D-branes on Stringy Calabi–Yau Manifolds

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We argue that D-branes corresponding to rational B boundary states in a Gepner model can be understood as fractional branes in the Landau–Ginzburg orbifold phase of the linear sigma model description. Combining this idea with the generalized McKay correspondence allows us to identify these states with coherent sheaves, and to calculate their K-theory classes in the large volume limit, without needing to invoke mirror symmetry. We check this identification against the mirror symmetry results for the example of the Calabi–Yau hypersurface in \( \mathbb{W}^{1,1,2,2} \).

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1. Introduction

D-branes in Calabi–Yau compactification of string theory have been the focus of a number of recent works. In this work we continue the study of D-branes at Gepner points initiated in \cite{33,4}. We will show how many results for the spectrum of rational boundary states and the corresponding brane world-volume theories can be derived starting from the linear sigma model. The basic idea is to realize the boundary states as fractional branes in the Landau–Ginzburg orbifold phase; we will show how recent mathematical work on the generalized McKay correspondence determines the identification of these boundary states as bundles in the large volume limit, and check this identification in an example against results obtained using mirror symmetry. As in \cite{15}, this framework allows identifying bound states of branes with bundles and provides explicit descriptions of their moduli spaces; we will pursue this in more detail in subsequent work.

For an overview of this line of work, we refer to \cite{12}. The starting point is Gepner’s identification of certain $\mathcal{N} = 2$ CFT’s as stringy Calabi–Yau manifolds (CYs). In \cite{33}, rational boundary states (those which can be easily obtained as orbifold products of boundary states in the individual $\mathcal{N} = 2$ minimal models) were constructed for Gepner models. This provides explicit CFT realizations of D-branes on these manifolds, and allows computing RR charges (in a natural basis at the Gepner point), as well as the number of marginal operators. It is also possible to compute superpotentials, as outlined in \cite{4} and as has been done in examples \cite{5}.

The natural extension of Gepner’s identification would be to identify these BPS boundary states with specific D-branes in the large volume limit of the same Calabi–Yau. The work \cite{4} made first steps towards such an identification. The “decoupling conjecture” made there gives strong reasons to think that B branes at any point in Kähler moduli space should be identifiable with specific holomorphic objects (bundles, coherent sheaves or complexes) in the large volume limit. Using a derivation of the Kähler moduli space from mirror symmetry \cite{4}, an explicit translation of the RR charges of the B boundary states in the $(3)^5$ Gepner model into Chern classes was made, which determines the topological type of the corresponding bundles in the large volume limit. Similar results for other Calabi–Yau manifolds have been obtained in \cite{11,25,38,32}.

In \cite{15} the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold was studied in detail, and a remarkable relation was found between the quiver gauge theory of \cite{36,37,16} and Beilinson’s construction \cite{2} of holomorphic vector bundles on $\mathbb{P}^2$: the quiver theory and mirror symmetry results reproduce this
construction, providing a very detailed correspondence between F-flat configurations of the gauge theory and holomorphic bundles in the large volume limit. It was also pointed out that the results of [4] for the quintic had a very similar relationship to Beilinson’s construction of bundles on $\mathbb{P}^4$.

The present work will explain and generalize this relationship. Besides the work above, it is inspired by a generalization of Beilinson’s construction developed in recent mathematical work on the generalized McKay correspondence [34, 23, 3].

The basic idea is to realize the Calabi–Yau threefold of interest as a submanifold of the resolution of a higher dimensional orbifold $\mathbb{C}^n/\Gamma$, define D-branes in the higher dimensional orbifold using the construction of Douglas and Moore [17], and then identify the D-branes of interest as the restriction of these to the original CY. As we explain, this procedure can also be directly motivated by the physics of boundary states in the linear sigma model construction of the CY [40, 20, 19, 21].

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2. Gepner models and quivers

2.1. Gepner models and linear sigma models

A Gepner model is a product of $r$ minimal models at level $k_i$ whose central charges $3k_i/(k_i+2)$ add to $3n$. As we will review shortly, this corresponds to a Fermat hypersurface in a weighted projective space, which if $n + r$ is even is $\mathbf{WP}(w_i)$, where $w_i = K/(k_i + 2)$ and $K = \text{lcm}\{k_i + 2\}$. If $n + r$ is odd, we adjoin $w_{r+1} = K/2$ to this list, and henceforth take $n + r$ even. One can show that for $r = n + 2$, these requirements imply that $K = \sum w_i$.

When $n = 3$, such a Gepner model can also be realized as a $(2, 2)$ linear sigma model [10]. It has a $U(1)$ gauge group, $r$ chiral superfields $Z^i$ with charges $w_i$, and a chiral superfield $P$ with charge $-K$. There is a superpotential $W_G = P \sum_i (Z^i)^{k_i+2}$. The D-flatness conditions are

$$\zeta = \sum_i w_i |Z^i|^2 - K|P|^2 \quad (2.1)$$

with an FI parameter $\zeta$, and the model has two phases depending on this parameter. The Gepner model is associated with the “Landau–Ginzburg” phase with $\zeta < 0$ and
\[ \langle P \rangle \neq 0; \] the action expanded around this configuration is a sum of \( \mathcal{N} = 2 \) Landau–Ginzburg models, while the \( U(1) \) symmetry is broken to \( \mathbb{Z}_K \). On the other hand, \( \zeta > 0 \) produces the “geometric” phase in which D-flat configurations with \( P = 0 \) parameterize the weighted projective space \( \mathbf{WIP}(w_i) \), while the condition \( 0 = \partial W_G/\partial P \) defines the CY as a hypersurface in this space.

It will be useful to have a picture of the space of D-flat configurations (in other words, the vacua of the corresponding theory with no superpotential) in the two phases. The general D-flat configuration in the geometric phase allows \( P \neq 0 \) and the total space is a line bundle over \( \mathbf{WIP}(w_i) \). For \( K = \sum w_i \) this is the anticanonical bundle, and the total space is itself a Calabi–Yau, generically singular because of the singularities of \( \mathbf{WIP}(w_i) \). Let us denote this Calabi–Yau as \( X(w_i) \) or simply \( X \).

Similarly, in the Landau–Ginzburg phase the general D-flat configuration has \( Z^i \neq 0 \). The condition (2.1) determines \( P \) and since \( \langle P \rangle \neq 0 \) always, the \( U(1) \) gauge symmetry is always broken to \( \mathbb{Z}_K \). Thus the D-flat moduli space in this phase is \( \mathbb{C}^r/\mathbb{Z}_K \) with the \( \mathbb{Z}_K \) action defined by the action of a generator

\[
g(Z^i) = e^{2\pi i w_i/K} Z^i.
\]

Note that this generates a discrete subgroup of \( SU(r) \), so this noncompact orbifold is also a CY.

In a later section, we will review the toric description of these configuration spaces and the relation between these two phases. The general idea is that the algebra of holomorphic functions on the configuration space is independent of the D-flatness conditions, and thus must be the same in the two phases. Thus we can consider the space \( X(w_i) \) as a (partial) resolution of the noncompact orbifold \( \mathbb{C}^r/\mathbb{Z}_K \). In both phases, the superpotential will confine the theory to the CY \( 3 \) as a hypersurface in the exceptional divisor, \( Z^i = 0 \) in \( \mathbb{C}^r/\mathbb{Z}_K \), and the resolution of this point \( \pi^{-1}(0) \) in \( X \).

2.2. B boundary states

Our basic claim is that the rational B boundary states can be thought of as the restriction of the “fractional brane” states of the \( \mathbb{C}^r/\mathbb{Z}_K \) orbifold to the CY \( 3 \).

A fractional brane state in a \( \mathbb{C}^r/\Gamma \) orbifold is a Dirichlet boundary state in \( \mathbb{C}^r \), with an additional choice of an irreducible representation of \( \Gamma \). A collection of fractional branes is labeled by a representation \( R \) of \( \Gamma \) or equivalently the multiplicities \( n_a \) of the irreps \( \gamma_a \).
The world-volume theory of the collection is then derived from the world-volume theory of \( \dim R \) branes in \( \mathbb{C}^r \) by projecting on invariants under the action of \( \Gamma \) on the fields twisted by the \( \gamma_a \)'s; in particular vectors \( Z^i \) in \( \mathbb{C}^r \) (such as those parameterizing transverse motion of the branes) are projected as

\[
\gamma_R(g)^{-1} Z^i \gamma_R(g) = (\gamma_{def})^i_j(g) Z^j.
\]

These definitions do not require \( \hat{c} \leq 10 \) or even that the bulk theory of interest be a conformal field theory. Thus we can apply them directly to the LG orbifold phase of the linear sigma model. It is known that Dirichlet boundary conditions are \( \mathcal{N} = 2 \) supersymmetric in the ungauged LG model \([39,20,21]\). If we work far below the scale of \( U(1) \) gauge symmetry breaking (set by the vev of \( P \) and thus the FI term), the full (B type) linear sigma model boundary conditions must reduce to conventional Dirichlet boundary conditions for \( Z^i \). We need only take the unbroken discrete gauge symmetry into account, which is what is done by the fractional brane prescription.

