A FAMILY OF DIGIT FUNCTIONS WITH LARGE PERIODS

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Abstract. For odd $n \geq 3$, we consider a general hypothetical identity for the differences $S_{n,0}(x)$ of multiples of $n$ with even and odd digit sums in the base $n - 1$ in interval $[0,x)$, which we prove in the cases $n = 3$ and $n = 5$ and empirically confirm for some other $n$. We give a verification algorithm for this identity for any odd $n$. The hypothetical identity allows to give a general recursion for $S_{n,0}(x)$ for every integer $x$ depending on the residue of $x$ modulo $p(n) = 2n(n - 1)^{n-1}$, such that $p(3) = 24$, $p(5) = 2560$, $p(7) = 653184$, etc.

1. Introduction

For $x \in \mathbb{N}$ and $n \geq 3$, denote by $S_n(x)$ the sum

$$S_{n,j}(x) = \sum_{0 \leq r < x: \ r \equiv j \pmod{n}} (-1)^{s_{n-1}(r)},$$

where $s_{n-1}(r)$ is the digit sum of $r$ in base $n - 1$.

Note that, in particular, $S_{3,0}(x)$ equals the difference between the numbers of multiples of 3 with even and odd binary digit sums (or multiples of 3 from sequences A001969 and A000069 in [7]) in interval $[0, x)$.

Leo Moser (cf. [3], Introduction) conjectured that always

$$S_{3,0}(x) > 0.$$  

Newman [3] proved this conjecture. Moreover, he obtained the inequalities

$$\frac{1}{20} < S_{3,0}(x)x^{-\lambda} < 5,$$

where

$$\lambda = \frac{\ln 3}{\ln 4} = 0.792481\ldots.$$  

In connection with this, the qualitative result (2) we call a weak Newman phenomenon (or Moser-Newman phenomenon), while an estimating result of the form (3) we call a strong Newman phenomenon.
In 1983, Coquet [1] studied a very complicated continuous and nowhere differentiable fractal function \( F(x) \) with period 1 for which

\[
S_{3,0}(3x) = x^\lambda F \left( \frac{\ln x}{\ln 4} \right) + \frac{\eta(x)}{3},
\]

where

\[
\eta(x) = \begin{cases} 
0, & \text{if } x \text{ is even}, \\
(-1)^{s_2(3x-1)}, & \text{if } x \text{ is odd}
\end{cases}
\]

He obtained that

\[
\limsup_{x \to \infty, x \in \mathbb{N}} S_{3,0}(3x)x^{-\lambda} = \frac{55}{3} \left( \frac{3}{65} \right)^\lambda = 1.601958421 \ldots,
\]

\[
\liminf_{x \to \infty, x \in \mathbb{N}} S_{3,0}(3x)x^{-\lambda} = \frac{2\sqrt{3}}{3} = 1.154700538 \ldots.
\]

In 2007, Shevelev [4] gave an elementary proof of Coquet’s formulas (7)-(8) and his sharp estimates in the form

\[
\frac{2\sqrt{3}}{3} x^{\lambda} \leq S_{3,0}(3x, 0) \leq \frac{55}{3} \left( \frac{3}{65} \right)^\lambda x^{\lambda}, \quad x \in \mathbb{N}.
\]

In [4] it was found the following simple identity

\[
S_{3,0}(4x) = 3S_{3,0}(x), \quad \text{where } x \text{ is even.}
\]

Since in the left hand side of (10) the argument \( 4x \equiv 0 \pmod{8} \) then (10) is not a recursion for evaluation of \( S_{3,0}(x) \). However, in the same work Shevelev found the following recursion for fast calculation of \( S_{3,0}(x) \):

\[
S_{3,0}(x) = 3S_{3,0} \left( \left\lfloor \frac{x}{4} \right\rfloor \right) + \nu(x),
\]

where

\[
\nu(x) = \begin{cases} 
0, & \text{if } x \equiv 0, 7, 8, 9, 16, 17, 18, 22, 23 \pmod{24}; \\
(-1)^{s_2(x)}, & \text{if } x \equiv 3, 4, 10, 12, 20 \pmod{24}; \\
(-1)^{s_2(x)+1}, & \text{if } x \equiv 1, 2, 5, 6, 11, 19, 21 \pmod{24}; \\
2(-1)^{s_2(x)}, & \text{if } x \equiv 15 \pmod{24}; \\
2(-1)^{s_2(x)+1}, & \text{if } x \equiv 13, 14 \pmod{24}.
\end{cases}
\]

