A NOTE ON HILBERT SCHEMES OF NODAL CURVES

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Abstract. We study the Hilbert scheme and punctual Hilbert scheme of a nodal curve, and the relative Hilbert scheme of a family of curves acquiring a node. The results are then extended to flag Hilbert schemes, parametrizing chains of subschemes. We find, notably, that if the total space $X$ of a family $X/B$ is smooth (over an algebraically closed field $k$), then the relative Hilbert scheme $\text{Hilb}_m(X/B)$ is smooth over $k$ and the flag Hilbert schemes are normal and locally complete intersection, but generally singular.

The Hilbert scheme parametrizes ideal sheaves or subschemes of Projective Space or more generally, or a fixed scheme $X$. Perhaps the simplest case is where $X$ is a smooth curve, for then the Hilbert scheme $\text{Hilb}_m(X)$ parametrizing length-$m$ subschemes of $X$ coincides with the symmetric product $\text{Sym}^m(X)$. Our purpose in this note is to study what is in a sense the next simplest case, where $X$ is essentially a curve with ordinary nodes, with planar equation formally equivalent to $xy = 0$. Besides $\text{Hilb}_m(X)$ itself, there are (at least) 2 other types of Hilbert scheme of natural interest here: the punctual one $\text{Hilb}^0_m(X)$, parametrizing length-$m$ subschemes supported at a node; and the relative one $\text{Hilb}_m(\tilde{X}/B)$, parametrizing length-$m$ subschemes in the fibres of the family $\tilde{X}/B$ that is the map $\mathbb{A}^2 \to \mathbb{A}^1$ given by $xy = t$ (both $\text{Hilb}_m(X)$ and $\text{Hilb}_m(\tilde{X}/B)$ may conveniently be viewed as germs, or formal schemes, along $\text{Hilb}^0_m(X)$). All these will be determined below. We find that $\text{Hilb}^0_m(X)$ is a connected chain of $(m - 1)$ nonsingular rational curves meeting transversely; $\text{Hilb}_m(X)$ is a connected chain of $(m + 1)$ nonsingular $m$-dimensional germs, whose first and last members are supported on points and each other member is supported on a component of $\text{Hilb}^0_m(X)$, and only consecutive members intersect (and those transversely); finally, $\text{Hilb}_m(\tilde{X}/B)$ is a smooth $(m + 1)$-fold.

To elucidate the relations between the $\text{Hilb}_m$ for different $m$, with an eye to enumerative applications based on recursion on $m$, we will also study some flag Hilbert schemes $\text{Hilb}_{m,r}$, parametrizing chains of ideals whose colengths form a given sequence $m_r = (m_1 > m_2 > \ldots)$. For some purposes, these are easier to work with than ordinary Hilbert schemes, due to the natural maps between the $\text{Hilb}_{m,r}$ for different $(m_r)$. For example, for $m = (m, m - 1)$, we find that the

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punctual Hilbert scheme $\text{Hilb}_{m}(X)$ is a chain of $2m - 3$ copies of $\mathbb{P}^{1}$ that alternate between those coming from $\text{Hilb}_{m}(X)$ and from $\text{Hilb}_{m-1}(X)$. The relative Hilbert scheme $\text{Hilb}_{m}(\tilde{X}/B)$ is still smooth. Things begin to change though with $m = (m, m-1, m-2)$. Here $\text{Hilb}_{m}(X)$ is still a chain of $2m - 3$ components, but now only the 2 external ones on each side are $\mathbb{P}^{1}$'s and the rest are $\mathbb{P}^{1} \times \mathbb{P}^{1}$. The relative Hilbert scheme $\text{Hilb}_{m}(\tilde{X}/B)$ is no longer smooth, but it is still normal with locally complete intersection singularities. A similar picture emerges for longer chains, and in particular for the ‘full flag’ case $\text{Hilb}_{m} = \text{Hilb}_{m, m-1, m-2}$.

In [R], which uses some of these results, we will develop an alternative and more ‘geometric’ approach to these objects. In particular, we will identify $\text{Hilb}_{m}(\tilde{X}/B)$ with a certain space $\mathcal{W}_{m}(\tilde{X}/B)$ constructed as an explicit blow-up of the relative Cartesian product $\tilde{X}^{m}/B$. The proof of this identification uses some of the results here. We will then apply the spaces $\mathcal{W}_{m}(\tilde{X}/B)$ to some enumerative problems. In these applications, the relatively simple relationship between $\text{Hilb}_{m}$ and $\text{Hilb}_{m-1}$ is critical.

We work over a fixed algebraically closed field $k$. We denote by $R$ localization of the ring $k[x, y]/(xy)$ and at its maximal ideal $(x, y)$. A typical element of $R$ can be written in the form

$$u(a + \sum_{i \geq 1} b_{i}x^{i} + c_{i}y^{i})$$

where $u$ is a unit and the sum is finite. The formal completion

$$\hat{R} = k[[x, y]]/(xy) = \{a + \sum_{i \geq 1} b_{i}x^{i} + c_{i}y^{i}\}$$

(sum not necessarily finite) is isomorphic to the formal completion of the local ring at any 1-dimensional ordinary node. We seek first to determine the punctual Hilbert scheme $\text{Hilb}_{m}(R)$ of colength-$m$ ideals in $R$ which, as is well known, is naturally isomorphic to $\text{Hilb}_{m}(\hat{R})$. At this point, we do not seek to define or compute a natural scheme structure on $\text{Hilb}_{m}(R)$ (so calling it the punctual Hilbert scheme is something of a misnomer)—this will be done later (see Corollary 8). For now we simply view $\text{Hilb}_{m}(R)$ as an algebraic set endowed with a flat family of ideals that yields a bijective correspondence between the (closed, $k$-valued) points of $\text{Hilb}_{m}(R)$ and the colength-$m$ ideals of $R$.

