Moduli of Admissible Pairs for Arbitrary Dimension, I: Resolution

Nadezhda V. Timofeeva*

Abstract. A procedure resolving a torsion-free coherent sheaf on a nonsingular $N$-dimensional projective algebraic variety into a locally free sheaf on a projective scheme of certain class is proposed. This is a higher-dimensional analog of the resolution (called the standard resolution in previous works of the author) of coherent sheaves on a surface.

Bibliography: 23 items.

Keywords: moduli space, algebraic coherent sheaves, admissible pairs, vector bundles, nonsingular algebraic variety, projective algebraic variety, moduli of vector bundles, compactification of moduli.

1 Introduction

In the present paper we do preparatory work for constructing a scheme of moduli of semistable admissible pairs for arbitrary dimension. In the series of articles [14]–[23] the construction was done in dimension two: admissible schemes had dimension two and were obtained by some transformation from a nonsingular projective algebraic surface $S$. The final result was an isomorphism between the moduli functor of Gieseker-semistable torsion-free coherent sheaves of rank $r$ and with Hilbert polynomial $r p(n)$ on the surface $S$ with a fixed polarization $L$ and the moduli functor of semistable admissible pairs $((\tilde{S}, \tilde{L}), \tilde{E})$. Each such pair consists of an admissible scheme $\tilde{S}$ with a distinguished polarization $\tilde{L}$ and a semistable locally free sheaf $\tilde{E}$ of rank $r$ and with Hilbert polynomial $r p(n)$. The advantage of the moduli functor of semistable admissible pairs is that its moduli scheme is isomorphic to the Gieseker–Maruyama moduli scheme for the same rank and Hilbert polynomial. In particular, this provides a compactification of the moduli space of stable vector bundles by vector bundles on some special (admissible) schemes instead of the classical compactification by attaching non-locally free coherent sheaves. The procedure is intended for constructing a moduli scheme of such vector bundles on projective schemes generalizing analogous results for the two-dimensional case ([23]).

The basement for the subject of interest is the Kobayashi–Hitchin correspondence to be discussed below. It allows one to apply algebraic geometrical methods to problems of differential geometrical or gauge theoretical setting by transferring consideration of the moduli of connections in a vector bundle (including vector bundles endowed with additional structures) to consideration of the moduli for vector bundles which are slope-stable.

Consider a compact complex algebraic variety $X$, a complex vector bundle $E$ on $X$, Hermitian metrics $g$ on $X$ and $h$ on $E$ respectively. A holomorphic vector bundle on a compact complex manifold with a Hermitian metric is stable [10] if and only if $E$ admits an irreducible $g$-Hermitian–Einstein metric. If $E$ is a differentiable vector bundle and $\hat{\delta}$ is a holomorphic structure in $E$ then $\hat{\delta}$ is called $g$-stable [10] if the holomorphic bundle $(E, \hat{\delta})$ has this property. Let $M_{g}^{st}(E)$ be the moduli space for isomorphy classes of...
Correspondence operates with the coefficient field dimension 2, we extend the "locally free" compactification to arbitrary dimension. The Kobayashi–Hitchin correspondence to the compact case. Since the Kobayashi–Hitchin correspondence holds not only in analog. [23] and references therein), and here we describe a first step in the construction of its multidimensional [11]. A different approach is developed by the author of the present paper for the 2-dimensional case (see constructed for rank 2 vector bundles by D. Markushevich, A. Tikhomirov and G. Trautmann in 2012 is the algebraic geometrical bubble-tree compactification announced for vector bundles on a surface and shevich and A. Tikhomirov in [2], and the Taubes–Uhlenbeck–Feehan compactification [5]. Also there V. Baranovsky [1], the algebraic geometrical version for framed sheaves done by U. Bruzzo, D. Markushevich and A. Tikhomirov in [2], and the Taubes–Uhlenbeck–Feehan compactification [5]. Also there is the algebraic geometrical bubble-tree compactification announced for vector bundles on a surface and constructed for rank 2 vector bundles by D. Markushevich, A. Tikhomirov and G. Trautmann in 2012 [11]. A different approach is developed by the author of the present paper for the 2-dimensional case (see and references therein), and here we describe a first step in the construction of its multidimensional analog.

The compactifications which do not involve non-bundles can be of use for extending the Kobayashi–Hitchin correspondence to the compact case. Since the Kobayashi–Hitchin correspondence holds not only in dimension 2, we extend the "locally free" compactification to arbitrary dimension. The Kobayashi–Hitchin correspondence operates with the coefficient field \( k \). We work with an arbitrary algebraically closed field \( k \) of zero characteristic, because the isomorphism of functors is not subject to special properties of \( k \).

Properties of Gieseker–Maruyama moduli schemes for Gieseker-semistable sheaves on an arbitrary nonsingular projective algebraic variety are not very well studied. But we know that these schemes are projective and that they can consist of several connected components. Their components can have non-equal dimensions and can carry nonreduced scheme structures. Also some components can contain no locally free sheaves. These phenomena (reducibility, nonreducedness, non-equidimensionality and presence of non-bundle components) occur for some combinations of numerical invariants even in dimension 2 case.

Let \( S \) be a nonsingular (irreducible) projective algebraic variety over an algebraically closed field \( \overline{k} \) of zero characteristic, \( \mathcal{O}_S \) be its structure sheaf, \( E \) be a coherent torsion-free \( \mathcal{O}_S \)-module, \( E^\vee := \mathcal{H}om_{\mathcal{O}_S}(E, \mathcal{O}_S) \) be its dual \( \mathcal{O}_S \)-module. A locally free sheaf and its corresponding vector bundle are canonically identified with each other, and both terms are used as synonyms. Let \( L \) be a very ample invertible sheaf on \( S \); it is fixed and is used as a polarization. The symbol \( \chi(\cdot) \) denotes the Euler–Poincaré characteristic of a coherent sheaf, \( c_i(\cdot) \) is its \( i \)-th Chern class.

Recall the definition of a sheaf of 0-th Fitting ideals known from commutative algebra. Let \( X \) be a
scheme and \( F \) be an \( \mathcal{O}_X \)-module with a finite presentation

\[
\widehat{F}_1 \xrightarrow{\varphi} \widehat{F}_0 \to F \to 0.
\]

Without loss of generality we assume that \( \text{rank } \widehat{F}_1 \geq \text{rank } \widehat{F}_0 \).

**Definition 1.** The sheaf of 0-th Fitting ideals of the \( \mathcal{O}_X \)-module \( F \) is defined as

\[
\mathcal{Fitt}^0 F = \text{im} \left( \bigwedge_{\text{rank } \widehat{F}_0} \widehat{F}_1 \otimes \bigwedge_{\text{rank } \widehat{F}_0} \widehat{F}_0 \xrightarrow{\varphi'} \mathcal{O}_X \right),
\]

where \( \varphi' \) is the morphism of \( \mathcal{O}_X \)-modules induced by \( \varphi \).

**Remark 1.** We work with invertible sheaves \( L \) and \( \tilde{L} \), where the expression for \( \tilde{L} \) involves an appropriate tensor power of \( L \). In further considerations we replace \( L \) by its big enough tensor power, if necessary, in order to make \( \tilde{L} \) very ample. This power can be chosen uniform and fixed, as shown in [18] for the case when the homological dimension of \( E \) equals one: \( \text{hd } E = 1 \). The same reasoning is true for arbitrary dimension of \( S \) and for arbitrary homological dimension of sheaves. All Hilbert polynomials are computed according to new \( L \) and \( \tilde{L} \) respectively.

As shown in [17], if in the case \( \text{dim } S = 2 \), the scheme

\[
\tilde{S} = \text{Proj } \bigoplus_{s \geq 0} (I[t] + (t))^s/(t^{s+1})
\]

is not isomorphic to \( S \), it is decomposed into the finite union of several components \( \tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \). It has a morphism \( \sigma: \tilde{S} \to S \) which is induced by the structure of \( \mathcal{O}_S \)-algebra on the graded ring \( \bigoplus_{s \geq 0} (I[t] + (t))^s/(t^{s+1}) \). In the 2-dimensional case the scheme \( \tilde{S} \) can be produced as follows. Take the product \( (\text{Spec } k[t]) \times S \) and its blowing up \( B_{I}(\text{Spec } k[t]) \times S \) in the sheaf of ideals \( I = (t) + I[t] \) corresponding to the subscheme with the ideal \( I \) in the zero-fiber \( 0 \times S \). If \( \sigma: B_{I}(\text{Spec } k[t]) \times S \to (\text{Spec } k[t]) \times S \) is the blowup morphism then \( \tilde{S} \) is the zero-fiber of the composite of morphisms

\[
pr_1 \circ \sigma: B_{I}(\text{Spec } k[t]) \times S \to (\text{Spec } k[t]) \times S \to \text{Spec } k[t].
\]

In the present paper we develop a method to resolve singularities of a torsion-free coherent sheaf in higher dimensions by an iterative process, which leads to an analog of an admissible scheme for any dimension. The standard resolution and the notion of admissible scheme for the 2-dimensional case are included in our method as the simplest nontrivial case.

First we recall the definition of \( S \)-stable and \( S \)-semistable pairs in the 2-dimensional case. The analogous notions and the construction for arbitrary dimension are natural generalizations of the 2-dimensional case. An essential condition is that \( S \) must be nonsingular.

**Definition 2 (18).** An \( S \)-stable (respectively, \( S \)-semistable) pair \((\tilde{S}, \tilde{L}, \tilde{E})\) in the case when \( \text{dim } S = 2 \) is the following data:

- \( \tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i \) is an admissible scheme, \( \sigma: \tilde{S} \to S \) is a morphism which is called canonical, \( \sigma_i: \tilde{S}_i \to S \) are its restrictions to the components \( \tilde{S}_i, i \geq 0 \);
- \( \tilde{E} \) is a vector bundle on the scheme \( \tilde{S} \);
- \( \tilde{L} \in \text{Pic } \tilde{S} \) is a distinguished polarization;

such that

- \( \chi(\tilde{E} \otimes \tilde{L}^n) = rp(n) \), where the polynomial \( p(n) \) and the rank \( r \) of the sheaf \( \tilde{E} \) are fixed;
• the sheaf $\tilde{E}$ on the scheme $\tilde{S}$ is stable (respectively, semistable) in the sense of Gieseker (see Definition 4 below);

• on each of the additional components $\tilde{S}_i, i > 0$, the sheaf $\tilde{E}_i := \tilde{E}|_{\tilde{S}_i}$ is quasi-ideal, i.e. it admits a description of the form
  \[ \tilde{E}_i = \sigma_i^* \ker q_0 / \text{tors}_i \]
  for some $q_0 \in \bigcup_{i \leq c_2} \text{Quot}^f \bigoplus^r \mathcal{O}_S$.

The definition of the subsheaf $\text{tors}_i$ is given below. The coefficients of the Hilbert polynomial $rp(n)$ of the sheaves $E$ and $\tilde{E}$ depend on their Chern classes. In particular, $c_2$ is the 2nd Chern class of a sheaf with Hilbert polynomial equal to $rp(n)$.

