On irreducibility of the family of ACM curves of degree 8 and genus 4 in $\mathbb{P}_k^4$

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Abstract

Let $C$ be an arithmetically Cohen-Macaulay curve of arithmetic genus 4. We prove that the family of such curves of degree 8 in $\mathbb{P}_k^4$ is irreducible.

Keywords: CI liaison, Gorenstein liaison, irreducible curves, ACM curves.

Liaison using complete intersection or Gorenstein schemes is widely used in algebraic geometry. An excellent reference book is [7]. We use the technique of resolving ideal sheaves of ACM curves by special type of sheaves (so called $\mathcal{E}$- and $\mathcal{N}$-type resolutions) to link the curve in question to a simpler curve. Using this technique we conclude that a family of $(8, 4)$ curves is irreducible. Also this paper demonstrates a usage of correspondance between ACM curves and ACM sheaves (as discussed in [3])

For convenience we recall here some definitions and results of liaison theory we will be using in this work. See [7] for reference.

Definition 1. A scheme $X$ of $\mathbb{P}_k^n$ is called arithmetically Cohen-Macaulay (ACM) if its homogeneous coordinate ring is a Cohen-Macaulay ring.

Definition 2. Let $Z$ be a subscheme of $\mathbb{P}_k^4$. Let $V_1, V_2$ be equidimensional subschemes of $\mathbb{P}_k^n$ of codimension $r$ and without embedded components. We say that $V_1$ and $V_2$ are linked by $Z$ if

1. $\mathcal{I}_Z \subset \mathcal{I}_{V_1} \cap \mathcal{I}_{V_2}$
2. $\mathcal{I}_{V_2}/\mathcal{I}_Z \cong \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{V_1}, \mathcal{O}_Z)$
3. $\mathcal{I}_{V_1}/\mathcal{I}_Z \cong \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{V_2}, \mathcal{O}_Z)$

If $Z$ is AG, we say $V_1$ is G-linked to $V_2$; if $Z$ is CI, we say $V_1$ is CI-linked to $V_2$. 

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**Definition 3.** On a nonsingular quadric hypersurface $Q$ a locally free sheaf $F$ with the property that $H^i_F(F) = 0$ for $i = 1, 2$ is called an ACM sheaf.

**Proposition 4.** Let $C$ be an ACM curve of degree 8 and arithmetic genus 4 in $\mathbb{P}^4$. Then $I_C$ is generated in degrees 2 and 3.

**Proof.** We need to compute the cohomology table of $I_C(n)$. From the Riemann-Roch theorem $h^0(\mathcal{O}_C(1)) - h^1(\mathcal{O}_C(1)) = 8 + 1 - 4 = 5$. Taking chomology in the exact sequence

$$0 \rightarrow I_C(n) \rightarrow \mathcal{O}_{\mathbb{P}^4}(n) \rightarrow \mathcal{O}_C(n) \rightarrow 0$$

we arrive at

$$0 \rightarrow H^0(I_C(n)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \rightarrow H^0(\mathcal{O}_C(n)) \rightarrow$$

$$\rightarrow H^1(I_C(n)) \rightarrow H^1(\mathcal{O}_{\mathbb{P}^4}(n)) \rightarrow H^1(\mathcal{O}_C(n)) \rightarrow$$

$$\rightarrow H^2(I_C(n)) \rightarrow H^2(\mathcal{O}_{\mathbb{P}^4}(n)) \rightarrow H^2(\mathcal{O}_C(n)) \rightarrow H^3(I_C(n)) \rightarrow \ldots . \ (1)$$

$H^1(I_C(n)) = 0$ since $C$ is ACM. Thus, map $H^0(\mathcal{O}_{\mathbb{P}^4}(n)) \rightarrow H^0(\mathcal{O}_C(n))$ is surjective and so, $h^0(\mathcal{O}_C(n)) \leq h^0(\mathcal{O}_{\mathbb{P}^4}(n))$. From the Riemann-Roch theorem we get $h^0(\mathcal{O}_C(1)) = 5 + h^1(\mathcal{O}_C(1))$ while $h^0(\mathcal{O}_{\mathbb{P}^4}(1)) = 5$. This implies that $h^1(\mathcal{O}_C(1)) = 0$ and $h^0(\mathcal{O}_C(1)) = 5$.

