SEMIREGULARITY MAPS AND DEFORMATIONS OF MODULES OVER LIE ALGEBROIDS

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Abstract. We determine a DG-Lie algebra controlling deformations of a locally free module over a Lie algebroid \( A \). Moreover, for every flat inclusion of Lie algebroids \( A \subseteq L \) we introduce semiregularity maps and prove that they annihilate obstructions, provided that the Leray spectral sequence of the pair \( (L, A) \) degenerates at \( E_1 \).

1. Introduction

Let \( X \) be a separated scheme of finite type over a field \( K \) of characteristic 0 and let \( E \) be a locally free sheaf on \( X \). Following Buchweitz and Flenner [5], the semiregularity maps of \( E \) are defined as

\[
\tau_k : \text{Ext}^2_X(E, E) \to H^{2+k}(\Omega^k_X), \quad \tau_k(x) = \frac{1}{k!} \text{Tr}(At(E)^k x), \quad k \geq 0,
\]

where \( At(E) \in \text{Ext}^1_X(E, E \otimes \Omega^1_X) \) is the Atiyah class of \( E \).

After [2, 5, 24] it is known that these semiregularity maps annihilate obstructions to deformations, provided that the Hodge to de Rham spectral sequence of \( X \) degenerates at \( E_1 \).

More generally, writing \( \Omega^k_X \leq \) for the algebraic de Rham complex truncated in degree \( \leq k \), it is known that the composition of \( \tau_k \) with the natural map \( H^{2+k}(\Omega^k_X) \to H^{2+2k}(\Omega^k_X) \) annihilates obstructions, regardless of degeneration properties of the aforementioned spectral sequence.

The main goal of this paper is to extend these results to locally free modules over a Lie algebroid \( A \) on \( X \), see Definition 3.1 below. By definition, a locally free \( A \)-module is a pair \((E, \nabla)\), where \( E \) is a locally free \( O_X \)-module, and \( \nabla : A \to \text{Hom}_K(E, E), \quad l \mapsto \nabla_l \), is an \( O_X \)-linear map such that:

1. \( \nabla \) is an \( A \)-connection; by definition, this means that \( \nabla_l(f e) = a(l)(f)e + f\nabla_l(e) \) for \( l \in A, \ f \in O_X \) and \( e \in E \), where \( a : A \to \Theta_X \) is the anchor map;
2. the \( A \)-connection \( \nabla \) is flat, i.e., its curvature \( \nabla^2(l, m) = [\nabla_l, \nabla_m] - \nabla_{[l,m]} \) vanishes identically.

When \( A = \Theta_X \) with anchor map the identity, then the notion of \( A \)-connection reduces to the usual definition of analytic connection.

Recall also that the Atiyah class of a locally free sheaf can be defined as the obstruction to the existence of an analytic connection. In other words, the Atiyah class of \( E \) can be defined as the obstruction to the lifting of the (unique) 0-connection on \( E \) to a \( \Theta \)-connection; in view of the generalisation considered in this paper we also write \( At(E) = At_{\Theta/0}(E) \).

By a straightforward generalisation, we can replace \( \Theta \) with \( A \) and define \( At_{A/0}(E) \) as the obstruction to the existence of an \( A \)-connection on \( E \); however, this generalisation does not lead to anything new from the point of view of semiregularity maps and deformation theory.

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Instead, we are here interested in the definition of a class \( \text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \) in the following situation:

1. \( \mathcal{A} \subset \mathcal{L} \) is an inclusion of Lie algebroids such that the quotient sheaf \( \mathcal{L}/\mathcal{A} \) is locally free;
2. \((\mathcal{E}, \nabla)\) is a locally free \( \mathcal{A} \)-module.

In the above situation the quotient sheaf \( \mathcal{L}/\mathcal{A} \) carries a natural structure of \( \mathcal{A} \)-module given by the Bott connection \( \nabla^B : \mathcal{A} \to \mathcal{E} \mathrm{nd}_k(\mathcal{L}/\mathcal{A}, \mathcal{E}/\mathcal{A}), \nabla^B_a(x) = [a, x] \mod \mathcal{A} \). Thus, for every \( r \geq 0 \), the sheaf \( \mathcal{M}_r := \bigwedge^r(\mathcal{L}/\mathcal{A})^\vee \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \) carries a natural structure of \( \mathcal{A} \)-module.

Denoting by \( \mathbb{H}^*(\mathcal{A}; \mathcal{M}_r) \) the Lie algebroid cohomology of \( \mathcal{A} \) with coefficients in \( \mathcal{M}_r \) (see Definition 3.11), in this paper we prove in particular that:

1. \( \mathbb{H}^{1}(\mathcal{A}; \mathcal{M}_0) \) is the space of first order deformations of \( \mathcal{E} \) as an \( \mathcal{A} \)-module;
2. \( \mathbb{H}^{2}(\mathcal{A}; \mathcal{M}_0) \) is a complete obstruction space for deformations of \( \mathcal{E} \) as an \( \mathcal{A} \)-module;
3. the Atiyah class \( \text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \in \mathbb{H}^{1}(\mathcal{A}; \mathcal{M}_1) \) is properly defined.

The first two items above are proved by showing that the DG-Lie algebra of derived sections of the sheaf of DG-Lie algebras \( \Omega^*(\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \) controls deformations of \( \mathcal{E} \) as an \( \mathcal{A} \)-module, where \( \Omega^*(\mathcal{A}) \) is the de Rham DG-algebra of \( \mathcal{A} \). The Atiyah class \( \text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \) is the primary obstruction to the extension of \( \nabla \) to a flat \( \mathcal{L} \)-connection. More precisely, \( \text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \) is the obstruction to the extension of \( \nabla \) to a flat \( \mathcal{L} \)-connection \( \nabla' : \mathcal{L} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \) such that \( [\nabla'_l, \nabla'_a] = \nabla'_{[l,a]} \) for every \( l \in \mathcal{L} \) and \( a \in \mathcal{A} \), cf. [6].

By analogy with the classical case, we define the semiregularity maps

\[
\tau_k : \mathbb{H}^{2}(\mathcal{A}; \mathcal{M}_0) \to \mathbb{H}^{2+k}(\mathcal{A}; \bigwedge^k(\mathcal{L}/\mathcal{A})^\vee), \quad \tau_k(x) = \frac{1}{k!} \text{Tr}(\text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),
\]

and we use the main result of [2] in order to prove that every \( \tau_k \) annihilates obstructions, provided that the Leray spectral sequence (Definition 5.2) of the pair \((\mathcal{L}, \mathcal{A})\) degenerates at \( E_1 \).

1.1. Notation. In this paper we work over a fixed field \( \mathbb{K} \) of characteristic 0; unless otherwise specified every (graded) vector space is intended over \( \mathbb{K} \).

Unless otherwise specified the term differential graded (DG) means graded over the integers and with differential of degree +1. The degree of a homogeneous element \( x \) in a graded vector space will be denoted \( |x| \). We adopt the Grothendieck–Verdier formalism for degree shifting: given a DG-vector space \((V = \oplus_n V^n, d_V)\) and an integer \( p \), we define the DG-vector space \((V[p], d_{V[p]})\) by setting \( V[p]^n = V^{n+p}, d_{V[p]} = (-1)^p d_V \).

2. Semiregularity maps for curved DG-algebras

We briefly review some definitions and results from [2]. By a graded algebra we intend a unitary graded associative algebra over a fixed field \( \mathbb{K} \) of characteristic 0. Every graded associative algebra is also a graded Lie algebra, with the bracket given by the graded commutator \( [a, b] = ab - (-1)^{|a||b|}ba \).

\textbf{Definition 2.1.} A curved DG-algebra is the datum \((A, d, \cdot, R)\) of a graded associative algebra \((A, \cdot)\) together with a degree one derivation \( d : A^* \to A^{*+1} \) and a degree two element \( R \in A^2 \), called curvature, such that
\[
d(R) = 0, \quad d^2(x) = [R, x] = R \cdot x - x \cdot R \quad \forall x \in A.
\]

For notational simplicity we shall write \((A, d, R)\) in place of \((A, d, \cdot, R)\) when the product \( \cdot \) is clear from the context. We denote by \([A, A] \subset A\) the linear span of all the graded commutators \([a, b] = ab - (-1)^{|a||b|}ba\). Notice that \([A, A]\) is a homogeneous Lie ideal and then \(A/[A, A]\) inherits a natural structure of DG-Lie algebra with trivial bracket.
Definition 2.2. Let \( A = (A, d, R) \) be a curved DG-algebra. A \textit{curved ideal} in \( A \) is homogeneous bilateral ideal \( I \subset A \) such that \( d(I) \subset I \) and \( R \in I \).

By a \textit{curved DG-pair} we mean the data \((A, I)\) of a curved DG-algebra \( A \) equipped with a curved ideal \( I \).

In particular, for every curved DG-pair \((A, I)\), the quotient \( A/I \) is a (non-curved) associative DG-algebra, and therefore also a DG-Lie algebra. Writing \( I^{(k)}, k \geq 0 \), for the \( k \)-th power of \( I \), we have that \( I^{(k)} \) is an associative bilateral ideal of \( A \) for every \( k \). The differential graded algebra \( \text{Gr}_I A = \bigoplus_{k \geq 0} \frac{I^{(k)}}{I^{(k+1)}} \) is non-curved, since \( d(I) \subset I \) and \( d^2(I) \subset I^{(2)} \), the derivation \( d \) factors through differentials

\[
d: \frac{I^{(k)}}{I^{(k+1)}} \to \frac{I^{(k)}}{I^{(k+1)}}, \quad d^2 = 0.
\]

Definition 2.3. Let \( A = (A, d, R) \) be a curved DG-algebra and \( I \subset A \) a curved ideal. The \textit{Atiyah cocycle} of the pair \((A, I)\) is the class of \( R \) in the DG-vector space \( \frac{I}{I^{(2)}} \). The \textit{Atiyah class} of the pair \((A, I)\) is the cohomology class of the Atiyah cocycle:

\[
\text{At}(A, I) = [R] \in H^2 \left( \frac{I}{I^{(2)}} \right).
\]

For every \( x \in I \) of degree 1, we can consider the twisted derivation \( d_x := d + [x, -] \) with curvature \( R_x = R + dx + \frac{1}{2}[x, x] \). Then \( I \) remains a curved ideal of the twisted curved DG-algebra \((A, d_x, R_x)\).

Lemma 2.4. The Atiyah class of the pair \((A, d_x, R_x, I)\) does not depend on the choice of \( x \in I \). The Atiyah class \( \text{At}(A, I) \) is trivial if and only if there exists \( x \in I \) of degree 1 such that \( R_x \) belongs to \( I^{(2)} \).

Proof. Firstly, notice that the differential on the algebra \( \text{Gr}_I A \) does not depend on the choice of \( x \in I \): since \( x \) belongs to \( I \) the adjoint operator \([x, -]\) sends \( I^{(k)} \) to \( I^{(k+1)} \), and so \( d = d_x := d + [x, -] \) in \( \frac{I^{(k)}}{I^{(k+1)}} \). In \( \frac{I}{I^{(2)}} \), one has that \([x, x] = 0\), so that

\[
R_x - R = R + dx + \frac{1}{2}[x, x] - R = dx,
\]

and the cohomology classes of \( R \) and \( R_x \) in \( H^*(\frac{I}{I^{(2)}}) \) coincide.

Let now \( x \in I \) be such that \( R_x = R + dx + \frac{1}{2}[x, x] \) belongs to \( I^{(2)} \). Then \( R + dx \) also belongs to \( I^{(2)} \) and \( R = -dx \) in \( \frac{I}{I^{(2)}} \), so that the Atiyah class is trivial. Conversely, let \( R = dx \) in \( \frac{I}{I^{(2)}} \), then \( R - dx \) belongs to \( I^{(2)} \), and so does \( R_x - R = R - dx + \frac{1}{2}[x, x] \).

Definition 2.5. A \textit{trace map} on a curved DG-algebra \((A, d, R)\) is the data of a complex of vector spaces \((C, \delta)\) and a morphism of graded vector spaces \( \text{Tr}: A \to C \) such that \( \text{Tr} \circ d = \delta \circ \text{Tr} \) and \( \text{Tr}(A, [A, A]) = 0 \).

Assume now there are given a curved DG-algebra \((A, d, R)\), a curved ideal \( I \) and a trace map \( \text{Tr}: A \to C \). Consider the decreasing filtration \( C_k = \text{Tr}(I^{(k)}) \) of subcomplexes of \( C \). By basic homological algebra, the spectral sequence associated to this filtration degenerates at \( E_1 \) if and only if for every \( k \) the inclusion \( C_k/C_{k+1} \subset C/C_{k+1} \) is injective in cohomology, see e.g. [22, Thm. C.6.6].

In the above situation we can define semiregularity maps

\[
\tau_k: H^2(A/I) \to H^{2+2k}(C_k/C_{k+1}), \quad \tau_k(x) = \frac{1}{k!} \text{Tr}(\text{At}(A, I)^k x).
\]
The composition of $\tau_k$ with the natural morphism $H^{2+2k}(C_k/C_{k+1}) \to H^{2+2k}(C/C_{k+1})$ is induced by the morphism of complexes

$$\sigma^k_1: \frac{A}{k} \to \frac{C}{C_{k+1}}[2k], \quad \sigma^k_1(x) = \frac{1}{k!} \text{Tr}(R^k x).$$

Considering $C/C_{k+1}$ as a DG-Lie algebra with trivial bracket, we can immediately see that $\sigma^k_1$ is a morphism of DG-Lie algebras for $k = 0$, while for $k > 0$ we have the following result.

**Theorem 2.6** ([2, Corollary 2.10]). In the above situation, the map $\sigma^k_1$ is the linear component of an $L_\infty$-morphism $\sigma_1: A/I \to C/C_{k+1}[2k]$. In particular, $\sigma^1_1$ annihilates obstructions for the deformation functor associated to the DG-Lie algebra $A/I$.

3. **Lie algebroid connections**

Throughout all this paper, $X$ will denote a smooth separated scheme of finite type over a field $K$ of characteristic 0.

We denote by $\Theta_X$ its tangent sheaf and by $\Omega^k_X$, $k \geq 0$, the sheaves of differential forms. For every pair of sheaves of $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we denote by $\text{Hom}_K(\mathcal{F}, \mathcal{G})$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ the sheaves of $K$-linear morphisms and $\mathcal{O}_X$-linear morphisms respectively. The $\mathcal{O}_X$-module structure on $\mathcal{G}$ induces an $\mathcal{O}_X$-module structure both on $\text{Hom}_K(\mathcal{F}, \mathcal{G})$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$. We also write $\text{End}_K(\mathcal{F})$ and $\text{End}_{\mathcal{O}_X}(\mathcal{F})$ for $\text{Hom}_K(\mathcal{F}, \mathcal{F})$ and $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$ respectively.

