Chiral symmetry in non-Hermitian systems: product rule, Clifford algebra and pseudo-chirality

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(Dated: March 7, 2019)

Chiral symmetry provides the symmetry protection for a large class of topological edge states. It exists in non-Hermitian systems as well, and the same anti-commutation relation between the Hamiltonian and a linear chiral operator, i.e., \{H, Π\} = 0, now warrants a symmetric spectrum about the origin of the complex energy plane. Here we show two general approaches to construct chiral symmetry in non-Hermitian systems, with an emphasis on lattices with detuned on-site potentials that can vary in both their real and imaginary parts. One approach relies on the simultaneous satisfaction of both non-Hermitian particle-hole symmetry and non-Hermitian bosonic anti-linear symmetry, while the other utilizes the Clifford algebra satisfied by the Dirac matrices. We also distinguish non-Hermitian chiral symmetry from pseudo-chirality, with the latter defined by \(\eta H^T \eta^{-1} = -H\) and belonging to a broadened definition of symmetry that maps between the left and right eigenspaces of a non-Hermitian Hamiltonian.

I. INTRODUCTION

Chiral symmetry plays an important role in the study of topological phases of matter \([17]\). It warrants that the energy spectrum of the system is symmetric about a well-defined energy level, such as the Fermi level or the energy of an uncoupled orbital, which is chosen as the zero energy. Recently \([8,9]\), the exploration of chiral symmetry in non-Hermitian systems \([10–13]\) has also attracted fast growing interests, especially with the advent of topological photonics \([14]\) and lasers \([15–21]\). In non-Hermitian systems, the energy spectrum is complex in general, and chiral symmetry, defined by the anti-commutation relation of the Hamiltonian and a linear operator, warrants that the complex spectrum is symmetric about the origin of the complex energy plane. As a consequence, a non-Hermitian zero mode, with its energy right at the origin of the complex plane \([8]\), can still exist similar to its Hermitian counterpart.

On the one hand, the easiest way to construct a non-Hermitian system with chiral symmetry is to maintain the sublattice symmetry of an underlying Hermitian system (such as in a tight-binding square or honeycomb lattice with no on-site detunings) and lift its Hermiticity by introducing asymmetric couplings \([8]\). While this approach can be applied to both periodic \([9]\) and finite-size systems, it does not utilize one important benefit provided by the non-Hermitian platforms in optics and photonics \([10]\), namely, the availability and tunability of gain and loss in optical cavities and waveguides, which are represented by an imaginary detuning between different lattice sites. On the other hand, if we directly apply such an imaginary detuning to a Hermitian system with chiral symmetry, its chiral symmetry will be lifted and we often obtain non-Hermitian particle-hole (NHPH) symmetry instead, which results in a spectrum symmetric about the imaginary axis of the complex energy plane \([8,9,22,24]\).

To overcome these obstacles and facilitate the exploration of topological phases of matter in non-Hermitian systems, we propose in this work two general approaches to construct non-Hermitian chiral symmetry. In the first approach we introduce a product rule where chiral symmetry, denoted by \(\Pi\) below, results from the simultaneous satisfaction of NHPH symmetry and non-Hermitian bosonic anti-linear symmetry. The former is defined similarly to its Hermitian counterpart, i.e., with the Hamiltonian \(H\) anticommuting with an antilinear operator \(\Sigma\); the latter is defined as a commutation relation between the Hamiltonian and an antilinear operator \(\Lambda\), with parity-time symmetry \([25]\) being a specific example. As we will exemplify below, the advantage of this approach lies in the fact that both NHPH symmetry \([15–20,26,27]\) and bosonic anti-linear symmetry \([28,38]\) are straightforward to implement in optics and related fields.

In the second approach, we discuss how the Clifford algebra can be used to generate non-Hermitian chiral symmetry for \(4 \times 4\) Hamiltonians, independent of NHPH and bosonic anti-linear symmetries. Through the discussion of several examples, we show that the Dirac matrices and their products can be arranged to satisfy chiral symmetry in the presence of complex detuning between on-site potentials.

