Characterization of Lipschitz Functions via the Commutators of Maximal Function on Stratified Lie Groups

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Abstract
In this paper, the main aim is to consider the boundedness of the Hardy-Littlewood maximal commutator \( M_b \) and the nonlinear commutator \([b, M]\) on the Lebesgue spaces and Morrey spaces over some stratified Lie group \( G \) when \( b \) belongs to the Lipschitz space, by which some new characterizations of the Lipschitz spaces on Lie group are given.

Keywords: stratified Lie group, maximal function, Lipschitz function, commutator, Morrey space

MSC Classification: 42B35, 43A80

1 Introduction and main results

Stratified groups appear in quantum physics and many parts of mathematics, including several complex variables, Fourier analysis, geometry, and topology [10, 28]. The geometry structure of stratified groups is so good that it inherits a lot of analysis properties from the Euclidean spaces [11, 27]. Apart from this, the difference between the geometry structures of Euclidean spaces and stratified groups makes the study of function spaces on them more complicated.

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However, many harmonic analysis problems on stratified Lie groups deserve a further investigation since most results of the theory of Fourier transforms and distributions in Euclidean spaces cannot yet be duplicated.

Let $T$ be the classical singular integral operator. The commutator $[b, T]$ generated by $T$ and a suitable function $b$ is defined by

$$[b, T]f = bT(f) - T(bf). \quad (1.1)$$

It is known that the commutators are intimately related to the regularity properties of the solutions of certain partial differential equations (PDE), see [4, 7, 25].

The first result for the commutator $[b, T]$ was established by Coifman et al [6], and the authors proved that $b \in \text{BMO}(\mathbb{R}^n)$ (bounded mean oscillation functions) if and only if the commutator (1.1) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In 1978, Janson [15] generalized the results in [6] to functions belonging to a Lipschitz functional space and gave some characterizations of the Lipschitz space $\Lambda_\beta(\mathbb{R}^n)$ via commutator (1.1), and the author proved that $b \in \Lambda_\beta(\mathbb{R}^n)$ if and only if $[b, T]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ where $1 < p < n/\beta$ and $1/p - 1/q = \beta/n$ (see also [24]).

In addition, using real interpolation techniques, Milman and Schonbek [21] established a commutator result that applies to the Hardy-Littlewood maximal function as well as to a large class of nonlinear operators. In 2000, Bastero et al [1] proved the necessary and sufficient conditions for the boundedness of the nonlinear commutator $[b, M]$ on $L^p$ spaces, and the similar problems for $[b, M_\alpha]$ were also studied by Zhang and Wu [31]. In 2017, Zhang [30] considered some new characterizations of the Lipschitz spaces via the boundedness of maximal commutator $M_b$ and the (nonlinear) commutator $[b, M]$ in Lebesgue spaces and Morrey spaces on Euclidean spaces. In 2018, Zhang et al [32] gave necessary and sufficient conditions for the boundedness of the nonlinear commutator $[b, M_\alpha]$ on Orlicz spaces when the symbol $b$ belongs to Lipschitz spaces, and obtained some new characterizations of non-negative Lipschitz functions. And Guliyev [12] recently gave necessary and sufficient conditions for the boundedness of the fractional maximal commutators in the Orlicz spaces $L^\Phi(G)$ on stratified Lie group $G$ when $b$ belongs to $\text{BMO}(G)$ spaces, and obtained some new characterizations for certain subclasses of $\text{BMO}(G)$ spaces.

Inspired by the above literature, the purpose of this paper is to study the boundedness of the Hardy-Littlewood maximal commutator $M_b$ and the nonlinear commutator $[b, M]$ in the Lebesgue spaces and Morrey spaces on some stratified Lie group $G$ when $b \in \Lambda_\beta(G)$, by which some new characterizations of the Lipschitz spaces are given.

Let $f \in L^1_{\text{loc}}(G)$, the Hardy–Littlewood maximal function $M$ is given by

$$M(f)(x) = \sup_{B \ni x} |B|^{-1} \int_B |f(y)|dy$$
where the supremum is taken over all balls $B \subset \mathbb{G}$ containing $x$, and $|B|$ is the Haar measure of the $\mathbb{G}$-ball $B$. And the maximal commutator $M_b$ generated by the operator $M$ and a locally integrable function $b$ is defined by

$$M_b(f)(x) = \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)|dy.$$  

On the other hand, similar to (1.1), we can define the (nonlinear) commutator of the Hardy-Littlewood maximal function $M$ with a locally integrable function $b$ is defined by

$$[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x).$$

Note that operators $M_b$ and $[b, M]$ essentially differ from each other. For example, $M_b$ is positive and sublinear, but $[b, M]$ is neither positive nor sublinear.

The first part of this paper is to study the boundedness of $M_b$ when the symbol $b$ belongs to a Lipschitz space. Some characterizations of the Lipschitz space via such commutator are given.