From the exact sequence (1) we obtain $h^2(I_C(n)) = h^1(\mathcal{O}_C(n))$ since $H^i(\mathcal{O}_{\mathbb{P}^4}(n)) = 0$ for $i = 1, 2$ \[III.5.1\]. Wherefrom $h^2(I_C(1)) = h^1(\mathcal{O}_C(1)) = 0$. Note also that $H^3(I_C) = 0$. Thus, the cohomology table is:

Thus, $h^i(I_C(3-i)) = 0$ for all $i > 0$. By definition 1.1.4 of [7] this implies that $I_C$ is 3-regular. This, in turn, by Castelnuovo-Mumford regularity [7] 1.1.5.(1), implies that $h^i(I_C(k)) = 0$ for $i > 0$, $k + i \geq 3$. Equivalently, $h^2(I_C(n)) = 0$ for $n \geq 1$. Thus, $h^1(\mathcal{O}_C(n)) = 0$ for all $n \geq 1$. Therefore, $I_C(k)$ is generated as $\mathcal{O}_{\mathbb{P}^4}$-module by its global sections for all $k \geq 3$ (by [7] theorem 1.1.5.(3)). \[\square\]
Corollary 5. Any ACM curve $C$ of degree 8 and genus 4 in $\mathbb{P}^4_k$ is contained in a quadric hypersurface.

Proof. Proposition above implies that $h^0(\mathcal{O}_C(n)) = nd + 1 - g$ for $n \geq 1$, or

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $h^0(\mathcal{O}_C(n))$ | 1 | 5 | 13 | 21 | 29. |

Recall that we have

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $h^0(\mathcal{O}_Q(n))$ | 1 | 5 | 15 | 35 | 70, |

wherefrom we obtain

| $n$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| $h^0(\mathcal{I}_C(n))$ | 0 | 0 | 2 | 14 | 41. |

Thus $h^0(\mathcal{I}_C(2)) = 2$, which implies that there is at least one quadric hypersurface containing $C$. 

Proposition 6. There is no ACM curve of degree 8 and genus 4 in $\mathbb{P}^3_k$.

Proof. Assume $C$ is an ACM curve of degree 8 and genus 4 in $\mathbb{P}^3_k$. Taking cohomology in the exact sequence

$$0 \to \mathcal{I}_C(1) \to \mathcal{O}_{\mathbb{P}^3_k}(1) \to \mathcal{O}_C(1) \to 0$$

we obtain:

$$0 \to H^0(\mathcal{I}_C(1)) \to H^0(\mathcal{O}_{\mathbb{P}^3_k}(1)) \to H^0(\mathcal{O}_C(1)) \to H^1(\mathcal{I}_C(1)) \to 0.$$ 

Thus $h^0(\mathcal{I}_C(1)) = h^0(\mathcal{O}_{\mathbb{P}^3_k}(1)) - h^0(\mathcal{O}_C(1))$ since $h^1(\mathcal{I}_C(1)) = 0$ for an ACM curve $C$. However $h^0(\mathcal{O}_{\mathbb{P}^3_k}(1)) = 4$ and $h^0(\mathcal{O}_C(1)) = 5 + h^1(\mathcal{O}_C(1)) \geq 5$. This would give $h^0(\mathcal{I}_C(1)) < 0$, which is impossible. Thus $h^1(\mathcal{I}_C(1)) \neq 0$, and $C$ is not an ACM curve. 

Corollary 7. Any ACM curve of degree 8 and genus 4 in $\mathbb{P}^4_k$ is nondegenerate.

