Elementary Factors and Reduced Minors for Linear Systems over Commutative Rings

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Abstract. In 1994, Sule presented the necessary and sufficient conditions of the feedback stabilizability of systems over unique factorization domains in terms of elementary factors and in terms of reduced minors. Recently, Mori and Abe have generalized his theory over commutative rings. They have introduced the notion of the generalized elementary factor, which is a generalization of the elementary factor, and have given the necessary and sufficient condition of the feedback stabilizability. In this paper, we present two generalization of the reduced minors. Using each of them, we state the necessary and sufficient condition of the feedback stabilizability over commutative rings. Further we present the relationship between the generalizations and the generalized elementary factors.

Keywords. Linear systems, Feedback stabilization, Factorization approach, Systems over rings

1. Introduction. This paper is concerned with the coordinate-free approach to control systems. The coordinate-free approach is a factorization approach but does not require the coprime factorizations of plants.

The factorization approach was patterned after Desoer et al. [4] and Vidyasagar et al. [21], which has the advantage that it embraces, within a single framework, numerous linear systems such as continuous-time as well as discrete-time systems, lumped as well as distributed systems, 1-D as well as n-D (multidimensional) systems, etc. [21]. In this approach, when problems such as feedback stabilization are studied, one can focus on the key aspects of the problem under study rather than be distracted by the special features of a particular class of linear systems. A transfer function of this approach is considered as the ratio of two stable causal transfer functions and the set of stable causal transfer functions forms a commutative ring.

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For a long time, the theory of the factorization approach had been founded on the coprime factorizability of transfer matrices, which is satisfied in the case where the set of stable causal transfer functions is such a commutative ring as a Euclidean domain, a principal ideal, or a Bézout domain.

However, Anantharam in [1] showed that there exist models in which some stabilizable plants do not have right-/left-coprime factorizations. He considered the case where $\mathbb{Z}[\sqrt{-5}]$ ($\mathbb{Z}[x]=\langle x^2 + 5 \rangle$) is the set of stable causal transfer functions, where $\mathbb{Z}$ is the ring of integers and $i$ the imaginary unit. Using it, he showed that there exists a stabilizable plant which does not have right-/left-coprime factorizations. Further Mori in [14] has recently considered the case where $\mathbb{R}[z^2; z^3]$ is the set of stable causal transfer functions, where $z$ denotes the unit delay operator and $\mathbb{R}$ the real field. This set is corresponding to the discrete finite-time delay system which does not have the unit delay. He has presented that in the model, some stabilizable plants do not have right-/left-coprime factorizations. Both $\mathbb{Z}[\sqrt{-5}]$ and $\mathbb{R}[z^2; z^3]$ are not unique factorization domains.

Sule in [18, 19] has presented a theory of the feedback stabilization of multi-input multi-output strictly causal plants over commutative rings with some restrictions. This approach to the stabilization theory is called “coordinate-free approach” in the sense that the coprime factorizability of transfer matrices is not required.

In the case where the set of stable causal transfer functions is a unique factorization domain, Sule in [18] introduced two notions, that is, elementary factors and reduced minors. Using each of them he gave the necessary and sufficient condition of the feedback stabilizability of the causal plants over commutative rings (Theorem 4 and Corollary 2 of [18]). Especially, using elementary factors, Sule presented a construction method of a stabilizing controller of a stabilizable plant. Recently, Mori and Abe in [15, 16] have generalized his theory over commutative rings. They have introduced the notion of the generalized elementary factor, which is a generalization of the elementary factor, and have given the necessary and sufficient condition of the feedback stabilizability. Further Lin in [11] has presented the necessary and sufficient condition of the (structural) stabilizability of the multidimensional systems with the construction method of a stabilizing controller. In the case of the structural stability[5], it is known that the set of stable causal transfer functions is a unique factorization domain. Lin in [11] introduced a notion “generating polynomial” about the plants and presented the necessary and sufficient condition of the stabilizability of the multidimensional systems with the construction method of a stabilizing controller. It is known that the notion of the generating polynomial is equivalent to the notion of the reduced minors.
In this paper we have two main objectives. The first one is to generalize the notion of the reduced minors and, using the generalizations, to state the necessary and sufficient condition of the feedback stabilizability over commutative rings since the original definition has been given on unique factorization domains. We will present two generalizations. The other is to present the relationship between the generalizations and the generalized elementary factors.

Historically the minors concerning the plants are much investigated (e.g. [3, 8, 9, 10, 12, 22, 23, 24, 25]). We will present that in the coordinate-free approach, the minors can play a role to state the feedback stabilizability, that is, the projectivity of the ideal generated by minors concerning the plant is a criterion of the feedback stability.

This paper is organized as follows. After this introduction, we begin on the preliminary in Section 2, in which we give mathematical preliminaries, set up the feedback stabilization problem and present the previous results. In Section 3, we present the previous results of the feedback stabilizability expressed with the elementary factors, its derivation, and the reduced minors. We present a generalization of the reduced minor in Section 4 and using it present the necessary and sufficient condition of the feedback stabilizability over commutative rings in Section 5. Then in Section 6 we present another generalization of the reduced minors and its relation to the generalized elementary factors.

2. Preliminaries. In the following we begin by introducing the notations of commutative rings, matrices, and modules used in this paper. Then we give the formulation of the feedback stabilization problem.

2.1. Notations.

Commutative Rings. In this paper, we consider that any commutative ring has the identity 1 different from zero. Let $\mathbb{R}$ denote a (unspecified) commutative ring. The total ring of fractions of $\mathbb{R}$ is denoted by $\mathbb{F}(\mathbb{R})$.

We will consider that the set of stable causal transfer functions is a commutative ring, which is denoted by $\mathbb{A}$ throughout this paper. Further, we will use the following rings of fractions.

(i) The first one appears as the total ring of fractions of $\mathbb{A}$, which is denoted by $\mathbb{F}(\mathbb{A})$ or simply by $\mathbb{F}$; that is, $\mathbb{F} = \mathbb{F}(\mathbb{A}) = \frac{f}{\mathbb{A}}$; $d \in \mathbb{A}$; $d$ is a nonzerodivisor. This will be considered as the set of all possible transfer functions.

(ii) Let $f$ denote a nonzero (but possibly nonzerodivisor) element of $\mathbb{A}$. Given a set $S_{f} = \{f, f^{2}, \ldots, \}$, which is a multiplicative subset of $\mathbb{A}$, we denote by $\mathbb{A}_{f}$
the ring of fractions of \( A \) with respect to the multiplicative subset \( S_\mathfrak{p} \); that is, \( A_\mathfrak{p} = \frac{fn=\mathfrak{d}}{jn} \mathfrak{2} A; d \mathfrak{2} S_\mathfrak{p}.g \).

(iii) Let \( \mathfrak{p} \) denote a prime ideal of \( A \) and \( S \) the complement of the prime ideal \( \mathfrak{p} \), that is, \( S = A \mathfrak{n} \mathfrak{p} \). Then \( S \) is a multiplicative subset of \( A \). We denote by \( A_\mathfrak{p} \) the ring of fractions of \( A \) with respect to the multiplicative subset \( S \); that is, \( A_\mathfrak{p} = \frac{fn=\mathfrak{d}}{jn} \mathfrak{2} A; d \mathfrak{2} Sg \).

(iv) The last one is the total ring of fractions of \( A_\mathfrak{f} \) or \( A_\mathfrak{p} \), which is denoted by \( F(A_\mathfrak{f}) \) and \( F(A_\mathfrak{p}) \); that is, \( F(A_\mathfrak{f}) = \frac{fn=\mathfrak{d}}{jn} \mathfrak{2} A_\mathfrak{f}; d \mathfrak{2} A_\mathfrak{f} \mathfrak{p} \mathfrak{g} \) and \( F(A_\mathfrak{p}) = \frac{fn=\mathfrak{d}}{jn} \mathfrak{2} A_\mathfrak{p}; d \mathfrak{2} A_\mathfrak{p} \mathfrak{g} \). If \( \mathfrak{f} \) is a nonzerodivisor of \( A_\mathfrak{f} \mathfrak{g} \) and \( \mathfrak{p} \) is a nonzerodivisor of \( A_\mathfrak{p} \mathfrak{g} \). If \( \mathfrak{f} \) is a nonzerodivisor of \( A, F(A_\mathfrak{f}) \) coincides with the total ring of fractions of \( A \). Otherwise, they do not coincide.

In the case where \( A \) is a unique factorization domain, we call \( a \) in \( A \) the radical of \( b \) in \( A \) if \( a \) has all nonunit factors of \( b \) and is squarefree, that is, \( a \) does not have duplicated nonunit factors. Note here that the radical defined here is unique up to any unit multiple.

For convenience, throughout the paper, if \( a \mathfrak{2} A \) (\( a \mathfrak{2} R \)), then \( a \) itself denotes \( a=1 \) in \( A_\mathfrak{f} \) and \( A_\mathfrak{p} \) (\( a=1 \) in \( F(R) \)). Moreover if \( a \mathfrak{2} A_\mathfrak{f} \) or \( A_\mathfrak{p} \) (\( a \mathfrak{2} R \)) and if there exists \( b \mathfrak{2} A \) such that \( a = b=1 \over A_\mathfrak{f} \) or \( A_\mathfrak{p} \) (over \( F(R) \)), then we regard \( a \) as an element of \( A \) (\( R \)).

In the rest of the paper, we will use \( R \) as an unspecified commutative ring and mainly suppose that \( R \) denotes one of \( A, A_\mathfrak{f}, \) and \( A_\mathfrak{p} \).

