A POSITIVE BOUND STATE FOR AN ASYMPTOTICALLY LINEAR OR SUPERLINEAR SCHRÖDINGER EQUATION IN EXTERIOR DOMAINS

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Abstract. We establish the existence of a positive solution for semilinear elliptic equation in exterior domains

\[-\Delta u + V(x)u = f(u), \quad \text{in } \Omega \subseteq \mathbb{R}^N\]  \hfill \text{(PV)}

where \(N \geq 2\), \(\Omega\) is an open subset of \(\mathbb{R}^N\) and \(\mathbb{R}^N \setminus \Omega\) is bounded and not empty but there is no restriction on its size, nor any symmetry assumption. The nonlinear term \(f\) is a non homogeneous, asymptotically linear or superlinear function at infinity. Moreover, the potential \(V\) is a positive function, not necessarily symmetric. The existence of a solution is established in situations where this problem does not have a ground state.

1. Introduction. Our goal in this paper is to show the existence of a positive bound state solution for problem \((PV)\) when a ground state can not be obtained. Using a new approach recently developed by Évéquoz and Weth [20], Clapp and Maia [14] and Maia and Pellacci [25] a positive solution is found, extending the existence results obtained in the celebrated papers of Benci and Cerami [5] and Bahri and Lions [3], for general non homogeneous nonlinearities, either superlinear or asymptotically linear at infinity in an exterior domain.

The study of solitary waves of nonlinear Schrödinger equations or of nonlinear Klein-Gordon equations is modeled by \((PV)\) with \(\Omega = \mathbb{R}^N\). Likewise, exterior boundary-value problems may be associated with models of steady-state flows in fluid dynamics (see [21]) and electrostatic problem of capacitors (see [16], Volume 1, Chapter II), for instance.

The primary works applying variational methods to find solutions of problems like \((PV)\) report to the 80’s and 90’s with the articles of Benci and Cerami [5] and Bahri and Lions [3]. The method applied in both works was finding critical points of a functional constrained on a manifold and absorbing a Lagrange multiplier by the homogeneity of the nonlinear term \(f(u) = |u|^{p-2}u\) where \(p \in (2, 2^*)\) and \(2^* = \frac{2N}{N-2}\) if \(N \geq 3\), \(2^* = \infty\) if \(N = 2\) in order to obtain a positive solution of the Euler equation in \((PV)\).

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One of the main challenges of trying to apply the usual variational method when \( \Omega \) is an unbounded domain is the lack of compactness of the Sobolev embeddings. In order to circumvent this difficulty, a deeper study of the obstruction for compactness was performed by Benci and Cerami in [5] and a clever description was obtained of what happens when a Palais-Smale sequence does not converge to its weak limit (for details see [11] and references therein). Problem \((P_V)\) with \( f(u) = |u|^{p-2}u, \ p \in (2, 2^*) \) was solved in the case that the ground state does not exist first in [5] in the autonomous case \( V(x) = \lambda \) a positive constant, proving the existence of a positive solution with some restriction on the size of the hole \( \mathbb{R}^N \setminus \Omega \), and posteriorly that condition was eliminated in [3] and existence was proved for potentials \( V \) which decay to a constant potential \( V_\infty \) at infinity. In the same spirit, this problem has been extensively studied if \( \Omega \) is an exterior domain for power non-linearity \( f(u) = |u|^{p-2}u \) in recent years (see [3]). If the non-linear term \( f \) is not a pure power with respect to \( u \), there are few contributions in the literature. In particular, the existence of solution is proved in [13], using topological methods, in the case that \( f \) is super-linear and depends on the spatial variable but the asymptotic non-linearity \( f_\infty \) of the autonomous problem, must satisfy a convexity assumption.

In the case \( \Omega \) is spherically symmetric about some point, benefiting from the strength of the symmetry property, this problem can be solved on \( H^1_{rad}(\Omega) \) (subspace of radial functions in \( \mathbb{R}^N \)) which embeds compactly in \( L^p(\Omega) \), if \( p \in (2, 2^*) \). This idea was exploited by Berestycki and Lions in [6], Coffman and Marcus in [15] and Esteban and Lions in [19] when \( \Omega \) is the complement of a ball. However, symmetry of \( \Omega \) does not help if we don’t have radial symmetry in \( V(x) \). This is the case in our problem \((P_V)\) where we do not assume any symmetry, neither in \( \Omega \) nor in \( V(x) \).

In the past five decades a different approach has been successfully applied in order to obtain solutions for this class of problems with no symmetry assumption. The so-called Nehari method, [27] and [28], which consists of finding solutions of \((P_V)\) which are critical points of a functional associated with the equation in \((P_V)\), restricted to the Nehari manifold. This method has been extensively used in the last years in order to find ground state solutions as well as sign changing solutions of nonlinear elliptic problems in \( \mathbb{R}^N \) and exterior domains (see [17, 24, 29] and references therein). When finding a solution which is a minimum of the functional restricted to the Nehari manifold, the Lagrange multiplier is proved to be zero, yielding that the constrained critical point is in fact a free critical point of the functional, and that the manifold is a natural constraint. This allows to solve the problem for non-homogeneous nonlinearities because the multiplier does not have to be absorbed in the construction of a solution for the equation. Most importantly, this approach enables to avoid the use of a technical algebraic inequality \((a+b)^p \geq a^p + b^p + (p-1)(a^{p-1}b + ab^{p-1})\) largely applied in the case \( f(u) = |u|^{p-2}u \) ([2, 3, 12]). We follow these ideas, closely related to the arguments found in [14] and [25], for general nonlinearities \( f \) which satisfy the assumption that \( f(s)/s \) is increasing. In this setting, not all functions \( u \neq 0 \) are projectable on the Nehari manifold, however the class of functions which are good for projections in this environment is enough to pursue the argument.

Our main contribution in this paper is extending the result of Bahri and Lions [3] for non-homogeneous \( f \), with no symmetry assumption on \( V \) or \( \Omega \). Note that here \( \Omega \) is a general open set in \( \mathbb{R}^N \), with a bounded complement set, which might included several holes in a ball or just a point set \( \{0\} \). Moreover, we allow the non-linear \( f \) to be a less smooth function just in \( C^1 \), improving the hypotheses in [14]
and [25] where it was considered in $C^3$ for technical reasons (see Lemma 3.3 in [14]). Likewise, the work of [20] provided some useful tools and insight for estimates, even though their problem is for super-linear $f$ in the whole $\mathbb{R}^N$ and uses the generalized Nehari manifold.

To our knowledge the results we present here are new and extend the previous works in the literature for a class of problems in exterior domains. We consider the elliptic problem

$$-\Delta u + V(x)u = f(u), \quad u \in H^1_0(\Omega) \quad (P_V)$$

where $N \geq 2$, $\Omega$ is open and $\mathbb{R}^N \setminus \Omega \subseteq B_K(0)$ the ball of radius $K$ and center at the origin in $\mathbb{R}^N$, in fact $\mathbb{R}^N \setminus \Omega$ is bounded, $u \in H^1_0(\Omega)$ and $V$ is a potential satisfying the conditions:

(V1) $V \in C^0(\Omega)$, $\inf_{x \in \Omega} V(x) > 0$ and $\lim_{|x| \to +\infty} V(x) = V_\infty$;

(V2) $V(x) \leq V_\infty + C e^{-\gamma |x|}$, where $C > 0$ and $\gamma > 2 \sqrt{\gamma}$. The conditions that we consider on the nonlinearity $f$ are the following:

(f1) $f \in C^1([0, \infty))$;

(f2) There exist $A_1 > 0$ and $1 < p_1 \leq p_2$ such that $p_1, p_2 < 2^* - 1$ and

$$|f^{(k)}(s)| \leq A_1(|s|^{p_1-k} + |s|^{p_2-k})$$

for $k \in \{0, 1\}$ and $s > 0$;

(f3) $\lim_{s \to +\infty} \frac{f(s)}{s^3} \geq m > V_\infty$;

(f4) If $F(s) := \int_0^s f(t)dt$ and $Q(s) := \frac{1}{2} f(s)s - F(s)$, then

$$\lim_{s \to +\infty} Q(s) = +\infty;$$

(f5) The function $\frac{f(s)}{s} < f'(s)$ for $s > 0$;

(U) The positive radially symmetric solution of limit problem

$$-\Delta u + V_\infty u = f(u), \quad u \in H^1_0(\mathbb{R}^N) \quad (P_\infty)$$

is unique, up to translations.

Remark 1. We have

$$Q(s) := \frac{1}{2} f(s)s - F(s) > 0, \quad \forall s > 0 \quad (1)$$

because from (f3), $(\frac{f(s)}{s})' = \frac{tf'(s) - f(s)}{t^2} > 0$ and hence

$$f(s)s - 2F(s) = \int_0^s (f(t)t)' - 2f(t)dt = \int_0^s tf'(t) - f(t)dt > 0.$$

Remark 2. Note that $f(s) > 0$ for $s > 0$, since by (f2), $f(0) = f'(0) = 0$. On the other hand $f'(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s} = \lim_{s \to 0} \frac{f(s)}{s}$ and so by (f5), $\frac{f(s)}{s} > 0$; now we can write $f(s) = \frac{f(s)}{s}s > 0$ for $s > 0$.

It is straightforward to verify that the superlinear model nonlinearity $f(s) = s^p$, $s > 0$ with $p \in (1, 2^* - 1)$, and the asymptotically linear model nonlinearity $f(s) = \frac{s^3}{1 + \kappa s}$ with $b \in (0, V_\infty^{-1})$ satisfy the hypotheses (f1) - (f5).

Remark 3. The assumption (U) $\psi(s) := \frac{-V_\infty s + f(s)}{s} f'(s) - f(s)$ is non decreasing in $s \in (\sigma, +\infty)$ where $\sigma$ is the unique positive number such that $\frac{f(\sigma)}{\sigma} = V_\infty$, guarantees that the positive solution to the problem $(P_\infty)$ is unique (see [26], theorem 1 or
[30], theorem 1). It may be replaced by any other assumption which guarantees the uniqueness of positive ground state solution.

The main result of this work is the following

**Theorem 1.1.** Under assumptions $(V_1) - (V_2)$, $(f_1) - (f_3)$ and $(U)$, problem $(P_V)$ has a positive solution $u$ in $H^1_0(\Omega)$.

The paper is organized as follows. In section 2, we formulate the variational setting and present some preliminary results. Section 3 is dedicated to compactness condition. In section 4, applying a topological argument, which involves the barycenter map, we show that $I_V$ has a positive critical value.

2. Variational setting and exponential decay estimate. Note that by Remark 2, $f(s) > 0$ for $s > 0$, and we shall consider the extended $f(s) := -f(-s)$ for $s < 0$, so without loss of generality we may suppose that $f$ is odd and establish the existence of positive solution for it, which in particular will be a positive solution of the problem with the original $f$. We will use the following notation:

$$
\langle u, v \rangle_\Omega = \int_\Omega (\nabla u \cdot \nabla v + V(x)uv)dx, \quad \|u\|^2_\Omega = \int_\Omega (|\nabla u|^2 + V(x)u^2)dx.
$$

Our assumptions on $V$ imply that $\| \cdot \|_\Omega$ is a norm in $H^1_0(\Omega)$ which is equivalent to the standard one. We write

$$
\langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V_\infty uv)dx, \quad \|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V_\infty u^2)dx
$$

and since $V_\infty > 0$, then $\| \cdot \|$ is a norm in $H^1(\mathbb{R}^N)$ which is equivalent to the standard one. If $u \in H^1_0(\Omega)$ we may define $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and in fact $H^1_0(\Omega) \subset H^1(\mathbb{R}^N)$ (see [9], Proposition 9.18).