**Theorem 1.1.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(\mathbb{G})$.
2. $M_b$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for all $p, q$ with $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$.
3. $M_b$ is bounded from $L^p(\mathbb{G})$ to $L^q(\mathbb{G})$ for some $p, q$ with $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$.
4. $M_b$ satisfies the weak-type $(1, Q/(Q - \beta))$ estimates, namely, there exists a positive constant $C$ such that

$$\left|\{ x \in \mathbb{G} : M_b(f)(x) > \lambda \} \right| \leq C \left(\lambda^{-1}\|f\|_{L^1(\mathbb{G})}\right)^{Q/(Q - \beta)} \quad (1.2)$$

holds for all $\lambda > 0$.

**Theorem 1.2.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$, $0 < \lambda < Q - \beta p$.

1. If $1/q = 1/p - \beta/(Q - \lambda)$. Then $b \in \dot{\Lambda}_\beta(\mathbb{G})$ if and only if $M_b$ is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\lambda}(\mathbb{G})$.
2. If $1/q = 1/p - \beta/Q$ and $\lambda/p = \mu/q$. Then $b \in \dot{\Lambda}_\beta(\mathbb{G})$ if and only if $M_b$ is bounded from $L^{p,\lambda}(\mathbb{G})$ to $L^{q,\mu}(\mathbb{G})$.

The second part of this paper aims to study the mapping properties of the (nonlinear) commutator $[b, M]$ when $b$ belongs to some Lipschitz space. To state our results, we recall the definition of the maximal operator with respect to a ball. For a fixed ball $B_0$, the Hardy-Littlewood maximal function with
respect to $B_0$ of a function $f$ is given by

$$M_{B_0}(f)(x) = \sup_{B_0 \supseteq B \ni x} |B|^{-1} \int_B |f(y)|dy,$$

where the supremum is taken over all the balls $B$ with $B \subseteq B_0$ and $x \in B$.

**Theorem 1.3.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(G)$ and $b \geq 0$.
2. $[b, M]$ is bounded from $L^p(G)$ to $L^q(G)$.
3. There exists a constant $C > 0$ such that

$$\sup_{B \ni x} |B|^{-\beta/Q} \left(|B|^{-1} \int_B |b(x) - M_B(b)(x)|^q dx\right)^{1/q} \leq C.$$(1.3)

**Theorem 1.4.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$, $0 < \lambda < Q - \beta p$ and $1/q = 1/p - \beta/(Q - \lambda)$. Then the following statements are equivalent:

1. $b \in \Lambda_\beta(G)$ and $b \geq 0$.
2. $[b, M]$ is bounded from $L^{p, \lambda}(G)$ to $L^{q, \lambda}(G)$.

**Theorem 1.5.** Let $b$ be a locally integrable function and $0 < \beta < 1$. Suppose that $1 < p < Q/\beta$, $0 < \lambda < Q - \beta p$, $1/q = 1/p - \beta/Q$ and $\lambda/p = \mu/q$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(G)$ and $b \geq 0$.
2. $[b, M]$ is bounded from $L^{p, \lambda}(G)$ to $L^{q, \mu}(G)$.

This paper is organized as follows. In the next section, we recall some basic definitions and known results. In Section 3, we will prove Theorems 1.1 and 1.2. Section 4 is devoted to proving Theorems 1.3 to 1.5.

Throughout this paper, the letter $C$ always stands for a constant independent of the main parameters involved and whose value may differ from line to line.

## 2 Preliminaries and lemmas

### 2.1 Lie group $G$

To prove the main results of this paper, we first recall some necessary notions and remarks. Firstly, we recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to [3, 10, 27].
Definition 2.1. We say that a Lie algebra $G$ is stratified if there is a direct sum vector space decomposition
\[ G = \bigoplus_{j=1}^{m} V_j = V_1 \oplus \cdots \oplus V_m \] (2.1)
such that $G$ is nilpotent of step $m$ if $m$ is the smallest integer for which all Lie brackets (or iterated commutators) of order $m + 1$ are zero, that is,
\[
[V_1, V_j] = \begin{cases} V_{j+1}, & 1 \leq j \leq m - 1 \\ 0, & j \geq m \end{cases}
\]
holds.

It is not difficult to find that the above $V_1$ generates the whole of the Lie algebra $G$ by taking Lie brackets.

Remark 1. [33] Let $G = G_1 \supset G_2 \supset \cdots \supset G_{m+1} = \{0\}$ denote the lower central series of $G$, and $\{X_1, \ldots, X_N\}$ be a basis for $V_1$ of $G$.

(i) The direct sum decomposition (2.1) can be constructed by identifying each $G_j$ as a vector subspace of $G$ and setting $V_m = G_m$ and $V_j = G_j \setminus G_{j+1}$ for $j = 1, \ldots, m - 1$.

(ii) The dimension of $G$ at infinity as the integer $Q$ is given by
\[ Q = \sum_{j=1}^{m} j \dim(V_j) = \sum_{j=1}^{m} \dim(G_j). \]

Definition 2.2. A Lie group $G$ is said to be stratified when it is a connected simply-connected Lie group and its Lie algebra $G$ is stratified.

If $G$ is stratified, then its Lie algebra $G$ admits a canonical family of dilations $\{\delta_r\}$, namely, for $r > 0$, $X_k \in V_k$ ($k = 1, \ldots, m$),
\[
\delta_r \left( \sum_{k=1}^{m} X_k \right) = \sum_{k=1}^{m} r^k X_k,
\]
which are Lie algebra automorphisms.

