AN ACCELERATED VARIANT OF SIMULATED ANNEALING THAT CONVERGES UNDER FAST COOLING

MICHAEL C.H. CHOI

ABSTRACT. Given a target function $U$ to minimize on a finite state space $X$, a proposal chain with generator $Q$ and a cooling schedule $T(t)$ that depends on time $t$, in this paper we study two types of simulated annealing (SA) algorithms with generators $M_{1,t}(Q, U, T(t))$ and $M_{2,t}(Q, U, T(t))$ respectively. While $M_{1,t}$ is the classical SA algorithm, we introduce a simple and greedy variant that we call $M_{2,t}$ which provably converges faster. Under any $T(t)$ that converges to 0 and mild conditions on $Q$, the Markov chain generated by $M_{2,t}$ is weakly ergodic. When $T(t) > c_{M_2} / \log(t+1)$ follows the logarithmic cooling schedule, our proposed algorithm is strongly ergodic both in total variation and in relative entropy, and converges to the set of global minima, where $c_{M_2}$ is a constant that we explicitly identify. If $c_{M_1}$ is the optimal hill-climbing constant that appears in logarithmic cooling of $M_{1,t}$, we show that $c_{M_1} \geq c_{M_2}$ and give simple conditions under which $c_{M_1} > c_{M_2}$. Our proposed $M_{2,t}$ thus converges under a faster logarithmic cooling in this regime. The other situation that we investigate corresponds to $c_{M_1} > c_{M_2} = 0$, where we give a class of fast and non-logarithmic cooling schedule that works for $M_{2,t}$ (but not for $M_{1,t}$). To the best of our knowledge this is the first instance where strong ergodicity and convergence in relative entropy are proved for faster than logarithmic cooling. Finally, we give an algorithm to simulate $M_{2,t}$ by uniformization of Markov chains.

AMS 2010 subject classifications: 60J27, 60J28

Keywords: Simulated annealing; non-homogeneous Markov chains; strong ergodicity; relative entropy; spectral gap

CONTENTS

1. Introduction 2
2. Preliminaries 3
2.1. Two types of simulated annealing: $X^{M_1}$ and its greedy variant $X^{M_2}$ 3
2.2. Review on ergodicity of non-homogeneous Markov chains 6
3. Main results 8
4. Proof of the main results 12
4.1. Proof of Theorem 3.1 13
4.2. Proof of Proposition 3.1 13
4.3. Proof of Theorem 3.2 13
4.4. Proof of Theorem 3.3 17
4.5. Proof of Theorem 3.4 18
4.6. Proof of Theorem 3.5 19
5. Algorithm to simulate $X^{M_2}$ 19
6. Conclusion and future work 20
Acknowledgements 21
References 21

Date: January 30, 2019.
1. Introduction

Given a target function $U$ to minimize on a finite state space $\mathcal{X}$, simulated annealing is a popular stochastic optimization algorithm that has found extensive empirical success in diverse disciplines ranging from image processing to statistics and combinatorial optimization problems, see for example Bertsimas and Tsitsiklis (1993); Geman and Geman (1984); Kirkpatrick (1984); Kirkpatrick et al. (1983) and the references therein. At the heart of the algorithm lies in constructing a non-homogeneous continuous-time Markov chain $X^{M_1} = (X^t_{M_1})_{t \geq 0}$ whose generator depends on the proposal chain as well as the so-called cooling schedule $T(t)$. Roughly speaking, simulated annealing can be considered as a non-homogeneous version of the Metropolis-Hastings algorithm in which the acceptance probability depends on $T(t)$. The cooling schedule $T(t)$ is carefully designed so that in the long run $X^{M_1}$ converges to the set of global minima of $U$. On one hand, $T(t)$ cannot converge too slowly as it is impractical to run $X^{M_1}$ for a long period of time; on the other hand however, $T(t)$ cannot converge too fast as there are well-documented instances in which $X^{M_1}$ may get stuck at a local minimum. Hajek (1988) proved the optimal cooling schedule for $X^{M_1}$ is of the form

$$T(t) = \frac{c_{M_1}}{\log(t + 1)},$$

where $c_{M_1}$ is the hill-climbing constant to be introduced in (3.2) below. In practice however, people adapt fast cooling schedule even though they usually do not come along with convergence guarantee.

In this paper, inspired by the recent work of the author Choi (2017); Choi and Huang (2018) who studied a new variant of the Metropolis-Hastings algorithm, we propose a promising accelerated variant of simulated annealing $X^{M_2} = (X^t_{M_2})_{t \geq 0}$ that enjoys superior mixing properties, provably converges under fast cooling and in some cases does not suffer from the drawbacks of $X^{M_1}$ as mentioned above. Precisely, our contributions are the following:

1. Derive basic yet important properties of $X^{M_2}$: We prove a few elementary properties and compare them with their $X^{M_1}$ counterparts in Lemma 2.1. These simple results turn out to be crucial in proving our main results. In particular, the difference in the convergence behaviour between $X^{M_1}$ and $X^{M_2}$ stems from the difference between their quadratic forms. The spectral gap lower bound of $X^{M_2}$ presented in Lemma 4.1 is also of independent interest.

2. Weak ergodicity of $X^{M_2}$ under general cooling and mild conditions on the proposal chain: For any cooling schedule that decreases to zero, we prove that $X^{M_2}$ is weakly ergodic under mild conditions on the proposal chain in Theorem 3.1. Intuitively this means $X^{M_2}$ forgets its initial state in the long run even under very fast cooling.

3. Strong ergodicity of $X^{M_2}$ under fast cooling: We split our ergodicity results into two regime according to $c_{M_2} > 0$ or $c_{M_2} \leq 0$, where $c_{M_2}$ is a constant that we explicitly identify in (3.3) below and depends on $U$ and $Q$. In the first case when $c_{M_2} > 0$ in Theorem 3.2, we establish rigorous convergence guarantee (in total variation and in relative entropy) of $X^{M_2}$ when the cooling schedule $T(t)$ is of the form

$$T(t) = \frac{c_{M_2} + \epsilon}{\log(t + 1)},$$

where $\epsilon > 0$. Thus the speed-up of the proposed variant depends on the difference $c_{M_1} - c_{M_2}$.

In the second situation when $c_{M_2} \leq 0$ in Theorem 3.3 and Theorem 3.4, we give a class of cooling schedule that are faster than logarithmic cooling and prove the strong ergodicity of $X^{M_2}$ in such setting. To the best of our knowledge, this is the first instance where strong ergodicity and convergence in relative entropy are proved for annealing algorithms under fast and non-logarithmic cooling.
(4) \( X^{M_2} \) effectively escapes local minima under general cooling: It is well-known that \( X^{M_1} \) may get trapped at a local minimum when one adapts fast cooling. As a result, one may have similar concern for \( X^{M_2} \). In Theorem 3.5, we prove that such concern is not valid and \( X^{M_2} \) effectively escapes local minima with probability tends to 1 as time goes to infinity under general cooling. Thus there is no issue in operating a fast cooling schedule on \( X^{M_2} \).

(5) Propose an algorithm to simulate \( X^{M_2} \): While \( X^{M_1} \) can be efficiently simulated using the idea of acceptance-rejection, it seems difficult to adapt such procedure to our proposed variant \( X^{M_2} \). Using the idea of uniformization of non-homogeneous Markov chains, we propose a method to simulate \( X^{M_2} \) in Algorithm 1.

The rest of this paper is organized as follow. In Section 2, we fix our notations and introduce the classical simulated annealing \( X^{M_1} \) as well as the proposed variant \( X^{M_2} \). This is then followed by a quick review on various notions of ergodicity for non-homogeneous Markov chains. We proceed to discuss the main results in Section 3, and their proofs can be found in Section 4. We present the algorithm for our proposed variant in Section 5, and conclude our paper with a list of possible future work in Section 6.

2. Preliminaries

2.1. Two types of simulated annealing: \( X^{M_1} \) and its greedy variant \( X^{M_2} \). In this section, we introduce the two types of simulated annealing that will be the focus of this paper. Along the way we will fix a few notations. Suppose that we are given the task to minimize a function \( U : \mathcal{X} \rightarrow \mathbb{R} \) living on a finite state space \( \mathcal{X} \). In the setting of simulated annealing, there is an ergodic homogeneous and continuous-time Markov chain with generator \( Q \) and stationary distribution \( \pi \) that acts as the proposal chain. We assume that \( Q \) is reversible with respect to \( \pi \), that is, the detailed balance condition is satisfied with \( \pi(x)Q(x,y) = \pi(y)Q(y,x) \) for all \( x, y \in \mathcal{X} \). Equivalently, \( Q \) is a self-adjoint operator in the Hilbert space \( \ell^2(\pi) \), endowed with the usual inner product

\[
(f, g)_{\pi} := \sum_{x \in \mathcal{X}} f(x)g(x)\pi(x), \quad f, g : \mathcal{X} \rightarrow \mathbb{R}.
\]

For any reversible \( Q \), it is well-known that the quadratic form of \(-Q\) can be written as

\[
(-Qf, f)_{\pi} = \frac{1}{2} \sum_{x,y \in \mathcal{X}} (f(y) - f(x))^2Q(x,y)\pi(x).
\]

For any reversible ergodic generator \(-Q\), we arrange its eigenvalues in non-decreasing order and write

\[0 = \lambda_{1}(-Q) < \lambda_{2}(-Q) \leq \lambda_{3}(-Q) \ldots \leq \lambda_{|\mathcal{X}|}(-Q)\].

