Proximal bundle algorithms for nonsmooth convex optimization via fast gradient smooth methods

Adam Ouorou

the date of receipt and acceptance should be inserted later

Abstract We propose new proximal bundle algorithms for minimizing a nonsmooth convex function. These algorithms are derived from the application of Nesterov fast gradient methods for smooth convex minimization to the so-called Moreau-Yosida regularization $F_\mu$ of $f$ w.r.t. some $\mu > 0$. Since the exact values and gradients of $F_\mu$ are difficult to evaluate, we use approximate proximal points thanks to a bundle strategy to get implementable algorithms. One of these algorithms appears as an implementable version of a special case of inertial proximal algorithm. We give their complexity estimates in terms of the original function values, and report some preliminary numerical results.

Mathematics Subject Classification (2000) 20E28, 20G40, 20C20

Keywords Fast gradient methods, proximal bundle methods, inertial proximal methods, nonsmooth convex optimization.

1 Introduction

We consider the problem

$$\min_{x \in \mathbb{R}^n} f(x),$$

(1.1)

where $f$ is a convex (non necessarily differentiable) function. We assume that the set $X^*$ of minimizers of $f$ is nonempty. It is well known that this problem can be transformed into a differentiable convex minimization problem

$$\min_{x \in \mathbb{R}^n} F_\mu(x),$$

(1.2)

where $\mu > 0$, $\|\cdot\|$ is the usual Euclidean norm, and $F_\mu$ is the Moreau-Yosida regularization of $f$ defined by

$$F_\mu(x) = \min_{z \in \mathbb{R}^n} \left\{ f(z) + \frac{\mu}{2} \|z - x\|^2 \right\}.$$ 

(1.3)
The parameter $\mu$ is usually termed as the proximity parameter. The function $F_\mu$ is a differentiable convex function defined on the whole space $\mathbb{R}^n$ and has $\mu$-Lipschitzian gradient without any further assumption \[21\]. i.e., $|\nabla F_\mu(x) - \nabla F_\mu(y)| \leq \mu|x - y|$, $x, y \in \mathbb{R}^n$. The unique minimizer in (1.3) is called the proximal point of $x$ and we denote it by $p_\mu(x)$, i.e.

$$p_\mu(x) = \arg\min_{z \in \mathbb{R}^n} \left\{ f(z) + \frac{\mu}{2} \|z - x\|^2 \right\}. \quad (1.4)$$

The derivative of $F_\mu$ is given by

$$\nabla F_\mu(x) = \mu(x - p_\mu(x)), \quad (1.5)$$

and $\nabla F_\mu(x) \in \partial f(p_\mu(x))$ where $\partial f$ is the subdifferential of $f$. Minimizing $f$ and $F_\mu$ are equivalent problems, in the sense that the minima of the two functions coincide, see \[21\], Theorem XV.4.1.7. Assuming a fast computation of its gradient (in fact $p_\mu(x)$), an efficient smooth minimization algorithm applied to $F_\mu(x)$ is appealing (attractive). This explains the motivation of developing quasi-Newton type algorithms for the minimization of $F_\mu$, see for instance \[7,20,29,10\]. The proximal point algorithm \[34\] for solving (1.1) is as follows.

**Algorithm 1.1 Proximal Point Algorithm (PPA)**

0. Choose $x^0 \in \mathbb{R}^n$ and set $k = 0$.
1. Compute $p_\mu(x^k)$.
2. If $p_\mu(x^k) = x^k$ stop: $x^k$ solves (1.1).
3. $x^{k+1} = p_\mu(x^k)$. Increase $k$ by 1 and loop to Step 1.

As already observed in the literature, the proximal point algorithm can be regarded as a standard gradient algorithm applied to the minimization of $F_\mu$. The classical gradient descent (CGDA) is one of the simplest method for smooth convex minimization. It writes $x^{k+1} = x^k - \alpha_k \nabla F_\mu(x^k)$ for (1.2) where $\alpha_k$ is a stepsize, and stops when $\nabla F_\mu(x^k) = 0$.

**Algorithm 1.2 Classical Gradient Descent Algorithm (CGDA)**

0. Choose $x^0 \in \mathbb{R}^n$ and set $k = 0$.
1. Compute $\nabla F_\mu(x^k)$.
2. If $\nabla F_\mu(x^k) = 0$ stop: $x^k$ solves (1.1).
3. $x^{k+1} = x^k - \alpha_k \nabla F_\mu(x^k)$. Increase $k$ by 1 and loop to Step 1.

There are different strategies of choosing the stepsize $\alpha_k$, leading to various versions of CGDA. Since $\nabla F_\mu(x^k) = \mu(x^k - p_\mu(x^k))$, the stopping criterion in this algorithm is exactly the same as in PPA. By setting $\alpha_k = \mu^{-1}$, $k \geq 0$, CGDA reduces to PPA. The differentiability of $F_\mu$ motivates us to investigate alternatives to classical gradient methods which are simple but not optimal \[31\]. In this paper, we consider fast gradient methods initiated by Nesterov in \[30,31\], which attain the optimal oracle complexity for smooth convex optimization. Their remarkable feature is that, as in a classical gradient method, they do not need more than one gradient evaluation at each iteration. The development of fast first-order methods for smooth problems is an active area of research \[9,18,22,35\], motivated by the need to solve large scale problems unsuited to second-order methods (so is Problem (1.2) as $F_\mu$ is not twice differentiable in general \[27\]). The idea of exploiting these fast methods for the optimization of nonsmooth convex functions is not new. There
is an increasing interest in the context of computing the zeros of the sum of a maximally monotone operator, resulting in the class of so-called inertial proximal algorithms, see for instance [14, 15] and references therein. In [20], Göüler extended the concept of estimate sequences (see [31], Definition 2.2.1) to the nonsmooth function \( f \) from which a main algorithm is established with a convergence rate estimate \( O(1/k^2) \). This algorithm is conceptual in the sense that it makes use of the exact solutions of the same type of problems as (1.4) for some \( x = x^k \). We already pointed out the difficulty to solve these problems in practice. A variant in which approximate proximal points can be used has been proposed by Göüler, according to the following criterion proposed by Rockafellar in [34] to compute an approximate proximal point \( z^{k+1} \) for a given point \( x^k \):

\[
\min \left\{ \| g \| : \ g \in \partial \phi_k(z^{k+1}) \right\} \leq \frac{\varepsilon_k}{\lambda_k},
\]

where \( \varepsilon_k = O(k^{-\sigma}) \) for some \( \sigma > 0 \) and \( \phi_k(z) = f(z) + \frac{1}{2\lambda_k} \| z - x^k \|^2 \). \( \lambda_k > 0 \). The l.h.s in (1.6) expresses the distance of 0 to the set \( \partial \phi_k(z^{k+1}) \). The criterion (1.6) is not always easy to check in practice since the l.h.s problem is not tractable most of the time. No numerical experiment has been conducted in [20] to have an idea on the practical efficiency of the proposed algorithms. Our approach is closed in spirit to that of Göüler in [20] but our purpose is to propose implementable algorithms for a wide range of problems of type (1.4), that make use of approximate function and gradient values of \( F_\mu \). To this aim, we simply use a bundle strategy to perform these approximate computations.

We use some standard notations throughout the paper. The symbol \( \langle \cdot, \cdot \rangle \) denotes the usual scalar product while the Euclidean norm is denoted by \( \| \cdot \| \). For any \( \epsilon \geq 0 \), the \( \epsilon \)-subdifferential of \( f \) at \( z \) is \( \partial \epsilon f(z) = \{ g : f(x) \geq f(z) + \langle g, x-z \rangle - \epsilon \forall x \in \mathbb{R}^n \} \). This set is identical with the subdifferential of \( f \) at \( z \) when \( \epsilon = 0 \).

The paper is organized as follows. In the next section, we present formally a new class of proximal bundle algorithms. In Section 3 we give their complexity estimates and analyze in Section 4 the accumulation of errors due to the inexact computation of gradients of \( F_\mu \) and propose practical tolerances for these computations. In Section 5 we report some preliminary computational results obtained with the proposed algorithms and conclude in Section 6.