Unless otherwise specified we write $\otimes$ for the tensor product over $\mathcal{O}_X$, in particular for two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we have $\mathcal{F} \otimes \mathcal{G} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$.

**Definition 3.1.** A Lie algebroid over $X$ is the data of $(\mathcal{L}, [-, -], a)$ where:

- $\mathcal{L}$ is a locally free coherent sheaf of $\mathcal{O}_X$-modules;
- $[-, -]$ is a $K$-linear Lie bracket on $\mathcal{L}$;
- $a: \mathcal{L} \to \Theta_X$ is a morphism of sheaves of $\mathcal{O}_X$-modules, called the anchor map, commuting with the brackets;
- finally, we require the Leibniz rule to hold

$$[l, fm] = a(l)(f)m + f[l, m], \quad \forall l, m \in \mathcal{L}, f \in \mathcal{O}_X.$$

**Example 3.2.** The trivial sheaf $\mathcal{L} = 0$ and the tangent sheaf $\mathcal{L} = \Theta_X$, with anchor map equal to the identity, are Lie algebroids. A Lie algebroid over Spec $K$ is exactly a Lie algebra over the field $K$. Every sheaf of Lie algebras with $\mathcal{O}_X$-linear bracket can be considered as a Lie algebroid over $X$ with trivial anchor map.

**Example 3.3** (see [16] for details). Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module, then the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol has a natural structure of Lie algebroid. Since $\Theta_X$ is the sheaf of $K$-linear derivations of $\mathcal{O}_X$, we can introduce the sheaf

$$P(\Theta_X, \mathcal{E}) = \{(\theta, \phi) \in \Theta_X \times \text{End}_K(\mathcal{E}) \mid \phi(fe) = f\phi(e) + \theta(f)e, \quad f \in \mathcal{O}_X, \quad e \in \mathcal{E}\}.$$

Denoting by $a: P(\Theta_X, \mathcal{E}) \to \Theta_X$ the projection on the first factor, we have an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \text{End}_{\mathcal{O}_X}(\mathcal{E}) \to P(\Theta_X, \mathcal{E}) \xrightarrow{a} \Theta_X \to 0$$

and it is immediate to check that $P(\Theta_X, \mathcal{E})$ is a Lie algebroid with anchor map $a$. Moreover, the map $P(\Theta_X, \mathcal{E}) \to \text{End}_K(\mathcal{E})$, $(\theta, \phi) \mapsto \phi$, is injective and its image is the sheaf of first order differential operators on $\mathcal{E}$ with principal symbol.

The de Rham algebra of $\mathcal{L}$ is defined as the sheaf of commutative graded algebras

$$\Omega^*(\mathcal{L}) = \bigoplus_{k \geq 0} \Omega^k(\mathcal{L}), \quad \Omega^k(\mathcal{L}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{L}[1] \otimes \mathcal{L}, \mathcal{O}_X),$$

equipped with the convolution product. Notice that $\mathcal{L}[1]$ is just $\mathcal{L}$ considered as a graded sheaf concentrated in degree $-1$, hence $\Omega^*(\mathcal{L})$ is a locally free graded sheaf with $\Omega^k(\mathcal{L})$ in degree $k$. 


By definition the convolution product is the dual of the coproduct $\Delta$ on the graded symmetric algebra $S(L[1]) = \bigoplus_k L[1]^\otimes k$, defined by
\[
\Delta(l_1, \ldots, l_n) = \sum_{a=0}^{n} \sum_{\sigma \in S(a,n-a)} \epsilon(\sigma)(l_{\sigma(1)}, \ldots, l_{\sigma(a)}) \otimes (l_{\sigma(a+1)}, \ldots, l_{\sigma(n)}),
\]
where $\epsilon(\sigma)$ is the Koszul sign and $S(a, n-a)$ is the subset of unshuffles. More concretely, for $\omega \in \Omega^k(L)$ and $\eta \in \Omega^l(L)$ we have
\[
(\omega \eta)(l_1, \ldots, l_{k+l}) = \sum_{\sigma \in S(k,l)} (-1)^{\epsilon(\sigma)} \omega(l_{\sigma(1)}, \ldots, l_{\sigma(k)}) \eta(l_{\sigma(k+1)}, \ldots, l_{\sigma(k+l)}).
\]
Notice that the contraction product
\[
\mathcal{L} \times \Omega^{k+1}(\mathcal{L}) \rightarrow \Omega^k(\mathcal{L}), \quad (l \cdot \omega)(l_1, \ldots, l_k) = \omega(l, l_1, \ldots, l_k),
\]
is $\mathcal{O}_X$-bilinear and satisfies the Koszul identity $l \cdot (\omega \eta) = (l \cdot \omega) \eta + (-1)^{|\omega|} \omega (l \cdot \eta)$.

More generally, if $C^*$ is a sheaf of graded associative $\mathcal{O}_X$-algebras, the same holds for
\[
\Omega^*(\mathcal{L}, C^*) = \Omega^*(\mathcal{L}) \otimes C^* = \bigoplus_{k \geq 0} \text{Hom}_{\mathcal{O}_X}(\mathcal{L}[1]^\otimes k, C^*).
\]

The de Rham differential of $\mathcal{L}$, denoted by $d_\mathcal{L} : \Omega^k(\mathcal{L}) \rightarrow \Omega^{k+1}(\mathcal{L})$, is defined by the formula (see e.g. [20]):
\[
d_\mathcal{L}(\omega)(l_0, \ldots, l_k) = \sum_{i=0}^{n} (-1)^i a(l_i)(\omega(l_0, \ldots, \hat{l}_i, \ldots, l_k)) + \sum_{i<j} (-1)^{i+j} \omega([l_i, l_j], l_0, \ldots, \hat{l}_i, \ldots, \hat{l}_j, \ldots, l_k).
\]

In particular for $\omega \in \Omega^0(\mathcal{L}) = \mathcal{O}_X$ we have $l \cdot d_\mathcal{L}(\omega) = d_\mathcal{L}(\omega)(l) = a(l)(\omega)$, for every $l \in \mathcal{L}$.

By definition $\Omega^k(\Theta_X)[k] = \mathcal{O}^k_X$ is the sheaf of $k$-differential forms on $X$ and the global formula for the exterior derivative implies that $d_\Theta$ is the usual de Rham differential.

For every sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we denote $\Omega^*(\mathcal{L}, \mathcal{F}) = \Omega^*(\mathcal{L}) \otimes \mathcal{F}$ and by
\[
\Omega^*(\mathcal{L}) \times \Omega^*(\mathcal{L}, \mathcal{F}) \rightarrow \Omega^*(\mathcal{L}, \mathcal{F}) : \quad \eta \cdot \left( \sum_i \mu_i \otimes e_i \right) = \sum_i \eta \mu_i \otimes e_i, \quad \mu_i \in \Omega^*(\mathcal{L}), \quad e_i \in \mathcal{F},
\]
\[
\mathcal{L} \times \Omega^*(\mathcal{L}, \mathcal{F}) \rightarrow \Omega^*(\mathcal{L}, \mathcal{F}) : \quad l \cdot \left( \sum_i \mu_i \otimes e_i \right) = \sum_i l \cdot \mu_i \otimes e_i, \quad \mu_i \in \Omega^*(\mathcal{L}), \quad e_i \in \mathcal{F}.
\]

**Definition 3.4.** Given a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, an $\mathcal{L}$-connection $\nabla$ on $\mathcal{F}$ is a $\mathbb{K}$-linear morphism of graded sheaves of degree 1
\[
\nabla : \mathcal{F} \rightarrow \Omega^1(\mathcal{L}, \mathcal{F}) = \Omega^1(\mathcal{L}) \otimes \mathcal{F},
\]
such that
\[
\nabla(f e) = d_\mathcal{L}(f) \cdot e + f \nabla(e), \quad \forall f \in \mathcal{O}_X, \quad e \in \mathcal{F}.
\]

As in the usual case, every $\mathcal{L}$-connection $\nabla$ admits a unique extension to $\mathbb{K}$-linear morphism of graded sheaves of $\mathcal{O}_X$-modules of degree 1
\[
\nabla : \Omega^*(\mathcal{L}, \mathcal{F}) \rightarrow \Omega^*(\mathcal{L}, \mathcal{F})
\]
such that
\[
\nabla(f e) = d_\mathcal{L}(f) \cdot e + (-1)^{|f|} f \nabla(e), \quad \forall f \in \Omega^*(\mathcal{L}), \quad e \in \Omega^*(\mathcal{L}, \mathcal{F}),
\]
and the connection is called flat if $\nabla^2 = 0$. 
Remark 3.5. Since the contraction product \( \omega : \mathcal{L} \times \Omega^1(\mathcal{L}) \to \mathcal{O}_X \) is nondegenerate, every \( \mathbb{K} \)-linear morphism of sheaves \( \nabla : \mathcal{F} \to \Omega^1(\mathcal{L}, \mathcal{F}) \) is completely determined by the morphism of \( \mathcal{O}_X \)-modules
\[
\mathcal{L} \to \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \quad l \mapsto \nabla_l : \quad \nabla_l(\xi) = \xi \cdot l, \quad \xi \in \mathcal{F}.
\]
It is straightforward to verify that \( \nabla \) is a connection if and only if
\[
\nabla_l(f \xi) = a(l)(f) \xi + f \nabla_l(\xi), \quad \forall f \in \mathcal{O}_X, \ l \in \mathcal{L}, \ \xi \in \mathcal{F}.
\]
A simple computation shows that the curvature is given by the formula
\[
\nabla^2(\xi_1, \xi_2) = \nabla_\xi_1(\nabla_\xi_2) - \nabla_{\xi_2}(\nabla_\xi_1) - \nabla_{[\xi_1, \xi_2]}, \quad \forall \xi_1, \xi_2 \in \mathcal{F}.
\]

For instance, if \( \mathcal{F} \) is locally free and \( \mathcal{L} = \mathcal{End}_{\mathcal{O}_X}(\mathcal{F}) \) (with trivial anchor map), then the natural inclusion \( \mathcal{L} \to \mathcal{End}_{\mathcal{O}_X}(\mathcal{F}) \) is a flat connection.

Since \( \mathcal{L} \) is locally free we have natural isomorphisms
\[
\Omega^* (\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) \cong \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \Omega^*(\mathcal{L}, \mathcal{F})) \cong \Omega^*(\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})),
\]
and, therefore, a natural identification of \( \Omega^*(\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) \) with the subset of morphisms of graded sheaves \( f : \Omega^*(\mathcal{L}, \mathcal{F}) \to \Omega^*(\mathcal{L}, \mathcal{F}) \) such that \( f(\alpha \cdot \beta) = (-1)^{|f| |\alpha|} \alpha \cdot f(\beta) \) for every \( \alpha \in \Omega^* \mathcal{L}, \beta \in \Omega^* \mathcal{L} \).

The following lemma is a completely straightforward generalisation of well known facts about connections and curvature.

Lemma 3.6. Let \( \mathcal{L} \to \Omega^* (\mathcal{L}, \mathcal{F}) \) be an \( \mathcal{L} \)-connection, then \( \nabla^2 \in \Omega^2(\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) \) and \( [\nabla, f] \in \Omega^*(\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) \) for every \( f \in \Omega^* \mathcal{L} \). 

In particular, \( \Omega^*(\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})), d = [\nabla, -], \nabla^2 \) is a properly defined sheaf of curved DG-algebras over \( X \).

If in addition \( \mathcal{F} \) admits a locally free resolution, then the trace map \( \text{Tr} : \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to \mathcal{O}_X \), which is a morphism of sheaves of Lie algebras, is properly defined. By an analogous calculation to that of [18, Lemma 2.6], its extension
\[
(3.1) \quad \text{Tr} : \Omega^*(\mathcal{L}, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})) \to \Omega^*(\mathcal{L}), \quad \text{Tr}(\omega \cdot f) = \omega \cdot \text{Tr}(f), \quad \omega \in \Omega^*(\mathcal{L}), \ f \in \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}),
\]
is a trace map in the sense of Definition 2.5.

Definition 3.7. An \( \mathcal{L} \)-module is a pair \((\mathcal{F}, \nabla)\) consisting of a sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{F} \) and a flat \( \mathcal{L} \)-connection \( \nabla \) on \( \mathcal{F} \). An \( \mathcal{L} \)-module \((\mathcal{F}, \nabla)\) is said to be coherent (resp.: torsion free, locally free) if \( \mathcal{F} \) is coherent (resp.: torsion free, locally free) as an \( \mathcal{O}_X \)-module.

Example 3.8. Every \( \mathcal{O}_X \)-module has a unique structure of module over the trivial Lie algebroid \( \mathcal{L} = 0 \).

Example 3.9. For every Lie algebroid \( \mathcal{L} \), the pair \((\mathcal{O}_X, d_{\mathcal{L}})\) is an \( \mathcal{L} \)-module. More generally every choice of a basis on a free \( \mathcal{O}_X \)-module gives an \( \mathcal{L} \)-module structure.

Every \( \mathcal{L} \)-connection \( \nabla \) on a locally free \( \mathcal{O}_X \)-module \( \mathcal{F} \) naturally induces \( \mathcal{L} \)-connections on the associated sheaves \( \mathcal{F}^\vee, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \mathcal{F}^{\wedge k} \) etc.. If \( \mathcal{F} \) is an \( \mathcal{L} \)-module, then also \( \mathcal{F}^\vee, \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}), \mathcal{F}^{\wedge k} \) etc. are \( \mathcal{L} \)-modules in a natural way.

