Anyonic behavior of quantum group gases

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Abstract

We first introduce and discuss the formalism of $SU_q(N)$-bosons and fermions and consider the simplest Hamiltonian involving these operators. We then calculate the grand partition function for these models and study the high temperature (low density) case of the corresponding gases for $N = 2$. We show that quantum group gases exhibit anyonic behavior in $D = 2$ and $D = 3$ spatial dimensions. In particular, for a $SU_q(2)$ boson gas at $D = 2$ the parameter $q$ interpolates within a wider range of attractive and repulsive systems than the anyon statistical parameter.

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1 Introduction

In the last few years, the search for new applications of quantum groups and quantum algebras \[1,2\], other than the theory of integrable models and the quantum inverse scattering method, has attracted the attention of mathematicians and physicists alike. The published literature on formulations based on quantum group theory includes studies in non-commutative geometry \[3,4\], quantum mechanics \[5\], field theory \[6\], molecular and nuclear physics \[7\]. Many of these approaches are attempts to develop more general formulations of quantum mechanics and field theory, and to look for small deviations from the standard value \(q = 1\) in nuclear and molecular physics.

In this article we study the high temperature (low density) behavior of two quantum group gases. In Section 2 we discuss the covariant \(SU_q(N)\) fermion and boson algebras, and specialize to the case \(N = 2\). In subsections 2.1 and 2.2 we introduce the \(SU_q(2)\) fermion and boson models respectively, and in each case we give a representation of these operators in terms of the corresponding standard fermion or boson oscillators. Section 3 contains the main results of this work. We obtain the equation of state as a virial expansion and discuss their anyonic behavior for both gases at \(D = 2\) and \(D = 3\). In \(D = 2\) we compare the parameter \(q\) with the anyon statistical parameter \(\alpha\).

2 Quantum group bosons and fermions

In this section we briefly discuss the quantum group field algebras introduced in Reference \[8\]. These algebras can be seen as generalizations of the standard bosonic and fermionic algebras. As it is well known, bosonic and fermionic
operators satisfy the algebraic relations

\[ \phi_i \phi_j^\dagger - \phi_j^\dagger \phi_i = \delta_{ij} \]
\[ \psi_i \psi_j^\dagger + \psi_j^\dagger \psi_i = \delta_{ij}, \] (1)

which, for \( i, j = 1, \ldots N \), are covariant under \( SU(N) \) transformations. The quantum group analogues of these equations are given by the following relations

\[ \Omega_j \Omega_i = \delta_{ij} \pm q^{\pm 1} R_{kijl} \Omega_l \Omega_k \] (2)
\[ \Omega_l \Omega_k = \pm q^{\mp 1} R_{ijkl} \Omega_j \Omega_i, \] (3)

where \( \Omega = \Phi, \Psi \) and the upper (lower) sign applies to quantum group bosons \( \Phi_i \) (quantum group fermions \( \Psi_i \)) operators. The \( N^2 \times N^2 \) matrix \( R_{ijkl} \) is explicitly written \[4]\n
\[ R_{ijkl} = \delta_{jk} \delta_{il} (1 + (q - 1) \delta_{ij}) + (q - q^{-1}) \delta_{ik} \delta_{jl} \theta(j - i), \] (4)

where \( \theta(j - i) = 1 \) for \( j > i \) and zero otherwise. Denoting the new fields as \( \Omega'_i = \sum_{j=1}^{N} T_{ij} \Omega_j \), the \( SU_q(N) \) transformation matrix \( T \) and the \( R \)-matrix satisfy the well known algebraic relations \[9\]

\[ RT_1 T_2 = T_2 T_1 R, \] (5)

and

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \] (6)

with the standard embedding \( T_1 = T \otimes 1 \), \( T_2 = 1 \otimes T \in V \otimes V \) and \( (R_{23})_{ijk,i'j'k'} = \delta_{i'i} \delta_{j'j} R_{jk,j'k'} \in V \otimes V \otimes V \).

