Induced Homeomorphism and Atsuji Hyperspaces

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Abstract—Given uniformly homeomorphic metric spaces $X$ and $Y$, it is proven that hyperspaces $C(X)$ and $C(Y)$ are uniformly homeomorphic, where $C(X)$ denotes the collection of all nonempty closed subsets of $X$, and is endowed with Hausdorff distance. Gerald Beer has proved that hyperspace $C(X)$ is an Atsuji space when $X$ is either compact or uniformly discrete. An Atsuji space is a generalization of compact metric spaces as well as of uniformly discrete spaces. In this paper, we investigate space $C(X)$ when $X$ is an Atsuji space, and a class of Atsuji subspaces of $C(X)$ is obtained. Using the results, some fixed point results for continuous maps on Atsuji spaces are obtained.

Keywords: metric space, Hausdorff distance, homeomorphism, Atsuji space, multivalued map

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INTRODUCTION

Let $(W, \tau)$ be a topological space in which every singleton set $\{x\}$ is closed. A hyperspace of space $W$ is the collection $C(W)$ of nonempty closed subsets of $W$, endowed with topology $\tau'$ such that mapping $I : (W, \tau) \to (C(W), \tau')$, defined as $I(x) = \{x\}$, is a homeomorphism onto its range. This suitable topology $\tau'$ is called hypertopology or hyperspace topology [1]. If $W$ is a metric space, then induced Hausdorff distance $H$, an extended real-valued metric on $W$, gives a hypertopology.

Metric space $X$ is said to be an Atsuji space if the set of its limit points $X'$ is compact and for any $\epsilon > 0$ the complement of the $\epsilon$-neighborhood of $X'$ is uniformly discrete; or equivalently, if each continuous map from $X$ to metric space $Y$ is uniformly continuous [2]. The property of being Atsuji lies in between the completeness and the compactness. For the last few decades, the theory of these spaces has attracted the attention of several researchers. These spaces demonstrate several interesting internal characterizations [2–6]. It is well known that if function $f$ from metric space $X$ to metric space $Y$ is uniformly continuous, then it maps fundamental sequences to fundamental sequences. The converse is true if and only if $X$ has an Atsuji completion. For metrizable space $X$, the set of its limit points $X'$ is compact if and only if the topology of space $X$ admits the Atsuji metric [7]; also, $X'$ is separable if and only if the topology admits metric $d$ such that completion $(\hat{X}, d)$ is an Atsuji space [8]. For metric space $(X, d)$, the following conditions are equivalent:

1. $X$ is an Atsuji space.
2. Any pseudo-Cauchy sequence with distinct terms from $X$ has a limit point (see Definition 2).
3. Any open covering of $X$ has a Lebesgue number.
4. Any continuous function defined on $X$ is bounded on set $\{x \in X : I(x) < \delta\}$ for some $\delta > 0$, where $I(x) = \inf_{y \neq x} d(x, y)$.
5. For a pair of disjoint nonempty closed subsets $A, B$ of space $X$, there is $\epsilon > 0$ such that $N_\epsilon(A) \cap N_\epsilon(B) = \emptyset$, where $N_\epsilon(A)$ denotes the $\epsilon$-neighborhood of $A$.
6. For sequence $\{A_n\}$ of subsets $X$, condition $\bigcap_{n=1}^{\infty} A_n = \emptyset$ implies equality $\bigcap_{n=1}^{\infty} N_\epsilon(A_n) = \emptyset$ for some $\epsilon > 0$, where $\overline{A_n}$ is the closure of $A_n$.
Atsuji spaces also have some interesting external characterizations and properties in hyperspace theory [7–10]. We mention only those of them in which Hausdorff distance $H$ is involved. The collection of all nonempty Atsuji subsets of metric space $X$ is a dense subset in Hausdorff hyperspace $(C(X), H)$. For metric space $(X, d)$ the following conditions are equivalent:

1. $X$ is an Atsuji space.
2. The topology of the Hausdorff metric is stronger than the Vietoris topology on $C(X)$.
3. The topology of the Hausdorff metric coincides with the locally finite topology on $C(X)$.