In 2008, Drmota and Stoll [2] proved a generalized weak Newman phenomenon, showing that (2) is valid for \( S_{n,0}(x) \) for every \( n \geq 3 \), at least beginning with \( x \geq x_0(n) \). A year before, Shevelev [5] proved a strong form
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of this generalization, but yet only in "full" intervals of the form \([0, (n-1)^{2p})\). Recently Shevelev and Moses \([6]\) in the case of odd \(n \geq 3\) and \(p \geq \frac{n-1}{2}\) found the relation

\[
\sum_{k=0}^{n/2} (-1)^k \binom{n}{2k} S_{n,0}((n-1)^{2p-2k}) = \begin{cases} 0, & \text{if } p \geq \frac{n+1}{2} \\ (-1)^n, & \text{if } p = \frac{n-1}{2}. \end{cases}
\]

In the case of \(p = \frac{n-1}{2}\), (13) could be rewrite in the form

\[
\sum_{j=0}^{n/2} (-1)^j \binom{n}{2j+1} S_{n,0}((n-1)^{2j}) = 1.
\]

Numerous experiments show that, most likely, the following more general relation takes place:

\[
\sum_{j=0}^{n/2} (-1)^j \binom{n}{2j+1} S_{n,0}((n-1)^{2j}) = \sum_{j=0}^{n-1} S_{n,j}(x), \quad x \geq 1, \; n \equiv 1 \pmod{2}.
\]

In particular, we verified (15) for \(n = 3, 5, 7, ..., 35\) and \(1 \leq x \leq 1000\). It is clear that (14) is a special case of (15) for \(x = 1\), since

\[
S_{n,j}(1) = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } 1 \leq j \leq n-1. \end{cases}
\]

Below we show that (15) allows with the uniform positions to find a recursion for \(S_{n,0}(x)\) for every odd \(n \geq 3\). In the two first sections we prove identity (15) in cases \(n = 3\) and \(n = 5\). In Section 4 we give a general verification algorithm for the identity (15) which allows to prove the identity (15) for \(n = 7, 9, ..., etc\). In Section 5 we give a simplification of the conjectural equality (15). In Section 6 we prove the recursion in case \(n = 3\) and in Section 7 we give the recursion in case \(n = 5\). After these sections, in supposition that (15) is true, it will be clear how to find the further recursions for odd \(n \geq 7\).

2. THE IDENTITY IN CASE \(n = 3\)

Note that, by (11),

\[
S_{3,j}(x) = \sum_{0 \leq r < x: \; r \equiv j \pmod{3}} (-1)^{s_2(r)}
\]

which yields that

\[
\sum_{0 \leq r < 2x: \; r \equiv 2j \pmod{6}} (-1)^{s_2(r)}, \; j = 0, 1, 2.
\]
On the other hand,
\[ S_{3, j}(2x) = \sum_{0 \leq r < 2x: \, r \equiv j \pmod{6}} (-1)^{s_2(r)} + \]
(18)
\[ \sum_{0 \leq r < 2x: \, r \equiv j + 3 \pmod{6}} (-1)^{s_2(r)}, \, j = 0, 1, 2. \]

Using (18), for \( j = 0, 1, 2 \), we consecutively find
\[ S_{3, 0}(2x) = \sum_{0 \leq r < 2x: \, r \equiv 0 \pmod{6}} (-1)^{s_2(r)} - \]
(19)
\[ \sum_{0 \leq r < 2x: \, r \equiv 2 \pmod{6}} (-1)^{s_2(r)}, \]
\[ S_{3, 1}(2x) = -\sum_{0 \leq r < 2x: \, r \equiv 0 \pmod{6}} (-1)^{s_2(r)} + \]
(20)
\[ \sum_{0 \leq r < 2x: \, r \equiv 4 \pmod{6}} (-1)^{s_2(r)}, \]
\[ S_{3, 2}(2x) = \sum_{0 \leq r < 2x: \, r \equiv 2 \pmod{6}} (-1)^{s_2(r)} - \]
(21)
\[ \sum_{0 \leq r < 2x: \, r \equiv 4 \pmod{6}} (-1)^{s_2(r)}. \]

Now the application of (17) to (19)-(21) yields the relations
\[ S_{3, 0}(2x) = S_{3, 0}(x) - S_{3, 1}(x), \]
(22)
\[ S_{3, 1}(2x) = -S_{3, 0}(x) + S_{3, 2}(x), \]
(23)
\[ S_{3, 2}(2x) = S_{3, 1}(x) - S_{3, 2}(x). \]
(24)