2 New algorithms

We recall that \( F_\mu \) is \( \mu \)-smooth i.e. \( \nabla F_\mu \) is Lipschitz continuous with constant \( \mu \). We also have by definition of \( F_\mu \), \( F_\mu(x) \leq f(x) \) for any \( x \in \mathbb{R}^n \). We will make use of the following properties relating the two problems (1.1) and (1.2), see for instance [21], Theorem XV.4.1.7.

**Proposition 2.1** The following statements are equivalent: a) \( x \) minimizes \( f \), b) \( x = p_\mu(x) \), c) \( \nabla F_\mu(x) = 0 \), d) \( x \) minimizes \( F_\mu \), e) \( f(x) = f(p_\mu(x)) \), f) \( f(x) = F_\mu(x) \).

The Moreau-Yosida regularization \( F_\mu \) provides a smooth lower approximation of \( f \) which coincides with \( f \) at optimality. One can then apply a fast gradient method to \( F_\mu \) in order to get a minimizer of \( f \). Based on the above proposition, a gap function may be defined as \( \delta(x) = f(x) - F_\mu(x) \) \( \geq 0 \), which gives \( \delta(x) = 0 \) iff \( x \) is optimal for (1.1).
The fast gradient method developed in [30] for smooth convex functions, uses the sequence of reals
\[ \lambda_0 = 1, \quad \lambda_{k+1} = \frac{1 + \sqrt{1 + 4 \lambda_k^2}}{2}, \quad k \geq 0, \]
which satisfies the following useful relations
\[ \lambda_{k-1}^2 = \lambda_k (\lambda_k - 1), \quad k \geq 1, \tag{2.1} \]
and
\[ \lambda_k^2 = \sum_{i=0}^{k} \lambda_i, \quad \lambda_k \geq \frac{k + 2}{2}, \quad k \geq 0. \tag{2.2} \]

Starting from an arbitrary initial point \( x^0 \), the fast gradient algorithm generates a sequence \( \{y^k\} \) of approximate solutions with \( y^0 = x^0 \), and a sequence \( \{x^k\} \) of search points according to the following rule
(when applied to \( F_\mu \)),
\[ y^{k+1} = x^k - \frac{1}{\mu} \nabla F_\mu(x^k) = p_\mu(x^k), \quad x^{k+1} = y^{k+1} + \alpha_k (y^{k+1} - y^k), \quad \alpha_k = \lambda_{k+1}^{-1} (\lambda_k - 1). \tag{2.3} \]

The above scheme is in fact a special result of Nesterov’s key idea of forming estimating sequences to devise optimal first-order methods for smooth optimization [31]. The improvement over the gradient descent relies on the introduction of the momentum term \( y^{k+1} - y^k \) as well as the particular coefficient from the sequence \( \{\lambda_k\} \). By considering one smooth optimization problem (1.2) on which Nesterov’s scheme is applied, our goal is not to have to tune the proximity parameter \( \mu \), looking for acceleration through the momentum term only. In terms of the minimization problem (1.1), it certainly makes sense to consider a varying parameter \( \mu \), say \( y^{k+1} = p_\mu(x^k) \). By doing so, the resulting scheme writes
\[ y^{k+1} = \arg \min_{x \in \mathbb{R}^n} \left\{ f(x) + \frac{\mu_k}{2} \| x - x^k \|^2 \right\}, \quad x^{k+1} = y^{k+1} + \alpha_k (y^{k+1} - y^k), \tag{2.4} \]
and can be cast into the recent class of so-called inertial proximal methods, the origins of which go back to [1]. There is a rich literature devoted to this class of methods, see for instance the recent papers [4,5]. They stem from the use of an implicit discretization of a differential system of second-order in time and give interesting insight into Nesterov’s scheme [38]. Applied to the original problem (1.1), an iteration of the inertial proximal algorithm with parameters \( \alpha_k > 0 \) and \( \tau_k = \mu_k^{-1} > 0 \) is given (with our notations) by (2.3) but with a more general nonnegative sequence \( \{\alpha_k\} \) that capture the inertial effect of the differential system. The sequence \( \{\tau_k\} \) is interpreted as a sequence of proximal parameters taking into account the temporal scale effects of the system. In fact, the main proximal algorithm proposed by Güler for (1.1) can also be written as an inertial proximal algorithm with some appropriate parameters, see [4,5]. Although taking inspiration from Nesterov’s

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1 The gradient method in [30] computes a steplength \( \alpha_k \) which can be taken as the inverse of the Lipschitz constant \( L \) for the gradient of the objective when it is available. Since \( F_\mu \) is \( \mu \)-smooth, the Lipschitz constant is \( \mu \). We avoid also the evaluations of \( F_\mu \)-values which would be necessary if a steplength has to be computed.
method in [20], the second algorithm proposed by Güler for (1.1) in Section 6 of
[20] is different from the scheme (2.3), in that the update of $x^{k+1}$ involves $x^k$ as follows,

$$x^{k+1} = y^{k+1} + \lambda_k^{-1} (y^{k+1} - y^k) + \frac{\lambda_k}{\lambda_{k+1}} (y^{k+1} - x^k).$$

(2.5)

This is the same rule for the update of the sequence $\{x^k\}$ proposed in the recent work by Kim and Fessler for their proposed optimized gradient method OGM1, see [22], page 99. The authors seem not to know the work of Güler [20], it is not referenced in their paper. Güler proposed this rule intuitively with no explanation, while in [22], it is shown that it corresponds to an optimal choice of parameters obtained through a relaxed performance estimation problem introduced by Drori and Teboulle in [13] to optimize first-order algorithms. We will consider the update (2.6) as well for a second algorithm through the following general rule,

$$x^{k+1} = y^{k+1} + \alpha_k (y^{k+1} - y^k) + \beta_k (y^{k+1} - x^k),$$

(2.6)

where $\alpha_k = \lambda_k^{-1} (\lambda_k - 1)$ is Nesterov’s extrapolation coefficient given in (2.3) and $\{\beta_k\}_{k \geq 0}$ is one of the two sequences:

- $\beta_k = 0$, $k \geq 0$ (then (2.6) reduces to the update of $x^{k+1}$ in (2.3)) or,
- $\beta_k = \lambda_k \lambda_k^{-1}$, $k \geq 0$ (to get (2.5)).

We are now ready to propose a conceptual fast algorithm for the minimization of the smooth function $F_\mu$ and consequently for solving (1.1).

**Algorithm 2.1 Fast Proximal Point Algorithm (FPPA)**

0. Choose $x^0 = y^0 \in \mathbb{R}^n$ and the sequence $\{\beta_k\}_{k \geq 0}$. Set $k = 0$.
1. Set $y^{k+1} = p_\mu(x^k)$.
2. If $x^{k+1} = x^k$ stop: $x^k$ solves (1.1).
3. Set $x^{k+1} = y^{k+1} + \alpha_k (y^{k+1} - y^k) + \beta_k (y^{k+1} - x^k)$.
4. Increase $k$ by 1 and loop to Step 1.

We refer to the two algorithms depending on the choice of $\beta_k$ respectively by FPPA1 and FPPA2. The main difference of FPPA with PPA is that the proximal point is not computed at the previous iterate but rather at a specific linear combination of the two previous proximal points for FPPA1, and a second momentum term $y^{k+1} - x^k$ with the coefficient $\lambda_k \lambda_k^{-1}$ for FPPA2.

