A Strictly NSOP
3 Theory

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We define the new notion of a $\mu$-uniformly-counter-collapsed structure using which we introduce a theory $T_{\mu}$ serving as the first example of a strictly NSOP
3 theory. This answers a question posed in [DS04] and reappeared in [SU08] and [MS16]. The existence of such a theory provides a better understanding of the hierarchy of unstable theories without the strict order property.

Keywords: Strong order property of the third kind (SOP
3), SOP
2, strict order property (SOP), generic structures, Hrushovski constructions, Fraïssé-Hrushovski limits, amalgamation classes.

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1. Introduction

The question of whether the strong order property of the second kind (SOP
2) implies, for a theory, the property of SOP
3 has been frequently asked in the literature since the introduction of the notion of SOP
2 in [DS04] (see 4.1 and 4.3 for definitions).

This question was addressed again in Malliaris and Shelah’s seminal work in [MS16] where the authors mentioned the need for a framework capable of studying the possible interactions between trees and orders in a comparative outlook (Discussion 10.12 in [MS16]). Finding a strictly NSOP
3 theory addresses this issue in a concrete way and provides an example of a theory inside which there are only found purely tree-like objects that are not built out of an order.

In the rest of this introduction, we further clarify the origins and motivations behind the problem together with explaining the techniques that are used in this paper.

Orders and Maximality. Since the introduction of the notion of Keisler’s order in [Kei67], a long-standing trend in model theory has been seeking to find a syntactical/combinatorial characterization for theories that are maximal with regard to this order. This study initiated in [Kei67] by introducing the notion of

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a “versatile” formula the existence of which makes a theory maximal in Keisler’s order.

Shelah showed in [She78] that the strict order property (SOP) is sufficient for maximality in Keisler’s order. Recall that a complete theory $T$ has the strict order property if there is a formula $\varphi(\bar{x}; \bar{y})$ with $|\bar{x}| = |\bar{y}|$ that defines in some model of $T$ a partial strict order with an infinite chain.

In [She96], Shelah introduced his hierarchy of strong order properties, SOP$_n$ for $n \geq 3$, arranged to establish an infinite spectrum of dividing lines inside the class of theories without the strict order property. Literally, a formula has SOP$_n$ when it can define an infinite chain while lacking the ability to define a cycle of length $n$ (Definition 4.1).

For every $n \geq 3$, it was shown that the following irreversible implications hold

$$\text{SOP} \implies \text{SOP}_n \implies \text{non-simple}.$$ 

In the same paper, Shelah remarkably improved his aforementioned result by showing that the weakest of these properties, namely SOP$_3$, which is far beyond SOP is sufficient for a theory to be maximal in Keisler’s order.

Maximality and Trees. The notions of SOP$_1$ and SOP$_2$ were introduced in [DS04] and further studied in [SU08] to refine the gap between non-simple theories and theories without SOP$_3$ (NSOP$_3$). Despite their names, these newly-arrived properties were actually about trees rather than orders. The following implications was proved in [DS04]

$$\text{SOP}_3 \implies \text{SOP}_2 \implies \text{SOP}_1 \implies \text{non-simple}.$$ 

A real breakthrough was made almost recently in [MS16] by lifting the sufficient condition for maximality in Keisler’s order up to SOP$_2$. This result sets a meaningful shift from the hierarchy of orders to that of the trees regarding our conception of maximality in Keisler’s order. Their approach amounts to posing the conjecture that having SOP$_2$ for a theory is equivalent to maximality in Keisler’s order (Conjecture 1.2 in [MS16]).

The latter result, comparing to the aforementioned Shelah’s theorem stating that maximality in Keisler’s order is implied by SOP$_3$, highlights again the importance of the question of whether the class of SOP$_2$ theories is identical to that of the theories having SOP$_3$; a question which we answer negatively in this paper.

Outline and History of the Method. Our technique in constructing a strictly NSOP$_3$ theory is essentially based on the machinery of amalgamation methods used for deriving Fraïssé-Hrushovski generic limits from a suitable class of finite structures. However, we do not exactly follow the routine of that method.

Our approach fits primarily into the theme of the methods used by Hrushovski in his approach towards characterizing existentially closed models of a natural theory derived from a certain amalgamation class ([Hru97]). In his previous works in
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[Pol02] and [Pol03], the first author distinguished the need for a stronger axiom that is necessary to fully characterize the class of existentially closed models of a natural theory; the meaning of a natural theory will be clarified in subsection 2.2.

Starting by a certain natural-valued function $\mu$, we introduce a class of infinite structures that will be called $\mu$-uniformly-counter-collapsed, or $\mu$-ucc in short. Based on this class of structures, we define a theory, $T_{\mu}$, that is axiomatized by a suitable adaptation of what is called semigenericity in the literature together with strengthening it by a new axiom scheme whose origin, as mentioned above, is traced back to the work of the first author in [Pol02].

We will show that $T_{\mu}$ is the model completion of an inductive theory, denoted by $T_{\text{in}}$, obtained from an expansion by definition of the theory that axiomatizes the class of $\mu$-ucc structures, namely $T_{\text{in}}^{\mu}$. As mentioned earlier, this approach finds its origins in Hrushovski's works in [Hru97].

It is worth reviewing some related results obtained using the routine method of Hrushovski constructions which can somewhat explain the need for exploring new domains beyond the scope of ordinary Hrushovski constructions.

By focusing on Hrushovski constructions that are equipped with a control function, Evans and Wong showed in [EW09] that theories obtained in this way are either simple or strictly NSOP₄. In addition, they proved that removing this control function would bring about combinatorial configurations giving rise to the strict order property. In fact, by results in [BL12], the latter theory interprets Robinson arithmetic and therefore is considered to be a wild theory.

In a different theme, there are varieties of Hrushovski constructions inside which the closure is uniformly bounded by the algebraic closure resulting in axiomatizable tame theories happened to be stable or simple. Therefore, these methods are not either capable of finding theories having SOP₂ or even SOP₁.

The class of $\mu$-ucc structures represents a twofold nature by letting closure exceed the bounds of the algebraic closure while still yielding a theory that is tame enough to be NSOP₃.

Historically, Hrushovski construction was introduced in [Hru93] to refute Zilber's trichotomy conjecture on the pregeometries available inside $\aleph_1$-categorical structures ([Zil84a] and [Zil84b]). As mentioned earlier, this method was also used to produce new examples in the hierarchy of classification theory which includes Baldwin's result on introducing an almost strongly minimal non-Desargusian projective plane in [Bal94], the study of stable generic structures in [BS96], and applying this method to the context of simplicity in [Eva02], [Pol02] and [Pol03].

The key notion of semigenericity was introduced in [BS97] by Baldwin and Shelah in which they generalized significantly Shelah and Spencer’s seminal results in [SS88] on proving the existence of 0-1 laws for a large family of random graphs. Semigenericity, considered as a first-order version of genericity, plays a central role in the results appearing in [BS97] towards finding a complete axiomatization for
the almost sure theory of random hypergraphs.

Organization of the Paper. In Section 2, we first provide a concise account on Hrushovski constructions built using predimension functions. We proceed in subsection 2.2 by recalling an expansion by definition of the language together with proving some basic properties for it. This expanded language will be denoted by $L^+$ and will be associated with a theory called $T_{\text{nat}}$ giving a natural interpretation to the new augmented predicates of $L^+$.

In Section 3, we define $T_\mu$ that is built upon the notion of a $\mu$-uniformly-counter-collapsed ($\mu$-ucc) structure. The class of $\mu$-ucc structures will be axiomatized by the $L$-theory $T_\mu$. One of the main goals of this section is to prove the consistency of $T_\mu$ addressed in subsection 3.1. In fact, via two main steps in Lemmas 3.11 and 3.13 we show that any existentially closed model of the inductive $L^+$-theory $T_{\text{nat}} \cup T_\mu$ is a model of $T_\mu$. We continue in subsection 3.2 by proving that $T_\mu$ admits a quantifier elimination down to the predicates available in $L^+$ (Theorem 3.16). This, using the results of subsection 3.1, ultimately implies that $T_\mu$ fully axiomatizes the class of existentially closed models of $T_{\text{nat}} \cup T_\mu$.

Section 4 is focused on showing that $T_\mu$ is strictly NSOP$_3$ based on Lemma 4.5 that distinguishes a key property of indiscernible sequences in the context of Hrushovski constructions built upon a predimension function. In a few number of remarks in Section 5, we state some other properties of $T_\mu$; we will close by suggesting some open problems.

2. Preliminaries

The necessary preliminaries on Hrushovski constructions will be provided in subsection 2.1. In subsection 2.2, we will review a natural expansion by definition of the language that forms the initial step towards proving quantifier elimination for $T_\mu$.

Throughout, we will be working in a language $L$ that only consists of a ternary relation $R$ whose interpretation $R^A$ is always irreflexive and symmetric; that is, $R^A$ holds only for tuples of distinct elements and whenever it holds for an ordered tuple $(x, y, z)$, then any other permutation of $x, y, z$ will also satisfy $R^A$.

