COMPUTING TROPICAL RESULTANTS

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Abstract. We fix the supports $A = (A_1, \ldots, A_k)$ of a list of tropical polynomials and define the tropical resultant $\mathcal{T}R(A)$ to be the set of choices of coefficients such that the tropical polynomials have a common solution. We prove that $\mathcal{T}R(A)$ is the tropicalization of the algebraic variety of solvable systems and that its dimension can be computed in polynomial time. The tropical resultant inherits a fan structure from the secondary fan of the Cayley configuration of $A$ and we present algorithms for the traversal of $\mathcal{T}R(A)$ in this structure. We also present a new algorithm for recovering a Newton polytope from the support of its tropical hypersurface. We use this to compute the Newton polytope of the sparse resultant polynomial in the case when $\mathcal{T}R(A)$ is of codimension 1. Finally we consider the more general setting of specialized tropical resultants and report on experiments with our implementations.

1. Introduction

We study generalizations of the problem of computing the Newton polytope of the sparse resultant combinatorially, without first computing the resultant polynomial. The input is a tuple $A = (A_1, A_2, \ldots, A_k)$ of integer point configurations in $\mathbb{Z}^n$. The sparse resultant $R(A)$ of $A$, or the variety of solvable systems, is the closure in $(\mathbb{C}^*)^{A_1} \times (\mathbb{C}^*)^{A_2} \times \cdots \times (\mathbb{C}^*)^{A_k}$ of the collection of tuples of polynomials $(f_1, f_2, \ldots, f_k)$ such that $f_1 = f_2 = \cdots = f_k = 0$ has a solution in $(\mathbb{C}^*)^n$ and each $f_i$ has support $A_i$. This variety is irreducible and defined over $\mathbb{Q}$ [Stu94]. If $R(A)$ is a hypersurface then it is defined by a polynomial, unique up to scalar multiple, called the (sparse) resultant polynomial of $A$. Its Newton polytope is called the resultant polytope of $A$.

In Ref. [Stu94], Sturmfels gave a combinatorial description of the resultant polytope, giving rise to a combinatorial algorithm for computing its vertices from the vertices of the secondary polytope of the Cayley configuration $\text{Cay}(A)$. A drawback of this construction is that the secondary polytope typically has far more vertices than the resultant polytope. There have been attempts to compute the resultant polytopes without enumerating all vertices of the secondary polytope [EFK10]. A main contribution of our paper is an algorithm (Section 2.5) for traversing the tropicalization of $R(A)$ as a subfan of the secondary fan of $\text{Cay}(A)$. This approach also allows us to compute tropicalizations of resultant varieties of arbitrary codimension.

The tropical resultant $\mathcal{T}R(A)$ consists of tuples of tropical polynomials having a common solution. We show in Theorem 2.4 that $\mathcal{T}R(A)$ coincides with the tropicalization of $R(A)$. The tropical resultant is combinatorial in nature, and we present in Theorem 2.9 a simple description of it as a union of polyhedral cones.
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each of which is the sum of a positive orthant and a linear space. The tropicalization of a variety is a polyhedral fan of the same dimension as the original variety. We derive a new formula for the codimension of the (tropical) resultant in Theorem 2.23 and show that it can be computed in polynomial time using the cardinality matroid intersection algorithm.

Specialized resultants are obtained by fixing some coefficients of \( f_i \)'s and considering the collection of other coefficients giving a polynomial system solvable in the algebraic torus. In other words, the specialized resultants are intersections of sparse resultants and subspaces parallel to coordinate subspaces. When the specialized coefficient values are generic, the tropicalization \( \mathcal{T}R_S(A) \) of the specialized resultant is the stable intersection of the tropical resultant \( \mathcal{T}R(A) \) with a coordinate subspace. This is a subfan of the restriction of the secondary fan of Cay(\( A \)) to the subspace and can be computed by a fan traversal. The algorithms are significantly more complex and are described in Section 3. Moreover, using the results from our concurrent work on tropical stable intersections [JY11], we describe the specialized tropical resultant as a union of cones, each of which is the intersection of a coordinate subspace and the sum of a positive orthant and a linear space.

Computation of resultants and specialized resultants, of which the implicitization problem is a special case, is a classical problem in commutative algebra that remains an active area. In the concurrent work Ref. [EFKP11] an algorithm for computing Newton polytopes of specialized resultant polynomials using Sturmfels’ formula and the beneath beyond method is presented and implemented, and the work is therefore highly relevant for our project. While the main focus of Ref. [EFKP11] is the efficiency of the computation of the Newton polytopes of specialized resultant polynomials, our main interest has been the geometric structure of secondary fans which allows traversal of tropical resultants of arbitrary codimension.

