Intersection cohomology and quantum cohomology of conical symplectic resolutions

Michael McBreen
Department of Mathematics, École Polytechnique Fédérale de Lausanne, CH-1015 Lausanne

Nicholas Proudfoot
Department of Mathematics, University of Oregon, Eugene, OR 97403

Abstract. For any conical symplectic resolution, we give a conjecture relating the intersection cohomology of the singular cone to the quantum cohomology of its resolution. We prove this conjecture for hypertoric varieties, recovering the ring structure on hypertoric intersection cohomology that was originally constructed by Braden and the second author.

1 Introduction

Let $\tilde{X}$ be a conical symplectic resolution of $X$; examples include the Springer resolution, Hilbert schemes of points on ALE spaces, quiver varieties, hypertoric varieties, and transverse slices to Schubert varieties in the affine Grassmannian. The purpose of this paper is to state a conjectural relationship between the intersection cohomology of $X$ and the quantum cohomology of $\tilde{X}$ (Conjecture 2.5), and to prove this conjecture for hypertoric varieties (Theorem 3.15).

Before describing the conjecture itself, we say a few words about the significance of the two sides. Intersection cohomology groups of quiver varieties were shown by Nakajima to coincide with multiplicity spaces of simple modules in standard modules over a specialized quantum loop algebra [Nak01, 3.3.2 & 14.3.10]. The equivariant intersection cohomology of a hypertoric variety is isomorphic to the Orlik-Terao algebra of a hyperplane arrangement [BP09, 4.5], which has been the subject of much recent study [Ter02, PS06, ST09, Sch11, VLR13, DGT14, Le14, Liu]. The equivariant intersection cohomology groups of slices in the affine Grassmannian, with the equivariant parameters specialized to generic values, are isomorphic via the geometric Satake correspondence to weight spaces of simple representations for the Langlands dual group [Gin 3.11 & 5.2].

On the quantum cohomology side, Okounkov and Pandharipande studied the Hilbert scheme of points in the plane [OP10], and Maulik and Oblomkov studied more generally the Hilbert scheme of points on an ALE space of type $A$ [MO09]. Braverman, Maulik, and Okounkov computed the quantum cohomology of the Springer resolution [BMO11] and gave some indication of how to proceed for arbitrary conical symplectic resolutions. This program was carried out for quiver varieties by Maulik and Okounkov [MO], who relate their quantum cohomology to the representation theory of the Yangian, and for hypertoric varieties by Shenfeld and the first author [MS13]. This last paper gives an explicit generators-and-relations presentation of the hypertoric quantum cohomology ring,

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which is a large part of what makes the hypertoric case of our conjecture more tractable than the others.

Our conjecture very roughly says that the intersection cohomology of $X$ is isomorphic to the quantum cohomology of $\tilde{X}$ specialized at $q = 1$. Of course, this cannot quite be correct as stated. The first problem is that quantum cohomology is an algebra over power series, not polynomials, so it does not make sense to set $q$ equal to 1. We address this problem simply by working with the subalgebra of quantum cohomology generated by ordinary cohomology and polynomials in $q$, which is tautologically an algebra over polynomials in $q$. The second problem is that we would expect any specialization of the quantum parameters to have the same dimension as the cohomology of $\tilde{X}$, and the intersection cohomology of $X$ is strictly smaller than that. Indeed, the actual statement involves taking a quotient of this specialization by the annihilator of $\hbar$, where $\hbar$ is the equivariant parameter for the conical action of the multiplicative group. A precise formulation of the conjecture appears in Section 2. We work out the example of $T^*\mathbb{P}^1$ in explicit detail, and give a heuristic reason why we would expect the conjecture to hold in general (Example 2.8).

One of the interesting consequences of our conjecture would be that the intersection cohomology of $X$ inherits a ring structure from the quantum cohomology of $\tilde{X}$. As mentioned above, the intersection cohomology of a hypertoric variety is already known to have a natural ring structure by work of Braden and the second author [BP09]. However, the techniques in that paper were very combinatorial, and it was never adequately explained why such a structure should exist. We regard the proof of our conjecture for hypertoric varieties as an explanation of where this mysterious ring structure comes from. See Section 3.5 for a more detailed discussion of the relationship between our results and those of [BP09]. For other conical symplectic resolutions, our (conjectural) ring structure on the intersection cohomology of $X$ appears to be new. In particular, when $X$ is a slice in the affine Grassmannian, our conjecture posits the existence of a natural ring structure on a weight space of an irreducible representation of the Langlands dual group. This may be related to the ring structure on an entire irreducible representation constructed by Feigin, Frenkel, and Rybnikov [FFR10] (Remark 2.9).

Section 2 is devoted to the statement of our conjecture, while the remainder of the paper is dedicated to the proof in the hypertoric case. The proof involves two technical results about Orlik-Terao algebras that we believe may be of independent interest, and we therefore placed them in an appendix that can be read independently from the rest of the paper.

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2 Statement of the conjecture

The purpose of this section is to make the necessary definitions to state our conjecture.
2.1 Conical symplectic resolutions

Let \((\tilde{X}, \omega)\) be a symplectic variety equipped with an action of \(\mathbb{C}^\times\), and let \(X = \text{Spec} \mathbb{C}[\tilde{X}]\). We say that \(\tilde{X}\) is a **conical symplectic resolution** if \(\mathbb{C}^\times\) acts on \(\omega\) with positive weight, \(\mathbb{C}[\tilde{X}]\) is non-negatively graded with only the constants in degree zero, and the natural map from \(\tilde{X}\) to \(X\) is a projective resolution of singularities. Examples of conical symplectic resolutions include the following:

- \(\tilde{X}\) is a crepant resolution of \(X = \mathbb{C}^2/\Gamma\), where \(\Gamma\) is a finite subgroup of \(\text{SL}(2; \mathbb{C})\). The action of \(\mathbb{C}^\times\) is induced by the inverse of the diagonal action on \(\mathbb{C}^2\).
- \(\tilde{X}\) is the Hilbert scheme of a fixed number of points on the crepant resolution of \(\mathbb{C}^2/\Gamma\), and \(X\) is the symmetric variety of unordered collections of points on the singular space.
- \(\tilde{X}\) and \(X\) are a hypertoric varieties (Section 3).
- \(\tilde{X}\) = \(T^* (G/P)\) for a reductive algebraic group \(G\) and a parabolic subgroup \(P\), and \(X\) is the affinization of this variety. (If \(G\) is of type A, then \(X\) is isomorphic to the closure of a nilpotent orbit in the Lie algebra of \(G\).) The action of \(\mathbb{C}^\times\) is the inverse scaling action on the cotangent fibers.
- \(X\) is a transverse slice between Schubert varieties in the affine Grassmannian, and \(\tilde{X}\) is a resolution constructed from a convolution variety (Remark 2.9).
- \(\tilde{X}\) and \(X\) are Nakajima quiver varieties [Nak94, Nak98].

**Remark 2.1.** The last class of examples overlaps significantly with each of the others. The first two classes are special cases of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to \(\Gamma\). A hypertoric variety associated to a cographical arrangement is a quiver variety for the corresponding graph, but not all hypertoric varieties are of this form. If \(G\) has type A, then \(T^* (G/P)\) is a quiver variety of type A, as are slices the affine Grassmannian for \(G\) (but neither of these statements holds in other types).

2.2 BBD decomposition

Let \(G\) be a reductive algebraic group acting on \(\tilde{X}\) via Hamiltonian symplectomorphisms that commute with the action of \(\mathbb{C}^\times\), and let \(G = G \times \mathbb{C}^\times\). Let \(Z := \tilde{X} \otimes_X \tilde{X}\) be the **Steinberg variety**, and let \(Z_0, Z_1, \ldots, Z_r\) be its irreducible components, with \(Z_0\) being the diagonal copy of \(\tilde{X}\). Let

\[
H := H^{2 \text{dim} X}_{BM}(Z; \mathbb{C}) = \mathbb{C}\{[Z_0], [Z_1], \ldots, [Z_r]\}
\]

be the top degree Borel-Moore homology group of \(Z\). Then \(H\) is an algebra under convolution with unit \([Z_0]\) [CG97, 2.7.41], and it acts on \(H_G^* (\tilde{X}; \mathbb{C})\). Explicitly, the action of \([Z_i]\) is the graded \(H_G^* (\ast; \mathbb{C})\)-linear endomorphism \(L_i\) of \(H_G^* (\tilde{X}; \mathbb{C})\) given by pulling and pushing along the two projections from \(Z_i\) to \(\tilde{X}\). The following results follow from [CG97, §8.9]; the main tool in the proof is the Beilinson-Bernstein-Deligne decomposition theorem, applied to the map \(\tilde{X} \to X\).
Theorem 2.2. For each pair \((S, \chi)\) consisting of a symplectic leaf of \(X\) and a local system \(\chi\) on \(S\), there is a vector space \(V_{(S,\chi)}\) such that the following statements hold.