And for completion $\hat{X}$ of metric space $X$ the following statements are equivalent:

1. $\hat{X}$ is an Atsuji space.
2. The topology of the Hausdorff metric is stronger than the Vietoris topology on collection $CP(X)$ of all nonempty complete subsets of $X$.
3. The topology of the Hausdorff metric coincides with the locally finite topology on $CP(X)$.

Atsuji spaces are also known as normal metric spaces [11] and as Lebesgue metric spaces [12, 13].

In [10], Gerald Beer showed that metric space $X$ is either compact or uniformly discrete if and only if its Hausdorff hyperspace $C(X)$ is Atsuji. A compact metric space and a uniformly discrete space are Atsuji spaces. In this paper, we take $X$ to be Atsuji and investigate space $\hat{X}$. It can be seen that space $\hat{X}$ fails to be Atsuji, however $\hat{X}$ contains a class of Atsuji subspaces. These Atsuji subspaces include the completions of those point-finite collections in $X$, which contain all singletons $\{x\}$, $x \in X$.

The paper is organized as follows. Section 1 contains some preliminaries required for the discussion in later sections. In Section 2 it is shown that if $X$ and $Y$ are uniformly homeomorphic spaces, then their corresponding hyperspaces $C(X)$ and $C(Y)$ are also uniformly homeomorphic. Section 3 presents the sufficient conditions for the subspace of $C(X)$ to be Atsuji. In this regard, an open problem for the existence of a maximal Atsuji subspace in $C(X)$ is also placed. Applying the obtained results, in Section 4, we acquire fixed point results for continuous maps on Atsuji spaces.

### 1. PRELIMINARY INFORMATION

Given subset $A$ in metric space $X$, we denote the set of all limit points of $A$ by $A'$, and the complement of $A$ in $X$ by $A^c$ (or $X\setminus A$). An open ball in $X$, centered at $x \in X$ with radius $\epsilon > 0$ is denoted by $B(x, \epsilon)$. Set $\bigcup_{x \in A} B(x, \epsilon)$ is called the $\epsilon$-neighborhood of $A$ and is denoted by $N_\epsilon(A)$.

**Definition 1.** The Hausdorff distance, $H$, of two nonempty subsets $A, B$ of metric space $(X, d)$ is defined as

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A) \right\} = \inf \{ \epsilon > 0 : A \subseteq N_\epsilon(B), B \subseteq N_\epsilon(A) \},$$

where $d(x, A) = \inf_{y \in A} d(x, y)$.

**Definition 2.** Sequence $\{x_n\}$ in metric space $(X, d)$ is called a pseudo-Cauchy sequence if $\forall \epsilon > 0$, $\forall n \in \mathbb{N}$ there are such different $i, j \in \mathbb{N}$ that $i, j > n$ and $d(x_i, x_j) < \epsilon$.

**Definition 3.** Subset $A$ in metric space $(X, d)$ is called uniformly discrete if there exists a number $\epsilon > 0$ such that $d(x, y) > \epsilon$ for all distinct $x, y \in A$.

**Definition 4** Metric space $X$ is said to be an Atsuji space if set of limit points $X'$ is compact and for each $\epsilon > 0$ set $[N_\epsilon(X')]^c$ is uniformly discrete.

**Theorem 1** ([8]). Let $(X, d)$ be a metric space. Completion $(\hat{X}, d)$ is an Atsuji space if and only if every sequence $\{x_n\}$ in $X$ with $\lim_{n \to \infty} I(x_n) = 0$ has a fundamental subsequence, where $I(x) = d(x, X \setminus \{x\})$, $x \in X$.

Topological vector space $X$ is said to be locally convex if there is local base $\mathcal{B}$ at $0$ (the zero vector) whose members are convex.