Example 3.10. Let \((X, \pi)\) be a smooth Poisson variety, and denote by \( \{-, -\} \) the Poisson bracket on the sheaf of functions \( \mathcal{O}_X \). The cotangent sheaf \( \Omega^1_X \) of holomorphic differential 1-forms on \( X \) has an induced structure of holomorphic Lie algebroid with the anchor \( a(df) := \{f, -\} \) and the bracket \([df, dg] := d\{f, g\}\) for all \( f, g \in \mathcal{O}_X \) (this defines \( a \) and \( \{-, -\} \) completely since \( \Omega^1_X \) is generated by exact forms as an \( \mathcal{O}_X \)-module), see e.g. [11] for more details. An \( \Omega^1_X \)-module is the same as a coherent sheaf \( \mathcal{E} \) together with a sheaf of Poisson modules structure on the sections of \( \mathcal{E} \). Namely, continuing to denote by \( \{-, -\} \) the Poisson bracket on \( \mathcal{E} \), the associated connection is defined by
\[
\nabla : \Omega^1_X \to \mathcal{End}_{\mathcal{O}_X}(\mathcal{E}), \quad df \mapsto \nabla df, \quad \nabla df e := \{f, e\} \quad \forall f \in \mathcal{O}_X, \ e \in \mathcal{E}.
\]
The fact that $\nabla$ is an $\Omega^1_X$-connection on $\mathcal{E}$ is equivalent to the Poisson identities
$$\{f, ge\} = \{f, g\} e + g\{f, e\}, \quad \{fg, e\} = f\{g, e\} + g\{f, e\},$$
while the flatness of $\nabla$ is equivalent to the Jacobi identity
$$\{\{f, g\}, e\} = \{f, \{g, e\}\} - \{g, \{f, e\}\}.$$

**Definition 3.11.** Let $\mathcal{L}$ be a Lie algebroid over $X$. The hypercohomology of the complex $(\Omega^*(\mathcal{L}), d_\mathcal{L})$ is called the *Lie algebroid cohomology of $\mathcal{L}$*, and it is denoted by $\mathbb{H}^*(\mathcal{L})$.

For an $\mathcal{L}$-module $(\mathcal{F}, \nabla)$ the complex $(\Omega^*(\mathcal{L}, \mathcal{F}), \nabla)$ is called the *standard complex* of $(\mathcal{F}, \nabla)$ and its hypercohomology, denoted by $\mathbb{H}^*(\mathcal{L}; \mathcal{F})$, is called the *Lie algebroid cohomology of $\mathcal{L}$ with coefficients in $\mathcal{F}$*.

Notice that $\mathbb{H}^*(\mathcal{L}) = \mathbb{H}^*(\mathcal{L}; \mathcal{O}_X)$, where $\mathcal{O}_X$ carries the $\mathcal{L}$-module structure of Example 3.9.

The notion of standard complex is borrowed from [20], while for Lie algebroid cohomology we follow the notation of [1, 4].

**Example 3.12.** The Lie algebroid cohomology of the tangent sheaf $\Theta_X$ is the de Rham cohomology of $X$. The Lie algebroid cohomology of a Lie algebroid $\mathfrak{g}$ over Spec $\mathbb{K}$ is the Chevalley–Eilenberg cohomology of the Lie algebra $\mathfrak{g}$.

4. **Infinitesimal deformations of locally free $\mathcal{L}$-modules**

In this section we describe a DG-Lie algebra controlling the infinitesimal deformations of a locally free $\mathcal{L}$-module. In order to do so, we give a brief review of the Thom–Whitney totalisation.

Let $\mathcal{L}$ be a Lie algebroid over $X$ and let $(\mathcal{E}, \nabla)$ be an $\mathcal{L}$-module, with $\mathcal{E}$ locally free as an $\mathcal{O}_X$-module. Let $B$ be an Artin local $\mathbb{K}$-algebra with residue field $\mathbb{K}$. We denote by $X_B = X \times \text{Spec}(B)$, by $p_X : X \times \text{Spec}(B) \to X$ the projection onto the first factor, and by $i_X : X \to X \times \text{Spec}(B)$ the inclusion induced by $B \to B/\mathfrak{m}_B = \mathbb{K}$. We notice that the pull-back sheaf $p_X^*\mathcal{L} = \mathcal{L} \otimes_{\mathbb{K}} B$ has a natural structure of Lie algebroid over $X_B$, with the Lie bracket extending $B$-bilinearly the one on $\mathcal{L}$. Moreover, it is easy to check that a $p_X^*\mathcal{L}$-module $\mathcal{F}$ on $X_B$ restricts to an $\mathcal{L}$-module $i_X^*\mathcal{F}$ on the central fibre $X$.

**Definition 4.1.** A deformation of the $\mathcal{L}$-module $(\mathcal{E}, \nabla)$ over Spec($B$) consists of the data of a deformation $\mathcal{E}_B$ of $\mathcal{E}$ over $X_B$ and a $p_X^*\mathcal{L}$-module structure
$$\nabla_B : \mathcal{E}_B \to \Omega^1(p_X^*\mathcal{L}; \mathcal{E}_B) = \Omega^1(\mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{E}_B$$
such that the restriction $i_X^*\mathcal{E}_B$ to $X$, with the naturally induced $\mathcal{L}$-module structure, coincides with $(\mathcal{E}, \nabla)$. An isomorphism of deformations $(\mathcal{E}_B, \nabla_B) \to (\mathcal{E}_B', \nabla_B')$ is an isomorphism of deformations of sheaves $\phi : \mathcal{E}_B \to \mathcal{E}_B'$ such that $\phi\nabla_B = \nabla_B'\phi$.

We want to describe a DG-Lie algebra controlling the infinitesimal deformations of $(\mathcal{E}, \nabla)$. To this end we first review the definition and some of the main properties of the Thom–Whitney totalisation functor Tot; for more details see e.g. [9, 10, 15, 22]. The Thom–Whitney totalisation is a functor from the category of semicosimplicial DG-vector spaces to the category of DG-vector spaces. For every $n \geq 0$ consider
$$A_n = \mathbb{K}[t_0, \ldots, t_n, dt_0, \ldots, dt_n] \quad \frac{1}{(1 - \sum_i t_i, \sum_i dt_i)}$$
the commutative differential graded algebra of polynomial differential forms on the affine standard $n$-simplex, and the maps
$$\delta^k_\ell : A_n \to A_{n-1}, \quad 0 \leq k \leq n \quad \delta^k_\ell(t_i) = \begin{cases} t_i & i < k \\ 0 & i = k \\ t_{i-1} & i > k. \end{cases}$$

**Definition 4.2.** The Thom–Whitney totalisation of a semicosimplicial DG-vector space $V$
Example 4.3. Let \((\mathcal{E}^*, \delta)\) be a bounded below complex of quasi-coherent sheaves on \(X\), and let \(\mathcal{U} = \{U_i\}\) be an open affine cover of \(X\). Denote by \(U_{i_1 \cdots i_n} = U_{i_1} \cap \cdots \cap U_{i_n}\), and consider the semicosimplicial DG-vector space of Čech cochains:

\[
\mathcal{E}^*(\mathcal{U}) : \prod_{i} \mathcal{E}^*(U_i) \xrightarrow{\delta_0} \prod_{i,j} \mathcal{E}^*(U_{ij}) \xrightarrow{\delta_0} \prod_{i,j,k} \mathcal{E}^*(U_{ijk}) \xrightarrow{\delta_2} \cdots.
\]

According to Whitney integration theorem, there exists a natural quasi-isomorphism

\[
I : \text{Tot}(\mathcal{U}, \mathcal{E}^*) \to C^*(\mathcal{U}, \mathcal{E}^*)
\]

where \(C^*(\mathcal{U}, \mathcal{E}^*) = \oplus_i C^*(\mathcal{U}, \mathcal{E}^i)[-i]\) is the hypercomplex of Čech cochains (see [26] for the \(C^\infty\) version, [12, 19, 22, 23] for the algebraic version used here). Therefore the cohomology of \(\text{Tot}(\mathcal{U}, \mathcal{E}^*)\) is isomorphic to the hypercohomology of the complex of sheaves \(\mathcal{E}^*\) and then the quasi-isomorphism class of \(\text{Tot}(\mathcal{U}, \mathcal{E}^*)\) does not depend on the affine open cover, since \(H^i(\text{Tot}(\mathcal{U}, \mathcal{E}^*)) = \mathbb{H}^i(X, \mathcal{E}^*)\) and the map \(I\) commutes with refinements of affine covers.

For our later application it is important to point out that there exists a natural inclusion of DG-vector spaces \(\Gamma(X, \mathcal{E}^*) \to \text{Tot}(\mathcal{U}, \mathcal{E}^*)\) such that the restriction of \(I\) to \(\Gamma(X, \mathcal{E}^*)\) is the natural inclusion map

\[
i : \Gamma(X, \mathcal{E}^*) \to \prod_{i} \mathcal{E}^*(U_i), \quad i(s) = \{s|_{U_i}\}.
\]

In fact, \(\delta_0 i = \delta_1 i\), therefore

\[
\delta_{j_1} \delta_{j_2} \cdots \delta_{i_1} i = \delta_0^k i, \quad \text{for every } 0 \leq j_s \leq s,
\]

and this implies that

\[
i : \Gamma(X, \mathcal{E}^*) \to \text{Tot}(\mathcal{U}, \mathcal{E}^*), \quad i(a) = (1 \otimes i(a), 1 \otimes \delta_0 i(a), 1 \otimes \delta_1^2 i(a), \ldots)
\]

is a properly defined injective morphism of DG-vector spaces.

For later use we point out that for every quasi-coherent sheaf \(\mathcal{F}\) and every affine open cover \(\mathcal{U}\), the inclusion \(\Gamma(X, \mathcal{F}) \subset \text{Tot}(\mathcal{U}, \mathcal{F})\) induces an isomorphism \(\Gamma(X, \mathcal{F}) \cong H^0(\text{Tot}(\mathcal{U}, \mathcal{F}))\).
Returning to our initial situation of a locally free $\mathcal{L}$-module $(\mathcal{E}, \nabla)$, since the $\mathcal{L}$-connection $\nabla$ is flat, by Lemma 3.6 $(\Omega^*(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), d = [\nabla, -])$ is a sheaf of locally free DG-algebras, which gives rise to a sheaf of locally free DG-Lie algebras $(\Omega^*(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})), d = [\nabla, -], [-,-])$.

**Theorem 4.4.** In the above situation, for every affine open cover $\mathcal{U} = \{U_i\}$, the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$ controls the infinitesimal deformations of $(\mathcal{E}, \nabla)$. In particular $\mathbb{H}^1(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is the space of first order deformations and $\mathbb{H}^2(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is an obstruction space.

**Proof.** This result is probably well known to experts, at least in the case $\mathcal{L} = \Theta_X$, cf. [13, Thm. 6.8], and follows easily from Hinich’s theorem on descent of Deligne groupoids. According to [14], it is sufficient to check that locally the Deligne groupoid of $\Omega^*(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is equivalent to the groupoid of deformations of $(\mathcal{E}, \nabla)$.

In order to check this, it is not restrictive to assume $X$ affine. Given an Artin ring $B$ as above, up to isomorphism every deformation of $\mathcal{E}$ is trivial, i.e. $\mathcal{E}_B = \mathcal{E} \otimes_B B$ and $\Hom_{\mathcal{O}_{X_B}}(\mathcal{E}_B, \mathcal{E}_B) = \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \otimes_B B$. Denoting by $\nabla_0 : \mathcal{E}_B \to \Omega^1(p^*_X \mathcal{L}, \mathcal{E}_B) = \Omega^1(\mathcal{L}, \mathcal{E}) \otimes_B B$ the natural $B$-linear extension of $\nabla$, every deformation of $\nabla$ over $B$ is of the form $\nabla_0 + x$, with $x \in \Gamma(X, \Omega^1(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))) \otimes_B \mathfrak{m}_B$, and the flatness condition $(\nabla_0 + x)^2 = 0$ is exactly the Maurer–Cartan equation $dx + \frac{1}{2}[x, x] = 0$.

To conclude the proof we only need to show that two solutions of the Maurer–Cartan equation $x, y$ are gauge equivalent if and only if there exists an isomorphism of deformations $\phi : \mathcal{E}_B \to \mathcal{E}_B$ such that $\phi(\nabla_0 + x)\phi^{-1} = \nabla_0 + y$. Every $\phi$ as above is of the form $\phi = e^a$, with $a \in \Gamma(X, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \otimes_B \mathfrak{m}_B$, and then the condition $\phi(\nabla_0 + x)\phi^{-1} = \nabla_0 + y$ is equivalent to

$$ \nabla_0 + y = e^{[a, -]}(\nabla_0 + x) = \nabla_0 + x + \sum_{n=0}^{\infty} \frac{[a, -]^n}{(n+1)!} ([a, x] - da), $$

which is the same as $y = e^a \ast x$, where $\ast$ denotes the gauge action.

**Remark 4.5.** One can consider a different deformation problem, namely the deformation of pairs (bundle, $\mathcal{L}$-connection) without requiring the vanishing of the curvature. Then the same argument as above shows that this deformation problem is controlled by the DG-Lie algebra $\mathrm{Tot}(\mathcal{U}, \Omega^{\leq 1}(\mathcal{L}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})))$, while it is well known that $\mathrm{Tot}(\mathcal{U}, \Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ controls the deformations of $\mathcal{E}$ [9].

5. Lie pairs

**Definition 5.1.** A Lie pair $(\mathcal{L}, \mathcal{A})$ of Lie algebroids over $X$ is a pair consisting of a Lie algebroid $\mathcal{L}$ over $X$ and a Lie subalgebroid $\mathcal{A} \subset \mathcal{L}$ such that the quotient sheaf $\mathcal{L}/\mathcal{A}$ is locally free.

Let $(\mathcal{L}, \mathcal{A})$ be a Lie pair. Since $\mathcal{L}/\mathcal{A}$ is assumed locally free we have a surjective restriction map $\rho : \Omega^*(\mathcal{L}) \to \Omega^*(\mathcal{A})$, which is a morphism of sheaves of commutative differential graded algebras. The powers of its kernel give a finite decreasing filtration of differential graded ideal sheaves

$$ \Omega^*(\mathcal{L}) = \mathcal{G}^0 = \mathcal{G}^1 = \ker(\rho) \supset \cdots \mathcal{G}^r = (\ker(\rho))^{(r)} \supset \cdots. $$

If we forget the de Rham differential, we can immediately see that $\mathcal{G}^r_{\mathcal{L}}$ is the image of the morphism of graded $\mathcal{O}_X$-modules

$$ \bigwedge^p (\mathcal{L}/\mathcal{A})^\vee \otimes [\mathcal{L}, \mathcal{A}] \to \Omega^*(\mathcal{L}), $$

and we have natural isomorphisms of graded sheaves

$$ \mathcal{G}^r_{\mathcal{L}} \to \mathcal{G}^r_{\mathcal{L}+1} \mid [\mathcal{L}, \mathcal{A}] \cong \bigwedge^p (\mathcal{L}/\mathcal{A})^\vee \otimes \Omega^*(\mathcal{A}). $$. 

(5.1)
In particular, \( G^i_p \neq 0 \) only for pairs \((i, p)\) such that \( p \leq i \leq \text{rank } \mathcal{L} \) and \( p \leq \text{rank } \mathcal{L} - \text{rank } \mathcal{A} \).