**Theorem 1**. (i) Every ideal $I < R$ of colength $m$ is of one of the following, said to be of type $(c^{m}_{i})$, $(q^{m}_{i})$, respectively:

$$I^{m}_{i}(a) = (y^{i} + ax^{m-i}), 0 \neq a \in k, i = 1, ..., m - 1;$$

$$Q^{m}_{i} = (x^{m-i+1}, y^{i}), i = 1, ..., m.$$

(ii) The closure $C^{m}_{i}$ in the Hilbert scheme of the set of ideals of type $(c^{m}_{i})$ is isomorphic to $\mathbb{P}^{1}$ and consists of the ideals of types $(c^{m}_{i})$ or $(q^{m}_{i})$ or $(q^{m}_{i+1})$. In fact, we have

$$\lim_{a \to 0} I^{m}_{i}(a) = Q^{m}_{i},$$
\[ \lim_{a \to \infty} I^m_i(a) = Q^m_{i+1}. \]

(iii) The punctual Hilbert scheme Hilb^0_m(R), as algebraic set, is a rational chain

\[ C^m_1 \cup Q^m_2 \cup C^m_2 \cup \cdots \cup Q^m_{m-1} \cup C^m_{m-1}; \]

it has ordinary nodes at \( Q^m_2, \ldots, Q^m_{m-1} \) and is smooth elsewhere.

proof. Recall that elements \( z, z' \in R \) are said to be associate if \( z = uz' \) for some unit \( u \). Note that any nonzero nonunit \( z \in R \) is associate to a uniquely determined element of the form \( x^\alpha \) or \( y^\beta \) or \( x^\alpha + ay^\beta, a \neq 0, \alpha, \beta > 0 \), in which case we will say that \( z \) is of type \((\alpha, 0) \) or \((0, \beta) \) or \((\alpha, \beta) \), respectively. Note also that for any ideal \( I \) of colength \( m \) we have

\[ x^m, y^m \in I. \]

Now given \( I \) of colength \( m \), pick \( z \in I \) of minimal type \((\alpha, \beta) \), with respect to the natural partial ordering on types. Suppose to begin with that \( \alpha, \beta > 0 \). Then note that

\[ x^{\alpha+1}, y^{\beta+1} \in I, \]

and consequently \((\alpha, \beta)\) is unique: indeed if \((\alpha', \beta')\) is also minimal then we may assume \( \alpha' > \alpha \), hence \( x^{\alpha'} \in I \), hence \( y^{\beta'} \in I \), contradicting minimality. Hence \((\alpha, \beta)\) is unique and it is then easy to see that the element \( z' = x^\alpha + ay^\beta \in I \) is unique as well, so clearly \( z' \) generates \( I \) and \( I \) is of type \((\hat{c}_\beta^m)\).

Thus we may assume that any minimal element of \( I \) is of type \((\alpha, 0) \) or \((0, \beta) \). Since \((x^m, y^m) \subseteq I \), \( I \) clearly contains minimal elements of type \((\alpha, 0) \) or \((0, \beta) \), and then it is easy to see that \( I \) is of type \((q^m_{\beta}) \). This proves assertion (i). Since \( I^m_i(a) \) contains \( y^{i+1}, x^{m-i+1} \), assertion (ii) is easy. As for (iii), let \( C = \bigcup_i C^m_i \) be an abstract nodal curve as in (1). It follows from (ii) that each \( C^m_i \) carries a flat family of ideals, i.e. admits a natural morphism- clearly injective- to Hilb_m(R).

Since these morphisms agree on the intersections \( C^m_i \cap C^m_{i+1} = Q^m_{i+1} \), they yield a morphism, again clearly injective, from \( C \) to Hilb_m(R), which identifies \( C \) with Hilb^0_m(R) as an algebraic set. □

Next we determine the structure of the full Hilbert scheme of \( R \) and \( \hat{R} \):

**Theorem 2.** The Hilbert scheme Hilb_m(R) (resp. Hilb_m(\( \hat{R} \))) is a chain

\[ D^m_0 \cup D^m_1 \cup \cdots \cup D^m_{m-1} \cup D^m_m \]

where each \( D^m_i \) is a smooth and \( m \)-dimensional germ (resp. formal scheme) supported on \( C^m_i \) for \( i = 1, \ldots, m - 1 \) or \( Q^m_i \) for \( i = 0, m \); for \( i = 1, \ldots, m - 1 \), \( D^m_i \) meets its neighbors \( D^m_{i \pm 1} \) transversely in dimension \( m - 1 \) and meets no other \( D^m_j \).

The generic point of \( D^m_i \) corresponds to subscheme of Spec(R) comprised of \( m - i \) points on the x-axis and \( i \) points on the y-axis.

**proof.** Clearly Hilb_m(R) (resp. Hilb_m(\( \hat{R} \))) is a germ (resp. formal scheme) supported on Hilb^0_m(R), so this is a a matter of determining the scheme structure of Hilb_m(R) and Hilb_m(\( \hat{R} \)) at each point of Hilb^0_m(R), which may be done formally
by testing on Artin local algebras. Again, we shall do so at $Q_i^{m}, i > 1$ as the cases of $P_i^m(a)$ and $Q_1^m$ are similar and simpler. Given $S$ artinian local augmented, a flat $S$-deformation of $I = Q_i^m$ is given by an ideal

$$I_S = (f, g),$$

(4) $f = x^{m+1-i} + f_1(x) + f_2(y),$

$g = y^i + g_1(x) + g_2(y),$

where $f_i, g_j$ have coefficients in $m_S$, and such that $R_S/I_S$ is $S$-free of rank $m$, in which case it is clear by Nakayama’s Lemma that

$$1, x, ..., x^{m-i}, y, ..., y^{i-1}$$

is a free basis for $R_S/I_S$. It is easy to see that we may assume $f_1, g_1$ are in fact polynomials of degree $\leq m - i$ and $f_2, g_2$ are of degree $< i$ and $f_2, g_2$ have no constant term. Let’s write

(5) $f_1(x) = \sum_{0}^{m-i} a_jx^j, f_2(y) = \sum_{1}^{i-1} b_jy^j,$

(6) $g_1(x) = \sum_{0}^{m-i} c_jx^j, g_2(y) = \sum_{1}^{i-1} d_jy^j.$

Now obviously

$$yf - b_{i-1}g \equiv 0 \equiv xg - c_{m-i}f \mod I_S.$$ 

Writing these elements out in terms of $1, x, ..., x^{m-i}, y, ..., y^{i-1}$ yields relations among $1, x, ..., x^{m-i}, y, ..., y^{i-1}$. Since the latter elements form an $S$-free basis of $R_S/I_S$, those relations must be trivial. In other words, we have exact equalities rather than congruences:

$$yf - b_{i-1}g = 0 = xg - c_{m-i}f.$$ 

Writing out this equality term by term yields the following identities

$$b_j = b_{i-1}d_{j+1}, j = 1, ..., i - 2,$$

$$b_{i-1}d_1 = a_0,$$

(7) $$b_{i-1}c_j = 0, j = 0, ..., m - i,$$

$$c_j = c_{m-i}a_{j+1}, j = 0, ..., m - i - 1,$$

$$c_{m-i}a_0 = 0,$$
\[ c_{m-i} b_j = 0, \ j = 1, \ldots, i - 1. \]