Notation 2.1. By $A, B, C, \ldots$ we denote finite $L$-structures while $M, N, \ldots$ represent arbitrary structures that might be infinite. By $A \subseteq \omega M$, we mean that $A$ is a finite substructure of $M$.

Definition 2.2. For structures $M_0, M_1$ and $M_2$ with $M_1 \cap M_2 = M_0$, the free join or the free amalgamation of $M_1$ with $M_2$ over $M_0$ is denoted by $M_1 \otimes M_2$ and is defined as the $L$-structure with universe $M_1 \cup M_2$ whose set of relations consists only of $R^{M_1} \cup R^{M_2}$. In other words, no relation having components both in $M_1 \setminus M_0$ and $M_2 \setminus M_0$ does exist in the free join.
2.1. Hrushovski Constructions Using a Predimension

An integer-valued function $\delta$, called the predimension function, is defined over all finite structures. This function measures in a way the sparsity of a finite $\mathcal{L}$-structure; in fact, the more a structure is dense, the less is its assigned value by $\delta$. Using this predimension function, the class of all finite $\mathcal{L}$-structures can be equipped by two binary relations $\leq$ and $<$ each strengthening the ordinary notion of a substructure.

By generalizing Fraïssé amalgamation methods, the machinery of Hrushovski constructions was first introduced in $[Hru93]$ to refute the Zilber’s trichotomy conjecture on the available pregeometries inside an $\aleph_1$-categorical structure. More details on these constructions can be found in $[Wag94]$ and $[BS96]$.

For any finite $\mathcal{L}$-structure $A$, let

$$\delta(A) := |A| - [R^A],$$

where $[R^A]$ denotes $\frac{|R^A|}{3!}$. In an $\mathcal{L}$-structure $M$, for two substructures $A$ and $B$, whether intersecting each other or not, the relative predimension of $B$ over $A$ is defined as

$$\delta(B/A) := \delta(AB) - \delta(A),$$

where $AB$ denotes the structure that is induced by $M$ on $A \cup B$.

**Definition 2.3.** Let $A \subseteq B$ be finite.

(i) $A$ is said to be weakly closed, or $w$-closed, in $B$ if for every structure $C$ with $A \subseteq C \subseteq B$ we have that $\delta(C/A) \geq 0$. This is denoted by $A \leq B$.

(ii) $A$ is called strictly closed, or closed, in $B$ if for every structure $C$ with $A \varsubsetneq C \subseteq B$ we have that $\delta(C/A) > 0$. In notations we write $A < B$.

(iii) We say that $A$ is $n$-weakly closed in $B$, denoted by $A \leq_n B$, if for every structure $C$ with $A \subseteq C \subseteq B$ and $|C\setminus A| \leq n$ we have that $A \leq C$.

(iv) For $A \subseteq_\omega M$, the substructure $A$ is said to be closed/w-closed in $M$ if $A$ is closed/w-closed in any finite substructure $B \subseteq M$ containing $A$. These notions are respectively denoted by $A < M$ and $A \leq M$. Similarly, we can define $A$ being $n$-weakly closed in $M$ which is written as $A \leq_n M$.

Given $A \subseteq_\omega M$, all intermediate finite structures $A \subseteq C \subseteq_\omega M$ that prevent $A$ from being closed/w-closed in $M$ are collected in a set that is called the closure/w-closure of $A$ in $M$. The following definition can provide a uniform description regarding the building blocks of the closure/w-closure in this context.

**Definition 2.4.** Let $A \subseteq B$.

(i) $B$ is called a zero minimal extension of $A$, if $A \not< B$ but $A \leq B$ and for every $C$ with $A \subseteq C \subseteq B$ we have that $A < C$.

(ii) We say that $B$ is a negative minimal extension of $A$, if $A \not\leq B$ and for every $C$ with $A \subseteq C \subseteq B$ we have that $A < C$. 

Remark 2.5. If $B$ is a zero minimal extension of $A$, then we have that $\delta(B/A) = 0$ while for a negatively minimal extension, we have that $\delta(B/A) < 0$.

Definition 2.6. Let $A \subseteq B$ be finite. $B$ is said to be an intrinsic extension of $A$, denoted by $A <_{i} B$, if it is the union of a chain of structures like

$$A = B_{0} \subseteq B_{1} \subseteq \cdots \subseteq B_{n} = B,$$

where for each $1 \leq i \leq n - 1$, the structure $B_{i}$ is a zero minimal extension over $B_{i-1}$ and the structure $B_{n}$, depending on the value of $\delta(B/A)$, is either a zero or a negative minimal extension of $B_{n-1}$.

Definition 2.7. Let $A \subseteq \omega M$.

(i) The closure of $A$ in $M$ is defined as

$$\cl_{M}(A) := \bigcup \left\{ B \subseteq \omega M \bigg| A <_{i} B \right\}.$$

(ii) The weak closure, or w-closure of $A$ in $M$ is defined as

$$\cl_{M}^{w}(A) := \bigcup \left\{ B \subseteq \omega M \bigg| A <_{i} B \text{ and } \delta(B/A) < 0 \right\}.$$

(iii) If $N \subseteq M$, the closure of $N$ in $M$ is defined as the union of the closures of all finite substructures of $N$, namely

$$\cl_{M}(N) := \bigcup_{A \subseteq \omega N} \cl_{M}(A).$$

The w-closure of $N$ in $M$ is defined in a similar way.

(iv) The structure $N \subseteq M$ is called closed/w-closed in $M$, denoted by $N <_{M} M$ and $N \leq_{M} M$ respectively, if the closure/w-closure of $N$ in $M$ is equal to $N$.

Remark 2.8. It can be verified that the closure/w-closure of a substructure $N \subseteq M$ is the smallest subset of $M$ that contains $N$ and is closed/w-closed in $M$.

The fact below is easily followed from the above definitions.

Fact 2.9. Let $A, B$ and $C$ be finite substructures of $M$.

(i) If $A <_{i} B$ and $A <_{i} C$, then $A <_{i} BC$. Moreover, we have that $B <_{i} BC$ and $C <_{i} BC$.

(ii) If $A <_{i} B$ and $A \subseteq C$, then $AC <_{i} BC$.

To obtain the suitable properties required for the machinery of amalgamation classes we restrict ourselves to the following subclass of finite $\mathcal{L}$-structures.

$$\mathcal{K}_{0} := \left\{ A \bigg| A \text{ is a finite } \mathcal{L}\text{-structure with } \emptyset < A \right\}.$$ 

Also, $\overline{\mathcal{K}}_{0}$ denotes the class of $\mathcal{L}$-structures $M$ whose finite substructures belong to $\mathcal{K}_{0}$.

Notation 2.10. Let $A \subseteq \omega M \in \overline{\mathcal{K}}_{0}$ and $A \subseteq B$. 
(i) The number of distinct realizations or copies of $B$ over $A$ in $M$ is denoted by $\chi_M(B/A)$.

(ii) The maximal number of the disjoint realizations or copies of $B$ over $A$ in $M$ is denoted by $\chi^*_M(B/A)$.

The value of $\chi_M$ and $\chi^*_M$ is considered to be either a natural number or $\infty$.

Fact 2.11. For any $A \in \mathcal{K}_0$ and for every structure $M \in \mathcal{K}_0$ with $A \subseteq M$, if $B$ is an intrinsic extension of $A$ with $\delta(B/A) < 0$, then we have that

$$\chi_M(B/A) \leq \frac{\delta(A)}{|R^{B/A}|} \leq \delta(A) \leq |A|,$$

where $R^{B/A}$ denotes the set of all relations with at least one of their components belonging to $A$ and at least another belonging to $B \setminus A$.

Remark 2.12. A particular consequence of Fact 2.11 is that for structures $A \subseteq \omega \ M \in \mathcal{K}_0$ and $A \preceq_i B$, if $\chi_M(B/A) = \infty$, then the relative predimension of $B$ over $A$ is necessarily zero or equivalently we have that $A \subseteq B$.

Fact 2.11 shows that in any structure $M \in \mathcal{K}_0$ the number of realizations of an intrinsic extension with a negative predimension over the base set is uniformly bounded above. In particular, we have the following fact which is well known in the literature.

Fact 2.13. Suppose that $A \subseteq \omega \ M \in \mathcal{K}_0$. Then $\text{cl}^w_M(A)$ is finite and is a subset of the algebraic closure of $A$ in $M$.

Lemma 2.14. Let $B$ and $C$ be distinct intrinsic extensions of $A$. In any structure $M \in \mathcal{K}_0$ with

$$\chi^*_M(B/A) = \infty \quad \text{and} \quad \chi^*_M(C/A) = \infty,$$

there exist infinitely many disjoint copies of $B$ and $C$ that are mutually in free amalgamation over $A$.

