Sums of two triangularizable quadratic matrices over an arbitrary field

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Abstract

Let \( K \) be an arbitrary field, and \( a, b, c, d \) be elements of \( K \) such that the polynomials \( t^2 - at - b \) and \( t^2 - ct - d \) are split in \( K[t] \). Given a square matrix \( M \in M_n(K) \), we give necessary and sufficient conditions for the existence of two matrices \( A \) and \( B \) such that \( M = A + B, \ A^2 = aA + bI_n \) and \( B^2 = cB + dI_n \). Prior to this paper, such conditions were known in the case \( b = d = 0, a \neq 0 \) and \( c \neq 0 \) [4] and in the case \( a = b = c = d = 0 \) [1]. Here, we complete the study, which essentially amounts to determining when a matrix is the sum of an idempotent and a square-zero matrix. This generalizes results of Wang [5] to an arbitrary field, possibly of characteristic 2.

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1 Introduction

1.1 Basic notations and aims

Let \( K \) be an arbitrary field, and \( \overline{K} \) an algebraic closure of it. We denote by \( \text{car}(K) \) the characteristic of \( K \). We denote by \( M_n(K) \) the algebra of square matrices with

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n rows and entries in \( \mathbb{K} \), and by \( I_n \) its identity matrix. Similarity of two square matrices \( A \) and \( B \) is denoted by \( A \sim B \). Given \( M \in M_n(\mathbb{K}) \), we denote by \( \text{Sp}(M) \) the set of eigenvalues of \( M \) in the field \( \mathbb{K} \). We denote by \( \mathbb{N} \) the set of non-negative integers, and by \( \mathbb{N}^* \) the set of positive ones.

A matrix of \( M_n(\mathbb{K}) \) is called **quadratic** when it is annihilated by a polynomial of degree two. More precisely, given a pair \( (a, b) \in \mathbb{K}^2 \), a matrix \( A \) of \( M_n(\mathbb{K}) \) is called \( (a, b) \)-**quadratic** when \( A^2 = aA + bI_n \). In particular, a matrix is \((1, 0)\)-quadratic if and only if it is idempotent, and it is \((0, 0)\)-quadratic if and only if it is square-zero.

Let \( (a, b, c, d) \in \mathbb{K}^4 \). A matrix is called an \((a, b, c, d)\)-**quadratic sum** when it may be decomposed as the sum of an \((a, b)\)-quadratic matrix and of a \((c, d)\)-quadratic one. Note that a matrix which is similar to an \((a, b, c, d)\)-quadratic sum is an \((a, b, c, d)\)-quadratic sum itself. Our aim here is to give necessary and sufficient conditions for a matrix of \( M_n(\mathbb{K}) \) to be an \((a, b, c, d)\)-quadratic sum.

In \[5\], Wang has expressed such conditions in terms of rational canonical forms when \( \mathbb{K} \) is the field of complex numbers, and his proof actually encompasses the more general case of an algebraically closed field of characteristic not 2. In our recent \[4\], we have worked out the case \( b = d = 0 \), \( a \neq 0 \) and \( c \neq 0 \), i.e., we have determined when a matrix may be written as \( aP + cQ \), where \( P \) and \( Q \) are idempotent matrices (this generalized earlier results of Hartwig and Putcha \[3\]). In \[1\], Botha has worked out the case \( a = b = c = d = 0 \) for an arbitrary field, generalizing results of Wang and Wu \[6\]; as in \[4\], fields of characteristic 2 yield somewhat different results than the others.

The purpose of this paper is to solve the remaining cases, assuming that the polynomials \( t^2 - at - b \) and \( t^2 - ct - d \) are split over \( \mathbb{K} \).

The basic strategy is to reduce the situation to a more elementary one. Assume, for the rest of the section, that \( t^2 - at - b \) and \( t^2 - ct - d \) are split over \( \mathbb{K} \), and let \( \alpha \) be a root of \( t^2 - at - b \) and \( \beta \) be one of \( t^2 - ct - d \). Then an \((a, b)\)-quadratic matrix is a matrix of the form \( \alpha I_n + P \), where \( P \) is \((a - 2\alpha, 0)\)-quadratic. We deduce that a matrix of \( M_n(\mathbb{K}) \) is an \((a, b, c, d)\)-quadratic sum if and only if it splits as \((\alpha + \beta)I_n + M \), where \( M \) is an \((a - 2\alpha, 0, c - 2\beta, 0)\)-quadratic sum.

