A REMARK ON THE CONVOLUTION WITH THE BOX SPLINE

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Abstract. The semi-discrete convolution with the Box Spline is an important tool in approximation theory. We give a formula for the difference between semi-discrete convolution and convolution with the Box Spline. This formula involves multiple Bernoulli polynomials.

Key words: polynomial interpolation, box splines, zonotopes, hyperplane arrangements, Bernoulli polynomials.

1. Box Splines and semi-discrete convolution

Let $V$ be a $n$-dimensional real vector space equipped with a lattice $\Lambda$. If we choose a basis of the lattice $\Lambda$, then we may identify $V$ with $\mathbb{R}^n$ and $\Lambda$ with $\mathbb{Z}^n$. We choose here the Lebesgue measure $dv$ associated to the lattice $\Lambda$.

Let $X = [a_1, a_2, \ldots, a_N]$ be a sequence (a multiset) of $N$ non zero vectors in $\Lambda$.

The zonotope $Z(X)$ associated with $X$ is the polytope

$$Z(X) := \{ \sum_{i=1}^{N} t_i a_i ; t_i \in [0, 1] \}.$$ 

In other words, $Z(X)$ is the Minkowski sum of the segments $[0, a_i]$ over all vectors $a_i \in X$.

We denote by $\mathbb{C}[V]$ the space of (complex valued) polynomial functions on $V$.

Recall that the Box Spline $B(X)$ is the distribution on $V$ such that, for a test function $test$ on $V$, we have the equality

$$\langle B(X), test \rangle = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} test(\sum_{i=1}^{N} t_i a_i) dt_1 \cdots dt_N.$$ 

We also note $\langle B(X), test \rangle = \int_{V} B(X)(v) test(v)$.

The distribution $B(X)$ is a probability measure supported on the zonotope $Z(X)$. If $X$ is empty, then $B(X)$ is the $\delta$ distribution on $V$. For the basic properties of the Box Spline, we refer to [5] (or [6], chapter 16).

If $D$ is any distribution on $V$, the convolution $B(X) \ast D$ is well defined and is again a distribution on $V$. If $D = f(v)dv$ is a smooth density, then
\[ \tau \]

Figure 1. Affine topes for \( X = [e_1, e_2, e_1 + e_2] \)

\[
B(X) * D = F(v)dv \text{ is a smooth density with }
F(v) = \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} f(v - \sum_{i=1}^{N} t_i a_i) dt_1 \cdots dt_N.
\]

If \( X \) generates \( V \), the zonotope is a full dimensional polytope, and \( B(X) \) is given by integration against a locally \( L^1 \)-function. Let us describe more precisely where this function is smooth.

We continue to assume that \( X \) generates \( V \). An hyperplane of \( V \) generated by a subsequence of elements of \( X \) is called admissible. An element of \( V \) is called (affine) regular, if no translate \( v + \lambda \) of \( v \) by any \( \lambda \) in the lattice \( \Lambda \) lies in an admissible hyperplane. We denote by \( V_{\text{reg,aff}} \) the open subset of \( V \) consisting of affine regular elements: the set \( V_{\text{reg,aff}} \) is the complement of the union of all the translates by \( \Lambda \) of admissible hyperplanes. A connected component \( \tau \) of the set of regular elements will be called a (affine) tope (see Figure 1).

The choice of the Lebesgue measure \( dv \) on \( V \) allows us to identify distributions and generalized functions: if \( F \) is a generalized function, \( Fdv \) is a distribution. If the distribution \( Fdv \) is given by \( \langle Fdv, \text{test} \rangle = \int_{V} f(v) \text{test}(v) dv \), with \( f(v) \) locally \( L^1 \), we say that \( F \) is locally \( L^1 \), and we use the same notation for \( F \) and the locally \( L^1 \) function \( f \).