In particular, for \( N = 2 \), Equations (2) and (3) are simply written
a) $SU_q(2)$ — fermions

\[
\{\Psi_2, \overline{\Psi}_2\} = 1 \quad (7) \\
\{\Psi_1, \overline{\Psi}_1\} = 1 - (1 - q^{-2})\overline{\Psi}_2\Psi_2 \quad (8) \\
\Psi_1\Psi_2 = -q\Psi_2\Psi_1 \quad (9) \\
\overline{\Psi}_1\Psi_2 = -q\Psi_2\overline{\Psi}_1 \quad (10) \\
\{\Psi_1, \Psi_1\} = 0 = \{\Psi_2, \Psi_2\}, \quad (11)
\]

b) $SU_q(2)$ — bosons

\[
\Phi_2\overline{\Phi}_2 - q^2\Phi_2\overline{\Phi}_2 = 1 \quad (12) \\
\Phi_1\overline{\Phi}_1 - q^2\overline{\Phi}_1\Phi_1 = 1 + (q^2 - 1)\overline{\Phi}_2\Phi_2 \quad (13) \\
\Phi_2\Phi_1 = q\Phi_1\Phi_2 \quad (14) \\
\Phi_2\overline{\Phi}_1 = q\overline{\Phi}_1\Phi_2 \quad (15)
\]

which for $q = 1$ become the fermion and boson algebras respectively.

According to Equation (5) the matrix \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) elements generate the algebra

\[
ab = q^{-1}ba \quad , \quad ac = q^{-1}ca \\
bc = cb \quad , \quad dc = qcd \\
db = qbd \quad , \quad da - ad = (q - q^{-1})bc \\
\det_q T \equiv ad - q^{-1}bc = 1,
\]

with the unitary conditions \[10\] $a = d, \overline{b} = q^{-1}c$ and $q \in \mathbb{R}$. Hereafter, we take $0 \leq q < \infty$. 

4
2.1 $SU_q(2)$ fermion model

The simplest Hamiltonian one can write in terms of the operators $\Psi_i$ is simply

$$H_F = \sum_\kappa \varepsilon_\kappa (\mathcal{M}_{1,\kappa} + \mathcal{M}_{2,\kappa}), \quad (17)$$

where $\mathcal{M}_{i,\kappa} = \overline{\Psi}_{i,\kappa} \Psi_{i,\kappa}$ and $\{\overline{\Psi}_{\kappa,i}, \Psi_{\kappa',j}\} = 0$ for $\kappa \neq \kappa'$. From Equation (11) we see that the occupation numbers are restricted to $m = 0, 1$ and therefore $SU_q(N)$ fermions satisfy the Pauli exclusion principle. For a given $\kappa$ a normalized state is simply written

$$\overline{\Psi}_2^n \Psi_1^m |0>, \quad n, m = 0, 1 \quad (18)$$

and the $\mathcal{M}_i$ operator satisfies

$$[\mathcal{M}_2, \Psi_1] = 0, \quad (19)$$

and

$$\mathcal{M}_1 \Psi_2 - q^2 \Psi_2 \mathcal{M}_1 = 0. \quad (20)$$

The grand partition function is given by

$$Z_F = Tr e^{-\beta\sum_\kappa \varepsilon_\kappa (\mathcal{M}_{1,\kappa} + \mathcal{M}_{2,\kappa})} e^{\beta \mu (\mathcal{M}_{1,\kappa} + \mathcal{M}_{2,\kappa})}, \quad (21)$$

where $M_{i,\kappa} = \psi_{i,\kappa}^\dagger \psi_{i,\kappa}$ are the standard fermion number operators, and the trace is taken with respect to the states in Equation (18). Since the pair $\Psi_2, \overline{\Psi}_2$ satisfies standard anticommutation relations we can identify it without any loss of generality with a fermion pair $\psi_2, \psi_2^\dagger$. In addition, from Equations (8) and (11) we see that the operator $\Psi_1(\overline{\Psi}_1)$ is a function of the operator $\psi_1(\psi_1^\dagger)$ times a function of $\mathcal{M}_2$. Therefore the grand partition function $Z_F$ becomes