Finally, we distinguish non-Hermitian chiral symmetry from pseudo-chirality, both of which warrant a symmetric spectrum about the origin in the complex energy plane. Pseudo-chirality is defined by \(\eta H^T \eta^{-1} = -H\), where \(\eta\) is a linear operator or an invertible matrix in its matrix form. Similar to pseudo-Hermiticity defined by \(\eta H^\dagger \eta^{-1} = H\) that can lead to a purely real energy spectrum for a non-Hermitian Hamiltonian \([39,40]\), pseudo-chirality belongs to a broadened sense of symmetry where the operator \(\eta\) maps between the left and right eigenspaces of a non-Hermitian Hamiltonian.
II. CONSTRUCTION OF NON-HERMITIAN CHIRAL SYMMETRY

A. Approach I: product rule

We first review two important concepts in the fundamental proposition regarding possible forms of symmetries in quantum systems, i.e., the Wigner theorem \([41]\). It states that any symmetry transformation is necessarily represented by a linear (and unitary) or anti-linear (and anti-unitary) transformation of the Hilbert space. A linear symmetry transformation \(U\) satisfies

\[
U(a\phi_1 + b\phi_2) = aU\phi_1 + bU\phi_2,
\]

where \(\phi_{1,2}\) are two arbitrary quantum states and the complex numbers \(a, b\) are their linear superposition coefficients. In contrast, an anti-linear symmetry operator \(A\) satisfies

\[
A(a\phi_1 + b\phi_2) = a^*A\phi_1 + b^*A\phi_2,
\]

where the asterisks denote the complex conjugation as usual. From this definition, it can be inferred that an anti-linear operator can be represented by the product of a linear operator and the complex conjugation.

As mentioned in the introduction, the first approach we employ to construct non-Hermitian chiral symmetry relies on the simultaneous satisfaction of NHPH symmetry and a non-Hermitian bosonic anti-linear symmetry:

\[
\{H, \Xi\} = 0, \quad [H, \Lambda] = 0.
\]

Bosonic anti-linear symmetry can be implemented conveniently using strategically placed photonic elements with balanced optical gain and loss \([10]\). Meanwhile, a probably more important and intriguing foundation of this approach is that imposing any arbitrary imaginary on-site potentials to an underlying Hermitian chiral lattice with real-valued couplings gives rise to NHPH symmetry automatically \([8]\). Therefore, NHPH symmetry can coexist nicely with bosonic anti-linear symmetry enabled by optical gain and loss, which in turn warrants non-Hermitian chiral symmetry as we show in detail below.

Since both \(\Xi\) and \(\Lambda\) in Eq. \((3)\) are anti-linear operators, they can be written as the product of a linear operator and the complex conjugation \(K\):

\[
\Xi \equiv CK, \quad \Lambda \equiv XK.
\]

\(K\) is often the manifestation of time-reversal operator for a finite-sized system \([25]\), and \(C\) for the NHPH symmetry mentioned above is given by the chiral operator of the underlying Hermitian lattice, i.e., \(C = P_A - P_B\) as in the SSH model \([15]\), where \(P_{A,B}\) are the projection operators onto the two sublattices [see Fig. \((1a)\)]. These two sublattices are defined such that there is no coupling between two sites on the same sublattice. \(X\), on the other hand, can take a variety of forms. For example, two convenient choices of \(X\) in two dimensions are mirror reflection and rotation, which lead to parity-time \([10, 11, 25]\) and rotation-time symmetry \([42, 44]\), respectively.

