Determination of the pion distribution amplitude

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Right now, we have not enough knowledge to determine the hadron distribution amplitudes (DAs) which are universal physical quantities in the high energy processes involving hadron for applying pQCD to exclusive processes. Even for the simplest pion, one can’t discriminate from different DA models. Inversely, one expects that processes involving pion can in principle provide strong constraints on the pion DA. For example, the pion-photon transition form factor (TFF) can get accurate information of the pion wave function or DA, due to the single pion in this process. However, the data from Belle and BABAR have a big difference on TFF in high $Q^2$ regions, at present, they are helpless for determining the pion DA. At the present paper, we think it is still possible to determine the pion DA as long as we perform a combined analysis of the most existing data of the processes involving pion such as $\pi \rightarrow \mu \nu$, $\pi^0 \rightarrow \gamma \gamma$, $B \rightarrow \pi l \nu$, $D \rightarrow \pi l \nu$, and etc. Based on the revised light-cone harmonic oscillator model, a convenient DA model has been suggested, whose parameter $B$ which dominates its longitudinal behavior for $\phi_\pi(x, \mu^2)$ can be determined in a definite range by those processes. A light-cone sum rule analysis of the semi-leptonic processes $B \rightarrow \pi l \nu$ and $D \rightarrow \pi l \nu$ leads to a narrow region $B = [0.01, 0.14]$, which indicate a slight deviation from the asymptotic DA. Then, one can predict the behavior of the pion-photon TFF in high $Q^2$ regions which can be tested in the future experiments. Following this way it provides the possibility that the pion DA will be determined by the global fit finally.

I. INTRODUCTION

In the perturbative QCD (pQCD) theory, the distribution amplitude (DA) provides the underlying links between the hadronic phenomena in QCD at the large distance (non-perturbative) and the small distance (perturbative). The pion DA is an important element for applying pQCD to exclusive processes in the high energy processes involving pion, and inversely, all of them can in principle provide strong constraints on the pion DA. The pion DA is usually arranged according to its different twist structures. There are processes in which the contributions from the higher twists are highly power suppressed at the short distance. For example, it has been found that the contribution to the pion-photon transition form factor (TFF) from higher helicity and higher twist structures is negligible [1, 2]. Thus, those processes will provide good platforms to learn the properties of the leading-twist pion DA. It is well-known that the leading-twist DA has the definite asymptotic form, $\phi_\pi(x, \mu^2)|_{\mu^2 \rightarrow \infty} = 6x(1-x)$, which is independent to its shape around some initial scale $\mu_0 \sim O(1GeV)$. However, in practical calculation, it is important to know what is the right shape of the pion DA at low and moderate scales.

The pion leading-twist DA at any scale $\mu$ can be expanded in Gegenbauer series in the following form [3, 4]

$$\phi_\pi(x, \mu^2) = 6x(1-x) \sum_{n=0}^{\infty} a_n(\mu^2) C_n^{3/2}(2x-1), \quad (1)$$

where $C_n^{3/2}(2x-1)$ are Gegenbauer polynomials and the nonperturbative coefficients $a_n(\mu^2)$ are Gegenbauer moments. Due to the isospin-symmetry, only the even moments are non-zero. Usually the Gegenbauer series is convergent, one can adopt the first several terms to analyze the experimental data. If the shape of the pion DA at an initial scale $\mu_0$ is known, then

- by using the orthogonality relations for the Gegenbauer polynomials, the Gegenbauer moments $a_n(\mu_0^2)$ can be obtained via the equation,

$$a_n(\mu_0^2) = \frac{\int_0^1 dx \phi_\pi(x, \mu_0^2) C_n^{3/2}(2x-1)}{\int_0^1 dx 6x(1-x)|C_n^{3/2}(2x-1)|^2}. \quad (2)$$

- by using the QCD evolution equation [3], one can derive the pion DA at any other scale from $\phi_\pi(x, \mu_0^2)$.

