ON SOME APPROXIMATION PROPERTIES OF THE
GAUSS-WEIERSTRASS OPERATORS

BAŞAR YILMAZ

Abstract. In this paper, we present some approximation properties of the
Gauss-Weierstrass operators in exponential weighted spaces including norm
convergence of them and Voronovskaya and quantitative Voronovskaya-type
theorems.

1. Preliminaries

The Gauss-Weierstrass singular integral operator

\[ (W_n f)(x) := \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} f(x + t) e^{-nt^2} dt, \] (1)

where \( x \in \mathbb{R} \), \( n \in \mathbb{N} \) and \( n \to \infty \), was examined in \([1, 3, 4, 8]\) for functions
belonging to the space \( L_p \) and the classical Hölder spaces.

In this paper we examine the Gauss-Weierstrass operators \( W_n \) for functions \( f \)
belonging to the exponential weighted spaces \( L_{p,q}^r(\mathbb{R}) \) and \( L_{p,q}^{r-r}(\mathbb{R}) \) which definitions
are given below. We give some elementary properties, the orders of approximation
and the Voronovskaya type theorem and quantitative Voronovskaya type theorem
for these operators. Also simultaneous approximation property is obtained.

Let \( q > 0 \) be a fixed number and let

\[ \nu_q(x) := e^{-qx^2}, \quad x \in \mathbb{R}. \] (2)

For a fixed \( 1 \leq p \leq \infty \) and \( q > 0 \) we denote by \( L_{p,q}^r(\mathbb{R}) \) the set of all real-valued
functions \( f \) defined on \( \mathbb{R} \) for which the \( p- \) th power of \( \nu_q f \) is Lebesgue-integrable
on \( \mathbb{R} \) if \( 1 \leq p < \infty \), and \( \nu_q f \) is uniformly continuous and bounded on \( \mathbb{R} \) if \( p = \infty \).
Let the norm in \( L_{p,q}^r \) be given below by the formula

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weighted space.
\[ \|f\|_{p,q} = \|f(\cdot)\|_{p,q} := \begin{cases} \left( \int_{-\infty}^{\infty} |\nu_q(x)f(x)|^p \, dx \right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \sup_{x \in \mathbb{R}} |\nu_q(x)f(x)|, & \text{if } p = \infty. \end{cases} \tag{3} \]

Also, let \( r \in \mathbb{N}_0 \) and \( L_{q,r}^p \equiv L_{q,r}^p(\mathbb{R}) \) be the class of all \( r \)-times differentiable functions \( f \in L_q^p \) having the derivatives \( f^{(k)} \in L_q^p, 1 \leq k \leq r \). The norm in \( L_{q,r}^p \) is given by \( (3) \). The spaces \( L_q^p \) and \( L_{q,r}^p \) are called exponential weighted spaces (see [2]).

For \( f \in L_q^p \) we define the modulus of smoothness of the order two (see [5])
\[ \omega_2(f, L_q^p; t) := \sup_{|h| \leq t} \|\Delta_h^2 f(\cdot)\|_{p,q} \text{ for } t \geq 0, \tag{4} \]
where
\[ \Delta_h^2 f(x) := f(x+h) - f(x-h) - 2f(x), \quad x, h \in \mathbb{R}. \tag{5} \]
From (3)-(5) for \( f \in L_q^p \) follows
\[ \|f(\cdot + h)\|_{p,q} \leq e^{qh^2} \|f(\cdot)\|_{p,q}, \quad h \in \mathbb{R}, \tag{6} \]
\[ 0 = \omega_2(f; L_q^p; 0) \leq \omega_2(f; L_q^p; t_1) \leq \omega_2(f; L_q^p; t_2) \text{ if } 0 \leq t_1 < t_2. \tag{7} \]

Using the identity (see [6])
\[ \Delta_{nh}^2 f(x) = \sum_{k=1}^{n} k \Delta_h^2 f(x - (n - k)h) + \sum_{k=1}^{n} (n - k) \Delta_h^2 f(x + kh), \]
\( x, h \in \mathbb{R}; \ n = 2, 3, \ldots \), and by (2) and (6) we can prove that
\[ \omega_2(f; L_q^p; \lambda t) \leq (1 + \lambda)^2 e^{q(t\lambda)^2} \omega_2(f; L_q^p; t) \text{ for } \lambda, t \geq 0. \tag{8} \]

2. AUXILIARY RESULTS

In this part, we shall give some fundamental properties of the Gauss-Weierstrass integral operators \( W_n \) in the spaces \( L_{p,2q}(\mathbb{R}) \).