By the Baker-Campbell-Hausdorff formula for sufficiently small elements $X$ and $Y$ of $G$ one has
\[
\exp X \exp Y = \exp H(X, Y) = X + Y + \frac{1}{2} [X, Y] + \cdots
\]
where $\exp : G \to G$ is the exponential map, $H(X, Y)$ is an infinite linear combination of $X$ and $Y$ and their Lie brackets, and the dots denote terms of order higher than two.
The following properties can be found in [26](see Proposition 1.1.1, or Proposition 2.1 in [29] or Proposition 1.2 in [10]).

**Proposition 2.1.** Let $G$ be a nilpotent Lie algebra, and let $G$ be the corresponding connected and simply-connected nilpotent Lie group. Then we have

1. The exponential map $\exp : G \to G$ is a diffeomorphism. Furthermore, the group law $(x, y) \mapsto xy$ is a polynomial map if $G$ is identified with $G$ via $\exp$.

2. If $\lambda$ is a Lebesgue measure on $G$, then $\exp \lambda$ is a bi-invariant Haar measure on $G$ (or a bi-invariant Haar measure $dx$ on $G$ is just the lift of Lebesgue measure on $G$ via $\exp$).

**Notations:**

- $y^{-1}$ represents the inverse of $y \in G$,
- $y^{-1}x$ stands for the group multiplication of $y^{-1}$ by $x$,
- Let the group identity element of $G$ be referred to as the origin denotes by $e$,
- $\chi_E$ denotes a characteristic function of a measurable set $E$ of $G$,
- $L^p (1 \leq p \leq \infty)$ denotes the standard $L^p$-space with respect to the Haar measure $dx$, with the $L^p$-norm $\| \cdot \|_p$.

A homogenous norm on $G$ is a continuous function $x \mapsto \rho(x)$ from $G$ to $[0, \infty)$, which is $C^\infty$ on $G \setminus \{0\}$ and satisfies

$$
\begin{cases}
\rho(x^{-1}) = \rho(x), \\
\rho(\delta t x) = t \rho(x) \text{ for all } x \in G \text{ and } t > 0, \\
\rho(e) = 0.
\end{cases}
$$

Moreover, there exists a constant $c_0 \geq 1$ such that $\rho(xy) \leq c_0 (\rho(x) + \rho(y))$ for all $x, y \in G$.

With the norm above, we define the $G$ ball centered at $x$ with radius $r$ by $B(x, r) = \{ y \in G : \rho(y^{-1}x) < r \}$, and by $\lambda B$ denote the ball $B(x, \lambda r)$ with $\lambda > 0$, let $B_r = B(e, r) = \{ y \in G : \rho(y) < r \}$ be the open ball centered at $e$ with radius $r$, which is the image under $\delta_r$ of $B(e, 1)$. And by $^c B(x, r) = G \setminus B(x, r) = \{ y \in G : \rho(y^{-1}x) \geq r \}$ denote the complement of $B(x, r)$. Let $|B(x, r)|$ be the Haar measure of the ball $B(x, r) \subset G$, and there exists $c_1 = c_1(G)$ such that

$$
|B(x, r)| = c_1 r^Q, \quad x \in G, r > 0.
$$

The most basic partial differential operator in a stratified Lie group is the sub-Laplacian associated with $X$ is the second-order partial differential operator
on $\mathbb{G}$ given by

$$\mathcal{L} = \sum_{i=1}^{n} X_i^2.$$  

### 2.2 Maximal function

Let $0 \leq \alpha < Q$ and $f : \mathbb{G} \to \mathbb{R}$ is a locally integrable function. The fractional maximal function is defined by

$$M_{\alpha}(f)(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/Q}} \int_B |f(y)| dy,$$

where the supremum is taken over all balls $B \subset \mathbb{G}$ containing $x$.

The fractional maximal function $M_{\alpha}(f)$ coincides for $\alpha = 0$ with the Hardy-Littlewood maximal function $M(f)(x) \equiv M_0(f)(x)$.

The following propositions can be found in [16].

**Proposition 2.2.** Let $0 \leq \alpha < Q$ and $1 < p < \gamma^{-1} = \frac{Q}{\alpha}$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then the following two conditions are equivalent:

1. There is a constant $C > 0$ such that for any $f \in L^p_\omega(\mathbb{G})$ the inequality

$$\left( \int_{\mathbb{G}} \left( M_{\gamma}(f \omega^{\gamma})(x) \right)^q \omega(x) dx \right)^{1/q} \leq C \left( \int_{\mathbb{G}} |f(x)|^p \omega(x) dx \right)^{1/p}$$

holds.

2. $\omega \in A_{1+q/p'}(\mathbb{G})$, $p' = \frac{p}{p-q}$.

**Proposition 2.3.** Let $0 < \alpha < Q$, $\gamma = \alpha/Q$, $q = (1 - \gamma)^{-1}$, and $f \in L^q(\mathbb{G})$. Then the following two conditions are equivalent:

1. $\omega \{x \in \mathbb{G} : M_{\gamma}(f \omega^{\gamma})(x) > \lambda \} \leq C \lambda^{-q} \left( \int_{\mathbb{G}} |f(x)| dx \right)^q$

with a constant $C > 0$ independent of $f$ and $\lambda > 0$.