The solutions of problem $(P_V)$ are critical points of the functional

$$
I_V(u) = \frac{1}{2} \|u\|^2_\Omega - \int_\Omega F(u)dx,
$$

with $u \in H^1_0(\Omega)$. Set

$$
J_V(u) = I'_V(u)u = \|u\|^2_\Omega - \int_\Omega f(u)udx,
$$

$$
N_V := \{ u \in H^1_0(\Omega) \setminus \{0\} : J_V(u) = 0 \},
$$

and

$$
c_V := \inf_{u \in N_V} I_V(u).
$$

Also we denote in the same way $I_\infty$, $J_\infty$, $N_\infty$ and $c_\infty$ for the definition with the constant $V_\infty$ in place of $V(x)$. Let $w$ be the unique positive radial solution of $(P_\infty)$, see [6, 7, 30]. It is well known, see [22] that there are positive constants $C_1$ and $C_2$,

$$
C_1(1 + |x|)^{-\frac{N-1}{2}e^{-\sqrt{\frac{N}{C_1}}|x|}} \leq |D^i w(x)| \leq C_2(1 + |x|)^{-\frac{N-1}{2}e^{-\sqrt{\frac{N}{C_1}}|x|}}, \quad i = 0, 1.
$$

(2)

Hereafter $C$ will denote a positive constant, not necessarily the same one. The following lemma gives informations about the Nehari manifold $N_V$ which are, by now, standard (see [14] Lemma 2.1). We include them here for the sake of completeness.
Lemma 2.1. (a) There exists $\varrho > 0$ such that $\|u\|_\Omega \geq \varrho$ for every $u \in \mathcal{N}_V$.

(b) $\mathcal{N}_V$ is a closed $C^1$-submanifold of $H^1_0(\Omega)$ and natural constraint for $I_V$.

(c) If $u \in \mathcal{N}_V$, the function $t \mapsto I_V(tu)$ is strictly increasing in $(0, 1]$ and strictly decreasing in $(1, \infty)$. In particular,

$$I_V(u) = \max_{t > 0} I_V(tu) > 0$$

Proof. (a) Property $(f_2)$ and the Sobolev embedding theorem imply that

$$J_V(u) \geq \|u\|^2_\Omega - C \int_\Omega |u|^{p_1}dx \geq \|u\|^2_\Omega - C\|u\|^{p_2}_{H^1_0(\Omega)}, \quad u \in H^1_0(\Omega).$$

If $u \in \mathcal{N}_V$ then $J_V(u) = 0$ and so $\frac{C\|u\|^{p_2}_{H^1_0(\Omega)}}{\|u\|^2_\Omega} > 1$ and as $p_2 > 1$ we have $\|u\|^{p_2-1} > \frac{1}{C}$.

(b) Since $J_V(u)$ is continuous, it follows from (a) that $\mathcal{N}_V := \{u \in H^1_0(\Omega) \setminus \{0\} : J_V(u) = 0\}$ is closed in $H^1_0(\Omega)$. Moreover, property $(f_5)$ yields

$$J'_V(u)u = 2\|u\|^2_\Omega - \int_\Omega f'(t)u^2 - \int_\Omega f(u)u = \int_\Omega [f(u) - f'(t)u]u < 0.$$

for every $u \in \mathcal{N}_V$. This implies that 0 is a regular value of $J_V : H^1_0(\Omega) \setminus \{0\} \to \mathbb{R}$. So, as $J_V$ is of class $C^1$, $\mathcal{N}_V$ is a $C^1$-submanifold of $H^1_0(\Omega)$. It also implies that $u$ is not on the tangent space of $\mathcal{N}_V$ at $u$ and, therefore, that $\mathcal{N}_V$ is a natural constraint for $I_V$.

(c) Let $u \in \mathcal{N}_V$. Set $\Omega^+ := \{x \in \Omega; u(x) > 0\}$, $\Omega^- := \{x \in \Omega; u(x) < 0\}$.

Then

$$\frac{d}{dt} I(tu) = \frac{1}{t} f(tu) = t\|u\|^2_\Omega - \int_\Omega f(tu)udu = \int_\Omega \left[ f(u) - \frac{f(tu)}{t} \right]u \left( \int_\Omega \frac{f(u)}{u} - \frac{f(tu)}{tu} \right)u^2 + \int_\Omega \left[ \frac{f(u)}{u} - \frac{f(tu)}{tu} \right]u^2.$$

By property $(f_5)$ we have that $\frac{f(u)}{u}$ is strictly increasing for $u \in (0, \infty)$ and strictly decreasing for $u \in (-\infty, 0)$. Therefore $\frac{d}{dt} I_V(tu) > 0$ if $t \in (0, 1)$ and $\frac{d}{dt} I_V(tu) > 0$ if $t \in (1, \infty)$. This proves (c).

The next lemma is the key that allows us to take the non-linear $f$ a less smooth function in $C^1$, which improved the hypotheses in [14] and [25] where it was considered in $C^3$.

Lemma 2.2. For every $0 < \nu < p_1 - 1$ and $\rho > 0$ there exists $C_\rho \geq 0$ such that for all $0 \leq u, v \leq \rho$ we have

$$F(u + v) - F(u) - F(v) - f(u)v - f(v)u \geq -C_\rho (uv)^{1 + \frac{\nu}{2}}.$$

Proof. The inequality is obviously satisfied if $u = 0$ or $v = 0$. By $(f_3)$, $f$ is increasing, which yields

$$F(u + v) - F(u) = \int_u^{u+v} f(w)dw \geq f(u)v.$$

Moreover by $(f_2)$ for every $0 < \nu < p_1 - 1$ we have

$$f(s) = o(|s|^{1+\nu}) \quad \text{as} \quad |s| \to 0,$$

and then $\bar{C}_\rho := \sup_{0 \leq u \leq \rho} \frac{f(u)}{u^{1+\nu}} < \infty$. Now for $0 < v \leq u \leq \rho$, we deduce

$$F(u + v) - F(u) - F(v) - f(u)v - f(v)u \geq -F(v) - f(v)u.$$
Lemma 2.3 (\cite{1}), Lemma 2.1. If \( \mu_2 > \mu_1 \geq 0 \), there exists \( C > 0 \) such that, for all \( x_1, x_2 \in \mathbb{R}^N \),
\[
\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} \, dx \leq Ce^{-\mu_1|x_1-x_2|}.
\]
If \( \mu_2 \geq \mu_1 > 0 \), and \( \mu_3 > \mu_1 \geq 0 \), there exists \( C > 0 \) such that, for all \( x_1, x_2, x_3 \in \mathbb{R}^N \),
\[
\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} e^{-\mu_3|x-x_3|} \, dx \leq Ce^{-\mu_1|x_1-x_2|}.
\]
Proof. Since \( \mu_1 |x_1 - x_2| + (\mu_2 - \mu_1) |x - x_2| \leq \mu_1 (|x - x_1| + |x - x_2|) + (\mu_2 - \mu_1) |x - x_2| = \mu_1 |x - x_1| + \mu_2 |x - x_2| \), we have
\[
\int_{\mathbb{R}^N} e^{-\mu_1|x-x_1|} e^{-\mu_2|x-x_2|} \, dx \leq \int_{\mathbb{R}^N} e^{-\mu_1|x_1-x_2|} \, dx = Ce^{-\mu_1|x_1-x_2|}.
\]
The second inequality is obtained in a similar way. \( \square \)

The next four lemmas present some description of \( w \) the positive radial solution of \((P_{\infty})\) and its translates. For \( \lambda \in (0,1) \) and \( r \in (0,\infty) \) set \( \phi : [0,1] \times (0,\infty) \to \mathbb{R} \) as
\[
\phi(\lambda, r) := \lambda^2 \left( \|w\|^2 - \int_{\mathbb{R}^N} \frac{f(r \lambda w)}{r \lambda w} \, w^2 \right),
\]
and for \( \lambda = 0 \) define \( \phi(0, r) := 0 \). By \((f_1)\) and \((f_2)\), \( \phi(\lambda, r) \) is continuous.

By \((f_3)\), \( \phi(\lambda, r) \) is decreasing with respect to \( r \). If \( r \lambda > 1 \) then
\[
\|w\|^2 - \int_{\mathbb{R}^N} \frac{f(r \lambda w)}{r \lambda w} \, w^2 < \|w\|^2 - \int_{\mathbb{R}^N} \frac{f(w)}{w} \, w^2 = 0
\]
and so \( \phi(\lambda, r) \) is decreasing with respect to \( \lambda \).

Lemma 2.4. There are \( S_0 < 0 \) and \( T_0 > 0 \) such that
\[
\phi(\lambda, r) + \phi(1 - \lambda, r) \leq S_0 < 0 \quad \forall \ r \geq T_0, \lambda \in [0,1].
\]
Proof. The function \( \phi \) is continuous and
\[
\phi(\lambda, r) \leq \|w\|^2 \lambda^2 =: A \lambda^2 \quad \forall \ r \in (0,\infty), \lambda \in [0,1].
\]
As \( w \) is a solution of problem \((P_{\infty})\) we have that
\[
\phi(\lambda, r) = \lambda^2 \left( \int_{\mathbb{R}^N} \left[ \frac{f(w)}{w} - \frac{f(r \lambda w)}{r \lambda w} \right] w^2 \right).
\]
There are two cases to study:

**case 1** if \( \lim_{u \to \infty} \frac{f(u)}{u} \to \infty \) (in fact \( f(u) \) is superlinear) then by \((f_5)\) and the Monotone Convergence Theorem

\[
\lim_{r \to \infty} \phi(\lambda, r) = -\infty, \quad \forall \lambda \in (0, 1]
\]

and if \( \lambda = 0, \lim_{r \to \infty} \phi(1, r) = -\infty \) and this case is settled.

**case 2** if \( \lim_{u \to \infty} \frac{f(u)}{u} \to a \) (the nonlinearity \( f(u) \) is asymptotically linear) then by \((f_5)\) and Lebesgue Dominated Convergence Theorem

\[
\lim_{r \to \infty} \phi(\lambda, r) = \lambda^2 \left( \int_{\mathbb{R}^N} \left[ \frac{f(w)}{w} - a \right] w^2 \right) =: -B \lambda^2 < 0, \quad \forall \lambda \in (0, 1].
\]

Due to the symmetry with respect to \( \lambda \), it suffices to consider \( \lambda \in [0, 1/2] \). Fix \( \lambda_0 \in (0, 1/2) \) such that \( A\lambda_0^2 < \frac{\beta}{T} (1 - \lambda_0)^2 \) and by the continuity of \( \phi \), there exists \( r_0 \in (0, \infty) \) such that

\[
\phi(1 - \lambda_0, r_0) = -\frac{B}{2} (1 - \lambda_0)^2
\]

Then, for all \( \lambda \in [0, \lambda_0] \) and all \( r \geq \max\{r_0, 2\} \) we have \( r(1 - \lambda) > 1 \) and

\[
\phi(\lambda, r) + \phi(1 - \lambda, r) \leq A\lambda_0^2 + \phi(1 - \lambda_0, r_0) = A\lambda_0^2 - \frac{B}{2} (1 - \lambda_0)^2 < 0.
\]

On the other hand, if \( \lambda \in [\lambda_0, 1/2] \), by fixing \( r_1 > 1/\lambda_0 \), we have that \( r(1 - \lambda) \geq r\lambda > 1 \) for all \( r > r_1 \). Hence,

\[
\phi(\lambda, r) + \phi(1 - \lambda, r) \leq \phi(\lambda_0, r) + \phi(1 - \lambda_0, r) < 0, \quad \forall \lambda \in [\lambda_0, 1/2] \text{ and } r > r_0.
\]

Set \( T_0 := \max\{2, r_0, r_1\} \) and

\[
S_0 := \max_{\lambda \in [0, 1]} \phi(\lambda, T_0) + \phi(1 - \lambda, T_0) < 0,
\]

we conclude that

\[
\phi(\lambda, r) + \phi(1 - \lambda, r) \leq \phi(\lambda, T_0) + \phi(1 - \lambda, T_0) \leq S_0
\]

for all \( r > T_0 \) and \( \lambda \in [0, 1] \), as claimed.

