On the inductive blockwise Alperin weight condition for classical groups

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Abstract

Recently, there has been substantial progress on the Alperin weight conjecture. As a step to establish the Alperin weight conjecture for all finite groups, we prove the inductive blockwise Alperin weight condition for simple groups of classical type under some additional assumption.

Keywords Alperin weight conjecture, inductive blockwise Alperin weight condition, classical groups

1 Introduction

On the 1986 Arcata conference on representations of finite groups, J. L. Alperin put forward his famous conjecture, which is now called the Alperin weight conjecture. To state it, let $G$ be a finite group and $\ell$ a prime, $B$ an $\ell$-block of $G$. As usual, we denote by $\text{Irr}(B)$ and $\text{IBr}_\ell(B)$ the sets of ordinary irreducible characters and irreducible $\ell$-Brauer characters of $B$ respectively. For an $\ell$-subgroup $R$ of $G$ and $\varphi \in \text{Irr}(N_G(R))$, the pair $(R, \varphi)$ is called an $\ell$-weight if $R \subseteq \ker \varphi$ is of $\ell$-defect zero viewed as a character of $N_G(R)/R$. Note that $R$ is necessarily an $\ell$-radical subgroup of $G$ for any $\ell$-weight $(R, \varphi)$. An $\ell$-weight $(R, \varphi)$ is called a $B$-weight if $\text{bl}_\ell(\varphi)^G = B$, where $\text{bl}_\ell(\varphi)$ is the $\ell$-block of $N_G(R)$ containing $\varphi$. We denote by $\mathcal{W}_\ell(B)$ the set of all $G$-conjugacy classes of $B$-weights so that the Alperin weight conjecture can be stated as follows.

Conjecture 1.1 (Alperin, \cite{Alperin}). Let $G$ be a finite group, $\ell$ a prime. If $B$ is an $\ell$-block of $G$, then

$$|\mathcal{W}_\ell(B)| = |\text{IBr}_\ell(B)|.$$
So far the Alperin weight conjecture have been achieved substantial progress. Specifically, it was shown to hold for groups of Lie type in defining characteristic by Cabanes [13], for symmetric groups and general linear groups by Alperin and Fong [2], and for certain groups of classical type by An [3] and [4].

A reduction theorem for the blockfree version of Alperin weight conjecture was obtained by Navarro and Tiep [45] in 2011. Soon afterwards, Späth [51] refined this result to achieve a reduction theorem for the blockwise version of Alperin weight conjecture; if all finite (quasi-)simple groups satisfy the so-called inductive blockwise Alperin weight (iBAW) condition, then the Alperin weight conjecture [1.1] holds for any finite group. Koshitani and Späth [34] obtained another version of the (iBAW) condition which will be used in this paper; see Definition 2.11 for details.

To the present, the (iBAW) condition has been verified for some cases, such as simple alternating groups, many of the sporadic groups, simple groups of Lie type in the defining characteristic, Suzuki groups and Ree groups, simple groups of type $G_2$ and $^3D_4$, and simple groups of type $A$ with cyclic outer automorphism groups; see for instance [8], [16], [36], [40], [49] and [51]. Unfortunately, it seems a far way to deal with the case of simple groups of Lie type in non-defining characteristic.

In this paper, we consider the classical groups. As a first step to verify the (iBAW) condition, we need to establish a blockwise equivariant bijection between $\ell$-Brauer characters and $\ell$-weights. In [4], the author has essentially given such a bijection. In this paper, we first consider the groups of type $B$ and prove the bijection given in [4] for $\text{SO}_{2n+1}(q)$ is equivariant under the field automorphism (see Theorem 4.9). From this, we obtain a blockwise $\text{Aut}(S)$-equivariant bijection between $\text{IBr}_\ell(S)$ and $\text{W}_\ell(S)$ under some assumption (see Theorem 4.13). Our first main result about groups of type $B$ is the following.

**Theorem 1.2.** Let $X = \text{Spin}_{2n+1}(q)$ with $q = p^f$ odd and $n \geq 2$, $\ell$ an odd prime and $B$ an $\ell$-block of $X$. Assume further $f$ is odd, $\ell$ is linear and $B$ dominates some $\ell$-block of $\Omega_{2n+1}(q)$. Then the inductive blockwise Alperin weight (iBAW) condition (cf. Definition 2.11) holds for $B$.

Recall that an odd prime $\ell$ not dividing $q$ is called linear (for $q$) if the multiplicative order of $q$ modulo $\ell$ is odd. In this paper, the assumption that $\ell$ is linear is always to ensure the unitriangular shape of decomposition matrices, which is due to Gruber and Hiss [26]. It is an open problem to show that decomposition matrices of finite groups of Lie type in non-defining characteristic have unitriangular shape (see for example [41, Problem 4.8]). If this is true, then the assumption that $\ell$ is linear can be removed from the main results of this paper.

For groups of type $C$, we first verify the Alperin weight conjecture [1.1] for every $\ell$-block of $\text{Sp}_{2n}(q)$ when both $\ell$ and $q$ are odd, and then we prove the (iBAW) condition for the simple group of symplectic type and a linear prime if the outer automorphism group is cyclic, which can be stated as follows.

**Theorem 1.3.** Let $q$ be a power of an odd prime, $n \geq 2$, $\ell$ an odd prime. Then the Alperin weight conjecture [1.1] holds for every $\ell$-block of $\text{Sp}_{2n}(q)$.

**Theorem 1.4.** Let $q = p^f$ be a power of an odd prime $p$, $\ell$ an odd prime and $n \geq 2$. Assume that $f$ is odd and $\ell$ is linear. Then the inductive blockwise Alperin weight (iBAW) condition (cf. Definition 2.12) holds for the simple group $\text{PSp}_{2n}(q)$ and prime $\ell$.

In order to prove Theorem 1.4, we need a parametrization of $\ell$-blocks of $\text{Sp}_{2n}(q)$, which may be of independent interest; see Theorem 5.10. In order to do this, we make use of both the parametrization of $\ell$-blocks of $\text{CSp}_{2n}(q)$ from Fong–Srinivasan [23], and the label of $\ell$-blocks of
an arbitrary finite groups of Lie type from Cabanes–Enguehard [15] for $\ell \geq 7$ and Kessar–Malle [31] for the largest possible generality. From this, we obtain a blockwise bijection between the irreducible $\ell$-Brauer characters and $\ell$-weights of $\text{Sp}_{2n}(q)$ which is equivariant under the action of automorphisms (see Theorem 5.16).

We should mention that Conghui Li has proved independently Theorem 5.16 in [35] with different methods.

In addition, we also determined a similar parametrization of $\ell$-blocks for $\text{SO}^\pm_{2n}(q)$, which is described in Appendix §B. From this, if a hypothesis for the action of $\text{GO}^\pm_{2n}(q)$ on the characters of $\text{SO}^\pm_{2n}(q)$ is true, then the Alperin weight conjecture [11] holds for every $\ell$-block of $\text{SO}^\pm_{2n}(q)$ when both $\ell$ and $q$ are odd; see Theorem B.6.

Is there an analogue of Jordan decomposition for weights? Malle proposed this problem in [41, Problem 4.9]. Furthermore, following Kessar–Malle [32, p. 28], we hope for a Bonnafé–Rouquier type reduction (cf. [7], see also [32, §3.3]) to a few special situations, i.e., quasi-isolated blocks. In this sense, the unipotent blocks would play a fundamental and important role when considering the (iBAW) condition for finite quasi-simple groups of Lie type. In [20], the author verified the (iBAW) condition for unipotent blocks of groups of type $\text{A}$, untwisted or twisted, under some additional assumption on the prime involved. Considering classical type, the following is our main result for unipotent blocks.

**Theorem 1.5.** Assume that both $\ell$ and $q$ are odd. Suppose that one of the following holds.

(i) $X \in \{\text{Spin}_{2n+1}(q), \text{Sp}_{2n}(q)\}$ with $n \geq 2$.

(ii) $X = \text{Spin}_{2n}^-(q)$ with $n \geq 4$.

(iii) $X = \text{Spin}_{2n}^+(q)$ with $n > 4$ and $\ell$ is linear.

Then the inductive blockwise Alperin weight (iBAW) condition (cf. Definition 2.71) holds for every unipotent $\ell$-block of $X$.

This paper is built up as follows. In Section §2 we introduce the general notation and state the (iBAW) condition. In Section §3 the action of automorphisms on the weights of classical groups for a special case was considered. Then we prove Theorem 4.9 and prove Theorem 1.5 for type $B$ in Section §4. In Section §5 we give a classification for blocks of symplectic groups and then prove Theorem 1.4 and prove Theorem 1.5 for type $C$. Finally, the (iBAW) condition for unipotent blocks of classical groups of type $D$ and $^2D$ have been verified in Section §6.

## 2 Preliminaries

### 2.1 General results

Let $G$ be a finite group. Concerning the block and character theory of $G$ we mainly follow the notation of [44], where for sets of $\ell$-Brauer characters or $\ell$-blocks we add a subscript to indicate the corresponding prime $\ell$ (e.g. $\text{IBr}_\ell(G), \text{Bl}_\ell(G)$). We denote the restriction of $\chi \in \text{Irr}(G) \cup \text{IBr}(G)$ to some subgroup $H \leq G$ by $\text{Res}_H^G\chi$, while $\text{Ind}_H^G\psi$ denotes the character induced from $\psi \in \text{Irr}(H) \cup \text{IBr}(H)$ to $G$. For $N \trianglelefteq G$ we sometimes identity the characters of $G/N$ with the characters of $G$ whose kernel contains $N$.

The cardinality of a set, or the order of a finite group, $X$, is denoted by $|X|$. If a group $A$ acts on a finite set $X$, we denote by $A_x$ the stabilizer of $x \in X$ in $A$, analogously we denote by $A_X$ the setwise stabilizer of $X' \subseteq X$. 


Let $\ell$ be a prime. If $A$ acts on a finite group $G$ by automorphisms, then there is a natural action of $A$ on $\text{Irr}(G) \cup \text{IBr}(G)$ given by $a \cdot \chi(g) = \chi(a^g)$ for every $g \in G$, $a \in A$ and $\chi \in \text{Irr}(G) \cup \text{IBr}(G)$. For $P \leq G$ and $\chi \in \text{Irr}(G) \cup \text{IBr}(G)$, we denote by $A_{P, \chi}$ the stabilizer of $\chi$ in $A_P$.

Let $\chi \in \text{Irr}(G)$, we denote by $\chi^\circ$ the restriction of $\chi$ to the set of all $\ell'$-elements of $H$ for $\chi \in \text{Irr}(H)$. Let $Y \subseteq \text{IBr}(G)$. A subset $X \subseteq \text{Irr}(G)$ is called a basic set of $Y$ if $\{\chi^\circ \mid \chi \in X\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}Y$. If $Y = \text{IBr}(B)$ for some $\ell$-block $B$ of $G$, then we also say $X$ a basic set of $B$.

Let $O$ denote the ring of algebraic integers in $\mathbb{C}$. Following [44, §2] we fix a maximal ideal $M$ of $O$ containing the ideal $\ell O$. Then by [44, Lemma 2.1] the field $\mathbb{F} := O/M$ is an algebraic closure of its prime field $\mathbb{F}_\ell$ of characteristic $\ell$, and we denote by $^* : O \to \mathbb{F}$ the natural epimorphism. Let $\chi \in \text{Irr}(G)$. Then the central character associated to $\chi$ is the algebra homomorphism $\omega_\chi : Z(\mathbb{C}G) \to \mathbb{C}$, $C \mapsto \omega_\chi(\hat{C}) = \frac{C(1)}{\chi(1)}$, where $C$ is a conjugacy class of $G$, $\hat{C} = \sum_{x \in C} x$ and $x \in C$. This yields an algebra homomorphism $\lambda_\chi : Z(\mathbb{F}G) \to \mathbb{F}$ such that $\lambda_\chi(\hat{C}) = \omega_\chi(\hat{C})^*$ for a conjugacy class $C$ of $G$. Then for $\chi, \psi \in \text{Irr}(G)$, they are in the same $\ell$-block of $G$ if and only if $\lambda_\chi = \lambda_\psi$. Let $B$ be an $\ell$-block of $G$, then we define $\lambda_B = \lambda_\psi$ for $\chi \in \text{Irr}(G)$.

A subgroup $R \leq G$ is $\ell$-radical if $R = O_\ell(N_G(R))$. We also say that $R$ is an $\ell$-radical subgroup of $G$. We denote by $\text{Rad}_\ell(G)$ the set of $\ell$-radical subgroups of $G$. Furthermore, $\text{Rad}_\ell(G)/\sim_G$ denotes a $G$-transversal of radical $\ell$-subgroups of $G$.

We denote the set of all $G$-conjugacy classes of $\ell$-weights of $G$ by $\mathcal{W}_\ell(G)$ while $\mathcal{W}_\ell(B)$ denotes the set of all $G$-conjugacy classes of $B$-weights for an $\ell$-block $B$ of $G$.

The following lemma is elementary.

**Lemma 2.1.** Let $G$ be a finite group, $Z$ a central subgroup of $G$ and $\pi : G \to \tilde{G} = G/Z$ be the canonical homomorphism. Suppose that $\tilde{B}$ is an $\ell$-block of $\tilde{G}$ which is dominated by $\ell$-block $B$ of $G$. Let $(\tilde{R}, \tilde{\varphi})$ be a $\tilde{B}$-weight and let $R = \pi^{-1}(\tilde{R})$ and $\varphi$ the inflation of $\tilde{\varphi}$ from $N_G(\tilde{R}) = N_G(R)$ to $N_G(R)$. Then $(R, \varphi)$ be a $B$-weight.

**Lemma 2.2.** Let $G$ be a finite group, $Z$ a central $\ell'$-subgroup of $G$ and $\tilde{G} = G/Z$. Then there is a bijection $\Theta : \text{Rad}_\ell(G) \to \text{Rad}_\ell(\tilde{G})$ given by $R \mapsto \tilde{R}$ with inverse given by $Q/Z \mapsto O_\ell(Q)$.

Moreover, $\Theta$ induces a bijection between $\text{Rad}_\ell(G)/\sim_G$ and $\text{Rad}_\ell(\tilde{G})/\sim_{\tilde{G}}$.

**Proof.** This follows by [45, Lem. 2.3 (c)].

Note that, we have $N_G(\tilde{R}) = N_G(R)$ in Lemma 2.2.

**Lemma 2.3.** Keep the hypothesis and notation of Lemma 2.2

(i) If $(\tilde{R}, \tilde{\varphi})$ is an $\ell$-weight of $\tilde{G}$, then $(R, \varphi)$ is an $\ell$-weight of $G$, where $R = \Theta^{-1}(\tilde{R})$ and $\varphi$ is the inflation of $\tilde{\varphi}$ to $N_G(R)$.

(ii) If $(R, \varphi)$ is an $\ell$-weight of $G$ such that $Z \leq \ker \varphi$, then $(\tilde{R}, \tilde{\varphi})$ is an $\ell$-weight of $\tilde{G}$, where $\tilde{\varphi}$ is the character of $N_G(\tilde{R})$ whose inflation is $\varphi$.

(iii) Let $\tilde{B}$ be an $\ell$-block of $\tilde{G}$ and $B$ an $\ell$-block of $G$ dominating $\tilde{B}$. Then the map $\mathcal{W}_\ell(B) \to \mathcal{W}_\ell(\tilde{B})$ given by $(R, \varphi) \mapsto (\tilde{R}, \tilde{\varphi})$ is a bijection.

**Proof.** Both (i) and (ii) are obvious. For (iii), by (i), (ii) and Lemma 2.2, it suffices to show that $Z \leq \ker \varphi$ holds for any $(R, \varphi) \in \mathcal{W}_\ell(B)$. Let $b = b_\ell(\varphi)$, then $b^G = B$. Thus $\lambda_b(z) = \lambda_\psi(z)$ for all $z \in Z$. By [44, Thm. (9.9)(c)], $Z \leq \ker \chi$ for all $\chi \in \text{Irr}(B)$. Hence $\omega_\chi(z) = \chi(z)/\chi(1) = 1$ and then $\lambda_\psi(z) = 1$ for all $z \in Z$. Thus $\lambda_b(z) = 1$, i.e. $\omega_\psi(z)^* = 1$ for all $z \in Z$. However, $\omega_\varphi(z)$
is an \( \ell' \)-root of unity and then by [44] Lem. (2.1), \( 1 = \omega_\varphi(z) = \varphi(z)/\varphi(1) \) for all \( z \in \mathbb{Z} \). Hence \( Z \leq \ker \varphi \), as desired.

\[ \square \]

**Lemma 2.4.** Keep the hypothesis and notation of Lemma 2.2. Let \( \sigma \in \text{Aut}(G) \) and \( \tilde{\sigma} \) the automorphism of \( \tilde{G} \) induced by \( \sigma \). Let \( \tilde{R}_i = \text{Rad}_i(\tilde{G}) \) and \( R_i = \Theta^{-1}(\tilde{R}_i) \) for \( i = 1, 2 \). If \( \tilde{\sigma}(\tilde{R}_1) = \tilde{R}_2 \), then \( \sigma(R_1) = R_2 \).

**Proof.** By the assumption, \( \sigma(R_1)Z = R_2Z \). Then, \( \sigma(R_1) = R_2 \) since it is the unique Sylow \( \ell \)-subgroup of \( \sigma(R_1)Z = R_2Z \), as stated.

\[ \square \]

**Lemma 2.5.** Keep the hypothesis and notation of Lemma 2.2 and Lemma 2.3 (iii). Let \( A \leq \text{Aut}(G) \) and \( \tilde{A} \) the subgroup of \( \text{Aut}(\tilde{G}) \) induced by \( A \). If there is an \( \tilde{A} \)-equivariant bijection between \( \text{IBr}_\ell(\tilde{B}) \) and \( \tilde{W}_i(\tilde{B}) \), then there is an \( A \)-equivariant bijection between \( \text{IBr}_\ell(B) \) and \( W_i(B) \).

**Proof.** This follows by Lemma 2.3 (iii) and Lemma 2.4 and the fact that \( \text{IBr}_\ell(B) = \text{IBr}_\ell(\tilde{B}) \) (see for example [44] Thm. (9.9)).

**Lemma 2.6.** Let \( N \) be a normal subgroup of finite group \( G \) such that \( G/N \) is cyclic and \( G/\mathbb{Z}(G) \) an \( \ell' \)-group and let \( b \) be an \( \ell \)-block of \( N \). Suppose that there are \( m \) \( \ell \)-blocks of \( G \) covering \( b \), where \( m = [G/N]_\ell \). Then the following statements hold.

(i) \( \text{Res}_N^G : \text{IBr}_\ell(B) \to \text{IBr}_\ell(B) \) is bijective for any \( \ell \)-block \( B \) of \( G \) covering \( b \).

(ii) Let \( A \) be a subgroup of \( \text{Aut}(G) \) stabilizing \( N \). Suppose that \( B \) is an \( \ell \)-blocks of \( G \) covering \( b \) such that \( B \) is \( A \)-invariant. If \( \phi \in \text{IBr}_\ell(B) \) which is \( A \)-invariant, then there is an extension \( \tilde{\phi} \in \text{IBr}_\ell(B) / \phi \) of \( \phi \) such that \( \tilde{\phi} \) is \( A \)-invariant.

**Proof.** Let \( \phi \in \text{IBr}_\ell(B) \) and \( \tilde{\phi} \in \text{IBr}_\ell(G) / \phi \). Since \( G/N \) is cyclic, by Clifford theory, each irreducible \( \ell \)-Brauer character covering \( \phi \) has form \( \tilde{\phi} \tau \) with \( \tau \in \text{IBr}_\ell(G/N) \). Then \( |\text{IBr}_\ell(G) / \phi| \leq m \). Now there are \( m \) \( \ell \)-blocks of \( G \) covering \( b \), so \( |\text{IBr}_\ell(G) / \phi| = m \). By Clifford theory, \( G/G_\phi \) is an \( \ell' \)-group. Since \( G/\mathbb{Z}(G) \) is an \( \ell' \)-group, we have \( G = G_\phi \), and then \( \tilde{\phi} \) is an extension of \( \phi \). Thus (i) follows easily.

For (ii), let \( \tilde{\phi} \in \text{IBr}_\ell(B) \) be the extension of \( \phi \). Since \( \phi \) is \( A \)-invariant and \( B \) is \( A \)-invariant too, we get that \( \tilde{\phi}^a \in \text{IBr}_\ell(B) \) is also an extension of \( \phi \) for any \( a \in A \). By the uniqueness of \( \tilde{\phi} \), we have \( \tilde{\phi}^a = \tilde{\phi} \). Then \( \tilde{\phi} \) is \( A \)-invariant.

\[ \square \]

**Lemma 2.7.** Let \( K \) be a subgroup of finite group \( G \) and \( b \) an \( \ell \)-block of \( K \), and \( \theta \) a linear character of \( G \) of \( \ell' \)-order. Assume that both \( b^G \) and \((\text{Res}_K^G \theta \otimes b)^G \) are defined. Then \((\text{Res}_K^G \theta \otimes b)^G = \theta \otimes b^G \).

**Proof.** Let \( B = b^G \). Then \( \lambda_\theta(\hat{C}) = \lambda_\theta(C \cap K) \) and \( \lambda_\theta((\text{Res}_K^G \theta \otimes b)^G) = \lambda_\theta(C \cap K) \), for any conjugacy class \( C \) of \( G \). It is easy to check that \( \lambda_{\text{Res}_K^G \theta \otimes b}(\hat{C}) = \theta(x)^* \lambda_\theta(C \cap K) \) and \( \lambda_{\text{Res}_K^G \theta \otimes b}(C \cap K) = \theta(x)^* \lambda_\theta(C \cap K) \) for \( x \in C \). Thus \( \lambda_{\text{Res}_K^G \theta \otimes b} = \lambda_{(\text{Res}_K^G \theta \otimes b)^G} \) and then \((\text{Res}_K^G \theta \otimes b)^G = \theta \otimes b^G \).

By Lemma 2.7 we have the following result immediately.
Corollary 2.8. Let $G$ be a finite group, $B$ an $\ell$-block and $(R, \varphi)$ a $B$-weight. Suppose that $\theta$ is a linear character of $G$ with $\ell'$-order. Then $(R, (\Res^G_{N_G(R)} \theta) \varphi)$ is a $\theta \otimes B$-weight.

We will make use of the following result.

Lemma 2.9. Let $A$ be a finite group, $G$ a normal subgroup of $A$ and $B$ an $\ell$-block of $G$. Suppose that there exists a basic set $X \subseteq \Irr(B)$ of $B$ such that the corresponding decomposition number matrix is unitriangular with a suitable order. If every $\chi \in X$ extends to $A$, then every $\phi \in \Irr(B)$ extends to $A$.

Proof. By [16, Lem. 7.5], there exists an $A$-equivariant bijection $\mathcal{D} : X \to \Irr(B)$ such that $\chi^o = \mathcal{D}(\chi) + \sum_{\phi \in \Irr(B) \setminus \mathcal{D}(\chi)} d_\phi \phi$ with $d_\phi \in \mathbb{Z}_{\geq 0}$. In particular, $A_\chi = A_{\mathcal{D}(\chi)}$. Now let $\tilde{\chi}$ be an extension of $\chi$ to $A$ and let $\tilde{\phi}$ be an irreducible constituent of $\tilde{\chi}^o$ such that $\tilde{\phi} \in \Irr(A, \mathcal{D}(\chi))$. Since $\mathcal{D}(\chi)$ is $A_\chi$-invariant, by Clifford theory, we know that $\tilde{\phi}$ is an extension of $\mathcal{D}(\chi)$ to $A_{\mathcal{D}(\chi)}$. \hfill $\square$

2.2 Background of the representations of finite groups of Lie type

We will need to view some finite classical groups as the groups of fixed points under some Frobenius endomorphisms of certain connected reductive algebraic groups. Let $q$ be a power of prime $p$ and let $\mathbb{F}_q$ be the field of $q$ elements. Also let $\overline{\mathbb{F}}_q$ be the algebraic closure of the field $\mathbb{F}_q$.

Algebraic groups are usually denoted by boldface letters. Suppose that $G$ is a connected reductive algebraic group over $\overline{\mathbb{F}}_q$ and $F : G \to G$ a Frobenius endomorphism endowing $G$ with an $\overline{\mathbb{F}}_q$-structure. The group of rational points $G^F$ is finite. Let $G^s$ be dual to $G$ with corresponding Frobenius endomorphism also denoted $F$.

Let $\ell$ be a prime number different from $p$. For a semisimple $\ell'$-element $s$ of $G^s$, we denote by $E_t(G^F, s)$ the union of the Lusztig series $E_t(G^F, st)$, where $t$ runs through semisimple $\ell$-elements of $G^F$ commuting with $s$. By [12], the set $E_0(G^F, s)$ is a union of $\ell$-blocks of $G^F$.

Also, we denote by $E(G^F, \ell')$ the set of irreducible characters of $G^F$ lying in a Lusztig series $E(G^F, s)$, where $s \in G^s$ is a semisimple $\ell'$-element. Considering the elements of $E(G^F, \ell')$ as a basic set is the main argument of [25] with the assumption that $\ell$ is good and $Z(G)$ is connected. It was generalized in [24 Thm. A], which can be stated as follows.

Theorem 2.10. Let $\ell$ be a prime good for $G$ and not dividing the defining characteristic of $G$. Assume that $\ell$ does not divide $(Z(G)/Z^{s}(G))_F$ (the largest quotient of $Z(G)$ on which $F$ acts trivially). Let $s \in G^s$ be a semisimple $\ell'$-element. Then $E(G^F, s)$ form a basic set of $E_0(G^F, s)$.

In this paper, any algebraic group $G$ involved is of classical type and the prime $\ell$ is always odd. Thus the hypothesis of Theorem [2.10] is always satisfied.

Let $d$ be a positive integer. We will make use of the terminology of Sylow $d$-theory (see for instance [10] and [11]). For an $F$-stable maximal torus $T$ of $G$, denotes $T_d$ its Sylow $d$-torus. An $F$-stable Levi subgroup $L$ of $G$ is called $d$-split if $L = C_G(Z^s(L)_d)$, and $\zeta \in \Irr(L^F)$ is called $d$-cuspidal if $\chi_{\text{M}}(\zeta) = 0$ for all proper $d$-split Levi subgroups $M < L$ and any parabolic subgroup $P$ of $L$ containing $M$ as Levi complement.

Let $s \in G^s$ be semisimple. Following [31 Def. 2.1], we say $\chi \in E(G^F, s)$ is $d$-Jordan-cuspidal if

- $Z^s(C^s_G(s))_d = Z^s(G^s)_d$, and
- $\chi$ corresponds under Jordan decomposition (cf. [38 Prop. 5.1]) to the $C_G(s)_F$-orbit of a $d$-cuspidal unipotent character of $C^s_G(s)_F$. 


If $\mathbf{L}$ is a $d$-split Levi subgroup of $\mathbf{G}$ and $\zeta \in \text{Irr}(\mathbf{L}^F)$ is $d$-Jordan-cuspidal, then $(\mathbf{L}, \zeta)$ is called a $d$-Jordan-cuspidal pair of $\mathbf{G}$.

Now we define an integer $e_0 = e_0(q, \ell)$, which is denoted by “$e$” in \cite{31} (in this paper, we will use “$e$” for another integer, see Section 2.4 page 10):

$$e_0 = e_0(q, \ell) = \text{multiplicative order of } q \text{ modulo } \begin{cases} \ell & \text{if } \ell > 2, \\ 4 & \text{if } \ell = 2. \end{cases} \quad (2.1)$$

The paper \cite{15} gave a label for arbitrary $\ell$-blocks of finite groups of Lie type for $\ell \geq 7$ and it was generalised in \cite{31} to its largest possible generality. Under the condition of \cite{31} Thm. A (e)], the set of $\mathbf{G}^F$-conjugacy classes of $e_0$-Jordan-cuspidal pairs $(\mathbf{L}, \zeta)$ of $\mathbf{G}$ such that $\zeta \in \mathcal{E}(\mathbf{L}^F, \ell')$, is a labeling set of the $\ell$-blocks of $\mathbf{G}^F$.

By \cite{6} Thm., the Mackey formula holds if $q > 2$, hence the Lusztig induction $R_{L\leq P}^G$ is independent of the ambient parabolic subgroup $P$ in this paper since we always assume that $q$ is odd. So throughout this paper we always omit the parabolic subgroups when considering Lusztig inductions since the “$q$” occurring in this paper is always odd.

### 2.3 The inductive blockwise Alperin weight conditions

#### Notation.
For a finite group $H$ and a prime $\ell$, we denote by

- $dz_\ell(H)$ the set of $\ell$-defect zero characters of $H$ and
- $bl_\ell(\varphi)$ the $\ell$-block of $H$ containing $\varphi$, for $\varphi \in \text{Irr}(H) \cup \text{IBr}_\ell(H)$.

If $Q$ is a radical $\ell$-subgroup of $H$ and $B$ an $\ell$-block of $H$, then we define the set

$$dz_\ell(N_H(Q)/Q, B) := \{ \chi \in dz_\ell(N_H(Q)/Q) | bl_\ell(\chi)^H = B \},$$

where we regard $\chi$ as an irreducible character of $N_Q(Q)$ containing $Q$ in its kernel when considering the induced $\ell$-block $bl_\ell(\chi)^H$.

There are several versions of the (iBAW) condition. Apart from the original version given in \cite{31} Def. 4.1], there is also a version treating only blocks with defect groups involved in certain sets of $\ell$-groups \cite{31} Def. 5.17], or a version handling single blocks \cite{34} Def. 3.2]. We shall consider the inductive condition for a single block here.

#### Definition 2.11 (\cite{34} Def. 3.2]).
Let $\ell$ be a prime, $S$ a finite non-abelian simple group and $X$ the universal $\ell$-covering group of $S$. Let $B$ be an $\ell$-block of $X$. We say the \textit{inductive blockwise Alperin weight (iBAW) condition} holds for $B$ if the following statements hold:

1. There exist subsets $\text{IBr}_\ell(B \mid Q) \subseteq \text{IBr}_\ell(B)$ for $Q \in \text{Rad}_\ell(X)$ with the following properties:
   - $\text{IBr}_\ell(B \mid Q)^a = \text{IBr}_\ell(B \mid Q^a)$ for every $Q \in \text{Rad}_\ell(X), a \in \text{Aut}(X)_b$,
   - $\text{IBr}_\ell(B) = \bigcup_{Q \in \text{Rad}_\ell(X)/\sim} \text{IBr}_\ell(B \mid Q)$.
2. For every $Q \in \text{Rad}_\ell(X)$ there exists a bijection

   $$\Omega^X_Q : \text{IBr}_\ell(B \mid Q) \to dz_\ell(N_X(Q)/Q, B)$$

   such that $\Omega^X_Q(\phi)^a = \Omega^X_Q(\phi^a)$ for every $\phi \in \text{IBr}_\ell(B \mid Q)$ and $a \in \text{Aut}(X)_b$.  