2. $\omega \in A_1(\mathbb{G})$.

The following strong and weak-type boundedness of $M_{\alpha}$ can be obtained from Propositions 2.2 and 2.3 when the weight $\omega = 1$, see Kokilashvili and Kufner [16] for more details. And the first part can also be obtained from Bernardis and Salinas [2].

**Lemma 2.1.** Let $0 < \alpha < Q$, $1 \leq p \leq Q/\alpha$ with $1/q = 1/p - \alpha/Q$, and $f \in L^p(\mathbb{G})$.  

(1) If $1 < p < \frac{Q}{\alpha}$, then there exists a positive constant $C$ such that
\[
\|M_\alpha(f)\|_{L^q(G)} \leq C\|f\|_{L^p(G)}
\]
(2) If $p = 1$, then there exists a positive constant $C$ such that
\[
|\{x \in G : M_\alpha(f)(x) > \lambda\}| \leq C(\lambda^{-1}\|f\|_{L^1(G)})^{Q/(Q-\alpha)}
\]
holds for all $\lambda > 0$.

2.3 Lipschitz spaces on $G$

Next we give the definition of the Lipschitz spaces on $G$, and state some basic properties and useful lemmas.

**Definition 2.3** (Lipschitz-type spaces on $G$).

(1) Let $0 < \beta < 1$, we say a function $b$ belongs to the Lipschitz space $\dot{\Lambda}_\beta(G)$ if there exists a constant $C > 0$ such that for all $x, y \in G$,
\[
|b(x) - b(y)| \leq C(\rho(y^{-1}x))^{\beta},
\]
where $\rho$ is the homogenous norm. The smallest such constant $C$ is called the $\dot{\Lambda}_\beta$ norm of $b$ and is denoted by $\|b\|_{\dot{\Lambda}_\beta(G)}$.

(2) (see Macías and Segovia [19]) Let $0 < \beta < 1$ and $1 \leq p < \infty$. The space $\text{Lip}_{\beta,p}(G)$ is defined to be the set of all locally integrable functions $b$, i.e., there exists a positive constant $C$, such that
\[
\sup_{B \ni x} \frac{1}{|B|^\beta/Q} \left(\frac{1}{|B|} \int_B |b(x) - b_B|^p dx\right)^{1/p} \leq C,
\]
where the supremum is taken over every ball $B \subset G$ containing $x$ and $b_B = \frac{1}{|B|} \int_B b(x) dx$. The least constant $C$ satisfying the conditions above shall be denoted by $\|b\|_{\text{Lip}_{\beta,p}(G)}$.

(3) (see Macías and Segovia [19]) Let $0 < \beta < 1$. When $p = \infty$, we shall say that a locally integrable functions $b$ belongs to $\text{Lip}_{\beta,\infty}(G)$ if there exists a constant $C$ such that
\[
\text{ess sup}_{x \in B} \frac{|b(x) - b_B|}{|B|^\beta/Q} \leq C
\]
holds for every ball $B \subset G$ with $b_B = \frac{1}{|B|} \int_B b(x) dx$. And $\|b\|_{\text{Lip}_{\beta,\infty}(G)}$ stand for the least constant $C$ satisfying the conditions above.

**Remark 2.** (i) Similar to the definition of Lipschitz space $\dot{\Lambda}_\beta(G)$ in (1), we also have the definition form as following (see Chen and Liu [5], Fan and
Xu [9], Krantz [17] et al.)

\[ \|b\|_{\hat{\Lambda}_\beta(G)} = \sup_{x, y \in G, \, y \neq e} \frac{|b(xy) - b(x)|}{(\rho(y))^{\beta}} = \sup_{x, y \in G, \, x \neq y} \frac{|b(x) - b(y)|}{(\rho(y^{-1}x))^{\beta}}. \]

And \( \|b\|_{\hat{\Lambda}_\beta(G)} = 0 \) if and only if \( b \) is constant.

(ii) In (2), when \( p = 1 \), we have

\[ \|b\|_{\text{Lip}_{\beta, 1}(G)} = \sup_{B \ni x} \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - b_B| \, dx \right) := \|b\|_{\text{Lip}_\beta(G)}. \]

(iii) There are two basically different approaches to Lipschitz classes on the n-dimensional Euclidean space. Lipschitz classes can be defined via Poisson (or Weierstrass) integrals of \( L^p \)-functions, or, equivalently, by means of higher order difference operators (see Meda and Pini [20]).

Lemma 2.2. (see [5, 18, 19] ) Let \( 0 < \beta < 1 \) and the function \( b(x) \) integrable on bounded subsets of \( G \).

1. When \( 1 \leq p < \infty \), then

\[ \|b\|_{\hat{\Lambda}_\beta(G)} = \|b\|_{\text{Lip}_\beta(G)} \approx \|b\|_{\text{Lip}_{\beta, p}(G)}. \]

2. Let balls \( B_1 \subset B_2 \subset G \) and \( b \in \text{Lip}_{\beta, p}(G) \) with \( p \in [1, \infty] \). Then there exists a constant \( C \) depends on \( B_1 \) and \( B_2 \) only, such that

\[ |b_{B_1} - b_{B_2}| \leq C\|b\|_{\text{Lip}_{\beta, p}(G)} |B_2|^{\beta/Q}. \]

3. When \( 1 \leq p < \infty \), then there exists a constant \( C \) depends on \( \beta \) and \( p \) only, such that

\[ |b(x) - b(y)| \leq C\|b\|_{\text{Lip}_{\beta, p}(G)} |B|^{\beta/Q} \]

holds for any ball \( B \) containing \( x \) and \( y \).