For instance, whenever \( i = 2 \) we have \( G^2_0 = \Omega^2(\mathcal{L}), \ G^2_2 = 0 \),
\[
\begin{align*}
G^2_1 &= \{ \phi \in \Omega^2(\mathcal{L}) \mid \phi(a, b) = 0 \ \forall a, b \in \mathcal{A} \}, \\
G^2_2 &= \{ \phi \in \Omega^2(\mathcal{L}) \mid \phi(a, l) = 0 \ \forall a \in \mathcal{A}, l \in \mathcal{L} \}.
\end{align*}
\]

Recall that \( H^{**}(\mathcal{L}) = H^{**}(X, \Omega^{**}(\mathcal{L})) \) denotes the Lie algebroid cohomology of \( \mathcal{L} \), as in Definition 3.11.

**Definition 5.2.** In the above notation, the filtration \( \Omega^*\mathcal{L} = G^0_0 \supseteq G^0_1 \cdots \) is called the Leray filtration of the Lie pair \((\mathcal{L}, \mathcal{A})\). We shall call the associated spectral sequence in hypercohomology
\[
E_1^{p, q} = H^q(X, G^p_r / G^p_{r+1}[p]) \Rightarrow H^{p+q}(\mathcal{L})
\]
the Leray spectral sequence of the Lie pair \((\mathcal{L}, \mathcal{A})\).

The name Leray filtration is motivated by Example 5.4 below. Notice however that for the Lie pair \((\Theta_X, 0)\) the Leray filtration coincides with the Hodge filtration on differential forms.

Given an \( \mathcal{A}\)-module \((\mathcal{E}, \nabla)\), we can also define a filtration \( G^*_\mathcal{E}(\mathcal{E}) = G^*_\mathcal{E} \otimes \mathcal{E} \) of the graded sheaf \( \Omega^*(\mathcal{L}, \mathcal{E}) \); equivalently, \( G^*_\mathcal{E}(\mathcal{E}) \) may be defined as the image of the multiplication map
\[
G^*_\mathcal{E} \otimes \Omega^*(\mathcal{L}, \mathcal{E}) \to \Omega^*(\mathcal{L}, \mathcal{E}).
\]

If \( \nabla' \) is an \( \mathcal{L}\)-connection on \( \mathcal{E} \) extending \( \nabla \), then by Leibniz rule the filtration \( G^*_\mathcal{E}(\mathcal{E}) \) is preserved by \( \nabla' \) and we can immediately see that the maps induced on the quotients \( G^*_\mathcal{E}(\mathcal{E}) / G^*_{\mathcal{E}, r+1}(\mathcal{E}) \) are independent of \( \nabla' \) and square-zero operators. Notice also that the curvature of \( \nabla' \) belongs to \( G^2_{\mathcal{E}, \mathcal{L}}(\mathcal{E}, d_{\mathcal{X}}X(\mathcal{E})) \) if and only if \( [\nabla', \nabla'] = \nabla'_{[r, a]} \) for every \( r \in \mathcal{L} \) and \( a \in \mathcal{A} \).

Since \( \nabla \) always admits extensions locally (see Remark 7.3 below), for every \( r \) there is a properly defined structure of differential graded sheaf on \( G^*_\mathcal{E}(\mathcal{E}) / G^*_{\mathcal{E}, r+1}(\mathcal{E}) \).

It is interesting to point out that the groups \( E_1^{p, q} = H^q(X, G^p_r / G^p_{r+1}[p]) \), and more generally the hypercohomology groups of \( G^*_\mathcal{E}(\mathcal{E}) / G^*_{\mathcal{E}, r+1}(\mathcal{E}) \), are cohomology groups of \( \mathcal{E} \) with coefficients in suitable \( \mathcal{A}\)-modules. In fact, there is a canonical \( \mathcal{A}\)-module structure on the quotient sheaf \( \mathcal{L} / \mathcal{A} \) given by the Bott connection: denoting by \( \pi : \mathcal{L} \to \mathcal{L} / \mathcal{A} \) the projection, the connection is defined by the formula
\[
\nabla^B\pi(b) = \pi([a, b]), \quad \forall a \in \mathcal{A}, \ b \in \mathcal{L}.
\]

Therefore, there is a canonical \( \mathcal{A}\)-module structure on \( \Lambda^r(\mathcal{L} / \mathcal{A})^\vee \) for every \( r \).

**Lemma 5.3.** Let \((\mathcal{L}, \mathcal{A})\) be a Lie pair and let \( \mathcal{E} \) be an \( \mathcal{A}\)-module. Then for every \( r \geq 1 \), the differential graded sheaf \( \frac{G^*_\mathcal{E}(\mathcal{E})}{G^*_{\mathcal{E}, r+1}(\mathcal{E})}[r] \) is isomorphic to the standard complex of the \( \mathcal{A}\)-module \( \Lambda^r(\mathcal{L} / \mathcal{A}) \otimes \mathcal{E} \). In particular, the Leray spectral sequence of the pair \((\mathcal{L}, \mathcal{A})\) is
\[
E_1^{p, q} = H^q(\mathcal{A}; \Lambda^p(\mathcal{L} / \mathcal{A})^\vee).
\]

**Proof.** For every \( r \geq 1 \), consider the isomorphism of graded sheaves \( \varphi : \frac{G^*_\mathcal{E}(\mathcal{E})}{G^*_{\mathcal{E}, r+1}(\mathcal{E})}[r] \to \Lambda^r(\mathcal{L} / \mathcal{A})^\vee \otimes \Omega^*(\mathcal{A}) \) of (5.1). We begin by showing that this is an isomorphism of complexes, where the differential on the left is induced by \( d_{\mathcal{E}} \), and the differential on the right is given by the dual connection to the Bott connection.

Denote by \( \nabla^B \) the Bott connection on \( \mathcal{L} / \mathcal{A} \), and by \( \nabla^{B, \vee} \) the induced connection on \( \Lambda^r(\mathcal{L} / \mathcal{A})^\vee \) for every \( r \geq 0 \). We denote by \( a_{\mathcal{L}} \) and \( a_{\mathcal{A}} \) the anchor maps of \( \mathcal{L} \) and \( \mathcal{A} \) respectively. Finally, denote by \( j \) the inclusion \( j : (\frac{\mathcal{E}}{\mathcal{A}})^\vee [-1] \to \Omega^1(\mathcal{L}) \), and by \( \pi \) the projection \( \pi : \mathcal{L} \to \frac{\mathcal{E}}{\mathcal{A}} \), so that for \( m \in \mathcal{L} \) and \( \eta \in (\frac{\mathcal{E}}{\mathcal{A}})^\vee [-1] \) one has that \( m_{\mathcal{L}}j(\eta) = (j(\eta))(m) = \eta(\pi(m)) = \pi(m) \cdot \eta \).

For every \( \eta \in G^r / G^*_{r+1}[r] \), we prove that
\[
\varphi(d_{\mathcal{E}} \eta) = \nabla^{B, \vee} \varphi(\eta).
\]
Firstly, consider \( \omega \in \mathcal{G}^1_1/\mathcal{G}^2_1[1] \cong (\mathcal{L}/\mathcal{A})^\vee \otimes \Omega^*(\mathcal{A}) \) of degree zero, so that \( \omega \) belongs to \( \mathcal{G}^1_1/\mathcal{G}^2_1[1] = \mathcal{G}^1_1[1] \cong (\mathcal{L}/\mathcal{A})^\vee \). Then \( d\mathcal{L}\omega \) belongs to \( \mathcal{G}^2_1[1] \), but we consider its projection to \( \mathcal{G}^2_1[1] \cong (\mathcal{L}/\mathcal{A})^\vee \otimes \Omega^*(\mathcal{A}). \) Hence we calculate it on \( b \in \mathcal{A} \) and \( \pi(l) \in \mathcal{L}/\mathcal{A} \), obtaining

\[
d\mathcal{L}\omega(b, \pi(l)) = a\mathcal{L}(b)(j(\omega)(l)) - a\mathcal{L}(l)(j(\omega)(b)) - j(\omega)([b, l])
= a\mathcal{A}(b)(\omega(\pi(l))) - a\mathcal{L}(l)(\omega(\pi(b))) - \omega([b, l])
= a\mathcal{A}(b)(\omega(\pi(l))) - \omega(\pi([b, l])),
\]

since \( \pi(b) = 0 \). The connection \( \nabla^\mathcal{L}/\mathcal{A}_b \omega \) for \( \omega \in (\mathcal{L}/\mathcal{A})^\vee \), \( b \in \mathcal{A} \) and \( \pi(l) \in \mathcal{L}/\mathcal{A} \) is given by

\[
\pi(l)\nabla^\mathcal{L}/\mathcal{A}_b \omega = d\mathcal{L}(\pi(l)\omega)(b) - (\nabla^\mathcal{L}/\mathcal{A}_b \omega)(\pi(l)\omega) = a\mathcal{L}(b)(\pi(l)\omega) - (\pi([b, l]))\omega
= a\mathcal{A}(b)(\omega(\pi(l))) - \omega(\pi([b, l])),
\]

therefore \( d\mathcal{L}\omega = \nabla^\mathcal{L}/\mathcal{A}_b \omega \).

Consider now \( \eta \in \mathcal{G}^r_1/\mathcal{G}^{r+1}_1[1] \) of degree \( k - r \geq 0 \), which we can assume to be of the form

\[
\eta = \omega_1 \cdots \omega_k,
\]

with \( \omega_i \in \Omega^1(\mathcal{L}/\mathcal{A})[1] \) for \( i = 1, \ldots, r \) such that \( \rho(\omega_1) = \cdots = \rho(\omega_r) = 0\) (i.e., \( \omega_i \in (\mathcal{L}/\mathcal{A})^\vee \) for \( i = 1, \ldots, r \)) and \( \omega_j \in \Omega^1(\mathcal{L}) \) for \( j = r + 1, \ldots, k \) such that \( \rho(\omega_{r+1}), \ldots, \rho(\omega_k) \neq 0 \).

Then we have that

\[
\varphi: \mathcal{G}^r_1/\mathcal{G}^{r+1}_1[1] \to \bigwedge^r \left( \frac{\mathcal{L}}{\mathcal{A}} \right)^\vee \otimes \Omega^*(\mathcal{A}), \quad \varphi(\eta) = \omega_1 \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(\omega_k),
\]

and so

\[
\nabla^\mathcal{L}/\mathcal{A}_b \varphi(\eta) = \nabla^\mathcal{L}/\mathcal{A}_b (\omega_1 \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(\omega_k))
= \sum_{i=1}^r \omega_1 \cdots \nabla^\mathcal{L}/\mathcal{A}_b (\omega_i) \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(\omega_k)
+ \sum_{i=r+1}^k (-1)^{i-r} \omega_1 \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(d\mathcal{L}(\omega_i)) \cdots \rho(\omega_k)
= \sum_{i=1}^r \omega_1 \cdots \nabla^\mathcal{L}/\mathcal{A}_b (\omega_i) \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(\omega_k)
+ \sum_{i=r+1}^k (-1)^{i-r} \omega_1 \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(d\mathcal{L}(\omega_i)) \cdots \rho(\omega_k)
= \sum_{i=1}^r \omega_1 \cdots d\mathcal{L}(\omega_i) \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(\omega_k)
+ \sum_{i=r+1}^k (-1)^{i-r} \omega_1 \cdots \omega_r \otimes \rho(\omega_{r+1}) \cdots \rho(d\mathcal{L}(\omega_i)) \cdots \rho(\omega_k)
= (\text{id} \otimes \rho) \left( \sum_{i=1}^r \omega_1 \cdots d\mathcal{L}(\omega_i) \cdots \omega_k + \sum_{i=r+1}^k (-1)^{i-r} \omega_1 \cdots d\mathcal{L}(\omega_i) \cdots \omega_k \right)
= \varphi(d\mathcal{L}(\omega_1 \cdots \omega_k)) = \varphi(d\mathcal{L}(\eta)).
\]

For every \( r \geq 1 \), it follows by (5.1) and by the definition of \( \mathcal{G}^r_1(\mathcal{E}) \) that there is an isomorphism of graded sheaves

\[
\varphi \otimes \text{id}_{\mathcal{E}}: \frac{\mathcal{G}^r_1(\mathcal{E})}{\mathcal{G}^{r+1}_1(\mathcal{E})}[r] \to \bigwedge^r (\mathcal{L}/\mathcal{A})^\vee \otimes \Omega^*(\mathcal{A}) \otimes \mathcal{E}.
\]

Denote by \( \nabla \) the flat \( \mathcal{A} \)-connection on \( \mathcal{E} \), and by \( \nabla' \) a local extension of \( \nabla \) to an \( \mathcal{L} \)-connection on \( \mathcal{E} \), which is such that \( (\rho \otimes \text{id})\nabla' = \nabla \) and which induces a differential on \( \mathcal{G}^*_{\mathcal{E}}(\mathcal{E})/\mathcal{G}^{*+1}_{\mathcal{E}}(\mathcal{E})[r] \).
Take now $\eta \otimes e \in \mathcal{G}_r^*(\mathcal{E})[r] = (\mathcal{G}_r^* \otimes \mathcal{E})[r]$, then $\nabla'(\eta \otimes e) = d_{\mathcal{L}} \eta \otimes e + (-1)^{|\eta|} \eta \otimes \nabla'(e)$, and

$$(\varphi \otimes \text{Id}_{\mathcal{E}})(\nabla'(\eta \otimes e)) = \varphi(d_{\mathcal{L}} \eta) \otimes e + (-1)^{|\eta|} \varphi(\eta) \otimes (\varphi \otimes \text{Id}_{\mathcal{E}}) \nabla'(e)$$

$$= \nabla^{B,V}(\varphi(\eta)) \otimes e + (-1)^{|\eta|} \varphi(\eta) \otimes (\varphi \otimes \text{Id}_{\mathcal{E}}) \nabla'(e).$$

Since

$$(\nabla^{B,V} \otimes \nabla)((\varphi \otimes \text{Id}_{\mathcal{E}})(\eta \otimes e)) = (\nabla^{B,V} \otimes \nabla)(\varphi(\eta) \otimes e) = \nabla^{B,V}(\varphi(\eta)) \otimes e + (-1)^{|\eta|} \varphi(\eta) \otimes \nabla(e),$$

it remains only to show that $(\varphi \otimes \text{Id}_{\mathcal{E}}) \nabla'(e) = \nabla(e)$ for every $e \in \mathcal{E}$, which follows by the definition of $\varphi$ and by the fact that $(\varphi \otimes \text{Id}) \nabla' = \nabla$, since $\nabla'$ is a local extension of $\nabla$. \[\square\]

**Example 5.4.** Let $f: X \to Y$ be a smooth morphism of irreducible smooth schemes. Then a Lie pair on $X$ is given by $(\Theta_X, \Theta_f)$, where $\Theta_f = \text{Homo}_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ is the subsheaf of relative vector fields: since $f$ is smooth there exists an exact sequence of sheaves

$$0 \to \Theta_f \to \Theta_X \to f^*\Theta_Y \to 0.$$  

In this case $\Omega^*(\mathcal{L}) = \Omega^*_X$ is the usual de Rham complex of $X$, while $\Omega^*(\mathcal{A}) = \Omega^*_X{\downharpoonright}_{Y}$ is the relative de Rham complex and the filtration $\mathcal{G}_r^*$ is the algebraic analogue of the holomorphic Leray filtration, see [25, 17.2], [27, 2.16].

