Covariant Wigner Function Approach to the Boltzmann Equation for Mixing Fermions in Curved Spacetime

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Based on the covariant Wigner function approach we derive the quantum Boltzmann equation for fermions with flavor mixing in general curved spacetime. This work gives a rigorous theoretical framework to investigate the flavor oscillation phenomena taking the gravitational effect into account. It is shown that the Boltzmann equation of the lowest order of the expansion with respect to $\hbar$ reproduces the previous result which was derived in the relativistic limit on the Minkowski background spacetime. It is demonstrated that the familiar formula for the vacuum neutrino oscillation can be obtained by solving the Boltzmann equation. Higher order effects of the $\hbar$-expansion is also briefly discussed.

I. INTRODUCTION

The Boltzmann equation, which describes non-equilibrium systems of many particles, is one of the most fundamental to investigate physical processes in the early universe. For example, the statistics of the angular anisotropies in the cosmic microwave background is predicted by solving the Boltzmann equation for photons in curved spacetime (see, e.g., [1]). The decoupling of heavy particles from thermal medium in the expanding universe, which is known as the generalized Lee-Weinberg problem, is investigated with the use of the Boltzmann equation [2, 3]. The Boltzmann equation for leptogenesis is a recent topics of such early universe physics [4, 5]. Because field theory is more fundamental to describe particle processes, it is useful to find the method to derive the Boltzmann equation from the field theory. There are two different approaches to derive the Boltzmann equation from the field theory. One is the method with the use of the density matrix and the other is the Wigner function approach, which we will adopt in the present paper. Thus the Boltzmann equation is regarded as an effective theory of the field theory in a many-particle limit. (see e.g. the text book [6].)

On the other hand, recently, the flavor mixing of the neutrinos has been experimentally established by several groups [7]. These results strongly suggest that the neutrinos have non-zero mass and that their interaction basis is not the mass eigenstate. The propagation of particles with mass mixing might not be trivial in general curved spacetime. Actually the gravity effect on the neutrino oscillation has been investigated by several authors ([8, 9, 10, 11] and references therein). However, these works focus on how the quantum-mechanical phase of the mixing is affected due to the gravity. We might claim that the Boltzmann equation is needed to investigate collective phenomena of many-particle systems. The Boltzmann equation of the neutrinos has been investigated by several authors (see [12, 13, 14] and references therein). Sigl and Raffelt developed a general formula of the Boltzmann equation for the neutrinos with flavor mixing using the density matrix approach [12]. It has been shown that the same result can be reproduced with the Wigner function approach [13, 14]. The Boltzmann equation of the neutrinos is very important to understand the early universe. For example, Dolgov et al. have claimed that the flavor equilibrium can be achieved due to the flavor mixing effect before the neutrino decoupling, which might lead a significant constraint on the lepton number asymmetry of the background neutrinos by combining a constraint from the successful primordial nucleosynthesis [15, 16].

The Boltzmann equation of the neutrinos, in the previous works, have been derived assuming the Minkowski background spacetime. The gravity effect on the Boltzmann equation is usually taken into account by replacing the Liouville operator in the Minkowski spacetime with that in curved spacetime. The validity of this procedure might be worth being checked. The primary aim of the present paper is to derive the Boltzmann equation for fermions with flavor mixing in general curved spacetime in a rigorous manner starting from the Dirac equation on curved spacetime. For that purpose, we take the covariant Wigner function approach. It is known that a simple Wigner function approach encounters the problem of the covariance of the general coordinate transformation. To avoid this problem, several authors considered the covariant Wigner function approach. Winter defined the covariant Wigner function by introducing the covariant geodesic distance [17], while Calzetta, Habib and Hu defined the covariant Wigner function using the Riemann normal coordinate [18]. Fonarev developed a very elegant framework for the covariant Wigner function using the tangent space. He checked that these three methods yield the locally same equation [19, 20]. Here we adopt the framework by Fonarev to apply it to the fermions with flavor mixing.

This paper is organized as follows: In section 2 we introduce the covariant Wigner function for mixing fermions, and derive its equation of motion. The equation of motion can be regarded as the Boltzmann equation for mixing
fermions in curved spacetime. Some details of the derivation are described in the Appendix. In section 3, we show that the same equation is reproduced at the lowest order of the expansion with respect to $\hbar$ in the ultra-relativistic limit, which is a useful test of the previous result using the density matrix formalism. In section 4, we demonstrate that the familiar formula of the transition probability of the neutrino oscillation is obtained by solving the Boltzmann equation. Higher order effects of the $\hbar$-expansion on the Boltzmann equation is discussed in section 5. Section 6 is devoted to summary and conclusions.

In the present paper we use the unit in which the light velocity $c$ equals 1. We follow the convention of $(+---)$ for the metric $\eta_{\mu\nu}$. We use $\alpha, \beta, \mu, \nu, \cdots$ to denote the index of coordinate, $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol, and the Riemann tensor is defined $R^\alpha_{\beta\mu\nu} = \partial_\beta \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\beta\nu} + \Gamma^\alpha_{\rho\nu} \Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\rho\mu} \Gamma^\rho_{\beta\nu}$. In this paper, the superscript, $A, B, C, \cdots$ are used to denote the flavor index, while spinor index is omitted.