For \( n = 3 \), the left hand side of (15) is \( 3S_{3, 0}(x) - S_{3, 0}(4x) \) and, using (22)-(24), we have
\[ 3S_{3, 0}(x) - S_{3, 0}(4x) = 3S_{3, 0}(x) - S_{3, 0}(2x) + S_{3, 1}(2x) = \]
\[ 3S_{3, 0}(x) - S_{3, 0}(x) + S_{3, 1}(x) - S_{3, 0}(x) + S_{3, 2}(x) = \]
\[ S_{3, 0}(x) + S_{3, 1}(x) + S_{3, 2}(x) \]
which proves (15) in the case \( n = 3 \).
3. The identity in case $n = 5$

In the same way, instead of (22)-(24), we find the following relations

(25) \[ S_{5,0}(4x) = S_{5,0}(x) - S_{5,1}(x) + S_{5,2}(x) - S_{5,3}(x), \]

(26) \[ S_{5,1}(4x) = -S_{5,0}(x) + S_{5,1}(x) - S_{5,2}(x) + S_{5,4}(x), \]

(27) \[ S_{5,2}(4x) = S_{5,0}(x) - S_{5,1}(x) + S_{5,3}(x) - S_{5,4}(x), \]

(28) \[ S_{5,3}(4x) = -S_{5,0}(x) + S_{5,2}(x) - S_{5,3}(x) + S_{5,4}(x), \]

(29) \[ S_{5,4}(4x) = S_{5,1}(x) - S_{5,2}(x) + S_{5,3}(x) - S_{5,4}(x). \]

For $n = 5$, the left hand side of (15) is

(30) \[ 5S_{5,0}(x) - 10S_{5,0}(16x) + S_{5,0}(256x). \]

Using (25)-(29), we easily find

(31) \[ S_{5,0}(16x) = 4S_{5,0}(x) - 3S_{5,1}(x) + S_{5,2}(x) + S_{5,3}(x) - 3S_{5,4}(x), \]

(32) \[ S_{5,1}(16x) = -3S_{5,0}(x) + 4S_{5,1}(x) - 3S_{5,2}(x) + S_{5,3}(x) + S_{5,4}(x), \]

(33) \[ S_{5,2}(16x) = S_{5,0}(x) - 3S_{5,1}(x) + 4S_{5,2}(x) - 3S_{5,3}(x) + S_{5,4}(x), \]

(34) \[ S_{5,3}(16x) = S_{5,0}(x) + S_{5,1}(x) - 3S_{5,2}(x) + 4S_{5,3}(x) - 3S_{5,4}(x), \]

(35) \[ S_{5,4}(16x) = -3S_{5,0}(x) + S_{5,1}(x) + S_{5,2}(x) - 3S_{5,3}(x) + 4S_{5,4}(x). \]

Now using (31)-(35), we find

\[
S_{5,0}(256x) = 36S_{5,0}(x) - 29S_{5,1}(x) + 11S_{5,2}(x) + 11S_{5,3}(x) - 29S_{5,4}(x).
\]

Finally, for the expression (30), using (31) and (36), we have

(37) \[ 5S_{5,0}(x) - 10S_{5,0}(16x) + S_{5,0}(256x) = S_{5,0}(x) + S_{5,1}(x) + S_{5,2}(x) + S_{5,3}(x) + S_{5,4}(x). \]

It is the identity (15) in the case $n = 5$. 

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4. General problem

Quite analogously to systems (22)-(24), (25)-(29) we can write the system for any \( n \geq 3 \). For odd \( n \), we have

\[
S_{n,0}((n-1)x) = S_{n,0}(x) - S_{n,1}(x) + \cdots + S_{n,n-3}(x) - S_{n,n-2}(x),
\]

\[
S_{n,1}((n-1)x) = S_{n,0}(x) + S_{n,1}(x) - \cdots - S_{n,n-3}(x) + S_{n,n-1}(x),
\]

\[
S_{n,2}((n-1)x) = S_{n,0}(x) - S_{n,1}(x) + \cdots - S_{n,n-4}(x) + S_{n,n-2}(x) - S_{n,n-1}(x),
\]

.................................

\[
S_{n,n-2}((n-1)x) = -S_{n,0}(x) + S_{n,2}(x) - \cdots - S_{n,n-2}(x) + S_{n,n-1}(x),
\]

(38)

\[
S_{n,n-1}((n-1)x) = S_{n,1}(x) - S_{n,2}(x) + \cdots + S_{n,n-1}(x).
\]

It is easy to see that the right hand side of the \( i \)-th equality for \( S_{n,i}((n-1)x) \), \( i = 0, 1, \ldots, n-1 \), of the system (38) satisfies the rules: 1) the signs alternate, beginning with \((-1)^{i}\); 2) there is no summand \( S_{n,n-1-i}(x) \). Using, as usual, the convention \( \sum_{a}^{b} = 0 \), if \( b < a \), one can write the system (38) in the form

\[
(-1)^{i}S_{n,i}((n-1)x)) = \sum_{j=0}^{n-i-2} (-1)^{j}S_{n,j}(x) - \sum_{j=n-i}^{n-1} (-1)^{j}S_{n,j}(x).
\]

(39)

Thus the general problem is to prove that (39) yields (15).