1. The convolution algebra \(H\) is semisimple with
\[
H \cong \bigoplus_{(S,\chi)} \text{End}(V_{(S,\chi)}).
\]

2. Let \(\hat{X}\) be the dense symplectic leaf and \(\text{triv}\) the trivial local system on \(\hat{X}\). Then \(V_{(\hat{X},\text{triv})} \cong \mathbb{C}\).

3. There is a canonical isomorphism
\[
\text{IH}^*_G(X; \mathbb{C}) \cong \text{Hom}_H(V_{(\hat{X},\text{triv})}, H^*_G(\tilde{X}; \mathbb{C})).
\]

4. The kernel of the map from \(H\) to \(\text{End}(V_{(\hat{X},\text{triv})})\) is equal to \(\mathbb{C}\{[Z_1], \ldots, [Z_r]\}\).

From this we may deduce the following corollary.

Corollary 2.3. There is a canonical decomposition of graded \(H^*_G(\mathbb{C})\)-modules
\[
H^*_G(\hat{X}; \mathbb{C}) \cong \bigcap_{i=1}^r \text{Ker}(L_i) \oplus \sum_{i=1}^r \text{Im}(L_i),
\]
and the first summand is canonically isomorphic to \(\text{IH}^*_G(X; \mathbb{C})\).

Proof. From part (1) of Theorem 2.2 we have
\[
H^*_G(\hat{X}; \mathbb{C}) \cong \bigoplus_{(S,\chi)} \text{Hom}(V_{(S,\chi)}, H^*_G(\tilde{X}; \mathbb{C})) \otimes V_{(S,\chi)}.
\]

From parts (2) and (3), the summand corresponding to the pair \((\hat{X},\text{triv})\) is canonically isomorphic to \(\text{IH}^*_G(X; \mathbb{C}) \otimes \mathbb{C} \cong H^*_G(X; \mathbb{C})\). From part (4), the complementary summand is equal to \(\mathbb{C}\{[Z_1], \ldots, [Z_r]\}\cdot H^*_G(\hat{X}; \mathbb{C}) = \sum \text{Im}(L_i)\).

\[\square\]

2.3 Quantum cohomology

Let \(C \subset H_2(\bar{X}; \mathbb{Z})/H_2(\bar{X}; \mathbb{Z})_{\text{torsion}}\) be the semigroup of effective curve classes. Let
\[
\Lambda := \mathbb{C}[C] = \mathbb{C}\{q^\beta \mid \beta \in C\}
\]
be the semigroup ring of \(C\), and let \(\hat{\Lambda}\) be the completion of \(\Lambda\) at the augmentation ideal.

Assume that we are given a class \(\kappa \in H^2(\bar{X}; \mathbb{Z}/2\mathbb{Z})\) with the property that the restriction of \(\kappa\) to any smooth Lagrangian subvariety of \(\bar{X}\) is equal to its second Steifel-Whitney class. If \(\bar{X}\) is a cotangent bundle, this condition uniquely determines \(\kappa\). If \(\bar{X}\) is a Hamiltonian reduction of a
symplectic vector space by the linear action of a reductive group, then there is a natural choice for
\( \kappa \) \cite[§2.4]{BLPW}. These two cases cover all but the fifth class of examples in Section 2.1.

Let \( QH_G^{*}(\tilde{X};\mathbb{C}) \) denote the \( G \)-equivariant quantum cohomology ring of \( \tilde{X} \), modified in the sense of \cite[§1.2.5]{MO}. More precisely, the element \( q^\beta \) in our ring corresponds to the element \((-1)^{(\beta,\kappa)}q^\beta\) in the usual quantum cohomology ring. As a graded vector space, we have

\[
QH_G^*(\tilde{X};\mathbb{C}) := H_G^*(\tilde{X};\mathbb{C}) \otimes_{\mathbb{C}} \hat{\Lambda},
\]

where \( \hat{\Lambda} \) lies in degree zero. Let \( QH_G^*(\tilde{X};\mathbb{C})_{\text{pol}} \subset QH_G^*(\tilde{X};\mathbb{C}) \) be the \( \Lambda \)-subalgebra generated by the subspace \( H_G^*(\tilde{X};\mathbb{C}) \otimes_{\mathbb{C}} \Lambda \).

Consider the maximal ideal

\[
m := \langle 1 - q^\beta \mid \beta \in \mathbb{C} \rangle \subset \Lambda.
\]

Remark 2.4. Philosophically, \( m \) should be the “worst possible” maximal ideal, in that the formula for modified quantum multiplication for the Springer resolution \cite[1.1]{BMO11}, hypertoric varieties \cite[4.2]{MS13}, and quiver varieties \cite[1.3.2]{MO} all involve rational functions with denominators of the form \( 1 - q^\beta \) for some effective curve class \( \beta \).

Consider the ring

\[
R_G(\tilde{X}) := QH_G^*(\tilde{X};\mathbb{C})_{\text{pol}} \otimes_{\Lambda} \Lambda/m,
\]

which is a graded algebra over \( \mathbb{C}[\hbar] = H^*_C(\ast;\mathbb{C}) \). Let

\[
R'_G(\tilde{X}) := R_G(\tilde{X})/\text{Ann}(\hbar).
\]

We are now prepared to state our main conjecture.

Conjecture 2.5. Consider the natural map \( \psi_G : H^*_G(\tilde{X};\mathbb{C}) \to R'_G(\tilde{X}) \) of graded \( \mathbb{C}[\hbar] \)-modules given by the composition

\[
H^*_G(\tilde{X};\mathbb{C}) \hookrightarrow QH_G^*(\tilde{X};\mathbb{C})_{\text{pol}} \twoheadrightarrow R'_G(\tilde{X}).
\]

1. The map \( \psi_G \) is surjective.

2. The kernel of \( \psi_G \) is equal to \( \sum_{i=1}^r \text{Im}(L_i) \).

Proposition 2.6. If Conjecture 2.5 holds, then \( \psi_G \) descends to isomorphism of graded \( \mathbb{C}[\hbar] \)-modules from \( IH^*_G(X;\mathbb{C}) \) to \( R'_G(\tilde{X}) \).

Proof. By Corollary 2.3 we have a canonical isomorphism \( IH^*_G(X;\mathbb{C}) \cong H^*_G(\tilde{X};\mathbb{C})/\sum \text{Im}(L_i) \).

It is not known whether such a class \( \kappa \) exists in general, or whether there is always a canonical choice.
Remark 2.7. Since $H^*(\tilde{X}; \mathbb{C})$ and $H^*_G(\ast; \mathbb{C})$ both vanish in odd degree [BPW 2.5], the Leray-Serre spectral sequence for the fibration $X_G \hookrightarrow X_C \to B\Gamma$ tells us that $H^*_C(\tilde{X}; \mathbb{C})$ is a free module over $H^*_G(\ast; \mathbb{C})$ and $H^*_G(\tilde{X}; \mathbb{C}) \cong H^*_C(\tilde{X}; \mathbb{C}) \otimes H^*_G(\ast; \mathbb{C}) \mathbb{C}[\hbar]$. Similar statements hold for quantum cohomology of $\tilde{X}$ and intersection cohomology of $X$. For this reason, if Conjecture 2.5 holds for $\psi^*_G$, then it also holds for $\psi^*_C$.

