Oscillating decay of an unstable system

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We study the short-time and medium-time behavior of the survival probability in the frame of the $N$-level Friedrichs model. The time evolution of an arbitrary unstable initial state is determined. We show that the survival probability may oscillate significantly during the so-called exponential era. This result explains qualitatively the experimental observations of the NaI decay.

I. INTRODUCTION

Recent developments in femtosecond laser optics, see for example the XXth Solvay Conference on Chemistry, opened new possibilities for the study of quantum transitions, which are a very important subject of the quantum theory. In a series of works, Zewail et al. \cite{2–5} applied femtosecond transition-time spectroscopy for the probing of chemical reactions. Following the work of Kinsey et al. \cite{6}, they attempted in paper \cite{5} to track wave packet trajectories in the dissociation of NaI.

The shapes of the ground state potential for NaI and of the quasi-bound potential of the Na($^2S_{1/2}$)+I($^2P_{3/2}$) system suggest a mechanism of the induced dissociation process. The femtosecond laser pulse brings the NaI molecule to the state of quasi-bound ions. The distance between the ions reaches the region where two potentials have similar values due to vibrations of a NaI excited state. Then the transition from Na($^2S_{1/2}$)+I($^2P_{3/2}$) quasi-bound state to NaI continuum state occurs resulting in the dissociation of the molecule.

After an initial exciting laser pulse, the experiment shows oscillations of the Na($^2S_{1/2}$)+I($^2P_{3/2}$) population, which are explained in \cite{5} by wave packet propagation. The direction of the wave packet propagation is correlated with the oscillation (extension and contraction) of the NaI bond. The quantum dynamics calculations are based on a time-dependent perturbation formalism.

This problem is an example of the interaction of the discrete spectrum with the continuous spectrum, which was extensively discussed in the literature starting from the work of Friedrichs. Indeed, the energy states of Na($^2S_{1/2}$)+I($^2P_{3/2}$) are the excited state embedded into the continuum states of the decay products. Therefore, the time dependence of the survival probability of the excited state is described by the survival probability of the excited state prepared by the laser pulse.

The original Friedrichs model \cite{7} contains two discrete energy level, a ground state and an excited state, coupled with the continuum, being bounded from below. The time dependence of the survival probability of the excited state has been studied both theoretically \cite{9–14} and experimentally \cite{15–17}. It is exponential with a short non-exponential initial era and a non-exponential long tail. As a result, Friedrichs models are very appropriate for the discussion of the particle decay and for the description of dressed unstable states. The analytical structure of the $N$-level Friedrichs model has been widely discussed \cite{18–20}, and the possibility of the oscillations of the survival probability was pointed out in \cite{13,21}.

In the present paper we shall show that the $N$-level Friedrichs model can also explain the oscillations of the survival probability of the excited state observed by Zewail and co-workers \cite{5}. Several excited levels are necessary in order to construct a wave packet, which can exhibit localization and nonconventional time evolution. In Section 2 we present the model and describe the exact solution diagonalising the Hamiltonian. Using the relation between eigenstates of the unperturbed Hamiltonian and the total Hamiltonian, we describe in Section 3 the time evolution of the basis states. Specifying the formfactor of the interaction, we show in Section 4 the appearance of oscillations already for the two level Friedrichs model. In Section 5 we demonstrate that the survival probability of unstable states in the $N$-level Friedrichs model is in fact very close to the one obtained in the experiment.
II. MODEL AND EXACT SOLUTION

The Hamiltonian of the Friedrichs model \( \mathcal{H} \) generalized to \( N \)-level is:
\[
H = H_0 + \lambda V,
\]
where
\[
H_0 = \sum_{k=1}^{N} \omega_k |k\rangle \langle k| + \int_0^{\infty} d\omega \omega |\omega\rangle \langle \omega|,
\]
\[
V = \sum_{k=1}^{N} \int_0^{\infty} d\omega f_k(\omega) (|k\rangle \langle \omega| + |\omega\rangle \langle k|).
\]

Here \( |k\rangle \) represent states of the discrete spectrum with the energy \( \omega_k, \omega_k > 0 \). We assume the simplest case that \( \omega_k \neq \omega_{k'} \) for \( k \neq k' \). The vectors \( |\omega\rangle \) represent states of the continuous spectrum with the energy \( \omega \), \( f_k(\omega) \) are the formfactors for the transitions between the discrete and the continuous spectrum, and \( \lambda \) is the coupling parameter. The vacuum energy is chosen to be zero. The states \( |k\rangle \) and \( |\omega\rangle \) form a complete orthonormal basis:
\[
|k\rangle \langle k| = \delta_{kk}, \quad \langle \omega|\omega\rangle = \delta(\omega - \omega'), \quad \langle \omega|k\rangle = 0, \quad k, k' = 1 \ldots N,
\]
\[
\sum_{k=1}^{N} \omega_k |k\rangle \langle k| + \int_0^{\infty} d\omega |\omega\rangle \langle \omega| = I,
\]
where \( \delta_{kk'} \) is the Kronecker symbol, \( \delta(\omega - \omega') \) is the Dirac’s delta function and \( I \) is the unity operator. The Hamiltonian \( H_0 \) has the continuous spectrum on the interval \([0, \infty)\) and the discrete spectrum \( \omega_1, \ldots, \omega_N \) embedded in the continuous spectrum.