2.2 An inexact first-order oracle for $F_\mu$

Now, FPPA is not implementable as is since obtaining the exact proximal point $p_\mu(x)$ for any given $x \in \mathbb{R}^n$ is as difficult as solving the original problem (1.1). Hopefully, bundle methods offer a practical mean to compute a proximal point approximately as follows, see [21], Section XV.4.3 and the recent survey [16] on bundle methods. Assume a first-order (exact) oracle for $f$ is available, that given, $z \in \mathbb{R}^n$ computes $f(z)$ and a subgradient $g(z) \in \partial f(z)$. At a given step $j$, after a given number of calls to the oracle at different points $z^i$, $i = 1, \ldots$, with $g^i \in \partial f(z^i)$, we can form the so-called bundle $B_j = \{ (z^i, f(z^i), g^i) \}$ and built the following approximation function of $f$ defined by,

$$\hat{f}_{B_j}(x) = \max \{ f(z^i) + \langle g^i, x - z^i \rangle : (z^i, f(z^i), g^i) \in B_j \}. $$
This lower approximation \( \tilde{f}_{B_j}(z) \leq f \) replaces \( f \) in (1.9) to yield the following quadratic problem

\[
F_{\mu,B_j}(x) = \min_{z \in \mathbb{R}^n} \left\{ \tilde{f}_{B_j}(z) + \frac{\mu}{2}||z - x||^2 \right\}.
\]

whose solution \( z^j \) tends to \( p_\mu(x) \) as the bundle grows, see [17]. In practice, \( z^j \) is considered as an approximation of \( p_\mu(x) \) when the following condition is met [21], Chapter XV,

\[
f(z^j) - \tilde{f}_{B_j}(z^j) \leq \varepsilon, \quad \varepsilon > 0.
\]

In proximal bundle methods, \( \varepsilon \) in (2.8) is usually taken as \( \varepsilon = (1-\sigma)[f(x) - \tilde{f}_{B_j}(z^j)] \) for some \( 0 < \sigma < 1 \), in which case the condition writes \( f(z^j) \leq f(x) - \sigma(f(x) - \tilde{f}_{B_j}(z^j)) \), resulting in a decrease of the objective function \( f \) from \( x \) to \( z^j \). We note in passing that the criterion (2.8) used here to identify an approximate proximal point is clearly much easier to check than (1.6). As pointed out in [17], it does not imply those of [34], in particular (1.6) used in [20].

An algorithm to compute an approximation \( \tilde{p}_\mu(x) \) of the proximal point of a given \( x \in \mathbb{R}^n \) with a tolerance \( \varepsilon \) is as follows. We consider it as the (inexact) first-order oracle for \( F_\mu \).

**Algorithm 2.2 Approximate Proximal Point Oracle (APPO) at \( x \in \mathbb{R}^n \)**

0. Initialize the bundle \( B_j, j = 1 \).
1. Compute the solution \( z^j \) of (2.7)
2. If (2.8) holds, set \( \tilde{p}_\mu(x) = z^j \) and exit.
3. Compute \( f(z^j), g^j \in \partial f(z^j) \) and incorporate \( (z^j,f(z^j),g^j) \) to the bundle
   Increase \( j \) by 1 and loop to Step 1.

Efficient algorithms have been proposed by Frangioni [15] and Kiwiel [24] for solving the special quadratic problem (2.7). We review some basic results of the sequence generated by APPO useful for our subsequent analysis. Let us introduce the functions

\[
\tilde{F}_{x,j}(z^j) = \tilde{F}_{x,j+1}(z^{j+1}) \leq F_\mu(x).
\]

By definition, \( F_\mu(x) = \min_{z \in \mathbb{R}^n} F_\mu(z) = F_\mu(p_\mu(x)) \). The properties of the sequence \( \{z^j\} \) generated by the iterative procedure APPO can be found in [17, Proposition 3], namely the following

\[
\tilde{F}_{x,j}(z^j) \leq \tilde{F}_{x,j+1}(z^{j+1}) \leq F_\mu(x).
\]

As the bundle \( B_j \) grows, \( f(z^j) \) and \( \tilde{f}_{B_j}(z^j) \) get closer to each other i.e. \( \lim_{j \to \infty} \{f(z^j) - \tilde{f}_{B_j}(z^j)\} \to 0 \). The condition (2.8) is satisfied for large \( j \) when \( z^j \) becomes close to \( p_\mu(x) \), justifying the fact that we consider \( z^j \) as an approximate proximal point of \( x \) when (2.8) occurs. APPO then provides an approximate gradient as \( \mu(x - \tilde{p}_\mu(x)) \) and an approximate function value as \( F_\mu(\tilde{p}_\mu(x)) \) since at stop we get

\[
F_\mu(x) \leq F_\mu(\tilde{p}_\mu(x)) \leq F_\mu(x) + \varepsilon.
\]

Indeed, clearly \( F_\mu(x) \leq F_\mu(\tilde{p}_\mu(x)) \). Next,

\[
F_\mu(\tilde{p}_\mu(x)) = f(\tilde{p}_\mu(x)) + \frac{\mu}{2}|\tilde{p}_\mu(x) - x|^2 \leq \tilde{f}_{B_j}(\tilde{p}_\mu(x)) + \frac{\mu}{2}|\tilde{p}_\mu(x) - x|^2 + \varepsilon = \tilde{F}_{x,j}(\tilde{p}_\mu(x)) + \varepsilon.
\]
We then get (2.10) from the fact that \( \tilde{F}_{x,j}(\tilde{p}_\mu(x)) \leq F_\mu(x) \), see (2.9). The necessary and sufficient optimality condition for the quadratic problem (2.7) at the stop of APPO (with the bundle set \( B_j \)) writes \( 0 \in \partial \tilde{F}_{B_j}(\tilde{p}_\mu(x)) - \mu(x - \tilde{p}_\mu(x)). \) Hence, for any \( z \in \mathbb{R}^n \), we have
\[
f(z) \geq \tilde{F}_{B_j}(z) \geq f(\tilde{p}_\mu(x)) + \langle \mu(x - \tilde{p}_\mu(x)), z - \tilde{p}_\mu(x) \rangle - [f(\tilde{p}_\mu(x)) - \tilde{F}_{B_j}(\tilde{p}_\mu(x))].
\]
and from (2.8), \( f(z) \geq f(\tilde{p}_\mu(x)) + \langle \mu(x - \tilde{p}_\mu(x)), z - \tilde{p}_\mu(x) \rangle - \varepsilon. \) In other words,
\[
\mu(x - \tilde{p}_\mu(x)) \in \partial \varepsilon f(\tilde{p}_\mu(x)). \quad (2.11)
\]
It is worth mentioning that APPO is not an inexact first-order oracle in the sense of [9]. It is also different from the procedure given in Section 3.3 for computing approximate solutions for the Moreau-Yosida regularization. The inexact oracle for \( F_\mu \) proposed in [9] computes a pair \((F_{\mu,\delta}(x), g_\delta(x))\) which satisfies the following two inequalities within a tolerance \( \delta \geq 0 \):
\[
0 \leq F_\mu(z) - (F_{\mu,\delta}(x) + \langle g_\delta(x), z - x \rangle) \leq \frac{\mu}{2} \|z - x\|^2 + \delta, \quad x, z \in \mathbb{R}^n,
\]
which are relaxations of the inequalities
\[
0 \leq F_\mu(z) - (F_\mu(x) + \langle \nabla F_\mu(x), z - x \rangle) \leq \frac{\mu}{2} \|z - x\|^2, \quad x, z \in \mathbb{R}^n,
\]
which result from the fact that \( F_\mu \) has Lipschitz continuous gradient. The provided pair has the following properties. \( F_{\mu,\delta}(x) \) is a lower approximation of \( F_\mu(x) \) in the following sense \( F_{\mu,\delta}(x) \leq F_\mu(x) \leq F_{\mu,\delta}(x) + \delta \), while \( g_\delta(x) \) is a \( \delta \)-subgradient of \( F_\mu \) at \( x \) i.e. \( F_\mu(z) \geq F_\mu(x) + \langle g_\delta(x), z - x \rangle - \delta, \quad z \in \mathbb{R}^n \). Even setting \( \varepsilon = \delta \), these features are different from what we have with (2.10) and (2.11) which are satisfied by the output \((F_\varepsilon(\tilde{p}_\mu(x)), \mu(x - \tilde{p}_\mu(x)))\) from APPO. Several papers e.g. [37,39] have been devoted to the study of errors (in different ways as in the present work) in accelerated proximal gradient methods proposed for the case \( f = g + h \) where \( g \) and \( h \) are convex but \( h \) is differentiable, taking advantage of this structure.