Pairs $((\tilde{S}, \tilde{L}), \tilde{E})$ such that $(\tilde{S}, \tilde{L}) \cong (S, L)$ are called $S$-pairs.

In the series of articles of the author [14]—[18] a projective algebraic scheme $\tilde{M}$ is built up. It is a reduced scheme corresponding to (reduction of) the moduli scheme of $\text{S}$-semistable admissible pairs. In [19] the moduli scheme of $S$-semistable admissible pairs is constructed as a (possibly) nonreduced moduli space. In both cases the same notation $\tilde{M}$ has been used but in [14]—[18] the scheme which was constructed was $\tilde{M}_{\text{red}}$.

The scheme $\tilde{M}$ contains an open subscheme $\tilde{M}_0$ which is isomorphic to the subscheme $M_0$ of Gieseker-semistable vector bundles in the Gieseker–Maruyama moduli scheme $\tilde{M}$ of torsion-free semistable sheaves whose Hilbert polynomial is equal to $\chi(E \otimes L^n) = rp(n)$. The following definition of Gieseker-semistability is used.

**Definition 3** ([12]). A coherent $\mathcal{O}_S$-sheaf $E$ is stable (respectively, semistable) if for any proper subsheaf $F \subset E$ of rank $r' = \text{rank} F$ for $n \gg 0$ one has

\[ \frac{\chi(F \otimes L^n)}{r'} < \frac{\chi(E \otimes L^n)}{r} \quad (\text{respectively,} \quad \frac{\chi(F \otimes L^n)}{r'} \leq \frac{\chi(E \otimes L^n)}{r}). \]

Let $E$ be a semistable locally free sheaf. Then the sheaf $I = \mathcal{Fitt}^0 \mathcal{E}x^1(E, \mathcal{O}_S)$ is trivial and $\tilde{S} \cong S$. In this case $((\tilde{S}, \tilde{L}), \tilde{E}) \cong ((S, L), E)$, and we have a bijective correspondence $\tilde{M}_0 \cong M_0$. It is a scheme isomorphism.

Let $E$ be a semistable non-locally free coherent sheaf; then the scheme $\tilde{S}$ contains the reduced irreducible component $\tilde{S}_0$ such that the morphism $\sigma_0 := \sigma|_{\tilde{S}_0} : \tilde{S}_0 \to S$ is a morphism of blowing up of the scheme $S$ in the sheaf of ideals $I = \mathcal{Fitt}^0 \mathcal{E}x^1(E, \mathcal{O}_S)$. Formation of a sheaf $I$ is a way to characterize the singularities of the sheaf $E$, i.e. its difference from a locally free sheaf. Indeed, the quotient sheaf $\pi := E^{\vee \vee} / E$ is Artinian and its length is not greater than $c_2 = c_2(E)$, and we have $\mathcal{E}x^1(E, \mathcal{O}_S) \cong \mathcal{E}x^2(\pi, \mathcal{O}_S)$. Let $Z$ be the subscheme corresponding to the sheaf of ideals $\mathcal{Fitt}^0 \mathcal{E}x^2(\pi, \mathcal{O}_S)$. In general, $Z$ is nonreduced and has bounded length [19]. The subscheme $Z$ is supported at a finite set of points on the surface $S$. As shown in [17], in general, the other irreducible components $\tilde{S}_i, i > 0$, of the scheme $\tilde{S}$ can carry nonreduced scheme structures.

Each semistable coherent torsion-free sheaf $E$ corresponds to a pair $((\tilde{S}, \tilde{L}), \tilde{E})$, where $((\tilde{S}, \tilde{L}), \tilde{E})$ is defined as described above.

Now we turn to the construction of the subsheaf $\text{tors}_i$ in [11]. Let $U$ be a Zariski-open subset in one of the components $\tilde{S}_i, i \geq 0$, and $\sigma^* E|_{\tilde{S}_i}(U)$ be the corresponding group of sections. This group is an $\mathcal{O}_{\tilde{S}_i}(U)$-module. Let $\text{tors}_i(U)$ be the submodule in $\sigma^* E|_{\tilde{S}_i}(U)$ which consists of the sections $s \in \sigma^* E|_{\tilde{S}_i}(U)$ such that $s$ is annihilated by some prime ideal of positive codimension in $\mathcal{O}_{\tilde{S}_i}(U)$. The correspondence $U \mapsto \text{tors}_i(U)$ defines a subsheaf $\text{tors}_i \subset \sigma^* E|_{\tilde{S}_i}$. Note that the associated prime ideals of positive codimensions which annihilate the sections $s \in \sigma^* E|_{\tilde{S}_i}(U)$ correspond to the subschemes which are supported in the preimage $\sigma^{-1}(\text{Supp } \pi) = \bigcup_{i > 0} \tilde{S}_i$. Since the scheme $\tilde{S} = \bigcup_{i \geq 0} \tilde{S}_i$ is connected [17], the subsheaves $\text{tors}_i, i \geq 0$, allow one to construct a subsheaf $\text{tors}_i \subset \sigma^* E$. The subsheaf $\text{tors}_i$ is defined as follows. A section $s \in \sigma^* E|_{\tilde{S}_i}(U)$ satisfies the condition $s \in \text{tors}_i|_{\tilde{S}_i}(U)$ if and only if
• there exists a section \( y \in O_{\widetilde{S}_i}(U) \) such that \( ys = 0 \),

• at least one of the following two conditions is satisfied: either \( y \in p \), where \( p \) is a prime ideal of positive codimension, or there exist a Zariski-open subset \( V \subset \widetilde{S} \) and a section \( s' \in \sigma^*E(V) \) such that \( V \supset U \), \( s'|_U = s \), and \( s'|_{V \cap \widetilde{S}_0} \in \text{tors} (\sigma^*E|_{\widetilde{S}_0})(V \cap \widetilde{S}_0) \). In the latter expression the torsion subsheaf \( \text{tors} (\sigma^*E|_{\widetilde{S}_0}) \) is understood in the usual sense.

The role of the subsheaf tors \( \subset \sigma^*E \) in our construction is analogous to the role of the torsion subsheaf in the case of a reduced and irreducible base scheme. Since no confusion occurs, the symbol tors is understood everywhere in the described sense. The subsheaf tors is called the torsion subsheaf (in the modified sense).

In [18] it is proven that the sheaves \( \sigma^*E/\text{tors} \) are locally free. The sheaf \( \widetilde{E} \) included in the pair \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \) is defined by the formula \( \widetilde{E} = \sigma^*E/\text{tors} \). In this situation there is an isomorphism \( H^0(S, E \otimes \widetilde{L}) \cong H^0(S, E \otimes L) \).

Also in [18] it was proven that the restriction of the sheaf \( \widetilde{E} \) to each of the components \( \widetilde{S}_i \), \( i > 0 \), is given by the quasi-ideality relation (1.1), where \( q_0 : O^\text{pr}_S \rightarrow \varkappa \) is an epimorphism defined by the exact triple \( 0 \rightarrow E \rightarrow E^\vee \rightarrow \varkappa \rightarrow 0 \), since the sheaf \( E^\vee \) is locally free.

The mechanism described was called in [18] the standard resolution.

Since our main goal is compactification of moduli of vector bundles, in the present paper we assume \( E \) to be deformation equivalent to a locally free sheaf in the following strict sense. The sheaf \( E \) includes in a flat family \( E \) of coherent sheaves with one-dimensional regular base \( T \) (i. e. \( E \) is a \( T \)-flat sheaf of \( \mathcal{O}_{T \times S} \)-modules), where the sheaf \( E|_{t \times S} \) is locally free whenever \( t \in T \) is general enough.

In the present paper we prove the following result.

**Theorem 1.** Let \( S \) be a non-singular projective algebraic variety of dimension not less than 2 over the algebraically closed field \( k \) of characteristic zero, \( L \) a (big enough) very ample invertible sheaf on \( S \), \( E \) a coherent torsion-free sheaf of rank \( r \) and the its Hilbert polynomial equals \( \chi(E \otimes L^n) = rp(n) \). Then there exist: (i) an equidimensional projective scheme of finite type \( \widetilde{S} \) together with a morphism \( \sigma : \widetilde{S} \rightarrow S \), (ii) an ample invertible \( \mathcal{O}_{\widetilde{S}} \)-sheaf \( \widetilde{L} \), such that: (1) \( \widetilde{E} = \sigma^*E/\text{tors} \) is a locally free \( \mathcal{O}_{\widetilde{S}} \)-module, (2) \( \chi(\widetilde{L}^n) = \chi(L^n) \), (3) if \( E \) is deformation equivalent to a locally free sheaf then \( \chi(\widetilde{E} \otimes \widetilde{L}^n) = rp(n) \).

The subsheaf tors is defined in Section 5. The resolution described in the present paper takes any coherent torsion-free \( \mathcal{O}_S \)-sheaf \( E \) on a polarized projective scheme \( S \) to an admissible pair of the form \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \), which is a generalization of the one in the dimension two (resp., homological dimension 1) case. Here \( \widetilde{S} \) is an admissible scheme; its structure depends on the structure of the sheaf \( E \) under the resolution. In Section 2 we give general remarks on the resolution and families to be constructed. Section 3 plays the central role and contains the motivation and the description of the resolution. We give a transformation of a coherent torsion-free sheaf \( E \) into an admissible semistable pair \( ((\widetilde{S}, \widetilde{L}), \widetilde{E}) \). This transformation generalizes the procedure of the standard resolution to the case when the initial family is not obliged to contain locally free sheaves. Section 4 is devoted to the structure of admissible schemes and to their polarizations. Finally, Section 5 contains an additional result concerning admissible pairs. It clarifies the structure of admissible schemes and sheaves on them. Also in Section 5 we give a description for the subsheaf tors in the multidimensional case.

**Acknowledgements.** This work was carried out within the framework of a development program for the Regional Scientific and Educational Mathematical Center of the P.G. Demidov Yaroslavl State University with financial support from the Ministry of Science and Higher Education of the Russian Federation (Agreement on provision of subsidies from the federal budget No. 075-02-2022-886).

I would like to thank my student Yegor Medvedev for checking algebraic (ring-theoretic) counterparts of my conjectures concerning the behavior of torsions and modified torsions of sheaves under the morphisms arising in the resolution.
2 The resolution of a sheaf

Let $S$ be a scheme over a field $k$ and $E$ be a coherent $\mathcal{O}_S$-module. The following procedure is mentioned as a resolution of $E$:

- a surjective morphism of $k$-schemes $\sigma: \tilde{S} \to S$. In this paper $\sigma$’s will be birational morphisms;
- a transformation taking the $\mathcal{O}_S$-module $E$ to a locally free $\mathcal{O}_S$-module $\tilde{E}$. In the present paper the transformations are of the form $E \mapsto \sigma^*E/\text{tors}$, where tors is either the usual torsion sheaf or some generalization of the torsion to be defined below (Section 5).

