Gravitational lensing by point masses on regular grid points

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ABSTRACT

It is shown that gravitational lensing by point masses arranged in an infinitely extended regular lattice can be studied analytically using the Weierstrass functions. In particular, we draw the critical curves and the caustic networks for the lenses arranged in regular-polygonal – square, equilateral triangle, regular hexagon – grids. From this, the mean number of positive parity images as a function of the average optical depth is derived and compared to the case of the infinitely extended field of randomly distributed lenses. We find that the high degree of the symmetry in the lattice arrangement leads to a significant bias towards canceling of the shear caused by the neighboring lenses on a given lens position and lensing behaviour that is qualitatively distinct from the random star field. We also discuss some possible connections to more realistic lensing scenarios.

1 INTRODUCTION

Beyond the Galactic and Local-Group (in particular, Magellanic Clouds and M31) microlensing experiments (Alcock et al. 2000; Afonso et al. 2003; Calchi Novati et al. 2005; de Jong et al. 2006), the main interest on the point-mass lenses lies in the effect of the individual stars in a lensing galaxy (Chang & Refsdal 1979, 1984; Nityananda & Ostriker 1984; Irwin et al. 1989; Witt et al. 1995; Schechter & Wambsganss 2002; Keeton et al. 2006). For these cases, each point-mass lens cannot be treated individually and their collective effects usually differ drastically from that of the simple linear superposition of them. The usual approach to this “high-optical-depth microlensing” problem is “inverse ray tracing” (Kayser et al. 1986; Schneider & Weiss 1987; Wambsganss et al. 1990, 1992), that is, examining statistical properties of lensing observables using Monte-Carlo realizations of the lensing system. However, as the number of the individual point-mass lenses to reproduce the realistic scenario well approaches the number of stars in the galaxy, this becomes very expensive rather quickly in terms of the required resources. Another complementary approach to the problem is from the random walk and the probability theory together with the thermodynamic approximation (that is, effectively considering the limit at the infinite number of the stars), which has been quite successful in certain well-defined problems (Nityananda & Ostriker 1984; Schneider 1987; Schneider et al. 1992).

In the following, we consider a different approach to the problem: seeking complete analytic solutions. In many complex systems, imposing high level of symmetry can reduce the problem to a simpler one, which may be otherwise too complicated to analyze. In a similar spirit, in this paper we consider gravitational lensing by equal point masses that are arranged in an infinitely extended regular lattice. After we derive the appropriate lens equation in Sect. 2, they are examined in Sect. 3 to find the corresponding critical curves and the caustic networks. In Sect. 4, we study the mean number of positive parity images, which is compared to the case of lensing by an infinite random star field. In Sect. 5, we consider the effect of the external potential by mean of adding external shear (and convergence), and discuss some possible connections to more realistic lensing scenarios.

We argue that, while the system considered here may be somewhat artificial in its construct and rather abstract in its nature, the study presented in this paper can lead us to some insight on microlensing at high optical depth.

2 LENS EQUATION

In most astronomical situations, gravitational lensing is described by the lens equation \( y = x - \nabla \psi \), which is derived from the lowest-order nontrivial approximation to the path of light propagation (Schneider et al. 1992; Petters et al. 2001; Kochanek et al. 2005). Here, \( y \) and \( x \) are the (2-dimensional) vectors representing the lines of sight toward, respectively, the angular positions of the source under the absence of lensing and the lensed image. The lensing potential \( \psi \) is the gravitational potential (equivalently, the gravitational ‘Shapiro’ time delay) integrated along the line of sight up to a scale constant (Blandford & Narayan 1986).

In particular, gravitational lensing by a point mass is described by the potential \( \psi(x) = \theta_E^2 \ln |x - l| \) if the point-mass lens is located along the line of sight indicated by the vector \( l \) (Paczyński 1986). Here, the scale constant \( \theta_E \) is known as the ‘Einstein (ring) radius’ and determined by the mass of the lens \( (\theta_E^2 \propto M) \) and the distances among the source, the lens, and the observer. With this potential, we find the lens equation for point-mass lensing,

\[
y = x - \frac{x - l}{|x - l|^2}
\]

where the unit of angular measurements has been chosen such that...
\( \theta_e = 1 \). The equation is generalized to the case of multiple point-mass lenses with a smoothly-varying ‘large scale’ potential (Young 1981) such that
\[
y = x = \sum_{i=1}^{\infty} \frac{m_i (x - l_i)}{M (x - l_i^e)} = \nabla \psi_{\text{ext}}.
\]  
(2)

Here, the unit of angular measurements is now given by \( \theta_e \) corresponding to a fiducial mass scale \( M \) and subsequently the ‘external’ lensing potential \( \psi_{\text{ext}} \) should be rescaled appropriately. In addition, the masses of individual lenses \( m_i \) enter into the equation as the ratio to the fiducial scale.

Next, let us think of the case that the equal point-mass lenses are located at the lattice points on an infinitely extended square grid. In addition, we ignore the effects of the deflection but the distortion of the deflection function; the lensing described by equation (3).

For this, we first write down the (formal) second-order complex derivative of the deflection function
\[
f''(z) = \sum_{m,n=-\infty}^{\infty} \frac{2}{(z - 2(m+n)\omega)^3} = -\psi''(z; g, 0)
\]  
(4)

where \( \eta, z, \lambda \) are the complexified variables corresponding to \( y, x, I \), respectively, and the over-bar notation is used to indicate complex conjugation. We further assume that the lens positions are given by \( \lambda = 2(m+n)\omega \) where \( m \) and \( n \) are any pair of integers (both running from the negative infinity to the positive infinity) and \( \omega \) is the half distance between adjacent grid points. Without any loss of generality, \( \omega \) can be restricted to be a positive real. Strictly speaking, the ‘deflection function’ \( f(z) \) given formally in terms of the infinite sum in equation (3) is not well-defined as it is not convergent. This is in fact a natural consequence of having an infinite total lensing mass, which is unphysical. However, what is important in our understanding of the lensing is not the absolute amount of the deflection but the difference of the deflections between neighbouring lines of sight. That is to say, if we add any constant to the deflection function in the lens equation, the resulting lens equation is reduced to the equivalent one without the constant by introducing the “offset” to the source position to cancel the constant. Two equations then are observationally equivalent because there is no a priori information regarding the source position in the absence of the lens. Similarly, despite the seeming global failure of equation (3), we can still proceed by focusing on the local properties of the lensing described by equation (3).

For this, we first write down the (formal) second-order complex derivative of the deflection function;
\[
f''(z) = \sum_{m,n=-\infty}^{\infty} \frac{2}{(z - 2(m+n)\omega)^3} = -\psi''(z; g, 0)
\]  
(4)

where
\[
g_z = (\frac{6\lambda_0}{\omega})^4 = \frac{3\pi^4}{16\omega^4} \omega^2.
\]  
(5)

We find that this is related to the definition of the special function known as the Weierstrass elliptic function \(^1\) (Abramowitz & Stegun 1972, or see also Appendix). Here, \( \Gamma_s = \Gamma(x) \) is the gamma function, \( \psi(z) \) is the Weierstrass elliptic function and \( \psi'(z) \) is its complex derivative. We also use an abbreviated notation for (the power to) the gamma function such that \( [\Gamma(x)]^\nu = \Gamma^\nu \) in order to avoid proliferation of parentheses and brackets. In addition, \( \omega_{0h} \approx 1.854 \) is the real half-period of the lemniscatic case of \( \nu \)-function (see Appendix) and \( \kappa_\nu = \pi / (2\omega)^2 \) is the optical depth (Vietri & Ostriker 1983; Paczynski 1986) of the lenses. If we consider the square grid given by the centres of each lensing lattice, it is straightforward to see that there exists a lens corresponding to each cell with side length of \( 2\omega \). Hence, the mean surface density of point masses is in fact given by \( \sigma = (2\omega)^{-2} \); and therefore, the optical depth by \( \kappa_\nu = \pi \sigma = \pi / (2\omega)^2 \). Moreover, the 90°-rotational symmetry of the system implies that the total shear due to all the other lenses on any given lens cancels out to be zero [mathematically this indicates the constant term in the Laurent-series expansion of \( f'(z) \) is null]. Consequently, we find that \( f'(z) = -\psi(z; g, 0) \) and \( f(z) = \zeta(z; g, 0) + C \). Here, \( \zeta(z) \) is the Weierstrass zeta function (Abramowitz & Stegun 1972) and \( C \) is an arbitrary complex constant. If we choose the ‘centre’ of the system to coincide with the lens at the coordinate origin, then we have \( C = 0 \). We note that, while the constant \( C \) cannot be independently determined, its choice is physically inconsequential. The situation is analogous to lensing by an infinitely extended mass screen. While a naive expectation from the symmetry appears to indicate null deflections along every line of sight, detailed physical consideration leads to the uniform focusing with respect to an arbitrary choice of the centre. In the current situation, the discrete translation symmetry implies the arbitrariness in the choice of the centre and consequently that of \( C \), which results in the infinite sum in equation (3) not being well-determined. However, the constant-deflection term in the lens equation again only leads to a constant offset between the source and the image/lens plane, which has no observable consequence and therefore can be ignored.

Some results are immediate from the resulting lens equation
\[
\eta = z - \zeta(z; g, 0).
\]  
(6)

For instance, the property of \( \zeta(z) \) such that
\[
\zeta(z + 2(m+n)\omega; g, 0) = \zeta(z; g, 0) + \frac{\pi}{2\omega}(m-ni).
\]  
(7)

for arbitrary integers \( m \) and \( n \) indicates that the vertices \( z, z + 2\omega, z + 2(1+i)\omega, \) and \( z + 2i\omega \) of a square in the lens plane map to the points on the source plane: \( \eta = z - \zeta(z), \eta + N, \eta + (1+i)N, \) and \( \eta + iN \) where \( N = 2\omega - \pi i(2\omega) \), and therefore the mean inverse magnification is given by \( N/(2\omega)^2 = [1 - \pi i(2\omega)^2]^2 \). The result confirms the expectation that the sufficiently large beam of light would see the system as if it were a uniform screen of mass with a convergence (or the optical depth) \( \kappa = \pi \sigma = \pi / (2\omega)^2 \). Along the real line, equation (6) maps each segment between two adjacent lenses onto the whole real line, and thus it is easily argued that there are infinite number of images. However, unless \( \kappa_\nu = 1 \), all but a finite number of images have negative parity.