We will denote by \( \text{Spec}(R) \) the set of all prime ideals of \( R \) and by \( \text{Max}(R) \) the set of all maximal ideals of \( R \). Suppose that \( a \) is an ideal of \( R \). Then we denote by \( a_\mathfrak{f} \) the ideal of fractions of \( a \) with respect to \( f_1; f_2; f_3; \ldots ; g \) with \( f \mathfrak{2} R \) (that is, \( a_\mathfrak{f} = \frac{fn=\mathfrak{d}}{jn} \mathfrak{2} a; d \mathfrak{2} f_1; f_2; f_3; \ldots ; g \)) and by \( a_\mathfrak{p} \) the ideal of fractions of \( a \) with respect to \( R \mathfrak{n} \mathfrak{p} \) with \( p \mathfrak{2} \text{Spec}(R) \) (that is, \( a_\mathfrak{p} = \frac{fn=\mathfrak{d}}{jn} \mathfrak{2} a; d \mathfrak{2} R \mathfrak{n} \mathfrak{p}g \)). If \( a \) is an ideal of \( R \) and if \( S \) is a subset of \( R \), then we denote by \( (a : S) \) the quotient ideal which is the set \( f \mathfrak{2} R \mathfrak{j} \mathfrak{f} S \mathfrak{a}g \).

The reader is referred to Chapter 3 of \([2]\) for the ring of fractions.

Matrices. The set of matrices over \( R \) of size \( x \times y \) is denoted by \( R^{x \times y} \). Further, the set of square matrices over \( R \) of size \( x \) is denoted by \( R^{x \times x} \). The identity and the zero matrices are denoted by \( E_x \) and \( O_x \), respectively, if the sizes are required, otherwise they are denoted by \( E \) and \( O \).

Matrix \( A \) over \( R \) is said to be nonsingular (singular) over \( R \) if the determinant of the matrix \( A \) is a nonzerodivisor (a zerodivisor) of \( R \). Matrices \( A \) and \( B \) over \( R \) are right- (left-) coprime over \( R \) if there exist matrices \( X \) and \( Y \) over \( R \).
such that $X A + Y B = E$ ($A X + B Y = E$) holds. Note that, in the sense of the above definition, two matrices which have no common right- (left-)factors except invertible matrices may not be right- (left-)coprime over $R$. Further, an ordered pair $(N ; D)$ of matrices $N$ and $D$ is said to be a right-coprime factorization over $R$ of $P$ if (i) $D$ is nonsingular over $R$, (ii) $P = N D^{-1}$ over $F (R)$, and (iii) $N$ and $D$ are right-coprime over $R$. As the parallel notion, the left-coprime factorization over $R$ of $P$ is defined analogously. That is, an ordered pair $(P ; N)$ of matrices $P$ and $N$ is said to be a left-coprime factorization over $R$ of $P$ if (i) $P$ is nonsingular over $R$, (ii) $P = N P^{-1}$ over $F (R)$, and (iii) $P$ and $N$ are left-coprime over $R$. Note that the order of the “denominator” and “numerator” matrices is interchanged in the latter case. This is to reinforce the point that if $(N ; D)$ is a right-coprime factorization over $R$ of $P$, then $P = N D^{-1}$, whereas if $(P ; N)$ is a left-coprime factorization over $R$ of $P$, then $P = N P^{-1}$ according to [20]. For short, we may omit “over $R$” when $R = A$, and “right” and “left” when the size of matrix is $1$. In the case where matrices are potentially used to express left fractional form and/or left coprimeness, we usually attach a tilde ‘e’ to symbols; for example $E$, $P$ for $P = N P^{-1}$ and $D$, $M$ for $[P N + D E] = E$.

Modules. Let $M_X (X)$ ($M_e (X)$) denote the $R$-module generated by rows (columns) of a matrix $X$ over $R$. Let $X = A B^{-1} = E^{-1} F$ be a matrix over $F (R)$, where $A$, $B$, $E$, $F$ are matrices over $R$. It is known that $M_x ([A^t B^t]) = M_e ([E F])$ is unique up to an isomorphism with respect to any choice of fractions $A B^{-1}$ of $X$ ($E^{-1} F$ of $X$) (Lemma 2.1 of [15]). Therefore, for a matrix $X$ over $R$, we denote by $X_{R}$ and $X_{E}$ the modules $M_x ([A^t B^t])$ and $M_e ([E F])$, respectively.

An $R$-module $M$ is called free if it has a basis, that is, a linearly independent system of generators. The rank of a free $R$-module $M$ is equal to the cardinality of a basis of $M$, which is independent of the basis chosen. An $R$-module $M$ is called projective if it is a direct summand of a free $R$-module, that is, there is a module $N$ such that $M \cong N$ is free. The reader is referred to Chapter 2 of [2] for the module theory.

We will consider occasionally ideals as modules in this paper. So, we will apply the words “projective,” “free,” and “isomorphic” to ideals. It is easy to check that an ideal which is free as a module is equivalent to a principal ideal whose generator is a nonzerodivisor.

2.2. Feedback Stabilization Problem. The stabilization problem considered in this paper follows that of Sule in [18], and Mori and Abe in [15], who consider the feedback system [20, Ch.5, Figure 5.1] as in Figure 2.1. For further details the reader is referred to [20]. Throughout the paper, the plant we consider has $m$
inputs and \( n \) outputs, and its transfer matrix, which is also called a \textit{plant} itself simply, is denoted by \( P \) and belongs to \( \mathcal{F}^{n \times m} \). We can always represent \( P \) in the form of a fraction \( P = ND^{-1} \) (\( P = \mathcal{F}^{-1} \)), where \( N \in \mathcal{A}^{n \times m} \) (\( \mathcal{F}^{2} \mathcal{A}^{n \times m} \)) and \( D \in (\mathcal{A})_{m} \) (\( \mathcal{F}^{2} (\mathcal{A})_{n} \)) with nonsingular \( D \) (\( \mathcal{F} \)).

**Definition 2.1.** For \( P \in \mathcal{F}^{n \times m} \) and \( C \in \mathcal{F}^{m \times n} \), a matrix \( H(P; C) \in (\mathcal{F})_{m+n} \) is defined as

\[
H(P; C) = \begin{bmatrix} (E_n + PC)^{-1} & P(E_m + CP)^{-1} \\ C(E_n + PC)^{-1} & (E_m + CP)^{-1} \end{bmatrix}
\]

provided that \( \det (E_n + PC) \) is a nonzerodivisor of \( \mathcal{A} \). This \( H(P; C) \) is the transfer matrix from \( [u_1^T \ u_2^T]^T \) to \( [e_1^T \ e_2^T]^T \) of the feedback system. If (i) \( \det (E_n + PC) \) is a nonzerodivisor of \( \mathcal{A} \) and (ii) \( H(P; C) \in (\mathcal{A})_{m+n} \), then we say that the plant \( P \) is stabilizable, \( P \) is stabilized by \( C \), and \( C \) is a stabilizing controller of \( P \).

Since the transfer matrix \( H(P; C) \) of the stable causal feedback system has all entries in \( \mathcal{A} \), we call the above notion \( \mathcal{A} \)-stabilizability. One can further introduce the notion of \( \mathcal{R} \)-stabilizability with either \( \mathcal{R} = \mathcal{A}_f \) or \( \mathcal{R} = \mathcal{A}_p \) as follows.

**Definition 2.2.** Suppose that \( \mathcal{R} \) is either \( \mathcal{A}_f \) with \( f \in \mathcal{A} \) \( f \notin \text{Spec}(\mathcal{A}) \) or \( \mathcal{R} = \mathcal{A}_p \) with \( p \in \text{Spec}(\mathcal{A}) \). If (i) \( \det (E_n + PC) \) is a nonzerodivisor of \( \mathcal{R} \) and (ii) \( H(P; C) \in (\mathcal{R})_{m+n} \), then we say that the plant \( P \) is \( \mathcal{R} \)-stabilizable, \( P \) is \( \mathcal{R} \)-stabilized by \( C \), and \( C \) is an \( \mathcal{R} \)-stabilizing controller of \( P \).

The causality of transfer functions is an important physical constraint. We employ, in this paper, the definition of the causality from Vidyasagar et al. \[21\], Definition 3.1.

**Definition 2.3.** Let \( Z \) be a prime ideal of \( \mathcal{A} \), with \( Z \notin \mathcal{A} \), including all zerodivisors. Define the subsets \( \mathcal{P} \) and \( \mathcal{P}_s \) of \( \mathcal{F} \) as follows:

\[
\mathcal{P} = \{ a = b^2 \mathcal{F} \ | \ a \in \mathcal{A} \ ; \ b \in \mathcal{A} \ \text{and} \ n \in \mathcal{Z} \} \\
\mathcal{P}_s = \{ a = b^2 \mathcal{F} \ | \ a \in \mathcal{Z} \ ; \ b \in \mathcal{A} \ \text{and} \ n \in \mathcal{Z} \} 
\]
Then every transfer function in \( P(s) \) is called causal (strictly causal). Analogously, if every entry of a transfer matrix \( F \) is in \( P(s) \), the transfer matrix \( F \) is called causal (strictly causal). A matrix over \( A \) is said to be \( \mathbb{Z} \)-nonsingular if the determinant is in \( A \cap \mathbb{Z} \), and \( \mathbb{Z} \)-singular otherwise.

Before proceeding the next section, we here introduce several symbols used throughout this paper. The symbol \( I \) denotes the family of all sets of \( m \) distinct integers between 1 and \( m + n \), and \( J \) the family of all sets of \( n \) distinct integers between 1 and \( m + n \) (recall that \( m \) and \( n \) are the numbers of the inputs and the outputs, respectively). Normally, elements of \( I \) (\( J \)) will be denoted by \( I(J) \) possibly with suffices. They will be used as suffices as well as sets. If \( I \) is an element of \( I \) and if \( i_1 < \cdots < i_m \) are elements of \( I \) with ascending order, that is, \( i_a < i_b \) if \( a < b \), then the symbol \( i \) denotes the \( m \times (m + n) \) matrix whose \( (k; i_k) \)-entry is 1 for \( i_k \in I \) and zero otherwise. Analogously if \( J \) is an element of \( J \) and if \( j_1 < \cdots < j_n \) are elements of \( J \) with ascending order, then the symbol \( j \) denotes the \( n \times (m + n) \) matrix whose \( (k; j_k) \)-entry is 1 for \( j_k \in J \) and zero otherwise.

3. Previous Results. In this section, we recall the previous results about the necessary and sufficient condition of the feedback stabilizability. First one is stated in terms of the elementary factors and the other in terms of the reduced minors.

3.1. Feedback Stabilizability in terms of Elementary Factors. To state the result, we first recall the notion of the elementary factors, which was defined under the assumption that \( A \) is a unique factorization domain.

