BRESSOUD’S IDENTITIES FOR EVEN MODULI. NEW COMPANIONS AND RELATED POSITIVITY RESULTS.

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To the Memory of Omar Foda

Abstract. We revisit Bressoud’s generalised Borwein conjecture. Making use of certain positivity-preserving transformations for \( q \)-binomial coefficients, we establish the truth of infinitely many new cases of the Bressoud conjecture. In addition, we prove new doubly-bounded refinement of the Foda-Quano identities. Finally, we discuss new companions to the Bressoud even moduli identities. In particular, all 10 mod 20 identities are derived.

1. Introduction and Background

In 1980, Bressoud [6] discovered even moduli identities:

\[
\sum_{n_1, n_2, \ldots, n_{v-1} \geq 0} q^{N_1^2 + \cdots + N_{v-1}^2 + N_1 + \cdots + N_{v-1}} (q)_{n_1} (q)_{n_2} \cdots (q)_{n_{v-1}} (q^2 ; q^2)_{n_{v-1}} = \frac{(q^{2v+1}, q^i, q^{2v-i} ; q^{2v+1})_\infty}{(q)_\infty},
\]

with \( v \geq 2, 1 \leq i \leq v, |q| < 1 \) and \( N_i = \sum_{j=i}^{v-1} n_j \). In (1.1) we use the standard notations

\[
(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),
\]

\[
(a; q)_\infty = \prod_{j \geq 0} (1 - aq^j),
\]

\[
(q)_n = (q; q)_n,
\]

\[
(q)_\infty = (q; q)_\infty,
\]

\[
(a_1, a_2, \ldots, a_k; q)_n = \prod_{i=1}^k (a_i; q)_n,
\]

where \( n \geq 0 \).

We remark that case \( i = v \) was not considered in [6]. The identities in (1.1) are closely related to the celebrated Andrews-Gordon identities [1]:

\[
\sum_{n_1, n_2, \ldots, n_{v-1} \geq 0} q^{N_1^2 + \cdots + N_{v-1}^2 + N_1 + \cdots + N_{v-1}} (q)_{n_1} (q)_{n_2} \cdots (q)_{n_{v-1}} (q^2 ; q^2)_{n_{v-1}} = \frac{(q^{2v+1}, q^i, q^{2v-1-i} ; q^{2v+1})_\infty}{(q)_\infty},
\]

with \( v \geq 2, 1 \leq i \leq v \).
In 1995, O. Foda, Y.H. Quano [10] found polynomial refinement of (1.1)
\[
\sum_{n_1, n_2, \ldots, n_{v-1} \geq 0} q^{N_1^2 + \cdots + N_{v-1}^2 + N_v + \cdots + N_{v-1}}
\]
(1.2)
\[
\left[ L - \sum_{i=1}^{v-1} N_i \right] \prod_{j=1}^{v-2} [n_j] \left[ n_j + 2L - 2 \sum_{l=1}^{j} N_l + \min(v-i, v-1-j) \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(v+1)} \left[ 2L + v - j \right] \left[ L - v j \right]_q,
\]
with \(1 \leq i \leq v\). For \(m, n \in \mathbb{Z}\) the q-binomial coefficient is defined as
\[
\left[ \begin{array}{c} n + m \\ n \end{array} \right]_q = \begin{cases} \frac{(q)_{n+m}}{(q)_n (q)_m}, & \text{if } m, n \geq 0 \\ 0, & \text{otherwise.} \end{cases}
\]
It is well known [2] that
(1.3)
\[
\left[ \begin{array}{c} n + m \\ n \end{array} \right]_q \geq 0.
\]
Here and everywhere
\(P(q) \geq 0\),
means that a polynomial in \(q, P(q)\), has non-negative coefficients. Obviously, (1.2), (1.3) imply that for \(v \geq 1\) and \(i = 1, \ldots, v\)
(1.4)
\[
G(L, L + v - i, 2 - \frac{i}{v}, \frac{i}{v}, v, q) \geq 0,
\]
where
\[
G(N, M, \alpha, \beta, K, q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(K+1) + j(K-1)} \left[ N + M \\ N - K j \right]_q.
\]
Note that (1.4) is consistent with the Bressoud positivity conjecture [8].

**Conjecture 1.1.** *Bressoud* Let \(K \in \mathbb{Z}_{>0}, N, M, \alpha K, \beta K \in \mathbb{N}\) such that
(1.5)
\[
1 \leq \alpha + \beta \leq 2K - 1,
\]
(1.6)
\[
\beta - K \leq N - M \leq K - \alpha,
\]
(strict inequality when \(K = 2\)),
then
(1.7)
\[
G(N, M, \alpha, \beta, K, q) \geq 0.
\]
We remark that when \(\alpha, \beta\) are integers, (1.7) becomes a theorem in [3]. Many cases of Conjecture 1.1 were settled in the literature [4, 5, 7, 12, 14–16].