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(iii) For every \( Q \in \text{Rad}_\ell(X) \) and every \( \phi \in \text{IBr}_\ell(B \mid Q) \) there exist a finite group \( A := A(\phi, Q) \) and \( \tilde{\phi} \in \text{IBr}_\ell(A) \) and \( \tilde{\phi}' \in \text{IBr}_\ell(N_A(Q)) \), where we use the notation

\[
\overline{Q} := QZ/Z \text{ and } Z := Z(X) \cap \ker(\phi),
\]

with the following properties:

1. for \( \overline{X} := X/Z \) the group \( A \) satisfies \( \overline{X} \leq A, A/C_A(\overline{X}) \cong \text{Aut}(X)_\phi, C_A(\overline{X}) = Z(A) \) and \( \ell \nmid |Z(A)| \),
2. \( \tilde{\phi} \in \text{IBr}_\ell(A) \) is an extension of the \( \ell \)-Brauer character of \( \overline{X} \) associated with \( \phi \),
3. \( \tilde{\phi}' \in \text{IBr}_\ell(N_A(Q)) \) is an extension of the \( \ell \)-Brauer character of \( N_X(Q) \) associated with the inflation of \( \Omega_X^\ell(\phi)^\circ \in \text{IBr}_\ell(N_X(Q)/Q) \) to \( N_X(Q) \),
4. \( \text{bl}_\ell(\text{Res}_A^X(\tilde{\phi})) = \text{bl}_\ell(\text{Res}^N_{N_Q(Q)}(\tilde{\phi}'))^{J} \) for every subgroup \( J \) satisfying \( \overline{X} \leq J \leq A \).

**Definition 2.12.** Let \( \ell \) be a prime, \( S \) a finite non-abelian simple group and \( X \) the universal \( \ell' \)-covering group of \( S \). We say that the inductive blockwise Alperin weight (iBAW) condition holds for \( S \) and \( \ell \) if the (iBAW) condition holds for every \( \ell \)-block of \( X \).

**Lemma 2.13.** Let \( \ell \) be a prime, \( S \) a finite non-abelian simple group and \( X \) the universal \( \ell' \)-covering group of \( S \). Let \( B \) be an \( \ell' \)-block of \( X \). If there is an \( \text{Aut}(X)_B \)-equivariant bijection between \( \text{IBr}_\ell(B) \) and \( \mathcal{W}_\ell(B) \), then there are natural defined sets \( \text{IBr}_\ell(B \mid Q) \) and bijections \( \Omega_Q(X) \) such that (i) and (ii) of Definition 2.11 holds for \( B \).

**Proof.** This is [49, Lem. 2.10].

**Corollary 2.14.** Let \( \ell \) be a prime, \( S \) a finite non-abelian simple group such that \( \text{Aut}(S)/S \) is cyclic and \( X \) the universal \( \ell' \)-covering group of \( S \). Let \( B \) be an \( \ell' \)-block of \( X \). If there is an \( \text{Aut}(X)_B \)-equivariant bijection between \( \text{IBr}_\ell(B) \) and \( \mathcal{W}_\ell(B) \), then the (iBAW) condition holds for \( B \).

**Proof.** By the proof of [51, Lem. 6.1], it suffices to prove (i) and (ii) of Definition 2.11 (for details, see [49, Rmk. 2.7]), which follows by Lemma 2.13.

The following lemma is [51, Thm. C]. From this, for finite simple group of Lie type, we only need to consider non-defining characteristic.

**Lemma 2.15.** Suppose that \( S \) is a finite simple group of Lie type defined over a field of characteristic \( p \). Then the (iBAW) condition holds for \( S \) and prime \( p \).

### 2.4 Some notations and conventions for classical groups

In this paper, we always assume that \( p \) is an odd prime, \( q = p^f \) with a positive integer \( f \), and \( \ell \) is an odd prime number different from \( p \). Let \( \mathbb{F}_q \) be the field of \( q \) elements.

We follow mainly the notation from [23] and [4]. Let \( V \) be a finite dimensional symplectic or orthogonal space over the field \( \mathbb{F}_q \). We denote by \( I(V) \) the group of isometries of \( V \), \( I_0(V) \) the subgroups of \( I(V) \) of determinant 1, and \( \eta(V) = \pm 1 \) the type of \( V \) if \( V \) is orthogonal. For simplicity, we set \( \eta(V) = 1 \) if \( V \) is symplectic. Furthermore, we identify 1, \(-1\) with \(+\), \(-\) respectively when considering the type of spaces and groups. Obviously, \( I(V) = I_0(V) = \text{Sp}(V) \) if \( V \) is a symplectic space and \( I(V) = \text{GO}(V), I_0(V) = \text{SO}(V) \) if \( V \) is an orthogonal space.
We recall that there exists a set \( \mathcal{F} \) of polynomials serving as elementary divisors for all semisimple elements of each of these groups. We denote by \( \text{Irr}(\mathbb{F}_q[x]) \) the set of all monic irreducible polynomials over the field \( \mathbb{F}_q \). For each \( \Delta \) in \( \text{Irr}(\mathbb{F}_q[x]) \), we define \( \Delta^* \) be the polynomial in \( \text{Irr}(\mathbb{F}_q[x]) \) whose roots are the inverses of the roots of \( \Delta \). Now, we denote by

\[
\mathcal{F}_0 = \{ x - 1, x + 1 \},
\]

\[
\mathcal{F}_1 = \{ \Delta \in \text{Irr}(\mathbb{F}_q[x]) \mid \Delta \notin \mathcal{F}_0, \Delta \neq x, \Delta = \Delta^* \},
\]

\[
\mathcal{F}_2 = \{ \Delta \Delta^* \mid \Delta \in \text{Irr}(\mathbb{F}_q[x]) \setminus \mathcal{F}_0, \Delta \neq x, \Delta \neq \Delta^* \}.
\]

Let \( \mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \). Given \( \Gamma \in \mathcal{F} \), denote \( d_\Gamma \) its degree and \( \delta_\Gamma \) its reduced degree defined by

\[
\delta_\Gamma = \begin{cases} 
\frac{d_\Gamma}{2} & \text{if } \Gamma \in \mathcal{F}_0; \\
\frac{1}{2}d_\Gamma & \text{if } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2.
\end{cases}
\]

Since the polynomial in \( \mathcal{F}_1 \cup \mathcal{F}_2 \) have even degree, \( \delta_\Gamma \) is an integer. In addition, we mention a sign \( \varepsilon_\Gamma \) for \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \) defined by

\[
\varepsilon_\Gamma = \begin{cases} 
-1 & \text{if } \Gamma \in \mathcal{F}_1; \\
1 & \text{if } \Gamma \in \mathcal{F}_2.
\end{cases}
\]

Given a semisimple element \( s \in I(V) \), there exists a unique orthogonal decomposition

\[
V = \sum_\Gamma V_\Gamma(s), \quad s = \prod_\Gamma s(\Gamma),
\]

(2.2)

where the \( V_\Gamma(s) \) are non-degenerate subspaces of \( V \), \( s(\Gamma) \in I(V_\Gamma(s)) \), and \( s(\Gamma) \) has minimal polynomial \( \Gamma \). The decomposition (2.2) is called the primary decomposition of \( s \) in \( I(V) \). Let \( m_\Gamma(s) \) be the multiplicity of \( \Gamma \) in \( s(\Gamma) \). If \( m_\Gamma(s) \neq 0 \), then we say \( \Gamma \) an elementary divisor of \( s \). Then the centralizer of \( s \) in \( I(V) \) has a decomposition \( C_{I(V)}(s) = \prod_\Gamma C_\Gamma(s) \), where \( C_\Gamma(s) = C_{I(V_\Gamma(s))}(s(\Gamma)) \). Moreover, by [23, (1.13)],

\[
C_\Gamma(s) = \begin{cases} 
I(V_\Gamma(s)) & \text{if } \Gamma \in \mathcal{F}_0; \\
\text{GL}_{m_\Gamma(s)}(q^{\delta_\Gamma}) & \text{if } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2.
\end{cases}
\]

Here, \( \text{GL}_m(q) \) means \( \text{GU}_m(q) \). Note that \( C_\Gamma(s) \leq I_0(V_\Gamma(s)) \) for \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \).

Let \( \eta_\Gamma(s) \) be the type of \( V_\Gamma(s) \). Here \( \eta_\Gamma(s) = 1 \) for all \( \Gamma \in \mathcal{F} \) if \( V \) is symplectic. By [23, (1.12)], the multiplicity and type functions \( \Gamma \mapsto m_\Gamma(s), \Gamma \mapsto \eta_\Gamma(s) \) satisfy the following relations

\[
\dim V = \sum_\Gamma d_\Gamma m_\Gamma(s),
\]

\[
\eta(V) = (-1)\frac{\sum_{m_\Gamma(s) \leq 1} d_\Gamma m_\Gamma(s)}{\prod_\Gamma \eta_\Gamma(s)},
\]

(2.3)

\[
\eta(V_\Gamma(s)) = \varepsilon_\Gamma^{m_\Gamma(s)} \text{ for } \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2.
\]

Conversely, if \( \Gamma \mapsto m_\Gamma(s), \Gamma \mapsto \eta_\Gamma \) are functions from \( \mathcal{F} \) to \( \mathbb{N}, \{ \pm 1 \} \) respectively satisfying (2.3), then there exists a semisimple element \( s \) of \( I(V) \) with these functions as multiplicity and type functions. Moreover, two semisimple elements \( s \) and \( s' \) of \( I(V) \) are \( I(V) \)-conjugate if and only if \( m_\Gamma(s) = m_\Gamma(s') \) and \( \eta_\Gamma(s) = \eta_\Gamma(s') \) for all \( \Gamma \in \mathcal{F} \).

For \( \Gamma \in \mathcal{F} \), let \( (\Gamma) \) be the companion matrix of \( \Gamma \). Denote \( m(\Gamma) = \text{diag}(\Gamma, \ldots, \Gamma) \) with \( m \) copies. So if \( s \) is a semisimple element of \( I(V) \) with a unique elementary divisor \( \Gamma \) of multiplicity \( m \), then \( s \) is \( I(V) \)-conjugate to \( m(\Gamma) \).
Now assume that \( V \) is orthogonal. A semisimple element \( s \) lies in \( I_0(V) \) if and only if \( m_{x+1}(s) \) is even. If \( s \in I_0(V) \), then
\[
|C_{\ell_0}(V)(s) : \prod_{\Gamma} C_{\ell_0(V_{\Gamma}(s))(s(\Gamma))}| = 1 \text{ or } 2,
\]
and index 2 occurs if and only if \( m_{x-1}(s) \) and \( m_{x+1}(s) \) are both non-zero. For more details, see [23, §1].

For a semisimple element \( s \in I_0(V) \), we define \( \Psi_I(s) \) to be the set of partitions of \( m_I(s) \) if \( \Gamma \in \mathcal{F}_I \cup \mathcal{F}_2 \). If \( \Gamma \in \mathcal{F}_0 \), then \( \Psi_I(s) \) is defined to be the set of symbols of rank \( \left\lfloor \frac{m_I(s)}{2} \right\rfloor \) such that
- If \( \Psi_I(s) \) is symplectic or orthogonal of odd dimension, then the symbols have odd defect.
- If \( \Psi_I(s) \) is orthogonal of even dimension and type +, then the symbols have defect divided by 4. Moreover, degenerate symbols are counted twice.
- If \( \Psi_I(s) \) is orthogonal of even dimension and type −, then the symbols have defect congruent to 2 modulo 4.

Let
\[
\Psi(s) = \prod_{\Gamma} \Psi_I(s).
\] (2.4)
Follow [23, p. 132], we define an operate \( ' \) on the sets \( \Psi(s) \) and \( \Psi_I(s) \) as follows. Let \( \mu_I \in \Psi_I(s) \).
Then define \( (\mu_I)' = \mu_I \) if \( \mu_I \) is a partition or a non-degenerate symbol, and define \( (\mu_I)' \) to be the other copy of \( \mu_I \) in \( \Psi_I(s) \). If \( \mu = \prod_I \mu_I \in \Psi(s) \), then we define \( \mu' = \prod_I (\mu_I)' \).

In this paper, we let \( e \) be the multiplicative order of \( q^2 \) modulo \( \ell \). Then \( e = e_0/\gcd(2, e_0) \), where \( e_0 \) is defined as in (2.1). We say the prime \( \ell \) is linear if \( e_0 \) is odd while \( \ell \) is unitary if \( e_0 \) is even.

Let \( \mathcal{F}' \) be the subset of \( \mathcal{F} \) consisting of polynomials whose roots are of \( \ell' \)-order. For \( \Gamma \in \mathcal{F}' \), we define \( e_\Gamma \) to be the multiplicative order of \( q^2 \) or \( e_\Gamma q^{2r} \) modulo \( \ell \) according as \( \Gamma \in \mathcal{F}_0 \) or \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \). Then \( e_\Gamma = e \) for \( \Gamma \in \mathcal{F}_0 \). Let \( s \) be a semisimple \( \ell' \)-element of \( I_0(V) \) and \( \mu = \prod_I \mu_I \in \Psi(s) \). Now we define the \( e_\Gamma \)-core of \( \mu_I \) for every \( \Gamma \in \mathcal{F}' \). If \( \Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2 \), the \( e_\Gamma \)-core of \( \mu_I \) is defined in the usual way for partitions (see for example [47, §3]). For \( \Gamma \in \mathcal{F}_0 \), the \( e_\Gamma \)-core of \( \mu_I \) is defined in [23, p. 159] which we state as follows. Let \( \kappa_I \) be the symbols which is gotten by actually removing \( w_I \) \( e_I \)-hooks (or \( e_I \)-cohooks, resp.) from \( \mu_I \) and there is no \( e_I \)-hooks (or \( e_I \)-cohooks, resp.) in \( \kappa_I \) if \( \ell \) is linear (or unitary, resp.). If \( \kappa_I \) is degenerate and \( w_I > 0 \), then both copies of \( \kappa_I \) are considered as the \( e_\Gamma \)-core of \( \mu_I \) (i.e., the \( e_\Gamma \)-core of \( \mu_I \) is defined to be the doubleton \( \{\kappa_I, \kappa'_I\} \)). If \( \kappa_I \) is degenerate and \( w_I = 0 \) (i.e., \( \kappa_I = \mu_I \)), the only \( \kappa_I \), but not its copy, is the \( e_\Gamma \)-core of \( \mu_I \). If \( \kappa_I \) is non-degenerate, then the \( e_\Gamma \)-core of \( \mu_I \) is \( \kappa_I \).

Note that the definition of \( e_\Gamma \)-core of a symbol here (as in [23]) is the same with the definition in [22, p. 307], and is slightly different from those used in [11], [46] and [47]. We follow [22] and [23] and say \( e_\Gamma \)-core for both \( e_\Gamma \)-core (when \( \ell \) is linear) and \( e_\Gamma \)-cocore (when \( \ell \) is unitary) in [11], [46] and [47].

Let \( s \) be a semisimple \( \ell' \)-element. For \( \Gamma \in \mathcal{F}_I \), we define \( C_I(s) \) the set of \( \kappa_I \) such that there exists \( \mu_I \in \Psi_I(s) \) satisfy that \( \kappa_I \) is an \( e_\Gamma \)-core of \( \mu_I \). Denote
\[
C(s) = \prod_{\Gamma} C_I(s).
\] (2.5)
In particular, \( |\kappa| \in \{1, 2, 4\} \) for each \( \kappa \in C(s) \). We also define an operate \( ' \) on the sets \( C(s) \) and \( C_I(s) \) as follows. If \( \kappa_I \) is a doubleton, then we define \( \kappa_I = \kappa_I \) and if \( |\kappa_I| = 1 \), then we define \( \kappa_I = \kappa'_I \) as above. For \( \kappa = \prod_I \mu_I \in C(s) \), we define \( \kappa' = \prod_I (\kappa_I)' \).
Lemma 2.16. Let \( n = md \) and \( \varepsilon \in \{ \pm \} \). Let \( \iota \) be the natural embedding of \( \text{GL}_m(\mathbb{F}d) \) into \( \text{SO}^e_{2n}(q) \). Then \( \iota(\text{GL}_m(\mathbb{F}d)) \not\subseteq \Omega^e_{2n}(q) = [\text{SO}^e_{2n}(q), \text{SO}^e_{2n}(q)] \).

Proof. First assume that \( \varepsilon = 1 \). Now we give the structure of the embeddings. First note that

\[
\text{SO}^e_{2n}(q) = \{ A \in \text{SL}_{2n}(q) \mid A^r K_{2n} A = K_{2n} \},
\]

where \( K_{2n} = \begin{pmatrix} 1 & & & \cdots \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \).

Let \( \xi \) be a generator of
$\mathbb{F}_q^\times$ Let $\ell_1$ be induced by $\mathbb{F}_q^\times \hookrightarrow \text{GL}_d(q)$, $\xi \mapsto \Lambda_\xi$, where $\Lambda_\xi$ is the minimal polynomial of $\xi$ over $\mathbb{F}_q$. Let $A \in \text{GL}_n(q)$, then we take $\ell_2(A) = \text{diag}(A, K_n(A^{-1})^\text{tr} Z E)$. Then $\ell = \ell_2 \circ \ell_1$. Let $\theta : \text{SO}_d^n(q) \to \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$ be the spinor norm (see, for example, [28, §9]). Then $\Omega_{2n}^+(q)$ is the kernel of $\theta$. Thus it suffices to show that $\theta(\ell \text{GL}_m(\epsilon q^d)) = \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$.

Let $V$ be the orthogonal space with $\dim(V) = 2n$ with symmetric bilinear form $B$ and basis $v_1, \ldots, v_{2n}$ such that $B(v_i, v_j) = 1$ if $i + j = 2n + 1$ and $B(v_i, v_j) = 0$ otherwise. For $v \in V$, let $\sigma_v$ be the reflection along $v$, i.e., $\sigma_v(u) = u - \frac{2B(u, v)}{B(v, v)}v$ for any $u \in V$. For $\zeta \in \mathbb{F}_q^\times$ and $1 \leq i \leq n$ the matrix corresponding to $\sigma_{v_i + \zeta v_{2n-i+1}}$ with respect to the basis $v_1, \ldots, v_{2n}$ is

$$
\begin{pmatrix}
I_{i-1} & -\zeta \\
-\zeta^{-1} & I_{2n-2i} \\
& & I_{i-1}
\end{pmatrix}.
$$

Now let $\xi$ be a generator of $\mathbb{F}_q^\times$. Over the field $\mathbb{F}_q^\times$, $\ell(\text{diag}(\xi, I_{(m-1)}))$ is conjugate to

$$
\text{diag}(\xi, q^{d_1}, \ldots, q^{d_{m-1}}, I_{(2m-1)d}, q^{-d_{m-1}}, \ldots, q^{-d_1}, q^{-1}),
$$

which is $\prod_{i=0}^{d-1} \sigma_{\xi^{d_i} v_{2n-2i}} \sigma_{\xi^{d_i} v_{2n-i+1}}$. Let $Q(v) = B(v, v)$ for $v \in V$ be the associated quadratic form. Then

$$
\prod_{i=0}^{d-1} Q(v_i + \xi^{d_i} v_{2n-2i+1}) Q(v_i + v_{2n-i+1}) = \prod_{i=0}^{d-1} 2^2 \xi^{d_i} = 2^d \xi^{d-1} \in \mathbb{F}_q^\times. $$

By Remark 3 after [28, Cor. 9.9], the spinor norms can be determined from the Clifford groups, and then is independent of the fields. Then $\theta(\ell(\text{diag}(a, I_{(m-1)}))) = \xi^{d-1} \in (\mathbb{F}_q^\times)^2$. Note that $\xi^{d-1}$ is a generator of $\mathbb{F}_q^\times$, we have $\theta(\ell(\text{GL}_m(\epsilon q^d))) = \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2$.

The proof for $\epsilon = -1$ is completely analogous. □

3 A basic case of weights for classical groups

3.1 Radical subgroups of $I(V)$

We first give some more notations and conventions as in [4]. Let $\ell$ be defined as in §2.4. First, we define the integer $a$, and sign $\epsilon = \pm 1$ as follows: let $\ell^\circ$ be the exact power of $\ell$ dividing $q^{2\alpha} - 1$ and let $\epsilon$ be the sign chosen so that $\ell^\circ$ divides $q^\alpha - \epsilon$.

Let $a, \gamma$ be non-negative integers, $Z_a$ be the cyclic group of order $\ell^{a+\gamma}$ and $E_\gamma$ be an extraspecial $\ell$-group of order $\ell^{2\gamma+1}$. We may assume the exponent of $E_\gamma$ is $\ell$ by [4, (1B)]. Denote by $Z_a E_\gamma$ the central product of $Z_a$ and $E_\gamma$ over $\Omega_1(Z_a) = Z(E_\gamma)$. Let $V_{a, \gamma}$ be a symplectic or orthogonal space over $\mathbb{F}_q$ of dimension $2d_\ell a + \gamma$ and $\eta(V_{a, \gamma}) = \epsilon$ if $V_{a, \gamma}$ is orthogonal. By [4, (1A)], the group $Z_a E_\gamma$ can be embedded into $\text{GL}(d_\ell, q^{d_\ell + \epsilon})$ uniquely up to conjugacy in the sense that $Z_a$ is identified with $O(d_\ell(Z(\text{GL}(d_\ell, q^{d_\ell + \epsilon}))))$. We denote by $R_{a, \gamma}$ the image of $Z_a E_\gamma$ under the composition $Z_a E_\gamma \hookrightarrow \text{GL}(d_\ell, q^{d_\ell + \epsilon}) \hookrightarrow I(V_{a, \gamma})$. Then by [4, (1C)], $R_{a, \gamma}$ is uniquely determined by $Z_a E_\gamma$ up to conjugacy.

For integer $m \geq 1$, let $V_{m, a, \gamma} = V_{a, \gamma} \perp \cdots \perp V_{a, \gamma}$ with $m$ terms and let $R_{m, a, \gamma} = R_{a, \gamma} \otimes I_{(m)}$. For each positive integer $c$, let $A_c$ denote the elementary abelian group of order $\ell^c$. For a sequence of positive integers $c = (c_1, \ldots, c_t)$ with $t \geq 0$, we denote by $A_c = A_{c_t} \times \cdots \times A_{c_1}$ and $|c| = c_1 + \cdots + c_t$. Then $A_c$ can be regarded as an $\ell$-subgroup of the symmetric group $\Sigma(\ell^c)$. Let $V_{m, a, \gamma, c} = V_{m, a, \gamma} \perp \cdots \perp V_{m, a, \gamma}$ with $\ell^c$ terms. Groups of the form $R_{m, a, \gamma, c} = R_{m, a, \gamma} \rtimes A_c$ are called
the basic subgroups of $I(V_{m,a,γ,c})$. Then $R_{m,a,γ,c}$ is determined up to conjugacy in $I(V_{m,a,γ,c})$ and $η(V_{m,a,γ,c}) = ϵ^m$ if $V_{m,a,γ,c}$ is orthogonal. By [4] (2D), any $ℓ$-radical subgroup $R$ of $I(V)$ is conjugate to $R_0 × R_1 × ... × R_ℓ$, where $R_0$ is a trivial group and $R_i$ ($i \geq 1$) is a basic subgroup. Moreover, by the construction in [4],

$$R_{m,a,γ,c}C_{I(V_{m,a,γ,c})}(R_{m,a,γ,c}) \leq I_0(V_{m,a,γ,c})$$

(3.1)

and

$$N_{I(V_{m,a,γ,c})}(R_{m,a,γ,c}) \nsubseteq I_0(V_{m,a,γ,c}).$$

(3.2)

By [20, Lem.2.2(ii)], the map $Rad_ℓ(I(V)) \rightarrow Rad_ℓ(I_0(V))$ given by $R \mapsto R \cap I_0(V)$ is surjective. Since $ℓ$ is odd, we have that $Rad_ℓ(Ω(V)) = Rad_ℓ(I_0(V))$. Now assume that $V$ is orthogonal and let $Ω(V) = [I_0(V), I_0(V)]$. Then $|I_0(V), Ω(V)| = 2$. Similarly, $Rad_ℓ(Ω(V)) = Rad_ℓ(I_0(V))$.

**Lemma 3.1.** Assume that $V$ is orthogonal. Let $R$ be an $ℓ$-radical subgroup of $I_0(V)$, then $N_{I(V)}(R) \nsubseteq I_0(V)$ and $N_{I(V)}(R) \nsubseteq Ω(V)$.

**Proof.** $N_{I(V)}(R) \nsubseteq I_0(V)$ follows by (3.2). For the second assertion, it suffices to show that $θ(C_{I_0(V)}(R)) = FP/(FP)^2$, where $θ$ is the spinor norm. If $R = R_0$ is trivial, then $C_{I_0(V)}(R) = I_0(V)$. Now we assume that $R \neq R_0$. And then it suffices to show that $θ(C_{I_0(V)}(R_{m,a,γ,c})) = FP/(FP)^2$ for any $m,a,γ,c$. Note that $C_{I_0(V)}(R_{m,a,γ,c}) = C_{I(V)}(R_{m,a,γ,c}) ≅ GL(m, εq^{εc}) ⊗ I_γ ⊗ I_c$ by [4] p.12-13. Thus $θ(C_{I_0(V)}(R_{m,a,γ,c})) = FP/(FP)^2$ follows by Lemma 2.16.

The following lemma follows from Lemma 3.1 immediately.

**Lemma 3.2.** Let $V$ be orthogonal and $R$ an $ℓ$-radical subgroup of $I_0(V)$. Then $N_{I(V)}(R)/N_{I_0(V)}(R) \cong I(V)/I_0(V)$ and $N_{I_0(V)}(R)/N_{Ω(V)}(R) \cong I_0(V)/Ω(V)$.

Let $R$ be an $ℓ$-radical subgroup of $I(V)$, by Lemma 3.1, $I(V) = I_0(V)N_{I(V)}(R)$ and $I_0(V) = Ω(V)N_{I_0(V)}(R)$. So if two $ℓ$-radical subgroups of $I(V)$ are $I(V)$-conjugate, then they are $I_0(V)$-conjugate and $Ω(V)$-conjugate. Thus we have:

**Corollary 3.3.** Assume that $V$ is orthogonal. Then $Rad_ℓ(I(V))/ ~_I(V) = Rad_ℓ(I_0(V))/ ~_I_0(V) = Rad_ℓ(Ω(V))/ ~_Ω(V)$.

**Remark 3.4.** By the uniqueness of $R_{m,a,γ,c}$ proved in [4] (1C), we know that $Aut(I(V))$ acts trivially on $Rad_ℓ(I(V))/ ~_I(V)$.

Let $V$ be a symplectic or orthogonal space over $G = I(V)$, and $R$ an $ℓ$-radical subgroup of $G$. Then there exists a corresponding decomposition $V = V_0 ⊥ V_1 ⊥ ... ⊥ V_ℓ$, $R = R_0 × R_1 × ... × R_ℓ$, such that $R_0$ is the trivial subgroup of $I(V_0)$ and $R_i$ is a basic subgroup of $I(V_i)$ for $i > 1$. Let $σ$ be an automorphism of $G$. Then there is an automorphism $σ'$ of $G$, which is a composition of $σ$ by some suitable inner automorphism, such that $σ'$ stabilizes $V_i$ and $R_i$ for $0 ≤ i ≤ ℓ$.

### 3.2 Action of automorphisms on weights of $I(V)$

Given $Γ ∈ F$, let $e_Γ$ and $β_Γ$ be defined as in [2,4] and let $α_Γ$ and $m_Γ$ be the following: $ℓα_Γ$ is the exact power of $ℓ$ dividing $d_Γ$, and $m_Γ$ satisfies $m_Γℓα_Γe_Γ = e_Γβ_Γ$. Remind that there is no direct connection between $m_Γ$ and $m_Γ(s)$.

In this section, we let $σ = F_ρ$ be the field automorphism of $G = I_0(V)$ which sends $(a_{ij})$ to $(a_ρ^{ij})$ and let $σ^*$ be the $G^*$ such that $σ$ is dual to $σ^*$ as in [52, §5.3]. Then $σ^*$ is also the field automorphism which sends $(a_{ij})$ to $(a_ρ^{ij})$.
Recall that $F'$ is denoted to be the subset of $F$ consisting of polynomials whose roots are of $\ell'$-order. Given $\Gamma \in F'$, we define $G_\Gamma$, $R_\Gamma$, $C_{\Gamma}$, $\theta_\Gamma$ and $s_\Gamma$ as follows: let $V_\Gamma$ denote a symplectic or orthogonal space of dimension $2e_\ell\theta_\Gamma$ over $\mathbb{F}_q$ and of type $e$ or $e^{k_\ell}$ according as $\Gamma \in F_0$ or $\Gamma \in F_1 \cup F_2$ if $V_\Gamma$ is orthogonal. Let $\tilde{G}_\Gamma = I(V_\Gamma)$ and $G_\Gamma = I_0(V_\Gamma)$. Thus $\tilde{G}_\Gamma$ has a primary element $s_\Gamma$ with a unique elementary divisor $\Gamma$ of multiplicity $\beta_\Gamma$. $\tilde{G}_\Gamma$ has a basic subgroup $R_\Gamma$ of form $R_{m,\alpha,\gamma,\delta}$ by \cite[(1.12) and (5.2)]{23}. Let $\tilde{C}_\Gamma = C_{\Gamma}(R_\Gamma)$ and $\tilde{N}_\Gamma = N_{\Gamma}(R_\Gamma)$. Then $s_\Gamma \in G_\Gamma$, $\tilde{C}_\Gamma \subseteq G_\Gamma$ and $\tilde{C}_\Gamma \cong {\text{GL}}_m(e\varepsilon_{\ell'}e^{k_\ell})$, so that a Coxeter torus $T_\Gamma$ of $\tilde{C}_\Gamma$ has order $q^{m\alpha\delta\gamma} - e^{k_\ell}$. The dual $\tilde{T}_\Gamma$ is embedded as a regular subgroup of $\tilde{C}_\Gamma$, and $\tilde{C}_\Gamma$ is embedded as a regular subgroup of $G_\Gamma^\ast$. By \cite[p. 22]{4}, there exists an element $s_\Gamma$ in $\tilde{T}_\Gamma$ such that $C_{\Gamma}(s_\Gamma) = \tilde{T}_\Gamma$ and as an element of $G_\Gamma^\ast$, $s_\Gamma$ and $s_\Gamma$ are dual each other in the sense of \cite[(3E)]{4}. Here, $s_\Gamma$ has unique elementary divisor $\Gamma$ and $s_\Gamma$ is uniquely determined by $\Gamma$ up to $I(V_\Gamma^\ast)$-conjugacy. We denote by $\tilde{\tau}_\Gamma$ the character of $T_\Gamma$ corresponding to $s_\Gamma$ and let $\tilde{\theta}_\Gamma = \pm R_{\Gamma,\delta}^\ast(\tilde{\tau}_\Gamma)$ where the sign is chosen so the $\tilde{\theta}_\Gamma$ is an irreducible character of $\tilde{C}_\Gamma$. The block $\tilde{\theta}_\Gamma$ of $\tilde{C}_\Gamma$ containing $\tilde{\theta}_\Gamma$ then has defect group $R_\Gamma$ by \cite[(4C)]{21} or \cite[(3.2)]{9}.

Let $\mathcal{C}$ be a group with an integer $\delta$ and $\gamma$ be a tuple and an integer as in the previous sections, and $\delta = |\gamma| + \gamma$. Let $V_{\Gamma,\delta} = V_\Gamma \perp \cdots \perp V_\Gamma$, where there are $\ell'$ terms $V_\Gamma$ on the right-hand side. Then if $V_\Gamma$ is orthogonal, $V_{\Gamma,\delta}$ has type $e^{k_\ell} = e$ or $e^{k_\ell} = e^{k_\ell'}$ according as $\Gamma \in F_0$ or $\Gamma \in F_1 \cup F_2$. Let $\tilde{G}_{\Gamma,\gamma,\delta} = I(V_{\Gamma,\delta})$, $G_{\Gamma,\gamma,\delta} = I_{\Gamma,\delta}(V_{\Gamma,\delta})$, $\tilde{N}_{\Gamma,\gamma,\delta} = N_{\Gamma,\gamma,\delta}(R_{\Gamma,\gamma,\delta})$ and $\tilde{C}_{\Gamma,\gamma,\delta} = C_{\Gamma,\gamma,\delta}(R_{\Gamma,\gamma,\delta})$. Then $\tilde{C}_{\Gamma,\gamma,\delta} \subseteq C_{\Gamma}(G_{\Gamma,\gamma,\delta})$ can be viewed as a canonical character of $C_{\Gamma,\gamma,\delta}$, $\gamma$, $\delta$ such that $\gamma_{\Gamma,\gamma,\delta}$ is embedded in the kernel and all canonical characters are of this form. Let $B_{\Gamma,\gamma,\delta} = \tilde{C}_{\Gamma,\gamma,\delta}$. Then $B_{\Gamma,\gamma,\delta} \subseteq E_{\ell'}(G_{\Gamma,\delta}, x_{\Gamma})$, where $x_{\Gamma} = s_\Gamma \otimes I_\delta$ by the proof of \cite[(4A)]{4}.