(If \( i = 1 \) only lines 4 and 5 of display (7) are operational.) Conversely, suppose the relations (7) are satisfied, or equivalently

\[ y f - b_{i-1} g = 0 = x g - c_{m-i} f. \]

By Nakayama’s Lemma, 1, \( x, \ldots, x^{m-i}, y, \ldots, y^{i-1} \) generate \( R_S/I_S \), hence to show (4) defines a flat family it suffices to show these elements admit no nontrivial \( S \)-relations mod \( I_S \). To this end, suppose

\[ u_{m-i}(x) + v_{i-1}(y) = A(x, y) f + B(x, y) g \]

where \( u, v, A, B \) are all polynomials in the indicated variables and of the indicated degrees (if any) with coefficients in \( S \) and \( v \) has no constant term; in fact it clear a priori that then \( A, B \) must have coefficients in \( m_S \). Then the relations (8) allow us to rewrite (9) as

\[ u_{m-i}(x) + v_{i-1}(y) = A'(x) f + B'(y) g, \]

with \( A'(x) = \sum a'_k x^k \in m_S[[x]], B'(x) = \sum b'_k y^k \in m_S[[y]]. \) Comparing coefficients of \( x^{m-i+1}, \ldots, y', \ldots \) in (9') we get relations

\[ -a'_0 = a'_1 a_{m-i} + \ldots, -b'_0 = b'_1 b_{i-1} + \ldots \]

\[ \ldots \]

\[ -a'_k = a'_{k+1} a_{m-i} + \ldots, -b'_k = b'_{k+1} b_{i-1} + \ldots \]

Starting from the fact that \( a_j, a'_k \in m_S, \forall j, k, \) we infer from these first that, in fact, \( a'_k \in m_S^2, \forall k; \) plugging the latter fact back into the relations (10) we then infer \( a'_k \in m_S^3, \forall k, \) and so on. Since \( S \) is artinian it follows that \( a'_k = 0, \forall k \) and likewise for \( b'_k. \) Thus

\[ A' = B' = u_{m-i} = v_{i-1} = 0, \]

hence there are no nontrivial relations, as claimed.

Thus the Hilbert scheme is embedded in the space of the variables

\[ a_{1}, \ldots, a_{m-i}, d_{1}, \ldots, d_{i-1}, b_{i-1}, c_{m-i}, \]

i.e. \( \mathbb{A}^{m+1} \), and defined by the relation

\[ b_{i-1} c_{m-i} = 0. \]

Thus it is a union of 2 smooth \( m- \)dimensional components meeting transversely in a smooth \( (m-1) \)-dimensional subvariety. The generic point on the component where \( b_{i-1} = 0 \) (resp. \( c_{m-i} = 0 \)) is clearly an ideal generated by \( g \) (resp. \( f \)), which has the properties as claimed.
In the case $I = I^m_1(a) = (y^i + ax^{m-i})$, a similar analysis shows that an $S$-deformation of $I$ is given by the principal ideal

$$I_S = (y^i + \tilde{a}x^{m-i} + f_1(x) + f_2(y))$$

where

$$\tilde{a} \in S, \tilde{a} \equiv a \mod m_S,$$

$$f_1(x) = \sum_{j=0}^{m-i-1} a_j x^j, f_2(y) = \sum_{j=1}^{i-1} b_j y^j,$$

and via $(\tilde{a}, a_0, ..., a_{m-i-1}, b_1, ..., b_{i-1})$ we may identify Hilbert scheme locally with $A^m$. \hfill \square

**Remark 2.1.** We note that in terms of the above coordinates the subset $\text{Hilb}^0_m(R) \subset \text{Hilb}_m(R)$ is defined by

$$a_1 = ... = a_{m-i} = d_1 = ... = d_{i-1} = 0,$$

i.e. by the conditions

$$f(x, 0) = x^{m-i+1}, g(0, y) = y^i.$$

We shall see below that this, in fact, defined the 'natural' scheme structure on $\text{Hilb}^0_m(R)$. \hfill \square

Next we consider the relative local situation, i.e. that of a germ of a (1-parameter) family of curves with smooth total space specializing to a node. Thus set

$$\tilde{R} = \mathfrak{t}[x, y]_{(x, y)}, B = \mathfrak{t}[\mathfrak{t}]_{(t)},$$

and view $\tilde{R}$ as a $B$-module via $xy = t$. As is well known, this is the versal deformation of the node singularity $xy = 0$, so any family of nodal curves is locally a pullback of this.

**Theorem 3.** The relative Hilbert scheme $\text{Hilb}^m(\tilde{R}/B)$ is formally smooth, formally $(m + 1)$-dimensional over $\mathfrak{t}$.

**proof.** The relative Hilbert scheme parametrizes length-$m$ schemes contained in fibres of $\text{Spec}(\tilde{R}) \to \text{Spec}(B)$. This means ideals $I_S < \tilde{R}_S$ of colength $m$ containing $xy = s$ for some $s \in m_S$, such that $\tilde{R}_S/I_S$ is $S$-free. The analysis of these is virtually identical to that contained in the proof of Theorem 2, except that the relation $b_{i-1}c_{m-i} = 0$ gets replaced by

$$(*) \quad b_{i-1}c_{m-i} = s$$

and lines 3,5,6 of display (7) are replaced, respectively, by

$$b_{i-1}c_j = sa_{j+1}, j = 0, ..., m - i - 1$$
Set $L \equiv \text{equations and that in a neighborhood of } \tilde{C}
abla \text{be the subscheme defined by the affine space of the variables } a_1, \ldots, a_{m-i}, d_1, \ldots, d_{i-1}, b_{i-1}, c_{m-i}, t \text{ defined by the relation (12)}$

$$b_{i-1}c_{m-i} = t,$$

hence is smooth as claimed. □

Remark 3.1. After this was written, the author was informed by Prof. I. Smith of some related work by himself and Prof. S. Donaldson [DS, Sm] which considers the relative Hilbert scheme of a pencil of nodal curves on a smooth surface from a rather different viewpoint, valid in the symplectic category over $\mathbb{C}$; in particular, they prove in this context an analogue of Theorem 3 (smoothness of the total space of the relative Hilbert scheme).