In Section 2, we will show that (1.2) implies

**Theorem 1.2.** For \(v \geq 2\), \(0 \leq \Delta < v\)
\[
\sum_{m, k, n_1, n_2, \ldots, n_{v-1} \geq 0} q^{(m+k)^2 + k^2 + \Delta(m+2k) + \sum_{j=1}^{v-1} N_j^2 + \sum_{j=v+1}^{v-1} N_j} \left( \frac{L}{m, 2k + \Delta} \right)_q
\]
(1.8)
\[
\left[ k - \sum_{j=1}^{v-1} N_i \right] \prod_{j=1}^{v-2} \left[ n_j + 2k - 2 \sum_{l=1}^{j} N_l + \min(\Delta, v-1-j) \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(2v+1) \Delta j + (2v+1) \Delta j} \left[ \frac{2L}{L - \Delta - 2v j} \right]_q,
\]
where
\[
\binom{L}{m,n}_q = \binom{L}{m}_q \binom{L-m}{n}_q \geq 0.
\]

Hence, (1.8) implies a new positivity result.
For \( v \geq 2, 0 \leq \Delta < v \)
\[
(1.9) \quad G(L - \Delta, L + \Delta, (v + \Delta)(1 + \frac{1}{2v}), (v - \Delta)(1 + \frac{1}{2v}), 2v, q) \geq 0.
\]

Again (1.9) agrees with Conjecture 1.1. In Section 5, we will discuss an extra parameter generalization of (1.9), which is given in the Theorem 5.8.

As \( L \to \infty \) in (1.8) we get for \( v \geq 2, 0 \leq \Delta < v \)
\[
(1.10) \quad \sum_{m,k,n_1,n_2,...,n_{v-1} \geq 0} \frac{q^{(m+k)^2+k^2+2k+(m+2k)\Delta+N_j^2+N_{j+1}^2 \Delta}}{(q)_m(q)_n} v^j \prod_{j=1}^{v-2} \frac{v_j^2}{q^j} q^{N_{j+1}^2 \Delta - (2j+1)(v+\Delta)} q^{2v^2(q^2(q_2(v+\Delta)) \cdot (q^2(q^2(2v+1))))_\infty}.
\]

We will prove in Section 5, that for \( 1 \leq i \leq v \)
\[
(1.11) \quad \sum_{n_1,...,n_{v-1} \geq 0} q^{\sum_{j=1}^{v-1} N_j^2} \left[ \frac{L - \sum_{i=1}^{v-1} N_i}{q} \prod_{j=1}^{v-2} \frac{v_j^2}{q^j} q^{N_{j+1}^2 \Delta - (2j+1)(v+\Delta)} q^{2v^2(q^2(q^2(2v+1))))_\infty} \right].
\]

This is a new family of polynomial refinements of (1.1) with \( i = v \). We will show that it implies for \( 0 \leq \Delta < v \)
\[
(1.12) \quad \sum_{m,k,n_1,n_2,...,n_{v-1} \geq 0} q^{k^2+(m+k)^2+(m+2k)\Delta+\sum_{j=1}^{v-1} N_j^2} \left[ \frac{L}{q} \binom{L}{m,2k+\Delta}_q \right] v^j \prod_{j=1}^{v-2} \frac{v_j^2}{q^j} q^{(2j+1)(v+\Delta)} q^{2v^2(q^2(q^2(2v+1))))_\infty}.
\]

And so, for \( 0 \leq \Delta < v \)
\[
(1.13) \quad G(L - \Delta, L + \Delta, v + \Delta + \frac{1}{2}, v - \Delta + \frac{1}{2}, 2v, q) \geq 0.
\]

Again, this new inequality agrees with (1.7). In Section 5, we will discuss an extra parameter generalization of (1.13), which is given in the Theorem 5.9.
As \( L \to \infty \) in (1.12) we get for \( 0 \leq \Delta < v \)

\[
\sum_{m,k,n_1,\ldots,n_{v-1} \geq 0} \frac{q^{k^2+(m+k)^2+(m+2k)\Delta + \sum_{j=1}^{v-1} N_j^2}}{(q)_{m(q)2k+\Delta}} \left[ \frac{k - \sum_{j=1}^{v-2} N_j}{n_{v-1}} \right] \prod_{j=1}^{v-2} \left[ \frac{n_j+2k-2\sum_{i=1}^{j} N_i + \min(\Delta, v-1-j)}{n_j} \right]_q = \\
(q^{2v(2v+1)}, q^{n(2v+1-2\Delta)}, q^{v(2v+1+2\Delta)}; q^{2v(2v+1)})_\infty
\]

(1.14)

We remark that (1.10) and (1.14) are the new companions to the Bre sssoud identities mod \( 2v(2v+1) \).

Another identity of that type is given by (5.15).