As a consequence of Eq. \((3)\), the following symmetry transformations hold for the eigenstates of \(H\):

\[
\Xi\psi_\mu = \psi_\nu, \quad \Lambda\psi_\nu = \psi_\nu',
\]

where the subscripts \(\mu, \nu, \nu'\) are not necessarily the same. The corresponding energy eigenvalues satisfy

\[
\varepsilon_\mu = -\varepsilon_\nu^*, \quad \varepsilon_\nu = \varepsilon_\nu',
\]

i.e., they are symmetric about the imaginary and real energy axis of the complex energy plane, respectively. It is then straightforward to see

\[
\Lambda\Xi\psi_\mu = \psi_{\nu'}, \quad \varepsilon_\mu = -\varepsilon_{\nu'},
\]

which indicates the existence of chiral symmetry, i.e., \(\{H, \Pi\} = 0\), with the linear operator \(\Pi = \Lambda\Xi = XC^*\). A non-Hermitian zero mode occurs when all the subscripts are the same, leading to \(\varepsilon_\mu = 0\).

This product rule is similar in construction to how chiral symmetry in a Hermitian system can be generated as the product of particle-hole symmetry and time-reversal symmetry \([1]\). However, we note that since the non-Hermitian chiral operator \(\Pi\) is no longer given by the

FIG. 1. Construction of non-Hermitian chiral symmetry by the product rule. (a) Schematic of a finite-size honeycomb lattice with only nearest neighbor couplings. The two sublattices \(A\) and \(B\) are marked by filled and open dots, and both NHPH and parity-time symmetry are present. (b) Real part of the complex spectrum as a function of the gain and loss strength \(\tau\). Thick black line shows the existence of a zero mode with \(\varepsilon_\mu = 0\). Black dot marks its exceptional point. (c) and (d) Spatial profile of this zero mode at \(\tau = 0\) and the exceptional point in (b), respectively.
difference of the sublattice projection operators, the wave function of a non-Hermitian zero mode with \( \varepsilon_\mu = 0 \) does not necessarily vanish on one sublattice. More importantly, this zero energy can be an exceptional point, i.e., a non-Hermitian degeneracy where two or more eigenstates coalesce \([46, 53]\).

To illustrate the approach outlined above and these general properties of the non-Hermitian zero mode at the origin of the complex energy plane, we consider a tight-binding honeycomb lattice with three rings and the same nearest-neighbor coupling \( g \in \mathbb{R} \) everywhere [Fig. 1(a)]. To realize non-Hermitian chiral symmetry, we first impose parity-time symmetry by constructing three pairs of lattice sites with equal gain and loss (i.e., \( \pm i \tau \) for the imaginary parts of the on-site potentials). As a result, the system is described by the generalized dihedral group \( \mathcal{M}_2 \) (\( v = 3 \)) \([42]\), with all \( v \) reflections in the dihedral group \( D_2 \), now combined with time reversal. Therefore, we actually have three different parity-time operators \( \Lambda_j = \mathcal{P}_j \mathcal{R}_n \) (\( j = 1, 2, 3 \)), which are about the vertical, 30°(210°), and 150°(330°) axes, respectively. Since the underlying Hermitian honeycomb lattice with real-valued couplings has chiral symmetry, NHPH symmetry arises automatically with the introduced gain and loss modulation as mentioned previously, with \( \Xi \equiv (\mathcal{P}_A - \mathcal{P}_B) \mathcal{K} \).

Together they lead to three related non-Hermitian chiral operators specified by the product rule in Eq. (7), i.e.,

\[
\Pi_j = \mathcal{P}_j (P_A - P_B),
\]

with \( \Pi_2 = \mathcal{R}_3 \Pi_1 \) and \( \Pi_1 = \mathcal{R}_3 \Pi_2 \). Here \( \mathcal{R}_n \) is the counterclockwise rotation by an angle equal to 360°/\( n \).