Proposition 8. Let $C$ be an ACM curve of degree 8 and genus 4 on a nonsingular quadric hypersurface $Q$. Then, there is an $\mathcal{E}$-type resolution of $\mathcal{I}_C$ of the form

$$0 \to \mathcal{E}_0^2(-2) \to \mathcal{O}_Q(-2) \oplus \mathcal{O}_Q^4(-3) \to \mathcal{I}_C \to 0$$

(2)

Proof. Since $0 \to \mathcal{I}_C(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_C(n) \to 0$ is exact we obtain

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-----|---|---|---|---|---|---|---|
| $h^0(\mathcal{I}_C(n))$ | 0 | 0 | 1 | 9 | 26 | 54 | 95. |
We know that $\mathcal{I}_C$ is generated in degrees 2 and 3 and also the generator of $\mathcal{I}_C$ in degree 2 multiplied by linear functions gives a 5-dimensional subspace of $H^0(Q, \mathcal{I}_C(3))$. Therefore we need 4 generators in degree 3, which are not products of linear form and the degree two generator. Thus, there is an $\mathcal{E}$-type resolution of $\mathcal{I}_C$ of the form

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Q(-2) \oplus \mathcal{O}_Q^4(-3) \rightarrow \mathcal{I}_C \rightarrow 0 \quad (3)$$

where $\mathcal{E}$ is ACM sheaf by [3, Theorem 2] and rank $\mathcal{E} = 4$.

[3] gives the following table of cohomology:

| $n$ | $h^0(\mathcal{E}(n))$ | $h^0(\mathcal{O}_Q(-2 + n) \oplus \mathcal{O}_Q^4(-3 + n))$ | $h^0(\mathcal{I}_C(n))$ |
|-----|-----------------|---------------------------------|---------------------|
| 0   | 0               | 0                               | 0                   |
| 1   | 0               | 0                               | 0                   |
| 2   | 0               | 1                               | 1                   |
| 3   | 0               | 9                               | 9                   |
| 4   | 8               | 34                              | 26                  |
| 5   | 32              | 86                              | 54                  |
| 6   | 80              | 175                             | 95                  |

Thus, by [3, Corollary 3] $\mathcal{E}$ must be one of the following:

1. $\mathcal{E}_0(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$, or
2. $\mathcal{E}_0(a) \oplus \mathcal{E}_0(b)$, or
3. $\bigoplus_{i=1}^4 \mathcal{O}(a_i)$,

where the sheaf $\mathcal{E}_0$ is given by cite[Definition 5]drozd1 .

Comparing

$$\begin{array}{c|ccccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 h^0(\mathcal{E}(n)) & 0 & 0 & 0 & 8 & 32 & 80 \\
\end{array}$$

with cohomology tables for $\mathcal{E}_0(n)$ and $\mathcal{O}(n)$ we obtain $\mathcal{E} = \mathcal{E}_0^2(-2)$. This gives us

$$0 \rightarrow \mathcal{E}_0^2(-2) \rightarrow \mathcal{O}_Q(-2) \oplus \mathcal{O}_Q^4(-3) \rightarrow \mathcal{I}_C \rightarrow 0$$

as an $\mathcal{E}$-type resolution of $\mathcal{I}_C$.

**Proposition 9.** Let $C$ be an ACM curve of degree 8 and genus 4 on $Q$. Then $C$ is CI-linked to an ACM curve $C'$ of degree 4 and genus 0 (possibly reducible).

**Proof.** Let $C$ be an ACM curve of degree 8 and genus 4 on a quadric hypersurface $Q$ in $\mathbb{P}_k^4$.

Then by $\square C$ is nondegenerate. Note that $h^0(P_k^4, \mathcal{I}_C(2)) = 2$, therefore $h^0(Q, \mathcal{I}_C(2)) = 1$. Thus the generator of $\mathcal{I}_C$ in degree 2 cuts out a surface $Y$ of degree 4 on $Q$. We claim
that \( Y \) is irreducible. To prove this let \( Y \) be a union of two surfaces \( Q_1 \) and \( Q_2 \). Then 
\[
\deg Q_1 = \deg Q_2 = 2 \text{ since } Y \text{ is a degree } 4 \text{ surface on a nonsingular quadric hypersurface and thus Klein’s theorem [5, ex.II.6.4.(d)] implies that } \deg Q_i, \ i = 1,2 \text{ must be even. However a quadric surface lies in } \mathbb{P}^3_k, \text{ which contradicts } C \text{ is nondegenerate. Thus } Y \text{ is irreducible.}
\]

Let \( F \) be a hypersurface of degree 3 containing \( C \), but not containing \( Y \) completely. Such a hypersurface exists since \( h^0(Q, \mathcal{I}_C(3)) - \dim V = 14 - 5 = 9 \), where \( V \) is a subspace of \( H^0(Q, \mathcal{I}_C(3)) \) generated by elements of the form \( l \cdot s \), where \( l \) is a linear form and \( s \in H^0(Q, \mathcal{I}_C(2)) \). Let \( Z \) be a complete intersection of \( Y \) and \( F \). Then \( Z \) has degree 12 and it contains \( C \). Let curve \( C' \) be CI-linked to \( C \) via \( Z \).