A generalized function \( b \) on \( V \) will be called piecewise polynomial (relative to \( X, \Lambda \)) if:

- the function \( b \) is locally \( L^1 \),
- on each tope \( \tau \), there exists a polynomial function \( b(\tau) \) on \( V \) such that the restriction of \( b \) to \( \tau \) coincides with the restriction of the polynomial \( b(\tau) \) to \( \tau \).

If \( F \) is a piecewise polynomial function, we will say that the distribution \( Fdv \) is piecewise polynomial.

If \( X \) generates \( V \), the Box Spline \( B(X) \) is a piecewise polynomial (relative to \( (X, \Lambda) \)) distribution supported on the zonotope \( Z(X) \).
Let $f$ be a smooth function on $V$. Then there are two distributions naturally associated to $X, \Lambda, f$:

- the piecewise polynomial distribution $B(X) * d f$:
on a test function $\text{test}$,

$$\langle B(X) * d f, \text{test} \rangle = \sum_{\lambda \in \Lambda} f(\lambda) \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} \text{test}(\lambda + \sum_{i=1}^{N} t_i a_i) dt_1 \cdots dt_N.$$  

- the smooth density $B(X) * c f$:
on a test function $\text{test}$,

$$\langle B(X) * c f, \text{test} \rangle = \int_{V} f(v) \int_{t_1=0}^{1} \cdots \int_{t_N=0}^{1} \text{test}(v + \sum_{i=1}^{N} t_i a_i) dt_1 \cdots dt_N dv.$$  

The notations $* d$ and $* c$ mean discrete, versus continuous. $B(X) * d f$ is the convolution of $B(X)$ with the discrete measure $\sum_{\lambda} f(\lambda) \delta_\lambda$, while $B(X) * c f$ is the usual convolution of $B(X)$ with the smooth density $f(v) dv$. The subscript $* c$ is just for emphasis. The operation $* d$ is denoted $*'$ in [5], [6] and is called semi-discrete convolution.

Our aim is to write an explicit formula for the difference $B(X) * d f - B(X) * c f$.

We also associate to $a \in X$ three operators:

- the partial differential operator

$$(\partial_a f)(v) = \frac{d}{d\epsilon} f(v + \epsilon a),$$

- the difference operator

$$(\nabla_a f)(v) = f(v) - f(v - a),$$

- the integral operator

$$(I_a f)(v) = \int_{0}^{1} f(v - ta) dt.$$  

The operator $I_a$ is the convolution $B([a]) * c f$ with the Box Spline associated to the sequence with a single element $a$.

These three operators respects the space of polynomial functions $\mathbb{C}[V]$ on $V$. The Taylor series formula implies that, on the space $\mathbb{C}[V]$, the operator $I_a$ is the invertible operator given by

$$I_a = \frac{1 - e^{-\partial_a}}{\partial_a} = \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)!} \partial_a^j.$$  

In particular, if $f \in \mathbb{C}[V]$ is a polynomial

$$(2) \quad B(X) * c f = \left( \prod_{a \in X} \frac{1 - e^{-\partial_a}}{\partial_a} f \right) dv.$$
If \( I, J \) are subsequences of \( X \), we define the operators \( \partial_I = \prod_{a \in I} \partial_a \) and \( \nabla_J = \prod_{b \in J} \nabla_b \). They are defined on distributions.

Recall that \( \partial_Y B(X) = \nabla_Y B(X \setminus Y) \), if \( Y \) is a subsequence of \( X \).

A subsequence \( Y \) of \( X \) will be called long if the sequence \( X \setminus Y \) do not generate the vector space \( V \). A long subsequence \( Y \), minimal along the long subsequences, is also called a cocircuit: then \( Y = X \setminus H \) where \( H \) is an admissible hyperplane.

In our formula, when \( f \) is a polynomial, \( B(X) \ast_d f - B(X) \ast_c f \) is naturally expressed in function of the derivatives \( \partial_Y f \) with respect to long subsequences \( Y \).

2. Piecewise smooth distributions

Our aim is to write an explicit formula for the difference of the two distributions \( B(X) \ast_d f - B(X) \ast_c f \). As the first one is a piecewise polynomial distribution, the second a smooth density, we will need to introduce an intermediate space of distributions. We will use "piecewise smooth distributions". Let us give a definition.