Since the relative de Rham differential is $f^{-1}\Omega_Y^*$-linear and $\mathcal{G}_r^*$ is the ideal sheaf generated by $f^{-1}\Omega_Y^*$, for every $r$ we have a natural isomorphism of differential graded sheaves

$$\frac{\mathcal{G}_r^*}{\mathcal{G}_{r+1}^*} \cong f^{-1}\Omega_Y^r \otimes_{f^{-1}\mathcal{O}_Y} \Omega^*_X{\downharpoonright}_{Y}$$

and therefore the first page of the Leray spectral sequence is

$$E_1^p = H^p(X, \mathcal{G}_r^*/\mathcal{G}_{r+1}^*) = H^p\left(X, f^{-1}\Omega_Y^p \otimes_{f^{-1}\mathcal{O}_Y} \Omega^*_X{\downharpoonright}_{Y}\right) = H^p\left(Y, \Omega_Y^p \otimes_{Rf^*\mathcal{O}_Y} Rf_*\Omega_X^*_Y\right).$$

It is an easy consequence of Deligne’s results on Hodge theory that if $X$ and $Y$ are complex projective manifolds, then the Leray spectral sequence of the Lie pair $(\Theta_X, \Theta_f)$ degenerates at $E_1$. In fact, by Hodge decomposition we have

$$Rf_*\Omega_X^*_Y = \oplus_q R^qf_*\Omega_X^*_Y[^{-q}] \cong \oplus_q \mathcal{O}_Y \otimes_{\mathcal{O}_X} R^qf_*\mathcal{C}[^{-q}],$$

and then $E_1^p = \oplus_q H^*(Y, \Omega_Y^p \otimes_{\mathcal{O}_X} R^qf_*\mathcal{C}[^{-q}]$. Since $R^qf_*\mathcal{C}$ is a local system with real structure and $Y$ is compact Kähler, according to [27, 2.11] (see also [13, 8.5]), the cohomology of $\Omega_Y^p \otimes_{\mathcal{O}_X} R^qf_*\mathcal{C}$ is a direct summand of the cohomology of $R^qf_*\mathcal{C}$. Since the (topological) Leray spectral sequence of $Rf_*\mathcal{C}$ degenerates at $E_2$, we have that $E_1^p$ is a direct summand of $H^*(Y, Rf_*\mathcal{C}) = H^*(X, \mathcal{O}_X^*)$.

For every locally free sheaf $\mathcal{E}$ on $Y$ its pull-back $f^*\mathcal{E} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{E}$ has a natural structure of $\Theta_f$-module with connection

$$\nabla_\eta(g \otimes e) = \eta(g) \otimes e.$$  

More generally, every $\Theta_f$-module can be interpreted, as in [3], as a locally free sheaf on $X$ which is endowed with a connection relative to $f$ that is flat.

6. **Reduced Atiyah classes**

For every Lie algebroid $\mathcal{L}$ and every $\mathcal{O}_X$-module $\mathcal{F}$ we define the sheaf of $\mathcal{O}_X$-modules

$$P(\mathcal{L}, \mathcal{F}) = \{(l, \phi) \in \mathcal{L} \times \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \mid \phi(f e) = f \phi(e) + a(l)(f) e, \ f \in \mathcal{O}_X, \ e \in \mathcal{F}\}.$$  

If $\mathcal{F}$ is coherent then also $P(\mathcal{L}, \mathcal{F})$ is coherent. This has been proved in [16, Prop. 5.1] in the case $\mathcal{L} = \Theta_X$, while for the general case it is sufficient to observe that $P(\mathcal{L}, \mathcal{F}) = P(\Theta_X, \mathcal{F}) \times_{\Theta_X} \mathcal{L}$. 

Denoting by $p: P(\mathcal{L}, \mathcal{F}) \to \mathcal{L}$ the projection on the first factor, we have two exact sequences of (graded) $\mathcal{O}_X$-modules

$$0 \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to P(\mathcal{L}, \mathcal{F}) \xrightarrow{\nabla} \mathcal{L},$$

(6.1)

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}) \to \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{F}) \xrightarrow{\nabla} \Omega^1(\mathcal{L}) \otimes \mathcal{L} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})[-1],$$

where the second sequence is obtained by applying the exact functor $\Omega^1(\mathcal{L}) \otimes -$ to the first. Now and in the sequel, we will consider $\text{Id}_\mathcal{L}$ as a global section of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})[-1]$, a graded sheaf concentrated in degree 1.

**Lemma 6.1.** In the above setup, there exists a natural bijection between the set of $\mathcal{L}$-connections on $\mathcal{F}$ and global sections $D \in \Gamma(X, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{F}))$ such that $p(D) = \text{Id}_\mathcal{L}$.

**Proof.** Let $l_1, \ldots, l_r$ be a local frame of $\mathcal{L}$ with dual frame $\phi_1, \ldots, \phi_r \in \Omega^1(\mathcal{L})$. Every $\mathbb{K}$-linear morphism $\nabla: \mathcal{F} \to \Omega^1(\mathcal{L}, \mathcal{F})$ can be written locally as $\nabla = \sum_{i=1}^r \phi_i \cdot D_i$, with $D_i \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$. By definition, $\nabla$ is a connection if and only if for every $f \in \mathcal{O}_X$, $e \in \mathcal{F}$, and every $i$ we have

$$D_i(f(e)) = l_i \cdot \nabla(f(e)) = a(l_i)(f)e + f D_i(e)$$

and this is equivalent to the fact that $\sum_{i=1}^r \phi_i \otimes (l_i, D_i) \in \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{F})$. □

**Lemma 6.2.** If $\mathcal{F}$ is a locally free sheaf, then the morphism $p: P(\mathcal{L}, \mathcal{F}) \to \mathcal{L}$ is surjective.

**Proof.** We show this locally, with a proof similar to [16, Lemma 3.1]. Let $R$ be a $\mathbb{K}$-algebra, let $(L, [-, -], a)$ be a Lie algebroid over $R$ with anchor map $a: L \to \text{Der}_R(R, R)$, and let $\mathcal{F}$ be a free $R$-module with basis $\{e_i\}$. We set

$$P(L, F) = \{(l, \phi) \in L \times \mathcal{H}om_{\mathcal{O}_X}(F, F) \mid \phi(re) = r\phi(e) + a(l)(r)e, \forall r \in R, e \in F\},$$

and show that the projection $p: P(L, F) \to L$ is surjective. For every $x \in L$, consider the derivation $a(x) \in \text{Der}_R(R, R)$, and set

$$w(\sum_i r_i e_i) := \sum_i a(x)(r_i)e_i, \quad r_i \in R.$$ 

Then the pair $(x, w)$ belongs to $P(L, F)$. □

Assume now that $\mathcal{F}$ is a locally free sheaf, so that the morphism $p: P(\mathcal{L}, \mathcal{F}) \to \mathcal{L}$ is surjective and we have an exact sequence of locally free graded sheaves of $\mathcal{O}_X$-modules

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}) \xrightarrow{\nabla} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})[-1] \to 0.$$

We can rewrite the above short exact sequence of graded sheaves concentrated in degree 1 as a sequence of sheaves in degree 0:

$$0 \to \Omega^1(\mathcal{L})[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to \Omega^1(\mathcal{L})[1] \otimes P(\mathcal{L}, \mathcal{E}) \xrightarrow{\nabla} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to 0.$$

By Lemma 6.1, there exists an $\mathcal{L}$-connection on $\mathcal{E}$ if and only if the identity on $\mathcal{L}$ lifts to a global section of $\Omega^1(\mathcal{L})[1] \otimes P(\mathcal{L}, \mathcal{E})$. Writing

$$\text{At}_\mathcal{L}(\mathcal{E}) = \partial(\text{Id}_\mathcal{L}) \in H^1(X, \Omega^1(\mathcal{L})[1] \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) = \text{Ext}^1_X(\mathcal{L} \otimes \mathcal{E}, \mathcal{E}),$$

where $\partial$ is the connecting morphism in the cohomology long exact sequence, we have that $\text{At}_\mathcal{L}(\mathcal{E}) = 0$ if and only if there exists an $\mathcal{L}$-connection on $\mathcal{E}$.

Equivalently, we can define $\text{At}_\mathcal{L}(\mathcal{E})$ as the extension class of the short exact sequence

$$0 \to \Omega^1(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to Q(\mathcal{L}, \mathcal{E}) \xrightarrow{\nabla} \mathcal{O}_X[-1] \to 0,$$

where, by definition, $Q(\mathcal{L}, \mathcal{E}) = p^{-1}(\mathcal{O}_X[-1] \cdot \text{Id}_\mathcal{L})$. More explicitly, in a local frame $l_1, \ldots, l_r$ of $\mathcal{L}$, with dual frame $\phi_1, \ldots, \phi_r \in \Omega^1(\mathcal{L})$, the elements of $Q(\mathcal{L}, \mathcal{E})$ are those of $\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E})$ of the form $\sum_{i=1}^r \phi_i \otimes (f l_i, D_i)$ for some $f \in \mathcal{O}_X$. 

Let now \((\mathcal{L}, \mathcal{A})\) be a Lie pair on \(X\). Given an \(\mathcal{A}\)-connection \(\nabla : \mathcal{E} \to \Omega^1(\mathcal{A}, \mathcal{E})\) on \(\mathcal{E}\) locally free it makes sense to ask whether \(\nabla\) lifts to an \(\mathcal{L}\)-connection or not. We prove that the solution to this problem is completely determined by an obstruction
\[
\partial(\nabla) \in \text{Ext}^1_X \left( \frac{\mathcal{L}}{\mathcal{A}} \otimes \mathcal{E}, \mathcal{E} \right) = \text{Ext}^1_X \left( \mathcal{E}, \mathcal{E} \otimes \mathcal{G}_1^1[1] \right).
\]

It is possible to prove, by applying the results of [16, Section 3] to an injective resolution, that the same holds also if \(\mathcal{E}\) is not locally free; however we don’t need this result.

The case \(\mathcal{A} = 0\) has been already considered. Suppose \(\mathcal{A} \neq 0\), then we have a commutative diagram with exact rows
\[
\begin{array}{cccccc}
0 & \rightarrow & \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) & \rightarrow & Q(\mathcal{L}, \mathcal{E}) & \rightarrow & \mathcal{O}_X[-1] & \rightarrow & 0 \\
\alpha & & \downarrow & & \beta & & & & \\
0 & \rightarrow & \Omega^1(\mathcal{A}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) & \rightarrow & Q(\mathcal{A}, \mathcal{E}) & \rightarrow & \mathcal{O}_X[-1] & \rightarrow & 0
\end{array}
\]

where \(\alpha, \beta\) are the natural restriction maps. In a local frame, \(\phi_1, \ldots, \phi_r \in \Omega^1(\mathcal{L})\) and such that \(l_1, \ldots, l_r\) is a local frame for \(\mathcal{A}\), we have
\[
\alpha(\sum_{i=1}^r \phi_i \otimes g_i) = \sum_{i=1}^s \phi_i \otimes g_i, \quad \beta(\sum_{i=1}^r \phi_i \otimes (fl_i, D_i)) = \sum_{i=1}^s \phi_i \otimes (fl_i, D_i).
\]

Since \(\alpha\) and \(\beta\) are surjective, by the snake lemma we have an exact sequence
\[
0 \rightarrow \mathcal{G}_1^1 \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \rightarrow Q(\mathcal{L}, \mathcal{E}) \xrightarrow{\delta} Q(\mathcal{A}, \mathcal{E}) \rightarrow 0,
\]
since \(\mathcal{G}_1^1\) is by definition the kernel of the surjective map \(\Omega^1(\mathcal{L}) \rightarrow \Omega^1(\mathcal{A})\). For simplicity we can rewrite the above short exact sequence of graded sheaves living in degree 1 as a short exact sequence of sheaves in degree 0:
\[
0 \rightarrow \mathcal{G}_1^1[1] \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \rightarrow Q(\mathcal{L}, \mathcal{E})[1] \xrightarrow{\delta} Q(\mathcal{A}, \mathcal{E})[1] \rightarrow 0.
\]

Then the \(\mathcal{A}\)-connection \(\nabla\) is an element of \(H^0(Q(\mathcal{A}, \mathcal{E})[1])\) such that \(p(\nabla) = 1\), and the element
\[
\overline{\nabla}_{\mathcal{E}/\mathcal{A}}(\mathcal{E}, \nabla) = \partial(\nabla) \in H^1 \left( X, \mathcal{G}_1^1[1] \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \right) = \text{Ext}^1_X \left( \mathcal{E}, \mathcal{E} \otimes \mathcal{G}_1^1[1] \right),
\]
is the obstruction to lifting \(\nabla\) to an \(\mathcal{L}\)-connection. We will call this the reduced Atiyah class of \((\mathcal{E}, \nabla)\).

7. Simplicial \(\mathcal{L}\)-connections

In this section, following [17], we define simplicial \(\mathcal{L}\)-connections for a Lie algebroid \(\mathcal{L}\), and simplicial extensions of an \(\mathcal{A}\)-connection for a Lie pair \((\mathcal{L}, \mathcal{A})\). We prove that the adjoint operator of a simplicial \(\mathcal{L}\)-connection on a locally free sheaf \(\mathcal{E}\) induces a curved DG-algebra structure on \(\text{Tot}(\mathcal{U}, \Omega^r(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))\). In the case of a Lie pair \((\mathcal{L}, \mathcal{A})\) and of a simplicial extension of a flat \(\mathcal{A}\)-connection \(\nabla\) on \(\mathcal{E}\), we obtain the data of a curved DG-pair. Simplical connections allow us to give representatives of the classes \(\text{At}_\mathcal{L}(\mathcal{E})\) and \(\overline{\nabla}_{\mathcal{E}/\mathcal{A}}(\mathcal{E}, \nabla)\), and a representative of the obstruction to extending a flat \(\mathcal{A}\)-connection on \(\mathcal{E}\) to a \(\mathcal{L}\)-connection on \(\mathcal{E}\) with curvature in \(\mathcal{G}_2^2 \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})\).