**Definition 3.1.** (Elementary Factors, [18, p.1689]) Suppose that \( A \) is a unique factorization domain. Denote by \( T \) and \( W \) the matrices \( [N^T \in \mathbb{E}_m] \) and \( [N \in \mathbb{E}_n] \) over \( A \) with \( P = N \cap \mathbb{Z} \). Further denote by \( I \) (\( J \)) the set of \( I \)'s in \( I \) (\( J \)'s in \( J \)) such that \( i \) \( T \) \( j \) \( W \) is nonsingular. Then for each \( I \in I \), let \( f_i \) be the radical of the least common multiple of all the denominators of the matrix \( T \ (i \ ) \) and for each \( J \in J \), \( g_j \) be the radical of the least common multiple of all the denominators of the matrix \( W \ (j \ ) \). Then \( f_i \) (\( g_j \)) is called the elementary factor of the matrix \( T \ (i \ ) \) with respect to \( I \in I \) (\( J \in J \)), \( F = \{ f_i \}_{i \in I} \) \( J \in J \) \( g \) the family of elementary factors of the matrix \( T \), \( G = \{ g_j \}_{j \in J} \) \( I \in I \) \( g \) the family of elementary factors of the matrix \( W \), and \( H = \{ f_i \}_{i \in I} \) \( J \in J \) \( g \) the family of elementary factors of \( P \).

Then the necessary and sufficient condition of the feedback stabilizability is given as follows.

**Theorem 3.2.** (Theorem 4 of [18]) Suppose that \( A \) is a unique factorization
domain. Then the plant $P$ is stabilizable if and only if the elementary factors of $P$ are coprime, that is, $I \cap J = A$.

In the proof of this theorem, Sule gave a method to construct a stabilizing controller of the plant.

The result above has been extended to include systems over commutative rings by Mori and Abe in [16] as follows. They introduced the notion of the generalized elementary factors, which is a generalization of the elementary factors, and using it, stated the necessary and sufficient conditions of the feedback stabilizability over commutative rings.

**Definition 3.3.** (Generalized Elementary Factors, Definition 3.1 of [16]) Denote by $T$ the matrix $[N \ D \ \frac{t}{j}]$ over $A$ with $P = ND^{-1}$. For each $I \subset J$, an ideal $P_I$ over $A$ is defined as

$$P_I = \{ f \in A | \exists k \in A, \text{ such that } kf \in I \}.$$

We call the ideal $P_I$ the generalized elementary factor of the plant $P$ with respect to $I$. Further, the set of all $P_I$'s is denoted by $L_P$, that is, $L_P = \{ P_I | I \subset J \}$.

In the case where $A$ is a unique factorization domain, a generalized elementary factor with respect to $I \subset J$ is a principal ideal and the radical of its generator is an elementary factor of $T$ with respect to $J$ up to a unit multiple.

It is known that the generalized elementary factor of a plant $P$ is independent of the choice of fractions $ND^{-1} = P$ (Lemma 3.3 of [16]).

The following is the necessary and sufficient conditions of the feedback stabilizability.

**Theorem 3.4.** (Theorem 3.2 of [16]) Consider a causal plant $P$. Then the following statements are equivalent:

(i) The plant $P$ is stabilizable.

(ii) $A$-modules $T_{P_A}$ and $W_{P_A}$ are projective.

(iii) The set of all generalized elementary factors of $P$ generates $A$; that is, $L_P$ satisfies:

$$\bigcap_{P_I \in L_P} P_I = A.$$  

Provided that we can check (3.1) and that we can construct the right-coprime factorizations over $A \setminus I$ of the given causal plant, where $I$ is a nonzero element of $A$, Mori and Abe [16] have given a method to construct a causal stabilizing controller of a causal stabilizable plant, which has been given in the proof of “(iii) → (i)” of Theorem 3.2 of [16].
3.2. Feedback Stabilizability in terms of Reduced Minors. We first recall the definition of the reduced minors and then state the necessary and sufficient conditions of the feedback stabilizability in terms of the reduced minors. We suppose in this subsection that \( A \) is a unique factorization domain.

**Definition 3.5.** (Reduced Minors, [18, p.1690]) Let \( P \) be a plant of \( F^{n \times m} \), \( N \) a matrix of \( A^{n \times m} \), and \( d \) an element of \( A \) such that \( P = N \cdot d \). Denote by \( T \) and \( W \) the matrices \( [N \cdot t, \cdot d \cdot E] \) and \( [N \cdot \cdot d \cdot E] \). Let \( t_1 = \det (T) \) (\( w_1 = \det (W) \)), which is a full-size minor of the matrix \( T \) (\( W \)), for \( I \subseteq I \) (\( J \subseteq J \)). Let \( d_1, \ldots, d_n \) be the greatest common factor of \( t_1 \)'s (\( w_1 \)'s) and \( a_1 = t_1 \cdot d_1 \) for \( I \subseteq I \) (\( b_1 = w_1 \cdot d_1 \) for \( J \subseteq J \)). Then \( a_1 \cdot b_1 \) is called the reduced minor of the matrix \( T \) (\( W \)) with respect to \( I \subseteq I \) (\( J \subseteq J \)), the set \( \{ a_I \mid I \subseteq I \} \) (\( \{ b_J \mid J \subseteq J \} \)) the family of reduced minors of \( T \) (\( W \)).

It is known that the families of reduced minors of \( T \) and of \( W \) are identical modulo units (Lemma 5 of [18]).

Now, Corollary 2 of [18] including its comments can be stated as follows:

**Theorem 3.6.** (cf. Corollary 2 of [18]) Suppose that \( A \) is a unique factorization domain. A plant \( P \in F^{m \times n} \) is stabilizable if and only if the family of the reduced minors of \( T \) (and also of \( W \)) generates \( A \).

The theorem above can be rewritten directly as follows.

**Corollary 3.7.** Let \( t_I \) and \( w_J \) be as in Definition 3.5. Then the following are equivalent:

(i) A plant \( P \in F^{m \times n} \) is stabilizable.

(ii) The ideal \( \langle t_I \rangle \) is principal, or equivalently free as an \( A \)-module.

(iii) The ideal \( \langle w_J \rangle \) is principal, or equivalently free as an \( A \)-module.

4. Full-Size Minor Ideal. On the statements concerning the elementary factors and the reduced minors in Subsections 3.1 and 3.2, we have considered that the denominator matrices of the plant is expressed as \( dE_m \) or \( dE_n \) rather than general nonsingular matrices. This may be considered as a restriction on the expression of the plant. Thus we rather consider that \( P \) is expressed as either \( P = N D \) with \( N \subseteq A^{n \times m} \) and \( D \subseteq (A)_m \) or \( P = D F \) with \( D \subseteq A^{m \times n} \) and \( F \subseteq (A)_m \). Now we redefine the matrices \( T, W \) as \( T = [N \cdot t, \cdot D] \) and \( W = [D \cdot F] \). Further we consider that \( t_I \)'s and \( w_J \)'s are defined with the matrices \( T \) and \( W \) here. In the rest of this paper, we will use these notations unless otherwise stated.

We now introduce a notion to state the feedback stabilizability over commutative rings.
DEFINITION 4.1. (Full-Size Minor Ideals)  The ideal generated by \( P \)'s for \( I \) is called the full-size minor ideal of the plant \( P \). We denote it by \( _{I2I}(t_I) \) or simply \( t \).

We can also consider the ideal generated by \( w_J \)'s for \( J \), denoted by \( _{J2J}(w_J) \) or simply \( w \). The ideals \( t \) and \( w \) depend on the fractional representation of the plant \( P = ND^{-1} = FR^{-1}F \). However, this is not a problem from the following reason. To state the feedback stabilizability in terms of the full-size minor ideals, we will regard them as modules. Further, when these ideals are considered as modules, both the ideals \( t \) and \( w \) are uniquely determined as modules up to isomorphism with respect to any choice of fractions \( ND^{-1} \) and \( FR^{-1}F \) of \( P \) as shown below.

LEMMA 4.2. Let \( P \) be in \( FR^n \), where \( R \) is one of \( A, A_p \) with a nonzero \( f \) 2 \( A \), and \( A_p \) with a prime ideal \( p \) in \( Spec(A) \). For \( x = 1 \) 2 \( N, D, fR, fA \) be matrices over \( R \) with \( P = ND^{-1} = FR^{-1}F \) over \( F \). \( T_x = [N^T \ D^{-1}_x] \) and \( W_x = [fN \ fD] \}. Further for \( x = 1 \) 2 \( P \) and for \( I = 1 \) 2 \( J \) 2 \( P \), let \( t_x : = det(\ _IT_x) \) and \( W \). Then the ideals \( _{I2I}(t_I) \), \( _{I2I}(t_{1I}) \), \( _{J2J}(w_J) \), \( _{J2J}(w_{1J}) \) are isomorphic to one another as \( R \)-modules.

Proof. We show first (i) \( _{I2I}(t_I) \)' \( _{I2I}(t_{1I}) \)' and then (ii) \( _{I2I}(t_{1I}) \)' \( _{J2J}(w_J) \) \( _{J2J}(w_{1J}) \). The isomorphism \( _{J2J}(w_{1J}) \)' \( _{J2J}(w_{2J}) \) can be proved analogously to (i) and so is omitted.

(i). Observe that in the case where \( [N \ D] = [N \ D] \) holds with some nonsingular matrix \( X \) over \( R \), the statement of the lemma obviously holds. Hence by considering \( [N \ D] = [N \ D] \) as \( [N \ D] \) we can assume without loss of generality that \( D \) is expressed as \( \underline{d}E_m \) with nonzero \( \underline{d} \). Observe now that \( [N \ D] = [N \ d] \) holds. From this relation and the first observation, we now have (i).

(ii). It is sufficient to consider the case \( P = ND^{-1} = ND \) with \( ND \) and \( ND \) as in (i). In the case \( P = ND^{-1} = ND \), one can consider \( P = (N \ adj\ D) \) det\( D \).
Since $I_N$ and $I_d$ can be expressed by $J_N$ and $J_d$ as $I_N = [1; n]J_f m$ $J_f m$ $J_f m$ $J_d g$, $I_d = f^2 n + n j 2 [1; m]J_N g$, the inverse mapping $^1 : J^! : I$ can be defined naturally. Hence, the map is bijective.