We start with the trivial family $\Sigma = T \times S$ and the projection $p: \Sigma \to T$ forgetting the second factor and come to a flat family of schemes $\pi: \widetilde{\Sigma} \to T$, which fits into the commutative triangle

$$
\begin{array}{ccc}
\widetilde{\Sigma} & \xrightarrow{\sigma} & \Sigma \\
\downarrow \pi & & \downarrow p \\
T & & T \end{array}
$$

Set $\Sigma = T \times S$ for $T = \mathbb{P}^1$, $I \subset \mathcal{O}_S$ any sheaf of ideals. Fix a closed point $0 \in T$ and denote the closed immersion $0 \times S \hookrightarrow \Sigma$ by $i_0$. Let $\tilde{\Sigma}$ be the scheme obtained by blowing up of $\Sigma$ in $\Sigma \leftarrow \ker (\mathcal{O}_\Sigma \to i_{0*} (\mathcal{O}_S/I))$. The scheme $\tilde{\Sigma}$ is included into the diagram (2.1). The morphism $\pi$ is projective. It factors through a closed immersion into an appropriate relative projective space $\pi: \tilde{\Sigma} \hookrightarrow \mathbb{P}_T \to T$. For $\Sigma_0 := \Sigma \setminus (0 \times S)$ we have $\tilde{\Sigma} = \overline{i(\Sigma_0)}$. Then by [17, Ch.III, Prop. 9.8] $\tilde{\Sigma}$ is flat relative to $T$. Let $\mathbb{L}$ be an invertible $\mathcal{O}_\Sigma$-sheaf which is very ample relative to $T$ and let $\mathbb{L}$ be big enough such that $\mathbb{L} = \sigma^*\mathbb{L} \otimes \sigma^{-1}I: \mathcal{O}_{\tilde{\Sigma}}$ is also very ample relative to $T$. Then $\pi_*\mathbb{L}^n$ is locally free and rank $\pi_*\mathbb{L}^n = \text{rank } p_*\mathbb{L}^n$.

Also if $t$ is a parameter along $T$ then $\pi^{-1}0 = \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^s / (t)^{s+1}$. The computation of the fiber $\pi^{-1}0$ is directly transferred from [17], where it is done for $S$ being a surface but remains true for any scheme $S$ over a field $k$.

The same construction and conclusions from it remain true if blowing ups are iterated with centers in fibers over one or several distinct points in $T$.

From this we conclude that if any (non necessarily one-parameter) flat family $\pi: \tilde{\Sigma} \to T$ of admissible schemes is supplied with a locally free $\mathcal{O}_{\tilde{\Sigma}}$-sheaf $\tilde{E}$ and with an invertible sheaf $\mathbb{L}$ very ample relatively to $T$, then $\pi_* (\mathbb{E} \otimes \mathbb{L}^n)$ is locally free for $n \gg 0$ and hence $\mathbb{E}$ has fiberwise uniform Hilbert polynomial. Fiberwise Hilbert polynomial is computed with respect to $\mathbb{L}$.

3 The Sheaves-to-Pairs transformation (the standard resolution)

The aim of this section is to modify the transformation of a coherent torsion-free sheaf to a pair $((\tilde{S}, \tilde{L}), \tilde{E})$ of admissible pairs done for the two-dimensional case in previous papers to make it valid for an $N$-dimensional ground variety $S$.

To resolve the singularities of the $\mathcal{O}_S$-sheaf $E$ we repeat the manipulations from [23], with necessary modifications. To prove Theorem 1 we will need two cases. The first case is the procedure of resolution of a single sheaf $E$ on the variety $S$. In this case we start with the product $\mathbb{A}^1 \times S$ and perform blowup (see below) $\sigma_i: \tilde{\Sigma} \to \Sigma$ of the sheaf of ideals $\mathbb{I}_i = I[t] + (t)$ ($t$ is a parameter along $\mathbb{A}^1$). The morphism $\sigma_i: \Sigma_i \to \Sigma_{i-1}$ below is understood as the restriction of $\sigma_i$ to the zero fiber $S_i = \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^s / (t)^{s+1}$ of $\tilde{\Sigma}_i$. These morphisms are understood as blowup morphisms for the sheaves of ideals $\mathbb{I}_i$, $i = 1, \ldots, \ell$. For the both cases the manipulations are almost similar and will be described together.
If $S$ is a surface, $T$ an algebraic scheme of finite type and $\mathcal{E}$ is a flat family of coherent torsion-free sheaves on $S$ over the base $T$ then the homological dimension of $\mathcal{E}$ as an $\mathcal{O}_{T \times S}$-module is not greater than 1. In our case $S$ has the bigger dimension and we have to work with a locally free resolution of the higher length. Set $\Sigma = T \times S$. Start with a shortest locally free resolution of the family of sheaves $\mathcal{E}$ and cut the corresponding exact $\mathcal{O}_\Sigma$-sequence of length $\ell$

$$0 \to \widehat{E}_\ell \to \widehat{E}_{\ell-1} \to \cdots \to \widehat{E}_0 \to \mathcal{E} \to 0$$

with locally free $\mathcal{O}_\Sigma$-modules $\widehat{E}_\ell, \ldots, \widehat{E}_0$ into triples:

$$0 \to W_i \to \widehat{E}_{i-1} \to W_{i-1} \to 0. \quad (3.1)$$

Here $W_\ell = \widehat{E}_\ell$, $W_1 = \ker (\widehat{E}_0 \to \mathcal{E})$ and $W_i = \ker (\widehat{E}_{i-1} \to \widehat{E}_{i-2}) = \text{coker} (\widehat{E}_{i+1} \to \widehat{E}_i)$ for $i = 2, \ldots, \ell - 1$. Also we keep in mind that all the sheaves $W_i$ except $W_\ell$ are not locally free (otherwise the resolution can be shorter). Since $S$ is assumed to be regular, $E$ possesses a locally free resolution of length not greater than $\text{dim} S$.

For the first case $\Sigma = S$, $\mathcal{E} = E$ we need a morphism $\sigma_1 : S_1 \to S$ which is induced by a blowing up morphism $\sigma'_1 : \Sigma_1' \to S \times \mathbb{A}^1$ of the product $S \times \mathbb{A}^1$ in the sheaf of ideals $\Pi_i' = \Pi_1[t] + (t)$ for $\Pi_1 = \mathcal{Fitt}_0^0 \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_S)$. For the second case we perform a blowing up $\sigma_i : \Sigma_1 \to \Sigma$ of the sheaf of ideals $\Pi_i = \mathcal{Fitt}_0^0 \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma)$. Whereas for $h_d E = 1$ one morphism is needed, for $h_d E = \ell$ we need a sequence of $\ell$ consequent morphisms $\sigma_i, i = 1, \ldots, \ell$.

We work with the sheaves of ideals

$$\Pi_1 = \mathcal{Fitt}_0^0 \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_S), \quad \Pi_2 = \mathcal{Fitt}_0^0 \mathcal{E}xt^1(\sigma'_1 W_{\ell-2}, \mathcal{O}_{\Sigma_1}), \quad \Pi_3 = \mathcal{Fitt}_0^0 \mathcal{E}xt^1(\sigma'_2 \sigma'_1 W_{\ell-3}, \mathcal{O}_{\Sigma_2}), \ldots, \quad \Pi_\ell = \mathcal{Fitt}_0^0 \mathcal{E}xt^0(\sigma'_1 \cdots \sigma'_1 W_0, \mathcal{O}_{\Sigma_\ell}).$$

Each morphism $\sigma_i : \Sigma_i \to \Sigma_{i-1}$ is induced by the sheaf of ideals $\Pi_i$, $i = 1, \ldots, \ell$, and these morphisms form the sequence

$$\widetilde{\Sigma} := \Sigma_\ell \xrightarrow{\sigma_\ell} \cdots \xrightarrow{\sigma_1} \Sigma_0 := \Sigma.$$

Each segment

$$0 \to W_i \to \widehat{E}_{i-1} \to W_{i-1} \to 0$$

is resolved by the morphism $\sigma_{\ell-i+1}$ at $(\ell - i + 1)^{\text{st}}$ step of the process.

We do computations as in (23). Start with the exact $\mathcal{O}_\Sigma$-triple $(i = \ell)$:

$$0 \to W_\ell \to \widehat{E}_{\ell-1} \to W_{\ell-1} \to 0. \quad (3.2)$$

Since $h_d W_{\ell-1} = 1$, this sheaf can be resolved as in (23) by the morphism $\sigma_1 : \Sigma_1 \to \Sigma$. Denote

$$H_\ell := \ker (W_\ell^\vee \to \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma)) = \text{coker} (W_{\ell-1}^\vee \to \widehat{E}_{\ell-1}^\vee)$$

in the dual sequence of (3.2):

$$0 \to W_\ell^\vee \to \widehat{E}_{\ell-1}^\vee \to W_{\ell-1}^\vee \to \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma) \to 0.$$

Apply the inverse image $\sigma_1^*$:

$$\sigma_1^* W_\ell^\vee \to \sigma_1^* \widehat{E}_{\ell-1}^\vee \to \sigma_1^* H_\ell \to 0,$$

$$\sigma_1^* H_\ell \to \sigma_1^* W_\ell^\vee \to \sigma_1^* \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma) \to 0. \quad (3.3)$$

In (3.3) we denote $N_\ell := \ker (\sigma_1^* W_\ell^\vee \to \sigma_1^* \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma))$. The sheaf $\mathcal{Fitt}_0^0(\sigma_1^* \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma))$ is invertible by the functorial property of $\mathcal{Fitt}$.

In the first case

$$\mathcal{Fitt}_0^0(\sigma_1^* \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma)) = (\sigma_1^{-1} \mathcal{Fitt}_0^0(\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma))) \cdot \mathcal{O}_{\Sigma_1} = (\sigma_1^{-1} \Pi_1) \cdot \mathcal{O}_{\Sigma_1} = (\sigma_1^{-1} l_0^{-1} \Pi_1^0) \cdot \mathcal{O}_{\Sigma_1} = l_0^{-1} (\sigma_1^{-1} \Pi_1') \cdot \mathcal{O}_{\Sigma_1}. \quad (3.4)$$
Here we take into account that \( \mathbb{I}_1' = pr_1^*\mathbb{I}_1 + (t) = \mathbb{I}_1[t] + (t) \), where \( \mathbb{A}^1 = \text{Spec } k[t] \) and \( pr_1: \Sigma \times \mathbb{A}^1 \to \Sigma \) is the natural projection. The closed immersion \( \bar{i}_0 : \Sigma_1 \hookrightarrow \Sigma_1' \) is fixed by the fibered square

\[
\begin{array}{ccc}
\bar{\Sigma}_1 & \xrightarrow{\sigma_1'} & \Sigma \\
\downarrow{\bar{i}_0} & & \downarrow{\bar{i}_0} \\
\Sigma_1 & \xrightarrow{\sigma_1} & \Sigma \\
\end{array}
\]

In the second case

\[
\mathcal{F}itt^0(\sigma_1^*\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma)) = (\sigma_1^{-1}\mathcal{F}itt^0(\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma))) \cdot \mathcal{O}_{\Sigma_1} = (\sigma_1^{-1}\mathbb{I}_1) \cdot \mathcal{O}_{\Sigma_1}.
\]

Now we need the following lemma.