Let \(\mathcal{L}\) be a Lie algebroid on \(X\) and \(\mathcal{E}\) a locally free sheaf. We have seen that \(\mathcal{L}\)-connections on \(\mathcal{E}\) exist locally but in general it does not exist any globally defined connection. However we can define a weaker notion of connection, which always exists and equally gives a significative example of curved DG-algebra.

In the notation of Sections 3 and 6, consider the short exact sequence
\[
0 \rightarrow \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \rightarrow \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}) \xrightarrow{\delta} \Omega^1(\mathcal{L}) \otimes \mathcal{L} \rightarrow 0,
\]
and recall that by Lemma 6.1 an \(\mathcal{L}\)-connection on \(\mathcal{E}\) is a global section \(D\) of \(\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E})\) such that \(p(D) = \text{Id}_\mathcal{L}\), where \(\text{Id}_\mathcal{L}\) is considered as a global section of \(\Omega^1(\mathcal{L}) \otimes \mathcal{L}\). Fix an affine
At (4.1) to assume can define an analogous notion of simplicial by the inclusion of global sections in the totalisation, and one has that

$$A \xrightarrow{\Id} \text{Tot}(U, \Omega^1(L) \otimes \mathcal{L})$$

Because of the natural inclusion (4.1) of global sections in the totalisation, we can consider $\Id_L$ as an element of $\text{Tot}(U, \Omega^1(L) \otimes \mathcal{L})$.

**Definition 7.1.** A simplicial $\mathcal{L}$-connection on $\mathcal{E}$ is a lifting $\nabla$ in $\text{Tot}(U, \Omega^1(L) \otimes \mathcal{P}(\mathcal{L}, \mathcal{E}))$ of $\Id_L$ in $\text{Tot}(U, \Omega^1(L) \otimes \mathcal{L})$.

It is clear that a simplicial $\mathcal{L}$-connection on $\mathcal{E}$ always exists.

In the case of a Lie pair $(\mathcal{L}, \mathcal{A})$ and of an $\mathcal{A}$-connection $\nabla^A$ on the locally free sheaf $\mathcal{E}$, we can define an analogous notion of simplicial $\mathcal{L}$-connection extending $\nabla^A$. It is not restrictive to assume $\mathcal{A} \neq 0$; then the exact sequence of locally free graded sheaves (6.2)

$$0 \to G_1 \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \to Q(\mathcal{L}, \mathcal{E}) \xrightarrow{\beta} Q(A, \mathcal{E}) \to 0$$

induces the short exact sequence of DG-vector spaces

$$(7.2) \quad 0 \to \text{Tot}(U, G_1 \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \to \text{Tot}(U, Q(\mathcal{L}, \mathcal{E})) \xrightarrow{\beta} \text{Tot}(U, Q(A, \mathcal{E})) \to 0.$$  

We have already observed that an $\mathcal{A}$-connection $\nabla^A$ on $\mathcal{E}$ is a global section of $Q(\mathcal{A}, \mathcal{E})$ such that $p(\nabla^A) = 1$, where $p : Q(\mathcal{A}, \mathcal{E}) \to \mathcal{O}_X[-1]$ is induced by the map $p$ of (7.1). By the inclusion of global sections in the totalisation, $\nabla^A$ belongs to $\text{Tot}(U, Q(\mathcal{A}, \mathcal{E}))$.

**Definition 7.2.** By a simplicial extension of an $\mathcal{A}$-connection $\nabla^A$ on $\mathcal{E}$ we mean a lifting $\nabla$ in $\text{Tot}(U, Q(\mathcal{L}, \mathcal{E}))$ of $\nabla^A$ in $\text{Tot}(U, Q(A, \mathcal{E}))$.

**Remark 7.3.** Notice that the exact sequence (6.2) implies that a local extension of an $\mathcal{A}$-connection to an $\mathcal{L}$-connection always exists.

Since maps on the totalisation are induced locally, a similar argument to that of Lemma 5.3 shows that every simplicial extension $\nabla'$ of a flat $\mathcal{A}$-connection $\nabla^A$ on $\mathcal{E}$ induces a differential on the complex $\text{Tot}(U, G_{r+1}(\mathcal{E}) \otimes E_{r+1}(\mathcal{E}))$. We then have that $H^*(\text{Tot}(U, G_r(\mathcal{E}) \otimes E_{r}(\mathcal{E})) \otimes \mathcal{E})$ is isomorphic to the Lie algebroid cohomology of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-module $\mathcal{L} \wedge \ldots \wedge \mathcal{L} \otimes \mathcal{E}$, again by Lemma 5.3.

**Lemma 7.4.** For a Lie algebroid $\mathcal{L}$ and a simplicial $\mathcal{L}$-connection $\nabla$ on $\mathcal{E}$, the cohomology class of $d_{\text{Tot}} \nabla$ in $\text{Tot}(U, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is the obstruction $\mathcal{A}_L(\mathcal{E})$ to the existence of an $\mathcal{L}$-connection on $\mathcal{E}$.

For a Lie pair $(\mathcal{L}, \mathcal{A})$ and a simplicial extension $\nabla$ of an $\mathcal{A}$-connection $\nabla^A$ on $\mathcal{E}$, the cohomology class of $d_{\text{Tot}} \nabla$ in $\text{Tot}(U, G_{r+1}(\mathcal{E}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ is the obstruction $\overline{\mathcal{A}}_{L/A}(\mathcal{E}, \nabla^A)$ to the extension of $\nabla^A$ to an $\mathcal{L}$-connection.

**Proof.** According to Example 4.3 we have natural isomorphisms

$$H^0(\text{Tot}(U, \Omega^1(L) \otimes \mathcal{P}(\mathcal{L}, \mathcal{E}))) = \Gamma(X, \Omega^1(L) \otimes \mathcal{P}(\mathcal{L}, \mathcal{E})).$$

$$H^0(\text{Tot}(U, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))) = \Gamma(X, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})).$$

Consider first the case of a simplicial $\mathcal{L}$-connection $\nabla$ on $\mathcal{E}$; notice that $d_{\text{Tot}} \nabla$ belongs to $\text{Tot}(U, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$, because $p(d_{\text{Tot}} \nabla) = d_{\text{Tot}} p(\nabla) = d_{\text{Tot}} \Id_L = 0$, since $\Id_L$ is a global section. If there exists an $\mathcal{L}$-connection $\nabla'$ on $\mathcal{E}$ it belongs to $\text{Tot}(U, \Omega^1(L) \otimes \mathcal{P}(\mathcal{L}, \mathcal{E}))$ by the inclusion of global sections in the totalisation, and one has that $d_{\text{Tot}} \nabla' = 0$. Then for any simplicial connection $\nabla$, the difference $\nabla - \nabla'$ belongs to $\text{Tot}(U, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$ and $d_{\text{Tot}}(\nabla - \nabla') = d_{\text{Tot}} \nabla$, so that $d_{\text{Tot}} \nabla$ is trivial in the cohomology of $\text{Tot}(U, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$. Conversely, if $d_{\text{Tot}} \nabla = d_{\text{Tot}} \varphi$, with $\varphi \in \text{Tot}(U, \Omega^1(L) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$, then $\nabla - \varphi$ is a global $\mathcal{L}$-connection on $\mathcal{E}$.

In the case of a Lie pair $(\mathcal{L}, \mathcal{A})$ and a simplicial extension $\nabla$ of an $\mathcal{A}$-connection $\nabla^A$ on $\mathcal{E}$, notice that $d_{\text{Tot}} \nabla$ belongs to $\text{Tot}(U, G_1 \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))$; in fact, $\beta(d_{\text{Tot}} \nabla) = \beta(d_{\text{Tot}} \varphi) = 0$.
induces a graded Lie bracket on the totalisation \( \text{Tot}(\mathcal{L}, E) \) such that \( \beta(\nabla') = \nabla^A \), which is such that \( d_{\text{Tot}} \nabla' = 0 \) in \( \text{Tot}(\mathcal{U}, Q(\mathcal{L}, E)) \), because it is a global section. Then for every simplicial connection \( \nabla' \) lifting \( \nabla^A \), \( \nabla - \nabla' \) belongs to the kernel of \( \beta \), which is \( \text{Tot}(\mathcal{U}, G^\omega_1 \otimes \text{Hom}_{\mathcal{O}_X}(E, E)) \), and \( d_{\text{Tot}}(\nabla - \nabla') = d_{\text{Tot}} \nabla \), so that \( d_{\text{Tot}} \nabla \) is trivial in cohomology. Vice versa, if \( d_{\text{Tot}} \nabla = d_{\text{Tot}} \phi \) is trivial in the cohomology of \( \text{Tot}(\mathcal{U}, G^\omega_1 \otimes \text{Hom}_{\mathcal{O}_X}(E, E)) \), it is easy to see that \( \nabla - \phi \) is a connection lifting \( \nabla^A \).

A simplicial \( \mathcal{L} \)-connection on a locally free sheaf \( E \) induces a curved DG-algebra structure on the DG-vector space \( \text{Tot}(\mathcal{U}, \Omega^\bullet(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(E, E))) \). To see this, the first step is the construction of an adjoint operator for the simplicial connection, which is done via the following lemma.

**Lemma 7.5.** In the above situation, the \( \mathcal{O}_X \)-bilinear map

\[
[-, -]: (\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, E)) \times \text{Hom}_{\mathcal{O}_X}(E, E) \to \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E),
\]

is well-defined.

**Proof.** For \( r \in \mathcal{O}_X \),

\[
[v, g](re) = v(rg(e)) - g(rv(e) + a(l)(r)e) = reg(e) + a(l)(r)g(e) - rgv(e) - a(l)(r)g(e) = r[v, g](e),
\]

so \([v, g] \) belongs to \( \text{Hom}_{\mathcal{O}_X}(E, E) \). The bracket is well-defined: for \( r \in \mathcal{O}_X \),

\[
[\eta \otimes (rl, rv), g] = \eta \otimes [rv, g] = \eta \otimes r[v, g] = r[\eta \otimes (l, v), g] = [r\eta \otimes (l, v), g].
\]

The bracket defined in Lemma 7.5 induces a graded Lie bracket on the totalisation

\[
[-, -]: \text{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, E)) \times \text{Tot}(\mathcal{U}, \text{Hom}_{\mathcal{O}_X}(E, E)) \to \text{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E)),
\]

which allows to define the adjoint operator to a simplicial \( \mathcal{L} \)-connection \( \nabla \) on \( E \):

\[
(7.3) \quad d_{\nabla} := [\nabla, -]: \text{Tot}(\mathcal{U}, \text{Hom}_{\mathcal{O}_X}(E, E)) \to \text{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E)).
\]

Recall that since \( \Omega^\bullet(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(E, E)) \) is a sheaf of graded algebras and the Tot functor preserves multiplicative structures, \( \text{Tot}(\mathcal{U}, \Omega^\bullet(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(E, E))) \) is a differential graded algebra, with differential denoted by \( d_{\text{Tot}} \).

**Lemma 7.6.** The adjoint operator

\[
d_{\nabla} = [\nabla, -]: \text{Tot}(\mathcal{U}, \text{Hom}_{\mathcal{O}_X}(E, E)) \to \text{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E))
\]

extends for every \( i \geq 0 \) to a \( k \)-linear operator

\[
d_{\nabla}^i := \text{Tot}(\mathcal{U}, \Omega^i(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E)) \to \text{Tot}(\mathcal{U}, \Omega^{i+1}(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E)).
\]

Then \( (\text{Tot}(\mathcal{U}, \Omega^\bullet(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(E, E))), d_{\text{Tot}} + d_{\nabla}) \) is a curved DG-algebra with curvature

\[
d_{\text{Tot}} \nabla + C,
\]

with \( d_{\text{Tot}} \nabla \in \text{Tot}(\mathcal{U}, \Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E)) \) and \( C \in \text{Tot}(\mathcal{U}, \Omega^2(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(E, E)) \) such that \( d_{\nabla}^2 = [C, -] \).

**Proof.** Consider first the case of a germ of an \( \mathcal{L} \)-connection, i.e., an element \( Y \) of \( \Gamma(V, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, E)) \) such that \( p(Y) = \text{Id}_{\mathcal{L}} |_V \), for some open set \( V \subset X \). As usual, \( Y \) extends uniquely to a \( k \)-linear morphism of degree 1

\[
Y: \Omega^\bullet(\mathcal{L}, E) |_V \to \Omega^\bullet(\mathcal{L}, E) |_V
\]

such that

\[
Y(\eta \otimes e) = d_{\mathcal{L}}(\eta) \otimes e + (-1)^{\deg \eta} \eta \otimes Y(e).
\]

for all \( \eta \in \Omega^\bullet(\mathcal{L}) |_V \), \( e \in E |_V \). It is easy to see that the map \( Y^2 \) is \( \mathcal{O}_X \)-linear, so it can be identified with a section of \( \Omega^2(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(E, E)) |_V \).
One can define an adjoint operator

\[ d_Y := [Y, -]: \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})|_{\mathcal{V}} \to \Omega^1(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))|_{\mathcal{V}}, \]

which can be extended for all \( i \geq 0 \) to an operator

\[ d_Y: \Omega^i(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))|_{\mathcal{V}} \to \Omega^{i+1}(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))|_{\mathcal{V}} \]

by setting

\[ (7.4) \quad d_Y(\eta \otimes f) := d_\mathcal{L}(\eta) \otimes f + (-1)^{|\eta|} \eta \otimes [Y, f], \]

where \([Y, f]\) denotes the Lie bracket of Lemma 7.5.

As in the classical case, one can see that

\[ (7.5) \quad d_Y^2(\eta \otimes f) = [Y^2, \eta \otimes f] \]

for all \( \eta \in \Omega^*(\mathcal{L})|_{\mathcal{V}} \) and \( f \in \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})|_{\mathcal{V}} \).