Proof. By Remark 2.12 we know that $\delta(B/A) = \delta(C/A) = 0$. Fix a copy of $B$ over $A$, say $B_1$ and note that for another copy of $B$ that is disjoint from $B_1$ over $A$, say $B_2$, if there are relations preventing $B_2$ from being freely amalgamated with $B_1$ over $A$, then we will have that

$$\delta(B_2/AB_1) < \delta(B_2/A) = 0.$$

This shows that there exist at most $\delta(B_1)$-many of such copies over $A$ since otherwise, a finite structure with a negative predimension will appear inside $M$ which is impossible due to the fact that $M \in \mathcal{K}_0$. Therefore, there are infinitely many disjoint copies of $B$ that are in free amalgamation with $B_1$ over $A$. Iterating this argument leads to finding infinitely many copies of $B$ that are mutually in free amalgamation over $A$; let $\{B_i\}_{i \in \omega}$ enumerate them.
Now, by fixing a copy of $C$, say $C_1$, and considering the fact that the structures $B_i$ are mutually disjoint over $A$, we obtain an infinite subsequence of $\{B_i\}_{i \in \omega}$ that are disjoint from $C_1$ over $A$. Again, this argument can be iterated in order to find the desired disjoint copies in $M$.

**Notation 2.15.** The first-order theory $T_0$ consists of all formulas of the form $\forall \bar{x} \neg \text{Diag}_A(\bar{x})$ where $A$ ranges over all finite structures $A \notin K_0$.

The following has fairly an easy proof but since it is used several times in this work we indicate it as a lemma.

**Lemma 2.16.** If $A \subseteq \omega, M \in \overline{K_0}$ is weakly closed, then any intrinsic extension of $A$ in $M$ is also weakly closed.

**Proof.** Suppose that $B$ is an intrinsic extension of $A$ that is not weakly closed. Then, there exists a finite structure $C \subseteq \omega, M$ satisfying $\delta(C/B) < 0$. Hence, we have that

$$\delta(C/A) = \delta(C/B) + \delta(B/A) < \delta(B/A) \leq 0$$

which contradicts the assumption that $A$ is weakly closed in $M$.

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**2.2. A Natural Expansion of the Language**

Following Hrushovski’s works in [Hru97] and the results of the first author in [Pou02] and [Pou03], we will define an expansion by definition $L^+$ of $L$ to make the notion of a substructure capable of preserving a notable part of closures in $L^+$-extensions. More precisely, given two arbitrary $L^+$-structures $M \subseteq N$, the new augmented predicate symbols together with their supporting axioms would make every finite part of $\text{cl}_N(M)$ realizable in $M$; in particular, $M \subseteq N$ would imply that $M \leq N$. Let

$$L^+ := L \cup \left\{ R_{(A,B)}(\bar{x}) \mid A <_i B \in K_0, |\bar{x}| = |A| \right\},$$

and define $T_{\text{nat}}$ to be the following inductive theory ensuring that the new predicates are interpreted in a natural way:

$$T_{\text{nat}} := T_0 \cup \left\{ \forall \bar{x} \left[ R_{(A,B)}(\bar{x}) \leftrightarrow \left( \text{Diag}_A(\bar{x}) \land \exists \bar{y} \text{Diag}_B(\bar{x}; \bar{y}) \right) \right] \mid A <_i B \in K_0 \right\}.$$

**Remark 2.17.** Any $L$-structure $M \in \overline{K_0}$ can be uniquely expanded to an $L^+$-structure that is a model of $T_{\text{nat}}$. However, to ease notation, in dealing with models of $T_{\text{nat}}$ we identify $M$ with its expansion.

**Lemma 2.18.** Suppose that $M, N \models T_{\text{nat}}$.

(i) If $M < N$, then $M \subseteq N$. 

Hence, by part (ii) of this lemma, we have
\[ \chi_M(B/A) = \chi_N(B/A) \]
and every copy of $B$ over $A$ in $N$ lies inside $M$. Moreover, if $\chi_N(B/A) = \infty$, then $\chi_M(B/A) = \infty$.

(iii) If $M \subset N$, then $M \leq N$.

**Proof.** Part (i) directly follows from the definitions. For part (ii), by Fact 2.9 any finite union of intrinsic extensions is again an intrinsic extension. This, using the hypothesis that $M \subset N$ yields the result. For part (iii), by Fact 2.11 for any $A \subseteq M$ and any intrinsic extension $B$ of $A$ with $\delta(B/A) < 0$ we have that
\[ \chi_M(B/A) < \infty. \]

Hence, by part (ii) of this lemma, we have $B \subseteq M$ which proves that $M \leq N$. \(\square\)

A more detailed account of the idea of expanding $\mathcal{L}$ to $\mathcal{L}^+$ can be found in [Pou02] and [Pou03] where it was used to study Hrushovski constructions in the context of simplicity.

**Lemma 2.19.**

(i) Let $\langle K, \subseteq \rangle \in \{ \langle \mathbb{K}_\omega, \leq \rangle, \langle \mathbb{K}_\omega, < \rangle \}$. Then $\langle K, \subseteq \rangle$ has the full amalgamation property; i.e. if $M_0, M_1$ and $M_2$ are $\mathcal{L}$-structures in $K$ with $M_0 \subseteq M_1$ and $M_0 \subseteq M_2$, then $M = M_1 \otimes M_2$ belongs to $K$ and $M_1 \subseteq M$. Furthermore, if $M_0 \subseteq_n M_1$ then $M_2 \subseteq_n M$.

(ii) $\langle \text{Mod}(T_{\text{nat}}), \subseteq_+ \rangle$ has the full amalgamation property.

**Proof.** Part (i) is well known in the literature.

For part (ii), Suppose that $M_0 \subseteq M_1, M_0 \subseteq M_2$ and let $M := M_1 \otimes M_2$. Part (i) of Lemma 2.18 implies that $M_0 \leq M_2$ and hence, by part (i), $M$ belongs to $\mathbb{K}_\omega$.

To show that $M_1 \subseteq M$, let $A \prec B$ with $A \subseteq M_1$ and $B \subseteq M$. Since $B \cap M_1 \leq B$ and $B \cap M_1$ is already a subset of $M_1$, without loss of generality, we may assume that $B \cap M_1 = \emptyset$ and $B \subseteq M_2 \setminus M_1$. Let $A_0 := A \cap M_0$ and $A_1 := A \setminus M_2$. Since $B$ is in free amalgamation with $A \setminus M_2$ over $A \cap M_0$, we have that $A_0 \prec B$.

If $\chi_{M_2}(B/A_0) < \infty$, part (ii) of Lemma 2.18 shows that $B \subseteq M_0$.

In the case that $\chi_{M_2}(B/A_0) = \infty$, by part (ii) of Lemma 2.18 we have that $\chi_{M_0}(B/A_0) = \infty$. Since $B$ is an intrinsic extension of $A_0$, the relative predimension of $B$ over $A_0$ is less than or equal to zero. Hence, for a given copy of $B$ in $M_0$ that is not in free amalgamation with $A_1$ over $A_0$, say $B'$, we have that
\[ \delta(B'/A) < \delta(B'/A_0) \leq 0. \]

Therefore, by Fact 2.11 the number of distinct copies of such a structure $B'$ over $A$ in $M_0$ is bounded by $\delta(A)$. 

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On the other hand, considering the digram of $B$ over $A$, there are only finitely many possibilities available for $B$ to be not in free amalgamation with $A_1$ over $A_0$. Therefore, there exist infinitely many copies of $B$ inside $M_0$ that are in free amalgamation with $A_1$ over $A_0$. This, in particular, provides a copy of $B$ over $A$ in $M_1$ which shows that $M_1 \subseteq + M$.

3. Introducing $T_\mu$

By fixing a certain natural-valued function $\mu$ defined on pairs of intrinsic extensions, we consider a class of infinite structures called $\mu$-uniformly-counter-collapsed, or more briefly $\mu$-ucc structures. This class of structures will be axiomatized by an $\mathcal{L}$-theory $T^\infty_\mu$. Thereafter, we introduce an $\mathcal{L}^+$-theory $T_\mu$ that will eventually axiomatize the class of existentially closed models of $T^\infty_\mu \cup T_{\text{nat}}$. In subsection 3.1 we prove that $T_\mu$ is consistent by showing that every existentially closed model of $T^\infty_\mu \cup T_{\text{nat}}$ is a model of $T_\mu$. In subsection 3.2 we show that $T_\mu$ eliminates quantifiers in $\mathcal{L}^+$.

Axioms of $T_\mu$ are obtained by a suitable adaptation of what is called semigenericity in the literature together with strengthening it by a new axiom scheme that will be called the axiom of intrinsic semigenericity.

Convention 3.1. From now on, any finite structure under question would belong to $K^0_\mu$.

Assumption 3.2. We fix a function $\mu : \omega \setminus \{0\} \times \omega \setminus \{0\} \rightarrow \omega$ with the following properties:

(i) For any $m, n$ with $n \leq m$, we have that $\mu(m, n) = 0$.

(ii) For every $m, n$ with $n > m$, we have that $\mu(m, n) \geq n$.