**Lemma 3.1.** [21] Lemma 2] Let \( X \) be a Noetherian scheme such that its reduction \( X_{\text{red}} \) is irreducible, \( \mathcal{F} \) nonzero coherent \( \mathcal{O}_X \)-sheaf supported on a subscheme of codimension \( \geq 1 \). Then the sheaf of 0-th Fitting ideals \( \mathcal{F}itt^0(\mathcal{F}) \) is an invertible \( \mathcal{O}_X \)-sheaf if and only if \( \mathcal{F} \) has homological dimension equal to 1: \( \text{hd}_X \mathcal{F} = 1 \).

**Remark 2.** If the scheme \( X = \Sigma_1 \) has an irreducible reduction (it is true in the second case) then this lemma is applicable immediately and we conclude that \( \text{hd} \sigma_1^*\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma) = 1 \). If \( \Sigma_1 \) has a reducible reduction then there is a natural decomposition \( \Sigma_1 = \Sigma_0^0 \cup D_1 \), where \( \Sigma_0^0 = \text{Proj } \bigoplus_{s \geq 0} \mathbb{I}_1^s/\mathbb{I}_1^{s+1} \) is isomorphic to the scheme obtained by blowing up \( \Sigma \) in the sheaf of ideals \( \mathbb{I}_1 \) and \( D_1 \) is an exceptional divisor of the blowing up morphism \( \sigma_1': \Sigma_1' \to \Sigma \times \mathbb{A}^1 \). Their scheme-theoretic intersection equals the exceptional divisor \( D_1 \) of the blowing up morphism \( \sigma_1^0: \Sigma_1^0 \to \Sigma \):

\[
\Sigma_1^0 \cap D_1' = D_1.
\]

One comes to the decomposition of the morphism \( \sigma_1: \Sigma_1 \xrightarrow{\delta_1} \Sigma_1^0 \xrightarrow{\sigma_1^0} \Sigma \) where \( \delta_1 \) acts identically on \( \Sigma_1^0 \) and its action on \( D_1' \)

\[
\delta_1|_{D_1'}: D_1' \to \Sigma_1^0
\]

factors through the exceptional divisor \( D_1 = \text{Proj } \bigoplus_{s \geq 0} \mathbb{I}_1^s/\mathbb{I}_1^{s+1} \) of the morphism \( \sigma_1^0 \) and is defined by the structure of \( \bigoplus_{s \geq 0} \mathbb{I}_1^s/\mathbb{I}_1^{s+1} \)-algebra on the graded ring \( \mathbb{I}_1^s/\mathbb{I}_1^{s+1} \).

In this case all the manipulations hold if the morphism \( \sigma_1 \) in (3.3) can be replaced by \( \sigma_1^0 \) and the ideal sheaf \( \mathcal{F}itt^0(\sigma_1^0\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma)) \) is also invertible. Applying Lemma 3.1 we conclude that \( \text{hd } \sigma_1^0\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma) = 1 \) (and hence also \( \text{hd } \sigma_1^*\mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma) = 1 \)).

**Convention 1.** For uniformity of notation we use the decomposition \( \sigma_i = \sigma_i^0 \circ \delta_i \) for all \( i = 1, \ldots, \ell \) with no reference if \( \delta_i \) is either a non-identity or an identity morphism.

Hence the sheaf \( N_\ell = \ker (\sigma_1^0 W_{\ell-1}^\vee \to \sigma_1^0 \mathcal{E}xt^1(W_{\ell-1}, \mathcal{O}_\Sigma)) \) is locally free. For \( H_\ell = \text{coker } (W_{\ell-1}^\vee \to \hat{E}_{\ell-1}^\vee) \) there is a surjective morphism of \( \mathcal{O}_\Sigma \)-modules \( \sigma_1^0 H_\ell \twoheadrightarrow N_\ell \) and hence we get the isomorphism \( N_\ell = \sigma_1^0 H_\ell / \text{tors} \). Now we come to the surjective composite morphism

\[
\sigma_1^0 \hat{E}_{\ell-1}^\vee \twoheadrightarrow \sigma_1^0 H_\ell \twoheadrightarrow N_\ell.
\]

Its kernel is a locally free \( \mathcal{O}_\Sigma \)-module; denote it by

\[
W_{\ell-1}^\vee = \ker (\sigma_1^0 \hat{E}_{\ell-1}^\vee \twoheadrightarrow N_\ell).
\]

Now we have the following commutative diagram of \( \mathcal{O}_\Sigma \)-modules with exact rows

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \sigma_1^0 W_{\ell-1}^\vee / \text{tors} & \rightarrow & \sigma_1^0 \hat{E}_{\ell-1}^\vee & \rightarrow & \sigma_1^0 H_\ell & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & W_{\ell-1}^\vee & \rightarrow & \sigma_1^0 \hat{E}_{\ell-1}^\vee & \rightarrow & N_\ell & \rightarrow & 0
\end{array}
\]
Remark 3. When we return to $\sigma_1$ the morphism $\sigma_1^0$ is replaced by the composite morphism $\sigma_1 = \sigma_1^0 \circ \delta_1$ (when the inverse image under $\delta_1$ is taken) then the usual torsion subsheaf in $\sigma_1^0 W_{\ell-1}'$ becomes a modified torsion subsheaf. The sheaf
\[
\delta_1^* W_{\ell-1}' = \ker (\sigma_1^* E_{\ell-1}^\vee \to \delta_1^* N_\ell)
\]
remains locally free. Also in the case when $\sigma_i$ (and/or in the next steps $\sigma_j$) is a blowing up we suppose corresponding $\delta_1$ (resp., $\delta_i$) to be an identity morphism but mention it for the sake of uniformity of notation.

Since $W_\ell$ is a locally free sheaf, then $(\sigma_1^0 W_\ell')^\vee = \sigma_1^0 W_\ell$. We arrive to the inclusion of kernels in (3.4):

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma_1^0 W_\ell & \longrightarrow & \sigma_1^0 E_{\ell-1} & \longrightarrow & \sigma_1^0 W_{\ell-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_\ell^\vee & \longrightarrow & \sigma_0^0 E_{\ell-1} & \longrightarrow & W_{\ell-1} & \longrightarrow & 0
\end{array}
\]

which implies the epimorphism of cokernels and the isomorphism $W_{\ell-1}' = \sigma_1^0 W_{\ell-1}/\text{tors}$.

On the next step when passing to the morphism $\sigma_2: \Sigma_2 \to \Sigma_1$, one comes to the need for application of Lemma 3.1 to $\Sigma_2$ (and/or the same for the consequent morphisms $\sigma_{i+1}: \Sigma_{i+1} \to \Sigma_i$, $i = 2, \ldots, \ell - 1$.) Each of these morphisms is decomposed as $\sigma_i = \sigma_i^0 \circ \delta_i$, $i = 2, \ldots, \ell$, but if $\Sigma_\ell$ can have a reducible reduction (i.e. $\delta_j$ is not an identity for some $j \leq i$), then this decomposition will not validate Lemma 3.1.

To overcome this obstacle we return to the decomposition $\sigma_i = \sigma_i^0 \circ \delta_i$ where $\sigma_i^0$ is a blowup morphism.

We are interested in the composite morphisms $\delta_i \circ \sigma_i^0$. The idea is to “interchange” the morphisms in this composite by the commutative diagram

\[
\begin{array}{ccccccccc}
\Sigma_{i+1}^0 & \longrightarrow & \Sigma_i^0 & \longrightarrow & \Sigma_i \\
\downarrow \delta_i^{-1} & & \downarrow \delta_i & & \downarrow \\
\Sigma_{i+1} & \longrightarrow & \Sigma_i & \longrightarrow & \Sigma_i
\end{array}
\]

as it is explained below.

Remark 4. We use double indexing with lower and upper indices for schemes and morphisms. For example, $-1$ in notation $\delta_i^{-1}$ or $\sigma_i^{-1}$ is just index but neither a sign of an inverse of the map nor a notation for an inverse image of a sheaf.

Since $\Sigma_{i+1}$ is obtained by a blowing up of the scheme $\Sigma_i$ in the sheaf of ideals $\mathbb{I}_{i+1} \subset \mathcal{O}_{\Sigma_i}$ and it can be interpreted as a projective spectrum $\Sigma_{i+1}^0 = \text{Proj} A_{i+1}^0$ of the graded $\mathcal{O}_{\Sigma_i}$-algebra $A_{i+1}^0 = \bigoplus_{s_i \geq 0} \mathbb{I}_{i+1}^{s_i}$. But $\Sigma_i$ itself is a projective spectrum $\Sigma_i = \text{Proj} A_i$ of the $\mathcal{O}_{\Sigma_i}$-algebra $A_i = \bigoplus_{s_i \geq 0} (\mathbb{I}_i[t_i] + (t_i)^{s_i}/(t_i)^{s_i+1}$ for $\mathbb{I}_i \subset \mathcal{O}_{\Sigma_{i-1}}$, $\Sigma_0 = \text{Proj} A_0^0$ for the $\mathcal{O}_{\Sigma_0}$-algebra $A_0^0 = \bigoplus_{s_i \geq 0} \mathbb{I}_0^{s_i}$, and the morphism $\delta_i$ is induced by the structure of an $A_i^0 = \bigoplus_{s_i \geq 0} \mathbb{I}_i^{s_i}$-algebra on $A_i = \bigoplus_{s_i \geq 0} (\mathbb{I}_i[t_i] + (t_i)^{s_i}/(t_i)^{s_i+1}$.

Obviously, $A_i^0 = \bigoplus_{s_i \geq 0} \mathbb{I}_i^{s_i}$ is a subalgebra in $A_i$; it is formed by and only by those elements that have degree zero in $t_i$, i.e. $A_i^0 = A_i|_{t_i=0}$.

Define $\mathbb{I}_{i+1}^u := \mathbb{I}_{i+1} \cap A_i^0$ as an intersection with the subalgebra $A_i^0$ in $A_i$ of elements of zero degree in $t_i$.

Obviously, $(\mathbb{I}_{i+1}^u)^u = (\mathbb{I}_{i+1})^u \cap A_i^0$ (whereas $\mathbb{I}_{i+1}$ consists of polynomials in $t_i$, the left hand side and the right hand side of the latter equality consist of homogeneous terms of zero degree in $t_i$) for any $u \geq 0$.