Now, since we start with a non conformal theory, we must expect the IR spectrum of marginal operators to be rather different from the UV free theory spectrum, raising the question of what world-volume theory we should take for the branes.

We do know that the flow must preserve the massless Ramond states, as these are protected by the usual index considerations. We can thus compute the massless Ramond spectrum in the UV and carry it to the IR.

We then make the crucial assumption that—although the combinations of BPS branes we are considering together break all supersymmetry (they preserve different \( \mathcal{N} = 1 \) subalgebras of the original \( \mathcal{N} = 2 \))—this supersymmetry breaking is a spontaneous supersymmetry breaking in an effective \( \mathcal{N} = 1, d = 4 \) world-volume theory. In particular, combinations of BPS branes which together would break supersymmetry can lead to BPS bound states, which are simply described by (quasi)-supersymmetric vacua of the combined theory. This assumption is not completely obvious, especially as we will be discussing fields with string scale masses in the broken supersymmetry vacua, and as we will see it is literally true only for a subset of the theories. It is further discussed and motivated in \([13]\). In any case, we proceed to postulate an effective \( \mathcal{N} = 1 \) world-volume theory which is compatible with our information.

The massless open string Ramond sector for the CFT of \( r \) free superfields will simply be a spinor (of definite chirality) of \( SO(2r) \), or equivalently a sum of antisymmetric representations of \( SU(r) \). In the familiar case of \( \mathbb{C}^3 \) orbifolds, this leads to a singlet and a
vector of $SU(3)$, and supersymmetry incorporates these into space-time vector and chiral multiplets respectively. The resulting world-volume theory is the familiar $\mathcal{N} = 4$ super Yang–Mills and its dimensional reductions, to which the $\Gamma$ projection is applied.

In the case of $\mathbb{C}^5$ orbifolds, these considerations lead to a vector of $SU(5)$, a three index antisymmetric tensor, and a singlet (equivalently a five index antisymmetric tensor). We then assume that the flow to the IR leads to a $(2, 2)$ supersymmetric theory with an $\mathcal{N} = 1, d = 4$ world-volume interpretation. On general grounds, the nonsinglets will have to enter into chiral multiplets in this theory. The singlet might enter into either chiral or vector multiplets $a$ priori, but given that any boundary theory will contain the operator 1 which is the internal CFT part of the gauge boson vertex operator, there must be a vector multiplet in the space-time theory, whose fermion must be this singlet.

This motivates the claim that the world-volume theory of $N$ D-branes on $\mathbb{Z}^5/\Gamma$ (assuming $\mathcal{N} = 1$ supersymmetry) is a $U(N)$ gauge theory with 15 chiral multiplets in the adjoint of $U(N)$, transforming as $5 + \overline{10}$ of a global $SU(5)$. Let us denote these multiplets as $X^i$ and $Y^{[ij]}$ respectively.

Such a theory admits gauge invariant superpotentials, and the leading possible term is cubic:

$$W = \text{tr } X^i X^j Y^{[ij]} + \ldots.$$  \hfill (2.2)

Such a term in the superpotential is also natural from the CFT point of view and as discussed in [4], it can be computed in the topologically twisted model. The non-zero amplitudes are those in which the operators combine to saturate the fermion zero modes; in terms of the translation to forms on $\mathbb{C}^5$ given above they are the amplitudes in which the product of the forms involved produces a top form on $\mathbb{C}^5$, which produce exactly (2.2).

2.3. Quiver gauge theory

We now apply the orbifold projection to derive the world-volume theory of boundary states on $\mathbb{C}^5/\Gamma$. We will discuss $\Gamma \cong \mathbb{Z}_K$ in detail here, though similar considerations would apply to nonabelian groups, as for $\mathbb{C}^2/\Gamma$ in [24]. It will be a quiver theory with $K$ nodes, chiral superfields in the 5 of $SU(5)$ $X^i_{M,M+w_i}$, chiral superfields in the $\overline{10}$ $Y^{ij}_{M,M-w_i-w_j} \equiv -Y^{ji}_{M,M-w_i-w_j}$, and the restriction of (2.2),

$$W = \sum_{M,i,j} X^i_{M,M+w_i} X^j_{M+w_i,M+w_i+w_j} Y^{ij}_{M+w_i+w_j,M}.$$  \hfill (2.3)
Indeed all of this data agrees with the explicit CFT results of [33,4]. B boundary states in these models are characterized by labels \( L_i \) in each minimal model factor and 

\[ M = \sum w_j M_j \text{ in } [0, 2K - 1]. \]

In particular, if we define the states \( |M\rangle \) with a given \( M \) and all \( L_i = 0 \), and the operator \( g \) acting as \( M \rightarrow M + 2 \), then we can write the intersection matrix between the \( L = 0 \) states as

\[ I_G = \prod_j (1 - g^{w_j}) \]

\[ = \sum_{k_j = 0,1} (-)^{\sum k_j} g^{\sum k_j w_j} \]

which agrees with the massless Ramond spectrum we described. This term in the superpotential can be checked from CFT [3].

In general the superpotential will contain higher order terms as well. These are computable in CFT and are also topological, but not too much is known about them at present. Our results so far are consistent with the idea that in the theories in which the low energy description is justified (we will explain this point shortly), such terms are absent, but this remains to be seen.

The final item required to complete the specification of the world-volume gauge theories is the Fayet-Iliopoulos terms for the \( U(1)^K \) subgroup of the gauge group. As pointed out in [14,13], these can be determined from the masses of the bosonic superpartners of the massless fermions, which are also known from CFT. These superpartners can be defined using the spectral flow operator on either brane (they produce the same result up to a phase) and in the Gepner models under discussion are obtained by multiplying the top chiral primaries in a subset of the individual minimal model factors. The result is that a boson on a link from \( M \) to \( M + w \) has \( m^2 = -w \lambda \) where we consider \( X^i_{M,M+w} \) as “forward” links (so these are tachyonic) and \( Y^{ij}_{M,M-w-w} \) as “backward” links (so these are massive). \( \lambda \) is computable and order string scale.

The FI terms must then reproduce these masses

\[ m_{ij}^2 = \zeta_i - \zeta_j \]

where \( \zeta_i \) is the FI term for the \( U(1) \) of the \( i \)'th brane. In fact one can argue directly for this structure from general properties of \( \mathcal{N} = 2 \) CFT: the \( \zeta_i \) and thus \( m^2 \) are directly related to the overall \( U(1) \) charge and thus the phases of the central charges of the two branes [13].
This determines the FI terms to be $\zeta_k = k\lambda$, but one immediately notices that this cannot reproduce all the masses in all theories. The condition for it to work is that we do not have links going “all the way around the clock,” for example closed loops with only $X$ fields. It does not forbid closed loops involving both $X$ and $Y$.

This is a condition on the allowed fractional brane content: if all types of fractional branes are present it will fail, but it can be satisfied by excluding some types of fractional branes. If it fails, the masses cannot all be reproduced in this low energy field theory description, which probably signals its breakdown. As we will see, such configurations typically restrict to branes on the Calabi-Yau with zero RR charge (or with simpler realizations) and would thus be expected to decay to the vacuum (or the simpler realization); this process would then not have a description purely in the low energy theory.

In conclusion, we have derived the low energy theories of combinations of the $L = 0$ Gepner model branes by adapting the orbifold construction to $\mathbb{C}^5/\mathbb{Z}_K$ in a way suggested by linear sigma model considerations. The result is a quiver gauge theory very analogous to those for $\mathbb{C}^3/\Gamma$ orbifolds. The construction generalizes straightforwardly to general quotients $\mathbb{C}^r/\Gamma$.

2.4. Bound states

We now make the general claim that supersymmetric vacua of these theories with unbroken gauge symmetry $U(1)$ correspond to general (classical) bound states of these rational branes. If one keeps all the modes of the open string theory (e.g. by using string field theory), this claim seems difficult to dispute. A less obvious claim is that many bound states can be described purely within the theory obtained by keeping the chiral primaries, in other words the theories we just derived. The potential problem is that the FI terms and thus the vevs of the fields at the supersymmetric vacuum have string-scale values.

Nevertheless, as we discussed, in a large subset of the theories we discussed, those with chiral fields whose masses can be reproduced by FI terms, there is no a priori reason for this description to break down. A basic test of it which can be done is to construct non-rigid ($L > 0$) rational branes as bound states of the $L = 0$ branes and check that the dimension of the moduli space comes out right. Some non-trivial examples of this for the quintic were discussed in [15], and further examples will be discussed in [9].

Now, work on non-BPS brane configurations in flat space and other simpler examples does support the claim that in many cases a good qualitative description can be obtained just using tachyons and massless fields. Thus it should probably not be too surprising
that this works in some subset of the theories we just derived. Whether this works in all the theories which \textit{a priori} appear sensible, and whether string field theory or similar frameworks can provide a more general description, are important questions for further work.