Now let $T = [N \ t \ dE m]$ and $W = [N \ dE n]$. By the straightforward calculation with noting that $dE_m$ and $dE_n$ are diagonal, we obtain the following relations:

\[ \det ( \ t T ) = \det ( \ t w ) d^n n : \]

Thus $t_{I_1} = w_{I_1} d^n n$ for all $I \geq I$. It follows that the ideals $t_{I_1}$ and $w_{I_1}$ are isomorphic to each other.

**Note 4.3.** The reduced minors are derived from $t_i$’s and $w_j$’s in Definition 3.5. Thus $t_i$’s and $w_j$’s can be considered more primitive than the reduced minors. Nevertheless since we will present in Theorem 5.2 that $t_i$’s and $w_j$’s (or the ideals $t$ and $w$ generated by them) have the capability to state feedback stabilizability over commutative rings, we here consider that the full-size minor ideal $t$ (or the ideal $w$) is a generalization of the reduced minors.

**5. Feedback Stabilizability in terms of Full-Size Minor Ideal.** In this section, we present the necessary and sufficient condition of the feedback stabilizability over commutative rings in terms of the full-size minor ideal.

Let us consider the case where the set $A$ of the stable causal transfer functions is not a unique factorization domain. Then it is not sufficient to use the family of reduced minors in order to state the feedback stabilizability. To see this, let us consider the result given by Anantharam in [1].

**Example 5.1.** In [1], Anantharam considered the case where $Z[\alpha](\mathbb{Z}[\alpha]=\alpha^2 + 5)$ is the set of stable causal transfer functions, where $\mathbb{Z}$ is the ring of integers and $\mathbb{I}$ the imaginary unit; that is, $A = Z[\alpha]$. The set of all possible transfer functions is given as the field of fractions of $A$; that is, $\mathbb{F}_p = \mathbb{Q}(\alpha^2 + 5)$. In this case we have multiple factorizations $2 = (1 + 5i)(1 - 5i)$ of $2$ over $A$, so that $A$ is not a unique factorization domain. Anantharam in [1] considered the single-input single-output case and showed that the plant $p = (1 + 5i) = 2$ does not have its coprime factorization over $A$ but is stabilizable.

Now let $T = [1 + 5i \ 2]$. Since the plant $p$ is of the single-input single-output $(m = n = 1)$, we have $I = ff_1g; f_2g$. Thus let $I_1 = f_1g$ and $I_2 = f_2g$ so that $I = f_1g + f_2g$. The full-size minors of the matrix $T$ are $t_{I_1} = \det ( I_1 T ) = 1 + 5i$ and $t_{I_2} = \det ( I_2 T ) = 2$. If Theorem 3.6 (or equivalently Corollary 3.7)

\[ ^1 \text{The author wishes to thank to Dr. A. Quadrat (Centre d’Enseignement et de Recherche en Mathématiques, Informatique et Calcul Scientifique, ENPC, France) who introduced him to the paper of Anantharam[1].} \]
could be applied even over a general commutative ring, the ideal \( (t_{11}, t_{12}) \) should be principal. However, the ideal \( (t_{11}, t_{12}) \) is not principal since \( p \) does not have its coprime factorization. \[ \square \]

In order to involve even such an example as a system over commutative ring, we extend Theorem 3.6. Since we cannot use the reduced minors to state the feedback stabilizability in general, we alternatively employ the full-size minor ideal \( t \) rather than the reduced minors. The extension is the first main result of this paper and stated as follows.

**Theorem 5.2.** Let \( \mathcal{P} \) be a causal plant of \( \mathbb{P}^{n \times m} \). Then the plant \( \mathcal{P} \) is stabilizable if and only if the full-size minor ideal \( t \) of the plant \( \mathcal{P} \) is projective. Further when \( t \) is projective, it is of rank 1.

By virtue of Lemma 4.2, the above theorem can be also stated with the ideal \( w \) instead of the full-size minor ideal \( t \).

In the case where \( A \) is a unique factorization domain, as in Theorem 3.6, the condition of feedback stabilizability is that the full-size minor ideal is free. On the other hand, in Theorem 5.2, the condition is that the ideal is projective. They are equivalent to each other in the case where \( A \) is a unique factorization domain as follows.

**Proposition 5.3.** Let \( R \) be a unique factorization domain. Then the ideal generated by finite elements of \( R \) is projective if and only if it is free.

This proof will be given after finishing the proof of Theorem 5.2.

Now that we have presented the statement of Theorem 5.2, the main objective of the remainder of this section is to carry out the proof of Theorem 5.2. To do so, we prepare two main intermediate results. The first one is about the existence of right-/left-coprime factorizations of stabilizable plants over local rings, which will be presented in Subsection 5.1. The other is about the local-global principle of the feedback stabilizability, which will be presented in Subsection 5.2. Then we will prove Theorem 5.2. After the proof of Theorem 5.2, we will prove Proposition 5.3.

Before finishing this section, we will present the relationship among the full-size minor ideals of \( P \), \( C \), and \( H(P; C) \).

### 5.1. Right-/Left-Coprime Factorizations over Local Rings.

The following is the first intermediate result of Theorem 5.2 about the existence of right-/left-coprime factorizations of stabilizable plants over local rings.

**Proposition 5.4.** Let \( \mathcal{P} \) be a plant in \( \mathbb{F}^{n \times m} \). Suppose that \( R \) is \( \mathbb{A}_p \) with a prime ideal \( p \) in \( \text{Spec}(\mathbb{A}) \). Then the following statements are equivalent:

(i) The plant \( \mathcal{P} \) is \( R \)-stabilizable.
There exists a right-coprime factorization over $\mathbb{R}$ of $P$.

There exists a left-coprime factorization over $\mathbb{R}$ of $P$.

The proof of this proposition will be presented after giving several intermediate results.

We here recall the notion of Hermite used in [20], which can characterize the existence of both right-/left-coprime factorizations of transfer matrices.

**Definition 5.5.** ([20, p.345]) Let $\mathbb{R}$ be a commutative ring and $A$ a matrix over $\mathbb{R}$ of size $x \times y$ with $x < y$. Then we say that the matrix $A$ can be complemented if there exists a unimodular matrix in $(\mathbb{R})^y$ containing the matrix $A$ as a submatrix. A row $[a_1, \ldots, a_y] \in \mathbb{R}^1 \times y$ is said to be a unimodular row if $a_1, \ldots, a_y$ together generate $\mathbb{R}$. A commutative ring $\mathbb{R}$ is said to be Hermite if every unimodular row can be complemented.

The following result was given in [20] provided that $\mathbb{R}$ is an integral domain.

**Theorem 5.6.** (cf. Theorem 8.1.66 of [20]) Let $\mathbb{R}$ be a commutative ring. The following three statements are equivalent:

(i) The commutative ring $\mathbb{R}$ is Hermite.

(ii) If a matrix over $\mathbb{F}(\mathbb{R})$ has a right-coprime factorization over $\mathbb{R}$, it has also a left-coprime factorization over $\mathbb{R}$.

(iii) If a matrix over $\mathbb{F}(\mathbb{R})$ has a left-coprime factorization over $\mathbb{R}$, it has also a right-coprime factorization over $\mathbb{R}$.

The “integral domain” version of this theorem was given as Theorem 8.1.66 of [20]. Even in the case of commutative rings, the proof is similar with that of Theorem 8.1.66 of [20] and so is omitted.

The following result is the intermediate result of Proposition 5.4, which makes the result above applicable to the proof of the proposition.

**Lemma 5.7.** Any local ring is Hermite.

**Proof.** Suppose that $\mathbb{R}$ is a local ring and $[a_1, \ldots, a_y] \in \mathbb{R}^1 \times y$ is a unimodular row. Thus there exist $b_2, \ldots, b_y \in \mathbb{R}$ such that

$$a_1 b_1 + \ldots + b_y a_y = 1. \quad (5.1)$$

Since $\mathbb{R}$ is local, the set of all nonunits is an ideal. From (5.1), there exists an $\mathbb{i}$ with $1 \leq i \leq y$ such that $a_i$ is a unit. We assume without loss of generality that $a_1$ is a unit. It should be noted that this definition of “Hermite” is different from [6, 13].
is a unit. If \( y = 1 \), then \( a_1 \) is a unit, which can be considered as a unimodular matrix of \((\mathbb{R}_1)\). In the following we consider the case \( y > 1 \). Then we can construct a unimodular matrix \( U = (u_{ij}) 2 (\mathbb{R}_y) \):

\[
\begin{align*}
\emptyset & \quad \text{if } i = 1, \\
\emptyset & \quad a_{ij} \quad \text{if } i = j = 2, \\
\emptyset & \quad \vdots \quad a_i^1 \quad \text{if } i > j > 2, \\
\emptyset & \quad 1 \quad \text{if } i = j > 2, \\
\end{align*}
\]

This \( U \) contains the row \([a_1; \ldots; a_y]\) as a submatrix and hence every unimodular row can be complemented. Therefore \( \mathbb{R} \) is Hermite.

We prepare one more result which will help us present a nonsingular denominator matrix of a stabilizing controller.

**Lemma 5.8.** Let \( \mathbb{R} \) be a commutative ring and \( p \) a prime ideal of \( \mathbb{R} \). Suppose that there exist matrices \( A, B, C_1, C_2 \) over \( \mathbb{R} \) such that the determinant of the following square matrix is in \( \mathbb{R}^n p \):

\[
\begin{bmatrix}
A & C_1 \\
B & C_2
\end{bmatrix}
\]

where the matrix \( A \) is square and the matrices \( A \) and \( B \) have same number of columns. Then there exists a matrix \( R \) over \( \mathbb{R} \) such that the determinant of the matrix \( A + R B \) is in \( \mathbb{R}^n p \).

Before starting the proof, it is worth reviewing some easy facts about a prime ideal.