Let $\mathcal{R}_{\Gamma,\delta}$ be the set of all the basic subgroups of the form $R_{\Gamma,\gamma,\delta}$ with $|\gamma| = \delta$. Label the basic subgroups in $\mathcal{R}_{\Gamma,\delta}$ as $R_{\Gamma,\delta,1}$, $R_{\Gamma,\delta,2}$, $\cdots$ and we denote the canonical character associated to $R_{\Gamma,\delta,i}$ by $\theta_{\Gamma,\delta,i}$. It is possible that there exists $\Gamma' \in F'$ such that $m_{\Gamma'} = m_\Gamma =: m$ and $\alpha_{\Gamma'} = \alpha_\Gamma =: \alpha$. In this case, $\mathcal{R}_{\Gamma,\delta} = \mathcal{R}_{\Gamma',\delta}$ and naturally we may choose the labeling of $\mathcal{R}_{\Gamma,\delta}$ and $\mathcal{R}_{\Gamma',\delta}$ such that $R_{\Gamma,\delta,i} = R_{\Gamma',\delta,i}$ for $i = 1, 2, \cdots$. For convention, we denote $\mathcal{R}_{\Gamma,\delta}$ by $\mathcal{R}_{\Gamma,\delta,i}$ or $R_{\Gamma,\delta,i}$ depending on that the related canonical character of $\tilde{C}_{\Gamma,\delta} \Gamma R_{\Gamma,\delta,i}$ is $\theta_{\Gamma,\delta,i}$ or $\theta_{\Gamma'}$.

Let $\mathcal{G}_{\Gamma,\delta}$ be the set of characters of $(\tilde{N}_{\Gamma,\delta})_{\tilde{\tau}_\delta,j}$ lying over $\tilde{\theta}_{\Gamma,\delta,i}$ and of defect zero as characters of $((\tilde{N}_{\Gamma,\delta})_{\tilde{\tau}_\delta,j} / R_{\Gamma,\delta,i})$ for all $i$. By Clifford theory, this set is in bijection with the set of characters of $\tilde{N}_{\Gamma,\delta}$ lying over $\tilde{\theta}_{\Gamma,\delta,j}$ and of defect zero as characters of $\tilde{N}_{\Gamma,\delta} / R_{\Gamma,\delta,i}$ for all $i$. We assume $\mathcal{G}_{\Gamma,\delta} = \{ \tilde{\psi}_{\Gamma,\delta,j} \}$ with $\tilde{\psi}_{\Gamma,\delta,j}$ a character of $(\tilde{N}_{\Gamma,\delta})_{\tilde{\tau}_\delta,j}$. Then $|\mathcal{G}_{\Gamma,\delta}| = \beta_{\Gamma,\delta} e^{\ell'}$ by the proof of \cite[(4A)]{4}.

For $\Gamma \in F$, we define $\sigma^\Gamma$ to be the unique elementary divisor of $\sigma^\Gamma((\Gamma))$, where $(\Gamma)$ is the companion matrix of $\Gamma$. Obviously $m_{\sigma^\Gamma} = m_\Gamma$, $\alpha_{\sigma^\Gamma} = \alpha_\Gamma$ and $R_{\sigma^\Gamma} = R_{\Gamma,\delta,1}$. By Remark \[3.4\] we may assume $R_{\sigma^\Gamma} = R_{\Gamma,\delta,1}$ up to a composition to $\sigma$ by some suitable inner automorphism. Then we may assume $B_{\sigma^\Gamma} = B_{\sigma^{-1} \Gamma,\delta,1}$ since $B_{\Gamma,\delta,1} \subseteq E_{\ell'}(G_{\Gamma,\delta,1}, x_{\Gamma})$, $B_{\sigma^\Gamma} \subseteq E_{\ell'}(G_{\sigma^\Gamma,\delta,1}, \sigma^{-1} \Gamma, \sigma^{-1} \Gamma \sigma^{-1} \Gamma \sigma^{-1} \Gamma)$ and $\sigma^{-1} \Gamma = x_{\sigma^{-1} \Gamma}$ (see for instance \cite[Prop. 7.2]{52}). Since $\tilde{\theta}_{\Gamma,\delta,j}$ is the canonical character of a root block of $B_{\Gamma,\delta,1}$, we have $\tilde{\theta}_{\Gamma,\delta,j} = \tilde{\theta}_{\sigma^{-1} \Gamma,\delta,1}$ up to a composition to $\sigma$ by an inner automorphism. Then we may denote $R_{\sigma^\Gamma} = R_{\sigma^{-1} \Gamma,\delta,1}$, $\tilde{N}_{\sigma^\Gamma} = \tilde{N}_{\sigma^{-1} \Gamma,\delta,1}$ and $\tilde{C}_{\sigma^\Gamma} = \tilde{C}_{\sigma^{-1} \Gamma,\delta,1}$ although the corresponding terms indexed by $\Gamma$ and $\sigma^{-1} \Gamma$ are actually the same. Also $(N_{\Gamma,\delta,1} \tilde{\tau}_\delta,j)^{\sigma} = (\tilde{N}_{\Gamma,\delta,1} \tilde{\tau}_\delta,j)^{\sigma} = (\tilde{N}_{\sigma^{-1} \Gamma,\delta,1} \tilde{\tau}_\delta,j)^{\sigma} \tilde{\tau}_\delta,j$. We may choose the labeling of $\mathcal{G}_{\Gamma,\delta}$ and $\mathcal{G}_{\Gamma,\delta,1}^{\sigma^{-1} \Gamma}$ such that

$$\tilde{\psi}_{\Gamma,\delta,j} = \tilde{\psi}_{\sigma^{-1} \Gamma,\delta,1,j}^\sigma$$

(3.3)

**Remark 3.5.** We can assume \[3.3\] because $\tilde{\psi}_{\Gamma,\delta,j}$ is invariant under the action of $\sigma$ if $\sigma^{-1} \Gamma = \Gamma$. We prove this as follows. First note that $\sigma^{-1} \Gamma = \Gamma$ if and only if $\theta_{\Gamma,\delta,i} = \theta_{\Gamma,\delta,1}$ is invariant under the action of $\sigma$. Let $R_{\Gamma,\delta,i} = R_{m_{\sigma^\Gamma},\gamma,\delta}$ with $e = (c_1, \ldots, c_\ell)$. We also abbreviate $R = R_{\Gamma,\delta,i}$, $\tilde{N} = \tilde{N}_{\Gamma,\delta,i}$,
\[ \tilde{C} = \tilde{C}_{\Gamma, \delta}, \text{ and } \tilde{\theta} = \tilde{\theta}_{\Gamma, \delta}. \] By [4, (2E)], \( \tilde{N}/R = \tilde{N}_{m, \omega, \gamma}/R_{m, \omega, \gamma} \times \prod_{i=1}^{l} GL_{c_i} (\ell) \) and then we may assume that \(|e| = 0\), i.e., \( R = R_{m, \omega, \gamma} \). Let \( \tilde{N}^0 = \{ g \in \tilde{N} \mid [g, Z(R)] = 1 \} \), then by the remark after [4, (3I)], \( \tilde{N}^0 \leq \tilde{N}_b \) and Also, \( \tilde{N}^0 = CRL \), where \( L \leq \tilde{N}_b \) satisfies \( [L, C] = 1, L \cap C = Z(L) = Z(C) \) and \( L/Z(L)R \cong Sp_{2\gamma}(\ell) \). Then there is exactly one character \( \tilde{\theta} \) of \( \tilde{N}_b \) which lies over \( \tilde{\theta} \) and of \( \ell \)-defect zero when viewed as a character of \( \tilde{N}_b/R \) and we may write \( \tilde{\theta} = \tilde{\theta} \times \zeta \), where \( \zeta \) is the Steinberg character of \( Sp_{2\gamma}(\ell) \). Hence \( \tilde{\theta} \) is invariant under the action of \( \sigma \). On the other hand, we have \( \tilde{N}_b/\tilde{N}^0 \cong \tilde{N}_b/\tilde{C}_\Gamma \). From this we may assume further that \( \gamma = 0 \) and then \( \delta = 0 \), which means \( R = R_\Gamma \), \( \tilde{N} = \tilde{N}_\Gamma \), \( \tilde{C} = \tilde{C}_\Gamma \) and \( \tilde{\theta} = \tilde{\theta}_\Gamma \). Now \( \tilde{C} \cong GL_{m}(eq^{eq^{\rho}}) \). By [22, (1.14)], \( \tilde{N} = \langle \tilde{C}, D \rangle \), where \( D \) is generated by one or two element and every element of \( D \) acts on \( \tilde{C} \) as a field or graph automorphism. Then \( \tilde{\theta} \) extends to \( \langle \tilde{N}, \sigma \rangle \) by a result of Bonnafe [5, Thm. 4.3.1 and Lem. 4.3.2] (see [20, Pro. 4.17] for details). Thus \( \sigma \) acts on \( \text{Irr}(\tilde{N}, \sigma) \) trivially.

Now let \( V \) be a symplectic or even dimensional orthogonal space and let \( B \) be an \( \ell \)-block of \( \tilde{G} = I(V) \) with defect group \( D \) and root block \( b \) such that \( V = [V, D] \) and \( b^G \subseteq E_{\ell}(G, s) \) for some semisimple \( \ell^* \)-element \( s \in G^* \). Let \( s^* \) be a dual of \( s \). Then \( mt(s^*) = w_T \beta_T e_\ell \) for some positive integer \( w_T \). Similar with [36, p.145] for groups of type \( A \), now we define \( \gamma \mathcal{W}(\tilde{B}) \) be the set of elements \( K = K_\Gamma, \) where \( K_\Gamma : \bigcup_{\delta} \mathcal{E}_{a, b} \to \{ \ell \text{-cores} \} \) such that \( \sum_{\delta, i, j} t \ell_0[K_\Gamma(\psi_{\delta, i, j})] = w_T \). Here, an \( \ell \)-core mean an \( \ell \)-core of some partition.

A bijection between \( \gamma \mathcal{W}(\tilde{B}) \) and \( \gamma \mathcal{W}(\tilde{B}) \) has been constructed implicitly in [4, (E4)] and can be described as follows. Let \( (R, \tilde{\varphi}) \) be an \( \ell \)-weight of \( G \). Set \( C = C_{\tilde{G}}(R) \) and \( \tilde{N} = N_{\tilde{G}}(R) \). Thus there exists an \( \ell \)-block \( \tilde{b} \) of \( \tilde{CR} \) with \( R \) a defect group such that \( \tilde{\varphi} = \text{Ind}_{\tilde{N}(0)}^{\tilde{N}} \tilde{\varphi} \) where \( \tilde{\varphi} \) is the canonical character of \( \tilde{b} \) and \( \tilde{\varphi} \) is a character of \( \tilde{N}(\tilde{\varphi}) \) lying over \( \tilde{\varphi} \) and of \( \ell \)-defect zero as a character of \( \tilde{N}(\tilde{\varphi})/R \).

We may suppose that \( Z(D) \leq Z(R) \leq R \leq D \) so that \( V = [V, R] \). Assume we have the following decomposition \( \tilde{\theta}_\Gamma = \prod_{\delta, i, j} \tilde{\theta}_{\Gamma, \delta, i, j} \), \( R_\Gamma = \prod_{\delta, i, j} R_{\Gamma, \delta, i, j} \). Note that \( \tilde{\theta}_\Gamma \) determines a semisimple \( \ell^* \)-element with canonical form \( \beta_\ell e_\ell(\Gamma) \) in \( G_\Gamma \). Thus \( mt(s^*) = \prod_{\delta, i, j} t_{\Gamma, \delta, i, j} \beta_\ell e_\ell(\Gamma) \) for each \( \Gamma \).

Now we have
\[ \tilde{N}(\tilde{\varphi}) = \prod_{\Gamma, \delta, i} \tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i), \quad \tilde{\varphi} = \prod_{\Gamma, \delta, i} \tilde{\varphi}_{\Gamma, \delta, i}, \]
with \( \tilde{\varphi}_{\Gamma, \delta, i} \) a character of \( \tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i) \) covering \( \tilde{\theta}_{\Gamma, \delta, i}^\Gamma \) and of \( \ell \)-defect zero as a character of \( \tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i) \). By Clifford theory, \( \tilde{\varphi}_{\Gamma, \delta, i} \) is of the form
\[ \text{Ind}_{\tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i) \succ \mathcal{E}(\Gamma, \delta, i)}^{\tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i)} \tilde{\varphi}_{\Gamma, \delta, i} = \prod_{j \in \Phi_{K, \delta, i}} \tilde{\varphi}_{\Gamma, \delta, i} \]
with \( t_{\Gamma, \delta, i} = \sum_{j \in \Phi_{K, \delta, i}} \prod_{j \in \Phi_{K, \delta, i}} \tilde{\varphi}_{\Gamma, \delta, i} \)
\[ \text{where } \text{Ind}_{\tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i) \succ \mathcal{E}(\Gamma, \delta, i)}^{\tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i)} \tilde{\varphi}_{\Gamma, \delta, i} \text{ is an extension of } \prod_{j \in \Phi_{K, \delta, i}} \tilde{\varphi}_{\Gamma, \delta, i} \text{ from } \tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i) \succ \mathcal{E}(\Gamma, \delta, i) \text{ to } \tilde{N}_{\Gamma, \delta, i}(\tilde{\theta}_{\Gamma, \delta, i}) \succ \mathcal{E}(\Gamma, \delta, i) \succ \mathcal{E}(\Gamma, \delta, i). \]

We can define the action of \( \sigma^* \) on \( K \) by \( (\sigma^*K)^* = K_\Gamma \).

**Lemma 3.6.** With the notation above, if \( (R, \tilde{\varphi}) \) is a \( \tilde{B} \)-weight with label \( K \), \( (R, \tilde{\varphi})^* \) is a \( \tilde{B}^* \)-weight with label \( \sigma^{-1} K \).

**Proof.** Let \( K' \) be the label of \( (R, \tilde{\varphi})^* \). First note that \( R_\sigma = R, \tilde{C}_\sigma = \tilde{C}, \tilde{N}_\sigma = \tilde{N}, \) and \( \sigma \) stabilizes every \( \tilde{C}_{\Gamma, \delta, i} \) up to conjugacy.

By the argument above, we may denote \( R_{\Gamma, \delta, i}^\sigma = R_{\sigma^{-1} \Gamma, \delta, i}^\sigma, \tilde{N}_{\Gamma, \delta, i}^\sigma = \tilde{N}_{\sigma^{-1} \Gamma, \delta, i}^\sigma, \) and \( \tilde{C}_{\Gamma, \delta, i}^\sigma = \tilde{C}_{\sigma^{-1} \Gamma, \delta, i} \) although the corresponding terms indexed by \( \Gamma \) and \( \sigma^{-1} \Gamma \) are actually the same. To
determine $K'$, we note that $\tilde{\psi}^\sigma = \prod_{\Gamma, \delta, i} \tilde{\psi}_{\Gamma, \delta, i}^\sigma$. By (3.4), $\tilde{\psi}_{\Gamma, \delta, i}^\sigma$ is

$$\text{Ind} \frac{\tilde{\psi}_{\Gamma, \delta, i}^\sigma(1)}{\tilde{\psi}_{\Gamma, \delta, i}^\sigma(\Gamma_{\delta, i})} \prod_{\delta, j} \tilde{\psi}_{\Gamma, \delta, j}^\sigma \prod_{\delta, j} \phi_{\delta, \delta, j}$$

Here, we note that $\sigma$ acts trivially on $\Xi(\Gamma, \delta, i)$ and $\Xi(\Gamma, \delta, j, i)$ by Remark 3.4. Since $\tilde{\psi}_{\Gamma, \delta, i}^\sigma = \tilde{\psi}_{\Gamma, \delta, i}^{\sigma^{-1}}$, we have $\tilde{\psi}_{\Gamma, \delta, i}^{\sigma^{-1}}(\tilde{\psi}_{\Gamma, \delta, i}^{\sigma}) = \tilde{\psi}_{\Gamma, \delta, i}^{\sigma^{-1}}(\tilde{\psi}_{\Gamma, \delta, i}^{\sigma^{-1}})$. We can fix the way to extend $\prod_j \tilde{\psi}_{\Gamma, \delta, i, j}^\sigma$ as in [29 Lem. 25.5], then we have that

$$\left(\prod_j \tilde{\psi}_{\Gamma, \delta, i, j}^\sigma\right)^\sigma = \prod_j \left(\tilde{\psi}_{\Gamma, \delta, i, j}^\sigma\right)^{\sigma^{-1}}$$

Since $\tilde{\psi}_{\Gamma, \delta, i, j}^\sigma = \psi_{\sigma^{-1}}(\Gamma_{\delta, i})$ by (3.3), $\tilde{\psi}_{\Gamma, \delta, i}^\sigma$ would be

$$\text{Ind} \frac{\tilde{\psi}_{\Gamma, \delta, i}^\sigma(1)}{\tilde{\psi}_{\Gamma, \delta, i}^\sigma(\Gamma_{\delta, i})} \prod_{\delta, j} \tilde{\psi}_{\Gamma, \delta, j}^\sigma \prod_{\delta, j} \phi_{\delta, \delta, j}$$

Then $K^\sigma = K$ which is just $K' = \sigma^{-1} K$.

Thus, by a similar proof with [2] (1A), there is a canonical between $\mathcal{P}(\beta e_\Gamma, h_\Gamma)$ (defined as in (2.7)) and $K_\Gamma$ for every $\Gamma \in \mathcal{F}$ by [47 Prop.(3.7)]. Let $\mathcal{P}(\tilde{B}) = \prod_\Gamma \mathcal{P}(\beta e_\Gamma, h_\Gamma)$. Then by the argument above, we have a bijection between $\mathcal{P}(\tilde{B})$ and $\mathcal{W}_H(\tilde{B})$. We also define $\sigma^* \mu = \prod_{\Gamma} (\sigma^* \mu)_{\Gamma}$ with $\sigma^* \mu_{\sigma^{-1}} = \mu_{\Gamma}$. By Lemma 3.6 we have

**Corollary 3.7.** With the notation above, if $(R, \tilde{\varphi})$ is a $\tilde{B}$-weight with label $\mu \in \mathcal{P}(\tilde{B})$, then $(R, \tilde{\varphi})^\sigma$ is a $\tilde{B}^\sigma$-weight with label $\sigma^{-1} \mu$.

Now we consider the action of diagonal automorphism on the weights of $\tilde{B}$. The following two lemmas will be useful.

**Lemma 3.8.** Let $H$ be arbitrary finite group, $L, K \leq H$ and $M = L \cap K$ such that $|H/K| = 2$ and $H/L$ is cyclic. Suppose that $\varphi \in \text{Irr}(L)$ such that $\theta = \text{Res}_H^L \varphi \in \text{Irr}(M)$. Assume further $\theta$ is $H$-invariant and let $t = |H : H_{\varphi}|$. Then exactly one of the following statements holds.

(i) $t = 1$ and $\text{Res}_{K, H}^H \varphi$ is irreducible for every $\chi \in \text{Irr}(H_{\varphi})$.

(ii) $t = 2$ and $\text{Res}_{K, H}^H \varphi$ is not irreducible for every $\chi \in \text{Irr}(H_{\varphi})$.

**Proof.** Let $\chi \in \text{Irr}(H \mid \varphi)$. First by Clifford theory, we may write $\chi = \text{Ind}_{H_{\varphi}}^H \psi$ for some $\psi \in \text{Irr}(H_{\varphi} \mid \varphi)$ and then Mackey formula implies that $\text{Res}_{K, H}^H \chi = \text{Ind}_{K_{\varphi}}^K (\text{Res}_{K_{\varphi}}^H \psi)$. Note that $\psi$ is an extension of $\theta$. So $\text{Res}_{K, H}^H \chi$ is irreducible. Thus $\text{Ind}_{K_{\varphi}}^K (\text{Res}_{K_{\varphi}}^H \psi)$ is a sum of $t$ irreducible constituents. Hence $t \leq 2$ since $|H/K| = 2$ and the assertion follows easily.

**Lemma 3.9.** Let $H$ be arbitrary finite group and $M \leq H$ such that $H = C_H(M)M$. Suppose that $\theta \in \text{Irr}(M)$ and $\eta \in \text{Irr}(C_H(M))$ such that $\eta(1) = 1$ and $\text{Irr}(Z(M) \mid \theta) = \text{Irr}(Z(M) \mid \eta)$. Then there exists a unique extension $\varphi$ of $\theta$ to $H$ with $\text{Irr}(C_H(M) \mid \varphi) = \{\eta\}$. In particular, $\varphi(c) \neq 0$ for every $c \in C_H(M)$.
Proof. In fact, this follows from [20, 2.1]. But for convenience, we still give the details here. Let $D : M \to \text{GL}_n(\mathbb{C})$ be a $\mathbb{C}$-representation of $M$ affording $\theta$. Then define $D' : H \to \text{GL}_n(\mathbb{C})$, $cg \mapsto \eta(c)D(g)$ for $c \in C_H(M)$ and $g \in M$. It is easy to check that $D'$ is well-defined and is a $\mathbb{C}$-representation of $H$. Let $\varphi$ be the character afforded by $D'$. Then $\varphi$ is the unique extension of $\theta$ to $H$ with $\text{Irr}(C_H(M) | \varphi) = \{\eta\}$. Let $c \in C_H(M)$, then $\varphi(c) = \eta(c)\theta(1)$. Now $\eta$ is a linear character, then $\eta(c) \neq 0$ for every $c \in C_H(M)$. Thus $\varphi(c) \neq 0$ for every $c \in C_H(M)$. □

Following the notation of [23], we denote by $J(V)$ the group of all conformal endomorphisms of $V$ when $\dim(V)$ is even. Then $J(V) = \text{CSp}(V)$ or $\text{CGO}(V)$ according as $V$ is symplectic or orthogonal. We also let $J_0(V) = \text{CSp}(V)$ or $\text{CSO}(V)$ according as $V$ is symplectic or orthogonal. Then $|J_0(V)/J_0(V)\beta(J_0(V))| = 2$. Let $J_0(V) = \langle \tau, \tau \rangle$ where $\tau \in J_0(V)$. Obviously, $J(V) = \langle \tau \rangle$. Then for a basic subgroup $R_{\Gamma,\delta,i}$, up to a composition to $\tau$ by some suitable inner automorphism, which is denoted by $\tau'$, we have that $N_{\Gamma(V),\delta}(R_{\Gamma,\delta,i}) = \langle \tau, \tau' \rangle$, $C_{\Gamma(V),\delta}(R_{\Gamma,\delta,i}) = \langle \tau, \tau' \rangle$ and $\tau'$ commutes with $R_{\Gamma,\delta,i}(\tilde{C}_{\Gamma,\delta,i})$ by [23, §5].

Lemma 3.10. Keep the hypothesis and setup above.

(i) If $\Gamma \neq x + 1$, then every element of $\Xi_{\Gamma,\delta}$ is invariant under $\tau'$.

(ii) If $\Gamma = x + 1$, then any element of $\Xi_{\Gamma,\delta}$ is not invariant under $\tau'$.

Proof. Similar with the argument of Remark [3, 5] we may assume that $\delta = 0$. Then $R_{\Gamma,\delta,i} = R_{\Gamma,\delta,i}$, $\tilde{N}_{\Gamma,\delta,i} = \tilde{N}_{\Gamma}$, $\tilde{C}_{\Gamma,\delta,i} = \tilde{C}_{\Gamma}$ and $\tilde{C}_{\Gamma,\delta,i} = \tilde{C}_{\Gamma}$. In this way $\Xi_{\Gamma,\delta}$ is the set of extensions of $\tilde{C}_{\Gamma}$ to $\tilde{N}_{\Gamma}$. Recall that $\tilde{C}_{\Gamma} = \pm C_{\Gamma}(\tilde{\beta}_{\Gamma})$. We abbreviate $R = R_{\Gamma,\delta,i}$, $\tilde{N} = \tilde{N}_{\Gamma}$, $\tilde{C} = \tilde{C}_{\Gamma}$, $\tilde{\beta} = \tilde{\beta}_{\Gamma}$, $N' = N_{\Gamma(V),\delta}(R_{\Gamma})$ and $C' = C_{\Gamma(V),\delta}(R_{\Gamma})$. Also, there is an extension $\tilde{\beta}'$ of $\tilde{\beta}$ to $\tilde{C}'$. Claim that $\tilde{\beta}'(\tau') = 0$. In order to do this, we choose an canonical $\tilde{\beta}'$. First note that $C_{\Gamma}(\tilde{C}) = \langle Z(\tilde{C}), \tau' \rangle$ is abelian. Let $\{\eta\} = \text{Irr}(Z(\tilde{C}) | \tilde{\beta})$ and take $\eta'$ to be an extension of $\eta$ to $C_{\Gamma}(\tilde{C})$. By Lemma [3, 5] there is a unique extension $\tilde{\beta}'$ of $\tilde{\beta}$ to $\tilde{C}'$ with $\eta' = \text{Irr}(C_{\Gamma}(\tilde{C}) | \tilde{\beta}')$. In particular, $\tilde{\beta}'(\tau') = 0$, as claimed. Hence for $g \in \tilde{N}$, $(\tilde{\beta}')^g = \tilde{\beta}'$ if and only if $(\tilde{\beta})^g(\tau') = \tilde{\beta}'(\tau')$ by [48, Corr. 1.22]. Also by [23, (5A)], $[\tau', \tilde{N}] \leq Z(\tilde{C})$. So $\tilde{\beta}'(\tau') = \tilde{\beta}(g, \tau')\tilde{\beta}'(\tau')$ and thus $(\tilde{\beta}')^g = \tilde{\beta}'$ if and only if $[g, \tau'] \in \text{ker}(\tilde{\beta})$. Now we calculate the stabilizer of $\tilde{\beta}'$ in $\tilde{N}'$.

If $\Gamma = x - 1$, then $\tilde{\beta}'$ is the trivial character. Thus $\tilde{N}' = \tilde{N}$. If $\Gamma \in F_{1} \cup F_{2}$, then by [23, (6A)(2)], we also have $\tilde{N}' = \tilde{N}$. If $\Gamma = x + 1$, then by [23, (6A)(3)], we have $\tilde{N}' \neq \tilde{N}'$. Hence the assertion holds by Lemma [3, 8]. □

For $\Gamma = x + 1$, we recall that $|\Xi_{\Gamma,\delta}| = 2e\ell^e$. Thus follows from Lemma 3.10 we may rewrite $\Xi_{x+1,\delta} = \{\tilde{\psi}_{x+1,\delta,i,j} | 1 \leq i \leq 2e, 1 \leq j \leq \ell^e\}$ such that $\tilde{\psi}_{x+1,\delta,i,j} = \tilde{\psi}_{x+1,\delta,i,e+i,j}$ for every $1 \leq i \leq e$. By Lemma 3.10 again, we have the following result by a similar argument with Lemma 3.6 and 3.7.

Proposition 3.11. With the notation of Corollary 3.7 we let $(R, \tilde{\varphi})$ be a $\tilde{B}$-weight with label $\mu \in \mathcal{P}(\tilde{B})$ and write $\mu = \prod \mu_{\delta,i}$, where $\mu_{\delta,i} = (\mu_{\Gamma}^{(i)} = \mu_{\Gamma}^{(i)}(\beta_{\Gamma}))$. Then the image of $(R, \tilde{\varphi})$ under the non-trivial action of $J(V)/Z(J(V))\beta(J(V))$ is a $B$-weight with label $\mu^+ \in \mathcal{P}(B)$, where $\mu^+ = \prod \mu_{\Gamma}^{(i)}$ with $\mu_{\Gamma}^{(i)} = (\mu_{\Gamma}^{(i)}, \ldots, \mu_{\Gamma}^{(i)})$ such that $\mu_{\Gamma}^{(i)} = \mu_{\Gamma}$ if $\Gamma \neq x + 1$ and $\mu_{x+1}^{(i)} = \mu_{x+1}^{(e+i)} = \mu_{x+1}^{(i)}$ for every $1 \leq i \leq e$.

4 Type B

In this section, we let $G = \text{SO}_{2p+1}(q)$. As usual, we denote $G = \text{SO}_{2p+1}(F_q)$ (a connected reductive algebraic group). Throughout this paper, we always denote by $F_p$ the field automorphism which sends $(a_{ij})$ to $(a_{ij}^p)$ and we write $E = \langle F_p \rangle$. Let $F := F_p^l$ be the standard Frobenius
endomorphism over \( G \). We write \( G^F \) for the group of fixed points, then \( G = G^F \). As before, we denote by \( G^* = \text{Sp}_{2n}(q) \) the dual of \( G \) and \( G^* = G^* = \text{Sp}_{2n}(q) \).

### 4.1 Characters and \( \ell \)-Brauer characters of \( \text{SO}_{2n+1}(q) \) and \( \Omega_{2n+1}(q) \)

Let \( s \in G^F \) be a semisimple element and let \( \Psi(s) = \prod_i \Psi_i(s) \) be defined as \([7.4]\). Then the unipotent characters of \( C_{G^*}(s)^F \) are in bijection with \( \Psi(s) \). For \( \mu \in \Psi(s) \), we denote \( \psi_\mu \), the unipotent character of \( C_{G^*}(s)^F \) corresponding to \( \mu \). Now we define \( \iBr(G) \) to be the set of \( G^* \)-conjugacy classes of pairs \((s, \mu)\) where \( s \) is a semisimple element of \( G^F \) and \( \mu \in \Psi(s) \). Here two pairs \((s_1, \mu_1)\) and \((s_2, \mu_2)\) are called to be \( G^* \)-conjugate if there exists \( g \in G^* \) such that \( s_1 = g s_2 g^{-1} \) and \( \mu_1 = \mu_2 \). The irreducible characters of \( G \) have been classified by Lusztig \([37]\).

By the Jordan decomposition of characters of \( G \), there is a bijection between \( \mathcal{E}(C_{G^*}(s), 1) \) and \( \mathcal{E}(G^F, s) \) for every semisimple element \( s \) of \( G^* \). For \( \mu \in \Psi(s) \), we denote \( \chi_{s, \mu} \), the character in \( \mathcal{E}(G^F, s) \) corresponding to \( \psi_\mu \). So \( \iBr(G) \) is a labeling set of the characters of \( G \).

In this section, we assume that \( \sigma = F_p \) is the field automorphism, then \( \sigma^* \) is also the field automorphism \( F_p \) of \( G^* \). For \( \mu \in \Psi(s) \), we define \( \sigma \mu = \prod_i (\sigma \mu_i)_{\Gamma} \), with \( (\sigma \mu_i)_{\Gamma} = \mu_i \).

**Proposition 4.1.** With the above definitions, we have \( \chi_{s, \mu} = \chi_{s^\sigma, (s^\sigma)^{-1} \mu} \) in the sense that the pair \((s, \mu)\) in the subscription means in fact a \( G^* \)-conjugacy class.