We continue to assume that \( X \) generates \( V \).

**Definition 2.1.** A generalized function \( b \) on \( V \) will be called piecewise smooth (relative to \( X, \Lambda \)) if:

- the generalized function \( b \) is locally \( L^1 \),
- on each tope \( \tau \), there exists a smooth function \( b(\tau) \) on the full space \( V \) such that the restriction of \( b \) to \( \tau \) coincides with the restriction of the smooth function \( b(\tau) \) to \( \tau \).

In this definition, given a tope \( \tau \), the function \( b \) restricted to \( \tau \) (as well as all its derivatives) extends continuously to the closure of \( \tau \). However, these extensions do not always coincide on intersections of the closures of topes.

If \( b \) is piecewise smooth, we then say that the distribution \( B := b(v)dv \) (given by integration against the locally \( L^1 \) function \( b \)) is piecewise smooth.

It is clear that if we multiply a piecewise polynomial distribution \( B \) by a smooth function, we obtain a piecewise smooth distribution. Remark that the space of piecewise smooth distributions is stable by the operators \( \nabla_a \), and by convolution with Box Splines \( B(Y) \) (\( Y \) any subsequence of \( X \)). However, it is not stable under operators \( \partial_a \). For example, \( \partial_Y B(X) = \nabla_X B(\emptyset) \) is a linear combination of \( \delta \) distributions.

3. Multiple Bernoulli periodic polynomials

Let \( U \) be the dual vector space to \( V \) and \( \Gamma \subset U \) be the dual lattice to \( \Lambda \).

If \( Y \) is a subsequence of \( X \), we define

\[
U_{\text{reg}}(Y) = \{ u \in U \; : \; \langle a, u \rangle \neq 0, \text{ for all } a \in Y \}
\]

and

\[
\Gamma_{\text{reg}}(Y) = \Gamma \cap U_{\text{reg}}(Y).
\]
Consider the periodic function on $V$ given by the (oscillatory) sum

\begin{equation}
W(X)(v) = \sum_{\gamma \in \Gamma_{\text{reg}}(X)} e^{2i\pi \langle v, \gamma \rangle} \prod_{a \in X} e^{2i\pi \langle a, \gamma \rangle}.
\end{equation}

This is well defined as a generalized function on $V$. In the sense of generalized functions, we have

\begin{equation}
\partial_X W(X)(v) = \sum_{\gamma \in \Gamma_{\text{reg}}(X)} e^{2i\pi \langle \gamma, v \rangle}.
\end{equation}

We will use this equation to construct “primitives” of parts of the Poisson formula.

We will call the series $W(X)$ a multiple Bernoulli series. Multiple Bernoulli series have been extensively studied by A. Szenes [7]. They are natural generalizations of Bernoulli series: for $\Lambda = \mathbb{Z}\omega$ and $X_k := [\omega, \omega, \ldots, \omega]$, where $\omega$ is repeated $k$ times with $k > 0$, the series

\begin{equation}
W(X_k)(t\omega) = \sum_{n \neq 0} \frac{e^{2i\pi n t}}{(2i\pi n)^k}
\end{equation}

is equal to $-\frac{1}{k!}B(k, t-[t])$ where $B(k, t)$ denotes the $k$th Bernoulli polynomial in variable $t$. In particular, for $k = 1$, we have $W(X_1)(t\omega) = \frac{1}{2} - t + \lfloor t \rfloor$ (see Figure 2).

We recall the following proposition [7] (see also [2], [1]).

**Proposition 3.1.** If $X$ generates $V$, the generalized function $W(X)$ is piecewise polynomial (relative to $(X, \Lambda)$).

Thus we will also call $W(X)$ a multiple periodic Bernoulli polynomial.