The value of the Gegenbauer moments have been studied by using the non-perturbative approaches as the QCD sum rules [4, 5] or the lattice QCD [6]. However, at present, there is no definite conclusion on whether the pion DA $\phi_\pi(x, \mu_0^2)$ is asymptotic form [3] or CZ-form [6] or even flat-like [10]. It would be helpful to have a general pion DA model that can mimic all the DA behaviors suggested in the literature. For this purpose, one can first
construct a wavefunction (WF) model, since the pion DA is related to its WF $\Psi_\pi(x, k_\perp)$ via the following relation,

$$\phi_\pi(x, \mu_0^2) = \frac{2\sqrt{6}}{f_\pi} \int_{|k_\perp|^2 \leq \mu_0^2} \frac{d^2k_\perp}{16\pi^3} \Psi_\pi(x, k_\perp),$$  

(3)

where $f_\pi$ is the pion decay constant. It is noted that a proper way of constructing the pion WF/DA is also very important to derive a better end-point behavior at small $x$ and $k_\perp$ region for dealing with high energy processes within the $k_T$-factorization approach \cite{11}, and thus to provide a better estimation for the pion photon TFF, pion electromagnetic form factor, and etc.

The revised light-cone harmonic oscillator model for the pion leading-twist WF, and hence the model for the leading-twist DA, has been suggested in Refs.\cite{12,13}. It has been found that by a proper change of the pion DA parameters, one can conveniently simulate the shape of the DA from asymptotic-like to CZ-like. By comparing the theoretical estimations on the pionic processes with the corresponding experimental data, those undetermined parameters of the DA model can be fixed or at least be greatly restricted. This is the purpose of the present paper.

More explicitly, we shall make a combined analysis of the pion DA by using the pion decay channels $\pi^0 \rightarrow \gamma \gamma$ and $\pi \rightarrow \mu \nu$, the pion-photon TFF $F_{\pi\gamma}(Q^2)$, the semileptonic decays $B \rightarrow \pi l \nu$ and $D \rightarrow \pi l \nu$, and the exclusive process $B \rightarrow \pi \pi$. For example, the pion-photon TFF $F_{\pi\gamma}(Q^2)$ that relates pion with two photons provides the simplest example for the perturbative application to exclusive processes. In the lower energy region the data on the pion-photon TFF measured by CELLO, CLEO, BABAR and Belle are consistent with each other \cite{12,13}, so these data can be adopted for constraining the WF parameters. Based on the present DA model, the model parameter $B$ for $\phi_\pi(x, \mu^2)$ can be determined, then one can predict the behavior of the pion-photon TFF in high $Q^2$ regions which can be tested in the future experiments.

The remaining parts of the paper is organized as follows. In Sec.II, we give a brief review on the pion leading-twist WF/DA, properties of DA have also been presented there. In Sec.III, we show how DA parameters can be constrained, and present a detailed derivation of the parameter $B$ by using the $B/D \rightarrow \pi \pi$ transition form factors within the light-cone sum rule (LCSR). A discussion on the $B \rightarrow \pi \pi$ process and pion-photon TFF is presented in Sec.V. The final section is reserved for a summary.

II. A BRIEF REVIEW ON THE PION LEADING-TWIST WF/DA

One useful way for modeling the hadronic valence WF is to use the approximate bound state solution of a hadron in terms of the quark model as the starting point. The Brodsky-Huang-Lepage (BHL) prescription \cite{19} of the hadronic WF is rightly obtained in this way by connecting the equal-time WF in the rest frame and the WF in the infinite momentum frame. Based on this prescription, the revised light-cone harmonic oscillator model of the pion leading-twist WF has suggested in Refs.\cite{12,13}, which shows

$$\Psi_\pi(x, k_\perp) = \sum_{\lambda_1, \lambda_2} \chi_{\lambda_1, \lambda_2}(x, k_\perp) \Psi_\pi^R(x, k_\perp),$$  

(4)

where $\chi_{\lambda_1, \lambda_2}(x, k_\perp)$ stands for the spin-space WF, $\lambda_1$ and $\lambda_2$ being the helicity states of the two constituent quarks in pion. The $\chi_{\lambda_1, \lambda_2}$ comes from the Wigner-Melosh rotation whose explicit form can be found in Refs.\cite{20,21}. $\Psi_\pi^R(x, k_\perp)$ indicates the spatial WF, which can be divided into a $k_\perp$-dependent part and an $x$-dependent part. For the $k_\perp$-dependent part, Brodsky-Huang-Lepage suggests that there is possible connection between the rest frame WF $\Psi_{c.m.}(q)$ and the light-cone WF $\Psi_{LC}(x, k_\perp)$ \cite{19}:

$$\Psi_{c.m.}(q^2) \longleftrightarrow \Psi_{LC} \left[ \frac{k_\perp^2 + m_q^2}{4\sqrt{x}(1-x)} - m_q^2 \right],$$  