Define
\[
A_{i+1}^{-1} := \bigoplus_{s_i+1 \geq 0} \mathbb{I}_{i+1}^{s_i} (\mathbb{I}_{i+1}^u)^{s_i+1} = A_{i+1}^0|_{t_i=0}
\]
as a subalgebra of $A_{i+1}^0$ formed by elements of zero degree in $t_i$. The corresponding scheme is $\Sigma_{i+1}^{-1} := \text{Proj} A_{i+1}^{-1}$. Obviously, $A_{i+1}^0 = A_{i+1}^{-1} \otimes A_i^0 A_i$, and the corresponding square of schemes (3.5) is fibered. Also $\sigma_{i+1}^{-1}: \Sigma_{i+1} \to \Sigma_i$ is a morphism of blowing up in the sheaf of ideals $\mathbb{I}_{i+1}^u$.
The resolution of singularities of the sheaf $\mathcal{E}$ is done by the sequence of morphisms $\sigma_i = \delta_i \circ \sigma_i^0$, $i = 1, \ldots, \ell$. Their composite can be decomposed into the following diagram by iterating the square (3.5):

(3.6)

Each square of this diagram has a view

\[
\begin{array}{ccc}
\Sigma_i^0 & \xrightarrow{\sigma_i} & \Sigma_{i-1} \\
\downarrow & & \downarrow \\
\Sigma_i & \xrightarrow{\sigma_i^0} & \Sigma_{i-1} \\
\end{array}
\]

for $i = 0, \ldots, \ell$, $j = -1, \ldots, i - 1$, where $\Sigma_0 := \Sigma$, $\Sigma_i^0 := \Sigma_i$, $\delta_i := \delta_i$. Index rules are as follows:

- for $\Sigma$’s a lower index is constant along the columns and decreases in the direction of the horizontal arrows. An upper index is constant along the skew lines going parallel to the diagonal and decreases in the direction of the vertical arrows;
- for $\delta$’s a lower index is constant along the rows and decreases in the direction of the vertical arrows. An upper index is constant along the lines going parallel to the diagonal and decreases in the direction of the vertical arrows;
- for $\sigma$’s a lower index is constant along the columns and decreases in the direction of the horizontal arrows. An upper index is constant along the lines going parallel to the diagonal and increases in the direction of the horizontal arrows.

In this diagram the bottom horizontal row is a composite of consequent blowups. The left vertical column $\delta_{i-1}^{-\ell+1} \circ \cdots \circ \delta_0^0$ provides flatness of the scheme $\Sigma_\ell = \Sigma$ over its base $T$ by consequent “growings up” additional components of several levels. Application of the functor $(\delta_{i-1}^{-\ell+1} \circ \cdots \circ \delta_0^0)^*$ leads, as we will describe below, to an inverse image of a locally free sheaf produced from the initial $\mathcal{O}_\Sigma$-sheaf $\mathcal{E}$ by the morphisms of the bottom horizontal row. Each square of the diagram is an analog of (3.5). Also it is useful to keep in mind that each $\Sigma_i^{-i+1}$ is included into $\Sigma_i$ as a component ($i = 1, \ldots, \ell$).
Example 1. Let \( \ell = 2 \) and (3.6) consists of one square and two triangles:

![Diagram](image)

After taking an inverse image of the morphism \( \sigma_1 \) and the resolution of the intermediate sheaf \( W_1 \) one comes to the \( O_{\Sigma_1} \)-triple

\[
0 \to \delta_1^{0*} W' \to \sigma_1^1 \hat{E}_0 \to \sigma_1^1 E \to 0.
\]

Applying to it all the steps of the resolution again with the morphism \( \sigma_2^0 \) one comes to the \( O_{\Sigma_2} \)-sheaf \( N := \ker (\sigma_2^{0*} W'^{\vee} \to \sigma_2^{0*} \text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) \). In a standard way we deduce that 0th Fitting ideal

\[
\text{Fitt}^0(\sigma_2^{0*} \text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) = (\sigma_2^{0})^{-1} 1_2 \cdot O_{\Sigma_2}
\]

is invertible. But the scheme \( \Sigma_2^0 \) as well as \( \Sigma_1 \) is not obliged to have an irreducible reduction which is necessary for applicability of Lemma 3.1. However, the sheaf of Fitting ideals can be rewritten as

\[
\text{Fitt}^0(\sigma_2^{0*} \text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) = \text{Fitt}^0(\sigma_2^{0*} \delta_1^{0*} \text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) = (\sigma_2^{0} \circ \delta_1^{0})^{-1} \text{Fitt}^0(\text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) \cdot O_{\Sigma_2}
\]

and this sheaf is invertible on the scheme \( \Sigma_2^0 \) in whole. Hence this sheaf is invertible on its component \( \Sigma_2^{1} \), i.e. \( (\sigma_2^{1})^{-1} \text{Fitt}^0(\text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) \cdot O_{\Sigma_2^{1}} = \text{Fitt}^0(\sigma_2^{1*} \text{Ext}^1(\sigma_1^{*} E, O_{\Sigma_1})) \) is also invertible. Lemma 3.1 yields:

\[
\text{hd} (\sigma_2^{1*} \text{Ext}^1(\sigma_1^{0*} E, O_{\Sigma_2})) = 1.
\]

This means that it is possible to interchange the morphisms in the square in the diagram (3.7) and to do the resolution by \( (\sigma_2^{1})^* \) following \( (\sigma_1^0)^* \). Performing all the necessary steps we come to the new version of \( N_1 \):

\[
N_1 := \ker (\sigma_2^{1*} W'^{\vee} \to \sigma_2^{1*} \text{Ext}^1(\sigma_1^{0*} E, O_{\Sigma_2})).
\]

Lemma 3.1 works in this situation, the sheaf \( N_1 \) is hence locally free and one comes to the composite morphism

\[
\sigma_2^{1*} \hat{E}_0' \to H_1 \to N_1
\]

(where \( H_1 = \text{coker} (E'^{\vee} \to \hat{E}_0') \)). The kernel of this composite is also a locally free \( O_{\Sigma_2^{1}} \)-module. We denote its dual as \( \hat{E}' \) and \( \hat{E} := \delta_2^{0*} \delta_1^{-1*} \hat{E}' \). This completes the example.

To generalize this recipe to an arbitrary homological dimension we act inductively and do interchanging in the inductive step.

Now we pass to the next segment

\[
0 \to W_{\ell-1} \to \hat{E}_{\ell-2} \to W_{\ell-2} \to 0
\]

and to its “inverse image” under \( \sigma_1^0 \):

\[
0 \to W'_{\ell-1} \to \sigma_1^{0*} \hat{E}_{\ell-2} \to \sigma_1^{0*} W_{\ell-2} \to 0.
\]

The “inverse image” under \( \sigma_1 = \sigma_1^0 \circ \delta_1^0 \) is exact because the kernel \( W'_{\ell-1} \) is locally free. In this and other further segments the cokernel sheaves contain torsions (possibly, in the modified sense). This does not
cause any obstacle for the procedure of resolution because, as we will see later, the resolution leads to a factoring out of the torsion.

Next steps are similar to each other. They involve inverse images of the consequent segments

\[
0 \to W_{\ell-i}^\prime \to \sigma_{i}^{0} \sigma_{i}^{*} \hat{E}_{\ell-i-1} \to \sigma_{i}^{0} \sigma_{i-1} \sigma_{i-1}^{*} W_{\ell-i-1} \to 0, \\
\vdots \\
0 \to W_{\ell-i}^{\prime} \to \sigma_{i}^{0} \sigma_{i-1} \ldots \sigma_{i}^{1} \sigma_{i}^{*} \hat{E}_{\ell-i-1} \to \sigma_{i}^{0} \sigma_{i-1} \ldots \sigma_{i}^{1} \sigma_{i}^{*} W_{\ell-i-1} \to 0, \\
\vdots \\
0 \to W_{1}^{\prime} \to \sigma_{\ell-i}^{*} \sigma_{\ell-i}^{*} \sigma_{\ell-i}^{*} \sigma_{\ell-i}^{*} W_{0} \to 0,
\]

(3.8)

where \( W_{0} = \mathbb{E} \) and the kernel sheaf \( W_{\ell-i}^{\prime} \) in the next triple is a locally free \( \mathcal{O}_{\Sigma_{i-1}} \)-module which is produced by the resolution of \( \sigma_{i}^{*} \ldots \sigma_{i}^{*} W_{\ell-i} \) in the previous triple.

Denote the composite

\[
\sigma_{[i]} := \sigma_{1} \circ \cdots \circ \sigma_{i}.
\]

To describe the standard resolution of each triple in a uniform fashion we need composite morphisms

\[
\sigma_{[i-1]} \circ \sigma_{i}^{0} = \sigma_{[i-2]} \circ \sigma_{i-1} \circ \sigma_{i}^{0} = \sigma_{[i-2]} \circ \sigma_{i-1} \circ \sigma_{i}^{0} = \sigma_{[i-2]} \circ \sigma_{i-1} \circ \sigma_{i}^{0} = \sigma_{[i-2]} \circ \sigma_{i-1} \circ \sigma_{i}^{-1} \circ \sigma_{i-1}^{-1} = \\
\sigma_{[i-3]} \circ \sigma_{i-2} \circ \sigma_{i-1} \circ \sigma_{i}^{0} = \sigma_{[i-3]} \circ \sigma_{i-2} \circ \sigma_{i-1} \circ \sigma_{i}^{-1} \circ \sigma_{i-1}^{-1} = \\
\sigma_{[i-3]} \circ \sigma_{i-2} \circ \sigma_{i-1} \circ \sigma_{i}^{-1} \circ \sigma_{i-1}^{-1} = \\
\cdots = \sigma_{[i]} \circ \sigma_{2}^{-1} \circ \cdots \circ \sigma_{i}^{-1} \circ \sigma_{i+1}^{-1} \circ \sigma_{i+2}^{-1} \circ \cdots \circ \sigma_{i}^{-1}
\]

and

\[
\sigma_{[i]} = \sigma_{[i-1]} \circ \sigma_{i} = \sigma_{[i+1]} \circ \sigma_{i} = \sigma_{[i]} \circ \sigma_{[i]}.
\]

where we have introduced the notations

\[
\sigma_{[i]} := \sigma_{1} \circ \sigma_{2}^{-1} \circ \cdots \circ \sigma_{i}^{-1}, \quad \sigma_{[i]} := \sigma_{[i-1]} \circ \sigma_{i}^{-1} \circ \sigma_{i+2}^{-1} \circ \cdots \circ \sigma_{i}^{-1}.
\]

We perform the standard resolution of the triple

\[
0 \to \sigma_{[i]}^{0} \sigma_{i}^{*} W_{\ell-i} \to \sigma_{[i]}^{*} \hat{E}_{\ell-i-1} \to \sigma_{[i]}^{*} W_{\ell-i-1} \to 0
\]

for \( i = 1, \ldots, \ell - 1 \). Dualization leads to the exact \( \mathcal{O}_{\Sigma_{i}} \)-sequence

\[
0 \to (\sigma_{[i]}^{*} W_{\ell-i-1})^{\vee} \to \sigma_{[i]}^{*} \hat{E}_{\ell-i-1}^{\vee} \to \sigma_{[i]}^{*} W_{\ell-i-1}^{\vee} \to \mathcal{E}xt^{1}(\sigma_{[i]}^{*} W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i}}) \to 0.
\]

Denote

\[
H_{\ell-i} := \ker (\sigma_{[i]}^{0} \sigma_{[i]}^{*} W_{\ell-i}^{\vee} \to \mathcal{E}xt^{1}(\sigma_{[i]}^{*} W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i}}))
\]

\[
= \operatorname{cok} ((\sigma_{[i]}^{*} W_{\ell-i-1})^{\vee} \to \sigma_{[i]}^{*} \hat{E}_{\ell-i-1}^{\vee}).
\]

