Lévy stable noise-induced transitions: stochastic resonance, resonant activation and dynamic hysteresis

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Abstract. A standard approach to analysis of noise-induced effects in stochastic dynamics assumes a Gaussian character of the noise term describing interaction of the analyzed system with its complex surroundings. An additional assumption about the existence of timescale separation between the dynamics of the measured observable and the typical timescale of the noise allows external fluctuations to be modeled as temporally uncorrelated and therefore white. However, in many natural phenomena the assumptions concerning the above mentioned properties of ‘Gaussianity’ and ‘whiteness’ of the noise can be violated. In this context, in contrast to the spatiotemporal coupling characterizing general forms of non-Markovian or semi-Markovian Lévy walks, so called Lévy flights correspond to the class of Markov processes which can still be interpreted as white, but distributed according to a more general, infinitely divisible, stable and non-Gaussian law. Lévy noise-driven non-equilibrium systems are known to manifest interesting physical properties and have been addressed in various scenarios of physical transport exhibiting a superdiffusive behavior. Here we present a brief overview of our recent investigations aimed at understanding features of stochastic dynamics under the influence of Lévy white noise perturbations. We find that the archetypal phenomena of noise-induced ordering are robust and can be detected also in systems driven by memoryless, non-Gaussian, heavy-tailed fluctuations with infinite variance.

Keywords: driven diffusive systems (theory), stochastic particle dynamics (theory), transport processes/heat transfer (theory), stochastic processes (theory)
1. Introduction

Systems operating far from thermodynamic equilibrium are usually subjected to the action of noise which may profoundly influence their performance. Over the past two decades extensive theoretical and experimental studies in physics, information theory and biology have documented various phenomena of noise-induced order, noise-facilitated kinetics or noise-improved signal transmission and detection. Almost all research in this field assumes that the noise process involved can be characterized by finite variance. Yet, in many situations the external non-thermal noise can be described by distribution of impulses following the heavy-tail stable law statistics of infinite variance. Notably, infinite variance does not exclude finite spread or dispersion of the distribution around its modal value. In fact, for the family of \(\alpha\)-stable noises the interquantile distance can be used as a proper measure of the distribution width around the modal (or median) values.

Dynamical description of the system and its surroundings is typically carried out within a stochastic picture based on Langevin equations [1, 2]. The basic equation of this type reads

\[
\dot{x}(t) = f(x, t) + \zeta(t),
\]

where \(f(x, t)\) is the deterministic (and possibly time-dependent) ‘force’ acting on the system and \(\zeta(t)\) stands for the ‘noise’ contribution describing interaction between the system and its complex surrounding. If the noise can be considered as white and Gaussian, the above equation gives rise to the classical Langevin approach used in the analysis of Brownian motion. The whiteness of the noise (lack of temporal correlations) corresponds to the existence of timescale separation between the dynamics of a relevant variable of interest \(x(t)\) and the typical timescale of the noise. Hence white noise can be considered as a standard stochastic process that describes in the simplest fashion the effects of ‘fast’ surroundings [3, 4].

Although in various phenomena, the noise can be indeed interpreted as white (i.e. with stationary, independent increments), the assumption about its Gaussianity can be easily
violated. The examples range from the description of the dynamics in plasmas [5], diffusion in energy space, self-diffusion in micelle systems, exciton and charge transport in polymers under conformational motion and incoherent atomic radiation trapping to the spectral analysis of paleoclimatic [6] or economic data [7], motion in optimal search strategies among randomly distributed target sites [8], fluorophore diffusion as studied in photo-bleaching experiments, interstellar scintillations [9], ratcheting devices [10]–[12] and many others [13].

The present work overviews properties of Lévy flights in external potentials with a focus on astonishing aspects of noise-induced phenomena like resonant activation (RA) [14]–[18], stochastic resonance (SR) [2,13], [19]–[22] and dynamic hysteresis [20,23,24]. In particular, the persistency of the SR occurrence is examined within a continuous and a two-state description of the generic system [22] composed of a test particle moving in the double-well potential and subject to the action of deterministic, periodic perturbations and α-stable, white Lévy type noises. In the same system the appearance of dynamic hysteresis is documented. Moreover, an archetypal problem of escape over a fluctuating barrier is analyzed, revealing the RA phenomenon in the presence of a non-equilibrium, memoryless Lévy type bath.

2. The Lévy–Wiener process and Lévy white noise

The Langevin equation (1) is a basic tool for studying noise phenomena in systems coupled to a fluctuating environment. In its most common form the fluctuation term \( \zeta(t) \) represents white Gaussian noise defined as a time derivative (in the sense of generalized functions) of the Wiener process \( W(t) \), i.e. the process with independent, Gaussian distributed increments [25]:

\[
\int_t^{t+s} \zeta(t') \, dt' = \Delta W(s). \tag{2}
\]

