A MODEL WITH EVERYTHING EXCEPT FOR A WELL-ORDERING OF THE REALS

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Abstract. We construct a model of $\text{ZF} + \text{DC}$ containing a Luzin set, a Sierpiński set, as well as a Burstin basis but in which there is no a well ordering of the continuum.

1. Introduction

In this paper we study subsets of the real line $\mathbb{R}$ with specific properties whose classic constructions were performed by assuming various forms of the Axiom of Choice (AC). The first pathological set was constructed by F. Bernstein in 1908 (cf. [3]); he constructed a set $B \subset \mathbb{R}$ of cardinality the continuum such that neither $B$ nor $\mathbb{R} \setminus B$ contains a perfect subset of reals. Such a set can be obtained by assuming the existence of a well-ordering of $\mathbb{R}$. Later in 1914, Luzin constructed an uncountable set $\Lambda \subset \mathbb{R}$ having countable intersection with every meager set (cf. [15]). His construction required the continuum hypothesis ($\text{CH}$, in the strong form according to which $\mathbb{R}$ may be well-ordered in order type $\omega_1$). In 1924, Sierpiński developed a similar construction to the one given by Luzin; under the assumption of the same form of $\text{CH}$, he constructed an uncountable set $S \subset \mathbb{R}$ having countable intersection with every measure zero set (cf. [23]).

However $\text{CH}$ is not a necessary assumption for the existence of Luzin and Sierpiński sets (see [18]). Moreover a Luzin set may exist in a model in which the set of reals is not well-ordered. In fact, D. Pincus and K. Prikry [19] proved that in the Cohen-Halpern-Lévy model $H$, a model in which the reals cannot be well-ordered (in fact, in $H$ there is an uncountable set of reals with no countable subset), there is a Luzin set as well as a Vitali set. Additionally, Pincus and Prikry asked whether a Hamel basis, i.e., a basis for $\mathbb{R}$ construed as a vector space over the field of rational numbers $\mathbb{Q}$, exists in $H$ or, in general, if the existence of a Hamel basis is compatible with the non-existence of a well-ordering of the reals. Recently, M. Beriashvili, R. Schindler, L. Wu and L. Yu (cf. [2]) answered this question in the affirmative, by showing that in $H$ there is a Hamel basis and, furthermore, in $H$ there is also a Bernstein set (see [2, Theorems 1.7 and 2.1]). Thus the model $H$ has many pathological sets of reals, but in $H$ the continuum cannot be well ordered. There is no Sierpiński set in $H$, though (see [2, Lemma 1.6]).

Let us informally refer to a model $M$ as a “Solovay model” iff $M$ is obtained via a symmetric collapse over a model in which what is to become $\omega_1^M$ is either inaccessible or a limit of large cardinals (e.g., Woodin cardinals). The paper [20] shows that if $U$ is a selective ultrafilter on $\omega$ which was added by forcing over a Solovay model $M$, then $M[U]$ satisfies the Open Coloring Axiom (see [20, p. 247]), hence $M[U]$ inherits from $M$ the property that every uncountable set of reals that a perfect subset and in particular $M[U]$ does not contain a well–ordering of the reals, see [20, Theorem 5.1].

The paper [13] further explores this topic and studies which consequences of having a well–ordering of $\mathbb{R}$ remain false when adding certain ultrafilters on $\omega$ over a Solovay model or when adding a Vitali set. Also, [13] produces a model of $\text{ZF} + \text{DC}$ plus “there is a Hamel basis” plus “there is no well–ordering of the reals.” The verification in [13] that the extension of the Solovay model via forcing with countable linearly independent sets of reals (called $\mathbb{Q}_U$ in the current paper, see Definition 4.9 below) doesn’t have a well–ordering of its reals uses large cardinals, specifically Woodin’s stationary tower forcing. The forcing $\mathbb{Q}_U$ used by [13] does not work in the absence of large cardinals, though, see Corollary 4.11 below.

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The current paper improves the result obtained in [2] by showing that there is a model $W$ of ZF + DC such that in $W$ the reals cannot be well-ordered and $W$ contains Luzin as well as Sierpiński sets and also a Burstin basis, i.e., a set which is simultaneously a Hamel basis and a Bernstein set. Notice that from the existence of a Hamel basis one can derive that in $W$ there is also a Vitali set (see [2, Lemma 1.1]).

2. Basic definitions and results

2.1. Pathological sets within ZFC.

**Definition 2.1.** Let $A \subseteq \mathbb{R}$ uncountable. We say that $A$ is

(i) a *Vitali set* if $A$ is the range of a selector for the equivalence relation $\sim_{\mathbb{Q}}$ defined over $\mathbb{R} \times \mathbb{R}$ by $x \sim_{\mathbb{Q}} y \iff x - y \in \mathbb{Q}$;

(ii) a *Sierpiński set* if for every $N \in \mathcal{N}$ - the ideal of null-sets with respect to Lebesgue measure over $\mathbb{R}$- we have $|A \cap N| \leq \omega$;

(iii) a *Luzin set* if for every $M \in \mathcal{M}$ - the ideal of the Borel meager sets- we have $|A \cap M| \leq \omega$;

(iv) a *Berstein set* if for every perfect set $P \subseteq \mathbb{R}$ we have $A \cap P \neq \emptyset$ for no $P$;

(v) a *Hamel basis* if $A$ is a maximal linearly independent subset of $\mathbb{R}$ when we consider it as a vector space over $\mathbb{Q}$.

(vi) a *Burstin basis* if $A$ is a Hamel basis which has nonempty intersection with every perfect set.

The existence of a Hamel basis in a model of ZF + DC implies the existence of nonmeasurable sets and the existence of sets without the Baire property. In particular, we have the next result connecting Hamel bases and Vitali sets. For a proof, see [2, Lemma 1.1].

**Lemma 2.2. (Folklore)** Suppose $V \models \text{ZF}$ and suppose that a Hamel basis $H$ exists. Then there is a Vitali set.

**Lemma 2.3. (Luzin, 1914, and Sierpiński, 1924)** Assume $V$ is a model of ZFC + CH. Then, there are $\Lambda$ and $S$ in $V$ such that $\Lambda$ is a Luzin set and $S$ is a Sierpiński set.

**Proof.** Let $\{N_i : i < \omega_1\}$ be an enumeration of all $G_\delta$ null sets. Recursively define $\langle x_i : i < \omega_1 \rangle$ such that $x_i \notin \bigcup\{N_j : j < i\} \cup \{x_j : j < i\}$. Then, $S = \{x_i : i < \omega_1\}$ is a Sierpiński set.

The same procedure gives us a Luzin set, starting out with an enumeration $\{M_i : i < \omega_1\}$ of all $F_\sigma$-meager sets.

**Remark 2.4.** As we may write $\mathbb{R} = N \cup M$ where $N$ is null and $M$ is meager, no set can be both a Sierpiński set as well as a Luzin set.

The construction of a Bernstein set in $V$ is based on the enumeration of all perfect subsets of $\mathbb{R}$. We omit the proof and instead present below the construction of a Burstin basis in $V$ under AC (see Theorem 2.6).

**Proposition 2.5. (Folklore)** Every Burstin basis is a Bernstein set.

**Proof.** Suppose $B \subseteq \mathbb{R}$ is a Burstin basis such that $P \subseteq B$ for some perfect $P \subseteq \mathbb{R}$. As $B$ is linearly independent, the set $2P = \{2p : p \in P\}$ has empty intersection with $B$. On the other hand, $2P$ is a perfect set, so $2P \cap B \neq \emptyset$, which gives a contradiction. It follows that $B$ is totally imperfect, so $(\mathbb{R} \setminus B) \cap P \neq \emptyset$ as well, i.e., $B$ is a Bernstein set.

It is easy to construct a Hamel basis $H$ such that $H \cap P = \emptyset$ for some perfect set $P$; no such $H$ can then be a Burstin basis. It is also not hard to construct a Hamel basis $H$ which contains a perfect set (see e.g., [11, Example 1, p. 477f.]); no such $H$ can be a Burstin basis either.

**Theorem 2.6. (Burstin, 1916)** Assume $V \models \text{ZFC}$. Then there is a Burstin basis $B$.

**Proof.** Suppose $\{P_i : i \leq 2^{\aleph_0}\}$ is a enumeration of all perfect subsets of $\mathbb{R}$. By transfinite recursion we are going to define a set $\{b_\alpha : \alpha < 2^{\aleph_0}\} \subseteq \mathbb{R}$ such that

(i) $b_\alpha \in P_\alpha$ for every $\alpha < 2^{\aleph_0}$

(ii) for every $\beta < 2^{\aleph_0}$, the set $\{b_\alpha : \alpha < \beta\}$ is linearly independent
Suppose that $\beta < 2^{\aleph_0}$ and we already have defined the collection $\{b_\alpha : \alpha < \beta\}$ satisfying (i) and (ii) above.

Consider the set $\text{span}\{b_\alpha : \alpha < \beta\}$. Note that
$$|\text{span}\{b_\alpha : \alpha < \beta\}| \leq |\beta| + \omega < 2^{\aleph_0}$$

Thus, $P_{\beta} \setminus \text{span}\{b_\alpha : \alpha < \beta\} \neq \emptyset$ and we may pick an element $b_\beta$ from this set.