(iii) $\mu$ is increasing according to the lexicographic order on $\omega \setminus \{0\} \times \omega \setminus \{0\}$.

Definition 3.3. A structure $M \in K^\infty_\mu$ is said to be $\mu$-uniformly counter-collapsed, or briefly $\mu$-ucc, if for every pair of finite structures $(A, B)$ with $A \subseteq \omega \cdot M$ and $A <_I B$ either we have

$$\chi^*_M(B/A) \leq \mu(|A|, |B|),$$

or $\chi^*_M(B/A) = \infty$. The class of all $\mu$-ucc structures is denoted by $K^\infty_\mu$.

Remark 3.4. The following first-order extension of $T_0$ axiomatizes the class $K^\infty_\mu$:

$$T^\infty_\mu := T_0 \cup \left\{ \forall \bar{x} \left[ \text{Diag}_A(\bar{x}) \land \exists^2 n \bar{y} \, \text{Diag}_B(\bar{x}; \bar{y}) \rightarrow \exists^2 n + 1 \bar{y} \, \text{Diag}_B(\bar{x}; \bar{y}) \right] \right\},$$

where $(A, B)$ ranges over all pairs of finite structures with $A <_I B$ while $n$ ranges over all natural numbers strictly greater than $\mu(|A|, |B|)$.

Notation 3.5. We denote the $\mathcal{L}^+$-theory $T^\infty_\mu \cup T_{\text{nat}}$ by $T^\infty_{\text{nat}}$.

Definition 3.6. Let $B \subseteq C$ be finite.
(i) A model $N \models T^\infty_\mu$ containing an isomorphic copy of $B$, say $\bar{b}$, is said to omit $C$ over $\bar{b}$ if

$$N \models \neg \exists z \text{ Diag}_C(\bar{b}; z).$$

(ii) $C$ is called $\mu$-ometric over $B$, or briefly omittable, if there exists a model $N \models T^\infty_\mu$ with an embedding $f : B \rightarrow N$ that omits $C$ over $fB$.

Before proceeding further, we introduce a set of first-order formulas that will be used in the rest of this section.

**Remark 3.7.**

(i) Given a finite structure $A$ and a natural number $n$, there is a formula $\sigma_{(\cdot, n)}(\bar{x})$ with $|\bar{x}| = |A|$ such that for any $M \in \mathcal{K}_n$ and any tuple $\bar{a} \in M$ we have that $M \models \sigma_{(\cdot, n)}(\bar{a})$ if and only if $\bar{a} \approx A$ and $\bar{a} \leq^n M$. Namely, $\sigma_{(\cdot, n)}(\bar{x})$ is the following formula

$$\text{Diag}_A(\bar{x}) \land \bigwedge_{B \in \mathcal{K}_{(\cdot, n)}} \neg \exists y_B \text{ Diag}_B(\bar{x}; y_B),$$

where $\mathcal{K}_{(\cdot, n)}$ is the collection of all finite structures $B \in \mathcal{K}_n$ with $|B \setminus A| \leq n$ and $A \not\subseteq B$.

(ii) Let $B$ be a finite structure and $\Omega$ a finite set of omittable structures over $B$. Then, there is a first-order formula as below denoted by $\omega_{B, 0}(\bar{y})$ with $|\bar{y}| = |B|$ which claims in any model $N \models T^\infty_\mu$ with $N \models \omega_{B, 0}(\bar{b})$ that the tuple $\bar{b}$ is isomorphic to $B$ and that $N$ omits each $C \in \Omega$ over $\bar{b}$:

$$\text{Diag}_B(\bar{y}) \land \bigwedge_{C \in \Omega} \neg \exists z \text{ Diag}_C(\bar{y}; z).$$

**Definition 3.8.** Let $A \subseteq_\omega M \in \mathcal{K}_\infty^\mu$ and $n \in \omega$. Also, suppose that $B \in \mathcal{K}_n$ with $A \subseteq B$ and $\Omega$ is a set of omittable structures over $B$ such that each $C \in \Omega$ is an intrinsic extension of $B$ while $A(C \setminus B)$ being not an intrinsic extension of $A$.

(i) $M$ is said to satisfy the axiom of semigenericity for $A, B, \Omega$ and $n$, if whenever $A \leq_n M$ and $A \not\prec B$ there exists an embedding $f : B \rightarrow M$ with $f|_A = \text{id}_A$ and $fB \leq_n M$ such that $M$ omits each $C \in \Omega$ over $fB$. That is, $M$ satisfies the following formula

$$\text{SEM}_{(A, B, \Omega, n)} : \forall \bar{x} \left[ \sigma_{(\cdot, n)}(\bar{x}) \rightarrow \exists y \left( \sigma_{(\cdot, 0)}(\bar{x}; y) \land \omega_{(\cdot, 0)}(\bar{x}; y) \right) \right].$$

(ii) We say that $M$ satisfies the axiom of intrinsic semigenericity for $A, B, \Omega$ and $n$, if whenever $A \leq_n M$ and $A \not\prec B$ with $\chi_M(B/A) = \infty$, there exists an embedding $f : B \rightarrow M$ with $f|_A = \text{id}_A$ and $fB \leq_n M$ such that $M$ omits each $C \in \Omega$ over $fB$. In other words, $M$ satisfies the following formula

$$\text{ISEM}_{(A, B, \Omega, n)} : \forall \bar{x} \left[ \left( \sigma_{(\cdot, n)}(\bar{x}) \land \exists^* y \chi^*_{(\cdot, 0)}(\bar{x}; y) \right) \rightarrow \exists y \left( \sigma_{(\cdot, 0)}(\bar{x}; y) \land \omega_{(\cdot, 0)}(\bar{x}; y) \right) \right].$$
Definition 3.9. $T^\infty_\mu$ denotes the following $\Pi_2$-axiomatizable $\mathcal{L}^+$-theory

$$T^\infty_{\text{nat}} \cup \left\{ \text{SEM}_{(A,B,\Omega,n)} \right\} \cup \left\{ \text{ISEM}_{(A,B,\Omega,n)} \right\},$$

where $A, B, \Omega$ and $n$ are correspondingly subject to the conditions mentioned in Definition 3.8.

### 3.1. Consistency

Following Hrushovski’s approach in [Hru97], we will show that any existentially closed model of $T^\infty_{\text{nat}}$ is a model of $T^\mu_\mu$. Towards this goal, Lemma 3.11 forms a major step by enabling us to extend canonically a given structure $M \in \mathcal{K}_\mu$ to a model of $T^\infty_\mu$, called the $\mu$-ucc-hull of $M$. As a stronger consequence, in Proposition 3.18 below, we will see that $T^\mu_\mu$ is actually the model completion of the theory $T^\infty_{\text{nat}}$.

**Lemma 3.10.** Given a countable $M \models T_0$, there exists a countable model $\overline{M} \models T_0$ extending $M$ with $M \preceq \overline{M}$ and having the following properties:

(i) For any $A <_1 B$ with $A \subseteq_\omega M$ and $\chi_M^*(B/A) > \mu(|A|, |B|)$ we have that $\chi_{\overline{M}}^*(B/A) = \infty$.

(ii) For any $N \models T^\infty_\mu$ and any embedding $f : M \rightarrow N$ with $fM \preceq N$, there exists an embedding $\overline{f} : \overline{M} \rightarrow N$ extending $f$ such that $\overline{fM} \preceq N$.

(iii) For any $M' \subseteq M$ with $M' \models T^\infty_\mu$, if $M' \preceq_+ M$ ($M' < M$) then we have that $M' \preceq_+ \overline{M}$ ($M' < \overline{M}$).

**Proof.** Let $\{(A_n, B_n) : n \geq 1\}$ enumerate all pairs that violate the $\mu$-ucc condition in $\mathcal{M}$; this means that for each $n \geq 1$ the structure $B_n$ is an intrinsic extension of $A_n$, and $\chi_M^*(B_n/A_n)$ is finite and is strictly greater than $\mu(|A_n|, |B_n|)$.

Let $M_0 := M$ and for $n \geq 1$ define $M_n$ as the free amalgamation of $M_{n-1}$ (over $A_n$) with infinitely many disjoint copies of $B_n$ that are mutually in free amalgamation over $A_n$. Finally, define $\overline{M}$ as the union of all structures $M_n$. It is obvious that each $M_n$ and hence $\overline{M}$ belongs to $\mathcal{K}_\mu$. Also, condition (i) in the lemma is obvious by our way of constructing $\overline{M}$ and condition (ii) follows from the construction and Lemma 2.14.

For condition (iii), suppose that $M' \preceq_+ M$. For every $n \geq 1$ we show that $M' \preceq_+ M_n$.

Let $A_n := A_n \cap M'$ and $L_n := \text{cl}_{B_n}(\bar{A_n})$. The case of $L_n = \bar{A_n}$ leads trivially to the stronger property of $M' < M_n$. Hence, we suppose that $L_n \setminus \bar{A_n} \neq \emptyset$.