As the interaction \( \lambda V \) is switched on, the eigenstates \( |k\rangle \) become resonances of \( H \) as in the case of the one-level Friedrichs model \( \mathcal{H} \). Let us consider the eigenvalue problem for the \( N \)-level Friedrichs Hamiltonian \( \mathcal{H} \)
\[
H |\Psi_\omega\rangle = \omega |\Psi_\omega\rangle.
\]

We shall look for the solution of Eq. (3) in the form:
\[
|\Psi_\omega\rangle = \sum_k \psi_k(\omega)|k\rangle + \int_0^{\infty} d\omega' \psi(\omega, \omega')|\omega'\rangle,
\]
where \( \psi_k(\omega) \) and \( \psi(\omega, \omega') \) are unknown functions. Inserting (3) into (3) and making use of the orthogonality relations, we obtain for them a system of equations:
\[
\begin{cases}
(\omega_k - \omega)\psi_k(\omega) + \lambda \int_0^{\infty} d\omega' f_k(\omega')\psi(\omega, \omega') = 0, \\
(\omega' - \omega)\psi(\omega, \omega') + \lambda \sum_{k=1}^{N} f_k(\omega')\psi_k(\omega) = 0.
\end{cases}
\]

Eliminating \( \psi(\omega, \omega') \) from this system, we arrive at the following equation for \( \psi_k(\omega) \):
\[
\sum_{k'=1}^{N} G_{kk'}^{-1}(\omega)\psi_{k'}(\omega) = -C \lambda f_k(\omega),
\]
where \( C \) is an arbitrary constant. \( G_{kk'}(\omega) \) are the matrix elements of the partial resolvent which is:
\[
G_{kk'}^{\lambda}(\omega) = (\omega_k - \omega)\delta_{kk'} - \lambda^2 \int_0^{\infty} d\omega' \frac{f_k(\omega')f_{k'}(\omega')}{\omega' - \omega}.
\]
Under certain conditions (which will be specified below, see also [26]), the function $G_{kk'}(z)$ is analytic everywhere in the first sheet of the Riemann manifold except for the cut $[0, \infty)$. In this case, the Hamiltonian $H$ has no discrete spectrum. The solution of Eq. (8) is given by

$$\psi_k(\omega) = -C\lambda \sum_{k'=1}^{N} G_{kk'}(\omega \pm i0) f_{k'}(\omega).$$

(10)

With this equation we find $\psi(\omega, \omega')$ from the system (7):

$$\psi(\omega, \omega') = C \left[ \delta(\omega - \omega') + \lambda \sum_{k,k'=1}^{N} f_k(\omega) G_{kk'}(\omega \pm i0) f_{k'}(\omega) \right].$$

(11)

The eigenvalue problem (5) has two sets of solutions

$$|\Psi_\omega\rangle_{in} = |\omega\rangle + \lambda \sum_{k,l=1}^{N} f_l(\omega) G_{kl}(\omega \pm i0) \left\{ \int_0^\infty d\omega' \frac{\lambda f_k(\omega') G_{mk'}(\omega' \pm i0)}{\omega - \omega'} - |k\rangle \right\},$$

(12)

which correspond to the “in” and “out” asymptotic conditions. The value $C = 1$ corresponds to the orthonormalization condition

$$\langle \Psi_\omega | \Psi_\omega' \rangle_{in out} = \delta(\omega - \omega').$$

(13)

We can also prove the completeness condition

$$\int_0^\infty d\omega \langle \Psi_\omega | \psi_{in} \rangle_{out out} = \sum_{k=1}^{N} \omega_k |k\rangle \langle k| + \int_0^\infty d\omega |\omega\rangle \langle \omega|.$$

(14)

Hence the new states diagonalize the total Hamiltonian (1) as

$$H = \int_0^\infty d\omega \langle \Psi_\omega | \psi_{in} \rangle_{out out} |\Psi_\omega\rangle.$$

(15)

The proof of completeness is based on the matrix formula

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1},$$

from which we can derive:

$$G_{kk'}(\omega + i0) - G_{kk'}(\omega - i0) = 2\pi i\lambda^2 \sum_{l,m=1}^{N} G_{kl}(\omega + i0) f_l(\omega) f_m(\omega) G_{mk'}(\omega - i0).$$

(16)