$$Z_F = \prod_\kappa \left( \sum_{n=0}^{1} \sum_{m=0}^{1} e^{-\beta \varepsilon_\kappa(n+m-(1-q^{-2})mn)} e^{\beta \mu(n+m)} \right) \quad (22)$$

and

$$Z_F = \prod_\kappa \left( 1 + 2 e^{-\beta(\varepsilon_\kappa-\mu)} + e^{-\beta(\varepsilon_\kappa(q^{-2}+1)-2\mu)} \right). \quad (23)$$
which for \( q = 1 \) becomes the square of a single fermion type grand partition function. From Equation (23) we see that the original Hamiltonian becomes the interacting Hamiltonian

\[
H_F = \sum_\kappa \varepsilon_\kappa \left( M_{1,\kappa} + M_{2,\kappa} + (q^{-2} - 1)M_{1,\kappa}M_{2,\kappa} \right).
\]  

(24)

Therefore the parameter \( q \neq 1 \) mixes the two degrees of freedom in a non-trivial way through a quartic term in the Hamiltonian. The thermodynamics of this system will be discussed in section (3.1).

A simple check shows that Equations (8)-(11) and (24) are consistent with the following representation of \( \Psi_i \) operators in terms of fermions operators \( \psi_j \)

\[
\Psi_2 = \psi_2 \\
\overline{\Psi}_2 = \psi_2^\dagger \\
\Psi_1 = \psi_1 \left( 1 + (q^{-1} - 1)M_2 \right) \\
\overline{\Psi}_1 = \psi_1^\dagger \left( 1 + (q^{-1} - 1)M_2 \right),
\]

(25, 26, 27, 28)

and according to Equations (2) and (3) this result easily generalizes for arbitrary \( N \) to

\[
\Psi_m = \psi_m \prod_{l=m+1}^{N} \left( 1 + (q^{-1} - 1)M_l \right),
\]

(29)

and similarly for the adjoint equation.

It is interesting to remark the distinction between \( SU_q(2) \)-fermions with the so called \( q \)-fermions \( b_i \) and \( b_i^\dagger \). The \( q \)-fermionic algebra was introduced in [11]

\[
bb^\dagger + qb^\dagger b = q^{N_q} \\

b^\dagger b = [N_q]
\]

(30, 31)
where the bracket \([x] = \frac{x^q - q^x}{q - q^{-1}}\) and the number operator \(N_q|n\rangle = n|n\rangle\) with \(n = 0, 1\). Since the \(q\)-number \([x] = x\) for \(x = 0, 1\), it is obvious that the grand partition function for \(q\)-fermions is no different than the Fermi grand partition function, and therefore the \(q\)-fermions do not lead to new results as far as thermodynamics is concerned.

### 2.2 \(SU_q(2)\) boson model

In terms of \(SU_q(2)\)-bosons we introduce the following Hamiltonian

\[
\mathcal{H}_B = \sum_{\kappa} \varepsilon_\kappa (\mathcal{N}_{1,\kappa} + \mathcal{N}_{2,\kappa}),
\]

where \([\Phi_{i,\kappa}, \Phi_{\kappa',j}] = 0\) for \(\kappa \neq \kappa'\). The operator \(\mathcal{N}_{i,\kappa} = \overline{\Phi}_{i,\kappa} \Phi_{i,\kappa}\) satisfy the relations

\[
[\mathcal{N}_{2,\kappa}, \Phi_{1}] = 0,
\]

and

\[
\mathcal{N}_{1,\kappa} \Phi_2 - q^{-2} \Phi_2 \mathcal{N}_{1,\kappa} = 0.
\]