It is well-known that \( \lambda_2(-Q) \) is the spectral gap of \( Q \) and admits a variational formula given by

\[
\lambda_2(-Q) = \inf_{f \in \ell^2(\pi): \langle f, f \rangle_{\pi} = 0} \frac{\langle -Qf, f \rangle_{\pi}}{\langle f, f \rangle_{\pi}}.
\]

Apart from the proposal chain \( Q \), another critical component in simulated annealing is a monotonely decreasing function \( T(t) \) that we call the temperature or the cooling schedule. We assume that \( T(t) > 0 \) and decreases to 0 as \( t \rightarrow \infty \). We write \( \pi_{T(t)} \) to be the Gibbs distribution with probability mass function given by

\[
\pi_{T(t)}(x) = \frac{e^{-\frac{U(x)}{T(t)}}\pi(x)}{Z_{T(t)}}, \quad x \in \mathcal{X},
\]

where \( Z_{T(t)} = \sum_{x \in \mathcal{X}} e^{-\frac{U(x)}{T(t)}}\pi(x) \) is the normalization constant. For any \( x, y \in \mathbb{R} \), we also denote \( x \lor y := \max\{x, y\}, x \land y := \min\{x, y\} \) and \( x_+ := x \lor 0 \). At each time \( t \), classical simulated annealing
amounts to a Metropolis-Hastings or acceptance-rejection procedure: a move is proposed by the proposal
chain with generator $Q$, and is accepted with probability that depends on $T(t)$ and $U$ in a way such that
the Markov chain generated by the algorithm at time $t$ is reversible with respect to $\pi_{T(t)}$. Precisely, we
have

**Definition 2.1** (Classical simulated annealing $X^{M_1}$). Given a target function $U$ on finite state space $\mathcal{X}$,
a proposal continuous-time ergodic Markov chain with generator $Q$ and a cooling schedule $T(t)$, the
simulated annealing algorithm $X^{M_1} = (X^{M_1}_t)_{t \geq 0}$ is a non-homogeneous Markov chain with generator
given by $M_{1,t} = M_{1,t}(Q,U,T(t)) = (M_{1,t}(x,y))_{x,y \in \mathcal{X}}$ for $t \geq 0$, where

$$M_{1,t}(x,y) := \begin{cases} Q(x,y) \min \left\{ 1, e^{\frac{U(x) - U(y)}{T(t)}} \right\} = Q(x,y)e^{-\frac{(U(y) - U(x))}{T(t)}}, & \text{if } x \neq y; \\ -\sum_{z:z \neq x} M_{1,t}(x,z), & \text{if } x = y. \end{cases}$$

We write $P^{M_1} = (P_{s,t}^{M_1})_{0 \leq s \leq t} = (P_{s,t}^{M_1}(x,y))_{x,y \in \mathcal{X}, s \leq t}$ to be the transition semigroup of $X^{M_1}$, where
$P_{s,t}^{M_1}(x,y)$ is the transition probability of $X^{M_1}$ starting in state $x$ at time $s$ to state $y$ at time $t$.

Inspired by the recent work by the author Choi (2017); Choi and Huang (2018), we would like to
introduce a variant of $X^{M_1}$ that we call $X^{M_2}$. It can be constructed by mirroring the transition effect of
$X^{M_1}$ to capture the opposite movement. More precisely, we have

**Definition 2.2** (The greedy variant $X^{M_2}$). Given a target function $U$ on finite state space $\mathcal{X}$, a proposal
continuous-time ergodic Markov chain with generator $Q$ and a cooling schedule $T(t)$, the greedy variant
$X^{M_2} = (X^{M_2}_t)_{t \geq 0}$ is a non-homogeneous Markov chain with generator given by $M_{2,t} = M_{2,t}(Q,U,T(t)) =
(M_{2,t}(x,y))_{x,y \in \mathcal{X}}$ for $t \geq 0$, where

$$M_{2,t}(x,y) := \begin{cases} Q(x,y) \max \left\{ 1, e^{\frac{U(x) - U(y)}{T(t)}} \right\} = Q(x,y)e^{\frac{(U(y) - U(x))}{T(t)}}, & \text{if } x \neq y; \\ -\sum_{z:z \neq x} M_{2,t}(x,z), & \text{if } x = y. \end{cases}$$

We write $P^{M_2} = (P_{s,t}^{M_2})_{0 \leq s \leq t} = (P_{s,t}^{M_2}(x,y))_{x,y \in \mathcal{X}, s \leq t}$ to be the transition semigroup of $X^{M_2}$, where
$P_{s,t}^{M_2}(x,y)$ is the transition probability of $X^{M_2}$ starting in state $x$ at time $s$ to state $y$ at time $t$.

Comparing Definition 2.1 and 2.2, we say that $X^{M_2}$ is a greedy variant of $X^{M_1}$ in the following sense:
starting at a state $x$ the transition rate of $X^{M_2}$ to any other state $y \neq x$ is greater than that of $X^{M_1}$ at any
time $t$. Mathematically we see that in defining $M_1$ we take min while in $M_2$ we consider max for off-
diagonal entries. Intuitively, we can imagine $Q$ as the base transition rate. $X^{M_1}$ and $X^{M_2}$ both modify
this base rate $Q$ according to their own (and opposite) rules. If $U(y) \leq U(x)$, $X^{M_1}$ leaves $Q$ unchanged
while $X^{M_2}$ increases this rate to $Q(x,y)e^{\frac{U(x) - U(y)}{T(t)}}$. The larger the difference between $U(x)$ and $U(y)$,
the greater the “boost” on the base transition rate for $X^{M_2}$. On the other hand if $U(y) > U(x)$, $X^{M_1}$
lower this base rate to $Q(x,y)e^{-\frac{(U(y) - U(x))}{T(t)}}$ while $X^{M_2}$ leaves the base rate $Q$ unchanged. We will see
that these two key differences allow $X^{M_2}$ to converge under fast cooling schedule and is able to escape
local minima effectively.

The main goal of the paper is to derive convergence theorems for the greedy variant $X^{M_2}$ under perhaps
faster cooling schedule $T(t)$, and to compare the properties and behaviour between $X^{M_1}$ and $X^{M_2}$.
To allow for effective comparison between these generators, we recall the notion of Peskun ordering of
continuous-time Markov chains. This partial ordering was first introduced by Peskun (1973) for discrete
time Markov chains on finite state space. It was further generalized by Tierney (1998) and Leisen and
Mira (2008) to general state space and to continuous-time Markov chains respectively.
Similarly, the reversibility of $U$ where in the second equality we use the reversibility of $(\text{Peskun ordering})$ Definition 2.3 $Q$ and $M_2$, at a fixed time $t \geq 0$. These results will be repeatedly used to develop our main results in Section 3.

Lemma 2.1. Given a target function $U$ on finite state space $\mathcal{X}$, a proposal continuous-time ergodic Markov chain with generator $Q$ and a cooling schedule $T(t)$, at a fixed $t \geq 0$ we have

1. (Reversibility) $M_{1,t}$ and $M_{2,t}$ are reversible with respect to the Gibbs distribution $\pi_{T(t)}$.
2. (Peskun ordering) $M_{2,t} \succeq M_{1,t}$.
3. (Quadratic form) For any function $f : \mathcal{X} \rightarrow \mathbb{R}$ with $f \in \ell^2(\pi_{T(t)})$,
   $$\langle M_{2,t} f, f \rangle_{\pi_{T(t)}} \leq \langle M_{1,t} f, f \rangle_{\pi_{T(t)}},$$
   where
   $$\langle -M_{2,t} f, f \rangle_{\pi_{T(t)}} = \frac{1}{2Z_{T(t)}} \sum_{x,y \in \mathcal{X}} (f(y) - f(x))^2 e^{-\frac{\min\{U(x),U(y)\}}{T(t)}} Q(x,y)\pi(x),$$
   $$\langle -M_{1,t} f, f \rangle_{\pi_{T(t)}} = \frac{1}{2Z_{T(t)}} \sum_{x,y \in \mathcal{X}} (f(y) - f(x))^2 e^{-\frac{\max\{U(x),U(y)\}}{T(t)}} Q(x,y)\pi(x).$$
4. (Spectral gap)
   $$\lambda_2(-M_{2,t}) \geq \lambda_2(-M_{1,t}).$$

Proof. We first prove item 1. For any $x \neq y$, we have
   $$\pi_{T(t)}(x)M_{2,t}(x,y) = \frac{e^{-\frac{U(x)}{T(t)}}\pi(x)}{Z_{T(t)}} Q(x,y) e^{-\frac{(U(x)-U(y))_+}{T(t)}} = e^{-\frac{U(x)}{T(t)}}\pi(y) Q(y,x) e^{-\frac{(U(y)-U(x))_+}{T(t)}} = \pi_{T(t)}(y)M_{2,t}(y,x),$$
   where in the second equality we use the reversibility of $Q$ with respect to $\pi$ and $\min\{U(x),U(y)\} = U(x) - (U(x) - U(y))_+ = U(y) - (U(y) - U(x))_+$. $M_2$ is therefore reversible with respect to $\pi_{T(t)}$.
   Similarly, the reversibility of $M_{1,t}$ can be deduced via
   $$\pi_{T(t)}(x)M_{1,t}(x,y) = \frac{e^{-\frac{U(x)}{T(t)}}\pi(x)}{Z_{T(t)}} Q(x,y) e^{-\frac{-(U(x)-U(y))_+}{T(t)}} = e^{-\frac{U(y)}{T(t)}}\pi(y) Q(y,x) e^{-\frac{-(U(y)-U(x))_+}{T(t)}} = \pi_{T(t)}(y)M_{1,t}(y,x),$$
   where we use again the reversibility of $Q$ and $\max\{U(x),U(y)\} = U(x) + (U(y) - U(x))_+ = U(y) + (U(x) - U(y))_+$. $M_2$ is therefore reversible with respect to $\pi_{T(t)}$.