The above proposition is proved by reduction to the one variable case. Indeed, the function $\frac{1}{\prod_{a \in X} (a, z)^n}$ can be decomposed in a sum of functions $\frac{1}{\prod_{i=1}^r (a_{ij}, z)^{n_i}}$ with respect to a basis $a_{ji}$ of $V$ extracted from $X$. This reduces the computation to the one dimensional case. A. Szenes [7] gave an efficient multidimensional explicit residue formula to compute $W(X)$. 
Example 3.2. Let $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ with lattice $\Lambda = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$. Let $X = \{e_1, e_2, e_1 + e_2\}$. We write $v \in V$ as $v = v_1 e_1 + v_2 e_2$.

We compute the generalized function

$$W(v_1, v_2) = \sum_{n_1 \neq 0, n_2 \neq 0, n_1 + n_2 \neq 0} \frac{e^{2i\pi(n_1 v_1 + n_2 v_2)}}{(2i\pi n_1)(2i\pi n_2)(2i\pi(n_1 + n_2))}.$$

Then $W$ is a locally $L^1$-function on $V$, periodic with respect to $\mathbb{Z}e_1 + \mathbb{Z}e_2$. To describe it, it is sufficient to write the formulae of $W(v_1, v_2)$ for $0 < v_1 < 1$ and $0 < v_2 < 1$, which we compute (for example using the relation $1/n_1n_2(n_1 + n_2) = 1/n_1(n_1 + n_2)^2 + 1/n_2(n_1 + n_2)^2$) as:

$$W(v_1, v_2) = \begin{cases} 
\frac{1}{4}(1 + v_1 - 2v_2)(v_1 - 1 + v_2)(2v_1 - v_2), & v_1 < v_2 \\
\frac{1}{4}(v_1 - 2v_2)(v_1 - 1 + v_2)(2v_1 - 1 - v_2), & v_1 > v_2.
\end{cases}$$

Thus we see that $W$ is a piecewise polynomial function.

Remark 3.3. If $X$ do not generate $V$, $W(X)$ is not locally $L^1$: take $X = \emptyset$, then, by Poisson formula, $W(\emptyset)$ is the delta distribution of the lattice $\Lambda$.

Definition 3.4. A subspace $s$ of $V$ generated by a subsequence of elements of $X$ is called $X$-admissible. We denote by $\mathcal{R}$ the set of $X$-admissible subspaces of $V$. We denote by $\mathcal{R}'$ the set of proper $X$-admissible subspaces.

The spaces $s = V$ and $s = \{0\}$ are among the admissible subspaces of $V$. The set $\mathcal{R}'$ consists of all admissible subspaces of $V$, except $s = V$.

Let $s$ be an admissible subspace of $V$. Let us consider the list $X \setminus s$, where we have removed from the list $X$ all elements belonging to $s$. The projection of the list $X \setminus s$ on $V/s$ will be denoted by $X/s$. The image of the lattice $\Lambda$ in $V/s$ is a lattice in $V/s$. If $X$ generates $V$, $X/s$ generates $V/s$. Using the projection $V \to V/s$, we identify the piecewise polynomial function $W(X/s)$ on $V/s$ to a piecewise polynomial function on $V$ constant along the affine spaces $v + s$.

Define $U_{\text{reg}}(X/s) = U_{\text{reg}}(X \setminus s) \cap s^\perp$. Thus $\Gamma_{\text{reg}}(X/s) := \Gamma \cap U_{\text{reg}}(X/s)$ is the set of elements $\gamma \in \Gamma$ such that:

$$\langle \gamma, s \rangle = 0 \quad \text{for all} \quad s \in s; \quad \langle \gamma, a \rangle \neq 0 \quad \text{for all} \quad a \in X \setminus s.$$

Identifying the dual space to $V/s$ to the space $s^\perp$, we see that the function $W(X/s)$ is the function on $V$ given by the series (convergent in the sense of generalized functions)

$$W(X/s)(v) := \sum_{\gamma \in \Gamma_{\text{reg}}(X/s)} \frac{e^{2i\pi \langle v, \gamma \rangle}}{\prod_{a \in X \setminus s} 2i\pi \langle a, \gamma \rangle}.$$

This function is periodic with respect to the lattice $\Lambda$, piecewise polynomial on $V$ (relative to $X, \Lambda$) and constant along $v + s$. 
If \( s = V \), the function \( W(X/s) \) is identically equal to 1, while if \( s = \{0\} \), we obtain back our series \( W(X) \).