We are thus reduced to studying the case \( b = d = 0 \).

In the case \( b = d = 0 \) and \( a \neq 0 \), notice furthermore that an \((a, b, c, d)\)-quadratic sum is simply the product of \( a \) with a \((1, 0, \frac{c}{a}, 0)\)-quadratic sum. Therefore, the case \( b = d = 0 \) is essentially reduced to three cases:
(i) \( b = d = 0, \ a \neq 0 \) and \( c \neq 0 \);
(ii) \( a = b = c = d = 0 \);
(iii) \( a = 1 \) and \( b = c = d = 0 \).

Case (i) has been dealt with in [4], and case (ii) more recently in [1]. Therefore, only case (iii) remains to be studied in order to complete the case where both polynomials \( t^2 - at - b \) and \( t^2 - ct - d \) are split over \( K \). In other words, it remains to determine which matrices may be decomposed as the sum of an idempotent and a square-zero matrix. This has been done by Wang in [5] for the case \( K = \mathbb{C} \). Our aim is to generalize his results.

1.2 Main theorem

Definition 1. Let \((u_n)_{n \geq 1}\) and \((v_n)_{n \geq 1}\) be two non-increasing sequences of non-negative integers. Let \( p > 0 \) be a positive integer. We say that \((u_n)\) and \((v_n)\) are \( p \)-intertwined when
\[
\forall n \geq 1, \ u_{n+p} \leq v_n \quad \text{and} \quad v_{n+p} \leq u_n.
\]

Notation 2. Given \( A \in M_n(K) \), \( \lambda \in \mathbb{K} \) and \( k \in \mathbb{N}^* \), we set
\[
n_k(A, \lambda) := \dim \ker(A - \lambda I_n)^k - \dim \ker(A - \lambda I_n)^{k-1},
\]
and
\[
j_k(A, \lambda) := n_k(A, \lambda) - n_{k+1}(A, \lambda)
\]
i.e., \( n_k(A, \lambda) \) (respectively, \( j_k(A, \lambda) \)) is the number of blocks of size \( k \) or more (respectively, of size \( k \)) associated to the eigenvalue \( \lambda \) in the Jordan reduction of \( A \).

Our main theorem follows.

Theorem 1. Let \( M \in M_n(K) \). The following conditions are equivalent:

(i) \( M \) is a \( (1, 0, 0, 0) \)-quadratic sum.
(ii) \( \forall \lambda \in \mathbb{K} \setminus \{0, 1\}, \ \forall k \in \mathbb{N}^* , \ j_k(M, \lambda) = j_k(M, 1 - \lambda) \), the sequences \( (n_k(M, 0))_{k \geq 1} \) and \( (n_k(M, 1))_{k \geq 1} \) are 2-intertwined, and, if \( \text{car}(K) \neq 2 \), the Jordan blocks of \( M \) for the eigenvalue \( \frac{1}{2} \) are all even-sized.
(iii) There are matrices $A \in M_p(\mathbb{K})$ and $B \in M_{n-p}(\mathbb{K})$ such that $M \sim A \oplus B$, where all the invariant factors of $A$ are polynomials of $t(t-1)$ and $A$ has no eigenvalue in $\{0, 1\}$, the matrix $B$ is triangularizable with $\text{Sp}(B) \subset \{0, 1\}$, and the sequences $(n_k(B,0))_{k \geq 1}$ and $(n_k(B,1))_{k \geq 1}$ are 2-intertwined.

1.3 Structure of the proof

The equivalence between conditions (ii) and (iii) of Theorem 1 is a straightforward consequence of the kernel decomposition theorem and of Proposition 9 of [4], which we restate:

**Proposition 2.** Let $A \in M_n(\mathbb{K})$ and $\alpha \in \mathbb{K}$. The following conditions are equivalent:

(i) The invariant factors of $A$ are polynomials of $t(t-\alpha)$.