**Remark 5.9.** Suppose that \( p \) is a prime ideal of \( \mathbb{R} \). (i) If \( a \) is in \( \mathbb{R}^n p \) and expressed as \( a = b + c \) with \( b, c \in \mathbb{R} \), then at least one of \( b \) and \( c \) is in \( \mathbb{R}^n p \). (ii) If \( a \) is in \( \mathbb{R}^n p \) and \( b \) in \( p \), then the sum \( a + b \) is in \( \mathbb{R}^n p \). (iii) Every factor in \( \mathbb{R} \) of an element of \( \mathbb{R}^n p \) belongs to \( \mathbb{R}^n p \) (that is, if \( a; b \in \mathbb{R} \) and \( ab \in \mathbb{R}^n p \), then \( a; b \in \mathbb{R}^n p \)).

**Proof of Lemma 5.8.** This proof mainly follows that of Lemma 4.4.21 of [20].

If \( \det(\mathbb{A}) \) is in \( \mathbb{R}^n p \), then we can select the zero matrix as \( \mathbb{R} \). Thus we assume in the following that \( \det(\mathbb{A}) \) is in \( p \).

Since the determinant of (5.2) is in \( \mathbb{R}^n p \), there exists a full-size minor of \([A \; B \; \mathbf{f}] \) in \( \mathbb{R}^n p \) by Laplace’s expansion of (5.2) and by Remark 5.9(i,iii). Let \( \emptyset \) be such a full-size minor of \([A \; B \; \mathbf{f}] \) having as few rows from \( B \) as possible.

We here construct a matrix \( \mathbb{R} \) such that \( \det(\mathbb{A} + R B) = a + \mathbf{r} \) with \( a \mathbf{r} \in \mathbb{R}^n p \). Since \( \det(\mathbb{A}) \in p \), the full-size minor \( \emptyset \) must contain at least one row of \( B \) from the matrix \([A \; B \; \mathbf{f}] \). Suppose that \( \emptyset \) is obtained by excluding the rows
from the factorization by Proposition 2.1 of [15]. Further it is free by Corollary 3.5 of [7, Ch.IV].

show that the matrices full-size minor of (\[A \in \mathbb{R}\]) and \([A \in \mathbb{R}\]) from the factorization \(A + RB = [E \in \mathbb{R}] [A \in \mathbb{R}]\) by the Binet-Cauchy formula. Every minor of \([E \in \mathbb{R}]\) containing more than \(k\) columns of \(R\) is zero. By the method of choosing the rows from \([A \in \mathbb{R}]\) for the full-size minor \(a\), every full-size minor of \([A \in \mathbb{R}]\) having less than \(k\) rows of \(B\) is in \(p\). There is only one nonzero minor of \([E \in \mathbb{R}]\) containing exactly \(k\) columns of \(R\), which is obtained by excluding the columns \(i_1; \ldots; i_k\) of the identity matrix \(E\) and including the columns \(j_1; \ldots; j_k\) of \(R\); it is equal to \(1\). From the Binet-Cauchy formula the corresponding minor of \([A \in \mathbb{R}]\) is \(a\). As a result, \(\det(A + RB)\) is given as a sum of \(\det(a)\) and elements in \(p\). By Remark 5.9(ii), the sum is in \(\mathbb{R} p\) and so is \(\det(A + RB)\).

Now that we have the result above, we can prove Proposition 5.4.

**Proof of Proposition 5.4.** Since \(R\) is local, (ii) and (iii) are equivalent by Theorem 5.4 and Lemma 5.7. Thus we only prove (i)! (ii) and *vice versa.*

(i)! (ii). Suppose that \(P\) is \(\mathbb{R}\)-stabilizable. Then the \(\mathbb{R}\)-module \(T_{\mathbb{R}P}\) is projective by Proposition 2.1 of [13]. Further it is free by Corollary 3.5 of [7, Ch.IV]. Let \(N\) and \(D\) be matrices over \(\mathbb{R}\) with \(P = ND - 1\). Then the \(\mathbb{R}\)-module \(M_\mathbb{R}([N \in \mathbb{R}] D \in \mathbb{R})\) is free of rank \(m\), since \(D\) is nonsingular over \(\mathbb{R}\). Let \(v_1; v_2; \ldots; v_m\) be a basis of the module \(M_\mathbb{R}([N \in \mathbb{R}] D \in \mathbb{R})\) and \(V\) the matrix of \((\mathbb{R})_m\) whose rows are \(v_1; v_2; \ldots; v_m\). Then, the matrix \([N \in \mathbb{R}] D \in \mathbb{R}\) can be written in the form \([N \in \mathbb{R}] D \in \mathbb{R} = [N_0 \in \mathbb{R}] [D_0 \in \mathbb{R}] V\) by uniquely choosing the matrices \(N_0\) in \(\mathbb{R}^m\) and \(D_0\) in \((\mathbb{R})_m\). Because of \(\det(D) = \det(D_0)\), \(\det(D_0)\) is a nonzerodivisor. It follows that \(P = N_0\mathbb{D}^{-1} D\) over \(\mathbb{R}\). In the following we show that the matrices \(N_0\) and \(D_0\) are right-coprime over \(\mathbb{R}\). Since \(v_1; \ldots; v_m\) belong to \(M_\mathbb{R}([N \in \mathbb{R}] D \in \mathbb{R})\), there exist matrices \(\mathcal{P}, \mathcal{Q}\) in \(\mathbb{R}^m\) and \(\mathcal{X}\) in \((\mathbb{R})_m\) such that \(V = [\mathcal{P} \mathcal{Q}] [N \in \mathbb{R}] D \in \mathbb{R}\). So we have \(V = (\mathcal{P}N_0 + \mathcal{Q}D_0) V\). Since \(V\) is nonsingular, we obtain \(\mathcal{P}N_0 + \mathcal{Q}D_0 = E_m\) over \(\mathbb{R}\). Thus \((\mathbb{N}_0; D_0)\) is a right-coprime factorization over \(\mathbb{R}\) of \(P\).

(ii)! (i). Suppose that there exists a right-coprime factorization over \(\mathbb{R}\) of the plant \(P\); that is, there exist the matrices \(N, D, \mathcal{P}, \mathcal{Q}\) over \(\mathbb{R}\) with \(\mathcal{P}N + \mathcal{Q}D = E_m\) and \(P = ND^{-1}\). If \(\det(\mathcal{P})\) is a nonzerodivisor of \(\mathbb{R}\), it is obvious that \(\mathcal{X}^{-1}\mathcal{P}\) is an \(\mathbb{R}\)-stabilizing controller. Thus in the following we suppose that \(\det(\mathcal{P})\) is a zerdodivisor of \(\mathbb{R}\).

By the equivalence between (ii) and (iii), there also exists a left-coprime factorization over \(\mathbb{R}\) of \(P\); that is, there exist the matrices \(\mathcal{N}, \mathcal{Q}, \mathcal{Y}, \mathcal{X}\) over \(\mathbb{R}\) with
\( Y + X = E \), and \( P = E \). Thus we have the following matrix equation:

\[
\begin{bmatrix}
\mathcal{E} & \mathcal{F} & D & Y \\
\mathcal{E} & \mathcal{F} & N & X
\end{bmatrix} =
\begin{bmatrix}
E_m & \mathcal{E} & Y \\
E_n & O & E_n
\end{bmatrix}:
\]

(5.3)

Observe that the determinant of the right-hand side of the matrix equation above is in \( R nZ_p \), where \( Z_p \) denotes the localization of the prime ideal \( Z \) at \( p \) (Note that \( Z_p \) is also a prime ideal of \( R \)). Hence the determinant of the first matrix in (5.3) is in \( R nZ_p \) again. Applying Lemma 5.8 to the first matrix, we have a matrix \( R \) over \( R \) such that the determinant of the matrix \( \mathcal{E} + R \mathcal{F} \) is in \( R nZ_p \). Now \( \mathcal{E} + R \mathcal{F} \) is an \( R \)-stabilizing controller.

5.2. Local-Global Principle in Stabilizability. Next we present the local-global principle below about the feedback stabilizability as the second intermediate result of this section.

**Proposition 5.10.** Suppose that the plant \( \mathcal{P} \) is causal. Then the following statements are equivalent:

(i) \( \mathcal{P} \) is stabilizable.

(ii) \( \mathcal{P} \) is \( A_p \)-stabilizable for each prime ideal \( p \) in \( \text{Spec}(A) \).

(iii) \( \mathcal{P} \) is \( A_m \)-stabilizable for each maximal ideal \( m \) in \( \text{Max}(A) \).

(iv) For every prime ideal \( p \) in \( \text{Spec}(A) \), \( \mathcal{P} \) has either its right- or left-coprime factorization over \( A_p \).

(v) For every maximal ideal \( m \) in \( \text{Max}(A) \), \( \mathcal{P} \) has either its right- or left-coprime factorization over \( A_m \).

Further, if \( \mathcal{P} \) is stabilizable, then there exists a causal stabilizing controller of \( \mathcal{P} \).

Note here that by virtue of Proposition 5.4, if (iv) holds (if (v) holds), then the plant \( \mathcal{P} \) has both right-/left-coprime factorizations over \( A_p \) (over \( A_m \)).

We consider that this is a generalization of Proposition 2 of [18] in which the strict causality of the plant is assumed (see [19] for the definition of the strict causality). On the other hand, we assume only that the plant is causal.

Now we begin to prove Proposition 5.10.

**Proof of Proposition 5.10.** Since the following implications are obvious:

\[
\begin{array}{cccc}
(i) & \longrightarrow & (ii) & \longrightarrow \ \\
& & (iii) & \longrightarrow \ \\
& & (iv) & \longrightarrow \ \\
& & (v) & \longrightarrow
\end{array}
\]
by virtue of Proposition\textsuperscript{5.4}, we only show that (v) implies (i).