3. The large volume interpretation of the fractional branes

In [15], it was noticed that the large volume interpretation of the fractional branes of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold found using mirror symmetry [10] gave the same bundles which form the natural basis (due to Beilinson) for the general construction of bundles on $\mathbb{P}^2$, which is the exceptional divisor of the resolution of $\mathbb{C}^3/\mathbb{Z}_3$.

As it turns out, recent mathematical work has led to a very general conjecture for the higher dimensional analog of this correspondence, which we will be able to apply directly to our $\mathbb{C}^5/\Gamma$ theories [34,3,23].

The idea is to generalize the famous McKay correspondence [29] between discrete subgroups $\Gamma$ of $SU(2)$ and finite simply laced Dynkin diagrams to discrete subgroups $\Gamma \subset SU(N)$. More specifically, one has a relation between the representation ring of $\Gamma$ (which is directly encoded in the Dynkin diagram) and a basis for the exceptional cycles in a resolution of a $\mathbb{C}^2/\Gamma$ singularity. According to [34], the idea that a higher dimensional generalization should exist actually has its origins in the study of orbifolds in string theory, and the observation that the Euler number of a resolved $\mathbb{C}^3/\Gamma$ singularity equals the number of conjugacy classes of $\Gamma$.

The precise version of this idea which has been generalized is a duality between the category of sheaves on $X \cong \mathbb{C}^n/\Gamma$, and the category of sheaves with compact support, which for $X \cong \mathbb{C}^n/\Gamma$ will be sheaves supported at the origin (or, if a partial resolution has been performed, on the exceptional divisor).

In making this precise, one must work with specific categories. The duality can be made quite concrete, as was done by Ito and Nakajima in [23], where the dual objects are constructed as explicit complexes of line bundles. This should allow making a detailed identification between quiver representations and large volume sheaves along the lines of [13]. In [3], the duality was shown to be an equivalence between derived categories, which should allow proving the analog of Beilinson’s theorem for this case. Although less concrete, this is still a very strong statement about the relation between the two categories.
Here we will content ourselves with deriving the K-theory classes which correspond to the dual basis. We will then restrict these to the Calabi–Yau and compare with the predictions of mirror symmetry in a solved example.

The mathematics of the generalized McKay correspondence is clearly described in [34,3,23] and thus in the remainder of this section we concentrate on describing the ideas for physicists.

3.1. Orbifold resolution, tautological line bundles, and dual bases

The problem of resolving singularities \( X = \mathbb{C}^2/\Gamma \) has a long mathematical history, going back to Klein (see [35] for some of this background). Such a singular variety \( X \) can be resolved to a smooth space \( Y \) with non-trivial \( H^2(Y) \) and intersection form given by the Cartan matrix of the extended ADE Dynkin diagram associated to the subgroup \( \Gamma \in SU(2) \).

The most basic string theory application of this is to the duality between IIA strings on K3 and heterotic strings on \( T^4 \) [22], where IIA D2-branes wrapped on the resolved (or “exceptional”) cycles provide the nonabelian gauge bosons of the enhanced ADE gauge symmetry predicted by duality.

The most direct connection between this geometry and the structure of the group \( \Gamma \) appears in the McKay correspondence. The McKay quiver associated to \( \Gamma \) has a node for every irrep \( r_i \) of \( \Gamma \), and a link from \( r_i \) to \( r_j \) for every component \( r_j \) in \( r_{def} \otimes r_i \), where \( r_{def} \) is the representation by which \( \Gamma \) acts on \( \mathbb{C}^2 \). The result is a quiver which can be simply obtained from the ADE extended Dynkin diagram by replacing each link of the latter by a pair of links of opposite orientation.

This construction can also form the basis of an explicit construction of the resolved space, as was done by Kronheimer [26,27]. Physically the same construction appears in defining D-branes on the quotient space, and provides an explicit gauge theory description of the resolution [17]. It furthermore provides an explicit description of the branes wrapped on the exceptional cycles; these are “fractional branes” obtained by using irreducible representations in the quotient construction. One thus has the basic prediction that there should exist a natural basis for \( H^2(Y) \), or better the K-theory of the resolved singularity, labeled by irreducible representations of \( \Gamma \).

In this context the relation between the Dynkin diagram and the intersection form has a physical interpretation as well: each link corresponds to a hypermultiplet coming from
open strings stretched between a pair of fractional branes; their number can be computed from the index theorem and is equal to the intersection number.

An even more general physical system was studied in [17], containing both Dp-branes at points in X and Dp + 4-branes extending in X. It was found to reproduce a construction of general self-dual gauge fields on the resolved singularity due to Kronheimer and Nakajima [28]. Now both types of branes are labeled by a choice of group representation, and each can be associated to a quiver node. Let $R_i$ be the Dp + 4 node corresponding to $r_i$ and $S^j$ be the Dp node corresponding to $r_j$. The spectrum of $(p, p + 4)$-strings between a pair $(R_i, S^j)$ is also determined by the orbifold projection and one finds the number of hypermultiplets to be $\delta^j_i$. As in our previous discussion, this implies that the intersection form between the two types of branes should be

$$\langle R_i, S^j \rangle = \delta^j_i. \quad (3.1)$$

Now the interpretation of the Dp + 4 (extended) branes as bundles is rather clear, at least far from the singularity. The orbifold projection acts on the Yang–Mills connection as

$$\gamma A_i(z) \gamma^{-1} = r^i_j A_j(g(z)). \quad (3.2)$$

This tells us that scalar matter in the fundamental, i.e., a section of the associated bundle, must transform as

$$\gamma \phi(z) = \phi(g(z)). \quad (3.3)$$

A particularly simple case is to take $\gamma$ to be the regular representation, in which case we can consider $\phi(z)$ as a vector-valued field indexed by an element of $\Gamma$, so (3.3) becomes

$$\phi_{gh}(z) = \phi_h(g(z)). \quad (3.4)$$

This bundle is referred to as the “tautological bundle” over the quotient space. It can be decomposed as a direct sum over bundles $R_i$ associated to irreps $\gamma$ which if $\Gamma$ is abelian are line bundles; these are the tautological line bundles.

The dual relation (3.1) then determines the bundles $S_j$. On a noncompact space $X$, the natural duality for K-theory (just as for cohomology) is between $K(X)$ and the K-theory of bundles with compact support $K_c(X)$, meaning bundles over compact submanifolds of $X$. Thus the bundles $S^j$ naturally live in $K_c(X)$ and provide a preferred basis for it. These are the bundles associated to the fractional Dp-branes.
Given the intersection form in an explicit basis, we can make this definition quite concrete. For example, if we have

$$\langle R_i, R_j \rangle \equiv (I^{-1})_{ij}, \quad (3.5)$$

then we can write

$$S^j = I^{ij} R_i \quad (3.6)$$

for which

$$\langle S^j, S^k \rangle = I^{jk}. \quad (3.7)$$

As in [23], the relation (3.6) can be used to define the $S^j$ as complexes built from the bundles $R_i$. In terms of the K-theory classes, (3.6) becomes

$$[S^j] = I^{ij} [R_i], \quad (3.8)$$

a simple explicit formula for the K-theory classes of the fractional branes given those of the tautological line bundles.

Restricting these bundles or their classes to a subvariety, such as a Calabi–Yau embedded in the exceptional divisor, is a standard operation: let $V^j = S^j|_{CY}$ be these restrictions. Thus we can define an intersection form on the Calabi–Yau

$$I^{jk}_{CY} = \langle V^j, V^k \rangle_{CY} = \int_{CY} \text{ch}(V^j) \text{ch}(V^k) \text{Td}(CY), \quad (3.9)$$

and the conjecture is that

$$I_{CY} = I_G \quad (3.10)$$

where $I_G$ is the intersection form (2.4) of section 2.

The result is a physically motivated prediction for the K-theory classes of the rational B boundary states, which we will test against results derived using mirror symmetry. Indeed, the example of the quintic discussed in [15] is already a non-trivial test, as the procedure we just described leads to Beilinson’s dual bases in the case of $\mathbb{C}^n/\mathbb{Z}_n$, which as checked there agree with the results of [4].

Although (3.8) is the formula we will test in this paper, let us emphasize that (3.6) provides a definition of the fractional branes $S^j$ as holomorphic objects, not just K-theory classes. This is made quite explicit in [3,23], where the dual bases in (3.1) are used to construct a resolution of the diagonal, which can be used to prove Beilinson’s theorem for
these spaces. This leads to explicit large volume interpretations of general bound states of the fractional branes as complexes of sheaves, as we will discuss in future work [9].

So far, none of our definitions had any real dependence on the dimension of $X$; we could make the same discussion for $\mathbb{C}^n/\Gamma$ for any $n$. The point where such dependence will come in is when we discuss the resolution of the singular space $X$ in detail. Indeed, unless we can resolve $X$, it is not obvious in what sense the $R_i$ can be thought of as bundles or how to compute their K-theory classes. General theory [18] does tell us that given a resolution $Y$ of $X$, there will be a natural lift of these K-theory classes to $Y$, but we might expect this to depend on the particular resolution we choose.