(5)

where $m_q$ stands for the mass of the constituent quarks. From an approximate bound-state solution in the quark models for pion, the WF of the harmonic oscillator model in the rest frame can be obtained \cite{22}. Thus, for the $k_\perp$-dependent part of spatial WF $\Psi_\pi^R(x, k_\perp)$, we have:

$$\Psi_\pi^R(x, k_\perp) \propto \exp \left[ - \frac{k_\perp^2 + m_q^2}{8\beta^2(1-x)} \right].$$  

(6)

For the $x$-dependent part of $\Psi_\pi^R(x, k_\perp)$, we take $\varphi_\pi(x) = [1 + B \times C_2/(2x - 1)]$, which dominates the longitudinal distribution broadness of the WF and can be expanded in the Gegenbauer polynomials. Here we only keep the first two terms in $\varphi_\pi(x)$, in which the parameter $B \sim a_2$ can be regarded as an effective parameter to determine the broadness of the longitudinal part of the WF.

As a combination, the explicit form of the spatial WF can be obtained:

$$\Psi_\pi^R(x, k_\perp) = A \varphi_\pi(x) \exp \left[ - \frac{k_\perp^2 + m_q^2}{8\beta^2(1-x)} \right],$$  

(7)

where $A$ is the normalization constant. After integration over the transverse momentum dependence, one can obtain the pion DA with the help of Eq.(3).
\[ \phi_\pi(x, \mu_0^2) = \frac{\sqrt{3} \lambda_m \beta}{2\pi^{3/2} f_\pi} \sqrt{x(1-x)} \varphi_\pi(x) \times \left\{ \text{Erf} \left[ \frac{m_\pi^2 + \mu_0^2}{8 \beta^2 x (1-x)} \right] - \text{Erf} \left[ \frac{m_\pi^2}{8 \beta^2 x (1-x)} \right] \right\}, \]

where \( \text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \).

Except for the constituent quark mass \( m_q \), which can be taken as the conventional value about 0.30 GeV, there are three undetermined parameters, \( A, \beta \) and \( B \), in the above model. Two important constraints have been found in Ref. [11] to constrain those parameters: (1) the process \( \pi \to \mu \bar{\nu} \) provides the WF normalization condition

\[ \int_0^1 dx \int \frac{d^2 k}{16\pi^3} \Psi_\pi(x, k_\perp) = \frac{f_\pi}{2\sqrt{6}}; \quad (9) \]

(2) the sum rule derived from \( \pi^0 \to \gamma \gamma \) decay amplitude implies,

\[ \int_0^1 dx \Psi_\pi(x, k_\perp = 0) = \frac{\sqrt{6}}{f_\pi}. \quad (10) \]

In addition to these two basic constraints, one needs other processes involving pion to further constrain the parameters, especially to determine the value of the parameter \( B \). We put the DA shapes from asymptotic-like to CZ-like:

- The second moments \( a_2 \) varies from 0.03 to 0.68;
- The first inverse moments of the pion DA at energy scale \( \mu_0 \), \( \int_0^1 \left[ \phi_\pi(x, \mu_0^2)/x \right] dx \), varies from 3.0 to 5.0.

Thus, if we have precise measurements for certain processes, then by comparing the theoretical estimations derived under the DA model [8], one can conveniently fix the pion DA behavior.

\[ \begin{array}{cccc}
B & A(\text{GeV}^{-1}) & \beta(\text{GeV}) & a_2(\mu_0) & a_4(\mu_0^2) \\
-0.60 & 36.03 & 0.456 & -0.523 & 0.051 \\
-0.30 & 30.43 & 0.514 & -0.279 & 0.000 \\
0.00 & 24.80 & 0.589 & 0.028 & -0.027 \\
0.30 & 20.05 & 0.672 & 0.364 & -0.017 \\
0.60 & 16.46 & 0.749 & 0.681 & 0.022 \\
\end{array} \]

We put the WF parameters for several typical \( B \) in Table I where the region of the parameter \( B \) is broadened to be \([-0.60, 0.60]\). The value of \( B \) is close to the second Gegenbauer moment, \( B \sim a_2 \), and because of the fact that the longitudinal distribution is dominated by the second Gegenbauer moment, cf. Refs. [5, 9, 23–26], thus the parameter \( B \) dominantly determines the broadness of the longitudinal part of the wave function.

