IDENTIFICATION OF FRACTIONAL-ORDER DYNAMICAL SYSTEMS

Ľubomír Dorčák, Vladimír Leško, Imrich Koštial

Department of Management and Control Engineering
BERG Faculty, Technical University Košice
Boženy Němcovej 3, 042 00 Košice, Slovakia
e-mail: Lubomir.Dorcak@tuke.sk, phone: (+42155) 6025172

ABSTRACT: This contribution deals with identification of fractional-order dynamical systems. We consider systems whose mathematical description is a three-member differential equation in which the orders of derivatives can be real numbers. We give a discretization method and a numerical solution of differential equations of this type. An experimental method of identification is given which is based on evaluation of transfer characteristics. This is a combination of the method of derivatives of transfer characteristics and of the method of passive search. The verification was performed on systems with known parameters and also on a laboratory object.

1. INTRODUCTION

Real objects in general are fractional-order systems, although in some types of systems the order is very close to an integer order. Such systems are mainly electronic systems composed of quality electronic elements. So far, however, the systems were described as integer-order systems, regardless of the negative consequences caused by neglecting the real order of the system [4,5]. Disregarding the fractional order of the system was caused mainly by the nonexistence of simple mathematical tools for the description of such systems. Since major advances have been made in this area in the last few years [1,2,3,4,5,6] it is possible to consider also the real order of the dynamical systems. Such models are more adequate for the description of dynamical systems with distributed parameters than integer-order models with concentrated parameters. With regard to this, in the task of identification, it is necessary to consider also the fractional order of the dynamical system. In this contribution we will concentrate mainly on the identification of parameters (including the order of derivatives) for a chosen structure of the model with emphasis on the methods of evaluation of transfer characteristics. The verification of the correctness of parameter identification will be done by using it in systems with known parameters.

2. DEFINITION OF THE FRACTIONAL-ORDER SYSTEM MODEL

For the purposes of later development let us consider as the model of a dynamical system the following fractional-order three-member differential equation

\[ a_2 y^{(\alpha)}(t) + a_1 y^{(\beta)}(t) + a_0 y(t) = u(t) \]  \hspace{1cm} (1)

where \( \alpha \) and \( \beta \) are in general real, with initial conditions \( y^{(\beta)}(0) = 0 \) and \( y(0) = 0 \).  
3. NUMERICAL SOLUTION OF THE FRACTIONAL-ORDER DIFFERENTIAL EQUATION

For the numerical calculation of the fractional-order differential equation (1) we use for an approximation of fractional derivatives in the equation (1) the following formula taken from [6]

\[ y^{(\alpha)}(t) \approx (t-L)D_t^\alpha y(t) = h^{-\alpha}\sum_{j=0}^{N(t)} b_j y(t-jh), \] (2)

where \( L \) is the "memory length", \( h \) is the time step,

\[ N(t) = \min\left\{ \left\lfloor \frac{t}{h} \right\rfloor, \left\lfloor \frac{L}{h} \right\rfloor \right\}, \]

\([z]\) is the integer part of \( z \),

\[ b_j = (-1)^j \binom{\alpha}{j} \] (3)

where \( \binom{\alpha}{j} \) is a binomial coefficient.

The approximation of the equation (1) in discrete time steps \( t_m \) (\( m = 2, 3, \ldots \)) has the following form [4]

\[ a_2 h^{-\alpha}\sum_{j=0}^{m} b_j y_{m-j} + a_1 h^{-\beta}\sum_{j=0}^{m} c_j y_{m-j} + a_0 y_m = u_m \] (4)

From the approximation (1) we can derive [4] the following explicit recursive formula for the calculation of the values \( y_m \) (\( m = 2, 3, \ldots \))

\[ y_m = \frac{u_m - a_2 h^{-\alpha}\sum_{j=1}^{m} b_j y_{m-j} - a_1 h^{-\beta}\sum_{j=1}^{m} c_j y_{m-j}}{a_2 h^{-\alpha}b_0 + a_1 h^{-\beta}c_0 + a_0} \] (5)

with \( y_0 = 0 \), \( y_1 = 0 \). An analytical solution of fractional-order differential equations is described in detail in [2], and for closed regulation circuits also in [4,5].