Next, we prove item 2. For the off-diagonal entries with $x \neq y$,
   $$M_{2,t}(x,y) = Q(x,y) \max \left\{1, e^{\frac{U(x)-U(y)}{T(t)}} \right\} \geq Q(x,y) \min \left\{1, e^{\frac{U(x)-U(y)}{T(t)}} \right\} = M_{1,t}(x,y).$$
   Using both item 1 and item 2, item 3 follows readily from (Leisen and Mira, 2008, Theorem 5) since $M_{2,t} \succeq M_{1,t}$ and they are both reversible with respect to $\pi_{T(t)}$. In addition, using (2.1) and
   $$\pi_{T(t)}(x)M_{2,t}(x,y) = \frac{1}{Z_{T(t)}} e^{-\frac{\min\{U(x),U(y)\}}{T(t)}} Q(x,y)\pi(x),$$
\[
\pi_{T(t)}(x)M_{1,t}(x, y) = \frac{1}{Z_{T(t)}} e^{-\frac{\max\{U(x), U(y)\}}{T(t)}} Q(x, y) \pi(x),
\]
we have
\[
\langle -M_{2,t} f, f \rangle_{\pi_{T(t)}} = \frac{1}{2} \sum_{x,y \in \mathcal{X}} (f(y) - f(x))^2 M_{2,t}(x, y) \pi_{T(t)}(x)
\]
\[
= \frac{1}{2Z_{T(t)}} \sum_{x,y \in \mathcal{X}} (f(y) - f(x))^2 e^{-\frac{\min\{U(x), U(y)\}}{T(t)}} Q(x, y) \pi(x).
\]
Similar expression can be obtained for \(M_{1,t}\). Finally, we prove item (4). It follows easily from item (3) and the variational principle for spectral gap (2.2).

2.2. Review on ergodicity of non-homogeneous Markov chains. In this section, we will give a short detour on various notions of ergodicity of non-homogeneous Markov chains. As the classical simulated annealing \(X^{M_1}\) and its greedy variant \(X^{M_2}\) are non-homogeneous Markov chains, these notions of ergodicity are particularly important in order to properly understand our main results in Section 3. In particular, we will apply Lemma 2.2, 2.3 and 2.4 below to derive our main results. To keep our notations consistent with the previous section, we assume a non-homogeneous Markov chain with generator \(Q_t\) that depends on time \(t\) and Markov semigroup \((P_{s,t})_{0 \leq s \leq t}\) on state space \(\mathcal{X}\). For any two probability measures \(\mu\) and \(\nu\) on \(\mathcal{X}\), we write
\[
||\mu - \nu||_{TV} := \frac{1}{2} \sum_{x \in \mathcal{X}} |\mu(x) - \nu(x)|
\]
to be the total variation distance between \(\mu\) and \(\nu\). We now recall the notions of strong ergodicity and weak ergodicity of non-homogeneous Markov chains.

**Definition 2.4** (Strong ergodicity and weak ergodicity). Let \(X = (X_t)_{t \geq 0}\) be a non-homogeneous continuous-time Markov chain with generator \(Q_t\) and Markov semigroup \((P_{s,t})_{0 \leq s \leq t}\) on a finite state space \(\mathcal{X}\). \(X\) is said to be strongly ergodic if there exists a probability measure \(\mu\) on \(\mathcal{X}\) such that for all \(s \geq 0\), we have
\[
\lim_{t \to \infty} \sup_{x \in \mathcal{X}} ||P_{s,t}(x, \cdot) - \mu||_{TV} = 0.
\]
\(X\) is said to be weakly ergodic if for all \(s \geq 0\), we have
\[
\lim_{t \to \infty} \sup_{x,y \in \mathcal{X}} ||P_{s,t}(x, \cdot) - P_{s,t}(y, \cdot)||_{TV} = 0.
\]

It is easy to see that strong ergodicity implies weak ergodicity using the triangle inequality of total variation distance. While Definition 2.4 is perhaps hard to check in practice, we now state a sufficient condition on the transition rates \(Q_t\) that leads to weak ergodicity:

**Lemma 2.2** (Sufficient condition for weak ergodicity Griffeath (1975)). Let \(X = (X_t)_{t \geq 0}\) be a non-homogeneous continuous-time Markov chain with generator \(Q_t\) and Markov semigroup \((P_{s,t})_{0 \leq s \leq t}\) on a finite state space \(\mathcal{X}\). For \(x \neq y \in \mathcal{X}\), let
\[
\beta_{x,y}(t) := Q_t(x, y) + Q_t(y, x) + \sum_{k \in \mathcal{X}; k \notin \{x,y\}} Q_t(x, k) \land Q_t(y, k),
\]
\[
\beta(t) := \min_{x \neq y} \beta_{x,y}(t).
\]
If \(\int_s^\infty \beta(t) \, dt \to \infty\) as \(s \to \infty\), then \(X\) is weakly ergodic.
Similar sufficient condition for strong ergodicity in terms of the transition rates $Q_t$ can be found in Johnson and Isaacson (1988). We now give a sufficient condition for strong ergodicity in terms of the spectral gap of $Q_t$:

**Lemma 2.3** (Sufficient condition for strong ergodicity Gidas (1985)). Let $X = (X_t)_{t \geq 0}$ be a non-homogeneous continuous-time Markov chain with generator $Q_t$ and Markov semigroup $(P_{s,t})_{0 \leq s \leq t}$ on a finite state space $\mathcal{X}$. Assume that $Q_t$ is reversible with respect to a probability measure $\mu_t$ and its spectral gap is $\lambda_2(-Q_t)$. If there exists a function $\gamma(t)$ and a probability measure $\mu$ such that

$$\left| \frac{d\mu_t(x)}{dt} \right| \leq \gamma(t)\mu_t(x), \quad x \in \mathcal{X},$$

$$\int_0^\infty \lambda_2(-Q_t) \, dt = \infty,$$

$$\lim_{t \to \infty} \frac{\gamma(t)}{\lambda_2(-Q_t)} = 0,$$

$$\lim_{t \to \infty} \|\mu_t - \mu\|_{TV} = 0,$$

then $X$ is strongly ergodic and converges to $\mu$ in total variation distance.

Another notion of ergodicity that we will study is convergence in relative entropy. We write $\text{Ent}_\nu(\mu)$ to denote the relative entropy of $\mu$ with respect to $\nu$, that is,

$$\text{Ent}_\nu(\mu) := \sum_{x \in \mathcal{X}} \mu(x) \log \left( \frac{\mu(x)}{\nu(x)} \right).$$

**Definition 2.5** (Convergence in relative entropy). Let $X = (X_t)_{t \geq 0}$ be a non-homogeneous continuous-time Markov chain with generator $Q_t$ and Markov semigroup $(P_{s,t})_{0 \leq s \leq t}$ on a finite state space $\mathcal{X}$. Suppose further that the stationary measure of $Q_t$ is $\mu_t$ at a fixed $t \geq 0$. We say that $X$ converges in relative entropy if for every $x \in \mathcal{X}$,

$$\lim_{t \to \infty} \text{Ent}_{\mu_t}(P_{0,t}(x, \cdot)) = 0.$$

Note that convergence in relative entropy of the classical simulated annealing $X^{M_1}$ are discussed in Del Moral and Miclo (1999); Miclo (1992). We now state a sufficient condition for convergence in relative entropy in the setting of $X^{M_1}$ and $X^{M_2}$:

**Lemma 2.4** (Sufficient condition for convergence in relative entropy Del Moral and Miclo (1999); Miclo (1992)). For $i = 1, 2$, let $X^{M_i} = (X^{M_i}_t)_{t \geq 0}$ be the non-homogeneous continuous-time Markov chain introduced in Definition 2.1 and 2.2 with generator $M_{i,t}$ and Markov semigroup $(P_{s,t}^{M_i})_{0 \leq s \leq t}$ on a finite state space $\mathcal{X}$, where $M_{i,t}$ are both reversible with respect to $\pi_{T(t)}$. If there exists a constant $R \geq 0$, a function $a_{T(t)}$ and the cooling schedule is selected such that for $x \in \mathcal{X}$,

$$\frac{d}{dt} \text{Ent}_{\pi_{T(t)}}(P_{0,t}^{M_i}(x, \cdot)) \leq -a_{T(t)}^{-1} \text{Ent}_{\pi_{T(t)}}(P_{0,t}^{M_i}(x, \cdot)) + R \left| \frac{d}{dt} T(t) \right| \frac{1}{T(t)^2},$$

$$\int_0^\infty a_{T(t)}^{-1} \, dt = \infty,$$

$$\lim_{t \to \infty} \left| \frac{d}{dt} T(t) \right| \frac{a_{T(t)}}{T(t)^2} = 0,$$
then $X^{M_i}$ converges in relative entropy, i.e. for $x \in X$,

$$\lim_{t \to \infty} \text{Ent}_{\pi_{T(t)}}(P_{0,t}^{M_i}(x, \cdot)) = 0.$$ 

We remark that the proof of Lemma 2.3 and Lemma 2.4 are very similar: the conditions in these lemmas guarantee that, for some functions $g, h$ and differentiable function $f$ that satisfy

$$\frac{d}{dt} f(t) \leq -g(t) f(t) + h(t), \quad \int_0^\infty g(t) \, dt = \infty, \quad \lim_{t \to \infty} \frac{h(t)}{g(t)} = 0,$$

then $\lim_{t \to \infty} f(t) = 0$.