Note that \( C' \) is ACM since so is \( C \). To complete the proof we need to compute degree and genus of \( C' \): \( \deg C' = \deg Z - \deg C = 4 \). By [7] corollary 5.2.14, \[ g(C) - g(C') = \frac{1}{2} (\deg F + \deg Y - 5) \cdot (\deg C - \deg C') \]. Thus, \( g(C') = g(C) - 4 = 0 \), wherefrom \( C' \) is an ACM curve of degree 4 and genus 0.

In order to find an \( \mathcal{N} \)-type resolution of a nondegenerate ACM (8,4) curve, we will determine an \( \mathcal{E} \)-type resolution of a linked (4,0) ACM curve.

Now we compute an \( \mathcal{E} \)-type resolution of a nondegenerate ACM (4,0) curve \( C' \).

**Proposition 10.** There exists an \( \mathcal{E} \)-type resolution of any ACM (4,0) curve on a nonsingular quadric hypersurface \( Q \) of the form

\[
0 \longrightarrow \mathcal{E}^2_0(-1) \longrightarrow \mathcal{O}^5_Q(-2) \longrightarrow \mathcal{I}_C \longrightarrow 0.
\]

**Proof.** We claim that \( \mathcal{I}_C \) is generated in degree 2 and \( h^1(\mathcal{O}_C(n)) = 0 \) for \( n \geq 1 \).

To prove this we need to compute the cohomology table. From the Riemann-Roch theorem
\[
h^0(\mathcal{O}_C(1)) - h^1(\mathcal{O}_C(1)) = 4 + 1 - 0 = 5. \text{ Thus } h^3(\mathcal{O}_C(1)) \geq 5. \text{ Taking cohomology in the short exact sequence}
\]

\[
0 \longrightarrow \mathcal{I}_C(n) \longrightarrow \mathcal{O}_Q(n) \longrightarrow \mathcal{O}_C(n) \longrightarrow 0
\]

we obtain:

\[
0 \longrightarrow h^0(\mathcal{I}_C(n)) \longrightarrow h^0(\mathcal{O}_Q(n)) \longrightarrow h^0(\mathcal{O}_C(n)) \longrightarrow \\
\longrightarrow h^1(\mathcal{I}_C(n)) \longrightarrow h^1(\mathcal{O}_Q(n)) \longrightarrow h^1(\mathcal{O}_C(n)) \longrightarrow \\
\longrightarrow h^2(\mathcal{I}_C(n)) \longrightarrow h^2(\mathcal{O}_Q(n)) \longrightarrow h^2(\mathcal{O}_C(n)) \longrightarrow h^3(\mathcal{I}_C(n)) \longrightarrow \ldots \ldots (4)
\]

Note that \( h^1(\mathcal{I}_C(n)) = 0 \) since \( C \) is ACM. Thus the map \( h^0(\mathcal{O}_Q(n)) \longrightarrow h^0(\mathcal{O}_C(n)) \) is surjective, and \( h^0(\mathcal{O}_C(n)) \leq h^0(\mathcal{O}_Q(n)) \). For \( n = 0 \) this means that \( h^0(\mathcal{O}_C) \leq h^0(\mathcal{O}_Q) = 1 \). However, \( h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1 \), thus \( h^0(\mathcal{O}_C) = 1 \) and \( h^1(\mathcal{O}_C) = 0 \).

Also, since \( h^1(\mathcal{O}_Q(n)) = h^2(\mathcal{O}_Q(n)) = 0 \) [5, Ex.II.5.5(c)] we have \( h^1(\mathcal{O}_C(n)) = h^2(\mathcal{I}_C(n)) \). Thus \( h^2(\mathcal{I}_C) = h^1(\mathcal{O}_C) = 0 \). We obtain the following cohomology table:
Thus, by Castelnuovo-Mumford regularity $\mathcal{I}_C$ is generated in degree 2 and $\mathcal{O}_C(n)$ is nonspecial for $n \geq 1$.