Since $\chi_M^*(B_n/A_n) > \mu(|A_n|, |B_n|)$, we have that

$$\chi_M^*(L_n/\bar{A_n}) \geq \chi_M^*(B_n/A_n) > \mu(|A_n|, |B_n|).$$

This holds because any disjoint copy of $B_n$ over $A_n$ carries within itself a disjoint copy of $L_n$ over $\bar{A_n}$. 

On the other hand, by part (iii) in Assumption 3.2, we know that $\mu$ is an increasing function according to the lexicographic order. In particular, we have that $\mu(|A_n|, |B_n|) \geq \mu(|A_n|, |L_n|)$ which by the above inequality implies that

$$\chi_M^*(L_n/\bar{A}_n) > \mu(|A_n|, |L_n|).$$

Since $\bar{A}_n \subseteq M'$ and $M' \subseteq M$, there exist at least $\left(\mu(|\bar{A}_n|, |L_n|) + 1\right)$-many disjoint copies of $L_n$ over $\bar{A}_n$ inside $M'$. This, using the fact that $M' \models T_\mu^\infty$, leads to finding infinitely many disjoint copies of $L_n$ over $\bar{A}_n$ in $M'$. This shows that in the process of constructing $M_n$ we did not add any new finite intrinsic extension over $M'$ that had been already absent in $M'$; this proves that $M' \subseteq M_n$.

The hypothesis of $M' < M$ is reduced to the case of $L_n = \bar{A}_n$.

**Lemma 3.11.** Given a countable $M \models T_0$, there exists a countable extension $\mathcal{M} \models T_\mu^\infty$ with $M \leq \mathcal{M}$ that satisfies the properties (ii) and (iii) mentioned in Lemma 3.10. In particular, for any $\bar{b} \in M$ and $C \in K_\mu$ that is omittable over $\bar{b}$ we have that

$$\mathcal{M} \models \neg \exists \bar{z} \text{Diag}_C(\bar{b}; \bar{z}).$$

**Proof.** Let $M_0 := M$ and for each $n \geq 1$ define $M_n$ to be the structure $\overline{\mathcal{M}}_{n-1}$ obtained by applying Lemma 3.10 for $M_{n-1}$. Finally, let $\mathcal{M} := \bigcup_{n \in \omega} M_n$.

Property (i) in Lemma 3.10 guarantees that $\mathcal{M}$ belongs to $K_\mu^\infty$. It is easy to check that property (iii) is preserved under taking a union.

Given a model $N \models T_\mu^\infty$ with an embedding $f : M \xrightarrow{\leq} N$, by property (ii) of Lemma 3.10 we can appropriately embed each $M_n$ into $N$ over $M_{n-1}$ which ultimately gives the desired embedding of $\mathcal{M}$ into $N$. The fact that an omittable structure over a finite substructure of $M$ is not realized in $\mathcal{M}$ is a direct consequence of Definition 3.6 and property (ii) of the lemma.

**Notation 3.12.** For a countable model $M \models T_0$, the structure $\mathcal{M} \models T_\mu^\infty$ obtained in Lemma 3.11 is called the $\mu$-ucc-hull, or briefly the hull, of $M$ and is denoted by $\langle M \rangle^\infty_\mu$.

**Lemma 3.13.** Any existentially closed model of $T_\mu^\infty$ is a model of $T_\mu$.

**Proof.** Let $M$ be a countable existentially closed model of $T_\mu^\infty$. To show that $M$ satisfies the axiom SEM, suppose that $\bar{a} < \bar{a}\bar{b} \in K_\mu, \bar{a} \leq_n M$, and let $\Omega$ be a finite set of omittable structures over $\bar{a}\bar{b}$.

If $N_0$ denotes the free amalgamation of $M$ with $\bar{a}\bar{b}$ over $\bar{a}$, by Lemma 2.19 we know that $N_0$ belongs to $K_\mu$ and moreover $M < N_0$. Let $N$ be the $\mu$-ucc-hull of $N_0$, namely $N = \langle N_0 \rangle^\infty_\mu$, that is obtained by Lemma 3.11.

By Lemma 3.11 for each $C \in \Omega$ we have that

$$N \models \neg \exists \bar{z} \text{Diag}_C(\bar{a}\bar{b}; \bar{z}).$$
By Remark 3.7, this actually means that 
\[ N \models \omega_{(a,0)}(\bar{a} \bar{b}). \]

On the other hand, by Lemma 2.19 we have that \( \bar{a} \bar{b} \leq_n N \) which, using Lemma 3.11, implies that \( \bar{a} \bar{b} \leq_n N \). This, in particular, means that for any \( n \) we have 
\[ N \models a_{(a,0)}(\bar{a} \bar{b}). \]

By property (iii) in Lemma 3.11 we have that \( M \) is closed in \( N \). This, by part (i) of Lemma 2.18 implies that \( M \subseteq_+ N \).

By Lemma 3.11, we know that \( N \) is a model of \( T^\infty \). This model can readily be expanded to a model of \( T^\infty_{nat} \), and hence a model of \( T^\infty_{nat} \). Moreover, by the hypothesis, \( M \) is an existentially closed model of \( T_{nat}^\infty \) which altogether yields 
\[ M \models \text{SEM}_{(a, \bar{a}, n, \bar{a})}. \]

The proof of \( M \models \text{ISEM}_{(a, \bar{a}, n, \bar{a})} \) proceeds using a similar argument; the only difference is that the condition of \( M \subseteq_+ N \) follows directly from the hypothesis of \( \chi_{\mu}^*(b/\bar{a}) = \infty \) and not from Lemma 2.19.

**Corollary 3.14.** \( T_{\mu} \) is consistent.

**Proof.** Being an inductive theory, any model of \( T_{nat}^\infty \) can be extended to an existentially closed model. Therefore, by Lemma 3.13 any such model is a model of \( T_{\mu} \). \( \square \)

**Remark 3.15.** Despite all the subtleties it might involve, it seems possible to build a model of \( T_{\mu} \) completely out of scratch by adapting the usual techniques of Fraïssé-Hrushovski method into the context of infinite objects introduced here, namely the \( \mu \)-ucc structures. This means primarily to define a suitable notion of richness and then to constructively build a rich model using amalgamation over the structures in \( K_{\infty}^\mu \). In that way, the obtained rich model, supposed to be universal and ultrahomogeneous with regard to the structures in \( K_{\infty}^\mu \), would play the role of a model of \( T_{\mu} \). However, as mentioned earlier, we chose a different direction by following Hrushovski’s approach in [Hru97] that is based on techniques of Robinson model theory in axiomatizing existentially closed models.

### 3.2. Quantifier Elimination

In this subsection we prove that \( T_{\mu} \) eliminates quantifiers in \( L^+ \). This, using Lemma 3.13 will imply that \( T_{\mu} \) is a model companion for \( T_{nat}^\infty \). In fact, we will see in Proposition 3.18 that \( T_{\mu} \) is actually the model completion of \( T_{nat}^\infty \).

**Theorem 3.16.** \( T_{\mu} \) is complete and eliminates quantifiers in \( L^+ \).

**Proof.** To prove quantifier elimination, let \( M_1 \) and \( M_2 \) be two models of \( T_{\mu} \) with a common \( L^+ \)-substructure \( A = M_1 \cap M_2 \). Being a model of \( (T_{nat})_\nu \), the structure \( A \)
satisfies the following weak version of axiom schemes in $T_{\text{nat}}$ for all pair of structures $(B, C)$ with $B <_i C$

$\forall \bar{x} \left[ (\text{Diag}_B(\bar{x}) \land \exists \bar{y} \text{Diag}_C(\bar{x}; \bar{y})) \rightarrow R_{(B, C)}(\bar{x}) \right].$

Suppose that $\bar{a} \in A$ and $b$ is an element in $M_1 \setminus A$ satisfying a quantifier free $\mathcal{L}^+$-formula $\varphi(\bar{x}y)$ as

$$\bigwedge_{i=1}^{k_1} R_{(\bar{a}b, C_i)}(\bar{x}y) \land \bigwedge_{j=1}^{k_2} \neg R_{(\bar{a}b, D_j)}(\bar{x}y),$$

where for each $i$ and $j$ we have that $\bar{a}b <_i C_i$ and $\bar{a}b <_i D_j$. We will show that there exists an element $b' \in M_2$ satisfying the same formula.

By Fact 2.13 the weak closure of a finite subset of $A$, say $\bar{c}$, is finite in both $M_1$ and $M_2$. Using a suitable set of predicates $R(\bar{c}, E) \in \mathcal{L}^+$, we can easily show that the weak closure of $\bar{c}$ in $M_1$ is isomorphic to its weak closure in $M_2$. Therefore, without loss of generality, we may assume that $A$ is weakly closed in $M_1$ and $M_2$; in particular, we may assume that $\bar{a}$ is weakly closed in both $M_1$ and $M_2$.

Let

$$C = \bigcup_{i=1}^{k_1} C_i.$$

We partition the structures $D_j$ into two types of structures; recall that any structure $D_j$ is an intrinsic extension of $\bar{a}b$. The first type, or type-1, consists of all structures $D_j$ that are also an intrinsic extension of $\bar{a}$. All other structures $D_j$ would belong to the second type of structures, or type-2.