The parameters listed in Table I are for \( \mu_0 = 1 \) GeV. They can be run to any other scales by applying the evolution equation, i.e. to order \( \mathcal{O}(\alpha_s) \), we have [4]

\[ x_1 x_2 \frac{\partial \tilde{\phi}_\pi(x_i, \mu^2)}{\partial \ln \mu^2} = C_F \frac{\alpha_s(\mu^2)}{4\pi} \left\{ \int_0^1 dy V(x_i, y_i) \tilde{\phi}_\pi(y_i, \mu^2) - x_1 x_2 \tilde{\phi}_\pi(x_i, \mu^2) \right\}, \quad (11) \]
More explicitly, the explicit expression for forms the DA scale dependence to the determination of using the DA Gegenbauer expansion (1), which transforms the DA scale dependence to the determination of the scale dependent of the Gegenbauer moments [3, 4]. More explicitly, the explicit expression for \( a_n(\mu^2) \) to leading-logarithmic (LL) accuracy can be written as [27]:

\[
a_n(\mu^2) = a_n(\mu_0^2) \left( \frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} \right)^{\gamma_n/\beta_0},
\]

where the anomalous dimensions

\[
\gamma_n = C_F \left( 1 - \frac{2}{n+1} - \frac{4}{n+2} + 4 \sum_{m=2}^{n+1} \frac{1}{m} \right)
\]

with \( \beta_0 = (11N_c - 2N_f)/3 \). Usually, one truncates the Gegenbauer expansion with the first several terms \( n = 0, 2, 4, 6 \) respectively to derive the DA behavior at the high energy scales.

In this paper we solve the evolution equation (11) strictly to get the DA’s behavior at the higher energy scale. It is noted that if the Gegenbauer expansion converges quickly, these two evolution methods (11) and (12) are equivalent to each other. The solution of the evolution equation (11) can be done numerically. Here we suggest an equivalent but simpler and more effective way to get the DA after evolution, i.e. we transform the whole scale dependence of \( \phi(x, \mu^2) \) into the scale dependence of the undetermined parameters \( A, B \) and \( \beta \). The valence quark mass \( m_q \) is scale independent and we keep it to be 0.30 GeV. Its main idea is to take the second Gegenbauer moment \( a_2(\mu^2) \) as a ligament between the DA and the DA parameters. Firstly, from the initial DA \( \phi(x, \mu_0^2) \) with known \( A, B \) and \( \beta \) at the initial \( \mu_0 \), we derive its second Gegenbauer moment \( a_2(\mu_0^2) \) via Eq. (12), and get its value at any scale \( \mu \) by using the evolution equation (12). Secondly, we use the value of \( a_2(\mu^2) \) together with the two constraints (9) and (10) to determine the values of \( A, B \) and \( \beta \) at the scale \( \mu \). We put the parameters \( A, B \) and \( \beta \) at three typical scales \( \mu = 1, 1.5 \) and \( 3 \) GeV in Table I. From the table, one observes that the value of \( A \) increases and the value of \( \beta \) decreases with the increment of the scale.