### 3. MAIN RESULTS

This section contains the main results of this paper. We provide three category of main results:

- In Theorem 3.1, we prove weak ergodicity of $X^{M_2}$ under general cooling schedule and mild conditions on $Q$.
- In Theorem 3.2, we discuss strong ergodicity and convergence in relative entropy of $X^{M_2}$ under logarithmic cooling. In some cases depending on $U$ and $Q$, we provide convergence guarantee under faster than logarithmic cooling in Theorem 3.3 and Theorem 3.4.
- In Theorem 3.5, we show that $X^{M_2}$ escapes from local minimum under general cooling.

Our first result states that $X^{M_2}$ is weakly ergodic provided that some conditions on the proposal generator $Q$ are satisfied:

**Theorem 3.1 (Weak ergodicity of $X^{M_2}$ under general cooling).** Let $X^{M_2}$ be the greedy variant introduced in Definition 2.2, and recall that $Q$ is the (reversible) generator of the proposal chain. If there exists a reference state $x_0$ such that

$$\min_{x \in X, x \neq x_0} Q(x, x_0) > 0,$$

then $X^{M_2}$ is weakly ergodic under any cooling schedule $T(t)$ that decreases to 0 as $t \to \infty$.

**Remark 3.1.** (3.1) means there is a reference state $x_0$ which is reachable from any other state. One can easily find examples of $Q$ that satisfy (3.1). In the following, we list a few well-known examples that fulfill the criteria:

1. The class of reversible $Q$ such that its off-diagonal entries are all strictly positive, e.g. the generator associated with the random walk on the complete graph.
2. The generator for Metropolised independent sampling Liu (1996) also satisfies this assumption, where we recall that in this setting we take $Q = P - I$, $P(x, y) = \mu(y)$ for some probability measure $\mu$ and $I$ is the identity matrix.
3. The optimal reversible generator that minimizes the worst-case asymptotic variance Frigessi et al. (1992). See also Chen et al. (2012); Huang et al. (2018) for latest development in this direction.

Weak ergodicity of $X^{M_2}$ implies that the greedy variant tends to lose track of its initial state in the long run under any cooling schedule. Before we proceed to discuss strong ergodicity of $X^{M_2}$, we introduce a few parameters that will play a fundamental role in the cooling schedule. Similar to Holley and Stroock (1988); Löwe (1996), we say that a path from $x$ to $y$ is any sequence of points starting from $x_0 = x, x_1, x_2, \ldots, x_n = y$ such that $Q(x_{i-1}, x_i) > 0$ for $i = 1, 2, \ldots, n$. Irreducibility of $Q$ guarantees such path exists for any $x \neq y$. Let $\Gamma_{x,y}$ denote the set of paths from $x$ to $y$, and elements of $\Gamma_{x,y}$ are denoted by $\gamma = (\gamma_i)_{i=0}^n$. If the value of $U(x)$ are considered as the elevation at $x$, then the highest elevation along
a path $\gamma \in \Gamma^{x,y}$ is $\text{Elev}(\gamma) = \max \{U(\gamma_i) ; \gamma_i \in \gamma\}$, and the lowest possible highest elevation along any path from $x$ to $y$ is

$$H(x, y) := \min \{\text{Elev}(\gamma) ; \gamma \in \Gamma^{x,y}\}.$$

We also write

$$U_{\text{min}} := \{x \in X ; U(x) = \min_y U(y)\},$$

$$\pi_{\text{min}}(x) := \begin{cases} \pi(x), & \text{if } x \in U_{\text{min}}, \\ \pi(U_{\text{min}}), & \text{if } x \notin U_{\text{min}}. \end{cases}$$

With these notations in mind, the two parameters that we are interested in are

\begin{equation}
(3.2) \quad c_{M_1} = c_{M_1}(Q, U) := \max_{x,y \in X} \{H(x, y) - U(x) - U(y)\},
\end{equation}

\begin{equation}
(3.3) \quad c_{M_2} = c_{M_2}(Q, U) := \max_{x,y \in X} \left\{ \max_{z,w \in \gamma^{x,y}, z = \gamma^{x,y}_i, w = \gamma^{x,y}_{i+1}} \text{for some } i \right. \\
Elev(\gamma^{x,y}) = H(x,y) \left. \right\} U(z) \wedge U(w) - U(x) - U(y) \}. 
\end{equation}

We now state a few basic properties regarding these two constants:

**Proposition 3.1** (Comparison of $c_{M_1}$ and $c_{M_2}$). Suppose that $c_{M_i}$ are defined as in (3.2) and (3.3) for $i = 1, 2$. We have

1. $c_{M_1} \geq c_{M_2}$. In particular, when $U$ does not have repeated value, $c_{M_1} > c_{M_2}$.
2. $c_{M_1} \geq -\min_y U(y)$. In particular, if we assume $\min_y U(y) = 0$ then $c_{M_1} \geq 0$.
3. ($c_{M_2}$ can be negative) For $x \neq y \in X$, we write $s^{x,y}$ to be the second largest element along the path $\gamma^{x,y}$ that achieves the elevation $H(x,y)$, that is,

$$U_{\text{max}}^{x,y} := \{w \in \gamma^{x,y} ; U(w) = \text{Elev}(\gamma^{x,y}) = H(x,y)\}$$

$$s^{x,y} := \begin{cases} \max\{U(w) ; w \in \gamma^{x,y}\} \backslash U_{\text{max}}^{x,y} \}, & \text{if } |U_{\text{max}}^{x,y}| = 1, \\
\text{Elev}(\gamma^{x,y}) = H(x,y), & \text{if } |U_{\text{max}}^{x,y}| > 1. \end{cases}$$

If

$$s^{x,y} \leq U(x) + U(y),$$

then $c_{M_2} \leq 0$.

Intuitively speaking, $c_{M_1}$ is the largest hill to climb from a local minimum to a global minimum, while $c_{M_2}$ is bounded above by the second largest hill to climb along any path from $x$ to $y$. To gain a better understanding of these two parameters, we plot four graphs in Figure 1 as a simple illustration. In Figure 1, $U$ is a one-dimensional function consisting only of crosses on the graph, and we take the proposal chain $Q$ to be a birth-death process that explores neighbouring points on the left or on the right.
Recall that for cooling schedule of the form $T(t) = \frac{c_{M_2} + \epsilon}{\log(t + 1)}$, $\epsilon > 0$, Gidas (1985); Holley and Stroock (1988) shows the strong ergodicity of $X^{M_1}$. As we will see in the next result below, $c_{M_2}$ plays a similar role as that of $c_{M_1}$ in the cooling schedule. We split our ergodicity results into two cases according to whether $c_{M_2} > 0$ or $c_{M_2} \leq 0$. We first look at the case $c_{M_2} > 0$:

**Theorem 3.2 (Strong ergodicity of $X^{M_2}$ and convergence in relative entropy under logarithmic cooling and $c_{M_2} > 0$).** Let $X^{M_2}$ be the greedy variant introduced in Definition 2.2, and suppose that $c_{M_2} > 0$. Under the logarithmic cooling schedule of the form

$$T(t) = \frac{c_{M_2} + \epsilon}{\log(t + 1)},$$

where $\epsilon > 0$, then $X^{M_2}$ is
(1) strongly ergodic and converges to \(\pi_{\min}\) in total variation distance in the sense of Definition 2.4; 
(2) converges in relative entropy in the sense of Definition 2.5.

Thus, the greedy variant \(X^{M_2}\) converges at a faster logarithmic cooling compared to \(X^{M_1}\), and the speed-up depends on the difference \(c_{M_1} - c_{M_2}\).

At first glance, it is perhaps hard to see the reason why \(c_{M_2}\) appears in the cooling schedule for \(X^{M_2}\) while \(c_{M_1}\) is the corresponding constant for \(X^{M_1}\). The reason becomes clear when one looks at the proof of Theorem 3.2: it relies crucially on a spectral gap lower bound of \(M_{2,t}\), where the difference between \(c_{M_1}\) and \(c_{M_2}\) follows from the difference between the quadratic form of \(M_{1,t}\) and \(M_{2,t}\) in Lemma 2.1. This observation also leads us to the second case when \(c_{M_2} < 0\). In such case, the spectral gap of \(M_{2,t}\) is uniformly bounded away from 0 for all \(t > 0\), and so one expect that faster than logarithmic cooling should work for \(X^{M_2}\) in this regime. Our next two results tell us that it is indeed the case:

**Theorem 3.3** (Strong ergodicity of \(X^{M_2}\) under faster than logarithmic cooling and \(c_{M_2} \leq 0\)). Let \(X^{M_2}\) be the greedy variant introduced in Definition 2.2, and suppose that \(c_{M_2} < 0\). If the cooling schedule satisfies

\[
(3.4) \quad \lim_{t \to \infty} \left(\frac{d}{dt} T(t)\right) e^{\frac{c_{M_2}}{T(t)} T(t)^2} = 0,
\]

then \(X^{M_2}\) is strongly ergodic and converges to \(\pi_{\min}\) in total variation distance in the sense of Definition 2.4. If \(c_{M_2} = 0\), examples of fast cooling schedule that satisfy (3.4) are

(1) (power law cooling) \(T(t) = (t + 1)^{-\alpha}\), where \(\alpha \in (0, 1)\).
(2) (powers of logarithmic cooling) \(T(t) = (\log(t + 1))^{-k}\), where \(k > 1\).
(3) \(T(t) = (t + 1)^{-\alpha} (\log(t + 1))^{-1}\), where \(\alpha \in (0, 1)\).

If \(c_{M_2} < 0\), examples of fast cooling schedule that satisfy (3.4), in addition to those listed above, are

(4) (exponential cooling) \(T(t) = e^{-t}\).
(5) (power law cooling) \(T(t) = (t + 1)^{-\alpha}\), where \(\alpha > 0\).