According to this procedure, we have constructed a linearly independent family $\{b_\alpha : \alpha < 2^{\aleph_0}\}$ satisfying (i). We can extend this family to a maximal one, call it $B$, and in this way, $B$ will be a Hamel basis over $\mathbb{R}$.

By construction, $B$ intersects every perfect subset of $\mathbb{R}$, so $B$ is in fact a Burstin basis. □

2.2. The Marczewski ideal and new generic reals. Before the appearance of the forcing technique, in 1935 E. Marczewski introduced the $\sigma$-ideal $s^0$. This ideal is related to random forcing in much the same way that Cohen forcing is related with the ideal of meager subsets of $\mathbb{R}$ and Random forcing is related with the ideal of Lebesgue null subsets of $\mathbb{R}$.

**Definition 2.7.** (Marczewski, 1935) A set $X \subseteq \omega^2$ is in $s^0$ if and only if for every perfect tree $T \subseteq \omega^2$, there is a perfect subtree $S \subseteq T$ with $|S| \cap X = \emptyset$.

It is easy to see that $s^0$ is an ideal which does not contain any perfect set. Furthermore, any subset $X$ of the reals with $|X| < 2^{\aleph_0}$ is in the Marczewski ideal, as well as every universal measure zero set and every perfectly meager set$^1$. However, $s^0$ contains sets of size continuum (cf. [18, Theorem 5.10]). Moreover, by a “fusion” argument we can see that $s^0$ is a $\sigma$-ideal, i.e. closed under countable unions.

**Remark 2.8.** We say that $X \subseteq \omega^2$ is $s$-measurable if for each $T \in S$ there is $S \subseteq T$ such that either $|S| \cap X = \emptyset$ or $|S| \subseteq X$. Note that the algebra of the $s$-measurable sets modulo the ideal $s^0$ corresponds, in fact, to Sacks forcing.

**Definition 2.9.** Suppose that $M \subseteq N$ are models of ZFC. We say that the pair $(M, N)$ satisfies countable covering for reals if for every $A \subseteq \omega^{2M}$, $A \in N$, such that $A$ is countable in $N$, there is a set $B \subseteq \omega^{2M}$, $B \in M$, such that $A \subseteq B$ and $B$ is countable in $M$.

In the 1960’s, K. Prikry asked whether the existence of a non constructible real implies the existence of a perfect set of non constructible reals (cf. [16]). In order to find a solution to Prikry’s problem, Marcia J. Groszek and Theodore A. Slaman have shown the following result in [10, Theorem 2.4]$^2$:

**Theorem 2.10.** Suppose that $M \subseteq N$ are models of ZFC such that $(M, N)$ satisfies countable covering for reals. Then every perfect set $P \subseteq \omega^{2N}$ in $N$ has an element which is not in $M$.

In [10, §1], the authors state without proof that the conclusion in 2.10 can be strengthened to: for every perfect set $P \subseteq \omega^{2N}$ in $N$ there is a perfect set $P^* \subseteq P$ in $N$ such that $P^* \cap M = \emptyset$, which is equivalent to saying that $\omega^{2M} \in s^0_N (s^0_N$ being $s^0$ of $N$). In what follows we present a proof of this strengthened version of [10, Theorem 2.4].

**Theorem 2.11.** (Groszek-Slaman) Let $W \subseteq V$ be an inner model such that $W \models \text{CH}$. If $\omega^V \setminus \omega^W \neq \emptyset$ holds, we have
$$V \models \omega^W \in s^0$$

**Proof.** We may assume that $\omega^W = \omega^V$, as otherwise $W$ has only countably many reals and the result is trivial.

**Claim 1.** The pair $(W, V)$ satisfies countable covering for reals.

**Proof.** Suppose that $A \in V$ is a countable set such that $A \subseteq \omega^V$. Since $\omega^W = \omega^V$ and $W \models \text{CH}$ we can take a well-ordering of $\omega^W$ in $W$ of length $\omega_1$. Then, there is some $\alpha < \omega^W$ such that $A \subseteq \{a_i : i < \alpha\}$ where $\{a_i : i < \omega^M\}$ is an enumeration of $\omega^W$ according with the prefixed well-ordering. Therefore, $B = \{a_i : i < \alpha\} \in W$ is countable in $W$ and covers $A$. □

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$^1$A set $N^* \subseteq \omega^2$ has universal measure zero if for every measure $\mu$ defined on the Borel sets of $\omega^2$, there is $B$ a $\mu$-null Borel set such that $N^* \subseteq B$. Analogously, we say that $M^* \subseteq \omega^2$ is perfectly meager if for every perfect tree $T \subseteq \omega^2$, the set $M^* \cap \{T\}$ is meager relative to the topology of $\{T\}$.

$^2$See also [24, Theorem 3].
Let us fix a perfect set \( P \subseteq \omega_2 \) in \( V \). We aim to find a perfect subset \( \bar{P} \subseteq P \) such that \( \bar{P} \cap \omega_2^\omega = \emptyset \), or, equivalently \( \bar{P} \subseteq V \setminus W \). Let \( \bar{T} \subseteq \omega_2 \) be a perfect tree such that \( \bar{P} = [\bar{T}] \). We call \( x \in [\bar{T}] \) \( \textit{eventually trivial} \) if and only if there is some finite \( s \subseteq x \) such that \( x \) is the leftmost or the right most branch of \( T_s \). We consider two cases:

\textbf{Case 1.} Suppose that there is some \( s \in T \) such that if \( x \in [T_s] \) is not eventually trivial then \( x \in V \setminus W \). In this situation we have that \([T_s] \cap W\) is a subset of all eventually trivial elements of \([T_s]\); since the later set is countable there is some perfect set \( P \subseteq [T_s] \) consisting only of elements of \( V \setminus W \). But then \( \bar{P} \subseteq [T_s] \subseteq \bar{P} \).

\textbf{Case 2.} Now suppose that for all \( s \in T \), there is some \( x \in [T_s] \cap W \) which is not eventually trivial. For each \( s \in T \), pick \( x_s \in [T_s] \cap W \) not eventually trivial. Let \( \bar{g} = \langle g_n \mid n < \omega \rangle \in W \) be a sequence of elements of \( \omega_2 \cap W \) such that for all \( s \in T \), there is some \( n < \omega \) such that \( x_s = g_n \). \( \bar{g} \) exists by \( 1 \). We shall also assume that \( g_0 = x_{s_0} \) for some \( s_0 \in T \).

First, we prove \( P \cap (V \setminus W) \neq \emptyset \). Fix \( r \in (\omega_2 \cap V) \setminus W \) and construct \( x, y \in \omega_2 \) and subsequences \( \bar{g}^x, \bar{g}^y \) of \( \bar{g} \) such that \( x, y \in [\bar{T}] \) and

\[
\begin{align*}
(1^*) & \ r \leq_{\bar{T}} x, \bar{g}^x, \text{ and} \\
(2^*) & \ \bar{g}^x, \bar{g}^y \leq_{\bar{T}} x, y, \bar{g}.
\end{align*}
\]

Thus, we have that \( r \leq_{\bar{T}} x, y, \bar{g} \). But then, \( x \in V \setminus W \) or \( y \in V \setminus W \) and hence \( P \) will have a member in \( V \setminus W \). In a second round we shall actually produce a perfect \( \bar{P} \subseteq P \), \( \bar{P} \subseteq V \setminus W \).

We shall produce recursively strict initial segments of \( x \) given by \( \bar{g}^x = \langle g_n^x \mid n < \omega \rangle \), \( y \) and \( \bar{g}^y = (g_n^y \mid n < \omega) \) as follows.

We start with \( \bar{g}_0^x = g_0 = g_0^y \). We shall maintain inductively that \( m = m(n), k = k(n) \) are such that \( k \geq m \geq n \). Suppose we are given \( x \upharpoonright m(n), \bar{g}^x_n, y \upharpoonright k(n), \bar{g}^y_n \) such that

\[
\begin{align*}
(a) & \ x \upharpoonright m(n) \subseteq \bar{g}_n^x, \\
(b) & \ \bar{g}^x_n = x_s \text{ for some } s \in T, \\
(c) & \ y \upharpoonright k(n) \subseteq \bar{g}_n^y, \text{ and} \\
(d) & \ \bar{g}_n^y = x_{s'} \text{ for some } s' \in T.
\end{align*}
\]

For \( n = 0 \), we may just let \( m = 0 = k \) and then \( a \) through \( d \) will be satisfied.

Now say \( \bar{g}_n^x = \bar{g}_j \). Pick \( m' > m(n), k(n) \) such that \( \bar{g}_j \upharpoonright m' \neq \bar{g}_j \upharpoonright m' \) for all \( l < j \). By item \( b \), we may also assume that \( \bar{g}_n^x \upharpoonright m' \) is a splitting node in \( T \) and \( \bar{g}_n^x \upharpoonright (m' \setminus r(n)) \).

Then set

\[ x \upharpoonright m' + 1 = \bar{g}_n^x \upharpoonright m'-r(n) \]

and pick \( \bar{g}_{n+1}^x \) such that for \( s'' := x \upharpoonright m' + 1 \in T \) we have \( \bar{g}_{n+1}^x = x_{s''} \) and \( x \upharpoonright m' + 1 \subseteq x_{s''} \).