In the above, the integral exists in the mean square sense. The process \( W(t) \) is a mathematical idealization of a free \( (f(x,t) = 0) \) Brownian motion \( (dx(t) = dW(t), [dW(t)]^2 = dt) \) and its representation is introduced as a limit (in distribution) of independent, identically distributed (i.i.d.) random (Gaussian) jumps taken at infinitesimally short time intervals of non-random length. Other generalizations are also possible, e.g. \( W(t) \) can be defined as a limit of random Gaussian jumps performed at random Poissonian times:

\[
W(t) = \lim_{n \to \infty} \sum_{k=1}^{\lfloor nt \rfloor} W_k \left( \frac{1}{n} \right) = \lim_{n \to \infty} W \left( \frac{N(nt)}{n} \right) = \lim_{n \to \infty} \sum_{k=1}^{N(nt)} W_k, \tag{3}
\]

with a cumulative distribution

\[
\text{Prob} \{ W(t) \leq w \} = \lim_{n \to \infty} \sum_{k=0}^{\infty} \text{Prob} \left\{ W \left( \frac{k}{n} \right) \leq w \right\} \text{Prob} \{ N(nt) = k \}
\]

\[
= \lim_{n \to \infty} \int_{-\infty}^{w} \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi k/n}} e^{-nx^2/2nk} \frac{1}{k!} e^{-nt} \, dx
\]

\[
= \int_{-\infty}^{w} e^{-x^2/(2nt)} \, dx. \tag{4}
\]
Here \([tn]\) denotes the integer part of the number \(tn\). The characteristic feature of the Gaussian Wiener process is the continuity of its sample paths. In other words, realizations (trajectories) of the Wiener process are continuous (although nowhere differentiable) [25]. The process is also self-similar (scale invariant) which means that by rescaling \(t' = \lambda t\) and \(W'(t) = \lambda^{-1/2}W(\lambda t)\) another Wiener process with the same properties is obtained. Among scale invariant stable processes, the Wiener process is the only one which possesses finite variance [25]–[27]. Moreover, since the correlation function of increments \(\Delta W(s) = W(t + s) - W(t)\) depends only on time difference \(s\) and increments of non-overlapping times are statistically independent, the formal differentiation of \(W(t)\) yields a white, memoryless Gaussian process [28].

In more general terms, by dropping the white character of noise and the requirement of Gaussianity, the noise increments may be modeled as a random sum (a continuous time, random walk process) of i.i.d. random variables:

\[
\Delta \tilde{W}(t) = \Delta x(t) = \sum_{i=1}^{N(t)} X_i,
\]

where the number of summands \(N(t)\) is statistically independent from \(X_i\) and defined via a renewal process \(\sum_{i=1}^{N(t)} T_i \leq t < \sum_{i=1}^{N(t)+1} T_i\) with \(t > 0\). Let us assume that \(T_i, X_i\) belong to the domain of attraction of stable distributions, \(T_i \sim S_{\lambda,1}\) and \(X_i \sim S_{\alpha,\beta}\), whose corresponding characteristic functions \(\phi(k) = \langle \exp(i k S_{\alpha,\beta}) \rangle = \int_{-\infty}^{\infty} e^{ikx} l_{\alpha,\beta}(x; \sigma, \mu = 0) \, dx\), with the density \(l_{\alpha,\beta}(x; \sigma, \mu = 0)\), are given by

\[
\phi(k) = \exp \left[ -\sigma^\alpha |k|^\alpha \left( 1 - i \beta \operatorname{sgn} k \tan \frac{\pi \alpha}{2} \right) \right],
\]

for \(\alpha \neq 1\) and

\[
\phi(k) = \exp \left[ -\sigma |k| \left( 1 + i \frac{\beta}{2} \log |k| \right) \right],
\]

for \(\alpha = 1\). Here the parameter \(\alpha \in (0, 2]\) denotes the stability index, yielding the asymptotic long-tail power law for the \(x\)-distribution, which for \(\alpha < 2\) is of the \(|x|^{-(1+\alpha)}\) type. The parameter \(\sigma\) (\(\sigma \in (0, \infty)\)) characterizes the scale whereas \(\beta\) (\(\beta \in [-1, 1]\)) defines an asymmetry (skewness) of the distribution and \(\mu\) represents the shift, which for the strictly stable distributions [29, 30] is set to 0.

Note that for \(0 < \lambda < 1\), \(\beta = 1\), the stable variable \(S_{\lambda,1}\) is defined on the positive semi-axis. Within the above formulation the counting process \(N(t)\) satisfies

\[
\lim_{t \to \infty} \text{Prob} \left\{ \frac{N(t)}{(t/c)^{\lambda}} < x \right\} = \lim_{t \to \infty} \text{Prob} \left\{ N(t) < \left[ \frac{(t/c)^{\lambda}}{x} \right] x \right\}
\]

\[
= \lim_{t \to \infty} \text{Prob} \left\{ \sum_{i=1}^{\left[ \frac{(t/c)^{\lambda}}{x} \right]} T_i > t \right\}
\]

\[
= \lim_{n \to \infty} \text{Prob} \left\{ \sum_{i=1}^{n} T_i > \frac{cn^{1/\lambda}}{x^{1/\lambda}} \right\}
\]
\[ \lim_{n \to \infty} \text{Prob} \left\{ \frac{1}{c_1 n^{1/\lambda}} \sum_{i=1}^{[n]} T_i > \frac{1}{x^{1/\lambda}} \right\} = 1 - L_{\lambda,1}(x^{-1/\lambda}). \quad (8) \]