Construction 3.2. An explicit construction of $\text{Hilb}_m(R)$, globally along $\text{Hilb}^0_m(R)$, can be given as follows. Let $C_1, \ldots, C_{m-1}$ be copies of $\mathbb{P}^1$, with homogenous coordinates $u_i, v_i$ on the $i$-th copy. Let $\tilde{C} \subset C_1 \times \ldots \times C_{m-1} \times \mathbb{A}^1$ be the subscheme defined by

$$v_1u_2 = tu_1v_2, \ldots, v_{m-2}u_{m-1} = tu_{m-2}v_{m-1}.$$  

Thus $\tilde{C}$ is a reduced complete intersection of divisors of type $(1,1,0,\ldots,0), (0,1,1,0,\ldots,0), \ldots, (0,0,1,1)$ and it is easy to check that the fibre of $\tilde{C}$ over $0 \in \mathbb{A}^1$ is

$$\tilde{C}_0 = \bigcup_i [1,0] \times \ldots \times [1,0] \times C_i \times [0,1] \times \ldots \times [0,1]$$

and that in a neighborhood of $\tilde{C}_0$, $\tilde{C}$ is smooth and $\tilde{C}_0$ is its unique singular fibre over $\mathbb{A}^1$. We may identify $\tilde{C}_0$ is an obvious way with $\text{Hilb}^0_m(R)$. Next consider an affine space $\mathbb{A}^{2m}$ with coordinates $a_0, \ldots, a_{m-1}, d_0, \ldots, d_{m-1}$ and let $\tilde{H} \subset \tilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

$$a_0u_1 = tv_1, d_0u_{m-1} = tu_{m-1}$$

$$a_1u_1 = d_{m-1}u_1, \ldots, a_{m-1}u_{m-1} = d_1v_{m-1}.$$  

Set $L_i = p_{\tilde{C}_i}^* \mathcal{O}(1)$. Then consider the subscheme of $\tilde{H} \times_{\mathbb{A}^1} \text{Spec}(\tilde{R})$ defined by the equations

$$F_0 := x^m + a_{m-1}x^{m-1} + \ldots + a_1x + a_0 \in \Gamma(\mathcal{O}_{\tilde{H}} \otimes R)$$

$$F_1 := u_1x^{m-1} + u_1a_{m-1}x^{m-2} + \ldots + u_1a_2x + u_1a_1 + v_1y \in \Gamma(L_1 \otimes R)$$

$$\ldots$$

$$F_i := u_ix^{m-1} + u_ia_{m-1}x^{m-i-1} + \ldots + u_ia_{i+1}x + u_ia_i + v_id_{m-i+1}y + \ldots + v_id_{m-1}y^{i-1} + v_iy^i$$
\[ \in \Gamma(L_i \otimes R) \]

\[ F_m := d_0 + d_1 y_1 + \ldots + d_{m-1} y^{m-1} + y^m \in \Gamma(O_{\tilde{H}} \otimes R). \]

In view of the above results, it is easy to check that the ideal sheaf \( I \) generated by \( F_0, \ldots, F_m \) defines a subscheme of \( \tilde{H} \times \text{Spec}(R) \) that is flat over \( \tilde{H} \) and that may serve to identify \( \tilde{H} \) with \( \text{Hilb}(\tilde{R})/B \). Locally at a point of type \( c_i^m \) (resp. \( Q_i^m \)), \( I \) is generated by \( F_i \) (resp. \( F_{i-1}, F_i \)).

Now let \( E \) be the universal bundle on \( \tilde{H} \), whose fibre at a point corresponding to a length-\( m \) scheme \( z \) is \( \Gamma(O_z) \), and let \( V \) be the trivial rank \(- (2m + 1) \) bundle on the symbols \( 1, x, \ldots, x^m, y, \ldots, y^m \). Since \( 1, x, \ldots, x^m, y, \ldots, y^m \) generate \( \Gamma(O_z) \) for all \( z \in \text{Hilb}_m(\tilde{R}) \), we have a surjection \( V \to E \). Then \( F_0, \ldots, F_m \) together yield a map

\[ F : O_{\tilde{H}} \oplus \bigoplus_{i=1}^{m-1} L_i^{-1} \oplus O_{\tilde{H}} \to V \]

so we get an exact sequence

\[ 0 \to O_{\tilde{H}} \oplus \bigoplus_{i=1}^{m-1} L_i^{-1} \oplus O_{\tilde{H}} \to V \to E \to 0. \]

In particular, it follows that \( c_1(E) = L_1 + \ldots L_{m-1} \).

Note that \( \mathbb{A}_a^m, a_{m-1}, d_{0}, \ldots, d_{m-1} \) may be identified, respectively, with \( \text{Sym}^m \mathbb{A}_x^1 \), \( \text{Sym}^m \mathbb{A}_y^1 \) and accordingly it will be convenient to use the notation \( \sigma_i = a_{m-i}, \tau_i = d_{m-i} \), these being respectively the \( i \)-th elementary symmetric functions in the roots of \( F_0, F_m \). I claim, in fact, that the projection \( c : \tilde{H} \to \mathbb{A}^{2m+1} \) may be identified with the cycle map \( \tilde{H} \to \text{Sym}^m(\text{Spec}(\tilde{R})/B) \) : this is because the natural map

\[ \Phi : \text{Sym}^m(\text{Spec}(\tilde{R})/B) \to \mathbb{A}_d^m \times \mathbb{A}_v^m \times B \]

is an embedding. \( \Phi \) is an embedding because its restrictions on the generic fibre and the special fibre over \( B \) are embeddings. Moreover, it is not hard to see that the image of \( \Phi \) (which coincides with that of \( c \)) is scheme-theoretically defined by the equations

\[ \sigma_i \tau_j = t \sigma_{i-1} \tau_{j-1}, \quad \forall i, j = 1, \ldots, m, \quad i + j > m \]

where we set \( \sigma_0 = \tau_0 = 1 \). Setting \( S_m = \text{im}(\Phi) \), we have that \( \tilde{H} \) is a Weil divisor in \( S_m \times \mathbb{C} \). \( \square \)

The local analysis immediately yields some conclusions for the Hilbert scheme of a nodal curve:

**Corollary 4.** Let \( C_0 \) be a curve with only \( k \) nodes as singularities and \( c \) irreducible components. Then

(i) \( \text{Hilb}_m(C_0) \) is reduced and has precisely \( \binom{m+c-1}{m} \) components, the general element of each of which corresponds to a reduced subscheme of the smooth part of \( C_0 \);
(ii) let \( I \) be a point of \( \text{Hilb}_m(C_0) \) having colength \( m_i \) at the \( i \)-th node of \( C_0 \); then locally at \( I, \text{Hilb}_m(C_0) \) is a cartesian product of \( k \) factors, each of which is a 2-component normal crossing of dimension \( m_i, i = 1, ..., k \), or a point if \( m_i = 0 \), times a smooth factor.