We conclude this section with a list of five useful formulas, which can b e found in [2]:

\[
\lim_{L \to \infty} \left[ \frac{L}{m} \right]_q = \frac{1}{(q)_m},
\]

(1.15)

\[
\lim_{L,M \to \infty} \left[ \frac{L+M}{L} \right]_q = \frac{1}{(q)_\infty},
\]

(1.16)

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_q = \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right]_q + q^m \left[ \begin{array}{c} n-1 \\ n \end{array} \right]_q = \left[ \begin{array}{c} n-1 \\ n \end{array} \right]_q + q^n - m \left[ \begin{array}{c} n-1 \\ m-1 \end{array} \right]_q,
\]

(1.17)

\[
\sum_{n \geq 0} q^{\left( \begin{array}{c} 2 \\ n \end{array} \right)} z^n \left[ \frac{L}{n} \right]_q = (-z; q)_L,
\]

(1.18)

\[
\sum_{j=-\infty}^{\infty} (-1)^j z^j q^{2j} = \left( q^2, q, z, q^2 \right)_\infty,
\]

(1.19)

with \( L, M, m, n \in \mathbb{N} \). Observe that (1.15) implies

\[
\lim_{L \to \infty} \left( \frac{L}{m,n} \right) = \frac{1}{(q)_m(q)_n}.
\]

(1.20)

The rest of this paper is organized as follows. In Section 2, we review three positivity preserving transforma- tions for \( q \)-binomial coefficients and prove Theorem 1.2. Section 3 is dedicated to the Foda-Quano polynomials (1.2) with \( v = 2 \) and their variants. In Section 4 we convert the Section 3 polynomial iden- tities into ten identities mod 20. Finally, in Section 5 we derive doubly-bounded polynomial refinements of (1.1) with \( i = v \), prove (1.11) and (1.12) and establish three new positivity results.

2. Positivity-preserving Transformations. Proof of Theorem 1.2

We start with the following summation formula [4]

**Theorem 2.1.** For \( L \in \mathbb{N}, a \in \mathbb{Z} \)

\[
\sum_{k \geq 0} C_{L,k}(q) \left[ \frac{k}{k-a} \right]_q = q^{T(a)} \left[ \frac{2L+1}{L-a} \right],
\]

(2.1)

where

\[ T(j) := \frac{(j+1)j}{2} \]

and

\[
C_{L,k}(q) = \sum_{m=0}^{L} q^{T(m)+T(m+k)} \left[ \frac{L}{m,k} \right]_q.
\]

(2.2)
Observe that \( C_{L,k}(q) \geq 0 \). Using transformation (2.1) it is easy to check that identity of the form

\[
F_c(L, q) = \sum_{j=-\infty}^{\infty} \alpha(j) \left\lfloor \frac{L}{j^2} \right\rfloor q^j,
\]

implies that

\[
\sum_{k \geq 0} C_{L,k}(q) F_c(k, q) = \sum_{j=-\infty}^{\infty} \alpha(j) q^{T(j)} \left[ \frac{2L+1}{L-j} \right]_q.
\]

Hence, if \( F_c(L, q) \geq 0 \) then

\[
\sum_{j=-\infty}^{\infty} \alpha(j) q^{T(j)} \left[ \frac{2L+1}{L-j} \right]_q \geq 0.
\]

For that reason, we say that (2.1) is positivity-preserving.

Letting \( L \to \infty \) in (2.4) we derive

\[
\sum_{m,k \geq 0} q^{T(m)+T(m+k)} W_{L,k}(q) F_c(k, q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \alpha(j) q^{T(j)},
\]

Theorem 2.1 is closely related to the Warnaar transformation (Corollary 2.6 in [16]).

**Theorem 2.2.** For \( L \in \mathbb{N}, a \in \mathbb{Z} \)

\[
\sum_{k \geq 0} W_{L,k}(q) \left[ \frac{2k}{k-a} \right]_q = q^{2a^2} \left[ \frac{2L}{L-2a} \right]_q,
\]

where

\[
W_{L,k}(q) = \sum_{m=0}^{L} q^{(m+k)^2+k^2} \binom{L}{m,2k}_q \geq 0.
\]

Using transformation (2.7) it is easy to check that identity of the form

\[
F_w(L, q) = \sum_{j=-\infty}^{\infty} \alpha(j) \left[ \frac{2L}{L-j} \right]_q,
\]

implies

\[
\sum_{k \geq 0} W_{L,k}(q) F_w(k, q) = \sum_{j=-\infty}^{\infty} \alpha(j) q^{2j^2} \left[ \frac{2L}{L-2j} \right]_q.
\]

Hence, if \( F_w(L, q) \geq 0 \), then

\[
\sum_{j=-\infty}^{\infty} \alpha(j) q^{2j^2} \left[ \frac{2L}{L-2j} \right]_q \geq 0.
\]

Letting \( L \to \infty \) in (2.10), we derive

\[
\sum_{m,k \geq 0} q^{(m+k)^2+k^2} W_{L,k}(q) F_w(k, q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \alpha(j) q^{2j^2}.
\]

Observe that unlike (2.7), transformation (2.1) can not be iterated. Interestingly enough, there exists an odd companion to Theorem 2.2, discussed in [5].