III. DETERMINATION OF DA FROM \( B/D \to \pi \) TRANSITION FORM FACTORS

The semi-leptonic \( B \)-meson decay \( B \to \pi l \nu \) is usually used to extract the CKM matrix element \( |V_{ub}| \), whose differential cross section for massless leptons can be written as

\[
\frac{d\Gamma}{dq^2} (B \to \pi l \nu) = \frac{C_F^2 |V_{ub}|^2}{192\pi^3 m_B^3} \left[ (q^2 + m_B^2 - m_{\pi}^2)^2 - 4m_B^2m_{\pi}^2 \right]^{3/2} |f_{+}^{B \to \pi}(q^2)|^2,
\]

where the momentum transfer \( q = p_B - p_{\pi} \). The TFF \( f_{+}^{B \to \pi} \) is the key factor of the process, which has been deeply investigated by using several approaches, such as the pQCD approach [28, 29], the QCD LCSR approach [23, 27, 30, 36] and the lattice QCD approach [37, 38]. Different approaches are applicable for different energy regions. Among them, the QCD LCSR is reliable for the intermediate energy region, which can be extended to the whole physical region with proper extrapolation. So this approach is usually adopted for a detailed analysis in comparison with the experimental data.

Under LCSR, the expression for \( f_{+}^{B \to \pi} \) depends on how one chooses the correlator [33]; different choice of the currents in the correlation function shall result in different expressions, in which, the pionic different twist structures provide different contributions. Here we adopt the chiral correlator suggested in Ref. [31] to do our discussion, in which the leading-twist DA’s contribution have been amplified and it provides us a better chance to know the detail of the leading-twist DA in comparison with data. By using the chiral correlator, up to twist-4, the form factor \( f_{+}^{B \to \pi}(0) \) at the large recoil region can be obtained [33],

\[
\text{where}[dy] = dy_1 dy_2 \delta(1 - y_1 - y_2), \phi_\pi(x_i, \mu^2) = x_1 x_2 \phi_\pi(x_i, \mu^2) \text{ and } V(x_i, y_i) = 2 \left[ x_1 y_2 \theta(y_1 - x_1) \left( \delta_{h_1 h_2} + \frac{\Delta}{(y_1 - x_1)} \right) + (1 \leftrightarrow 2) \right].
\]
TABLE II: Typical pion WF parameters for $m_q = 0.30$ GeV at several typical energy scales, $\mu = 1, 1.5, 3$ GeV, respectively.

| $A(\text{GeV}^{-1})$ | $\mu = 1\text{GeV}$ | $\mu = 1.5\text{GeV}$ | $\mu = 3\text{GeV}$ |
|----------------------|-----------------------|-----------------------|-----------------------|
| $B$                  | $\beta(\text{GeV})$  | $B$                  | $\beta(\text{GeV})$  | $B$                  | $\beta(\text{GeV})$  |
| 24.63                | 0.01                  | 0.592                | 24.99                | 0.037                | 0.560                | 25.11                | 0.033                | 0.556                |
| 23.93                | 0.05                  | 0.603                | 24.40                | 0.073                | 0.567                | 24.63                | 0.062                | 0.562                |
| 23.09                | 0.10                  | 0.617                | 23.67                | 0.118                | 0.577                | 24.05                | 0.099                | 0.570                |
| 22.44                | 0.14                  | 0.628                | 23.11                | 0.154                | 0.585                | 23.59                | 0.128                | 0.576                |
| 22.28                | 0.15                  | 0.631                | 22.97                | 0.163                | 0.587                | 23.48                | 0.135                | 0.578                |
| 21.50                | 0.20                  | 0.645                | 22.30                | 0.208                | 0.597                | 22.93                | 0.171                | 0.585                |
| 20.76                | 0.25                  | 0.658                | 21.65                | 0.252                | 0.607                | 22.39                | 0.207                | 0.593                |
| 20.05                | 0.30                  | 0.672                | 21.03                | 0.296                | 0.617                | 21.88                | 0.242                | 0.601                |
| 19.37                | 0.35                  | 0.686                | 20.43                | 0.340                | 0.626                | 21.38                | 0.277                | 0.608                |
| 18.72                | 0.40                  | 0.699                | 19.87                | 0.383                | 0.636                | 20.90                | 0.311                | 0.616                |
| 18.47                | 0.42                  | 0.704                | 19.65                | 0.400                | 0.640                | 20.72                | 0.325                | 0.618                |

where $f_B$, $m_B$, $M^2$, $m_b$ and $s_0^B$ indicate the $B$ meson decay constant, the $B$ meson mass, the Borel parameter, the $b$ quark mass and the effective threshold parameter, respectively. The parameter $u_0 = m_b^2/s_0^B$. The functions $\phi_{4\pi}$ and $\psi_{4\pi}$ are pion two-particle-twist-4 DAs. $I_{4\pi}$ is a combination of pion three-particle-twist-4 DAs. The hard scattering amplitude $\text{Im}T(\eta m^2/s, s/m^2, \mu)$ involves the LO and NLO parts. The scale of the process $\mu = \sqrt{m_B^2 - m_c^2} \approx 3$ GeV.