(ii) For every $\lambda \in \mathbb{K}$,

- if $\lambda \neq \alpha - \lambda$, then $\forall k \in \mathbb{N}^*, j_k(A, \lambda) = j_k(A, \alpha - \lambda)$;
- if $\lambda = \alpha - \lambda$, then $\forall k \in \mathbb{N}, j_{2k+1}(A, \lambda) = 0$.

The equivalence of (i) and (iii) is much more involving and takes up the rest of the paper:

- In Section 2, we show that the equivalence (i) $\Leftrightarrow$ (iii) needs to be proven only in the following elementary cases:
  - (a) $M$ has no eigenvalue in $\{0, 1\}$;
  - (b) $M$ is triangularizable and $\text{Sp}(M) \subset \{0, 1\}$.

- In Section 3, we prove that (i) $\Leftrightarrow$ (iii) holds in case (a).

- In Section 4, we prove that (i) $\Leftrightarrow$ (iii) holds in case (b).

2 Reduction and reconstruction principles

2.1 A reconstruction principle

Let $M_1$ and $M_2$ be two $(1, 0, 0, 0)$-quadratic sums (respectively in $M_{n}(\mathbb{K})$ and $M_{p}(\mathbb{K})$). Split up $M_1 = A_1 + B_1$ and $M_2 = A_2 + B_2$, where $A_1, A_2$ are idempotent
and $B_1, B_2$ are square-zero. Then $M_1 \oplus M_2 = (A_1 \oplus A_2) + (B_1 \oplus B_2)$, while $A_1 \oplus A_2$ is idempotent and $B_1 \oplus B_2$ is square-zero. Therefore $M_1 \oplus M_2$ is a $(1, 0, 0, 0)$-quadratic sum.

### 2.2 The basic lemma

The following lemma is a key tool to analyze quadratic sums in general.

**Lemma 3.** Let $(a, b, c, d) \in \mathbb{K}^4$. Let $A$ and $B$ be respectively an $(a, b)$-quadratic and a $(c, d)$-quadratic matrix of $M_n(\mathbb{K})$. Then $A$ and $B$ both commute with $(A + B)((a + c)I_n - (A + B))$.

**Proof.** Let $C := (A + B)((a + c)I_n - (A + B))$ and note that $C = (a + c)(A + B) - A^2 - B^2 - AB - BA = -(b + d)I_n + cA + aB - AB - BA$.

Therefore

$$AC - CA = a(AB - BA) - A^2B + BA^2 = -bB + bB = 0$$

and by symmetry $BC - CB = 0$.

**Corollary 4.** Let $(A, B) \in M_n(\mathbb{K})^2$ such that $A^2 = A$ and $B^2 = 0$. Then $A$ and $B$ both commute with $(A + B)(A + B - I_n)$.

### 2.3 Reduction to elementary cases

Let $M \in M_n(\mathbb{K})$. The minimal polynomial $\mu$ of $M$ splits up as

$$\mu(t) = P(t) t^p (t - 1)^q,$$

where $P(t)$ has no root in $\{0, 1\}$ and $(p, q) \in \mathbb{N}^2$. Let $M_1$ (respectively, $M_2$) be a matrix associated to the endomorphism $X \mapsto MX$ on the vector space $\text{Ker} P(M)$ (respectively, on the vector space $\text{Ker} M^p(M - I_n)^q$). By the kernel decomposition theorem, one has

$$M \sim M_1 \oplus M_2,$$

while $P(M_1) = 0$ and $t^p(t - 1)^q$ annihilates $M_2$. If implication (iii) $\Rightarrow$ (i) holds for $M_1$ and $M_2$, then the reconstruction principle of Section 2.1 shows that it also holds for $M$.

Conversely, assume that $M = A + B$ for a pair $(A, B) \in M_n(\mathbb{K})^2$ with $A^2 = A$ and $B^2 = 0$. By Corollary 4, $A$ and $B$ both commute with $M(M - I_n)$, and
hence they stabilize the subspaces $\text{Im}(M(M - I_n))^n$ and $\text{Ker}(M(M - I_n))^n$ in the Fitting decomposition of $M(M - I_n)$. Using an adapted basis of $K^n$ for this decomposition, we find $P \in \text{GL}_n(K)$, an integer $p \geq 0$, matrices $A_1, B_1$ in $M_p(K)$ and matrices $A_2, B_2$ in $M_{n-p}(K)$ such that