Away from an exceptional point, this system has one zero mode with \( \varepsilon_\mu = 0 \) [mode 0; thick solid line in Fig. 1(b)] and four pairs of doubly degenerate modes [dashed lines in Fig. 1(b)]. The existence of this zero mode is warranted by a generalized Lieb theorem \([5]\), which applies here since the numbers of sites on the two sublattices differ by 1. When the gain and loss strength becomes \( \tau = \sqrt{2g} \), two other modes (1 and 2) reach the origin of the complex energy plane [Fig. 1(b)], and they coalesce with the original zero mode and form an exceptional point of order 3 \([24, 54, 57]\).

When \( \tau \) is further increased, mode 1 and 2 become a pair of non-Hermitian zero modes defined by \( \Re[\varepsilon_\mu] = 0 \) \([8, 24]\) and protected by NHPH symmetry. As mentioned before, even though the original Hermitian zero mode has a finite amplitude only on the \( A \) sublattice [filled dots in Fig. 1(c)], its non-Hermitian counterpart with \( \varepsilon_\mu = 0 \) does not inherit this property due to the different chiral operators now (again there are three of them), as we illustrate in Fig. 1(d) at the exceptional point. In fact, even though the gain and loss are only on the \( A \) sublattice and overlap exactly with the Hermitian zero mode, the intensity of the non-Hermitian zero mode at \( \varepsilon_\mu = 0 \) becomes stronger on the \( B \) sublattice when \( \tau > g \):

\[
\frac{\max I_B}{\max I_A} = \frac{\pi^2}{g^2}.
\]

If we do not impose bosonic anti-linear symmetry in the example above, for instance, by having only gain at the sites where the original Hermitian zero mode is localized, we no longer have non-Hermitian chiral symmetry or the zero mode at \( \varepsilon_\mu = 0 \) it protects. However, there are still non-Hermitian zero modes defined by \( \Re[\varepsilon_\mu] = 0 \) similar to mode 1 and 2 in Fig. 1(b). Since these non-Hermitian zero modes can only be generated or annihilated in pairs due to NHPH symmetry \([8]\), the original Hermitian zero mode persists in this system and merely acquires a finite \( \Im[\varepsilon_\mu] \), which is the largest among all modes. It was this “selective pumping” mechanism \([58–64]\) in the presence of NHPH symmetry that led to the observations of topological insulator lasers \([15–20]\).

Both \( \Xi \) and \( \Lambda \)‘s for NHPH and bosonic anti-linear symmetries are easy to identify in the example shown in Fig. 1. In some cases however, one may accidentally construct non-Hermitian chiral symmetry using this approach without realizing it \([65]\), because either \( \Xi \) or \( \Lambda \) is difficult to recognize. One such example is given in Fig. 2(a), where four sites on a tight-binding ring are coupled by two pairs of complex couplings. This model can be considered as a generalized Rice-Mele model \([66]\), and below we analyze it using the second approach mentioned in the introduction, i.e., from the perspective of the Clifford algebra and the Dirac matrices.
B. Approach 2: Clifford algebra

Given by
\[ \gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \quad (j = 1, 2, 3) \]  
(10)
in terms of the identity matrix and Pauli matrices, the Dirac matrices are used in the Dirac equation to describe relativistic quantum mechanics. Together with
\[ \gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \]
(11)
they satisfy the Clifford algebra
\[ \{\gamma^\mu, \gamma^\nu\} = 0 \quad (\mu \neq \nu), \]
(12)
\[ \{\gamma^\mu, \gamma^\nu\} = 2\xi^\mu\gamma^\nu \]  
(13)
where \(\xi^\mu = 1\) for \(\mu = 0, 5\) and \(-1\) for \(\mu = 1, 2, 3\). This defining property of the Clifford algebra is particularly appealing in the construction of non-Hermitian chiral symmetry: by defining the Hamiltonian as a superposition of the Dirac matrices and their products, we have a straightforward way to determine its chiral operators.