The states are built by the action of the \(\Phi\) operators on the vacuum state. For example, for a given \(\kappa\) a normalized state with \(n_1\) particles of species 1 and \(n_2\) particles of species 2 is defined by

\[
\frac{1}{\sqrt{\{n_1\}!\{n_2\}!}} \Phi_{n_2,1} \Phi_{n_1,1} |0\rangle,
\]

where the \(q\)-numbers \(\{n\} = \frac{1-q^{2n}}{1-q}\) and the \(q\)-factorials \(\{n\}!\) are defined \(\{n\}! = \{n\}\{n-1\}\{n-2\}...1\). The grand partition function \(Z_B\) is written

\[
Z_B = Tr e^{-\beta \varepsilon_\kappa (\overline{\Phi}_{1,\kappa} \Phi_{1,\kappa} + \overline{\Phi}_{2,\kappa} \Phi_{2,\kappa})} e^{-\beta \mu (\mathcal{N}_{1,\kappa} + \mathcal{N}_{2,\kappa})},
\]
where $N_{i,\kappa}$ are the ordinary boson number operators $N_{i,\kappa} = \phi_{i,\kappa}^\dagger \phi_{i,\kappa}$ and the trace is taken with respect to the states in Equation (37). For a given $\kappa$ the $SU_q(2)$ bosons are written in terms of boson operators $\phi_{i,\kappa}$ and $\phi_{i,\kappa}^\dagger$ with usual commutations relations $[\phi_i, \phi_j^\dagger] = \delta_{ij}$ as follows

\begin{align}
\Phi_2 &= (\phi_2^\dagger)^{-1}\{N_2\} \\
\overline{\Phi}_2 &= \phi_2^\dagger \\
\Phi_1 &= (\phi_1^\dagger)^{-1}\{N_1\}q^{N_2} \\
\overline{\Phi}_1 &= \phi_1^\dagger q^{N_2}
\end{align}

The grand partition function $Z_B$ then becomes

\[ Z_B = \prod_{\kappa} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-\beta \varepsilon_{\kappa}(n+m)} e^{\beta \mu (n+m)}, \]

with the corresponding interacting Hamiltonian

\[ H_B = \sum_{\kappa} \varepsilon_{\kappa}\{\phi_{1,\kappa}^\dagger \phi_{1,\kappa} + \phi_{2,\kappa}^\dagger \phi_{2,\kappa}\}, \]

with the bracket $\{x\}$ as defined below Equation (37). Therefore, the original Hamiltonian becomes a Hamiltonian in terms of ordinary boson interactions involving powers of the number operators $N_{i,\kappa}$ and $\log q$. Equations (39)-(42) are easily generalized for $N > 2$ to the set of equations

\begin{align}
\overline{\Phi}_m &= \phi_m^\dagger \prod_{l=m+1}^{N} q^{N_l}, \\
\Phi_m &= (\phi_m^\dagger)^{-1}\{N_m\} \prod_{l=m+1}^{N} q^{N_l},
\end{align}

and a $SU_q(N)$-boson state in terms of boson operators reads

\[ \frac{1}{\sqrt{\{n_1\}!\{n_2\}!...\{n_M\}!}} \phi_{M,\kappa M}^{n_M} \phi_{M-1,\kappa M-1}^{n_{M-1}} ... \phi_{1,\kappa 1}^{n_1} |0\rangle. \]
The normalization is consistent with the fact that the dual of the state in Equation (47) is obtained by applying the adjoint operation defined on Φ. The number operator 
\[ N_l = \phi_l^\dagger \phi_l \]
satisfies standard commutation relations with the operators Φₘ
\[ [N_l, \Phi_m] = \Phi_m \delta_l, m \]
and
\[ [N_l, \Phi_m'] = -\Phi_m' \delta_l, m, \]
such that
\[ N_l \Phi_m |0\rangle = m \Phi_m |0\rangle. \]
The difference between the operators Φ and the so called q-bosons is obvious. A set \((a_i, a_i^\dagger)\) of q-bosons satisfies the relation
\[ a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^N, \quad [a_i, a_j^\dagger] = 0 = [a_i, a_j], \]
where \(N|n\rangle = n|n\rangle\). By taking two commuting sets of q-bosons it has been shown that the operators
\[ J_+ = a_2^\dagger a_1, \quad J_- = a_1^\dagger a_2, \quad 2J_3 = N_2 - N_1 \]
generate the quantum Lie algebra \(su_q(2)\)
\[ [J_3, J_\pm] = \pm J_\pm, \quad [J_-, J_-] = [2J_3]. \]
In contrast to the algebraic relations involving the operators Φᵢ and Φⱼ, Equation (51) with \(i, j = 1, 2\) is not covariant under the action of the \(SU_q(2)\) quantum group matrices. The thermodynamics of q-bosons and similar operators called quons has been studied by several authors. In the following section we study the thermodynamics of the two \(SU_q(2)\) models described in this section.
3 Quantum group gases