4. METHOD OF DIFFERENTIATION OF TRANSFER CHARACTERISTICS

Assume the input of the system (1) with known values \( \alpha \) and \( \beta \) is acted upon by arbitrary function \( u(t) \), and \( y(t) \) is the output function of the system, while the initial conditions need not be zero in this method. If the input and output functions are known functions of time obtained, for example, by measurement, then for given discrete times \( t_m \) (\( m = 2, 3, \ldots \)) we can determine the values of the derivatives of the output quantity according to the formula (2). For each time instance \( t_m \) the equation (1) must be satisfied. Therefore after substituting the corresponding values into the equation (1) we obtain a system of three linear equations for unknown coefficients \( a_0, a_1, a_2 \). By solving this system of equations and obtaining the values of coefficients \( a_0, a_1, a_2 \) the system, from which we used experimental data in the calculations, is...
identified. In order to be able to use more measured values \( y_i, u_i \), and to make the identification more accurate, we consider for the estimate of the vector of unknown parameters \( \bar{a} \) as an error criterion the following functional

\[
E(\bar{a}) = \int_0^T [a_2 y^{(\alpha)}(t) + a_1 y^{(\beta)}(t) + a_0 y(t) - u(t)]^2 dt \approx \min
\]

A necessary condition for the minimum to be achieved is

\[
\frac{\partial E(\bar{a})}{\partial \bar{a}} = 0
\]

Hence, after manipulations we obtain a system of three linear equations for the vector of unknown parameters \( \bar{a} \)

\[
\begin{align*}
a_2 \int_0^T (y^{(\alpha)}(t))^2 dt + a_1 \int_0^T y^{(\alpha)}(t) y^{(\beta)}(t) dt + a_0 \int_0^T y^{(\alpha)}(t) y(t) dt &= \int_0^T y^{(\alpha)}(t) u(t) dt \\
a_2 \int_0^T y^{(\alpha)}(t) y^{(\beta)}(t) dt + a_1 \int_0^T (y^{(\beta)}(t))^2 dt + a_0 \int_0^T y^{(\beta)}(t) y(t) dt &= \int_0^T y^{(\beta)}(t) u(t) dt \\
a_2 \int_0^T y^{(\alpha)}(t) y(t) dt + a_1 \int_0^T y^{(\beta)}(t) y(t) dt + a_0 \int_0^T (y(t))^2 dt &= \int_0^T y(t) u(t) dt
\end{align*}
\]

where \( y(t) \) and \( u(t) \) are the measured output and input functions of the system, and the derivatives \( y^{(\alpha)}(t) \) and \( y^{(\beta)}(t) \) are the derivatives of the output function of the system, obtained by calculation from the formula (4). In the discretization of the system (5) we substitute the continuous functions \( y(t), y^{(\alpha)}(t), y^{(\beta)}(t), u(t) \) in discrete time instants \( t_m (m = 0, 1, 2, 3, ..., M) \) with their discrete values \( y_m, y_m^{(\alpha)}, y_m^{(\beta)}, u_m \) and the integration we substitute with the summation for the time interval \( T \) for which we have \( M + 1 \) measured values from the system. Then we get

\[
\begin{align*}
a_2 \sum_{m=0}^{M} (y_m^{(\alpha)})^2 + a_1 \sum_{m=0}^{M} y_m^{(\alpha)} y_m^{(\beta)} + a_0 \sum_{m=0}^{M} y_m^{(\alpha)} y_m &= \sum_{m=0}^{M} y_m^{(\alpha)} u_m \\
a_2 \sum_{m=0}^{M} y_m^{(\alpha)} y_m^{(\beta)} + a_1 \sum_{m=0}^{M} (y_m^{(\beta)})^2 + a_0 \sum_{m=0}^{M} y_m^{(\beta)} y_m &= \sum_{m=0}^{M} y_m^{(\beta)} u_m \\
a_2 \sum_{m=0}^{M} y_m^{(\alpha)} y_m + a_1 \sum_{m=0}^{M} y_m^{(\beta)} y_m + a_0 \sum_{m=0}^{M} (y_m)^2 &= \sum_{m=0}^{M} y_m u_m
\end{align*}
\]

