Isomorphisms of Hilbert C*-Modules and ∗-Isomorphisms of Related Operator C*-Algebras

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Abstract

Let $M$ be a Banach C*-module over a C*-algebra $A$ carrying two $A$-valued inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ which induce equivalent norms on $M$. Then the appropriate unital C*-algebras of adjointable bounded $A$-linear operators on the Hilbert $A$-modules $\{M, \langle \cdot, \cdot \rangle_1\}$ and $\{M, \langle \cdot, \cdot \rangle_2\}$ are shown to be ∗-isomorphic if and only if there exists a bounded $A$-linear isomorphism $S$ of these two Hilbert $A$-modules satisfying the identity $\langle \cdot, \cdot \rangle_2 \equiv \langle S(\cdot), S(\cdot) \rangle_1$. This result extends other equivalent descriptions due to L. G. Brown, H. Lin and E. C. Lance. An example of two non-isomorphic Hilbert C*-modules with ∗-isomorphic C*-algebras of "compact"/adjointable bounded module operators is indicated.

Investigations in operator and C*-theory make often use of C*-modules as a tool for proving, especially of Banach and Hilbert C*-modules. Impressing examples of such applications are G. G. Kasparov's approach to K- and KK-theory of C*-algebras [6, 15] or the investigations of M. Baillet, Y. Denizeau and J.-F. Havet [1] and of Y. Watatani [14] on (normal) conditional expectations of finite index on W*-algebras and C*-algebras. In addition, the theory of Hilbert C*-modules is interesting in its own.

Our standard sources of reference to Hilbert C*-module theory are the papers [12, 8, 4, 5], chapters in [6, 15] and the book of E. C. Lance [10]. We make the convention that all C*-modules of the present paper are left modules by definition. A pre-Hilbert $A$-module over a C*-algebra $A$ is an $A$-module $M$ equipped with an $A$-valued mapping $\langle \cdot, \cdot \rangle : M \times M \to A$ which is $A$-linear in the first argument and has the properties:

$$\langle x, y \rangle = \langle y, x \rangle^*, \quad \langle x, x \rangle \geq 0 \quad \text{with equality iff} \quad x = 0.$$  

The mapping $\langle \cdot, \cdot \rangle$ is called the $A$-valued inner product on $M$. A pre-Hilbert $A$-module $\{M, \langle \cdot, \cdot \rangle\}$ is Hilbert if and only if it is complete with respect to the norm $\| \cdot \| = \|\langle \cdot, \cdot \rangle\|_{A}^{1/2}$. We always assume that the linear structures of $A$ and $M$ are compatible.

One of the key problems of Hilbert C*-module theory is the question of isomorphism of Hilbert C*-modules. First of all, they can be isomorphic as Banach $A$-modules. But there is another natural definition: Two Hilbert $A$-modules $\{M_1, \langle \cdot, \cdot \rangle_1\}, \{M_2, \langle \cdot, \cdot \rangle_2\}$ over a fixed C*-algebra $A$ are isomorphic as Hilbert C*-modules if and only if there exists a bijective bounded $A$-linear mapping $S : M_1 \to M_2$ such that the identity $\langle \cdot, \cdot \rangle_1 \equiv \equiv \langle S(\cdot), S(\cdot) \rangle_2$ is valid on $M_1 \times M_1$. In 1985 L. G. Brown presented two examples of
Hilbert $C^*$-modules which are isomorphic as Banach $C^*$-modules but which are non-isomorphic as Hilbert $C^*$-modules, cf. [3, 11, 5]. This result was very surprising since Hilbert space theory, the classical investigations on Hilbert $C^*$-modules like [13, 8], G. G. Kasparov’s approach to KK-theory of $C^*$-algebras relying on countably generated Hilbert $C^*$-modules and other well-known investigations in this field did not give any indication of such a serious obstacle in the general theory of Hilbert $C^*$-modules. L. G. Brown obtained his examples from the theory of different kinds of multipliers of $C^*$-algebras without identity. Furthermore, making use of the results of the Ph.D. thesis of Nien-Tsu Shen [13] he proved the following: For a Banach $C^*$-module $\mathcal{M}$ over a $C^*$-algebra $A$ carrying two $A$-valued inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ which induce equivalent to the given one norms on $\mathcal{M}$ the appropriate $C^*$-algebras of ”compact” bounded $A$-linear operators on the Hilbert $A$-modules $\{\mathcal{M}, \langle \cdot, \cdot \rangle_1\}$ and $\{\mathcal{M}, \langle \cdot, \cdot \rangle_2\}$ are $*$-isomorphic if and only if there exists a bounded $A$-linear isomorphism $S$ of these two Hilbert $A$-modules satisfying $\langle \cdot, \cdot \rangle_2 \equiv \langle S(\cdot), S(\cdot) \rangle_1$, cf. [2, Thm. 4.2, Prop. 4.4] together with [3, Prop. 2.3], ([3]). By definition, the set of ”compact” operators $K_A(\mathcal{M})$ on a Hilbert $A$-module $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$ is defined as the norm-closure of the set $K_A^0(\mathcal{M})$ of all finite linear combinations of the operators 