The high and low temperature behavior of the $SU_q(2)$ fermion model has been studied in References [17, 18], and here we recall some results that will be compared with the $SU_q(2)$ boson case.

3.1 Quantum group fermion gas

The internal energy $U$ for this model is calculated from the grand potential $\Omega = -\frac{1}{\beta} \ln Z_F$ according to the equation

$$U = \left( \frac{\partial \beta \Omega}{\partial \beta} + \mu M \right) = V \int \frac{p^2}{2m} \frac{(2 + (q^{-2} + 1)e^{\beta(\mu - \frac{q^{-2}}{2m})})}{(2\pi\hbar)^3 f(\varepsilon, \mu, q)} d^3p, \quad (54)$$

where the function $f(\varepsilon, \mu, q) = e^{\beta(\varepsilon - \mu)} + 2 + e^{-\beta(q^{-2}\varepsilon - \mu)}$. The low temperature regime of a $SU_q(2)$ fermion gas exhibits the interesting feature that for every value of $q \neq 1$ the entropy lies below the Fermi entropy. For $q > 1$ and $q < 1$ the entropy functions are given respectively by the equations [18]

$$S(q > 1) \approx \lambda k \frac{1.28\sqrt{2\mu_0 k^2 T}}{(q^{-2} + 1)^{3/2}}, \quad (55)$$

and

$$S(q < 1) \approx \lambda k \sqrt{\mu_0 T} \left[ 1.08(q^3 + 1) \frac{(1 - q^3)^2}{2(1 + q^3)^2 \ln^2 3} \right], \quad (56)$$

where $\lambda = \frac{4\pi V(2m)^{3/2}}{(2\pi\hbar)^3}$. The lower bound to the entropy values corresponds to the limit $q \to 0$. Furthermore, systems described by a Hamiltonian with $q > 1$ share the same entropy function with systems with $q < 1$. Comparing Equation (53) with Equation (54) we obtain that two gases share the entropy function if the following relation is satisfied

$$(1 + q^{-2})^{3/2} = \frac{3.62(1 + q^3)}{2.16(1 + q^3)^2 - (1 - q^3)^2 \ln^2 3}, \quad (57)$$
where \( q' > 1 \) and \( q < 1 \). Specifically, the equality is satisfied in the interval \( 0.33 \leq q < 0.91 \).

The high temperature behavior of this model is also interesting. Starting with the grand partition function \( Z_F \)

\[
\ln Z_F = \frac{4\pi V}{h^3} \int_0^\infty p^2 \ln \left( 1 + 2e^{-\beta(\varepsilon-\mu)} + e^{-\beta(\varepsilon(q-2+1)-2\mu)} \right) dp \quad (58)
\]

it was shown in Reference [18] that in \( D = 3 \) spatial dimensions the virial expansion leads to the equation of state

\[
pV = kT \langle M \rangle \left( 1 + \frac{\alpha(q)}{2} \left( \frac{h^2}{2\pi mkT} \right)^{3/2} \frac{\langle M \rangle}{V} + \ldots \right), \quad (59)
\]