Using the asymptotics:

$$G_{kk'}(\omega) \sim \delta_{kk'} \frac{1}{\omega - \omega'} + o\left( \frac{1}{\omega - \omega'} \right),$$

(17)

we prove other useful relations for $G$:

$$G_{kk'}(\omega \pm i0) = \lambda^2 \int_0^\infty d\omega' \sum_{l,m=1}^{N} f_l(\omega') f_m(\omega') \frac{G_{kl}(\omega' + i0) G_{mk'}(\omega' - i0)}{\omega' - \omega \pm i0},$$

(18)

and

$$\lambda^2 \int_0^\infty d\omega \sum_{l,m=1}^{N} f_l(\omega) f_m(\omega) G_{kl}(\omega + i0) G_{mk'}(\omega - i0) = \delta_{kk'}. $$

(19)
Due to the completeness of the new basis [14] the old basis vectors may be expressed in terms of the new ones as:

$$|k\rangle = \int_0^\infty d\omega \langle \Psi_\omega |_\text{in} |\Psi_\omega |_\text{in} \rangle |k\rangle, \quad |\omega\rangle = \int_0^\infty d\omega' \langle \Psi_{\omega'} |_\text{in} |\Psi_{\omega'} |_\text{in} \rangle |\omega\rangle$$  \hspace{1cm} (20)

where $|\Psi_{\omega} |_\text{in}$ and $|\Psi_{\omega'} |_\text{in}$ are the complex conjugates of $(|k|_\text{in} \langle \Psi_{\omega} |_\text{in}$ and $(|\omega|_\text{in} \langle \Psi_{\omega'} |_\text{in}$ respectively, which may be obtained from [12]:

$$\langle k|_\text{in} \Psi_{\omega} \rangle = -\lambda \sum_{l=1}^{N} f_l(\omega) G_{kl}(\omega + i0) ,$$  \hspace{1cm} (21)  

$$\langle \omega|_\text{in} \Psi_{\omega'} \rangle = \delta(\omega - \omega') - \sum_{k,l=1}^{N} \frac{\lambda^2 f_k(\omega) f_l(\omega') G_{k,l}(\omega')}{\omega' - \omega - i0} .$$  \hspace{1cm} (22)

Inserting complex conjugate of (21) into (20) we obtain the inverse relation in the form:

$$|k\rangle = -\lambda \sum_{l=1}^{N} \int_0^\infty d\omega f_l(\omega) G_{kl}(\omega - i0) |\Psi_{\omega} |_\text{in}$$  \hspace{1cm} (23)

$$|\omega\rangle = |\Psi_{\omega} |_\text{in} - \sum_{k,l=1}^{N} \lambda f_k(\omega) \int_0^\infty d\omega' \frac{\lambda f_l(\omega') G_{k,l}(\omega')}{\omega' - \omega - i0} |\Psi_{\omega'} |_\text{in} .$$  \hspace{1cm} (24)

These inverse relations will be used for the calculation of the time evolution of $|k\rangle$ and $|\omega\rangle$ in the next section.

### III. TIME EVOLUTION

Using the known evolution of the state $|\Psi_\omega |_\text{in}$,  

$$e^{-iHt}|\Psi_\omega |_\text{in} = e^{-i\omega t}|\Psi_\omega |_\text{in},$$

we can find the evolution of the eigenstates of $H_0$:

$$|k\rangle_t = -\lambda \sum_{l=1}^{N} \int_0^\infty d\omega e^{-i\omega t} f_l(\omega) G_{kl}(\omega - i0) |\Psi_\omega |_\text{in} ,$$  \hspace{1cm} (25)

$$|\omega\rangle_t = e^{-i\omega t} |\Psi_\omega |_\text{in} - \sum_{k,l=1}^{N} \lambda f_k(\omega) \int_0^\infty d\omega' e^{-i\omega' t} \frac{\lambda f_l(\omega') G_{k,l}(\omega')}{\omega' - \omega - i0} |\Psi_{\omega'} |_\text{in} .$$  \hspace{1cm} (26)

Using (12), we obtain the representation

$$|k\rangle_t = \sum_{l=1}^{N} A_{kl}(t) |l\rangle + \lambda \sum_{l=1}^{N} \int_0^\infty d\omega f_l(\omega) g_{kl}(\omega, t) |\omega\rangle ,$$  \hspace{1cm} (27)

$$|\omega\rangle_t = e^{-i\omega t} |\omega\rangle - \lambda^2 \sum_{k,l=1}^{N} f_k(\omega) \int_0^\infty d\omega' f_l(\omega') \frac{g_{kl}(\omega', t) - g_{kl}(\omega, t)}{\omega' - \omega} + \sum_{k,l=1}^{N} \lambda f_k(\omega) g_{kl}(\omega, t) |l\rangle$$  \hspace{1cm} (28)

in terms of the time-dependent matrix functions $A_{kl}(t)$ and $g(\omega, t)$:
\[ A_{kl}(t) = \lambda^2 \sum_{l,m,n=1}^{N} \int_{0}^{\infty} d\omega e^{-i\omega t} f_m(\omega) f_n(\omega) G_{km}(\omega + i0) G_{ln}(\omega - i0) |l\rangle, \]  