Moreover, since \( \lim_{n \to \infty} \text{Prob} \{1/(c_1 n^{1/\alpha}) \sum_{i=1}^{n} X_i < x\} \to L_{\alpha,\beta}(x) \) and \( p(x, t) = \sum_{n} p(x|n)p_n(n(t)) \), asymptotically one gets (\( dL_{\alpha,\beta}(x)/dx = l_{\alpha,\beta}(x) \))

\[
p(x, t) \sim (c_2 t)^{-\lambda/\alpha} \int_{0}^{\infty} l_{\alpha,\beta}(c_2 t)^{-\lambda/\alpha} x^{\lambda/\alpha} l_{\lambda,1}(\tau) \tau^{\lambda/\alpha} \, d\tau, \quad (9)\]

where \( c_1 \) and \( c_2 \) are constants. The resulting (non-Markov) process becomes a \( \lambda/\alpha \) self-similar Lévy random walk \([3, 4, 26]\). In consequence, the time derivative of \( W(t) \) would result in a general, colored Lévy noise.

It should be stressed that in the majority of studies relating to the noise-induced phenomena, the additive noise at the level of Langevin equation (1) is taken as spectrally flat (i.e. white) with zero time correlations. On the other hand, colored noise \([31]–[33]\) with a (typically) algebraic spectrum \( 1/f^{\alpha} \) has been detected in many natural systems \([34]\), thus documenting that the white noise idealization may be insufficient for biological modeling.

In principle, non-equilibrium noises may be both correlated (colored) and non-thermal (non-Gaussian). This situation clearly extends investigations of positive effects of noises to more general scenarios and there is accumulating evidence for colored-noise-driven transport, stochastic resonance, synchronization and phase ordering \([35]–[37]\).

In order to explore the correspondence between phenomena induced by Gaussian and non-Gaussian noises, in the forthcoming paragraphs we will restrict ourselves solely to white (delta correlated in time) but, otherwise, Lévy distributed noise understood as a time derivative of the generalized Wiener process \( W_{\alpha,\beta}(s) \):

\[
\int_{t}^{t+s} dt' \zeta(t') = \Delta W_{\alpha,\beta}(s), \quad (10)
\]

i.e. a non-Gaussian stochastic process with stationary and independent increments \([38]–[42]\) distributed according to the \( \alpha \)-stable law defined in terms of the characteristic function \( \phi(k, s) = \exp[-s\sigma^\alpha |k|^\alpha (1-i\beta \text{sgn} k \tan(\alpha \pi/2))]; \) see section 3 and equation (13). Example probability density functions \( l_{\alpha,\beta}(x; \sigma, \mu = 0) \) for symmetric (\( \beta = 0 \)) and skewed (\( \beta \neq 0 \)) cases have been displayed in figure 1.

3. Model

The model system is described by the following (overdamped) Langevin equation driven by a Lévy stable, white noise \( \zeta(t) \):

\[
\frac{dx(t)}{dt} = -V'(x, t) + \zeta(t). \quad (11)
\]

To examine the occurrence of stochastic resonance and dynamic hysteresis, the generic double-well potential with periodic perturbation (see left panel of figure 2) has been chosen:

\[
V(x, t) = -\frac{a}{2} x^2 + \frac{b}{4} x^4 + A_0 x \cos \Omega t, \quad (12)
\]

with \( a = 128, b = 512, A_0 = 8 \).

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Figure 1. Symmetric stable distributions for $\alpha = 2.0, 1.5, 1.0, 0.5$ with $\beta = 0$ (from the bottom to the top; left panel) and asymmetric stable densities for $\beta = 1.0, 0.5, 0.0, -0.5, -1.0$ with $\alpha = 1.5$ (from left to right; right panel).

Figure 2. The generic double-well potential $V(x, t) = -(a/2)x^2 + (b/4)x^4 + A_0 x \cos \Omega t$ with $a = 128$, $b = 512$, $A_0 = 8$ for $t = \{0, \pi/2, \pi\}$ (left panel) and the generic linear potential slope dichotomously switching between two configurations characterized by different heights $H_{\pm}$ (right panel).
Lévy stable noise-induced transitions

In turn, for inspection of resonant activation, the action of the potential \( V(x,t) \) has been approximated by a linear slope switching dichotomously between two distinct configurations \( V_\pm(x) = H_\pm x \) (see right panel of figure 2). The detection of RA has been carried out for a relatively ‘high’ barrier \( H_+ = 8 \), which under the normal diffusion condition guarantees the proper separation of timescales (the actual times of escape events and time of diffusive motion within the potential well [14,43]) of the process. Following former studies on RA [14], the lower barrier height has been chosen to take one of two values, \( H_- = -8 \) or 0. A particle has been assumed to start its motion at the reflecting boundary \( x = 0 \) and continues as long as the position fulfills \( x(t) < 1 \).