$$A = P(A_1 \oplus A_2)P^{-1} \quad \text{and} \quad B = P(B_1 \oplus B_2)P^{-1},$$

the matrices $M_1 := A_1 + B_1$ and $M_2 := A_2 + B_2$ being both $(1, 0, 0, 0)$-quadratic sums, with $M_1(M_1 - I_p)$ non-singular and $M_2(M_2 - I_{n-p})$ nilpotent. In other words, $M_1$ has no eigenvalue in $\{0, 1\}$ and $M_2$ is triangularizable with $\text{Sp}(M_2) \subset \{0, 1\}$. If implication $(i) \Rightarrow (iii)$ holds for both $M_1$ and $M_2$, then it clearly holds for $M$.

We conclude that equivalence $(i) \Leftrightarrow (iii)$ needs to be proven only in the following special cases:

(a) $M$ has no eigenvalue in $\{0, 1\}$;
(b) $M$ is triangularizable with $\text{Sp}(M) \subset \{0, 1\}$.

### 3 The case $M$ has no eigenvalue in $\{0, 1\}$

#### 3.1 A lemma on companion matrices

**Notation 3.** Given a monic polynomial $P = t^n - a_{n-1} t^{n-1} - \cdots - a_1 t - a_0 \in K[t]$, we denote its *companion matrix* by

$$C(P) := \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & a_{n-2} \\ 0 & \cdots & \cdots & 0 & 1 & a_{n-1} \end{bmatrix} \in M_n(K).$$

**Notation 4.** For $E \in M_p(K)$, we set

$$U_E := \begin{bmatrix} I_p & E \\ I_p & 0_p \end{bmatrix} \in M_{2p}(K).$$

We start with two easy lemmas on the matrices of type $U_E$. 

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Lemma 5. Given two similar matrices \( E \) and \( E' \) of \( M_p(\mathbb{K}) \), the matrices \( U_E \) and \( U_{E'} \) are similar.

Proof. Choosing \( R \in \mathrm{GL}_p(\mathbb{K}) \) such that \( E' = R E R^{-1} \), a straightforward computation shows that 
\[
U_{E'} = (R \oplus R) U_E (R \oplus R)^{-1}.
\]

Conjugating by a well-chosen permutation matrix, the following result is straightforward:

Lemma 6. Given square matrices \( A \) and \( B \), one has \( U_{A \oplus B} \sim U_A \oplus U_B \).

We now examine the case \( E \) is a companion matrix. The following lemma generalizes Lemma 14 of [4] and is the key to equivalence (i) \( \iff \) (iii) in Theorem 1 for a matrix with no eigenvalue in \( \{0, 1\} \):

Lemma 7. Let \((\alpha, \beta) \in \mathbb{K}^2\). Let \( P(t) \) be a monic polynomial of degree \( n \). Then
\[
\begin{bmatrix}
\alpha I_n & C(P) \\
I_n & \beta I_n
\end{bmatrix} \sim C(P((t-\alpha)(t-\beta))).
\]

Lemma 7 was stated and proved in [4] with the extra condition that \( \alpha \neq 0 \) and \( \beta \neq 0 \), but an inspection of the proof shows that this condition is unnecessary.

Corollary 8. Let \( P \in \mathbb{K}[t] \) be a monic polynomial. Then the companion matrix \( C(P(t(t-1))) \) is a \((1,0,0,0)\)-quadratic sum.

Proof. Indeed, Lemma 7 shows, with \( n := \deg P \), that 
\[
C(P(t(t-1))) \sim A + B \quad \text{with} \quad A = \begin{bmatrix} I_n & 0_n \\ I_n & 0_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0_n & C(P) \\ 0_n & 0_n \end{bmatrix}.
\]

Obviously, \( A^2 = A \) and \( B^2 = 0 \), and hence \( C(P(t(t-1))) \) is the sum of an idempotent and a square-zero matrix.

3.2 Application to \((1,0,0,0)\)-quadratic sums

Let \( M \in M_n(\mathbb{K}) \).
• Assume that each invariant factor of $M$ is a polynomial of $t(t - 1)$. Then we may find monic polynomials $P_1, \ldots, P_p$ such that

$$M \sim C(P_1(t(t - 1))) \oplus \cdots \oplus C(P_p(t(t - 1))).$$

Using Corollary 8 and the reconstruction principle of Section 2.1, we deduce that $M$ is a $(1,0,0,0)$-quadratic sum.