Now, let \( y_0 \in \mathbb{R}^N \) with \( \|y_0\| = 1 \) and \( B_2(y_0) := \{x \in \mathbb{R}^N : |x - y_0| \leq 2\} \). We write for each \( y \in \partial B_2(y_0) \) and \( R > 0 \),

\[
w_0^R := w(\cdot - R y_0), \quad w_y^R := w(\cdot - R y).
\]

**Lemma 2.5.** Let \( q > 0 \) and \( R > 0 \) be large enough, then it holds

\[
\begin{align*}
\text{a)} & \quad \int_{B_{2K}(0)} |w_0^R|^q \leq CR^{-q \frac{N-1}{2}} e^{-q \sqrt{|x|} R} \text{ and } \int_{B_{2K}(0)} |w_y^R|^q \leq CR^{-q \frac{N-1}{2}} e^{-q \sqrt{|x|} R}, \\
\text{b)} & \quad \int_{B_{2K}(0)} |\nabla w_0^R|^q \leq CR^{-q \frac{N-1}{2}} e^{-q \sqrt{|x|} R} \text{ and } \int_{B_{2K}(0)} |\nabla w_y^R|^q \leq CR^{-q \frac{N-1}{2}} e^{-q \sqrt{|x|} R}.
\end{align*}
\]

**Proof.** In order to prove the first estimate let \( 2K < \frac{1}{2} R \), so that

\[
\frac{1}{2} R = R - \frac{1}{2} R < |R y_0| - |x| < |x - R y_0| < 1 + |x - R y_0|, \quad \forall x \in B_{2K}(0).
\]

(3)

Now by (2) and (3) we have

\[
\int_{B_{2K}(0)} |w_0^R|^q = \int_{B_{2K}(0)} |w(x - R y_0)|^q dx
\]
The proofs of the other estimates are similar. \hfill \Box

**Lemma 2.6.** Let \( p \geq q \geq 1 \) then

\[
\int_{\mathbb{R}^N} (w(x-Ry_0))^q(w_y^R)^p \leq CR^{-qN-1} e^{-2q\sqrt{\gamma R}}
\]

and

\[
\int_{\mathbb{R}^N} (w(x-Ry_0))^q(w_0^R)^p \leq CR^{-qN-1} e^{-2q\sqrt{\gamma R}}.
\]

**Proof.** Note that

\[
\{ \begin{align*}
&\text{if } |x| > R \text{ then } R < |x| + 1 \text{ and } \\
&\text{if } |x| < R \text{ then } 2R - R < |R(y - y_0)| - |x| < 1 + |x - R(y - y_0)|.
\end{align*} \tag{4}
\]

Now by (2), (4) and Lemma 2.3 we have

\[
\begin{align*}
\int_{\mathbb{R}^N} (w(x-Ry_0))^q(w_y^R)^p & = \int_{\mathbb{R}^N} (w(x-Ry_0))^q w(x-Ry)^p dx = \int_{\mathbb{R}^N} w(x)^q w(x-R(y-y_0))^p dx \\
& \leq \int_{\mathbb{R}^N} (1 + |x|)^{-qN-1} e^{-q\sqrt{\gamma |x|}} (1 + |x - R(y-y_0)|)^{-pN-1} e^{-p\sqrt{\gamma |x - R(y-y_0)|}} \\
& \leq \int_{B_R(0)} e^{-q\sqrt{\gamma |x|}} (1 + |x - R(y-y_0)|)^{-pN-1} e^{-p\sqrt{\gamma |x - R(y-y_0)|}} \\
& \quad + \int_{\mathbb{R}^N \setminus B_R(0)} e^{-q\sqrt{\gamma |x|}} (1 + |x|)^{-qN-1} e^{-p\sqrt{\gamma |x - R(y-y_0)|}} \\
& \leq CR^{-qN-1} e^{-2q\sqrt{\gamma R}}.
\end{align*}
\]

Similarly we can prove the second estimate. \hfill \Box

**Lemma 2.7.** For \( R > 0 \) sufficiently large we have

\[
\int_{\Omega} (V(x) - V_\infty)(w_0^R \psi)^2 \leq CR^{-(N-1)} e^{-2\sqrt{\gamma R}}, \tag{5}
\]

\[
\int_{\Omega} (V(x) - V_\infty)(w_y^R \psi)^2 \leq CR^{-(N-1)} e^{-2\sqrt{\gamma R}}, \tag{6}
\]

and

\[
\int_{\Omega} (V(x) - V_\infty)w_y^R \psi w_0^R \psi \leq CR^{-(N-1)} e^{-2\sqrt{\gamma R}}. \tag{7}
\]

**Proof.** In order to prove the first inequality, it follows from (V2) and estimative (2) that

\[
\begin{align*}
\int_{\Omega} (V(x) - V_\infty)(w_0^R \psi)^2 & \leq \int_{\mathbb{R}^N} (V(x) - V_\infty)(w_0^R)^2 \\
& \leq \int_{\mathbb{R}^N} e^{-\gamma|x|}(1 + |x - R_0|)^{-(N-1)} e^{-2\sqrt{\gamma |x - R_0|}}.
\end{align*} \tag{8}
\]

Now let \( \sigma = \frac{1}{2} - \frac{\sqrt{\gamma}}{\gamma} > 0 \), so we can write (8) as

\[
\begin{align*}
\int_{\mathbb{R}^N \setminus B_\sigma(R_0)} e^{-\gamma|x|}(1 + |x - R_0|)^{-(N-1)} e^{-2\sqrt{\gamma |x - R_0|}}
\end{align*} \tag{9}
\]
Lemma 2.8. \( U_{\lambda,y} \) and \( \psi \) as before. \( \lambda, y > 0 \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with smooth boundary. Define \( Z_{\lambda,y}^R := \lambda w^R \), \( \lambda \in [0,1] \), \( R > 0 \), and \( U_{\lambda,y}^R := Z_{\lambda,y}^R \psi \) where \( \psi \in C^\infty(\mathbb{R}^N) \) is continuous radially symmetric and increasing cut-off function

\[
\psi(x) = \begin{cases} 
0 & |x| \leq K \\
0 < \psi < 1 & K < |x| < 2K \\
1 & |x| \geq 2K.
\end{cases}
\]

Note that here as before \( K \) is the radius of the sphere \( B_K(0) \) which contains \( \mathbb{R}^N \setminus \Omega \). We can consider \( U_{\lambda,y}^R \) in \( H^1(\mathbb{R}^N) \) by extending \( U_{\lambda,y}^R \equiv 0 \) outside \( \Omega \).

Lemma 2.8. \( U_{\lambda,y}^R - Z_{\lambda,y}^R \to 0 \) in \( H^1(\mathbb{R}^N) \), as \( R \to \infty \).

As \( |x - R_y| > \sigma R \) in \( \mathbb{R}^N \setminus B_{\sigma R}(R_y) \), applying Lemma 2.3 with \( \mu_2 = \gamma > \mu_1 = 2\sqrt{\lambda} \), we get

\[
\int_{B_{\sigma R}(R_y)} |x - R_y|^2 \leq CR^{-N} \gamma |x - R_y|^2,
\]

(9) \( \leq CR^{-2\sqrt{\lambda}} \).

On the other hand, \( |x + R_y| \geq \sigma R \) for \( x \in B_{\sigma R}(0) \) and by making a change of variables, we have

\[
\int_{B_{\sigma R}(0)} e^{-\gamma |x+R_y|} = e^{-2\sqrt{\lambda} |x|} \\
\leq e^{-\gamma (1-\sigma) R} \int_{B_{\sigma R}(0)} e^{-2\sqrt{\lambda} |x|} \leq Ce^{-\gamma (1-\sigma) R} \int_0^{\sigma R} r^{N-1} \sqrt{\frac{1}{r}},
\]

\[
\leq Ce^{-\gamma (1-\sigma) R} R^{N-1} \leq CR^{-N} e^{-2\sqrt{\lambda} R},
\]

since by definition of \( \sigma, \gamma (1-\sigma) R > \gamma (\frac{1}{2} + \frac{1}{\sqrt{\lambda}}) R > (\frac{1}{2} + \sqrt{\lambda}) R > 2\sqrt{\lambda} R \). The proof of the first inequality is complete. Similarly we can prove the second estimate.

Finally, in order to prove (7) we may repeat the above argument for \( B_{\sigma R}(R_y) \cup B_{\sigma R}(R_y) \) rather than \( B_{\sigma R}(R_y) \). Note that \( |x - R_y|, |x - R_y| > \sigma R \) in \( \Gamma := \mathbb{R}^N \setminus (B_{\sigma R}(R_y) \cup B_{\sigma R}(R_y)) \); performing a change of variables and applying Lemma 2.3

\[
\int_{\Omega} (V(x) - V_{\infty}) w^R \psi \leq \int_{\mathbb{R}^N} (V(x) - V_{\infty}) w_0^R \psi + \int_{B_{\sigma R}(R_y)} \int_{B_{\sigma R}(R_y)} e^{-\gamma |x+R_y|} \leq Ce^{-\gamma (1-\sigma) R} \int_0^{\sigma R} r^{N-1} \sqrt{\frac{1}{r}},
\]

\[
\leq Ce^{-\gamma (1-\sigma) R} R^{N-1} \leq CR^{-N} e^{-2\sqrt{\lambda} R}.
\]

In what follows we exploit the ideas of Bahri and Li in [2] by working with a convex combination of two translated copies of \( w \), the ground state solution of (P \( \infty \)) (see also [14], [20] and [25]).

Define \( Z_{\lambda,y}^R := \lambda w^R + (1-\lambda)w^R \), \( \lambda \in [0,1] \), \( R > 0 \), and

\[
U_{\lambda,y}^R := Z_{\lambda,y}^R \psi
\]

where \( \psi \in C^\infty(\mathbb{R}^N) \) is continuous radially symmetric and increasing cut-off function

\[
\psi(x) = \begin{cases} 
0 & |x| \leq K \\
0 < \psi < 1 & K < |x| < 2K \\
1 & |x| \geq 2K.
\end{cases}
\]
Proof. First of all, for $R$ sufficiently large we claim that
\begin{equation}
|w^R_0 - \psi w^R_0|^2_{L^2(B_{2K}(0))} \leq CR^{-(N-1)}e^{-2\sqrt{\infty} R},
\end{equation}
\begin{equation}
|\nabla w^R_0 - \nabla \psi w^R_0|^2_{L^2(B_{2K}(0))} \leq CR^{-(N-1)}e^{-2\sqrt{\infty} R},
\end{equation}
\begin{equation}
|w^R_y - \psi w^R_y|^2_{L^2(B_{2K}(0))} \leq CR^{-(N-1)}e^{-2\sqrt{\infty} R},
\end{equation}
\begin{equation}
|w^R_y - \psi w^R_y|^2_{L^2(B_{2K}(0))} \leq CR^{-(N-1)}e^{-2\sqrt{\infty} R},
\end{equation}
therefore
\begin{align*}
\|U^R_{\lambda,y} - Z^R_{\lambda,y}\| &\leq \lambda\|w^R_0 - \psi w^R_0\| + (1 - \lambda)\|w^R_y - \psi w^R_y\| \\
&= \lambda\|w^R_0 - \psi w^R_0\|_{H^1(B_{2K}(0))} + (1 - \lambda)\|w^R_y - \psi w^R_y\|_{H^1(B_{2K}(0))} \\
&= \lambda\|w^R_0 - \psi w^R_0\|^2_{L^2(B_{2K}(0))} + |\nabla w^R_0 - \nabla \psi w^R_0|^2_{L^2(B_{2K}(0))}\frac{1}{2} \\
&+ (1 - \lambda)\|w^R_y - \psi w^R_y\|^2_{L^2(B_{2K}(0))} + |\nabla w^R_y - \nabla \psi w^R_y|^2_{L^2(B_{2K}(0))}\frac{1}{2}
\end{align*}
and by the claim we have
\begin{equation}
\|U^R_{\lambda,y} - Z^R_{\lambda,y}\|^2 \leq CR^{-(N-1)}e^{-2\sqrt{\infty} R}
\end{equation}
and this shows the lemma.