Say \( \bar{g}_{n+1}^x = \bar{g}_j \). Pick \( k' > m' + 1 \) such that \( \bar{g}_j \upharpoonright k' \neq \bar{g}_j \upharpoonright k' \) for all \( k < i \). By \( d \), we may also assume that \( \bar{g}_n^y \upharpoonright k' \) is a splitting node.

Then set

\[ y \upharpoonright k' + 1 = \bar{g}_n^y \upharpoonright k'-r(n) \]

and pick \( \bar{g}_{n+1}^y \) such that for \( s''' := y \upharpoonright k' + 1 \in T \) we have \( \bar{g}_{n+1}^y = x_{s'''} \) and \( y \upharpoonright k' + 1 \subseteq x_{s'''}. \)

Then, we are back to \( a \) through \( d \) with \( x \upharpoonright m+1, \bar{g}_{n+1}^x, s', y \upharpoonright k'+1, \bar{g}_{n+1}^y, s''' \) replacing \( x \upharpoonright m \), \( \bar{g}_n^x, s, y \upharpoonright k', \bar{g}_n^y, s' \), respectively.

This finishes the construction of \( x, \bar{g}^x, y, \bar{g}^y \). For every \( n < \omega \), \( r(n) = 1 - \bar{g}_n^x \upharpoonright (m'(n)) \), where \( m' \) is maximal such that \( x \upharpoonright m' = \bar{g}_n^x \upharpoonright m' \). This shows \( 1^* \) on p. 4.

To show \( 2^* \) on p. 4, notice that \( \bar{g}_n^y = \bar{g}_j \) for the least \( j \) such that \( y \upharpoonright m' = \bar{g}_j \upharpoonright m' \), where \( m' \) is maximal with \( x \upharpoonright m' = \bar{g}_j^x \upharpoonright m' \); also, \( \bar{g}_{n+1}^y = \bar{g}_j \) for the least \( i \) such that \( x \upharpoonright k' = \bar{g}_i \upharpoonright k' \), where \( k' \) is maximal with \( y \upharpoonright k' = \bar{g}_i^y \upharpoonright k' \).

We have shown that \( P \cap (V \setminus W) \neq \emptyset \).

Let us now prove the full theorem, varying the argument above. By recursion on the length of \( s \in \omega \setminus \omega_2 \) we construct \( x, y \in T \) and subsequences \( \bar{g}^x, \bar{g}^y \) of \( \bar{g} \) such that

\[
\begin{align*}
(1) & \ x^{s_0} \subseteq x, y^{s_0} \subseteq y \text{ and } y^{s_0} \cap y^{s_0} = \emptyset, \\
(2) & \ x^{s_0} \subseteq x^s, y^s \subseteq y^s \text{ for } s \subseteq s', \\
(3) & \ \bar{g}^x \upharpoonright s, \bar{g}^y \upharpoonright s \text{ are sequences of elements from } \bar{g}, \text{ in fact from } \{x_s : s \in T\}, \text{ of length } \text{lh}(s) + 1; \\
(4) & \ \bar{g}^x \upharpoonright s, \bar{g}^y \upharpoonright s \text{ for } s \subseteq s'.
\end{align*}
\]
(5) if \( z \in \omega^2 \) we write \( v^z = \bigcup \{ v^s : s \subseteq z \} \), where \( v \in \{ x, y \} \), we have also

(6) for all \( z, z' \in \omega^2 \): \( \bar{g}^x_z = \bigcup \{ \bar{g}^x_s : s \subseteq z \} \), \( \bar{g}^y_z = \bigcup \{ \bar{g}^y_s : s \subseteq z \} \)

(6-a) \( r \leq_T x^z, \bar{g}^x_z \), and

(6-b) \( \bar{g}^x_z, \bar{g}^y_z \leq_T x^z, y^z, \bar{g}^z \).

In particular, \( r \leq_T x^z, y^z, \bar{g}^z \) for all \( z, z' \in \omega^2 \). But then \( \{ x^z : z \in \omega^2 \} \subseteq V \setminus W \) or \( \{ y^z : z \in \omega^2 \} \subseteq V \setminus W \), because if \( x^z, y^z \in W \) we would have \( r \in W \). By (1), both \( \{ x^z : z \in \omega^2 \} \) and \( \{ y^z : z \in \omega^2 \} \) are perfect, so one of them is a perfect set \( P \subseteq P \) consisting entirely of reals in \( V \setminus W \), as desired.

The construction of \( x^z, \bar{g}^x_z, y^z, \bar{g}^y_z \) is basically as above, just building in (1). Again, we start out with \( x^0 = \emptyset = y^0, \bar{g}^x_0 = \bar{g}^y_0 \). Suppose we already have defined \( x^s, \bar{g}^x_s, y^s, \bar{g}^y_s \) for all \( s \in \omega^2 \) of length \( \leq n \).

Fix \( s \) of length \( n \), and let us define \( x^s, \bar{g}^x_s, x^s \setminus 1, \bar{g}^x_s \setminus 1 \). Let \( j = \max \{ i : \bar{g}^y_i = \bar{g}_i, \text{lh}(t) = \} \), and pick \( m' > \max \{ \text{lh}(x^i), \text{lh}(y^i) : \text{lh}(t) = n \} \) such that \( \bar{g}^y_i \upharpoonright m' \neq \bar{g}^y_i \upharpoonright m' \) for all \( l, l' \leq j, l \neq l' \) and \( m_1 > m_0 \geq m \) are both such that \( \bar{g}^x_i \upharpoonright m_0, \bar{g}^x_i \upharpoonright m_1 \) are splitting nodes in \( T \) and \( \bar{g}^x_i \upharpoonright (m_0) \neq \text{r}(n) \neq \bar{g}^x_i \upharpoonright (m_1) \).

Then set

\[
\begin{align*}
x^s &= \bar{g}^x_i \upharpoonright (m_0) \text{r}(n) \\
x^s &= \bar{g}^x_i \upharpoonright (m_1) \text{r}(n)
\end{align*}
\]

and pick \( \bar{g}^x_{n+1}, \bar{g}^x_{n+1} \) such that there are \( s'' \subseteq s'' \subseteq T \) with \( x^s \subseteq x^s \subseteq x^s = \bar{g}^x_{n+1}, x^s \subseteq x^s = \bar{g}^x_{n+1} \).

This defines all \( x^t, \bar{g}^x_t, \text{lh}(t) = n + 1 \). Again, fix \( s \) of length \( n \), and let us define \( y^s, \bar{g}^y_{n+1}, y^s \setminus 1, \bar{g}^y_s \setminus 1 \).

Let \( i = \max \{ i : \bar{g}^y_i = \bar{g}_i, \text{lh}(t) = n + 1 \} \) and pick \( k' > \max \{ \text{lh}(y^i), \text{lh}(x^i) : \text{lh}(t) = n \} \) such that \( \bar{g}^y_i \upharpoonright k' \neq \bar{g}^y_i \upharpoonright k' \) for all \( l, l' \leq j, l \neq l' \), and \( k_1 > k_0 \geq k \) are both such that \( \bar{g}^y_i \upharpoonright m, \bar{g}^y_i \upharpoonright m \) are splitting nodes in \( T \).

Then set

\[
\begin{align*}
y^s &= \bar{g}^y_i \upharpoonright (k_0 - (1 - \bar{g}^y_i(k_0))) \\
y^s &= \bar{g}^y_i \upharpoonright (k_1 - (1 - \bar{g}^y_i(k_1)))
\end{align*}
\]

and pick \( \bar{g}^y_{n+1}, \bar{g}^y_{n+1} \) such that there are \( s'' \subseteq s'' \subseteq T \) with \( y^s \subseteq y^s \subseteq y^s = \bar{g}^y_{n+1}, y^s \subseteq y^s = \bar{g}^y_{n+1} \).

This defines all \( y^t, \bar{g}^y_t \) where \( \text{lh}(t) = n + 1 \). This finishes the construction.

The proofs of items (6-a) and (6-b) on p. 5 are like the proofs of (1*) and (2*) on p. 4: for each \( n, r(n) = x^z(m) \), where \( m \) is largest such that \( x^z \upharpoonright m = \bar{g}^x_z \upharpoonright m \). This shows (6-a). Moreover, \( \bar{g}^y_i = \bar{g}_i \) for the least \( j \) such that \( y^i \upharpoonright m' = \bar{g}_j \upharpoonright m' \) where \( m' \) is maximal with \( x^z \upharpoonright m' = \bar{g}^x_z \upharpoonright m' \). Also, \( \bar{g}^x_{n+1} = \bar{g}_i \) for the least \( i \) such that \( x^z \upharpoonright k' = \bar{g}_i \upharpoonright k' \) where \( k' \) is maximal with \( y^s \upharpoonright k' = \bar{g}^y_i \upharpoonright k' \).

This shows item (6-b).

### 2.3. Side-by-side product of Sacks forcing and its properties

This section recapitulates well-known facts about Sacks forcing.

**Definition 2.12.** Sacks forcing \( S \) is defined in the following way.

\[ S = \{ T : \text{a perfect tree on } 2 \} \]

For \( S, T \in S \) we stipulate \( S \leq T \) if and only if \( S \subseteq T \). If \( S \in S \) and \( p \in S \), we define the subtree \( S_p = \{ t \in S : t \subseteq p \text{ or } p \subseteq t \} \).