**Proof.** This follows by \([16\text{Thm. 3.1]}\) (or \([19\text{Prop. 1.3.1(iv)}\]) and the fact that the unipotent characters of symplectic groups with odd defining characteristic and general linear and unitary groups are invariant under the automorphism groups (see, for example, \([39\text{Thm. 2.5}])\).

Let \( s \) be a semisimple \( \ell' \)-element and let \( C(s) = \prod C_{\Gamma}(s) \) as \([25]\). We define \( iB_l(G) \) to be the set of \( G^* \)-conjugacy classes of pairs \((s, \kappa)\) where \( s \) is a semisimple \( \ell' \)-element of \( G^* \) and \( \kappa \in C(s) \). Then by \([23\text{(10B)}]\), there is a bijection \((s, \kappa) \mapsto B(s, \kappa) \) from \( iB_l(G) \) to \( Bl_l(G) \). Also by \([23\text{(11A)}]\), \( \iBr(G(s, \kappa)) \cap \mathcal{E}(G, \ell') \) \( \chi_{s, \mu} \mid \mu \in \Psi(s, \kappa) \), where \( \Psi(s, \kappa) \) is as defined in \([2.6]\).

We define the action of \( \sigma \) on \( iB_l(G) \) similarly as the action on \( \iBr(G) \). The following result can be deduced directly from Proposition 4.1.

**Proposition 4.2.** \( B(s, \kappa)^\sigma = B((s^\sigma)^{-1}(s), (s^\sigma)^{-1} \kappa) \).

We define \( iBr_l(G) := \{ (s, \mu) \in \iBr(G) \mid s \text{ is of } \ell'\text{-order} \} \). Then for \((t, \mu) \in iBr_l(G), \chi_{t, \mu} \) lies in \( \ell \)-block \( B(s, \kappa) \) if and only if \( t \) is \( G^* \)-conjugate to \( s \) and \( \mu \in \Psi(s, \kappa) \).

**Proposition 4.3.** Assume that \( \ell \) is linear. There is a bijection \((s, \mu) \mapsto \phi_{s, \mu} \) from \( iBr_l(G) \) to \( IBr_l(G) \) such that \( \phi_{s^\sigma, (s^\sigma)^{-1}\mu} = \phi_{s, \mu} \).

**Proof.** By Theorem \([2.10]\), \( \mathcal{E}(G^F, \ell') \) is a basic set of \( IBr_l(G) \). By \([26]\), the decomposition matrix with respect to \( \mathcal{E}(G^F, \ell') \) is unitriangular since \( \ell \) is linear. Then by \([16\text{Lem. 7.5]}\), there is an \( \sigma \)-equivariant bijection from \( \mathcal{E}(G^F, \ell') \) to \( IBr_l(G) \) which preserves blocks. Thus the assertion follows from Proposition 4.1.

**Remark 4.4.** In fact, by the construction in \([16\text{Lem. 7.5]}\), the proof of Proposition 4.3 gives a bijection \( \mathcal{D} : \mathcal{E}(G^F, \ell') \rightarrow IBr_l(G) \) such that \( \phi_{s, \mu} = \mathcal{D}(\chi_{s, \mu}) \) for all \((s, \mu) \in iBr_l(G) \). In addition, there is a partial order relation \( \preceq \) on \( IBr_l(G) \), such that \( \chi_{s, \mu} = \phi_{s, \mu} + \sum_{\varphi \in \iBr_l(G), \varphi \preceq \varphi} d_{\varphi} \varphi \) with \( d_{\varphi} \in \mathbb{Z} \).

Note that \( Z(G^*) = \langle z \rangle \), where \( z = -I_{2n} \). For a semisimple element \( s \in G^* \), we write \( -s := zs = -I_{2n} \cdot s \). For \( \Gamma \in F \), let \( \xi \) be a root of \( \Gamma \). We define \( z \Gamma \) to be the unique polynomial in \( F \) such that \(-\xi \) is a root of \( z \Gamma \). For \( \mu \in \Psi(s) \), we define \( -\mu = (-\mu)_{z \Gamma} \), with \( (-\mu)_{z \Gamma} = \mu_{-\Gamma} \).
Let $\tilde{z} \in \mathcal{E}(G^F, z)$ be the character corresponding under Jordan decomposition to $1_G \in \mathcal{E}(G^F, 1)$. Then $\tilde{z}$ is the (unique) non-trivial linear character of $G$ (the definition of $\tilde{z}$ also follows from [18 Prop. 13.30]). Then by [19 Prop. 1.3.1(ii)], we have the following result.

**Proposition 4.5.** $\tilde{z} \chi_{s, \mu} = \chi_{-s, -\mu}$ in the sense that the pair $(s, \mu)$ in the subscription means in fact a $G^-$-conjugacy class.

Since $\ell$ is odd and $z$ has order 2, the character $\tilde{z}$ in Proposition 4.5 can be regarded as a linear $\ell$-Brauer character of $G$.

**Proposition 4.6.** With the notation of Proposition 4.3 and 4.5 if $\ell$ is linear, then $\tilde{z} \phi_{s, \mu} = \phi_{-s, -\mu}$ in the sense that the pair $(s, \mu)$ in the subscription means in fact a $G^+$-conjugacy class.

**Proof.** Here, we use [20 Lem. 2.4]. By its proof, $\tilde{z}$ induces an automorphism of the associated group algebra. Then it permutes the irreducible ordinary and $\ell$-Brauer characters in the way indicated. Thus the assertion follows from Remark 3.4, Proposition 4.3 and 4.5. $\square$

### 4.2 Weights of $SO_{2n+1}(q)$

Now we let $V$ be a odd dimensional orthogonal space, $\tilde{G} = I(V)$ and $G = I_0(V)$. Then $\tilde{G} = Z(\tilde{G}) \times G$. Define

$$i^* W'_i(G) = \left\{ (s, \kappa, K)^G \mid s \text{ is an semisimple } \ell'-\text{element of } G^*, \kappa \in C(s), \begin{array}{c}
K = K_\Gamma, \quad K_\Gamma : \bigcup_{b \in \mathcal{E}_\Gamma} \mathcal{C} \rightarrow \{ \ell\text{-cores} \} \text{ s.t. } \\
\sum_{i,j} \ell_0(\kappa_{\Gamma, (\delta, i,j)}) = w_\Gamma, \quad m_0(s)/\beta_\Gamma = |\kappa_\Gamma| + e_\Gamma w_\Gamma. \end{array}\right\}.$$ 

Here, $(s, \kappa, K)^G$ means a $G^*$-conjugacy class of $(s, \kappa, K)$.

A bijection between $W'_i(G)$ and $i^*W'_i(G)$ has been constructed implicitly in the proof of [4 (4G)] and can be described as follows. Let $(R, \varphi)$ be an $\ell$-weight of $G$. Then $(R, \varphi)$ is an $\ell$-weight of $\tilde{G}$, where $\varphi = 1_{Z(\tilde{G})} \times \varphi$. Set $\tilde{C} = C_{\tilde{G}}(\tilde{R})$ and $\tilde{N} = N_{\tilde{G}}(\tilde{R})$. Thus there exists an $\ell$-block $\tilde{b}$ of $\tilde{C}\tilde{R}$ with $\tilde{R}$ a defect group such that $\tilde{\varphi} = \text{Ind}^\tilde{G}_{\tilde{N}(\tilde{R})} \tilde{\psi}$ where $\tilde{\varphi}$ is the canonical character of $\tilde{b}$ and $\tilde{\psi}$ is a character of $\tilde{N}(\tilde{R})$ lying over $\varphi$ and of $\ell$-defect zero as a character of $\tilde{N}(\tilde{R})/\tilde{R}$.

Let $V_0 = C_V(R)$ and $V_+ = [V, R]$. Then $V = V_0 \perp V_+$ and $V_+$ is an even dimensional orthogonal space. Suppose that $\dim(V_0) = 2n_0 + 1$. In addition, let $\tilde{G}_0 = I(V_0)$, $G_0 = I_0(V_0)$, $\tilde{G}_+ = I(V_+)$ and $G_+ = I_0(V_+)$. Then $R = R_0 \times R_+ = b_0 \times b_+ = \tilde{b} \times \tilde{b}_+ = \tilde{b} \times \tilde{b}_+$, where $R_0 = \langle 1_{V_0} \rangle \leq \tilde{G}_0$, $R_+ \leq \tilde{G}_+$, $b_0$, $b_+$ are blocks of $\tilde{G}_0$, $C_{\tilde{G}_0}(R_0)$ respectively, and $\tilde{b}_0 \in \text{Irr}(\tilde{b}_0)$, $\tilde{b}_+ \in \text{Irr}(\tilde{b}_+)$. First, we let $\tilde{C}_0 = \tilde{N}_0 = \tilde{G}_0$, $C_+ = C_{I(V_+)}(R_+)$, and $N_+ = N_{I(V_+)}(R_+)$. Then $\tilde{\varphi}_0 = \tilde{\psi}_0 = \tilde{\varphi}_0$ is a character of $\tilde{G}_0$ of $\ell$-defect zero. Let $\tilde{b}_0 = 1_{(-1_{V_0})} \times \tilde{b}_0$, where $\tilde{b}_0$ is a character of $\ell$-defect zero of $\tilde{G}_0 \cong SO_{2n_0+1}(q)$. So it is of the form $\chi_{s_0, \kappa}$, where $s_0$ is a semisimple $\ell'$-element of $G_0^* \cong Sp_{2n_0}(q)$ and $\kappa \in \Psi(s_0)$ such that $\kappa_\Gamma$ is an $e_\Gamma$-core which affords the second component of the triple $(s, \kappa, K)$.

Secondly, assume we have the following decomposition $\tilde{b}_+ = \prod_{i,j} \tilde{b}_+^{(i,j)}$, $R_+ = \prod_{i,j} R_+^{(i,j)}$, $\theta_1$ determines a semisimple $\ell$'-element with canonical form $e_\Gamma(\Gamma)$ in $G_\Gamma$. Let $s_1$, be a semisimple element of $G_+$ such that $s_1$ has divisor $\Gamma$ with multiplicity $\prod_{i,j} t_{\Gamma, (i,j)} e_\Gamma(\ell)$. Then $s = s_0 \times s_+$ is the first component of the triple $(s, \kappa, K)$. We can view the block $\tilde{b}$ as a block of $C_{\tilde{G}_0}(R)$. Thus $(R, \varphi)$ belongs to an $\ell$-block of $G$ with label $(s, \kappa, K)$. In particular, $\kappa \in C(s)$.

Finally, the correspondence $(R, \text{Ind}^\tilde{G}_{\tilde{N}(\tilde{R})}(\tilde{\psi})) \mapsto (R_+, \text{Ind}^\tilde{G}_{\tilde{N}(\tilde{R})}(\tilde{\psi}_+))$ is a bijection from $\{(R, \text{Ind}^\tilde{G}_{\tilde{N}(\tilde{R})}(\tilde{\psi})) \mid \tilde{\psi} \in \text{Irr}(\tilde{G}(\tilde{R})) \times \tilde{b}_0 \})$ to $\{(R_+, \text{Ind}^\tilde{G}_{\tilde{N}(\tilde{R})}(\tilde{\psi}_+)) \mid \tilde{\psi}_+ \in \text{Irr}(\tilde{N}((\tilde{R}_+) \times \tilde{b}_+)) \}$. Then the third component $K = \prod_{\Gamma} K_\Gamma$ of the triple $(s, \kappa, K)$ is given as in the statement preceding Lemma 3.6.

Let $(R, \varphi)$ be the $\ell$-weight of $G$ with label $(s_1, \kappa_1, K)$. Then by the proof of [4 (4G)], $(R, \varphi)$ is a $B(s_2, \kappa_2)$-weight if and only if $s_1$ and $s_2$ are $G^*$-conjugate and $\kappa_1 = \kappa_2$. 

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Proposition 4.7. Let \((R, \varphi)\) be the \(\ell\)-weight of \(G\) with label \((s, \kappa, K)^G\). Then \((R, \varphi)^\sigma\) is the \(\ell\)-weight of \(G\) with label \((\sigma^{-1} s), \sigma^{-1} \kappa, \sigma^{-1} K)^G\).

Proof. Now we want to find which triple corresponds to \((R, \varphi)^\sigma\). Assume it be \((s', \kappa', K')\). First, \(R^\sigma = R, \tilde{C}_d^\sigma = \tilde{N}_0^\sigma = \tilde{C}_0 = \tilde{N}_0, \tilde{C}_e^\sigma = \tilde{C}_+ \) and \(\tilde{N}_e^\sigma = \tilde{N}_+\) by Remark 3.4. Now, \(\tilde{\varphi}^\sigma = \tilde{\varphi}_0^\sigma \times \tilde{\varphi}_e^\sigma\), \(\varphi_0 = \text{Res}_{C_0}^\mathbb{C}(\tilde{\varphi}_0)\) is of the form \(\chi_{s_0, \kappa}\) by construction. By Proposition 4.1, \(\chi_{s_0, \kappa}^\sigma = \chi_{s_0, \kappa} = \chi_{\sigma^{-1} s_0, \sigma^{-1} \kappa}\). Then we have \(\kappa' = \sigma^{-1} \kappa\).

Secondly, \(\sigma\) stabilizes every \(\tilde{C}_{T, \delta, i}\). Now \(\tilde{\theta}_{s, \delta, i}^\sigma = \tilde{\theta}_{s^{-1}, \sigma, \delta, i}\) corresponds to \(\beta_T e_\ell^\theta(\Gamma)\). Up to conjugacy, we have \(s' = \sigma(s)\).

Finally, by the argument above, we may denote \(R_{T, \delta, i}^\sigma = R_{s, \sigma, \delta, i}, \tilde{C}_{T, \delta, i}^\sigma = \tilde{N}_{s, \sigma, \delta, i}\) and \(\tilde{C}_{\sigma, T, \delta, i} = \tilde{C}_{s^{-1}, \sigma, \delta, i}\) although the corresponding terms indexed by \(\Gamma\) and \(s^{-1} \Gamma\) are actually the same. Then \(K' = \sigma^{-1} K\) follows by Lemma 3.6. \(\square\)

Let \(i \mathcal{W}_G^\ell\) be the set of \(G\)-conjugacy classes of triples \((s, \kappa, \mu)\) such that \(s\) is an semisimple \(\ell\)-element of \(G\), \(\kappa \in \mathcal{C}(S), \mu \in \prod_T \mathcal{P}(\beta_T e_\mathbb{R}, \psi_{\ell, T})\). Then by [47, Prop. (3.7)], \(i \mathcal{W}_G^\ell\) is also a labeling set for \(\mathcal{W}_G^\ell\). Now by Corollary 3.7 and Proposition 4.7 we have

Corollary 4.8. Let \((R, \varphi)\) be an \(\ell\)-weight of \(G\) corresponding to \((s, \kappa, \mu)\) in \(i \mathcal{W}_G^\ell\), then \((R, \varphi)^\sigma\) corresponds to \((s^{-1}, \sigma^{-1} \kappa, \sigma^{-1} \mu)\).

Theorem 4.9. Let \(G = \text{SO}_{2n+1}(q)\), where \(q = p^f\) is a power of an odd prime \(p\), \(\ell \neq p\) is an odd prime and \(n \geq 2\). Assume that \(\ell\) is linear. Then the blockwise bijection between \(\mathcal{B}_G(\Gamma)\) and \(\mathcal{W}_G^\ell\) given in [4] is equivariant under the action of field automorphisms.

Proof. By Theorem 2.10 the set \(\mathcal{E}(G, \ell')\) is a basic set of \(\mathcal{B}_G(\Gamma)\) and by [26], the corresponding \(\ell\)-decomposition matrix of \(G\) is lower unitriangular since \(\ell\) is linear. Then there is a canonical bijection \(\Xi\) from \(i \mathcal{B}_G(\Gamma)\) to \(i \mathcal{W}_G^\ell\). By the construction of \(\Xi\) there, \(\Xi\) is \(E\)-equivariant. Thus the assertion follows by Proposition 4.3 and Corollary 4.8. \(\square\)

4.3 Weights of \(\Omega_{2n+1}(q)\)

Recall that \(z = -I_{2n} \in Z(\text{Sp}_{2n}(q))\). We may identify \(\hat{z}\) with \(1_{G_2} \times \hat{z}\) and regard \(\hat{z}\) as a linear character of \(G\). We may assume \(R_{T, \delta, i} = R_{s, \sigma, \delta, i}, \tilde{N}_{T, \delta, i} = N_{s, \sigma, \delta, i}, \) and \(C_{T, \delta, i} = C_{s, \sigma, \delta, i}\). We also may regard \(\hat{z}\) a non-trivial linear character of \(C_{T, \delta, i}\) by Lemma 3.1. Then by [18, Prop. 12.6], \(\hat{z} \theta_T = \pm z R_{T, \delta}^\sigma(\hat{\theta}_T) = \pm R_{s, \sigma, \delta, i}^\sigma(\hat{\theta}_T)\), and then \(\hat{z} \theta_{T, \delta, i} = \hat{\theta}_{s, \sigma, \delta, i}\). Thus \(\hat{z} \theta_{T, \delta, i} = \theta_{s, \sigma, \delta, i}\). So we may choose the labeling of \(\tilde{C}_{T, \delta, i}\) and \(\tilde{C}_{s^{-1}, \sigma, \delta, i}\) such that

\[ \hat{\psi}_{T, \delta, i} = \psi_{s, \sigma, \delta, i}\]. (4.1)

We define \(-K = \prod_T (-K_{s, \Gamma})\) with \((-K)_{s, \Gamma} = K_{s, \Gamma}\). Since \(N_G(R)/N_S(R) \cong G/S\), we may regard \(\hat{z}\) a linear character of \(N_G(R)\) whose kernel is \(N_S(R)\).

Proposition 4.10. Let \((R, \varphi)\) be the \(\ell\)-weight of \(G\) with label \((s, \kappa, K)^G\). Then \((R, \varphi)\) is the \(\ell\)-weight of \(G\) with label \((-s, -\kappa, -K)^G\).

Proof. We want to find which triple corresponds to \((R, \varphi)\). Assume it be \((s', \kappa', K')\). Now \(\tilde{\varphi} = 1_{G_2} \times \varphi\), so \(\tilde{\varphi} = 1_{G_2} \times \tilde{\varphi}_e\). First, \(\tilde{\varphi}_e = \tilde{\varphi}_0 \times \tilde{\varphi}_e\). \(\varphi_0 = \text{Res}_{C_0}^G(\tilde{\varphi}_0)\) is of the form \(\chi_{s_0, \kappa}\) by construction. By Proposition 4.5, \(\tilde{\chi}_{s_0, \kappa} = \chi_{s_0, \kappa}\). Then we have \(\kappa' = -\kappa\).

Secondly, we have \(\hat{\theta}_{T, \delta, i} = \hat{\theta}_{s, \sigma, \delta, i}\) as above. Note that \(\hat{\theta}_{T, \delta, i}\) corresponds to \(\beta_T e_\ell^\theta(\Gamma)\) and \(\hat{\theta}_{s, \sigma, \delta, i}\) corresponds to \(e_\ell^\theta(s, \Gamma)\). Up to conjugacy, we have \(s' = -s\).
Finally, by the conventions above, we may assume $R_{\Gamma,\delta,i} = R_{\tilde{\Gamma},\delta,i}$, $\tilde{N}_{\Gamma,\delta,i} = \tilde{N}_{\tilde{\Gamma},\delta,i}$, and $\tilde{C}_{\Gamma,\delta,i} = \tilde{C}_{\tilde{\Gamma},\delta,i}$. To determine $K'$, we note that $\tilde{\psi}_+ = \prod_{\Gamma,\delta,i} \tilde{\psi}_{\Gamma,\delta,i}$. By (3.4), $\tilde{\psi}_{\Gamma,\delta,i}$ is

$$
\tilde{\psi}_{\Gamma,\delta,i} = \text{Ind}^{\tilde{\chi}_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i})}_{\chi_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i})} \left( \prod_j \tilde{\psi}_{\Gamma,\delta,i,j} \right) \cdot \prod_j \phi_{\delta,i,j}.
$$

Since $\tilde{\theta}_{\Gamma,\delta,i} = \tilde{\theta}_{\tilde{\Gamma},\delta,i}$, we have $\tilde{N}_{\Gamma,\delta,i}(\tilde{\theta}_{\Gamma,\delta,i}) = \tilde{N}_{\tilde{\Gamma},\delta,i}(\tilde{\theta}_{\Gamma,\delta,i})$. We can fix the way to extend $\prod_j \tilde{\psi}_{\Gamma,\delta,i,j}$ as in [22] Lem. 25.5, then we have that $\tilde{\psi}_{\Gamma,\delta,i} = \tilde{\psi}_{\Gamma,\delta,i}$. Since $\tilde{\psi}_{\Gamma,\delta,i} = \tilde{\psi}_{\Gamma,\delta,i}$ by (4.4), $\tilde{\psi}_{\Gamma,\delta,i}$ would be

$$
\text{Ind}^{\tilde{\chi}_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i})}_{\chi_{\Gamma,\delta,i}(\theta_{\Gamma,\delta,i})} \left( \prod_j \tilde{\psi}_{\Gamma,\delta,i,j} \right) \cdot \prod_j \phi_{\delta,i,j}.
$$

Then $K'_{\Gamma,\delta,i} = K'_{\tilde{\Gamma},\delta,i}$ which is just $K' = \pi K$. Thus we complete the proof.  

**Corollary 4.11.** Let $(R, \varphi)$ be an $\ell$-weight of $G$ corresponding to $(s, \kappa, \mu) \in \mathcal{I}'W'(G)$, then $(R, \varphi)$ corresponds to $(-s, -\kappa, -\mu)$. 

Let $\mathcal{W}_\ell(G)$ be a complete set of representatives of all $G$-conjugacy classes of $\ell$-weights of $G$. We may assume that for $(R_1, \varphi_1), (R_2, \varphi_2) \in \mathcal{W}_\ell(G)$, $R_1$ and $R_2$ are $G$-conjugate if and only if $R_1 = R_2$.

Let $S = \Omega_{2n+1}(q)$, then $\text{Rad}_\ell(S) = \Omega_{2n}(S)$. Now define a equivalence relation on $\mathcal{W}_\ell(G)$ such that for $(R_1, \varphi_1), (R_2, \varphi_2) \in \mathcal{W}_\ell(G)$, $(R_1, \varphi_1) \sim (R_2, \varphi_2)$ if and only if $R_1 = R_2$ and $\varphi_1 = \varphi_2 \eta$ for some $\eta \in \text{Irr}(N_S(R_1)/N_S(R_1))$. Then by [20] Lem. 2.4 and Corollary [3.3] the set $\{(R, \varphi)\}$, where $(R, \varphi)$ runs through a complete set of representatives of the equivalence classes of $\mathcal{W}_\ell(G)$, is a complete set of representatives of all $S$-conjugacy classes of $\ell$-weights of $S$.

Let $(R, \varphi)$ be an $\ell$-weight of $G$, $(R, \psi)$ an $\ell$-weight of $S$ such that $\varphi \in \text{Irr}(N_S(R) \mid \psi)$. Let $b = b(R, \varphi)$, $b_0 = b(R, \psi)$ and $B = b^G$ and $B_0 = b^G_0$. By [33] Lem. 2.3, if $b$ covers $b_0$, then $B$ covers $B_0$.

Let $B_0$ be an $\ell$-block of $S$. Denote by $B_0$ the union of the $\ell$-blocks of $S$ which are $G$-conjugate to $B_0$ and $B$ the union of the $\ell$-blocks of $G$ which cover $B_0$. Then

- if $(R, \varphi)$ is an $\ell$-weight of $G$ belonging to $B$, then for every $\psi \in \text{Irr}(N_S(R) \mid \varphi)$, $(R, \psi)$ is an $\ell$-weight of $X$ belonging to $B_0$, and
- if $(R, \psi)$ is an $\ell$-weight of $S$ belonging to $B_0$, then there exists $\varphi \in \text{Irr}(N_G(R) \mid \psi)$ such that $(R, \varphi)$ is an $\ell$-weight of $G$ belonging to $B$.

**Proposition 4.12.** Let $q$ be a power of and odd prime and $\ell$ an odd prime. Assume that $\ell$ is linear. Then the Alperin weight conjecture [17] holds for every $\ell$-block of group $S = \Omega_{2n+1}(q)$.

**Proof.** The proof here is analogous to [20] Thm. 1.2. Thanks to [13], we may assume that $\ell \nmid q$. Let $\Theta$ be the canonical blockwise bijection between $\text{IBr}_\ell(G)$ and $\mathcal{W}_\ell(G)$. For $\phi \in \text{IBr}_\ell(G)$, let $(R, \varphi) = \Theta(\phi)$. By Proposition [4.6] and Corollary [4.11] $\phi = \phi$ if and only if $\phi = \varphi$. Thus the assertion follows by the argument above.  

By [27] §2.5, $\text{Aut}(S) \cong G \rtimes E$, where $E = \langle F_p \rangle$. 

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Theorem 4.13. Let $S = \Omega_{2n+1}(q)$, where $q = p^\ell$ is a power of an odd prime $p$, $\ell \neq p$ is an odd prime and $n \geq 2$. Assume further that $f$ is odd and $\ell$ is linear. Then there exists a blockwise $\text{Aut}(S)$-equivariant bijection between $\text{IBr}_t(S)$ and $\mathcal{W}_t(S)$.

Proof. It is analogous to the proof of [20, Prop. 5.19]. By Theorem 4.9 and the proof of Proposition 4.12, it suffices to show that for any $\phi \in \text{IBr}_t(G)$ and any $(R, \varphi) \in \mathcal{W}_t(G)$, $E$ acts trivially on $\text{IBr}_t(S)$ and $N_{G_{N_{G_t}(R)}}(R)$ acts trivially on $\text{Irr}(N_{G_t}(R) | \varphi)$. Let $m_1$ be the length of an orbit of $E$ on $\text{IBr}_t(S)$ and $m_2$ the length of an orbit of $N_{G_{N_{G_t}(R)}}(R)$ on $\text{Irr}(N_{G_t}(R) | \varphi)$. Then $m_1, m_2 \leq 2$, $m_1 | f$ and $m_2 | f$. Now $f$ is odd, so $m_1 = m_2 = 1$. This completes the proof. □

Note that $X = \text{Spin}_{2n+1}(q)$ is the universal $\ell'$-covering group of the simple group $S = \Omega_{2n+1}(q)$ unless when $n = q = 3$ by [27, § 6.1].

Proof of Theorem 7.2 By Lemma 2.15 we may assume that $\ell \nmid q$. By assumption, $\text{Aut}(S)/S$ is cyclic for $S = X/Z(X) = \Omega_{2n+1}(q)$. Thus by Corollary 2.14 it suffices to show that there exists a blockwise $\text{Aut}(X)$-equivariant bijection between $\text{IBr}_t(X)$ and $\mathcal{W}_t(X)$.

By Lemma 2.5, it suffices to show that there exists a blockwise $\text{Aut}(S)$-equivariant bijection between $\text{IBr}_t(S)$ and $\mathcal{W}_t(S)$, which follows by Theorem 4.13. □

### 4.4 The unipotent blocks

Proposition 4.14. Let $G = \text{SO}_{2n+1}(q)$ and $S = \Omega_{2n+1}(q)$ with $n \geq 2$ and $q$ odd, and $\ell \nmid q$ an odd prime. Suppose $B$ is a unipotent $\ell$-block of $G$. Then $B$ covers a unique $\ell$-block $b$ of $S$ so that $\text{Res}_S^G : IBr_t(B) \to \text{IBr}_t(b)$ is bijective and $\mathcal{W}_t(B) \to \mathcal{W}_t(b)$, $(R, \varphi) \mapsto (R, \text{Res}_{N_{G_t}(R)}^{N_{G_t}(B)} \varphi)$ is a bijection.

Proof. For $\ell$-weights, this follows by Proposition 4.10. Now we prove that $\text{Res}_S^G : IBr_t(B) \to IBr_t(b)$ is bijective. First, $\bar{\ell} \otimes B$ is also an $\ell$-block of $G$ covering $b$ (for the definition of $\bar{\ell} \otimes B$, see Page 5). Also, $\bar{\ell} \otimes B \neq B$ since $\bar{\ell} \otimes B \subseteq \mathcal{E}_\ell(G, z)$. Thus there are two $\ell$-blocks covering $b$. So by Lemma 2.6, $\text{Res}_S^G : IBr_t(B) \to IBr_t(b)$ is bijective. □

Proof of Theorem 7.5 for type $B_n$. By Lemma 2.15 we may assume that $\ell \nmid q$. Let $b$ be a unipotent $\ell$-block of $X$. Then $b$ dominates an $\ell$-block $\bar{b}$ of $S = \Omega_{2n+1}(q)$. Thus there exists a unique unipotent $\ell$-block $\bar{B}$ of $G = \text{SO}_{2n+1}(q)$ which covers $\bar{b}$. By Proposition 4.12, $\text{Res}_S^G : IBr_t(\bar{B}) \to IBr_t(\bar{b})$ is bijective and $\mathcal{W}_t(\bar{B}) \to \mathcal{W}_t(\bar{b})$, $(\bar{R}, \bar{\varphi}) \mapsto (\bar{R}, \text{Res}_{N_{G_t}(\bar{B})}^{N_{G_t}(\bar{b})} \bar{\varphi})$ is a bijection. Thus by Theorem 4.9(i), there exists a blockwise $\text{Aut}(S)_b$-equivariant bijection between $\text{IBr}_t(\bar{b})$ and $\mathcal{W}_t(\bar{b})$. This gives a blockwise $\text{Aut}(X)_b$-equivariant bijection between $\text{IBr}_t(b)$ and $\mathcal{W}_t(b)$ by Lemma 2.5. Then it suffices to show Definition 2.11(iii) by Lemma 2.13.

Let $\bar{X}$ be the special Clifford group of $V$. Then $\bar{X}/Z(\bar{X}) \cong G$, $|\bar{X}/Z(\bar{X})| = 2$ and $\text{Aut}(X) \cong \bar{X}/Z(\bar{X}) \rtimes E$. Thus we may identify the non-trivial outer automorphism induced by $\bar{X}$ on $X$ with the non-trivial outer automorphism induced by $G$ on $S$. Let $B$ be the $\ell$-block of $\bar{X}$ which dominates $\bar{B}$.

Claim 1: Every $\phi \in \text{IBr}_t(b)$ extends to $\bar{X} \rtimes E$.

First note that all irreducible character of $b$ have $Z(X)$ in their kernel. By Theorem 2.10, $\text{Irr}(b) \cap \mathcal{E}(X, \ell')$ is a basic set of $b$, so all irreducible $\ell'$-Brauer characters of $b$ have $Z(X)$ in their kernel. We denote by $\hat{\phi}$ the corresponding $\ell'$-Brauer character of $S$. Then by Proposition 4.11, there exists a unique unipotent irreducible $\ell'$-Brauer character $\bar{\phi}$ of $G$ which is an extension of $\hat{\phi}$. Let $\phi$ be the inflation of $\bar{\phi}$ to $\bar{X}$. Then $\phi$ is an extension of $\phi$ to $\bar{X}$. By [39, Thm. 2.5], $\bar{\phi}$ is $E$-invariant. Thus $\phi$ is $E$-invariant. So $\phi$ extends to $\bar{X} \rtimes E$ and so does $\phi$ and finally claim 1 holds.
Claim 4.15. Now we give the relationship between the proof. □

and so does the proof above. Since φ and C

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Claim 2: If (R, φ) is a b-weight, then φ extends to (X ∗ E)_R.