Note that all these examples are faster than logarithmic cooling in the sense that \(\lim_{t \to \infty} T(t) \log(t + 1) = 0\).

**Theorem 3.4** (Convergence in relative entropy of \(X^{M_2}\) under faster than logarithmic cooling and \(c_{M_2} \leq 0\)). Let \(X^{M_2}\) be the greedy variant introduced in Definition 2.2, and suppose that \(c_{M_2} < 0\). If the cooling schedule satisfies

\[
(3.5) \quad \int_0^\infty (1 + 1/T(t))^{-1} \exp\{-c_{M_2}/T(t)\} \, dt = \infty,
\]

\[
(3.6) \quad \lim_{t \to \infty} \left| \frac{d}{dt} T(t) \right| \frac{\exp\{c_{M_2}/T(t)\}}{T(t)^2 + T(t)^3} = 0,
\]

then \(X^{M_2}\) converges in relative entropy in the sense of Definition 2.5. If \(c_{M_2} = 0\), an example of fast cooling schedule that satisfies (3.5) and (3.6) is

(1) (power law cooling) \(T(t) = (t + 1)^{-\alpha}\), where \(\alpha \in (0, 1)\).

If \(c_{M_2} < 0\), an example of fast cooling schedule that satisfies (3.5) and (3.6), in addition to those listed above, is

(2) (exponential cooling) \(T(t) = e^{-t}\).
Note that all these examples are faster than logarithmic cooling in the sense that \( \lim_{t \to \infty} T(t) \log(t + 1) = 0 \).

Note that the speed-up of the proposed variant is most prominent in the case when \( c_{M_1} > 0 \) while \( c_{M_2} \leq 0 \), see for instance the cases in Figure 1c and Figure 1d. When \( c_{M_1} > 0 \), the convergence guarantee of \( X^{M_1} \) is given by the logarithmic cooling
\[
\frac{c_{M_1}}{\log(t + 1)},
\]
while for \( X^{M_2} \) one can apply a fast and non-logarithmic cooling schedule as mentioned in Theorem 3.3 and Theorem 3.4. To the best of our knowledge, we provide the first instance in which strong ergodicity and convergence in relative entropy are proved under fast and non-logarithmic cooling for annealing algorithms.

There are simple examples in the literature which demonstrate \( X^{M_1} \) can get stuck at local minimum and does not converge to \( U_{\text{min}} \) under fast cooling, see e.g. (Brémaud, 1999, Example 8.10) and (Häggström, 2002, Example 13.4). As a result, one possible concern for \( X^{M_2} \) under fast cooling is that it may exhibit similar behaviour as \( X^{M_1} \) and get trapped at a local minimum as the temperature cools down quickly. Our next result shows that such concern is not valid.

Let us now introduce a few notations:

\[
N(x) := \{ y \in X; Q(x, y) > 0 \},
\]
\[
U_{\text{loc}}^{\text{min}} := \{ x \in X; U(y) \geq U(x), y \in N(x) \},
\]
\[
\delta := \min \{ U(y) - U(x); x \in U_{\text{loc}}^{\text{min}}, y \in N(x) \}.
\]

Here \( N(x) \) is the neighbourhood of \( x \) induced by the proposal generator \( Q \) and \( U_{\text{loc}}^{\text{min}} \) is the set of local minimum of \( U \). If all local minima are strict local minima, then \( \delta > 0 \).

**Theorem 3.5** (\( X^{M_2} \) effectively escapes local minimum while \( X^{M_1} \) may get trapped under fast cooling).

For \( i = 1, 2 \), let \( X^{M_i} = (X^{M_i}_t)_{t \geq 0} \) be the non-homogeneous continuous-time Markov chain introduced in Definition 2.1 and 2.2 with generator \( M_{i,t} \), proposal generator \( Q \) and target function \( U \). Suppose that \( x \in U_{\text{loc}}^{\text{min}} \) and under any cooling schedule,
\[
\mathbb{P}_x(X^{M_2}_t = x \ \forall t \geq 0) = \lim_{t \to \infty} e^{-\left( \sum_{y \neq x} Q(x, y) \right)t} = 0.
\]

If \( \delta > 0 \) (e.g. all local minima are strict) and under cooling schedule of the form
\[
T(t) = \frac{\delta - \epsilon}{\log(t + 1)},
\]
where \( \epsilon > 0 \) such that \( \delta > \epsilon \), then
\[
\mathbb{P}_x(X^{M_1}_t = x \ \forall t \geq 0) > 0.
\]

## 4. Proof of the Main Results

In this section we prove the main results presented in Section 3.
4.1. **Proof of Theorem 3.1.** To prove Theorem 3.1, our plan is to invoke Lemma 2.2, where we take $Q_t$ in that lemma to be the greedy variant $M_{2,t}$. Precisely, for any $x \neq y \in \mathcal{X}$ we have
\[
\beta_{x,y}(t) = M_{2,t}(x, y) + M_{2,t}(y, x) + \sum_{k \in \mathcal{X}: k \notin \{x,y\}} M_{2,t}(x, k) \land M_{2,t}(y, k),
\]
\[
M_{2,t}(x, y) = Q(x, y) \max \left\{1, e^{\frac{U(x) - U(y)}{t}} \right\} \geq Q(x, y).
\]
If $x_0 \notin \{x, y\}$, then
\[
\beta_{x,y}(t) \geq M_{2,t}(x, x_0) \land M_{2,t}(y, x_0) \geq Q(x, x_0) \land Q(y, x_0) \geq \min_{x \in \mathcal{X}, x \neq x_0} Q(x, x_0).
\]
If $x_0 \in \{x, y\}$, then without loss of generality assume $y = x_0$ and so we have
\[
\beta_{x,y}(t) \geq M_{2,t}(x, x_0) \geq Q(x, x_0) \geq \min_{x \in \mathcal{X}, x \neq x_0} Q(x, x_0).
\]
As a result
\[
\beta(t) = \min_{x \neq y} \beta_{x,y}(t) \geq \min_{x \in \mathcal{X}, x \neq x_0} Q(x, x_0),
\]
and so for every $s > 0$, $\int_s^\infty \beta(t) \, dt = \infty$. Weak ergodicity of $X^{M_2}$ thus follows from Lemma 2.2.

4.2. **Proof of Proposition 3.1.** We first prove item (1). Fix $x \neq y \in \mathcal{X}$. Along the path $\gamma^{x,y}$ that achieves the lowest elevation, we have
\[
H(x, y) = \text{Elev}(\gamma^{x,y})
\]
\[
\geq \max_{z, w \in \gamma^{x,y}, z=\gamma_i^{x,y}, w=\gamma_{i+1}^{x,y} \text{ for some } i} U(z) \lor U(w)
\]
\[
\geq \max_{z, w \in \gamma^{x,y}, z=\gamma_i^{x,y}, w=\gamma_{i+1}^{x,y} \text{ for some } i} U(z) \land U(w).
\]
Desired result follows from substracting both sides by $U(x) + U(y)$ and taking maximum over all $x \neq y$. In particular, when $U$ does not have repeated values, then the last inequality above becomes strict inequality since $U(z) \lor U(w) > U(z) \land U(w)$. To see item (2), first we note that $c_{M_2}$ is attained at some $x$ and $y_0 \in U_{\text{min}}$ according to Holley and Stroock (1988). For any $\gamma \in \Gamma^{x,y_0}$, using the definition of elevation we note that
\[
\text{Elev}(\gamma) \geq U(x),
\]
and so
\[
c_{M_2} = H(x, y_0) - U(x) - U(y_0) \geq -U(y_0) = -\min_y U(y).
\]
Finally, we prove item (3), which can be seen from the observation that
\[
\max_{z, w \in \gamma^{x,y}, z=\gamma_i^{x,y}, w=\gamma_{i+1}^{x,y} \text{ for some } i} U(z) \land U(w) \leq s^{x,y}.
\]
4.3. **Proof of Theorem 3.2.** In this proof, without loss of generality we assume $\min_y U(y) = 0$. In Subsection 4.3.1, we prove Theorem 3.2 item (1), while in Subsection 4.3.2, we prove Theorem 3.2 item (2).
Proof of Theorem 3.2 item (1). Assume for now that we have the following spectral gap lower bound for $M_{2,t}$. We will return to its proof at the end of this Subsection.

**Lemma 4.1.** [Spectral gap estimate for $M_{2,t}$] There is a constant $A > 0$ such that for $t \geq 0$,

$$\lambda_2(-M_{2,t}) \geq A \exp \left\{ - \frac{cM_2}{T(t)} \right\},$$

where we recall that $c_{M_2}$ is defined in (3.3).