$$\{\theta_{x,y} : \theta_{x,y}(z) = \langle z, x \rangle y \text{ for every } x, y, z \in \mathcal{M}\}.$$ 

It is a $C^*$-subalgebra and a two-sided ideal of $\text{End}_A^1(\mathcal{M})$, the set of all adjointable bounded $A$-linear operators on $\{\mathcal{M}, \langle \cdot, \cdot \rangle\}$, what is the multiplier $C^*$-algebra of $K_A(\mathcal{M})$ by [8, Thm. 1]. Note, that in difference to the well-known situation for Hilbert spaces, the properties of an operator to be ”compact” or to possess an adjoint depend heavily on the choice of the $A$-valued inner product on $\mathcal{M}$. These properties are not invariant even up to isomorphic Hilbert structures on $\mathcal{M}$, in general, cf. [6]. We make the convention that operators $T$ which are ”compact”/adjointable with respect to some $A$-valued inner product $\langle \cdot, \cdot \rangle_i$ will be marked $T^{(i)}$ to note where this property arises from. The same will be done for sets of such operators.

In 1994 E. C. Lance showed that two Hilbert $C^*$-modules are isomorphic as Hilbert $C^*$-modules if and only if they are isometrically isomorphic as Banach $C^*$-modules ([8]) opening the geometrical background of this functional-analytical problem and extending a central result for $C^*$-algebras: $C^*$-algebras are isometrically multiplicatively isomorphic if and only if they are $*$-isomorphic, [6, Thm. 7, Lemma 8].

At the contrary, non-isomorphic Hilbert structures on a given Hilbert $A$-module $\mathcal{M}$ over a $C^*$-algebra $A$ can not appear at all if $\mathcal{M}$ is self-dual, i. e. every bounded module map $r : \mathcal{M} \to A$ is of the form $\langle \cdot, a_r \rangle$ for some element $a_r \in \mathcal{M}$ (cf. [4, Prop. 2.2,Cor. 2.3]), or if $A$ is unital and $\mathcal{M}$ is countably generated, i. e. there exists a countably set of generators inside $\mathcal{M}$ such that the set of all finite $A$-linear combinations of generators is norm-dense in $\mathcal{M}$ (cf. [2, Cor. 4.8, Thm. 4.9] together with [6, Cor. 1.1.25] and [5, Prop. 2.3]).

Now, we come to the goal of the present paper: Whether for a Banach $C^*$-module $\mathcal{M}$ over a $C^*$-algebra $A$ carrying two $A$-valued inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ which induce equivalent to the given one norms on $\mathcal{M}$ the appropriate $C^*$-algebras $\text{End}_{A,1}(\mathcal{M})$ and $\text{End}_{A,2}(\mathcal{M})$ of all adjointable bounded $A$-linear operators on $\mathcal{M}$ are $*$-isomorphic, or not? This question is non-trivial since even non-$*$-isomorphic non-unital $C^*$-algebras can possess a common multiplier $C^*$-algebra: For example, on the closed interval $[0, 2] \subset \mathbb{R}$ consider the $C^*$-algebra of all continuous functions vanishing at zero together with the $C^*$-algebra of all continuous function vanishing at one. They are non-$*$-isomorphic, but the multiplier
The C*-algebra $C((0,2))$ of them consisting of all continuous functions on $[0,2]$ is the same in both cases. That is, additional arguments are needed to describe the relation between the multiplier C*-algebras of non-$\ast$-isomorphic C*-algebras of "compact" operators on some Banach C*-modules carrying non-isomorphic C*-valued inner products. One quickly realizes that the techniques of multiplier theory are not suitable to shed some more light on this general situation. One has to turn back to C*-theory and to the properties of $\ast$-isomorphisms, as well as to the theory of Hilbert C*-modules.