(29)

\[ g_{kl}(\omega, t) = -e^{-i\omega t} G_{kl}(\omega - i0) \]  

(30)

\[ + \lambda^2 \sum_{m,n=1}^{N} \int_{0}^{\infty} d\omega' e^{-i\omega' t} \frac{f_m(\omega') f_n(\omega') G_{km}(\omega' - i0) G_{ln}(\omega' + i0)}{\omega' - \omega + i0}. \]  

With the help of (16), we can rewrite (29) in the form

\[ A_{kl}(t) = \frac{1}{2\pi i} \int_{0}^{\infty} d\omega e^{-i\omega t} (G_{kl}(\omega + i0) - G_{kl}(\omega - i0)) = \frac{1}{2\pi i} \int_{C} d\omega e^{-i\omega t} G_{kl}(\omega), \]  

(31)

where the contour \( C \) is shown in Fig. 1. With the help of (18), we rewrite (30) in the form

\[ g_{kl}(\omega, t) = \lambda^2 \sum_{m,n=1}^{N} \int_{0}^{\infty} d\omega' f_m(\omega') f_n(\omega') G_{km}(\omega' - i0) G_{ln}(\omega' + i0) e^{-i\omega' t - e^{-i\omega t}} \frac{\omega' - \omega + i0}{\omega' - \omega + i0}. \]  

(32)

The integrand in (32) does not have any singularity at \( \omega' = \omega \), therefore \( i0 \) in the denominator becomes redundant. Then using (16) we obtain

\[ g_{kl}(\omega, t) = \frac{1}{2\pi i} \int_{0}^{\infty} d\omega' G_{kl}(\omega') \frac{e^{-i\omega' t} - e^{-i\omega t}}{\omega' - \omega}, \]  

(33)

where the contour \( C \) is shown in Fig. 1. For real \( \omega > 0 \) the term with the factor \( e^{-i\omega t} \) vanishes because it does not have any singularities outside the positive part of the real line. Then we have

\[ g_{kl}(\omega, t) = \frac{1}{2\pi i} \int_{C} d\omega' G_{kl}(\omega') \frac{e^{-i\omega' t}}{\omega' - \omega}. \]  

(34)

One can easily check the following relation between \( A_{kl}(t) \) and \( g_{kl}(\omega, t) \):

\[ A_{kl}(t) = \left( i \frac{d}{dt} - \omega \right) g_{kl}(\omega, t). \]  

(35)

The time evolution of a state \( |\Phi\rangle \), which is a superposition of the eigenstates of \( H_0 \),

\[ |\Phi\rangle = \sum_{k=1}^{N} a_k |k\rangle, \]  

(36)

may be obtained with the help of (27)

\[ |\Phi(t)\rangle = \sum_{k=1}^{N} a_k |k\rangle_t. \]  

(37)

The survival amplitude \( A(t) \) of this state is

\[ A(t) \equiv \langle \Phi | \Phi(t) \rangle = \sum_{k,k'=1}^{N} a_k a_{k'}^* \langle k|k'\rangle_t = \sum_{k,k'=1}^{N} a_k a_{k'}^* A_{kk'}(t). \]  

(38)
Changing the contour of the integration \( C \) to \( C_1 \) in \( A_{kk'}(t) \) as shown in Fig. 1, we arrive at

\[
A_{kk'}(t) = -\sum_j r_{kk'}^j e^{-iz_j t} + \frac{1}{2\pi i} \int_{C_1} d\omega e^{-i\omega t} G_{kk'}(\omega),
\]

where \( r_{kk'}^j \) is the residue of \( G_{kk'}(\omega) \) at the pole \( z_j \):

\[
r_{kk'}^j = \text{res} G_{kk'}(\omega)|_{\omega = z_j}.
\]

The first term in (39) corresponds to the contribution of the poles \( z_j \) while the second term is the background integral, which gives rise to so-called long tail behavior [27,30]. It is known that the integral term plays essential role for very long as well as very short times. In the case of very short times we have the well-known Zeno and anti-Zeno regions [9,11,14,37]. If we consider the intermediate “exponential decay” era, the integral term can be neglected because in this time scale, it is of the next order in \( \lambda^2 \) compared with the first term.

The same result for \( A_{kk'}(t) \) (39) is obtained in Appendix A in terms of Gamov vectors (A14). In the intermediate “exponential” era, the main contribution to the survival probability comes from the Gamov vectors as one may neglect the integral term arising from the background.