It should be stressed that with generally non-Gaussian white noise the knowledge of the boundary location alone cannot specify in full the corresponding boundary conditions for reflection or absorption, respectively [44]. The trajectories driven by non-Gaussian white noise display irregular, discontinuous jumps. As a consequence, the location of the boundary itself is not hit by the majority of discontinuous trajectories. This implies that regimes beyond the location of the boundaries must be properly accounted for when setting up the boundary conditions. In particular, multiple recrossings of the boundary location from excursions beyond the specified state space have to be excluded. As has been demonstrated elsewhere [44], incorporation of Lévy flights into kinetic description requires the use of nonlocal boundary conditions which, for the case of an absorbing boundary at, say, \( x = 1 \), calls for extension of the absorbing regime to the semiline beyond that point, \( x \geq 1 \).

The barrier fluctuations causing alternating switching between the high \( (H_+) \) and low \( (H_-) \) barrier configurations have been approximated by the Markovian dichotomous noise [28] with the exponential autocorrelation function \( \langle [\eta(t) - \langle \eta(t) \rangle][\eta(t') - \langle \eta(t') \rangle] \rangle = \frac{\gamma}{4}(H_+ - H_-)^2 \exp(-2\gamma|t - t'|) \). Here \( \gamma \) represents the rates of transition between (±) and (≠) states, and both Lévy and dichotomous noises have been assumed to be statistically independent.

The sample trajectories \( x(t) \) have been obtained by a direct integration of the Langevin equation (11) using the standard techniques of integration of stochastic differential equations with respect to the Lévy stable PDFs [29]

\[
x(t) = -\int_{t_0}^t V'(x(s),s) \, ds + \int_{t_0}^t \zeta(s) \, ds \approx -\int_{t_0}^t V'(x(s),s) \, ds + \sum_{i=0}^{N-1} \Delta s^{1/\alpha} \zeta_i. \tag{13}
\]

Here \( N \Delta s = t - t_0 \) and \( \zeta_i \) is distributed according to the stable probability density function \( l_{\alpha,\beta}(\zeta;\sigma,\mu = 0) \) which for \( \alpha = 2 \) yields the Gaussian law. Note that the time increments \( \Delta s \) contribute as a scaling parameter. In the limit of \( N \to \infty \) the sum \( \sum_{i=0}^{N-1} \Delta s^{1/\alpha} \zeta_i \) converges to a stable variable \( S_{\alpha,\beta} \) with \( l_{\alpha,\beta}(\zeta;\sigma(t - t_0)^{1/\alpha},\mu = 0) \) density. The formula given by equation (13) explains the superdiffusive character of the generalized Wiener process \( W_{\alpha,\beta}(s) \), since (for \( V'(x) = 0 \)) one observes \( \langle x^2(t) \rangle \propto t^{2/\alpha} \).

1 Here we recall that for independent copies \( \zeta_1,\zeta_2 \) of a stable random variable \( \zeta \sim S_{\alpha,\beta} \) with the density \( l_{\alpha,\beta}(\zeta;\sigma,\mu = 0) \) and any positive constants \( A, B \), the sum \( A^{1/\alpha} \zeta_1 + B^{1/\alpha} \zeta_2 \) tends to a stable random variable whose PDF is given by \( l_{\alpha,\beta}(\zeta;\sigma(A + B)^{1/\alpha},\mu = 0) \).

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4. Results

4.1. Stochastic resonance and dynamic hysteresis

The response of the ensemble average coordinate $\langle x(t) \rangle$ to external periodic perturbations (cf. equations (11) and (12)) gives rise to the phenomenon of ‘dynamic hysteresis’. In order to register the behavior, we have performed an analysis of trajectories $x(t)$ based on a two-state approximation. For this purpose, we have defined an occupation probability of being in the left (with the respect to the barrier position $x = x_b$) or right state according to the definition of the occupation probabilities:

$$p_{\text{left}}(t) = \text{Prob}\{x(t) < x_b\} = \int_{-\infty}^{x_b} p(x, t) \, dx = 1 - p_{\text{right}}(t).$$

Note that the relative occupation of either one of those states can be altered not only by the modulation of the potential (see equation (12)), but also by an appropriate tuning of the additive noise parameters. When plotted as a function of the periodic driving $\cos \Omega t$, the occupation probability $p_{\text{right}}(t)$ exhibits a characteristic hysteresis loop. Quite obviously, the non-zero skewness parameter $\beta$ introduces asymmetry to the dynamic hysteresis loops; cf. figure 3. As can be inferred from figure 3, the positive $\beta$ ‘biases’ the motion, causing the trajectories to stay more probably to the right of the barrier top $x_b$.