Thus we need to discuss the resolution of $X$ in more detail. One idea which has been used with great success in the math literature has been to use subspaces of the Hilbert scheme of $N = |\Gamma|$ points on $\mathbb{C}^n$ which are invariant under $\Gamma$. It has been shown for $n = 2$ and all $\Gamma$, and $n = 3$ and abelian $\Gamma$, that such a subspace provides a canonical complete resolution $Y$ of $X$. The definition of tautological bundle then lifts naturally to $Y$, and the story can be completed in this framework.

For $n > 3$ there are known examples in which this construction does not produce a complete resolution. Moreover, the Hilbert scheme becomes progressively more difficult to work with in higher dimensions.

An alternate approach is to define the quotient $X$ as the moduli space of a quiver gauge theory, and then find the resolution $Y$ by the usual procedure of turning on Fayet–Iliopoulos terms. This approach was successfully used for $\mathbb{C}^3/\Gamma$ by Ito and Nakajima and is clearly well motivated in our D-brane application, so we shall follow it below. One disadvantage of this approach is that the choice of Fayet–Iliopoulos terms generally translates into a choice of resolution; it is not obvious that any of these is preferred. However, as we argued, the Gepner models produce quiver gauge theories come with a natural choice of FI terms, so we should try to make the construction work with these.

4. Orbifolds via toric methods

A general procedure for analyzing abelian orbifolds as toric varieties was given in [16]. We will briefly review this, and give the definition of the tautological line bundles in this context.

D-branes in orbifold backgrounds are described by supersymmetric world-volume gauge theories, as was shown in [17] for orbifolds in flat space and as we have argued
here for Landau–Ginzburg orbifolds. The resolved orbifold will be the moduli space of
c supersymmetric vacua of the regular representation theory.

In physical terms, a toric variety can be defined as the moduli space of vacua for an
abelian $N = 1$ supersymmetric gauge theory with no superpotential. While the moduli
space of vacua for a general supersymmetric gauge theory does not admit a toric realization,
theories for which the F-flatness constraints can be written as relations between monomials
do.

In the general class of theories we described, the F-flatness conditions are indeed
relations between monomials: they are

$$X^i_{M,M+w_i} X^j_{M+w_i,M+w_i+j} = X^j_{M,M+w_j} X^i_{M+w_j+w_i}, \quad (4.1)$$

where $M = 1, \ldots, |\Gamma|$ labels the nodes of the quiver diagram and $X^i_{M,M+w_i}$ are the chiral
multiplets. As explained in [16], the solutions to these constraints are parameterized by
an affine toric variety $Z \subset \mathbb{C}^d|\Gamma|$, which has been called the variety of commuting matrices
in [1].

The idea which allows describing this as a toric variety can be illustrated with the
variety $X$ defined by the simple relation $xy = wz$. Let us solve for $z$ as $z = xy/w$. We
can then describe the space of functions on the variety, as the functions $f(w, x, y)$ which
are generated by multiplying the monomials $w, x$ and $y$ and the monomial $xy/w$. In other
words, the presence of $z$ is described by admitting more functions than we would on $\mathbb{C}^3$.
The set of exponents of these monomials is the cone $M_+$ generated by positive integral
combinations of the vectors $(1 0 0)$, $(0 1 0)$, $(0 0 1)$ and $(−1 1 1)$.

The same data can be described by giving the dual cone $N_+$ of vectors satisfying
$n \cdot m \geq 0$. In this case it would be generated by $n_a = (1 1 0)$, $n_b = (1 0 1)$, $n_c = (0 1 0)$
and $n_d = (0 0 1)$.

Now we can describe the space of functions on $X$ by associating variables with these
generators of $N_+$, say $a, b, c$ and $d$, and writing monomials in these variables. The non-
trivial data about $X$ is now expressed in the relations between the generators. In our
example there is a single relation, $n_a + n_d = n_b + n_c$.

---

1 We are only considering the special case $Y = 0$ here, as this is what makes direct contact
with [23] and the resolution to weighted projective spaces. The moduli $Y$ appear to be connected
with deformations of bundles which appear after restriction to the CY [3].
The important fact is now that **constraints are dual to gauge invariances**, where duality is in the linear algebra sense. This is fairly obvious on reflection but can be best seen by using the language of exact sequences. Consider a sequence

\[ 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0. \]

Its exactness means that \( g \cdot f = 0 \).

One interpretation we could make of this is that \( g = 0 \) expresses a set of constraints on the space \( B \), parameterized by elements of \( C \). The map \( f \) would then be an explicit set of solutions to the constraints, parameterized by elements of \( A \).

Another possible interpretation is that we have abelian gauge symmetries acting on the space \( B \), described by the image of the map \( f \) and parameterized by elements of \( A \). We could then regard \( C \) as the gauge invariant subspace or quotient \( B/A \). This formulation is the best when we can use it, as it avoids the need to make an explicit choice of a gauge slice; if we needed to exhibit a slice in \( B \) we would need to choose a partial inverse \( h \) of \( g \) satisfying \( g \cdot h = 1_{|C} \); \( h \) would then give a map from \( C \) to the slice.

The point now is that duality (in the linear algebra sense) reverses all the arrows and thus the roles of the maps \( f \) and \( g \). This leads to duality between constraints and gauge invariances.

In this example, the dual relation between \( M_+ \) and \( N_+ \) implies that constraints on the \( M \) monomials will lead to gauge invariances for the \( N \) monomials. This is formalized by writing the space \( X \) as the spectrum of the algebra of monomials. Defining this algebra as \( \text{Hom}(M, \mathbb{C}) \) allows applying the previous discussion; see for example [8].

Thus, the relation \( n_a + n_d = n_b + n_c \) of our example should translate into a gauge invariance, with \( U(1) \) acting on the four variables \( (a \ b \ c \ d) \) with the charges \((1 \ -1 \ -1 \ 1)\). We can test this claim by writing out the gauge invariant monomials and checking that they satisfy the relation. Indeed, these are \( ab, ac, db \) and \( dc \), which can be identified with the original \( w, x, y, \) and \( z \) satisfying the relation \( xy = wz \).

This type of realization, in which the relations between generators of \( N_+ \) are interpreted as gauge invariances, is completely general, and gives a method for turning the F-flatness constraints into abelian gauge invariances. Thus we can realize the final moduli space of vacua entirely as a toric variety. The main difference between the original abelian gauge invariances and the newly generated ones is that the former will typically come with FI terms, while the latter will not.
U(1) gauge groups with non-zero FI terms will typically be associated to non-trivial topology in the quotient, as is also clear in our example: turning on the FI term produces a bundle over \( \mathbb{P}^1 \) (the “small resolution” of the conifold). We will make a related precise statement in the next subsection, which we will use to justify associating the tautological line bundles with \( U(1) \)'s coming with non-zero FI terms and thus with nodes in the original quiver.

4.1. A concise presentation

We describe the variety of commuting matrices \( Z \) in terms of its lattice of monomials \( M \), and the positive cone \( M_+ \) within that lattice. The lattice \( M \) is given as the quotient

\[
0 \longrightarrow R \overset{\hat{R}^t}{\longrightarrow} \mathbb{Z}^{d|\Gamma|} \overset{K^t}{\longrightarrow} M \longrightarrow 0
\]

(4.2)

where \( R \) is the lattice of relations corresponding to the F-flatness constraints. It can be easily checked that

\[
\text{rk}(R) = (d - 1)(|\Gamma| - 1), \quad \text{rk}(M) = d + |\Gamma| - 1.
\]

(4.3)

The cone of monomials \( M_+ \) consists of all elements of \( M \) with nonnegative components \( M_+ = M \cap \mathbb{R}^{d|\Gamma|}_+ \). The linear map \( \hat{R} \) in (4.2) can be represented by a \((d - 1)(|\Gamma| - 1) \times d|\Gamma|\) matrix whose rows correspond to the relations (4.1). \( K \) is a \( d|\Gamma| \times (d + |\Gamma| - 1) \) matrix whose columns form an integral basis for the kernel of \( \hat{R} \). The column vectors of \( K \) form an integral basis of \( M \), which is therefore isomorphic to \( \mathbb{Z}^{d+|\Gamma|-1} \).

As explained in [16,1] and just above, the variety \( Z \) can be alternatively represented as a holomorphic quotient \( \mathbb{C}^c/(\mathbb{C}^*)^{c-d-|\Gamma|+1} \). Here \( c \) is the number of generators of the dual cone \( N_+ \subset N = \text{Hom}(M, \mathbb{Z}) \). In this case, the generators of \( N_+ \) correspond to \( c \) homogeneous variables and we have \( d + |\Gamma| - 1 \) relations generating the charge lattice. This data can be conveniently summarized in an exact sequence

\[
0 \longrightarrow S \overset{Q^t}{\longrightarrow} \mathbb{Z}^c \overset{T}{\longrightarrow} N \longrightarrow 0
\]

(4.4)

where \( S \) is the lattice of charges. \( T \) is a \((d + |\Gamma| - 1) \times c\) matrix whose column vectors generate \( N_+ \). \( Q \) is the transpose of the kernel of \( T \).