II. COVARIANT WIGNER FUNCTION

In this section we introduce the covariant Wigner function following Fonarev [19]. The covariance for the general coordinate transformation is manifest due to the definition with the use of the tangent space [21]. In order to introduce the Wigner function, it is instructive to start from considering the energy momentum operator for the fermion fields $\psi^A$ [21]:

$$T_{\mu \nu} = \frac{i}{2} \text{tr} [\bar{\psi} \gamma_{(\mu} \nabla_{\nu)} \psi^A - \nabla_{(\mu} \bar{\psi} \gamma_{\nu)} \psi^A],$$

where the subscript $(\mu \cdots \nu)$ denotes the symmetrization with respect to $\mu$ and $\nu$, $A$ and $B$ denote the flavor index, and $\text{tr}$ should be taken for the spinor and the flavor indices. The gamma matrix satisfies

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu \nu}.$$  \hfill (2)

Note that the above expression can be rephrased

$$T_{\mu \nu} = -\text{tr} \left[ \gamma_{(\mu} \bar{\psi} \left( -\frac{i\hbar \nabla}{2} \right) \psi_{\nu)} \right],$$

where $\nabla_{\nu} = \nabla_{\nu} - \nabla_{\nu}$. Thus, in general, we consider an operator $\hat{A}$ which can be written as

$$\hat{A} = -\text{tr} \left[ \Gamma \psi^A a \left( -\frac{i\hbar \nabla}{2} \right) \bar{\psi}^B \right],$$

where $\Gamma$ consists of $\gamma$-matrix and $a$ is a function of $-i\hbar \nabla / 2$. Then we find that $\hat{A}$ can be written as

$$\hat{A} = -\text{tr} \left[ \frac{\Gamma}{(2\pi \hbar)^4} \int d^4p \frac{1}{\sqrt{-g(x)}} \int d^4y \sqrt{-g(x)} e^{-2i\eta \cdot p, \eta / \hbar} a(p) \psi^A(x, -y) \bar{\psi}^B(x, y) \right],$$

where

$$\psi^A(x, -y) = \left( 1 - y^\alpha \nabla_\alpha + \frac{1}{2!} y^\alpha y^\beta \nabla_\alpha \nabla_\beta - \cdots \right) \psi^A(x),$$

$$\bar{\psi}^B(x, y) = \left( 1 + y^\alpha \nabla_\alpha + \frac{1}{2!} y^\alpha y^\beta \nabla_\alpha \nabla_\beta + \cdots \right) \bar{\psi}^B(x).$$

The expectation value of $\hat{A}$ is written

$$\langle \hat{A} \rangle = \text{tr} \left[ \frac{d^4p}{\sqrt{-g(x)}} \Gamma a(p) N^{AB}(x, p) \right],$$

where $N^{AB}(x, p)$ is the covariant Wigner function defined by

$$N^{AB}(x, p) = \frac{-1}{(2\pi \hbar)^4} \int d^4y \sqrt{-g(x)} e^{-2iy \cdot p, \eta / \hbar} \langle \psi^A(x, -y) \bar{\psi}^B(x, y) \rangle.$$
For example the expectation value of the energy momentum tensor is

\[ \langle T_{\mu\nu} \rangle = \text{tr} \int \frac{d^4 p}{\sqrt{-g(x)}} \gamma(\mu p_\nu) N^{AB}(x, p). \]  

(10)

In general curved spacetime, \( \langle T_{\mu\nu} \rangle \) diverges. However, how to regularize the divergence is not explicitly specified here (see also [20]). Equation of motion of \( N^{AB}(x, p) \) is the (generalized) Boltzmann equation, which we derive from the Dirac equation for \( \psi^A(x) \). In the present paper, we consider the Dirac equation:

\[ \sum_B \left[i\hbar \gamma^\mu \nabla_\mu \delta^{AB} - M^{AB}(x) - \hbar \gamma^\mu J^{AB}_\mu(x)\right] \psi^B(x) = 0, \]  

(11)

where \( J^{AB}_\mu(x) \) denotes an effective potential which arises from interaction with background particles [22]. Here we adopt the above expression motivated by the interaction of the neutrinos, which are mediated by the vector bosons.

We briefly summarized the derivation of equation of motion for \( N^{AB}(x, p) \) in Appendix for being self-contained, though the derivation is similar to that in the reference [20]. Equation of motion for \( N^{AB}(x, p) \) is formally obtained by the expansion with respect to the power index of \( \hbar \). The equation of motion up to the order of \( \hbar^2 \) is

\[ \gamma^\alpha \left(p_\alpha + \frac{i\hbar}{2} D_\alpha\right) N^{AB}(x, p) - \sum_C \left(M^{AC}(x) + \hbar \gamma^\mu J^{AC}_\mu(x)\right) \frac{\partial N^{CB}(x, p)}{\partial p_\alpha} + \hbar^2 \gamma^\alpha \sum \frac{\partial}{\partial p_\alpha} \frac{\partial}{\partial p_\beta} \frac{\partial}{\partial p_\gamma} N^{AB}(x, p) = -\gamma^\alpha \left(\frac{\hbar^2}{16} \frac{\partial N^{AB}(x, p)}{\partial p_\alpha} R_{\nu\alpha\mu\rho} \sigma^{\mu\rho} - \frac{\hbar^2}{24} R_{\nu\alpha\rho\mu} \frac{\partial}{\partial p_\nu} \frac{\partial}{\partial p_\rho} N^{AB}(x, p) - \frac{7\hbar^2}{48} R_{\alpha\mu} \frac{\partial}{\partial p_\mu} N^{AB}(x, p) \right), \]  

(12)

where we defined

\[ D_\alpha = \nabla_\alpha + \Gamma^\beta_{\alpha\nu} p_\beta \frac{\partial}{\partial p_\nu} \]  

(13)

and \( \sigma^{\mu\rho} = [\gamma^\mu, \gamma^\rho]/4 \). We first consider the equation up to the order of \( \hbar \) in the next section. The effect of the higher order terms in proportion to \( \hbar^2 \) is discussed in section 5.