The overall behavior of the dynamic hysteresis loops

$$HL \equiv \int_{t=0}^{2\pi/\Omega} \langle x(t) \rangle \, d(\cos \Omega t)$$

Figure 3. Dynamic hysteresis loops for various $\beta$ with fixed $\alpha = 1.9$. The time step of the integration $\Delta t = 10^{-4}$. Results were averaged over $N = 10^3$ realizations. Error bars represent standard deviation of the mean. Initial conditions $x(0)$ were sampled from the interval $[-1.25, 1.25]$. The particle is moving in the modulated double-well potential (12) with $a = 128$, $b = 512$, $A_0 = 8$, $\Omega = 1$, $\sigma = \sqrt{2}$.
under the $\alpha$-stable noises could be deduced from the inspection of the trajectories of the process presented in figure 5. For the symmetric stable noise, i.e. $\beta = 0$, the process spends, on average, the same amount of time in the left/right states and, consequently, the occupations of the two states are equal. For non-zero and increasing $|\beta|$ a larger asymmetry in the distribution of residence times in the right/left states is registered. Asymmetry of the occupation probability in either one of the states is reflected in the shape of the dynamic hysteresis loop, which with an increasing $|\beta|$ becomes distorted into the direction determined by the sign of the skewness parameter.

The probability of finding the process in the right/left state depends strongly on the stability index $\alpha$; cf. figure 4. For symmetric noises ($\beta = 0$) the area of the hysteresis loop decreases with decreasing stability index $\alpha$; see figures 4 and 6. This observation is a direct consequence of a heavy-tailed nature of the noise term in equation (11). With decreasing $\alpha$, larger excursions of the particle are possible and these occasional jumps of trajectories may be of the order of, or even larger than, the distance separating the two minima of the potential $V(x,t)$. A combined interplay of the two noise parameters $\alpha, \beta$ may result in a permanent locking of the process in one of its states; see the right panel of figure 4.

In order to quantify the SR phenomenon [1, 23] we have used the standard measures of the signal-to-noise ratio SNR and the spectral power amplification $\eta$ [1]. The SNR is

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Figure4.png}
\caption{Dynamic hysteresis loops for various $\alpha$ with $\beta = 0$ (left panel) and $\beta = -1$ (right panel). Parameters of the simulation: like in figure 3. Error bars represent standard deviation of the mean.}
\end{figure}
Figure 5. Trajectories of the $\alpha$-stable process for the two-state and the continuous model constructed by use of equation (11). The time step of the integration $\Delta t = 10^{-5}$; the scale parameter $\sigma = \sqrt{2}$. The frequency of the cycling voltage $\Omega = 1$. Noise parameters: as indicated in the figures.

defined as a ratio of a spectral content of the signal in the forced system to the spectral content of the noise $\zeta(t)$:

$$\text{SNR} = 2 \left[ \lim_{\Delta \omega \to 0} \int_{\Omega-\Delta \omega}^{\Omega+\Delta \omega} S(\omega) \, d\omega \right] / S_N(\Omega).$$

(16)

Here $\int_{\Omega-\Delta \omega}^{\Omega+\Delta \omega} S(\omega) \, d\omega$ represents the power carried by the signal, while $S_N(\Omega)$ estimates the background noise level. In turn, the spectral power amplification ($\eta$) is given by the ratio of the power of the driven oscillations to that of the driving signal at the driving frequency $\Omega$. Closer examination of these quantifiers reveals that $\eta$ and SNR behave in a typical way both in the continuous model (cf. figure 6) and in its two-state analogue (results not shown). The spectral amplification $\eta$ has a characteristic bell-shape form indicating detection of stochastic resonance within the interval of suitably chosen noise intensity $\sigma^2$.

For example, if the stability index is set to $\alpha = 1.9$ and trajectories are simulated with extremely weak noise intensities $\sigma^2$, the periodic signal is not well separated from the noisy background and consequently SNR stays negative.

However, SNR becomes positive with increasing noise intensity which indicates emerging separation of the signal from the noisy background. In figure 6 results for $\eta$ and SNR analysis (in a continuous SR model) with various stability indexes $\alpha$ are presented. Note that for a given $\alpha$ the power spectra for $\pm \beta$ are the same. Therefore, stochastic
Figure 6. Various quantifiers of stochastic resonance: spectral power amplification $\eta$ (top panel), signal-to-noise ratio SNR (middle panel) and hysteresis loop area $HL$ (bottom panel) for the periodically perturbed generic double-well potential driven by additive Lévy noises. The results presented were derived for Lévy noises with $\beta = 0$ (left panel), $\beta = -1.0$ (right panel) and various $\alpha$. The simulation parameters are the same as in figure 5. Lines are drawn to guide the eye.
resonance quantifiers derived from the power spectra are equivalent for \( \pm \beta \) with the same value of \( \alpha \).

Switching from the continuous model to the two-state approximation (cf. figure 5) does not change the qualitative behavior of the SR quantifiers\(^2\). However, the slope of the decaying part of the signal-to-noise ratio is flatter for the two-state description than for the continuous model (results not shown).