A node \( p \in T \) is called a splitting node if \( p^-0 \cdot p^-1 \). The set of splitting points of \( T \) is denoted by \( \text{split}(T) \). We define \( \text{stem}(T) \) as the unique element in \( \text{split}(T) \) comparable with any other node of \( T \). A node \( p \in T \) is in \( \text{split}_n(T) \) if \( p \in \text{split}(T) \) and \( p \) has exactly \( n \) predecessors in \( \text{split}(T) \). In particular, \( \text{split}_0(T) = \{ \text{stem}(T) \} \). Notice that for \( T \in S, |\text{split}_n(T)| = 2^n \).

For every \( n \in \omega \) and \( S \in S \) we write \( \text{Lev}_n(S) = \{ t \in S : 3 \text{ is } \epsilon \text{ in } \text{split}_n(S) t \subseteq s \} \), and for \( S, T \in S \) we stipulate \( S \leq_T T \) if and only if \( S \subseteq T \) and \( \text{Lev}_n(S) = \text{Lev}_n(T) \).
Definition 2.13. If $\kappa$ is an ordinal and $X \subseteq \kappa$ (e.g., $X = \kappa$), let $S_X$ be the $\kappa$-side-by-side countable support product of Sacks forcing, i.e., $S_X$ is the set of all functions $p : X \to S$ such that $\text{supp}(p) := \{ \alpha \in X : p(\alpha) \neq 1\}$ is at most countable. If $p, q \in S_X$, we stipulate

$$p \leq q \iff \forall \alpha < \kappa (p(\alpha) \leq q(\alpha)).$$

This implies in particular that $\text{supp}(q) \subseteq \text{supp}(p)$.

For now we are only interested in the case that $X = \kappa$ is a cardinal, the more general case will only show up in the proof of Lemma 5.1. If $g$ is $S_\kappa$-generic over $V$, and $\alpha < \kappa$, then

$$s_\alpha = \bigcup_{p \in g} \text{stem} (p(\alpha))$$

is a real which is $S$-generic over $V$. Therefore forcing with $S_\kappa$ adds $\kappa$-many Sacks reals which are independent over the ground model, i.e. for any $A \subseteq \kappa$ in $V$,

$$\omega_2^V[|\{ \alpha < \kappa : \alpha \in A \}|] = \omega_2^V.$$

The product forcing $S_\kappa$ has properties very similar to those of $S$. By defining a suitable notion of levels and fusion, it can be shown that $S_\kappa$ satisfies the Baumgartner Axiom $\Lambda$ and therefore it is proper and does not collapse $\omega_1$. For our purposes, the most remarkable property of $S_\kappa$ is that it inherits from $S$ also the so called Sacks property.

Definition 2.14. Let $g : \omega \to \omega$ be an increasing function. We say $F : \omega \to [\omega]^{<\omega}$ is a $g$-slalom if $|F(n)| \leq g(n)$ for all $n \in \omega$.

Definition 2.15. Let $P$ be a forcing notion and suppose $g \in \omega^{<\omega} \cap V$ is an increasing function. We say that $P$ has the Sacks property if whenever $G$ is $P$-generic over $V$, for every $f \in \omega^{<\omega} \cap V[G]$ there exists a $g$-slalom $F \in V$, such that $V[G] \models \forall n(f(n) \in F(n))$.

Lemma 2.16. Let $\kappa$ be a cardinal. Suppose that $p \in S_\kappa$ and for $\theta \geq \kappa$ let $X \subseteq \theta \cap \kappa$ be a countable elementary substructure with $p, S_\kappa \in X$. Let $\{ \tau_n : n < \omega \} \subseteq X$ (possibly but not necessarily $\{ \tau_n : n < \omega \} \subseteq X$). Then, there is some $q \leq p$ and some $F : \omega \to [X \cap \theta]^{<\omega}$, $F \in V$, such that for all $n < \omega$:

1. $q \models \tau_n \in (F(n))^\theta$,
2. $|F(n)| \leq 2^{\omega_1}$, and
3. $F(n) \subseteq X$.

Proof. Suppose that $\alpha = X \cap \omega_1$. Since $\text{supp}(p)$ is an element of $X$, $\text{supp}(p)$ also is a subset of $X$. Let $e : \omega \leftrightarrow \alpha$ be a fixed bijection. We aim to produce a sequence $\langle p_n : n < \omega \rangle$ such that $p_0 = p$ and $p_{n+1} \leq p_n, p_n \in X$ for all $n \in \omega$. In this way, we also will have $\text{supp}(p_n) \subseteq \alpha$ for every $n \in \omega$. Suppose $p_n$ is already defined. Working in $X$, we shall produce $p_{n+1} \leq p_n$ such that for all $k < n$,

(i) $p_{n+1}(e(k)) \leq p_n(e(k))$, and
(ii) there is some $a_n \in [X \cap \theta)^{<\omega}$ such that $p_{n+1} \models \tau_k \in a_n$.

The condition $q$ defined as $q(e(k)) = \bigcap_{n < \omega} p_n(e(k))$ for each $k < \omega$ and the function $F$ given by $F(n) = a_n$ satisfy the conclusion of our lemma.

We may produce $p_{n+1}$ by means of some sequence $\langle q_m : m \leq 2^{n} \rangle$ defined as follows inside $X$. Let $q_0 = p_n$. Fix some enumeration $\langle s_m : m \leq 2^{n} \rangle$ of all tuples $\bar{s} = (s_{e(0)}, \ldots, s_{e(n-1)})$ such that $s_{e(k)} \in \text{Lev}_n(p_n(e(k)))$ for all $k < n$.

Suppose $m < 2^{n}$ and $q_m$ has been chosen. We aim to define $q_{m+1}$. Write $\bar{s}_m = (s_{e(0)}, \ldots, s_{e(n-1)})$.

For each $k < n$, let $m_k \leq m$ be maximal such that $s_{e(k)} \in q_{m_k}$, and define $\bar{q}$ in such a way that $\text{supp}(\bar{q}) = \text{supp}(q_m)$ and

$$\bar{q}(\xi) = \begin{cases} (q_{m_k}(e(k)))_{\bar{s}_{m_k}} & \text{if } \xi = e(k) \\ q_m(\xi) & \text{if } \xi \neq e(k) \end{cases}$$

for all $k < n$.

Let $q_{m+1} \leq \bar{q}$ be a condition deciding $\tau_n$, and put the $\xi \in X \cap \theta$ with $q_{m+1} \models \tau_n = \xi$ into $a_n$. This defines $\langle q_m : m \leq 2^{n} \rangle$. Let us define $p_{n+1}$ as follows. For each $k < n$ and $s \in \text{Lev}_n(p_n(e(k)))$, let $m_k, s \leq m$ be maximal such that $s \in q_{m_k}(e(k))$. Then $(q_{m_k, s}(e(k)))_{s} = q_{m_k, s}(e(k))$. Let $p_{n+1}$ have the same support as $q_{2^{n}}$. and

\(^3\)For the details, see [8, §6]
\(^4\)For equivalent definitions of Sacks property, the reader can see [9, Fact 6.35].
Given any increasing function \( g : \omega \rightarrow \omega \), there exists some \( n \leq p \) with \( \text{supp}(q) \subseteq \omega \) such that
\[
q \models \forall \tau(n) \in F(n)^\nu.
\]
Therefore \( S_n \) has the Sacks property.

**Corollary 2.18.** For every cardinal \( \kappa \), the countable support product \( S_n \) is a proper forcing. If \( g \) is \( S_n \)-generic over \( V \) and if \( x \in V[\omega] \), then there exists some \( \tau \in V[S_n] \) which is countable in \( V \) such that \( x = \tau[\theta] \).

**Proof.** First part: Let \( p \in S_n \). Suppose that \( \theta \models S_n \) and let \( N \prec H_\theta \) be a countable substructure with \( S_n \subseteq N, p \in N \).

Let \( \{ \tau_n : n \in \omega \} \in V \) be an enumeration of all \( S_n \)-names for ordinals in \( N \). By lemma 2.16, there exists some \( q \leq p \) and some \( F : \omega \rightarrow [\omega]^{<\omega} \) in \( V \) such that for all \( n \in \omega \),
\[
q \models \tau_n \in F(n)^\nu \subseteq \bar{N}.
\]
I.e., \( q \models \hat{\alpha} \in \bar{N} \cap \text{OR} \) for every \( S_n \)-name \( \hat{\alpha} \in N \) for an ordinal. This implies that \( S_n \) is proper.

Second part: Let \( x = \sigma[\theta] \), where \( \sigma = \bigcup \{(n, h)^\nu : (n, h) \in \omega \times 2 \} \in V[S_n] \) and for each \( (n, h) \in \omega \times 2 \), \( A_n \) is a maximal antichain of \( p \in S_n \) such that \( p \models \sigma(\hat{\alpha}) = h \). In \( V[\omega] \), for each \( n < \omega \) there is some unique \( h = h_n \in 2 \) and \( p = p_n \in S_n \) such that \( p \in A_n \cap g \). Let \( X = \{ p_n : n < \omega \} \), where \( X \in V \) is countable in \( V \). Then \( \tau = \bigcup \{(n, h)^\nu \} \times (A_n \cap X) : (n, h) \in \omega \times 2 \) is as desired.

[12] gives more information on how reals in \( V[S_n] \) may be represented.