**Theorem 2.3.** For \( L \in \mathbb{N}, a \in \mathbb{Z} \)

\[
\sum_{k \geq 0} O_{L,k}(q) \left[ \frac{2k+1}{k-a} \right]_q = q^{4T(a)} \left[ \frac{2L}{L-2a-1} \right]_q,
\]
Letting $L$ be a polynomial, it is easy to check that identity of the form
\[
F_o(L, q) = \sum_{j=\infty}^{\infty} \alpha(j) \left[ \frac{2L+1}{L-j} \right]_q
\]
implies that
\[
\sum_{j=\infty}^{\infty} O_{L,k}(q)F_o(k, q) = \sum_{j=\infty}^{\infty} \alpha(j)q^{2j^2+2j} \left[ \frac{2L}{L-2j-1} \right]_q.
\]
Hence, if $F_o(L, q) \geq 0$, then
\[
\sum_{j=\infty}^{\infty} \alpha(j)q^{2j^2+2j} \left[ \frac{2L}{L-2j-1} \right]_q \geq 0.
\]
Letting $L \to \infty$ in (2.14), we obtain
\[
\sum_{m,k \geq 0} q^{2T(m+k)+2T(k)} \frac{(q)_m(q)_{2k+1}}{(q)_\infty} F_o(k, q) = \frac{1}{(q)_\infty} \sum_{j=\infty}^{\infty} \alpha(j)q^{2j^2+2j}.
\]

To prove Theorem 1.2, we replace $L \to L - \lfloor \frac{L}{2} \rfloor$ in (1.2) where $\Delta = v - i$. We then apply Theorem 2.2 if $\Delta$ is even, and Theorem 2.3 if $\Delta$ is odd. After simplification we derive (1.8).

3. Foda-Quano identities with $v = 2$ and their variants

We begin by noting two special cases of (1.2).

For $v = 2, i = 1$
\[
\sum_{n_1 \geq 0} q^{n_1^2+n_1} \left[ \frac{L}{n_1} \right]_{q^2} = \tilde{F}(L),
\]
where
\[
\tilde{F}(L) = \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2+j} \left[ \frac{2L+1}{L-2j} \right]_q.
\]

With the aid of (1.18) we get
\[
(-q^2; q^2)_L = \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2+j} \left[ \frac{2L+1}{L-2j} \right]_q.
\]

For $v = 2, i = 2$
\[
\sum_{n_1 \geq 0} q^{n_1^2} \left[ \frac{L}{n_1} \right]_{q^2} = \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L}{L-2j} \right]_q.
\]

Using (1.18) on the left we obtain
\[
(-q; q^2)_L = \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L+1}{L-2j} \right]_q = \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L+1}{L-2j} \right]_q.
\]
The equality for the RHS of (3.5) follows from (3.16) and (3.18) in [4]. Next, with the aid of (1.17) we obtain
\[
\sum_{j=\infty}^{\infty} (-1)^j q^{2j^2+2j} \left[ \frac{2L+1}{L-2j} \right]_q = q^L \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L}{L-2j} \right]_q + \sum_{j=\infty}^{\infty} (-1)^j q^{2j^2+2j} \left[ \frac{2L}{L-2j-1} \right]_q.
\]
Observe that the summand in last sum on the right of (3.6) negates under $j \to -j - 1$, and so this sum equals zero. Hence, with the aid of (3.5) we derive

$$q^L(-q; q^2)_L = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2 + 2j} \left[ \frac{2L + 1}{L - 2j} \right]_q.$$

Using (1.17) we get

$$-q^{L-1} A(L) + B(L) = q^L(-q; q^2)_L,$$

where

$$A(L) := \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L}{L - 2j - 1} \right]_q,$$

$$B(L) := \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2 - 2j} \left[ \frac{2L}{L - 2j} \right]_q \geq 0.$$

Using $q$-binomial recurrence (1.17) on the left of

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{2L + 1}{L - 2j} \right]_q = (-q; q^2)_L,$$

we get for $A(L)$ and $B(L)$

$$A(L) + q^L B(L) = (-q; q^2)_L.$$

Solving (3.8) and (3.12), we obtain

$$A(L) = (-q; q^2)_{L-1}(1 - q^{2L}),$$

$$B(L) = q^L(-\frac{1}{q}; q^2)_L \geq 0.$$

Using $q$-binomial recurrence (1.17) on (3.3) we have

$$q^L X(L) + Y(L) = X(L) - q^2 Y(L) = (-q^2; q^2)_L,$$

where

$$X(L) := \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2 - j} \left[ \frac{2L}{L - 2j} \right]_q$$

and

$$Y(L) := \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2 + j} \left[ \frac{2L}{L - 2j - 1} \right]_q.$$

Solving for $X(L), Y(L)$, we obtain:

$$X(L) = (1 + q^L)(-q^2; q^2)_{L-1} \geq 0$$

and

$$Y(L) = (1 - q^L)(-q^2; q^2)_{L-1}.$$

We remark that (3.18) was first proven in [12]. Let

$$Z(L) := \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2 - j} \left[ \frac{2L + 1}{L - 2j} \right]_q.$$