Our numerical analysis implies that \( \eta \) is more sensitive to the variation of noise parameters than SNR; see figure 6. At decreasing values of the stability index \( \alpha \), the maximum of \( \eta(\sigma^2) \) drops and shifts towards higher values of the noise intensity. Obviously, the decrease of \( \alpha \) weakens the stochastic resonance and reduces the system performance. The diminishment of the spectral amplification for \( \alpha < 2 \) indicates that the input signal is less well reproduced in the recorded output. This behavior is more easily detectable for symmetric noises (see left panel of figure 6) than for asymmetric ones (see right panel) and can be readily explained by the analysis of example trajectories; see figure 5. Sharp spikes clearly visible in the right corner panel of figure 5 are due to the heavy-tailed nature of the Lévy stable distribution and they become more pronounced the smaller the index \( \alpha \) becomes. Their presence indicates that for a sufficiently small \( \alpha \) the trajectory becomes discontinuous and, on average, switches between left and right wells of the potential \( V(x, t) \) are realized by sudden long-jump escape events fairly independent of the periodic driving. Although the SNR is less sensitive to variations in noise parameters (cf. figure 6), the shape of this function flattens for decreasing values of \( \alpha \), thus hampering detection of the resonant value of the noise optimal intensity for which the SR phenomenon is most probably perceived. Finally, the response of the system to periodic and random driving can be characterized by the area of the dynamic hysteresis loop (see equation (15) and figure 4) which can be considered an additional feature of the SR phenomenon [45]–[47]. For relatively low noise intensities, the system response is better synchronized with the input signal and the maximum of HL (as informative of the appearance of SR) is detected at some optimal noise intensity \( \sigma^2 \). Lowering stability index \( \alpha \) causes a significant diminishment of the loop area. Also, the additional skewness of the noise statistics (when the distribution of noise pushes is asymptotically ‘heavier’ to the left; cf. figure 1) affects the synchronized response and shifts the maximum of HL towards higher values of the noise intensity. This tendency can be contrasted with the effect of symmetric Lévy noises which, at the same value of \( \alpha \), induce the optimal response at lower values of \( \sigma^2 \). In general, inspection of the hysteresis loop area stays in line with conclusions deduced from the SNR and SPA analysis. However, maxima of HL are detected at smaller values of \( \sigma^2 \) than maxima of other SR quantifiers; cf. figure 6.

4.2. Resonant activation

As already discussed in the preceding sections, the stochastic kinetics driven by additive non-Gaussian stable white noises is very different from the Gaussian case. For \( \alpha < 2 \), a test particle moving in the linear potential can change its position via extremely long, jump-like excursions. This in turn requires the use of nonlocal boundary conditions in evaluation of the mean first-passage time (MFPT) [15, 44], [48]–[50]. In this section, as

\(^2\) In fact, some changes are observed but in the parameter region where barrier crossing events are not recorded, i.e. at relatively small noise intensity \( \sigma^2 \) with large \( \alpha \) (\( \alpha \approx 2 \)).
well as in our previous investigations, the above issue is taken care of when generating first-passage times (FPTs) using Monte Carlo simulations. Simulated trajectories are representative for a motion of a test particle over the interval \([0, 1]\) (see the right panel of figure 2) influenced by independent dichotomous switching of the potential slope and subjected to additive white Lévy noise. A particle starts its motion at \(x = 0\) where the reflecting boundary is located. The absorbing boundary is located at \(x = 1\), meaning that the whole semi-axis \([1, \infty)\) is assumed absorbing, yielding zero PDF \(p(x, t) = 0\) for all \(x \geq 1\); see the discussion in \([44]\) and \([15]\), \([48]\)–\([50]\).

From the ensembles of collected first-passage times we have evaluated the mean values of the distributions and our findings for MFPT are presented in figure 7. Integration of equation (11) was performed for a series of descending time steps of integration \(\Delta t = \{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}\) to ensure that the results are self-consistent.

The left panel of figure 7 displays behavior of derived MFPTs as a function of the stability index \(\alpha\) and the rate of the barrier modulation \(\gamma\). In the right panel

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Figure 7. MFPT(\(\alpha, \gamma\)) for \(H_\pm = \pm 8\) with \(\beta = 1, \sigma = 1\) (left panel) and sample cross-sections MFPT(\(\gamma\)) for various \(\alpha\): ‘+’: \(\alpha = 0.2\); ‘×’: \(\alpha = 0.8\); ‘∗’: \(\alpha = 0.9\); ‘□’: \(\alpha = 1.1\); ‘■’: \(\alpha = 1.7\) (right panel). The results were calculated by direct integration of equation (11) with the time step \(\Delta t = 10^{-5}\) and averaged over \(N = 10^3\) realizations. Horizontal lines represent asymptotic values of MFPT(\(\gamma \to 0\)) and MFPT(\(\gamma \to \infty\)). They have been evaluated by use of the Monte Carlo method with \(\Delta t = 10^{-5}\) and averaged over \(N = 5 \times 10^3\) realizations. Error bars represent standard deviations of the mean.
sample cross-sections of the surfaces $\text{MFPT}(\alpha, \gamma)$ are drawn. For $\alpha = 1$, or adequately $\alpha \approx 1$, the procedure of simulating skewed ($\beta \neq 0$) stable random variables becomes unstable. This can be well explained by examining the form of the additive noise term characteristic function $\phi(t)$—see equations (6) and (7). The exponential functions are no longer continuous functions of the parameters and exhibit discontinuities when $\alpha = 1$, $\beta \neq 0$. Therefore, for the clarity of presentation, these parameter sets have been excluded from consideration.