Let (R, φ) be the b-weight corresponding to (R, φ) (as in Lemma 2.3 (iii)). Then by Proposition 4.14 there exists a unique B-weight (R, ˜φ) such that ˜φ = Res^{N_b(R)}_{N_b(R)} φ. Let (R, ˜φ) be the B-weight corresponding to (R, ˜φ) as in Lemma 2.1. Then by Remark 4.15 R = X ∩ R, ˜R = ROI(Z( ˜X)) and φ = Res^{N_b(R)}{N_b(R)} ˜φ. By Proposition 4.17 the G-conjugacy class of (R, ˜φ) is E-invariant. Then (X ∗ E)_R = (X ∗ E)_R and ˜φ is (X ∗ E)_R-invariant. Thus ˜φ extends to (X ∗ E)_R and so does φ. So claim 2 holds.

For a set IBr(b | Q) as in Lemma 2.13 (for definition, see the proof of [49 Lemma 2.10]) and φ ∈ IBr(b | Q), we let A := A(φ, Q) = (X ∗ E)/Z(X)O_L(Z(X)). By a similar argument with the proof of [20 Prop. 7.1], Definition 2.11 (iii)(1)-(3) hold. For Definition 2.11 (iii)(4), the proof of [20 Lem. 7.2] applies here. Then the (iBAW) condition holds for b, which completes the proof. □

Remark 4.15. Now we give the relationship between ℓ-weights of ˜X and ℓ-weights of X in the proof above. Since |X/XZ( ˜X)| = 2, by the same argument with [20 §5.2 and §5.3], we have the following statements.

• ˜R → ˜R ∩ X gives a bijection from the Rad_b( ˜X) to Rad_b(X) with inverse given by R → R × O_L(Z( ˜X)).

• Let ˜R = ROI(Z( ˜X)) for R ∈ Rad_b(X). If ( ˜R, ˜φ) is an ℓ-weight of ˜X, then (R, φ) is an ℓ-weight of X for every φ ∈ Irr(N_b(R) | ˜φ). Conversely, if (R, φ) is an ℓ-weight of X, then there exists ˜φ ∈ Irr(N_b( ˜R) | φ) such that (R, ˜φ) is an ℓ-weight of ˜X.

• Let R ∈ Rad_b(X), ˜R = ROI(Z( ˜X)), ( ˜R, ˜φ) an ℓ-weight of ˜X and φ ∈ Irr(N_b(R) | ˜φ). Suppose that ˜B is an ℓ-block of ˜X and B is an ℓ-block of X. If ( ˜R, ˜φ) is an ℓ-weight and (R, φ) is an B-weight, then ˜B covers B.

5 Type C

In this section, we denote by G = Sp_{2n}(_F_q), G = CSp_{2n}(_F_q) with q odd and n ≥ 2. Let F_p : G → G is the raising of matrix entries to the p-th power, F = F^p, for some f such that q = p^f. Let G = G^F = Sp_{2n}(q), G = G^F = CSp_{2n}(q). Then |G/GZ(G)| = 2. We denote by V the underlying space of G and G^F.

5.1 The characters of symplectic groups

First note that G^F is the corresponding special Clifford group (then G^* = (G^*)^F is a special Clifford group over _F_q and G^* = SO_{2n+1}(_F_q). Thus there is a natural epimorphisms π : G^* → G^*. Clearly, π(G^*) = G^* = G^F = SO_{2n+1}(q). For a semisimple element s of G^*, we write ˜s = π(s). Note that m_{−1}( ˜s) is odd and m_{−1}( ˜s) is even. In particular, m_{−1}( ˜s) ≠ 0. Let irr( ˜G^*) be the set of G^*-conjugacy classes of pairs (s, µ), where s is a semisimple ℓ'-element of G^* and µ ∈ Ψ( ˜s) (where Ψ( ˜s) is defined as (2.4)). Here, (s_1, µ_2) and (s_1, µ_2) are conjugate if and only if s_1 and s_2 are G^*-conjugate and µ_1 = µ_2. With the parametrization of pairs involving semisimple elements and unipotent characters, the irreducible characters of ˜G were constructed by Lusztig [37]: by Jordan decomposition of characters, there is a bijection from irr( ˜G) onto Irr( ˜G) (see also [23 (4.5)]). We write ˜X_{s, µ} the character of ˜G corresponding to (s, µ).

In this section, we always assume that σ = F_p is the field automorphism and E = ⟨F_p⟩ as above. Then σ^* is also a field automorphism. Note that σ^* commutes with π.
Lemma 5.1. Let \((s, \mu) \in \text{Irr}(\tilde{G})\). Then \(\tilde{\chi}_{s,\mu}^\sigma = \tilde{\chi}_{s^{-1}(s),\sigma^{-1}\mu}^\sigma\).

Proof. Similar with Proposition 4.1, this follows [16, Thm. 3.1] and the fact that every unipotent character of groups of type \(A, \tilde{A}, B, D\) and \(\tilde{2}D\) is invariant under field automorphisms (see [39, Thm. 2.5]).

We will make use of the following result.

Theorem 5.2. Let \(\tilde{\chi} \in \text{Irr}(\tilde{G})\) and \(\Delta = \text{Irr}(G \mid \tilde{\chi})\). Then \(E_\Delta\) acts trivially on \(\Delta\).

Proof. This is [17, Thm. 3.1] (or [52, Thm. 16.2]).

If \(\chi \in \text{Irr}(G \mid \tilde{\chi}_{s,\mu})\), then we say \(\chi\) corresponds to the pair \((\tilde{s}, \mu)\).

Lemma 5.3. Let \((s, \mu) \in \text{iIrr}(\tilde{G})\), \(\tilde{s} = \pi(s)\) and \(\tilde{\chi} = \tilde{\chi}_{s,\mu}^\sigma\).

(i) If \(-1\) is not an eigenvalue of \(\tilde{s}\), then \(\text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}\) is irreducible.

(ii) If \(-1\) is an eigenvalue of \(\tilde{s}\), then

- if \(\mu_{s+1}\) is degenerate, then \(\text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}\) is irreducible, and
- if \(\mu_{s+1}\) is non-degenerate, then \(\text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}\) is a sum of two irreducible constituents.

Proof. First note that \(\tilde{G}\) is a regular embedding of \(G\). Let \(\psi_\mu\) be the unipotent character of \(C_G(\tilde{s})^F\) corresponding to \(\mu\). Then we may regard \(\psi_\mu\) as a unipotent character of \(C_G(\tilde{s})\) since \(E(C_G(\tilde{s})^F, 1) = E(C_G(\tilde{s})^F, 1)\). By Jordan decomposition (cf. [38, Prop. 5.1]), \(|\text{Irr}(G \mid \tilde{\chi})| = \text{Irr}(G(\tilde{s})^F | \psi_\mu)|

Let \(V^s\) be the underlying space of \(G^s\) and let \(\tilde{s} = \prod G^s(\Gamma)\) and \(V^s = \sum G^s(\tilde{s})\) be the primary decomposition. In particular, we abbreviate \(V^s_1 := V^s_{x-1}(\tilde{s})\), \(V^s_{-1} := V^s_{x+1}(\tilde{s})\) and \(V^s_{-2} := V^s_{x+2}(\tilde{s})\).

Here, \(V^s_1\) is of odd dimension and \(V^s_{-1}\) is of even dimension. Then \(C_G(\tilde{s})^F = (C_G(\tilde{s})^F, 1) \cap (GO(V^s_1) \times GO(V^s_{-1})) \times \prod C_{GO(V^s_1)}(\tilde{s}(\Gamma))\).

Now let \(-1\) be an eigenvalue of \(\tilde{s}\). Then \(|C_G(\tilde{s})^F/C_G(\tilde{s})^F| = 2\) and \(\text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}\) is irreducible if and only if \(\psi_\mu\) is not \(C_G(\tilde{s})^F/C_G(\tilde{s})^F\)-invariant. Let \(g = g_1 \times g_{-1} \times \prod g_{\Gamma}\) with \(g_1 \in \text{GO}(V^s_1)\), \(g_{-1} \in \text{GO}(V^s_{-1})\) and \(g_{\Gamma} \in \text{C}_{GO(V^s_1)}(\tilde{s}(\Gamma))\) such that \(g_1\) and \(g_{-1}\) are of determinant \(-1\). Then \(\psi_\mu = \psi_1 \times \psi_{-1} \times \prod \psi_{\Gamma}\) and the assertion holds. Thus the assertion holds.

Remark 5.4. In Lemma 5.3(ii), if \(\mu_{s+1}\) is degenerate, then by Jordan decomposition, \(\text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}_{s,\mu} = \text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}_{s,\mu'}\), where \(\mu'\) is defined as in [22, Thm. 5.2]. Thus \(\text{Irr}(\tilde{G} \mid \text{Res}_{\tilde{G}\tilde{s}}^G \tilde{\chi}_{s,\mu}) \cap \text{E}(\tilde{G}, s) = \{\tilde{\chi}_{s,\mu}, \tilde{\chi}_{s,\mu'}\}\).

5.2 The blocks of symplectic groups

Let \(\ell\) be an odd prime with \(\ell \not| q\) and \(e_0\) the multiplicative order of \(q\) modulo \(\ell\). Then \(e = \gcd(e_0, 2)\). The labeling of \(\ell\)-blocks of \(\tilde{G}^F\) and \(G^F\) (using \(e_0\)-Jordan-cuspidal pairs) described in [15] and [31] can be stated as follows.

Theorem 5.5. Let \(H \in \{\tilde{G}, G\}\) and \(e_0 = e_0(q, \ell)\) is defined as in Equation (2.7).
Proposition 5.6. (i) Let \((\tilde{L}, \tilde{\zeta})\) be an \(e_0\)-cuspidal pair of \(\tilde{G}\) and \(B\) an \(\ell\)-block of \(\tilde{G}^F\) covered by \(\tilde{B} = b_{\tilde{G}^F}(\tilde{L}, \tilde{\zeta})\), then \(B = b_{G^F}(L, \zeta)\), where \(L = \tilde{L} \cap G\) and \(\zeta\) is an irreducible constituent of \(\text{Res}^{\tilde{L}^F}_{L^F}\tilde{\zeta}\).

(ii) Moreover, the map \(\Xi : (L, \zeta) \mapsto b_{\tilde{G}^F}(L, \zeta)\) is a bijection from the set of \(H^F\)-conjugacy classes of \(e_0\)-Jordan-cuspidal pairs \((L, \zeta)\) of \(H\) such that \(\zeta \in \mathcal{E}(L^F, \ell')\) to the \(\ell\)-blocks of \(H^F\).

Now we give the relationship between the \(e_0\)-cuspidal pairs of \(\tilde{G}\) and the \(e_0\)-cuspidal pairs of \(G\).

**Proposition 5.6.** Let \((\tilde{L}, \tilde{\zeta})\) be an \(e_0\)-cuspidal pair of \(\tilde{G}\) and \(B\) an \(\ell\)-block of \(G^F\) covered by \(\tilde{B} = b_{\tilde{G}^F}(\tilde{L}, \tilde{\zeta})\), then \(B = b_{G^F}(L, \zeta)\), where \(L = \tilde{L} \cap G\) and \(\zeta\) is an irreducible constituent of \(\text{Res}^{\tilde{L}^F}_{L^F}\tilde{\zeta}\).

**Proof.** This follows by [23, Lem. 3.7 and 3.8] (see [20, Prop. 4.5] for details).

Note that we have \(\tilde{L} = Z(\tilde{G})L\) in Proposition 5.6. In fact, the \(F\)-stable Levi subgroups of \(\tilde{G}\) and \(G\) have been classified in [23] (3A) and (3B)).

**Lemma 5.7.** Let \(\tilde{L}\) be an \(F\)-stable Levi subgroup of \(\tilde{G}\), \(\tilde{\zeta} \in \text{Irr}(\tilde{L}^F)\), and \(L = \tilde{L} \cap G\). Let \(\Delta := \text{Irr}(L^F \mid \tilde{\zeta})\), then \(N_{\tilde{G}^F}(L)\Delta\) acts trivially on \(\Delta\).

**Proof.** Let \(L = L^F\) and \(\tilde{L} = \tilde{L}^F\). Follow [23, (3A) and (3B)], we may assume that there is the orthogonal decomposition \(V = V_0 \perp V_+\) of \(V\), where \(V_+ = \sum_{i=1}^t V_i\) such that

- \(L = L_0 \times L_+\), where \(L_0 = \text{Sp}(V_0)\), \(L_+ = \prod_{i=1}^t L_i\) such that \(L_i \leq \text{Sp}(V_i)\) isomorphic to some general linear or unitary group for \(1 \leq i \leq t\).
- \(\tilde{L} = \langle \tau, L \rangle\), where \(\tau\) satisfies \(\tilde{G} = (G, \tau)\) and \([\tau, L_+] = 1\). Moreover, \(\tau = \tau_0 \times \tau_+\) such that \(\tau_0 \in \text{CSp}(V_0)\) and \(\tau_+ \in \text{CSp}(V_+)\).

Thus, \(|L/L^2(L)| = 2\) and \(N_{\tilde{G}^F}(L) = N_0 \times N_+\), with \(N_0 = L_0\) and \(N_+ \leq \text{Sp}(V_+).\) So \(|\Delta| \leq 2\). If \(|\Delta| = 1\), then the assertion is obvious. Now we may assume that \(|\Delta| = 2\).

Let \(\Delta = \{\zeta, \zeta'\}\), then \(\zeta\) and \(\zeta'\) are \(\tilde{L}\)-conjugate. We write \(\tilde{\zeta} = \zeta_0 \times \zeta_+\) and \(\tilde{\zeta}' = \zeta_0' \times \zeta_+\) with \(\zeta_0, \zeta_0' \in \text{Irr}(L_0)\) and \(\zeta_+, \zeta_+ \in \text{Irr}(L_+)\). Since \([\tau, L_+] = 1\), we know \(\zeta_+ = \zeta_+\). Hence \(\zeta_0 \neq \zeta_0'\). For \(n \in N_{\tilde{G}^F}(L)\), we let \(n = n_0 \times n_+\), where \(n_0 \in N_0\) and \(n_+ \in N_+\). If \(\zeta_0 = \zeta_0'\), then \(\zeta_0'' = \zeta_0'\) and this is impossible since \(n_0 \in L_0\). So \(\zeta_0'' \neq \zeta_0'\), which implies that \(N_{\tilde{G}^F(L)\Delta}\) acts trivially on \(\Delta\).

**Remark 5.8.** Let \(L\) be a Levi subgroup of \(\tilde{G}\), and \(L = \tilde{L} \cap G\). Then by (4) of [23, (3B)] \(\tilde{G}^F/L^F \cong G^F/L^F\) and then \(G^F = G^F N_{\tilde{G}^F}(L)\). So the \(G^F\)-conjugacy classes of \(e_0\)-split Levi subgroups of \(G\) are just the \(G^F\)-conjugacy classes of \(e_0\)-split Levi subgroups of \(G\). We denote by \(\tilde{L}\) a complete set of representatives of the \(G^F\)-conjugacy classes of \(e_0\)-Jordan-cuspidal pairs of \(\tilde{G}\) such that \(\tilde{\zeta} \in \mathcal{E}(\tilde{L}^F, \ell')\). We may assume that for \((L, \zeta), (L', \zeta') \in \tilde{L}\), if \(L\) and \(L'\) are \(G^F\)-conjugate, then \(L = L'\). Now we define an equivalence relation on \(\tilde{L}\) : \((L, \zeta) \sim (L', \zeta')\) if and only if \(L = L'\) and \(\text{Res}^{\tilde{L}^F}_{L^F}\zeta = \text{Res}^{\tilde{L}^F}_{L^F}\tilde{\zeta}\) where \(L = L \cap G\). Then by Proposition 5.6 and Lemma 5.7, \(\{L \cap G, \tilde{\zeta}\}\) is a complete set of representatives of \(G^F\)-conjugacy classes of \(e_0\)-Jordan-cuspidal pairs of \(G\) such that \(\zeta_0 \in \mathcal{E}(L \cap G) F^F \ell'\), where \((L, \zeta)\) runs through a complete set of representatives of the equivalence classes of \(L / \sim\) and \(\zeta\) runs through \(\text{Irr}(L \cap G)^F (\ell')\).
Now we recall the classification of $\ell$-blocks of $\tilde{G}^F$ in [23, §11]. Let $i\mathrm{Bl}_I(\tilde{G})$ be the set of $\tilde{G}^*$-conjugacy classes of pairs $(s, \kappa)$ where $s$ is a semisimple $\ell^*$-element of $\tilde{G}$ and $\kappa \in C(\tilde{s})$ (where $C(\tilde{s})$ is defined as (2.5)). Here, $(s_1, \kappa_1)$ and $(s_2, \kappa_2)$ are $\tilde{G}^*$-conjugate if and only if $s_1$ and $s_2$ are $\tilde{G}^*$-conjugate and $\kappa_1 = \kappa_2$. Then [23] (11E) gives a bijection $(s, \kappa) \mapsto \tilde{B}(s, \kappa)$ from $i\mathrm{Bl}_I(\tilde{G})$ to $\mathrm{Bl}_I(\tilde{G})$.

For $(s, \kappa) \in i\mathrm{Bl}_I(\tilde{G})$, [23] (13B) also gave a criterion that when an irreducible character of $\tilde{G}$ lies in the $\ell$-block $\tilde{B} = \tilde{B}(s, \kappa)$. In particular, the irreducible characters of $\mathrm{Irr}(\tilde{B}) \cap \mathcal{E}(\tilde{G}, \ell^*)$ are of form $\tilde{\chi}_{s, \mu}$ with $\mu \in \Psi(\tilde{s}, \kappa)$ (where $\Psi(\tilde{s}, \kappa)$ is defined as in (2.6)). In addition, by (2.11), $\Psi_T(\tilde{s}, \kappa)$ is in bijection with $\mathcal{P}(\beta_T e_T, w_{\Gamma})$ if $\Gamma \neq x + 1$ or $\Gamma = x + 1$ and $\kappa_{x+1}$ is non-degenerate and in bijection with $\mathcal{P}'(2e, w_{x+1})$ if $\Gamma = x + 1$ and $\kappa_{x+1}$ is degenerate. Here, the sets $\mathcal{P}(\beta_T e_T, w_{\Gamma})$ and $\mathcal{P}'(2e, w_{x+1})$ are defined as in (2.7) and (2.10) respectively.

Fix $(s, \kappa) \in i\mathrm{Bl}_I(\tilde{G})$. Now we give an $e_0$-Jordan-cuspidal pair of $\tilde{G}^F$ corresponding to $\tilde{B}(s, \kappa)$. First, we define an $e_0$-split Levi subgroup $L$ of $G^F$. Let $(s, \mu) \in i\mathrm{Irr}(\tilde{G})$ such that $\chi_{s, \mu} \in B(s, \kappa)$. Recall that we have integers $w_\Gamma = e_\Gamma^{-1}(m_\Gamma(s) - |\kappa|)$ if $\Gamma \in F_1 \cup F_2$ and $w_\Gamma$ is the number of $e_\Gamma$-hooks (or $e_\Gamma$-cohooks) removed from $\mu$ to get $\kappa$ if $\Gamma \notin F_0$ (see §2.4). Note that $w_\Gamma$‘s do not depend on the choice of $\mu$ and are determined by $(s, \kappa)$. Let $\tilde{D}$ be a defect group of $\ell$-block $\tilde{B}(s, \kappa)$ and $D = \tilde{D} \cap G$. Then by [23] §11, there exist corresponding orthogonal decompositions $V = V_0 \perp V_1 \perp \cdots \perp V_t$ and $D = D_0 \times D_1 \times \cdots \times D_t$ such that $D_0 = \langle V_0 \rangle$ and for $i > 0$, $D_i = R_m, \alpha_i, \kappa_i$ for integers $m_i, \alpha_i, \beta_i$. Here, $R_m, \alpha_i, \kappa_i = R_m, \alpha_i, \kappa, \gamma$ with $\gamma = 0, e = (\beta_i)$ as defined in §3.1. In addition, we may write $D = D_0 \prod\delta_i(R_m, \alpha_i, \kappa_i)^{e_\delta_i}$, where $\delta_i$ are the coefficients occurring in the $\ell$-adic expansion $\sum_{\delta} \delta_i R_m, \alpha_i, \kappa_i$ for $\Gamma \in F_\ell$. Thus $V_0 = C_V(D)$ and $V_+ = [D, V]$, where $V_+ = V_1 \perp \cdots \perp V_t$.

Let $L$ be an $F$-stable Levi subgroup of $G$ (described as in [23, (3A)]) such that $L = L^F = L_0 \times L_+$ with $L_0 = \mathrm{Sp}(V_0)$, $L_+ = \prod \prod_{i=1}^{w_\Gamma} L_{i, j} \leq \mathrm{Sp}(V_+)$ and $L_{i, j} \cong \GL_{m_{i, j}}(\mathbb{F}_{q^i})$ for $1 \leq i \leq w_\Gamma$, where $\epsilon = 1$ if $\ell$ is linear and $\epsilon = -1$ if $\ell$ is unitary. Let $V = V_0 + \sum \sum_{i=1}^{w_\Gamma} V_{i, j}$ be the corresponding orthogonal decomposition of $V$. Obviously, $V_+ = \sum \sum_{i=1}^{w_\Gamma} V_{i, j}$. From this we obtain an $e_0$-split Levi subgroup $L$ of $G$. Then $\tilde{L} = L \tau(G)$ is an $e_0$-split Levi subgroup of $\tilde{G}$ and the structure of $L^*$ and $\tilde{L}^*$ are described in [23, (3A) and (3B)]. In fact, $\tilde{L}$ is a regular embedding of $L$. Clearly, $\tilde{s} \in \tilde{L}^* := L^{*F}$ and $s \in L^* := (L^*)^F$ up to conjugacy. Also, $\tilde{L} = \langle L_0, \tau \rangle \cdot L_+$ is the central product of $\langle L_0, \tau \rangle$ and $L_+$, where $\tau$ is as in the proof of Lemma 5.7. Write $\tilde{s} = \tilde{s}_0 \times \tilde{s}_+$, with $\tilde{s}_0 \in L_0^*$ and $\tilde{s}_+ \in L_+^*$.

Let $V^*$ be the underlying space of $\tilde{G}^*$ and $G^*$ and $V^* = V_0^* + \sum \sum_{i=1}^{w_\Gamma} V_{i, j}^*$, $L^* = L_0^* \times L_+^*$ with $L_+^* = \prod \prod_{i=1}^{w_\Gamma} L_{i, j}^*$. In addition, we have the primary decompositions $V_0^* = \sum V_i^*(\tilde{s}_0)$ and $\tilde{s}_0 = \prod \bar{s}_0(\Gamma)$ of $V_0^*$ and $s_0$. Thus $C_{L_0^*}(\tilde{s}) = \SO(V_{x, i}(\tilde{s}_0)) \times \SO(V_{x, i+1}(\tilde{s}_0)) \times \prod \GL_{m_{i, j}}(\mathbb{F}_{q^{i+1}}) \times C_{L_+^*}(\tilde{s}_0)$, where $C_{L_+^*}(\tilde{s}_0) = \prod \prod_{i=1}^{w_\Gamma} \GL_{1}(\mathbb{F}_{q^{i+1}})^{\kappa_i \delta_i}$. Let $\phi_\kappa = \prod \phi_\kappa \times 1_{C_{L_+^*}(\tilde{s}_0)}$ be the unipotent character of $C_{L_+^*}(\tilde{s})$, where $\phi_\kappa$ is the unipotent character of $\SO(V_{x, i}^*(\tilde{s}_0))$ corresponding to $\kappa_\Gamma$ if $\Gamma \in F_0$ and unipotent character of $\GL_{m_{i, j}}(\mathbb{F}_{q^{i+1}})$ corresponding to $\kappa_i$ if $\Gamma \in F_1 \cup F_2$. Then $\phi_\kappa$ is an $e_0$-cuspidal unipotent character of $C_{L_+^*}(\tilde{s})$. Now note that $\mathcal{E}(C_{L_+^*}(\tilde{s})^F, 1) = \mathcal{E}(C_{L_0^*}(\tilde{s})^F, 1)$ and then we may regard $\phi_\kappa$ as an $e_0$-cuspidal unipotent character of $C_{L_+^*}(\tilde{s})^F$. Let $\tilde{\zeta}$ be the character of $\tilde{L}^F$ corresponding under the Jordan decomposition to $\phi_\kappa \in \mathcal{E}(C_{L_0^*}(\tilde{s})^F, 1)$. Then (\tilde{L}, \tilde{\zeta}) is an
e₀-Jordan-cuspidal pair of \( \tilde{G} \).

**Lemma 5.9.** With the notation above, the \( \ell \)-block \( b_{G^\ell}(\tilde{L}, \tilde{\zeta}) \) of \( \tilde{G}^\ell \) corresponding to the e₀-Jordan-cuspidal pair \((\tilde{L}, \tilde{\zeta})\) is \( \tilde{B}(s, \kappa) \).

**Proof.** Now we prove that there is one irreducible constituent of \( R_{L}^G(\tilde{\zeta}) \) lies in \( \tilde{B}(s, \kappa) \). In fact, this is essentially contained in [23, §13]. Let \( Q \) and \( \tilde{Q} \) be the \( F \)-fixed point of some \( F \)-stable Levi subgroups (say, \( Q \) and \( \tilde{Q} \)) defined in [23, p. 178], centralizer of a certain \( \ell \)-element in \( Z(D) \) in \( G \) and \( \tilde{G} \) respectively. Then \( Q \cong Q_0 \times Q_+ \) and \( \tilde{Q} = \langle Q_0, \tau \rangle Q_+ \) with \( Q_0 = L_0 \) and \( L_+ \leq Q_+ \). Also, we let \( \hat{b} \) be the \( \ell \)-block of \( \tilde{Q} \) defined in [23, p. 179]. Now \( \hat{L} \leq \tilde{Q} \), so \( R_{L}^G(\tilde{\zeta}) = R_{\tilde{Q}}^G(\hat{R}_{L}^{\hat{b}}(\hat{\zeta})) \).

In addition \( C_{\tilde{G}}(s) \leq \hat{L}^\ell \), then \( R_{L}^G(\hat{\zeta}) \) lies in \( \mathcal{E}(\hat{Q}, s) \) and then by [23 (13A)], lies in \( \hat{b} \). Thus we conclude from the proof of [23 (13B)] that there exists one irreducible constituent of \( R_{L}^G(\tilde{\zeta}) \) lies in \( \tilde{B}(s, \kappa) \).

Now keep the hypotheses and setup above and we wish to investigate how many \( \ell \)-blocks of \( G \) covered by \( \tilde{B}(s, \kappa) \). This number is equal to the cardinality of the set \( \text{Irr}(L \mid \zeta) \) by Remark 5.8.

Let \( \varepsilon \) be the identity element of the clifford algebra over \( V^\ast \). Then \( \mathbb{Z}(G)^{\varepsilon} = \{ k\varepsilon \mid k \in \mathbb{F}_q^\ast \} \).

For \( z \in \mathbb{Z}(\tilde{G}) \), we denote by \( \hat{z} \) the corresponding linear character (by [18 Prop. 13.30]) of \( \tilde{G} \) as before. Moreover, we may regard \( \hat{z} \) as a linear character of \( \tilde{L}/L \) since \( \tilde{L}/L \cong \hat{G}/G \). From this, \((\tilde{L}, \hat{z})\) is also an e₀-Jordan-cuspidal pair of \( G \). Also, \( \hat{z} \in \mathcal{E}(\tilde{L}, \ell') \) if and only if \( z \) is of \( \ell' \)-order. Conversely, if \( B' \) is an \( \ell \)-block of \( \hat{G} \) such that \( B' \) and \( b_{G^\ell}(\tilde{L}, \tilde{\zeta}) \) cover the same \( \ell \)-blocks of \( G \), then \( B' = b_{G^\ell}(\tilde{L}, \tilde{\zeta}) \) for some \( z \in \text{O}_F(\mathbb{Z}(G^\ast)) \).

The relations between conjugacy classes of \( G^\ast \) and \( G^\ast \) are given in [37, §6.4] (or [23, (2D)]). Let \( \hat{G} \) be the conjugacy class of \( G^\ast \) containing \( \hat{s} \) and \( C = \pi^{-1}(\hat{G}) \). If \( \ell = 1 \) is not an eigenvalue of \( \tilde{s} \), then \( C \) is the union of \( \{Z(G^\ast)\} \) conjugacy classes of \( G^\ast \) and each class contains a unique element of \( \{z \in \mathbb{F}_q^\ast \mid k \in \mathbb{Z}(G^\ast) \} \). If \( \ell = 1 \) is an eigenvalue of \( \tilde{s} \), then \( C \) is the union of \( \frac{1}{2} \) conjugacy classes of \( G^\ast \) and each class contains exactly two elements \( z_1 \) and \( z_2 \) of \( \{z \in \mathbb{F}_q^\ast \mid k \in \mathbb{Z}(G^\ast) \} \) such that \( z_1 = -z_2 \).

If \( \ell = 1 \) is not an eigenvalue of \( \tilde{s} \), then \( \tilde{B}(\hat{z}, \kappa) \) are \( \ell \)-blocks of \( \hat{G} \) for \( z \in \text{O}_F(\mathbb{Z}(G^\ast)) \) and they cover the same \( \ell \)-blocks of \( G \). In addition, \( b_{G^\ell}(\tilde{L}, \hat{z}) = \tilde{B}(\hat{z}, \kappa) = \tilde{B}(\hat{z}, \kappa) \). Now suppose \( \ell = 1 \) is an eigenvalue of \( \tilde{s} \). Then \( \hat{B}(\hat{s}, \kappa) \) and \( \hat{B}(\hat{z}, \kappa) \) (where \( \kappa \) is defined as in page [10]) are \( \ell \)-blocks of \( \hat{G} \), where \( \hat{s} \) runs through a complete set of representatives of \( \{(-\varepsilon)-\text{cosets in} \text{O}_F(\mathbb{Z}(G^\ast)) \}, \) and they cover the same \( \ell \)-blocks of \( G \). If \( \kappa_{x+1} \neq 0 \) or \( \kappa_{x+1} = 0 \) and \( \kappa_{x+1} \) is degenerate, then \( \hat{B}(\hat{s}, \kappa) = \hat{B}(\hat{s}, \kappa) \) by Remark 5.4 and then \( b_{G^\ell}(\tilde{L}, \hat{z}) = \hat{B}(\hat{s}, \kappa) = \hat{z} \otimes \hat{B}(\hat{s}, \kappa) \).

Let \( \kappa_{x+1} = 0 \) and \( \kappa_{x+1} \) is degenerate. Fix \( z \in \text{O}_F(\mathbb{Z}(G^\ast)) \), then \( \hat{B}(\hat{s}, \kappa) \) and \( \hat{B}(\hat{s}, \kappa) \) are distinct \( \ell \)-blocks of \( \hat{G} \). In addition, if \( b_{G^\ell}(\tilde{L}, \hat{z}) = \hat{B}(\hat{s}, \kappa) = \hat{z} \otimes \hat{B}(\hat{s}, \kappa) \), then \( b_{G^\ell}(\tilde{L}, \hat{z}) = \hat{B}(\hat{s}, \kappa) = \hat{z} \otimes \hat{B}(\hat{s}, \kappa) \).