where the coefficient \( \alpha(q) = \frac{1}{2^{3/2}} - \frac{1}{2(q^2+1)^{3/2}} \). From equation (59) we see that the sign of the second virial coefficient depends on the value of \( q \), showing that the parameter \( q \) interpolates between attractive and repulsive behavior. The function \( \alpha(q) \) takes values in the interval \( 2^{-5/2} \geq \frac{\alpha(q)}{2} \geq -2^{-5/2}(\sqrt{2} - 1) \) as \( q \) varies from zero to \( \infty \) and vanishes at \( q = 1.96 \). Figure 1 shows a graph of the function \( B(q,T) = \alpha(q) \beta^{3/2} \) for large values of the temperature and \( q = 10, 1.96, 1, 0.3 \).

It is important to remark that the free boson limit \( B_b(T) = -2^{-7/2}\beta^{3/2} < B(\infty, T) = -2^{-5/2}(\sqrt{2} - 1)\beta^{3/2} \), and therefore free bosons are not described in this model. A natural question to address is whether a similar interpolation occurs at \( D = 2 \). The same procedure leads to the equation of state

\[
pA = kT \langle M \rangle \left( 1 + \frac{1}{1 + q^2} \frac{h^2}{8\pi mkT} \frac{\langle M \rangle}{A} + \ldots \right), \quad (60)
\]

wherein the second virial coefficient is positive for all values of \( q \), showing that this model, at \( D = 2 \), describes only repulsive systems.
3.2 Quantum group boson gas

The grand partition function $Z_B$ in Equation (43) can be simply rewritten as

$$Z_B = \prod_\kappa \sum_{m=0}^{\infty} (m + 1)e^{-\beta \varepsilon_\kappa m} z^m,$$  \hspace{1cm} (61)

where $z = e^{\beta \mu}$ is the fugacity. In $D = 3$ the first few terms in powers of $z$ read

$$\ln Z_B = \frac{4 \pi V}{h^3} \int_0^{\infty} dp p^2 \left(2 e^{-\beta \varepsilon_\kappa} z + (6 e^{-\beta \varepsilon_\kappa} - 4 e^{-\beta \varepsilon_\kappa^2}) \frac{z^2}{2} + (24 e^{-\beta \varepsilon_\kappa^3} - 36 e^{-\beta \varepsilon_\kappa^2} e^{-\beta \varepsilon_\kappa} + 16 e^{-\beta \varepsilon_\kappa^3}) \frac{z^3}{3!} + \ldots \right),$$  \hspace{1cm} (62)

such that performing the elementary integrations gives

$$\ln Z_B = \frac{4 \pi V}{h^3} \left( \frac{\sqrt{\pi}}{2} \left( \frac{2m}{\beta} \right)^{3/2} z + \sqrt{\pi} \left( \frac{2m}{\beta} \right)^{3/2} \delta(q) z^2 + \ldots \right),$$  \hspace{1cm} (63)

where $\delta(q) = \frac{1}{4} \left( \frac{3}{(1+q^2)^{3/2}} - \frac{1}{\sqrt{2}} \right)$.

Calculating the average number of particles $\langle N \rangle = \frac{1}{\beta} \left( \frac{\partial \ln Z_B}{\partial \mu} \right)_{T,V}$ and reverting the equation we find for the fugacity

$$z \approx \frac{1}{2} \left( \frac{h^2}{2m \pi kT} \right)^{3/2} \frac{\langle N \rangle}{V} - \delta(q) \left( \frac{h^2}{2m \pi kT} \right)^3 \left( \frac{\langle N \rangle}{V} \right)^2,$$  \hspace{1cm} (64)

leading to the following equation of state

$$pV = kT \langle N \rangle \left( 1 - \delta(q) \left( \frac{h^2}{2m \pi kT} \right)^{3/2} \left( \frac{\langle N \rangle}{V} \right) + \ldots \right).$$  \hspace{1cm} (65)