The depicted results indicate the appearance of the RA phenomenon which is most visible when the potential barrier is switching between two barrier configurations characterized by $H_{\pm} = \pm 8$. However, the dependence of RA on noise parameters is highly nontrivial. The overall tendency in kinetics is similar to cases described in earlier sections: heavier tails ($\alpha < 2$) in the distribution of noise increments $\zeta(t)$ result in stronger discontinuities of trajectories causing the RA effect to become inaudible and fade gradually. Notably, for totally skewed additive noise $\beta = 1$ which acts in favor of the motion to the right, the RA seems to disappear for $\alpha = 0.9$. It reappears again for smaller values of $\alpha$, when the driving Lévy white noise becomes a one-sided Lévy process with strictly positive increments which tend to push trajectories to the right.

To further examine the character and distribution of events of escape from $x = 0$, we have analyzed survival probabilities $G(\gamma, t)$ constructed from generated trajectories at fixed values of frequencies $\gamma$. For better comparison with the Gaussian RA scenario, the summary of results is displayed in figure 8 with sets of data relating to $\alpha = 2$ and 0.9. The left and right panels of figure 8 present the behavior of MFPT as a function of the barrier modulation parameter $\gamma$, the survival probability $G(\gamma, t)$ and example cross-sections of the survival probability surface. The insets depict the behavior of the survival probability $G(\gamma, t)$ at short timescales (small $t$).

At low rates of the barrier switching process (small $\gamma$), two distinct timescales of the barrier crossing events can be observed. The fast timescale corresponds to passages over the barrier in its lower state, while the large timescale is pertinent to the slower process, i.e. the passages over the barrier in its higher energetic state. This effect is well pronounced after the RA phenomenon sets up. The presence of the two timescales for small $\gamma$ and only one timescale for large $\gamma$ explains the asymptotic behavior of MFPTs. Namely, for low values of the switching rate $\gamma$, MFPT is an average value of MFPTs over both configuration of the barrier. With increasing frequency $\gamma$ the distinguishable timescales coalesce and disappear. This is due to the fact that the barrier changes its height multiple times during the particle’s motion, so the resulting MFPT describes kinetics over the average potential barrier.

5. Discussion

We have examined the influence of various kinds of stochastic $\alpha$-stable drivings on dynamic properties of a generic two-state model system. Although our primary interest was in understanding the effects of Lévy type drivings with the stability index $0 < \alpha < 2$, the numerical analysis as applied in these studies remains valid for any set of parameters characterizing $\alpha$-stable excitations.

The response to external or/and parametric perturbations has been examined by analyzing qualitative changes in the system’s dynamic behavior as expressed in the onset
Lévy stable noise-induced transitions

Figure 8. Comparison of the behavior of MFPTs for $\alpha = 2$ (left panel) and $\alpha = 0.9$ (right panel) for $H_{\pm} = \pm 8$. MFPT curves (upper panel), survival probability surfaces $G(\gamma,t)$ (middle panel) and sample cross-sections of $G(\gamma,t)$ surface (lower panel) for small (‘+’), resonant (‘×’) and large (‘∗’) $\gamma$-values are depicted. Simulation parameters: as in figure 7. Note the log scale on the $z$-axis (middle panel) and on the $y$-axis (lower panel). Error bars represent standard deviations of the mean.

of resonant activation, stochastic resonance and dynamic hysteresis. Due to the inherent symmetry of the potential, the SR quantifiers constructed at a given value of the stability index $\alpha$ have been the same for $\pm \beta$. On the other hand, the asymmetry of the driving noise has been shown to influence strongly the population of states and therefore has affected mostly the appearance and performance of the dynamic hysteresis.

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The system efficiency has been described using standard SR measures, i.e. signal-to-noise ratio (SNR), spectral power amplification ($\eta$) and dynamical hysteresis loop area (HL). Those quantifiers have been shown to behave in a typical way. In other words, by tuning the noise intensity $\sigma^2$, the maximum in the SNR($\sigma^2$), $\eta(\sigma^2)$ and HL($\sigma^2$) could be detected, thus proving that stochastic resonance is a robust phenomenon which may be observed also in systems subjected to the action of impulsive jump noise stochastic processes. Decrease in stability index $\alpha$ results in larger jump-like excursions of the process $x(t)$ and consequently causes weakening of SR. Out of three SR measures, the spectral power amplification and area of the dynamical hysteresis loop have turned out to be more sensitive to variations of $\alpha$ than SNR. A more pronounced drop in system efficiency has been observed for symmetric noises with $\beta = 0$.

The non-equilibrated, non-thermal Lévy white noise affects also a paradigm scenario of escape kinetics. The behavior of a Lévy flier with jump length statistics described by $\alpha < 2$ is drastically different from that of a traditional Gaussian walker. The increments of the normal diffusion process are characterized by a statistics which excludes the occurrence of long jumps. Therefore the walker is more likely to approach and eventually hit a pointlike, local boundary. In contrast, the heavy-tail jump statistics ($\alpha < 2$) of meandering Lévy fliers results in possible multiple long jumps over the boundary and multiple recrossings into the finite interval [44, 49, 50]. This brings about a formulation of the boundary condition that necessarily has to be nonlocal in nature. Accordingly, for Lévy flights with $\alpha < 2$, the absorbing boundary condition is imposed by stopping the trajectory (removing the particle) whenever a jump takes it to a location outside the interval $[0, 1]$ (see figure 2, right panel). By numerically implementing the boundary conditions set up for the problem [15, 44], we have investigated the statistics of escape times over the dichotomously fluctuating barrier. The manifestation of the RA phenomenon has been analyzed within a certain frequency $\gamma$ regime, indicating that by a continuous readjustment of the external noise parameters the resonant activation can be either suppressed or re-induced.