Example 2.8. Consider $T^*\mathbb{P}^1$, equipped with the inverse scaling action of $\mathbb{C}^\times$ on the fibers and the natural action of a maximal torus $T \subset \text{PGL}(2)$. We have $H^*_T(T^*\mathbb{P}^1; \mathbb{C}) = \mathbb{C}[x, y, \hbar]/\langle xy \rangle$, where $x = [T^*_0\mathbb{P}^1]$ and $y = [T^*_\infty\mathbb{P}^1]$.

In quantum cohomology, we have

$$x \ast y = \frac{q}{1 - q} h L_1(y).$$

Here $L_1(y) = \mathbb{P}^1 = h - x - y$, but it will not be necessary to know this for the discussion that follows. We have presentations

$$QH^*_T(T^*\mathbb{P}^1; \mathbb{C}) = \mathbb{C}[x, y, h][[q]]/\left\langle xy - \frac{q}{1 - q} h L_1(y) \right\rangle$$

and

$$QH^*_T(T^*\mathbb{P}^1; \mathbb{C})_{\text{pol}} = \mathbb{C}[x, y, h, q]/\left\langle (1 - q)xy - q h L_1(y) \right\rangle.$$

Setting $q = 1$ gives us

$$R_T(T^*\mathbb{P}^1) \cong \mathbb{C}[x, y, h, q]/\langle h L_1(y) \rangle,$$

and killing the annihilator of $h$ gives us

$$R'_T(T^*\mathbb{P}^1) \cong \mathbb{C}[x, y, h, q]/\langle L_1(y) \rangle.$$
while $R'_G(\tilde{X})$ is an algebra. One of the interesting consequences of Conjecture 2.5 and Proposition 2.6 is that it would endow $IH^*_G(X;\mathbb{C})$ with an algebra structure. In the case of hypertoric varieties, the module $IH^*_T(\tilde{X};\mathbb{C})$ was given an algebra structure by Braden and the second author [BP09] via completely different means, and this coincides with the algebra structure that we obtain in this paper after setting $\hbar$ to zero (Proposition 3.16).

Another intriguing class of examples is the fifth one mentioned in Section 2.1. Fix a simple, simply laced algebraic group $G$ with maximal torus $T \subset G$. Let $Gr$ be the affine Grassmannian for $G$, and for any dominant coweight $\lambda \in \text{Hom}(\mathbb{C}^\times, T)$, consider the Schubert variety $Gr^\lambda \subset Gr$. Fix dominant coweights $\lambda \geq \mu$, and let $X$ be a normal slice to $Gr^\mu$ inside of $Gr^\lambda$. Using the geometric Satake correspondence [Gin, MV07], Ginzburg produces an isomorphism between a quotient of $IH^*_T(X;\mathbb{C})$ (obtained by choosing generic values for the equivariant parameters) and the $\mu$ weight space of the irreducible representation $V(\lambda)$ of $GL$ [Gin, 3.11 & 5.2]. If $\lambda$ is a sum of minuscule coweights (for example, if $G$ is of type A), then $X$ admits a conical symplectic resolution [KWWY14, 2.9]. Thus our Conjecture 2.5 and Proposition 2.6 would endow the weight space $V(\lambda)_\mu$ with a ring structure.

Since this construction involves setting equivariant parameters equal to generic values, this conjectural ring would be filtered rather than graded. It is natural to guess that the associated graded of the filtered ring $V(\lambda) = \oplus_\mu V(\lambda)_\mu$ would be isomorphic to the graded ring structure on $V(\lambda)$ constructed by Feigin, Frenkel, and Rybnikov [FFR10].

3 Hypertoric varieties

In this section we prove Conjecture 2.5 for hypertoric varieties.

3.1 Definitions

We begin by reviewing the constructive definition of a projective hypertoric variety, which was first introduced in [BD00]. An intrinsic approach to hypertoric varieties can be found in [AP].

Fix a finite-rank lattice $N$, along with a list of (not necessarily distinct) nonzero primitive vectors $a_1, \ldots, a_n \in N$ and integers $\theta_1, \ldots, \theta_n$. Consider the hyperplanes

$$H_i := \{ x \in N^\vee_\mathbb{R} \mid \langle a_i, x \rangle + \theta_i = 0 \}$$

along with the associated half-spaces

$$H_i^+ := \{ x \in N^\vee_\mathbb{R} \mid \langle a_i, x \rangle + \theta_i \geq 0 \} \quad \text{and} \quad H_i^- := \{ x \in N^\vee_\mathbb{R} \mid \langle a_i, x \rangle + \theta_i \leq 0 \}.$$

We make the following assumptions on our data:

- **Full rank**: The lattice $N$ is spanned by $\{a_1, \ldots, a_n\}$.
- **No co-loops**: For all $i$, the lattice $N$ is spanned by $\{a_1, \ldots, a_n\} \smallsetminus \{a_i\}$.
• **Unimodular:** For any $S \subset [n]$, if $\{a_i \mid i \in S\}$ spans $N_\mathbb{Q}$ over $\mathbb{Q}$, then it spans $N$ over $\mathbb{Z}$.

• **Simple:** For any $S \subset [n]$, $\text{codim } \bigcap_{i \in S} H_i = |S|$ (note that the empty set has every codimension).

Consider the short exact sequence

$$0 \to P \xrightarrow{i} \mathbb{Z}^n \xrightarrow{\pi} N \to 0,$$

where $\pi$ takes the $i^{th}$ coordinate vector to $a_i$ and $P := \ker(\pi)$. Dualizing and then taking homomorphisms into $\mathbb{C}^\times$, we obtain an exact sequence of tori

$$1 \to K \to T^n \to T \to 1.$$

The torus $T^n$ acts symplectically on $T^*\mathbb{C}^n$ with moment map

$$\mu_n : T^*\mathbb{C}^n \to \text{Lie}(T^n)^\vee \cong \mathbb{C}^n$$

given by the formula

$$\mu_n(z_1, w_1, \ldots, z_n, w_n) = (z_1 w_1, \ldots, z_n w_n).$$

Composing with $\iota^\vee : \mathbb{C}^n \to P_\mathbb{C}^\vee$, we obtain a moment map

$$\mu_K : T^*\mathbb{C}^n \to P_\mathbb{C}^\vee$$

for the action of $K$. The element $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{Z}^n \cong \text{Hom}(T^n, \mathbb{C}^\times)$ is a character of $T^n$, which we also regard as a character of $K$ by restriction. Consider the symplectic quotients

$$X := \mu_K^{-1}(0)/_{0} K = \text{Spec} \mathbb{C}[\mu_K^{-1}(0)]^K,$$

and

$$\tilde{X} := \mu_K^{-1}(0)/_{\theta} K = \text{Proj} \left( \mathbb{C}[\mu_K^{-1}(0)] \otimes \mathbb{C}[t] \right)^K,$$

where $K$ acts on $t$ via the character $\theta$. The assumptions of simplicity and unimodularity imply that the natural map from $\tilde{X}$ to $X$ is a projective symplectic resolution [BD00, 3.2 & 3.3].

The action of $\mathbb{C}^\times$ on $T^*\mathbb{C}^n$ via inverse scaling of the cotangent fibers descends to an action on $\tilde{X}$, and the symplectic form has weight 1 with respect to this action. The assumption of no co-loops implies that $\mathbb{C}[X]^\mathbb{C}^\times = \mathbb{C}$, and therefore that $\tilde{X}$ is a conical symplectic resolution of $X$. The Hamiltonian action of $T^n$ on $T^*\mathbb{C}^n$ induces an action on $\tilde{X}$, and this descends to an effective Hamiltonian action of $T$ that commutes with the action of $\mathbb{C}^\times$. Let $T = T \times \mathbb{C}^\times$.

### 3.2 Cohomology

We next review some basic facts about the cohomology of hypertoric varieties. A minimal set $C \subset [n]$ such that $\bigcap_{i \in C} H_i = \emptyset$ is called a **circuit**. If $C$ is a circuit, then there exists a unique
decomposition
\[ C = C^+ \sqcup C^- \quad \text{such that} \quad \bigcap_{i \in C^+} H^+_i \cap \bigcap_{i \in C^-} H^-_i = \emptyset. \]

Let \( A := \text{Sym} N^\vee \cong H^*_T(\ast; \mathbb{Z}) \) be the \( T \)-equivariant cohomology ring of a point. The \( T \)-equivariant cohomology ring of \( \tilde{X} \) was computed by Harada and the second author \([\text{HP}04, 4.4]\), building on the \( T \)-equivariant computation in \([\text{Kon}99]\).