From the system of equations (9) we can calculate the unknown parameters \( \bar{a} \), and the task of parameter identification of the system is thus solved provided we know the values \( \alpha \) and \( \beta \).
5. METHOD OF IDENTIFICATION OF FRACTIONAL-ORDER DYNAMICAL SYSTEMS

In general in the systems of the type \( (1) \) it is necessary to identify the orders of derivatives \( \alpha \) and \( \beta \) which are non-integer and real. If, under this assumption, we used the given method of differentiation of transfer characteristics for the system \( (1) \), we would obtain a system of five nonlinear equations. It is difficult to solve such systems of equations, however, therefore we chose a combination of the method of differentiation of transfer characteristics for the calculation of the coefficients \( a_0, a_1, a_2 \) and the method of simple search for the determination of the coefficients \( \alpha \) and \( \beta \). While the first method is analytical, in the second the coefficients are calculated iteratively.

In the identification of the parameters of the model we proceed as follows:

(a) For the parameters \( \alpha \) and \( \beta \) we determine the intervals \( < \alpha_{\text{min}}, \alpha_{\text{max}} > \) and \( < \beta_{\text{min}}, \beta_{\text{max}} > \), in which we look for the parameters. We divide each of these intervals into \( n_{\alpha} \) and \( n_{\beta} \) subintervals

\[
n_{\alpha} = \frac{2(\alpha_{\text{max}} - \alpha_{\text{min}})}{\varepsilon} \quad (10)
\]

\[
n_{\beta} = \frac{2(\beta_{\text{max}} - \beta_{\text{min}})}{\varepsilon} \quad (11)
\]

where \( \varepsilon \) denotes the fineness of division. We choose the starting values \( \alpha \) and \( \beta \) from the chosen intervals.

(b) We use the method of differentiation of the transfer characteristic to calculate for given \( \alpha \) and \( \beta \) the corresponding values of the parameters \( a_0, a_1, a_2 \).

(c) We calculate the criterion of the approximation of experimental output values of the real system \( ye_i \) and the outputs of the model \( y_i \)

\[
Q = \frac{1}{M+1} \sum_{i=0}^{M} [ye_i - y_i]^2
\]

(12)

which serves for the choice of optimum values of the coefficients \( \alpha \) and \( \beta \).

(d) If the value of the actually computed approximation criterion is less than the value of the criterion from the preceding iteration step, we choose for the optimum subintervals those containing last values of the coefficients \( \alpha \) and \( \beta \).

(e) After the calculation of the approximation criterion for all subintervals, we determine from the optimum subintervals the intervals \( < \alpha_{\text{min}}, \alpha_{\text{max}} > \) and \( < \beta_{\text{min}}, \beta_{\text{max}} > \) and for these subintervals the new subintervals. We repeat this procedure until the required accuracy of the calculation of parameters \( \alpha \) and \( \beta \) is achieved, where we have always calculated the values of the parameters \( a_0, a_1, a_2 \) by using the method of differentiation of transfer characteristics.
6. VERIFICATION OF THE METHOD

The verification of the correctness of the described method of parameter identification will be done by using it in systems with known parameters.

Consider an integer-order system [4] with coefficients \( a_2 = 1, a_1 = 3, a_0 = 2, \alpha = 2, \beta = 1 \). The transfer characteristic of this system has aperiodic behavior and is shown in Fig.1. For the chosen intervals \( \alpha \in < 2; 2 > \) and \( \beta \in < 1; 1 > \) the precise values of the coefficients \( a_2 = 1, a_1 = 3, a_0 = 2 \) were calculated with the method of differentiation of transfer characteristics. For the intervals \( \alpha \in < 1,5; 2,55 > \) and \( \beta \in < 0,7; 1,33 > \) the following values of the coefficients were calculated with both methods \( a_2 = 1,0001, a_1 = 2,99987, a_0 = 1,99998, \alpha = 1,99993, \beta = 0,99996 \), which values are very close to the parameters of the identified system.