4. A Formula

Let us now state our formula. We assume, as before, that \( X \) generates \( V \).

For each \( s \in \mathcal{R} \), we consider all possible decompositions of the list \( X \setminus s \) in disjoint lists \( I \sqcup J \). If \( f \) is a smooth function, the function

\[
F(v) = W(X/s)(v)(\partial_I \nabla_J f)(v)
\]

is a piecewise smooth function on \( V \). If \( Y \) is a subsequence of \( X \), the convolution \( B(Y) * Fdv \) is well defined and the result is a piecewise smooth distribution on \( V \) that we denote by \( B(Y) * c (W(X/s)\partial_I \nabla_J f) \).

**Theorem 4.1.** Let \( f \) be a smooth function on \( V \). We have

\[
B(X) * d f - B(X) * c f = \sum_{s \in \mathcal{R}} \sum_{I \subset X \setminus s} (-1)^{|I|} B((X \cap s) \sqcup I) * c (W(X/s)\partial_I \nabla_J f).
\]

In this formula \( J \) is the complement of the sequence \( I \) in \( X \setminus s \).

This equality holds in the space of piecewise (relative to \( (X, \Lambda) \)) smooth distributions on \( V \), relative to \( (X, \Lambda) \).

**Remark 4.2.** If \( f \) is a polynomial, the term \( B(X) * c f \) is a polynomial density and all terms of the difference formula are locally polynomial distributions on \( V \).

Before proceeding, let us comment on the proof. As in [3] (see also [5]), we use the Poisson formula to compute \( B(X) * d f \). Then we group the terms in the dual lattice \( \Gamma \) in strata according to the hyperplane arrangement \( \cup_{a \in X} \{a = 0\} \). We then use the Bernoulli series as primitives of the corresponding sums. This way, we introduce the needed derivatives of the function \( f \).

**Proof.** Let \( \mathcal{R} \) be the set of admissible subspaces of \( V \). We have the disjoint decomposition:

\[
U = \bigsqcup_{s \in \mathcal{R}} U_{\text{reg}}(X/s).
\]

Let \( \text{test} \) be a test function on \( V \). We compute

\[
S := \int_V (B(X) * d f)(v) \text{test}(v) = \sum_{\lambda \in \Lambda} f(\lambda) \int_V B(X)(v) \text{test}(\lambda + v).
\]

We apply Poisson formula to the compactly supported smooth function

\[
q(w) = f(w) \int_V B(X)(v) \text{test}(w + v)
\]

as our sum \( S \) is equal to \( \sum_{\lambda \in \Lambda} q(\lambda) \). We obtain