**IV. TWO LEVEL MODEL**

The rich structure of the model involving more than one level, will be first illustrated with example with two excited levels by choosing the formfactor in the form similar to [27]

\[
f_k(\omega) = \frac{\omega^{1/4}}{\omega + \rho_k^2}.
\]

For this formfactor the matrix element \( G^{-1}_{kk'}(\omega) \) (4) is

\[
G^{-1}_{kk'}(\omega) = (\omega_k - \omega)\delta_{kk'} + \frac{\pi\lambda^2}{\rho_k + \rho_k'} (\sqrt{\omega} + i\rho_k)(\sqrt{\omega} + i\rho_k'),
\]

where the first sheet of the complex \( \omega \) plane corresponds to the upper half of the complex \( \sqrt{\omega} \) plane. The square root is defined with the cut \([0, +\infty)\) such that \( \sqrt{\omega} > 0 \) at the upper rim of the cut. For \( \rho_k > 0 \) all singularities of the integral in expression (3) are on the second sheet. In the case of two levels the matrix is

\[
G^{-1}(\omega) = \begin{pmatrix}
(\omega_k - \omega) + \frac{\pi\lambda^2}{2\rho_1(\sqrt{\omega} + i\rho_1)^2} & \frac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} \\
\frac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} & (\omega_k - \omega) + \frac{\pi\lambda^2}{2\rho_2(\sqrt{\omega} + i\rho_2)^2}
\end{pmatrix}
\]

(43)

The \( 2 \times 2 \) matrix representing the partial resolvent is:

\[
G(\omega) = \text{det} G(\omega) = \begin{pmatrix}
(\omega_k - \omega) + \frac{\pi\lambda^2}{2\rho_2(\sqrt{\omega} + i\rho_2)^2} & -\frac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} \\
-\frac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} & (\omega_k - \omega) + \frac{\pi\lambda^2}{2\rho_1(\sqrt{\omega} + i\rho_1)^2}
\end{pmatrix}.
\]

(44)

The determinant \( \text{det} G(\omega) \) is

\[
(\text{det} G(\omega))^{-1} = \left[\frac{\pi\lambda^2}{2\rho_1(\sqrt{\omega} + i\rho_1)^2}\right] \left[\frac{\pi\lambda^2}{2\rho_2(\sqrt{\omega} + i\rho_2)^2}\right] - \left[\frac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)}\right]^2.
\]

(45)

Here we can formulate necessary conditions for the analyticity of the function \( G^{-1}_{kk'} \) on the first sheet:
1. $\omega_1^2 - \frac{\lambda^2}{2\rho_1} > 0$, $i = 1, 2$
2. $\left(\omega_1^2 - \frac{\lambda^2}{2\rho_1}\right) \left(\omega_1^2 - \frac{\lambda^2}{2\rho_2}\right) > \left(\frac{\pi\lambda^2}{\rho_1+\rho_2}\right)^2$.

These conditions are definitely satisfied in the weak coupling regime, because $\omega_1$, $\rho_1$, and $\lambda$ are independent parameters and for any fixed $\omega_1$ and $\rho_1$, in the limit $\lambda \to 0$ (46) becomes
1. $\omega_1^2 > 0$, $i = 1, 2$
2. $\omega_1 \rho_2^2 > 0$.

which is obviously true as $\omega_j$ and $\rho_j$ are positive for any $i$.

In order to find out the analytic structure of $G(\omega)$, we analyse the poles of the determinant:

$$(\text{det } G(\omega))^{-1} = \left(\omega + x^2)(x + \rho_1)^2 - \frac{\pi\lambda^2}{2\rho_1}\right) \left(\omega + x^2)(x + \rho_2)^2 - \frac{\pi\lambda^2}{2\rho_2}\right) - \left(\frac{\pi\lambda^2}{(\rho_1 + \rho_2)}\right)^2 = 0,$$  

where we substitute $\sqrt{\omega} = ix$. This is an algebraic equation of 8th degree with real coefficients, so all the roots of this equation are either real or complex conjugated pairs. All roots are on the second Riemann sheet, and there can be $k (k = 0 \ldots 4)$ pairs of complex conjugated roots and $(8 - 2k)$ real roots corresponding to virtual states, i.e. negative energy states on the second sheet. For weak coupling $\lambda \to 0$, the third possibility is realized and we have two pairs of complex conjugated roots $z_j$, $z_j^*$, which can be evaluated perturbatively as:

$$z_j = \omega_j + \frac{\pi\lambda^2}{2\rho_j} \frac{(\sqrt{\omega_j} - i\rho_j)^2}{(\sqrt{\omega_j} + i\rho_j)^2} + \frac{\pi^2\lambda^4}{4\rho_j^2(\sqrt{\omega_j} + i\rho_j)^3} + O(\lambda^6), \quad j = 1, 2, \quad k \neq j.$$  