5. A simplification of the conjecture

Note that in the sum \( \sum_{j=0}^{n-1} S_{n,j}(x) \) the index of summing \( j \) runs all residues modulo \( n \). Therefore, we have

\[
\sum_{j=0}^{n-1} S_{n,j}(x) = S_{1,0}(x) = \sum_{0 \leq i < x} (-1)^{s_{n-1}(i)} =
\]

(40)

\[
\begin{cases}
0, & \text{if } x \text{ is even}, \\
(-1)^{s_{n-1}(x-1)}, & \text{if } x \text{ is odd}.
\end{cases}
\]

Thus the conjectural relation (15) is equivalent to the equality

\[
\sum_{j=0}^{n-1} (-1)^{j} \binom{n}{2j+1} S_{n,0}((n-1)^{2j}x) =
\]

(41)

\[
\begin{cases}
0, & \text{if } x \text{ is even}, \\
(-1)^{s_{n-1}(x-1)}, & \text{if } x \text{ is odd}.
\end{cases}
\]

In particular, for \( x = 1 \), we again have (14). Note that (41) means that its left hand side taken with sign \((-1)^{s_{n-1}(x-1)}\) is periodic with period 2:
\[
(-1)^{s_{n-1}(x-1)} \sum_{j=0}^{n-1} (-1)^j \left( \frac{n}{2j+1} \right) S_{n,0}((n-1)^{2j}x) = \\
\begin{cases} 
0, & \text{if } x \text{ is even,} \\
1, & \text{if } x \text{ is odd.}
\end{cases}
\]

(42)

6. Recursion for \( S_{3,0}(x) \)

Here we prove (11)-(12). Let us write (42) for \( n = 3 \) and \( x := \lfloor \frac{x}{4} \rfloor \). We have

\[
(-1)^{s_2(\lfloor \frac{x}{4} \rfloor - 1)} (3S_{3,0}(\lfloor \frac{x}{4} \rfloor) - S_{3,0}(4\lfloor \frac{x}{4} \rfloor)) = \\
\begin{cases} 
0, & \text{if } \lfloor \frac{x}{4} \rfloor \text{ is even,} \\
1, & \text{if } \lfloor \frac{x}{4} \rfloor \text{ is odd.}
\end{cases}
\]

(43)

Note that \( \lfloor \frac{x}{4} \rfloor \) is even, if \( x = 0, 1, 2, 3, 8, 9, 10, 11, \ldots \) and odd for other integers. Thus we obtain

**Lemma 1.** The sequence \( \{A_3(x)\} \), where

\[
A_3(x) = (-1)^{s_2(\lfloor \frac{x}{4} \rfloor - 1)} (3S_{3,0}(\lfloor \frac{x}{4} \rfloor) - S_{3,0}(4\lfloor \frac{x}{4} \rfloor)),
\]

is periodic with the period 8, such that

\[
A_3(x) = \begin{cases} 
0, & \text{if } x \equiv 0, 1, 2, 3, \pmod{8}, \\
1, & \text{if } x \equiv 4, 5, 6, 7 \pmod{8}.
\end{cases}
\]

(45)

Consider the difference

\[
\Delta_3(x) = S_{3,0}(x) - S_{3,0}(4\lfloor \frac{x}{4} \rfloor).
\]

**Lemma 2.** We have

\[
\Delta_3(x) = \begin{cases} 
(-1)^{s_2(x-1)}, & \text{if } x \equiv 1, 7 \text{ or } 10 \pmod{12} \\
(-1)^{s_2(x-2)}, & \text{if } x \equiv 2 \text{ or } 11 \pmod{12} \\
(-1)^{s_2(x-3)}, & \text{if } x \equiv 3 \pmod{12} \\
0, & \text{otherwise.}
\end{cases}
\]

(47)

**Proof.** Let \( x = 12t + j \), \( j = 0, 1, \ldots, 11 \). Consider 3 cases.

a) \( j = 0, 1, 2 \) or 3.