Lévy white noise with $\alpha < 2$ extends a standard Brownian noise to a vast family of impulsive jump-like stochastic processes. Our studies document that dynamical systems driven by such sources can also benefit and display a noise-enhanced order. Further investigations in this field should address also robustness of the noise-induced order under the action of non-Markov, Lévy fluctuations.

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References

[1] Gammaitoni L, Hänggi P, Jung P and Marchesoni F, 1998 Rev. Mod. Phys. 70 223
[2] Anishchenko V S, Neiman A B, Moss F and Schimansky-Geier L, 1992 Sov. Phys. Usp. 42 7
[3] Shlesinger M F, Zaslavsky G M and Frisch J (ed), 1995 Lévy Flights and Related Topics in Physics (Berlin: Springer)
[4] Barndorff-Nielsen O E, Mikosch T and Resnick S I (ed), 2001 Lévy Processes: Theory and Applications (Boston, MA: Birkhäuser)
Lévy stable noise-induced transitions

[5] Chechkin A V, Gonchar V Y and Szyd/ọkowski M, 2002 Phys. Plasmas 9 78
[6] Dittevens P D, 1999 Geophys. Res. Lett. 26 1441
[7] Mantegna R N and Stanley H E, 2000 An Introduction to Econophysics. Correlations and Complexity in Finance (Cambridge: Cambridge University Press)
[8] Viswanathan G M et al, 1999 Nature 401 911
[9] Ditlevsen P D, 1999 Geophys. Res. Lett. 26 1441
[10] Mantegna R N and Stanley H E, 2000 An Introduction to Econophysics. Correlations and Complexity in Finance (Cambridge: Cambridge University Press)
[11] Viswanathan G M et al, 1999 Nature 401 911
[12] Boldyrev S and Gwinn C R, 2003 Phys. Rev. Lett. 91 051101
[13] Dybiec B, Gudowska-Nowak E and Sokolov I M, 2008 Phys. Rev. Lett. 101 010601
[14] Dybiec B, 2008 Phys. Rev. E 78 061120
[15] Kosko B and Mitaim S, 2001 Phys. Rev. E 64 051110
[16] Doering C R and Gadoua J C, 1992 Phys. Rev. Lett. 69 2318
[17] Dybiec B and Gudowska-Nowak E, 2004 Phys. Rev. E 70 041122
[18] Dybiec B and Gudowska-Nowak E, 2004 Fluct. Noise Lett. 4 L273
[19] Dybiec B and Gudowska-Nowak E, 2004 Proc. SPIE 5467 411
[20] Dybiec B and Gudowska-Nowak E, 2004 Fluct. Noise Lett. 4 L273
[21] Horsthemke W and Lefever R, 1984 Noise-Induced Transitions. Theory and Applications in Physics, Chemistry, and Biology (Berlin: Springer)
[22] Janicki A and Weron A, 1994 Simulation and Chaotic Behavior of α-Stable Stochastic Processes (New York: Marcel Dekker)
[23] Nolan J P, 2007 Stable Distributions—Models for Heavy Tailed Data (Boston, MA: Birkhäuser) chapter 1 (online at academic2.american.edu/~jpnolan in progress)
[24] Gammaitoni L, Marchesi F and Santucci S, 1995 Phys. Rev. Lett. 74 1052
[25] Dubkov A A and Spagnolo B, 2005 Fluct. Noise Lett. 5 L267
[26] Horsthemke W and Lefever R, 1984 Noise-Induced Transitions. Theory and Applications in Physics, Chemistry, and Biology (Berlin: Springer)
[27] Janicki A and Weron A, 1994 Simulation and Chaotic Behavior of α-Stable Stochastic Processes (New York: Marcel Dekker)
[28] Nolan J P, 2007 Stable Distributions—Models for Heavy Tailed Data (Boston, MA: Birkhäuser) chapter 1 (online at academic2.american.edu/~jpnolan in progress)
[29] Gammaitoni L, Marchesi F and Santucci S, 1995 Phys. Rev. Lett. 74 1052
[30] Dubkov A A and Spagnolo B, 2005 Fluct. Noise Lett. 5 L267
[31] Horsthemke W and Lefever R, 1984 Noise-Induced Transitions. Theory and Applications in Physics, Chemistry, and Biology (Berlin: Springer)
[32] Janicki A and Weron A, 1994 Simulation and Chaotic Behavior of α-Stable Stochastic Processes (New York: Marcel Dekker)
[33] Nolan J P, 2007 Stable Distributions—Models for Heavy Tailed Data (Boston, MA: Birkhäuser) chapter 1 (online at academic2.american.edu/~jpnolan in progress)