**Lemma 1.** The equality
\[ \int_0^\infty t^r e^{-nt^2} \, dt = \frac{1}{2n^\frac{r+1}{2}} \Gamma\left(\frac{r+1}{2}\right) \]
holds for every \( r \in \mathbb{N}_0 \) and \( n > 0 \).
Lemma 2. Let \( e_0(x) = 1, \ e_1(x) = x \) and let \( \varphi_x(t) = t - x \) for \( x, t \in \mathbb{R} \) and \( k \in \mathbb{N} \). Then,
\[
W_n(e_i; x) = e_i(x), \text{ for } x \in \mathbb{R}, n \in N, \ i = 0, 1
\]
\[
W_n \left( \varphi_x^k(t); x \right) = \frac{((-1)^k + 1) \Gamma \left( \frac{k+1}{2} \right)}{2 \sqrt{\pi n^2}}
\]
\[
W_n \left( \left| \varphi_x(t) \right|^k \exp(q |\varphi_x(t)|^2); x \right) = \frac{n}{\pi} \left[ \Gamma \left( \frac{k+1}{2} \right) \right], \ n > q + 1
\]

Lemma 3. Let \( f \in L_{p,q} (\mathbb{R}) \), with fixed \( 1 \leq p \leq \infty, q > 0 \). Then for \( n > 2q + 1 \), we have
\[
\| W_n f \|_{p,2q} \leq \sqrt{\frac{n}{n - 2q}} \| f \|_{p,q}
\]

Lemma 4. Let \( f \in L_{p,q} (\mathbb{R}) \) with fixed \( 1 \leq p \leq \infty \) and \( q > 0 \) and let \( n \in N \). Let \( f \in L_{\infty,q} (\mathbb{R}) \) with a fixed \( r \in \mathbb{N} \). Then \( W_n f \in L_{\infty,q} (\mathbb{R}) \) and for derivatives of \( W_n f \) there holds
\[
\left\| (W_n f)^{(k)} \right\|_{\infty,2q} = \left\| W_n f^{(r)} \right\|_{\infty,2q} \leq \sqrt{\frac{n}{n - 2q}} \left\| f^{(k)} \right\|_{\infty,q}
\]

Proof. For details see [9].

3. APPROXIMATION RESULTS

Theorem 5. Let \( f \in L_{p,q} (\mathbb{R}) \) with fixed \( 1 \leq p \leq \infty \), \( q > 0 \) and \( n > q + 1 \). Then we have
\[
\| W_n (f) - f \|_{p,2q} \leq \omega_2 \left( f, L_p^2; \frac{1}{\sqrt{n}} \right) \left[ \frac{1}{2} \sqrt{\frac{n}{n - q}} + \frac{2n}{\sqrt{\pi (n - q)}} + \frac{n^{3/2}}{4 (n - q)^2} \right].
\]

Proof. From [1] and [5] we get
\[
W_n (f; x) - f(x) = \sqrt{\frac{n}{\pi}} \int_0^\infty \Delta_x^2 f(x)e^{-nt^2} dt
\]
for \( x \in \mathbb{R} \) and \( n > q + 1 \). By [4] and [8], we get
\[
\| W_n (f) - f \|_{p,2q} \leq \sqrt{\frac{n}{\pi}} \int_0^\infty \| \Delta_x^2 f(x) \|_{p,q} e^{-nt^2} dt
\]
\[ \left( f, L_p^q; \frac{1}{\sqrt{n}} \right) \int_0^{\infty} \left( 1 + \sqrt{n}t \right)^2 e^{-t^2(n-q) dt}. \]

Using Lemma 1, we obtain

\[ \| W_n(f) - f \|_{p,2q} = \omega_2 \left( f, L_p^q; \frac{1}{\sqrt{n}} \right) \left[ \frac{1}{2} \sqrt{n-2} \frac{2n}{\sqrt{\pi} (n-q)} + \frac{n^3}{4(n-q)^{3/2}} \right]. \]

Thus the theorem is completed.