**III. EQUATION OF O(\( \hbar \))**

In this section we focus on the equation of order \( \hbar \):

\[ \gamma^\alpha \left(p_\alpha + \frac{i\hbar}{2} D_\alpha\right) N(x, p) - \left( M(x) + \hbar \gamma^\alpha J_\alpha(x) - \frac{i\hbar}{2} \nabla_\alpha M(x) \frac{\partial}{\partial p_\alpha} \right) N(x, p) = 0, \]  

(14)

where we omit the flavor index of \( N^{AB}(x, p) \), for simplicity. Note that we omit the spinor index too. Following the procedure in [13], we separate the left handed chirality component from the right handed component. Introducing the left- and right- handed chirality projection operators

\[ P_L = \frac{1 - \gamma^5}{2}, \]  

(15)

\[ P_R = \frac{1 + \gamma^5}{2}, \]  

(16)

respectively, where

\[ \gamma^5 = -\frac{1}{4!} \sqrt{-g(x)} \epsilon_{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu, \]  

(17)
we define

\[ N_{RL}(x,p) = P_R N(x,p) P_R, \]  
\[ N_L(x,p) = P_L N(x,p) P_R, \]  
\[ N_R(x,p) = P_R N(x,p) P_L, \]  
\[ N_{LR}(x,p) = P_L N(x,p) P_L. \]  

In the present paper, in general, we consider \( J_\mu \) written in the form:

\[ J_\mu(x) = P_L J_{\mu L}(x) + P_R J_{\mu R}(x), \]  

where \( J_{\mu L}(x) \) and \( J_{\mu R}(x) \) are the vector quantities. Projecting \( P_L \) and \( P_R \) from left and right on equation (14), respectively, we have

\[ \gamma^\alpha \left( p_\alpha + \frac{i\hbar}{2} D_\alpha - \hbar J_{\alpha L}(x) \right) N_{RL} - \left( M(x) - \frac{i\hbar}{2} \nabla_\alpha M(x) \frac{\partial}{\partial p_\alpha} \right) N_L(x,p) = 0. \]  

In a similar way, projecting \( P_R \) and \( P_R \) from left and right on equation (14), we have

\[ \gamma^\alpha \left( p_\alpha + \frac{i\hbar}{2} D_\alpha - \hbar J_{\alpha R}(x) \right) N_L - \left( M(x) - \frac{i\hbar}{2} \nabla_\alpha M(x) \frac{\partial}{\partial p_\alpha} \right) N_{RL}(x,p) = 0. \]  

Combining (20) and (21), we eliminate \( N_{RL}(x,p) \), and obtain the equation for \( N_L \) up to the order of \( \hbar \)

\[ \left( p^\alpha p_\alpha - M^2(x) + i\hbar p^\alpha D_\alpha + \frac{i\hbar}{2} (\nabla_\alpha M^2(x)) \frac{\partial}{\partial p_\alpha} - \hbar p_\alpha J_{\alpha L}(x) \gamma^\alpha \gamma^\beta \right) 
\[ - \hbar M(x) J_{\alpha R}(x) M^{-1}(x) p_\beta \gamma^\alpha \gamma^\beta - i\hbar (\nabla_\alpha M(x)) M^{-1}(x) p_\beta \gamma^\alpha \gamma^\beta \right) N_L(x,p) = 0, \]  

where \( M^{-1}(x) \) is the inverse of the mass matrix \( M(x) \). Similar to the way to derive (25), equations for \( N_R(x,p) \) can be derived, which have the same expression as (26) but with replacing \( R(L) \) by \( L(R) \).

Now we focus on equation (25). Let us consider a solution of the form:

\[ N_L(x,p) = F_\mu(x,p) P_L \gamma^\mu. \]  

Note that \( N_L(x,p) \) has the spinor index, but \( F_\mu(x,p) \) does not have it. Inserting (26) into (25) we finally have

\[ \left( p^\alpha p_\alpha - M^2(x) + i\hbar p^\alpha D_\alpha + \frac{i\hbar}{2} (\nabla_\alpha M^2(x)) \frac{\partial}{\partial p_\alpha} \right) F_\nu(x,p) 
\[ - K_\alpha^\nu F_\nu(x,p) + K_{\mu \nu} F^\mu(x,p) - K_{\nu \mu} F^\mu(x,p) - i\epsilon^{\alpha \beta \mu} K_{\alpha \beta} F_\mu(x,p) = 0, \]  

where we defined

\[ K_{\alpha \beta} = \hbar p_\alpha J_{\alpha \beta} + \hbar M(x) J_{\alpha R}(x) M^{-1}(x) p_\beta + i\hbar (\nabla_\alpha M(x)) M^{-1}(x) p_\beta. \]  

**IV. APPLICATION TO NEUTRINOS**

In this section, we apply the result of the previous section to the neutrinos. Setting \( \nabla_\alpha M(x) = 0 \) and \( J_{\alpha R}(x) = 0 \), equation (27) reduces to

\[ \left( p^\alpha p_\alpha - M^2 + i\hbar p^\alpha D_\alpha \right) F_\nu - \hbar \left( p_\alpha J_{\alpha L}^\nu F_\nu - p_\alpha J_{\alpha R}^\nu F_\alpha + p_\nu J_{\alpha L} F^\alpha + i\epsilon^{\alpha \beta \mu} p_\alpha J_{\alpha \beta} F_\mu \right) = 0. \]  