(iii) the fibre of the cycle map

\[ \text{cyc}: \text{Hilb}_m(C_0) \rightarrow \text{Sym}^m(C_0) \]

(cf. [A]) over a cycle having multiplicity \( m_i \) at the \( i \)-th node is a product of \( 1 \)-dimensional rational chains of length \( m_i - 1 \).

**proof.** It is clear from the explicit analysis in the proof of Theorem 2 that any sub-scheme of \( C_0 \) deforms to a reduced subscheme supported on the smooth part. Such subschemes are parametrized by an open dense subset of the symmetric product \( \text{Sym}^m(C_0) \). This clearly yields (i) and (ii), while (iii) follows from the fact that the fibres of \( \text{cyc} \) are products of punctual Hilbert schemes. \( \Box \)

Next we extend (most of) the above results to the case of flag Hilbert schemes (see [Se] for a general discussion of those). By definition, for any decreasing sequence of positive integers

\[ m_1 = (m_1 > ... > m_k), \]

the flag Hilbert scheme \( \text{Hilb}_{m_1}(R) \) parametrizes nested chains of ideals

\[ I_1^1 < ... < I_k^1 < R_S \]

such that \( R_S/I_S \) is \( S \)-free of rank \( m_j, j = 1, ..., k \) (i.e. such that each \( I_S^j \) is an \( S \)-point of \( \text{Hilb}_{m_j}(R) \)); similarly for punctual and relative Hilbert schemes. Thus a flag Hilbert scheme is by definition a subscheme of a product (or, in the relative case, a fibre product) of its ’constituent’ ordinary Hilbert schemes and as such comes equipped with forgetful projection morphisms to those constituents; moreover the defining equations for \( \text{Hilb}_{m_1}(R) \) in \( \prod_{j=1}^{k} \text{Hilb}_{m_j}(R) \) are just the conditions that the inclusions (13) hold, and these equations involve only pairs of successive factors \( \text{Hilb}_{m_j}(R), \text{Hilb}_{m_{j+1}}(R), j = 1, ..., k - 1 \). In the case of ’full flags’, i.e. the case

\[ m_1 = (m > m - 1 > ... > 1), \]

we will denote \( \text{Hilb}_{m_1}(R) \) by \( \text{fHilb}_{m_1}(R) \). We begin by analyzing the case of pairs of ideals of relative colength 1:

**Theorem 5.** (i) The only colength-\((m - 1)\) ideal containing \( I_i^m(a) \) is \( Q_i^{m-1} \); the only colength-\((m - 1)\) ideals containing \( Q_i^m \) are the \( I_i^{m-1}(a) \) for \( a \neq 0 \) and their limits \( Q_i^{m-1} \) and \( Q_i^{m-1} \).

(ii) The punctual flag Hilbert scheme \( \text{Hilb}^0_{m,m-1}(R) \), as algebraic set, is a chain of nonsingular rational curves of the form

\[ C_1^m \cup (Q_1^m, Q_1^{m-1}) C_1^{m-1} \cup (Q_2^m, Q_2^{m-1}) ... C_{m-1}^m \cup \sum_{(Q_{m-1}^m, Q_{m-1}^{m-1})} C_{m-1}^m; \]
it has ordinary nodes at \( Q_2^m, ..., Q_{m-1}^m \) and is smooth elsewhere. Each component \( C_i^m \) projects isomorphically to its image in \( \text{Hilb}^0_m(R) \) and to a point \( Q_i^m \) in \( \text{Hilb}_{m-1}(R) \), and vice-versa for \( C_i^{m-1} \).

(iii) The flag Hilbert scheme \( \text{Hilb}_{m,m-1}(R) \), as formal scheme along \( \text{Hilb}^0_{m,m-1}(R) \), has normal crossing singularities and at most triple points. Each of its components is formally smooth, \( m \)-dimensional and of the form

\[
D_{i,i'}^{m,m-1}, i = 0, ..., m, i - 1 \leq i' \leq i,
\]

and projects to \( D_i^m, D_{i'}^{m-1} \) respectively (cf. Theorem 2); \( D_{i,i'}^{m,m-1} \) meets \( D_{j,j'}^{m,m-1} \) nontrivially iff

\[
|i - j| + |i' - j'| \leq 1;
\]

the components of \( \text{Hilb}^0_{m,m-1}(R) \) contained in \( D_{i,i'}^{m,m-1} \) (resp. \( D_{i,i'-1}^{m,m-1} \)) are \( C_i^m \) and \( C_i^{m-1} \) (resp. \( C_{i-1}^m \) and \( C_i^{m} \)).

(iv) The relative flag Hilbert scheme \( \text{Hilb}_{m,m-1}(\tilde{R}/B) \), as formal scheme along \( \text{Hilb}^0_{m,m-1}(R) \), is formally smooth and \((m+1)\)-dimensional over \( \mathfrak{k} \). The natural map

\[
\text{Hilb}_{m,m-1}(\tilde{R}/B) \to \text{Hilb}_{m-1}(\tilde{R}/B)
\]

is a flat, locally complete intersection morphism of relative dimension 1.