At the next stage, we have to solve for D-flatness constraints, which is equivalent to taking a symplectic quotient of \( Z \) by the effective gauge group \( G \) of the quiver. Note that
the diagonal $U(1)$ is ungauged, therefore we have $G = U(1)^{|\Gamma|-1}$. The action of $G$ on the monomials can be represented as a linear map $\Delta : \mathbb{Z}^{d|\Gamma|} \rightarrow \mathbb{Z}^{d|\Gamma|-1}$ assigning to each monomial a vector of charges. Since the relations (4.1) are invariant under $G$, this map factors through $p : \mathbb{Z}^{d|\Gamma|} \rightarrow M$, i.e., there exists $V : M \rightarrow \mathbb{Z}^{d|\Gamma|-1}$ such that $\Delta = V \circ p$. By dualizing, one obtains the maps $\Delta^t : (\mathbb{Z}^{d|\Gamma|-1})^* \rightarrow (\mathbb{Z}^{d|\Gamma|})^*$ and $V^t : (\mathbb{Z}^{d|\Gamma|-1})^* \rightarrow N$ which fit in the following diagram

\[ 
\begin{array}{ccccccc}
0 & \rightarrow & N & \rightarrow & (\mathbb{Z}^{d|\Gamma|})^* & \rightarrow & R^* & \rightarrow & 0 \\
& & & & & & & & \\
& & & & & & & & T \downarrow \quad V^t \downarrow \quad \Delta^t \downarrow \quad U^t \downarrow \quad U^tV^t \downarrow \\
& & & & & & & & Z^c \quad (\mathbb{Z}^{d|\Gamma|-1})^* \quad N \quad 0 \\
& & & & & & & & Q^t \downarrow \\
& & & & & & & & S \downarrow \\
& & & & & & & & 0 \\
\end{array}
\]

The map $U^t$ is a quasi-inverse of $T$ satisfying $TU^t = I_{d+|\gamma|-1}$ which is required in order to express the action of $G$ on $Z$. More precisely, the final moduli space is represented by the toric data

\[ 
0 \rightarrow S \oplus (\mathbb{Z}^{d|\Gamma|-1})^* \xrightarrow{(Q^t, (VU)^t)} \mathbb{Z}^c \xrightarrow{\tilde{N}} 0 \quad (4.5)
\]

where $\tilde{N}$ is a rank $d$ lattice. Note that the matrix of charges $\tilde{Q}$ is obtained by concatenating $Q$ and $VU$

\[ 
\tilde{Q} = \begin{bmatrix} Q \\ VU \end{bmatrix}. \quad (4.6)
\]

To summarize, note that we have interpreted the $U(1)$ gauge groups of the quiver theory as a subset of the generators of the charge lattice of the toric moduli space $X$. It is known that each such generator corresponds to a Weil divisor class on $X$ (\cite{18} 3.4 or \cite{8} section 3.16)
the basic idea is that (4.4) is dual to the exact sequence expressing divisor classes as divisors (rays in $N_+$) modulo functions (elements of $M$).

Since these $U(1)$ gauge groups are labeled by a choice of $\Gamma$ irrep, this construction produces a canonical divisor class for each irrep. This will be our definition of the tautological line bundles $R_k$.

The question of whether these divisors actually correspond to line bundles (live in $\text{Pic}(X)$) \textit{a priori} depends on the particular FI terms and subdivision of the toric fan which we take. We will check this explicitly for our example; a more general treatment might use the methods of [18].

### 4.2. Limitations of the method

As we discussed in section 3, the identification of the dual bases with actual K theory classes implicitly requires that the space $X$ be completely resolved. It has not been proven that the procedure of [16] will always produce a complete resolution, even if one exists.

Even worse, in dimensions four and above, not all $SU(n)$ orbifold singularities are crepant (admit Calabi-Yau resolutions), and it is not clear how to define the generalized McKay correspondence in these cases. This is a problem as many Gepner models construct Calabi-Yaus as hypersurfaces in weighted projective spaces containing such singularities. If the hypersurface avoids the singularity, one has a completely sensible Calabi-Yau and geometrical interpretation, but the method presented here does not immediately apply. It is an important open problem to find a method which handles these cases.

There are also many cases for which this is not a problem, and we proceed to test the method for one of them.

### 5. An example: $\mathbf{WIP}^{1,1,2,2,2}$

In the previous section we have proposed a general toric algorithm for determining the tautological line bundles $R_k$ on a (resolved) weighted projective space. We have also conjectured that the resulting line bundles $R_k$ are K-theory generators dual to the classes $S_k$ corresponding to the fractional branes. Here we test this conjecture for the two parameter model $\mathbf{WIP}^{1,1,2,2,2}$ discussed in [16]. The B boundary states for this model have been considered in [25].
Before presenting the details of the computation, some background on the geometry of $\mathbf{WIP}^{1,1,2,2,2}$ may be helpful. This is a singular weighted projective space whose toric resolution is defined by the following data \[31\]

\[
\begin{array}{cccccc}
  & x_0 & x_1 & x_2 & x_3 & x_4 \\
 g_1(\lambda) & 1 & 1 & 1 & 1 & 0 \\
 g_2(\lambda) & 0 & 0 & 0 & -2 & 1 \\
\end{array}
\]  

with disallowed set

\[ F = \{ x_0 = x_1 = x_2 = x_3 = 0 \} \cup \{ x_4 = x_5 = 0 \}. \]  

The resulting smooth toric variety will be denoted by $W$ for simplicity. The Picard group $\text{Pic}(W)$ is generated by two divisors $H, L$ with relations

\[ H^3(H - 2L) = 0, \quad L^2 = 0 \]  

and intersection numbers

\[ H^4 = 2, \quad H^3L = 1, \quad H^2L^2 = HL^3 = L^4 = 0. \]  

The Calabi–Yau hypersurface $M \subset W$ is the zero locus of a generic section of the anticanonical line bundle $-K_W = \mathcal{O}(4H)$.

Certain details on the geometry of $M$ will also be needed in the following \[8\]. For simplicity, let $H, L$ also denote the restrictions of the divisor classes to $M$. The meaning will be clear from the context. Then we have the intersection numbers

\[ (H^3)_M = 8, \quad (H^2L)_M = 4, \quad (HL^2)_M = (L^3)_M = 0. \]  

The cone of curves on $M$ is generated by $(h, l)$ \[8\] such that

\[ (H \cdot h)_M = 1, \quad (H \cdot l)_M = 0 \]

\[ (L \cdot h)_M = 0, \quad (L \cdot l)_M = 1. \]  

Moreover, we have the intersection relations

\[ 4l = (H^2 - 2HL)_M \Rightarrow H(H^2 - 2HL) = l \]

\[ 4h = (HL)_M \Rightarrow H^2L = h. \]
Finally, the second Chern class of $M$ is
\[ c_2(M) = 56h + 24l. \] (5.8)

5.1. Tautological line bundles

We start by determining the quiver moduli space and the tautological line bundles for $\mathbb{C}^5/\Gamma$, where $\Gamma = \mathbb{Z}_8$ acts on $\mathbb{C}^5$ as
\[ (Z^1, Z^2, Z^3, Z^4, Z^5) \rightarrow (\omega Z^1, \omega^2 Z^2, \omega^2 Z^3, \omega^2 Z^4, \omega^2 Z^5), \quad \omega = e^{\frac{2\pi i}{8}}. \] (5.9)

As discussed in section 2.3, the associated quiver theory has eight nodes labeled by $M = 0, \ldots, 7$. For each node, we have five chiral multiplets $X_{M,M+1}^{1,2}$, $X_{M,M+2}^{3,4,5}$. The superpotential (2.2) yields eighty F-flatness conditions (4.1), out of which only twenty-eight are independent. We first determine the moduli space of the quiver gauge theory using toric methods as explained in the previous section. Then we find similarly the tautological line bundles $R_k$.

The equations (4.1) can be solved in terms of the twelve independent variables $X_{01}^1, X_{01}^2, X_{02}^3, X_{02}^4, X_{05}^5, X_{12}^1, \ldots, X_{70}^1$

\[ X_{M,M+1}^2 = \frac{X_{01}^2}{X_{01}^1} X_{M,M+1}^1, \quad M = 1, \ldots, 7 \]
\[ X_{M,M+2}^i = \frac{X_{02}^i}{X_{01}^1} \frac{X_{M,M+1}^1}{X_{12}^1} X_{M+1,M+2}^1, \quad i = 3, 4, 5, M = 1, \ldots, 7. \] (5.10)

We obtain therefore twenty-eight vectors in $\mathbb{R}^{12}$ spanning the cone of monomials $M_+$. The dual cone $N_+$ is spanned by twenty-one twelve dimensional vectors, whose coordinates form a $12 \times 21$ matrix $T$

\[
T = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix}.
\] (5.11)
The columns of $T$ correspond to homogeneous coordinates of the variety of commuting matrices $Z$. The transpose of the kernel of $T$ determines a charge matrix $Q$

$$Q =
\begin{bmatrix}
1 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(5.12)

In order to find the action of the quiver $U(1)$ gauge groups on $Z$, we have to choose a $12 \times 21$ matrix $U$ such that

$$TU^t = I_{12}.
$$

(5.13)

We pick $U$ of the form

$$U =
\begin{bmatrix}
0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

(5.14)

The quiver gauge group consists of eight $U(1)$ factors, the diagonal $U(1)$ leaving the chiral multiplets $X^t_{M,M+w_i}$ invariant. Therefore, in solving the D-flatness constraints we have to divide by the effective group $G = U(1)^7$. We choose the seven independent $U(1)$ factors to correspond to the nodes $0,1,\ldots,6$ of the quiver. The charges of the twelve independent
variables considered above are given by a $7 \times 12$ matrix

$$V = \begin{bmatrix}
-1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}. \quad (5.15)$$

The total charge matrix is obtained by concatenating $Q$ and $VU$ in a single $16 \times 21$ matrix

$$\tilde{Q} = \begin{bmatrix} Q \\ VU \end{bmatrix}. \quad (5.16)$$

Note that the quiver FI terms can also be included as an extra column $[\tilde{Q} \, \xi]$ where $\xi = [0^9, \xi_1, \ldots, \xi_7]^t$.