Applying (1.17) we have

$$Z(L) = X(L) + q^{L+1} Y(L) = \frac{(-1; q^2)_L}{2} (1 + q^L + q^{L+1} - q^{2L+1}).$$
Next, let
\[ (3.22) \quad C(L) = \sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2-j} \left[ \frac{2L}{L-2j-1} \right]_q. \]

We conclude this section with

**Theorem 3.1.** For \( L \in \mathbb{N} \)
\[ (3.23) \quad C(L) = (-q^2; q^2)_{L-2} \{ (1 + q^L)(1 - q^{2L}) + q^{L-1}(1 - q^L)(1 + q^2) \}. \]

**Proof.** We start with the relation discussed by Prodinger in [13]
\[ (3.24) \quad \left[ \frac{L}{k} \right]_q = (1 + q - q^L) \left[ \frac{L-1}{k} \right]_q + q^{2L-2} \left[ \frac{L-1}{k-1} \right]_q + (q^L - q) \left[ \frac{L-2}{k} \right]_q, \]
with \( L > 0 \).

Replacing \( L \) by \( 2L \), \( k \) by \( L - 2j - 1 \) in (3.24) and multiplying both sides by \((-1)^j q^{2j^2-j}\) we obtain after summing over \( j \)
\[ (3.25) \quad C(L) = (1 + q - q^{2L}) Z(L-1) - q^{2L+1} \tilde{F}(L-1) + (q^{2L} - q) X(L-1) \]
\[ = (-q^2; q^2)_{L-2} \{ (1 + q^L)(1 - q^{2L}) + q^{L-1}(1 - q^L)(1 + q^2) \}, \]
where we used (3.3), (3.18) and (3.21). Finally, we observe that \( C(0) = 0 \). Hence, (3.23) is valid for \( L \geq 0 \). \( \square \)

4. 10 identities \( \mod 20 \)

In [4] we derived three new identities \( \mod 20 \). Here, we will employ the polynomial identities proven in Section 3, together with (2.11), (2.15), and (1.19) to derive all 10 identities \( \mod 20 \).

For example, applying (2.11) to (3.19) we derive, with the aid of (1.19),
\[ (4.1) \quad \sum_{k,m \geq 0} \frac{q^{k^2+(m+k)^2+2(m+2k)}(-q^2)_k}{(q)_m(q)_{2k+1}} = \frac{(q^{20}; q^1, q^{19}; q^{20})_\infty}{(q)_\infty}. \]

Applying (2.11) to (3.13) we derive, with the aid of (1.19),
\[ (4.2) \quad \sum_{k,m \geq 0} \frac{q^{k^2+(m+k)^2+2(m+2k)}(-q;q^2)_k}{(q)_m(q)_{2k+1}} = \frac{(q^{20}; q^2, q^{18}; q^{20})_\infty}{(q)_\infty}. \]

Applying (2.11) to (3.23) we derive, with the aid of (1.19),
\[ (4.3) \quad \sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+2(m+2k)}(1 + q^{k+1})(1:-q^2)_k}{2(q)_m(q)_{2k+1}} \]
\[ + \sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+2(m+2k)+k(1+q^{k+1})}(1:-q^2)_k}{2(q)_m(q)_{2k+1}(1 + q^{k+1})} = \frac{(q^{20}; q^3, q^{17}; q^{20})_\infty}{(q)_\infty}. \]

Applying (2.15) to (3.7) we derive, with the aid of (1.19),
\[ (4.4) \quad \sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+(m+3k)}(-q;q^2)_k}{(q)_m(q)_{2k+1}} = \frac{(q^{20}; q^4, q^{16}; q^{20})_\infty}{(q)_\infty}. \]

Applying (2.15) to (3.3) we derive, with the aid of (1.19),
\[ (4.5) \quad \sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+(m+2k)}(-q^2;q^2)_k}{(q)_m(q)_{2k+1}} = \frac{(q^{20}; q^5, q^{15}; q^{20})_\infty}{(q)_\infty}. \]
Applying (2.15) to (3.5) we derive, with the aid of (1.19),

\[
\sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+(m+2k)}}{(q)_m(q)_{2k+1}^2} (-q^2)_k = \frac{(q^{20}, q^6, q^{14}; q^{20})_\infty}{(q)_\infty}.
\]

Applying (2.15) to (3.21) we derive, with the aid of (1.19),

\[
\sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+(m+3k)}}{(q)_m(q)_{2k+1}^2} (-1; q^2)_k = \frac{(q^{20}, q^7, q^{13}; q^{20})_\infty}{(q)_\infty}.
\]

Applying (2.11) to (3.14) we derive, with the aid of (1.19),

\[
\sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2+k}}{(q)_m(q)_{2k}^2} \left(\frac{1}{q}\right)^{m,k} (-1; q^2)_k = \frac{(q^{20}, q^8, q^{12}; q^{20})_\infty}{(q)_\infty}.
\]

This is a bit different than the similar identity in [4].