Our plan is to invoke Lemma 2.3 by taking

$$Q_t = M_{2,t}, \quad \mu_t = \pi_{T(t)}, \quad T(t) = \frac{cM_2 + \epsilon}{\log(t+1)},$$

$$\gamma(t) = -\left( \frac{d}{dt} T(t) \right) \frac{1}{T(t)^2} \left( \max_y U(y) - \min_y U(y) \right) = \frac{1}{(cM_2 + \epsilon)(t+1)} \left( \max_y U(y) - \min_y U(y) \right),$$

where the form of $\gamma(t)$ follows from (Gidas, 1985, equation (2.18)). Using Lemma 4.1 twice leads to

$$\int_0^\infty \lambda_2(-M_{2,t}) dt \geq \int_0^\infty A \exp \left\{ - \frac{cM_2}{T(t)} \right\} dt = A \int_0^\infty \frac{1}{t} \frac{e^{cM_2}}{t+1} dt \geq A \int_0^\infty \frac{1}{t+1} dt = \infty,$$

$$\lim_{t \to \infty} \frac{\gamma(t)}{\lambda_2(-M_{2,t})} \leq \frac{A \left( \max_y U(y) - \min_y U(y) \right)}{cM_2 + \epsilon} \lim_{t \to \infty} \frac{1}{(t+1)^{\frac{\epsilon}{cM_2 + \epsilon}}} = 0.$$
Let \( \alpha(x, y) := \pi(x)Q(x, y) \). Reversibility of \( Q \) implies \( \alpha \) is symmetric. If \( \alpha(z, w) = 0 \), then \( \chi_{z,w}(x, y) = 0 \) for all \( x, y \). In the following we take \( \chi_{z,w}(x, y)/\alpha(z, w) = 0 \) if \( \chi_{z,w}(x, y) = 0 \). We see that

\[
2\langle f, f \rangle_{\pi_T(t)} = \sum_{x,y} (f(y) - f(x))^2 \pi_T(t)(x)\pi_T(t)(y)
\]

\[
= \sum_{x,y} \left( \sum_{i=1}^{n(x,y)} f(\gamma_i^{x,y}) - f(\gamma_{i-1}^{x,y}) \right)^2 \pi_T(t)(x)\pi_T(t)(y)
\]

\[
\leq \sum_{x,y} \sum_{i=1}^{n(x,y)} (f(\gamma_i^{x,y}) - f(\gamma_{i-1}^{x,y}))^2 \pi_T(t)(x)\pi_T(t)(y)
\]

\[
\leq N \sum_{x,y} \sum_{w,z} \chi_{z,w}(x, y)(f(z) - f(w))^2 \left( \frac{\alpha(z, w)}{Z_T(t)} \right) e^{-\frac{\min(U(z), U(w))}{T(t)}} \frac{\pi_T(t)(x)\pi_T(t)(y)Z_T(t)}{\alpha(z, w)e^{-\frac{\min(U(z), U(w))}{T(t)}}}
\]

\[
\times \sum_{w,z} (f(z) - f(w))^2 \left( \frac{\alpha(z, w)}{Z_T(t)} \right) e^{-\frac{\min(U(z), U(w))}{T(t)}}
\]

\[
= 2N \left( \max_{z,w} \sum_{x,y} \chi_{z,w}(x, y) \frac{\pi_T(t)(x)\pi_T(t)(y)Z_T(t)}{\alpha(z, w)e^{-\frac{\min(U(z), U(w))}{T(t)}}} \right) \langle -M_{2,t}f, f \rangle_{\pi_T(t)}
\]

where the first inequality follows from Cauchy-Schwartz inequality, the second inequality follows from \( n(x, y) \leq N \), and the last equality follows from the quadratic form given in Lemma 2.1. Now note that

\[
\chi_{z,w}(x, y) \frac{\pi_T(t)(x)\pi_T(t)(y)Z_T(t)}{\alpha(z, w)e^{-\frac{\min(U(z), U(w))}{T(t)}}} = \frac{\chi_{z,w}(x, y)\pi(x)\pi(y)}{\alpha(z, w)} e^{-\frac{\min(U(z), U(w))}{T(t)}} \frac{Z_T(t)}{\pi(U_{\min})}
\]

\[
\leq \exp \left\{ \frac{cM_2}{T(t)} \right\} \frac{\chi_{z,w}(x, y)\pi(x)\pi(y)}{\alpha(z, w)} \frac{1}{\pi(U_{\min})}
\]

Desired result follows by taking

\[
A^{-1} = N \left( \max_{z,w} \sum_{x,y} \frac{\chi_{z,w}(x, y)\pi(x)\pi(y)}{\alpha(z, w)} \frac{1}{\pi(U_{\min})} \right).
\]

\[
\square
\]

4.3.2. Proof of Theorem 3.2 item (2). Our plan is to invoke Lemma 2.4 and identify the appropriate \( a_{T(t)} \). The spectral gap estimate of \( M_{2,t} \) in Lemma 4.1 will again play a crucial role. First, we compute

\[
\frac{d}{dt} \text{Ent}_{\pi_T(t)} \left( P_{M_2}^{M_2}(x, \cdot) \right) = \sum_y \log \left( \frac{P_{0,t}^{M_2}(x, y)}{\pi_T(t)(y)} \right) \frac{d}{dt} P_{0,t}^{M_2}(x, y) - \frac{P_{0,t}^{M_2}(x, y)}{\pi_T(t)(y)} \frac{d}{dt} \pi_T(t)(y).
\]

Using the Kolmogorov forward equation we see that
\[
\frac{d}{dt} P_{0,t}^{M_2}(x, y) = \sum_k P_{0,t}^{M_2}(x, k) M_{2,t}(k, y) \\
= \sum_{k \neq y} M_{2,t}(k, y) P_{0,t}^{M_2}(x, k) - \sum_{k \neq y} P_{0,t}^{M_2}(x, y) M_{2,t}(y, k) \\
= \sum_{k \neq y} \pi_{T(t)}(k) M_{2,t}(k, y) \left( \frac{P_{0,t}^{M_2}(x, k)}{\pi_{T(t)}(k)} - \frac{P_{0,t}^{M_2}(x, y)}{\pi_{T(t)}(y)} \right) \\
= \sum_{k \neq y} \pi_{T(t)}(k) M_{2,t}(k, y) (f(k) - f(y)),
\]

where in the last equality we let

\[(4.2) \quad f(k) := \frac{P_{0,t}^{M_2}(x, k)}{\pi_{T(t)}(k)}.
\]

Substituting these back into (4.1) yields

\[
\frac{d}{dt} \text{Ent}_{\pi_{T(t)}} (P_{0,t}^{M_2}(x, \cdot)) = \sum_y \log f(y) \sum_{k \neq y} \pi_{T(t)}(k) M_{2,t}(k, y) (f(k) - f(y)) - \frac{P_{0,t}^{M_2}(x, y)}{\pi_{T(t)}(y)} \frac{d}{dt} \pi_{T(t)}(y).
\]

\[
\leq -2C \sum_{k \neq y} \pi_{T(t)}(k) M_{2,t}(k, y) \left( \sqrt{f(k)} - \sqrt{f(y)} \right)^2 - \frac{P_{0,t}^{M_2}(x, y)}{\pi_{T(t)}(y)} \frac{d}{dt} \pi_{T(t)}(y)
\]

\[(4.3) \quad = -4C \langle -M_{2,t} \sqrt{f}, \sqrt{f} \rangle_{\pi_{T(t)}} - \frac{P_{0,t}^{M_2}(x, y)}{\pi_{T(t)}(y)} \frac{d}{dt} \pi_{T(t)}(y),
\]

where the first inequality follows from (Miclo, 1992, Proposition 3) and \(C\) is a constant that depends on the size of \(\mathcal{X}\). Now, for this choice of \(f\) in (4.2), \(p \in (2, \infty)\) and Lemma 4.1,

\[
||\sqrt{f} - \pi(\sqrt{f})||_{L^p(\pi)}^2 \leq ||\sqrt{f} - \pi(\sqrt{f})||_{L^2(\pi)}^2 \leq \frac{\langle -Q \sqrt{f}, \sqrt{f} \rangle_\pi}{\lambda_2(-Q)},
\]

\[
||\sqrt{f} - \pi(\sqrt{f})||_{L^2(\pi_{T(t)})}^2 \leq \frac{\langle -M_{2,t} \sqrt{f}, \sqrt{f} \rangle_{\pi_{T(t)}}}{\lambda_2(-M_{2,t})} \leq A \exp \left\{ \frac{c_{M_2}}{T(t)} \right\} \langle -M_{2,t} \sqrt{f}, \sqrt{f} \rangle_{\pi_{T(t)}},
\]

where \(A\) is the constant that appears in Lemma 4.1. As a result, equation (3.12) and (3.15) in Holley and Stroock (1988) are fulfilled. By (Holley and Stroock, 1988, Theorem 3.21), there is a constant \(D = D(p, c_{M_2}, \lambda_2(-Q), \max_y U(y) - \min_y U(y)) < \infty\) such that

\[(4.4) \quad \langle -M_{2,t} \sqrt{f}, \sqrt{f} \rangle_{\pi_{T(t)}} \geq \frac{1}{D} \frac{1 + 1/T(t)}{1 - c_{M_2}/T(t)} \text{Ent}_{\pi_{T(t)}} (P_{0,t}^{M_2}(x, \cdot)).
\]

Combining (4.4) and (4.3) yields

\[
\frac{d}{dt} \text{Ent}_{\pi_{T(t)}} (P_{0,t}^{M_2}(x, \cdot)) \leq - \frac{4C(1 + 1/T(t))^{-1} \exp\{-c_{M_2}/T(t)\}}{D} \text{Ent}_{\pi_{T(t)}} (P_{0,t}^{M_2}(x, \cdot))
\]

\[
+ (\max_y U(y) - \min_y U(y)) \left| \frac{d}{dt} T(t) \right| \frac{1}{T(t)^2},
\]
where the second term follows from (Gidas, 1985, equation 2.18). This is exactly the form presented in Lemma 2.4 with the choice of
\[ a_{T(t)} = \frac{D(1 + 1/T(t))}{4C} \exp\{c_{M_2}/T(t)\}, \quad R = \max_y U(y) - \min_y U(y). \]

Under the logarithmic cooling schedule of the form \( T(t) = \frac{c_{M_2} + \epsilon}{\log(t + 1)} \), we calculate that
\[
\int_0^\infty a_{T(t)}^{-1} dt \geq \frac{4C}{D} \int_0^\infty \frac{c_{M_2} + \epsilon}{c_{M_2} + \epsilon + \log(t + 1)} \frac{1}{t + 1} dt = \infty.
\]

Desired result follows from Lemma 2.4.