As a simple example, let us first consider
\[ H = g_1\gamma^5 - g_2\gamma^1 \]  
(14)
depicted schematically in Fig. 3(a), featuring one symmetric coupling \((g_1)\) and one asymmetric coupling \((g_2)\). Using Eq. (13), we know immediately that its chiral operators can be \(\gamma^0, \gamma^2, \gamma^3\) as well as \(g_2\gamma^5 - g_1\gamma^1\) upon proper normalization. In addition, we also note that
\[ \{\gamma^j\gamma^k, \gamma^l\} = 0 \]  
(15)
holds when \(j \neq k\) and \(l = j \text{ or } k\). Therefore, \(\gamma^1\gamma^5\) is also a non-Hermitian chiral symmetry in this case.

If the Hamiltonian contains the product of two Dirac matrices, e.g.,
\[ H = g_1\gamma^5 + g_2\gamma^0\gamma^1 \]  
(16)
as shown in Fig. 3(b), it remains a simple task to verify that the chiral operators can be \(\gamma^0, \gamma^1\) using Eq. (15). In addition, by noticing the relation
\[ \{\gamma^j\gamma^k, \gamma^l\} = 0 \quad (j \neq l \neq k), \]  
(17)
we also identify \(\gamma^1\gamma^5, \gamma^0\gamma^3\) as additional non-Hermitian chiral symmetries.

Before we discuss more complicated constructions using the Clifford algebra, we note that the two examples given in Fig. 3 still have two sublattices without detuning. Therefore, we naturally expect them to have chiral symmetry with respect to the operator \(P_A - P_B\), and the difference of these two sublattice projection operators is exactly \(\gamma_0\) in this case. The power of the more general analysis based on the Clifford algebra lies in the revelation of more chiral operators of different forms, including \(\gamma^1\gamma^5\) shared by both examples.

Now let us revisit the system shown in Fig. 2(a). Its Hamiltonian can be written as
\[ H = \beta\gamma^0 + g_{1r}\gamma^5 + \gamma^0(g_{2r}\gamma^1 + ig_{1i}\gamma^3) + g_{2r}\gamma^2, \]  
(18)
where \(g_1 = g_{1r} + ig_{1i}\) and \(g_2 = g_{2r} + ig_{2i}\). In the absence of detuning (i.e., \(\beta = 0\)), we again find \(\gamma^0\) as a chiral operator. Note however, it is not the only chiral operator even in this more complicated case. Using the extension of Eq. (15), i.e.,
\[ \{\gamma^j\tilde{\gamma}, \tilde{\gamma}\} = 0, \quad \tilde{\gamma} = \sum_{k \neq j} a_k\gamma^k, \]  
(19)
we find another chiral operator given by
\[ \Pi = g_{2r}\gamma^1 + ig_{1i}\gamma^3 \]  
(20)
upon proper normalization.

In fact, the non-Hermitian chiral symmetry defined by this \(\Pi\) operator holds even when \(\beta\) is finite, whereas that defined by \(\gamma^0\) is lifted. When \(\beta\) is real, the system also has rotation-time symmetry \(\Lambda = R_2K\), a form of bosonic anti-linear symmetry mentioned before. This observation explains the symmetric spectrum about both the real and imaginary axes in Fig. 2(b) following Eq. (6). From the perspective of the product rule discussed previously, i.e., \(\Pi = \Lambda\Xi\), one can then attribute the existence of non-Hermitian chiral symmetry in Fig. 2(a) to the simultaneous satisfaction of rotation-time symmetry and a hidden \(NHPH\) symmetry
\[ \Xi = R_2K(g_{2r}\gamma^1 + ig_{1i}\gamma^3) = R_2(g_{2r}\gamma^1 - ig_{1i}\gamma^3)K. \]  
(21)
Importantly, the non-Hermitian chiral symmetry here holds even when a complex \(\beta\) lifts the rotation-time symmetry and \(NHPH\) symmetry [Fig. 2(c)], which shows that the approach based on the Clifford algebra is more fundamental in this case.