• Conversely, assume that $M = A + B$ for some pair $(A, B) \in M_n(\mathbb{K})^2$ such that $A^2 = A$ and $B^2 = 0$. Assume furthermore that $M$ has no eigenvalue in $\{0, 1\}$. This last assumption yields

$$\text{Ker } A \cap \text{Ker } B = \text{Ker}(A - I_n) \cap \text{Ker } B = \{0\}.$$  

Therefore

$$\dim \text{Ker } A \leq n - \dim \text{Ker } B = \text{rk } B \quad \text{and} \quad \dim \text{Ker}(A - I_n) \leq \text{rk } B.$$  

Adding these inequalities yields $n \leq 2 \text{rk } B$. However $2 \text{rk } B \leq \text{rk } B + \dim \text{Ker } B = n$ since $\text{Im } B \subset \text{Ker } B$. It follows that

$$\dim \text{Ker } A = \dim \text{Ker}(A - I_n) = \dim \text{Ker } B = \text{rk } B = \frac{n}{2}$$

and hence

$$\mathbb{K}^n = \text{Ker } A \oplus \text{Ker } B.$$  

Set now $p := \frac{n}{2}$. Using a basis of $\mathbb{K}^{2p}$ which is adapted to the decomposition $E = \text{Ker } B \oplus \text{Ker } A$, we find $P \in \text{GL}_n(\mathbb{K})$ and matrices $C, D$ in $M_p(\mathbb{K})$ such that

$$A = P \begin{bmatrix} I_p & 0_p \\ C & 0_p \end{bmatrix} P^{-1} \quad \text{and} \quad B = P \begin{bmatrix} 0_p & D \\ 0_p & 0_p \end{bmatrix} P^{-1}.$$  

Using $\text{Ker}(A - I_n) \cap \text{Ker } B = \{0\}$, we find that $C$ is non-singular. Setting $Q := \begin{bmatrix} I_p & 0_p \\ 0 & C \end{bmatrix}$, we finally find some $D' \in M_p(\mathbb{K})$ such that

$$M = (PQ) \begin{bmatrix} I_p & D' \\ I_p & 0_p \end{bmatrix} (PQ)^{-1} \sim U_{D'}.$$  

The rational canonical form of $D'$ yields monic polynomials $P_1, \ldots, P_q$ such that $D' \sim C(P_1) \oplus \cdots \oplus C(P_q)$ and $P_k$ divides $P_{k+1}$ for every $k \in \{1, \ldots, q - 1\}$. By Lemmas 5 and 6 this yields

$$M \sim U_{C(P_1)} \oplus \cdots \oplus U_{C(P_q)}.$$  

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Using Corollary 8, it follows that

\[ M \sim C(P_1(t(t-1))) \oplus \cdots \oplus C(P_q(t(t-1))). \]

Finally, \( P_k(t(t-1)) \) divides \( P_{k+1}(t(t-1)) \) for every \( k \in \{1, \ldots, q-1\} \), and hence \( P_1(t(t-1)), \ldots, P_q(t(t-1)) \) are the invariant factors of \( M \). Since \( M \) has no eigenvalue in \( \{0,1\} \), we conclude that \( M \) satisfies condition (iii) in Theorem 1.

We conclude that equivalence (i) \( \Leftrightarrow \) (iii) of Theorem 1 holds for any square matrix with no eigenvalue in \( \{0,1\} \).

4 The case \( M \) is triangularizable with eigenvalues in \( \{0,1\} \)

4.1 A review of Wang’s results

In [5, Lemma 2.3], Wang proved the following characterization of pairs of nilpotent matrices \((M, N)\) for which the sequences \((n_k(M,0))_{k \geq 1}\) and \((n_k(N,0))_{k \geq 1}\) are \(p\)-intertwined (generalizing a famous theorem of Flanders [2]).

**Theorem 9** (Wang). Let \( p \in \mathbb{N}^* \) and \((M, N) \in M_r(\mathbb{K}) \times M_s(\mathbb{K})\) be a pair of nilpotent matrices. The following conditions are equivalent:

(i) The sequences \((n_k(M,0))_{k \geq 1}\) and \((n_k(N,0))_{k \geq 1}\) are \(p\)-intertwined.