Furthermore, the semi-leptonic $D$-meson decay $D \to \pi l \nu$ can also be used to extract the CKM matrix element $V_{cd}$ if we know the $D \to \pi$ TFF $f^{D \to \pi}$ well. The TFF $f^{D \to \pi}$ has been studied in Refs. [33, 40, 41]. Replacing all the $B$ meson parameters in (15) by those of $D$ meson, we can obtain the LCSR expression for $f^{D \to \pi}(0)$. For example, the scale for $f^{D \to \pi}$ now equals $\mu = \sqrt{m_D^2 - m_c^2} \approx 1.5$ GeV.

Using the formula (16), we obtain that the contributions from pion twist-4 DAs terms are less than 1% for $f^{B \to \pi}(0)$ and less than 5% for $f^{D \to \pi}(0)$. Thus this provides a good platform to study the properties of the pion leading-twist DA. In Ref. [32], the authors have made use of this platform to determine the DA parameter $B$ with experimental data of $f^{B \to \pi}(0)$ by taking the input parameters as same as in Ref. [32]. At the present section, we update the analysis there by using the input parameters to be those given by the Particle Data Group [42], and simultaneously we make use of $f_{D^+}$, $f^{D \to \pi}(0)$ as a further constrain to determine the pion DA parameters.

The input parameters are listed in the following. The $\overline{MS}$-running $b$ and $c$ masses, the $B^+$ and $D^+$ meson masses are $[42]: m_b(m_b) = 4.18 \pm 0.03$ GeV, $m_c(m_c) = 1.275 \pm 0.025$ GeV, $m_{B^+} = 5279.25 \pm 0.17$ MeV and $m_{D^+} = 1869.62 \pm 0.15$ MeV. The $B^+$ meson decay constant $f_{B^+} = 214.55\text{MeV}$ [24]. Because there is large discrepancy for the estimation of the $D^+$ meson decay constant $f_{D^+}$ [43, 44], instead of using $f^{D \to \pi}(0)$ as a criteria, we adopt the combined value of $f_{D^+}$, $f^{D \to \pi}(0)$ to constrain the pion DA. The pion decay constant $f_{\pi}$ is set to be $f_{\pi} = 208$, which is $130.41 \pm 0.03 \pm 0.20$ MeV [42]. As for the effective threshold and Borel variables, we take them to be same as those of Ref. [33].

Experimentally, from the processes $B^+ / D^+ \to \pi^0 l^+ \nu_l$, it has been shown that the multiplication of the form factor and the corresponding CKM matrix element by the BABAR [45] and CLEO [50] collaborations are,

$$f^{B \to \pi}(0)|V_{ub}| = (9.4 \pm 0.3 \pm 0.3) \times 10^{-4} \quad (16)$$

and

$$f^{D \to \pi}(0)|V_{cd}| = 0.146 \pm 0.007 \pm 0.002. \quad (17)$$

From a simultaneous fit to the experimental partial rates and lattice points on the exclusive process $B \to \pi l \nu$ versus $q^2$, the CKM matrix element $|V_{ub}|$ is derived as $(3.23 \pm 0.31) \times 10^{-3}$ [38]. As a combination, we can obtain the experimental value for $f^{B \to \pi}(0)$:

$$f^{B \to \pi}(0) = 0.291^{+0.010}_{-0.009}. \quad (18)$$

Comparing this value with the estimated one from the LCSR (15), as indicated by Fig. (2), we obtain the first reasonable region for the parameter $B$:

$$B_{(B \to \pi l \nu)} = [0.01, 0.42], \quad (19)$$
where all the input parameters are varied within their reasonable regions listed above. Our present value for $B$ is different from that of Ref. [32], which is because we have adopted a different $\mpi$ b-quark mass. Fig. 2 gives the value of $f_{D^+}^{B\rightarrow\pi}(0)$ versus the parameter $B$. Where the lighter shaded band indicates the experimental value [13], the solid, dashed and dotted lines stand for the central, upper and lower ones calculated by the LCSR [15], and the thicker shaded band is the result of Ref. [31].