**Discussion.** If a type-1 structure $D_j$ is not realized in $M_1$ over $\bar{a}$ at all, then, by the fact that $M_1$ is a model of $T_{\text{nat}}$, this assumption implies that $M_1$ satisfies $\neg R_{(\bar{a}, D_j)}(\bar{a})$ which easily leads to

$$M_2 \models \neg R_{(\bar{a}, D_j)}(\bar{a}).$$

Consequently, there would not exist any copy of $D_j$ over $\bar{a}$ in $M_2$. Hence, this kind of type-1 structures are automatically omitted in $M_2$ and hence, in the rest of the proof we assume any type-1 structure to be realized in $M_1$. □

Moreover, in all the cases below, except for Case B.2, we deal solely with type-1 structures because the very formalism of the axioms SEM and ISEM would guarantee the omission of all type-2 structures over any embedding obtained by those axioms.

**Case A:** $b \in \text{cl}_{M_1}^*(\bar{a})$. In this case, for all the structures $C$ and $D_j$ appeared in $\varphi(\bar{x}y)$ we actually have that

$$\bar{a} <_i bC \quad \text{and} \quad \bar{a} <_i bD_j.$$
If $bC$ is algebraic over $\bar{a}$ with $m$-many conjugates in $M_1$, using a suitable set of $\mathcal{L}^+$-predicates, we can easily show the existence of a set $b'C'$ in $\text{acl}_{M_2}(\bar{a})$ that is isomorphic to $bC$ over $\bar{a}$ with exactly $m$-many number of copies in $M_2$. Moreover, the existence/non-existence of the structures $D_j$ over these $m$-many conjugates of $bC$ can be described by a finite number of $\mathcal{L}^+$-predicates. Therefore, at least one of the conjugates of $b'C'$ in $M_2$ will satisfy the formula $\varphi$.

Now suppose that $bC$ is not algebraic over $\bar{a}$ in $M_1$. To ease notation, we drop the subscript $j$ from any structure $D_j$ under consideration below.

For a type-1 structure $D$, by the Discussion above, we can assume that there is a realization of this structure over $\bar{a}$ in $M_1$; let $D'$ denote this realization.

**Claim A.** $\delta(D/\bar{a}bC) < 0$.

**Proof of the claim.** First, we show that $D'$ is in free amalgamation with $bC$ over $\bar{a}$. If it was not the case and there were relations violating this free amalgamation, then we would have that

$$\delta(D'/\bar{a}bC) < \delta(D'/\bar{a}) \leq 0$$

which shows that $\bar{a}bC$ is not weakly closed in $M_1$. By Lemma 2.16, this would contradict the assumption of $A \leq M_1$.

Now, note that the diagram of $D$ is omitted over $\bar{a}b$ in $M_1$. On the other hand, we just have proved that there exists a realization of the diagram of $D$, with respect to $\bar{a}$, that is in free amalgamation with $bC$ over $\bar{a}$, namely $D'$. This shows that the diagram of $D$, with respect to $\bar{a}bC$, contains at least one relation whose components intersects $bC$. Hence, we have that

$$\delta(D/\bar{a}bC) < \delta(D/\bar{a}) = 0,$$

where the latter equality holds because $D$ is a type-1 structure; that is $D$ is an intrinsic extension of $\bar{a}$.

---

**Claim A**

A consequence of the above claim is that any type-1 structure $D$ with at least one realization over $\bar{a}$ in $M_1$ will be omitted over any sufficiently weakly closed embedding of $bC$ into $M_2$.

Finally, in Case A, let $n$ be the maximum of the numbers $|D\backslash \bar{a}|$ with $D$ ranging over all type-1 structures. Also, let $\Omega$ consist of all structures $D$ of the second type. Now, an application of ISEM for $\bar{a}, bC, n$ and $\Omega$ leads to finding an embedding of $bC$ into $M_2$ that omits all type-2 structures $D$ and that is $n$-weakly closed in $M_2$. The latter property, by the discussion after Claim A, implies that all type-1 structures are also omitted over this embedding.

**Case B:** $b \notin \text{cl}_{M_1}^*(\bar{a})$. Let

$$E_0 := \text{cl}_{bc}^*(\bar{a}), \quad \text{and} \quad E_1 := bC \backslash E_0.$$

By definition of a closure, we have that $\bar{a} \preceq_i E_0$ and $E_0 \prec E_0 E_1$. Our aim would be to find a suitable copy of $E_0$ over $\bar{a}$ in $M_2$, say $E_0'$, and next to apply the
axiom SEM in order to embed $E_1$ over $E'_0$ while omitting all structures $D_j$ over that embedding.

We stick to the convention made in Case A by omitting subscript of the structures $D_j$ under consideration.

**Case B.1:** $E_0 = \bar{a}$. Given a structure $D$ of the first type, that is an intrinsic extension of $\bar{a}$, by Discussion above, we can assume that there exists a copy of $D$ over $\bar{a}$ in $M_1$; let $D'$ denote this realization.

**Claim B.** $\delta(D/b) < 0$.

**proof of the claim.** First, we show that $D'$ is in free amalgamation with $b$ over $\bar{a}$. If it was not the case and there were relations violating this free amalgamation, $b$ would belong to the closure of $\bar{a}D'$ which, using the fact that $D'$ is already in the closure of $\bar{a}$, implies that $b$ belongs to the closure of $\bar{a}$. This contradicts our assumption of $b \notin \text{cl}_{M_1}(\bar{a})$. The rest of the proof proceeds exactly similar to the proof of Claim A.

\[\blacksquare\text{Claim B}\]

Hence, in Case B.1, let $n$ and $\Omega$ be exactly as they were considered in Case A. Then, an application of SEM for $\bar{a}, E_1, n$ and $\Omega$ leads to finding an embedding of $E_1$ into $M_2$ that omits all type-2 structures $D$ and that is $n$-weakly closed in $M_2$.

The latter property, by Claim B, implies that all type-1 structures are also omitted over this embedding.

**Case B.2:** $E_0 \supseteq \bar{a}$. Since $E_0$ is an intrinsic extension of $\bar{a}$, the structure $M_1$ satisfies $R(\bar{a}, E_0)(\bar{a})$. Hence, it is easily seen that there exists a copy of $E_0$ over $\bar{a}$ in $M_2$; let $E'_0$ denote this copy. Lemma 2.16 implies that $\bar{a}E'_0$ is weakly closed in $M_2$.

We handle all type-1 structures exactly in the same way as we did in Case B.1. For a type-2 structure $D$, recall that such a structure is an intrinsic extension of $\bar{ab}$ but not an intrinsic extension of $\bar{a}$. Also, notice that for all structures $D$, the formula $\phi(\bar{xy})$ only contains information concerning the diagram of $D$ over $\bar{xy}$ while remaining silent on the possible interactions between $D$ and $C$. In particular, $\phi(\bar{xy})$ contains no information about the diagram of $D$ over $E_0$.

Based on the latter observation, we define a new diagram $D'$ that extends $D$ to a new structure over $E_0$ without adding any new relation; that is, in $D'$ the structure $D$ is in free amalgamation with $E_0$ over $\bar{ab}$.

Defined in this way, $D'$ is actually an intrinsic extension of $\bar{ab}E_0$ while being not an intrinsic extension over $\bar{a}E_0$. Hence, if we let $n = |D\setminus\bar{a}|$ and $\Omega = \{D'\}$ and then apply SEM for $E'_0, E_1, n$ and $\Omega$, we can find an embedding $f$ of $E_1$ over $E'_0$ into $M_2$ while omitting $D'$ over this embedding.

Therefore, the only remaining chance for $D$ to be realized over $\bar{a}E'_0, f(E_1)$ is to be not isomorphic to $D'$ by having some extra relations over $E'_0$. If $D''$ is such a realization, then we have that

$$\delta(D''/\bar{a}E'_0, f(E_1)) < \delta(D''/\bar{a}f(E_1)) \leq \delta(D''/\bar{a}f(b)) = 0,$$
where the last equality holds because $D$ is an intrinsic extension of $\bar{a}b$. But, the above inequality would contradict the fact that $\bar{a}E_0 f(E_1)$ is $n$-weakly closed in $M_2$ and hence there would not exist any realization of $D$ over this embedding.

The above discussion about type-2 structures shows that in Case B.2, we need to let $\Omega$ consist of all type-2 structures $D'_j$, as defined above, and to let $n$ be the maximum of all numbers $|D_j \setminus \bar{a}|$ with $D_j$ ranging over all structures, either being type-1 or type-2. Then, an application of the axiom SEM for $E_0', E_1, n$ and $\Omega$ yields an embedding $f$ of $E_1$ over $E_0'$ into $M_2$ such that all structures $D_j$ in $\varphi(\bar{x}y)$ get omitted over $f(E_1)$.