Let now \( \nabla \) be a simplicial \( \mathcal{L} \)-connection on \( \mathcal{E} \), namely an element of \( \text{Tot}(U, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E})) \) such that \( p(\nabla) = \text{Id}_{\mathcal{E}} \in \text{Tot}(U, \Omega^1(\mathcal{L}) \otimes \mathcal{L}) \). Then for every \( i \geq 0 \) the extension of the operator \( d_\nabla = [\nabla, -] \), defined in (7.3), to an operator

\[ d_\nabla = [\nabla, -]: \text{Tot}(U, \Omega^i(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \to \text{Tot}(U, \Omega^{i+1}(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})) \]

can be defined by using the map induced by (7.4) on the totalisation, and one obtains a degree one operator

\[ d_\nabla = [\nabla, -]: \text{Tot}(U, \Omega^*(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))) \to \text{Tot}(U, \Omega^*(\mathcal{L}, \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))). \]

In detail, let \( \nabla = (D_n) \) with \( D_n \in A_n \otimes \prod_{i_1, \ldots, i_n} (\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}))(U_{i_1}, \ldots, i_n) \) such that \( p(D_n) = 1 \otimes (\text{Id}_{\mathcal{E}}|_{\mathcal{V}(U_{i_1}, \ldots, i_n)}) \) for every \( n \geq 0 \). Since maps on the totalisation are defined componentwise, it is enough to define the bracket

\[ [D_n, \phi_n \otimes (\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n})], \]

for \( \phi_n \otimes (\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n}) \) in \( A_n \otimes \prod_{i_1, \ldots, i_n} (\Omega^1(\mathcal{L}) \otimes \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}))(U_{i_1}, \ldots, i_n) \). Let

\[ (7.6) \quad D_n = \sum_j \eta_{j,n} \otimes (t_{j,i_1, \ldots, i_n}), \quad \eta_{j,n} \in A_n, \quad t_{j,i_1, \ldots, i_n} \in (\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}))(U_{i_1}, \ldots, i_n); \]

then the bracket can be defined as

\[ [D_n, \phi_n \otimes (\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n})] = \left[ \sum_j \eta_{j,n} \otimes (t_{j,i_1, \ldots, i_n}), \phi_n \otimes (\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n}) \right] \]

\[ = p(D_n)(\phi_n \otimes (\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n})) \]

\[ + (-1)^{|\phi_n| + |\omega_{i_1, \ldots, i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1, \ldots, i_n} \otimes [t_{j,i_1, \ldots, i_n}, f_{i_1, \ldots, i_n}]) \]

\[ = (1 \otimes (\text{Id}_{\mathcal{L}}|_{\Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E})}(U_{i_1, \ldots, i_n}))(\phi_n \otimes (\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n})) \]

\[ + (-1)^{|\phi_n| + |\omega_{i_1, \ldots, i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1, \ldots, i_n} \otimes [t_{j,i_1, \ldots, i_n}, f_{i_1, \ldots, i_n}]) \]

\[ = (-1)^{|\phi_n|} \phi_n \otimes (d_\mathcal{L}\omega_{i_1, \ldots, i_n} \otimes f_{i_1, \ldots, i_n}) \]

\[ + (-1)^{|\phi_n| + |\omega_{i_1, \ldots, i_n}|} \sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1, \ldots, i_n} \otimes [t_{j,i_1, \ldots, i_n}, f_{i_1, \ldots, i_n}]), \]

where the bracket \([t_{j,i_1, \ldots, i_n}, f_{i_1, \ldots, i_n}]\) is induced by the one of Lemma 7.5.

For every \( i \geq 0 \) the simplicial \( \mathcal{L} \)-connection \( \nabla \) also induces a map

\[ \nabla: \text{Tot}(U, \Omega^i(\mathcal{L}) \otimes \mathcal{E}) \to \text{Tot}(U, \Omega^{i+1}(\mathcal{L}) \otimes \mathcal{E}) \]
Lemma 7.7. \( C \) \( \nabla \) so the curvature is \( d \), the curvature of the simplicial connection \( \nabla \) belongs to \( \Omega \) \( \Gamma(L) \) \( \otimes \) \( \mathcal{E} \) \( (U_{i_1,...,i_n}) \). Then the operator can be defined as
\[
D_n(\phi_n \otimes (\omega_{i_1,...,i_n} \otimes e_{i_1,...,i_n})) = (\sum_j \eta_{j,n} \otimes (t_{j,i_1,...,i_n})(\phi_n \otimes (\omega_{i_1,...,i_n} \otimes e_{i_1,...,i_n}))
\]
\[
= p(D_n)(\phi_n \otimes (\omega_{i_1,...,i_n} \otimes e_{i_1,...,i_n})) +
- (1)^{\phi_n}(\omega_{i_1,...,i_n} \otimes \eta_{j,n} \otimes (\omega_{i_1,...,i_n} \otimes t_{i_1,...,i_n}(e_{i_1,...,i_n}))
\]
\[
= (1 \otimes (\text{Id}_{\mathcal{E}} |_{U_{i_1,...,i_n}}))(\phi_n \otimes (\omega_{i_1,...,i_n} \otimes e_{i_1,...,i_n})) +
- (1)^{\phi_n}(\omega_{i_1,...,i_n} \otimes \eta_{j,n} \otimes (\omega_{i_1,...,i_n} \otimes t_{i_1,...,i_n}(e_{i_1,...,i_n}))
\]
\[
= (1)^{\phi_n}(\phi_n \otimes (d_{\mathcal{L}}\omega_{i_1,...,i_n} \otimes e_{i_1,...,i_n})
+ (1)^{\omega_{i_1,...,i_n}}(\sum_j \phi_n \eta_{j,n} \otimes (\omega_{i_1,...,i_n} \otimes t_{i_1,...,i_n}(e_{i_1,...,i_n})))
\].

Since all the maps considered on the totalisation are induced by the ones defined locally on the complexes of sheaves, for \( d_{\nabla} = [\nabla, -] \) one has that, by (7.5),
\[
d^2_{\mathcal{E}} = [C, -], \quad C \in \text{Tot}(U, \Omega^2(L, \mathcal{E}))
\].

Then \( d_{\text{Tot}} + d_{\nabla} \) is a degree one derivation of \( \text{Tot}(U, \Omega^\ast(L) \otimes \mathcal{E}) \), with square
\[
(d_{\text{Tot}} + d_{\nabla})^2 = d_{\text{Tot}}^2 + d_{\text{Tot}}d_{\nabla} + d_{\nabla}d_{\text{Tot}} = [d_{\nabla}d_{\text{Tot}}, -] + [C, -] = [d_{\text{Tot}} \nabla + C, -],
\]
so the curvature is \( d_{\text{Tot}} \nabla + C \). We have already seen in Lemma 7.4 that \( d_{\text{Tot}} \nabla \) belongs to \( \text{Tot}(U, \Omega^\ast(L) \otimes \mathcal{E}) \).

The last thing to prove is that \( (d_{\text{Tot}} + d_{\nabla})(d_{\text{Tot}} \nabla + C) = 0 \). One has that
\[
(d_{\text{Tot}} + d_{\nabla})(d_{\text{Tot}} \nabla + C) = d_{\text{Tot}}^2 \nabla + d_{\text{Tot}}d_{\nabla} \nabla + d_{\text{Tot}} C + d_{\nabla} d_{\text{Tot}} \nabla + d_{\nabla} C = d_{\nabla}d_{\text{Tot}} \nabla + d_{\text{Tot}} C.
\]

Then
\[
d_{\text{Tot}}d_{\text{Tot}} \nabla = [\nabla, -] = -d_{\text{Tot}} \nabla,
\]
so that \( (d_{\text{Tot}} + d_{\nabla})(d_{\text{Tot}} \nabla + C) = 0 \). \( \Box \)

In the case of a Lie pair \( (L, A) \) and a locally free sheaf \( \mathcal{E} \), the natural surjective restriction maps
\[
\rho: \Omega^\ast(L) \to \Omega^\ast(A), \quad \rho \otimes \text{Id}: \Omega^\ast(L, \mathcal{E}) \to \Omega^\ast(A, \mathcal{E}),
\]
induce morphisms on the totalisation
\[
\rho: \text{Tot}(U, \Omega^\ast(L)) \to \text{Tot}(U, \Omega^\ast(A)),
\]
\[
\rho \otimes \text{Id}: \text{Tot}(U, \Omega^\ast(L, \mathcal{E})) \to \text{Tot}(U, \Omega^\ast(A, \mathcal{E})),
\]
whose kernels define bilateral ideals
\[
\text{Tot}(U, G_1^\ast) = \ker(\rho) \subset \text{Tot}(U, \Omega^\ast(L)),
\]
\[
\text{Tot}(U, G_1^\ast \otimes \mathcal{E}) = \ker(\rho \otimes \text{Id}) \subset \text{Tot}(U, \Omega^\ast(L, \mathcal{E})).
\]

Lemma 7.7. Let \( (\mathcal{E}, \nabla^A) \) be a locally free \( A \)-module, and let \( \nabla \) be a simplicial extension of \( \nabla^A \) to an \( L \)-connection. Then \( I := \text{Tot}(U, G_1^\ast \otimes \mathcal{E}) \) is a curved ideal of the curved DG-algebra
\[
(\text{Tot}(U, \Omega^\ast(L, \mathcal{E})), d_{\text{Tot}} + d_{\nabla}, d_{\text{Tot}} \nabla + C),
\]
where \( C \), the curvature of the simplicial connection \( \nabla \), belongs to \( \text{Tot}(U, G_1^\ast \otimes \mathcal{E}) \) and \( d_{\text{Tot}} \nabla \) belongs to \( \text{Tot}(U, G_1^\ast \otimes \mathcal{E}) \).
Proof. It is clear that the ideal $I = \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$ is $d_{\operatorname{Tot}}$-closed. Let $x$ be an element of $I$, so that $(\varrho \otimes \operatorname{Id})(x) = 0$, then
\[
 (\varrho \otimes \operatorname{Id})(d_{\nabla}x) = d_{\nabla} \varrho (\varrho \otimes \operatorname{Id})(x) = 0,
\]
so $I$ is also $d_{\nabla}$-closed. Since the $\mathcal{A}$-connection $\nabla^A$ is flat, the curvature $C$ of $\nabla$ belongs to $\operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})) \subset I$, which is the kernel of the surjective map
\[
 \varrho \otimes \operatorname{Id} : \operatorname{Tot}(\mathcal{U}, \Omega^2(\mathcal{L}) \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})) \to \operatorname{Tot}(\mathcal{U}, \Omega^2(\mathcal{A}) \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})).
\]
By Lemma 7.4, $d_{\operatorname{Tot}} \nabla$ belongs to $\operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$, therefore it belongs to the ideal $I$. \hfill \qed

For the ideal $I = \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$ we have that
\[
 I^{(n)} = \operatorname{Tot}(\mathcal{U}, G^*_n \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})).
\]
In fact, the inclusion $I^{(n)} \subset \operatorname{Tot}(\mathcal{U}, G^*_n \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$ is clear. For the other one, it suffices to notice that the multiplication map $G^*_1 \otimes \cdots \otimes G^*_1 \to G^*_n$ is surjective on all affine open sets.

According to Definition 2.3, the Atiyah cocycle of the curved DG-pair
\[
 (A = \operatorname{Tot}(\mathcal{U}, \Omega(\mathcal{L}, \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))), I = \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})))
\]
is the class of the curvature $R = d_{\operatorname{Tot}} \nabla + C$ in
\[
 I = \frac{I^{(2)}}{I^{(1)}} = \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})).
\]

Theorem 7.8. Given a Lie pair $(\mathcal{L}, \mathcal{A})$ and a locally free $\mathcal{A}$-module $(\mathcal{E}, \nabla^A)$, the Atiyah class $\operatorname{At}(A, I)$ of the curved DG-pair
\[
 (A = \operatorname{Tot}(\mathcal{U}, \Omega(\mathcal{L}, \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))), I = \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})))
\]
does not depend on the choice of the simplicial $\mathcal{L}$-connection extending $\nabla^A$. Moreover, it is the obstruction to the existence of a $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^A$ with curvature in $G^*_2 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})$.

Proof. The difference of two simplicial extensions $\nabla$ and $\nabla'$ of the $\mathcal{A}$-connection $\nabla^A$ belongs to the ideal $I$. In fact, considering the short exact sequence (7.2),
\[
 0 \to \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})) \to \operatorname{Tot}(\mathcal{U}, Q(\mathcal{L}, \mathcal{E})) \xrightarrow{\beta} \operatorname{Tot}(\mathcal{U}, Q(\mathcal{A}, \mathcal{E})) \to 0,
\]
we have that $\beta(\nabla - \nabla') = \nabla^A - \nabla^A = 0$ and therefore, writing $\phi := \nabla - \nabla'$, we have $\phi \in \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})) \subset I$. Then $d_{\nabla'} = d_{\nabla} + [\phi, -]$ and the first claim follows from Lemma 2.4.

Next, we show that the Atiyah class $\operatorname{At}(A, I)$ of the curved DG-pair is the obstruction to the existence of a $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^A$, with curvature in $G^*_2 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})$. By Lemma 2.4, $\operatorname{At}(A, I)$ is the obstruction to the existence of $x \in I = \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$ of degree 1 such that $R + (d_{\operatorname{Tot}} + d_{\nabla})x$ belongs to $I^{(2)} = \operatorname{Tot}(\mathcal{U}, G^*_2 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$. Assume that there exists such $x$, and notice that by degree reasons it belongs to $\operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E}))$, since $G^*_0 = 0$. Then, since $G^*_2 = 0$,
\[
 d_{\operatorname{Tot}} \nabla + d_{\operatorname{Tot}}x \in I^{(2)} \cap \operatorname{Tot}(\mathcal{U}, G^*_1 \otimes \mathcal{H}om_{\mathcal{O}X}(\mathcal{E}, \mathcal{E})) = 0
\]
and by the first equation $\nabla + x$ is a global $\mathcal{L}$-connection on $\mathcal{E}$ extending $\nabla^A$.

We denote by $R_x = d_{\operatorname{Tot}}(\nabla + x) + C_x = C_x$ the curvature of the curved DG-algebra $(A, d_{\operatorname{Tot}} + d_{\nabla} + x)$. Then
\[
 R_x = R + (d_{\operatorname{Tot}} + d_{\nabla})x + \frac{1}{2}[x, x] = d_{\operatorname{Tot}} \nabla + C + d_{\operatorname{Tot}}x + d_{\nabla}x + \frac{1}{2}[x, x] = C + d_{\nabla}x + \frac{1}{2}[x, x],
\]
so that the curvature of $\nabla + x$ is equal to $C_x = C + d\nabla x + \frac{1}{2}[x, x]$, which belongs to $\text{Tot}(\mathcal{U}, \mathcal{G}_2^2 \otimes \text{Hom}_{O_X}(\mathcal{E}, \mathcal{E}))$. Finally, since $d\nabla x + (R_x) = 0$, one has that
\[ 0 = (d\text{Tot} + d\nabla x)(R_x) = (d\text{Tot} + d\nabla x)(C_x) = d\text{Tot}C_x, \]
and $C_x$ is a global section of $\mathcal{G}_2^2 \otimes \text{Hom}_{O_X}(\mathcal{E}, \mathcal{E})$.