### 2.1 lenses on triangular and hexagonal grid

Analogous to the preceding case, we can also set up the lens equation for the equal point-mass lenses located at the lattice points on an infinitely extended equilateral-triangular grid. The 60°-rotational symmetry of the system again implies that the total shear on any given lens caused by the remaining set of lenses also cancels out. This fact, combined with the general definition of \( \psi \)-function indicates that the lens equation can again be written

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\(^1\) Throughout this paper, the arguments of the Weierstrass functions followed by a vertical bar – | – denote the half-periods whereas the elliptic invariants are indicated by those followed by a semicolon (;). For details, see the listed references or Appendix. Whenever it is appropriate, these arguments may be suppressed, provided that there is little danger of confusion.
Lensing by regular point-mass lattices

Figure 1. Contour plots of the equipotential lines of the lensing potential for the lattice lenses. Top Left: square lattice. Top Right: triangular lattice. Bottom Left: hexagonal lattice with equation (14). Bottom Right: hexagonal lattice with a lens being at the center.

down using the Weierstrass functions. If the lenses are placed over the grid given by \( \lambda = 2 \omega (m + ne^{\pi i/3}) \) where \( m \) and \( n \) are integers, the corresponding lens equation is given by

\[ \eta = z - f(z) ; \quad f(z) = \zeta(z, \omega, e^{\pi i/3} \omega) = \zeta(z; 0, g_3) \]

where

\[ g_3 = \left( \frac{\omega}{\omega_2} \right)^6 = \left( \frac{\sqrt{3} \Gamma_1^{1/3}}{8 \pi^3} \right)^3 \kappa^2 \]

and \( \omega_2 \approx 1.530 \) is the real half-periods of the equiharmonic case of \( \wp \)-function (see Appendix) whereas \( \kappa_0 = \pi / (2 \sqrt{3} \omega^2) \) is the corresponding optical depth. Likewise, the quasiperiodicity of \( \zeta(z; 0, g_1) \),

\[ \zeta[z + 2(m + ne^{\pi i/3}) \omega; 0, g_1] = \zeta(z; 0, g_1) + \frac{\pi}{\sqrt{3} \omega} (m + ne^{-\pi i/3}) \]

where \( m \) and \( n \) are arbitrary integers, indicates that the vertex points \( z, z + 2 \omega, \) and \( z + 2e^{\pi i/3} \omega \) of the equilateral triangle in the lens plane map onto the source plane points: \( \eta = z - \zeta(z), \eta + \Xi \) and \( \eta + \Xi e^{\pi i/3} \)

where \( \Xi = 2 \omega - \pi i / (\sqrt{3} \omega) \) so that the mean inverse magnification is again given by \( (1 - \kappa)^2 \) where the optical depth or the mean convergence is \( \kappa_* = \pi / (2 \sqrt{3} \omega^2) \) (and the mean surface density \( \sigma \) of the point masses on the triangular grid is given by \( \sigma^{-1} = 2 \sqrt{3} \omega^2 \) where \( 2 \omega \) is the side length of the unit triangular cell).

We note that the preceding two cases of the lens arrangement correspond to two of the three possible regular tessellations of the two-dimensional plane. It may be of some interest to consider the regular lens placement corresponding to the remaining regular tessellation – the hexagonal or honeycomb tiling. It turns out that the regular hexagonal grid case is closely related to the equilateral triangular grid. By removing every third of the lens that forms a (30°-rotated) larger triangular grid of side length of \( 2 \sqrt{3} \omega \) from the base triangular grid with side length of \( 2 \omega \), the remaining lenses form a hexagonal grid (or the vertices of honeycomb cells).

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\[ ^2 \text{Note that they are in fact so-called dual tiling of each other. That is, if we consider a grid with vertex points at the centres of each triangular cell of the equilateral triangular grid, then the resulting lattice forms a regular-hexagonal (honeycomb) grid and also vice versa.} \]
Consequently, the corresponding lens equation may be written as
\[
\eta = z - b(z; g_3);
\]
\[
b(z; g_3) = \zeta(\zeta(\alpha, e^{-\pi i/3}) - \zeta(\zeta(\alpha, e^{\pi i/6} \omega, \sqrt{3} \omega))
\]
(11)
where \(g_3\) is related to \(\omega\) through equation (9) provided that \(2\omega\) is still the side length of the base triangular grid (and also that of the unit honeycomb cell). However, since every third of lens has been removed from the base triangular cell, the optical depth is reduced to \(\kappa_\pi = \pi/(3 \sqrt{3} \omega^2)\), and thus
\[
g_3 = \left(\frac{3^{1/2} \xi_0^4}{4\pi^2}\right) \kappa_\pi^3.
\]
(12)

In addition, \(b'(z; g_3) = -b(z; g_3)\) (the primed symbol indicates the complex differentiation, i.e., the derivative with respect to \(z\)). The function \(b(z; g_3)\) is studied in Appendix A2.1. It is again straightforward to establish that the mean inverse magnification is \((1 - \kappa_\pi)^3\).

Unlike the preceding two cases, the system described by lens equation (11) does not have the lens at the centre, but the centre of the system corresponds to the location on the lens plane such that \(b'(0) = b(0) = 0\). However, we note that this is deliberately chosen for mathematical simplicity, and does not imply any intrinsic difference of the honeycomb grid from the square or triangular grid. In particular, we note the 120°-rotational symmetry of the system with respect to any lens location, which actually implies that any lens on the honeycomb grid experiences zero shear from the remaining point masses. In other words, for all three cases of the point masses, the total shear from all the adjacent lenses exactly cancels out at any lens position.

2.2 lensing potential

While most of the lensing properties of the lattice lens can be studied using the lens equations, it is still useful to have an expression for the lensing potential for some purposes such as the time delay. Although the direct infinite sum of the individual logarithmic potential of the point-mass lens is divergent everywhere, one can get around this difficulty through the known antiderivative of the Weierstrass zeta function. First, we note the relation between the real potential and the complex deflection function (An 2005; An & Evans 2006): the complex lens equation is given by \(\eta = z - 20 \psi\) if the real potential is given by \(\psi\). Here, the operator \(\partial_z\) indicates the ‘Wirtinger derivative’ (e.g., Schramm & Kayser 1995) with respect to \(\bar{z}\), that is, if \(\psi = \psi(x, y)\) and \(z = x + yi\), then
\[
\partial_z \psi = \frac{\partial \psi}{\partial x} + \frac{i \partial \psi}{\partial y} = \frac{1}{2} \frac{\partial}{\partial x} - \frac{i}{2} \frac{\partial}{\partial y}.
\]
(13)
because \(x = (z + \bar{z})/2\) and \(y = i(\bar{z} - z)/2\). For the current scenario, we have \(f(z) = 20\psi = 20\bar{\psi} = 20\psi(\text{note that } \psi = \psi \text{ since } \psi \text{ is real})\). Moreover, \(f(z)\) is a complex analytic function and so there exists a complex analytic function \(\psi(z)\) such that \(f(z) = \psi(z) = \partial_z \psi\) and \(\partial_z \psi = 0\). Then, \(\partial_z (\psi + \bar{\psi}) = 0\), \(\partial_z \bar{\psi} = 0\), \(\partial_z \psi + \partial_z \bar{\psi} = f(z)\), and thus \(2\psi = \psi + \bar{\psi}\), that is, \(\psi(x, y) = \Re[\psi(x + yi)]\) up to an additive constant (An 2005). Here, \(\Re[f]\) is the real part of a complex-valued function \(f\). Consequently, we find that the real potential up to an additive constant for the regular lensing lattice is given by
\[
\psi(x, y) = \begin{cases} 
\ln |\sigma(\zeta; g_2, 0)| & \square \\
\ln |\sigma(\zeta; 0, g_3)| & \triangle \\
\ln |\sigma(\zeta; 0, g_3)| / |\sigma(\zeta; 0, -g_3/2)| & \bigcirc 
\end{cases}
\]
(14)
where \(z = x + yi\) and \(\sigma(\zeta; g_2, g_3)\) is the Weierstrass sigma function (Abramowitz & Stegun 1972), which can be defined as the anti-log-derivative of the Weierstrass zeta function, that is, \(\partial_z \ln \sigma(z) = \zeta(z)\). Here, we have also used the property of the complex logarithm function such that \(\Re[\ln g] = \ln |g|\) where \(g\) is any complex-valued function. Note that an additional linear term may be required to be added in the expression for the potential if the linear term of the system is chosen differently from those of the lens equations discussed earlier. Fig. 1 shows contour plots for the equipotential lines for the potential given in equations (14).
though they can all be recovered from the discrete grid translation (odic), there exists an infinite set of critical curves and caustics of a 'unit' curve. As usual, the Jacobian determinant of equation (6), (8) or (11) given by the lens equation. Since $|\phi(z;1,0)|$ (square grid) and $|\phi(z;0,1)|$ (triangular and hexagonal grid). In particular, the critical curves for the square (or triangular) lens lattice with the half-period $\omega$ are basically the curves of constant $|\phi(z;1,0)| = \omega^2/\omega_0^2$ or $|\phi(z;0,1)| = \omega^2/\omega_2^2$ rescaled by a factor of $\omega/\omega_0$ (or $\omega/\omega_2$). On the other hand, those for the honeycomb lattice are found from the curves of constant $|\phi(z;0,1)| = \omega_2/(\sqrt{3}\omega)$ scaled by $\sqrt{3}\omega/\omega_2$ and rotated by $-30^\circ$. Since $\phi$-functions are well-defined elliptic functions, for all three cases the critical curves are an infinite set of nonintersecting closed curves except when $\omega$ assumes a particular value at which the curves are given by infinitely extended boundary lines that divide the lens plane.

### 3.1 square lattice

For the square grid case, if $\omega < (\omega_0/\sqrt{2}) \approx 1.311$ [i.e., $\kappa_* > \pi/(2\omega_0^2) \approx 0.457$], each of the critical curves is centred at the centre of a square cell defined by the four adjacent lenses (i.e., $k\omega + ip\omega$ where $k$ and $p$ are odd integers). Any image in the region ‘within’ the critical curves has positive parity and vice versa. On the other hand, if $\omega > 2^{-1/2}\omega_0$ [i.e., $\kappa_* < \pi/(2\omega_0^2)$], the critical curves are centred around each lens location. For this case, the images ‘within’ the critical curves now have negative parity. Finally, if $\omega = 2^{-1/2}\omega_0$ [i.e., $\kappa_* = \pi/(2\omega_0^2)$], the critical curves are given by two infinite sets of parallel diagonal lines, and the whole lens plane is evenly divided.
into checker-like tiled regions according to the parity of the images.