This is consistent with the result in the reference [13], in which the effect of the gravity is not taken into account. As we will show below, in the ultra-relativistic regime, we may set

\[ F_\nu = p_\nu F, \]  

as
where $F = F(x, p)$ is a scalar function. Substituting (30) into (29), we have

\[ p_\alpha (p^\alpha p_\alpha - M^2 + i\hbar p^\alpha D_\alpha - 2\hbar p^\alpha J_{L\alpha}) F + \hbar p^\alpha p_\alpha J_{L\alpha} F = 0. \]  

(31)

The last term of the left hand side of the above equation can be neglected in the ultra-relativistic limit of $(p^0)^2 \gg O(M^2)$. In this limit, equation of $F$ is

\[ (p^\alpha p_\alpha - M^2 - 2\hbar p^\alpha J_{L\alpha} + i\hbar p^\alpha D_\alpha) F = 0. \]  

(32)

Note that $F$ has the flavor index and that $F$ is hermite. Taking the dagger of $F$, $\bar{F}(p^\alpha p_\alpha - M^2 - 2\hbar p^\alpha J_{L\alpha} - i\hbar p^\alpha D_\alpha) F = 0$.  

(33)

Here we assumed $J_{L\alpha}^\dagger = J_{L\alpha}$. From (32) and (33) we have

\[ \{ p^\alpha p_\alpha - M^2 - 2\hbar p^\alpha J_{L\alpha}, F \} = 0, \]  

(34)

\[ i\hbar p^\alpha D_\alpha F = -\frac{1}{2} [F, M^2 + 2\hbar J_{L\alpha} p^\alpha], \]  

(35)

where $\{ \cdots, \cdots \}$ and $[\cdots, \cdots]$ denote the anticommutator and commutator, respectively. Equation (34) is the constraint equation to describe the on-shell condition or the dispersion relation, while (35) is the Boltzmann equation. This result is consistent with the previous result in the flat spacetime limit \[12, 13, 14\]. The gravity effect of this Boltzmann equation arises through the differential operator $p^\alpha D_\alpha$, with which we can write

\[ p^\alpha D_\alpha F(x, p) = p^\alpha \left( \frac{\partial}{\partial x^\alpha} + \Gamma^\beta_{\alpha\gamma} p_\beta \frac{\partial}{\partial p_\gamma} \right) F(x, p) \]  

\[ = \frac{d}{d\lambda} F(x(\lambda), p(\lambda)), \]  

(36)

where $\lambda$ is the affine parameter. This suggests that the affine parameter will be fundamental to describe the oscillation in general curved spacetime. From now on, we use the unit $\hbar = 1$ for convenience, excepting the case we write $\hbar$ explicitly to avoid confusion.

Here we demonstrate that the familiar formula of the neutrino oscillation probability is obtained by solving the Boltzmann equation. For simplicity, we consider the two flavor system with the mass matrix

\[ M^2 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]  

(37)

and the source term $J_{L\alpha}$

\[ J_{L\alpha} = \begin{pmatrix} J_{L\alpha}^{ee} \\ J_{L\alpha}^{e\mu} \end{pmatrix}. \]  

(38)

In the ultra-relativistic limit, we assume that $F$ might be written

\[ F = \begin{pmatrix} f_{ee} \\ f_{e\mu} \\ f_{\mu e} \end{pmatrix}. \]  

(39)

Writing the off-diagonal component $f_{e\mu}$ as

\[ f_{e\mu} = \text{Re}[f_{e\mu}] + i\text{Im}[f_{e\mu}], \]  

(40)

\[ \implies \] yields

\[ p^\alpha D_\alpha f_{ee} = 2\xi \text{Im}[f_{e\mu}], \]  

(41)

\[ p^\alpha D_\alpha f_{e\mu} = -2\xi \text{Im}[f_{e\mu}], \]  

(42)

\[ p^\alpha D_\alpha \text{Im}[f_{e\mu}] = -\zeta (f_{ee} - f_{\mu\mu}) + 2\xi \text{Re}[f_{e\mu}], \]  

(43)

\[ p^\alpha D_\alpha \text{Re}[f_{e\mu}] = -2\xi \text{Im}[f_{e\mu}], \]  

(44)

where we defined

\[ \xi = \frac{1}{4} (m_2^2 - m_1^2) \cos 2\theta - \frac{1}{2} (J_{L\alpha}^{ee} - J_{L\alpha}^{\mu\mu}) p^\alpha; \]  

(45)

\[ \zeta = \frac{1}{4} (m_2^2 - m_1^2) \sin 2\theta. \]  

(46)
We next demonstrate that the above equations reproduce the well-known formula of the neutrino oscillation on the flat spacetime background. We consider one dimensional stationary flux of neutrinos along the $z$ axis, where the momentum of the relativistic neutrinos has the component $p^\nu = (p, 0, 0, p)$. In this case, combining equations (11, 14), we have (see also [12])

\[
\frac{\partial^3}{\partial z^3}(f_{ee} - f_{\mu\mu}) - \frac{1}{\xi} \frac{\partial^2}{\partial z \partial \xi^2}(f_{ee} - f_{\mu\mu}) + 4 \frac{\xi^2 + \zeta^2}{p^2} \frac{\partial}{\partial z}(f_{ee} - f_{\mu\mu}) - 4 \frac{\partial \xi \partial \zeta}{p^3}(f_{ee} - f_{\mu\mu}) = 0
\]

and

\[
\frac{\partial}{\partial z}(f_{ee} + f_{\mu\mu}) = 0.
\]

