The lattice Schwinger model with the SW action

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Abstract

We perform a model study on the 2-flavor lattice Schwinger model using standard Wilson and \(O(a)\) improved Sheikholeslami-Wohlert (SW) action. We find, that the phase diagram is altered, the critical line shifted closer towards its continuum value \(\kappa_c = 0.25\). We find no improvement in the rotation invariance of meson propagators; the scaling of the Schwinger mass is considerably improved, high momentum states are not. The additional cost of \(\approx 30\%\) CPU-time is highly justified when calculating masses, but not for high momentum observables.

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1 Introduction

In recent years, there has been an increasing interest in possible use of improved actions to lattice QCD. There exist different improvement schemes for lattice actions and observables, all which aim at the extraction of continuum physics from simulations on coarse lattices.

The simplest of these improvement schemes is the clover action, an $O(a)$ improvement of the Wilson fermion action suggested by Sheikholeslami and Wohlert [1], who implemented the Symanzik improvement [2] for the Dirac operator. Several calculations in quenched and some in full QCD have been performed using this action especially to calculate both heavy and light hadron masses. The results obtained seem promising, so that one might want to check the improvement effects of the clover action in a simple and well-known toy model. Here we study the effects of the $O(a)$ improvement of the Wilson action in the full lattice Schwinger model. This model seems optimally suited for this purpose.

Massless QED in 1 + 1 dimension was first studied by Schwinger [3] as an example of an explicitly solvable QFT. In subsequent papers [4] the model was generalized to allow for fermion masses and include different flavors.

The Schwinger model became of considerable interest during the early 1970’s, because it shows some remarkable features – like fermion confinement or a $\Theta$-vacuum structure – which are reminiscent of QCD. For the purpose of this paper, it is sufficient to note, that the massless $N$-flavor Schwinger model has a spectrum which consists of one massive isosinglet boson and a isomultiplet of $(N^2 - 1)$ massless fermion-antifermion bound states (for further discussion cf. e.g. [5]). The mass of the isosinglet state, the 'Schwinger mass' (in lattice units) is given by

$$m_s = \sqrt{\frac{N}{\beta \pi}}. \quad (1)$$

where $\beta = 1/(e^2 a^2)$ denotes the dimensionless gauge coupling.

In the next section some technical details about the definition and the numerical simulation will be given and in Sec. 3 we discuss the effect of the improvement on the rotation invariance of the propagator, the phase diagram and the dispersion relations of the Schwinger boson.
2 \( O(a) \) improvement and simulation

The fermionic part of the Wilson action for the massive, N-flavor Schwinger model is

\[
S_W = \sum_x \bar{\Psi}_x \Psi_x - \kappa \sum_{x, \mu} \bar{\Psi}_{x+\hat{\mu}} (1 + \sigma_\mu) U_\mu^\dagger \Psi_x + \bar{\Psi}_x (1 - \sigma_\mu) U_\mu \Psi_{x+\hat{\mu}}.
\] (2)

The fields have 2\( N \) components, \( \sigma_\mu \) are the Pauli matrices. This action has a classical \( O(a) \) error introduced by the Wilson term, which can be removed by rotating the fermion fields

\[
\Psi \rightarrow (1 - \frac{1}{2} \hat{\mathcal{D}}) \Psi, \quad \bar{\Psi} \rightarrow \bar{\Psi} (1 + \frac{1}{2} \hat{\mathcal{D}})
\] (3)

and adding a term

\[
S_{SW} = -\kappa c_{SW} \frac{i}{2} \sum_x F_{\mu\nu} \sigma_{\mu\nu} \bar{\Psi}_x \Psi_x
\] (4)

to the action. \( F_{\mu\nu} \) denotes the clover-leaf discretization of the field strength tensor. The resulting lattice Dirac operator now has discretization errors of \( O(a^2) \).

We note in passing that for this model one may introduce a topological charge in its ‘geometric’ definition \( \nu \propto \sum_{\mu, \nu} F_{\mu, \nu} \) [6]. Thus \( S_{SW} \) introduces a local coupling of \( \bar{\Psi} \Psi \) to \( \nu \sigma_3 \).