**Corollary 6.** Let \( f \in L_{p,q}(\mathbb{R}) \) with fixed \( 1 \leq p \leq \infty, q > 0 \) and \( n > q + 1 \). Then

\[ \lim_{n \to \infty} \| W_n(f) - f \|_{p,2q} = 0. \] (13)

Applying Corollary 1, we shall prove the Voronovskaya-type theorem for \( W_n \).

**Theorem 7.** Let \( f \in L_{q,2}^\infty(\mathbb{R}) \) has second derivate at a point \( x \in \mathbb{R} \) and with a fixed \( q > 0 \). Then we have

\[ \lim_{n \to \infty} n | W_n(f;x) - f(x) | = \frac{f''(x)}{4}. \]

**Proof.** For \( f \in L_{q,2}^\infty \) and \( x \in \mathbb{R} \). Then we can use Taylor formula in the form

\[ f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \mu(t;x)(t-x)^2 \]

for \( t \in \mathbb{R} \),

where \( \mu(t) = \mu(t;x) \) is a function belonging to \( L_{q}^{\infty} \) and

\[ \lim_{t \to x} \mu(t;x) = \mu(x) = 0. \]

Using the operator \( W_n \), (9) and (10), we get

\[ W_n(f(t);x) = f(x) + f'(x)W_n(t-x;x) \]
\[ + \frac{1}{2} f''(x)W_n((t-x)^2;x) + W_n(\mu(t)\varphi_x^2(t);x) \]
\[ = f(x) + \frac{1}{4n} f''(x) + W_n(\mu(t)\varphi_x^2(t);x) \]

and by the Hölder inequality and (10), we have

\[ | W_n(\mu(t)\varphi_x^2(t);x) | \leq (W_n(\mu^2(t);x)^{\frac{1}{2}} W_n(\varphi^4(t);x)^{\frac{1}{2}} \]
\[ = n^{-1} \left( \frac{3}{4} W_n(\mu^2(t);x) \right)^{\frac{1}{2}}. \]

From properties of \( \mu \) and (13) there result that

\[ \lim_{n \to \infty} W_n(\mu^2(t);x) = \mu^2(x) = 0. \]
Thus we have
\[
\lim_{n \to \infty} n W_n (\mu(t) \varphi_x^2(t); x) = 0
\]
from \([14]\) we have desired result. \(\square\)

**Theorem 8.** Let \(f \in L_q^{\infty,2}(\mathbb{R})\) with a fixed \(q > 0\). Then

\[
\| 4n [W_n(f) - f] - f'' \|_{\infty,2q} \leq \omega_1 \left( f''; L_q^{\infty} \right) \left[ \frac{1}{4} \left( \frac{n}{n-q} \right)^{\frac{3}{2}} + \frac{1}{2\sqrt{\pi}} \left( \frac{n}{n-q} \right)^2 \right].
\]  

(15)

**Proof.** For \(f \in L_q^{\infty,2}\) and \(x, t \in \mathbb{R}\) there holds the Taylor-type formula

\[
f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + (t-x)^2 I(t,x),
\]

where

\[
I(t,x) := \int_0^1 (1-u) \left[ f''(x+u(t-x)) - f''(x) \right] du.
\]  

(16)

Using operator \(W_n\), and \([9]-[11]\), we get

\[
W_n(f(t); x) = f(x) + \frac{1}{4n} f''(x) + W_n (\varphi_x^2(t) I(t,x); x),
\]

which implies that

\[
4n [W_n(f; x) - f(x)] - f''(x) = n W_n (\varphi_x^2(t) I(t,x); x)
\]

for \(x \in \mathbb{R}\). Now, applying \([4], [7]\) and \([8]\), we get

\[
|I(t,x)| \leq \int_0^1 (1-u) \omega_1 \left( f''; L_q^{\infty} \right) e^{qx^2} du
\]

\[
\leq \frac{1}{2} \omega_1 \left( f''; L_q^{\infty} \right) e^{qx^2}
\]

\[
\leq \frac{1}{2} \omega_1 \left( f''; L_q^{\infty} \right) \left( 1 + \sqrt{n} |t-x| \right) e^{q x^2 + q|t-x|^2}
\]