2.3 Fast proximal bundle algorithms

An implementable version of Algorithm 2.3 is obtained by using APPO for the approximate computation of \( p_\mu(x^k) \) in its step 1. It is described as follows.

**Algorithm 2.3 Fast Proximal Bundle Algorithm (FPBA)**

0. Choose \( x^0 = y^0 \in \mathbb{R}^n \) and the sequence \( \{\beta_k\}_{k \geq 0}. \) Define the sequence \( \{\varepsilon_k\}_{k \geq 0}. \)

Set \( k = 0. \)

1. Call APPO at \( x = x^k \) with \( \varepsilon = \varepsilon_k \) and set \( y^{k+1} = \tilde{p}_\mu(x^k). \)

2. Set \( x^{k+1} = y^{k+1} + \alpha_k(y^{k+1} - y^k) + \beta_k(y^{k+1} - x^k). \)

3. Increase \( k \) by 1 and loop to Step 1.

As for FPBA, we refer the two versions of FPBA according to the choice of the sequence \( \{\beta_k\} \) to FPBA1 and FPBA2 respectively. The latter can be viewed as an implementable version of G"uler second algorithm if a fixed parameter \( \mu \) is considered (in [20], it is allowed to depend on \( k \)). The work performed at a previous call to APPO can be exploited in the initialization of the bundle at a next call. The
The subgradient selection taken from (in our notations) the sequence \( t \) standard proximal bundle algorithm lies in the stability center \( x^k \) which is usually taken from (in our notations) the sequence \( \{z_j\}_{j \in k} \) even if this is not necessary to get convergence, see [2,16]. Here, \( x^k \) is obtained from a fast gradient iteration. Also, to the contrary of a classical proximal bundle algorithm, at each serious step there is no guarantee of decrease in the objective function value between two successive approximate solutions \( y^k \) and \( y^{k+1} \) (in [2] a serious step does not correspond to a decrease in the objective value as well).

**Algorithm 2.4**

0. Choose an initial point \( x^0 \in \mathbb{R}^n \) and the sequence \( \{\beta_k\}_{k \geq 0} \). Define the sequence \( \{\varepsilon_k\}_{k \geq 0} \). Set \( y^0 = x^0 \), \( k = 0 \) and \( \lambda_0 = 1 \).

1. Set \( z^0 = x^k \) and set \( j = 0 \). Compute \( f(z^j) \), \( g^j \in \partial f(z^j) \) and initialize \( B_j \).

2. If \( g^j = 0 \), terminate; \( z^j \) solves (1.1).

3. Get the solution \( z^{j+1} \) of the quadratic problem

\[
\min_{z \in \mathbb{R}^n} \left\{ \tilde{f}_{B_j}(z) + \frac{\mu}{2} \|z - x^k\|^2 \right\},
\]

Compute \( f(z^{j+1}) \) and \( g^{j+1} \in \partial f(z^{j+1}) \).

- If \( f(z^{j+1}) - \tilde{f}_{B_j}(z^{j+1}) \leq \varepsilon_k \) then \( \tilde{y}_k(x^k) \) is computed.

- Set \( \lambda_{k+1} = 1 + \sqrt{1 + 4\lambda_k^2} \).

- Set \( y^{k+1} = z^{j+1} \), \( x^{k+1} = y^{k+1} + \alpha_k(y^{k+1} - y^k) + \beta_k(y^{k+1} - x^k) \).

- Set \( B_k = B_j \), \( k = k + 1 \) and go to Step 1.

Otherwise, set \( B_{j+1} = B_j \cup \{(z^{j+1}, f(z^{j+1}), g^{j+1})\} \), increase \( j \) by 1 and loop to Step 2.

The subgradient selection or subgradient aggregation techniques may be used to maintain the size of the bundle reasonable, see for instance [23]. Using the fact that \( y^{k+1} \) solves the quadratic problem in Step 5 we have

\[
\tilde{f}_{B_k}(y^{k+1}) \leq \tilde{f}_{B_k}(y^{k+1}) + \frac{\mu}{2}\|y^{k+1} - x^k\|^2 \leq F_\mu(x^k) \leq f(x^k). \tag{2.12}
\]

If it happens that

\[
f(x^k) - \tilde{f}_{B_k}(y^{k+1}) \leq \eta, \tag{2.13}
\]

for some \( \eta \geq 0 \), then

\[
F_\mu(x^k) \leq f(x^k) \leq \tilde{f}_{B_k}(y^{k+1}) + \eta \leq F_\mu(x^k) + \eta. \tag{2.14}
\]

So, when (2.13) holds and \( \eta \) is sufficiently small, we may conclude that \( f(x^k) \approx F_\mu(x^k) \) and then \( x^k \) solves approximately (1.1) according to Proposition 2.1. Another consequence of (2.13) is the following relation,

\[
f(x^k) \leq f(z) + \sqrt{2\mu \|z - x^k\| + \eta} \text{ for all } z \in \mathbb{R}^n,
\]
which is used sometimes to show that (2.13) implies the (approximate) optimality of $x^k$ if $\eta$ is small, see for instance [17,10]. However, for large $\mu$, $\sqrt{2\eta\mu}$ may not be negligible even if $\eta$ is very small. The relation (2.13) is enough on its own as shown by (2.14). Note also that (2.13) implies
\[
\tilde{f}_{B_k}(y^{k+1}) \leq f(x^k) \leq \tilde{f}_{B_k}(y^{k+1}) + \eta \leq \tilde{f}_{B_k}(y^{k+1}) + \varepsilon_k + \eta,
\]
where the last inequality comes from the definition of $y^{k+1}$. Hence, if $\varepsilon_k$ is small as well, $y^{k+1}$ could also be considered as an approximate solution.

We finally observe that, if we discard the momentum (i.e. $\alpha_k = \beta_k = 0$), Algorithm 2.4 becomes a proximal bundle algorithm with a fixed penalty parameter. The present approach can be extended to convex optimization methods related to proximal point algorithms such as those proposed in [14,28,36].

3 Convergence analysis

In this section, we consider the global convergence and rate of convergence of FPBA in its two variants. Let $x^* \in X^*$ and denote $R = |x^0 - x^*| = |y^0 - x^*|$. The application of [30, Theorem 1] in a straightforward manner to FPPA1 (which aims at solving (1.2)) gives the following convergence estimate
\[
F_\mu(y^k) - f^* \leq \frac{4\mu R^2}{(k+2)^2},
\]
where $x^*$ is any optimal solution of (1.2) (and so that of (1.1)) and $f^* = f(x^*)$. This estimate has been improved by Beck and Teboulle in [3] to
\[
F_\mu(y^k) - f^* \leq \frac{2\mu R^2}{(k+1)^2}.
\]
(3.1)
The second algorithm proposed by G"uler [20, Section 6] applies to (1.1) and requires exact proximal points. It uses the relation (2.5) with proximity parameters depending on $k$ and decreasing i.e. $\mu_{k+1} \leq \mu_k$, $k \geq 0$ starting from some $\mu_0 > 0$. The complexity estimate of this algorithm involves the original function as,
\[
f(y^k) - f^* \leq \frac{\mu_0 R^2}{(k+1)^2}.
\]
(3.2)
Since $F_\mu \leq f$ for any $\mu > 0$, the same bound holds for $F_{\mu_0}(y^k) - f^*$, so this result improves over (3.1). The bound obtained in (3.2) improves slightly on (3.1) since in their complexity estimate, $(k+1)(k+1 + \sqrt{2})$ replaces $(k+1)^2$ in the r.h.s. of (3.1).

Those bounds do not apply to FPBA since, to the contrary of FPPA, it uses approximate proximal points. We now give the convergence results for the two versions of FPBA. For a given iterate $y^k$, let $\delta_k = f(y^k) - f^*$ The complexity estimate of the algorithms hinges on a lower bound on $\lambda_1^{k+1} \delta_k - \lambda_1^k \delta_{k+1}$. We start by giving a common one for the sequences $\{y^k\}$ generated by FPBA1 and FPBA2.
Lemma 3.1 Assume that the sequence \( \{(x^k, y^k)\} \) is generated by FPBA. Then,
\[
\lambda_{k-1}^2 \delta_k - \lambda_k^2 \delta_{k+1} \geq \mu \langle u^k, v^k \rangle - \lambda_k^2 \varepsilon_k,
\]
where \( u^k = \lambda_k(y^{k+1} - x^k) \) and \( v^k = \lambda_k(y^{k+1} - y^k) + y^k - x^* \).