Now \( \tilde{L} \) is a regular embedding of \( L \), so by Jordan decomposition, we have \( \text{Irr}(L \mid \tilde{\zeta}) = \text{Irr}(C_{L^\ell}(\tilde{s}) \mid \phi_\tilde{s}) \). Thus if \( \ell = 1 \) is not an eigenvalue of \( \tilde{s} \), then \( \text{Irr}(L \mid \tilde{\zeta}) = 1 \). Now suppose that \( \ell = 1 \) is an eigenvalue of \( \tilde{s} \). Then by Clifford theory, \( \text{Res}_{L^\ell}^{\tilde{L}^\ell} \tilde{\zeta} \) is not irreducible if and only if \( \phi_\tilde{s} \) is \( \text{Irr}(C_{L^\ell}(\tilde{s})) \)-invariant. Note that \( C_{L^\ell}(\tilde{s}) = \mathbb{SO}(V^\ast_{x+1}(\tilde{s}_0) \cap V^\ast_{x+1}(\tilde{s}_0)) \cap (\mathbb{GO}(V^\ast_{x+1}(\tilde{s}_0)) \times \mathbb{GL}_{m_1}(\tilde{s}_0)) \times C_{L_{x+1}}(\tilde{s}_{x+1}) \).

Here, \( V^\ast_{x+1} \) has even dimension and \( V^\ast_{x+1} \) has odd dimension. In this way, by a similar proof of Lemma 5.3, \( \text{Res}_{L^\ell}^{\tilde{L}^\ell} \tilde{\zeta} \) is irreducible if and only if \( \kappa_{x+1} \) is degenerate.

By the argument above, we have the following result.

**Theorem 5.10.** Let \((s, \kappa) \in \text{Irr}(\tilde{G}) \), \( \tilde{s} = \pi(s) \), \( \hat{B} = \hat{B}(s, \kappa) \) and \( B \) an \( \ell \)-block of \( G \) covered by \( \hat{B} \).
(i) $-1$ is not an eigenvalue of $\tilde{s}$, then $B$ is the unique $\ell$-blocks of $G$ covered by $\tilde{B}$ and there are $|O_{\ell}(F_\tilde{G})|$ $\ell$-blocks of $\tilde{G}$ covering $B$. In addition, the $\ell$-blocks covering $B$ of $\tilde{G}$ are $\tilde{B}(z,\kappa)$, where $z$ runs through $O_{\ell}(Z(\tilde{G}^*))$.

(ii) If $-1$ is an eigenvalue of $\tilde{s}$, $w_{x+1} = 0$ and $\kappa_{x+1}$ is degenerate, then $B$ is the unique $\ell$-blocks of $G$ covered by $\tilde{B}$ and there are $|O_{\ell}(F_\tilde{G})|$ $\ell$-blocks of $\tilde{G}$ covering $B$. In addition, the $\ell$-blocks covering $B$ of $\tilde{G}$ are $\tilde{B}(z,\kappa)$ and $\tilde{B}(z,\kappa')$, where $z$ runs through a complete set of representatives of $(-\infty)$-cosets in $O_{\ell}(Z(\tilde{G}^*))$.

(iii) If $w_{x+1} \neq 0$ or $\kappa_{x+1}$ is non-degenerate, then there are $\frac{1}{2}|O_{\ell}(F_\tilde{G})|$ $\ell$-blocks of $\tilde{G}$ covering $B$ and they are $\tilde{B}(z,\kappa)$, where $z$ runs through a complete set of representatives of $(-\infty)$-cosets in $O_{\ell}(Z(\tilde{G}^*))$.

Moreover,

- if $w_{x+1} \neq 0$ and $\kappa_{x+1}$ is degenerate, then $B$ is the unique $\ell$-blocks of $G$ covered by $\tilde{B}$ and
- if $\kappa_{x+1}$ is non-degenerate, then there are two $\ell$-blocks of $G$ covered by $\tilde{B}$.

Now let $iB^{(1)}_{\ell}(G)$ be the set of $G^*$-conjugacy classes of pairs $(s,\kappa)$, where $s \in G^*$ is a semisimple $\ell'$-element and $\kappa \in C(s)$ such that $-1$ is not an eigenvalue of $s$ or $\kappa_{s+1}$ is degenerate. Here, we identify $(s,\kappa)$ with $(s',\kappa')$. Let $iB^{(2)}_{\ell}(G)$ be the set of $G^*$-conjugacy classes of pairs $(s,\mu)$, where $s \in G^*$ is a semisimple $\ell'$-element and $\kappa \in C(s)$ such that $w_{x+1}(s) \neq 0$ and $\kappa_{s+1}$ is non-degenerate. Also $(s_1,\kappa_2)$ and $(s_2,\kappa_2)$ are $G^*$-conjugate means $s_1$ and $s_2$ are $\ell'$-conjugate and $\kappa_1 = \kappa_2$. Then $iB_{\ell}(G) := iB^{(1)}_{\ell}(G) \cup iB^{(2)}_{\ell}(G)$, where the elements of $iB^{(2)}_{\ell}(G)$ counting twice, is a labeling set for $B_{\ell}$ by Theorem 5.10. If $(s,\kappa) \in iB^{(1)}_{\ell}(G)$, we denote by $B(s,\kappa)$ the $\ell$-blocks of $G$ corresponding to $(s,\kappa)$. If $(s,\kappa) \in iB^{(2)}_{\ell}(G)$, then $B^{(1)}(s,\kappa)$ and $B^{(1)}(s,\kappa)$ denote the two $\ell$-blocks of $G$ corresponding to $(s,\kappa)$.

5.3 The action of $\text{Aut}(G)$ on the Brauer characters and weights

Now let $iB^{(1)}_{\ell}(G)$ be the set of $G^*$-conjugacy classes of pairs $(s,\mu)$, where $s \in G^*$ is a semisimple $\ell'$-element and $\mu \in \Psi(s)$ such that $-1$ is not an eigenvalue of $s$ or $\kappa_{s+1}$ is degenerate. Here, we identify $(s,\mu)$ with $(s,\mu)$ in $iB^{(1)}_{\ell}(G)$, which means degenerate symbols are not counted twice in any case. Let $iB^{(2)}_{\ell}(G)$ be the set of $G^*$-conjugacy classes of pairs $(s,\mu)$, where $s \in G^*$ is a semisimple $\ell'$-element and $\mu \in \Psi(s)$ such that $-1$ is an eigenvalue of $s$ and $\mu_{s+1}$ is non-degenerate. Then $iB_{\ell}(G) := iB^{(1)}_{\ell}(G) \cup iB^{(2)}_{\ell}(G)$, where the elements of $iB^{(2)}_{\ell}(G)$ counting twice, is a labeling set for $\text{Irr}(G) \cap \mathcal{E}(G,\ell')$ by Lemma 5.3. If $(s,\mu) \in iB^{(1)}_{\ell}(G)$, we denote by $x_{s,\mu}$ the character of $G$ corresponding to $(s,\mu)$. If $(s,\mu) \in iB^{(2)}_{\ell}(G)$, then $x_{s,\mu}^{(1)}$ and $x_{s,\mu}^{(1)}$ denote the two characters of $G$ corresponding to $(s,\mu)$.

Furthermore, if $(s,\kappa) \in iB^{(1)}_{\ell}(G)$ and $B = B(s,\kappa)$, then $\text{Irr}(B) \cap \mathcal{E}(G,s) = \{|x_{s,\mu}^{(1)}| \mu \in \Psi(s,\kappa),(s,\mu) \in iB^{(2)}_{\ell}(G)\}$. Here, the second set is non-empty if and only if both $1$ is an eigenvalue of $s$ and $\kappa_{s+1}$ is degenerate. If $(s,\kappa) \in iB^{(2)}_{\ell}(G)$, $B^{(1)} = B^{(1)}(s,\kappa)$ and $B^{(1)} = B^{(1)}(s,\kappa)$, then $\text{Irr}(B^{(1)} \cup B^{(1)}) \cap \mathcal{E}(G,s) = \{|x_{s,\mu}^{(1)}| \mu \in \Psi(s,\kappa)\}$. We may assume that $\text{Irr}(B^{(0)}) \cap \mathcal{E}(G,s) = \{|x_{s,\mu}^{(0)}| \mu \in \Psi(s,\kappa)\}$ for $i = \pm 1$. Note that, if $\kappa_{s+1}$ is non-degenerate, then $\mu_{s+1}$ is also non-degenerate, and then we always have $(s,\mu) \in iB^{(2)}_{\ell}(G)$.

Remark 5.11. Let $g \in \tilde{G} \setminus GZ(\tilde{G})$. Then $g$ induces the non-trivial diagonal automorphism on $G$. By Lemma 5.1 and 5.3 we have $x_{s,\mu}^{(0)} = x_{s,\mu}^{(0)}(s,\sigma^{-1})\mu$ and $x_{s,\mu}^{(0)} = x_{s,\mu}^{(0)}(s,\sigma)$ if $(s,\mu) \in iB^{(1)}_{\ell}(G)$. By Theorem 5.2, we may assume that $(x_{s,\mu}^{(1)})^{(0)} = x_{s,\mu}^{(1)}(s,\sigma^{-1})\mu$ and $(x_{s,\mu}^{(1)})^{(0)} = x_{s,\mu}^{(1)}(s,\sigma)$ for $(s,\mu) \in iB^{(2)}_{\ell}(G)$ and $i = \pm 1$. 

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For \((s, \kappa) \in i \text{Bl}_1(G)\) and an \(\ell\)-block \(B\) of \(G\) corresponding to \((s, \kappa)\), then we define \(\mathcal{P}(B) := \prod \mathcal{P}(\beta_\ell r_\ell, w_\ell)\), where the sets \(\mathcal{P}(\beta_\ell r_\ell, w_\ell)\) defined as in (2.7).

**Proposition 5.12.** With the preceding notation, \(\mathcal{P}(B)\) is a labeling set for \(\text{Irr}(B) \cap E(G, \ell')\).

**Proof.** Let \(\tilde{B}\) be an \(\ell\)-block of \(\tilde{G}\) covering \(B\). If \((s, \kappa) \in i \text{Bl}_1^{(2)}(G)\), then every character of \(\text{Irr}(\tilde{B}) \cap E(\tilde{G}, \ell')\) is parametrized by an element of \(i \text{Bl}_1^{(2)}(G)\). Thus the map \(\text{Res}_{\ell'}^{\ell} : \text{Irr}(\tilde{B}) \cap E(\tilde{G}, \ell') \to \text{Irr}(B) \cap E(G, \ell')\) is bijective. So \(\Psi(s, \kappa)\) is a labeling set for \(\text{Irr}(B) \cap E(G, \ell')\). From this there is a canonical bijection between \(\Psi(s, \kappa)\) and \(\mathcal{P}(B)\) by (2.11). So \(\mathcal{P}(B)\) is a labeling set for \(\text{Irr}(B) \cap E(G, \ell')\).

Now we assume that \((s, \kappa) \in i \text{Bl}_1^{(1)}(G)\). If \(-1\) is not an eigenvalue of \(s\), then every character of \(\text{Irr}(\tilde{B}) \cap E(\tilde{G}, \ell')\) is parametrized by an element of \(i \text{Bl}_1^{(1)}(G)\). From this the map \(\text{Irr}(\tilde{B}) \cap E(\tilde{G}, \ell') \to \text{Irr}(B) \cap E(G, \ell')\), which sends \(\tilde{\chi}\) to the unique element of \(\text{Irr}(B) \cap E(G, \ell')\). So \(\Psi(s, \kappa)\) is a labeling set for \(\text{Irr}(B) \cap E(G, \ell')\). Thus \(\mathcal{P}(B)\) is a labeling set for \(\text{Irr}(B) \cap E(G, \ell')\) as in last paragraph.

Now we assume further \(-1\) is an eigenvalue of \(s\). Let \(\mu \in \Psi(s, \kappa)\). If \((s, \mu) \in i \text{Bl}_1^{(1)}(G)\), then \(\mu_{\ell+1}\) is degenerate and if \((s, \mu) \in i \text{Bl}_1^{(2)}(G)\), then \(\mu_{\ell+1}\) is non-degenerate. By the proof of [46 Prop. 15 (2)], if \(\mu_{\ell+1}\) is degenerate, then it corresponds to an element of \(\mathcal{P}(\phi(2e, w_{\ell+1}))\) (defined as in (2.8)) and if \(\mu_{\ell+1}\) is non-degenerate, then \(\mu_{\ell+1}\) and its copy correspond to the two element of \(\mathcal{P}(2e, w_{\ell+1})\) \(\setminus \mathcal{P}(2e, w_{\ell+1})\) which are equivalent in the sense of (2.9). Thus we have a natural bijection between \(\mathcal{P}(B)\) and \(\text{Irr}(B) \cap E(G, \ell')\).

By the proof of Proposition 5.12 and Remark 5.11, we have the following result immediately.

**Proposition 5.13.** Let \(B\) be an \(\ell\)-block of \(G\), \(\mu \in \mathcal{P}(B)\) and \(\chi\) be the irreducible character in \(\text{Irr}(B) \cap E(G, \ell')\) corresponding to \(\mu\). Let \(g\) be an element of \(\tilde{G}\) which induces a non-trivial diagonal automorphism and \(\sigma\) be a field automorphism such that \(g\) and \(\sigma\). Then

(i) \(\chi^{\sigma}\) is a character of \(B^{\sigma}\) corresponding to \(\sigma^{-1}\mu\).

(ii) \(\chi^g\) a character of \(B^g\) corresponding to \(\mu^G\), which is defined as in Proposition 3.77.

Now assume that \(V\) be a \(2n\) dimensional symplectic space over \(\mathbb{F}_q\) with \(n \geq 2\) and let \(G = I(V) = I_0(V) = \text{Sp}(V)\) throughout this section. Let \(B\) be an \(\ell\)-block of \(G\) covered by \(\tilde{B} = \tilde{B}(s, \kappa)\) and define \(i \text{W}_{\ell'}(B)\) to be the set of \(K = \prod_{T} K_T\) where \(K_T : \bigcup \theta_{T, \delta} \to \{\ell\text{-cores}\}\) such that \(\sum_{i,j} \ell^{d} | K_T(\varphi_{T, i, j})| = w_T\).

A bijection between \(\text{W}_{\ell'}(B)\) and \(i \text{W}_{\ell'}(B)\) has been constructed implicitly in the proof of [4 (4F)] and can be described as follows. Let \(D\) be a defect group of \(B\), \(V_0 = C_V(D)\) and \(V_+ = [V, D]\) so that \(V = V_0 \perp V_+\) as above. Let \((D, b)\) be a maximal Brauer pair of \(G\) containing \((1, B)\), and \(\theta\) be the canonical character of \(b\). Then \(D = D_0 \times D_+, b = b_0 \times b_+\) and \(\theta = \theta_0 \times \theta_+\), where \(D_0 = \langle 1_{V_0} \rangle \leq \text{Sp}(V_0), D_+ \leq \text{Sp}(V_+), b_0, b_+\) are \(\ell\)-blocks of \(\text{Sp}(V_0)\) and \(C_{\text{Sp}(V_+)}(D_+)\) respectively, and \(\theta_0 \in \text{Irr}(b_0), \theta_+ \in \text{Irr}(b_+).\)

Let \((R, \varphi)\) be a \(B\)-weight, \(C = C_G(R)\) and \(N = N_G(R)\). Then there is an \(\ell\)-block \(b\) of \(\text{CR}\) with defect group \(R\) and canonical character \(\theta\) such that \(b^G = B\) and \(\varphi = \text{Ind}_{N_\ell}^N \psi\) for some \(\psi \in \text{Irr}^0(N_\ell | \theta)\). We may suppose \(Z(D) \leq Z(R) \leq R \leq D\). Thus \(V_0 = C_V(R)\) and \(V_+ = [V, R]\), so that \(R = R_0 \times R_+\), \(C = C_0 \times C_+, N = N_0 \times N_+\), where \(R_0 = D_0, R_+ \leq \text{Sp}(V_+), C_0 = \text{Sp}(V_0), C_+ = C_{\text{Sp}(V_+)}(R_+)\) and \(N_+ = N_{\text{Sp}(V_+)}(R_+)\). Let \(b = b_0 \times b_+\) and \(\theta = \theta_0 \times \theta_+\) be the corresponding decompositions. Then \(\theta_0 = \theta_0\) and \(b_+^{\text{Sp}(V_+)} = b_+^{\text{Sp}(V_+)}\). Note that \(N_\ell = N_0 \times N_\ell\). If \(\psi \in \text{Irr}^0(N_\ell | \theta),\)
then \( \psi = \vartheta_0 \times \psi_+ \), where \( \psi_+ \in \text{Irr}_0(N_0, | \theta_+|) \). The map \((R, \text{Ind}^N_{N_0} \psi) \mapsto (R_+, \text{Ind}_{(N_0)_{+}}^N \psi_+)\) is a bijection from \(((R, \text{Ind}^N_{N_0} \psi) | \psi \in \text{Irr}^0(N_0, | \theta|))\) to \(((R_+, \text{Ind}^N_{(N_0)_+} \psi_+) | \psi_+ \in \text{Irr}^0(N_0, | \theta_+|))\). Then the bijection between \(\mathcal{W}_\ell(B)\) and \(i' \mathcal{W}_\ell(B)\) can be given as in §3.2.

Then there is a canonical bijection between \(i' \mathcal{W}_\ell(B)\) and \(\mathcal{P}(B) := \prod \mathcal{P}(\beta_1 \epsilon_1, w_1)\). So \(\mathcal{P}(B)\) is also a labeling set for \(\mathcal{W}_\ell(B)\) by [4, (4F)].

**Proof of Theorem 5.13** Thanks to [13], we may assume that \(\ell \nmid q\). Let \(B\) be an \(\ell\)-block of \(G = \text{Sp}_{2n}(q)\). Then \(\mathcal{P}(B)\) is a labeling set for both \(\text{Irr}(B) \cap \mathcal{E}(G, \ell')\) and \(\mathcal{W}_\ell(B)\) by above argument. So \(|\text{Irr}(B) \cap \mathcal{E}(G, \ell')| = |\mathcal{W}_\ell(B)|\). Thus the assertion follows by Theorem 2.10.

By Corollary 3.7 and Proposition 3.11 we have the following result.

**Proposition 5.14.** Let \(B\) be an \(\ell\)-block of \(G\), \((R, \varphi)\) be a \(B\)-weight corresponding to \(\mu \in \mathcal{P}(B)\) let \(g\) be an element of \(\text{Aut}(\mathcal{P}(B))\). Then

(i) \((R, \varphi)^\sigma\) is a \(B^\sigma\)-weight corresponding to \(\sigma^{-1} \mu\),

(ii) \((R, \varphi)^{\epsilon}\) a \(B^\epsilon\)-weight corresponding to \(\mu^{\epsilon}\), which is defined as in Proposition 3.11.

Note that \(X = G = \text{Sp}_{2n}(q)\) is the universal \(\ell'\)-covering group of the simple group \(S = \text{PSp}_{2n}(q)\) by [27] §6.1. By [27] §2.5, the automorphism induced by \(\tilde{G} \rtimes E\) equal \(\text{Aut}(G)\). Recall that \(\epsilon = \langle F_p \rangle\).

By Proposition 5.13 and 5.14 we have

**Theorem 5.15.** Let \(G = \text{Sp}_{2n}(q)\), \(B\) an \(\ell\)-block of \(G\) where \(q = p^{f}\) be a power of an odd prime \(p\), \(n \geq 2\) and \(\ell \nmid q\) is an odd prime. Then there is an \(\text{Aut}(G)_B\)-equivariant bijection between \(\text{Irr}(B) \cap \mathcal{E}(G, \ell')\) and \(\mathcal{W}_\ell(B)\).

**Theorem 5.16.** Keep the hypothesis and setup of Theorem 5.15. Assume further that \(\ell\) is linear. Then there exists an \(\text{Aut}(G)_B\)-equivariant bijection between \(\text{IBr}_\ell(B)\) and \(\mathcal{W}_\ell(B)\).

**Proof.** By Theorem 2.10 \(\text{Irr}(B) \cap \mathcal{E}(G, \ell')\) is a basic set for \(B\). Since \(\ell\) is linear, the decomposition matrix is unitriangular by [26]. Hence there is an \((\tilde{G} \rtimes E)_B\)-equivariant bijection between \(\text{Irr}(B) \cap \mathcal{E}(G, \ell')\) and \(\text{IBr}_\ell(B)\) by [16] Lem. 7.5. Thus the assertion follows from Theorem 5.15.

Now we prove the main result of this paper for simple groups of type \(C_n\).

**Proof of Theorem 7.4** With the assumption that \(f\) is odd, we know \(\text{Aut}(S)/S\) is cyclic. Then by Corollary 2.14 it suffice to show that there exists an \(\text{Aut}(G)_B\)-equivariant bijection between \(\text{IBr}_\ell(B)\) and \(\mathcal{W}_\ell(B)\) for every \(\ell\)-block \(B\) of \(G\). By Lemma 2.15 we may assume that \(\ell \nmid q\). Then the assertion follows by Theorem 5.16.

### 5.4 The unipotent blocks

We first summarize the description for the unipotent \(\ell\)-blocks of symplectic groups above.

Let \(G = \text{Sp}_{2n}(q)\), with \(n \geq 2\) and \(q\) odd. Then the unipotent \(\ell\)-blocks of \(G\) are parametrised by \(C(1)\). Let \(w = w_{i-1}\). We also write \(B(\kappa, w)\) for \(B(1, \kappa)\) to emphasize \(w\). Then \(\mathcal{P}(2e, w)\) is a labeling set for the unipotent characters of \(B(\kappa, w)\).
Now assume that $V$ be a $2n$ dimensional symplectic space over $\mathbb{F}_q$ with $n \geq 2$ and let $G = I(V) = I_0(V) = \text{Sp}(V)$ throughout this section. Let $B = B(\kappa, w)$ be a unipotent $\ell$-block of $G$. Then by the argument in the preceding section, there is a bijection between $\mathcal{W}_r(B)$ and $\mathcal{P}(B) = \mathcal{P}(2e, w)$. By Proposition 5.14 we have

**Lemma 5.17.** Let $B$ be a unipotent $\ell$-block of $G = \text{Sp}_{2n}(q)$. Then every $B$-weight (in the sense of $G$-conjugacy class) is invariant under the action of $\tilde{G} \rtimes E$.

Since $|\tilde{G}/GZ(\tilde{G})| = 2$, by the same argument, the relationship between $\ell$-weights of $\tilde{G}$ and $\ell$-weights of $G$ is with completely analogous to Remark 4.15 with $X, \tilde{X}$ replaced by $G, \tilde{G}$ respectively throughout.

**Lemma 5.18.** Let $R \in \text{Rad}_r(\tilde{G}), \tilde{R} = RO_r(Z(\tilde{G}))$ and $(R, \varphi)$ an $\ell$-weight of $G$. Then the cardinality of the set $\{\tilde{\varphi} \in \text{Irr}(N_C(\tilde{R}) \mid \varphi) \mid (\tilde{R}, \tilde{\varphi})$ is an $\ell$-weight of $\tilde{G}\}$ is not greater than $|O_r(\tilde{G}/G)|$.

**Proof.** Let $\tilde{\varphi}_0 \in \text{Irr}(N_C(\tilde{R}) \mid \varphi)$ such that $(\tilde{R}, \tilde{\varphi})$ is an $\ell$-weight of $\tilde{G}$. Since $N_C(\tilde{R})/N_C(R) \leq \tilde{G}/G$ is cyclic, by Clifford theory we have that $\text{Irr}(N_C(\tilde{R}) \mid \varphi) = \{\tilde{\varphi}_0\tau \mid \tau \in \text{Irr}(N_C(\tilde{R})/N_C(R))\}$. If $(\tilde{R}, \tilde{\varphi}_0\tau)$ is an $\ell$-weight of $\tilde{G}$, then $\tilde{R}$ is contained in the kernel of $\varphi_0\tau$, and then $\sigma(\tilde{Z}(\tilde{G}))$ is contained in the kernel of $\tau$. Thus $\tau \in \text{Irr}(N_C(\tilde{R})/O_r(Z(\tilde{G})))N_C(R))$. Now $N_C(\tilde{R})/O_r(Z(\tilde{G}))N_C(R)$ is an $\ell$-subgroup of $\tilde{G}/G$, thus the assertion holds.

**Proof of Theorem 1.3 for type $C_n$.** By Lemma 2.15 we may assume that $\ell \nmid q$. Let $B = B(\kappa, w)$ be a unipotent $\ell$-block of $X = G = \text{Sp}_{2n}(q)$. Then by the argument above, $\mathcal{P}(2e, w)$ is a labeling set of $\mathcal{W}_r(B)$. By Lemma 5.17 every element of $\mathcal{W}_r(B)$ is $\tilde{G} \rtimes E$-invariant. On the other hand, $\text{Irr}(B) \cap \mathcal{E}(G, 1)$ is a basic set of $B$ by Theorem 2.10. By [39] Thm. 2.5], every unipotent character of $G$ is $\tilde{G} \rtimes E$-invariant. Thus $\mathcal{P}(2e, w)$ is a labeling set of $\text{IBr}_r(B)$ and every element of $\text{IBr}_r(B)$ is $\tilde{G} \rtimes E$-invariant. Hence there exists a bijection $\text{Aut}(G)$-equivariant bijection between $\text{IBr}_r(B)$ and $\mathcal{W}_r(B)$. Then it suffices to show condition (iii) of Definition 2.11. Note that the number of $\tilde{G}$ which cover $B$ is $|O_r(\mathbb{F}_q^{2n})|$ and there is a unique $\ell$-block $\tilde{B}$ of $\tilde{G}$ covering $B$ by Theorem 5.10.

Then $\tilde{B}_z := z \otimes \tilde{B}$ for $z \in O_r(Z(\tilde{G}^*))$ are all the $\ell$-blocks of $\tilde{G}$ covering $B$.

**Claim 1:** Every $\tilde{\varphi} \in \text{IBr}_r(B)$ extends to $\tilde{G} \rtimes E$.

Let $\tilde{\varphi} \in \text{IBr}_r(\tilde{B})$ be an extension of $\varphi$. Note that every element of $\text{IBr}_r(\tilde{B})$ is $\tilde{G} \rtimes E$-invariant since $\text{Irr}(\tilde{B}) \cap \mathcal{E}(\tilde{G}, 1)$ is a basic set of $\tilde{B}$ by Theorem 2.10 and every unipotent character of $G$ is $E$-invariant. Thus $\tilde{\varphi}$ extends to $\tilde{G} \rtimes E$ and so does $\varphi$.

**Claim 2:** If $(R, \varphi)$ is a $B$-weight, then $\varphi$ extends to $(\tilde{G} \rtimes E)_R$.

By Lemma 5.17, $\varphi$ is $(\tilde{G} \rtimes E)_R$-invariant. For $z \in O_r(Z(\tilde{G}^*))$, there exists $\tilde{\varphi}_z \in \text{Irr}(N_C(\tilde{R}) \mid \varphi)$ such that $(\tilde{R}, \tilde{\varphi}_z)$ is a $\tilde{B}$-weight by Corollary 2.8. Now the number of $\tilde{G}$ which cover $B$ is $|O_r(\mathbb{F}_q^{2n})|$, by Lemma 5.18 $\varphi_z$ is unique. Let $\tilde{\varphi} = \tilde{\varphi}_1$, that is, $\tilde{\varphi} \in \text{Irr}(N_C(\tilde{R}) \mid \varphi)$ and $(\tilde{R}, \tilde{\varphi})$ is a $B$-weight. Now $\tilde{B}$ is $D$-invariant, so we have $\tilde{\varphi}^x \in \text{Irr}(N_C(\tilde{R}) \mid \varphi)$ and $(\tilde{R}, \tilde{\varphi}^x)$ is a $\tilde{B}$-weight for all $x \in (\tilde{G} \rtimes E)_R$. Thus $\tilde{\varphi}^x = \tilde{\varphi}$ and then $\tilde{\varphi}$ is $(\tilde{G} \rtimes E)_R$-invariant. Note that $E$ acts trivially on $\text{Rad}_r(\tilde{G})/\sim_{G}$. Thus there exists $g \in G$ such that $g^{-1}\sigma$ stabilizes $R$, and then $(\tilde{G} \rtimes E)_R = N_G(R)(g^{-1}\sigma)$, which implies $(\tilde{G} \rtimes E)_R/N_G(\tilde{R})$ is cyclic. Hence $\tilde{\varphi}$ extends to $(\tilde{G} \rtimes E)_R$ and so does $\varphi$.

The remaining process is similar with the case of type $B$. For a set $\text{IBr}_r(B \mid Q)$ as in Lemma 2.13 (for definition, see the proof of [49] Lemma 2.10) and $\phi \in \text{IBr}_r(B \mid Q)$, we let $A := A(\phi, Q) = (\tilde{G} \rtimes E)/Z(\tilde{G})$. By a similar argument with the proof of [20], Prop. 7.1, Definition 2.11 (iii)(1)-(3) holds. For Definition 2.11 (iii)(4), the proof of [20], Lem. 7.2 applies here. □
6 Type D

Let $V$ be an $2n$ dimensional orthogonal space over $\mathbb{F}_q$ with $n \geq 4$ and let $\tilde{G} = I(V) = GO(V) = GO_{2n}(q)$ and $G = I_0(V) = SO(V) = SO_{2n}(q)$ with $\epsilon \in \{\pm\}$. In this section, we use $J = CSO_{2n}^\epsilon(q)$ for the special conformal orthogonal groups and $\tilde{G} = GO_{2n}(q)$ for the general orthogonal groups for convention (of description of $\ell$-weight of special orthogonal groups in \([4]\)), which is not the same with the notation in Appendix \(\S 3\).

The blocks of $SO_{2n}^\epsilon(q)$ can be obtained from $CSO_{2n}^\epsilon(q)$ as what we do for $Sp_{2n}(q)$ in \(\S 5.2\); see Appendix \(\S 3\). But in this section, we only consider unipotent $\ell$-blocks, which are classified by Cabanes and Enguehard \([14]\), which is easier to describe. So we do not use the results of Appendix \(\S 3\) in this section. Let $\ell$ be an odd prime with $\ell \nmid q$ and $e_0$ and $e$ defined as before. As in \(\S 5.4\) by \([14]\) Thm.], the unipotent $\ell$-blocks of $G = SO_{2n}^\epsilon(q)$ are parametrised by the $G_{\ell}(q)$-conjugacy classes of $e_0$-cuspidal pairs $(L, \lambda)$. Here $L$ satisfies that $L = \mathbf{L}^F = SO_{2n-e}(q) \times T_{e_0}$, with either $T_{e_0} = GL_1(q^2)$ if $e_0$ is odd, or $T_{e_0} = GU_1(q^2)$ if $e_0$ is even, and $\delta = \epsilon$ if $e_0$ is odd or $w$ is even, and $\delta = -\epsilon$ else, and $\lambda$ is an $e_0$-cuspidal unipotent character of $L$ (for the structure of $e_0$-cuspidal pairs, see \([11]\)). Follow the notation of \([42]\) \$5.3\), we write $B = b(L, \lambda)$ for the corresponding (unipotent) $\ell$-block.

Write $\lambda = \lambda_0 \times 1_{T_{e_0}}$, where $\lambda_0$ is an $e_0$-cuspidal unipotent character of $G_{n-e}(q)$. Let $\kappa$ be the symbol corresponding to $\lambda_0$. Then by \([11]\) \$3\), $\kappa$ is an e-core. Moreover, the unipotent characters in the block $B(L, \lambda)$ are then the members of the $e_0$-Harish-Chandra series above $(L, \lambda)$, and then the ones parametrised by the symbols of rank $n$ and having $e$-core $\kappa$. Thus we also write $B(\kappa, w)$ for $B(L, \lambda)$.

