{1}{6}} \log t$ throughout the range. Our results are summarised in Table 1.

It seems that the bound in Theorem 1 is still very far from optimal.

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| t       | A    |
|---------|------|
| [2, 200]| 0.7090 |
| [200, 10^3]| 0.4873 |
| [10^3, 10^4]| 0.4682 |
| [10^4, 10^5]| 0.4217 |
| [10^5, 10^6]| 0.3765 |
| [10^6, 10^7]| 0.3238 |
| [10^7, 10^8]| 0.2854 |

Table 1: Bounds on $|\zeta(\frac{1}{2} + it)| \leq At^\frac{\delta}{2} \log t$ for ranges of $t$. 

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