Proof Using (2.11) with \( x = x^k \) and \( \varepsilon = \varepsilon_k \), we have for any \( x \in \mathbb{R}^n \)
\[
f(x) \geq f(y^{k+1}) + \mu \langle x^k - y^{k+1}, x - y^{k+1} \rangle - \varepsilon_k.
\]
We use this inequality with \( u \) where
\[
\|x - y^k\| = \|x^k - y^{k+1}\|.
\]
Hence, depending on the rule used, \( u \) and \( v \) are given by
\[
\begin{cases}
\lambda_k(y^{k+1} - y^k) + y^k - x^* = u^k & \text{for FPBA1} \\
\lambda_k(2y^{k+1} - y^k - x^k) + y^k - y^{k+1} & \text{for FPBA2}.
\end{cases}
\]
Based on this and the common lower bound given in Lemma 3.1, we derive other lower bounds for FPBA1 and FPBA2 involving only the sequence \( \{w^k\}_{k \geq 0} \).
**Lemma 3.2** Assume that the sequence \( \{ (x^k, y^k) \} \) is generated by FPBA1. Then
\[
\lambda_{k-1}^2 \delta_k - \lambda_k^2 \delta_{k+1} \geq \frac{\mu}{2} \| w^{k+1} \|^2 - \frac{\mu}{2} \| w^k \|^2 - \lambda_k^2 \varepsilon_k. \tag{3.7}
\]

**Proof** Using (3.5), we get
\[
\langle u^k, v^k \rangle = \frac{1}{2} \left( \| u^k \|^2 + \| v^k \|^2 - \| u^k - v^k \|^2 \right) \geq \frac{1}{2} \left( \| v^k \|^2 - \| u^k - v^k \|^2 \right).
\]
Therefore,
\[
\lambda_{k-1}^2 \delta_k - \lambda_k^2 \delta_{k+1} \geq \mu \langle u^k, v^k \rangle - \lambda_k^2 \varepsilon_k \geq \frac{\mu}{2} \left( \| v^k \|^2 - \| u^k - v^k \|^2 \right) - \lambda_k^2 \varepsilon_k.
\]
But \( u^k - v^k = -w^k \) and for FPBA1, we have \( w^{k+1} = u^k \) (see (3.6)). This gives the desired result.

An analogue result for FPBA2 is as follows.

**Lemma 3.3** Assume that the sequence \( \{ (x^k, y^k) \} \) is generated by FPBA2. Then
\[
\lambda_{k-1}^2 \delta_k - \lambda_k^2 \delta_{k+1} \geq \frac{\mu}{4} \| w^{k+1} \|^2 - \frac{\mu}{4} \| w^k \|^2 - \lambda_k^2 \varepsilon_k. \tag{3.8}
\]

**Proof** Noting that for FPBA2, \( w^{k+1} = u^k + v^k \), we get from (3.4),
\[
\langle u^k, v^k \rangle = \frac{1}{4} \left( \| u^{k+1} \|^2 - \| u^k \|^2 \right).
\]
Apply then Lemma 3.1.

We are now ready to give the complexity estimate for FPBA1, using Lemma 3.3.

**Theorem 3.1** The sequence \( \{ (x^k, y^k) \} \) generated by FPBA1 satisfies the following bound
\[
f(y^k) - f^* \leq \frac{2\mu R^2}{(k+1)^2} + \frac{1}{\lambda_{k-1}^2} \sum_{i=0}^{k-1} \lambda_i^2 \varepsilon_i, \quad k \geq 1.
\]

**Proof** Summing the inequalities (3.7) for \( i = 1, \ldots, k-1 \), gives (recall that \( \lambda_0 = 1 \))
\[
\lambda_{k-1}^2 \delta_k \leq \delta_1 + \frac{\mu}{2} \| w^1 \|^2 + \sum_{i=1}^{k-1} \lambda_i^2 \varepsilon_i - \frac{\mu}{2} \| w^k \|^2 \leq \delta_1 + \frac{\mu}{2} \| w^1 \|^2 + \sum_{i=1}^{k-1} \lambda_i^2 \varepsilon_i.
\]
From the second inequality of (3.3) with \( k = 0 \), we get
\[
\delta_1 \leq \mu \langle x^0 - y^1, y^1 - x^* \rangle + \varepsilon_0 = -\mu \langle x^0 - y^1, x^* - y^1 \rangle + \varepsilon_0 \leq -\frac{\mu}{2} \| x^0 - y^1 \|^2 + \frac{\mu}{2} \| y^1 - x^* \|^2 + \varepsilon_0 \leq -\frac{\mu}{2} \| y^1 - x^* \|^2 + \frac{\mu}{2} \| x^0 - x^* \|^2 + \varepsilon_0.
\]
Note that \( x^1 = \lambda_0(y^1 - y^0) + y^0 - x^* = y^1 - x^* \) since \( \lambda_0 = 1 \). Therefore
\[
\delta_1 \leq -\frac{\mu}{2} \| w^1 \|^2 + \frac{\mu}{2} \| x^0 - x^* \|^2 + \varepsilon_0.
\]
and
\[ \lambda_{k-1}^2 \delta_k \leq \frac{\mu}{2} \|x^0 - x^*\|^2 + \sum_{i=0}^{k-1} \lambda_i^2 \varepsilon_i, \]  
(3.9)

which combined with the fact that \( \lambda_{k-1} \geq (k+1)/2 \) gives the desired result.  
\[ \square \]

Thanks to a better lower bound obtained in Lemma 3.3 for the sequence generated by FPBA2, its complexity estimate appears better.

\textbf{Theorem 3.2} The sequence \( \{y^k\} \) generated by FPBA2 satisfies the following bound
\[ f(y^k) - f^* \leq \frac{\mu R^2}{(k+1)^2} + \frac{1}{\lambda_{k-1}^2} \sum_{i=0}^{k-1} \lambda_i^2 \varepsilon_i, \quad k \geq 1. \]

\[ \text{Proof} \] As in the proof of Theorem 3.1, we sum the inequalities (3.8) for \( i = 1, \ldots, k-1 \) and get
\[ \lambda_{k-1}^2 \delta_k \leq \delta_1 + \frac{\mu}{4} \|w^1\|^2 + \sum_{i=1}^{k-1} \varepsilon_i \lambda_i^2 - \frac{\mu}{4} \|w^k\|^2 \leq \delta_1 + \frac{\mu}{4} \|w^1\|^2 + \sum_{i=1}^{k-1} \lambda_i^2 \varepsilon_i. \]

We use again the second inequality of (3.3) for \( k = 0 \) to obtain
\[ \delta_1 \leq \mu \langle x^0 - y^1, y^1 - x^* \rangle + \varepsilon_0 = \frac{\mu}{4} \|x^0 - x^*\|^2 - \frac{\mu}{4} \|x^0 - 2y^1 + x^*\|^2 + \varepsilon_0 = \frac{\mu}{4} \|x^0 - x^*\|^2 - \frac{\mu}{4} \|w^1\|^2 + \varepsilon_0, \]
noting that \( w^1 = \lambda_0 (x^0 + y^0 - 2y^1) + x^* - y^0 = x^0 - 2y^1 + x^* \). Putting together the above bound on \( \delta_1 \) and the previous inequality, one gets
\[ \delta_k \leq \frac{\mu}{4 \lambda_{k-1}^2} \|x^0 - x^*\|^2 + \frac{1}{\lambda_{k-1}^2} \sum_{i=0}^{k-1} \lambda_i^2 \varepsilon_i \]  
(3.10)

It remains to use the fact that \( \lambda_{k-1} \geq (k+1)/2 \), see (2.2).  
\[ \square \]

We have fixed a parameter \( \mu > 0 \) and get one smooth optimization problem \((1.2)\) on which the fast gradient concept has been applied. In this way, the number of calls to the \( F_\mu \)-oracle APPO is optimized. Of course, \( \mu \) has an impact in the efficiency of solving the quadratic subproblems \((2.7)\) as well as the number of calls to the first-order oracle for \( f \), which is better to be minimized. There comes the need to adapt \( \mu \) at each step although this breaks the philosophy of our approach. Following a different approach, the algorithms proposed by Güler in [20] use proximity parameters depending on \( k \) satisfying the condition (with our notations)

\[ \mu_0 = \mu \text{ for some given } \mu > 0 \text{ and } \mu_k \leq \mu_{k-1}, \quad k \geq 1. \]  
(3.11)

In the present setting, it is also possible to use different parameters under the same condition. In this case, Step 1 of FPBA is modified as follows.