2.4 Morrey spaces on \( G \)

Morrey spaces were originally introduced by Morrey in [22] to study the local behavior of solutions to second-order elliptic partial differential equations.

Definition 2.4 (Morrey-type spaces on \( G[8] \)).

1. Let \( 1 \leq p < \infty \) and \( 0 \leq \lambda \leq Q \). The Morrey-type spaces \( L^{p, \lambda}(G) \) is defined by

\[ L^{p, \lambda}(G) = \{ f \in L^p_{\text{loc}}(G) : \|f\|_{L^{p, \lambda}(G)} < \infty \}. \]
\[ \|f\|_{L^p,\lambda(G)} = \sup_{B \ni x, B \subset G} \left( \frac{1}{|B|^{\lambda/Q}} \int_B |f(y)|^p \, dy \right)^{1/p}, \]

where the supremum is taken over every ball \( B \subset G \) containing \( x \).

(2) Let \( 1 \leq p < \infty \) and \( \varphi(x, r) \) be a positive measurable function on \( G \times (0, \infty) \). The generalized Morrey space \( L^{p,\varphi}(G) \) is defined for all functions \( f \in L^p_{\text{loc}}(G) \) by the finite norm
\[ \|f\|_{L^{p,\varphi}(G)} = \sup_{B \ni x, B \subset G} \frac{1}{\varphi(x, r)} \left( \frac{1}{|B|^{\lambda/Q}} \int_B |f(y)|^p \, dy \right)^{1/p}, \]

where the supremum is taken over every ball \( B \subset G \) containing \( x \).

**Remark 3** (Guliyev [14]). (i) It is well known that if \( 1 \leq p < \infty \) then
\[ L^{p,\lambda}(G) = \begin{cases} L^p(G) & \text{if } \lambda = 0, \\ L^\infty(G) & \text{if } \lambda = Q, \\ \Theta & \text{if } \lambda < 0 \text{ or } \lambda > Q, \end{cases} \]

where \( \Theta \) is the set of all functions equivalent to 0 on \( G \).

(ii) In (2), when \( 1 \leq p < \infty \) and \( 0 \leq \lambda \leq Q \), we have \( L^{p,\varphi}(G) = L^{p,\lambda}(G) \) if \( \varphi(x, r) = |B|^{(\lambda/Q-1)/p} \) and \( B \subset G \) denotes the ball with radius \( r \) and containing \( x \).

We now recall the result on the boundedness of the fractional maximal operator in the generalised Morrey spaces, which can be found in [13] (theorem 3.2 and 3.3, see also [23]).

**Proposition 2.4** (Spanne-type). Let \( 1 \leq p < \infty \), \( 0 \leq \alpha < \frac{Q}{p} \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q} \) and \( (\varphi_1, \varphi_2) \) satisfy the condition
\[ \sup_{r < t < \infty} t^{\alpha-Q/p} \inf_{t < s < \infty} \varphi_1(x, s)s^{Q/p} \leq C \varphi_2(x, r), \]

where \( C > 0 \) does not depend on \( r \) and \( x \in G \).

(1) Then, for \( 1 < p < \infty \) and any \( f \in L^{p,\varphi_1}(G) \), there exists some positive constant \( C \) such that
\[ \|M_\alpha f\|_{L^{q,\varphi_2}(G)} \leq C\|f\|_{L^{p,\varphi_1}(G)}. \]

(2) Then, for \( p = 1 \) and any \( f \in L^{1,\varphi_1}(G) \), there exists some positive constant \( C \) such that
\[ \|M_\alpha f\|_{W^{q,\varphi_2}(G)} \leq C\|f\|_{L^{1,\varphi_1}(G)}. \]
In the case \( \alpha = 0 \) and \( p = q \), the conclusions of Proposition 2.4 are also valid.

**Proposition 2.5** (Adams-type). Let \( 1 \leq p < q < \infty \), \( 0 < \alpha < \frac{Q}{p} \), and let \( \varphi(x, \tau) \) satisfy the condition

\[
\sup_{r < t < \infty} t^{-Q} \inf_{t < s < \infty} \varphi(x, s)s^Q \leq C \varphi(x, r)
\]

and

\[
\sup_{r < t < \infty} t^\alpha \varphi(x, \tau)^{1/p} \leq C r^{-\alpha p/(q-p)},
\]

where \( C > 0 \) does not depend on \( r \) and \( x \in G \).