Suppose that (v) holds. Let $N$, $D$, $\mathcal{P}$, and $\mathcal{Q}$ be matrices over $\mathbb{A}$ with $P = N D^{-1} = \mathcal{P}^{-1} \mathcal{Q}$ such that $D$ and $\mathcal{Q}$ are $\mathbb{Z}$-nonsingular (recall that $P$ is causal). By Proposition\textsuperscript{5.4}, $P$ has both right-/left-coprime factorizations over $\mathbb{A}_m$ with $m \geq 2$ $\mathbb{M}$ $\text{ax}(\mathbb{A})$. As in the proof of Proposition\textsuperscript{5.4}, for each $m$ in $\mathbb{M}$ $\text{ax}(\mathbb{A})$, there exist matrices $Y_m$, $X_m$, $\mathcal{P}_m$, $\mathcal{Q}_m$, $N_m$, $D_m$, $\mathcal{N}_m$, $\mathcal{Q}_m$, $V_m$, and $W_m$ over $\mathbb{A}_m$ such that

\begin{equation}
N D = N_m V_m; \quad \mathcal{P} \mathcal{Q} = W_m \mathcal{P}_m \mathcal{Q}_m ;
\end{equation}

\begin{equation}
\mathcal{P}_m N_m + \mathcal{Q}_m D_m = E_m; \quad \mathcal{P}_m V_m + \mathcal{Q}_m X_m = E_n
\end{equation}

hold over $\mathbb{A}_m$. For each $m \geq 2$ $\mathbb{M}$ $\text{ax}(\mathbb{A})$ let $q_h$ be an arbitrary but fixed element of $\mathbb{A}_m$ such that the six matrices $q_h N_m$, $q_h N_m \mathcal{P}_m$, $q_h D_m$, $q_h D_m \mathcal{P}_m$, $q_h$, and $q_h \mathcal{Q}_m$ are over $\mathbb{A}$.

For a subset $B$ of $\mathbb{A}$, denote by $\mathcal{B}$ the set of all maximal ideals of $\mathbb{A}$ with $B \subseteq m$, that is, $\mathcal{B} = \{m \in 2 \mathbb{M} \text{ax}(\mathbb{A}) : B \subseteq m \}$. Since $q_h 2 \mathbb{A}_m$, we have $m \geq 2$ $\mathbb{M}$ $\text{ax}(\mathbb{A})$. Thus $\mathbb{M}$ $\text{ax}(\mathbb{A}) = \bigcup_{m \geq 2} \mathbb{M}$ $\text{ax}(\mathbb{A})$, and it follows that $\mathbb{M}$ $\text{ax}(\mathbb{A}) = \{ \mathcal{B} \}$. Therefore there exist $r_1; \ldots; r_t$ in $\mathbb{A}$ with $1 = r_1 q_h + \mathcal{Q}_m E_n$.

Next we want to consider that at least one of $m_1; \ldots; m_t$ contains $\mathbb{Z}$. In the case where every $m_i$ in $m_1; \ldots; m_t$ does not contain $\mathbb{Z}$, we reconstruct $t$, $r_1$, and $q_{h_i}$ as follows. We first pick an $m_i$ in $m_1; \ldots; m_t$ and reconstruct $t$, $r_1$, and $q_{h_i}$ as follows. We first pick an $m_i$ in $m_1; \ldots; m_t$ and then let $r_1$ be $1 = r_1 q_{h_i} + r_1$. Then we have $r_1 = r_1 q_{h_i} + r_1$. In the case, $m_i \subseteq \mathbb{Z}$. Hence we can assume without loss of generality that at least one of $m_1; \ldots; m_t$, say $m_1$, contains $\mathbb{Z}$.

Observe then that the following equality holds:

\begin{equation}
1 = (r_1 q_{h_1} + r_1) q_{h_1} + (r_2 q_{h_1} + r_2) q_{h_2} + \cdots + (r_t q_{h_1} + r_t) q_{h_t};
\end{equation}

At least one of $r_1 q_{h_1} + r_1 = 1$ and $r_1$ must be in $\mathbb{A}_n \mathbb{Z}$. Thus in the case $r_1 \subseteq \mathbb{Z}$, we can reassign $r_1$ as in (5.6), so that $r_1$ is in $\mathbb{A}_n \mathbb{Z}$. Therefore we can assume without loss of generality that $r_1 q_{h_1} 2 \mathbb{A}_n \mathbb{Z}$.

Consider here the following matrix

\begin{equation}
\begin{bmatrix}
\mathcal{P} & \mathcal{P} & \cdots & \mathcal{P} \\
\mathcal{P} & \mathcal{P} & \cdots & \mathcal{P} \\
\mathcal{P} & \mathcal{P} & \cdots & \mathcal{P} \\
\mathcal{P} & \mathcal{P} & \cdots & \mathcal{P}
\end{bmatrix}
\end{equation}
which is over A. For short we partition (5.7) as
\[
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\]
In the case where \( H_{22} \) is \( \mathbb{Z} \)-nonsingular, letting \( C = H_{22}^{-1} H_{21} \) \( \mathbb{P}^m \) we can check that \( H (\mathbb{P}^m \cap C) \) is equal to (5.7), which implies that \( \mathbb{P}^m \) is stabilized by \( C \). Hence in the rest of this proof we show that if \( H_{22} \) is \( \mathbb{Z} \)-singular, then \( H_{22} \) can be made \( \mathbb{Z} \)-nonsingular by reassignment \( \approx_{m_1} \) and \( \approx_{m_1} \) for an i.

First we show the \( \mathbb{Z} \)-nonsingularity of the matrices \( r_1 q_{m_1} D_{m_1} \) and \( r_1 q_{m_1} \approx_{m_1} \). Since \( r_1 q_{m_1} \in \mathbb{A} n \mathbb{Z} \), we have \( \det (r_1 q_{m_1} D) \in \mathbb{A} n \mathbb{Z} \). From the first matrix equation of (5.4), we have \( \det (r_1 q_{m_1} D) = \det (r_1 q_{m_1} D_{m_1}) \det (V_{m_1}) \). Hence \( r_1 q_{m_1} D_{m_1} \) is \( \mathbb{Z} \)-nonsingular. Analogously, from the second matrix equation of (5.4), \( r_1 q_{m_1} \approx_{m_1} \) is \( \mathbb{Z} \)-nonsingular.

Next consider the following matrix equation over \( A \):
\[
\begin{align*}
& P \left( \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right) r_1 q_{m_1} D_{m_1} \approx_{m_1} \\
& r_1 q_{m_1} \det (r_1 q_{m_1} D_{m_1}) \approx_{m_1} \\
& \det (r_1 q_{m_1} D_{m_1}) \approx_{m_1} \\
& D \quad O \\
& N \quad E_n \quad O \\
& r_1 q_{m_1} \det (r_1 q_{m_1} D_{m_1}) \approx_{m_1}
\end{align*}
\]
(5.8)

Since the matrices \( D \), \( r_1 q_{m_1} D_{m_1} \), and \( r_1 q_{m_1} \approx_{m_1} \) are \( \mathbb{Z} \)-nonsingular, so is the right-hand side of (5.8). Thus the first matrix of (5.8) is also \( \mathbb{Z} \)-nonsingular. By Lemma 5.8 and the first matrix of (5.8), there exists a matrix \( R_{m_1}^0 \) of \( \mathbb{A}^m \) such that the following matrix is \( \mathbb{Z} \)-nonsingular:
\[
\begin{align*}
& X^t \\
& r_1 q_{m_1} D_{m_1} \approx_{m_1} \\
& r_1 q_{m_1} \det (r_1 q_{m_1} D_{m_1}) R_{m_1}^0 \approx_{m_1}
\end{align*}
\]
(5.9)

Now let \( R_{m_1} \) be \( r_1 a_{m_1} \text{adj}(r_1 a_{m_1} D_{m_1}) R_{m_1}^0 \). Further we let \( \approx_{m_1} \) be the matrix \( \approx_{m_1} \), \( R_{m_1} \approx_{m_1} \), and \( \approx_{m_1} \) the matrix \( \approx_{m_1} + R_{m_1} \approx_{m_1} \), which are consistent with (5.5). Thus we can now consider without loss of generality that the matrix \( t_{i=1}^t r_1 q_{m_1} D_{m_1} \approx_{m_1} \) is \( \mathbb{Z} \)-nonsingular and so is \( H_{22} \).

5.3. Proof of Theorem 5.2. Before proving Theorem 5.2, we should prepare a small result.

Lemma 5.11. Let a 2 \( A \) and \( p \in \text{Spec}(A) \). Then \( (a)_p \) and \( (a=1) \) are isomorphic to each other as \( A_p \)-modules, where \( (a)_p \) denotes the localization,
at \( p \), of the principal ideal generated by \( a \), and \( (a=1) \) the principal ideal generated by \( a=1,2 \) \( A_p \).

The proof of the lemma is elementary and is omitted.

Now we start to prove the first result of this paper.

**Proof of Theorem 5.2.** We show first the “Only If” part and then the “If” part. (Only If) Suppose that \( P \) is stabilizable. Then by Proposition 5.10, for every prime ideal \( p \) in \( \text{Spec}(A) \), \( P \) is \( A_p \)-stabilizable. By Proposition 5.4, \( P \) has both its right-/left-coprime factorizations over \( A_p \). Suppose that \( \mathcal{I}_p N_p + \mathcal{I}_p D_p = E_m \) holds over \( A_p \) with \( P = N_p D_p^{-1} \), where the matrices \( N_p, D_p, \mathcal{I}_p, \) and \( \mathcal{I}_p \) are over \( A_p \). Then let \( t_p = [N_p \ t_p \ D_p \ t_p] \). By Binet-Cauchy formula we have 
\[
\det( t_p T_p ) = \mathcal{I}_p,
\]
which is a right-coprime factorization of \( t_p \). Thus by virtue of Lemmas 4.2 and 5.11, the ideal \( t_p \) is free (recall that \( t_p \) denotes the localization of the full-size minor ideal \( t \) at \( p \)), which is also finitely generated. This holds for every prime ideal \( p \). From Theorem IV.32 of [13], the full-size minor ideal \( t \) is projective.