To show that $T_\mu$ is complete, let $A = \emptyset$ and the proof of quantifier elimination shows that $T_\mu$ has the joint embedding property. This together with quantifier elimination prove the completeness of $T_\mu$.

**Corollary 3.17.** $T_\mu$ is a model companion for $T_{\text{nat}}^\infty$. It actually axiomatizes the class of existentially closed models of $T_{\text{nat}}^\infty$.

**Proof.** By Lemma 3.13, any existentially closed model of $T_{\text{nat}}^\infty$ is a model of $T_\mu$. On the other hand, $T_{\text{nat}}^\infty$ is an inductive theory and each of its models can be extended to an existentially closed model. Hence, $T_\mu$ and $T_{\text{nat}}^\infty$ are cotheories. Moreover, $T_\mu$, by Theorem 3.16, has quantifier elimination which shows that it is a model companion for $T_{\text{nat}}^\infty$. Since $T_{\text{nat}}^\infty$ is inductive, the latter is equivalent to saying that $T_\mu$ axiomatizes the class of all existentially closed models of $T_{\text{nat}}^\infty$.

**Proposition 3.18.** $T_\mu$ is the model completion of $T_{\text{nat}}^\infty$.

**Proof.** By Corollary 3.17, $T_\mu$ is a model companion for $T_{\text{nat}}^\infty$ and we only need to show that $T_{\text{nat}}^\infty$ has the amalgamation property. Let $M_0, M_1, M_2$ be models of $T_{\text{nat}}^\infty$ with $M_0 \subseteq_+ M_1$ and $M_0 \subseteq_+ M_2$. If $M$ denotes the free join of $M_1$ and $M_2$ over $M_0$, by part (ii) of Lemma 2.19, the structure $M$ would be a model of $T_{\text{nat}}$ with $M_1 \subseteq_+ M$ and $M_2 \subseteq_+ M$. By Lemma 3.11, the structure $\langle M \rangle_\mu^\infty$ is a model of $T_\mu^\infty$ with $M_1 \subseteq_+ \langle M \rangle_\mu^\infty$ and $M_2 \subseteq_+ \langle M \rangle_\mu^\infty$. Now, it is easy to expand $\langle M \rangle_\mu^\infty$ to a model of $T_{\text{nat}}^\infty$ which completes the proof.

4. **$T_\mu$ is Strictly NSOP$_3$**

Towards proving that $T_\mu$ is strictly NSOP$_3$, we distinguish in Lemma 4.5 a key property of indiscernible sequences in the models of $T_\mu$. This lemma seems more direct and economic by avoiding the challenges that usually appear in deploying the dimension theory available in Hrushovski constructions built using a predimension function; hence, it can be easily applied as long as one concerns with investigating a combinatorial property in that context.

**Definition 4.1.** Let $n \geq 3$ and $T$ be a complete first order theory.
A formula $\varphi(\bar{x}; \bar{y})$ is said to have the strong order property of the $n$-th kind (SOP$_n$) if there is a model $M \models T$ satisfying

$$\neg \exists \bar{x}_0 \ldots \bar{x}_{n-1} \left[ \varphi(\bar{x}_0; \bar{x}_1) \land \varphi(\bar{x}_1; \bar{x}_2) \land \cdots \land \varphi(\bar{x}_{n-1}; \bar{x}_0) \right],$$

and an infinite sequence of distinct tuples $\{\bar{a}_i\}_{i \in \omega}$ in $M$ such that for any $i < j \in \omega$ we have $M \models \varphi(\bar{a}_i; \bar{a}_j)$.

(i) A formula $\varphi(\bar{x}; \bar{y})$ is said to have the strong order property of the $n$-th kind (SOP$_n$) if there is a model $M \models T$ satisfying

$$\neg \exists \bar{x}_0 \ldots \bar{x}_{n-1} \left[ \varphi(\bar{x}_0; \bar{x}_1) \land \varphi(\bar{x}_1; \bar{x}_2) \land \cdots \land \varphi(\bar{x}_{n-1}; \bar{x}_0) \right],$$

and an infinite sequence of distinct tuples $\{\bar{a}_i\}_{i \in \omega}$ in $M$ such that for any $i < j \in \omega$ we have $M \models \varphi(\bar{a}_i; \bar{a}_j)$. (ii) $T$ has SOP$_n$ if there is a formula in $T$ with SOP$_n$.

(iii) For a formula or a theory, the property of NSOP$_n$ is naturally defined as the negation of the notions above.

**Remark 4.2.** In a theory and for any $n \geq 3$, it is easy to see that being NSOP$_n$ for all parameter-free formulas implies NSOP$_n$ for all formulas with a parameter. Moreover, using a Ramsey argument, we can replace the sequence $\{\bar{a}_i\}_{i \in \omega}$ in part (i) of Definition 4.1 by an indiscernible sequence over a parameter set.

As already mentioned in the introduction, the property of SOP$_2$, despite its name, is about trees rather than orders.

**Definition 4.3.** Let $T$ be a complete first order theory. A formula $\varphi(\bar{x}; \bar{y})$ has the strong order property of the second kind (SOP$_2$) if there is a model $M \models T$ and a set of tuples $\{\bar{b}_\eta : \eta \in <\omega^2\}$ such that

(i) for every $\sigma \in <\omega^2$ the set of formulas $\{\varphi(\bar{x}; \bar{b}_{\eta(n)}) : n \in \omega\}$ is consistent, and

(ii) for every two incomparable $\eta, \gamma \in <\omega^2$, meaning that they do not belong to the same maximal branch, the set $\{\varphi(\bar{x}; \bar{b}_\eta) \land \varphi(\bar{x}; \bar{b}_\gamma)\}$ is not consistent.

For a theory, the notions of SOP$_2$ and NSOP$_2$ can be naturally defined as in Definition 4.1.

**Lemma 4.5.** Suppose that $\{\bar{a}_i\}_{i \in \omega}$ is an $A$-indiscernible sequence. Then

(i) the tuples $\bar{a}_i$ are mutually in free amalgamation over $A$.

(ii) The sequence $\{\text{cl}^w(\bar{a}_i)\}_{i \in \omega}$ is an indiscernible sequence of finite tuples over $\text{cl}^w(A)$ which are mutually in free amalgamation over $\text{cl}^w(A)$.

(iii) If $\bar{A}$ denotes the weak closure of $A$, then for any $n \in \omega$ the weak closure of the structure $\bar{A}\bar{a}_0 \cdots \bar{a}_n$ is equal to

$$\text{cl}^w(\bar{a}_0) \otimes \cdots \otimes \text{cl}^w(\bar{a}_n).$$

**Proof.** (i) Without loss of generality, we may assume that the intersection of tuples $\bar{a}_i$ lies inside $A$ and they are mutually disjoint over $A$. 

Assumption 4.4. We fix a big model $\mathcal{M}$ for the theory $T_p$ and will omit all subscripts referring to the ambient model; therefore, instead of notations like $\chi^*_{\mathcal{M}}(B/A)$ or $\text{cl}_{\mathcal{M}}(A)$ we only write $\chi^*(B/A)$ and $\text{cl}(A)$.
Let \( r > 0 \) denote the number of relations preventing \( \bar{a}_0 \) from being freely amalgamated with \( \bar{a}_1 \) over \( A \). A particular consequence of indiscernibility is that all tuples \( \bar{a}_i \) have an identical predimension. Hence, an easy induction on \( n \) shows that

\[
\delta(\bar{a}_0 \cdots \bar{a}_{n-1}) = n\delta(\bar{a}_0) - \frac{n(n - 1)}{2} r.
\]

Therefore, for every number \( n > \frac{2\delta(\bar{a}_0)}{r} + 2 \) we have that \( \delta(\bar{a}_0 \cdots \bar{a}_{n-1}) < 0 \). This would contradict the fact that \( M \in \bar{A} \).

(ii) It is easy to see that \( \{ \text{cl}^w(\bar{a}_i) \}_{i \in \omega} \) is indiscernible over the weak closure of \( A \). The claimed free amalgamation then follows directly from part (i).

(iii) If any \( i \in \omega \) let \( \bar{c}_i := \text{cl}^w(\bar{a}_i) \). Using parts (i) and (ii), we know that the sequence \( \{ \bar{c}_i \}_{i \in \omega} \) is \( \bar{A} \)-indiscernible with its elements being in free amalgamation over \( \bar{A} \). Hence, we only need to show that the structure \( \bar{c}_0 \otimes \cdots \otimes \bar{c}_n \) is weakly closed. We show this for \( n = 2 \) which completely resembles the proof of the general case.