The converse is clear. \hfill \square

By the above, the Atiyah class $\text{At}(A, I)$ of the curved DG-pair
\[ (A = \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \text{End}_{O_X}(\mathcal{E}))), I = \text{Tot}(\mathcal{U}, \mathcal{G}_1^* \otimes \text{End}_{O_X}(\mathcal{E}))) \]
is well-defined:
\[ \text{At}(A, I) \in \mathbb{H}^2\left( X, \mathcal{G}_1^* \otimes \text{End}_{O_X}(\mathcal{E}) \right). \]

**Definition 7.9.** In the above situation, via the isomorphisms of Lemma 5.3, we call
\[ \text{At}_{\mathcal{L}/A}(\mathcal{E}) := \text{At}(A, I) \in \mathbb{H}^1(\mathcal{X} ; (\mathcal{L}/A)^{\vee} \otimes \text{End}_{O_X}(\mathcal{E})). \]
the $(\mathcal{L}, A)$-Atiyah class of $\mathcal{E}$.

**Remark 7.10.** Recalling that $\mathcal{G}_1^2 = 0$, the morphism of graded sheaves $t: \frac{\mathcal{G}_1^*}{\mathcal{G}_2^2} \to \mathcal{G}_1^1$ with kernel $\frac{\mathcal{G}_2^2}{\mathcal{G}_2^2}$ induces a morphism of DG-vector spaces
\[ t: \text{Tot}(\mathcal{U}, \frac{\mathcal{G}_1^*}{\mathcal{G}_2^2} \otimes \text{End}_{O_X}(\mathcal{E})) \to \text{Tot}(\mathcal{U}, \mathcal{G}_1^1 \otimes \text{End}_{O_X}(\mathcal{E})), \]
which sends the class of $R = d\text{Tot} \nabla + C$ to $d\text{Tot} \nabla$. The reduced Atiyah class $\overline{\text{At}}_{\mathcal{L}/A}(\mathcal{E}, \nabla^A)$ is then the image of the Atiyah class $\text{At}_{\mathcal{L}/A}(\mathcal{E})$ of the curved DG-pair
\[ (A = \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \text{End}_{O_X}(\mathcal{E}))), I = \text{Tot}(\mathcal{U}, \mathcal{G}_1^* \otimes \text{End}_{O_X}(\mathcal{E}))) \]
via the map induced by $t$ in hypercohomology
\[ t: \mathbb{H}^*(X, \frac{\mathcal{G}_1^*}{\mathcal{G}_2^2} \otimes \text{End}_{O_X}(\mathcal{E})) \to \mathbb{H}^*(X, \mathcal{G}_1^1 \otimes \text{End}_{O_X}(\mathcal{E})) \]
\[ \text{At}_{\mathcal{L}/A}(\mathcal{E}) \mapsto \overline{\text{At}}_{\mathcal{L}/A}(\mathcal{E}, \nabla^A). \]
In particular if $\text{At}_{\mathcal{L}/A}(\mathcal{E})$ is trivial, then so is $\overline{\text{At}}_{\mathcal{L}/A}(\mathcal{E}, \nabla^A)$.

If we consider the Lie pair $(\mathcal{L}, 0)$, both the obstructions $\text{At}_{\mathcal{L}/A}(\mathcal{E})$ and $\overline{\text{At}}_{\mathcal{L}/A}(\mathcal{E}, \nabla^A)$ reduce to the obstruction $\text{At}_{\mathcal{L}}(\mathcal{E})$ to the existence of an $\mathcal{L}$-connection on $\mathcal{E}$.

**Corollary 7.11.** Let $(\mathcal{L}, A)$ be a Lie pair on $X$ such that there exists an $O_X$-linear projection $p: \mathcal{L} \to A$ which commutes with anchor maps and with adjoint Lie actions of $A$. Then for every $A$-module $\mathcal{E}$ the Atiyah class $\text{At}_{\mathcal{L}/A}(\mathcal{E})$ is trivial.

**Proof.** The assumption that $p: \mathcal{L} \to A$ commutes with adjoint Lie actions of $A$ means that $p([x, y]) = [x, p(y)]$ for every $x \in A$ and $y \in \mathcal{L}$.

Let $\nabla: A \to \text{End}_k(\mathcal{E})$ be a flat $A$-connection on $\mathcal{E}$. The existence of an $O_X$-linear projection $p: \mathcal{L} \to A$ commuting with anchor maps ensures that the composition $\overline{\nabla} := \nabla p: \mathcal{L} \to \text{End}_k(\mathcal{E})$ is a connection. In fact, for $l \in \mathcal{L}$, $f \in O_X$ and $e \in \mathcal{E}$,
\[ \overline{\nabla}_l(fe) = \nabla_p(l)(fe) = a_A(p(l))(fe) + f\nabla_p(l)(e) = a_A(l)(fe) + f\overline{\nabla}_l(e). \]
For every $a \in A$ and every $l \in \mathcal{L}$ we have
\[ [\overline{\nabla}_a, \overline{\nabla}_l] = [\nabla_a, \nabla_p(l)] = \nabla_{a, p(l)} = \nabla_{p(a, l)} = \overline{\nabla}_{[a, l]}, \]
and this implies that the curvature of $\overline{\nabla}$ belongs to $\mathcal{G}_2^2 \otimes \text{End}_{O_X}(\mathcal{E})$, so that by Theorem 7.8 the Atiyah class of $\mathcal{E}$ is trivial. \hfill \square

Notice that Corollary 7.11 applies in particular in the case $X = \text{Spec}(k)$ and $A$ a semisimple Lie algebra. On the other hand, the Examples 2.10 and 2.11 of [6] give explicit situations where $X$ is a single point and the Atiyah class does not vanish.
8. Semiregularity maps and obstructions

Let \((\mathcal{L}, \mathcal{A})\) be a Lie pair on a smooth separated scheme \(X\) of finite type over a field \(K\) of characteristic 0. Given a locally free \(\mathcal{A}\)-module \((\mathcal{E}, \nabla^A)\) we introduced the Atiyah class

\[
\text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \in \mathbb{H}^1(\mathcal{A}; (\mathcal{L}/\mathcal{A})^\vee \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E})),
\]

which is the primary obstruction to the extension of the \(\mathcal{A}\)-connection \(\nabla^A\) to a flat \(\mathcal{L}\)-connection; more precisely the Atiyah class is a complete obstruction to the extension of \(\nabla^A\) to an \(\mathcal{L}\)-connection with curvature in \(\mathcal{G}^2_2 \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E})\).

Taking exterior cup products in \(\mathcal{A}\)-cohomology it makes sense to consider the exterior powers

\[
\text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k \in \mathbb{H}^k\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E})\right)
\]

together with the morphisms of graded vector spaces

\[
\mathbb{H}^* (\mathcal{A}; \text{End}_{\mathcal{O}_X}(\mathcal{E})) \to \mathbb{H}^*\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E})\right)[k] \to \mathbb{H}^*\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right)[k],
\]

\[
x \mapsto \frac{1}{k!} \text{Tr}(\text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x).
\]

The following definition is a clear natural extension of the definition of semiregularity maps for coherent sheaves \([2, 5]\).

**Definition 8.1.** In the above situation, for every \(k \geq 0\) the map

\[
\tau_k : \mathbb{H}^2 (\mathcal{A}; \text{End}_{\mathcal{O}_X}(\mathcal{E})) \to \mathbb{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right), \quad \tau_k(x) = \frac{1}{k!} \text{Tr}(\text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),
\]

is called the \(k\)-semiregularity map of the \(\mathcal{A}\)-module \((\mathcal{E}, \nabla^A)\), (with respect to the Lie pair \((\mathcal{L}, \mathcal{A})\)).

If \(\mathcal{G}^*_k\) is the Leray filtration of the Lie pair \((\mathcal{L}, \mathcal{A})\) we have proved in Lemma 5.3 that there exist canonical isomorphisms \(\mathbb{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right) \cong \mathbb{H}^{2+2k}\left(X, \mathcal{G}^*_k/\mathcal{G}^*_{k+1}\right)\) and therefore there exist natural maps

\[
i_k : \mathbb{H}^{2+k}\left(\mathcal{A}; \bigwedge^k (\mathcal{L}/\mathcal{A})^\vee\right) \to \mathbb{H}^{2+2k}\left(X, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}\right),
\]

which are injective whenever the Leray spectral sequence degenerates at \(E_1\).

We are now ready to apply the abstract general results of \([2]\) to our situation in order to obtain the following result.

**Theorem 8.2.** Let \((\mathcal{L}, \mathcal{A})\) be a Lie pair on a smooth separated scheme \(X\) of finite type over a field \(K\) of characteristic 0. Given a locally free \(\mathcal{A}\)-module \((\mathcal{E}, \nabla^A)\), for every \(k \geq 0\) the composite map

\[
i_k \tau_k : \mathbb{H}^2 (\mathcal{A}; \text{End}_{\mathcal{O}_X}(\mathcal{E})) \to \mathbb{H}^{2+2k}\left(X, \frac{\Omega^*(\mathcal{L})}{\mathcal{G}^*_{k+1}}\right)
\]

annihilates every obstruction to deformations of \((\mathcal{E}, \nabla^A)\) as an \(\mathcal{A}\)-module. In particular, if the Leray spectral sequence of the Lie pair \((\mathcal{L}, \mathcal{A})\) degenerates at \(E_1\), then every semiregularity map annihilates obstructions.

**Proof.** We take an affine cover \(U\) of \(X\) and we choose a simplicial connection \(\nabla \in \text{Tot}(U, \Omega^1(\mathcal{L}) \otimes P(\mathcal{L}, \mathcal{E}))\) extending \(\nabla^A\). By Lemma 7.7, the ideal \(I := \text{Tot}(U, \mathcal{G}^*_1 \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E}))\) is a curved ideal of the curved DG-algebra

\[
 \mathcal{A} := \text{Tot}(U, \Omega^*(\mathcal{L}) \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E})), \quad d_{\text{Tot}} + d_\nabla, \quad d_{\text{Tot}} \nabla + C,
\]

so that the quotient

\[
 \mathcal{B} := \mathcal{A}/I = \text{Tot}(U, \Omega^*(\mathcal{L}) \otimes \text{End}_{\mathcal{O}_X}(\mathcal{E}))
\]
is a non-curved DG-Lie algebra, with differential given by \( d_{\mathrm{Tot}} + d_{\mathcal{A}} \). This is precisely the DG-Lie algebra controlling deformations of the \( \mathcal{A} \)-module \((\mathcal{E}, \nabla^A)\) of Theorem 4.4.

The trace morphism

\[
\text{Tr}: \Omega^*(\mathcal{L}, \mathcal{E}_{\mathcal{O}_X}(\mathcal{E})) \rightarrow \Omega^*(\mathcal{L})
\]

of (3.1) induces

\[
\text{Tr}: \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E}_{\mathcal{O}_X}(\mathcal{E}))) \rightarrow \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{L})),
\]

which is a trace map in the sense of Definition 2.5. It is plain that

\[
\text{Tr}(\text{Tot}(\mathcal{U}, g_k^i \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E}))) \subset \text{Tot}(\mathcal{U}, g_k^i),
\]

for every \( k \geq 0 \). Finally, according to (7.7) and the exactness properties of \( \text{Tot} \), for every \( i \leq j \) we have

\[
\frac{I(i)}{I(j)} = \frac{\text{Tot}(\mathcal{U}, g_j^* \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E}))}{\text{Tot}(\mathcal{U}, g_j^* \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E}))} = \text{Tot}\left(\mathcal{U}, \frac{g_j^*}{g_j^*} \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E})\right).
\]

Now, by Theorem 2.6, there exists an \( L_{\infty} \) morphism between DG-Lie algebras

\[
\sigma^k: \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{A}) \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E})) \rightarrow \text{Tot}\left(\mathcal{U}, \frac{\Omega^*(\mathcal{L})}{g_k^*+1}[2k]\right)
\]

whose linear component is given by

\[
\sigma^k_1: \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{A}) \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E})) \rightarrow \text{Tot}\left(\mathcal{U}, \frac{\Omega^*(\mathcal{L})}{g_k^*+1}[2k]\right), \quad \sigma^k_1(x) = \frac{1}{k!} \text{Tr}(R^k x),
\]

where \( R = d_{\text{tot}} \nabla + C \) denotes the curvature of the DG-algebra \( \mathcal{A} \).

In cohomology the above maps \( \sigma^k_1 \) may be written as

\[
\sigma^k_1: \mathbb{H}^2(\mathcal{A}; \mathcal{E}_{\mathcal{O}_X}(\mathcal{E})) \rightarrow \mathbb{H}^{2k+2}\left(X, \frac{\Omega^*(\mathcal{L})}{g_k^*+1}\right), \quad \sigma^k_1(x) = \frac{1}{k!} \text{Tr}(\text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E})^k x),
\]

and then \( \sigma^k = i_k \tau_k \).

Then the theorem is a consequence of the fact that the DG-Lie algebra \( \text{Tot}\left(\mathcal{U}, \frac{\Omega^*(\mathcal{L})}{g_k^*+1}[2k]\right) \) is abelian and then, by general facts (see e.g. [21, 22]), every obstruction of the deformation functor associated to the DG-Lie algebra \( \mathcal{B} \) is annihilated by the maps \( \sigma^k_1 \).

\[\square\]

**Remark 8.3.** The induced map in hypercohomology \( \sigma^k_1 \) depends only on the \( \mathcal{A} \)-module \((\mathcal{E}, \nabla^A)\) and not on the choice of a simplicial \( \mathcal{L} \)-connection \( \nabla \) extending \( \nabla^A \). In fact, \( \sigma^k_1 \) depends only on the Atiyah class \( \text{At}_{\mathcal{L}/\mathcal{A}}(\mathcal{E}) \) of the curved DG-pair

\[
(A = \text{Tot}(\mathcal{U}, \Omega^*(\mathcal{L}, \mathcal{E}_{\mathcal{O}_X}(\mathcal{E}))), I = \text{Tot}(\mathcal{U}, g_k^* \otimes \mathcal{E}_{\mathcal{O}_X}(\mathcal{E}))),
\]

which we proved in Theorem 7.8 does not depend on the choice of \( \nabla \).

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