Despite their different levels of complexity, the three examples specified by Eqs. (14), (16) and (18) feature only nearest neighbor couplings, which is important for
the existence of chiral symmetry in Hermitian systems with two sublattices (such as in the SSH model). Now consider the system shown schematically in Fig. 4(a), i.e.,

$$H = g_1 \gamma^5 + g_2 \gamma^0 \gamma^1 + g_3 \gamma^0 \gamma^1 \gamma^5,$$

where all four lattice sites are coupled to each other. Clearly the introduction of the next nearest neighbor (NNN) coupling $g_3$ lifts the chiral symmetry associated with the two sublattices (i.e., $\gamma^0$) and no other chiral symmetry exists.

However, if we change the sign of one NNN coupling [e.g., between site 3 and 4 as shown in Fig. 4(b)], the Hamiltonian of the modified system is now given by

$$H = g_1 \gamma^5 + g_2 \gamma^0 \gamma^1 + g_3 \gamma^1 \gamma^5,$$

and it still has the chiral symmetries specified by either $\Pi = \gamma^1$ or $\gamma^0 \gamma^5$, which can be checked using Eqs. (15) and (17). $\gamma^1$ tolerates detuning of the forms $\gamma^0$ and $\gamma^5$, which are diagonal matrices with elements 1, 1, −1, −1 and $-i, i, -i, i$; $\gamma^0 \gamma^5$ can accommodate detuning given by $\gamma^0$ as well as $\gamma^5$ with diagonal elements 1, −1, −1, 1 [see Fig. 4(c)]. By first glance, a detuning consisting of a superposition of $\gamma^3 \gamma^5$ and $\gamma^1 \gamma^2$ (i.e., $H \rightarrow H + d_1 \gamma^1 \gamma^2 + d_2 \gamma^3 \gamma^5$) would then lift all chiral symmetries of the system. However, one of the chiral operators actual evolves with the detuning, i.e.,

$$\gamma^1 \rightarrow \gamma^1 + d_2 \gamma^1 \gamma^3,$$

and the non-Hermitian Hamiltonian still has a symmetric spectrum about the origin of the complex energy plane [see Fig. 4(d)]. Only by mixing all three forms of detuning do we lift the chiral symmetry of the system.

### III. PSEUDO-CHIRALITY

Besides non-Hermitian chiral symmetry, pseudo-chirality also warrants a symmetric spectrum about the origin of the complex energy plane, i.e., $\epsilon_{\mu} = -\epsilon_{\nu}$, as we will show below. From its definition mentioned in the introduction, it is clear that its difference from non-Hermitian chiral symmetry lies in a matrix transpose on the Hamiltonian: instead of $\Pi \Pi^{-1} = -H$ required by non-Hermitian chiral symmetry, it demands $\eta H^T \eta^{-1} = -H$.

This transpose, however, has a profound consequence: unlike $\Pi$ in non-Hermitian chiral symmetry and all symmetry operators we have discussed so far, $\eta$ does not map between states in the (right) eigenspace of $H$ as [see Eq. (7)]. Instead, it maps a state in the left eigenspace of $H$ to a state in its right eigenspace. The left and right eigenstates of a non-Hermitian Hamiltonian are defined by:

$$H \psi_{\mu} = \epsilon_{\mu} \psi_{\mu}, \quad \bar{\psi}_{\mu}^T H = \epsilon_{\mu} \bar{\psi}_{\mu}^T.$$

We note that a pair of $\psi_{\mu}$ and $\bar{\psi}_{\mu}$ are different in general but share the same eigenvalue $\epsilon_{\mu}$. The second relation in Eq. (25) is also often written as $H^T \bar{\psi}_{\mu} = \epsilon_{\mu} \bar{\psi}_{\mu}$, which gives

$$H \eta \bar{\psi}_{\mu} = -\eta H^T \bar{\psi}_{\mu} = -\epsilon_{\mu} \eta \bar{\psi}_{\mu}$$

when combined with $H \eta = -\eta H^T$. Equation (26) indicates

$$\eta \bar{\psi}_{\mu} \propto \psi_{\nu}, \quad -\epsilon_{\mu} = \epsilon_{\nu}.$$

In other words, $\eta$ maps a left eigenstate of $H$ to a right eigenstate of $H$, and the spectrum is symmetric about the origin of the complex energy plane. The difference between non-Hermitian chiral symmetry and pseudo-chirality disappears only when $H$ is symmetric, under which they are defined by the same operator and the left and right eigenspaces of $H$ also become the same.