**Theorem 2** (Tychonoff’s fixed point theorem [14]). Let $A$ be a compact convex subset of a locally convex topological vector space. If $f : A \to A$ is a continuous map, then $f$ has a fixed point.
2. INDUCED MAP ON HYPERSPACES

Consider map \( f : X \rightarrow Y \). Let \( P(X) \) denote the collection of all nonempty subsets of \( X \). Then, induced map \( F : P(X) \rightarrow P(Y) \) is defined as \( A \mapsto \{ f(x) : x \in A \} \), \( A \in P(X) \). For convenience, the induced map, generated by given map \( f \), will be denoted by corresponding capital letter \( F \).

The induced map plays some roles in fixed point theory. In this paper we use this map to find a fixed point for its inducing map. The induced map has been used by Nadler in [15], to discuss the fixed point property of some hyperspaces of certain continua.

For metric spaces \((X, d), (Y, d')\), we denote their corresponding Hausdorff hyperspaces of nonempty closed subsets by \((C(X), H), (C(Y), H')\), respectively; where \( H \) and \( H' \) are Hausdorff distances induced by metrics \( d \) and \( d' \), respectively.

Uniform homeomorphism \( f \) from metric space \( X \) to metric space \( Y \) is a bijective map such that \( f \) and \( f^{-1} \) are uniformly continuous. It can be noted that this definition also makes sense in the case of extended real-valued metrics.

The following result provides a sufficient condition for hyperspace \( C(X) \) of metric space \( X \) to be uniformly homeomorphic to hyperspace \( C(Y) \), as well as is used to derive a fixed point result in Section 4.

**Theorem 3.** Let \( X, Y \) be metric spaces. Then, \( f \) is a uniform homeomorphism from \( X \) to \( Y \) if and only if induced map \( F \) defined on hyperspace \( C(X) \), is a uniform homeomorphism from \( C(X) \) to hyperspace \( C(Y) \).

**Proof.** By continuity of \( f^{-1} \), we have \( F(A) \) closed in \( Y \) for each \( A \in C(X) \).

We show that \( F \) is uniformly continuous. By uniform continuity of \( f \), for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( d'(f(x), f(y)) < \epsilon \) whenever \( d(x, y) < \delta \) for all \( x, y \in X \). Consider \( H(A, B) < \delta \) for some \( A, B \in C(X) \). This implies, \( B(a, \delta) \cap B \neq \emptyset \) for all \( a \in A \), and therefore,

\[ d'(f(a), f(y)) < \epsilon \quad \forall y \in B(a, \delta) \cap B \quad \forall a \in A, \]

which implies, \( \inf_{y \in B(a, \delta) \cap B} d'(f(a), f(y)) < \epsilon \forall a \in A \). And hence, \( \sup \inf_{a \in A} d'(f(a), f(y)) \leq \epsilon \). Similarly, we can prove, \( \sup \inf_{b \in B} d'(f(b), f(y)) \leq \epsilon \). Thus we proved, for each \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( H'(F(A), F(B)) \leq \epsilon \) whenever \( H(A, B) < \delta \forall A, B \in C(X) \).

By continuity and surjectivity of \( f \), for each \( P \in C(Y) \) there is set \( S = \{ x \in X : f(x) \in P \} \), which is nonempty and closed in \( X \). Thus, \( F^{-1} \) exists. Using the uniform continuity of \( f^{-1} \), we can show, as above, that \( F^{-1} \) is uniformly continuous.

Conversely, from the uniform continuities of \( F \) and \( F^{-1} \), it follows that \( f \) and \( f^{-1} \) are uniformly continuous.

\[ \square \]

In [16, 17], Aram Arutyunov with colleagues studied quasi-metrics, \((q_0, q_1)\) quasi-metrics, \( f \)-quasi-metrics, and their topological and geometric properties. In accordance with Corollary 1.3 from [17], subset \( A \) of quasimetric space \((X, \rho)\) belongs to \( C(X) \) if and only if for any sequence with property \( \{ x_n \} \subset A \) with property \( \rho(x_0, x_n) \rightarrow 0 \) we have \( x_0 \in A \). Induced Hausdorff distance \( H_\rho \) on \( C(X) \), given by