Consider a fractional-order system with coefficients [4] \( a_2 = 0,8, a_1 = 0,5, a_0 = 1, \alpha = 2,2, \beta = 0,9 \). The transfer characteristic of this system computed according to the relation (5), has periodic behavior and is shown in Fig.2. For the chosen intervals \( \alpha \in < 2,2; 2,2 > \) and \( \beta \in < 0,9; 0,9 > \) the precise values of the coefficients \( a_2 = 0,8, a_1 = 0,5, a_0 = 1 \) were computed with the method of differentiation of transfer characteristics. For the intervals \( \alpha \in < 1,5; 2,55 > \) and \( \beta \in < 0,7; 1,33 > \) the following values were computed by using both methods \( a_2 = 0,80005, a_1 = 0,49996, a_0 = 0,99998, \alpha = 2,19996, \beta = 0,89989 \), which values are very close to the parameters of the identified system.

Let us consider again the previous fractional-order system with coefficients \( a_2 = 0,8, a_1 = 0,5, a_0 = 1, \alpha = 2,2, \beta = 0,9 \), whose behavior is given in Fig.2.
Let us identify it as a second-order system, which was the classical procedure up until now. If we want to identify the system as a second-order system, we have to specify the same upper and lower bounds $\alpha \in < 2; 2 >$ and $\beta \in < 1; 1 >$. With the method of differentiation of transfer characteristics the following coefficient values were calculated: $a_2 = 0.76639$, $a_1 = 0.23184$, $a_0 = 1$. The system with these parameters is actually a classical integer-order model of a fractional-order system; their graphical comparison is shown in Fig.3. Although optically both characteristics are rather close to each other, the use of such model in the synthesis of the parameters of the regulator is associated with unfavorable consequences for the dynamics of the regulation circuit as shown e.g. in [4,5].

Let us now identify the parameters of an experimental heating furnace whose transfer characteristic is shown in Fig. 4. Assuming it is a second order system with $\alpha = 2$, $\beta = 1$, we obtain parameters $a_0 = 1.928$, $a_1 = 4892.733$, $a_2 = -73043.36$ and the value $1.02 \times 10^{-3}$ of the criterion (12). For the chosen intervals $\alpha \in < 1.1; 2.55 >$ and $\beta \in < 0.33; 1.3 >$, the parameters $a_2 = -14994.3$, $a_1 = 6009.52$, $a_0 = 1.69$, $\alpha = 1.31$, $\beta = 0.97$ were identified for this system, and the criterion (12) had the value $2.7 \times 10^{-4}$ (see Fig. 4). As we can see ($a_2$) the previous two approximations are unsuitable. Assuming a two-member differential equation, the parameters take on the values $a_1 = 788.35$, $a_0 = 1.39$, $\beta = 0.73$ and the value of the criterion (12) is $6.3 \times 10^{-4}$.

7. CONCLUSION

The above results of parameter identification for fractional- or integer-order dynamical systems confirm that in known systems the agreement of calculated and actual parameters was very good. In the case of real objects identification it is necessary to pay attention to the accuracy of the output quantity measurement in the identified system, and to the time step of the measurement from the point of view of algorithms.

When identifying the parameters of fractional-order dynamical systems in which real orders of derivatives $\alpha$ and $\beta$ are identified, it is suitable, considering the existence of local minima of the functional (10), to repeat the calculations for different intervals $< \alpha_{\min}, \alpha_{\max} >$ and $< \beta_{\min}, \beta_{\max} >$, in which we look for the parameters, and for different fineness of division $\varepsilon$ in the formulas (10) and (11).

8. REFERENCES

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