Then

\[
\Delta_3(x) = S_{3,0}(12t + j) - S_{3,0}(12t) = \\
\begin{cases} 
0, & \text{if } j = 0, \\
(-1)^{s_2(x-j)}, & \text{if } j = 1, 2, 3.
\end{cases}
\]
Then
\[ \Delta_3(x) = S_{3,0}(12t + j) - S_{3,0}(12t + 4) = \]
\[
\begin{cases}
0, & \text{if } j = 4, 5, 6, \\
\left(-1\right)^{s_2(x-1)}, & \text{if } j = 7.
\end{cases}
\]

Then
\[ \Delta_3(x) = S_{3,0}(12t + j) - S_{3,0}(12t + 8) = \]
\[
\begin{cases}
0, & \text{if } j = 8, 9, \\
\left(-1\right)^{s_2(x-1)}, & \text{if } j = 10, \\
\left(-1\right)^{s_2(x-2)}, & \text{if } j = 11
\end{cases}
\]

and (47) follows. ■

Now from (44)-(47) we easily deduce the following result.

**Theorem 3.**

\[ S_{3,0}(x) = 3S_{3,0}(\lfloor \frac{x}{4} \rfloor) + \Delta_3(x) - (-1)^{s_2(\lfloor \frac{x}{4} \rfloor) - 1}A_3(x), \]

where \( A_3(x) \) and \( \Delta_3(x) \) are defined by (45) and (47) respectively.

Formula (48) gives a recursion for \( S_{3,0}(x) \). Let us show that it coincides with the recursion (11)-(12), i.e.,

\[ \Delta_3(x) - (-1)^{s_2(\lfloor \frac{x}{4} \rfloor) - 1}A_3(x) = \nu(x), \]

where \( \nu(x) \) is defined by (12). This follows from the following two lemmas.

**Lemma 4.** The sequence

\[ \{-(-1)^{s_2(x)+s_2(\lfloor \frac{x}{4} \rfloor) - 1} A_3(x)\} \]

is periodic with period 8.

**Proof.** In cases \( x \equiv i \pmod{8}, \ i = 0, 1, 2, 3 \) the terms of the sequence are zeros. If \( x \equiv i \pmod{8}, \ i = 4, 5, 6, 7 \), put \( x = 8t + i \). Then \( A_3(x) = 1 \) and we have

\[ (-1)^{s_2(x)+s_2(\lfloor \frac{x}{4} \rfloor) - 1} = (-1)^{s_2(8t+i)+s_2(2t)} = \]

\[ (-1)^{s_2(8t+i)+s_2(8t)} = (-1)^{s_2(i)} \]

and the lemma follows. ■

Note that period of sequence (50) is
Lemma 5. The sequence

\[
\{(-1)^{s_2(x)} \Delta_3(x)\}
\]

is periodic with period 12.

Proof. According to (47), we have

\[
(-1)^{s_2(x)} \Delta_3(x) = \begin{cases} 
(-1)^{s_2(x)+s_2(x-1)}, & \text{if } x \equiv 1, 7 \text{ or } 10 \pmod{12} \\
(-1)^{s_2(x)+s_2(x-2)}, & \text{if } x \equiv 2 \text{ or } 11 \pmod{12} \\
(-1)^{s_2(x)+s_2(x-3)}, & \text{if } x \equiv 3 \pmod{12} \\
0, & \text{otherwise.}
\end{cases}
\]

Let \( x = 12t + i, \ 0 \leq i \leq 11 \). Let, firstly, \( i = 1, 7, 10 \). In cases \( i = 1 \) and \( i = 7 \), we, evidently, have \((-1)^{s_2(x)+s_2(x-1)} = -1\), while in case \( i = 10 \),

\[
(-1)^{s_2(12t+10)+s_2(12t+9)} = (-1)^{s_2(12t+1010_2)+s_2(12t+1001_2)} = 1.
\]

Let now \( i = 2, 11 \). In case \( i = 2 \), we, evidently, have \((-1)^{s_2(x)+s_2(x-2)} = -1\) and also in case \( i = 11 \), we find

\[
(-1)^{s_2(12t+11)+s_2(12t+9)} = (-1)^{s_2(12t+1011_2)+s_2(12t+1001_2)} = -1;
\]

finally, if \( i = 3 \), then, evidently, we have \((-1)^{s_2(x)+s_2(x-3)} = 1\). In other cases, the terms of the sequence are zeros. \(\blacksquare\)

Thus period of sequence (52) is

\[
\{0, -1, -1, 1, 0, 0, -1, 0, 0, 1, -1\}.
\]

Subtracting the tripled period (51) from the doubled period (54), we obtain the period of length 24 of the left hand side of (49) multiplied by \((-1)^{s_2(x)}\).

