Unbalanced fractional elliptic problems with exponential nonlinearity in $\mathbb{R}^N$

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Abstract
This article deals with the existence of a solution of following $(p,q)$-fractional equation:

\[
\begin{cases}
(\Delta)_p^{s_1}u + (\Delta)_q^{s_2}u + V(x)(|u|^p-2u + |u|^q-2u) = K(x)\frac{f(u)}{|x|^\beta} \quad \text{in } \mathbb{R}^N,
\end{cases}
\]

where $1 < q < p$, $0 < s_2 < s_1 < 1$, $ps_1 = N$, $\beta \in [0,N)$, $V, K : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying certain conditions. We prove the existence of a nontrivial nonnegative solution for both the cases when $f$ possesses subcritical as well as critical growth conditions with respect to the exponential nonlinearity.

Key words: Nonlocal operators, fractional $(p,q)$-equation, singular exponential type nonlinearity.

Mathematics Subject Classification 2010: 35J35, 35J60, 35R11

1 Introduction

In this article, we are concerned with existence result for the following singular $(p,q)$-fractional equation

\[
(\mathcal{P}) \quad \begin{cases}
(\Delta)_p^{s_1}u + (\Delta)_q^{s_2}u + V(x)(|u|^p-2u + |u|^q-2u) = K(x)\frac{f(u)}{|x|^\beta} \quad \text{in } \mathbb{R}^N,
\end{cases}
\]

where $1 < q < p$, $0 < s_2 < s_1 < 1$, $ps_1 = N$, $\beta \in [0,N)$, $V, K : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying certain conditions. $(\Delta)_p^s$ is the fractional $p$-Laplace operator defined as

\[
(\Delta)_p^s u(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(y) - u(x))}{|x - y|^{N+ps}} \, dy.
\]

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Problems of the type (P) arise from a wide range of real-world applications such as optimization, phase transition, anomalous diffusion, image processing, soft thin films, conservation laws, water waves and many more, for a list of more bibliography and other details on this topic we refer to [7]. The main motivation to study problems with leading operator given in (P) comes when \( s_1 = s_2 = 1 \), that is the local case. Here the leading operator, known as \((p,q)\)-Laplacian, arises while studying the stationary solutions of general reaction-diffusion equation

\[ u_t = \nabla \cdot [A(u)\nabla u] + r(x, u), \tag{1.1} \]

where \( A(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2} \). The problem (1.1) has applications in biophysics, plasma physics and chemical reactions, where \( u \) corresponds to the concentration term, the first term on the right hand side represents diffusion with a diffusion coefficient \( A(u) \) and the second term is the reaction which relates to sources and loss processes. For more details, readers are referred to [14].

In the local case, that is when \( s_1 = s_2 = 1 \), problem (P) is motivated by the famous Moser-Trudinger inequality. This comes into the picture because of the fact that \( W^{1,N}(\mathbb{R}^N) \) is embedded into \( L^p(\mathbb{R}^N) \) for all \( N \leq p < \infty \) but not in \( L^\infty(\Omega) \), hence in this case the critical nonlinearity is considered to have exponential type growth condition. These kinds of problems were studied by several authors Adimurthi [11, 13, 14]. As far as problems with singular exponential nonlinearity is concerned, Adimurthi and Sandeep [2] proved that the embedding \( W^{1,N}_0(\Omega) \ni u \mapsto \frac{1}{|x|^\alpha}e^{\frac{\alpha}{\alpha N}u^{N/(N-1)}} \in L^1(\Omega) \) is compact if \( \frac{\alpha}{\alpha N} + \frac{\beta}{N} < 1 \) and is continuous if \( \frac{\alpha}{\alpha N} + \frac{\beta}{N} = 1 \). Using this result they studied problems having singular exponential type nonlinearity in a bounded domain. In the case of \( \mathbb{R}^N \), Adimuthi and Yang [3] considered the following singular problem

\[ (-\Delta_N)u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^\beta} + \epsilon h(x) \text{ in } \mathbb{R}^N, \]

where among other assumptions, \( f \) has exponential growth condition and \( h \) is in the dual of \( W^{1,N}(\mathbb{R}^N) \). Here authors established singular Moser-Trudinger type inequality for whole \( \mathbb{R}^N \) and obtained the existence result for a mountain pass solution when \( \epsilon > 0 \) is small. Subsequently, Yang [21] and Goyal and Sreenadh [9] studied similar singular problems in the whole of \( \mathbb{R}^N \). In the later work, authors proved the existence and multiplicity results using Nehari manifold method.

Regarding the problems involving operators with unbalanced growth conditions, we mention the recent work of Figueiredo and Nunes [10]. Using the method of Nehari manifold authors proved the existence of a solution for \((N,p)\) type equations in bounded domain. In [11], Fiscella and Pucci studied the following \((N,p)\) equation:

\[ -\Delta_p u - \Delta_N u + |u|^{p-2}u + |u|^{N-2}u = \lambda h(x)u_{-1}^{q-1} + \gamma f(x,u) \text{ in } \mathbb{R}^N, \]
where \(1 < q, p < N < \infty\), \(N \geq 2\), \(h(x) \geq 0\), \(\lambda, \gamma > 0\) are parameters and the function \(f\) has exponential type growth condition. In this work authors proved the existence of multiple solutions for small \(\lambda\) and large \(\gamma\).

In the nonlocal setting, we mention the work of Giacomoni et al. [12]. Here authors proved existence of multiple solutions using Nehari manifold for the 1/2-Laplacian problem in a bounded domain of \(\mathbb{R}\). Zhang [22] established Moser-Trudinger type inequality in fractional Sobolev-Slobodeckij spaces \(W^{s,p}(\mathbb{R}^N)\) and proved existence and multiplicity of solutions for the following fractional Laplacian equation
\[
(-\Delta)^s_p u + V(x)|u|^{p-2}u = f(x, u) + \epsilon h(x) \quad \text{in } \mathbb{R}^N,
\]
when \(\epsilon > 0\) is sufficiently small. Recently, Mingqi et al. in [16] and Xiang et al. in [20] studied fractional Kirchhoff problems with exponential nonlinearity in bounded domain and in \(\mathbb{R}^N\), respectively.

Problem of the type \((P)\) involving potential \(K\) and exponential type nonlinearity was studied by do Ó et al. [8] for the case \(N = 1\) and \(s = 1/2\). They considered,
\[
(-\Delta)^{1/2} u + u = K(x)g(u) \quad \text{in } \mathbb{R}.
\]
In this work, by assuming certain condition on \(K\), authors proved compactness results, which was lacking due to unboundedness of the domain, and obtained existence of a nontrivial nonnegative solution in the cases when \(g\) possesses subcritical or critical growth condition. Subsequently, this work was generalized by Miyagaki and Pucci [17] for Kirchhoff problem in 1-dimension.

Inspired by these works, we plan to obtain the existence of a nontrivial nonnegative solution of \((P)\) under the following assumptions on \(V, K : \mathbb{R}^N \to \mathbb{R}\).

(i) The functions \(V\) and \(K\) are continuous and positive in \(\mathbb{R}^N\).

(ii) \(K \in L^\infty(\mathbb{R}^N)\) and there exists a constant \(V_0 > 0\) such that \(V(x) \geq V_0 > 0\) for all \(x \in \mathbb{R}^N\).

(iii) If \(\{A_n\}\) is a sequence of measurable sets of \(\mathbb{R}^N\) with \(|A_n| \leq R\) for all \(n \in \mathbb{N}\) and some \(R > 0\), then
\[
\lim_{r \to \infty} \int_{A_n \cap B_r(0)} K(x)dx = 0 \quad \text{uniformly w.r.t. } n \in \mathbb{N}. \quad (1.2)
\]

To define the natural space which contains all the solutions of problem \((P)\), we first recall the notion of following spaces. For \(1 < p < \infty\) and \(0 < s < 1\), the fractional Sobolev space is defined as
\[
W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < +\infty\}
\]
endowed with the norm \( \|u\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p} \), where
\[
[u]_{s,p}^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}}
\]
dx dy.