For \( \omega \neq 2^{-1/2 \kappa_\star} \), the lens equation maps each connected and closed ‘unit’ critical curve to a caustic that is also connected and closed. Fig. 2 shows the critical curves and the caustic network for \( \omega = 1.4 \). Like the critical curves, the whole caustic network is composed of the lattice arrangement of ‘unit’ caustic curves, but the unit curve exhibits self-crossing in contrast to the critical curves. The shape of the unit caustic curve, which is roughly similar to a variant of (irregular) octagram (i.e., \( [8/3] \)-star-polygon\(^3\)), is actually generic for the case that \( \omega > 2^{-1/2 \kappa_\star} \). The ‘centre’ of each unit caustic curve forms a similar square grid to the lens lattice but the side length of the grid on the source plane is given by \( 2 \omega (1 - \kappa_\star) \). As \( \omega \) gets larger (or equivalently \( \kappa_\star \) gets smaller), the size of the caustics shrinks and they asymptotically reduce to degenerate points. For \( \omega < 2^{-1/2 \kappa_\star} \), the caustic network can still be understood as the lattice arrangement of ‘unit’ caustic curves, the shape of which is simply described as being a diagonally stretched square. Again, the separation between the centres of the unit caustics are given by \( 2 \omega |1 - \kappa_\star| \). However, this is smaller than the ‘size’ of each unit caustic if \( \kappa_\star \sim 1 \) so that the network can exhibit overlapping among the neighboring caustics, which leads to a rather complex network (albeit regular thanks to the symmetry of the system). The critical curves and caustic network for \( \omega = 1.2 \) are shown in Fig. 3. The transition of the caustic network over \( \omega = 2^{-1/2 \kappa_\star} \) can be understood as the contact between (the vertices of) adjacent quasi-octagram caustics leading to their connection. The further evolution of the caustic topography for \( \omega < 2^{-1/2 \kappa_\star} \) is presented in Fig. 4. Since the separation between the unit caustics reduces to nil as \( \kappa_\star \to 1 \), the network is the densest around \( \kappa_\star = 1 \). On the other hand, as \( \kappa_\star \) increases past unity, the separations between unit caustics coincide with the separations between them is found approximately to be \( \omega \approx 0.748 \ (\kappa_\star \approx 1.405) \).

One physical interpretation of the caustics in gravitational lensing is that they are boundaries between regions in the source plane that produce different number of images. In addition, it is also known that a pair of images of opposite parity appears or disappears whenever the source crosses the caustics, and that the number of positive parity images, if the source lies outside any caustics, is one or nil depending on the characteristics of the system. Hence, the maximum number of positive parity images can be determined from examination of the caustics network. For example, the quasi-octagram caustics for \( \omega > 2^{-1/2 \kappa_\star} \) allow at most four positive parity images in the region around the centres of caustics.

It is easy to convince oneself that the minimum number of positive parity images for the corresponding scenario is one when the source lies outside any caustic. As for the case \( \omega \leq 2^{-1/2 \kappa_\star} \), the examination of the caustic network leads us to conclude that, while the maximum number of positive images approaches infinity at \( \kappa_\star = 1 \) (i.e., \( \omega = \pi^{1/2}/2 \approx 0.886 \)), it is greater than four only if \( 0.803 \leq \omega \leq 1.039 \). In other words, the number of positive parity images is bounded by four provided that \( \omega \geq 1.039 \ (\kappa_\star \leq 0.727) \) or that \( \omega \leq 0.803 \ (\kappa_\star \geq 1.217) \). The source positions with no positive parity image start to exist if \( \omega \leq 0.776 \) or equivalently \( \kappa_\star \geq 1.039 \). Finally, if \( \omega \leq 0.748 \ (\kappa_\star \geq 1.405) \), there is no overlap between neighboring caustics, and thus the number of positive parity images is one if the source lies in the caustic, and nil if otherwise.

### 3.2 equilateral triangular lattice

The lenses on an equilateral triangular lattice produce the critical curves and the caustic network in a basically consistent pattern as the square lattice lens although many details are quite different. For triangular lattices, the unit critical curves for \( \omega < (\omega_\star/\sqrt{3}) \approx 1.214 \)
Figure 7. Critical curves (dotted lines) and caustic networks (solid lines) for lenses on a honeycomb lattice with $\omega = 1.45 \ [\kappa_\star = \pi/(3\sqrt{3}\omega^2) \approx 0.286]$.  

Figure 8. Same as Fig. 7 except for $\omega = 1.3 \ [\kappa_\star = \pi/(3\sqrt{3}\omega^2) \approx 0.358]$.  

3.3 regular hexagonal lattice

In qualitative terms, the critical curves of the honeycomb lattice are basically the ‘dual’ of that of the triangular case. The critical curves for $\omega < (2^{1/3}3^{-1/2}\omega_2) \approx 1.402 \ [\kappa_\star > \pi/(3^{1/2}2^{3/4}\omega_2^2) \approx 0.308]$ are analogous to those for the triangular lattice with $\omega > 2^{-1/3}\omega_2$ and located within each hexagonal cell centred at the positions of null shear. On the other hand, if $\omega > 2^{1/3}3^{-1/2}\omega_2$, the critical curves are like those for the triangular lattice with $\omega < 2^{-1/3}\omega_2$ and centred around each lens positions. The dual characteristic further indicates that there are twice more poles than the locations of null shear (which are in fact fourth-order zeros), and thus there is one-to-two correspondence between unit critical curves of the former to the latter (formally, the ‘winding number’ of the former is twice that of the latter).

On the other hand, the caustics of a honeycomb lattice are qualitatively distinct from those of a triangular lattice. Fig. 7 shows an example for a sparse honeycomb lattice with $\omega = 1.45$, which is in fact archetypal for $\omega > 2^{1/3}3^{-1/2}\omega_2$. The unit caustics are given by a six-sided closed self-intersecting curve, which may be labeled a bi-triangle, and they are arranged in a similar hexagonal pattern as the lattice of the lens with side length $2\omega(1 - \kappa_\star)$. We also note that the maximum number of positive parity images associated with these caustics is three. For a dense honeycomb lens lattice ($\omega < 2^{1/3}3^{-1/2}\omega_2$), the unit caustics are found to be in the shape of a stretched hexagon (an example of which for $\omega = 1.3$ is given in Fig. 8) and arranged in a triangular grid with side length of $2\sqrt{3}\omega(1 - \kappa_\star)$. They also exhibit trends of varying overlap similar to the square or triangular lattice, as $\kappa_\star$ gets larger past the unity. We find that the critical value for no overlapping caustics for the honeycomb lattice is approximately given by $\omega \approx 0.622 \ (\kappa_\star \approx 1.565)$.

Despite their variety, we also notice some common characteristics of the caustic networks of three kinds of regular lensing lattices. For example, with an optical depth that is greater than a certain critical value, the unit curves are more or less in a similar shape as the lensing lattice except for the fact that they are distorted in a way stretching the vertices radially outward. They densely overlap with one another around $\kappa_\star \sim 1$. However, as $\kappa_\star$ gets larger past the
unity, their size shrink whereas the separations among them grow, and so the networks eventually reduce to a simple array of tiles. Likewise, towards the lower optical depths below the critical value, the networks are comprised of star-shaped unit curves whose exact characteristics are related to meeting pattern of the lensing lattice at the common vertex points.

4 MEAN NUMBER OF POSITIVE PARITY IMAGES

The area in the caustic accounting for its multiplicity can be understood as the covering fraction of the source plane by the region with ‘extra image’ pairs once it is properly normalized. In general, however, the normalization is ill-defined for localized lenses. This difficulty is ameliorated with the introduction of the lattice of point masses since the whole plane is now evenly divided into identical cells. With the regular lensing lattice, the area under the unit caustic can be properly normalized with respect to the area of the mapped image (on the source plane) of the unit cell comprising the lensing lattice. The result, if the multiplicity is properly accounted for, is actually the mean number of extra image pairs, which has been calculated for the case of the random star field using techniques from the probability theory (Wambsganss et al. 1992; Granot et al. 2003).

For ‘dense’ lattices that produce nonintersecting (but possibly overlapping) unit caustics, the relevant area under the unit caustics is straightforward to calculate since, despite overlapping, they are simply reduced to a set of closed loops. Then, the area under one such loop normalized to the area of a unit cell in the source plane, \( \omega_n (1 - \kappa)^2 \) where \( \omega_n \) is the area under the unit cell defined by the adjacent lens positions in the lens plane, is the mean number both of extra image pairs and of positive parity images for a given source position because the corresponding critical curves in the image plane completely enclose the region where any positive images reside and thus there are no positive images formed ‘outside’ of any caustics. On the other hand, for ‘sparse’ lattices that produce the network composed of a grid set of self-intersecting ‘star-shaped’ curves, the self-crossing of the unit structures requires some consideration on the meaning of ‘area’ that properly accounts for the image multiplicity. We argue that the solution is rather simple. Once the area calculation is reduced to the line integral along its boundary (see e.g., An 2005; An & Evans 2006) using the fundamental theorem of multivariate calculus\(^4\), the corresponding line integral along the caustic actually results in the area counted with multiplicity. That is to say, they can be considered as the ‘area-excess’ or the ‘over-covering factor’ of the source positions that produce positive parity images. Unlike the previous case, the normalization cell in the lens plane is the unit cell of the dual of the lensing lattice, i.e., the cell defined by the adjacent null shear locations (which is in fact the centre of each cell of the lensing lattice). The corresponding critical curves lie completely within this unit normalization cell and furthermore enclose the regions of the image plane where negative parity images reside. Subsequently, there is at least one positive parity image for any given source position – that is, the source ‘outside’ caustics forms one positive parity image. As a result, the mean number of extra image pairs is less than the number of positive parity images by one. We note that the mean number of positive parity images actually varies continuously over the critical value dividing the dense lattice from the sparse one – that is, the mean number of extra image pairs jumps by one. In fact, the theorem is usually known as Green’s theorem in elementary multivariate calculus when referring to the relation between two-dimensional integral over a domain in two-dimensional space and one-dimensional integral along the boundary of the domain. This is also a special case of so-called (generalized) Stoke’s theorem, the name under which the theorem is usually known as.

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designating a random image pair as an extra involves some degrees of arbitrariness whereas counting the number of positive parity images is well-defined. For these reasons, henceforth, we shall discuss the mean number of positive parity images exclusively.