It is

\[
\{0, -1, -1, 1, 1, -1, -1, 0, 0, 1, -1, -1, 0, 0, 1, -1, 1, -1, 0, 0, 1, 0\}.
\]

It is left to note that, according to (12), \((-1)^{s_2(x)\nu(x)}\) is periodic with the same period. \(\blacksquare\)
7. On recursion for $S_{n,0}(x)$

Let \((12)\) be true. Let us write \((12)\) for $x := \lfloor \frac{x}{(n-1)^{n-1}} \rfloor$. We have
\[
(-1)^{s_{n-1}}((-1)^{n-1}\frac{x}{(n-1)^{n-1}})^{-1}(-1)^{\frac{n-1}{2}}S_{n,0}((n-1)^{n-1}\lfloor \frac{x}{(n-1)^{n-1}} \rfloor) +
\sum_{j=0}^{\frac{n}{2}}(-1)^{j}\binom{n}{2j+1}S_{n,0}((n-1)^{2j}\lfloor \frac{x}{(n-1)^{n-1}} \rfloor) =
\]
\[
\begin{cases}
0, & \text{if } \lfloor \frac{x}{(n-1)^{n-1}} \rfloor \text{ is even},
1, & \text{if } \lfloor \frac{x}{(n-1)^{n-1}} \rfloor \text{ is odd}.
\end{cases}
\]

Denote the left hand side of (56) by $A_n(x)$. Then, similar to (45), we have
\[
A_n(x) =
\]
\[
\begin{cases}
0, & \text{if } x \equiv 0, ..., (n-1)^{n-1} - 1, \pmod{2(n-1)^{n-1}},
1, & \text{if } x \equiv (n-1)^{n-1} - 1, 2(n-1)^{n-1} - 1, \pmod{2(n-1)^{n-1}}.
\end{cases}
\]

Furthermore, we consider the difference
\[
\Delta_n(x) = S_{n,0}(x) - S_{n,0}((n-1)^{n-1}\lfloor \frac{x}{(n-1)^{n-1}} \rfloor).
\]

**Lemma 6.** $(-1)^{s_{n-1}(x)}\Delta_n(x)$ is periodic with period $(n-1)^{n-1}$.

**Proof.** Indeed, let
\[
x = n(n-1)^{n-1}t + j, \quad j = 0, 1, ..., n(n-1)^{n-1} - 1.
\]

Let $j$ such that
\[
\lfloor \frac{j}{(n-1)^{n-1}} \rfloor = m, \quad 0 \leq m \leq n - 1.
\]

Then
\[
j = (n-1)^{n-1}m + k, \quad 0 \leq k \leq (n-1)^{n-1} - 1.
\]

We have
\[
\Delta_n(x) = S_{n,0}(n(n-1)^{n-1}t + j) - S_{n,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m) =
S_{n,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m + k) - S_{n,0}(n(n-1)^{n-1}t + (n-1)^{n-1}m) =
\]
\[
\sum_{i:(n-1)^{n-1}m + 1 \leq 5i \leq (n-1)^{n-1}m + k-1}(-1)^{s_4(n(n-1)^{n-1}t+5i)}.
\]

Note that
\[
5i = (n-1)^{n-1}m + l, \quad 1 \leq l \leq k - 1 \leq (n-1)^{n-1} - 2.
\]

Therefore, the summands in (59) multiplied by $(-1)^{s_{n-1}}$ have the form
Lemma 7. The sequence

\[(60) \{(-1)^{s_n-1(x)} + s_n-1(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor-1) A_n(x) \}\]

is periodic with period $2(n-1)^{n-1}$.

Proof. In cases $x \equiv i \pmod{2(n-1)^{n-1}}$, $i = 0, 1, ..., (n-1)^{n-1} - 1$ the terms of the sequence are zeros. If $x \equiv i \pmod{2(n-1)^{n-1}}$, $i = (n-1)^{n-1}, ..., 2(n-1)^{n-1} - 1$, put $x = 2(n-1)^{n-1} t + i$. Then $A_n(x) = 1$ and we have

\[
(-1)^{s_n-1(x)} + s_n-1(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor-1) = (-1)^{s_n-1(2(n-1)^{n-1} t+i) + s_n-1(2t)} =
\]

\[
(-1)^{s_n-1(2(n-1)^{n-1} t+i) + s_n-1(2(n-1)^{n-1} t)} = (-1)^{s_n-1(i)}
\]

and the lemma follows.