The tropical description of a polytope \( P \) is a collection of cones whose union is the support of the codimension one skeleton of the normal fan of \( P \), with multiplicities carrying lengths of the edges of \( P \). That is, the union is the tropical hypersurface defined by \( P \). For example, the tropical hypersurface of a zonotope is the union of the dual hyperplanes (zones), and the tropical hypersurface of the secondary polytope of a point configuration contains codimension one cones spanned by vectors in the Gale dual. See Section 2.3. The tropical description uniquely identifies the polytope up to translation, and we consider it to be an equally important representation of a polytope as the V- and H-descriptions. Furthermore, the conversion algorithms between these representations deserve the same attention as other fundamental problems in convex geometry. A contribution of this paper is an algorithm (Algorithm 4.1) for reconstructing polytopes from their tropical descriptions. We apply the algorithm to the tropical description of resultant polytopes in Theorem 2.9 to recover the combinatorics of the resultant polynomial.

All the algorithms described in this paper have been implemented in the software Gfan [Jen]. Computational experiments and examples are presented in Section 5.

2. Resultants

Let \( A = (A_1, A_2, \ldots, A_k) \) where each \( A_i = \{a_{i,1}, a_{i,2}, \ldots, a_{i,m_i}\} \) is a multi-subset of \( \mathbb{Z}^n \), and let \( m = m_1 + m_2 + \cdots + m_k \). Throughout this paper, we will assume that \( m_i \geq 2 \) for all \( i \). Let \( Q_1, Q_2, \ldots, Q_k \) be the convex hulls of \( A_1, A_2, \ldots, A_k \) respectively. Let \( (\mathbb{C}^*)^{A_i} \) denote the set of polynomials of the form \( \sum_{j=1}^{m_i} c_j x^{a_{i,j}} \) in
\[ C[x_1, x_2, \ldots, x_n], \text{ where each } c_j \text{ is in } C^* := C \setminus \{0\}. \text{ Let } Z \subseteq \prod_{i=1}^{k} (C^*)^{A_i} \text{ be the set consisting of tuples } (f_1, f_2, \ldots, f_k) \text{ such that the system of equations } f_1 = f_2 = \cdots = f_k \text{ has a solution in } (C^*)^n. \]

**Definition 2.1.** The resultant variety, or the variety of solvable systems, is the closure \( \overline{Z} \) of \( Z \) in \( \prod_{i=1}^{k} (C^*)^{A_i} \) and is denoted \( \mathcal{R}(\mathcal{A}) \).

The resultant variety is usually defined as a subvariety of \( \prod_{i=1}^{k} C^{A_i} \) or its projectivization \( [GKZ94, Stu94] \), but we chose to work in \( \prod_{i=1}^{k} (C^*)^{A_i} \) as tropicalizations are most naturally defined for subvarieties of tori.

### 2.1 A simple description of the tropical resultant and its multiplicities.

The tropical semiring \( T = (\mathbb{R}, \oplus, \odot) \) is the set of real numbers with minimum as tropical addition \( \oplus \) and usual addition as tropical multiplication \( \odot \). A tropical (Laurent) polynomial \( F \) in \( n \) variables \( x = (x_1, x_2, \ldots, x_n) \) is a multiset of terms \((c, a)\) or \( c \odot x^a \) where \( c \in \mathbb{R} \) is the coefficient and \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \) is the exponent. We will also write \( F = \bigoplus_{(c, a) \in F} (c \odot x^a) \). The support of \( F \) is the multiset of \( a \)'s, and the Newton polytope of \( F \) is the convex hull of its support.

The tropical solution set \( \mathcal{T}(F) \) of a tropical polynomial \( F \) is the locus of points \( x \in \mathbb{R}^n \) such that the minimum is attained at least twice in the expression

\[
\bigoplus_{(c, a) \in F} (c \odot x^a) = \min_{(c, a) \in F} (c + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n).
\]

In other words, a point \( x \in \mathbb{R}^n \) is in \( \mathcal{T}(F) \) if and only if the minimum for \( (1, x) \cdot (c, a) \) is attained for two terms in \( F \), which may be repeated elements. Therefore, \( \mathcal{T}(F) \) is a (not necessarily pure dimensional) subcomplex of a polyhedral complex dual to the (marked) regular subdivision of the support of \( F \) induced by the coefficients \( c \), consisting of duals of cells with at least two marked points. See Section 2.2 for definitions of subdivisions and marked points.