1'. Call APPO at \( x = x^k \) and \( \mu = \mu_k \). Set \( y^{k+1} = \tilde{y}_\mu(x^k) \).
Proposition 3.1 The complexity estimates of Theorems 3.1 and 3.2 hold if instead of a fixed proximity parameter, we consider a sequence of positive numbers \( \{\mu_k\}_{k \geq 0} \) satisfying (3.11).

Proof. We consider only Theorem 3.1 and show that it remains valid with the above modification (the proof for Theorem 3.2 is similar). It easily seen that Lemma 3.1 and Lemma 3.2 hold with \( \mu_k \) in place of \( \mu \). Based on the fact that \( \mu_k \approx \mu_k \leq 1 \), inequality (3.7) yields

\[
\lambda_k^2 \delta_k - \lambda_k^2 \delta_{k+1} \geq \frac{\mu_k}{2} \|w_{k+1}\|^2 - \frac{\mu_k}{2} \|w_k\|^2 - \lambda_k^2 \epsilon_k
\]

Summing these inequalities for \( i = 1, \ldots, k-1 \), yields

\[
\lambda_k^2 \delta_k \leq \delta_1 + \frac{\mu_0}{2} \|w_1\|^2 + \sum_{i=1}^{k-1} \lambda_i^2 \epsilon_i - \frac{\mu_{k-1}}{2} \|w_k\|^2
\]

\[
\leq \delta_1 + \frac{\mu_0}{2} \|w_1\|^2 + \sum_{i=1}^{k-1} \lambda_i^2 \epsilon_i \quad \text{(we use } \mu_0 = \mu) \]

In the present context, (3.3) with \( k = 0 \) and (3.5) give

\[
\delta_1 \leq -\frac{\mu_0}{2} \|y^0 - x^*\|^2 + \frac{\mu_0}{2} \|x^0 - x^*\|^2 + \epsilon_0 = -\frac{\mu}{2} \|w_1\|^2 + \frac{\mu R^2}{2} + \epsilon_0,
\]

and then

\[
\lambda_k^2 \delta_k \leq \frac{\mu R^2}{2} + \sum_{i=0}^{k-1} \lambda_i^2 \epsilon_i.
\]

Remark 3.1 Observe that (3.9) and (3.10) remain valid if the sequence \( \{\lambda_k\} \) satisfies the relation

\[
\lambda_k^2 - \lambda_{k-1}^2 \leq \lambda_k, \quad k \geq 1, \quad (3.12)
\]

used in [3,4,5] to generalize the extrapolation coefficients \( \alpha_k = \lambda_k^{-1} (\lambda_k - 1) \) for inertial proximal methods. Equality holds in (3.12) for Nesterov’s sequence, cf (2.1) used in Lemma 3.1 (which holds with (3.12)).

Remark 3.2 Initially, FPBA intends to solve the minimization problem of \( F_\mu \). However, the complexity estimates are expressed in terms of \( f \)-values. If we discard the errors in these complexity estimates, we recover the known ones given at the beginning of this section for FPPA1 and FPPA2 using exact proximal points. One cannot draw a conclusion of the superiority of a scheme to the other from the above complexity estimates. These worst-case convergence bounds are the ones we were able to establish. We cannot exclude that it is possible to get tighter bounds.

Remark 3.3 Complexity estimates for classical proximal bundle methods have been established requiring \( O(\varepsilon^{-2}) \) outer iterations and \( O(\varepsilon^{-3}) \) iterations while taking into account the number of inner iterations. For proximal level bundle methods, the complexity estimate is \( O(\varepsilon^{-2}) \). See for instance [10,25]. A subsequent work is needed to include inner iterations in the complexity analysis of FPBA.
4 Error accumulation

It is pointed out in [9] that fast first-order methods suffer from accumulation of errors to the contrary of classical gradient methods (see also the inexact approach in [20]). The accumulation of errors at step \( k \), \( \vartheta_k = \lambda_{k-1}^{-2} \sum_{i=0}^{k-1} \lambda_i^2 \varepsilon_i \), is identical in both schemes FPBA1 and FPBA2 and similar to that of the fast gradient method with the inexact oracle proposed in [9]. Since \( \lambda_{k-1} \geq (k+1)/2 \), we have,

\[
\vartheta_k \leq \frac{4}{(k+1)^2} \sum_{i=0}^{k-1} \lambda_i^2 \varepsilon_i.
\]

(4.1)

4.1 Error weights

Let \( \omega_{i,k} = \lambda_i^2 \lambda_{k-1}^{-2} \), \( i = 0, \ldots, k-1 \), be the weight of the error \( \varepsilon_i \) in \( \vartheta_k \) (note that it depends on \( k \)). Using the first relation (2.1), we have for \( i = 0, \ldots, k-2 \),

\[
\omega_{i,k} = \frac{\lambda_i^2}{\lambda_{k-1}^2} = \frac{\lambda_i^2 - \lambda_{i+1}}{\lambda_{k-1}^2} = \omega_{i+1,k} - \frac{\lambda_{i+1}}{\lambda_{k-1}^2} i.e. \omega_{i+1,k} = \omega_{i,k} + \frac{\lambda_{i+1}}{\lambda_{k-1}^2}.
\]

Hence, \( \omega_{i,k} \) increases strictly with \( i \) but is bounded by 1,

\[
0 < \omega_{0,k} = \frac{1}{\lambda_{k-1}^2} < \omega_{1,k} < \omega_{2,k} < \ldots < \omega_{k-1,k} = 1.
\]

However, \( \omega_{i,k} \) decreases with \( k \) as \( \lambda_{k-1} \) is increasing. But for a given \( k \), we have \( \omega_{i,k} > \omega_{i+1,k} \) for \( i = 0, \ldots, k-2 \) i.e the weight increases from \( \lambda_{k-1}^{-2} \) to the maximum \( \omega_{k-1,k} = 1 \) (with the weights in the r.h.s of (4.1), the last ones exceed 1). With this observation, one can tolerate large errors in early iterations but require smaller and smaller errors in the progress of the algorithms.