**Theorem 3.1.** The ring \( H^*_T(\tilde{X}; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}[u_1, \ldots, u_n, h]/J_0 \), where \( J_0 \) is the ideal generated by
\[ \prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (h - u_j) \]
for each circuit \( C \subset [n] \). We have \( \deg(u_i) = \deg(h) = 2 \) for all \( i \), and the \( A \)-algebra structure is given by the natural inclusion
\[ \pi^\vee : N^\vee \to \mathbb{Z}^n \cong \mathbb{Z}\{u_1, \ldots, u_n\}. \]

**Corollary 3.2.** We have a canonical isomorphism \( H_2(\tilde{X}; \mathbb{Z}) \cong P \).

**Proof.** Since the generators of \( J \) all have degree at least 4, we have
\[ H^2_T(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z}\{u_1, \ldots, u_n, h\} = \mathbb{Z}^n \oplus \mathbb{Z}, \]
and therefore \( H^2(\tilde{X}; \mathbb{Z}) \cong \mathbb{Z}^n/N^\vee \cong P^\vee \). Dualizing, we obtain our result. \( \square \)

For any set \( S \subset [n] \), let \( u_S := \prod_{i \in S} u_i \). The set \( S \) is called **independent** if it contains no circuits.

**Corollary 3.3.** The ring \( H^*_T(\tilde{X}; \mathbb{Z}) \) is spanned over \( A[h] \) by monomials of the form \( u_S \), where \( S \subset [n] \) is independent.

**Proof.** It is sufficient to prove that \( H^*_T(\tilde{X}; \mathbb{Z}) \) is spanned over \( A \) by monomials of the form \( u_S \), where \( S \subset [n] \) is independent. This is shown in the appendix (Lemma \([\text{A.4}]\)). \( \square \)

3.3 Quantum cohomology

We continue by describing the various versions of the quantum cohomology ring of \( \tilde{X} \). For any circuit \( C \subset [n] \), unimodularity implies that
\[ \sum_{i \in C^+} a_i - \sum_{j \in C^-} a_j = 0. \]

Let \( C_i = 1 \) if \( i \in C^+ \), \( -1 \) if \( i \in C^- \), and 0 otherwise, and consider the element
\[ \beta_C := \sum_{i=1}^n C_i e_i \in \ker(\mathbb{Z}^n \to N) = P \cong H_2(\tilde{X}; \mathbb{Z}). \]
To compute $QH_T^*(\tilde{X}; \mathbb{C})_{pol}$, we need the following lemma, which is implicit in [MS13].

**Lemma 3.4.** For any independent subset $S \subset [n]$, the quantum product of $\{u_i \mid i \in S\}$ is equal to the ordinary product $u_S$.

**Proof.** We proceed by induction on the size of $S$. Let $j$ be the maximal element of $S$, and let $\bar{S} = S \setminus \{j\}$. By our inductive hypothesis, the quantum product of the elements $\{u_i \mid i \in S\}$ is equal to the quantum product of $u_j$ with $u_{\bar{S}}$. By [MS13, 4.2], we have

$$u_j \cdot u_{\bar{S}} = u_S + \hbar \sum_C C_j \frac{q^{\beta_C}}{1 - q^{\beta_C}} L_C(u_{\bar{S}}),$$

where $L_C$ is a certain linear combination of $L_1, \ldots, L_r$. Thus it is sufficient to show that $L_C(u_{\bar{S}}) = 0$ for all circuits $C$ containing $j$.

Let $\mu : \tilde{X} \to N^\vee_C$ be the moment map induced by $\mu_n$ for the action of $T$ on $\tilde{X}$. The operator $L_C$ is given by a correspondence $Z_C \subset Z = \tilde{X} \times_{\hat{X}} \tilde{X}$ that lies over the locus

$$H_C := \bigcap_{i \in C} H^C_i \subset N^\vee_C.$$

On the other hand, the element $u_{\bar{S}}$ may be represented by a cycle that lies over $H_{\bar{S}}$, thus $L_C(u_{\bar{S}})$ may be represented by a cycle that lies over $H_{C \cup \bar{S}} \subset H_S$. Since $S$ is independent, we have

$$\text{codim } H_S = |S| > |\bar{S}| = \frac{1}{2} \deg u_{\bar{S}} = \frac{1}{2} \deg L_C(u_{\bar{S}}),$$

which implies that $L_C(u_{\bar{S}}) = 0$.

**Theorem 3.5.** The ring $QH_T^*(\tilde{X}; \mathbb{C})_{pol}$ is isomorphic to $\Lambda[u_1, \ldots, u_n, \hbar]/J$, where $J$ is the ideal generated by

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) - q^{\beta_C} \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j$$

for each circuit $C \subset [n]$. The $A_C$-algebra structure is as in Theorem 3.1.

**Proof.** The fact that $QH_T^*(\tilde{X}; \mathbb{C})_{pol}$ is generated over $\Lambda$ by $H^2_T(\tilde{X}; \mathbb{C})$ follows from Corollary 3.3 and Lemma 3.4. The rest of the theorem appears in [MS13, 1.1].

**Corollary 3.6.** The ring $R_T(\tilde{X})$ is isomorphic to $\mathbb{C}[u_1, \ldots, u_n, \hbar]/J_1$, where $J_1$ is the ideal generated by

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - \hbar) - \prod_{i \in C^+} (u_i - \hbar) \cdot \prod_{j \in C^-} u_j$$

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3The formula in [MS13] has $q^{\beta_C}$ replaced with $(-1)^{|C|} q^{\beta_C}$ because that paper uses the unmodified quantum product.
for each circuit $C \subset [n]$. The ring $R'_T(\tilde{X}) := R_T(\tilde{X}) / \text{Ann}(h)$ is isomorphic to $\mathbb{C}[u_1, \ldots, u_n, h]/J'_1$, where $J'_1$ is the ideal generated by

$$f_C := h^{-1} \left( \prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - h) - \prod_{i \in C^+} (u_i - h) \cdot \prod_{j \in C^-} u_j \right)$$

for each circuit $C \subset [n]$.

Proposition 3.7. The ring $R'_T(\tilde{X})$ is spanned over $A_C[h]$ by monomials of the form $u_S$, where $S \subset [n]$ is independent.

Proof. It is sufficient to prove that $R'_T(\tilde{X}) / \langle h \rangle$ is spanned over $A_C$ by monomials of the form $u_S$, where $S \subset [n]$ is independent. This is shown in the appendix (Theorem A.3 and Remark A.8).

Corollary 3.8. The map $\psi_T$ from Conjecture 2.5 is surjective.

Proof. Lemma 3.4 tells us that $\psi_T$ takes the image of $u_S$ in $QH^*_T(\tilde{X}; \mathbb{C})_{\text{pol}}$ to the image of $u_S$ in $R'_T(\tilde{X})$ for all independent $S \subset [n]$. By Proposition 3.7, this implies that $\psi_T$ is surjective.

3.4 The kernel of $\psi_T$

In this section we prove the second half of Conjecture 2.5 for hypertoric varieties. Let

$$U := \ker(\psi_T) \subset H^*_T(\tilde{X}; \mathbb{C}) \supset \text{Im}(L_1) + \ldots + \text{Im}(L_r) =: V;$$

the conjecture says that $U = V$.

For any circuit $C$, let $\overline{C}$ be the set obtained from $C$ by deleting the maximal element $j_{\text{max}} \in C$, and consider the graded vector subspace

$$W := A_C[h] \cdot \{ u_S f_C \mid S \cap \overline{C} = \emptyset \text{ and } S \cup \overline{C} \text{ is independent} \}.$$ 

Lemma 3.9. $W \subset U$.

Proof. Each term of $u_S f_C$ is equal to plus or minus a power of $h$ times a square-free monomial of independent support. By Lemma 3.4, such a monomial is taken to itself by $\psi_T$. This means that $\psi_T$ takes $u_S f_C$ to itself, and $u_S f_C$ represents the zero element of $R'_T(\tilde{X})$ by Corollary 3.6.