Apply the inverse image \( \sigma_{i+1}^{0} \cdot \):

\[
\sigma_{i+1}^{0} \sigma_{i}^{*} (\sigma_{i}^{*} W_{\ell-i-1})^{\vee} \to \sigma_{i+1}^{0} \sigma_{i}^{*} \hat{E}_{\ell-i-1}^{\vee} \to \sigma_{i+1}^{0} H_{\ell-i} \to 0,
\]

\[
\sigma_{i+1}^{0} H_{\ell-i} \to \sigma_{i+1}^{0} \sigma_{i}^{*} W_{\ell-i}^{\vee} \to \sigma_{i+1}^{0} \mathcal{E}xt^{1}(\sigma_{[i]}^{*} W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i}}) \to 0.
\]

(3.9)

In (3.9) denote \( \mathcal{N}_{\ell-i} := \ker (\sigma_{[i]}^{0} \sigma_{[i]}^{*} W_{\ell-i}^{\vee} \to \sigma_{i+1}^{0} \mathcal{E}xt^{1}(\sigma_{[i]}^{*} W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i}})). \) The sheaf \( \mathcal{F}it^{0}(\sigma_{i+1}^{0} \mathcal{E}xt^{1}(\sigma_{[i]}^{*} W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i}})) \) is invertible by the functorial property of \( \mathcal{F}it^{0} \):

\[
\mathcal{F}it^{0}(\sigma_{i+1}^{0} \mathcal{E}xt^{1}(\sigma_{[i]}^{*} W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) = (\sigma_{i+1}^{0})^{-1} \mathbb{I}_{i+1} \cdot \mathcal{O}_{\Sigma_{i+1}}.
\]

(3.10)

Now we need the following easy
Lemma 3.2. There is an isomorphism
\[ \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_i}) = \mathcal{E}xt^1(\sigma_{i+1}^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}}). \] (3.11)

Proof. Start with the exact triple
\[ 0 \to W_{\ell-i} \to \sigma_{i+1}^* \hat{E}_{\ell-i-1} \to \sigma_i^* W_{\ell-i-1} \to 0 \] (3.12)
and with its inverse \( \mathcal{O} \)-image
\[ 0 \to \delta_{i+1}^* W_{\ell-i} \to \delta_{i+1}^* \sigma_i^* \hat{E}_{\ell-i-1} \to \delta_i^* \sigma_i^* W_{\ell-i-1} \to 0. \] (3.13)

The triple (3.13) is also exact because the sheaf \( W_{\ell-i} \) is locally free (locally free sheaves \( W_{\ell-i} \) are obtained inductively on \( i \)). Dualization of both the triples (3.12,3.13) and taking an inverse image of the dual of (3.12) lead to the diagram with exact rows and vertical isomorphisms
\[
\begin{array}{c}
\delta_i^*(\sigma_i^* \hat{E}_{\ell-i-1})^\vee & \to & \delta_i^*(W_{\ell-i})^\vee & \to & \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}}) & \to & 0 \\
\mathcal{E}xt^1(\mathcal{O}_{\Sigma_i}) & \to & \mathcal{E}xt^1(\sigma_{i+1}^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}}) & \to & 0
\end{array}
\]

The right hand side vertical isomorphism completes the proof. \( \square \)

Now (3.10) can be rewritten as
\[
\mathcal{F}\ell^0(\sigma_{i+1}^* \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) = \mathcal{F}\ell^0(\sigma_{i+1}^* \delta_i^* \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) \\
= (\sigma_{i+1}^* \delta_i^*)^{-1} \mathcal{F}\ell^0(\mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) \cdot \mathcal{O}_{\Sigma_{i+1}} \\
= (\sigma_{i+1}^* \delta_i^*)^{-1} \mathcal{F}\ell^0(\mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) \cdot \mathcal{O}_{\Sigma_{i+1}}
\]

(where we use the notation \( \delta_i^{-1} := \delta_{i-1} \circ \cdots \circ \delta_1 \)). This sheaf is invertible on the scheme \( \Sigma_{i+1}^0 \) in whole. Hence it is invertible on the component \( \Sigma_{i+1}^{-1} \) of the scheme \( \Sigma_{i+1}^0 \), i.e. the sheaf
\[
(\sigma_{i+1}^* \delta_i^*)^{-1} \mathcal{F}\ell^0(\mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) \cdot \mathcal{O}_{\Sigma_{i+1}}
\]
is also invertible. The scheme \( \Sigma_{i+1}^{-1} \) has an irreducible reduction and application of Lemma 3.1 yields:
\[
\mathcal{H}d(\sigma_{i+1}^* \delta_i^*) \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}}) = 1. \] (3.14)

This means that it is possible to interchange the morphisms
\[
\delta_i^* \circ \sigma_i^* = \sigma_{i+1} \circ \delta_i^{-1}
\]
and to do the resolution by \( (\sigma_{i+1}^*)^{-1} \) following \( (\sigma_i^*)^* \). We perform all the necessary steps of the resolution beginning with the triple
\[
0 \to W_{\ell-i} \to \sigma_{i+1}^* \hat{E}_{\ell-i-1} \to \sigma_i^* W_{\ell-i-1} \to 0
\]
(generally, over all \( i \) inductively along the bottom of the diagram (3.6)). Denoting
\[
H_{\ell-i} := \ker (W_{\ell-i}^\vee \to \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})) \\
= \text{coker} ((\sigma_i^* W_{\ell-i-1})^\vee \to (\sigma_i^* \hat{E}_{\ell-i-1})^\vee)
\]
and applying \( \sigma_{i+1}^{-1} \) in view of (3.14), we come to the locally free sheaf \( N_{\ell-i} \):
\[
N_{\ell-i} := \ker (\sigma_{i+1}^* W_{\ell-i}^\vee \to \sigma_i^* \mathcal{E}xt^1(\sigma_i^* W_{\ell-i-1}, \mathcal{O}_{\Sigma_{i+1}})). \] (3.15)
Now we have a composite morphism
\[
(\sigma_{i+1}^0)_{\ell-i-1} \to \sigma_{i+1}^{-i} H_{\ell-i} \to N_{\ell-i}
\]
of locally free $O_{\Sigma_{\ell-i+1}}$-sheaves. The kernel of this composite morphism is also locally free. We denote it as $W'_{\ell-i-1}$ and from (3.12) come to the monomorphism $\sigma_{i+1}^{-i} W'_{\ell-i} \hookrightarrow N'_{\ell-i}$. This leads to the commutative diagram with exact rows
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \sigma_{i+1}^{-i} W'_{\ell-i} & \longrightarrow & \sigma_{i+1}^0 \hat{E}_{\ell-i-1} & \longrightarrow & \sigma_{i+1}^0 W_{\ell-i-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N'_{\ell-i} & \longrightarrow & \sigma_{i+1}^0 \hat{E}_{\ell-i-1} & \longrightarrow & W'_{\ell-i-1} & \longrightarrow & 0 \\
\end{array}
\]
and $W'_{\ell-i-1} = (\sigma_{i+1}^0 W_{\ell-i-1})/\text{tors}$. This completes the inductive step of the resolution procedure.

After all the steps one comes to a locally free $O_{\Sigma_{\ell}}$-sheaf
\[
\hat{\mathcal{E}} := \sigma_{\ell}^0 \mathcal{E}/\text{tors}
\]
and an $O_{\Sigma_{\ell}}$-sheaf
\[
\mathcal{E} := \sigma_{\ell}^0 \mathcal{E}.
\]
We will use the notation $\tilde{\Sigma} := \Sigma_{\ell}$ which is analogous to one used for a family of admissible schemes in the previous papers.

Remark 5. Since $\hat{\mathcal{E}}$ is locally free as an $O_{\Sigma}$-module then $\mathcal{E}$ is also flat over $T$ whenever $O_{\Sigma}$ is an $O_T$-flat $O_{\Sigma}$-module. Hence in the second case we get the equality of Hilbert polynomials on fibres of the family $\tilde{\Sigma}$: $\chi(\mathcal{E} \otimes \mathbb{L}^m|_{\pi^{-1}(t)})$ does not depend on the choice of $t \in T$ and in general points $t \in T$ $\hat{\mathcal{E}}|_{\pi^{-1}(t)} \cong \mathcal{E}|_{t \times S}$. This proves (iii) of Theorem [1].

Remark 6. The procedure described involves a choice of locally free resolution of the sheaf under the resolution. But the sequence of morphisms does not depend on this choice due to the following theorem from commutative algebra.

Theorem 2 ([1], Theorem 20.2). Let $R$ be a local ring and $M$ be a finitely generated $R$-module. If $F$ is a minimal free resolution of $M$ then any free resolution of $M$ is isomorphic to the direct sum of $F$ and a trivial complex. In particular, there is up to isomorphism only one minimal free resolution of $M$.

From this theorem we conclude inductively that the sheaves
\[
\mathcal{E}^x t^1(W_{\ell-1}, O_{\Sigma_{\ell}}), \mathcal{E}^x t^1(\sigma_{\ell}^1 W_{\ell-2}, O_{\Sigma_{\ell-1}}), \ldots, \mathcal{E}^x t^1(\sigma_2^1 \ldots \sigma_{\ell}^1 W_0, O_{\Sigma_{\ell-1}})
\]
are independent of the choice of a locally free resolution.

4 The structure of an admissible scheme and the fiberwise resolution

In this section we discuss the structure of fibers of the morphism $\tilde{\Sigma} \to T$. The resolution procedure leads to the composite of morphisms of $T$-schemes $\pi_i : \Sigma_i \to T, i = 0, \ldots, \ell$:
When restricted to the fibers over any fixed closed point \( t \in T \) (or, equivalently, one can set \( T \) to be a reduced point) this chain gives rise to the chain of morphisms among fibers \( S_i = \pi_i^{-1}(t) \):

\[
\begin{array}{cccccccc}
\widetilde{S} & \xrightarrow{\sigma_\ell} & S_\ell & \xrightarrow{\sigma_{\ell-1}} & S_{\ell-1} & \cdots & \xrightarrow{\sigma_2} & S_1 & \xrightarrow{\sigma_1} & S_0 & \xrightarrow{=} & S
\end{array}
\]

Each of the morphisms \( \sigma_i^0 \) is a blowing up morphism and each of the morphisms \( \delta_i^0 \) contracts the corresponding additional component of the scheme \( S_i \) onto the exceptional divisor of the morphism \( \sigma_i^0 \). Moving against arrows we can say that the morphisms \( \sigma_i^0 \) blow up and the morphisms \( \delta_i^0 \) grow up the corresponding additional components. Since these two types of morphisms alternate, next blowup is applied to the scheme which consists of several connected components.

**Convention 2.** We use the notations for single fibers which are completely parallel to the ones for families, preserving the indexing rules as they were used in (3.6). The fiberwise version for (3.6) can be obtained when one replaces the symbol \( \Sigma \) by the symbol \( S \) with all indices preserved. Also, when passing to the morphisms for single fibers, double letters \( \sigma, \delta \) are replaced by respective usual “single” letters \( \sigma, \delta \) with all indices preserved for morphisms. The reader should keep in mind that each \( \delta \) projects its source scheme to its component.