When \( F \) contains no repeated elements, the tropical solution set coincides with the non-smooth locus of the piecewise-linear function from \( \mathbb{R}^n \) to \( \mathbb{R} \) given by \( x \mapsto F(x) = \bigoplus_{(c, a) \in F} (c \odot x^a) \), which is also called a tropical hypersurface. In particular, if all coefficients of \( F \) are the same and if \( F \) contains no repeated elements, then the tropical hypersurface is the codimension one skeleton of the inner normal fan of the Newton polytope of \( F \).

Let \( \mathcal{A} = (A_1, A_2, \ldots, A_k) \) be as before, and let \( \mathbb{R}^{A_i} \) denote the set of tropical polynomials of the form \( \bigoplus_{j=1}^{m} (c_{ij} \odot x^{a_{ij}}) \).

**Definition 2.2.** The tropical resultant \( \mathcal{TR}(\mathcal{A}) \) of \( \mathcal{A} \) is the subset of \( \mathbb{R}^m \), or \( \mathbb{R}^{A_1} \times \mathbb{R}^{A_2} \times \cdots \times \mathbb{R}^{A_k} \), consisting of tuples \((F_1, F_2, \ldots, F_k)\) such that the tropical solution sets of \( F_1, F_2, \ldots, F_k \) have a nonempty common intersection in \( \mathbb{R}^n \).

We can also consider the tropical resultant as a subset of \( \prod_{i=1}^{k} \mathbb{TP}^{m_i-1} \) or \( \prod_{i=1}^{k} \mathbb{R}^{A_i}/(1, 1, \ldots, 1) \mathbb{R} \), but we prefer to work with \( \mathbb{R}^m \) in this paper.

**Definition 2.3.** Let \( k \) be a field and \( I \subseteq k[x_1, \ldots, x_n] \) an ideal. The tropical variety \( \mathcal{T}(I) \) of \( I \), or the tropicalization of \( V(I) \), is a polyhedral fan with support

\( \mathcal{T}(I) := \{ \omega \in \mathbb{R}^n : \text{ the initial ideal } \text{in}_\omega(I) \text{ contains no monomials} \} \).

For \( \omega \) in the relative interior of a cone \( C_\omega \in \mathcal{T}(I) \) we define the multiplicity as

\[
\text{mult}_\omega(\mathcal{T}(I)) := \dim_k(k[\mathbb{Z}^n \cap C_\omega]/(\text{in}_\omega(I))).
\]
when the right hand side is finite, in particular when $\omega$ is in the relative interior of a Gröbner cone of the same dimension as $\mathcal{T}(I)$.

In this definition we refer to the “constant coefficient” initial ideal as in Ref. [BJS+07], except that we are picking out the terms with smallest $\omega$-degree. If the ideal $I$ is homogeneous, $\mathcal{T}(I)$ gets a fan structure from the Gröbner fan of $I$. When $C_\omega$ is the smallest Gröbner cone in $\mathcal{T}(I)$ containing $\omega$, the initial ideal $\text{in}_\omega(I)$ is homogeneous with respect to any weight in the linear span of $C_\omega$. Hence after multiplying each homogeneous element of $\text{in}_\omega(I)$ by a Laurent monomial they generate an ideal $\langle \text{in}_\omega(I) \rangle$ in the Laurent polynomial ring $k[\mathbb{Z}^n \cap C_\omega^\perp]$.

**Theorem 2.4.** The support of the tropicalization of the resultant variety $\mathcal{R}(A)$ coincides with the tropical resultant $\mathcal{TR}(A)$.

A consequence is that we may identify $\mathcal{TR}(A)$ with the tropicalization of $\mathcal{R}(A)$ and we define its multiplicities accordingly.

We will use incidence varieties to give a proof of Theorem 2.4. Let the incidence variety be

$$W := \{(f_1, f_2, \ldots, f_k, x) : \forall i : f_i(x) = 0\} \subseteq \prod_{i=1}^k (\mathbb{C}^*)^{A_i} \times (\mathbb{C}^*)^n,$$

and let the tropical incidence variety be the set

$$\mathcal{T}W := \{(F_1, F_2, \ldots, F_k, X) : \forall i \in \mathcal{T}(F_i)\} \subseteq \prod_{i=1}^k \mathbb{R}^{A_i} \times \mathbb{R}^n.$$

The tropical incidence variety is the tropical prevariety [BJS+07] defined by the tropicalization of the polynomials $f_1, f_2, \ldots, f_k$, where $f_i$ is considered as a polynomial in $m_i + n$ variables whose support in the $n$ variables is $A_i$ and whose $m_i$ terms have indeterminate coefficients. Even if $A_i$ contains repeated points, the support of $f_i$ in $m_i + n$ variables has no repeated points.