**Theorem:** Let $A$ be a C*-algebra and $\mathcal{M}$ be a Banach $A$-module carrying two $A$-valued inner products $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ which induce equivalent to the given one norms. Then the following conditions are equivalent:

(i) The Hilbert $A$-modules $\{\mathcal{M}, \langle \cdot, \cdot \rangle_1\}$ and $\{\mathcal{M}, \langle \cdot, \cdot \rangle_2\}$ are isomorphic as Hilbert C*-modules.

(ii) The Hilbert $A$-modules $\{\mathcal{M}, \langle \cdot, \cdot \rangle_1\}$ and $\{\mathcal{M}, \langle \cdot, \cdot \rangle_2\}$ are isometrically isomorphic as Banach $A$-modules.

(iii) The C*-algebras $K^{(1)}_A(\mathcal{M})$ and $K^{(2)}_A(\mathcal{M})$ of all "compact" bounded $A$-linear operators on both these Hilbert C*-modules, respectively, are $\ast$-isomorphic.

(iv) The unital C*-algebras $\operatorname{End}^{(1,\ast)}_A(\mathcal{M})$ and $\operatorname{End}^{(2,\ast)}_A(\mathcal{M})$ of all adjointable bounded $A$-linear operators on both these Hilbert C*-modules, respectively, are $\ast$-isomorphic.

Further equivalent conditions in terms of positive invertible quasi-multipliers of $K^{(1)}_A(\mathcal{M})$ can be found in [5].

**Proof.** The equivalence of (i) and (ii) was shown by E. C. Lance [9], and the equivalence of (i) and (iii) turns out from a result for C*-algebras of L. G. Brown [2, Thm. 4.2, Prop. 4.4] in combination with [5, Prop. 2.3]. Referring to G. G. Kasparov [8, Thm. 1] the implication (iii) $\rightarrow$ (iv) yields naturally.

Now, suppose the unital C*-algebras $\operatorname{End}^{(1,\ast)}_A(\mathcal{M})$ and $\operatorname{End}^{(2,\ast)}_A(\mathcal{M})$ are $\ast$-isomorphic. Denote this $\ast$-isomorphism by $\omega$. One quickly checks that the formula

$$x \in \mathcal{M} \rightarrow \langle x, x \rangle_{1,0p} = \theta^{(1)}_{x,x} \in K^{(1)}_A(\mathcal{M})$$

defines a $K^{(1)}_A(\mathcal{M})$-valued inner product on the Hilbert $A$-module $\mathcal{M}$ regarding it as a right $K^{(1)}_A(\mathcal{M})$-module. Moreover, the set $\{K(x) : x \in \mathcal{M}, K \in K^{(1)}_A(\mathcal{M})\}$ is norm-dense inside $\mathcal{M}$ since the limit equality

$$x = \|x\|_\mathcal{M} - \lim_{n \to \infty} (\theta^{(1)}_{x,x}(\theta^{(1)}_{x,x} + n^{-1})^{-1}(x)$$

holds for every $x \in \mathcal{M}$.

As a first step we consider the intersection of the two $C^*$-subalgebras and two-sided ideals $\omega(K^{(1)}_A(\mathcal{M}))$ and $K^{(2)}_A(\mathcal{M})$ inside the unital C*-algebra $\operatorname{End}^{(2,\ast)}_A(\mathcal{M})$. The intersection of them is a C*-subalgebra and two-sided ideal of $\operatorname{End}^{(2,\ast)}_A(\mathcal{M})$ again. It contains the operators