We start with the initial nonsingular variety \( S_0 = S \). When passing to \( S_1^0 \) and after that to \( S_1 \), we arrive to the scheme \( S_1 \) consisting of a principal component \( S_1^0 \) and an additional component(s) \( S_1^{add} \). The principal component \( S_1^0 = (\sigma_1^0)^{-1}S \) is an algebraic variety which is obtained by blowing up of \( S \). The additional component \( S_1^{add} \) can carry a nonreduced scheme structure and it also can have a reducible reduction. In the previous papers this closed subscheme appeared as a union of additional components of the admissible scheme.

Passing to \( S_2^0 \) leads to a transformation of both \( S_1^0 \) and \( S_1^{add} \). We come to an algebraic variety \( S_2^0 = (\sigma_2^{-1})S_1^0 = (\sigma_2^{-1})^{-1}(\sigma_1^0)^{-1}S \) obtained by blowing up of the principal component \( S_1^0 \) of \( S_1 \) and to the scheme \( (\sigma_2^0)^{-1}S_1^{add} \). Passing to \( S_2 \) against \( \delta_2^0 \) grows up an additional component \( S_2^{add} \), and we can write 
\[ S_2 = S_0^0 \cup (\sigma_2^0)^{-1}S_1^{add} \cup S_2^{add} \].

Analogously, on \( \ell \)-th step we have the following scheme

\[ \widetilde{S} := S_\ell = (\sigma_1^0 \circ \cdots \circ \sigma_\ell \cdots \circ \sigma_0^{-\ell+1})^{-1}S \cup (\sigma_2^0 \circ \cdots \circ \sigma_\ell \cdots \circ \sigma_0^{-\ell+2})^{-1}S_1^{add} \cup \cdots \cup (\sigma_0^{-\ell})^{-1}S_{\ell-1}^{add} \cup S_0^\ell . \]

Depending on the structure of the initial \( \mathcal{O}_S \)-sheaf \( E \), several morphisms \( \sigma_i \) can turn to be identities. Hence, the actual length of the chain \( S_1, \ldots, S_\ell \) can vary from 0 (for the case when \( E \) is locally free) to the maximal value equal to \( \ell \).

To measure numerical invariants of the objects we obtained in the standard resolution we need to fix an appropriate ample invertible sheaf \( L \) on each \( \widetilde{S} \). The sheaf playing an analogous role was called a distinguished polarization in previous papers, where the procedure of standard resolution in the h1-one case was developed. These invertible sheaves provide, in particular, fiberwise uniform Hilbert polynomials in flat families of admissible schemes. Strictly speaking, if \( L \) is an invertible \( \mathcal{O}_{\Sigma} \)-sheaf which is very ample relatively to the base \( T \), then for any closed point \( t \in T \) and for any integer \( n \gg 0 \) the Hilbert polynomial of the fiber \( \pi^{-1}(t) \) compute as \( \chi(\tilde{L}^n |_{\pi^{-1}(t)}) \) is independent of the choice of \( t \in T \).

Let \( \Sigma \) carries an invertible sheaf \( L \) which is very ample relatively to \( T \). In h1-one case there is a one-step standard resolution by \( T \)-morphism \( \sigma: \Sigma \to \Sigma \). This resolution procedure is associated to the sheaf of ideals \( I \subset \mathcal{O}_\Sigma \). In this case it was shown that a distinguished polarization can be chosen as \( \tilde{L} = \sigma^*L^m \otimes \sigma^{-1}I \cdot \mathcal{O}_\Sigma \), when \( m \) is sufficiently big to provide ampleness of \( \tilde{L} \) relatively to \( T \).

In the case of the bigger homological dimension \( \ell \) this step of constructing a family of polarizations on \( T \)-scheme \( \Sigma \) is iterated \( \ell \) times until one comes to

\[ \tilde{L} := L_\ell = [\sigma_\ell^* \cdots [\sigma_2^* \sigma_1^* L^m \otimes (\sigma_1)^{-1}I_1 \cdot \mathcal{O}_\Sigma_1]^{m_2} \otimes (\sigma_2)^{-1}I_2 \cdot \mathcal{O}_\Sigma_2]^{m_3} \cdots ]^{m_\ell} \otimes (\sigma_\ell)^{-1}I_\ell \cdot \mathcal{O}_\Sigma_\ell \]  \quad (4.1)
The corresponding $T$-scheme $\Sigma_i$ arising on the $i$-th step of the standard resolution has the fiberwise constant Hilbert polynomial with respect to

$$L_i := [\sigma_i^* \ldots [\sigma_1^* L^{m_1} \otimes (\sigma_1)^{-1} I_1 \cdot O_{\Sigma_i}]^m \otimes (\sigma_2)^{-1} I_2 \cdot O_{\Sigma_2}]^m \ldots ]^m \otimes (\sigma_i)^{-1} I_i \cdot O_{\Sigma_i}. $$

A distinguished polarization on a single admissible scheme $\widetilde{S}$ has a view

$$\tilde{L} := L_i = [\sigma_i^* \ldots [\sigma_1^* L^{m_1} \otimes (\sigma_1)^{-1} I_1 \cdot O_{\Sigma_1}]^m \otimes (\sigma_2)^{-1} I_2 \cdot O_{\Sigma_2}]^m \ldots ]^m \otimes (\sigma_i)^{-1} I_i \cdot O_{\Sigma_i}.  \quad (4.2)$$

The distinguished polarization $\tilde{L}$ is assumed to be fixed for each admissible scheme $\widetilde{S}$. If $\widetilde{S} = S$, then $\tilde{L} = L$.

Now we redenote $\mathbb{L}^{m_1, m_2, \ldots , m_\ell}$ as $\mathbb{L}$ and $L^{m_1, m_2, \ldots , m_\ell}$ as $L$ and from now we work with these new polarization. Also we come to the following shorthand notations for (4.1) and (4.2) respectively:

$$\mathbb{L} = \sigma^* \mathbb{L} \otimes \text{Exc};  \quad (4.3)$$

$$\text{Exc} := [\ldots [[[\sigma_1^{-1} I_1 \cdot O_{\Sigma_1}]^m \otimes (\sigma_2)^{-1} I_2 \cdot O_{\Sigma_2}]^m \ldots ]^m \otimes (\sigma_\ell)^{-1} I_\ell \cdot O_{\Sigma_\ell}];  \quad (4.4)$$

$$\text{Exc}_{\Sigma_i} := [\ldots [[[\sigma_1^{-1} I_1 \cdot O_{\Sigma_1}]^m \otimes (\sigma_2)^{-1} I_2 \cdot O_{\Sigma_2}]^m \ldots ]^m \otimes (\sigma_\ell)^{-1} I_\ell \cdot O_{\Sigma_\ell}.  \quad (4.6)$$

Now we come to the equality for the fiberwise Hilbert polynomials

$$\chi(\tilde{L}^n) = \chi(L^n) = rp(n), \quad n \gg 0.$$

Now we recall the following

**Definition 4 ([8], Definition 1.7,5).** A set of isomorphism classes of coherent sheaves on a projective $k$-scheme $S$ is bounded if there is a $k$-scheme $B$ of finite type and a coherent $O_{B \times S}$-sheaf $F$ such that the given set is contained in the set $\{F|_{B \times S}| b \in B \text{ is a closed point}\}.$

**Proposition 1.** Let some set $Q$ of coherent $O_S$-sheaves $E$ with Hilbert polynomials (computed with respect to the polarization $L$) equal to $P(n)$ be bounded. Then polarizations $\tilde{L}$ for all possible schemes $\tilde{S}$ obtained by standard resolutions of all the sheaves $E \in Q$ can be chosen in such a way that all the admissible schemes $(\tilde{S}, \tilde{L})$ have Hilbert polynomials also equal to $P(n)$.

**Proof.** Since the definition of bounded set of sheaves involves only closed points of the scheme $B$, we can assume this scheme to be reduced. We are interested in $O_S$-sheaves with equal Hilbert polynomials. Hence, after excluding, if necessary, some closed subscheme $B_0$ from $B$ we can suppose the sheaf $F$ to be flat over $B \setminus B_0$. Reduceness of the scheme $B \setminus B_0$ is needed to conclude flatness from fibrewise uniform Hilbert polynomials. The open subscheme $B \setminus B_0$ contains all the closed points $b \in B$ such that $F|_{B \times S} \in Q$. Now one can set $T := B \setminus B_0$ and perform the standard resolution of the family $E = F|_{T \times S}$. This resolution leads to the family of schemes $\pi_i: \Sigma_i \to T$ with the family of distinguished polarizations $\tilde{L}$ such that the fibrewise Hilbert polynomials $\chi(\tilde{L}^n)|_{\pi_i^{-1}(b)}$ is uniform over $b \in T$ and equals $P(n)$. \[\square\]

5 Further remarks on the additional components

Each admissible scheme $\widetilde{S} = S_i$ is built iteratively by alternating transformations of two types: these are blowing ups $\sigma_i^0$ followed by growing ups additional components $\delta_i = \delta_i^0, \ i = 1, \ldots, \ell$. Additional component $S_i^{\text{add}} = \text{Proj} \bigoplus_{s \geq 0} (I[t] + (t))^s/(I[t] + (t))^{s+1}$ grown by $\delta_i$ intersects the union $S_0^0$ of all the components of the previous $(i-1)^{st}$ level along an exceptional divisor $D_i = \text{Proj} \bigoplus_{s \geq 0} I_i^s/I_i^{s+1}$ of the blowing up morphism $\sigma_i^0$.

For now we work with a commutative ring $A$ and a proper ideal $I \subset A$; denote $\widehat{A} := \bigoplus_{s \geq 0} (I^s/I^{s+1})$. Also set $M = \bigoplus_{s \geq 0} (I[t] + (t))^s/(I[t] + (t))^{s+1} = \bigoplus_{s \geq 0} M_s$ for $M_s = (I[t] + (t))^s/(I[t] + (t))^{s+1}$. 

16
Proposition 2. $M = \bigoplus_{s \geq 0} (I[t] + (t))^s/(I[t] + (t))^{s+1}$ is flat as an $\hat{A}$-module.

Proof. Now write down explicit forms for every direct summand of $M$ keeping in mind that $t$ is a transcendental element over $A$ of graded degree 1. There are $A$-isomorphisms

$M_0 = A[t]/(I[t] + (t)) \cong A/I$;

$M_1 = (I[t] + (t))/(I[t] + (t))^2 \cong I/I^2 \oplus (A/I)t$;

$M_2 = (I[t] + (t))^2/(I[t] + (t))^3 \cong I^2/I^3 \oplus (I/I^2)t \oplus (A/I)t^2$;

\[ \vdots \]

$M_s = (I[t] + (t))^s/(I[t] + (t))^{s+1} \cong I^s/I^{s+1} \oplus (I^{s-1}/I^s)t \oplus \cdots \oplus (A/I)t^s$;

\[ \vdots \]

Here the expressions $(I^s/I^{s+1})t^q$ mean additive subgroups of the view $\{ \overline{t}^q t \in I^s/I^{s+1} \}$.