Fig. 9 shows the resulting mean number of positive parity images as a function of $\kappa_s$. Comparison to the case of the random star field (with zero external shear) (Wambsganss et al. 1992) reveals some significant differences between the regular lensing lattices and the random field, particularly in the behaviour toward $\kappa_s \to \infty$. While the mean number falls off as $k_4^2$, for the random star field and triangular lensing lattices (albeit the normalization for the random field being larger by a factor of ~4), its behaviours for the lenses on square lattices and honeycomb lattices are characterized by slower fall-offs given by $\sim k_2^2$ and $\sim k_2^{-2}$, respectively. At first, this is somewhat counterintuitive. That is, in the random star field, the distance $s$ to the closest neighboring lens is distributed according to

$$\mathcal{P}(s)ds = 2\pi s\sigma e^{-\pi s^2/\sigma^2}ds$$

where $\sigma = \kappa_s/\pi$ is the mean surface number density of the stars. The mean and the variance are given by $\mu = 1/(2\sqrt{\pi})$ and $\sigma^2 = 1/(4\pi/\kappa^2)$. In other words, as the density of stellar field increases, the random fluctuations around the mean decrease and one would expect that the random stellar field may approach to a regular field.

The actual result can be understood using the distribution of the (internal) lensing shear $\gamma$ due to stars experienced by the given line of sight towards a lensed image. For a field of point masses, once this distribution is known, the mean number of positive parity images is recovered through (Wambsganss et al. 1992)

$$\langle N_s \rangle = \langle \mu \rangle \int_0^\infty (1 - \gamma^2) \mathcal{P}(\gamma) d\gamma.$$  \hfill (19)

With lenses on a regular lattice, we have shown that the shear at the given image plane location is given by the Weierstrass elliptic function, and hence, the distribution on the whole image plane reduces to that on the unit cell in the image plane. Then, the cumulative distribution of shear is found by the fraction of area in the unit cell where the shear is smaller than the given value, and consequently, its derivative gives the differential distribution of the shear. Furthermore, the homogeneity relation of $\gamma$-function implies the similarity relation between the distribution corresponding to different $\omega$. For example, with a square lensing lattice, we have that

$$\mathcal{P}(\gamma |k_x) d\gamma' = \frac{1}{\mathcal{A}_0} \left[ \mu \int_{\omega_0} \mu' \gamma' \right]_{\omega_0' \omega_0} = \frac{1}{\mathcal{A}_0} \left[ \int_{\omega_0 \gamma} \mathcal{P}(\gamma |k_x) d\gamma \right]$$

$$\mathcal{P}(\gamma |k_x) = \frac{1}{\mathcal{A}_0} \left[ \int_{\omega_0} \mu' \gamma' \right]_{\omega_0' \omega_0} = \frac{1}{\mathcal{A}_0} \left[ \int_{\omega_0} \mu' \gamma' \right]_{\omega_0' \omega_0} = \frac{k_0}{k_x} \mathcal{P}(\gamma |k_0),$$  \hfill (20)

where $k_0 = \pi/(2\omega_0^2)$ and $\mathcal{A}_0 = \mu_0 |\mathcal{P}(\mathcal{P}(\gamma |k_x) d\gamma |k_0)$. For $\gamma < \gamma'$ is the area of the unit cell where $|\mathcal{P}(\gamma |k_0) d\gamma |k_0 \leq \gamma$ for a square lensing lattice with side length of $\omega$ (see eq. [5] for $\gamma$ as a function of $\omega$ and $k_s$). In fact, any value of $\kappa_s$ instead of $k_0$ can be chosen as the normalization and the resulting similarity relation is valid for all three regular lattices.

The resulting distributions for all three regular lattices are shown in Fig. 10 together with the same distribution for the random star field (Niyananda & Ostriker 1984; Schneider 1987; Wambsganss et al. 1992), which is given by

$$\mathcal{P}(\gamma |k_s) = \frac{k_2^2 \gamma}{(k_2^2 + \gamma^2)^{5/2}} = \frac{1}{k_2^2} \frac{\gamma}{(1 + |\gamma/k_2|^2)^{5/2}}.$$  \hfill (22)

Unlike the random field case, we find that there is nonzero probability that the line of sight experiences no net shear for the square lattice lens. By contrast, the same distribution linearly tends to zero as $\gamma \to 0$ for the triangular lattice (and also the random field) and it diverges as $\sim \gamma^{-1/2}$ for the honeycomb lattice. These behaviours are related to the fact that the zeros of $\mathcal{P}(\mathcal{P}(\gamma |k_0) d\gamma |k_0)$ are second, first, and fourth-order, respectively; that is to say, their Taylor-series expansions at their zeros are given by

$$\mathcal{P}(\gamma |k_0) = \frac{k_0^2 \gamma}{k_2^2 + \gamma^2} + \delta(\gamma/k_2) \approx \frac{2088}{k_s} \gamma,$$

$$\mathcal{P}(\gamma |k_0) = \frac{k_0^2 \gamma}{k_2^2 + \gamma^2} + \delta(\gamma/k_2) \approx \frac{2326}{k_s^2} \gamma,$$

$$\mathcal{P}(\gamma |k_0) = \frac{k_0^2 \gamma}{k_2^2 + \gamma^2} + \delta(\gamma/k_2) \approx \frac{1705}{k_s^2} \gamma^{1/2},$$

where $z_0$ is a zero of the corresponding function (note that zero of $\gamma$ is at $z_0 = 0$). For $\gamma < 1$, the discussion in the preceding paragraph indicates that $\mathcal{P}(\gamma |k_0) d\gamma \approx \pi z_0 |z_0|^2$ where $z_0$ is the null shear position and $z$ traces the locations at which the shear is given by $\gamma$. By noticing the shear is given by $|\mathcal{P}(\gamma |k_0) d\gamma |k_0 = |\mathcal{P}(\gamma |k_0) d\gamma |k_0 = \mathcal{P}(\gamma |k_0) d\gamma |k_0$ for square, triangular, and honeycomb lattices, we can then find the leading-order behaviours for the shear distributions as $\gamma \to 0$ from the above Taylor expansions$^5$:

$$\mathcal{P}(\gamma |k_s) = \frac{1}{k_s} \left( \frac{2088}{k_s} \gamma \right),$$

$$\mathcal{P}(\gamma |k_s) = \frac{1}{k_s} \left( \frac{2326}{k_s^2} \gamma \right),$$

$$\mathcal{P}(\gamma |k_s) = \frac{1}{k_s} \left( \frac{1705}{k_s^{1/2}} \gamma^{1/2} \right),$$

which agree with the numerical results shown in Fig. 10 (note that $k_s = \pi$ for Fig. 10). In other words, the higher-than-linear-order (i.e., degenerate) zeros of the shear lead to nonzero finite or even divergent probability for the zero net shear. Physically speaking, accidental canceling of the net shear in the random star field is much less likely to occur than in the highly symmetric arrangement of the lenses, for which there may be significant chances that the image is located at an exactly balanced position where the net shear is null. This is basically the reason for slower fall-offs of the mean number of positive parity images in dense square or honeycomb lattices. From equation (21), if the shear distribution behaves like $\mathcal{P}(\gamma |k_s) \approx S \kappa^{-1}(\gamma/k_s)^{\alpha}$ as $\gamma/k_s \to 0$ where $S$ is a constant, $\alpha = (2/n) - 1 > -1$ is the asymptotic power index for the shear distribution and $n$ is the order of the zeros of the shear, then equation (19) as $k_s \to \infty$ reduces to

$$\langle N_s \rangle = \frac{1}{(1 - k_s^2)^2} \int_0^1 dy (1 - \gamma^2) \mathcal{P}(\gamma |k_s),$$

$$\approx \frac{S}{(k_s - 1)^2 \kappa^2 + 1} \int_0^1 dy (1 - \gamma^2) \mathcal{P}(\gamma |k_s).$$  \hfill (25)

$^5$ It is actually easier to work out from the series for the inverse functions, which are in fact hypergeometric function (see also Appendix).
Since the last integral is a finite constant for \( \alpha > -1 \), we find that \( \langle N_s \rangle \sim \kappa_*^{-(\alpha+3)} \) as \( \kappa_* \to \infty \), which is consistent with the results found earlier (Fig. 9).

In fact, the shear distributions for regular lensing lattices are quite distinct from that of the random field, and it is somewhat remarkable that the behaviours of \( \langle N_s \rangle \) are generally similar for all cases at least for small \( \kappa_* \). This is related to the fact that \( \langle N_s \rangle \) is basically an integration (or convolution) of the shear distribution, and therefore any ‘roughness’ in the shear distribution is ‘smoothed’ out in \( \langle N_s \rangle \). In particular, we note that the similarity relation of the shear distribution is essentially the result of the scale invariance and so should be generic. Then, \( \mathcal{P}(\gamma/\kappa_*) = \kappa_*^2 \mathcal{P}(\gamma/\kappa_*) \) where \( \mathcal{P}(\gamma/\kappa_*) \) is a function of \( \gamma/\kappa_* \), and consequently we have

\[
\langle N_s \rangle = \frac{1}{(1-\kappa_*^2)} \int_0^{1/\kappa_*} \mathrm{dr} (1-\kappa_*^2 \mathcal{P}(t)).
\]

Furthermore, since the shear for the lines of sight that pass sufficiently close to any of the lenses is simply given by \( \gamma \approx d^{-2} \) where \( d \) is the separation between the given line of sight and that to the lens, it is straightforward to establish that the generic asymptotic behaviour of the shear distribution as \( \gamma/\kappa_* \to \infty \) to be \( \mathcal{P}(\gamma/\kappa_*) \approx \kappa_*^2/\gamma^2 \) [i.e., \( \mathcal{P}(t) \approx t^{-2} \) as \( t \to \infty \)]. Suppose that power-series expansion of \( \mathcal{P}(t) \) at \( t = \infty \) is given by \( \mathcal{P}(t) \approx t^{-2}(1 + \sum_{n=1}^{\infty} \epsilon_n t^n) \) (and it is uniformly convergent in an interval containing \( t = \infty \)). Then, for \( \kappa_* \to 0 \), we find that (assuming \( p \neq 1/n \) for any positive integer \( n \));

\[
\int_0^{1/\kappa_*} \mathrm{dr} \mathcal{P}(t) \approx 1 - \kappa_* - \sum_{n=1}^{\infty} \frac{\epsilon_n \kappa_*^{n+1}}{pn + 1}
\]

\[
\int_0^{1/\kappa_*} \mathrm{dr}^2 \mathcal{P}(t) \approx \frac{1}{\kappa_*} - P_0 - \sum_{n=1}^{\infty} \frac{\epsilon_n \kappa_*^{n+1}}{(pn - 1)(pn + 1)}
\]

\[
\frac{\langle N_s \rangle}{\langle \mu \rangle} \approx 1 - 2\kappa_* + P_0\kappa_*^2 + \sum_{n=1}^{\infty} \frac{2\epsilon_n \kappa_*^{n+1}}{(pn - 1)(pn + 1)}
\]

where \( P_0 \) is a finite constant that weakly depends on the global (that is, far from \( t = \infty \)) behaviour of \( \mathcal{P}(t) \), and thus \( \langle N_s \rangle/\langle \mu \rangle \simeq 1 - 2\kappa_* + \mathcal{O}(\kappa_*^2) \) and \( \langle N_s \rangle = 1 + \mathcal{O}(\kappa_*^2) \), regardless of the higher-order behaviour of \( \mathcal{P}(t) \), provided that \( p > 1 \). In fact, we have \( \mathcal{P}_{\text{ran}}(t) = t(1 + t^{-2})^{-3/2} = t^{-2}(1 + t^{-2})^{-3/2} \) so that \( p = 2 \) for \( \mathcal{P}_{\text{ran}}(t) \), and also the power-series expansions of \( \varphi \)-functions at their poles and zeros.