The coefficient \( c_{SW} \), which generally has to be determined non-perturbatively at finite \( \beta \), is just \( c_{SW} = 1 \) in 2D, since the gauge coupling \( e \) is dimensionful and therefore \( c_{SW} = 1 + O(e^2 a^2) \), which only gives corrections to higher order in \( a \).

Since the gauge part of the Wilson action

\[
S_G = \frac{1}{2} \sum_p \text{Tr}(U_p + U_p^\dagger)
\] (5)

has \( O(a^2) \) discretization errors, the total action

\[
S = S_W + S_{SW} + S_G
\] (6)

together with [6] gives a lattice version of the Schwinger model with discretization errors of \( O(a^2) \).
We have done a MC simulation of the Schwinger model, using both the Wilson and the SW action, with two degenerate flavors of dynamical fermions. We used a hybrid Monte Carlo algorithm with 10 integration steps and a trajectory length tuned for a 0.8 acceptance rate of the MC step. For fermion matrix inversion we used a standard conjugate gradient algorithm, and there was no problem with its convergence or stability. In that algorithm the doubling of the fermionic species is required in order to guarantee positivity of the fermionic determinant – one of our reasons to study the Schwinger model with two flavors.

In our implementation, the speed loss when including the clover term into the action is consistently \( \approx 30\% \).

Depending on the lattice size (from \( 8 \times 8 \) up to \( 18 \times 18 \)) and couplings, we skipped 20 to 100 configurations between measurements and finally had between \( 2 \times 10^3 \) and \( 10^4 \) independent configurations. To check for ergodicity, we measured the topological charge (using the geometric definition) and observed, that in each run the system tunneled several times between different topological sectors.

3 Results

3.1 Phase diagram

The additional term in the action affects the critical \( \kappa \) value where the theory has effectively massless fermions. Since the clover action is supposedly closer to the continuum (has smaller corrections to leading scaling behavior), we might expect less renormalization of \( \kappa_c \) with a value closer to its continuum value of 0.25 even for small \( \beta \).

In order to determine \( \kappa_c(\beta) \) we follow suggestions [7] to utilize the PCAC relation as discussed in [8] for the Schwinger model. At each \( \beta \) we measured an observable proportional to the effective fermion mass for 5 different \( \kappa \) values and determined \( \kappa_c \). Our result for a \( 8 \times 8 \) lattice is compared with the result for the original Wilson fermion action in Fig. [1].

The values \( \kappa_c \) for the clover action are in fact closer to 0.25. However, both, Wilson and clover action have corrections

\[
\kappa_c = 0.25 + \mathcal{O}(a^2) = 0.25 + \mathcal{O}(1/\beta). \tag{7}
\]

The dependence on the finite lattice volume is weak and – for the Wilson action – is discussed elsewhere [8].
Figure 1: Phase diagram for Wilson and clover action on an $8 \times 8$ lattice. The diamonds indicate the results (Wilson action) of an extrapolation to infinite lattice volume from results at various lattice sizes [8]. We find that $\kappa_c$ is closer to its continuum value 0.25 for the clover action.

3.2 Spectrum of the Dirac operator

The spectral distribution of the Dirac operator is of some interest with regard to topological and chiral properties of the system. Recent studies for the Schwinger model [9, 10, 11, 12] and for the SW-action in 4D [13] have emphasized these aspects. Fig. 2 should be compared with typical spectra for the pure Wilson Dirac operator (cf. e.g. [11]).

As compared to the Wilson action with its $\lambda \leftrightarrow \bar{\lambda}$ and $(1 - \lambda) \leftrightarrow (\lambda - 1)$ symmetries we observe for the SW operator an agglomeration of eigenvalues at larger distances from 0. This feature may improve somewhat the separation of the low-lying states from the doubling states already at moderate $\beta$. For the continuum limit the distribution density of small eigenvalues is of relevance. However, the distribution is not much closer to the circular shape one obtains for fixed point actions [14] or Neuberger-projected actions [15]. We cannot identify a clear signal of improvement in this respect for the SW Dirac operator.
Figure 2: The typical shape of the eigenvalue spectrum for the SW action; here shown for a configuration on a $16 \times 16$ lattice determined at $\beta = 2$ and $\kappa_c$.