Applying (2.11) to (3.18) we derive, with the aid of (1.19),

\[
\sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2}}{(q)_m(q)_{2k}^2} \left(\frac{1 + q^k}{2}\right) (-1; q^2)_k = \frac{(q^{20}, q^9, q^{11}; q^{20})_\infty}{(q)_\infty}.
\]

Applying (2.11) to (3.5) we derive, with the aid of (1.19),

\[
\sum_{m,k \geq 0} \frac{q^{k^2+(m+k)^2}}{(q)_m(q)_{2k}^2} (-q^2)_k = \frac{(q^{20}, q^{10}, q^{10}; q^{20})_\infty}{(q)_\infty}.
\]

We remark that identity (4.10) appeared in disguise in [15].

5. New Family of Doubly-Bounded Polynomial Refinements of (1.1) with i = v.

Proof of (1.11) and (1.12). Three New Positivity Results

In [9] Burge recognized the iterative power of the following polynomial identity.

Lemma 5.1.

\[
\sum_{i=0}^{M} q^{2L + b + M - i} \left[ m_1 + m_2 + M - i \right] \left[ m_1 \right]_q \left[ m_2 \right]_q = q^{2L + b + (a - 1)j} \left[ m_1 + M - j + \beta \right] \left[ m_2 + M + j + \alpha \right]_q,
\]

with \( \alpha, \beta, j \in \mathbb{Z} \), \( m_1, m_2, M \in \mathbb{N} \).

This lemma was further explored in [5, 11, 15].

Next, setting \( \alpha = \beta = 0 \), \( m_1 = L - (a - 1)j \), \( m_2 = L + b + (a - 1)j \) in the above lemma we derive

Theorem 5.2.

\[
\sum_{i=0}^{M} q^{2L + b + M - i} \left[ L - (a - 1)j \right] \left[ L + b + (a - 1)j \right] = q^{2L + b + (a + 1)j} \left[ L + b + (a + 1)j \right]_q,
\]

where \( L, M, a - 1, b \in \mathbb{N}, j \in \mathbb{Z} \).
Theorem 5.3.

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M - (v - 1)j}{L - vj} \right] \left[ \frac{L + v + M + (v - 1)j}{L + v + vj} \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M - (v - 1)j}{L - vj} \right] \left[ \frac{L + v - 1 + M + (v - 1)j}{L + v - 1 + vj} \right]_q. \]  

(5.2)

Proof. With the aid of (1.17) we have

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M - (v - 1)j}{L - vj} \right] \left[ \frac{L + v + M + (v - 1)j}{L + v + vj} \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M - (v - 1)j}{L - vj} \right] \left[ \frac{L + v - 1 + M + (v - 1)j}{L + v - 1 + vj} \right]_q + q^{L+v} \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2+vj} \left[ \frac{L + M - (v - 1)j}{L - vj} \right] \left[ \frac{L + v - 1 + M + (v - 1)j}{L + v + vj} \right]_q. \]

The summand in the last sum on the right negates under \( j \to -j - 1 \), and so the last sum equals zero. \( \square \)

Theorem 5.4. For \( b \geq 0, v \geq 1; L, M \geq 0 \) if

\[ F_b(L, M, v) = \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M - (v - 1)j}{L - vj} \right] \left[ \frac{L + b + M + (v - 1)j}{L + b + vj} \right]_q. \]

(5.4)

Then,

\[ \sum_{i=0}^{M} q^2 \left[ \frac{2L + b + M - i}{M - i} q \right] F_b(L - i, i, v) = \sum_{j=-\infty}^{\infty} (-1)^j q^{(v+1)j^2} \left[ \frac{L + M - vj}{L - (v + 1)j} q \right] \left[ \frac{L + b + M + vj}{L + b + (v + 1)j} \right]_q \]

(5.5)

and

\[ \sum_{j=0}^{M} q^2 \left[ \frac{2L + v + M - i}{M - i} q \right] F_v(L - i, i, v) = \sum_{j=-\infty}^{\infty} (-1)^j q^{(v+1)j^2} \left[ \frac{L + M - vj}{L - (v + 1)j} q \right] \left[ \frac{L + v + 1 + M + vj}{L + v + 1 + (v + 1)j} \right]_q, \]

(5.6)

where we used Theorem 5.2 and Theorem 5.3 with \( v \to v + 1 \).