Proof. Assertion (i) is an elementary consequence of the analysis in the proof of Theorem 1 and its proof will be omitted. Assertion (ii) and the set-theoretic portion of Assertion (iii) follow from this. To complete the proof of (iii),(iv), it remains to analyze the situation locally at each pair \((I, I') \in \text{Hilb}^0_{m,m-1}(R)\). We will consider the case of \((Q_i^m, Q_i^{m-1}), 1 < i < m\), as other cases are similar or simpler. There we will focus mainly on Assertion (iv), as (iii) is essentially a special case of this (as will be indicated below). Consider then a pair \((I_S < I'_S)\) flatly deforming \((Q_i^m, Q_i^{m-1})\) relative to \( B \). Then we may assume that for some \( s \in m_S, I_S \) and \( I'_S \) are generated by \( xy - s \) and \( f, g \) (resp. \( f', g' \)) with

\[
f = x^{m+1-i} + \sum_{j=0}^{m-i} a_j x^j + \sum_{j=1}^{i-1} b_j y^j,
\]

\[
g = y^i + \sum_{j=0}^{m-i} c_j x^j + \sum_{j=1}^{i-1} d_j y^j,
\]

\[
f' = x^{m-i} + \sum_{j=0}^{m-i} a'_j x^j + \sum_{j=1}^{i-1} b'_j y^j,
\]

\[
g' = y^i + \sum_{j=0}^{m-i-1} c'_j x^j + \sum_{j=1}^{i-1} d'_j y^j.
\]

(For the non-relative case we take \( s = 0 \).)
As we saw above, the relations as in (7), or equivalently (8), are necessary and sufficient so that \( I_S, I'_S \) are \( S \)-flat deformations of \( I, I' \) respectively. In particular, these include

(14) \[ b_{i-1}c_{m-i} = s = b'_{i-1}c'_{m-i-1}. \]

The other relations can be used to eliminate some of the parameters. It remains to account for the condition that \( I_S < I'_S \). To this end it suffices to note that

\[ 1, x, \ldots, x^{m-i-1}, y, \ldots, y^{i-1} \]

form an \( S \)-free basis of \( R_S/I'_S \), then express \( f, g \) in terms of this basis and equate the coefficients to 0. This yields the coefficient relations

\[
\begin{align*}
a_0 &= -(a_{m-i} - a'_{m-i-1})a'_0 + sb'_1, \\
a_j &= a'_{j-1} - (a_{m-i} - a'_{m-i-1})a'_j, j = 1, \ldots, m-i-1, \\
b_j &= ((a_{m-i} - a'_{m-i-1})b'_j + sb'_{j+1}, j = 1, \ldots, i-2, \\
b_{i-1} &= (a_{m-i} - a'_{m-i-1})b'_{i-1}, \\
c_j &= c'_j + c_{m-i}a'_j, j = 0, \ldots, m-i-1, \\
d_j &= d'_j + c_{m-i}b'_j, j = 1, \ldots, i-1. \\
\end{align*}
\]

These coefficient relations are equivalent to

(14.1) \[ f = (x + a_{m-i} - a'_{m-i-1})f', \quad g = g' + c_{m-i}f'. \]

By formal manipulations, these relations imply that

(15) \[ c'_{m-i-1} = c_{m-i}(a_{m-i} - a'_{m-i-1}). \]

Therefore the 2 relations (14) are replaced by the single relation

(16) \[ (a_{m-i} - a'_{m-i-1})b'_{i-1}c_{m-i} = s. \]

Consequently the relative flag Hilbert scheme is smooth here, with regular parameters

\[ a'_1, \ldots, a'_{m-i-1}, a_{m-i}, d'_1, \ldots, d'_{i-1}, b'_{i-1}, c_{m-i} \]

and its fibre, i.e. \( \text{Hilb}_{m-i}(R) \), is the normal-crossing triple point

\[ (a_{m-i} - a'_{m-i-1})b'_{i-1}c_{m-i} = 0. \]

The 3 components are: \( D^{m,m-1}_{i-1,i-1} \), defined by \( a_{m-i} - a'_{m-i-1} = 0 \) (which implies \( b_{i-1} = c'_{m-i-1} = 0 \); \( D^{m,m-1}_{i,i} \), defined by \( b'_{i-1} = 0 \) (which implies \( b_{i-1} = 0 \);
Therefore the value of \( A_m \), defined by \( c_{m-i} = 0 \), (which implies \( c'_{m-i-1} = 0 \)). Finally the relation (15) exhibits Hilb\(_{m,m-1}(\mathbb{R}/B)\) locally as a conic in an \( \mathbb{A}^2 \) with coordinates \( a_{m-i}, i_{m-i} \) over Hilb\(_{m-1}(\mathbb{R}/B)\), and therefore the projection is a flat locally complete intersection morphism. \( \square \)

**Remark 5.1.** In the case of the punctual flag Hilbert scheme, we have by Remark 2.1 that the only coefficients in the above calculation not automatically 0 are \( b_{i-1}, c_{m-i}, b'_{i-1}, c_{m-1-i} \), and the vanishing of \( a_{m-i}, a'_{m-i-1}, d_1, d'_1 \) yield the relations

\[
(17) \quad b_{i-1} = c'_{m-i-1} = c_{m-i}b'_{i-1} = 0.
\]

Thus Hilb\(_{m,m-1}(R)\) has two components at \( (Q_i^m, Q_i^{m-1})\): one where \( c_{m-i} = 0 \), which projects to \( \{Q_i^m\} \subset \text{Hilb}_m(R) \) and to \( C_{m-1}^{m-1} \subset \text{Hilb}_{m-1}(R) \); the other where \( b'_{i-1} = 0 \) which projects to \( C_i^m \subset \text{Hilb}_m(R) \) and to \( \{Q_i^{m-1}\} \subset \text{Hilb}_{m-1}(R) \). Thus we recover in terms of equations the set-theoretic picture presented in Theorem 5(ii). \( \square \)

**Remark 5.2.** For future reference we note that in the analogous case of (relative) deformations of \( (Q_i^m, Q_i^{m-1}) \), \( f', g' \) take the form

\[
f' = x^{m-i+1} + \sum_{j=0}^{m-i} a_j' x^j + \sum_{j=1}^{i-2} b_j' y^j, \quad g' = y^{i-1} + \sum_{j=0}^{m-i} c_j' x^j + \sum_{j=1}^{i-2} d_j' y^j.
\]

The relations (15) and \( b_{i-1} = (a_{m-i} - a'_{m-i-1})b'_{i-1} \) (see just above (15)) are replaced by

\[
(18) \quad c_{m-i} = (d_{i-1} - d'_{i-2})c'_{m-i}, \quad b_{i-2}' = (d_{i-1} - d'_{i-2})b_{i-1}. \quad \text{As we shall see, this remark is useful in studying flag Hilbert schemes with more than 2 constituents.} \quad \square
\]

**Remark 5.3.** Note that for any \( (I, I') \in \text{Hilb}_{m,m-1}(R) \), the annihilator Ann(\( I'/I \)) \( \subset R \) is an ideal of colength 1. This yields a morphism

\[
A_m : \text{Hilb}_{m,m-1}(R) \to \text{Hilb}_1(R) = \text{Spec}(R).
\]

