On measurability of Banach indicatrix

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Abstract

We prove measurability of the multiplicity function for a measurable mapping of metric measure spaces.

1. Introduction. Given two metric measure spaces \( X, Y \). Let \( f : X \to Y \) be a measurable mapping and \( A \subset X \). The Banach indicatrix (multiplicity function) is defined as

\[
N(y, f, A) = \#\{ x \in A \mid f(x) = y \},
\]

i.e. the number of elements of \( f^{-1}(y) \) in \( A \) (possible \( \infty \)). In case \( A = X \) note \( N(y, f, X) = N(y, f) \). The question under our consideration is following: is the function \( N(y, f, A) \) measurable?

Let us briefly discuss some results and examples. The measurability of the multiplicity function for a continuous function \( f : [a, b] \to \mathbb{R} \) was proved by Banach in \([B, \text{Théorème 1.1}]\). Whereas \([B, \text{Théorème 1.2}]\) states that \( \int_a^b N(y, f) \, dy \) is equal to the total variation \( TV(f, [a, b]) \). Together Théorèmes 1.1 and 1.2 are named the Banach indicatrix theorem (see \([N, \text{p. 225–227}], [L, \text{p. 66–72}], [BC, 177-178]\)). There are further generalizations of this result, see for example \([TS, WS, RL]\) and the bibliography therein.

The Banach indicatrix play a role in the change of variables formula

\[
\int_A (u \circ f)|J(x, f)| \, dx = \int_{\mathbb{R}^n} u(y)N(y, f, A) \, dy.
\]

In \([H]\) the formula was obtained under minimal assumptions: the a.e. existence of approximative partial derivatives. In particular, the measurability of \( N(y, f, A) \) was proved.
In [RR] IV.1.2 the multiplicity function of a continuous transform was studied in detail. See also [GR] p. 272 for further investigation. The treatment in the setting of metric spaces is given in [F] 2.10.10–15.

This note aims to show the measurability of the Banach indicatrix for a measurable mapping (Theorem 2.1). The proof of Lemma 3.1 is based upon ideas of the original proof of [B] Théorème 1. While Lemma 3.2 is from author’s joint work with Professor S. K. Vodopyanov.

2. Assumptions and result. Let \((X, d_X, \mu_X)\) is a complete, separable metric space with a measure. Additionally \(X\) is supposed to be geometrically doubling: there is a constant \(\lambda \in \mathbb{N}\) such that every ball \(B(x, r) = \{z \in X \, | \, d_X(x, z) < r\}\) can be covered by at most \(\lambda\) balls \(B(x, r/2)\) of half radius. Measure \(\mu_X\) is a Borel regular measure such that each ball has finite measure. Assume \((Y, d_Y, \mu_Y)\) is a separable metric measurable space.

The mapping \(f : X \to Y\) is a \(\mu_X\)-measurable if and only if \(f\) is defined \(\mu_X\)-almost everywhere on \(X\) and \(f^{-1}(E)\) is \(\mu_X\)-measurable whenever \(E\) is open subset of \(Y\) [F, 2.3.2].

**Theorem 2.1.** Let \(f : X \to Y\) be a \(\mu_X\)-measurable mapping, and \(A \subset X\) be a Borel set. Then \(f\) can be redefined on a set of \(\mu_X\)-measure zero in such a way that the Banach indicatrix \(N(y, f, A)\) is a \(\mu_Y\)-measurable function.

**Example 2.2.** Let \(C \subset \mathbb{R}\) denotes the Cantor set and \(V \subset \mathbb{R}\) denotes the Vitaly non-measurable set. There is a bijection \(f : C \to V\). Define the function
\[
\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in C, \\ 0, & \text{if } x \notin C, \end{cases}
\]
which is measurable. But at the same time the multiplicity function \(N(y, \tilde{f}, A)\) can not be measurable as it coincides with characteristic function of the non-measurable set \(V\) on \(\mathbb{R} \setminus \{0\}\).

**Dyadic system.** We involve a system of dyadic cubes. Namely a family
\[
\{Q^k_\alpha \mid k \in \mathbb{Z}, \alpha \in \mathcal{A}_k \subset \mathbb{N}\}
\]
of Borel sets with parameters \(\delta \in (0, 1), 0 < c \leq C < \infty\) and centres \(\{x^k_\alpha\}\), meeting the following properties:

1) If \(l \geq k\) then either \(Q^l_\beta \subset Q^k_\alpha\) or \(Q^l_\beta \cap Q^k_\alpha = \emptyset\);
2) For each \(k \in \mathbb{Z}\) \(X = \bigcup_{\alpha \in \mathcal{A}_k} Q^k_\alpha\) is a disjoint union;
3) \( B(x^k_\alpha, c\delta^k) \subset Q^k_\alpha \subset B(x^k_\alpha, C\delta^k) \);  
4) If \( l \geq k \) and \( Q^l_\beta \subset Q^k_\alpha \) then \( B(x^l_\beta, C\delta^l) \subset B(x^k_\alpha, C\delta^k) \).

This specific dyadic system in doubling quasi-metric spaces was constructed in [HK] and generalize the dyadic cubes in the Euclidean space.

3. Measurability establishing. Before proceeding with the Theorem 2.1 we need following tow lemmas.

**Lemma 3.1.** Let \( A \subset X \) is a Borel set and \( f : X \to Y \) is a \( \mu_X \)-measurable mapping possessing the following property: \( f(B) \) is \( \mu_Y \)-measurable whenever \( B \subset A \) is a Borel set. Then \( N(y, f, A) \) is a \( \mu_Y \)-measurable function.