Let \( \widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \) be the space defined as
\[
\widetilde{W}_V^{s_1,p}(\mathbb{R}^N) := \left\{ u \in W^{s_1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u(x)|^p dx < \infty \right\},
\]
which is a reflexive Banach space when endowed with the norm
\[
\|u\|_{s_1,p} = \left( [u]_{s_1,p}^p + \int_{\mathbb{R}^N} V(x)|u(x)|^p dx \right)^{1/p}
\]
and analogously we define \( \widetilde{W}_V^{s_2,q}(\mathbb{R}^N) \). From [7, 19], we get the following continuous embedding result
\[
\widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \hookrightarrow W^{s_1,p}(\mathbb{R}^N) \hookrightarrow L^m(\mathbb{R}^N), \text{ for all } m \geq p. \tag{1.3}
\]

Let \( X := \widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \cap \widetilde{W}_V^{s_1,p}(\mathbb{R}^N) \) endowed with the norm
\[
\|u\| := \|u\|_{s_1,p} + \|u\|_{s_2,q}.
\]

In order to deal with problem \((\mathcal{P})\), we prove the following singular version of Moser-Trudinger type inequality for fractional Sobolev spaces in whole \( \mathbb{R}^N \). For this we first obtain similar inequality for bounded domain much in the spirit of Adimurthi-Sandeep [2, Theorem 2.1]. Then, using Schwarz symmetrization technique we prove our theorem. For convenience, we denote
\[
\Phi_\alpha(t) = e^{\alpha|t|^\frac{N}{N-s}} - \sum_{0 \leq j < N/s - 1} \frac{\alpha^j}{j!} |t|^{j\cdot\frac{N}{N-s}}, \text{ for } t \in \mathbb{R}.
\]

We state the theorem as follows.

**Theorem 1.1.** Let \( N \geq 2, \ s \in (0,1) \) and \( p = N/s \). For all \( \alpha > 0, \ \beta \in [0,N) \) and \( u \in W^{s,p}(\mathbb{R}^N) \), the following holds
\[
\int_{\mathbb{R}^N} \Phi_\alpha(u) dx < \infty.
\]

Furthermore, for all \( \alpha < (1 - \beta/N) \alpha_{N,s} \) and \( \tau > 0 \),
\[
\sup \left\{ \int_{\mathbb{R}^N} \frac{\Phi_\alpha(u)}{|x|^{\beta}} dx : u \in W^{s,p}(\mathbb{R}^N), \|u\|_{s,p,\tau} \leq 1 \right\} < \infty,
\]
where \( \|u\|_{s,p,\tau} = (\|u\|_{s,p}^p + \tau \int_{\mathbb{R}^N} |u|^p)^{1/p} \) and \( \alpha_{N,s} > 0 \), is defined in section 2 (see Theorem 2.3).
Now we mention some assumptions on the function $f$. Here $f$ is said to have subcritical growth condition with respect to the exponential nonlinearity if it satisfies (f2) and the growth is critical if it satisfies (f2)'.

(f1) The function $f : \mathbb{R} \to \mathbb{R}^+$ is continuous with $f(t) = 0$ for all $t \leq 0$ and

$$
\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = 0.
$$

(f2) (Subcritical growth condition). For all $\alpha > 0$, the following holds

$$
\lim_{t \to \infty} \frac{f(t)}{\Phi_\alpha(t)} = 0.
$$

(f3) $t^{1-p}f(t)$ is nondecreasing in $\mathbb{R}^+$ and $\lim_{t \to \infty} t^{-p}F(t) = \infty$.

For the critical growth condition, we assume $f$ satisfies the following conditions in addition to (f1).

(f2)' (Critical growth). There exists $\alpha_0 > 0$ such that

$$
\lim_{t \to \infty} \frac{f(t)}{\Phi_\alpha(t)} = 0 \quad \forall \alpha > \alpha_0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{f(t)}{\Phi_\alpha(t)} = \infty \quad \forall \alpha < \alpha_0.
$$

(f3)' There exists $\delta > p$ such that $t^{1-p}f(t)$ is nondecreasing in $\mathbb{R}^+$ and $F(t) \geq C_\delta t^\delta$ for all $t \in \mathbb{R}^+$, for $C_\delta > 0$ sufficiently large.

(AR) (Ambrosetti-Rabinowitz condition). There exists $\nu > p$ such that $\nu F(t) \leq tf(t)$ for all $t \in \mathbb{R}^+$.

Due to unboundedness of the domain, Cerami sequences do not have the compactness property. We restore this compactness by exploiting the special property of the potential $K$, namely \textbf{(1.2)} (see Lemma 2.6). This helps us to prove the strong convergence of Cerami sequences and hence to obtain nontrivial solutions. The existence of such sequences are obtained by using mountain pass lemma. In the subcritical case, we do not assume Ambrosetti-Rabinowitz type condition on $f$, which makes little difficult to prove boundedness of Cerami sequences. Now, we state our main theorem as follows.

**Theorem 1.2.** There exists a nonnegative solution of problem $(\mathcal{P})$ in the following cases

(i) If (f1), (f2) and (f3) are satisfied.

(ii) If (f1), (f2)', (f3)' and (AR) are satisfied with $C_\delta$, appearing in (f3)', is sufficiently large.

We remark that, to the best of our knowledge, the results of above theorem is new even for the case $\beta = 0$. 
Notations: For notational convenience, we will use the following

\[ A_1(u, v) = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+p_s+1}} \, dx dy, \] for all \( u, v \in W^{s_1,p}(\mathbb{R}^N) \),

and analogously \( A_2 \) is defined in \( W^{s_2,q}(\mathbb{R}^N) \).

Definition 1.3. A function \( u \in X \) is said to be a solution of problem \((P)\), if for all \( v \in X \)

\[ A_1(u, v) + A_2(u, v) + \int_{\mathbb{R}^N} V(x)(|u|^{p-2} + |u|^{q-2})uv \, dx - \int_{\mathbb{R}^N} K(x)f(u)v \, dx = 0. \]

The Euler functional \( J : X \to \mathbb{R} \) associated to the problem \((P)\) is defined as

\[ J(u) = \frac{1}{p} ||u||_{s_1,p}^p + \int_{\mathbb{R}^N} V(x)|u|^p + \frac{1}{q} ||u||_{s_2,q}^q + \int_{\mathbb{R}^N} V(x)|u|^q - \int_{\mathbb{R}^N} \frac{K(x)F(u(x))}{|x|^\beta} \, dx, \]

where \( F(t) = \int_0^t f(\tau) d\tau \).

2 Some technical results

In this section, we first establish some compact embedding results for space \( X \). We have the following notion of weighted Lebesgue spaces

\[ L^p_V(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \to \mathbb{R} : ||u||_{p,V} = \int_{\mathbb{R}^N} V(x)|u(x)|^p \, dx < \infty \right\}, \]

which is a Banach space when equipped with the norm \( || \cdot ||_{p,V} \), for \( 0 < V \in C(\mathbb{R}^N) \).

Remark 2.1. (A) Due to the fact that \( 0 \leq \beta < N \), one can easily get that the embedding

\[ X \hookrightarrow L^m(\mathbb{R}^N; |x|^{-\beta}) \]

is continuous for all \( m \geq p \), that is, for all \( m \geq p \) there exists \( C_m > 0 \) such that for all \( u \in X \),

\[ \int_{\mathbb{R}^N} |u(x)|^m |x|^{-\beta} \, dx \leq C_m ||u||^m. \]

(B) Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain. Arguments similar to that of [3, Remark 2.1] gives us

\( X \) is compactly embedded into \( L^m(\Omega) \). Indeed, by [7, Theorem 7.1] we have \( W^{s_1,p}(\mathbb{R}^N) \) is compactly embedded into \( L^p(\Omega) \) and then using (1.3) and interpolation we can prove \( X \) is compactly embedded into \( L^m(\Omega) \) for all \( m \geq p \). An easy verification together with the aforementioned compact embedding yields \( X \) is compactly embedded into \( L^m(\Omega; |x|^{-\beta}) \) for all \( m \geq p \).

Proposition 2.2. The space \( X \) is compactly embedded into \( L^m_K(\mathbb{R}^N) \) for all \( m \in (p, \infty) \).