As expected, at $q = 1$ the coefficient $\delta(1) = 2^{-7/2}$, which is the numerical factor in the second virial coefficient for a free boson gas with two species. The free fermion $\delta(q) = -2^{7/2}$ and ideal gas $\delta(q) = 0$ cases are reached at $q \approx 1.78$ and $q \approx 1.27$ respectively.
A very similar calculation for $D = 2$ gives the equation of state

$$pA = kT \langle N \rangle \left(1 - \eta(q) \frac{h^2}{2\pi mkT} \frac{\langle N \rangle}{A} + \ldots\right),$$

(66)

with $\eta(q) = \frac{(2-q^2)}{4(1+q^2)}$. At $D = 2$ this model behaves as a fermion gas for $q = \sqrt{5}$. Figure 2 shows a graph of the coefficient $\eta(q)$ as a function of the parameter $q$ for $D = 2$.

Since the $SU_q(2)$ boson gas at $D = 2$ also interpolates completely between bosons and fermions, we can find a relation between the parameter $q$ and the statistical parameter $\alpha$ for an anyon gas [19, 20] of two species. This relation is given by

$$\alpha = 1 - \sqrt{\frac{5 - q^2}{2(1 + q^2)}},$$

(67)

where $0 \leq \alpha \leq 1$, with the boson and fermion limits $\alpha = 0$ ($q = 1$) and $\alpha = 1$ ($q = \sqrt{5}$) respectively. The second virial coefficient in Equation (66) takes values in the interval $[-\frac{\lambda_T^2}{2}, \frac{\lambda_T^2}{4}]$, with $\lambda_T = \frac{\hbar^2}{2\pi mkT}$, and therefore the parameter $q$ interpolates within a larger range of systems than the $\alpha$ parameter does.

4 Conclusions

In this article we have studied the high temperature behavior of quantum group gases. Our approach is mainly based on promoting the $SU(N)$ covariant fermion and boson algebras to the corresponding algebraic relations covariant under $SU_q(N)$ transformations. For purposes of simplicity we have considered the $N = 2$ case. Starting with the simplest Hamiltonian we have calculated the partition function and obtained the equation of state for the two $SU_q(2)$ gases. Certainly, for $q = 1$ our results become those for two
species of free fermion or boson gases. For $q \neq 1$ this degeneracy is broken and the corresponding Hamiltonian written in terms of standard operators acquires an interaction term. Our results indicate that the $q$ parameter interpolates between repulsive and attractive behavior. In particular, for a $SU_q(2)$-fermion gas and $D=3$ the sign of the second virial coefficient depends on the value of $q$. The ideal gas case corresponds to $q = 1.96$ and the system becomes repulsive for $q < 1.96$. For $q > 1.96$ the system becomes attractive, but as $q \to \infty$ the free boson limit is not reached, and therefore this model does not interpolate completely between the free fermion and free boson cases. For $D = 2$ the second virial coefficient of this gas is positive for every value of $q$ and vanishes in the $q \to \infty$ limit. For $SU_q(2)$-bosons the results are more interesting. For $D = 2$ and $D = 3$ the parameter $q$ interpolates completely between a wide range of attractive and repulsive systems including the free fermion and boson cases. For $D = 2$ we have found a relation between $q$ and the statistical parameter $\alpha$ for an anyon gas. Therefore, the simple models studied here, and in particular the $SU_q(2)$-boson model, offer an alternative approach in describing systems obeying fractional statistics in two and three spatial dimensions.

References

[1] See for example:M.Jimbo ed., *Yang-Baxter equation in integrable systems*, Advanced series in Mathematical Physics V.10 (World Scientific,1990).

[2] V. Chari and A. Pressley, *A Guide to Quantum Groups*, (Cambridge Univ. Press, 1994).
[3] S. L. Woronowicz, Publ. RIMS 23, 117 (1987). Yu I. Manin, Comm. Math. Phys. 123, 163 (1989).

[4] J. Wess and B. Zumino, Nucl. Phys. B (Proc. Suppl.) 18, 302 (1990).