4.2 Special cases

4.2.1 Equal errors

Assume that \( \varepsilon_i = \varepsilon \) for all \( i \geq 0 \). Based on the first relation in (2.2), we have

\[
\theta_k = \theta_k \varepsilon
\]

where

\[
\theta_k = \sum_{i=0}^{k-1} \omega_{i,k} = \frac{1}{\lambda_{k-1}^2} \sum_{i=0}^{k-1} \sum_{l=0}^{i} \lambda_l = \frac{1}{\lambda_{k-1}^2} \sum_{i=0}^{k-1} (k-i) \lambda_i = k - \frac{1}{\lambda_{k-1}^2} \sum_{i=1}^{k-1} i \lambda_i.
\]

Hence, \( \theta_k \) is far away from \( k \) and so is the accumulated error \( \vartheta_k = \theta_k \varepsilon \) from \( k \varepsilon \). But it is asymptotically divergent with the first terms in the complexity bounds as it is the case for the fast gradient method of [9]. Indeed, starting from \( \theta_1 = 1 \), \( \theta_k \) is increasing with \( k \) as it is shown next. We have for any \( k \geq 1 \),

\[
\theta_{k+1} = \frac{1}{\lambda_k^2} \sum_{i=0}^{k} \lambda_i^2 = 1 + \frac{\lambda_{k-1}^2}{\lambda_k^2} \sum_{i=0}^{k-1} \frac{\lambda_i^2}{\lambda_{k-1}^2} = 1 + (1 - \frac{1}{\lambda_k}) \theta_k.
\]
Hence, \( \theta_{k+1} - \theta_k = 1 - \lambda_k^{-1} \theta_k \). We prove by induction that this difference is positive. It is true for \( k = 1 \) since \( 1 - \lambda_1^{-1} \theta_1 = 1 - \lambda_1^{-1} > 0 \). Assume that it holds for \( k \) i.e. \( 1 - \lambda_k^{-1} \theta_k \geq 0 \) and let show that it holds for \( k + 1 \). We have,

\[
1 - \frac{1}{\lambda_{k+1}} \theta_{k+1} = 1 - \frac{1}{\lambda_{k+1}} \left[ 1 + \left( 1 - \frac{1}{\lambda_k} \right) \theta_k \right] = 1 - \frac{1}{\lambda_{k+1}} \theta_k - \frac{1}{\lambda_{k+1}} \left( 1 - \frac{1}{\lambda_k} \right) \theta_k \\
\geq 1 - \frac{1}{\lambda_k} \theta_k - \frac{1}{\lambda_{k+1}} \left( 1 - \frac{1}{\lambda_k} \right) \theta_k \quad \text{(as } \lambda_k < \lambda_{k+1})
\]

\[
= \left( 1 - \frac{1}{\lambda_{k+1}} \right) \left( 1 - \frac{1}{\lambda_k} \right) \theta_k \\
\geq 0 \quad \text{(as } 1 < \lambda_{k+1}).
\]

The divergence between the two terms in the complexity estimates may be avoided if \( \varepsilon_k = O(k^{-\sigma}) \) for some parameter \( \sigma > 0 \), see [20].

### 4.2.2 Step dependent errors

For the case where \( \varepsilon_k \) is different for each step \( k \), the sequence \( \{\vartheta_k\}_{k \geq 0} \) satisfies the relation

\[
\vartheta_{k+1} = \varepsilon_k + \frac{\lambda_k^2}{\lambda_k^2 - 1} \sum_{i=0}^{k-1} \frac{\lambda_i^2}{\lambda_i^2 - 1} \vartheta_i = \varepsilon_k + \left( 1 - \lambda_k^{-1} \right) \vartheta_k.
\]

Note that for \( k \geq 1 \), we have \( 1 - \lambda_k^{-1} > 0 \). Since \( \vartheta_{k+1} - \vartheta_k = \varepsilon_k - \lambda_k^{-1} \vartheta_k \), the sequence \( \{\vartheta_k\}_{k \geq 0} \) may be made decreasing by choosing \( \varepsilon_k \leq \lambda_k^{-1} \vartheta_k \) for \( k \geq 1 \). In this case, as \( \vartheta_1 = \varepsilon_0 \), we have \( \vartheta_k \leq \varepsilon_0 \), \( k \geq 1 \) and the complexity estimates of FPBA1 and FPBA2 write respectively

\[
f(y^k) - f^* \leq \frac{2\mu R^2}{(k+1)^2} + \varepsilon_0 \quad \text{and} \quad f(y^k) - f^* \leq \frac{\mu R^2}{(k+1)^2} + \varepsilon_0.
\]

In other words, there is no accumulation error in this case, and \( f(y^k) - f^* \) tends asymptotically to \( \varepsilon_0 \). In particular, if we set \( \varepsilon_k = \lambda_k^{-1} \vartheta_k \), \( k \geq 1 \), we have \( \vartheta_k = \varepsilon_0 \) for any \( k \geq 1 \) and therefore

\[
\varepsilon_k = \frac{1}{\lambda_k} \varepsilon_0 \leq \frac{2}{k+2} \varepsilon_0.
\]

(4.2)

If we wish the residual \( f(y^k) - f^* \) to reach an accuracy \( \varepsilon \) with FPBA2 for instance, then we set \( \varepsilon_0 = \frac{\varepsilon}{2} \) and the number \( k \) of steps to perform should satisfy

\[
\frac{\mu R^2}{(k+1)^2} \leq \frac{\varepsilon}{2},
\]

which gives \( k \geq R \sqrt{\frac{2\mu}{\varepsilon}} - 1 \). It easy to check that for FPBA1, the condition on \( k \) is \( k \geq 2R \sqrt{\frac{\mu}{2\varepsilon}} - 1 \).
The choice \((4.2)\) results in a strictly decreasing errors sequence and the approach FPBA is asymptotically an “almost exact” fast gradient method. It is much interesting to exploit the fact that the weights of former errors are decreasing to zero as the iterations progress, and then choose the errors in order to escape from the bundle mechanism as soon as possible as in classical proximal bundle algorithms. For instance, it is still possible to use the condition of classical proximal bundle algorithms,

\[
f_{p}z_{j}^{k} \leq f_{p}x^{k} - \sigma, \quad j \geq 0,
\]

and set \(y^{k+1} = z^{j+1}\) when it is satisfied, implying \(f(y^{k+1}) \leq f(x^{k})\). This would mean setting \(\varepsilon_{k} = (1 - \sigma)[f(x^{k}) - \tilde{f}_{B_{k}}(y^{k+1})], \quad k \geq 0\).

5 Numerical experiments

In this section, we present some numerical results to provide a first idea about the performance of the proposed algorithms as compared to some previous proximal algorithms. To this aim, we consider fifteen of the academic test problems already used in [33]. For the proximity parameter, we consider in all our runs the standard choice \(\mu = 1\) which usually suits for well-scaled problems. As the optimal values of the test problems are available, we stop the algorithms when

\[
f_{k}^{\text{best}} - f^{*} \leq 10^{-6}(1 + |f_{k}^{\text{best}}|),
\]

where \(f_{k}^{\text{best}}\) is the best function value recorded during the \(k\) steps, or when \(|g_{j}^{j}| \leq 10^{-6}\) for some \(j\). Clearly, there is a need for a practical condition identifying \(y^{k}\) as an approximate solution of (1.1) other than fixing a number of steps to perform as in [23, 22] or considering the r.h.s in the complexity estimates of Theorems 3.1 and 3.2 which correspond to the worst case performance of the algorithms. One possibility is to use the following upper approximation of the gap function \(\delta(x)\),

\[
0 \leq \delta(x) \leq \delta_{B_{k}}(x) = f(x) - F_{\mu,B_{k}}(x), \quad \text{by checking } \delta_{B_{k}}(y^{k+1}) \leq \varepsilon \text{ for a given precision } \varepsilon > 0,
\]

but at the cost of computing \(F_{\mu,B_{k}}(y^{k+1})\) at each step \(k\).

We have implemented the algorithms using Python 3.5 and Cplex 12.7.1 as the solver of the quadratic problem (2.7) which has been reformulated as

\[
\min \left\{ w + \frac{\mu}{2}||x - x^{k}||^2 : f(z^{i}) + \langle g^{i}, x - z^{i} \rangle \leq w, \quad i \in B_{j}, \quad w \in R, \quad x \in R^n \right\}
\]

We fix the maximum number of \(k\)-steps to 250 in all the runs. The results obtained by the two versions of FPBA are collected on Table 5.2 with different values for \(c_{0}\) in (4.2) whose r.h.s is taken as \(\varepsilon_{k}\). We reported the number of calls \((\# f g)\) to the first-order \(f\)-oracle for function and subgradient evaluations at trial solutions \(z^{i}\), the number of steps used by the algorithms to reach the above stopping criterion \((\# k)\). Column \(f - f^{*}\) gives respectively the (absolute) difference between the best function value found by the algorithms at termination and the optimal value. The numerical experiments tend to confirm our observation at the end of Remark 3.2. At a first glance on the complexity estimates of Theorems 3.1 and 3.2, one would expect FPBA2 to outperform FPBA1. We can observe that this is not the case since there is no clear superior algorithm among the two versions of FPBA, in terms
of number of calls to APPO as well as the number of calls to the first-order oracle for $f$. The latter seems to increase with $\varepsilon_0$ for most of the test problems (we didn’t include the results obtained with $\varepsilon_0$ for space limitation). We also experiment the condition (4.3) used in classical proximal bundle algorithms with $\sigma = 0.5$. The results are given on Table 5.3 and show that this condition may be a good choice as well in the present setting, at the cost of an additional partial call to the $f$-oracle for the computation of $f(x_k)$. Even by including these calls to count the number of requests to the oracle, escaping from the bundling mechanism as soon as possible may be a winning strategy on some test problems. There is certainly a room for improving the practical efficiency of FPBA by devising practical rules for the management of the parameter $\mu$ in the lines suggested by Proposition 5.1 and the popular sequences of the literature on inertial proximal algorithms.