Equation (48) yields $f_{ee} + f_{\mu\mu} = \text{constant}$, which means the total number of the neutrinos conserves. We must specify initial condition to find a solution. We adopt the initial condition so that the electron neutrinos are emitted at $z = 0$, namely, $f_{ee}(z = 0) = f_{ee}(0)$, $f_{\mu\mu}(z = 0) = 0$, and $f_{e\mu}(z = 0) = 0$. In the vacuum case $J_{L\alpha} = 0$, integration of (17) yields

\[
\frac{\partial^2}{\partial z^2} f_{\mu\mu} + \omega^2 f_{\mu\mu} = 2 \frac{\xi^2}{p^2} f_{ee}(0),
\]

where $\omega = (m_2^2 - m_1^2)/2p$. Further integration gives the solution

\[
\frac{f_{\mu\mu}(z)}{f_{ee}(0)} = \sin^2 2\theta \sin^2 \left(\frac{(m_2^2 - m_1^2)z}{4p}\right),
\]

which gives the familiar oscillation probability from the electron neutrino to the mu neutrino.

The oscillation probability is given by considering the phase evolution of the mass eigenstates in the familiar approach. Namely, assuming a wave function that can be decomposed into a coherent superposition of the mass eigenstate, the oscillation probability is given by evaluating the transition amplitude from an initial state to other eigenstate, the oscillation probability is given by evaluating the transition amplitude from an initial state to other.

Consider a trajectory of a particle $x^\alpha(\lambda)$ parameterized by the affine parameter $\lambda$. Because we are considering the ultra-relativistic limit, we assume that the trajectory is a null geodesic. Then, from equation (36), the Boltzmann equation in the vacuum is written

\[
\frac{d^3}{d\lambda^3}(f_{ee} - f_{\mu\mu}) + \left(\frac{m_2^2 - m_1^2}{2}\right)^2 \frac{d}{d\lambda}(f_{ee} - f_{\mu\mu}) = 0
\]

and

\[
\frac{d}{d\lambda}(f_{ee} + f_{\mu\mu}) = 0,
\]

where $f_{ee}$ and $f_{\mu\mu}$ are understood as the functions along the trajectory $f_{ee} = f_{ee}(x(\lambda), p(\lambda))$ and $f_{\mu\mu} = f_{\mu\mu}(x(\lambda), p(\lambda))$, respectively. Along the trajectory we have the formal solution

\[
f_{ee} - f_{\mu\mu} = C_1 + C_2 \sin \left(\frac{m_2^2 - m_1^2}{2} \lambda(x, p) + \delta\right)
\]

and

\[
f_{ee} + f_{\mu\mu} = C_3,
\]

where $C_1$, $C_2$, $C_3$ and $\delta$ are the constants which should be determined by initial conditions. Thus the phase of the general solution is proportional to the affine parameter, which may contain the gravitationally induced contribution along the geodesic. This result can be consistent with (10), in which a resolution for the controversy in the references [8, 3] is presented (see also [11]). However, our argument assumes that the wave function of the different mass eigenstates overlap along the null geodesics. To incorporate the on-shell condition of the different mass eigenstates explicitly in our framework, we need to introduce the distribution function using the basis of the mass eigenstate [13]. We will revisit this problem beyond the ultra-relativistic limit as a future work.
V. HIGHER ORDER EFFECT

Here we briefly discuss the higher order effect of the ℏ-expansion of the Boltzmann equation. As we have shown in section 2, the new terms of the order of ℏ² are

\[
\frac{iℏ^2}{2} γ^μ \nabla_α J_μ(x) \frac{∂}{∂p_α} N(p, x)
\]

(55)

and

\[
ℏ^2 γ^α R_α[N(p, x)],
\]

(56)

where we defined

\[
R_α[N(p, x)] = \frac{1}{64} \frac{∂N(p, x)}{∂p^ρ} R_{ναμρ}[γ^μ, γ^ρ] - \frac{1}{24} ℏ^μ R_{ρτνα}[\frac{∂}{∂p_ν}, \frac{∂}{∂p_τ}] N(p, x) - \frac{7}{48} R_{αμ} \frac{∂}{∂p_μ} N(p, x).
\]

(57)

Similar to section 3, the left- and right-handed components can be separated. Equations corresponding to (55) and (56) are

\[
γ^α \left( p_α + \frac{iℏ}{2} D_α \right) N_{RL} + ℏ^2 R_α[N_{RL}] - MN_L = 0,
\]

(58)

\[
γ^α \left( p_α + \frac{iℏ}{2} D_α - ℏ J_α + \frac{iℏ^2}{2} \nabla_μ J_μ(x) \frac{∂}{∂p_α} \right) N_L + ℏ^2 γ^α R_α[N_L] - MN_{RL} = 0,
\]

(59)

where we consider the case \( \nabla_α M(x) = 0 \) and \( J_{RL}(x) = 0 \). Combining these equations,

\[
(p^α p_α - M^2 + ℏγ^α p_α J_β) N_L + \frac{iℏ^2}{2} γ^α γ^β \left( p_α(\nabla_μ J_β) \frac{∂N_L}{∂p_μ} - D_α(J_β N_L) \right)
\]

- \( \frac{ℏ^2}{4} γ^α γ^β D_α D_β N_L + ℏ^2 (γ^α γ^β p_α R_β[N_L] + γ^α R_α[γ^β p_β N_L]) = 0. \)

(60)

Note that the quantum effect due to spacetime curvature appears as the correction of the order of ℏ², i.e., expression (56). According to the analogy with the equation of the order of ℏ, the effect of the spacetime-curvature alters the constraint equation (on-shell condition) [20], while the variation of the effective potential, (55), affects the dynamical equation. However, the correction of the higher order terms are small in general cases because the correction is of order of ℏ/pL, where L is a typical length scale of spacetime curvature or the gradient of the effective potential, and p is the momentum of a particle. Thus ℏ/pL is the ratio of the de Broglie wave length to the curvature length scale, which is generally small, except for e.g., very early epoch of the universe.