[5] U. Carow-Watamura, M. Schlieker and S. Watamura, Z. Phys. C 49, 439 (1991). M. R. Ubriaco, Mod. Phys. Lett. A 8, 89 (1993). S. Shabanov, J. Phys. A 26, 2583 (1993). A. Lorek and J. Wess, Z. Phys. C 67, 671 (1995).

[6] I. Aref’eva and I. Volovich, Phys. Lett B264, 62 (1991). A. Kempf, J. Math. Phys. 35, 4483 (1994). T. Brzezinski and S. Majid, Phys. Lett. B298, 339 (1993). L. Castellani, Mod. Phys. Lett. A 9, 2835 (1994). M. R. Ubriaco, Mod. Phys. Lett. A 9, 1121 (1994). A. Sudbery, SU_q(n) Gauge Theory, hep-th/9601033.

[7] S. Iwao, Prog. Theor. Phys. B83, 363 (1990). D. Bonatsos, E. Argyres and P. Raychev, J. Phys. A 24, L403 (1991). R. Capps, Prog. Theor. Phys. 91, 835 (1994).

[8] M. R. Ubriaco, Mod. Phys. Lett. A 8, 2213 (1993); erratum A 10, 2223 (1995).

[9] L. A. Takhatajan, Advanced Studies in Pure Mathematics 19 (1989) and references therein.

[10] S. Vokos, B. Zumino and J. Wess, Properties of Quantum 2x2 Matrices, in: Symmetry in Nature, Quaderni della Scuola.

[11] Y.J. Ng, J. Phys. A 23, 1203 (1990).
[12] O. W. Greenberg, Phys. Rev. Lett. 64, 705 (1990); Phys. Lett. A 209, 137 (1995).

[13] A. J. Macfarlane, J. Phys. A 22, 4581 (1989).

[14] L. C. Biedenharn, J. Phys. A 22, L873 (1989).

[15] M. Martín-Delgado, J. Phys. A 24, L1285 (1991). Gang Su and Mo-lin Ge, Phys. Lett. A 173, 17 (1993). J. Tuszynski, J. Rubin, J. Meyer and M. Kibler, Phys. Lett. A 175, 173 (1993). I. Lutzenko and A Zhedanov, Phys. Rev. E 50, 97 (1994). M. Salerno, Phys. Rev. E 50, 4528 (1994). P. Angelopoulou, S. Baskoutas, L. de Falco, A. Jannussis, R. Mignani and A. Sotiropoulou, J. Phys. A 27, L605 (1994). S. Vokos and C. Zachos, Mod. Phys. Lett. A 9, 1 (1994). J. Goodison and D. Toms, Phys. Lett. A 195, 38 (1994). 198, 471 (1995). M. R-Monteiro, I. Roditi and L. Rodrigues, Mod. Phys. Lett. B 9, 607 (1995).

[16] D. Fivel. Phys. Rev. Lett. 65, 3361 (1990). R. Campos, Phys. Lett. A 184, 173 (1994). S. Dalton and A. Inomata, Phys. Lett. A 199, 315 (1995). A. Inomata, Phys. Rev. A 52, 932 (1995).

[17] M. R. Ubriaco, Phys. Lett. A 219, 205 (1996).

[18] M. R. Ubriaco, hep-th/9607173, Mod. Phys. Lett. A (in press).

[19] F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982); 49, 957 (1982).

[20] D. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. B251, 117 (1985). D. Arovas, in Geometric Phases in Physics, A. Shapere and F. Wilczek eds. (World Scientific, 1989).
FIG.1: The function $B(q, T)$ as defined in the text in the interval $0 \leq \beta \leq 5/eV$ and four values of $q$.

FIG.2: The coefficient $\eta(q)$ for the interval $0 \leq q \leq 5$. At the values $q = 1$ and $q = 5^{1/2}$ the system behaves as a free boson and fermion gas respectively. The second virial coefficient vanishes at $q = 2^{1/2}$. 