$$\theta^{(2)}_{x,y} \cdot \omega(\theta^{(1)}_{z,t}) = \theta^{(2)}_{\omega(\theta^{(1)}_{z,t})} \omega(\theta^{(1)}_{t,z},y) = \theta^{(2)}_{\omega(\theta^{(1)}_{z,t})(x),y}$$
for every \( x, y, z, t \in \mathcal{M} \). Since the set of all finite linear combinations of special operators \( \{ \theta^{(1)}_{z,t} : z, t \in \mathcal{M} \} \) is norm-dense inside \( K^{(1)}_A(\mathcal{M}) \) by definition the intersection of \( \omega(K^{(1)}_A(\mathcal{M})) \) and \( K^{(2)}_A(\mathcal{M}) \) contains the set
\[
\{ \theta^{(2)}_{\omega(K^{(1)}_A(x), y)} : K^{(1)} \in K^{(1)}_A(\mathcal{M}), x, y \in \mathcal{M} \}.
\]
Because of the limit equality
\[
x = \|\cdot\|_\mathcal{M} - \lim_{n \to \infty} \omega(\theta^{(1)}_{x,x}(\theta^{(1)}_{x,x} + n^{-1})(x)) = \|\cdot\|_\mathcal{M} - \lim_{n \to \infty} \omega(\theta^{(1)}_{x,x})\omega((\theta^{(1)}_{x,x}(1) + n^{-1})(x))
\]
the set \( \{ \omega(K^{(1)}_A(x)) : K^{(1)} \in K^{(1)}_A(\mathcal{M}), x \in \mathcal{M} \} \) is norm-dense inside \( \mathcal{M} \). Consequently, the intersection of \( \omega(K^{(1)}_A(\mathcal{M})) \) and \( K^{(2)}_A(\mathcal{M}) \) inside the unital C*-algebra \( \text{End}^{(2,\ast)}_A(\mathcal{M}) \) contains the set of ”compact” operators \( \{ \theta^{(2)}_{x,y} : x, y \in \mathcal{M} \} \) generating one of the intersecting sets, \( K^{(2)}_A(\mathcal{M}) \), completely, and the inclusion relation \( K^{(2)}_A(\mathcal{M}) \subseteq \omega(K^{(1)}_A(\mathcal{M})) \) holds.

Secondly, by the symmetry of the situation and of the arguments the inclusion relation \( K^{(1)}_A(\mathcal{M}) \subseteq \omega^{-1}(K^{(2)}_A(\mathcal{M})) \) holds, too, inside the unital C*-algebra \( \text{End}^{(1,\ast)}_A(\mathcal{M}) \). Both inclusions together prove that \( \omega \) realizes a \( \ast \)-isomorphism of the C*-algebras \( K^{(1)}_A(\mathcal{M}) \) and \( K^{(2)}_A(\mathcal{M}) \) automatically, what implies (iii) and hence, (i).

Whether the \( \ast \)-isomorphism of the C*-algebras of ”compact” bounded \( A \)-linear operators of two different Hilbert \( A \)-modules \( \mathcal{M} \) and \( \mathcal{N} \) over some C*-algebras \( A \) implies their isomorphism as Hilbert C*-modules, or not? The answer is negative, even in the quite well-behaved cases. Counterexamples appear because of nontrivial \( K_0 \)-groups of \( A \), for instance. Let \( A \) be the hyperfinite type \( \Pi_1 \) \( W \)-factor. Set \( \mathcal{M} = A \) and \( \mathcal{N} = A^2 \) with the usual \( A \)-valued inner products. Both these Hilbert \( A \)-modules are self-dual and finitely generated. Obviously, \( K_A(\mathcal{M}) \) and \( K_A(\mathcal{N}) \) are \( \ast \)-isomorphic to \( A \) as C*-algebras. Nevertheless, \( \mathcal{M} \) and \( \mathcal{N} \) are not isomorphic as Banach \( A \)-modules because of the non-existence of non-unitary isometries for the identity caused by the existence of a faithful trace functional on \( A \). The \( K_0 \)-group of \( A \) equals \( \mathbb{R} \), i.e., it is non-trivial, and \( A \cong A \otimes M_2(\mathbb{C}) \).

In general, one could search for some special unital C*-algebra \( A \) with non-trivial \( K_0 \)-group, a natural number \( n \geq 1 \) and two projections \( p, q \in M_n(A) \) such that for every \( N \geq n \) the finitely generated Hilbert \( A \)-modules \( A^Np \) and \( A^Nq \) are non-isomorphic (i.e., \([p] \neq [q] \in K_0(A)\)), but the C*-algebras \( pM_n(A)p \) and \( qM_n(A)q \) are \( \ast \)-isomorphic.

Closing, we pose the problem whether for a Banach C*-module \( \mathcal{M} \) over a C*-algebra \( A \) carrying two \( A \)-valued inner products \( \langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2 \) which induce equivalent to the given one norms on \( \mathcal{M} \) the appropriate Banach algebras of all (not necessarily adjointable) bounded \( A \)-linear operators on \( \mathcal{M} \) are isometrically multiplicatively isomorphic, or not, especially in the case of non-isomorphic Hilbert structures. Those properties of all these kinds of operator algebras which are preserved switching from one \( A \)-valued inner product on \( \mathcal{M} \) to another have to be investigated in the future extending results for the ”compact” case of [3, 8].
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