Now, in order to complete the proof we have to show the claim. To obtain the first estimate (12) we use Lemma 2.5 with $q = 2$,
\begin{align*}
|w^R_0 - \psi w^R_0|^2_{L^2(B_{2K}(0))} &= \int_{B_{2K}(0)} |1 - \psi| |w^R_0|^2 \, dx \\
&\leq \int_{B_{2K}(0)} |w^R_0|^2 \, dx \leq R^{-(N-1)}e^{-2\sqrt{\infty} R}.
\end{align*}
To prove the second estimate (13) we have $\psi \in C^\infty$, then there exists positive constants $C_3$ and $C_4$ such that
\begin{equation}
|\nabla \psi w^R_0| = |(\nabla \psi) w^R_0 + (\nabla w^R_0) \psi| \leq C_3|w^R_0| + C_4|\nabla w^R_0| \quad \text{in} \quad B_{2K}(0)
\end{equation}
and so by Lemma 2.5 with $q = 2$,
\begin{align*}
|\nabla w^R_0 - \nabla \psi w^R_0|^2_{L^2(B_{2K}(0))} &\leq \int_{B_{2K}(0)} [(C_3 + 1)|w^R_0| + C_4|\nabla w^R_0|^2] \, dx \\
&\leq CR^{-(N-1)}e^{-2\sqrt{\infty} R}
\end{align*}
as claimed. The proof of (14) and (15) are similar. \hfill \Box

Lemma 2.9. For any $r > 0$, $J_\infty(rU^R_{\lambda,y}) - J_\infty(rZ^R_{\lambda,y}) \rightarrow 0$ as $R \rightarrow \infty$.

Proof. By the definition of $J_\infty$ we have
\begin{align*}
|J_\infty(rU^R_{\lambda,y}) - J_\infty(rZ^R_{\lambda,y})| &= \left|\int_{\mathbb{R}^N} f(rU^R_{\lambda,y}) u_{\lambda,y} - \|rZ^R_{\lambda,y}\|^2 + \int_{\mathbb{R}^N} f(rZ^R_{\lambda,y}) rZ^R_{\lambda,y} \right| \\
&\leq \|rU^R_{\lambda,y} - rZ^R_{\lambda,y}\|^2 + \left|\int_{\mathbb{R}^N} f(rZ^R_{\lambda,y}) rZ^R_{\lambda,y} - f(rU^R_{\lambda,y}) rU^R_{\lambda,y} \right|. \quad (17)
\end{align*}
By Lemma 2.8 the first part of (17) is equal to \( o_R(1) \), then it is enough to show that

\[
\int_{\mathbb{R}^N} f(rZ_{\lambda,y}^R)Z_{\lambda,y}^R - f(rU_{\lambda,y}^R)U_{\lambda,y}^R = \int_{B_{2K}(0)} f(rZ_{\lambda,y}^R)Z_{\lambda,y}^R - f(rU_{\lambda,y}^R)U_{\lambda,y}^R = o_R(1).
\]

By \((f_2)\), Lemma 2.5 and the inequality \((a + b)^p \leq 2^p(a^p + b^p)\) for \(a, b \geq 0\) we have

\[
|\int_{B_{2K}(0)} f(rZ_{\lambda,y}^R)Z_{\lambda,y}^R - f(rU_{\lambda,y}^R)U_{\lambda,y}^R|
\leq A_1 \int_{B_{2K}(0)} |(|rZ_{\lambda,y}^R|^{p_1} + |rZ_{\lambda,y}^R|^{p_2})Z_{\lambda,y}^R - (|rU_{\lambda,y}^R|^{p_1} + |rU_{\lambda,y}^R|^{p_2})U_{\lambda,y}^R|
\leq A_1 \int_{B_{2K}(0)} |1 - \psi(||rZ_{\lambda,y}^R||^{p_1} + |rZ_{\lambda,y}^R|^{p_2+1})| \leq C \int_{B_{2K}(0)} (|Z_{\lambda,y}^R|^{p_1+1} + |Z_{\lambda,y}^R|^{p_2+1})
\leq C \int_{B_{2K}(0)} |\lambda w_0^R + (1 - \lambda)w_y^R|^{p_1+1} + |\lambda w_0^R + (1 - \lambda)w_y^R|^{p_2+1}
\leq C \int_{B_{2K}(0)} |w_0^R|^{p_1+1} + |w_y^R|^{p_2+1} + |w_0^R|^{p_1+1} + |w_y^R|^{p_2+1}
\leq CR^{-N(1)} e^{-2\sqrt{\kappa}R} = o_R(1).
\]

Our assumptions do not guarantee that every \( u \in H^1_\lambda(\Omega) \setminus \{0\} \) admits a projection onto \( N_\nu \). However, the following lemma says that \( U_{\lambda,y}^R \) does admit a projection onto \( N_\nu \) if \( R \) is sufficiently large.

**Theorem 2.10.** There exist \( R_0 > 0, T_0 > 2 \) such that for each \( R \geq R_0, y \in \partial B_2(y_0) \) and \( \lambda \in [0, 1] \), there exists a unique \( T_{\lambda,y}^R \) such that

\[
T_{\lambda,y}^R U_{\lambda,y}^R \in N_\nu,
\]

\( T_{\lambda,y}^R \in [0, T_0] \) and \( T_{\lambda,y}^R \) is a continuous function of the variables \( \lambda, y \) and \( R \). In particular for \( \lambda = 1/2 \) we have \( T_{1,y}^R \rightarrow 2 \) as \( R \rightarrow \infty \) uniformly in \( y \in \partial B_2(y_0) \).

**Proof.** First note that, for each \( u \in H^1_\lambda(\Omega), u > 0 \), property \((f_5)\) implies that

\[
\frac{J_\nu(ru)}{r^2} = \|u\|^2 - \int_{\Omega} \frac{f(ru)}{ru} u^2
\]

is strictly decreasing in \( r \in (0, \infty) \). Therefore, if there exists \( r_u \in (0, \infty) \) such that \( J_\nu(r_u u) = 0 \), this number will be unique. Observe also that \( J_\nu(r_u) > 0 \) for \( r \) small enough. Next, we will show that, for \( R \) large enough and some \( T_0 > 0 \),

\[
J_\nu(rU_{\lambda,y}^R) < 0 \quad \forall r \geq T_0, \quad y \in \partial B_2(y_0), \quad \lambda \in [0, 1]. \tag{18}
\]

This implies that there exists \( T_{\lambda,y}^R \in [0, T_0] \) such that \( J_\nu(T_{\lambda,y}^R U_{\lambda,y}^R) = 0 \), i.e. \( T_{\lambda,y}^R U_{\lambda,y}^R \in N_\nu \). Let us prove (18). For \( u, v \in H^1(\mathbb{R}^N), u, v > 0 \), and \( r \in (0, \infty) \), by using \((f_5)\) we have

\[
J_\infty(ru + rv) = r^2(\|u\|^2 + \|v\|^2 + 2\langle u, v \rangle) - \int_{\mathbb{R}^N} \frac{f(ru + rv)}{ru + rv} (ru + rv)^2
\leq r^2(\|u\|^2 - \int_{\mathbb{R}^N} \frac{f(ru)}{ru} u^2 + \|v\|^2 - \int_{\mathbb{R}^N} \frac{f(rv)}{rv} v^2 + 2\langle u, v \rangle).
\]
Setting \( u := \lambda w_y^R \) and \( v := (1-\lambda)w_y^R \), performing a change of variable and applying Lemma 2.4 we conclude that

\[
J_\infty(r\lambda w_y^R + r(1-\lambda)w_y^R) = \phi(\lambda, r) + \phi(1-\lambda, r) + 2\lambda(1-\lambda)(w_y^R, w_y^R)
\]

\[
\leq S_0 + \frac{1}{2}(w_y^R, w_y^R) = S_0 + o_R(1) \quad \forall r \geq T_0, \quad \lambda \in [0,1],
\]

where \( o_R(1) \to 0 \) as \( R \to \infty \), uniformly in \( y \in \partial B_2(y_0) \) and \( \lambda \in [0,1] \). Also \( S_0 < 0 \) as in Lemma 2.4. Now since \( H \) as in Lemma 2.6. Now since \( H \)

and from the two previous estimates we have

\[
\int_{\partial B_2(y_0)} \phi(0, y) \leq \int_{\Omega} (V(x) - V_\infty)(U_y^R)\]

and by Lemma 2.7 and Lemma 2.9

\[
\frac{J_V(rU_y^R)}{r^2} \leq S_0 + o_R(1) \quad \forall r \geq T_0, \quad \lambda \in [0,1].
\]

Hence, there exists \( R_0 > 0 \) such that

\[
\frac{J_V(rU_y^R)}{r^2} \leq \frac{S_0}{2} < 0 \quad \forall r \geq T_0, \quad \lambda \in [0,1], \quad R > R_0.
\]

This proves (18), and so we have shown that \( N_V \neq \emptyset \).

Now let \( \varphi(u, v) = f(u + v) - f(u) - f(v) \), by the mean value theorem

\[
-Cv^{p_1} \leq -f(v) \leq \varphi(u, v) \leq f(u + v) - f(u) \leq f'(u + tv)v \leq Cv
\]

and by Lemma 2.6

\[
-o_R(1) = -C \int_{\mathbb{R}^N} (w_y^R)^{p_1} w_y^R \leq \int_{\mathbb{R}^N} \varphi(w_y^R, w_y^R)w_y^R \leq C \int_{\mathbb{R}^N} (w_y^R)w_y^R = o_R(1)
\]

or

\[
\int_{\mathbb{R}^N} |\varphi(w_y^R, w_y^R)w_y^R| = o_R(1),
\]

also by the symmetry of \( \varphi(u, v) \) in \( u \) and \( v \), we get similarly

\[
\int_{\mathbb{R}^N} |\varphi(w_y^R, w_y^R)w_y^R| = o_R(1),
\]

and from the two previous estimates we have

\[
\int_{\mathbb{R}^N} |\varphi(w_y^R, w_y^R)(w_y^R + w_y^R)| = o_R(1).
\]

Now, by Lemma 2.6 and (19) we can write

\[
J_\infty(w_y^R) = \|w_y^R\|^2 - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R)
\]

\[
= \|w_y^R\|^2 + 2\|w_y^R\|^2 - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R)
\]

\[
- \int_{\mathbb{R}^N} f(w_y^R)(w_y^R) - \int_{\mathbb{R}^N} f(w_y^R)(w_y^R) + \int_{\mathbb{R}^N} \varphi(w_y^R, w_y^R)(w_y^R + w_y^R)
\]

\[
= J_\infty(w_y^R) + o_R(1) = o_R(1)
\]

since \( w \) is a solution of \( (P_\infty) \). So, by Lemma 2.9 we have

\[
J_\infty((w_y^R)^{p_1} \psi) = J_\infty(w_y^R) + o_R(1) = o_R(1) \quad \text{as} \quad R \to \infty.
\]
Therefore, by (20), Lemma 2.7 and \( H^1_0(\Omega) \subset H^1(\mathbb{R}^N) \)
\[
J_V(2U^R_{\frac{y}{|y|}}) = J_V((w_0^R + w_y^R)\psi)
= J_\infty((w_0^R + w_y^R)\psi) + \int_\Omega (V(x) - V_\infty)(w_0^R + w_y^R)^2\psi^2 = o_R(1)
\]
since by Lemma 2.7
\[
\int_\Omega (V(x) - V_\infty)(w_0^R + w_y^R)^2\psi^2 \leq \int_{\mathbb{R}^N} (V(x) - V_\infty)(w_0^R + w_y^R)^2 = o_R(1)
\]
and the lemma is proved. 

\[ \square \]

3. Compactness results. Our goal in this section is to investigate the lack of compactness due to the fact that the problem is in an unbounded domain \( \Omega \). In order to do so, we present a sequence of lemmas that will culminate in Corollary 1, which presents a range of levels where \( I_V \) satisfies the Palais-Smale condition. Also note that the uniqueness of the positive solution of the limit problem is going to play a role in the arguments.