For the weak coupling regime the expressions for the real and imaginary parts of $z_j$ are:

$$\bar{\omega}_j = \text{Re}z_j = \omega_j + \frac{\pi\lambda^2}{2\rho_j} \frac{\omega_j - \rho_j^2}{(\omega_j + \rho_j)^2} + O(\lambda^4), \quad j = 1, 2,$$

$$\gamma_j = -\text{Im}z_j = \frac{\pi\lambda^2}{\sqrt{\omega_j}} + O(\lambda^4), \quad j = 1, 2.$$  

Neglecting the integral term in the representation (49), we can write:

$$A(t) \approx \sum_{k,k'=1,2} a_k a_{k'} \sum_{j=1,2} e^{-\gamma_j t} e^{-i\omega_j t} \rho_j^2,$$  

$$= \sum_{k,k'=1,2} a_k a_{k'} e^{-i\frac{\gamma_1 + \gamma_2}{2} t} \left\{ (r_{kk'}^1 e^{-\gamma_1 t} + i r_{kk'}^2 e^{-\gamma_2 t}) e^{\nu t} + i \left( r_{kk'}^1 e^{-\gamma_1 t} - r_{kk'}^2 e^{-\gamma_2 t} \right) \sin \nu t \right\}$$  

$$= \sum_{j=1}^2 2 |a_j|^2 e^{-\gamma_j t} - \lambda^2 \sum_{j=1}^2 \left( \frac{i \pi |a_j|^2 e^{-i\omega_j t}}{2\rho_j (\rho_j - i\sqrt{\omega_j})^3 \sqrt{\omega_j}} \right.$$

$$\left. + \frac{2\pi \text{Re}(a_j^*/a_j^2) e^{-i\omega_j t}}{(\rho_1 + \rho_2)(\rho_j - i\sqrt{\omega_j})(\rho_j - i\sqrt{\omega_j})(\omega_j - \omega_l)} \right) + O(\lambda^4), \quad l \neq j,$$

where

$$\nu = \frac{\bar{\omega}_1 - \bar{\omega}_2}{2}.$$  

We would like to notice that both expressions (45) and (49) contain the term $1/(\omega_k - \omega_l)$ and, therefore, cannot be directly used in the case of degenerate levels in the initial Hamiltonian $H_0$. Also, the case of the continuous spectrum of $H_0$ requires a special consideration.

For the initial conditions $a_1 = 1$, $a_2 = 0$, the survival amplitude (45) does not have any oscillations. However, such oscillations appear in the next order $\lambda^4$ in expression (44). The survival probability $p(t)$ in the lowest order of $\lambda^2$ can be now expressed as:

$$p(t) = |A(t)|^2 = |a_1|^2 e^{-\gamma_1 t} + |a_2|^2 e^{-\gamma_2 t} e^{-2i\nu t}|^2.$$  

We illustrate the possible behavior of the survival probability in Fig. 2. One can see that depending on the initial conditions, the decay can either mimic the behavior of the usual one level model (11) or display considerable oscillations.
V. N-LEVEL MODEL

In the weak coupling regime we can also analyze the N-level model with an arbitrary formfactor \( f_k(\omega) \). Using the representation (33), we find

\[
(\det G(\omega))^{-1} = \prod_{k=1}^{N} (\omega_k - \omega) - \lambda^2 \sum_{k=1}^{N} I_{kk}(\omega) \prod_{m \neq k}^{N} (\omega_m - \omega) + O(\lambda^4),
\]

(51)

where

\[
I_{kk}(\omega) = \int_{0}^{\infty} d\omega' \frac{f_k(\omega') f_k(\omega')}{\omega' - \omega - i\delta}.
\]

The zeros of this expression give the position of resonances:

\[
z_k = \omega_k - \lambda^2 I_{kk}(\omega_k) + O(\lambda^4) = \bar{\omega}_k - i\gamma_k, \quad j = 1 \ldots N.
\]

(52)

In the first non-trivial order of the perturbation theory with respect to \( \lambda^2 \) we have:

\[
\bar{\omega}_k = \omega_k, \quad \gamma_k = \pi \lambda^2 f_k^2(\omega_k).
\]

The partial resolvent \( G \) can also be calculated:

\[
G_{kk'}(\omega) = (\omega_k - \omega - \lambda^2 I_{kk'}(\omega))^{-1} \delta_{kk'} + O(\lambda^2).
\]

(53)

From this representation we obtain the expression for the residues (40):

\[
r_{kk'}^j = -\delta_{kk'} \delta_{kj} + O(\lambda^2).
\]

(54)

We derive the survival amplitude (33) in the first non-vanishing term of the perturbation expansion with respect to \( \lambda^2 \):

\[
A(t) = \sum_{k=1}^{N} |a_k|^2 e^{-i\omega_k t} e^{-\pi \lambda^2 f_k^2(\omega_k)t}.
\]

(55)

In the case of the \( N \)-level model, the behavior of the survival probability is much more complicated than in two level model. In order to illustrate this, we plot in Fig. 3 few examples of the survival probability corresponding to different initial conditions for the three level model with different parameters. In this case, the behavior is not necessarily "self-similar" even for the very slow decay. One can see that our curves reproduce fairly well the experimental results. Hence we can suggest here an explanation of the results [5] which does not refer to the semiclassical description invoked “self-similar” even for the very slow decay. One can see that our curves reproduce fairly well the experimental results.