Lemma 3.10. $W \subset V$.

Proof. Fix a circuit $C$, and assume that $j_{\text{max}} \in C^+$. Let

$$g_C := \prod_{i \in C^+ \setminus \{j_{\text{max}}\}} u_i \cdot \prod_{j \in C^-} (u_j - h) \in H^*_T(\tilde{X}; \mathbb{C}) \subset QH^*_T(\tilde{X}; \mathbb{C})_{\text{pol}}.$$
We may think of elements of $QH^*$ as functions on the space $\tilde{\mathcal{X}}$. By Theorem 3.5, we have

\[ u_{j_{\text{max}}} g_C = \prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - h) = q^{\beta C} \prod_{i \in C^+} (u_i - h) \cdot \prod_{j \in C^-} u_j \in QH_T^*(\tilde{\mathcal{X}}; \mathbb{C})_{\text{pol}}. \]

By definition of $f_C$, we have

\[ q^{\beta C} h f_C = q^{\beta C} \prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - h) - q^{\beta C} \prod_{i \in C^+} (u_i - h) \cdot \prod_{j \in C^-} u_j \]

\[ = q^{\beta C} u_{j_{\text{max}}} g_C - u_{j_{\text{max}}} g_C \]

\[ = (q^{\beta C} - 1) u_{j_{\text{max}}} g_C \in QH_T^*(\tilde{\mathcal{X}}; \mathbb{C})_{\text{pol}}. \]

Fix a set $S$ such that $S \cap C = \emptyset$ and $S \cup \overline{C}$ is independent. Multiplying both sides of the above equation by $u_S$, we obtain

\[ q^{\beta C} h u_S f_C = (q^{\beta C} - 1) u_{j_{\text{max}}} u_S g_C. \]

Since the classical product of $u_{j_{\text{max}}}$ with $g_C$ vanishes and $u_S g_C \in H_T^*(\tilde{\mathcal{X}}; \mathbb{C})$ by Lemma 3.4, we have[MS13 4.2]

\[ u_{j_{\text{max}}} \cdot u_S g_C = \hbar \sum_D D_{j_{\text{max}}} \frac{q^{\beta D}}{1 - q^{\beta D}} L_D(u_S g_C), \]

where $D$ ranges over all circuits.

Let $\mathbb{C}(\Lambda)$ be the field of fractions of $\Lambda$, and let $QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}}$ be the ring generated by $H^*_G(\tilde{\mathcal{X}}; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(\Lambda)$ under the quantum product. It follows easily from[MS13 4.2] that in fact $QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}} = H^*_G(\tilde{\mathcal{X}}; \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}(\Lambda)$ as a vector space, and that $QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{pol}} \subset QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}}$.

We may think of elements of $QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}}$ as meromorphic sections of the vector bundle with fiber $H^*_G(\tilde{\mathcal{X}}; \mathbb{C})$ over $\text{Spec} \Lambda$. In particular they have well-defined loci of poles.

We can now combine the two above equations to obtain

\[ \frac{q^{\beta C}}{q^{\beta C} - 1} h u_S f_C = \hbar \sum_D D_{j_{\text{max}}} \frac{q^{\beta D}}{1 - q^{\beta D}} L_D(u_S g_C) \in QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}}. \]

Since the left-hand side has poles only at $q^{\beta C} = 1$, so does the right-hand side. We conclude that all summands such that $D \neq C$ vanish, and we are left with

\[ \frac{q^{\beta C}}{q^{\beta C} - 1} h u_S f_C = \hbar \frac{q^{\beta C}}{1 - q^{\beta C}} L_C(u_S g_C). \]

Dividing by $\frac{q^{\beta C}}{1 - q^{\beta C}}$, we have

\[ h u_S f_C = -\hbar L_C(u_S g_C). \]

\textit{A priori}, this equation lives in $QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}}$. However, it is clear that both sides live in the subspace $QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{pol}} \subset QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{rat}}$. Furthermore, since $u_S f_C$ is a sum of powers of $h$ times independent square-free monomials, Lemma 3.4 tells us that $u_S f_C$ lies in $H_T^*(\tilde{\mathcal{X}}; \mathbb{C}) \subset QH^*_G(\tilde{\mathcal{X}}; \mathbb{C})_{\text{pol}}$. 

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Since $H_T^\star(\tilde{X}; \mathbb{C})$ is a free module over $\mathbb{C}[h]$, we may divide by $h$ to obtain
\[ u_S f_C = -L_C(u_S g_C) \in H_T^\star(\tilde{X}; \mathbb{C}). \]

Thus we see that $u_S f_C$ is in the image of $L_C$, and is therefore in the span of the images of $L_1, \ldots, L_r$. A similar argument can be applied if $j_{\text{max}} \in C^-$. \(\square\)

For any $\mathbb{N}$-graded vector space $Y = \bigoplus Y^k$ with finite-dimensional graded pieces, let
\[ \text{Hilb}(Y; t) := \sum_{k=0}^{\infty} \dim Y^k t^k \in \mathbb{N}[[t]]. \]

**Lemma 3.11.** Hilb($U; t$) = Hilb($V; t$).

**Proof.** Since $\psi_T$ is surjective (Corollary A.8), we have $R'_T(\tilde{X}) \cong H_T^\star(\tilde{X}; \mathbb{C})/U$. By Corollary 2.3, we also have $IH^\star_T(X; \mathbb{C}) \cong H_T^\star(\tilde{X}; \mathbb{C})/V$. Thus the statement that Hilb($U; t$) = Hilb($V; t$) is equivalent to the statement that Hilb($R'_T(\tilde{X}); t$) = Hilb($IH^\star_T(X; \mathbb{C}); t$).

By Remark A.8 and Theorem A.9 we have
\[ \text{Hilb}(R'_T(\tilde{X}); t) = \text{Hilb}(OT_h; t) = (1 - t)^{-1} \text{Hilb}(OT; t). \]

By Proposition A.2, Hilb($OT; t$) is equal to $(1 - t)^{-\text{rk} N}$ times the $h$-polynomial of the broken circuit complex of the matroid represented by the vectors $a_1, \ldots, a_n$. On the other hand, we have
\[ \text{Hilb}(IH^\star_T(X; \mathbb{C}); t) = (1 - t)^{-\text{rk} N - 1} \text{Hilb}(IH^\star(X; \mathbb{C}); t), \]
and Hilb($IH^\star(X; \mathbb{C}); t$) is itself equal to the $h$-polynomial of the broken circuit complex \cite[4.3]{PW07}. Thus Hilb($U; t$) = Hilb($V; t$). \(\square\)

Let $V_0 = V \otimes_{\mathbb{C}[h]} \mathbb{C}$, and let $W_0$ be the image of $W \subset V$ in $V_0$. More concretely, $V_0$ is the complement of $IH^\star_T(X; \mathbb{C})$ in
\[ H_T^\star(\tilde{X}; \mathbb{C}) \cong \mathbb{C}[u_1, \ldots, u_n]/\langle u_C \mid C \text{ a circuit} \rangle, \]
and $W_0$ is the $A_C$-submodule of $H_T^\star(\tilde{X}; \mathbb{C})$ spanned by $\{u_S f_C\}$, where $f_C$ is obtained from $f_C$ by setting $h$ equal to zero.

**Lemma 3.12.** Let $C$ be a circuit and let $S$ be a set disjoint from $C$. For any collections of non-negative integers $\underline{d} = (d_i \mid i \in S)$ and $\underline{e} = (e_j \mid j \in \overline{S})$, we have
\[ f(S, C, \underline{d}, \underline{e}) := u_S f_C \cdot \prod_{i \in S} u_i^{d_i} \cdot \prod_{j \in \overline{C}} (C_j u_j - C_{j_{\text{max}}} u_{j_{\text{max}}})^{e_j} \in W_0. \]

**Proof.** We proceed via a double induction. First, we fix $C$ and induct downward on the size of $S$. If $|S| > \text{rk} N - |\overline{S}|$, then every term of $u_S f_C$ contains a monomial supported on a dependent set,
so \( u_S f_C = 0 \). Thus we will fix \( S \) and assume that the lemma holds for all sets \( S' \supseteq S \) disjoint from \( C \). By the same reasoning, we may assume that \( S \cup C \) is independent.