(If). Suppose that the full-size minor ideal \( t \) is projective. Let \( p \) be a prime ideal in \( \text{Spec}(A) \). Then \( t_p \) is free by Theorem IV.32 of [13] again. Thus there exist \( g, a_i, \) and \( r_i \) in \( A_p \) with 
\[
g = \mathcal{I}_p r_i t_p \quad \text{and} \quad t_i = a_i g \quad \text{for every } i \in I.
\]
Since \( g = \mathcal{I}_p r_i a_i g \) and \( g \) is a nonzerodivisor, we have \( \mathcal{I}_p r_i a_i = 1 \). Recall here that \( A_p \) is local. Hence the set of all nonunits in \( A_p \) is an ideal. Thus there exists \( I_0 \) such that \( r_i a_i \) is a unit of \( A_p \). This implies that \( a_i \) is a unit of \( A_p \) and further that every \( t_i \) has a factor \( t_{i0} \) over \( A_p \) (that is, \( t_{i0} \) and \( g \) are associate). Now let \( T^{0} = T \det( i_0) \) and \( t_i = \det( i_0 T) \) for every \( i \in I \). Then 
\[
t_i = t_{i0} \det( \det( i_0 T) ) \quad \text{and} \quad \det( i_0 T) = t_{i0} E_m \quad \text{hold. Since} \quad \det( \det( i_0 T) ) = t_{i0}^{-1},
\]
every \( t_{i0} \) has a common factor \( t_{i0}^0 \).

Suppose that \( i \) is an integer with \( i \in I_0 \) and \( 1 \leq i \leq m + n \). Suppose further that \( i_0, \ldots, i_{k0} \) are elements in \( I_0 \) with ascending order. Now let \( I = \{ i_0, i_0 + 1, \ldots, i_{k0} - 1, i_{k0}, i_{k0} + 1, \ldots, i_{k0} - \} \). Then \( t_i \) is expressed as \( t_{ik} t_{i0}^{-1} \) where \( t_{ik} \) is the \( (i,k) \)-entry of the matrix \( T^{0} \). Since \( t_{i0} \) has a factor \( t_{i0}^0 \), \( t_{ik} \) has a factor \( t_{i0}^0 \). This fact holds for all \( i \) between 1 and \( m + n \) but \( i \in I_0 \). As a result, \( t_{i0} \) is a common factor of all entries of \( T^{0} \).

Let \( \mathcal{T}^{0} = \mathcal{T}^{0} T_{i0} \) over \( A_p \). Since \( \mathcal{T}^{0} \) is the identity matrix, the matrix \( \mathcal{T}_{i0} \) itself is a left inverse of \( T^{0} \). Let \( \mathcal{E}_{i0} \) and \( \mathcal{E}_{i0} \) be matrices with \( \mathcal{E}_{i0} \mathcal{E}_{i0} = \mathcal{I}_0 \). Further we let \( N_{i0} \) and \( D_{i0} \) be matrices over \( A_p \) with \( \mathcal{T}^{0} = [N_{i0} \mathcal{E}_{i0} \mathcal{E}_{i0} D_{i0}] \). Then we obtain \( \mathcal{E}_{i0} N_{i0} + \mathcal{E}_{i0} D_{i0} = E_m \) over \( A_p \), which is a right-coprime factorization over \( A_p \) of the plant \( P \). Therefore by Proposition 5.10, \( P \) is stabilizable. \( \square \)

**5.4. Proof of Proposition 5.3.** Now we prove Proposition 5.3. We first prepare the following local-global principle on ideals.
**Lemma 5.12.** Let $\mathbb{R}$ be a commutative ring. Let $a_1, \ldots, a_k$ be ideals of $\mathbb{R}$. Then the following statements are equivalent:

(i) $a_1 + \ldots + a_k = \mathbb{R}$.

(ii) $a_{1p} + \ldots + a_{kp} = \mathbb{R}_p$ for all prime ideal $p \in \text{Spec}(\mathbb{A})$.

(iii) $a_{1m} + \ldots + a_{km} = \mathbb{R}_m$ for all maximal ideal $m \in \text{Max}(\mathbb{A})$.

**Proof.** It is obvious that (i) implies (ii) and (ii) implies (iii). Hence we only show that (iii) implies (i).

(iii) $\Rightarrow$ (i). Suppose that (iii) holds. Let $m$ be a maximal ideal of $\mathbb{A}$. Since $\mathbb{R}_m$ is local, the set of all nonunits in $\mathbb{R}_m$ is an ideal. Hence there exists an $i_m$ with $1 \leq i_m \leq k$ such that $a_{i_m} = \mathbb{R}_m$. Thus there exists $s_m$ in $\mathbb{R}_m$ such that $s_m = a_{i_m}$.

Recalling the proof of Proposition 5.10, we have a finite number of $m_1, \ldots, m_t$ in $\text{Max}(\mathbb{R})$ and $r_1, \ldots, r_t$ in $\mathbb{R}$ such that $1 = r_1 s_{m_1} + \ldots + r_t s_{m_t}$ over $\mathbb{R}$. For every $1 = 1$ to $t$, $r_2 s_{m_2}$ is an element of $a_i$ with $i = i_{m_2}$. Therefore we have (i).

**Proof of Proposition 5.13.** Suppose that $\mathbb{R}$ is a unique factorization domain. Since the “If” part is obvious, we prove only the “Only If” part.

(Only If). Let $a_1, \ldots, a_k$ be in $\mathbb{R}$. Suppose that $(a_1, \ldots, a_k)$ is projective. If all $a_1, \ldots, a_k$ are zero, the proof is obvious. Thus in the following we suppose that at least one of $a_1, \ldots, a_k$ is nonzero. Since $\mathbb{R}$ is a unique factorization domain, there exists a nonzero greatest common factor of $a_i$'s, denoted by $g$. Thus there exist $b_i$'s in $\mathbb{A}$ with $b_i g = a_i$. Then $(b_1, \ldots, b_k)$ is projective again. For any prime ideal $p$ in $\text{Spec}(\mathbb{R})$, $(b_1, \ldots, b_k)_p$ is free of rank 1. Since there is no nonunit common factor among $b_i$'s over $\mathbb{R}$, $(b_1, \ldots, b_k)_p = \mathbb{R}_p$. By Lemma 5.12, $(b_1, \ldots, b_k)_p = \mathbb{R}$. Hence $(a_1, \ldots, a_k) = (g)$, which is free.

**5.5. Full-Size Minor Ideals of $P$, $C$, and $H (P; C)$.** Now that we have obtained Theorem 5.2, we know that the projectivity of the full-size minor ideal of the plant connects with the feedback stabilizability of the plant. Since $P$, $C$, and $H (P; C)$ are transfer matrices over $\mathbb{F}$, we can define the full-size minor ideals of $C$ and $H (P; C)$ analogously to that of $P$.

We present here the relationship among the full-size minor ideals of $P$, $C$, and $H (P; C)$.

**Proposition 5.13.** Let $t_P$, $t_C$, $t_H (P; C)$ be the full-size minor ideals of $P$, $C$, and $H (P; C)$, respectively. Then $t_H (P; C)$ is isomorphic (as an $\mathbb{A}$-module) to the ideal generated by $t_i t_j$'s for all $t_1, t_2$ and all $t_2, t_3$. 

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This proposition holds even if \( C \) is not a stabilizing controller of \( P \). Before proving this proposition, we present a preliminary lemma.

**Lemma 5.14.** Let \( A \) and \( B \) are matrices over \( \mathbb{R} \) such that \( B = U A \), where \( U \) is a unimodular matrix over \( \mathbb{R} \). Then the ideal generated by the full-size minors of \( A \) is equal to that of \( B \).

The proof of this lemma is straightforward and omitted.

**Proof of Proposition 5.13.** By virtue of Lemma 4.2, we suppose without loss of generality that \( N \) and \( N_c \) are matrices over \( A \) and \( d \) and \( d_c \) in \( A \) with \( P = N d \) and \( C = N_c d_c \). Let \( A \) and \( B \) be the following matrices:

\[
A = \begin{pmatrix}
N_c & 0 & 3 \\
\delta E_n & 0 & 7 \\
0 & N & 5 \\
0 & \delta E_m & \\
\end{pmatrix}; \\
B = \begin{pmatrix}
Q \\
S \\
\end{pmatrix};
\]

where \( Q = \delta E_n N ; S = \delta E_n \) \( \delta E_m \) : \( \delta E_m \) :

Then we can see that there exists a unimodular matrix \( U \) with \( B = U A \) and that \( H(P; C) = SQ \). Let \( a \) be the ideal generated by the full-size minors of \( A \) and \( t_{p,c} \) be the ideal generated by \( t_{1} t_{2} \)'s for all \( t_{1} \ 2 \ \ t_{p} \) and all \( t_{2} \ 2 \ t_{c} \). Then by Lemma 5.14, \( t_{p,c} \) is isomorphic to \( a \) as \( A \) -modules. Also by Binet-Cauchy formula, \( a \) \( \delta t_{p,c} \). Hence we obtain \( t_{p,c} \) \( \delta t_{p,c} \). \( \square \)

6. Stabilizability in terms of Coprimeness of Quotient Ideals. In this section, we present one more necessary and sufficient condition of the feedback stabilizability which is given in terms of quotient ideals.

**Theorem 6.1.** Let \( P \) be a causal plant of \( P \ n \ m \). Then the plant \( P \) is stabilizable if and only if the ideal \( X \)

\[(6.1) \ (t_{1}) : t \]

is equal to \( A \).

The ideal of \( (6.1) \) will be considered as another generalization of the reduced minors. This will be presented later as Proposition 6.5.

We note that the result above can be considered as a generalization of Theorem 2.1.1 in [17] given by Shankar and Sule as well as a generalization of Theorem 3.6. They considered the single-input single-output case. In Theorem 2.1.1 of [17], they stated the feedback stabilizability of the given plant in terms of the coprimeness of the ideal quotients as \( (6.1) \). As a result, Theorem 6.1 can be considered as a multi-input multi-output version of Theorem 2.1.1 of [17].
In order to prove Theorem 6.1, we prepare a relationship between projective modules and quotient ideals as follows.

**Theorem 6.2.** Let \( R \) be a commutative ring and \( a_1, \ldots, a_k \in R \). Then \((a_1, \ldots, a_k)\), that is, the ideal generated by \( a_1, \ldots, a_k \) is projective if and only if the following equation holds:

\[
\chi^k \left( (a_i) : (a_1, \ldots, a_k) \right) = R : \quad i = 1
\]

(6.2)

Once we obtain Theorem 6.2, the proof of Theorem 6.1 is directly obtained from Theorems 5.2 and 6.2. Thus we will present only the proof of Theorem 6.2, which will be given after showing intermediate results (Lemmas 6.3 and 6.4).