---

**Figure 11.** Locations of lensed images with a square lensing lattice with \( \omega = 1.35 \) (\( \kappa_* \equiv 0.431 \)). The locations of the source with respect to the caustic networks are indicated in the top right panel. In the remaining three panels, the locations of the individual lensed images are marked by open circles (positive parity images) or closed dots (negative parity images). Also drawn in dotted lines are the critical curves and the locations of the lenses are marked by crosses.
Lensing by regular point-mass lattices

Figure 12. Same as Fig. 11 except \( \omega = 0.8 (\kappa_* \approx 1.23) \).

(or equivalently the hypergeometric function form of their inverse functions) imply that the power-series expansions of \( P(t) \) at \( t = \infty \) for regular lensing lattices are given in the above form with \( p = 2 \), \( p = 3 \), and \( p = 3/2 \) for the square, triangular, and honeycomb lattices, respectively. Therefore, all four cases exhibit the common asymptotic behaviour of \( \langle N_i \rangle \) for \( \kappa_* \ll 1 \).

4.1 image locations

While the overall characteristics of the lensing situation can be understood simply by studying the critical curves and caustics, one still needs to invert the lens equation to find the image locations for a given source position. With transcendental functions involved, this must be done numerically, but some analytical insights can ease the task.

Since the real axis of the system for the lens equations (6), (8), and (11) is chosen such that the lensing lattice exhibits reflection symmetry with respect to this axis, the deflection functions for all cases should behave \( f(z) = f(\bar{z}) \), which is indeed confirmed by the symmetry of the Weierstrass functions. Then, the lens equations (6), (8), and (11) can also be written to be

\[
\eta = z - f(\bar{z}) ; \quad f(z) = \begin{cases} \bar{\zeta}(z; g_2, 0) & \square \\ \bar{\zeta}(z; 0, g_3) & \triangle \\ h(z; g_3) & \circ \end{cases}
\]

(28)

Applying (two-dimensional) Newton-Raphson method to equation (28) indicates that the solutions are given by the \( (n \to \infty) \)-limits of the sequences given by the recursion relation

\[
z_{n+1} = \frac{\eta + f(\bar{z}_n) - [\bar{\eta} - f(z_n) + z_n f'(z_n)]f'(\bar{z}_n)}{1 - f'(\bar{z}_n)f'(z_n)}.
\]

(29)

Alternatively, we can eliminate \( \bar{z} \) from equation (28) by means of their complex conjugate (e.g., Witt & Mao 1995; An & Evans 2006)

\[
\bar{\eta} = \bar{\eta} + f(z).
\]

(30)

and consequently, the image positions corresponding to the source position of \( \eta \) are the fixed points of the ‘conformal’ mapping given by

\[
h(z) = \eta + f(\bar{\eta} + f(z) ; \quad h'(z) = f'[\bar{\eta} + f(z)] f'(z)
\]

(31)
Figure 13. Same as Fig. 11 except $\omega = 0.7$ ($\kappa_\star \approx 1.60$). Shown in the top left panel, there is no positive parity images for the source lying 'outside' the caustics.

where

$$f'(z) = \begin{cases} \bar{\nu}(z; g_2, 0) & \square \\ -\nu(z; 0, g_3) & \triangle \\ -b(z; g_3) & \bigcirc \end{cases} \quad (32)$$

In principle, all image positions can again be found applying the (complex) Newton-Raphson method to zeros of $z - h(z)$ (which include all image positions and possibly some spurious solutions) unless the image is right on the critical curve. Specifically, the zeros can be located from the limits of the sequences given by

$$z_{n+1} = z_n - \frac{z_n - h(z_n)}{1 - h'(z_n)} = \frac{z_nh'(z_n) - h(z_n)}{h'(z_n) - 1} \quad (33)$$

as $n \to \infty$. While equation (33) is basically the same as equation (29) except for the fact that $\bar{\eta}_n$ in equation (33) is replaced by $\bar{\eta} + f(z_n)$ in equation (29), the latter relation in general is slightly better-behaved and also converges faster than the former. The iteration (with infinite precision) given by equation (33) necessarily converges to a zero starting from anywhere in the complex plane except for a set of points with zero measure ('Julia set'), and furthermore any solution that is not on the critical curve has a basin of attraction that contains an open neighborhood of the solution. However, where the iteration converges to for a given starting point is nontrivial to figure out a priori.

In fact, if we want to locate all of positive parity images, there actually exists an alternative route which is much more economical. Application of the results from the complex dynamics (see also Appendix of An & Evans 2006) implies that, for any solution $z_0$ of equation (30) that is also $|f'(z_0)| < 1$, there exists a point $z_c$ such that $f'(z_c) = 0$ and $\lim_{n \to \infty} h^n(z_c) = z_0$. Here, $h(z) = h(z)$ and $h^n(z) = h(h^{n-1}(z))$, that is, $h^n(z)$ is the $n$-times iterations of the mapping $h$ starting from $z$. In other words, all the positive parity image positions are the limit points of the iterations of the mapping $h$ given by equation (31) starting from a zero of $f'(z)$ (eq. [32]). Conversely, the iterations of the mapping $h(z)$ starting from any zero of $f'(z)$ necessarily converge to one of the fixed points of $h(z)$, the set of which contains all the positive parity image positions. Since the number of positive parity images can be determined by examining the caustic network, one only needs, in principle, to perform the iterations finitely many times until all images are located (unless $\kappa_\star = 1$). In addition, all the zeros of $f'(z)$ for regular lensing lattices occur at the centre of each unit cell defined by the adjacent
Figure 14. Same as Fig. 11 except for an equilateral triangular lensing lattice with \( \omega = 1.15 \) (\( \kappa_* \approx 0.685 \)).

poles (here, the poles are simply the lens positions). In summary, all the positive parity image positions for a given source position can be located by the iterations of \( h(z) \) starting from the centres of each unit cell (corresponding to the null shear positions) until the set of distinct limit points contains the number of image positions of positive parity that is predicted by the source position and the caustic network.

As for negative parity images, although there are infinitely many of them, we argue that most of them are insignificant. First, the lens equations indicate that \( |z - \eta| = |f(z)| = |f(\delta z)| \), which implies that, in order to have an image that is arbitrarily far from the source, the image position should be rather close to the lens, that is, the poles of \( f(z) \), at which \( f(z) \) is divergent. If we consider the lens mapping from the point \( z = \delta z + \Omega_{mn} \) where \( \Omega_{mn} \) is one of the lens locations, then the lens equation (28) reduces to

\[
\eta = \delta z + \Omega_{mn} - f(\delta z + \Omega_{mn}) = \delta z + (1 - \kappa_*)\Omega_{mn} - f(\delta z) \quad (34)
\]

where \( \delta z = \overline{\delta z} \). Here, we use the quasiperiodic relation of \( f(z) \) [which is inherited from that of \( \zeta(z) \)] and some additional properties of the Weierstrass zeta function. Note also that \( \kappa_* = \pi/(2\omega^2) \), \( \kappa_* = \pi/(2\sqrt{3}\omega^2) \), and \( \kappa_* = \pi/(3\sqrt{3}\omega^2) \) for square, triangular, and honeycomb lattices with side length of \( \omega \). Assuming \( |\delta z| \ll \omega \), equation (34) can be solved approximately from

\[
(1 - \kappa_*)\Omega_{mn} - \eta = f(\delta z) - \delta z \approx \frac{1}{\delta z} - \delta z + O
\]

where the remainder term \( O \) is the order of \( (\delta z)^j \) with \( j = 3 \) (square), \( j = 5 \) (triangle), or \( j = 2 \) (hexagon). Provided that \( |\mathcal{Z}| \gg \omega^{-1} \) where \( \mathcal{Z} = |z| e^{i\phi_L} = (1 - \kappa_*)\Omega_{mn} - \eta \), we have the image at \( z = \Omega_{mn} + \delta z \) where

\[
\delta z = \overline{\delta z} \left(1 - \frac{1}{|\mathcal{Z}|^2}\right) + O\left(\frac{1}{|\mathcal{Z}|^3}\right) \quad (36)
\]

and its signed magnification is found to be

\[
\mu = \frac{1}{|\mathcal{Z}|} \left(1 - \frac{1}{|f'|^2}\right) = \frac{1}{|\mathcal{Z}|} - \frac{4}{|\mathcal{Z}|^3} + O\left(\frac{1}{|\mathcal{Z}|^5}\right) \quad (37)
\]

Since \( \Omega_{mn} - \hat{\eta} = \mathcal{Z}/(1 - \kappa_*) \) where \( \hat{\eta} = \eta/(1 - \kappa_* \) with \( \kappa_* \neq 1 \), the condition that \( |\mathcal{Z}| \gg L_0 \) is sufficiently large translates to the lens positions (\( \Omega_{mn} \)) that lie outside the circular region of radius
and caustics of the Chang-Refsdal lens (a single point-mass lens) with the \( n_{\text{f}} \) shear (solid lines). Also plotted in dashed lines are the critical curves.

\[ L_0/(1 - \kappa_s) \text{ centred at } \eta. \]

Then, the images near all of those lens positions are simply found by the approximation (eq. [36]). More importantly, the contribution to the total magnification from all of those images is approximately

\[
\sum_{m,n} |\mu| \approx \sum_{m,n} \frac{1}{|Z|^2} \int_{L_0/(1 - \kappa_s)}^{\infty} \frac{2\pi\sigma R dR}{|Z|^2} = \frac{2\kappa_s R dR}{R^2(1 - \kappa_s)^2} = \frac{\kappa_s}{L_0(1 - \kappa_s)^2}
\]

where the summation is over the lens positions \((m,n)\) such that \(|\Omega_{mn} - \eta| > L_0/(1 - \kappa_s)\) and \(\sigma = \kappa_s/\pi\) is the number density of the lenses. Here the polar coordinate centred at \(\eta\) is used for the integral and so \(R = |\Omega_{mn} - \eta|\). The result indicates that, unless \(\kappa_s = 1\), the contribution from infinitely many negative parity images to the total magnification is finite (note that the integrand is convex so that the sum is bounded by the integral) and with sufficiently large \(L_0\), they are negligible.