**Lemma 2.5.** The polynomials $f_1, f_2, \ldots, f_k$ form a tropical basis for the incidence variety $W$, i.e. the tropical incidence variety coincides with the tropicalization of the incidence variety.

**Proof.** Let $K$ be the field of Puiseux series in $t$ with complex coefficients. Let $(F_1, F_2, \ldots, F_k, X)$ be an element in the tropical prevariety, i.e. $F_1, F_2, \ldots, F_k$ are tropical polynomials with support sets $A_1, A_2, \ldots, A_k$, and $X \in \mathbb{R}^n$ is a tropical solution for each $F_i$. Let $x_0 = (t^{X_1}, t^{X_2}, \ldots, t^{X_k}) \in (K^*)^n$. Then $F_i \in \mathbb{R}^{m_i}$ is contained in the tropical hypersurface of $f_i(x_0)$ considered as a polynomial in the indeterminate coefficients, so by Kapranov’s Theorem there is a point $c \in (K^*)^{m_i}$ with $\text{val}(c) = F_i$ giving $f_i(x_0) = 0$. Therefore $(F_1, F_2, \ldots, F_k, X)$ can be lifted to the incidence variety and lies in the tropical incidence variety. $\square$

A consequence of Lemma 2.5 is that we may identify the tropical incidence variety with the tropicalization of $W$ and we define its multiplicities accordingly.

The following lemma follows immediately from the definitions.

**Lemma 2.6.** The tropical resultant is the projection of the tropical incidence variety onto the first factor.

We can now prove Theorem 2.4.
Proof of Theorem 2.4. The resultant variety \( \mathcal{R}(A) \) is obtained from the incidence variety \( W \) by projecting onto the first factor \( \prod_{i=1}^{k} (C^*)^{A_i} \) and taking the closure. This proves the first of the following equalities.

\[
\mathcal{T}(\mathcal{R}(A)) = \mathcal{T}(\pi(W)) = \pi(\mathcal{T}(W)) = \pi(W) = \mathcal{T}\mathcal{R}(A)
\]

The second follows from Ref. [ST08] which says that the tropicalization of the closure of a projection of \( W \) is the projection of the tropicalization of \( W \). The third is Lemma 2.5 and the last Lemma 2.6.

For each \( i = 1, 2, \ldots, k \) let \( \tilde{P}_i \) be the Newton polytope of \( f_i \) in \( \mathbb{R}^{m_i} \times \mathbb{R}^n \), which is in turn embedded in \( \mathbb{R}^n \times \mathbb{R}^n \). Each \( \tilde{P}_i \) is a simplex; in particular, the exponent of every term in \( f_i \) is a vertex, and there is an edge between every pair of vertices. Every cone in the tropical incidence variety is a transverse intersection of normal cones to edges of \( \tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_k \).

The tropical incidence variety is

\[
\bigcup_{(E_1, E_2, \ldots, E_k)} \left( \bigcap_{i=1}^{k} \mathcal{N}(\tilde{E}_i) \right)
\]

where the union runs over all choices of pairs \( E_i \) of points from \( A_i \) and \( \mathcal{N}(\tilde{E}_i) \) denotes the inner normal cone of the corresponding edge \( \tilde{E}_i \) in \( \tilde{P}_i \).

Lemma 2.7. Every maximal cone in the tropical incidence variety \( \mathcal{T}W = \mathcal{T}(W) \) has multiplicity one.