3. **Lusin and Sierpiński sets in the Sacks model**

Let \( S_{\omega_1} \) be the countable support product of \( \omega_1 \)-many copies of Sacks forcing. From the fact that \( S_{\omega_1} \) has the Sacks property we shall show that in the generic extension obtained after forcing with \( S_{\omega_1} \), the Lusin and Sierpiński sets in the ground model are also Lusin and Sierpiński sets in the generic extension.

We use the following result.

**Lemma 3.1.** Let \( N \subseteq \omega \) be null and let \( \{ \varepsilon_n : n \in \omega \} \) be a sequence of positive reals. Then there is a sequence \( C_n \subseteq \omega_2 : n \in \omega \) of finite unions of basic open sets such that

(i) for all \( n < \omega \), \( \mu(C_n) < \varepsilon_n \) and

(ii) \( N \subseteq \bigcup_{n \in \omega} C_n \)

**Proof.** Since \( N \) is null, there is a collection of basic open sets \( \{ O_n : n \in \omega \} \) such that \( N \subseteq \bigcup \{ O_n : n \in \omega \} \) and \( \mu(\bigcup_{n \in \omega} O_n) < \varepsilon_0 \).

Then let \( k(n) = \min \{ m : \mu(\bigcup_{i \leq m} O_i) < \varepsilon_n \} \). Without loss of generality, we can assume that the sequence \( \{ \varepsilon_n : n \in \omega \} \) is decreasing, so \( k \) is monotone. We have \( k(0) = 0 \). Then for each \( n \) set
\[
C_n = \bigcup \{ O_i : k(n) \leq i < k(n + 1) \}.
\]
It is straightforward to see that the collection \( \{ C_n : n \in \omega \} \) satisfies (i) and (ii).

**Lemma 3.2.** Let \( P \) be a forcing notion satisfying the Sacks property and let \( G \) be a \( P \)-generic filter over \( V \). Then:
(1) For every null set $N \subseteq \omega$ in $V[G]$ there is a $G_{3}\text{-null set }\bar{N} \subseteq \omega$ coded in $V$ such that $\bar{N} \subseteq N$.

(2) Similarly, for every meager set $M \subseteq \omega$ in $V[G]$, there is a meager set $\bar{M} \subseteq \omega$ coded in $V$ such that $\bar{M} \subseteq M$.

Proof. We prove the statement (1). Let $\varepsilon > 0$. First, let us fix in $V$ an enumeration $\{C_n : n < \omega\}$ of all finite unions of basic open sets in $^{\omega}2$.

Let $N \subseteq \omega$ be a null set in $V[G]$. By 3.1 there is a function $f : \omega \to \omega$ in $V[G]$ such that

$$N \subseteq \bigcup_{n \in \omega} C_{f(n)}$$

and

$$\mu(C_{f(n)}) \leq \frac{\varepsilon}{2^{2n+1}} : n \in \omega$$

Since $\mathbb{P}$ has the Sacks property, there is a $2^{\omega}$-slalom $F : \omega \to [\omega]^{<\omega}$ such that for every $n \in \omega$, $f(n) \in F(n)$. Set

$$\bar{N} = \bigcup_{n \in \omega} \bigcup \{C_k : k \in F(n) \text{ and } \mu(F_k) \leq \frac{\varepsilon}{2^{2n+1}}\}$$

Since only ground model parameters are used in the definition of $\bar{N}$, $\bar{N}$ is an open set coded in the ground model. Note also that $N \subseteq \bar{N}$. Now since $|F(n)| \leq 2^n$ for each $n \in \omega$ it follows that

$$\mu\left(\bigcup \{C_k : k \in F(n) \text{ and } \mu(F_k) \leq \frac{\varepsilon}{2^{2n+1}}\}\right) \leq 2^n \cdot \frac{\varepsilon}{2^{2n+1}} = \frac{\varepsilon}{2^{n+1}}$$

Therefore $\mu(\bar{N}) \leq \sum_{n \in \omega} \frac{\varepsilon}{2^{2n+1}} = \varepsilon$. Since $\varepsilon$ was taken arbitrarily, it follows that $\bar{N}$ is a null subset coded in $V$. $\square$

Remark 3.3. Let $\mathcal{N}$ and $\mathcal{M}$ stand for the null and meager ideals over $^{\omega}\omega$ respectively. Since $\text{add}(\mathcal{N}) \leq \text{add}(\mathcal{M})$ and $\text{cof}(\mathcal{M}) \leq \text{cof}(\mathcal{N})$, if a forcing notion $\mathbb{P}$ satisfies item (1) above, then $\mathbb{P}$ satisfies (2) as well.

Corollary 3.4. If $\mathbb{P}$ has the Sacks property, then $\mathbb{P}$ preserves Luzin and Sierpiński sets.

Proof. Suppose that there is a Luzin set $\Lambda$ in $V$ and let $G$ be $\mathbb{P}$-generic over $V$. First, observe that, since $\omega_1$ is not collapsed by $\mathbb{P}$, $\Lambda$ remains to be non countable in $V[G]$. Now, let $M$ be a (Borel code for a) meager set in $V[G]$. In view of Lemma 3.2, there is a (Borel code for a) $G_{3}\text{-null set }\bar{M}$ in $V$ such that $V[G] \models M \subseteq \bar{M}$. Thus, since $V \models |\Lambda \cap M| \leq \omega$, it follows that $V[G] \models |\Lambda \cap M| \leq \omega$. Hence,

$$V[G] \models \bar{M}$$

is a Luzin set.

The proof of the preservation of Sierpiński sets is completely analogous. $\square$

4. ADDING GENERICALLY A BURSTIN BASIS

We now define a partial order $\mathbb{P}_B$ generically adding a Burstin basis.

Definition 4.1. We say $p \in \mathbb{P}_B$ if and only if there exists $x \in \mathbb{R}$ such that

(1) $p \in L[x]$, and

(2) $L[x] \models "p \text{ is a Burstin basis}."

We stipulate $p \leq_B q$ iff $p \geq q$.

Notice that by Theorem 2.6 we have $\mathbb{P}_B \neq \emptyset$.

If $\mathbb{R} \cap V \subseteq L[x]$ for some real $x$, then $\mathbb{P}_B$ has a dense set of atoms. We are interested in situations where the set of all reals is not constructible from a single real. Variants of $\mathbb{P}_B$ will be discussed at the end of this chapter.

The following is an immediate consequence of Theorem 2.11.

Lemma 4.2. Let $x$, $y$ be reals such that $y \not\in L[x]$, and let $\{z_0, z_1, \ldots\} \in L[x, y] \cap [\mathbb{R}]^\omega$. Then

$$\text{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \ldots\}) \in (s^0)^{L[x, y]}$$

i.e., for every perfect set $P$ in $L[x, y]$ there is a perfect set $\bar{P} \subseteq P$, $\bar{P} \in L[x, y]$ such that

$$P \cap \text{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \ldots\}) = \emptyset$$

Proof. We may assume that if $z \in \text{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \ldots\})$, then $z \in (\mathbb{R} \cap L[x]) + z_n$, for some $n < \omega$. Given $P \in L[x, y]$ a perfect set, we shall construct recursively a sequence $T_0 \supseteq T_1 \supseteq \cdots T_n \supseteq T_{n+1} \supseteq \cdots$ of perfect trees, such that
(1) $P = [T_0]$,
(2) $\text{Lev}_n(T_{n+1}) = \text{Lev}_n(T_n)$ and,
(3) $[T_{n+1}] \cap (\mathbb{R} \cap L[x] + z_n) = \emptyset$.

Let $T_0$ be the perfect tree such that $P = [T_0]$. By Theorem 2.11 we have that $L[x, y] \models \omega 2 \cap L[x] \in \delta^0\omega$. Since $P - z_0 = \{x - z_0 : x \in P\}$ is also perfect in $L[x, y]$, there is some $\bar{P} \subseteq P$ countable, $\bar{P} \in L[x, y]$, such that $\bar{P} \subseteq L[x, y] \setminus L[x]$. Therefore $P' := \bar{P} + z_0 \subseteq P$ is perfect and if $u \in \bar{P}$ (equivalently, $u + z_0 \in \bar{P} + z_0 = P'$), then $u \not\in L[x]$, so $u + z_0 \not\in (\mathbb{R} \cap L[x]) + z_0$. Thus, $P' \cap (\mathbb{R} \cap L[x] + z_0) = \emptyset$. Take then $T_1$ as the perfect tree such that $P' = [T_1]$.

Now suppose that we have constructed $T_0, T_1, \ldots, T_n$ satisfying (1)-(3) above. For any $s \in \text{Lev}_n(T_n)$ let us consider the subtree $(T_n)_s$ of $T_n$. By the argument from the previous paragraph, there is some perfect set $P_{n, s} \subseteq [T_n]$ such that $P_{n, s} \cap (\mathbb{R} \cap L[x] + z_n) = \emptyset$. Let

$$P_{n+1} = \bigcup \{P_{n, s} : s \in \text{Lev}_n(T_n)\}.$$  

Notice that $P_{n+1} \cap (\mathbb{R} \cap L[x] + z_n) = \emptyset$, hence by taking $T_{n+1}$ as the perfect tree such that $P_{n+1} = [T_{n+1}]$ condition (3) holds. Also, by construction, $\text{Lev}_n(T_{n+1}) = \text{Lev}_n(T_n)$.