Example 2. In $M_1$ one has $i_0 + i_1 t + \cdots + i_s t^s + t(f_0 + f_1 t + \cdots + f_q t^q) \mod (I^2[t] + tI[t] + (t^2)) \cong i_0 \mod I^2 + t f_0 \mod I$ for $i_r \in I$, $r = 0, \ldots, s$; $f_u \in A$, $u = 0, \ldots, q$.

Taking a direct sum over all $s \geq 0$ one comes to the expression

$M = \bigoplus_{s \geq 0} (I[t] + (t))^s/(I[t] + (t))^{s+1}$

$\cong \left( \bigoplus_{s \geq 0} I^s/I^{s+1} \right) \oplus \left( \bigoplus_{s \geq 0} I^s/I^{s+1} \right)t \oplus \left( \bigoplus_{s \geq 0} I^s/I^{s+1} \right)t^2 \oplus \cdots \oplus \left( \bigoplus_{s \geq 0} I^s/I^{s+1} \right)t^q \oplus \cdots$

$\cong \hat{A} \oplus \hat{A}t \oplus \hat{A}t^2 \oplus \cdots \oplus \hat{A}t^q \oplus \cdots$

This expression shows that $M$ is an $\hat{A}$-flat module.

Set in (3.16 3.17) the base scheme $T$ to be a single reduced point and hence $\Sigma = S$, $\Sigma_i = S_i$ is an admissible scheme with the natural decomposition $S_i = S_i^0 \cup S_i^{add}$. The morphism $\delta_i$ of additional component(s) factors through an exceptional divisor $D_i$ of the blowing up morphism $\sigma_i: S_i^0 \to S_{i-1}$

$\delta_i: S_i^{add} \to D_i \hookrightarrow S_i^0$.

Now we know that $\delta_i$ is flat over its scheme-theoretic image $D_i$. Then we can rewrite (3.16 3.17) as $\tilde{E} = \sigma^* E/tors$ where the symbol tors denotes the torsion in the modified sense. It is defined inductively as described below (for the convenience we keep the notations $\tilde{S}_0 := (\sigma_1^0 \circ \cdots \circ \sigma_{\ell-1}^0)^{-1} S$ for the principal component and $\tilde{S}_i := (\sigma_{i+1}^0 \circ \cdots \circ \sigma_{\ell-1}^0)^{-1} S_i^{add}$ for all $i = 1, \ldots, \ell - 1$).

Let $U$ be a Zariski-open subset in one of the components $\tilde{S}_0$ or $\tilde{S}_j$, $j > 0$, and $\sigma^* E|_{\tilde{S}_j}(U)$ a corresponding group of sections. This group is an $O_{\tilde{S}_j}(U)$-module. Let tors $j(U)$ be the submodule in $\sigma^* E|_{\tilde{S}_j}(U)$ which consists of the sections $s \in \sigma^* E|_{\tilde{S}_j}(U)$ such that $s$ is annihilated by some prime ideal of positive codimension in $O_{\tilde{S}_j}(U)$. The correspondence $U \mapsto tors_j(U)$ defines a subsheaf $tors_j \subset \sigma^* E|_{\tilde{S}_j}(U)$. Note that associated primes of positive codimensions which annihilate the sections $s \in \sigma^* E|_{\tilde{S}_j}(U)$ correspond to subschemes supported in $\tilde{S}_j^{add} = \bigcup_{j > 0} \tilde{S}_j$. Since the scheme $\tilde{S} = \tilde{S}_0 \cup \tilde{S}_j^{add}$ is connected by the construction done in (17), the subsheaves $j$, $j \geq 0$, allow us to construct a subsheaf $tors \subset \sigma^* E$. The latter subsheaf is defined as follows. A section $s \in \sigma^* E|_{\tilde{S}_j}(U)$ satisfies the condition $s \in tors|_{\tilde{S}_j}(U)$ if and only if

(i) there exists a section $y \in O_{\tilde{S}_j}(U)$ such that $ys = 0$,

(ii) for $j > 1$ at least one of the following two requirements is satisfied: either $y \in \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of positive codimension, or there exists a Zariski-open subset $V \subset \tilde{S}$ and a section $s' \in \sigma^* E(V)$ such that $V \supset U$, $s'|_U = s$ and for $s'|_{V \cap \tilde{S}_{j-1}}$ also conditions analogous to (i), (ii) hold. For $j = 1$ conditions (ii) take the view: either $y \in \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of positive codimension, or there exists a Zariski-open
subset \( V \subset \tilde{S} \) and a section \( s' \in \sigma^*E(V) \) such that \( V \supset U, \ s'|_U = s \) and \( s'|_{V \cap \tilde{S}_0} \in \text{tors}((\sigma^*E|_{\tilde{S}_0})(V \cap \tilde{S}_0)) \).

In the latter expression the torsion subsheaf \( \text{tors}(\sigma^*E|_{\tilde{S}_0}) \) is understood in the usual sense.

The modified torsion subsheaf in each \( (\sigma_1 \circ \cdots \circ \sigma_i)^*E \) is defined in the analogous fashion. Also \( \tilde{E}_i = (\sigma_1 \circ \cdots \circ \sigma_i)^*E/\text{tors} \).

Now we address to the analog of (3.6), where the base \( T \) is a single reduced point \( T = \text{Spec} \, k \), and to the morphism growing up additional component(s)

\[ \delta_i: S_i^{\text{add}} \to D_i, \]

where \( D_i \) is the exceptional divisor of the blowing up morphism \( \sigma_i^0: S_i^0 \to S_{i-1} \). Now we know that \( \delta_i \) is flat over \( D_i \).

Consider the sheaf \( \tilde{E}_{i-1} \) on \( \tilde{S}_{i-1} \) and its transform on \( \tilde{S}_i \) under \( \sigma_i = \sigma_i^0 \circ \delta_i \):

\[ \tilde{E}_i = \sigma_i^*\tilde{E}_{i-1} = \delta_i^*\sigma_i^0*\tilde{E}_{i-1}/\text{tors}, \]

where \( \text{tors} \) means the torsion subsheaf in the modified sense. For its direct image \( \delta_i*\tilde{E}_i \) under \( \delta_i \) i.e. we analyse the natural morphism of \( \mathcal{O}_{\tilde{S}_i} \)-modules

\[ \delta_{i*}\tilde{E}_i = \delta_{i*}(\delta_i^*\sigma_i^0*\tilde{E}_{i-1}/\text{tors}) \xrightarrow{ev_{i-1}^0} \sigma_i^0*\tilde{E}_{i-1}/\text{tors}. \]

Its kernel \( \ker ev_{i-1}^0 \) is concentrated on \( D_i \) and hence it contributes to the modified torsion \( \text{tors} \subset \delta_{i*}\tilde{E}_i \).

Now \( \delta_i^0*\tilde{E}_{i-1}/\text{tors} = \delta_i^0*\tilde{E}_{i-1}/\text{tors} \) and hence the latter sheaf is modified torsion-free. Consequently, \( \delta_{i*}\tilde{E}_i = \delta_{i*}(\delta_i^*\sigma_i^0*\tilde{E}_{i-1}/\text{tors}) \) is also modified torsion-free and \( ev_{i-1}^0 \) is a monomorphism.

References

[1] V. Baranovsky, *Uhlenbeck Compactification As a Functor* // Int. Math. Res. Notices, V. 2015. No. 23. P. 12678–12712.

[2] U. Bruzzo, D. Markushevich, A. Tikhomirov, *Uhlenbeck–Donaldson compactification for framed sheaves on projective surfaces* // Math. Z. 2013. V. 275. No. 3-4. P. 1073–1093.

[3] S.K. Donaldson, *Compactification and completion of Yang–Mills moduli spaces* in: *Differential Geometry, Peniscola, 1988*, Springer, Berlin, 1989. P. 145–160 (Lecture Notes in Math., vol.1410)

[4] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Springer-Verlag, New York–Berlin, 1995. Grad. Texts in Math., 150.

[5] P.M.N. Feehan, *Geometry of the ends of the moduli space of anti-self-dual connections* // J. Diff. Geom. 1995. V. 42. No. 3. P. 465–553.

[6] D. Gieseker, *On the moduli of vector bundles on an algebraic surface* // Annals of Math. 1977. V. 106. P. 45–60.

[7] R. Hartshorne, *Algebraic Geometry*. New York–Heidelberg – Berlin: Springer-Verlag, 1977. Graduate Texts in Mathematics, 52.

[8] D. Huybrechts, M. Lehn, *The geometry of moduli spaces of sheaves*, Vieweg, Braunschweig, 1997. Aspects Math., E31.

[9] J. Li, *Algebraic geometric interpretation of Donaldson polynomial invariants* // J. Diff. Geom., 1993, V. 37. No. 2. P. 417–466.

[10] M. Lübke, A. Teleman, *The Kobayashi – Hitchin correspondence* World Scient. Publ. Co., 1995.
[11] D. Markushevich, A. Tikhomirov, G. Trautmann, Bubble tree compactification of moduli spaces of vector bundles on surfaces // Cent. Eur. J. Math. 2012. V. 10. No. 4 P. 1331–1355.

[12] M. Maruyama, Moduli of stable sheaves, I // J. Math. Kyoto Univ. (JMKYAZ), 1977, V. 17. No. 1. P. 91–126.

[13] M. Maruyama, Moduli of stable sheaves, II // J. Math. Kyoto Univ. (JMKYAZ), 1978, V. 18. No. 3. P. 557–614.

[14] N. V. Timofeeva, Compactification in Hilbert scheme of moduli scheme of stable 2-vector bundles on a surface // Math. Notes, 82:5 (2007), 677–690.

[15] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface // Sb. Math., 199:7 (2008), 1051–1070.

[16] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. II // Sb. Math. 200:3 (2009), 405–427.

[17] N. V. Timofeeva, On degeneration of surface in Fitting compactification of moduli of stable vector bundles // Math. Notes, 90 (2011), 142–148.

[18] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. III: Functorial approach // Sb. Math., 202:3 (2011), 413 – 465.

[19] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. IV: Nonreduced moduli // Sb. Math., 204:1 (2013), 133–153.

[20] N. V. Timofeeva, On a new compactification of the moduli of vector bundles on a surface. V: Existence of universal family // Sb. Math., 204:3 (2013), 411–437.

[21] N. V. Timofeeva, On a morphism of compactifications of moduli scheme of vector bundles // Siberian Electronic Mathematical Reports (SEMR) – 2015. – V. 12. – P. 577–591.

[22] N. V. Timofeeva, Isomorphism of compactifications of moduli of vector bundles: nonreduced moduli arXiv:1411.7872 [math.AG]. 2014. Russian original in: Modelirovanie i Analiz Informatsionnykh Sistem (Modeling and Analysis of Information Systems). 2015. V. 22. No. 5, P. 629 - 647.

[23] N. V. Timofeeva, Admissible pairs vs Gieseker-Maruyama // Sb. Math., 2019, V. 210. No. 5. P. 731–755.