4.4. Proof of Theorem 3.3. Similar to Subsection 4.3.1, our plan is to invoke Lemma 2.3 by taking
\[ Q_t = M_{2,t}, \quad \mu_t = \pi_{T(t)}, \]
\[ \gamma(t) = -\left( \frac{d}{dt} T(t) \right) \frac{1}{T(t)^2} (\max_y U(y) - \min_y U(y)) = \frac{1}{(c_{M_2} + \epsilon)(t + 1)} (\max_y U(y) - \min_y U(y)), \]
Using Lemma 4.1 leads to
\[
\int_0^\infty \lambda_2(-M_{2,t}) dt \geq \int_0^\infty A \exp\left\{ -\frac{c_{M_2}}{T(t)} \right\} dt \\
\geq A \int_0^\infty 1 dt \\
= \infty,
\]
where we use the assumption that \( c_{M_2} \leq 0 \) in the second inequality above. It remains to check
\[
\lim_{t \to \infty} \frac{\gamma(t)}{\lambda_2(-M_{2,t})} \leq -A(\max_y U(y) - \min_y U(y)) \lim_{t \to \infty} \left( \frac{d}{dt} T(t) \right) \frac{e^{c_{M_2}T(t)}}{T(t)^2} = 0,
\]
where we use again Lemma 4.1 in the first inequality, and the equality follows from the assumption on the cooling schedule (3.4). We proceed to show the cooling schedules proposed satisfy (3.4):

1. \( T(t) = (t + 1)^{-\alpha} \)
\[
\lim_{t \to \infty} \left( \frac{d}{dt} T(t) \right) \frac{1}{T(t)^2} = \lim_{t \to \infty} \frac{-\alpha}{(1 + t)^{1+\alpha}} = 0.
\]

2. \( T(t) = (\log(t + 1))^{-k} \)
\[
\lim_{t \to \infty} \left( \frac{d}{dt} T(t) \right) \frac{1}{T(t)^2} = \lim_{t \to \infty} \frac{-k(\log(t + 1))^{k-1}}{t + 1} = 0.
\]

3. \( T(t) = (t + 1)^{-\alpha} (\log(t + 1))^{-1} \)
\[
\lim_{t \to \infty} \left( \frac{d}{dt} T(t) \right) \frac{1}{T(t)^2} = \lim_{t \to \infty} \frac{1}{(1 + t)^{1+\alpha}} - \frac{\alpha \log(t + 1)}{(1 + t)^{1+\alpha}} = 0.
\]
\( T(t) = e^{-t} \)

\[
\lim_{t \to \infty} \left( \frac{d}{dt} T(t) \right) \frac{e^{c_{M_2}}}{T(t)^2} = \lim_{t \to \infty} -e^t e^{c_{M_2} t} = 0,
\]

where we use \( c_{M_2} < 0 \) in the last equality.

\( T(t) = (t + 1)^{-\alpha} \)

\[
\lim_{t \to \infty} \left( \frac{d}{dt} T(t) \right) \frac{e^{c_{M_2}}}{T(t)^2} = \lim_{t \to \infty} -\alpha(t + 1)^{\alpha-1} e^{c_{M_2} (t+1)^\alpha} = 0,
\]

where we use \( c_{M_2} < 0 \) in the last equality.

4.5. **Proof of Theorem 3.4.** Our proof follows from the proof of Theorem 3.2 item (2) in Subsection 4.3.2. Recall that we have showed

\[
d \frac{\text{Ent}_{\pi_{T(t)}} (P_{0,t} (x, \cdot))}{\text{Ent}_{\pi_{T(t)}} (P_{0,t} (x, \cdot))} \leq -a_{T(t)}^{-1} \frac{d}{dt} \text{Ent}_{\pi_{T(t)}} (P_{0,t} (x, \cdot)) + R \left| \frac{d}{dt} T(t) \right| \frac{1}{T(t)^2},
\]

\[
a_{T(t)} = \frac{D(1 + 1/T(t)) \exp\{c_{M_2}/T(t)\}}{4C}, \quad R = \max_y U(y) - \min_y U(y).
\]

Equations (3.5) and (3.6) lead to

\[
\int_0^{\infty} a_{T(t)}^{-1} dt = \infty, \quad \lim_{t \to \infty} \left| \frac{d}{dt} T(t) \right| \frac{a_{T(t)}}{T(t)^2} = 0.
\]

Desired result follows from Lemma 2.4. We proceed to show the cooling schedules proposed satisfy (3.5) and (3.6):

1. \( T(t) = (t + 1)^{-\alpha}, \alpha \in (0, 1), \)

\[
\int_0^{\infty} (1 + 1/T(t))^{-1} \exp\{-c_{M_2}/T(t)\} dt = \int_0^{\infty} \frac{1}{1 + (1 + t)^\alpha} dt \geq \int_0^{\infty} \frac{1}{2 + t} dt = \infty,
\]

\[
\lim_{t \to \infty} \left| \frac{d}{dt} T(t) \right| \frac{\exp\{c_{M_2}/T(t)\}}{T(t)^2 + T(t)^3} = \lim_{t \to \infty} \frac{\alpha}{(1 + t)^{1-\alpha} + (1 + t)^{1-2\alpha}} = 0,
\]

2. \( T(t) = e^{-t} \)

First, we note that the integrand

\[
(1 + 1/T(t))^{-1} \exp\{-c_{M_2}/T(t)\} = \frac{e^{-c_{M_2}e^t}}{1 + e^t} \to \infty \quad \text{as} \quad t \to \infty,
\]

where we use \( c_{M_2} < 0 \). As a result,

\[
\int_0^{\infty} (1 + 1/T(t))^{-1} \exp\{-c_{M_2}/T(t)\} dt = \infty,
\]

\[
\lim_{t \to \infty} \left| \frac{d}{dt} T(t) \right| \frac{\exp\{c_{M_2}/T(t)\}}{T(t)^2 + T(t)^3} = \lim_{t \to \infty} \frac{e^{c_{M_2} e^t}}{e^{-t} + e^{-2t}} = 0.
\]
4.6. Proof of Theorem 3.5. We first prove (3.7). When $x \in U_{min}^{loc}$, then $(U(x) - U(y))_+ = 0$ for $y \in N(x)$, and so by the definition of $M_{2,s}$ in Definition 2.2 we have, for $s \geq 0$,

$$M_{2,s}(x, y) = Q(x, y),$$

$$M_{2,s}(x, x) = -\sum_{y \neq x} Q(x, y),$$

where we note that the right hand side of these two equations are independent of $s$. According to (van Dijk, 1992, equation (2.6))

$$\mathbb{P}_x(X_s^{M_2} = x \forall s \in [0, t]) = \exp \left\{ \int_0^t M_{2,s}(x, x) \, ds \right\} = \exp \left\{ -\sum_{y \neq x} Q(x, y)t \right\}.$$  

Taking $t \to \infty$ gives (3.7). Next, we prove (3.8). Using the definition of $\delta > 0$ and $M_{1,t}$, we see that for $x \in U_{min}^{loc}, y \in N(x)$,

$$M_{1,s}(x, y) \leq e^{-\frac{\delta}{T(s)}} Q(x, y),$$

$$M_{1,s}(x, x) \geq -\sum_{y \neq x} e^{-\frac{\delta}{T(s)}} Q(x, y).$$  

It follows again from (van Dijk, 1992, equation (2.6)) that

$$\mathbb{P}_x(X_s^{M_1} = x \forall s \in [0, t]) = \exp \left\{ \int_0^t M_{1,s}(x, x) \, ds \right\} \geq \exp \left\{ -\left( \sum_{y \neq x} Q(x, y) \right) \int_0^t e^{-\frac{\delta}{T(s)}} \, ds \right\}$$

$$= \exp \left\{ -\left( \sum_{y \neq x} Q(x, y) \right) \int_0^t \frac{1}{(s + 1)^{\frac{\delta}{s-\epsilon}}} \, ds \right\}.$$  

Taking $t \to \infty$, the equation becomes

$$\mathbb{P}_x(X_s^{M_1} = x \forall s \geq 0) \geq \exp \left\{ -\left( \sum_{y \neq x} Q(x, y) \right) \int_0^\infty \frac{1}{(s + 1)^{\frac{\delta}{s-\epsilon}}} \, ds \right\} > 0,$$

where we use the fact that $\int_0^\infty \frac{1}{(s + 1)^{\frac{\delta}{s-\epsilon}}} \, ds < \infty$ as

$$\sum_{k=1}^\infty \frac{1}{k^{\frac{\delta}{s-\epsilon}}} < \infty.$$

5. Algorithm to Simulate $X^{M_2}$

The objective of this section is to present an algorithm for simulating $X^{M_2}$, relying on the idea of uniformization of non-homogeneous continuous-time Markov chains. On one hand, $X^{M_1}$ is easy to simulate by means of acceptance-rejection: the proposal chain $Q$ proposes a move which is accepted according to certain probability that depends on the target function $U$ and temperature $T(t)$. Such acceptance-rejection procedure seems to be hard to adapt to the setting of $X^{M_2}$, as we are taking max instead of min in its Definition. As a result, we resort to the idea of uniformization of non-homogeneous Markov chain as introduced in van Dijk (1992). Roughly speaking, a non-homogeneous continuous-time Markov chain
can be thought as a discrete-time non-homogeneous Markov chain time-changed by an associated Poisson process. This allows us to simulate $X^{M_2}$ by its discrete-time counterpart and its affiliated Poisson process.