and next by \([2]\) and \([11]\), we can write for \(x \in \mathbb{R}\) and \(n > q + 1\),

\[
n \nu_q(x) W_n (\varphi_x^2(t) I(t,x); x) \leq \frac{n}{2} \omega_1 \left( f''; L_q^{\infty} \right) \left[ \frac{1}{\sqrt{n}} \right] \times \left\{ W_n ((t-x)^2 e^{q|t-x|^2}; x) + \sqrt{n} W_n ((t-x)^3 e^{q|t-x|^2}; x) \right\}
\]

\[
= \omega_1 \left( f''; L_q^{\infty} \right) \left[ \frac{1}{4} \left( \frac{n}{n-q} \right)^{\frac{3}{2}} + \frac{1}{2\sqrt{\pi}} \left( \frac{n}{n-q} \right)^2 \right].
\]

Now the estimate \([15]\) is obtained by \([16]\), the last inequality and \([3]\). \(\square\)
Theorem 9. Let $f \in L_{q}^{\infty,r}$, with fixed $q > 0$ and $r \in \mathbb{N}$. Then

$$
\left\| W_{n}^{(r)}(f) - f^{(r)} \right\|_{\infty,2q} \leq \omega_{2} \left( f^{(r)}; L_{q}^{\infty}; \frac{1}{\sqrt{n}} \right) \times \left( \sqrt{\frac{n}{n - 2q}} + \frac{n}{(n - 2q) \sqrt{\pi}} + \frac{1}{4} \left( \frac{n}{n - 2q} \right)^{\frac{3}{2}} \right)
$$

for $n > 2q + 1$.

Proof. If $f \in L_{q}^{\infty,r}$, then for $r$-th derivative of $W_{n}(f)$ we have by Lemma 4, (9) and (10):

$$
W_{n}^{(r)}(f;x) - f^{(r)}(x) = \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} \left[ f^{(r)}(x + t) - f^{(r)}(x) \right] e^{-nt^{2}} dt
$$

$$
= \sqrt{\frac{n}{\pi}} \int_{0}^{\infty} \left[ \Delta^{2}_{r} f^{(r)}(x - t) \right] e^{-nt^{2}} dt.
$$

from this and by (4), (8) and Lemma 1 we deduce that

$$
\left\| W_{n}^{(r)}(f;x) - f^{(r)}(x) \right\|_{\infty,2q} \leq \sqrt{\frac{n}{\pi}} \int_{0}^{\infty} \omega_{2} \left( f^{(r)}; L_{q}^{\infty}; t \right) e^{-(n-q)t^{2}} dt
$$

$$
\leq \omega_{2} \left( f^{(r)}; L_{q}^{\infty}; \frac{1}{\sqrt{n}} \right) \sqrt{\frac{n}{\pi}} \int_{0}^{\infty} \left( 1 + \sqrt{n}t \right)^{2} e^{-t^{2}(n-2q)} dt
$$

$$
= \omega_{2} \left( f^{(r)}; L_{q}^{\infty}; \frac{1}{\sqrt{n}} \right) \times \sqrt{\frac{n}{\pi}} \left[ \int_{0}^{\infty} e^{-t^{2}(n-2q)} dt + 2\sqrt{n} \int_{0}^{\infty} te^{-t^{2}(n-2q)} dt + \frac{\sqrt{n}}{2} \int_{0}^{\infty} t^{2} e^{-t^{2}(n-2q)} dt \right]
$$

$$
= \omega_{2} \left( f^{(r)}; L_{q}^{\infty}; \frac{1}{\sqrt{n}} \right) \times \left( \sqrt{\frac{n}{n - 2q}} + \frac{n}{(n - 2q) \sqrt{\pi}} + \frac{1}{4} \left( \frac{n}{n - 2q} \right)^{\frac{3}{2}} \right),
$$

for $n > 2q + 1$, which yields the estimate (17). \( \square \)

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Current address: Department of Mathematics, Faculty of Science and Arts, Kirikkale University, Yahşihan, 71450 Kiriklale, Turkey
E-mail address: basaryilmaz77@yahoo.com
ORCID Address: http://orcid.org/0000-0003-3937-992X