Now we obtain the following result.

Theorem 8. If the conjectural relation (13) is true, then we have

\[(61) \ S_{n,0}(x) = \sum_{j=0}^{n-3} (-1)^{n-3-j} \binom{n}{2j+1} S_{n,0}((n-1)^{2j}[\frac{x}{(n-1)^{n-1}}]) + \nu_n(x),\]

where $\nu_n(x)$ multiplied by $(-1)^{s_n-1(x)}$ is periodic with period $2n(n-1)^{n-1}$.

Proof. Indeed, by (56)-(58), we obtain (61) with

\[
\nu_n(x) = \Delta_n(x) + (-1)^{s_n-1(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor-1)} A_n(x).
\]

Then, by Lemmas 6-7, $(-1)^{s_n-1(x)} \nu_n(x)$ is periodic with period equal the least common multiple of numbers $2(n-1)^{n-1}$ and $n(n-1)^{n-1}$.

As a corollary, in the case $n = 3$ we again obtain Theorem 3 for $\nu(x) = \nu_3(x)$ but without detailed representation of $\Delta_3(x)$ and $\nu(x)$.

Remark 9. It follows from the proof that, if for some

\[
j = j_i, \ i = 1, ..., k, \ 1 \leq j_1 < j_2 < ... < j_k \leq \frac{n-3}{2},
\]

to replace in (61) $S_{n,0}((n-1)^{2j}[\frac{x}{(n-1)^{n-1}}])$ by $S_{n,0}(\lfloor \frac{x}{(n-1)^{n-1}} \rfloor)$ and to
digit functions with large periods denote the new sum by \( \Sigma(j_1, ..., j_k) \), then also the following form of Theorem 8 is valid

**Theorem 10.** If the conjectural relation (15) is true, then we have

\[
S_{n,0}(x) = \Sigma(j_1, ..., j_k) + \nu_n^{(j_1, ..., j_k)}(x),
\]

where \( \nu_n^{(j_1, ..., j_k)}(x) \) multiplied by \((-1)^{s_{n-1}(x)}\) is periodic with period \(2n(n-1)^{n-1}\).

Thus we have \(2^{n-3}\) different formulas of type (62). In particular, in case \(n = 3\) we have only formula, in case \(n = 5\) we have two different formulas, etc.

8. **Application of Theorem 8 in case \(n = 5\)**

Since the conjectural identity (15) was proved in case \(n = 5\), then, by Theorem 8 we conclude that

\[
(-1)^{s_5(x)}\nu_5(x) = (-1)^{s_5(x)}(S_{5,0}(x) - 10S_{5,0}(\frac{x}{256}) + 5S_{5,0}(\lfloor \frac{x}{256} \rfloor))
\]

is periodic with period 2560. If to write the period, then (63) gives a recursion for \(S_{5,0}(x)\). The computer calculations show that the period with positions \(\{0, ..., 2559\}\) contains all numbers from interval \([-35, 35]\). Here we give several sequences of positions in \([0, 2559]\) with these numbers \(g \in [-35, 35]\).

- \(g = 35 : \{251, 252, 254\}\),
- \(g = 34 : \{246, 249, 1531, 1532, 1534\}\),
- \(g = 33 : \{241, 243, 244, 1526, 1529\}\),
- \(g = 32 : \{237, 239, 1521, 1523, 1524\}\),
- \(g = 31 : \{231, 232, 234, 1517, 1519\}\),
- \(g = 30 : \{197, 199, 200, 217, 219, 220, 226, 229, 511, 1511, 1512, 1514, 2497, 2499, 2500, 2557, 2559\}\),
- \(g = -30 : \{196, 198, 216, 218, 227, 228, 230, 1513, 1515, 2496, 2498, 2556, 2558\}\),
- \(g = 31 : \{233, 235, 1516, 1518, 1520\}\),
- \(g = 32 : \{236, 238, 240, 1522, 1525\}\),
- \(g = 33 : \{242, 245, 1527, 1528, 1530\}\),
- \(g = 34 : \{247, 248, 250, 1533, 1535\}\),
- \(g = 35 : \{253, 255\}\).