VI. SUMMARY AND CONCLUSIONS

In this paper we have derived the quantum kinetic equation for the fermions with flavor mixing in curved spacetime. This derivation is based on the covariant Wigner function approach developed by Fonarev [21]. The result presents a rigorous theoretical framework to investigate flavor oscillation phenomena taking the gravitational effect in general curved-spacetime into account. The formula is expressed in terms of the expansion with respect to the power index of ℏ. The new terms of the order of ℏ² are the quantum effect due to the gravity, which alter the on-shell condition. At present the physical consequence of the correction on the on-shell condition is not clear (cf. see [23, 24]). The equation of the order of ℏ is consistent with the previous results. We have shown that the equation of the order of ℏ reduces to the previous result [12, 13, 14], with simply replacing the Liouville operator in curved spacetime to that in the Minkowski spacetime. It is also demonstrated that the familiar formula for the vacuum neutrino oscillation probability can be obtained by solving the Boltzmann equation. Then it is shown that our approach gives the consistent results which relies on the phase evolution. However, these results assume the ultra-relativistic limit. We have also derived the general Boltzmann equation which does not assume the ultra-relativistic limit in section 3. Analysis of the equation will clarify the effects of the finite mass on the neutrino’s propagation in general curved spacetime. We will revisit this problem in future work.
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APPENDIX A: MATHEMATICAL FORMULAS

Here we derive equation for the covariant Wigner function $N^{AB}$ defined in (9). The derivation is similar to that in the reference [20], excepting our generalization of the mass term and the flavor index, however, we write this Appendix for being self-contained. It is useful to define the derivative operator:

$$\hat{\nabla}_\alpha = \nabla_\alpha - \Gamma_\alpha^\beta y^\gamma \frac{\partial}{\partial y^\beta},$$  \hspace{1cm} (A1)

then we find

$$\hat{\nabla}_\alpha y^\beta = 0. \hspace{1cm} (A2)$$

Equations (6) and (7) are, respectively, written as

$$\Psi^A(x,-y) = \exp(-y^\alpha \hat{\nabla}_\alpha)\psi^A(x) \hspace{1cm} (A3)$$

and

$$\bar{\Psi}^B(x,y) = \exp(y^\alpha \hat{\nabla}_\alpha)\bar{\psi}^B(x). \hspace{1cm} (A4)$$

For arbitrary operators $\hat{A}$ and $\hat{B}$, the identities

$$[\hat{A}, e^{\hat{B}(t)}] = -\sum_{n=1}^{\infty} \frac{1}{n!} [\hat{B}, \cdots, [\hat{B}, \hat{A}] \cdots] e^{\hat{B}(t)} \hspace{1cm} (A5)$$

and

$$\frac{d}{dt} e^{\hat{B}(t)} = \left( \frac{d\hat{B}(t)}{t} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\hat{B}(t), \cdots, [\hat{B}(t), \frac{d\hat{B}(t)}{dt}] \cdots] \right) e^{\hat{B}(t)} \hspace{1cm} (A6)$$

are satisfied. Using these relation we have

$$\hat{\nabla}_\alpha \Psi^A(x,\pm y) = e^{\pm y^\nu \hat{\nabla}_\nu} \Psi^A(x) - \hat{H}_\alpha(x,\pm y)\Psi^A(x,\pm y) \hspace{1cm} (A7)$$

and

$$\frac{\partial}{\partial y^\alpha} \Psi^A(x,\pm y) = \pm \hat{\nabla}_\alpha \Psi^A(x,\pm y) \pm \hat{G}_\alpha(x,\pm y)\Psi^A(x,\pm y), \hspace{1cm} (A8)$$

where we defined

$$\hat{H}_\alpha(x,\pm y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} y^{\nu_1} \cdots y^{\nu_n} [\hat{\nabla}_{\nu_1}, \cdots, [\hat{\nabla}_{\nu_n}, \hat{\nabla}_\alpha] \cdots] \hspace{1cm} (A9)$$

and

$$\hat{G}_\alpha(x,\pm y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} y^{\nu_1} \cdots y^{\nu_n} [\hat{\nabla}_{\nu_1}, \cdots, [\hat{\nabla}_{\nu_n}, \hat{\nabla}_\alpha] \cdots], \hspace{1cm} (A10)$$
respectively. We also have
\[ e^{\pm y^* \nabla_{\alpha}} \sum_B \left( M^{AB}(x) + \hbar \gamma^\mu J^A_{\mu}(x) \right) \psi^B(x) \]
\[ = \sum_B \left( M^{AB}(x) + \hbar \gamma^\mu J^A_{\mu}(x) \right) \psi^B(x, \pm y) + \sum_B \hat{L}^{AB}(x, \pm y) \psi^B(x, \pm y). \]
(A11)

with defining
\[ \hat{L}^{AB}(x, \pm y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} y^{\nu_1} \cdots y^{\nu_n} \nabla_{\nu_1} \cdots \nabla_{\nu_n} \left( M^{AB}(x) + \hbar \gamma^\mu J^A_{\mu}(x) \right)^n. \]
(A12)