(ii) There is a pair \((X, Y) \in M_{r,s}(\mathbb{K}) \times M_{s,r}(\mathbb{K})\) such that \(M^p = XY\), \(N^p = YX\), \(MX = XN\) and \(YM = NY\).

Wang only considered the field of complex numbers but an inspection of his proof reveals that it holds for an arbitrary field.

In [5], implication (i) \( \Rightarrow \) (ii) of Theorem 9 is used, with \( p = 2 \), to obtain the following result:

**Proposition 10.** Let \( M \in M_n(\mathbb{K}) \) be a triangularizable matrix with eigenvalues in \( \{0,1\} \) and assume that the sequences \((n_k(M,0))_{k \geq 1}\) and \((n_k(M,1))_{k \geq 1}\) are 2-intertwined. Then \( M \) is a \((1,0,0,0)\)-quadratic sum.

Again, Wang’s proof in [5, Lemma 2.2, “Sufficiency” paragraph] holds for an arbitrary field and we shall not reproduce it. We deduce that implication (iii) \( \Rightarrow \) (i) in Theorem 1 holds when \( M \) is triangularizable with eigenvalues in \( \{0,1\} \).
4.2 A necessary condition for being a \((1, 0, 0, 0)\)-quadratic sum

Here, we prove the converse of Proposition 10:

**Proposition 11.** Let \(M \in M_n(\mathbb{K})\) be a triangularizable matrix with eigenvalues in \(\{0, 1\}\). Assume that \(M\) is a \((1, 0, 0, 0)\)-quadratic sum. Then the sequences \((n_k(M, 0))_{k \geq 1}\) and \((n_k(M, 1))_{k \geq 1}\) are 2-intertwined.

Proving this will complete our proof of Theorem 1.

In [5], Wang proved Proposition 11 in the special case \(\mathbb{K} = \mathbb{C}\). An inspection shows that his proof works for an arbitrary field of characteristic not 2, but fails for a field of characteristic 2 (due to Wang’s systematic use of the division by 2). Our aim is to give a proof that works regardless of the characteristic of \(\mathbb{K}\). In order to do this, we will reduce the situation to the one where no Jordan block of \(M\) has a size greater than 3 (in other words \(M^3(M - I_n)^3 = 0\)). Let us start by considering that special case:

**Lemma 12.** Let \(M \in M_n(\mathbb{K})\) be a \((1, 0, 0, 0)\)-quadratic sum such that \(M^3(M - I_n)^3 = 0\). Then \(n_3(M, 0) \leq n_1(M, 1)\) and \(n_3(M, 1) \leq n_1(M, 0)\).

**Proof.** We lose no generality in assuming that

\[
M = \begin{bmatrix} I_p + N & 0 & 0 \\ 0 & N' \end{bmatrix},
\]

where \(p + q = n\), \((N, N') \in M_p(\mathbb{K}) \times M_q(\mathbb{K})\), and \(N^3 = 0\) and \((N')^3 = 0\).

With the same block sizes, we may find some \(B = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix} \in M_n(\mathbb{K})\) such that \(B^2 = 0\) and \((M - B)^2 = M - B\). By Corollary 4 \(B\) commutes with \(M(M - I_n) = \begin{bmatrix} N^2 + N & 0 \\ 0 & (N')^2 - N' \end{bmatrix}\). It follows that \(B_1\) commutes with \(N + N^2\), whilst \(B_4\) commutes with \(N' - (N')^2\).

However \(N = (N + N^2) - (N + N^2)^2\) and \(N' = (N' - (N')^2) + (N' - (N')^2)^2\).

Therefore \(B_1\) commutes with \(N\), and \(B_4\) commutes with \(N'\).