The transpose of the kernel of $\tilde{Q}$ gives the presentation of the quiver moduli space as a toric variety. The columns of $\left(\text{Ker}\tilde{Q}\right)^t$, interpreted as vectors in a linear space of appropriate dimension, generate the toric fan of the moduli space. In the present case, we obtain

$$\left(\text{Ker}\tilde{Q}\right)^t = \begin{bmatrix}
-5 & 0 & 0 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}. \quad (5.17)$$

Let $\tilde{T}$ be the matrix obtained after eliminating the redundant columns. Given this matrix, we can find the associated charge matrix by taking again the transpose of its kernel. This yields

$$\begin{bmatrix} 1 & 1 & 1 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 & 0 & 1 \end{bmatrix}. \quad (5.18)$$

As stated before, the columns of this matrix correspond to homogeneous coordinates of the toric moduli space. We will denote them by $p_0, \ldots, p_7$. The presentation can be further simplified by using the middle charge vector to eliminate $p_6$ in terms of $p_4$ and $p_5$. Note that this is justified in a toric phase where $p_6$ is not allowed to vanish, since then it can
be gauged away using the \( \mathbb{C}^* \) action. Assuming that we are working in such a phase, we obtain a simplified charge matrix

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & -2 & 1
\end{bmatrix}.
\]

(5.19)

We can set this data in more familiar form by permuting the columns

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & -4 \\
0 & 0 & 0 & -2 & 1 & 1 & 0
\end{bmatrix}.
\]

(5.20)

This is easily recognizable as the toric data of the total space of the canonical line bundle \( K_W \). The first five columns, describe the smooth compact toric variety \( W \) discussed in the beginning of the section. The last column corresponds to an extra homogeneous coordinate representing the fiber the canonical line bundle \( K_W \).

In order to justify such an interpretation, we have to make sure we are in the right toric phase, i.e., the disallowed locus is \((5.2)\), while \( p_7 \) is allowed to vanish. This turns out to be true for a suitable interpretation of the Fayet–Iliopoulos terms arising from the Gepner model construction. Remember that these were \( \xi_k = k\lambda \), where \( \lambda \) is a string-scale (known) mass parameter, and the origin of the \( k \) index is set arbitrarily (as we commented, the full spectrum cannot in general be reproduced by a low energy supersymmetric theory).

The matrix \( \tilde{Q} \) with the associated FI terms can be set in canonical form [16, 1] by invertible row operations. We record the result in appendix A. Note that the first two rows reproduce the charge matrix of \( X \) (5.21). The other rows can be used to eliminate redundant variables as explained below. If all of \( \xi_2 \) through \( \xi_8 \) are positive, each row following the first two can be used to eliminate a unique variable (the one appearing with a negative charge), resulting in \((5.19)\). (The last row may appear problematic as \( \xi_7 \) appears with the wrong sign, but if one writes out the immediately preceding moment map equations \(|p_i|^2 - |p_{i+1}|^2 = -\xi\), one sees that so many negative FI terms (including \(-\xi_7\) have appeared that the final variable is guaranteed to be nonzero).

Thus we have obtained the expected result—a smooth noncompact toric fivefold \( X \) with vanishing canonical class.

We now determine in a similar fashion the tautological line bundles. The construction explained in the previous section can be implemented in practice by certain simple modifications of the quiver diagram. More precisely, we will consider a different gauge theory, obtained from the previous one by adding an extra chiral multiplet, corresponding to an extra leg in the diagram. The extra leg is attached to a single node, resulting in a multiplet
charged under the corresponding $U(1)$ factor. Therefore we obtain eight distinct quiver theories which can be labeled by the charge vector $v_M$ of the extra multiplet. Recall that we have fixed the gauge group to be $G = U(1)_0 \times U(1)_1 \times \ldots \times U(1)_6$. We have

$$v_M^i = \delta_i^M, \quad M = 0, \ldots, 7, \quad i = 0, \ldots, 6.$$  \hfill (5.21)

Each quiver theory has a moduli space, whose toric presentation can be identified (in a certain phase) to the total space of a certain line bundle over the variety $X$ determined above. We claim that the line bundles obtained this way are precisely the K-theory generators $R_k$ introduced in section two. The precise correspondence between $R_k$ and the charge vectors $v_M$ is given by $k = M + 1$.

In order to prove this, we proceed as before. The extra multiplet $\psi$ corresponds to an extra variable which is not related to $X_{101}, \ldots, X_{70}$. Therefore, the cone of monomials $M'_+ \subset \mathbb{R}^{12}$ can be obtained by embedding $M_+ \subset \mathbb{R}^{13}$ and adding an extra generator corresponding to the normal direction. Then, the dual cone $N'_+$ is characterized by the augmented matrix

$$T' = \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix}.$$ \hfill (5.22)

Similarly, $Q' = (\text{Ker}T')^\perp$ and $U'$ can be easily related to $Q,U$

$$Q' = \begin{bmatrix} Q & 0 \end{bmatrix}, \quad U' = \begin{bmatrix} U & 0 \\ 0 & 1 \end{bmatrix}.$$ \hfill (5.23)

The matrix $V$ is augmented by the charge vector of the extra multiplet$\footnote{Note that we have in fact eight different charge matrices $V'$ which should be labeled by an index $M$. In order to keep the notation simple, we will not write down this extra index explicitly.}$

$$V' = \begin{bmatrix} V & v_M \end{bmatrix}$$ \hfill (5.24)

resulting in a total charge matrix

$$\tilde{Q}' = \begin{bmatrix} Q & 0 \\ 0 & v_M \end{bmatrix}.$$ \hfill (5.25)

This is an important change, since the matrix $\tilde{Q}'$ determines the new moduli space.

Following the general algorithm, the next step is to determine the transpose of the kernel of $\tilde{Q}'$. This can be done by a straightforward computation for a generic vector $v_M$. We do not record the result here for reasons of space. After eliminating the redundant
column, we are left with a matrix $\tilde{T}'$ as before. At the last stage, we can determine the
charge matrix of the resulting toric variety by taking the transpose of the kernel of $\tilde{T}'$. Let
us carry out this procedure explicitly for

$$v_0 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0] \,^t.$$  \hspace{1cm} (5.26)

The matrix $\tilde{T}'$ reads in this case

$$\tilde{T}' = \begin{bmatrix}
-5 & 0 & 0 & 2 & -1 & -1 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 
\end{bmatrix}. \hspace{1cm} (5.27)$$

The associated charge matrix is

$$\begin{bmatrix}
1 & 1 & 1 & 0 & -4 & 0 & 1 & 0 & 0 & 0 & -3 \\
0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 
\end{bmatrix}. \hspace{1cm} (5.28)$$

Note again that the last three charge vectors can be used to gauge away $p_5, p_7, p_8$ in a
phase where they are not allowed to vanish. Assuming that we are working in such a
phase, and permuting again the columns, the charge matrix can be rewritten as

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & -4 & -3 \\
0 & 0 & 0 & -2 & 1 & 1 & 0 & -1 
\end{bmatrix}. \hspace{1cm} (5.29)$$

The first seven columns constitute the toric data of $X$ determined before. By adding the
last column, we find that the new toric variety $X_1$ can be interpreted in a certain phase as
the total space of a line bundle $R_1$ over $X$. The fiber is described by the last homogeneous
variable $p_{10}$, so this interpretation is justified if $p_{10}$ is allowed to vanish. The existence of
a suitable phase can be proved by a direct computation of the FI terms, as before. In this
case, the line bundle $R_1$ is determined by the entries in the last column to be

$$R_1 = \mathcal{O}(-3H - L),$$  \hspace{1cm} (5.30)

where $(H, L)$ generate the Picard group of the resolved $\text{WIP}^{1,1,2,2,2}$. The bundle in the
r.h.s. of (5.30) is pulled back to $X$. 