Second, we induct upward on the exponents. The base case is where \( d_i = e_j \) for all \( i \) and \( j \), in which case \( f(S, C, d, e) = u_S f_C \in W_0 \) by definition of \( W_0 \). Thus we may fix \( d \) and \( e \) such that \( f(S, C, d, e) \in W_0 \) and prove that for all \( i \in S \) and \( j \in C \), we have \( u_i f(S, C, d, e) \in W_0 \) and \( (C_j u_j - C_{j_{\text{max}}} u_{j_{\text{max}}}) f(S, C, d, e) \in W_0 \).

Let \( i \in S \) be given. Since \( S \cup C \) is independent, there exists \( x \in N^\vee \) such that \( \pi^\vee(x) = \sum \gamma_k u_k \) with \( \gamma_i = 1 \) and \( \gamma_k = 0 \) for all \( k \in S \cup C \setminus \{i\} \). Then

\[
\pi^\vee(x) \cdot f(S, C, d, e) = \sum_{k=1}^n \gamma_k u_k \cdot f(S, C, d, e) = u_i \cdot f(S, C, d, e) + \sum_{k \in S \cup C} \gamma_k u_k \cdot f(S, C, d, e) = u_i \cdot f(S, C, d, e) + \sum_{k \notin S \cup C} \gamma_k \cdot f(S \cup \{k\}, C, d, e).
\]

Our first inductive hypothesis tells us that \( f(S \cup \{k\}, C, d, e) \in W_0 \), and \( W_0 \) is by definition closed under multiplication by elements of \( A \), so we also have \( \pi^\vee(x) \cdot f(S, C, d, e) \in W_0 \). This implies that \( u_i \cdot f(S, C, d, e) \in W_0 \).

Let \( j \in C \) be given. Since \( S \cup C \) is independent, there exists \( x \in N^\vee \) such that \( \pi^\vee(x) = \sum \gamma_k u_k \) with \( \gamma_j = C_j \), \( \gamma_{j_{\text{max}}} = -C_{j_{\text{max}}} \), and \( \gamma_k = 0 \) for all \( k \in S \cup C \setminus \{j, j_{\text{max}}\} \). Then

\[
\pi^\vee(x) \cdot f(S, C, d, e) = \sum_{k=1}^n \gamma_k u_k \cdot f(S, C, d, e) = (C_j u_i - C_{j_{\text{max}}} u_{j_{\text{max}}}) \cdot f(S, C, d, e) + \sum_{k \notin S \cup C} \gamma_k u_k \cdot f(S, C, d, e) = (C_j u_i - C_{j_{\text{max}}} u_{j_{\text{max}}}) \cdot f(S, C, d, e) + \sum_{k \notin S \cup C} \gamma_k \cdot f(S \cup \{k\}, C, d, e).
\]

By the same reasoning as above, this implies that \( (C_j u_i - C_{j_{\text{max}}} u_{j_{\text{max}}}) \cdot f(S, C, d, e) \in W_0 \).

\[\square\]

**Lemma 3.13.** \( V = W \).

**Proof.** We will start by proving that \( \text{Hilb}(W_0; t) = \text{Hilb}(V_0; t) \). Consider the degree-lexicographic monomial order on \( H_T^*(X; \mathbb{C}) \) with \( u_1 > u_2 > \ldots > u_n \). Given \( C, S, d, \) and \( e \) as in Lemma 3.12. The initial term of \( f(S, C, d, e) \) with respect to this order is \( \pm u_{S \cup C} \Pi_S u_i^d \Pi_C u_j^e \). These monomials span the kernel of the projection

\[
H_T^*(X; \mathbb{C}) \cong \mathbb{C}[u_1, \ldots, u_n]/\langle u_C \mid C \text{ a circuit} \rangle \to \mathbb{C}[u_1, \ldots, u_n]/\langle u_C \mid C \text{ a circuit} \rangle =: \text{SR}_{bc}.\]

\( ^4\)This is the Stanley-Reisner ring of the broken circuit complex.
thus Lemma 3.12 tells us that \( \text{in}(W_0) \) contains this kernel. We therefore have

\[
\text{Hilb}(H^*_T(\tilde{X};\mathbb{C})/W_0; t) = \text{Hilb}(H^*_T(\tilde{X};\mathbb{C})/\text{in}(W_0); t) \\
\leq \text{Hilb}(SR_{bc}; t) \\
= \text{Hilb}(IH^*_T(X;\mathbb{C})) \quad \text{by [PW07, 4.3]} \\
= \text{Hilb}(H^*_T(\tilde{X};\mathbb{C})/V_0; t).
\]

Since \( W_0 \subset V_0 \), this implies that \( W_0 = V_0 \).

We would like to use this to conclude that \( W = V \). Suppose not, and let \( v \in V \) be a homogeneous element of minimal degree that is not contained in \( W \). Let \( v_0 \) be the image of \( v \) in \( V_0 \). Since \( W_0 = V_0 \), there exists a homogeneous \( w \in W \) such that \( w_0 = v_0 \). This means that \( v - w \) is in the kernel of the projection from \( V \) to \( V_0 \), so there exists a homogeneous \( v' \in V \) with \( v - w = hv' \). By minimality of the degree of \( v \), we have \( v' \in W \), and therefore \( v = hv' + w \in W \), which is a contradiction.

Corollary 3.14. \( U = V \).

Proof. Lemmas 3.9 and 3.13 imply that \( V \subset U \), and they have the same Hilbert series by Lemma 3.11, so they must be equal.

Theorem 3.15. Conjecture 2.5 holds for hypertoric varieties.

Proof. The first part of the conjecture is Corollary 3.8 while the second is Corollary 3.14. 

3.5 Comparison with previous work

Let \( OT := R'_T(\tilde{X})/(\hbar); \) this algebra is called the Orlik-Terao algebra. In an earlier paper, Braden and the second author showed that \( IH^*_T(X;\mathbb{C}) \) is canonically isomorphic to \( OT \) [BP09, 4.5]. The first thing we want to establish is that the isomorphism in this paper is the same as the isomorphism in that paper.

Proposition 3.16. The isomorphism from \( IH^*_T(X;\mathbb{C}) \) to \( OT \) induced by \( \psi_T \) (after setting \( \hbar \) equal to zero) coincides with the isomorphism in [BP09].

Proof. Let \( F \subset [n] \) be a flat of the matroid represented by the vectors \( a_1, \ldots, a_n \). Working only with the vectors \( \{a_i \mid i \in F\} \), we obtain an algebra \( OT_F \) which is isomorphic to the quotient of \( OT \) by the ideal generated by \( \{u_i \mid i \notin F\} \). We also obtain a hypertoric variety \( X_F \) which is equipped with a normally nonsingular inclusion into \( X \) [PW07, 2.4], inducing a map \( IH^*_T(X;\mathbb{C}) \to IH^*_T(X_F;\mathbb{C}) \). The isomorphisms in [BP09] are the unique isomorphisms such that the diagrams

\[
\begin{array}{ccc}
\text{IH}^*_T(X;\mathbb{C}) & \longrightarrow & OT \\
\downarrow & & \downarrow \\
\text{IH}^*_T(X_F;\mathbb{C}) & \longrightarrow & OT_F
\end{array}
\]
commute for all $F$. Thus it is sufficient to show that this diagram commutes using the isomorphisms constructed in this paper for the horizontal arrows. This follows from the fact that the inclusion of $X_F$ into $X$ lifts to an inclusion of $\tilde{X}_F$ into $\tilde{X}$ [PW07, 2.5], and the induced map from $H^*_T(\tilde{X}; \mathbb{C})$ to $H^*_T(\tilde{X}_F; \mathbb{C})$ is given by setting $u_i$ to zero for all $i \notin F$.

We conclude by discussing some of the advantages and disadvantages of the two approaches. The main advantage of [BP09] is that the ring structure is defined at a higher categorical level: it is shown there that the intersection cohomology sheaf $IC_X$ admits the structure of a ring object in the equivariant derived category of constructible sheaves on $X$, and that the isomorphism from $IH^*_T(X; \mathbb{C})$ to $OT$ is compatible with this structure.