To construct non-Hermitian Hamiltonians with pseudo-chirality, three approaches can be found that parallel our discussions of the non-Hermitian chiral symmetry. The first approach is to satisfy anti-pseudo-Hermiticity and non-Hermitian chiral symmetry lies in a matrix transpose on the Hamiltonian: instead of $\Pi \Pi^{-1} = -H$ required by non-Hermitian chiral symmetry, it demands $\eta H^T \eta^{-1} = -H$.

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where $\zeta$ and $\Lambda$ are the same operators that previously defined anti-pseudo-Hermiticity and bosonic anti-linear symmetry in this section. This is our second approach to construct pseudo-chirality, which bears a formal analogy to the Wick rotation that changes time to its imaginary counterpart $\text{Im}$. This approach does not apply in the construction of non-Hermitian chiral symmetry, because applying the same technique there would simply exchange the roles of the two anti-linear operators in NPHS symmetry and bosonic anti-linear symmetry.

In the third approach to realize pseudo-chirality, we again utilize the Clifford algebra and the following properties of the Dirac matrices: $(\gamma^\mu)^T = \gamma^\mu$ ($\mu = 0, 2, 5$) and $-\gamma^\mu$ ($\mu = 1, 3$). Again if the Hamiltonian is symmetric (or anti-symmetric), then the pseudo-chiral operator also defines a chiral symmetry (or a linear symmetry $[\eta, H] = 0$). Therefore, we look for a general asymmetric Hamiltonian that has a unique pseudo-chiral operator.

For this purpose we write $H = S + A$, where $S = ST = (H + H^T)/2$ is symmetric while $A = -A^T = (H - H^T)/2$ is antisymmetric. If $S$ and $A$ anticommute, then we find $\eta = A$ gives rise to pseudo-chirality:

$$AH^T + HA = A(S - A) + (S + A)A = \{A, S\} = 0.$$  

The Hamiltonian given by Eq. (14) meets this requirement, where $S = g_1\gamma^5$ and $A = -g_2\gamma^1$, and hence this Hamiltonian has pseudo-chirality specified by $\gamma^1$. We do note that the system constructed this way also has chiral symmetry specified by $\Pi = SA$:

$$\{H, SA\} = \{A + S, S\} = \{A, S\} + \{S, SA\} = 0.$$ 

In the case of Eq. (14), this is just the chiral operator $\gamma^1\gamma^5$ we have seen.

Now to eliminate non-Hermitian chiral symmetry, we require $\{A, S\} \neq 0$ and

$$\{\eta, S\} = 0, \quad [\eta, A] = 0.$$ 

These two conditions appear naturally in the definition of pseudo-chirality: $\eta H^T + H\eta = \eta(S - A) + (S + A)\eta = \{\eta, S\} - [\eta, A]$. Together they form a sufficient condition for $\eta$ to be a pseudo-chiral operator. One example is

$$S = \gamma^0 + \gamma^5, \quad A = \gamma^1 + \gamma^3 + \gamma^1\gamma^3 + (1 + i)\gamma^0\gamma^5,$$

with $\eta = \gamma^0\gamma^5$ satisfying both relations in Eq. (36). This Hamiltonian will be difficult to realize, and the purpose of this example is just to show that pseudo-chirality is independent from non-Hermitian chiral symmetry.