**Proof.** Take a system \( \{Q^k_\alpha\} \) of dyadic cubes on \( X \), and define a family of functions

\[
L^k_\alpha(y) = \chi_{f(Q^k_\alpha \cap A)}(y).
\]

Functions \( L^k_\alpha(y) \) are non-negative and \( \mu_Y \)-measurable (as characteristic functions of \( \mu_Y \)-measurable sets \( f(Q^k_\alpha \cap A) \)). Therefore the sum

\[
N_k(y) = \sum_{\alpha \in A_k} L^k_\alpha(y)
\]

is also measurable. Thus the sequence of measurable functions \( \{N_k(y)\} \) is non-decreasing and the pointwise limit

\[
N^*(y) = \lim_{k \to \infty} N_k(y)
\]
exists and is a \( \mu_Y \)-measurable function.

Note that \( N_k(y) \) counts on how many of the sets \( Q^k_\alpha \cap A \) the function \( f \) attains the value \( y \) at least once. So for each \( k \) \( N(y, f, A) \geq N_k(y) \) and

\[
N(y, f, A) \geq N^*(y).
\]

Prove the reverse inequality. Let \( q \) be an integer such that \( N(y, f, A) \geq q \). Then there exist \( q \) different points \( x_1, \ldots, x_q \subset A \) such that \( f(x_j) = y \). If \( k \) is large enough so that points \( x_1, \ldots, x_q \) are in separated cubes \( \{Q^k_{\alpha_j}\}, j = 1, \ldots, q \), then \( N_k(y) \geq q \). This shows \( N^*(y) \geq N(y, f, A) \) and

\[
N^*(y) = N(y, f, A).
\]

**Lemma 3.2.** Let \( f : X \to Y \) be a \( \mu_X \)-measurable mapping. Then there is an increasing sequence of closed sets \( \{T_k\} \subset X \) such that \( f \) is continuous on every \( T_k \) and \( \mu_X \left( X \setminus \bigcup_k T_k \right) = 0 \).
Proof. Let \( \{Q_\alpha\} \) be a collection of dyadic cubes of one generation and
\[
X = \bigcup_{\alpha=1}^{\infty} Q_\alpha \quad \text{disjoint union.}
\]

By Luzin’s theorem \([F, 2.3.5]\) there is a closed set \( C_1^\alpha \subset Q_\alpha \) such that \( f \) is continuous on \( C_1^\alpha \) and \( \mu_X(Q_\alpha \setminus C_1^\alpha) < 1 \). Similarly \( f \) continuous on \( C_2^\alpha \subset Q_\alpha \setminus C_1^\alpha \) and \( \mu_X((Q_\alpha \setminus C_1^\alpha) \setminus C_2^\alpha) < \frac{1}{2} \) and so on. This yields a sequence \( \{C_2^\alpha\} \) of closed sets.

Put
\[
P_k^\alpha = \bigcup_{i=1}^{k} C_i^\alpha,
\]
then \( P_k^\alpha \subset P_{k+1}^\alpha \) and the mapping \( f \) is continuous on each \( P_k^\alpha \). Furthermore \( \mu_X(Q_\alpha \setminus P_k^\alpha) < 1/k \) and hence \( \mu_X(Q_\alpha \setminus \bigcup_k P_k^\alpha) = 0 \).

Now defining
\[
T_j = \bigcup_{\alpha=1}^{j} P_j^\alpha,
\]
we get an increasing sequence of closed sets. In particular, \( \mu_X(Q_\alpha \setminus \bigcup_j T_j) = 0 \) since \( \bigcup_j P_j^\alpha \subset \bigcup_j T_j \). Then
\[
X \setminus \bigcup_{j=1}^{\infty} T_j = \bigcup_{\alpha=1}^{\infty} (Q_\alpha \setminus \bigcup_{j=1}^{\infty} T_j).
\]
Consequently the set \( X \setminus \bigcup_j T_j \) is of \( \mu_X \)-measure zero as a countable union of negligible sets.

Proof of Theorem 2.1. Let \( \{T_k\} \) be a sequence of closed sets from Lemma 3.2. Observe that an image of each Borel set \( B \subset T_k \) is \( \mu_Y \)-measurable since \( f \) is continuous on \( T_k \) \([F, 2.2.13]\). This puts us in a position to apply Lemma 3.1 to deduce that \( N(y, f, A \cap T_k) \) is a \( \mu_Y \)-measurable function. The sequence \( N(y, f, A \cap T_k) \) is non-decreasing and hence
\[
N\left(y, f, A \cap A \cap \bigcup_k T_k\right) = \lim_{k \to \infty} N(y, f, A \cap T_k)
\]
is a \( \mu_Y \)-measurable function.

Take a point \( y_0 \in Y \) and redefine \( f(x) = y_0 \) for \( x \in X \setminus \bigcup_k T_k \).

\( \square \)
Remark 3.3. Note that Theorem 2.1 requires that the set $A$ be a Borel set. On the other hand one can prove an analogous assertion for measurable set $A$ however assuming that mapping $f$ satisfies the Luzin $N$-property (because in this case the continuous image of every measurable set is measurable and Lemma 3.1 is applicable).

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