In Figs. 11-14, we show the image locations for some specific cases of lattice lenses. Here, all positive parity images are located through the iteration scheme outlined earlier. Most of negative parity images are found via the Newton-Raphson method (eq. [33]) with the starting points given by a grid of positions near \(z = \eta/(1 - \kappa_s)\).

5 EXTERNAL SHEAR

Next, we consider the effect of the large-scale potential. It is a usual practice that the external potential \(\psi_{\text{ext}}\) in equation (2) is approximated by a quadratic function and consequently the corresponding deflection by a linear function (i.e., tensor) – that is, keeping only the linear terms in its Taylor-series expansion (Young 1981; Nityananda & Ostriker 1984; Subramanian et al. 1985). The resulting linear function is symmetric, thanks to the mixed coordinate differential being commutative. It is also customary that the linear function is decomposed into a scalar dilation and a traceless tensor. The former, usually referred to as a convergence, is directly related to the surface mass density (which is not in the form of point masses) whereas the latter, known as an external shear, physically models the tidal effects from the external mass distribution.

In complex number notation, then the lens equation generalizes to

\[
\eta = (1 - \kappa_c)z - \gamma_\epsilon \bar{\gamma} - f(z); \quad \mathcal{J} = (1 - \kappa_c)^2 - \left|f'(z) - \bar{\gamma} \right|^2
\]

where \(\kappa_c\), which must be positive real, is the convergence due to the local surface mass density that is the source of the large scale potential (i.e., ‘continuous mass density’) and \(\gamma_\epsilon\), which can be any complex number, is the shear that models the tidal effect. The effect of the convergence term is uniform focusing, which basically results in a scaling difference between the image and source planes. It is usually incorporated into the analysis by the redefinition of the variables (Paczyński 1986). For example, with the rescaled position variables for the image plane \(w = |e|^{1/2}z\) and the source plane \(\xi = \text{sgn}(\epsilon)|e|^{1/2}\eta\) where \(\epsilon = 1 - \kappa_c\), equation (39) for the square lattice with side length of \(\omega\) (i.e., \(f(z) = \zeta(z; g_2, \omega_0) = \zeta(\omega_\epsilon, \omega_0)\)) reduces to

\[
\mathcal{J} = w - \mathcal{J} - \text{sgn}(\epsilon) \bar{\zeta}(w) \left| |e|^{1/2} \omega_\epsilon i |e|^{1/2} \omega_0 \right| - \text{sgn}(e)\xi^2
\]

where \(\zeta = \gamma_\epsilon/\epsilon\) is the reduced shear. We note that the deflection function is basically given by that for the square lattice with side length of \(|e|^{1/2} \omega\) (i.e., corresponding to the reduced optical depth \(\kappa = \kappa_s/\epsilon\)). In general, if \(\epsilon > 0\) (i.e., \(0 < \kappa_s < 1\)), the qualitative lensing behaviour is basically same as the case that \(\kappa_s = 0\) except the additional scaling difference between the image and the source planes, which leads to the use of the ‘reduced’ values of the external shear and the optical depth (Paczyński 1986; Granot et al. 2003; An \& Evans 2006). For the following discussion, we only consider the lens equation (39) for the case \(\kappa_s = 0\).

The critical curves and the caustic networks of the lattice lens with external shear can be found in the same way as the case without external shear. However, the shear introduces two further degrees of freedom in the parameter space to be explored, which makes the task quite extensive and difficult to generalize. Nevertheless, we can draw a rough generalization such that the patterns reduce to the case of the Chang-Refsdal lens (Chang & Refsdal 1979, 1984; An \& Evans 2006) if \(|\gamma_\epsilon| \gg \kappa_s\) and to the simple lattices with no (or negligible) shear if \(|\gamma_\epsilon| \ll \kappa_s\). Figs. 15-17 show some examples of the critical curves and the caustic networks for the square lattices with moderate values of \(|\gamma_\epsilon|\) and \(\kappa_s\) to illustrate the effects of varying \(\kappa_s\) at fixed \(\gamma_\epsilon\) or varying \(\gamma_\epsilon\) (both the magnitude and the orientation) at fixed \(\kappa_s\). The situation becomes quite complicated if \(|\gamma_\epsilon| \approx \kappa_s \approx 1\). As in the case of lattices with zero shear, with \(\kappa_s = 1\), there are overlaps among neighboring caustics. Furthermore, with nonzero shear, neighboring caustics can be connected with one another, and the resulting dense networks of caustics

6 Note that, as \(\kappa_s \to 1\), the radius of the region within which the approximation is not valid also increases.

7 The sum \(\sum_{|\delta r| > \tau} |\delta r|^2\) where the summation is over the lattice points except the points at which \(\delta r = 0\), is ‘absolutely convergent’ for \(\tau > 2\). For the rigorous proof, see the mathematical references on elliptic functions listed in Appendix.

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metrics are even more complicated to analyze properly. More detailed analysis of these will be attempted in a subsequent work.

Instead, here we speculate on some possible connections of lattice lensing to more realistic lensing scenarios (particularly, that of the random star field). One of the most significant differences of lattice lensing from that by a random field is the fact that the shear experienced at any given lens position due to the remaining lenses cancels out to be null. Adding external shear only alters the situation minimally, that is, the total shear at any lens position is shifted to the value of external shear but every lens still experiences the identical shear. By contrast, in the random star field, the shear experienced by a given star is distributed according to the same distribution as that of the shear experienced by the random lines of sight. However, we note that the latter distribution has been calculated (Nityananda & Ostriker 1984; Schneider 1987; Granot et al. 2003) using the straightforward application of the characteristic function in the probability theory and the random-walk process. If lattice lensing were to model the effect of the shear ‘variations’ in the random field reasonably well, each individual lens in the random field might be approximated as a lattice lens with the ‘external shear’ being equal to the actual total shear experienced by the given lens. Then, the lensing behaviour of the whole of the random star field could be understood as the sort of ‘convolution’ of the lattice lens (of the corresponding optical depth) with the external shear distributed as the distribution of the random shear in the random star field. If this indeed be the case, one could study more realistic scenarios of the microlensing effects due to the stars in the lensing galaxy in an alternative way to the traditional inverse ray tracing approach.

At this point, however, this remains a speculative possibility. As it has been noted widely, the effects of many lenses combine in a highly nonlinear fashion and the role of the individual lenses even in the simplest phenomenology is nontrivial (see e.g., Gil-Merino & Lewis 2005). While one could expect that the approach of decomposing the star field into lattices outlined in the previous paragraph might fare better than the simplistic approach of seeing the star field as the combination of individual stars (since it takes care of some of the nonlinear effects), no compelling argument can yet be made that this is the case.

6 CONCLUSION

Microlensing at high optical depth is not only of a theoretical interest but also of an importance in understanding a number of lensed systems (e.g., Keeton et al. 2006; Morgan et al. 2006). However, at present time, the only way to get any detailed quantitative grip on such problem is via Monte-Carlo simulations of each individual point corresponding specific values of the optical depth and the external shear (e.g., Granot et al. 2003; Schechter et al. 2004), which is rather expensive in its requirement for resources. We argue that a grid of point masses discussed in this paper, while artificial and abstract so that it may appear to be entirely divorced from any physically realistic problem, can provide us with some insight, albeit indirect, into what is at work in microlensing at high optical depth, in particular for the numbers and magnifications of microimages. Together with with various analytical methods and probabilistic approaches to the problem, this help us build pictures of phenomena complement to more traditional routes with comparatively inexpensive means.

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Elliptic functions are the class of complex-valued functions of a complex variable that are meromorphic (i.e., analytic everywhere in the complex plane except isolated poles) and biperiodic (or " doubly-periodic"). The biperiodicity means that there exist two complex numbers \( \omega_1 \) and \( \omega_2 \) such that \( \omega_1/\omega_2 \) is not real and \( f(z + 2\omega_1 + 2n\omega_2) = f(z) \) for all pairs of integers \( m \) and \( n \) and any complex number \( z \). Here, the two complex numbers \( \omega_1 \) and \( \omega_2 \) are referred to as the half-periods of the elliptic function if they are a pair of two ‘smallest’ such numbers.\(^{10}\) We note that the biperiodicity further implies that, once the behaviour of the function is known in the domain that is the parallelogram ‘cell’ whose vertices are 0, \( 2\omega_1 \), \( 2\omega_2 \), and \( 2\omega_1 + 2\omega_2 \) (or equivalently \( \omega_1 - \omega_2 \), \( \omega_1 + \omega_2 \), \( -\omega_1 + \omega_2 \), \( -\omega_1 - \omega_2 \)), it is completely specified everywhere in the complex plane. The cell is sometime referred to as the fundamental period parallelogram (FPP) of the elliptic function. According to the general theory of elliptic functions, any elliptic function can be expressed using a set of standard functions, which are usually chosen to be one of two classes: Jacobi functions (\( \sin x, \cos x, \sinh x \) and \( \cosh x \)) or the Weierstrass \( \wp \)-function and its derivative.

The Weierstrass \( \wp \)-function (or the ‘Weierstrass elliptic function’) is the elliptic function that has one second-order pole in its FPP. The standard definition locates a pole at \( z = 0 \) whose principal part is exactly \( z^{-2} \) and constant part is nil (i.e., \( \lim_{z \to 0} [\wp(z) - z^{-2}] = 0 \)). The formal definition is given by

\[
\wp(z; \omega_1, \omega_2) \equiv \frac{1}{z^2} + \sum_{m \neq 0, n \neq 0} \left[ \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{\omega_1^2} \right]
\]

where \( \Omega_{\text{mon}} = 2\omega_1 + 2n\omega_2 \), and the summation is over all integer grid points of \( (m,n) \) except \( (m,n) = (0,0) \). Here, the set of all complex numbers in the form of \( \Omega_{\text{mon}} \) is usually referred to as a lattice. Note that the function \( \wp(z) \) has second-order poles at every point of the lattice. In addition, \( \wp(z) \) is an even function, i.e., \( \wp(-z) = \wp(z) \). Note that, for a given lattice, the choice of the pair of the half-periods is not unique – that is to say, the lattice forms a module over the ring of integers and the set of two periods (i.e., \( (2\omega_1, 2\omega_2) \)) is the basis of this module. For this reason, the two secondary arguments of \( \wp(z) \) are usually replaced by the elliptic invariants defined by \( g_2 = 60 \sum_{n \neq 0} \Omega_{\text{mon}}^{-4} \) and \( g_3 = 140 \sum_{n \neq 0} \Omega_{\text{mon}}^6 \) such that \( \wp(z; g_2, g_3) = \wp(z; \omega_1, \omega_2) \). Unlike a pair of the half-periods, the elliptic invariants are in fact uniquely specified by the given lattice, and vice versa.