If the structure \( \bar{c}_0 \otimes \bar{c}_1 \) is not weakly closed, then there will exist a finite structure \( B \) and a finite substructure \( A_0 \subseteq \omega \bar{A} \) satisfying

\[
\delta(B/A_0\bar{c}_0\bar{c}_1) < 0.
\]

Hence, by indiscernibility, for any \( n \in \omega \) there are \( \frac{n(n - 1)}{2} \)-many realizations of \( B \) over the structure \( A_0\bar{c}_0 \cdots \bar{c}_{n-1} \). If \( S_n \) denotes the union of \( A_0\bar{c}_0 \cdots \bar{c}_{n-1} \) with all these realizations of \( B \), we will have that

\[
\delta(S_n) = \delta(A_0\bar{c}_0 \cdots \bar{c}_{n-1}) + \frac{n(n - 1)}{2} \delta(B/A_0\bar{c}_0\bar{c}_1) \\
\leq n\delta(A_0\bar{c}_0\bar{c}_1) + \frac{n(n - 1)}{2} \delta(B/A_0\bar{c}_0\bar{c}_1).
\]

This will lead to a similar contradiction appeared in the proof of part (i) of this lemma.

\[\square\]

Lemma 4.6. The theory \( T_\mu \) is NSOP_3.

Proof. By Remark 4.2 it suffices to show that for any formula \( \varphi(\bar{x}; \bar{y}) \) that is in the EM-type of an indiscernible sequence \( \{\bar{a}_i\}_{i \in \omega} \) we have that

\[
M \models \exists \bar{x}_0 \bar{x}_1 \bar{x}_2 [\varphi(\bar{x}_0; \bar{x}_1) \land \varphi(\bar{x}_1; \bar{x}_2) \land \varphi(\bar{x}_2; \bar{x}_0)].
\]

(4.1)

It is easy to see that NSOP_3 formulas are closed under disjunction. Hence, by Theorem 3.16 we can assume \( \varphi(\bar{x}; \bar{y}) \) to be in the following form

\[
\exists \bar{x} \text{Diag}_{(A,C)}(\bar{x}\bar{y}, \bar{z}) \land \bigwedge_{i=1}^{k} \neg \exists \bar{u}_i \text{Diag}_{(A,D_i)}(\bar{x}\bar{y}, \bar{u}_i).
\]

For any \( i \in \omega \), let \( b_i \) denote the weak closure of the tuple \( \bar{a}_i \) and \( \bar{a} \) denote the weak closure of the intersection of the sequence \( \{\bar{a}_i\}_{i \in \omega} \). Define \( B_0 \) as the structure
that is isomorphic to the mutual free amalgamation of \( \bar{b}_0, \bar{b}_1 \) and \( \bar{b}_2 \) over \( \bar{a} \). Then, let \( B \) denote the structure obtained by gluing three copies of \( C \) over the tuples \( \bar{a}_0\bar{a}_1, \bar{a}_1\bar{a}_2 \) and \( \bar{a}_2\bar{a}_0 \) appearing in \( B_0 \).

To show that (4.1) holds, we will apply the axiom SEM to embed \( B \) into \( M \) over the empty set while omitting all the structures \( D_j \) over this embedding. In doing so, we only need to check that the structure \( B \) belongs to \( K \) and that each \( D_j \) is omittable over \( B \).

**Case 1:** \( \delta(C/A) < 0 \).

In this case, for any \( i < j \in \omega \), the realization of \( C \) over \( \bar{b}_i\bar{b}_j \), that is forced by \( \varphi \), is a subset of the weak closure of \( \bar{b}_i\bar{b}_j \). On the other hand, by Lemma 4.5, we have that

\[
\text{cl}^w(\bar{b}_i\bar{b}_j) = \bar{b}_i \otimes \bar{b}_j.
\]

Now, it is easy to check that the structure \( B \) is identical with \( B_0 \) which is isomorphic to the free amalgamation of \( \bar{b}_0, \bar{b}_1 \) and \( \bar{b}_2 \) over \( \bar{a} \). Therefore, \( B \) is already a substructure of \( M \) and hence belongs to \( K \).

**Case 2:** \( \delta(C/A) = 0 \).

Note that \( B_0 \) is a substructure of \( M \) and hence belongs to \( K \). Moreover, it is easy to check that \( B \) is an intrinsic extension of \( B_0 \) with \( \delta(B/B_0) = 0 \) which easily implies that \( B \in K \).

\[\]

**Lemma 4.7.** \( T_\mu \) has SOP-2.

**Proof.** Let \( \varphi(x; y_1 y_2 y_3) \) be the following formula

\[
R(x, y_1, y_2) \land \exists z R(x, y_3, z).
\]

By constructing a finite structure \( A \) and suitably embedding it into \( M \), our aim would be to realize a given finite fragment of a tree that witnesses SOP-2 for \( \varphi \).

Fix a tuple \( \bar{b} = b^0b^1 \) and let \( \{\bar{b}_\eta : \eta \in <^n2\} \) be a finite set of tuples, each being of cardinality three, with

\[
\bigcap_{\eta \in <^n2} \bar{b}_\eta = \bar{b}.
\]

For any \( \eta \in <^n2 \), let \( b_\eta \) denote the singleton \( \bar{b}_\eta \setminus \bar{b} \).

For each maximal branch \( \sigma \in 2^n \), let \( A_\sigma \) denote a structure containing \( \bar{b} \) and a new element \( a_\sigma \) satisfying \( R(a_\sigma, b^0, b^1) \). Also, for each \( i \in n \), let \( C_{\sigma i} \) be the structure that contains \( A_\sigma b_{\sigma i} \), together with a new element \( c_{\sigma i} \) satisfying \( R(a_\sigma, b_{\sigma i}, c_{\sigma i}) \).

Finally, let \( A \) denote the following structure

\[
\bigcup_{\sigma \in 2^n} \bigcup_{i \in n} C_{\sigma i}.
\]
It is easy to check that $A$ belongs to $K_0$. Now, it suffices to apply SEM to embed $A$ into $M$ while omitting all the structures that realize $\varphi(x; \bar{b}_\eta) \land \varphi(x; \bar{b}_\gamma)$ for incomparable sequences $\eta, \gamma \in \langle n \rangle$. Any structure $C(\eta, \gamma) \in K_0$ that realizes $\varphi(x; \bar{b}_\eta) \land \varphi(x; \bar{b}_\gamma)$ must contain $\bar{b}_\eta \bar{b}_\gamma$ as well as elements like $a, c_1$ and $c_2$ satisfying $R(a, b_0, b_1) \land R(a, b_\eta, c_1) \land R(a, b_\gamma, c_2)$. But, it is obvious that such a structure $C(\eta, \gamma)$ is omittable over $A$ since for any incomparable sequences $\eta$ and $\gamma$, the structure $A$ does not contain even a single copy of $C(\eta, \gamma)$ over $\bar{b}_\eta \bar{b}_\gamma$.

**Theorem 4.8.** $\mathbb{T}_\mu$ is an NSOP$_3$ theory possessing SOP$_2$.

**Proof.** The theorem is followed from Lemmas 4.6 and 4.7.

5. Concluding Remarks

Finally, we close by stating some other aspects of $\mathbb{T}_\mu$ together with suggesting a few number of interesting open questions.

**Remark 5.1.**

(i) Based on the axiomatization given for $\mathbb{T}_\mu$, it can be easily seen that this theory is decidable.

(ii) Given a number $n \in \omega$ and working in a saturated model of $\mathbb{T}_\mu$, one can use SEM to realize infinitely many tuples $\bar{a}$ of cardinality $n$ with mutually non-isomorphic weak closures. This shows that $\mathbb{T}_\mu$ is not $\aleph_0$-categorical.

(iii) Essentially the same argument as in the proof of Lemma 4.7 can be used to show that $\mathbb{T}_\mu$ is a theory with TP$_2$.

Another interesting aspect for $\mathbb{T}_\mu$ might be the property of pseudofiniteness; hence, we pose the following question:

**Question 5.1.** Is $\mathbb{T}_\mu$ pseudofinite?

Also, without giving the definition of $\triangleleft^*$-order over the theories introduced in [DS04], we pose another question whose negative answer can further clarify the relation between $\triangleleft^*$-maximality and maximality in Keisler’s order. Roughly speaking, a theory $T$ is $\triangleleft^*$-maximal if for any other theory $T'$, there exists a certain fusion of $T$ and $T'$ in every model of which, say $M$, saturation of $M\upharpoonright T$ implies the saturation of $M|_{T'}$.

Recall that, by the results appeared in [DS04], maximality in $\triangleleft^*$ implies SOP$_2$ for a theory and by the aforementioned theorem of Malliaris ans Shelah in [MS16], a SOP$_2$ theory is maximal in Keisler’s order.

**Question 5.2.** Is the theory $\mathbb{T}_\mu$ maximal in $\triangleleft^*$-order?

The property of SOP$_1$ forms another major dividing line inside the spectrum of model-theoretic tree properties with which the present paper did not directly
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However, we would like to recall another important open problem asking whether the property of SOP\(_1\) implies, at the level of theories, that of SOP\(_2\).

A natural attitude would be seeking to know if the methods developed here, by making suitable adaptations, are capable of finding a strictly NSOP\(_2\) theory. Finally, we pose the mentioned open problem first asked as Question 2.4 in \[DS04\].

**Question 5.3.** Does there exist a strictly NSOP\(_2\) theory?

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