\[ H_\rho(A, B) = \inf\{ \epsilon > 0 : A \subset N_\epsilon(B), B \subset N_\epsilon(A) \}, \quad A, B \in C(X), \]

where \( N_\epsilon(A) = \bigcup_{a \in A} \{ y \in X : \rho(a, y) < \epsilon \} \) is an extended real-valued pseudometric on \( C(X) \), i.e., the distance between two distinct points in \((C(X), H_\rho)\) can be zero. In the case of \( q_0 \)-symmetric quasimetric \( \rho \), distance function \( H_\rho \) is an extended real-valued metric on \( C(X) \), and Theorem 3 can be extended to \( q_0 \)-symmetric quasimetric spaces by applying a technique similar to the proof above.

We note that, if \( f \) is a homeomorphism between two metric spaces, then induced map \( F \) need not be a homeomorphism between their Hausdorff hyperspaces.

**Example 1.** Consider spaces \( X = (-1, 1) \) and \( Y = \mathbb{R} \), both endowed with usual metric of \( \mathbb{R} \). Map \( f : (-1, 1) \rightarrow \mathbb{R} \), defined by \( f(x) = x/(1 - |x|) \), is a homeomorphism, but induced map \( F : C(X) \rightarrow C(Y) \) is not. Indeed, the sequence of closed intervals \( \{ [-n/(n+1), n/(n+1)] \} \) is convergent to \((1, -1) \) in \( C(X) \), while sequence \( \{ F([-n/(n+1), n/(n+1)]) \} \) is not convergent to \( F((-1, 1)) \).
The above theorem also deduces a necessary and sufficient condition for two Hausdorff metrics to be uniformly equivalent.

**Corollary 1.** Let \((X, d)\) be a metric space, and \(d'\) be another compatible metric. Then, \(d\) and \(d'\) are uniformly equivalent if and only if Hausdorff distances \(H\) and \(H'\) on \(C(X)\) are uniformly equivalent.

A similar result was mentioned by Beer ([18], Theorem 3.3.2).

### 3. ATSUJI HYPERSPACES

In this section we explore the conditions under which a subspace of hyperspace \(C(X)\) is an Atsuji space. A useful tool to derive some of our results in this section is:

**Lemma 1.** Let two metric spaces be uniformly homeomorphic. If one of the spaces is Atsuji, then other is too.

**Proof.** Let \(f : (X, d) \to (Y, d')\) be a uniform homeomorphism, and \(X\) be an Atsuji space. Clearly, \(Y\) is compact in \(Y\). Denote subset \(N_{\varepsilon}(Y') \subset Y\) by \(S\). If possible, suppose \(S'\) in \(Y\) is not uniformly discrete. Then, there are \(x_n, y_n\) in \(S'\) such that \(d'(x_n, y_n) \to 0\). Since \(f^{-1}(S)\) is open in \(X\), and contains compact set \(X'\), there is \(\varepsilon_1 > 0\) such that \(N_{\varepsilon_1}(X') \subset f^{-1}(S)\). By the uniform continuity of \(f^{-1}\) provision \(d'(x_n, y_n) \to 0\) implies \(d(f^{-1}(x_n), f^{-1}(y_n)) \to 0\), and therefore \([N_{\varepsilon_1}(X')]\) in \(X\) is not uniformly discrete, which is a contradiction.

The above lemma also implies that uniformly equivalent metrics on set \(X\) generate the same Atsuji subspaces of \(X\).

It is to be noted that Atsujierness is not preserved by homeomorphisms.

**Example 2.** Consider set of natural numbers \(\mathbb{N}\) and set \(\mathbb{N}\), both endowed with usual metric \(d\) of \(\mathbb{R}\). Map \(f : (\mathbb{N}, d) \to (\mathbb{N}, d)\), defined as \(f(n) = 1/n\), is a homeomorphism. Space \(\mathbb{N}\) is an Atsuji space while \(\mathbb{N}\) is not.