Now, set $T = \bigcap\{T_n : n \in \omega\}$. By condition (2), we have that $T$ is a perfect tree. Thus $\bar{P} := [T]$ is a perfect set such that $\bar{P} \subseteq P$ and $\text{span}((\mathbb{R} \cap L[x]) \cup \{z_0, z_1, \ldots\}) = \emptyset$, as required.

**Lemma 4.3.** Let $b \in L[x]$ be linearly independent, $x \in \mathbb{R}$. Let $y \in \mathbb{R} \setminus L[x]$. There is then some $p \supset b, p \subseteq L[x, y]$ such that $L[x, y] \models \text{"p is a Burstin basis".}$

**Proof.** Let $(P_i : i \in \omega_1)$ be an enumeration of all perfect sets of $L[x, y]$. Working in $L[x, y]$ we define recursively $(b_i : i < \omega_1)$ as follows. Let $(y_i : i < \omega_1) \subseteq L[x, y]$ enumerate the reals of $L[x, y]$. Given $(y_i : j < i)$, we will have that $b_i = \bigcup\{b_j : j < i\}$ is at most countable. By Lemma 4.2 there is some $\bar{P} \subseteq P_i$ perfect such that $\text{span}((\mathbb{R} \cap L[x]) \cup b) = \emptyset$. Pick $x \in \bar{P}$ and set

$$b_i = \begin{cases} 
\bar{b} \cup \{x\} & \text{if } y_i \in \text{span}((\mathbb{R} \cap L[x]) \cup b \cup \{x\}) \\
\bar{b} \cup \{x, y_i\} & \text{otherwise}
\end{cases}$$

Finally, if $c \in L[x]$ is such that $c \ni b$ and $L[x] \models \text{"c is a Hamel basis"}$, take

$$p := c \cup \bigcup\{b_i : i < \omega_1\}$$

By construction $p$ is a Hamel basis for $L[x, y]$. Moreover for each $i < \omega_1$, $b_i \subseteq p$ hence $P_i \cap p = \emptyset$. This shows that $p$ is a Burstin basis in $L[x, y]$.

**Lemma 4.4.** If $p \in \mathbb{P}_B$, say $L[x, y] \models \text{"p is a Burstin basis,"}$ and if $y$ is a real not in $L[x]$, then there is some $q \leq_{\mathbb{P}_B} p$ such that $q$ is a Burstin basis in $\mathbb{R} \cap L[x, y]$.

Also, Lemma 4.3 shows that $\mathbb{P}_B$ is countably closed under favourable circumstances. What is more than enough for our purposes is the following. Hypothesis (1) of Lemma 4.5 is satisfied e.g. if $V$ is a forcing extension of $L$ via some proper forcing. Hypotheses (1) and (2) are certainly satisfied in $V = L[g]$, where $g$ is $S_{\omega_1}$-generic over $L$, cf. Corollary 2.18.

**Lemma 4.5.** Assume that

(1) for every countable set of ordinals there is a set $Y \supset X, Y \in L$, such that $Y$ is countable in $L$, and

(2) there is no real $x$ such that $\mathbb{R} \subseteq L[x]$.

Then $\mathbb{P}_B$ is $\omega$-closed. In particular, forcing with $\mathbb{P}_B$ does not add any new reals.

**Proof.** Consider a sequence $(p_n : n < \omega)$ of conditions in $\mathbb{P}_B$ such that $p_{n+1} \leq_{\mathbb{P}_B} p_n$ for all $n < \omega$. For each $n < \omega$, let $x_n \in \mathbb{R}$ be such that $p_n \in L[x_n]$ is a Burstin basis for $\mathbb{R} \cap L[x_n]$. Pick $z \in \mathbb{R}$ such that $x_n \in L[z]$ for all $n < \omega$.

**Claim.** There is some $x \in \mathbb{R}$ such that $\{p_n : n < \omega\} \in L[x]$.

To prove the claim, notice that $\{p_n : n < \omega\} \subseteq L[z]$. Let $F : \text{OR} \to L[z]$ be bijective and definable over $L[z]$, and let $X = \{\xi : 3n < \omega F(\xi) = p_n\}$. By hypothesis (1) there is some $Y \supset X$, $Y \in L$, and $Y$ is countable in $L$. Let $f : \omega \to Y$ be bijective, $f \in L$, and write $x^* = f^{-1}(X)$. Then
$x^* \subset \omega$ and $X = f^* x^* \in L[x^*]$. But then \{\(p_n : n < \omega\) \(\in L[z, x^*]\), and if $x \in \mathbb{R}$ is such that $L[z, x^*] \subset L[x]$, then $x$ verifies the Claim.

Now let $b = \bigcup \{p_n : n < \omega\}$, let $x$ be as in the Claim, and let us make use of hypothesis (2) to pick some $y \in \mathbb{R} \setminus L[x]$. We have that $b \in L[x]$, so that by Lemma 4.3 we can extend the linearly independent set $b$ to a Burstin basis $p$ over $L[y]$. Then, for every $n < \omega$ we have that $p \leq_{\mathbb{P}_n} p_n$, so $\mathbb{P}_B$ is $\omega$-closed. \hfill \Box

**Notation.** For $\vec{x}$, $\vec{y}$ two real vectors of the same length, let $\vec{x} \cdot \vec{y} := \sum_{i \in \text{lh}(x)} x_i y_i$.

**Remark 4.6.** We have that

\[ p \in \mathbb{P}_B \iff \exists x (L[x] \models \text{"} p \text{ is a Burstin basis"}) \]

\[ \iff \exists x \in [p]^{< \omega} \exists \vec{q} \in [Q]^{< \omega} (\forall y \in R^{|x|} \exists \vec{p}_y \in [p]^{< \omega} \exists \vec{q}_y \in [Q]^{< \omega} \psi(x, \vec{y}, \vec{q}_y, p) \]

\[ y = \vec{q}_y \cdot \vec{p}_y \land \forall \vec{z} \in [p]^{< \omega} \forall \vec{q} \in [Q]^{< \omega} (\vec{q} \cdot \vec{z} = 0 \rightarrow \vec{q} = \vec{0}) \land \]

\[ L[\vec{q} \cdot \vec{x}] \models \text{"} p \cap \vec{p} \neq \emptyset \text{ for every perfect set } \vec{P} \text{"} \]

Since the matrix of this formula is $\Pi^1_1$ we have that

\[ (1) \quad p \in \mathbb{P}_B \iff \exists x \in [p]^{< \omega} \exists \vec{q} \in [Q]^{< \omega} \psi(x, \vec{q}, p) \]

where $\psi$ is $\Pi^1_2$.

**Remark 4.7.** In what follows, we will call

\[ \vec{b} := \{(\vec{x}, p) : x \in p \in \mathbb{P}_B\} \]

the *canonical name* for the generic Burstin basis $b$. By the previous remark,

\[ (\vec{x}, p) \in \vec{b} \iff \exists x \in p \land \exists \vec{x} \in [p]^{< \omega} \exists \vec{q} \in [Q]^{< \omega} \psi(x, \vec{q}, p) \]

\[ \iff \theta(x, p), \]

where $\theta$ is $\Sigma^1_1$, and \("(\vec{x}, p) \in \vec{b}\" is absolute between transitive class sized models of set theory.

Let us discuss some variants of $\mathbb{P}_B$.

**Definition 4.8.** We say $p \in \mathbb{P}_H$ if and only if there exists $x \in \mathbb{R}$ such that

\[ (1) \quad p \in L[x], \text{ and} \]

\[ (2) \quad L[x] \models \text{"} p \text{ is a Hamel basis."} \]

We stipulate $p \leq_{\mathbb{P}_H} q$ iff $p \supseteq q$.

If $\mathbb{R} \cap V \subset L[x]$ for some real $x$, then like $\mathbb{P}_B$, $\mathbb{P}_H$ has a dense set of atoms. If there is no real $x$ with $\mathbb{R} \cap V \subset L[x]$, then the content of Lemma 4.3 is exactly that $\mathbb{P}_B$ is dense in $\mathbb{P}_H$, which implies that $\mathbb{P}_H$ and $\mathbb{P}_B$ will be forcing equivalent and forcing with $\mathbb{P}_H$ will not just add a Hamel basis but in fact a Burstin basis.

Hence if we aim to generically add a Hamel basis which in the extension contains a perfect set, then forcing with $\mathbb{P}_H$ won’t work. E.g., let $P \in L$ be a perfect set in $L$ which is also linearly independent, see [11, Example 1, p. 477f.]. If $M \supset L$ is any inner model, then let us write $P_M$ for $M$’s version of $P$. Then $P_M$ is perfect in $M$, $P_M \cap L = P$, and by $\Pi^1_1$ absoluteness, $P_M$ is linearly independent in $M$. We may then let $p \in \mathbb{P}_H$ if and only if there exists $x \in \mathbb{R}$ such that $p \in L[x]$, $p \supseteq P_M[x]$, and $L[x] \models \text{"} p \text{ is a Hamel basis"}; p \leq_{\mathbb{P}_H} q$ iff $p \supseteq q$. If $p \in \mathbb{P}_H \cap L[x] \subset L[y]$, $x, y \in \mathbb{R}$, then $p \cup P_L[y]$ is linearly independent by $\Pi^1_1$ absoluteness, so that $\mathbb{P}_H$ will generically add a Hamel basis which contains the version of $P$ of the model over which we force. The proof of Lemma 5.1 will go through for $\mathbb{P}_H$ instead of $\mathbb{P}_B$.