For example, in the situation of (7), we have by (the analogue of) (8),

\[
yf' = b_{i-1}' g' = b_{i-1}'g - b_{i-1}'c_{m-i}f',
\]

hence \( J := (x + a_{m-i} - a'_{m-i-1}, y + b_{i-1}'c_{m-i}) \) annihilates \( f' \) mod \( I_S \). Hence by (7) again, \( J \) annihilates \( I'_S \) mod \( I_S \). Note that by (7), we have \( b_{i-1}'c_{m-i} = d_{i-1} - d'_{i-1} \). Therefore the value of \( A_m \) on the \( S \)-valued point \( (I_S, I'_S) \) (which extends the closed point \( (Q_i^m, Q_i^{m-1}) \)) is

\[
A_m(I_S, I'_S) = (x + a_{m-i} - a'_{m-i-1}, y + d_{i-1} - d'_{i-1}).
\]
Consequently, the closed (special) fibre of $A_m$ is defined in terms of $f, g, f', g'$ by

$$f(x,0) = xf'(x,0), g(0,y) = g'(0,y). \Box$$

Construction 5.4. An analogue of Construction 3.2 in the flag case may be given as follows. Let $a_i, d_i$ etc. be as there and let $a'_i, d'_i$ etc. be the analogous objects for $\text{Hilb}_{m-1}$. Set

$$r = a_{m-1} - a'_{m-2}, s = d_{m-1} - d'_{m-2},$$

so that

$$F_0 = F'_0(x + r), F_m = F'_{m-1}(y + s)$$

and we have the relation

$$rs = t,$$

so that $(r,s)$ give a copy of Spec$(\bar{R})/B$. Then $\text{Hilb}_{m,m-1}(\bar{R}/B)$ may be realized as the subscheme of $\tilde{C} \times_B \text{Spec}(\bar{R}) \times_B \text{Hilb}_{m-1}(\bar{R}/B)$ defined by

$$u'_iv_i = ru'_iv'_i, \quad v'_i u_{i+1} = sv_i u'_i$$

$$a'_i u_{i+1} = sd'_{m-1 - i} v_{i+1}, \quad d'_{m-1 - i} v_i = ra'_i u_i, i = 1, \ldots, m - 1.$$

$\Box$

In extending these results to the case of longer- a fortiori, full- flags, the same methods apply. But in the conclusions, a couple of new twists come up, already for flags of type $m = (m, m - 1, m - 2)$. First, the punctual Hilbert scheme $\text{Hilb}^0_m(R)$ will have components of varying dimensions (in this case, 1 and 2), roughly speaking because the parameters in $\text{Hilb}^0_m(R)$ and $\text{Hilb}^0_{m-2}(R)$ can vary independently. Second, the relative flag Hilbert scheme $\text{Hilb}_m(\bar{R}/B)$ is not the same locally at $(Q^m_i, Q^{m-1}_i, Q^{m-2}_i)$ as at $(Q^m_i, Q^{m-1}_i, Q^{m-2}_i)$. The following proof contains the relevant computation.

Lemma 6. Set $m = (m, m - 1, m - 2)$. Then (i) as algebraic set, $\text{Hilb}^0_m(R)$ is of the form

$$C^m_1 \cup C^{m-1}_1 \cup C^{m-2}_{2,1} \cup C^{m-1}_2 \cup \ldots \cup C^{m-2}_{m-2, m-3} \cup C^{m-1}_{m-2} \cup C^{m-1}_m.$$  

Each component $C^{m-2}_{i, i-1}$ projects isomorphically to $C^m_i \times C^{m-2}_{i-1} \subset \text{Hilb}_{m,m-2}(R)$ and to $\{Q^{m-1}_i\} \subset \text{Hilb}_{m-1}(R)$.

(ii) $\text{Hilb}_m(\bar{R}/B)$ is irreducible and is smooth except at points $(Q^m_i, Q^{m-1}_i, Q^{m-2}_i)$, where is has a rank-4 quadratic hypersurface singularity with local equation

$$a_{m-i} - a'_{m-i-1}c_{m-i} = (d'_{i-1} - d'_{i-2})c'_{m-i-1}.$$

Proof. The set-theoretic assertion (i) follows from Theorem 5. For (ii), we will analyze $\text{Hilb}_m(\bar{R}/B)$ locally at points of the form $(Q^m_i, Q^{m-1}_i, Q^{m-2}_i)$ or $(Q^m_i, Q^{m-1}_i, Q^{m-2}_i)$ as other cases are similar or simpler. Beginning with the former case, consider a
deformation \((I_S'' < I_S' < I_S)\) of \((Q_i^m, Q_i^{m-1}, Q_i^{m-2})\) where \(I_S, I_S'\) are as in the proof of Theorem 5 and \(I_S''\) is analogously defined by

\[
f'' = x^{m-i-1} + \sum_{j=0}^{m-i-2} a_j'' x^j + \sum_{j=1}^{i-1} b_j'' y^j,
\]

\[
g'' = y^i + \sum_{j=0}^{m-i-2} c_j'' x^j + \sum_{j=1}^{i-1} d_j'' y^j.
\]

Then working as in the proof of Theorem 5 we find the the \((m+1)\) parameters

\[a_1'', ..., a_{m-i-2}'', d_1'', ..., d_{i-1}'', b_{i-1}', c_{m-i}, a_{m-i-1}', a_{m-i}
\]

such that all the coefficients of all our polynomials \(f, ..., g''\), as well as the parameter \(s\), are regular expressions in these and there are no relations. This shows that \(\text{Hilb}_m(R/B)\) is smooth at \((Q_i^m, Q_i^{m-1}, Q_i^{m-2})\).

In the case of \((Q_i^m, Q_i^{m-1}, Q_i^{m-2})\), we may assume

\[
f'' = x^{m-i} + \sum_{j=0}^{m-i-1} a_j'' x^j + \sum_{j=1}^{i-2} b_j'' y^j,
\]

\[
g'' = y^{i-1} + \sum_{j=0}^{m-i-1} c_j'' x^j + \sum_{j=1}^{i-2} d_j'' y^j
\]

and analogous considerations yield \((m+2)\) parameters

\[a_1'', ..., a_{m-i-1}'', d_1'', ..., d_{i-2}'', b_{i-1}', c_{m-i}, c_{m-i-1}', d_{i-1}', a_{m-i}
\]

such that all the coefficients of all our polynomials, as well as the parameter \(s\), are regular expressions in these, and satisfying the relation

\[c_{m-i-1}' = (a_{m-i} - a_{m-i-1}')c_{m-i} = (d_{i-1}' - d_{i-2}'')c_{m-i-1}'
\]

which yields the equation (19).