Subset \(S\) in hyperspace \(C(\mathbb{N})\) is said to be point-finite if each point \(x \in X\) belongs to at most finite number of elements of \(\mathcal{C}\). It is known that a star-finite collection and a locally finite collection of subsets both are point-finite.

We denote a point-finite subset of \(C(\mathbb{N})\) by \(\mathcal{C}\), a point-finite subset of containing all singletons \(\{x\}\), \(x \in \mathbb{N}\), by \(\mathcal{C}\), and a subset of \(C(\mathbb{N})\) containing all singletons \(\{x\}\), \(x \in \mathbb{N}\), by \(\mathcal{C}(\mathbb{N})\).

Given subset \(S\) in hyperspace \(C(\mathbb{N})\), let us denote the \(\varepsilon\)-neighborhood of \(S\), \(\bigcup_{\varepsilon > 0} \{B \in C(\mathbb{N}) : H(A, B) < \varepsilon\}\), by \(N_{\varepsilon}(S)\). Then, space \(C(\mathbb{N})\) is an Atsuji space if the set of limit points \([C(\mathbb{N})]\) is compact, and for each \(\varepsilon > 0\), \([N_{\varepsilon}([C(\mathbb{N})])]\) is uniformly discrete.

For metric space \(X\) and its hyperspace \(C(X)\), Gerald Beer [10] proved that the following are equivalent:

1. \(X\) is either compact or uniformly discrete.
2. Set \([C(X)]\) is compact, and for each \(\varepsilon > 0\), \([N_{\varepsilon}([C(X)])]\) is uniformly discrete.

The class of Atsuji spaces contains compact metric spaces as well as uniformly discrete spaces. Replacing compactness and uniform discreteness of \(X\) in Beer’s result with the Atsujiiness of \(X\), the set of limit points \([C(X)]\) of hyperspace \(C(X)\) may fail to be compact. This is evident from the following:

**Example 3.** Subset \(P = \{e_n/n : m, n \in \mathbb{N}\} \cup \{0\}\) of normed space \((l_2, \|\cdot\|_2)\) is an Atsuji subspace of \(X\). Sequence \(\{0, e_n\}_{n=1}^{\infty}\) is in \([C(P)]\) with no convergent subsequence.

However, set \([N_{\varepsilon}([C(X)])]\) in \(C(X)\) remains uniformly discrete for all \(\varepsilon > 0\), as follows:

**Theorem 4.** Let \(X\) be an Atsuji space. Then, for each space \(C^\varepsilon(X)\) and \(\varepsilon > 0\), the set \([N_{\varepsilon}([C^\varepsilon(X)])]\) is uniformly discrete.
Proof. Suppose for some \( \varepsilon > 0 \), set \([N_\varepsilon((C'(X)))']\) in \( C'(X) \) is not uniformly discrete. So, for each \( n \in \mathbb{N} \), there are \( P_n, Q_n \in [N_\varepsilon((C'(X)))'] \) such that \( H(P_n, Q_n) < 1/n \). Since \( P_n \in [N_\varepsilon((C'(X)))'] \), so \( H(P_n, \{l\}) \geq \varepsilon > \varepsilon - \delta > 0 \). This implies, there exists sequence \( \{p_n\} \), with \( p_n \in P_n \cap [N_\varepsilon(X)'\cap' \] where \( \varepsilon' = \varepsilon - \delta \). Since \( H(P_n, Q_n) < 1/n \), so for sequence \( \{p_n\} \), there is sequence \( \{q_n\} \), with \( q_n \in Q_n \) such that \( d(p_n, q_n) < 1/n \to 0 \). For sequence \( \{q_n\} \), two cases arise: infinitely many points of \( \{q_n\} \) are either in \( X' \) or in \( [X']' \). If \( \{q_n\}^{\infty}_{n=1} \) is in \( X' \), then by the compactness of \( X' \), \( \{q_n\} \) has a limit point in \( X' \). So, sequence \( \{p_n\} \) will have the same limit point as well, in \( [N_\varepsilon(X')]' \), and hence \([N_\varepsilon(X')]'\) is not uniformly discrete, which is a contradiction. If \( \{q_n\}^{\infty}_{n=1} \) is in \( [X']' \) and it has no limit point, then by the compactness of \( X' \) there is \( \delta > 0 \) such that \( \{q_n\}^{\infty}_{n=1} \subset [N_\delta(X')]' \). Indeed, if \( \{q_n\}^{\infty}_{n=1} \subset [N_\delta(X')]' \), then for each \( k \in \mathbb{N} \), there is \( x_k \in X' \) such that \( B(x_k, 1/k) \cap \{q_n\}^{\infty}_{n=1} \neq \emptyset \), and since \( \{x_k\} \) has a convergent subsequence, so \( \{q_n\}^{\infty}_{n=1} \) will have a limit point. Hence, choosing \( \varepsilon_k = \min(\varepsilon', \delta') \), we get subsequences \( \{q_n\}^{\infty}_{n=1} \) and \( \{p_n\}^{\infty}_{n=1} \) are in \( [N_\varepsilon(X')]' \), and so \([N_\varepsilon(X')]'\) is not uniformly discrete, which is a contradiction.