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Proceeding similarly we can determine all eight line bundles \( R_k, k = 1, \ldots, 8 \)

\[
\begin{align*}
R_1 &= \mathcal{O}(-3H - L) & R_2 &= \mathcal{O}(-3H) \\
R_3 &= \mathcal{O}(-2H - L) & R_4 &= \mathcal{O}(-2H) \\
R_5 &= \mathcal{O}(-H - L) & R_6 &= \mathcal{O}(-H) \\
R_7 &= \mathcal{O}(-L) & R_8 &= \mathcal{O}.
\end{align*}
\]

(5.31)

5.2. A dual basis

We now test the main conjecture in this paper by determining a set of dual K-theory classes \( S_l \). Strictly speaking, we will only determine the Chern characters \( \text{ch}(S_l) \) so that the following orthogonality relation holds

\[
\int_X \text{ch}(R_k)\text{ch}(S_l)\text{Td}(X) = \delta_{kl}.
\]

(5.32)

The result is conveniently expressed in terms of a K-theory class \( S \) defined by

\[
\text{ch}(S) = 3 - (2H - L) + \frac{1}{12}(2H - L)^3
\]

(5.33)

and its conjugate \( \overline{S} \). Then the eight dual classes are given by

\[
\begin{align*}
S_1 &= \mathcal{O}(-H + L) & S_2 &= -\mathcal{O}(-H + 2L) \\
S_3 &= -S & S_4 &= S \otimes \mathcal{O}(L) \\
S_5 &= \overline{S} \otimes \mathcal{O}(-H) & S_6 &= -\overline{S} \otimes \mathcal{O}(-H + L) \\
S_7 &= -\mathcal{O}(-L) & S_8 &= \mathcal{O}.
\end{align*}
\]

(5.34)

It can be checked by a direct computation that the classes \( R_k, S_l \) defined above satisfy (3.5)–(3.7) and (5.32).

In order to establish a relation to Gepner model boundary states, we restrict the classes \( S_l \) to the Calabi–Yau hypersurface \( M \). Let \( V_l \) denote the restriction of \( S_l \) to \( M \). Using the intersection relations (5.7), it is straightforward to compute the Chern characters
In the remaining part of this section, we will compare (5.35) to the topological invariants of the fractional branes, finding a precise agreement.

5.3. Fractional branes

The geometric interpretation of Gepner model boundary states for Calabi–Yau models has been considered in [4,11,25,38]. In particular, the present two-parameter model has been studied in [25]. Here we determine a complete list of K-theory classes in $K_0(M)$ corresponding to the fractional branes.

Let us give some background on the special Kähler geometry of $\mathbf{WIP}^{1,1,2,2,2}$ [6]. We adopt the conventions of [25] for the basis of periods. The Kähler moduli space is parameterized near the large radius limit by

$$J = t_1 H + t_2 L. \quad (5.36)$$

The asymptotic expression of the prepotential (ignoring exponentially small corrections) is

$$F = -\frac{4}{3} t_1^3 - 2t_1^2 t_2 + \frac{7}{3} t_1 t_2 + t_2 \quad (5.37)$$

which yields the following vector of periods

$$\Pi(t) = \begin{bmatrix} \frac{4}{3} t_1^3 + 2t_1^2 t_2 + \frac{7}{3} t_1 + t_2 \\ -4t_1^2 - 4t_1 t_2 + \frac{7}{3} t_2 \\ -2t_1^2 + 1 \\ 1 \\ t_1 \\ t_2 \end{bmatrix}. \quad (5.38)$$
The monodromy matrix corresponding to the \(Z_8\) quantum symmetry of the Gepner model is [25]

\[
A = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
\frac{3}{2} & \frac{3}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 \\
\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} & \frac{1}{4}
\end{bmatrix}
\] (5.39)

We label BPS states by a six dimensional charge vector \(n = (n_6, n_4, n_2, n_0, n_2, n_2)\), with central charge

\[
Z(n) = n \cdot \Pi(t) \\
= \frac{4}{3}n_6t_1^3 + 2n_6t_1^2t_2 - (4n_4^1 + 2n_4^2)t_1^2 \\
- 4n_4^1t_1t_2 + (n_2 + \frac{7}{3}n_6)t_1 + (n_2^2 + n_6)t_2 \\
+ n_0 + \frac{7}{3}n_4^1 + n_4^2.
\] (5.40)

This is to be compared with the central charge of a D-brane configuration described by a bundle \(V \to M\)

\[
Z(V) = \int_X e^{-(t_1H + t_2L)} ch(V) \sqrt{\text{Td}(M)} \\
= \int_X e^{-(t_1H + t_2L)} ch(V) \left(1 + \frac{c_2(M)}{24}\right).
\] (5.41)

By identifying (5.40) and (5.41) we obtain the conversion formulae

\[
\begin{align*}
\text{ch}_0(V) &= -n_6 \\
\text{ch}_1(V) &= -n_4^1H - n_4^2L \\
\text{ch}_2(V) &= -n_2^1h - n_2^2l \\
\text{ch}_3(V) &= n_0 + \frac{14}{3}n_4^1 + 2n_4^2.
\end{align*}
\] (5.42)

The fractional branes generically correspond to Gepner model boundary states with \(L = (0,0,0,0,0)\) and they form an orbit of the \(Z_8\) quantum symmetry. Typically, one of these states corresponds to the pure D6-brane wrapping the CY hypersurface \(M\), with charge vector\(^3\)

\[
n_8 = (-1, 0, 0, 0, 0, 0).
\] (5.43)

\(^3\) The label \(n_8\) is for further convenience.
The other charge vectors are obtained by multiplying by $A^{-1}$ to the right. We obtain the following charges

\begin{align*}
n_1 &= (-1, 1, -1, -2, 0, -2) \\
n_2 &= (1, -1, 2, -2, -4, 2) \\
n_3 &= (3, -2, 1, 6, 0, 0) \\
n_4 &= (-3, 2, -4, 0, 8, 0) \\
n_5 &= (-3, 1, 1, -6, 0, 2) \\
n_6 &= (3, -1, 2, 2, -4, -2) \\
n_7 &= (1, 0, -1, 2, 0, 0).
\end{align*}

Using (5.44), we can now determine the topological invariants of the K-theory classes associated to these BPS states. A straightforward computation shows that they are in precise agreement with (5.33). This proves the claim.

6. Conclusions

In this paper we have continued the investigation of D-branes in Gepner models and their geometric counterparts initiated in [33,4]. One of the central problems in this area is to find a general geometric interpretation for Gepner model B boundary states in terms of holomorphic objects. So far, this question has been answered in several particular models using mirror symmetry techniques, but no general picture had been found.

The present work fills this gap by proposing a simple construction of the K theory elements associated to boundary states. The power of this new construction is that it does not make use of mirror symmetry results. The approach is inspired by the description of fractional branes in lower dimensional orbifold models [17,16]. In these cases, the geometric interpretation of fractional branes essentially reduces to the celebrated McKay correspondence [29,34,35]. The latter establishes a duality between orbifold (equivariant) K theory and the K theory of the resolved space. Given the K theoretic interpretation of branes [30,31], this is precisely our problem, formulated in a slightly different language.

We argued that our problem, although superficially different, is essentially a higher dimensional version of the McKay correspondence, less well studied in the mathematical literature. This follows naturally from the Landau-Ginzburg orbifold description of the Gepner model, which is a familiar point of view in the linear sigma model approach of [40]. By shifting perspective from the conformal field theory approach of [33] to the linear sigma
model, we obtain a simple and efficient description of boundary states as fractional branes in a $\mathbb{C}^5/\Gamma$ orbifold. Indeed, if we keep in mind that only quantities which are topological or protected in some way should be directly computable in the UV, we can derive the Gepner model results we use, namely the $\mathcal{N} = 1$ effective Lagrangian which describes combinations of rational boundary states, from the LG orbifold description.

It is well known from the work of [10,16] that orbifold boundary states are described by quiver gauge theories whose moduli space typically reproduces the orbifold resolution. Moreover, in this picture, one can naturally define a preferred set of K theory generators of the resolution – the tautological line bundles $R_k$ [26,28,23]. These are dual to another set of K theory generators $S_k$ supported on the compact exceptional locus of the blownup singularity. For $\mathbb{C}^5/\Gamma$ orbifolds, the resolved space is a noncompact Calabi-Yau variety isomorphic to the total space of the canonical line bundle of a weighted projective space.

The main result of the present work establishes a direct correspondence between fractional branes and the dual set of K theory generators $S_k$, which can be thought of as classes in the K theory of the weighted projective space. This shows that the Landau-Ginzburg orbifold is intimately related to the geometry of the ambient toric variety which contains the Calabi-Yau hypersurface.

In practice we have developed a systematic toric algorithm for determining the tautological line bundles $R_k$, starting from the quiver diagram. The basis of fractional branes is then determined by inverting the K theoretic intersection pairing on the ambient toric variety. Finally, these are restricted to the hypersurface to obtain the classes of Gepner model boundary states.