Proof. The proof given here is an adaptation of the proof of [17, Proposition 2.1] for \( N = 1 \). Here we provide only sketch of the proof. Let \( m > p \) and fix \( r > m \) and \( \epsilon > 0 \). Then, there
exists \( \tau_0 = \tau_0(\epsilon) \), \( \tau_1 = \tau_1(\epsilon) \) with \( 0 < \tau_0 < \tau_1 \), \( C = C(\epsilon) > 0 \) and \( C_0 \) depending only on \( K \), such that for all \( x \in \mathbb{R}^N \) and \( t \in \mathbb{R} \),

\[
K(x)|t|^m \leq cC_0(V(x)|t|^p + |t|^r) + CK(x)\chi_{[\tau_0,\tau_1]}(|t||t|^m). \tag{2.1}
\]

Let \( \{u_n\} \subset X \) be such that \( u_n \rightharpoonup u \) weakly in \( X \), for some \( u \in X \). Then, by the continuous embedding of \( X \) into \( L^r(\mathbb{R}^N) \), together with the fact that the sequence \( \{\|u_n\|\} \) is bounded, we get

\[
\|u_n\|^p_{p,V} \leq M \text{ and } \|u_n\|_\gamma^\gamma \leq M \text{ for all } n \in \mathbb{N} \text{ and } \gamma \in \{m, r\}.
\]

Therefore, \( Q(u_n) := C_0(\|u_n\|^p_{p,V} + \|u_n\|_r^r) \leq 2C_0M \) for all \( n \in \mathbb{N} \). Set

\[
A^n \sigma := \{x \in \mathbb{R}^N : \tau_0 \leq |u_n(x)| \leq \tau_1\}.
\]

Then, by the fact that \( \{u_n\} \) is bounded in \( L^m(\mathbb{R}^N) \), it is easy to observe that \( \{|A_n^\sigma|\} \) is bounded w.r.t. \( n \). Again, by \( \|u_n\| \leq \|u\| \), for \( \epsilon > 0 \), there exists \( r_\epsilon > 0 \) such that

\[
\int_{A^n \sigma \cap B_{r_\epsilon}(0)} K(x)dx < \frac{\epsilon}{C\tau_1^m}, \text{ for all } n \in \mathbb{N}.
\]

Using this together with the observation that \( Q(u_n) \) is bounded, \( 2.1 \) gives us

\[
\int_{B_{r_\epsilon}(0)} K(x)|u_n|^m \leq 2C_0M\epsilon + C\tau_1^m \int_{A^n \sigma \cap B_{r_\epsilon}(0)} K(x)dx < (2C_0M + 1)\epsilon, \text{ for all } n \in \mathbb{N}. \tag{2.2}
\]

Moreover, by compact embedding of the space \( X \) into \( L^\gamma(B_{r_\epsilon}(0)) \) for all \( \gamma \geq p \) (see Remark \ref{2.1}), we get

\[
\lim_{n \to \infty} \int_{B_{r_\epsilon}(0)} K(x)|u_n|^m = \int_{B_{r_\epsilon}(0)} K(x)|u|^m. \tag{2.3}
\]

Therefore, \( 2.2 \) and \( 2.3 \), implies

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)|u_n|^m = \int_{\mathbb{R}^N} K(x)|u|^m.
\]

We state the following Moser-Trudinger type inequality for fractional Sobolev spaces in case of bounded domain.

**Theorem 2.3.** \( \text{\cite{13}} \) Let \( \Omega \) be a bounded, open domain of \( \mathbb{R}^N \) \((N \geq 2)\) with Lipschitz boundary, and let \( s_1 \in (0, 1) \), \( ps_1 = N \). Let \( \widetilde{W}^{s_1,p}_0(\Omega) \) be the space defined as the completion of \( C_0^\infty(\Omega) \) with respect to \( \| \cdot \|_{W^{s_1,p}(\mathbb{R}^N)} \) norm. Then there exists \( \alpha_{N,s_1} > 0 \) such that

\[
\sup \left\{ \int_{\Omega} \exp \left( \frac{\alpha}{2} |u|^{N-s_1} \right) : u \in \widetilde{W}^{s_1,p}_0(\Omega), \|u\|_{ps_1} \leq 1 \right\} < \infty \text{ for } \alpha \in [0, \alpha_{N,s_1}).
\]
Moreover,
\[
\sup \left\{ \int_\Omega \exp \left( \alpha |u|^{\frac{N}{N-s_1}} \right) : u \in \dot{W}^{s_1,p}_0(\Omega), \|u\|_{p,s_1} \leq 1 \right\} = \infty \quad \text{for } \alpha \in (\alpha_{N,s_1}^*, \infty),
\]
where
\[
\alpha_{N,s_1}^* = N \left( \frac{2(N\omega_N)^2 \Gamma(p+1)}{N!} \sum_{k=0}^{\infty} \frac{(N+k-1)!}{k! (N+2k)^p} \right)^{\frac{s_1}{N-s_1}},
\]
with \(\omega_N\) as the volume of the \(N\)-dimensional unit ball.

Similar to the result of [2], we prove singular Moser-Trudinger inequality for fractional Sobolev spaces in bounded domains, which will help us to prove our Theorem 1.1.

**Lemma 2.4.** Let \(\Omega \subset \mathbb{R}^N (N \geq 2)\) be a bounded domain and let \(u \in W^{s_1,p}_0(\Omega)\). Then, for every \(\alpha > 0\) and \(\beta \in [0,N)\),
\[
\int_\Omega e^{\alpha |u|^{N/(N-s_1)}} \frac{1}{|x|^\beta} dx < \infty.
\]
Moreover, if \(\frac{\alpha}{\alpha_{N,s_1}} + \frac{\beta}{N} < 1\), then
\[
\sup_{\|u\|_{W^{s_1,p}(\Omega)} \leq 1} \int_\Omega e^{\alpha |u|^{N/(N-s_1)}} \frac{1}{|x|^\beta} dx < \infty, \tag{2.4}
\]

**Proof.** Let \(t > 1\) be such that \(\beta t < N\), then using Hölder inequality and Theorem 2.3, we deduce that
\[
\int_\Omega e^{\alpha |u|^{N/(N-s_1)}} \frac{1}{|x|^\beta} dx \leq \left( \int_\Omega e^{\alpha t' |u|^{N/(N-s_1)}} dx \right)^{\frac{1}{t'}} \left( \int_\Omega \frac{1}{|x|^\beta} dx \right)^{\frac{1}{t}} < \infty.
\]
For the second part of the theorem, first we observe that there exist \(\tilde{\alpha} < (\alpha, \alpha_{N,s_1})\) and \(t > 1\) such that \(\frac{\tilde{\alpha}}{\alpha} + \frac{\beta}{N} = 1\) (this can be done by first choosing \(\tilde{\alpha} < \alpha_{N,s_1}\) such that \(\frac{\tilde{\alpha}}{\alpha_{N,s_1}} + \frac{\beta}{N} < \frac{\alpha}{\alpha_{N,s_1}} + \frac{\beta}{N} < 1\)). Now by Hölder inequality, we have
\[
\sup_{\|u\|_{W^{s_1,p}(\Omega)} \leq 1} \int_\Omega e^{\tilde{\alpha} |u|^{N/(N-s_1)}} \frac{1}{|x|^\beta} dx \leq \sup_{\|u\|_{W^{s_1,p}(\Omega)} \leq 1} \left( \int_\Omega e^{\tilde{\alpha} |u|^{N/(N-s_1)}} dx \right)^{\frac{1}{t'}} \left( \int_\Omega \frac{1}{|x|^{N/t}} dx \right)^{\beta/t} \left( \int_\Omega \frac{1}{|x|^N} \right)^{\beta/N},
\]
since \(\tilde{\alpha} < \alpha_{N,s_1}\) and \(t > 1\), by Theorem 2.3 we get that the above quantity is finite. \(\Box\)

Before proving Theorem 1.1 we state the following radial lemma.