On the other hand, there are two advantages to the approach we take in this paper. The first is that we work $T$-equivariantly rather than $T$-equivariantly. This may not seem like a big deal, but it is not so easy to modify the techniques of [BP09] to account for the extra $\mathbb{C}^\times$-action. Any attempt in this direction would have to begin with a proof of Theorem A.9.

The second, and more significant, advantage of our approach is that the isomorphism in [BP09] comes out of nowhere: one simply shows that the ring $OT$ has the same Hilbert series and functorial properties as $IH^*_T(X; \mathbb{C})$, and that these functorial properties are sufficiently rigid to ensure that the two groups are canonically isomorphic. In contrast, the isomorphism in this paper is induced by the natural map $\psi_T$, and can be (at least conjecturally) generalized to arbitrary conical symplectic resolutions.

A Appendix: The Orlik-Terao algebra

In this paper we have required two technical results about the Orlik-Terao algebra of a collection of vectors (Theorems A.3 and A.9). Since we believe that these two statements may be of general interest in the theory of hyperplane arrangements, we put them in an appendix which may be read independently from the rest of the paper.

A.1 A spanning set

Let $k$ be a field of characteristic zero. Let $V$ be a vector space over $k$, and let $a_1, \ldots, a_n$ be nonzero linear functions on $V$ that span $V^*$. Let $I \subset k[u_1, \ldots, u_n]$ be the kernel of the map taking $u_i$ to $a_i^{-1}$. The graded $k$-algebra $OT := k[u_1, \ldots, u_n]/I$ is called the Orlik-Terao algebra.

For any subset $S \subset [n]$, let $u_S := \prod_{i \in S} u_i$. A set $C \subset [n]$ is called dependent if there exist constants $\{\eta_i \mid i \in C\}$, not all zero, such that $\sum \eta_i a_i = 0$. In this case, we have a nontrivial element

$$f_{C,0} := \sum_{i \in C} \eta_i u_{C \setminus \{i\}} \in I.$$  

This notation is somewhat sloppy, as $f_{C,0}$ depends not only on $C$, but also on the constants $\eta_i$. However, if $C$ is a circuit (a minimal dependent set), then the constants are determined up to a global nonzero scalar, thus the same is true for $f_{C,0}$.
Note that if our collection of vectors is unimodular and \( C \) is a circuit, then we may take \( \eta_i = \pm 1 \) for all \( i \), and then \( f_{C,0} \) will be the polynomial obtained from the polynomial \( f_C \) of Corollary \[\text{L.6}\] (and also of Section \[A.2]\) by setting \( h \) equal to zero; this explains our funny notation.

The following result is proved in \cite[Theorem 4]{PS06}.

**Theorem A.1.** The set \( \{ f_{C,0} \mid C \text{ a circuit} \} \) is a universal Gröbner basis for \( I \).

For any circuit \( C \), let \( \overline{C} \) be the set obtained from \( C \) by deleting the maximal element. Let

\[
SR_{\text{ind}} := k[u_1, \ldots, u_n]/\langle u_C \mid C \text{ a circuit} \rangle \quad \text{and} \quad SR_{\text{bc}} := k[u_1, \ldots, u_n]/\langle u_{\overline{C}} \mid C \text{ a circuit} \rangle.
\]

These algebras are called the **Stanley-Reisner rings** of the independence complex and the broken circuit complex, respectively. Note that \( u_{\overline{C}} \) is (up to scale) the initial term of \( f_{C,0} \), hence Theorem \[A.1\] says exactly that \( SR_{\text{bc}} \) is a flat degeneration of \( OT \).

Consider the map \( \text{Sym}(V) \to k[u_1, \ldots, u_n] \) taking \( v \in V \) to \( \sum a_i(v)u_i \). This makes the algebras \( OT, SR_{\text{ind}}, \) and \( SR_{\text{bc}} \) into graded \( \text{Sym}(V) \)-algebras. The following result is proved in \cite[Propositions 1 & 7]{PS06}.

**Proposition A.2.** The rings \( OT, SR_{\text{ind}}, \) and \( SR_{\text{bc}} \) are free as graded \( \text{Sym}(V) \)-modules. The graded rank of \( SR_{\text{ind}} \) is given by the \( h \)-numbers of the independence complex, while the graded ranks of \( OT \) and \( SR_{\text{bc}} \) are given by the \( h \)-numbers of the broken circuit complex.

The main result of this subsection is the following.

**Theorem A.3.** The ring \( OT \) is spanned over \( \text{Sym}(V) \) by elements of the form \( u_S \) where \( S \subset [n] \) is independent.

We begin by proving the analogous statement for \( SR_{\text{ind}} \) and \( SR_{\text{bc}} \), which will be used in the proof of Theorem \[A.3\].

**Lemma A.4.** The rings \( SR_{\text{ind}} \) and \( SR_{\text{bc}} \) are spanned over \( \text{Sym}(V) \) by elements of the form \( u_S \) where \( S \subset [n] \) is independent.

**Proof.** First note that \( SR_{\text{bc}} \) is a quotient of \( SR_{\text{ind}} \), so it is sufficient to prove the lemma only for \( SR_{\text{ind}} \). Since \( u_S \) vanishes whenever \( S \) contains a circuit, it is sufficient to prove that \( SR_{\text{ind}} \) is spanned over \( \text{Sym}(V) \) by square-free monomials. This is equivalent to showing that \( SR_{\text{ind}} \otimes_{\text{Sym}(V)} k \) is spanned over \( k \) by square-free monomials.

Consider an arbitrary monomial \( u^\sigma \) for some \( \sigma \in \mathbb{N}^n \) with independent support. This means that there exists a set \( B \subset [n] \) containing the support of \( \sigma \) such that \( \{ a_i \mid i \in B \} \) is a basis for \( V^* \). If \( \sigma_i \leq 1 \) for all \( i \), then we are already done, so let us suppose that there exists an index \( i \in [n] \) for which \( \sigma_i > 1 \). Consider the element \( v \in V \) that pairs to 1 with \( a_i \) and to 0 with \( a_j \) for all \( j \in B \setminus \{ i \} \), and let \( u_v := v \cdot 1 \in SR_{\text{ind}} \). By replacing \( u_i \) with \( u_i - u_v \) (which has the same image in \( SR_{\text{ind}} \otimes_{\text{Sym}(V)} k \)), we replace \( u^\sigma \) with a sum of monomials of the form \( u^\tau \), where \( \tau_i = \sigma_i - 1 \),

\[\text{17}\]

\footnote{The proof of this lemma does not require \( k \) to be a field; in particular, it holds over the integers.}
\( \tau_j = \sigma_j \) for all \( j \in B \setminus \{i\} \), and \( \tau_k \leq 1 \) for all \( k \notin B \). Applying this procedure recursively, we may express the image of \( u^\sigma \) as a sum of square-free monomials.

Given a subset \( S \subset [n] \), let \( \langle S \rangle \) be the set of all \( i \) such that \( a_i \) is contained in the \( k \)-linear span of \( \{a_j \mid j \in S\} \). We always have \( S \subset \langle S \rangle \); if \( \langle S \rangle = S \), then \( S \) is called a flat. Given any flat \( F \), let \( V_F \) be the quotient of \( V \) by the elements that vanish on \( a_i \) for all \( i \in F \). Then we can regard \( \{a_i \mid i \in F\} \) as a set of linear functionals on \( V_F \) that span \( V_F^* \). When \( F = [n] \), we have \( V_F = V, (SR_{bc})_F = SR_{bc}, \) and \( OT_F = OT \).

We have canonical maps

\[
\mu_F : SR_{bc} \rightarrow (SR_{bc})_F \quad \text{and} \quad \nu_F : OT \rightarrow OT_F
\]

given by setting the variables not in \( F \) to zero, as well as sections

\[
\alpha_F : (SR_{bc})_F \rightarrow SR_{bc} \quad \text{and} \quad \beta_F : OT_F \rightarrow OT
\]

taking \( u_i \) to \( u_i \) for all \( i \in F \). The following result is proved in [BP09, 3.12].