Next we consider \( D_\alpha N^{AB}(x, p) \), where \( D_\alpha \) is defined by (13). Using eqs. (A7) and (A8), we have the expression:
\[ D_\alpha N^{AB}(x, p) = \frac{2i}{\hbar} p_\alpha N^{AB}(x, p) - \frac{\sqrt{-g(x)}}{2(\pi \hbar)^2} \int d^4y e^{-2iy^\beta p_\beta/\hbar} \]
\[ \times \left( 2(e^{-y^\nu \nabla_{\nu}} \psi^A(x)) \bar{\psi}^B(x, y) - G^{AB}_\alpha(x, y) \right). \]
(A13)

where
\[ G^{AB}_\alpha(x, y) = \psi^A(x, -y) \left( \hat{G}_\alpha(x, y) \bar{\psi}^B(x, y) \right) - \left( \hat{G}_\alpha(x, -y) \psi^A(x, -y) \right) \bar{\psi}^B(x, y) \]
\[ + 2 \left( \hat{H}_\alpha(x, -y) \psi^A(x, -y) \right) \bar{\psi}^B(x, y). \]
(A14)

Projecting \( i\hbar \gamma^\alpha/2 \) on the both side of (A13), we have
\[ \gamma^\alpha(x) \left( p_\alpha + \frac{i\hbar}{2} D_\alpha \right) N^{AB}(x, p) = \frac{i\hbar}{2} \sqrt{-g(x)} \int d^4y e^{-2iy^\beta p_\beta/\hbar} \]
\[ \times \gamma^\alpha(x) \left( 2(e^{-y^\nu \nabla_{\nu}} \psi^A(x)) \bar{\psi}^B(x, y) - G^{AB}_\alpha(x, y) \right). \]
(A15)

Using the Dirac equation (11) and
\[ \nabla_{\alpha} \gamma^\beta(x) = 0, \]
(A16)

we have
\[ \gamma^\alpha(x) \left( p_\alpha + \frac{i\hbar}{2} D_\alpha \right) N^{AB}(x, p) = \sum_C M^{AC}(x, -\frac{i\hbar}{2} \partial/\partial p) N^{CB}(x, p) \]
\[ + \frac{i\hbar}{2} \sqrt{-g(x)} \int d^4y e^{-2iy^\beta p_\beta/\hbar} \gamma^\alpha(x) \left( G^{AB}_\alpha(x, y) \right), \]
(A17)

where
\[ M^{AB}(x, -\frac{i\hbar}{2} \partial/\partial p) = M^{AB}(x) + \hbar \gamma^\mu J^A_{\mu} \]
\[ + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{i\hbar}{2} \right)^n \nabla_{\nu_1} \cdots \nabla_{\nu_n} \left( M^{AB}(x) + \hbar \gamma^\mu J^A_{\mu} \right) \partial/\partial p_{\nu_1} \cdots \partial/\partial p_{\nu_n}. \]
(A18)

Then we focus on the last term of the right hand side of (A17). Using \( C_k^n \) to denote the binomial coefficient, one can derive the following recursion formula:
\[ G^{A(n)}_\alpha = -A^{(n-1)}_\alpha \psi^A(x, -y) + R^{(n)}_\alpha \psi^A(x, -y) \]
\[ - \sum_{k=1}^{n-2} C_k^{n-2} R^{(n-k)}_\alpha \psi^A(x, -y) \]
(A19)
where we defined

\[ G^{A(n+1)}_\alpha = y^{\nu_1} \cdots y^{\nu_n} \left[ \nabla_{\nu_1}, \cdots, [\nabla_{\nu_n}, \nabla_{\alpha}] \cdots \right] \Psi^A(x, -y) \quad (n \geq 1), \]

\[ G^{A(1)}_\alpha = \nabla_\alpha \Psi^A(x, -y), \]

\[ R^{(n)}_\alpha = y^{\nu_1} \cdots y^{\nu_n} R^{(n)}_{\alpha \nu_1 \cdots \nu_n} \quad (n \geq 3), \]

\[ R^{(2)}_\alpha = y^{\nu_1} y^{\nu_2} R^{(2)}_{\alpha \nu_1 \nu_2}, \]

\[ A^{(n)}_\alpha = y^{\nu_1} \cdots y^{\nu_n} A_{\alpha \nu_1 \cdots \nu_n} \quad (n \geq 2), \]

\[ A^{(1)}_\alpha = y^{\nu} A_{\alpha \nu}, \]

with

\[ A_{\alpha \beta} = \frac{1}{4} R_{\alpha \beta \mu \nu} \gamma^\mu \gamma^\nu, \]

which satisfies the commutator relation \( [\nabla_\alpha, \nabla_\beta] \Psi^A(x) = A_{\alpha \beta} \Psi^A(x) \). In a similar way, we also have the following recursion formula

\[ \tilde{G}^{B(n)}_\alpha = \tilde{\Psi}^B(x, y) A^{(n-1)}_\alpha + R^{(n)\nu \alpha} \frac{\partial}{\partial y^\nu} \tilde{G}^{B(n)}_\alpha - \sum_{k=1}^{n-2} C^{(n-k)\nu}_k R^{(n-k)\nu}_\alpha \tilde{G}^{B(k)}_\nu, \]

with

\[ \tilde{G}^{B(n+1)}_\alpha = y^{\nu_1} \cdots y^{\nu_n} \left[ \nabla_{\nu_1}, \cdots, [\nabla_{\nu_n}, \nabla_\alpha] \cdots \right] \tilde{\Psi}^B(x, y) \quad (n \geq 1) \]

\[ \tilde{G}^{B(1)}_\alpha = \nabla_\alpha \tilde{\Psi}^B(x, y). \]