Next, the identities \((M - B)^2 = M - B\) and \(B^2 = 0\) yield:

\[
M^2 - MB - BM = M - B.
\]

We deduce:

\[
N'B_2 + B_2N = 0; \quad NB_3 + B_3N' = 0,
\]
\[ N^2 + N = NB_1 + B_1N + B_1 = (2N + I_p)B_1 \quad \text{and} \quad \left(N'ight)^2 - N' = (2N' - I_q)B_4. \]

Therefore
\[ B_1 = (I_p + 2N)^{-1}(N + N^2) = (I_p - 2N + 4N^2)(N + N^2) = N - N^2 \]
and
\[ B_4 = (I_q - 2N')^{-1}(N' - (N')^2) = (I_q + 2N' + 4(N')^2)(N' - (N')^2) = N' + (N')^2. \]

Using this, we compute
\[ B_2^2 = \begin{bmatrix} N^2 + B_3B_2 & \quad ? \\ \quad ? & \quad (N')^2 + B_2B_3 \end{bmatrix}. \]

Since \( B_2 = 0 \), we deduce that
\[ N^2 = (-B_3)B_2 \quad \text{and} \quad (-N')^2 = B_2(-B_3). \]

Recalling that
\[ (-N')B_2 = B_2N \quad \text{and} \quad N(-B_3) = (-B_3)(-N'), \]
Theorem 9 yields \( n_3(N, 0) \leq n_1(-N', 0) \) and \( n_3(-N', 0) \leq n_1(N, 0) \), i.e., \( n_3(M, 1) \leq n_1(M, 0) \) and \( n_3(M, 0) \leq n_1(M, 1) \).

We finish by deducing the general case from the above special one:

**Proof of Proposition 11.** We think in terms of endomorphisms of the space \( \mathbb{K}^n \).

Let \( u \) be an endomorphism of \( \mathbb{K}^n \) such that \( u^n(u - \text{id})^n = 0 \), and assume that there is an idempotent endomorphism \( a \) and a square-zero endomorphism \( b \) such that \( u = a + b \).

By Corollary 4, \( E_k := \ker(u^k(u - \text{id})^k) \) is stabilized by \( a \) and \( b \) for every \( k \in \mathbb{N} \).

Let \( k \in \mathbb{N} \). Then \( a \), \( b \), and \( u \) induce endomorphisms \( a' \), \( b' \), and \( u' \) of \( E_{k+3}/E_k \), with \( (a')^2 = a' \), \( (b')^2 = 0 \), and \( (u')^3(u' - \text{id})^3 = 0 \) (as \( u^3(u - \text{id})^3 \) maps \( E_{k+3} \) into \( E_k \)). Applying Lemma 12 to \( u' \), we find that \( n_3(u', 1) \leq n_1(u', 0) \) and \( n_3(u', 0) \leq n_1(u', 1) \).

In order to conclude, it suffices to note that
\[ \forall i \in \{1, 2, 3\}, \ n_i(u', 0) = n_{k+1}(u, 0) \quad \text{and} \quad n_i(u', 1) = n_{k+1}(u, 1). \]

Note indeed, using the kernel decomposition theorem, that the characteristic subspace of \( u' \) for the eigenvalue 0 is \( (\ker u^{k+3} \oplus \ker(u - \text{id})^k)/(\ker u^k \oplus \ker(u - \text{id})^k) \).
id)^k), and hence the nilpotent part of \( u' \) is similar to the endomorphism \( v : x \mapsto u(x) \) of \( \text{Ker}\ u^{k+3}/\text{Ker}\ u^k \). However \( \text{Ker}\ v^i = \text{Ker}\ u^{k+i}/\text{Ker}\ u^k \) for every \( i \in \{0, 1, 2, 3\} \). Therefore

\[
n_i(u', 0) = n_i(v, 0) = \left( \dim \text{Ker}\ u^{k+i} - \dim \text{Ker}\ u^k \right) - \left( \dim \text{Ker}\ u^{k+i-1} - \dim \text{Ker}\ u^k \right) = n_{k+i}(u, 0)
\]

for every \( i \in \{1, 2, 3\} \). In the same way, one proves that \( n_i(u', 1) = n_{k+i}(u, 1) \) for every \( i \in \{1, 2, 3\} \).

The special cases \( i = 1 \) and \( i = 3 \) yield \( n_{k+3}(u, 1) \leq n_{k+1}(u, 0) \) and \( n_{k+3}(u, 0) \leq n_{k+1}(u, 1) \).

This completes our proof of Theorem 1.