**Lemma 3.1.** Any sequence \((u_k)\) satisfying
\[
(u_k) \in \mathcal{N}_V \quad \text{and} \quad I_V(u_k) \to d
\]
is bounded in \( H^1_0(\Omega) \).

**Proof.** First of all note that \( d \geq 0 \), since
\[
I_V(u_k) = I_V(u_k) - I'_V(u_k)u_k = \frac{1}{2} f(u_k)u_k - F(u_k) \geq 0.
\]
Now fix \( D > d \). Assume, by contradiction, that \( \|u_k\| \to \infty \) and set \( v_k := t_k u_k \) with \( t_k = \frac{2\sqrt{D}}{\|u_k\|} \). By Lemma 2.1 (c), for \( k \) large enough we have that
\[
D \geq I_V(u_k) \geq I_V(v_k) = \frac{1}{2} t^2 \|u_k\|^2 - \int_\Omega F(v_k) = 2D - \int_\Omega F(v_k).
\]
By using hypothesis \((f_2)\), we get that
\[
D \leq \int_\Omega F(v_k) \leq c(\|v_k\|_{L^p_1} + |v_k|_{L^p_2}).
\]
As \( D > d \geq 0 \) and \((v_k)\) is bounded in \( H^1_0(\Omega) \subset H^1(\mathbb{R}^N) \), this lower bound, together with Lions lemma [32], Lemma 2.1], implies that there exist \( \delta > 0 \) and a sequence \((y_k)\) in \( \mathbb{R}^N \) such that
\[
\int_{B_1(y_k)} v_k^2 = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} v_k^2 \geq \delta.
\]
Set \( \tilde{u}_k(x) := u_k(x + y_k) \) and \( \tilde{v}_k(x) := v_k(x + y_k) \). After passing to a subsequence \( \tilde{v}_k \rightharpoonup v \) weakly in \( H^1(\mathbb{R}^N) \), \( \tilde{v}_k \to v \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \) and \( \tilde{v}_k(x) \to v(x) \) a.e. in \( \mathbb{R}^N \). Therefore,
\[
\int_{B_1(0)} v^2 = \lim_{k \to \infty} \int_{B_1(0)} \tilde{v}_k^2 = \lim_{k \to \infty} \int_{B_1(y_k)} v_k^2 \geq \delta.
\]
Hence, \( v \neq 0 \) and there exists a subset \( \Lambda \) of positive measure in \( B_1(0) \) such that \( v(x) \neq 0 \) for every \( x \in \Lambda \). It follows that \( |\tilde{u}_k(x)| \to \infty \) for every \( x \in \Lambda \). Property
(f_5) implies that \( \frac{1}{2} f(u)u - F(u) \geq 0 \) if \( u \in \mathbb{R} \setminus \{0\} \). So, from property (f_3) and Fatou’s lemma, we conclude that

\[
D > \lim_{k \to \infty} I_V(u_k) = \lim_{k \to \infty} \int_{\Omega} \left[ \frac{1}{2} f(u_k)u_k - F(u_k) \right] = \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[ \frac{1}{2} f(u_k)u_k - F(u_k) \right] = \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[ \frac{1}{2} f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k) \right] \\
\geq \liminf_{k \to \infty} \int_{\Lambda} \left[ \frac{1}{2} f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k) \right] \geq \int_{\Lambda} \liminf_{k \to \infty} \left[ \frac{1}{2} f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k) \right] = \infty
\]

This is a contradiction. \( \square \)

**Lemma 3.2.** \( c_V, c_\infty > 0 \).

**Proof.** Let \( u_k \in \mathcal{N}_V \) be such that \( I_V(u_k) \to c_V \). By Lemma 3.1, after passing to a subsequence, we have that \( (u_k) \) is bounded in \( H_0^1(\Omega) \). From Lemma 2.1(a) and by property (f_2) we obtain

\[
0 < \vartheta^2 \leq \|u_k\|^2_{\Omega} = \int_{\Omega} f(u_k)u_k \leq c(\|u_k\|_{p_1+1} + |u_k|_{p_2+1}).
\]

This inequality, together with Lion lemma (considering the extention of \( u_k \) to \( H^1(\mathbb{R}^N) \)), implies that there exist \( \delta > 0 \) and a sequence \( (y_k) \) in \( \mathbb{R}^N \) such that

\[
\int_{B_1(y_k)} u_k^2 = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_k^2 \geq \delta.
\]

Set \( \tilde{u}_k(x) := u_k(x+y_k) \) After passing to a subsequence \( \tilde{u}_k \rightharpoonup u \) weakly in \( H^1(\mathbb{R}^N) \), \( \tilde{u}_k \to u \) in \( L^2_{\text{loc}}(\mathbb{R}^N) \) and \( \tilde{u}_k(x) \to u(x) \) a.e. in \( \mathbb{R}^N \). Therefore,

\[
\int_{B_1(0)} u^2 = \lim_{k \to \infty} \int_{B_1(y_k)} \tilde{u}_k^2 = \lim_{k \to \infty} \int_{B_1(y_k)} u_k^2 \geq \delta.
\]

Hence, \( u \neq 0 \) and there exists a subset \( \Lambda \) of positive measure in \( B_1(0) \) such that \( u(x) \neq 0 \) for every \( x \in \Lambda \). Property (f_5) implies that \( \frac{1}{2} f(u)u - F(u) > 0 \) if \( u \in \mathbb{R} \setminus \{0\} \). So, from the Fatou’s lemma, we conclude that

\[
c_V = \lim_{k \to \infty} I_V(u_k) = \lim_{k \to \infty} \int_{\Omega} \left[ \frac{1}{2} f(u_k)u_k - F(u_k) \right] = \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[ \frac{1}{2} f(u_k)u_k - F(u_k) \right] = \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[ \frac{1}{2} f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k) \right] \\
\geq \int_{\Lambda} \liminf_{k \to \infty} \left[ \frac{1}{2} f(\tilde{u}_k)\tilde{u}_k - F(\tilde{u}_k) \right] = \int_{\Lambda} \left[ \frac{1}{2} f(u)u - F(u) \right] > 0
\]

as claimed. By repeating this argument we obtain \( c_\infty > 0 \). \( \square \)

Note that if \( c_V \) is attained, then problem \( (P_V) \) has a non-zero ground state solution from this previous lemma.

**Lemma 3.3.** If \( u \) is a solution of \( (P_V) \) with \( I_V(u) \in [c_V, 2c_V] \), then \( u \) does not change sign.
Proof. If \( u \) is a solution of \( P_V \) then
\[
0 = J_V(u)u^\pm = J_V(u^\pm),
\]
where \( u^+ := \max\{u,0\} \) and \( u^- := \min\{u,0\} \) and so \( u^\pm \in \mathcal{N}_V \). Now if \( u^+ \neq 0 \) and \( u^- \neq 0 \) then
\[
I_V(u) = I_V(u^+) + I_V(u^-) \geq 2c_V.
\]
This proves the lemma.

Note that \( \nabla_{\mathcal{N}_V} I_V(u) \) is the orthogonal projection of \( \nabla I_V(u) \) onto the tangent space of \( \mathcal{N}_V \) at \( u \), that is defined by \( T_u(\mathcal{N}_V) := \{ v \in H^1_0(\Omega) \mid J_V'(u)v = 0 \} \). Recall that a sequence \( (u_k) \) in \( H^1_0(\Omega) \) is said to be a \( (PS)_d \)-sequence for \( I_V \) on \( \mathcal{N}_V \) if \( I_V(u_k) \to d \) and \( \|\nabla_{\mathcal{N}_V} I_V(u_k)\| \to 0 \). The functional \( I_V \) satisfies the Palais-Smale condition on \( \mathcal{N}_V \) at the level \( d \) if every \( (PS)_d \)-sequence for \( I_V \) on \( \mathcal{N}_V \) contains a convergent subsequence.

Remark 4. We can write \( \nabla I_V(u) \) for the gradient of \( I_V \) at \( u \), as
\[
\nabla I_V(u) = \nabla_{\mathcal{N}_V} I_V(u) + t\nabla J_V(u).
\]
Indeed, by the definition \( \langle \nabla_{\mathcal{N}_V} I_V(u), v \rangle = \langle \nabla I_V(u), v \rangle \) for all \( v \in T_u(\mathcal{N}_V) \) or
\[
(\nabla_{\mathcal{N}_V} I_V(u) - \nabla I_V(u), v) = 0, \quad \forall v \in T_u(\mathcal{N}_V) := \{ v \in H^1_0(\Omega) \mid J_V'(u)v = 0 \}.
\]
On the other hand \( T_u(\mathcal{N}_V) \) is of codimension one and so
\[
H^1_0(\Omega) = E = T_u(\mathcal{N}_V) \oplus <J_V'(u) >.
\]
Now by the Hahn-Banach Theorem, there is a continuous linear functional \( \nabla_{\mathcal{N}_V} I_V(u) \) on \( E \) such that
\[
\nabla_{\mathcal{N}_V} I_V(u) - \nabla I_V(u) = tJ_V'(u) \quad \text{or}
\]
\[
\nabla I_V(u) = \nabla_{\mathcal{N}_V} I_V(u) + t\nabla J_V(u),
\]
as we want.

The next Lemma shows that \( \mathcal{N}_V \) is a natural constraint.

Lemma 3.4. Every \((PS)_d\)-sequence \((u_k)\) for \( I_V \) restricted to \( \mathcal{N}_V \) contains a subsequence which is a \((PS)_d\)-sequence for \( I_V \) in \( H^1_0(\Omega) \).

Proof. Let \((u_k)\) be \((PS)_d\)-sequence for \( I_V \) on \( \mathcal{N}_V \). By Lemma (3.1), after passing to a subsequence, we have that \((u_k)\) is bounded in \( H^1_0(\Omega) \). Write
\[
\nabla I_V(u_k) = \nabla_{\mathcal{N}_V} I_V(u_k) + t_k \nabla J_V(u_k) \tag{21}
\]
By assumption \((f_k)\), the Sobolev embedding and Hölder’s inequality, for any \( v \) in \( H^1_0(\Omega) \),
\[
\left| \int_\Omega [f'(u_k)u_k - f(u_k)]v \right| \leq C \int_\Omega (|u_k|^{p_1} + |u_k|^{p_2})|v| \leq C(|u_k|^{p_1}_{p_1+1}|v|^{p_1+1}_{p_1+1} + |u_k|^{p_2}_{p_2+1}|v|^{p_2+1}_{p_2+1}) \leq C\|u_k\|^{p_1}_{p_1} + \|u_k\|^{p_2}_{p_2} \|v\|_\Omega \leq C\|v\|_\Omega.
\]
Therefore
\[
|\langle \nabla J_V(u_k), v \rangle_\Omega| = |2(u_k,v)_{\Omega} - \int_\Omega [f'(u_k)u_k + f(u_k)]v| \leq C\|v\|_\Omega \quad \forall v \in H^1_0(\Omega).
\]
This proves that \((\nabla J_V(u_k))\) is bounded.
As $|\nabla J_V(u_k)u_k| \leq \|\nabla J_V(u_k)\|u_k < C$, after passing to a subsequence, we have that $|J'_V(u_k)u_k| \to \rho \geq 0$. We show that $\rho > 0$.