**Theorem A.5.** We may choose a \( \text{Sym}(V) \)-module isomorphism \( \varphi : SR_{bc} \rightarrow OT \) and a \( \text{Sym}(V_F) \)-module isomorphism \( \varphi_F : (SR_{bc})_F \rightarrow OT_F \) for every flat \( F \) such that the diagram

\[
\begin{array}{ccc}
SR_{bc} & \xrightarrow{\varphi} & OT \\
\mu_F \downarrow & & \downarrow \nu_F \\
(SR_{bc})_F & \xrightarrow{\varphi_F} & OT_F
\end{array}
\]

commutes. Furthermore, these choices are unique if we require that \( \varphi(1) = 1 \).

**Lemma A.6.** Let \( S \subset [n] \) be independent. There exist constants \( c_S' \in k \) for each independent set \( S' \subset [n] \) with \( \langle S' \rangle = \langle S \rangle \) such that

\[
\varphi(u_S) = \sum_{S'} c_{S'} u_{S'} \in OT.
\]

**Proof.** Start by choosing any constants \( c_\sigma \) such that \( \varphi(u_S) = \sum_\sigma c_\sigma u^\sigma \in OT \), where the sum runs over \( \sigma \in \mathbb{N}^n \). We will show that we can kill all those \( c_\sigma \) with \( S \notin \langle \text{Supp}(\sigma) \rangle \) without changing the class that it represents in \( OT \). Since \( S \) is independent, this will imply that \( u^\sigma = u_{S'} \) for some independent set \( S' \) with \( \langle S' \rangle = \langle S \rangle \).

Suppose that \( F \) is a flat that does not contain \( S \). Then \( \mu_F(u_S) = 0 \), so

\[
\nu_F \circ \varphi(u_S) = \varphi_F \circ \mu_F(u_S) = \varphi_F(0) = 0.
\]
This means that
\[ \sum_{\langle \sigma \rangle \subset F} c_\sigma u^\sigma = 0 \in OT_F. \]

Applying \( \beta_F \), this implies that
\[ \sum_{\langle \sigma \rangle \subset F} c_\sigma u^\sigma = 0 \in OT. \]

Hence we may assume that \( c_\sigma = 0 \) for all \( \sigma \) such that \( \langle \text{Supp}(\sigma) \rangle \subset F \). Since we chose \( F \) to be an arbitrary flat not containing \( S \), this means that we can assume \( c_\sigma = 0 \) for all \( \sigma \) such that \( S \not\subset \langle \text{Supp}(\sigma) \rangle \).

Proof of Theorem A.3: By Lemma A.4 it is sufficient to show that, for all independent \( S \subset [n] \), \( \phi(u_S) \) may be expressed as a linear combination of elements of the form \( u_{S'} \) where \( S' \subset [n] \) is independent. This is exactly the content of Lemma A.6.

A.2 A flat deformation (in the unimodular case)

As in Section 3.1 fix a finite-rank lattice \( N \), along with a list of (not necessarily distinct) nonzero primitive vectors \( a_1, \ldots, a_n \in N \) that span \( N \). Let \( V = N \otimes \mathbb{Z} k \), and consider the associated Orlik-Terao algebra \( OT \). Again as in Section 3.1 we assume that our collection of vectors is unimodular. This implies that, for any circuit \( C \), there is a decomposition \( C = C^+ \cup C^- \) (unique up to swapping \( C^+ \) and \( C^- \)) such that \( \sum_{i \in C^+} a_i - \sum_{j \in C^-} a_j = 0 \). In other words, we may always take the constants \( \eta_i \) from the previous section to be \( \pm 1 \). We define a signed circuit to be a circuit equipped with a choice of decomposition.

Remark A.7. In Section 3.1 we chose a simple affine hyperplane arrangement, and used this to choose a distinguished signed circuit for each circuit. Here we have no such affine arrangement, and there is no distinguished choice.

For each signed circuit \( C \), let
\[ f_C := h^{-1} \left( \prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} (u_j - h) - \prod_{i \in C^+} (u_i - h) \cdot \prod_{j \in C^-} u_j \right), \]
and consider the \( k[h] \)-algebra
\[ OT_h := k[u_1, \ldots, u_n, h]/I_h, \]
where
\[ I_h = \langle f_C \mid C \text{ a signed circuit} \rangle. \]

For any \( t \in k \), let \( I_t \subset k[u_1, \ldots, u_n] \) be the ideal obtained from \( I_h \) by setting \( h \) equal to \( t \), and let
\[ OT_t := k[u_1, \ldots, u_n]/I_t. \]

In particular, we have \( I_0 = I \), and therefore \( OT_0 \) is equal to the Orlik-Terao algebra \( OT \).
Remark A.8. If $C$ and $C'$ are opposite signed circuits, then $f_C = -f_{C'}$, thus it is enough to pick one signed circuit for each circuit in the definition of $I_h$. In particular, this means that $I_h$ coincides with the ideal $J_1'$ defined in Corollary 3.6 when $k = \mathbb{C}$, and therefore $OT_h$ coincides with $R_T(\hat{X})$. Furthermore, the Sym($V$)[$h$]-algebra structure on $OT_h$ coincides with the $AC[h]$-algebra structure on $R_T(\hat{X})$.

Theorem A.9. The algebra $OT_h$ is a free module over $k[h]$, and is thus a flat deformation of $OT_0$.

Proof. Let

$$I'_h := \{ f \mid h^k f \in I_h \text{ for some } k \in \mathbb{N} \} \quad \text{and} \quad OT'_h := k[u_1, \ldots, u_n, h]/I'_h;$$

then $OT'_h$ is a flat deformation of $OT'_0$. It is clear that $OT_1 = OT'_1$ and that we have a surjection $OT_0 \rightarrow OT'_0$, and therefore a closed inclusion Spec $OT'_0 \subset$ Spec $OT_0$. Theorem A.9 is equivalent to the statement that this inclusion is an isomorphism. Since Spec $OT_0$ is reduced and irreducible of dimension $\text{rk } N$, it is sufficient to show that dim Spec $OT'_0 = \text{rk } N$. Since $OT'_h$ is flat, this is equivalent to showing that dim Spec $OT_1 = \text{rk } N$. Since we already know one inequality, we need only show that dim Spec $OT_1 \geq \text{rk } N$.

Consider the ideal $\bar{I}_1 \subset k[u_1, v_1, \ldots, u_n, v_n]$ generated by elements of the form

$$\prod_{i \in C^+} u_i \cdot \prod_{j \in C^-} v_j - \prod_{i \in C^+} v_i \cdot \prod_{j \in C^-} u_j$$

for each signed circuit $C$, and let $\overline{OT}_1 := k[u_1, v_1, \ldots, u_n, v_n]/\bar{I}_1$. Since Spec $OT_1$ is cut out of Spec $\overline{OT}_1$ by the $n$ equations $v_i = u_i - 1$ and intersects the regular locus of Spec $\overline{OT}_1$ nontrivially, it is sufficient to show that dim Spec $\overline{OT}_1 \geq \text{rk } N + n$.

Consider the lattice $L := N \oplus \mathbb{Z}^n$ and the elements $r_i = (a_i, e_i) \in L$ and $s_i = (-a_i, e_i) \in L$. Define a map from $\overline{OT}_1$ to $k\{q^\ell \mid \ell \in L\}$ by sending $u_i$ to $q^{r_i}$ and $v_i$ to $q^{s_i}$. Since $\{r_1, s_1, \ldots, r_n, s_n\}$ spans a finite index sublattice of $L$, the induced map from the torus $T_L := \text{Spec } k\{q^\ell \mid \ell \in L\}$ to Spec $\overline{OT}_1$ is finite-to-one. Since dim $T_L = \text{rk } N + n$, this completes the proof. \qed

Remark A.10. The quotient

$$AOT_0 := k[u_1, \ldots, u_n] / I + (u_1^2, \ldots, u_n^2)$$

is called the Artinian Orlik-Terao algebra. Moseley [Mos 4.5] studied the ring

$$AOT_h := k[u_1, \ldots, u_n, h] / I_h + (u_1(u_1 - h), \ldots, u_n(u_n - h)),$$

and showed that $AOT_h$ is a flat deformation of $AOT_0$ into the Varchenko-Gelfand algebra. This is the Artinian analogue of Theorem A.9 but neither result follows from the other.
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