The following identity was discussed by Burge [9]

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M}{L - j} q \right] \left[ \frac{L + M}{L + j} q \right] = \left[ \frac{L + M}{L} q \right]_q^{j^2}. \]

(5.7)

With the aid of Theorem 5.3 with \( v = 1 \) we can transform (5.7) into

\[ \sum_{j=-\infty}^{\infty} (-1)^j q^{vj^2} \left[ \frac{L + M}{L - j} q \right] \left[ \frac{L + 1 + M + 1}{L + 1 + j} q \right] = \left[ \frac{L + M}{L} q \right]_q^{j^2}. \]

(5.8)

Lemma 5.5. For \( i = 1, 2 \)

\[ \sum_{n_1 \geq 0} q^{n_1^2} \left[ \frac{2L + 2 - i + M - n_1}{M - n_1} q \right] \left[ \frac{L}{n_1} q \right] = \sum_{j=-\infty}^{\infty} (-1)^j q^{2j^2} \left[ \frac{L + M - j}{L - 2j} q \right] \left[ \frac{L + 2 - i + M + j}{L + 2 - i + 2j} q \right]. \]

(5.9)

Proof. For \( i = 2 \) it follows from (5.7) and (5.5) with \( b = 0, v = 1 \). For \( i = 1 \) it follows from (5.8) and (5.6) with \( v = 1 \). \( \square \)
Using (5.6) repeatedly on (5.8) we get with the aid of (5.2)

\[
\sum_{n_1, \ldots, n_{v-1} \geq 0} q^{\sum_{j=1}^{v-1} N_j^2} \left[ \frac{2L + v - 1 + M - N_1}{M - N_1} \right]_q \\
\left[ L - \sum_{n_v}^{v-2} N_t \right] \prod_{j=1}^{v-2} \left[ n_j + 2L - 2 \sum_{t=1}^{i_1} N_t + v - 1 - j \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ L + M - (v - 1)j \right]_q \left[ L + v + M + (v - 1)j \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ L + M - (v - 1)j \right]_q \left[ L + v - 1 + M + (v - 1)j \right]_q.
\]

(5.10)

We remark that (5.10) is a special case \( i = 1 \) of the following

**Theorem 5.6.** For \( 1 \leq i \leq v, v \geq 2 \)

\[
\sum_{n_1, \ldots, n_{v-1} \geq 0} q^{\sum_{j=1}^{v-1} N_j^2} \left[ \frac{2L + v - i + M - N_1}{M - N_1} \right]_q \\
\left[ L - \sum_{n_v}^{v-2} N_t \right] \prod_{j=1}^{v-2} \left[ n_j + 2L - 2 \sum_{t=1}^{i_1} N_t + \min(v - i, v - 1 - j) \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ L + M - (v - 1)j \right]_q \left[ L + (v - i) + M + (v - 1)j \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ L + M - (v - 1)j \right]_q \left[ L + v - 1 + M + (v - 1)j \right]_q.
\]

(5.11)

**Proof.** We proceed by Mathematical Induction on \( v \geq 2 \). Base case is established in (5.9). Suppose (5.11) is true for some \( v \geq 2 \). Applying (5.5) with \( b = v - i \) to (5.11) we derive

\[
\sum_{n_1, \ldots, n_{v-1} \geq 0} q^{\sum_{j=1}^{v-1} N_j^2} \left[ \frac{2L + v - i + M - N_1}{M - N_1} \right]_q \\
\left[ L - \sum_{n_v}^{v-2} N_t \right] \prod_{j=1}^{v-2} \left[ n_j + 2L - 2 \sum_{t=1}^{i_1} N_t + \min(v - i, v - 1 - j) \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ L + M - (v - 1)j \right]_q \left[ L + (v - i) + M + (v - 1)j \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ L + M - (v - 1)j \right]_q \left[ L + v - 1 + M + (v - 1)j \right]_q,
\]

which is (5.11) with \( (i, v) \to (i + 1, v + 1) \) and \( 1 \leq i \leq v \). Recall that \( i = 1 \) case was established for all \( v \). Therefore, if (5.11) is true for \( v \) then (5.11) is true for \( v + 1 \). Hence, (5.11) is true for \( v \geq 2 \) by Mathematical Induction. We remark that (5.12) with \( i = v \) was first established in [15].

Letting \( M \to \infty \) in (5.11) we arrive at (1.11). To prove (1.12), we replace \( L \to L - \left\lfloor \frac{\Delta}{2} \right\rfloor \) in (1.11) where \( \Delta = v - i \). We then apply Theorem 2.2 if \( \Delta \) is even, and Theorem 2.3 if \( \Delta \) is odd. After simplification we derive (1.12).

Cases \( i = v \) and \( i = v - 1 \) in (1.11) can be combined as follows. For \( v \geq 2 \)

\[
\sum_{n_1, \ldots, n_{v-1} \geq 0} q^{\sum_{j=1}^{v-1} N_j^2} \left[ \frac{L}{L - 2M} \right] - \sum_{n_v}^{v-2} N_t \right] \prod_{j=1}^{v-2} \left[ n_j + 2L - 2 \sum_{t=1}^{i_1} N_t \right]_q \\
= \sum_{j=-\infty}^{\infty} (-1)^i q^{v^2} \left[ \frac{L}{L - 2M} \right]_q \geq 0.
\]

(5.13)
With the aid of (2.4) we can transform (5.13) as

\[
\sum_{m,k,n_1,...,n_{v-1} \geq 0} q^{(m+1)+(m+k+1)+\sum_{j=1}^{v-1} N_j^2 } \left( \frac{L}{m,k} \right)_q
\]