Although, in case of Atsuji space \( X \), there is a class of subsets of \( C(X) \), such that the set of limit points of each subset from this class is compact.

**Theorem 5.** If \( X \) is a metric space for which \( X' \) is complete, then for each subset \( C_f(X) \subset C(X) \), set \([C_f(X)]'\) is complete.

Proof. For \( C \in [C_f(X)]' \), there is a sequence of distinct terms \( \{c_n\} \) in \( C_f(X) \) such that
\[
\forall n \in \mathbb{N} \exists Z_n \in \mathbb{N} : H(C_n, C) < 1/n \ \forall r \geq Z_n.
\]
This implies, \( C \subset N_{1/n}(C_{Z_n}) \) for all \( n \in \mathbb{N} \), and so for all \( c \in C \) for all \( n \in \mathbb{N} \), there is \( c_n \in C_{Z_n} \) such that \( d(c_n, c) < 1/n \). Since collection \( C_f(X) \) is point-finite, so at most finitely many \( c_n \)'s can be equal. Thus, \( c \in X' \), and so \( C \subset C(X') \). Hence, \([C_f(X)]' \subset C(X') \). We know that, for complete metric space \( W \), Hausdorff hyperspace \( C(W) \) is complete ([18], Theorem 3.2.4). Since \( X' \) is complete, set \( C(X') \) is complete in \( C(X) \). And, because set \([C_f(X)]'\) is a closed subset in \( C(X) \), \([C_f(X)]'\) is complete.

A similar proof gives the following.

**Theorem 6.** If \( X \) is a metric space for which \( X' \) is compact, then for each subset \( C_f(X) \subset C(X) \), set \([C_f(X)]'\) is compact.

Subset \( A \) in metric space \( X \) is said to be **totally bounded** if for each \( \varepsilon > 0 \), there are finitely many points \( x_1, x_2, \ldots, x_n \) in \( X \) such that \( A \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon) \).

The following lemma is immediate.

**Lemma 2.** Subset \( A \) in metric space \( X \) is totally bounded if and only if for each \( \varepsilon > 0 \), there are finitely many points \( a_1, a_2, \ldots, a_n \) in \( A \) such that \( A \subset \bigcup_{i=1}^{n} B(a_i, \varepsilon) \).

**Theorem 7.** If \( X \) is an Atsuji space, then a completion of each subspace \( C_{f_{\beta}}(X) \) of \( C(X) \) is Atsuji.