For the $D$ meson case, whose lifetime is $1040 \pm 71$ fs [51], we can adopt the measurement of $B(D^+ \rightarrow \pi^+\nu\overline{\nu})$. Then using the formulae

$$\Gamma(D^+ \rightarrow l^+\nu) = \frac{C_F^2 f_{D^+}^2 f_{\pi l}^2 m_{D^+}^2}{8 \pi} \left(1 - \frac{m_l^2}{m_{D^+}^2}\right)^2 |V_{cd}|^2,$$

where $m_l$ is the mass of the lepton, we can inversely obtain

$$f_{D^+}|V_{cd}| = 46.4 \pm 2.0 \text{ MeV.} \quad (20)$$

Furthermore, using the PDG value for $|V_{cd}| = 0.230 \pm 0.011$ [42], together with Eqs. (17,20), we can obtain an experimental constrain for the multiplication of $f_{D^+}$ with $f_{D^+}^{B\rightarrow\pi}(0)$, i.e.,

$$f_{D^+} f_{D^+}^{B\rightarrow\pi}(0) = 0.128 \pm 0.012 \text{ GeV.} \quad (21)$$

Combining this experimental values (21) of $f_{D^+} f_{D^+}^{B\rightarrow\pi}(0)$ with the theoretical one calculated by sum rules [15], as shown by Fig. (3), we obtain the second reasonable region for the parameter $B$:

$$B(D\rightarrow\pi l\nu) = [0.00, 0.14]. \quad (22)$$

Fig. (3) gives the value of $f_{D^+} f_{D^+}^{B\rightarrow\pi}(0)$ versus the parameter $B$, where the shaded band indicates the experimental values [21], the solid, the dashed and the dotted lines stand for the central, upper and lower edge of the theoretical values calculated by the LCSR [15] with slight parameter changes to agree with the $D$-meson case. Here we have implicitly set the value of $B$ to be biger than 0, which is reasonable, since as shown in Fig. (1), by varying $B \in [0, 0.6]$ the DA can mimic all of its known behaviors suggested in the literature.

As a final remark, the $D$-meson mass may be not large enough, the energy scale is about 1.5 GeV, thus, the reliability of the LCSR for the form factor $f_{D^+}^{B\rightarrow\pi}$ may be less reliable than the $B$-meson case. So we give two schemes for setting the region of parameter $B$:

- Scheme A: If we believe the LCSR has the same importance as that of $B \rightarrow \pi l\nu$, then the range of $B$ is

$$B = [0.01, 0.14]. \quad (23)$$

- Scheme B: If only the LCSR for $B \rightarrow \pi l\nu$ is acceptable, we have a broader region as shown in Eq. (19).

IV. DISCUSSION

If the parameter $B$ is determined, the shape of the pion leading-twist DA can be fixed. Under the scheme A, the second and fourth moments of the pion twist-2 DA can be calculated as $a_2(1 \text{GeV}) = [0.039, 0.184]$ and $a_4(1 \text{GeV}) = [-0.027, -0.028]$. Under the scheme B, the first two moments changes to $a_2(1 \text{GeV}) = [0.039, 0.495]$ and $a_4(1 \text{GeV}) = [-0.027, -0.004]$. We present our DA model with different values of $B$ at $\mu = 1$ GeV in Fig. (4), where the thin-solid line, the dashed line, the dotted line, the dash-dotted line and the thick-solid line are for $B = 0.01, 0.14, 0.2, 0.3$ and 0.42, respectively.
The result is shown in Fig. 5, where we vary the parameters and adopt the same twist-3 DAs used in the calculation, we take the parameters to be their central values and adopt the same twist-3 DAs used in Refs. [52–55] to do our calculation. The formulas can be found there. In doing the numerical calculation, we take the parameters to be their central values and adopt the same twist-3 DAs used in Refs. [52, 53, 55], but with our present leading-twist DA. The result is shown in Fig. 6, where we vary the parameter B within the region [0.01, 0.14]. The branching ratio B(B → ππ) increases with increment of the parameter B. The value of the branching ratio B(B → ππ) increases and is closing to the experimental data for a larger value of B.