From Lemma (2.1)(a) and by property (f2) we obtain

$$0 < \delta^2 \leq \|u_k\|^2 = \int_\Omega f(u_k)u_k \leq c(|u_k|_{p_1+1} + |u_k|_{p_2+1}).$$

This inequality, together with Lions’ lemma, implies that there exist $\delta > 0$ and a sequence $(y_k)$ in $\mathbb{R}^N$ such that

$$\int_{B_1(y_k)} u_k^2 = \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} u_k^2 \geq \delta.$$

Set $\tilde{u}_k(x) := u_k(x + y_k)$. After passing to a subsequence $\tilde{u}_k \rightharpoonup u$ weakly in $H^1(\mathbb{R}^N)$, $\tilde{u}_k \to u$ in $L^2_{loc}(\mathbb{R}^N)$ and $\tilde{u}_k(x) \to u(x)$ a.e. in $\mathbb{R}^N$. Therefore,

$$\int_{B_1(0)} u^2 = \lim_{k \to \infty} \int_{B_1(0)} \tilde{u}_k^2 = \lim_{k \to \infty} \int_{B_1(y_k)} u_k^2 \geq \delta.$$

Hence, $u \neq 0$ and there exists a subset $\Lambda$ of positive measure in $B_1(0)$ such that $u(x) \neq 0$ for every $x \in \Lambda$. Property (f5) implies that $f'(u(x))(u(x))^2 - f(u(x))u(x) > 0$ if $u(x) \neq 0$. So, from the Fatou’s lemma, we conclude that

$$\rho = \lim_{k \to \infty} \|\nabla J_V(u_k)u_k\| = \lim_{k \to \infty} \frac{1}{2}\|u_k\|^2|_\Omega - \int_\Omega \left[f'(u_k)u_k^2 + f(u_k)u_k\right]$$

$$= \lim_{k \to \infty} \int_\Omega f'(u_k)u_k^2 - f(u_k)u_k$$

$$= \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[f'(u_k)u_k^2 - f(u_k)u_k\right] = \lim_{k \to \infty} \int_{\mathbb{R}^N} \left[f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k\right]$$

$$\geq \liminf_{k \to \infty} \int_{\Lambda} \left[f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k\right] \geq \int_{\Lambda} \liminf_{k \to \infty} f'(\tilde{u}_k)\tilde{u}_k^2 - f(\tilde{u}_k)\tilde{u}_k \geq 0.$$

Taking the inner product of (21) with $u_k$ we obtain

$$0 = I'_V(u_k)u_k = \langle \nabla \mathcal{N}_{\Lambda} I_V(u_k), u_k \rangle + t_k \nabla J_V(u_k)u_k = o_k(1) + t_k \nabla J_V(u_k)u_k$$

and so $t_k \to 0$ and from (21) we deduce $\nabla I_V(u_k) \to 0$ as $\nabla \mathcal{N}_{\Lambda} I_V(u_k) \to 0$ and this proves the lemma.

**Lemma 3.5 (Splitting).** Let $(u_k)$ be a bounded sequence in $H^1_0(\Omega)$ such that

$I_V(u_k) \to d$ and $I'_V(u_k) \to 0$ in $H^{-1}(\Omega)$.

Replacing $u_k$ by a subsequence if necessary, there exist a solution $u_0$ de $(P_V)$, a number $m \in \mathbb{N}$, $m$ functions $w_1, \ldots, w_m$ in $H^1(\mathbb{R}^N)$ and $m$ sequences of points $(y'_k) \in \mathbb{R}^N$, $1 \leq j \leq m$, satisfying:

a) $u_k \to u_0$ in $H^1_0(\Omega)$ or
b) $w_i$ are nontrivial solutions of the limit problem $(P_\infty)$;

c) $|y'_k| \to +\infty$ and $|y'_i - y'_k| \to +\infty$ i $\neq j$;

d) $u_k - \sum_{i=1}^m w_j \to u_0$ in $H^1(\mathbb{R}^N)$.

e) $d = I_V(u_0) + \sum_{i=1}^m I_\infty(w_j)$.
Proof. The proof is almost the same as that in [25] Lemma 4.5, just observing that 
\[ u_k^1 := u_k - u_0, \quad I_0(u_k^1) = I_V(u_k^1) + \int_{\mathbb{R}^N} (V_{\infty} - V(x))(u_k^1)^2 \] and using (V2).

\[ \Box \]

Lemma 3.6. Problem (P_\infty) does not have a solution u such that \( I_\infty(u) \in (c_\infty, 2c_\infty) \)

Proof. Under our assumptions on f including that f is odd, the limit problem (P_\infty) has a positive solution w with \( I_\infty(w) = c_\infty \) [6]. If u is a solution of P_\infty such that \( I_\infty(u) \in [c_\infty, 2c_\infty) \) then, by Lemma (3.3), u does not change sign and, by [7], it is radially symmetric. By assumption (U) problem (P_\infty) has a unique positive solution and therefore u = ±w, up to a translation. Hence, \( I_\infty(u) = c_\infty \). \[ \Box \]

Corollary 1 (Compactness). If \( c_V \) is not attained, then \( c_V \geq c_\infty \) and \( I_V \) satisfies the Palais-Smale condition on \( N_V \) at every level \( d \in (c_\infty, 2c_\infty) \).

Proof. Let \( (u_k) \) be a (PS)_d-sequence for \( I_V \) on \( N_V \). By Lemmas 3.1 and Lemmas 3.4, after passing to a subsequence, we have that \( (u_k) \) is a bounded (PS)_d-sequence for \( I_V \). By the definition \( c_V := \inf_{u \in N_V} I_V(u) \), there exists \( \{u_j\} \subset N_V \) such that \( I_V(u_j) \rightarrow c_V \). Now by the Ekeland variational principle there exists \( \{\tilde{u}_j\} \subset N_V \) such that \( I_V(\tilde{u}_j) \rightarrow c_V \) and \( I'_V(\tilde{u}_j) \rightarrow 0 \) (Theorem 8.5 [32]). Now by the Splitting lemma if \( d = c_V \) is not attained, we have \( c_V = I_V(u_0) + \sum_{i=1}^m I_\infty(w_j) \) and so \( c_V \geq c_\infty \). If \( d \in (c_\infty, 2c_\infty) \) and \( (u_k) \) does not have a convergent subsequence then, by the Splitting lemma,

\[ 2c_\infty > d = I_V(u_0) + \sum_{i=1}^m I_\infty(w_j) \geq \begin{cases} mc_\infty & \text{if } u_0 = 0 \\ c_V + mc_\infty \geq (m+1)c_\infty & \text{if } u_0 \neq 0 \end{cases} \]

(22)

then in both cases, \( m < 2 \) and so \( m = 1 \). The hypothesis \( 2c_\infty > d \geq (m+1)c_\infty \) implies that it is not possible to occur \( m = 1 \) and \( u_0 \neq 0 \), there for \( u_0 = 0 \), which implies \( I_V(u_n) \rightarrow I_\infty(w_1) = d \). By (U) the solution is unique and so \( w_1 = w \) that yields there exists a solution w of \( P_\infty \) with \( d = I_\infty(w) \), which contradicts Lemma 3.6. Hence, \( I_V \) satisfies the Palais-Smale condition on \( N_V \) at every \( d \in (c_\infty, 2c_\infty) \). \[ \Box \]

Remark 5. If \( 0 < V_0 < V(x) < V_\infty \), then \( c_V < c_\infty \) and by Lemma 3.5, \( u_k \rightarrow u_0 \) in \( H_0^1(\Omega) \). Therefore, \( I'(u_0) = 0 \) and \( I(u_0) = c_V > 0 \), here \( u_0 \) is a solution of (P_V) and \( c_V \) is attained. If \( V(x) = V_\infty \), then by [25] \( c_V \) is not attained.

4. Existence of a positive bound state solution. Let us define, for \( R > 0 \), \( \|y_0\| = 1 \) and \( y \in \partial B_2(y_0) \),

\[ \varepsilon_R := \int_{\mathbb{R}^N} f(w_0^R)w_y^R = \int_{\mathbb{R}^N} f(w_y^R)w_0^R. \]

Note that in principle \( \varepsilon_R = \varepsilon_R(y) \) is dependent on \( y \), but we are going to show that the estimates on \( \varepsilon_R \) are independent of \( y \).

Hereafter, we present three lemmas that prepare for the essential proposition 1, which gives an upper bounded for the functional \( I_V \) on a convenient subset of \( N_V \).

Lemma 4.1. There exists \( C > 0 \) such that

\[ \varepsilon_R \leq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty R}}, \]

for all \( y \in \partial B_2(y_0) \).

Proof. From property (f_2), performing a change of variable, we have that

\[ \varepsilon_R \leq C \left( \int_{\mathbb{R}^N} |w(x)|^p |w(x-R(y-y_0))| + \int_{\mathbb{R}^N} |w(x)|^p |w(x-R(y-y_0))| \right) \]
As \( p_2 \geq p_1 > 1 \), using estimates (2), Lemma 2.3 and Lemma 2.6 with \( p = p_1 \) and \( q = 1 \) we obtain that
\[
\int_{\mathbb{R}^N} |w(x)|^{p_1} |w(x - R(y - y_0))| \leq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}
\]
and
\[
\int_{\mathbb{R}^N} |w(x)|^{p_2} |w(x - R(y - y_0))| \leq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}
\]
so the lemma is proved. \( \Box \)

Note that above lemma implies that
\[
\varepsilon_R \rightarrow 0 \quad \text{as } R \rightarrow \infty, \quad \text{uniformly in } \ y \in \partial B_2(y_0).
\]

Lemma 4.2. There exists \( C > 0 \) such that for all \( s, t \geq \frac{1}{2} \), \( y \in \partial B_2(y_0) \) and \( R \) large enough,
\[
\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R \geq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}.
\]

Proof. For \( |x| < 1 \) and \( R \) large enough we have
\[
1 + |x - R(y - y_0)| < 1 + |x| + |R(y - y_0)| < 4R.
\]
Now, by \((f_3), (24)\) and the decay estimates \((2)\) there exists \( C > 0 \) such that
\[
\int_{\mathbb{R}^N} f(sw_0^R)tw_y^R = st \int_{\mathbb{R}^N} \left[ \frac{f(sw_0^R)}{sw_0^R} \right] w_y^R \geq \frac{1}{4} \int_{\mathbb{R}^N} \left[ \frac{f(\frac{1}{2} w_0^R)}{\frac{1}{2} w_0^R} \right] w_y^R
\]
\[
\geq \frac{1}{4} \int_{B_1(R(y_0))} \left[ \frac{f(\frac{1}{2} w_0(x))}{\frac{1}{2} w_0(x)} \right] w_0^R w_y^R
\]
\[
\geq \frac{1}{4} \min_{x \in B_1(0)} f(\frac{1}{2} w_0(x)) \int_{B_1(0)} w(x)w(x - R(y - y_0))
\]
\[
\geq C \int_{B_1(0)} (1 + |x|)^{-\frac{N+1}{2}} e^{-\sqrt{V_\infty}|x|} (1 + |x - R(y - y_0)|)^{-\frac{N+1}{2}} e^{-\sqrt{V_\infty}|x - R(y - y_0)|}
\]
\[
\geq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}.
\]
\( \Box \)

If we set \( s, t = 1 \) in above lemma we have
\[
\varepsilon_R \geq CR^{-\frac{N-1}{2}} e^{-2\sqrt{V_\infty} R}.
\]

Lemma 4.3. For every \( b > 1 \) there is a constant \( C \), such that
\[
\left| \int_R [sf(w_0^R \psi) - f(sw_0^R \psi)]w_y^R \right| \leq C|s - 1|\varepsilon_R,
\]
for all \( s \in [0, b] \), \( y \in \partial B_2(y_0) \) and \( R \) large enough.