1. Then, for \( 1 < p < \infty \) and any \( f \in \mathcal{L}^{p, \varphi^{1/p}}(G) \), there exists some positive constant \( C \) such that

\[
\|M_\alpha f\|_{\mathcal{L}^{q, \varphi^{1/q}}(G)} \leq C\|f\|_{\mathcal{L}^{p, \varphi^{1/p}}(G)}.
\]

2. Then, for \( p = 1 \) and any \( f \in \mathcal{L}^{1, \varphi^{1/p}}(G) \), there exists some positive constant \( C \) such that

\[
\|M_\alpha f\|_{W^{1, q}_{\mathcal{L}^{q, \varphi^{1/q}}}(G)} \leq C\|f\|_{\mathcal{L}^{1, \varphi}(G)}
\]

When \( \varphi_1(x, r) = |B|^{(\lambda/Q-1)/p} = c_1 r^{Q(\lambda/Q-1)/p} \) and \( \varphi_2(x, r) = |B|^{(\mu/Q-1)/q} = c_2 r^{Q(\mu/Q-1)/q} \), we can summarize the results as follows from Propositions 2.4 and 2.5 (see also Corollary 3.3 in [13]).

**Lemma 2.3.** Let \( 0 < \alpha < Q \), \( 1 < p < Q/\alpha \) and \( 0 < \lambda < Q - \alpha p \).

1. If \( 1/q = 1/p - \alpha/(Q - \lambda) \), then there exists a positive constant \( C \) such that

\[
\|M_\alpha f\|_{L^{q, \lambda}(G)} \leq C\|f\|_{L^{p, \lambda}(G)}
\]

for every \( f \in L^{p, \lambda}(G) \).

2. If \( 1/q = 1/p - \alpha/Q \) and \( \lambda/p = \mu/q \). Then there exists a positive constant \( C \) such that

\[
\|M_\alpha f\|_{L^{q, \mu}(G)} \leq C\|f\|_{L^{p, \lambda}(G)}
\]

for every \( f \in L^{p, \lambda}(G) \).

### 3 Proofs of Theorems 1.1 and 1.2

We now give the proof of the main results. First, we prove Theorem 1.1.
Proof of Theorem 1.1 If $b \in \mathring{\Lambda}_\beta(G)$, then, using (1) in Definition 2.3, we have

$$M_b(f)(x) = \sup_{B \ni x} \left| B \right|^{-1} \int_B |b(x) - b(y)||f(y)|dy$$

$$\leq C\|b\|_{\mathring{\Lambda}_\beta(G)} \sup_{B \ni x} \left| B \right|^{-1} \int_B |\rho(y^{-1}x)|^\beta\|f(y)\|dy$$

$$\leq C\|b\|_{\mathring{\Lambda}_\beta(G)} \sup_{B \ni x} \frac{1}{\left| B \right|^{1-\beta/Q}} \int_B |f(y)|dy$$

$$\leq C\|b\|_{\mathring{\Lambda}_\beta(G)} M_b(f)(x).$$

(3) follows from Lemma 2.1 and above estimate.

(3) $\implies$ (1): Suppose $M_b$ is bounded from $L^p(G)$ to $L^q(G)$ for some $p, q$ with $1 < p < Q/\beta$ and $1/q = 1/p - \beta/Q$. For any ball $B \subset G$ containing $x$, using the Hölder’s inequality and noting that $1/p + 1/q' = 1 + \beta/Q$, one obtains

$$\frac{1}{\left| B \right|^{1+\beta/Q}} \int_B |b(x) - b_B|dx \leq \frac{1}{\left| B \right|^{1+\beta/Q}} \int_B \left( \frac{1}{\left| B \right|} \int_B |b(x) - b(y)|dy \right)dx$$

$$= \frac{1}{\left| B \right|^{1+\beta/Q}} \int_B \left( \frac{1}{\left| B \right|} \int_B |b(x) - b(y)|\chi_B(y)dy \right)dx$$

$$\leq \frac{1}{\left| B \right|^{1+\beta/Q}} \int_B M_b(\chi_B)(x)dx$$

$$\leq \frac{1}{\left| B \right|^{1+\beta/Q}} \left( \int_B (M_b(\chi_B)(x))^q dx \right)^{1/q} \left( \int_B \chi_B(x)dx \right)^{1/q'}$$

$$\leq C \|\chi_B\|_{L^p(G)} \|\chi_B\|_{L^q(G)}$$

$$\leq C.$$

This together with Lemma 2.2 gives $b \in \mathring{\Lambda}_\beta(G)$.

(4) $\implies$ (1): Assume $M_b$ satisfies the weak-type $(1, Q/(Q - \beta))$ estimates and (1.2) is true. In order to verify $b \in \Lambda_\beta(G)$, for any fixed ball $B_0 \subset G$, since for any $x \in B_0$,

$$|b(x) - b_{B_0}| \leq \frac{1}{|B_0|} \int_{B_0} |b(x) - b(y)|dy,$$

then, for all $x \in B_0$,

$$M_b(\chi_{B_0})(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)|\chi_{B_0}(y)dy$$

$$\geq \frac{1}{|B_0|} \int_{B_0} |b(x) - b(y)|\chi_{B_0}(y)dy$$

$$= \frac{1}{|B_0|} \int_{B_0} |b(x) - b(y)|dy$$

$$\geq |b(x) - b_{B_0}|.$$

Thus, combined with (1.2), we have

$$\left| \{x \in B_0 : |b(x) - b_{B_0}| > \lambda \} \right| \leq \left| \{x \in B_0 : M_b(\chi_{B_0})(x) > \lambda \} \right|$$

$$\leq C \left( \lambda^{-(1/\|\chi_{B_0}\|_{L^1(G)})} \right)^{Q/(Q - \beta)}$$
\[
\leq C \left( \lambda^{-1} |B_0| \right)^{Q/(Q-\beta)}.
\]