Thus $h^0(\mathcal{O}_C(n)) = nd + 1 - g = 4n + 1$ for $n \geq 1$ or

\[
\begin{array}{c|cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 h^0(\mathcal{O}_C(n)) & 1 & 5 & 9 & 13 & 21 & 25 \\
\end{array}
\]

and

\[
\begin{array}{c|cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 h^0(\mathcal{O}_Q(n)) & 1 & 5 & 14 & 30 & 55 & 91 & 140 \\
\end{array}
\]

which implies the following table

\[
\begin{array}{c|cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 h^0(\mathcal{I}_C(n)) & 0 & 0 & 5 & 17 & 38 & 70 & 115 \\
\end{array}
\]

since $0 \to \mathcal{I}_C(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_C(n) \to 0$ is exact.

Thus we have the following exact sequence:

\[
0 \to \mathcal{E} \to \mathcal{O}_Q^*(-2) \to \mathcal{I}_C \to 0
\]

with $\mathcal{E}$ ACM sheaf of rank 4. Thus, by [3, Corollary 3] $\mathcal{E}$ is one of the following:

1. $\mathcal{E}_0(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c)$, or
2. $\mathcal{E}_0(a) \oplus \mathcal{E}_0(b)$, or
3. $\bigoplus_{i=1}^4 \mathcal{O}(a_i)$.

$h^0(\mathcal{E}(n)) = h^0(\mathcal{O}_Q(n-2)) - h^0(\mathcal{I}_C(n))$ since $h^1(\mathcal{E}) = 0$. Comparing cohomology table:

\[
\begin{array}{c|cccccc}
 n & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 h^0(\mathcal{E}(n)) & 0 & 0 & 8 & 32 & 80 & 160 & \\
\end{array}
\]

with cohomology tables $\mathcal{E}_0(n)$ and $\mathcal{O}(n)$ we obtain $\mathcal{E} = \mathcal{E}_0^2(-1)$, proving the proposition. \qed
This proposition together with [3, Corollary 1] give us the following

**Corollary 11.** All ACM (4,0) curves on a nonsingular quadric hypersurface Q form an irreducible family.

**Proposition 12.** There exists an \( N \)-type resolution of an ACM (8,4) curve \( C \) on a nonsingular quadric hypersurface \( Q \) in \( \mathbb{P}^4_k \) of the form

\[
0 \longrightarrow \mathcal{O}_Q^5(-5) \longrightarrow \mathcal{O}_Q(-4) \oplus \mathcal{O}_Q(-3) \oplus \mathcal{E}_0^2(-3) \longrightarrow \mathcal{I}_C \longrightarrow 0.
\]

**Proof.** By proposition [3] an ACM curve of degree 8 and genus 4 on a nonsingular quadric hypersurface \( Q \) in \( \mathbb{P}^4_k \) can be CI-linked to an ACM curve \( C' \) of degree 4 and genus 0 by a complete intersection curve \( Z \) formed by two divisors \( \mathcal{O}_Q(4) \) and \( \mathcal{O}_Q(3) \). By proposition [10] there exists an \( \mathcal{E} \)-type resolution of \( \mathcal{I}_{C'} \) of the form:

\[
0 \longrightarrow \mathcal{E}_0^2(-1) \longrightarrow \mathcal{O}_Q^5(-2) \longrightarrow \mathcal{I}_{C'} \longrightarrow 0.
\]

However, by [3, Proposition 2],

\[
(\mathcal{E}_0^2(-1))^\vee = (\mathcal{E}_0^2(1))^2 = \mathcal{E}_0^2(4).
\]

Thus, there exists an \( N \)-type resolution of \( \mathcal{I}_C \) of the form:

\[
0 \longrightarrow \mathcal{O}_Q^5(-5) \longrightarrow \mathcal{O}_Q(-4) \oplus \mathcal{O}_Q(-3) \oplus \mathcal{E}_0^2(-3) \longrightarrow \mathcal{I}_C \longrightarrow 0.
\]

\( \Box \)

The above proposition together with [2, Corollary 1] imply

**Corollary 13.** All ACM (8,4) curves on a nonsingular quadric hypersurface \( Q \) form an irreducible family.

We note here that any nonsingular curve of degree 8 and genus 4 on \( Q \) is ACM.