Besides, by Theorem 10, also
(64) \((-1)^{s_2(x)} \nu_5^{(1)}(x) = \frac{-1}{10} (S_{5, \nu_0(x)} - 10 S_{5, \nu_0(\lfloor \frac{x}{16} \rfloor)} + 5 S_{5, \nu_0(\lfloor \frac{x}{256} \rfloor)})\)

is periodic with period 2560. Again, if to write the period, then (64) gives another recursion for \(S_{5, \nu_0(x)}\). The computer calculations show that the period with positions \(\{0, \ldots, 2559\}\) contains all numbers from interval \([-9, 9]\). Several sequences of positions in \([0, 2559]\) with these numbers \(h \in [-9, 9]\) are the following:

\[
\begin{align*}
  h &= -9 : \{2411, 2412, 2414, 2491, 2492, 2494\}, \\
  h &= -8 : \{1131, 1132, 1134, 1211, 1212, 1214, 2406, 2409, 2486, 2489\}, \\
  h &= 8 : \{1133, 1135, 1213, 1215, 2407, 2408, 2410, 2487, 2488, 2490\}, \\
  h &= 9 : \{2413, 2415, 2493, 2495\}. 
\end{align*}
\]

Finally, note that the sequence of the numbers of different values of \(\nu_3(x), \nu_5^{(1)}(x), \nu_5(x)\), etc. begins with \(5, 19, 71, \ldots\).

9. Recursions for \(S_{3,1}(x)\) and \(S_{3,2}(x)\)

Using (22)-(24), it is easy to show that the form \(3y(x) - y(4x)\) is invariant with respect to \(S_{3,i}(x)\), \(i = 0, 1, 2\). This means that together with

(65) \[3S_{3,0}(x) - S_{3,0}(4x) = S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x),\]

we have also

(66) \[3S_{3,1}(x) - S_{3,1}(4x) = S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x),\]

(67) \[3S_{3,2}(x) - S_{3,2}(4x) = S_{3,0}(x) + S_{3,1}(x) + S_{3,2}(x).\]

Using (66)-(67), as in Section 6, we can prove that the expressions

(68) \[(-1)^{s_2(x)} (S_{3,1}(x) - 3S_{3,1}(\lfloor \frac{x}{4} \rfloor)),\]

and

(69) \[(-1)^{s_2(x)} (S_{3,2}(x) - 3S_{3,2}(\lfloor \frac{x}{4} \rfloor)),\]

are eventually periodic with the same period as \((-1)^{s_2(x)} \nu(x)\) (12), i.e., the period (55), such that for \(S_{3,2}(x)\) the period starts at \(x = 8\), while for \(S_{3,1}(x)\) the period starts at \(x = 16\). This means that, for \(S_{3,i}(x)\), \(i = 1, 2\), the same recursions hold as the recursion for \(S_{3,0}(x)\) (11) with the same function \(\nu(x)\) (12):

(70) \[S_{3,1}(x) = 3S_{3,1}(\lfloor \frac{x}{4} \rfloor) + \nu(x), \quad x \geq 16,\]
with the initials

\[ S_{3,1}(x) = \begin{cases} 
0, & \text{if } x = 0, 1, \\
-1, & \text{if } x = 2, 3, 4, \\
-2, & \text{if } x = 5, 6, 7, 11, 12, 13, \\
-3, & \text{if } x = 8, 9, 10, 14, 15.
\end{cases} \tag{71} \]

\[ S_{3,2}(x) = 3S_{3,2} \left( \left\lfloor \frac{x}{4} \right\rfloor \right) + \nu(x), \quad x \geq 8, \tag{72} \]

with the initials

\[ S_{3,2}(x) = \begin{cases} 
0, & \text{if } x = 0, 1, 2, 6, 7, \\
-1, & \text{if } x = 3, 4, 5.
\end{cases} \tag{73} \]

For example, by (70), (71) and (12), we have

\[ S_{3,1}(20) = 3S_{3,1}(5) + \nu(20) = 3 \cdot (-2) + (-1)^{s_2(20)} = -5; \]

analogously, by (72), (73) and (12), we find

\[ S_{3,2}(20) = 3S_{3,2}(5) + \nu(20) = 3 \cdot (-1) + (-1)^{s_2(20)} = -2. \]

10. A generalization

A generalization of the conjectural equality (15) is the following

\[ \sum_{j=0}^{n-1} (-1)^j \binom{n}{2j+1} S_{n,j}((n-1)^{2j} x) = \]

\[ \sum_{j=0}^{n-1} S_{n,j}(x), \quad i = 0, \ldots, n-1, \quad x \geq 1, \quad n \equiv 1 \pmod{2}. \tag{74} \]

If this conjecture is valid, then, as in the previous sections, we can obtain the same recursions for every digit function \( S_{n,i}(x) \), \( i = 1, \ldots, n-1 \), as for \( S_{n,0}(x) \) (cf. Theorems 8, 10). The question on initials in cases \( i \geq 1 \) we here remain open.

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