**Lemma 2.5.** Let \(N \geq 2\) and \(u \in L^p(\mathbb{R}^N), 1 \leq p < \infty\), be a radially symmetric non-increasing function. Then
\[
|u(x)| \leq |x|^{-N/p} \left( \frac{N}{\omega_{N-1}} \right)^{1/p} \|u\|_p, \text{ for } x \neq 0,
\]
where \(\omega_{N-1}\) is the \((N-1)\)-dimensional measure of \((N-1)\) sphere.
Proof of Theorem 1.1] Without loss of generality, we assume \( u \geq 0 \) and let \( u^* \) be the Schwarz symmetrization of \( u \). Then by (4) and (5), for any continuous and increasing function \( G : [0, \infty) \to [0, \infty) \), there holds
\[
\int_{\mathbb{R}^N} G(u^*(x))dx = \int_{\mathbb{R}^N} G(u(x))dx.
\]
Moreover, for all \( u \in W^{s,p}(\mathbb{R}^N) \) and \( 1 \leq m < \infty \), \( u^* \in W^{s,p}(\mathbb{R}^N) \) with
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{2N}}dxdy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u^*(x) - u^*(y)|^p}{|x-y|^{2N}}dxdy \quad \text{and} \quad \|u^*\|_p = \|u\|_p. \tag{2.5}
\]
Therefore, by Hardy-Littlewood inequality for symmetrization and the fact that \((1/|x|)\)' = \(1/|x|\), we get
\[
\int_{\mathbb{R}^N} \frac{\Phi_\alpha(u)}{|x|^\beta} \leq \int_{\mathbb{R}^N} \frac{\Phi_\alpha(u^*)}{|x|^\beta}. \tag{2.6}
\]
Fix \( R > 0 \) (to be specified later), then
\[
\int_{|x|>R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} = \int_{|x|>R} \frac{1}{|x|^\beta} \sum_{j=k_0}^{\infty} \frac{\alpha_j}{j!} u^*|j|p', \tag{2.7}
\]
where \( k_0 \) is the smallest integer such that \( k_0 \geq p-1 \) and \( p' = p/(p-1) \) is the Hölder conjugate of \( p \). Now we consider the following cases:

**Case (i):** For all \( j \geq k_0 > p-1 \).

Using Lemma 2.5 and (2.5), we obtain
\[
\int_{|x|>R} \frac{|u^*|^j|p'|}{|x|^\beta} \leq \int_{|x|>R} \frac{1}{|x|^\beta} \sum_{j=k_0}^{\infty} \frac{\alpha_j}{j!} u^*|j|p' \leq \left( \frac{N}{\omega_{N-1}} \right)^{\frac{1}{p-1}} \|u\|_p \|u^*\|_{\beta}^{\frac{p}{p-1}} R^{N-\frac{N}{p-1}(j-\beta)}. \tag{2.8}
\]

**Case (ii):** If \( k_0 = p-1 \).

Using (2.5), we obtain
\[
\int_{|x|>R} \frac{|u^*|^{|k_0|}|p'|}{|x|^\beta} \leq \int_{|x|>R} \frac{|u^*|^{|p'|}}{|x|^\beta} \leq \frac{1}{R^\beta} \int_{\mathbb{R}^N} |u^*(x)|^pdx = \frac{\|u\|_p^p}{R^\beta}. \tag{2.9}
\]

Then, coupling (2.8) and (2.9) in (2.7), we get
\[
\int_{|x|>R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \leq \frac{1}{R^\beta} \left( C_\alpha \|u\|_p^p + \sum_{j=k_0+1}^{\infty} \frac{\alpha_j}{j!} \left( \frac{N}{\omega_{N-1}} \right)^{\frac{1}{p-1}} \|u\|_p^p R^{N-\frac{N}{p-1}(j-\beta)} \right).
\]

For fixed \( u \in W^{s,p}(\mathbb{R}^N) \), we choose \( R > 0 \) such that
\[
R^{\frac{N}{p-1}} \left( \frac{N}{\omega_{N-1}} \right)^{1/(p-1)} \|u\|_p^p = 1,
\]
this implies

\[ \int_{|x| > R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \leq \frac{1}{R^\beta} C(N, s, \alpha, \|u\|_p) < \infty. \quad (2.10) \]

Next, for \( p' = N/(N - s) \), there exists \( B = B(N, s) > 0 \) such that for all \( \epsilon > 0 \),

\[ (u + v)^{p'} \leq u^{p'} + Bu^{p' - 1}v + v^{p'}, \quad \text{and} \quad u^\gamma v^\gamma \leq \epsilon u + \epsilon^{-\gamma'/\gamma'} v \quad (2.11) \]

for all \( u, v \geq 0 \) and \( \gamma, \gamma' > 0 \) satisfying \( \gamma + \gamma' = 1 \). For fixed \( x_0 \in \mathbb{R}^N \) with \( |x_0| = 1 \), define

\[ v(x) = \begin{cases} u^*(x) - u^*(Rx_0), & \text{if} \ x \in BR(0) \\ 0, & \text{if} \ x \in \mathbb{R}^N \setminus BR(0). \end{cases} \]

Then, since \( u^* \) is radially decreasing function, we have \( v \geq 0 \) and by [20, Lemma 2.2],

\[ [v]_{s,p}^p \leq [u^*]_{s,p}^p \leq [u]_{s,p}^p < \infty. \]

Therefore, \( v \in W^{s,p} (\mathbb{R}^N) \) with \( v = 0 \) a.e. in \( \mathbb{R}^N \setminus BR(0) \). Using (2.11), for \( x \in BR(0) \), we deduce that

\[ |u^*(x)|^{p'} = |v + u^*(Rx_0)|^{p'} \leq v^{p'} + Bu^{p' - 1}u^*(Rx_0) + u^*(Rx_0)^{p'}, \]

and

\[ v^{p' - 1}u^*(Rx_0) = (v^{p'/(p' - 1)})^{(p' - 1)/p'} (u^*(Rx_0)^{p'})^{1/p'} \leq \frac{\epsilon}{A} v^{p'} + \left( \frac{\epsilon}{A} \right)^{-1/(p' - 1)} u^*(Rx_0)^{p'}. \]

Thus,

\[ |u^*(x)|^{p'} \leq (1 + \epsilon)v^{p'} + C(\epsilon, s, N)u^*(Rx_0)^{p'}, \]

where \( C(\epsilon, s, N) = 1 + (A/\epsilon)^{1/(p' - 1)} \). Therefore, using Lemma 2.1, we obtain

\[ \int_{|x| \leq R} \Phi_\alpha(u^*) \leq \int_{|x| \leq R} \epsilon^{\alpha|u^*|^{p'}} \leq \epsilon^{\alpha C(\epsilon, s, N)|u^*(Rx_0)|^{p'}} \int_{|x| \leq R} \frac{\epsilon^{\alpha(1 + \epsilon)|u|^{p'}}}{|x|^\beta} < \infty. \quad (2.12) \]

This together with (2.10) and (2.6) proves the first part of the Theorem.

For the second part, we consider \( u \in W^{s,p} (\mathbb{R}^N) \) such that \( \|u\|_{s,p,\tau} \leq 1 \). From (2.8), we have

\[ \int_{|x| > R} \frac{|u^*|^{p'}}{|x|^\beta} \leq \left( \frac{N}{\omega_{N - 1}} \right)^{1/\tau} \|u\|_{s,p,\tau}^{p'} R^{N - \frac{N}{p - \tau} j - \beta} \]

\[ \leq R^{N - \beta} \left( \frac{N}{\omega_{N - 1}} \right)^{1/\tau} R^{- \frac{N}{p - \tau} j}, \]

where in the last inequality we used the fact \( \|u\|_{s,p,\tau} \leq 1 \). Now choosing \( R > 0 \) such that

\[ R^{N - \frac{N}{\omega_{N - 1} j}} = 1, \]
from (2.7), we obtain
\[
\int_{|x| > R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \leq R^{N-\beta} \sum_{j=k_0}^{\infty} j! \leq C(N, s, \alpha, \beta, \tau).
\] (2.13)

Now due to the fact that \(\|u\|_{s,p,\tau} \leq 1\) and \(v(x) \leq u^*(x)\) in \(B_R(0)\), we have
\[
\|v\|_{s,p,\tau}^p = [v]_{s,p}^p + \tau \|v\|_p^p \leq [u^*]_{s,p}^p + \tau \|u^*\|_p^p \leq [u]_{s,p}^p + \tau \|u\|_p^p \leq 1,
\]
and by using the radial lemma 2.5, we have
\[
u^*(Rx_0)^p \leq |Rx_0|^{-N/(p-1)} \left( \frac{N}{\omega_{N-1}} \right)^{1/(p-1)} \|u^*\|_p^p \leq R^{-N/(p-1)} \left( \frac{N}{\omega_{N-1}} \right)^{1/(p-1)}.
\]

Then, (2.12) and (2.4) yield
\[
\int_{|x| \leq R} \frac{\Phi_\alpha(u^*)}{|x|^\beta} \leq e^{C(s,N,\tau)} \int_{|x| \leq R} \frac{1}{|x|^\beta} \exp\{\alpha(1 + \epsilon) \|v\|_{s,p,\tau}^p \|v\|_{s,p,\tau} \} \leq C(N, s, \tau, \alpha, \beta)
\] if we choose \(\epsilon > 0\) such that \(\alpha(1 + \epsilon) < (1 - \beta/N)\alpha_{N,s}\). Taking into account (2.13), (2.14) and (2.6), we get the second part of the Theorem. \(\square\)

Next, we establish the compactness result in the subcritical case.