Proof. Let \( \bar{C}_{f_{\beta}}(X) \) denote the closure of \( C_{f_{\beta}}(X) \) in \( C(X) \). Since \( C(X) \) is complete, space \( \bar{C}_{f_{\beta}}(X) \) is a completion of \( C_{f_{\beta}}(X) \). By the proof of Theorem 5, we have \([C_{f_{\beta}}(X)]' \subset C(X') \). Since \( X' \) is totally bounded, \( C(X') \) is totally bounded ([18], Theorem 3.2.4). Hence, by Lemma 2, set \([C_{f_{\beta}}(X)]' \) is totally bounded in space \( \bar{C}_{f_{\beta}}(X) \), which implies \([C_{f_{\beta}}(X)]' \) is compact. And, by Theorem 4, set \( \bar{C}_{f_{\beta}}(X) \cap [N_\varepsilon([C_{f_{\beta}}(X)]')] \) is uniformly discrete. Thus, we proved, \( \bar{C}_{f_{\beta}}(X) \) is an Atsuji space. Since any two completions of a metric space are isometric, so by Lemma 1, each completion of \( C_{f_{\beta}}(X) \) is Atsuji.
Theorem 8. If some $C_{\beta}(X)$ is an Atsuji space, then $X$ is Atsuji.

Proof. Since space $C_{\beta}(X)$ is Atsuji, and set
\[ S(X) = \{x : x \in X\} \]
is a closed subset in $C_{\beta}(X)$, so $S(X)$ is an Atsuji subspace of $C_{\beta}(X)$. Because space $X$ is uniformly homeomorphic to $S(X)$, using Lemma 1, space $X$ is an Atsuji space.

Remark. For metric space $X$ with compact $X'$, collection $A(X)$ of nonempty Atsuji subsets of $X$ need not be a point-finite collection. For, if set $X'$ is compact and collection $A(X)$ is point-finite, then by Theorem 6, $[A(X)]'$ is compact. Since $A(X)$ is dense in $C(X)$ (see [10], p. 657), so $[C(X)]'$ is compact, which is not true in general by Example 3.

Theorem 7 gives rise to a question: What are the maximal Atsuji subspaces in $C(X)$, provided $X$ is Atsuji? We elaborate the question as follows.

Open problem. For given Atsuji space $X$, consider collection $\mathcal{F}$ of the closures of all subsets $C_{\beta}(X)$ of $C(X)$. We endow $\mathcal{F}$ with partial order relation `$\leq$' as follows: For $A, B \in \mathcal{F}$ we put $A \leq B$ if and only if $A \subset B$. Does partially ordered set $(\mathcal{F}, \leq)$ have a maximal element? If yes, can one explicitly find the maximal element?

Theorem 9. Let $(X, d)$ be an Atsuji space. If $d'$ is another compatible Atsuji metric on $X$, then hyperspaces $(C(X), H)$ and $(C(X), H')$ have the same Atsuji subsets.

Proof. By Theorem 2.2 in [9], we get $\tau_H = \tau_{H'}$, where $\tau_H$ is the topology generated by $H$. This implies metrics $d, d'$ are uniformly equivalent. Then, using Corollary 1, we get that metrics $H, H'$ are uniformly equivalent. And, so by Lemma 1, hyperspaces $(C(X), H)$ and $(C(X), H')$ have the same Atsuji subsets.

4. FIXED POINT RESULTS

Here, we discuss the fixed point results for continuous mappings of Atsuji spaces. The fixed point theory has numerous applications in the theory of solvability of differential equations, mathematical analysis, economics, game theory, etc. One of the popular fixed point theorems is Schauder’s theorem. It is used in various ways to solve differential equations (see, for example, [19], p. 154; [14]). It is possible that in all versions of the Schauder fixed point theorem (see [19], Theorem 3.2; [14]) the domain of the function is convex. In the results on fixed points presented by us in this section, for example, in Theorem 12, the domain of the function need not be convex, but additional restrictions are imposed on the function itself. Thus, under the necessary restrictions, these results can be applied in other areas of mathematics and related fields.

For metric space $X$, let set $P(X)$ be endowed with Hausdorff distance $H$. Then, $(P(X), H)$ is an extended-real valued pseudo metric space.

Lemma 3. Let $\{x_n\}$ be a sequence converging to $x$ in metric space $X$. If sequence $\{A_n\}$ converges to $A$ in $P(X)$, then sequence $\{d(x_n, A_n)\}$ converges to $d(x, A)$.