\[
S = \sum_{\gamma \in \Gamma} \int_V e^{2\pi i \langle w, \gamma \rangle} q(w) dw.
\]
Thus we obtain
\[ W(X/s) \colon= \int e^{2\pi \langle w, \gamma \rangle} = \partial_{X\setminus s} W(X/s)(w), \]
we obtain
\[ S = \sum_{s \in R} \int_{V} W(X/s)(w)(-1)^{|X\setminus s|} \partial_{X\setminus s} q(w) dw. \]
The function \( q(w) \) is product of the two smooth functions \( f(w) \) and \( \int_{V} B(X) test(w + v) \). By Leibniz rule,
\begin{align*}
S := \sum_{s} (-1)^{|X\setminus s|} \sum_{I \cup J = X\setminus s} \int_{V} \int_{V} W(X/s)(w) \partial_{I} f(w) B(X/J)(v) \partial_{J} test(w+v) dw.
\end{align*}
We first integrate in \( v \) and use the equation satisfied by the Box Spline
\[ \langle B(X), \partial_{b} h \rangle = -\langle B(X \setminus \{ b \}), \nabla_{-b} h \rangle. \]
Thus we obtain
\begin{align*}
S = \sum_{s} \sum_{I \cup J = X\setminus s} (-1)^{|I|} \int_{V} \int_{V} W(X/s)(w) \partial_{I} f(w) B(X/J)(v) (\nabla_{-J} test)(w+v) dw.
\end{align*}
Let us integrate in \( w \). We use the invariance of the integral by \( \nabla_{b} : \int_{V} (\nabla_{b} f_{1})(w)f_{2}(w) dw = \int_{V} f_{1}(w)(\nabla_{-b} f_{2})(w) dw \). As \( b \in J \) is in \( \Lambda \), and \( W(X/s)(w) \) is periodic,
\begin{align*}
S = \sum_{s \in R} \sum_{I \cup J = X\setminus s} (-1)^{|I|} \int_{V} \int_{V} B(X/J)(v) W(X/s)(w) \nabla_{J} \partial_{I} f(w) test(w+v) dw.
\end{align*}
Writing \( R = \{ V \} \sqcup R' \), we obtain the formula of the theorem.
\[ \square \]
On the space of polynomials, one has
\[ \nabla_{J} \partial_{I} f = \prod_{b \in J} \frac{1 - e^{-\partial_{b}}}{\partial_{b}} \partial_{X\setminus s} f \]
if \( I \sqcup J = X \setminus s \).
Recall that the space \( D(X) \) of Dahmen-Micchelli polynomials is the space of polynomials on \( V \) such that \( \partial_{Y} f = 0 \) for all long subsequences \( Y \). In particular, if \( s \) is a proper subspace, the sequence \( X \setminus s \) is a long subsequence. So if \( I, J \) are such that \( I \sqcup J = X \setminus s \) and \( f \in D(X) \), then \( \nabla_{J} \partial_{I} f = 0 \).
As a corollary of our formula, if \( p \in D(X) \), we see that \( B(X) *_{d} p = B(X) *_{c} p \). Let us state more precisely this result of Dahmen-Micchelli \[ 4 \] (see also \[ 3 \], chapter 16).
Corollary 4.3. If \( p \in D(X) \), then
\[
P(v) := B(X) *_d p = \sum_{\lambda} p(\lambda)B(X)(v - \lambda)
\]
is a polynomial function on \( V \), equal to \( (\prod_{a \in X} \frac{1-e^{-\partial a}}{\partial a})p = B(X) *_c p \).
In this formula, we have identified \( B(X), B(X) *_d p \), and \( B(X) *_c p \) to piecewise polynomial functions.

5. Vertices of the arrangement and semi-discrete convolutions.

We now give a twisted version of Theorem 4.1, where we twist \( f \) by an exponential function \( e^{2i\pi \langle G,v \rangle} \).

The set of characters on \( \Lambda \) is the torus \( T := U/\Gamma \). If \( g \in T \), we denote by \( g^a \) the corresponding character on \( \Lambda \). More precisely if \( g \) has representative \( G \in U \), then by definition \( g^a = e^{2i\pi \langle G, a \rangle} \). Define
\[
X(g) := \{ a \in X; g^a = 1 \}.
\]
If \( g \in T = U/\Gamma \) has representative \( G \in U \), we denote by \( g + \Gamma \) the set \( G + \Gamma \).

For \( a \in X \), introduce the operator
\[
(\nabla(a, g)f)(v) = f(v) - g^{-a}f(v - a).
\]
If \( Y \) is a subsequence of \( X \), define
\[
\nabla_Y^g = \prod_{a \in Y} \nabla(a, g).
\]

We introduce a subset \( R(g) \) of admissible subspaces, depending on \( g \).

Definition 5.1. The admissible space \( s \) is in \( R(g) \) if the space \( (g + \Gamma) \cap s^\perp \) is non empty

Remark that if \( G \) is not in \( \Gamma \), then \( V \) is not in the set \( R(g) \).