The following forcing, $\mathbb{Q}_H$, is the obvious candidate for adding a Hamel basis.

**Definition 4.9.** We say $p \in \mathbb{Q}_H$ if and only if $p$ is a countable linearly independent set of reals. We stipulate $p \leq_{\mathbb{Q}_H} q$ iff $p \supseteq q$.

It is clear that in $\omega_1$ is inaccessible to the reals (i.e., $\mathbb{R} \cap L[x]$ is countable for all reals $x$), then $\mathbb{Q}_H$ is dense in $\mathbb{P}_H$ (and hence also in $\mathbb{P}_B$), so that under this hypothesis all the three forcings are forcing equivalent with each other. On the other hand, in the absence of large cardinals, in contrast to $\mathbb{P}_B$ and $\mathbb{P}_H$ (see Lemma 5.1 below) forcing with $\mathbb{Q}_H$ over $L(\mathbb{R})$ will add a well-ordering of $\mathbb{R}$, see Corollary 4.11 below, so that $\mathbb{Q}_H$ would be the wrong candidate for forcing a Hamel
basis for our purposes. (The forcing \( Q_H \) would be called \( P_\psi \) in [13], where \( \psi \) expresses linear independence, see [13, Introduction].)

**Lemma 4.10.** Let \( \vec{x} = (x_\alpha : \alpha < \omega_1) \) be a sequence of pairwise distinct reals such that \( \{x_\alpha : \alpha < \omega_1\} \) is linearly independent. Let \( g \) be \( Q_H \)-generic over \( V \), and let \( h = \bigcup g \). Then inside \( L(\mathbb{R}, \vec{x}, h) \), there is a well-order of \( \mathbb{R} \) of order type \( \omega_1 \). In particular, \( L(\mathbb{R}, \vec{x}, h) \) is a model of ZFC.

**Proof.** Of course \( Q_H \) is \( \omega \)-closed, so that \( V \) and \( V[g] \) have the same reals. Hence \( h \) is a Hamel basis inside \( L(\mathbb{R}, h) \).

Let \( p \in Q_H \), and let \( x \subset \omega \). There is a countable limit ordinal \( \lambda \) such that \( p \cup \{x_{\lambda+n} : n < \omega\} \) is linearly independent. Let \( q = p \cup \{x_{\lambda+n} : n \in x\} \cup \{2 : x_{\lambda+n} : n \in \omega \setminus x\} \).

Then \( q \in Q_H \), \( q \leq Q_H p \), and \( x = \{n < \omega : x_{\lambda+n} \in q\} \).

In \( L(\mathbb{R}, \vec{x}, h) \) let us define \( f : \mathcal{P}(\omega) \to \omega_1 \) by \( f(x) = \) the least countable limit ordinal such that \( x = \{n < \omega : x_{\lambda+n} \in h\} \). Trivially, \( f \) is injective, and by the density argument from the previous section \( f \) is a well-defined total function. This shows that in \( L(\mathbb{R}, \vec{x}, h) \), there is a well-order of \( \mathbb{R} \) of order type \( \omega_1 \).

As there is a surjection \( F : \mathbb{R} \times \text{OR} \to L(\mathbb{R}, \vec{x}, h) \) which is \( \Sigma_1 \)-definable over \( L(\mathbb{R}, \vec{x}, h) \) from the parameters \( \mathbb{R}, \vec{x}, h \), the existence of a well-order of \( \mathbb{R} \) inside \( L(\mathbb{R}, \vec{x}, h) \) yields that \( L(\mathbb{R}, \vec{x}, h) \) is a model of ZFC.

**Corollary 4.11.** Assume that \( \omega_1 \) is not inaccessible the reals, let \( g \) be \( Q_H \)-generic over \( V \), and let \( h = \bigcup g \). Then in \( L(\mathbb{R}, h) \), there is a well-order of \( \mathbb{R} \) of order type \( \omega_1 \) and \( L(\mathbb{R}, h) \) is a model of ZFC.

**Proof.** By our hypothesis, there is a real \( x \) such that we may pick \( \vec{x} \in L[x] \) and \( \vec{x} \) is as in the hypothesis of Lemma 4.10.

---

5. The main theorem

The following Lemma is dual to Corollary 4.11.

**Lemma 5.1.** Let \( g \) be \( S_{\omega_1} \)-generic over \( L \), let \( h \) be \( \mathbb{P}_B \)-generic \( h \) over \( L[g] \) and let \( b = \bigcup h \) be the Burstin basis added by \( h \). Let

\[
W = L(\mathbb{R}, b)^{L[g,h]}
\]

Then \( W \models \text{“There is no well-ordering of } \mathbb{R} \text{”}.\)

**Proof.** That \( b \) is indeed a Burstin basis in \( L[g,h] \) as well as in \( W \) follows from Lemmas 4.4 and 4.5.

Let us assume for contradiction that

\[
L[g,h] \models \text{“} \varphi(\cdot, \cdot, \vec{x}, \vec{\alpha}, b) \text{ defines a well-ordering of } \omega^2 \text{“}
\]

where \( \vec{x} \in \mathbb{R} \cap L[g,h] = \mathbb{R} \cap L[g] \) and \( \vec{\alpha} \in \text{OR}. \)

Then, there is some \( p \in h \subset \mathbb{P}_B \) such that

\[
p \forces_{L[g]} \varphi(\cdot, \cdot, \vec{x}, \vec{\alpha}, b) \text{ defines a well-ordering of } \omega^2,
\]

where \( \vec{b} \) is the canonical \( \mathbb{P}_B \)-name for the generic Burstin basis \( b \) as defined in Remark 4.7; but then we may rewrite this as

\[
p \forces_{L[g],\vec{b}} \varphi(\cdot, \cdot, \vec{x}, \vec{\alpha}, \{\langle y, q \rangle : \theta(y,q)\}) \text{ defines a well-ordering of } \omega^2,
\]

with \( \theta \) being the \( \Sigma_3^1 \) formula from Remark 4.7. We may pick \( \xi < \omega_1 \) with \( p, \vec{x} \in L[g \upharpoonright \xi] \), see Corollary 2.18. Now since \( S_2 \times S_{\omega_1} \upharpoonright \xi \) is isomorphic to \( S_{\omega_1} \), via the the isomorphism \( (p_0, p_1) \mapsto p_0 \cup p_1 \), standard arguments show that \( g \upharpoonright (\xi, \omega_1) \) is \( S_{\omega_1} \upharpoonright \xi \)-generic over \( L[g \upharpoonright \xi] \) and so we can write

\[
(2) \quad p \forces_{L[\xi,g[\xi]]} \varphi(\cdot, \cdot, \vec{x}, \vec{\alpha}, \{\langle y, q \rangle : \theta(y,q)\}) \text{ defines a well-ordering of } \omega^2
\]

The following only uses that \( S_{\omega_1} \) is a countable support product of uncountably many copies of the same forcing.
Claim 2. $S_{\omega_1}$ is weakly homogeneous, i.e., given $p, p' \in S_{\omega_1}$ there is an isomorphism $\pi : S_{\omega_1} \to S_{\omega_1}$ such that $p[\pi(p')]$.

Proof. Let $p, p' \in S_{\omega_1}$. Since $\text{supp}(p)$ is countable there is some $\gamma < \omega_1$ such that $\text{supp}(p) \subset \gamma$. Set $\pi : S_{\omega_1} \to S_{\omega_1}$ defined as follows:

$$\pi(r)(\beta) = \begin{cases} 1_s & \text{if } \beta < \gamma \\ r(\alpha) & \text{if } \beta = \gamma + \alpha \end{cases}$$

Note that $\text{supp}(p) \cap \text{supp}(\pi(p')) = \emptyset$, hence $p[\pi(p')]$. \hfill \Box

Since $S_{\omega_1}$ is weakly homogeneous and $S_{\omega_1 \setminus \xi} \cong S_{\omega_1}$, (2) gives us

$$1 \leftover{L[\xi]}{L[\xi]} \leftover{\mathcal{P}_n}{\mathcal{P}_n} \leftover{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\})}{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\})} \rightover{\text{defines a well-ordering of } \omega^2.}$$

Let $g^*$ be $S_{\omega_1}$-generic over $L[g]$ so that $g \models [\xi, \omega_1)$ and $g^*$ are (or may be construed as) mutually $S_{\omega_1}$-generics over $L[g \restriction [\xi, \omega_1)]$, and let $h^*$ be $\mathcal{P}_n$-generic over $L[g \restriction [\xi, g^*]$ with $p \in h^*$. We have that

$$L[g \restriction [\xi, g^*][h^*] = \leftover{L[g \restriction [\xi, g^*][h^*] = \leftover{L[g \restriction [\xi, \omega_1)] \leftover{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, b^*) \text{ defines a well-ordering of } \omega^2,}{\text{where } b^* := \bigcup h^* \text{ is the Burstin basis added by } h^*.}$$

we can find some $\beta$, some $n < \omega$, and $i \in \{0, 1\}$ such that

(i) $L[g, h] \models \text{the } n^{th} \text{ digit of the } \beta^{th} \text{ element of } \omega^2 \text{ given by } \varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, b)$ is $i^*$