**Lemma 2.6.** Let \(\{u_n\} \subseteq X\) be a sequence such that \(u_n \rightharpoonup u\) weakly in \(X\), for some \(u \in X\). Then up to a subsequence, the following hold
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)F(u_n(x)) dx = \int_{\mathbb{R}^N} K(x)F(u(x)) dx
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)f(u_n(x))u_n(x) dx = \int_{\mathbb{R}^N} K(x)f(u(x))u(x) dx
\]
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} K(x)f(u_n(x))v(x) dx = \int_{\mathbb{R}^N} K(x)f(u(x))v(x) dx, \text{ for all } v \in X.
\]

**Proof.** Set \(M = \sup \|u_n\|\). Fix \(\epsilon > 0\), \(\delta > 0\) and \(0 < \alpha < (1 - \beta/N)\alpha_{N,s}\). Due to (f1) and (f2), we get
\[
\limsup_{t \to \infty} \frac{f(t)}{\Phi_\alpha(t)} = \limsup_{t \to \infty} \frac{F(t)}{\Phi_\alpha(t)} = 0, \text{ and } \limsup_{t \to 0} \frac{f(t)}{|t|^\beta} = \limsup_{t \to 0} \frac{F(t)}{|t|^\beta} = 0.
\]

Then, there exists \(\rho_0 = \rho_0(\epsilon), \rho_1 = \rho_1(\epsilon)\) with \(0 < \rho_0 < \rho_1\), \(C = C(\epsilon) > 0\) and \(C_0 > 0\) depending only on \(K\), such that for all \(x \in \mathbb{R}^N\) and \(t \in \mathbb{R},
\[
|K(x)F(t)| \leq \epsilon C_0(|t|^p + \Phi_\alpha(t)) + CK(x)\chi_{[\rho_0,\rho_1]}(|t|)|t|^\delta
\]
\[
|K(x)f(t)t| \leq \epsilon C_0(|t|^p + \Phi_\alpha(t)) + CK(x)\chi_{[\rho_0,\rho_1]}(|t|)|t|^\delta.
\] (2.15)
By the embedding results of $X$ into $L^m(\mathbb{R}^N)$ (and hence into $L^m(\mathbb{R}^N; |x|^{-\beta})$, for $0 \leq \beta < N$), we have

$$
\sup_n \int_{\mathbb{R}^N} \frac{|u_n|^p}{|x|^\beta} \leq \tilde{M},
$$

(2.16)

for some $\tilde{M} \geq M > 0$. Now, for $\alpha < (1 - \frac{\beta}{N}) \frac{\alpha_{N,s}}{M^{N/N-s_1}}$, we have

$$\alpha \|u_n\|^{N/(N-s_1)} \leq \alpha M^{N/(N-s_1)} < (1 - \frac{\beta}{N})\alpha_{N,s_1},$$

therefore, by Theorem 1.1 and the fact that $\Phi_\alpha$ is increasing with respect to $\alpha$, we obtain

$$
\sup_n \int_{\mathbb{R}^N} \frac{\Phi_\alpha(u_n)}{|x|^\beta} \leq \sup_n \int_{\mathbb{R}^N} \frac{\Phi_{\alpha M^{N/(N-s_1)}}(\frac{u_n}{\|u_n\|})}{|x|^\beta} \leq \tilde{M}.
$$

(2.17)

Let $A_n := \{x \in \mathbb{R}^N : \rho_0 \leq |u_n(x)| \leq \rho_1\}$. Then, as $\{|A_n|\}$ is bounded w.r.t. $n$, taking into account (1.2), we deduce that

$$
\lim_{r \to \infty} \left| \int_{A_n \cap B_r(0)} \frac{K(x)}{|x|^\beta} dx \right| \leq \lim_{r \to \infty} \frac{1}{r^\beta} \left| \int_{A_n \cap B_r(0)} K(x) dx \right| = 0, \text{ uniformly w.r.t. } n \in \mathbb{N}.
$$

Therefore, for $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$
\int_{A_n \cap B_{R_\epsilon}(0)} \frac{K(x)}{|x|^\beta} dx < \frac{\epsilon}{C\rho_1^\delta} \text{ for all } n \in \mathbb{N}.
$$

(2.18)

Taking into account (2.15) through (2.18), we obtain

$$
\int_{B_{R_\epsilon}(0)} \frac{K(x)F(u_n(x))}{|x|^\beta} dx \leq 2C_0 \tilde{M} \epsilon + C \rho_1^\delta \int_{A_n \cap B_{R_\epsilon}(0)} \frac{K(x)}{|x|^\beta} dx < (2C_0 \tilde{M} + 1) \epsilon,
$$

$$
\int_{B_{R_\epsilon}(0)} \frac{K(x)f(u_n(x))u_n(x)}{|x|^\beta} dx \leq 2C_0 \tilde{M} \epsilon + C \rho_1^\delta \int_{A_n \cap B_{R_\epsilon}(0)} \frac{K(x)}{|x|^\beta} dx < (2C_0 \tilde{M} + 1) \epsilon,
$$

(2.19)

for all $n \in \mathbb{N}$. Furthermore, by (f1) and (f2), it is easy to observe that

$$
|f(t)| \leq C_1 (|t|^p + \Phi_\alpha(t)), \text{ for all } t \in \mathbb{R},
$$

where $C_1 > 0$ is a constant. Therefore, using the fact that $K \in L^\infty(\mathbb{R}^N)$, get

$$
\left| \int_{B_{R_\epsilon}(0)} \frac{K(x)\Phi(u_n)}{|x|^\beta} dx \right| \leq C \left( \int_{B_{R_\epsilon}(0)} \frac{|u_n|^p|u_n-u|}{|x|^\beta} dx + \int_{B_{R_\epsilon}(0)} \frac{\Phi_\alpha(u_n)|u_n-u|}{|x|^\beta} dx \right).
$$

We choose $\gamma \geq 1$ close to 1 so that $\gamma' > p$ and $\gamma \alpha < (1 - \frac{\beta}{N}) \frac{\alpha_{N,s}}{M^{N/N-s_1}}$. Then, using Hölder inequality, Theorem 1.1 and the fact that $\{|u_n|\}$ is bounded, we deduce that

$$
\left| \int_{B_{R_\epsilon}(0)} \frac{K(x)f(u_n)\Phi(u_n)}{|x|^\beta} dx \right| \leq C \left[ \left( \int_{\mathbb{R}^N} \frac{|u_n|^{\gamma p}}{|x|^\beta} \right)^{1/\gamma} \left( \int_{\mathbb{R}^N} \frac{\Phi_\alpha(u_n)^\gamma}{|x|^\beta} \right)^{1/\gamma} + \left( \int_{B_{R_\epsilon}(0)} \frac{|u_n-u|^{\gamma p}}{|x|^\beta} \right)^{1/\gamma} \left( \int_{B_{R_\epsilon}(0)} \frac{\Phi_\alpha(u_n)^\gamma}{|x|^\beta} \right)^{1/\gamma} \right]
$$

$$
C \left( \int_{B_{R_\epsilon}(0)} \frac{|u_n-u|^{\gamma p}}{|x|^\beta} \right)^{1/\gamma} \to 0 \text{ as } n \to \infty.
$$
where in the last line we used the compact embedding result of $X$ given in Remark 2.1. Hence,

$$\lim_{n \to \infty} \int_{B_{R_s}(0)} \frac{K(x)f(u_n)u_n}{|x|^{\beta}} dx = \int_{B_{R_s}(0)} \frac{K(x)f(u_n)u_n}{|x|^{\beta}} dx.$$ 

Using (f3), one can easily verify that $pF(t) \leq f(t)t$ for all $t \in \mathbb{R}^N$. Therefore, by generalized Lebesgue dominated convergence theorem, similar result holds for $F$ also. Thus, using (2.19), we get the required convergence result of the first two integrals of the Lemma.