While the values (56), (57) are chosen quite arbitrarily and can only be used for illustrative purposes, we would like to notice that the excitation process in experiments similar to Figs. 3, 4 in paper [5] is usually well-defined and well-reproduced. Hence the initial wavepacket may also be well-correlated.
VI. CONCLUSIONS

Dissociation processes like the dissociation of NaI, which is a kind of tunneling/decay process, may be described by the simple quantum mechanical model of the interaction of the $N$-level discrete spectrum with the continuous spectrum. Already the model with two levels displays decaying oscillations of the survival probability in the “exponential” era, while one-level model exhibit the purely exponential decay. The amplitude of the oscillation is determined by the initial state, which is a superposition of two excited levels. The model with tree levels may illustrate qualitatively the experimental curve of the NaI dissociation. In the $N$-level system, the decay is equally defined by both the parameters of the system and the distribution of the initial wavepacket.

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APPENDIX A: TIME EVOLUTION IN TERMS OF GAMOV VECTORS

By analytic continuation to the second sheet, we obtain the extended distributions $G_{kl}^d(\omega \pm i0)$ and $1/|\omega - z_k|^+$ defined as functionals, which act on a suitable test function $h(\omega)$ as:

$$\int_0^{\infty} d\omega h(\omega)G_{kl}^d(\omega \pm i0) \equiv \int_{\Gamma} h(\omega)G_{kl}(\omega \pm i0)$$

$$= \int_0^{\infty} d\omega h(\omega)G_{kl}(\omega \pm i0) + 2\pi i \sum_j \int_{C_{z_j}} h(\omega)G_{kl}(\omega \pm i0),$$

$$\int_0^{\infty} d\omega \frac{h(\omega)}{|\omega - z_j|^+} \equiv \int_{\Gamma} \frac{h(\omega)}{\omega - z_j} = \int_0^{\infty} d\omega \frac{h(\omega)}{\omega - z_j} + 2\pi i \int_{C_{z_j}} \frac{h(\omega)}{\omega - z_j}. \quad (A2)$$

The contours $\Gamma$ and $C_{z_k}$ are presented in Fig. 1. Using [(A1)] and [(A2)], we obtain from [(12)] the Gamov vectors [(18,19,28)] in the form

$$|\phi_j^G\rangle = N_j \sum_{k,l=1}^{N} \lambda f_l(z_j) r_{kl}^j \left[|k\rangle - \int_0^{\infty} d\omega \frac{\lambda f_k(\omega)}{|\omega - z_j|^+} |\omega\rangle \right]. \quad (A3)$$

$$\langle \phi_j^G | = N_j \sum_{k,l=1}^{N} \lambda f_l(z_j) r_{kl}^j \left[\langle k| - \int_0^{\infty} d\omega \frac{\lambda f_k(\omega)}{|\omega - z_j|^+} \langle \omega| \right]. \quad (A4)$$

$$|\Psi_j^G\rangle = |\omega\rangle + \lambda \sum_{k,l=1}^{N} f_l(\omega)G_{kl}^d(\omega + i0) \left\{ \int_0^{\infty} d\omega' \frac{\lambda f_k(\omega')}{\omega' - \omega - i0} |\omega'\rangle - |k\rangle \right\}. \quad (A5)$$

$$\langle \Psi_j^G | = \langle \omega| + \lambda \sum_{k,l=1}^{N} f_l(\omega)G_{kl}(\omega - i0) \left\{ \int_0^{\infty} d\omega' \frac{\lambda f_k(\omega')}{\omega' - \omega + i0} \langle \omega'| - \langle k| \right\}. \quad (A6)$$

We recall that $r_{kl}^j$ is the residue of $G_{kl}(\omega + i0)$ at the pole $z_j$. The normalization constants $N_k$ are:

$$N_j^{-2} = \sum_{k,l,m,n=1}^{N} \lambda^2 f_l(z_j)f_m(z_j)r_{kl}^j r_{mn}^j \left[ \delta_{km} + \int_0^{\infty} d\omega \frac{\lambda^2 f_k(\omega)f_m(\omega)}{|\omega - z_j|^+} \right]. \quad (A7)$$