Disregarding the way FPBA has been derived, other variants of FPBA can be considered as for proximal bundle algorithm, based on alternative (equivalent) subproblems of (2.7). First, from Proposition 2.2.3 in Chapter XV of [21], the exists $\kappa > 0$ such that any solution of the problem

$$\min_{z \in \mathbb{R}^n} \{ \tilde{f}_B_i(z) : \|z - x_k\|^2 \leq \kappa \mu^2 \},$$

also solves (2.7). Second, by interpreting $w$ in (5.3) (the below equivalent reformulation of (2.7)) as the dualization of a constraint $w \leq l(\mu)$, a level stabilization variant of FPBA consists in solving

$$\min \left\{ \|x - x^k\|^2 : f(z^i) + \langle g^i, x - z^i \rangle \leq l(\mu), \ i \in B_j, \ x \in \mathbb{R}^n \right\}.$$  

With a suitable choice of $l(\mu)$, the solution of this problem is that of (2.7). These equivalences are only theoretical as pointed out in [10,21], finding $\kappa(\mu)$ or $l(\mu)$ for a given $\mu$ is not trivial.

Finally, it could be interested to analyze if some improvement on inertial proximal algorithms may be obtained using a second momentum term, yielding a gen-
### Table 5.2 Results with different values of $\varepsilon_0$

| $\varepsilon_0 = 10^{-3}$ | FPBA1 | FPBA2 |
|---------------------------|--------|--------|
| #k | $f - f^*$ | $f - f^*$ |
| 1 | 11 | 20 | 1.04E-06 | 15 | 24 | 4.74E-08 |
| 2 | 9 | 13 | 6.66E-09 | 6 | 11 | 5.87E-09 |
| 3 | 5 | 10 | 4.37E-11 | 4 | 9 | 1.98E-08 |
| 4 | 9 | 20 | 5.47E-06 | 13 | 24 | 2.70E-06 |
| 5 | 2 | 6 | 1.20E-07 | 3 | 8 | 4.31E-08 |
| 6 | 11 | 26 | 1.27E-06 | 14 | 28 | 8.60E-07 |
| 7 | 15 | 27 | 1.05E-06 | 19 | 31 | 5.04E-08 |
| 8 | 22 | 48 | 2.33E-05 | 22 | 50 | 2.87E-05 |
| 9 | 22 | 52 | 2.33E-05 | 23 | 59 | 2.19E-05 |
| 10 | 106 | 182 | 8.63E-07 | 140 | 254 | 1.64E-06 |
| 11 | 222 | 491 | 9.30E-07 | 217 | 429 | 9.36E-07 |
| 12 | 37 | 77 | 1.22E-08 | 65 | 105 | 2.50E-09 |
| 13 | 11 | 62 | 6.26E-09 | 13 | 64 | 1.50E-08 |

### Table 5.3 Results obtained by FPBA algorithms with the rule (4.3)

| $\varepsilon_0 = 10^{-3}$ | FPBA1 | FPBA2 |
|---------------------------|--------|--------|
| #k | $f - f^*$ | $f - f^*$ |
| 1 | 8 | 25 | 6.33E-07 | 7 | 29 | 1.33E-06 |
| 2 | 9 | 13 | 5.69E-09 | 3 | 13 | 5.04E-09 |
| 3 | 4 | 10 | 2.91E-09 | 3 | 9 | 3.56E-06 |
| 4 | 11 | 37 | 4.52E-06 | 7 | 30 | 2.96E-06 |
| 5 | 1 | 7 | 1.28E-07 | 2 | 8 | 3.80E-08 |
| 6 | 11 | 37 | 1.22E-06 | 7 | 41 | 1.84E-06 |
| 7 | 9 | 35 | 4.06E-06 | 14 | 39 | 8.54E-07 |
| 8 | 2 | 13 | 1.60E-05 | 9 | 17 | 3.54E-06 |
| 9 | 5 | 78 | 1.22E-05 | 8 | 96 | 2.92E-05 |
| 10 | 187 | 9.09E-07 | 149 | 175 | 9.74E-07 |
| 11 | 56 | 202 | 9.36E-07 | 179 | 250 | 9.74E-07 |

### Table 5.3 Results obtained by FPBA algorithms with the rule (4.3)

| $\varepsilon_0 = 10^{-3}$ | FPBA1 | FPBA2 |
|---------------------------|--------|--------|
| #k | $f - f^*$ | $f - f^*$ |
| 1 | 10 | 15 | 8.82E-08 | 19 | 22 | 2.82E-06 |
| 2 | 5 | 13 | 2.24E-06 | 9 | 13 | 1.05E-06 |
| 3 | 5 | 10 | 4.37E-11 | 5 | 9 | 4.06E-08 |
| 4 | 10 | 16 | 6.17E-06 | 23 | 25 | 3.87E-06 |
| 5 | 3 | 7 | 1.29E-07 | 4 | 8 | 3.91E-08 |
| 6 | 7 | 24 | 1.11E-06 | 11 | 29 | 1.30E-09 |
| 7 | 18 | 30 | 5.88E-07 | 18 | 32 | 8.51E-07 |
| 8 | 13 | 37 | 3.71E-05 | 40 | 51 | 4.31E-05 |
| 9 | 15 | 38 | 2.30E-05 | 48 | 53 | 1.96E-05 |
| 10 | 147 | 1.53E-06 | 251 | 265 | 1.72E-06 |
| 11 | 56 | 202 | 9.67E-07 | 190 | 409 | 6.19E-07 |
| 12 | 35 | 76 | 2.22E-05 | 30 | 82 | 1.27E-05 |
| 13 | 14 | 25 | 1.11E-09 | 22 | 64 | 1.95E-08 |
| 14 | 203 | 209 | 9.09E-07 | 179 | 184 | 1.95E-09 |
| 15 | 57 | 105 | 9.68E-07 | 54 | 57 | 9.37E-07 |
eralized algorithm

\[ y^{k+1} = \underset{x \in \mathbb{R}^n}{\arg \min} \left\{ f(x) + \frac{1}{2\tau_k} \| x - x^k \|^2 \right\}, \quad x^{k+1} = y^{k+1} + \alpha_k (y^{k+1} - y^k) + \beta_k (y^k - y^k). \]

The sequence \( \{\alpha_k\} \) is general (including Nesterov’s extrapolation coefficients) while \( \beta_k \) may be the one we use in this paper i.e. \( \beta_k = \lambda_k \lambda_{k+1}^{-1} \) since it is shown in [13] to correspond to some optimal choice in first-order algorithms, or any other value that ensures convergence of the scheme.

6 Conclusion

We proposed new proximal bundle algorithms for the minimization of nonsmooth convex functions, by exploiting fast gradient smooth methods on Moreau-Yosida regularization. The difference with the proximal bundle algorithm is the generation of an additional sequence \( \{x^k\} \) from which a sequence \( \{y^k\} \) of proximal points is computed. The computation of \( x^k \) is trivial, so the main work is almost the same as in the classical proximal bundle algorithm. We derive complexity estimates of the proposed implementable algorithms which suffer from an error accumulation due to the use of approximate proximal points.

Acknowledgements I’m very grateful to Philippe Mahey for his useful comments on a previous version of the paper.

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