Next, to prove the last convergence result of the Lemma, we set $E_n := \{x \in \mathbb{R}^N : |u_n(x)| \leq 1 \}$ and $E := \{x \in \mathbb{R}^N : |u(x)| \leq 1 \}$. We first claim that the sequence $\{K(x)f(u_n)\chi_{E_n}\}$ is uniformly bounded in $L^{r'}(\mathbb{R}^N; |x|^{-\beta})$, where $r'$ is the Hölder conjugate of $r$. By (f1), it is easy to see that $|f(t)| \leq C|t|^{r-1}$ for all $|t| \leq 1$ and some $C > 0$. Therefore,

$$|f(u_n)| \leq C|u_n|^{p-1} \text{ in } E_n, \text{ for all } n \in \mathbb{N}.$$ 

By the fact that $X \hookrightarrow L^p(\mathbb{R}^N; |x|^{-\beta})$ is continuous and $\{\|u_n\|\}$ is bounded, we obtain

$$\int_{E_n} \frac{|K(x)f(u_n)|^{r'}}{|x|^{\beta}} \leq C \int_{E_n} \frac{|u_n|^{p}}{|x|^{\beta}} \leq C\|u_n\|^p \leq C, \text{ for all } n \in \mathbb{N}.$$ 

This together with the pointwise convergence gives us

$$\lim_{n \to \infty} \int_{E_n} \frac{K(x)f(u_n)\phi}{|x|^{\beta}} dx = \int_{E} \frac{K(x)f(u_n)\phi}{|x|^{\beta}} dx \text{ for all } \phi \in L^p(\mathbb{R}^N; |x|^{-\beta}).$$ 

Now, for any $v \in X$, we have $v \in L^r(\mathbb{R}^N; |x|^{-\beta})$ and hence

$$\lim_{n \to \infty} \int_{E_n} \frac{K(x)f(u_n)v}{|x|^{\beta}} dx = \int_{E} \frac{K(x)f(u_n)v}{|x|^{\beta}} dx.$$ 

Similarly, by (f2), for $m \geq 1$, we obtain

$$|f(u_n)\chi_{E_n}^m| \leq C(\Phi_{\alpha}(u_n)\chi_{E_n})^m \leq C\Phi_{m\alpha}(u_n)\chi_{E_n} \text{ for all } n \in \mathbb{N}.$$ 

We choose $m > 1$ close to 1 so that $m' > p$ and $m\alpha < (1 - \frac{\beta}{N})\frac{\alpha_N}{M_N/(N-\alpha)}$. Then, by Theorem 1.1 we get

$$\int_{E_n} \frac{|K(x)f(u_n)|^m}{|x|^{\beta}} \text{ is uniformly bounded.}$$ 

Therefore, for $v \in X$, we have $v \in L^{m'}(\mathbb{R}^N; |x|^{-\beta})$ and pointwise convergence yields

$$\lim_{n \to \infty} \int_{E_n} \frac{K(x)f(u_n)v}{|x|^{\beta}} dx = \int_{E} \frac{K(x)f(u_n)v}{|x|^{\beta}} dx.$$ 

This completes proof of the lemma. $\square$
Without loss of generality, we may assume $\alpha_0 = (1 - \beta/N)\alpha_{N,s_1}$, appearing in $(f2)'$. Then, we have similar compactness result in the critical case.

**Corollary 2.7.** Let $\{v_n\} \subset X$ be a sequence such that $v_n \rightharpoonup v$ weakly in $X$, for some $v \in X$ and

$$L := \sup_n \|v_n\| \in (0, 1).$$

Then, the convergence results of the Lemma 2.6 holds in this case also.

**Proof.** Since $L \in (0, 1)$, there exists $\alpha_L > \alpha_{N,s_1} (1 - \frac{\beta}{N})$ such that $\alpha_L < (1 - \frac{\beta}{N}) \frac{\alpha_{N,s_1}}{\rho_0^{(N-s_1)}}$. Then, by (f1) and (f2)', results similar to (2.15) hold in this case with $\alpha$ replaced by $\alpha_L$. Furthermore, Theorem 1.1 can be applied to obtain boundedness type result (2.16) and (2.17).

Now, rest of the proof follows similar to that of the Lemma with $\alpha$ replaced by $\alpha_L$. \(\square\)

It is easy to verify that the functional $J$ is of class $C^1(X)$. Now, we verify the mountain pass geometry for $J$.

**Lemma 2.8.** The functional $J$ satisfies the following

(I) there exists $e \in X \setminus \{0\}$ with $\|e\| \geq 2$ such that $J(e) < 0$.

(II) There exists $\eta > 0$ and $\rho \in (0, 1)$ such that $J(v) \geq \eta$ for all $v \in X$ with $\|v\| = \rho$.

**Proof.** (I) Proof of this part is a standard procedure and follows by the super linear nature of the nonlinearity $F$ with respect to $p$.

(II) For this, we fix $\rho_0 \in (0, 1)$. We choose $\alpha > 0$ such that $\alpha_{N,s_1} (1 - \frac{\beta}{N}) < \alpha < (1 - \frac{\beta}{N}) \frac{\alpha_{N,s_1}}{\rho_0^{(N-s_1)}}$. Now, by the fact that $K \in L^\infty(\mathbb{R}^N)$, (f1) and (f2) or (f2)', for $\delta > p$, we have

$$K(x)F(t) \leq \frac{1}{2^p p^p C_p} t^p + C_2 \Phi_\alpha(t) t^\delta \quad \text{for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^N,$$

where $C_2 > 0$ is a constant and $C_p$ appears in Remark 2.1 (A). We choose $m > 1$ close to $1$ such that $m' > p$ and $m \alpha < (1 - \frac{\beta}{N}) \frac{\alpha_{N,s_1}}{\rho_0^{(N-s_1)}}$, then by Theorem 1.1 and embeddings of $X$ into $L^\gamma(\mathbb{R}^N; |x|^{-\beta})$, for $\gamma \geq p$, for all $w \in X$ with $\|w\| = \rho \leq \rho_0$, we obtain

$$\int_{\mathbb{R}^N} \frac{\Phi_\alpha(w) w^\delta}{|x|^{\beta}} \leq \left( \int_{\mathbb{R}^N} \frac{\Phi_\alpha(w)^m}{|x|^{\beta}} \right)^{1/m} \left( \int_{\mathbb{R}^N} \frac{|w|^{m'} |x|^{\beta}}{|x|^{\beta}} \right)^{1/m'} \leq C_3 \left( \int_{\mathbb{R}^N} \frac{\Phi_{\alpha m}(w)}{|x|^{\beta}} \right)^{1/r} \|w\|^\delta \leq C_4 \|w\|^\delta,$$

where $C_3, C_4 > 0$ are constants independent of $w$. Therefore, using (2.20), (2.21) and Remark 2.1 (A), we deduce that

$$J(w) \geq \frac{1}{p} \|w\|_{s_1,p}^p + \frac{1}{q} \|w\|_{s_2,q}^q - \frac{1}{p 2^p C_p} \int_{\mathbb{R}^N} \frac{|w|^p}{|x|^{\beta}} - C_2 \int_{\mathbb{R}^N} \frac{\Phi_\alpha(w) |w|^\delta}{|x|^{\beta}} \geq \frac{2^{1-p}}{p} \|w\|^p - \frac{1}{p 2^p C_p} \|w\|_{p}^p - C_4 \|w\|^\delta = \frac{2-p}{p} \|w|_p^p - C_4 \|w\|^\delta,$$
where we have used the fact that \(\|w\|_{s_1,p}, \|w\|_{s_2,q} \leq \|w\| < 1\) and \(C_i\)'s are positive constants. By the fact that \(\delta > p\), there exists \(\eta > 0\) and \(\rho\) small enough such that \(J(w) \geq \eta\) for all \(w \in X\) with \(\|w\| = \rho\). \(\square\)

The mountain pass lemma ensures the existence of a Cerami sequence at the mountain pass level, that is, there exists a sequence \(\{u_n\} \subset X\) such that

\[
J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|)\|J'(u_n)\| \to 0, \quad \text{as} \ n \to \infty,
\]

where \(c := \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t))\) with \(\Gamma = \{g \in C([0,1], X) : g(0) = 0 \text{ and } J(g(1)) < 0\}\).