(ii) $L[g \restriction [\xi, \omega_1)] \models \text{the } n^{th} \text{ digit of the } \beta^{th} \text{ element of } \omega^2 \text{ given by } \varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\}) \text{ is } 1 - i^*$

Thus there exist two conditions $p_0 \in h$ and $p_1 \in h^*$ below $p$ such that

(i)* $p_0 \leftover{L[g \restriction [\xi, \omega_1)]}{L[g \restriction [\xi, \omega_1)]} \leftover{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\}) \text{ is } i^*}{\text{the } n^{th} \text{ digit of the } \beta^{th} \text{ element of } \omega^2 \text{ given by } \varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\}) \text{ is } i^*}$

(ii)* $p_1 \leftover{L[g \restriction [\xi, \omega_1)]}{L[g \restriction [\xi, \omega_1)]} \leftover{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\}) \text{ is } 1 - i^*}{\text{the } n^{th} \text{ digit of the } \beta^{th} \text{ element of } \omega^2 \text{ given by } \varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\}) \text{ is } 1 - i^*}$

Pick $\zeta \geq \xi, \xi < \omega_1$ such that $p_0 \in L[g \restriction [\xi]$ and $p_1 \in L[g \restriction [\xi, g^* \restriction [\zeta], \text{ say } \xi + \zeta = \zeta$. Then (i)* and (ii)* above give us

$$\left\{ \begin{array}{c}
\leftover{L[\xi]}{L[\xi]} \leftover{\mathcal{P}_n}{\mathcal{P}_n} \leftover{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\})}{\varphi(\cdot, \cdot, \bar{x}, \bar{\alpha}, \{\bar{y}, q : \theta(y, q)\})} \leftover{\text{the } n^{th} \text{ digit of the } \beta^{th} \text{ element of } \omega^2 \text{ given by }}{\text{the } n^{th} \text{ digit of the } \beta^{th} \text{ element of } \omega^2 \text{ given by }}
\end{array}\right. \right.$$ (1)

Now we want to make sure that the conditions $p_0$ and $p_1 \in L[g, g^*]$ are compatible.

Claim 3. $p_0 \cup p_1$ is linearly independent.

Proof. We may assume without loss of generality that

$$L[g \restriction [\xi] = \leftover{L[g \restriction [\xi] = \leftover{L[g \restriction [\xi] \models p_0 \text{ is a Burstin basis.}$$

In particular, it is true in $L[g \restriction [\xi]$ that $p_0$ is a Hamel basis. Suppose that there are $\bar{y} \in p$, $\bar{y}_0 \in p_0 \setminus p$, $\bar{y}_1 \in p_1 \setminus p$ and some vectors of rational numbers $\bar{q}, \bar{q}_0, \bar{q}_1$ such that

$$\bar{q} \cdot \bar{y} + \bar{q}_0 \cdot \bar{y}_0 + \bar{q}_1 \cdot \bar{y}_1 = 0$$

By mutual genericity we have

$$\bar{q} \cdot \bar{y} + \bar{q}_0 \cdot \bar{y}_0 = -\bar{q}_1 \cdot \bar{y}_1 \in L[g \restriction [\xi] \cap L[g \restriction [\xi, g^* \restriction [\zeta] \cap L[g \restriction [\xi]$$

Since $p$ is a Hamel basis for the reals of $L[g \restriction [\xi]$, there exists some $\bar{z} \in [p]^{<\omega}, \bar{r}_1 \in [Q]^{<\omega}$ such that

$$\bar{r}_1 \cdot \bar{z} = -\bar{q}_1 \cdot \bar{y}_1$$

Since $p_1 \supseteq p$ is linearly independent it follows that $\bar{r}_1 = 0 = \bar{q}_1$. Coming back to the equation (3), we now have that

$$\bar{q} \cdot \bar{y} + \bar{q}_0 \cdot \bar{y}_0 = 0$$
Since \( q_0 \supset p \) is also linearly independent, we conclude in that \( \bar{q} = 0 = \bar{q}_0 \). Hence \( p_0 \cup p_1 \) is linearly independent. \( \square \)

We may construe \( g \upharpoonright [\zeta, \omega_1) \) as \( S_{\omega_1} \)-generic over \( L[g \upharpoonright \xi, g^* \upharpoonright \zeta] \) as well as over \( L[g \upharpoonright \zeta] \). Therefore by \((*)\) it follows that

\[
\begin{align*}
(\ast\ast) & \begin{cases} 
p_0 \upmodels P_{L[g,B]} & \text{the \( n \)th digit of the \( \beta \)th element of} \omega^2 \text{ given by} \\
p_1 \upmodels P_{L[g,B]} & \text{the \( n \)th digit of the} \ \beta \text{th element of} \omega^2 \text{ given by} \\
 & \varphi(\cdot \cdot \cdot, \bar{x}, \bar{\alpha}, \{(\bar{y}, q) : \theta(y, q)\}) \text{ is} \bar{i}^n \\
 & \varphi(\cdot \cdot \cdot, \bar{x}, \bar{\alpha}, \{(\bar{y}, q) : \theta(y, q)\}) \text{ is} 1 - \bar{i}^n 
\end{cases}
\end{align*}
\]

By claim 3 and lemma 4.3, there is some \( q \leq p_0, p_1, q \in \mathbb{P}_{B}^{L[g, g^*]} \). But then, \( q \) forces the contradictory statements from the matrices of \((\ast\ast)\). This concludes the proof. \( \square \)

The previous proof in fact shows the following.

**Lemma 5.2.** Let \( g \) be \( S_{\omega_1} \)-generic over \( L \), let \( h \) be \( \mathbb{P}_B \)-generic \( h \) over \( L[g] \) and let \( b = \bigcup h \) be the Burstin basis added by \( h \). Inside \( L[g, h] \), there are Turing-cofinally many \( x \in \mathbb{R} \) such that if \( X \subset L[x] \), \( X \in OD_{x,b} \), then \( X \in L[x] \).

By standard arguments, Lemma 5.2 then implies.

**Lemma 5.3.** Let \( g \) be \( S_{\omega_1} \)-generic over \( L \), let \( h \) be \( \mathbb{P}_B \)-generic \( h \) over \( L[g] \) and let \( b = \bigcup h \) be the Burstin basis added by \( h \). Let \( W = L(\mathbb{R}, b)^{L[g,h]} \). Then

\[ \omega W \cap L[g,h] \subset W. \]

In particular, \( W \) is a model of DC, the principle of dependent choice.

**Theorem 5.4.** Let \( g \) be \( S_{\omega_1} \)-generic over \( L \), and let \( b \) be \( \mathbb{P}_B \) generic over \( L[g] \). Let

\[ W = L(\mathbb{R}, b)^{L[g,b]}. \]

Then, \( W \models ZF + DC \) and in \( W \) there are Luzin, Sierpiński, Vitali and a Burstin basis but in \( W \) there is no a well-ordering of \( \mathbb{R} \).

**Proof.** Clearly Lemma 5.3 gives \( W \models ZF + DC \). Now, as \( \mathbb{P}_B \) is \( \omega \)-closed, \( \mathbb{R} \cap W = \mathbb{R} \cap L[g] \), so that \( W \models \text{"}b\text{ is a Burstin basis"} \). This means that in \( W \), we have a Bernstein set and a Hamel basis. Hence, in view of 2.2, there is a Vitali in \( W \) set induced by \( b \). By Corollary 3.4, \( W \) has a Luzin as well as a Sierpiński set. Finally, by 5.1, in \( W \) there is no a well-ordering of the reals, as required. \( \square \)

6. Further remarks: ultrafilters on \( \omega \), Mazurkiewicz sets, etc.

Let \( g \) be \( S_{\omega_1} \)-generic over \( L \).

By [14, Theorem 6], in \( L[g] \) there is an ultrafilter on \( \omega \) which is generated by an ultrafilter in \( L \). In fact, if \( U \subset L \) is a selective ultrafilter on \( \omega \), then \( U \) generates an ultrafilter in \( L[g] \) (see [25]). This implies that the model \( W = L(\mathbb{R}, b)^{L[g,b]} \) from Theorem 5.4 has ultrafilters on \( \omega \).

A set \( M \subset \mathbb{R}^2 \) is a Mazurkiewicz set if \( M \) intersects every straight line in exactly two points. Mazurkiewicz showed in ZFC that Mazurkiewicz sets exist, see [17]. We may force with a poset \( \mathbb{P}_M \) consisting of “local” Mazurkiewicz sets over \( L[g] \) in much the same way as Definition 4.1 gave a forcing whose conditions are “local” Burstin bases. If \( m \) is the set added by \( \mathbb{P}_M \), then \( m \) will be a Mazurkiewicz set in \( L(\mathbb{R}, m)^{L[g,m]} \) and this model will not have a well-ordering of the reals. This result is produced in [1].

We may in fact force with the product \( \mathbb{P}_B \times \mathbb{P}_M \) over \( L[g] \) and get a model with a Burstin base and a Mazurkiewicz set with no well-order of the reals.

In the same fashion, one may add further “maximal independent” sets generically over \( L[g] \), e.g. selectors for \( \Sigma_1^1 \) definable equivalence relations, without adding a well-ordering of \( \mathbb{R} \). (Cf. [5].)
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