On top of these three approaches, we also note the following properties of a pseudo-chiral operator $\eta$:

- $\eta^T$ is also a pseudo-chiral operator;
• If another operator \( \varpi \) commutes with \( H \), then \( \varpi \eta \) is also a pseudo-chiral operator for \( H \), independent of whether \( \varpi \) is linear or anti-linear;

• \( \eta \) can simply be an identity matrix, with which pseudo-chirality reduces to anti-symmetric, i.e., \( H^T = -H \);

• Pseudo-chirality is equivalent to particle-hole symmetry in the Hermitian case, i.e., \( \{ H, \eta K \} = 0 \). However, they are clearly different in the non-Hermitian case, giving rise to a spectrum symmetric about the origin and the imaginary axis of the complex energy plane, respectively.

IV. CONCLUSION AND DISCUSSION

In summary, we have presented two general approaches to construct systems with non-Hermitian chiral symmetry, which we believe can facilitate the exploration of topological phases of matter in non-Hermitian systems, especially on optical and photonic platforms. The first approach relies on the simultaneous satisfaction of NHPH symmetry and non-Hermitian bosonic anti-linear symmetry. As a result, the complex energy spectrum is also symmetric about the real and imaginary axes. The second approach utilizes the Clifford algebra, and the examples we have discussed are based on the Dirac matrices.

This second approach can be applied to non-Hermitian Hamiltonians of other dimensions as well. For instance, to identify possible non-Hermitian extensions of the SSH model with non-Hermitian chiral symmetry, we can utilize the Clifford algebra obeyed by the Pauli matrices. Introducing asymmetric couplings, e.g., \( t_1 \to t_1 \pm \tau \), retains \( \sigma_z \) as the chiral operator of the now non-Hermitian SSH chain:

\[
H = \begin{pmatrix}
0 & t_1 + \tau + t_2 e^{-ika} \\
t_1 - \tau + t_2 e^{ika} & 0
\end{pmatrix}
= (t_1 + t_2 \cos ka)\sigma_x + (i\tau + t_2 \sin ka)\sigma_y. \tag{38}
\]

Adding imaginary on-site potentials, on the other hand, lifts this chiral symmetry:

\[
H = \begin{pmatrix}
i\tau & t_1 + t_2 e^{-ika} \\
t_1 + t_2 e^{ika} & -i\tau
\end{pmatrix}
= i\tau \sigma_z + (t_1 + t_2 \cos ka)\sigma_x + t_2 \sin ka \sigma_y. \tag{39}
\]

We have also discussed the differences between non-Hermitian chiral symmetry and pseudo-chirality, with the latter belonging to a broadened definition of symmetry that maps between the left and right eigenspaces of a non-Hermitian Hamiltonian. These two symmetries are independent as we have shown, and in the example given by Eq. (39), the system in fact has pseudo-chirality given by \( \eta = \sigma_y \) [see the discussion of Eq. (41)], even though it does not possess non-Hermitian chiral symmetry. For both symmetries we have exemplified a protected non-Hermitian zero mode with \( \varepsilon_\mu = 0 \), and these zero modes no longer have a vanished amplitude on one sublattice as its Hermitian counterpart does.

The role of \( \eta \) as a mapping between the left and right eigenspace of \( H \) also holds for pseudo-Hermiticity, defined by \( \eta H^\dagger \eta^{-1} = H \). Therefore, while pseudo-Hermiticity and parity-time symmetry (or non-Hermitian bosonic anti-linear symmetry in general) can both lead to a purely real spectrum, they are not equivalent and their difference is the same as that between pseudo-chirality and non-Hermitian chiral symmetry. This difference again only vanishes when the Hamiltonian is symmetric, which is true in both cases.

Finally, we note that chiral symmetry in optics and photonics can also refer to the symmetry between clockwise and counterclockwise modes of motion [68–71], which should be distinguished from our discussion here.

We thank Ali Mostafazadeh for helpful discussion. This project is supported by the NSF under Grant No. DMR-1506987.

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