Set \( t > 0 \) be a constant to be determined later, then applying Fubini’s theorem, one get
\[
\int_{B_0} |b(x) - b_{B_0}| \, dx = \int_0^\infty |\{ x \in B_0 : |b(x) - b_{B_0}| > \lambda \}| \, d\lambda \\
= \int_0^t |\{ x \in B_0 : |b(x) - b_{B_0}| > \lambda \}| \, d\lambda \\
+ \int_t^\infty |\{ x \in B_0 : |b(x) - b_{B_0}| > \lambda \}| \, d\lambda \\
\leq t|B_0| + C \int_t^\infty (\lambda^{-1} |B_0|)^{Q/(Q-\beta)} \, d\lambda \\
\leq t|B_0| + C|B_0|^{Q/(Q-\beta)} \int_t^\infty \lambda^{-Q/(Q-\beta)} \, d\lambda \\
\leq C \left( t|B_0| + |B_0|^{Q/(Q-\beta)} t^{1-Q/(Q-\beta)} \right).
\]

Let \( t = |B_0|^{\beta/Q} \) in the above estimate, we get
\[
\int_{B_0} |b(x) - b_{B_0}| \, dx \leq C|B_0|^{1+\beta/Q}.
\]

It follows from Lemma 2.2 that \( b \in \dot{\Lambda}_\beta(G) \) since \( B_0 \) is an arbitrary ball in \( G \).

The proof of Theorem 1.1 is completed since \((2) \implies (1)\) follows from \((3) \implies (1)\). \( \square \)

**Proof of Theorem 1.2** (1): We first prove that the necessary condition. Assume \( b \in \dot{\Lambda}_\beta(G) \), using (3.1) and Lemma 2.3, we obtain
\[
\| M_b(f) \|_{L^{q,\lambda}(G)} \leq C\| b \|_{\dot{\Lambda}_\beta(G)} \| M_{\beta f} \|_{L^{q,\lambda}(G)} \leq C\| b \|_{\dot{\Lambda}_\beta(G)} \| f \|_{L^{p,\lambda}(G)}.
\]

We now prove that the sufficient condition. If \( M_b \) is bounded from \( L^{p,\lambda}(G) \) to \( L^{q,\lambda}(G) \), then for any ball \( B \subset G \),
\[
|B|^{-\beta/Q} \left( |B|^{-1} \int_B |b(x) - b_B|^q \, dx \right)^{1/q} \leq |B|^{-\beta/Q} \left( |B|^{-1} \int_B (M_b(\chi_B)(x))^q \, dx \right)^{1/q} \\
\leq |B|^{-\beta/Q-1/q+\lambda/(Qq)} \| M_b(\chi_B) \|_{L^{q,\lambda}(G)} \\
\leq C |B|^{-\beta/Q-1/q+\lambda/(Qq)} \| \chi_B \|_{L^{p,\lambda}(G)} \\
\leq C,
\]
where in the last step we have used \( 1/q = 1/p - \beta/(Q-\lambda) \) and the fact
\[
\| \chi_B \|_{L^{p,\lambda}(G)} \leq |B|^{(1-\lambda/Q)/p}.
\] (3.2)

It follows from Lemma 2.2 that \( b \in \dot{\Lambda}_\beta(G) \). This completes the proof.

(2): By a similar proof to (1) in Theorem 1.2, we can obtain the desired result. \( \square \)
4 Proofs of Theorems 1.3 to 1.5

Now, we prove Theorem 1.3.

Proof of Theorem 1.3 (1) $\Rightarrow$ (2): For any fixed $x \in \mathcal{G}$ such that $M(f)(x) < \infty$, since $b \geq 0$ then

$$||b, M||_1 = |b(x) M(f)(x) - M(bf)(x)|$$

$$\leq \sup_{B \ni x} |B|^{-1} \int_B |b(x) - b(y)||f(y)|dy$$

$$= M_b(f)(x).$$

(3) $\Rightarrow$ (2): For any fixed ball $B \subset \mathcal{G}$ and all $x \in B$, one have

$$M(\chi_B)(x) = \chi_B(x) \quad \text{and} \quad M(b \chi_B)(x) = M_B(b)(x).$$

Then

$$|B|^{-\beta/Q} \left(|B|^{-1} \int_B |b(x) - M_B(b)(x)|^q dx \right)^{1/q}$$

$$= |B|^{-\beta/Q} \left(|B|^{-1} \int_B ||b, M(\chi_B)(x)||^q dx \right)^{1/q}$$

$$\leq |B|^{-\beta/Q - 1/q} ||b, M(\chi_B)||_{L^q(\mathcal{G})}$$

$$\leq C |B|^{-\beta/Q - 1/q} ||\chi_B||_{L^p(\mathcal{G})} \leq C,$$

which implies (3) since the ball $B \subset \mathcal{G}$ is arbitrary.