Finally, irreducibility of \(\text{Hilb}_m(R/B)\) follows from the fact that its natural map to \(\text{Hilb}_{m-1}(R/B)\) is flat with irreducible generic fibre and \(\text{Hilb}_{m-1}(R/B)\) is irreducible by Theorem 5(iv). \(\square\)

**Theorem 7.** The (full) flag Hilbert scheme \(f\text{Hilb}_m(R/B)\) has locally complete intersection singularities and its natural map to \(B\) is a local complete intersection morphism. In particular \(f\text{Hilb}_m(R/B)\) is reduced and is flat over \(B\).

**proof.** The fact that \(f\text{Hilb}_m(R/B)\) has locally complete intersection singularities follows by a downwards induction as in the proof of Lemma 6. A similar argument shows that each map

\[f\text{Hilb}_m(R/B) \to f\text{Hilb}_{m-1}(R/B)
\]
is a locally complete intersection morphism, hence so is $f \text{Hilb}_m(\tilde{R}/B) \to B$. □

As in Remark 5.3, we have a natural map

$$fA_m = A_m \times A_{m-1} \times \ldots \times A_1 : f\text{Hilb}_m(R) \to \text{Hilb}_1(R)^m = \text{Spec}(R)^m$$

which fits in the diagram

$$
\begin{array}{ccc}
\text{fHilb}_m(R) & \to & \text{Hilb}_m(R) \\
\downarrow fA_m & & \downarrow \text{cyc} \\
\text{Spec}(R)^m & \to & \text{Sym}^m(\text{Spec}(R))
\end{array}
$$

Then the closed fibre of $fA_m$ is called the $m$th punctual flag-Hilbert scheme of $R$, denoted $f\text{Hilb}_m^0(R)$, and the (scheme-theoretic) projection of $f\text{Hilb}_m^0(R)$ to $\text{Hilb}_m(R)$, i.e. the closed fibre of the cycle map $\text{cyc}$, is called the $m$th punctual Hilbert scheme of $R$, denoted $\text{Hilb}_m^0(R)$. It is clear that $f\text{Hilb}_m^0(R)$ and $\text{Hilb}_m^0(R)$ endow the similarly denoted algebraic sets discussed previously with a scheme structure.

**Corollary 8.** The full flag punctual Hilbert scheme $f\text{Hilb}_m^0(R) = \text{Hilb}_m^0,1(R)$ is reduced and is the transverse union of components of the form

$$C_{i-1}^{m-2j}, \forall i, 1 \leq i \leq m-1, j = \min\left(\left\lfloor \frac{m}{2}\right\rfloor, i-1, m-i-1\right) \geq 0,$$

which projects isomorphically to $C_i^m \times \ldots \times C_{i-j}^{m-2j}$, and to a point in the other factors;

$$C_{i-1}^{m-1,2j-1}, \forall i, 1 \leq i \leq m-2, j = \min\left(\left\lfloor \frac{m-1}{2}\right\rfloor, i-1, m-i-1-2\right) \geq 0,$$

which projects isomorphically to $C_i^{m-1} \times \ldots \times C_{i-j}^{m-2j-1}$ and to a point in the other factors. The punctual Hilbert scheme $\text{Hilb}_m^0(R)$, with the scheme structure as above is a reduced nodal curve.

**proof.** Except for the assertion that $f\text{Hilb}_m^0(R)$, hence also $\text{Hilb}_m^0(R)$ are reduced, this is a straightforward extension of Lemma 6; the constraints on $j$ come simply from the fact that the components of $\text{Hilb}_k^0(R)$ are $C_j^k, j = 1, \ldots, k - 1$. For the reducedness assertion, we argue by induction on $m$. We may work locally at a point of the form $(Q_i^m, Q_i^{m-1}, \ldots)$, as other cases are similar or simpler. Consider a deformation of the form $(I_S = (f, g) < I'_S = (f', g') < \ldots)$ with $I_S, I'_S$ as in ??, that yields an $S$-valued point of $f\text{Hilb}_m^0(R)$. By induction, we may assume

$$f'(x, 0) = x^{m-i}, g'(0, y) = y^i,$$

which implies that

$$f' = x^{m-i} + b_{i-1} y^{i-1}, g' = y^i + c_{m-i} x^{m-i-1}.$$
subscheme $\text{Hilb}_{m}^{0}(R) \subset \text{Hilb}_{m}(R)$ is simply defined by the vanishing of $a_{1}, \ldots, d_{i-1}$, hence is reduced as claimed, and likewise for $f\text{Hilb}_{m}^{0}(R)$. □

Remark 8.1. To 'picture' the configuration in Corollary 7, it is amusing to display the components of $\text{Hilb}_{k}^{0}(R), k = 2, \ldots, m$ as segments arranged as an isosceles triangle:

\[ \cdots \]

Then the components of $f\text{Hilb}_{m}^{0}(R)$ are the columnwise products of these components.

Remark 9. in [R] we construct, based on 'geometric' considerations, a space $W^{m}(X/B)$ together with a morphism $J_{m} : W^{m}(X/B) \to f\text{Hilb}_{m}(X/B)$, which we will prove is an isomorphism. This proof requires that we know a priori that $f\text{Hilb}_{m}(X/B)$ is reduced.

Remark 10. It seems likely that the above results go through without the assumption that the base field $\mathbb{k}$ is algebraically closed, provided the fibre nodes of $X/B$ are of 'split' type, i.e. each node $p$, and each of the 2 tangent directions at $p$, are defined over $\mathbb{k}$. Beyond this however, it seems some further analysis is needed. For instance, if $p$ is a node defined over $\mathbb{k}$ of nonsplit type, i.e. with equation formally equivalent to $x^{2} + y^{2}, \text{char}(\mathbb{k}) \neq 2$, the punctual Hilbert scheme $\text{Hilb}_{n}^{0}$ at $p$ apparently has just 1 or 0 components defined over $\mathbb{k}$ depending on whether $m$ is even or odd; the other components occur in conjugate pairs.

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