**Lemma 6.3.** Let \( R \) be a commutative ring and \( a_1, \ldots, a_k \in R \). If \((a_1, \ldots, a_k)\) is free, then (6.2) holds.

**Proof.** As in the proof of Proposition 5.3, if all \( a_1, \ldots, a_k \) are zero, the proof is obvious. Thus in the following we assume that at least one of \( a_1, \ldots, a_k \) is nonzero. Then there exist a nonzero \( g \in R \) and \( b_i \in R \) for \( i = 1 \) to \( k \) such that

\[
\left< g \right> = (a_1, \ldots, a_k) \quad \text{and} \quad a_i = g b_i \quad \text{for all} \quad i = 1 \to k.
\]

If \( g \) was a zerodivisor, the principal ideal \( \left< g \right> \) could not be free. Hence \( g \) is a nonzerodivisor. Now we have

\[
r_1 b_1 + \ldots + r_k b_k = 1 \quad \text{(6.3)}
\]

Since \( b_i (a_1, \ldots, a_k) \) \( (a_i) \) for all \( i \), we have \( b_2 \) \( (a_1, \ldots, a_k) \left( (a_i) : (a_1, \ldots, a_k) \right) \). It follows from (6.3) that we now have (6.2). \( \Box \)

**Lemma 6.4.** Let \( R \) be a commutative ring, \( a; b \) ideals of \( R \), and \( p \) a prime ideal of \( R \). Denote by \( (a : b)_p \) the localization of the quotient ideal \( (a : b) \) at \( p \). Further let \( (a_p : b_p) \) be the quotient ideal of \( R_p \), where \( a_p \) and \( b_p \) are localizations of ideals \( a \) and \( b \) at \( p \), respectively. Then \( (a : b)_p = (a_p : b_p) \) holds.

Now we are in a position to prove Theorem 6.2.

**Proof of Theorem 6.2.** By the same reason as in the proofs of Proposition 5.3 and Lemma 6.3, we assume that at least one of \( a_1, \ldots, a_k \) is nonzero. (If). Suppose that (6.2) holds. Then there exist \( x_i \in (a_i) : (a_1, \ldots, a_k) \) for \( i = 1 \) to \( k \) such that

\[
1 = \sum_{i=1}^k x_i.
\]

By appropriate changes of \( a_1, \ldots, a_k \), we assume without loss of generality that all \( x_1, \ldots, x_k \) are nonzero with \( 1 \leq k \) and all \( x_{k+1}, \ldots, x_k \) are zero subject to \( k_0 < k \). Observe that for each \( i \) between 1 and \( k'_0 \), \( (a_1, \ldots, a_k)_{x_i} = (a_i)_{x_i} \) over \( A_{x_i} \), where \( (a_1, \ldots, a_k)_{x_i} \) and \( (a_i)_{x_i} \) denote the localizations of \( (a_1, \ldots, a_k) \) and \( (a_i) \) at \( x_i \), respectively. Hence for each \( i \)
between 1 and $k$, $(a_1; \ldots; a_k)_{x_i}$ is free over $A_{x_i}$. Therefore by Theorem IV.32 of [13], $(a_1; \ldots; a_k)$ is projective as $R$-module.

(Only If). Suppose that $(a_1; \ldots; a_k)$ is projective. Then again by Theorem IV.32 of [13], for each $p$ in $\text{Spec}(R)$, $(a_1; \ldots; a_k)_p$ is free over $R_p$. By Lemma 6.3 we have

$$\chi^k \left( \left( (a_1)_p : (a_1; \ldots; a_k)_p \right) = R_p \right)_{i=1}$$

for each $p$ in $\text{Spec}(R)$. Then (6.4) can be rewritten as follows by Lemma 6.4:

$$\chi^k \left( (a_1 : (a_1; \ldots; a_k))_p = R_p : \right)_{i=1}$$

Since this holds for every $p$ in $\text{Spec}(R)$, applying Lemma 5.12 to (6.5) we obtain (6.2).

We now connect the reduced minors with the quotient ideal of (6.1) provided that $A$ is a unique factorization domain.

**Proposition 6.5.** Suppose that $A$ is a unique factorization domain. Let $a_\mathbb{Z}$ denote the reduced minor of the matrix $T$ with respect to $I \leq I$. Then $(a_\mathbb{Z}) = (\langle t_\mathbb{Z} \rangle : t)$ holds for every $I \leq I$.

**Proof.** We first show (i) $(a_\mathbb{Z}) = \left( (t_\mathbb{Z}) : t \right)$ and then (ii) the opposite inclusion. (i). For every $I \leq I$, $a_\mathbb{Z} t_\mathbb{Z} = a t_\mathbb{Z}$ holds, which implies that $a_\mathbb{Z} \leq (\langle t_\mathbb{Z} \rangle : (t_\mathbb{Z}))$. Hence $a_\mathbb{Z} = (\langle t_\mathbb{Z} \rangle : t)$.

(ii). Suppose that $t_\mathbb{Z}$ is an element of the quotient ideal $(\langle t_\mathbb{Z} \rangle : t)$. Then for every $I \leq I$, there exists $t_\mathbb{Z} \in A$ such that $t_\mathbb{Z} t_\mathbb{Z} = t_\mathbb{Z} t_\mathbb{Z}$ holds and so $t_\mathbb{Z} t_\mathbb{Z} = t_\mathbb{Z} a_\mathbb{Z}$. Since this equality holds for every $I \leq I$, $t_\mathbb{Z}$ has a factor $a_\mathbb{Z}$. Hence $a_\mathbb{Z}$ has a factor $a_\mathbb{Z}$. $\square$

From the result above, the reduced minor of the matrix $T$ with respect to $I \leq I$ is equal to the quotient ideal $(\langle t_\mathbb{Z} \rangle : t)$ up to a unit multiple of $A$ provided that $A$ is a unique factorization domain.

Now that we have shown a new criterion (6.1) of the feedback stabilizability, in the following we present the relationship between generalized elementary factors and (6.1) by using radicals of ideals.

**Theorem 6.6.** Let $p$ denote the generalized elementary factor of the plant $P$ with respect to $I$ in $I$. Then the radical of $p$ is equal to the radical of $(\langle t_\mathbb{Z} \rangle : t)$.

Before proving this result, we present an analogous result of Lemma 6.4.

**Lemma 6.7.** Let $R$ be a commutative ring, $a;b$ ideals of $R$, and $f \leq f$ R. Denote by $(a : b)_{f}$ the localization of the quotient ideal $(a : b)$ at $f$. Further
let \((a_f : b_f)\) be the quotient ideal of \(R_f\), where \(a_f\) and \(b_f\) are localizations of principal ideals \(a\) and \(b\) at \(f\), respectively. Then \((a : b)_f = (a_f : b_f)\) holds.

Analogously to Lemma 6.4, the proof of this lemma is omitted.

**Proof of Theorem 6.6.** Let \(I\) be fixed. We first show (i) \(\prod_I \check{(t_2)} : t\) and then (ii) \(\prod_I \check{(t_2)} : t\). They are sufficient to prove this theorem.

(i). Let \(a\) be an arbitrary but fixed element of \(\prod_I\). Then there exists a matrix \(K\) over \(A\) with \(T = K \cdot I\). Then for every \(I^0 \supseteq I\), we have \(\prod_{\check{I}} T = \prod_{\check{I}} K \cdot I\), so that \(m_t^2 = \det(\prod_{\check{I}} K) t_2\). This implies \(m_t^2 \cdot (t_2) : (t_2)^2\). Hence we have \(m_2 \cdot t^2_2 \cdot (t_2) : (t_2)^2\).

(ii). Let \(t\) be an arbitrary but fixed element of \((t_2\) : \(t\)). Then \((t_2) : (t_2) = A\) and hence \((t_2) : t = A\) by Lemma 6.7. This implies that \((t_2) : t = A\) and further that every full-size minor of \(T\) has a factor \(t\) over \(A\). Since \(t\) is a factor of \(\det(\prod_I)\), it is a nonzerodivisor of \(A\). Now let \(T^0 = T(\text{adj}(\prod_I))\) and \(T^0_0 = \det(\prod_{\check{I}} T^0)\) for every \(I^0 \supseteq I\). Then \(T^0 = t_2 \cdot \det(\text{adj}(\prod_I))\) and \(\prod_{\check{I}} T^0 = t_2 E_m\).

Analogously to the proof of Theorem 5.2, we can show that every entry of \(T^0\) has a factor \(t_2\). Let \(T^0 = T_0 \cdot t_2\) over \(A\). Then \(T^0 = T_0 \cdot I\) holds over \(A\). Hence there exists an integer \(l\) such that \(l T^0\) can be considered over \(A\) and further \(l T = l T^0 \cdot I\) holds over \(A\). Now letting \(K = l T^0 \cdot I\), we have that \(l\) is an element of \(\prod_I\) and hence \(l T = \prod_I T^0\).

In the case where \(A\) is a unique factorization domain, we obtain the following result which connects Theorems 5.4 and 5.6.

**Theorem 6.8.** Suppose that \(A\) is a unique factorization domain. Let \(\mathcal{P}\) be a causal plant and \(I\) in \(\mathcal{I}\). Then the radical of the elementary factor of the matrix \(T\) with respect to \(I\) is equal to the radical of the reduced minor of \(T\) with respect to \(I\) up to a unit multiple.

**Proof.** Let \(\ell_2\) denote the elementary factor of the matrix \(T\) with respect to \(I\). Also let \(\ell_0\) denote the reduced minor of \(T\) with respect to \(I\).

In the case where \(A\) is a unique factorization domain, the generalized elementary factor of the plant \(\mathcal{P}\) with respect to \(I\) is equal to the principal ideal \((\ell_0)\). Thus, by Theorem 5.6, \((\ell_0) = (\ell_2) : (t_2)\). By virtue of Proposition 6.5, we have \(\prod_I (\ell_2) = (a_2)\).

**7. Concluding Remarks.** We have presented two generalization of the reduced minors. One is the full-size minor ideal. Its projectivity is a criterion of the feedback stabilizability (Theorem 5.2). The other is quotient ideals in (6.2). Their coprimeness is a criterion of the feedback stabilizability (Theorem 6.1).
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