**Lemma 2.9.** Every solution \(u\) of \((P)\) is nonnegative and if \(\{u_n\} \subset X\) is a Cerami sequence, then \(\|u_n^-\| \to 0\), as \(n \to \infty\).

**Proof.** Proof follows using the inequalities,

\[
(u(x) - u(y))(u^-(x) - u^-(y)) \leq - |u^-(x) - u^-(y)|^2 \quad \text{and} \quad |u(x) - u(y)| \geq |u^-(x) - u^-(y)|.
\]

Using these one can deduce that \(A_2(u, u^-) + \int_{\mathbb{R}^N} V(x)|u|^{q-2}uu^- \leq 0\). Then, rest of the proof follows similar to [17, Lemma 2.9]. \(\square\)

Following the standard procedure, we can prove the following result.

**Lemma 2.10.** Suppose the function \(f\) satisfies \((f1)\), \((f3)'\) and \((AR)\). Then, any Cerami sequence of \(J\) at level \(c\) is bounded.

**Lemma 2.11.** Assume the function \(f\) satisfies \((f1)\), \((f2)'\), \((f3)'\) and \((AR)\). Let \(\{u_n\} \subset X\) be a Cerami sequence for \(J\) at level \(c\). Then

\[
\sup_{n \in \mathbb{N}} \|u_n\| \in (0, 1),
\]

provided the constant \(C_\delta\), appearing in \((f3)'\), is sufficiently large.

**Proof.** Fix \(\psi \in C_c^\infty(\mathbb{R}^N)\) with \(\|\psi\| > 0\). Set \(K_0 := \inf_{\text{supp}(\psi)} K > 0\) and \(S_\delta = \|\psi\|_\delta > 0\). Then, using \((f3)\)', for \(l > 1\), we get

\[
J(l\psi) = \frac{l^p}{p} \|\psi\|^p_{s_1,p} + \frac{l^q}{q} \|\psi\|^q_{s_2,q} - \int_{\mathbb{R}^N} \frac{K(x)F(l\psi(x))}{|x|^\beta} \leq \frac{l^p}{q} \|\psi\|^p - K_0C_\delta l^\delta S_{\delta}^{-\delta} \|\psi\|^\delta.
\]

Since \(\delta > p\), there exits \(l_\delta > 0\) sufficiently large such that \(J(l_\delta \psi) < 0\). Therefore, \(g(t) = tl_\delta \psi\), for \(t \in [0,1]\) is of class \(C([0,1], X)\) and belongs to \(\Gamma\). Hence,

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \leq \max_{t \in [0,1]} J(tl_\delta \psi) \leq \sup_{t \in \mathbb{R}^+} J(t\psi). \quad (2.22)
\]
Consider $h : [0, \infty) \to \mathbb{R}$ defined by $h(t) := \frac{t^p}{p} \|\psi\|^p + \frac{t^q}{q} \|\psi\|^q - C(\delta)t^\delta \|\psi\|^\delta$, where $C(\delta) = K_0 C_\delta S_\delta^{-\delta}$. An easy computation, yields

$$
\sup_{t \geq 0} h(t) \leq \sup_{t \geq 0} \left( \frac{t^p}{p} \|\psi\|^p - \frac{1}{2} C(\delta)t^\delta \|\psi\|^\delta \right) + \sup_{t \geq 0} \left( \frac{t^q}{q} \|\psi\|^q - \frac{1}{2} C(\delta)t^\delta \|\psi\|^\delta \right)
$$

\[
= \left( \frac{2}{C(\delta)} \right)^{\frac{p}{p-\delta}} \left( \frac{1}{p} \right) (\|\psi\|^{p-\delta})^{\frac{\delta}{p-\delta}} + \left( \frac{2}{C(\delta)} \right)^{\frac{q}{q-\delta}} \left( \frac{1}{q} \right) (\|\psi\|^{q-\delta})^{\frac{\delta}{q-\delta}}
\]

where we assumed $C_\delta > 1$ and $(K_0 S_\delta^{-\delta})^\gamma = \min \{(K_0 S_\delta^{-\delta})^p, (K_0 S_\delta^{-\delta})^q\}$.

Therefore, from (2.22), we observe that

$$
c \leq \sup_{t \geq 0} \mathcal{J}(tv) \leq \sup_{t \geq 0} h(t) \leq \left( \frac{1}{q} - \frac{1}{\delta} \right) \frac{2^{p/(\delta-p)}}{(K_0 S_\delta^{-\delta})^\gamma} \frac{\|\psi\|^{-\delta}}{C_\delta^{\delta/(\delta-q)}},
$$

By (AR) and the fact that $\{u_n\}$ is a Cerami sequence, we get

$$
c = \lim_{n \to \infty} \left( \mathcal{J}(u_n) - \frac{1}{\nu} \mathcal{J}'(u_n)u_n \right) \geq \limsup_{n \to \infty} \left( \left( \frac{1}{p} - \frac{1}{\nu} \right) \|u_n\|_{p} + \left( \frac{1}{q} - \frac{1}{\nu} \right) \|u_n\|_{q} \right)
$$

\[
\geq \limsup_{n \to \infty} \left( \frac{1}{p} - \frac{1}{\nu} \right) \|u_n\|_{\gamma},
\]

where $\gamma$ is such that $L^\gamma = \min\{L^p, L^q\}$ with $L := \sup_{n} \|u_n\|$. Then, using (2.23), we obtain

$$
\limsup_{n \to \infty} \|u_n\|_{\gamma} \leq \frac{\nu}{\nu - p} c \leq \frac{\nu}{\nu - p} \left( \frac{1}{q} - \frac{1}{\delta} \right) \frac{2^{p/(\delta-p)}}{(K_0 S_\delta^{-\delta})^\gamma} \frac{\|\psi\|^{-\delta}}{C_\delta^{\delta/(\delta-q)}} < 1,
$$

provided $C_\delta$ is sufficiently large.

In the subcritical case, we prove the boundedness of Cerami sequences. The proof differs from the critical case due to absence of Ambrosetti-Rabinowitz type condition for this case.

**Lemma 2.12.** Assume (f1)-(f3) hold. Then, any Cerami sequence of $\mathcal{J}$ at level $c$ is bounded.

**Proof.** Let $\{v_n\} \subset X$ be a Cerami sequence of $\mathcal{J}$ at level $c$. Let $t_n \in [0, 1]$ be such that

$$
\mathcal{J}(tv_n) = \max_{t \in [0, 1]} \mathcal{J}(tv_n).
$$

(2.24)

We claim that $\{\mathcal{J}(tv_n)\}$ is bounded.

The claim is obvious if $t_n = 0$ or 1, therefore we assume $t_n \in (0, 1)$. Also, we assume $v_n \geq 0$.