(5.14)

\[
\left( \frac{N}{n_{v-1}} - \sum_{t=1}^{v-2} N_t \right)_q \prod_{j=1}^{v-2} \frac{\left[ n_j + k - 2 \sum_{t=1}^{j} N_t \right]}{n_j}_q
\]

\[
= \sum_{j=-\infty}^{\infty} (-1)^j q^{v(2v+1)j^2+vj} \left[ \frac{2L+1}{L-2vj} \right]_q
\]

Letting \( L \to \infty \) in (5.14), we get with the aid of (1.16), (1.19), (1.20)

\[
\sum_{m,k,n_1,...,n_{v-1} \geq 0} q^{(m+1)+(m+k+1)+\sum_{j=1}^{v-1} N_j^2 } \frac{1}{(q)_m(q)_k}
\]

(5.15)

\[
\left( \frac{N}{n_{v-1}} - \sum_{t=1}^{v-2} N_t \right)_q \prod_{j=1}^{v-2} \frac{\left[ n_j + k - 2 \sum_{t=1}^{j} N_t \right]}{n_j}_q
\]

\[
= \left( q^{2v(2v+1)}, q^{2v^2}, q^{2v(v+1)}, q^{2v(2v+1)} \right)_{\infty}
\]

Recalling (1.10) and (1.14), we derived new companion identities for \( 2v \) products moduli \( 2v(2v+1) \) out of possible \( 2v^2 + v \) Bressoud's products in (1.1) with \( v \to v(2v+1) \).

Next, (5.14) implies that

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{v(2v+1)j^2+vj} \left[ \frac{2L+1}{L-2vj} \right]_q \geq 0
\]

(5.16)

Hence, with the aid of (2.4) we find

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{v(1+10v)j^2+5vj} \left[ \frac{2L}{L-4vj} \right]_q \geq 0
\]

(5.17)

Repeatedly using (2.10), we derive

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{v(2n^2-1)vj^2+4n^2vj} \left[ \frac{2L}{L-2n^2-2n^2vj} \right]_q \geq 0,
\]

where \( v \geq 2, n \geq 2 \). The above inequality can be stated as

**Theorem 5.7.** For \( v \geq 2, n \geq 2 \)

\[
G \left( L-2^{n-2}, L+2^{n-2}, 2v \left( \frac{2^n-2^{-n}}{3}, \frac{2^n+2^{1-n}}{3}, 2v \left( \frac{2^n-2^{-n}}{3} \right) + \frac{2^{2-n}-2^n}{3} \right), L-q \right) \geq 0.
\]

(5.19)

Next, (1.8) implies that

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{(2v+1)vj^2+(2v+1)\Delta j} \left[ \frac{2L}{L-\Delta-2vj} \right]_q \geq 0.
\]

(5.20)

Repeatedly using (2.10), we derive

\[
\sum_{j=-\infty}^{\infty} (-1)^j q^{(2v+1)vj^2+(2v+1)\Delta j} \left[ \frac{2L}{L-\Delta 2^{n-1}-2n^2vj} \right]_q \geq 0,
\]

where \( n \geq 1, 0 \leq \Delta < v, v \geq 2 \). Hence we proved
Theorem 5.8. For $v \geq 2$, $n \geq 1$, $0 \leq \Delta < v$

$$G(L - \Delta 2^{n-1}, L + \Delta 2^{n-1}, (v + \Delta) \left(\frac{2^{2n} - 2^{-n}}{3} + \frac{1}{2^n v}\right),$$

$$(v - \Delta) \left(\frac{2^{2n} - 2^{-n}}{3} + \frac{1}{2^n v}\right), 2^n v, q) \geq 0.$$ (5.22)

Note that (5.22) with $n = 1$ reduces to (1.9).

Next, (1.12) implies that

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{(2v+1)vj^2 + 2v\Delta j} \left[\frac{2L}{L - \Delta - 2vj}\right]_q \geq 0$$

(5.23)

Repeatedly using (2.10), we derive

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{(2v+1)vj^2 + 2v\Delta j} \left[\frac{2L}{L - \Delta 2^n - 2^n vj}\right]_q \geq 0,$$

(5.24)

where $n > 0$, $0 \leq \Delta < v$. Hence we proved

Theorem 5.9. For $n \geq 1$, $v \geq 2$, $0 \leq \Delta < v$

$$G \left( L - \Delta 2^{n-1}, L + \Delta 2^{n-1}, \frac{2^{2n} - 2^{-n}}{3}(v + \Delta) + 2^{-n}, \frac{2^{2n} - 2^{-n}}{3}(v - \Delta) + 2^{-n}, 2^n v, q \right) \geq 0.$$ (5.25)

Note that (5.25) with $n = 1$ reduces to (1.13).

We conclude with the remark that (5.19), (5.22), and (5.25) are in agreement with Conjecture 1.1.

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