The elliptic invariants are directly related to the coefficients of the Laurent-series expansion of \( \wp \)-function at origin via simple algebraic relations, that is, the coefficients are given by the polynomials of \( g_2 \) and \( g_3 \). Particularly, we have \( \wp(z; g_2, g_3) \approx z^{-2} + (g_2/20)z^2 + (g_2/28)z^4 + O(z^6) \). This series expansion further indicates that \( \{\wp(z; g_2, g_3)\}^2 - \{4\wp(z; g_2, g_3)^3 - g_3\wp(z; g_2, g_3) + g_2/3\} \). This combined with the results from the general theory of elliptic functions implies that \( y(z) = \wp(z; g_2, g_3) \) is a particular solution of the differential equation given by \( (y')^2 = 4y^3 - gy - 1 \). Since this is a first-order differential equation that does not involve the independent variable, its general solution is given by \( y(z) = \wp(z + C; g_2, g_3) \) where \( C \) is an integration constant (which is an arbitrary complex constant). Note that, if the polynomial \( 4y^3 - gy - 1 \) can be factored with a perfect square factor (so that \( g_2/3 = 27g_3^2 \)), the solution of the differential equation then can be expressed in terms of ‘elementary’ functions such as trigonometric or

8 http://mathworld.wolfram.com/
9 http://functions.wolfram.com/
The derivation of \( \wp \)-function

\[
\wp'(z; g_2, g_3) = \wp'(\zeta(z_0, \zeta_0)) \equiv \frac{d}{dz} \wp(\zeta(z_0, \zeta_0)) = \sum_{m, n \neq (0, 0)} \frac{2}{z - \zeta_m - \zeta_n}
\]

is also an elliptic function with the same half-periods as \( \wp(\zeta(z_0, \zeta_0)) \) but it has third-order poles at the lattice points and is an odd function. By contrast, the antiderivative (or the indefinite integral) of \( \wp \)-function cannot be an elliptic function. According to the general theory of elliptic functions, the sum of complex residues in FPP of any elliptic function must vanish. However, if the antiderivative of \( \wp \)-function were an elliptic function, it would have a single simple pole in its FPP and thus the sum of its residues could not vanish. In fact, the Weierstrass \( \zeta \)-function\(^1\), which is defined to be a particular antiderivative of \( \wp \)-function;

\[
\zeta(z; g_2, g_3) = \zeta(z_0, \zeta_0) \equiv \frac{1}{2} \sum_{m, n \neq (0, 0)} \int_0^z \left( \frac{1}{w^2} - \wp(w, \zeta_0) \right) dw
\]

is a quasiperiodic function such that \( \zeta(z + 2m\zeta_0 + 2n\omega_0) = \zeta(z) + 2m\zeta(\omega_0) + 2n\zeta(\omega_0) \) for all pairs of integers \( m \) and \( n \) and any complex number \( z \) with \( \omega_0 \) and \( \omega_0 \) being the half-periods of the corresponding \( \wp \)-function (secondary arguments suppressed for brevity). We also note that the Weierstrass \( \zeta \)-function is an odd function.

In addition, the theory of the Weierstrass functions also introduces another auxiliary function referred to as the Weierstrass \( \sigma \)-function;

\[
\sigma(z; g_2, g_3) = \sigma(z_0, \zeta_0) \equiv \prod_{m, n \neq (0, 0)} \left( 1 - \frac{z}{\zeta_m + \zeta_n} + \frac{z^2}{6\zeta_m \zeta_n} \right)
\]

where the product is again over all integer pairs \( (m, n) \) except \( (m, n) = (0, 0) \). From this definition, it is straightforward to show that \( \zeta(z) \) is the logarithmic-derivative of \( \sigma(z) \), that is,

\[
\frac{\sigma'(z)}{\sigma(z)} = \frac{d}{dz} \ln \sigma(z; g_2, g_3) = \zeta(z; g_2, g_3)
\]

We note that \( \sigma(z) \) is also a quasiperiodic function although the relation involves the successive ratios between the functional values at the points separated by the ‘period’, rather than the differences as in \( \zeta(z) \);

\[
\frac{\sigma(z + 2m\zeta_0 + 2n\omega_0)}{\sigma(z)} = (-1)^p \exp \left[ 2m\zeta(\omega_0) + n\zeta(\omega_0) \right] \frac{z + m\omega_0 + n\omega_0}{z}
\]

where \( p = m + n + mn \) and \( m \) and \( n \) are any pairs of integers. The function \( \sigma(z) \) is in fact an entire function, that is, it is analytic everywhere in the complex plane without any pole or singularity (except the essential singularity at infinity). In particular, it has zeros at the origin and the lattice points, that is, \( \sigma(0) = \sigma(\Omega_{mn}) = 0 \), which is easily verifiable from its definition.

We also note that the set of Weierstrass functions satisfy the homogeneity relations, which can be easily derived from their respective definitions in terms of the infinite sum or product over the lattice points. In particular,

\[
\begin{align*}
\sigma(az; a\omega_0, a\omega_0) &= \sigma(\zeta(az_0, \zeta_0)); \\
\sigma(az^2; a^2g_2, a^6g_3) &= \sigma(\zeta(z_0, \zeta_0)); \\
\zeta(az\omega_0, a\omega_0) &= -\zeta(\zeta_0, \zeta_0); \\
\zeta(az^2; a^2g_2, a^6g_3) &= \zeta(\zeta(z_0, \zeta_0)); \\
\wp(az\omega_0, a\omega_0) &= \zeta(\zeta(z_0, \zeta_0)); \\
\wp'(az^2; a^2g_2, a^6g_3) &= \zeta(\zeta(z_0, \zeta_0)); \\
g_2(az\omega_0, a\omega_0) &= a^4 g_2(z_0, \omega_0); \\
g_3(az\omega_0, a\omega_0) &= a^6 g_3(z_0, \omega_0),
\end{align*}
\]

where \( a \neq 0 \) is a complex constant.

### A1 The lemniscatic case

If \( F \) is a square (i.e., \( \omega_0 = i\omega_0 \)), then \( \Omega_{mn} = 2m\omega_0 + 2n\omega_0 = 2(m + n)\omega_0 \) and, from the symmetry, we have \( \sum_{m, n} \zeta_m = (2\omega_0)^{-1} \sum_{m, n} (m + n)^{-1} = 0 \), and so that \( g_3 = 0 \). Then, after \( g_2 \) is reduced to the unity using the homogeneity relation, the behaviour of the corresponding Weierstrass elliptic function can be studied through that of \( \wp(z; 1, 0) \), which is called the lemniscatic case. The corresponding half-periods of \( \wp(z; 1, 0) = \wp(z; \zeta_0, i\zeta_0) \) are given by \( \omega_{0b} = T^2/(4\sqrt{7}) \approx 1.854 \) and \( \omega_{0a} \) (The constant \( \sqrt{7} \approx 2.622 \) is sometimes known as the lemniscate constant). Some basic results known for the lemniscatic case are

\[
\begin{align*}
\wp(0; 1, 0) &= \frac{1}{2}; \\
\wp(\zeta_0; 1, 0) &= \frac{\pi}{4\omega_0}; \\
\zeta(0; 1, 0) &= \frac{1}{4\omega_0}; \\
\zeta(\zeta_0; 1, 0) &= \frac{1}{4\omega_0}; \\
\wp(\omega_{0b}; 1, 0) &= \frac{\pi i}{4\omega_{0b}}.
\end{align*}
\]

Here, \( \omega_0 = (1 + i)\omega_{0b} = \sqrt{2} e^{\pi i/4} \omega_{0a} \) is in fact the only zero of \( \wp(z; 1, 0) \) within its FPP (which is actually a degenerate second-order zero). Note that these zeros are located at the centres of the square cells defined by the poles. We further note that the behaviours of these functions within FPP exhibit high degrees of symmetry and their study can be reduced to the one-eighth of FPP, that is, the isosceles-right-triangle region defined by \( z = 0 \) (the pole), \( \omega_{0b} \) and \( \zeta_0 \) (the zero).

Although the standard reference on the numerical calculation, Press et al. (1992), does not list the routine for the Weierstrass functions, their numerical evaluation is available in commercial softwares such as Mathematica\(^6\) or Maple\(^7\) or subroutine libraries such as IMSL\(^12\). However, for most purposes, they can be evaluated in a pretty straightforward manner without using any black-box routine (see e.g., Abramowitz & Stegun 1972). While there exist more involved algorithms that are valid for an arbitrary pair of complex numbers \((g_2, g_3)\), the lemniscatic case of the Weierstrass function can be easily evaluated through its Laurent-series expansion at \( z = 0 \) or Taylor-series expansion at \( z = \omega_{0b} \) or \( z = \zeta_0 \) (after the reduction of the domain using biperiodicity and additional symmetry properties). The relevant series expansions up to the first several terms are available in Abramowitz & Stegun (1972). In particular, the Laurent series at \( z = 0 \) is given in the form of \( \wp(z; 1, 0) = \frac{z^2}{2} \sum_{n = 0}^{\infty} \left( a_{2n}/20^n \right) e^{2ik} \) (the first few coefficients \( a_{2n}'s \) of which are given in Table A1) so that it converges

\[ \text{http://www.vni.com/products/imsl/} \]

---

\(^1\) The Weierstrass \( \zeta \)-function should not be confused with several other mathematical functions conventionally denoted by the same symbols, the most important among which is the Riemann \( \zeta \)-function and its generalization, the Hurwitz \( \zeta \)-function.

\(^2\) http://www.vni.com/products/imsl/
relatively quickly for $|z| < 1$ – the radius of the convergence of the series is in fact $2\omega_0$. Note that the coefficients for the Laurent series at $z = 0$ can in fact be derived using a total recurrence relation.