As two simple applications, we apply our pion leading-twist DA to deal with the branching ratio of the B-meson exclusive decay B → ππ and the pion-photon TFF F_{πγ}(Q^2).

As a first application, we discuss with the process B → ππ, which has been calculated within the pQCD approach [52, 53]. There is a large discrepancy between the theoretical estimation and the experimental data. At present, we adopt the same calculation technology as described in Refs. [52, 53] to do our calculation. The formulas can be found there. In doing the numerical calculation, we take the parameters to be their central values and adopt the same twist-3 DAs used in Refs. [52, 53, 55], but with our present leading-twist DA. The result is shown in Fig. 4, where we vary the parameter B within the region [0.01, 0.42]. The branching ratio B(B → ππ) increases with increment of the parameter B. The value of the branching ratio B(B → ππ) increases and is closing to the experimental data for a large value of B.

As a second application, we revisit the pion-photon TFF. Generally, the pion-photon TFF F_{πγ}(Q^2) can be written as a sum of the valence part F_{πγ}^{V}(Q^2) and the non-valence quark part F_{πγ}^{NV}(Q^2):

\[ F_{πγ}(Q^2) = F_{πγ}^{V}(Q^2) + F_{πγ}^{NV}(Q^2). \]  

The expressions for F_{πγ}^{V}(Q^2) and F_{πγ}^{NV}(Q^2) can be found in Ref. [12]. Taking all of the input parameters to be same as those in Ref. [12], but with our present DA model by taking B ∈ [0.01, 0.42], we draw the pion-photon TFF F_{πγ}(Q^2) in Fig. 5. It shows that in the small Q^2 region, Q^2 \lesssim 15 GeV^2, the pion-photon TFF can explain the CELLO [13], CLEO [14], BABAR [15] and Belle [16] experimental data simultaneously. While for the large Q^2 region, our present estimation favors the Belle data and disfavors the BABAR data. If taking B ∈ [0.01, 0.42], the calculated curve [39] for the pion-photon TFF with the upper limit of the parameter (B = 0.42) will be between the Belle and BABAR data.

V. SUMMARY

In the present paper, based on the revised LC harmonic oscillator model for the pion leading-twist DA, we have made a combined analysis of the pion DA by using the channels π^0 → γγ, π → μν, the semi-leptonic

\[ Q^2 F_{πγ}(Q^2) \] with our WF model [41] by varying the model parameter B within the region [0.01, 0.14]. The shaded band is our theoretical estimation.

Our present estimation, being consistent with those obtained within the pQCD approach, is still smaller than the experimental data [12]. There may have some important factors need to be considered in pQCD calculation, such as the next-to-order correction may be big or there may have large non-perturbative contributions, which is beyond the scope of the present paper.
decays $B \rightarrow \pi l \nu$ and $D \rightarrow \pi l \nu$ in comparison with the experimental data. Based on the constraints from these processes, typical parameters for the pion leading-twist DA are presented in Table [4].

In addition to the two constraints (10–11), by using the constraint from the process $B \rightarrow \pi l \nu$, the parameter $B$ is restricted in $[0.01, 0.42]$. If taking the process $D \rightarrow \pi l \nu$ as a further constrain, we can obtain a more narrow region $B = [0.00, 0.14]$. Using the pion leading-twist DA model, we recalculate the branching ratio $B(B \rightarrow \pi \pi)$ and the pion-photon TFF. The branching ratio $B(B \rightarrow \pi \pi)$ increases with increment of the parameter $B$. For the pion-photon TFF, our present result with the parameter $B = [0.01, 0.14]$ favors the Belle data and the corresponding pion DA has the slight difference from the asymptotic form. Then, one can predict the behavior of the pion-photon TFF in high $Q^2$ regions which can be tested in the future experiments. It is expected that BABAR and Belle can obtain more accurate and consistent data in the future, then the behavior of the pion DA can be further determined completely. On the other hand, we can adopt more pionic processes, such as the pion electromagnetic form factor, to make a further constrain to the pion DA, which is in progress. It is believed that the pion DA will be determined by the global fit to the exclusive processes involving the pion in the coming future.

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