### 3.3 Rotation invariance

We measured the propagators of the pseudoscalar isotriplet ($\pi$) mesonic bound states. In the 2-flavor model these modes are expected to be massless at $\kappa_c$. For short distances on the lattice, the rotation invariance of these propagators is clearly broken when using Wilson action. Generally, improved actions are expected to show better rotation invariance (cf. the Schwinger model FP action study in [16]).

One must be careful not to compare these quantities simply at the same values of $\beta$ and $\kappa$, but at values corresponding to comparable discretization of the same continuum theory. We choose to compare the propagators at $\beta = 2$ and the respective $\kappa_c$, so that we have a discretization of the massless (fermion) theory in both cases.

The results are shown in Fig.3 We find no noticeable improvement of rotation invariance when using the clover action. The relative error of the diagonal $(1,1)$ propagator compared to the $(1,0)$ and $(2,0)$ propagators is $\approx (18 \pm 1)\%$ for both actions. This is the first indication, that short distance (high momentum) observables are not improved.
Figure 3: Plot of $\pi$ propagator as measured at $\beta = 2$ on a $16 \times 16$ lattice at the respective $\kappa_c$. Using the clover action gives no improvement in the rotation invariance of the propagator.

### 3.4 Meson dispersion relations

Fig. 4 shows the dispersion relations for the massive (isosinglet vector) and the massless (isotriplet vector) meson at $\beta = 2$ and $\kappa = \kappa_c$ on a $12 \times 12$ lattice. No improvement for the high momentum behavior is observed. Concerning the mass of the massive boson, however, the improvement is significant. Fig. 5 shows the ratio of the observed Schwinger mass over its continuum value at $\beta = 2, \kappa = \kappa_c$ on a $12 \times 12$ and $\beta = 4.5, \kappa = \kappa_c$ on a $18 \times 18$ lattice. Assuming the continuum scaling behavior $a(\beta) \propto 1/\sqrt{\beta}$ these are different discretizations of the same physical system size $L a \propto L/\sqrt{\beta} \simeq 8.5$. The improvement when including the clover term is quite obvious. The scaling corrections are remarkably smaller for the clover results.

This behavior, the considerable improvement of low momentum states and the lack thereof for high momentum states, is no surprise when one considers, that the construction of Symanzik improved actions aims at improving the fermion dispersion relations for low momenta (cf. [1]). The overall improvement of the fixed point action in [16] (which, however has more than hundred terms per site in the action) is definitely more pronounced and extends to high momenta as well.
Figure 4: Dispersion relation (for $\beta = 2$, lattice size $12 \times 12$) for a massless (isotriplet-vector) and the massive (isosinglet-vector) meson. The dashed curves denote the continuum dispersion relations. High momentum values are not improved by the clover action.

Figure 5: The Schwinger mass as measured at $\beta = 2$ on a $12 \times 12$ and $\beta = 4.5$ on a $18 \times 18$ lattice over the continuum Schwinger mass. For this low momentum observable, the improvement is significant. The lattice spacing is given in multiples of the value at $\beta = 2$, i.e. $\sqrt{2/\beta}$.
4 Conclusion

We performed a Monte Carlo simulation on the 2-flavor lattice Schwinger model, comparing standard Wilson to an $O(a)$ improved Sheikholeslami-Wohlert action. We found the following results:

- The critical value of the hopping parameter $\kappa_c$ moves closer to its continuum value 0.25.
- The eigenvalue spectrum of the lattice SW Dirac operator changes but does not appear to improve significantly towards, e.g., a circular shape.
- There was no improvement in the rotation invariance of meson propagators. For the meson dispersion relations, there is no improvement in the high momentum states.
- There is considerable improvement in the scaling behavior of the Schwinger mass of the massive boson.

These observations are consistent with the fact that the SW action is constructed as an improved action for low momentum fermionic states $p \ll 1/a$. While there is considerable improvement in low momentum observables, high momentum observables are largely unaffected by the addition of the clover term.

For the full fermion QCD simulations this suggests that it is preferable to use the clover action as long as one is only interested in the mass spectrum of the theory. On the other hand, some observables will likely show no improvement at all, especially those connected with high momentum states.

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