A repeated use of the recursion formula \( A19 \) or \( A23 \) allows one to rewrite \( \tilde{G}_\alpha(x, -y) \Psi^A(x, -y) \) and \( \tilde{H}_\alpha(x, -y) \Psi^A(x, -y) \) or \( \tilde{G}_\alpha(x, y) \tilde{\Psi}^B(x, y) \) in equation \( A14 \), in terms of \( \partial \Psi^A(x, -y) / \partial y^\nu, \nabla_\nu \Psi^A(x, -y) \) and \( \tilde{\Psi}^B(x, -y) \), eliminating the derivative terms higher than the second derivatives. Then the formal expression for \( G^{AB}_\alpha \) is written in the form,

\[
G^{AB}_\alpha(x, y) = \left[ \sum_{k=2}^{\infty} K^{(1)\beta}_{(k)\alpha} R^{(k)\nu}_{\beta} \nabla_\nu - \sum_{k=2}^{\infty} K^{(2)\beta}_{(k)\alpha} R^{(k)\nu}_{\beta} \frac{\partial}{\partial y^\nu} \right] \left( \Psi^A(x, -y) \tilde{\Psi}^B(x, y) \right) \\
+ \sum_{k=2}^{\infty} K^{(3)\beta}_{(k)\alpha} A^{(k-1)}_\beta \left( \Psi^A(x, -y) \tilde{\Psi}^B(x, y) \right) \\
+ \left( \Psi^A(x, -y) \tilde{\Psi}^B(x, y) \right) \sum_{k=2}^{\infty} K^{(4)\beta}_{(k)\alpha} A^{(k-1)}_\beta,
\]

where

\[ K^{(i)\beta}_{(k)\alpha} = \delta^{(i)\beta}_{(k)\alpha} C^{(i)}_{(k)} + \sum_{n=1}^{\infty} \sum_{k_1=2}^{\infty} \cdots \sum_{k_n=2}^{\infty} C^{(i)}_{(k)k_1 \cdots k_n} R^{(k_n)\beta}_{\alpha} \cdots R^{(k_1)\beta}_{\beta}, \]

and \( C^{(i)}_{(k)} \) and \( C^{(i)}_{(k)k_1 \cdots k_n} \) are numerical coefficients, which can explicitly be found using generating functional method \( 20 \). We omit the general expression, however, the lowest order coefficients are, for example,

\[ C^{(1)}_{(k)} = \frac{(-1)^{k}}{k!} + \frac{1 - (-1)^{k}}{(k + 1)!}, \]

\[ C^{(2)}_{(k)} = \frac{(-1)^{k}}{k!} + \frac{1 + (-1)^{k}}{(k + 1)!}, \]

\[ C^{(3)}_{(k)} = \frac{(-1)^{k}(2k - 1)}{k!}, \]

\[ C^{(4)}_{(k)} = \frac{1}{k!}. \]
Using (A25) we have

\[
\frac{i\hbar}{2} \sqrt{-g(x)} \int d^4y e^{-2iy^4} \left( G^{AB}_\alpha(x, y) \right) = \\
- \sum_{k=2}^{\infty} \left[ \left( K^{(2)\beta}_{(k)\alpha} R^{(k)\nu}_{\beta} p_\nu + \frac{i\hbar}{2} K^{(1)\beta}_{(k)\alpha} R^{(k)\nu}_{\beta} D_\nu \right) N^{AB}(x, p_\nu) \right] + \frac{i\hbar}{2} K^{(3)\beta}_{(k)\alpha} A^{(k-1)}_{\beta},
\]

where

\[
K^{(i)\beta}_{(k)\alpha} = \delta^{\beta}_{\beta} C^{(i)}_{(k)\alpha} + \sum_{n=1}^{\infty} \sum_{k_1=2}^{\infty} \cdots \sum_{k_n=2}^{\infty} C^{(i)}_{(k_1)\alpha_{k_1} \cdots k_n} R^{(k_n)\beta_{k_n} \cdots k_n} \cdots R^{(k_1)\beta},
\]

\[
R^{(k)\alpha}_{\beta} = \left( \frac{i\hbar}{2} \right)^k \mathcal{R}^{(k\alpha)\beta}_{\nu_1 \nu_2 \cdots \nu_k} \partial_{\nu_1} \cdots \partial_{\nu_k},
\]

\[
A^{(k)}_{\alpha} = \left( \frac{i\hbar}{2} \right)^k A^{(k\alpha)}_{\nu_1 \nu_2 \cdots \nu_k} \partial_{\nu_1} \cdots \partial_{\nu_k}.
\]

Finally we find that equation \( N^{AB}(x, p) \) follows

\[
\gamma^{\alpha}(x) \left( p_\alpha + \frac{i\hbar}{2} D_\alpha \right) N^{AB}(x, p) - \sum_{C} M^{AC}(x, -\frac{i\hbar}{2} \partial_{p_\alpha}) N^{CB}(x, p) = -\gamma^{\alpha}(x) \sum_{k=2}^{\infty} \left[ \left( K^{(2)\beta}_{(k)\alpha} R^{(k)\nu}_{\beta} p_\nu + \frac{i\hbar}{2} K^{(1)\beta}_{(k)\alpha} R^{(k)\nu}_{\beta} D_\nu \right) N^{AB}(x, p) \right] + \frac{i\hbar}{2} N^{AB}(x, p) K^{(3)\beta}_{(k)\alpha} A^{(k-1)}_{\beta}.
\]

\[\text{(A31)}\]

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