Remark 5.2. If \( s \in R(g) \), then all elements of \( X \cap s \) are in \( X(g) \). Thus \( R(g) \) is contained in the set of admissible spaces for \( X(g) \). However the converse does not hold: take \( V = \mathbb{R} \omega \ X = [2\omega], \Lambda = \mathbb{Z} \omega, \) and \( G = \frac{1}{2} \omega^* \). Then \( X(g) = X \), so that \( V \) is an admissible subspace for \( X(g) \). However, \( V \) is not in \( R(g) \).

If \( s \in R(g) \), take \( g_s \in (g + \Gamma) \cap s^\perp \). Then \( (g + \Gamma) \cap s^\perp \) is the translate by \( g_s \) of the lattice \( \Gamma \cap s^\perp \).

Define
\[
\Gamma_{reg}(X/s, g) = (g + \Gamma) \cap U_{reg}(X/s)^\perp.
\]
Thus \( \Gamma_{reg}(X/s, g) \) consists of elements \( \xi \in g + \Gamma \) such that
\[
\langle \xi, s \rangle = 0 \quad \text{for all} \quad s \in s; \quad \langle \xi, a \rangle \neq 0 \quad \text{for all} \quad a \in X \setminus s.
\]
The following series
\[
W(X/s, g)(v) = \sum_{\xi \in \Gamma_{\text{reg}}(X/s, g)} e^{2\pi i \langle v, \xi \rangle} \prod_{a \in X} e^{2\pi i \langle a, \xi \rangle}
\]
is well defined as a generalized function on \(V\).

The function \(W(X/s, g)(v)\) is not periodic with respect to \(\Lambda\). We have instead the covariance formula
\[
W(X/s, g)(v - \lambda) = g^{\lambda} W(X/s, g)(v).
\]

In the sense of generalized functions, we have
\[
\partial_{X \setminus s} W(X/s, g)(v) = \sum_{\xi \in \Gamma_{\text{reg}}(X/s, g)} e^{2\pi i \langle \xi, v \rangle}.
\]

We recall the following proposition [7] (see also [2],[1]).

**Proposition 5.3.** The generalized function \(W(X/s, g)\) is a piecewise polynomial (relative to \((X, \Lambda)\)) function on \(V\).

It is proven similarly by reduction to one variable.

**Example 5.4.** Let \(V = \mathbb{R} \omega\), \(\Lambda = \mathbb{Z} \omega\) and \(X_k := [\omega, \omega, \ldots, \omega]\), where \(\omega\) is repeated \(k\) times with \(k > 0\). Then \(\Gamma = \mathbb{Z} \omega^*\). Then if \(z\) is not an integer
\[
W(X_k, z\omega^*)(t\omega) = \sum_{n \in \mathbb{Z}} e^{2\pi i (n+z)t} \frac{1}{(2\pi (n+z))^k}.
\]

We have, for example, (see [2])
\[
W(X_1, z\omega^*)(t\omega) = e^{2\pi i [t]z} \frac{1}{1 - e^{-2\pi i z}}.
\]
\[
W(X_2, z\omega^*)(t\omega) = e^{2\pi i [t]z} \left( \frac{t - [t]}{1 - e^{-2\pi i z}} + \frac{1}{(1 - e^{-2\pi i z})(1 - e^{2\pi i z})} \right).
\]

Here \([t]\) is the integral part of \(t\). This function \([t]\) is a constant on each interval \([\ell, \ell + 1[\), and \(W(X_k, z\omega^*)\) is a locally polynomial function of \(t\).

**Theorem 5.5.** Let \(G \in U\), and \(g\) its image in \(U/\Gamma\). Let \(f(v) = e^{2\pi i \langle v, G \rangle} h(v)\), where \(h\) is a smooth function. Then
\[
B(X) *_d f = \sum_{s \in R(g)} \sum_{I \subset X \setminus s} (-1)^{|I|} B((X \cap s) \sqcup I) *_c (W(X/s, g)\partial_I \nabla_j^g h).
\]

In this formula, \(J\) is the complement of \(I\) in \(X \setminus s\).