The Gamov vectors [(A3,A6)] are left and right eigenfunctions of the extended Hamiltonian, which can be written as
\[ H^+ = \sum_j z_j |\phi_j^G\rangle\langle \phi_j^G| + \int_0^\infty d\omega \omega |\Psi_\omega^G\rangle\langle \tilde{\Psi}_\omega^G|. \] (A8)

The Gamov vectors form a biorthonormal set:
\[ \langle \tilde{\phi}_j^G | \phi_{j'}^G \rangle = \delta_{jj'}, \quad \langle \tilde{\Psi}_\omega^G | \Psi_{\omega'}^G \rangle = \delta(\omega - \omega'), \quad \langle \tilde{\Psi}_\omega^G | \phi_{j'}^G \rangle = 0, \] (A9)

which is complete. The completeness follows from the extension of (14):
\[ I = \sum_{k=1}^N |\phi_k^G\rangle\langle \phi_k^G| + \int_0^\infty d\omega |\Psi_\omega^G\rangle\langle \tilde{\Psi}_\omega^G|. \] (A10)

The time evolution of the vector \(|k\rangle\) in the new extended representation is
\[ |k\rangle_t = \sum_j e^{-iz_j t} |\phi_j^G\rangle\langle \phi_j^G| k\rangle + \int_0^\infty d\omega e^{-i\omega t} |\Psi_\omega^G\rangle\langle \tilde{\Psi}_\omega^G| k\rangle. \] (A11)

Using (A3-A6), we express the transition amplitude
\[ \langle k|k'\rangle_t = \sum_j e^{-iz_j t} N_j^2 \sum_{l,l'=1}^N \lambda^2 f_l(z_j) f_{l'}(z_j) r_{klk'l'}^j + \int_0^\infty d\omega e^{-i\omega t} \lambda^2 f_l(z_j) f_{l'}(z_j) G_{kk'}^d(\omega + i0) G_{kk'}^d(\omega - i0). \] (A12)

The integral terms of (A12) can be rewritten in the form
\[ \frac{1}{2\pi i} \int_0^\infty d\omega e^{-i\omega t} \left( G_{kk'}^d(\omega + i0) - G_{kk'}^d(\omega - i0) \right). \] (A13)

Taking into account that \( G_{kk'}^d(\omega + i0) \) implies integration along the contour \( \Gamma^* \), which goes to the second Riemann sheet below all the singularities of \( G_{kk'}(\omega + i0) \), we obtain the transition amplitude in the form
\[ \langle k|k'\rangle_t = \sum_j e^{-iz_j t} N_j^2 \sum_{k,k',l,l'=1}^N \lambda^2 f_l(z_j) f_{l'}(z_j) r_{klk'l'}^j + \frac{1}{2\pi i} \int_{C_1} d\omega e^{-i\omega t} G_{kk'}, \] (A14)

which must coincide with the result obtained using Friedrichs solution (39). In order to fulfill this requirement the following formula must hold:
\[ N_j^2 \sum_{k,k',l,l'=1}^N \lambda^2 f_l(z_j) f_{l'}(z_j) r_{klk'l'}^j = -r_{kk'}^j. \] (A15)

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Figure captions

Fig. 1. The contours of integration C and C_1.

Fig. 2. The survival probability p(t) for the two level model. The parameters are chosen to be \( \gamma_1 = \gamma_2 = 10^{-3} \), \( \omega_1 = 1.0, \omega_1 = 1.06 \). The initial conditions are \( a_1 = 0.5, a_2 = 0 \) (the solid line), \( a_1 = 0.5, a_2 = 0.2 \) (the long-dashed line), \( a_1 = 0.5, a_2 = 0.5 \) (the short-dotted line).

Fig. 3. The survival probability p(t) for the three level model. The parameters are chosen to be \( \gamma_1 = \gamma_2 = \gamma_3 = 10^{-3} \), \( \omega_1 = 1, \) the initial conditions are \( a_1 = 0.3, a_2 = 0.5, a_2 = 0.3 \). The energies are \( \omega_2 = 1.04, \omega_3 = 1.15 \) (the long-dashed line), \( \omega_2 = 1.06, \omega_3 = 1.15 \) (the solid line), \( \omega_2 = 1.064, \omega_3 = 1.15 \) (the short-dashed line).

Fig. 4. The survival probability p(t) for the N-level model. The parameters are chosen to be \( \gamma_k = 10^{-3}, \omega_k = \omega_0 + k\Delta\omega/N, k = -N \ldots N \), where \( \omega_0 = 1, \Delta\omega = 0.1 \). The initial conditions are \( \tilde{a}_k = \exp(-(k/N)^2) \), \( k = -N \ldots N \). The results for \( N = 2 \) (the solid line), \( N = 3 \) (the short-dashed line), and \( N = 5 \) (the long-dashed line) are presented.
The graph shows the function $p(t)$ over time $t$, with $p(t)$ decreasing as $t$ increases.