Set

$$
H(t) := tf(t) - rF(t) \quad \text{for } t \in \mathbb{R}.
$$

In account of (f1) and (f3), $t^{1-r}f(t)$ is nondecreasing and differentiable, therefore $H$ is non-decreasing in $\mathbb{R}$. Then, from (2.24), we have

$$
\frac{d}{dt} \mathcal{J}(tv_n) \big|_{t=t_n} = 0,
$$

where we assumed $C_\delta > 1$. Therefore we have

$$
\frac{d}{dt} \mathcal{J}(tv_n) \big|_{t=t_n} = 0.
$$

(2.25)
and hence
\[ p\mathcal{J}(t_nv_n) = \left(\frac{p}{q} - 1\right) t_n^q \|v_n\|_{s_2,q}^q + \int_{\mathbb{R}^N} \frac{K(x)H(t_nv_n)}{|x|^\beta} \]
\[ \leq \left(\frac{p}{q} - 1\right) \|v_n\|_{s_2,q}^q + \int_{\mathbb{R}^N} \frac{K(x)H(v_n)}{|x|^\beta} \]
\[ = p\mathcal{J}(v_n) - \mathcal{J}'(v_n)v_n = pc + o_n(1), \]
this proves the claim. Now we prove that \( \{v_n\} \) is bounded in \( X \). On the contrary assume that up to a subsequence \( \|v_n\| \to \infty \) as \( n \to \infty \) and \( \|v_n\| \geq 1 \) for all \( n \in \mathbb{N} \). Set \( w_n = \frac{v_n}{\|v_n\|} \). Then, since \( \{w_n\} \) is bounded in \( X \), there exists \( w \in X \) such that \( w_n \rightharpoonup w \) weakly in \( X \). We claim that \( w = 0 \) a.e. in \( \mathbb{R}^N \). Since \( \mathcal{J}(v_n) = c + o_n(1) \) and \( \|v_n\| \to \infty \), we get
\[ \frac{1}{p} \|v_n\|_{s_1,p}^p + \frac{1}{q} \|v_n\|_{s_2,q}^q - \int_{\mathbb{R}^N} \frac{K(x)F(v_n)}{|v_n|^p|x|^\beta} = o_n(1). \quad (2.25) \]
Since \( \lim_{t \to \infty} t^{-p}F(t) = \infty \), for every \( \tau > 0 \), there exists \( \xi > 0 \) such that
\[ F(t) \geq \tau |t|^p \quad \text{for all } |t| \geq \xi. \]
Therefore, from (2.25), we obtain
\[ o_n(1) + \frac{1}{p} \int_{\|v_n\| \geq \xi} \frac{K(x)F(v_n)w_n(x)^p}{|v_n(x)|^p|x|^\beta} \geq \tau \int_{\mathbb{R}^N} \frac{K(x)w_n(x)^p}{|x|^\beta} \chi_{\{|v_n| \geq \xi \|v_n\|\}}. \]
By Fatou’s lemma, for all \( \tau > 0 \), we have
\[ \tau \int_{\mathbb{R}^N} \frac{K(x)w(x)^p}{|x|^\beta} \leq \frac{1}{p}, \]
which implies \( w = 0 \) a.e. in \( \mathbb{R}^N \). Let \( T > 0 \), then there exists \( n_T \in \mathbb{N} \) such that for all \( n \geq n_T \), \( T\|v_n\|^{-1} \in (0,1) \). Then, from (2.24), we get
\[ \mathcal{J}(t_nv_n) \geq \mathcal{J}(Tw_n) = \frac{T^p}{p} \|w_n\|_{s_1,p}^p + \frac{T^q}{q} \|w_n\|_{s_2,q}^q - \int_{\mathbb{R}^N} \frac{K(x)F(Tw_n)}{|x|^\beta} \]
\[ \geq \frac{2^{1-p}T^\gamma}{p} \|w_n\|^p - \int_{\mathbb{R}^N} \frac{K(x)F(Tw_n)}{|x|^\beta}, \]
where \( T^\gamma = \min\{T^p, T^q\} \). Now, by the compactness lemma 2.6, we have
\[ \int_{\mathbb{R}^N} \frac{K(x)F(Tw_n)}{|x|^\beta} \to 0, \quad \text{as } n \to \infty. \]
Thus
\[ \liminf_{n \to \infty} \mathcal{J}(t_nv_n) \geq \frac{2^{1-p}T^\gamma}{p}, \]
which is a contradiction, if we choose \( T \) such that \( T = \left(2^p p \sup_n \{\mathcal{J}(t_nv_n)\}\right)^{1/\gamma} \). This proves the lemma. \( \square \)
3 Proof of Main Theorem

Proof of Theorem 1.2: The functional $\mathcal{J}$ satisfies mountain pass geometry in both the cases. Therefore, there exist Cerami sequences $\{u_n\} \subset X$ and $\{v_n\} \subset X$ in the subcritical and critical cases, respectively. Furthermore, $\{u_n\}$ and $\{v_n\}$ are bounded in $X$. Therefore, up to a subsequence $u_n \to u$ and $v_n \to v$ weakly in $X$, for some $u, v \in X$.

The subcritical case (I): By the compactness lemma 2.6, we see that $\int_{\mathbb{R}^N} \frac{K(x)f(u_n)}{|x|^p}(u_n-u) \to 0$ as $n \to \infty$. Moreover, since $\langle \mathcal{J}'(u_n), u_n - u \rangle \to 0$ as $n \to \infty$, it follows that

$$\mathcal{A}_1(u_n, u_n - u) + \mathcal{A}_2(u_n, u_n - u) + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n + |u_n|^{q-2}u_n)(u_n - u) = o_n(1).$$

On the other hand for fixed $u \in X$, it is easy to observe that $\Theta_{u,p} + \Theta_{u,q} \in X'$, where $\Theta_{u,p}(v) := \mathcal{A}_1(u, v) + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv$ for all $v \in X$ and $\Theta_{u,q}$ is analogously defined. Then, using the fact that $u_n \to u$ weakly in $X$, we get

$$\mathcal{A}_1(u, u_n - u) + \mathcal{A}_2(u, u_n - u) + \int_{\mathbb{R}^N} V(x)(|u|^{p-2}u + |u|^{q-2}u)(u_n - u) = o_n(1).$$

Coupling these, we obtain

$$\mathcal{A}_1(u_n, u_n - u) - \mathcal{A}_1(u, u_n - u) + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)$$

$$+ \mathcal{A}_2(u_n, u_n - u) - \mathcal{A}_2(u, u_n - u) + \int_{\mathbb{R}^N} V(x)(|u_n|^{q-2}u_n - |u|^{q-2}u)(u_n - u) = o_n(1). \quad (3.1)$$

Now, we consider the cases when $q \geq 2$ and $1 < q < 2$ (note that $p \geq 2$).

Case (i): $q \geq 2$.

Using the inequality $|a-b|^l \leq 2^{l-2}(|a|^{l-2}a - |b|^{l-2}b)(a-b)$ for $a, b \in \mathbb{R}^n$ and $l \geq 2$, from (3.1), it follows that

$$|u_n - u|_{s_1,p}^p + \int_{\mathbb{R}^N} V(x)|u_n - u|^p + |u_n - u|_{s_2,q}^q + \int_{\mathbb{R}^N} V(x)|u_n - u|^q \leq o_n(1),$$

that is

$$\|u_n - u\|_{s_1,p}^p + \|u_n - u\|_{s_2,q}^q = o_n(1)$$

this implies that $u_n \to u$ in $X$.

Case (ii): $1 < q < 2$.

As we know that for $a, b \in \mathbb{R}^n$ and $1 < m < 2$, there exists $C_m > 0$ a constant such that

$$|a-b|^m \leq C_m((|a|^{m-2}a - |b|^{m-2}b)(a-b))^\frac{m}{2} (|a|^m + |b|^m)^\frac{2-m}{2}.$$

Set $a = u_k(x) - u_k(y)$, $b = u_k(x) - u_k(y)$ and then using Hölder inequality, we deduce that

$$|u_n - u|_{s_2,q}^q \leq C(A_2(u_n, u_n - u) - A_2(u, u_n - u))^\frac{q}{2} \left( [u_n]_{s_2,q}^q + [v]_{s_2,q}^q \right)^\frac{2-q}{2}.$$
and boundedness of \( \{u_n\} \) in \( X \), implies
\[
[u_n - u]_{s_2,q}^2 \leq C(A_2(u_n, u_n - u) - A_2(u, u_n - u)).
\]
Therefore, using Lemma 2.9 and proceeding similarly, we obtain \( u_n \to u \) in \( \tilde{W}^{s_2,p}_V(\mathbb{R}^N) \) as well as in \( \tilde{W}^{s_2,q}_V(\mathbb{R}^N) \), which gives us the required strong convergence of \( u_n \) to \( u \) in \( X \). Using the fact that \( c > 0 \) and strong convergence, we get that \( u \neq 0 \). By Lemma 2.9 \( u \) is a nontrivial nonnegative solution of \( (P) \).

**The critical case (II):** We observe that if we choose \( C_0 > 0 \) such that Lemma 2.11 is satisfied, then the compactness results of corollary 2.7 hold. Now, we can proceed similarly to prove that \( v_n \to v \) in \( X \) and \( v \neq 0 \), hence \( v \) is a nontrivial weak solution of \( (P_\lambda) \). □

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