(3) $\Rightarrow$ (1): To prove $b \in \dot{A}_\beta(\mathcal{G})$, by Lemma 2.2, it suffices to verify that there is a constant $C > 0$ such that for all balls $B \subset \mathcal{G}$, one get

$$|B|^{-1 - \beta/Q} \int_B |b(x) - b_B| dx \leq C. \quad (4.3)$$

For any fixed ball $B \subset \mathcal{G}$, let $E = \{x \in B : b(x) \leq b_B\}$ and $F = \{x \in B : b(x) > b_B\}$. The following equality is trivially true (modifying the argument in [1], page 3331):

$$\int_E |b(x) - b_B| dx = \int_F |b(x) - b_B| dx.$$

Since for any $x \in E$ we have $b(x) \leq b_B \leq M_B(b)(x)$, then for any $x \in E$,

$$|b(x) - b_B| \leq |b(x) - M_B(b)(x)|.$$

Therefore

$$\frac{1}{|B|^{1 + \beta/Q}} \int_B |b(x) - b_B| dx = \frac{1}{|B|^{1 + \beta/Q}} \int_{E \cup F} |b(x) - b_B| dx$$

$$= \frac{2}{|B|^{1 + \beta/Q}} \int_E |b(x) - b_B| dx$$

$$\leq \frac{2}{|B|^{1 + \beta/Q}} \int_E |b(x) - M_B(b)(x)| dx$$

$$\leq \frac{2}{|B|^{1 + \beta/Q}} \int_B |b(x) - M_B(b)(x)| dx. \quad (4.4)$$
On the other hand, it follows from Hölder’s inequality and (1.3) that
\[
\frac{1}{|B|^{1+\beta/Q}} \int_B |b(x) - M_B(b)(x)|\,dx
\leq \frac{1}{|B|^{1+\beta/Q}} \left( \int_B |b(x) - M_B(b)(x)|^q\,dx \right)^{1/q} |B|^{1/q'}
\leq \frac{1}{|B|^\beta/Q} \left( |B|^{-1} \int_B |b(x) - M_B(b)(x)|^q\,dx \right)^{1/q}
\leq C.
\]
This together with (4.4) gives (4.3), and so we achieve \( b \in \dot{A}_\beta(G) \).

In order to prove \( b \geq 0 \), it suffices to show \( b^- = 0 \), where \( b^- = -\min\{b, 0\} \). Let \( b^+ = |b| - b^- \), then \( b = b^+ - b^- \). For any fixed ball \( B \subset G \), observe that
\[
0 \leq b^+(x) \leq |b(x)| \leq M_B(b)(x)
\]
for \( x \in B \) and thus we have that, for \( x \in B \),
\[
0 \leq b^-(x) \leq M_B(b)(x) - b^+(x) \leq M_B(b)(x) - b^+(x) + b^-(x) = M_B(b)(x) - b(x).
\]
Then, it follows from (1.3) that, for any ball \( B \subset G \),
\[
\frac{1}{|B|} \int_B b^-(x)\,dx \leq \frac{1}{|B|} \int_B |M_B(b)(x) - b(x)|\,dx
\leq \left( \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)|^q\,dx \right)^{1/q}
= |B|^{\beta/Q} \left( \frac{1}{|B|^{\beta/Q}} \left( \frac{1}{|B|} \int_B |b(x) - M_B(b)(x)|^q\,dx \right)^{1/q} \right)
\leq C|B|^\beta/Q.
\]
Thus, \( b^- = 0 \) follows from Lebesgue’s differentiation theorem.

The proof of Theorem 1.3 is completed. \( \Box \)

**Proof of Theorem 1.4** (1) \( \implies \) (2): We first prove that the necessary condition. Assume \( b \in \dot{A}_\beta(G) \) and \( b \geq 0 \). Using (4.1) and (1) in Theorem 1.2, it is not difficult to find that \([b, M]\) is bounded from \( L^{p,\lambda}(G) \) to \( L^{q,\lambda}(G) \).

(2) \( \implies \) (1): We now prove that the sufficient condition. Assume that \([b, M]\) is bounded from \( L^{p,\lambda}(G) \) to \( L^{q,\lambda}(G) \). Similarly to (4.2), for any ball \( B \subset G \), we obtain
\[
|B|^{-\beta/Q} \left( |B|^{-1} \int_B |b(x) - M_B(b)(x)|^q\,dx \right)^{1/q}
= |B|^{-\beta/Q} \left( |B|^{-1} \int_B ||[b, M](\chi_B)(x)||^q\,dx \right)^{1/q}
\leq |B|^{\lambda/(Qq) - \beta/Q - 1/q} \|[b, M](\chi_B)||_{L^{q,\lambda}(G)}
\leq C|B|^{\lambda/(Qq) - \beta/Q - 1/q} \|\chi_B||_{L^{q,\lambda}(G)} \leq C,
\]
where in the last step we have used \( 1/q = 1/p - \beta/(Q - \lambda) \) and (3.2).

Using Theorem 1.3, we can obtain that \( b \in \dot{A}_\beta(G) \) and \( b \geq 0 \). \( \Box \)
**Proof of Theorem 1.5** By the same way of the proof of Theorem 1.4, Theorem 1.5 can be proven. We omit the details.

**Declarations**

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