Proof. Fix \( u \in \mathbb{R} \) and consider the function \( g(s) := sf(u) - f(su) \). By assumption \((f_2), \)
\[
g'(s) := f(u) - f'(su)u \leq |f(u)| + (s^{p_1-1}|u|^{p_1} + s^{p_2-1}|u|^{p_2})
\]
\[
\leq |f(u)| + C(|u|^{p_1} + |u|^{p_2}) \quad \forall s \in [0, 1],
\]
hence, by the mean value theorem,
\[
|sf(u) - f(su)| = |g(u) - g(1)| = |g'(t)||s - 1|
\]
It follows that
\[ \int_{\Omega} |sf(w_0^R \psi) - f(sw_0^R \psi)w_y^R \psi| \leq |s - 1||f(u)| + C(|u|^{p_1} + |u|^{p_2})||s - 1|. \]

Now applying Lemma 2.6 and since $|\psi| \leq 1$, we have
\[ \int_{\mathbb{R}^N} |sf(w_0^R \psi) - f(sw_0^R \psi)|w_y^R \psi \leq C|s - 1|\varepsilon_R \]
for all $s \in [0, b]$, $y \in \partial B_2(y_0)$, as claimed. \hfill \Box

**Proposition 1.** There exists $R_1 > 0$ and, for each $R > R_1$, a number $\eta = \eta_R > 0$, $\eta_R = o_R(1)$ such that
\[ I_V(T_{\lambda,y}^RU_{\lambda,y}^R) \leq 2c_\infty - \eta, \]
for all $\lambda \in [0, 1]$, $y \in \partial B_2(y_0)$.

**Proof.** Let us denote, for simplicity,
\[ s := T_{\lambda,y}^R \lambda, \quad t := T_{\lambda,y}^R (1 - \lambda). \]
Recall that, by Lemma 2.10, $s, t \in (0, T_0)$ if $R$ is large enough.

We have that
\[ I_V(sw_0^R \psi + tw_y^R \psi) \]
\[ = \frac{s^2}{2} \int_{\Omega} \|
abla (w_0^R \psi)\|^2 + \frac{s^2}{2} \int_{\Omega} V(x)(w_0^R \psi)^2 + \frac{t^2}{2} \int_{\Omega} \|
abla (w_y^R \psi)\|^2 + \frac{t^2}{2} \int_{\Omega} V(x)(w_y^R \psi)^2 \]
\[ + st \int_{\Omega} \nabla (w_0^R \psi) \cdot \nabla (w_y^R \psi) + st \int_{\Omega} V(x)w_0^R \psi w_y^R \psi - \int_{\Omega} F(sw_0^R \psi + tw_y^R \psi) \]
\[ = \frac{s^2}{2} \int_{\Omega} \|
abla (w_0^R \psi)\|^2 + \frac{s^2}{2} \int_{\Omega} V_\infty(w_0^R \psi)^2 - \int_{\Omega} F(sw_0^R \psi) \]
\[ + \frac{t^2}{2} \int_{\Omega} \|
abla (w_y^R \psi)\|^2 + \frac{t^2}{2} \int_{\Omega} V_\infty(w_y^R \psi)^2 - \int_{\Omega} F(tw_y^R) \]
\[ + \frac{s^2}{2} \int_{\Omega} (V(x) - V_\infty)(w_0^R \psi)^2 + \frac{t^2}{2} \int_{\Omega} (V(x) - V_\infty)(w_y^R \psi)^2 \]
\[ + st \int_{\Omega} \nabla (w_0^R \psi) \cdot \nabla (w_y^R \psi) + st \int_{\Omega} V_\infty w_y^R \psi w_y^R \psi \]
\[ + st \int_{\Omega} (V(x) - V_\infty)w_y^R \psi w_y^R \psi \]
\[ - \int_{\Omega} F(sw_0^R \psi + tw_y^R \psi) - F(sw_0^R \psi) - F(tw_y^R \psi) - f(sw_0^R \psi)tw_y^R \psi - f(tw_y^R \psi)sw_0^R \psi \]
\[ - \int_{\Omega} f(sw_0^R \psi)tw_y^R \psi - \int_{\Omega} f(tw_y^R \psi)sw_0^R \psi. \]
The sum in line (26) is equal to \( I_\infty(su_0^R) + o(\varepsilon_R) \). Indeed,
\[
(26) = I_\infty(su_0^R) + \frac{s^2}{2} \int_{B_{2k}(0)} |\nabla(w_0^R\psi)|^2 - |\nabla w_0^R|^2 \\
+ \frac{s^2}{2} \int_{B_{2k}(0)} V_{w_0^R}(w_0^R\psi)^2 - V_{w_0^R}(w_0^R)^2 - \int_{B_{2k}(0)} F(su_0^R) - F(su_0^R\psi).
\]
But by Lemma 2.5 with \( q = 2 \) and (16) we have
\[
\frac{s^2}{2} \int_{B_{2k}(0)} |\nabla(w_0^R\psi)|^2 - |\nabla w_0^R|^2 + \frac{s^2V_{\infty}}{2} \int_{B_{2k}(0)} (w_0^R\psi)^2 - (w_0^R)^2 = o(\varepsilon_R).
\]
On the other hand, the mean value theorem, (f2) and Lemma 2.5 yield
\[
\int_{B_{2k}(0)} F(su_0^R) - F(su_0^R\psi) = \int_{B_{2k}(0)} f(sw_0^R + A(x)sw_0^R\psi)(sw_0^R - sw_0^R\psi) \\
\leq C \int_{B_{2k}(0)} (|sw_0^R|^{p_1} + |sw_0^R|^{p_2})sw_0^R \\
= C \int_{B_{2k}(0)} (|sw_0^R|^{p_1+1} + |sw_0^R|^{p_2+1}) = o(\varepsilon_R),
\]
thus
\[
(26) = I_\infty(su_0^R) + o(\varepsilon_R).
\]
Since \( w_0^R \) is the least energy solution of the limit problem \( (P_\infty) \) and by Lemma 2.1 (c) we have \( I_\infty(su_0^R) \leq c_\infty \) and similarly for the sum in line (27), so
\[
(26) + (27) \leq 2c_\infty + o(\varepsilon_R).
\]
By Lemma 2.7 we have
\[
(28) + (30) = o(\varepsilon_R).
\]
As to (31), by Lemma 2.2 there is \( 0 < \nu < p_1 - 1 \), now by Lemma 2.6 we have
\[
- \int_{\mathbb{R}^N} F(sw_0^R\psi + tw_y^R\psi) - F(sw_0^R\psi) - F(tw_y^R\psi) - f(sw_0^R\psi)tw_y^R\psi - f(tw_y^R\psi)sw_0^R\psi \\
\leq C(st)^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (w_y^R\psi w_0^R\psi)^{1+\frac{\nu}{2}} \leq C(st)^{1+\frac{\nu}{2}} \int_{\mathbb{R}^N} (w_y^R w_0^R)^{1+\frac{\nu}{2}} \\
\leq CR^{\frac{\nu}{2}+2}(1+\frac{\nu}{2})e^{-2(1+\frac{\nu}{2})\sqrt{\varepsilon_R}}
\]
so this yields that
\[
(31) \leq o(\varepsilon_R).
\]
We write the sum of the remaining terms as
\[
(29) + (32) = \frac{t}{2} \int_{\Omega} |sf(w_0^R\psi) - f(sw_0^R\psi)|w_y^R\psi + \frac{s}{2} \int_{\Omega} |tf(w_0^R\psi) - f(tw_y^R\psi)|w_y^R\psi \\
- \frac{t}{2} \int_{\Omega} f(sw_0^R\psi)tw_y^R\psi - \frac{1}{2} \int_{\Omega} f(tw_y^R\psi)sw_0^R\psi.
\]
By Lemma 4.3 there is a constant \( C > 0 \) such that
\[
\frac{t}{2} \int_{\Omega} |sf(w_0^R\psi) - f(sw_0^R\psi)|w_y^R\psi + \frac{s}{2} \int_{\Omega} |tf(w_0^R\psi) - f(tw_y^R\psi)|w_y^R\psi \leq C(|s-1|+|t-1|)\varepsilon_R
\]
for all \( s, t \in [0, T_0], y \in \partial B_2(y_0) \) and \( R \) large enough. Similar to the sum (26) we have
\[
\frac{1}{2} \int_{\Omega} f(sw_0^R\psi)tw_y^R\psi + \frac{1}{2} \int_{\Omega} f(tw_y^R\psi)sw_0^R\psi
\]
\begin{equation}
= \frac{1}{2} \int_{\mathbb{R}^N} f(sw_0^R t w_y^R) + \frac{1}{2} \int_{\mathbb{R}^N} f(t w_y^R) sw_0^R + o(\varepsilon_R)
\end{equation}
and by Lemma 4.2, there is a constant $C_0 > 0$ such that
\[\frac{1}{2} \int_{\mathbb{R}^N} f(sw_y^R t w_y^R) + \frac{1}{2} \int_{\mathbb{R}^N} f(t w_y^R) sw_0^R \psi \geq C_0 \varepsilon_R\]
for all $s, t \geq \frac{1}{2}, y \in \partial B_2(y_0)$ and $R$ large enough. By Lemma 2.10, if $\lambda = 1/2$, then $s, t \to 1$ as $R \to \infty$. So taking $R_0 > 0$ sufficiently large and $\delta \in (0, 1/2)$ sufficiently small such that $C(|s - 1| + |t - 1|) \leq \frac{C_0}{2}$, we have
\[
(29) + (32) \leq -\frac{C_0}{2} \varepsilon_R + o(\varepsilon_R)
\]
for all $\lambda \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta], y \in \partial B_2(y_0)$ and $R > R_0$. Summing up, we have obtained
\begin{equation}
I_V(sw_0^R + tw_y^R) \leq 2c_\infty - \frac{C_0}{2} \varepsilon_R + o(\varepsilon_R)
\end{equation}
for all $\lambda \in [\frac{1}{2} - \delta, \frac{1}{2} + \delta], y \in \partial B_2(y_0)$ and $R > R_0$.
On the other hand, for all $\lambda \in [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1], y \in \partial B_2(y_0)$ and $R$ sufficiently large, since if $T_{\lambda,y}^R \leq 2$ then $s = T_{\lambda,y}^R$, $\lambda \in [0, 1 - 2\delta]$ or $t = T_{\lambda,y}^R(1 - \lambda) \in [1, 1 - 2\delta]$ and $T_{\lambda,y}^R \geq 2$ then $s = T_{\lambda,y}^R$, $\lambda \in [1 + 2\delta, \infty]$ or $t = T_{\lambda,y}^R(1 - \lambda) \in [1 + 2\delta, \infty]$, in fact one of $s$ or $t$ is in $[0, 1 - 2\delta] \cup [1 + 2\delta, \infty]$ and so by Lemma 2.1(c) applied to $V_\infty$, there exists $\gamma \in (0, c_\infty)$ such that
\[
I_\infty(rw_0^R) \leq c_\infty - \gamma \quad \forall r \in [0, 1 - 2\delta] \cup [1 + 2\delta, \infty]
\]
also with our previous estimates we have $(28) + \ldots + (32) = O(\varepsilon_R)$, and so
\begin{equation}
I_V(sw_0^R + tw_y^R) \leq 2c_\infty - \gamma + O(\varepsilon_R).
\end{equation}
Inequalities (33) and (34), together, yield the statement of the proposition. 

\textbf{Remark 6.} Note that assumption $\gamma > 2\sqrt{\varepsilon_\infty}$ in (V2) is sharp if one looks for an upper bound in Lemma 2.7 with exponential decay of order $e^{-2\sqrt{\varepsilon_\infty} R}$. In [14] there is a constant 2 missing in the exponential term which bounds $\varepsilon_R$.

\textbf{Lemma 4.4.} For any $\delta > 0$, there exists $R_2 > 0$ such that
\[I_V(T_{\lambda,y}^R U_{\lambda,y}^R) < c_\infty + \delta,
\]
for $\lambda = 0$ and every $y \in \partial B_2(y_0)$ and $R \geq R_2$. In particular, $c_V \leq c_\infty$.