**Proposition 14.** Let \( C \) be a nonsingular curve of degree 8 and genus 4 on a nonsingular quadric hypersurface \( Q \) in \( \mathbb{P}^4_k \). Then \( C \) is ACM.

**Proof.** \( h^1(\mathcal{O}_C(n)) = 0 \) for all \( n \geq 1 \) since \( 2g - 2 = 6 \leq \deg C \). By the Riemann-Roch theorem \( h^0(\mathcal{O}_C(1)) = 8 + 1 - 4 + h^1(\mathcal{O}_C(1)) \). Thus \( h^0(\mathcal{O}_C(1)) = 5 \). Taking cohomology in the exact sequence \( 0 \longrightarrow \mathcal{I}_C(1) \longrightarrow \mathcal{O}_Q(1) \longrightarrow \mathcal{O}_C(1) \longrightarrow 0 \) we obtain

\[
0 \longrightarrow H^0(\mathcal{I}_C(1)) \longrightarrow H^0(\mathcal{O}_Q(1)) \longrightarrow H^0(\mathcal{O}_C(1)) \longrightarrow H^1(\mathcal{I}_C(1)) \longrightarrow 0
\]

If \( H^1(\mathcal{I}_C(1)) \neq 0 \) then \( H^0(\mathcal{I}_C(1)) \neq 0 \). Therefore there exists a hyperplane \( H \) such that \( C \subset H \cap Q \) and \( H \cap Q \) is a surface of degree 2 in \( \mathbb{P}^3_k \). Thus \( H \cap Q \) is one of the following:
• two planes, or
• double plane, or
• quadric cone, or
• nonsingular quadric surface.

Two planes and double plane are impossible since there is no (8,4) curve in $\mathbb{P}^2_k$ (plane curve of degree 8 has genus 21). The set of possible pairs $(d, g)$ on a quadric cone is a subset of the set of possible pairs $(d, g)$ on a nonsingular quadric surface. But there is no (8,4) curve on a quadric surface in $\mathbb{P}^3_k$. Thus, $H^1(\mathcal{I}_C(1))$ must be zero. Similarly $H^1(\mathcal{I}_C(2)) = 0$. From the exact sequence

$$0 \to H^0(\mathcal{I}_C(2)) \to H^0(\mathcal{O}_Q(2)) \to H^0(\mathcal{O}_C(2)) \to H^2(\mathcal{I}_C(2)) \to 0$$

we obtain $h^0(\mathcal{O}_C(2)) = 13$ and $h^0(\mathcal{O}_Q(2)) = 14$. Thus $h^0(\mathcal{I}_C(2)) \geq 1$. We claim that $h^0(\mathcal{I}_C(2)) = 1$. If $h^0(\mathcal{I}_C(2)) \geq 2$ then $h^0(\mathbb{P}^3_k, \mathcal{I}_C(2)) \geq 3$. Thus $C$ must be contained in the intersection $Z$ of three quadric surfaces, wherefrom $Z$ must be one of the following:

1. A curve. Then it is of degree 8 and genus 5, or
2. A surface of degree $\leq 4$.

Neither of these is possible, therefore $h^1(\mathcal{I}_C(2)) = 0$. Note that for any curve $C$ $h^0(\mathcal{O}_Q) \cong k \cong h^0(\mathcal{O}_C)$. Therefore $h^1(\mathcal{I}_C) = 0$. Also, $h^2(\mathcal{I}_C(1)) = h^1(\mathcal{O}_C(1)) = 0$ and $h^3(\mathcal{I}_C) = 0$. Thus we have the following cohomology table for $\mathcal{I}_C(n)$:

```
+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+-------+
|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 0     | 0     |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 0     | 0     |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 0     | 0     |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
| 0     |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |
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Thus $\mathcal{I}_C$ is 2-regular and $h^1(\mathcal{I}_C(n)) = 0$ for $n \geq 2$ and for $n < 0$. However $h^1(\mathcal{I}_C) = h^1(\mathcal{I}_C(1)) = h^1(\mathcal{I}_C(2)) = 0$, therefore $C$ is ACM.
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