Proof. The proof follows from the continuity of functional $d(., K_i) : X \to \mathbb{R}$ and inequality $d(x, K_i) \leq d(x, K_2) + H(K_1, K_2)$, where $K_1, K_2 \in P(X)$.

Given multivalued map $f$ from metric space $X$ to $P(X)$, point $x \in X$ is said to be an almost fixed point of $f$ if $\inf d(x, y) : y \in f(x) = 0$.

Let $X$ be a complete metric space and let $f : X \to P(X)$ be a continuous map such that for each $\epsilon > 0$ there is an $x \in X$ satisfying $d(x, f(x)) < \epsilon$; then $f$ does not have an almost fixed point, in general. Although, in case of Atsuji space $X$, we have the following.

Theorem 10. Let $X$ be an Atsuji space, and $f : X \to P(X)$ be a continuous map such that for each $\epsilon > 0$, there is an $x \in X$ satisfying $d(x, f(x)) < \epsilon$. Then, $f$ has an almost fixed point.
Proof. The given hypothesis implies, for each \( n \in \mathbb{N} \), there is \( x_n \in X \) such that \( d(x_n, f(x_n)) < 1/n \). If for some \( n_0 \), \( d(x_{n_0}, f(x_{n_0})) = 0 \), then \( f \) has an almost fixed point. Otherwise, for each \( n \in \mathbb{N} \), there is \( y_n \in f(x_n) \) such that \( 0 < d(x_n, y_n) < 1/n + 1/n \). Since \( X \) is an Atsuji space, sequence \( \{x_n\} \) has convergent subsequence \( \{x_{n_i}\}_{i=1}^{\infty} \) converging to some \( x \) in \( X \). Then, by continuity of \( f \), \( f(x_{n_i}) \to f(x) \). Using Lemma 3, we have \( d(x_{n_i}, f(x_{n_i})) \to d(x, f(x)) \). This implies \( d(x, f(x)) = 0 \).

\[ \square \]

In [2], Gerald Beer proved that: If \( X \) is an Atsuji space and \( f : X \to X \) is a continuous map such that for some \( x \in X \), \( \lim\inf_{n \to \infty} d(f^n(x), f^{n+1}(x)) = 0 \), then \( f \) has a fixed point. The following corollary to Theorem 10 provides a generalization of his result.

Corollary 2. Let \( X \) be an Atsuji space, and \( f : X \to C(X) \) be a continuous map such that for each \( \varepsilon > 0 \), there is an \( x \in X \) satisfying \( d(x, f(x)) < \varepsilon \). Then, \( f \) has a fixed point.

Theorem 11. Let \( X \) be an Atsuji space, and for \( C_{\beta}(X) \subset C(X) \), \( f : X \to \overline{C_{\beta}(X)} \) be a map such that for each \( \varepsilon > 0 \), there is an \( x \in X \) satisfying \( H(\{x\}, f(x)) < \varepsilon \). If \( f^{-1} \) exists and is continuous, then \( f \) has a fixed point.

Proof. Since \( X \) is an Atsuji space, so by Theorem 7, \( \overline{C_{\beta}(X)} \) is an Atsuji space. By the given hypothesis we have, for each \( n \in \mathbb{N} \), there is an \( x_n \in X \) such that \( H(\{x_n\}, f(x_n)) < 1/n \). If for some \( n_0 \in \mathbb{N} \), \( H(\{x_{n_0}\}, f(x_{n_0})) = 0 \), then \( x_{n_0} \) is a fixed point for \( f \). Otherwise, due to Atsujianness of \( \overline{C_{\beta}(X)} \), sequence \( \{f(x_n)\} \) has some convergent subsequence \( \{f(x_{n_i})\}_{i=1}^{\infty} \) converging to some \( A \in \overline{C_{\beta}(X)} \). Then, by continuity of \( f^{-1} \), sequence \( \{x_{n_i}\}_{i=1}^{\infty} \subset X \) is convergent to \( x := f^{-1}(A) \). Using Lemma 3, we have \( d(x, A) = 0 \), which implies \( x \in A = f(x) \).

CORFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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