\textbf{Proof.} By Lemma 2.10, $T_{\lambda,y}^R$ is bounded uniformly in $\lambda, y$ and $R$. As $w_y^R$ is a ground state solution of problem $(P_\infty)$, using Lemma 2.1(c) and Lemma 2.7, we obtain
\[
I_V(T_{0,y}^R U_{0,y}^R) = I_\infty(T_{0,y}^R w_y^R \psi) + (T_{0,y}^R)^2 \int_{\Omega} (V(x) - V_\infty)(w_y^R \psi)^2 \leq I_\infty(T_{0,y}^R w_y^R) + o(\varepsilon_R) + (T_{0,y}^R)^2 \int_{\mathbb{R}^N} (V(x) - V_\infty)(w_y^R)^2 \leq \max_{s > 0} I_\infty(sw_0^R) + o(1) \leq c_\infty + o_R(1),
\]
where $o_R(1) \to 0$ as $R \to \infty$, uniformly in $y \in \partial B_2(y_0)$.

Let $\beta : L^2(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ be a barycenter map, i.e. a continuous map such that, for every $u \in L^2(\mathbb{R}^N)$, every $y \in \mathbb{R}^N$ and every linear isometry $A$ of $\mathbb{R}^N$,
\[
\beta(u(-y)) = \beta(u) + y \quad \text{and} \quad \beta(u \circ A^{-1}) = A(\beta(u)).
\]
Note that $\beta(u) = 0$ if $u$ is radial. Barycenter maps have been constructed in [4].
Lemma 4.5. If $c_V$ is not attained then $c_V = c_\infty$ and there exists $\delta > 0$ such that

$$\beta(u) \neq 0, \quad \forall u \in \mathcal{N}_V \cap I_V^{c_\infty + \delta},$$

where $I_V = \{ u \in H_0^1(\Omega), \quad I_V(u) \leq c \}$. Proof. If $c_V$ is not attained, Corollary 1 and Lemma 4.4 imply that $c_V = c_\infty$. Arguing by contradiction, assume that for each $k \in N$ there exists $v_k \in \mathcal{N}_V$ such that $I_V(v_k) < c_V + \frac{1}{k}$ and $\beta(v_k) = 0$. By Ekeland variational principle [18], there exists a $(PS)_d$-sequence $(u_k)$ for $I_V$ on $\mathcal{N}_V$ at the level $d = c_V$ such that $\|u_k - v_k\| \to 0$ [32, Theorem 8.5]. By Lemmas 3.4 and 3.1, after passing to a subsequence, we have that $(u_k)$ is a bounded $(PS)_d$-sequence for $I_V$. As $c_V$ is not attained, Lemma 3.5 (splitting) implies that there exists a sequence $(y_k)$ in $\mathbb{R}^N$ such that $\|y_k\| \to \infty$ and $\|u_k - w(-y_k)\| \to 0$, where $w$ is the (positive or negative) radial ground state of $(P_\infty)$. Setting $\tilde{v}_k(x) := v_k(x + y_k)$, and using properties (35) and the continuity of the barycenter, we conclude that

$$-y_k = \beta(v_k) - y_k = \beta(\tilde{v}_k) \to \beta(w) = 0$$

yielding a contradiction.

We have constructed all the tools in order to apply a topological argument analogous to that found in [14] to prove our main result. For the sake of completeness we recall the argument and prove theorem 1.1.

Proof of Theorem 1.1. If $c_V$ is attained by $I_V$ at some $u \in \mathcal{N}_V$ then, by Lemma 3.2, $u$ is a nontrivial solution of problem $(P_V)$. So assume that $c_V$ is not attained. Then, by Lemma 4.5, $c_V = c_\infty$. We will show that $I_V$ has a critical value in $(c_\infty, 2c_\infty)$. By Lemma 4.5, we may fix $\delta \in (0, \frac{1}{4})$ such that

$$\beta(u) \neq 0, \quad \forall u \in \mathcal{N}_V \cap I_V^{c_\infty + \delta}.$$

Proposition 1 and Lemma 4.4 allow us to choose $R > 0$ sufficiently large and its corresponding $\eta_R = \eta \in (0, \frac{1}{4})$ such that

$$I_V(T^{R}_{\lambda, y} U^{R}_{\lambda, y}) \leq \begin{cases} 2c_\infty - \eta & \text{for all } \lambda \in [0, 1] \text{ and all } y \in \partial B_2(y_0) \\ c_\infty + \delta & \text{for } \lambda = 0 \text{ and all } y \in \partial B_2(y_0). \end{cases}$$

For this fixed $R > 0$, define $\alpha : B_2(y_0) \to \mathcal{N}_V \cap I_V^{2c_\infty - \eta}$ by

$$\alpha(\lambda y_0 + (1 - \lambda)y) := T^{R}_{\lambda, y} U^{R}_{\lambda, y} \quad \text{with } \lambda \in [0, 1], \quad y \in \partial B_2(y_0).$$

Arguing by contradiction, assume that $I_V$ does not have a critical value in $(c_\infty, 2c_\infty)$. As, by Corollary 1, $I_V$ satisfies the Palais-Smale condition on $\mathcal{N}_V$ at every level in $(c_\infty, 2c_\infty)$, there exists $\varepsilon > 0$ such that

$$\|\nabla \mathcal{N}_V I_V(u)\| \geq \varepsilon, \quad \forall u \in \mathcal{N}_V \cap I_V^{-1}[c_\infty + \delta, 2c_\infty - \eta].$$

At this point we may use a Deformation Lemma for $C^1$ manifold found in [8], which yields a continuous function

$$\rho : \mathcal{N}_V \cap I_V^{2c_\infty - \eta} \to \mathcal{N}_V \cap I_V^{c_\infty + \delta}$$

such that $\rho(u) = u$ for all $u \in \mathcal{N}_V \cap I_V^{c_\infty + \delta}$ (see [8]). Now we define $\Gamma(x) := (\beta \circ \rho \circ \alpha)(x)$. By Lemma 4.5 $\Gamma(x) \neq 0$ and so the function $h : B_2(0) \to \partial B_2(0)$ given by

$$h(x) := 2 \frac{\Gamma(x)}{|\Gamma(x)|}$$
is well defined and continuous. Furthermore, if \( y \in \partial B_2(y_0) \), then it holds
\[
\alpha(y) = T_{0,y}^R U_{0,y}^R = T_{0,y} R U_{0,y}^R \in N_\Omega \cap I_{\Omega}^{c_\infty + \delta}
\]
and hence
\[
(\beta \circ \rho \circ \alpha)(y) = \beta(T_{0,y}^R U_{0,y}^R) = R_y.
\]
Therefore, if we consider the homeomorphism \( \tilde{h} : \partial B_2(y_0) \to \partial B_2(0) \) defined by \( \tilde{h}(y) := \frac{y}{|y|} \), then \( (\tilde{h}^{-1} \circ \tilde{h})(y) = y \) for every \( y \in \partial B_2(y_0) \) and by Brouwer Fixed Point Theorem such a retraction does not exist, thus \( I_V \) must have a critical point \( u \in N_V \) with \( I_V(u) \in (c_\infty, 2c_\infty) \). By Lemma 3.3, \( u \) does not change sign and, since \( f \) is odd, \( -u \) is also a solution of \( (P_V) \). This proves that problem \( (P_V) \) has a positive solution.

\[\square\]

REFERENCES

[1] N. Ackermann, M. Clapp and F. Pacella, Alternating sign multibump solutions of nonlinear elliptic equations in expanding tubular domains, Comm. Partial Differential Equations, 38 (2013), 751–779.
[2] A. Bahri and Y.Y. Li, On a min-max procedure for the existence of a positive solution for certain scalar field equations in \( \mathbb{R}^N \), Rev. Mat. Iberoamericana 6, 1/2 (2013), 751–779.
[3] A. Bahri and P.L. Lions, On the existence of a positive solution of semilinear elliptic equations in unbounded domains, Ann. Inst. H. Poincare Anal. Non Lineaire, 14 (1997), 365–413.
[4] T. Bartsch and T. Weth, Three nodal solutions of singularly perturbed elliptic equations on domains without topology, Ann. Inst. H. Poincare Anal. Non Lineaire, 22 (2005), 259–281.
[5] V. Benci and G. Cerami, Positive solutions of some nonlinear elliptic problems in exterior domains, Arch. Rational Mech. Anal., 99 (1987), 283–300.
[6] H. Berestycki and P. L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal., 82 (1983), 313–345.
[7] H. Berestycki, T. Gallouet and O. Kavian, Equations de champs scalaires euclidiens non lineaires dans le plan, C. R. Math. Acad. Sci., 297 (1983), 307–310.
[8] A. Bonnet, A deformation lemma on \( C^1 \) manifold, Manuscripta Math., 81 (1993), 339–359.
[9] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer Science+Business Media, LLC 2011.
[10] G. Cerami, Un criterio di esistenza per i punti critici su varietà illimitate, Rend. Accad. Sc. Lett. Inst. Lombardo, 112 (1978), 332–336.
[11] G. Cerami, Some nonlinear elliptic problems in unbounded domains, Milan J. Math., 74 (2006), 47–77.
[12] G. Cerami and D. Passaseo, Existence and multiplicity results for semilinear elliptic dirichlet problems in exterior domains, Nonlinear Analysis TMA 24, 11 (1995), 1533–1547.
[13] G. Citti, On the exterior Dirichlet problem for \( \Delta u - u + f(x, u) = 0 \), Rendiconti del seminario matematico dell’università di Padova, 88 (1992), 83–110.
[14] M. Clapp and L. A. Maia, A positive bound state for an asymptotically linear or superlinear Schrödinger equation, J. Differential Equation, 260 (2016), 3173–3192.
[15] C. V. Coffman and M. Marcus, Superlinear elliptic Dirichlet problems in almost spherically symmetric exterior domains, Arch Rational Mech. Anal., 96 (1986), 167–196.
[16] R. Dautray and J-L. Lions, Mathematical Analysis and Numerical Methods for Science and Technology. Vol. 1. Physical Origins and Classical Methods, Springer-Verlag, Berlin, 1990.
[17] W.-Y. Ding and W-M. Ni, On the existence of positive entire solution of semilinear elliptic equation, Arch Rational Mech. Anal., 91 (1986), 283–308.
[18] I. Ekeland, On the variational principle, J. Math. anal. Appl., 47 (1974), 324–353.
[19] M. Esteban and P. L. Lions, Existence and nonexistence results for semi-linear elliptic problems in unbounded domains, Proc. Roy. Soc. Edinburgh Sect. A 93, 1-2 (1982), 1–14.
[20] G. Evéquoz and T. Weth, Entire solutions to nonlinear scalar field equations with indefinite linear part, Adv. Nonlinear Stud., 12 (2012), 281–314.
[21] G. P. Galdi and C. R. Grisanti, Existence and regularity of steady flows for shear-thinning liquids in exterior two-dimensional, Arch. Rational Mech. Anal., 200 (2011), 533–559.
[22] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, *Adv. in Math. Suppl. Stud.*, **7a** (1981), 369–402.

[23] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. Parts I and II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145 and 223–283.

[24] L. A. Maia, O. H. Miyagaki and S. M. Soares, A sign changing solution for an asymptotically linear Schrödinger equation, *Proc. Edinb. Math. Soc.*, (2015), 697–716.

[25] L. A. Maia and B. Pellacci, Positive solutions for asymptotically linear problems in exterior domains, *Annali di Matematica Pura ed Applicata*, (2016), **196** (2017), 1399–1430.

[26] K. McLeod, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$ II, *Trans. Amer. Math. Soc.*, **339** (1993), 495–505.

[27] Z. Nehari, Characteristic values associated with a class of non-linear second-order differential equations, *Acta Math.*, **105** (1961), 141–175.

[28] Z. Nehari, A nonlinear oscillation theorem, *Duke Math. J.*, **42** (1975), 183–189.

[29] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.*, **43** (1992), 270–291.

[30] J. Serrin and M. Tang, Uniqueness of ground states for quasilinear elliptic equations, *Indiana Univ. Math. J.*, **49** (2000), 897–923.

[31] C. A. Stuart, *An Introduction to Elliptic Equation in $\mathbb{R}^N$*, Trieste Notes, 1998.

[32] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhauser Boston, Inc., Boston, MA, 1996.

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