**Remark 5.6.** If \(G \in \Gamma\), then \(B(X) *_d f = B(X) *_d h\), and the formula of the theorem above coincide with the formula of Theorem 4.1 for \(h\): the set \(R(g)\) coincide with the set \(R\), and the term corresponding to \(V\) in the formula of Theorem 5.5 is \(B(X) *_c h\).
Proof. We proceed in the same way than the proof of Theorem 3.1. Let test be a test function on V. We compute $S := \int_V (B(X) * f)(v) test(v)$ by Poisson formula. If

$$q(w) = h(w) \int_V B(X)(v) test(w + v),$$

we obtain

$$S = \sum_{\gamma \in \Gamma} \int_V e^{2i\pi \langle w, \gamma \rangle} e^{2i\pi \langle w, G \rangle} q(w) dw.$$

Thus

$$S = \sum_{\xi \in (g + \Gamma)} \int_V e^{2i\pi \langle w, \xi \rangle} q(w) dw.$$

The set $g + \Gamma$ is a disjoint union over $s \in \mathcal{R}(g)$ of the sets $\Gamma_{\text{reg}}(X/s, g) = (g + \Gamma) \cap U_{\text{reg}}(X/s)$, so

$$S = \sum_{s \in \mathcal{R}(g)} \int_V W(X/s, g)(w)(-1)^{|X| \setminus s} \partial_{X/s} q(w) dw.$$

Then, using Leibniz rule for $\partial_a$, and equation for the Box Spline, we obtain that $S$ is equal to

$$\sum_{s \in \mathcal{R}(g)} \sum_{I \cup J = X \setminus s} (-1)^{|I|} \int_V \int_V W(X/s, g)(w) \partial_I f(w) B(X\setminus J)(v)(\nabla_{-J} test)(w + v) dw.$$

Using the covariance formula (7) for $W(X \setminus s, g)$, we see that

$$\int_V W(X \setminus s, g)(w) f_1(w)(\nabla_{-b} f_2)(w) dw = \int_V W(X \setminus s, g)(w) (\nabla (b, g) f_1)(w) f_2(w) dw$$

and we obtain the formula of the theorem. \hfill \Box

Let us point out a corollary of this formula.

**Definition 5.7.** We say that a point $g \in U/\Gamma$ is a toric vertex of the arrangement $X$, if $X(g)$ generates $V$. We denote by $\mathcal{V}(X)$ the set of toric vertices of the arrangement $X$.

If $g$ is a vertex, there is a basis $\sigma$ of $V$ extracted from $X$ such that $g^a = 1$, for all $a \in \sigma$. We thus see that the set $\mathcal{V}(X)$ is finite.

If $X$ is unimodular, then $\mathcal{V}(X)$ is reduced to $g = 0$.

**Corollary 5.8.** (Dahmen-Micchelli)

Let $g \in \mathcal{V}(X)$ be a toric vertex of the arrangement $X$ and let $p \in D(X(g))$ be a polynomial in the Dahmen-Micchelli space for $X(g)$. Assume that $g \neq 0$. Let $f(\lambda) = g^\lambda p(\lambda)$. Then $B(X) * f = 0$. 

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Proof. We apply the formula of Theorem \[5.5\] with \( h = p \). As \( g \neq 0 \), all terms \( s \in \mathcal{R}(g) \) are proper subspaces of \( V \). Let us show that all the terms in our formula are 0. Indeed let \( I \sqcup J = X \setminus s \). Let \( I' = I \cap X(g) \) and \( J' = J \cap X(g) \). Then \( I' \sqcup J' = X(g) \setminus s \) is a long subset of \( X(g) \). As \( \nabla_I^{g} = \nabla_I' \), we see that \( \partial_I \nabla_I^{h} \) is already equal to 0.

\[ \square \]

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