One surprising aspect of this is that the intersection form on the ambient variety is different from the intersection form on the Calabi-Yau hypersurface itself, which reproduces the CFT results. Although this seems to be a natural aspect of the whole picture, the objects $S_k$ defined above are not really physical, and neither is their intersection form: only their restrictions to the CY manifold are physical. Nevertheless, one gets correct results by working with the unphysical $S_k$’s. This seems to be another manifestation of the decoupling between D-flatness conditions (which define the ambient toric space) and F-flatness conditions (which define the CY), but remains somewhat mysterious.

We feel that the ideas presented here are the beginnings of a satisfactory picture both of the world-sheet and space-time interpretations of the rational boundary states, but there is much work still to do in this direction. It would be very useful to further develop the linear sigma model technology and complete the computations of all the “topological”
results: spectrum, superpotentials, and D-flatness conditions. The conditions under which a low energy theory treatment of bound states of branes is valid, and what must replace it in general, remain to be explored.

The central point for further development is that, as is clear from \([23,3]\) and the other references, the generalized McKay correspondence can be used not just to compute K-theory classes but to determine explicit holomorphic objects (sheaves or complexes of sheaves) representing the fractional branes. Let us recall what appears to be the central lesson of \([15]\): the detailed \(\mathcal{N} = 1\) world-volume theories describing combinations and bound states of fractional branes on a CY, have a direct correspondence to natural mathematical constructions of stable holomorphic objects. For \(\mathbb{C}^3/\mathbb{Z}_3\), the core of this was the identification of Beilinson’s construction of sheaves on \(\mathbb{P}^2\) within the \(\mathbb{Z}_3\) quiver gauge theory. It is precisely this identification which is generalized in the works \([23,3]\).

Thus, we can hope to generalize this lesson to fairly general CY’s. The simple conjecture (an oversimplification, in ways described in detail in \([15,13]\) and elsewhere) which can guide further developments is that moduli spaces of \(\mathcal{N} = 1\) theories describing bound states of fractional branes naturally correspond to moduli spaces of stable coherent sheaves on the CY, and furthermore the internal structure (F and D flatness conditions) of these theories corresponds to natural mathematical descriptions of the moduli space. If so, the constructions we described could then eventually lead to a construction of all sheaves on a CY which can be obtained by restriction from the ambient toric variety and then deformation. This is a sublattice of finite index in the K-theory of the CY (for example the D0 did not appear on the quintic) but still gives a very large subset of the possibilities.

The most interesting physical application of these results may be to constructing type I compactifications on Calabi–Yau manifolds, by finding combinations of branes and orientifolds which cancel tadpoles and anomalies, and finding the supersymmetric vacua of these theories. A geometric study of type I-heterotic duality should then be possible; indeed the \(\mathbb{C}^3/T\) results should already be quite useful for this purpose. A natural next step for both flat space and Landau–Ginzburg orbifolds would be to show that the tadpole cancellation conditions are equivalent to the familiar anomaly cancellation conditions for heterotic strings on the corresponding bundles.

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References

[1] C. Beasley, B. R. Greene, C. I. Lazaroiu, and M. R. Plesser, “D3-branes on partial resolutions of abelian quotient singularities of Calabi–Yau threefolds”, *Nucl. Phys. B* **566** (2000) 599, hep-th/9907186.

[2] A. A. Beilinson, “Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra”, *Funct. Anal. Appl.* **12** (1978) 214–216.

[3] T. Bridgeland, A. King, and M. Reid, “Mukai implies McKay”, math.AG/9908027.

[4] I. Brunner, M. R. Douglas, A. Lawrence, and C. Römelsberger, “D-branes on the quintic,” hep-th/9906200.

[5] I. Brunner and V. Schomerus, to appear.

[6] P. Candelas, X. de la Ossa, A. Font, S. Katz, and D. R. Morrison, “Mirror symmetry for two parameter models – I”, *Nucl. Phys. B* **416** (1994) 481, hep-th/9308083.

[7] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes, “A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory”, *Nucl. Phys. B* **359** (1991) 21.

[8] D. Cox, “Recent developments in toric geometry”, *Proc. Symp. Pure Math.* **62:2** (1997) 389–436, alg-geom/9606016.

[9] D.-E. Diaconescu and M. R. Douglas, work in progress.

[10] D.-E. Diaconescu and J. Gomis, “Fractional branes and boundary states in orbifold theories,” hep-th/9906242.

[11] D.-E. Diaconescu and C. Römelsberger, “D-branes and bundles on elliptic fibrations”, *Nucl. Phys. B* **574** (2000) 245, hep-th/9910172.

[12] M. R. Douglas, “Topics in D-geometry”, *Class. Quant. Grav.* **17** (2000) 1057, hep-th/9910170.

[13] M. R. Douglas, “D-branes and categories”, to appear.

[14] M. R. Douglas, B. Fiol, and C. Römelsberger, “Stability and BPS branes”, hep-th/0002037.

[15] M. R. Douglas, B. Fiol, and C. Römelsberger, “The spectrum of BPS branes on a noncompact Calabi–Yau”, hep-th/0003263.

[16] M. R. Douglas, B. R. Greene, and D. R. Morrison, “Orbifold resolution by D-branes,” *Nucl. Phys. B* **506** (1997) 84, hep-th/9704151.

[17] M. R. Douglas and G. Moore, “D-branes, quivers, and ALE instantons,” hep-th/9603167.

[18] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, The William H. Roever Lectures in Geometry, Princeton University Press, 1993.

[19] S. Govindarajan and T. Jayaraman, “On the Landau–Ginzburg description of boundary CFTs and special Lagrangian submanifolds”, hep-th/0003242.

[20] S. Govindarajan, T. Jayaraman, and T. Sarkar, “World sheet approaches to D-branes on supersymmetric cycles”, hep-th/9907131.
[21] K. Hori, A. Iqbal, and C. Vafa, “D-branes and mirror symmetry”, [hep-th/0005247].
[22] C. M. Hull and P. K. Townsend, “Enhanced gauge symmetries in superstring theories”, Nucl. Phys. B451 (1995) 525.
[23] Y. Ito and H. Nakajima, “McKay correspondence and Hilbert schemes in dimension three”, math.AG/9803120.
[24] C. V. Johnson and R. C. Meyers, “Aspects of type IIB theory on ALE spaces”, Phys. Rev. D55 (1997) 6382, [hep-th/9610140].
[25] P. Kaste, W. Lerche, C. A. Lütken, and J. Walcher, “D-branes on K3 fibrations,” hep-th/9912147.
[26] P. B. Kronheimer, “The construction of ALE spaces as hyper-Kähler quotients”, J. Differential Geom. 29 (1989) 665.
[27] P. B. Kronheimer, “A Torelli-type theorem for gravitational instantons”, J. Differential Geom. 29 (1989) 685.
[28] P. B. Kronheimer and H. Nakajima, “Yang–Mills instantons on ALE gravitational instantons”, Math. Ann. 288 (1990) 263.
[29] J. McKay, “Graphs, singularities, and finite groups”, Proc. Symp. in Pure Math. 37 (1980) 183.
[30] R. Minasian and G. Moore, “K-theory and Ramond-Ramond charge”, JHEP 11 (1997) 002, [hep-th/9710230].
[31] D. R. Morrison and M. R. Plesser, “Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties”, Nucl. Phys. B440 (1995) 279, [hep-th/9412236].
[32] M. Naka and M. Nozaki, “Boundary states in Gepner models,” JHEP 0005 (2000) 027, [hep-th/0001037].
[33] A. Recknagel and V. Schomerus, “D-branes in Gepner models”, Nucl. Phys. B531 (1998) 185, [hep-th/9712186].
[34] M. Reid, “McKay correspondence”, alg-geom/9702016.
[35] M. Reid, “La correspondance de McKay”, Séminaire Bourbaki (novembre 1999), no. 867, math.AG/9911165.
[36] A. V. Sardo Infirri, “Partial resolutions of orbifold singularities via moduli spaces of HYM-type bundles,” alg-geom/9610004.
[37] A. V. Sardo Infirri, “Resolutions of orbifold singularities and flows on the McKay quiver,” alg-geom/9610003.
[38] E. Scheidegger, “D-branes on some one- and two-parameter Calabi–Yau hypersurfaces,” JHEP 0004 (2000) 003, [hep-th/9912188].
[39] N.P. Warner, “Supersymmetry in Boundary Integrable Models,” Nucl. Phys. B450 (1995) 663-694; hep-th/9506064.
[40] E. Witten, “Phases of N=2 theories in two dimensions,” Nucl. Phys. B403 (1993) 159, [hep-th/9301042].
[41] E. Witten, “D-Branes and K-Theory”, JHEP 12 (1998) 019, [hep-th/9810188].
| 1 | 1 | 1 | 1 | 1 | 0 | -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
|---|---|---|---|---|---|----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

\[\xi_3 + \xi_4 + 2\xi_5 + 2\xi_6 + 3\xi_7 + 3\xi_8 \]

\[\xi_2 + \xi_4 + \xi_6 + \xi_8 - \xi_7 - \xi_6 - \xi_8 - \xi_5 - \xi_6 - \xi_8 - \xi_4 - \xi_4 - \xi_3 - \xi_4 - \xi_2 - \xi_7 - \xi_2 - \xi_5 - \xi_7 \]