Since the lemniscatic case of $\varphi$-function satisfies the differential equation $(y')^2 = 4y^3 - y$, its inverse function is expressible through elliptic integrals. However, the relevant integral in fact more easily reduces to an incomplete beta function

$$z = \varphi^{-1}(\varphi; 1, 0) = -\int_0^\infty \frac{dy}{\sqrt{4y^3 - y}} = \frac{1}{2^{3/2}} B_1(4ip_1, 1) \left( \frac{1}{4} \right),$$

or a hypergeometric function,

$$z = \varphi^{-1}(\varphi; 1, 0) = \frac{1}{\sqrt{\pi}} 2F_1 \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}, 4i\sigma^2 \right),$$

where $B_1(a, b)$ is the incomplete Beta function and $2F_1(a, b; c; x)$ is the Gaussian $(2, 1)$-hypergeometric function. [c.f., $B_1(a, b) = (\Gamma(a)\Gamma(b)/\Gamma(a+b))F_1(1-b, a; 1; x)$.] Here, the last expression may be used for a convergent hypergeometric power-series expansion for the inverse $\varphi$-function for $|\varphi| \gg 2^{-1}$. We note that, because we are interested in the case when $\varphi$ takes an arbitrary complex value, there is no particular advantage using the Legendre/Jacobi elliptic integrals or the incomplete Beta function, numerical or otherwise, compared to the hypergeometric series expression given above. Alternative convergent hypergeometric series for $|\varphi| \ll 2^{-1}$ also exist;

$$z = \varphi^{-1}(\varphi; 1, 0) = \zeta_0 + \frac{2ip_1^{1/2}}{2 F_1 \left( \frac{1}{2}, \frac{1}{4}, \frac{5}{4}, 4i\sigma^2 \right)},$$

where $\zeta_0 = \sqrt[3]{2\pi e^{-ni/2}\omega_0}$ is again the zero of $\varphi(z; 0, 1)$. The inverse function can be effectively evaluated almost everywhere using these hypergeometric series together with some further refinement methods such as the (complex) Newton-Raphson algorithm if necessary.

### A2 equiharmonic case

We note that FPP can be considered as a tile, the set of which covers the whole (complex) plane and the locations of the poles are vertices of individual tiles. Then, the preceding lemniscatic case corresponds to the tile given by a square (i.e. a regular tetra-gon), which is one of the three possible regular tessellations of the plane. The elliptic function with FPP corresponding to the tile pattern of the remaining two regular tessellations can also be studied similarly. The half-periods for the more basic of the two, that is, the tiles given by equilateral triangles, are related to each other via

$$\omega_1 = e^{ni/3}\omega_2$$

and then from the symmetry of the system, it follows that

$$g_2 = 60\sum_{\alpha_0} \Omega_{\alpha_0} = 0$$

(note that, while the choice of the two half-periods is not unique, the elliptic invariants are fixed for the given arrangement of poles). As before, the remaining invariant $g_3$ can be reduced to the unity using the homogeneity relation, and the resulting Weierstrass elliptic function, $\varphi(z; 0, 1)$ is referred to as the equiharmonic case. It is known that the real half-period of

$$\varphi(z; 0, 1) = \varphi(z|\omega_2, e^{ni/3}\omega_2)$$

is $\omega_2 = \Gamma_{1/3}/(4\pi) \approx 1.530$. Additionally, it can be derived that

$$\varphi(e^{-ni/3}\omega_2; 0, 1) = 2^{-2/3} e^{2\pi i/3}; \quad \zeta(e^{-ni/3}\omega_2; 0, 1) = \frac{\pi e^{ni/3}}{2 \sqrt[3]{\omega_2}},$$

$$\varphi(2\omega_2; 0, 1) = 2^{-2/3}; \quad \zeta(2\omega_2; 0, 1) = \frac{2}{9 \pi \omega_2},$$

$$\varphi(2\omega_2; 0, 1) = 2^{-2/3}; \quad \zeta(2\omega_2; 0, 1) = \frac{\pi e^{ni/3}}{2 \sqrt[3]{\omega_2}}.$$

The zeros of $\varphi(z; 0, 1)$ are found at the centres of each equilateral triangular cells so that there are actually two zeros within its FPP, and unlike the lemniscatic case, each zero is simple (i.e., the first-order zero). The particular zero within the cell defined by $z = 0$, $z = 2\omega_2$, and $z = 2e^{ni/3}\omega_2$ is located at $z_0 = (2/\sqrt[3]{}\pi \omega_2)$ with

$$\zeta(z_0; 0, 1) = \frac{\pi e^{ni/3}}{3(\omega_2)}.$$

Like the lemniscatic case, the equiharmonic case of the Weierstrass function can also be readily evaluated via the power-series expansions at $z = 0$, $z = \omega_2$, or $z = z_0$. Their coefficients are again found in the references such as Abramowitz & Stegun (1972). In fact, the Laurent-series expansion at $z = 0$ of the equiharmonic case (whose radius of convergence is $2\omega_2$) is in the form of

$$\varphi(z; 0, 1) = z^2(1 + \sum_{|n|} b_n(z/28)^{-2|n|})$$

which converges even faster than that of the lemniscatic case for $|z| < 1$. The first few coefficient $b_n$’s are again given in Table A1.

Analogue to the lemniscatic case, the inverse of $\varphi$-function can be obtained through the integration of the differential equation $(y')^2 = 4y^3 - y$. The resulting expression can again be written down in terms of elliptic integrals (which is rather complicated, incomplete beta functions, or hypergeometric functions,

$$z = \varphi^{-1}(\varphi; 0, 1) = -\int_0^\infty \frac{dy}{\sqrt{4y^3 - 1}} = \frac{1}{2\pi 2} B_1(4ip_1, 1) \left( \frac{1}{4} \right).$$

Similarly, the last expression can be used for the convergent hypergeometric power-series expansion of the inverse $\varphi$-function for $|\varphi| \gg 2^{-2/3}$. For the case when $|\varphi| \ll 2^{-2/3}$, an alternative expression involving a hypergeometric function can be derived as before;

$$z = \varphi^{-1}(\varphi; 0, 1) = \zeta_0 + i2 F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, 4i\sigma^2 \right).$$

Here, $\zeta_0 = (2/\sqrt[3]{\pi \omega_2}$ is again the zero of $\varphi(z; 0, 1)$. The sign in front of the hypergeometric function is actually dependent upon the choice of $\zeta_0$. For example, with an alternative zero $\zeta_0 = (2/\sqrt[3]{\pi \omega_2}$, it should be positive. Again, these hypergeometric series supplemented by the Newton-Raphson algorithm provide an efficient way to evaluate the inverse $\varphi$-function for the equiharmonic case.

---

**Table A1. Laurent-Series Coefficients of $\varphi(z)$ at $z = 0$.**

| $k$ | $a_{k+1}/a_k$ | $b_{k+1}/b_k$ |
|-----|---------------|---------------|
| 1   | 1/3           | 1/13          |
| 2   | 2/13          | 1/19          |
| 3   | 5/(2 - 17) = 5/34 | 3/(5 - 13) = 3/65 |
| 4   | 2/(3 - 5) = 2/15 | 2²/(3 - 31) = 4/93 |
| 5   | 5/(3 - 13) = 5/39 | (5 - 7 - 43)/(2² - 13 - 19 - 37) = 1505/36556 |
| 6   | (2 - 3⁵)/(5 - 29) = 18/145 | (2 - 3 - 431)/(5 - 7 - 43³) = 2586/64715 |

*For both cases, $a_1 = b_1 = 1.*
A2.1 hexagonal grid

Let us consider an elliptic function,

\[ b(z; g_3) = \wp(z; 0, g_3) - \wp(z; 0, -\frac{g_3}{27}). \]  

(A1)

Note that every pole of \( \wp(z; 0, -g_3/27) \) coincides with that of \( \wp(z; 0, g_3) \) and they cancel each other leaving a fourth-order zero at the location. On the other hand, the remaining poles of \( \wp(z; 0, g_3) \) are located at the vertices of regular hexagonal tiles. Hence, equation (A1) corresponds to the remaining case of the elliptic function with their (second-order) poles forming vertex points of the regular tessellation of the space.

While it is a straightforward calculation, equation (A1) is unsuitable for the accurate numerical evaluation near \( z = 0 \) (and also near the other zeros of the functions) because of the canceling of two divergent terms leading to the zero. Furthermore, the inversion of equation (A1) is also rather nontrivial. This difficulty can be overcome as follows. We first note that

\[
[q'(z)]^2 = [q(z)]^2 \left( 1 - \frac{2g_3}{27[q(z)]^3} \right)^2 = \left( 4[q(z)]^3 + \frac{g_3}{27} \right) \left( 1 - \frac{2g_3}{27[q(z)]} \right)^2 = 4[q(z)]^3 - g_3
\]

where \( q(z) = \wp(z) + (g_3/27)[wp(z)]^{-2} \) and \( \wp(z) = \wp(z; 0, -g_3/27) \). In other words, \( q(z) \) is the solution of the differential equation \( (y')^2 = 4y^3 - g_3 \), whose general solution is in the form of \( y = q(z + C; 0, g_3) \) where \( C \) is the (complex) integration constant. Since the principal part of the Laurent-series expansion of \( q(z) \) at \( z = 0 \) is \( z^{-2} \), it is in fact the case that \( q(z) = q(0, g_3) \), and consequently we have

\[
b(z; g_3) = \frac{g_3}{27} \left[ q(z; 0, -\frac{g_3}{27}) \right]^{-2} = \frac{1}{u^6} \left( \frac{g_3}{27} \right)^{1/6} \left[ q(\frac{g_3}{27})^{1/6}; 0, 1 \right]^{-2}
\]

(A2)

where \( u^6 = -1 \). Subsequently, we find that the inverse function of equation (A1) or equivalently that of equation (A2) is given by

\[
z = \frac{b^{1/4}}{(g_3/27)^{1/4}} F_1 \left( \frac{1}{2}, \frac{1}{6}, \frac{7}{6}; -\frac{b^{3/2}}{4(g_3/27)^{1/2}} \right)
\]

or

\[
z = \frac{\Gamma^{1/3}}{2\pi g_3^{1/6}} \frac{1}{b^{1/2}} F_1 \left( \frac{1}{2}, \frac{1}{3}; 4; \frac{4(g_3/27)^{1/2}}{b^{1/2}} \right)
\]

where some of signs are chosen so that the principal value for positive real values of \( b \) returns with being positive real (in particular, \( b \geq 0 \rightarrow z \in [0, 2\omega]) \).

On the other hand, if one wants to find the Taylor-series expansion of \( b(z; g_3) \) at \( z = 0 \), it may be easier to use equation (A1) at \( z = 0 \) directly;

\[
b(z; g_3) = \sum_{k=1}^{\infty} \left[ 1 - \left( -\frac{1}{27} \right)^k \right] \frac{g_3}{27} b_k z^{k-2}
\]

\[= \frac{g_3}{27} z^2 \left( 1 + \sum_{k=1}^{\infty} \left( -\frac{1}{27} \right)^k \sum_{p=0}^{k-2} b_{k+1} \left( \frac{g_3}{27} \right)^{k-2} \right)\]

where \( b_k \) is the Laurent-series coefficient of the equiharmonic case of \( \wp \)-function (see Table A1). In particular, \( b_1 = 1 \) which is actually used for the second equality of the preceding equation.

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