5 Addendum: a simplified proof of a result on linear combinations of idempotent matrices

In this last section, we wish to show how the strategy of Section 4.2 may be adapted so as to yield a simplified proof of the following result of [4]:

**Proposition 13.** Let \( \alpha, \beta \) be distinct elements of \( \mathbb{K} \setminus \{0\} \). Let \( M \in M_n(\mathbb{K}) \) be an \((\alpha, 0, \beta, 0)\)-quadratic sum such that \((M - \alpha I_n)^n(M - \beta I_n)^n = 0\). Then the sequences \((n_k(M, \alpha))_{k \geq 1}\) and \((n_k(M, \beta))_{k \geq 1}\) are 1-intertwined.

**Proof.** As in the proof of Proposition 11 one can use the commutation with \((M - \alpha I_n)(M - \beta I_n) = M(M - (\alpha + \beta)I_n) + \alpha\beta I_n\) (see Lemma 3) to reduce the situation to the one where \((M - \alpha I_n)^2(M - \beta I_n)^2 = 0\). In that case, we lose no generality in assuming that

\[
M = (\alpha I_p + N) \oplus (\beta I_q + N'),
\]

where \( p + q = n \), \( N \in M_p(\mathbb{K}) \) and \( N' \in M_q(\mathbb{K}) \) satisfy \( N^2 = 0 \) and \( (N')^2 = 0 \). Note that

\[
(M - \alpha I_n)(M - \beta I_n) = (\alpha - \beta)(N \oplus (-N')).
\]

Let then \( A \) and \( B \) be idempotent matrices such that \( M = \alpha A + \beta B \). Split

\[
A = \begin{bmatrix} A_1 & A_3 \\ A_2 & A_4 \end{bmatrix},
\]

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where $A_1, A_2, A_3, A_4$ are respectively $p \times p$, $q \times p$, $p \times q$ and $q \times q$ matrices. By Lemma 3 $A$ commutes with $(M - \alpha I_n)(M - \beta I_n)$; as $\alpha \neq \beta$, we deduce that $A_1$ commutes with $N$.

On the other hand, the identity $(M - \alpha A)^2 = \beta(M - \alpha A)$ yields:

$$\alpha(\alpha + \beta)A = \alpha (AM + MA) + \beta M - M^2.$$ 

Evaluating the upper-left blocks on both sides and using the commutation $A_1N = NA_1$, we deduce:

$$\alpha(\alpha + \beta)A_1 = 2\alpha(\alpha I_n + N)A_1 + \beta(\alpha I_n + N) - (\alpha I_n + N)^2$$

and hence

$$\alpha((\beta - \alpha)I_n - 2N)A_1 = \alpha(\beta - \alpha)I_n + (\beta - 2\alpha)N.$$

As $\alpha(\beta - \alpha) \neq 0$ and $N^2 = 0$, we deduce that

$$A_1 = \left( I_n + \frac{\beta - 2\alpha}{\alpha(\beta - \alpha)} N \right) \left( I_n - \frac{2}{\beta - \alpha} N \right)^{-1} = I_n + \frac{\beta}{\alpha(\beta - \alpha)} N,$$

and it follows that the upper-left block of $B$ is $\frac{1}{\beta}(\alpha I_n + N - \alpha A_1) = \frac{\alpha}{\beta(\alpha - \beta)} N$.

By symmetry, one has $A_4 = \frac{\beta}{\alpha(\beta - \alpha)} N'$. We deduce that

$$A - A^2 = \begin{bmatrix} \frac{\beta}{\alpha(\alpha - \beta)} N - A_3 A_2 & ? \\ ? & \frac{\beta}{\alpha(\beta - \alpha)} N' - A_2 A_3 \end{bmatrix}.$$  

Setting $X := \alpha(\alpha - \beta)A_2$ and $Y := \frac{1}{\beta}A_3$, we find:

$$N = XY \quad \text{and} \quad -N' = YX.$$

The main theorem of [2] (or Theorem 9 for $p = 1$, noting that $NX = XYX = X(-N')$ and $YN = YXY = (-N')Y$) then shows that the sequences $(n_k(N, 0))_{k \geq 1}$ and $(n_k(-N', 0))_{k \geq 1}$ are 1-intertwined, i.e., the sequences $(n_k(M, \alpha))_{k \geq 1}$ and $(n_k(M, \beta))_{k \geq 1}$ are 1-intertwined. 

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