If $B = B(\kappa, w)$, then we let $\mathcal{P}(B) := \mathcal{P}(2e, w)$ if $\kappa$ is non-degenerate, and $\mathcal{P}(B) := \mathcal{P}^\prime(2e, w)$ if $\kappa$ is degenerate. Thus by \([2.11]\), $\mathcal{P}(B)$ is a labeling set for Irr$(B) \cap E(G, 1)$ . Also, $B$ is a defect zero $\ell$-block if and only if $w = 0$. So we always assume that $w > 0$ from now on.

A defect group of $B(\kappa, w)$ is then obtained as a Sylow $\ell$-subgroup of $C_G(L, L)$, which is isomorphic to a Sylow $\ell$-subgroup of $GL_{e_0}(q)$ (if $e_0$ is odd) or $GU_{e_0}(q)$ (if $e_0$ is even) by \([43]\) \$5.6\).

Denote $\tilde{J} := CGO_{2n}^\epsilon(q)$, $J := CSO_{2n}^\epsilon(q)$. Let $S = P\Omega_{2n}(q)$. Then $S$ is simple. By \([27]\) \$2.5\], the automorphisms induced by $A := \tilde{J} \rtimes E$ on $S$ equal $\text{Aut}(S)$ except when $n = 4$ and $\epsilon = +$. Recall that $E = (F_p)$ is the group generated by the field automorphism $F_p$ which sends $(a_{ij})$ to $(a_{ij}^p)$. The following lemma is elementary.

**Lemma 6.1.** (i) Every unipotent character of $\tilde{G}$ is $\tilde{A}$-invariant.

(ii) Any element $g \in \tilde{A}$ fixes every unipotent character of $G$ except when $\epsilon = +$, the action of $g$ on $G$ can be induced by some element of $\tilde{G} \setminus G$ and the unipotent character is labelled by a degenerate symbol (or a element of $\mathcal{P}_0(2e, w)$). Furthermore, such $g$ interchanges the two unipotent characters in all pairs labeled by the same degenerate symbol.

**Lemma 6.2.** Let $\phi \in \text{IBr}_1(B)$ where $B$ is a unipotent $\ell$-block of $G$.

(i) If $\epsilon = -$, then $\phi$ extends to $\tilde{A}$.

(ii) If $\epsilon = +$ and $\ell$ is linear, then $\phi$ extends to $\tilde{A}_\delta$.

**Proof.** Let $B = B(\kappa, w)$. If $w = 0$, then $B$ is a defect zero $\ell$-block, and then the assertion also follows from \([39]\) Thm. 2.4]. Now we assume that $w > 0$. For every $z \in O_{\ell}(Z(J^*))$ (note that $J^*$ is the special Clifford theory), we let $\tilde{z}$ be the corresponding linear character of $J$ (cf. \([18]\) Prop. 13.30) for $z \in O_{\ell}(Z(J^*))$. Then by \([18]\) Prop. 13.30, $\tilde{z} \otimes \mathcal{E}(J, 1) = \mathcal{E}(J, z)$. Thus there exists an $\ell$-block $B_z$ of $J$ covering $B$ such that $B_z \subseteq \mathcal{E}_{\ell}(J, z)$ (see also Theorem \([B.2]\)). Moreover,
$B_1$ is unique. In this way, there are $|O_{F}(F_q)|\ell$-blocks of $J$ covering $B$. We denote $B = B_1$. Then $B_1 = \hat{\vartheta} \circ B$. Since $z$ is of $\ell'$-order, we may regard $\hat{\vartheta}$ as a linear $\ell'$-Brauer character of $J$. From this, $\text{IBr}_\ell(B_1) = \{\hat{\vartheta} | \varphi \in \text{IBr}_\ell(B)\}$ (see [26, Lem. 2.4]).

If $\kappa$ is non-degenerate, then there are two unipotent $\ell$-blocks $\tilde{B}^{(1)}, \tilde{B}^{(2)}$ of $\tilde{G}$ covering $B$. Let $\tilde{B}^{(i)}$ be the unique unipotent $\ell$-block of $\tilde{J}$ covering $\tilde{B}^{(i)}$ for $i = 1, 2$. Since $J/G \cong J/\tilde{G}$, we may regard $\hat{\vartheta}$ a linear character (or linear $\ell'$-Brauer character) of $J$ for $\varphi \in \text{IBr}_\ell(Z(J'))$. Note that $\tilde{G}$ acts trivially on $J/G$, and then $\hat{\vartheta}$ is $\tilde{G}$-invariant. Thus $B_1$ is $\tilde{J}$-invariant for every $z \in O_{\ell}(Z(J'))$. Let $\tilde{B}^{(1)}$ covers $B_1$. From this, $\tilde{B}^{(i)}$ for $i = 1, 2$ and $z \in O_{\ell}(Z(J'))$ are distinct $\ell$-blocks of $\tilde{J}$.

Now every character of $\text{Irr}(B) \cap \mathcal{E}(G, 1)$ is $\tilde{J}$-invariant by Lemma [6,1]. By Theorem [2.10] $\text{Irr}(B) \cap \mathcal{E}(G, 1)$ is a basic set of $B$. So every irreducible $\ell$-Brauer character of $B$ is $\tilde{A}$-invariant. Then there is an extension $\tilde{\varphi} \in \text{IBr}_\ell(\tilde{B}^{(1)})$ of $\tilde{\vartheta}$ to $\tilde{G}$. By Lemma [6,1] again, $\tilde{B}^{(1)}$ is $\tilde{A}$-invariant. So $\tilde{\varphi}$ is $\tilde{A}$-invariant by Lemma [2,6] Then there exists an extension $\tilde{\varphi}' \in \text{IBr}_\ell(\tilde{B}^{(1)})$ of $\tilde{\varphi}$. Note that the number of extensions of $\tilde{\varphi}$ to $\tilde{J}$ is at most $|O_{\ell}(F_q)|$. By Lemma [2,6] again, $\tilde{\varphi}'$ is $\tilde{A}$-invariant and then extends to $\tilde{A}$ since $\tilde{A}/\tilde{J}$ is cyclic.

If $\epsilon = -$, then every $\kappa$ is non-degenerate, and then this assertion holds.

Now we let $\epsilon = +$. Then $\ell$ is linear. By [26], with a suitable order, the decomposition number matrix of $B$ with respect to basic set $\text{Irr}(B) \cap \mathcal{E}(G, 1)$ is unitriangular. By [39, Thm. 2.4], every $\chi \in \text{Irr}(B) \cap \mathcal{E}(G, 1)$ extends to $\tilde{A}_\chi$. Then by Lemma [2.9] every $\varphi \in \text{IBr}_\ell(B)$ extends to $\tilde{A}_\varphi$. This completes the proof.

Now let $B = B(\kappa, w)$ be a unipotent $\ell$-block of $G$ and $D$ a defect group of $B$. Now we state the results for $B$-weights which follows from the proof of [4, (4H)]. Let $V_0 = \text{C}_V(D)$ and $V_\varphi = [V, D]$ so that $V = V_0 \perp V_\varphi$. Then $\dim(V_\varphi) = 2\varphi w$. Let $(D, b)$ be a maximal Brauer pair of $G$ containing $(1, B)$, and $\vartheta$ be the canonical character of $b$. Let $\tilde{G}_0 = \text{GO}(V_0)$, $G_0 = \text{SO}(V_0)$, $\tilde{G}_+ = \text{GO}(V_\varphi)$ and $G_+ = \text{SO}(V_\varphi)$. Then $G = D_0 \times D_+$, $b = b_0 \times b_+$ and $\vartheta = \vartheta_0 \times \vartheta_+$, where $D_0 = (V_0) \leq G_0$, $D_+ \leq G_+$, $b_0$, $b_+$ are $\ell$-blocks of $G_0$ and $C_{G_+}(D_+)$ respectively, and $\vartheta_0 \in \text{Irr}(b_0)$, $\vartheta_+ \in \text{Irr}(b_+)$. Let $(R, \varphi)$ be a $B$-weight, $\tilde{C} = C_G(R)$, $C = C_G(R)$, $\tilde{N} = N_G(R)$ and $N = N_G(R)$. Then there is an $\ell$-block $b$ of $C$ with respect to $G$ and canonical character $\vartheta$ such that $b^G = B$ and $\varphi = \text{Ind}_{N}^{G}(\psi)$ for some $\psi \in \text{Irr}(\tilde{N}_0 | \vartheta)$. We may suppose $Z(D) \leq Z(R) \leq R \leq D$. Thus $V_0 = \text{C}_V(R)$ and $V_\varphi = [V, R]$, so that $R = R_0 \times R_\varphi$, $C = G_0 \times C_\varphi$, $\tilde{C} = \tilde{G}_0 \times \tilde{C}_\varphi$, $\tilde{N} = N_0 \times N_\varphi$, and $N = (\tau, G_0 \times N_\varphi)$ and $\tilde{N} = \tilde{G}_0 \times \tilde{N}_\varphi$, where $R_0 = D_0$, $R_\varphi \leq G_\varphi$, $C_\varphi = C_G(R_\varphi)$, $N_\varphi = N_G(R_\varphi)$, $\tilde{N}_\varphi = C_{G_\varphi}(R_\varphi)$, $\tilde{N}_\varphi = N_{G_\varphi}(R_\varphi)$ and $\tau = \tau_0 \times \tau_\varphi$ with $\tau_0 \in \tilde{G}_0$, $\tau_\varphi \in \tilde{G}_\varphi$ of determinants $-1$. Then $\tilde{N} = (\tau, N_0 \times N_\varphi)$. Let $b = b_0 \times b_+$ and $\vartheta = \vartheta_0 \times \vartheta_+$ be the corresponding decompositions. Then $\vartheta_0 = \vartheta_0^G$ and $b_+^G = b^G_+$ and we suppose that $(R, b) \leq (D, b)$.

Case 1. $\vartheta_0 = \vartheta_0^G = \vartheta_0$ for any $\sigma_0 \in \tilde{G}_0$ of determinant $-1$, i.e., $\kappa$ is non-degenerate. There are two irreducible characters $\vartheta_0'$ and $\vartheta_0''$ of $\tilde{G}_0$ covering $\vartheta_0$. Let $\vartheta' = \vartheta_0' \times \vartheta_+$, $\vartheta'' = \vartheta_0'' \times \vartheta_+$, and $b'$, $b''$ be the $\ell$-blocks of $C_G(D)$ containing $\vartheta'$, $\vartheta''$ respectively. Then $\vartheta_0^G$ and $\vartheta_0^{\varphi}$ are two $\ell$-blocks of $\tilde{G}$. Let $\tilde{B} = \vartheta_0^G$.

Case 2. $V_0 = 0$ or $\vartheta_0 = \vartheta_0^G = \vartheta_0$ for any $\sigma_0 \in \tilde{G}_0$ of determinant $-1$, i.e., $\kappa$ is degenerate. If $V_0 = 0$, then it is the case in Lemma [A.1]. Now assume that $V_0 \neq 0$ and $\vartheta_0 = \vartheta_0^G = \vartheta_0$ for any $\sigma_0 \in \tilde{G}_0$ of determinant $-1$. Also, we may assume that $\vartheta = \vartheta_0 \times \vartheta_+$ for some character $\vartheta_0$ of $C_\varphi$. Then $\tilde{N}_\varphi = G_0 \times N_\varphi$ and $N_0 = G_0 \times N_\varphi$. Thus each character $\tilde{\vartheta}_0$ in $\text{Irr}(\tilde{N}_\varphi | \vartheta_0)$ and each $\psi \in \text{Irr}(N_0 | \vartheta)$ decompose as $\tilde{\vartheta}_0 = \vartheta_0 \times \psi_\varphi$ and $\psi = \vartheta_0 \times \psi_\varphi$ for some $\psi \in \text{Irr}(N_\varphi | \vartheta_\varphi)$ and $\psi \in \text{Irr}(N_\varphi | \vartheta_\varphi)$. Then $\text{Res}^{\tilde{N}_\varphi}N_\varphi \tilde{\vartheta}_0$ is irreducible if and only if $\text{Res}^{\tilde{N}_\varphi}N_\varphi \psi_\varphi$ is irreducible (which is the case in Lemma [A.1]). Let $b'$ be the $\ell$-block of $\tilde{C}_G(D)D$ containing $\vartheta + \vartheta_0 \times \vartheta_\varphi$. Then $\tilde{B} = b^G$ is the unique $\ell$-block of $\tilde{G}$ covering $B$.
For both cases, we define $iW(\tilde{B})$ to be the set of $K : \bigcup_{0} \mathcal{C}_{x-1,\delta} \to \{\ell\text{-cores}\}$ such that $\sum_{i,j} \ell \upsilon_{i,0}(K(\psi_{x,\delta_1,\iota})) = w$.

Note that $N_{\psi} = G_{0} \times N_{\psi}$. If $\psi \in \text{Irr}(N_{\psi} | \theta')$, then $\psi = \psi_{+} \times \psi_{\pm}$, where $\psi_{\pm} \in \text{Irr}(N_{\psi} | \theta')$.

The map $(R, \text{Ind}_{N_{\psi}}^{R} \psi) \mapsto (R, \text{Ind}_{N_{\psi}}^{N_{\omega}} \psi_{+})$ is a bijection from $\{(R, \text{Ind}_{N_{\psi}}^{R} \psi) | \psi \in \text{Irr}(N_{\psi} | \theta')\}$ to $\{(R_{+}, \text{Ind}_{N_{\omega}}^{N_{\omega}} \psi_{+}) | \psi_{\pm} \in \text{Irr}(N_{\psi} | \theta'))\}$. Then the bijection between $W(\tilde{B})$ and $iW(\tilde{B})$ can be given as in (3.2). As in the proof of [4 (4E)], there is a canonical bijection between $iW(\tilde{B})$ and $P(2e, w)$, so $P(2e, w)$ is also a labeling set of $W(\tilde{B})$ (see also Appendix A.2.1).

Similar with Lemma 5.17 (using Corollary 3.7 and Proposition 3.11), we have

**Lemma 6.3.** Every $\tilde{B}$-weight (in the sense of $G$-conjugacy class) is invariant under the action of $\tilde{A} = \text{CGO}_{2n}(q) \rtimes E$.

Now we give $\ell$-weights of $G$ by the argument above and Lemma A.1. If $k$ is non-degenerate, then $(R, \varphi) \mapsto (R, \text{Res}_{N_{G}(R)}^{G} \varphi)$ is a bijection from $W(\tilde{B})$ to $W(\tilde{B})$. If $k$ is degenerate and $w$ is odd, then $(R, \varphi) \mapsto (R, \text{Res}_{N_{G}(R)}^{G} \varphi)$ is also a bijection from $W(\tilde{B})$ to $W(\tilde{B})$.

Let $g$ be degenerate and let $w$ be even. If $(R, \varphi) \in W(\tilde{B})$ corresponds to some element in $P(2e, w) \setminus P_{0}(2e, w)$, then $\text{Res}_{N_{G}(R)}^{G} \varphi$ is irreducible. If $(R, \varphi) \in W(\tilde{B})$ corresponds to some element in $P_{0}(2e, w)$, then $\text{Res}_{N_{G}(R)}^{G} \varphi$ is a sum of two irreducible constituents (for the construction of these two irreducible constituents, see the proof of Lemma A.1). In addition, by the argument above, $P(2e, w)$ is a labeling set for $W(\tilde{B})$.

**Corollary 6.4.** Suppose that $g \in \tilde{A}$ and $(R, \varphi)$ be a $\tilde{B}$-weight. Let $\varphi \in \text{Irr}(N_{G}(R) | \varphi)$ such that $(R, \varphi)$ be a $\tilde{B}$-weight. Then the $G$-conjugacy class of $(R, \varphi)$ is invariant under the action of $g$ except when $e = +$, the action of $g$ on $G$ can be induced by some element of $\tilde{G}$ and $G$ and $(R, \varphi)$ corresponds to some element of $P_{0}(2e, w)$.

Furthermore, when $(R, \varphi)$ corresponds to some element of $P_{0}(2e, w)$, $g$ interchanges the two $G$-conjugacy classes of $(R, \varphi)$ and $(R, \varphi^{2})$, where $\varphi$ and $\varphi^{2}$ are the irreducible constituents of $\text{Res}_{N_{G}(R)}^{G} \varphi$.

**Proof.** By Lemma 6.3, it suffice to show that the $G$-conjugacy class of $(R, \varphi)$ is invariant under the action of $g$ on $G$ can be induced by some element of $\tilde{G}$ and $G$ and $(R, \varphi)$ corresponds to some element of $P_{0}(2e, w)$. It is similar with the proof of Lemma 6.3. In fact, by the remark after [4 (4A)], for every $\delta$, the restriction of every weight character in $\bigcup_{0} \mathcal{C}_{x-1,\delta}$ to the subgroup of $N_{x-1,\delta_{1}}$ has determinant 1 and is irreducible and then is invariant under the action of $\text{CSCO}_{2n}(q) \rtimes E$ by Lemma 5.10 (i). So we conclude from Lemma A.1 a similar result with Proposition 3.6 and 3.11 (i), hence the proof of Lemma 5.17 also applies here, and finally that this assertion holds.

**Lemma 6.5.** Let $(R, \varphi)$ be a $\ell$-weight, where $B$ is a $\ell$-unipotent block of $G$. Then $\varphi$ extends to $\tilde{A}_{R,\varphi}$.

**Proof.** The proof here is similar to the proof of Claim 2 in the proof of Theorem 1.5 for type C, using the argument in the proof of Lemma 6.2.

**Lemma 6.6.** Let $T = [G, G] = \Omega_{2n}'(q)$. $B$ a unipotent $\ell$-block of $G$ and $b$ an $\ell$-block covered by $B$. Then

(i) $\text{Res}_{T}^{G} : \text{IBr}_{\ell}(B) \to \text{IBr}_{\ell}(b)$ is a bijection, and

(ii) $(R, \varphi) \mapsto (R, \text{Res}_{N_{G}(R)}^{N_{G}} \varphi)$ is bijection from $W(\ell)B$ to $W(\ell)b$.

**Proof.** (i) is similar with Proposition 4.10 For (i), note that there exists another $\ell$-block of $G$ covering $b$. Indeed, let $1 \neq z \in Z(G') = Z(G)$ and $\tilde{\varphi}$ be the corresponding linear character (cf. [18 Prop. 13.30]), then $\tilde{\varphi} \otimes B$ is an $\ell$-block of $G$ covering $b$ and is contained in $E_{\ell}(G, z)$. So $\tilde{\varphi} \otimes B \neq B$. Thus (i) holds by Lemma 2.6.

□
Note that $X = \text{Spin}_{2n}^\ell(q)$ is the universal $\ell'$-covering group of the simple group $S = P\Omega_{2n}^\ell(q)$ by [27] § 6.1.

**Proof of Theorem 1.5 (ii) and (iii).** By Lemma 2.15, we may assume that $\ell \nmid q$. Let $b$ be a unipotent $\ell$-block of $X = \text{Spin}_{2n}^\ell(q)$. Then $b$ dominants an $\ell$-block $\tilde{b}$ of $T = \Omega_{2n}^\ell(q)$. Thus there exists a unique unipotent $\ell$-block $\tilde{B}$ of $G = \text{SO}_{2n}^\ell(q)$ covering $\tilde{b}$. By Lemma 6.4 and Proposition 6.1, there is an $\tilde{A}$-equivariant bijection between $\mathcal{W}_{\ell}(\tilde{B})$ and $\text{Irr}(\tilde{B}) \cap \mathcal{E}_{\ell}(G, 1)$. If $\epsilon = -$, then every character of $\text{Irr}(\tilde{B}) \cap \mathcal{E}_{\ell}(G, 1)$ is $\tilde{A}$-invariant. If $\epsilon = +$, then by assumption, $\ell$ is linear and thus by [16] Lem. 7.5, there is an $\tilde{A}$-equivariant bijection between $\text{Irr}(\tilde{B}) \cap \mathcal{E}_{\ell}(G, 1)$ and $\text{IBr}_\ell(\tilde{B})$ (since the corresponding decomposition matrix is unitriangular by [26]). In both cases, there exists $\tilde{A}$-equivariant bijection between $\mathcal{W}_{\ell}(\tilde{B})$ and $\text{IBr}_\ell(\tilde{B})$. From this, by Lemma 6.6 we get an $\tilde{A}$-equivariant bijection between $\mathcal{W}_{\ell}(\tilde{b})$ and $\text{IBr}_\ell(\tilde{b})$. Then by Lemma 2.5, there exists an $\text{Aut}(X)$-equivariant bijection between $\mathcal{W}_{\ell}(b)$ and $\text{IBr}_\ell(b)$. By Lemma 2.13 we only need to verify Definition 2.11 (iii).

The remaining process is similar with the case of type $B$. If $\epsilon = -$, we let $\tilde{\Gamma}$ be the regular embedding of $X$. If $\epsilon = +$, let $\tilde{\Gamma}$ be the extension of the Clifford group such that $\Gamma / Z(\Gamma) \cong \tilde{J} = \text{CGO}_{2n}(\ell)$ (in fact, let $\tilde{\Gamma}$ be the regular embedding of $X$, $\Gamma$ the special Clifford group, then we may take $\tilde{\Gamma} = \tilde{\Gamma}_1 \times \tilde{\Gamma}_2 / \Delta(\Gamma)$, where $\Delta(\Gamma) = \{(g, g^{-1} | g \in \Gamma) \subseteq \tilde{\Gamma}_1 \times \tilde{\Gamma}_2\}$. Thus $X \leq \tilde{\Gamma}$ and $\text{Aut}(X) = \tilde{\Gamma} \rtimes E / Z(\tilde{\Gamma})$. For a set $\text{IBr}_\ell(B | Q)$ as in Lemma 2.10 and Proposition 7.1, Definition 2.11 (iii)(1)-(3) holds. For Definition 2.11 (iii)(4), the proof of [20] Lem. 7.2 applies here. □

### A Appendix A: Remarks for [4]

In line 35 of [4, p. 33], after “Then $w \in X_0$ and so $\xi(w) = \xi^e(w)$”, we can only get $\xi_i(h)\xi_j(h) = \xi^e_i(h)\xi^e_j(h)$ but cannot conclude $\xi_i(h) = \xi^e_i(h)$, nor does the claim $X_0S_m = K$ in line 29 of [4, p. 33]. From this, we give a new proof for [4] (4C).

We will completely follow the notation in [4] and all references in this proof are to this paper throughout Appendix A

#### A.1 For (4C)

**A.1.1**

Now we give some convenient for orthogonal cases first. Let $\Gamma \in \mathcal{F}_{0}$ and

$\mathcal{G}_{\Gamma,d} = \{\varphi_{\Gamma,d,i,j} | 1 \leq i \leq 2e, 1 \leq j \leq r^d\}$

be the set as page 32.

Let $(R, \theta)$ be a pair of type $\Gamma$ and $R = R_{1,0,\gamma,e} (e = (c_1, \ldots, c_l))$ a basic subgroup such that $\gamma + c_1 + \cdots + c_l = d$. Let $V$ be the underlying (orthogonal) space of $R$. Then $|\text{Irr}^0(N(\theta) | \theta)| = 2e(r-1)^l$. Let $N_0(\theta) = N(\theta) \cap \text{SO}(V)$. Then $|N(\theta) : N_0(\theta)| = 2$ and the restriction of each character of $\text{Irr}^0(N(\theta) | \theta)$ to $N_0(\theta)$ is irreducible by the remark of (4A). For $\varphi_1, \varphi_2 \in N(\theta)$, we write $\varphi_1 \sim \varphi_2$ if $\varphi_1|_{N_0(\theta)} = \varphi_2|_{N_0(\theta)}$. Then if $\varphi_1 \sim \varphi_2$ and $\varphi_1 \neq \varphi_2$, then

$$\varphi_1(n) = -\varphi_2(n) \quad \text{for any } n \in N(\theta) \text{ with determinant } -1. \quad (A.1)$$

We may assume that

$$\varphi_{\Gamma,d,i,j} \sim \varphi_{\Gamma,d,e+i,j} \quad \text{for all } 1 \leq i \leq e, 1 \leq j \leq r^d. \quad (A.2)$$
Now we keep the notation and assumption preceding (4B). Let $B$ be the block in (4B). Then the proof of (4B) gives a bijection between $B$-weights and the assignments

$$\bigcup_{d \geq 0} \mathcal{C}_{d,d} \rightarrow \{r\text{-cores}\}, \ \varphi_{d,d,i,j} \mapsto \kappa_{d,d,i,j}$$

(A.3)

such that $\sum_{d \geq 0} r^d \beta_{d,r} \sum_{i=1}^{r^d} \sum_{j=1}^{r^d} |\kappa_{d,d,i,j}| = w_t$.

(4C) should be as follows.

**Lemma A.1.** With the notation and hypothesis preceding (4C), let $G = O(V)$ be an orthogonal group, $G_0 = SO(V)$, and $R$ a radical subgroup of $G$ such that $[V,R] = V$. Let $(R, b)$ a Brauer pair of $G_0$ labeled by $(R, s, -)$ and $\theta$ the canonical character of $b$. Then $|N(\theta) : N_b(\theta)| = \beta_{\Gamma}$. Moreover, if we write $N_0(\theta) = N(\theta) \cap G_0$, then the restriction $\psi|_{N_0(\theta)}$ of each $\psi \in \text{Irr}^G(N(\theta), \theta)$ to $N_0(\theta)$ is irreducible unless when $\Gamma \in \mathcal{F}_0$, $w_t$ is even and the associated assignment (A.3) of $\psi$ satisfies that $\kappa_{d,d,i,j} = \kappa_{d,d,e+i,j}$ for all $1 \leq i \leq e, 1 \leq j \leq r^d$.

A.1.2 Proof

**Lemma A.2.** Let $M$ be arbitrary finite group, $M_1, M_2 \leq M$ with $M_1M_2 = M$ and $L = M_1 \cap M_2$. Let $\xi \in \text{Irr}(M_1)$ such that $\varphi = \xi_L$ is irreducible. Let $\psi \in \text{Irr}(M(\xi) \mid \xi)$ and $\eta = \psi|_{M_1(\xi) \cap M_2}$. Then $\eta$ is irreducible and the following statements hold.

(i) If $M_2(\eta) = M(\xi) \cap M_2$, then the restriction of $\text{Ind}^{M}_{M(\xi)}(\psi)$ to $M_2$ is irreducible.

(ii) If $|M_2(\eta) : M(\xi) \cap M_2| = 2$, then the restriction of $\text{Ind}^{M}_{M(\xi)}(\psi)$ to $M_2$ is a sum of two irreducible characters.

(iii) If $M_2(\varphi) = M(\xi) \cap M_2$, then the restriction defines a bijection from $\text{Irr}(M \mid \xi)$ onto $\text{Irr}(M_2 \mid \varphi)$.

**Proof.** By [20, Cor. (4.2)], the restriction defines a bijection from $\text{Irr}(M(\xi) \mid \xi)$ onto $\text{Irr}(M(\xi) \cap M_2 \mid \varphi)$. So $\eta$ is irreducible.

(i) By Mackey formula, $(\text{Ind}^{M}_{M(\xi)}(\psi))|_{M_2} = \text{Ind}^{M_2}_{M(\xi) \cap M_2}((\psi|_{M(\xi) \cap M_2}) = \text{Ind}^{M_2}_{M_2(\eta)}(\eta)$. Then (i) follows by Clifford theory.

(ii) By Gallagher’s theorem, $\text{Ind}^{M_2(\eta)}_{M(\xi) \cap M_2}(\eta) = \tilde{\eta} + \tilde{\eta}'$, where $\tilde{\eta}$ and $\tilde{\eta}'$ are two extensions of $\eta$ to $M_2(\eta)$ (and $\tilde{\eta} \neq \tilde{\eta}'$). Thus $\text{Ind}^{M_2}_{M_2(\eta)}(\eta) = \text{Ind}^{M_2}_{M_2(\eta)}(\tilde{\eta}) + \text{Ind}^{M_2}_{M_2(\eta)}(\tilde{\eta}')$ and both $\text{Ind}^{M_2}_{M_2(\eta)}(\tilde{\eta})$ and $\text{Ind}^{M_2}_{M_2(\eta)}(\tilde{\eta}')$ are irreducible (and $\text{Ind}^{M_2}_{M_2(\eta)}(\tilde{\eta}) \neq \text{Ind}^{M_2}_{M_2(\eta)}(\tilde{\eta}')$) by Clifford theory. Then (ii) follows by Mackey formula.

(iii) follows from (i) immediately. \(\square\)

**Proof of Lemma A.7** Note that the first and second paragraph of (4C) also apply here. Keep the notation in (4C) (and in the first and second paragraph of its proof). First, we have $|N(\theta) : N_0(\theta)| = \beta_1$ and then we may assume that $\Gamma \in \mathcal{F}_0$. Also, we suppose that $u = 1$ and $d = d_1$.

For $\psi \in \text{Irr}^G(N(\theta), \theta)$, we write $\psi = \text{Ind}^{\mu}_{X_0S_m}(\tilde{\xi})$ (as in the second paragraph of the proof of (4C), where $\chi$ is some character of $X_0S_m$ trivial on $X$). Let $\xi_0 = \xi|_{X_0}$ and let $\xi = \mathcal{B}_{k=1}^d \xi_k$, where $\xi_k \in \text{Irr}(T)$. By the remark of (4A), $\xi|_{R_0}$ is irreducible for every $1 \leq k \leq d$ and hence $\xi|_{R_0}$ is irreducible. So $\xi_0$ is irreducible. Let $K$ be the stabilizer of $\xi_0$ in $H_0$ and let $\xi_0 = \xi|_{X_0S_m}$. Then $\xi_0$ is an extension of $\xi_0$ to $X_0S_m$. By Clifford theory, each irreducible character of $X_0S_m$ covering $\xi_0$ has the form $\tilde{\xi}_0\chi$, where $\chi$ is an irreducible character of $X_0S_m$ trivial on $X_0$. Then by Lemma A.2(iii), $\psi|_{N_0(\theta)}$ is irreducible if $K = X_0S_m$. 

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Now we assume that $K \neq X_0S_m$. Let $x \in K$ such that $x \not\in XS_m = H(\xi)$ first. Then $d > 1$. Also, we may assume that $x \in S(d)$. Write $\xi' = \xi' = \xi_d = \xi_{k_1} \cdots \xi_{k_n}$, where $\xi_{k_i} \in \text{Irr}(T)$. Then $\xi' \neq \xi$ and $\xi'|_{X_0} = \tilde{\xi}|_{X_0} = 0$. Thus both $\xi_{k_1}$ and $\xi_{k_2}$ are extensions of the irreducible character $\xi_{k_1}|_{T_0}$ to $T$ for every $1 \leq k \leq d$ (i.e., $\xi_{k_1} \sim \xi_{k_2}$). Let $1 \leq i, j \leq n$ with $i \neq j$ and $h \in T$ with determinant $-1$ and $w = \text{diag}(w_1, \ldots, w_d)$ such that $w_i = h = w_j$ and $w_k = 1$ if $k \neq i, j$. Then $w \in X_0$ and so $\xi(w) = \xi'(w)$ and then $\xi_{i_1}(h)\xi_{i_2}(h) = \xi_{i_1}(h)\xi_{i_2}(h)$. Thus if $\xi_{i_1} \neq \xi_{i_2}$, then $\xi_{i_1} \neq \xi_{i_2}$ by (A.1). Hence either $\xi_{i_1} = \xi_{i_2}$ for all $1 \leq k \leq d$ or $\xi_{k_1} \sim \xi_{k_2}$ and $\xi_{k_1} \neq \xi_{k_2}$ for all $1 \leq k \leq d$ holds. On the other hand, if this holds, it is easy to check that $K \neq X_0S_m$. Also, by the argument above, $|K : X_0S_m| \leq 2$.