Suppose that we would like to simulate $X^{M_2} = (X_t^{M_2})_{t=t_0}$ with transition semigroup $P^{M_2} = (P_{s,t}^{M_2})_{0 \leq s \leq t}$ on the interval $(t_0, t_1)$. As we assume the cooling schedule $T(t)$ is decreasing toward 0, by writing
\begin{equation}
R := \max_y U(y) - \min_y U(y),
\end{equation}
the transition rate at any given state $x$ and any $t \in (t_0, t_1)$ is bounded above by
\begin{equation}
|M_{2,t}(x, x)| \leq e^{\frac{R}{\int_{t_0}^{t_1}}} \sum_{y \neq x} Q(x, y) \leq e^{\frac{R}{\int_{t_0}^{t_1}}} \max_x |Q(x, x)| =: M.
\end{equation}
Now, note that
\begin{equation}
P_t^{M_2} := I + \frac{1}{M} M_{2,t}
\end{equation}
is a valid reversible (with respect to $\pi_{T(t)}$) stochastic matrix. According to (van Dijk, 1992, Theorem 3.1), we have

**Theorem 5.1** (Continuous-time non-homogeneous Markov chain as time-changed discrete-time non-homogeneous Markov chain).

\[
P_{t_0, t_1}^{M_2} = \sum_{k=0}^{\infty} \frac{e^{-(t_1-t_0)M}((t_1-t_0)M)^k}{k!} \int_{t_0}^{t_1} \ldots \int_{t_0}^{t_1} P_{n_1}^{M_2} P_{n_2}^{M_2} \ldots P_{n_k}^{M_2} dH(n_1, \ldots, n_k),
\]
where $dH(n_1, \ldots, n_k)$ is the density of the order statistics $n_1 \leq n_2 \leq \ldots \leq n_k$ of a $k$-dimensional uniform distribution in $[t_0, t_1] \times \ldots \times [t_0, t_1]$, and $P_{n_1}^{M_2} P_{n_2}^{M_2} \ldots P_{n_k}^{M_2}$ is the standard matrix product.

**Algorithm 1:** An algorithm to simulate $(X_t^{M_2})_{t=t_0}$ by uniformization

**Input:** Proposal reversible generator $Q$, cooling schedule $T(t)$, target function $U$, an interval $(t_0, t_1)$ to simulate and an initial state $x_0$

1. Calculate $M_{2,t}$ as in Definition 2.2, $R$ as in (5.1), $M$ as in (5.2) and $P_t^{M_2}$ as in (5.3).
2. Sample a number $N$ from the Poisson distribution with mean $(t_1-t_0)M$.
3. Draw $N$ random numbers uniformly distributed on the interval $(t_0, t_1)$, and sort them in ascending order. Label them as $n_1 \leq n_2 \leq \ldots \leq n_N$.
4. for $i = 1; i \leq N; i = i + 1$
   5. \hspace{1cm} With initial state $x_{i-1}$, simulate one-step of a Markov chain with matrix $P_{n_i}^{M_2}$.
   6. \hspace{1cm} Set $x_i$ to be the state after one-step.

**Output:** $x_N$

6. CONCLUSION AND FUTURE WORK

In this paper we study a theoretically promising and accelerated variant of simulated annealing that we call $X^{M_2}$. This non-homogeneous continuous-time Markov chain enjoys a few desirable properties compared with its classical counterpart $X^{M_1}$: under general cooling schedule it is weakly ergodic and escapes from the local minimum in the long run. As for strong ergodicity, we provide convergence guarantee in the two cases $c_{M_2} > 0$ and $c_{M_2} \leq 0$. In the former case one can adapt a fast logarithmic
cooling, while in the latter case faster than logarithmic cooling are available and the algorithm still converges to the set of global minima.

We believe this work opens the door for studying a new type of annealing algorithms. Future work includes empirical investigation of $X^M_2$, extending $X^M_2$ to time-dependent target function Löwe (1996), incorporating non-reversibility to speed up mixing Bierkens (2016); Chen and Hwang (2013; Hwang et al. (1993, 2005); Löwe (1997), investigating the analog of $X^M_2$ in generalized simulated annealing Del Moral and Miclo (1999) and extending the work to more general state space and its connection with the Langevin equation Andrieu et al. (2001); Chiang et al. (1987); Geman and Hwang (1986).

Acknowledgements. The author would like to thank Lu-Jing Huang, Chii-Ruey Hwang and Laurent Miclo for inspiring discussions that lead to this work. The author acknowledges the support from the Chinese University of Hong Kong, Shenzhen grant PF01001143.

REFERENCES
C. Andrieu, L. A. Breyer, and A. Doucet. Convergence of simulated annealing using Foster-Lyapunov criteria. J. Appl. Probab., 38(4):975–994, 2001.
D. Bertsimas and J. Tsitsiklis. Simulated annealing. Statist. Sci., 8(1):10–15, 02 1993.
J. Bierkens. Non-reversible Metropolis-Hastings. Stat. Comput., 26(6):1213–1228, 2016.
P. Brémaud. Markov chains, volume 31 of Texts in Applied Mathematics. Springer-Verlag, New York, 1999. Gibbs fields, Monte Carlo simulation, and queues.
T.-L. Chen and C.-R. Hwang. Accelerating reversible Markov chains. Statist. Probab. Lett., 83(9):1956–1962, 2013.
T.-L. Chen, W.-K. Chen, C.-R. Hwang, and H.-M. Pai. On the optimal transition matrix for Markov chain Monte Carlo sampling. SIAM J. Control Optim., 50(5):2743–2762, 2012.
T.-S. Chiang, C.-R. Hwang, and S. J. Sheu. Diffusion for global optimization in $\mathbb{R}^n$. SIAM J. Control Optim., 25(3):737–753, 1987.
M. C. Choi. Metropolis-Hastings reversiblizations of non-reversible Markov chains. arXiv:1706.00068, 2017.
M. C. Choi and L.-J. Huang. On hitting time, mixing time and geometric interpretations of Metropolis-Hastings reversiblizations. arXiv preprint arXiv:1810.11763, 2018.
P. Del Moral and L. Miclo. On the convergence and applications of generalized simulated annealing. SIAM J. Control Optim., 37(4):1222–1250, 1999.
A. Frigessi, C.-R. Hwang, and L. Younes. Optimal spectral structure of reversible stochastic matrices, Monte Carlo methods and the simulation of Markov random fields. Annals of Applied Probability, 2(3):610–628, 1992.
S. Geman and D. Geman. Stochastic relaxation, Gibbs distributions, and the Bayesian restoration of images. IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-6(6):721–741, Nov 1984.
S. Geman and C.-R. Hwang. Diffusions for global optimization. SIAM J. Control Optim., 24(5):1031–1043, 1986.
B. Gidas. Global optimization via the Langevin equation. In 1985 24th IEEE Conference on Decision and Control, pages 774–778, Dec 1985.
D. Griffeth. Uniform coupling of non-homogeneous Markov chains. J. Appl. Probability, 12(4):753–762, 1975.
O. Häggström. Finite Markov chains and algorithmic applications, volume 52 of London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2002.
B. Hajek. Cooling schedules for optimal annealing. *Math. Oper. Res.*, 13(2):311–329, 1988.

R. Holley and D. Stroock. Simulated annealing via Sobolev inequalities. *Comm. Math. Phys.*, 115(4):553–569, 1988.

L.-J. Huang, Y.-T. Liao, T.-L. Chen, and C.-R. Hwang. Optimal variance reduction for Markov chain Monte Carlo. *SIAM J. Control Optim.*, 56(4):2977–2996, 2018.

C.-R. Hwang, S.-Y. Hwang-Ma, and S. J. Sheu. Accelerating Gaussian diffusions. *Ann. Appl. Probab.*, 3(3):897–913, 1993.

C.-R. Hwang, S.-Y. Hwang-Ma, and S.-J. Sheu. Accelerating diffusions. *Ann. Appl. Probab.*, 15(2):1433–1444, 2005.

J. Johnson and D. Isaacson. Conditions for strong ergodicity using intensity matrices. *J. Appl. Probab.*, 25(1):34–42, 1988.

S. Kirkpatrick. Optimization by simulated annealing: quantitative studies. *J. Statist. Phys.*, 34(5-6):975–986, 1984.

S. Kirkpatrick, C. D. Gelatt, and M. P. Vecchi. Optimization by simulated annealing. *Science*, 220(4598):671–680, 1983.

F. Leisen and A. Mira. An extension of Peskun and Tierney orderings to continuous time Markov chains. *Statist. Sinica*, 18(4):1641–1651, 2008.

J. S. Liu. Metropolized independent sampling with comparisons to rejection sampling and importance sampling. *Statistics and Computing*, 6(2):113–119, Jun 1996.

M. Löwe. Simulated annealing with time-dependent energy function via Sobolev inequalities. *Stochastic Process. Appl.*, 63(2):221–233, 1996.

M. Löwe. On the invariant measure of non-reversible simulated annealing. *Statist. Probab. Lett.*, 36(2):189–193, 1997.

L. Miclo. Recuit simulé sans potentiel sur un ensemble fini. In *Séminaire de Probabilités, XXVI*, volume 1526 of *Lecture Notes in Math.*, pages 47–60. Springer, Berlin, 1992.

P. H. Peskun. Optimum Monte-Carlo sampling using Markov chains. *Biometrika*, 60:607–612, 1973.

L. Tierney. A note on Metropolis-Hastings kernels for general state spaces. *Ann. Appl. Probab.*, 8(1):1–9, 1998.

N. M. van Dijk. Uniformization for nonhomogeneous Markov chains. *Oper. Res. Lett.*, 12(5):283–291, 1992.