Now we write $\xi = \xi_{\Gamma,d,i,j}(\xi_{\Gamma,d,i,j})$. By the argument above, $K \neq X_0S_m$, if and only if

$$I_{\Gamma,d,i,j} = I_{\Gamma,d,e+i,j} \text{ for all } 1 \leq i \leq e, 1 \leq j \leq r.$$

This occurs only when $w_{\Gamma}$ is even.

Now we assume that $K \neq X_0S_m$ and thus $|K : X_0S_m| = 2$ and (A.4) holds. Let $\xi = \xi_{\Gamma,d,i,j}(\xi_{\Gamma,d,i,j})$ and $\zeta_0 = \xi_{\Gamma,d,i,j}(\xi_{\Gamma,d,i,j})$, then $\zeta_0 = \zeta_{\Gamma,d,i,j}(\xi_{\Gamma,d,i,j})$. Then $X_0S_m \leq H(\zeta_0) \leq K$. Then by (i) and (ii) of Lemma A.2, $\psi|_{N_0(\theta)}$ is irreducible if $H_0(\zeta_0) = X_0S_m$, and $\psi|_{N_0(\theta)}$ is a sum of two irreducible characters if $H_0(\zeta_0) = K$.

Now let $x \in K \setminus X_0S_m$ and without loss of generality we assume that $x \in S(d)$. Now we may write $\xi = \prod_{i=1}^e (\xi_{i}^{\text{t}} \times \xi_{i}^{\text{r}})$ with $\xi_{i}^{\text{t}} \sim \xi_{i}^{\text{t}}$ and $\xi_{i} \neq \xi_{i}^{\text{r}}$ for all $1 \leq i \leq s$. Then $\xi^{\text{t}} = \prod_{i=1}^e (\xi_{i}^{\text{t}} \times \xi_{i}^{\text{r}})$. Note that the values of the extension of $\xi^{\text{t}}$ to $T \rtimes S(t_1)$ on $S(t_1)$ only depend on $\xi_{i}^{\text{t}}(1)$ (see, for example, [29], Lem. 25.5)), i.e., there exists extension $\eta_i$ (resp. $\eta_i^{\text{r}}$) of $\xi_{i}^{\text{t}}$ (resp. $\xi_{i}^{\text{r}}$) to $T \rtimes S(t_1)$ such that $\eta_i|_{S(t_1)} = \eta_i^{\text{r}}|_{S(t_1)}$. So we may assume that $\tilde{\xi}_{\Gamma,d,i,j} = \tilde{\xi}_{\Gamma,d,i,j}$. On the other hand, $\tilde{\xi}_{\Gamma,d,i,j} = \tilde{\xi}_{\Gamma,d,i,j}$, and then $\tilde{\xi}_{\Gamma,d,i,j} = \tilde{\xi}_{\Gamma,d,i,j}$. Thus $\tilde{\xi}_{\Gamma,d,i,j} = \tilde{\xi}_{\Gamma,d,i,j}$. Hence

$$H_0(\zeta_0) = K \Rightarrow \tilde{\xi}_{\Gamma,d,i,j} = (\tilde{\xi}_{\Gamma,d,i,j})^{\text{t}} = \tilde{\xi}_{\Gamma,d,i,j} \Rightarrow \chi = \chi^{\text{t}}.$$

So $H_0(\zeta_0) = K$ if and only if $\kappa_{\Gamma,d,i,j} = \kappa_{\Gamma,d,e+i,j}$ for all $1 \leq i \leq e, 1 \leq j \leq r$. This completes the proof.

\section{4E and 4H}

The results used (4C) are the remark of (4E) and (4H).

\subsection{The remark of (4E)}

By Lemma A.1, the remark of (4E) should be stated as follows.

With the assumption of (4E), we have a bijection between $B$-weights and $\prod \mathcal{R}$, where $\mathcal{R}$ is the set of $\beta_{\Gamma,e_1}$-tuples $(\kappa_1, \kappa_2, \ldots, \kappa_{\beta_{\Gamma,e_1}})$ of partitions $\kappa_i$ such that $\sum_{i=1}^{\beta_{\Gamma,e_1}} |\kappa_i| = w_\Gamma$ (by the proof of (4E)).

Let $G = O(V)$, $G_0 = SO(V)$, $(R, \varphi)$ a $B$-weight of $G$, and $\theta$ an irreducible character of $C = C_{G}(R)$ covered by $\varphi$. Then $N(\theta) : N(\theta) | \theta$ as $m_{X,1} = 0$ or $m_{X,1} \neq 0$. Moreover, for each $\psi \in \text{Irr}^d(N(\theta) | \theta)$, the restriction $\psi|_{N(\theta)G_0}$ is irreducible unless when $m_{X,1} \neq 0$, $w_{X,1}$'s are even and the element $\kappa = \prod \kappa_{\Gamma}$ (with $\kappa_{\Gamma} = (\kappa_{\Gamma,1}^{\Gamma}, \kappa_{\Gamma,2}^{\Gamma}, \ldots, \kappa_{\Gamma,\beta_{\Gamma,e_1}}^{\Gamma})$ in $\prod \mathcal{R}$ corresponding to $\psi$ satisfies that $\kappa_{\Gamma,1}^{\Gamma} = \kappa_{\Gamma,1}^{\Gamma} \text{ for every } \Gamma \in \mathcal{F}$ and every $1 \leq i \leq e$.

\subsection{4H}

For $\Gamma \in \mathcal{F}$, we recall that the integer $f_\Gamma$ is defined to be the number of $\beta_{\Gamma,e_1}$-tuples $(\kappa_1, \ldots, \kappa_{\beta_{\Gamma,e_1}})$ of partitions such that $\sum_{i=1}^{\beta_{\Gamma,e_1}} |\kappa_i| = w_\Gamma$. 37
If \( \Gamma \in \mathcal{F}_0 \) and \( w_{\Gamma} \) is even, we define \( f'_{\Gamma} \) to be the number of \( e \)-tuples \((\kappa_1, \ldots, \kappa_e)\) of partitions such that \( \sum_{i=1}^{e} |\kappa_i| = \frac{1}{2} w_{\Gamma} \).

The conclusion of (4E) should be as follows.

1. The number of \( B \)-weights is \( \prod_{\Gamma} f_{\Gamma} \) if one of the following statements holds:
   
   a. \( m_{X,1}(s_+) = 0 \),
   
   b. \( \sigma_0^{\sigma_0} = \sigma_0 \) for some \( \sigma_0 \in O(V_0) \) of determinant \(-1\),

2. Suppose that \( m_{X,1}(s_+) \neq 0 \) and either \( V_0 = 0 \) or \( \sigma_0^{\sigma_0} \neq \sigma_0 \) for any \( \sigma_0 \in O(V_0) \) of determinant \(-1\).
   
   a. If either \( w_{x-1} \) or \( w_{x+1} \) is odd, then the number of \( B \)-weights is \( \frac{1}{2} \prod_{\Gamma \neq \mathcal{F}_0} f_{\Gamma} \).
   
   b. If both \( w_{x-1} \) and \( w_{x+1} \) are even, then the number of \( B \)-weights is \( \frac{f_{x-1}f_{x+1}f_{x-1}f_{x+1}}{2} \prod_{\Gamma \neq \mathcal{F}_0} f_{\Gamma} \).

B Appendix B: The blocks of special orthogonal groups in even dimension

Let \( G = SO_n^\varepsilon(q) \) with \( \varepsilon \in \{\pm\} \), \( q \) odd and \( n \geq 4 \). Now we give a classification for \( \ell \)-blocks of \( G \) for an odd prime \( \ell \nmid q \), which is completely analogous with the case of \( Sp_{2n}(q) \) in §5.2. Let \( V \) be the underlying space of \( G \) and \( \tilde{G} = \text{CSO}(V) \). Then \( \tilde{G}^* \) is the special Clifford group over \( V^* \) and \( G^* = G \). Let \( \pi: \tilde{G}^* \rightarrow G^* \) be the natural epimorphism. As usual, we let \( G = \text{SO}_n^\varepsilon(\mathbb{P}_q) \) and \( \tilde{G} = \text{CSO}_n^\varepsilon(\mathbb{P}_q) \) for the corresponding algebraic groups and \( F \) the Frobenius endomorphism.

Note that the notation of special conformal orthogonal groups and general orthogonal groups are not the same with those in §6 since the relations of \( \ell \)-blocks of \( \text{SO}_n^\varepsilon(q) \) and \( \text{CSO}_n^\varepsilon(q) \) is similar with the relations of \( \ell \)-blocks of \( Sp_{2n}(q) \) and \( \text{CSp}_{2n}(q) \). So we use the notation which is analogous to those in §5.2.

The irreducible characters of \( \tilde{G} \) has been classified by Lusztig [37]. For a semisimple element \( s \) of \( \tilde{G}^* \), we write \( \tilde{s} = \pi(s) \). Note that both \( m_{s^{-1}}(\tilde{s}) \) and \( m_{s+1}(\tilde{s}) \) are even. Let \( \text{Irr}(\tilde{G}) \) be the set of \( \tilde{G}^* \)-conjugacy classes of pairs \((s, \mu)\), where \( s \) is a semisimple \( \ell' \)-element of \( G^* \) and \( \mu \in \Psi(\tilde{s}) \) (where \( \Psi(\tilde{s}) \) is defined as (2.4)). Here, \( (s, \mu) \) and \( (s', \mu') \) are conjugate if and only if \( s \) and \( s' \) are \( \tilde{G}^* \)-conjugate and \( \mu = \mu' \). By Jordan decomposition of characters, there is a bijection from \( \text{Irr}(\tilde{G}) \) onto \( \text{Irr}(\tilde{G}) \) (see also [23] (4.5)). We write \( \tilde{\chi}_{s,\mu} \) the character of \( \tilde{G} \) corresponding to \((s, \mu)\).

If \( \chi \in \text{Irr}(G \mid \tilde{\chi}_{s,\mu}) \), then we say \( \chi \) corresponds to the pair \((\tilde{s}, \mu)\). Furthermore, we have the following result about the characters of \( G \), which is similar with Lemma 5.3.

Lemma B.1. Let \((s, \mu) \in \text{Irr}(\tilde{G}) \), \( \tilde{s} = \pi(s) \) and \( \tilde{\chi} = \tilde{\chi}_{s,\mu} \).

(i) If 1 or \(-1\) is not eigenvalue of \( \tilde{s} \), then \( \text{Res}^G_{\tilde{G}} \tilde{\chi} \) is irreducible.

(ii) If both 1 and \(-1\) are eigenvalue of \( \tilde{s} \), then

- if \( \mu_{k-1} \) or \( \mu_{k+1} \) is degenerate, then \( \text{Res}^G_{\tilde{G}} \tilde{\chi} \) is irreducible, and
- if both \( \mu_{k-1} \) and \( \mu_{k+1} \) are non-degenerate, then \( \text{Res}^G_{\tilde{G}} \tilde{\chi} \) is a sum of two irreducible constituents.

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Let $I = \text{GO}_{2n}^\circ(q)$. We recall the action of $I$ on $\text{Irr}(G)$ which was given in [23, (4D)]. Let $(s, \mu) \in i\text{Irr}(\tilde{G})$, $\tilde{\chi} = \tilde{\chi}_{s\mu}$ and $g \in I$ of determinant $-1$. Then $\tilde{\chi}^g$ corresponds to the pair $(g^*s g^{-1}, \mu^*)$ (the operate $^*$ is defined as in [23 §2, p. 132]). More precisely, we have

(a) $\tilde{\chi}^g = \tilde{\chi}$ if

- 1 is an eigenvalue of $\bar{s}$ and $-1$ is not an eigenvalue of $\bar{s}$, or
- both 1 and $-1$ are eigenvalues of $\bar{s}$ and $\mu_{s-1}$ is non-degenerate, and

(b) $\tilde{\chi}^g \neq \tilde{\chi}$ and $\text{Res}^G_G (\tilde{\chi}^g) = \text{Res}^G_G \tilde{\chi}$ if

- 1 is not an eigenvalue of $\bar{s}$ and $-1$ is an eigenvalue of $\bar{s}$, or
- both 1 and $-1$ are eigenvalues of $\bar{s}$ and $\mu_{s-1}$ is degenerate and $\mu_{s+1}$ is non-degenerate, and

(c) $\tilde{\chi}^g \neq \tilde{\chi}$ and $\text{Res}^G_G (\tilde{\chi}^g) \neq \text{Res}^G_G \tilde{\chi}$ if

- both 1 and $-1$ are not eigenvalues of $\bar{s}$, or
- both 1 and $-1$ are eigenvalues of $\bar{s}$ and both $\mu_{s-1}$ and $\mu_{s+1}$ are degenerate.

Now let $\chi \in \text{Irr}(G | \bar{\chi})$. By Lemma [B.1] if we are in case (b) or (c), then $\chi = \text{Res}^G_G \tilde{\chi}$. Moreover, $\chi$ is $I$-invariant in case (b) and $\chi$ is not $I$-invariant in case (c). If 1 is an eigenvalue of $\bar{s}$ and $-1$ is not an eigenvalue of $\bar{s}$ or both 1 and $-1$ are eigenvalues of $\bar{s}$ and $\mu_{s-1}$ is non-degenerate and $\mu_{s+1}$ is degenerate, then we also have $\chi = \text{Res}^G_G \tilde{\chi}$ and thus $\chi$ is $I$-invariant. If both 1 and $-1$ are eigenvalues of $\bar{s}$ and both $\mu_{s-1}$ and $\mu_{s+1}$ are non-degenerate, then $\text{Res}^G_G \tilde{\chi}$ is a sum of two irreducible constituents, and then $\tilde{\chi}$ is $I$-invariant but we do not know whether $\chi$ is $I$-invariant or not in this case now.

Now we recall the classification of $\ell$-blocks of $\tilde{G}$ given in [23 §11]. Let $i\text{Bl}_I(\tilde{G})$ be the set of $\tilde{G}^\ell$-conjugacy classes of pairs $(s, \kappa)$ where $s$ is a semisimple $\ell$-element of $\tilde{G}^\ell$ and $\kappa \in C(\bar{s})$, where $\bar{s} = \pi(s)$ and $C(\bar{s})$ is defined as [23, (5)]. Here, $(s, \kappa)$ and $(s', \kappa')$ are $\tilde{G}^\ell$-conjugate if and only if $s$ and $s'$ are $\tilde{G}^\ell$-conjugate and $\kappa = \kappa'$. Also note that both $m_{s-1}(\bar{s})$ and $m_{s+1}(\bar{s})$ are even and $|\kappa| = 1, 2, 4, 8$. By [23, (11E)], there is a bijection $(s, \kappa) \mapsto \bar{B}(s, \kappa)$ from $i\text{Bl}_I(\tilde{G})$ to $\text{Bl}_I(\tilde{G})$.

For $(s, \kappa) \in i\text{Bl}_I(\tilde{G})$, [23, (13B)] also gave a criterion that when an irreducible character of $\tilde{G}$ lies in the $\ell$-block $\bar{B} = \bar{B}(\bar{s}, \kappa)$. In particular, the irreducible characters of $\text{Irr}(\bar{B}) \cap \mathcal{E}(\tilde{G}, \ell^\prime)$ are of form $\tilde{\chi}_{s\mu}$ with $\mu \in \Psi(\bar{s}, \kappa)$ (where $\Psi(\bar{s}, \kappa)$ is defined as in [23, (5)]). In addition, by [23, (11)], $\Psi(\bar{s}, \kappa)$ is in bijection with $\mathcal{P}(\beta_{\ell\mathcal{I}}, w_{\ell\mathcal{I}})$ if $\Gamma \in \mathcal{F}_1 \cup \mathcal{F}_2$ or $\Gamma \in \mathcal{F}_0$ and $\kappa_{\ell\mathcal{I}}$ is non-degenerate and in bijection with $\mathcal{P}(2e, w_{\ell\mathcal{I}})$ if $\Gamma \in \mathcal{F}_0$ and $\kappa_{\ell\mathcal{I}}$ is degenerate. Here, the sets $\mathcal{P}(\beta_{\ell\mathcal{I}}, w_{\ell\mathcal{I}})$ and $\mathcal{P}(2e, w_{\ell\mathcal{I}})$ are defined as in [23, (7)] and (21.10) respectively.

Let $e_\pi$ be the identity element of the Clifford algebra over $V^\ell$. With the similar argument with [5.2] we may construct the $e_0$-Jordan-cuspidal pair for an $\ell$-block $\bar{B}(s, \kappa)$ of $\tilde{G} = \text{CSO}_{2n}^\circ(q)$, which is completely analogous with the case of $\tilde{G} = \text{CSO}_{2n}^\circ(q)$ and then we have the following result which is completely analogous with Theorem 5.10.

**Theorem B.2.** Let $(s, \kappa) \in i\text{Bl}_I(\tilde{G})$, $\bar{s} = \pi(s)$, $\bar{B} = \bar{B}(s, \kappa)$ and $B$ an $\ell$-block of $G$ covered by $\bar{B}$.

(i) If 1 or $-1$ is not eigenvalue of $\bar{s}$, then $B$ is the unique $\ell$-blocks of $G$ covered by $\bar{B}$ and there are $|\mathcal{O}(\mathcal{F}_0^\ell)|$ $\ell$-blocks of $\tilde{G}$ covering B. In addition, the $\ell$-blocks covering $B$ of $\tilde{G}$ are $\bar{B}(zs, \kappa)$, where $z$ runs through $\mathcal{O}(\mathcal{Z}(\tilde{G}^\ell))$. 

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(ii) If both 1 and −1 are eigenvalues of $\tilde{s}$, and there exists $\Gamma \in \mathcal{F}_0$ such that $w_{\Gamma} = 0$ and $\kappa_{\Gamma}$ is degenerate, then $B$ is the unique $\ell$-blocks of $G$ covered by $\tilde{B}$ and there are $|O_{e}(\mathbb{P}^{w}_{\Gamma})|$ $\ell$-blocks of $\tilde{G}$ covering $B$. In addition, the $\ell$-blocks covering $B$ of $\tilde{G}$ are $\tilde{B}(z,s,\kappa)$ and $\tilde{B}(z,s,\kappa')$, where $z$ runs through a complete set of representatives of $(-\infty)$-cosets in $O_{e}(Z(\tilde{G}^{*}))$.

(iii) Suppose that both 1 and −1 are eigenvalues of $\tilde{s}$ and $w_{\Gamma} \neq 0$ if $\Gamma \in \mathcal{F}_0$ and $\kappa_{\Gamma}$ is degenerate. Then there are $\frac{1}{2}|O_{e}(\mathbb{P}^{w}_{\Gamma})|$ $\ell$-blocks of $\tilde{G}$ covering $B$ and they are $\tilde{B}(z,s,\kappa)$, where $z$ runs through a complete set of representatives of $(\infty)$-cosets in $O_{e}(Z(\tilde{G}^{*}))$. Moreover,

- if $\kappa_{x-1}$ or $\kappa_{x+1}$ is degenerate, then $B$ is the unique $\ell$-blocks of $G$ covered by $\tilde{B}$, and
- if both $\kappa_{x-1}$ and $\kappa_{x+1}$ are non-degenerate, then there are two $\ell$-blocks of $G$ covered by $\tilde{B}$.

Now let $iB_{\ell}^{(1)}(G)$ be the set of $G$-conjugacy classes of pairs $(s, \kappa)$, where $s \in G$ is a semisimple $\ell'$-element and $\kappa \in C(s)$ such that either (1) 1 or −1 is not an eigenvalue of $s$ or (2) $\kappa_{x-1}$ or $\kappa_{x+1}$ is degenerate. Here, we identify $(s, \kappa)$ with $(s, \kappa')$. Let $iB_{\ell}^{(2)}(G)$ be the set of $G^{*}$-conjugacy classes of pairs $(s, \mu)$, where $s \in G^{*}$ is a semisimple $\ell'$-element and $\mu \in C(s)$ such that both 1 and −1 are eigenvalues of $s$ and both $\kappa_{x-1}$ and $\kappa_{x+1}$ are non-degenerate. Then $iB_{\ell}^{(1)}(G) := iB_{\ell}^{(1)}(G) \cup iB_{\ell}^{(2)}(G)$, where the elements of $iB_{\ell}^{(2)}(G)$ counting twice, is a labeling set for $iB_{\ell}(G)$ by Theorem B.2.

Now let $i\text{IrB}_{\ell}^{(1)}(G)$ be the set of $G^{*}$-conjugacy classes of pairs $(s, \mu)$, where $s \in G^{*}$ is a semisimple $\ell'$-element and $\mu \in \Psi(s)$ such that either 1 or −1 is not an eigenvalue of $s$ or both 1 and −1 are eigenvalues of $s$ and $\mu_{x-1}$ or $\mu_{x+1}$ is degenerate. Let $i\text{IrB}_{\ell}^{(2)}(G)$ be the set of $G^{*}$-conjugacy classes of pairs $(s, \mu)$, where $s \in G^{*}$ is a semisimple $\ell'$-element and $\mu \in \Psi(s)$ such that both 1 and −1 are eigenvalues of $s$ and both $\mu_{x-1}$ and $\mu_{x+1}$ are non-degenerate. Then $i\text{IrB}_{\ell}(G) := i\text{IrB}_{\ell}^{(1)}(G) \cup i\text{IrB}_{\ell}^{(2)}(G)$, where the elements of $i\text{IrB}_{\ell}^{(2)}(G)$ counting twice, is a labeling set for $\text{Irr}(G) \cap \mathcal{E}(G, \ell')$ by Lemma B.2. If $(s, \mu) \in i\text{IrB}_{\ell}^{(1)}(G)$, we denote by $\chi_{s,\mu}$ the character of $G$ corresponding to $(s, \mu)$. If $(s, \mu) \in i\text{IrB}_{\ell}^{(2)}(G)$, then $\chi_{s,\mu}^{(1)}$ and $\chi_{s,\mu}^{(-1)}$ denote the two characters of $G$ corresponding to $(s, \mu)$.

Furthermore, if $(s, \kappa) \in iB_{\ell}^{(1)}(G)$ and $B = B(s, \kappa)$, then $\text{Irr}(B) \cap \mathcal{E}(G, \ell') = \{\chi_{s,\mu} \mid \mu \in \Psi(s, \kappa), (s, \mu) \in i\text{IrB}_{\ell}^{(1)}(G)\} \cup \{\chi_{s,\mu}^{(1)} \mid \mu \in \Psi(s, \kappa), (s, \mu) \in i\text{IrB}_{\ell}^{(2)}(G)\}$. Here, the second set is non-empty if and only if both 1 and −1 are eigenvalues of $s$ and $\kappa_{x-1}$ or $\kappa_{x+1}$ is degenerate. If $(s, \kappa) \in iB_{\ell}^{(2)}(G)$, $B^{(1)} = B^{(1)}(s, \kappa)$ and $B^{(-1)} = B^{(-1)}(s, \kappa)$, then $\text{Irr}(B^{(1)} \cup B^{(-1)}) \cap \mathcal{E}(G, s) = \{\chi_{s,\mu} \mid \mu \in \Psi(s, \kappa)\}$. We may assume that $\text{Irr}(B^{(0)}, \mathcal{E}(G, s) = \{\chi_{s,\mu}^{(i)} \mid \mu \in \Psi(s, \kappa)\}$ for $i = \pm 1$. Note that, if both 1 and −1 are eigenvalue of $s$, and $\kappa_{x-1}$ and $\kappa_{x+1}$ are non-degenerate, then both $\mu_{x-1}$ and $\mu_{x+1}$ are non-degenerate, and then we always have $(s, \mu) \in i\text{IrB}_{\ell}^{(2)}(G)$.

Now we give a labeling set for $\text{Irr}(B) \cap \mathcal{E}(G, \ell')$. First, we define a set $\mathcal{P}(2e, w_{1}, w_{2}) := \mathcal{P}(2e, w_{1}) \times \mathcal{P}(2e, w_{2})$ for integers $e \geq 1$, $w_{1}, w_{2} \geq 0$, where $\mathcal{P}(2e, w_{1})$ and $\mathcal{P}(2e, w_{2})$ are defined as in (2.7). Now we define $\mathcal{P}_{0}(2e, w_{1}, w_{2}) := \mathcal{P}_{0}(2e, w_{1}) \times \mathcal{P}_{0}(2e, w_{2})$, where $\mathcal{P}_{0}(2e, w_{1})$ and $\mathcal{P}_{0}(2e, w_{2})$ are defined as in (2.8). First we define an equivalent relation on the set $\mathcal{P}(2e, w_{1}, w_{2})$. For $\mu^{(k)} = \mu^{(k)}(1) \times \mu^{(k)}(2)$, where $\mu^{(k)}(1) = (\kappa_{1}, \ldots, \kappa_{2}) \in \mathcal{P}(2e, w_{1}), \mu^{(k)}(2) = (\kappa_{1}, \ldots, \kappa_{2}) \in \mathcal{P}(2e, w_{2})$ and $k = 1, 2$, we let $\mu^{(1)} \sim \mu^{(2)}$ if $\mu^{(1)}(1) \sim \mu^{(2)}(1)$ and $\mu^{(1)}(2) \sim \mu^{(2)}(2)$ in the sense of (2.9). Then we define $\mathcal{P}_{d}(2e, w_{1}, w_{2}) = (\mathcal{P}(2e, w_{1}, w_{2}) \setminus \mathcal{P}_{0}(2e, w_{1}, w_{2})) / \sim$. Let $\mathcal{P}^{(2e, w_{1}, w_{2})} := \mathcal{P}_{d}(2e, w_{1}, w_{2}) \cup \mathcal{P}_{0}(2e, w_{1}, w_{2})$, where the elements of $\mathcal{P}_{0}(2e, w_{1}, w_{2})$ are counted twice.

Define $\mathcal{P}(B) :=$

(a) $\prod_{\Gamma} \mathcal{P}(\beta_{1}e_{\Gamma}, w_{\Gamma})$ if one of the following holds,

- $w_{x-1} = w_{x+1} = 0$, or
\* \* \* 

- \( \kappa_{i-1} \) or \( \kappa_{i+1} \) is non-degenerate,

(b) \( \mathcal{P}'(2e, w_{T_0}) \times \prod_{\Gamma \in \mathcal{I}_{T_0}} \mathcal{P}(\beta_{\Gamma} e_{\Gamma}, w_{\Gamma}) \) if both \( \kappa_{i-1} \) and \( \kappa_{i+1} \) are degenerate and there exists a unique \( \Gamma_0 \in \mathcal{F}_0 \) such that \( w_{T_0} \) is odd, where \( \mathcal{P}'(2e, w_{T_0}) \) is defined as in (2.10).

(c) \( \mathcal{P}'(2e, w_{x-1}, w_{x+1}) \times \prod_{\Gamma \not\in \mathcal{F}_0} \mathcal{P}(e_{\Gamma}, w_{\Gamma}) \), if one of the following holds,

- both \( \kappa_{x-1} \) and \( \kappa_{x+1} \) are degenerate and both \( w_{x-1} \) and \( w_{x+1} \) are odd, or
- \( w_{x-1} \) or \( w_{x+1} \) is non-zero, both \( \kappa_{x-1} \) and \( \kappa_{x+1} \) are degenerate and both \( w_{x-1} \) and \( w_{x+1} \) are even.

Similar with Proposition 5.12 we have

**Proposition B.3.** With the preceding notation, \( \mathcal{P}(B) \) is a labeling set for \( \text{Irr}(B) \cap \mathcal{E}(G, \ell') \).

Let \( f_{\Gamma} \) and \( f'_{\Gamma} \) be defined as in Appendix §A.2.2. Then \( f_{\Gamma} = |\mathcal{P}(\beta_{\Gamma} e_{\Gamma}, w_{\Gamma})| \) and \( f'_{\Gamma} = |\mathcal{P}_0(\beta_{\Gamma} e_{\Gamma}, w_{\Gamma})| \). We end the appendix by giving the number of irreducible \( \ell \)-Brauer characters in an \( \ell \)-block of \( G = \text{SO}^+_2(q) \), which follows by Proposition B.3 immediately.

**Theorem B.4.** Let \( B \) be an \( \ell \)-block corresponding to \( (s, \kappa) \in i \text{Br}_\ell(G) \) and \( l(B) = |\text{IBr}_\ell(B)| \).

(i) \( l(B) = \prod_{\Gamma} f_{\Gamma} \) if one of the following statements holds.

- \( w_{x-1} = w_{x+1} = 0 \).
- \( \kappa_{x-1} \) or \( \kappa_{x+1} \) is non-degenerate.

(ii) Suppose that \( w_{x-1} \) or \( w_{x+1} \) is non-zero and both \( \kappa_{x-1} \) and \( \kappa_{x+1} \) are degenerate.

- If either \( w_{x-1} \) or \( w_{x+1} \) is odd, then \( l(B) = \frac{1}{2} \prod_{\Gamma} f_{\Gamma} \).
- If both \( w_{x-1} \) and \( w_{x+1} \) are even, then \( l(B) = \frac{f_{\Gamma_0} f_{\Gamma_0} f_{\Gamma_0} f_{\Gamma_0}}{2} \prod_{\Gamma \not\in \mathcal{F}_0} f_{\Gamma} \).

Now we consider the Alperin weight conjecture for \( G = \text{SO}^+_2(q) \).

**Proposition B.5.** Let \( B \) be an \( \ell \)-block corresponding to \( (s, \kappa) \in i \text{Br}_\ell(G) \). If

- 1 or \(-1\) is not an eigenvalue of \( s \), or
- \( \kappa_{x-1} \) or \( \kappa_{x+1} \) is degenerate,

then the Alperin weight conjecture holds for \( B \), i.e., \( l(B) = |\text{W}_\ell(B)| \).

**Proof.** Let \( V_0, \vartheta_0 \) and \( s_0 \) be defined as in §4 (4H). Then by the proof of §4 (4H), \( \vartheta_0 \in G_0 \) and \( \tilde{\vartheta}_0 \in \text{Irr}(\tilde{G}_0) \) with \( G_0 = \text{SO}(V_0) \) and \( \tilde{G}_0 = \text{CSO}(V_0) \) such that \( \tilde{\vartheta}_0 = \tilde{\chi}_{\ell_0, \kappa} \), where \( \ell_0 \) satisfies that \( s_0 = \pi(\ell_0) \). Then the assertion follows by §A.2.2 and Theorem B.4 and the criterion that when \( \vartheta_0 \) is \( \text{GO}(V_0) \)-invariant given in the statements after Lemma B.1. \( \square \)

Now we consider the following properties about the action of \( I = \text{GO}^+_2(q) \) on the characters of \( G \).

\( \dagger \) Let \( \tilde{\chi} \in \text{Irr}(\tilde{G}) \) and \( \Delta = \text{Irr}(G \mid \tilde{\chi}) \). Then \( I_{\Delta} \) acts trivially on \( \Delta \).
Let $s$ be a semisimple element of $G$, $\mu \in \Psi(s)$ and $\chi$ be a character of $G$ corresponding to $(s, \mu)$. Suppose that both 1 and $-1$ are eigenvalues of $s$ and both $\mu_{s^{-1}}$ and $\mu_{s+1}$ are non-degenerate. Then $\chi$ is $I$-invariant.

Then by the proof of Proposition B.5 and the statements after Lemma B.1, we have

Theorem B.6.  
(i) The Alperin weight conjecture \([7,7]\) holds for every $\ell$-block of the special orthogonal group $G = \text{SO}_{2n}(q)$ with every $n \geq 4$, odd $q$ and $\epsilon = \pm$ if \((\dagger)\) is true for the special orthogonal group $G = \text{SO}_{2n}(q)$ with every $n \geq 4$, odd $q$ and $\epsilon = \pm$.

(ii) \((\dagger)\) holds if and only if \((\ddagger)\) holds.

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