Proof. Since every vertex of every \( \tilde{P}_i \) has its own coordinate, the dimension of a face of the Minkowski sum \( \tilde{P}_1 + \tilde{P}_2 + \cdots + \tilde{P}_k \) minimizing an \( \omega \) is the sum of the dimensions of the faces of each \( \tilde{P}_i \) with respect to \( \omega \). The dimension of the incidence variety is \( m + n - k \) and therefore, for a generic \( \omega \in \mathcal{T}(W) \), the face of \( \tilde{P}_1 + \tilde{P}_2 + \cdots + \tilde{P}_k \) minimizing \( \omega \) has dimension \( k \) and must be a zonotope. Consequently the forms \( \mathfrak{m}_\omega(f_1), \mathfrak{n}_\omega(f_2), \ldots, \mathfrak{n}_\omega(f_k) \) are binomials, each with an associated edge vector \( v_i \in \mathbb{Z}^{m+n} \). The vectors \( v_1, v_2, \ldots, v_k \) generate \( C^n_\omega \) and after multiplying each \( \mathfrak{m}_\omega(f_i) \) by a monomial it ends up in \( C[Z^{m+n} \cap C^n_\omega] \). Hence using the binomials to rewrite modulo \( \langle \mathfrak{m}_\omega(I) \rangle \) we get that \( \dim C[Z^{m+n} \cap C^n_\omega]/\langle \mathfrak{m}_\omega(I) \rangle \) is bounded by the index of the sublattice generated by \( v_1, v_2, \ldots, v_n \) in \( \mathbb{Z}^{m+n} \cap C^n_\omega \). If we write the edge vectors as columns of a matrix, then the matrix contains a full-rank identity submatrix, so the sublattice has index one.

Let \( \pi \) be the projection from \( \mathbb{R}^{m} \times \mathbb{R}^{n} \), where the incidence variety lies, to the first factor \( \mathbb{R}^{m} \). The tropical resultant is the projection of the tropical incidence variety, so

\[
\mathcal{T}\mathcal{R}(A) = \bigcup_{(E_1, E_2, \ldots, E_k)} \pi \left( \bigcap_{i=1}^{k} \mathcal{N}(\tilde{E}_i) \right).
\]

The Cayley configuration \( \text{Cay}(A) \) of a tuple \( A = (A_1, A_2, \ldots, A_k) \) of point configurations in \( \mathbb{Z}^{n} \) is defined to be \( A_1 \times \{e_1\} \cup \cdots \cup A_k \times \{e_k\} \) in \( \mathbb{Z}^{n} \times \mathbb{Z}^{k} \). We will also use \( \text{Cay}(A) \) to denote a matrix whose columns are points in the Cayley configuration.
Lemma 2.8. Let $E = (E_1, E_2, \ldots, E_k)$ be a tuple of pairs from $A_1$, $A_2$, \ldots, $A_k$ respectively. Then the following cones coincide:

$$\pi \left( \bigcap_{i=1}^{k} \mathcal{N}(E_i) \right) = \mathbb{R}_{\geq 0} \{ e_{ij} : a_{ij} \notin E_i \} + \text{rowspan}(\text{Cay}(A)).$$

Proof. Let $E$ be fixed. The left hand side consists of tuples of tropical polynomials $(F_1, F_2, \ldots, F_k) \in \prod_{i=1}^{k} \mathbb{R}^{A_i}$ for which there is a point $w \in \mathbb{R}^n$ attaining the minimum for $F_i$ at $E_i$ for every $i$.

On the other hand, the cone $\mathbb{R}_{\geq 0} \{ e_{ij} : a_{ij} \notin E_i \} + \mathbb{R} \sum_j e_{ij}$ consists of all $F = (F_1, F_2, \ldots, F_k)$ such that the minimum for $F_i$ evaluated at the point $0 \in \mathbb{R}^n$ is attained at $E_i$ for every $i$. For $w \in \mathbb{R}^n$ and $F \in \mathbb{R}^A$,

$$F(x - w) = \min_{(c, a) \in F} c + a \cdot (x - w) = (F - w \cdot A)(x),$$

where $A$ denotes the matrix whose columns are points in $A$, so

$$T(F) + w = T(F - wA).$$

Moreover, the tropical solution set remains the same if coefficients of $F$ are changed by a tropical scalar multiple. Therefore, changing the coefficients of $F_1$, $F_2$, \ldots, $F_k$ simultaneously by an element in the row space of Cay($A$) has the effect of translating all the tropical solution sets together, so the set on the right hand side $\mathbb{R}_{\geq 0} \{ e_{ij} : a_{ij} \notin E_i \} + \text{rowspan}(\text{Cay}(A))$ consists of all tuples $(F_1, F_2, \ldots, F_k)$ having a point $w \in \mathbb{R}^n$ achieving the minimum for $F_i$ at $E_i$ for every $i$. \qed

The following result gives a simple description of the tropical resultant as a union of cones with multiplicities.

Theorem 2.9. The tropical resultant of $A$ is the set

$$\mathcal{TR}(A) = \bigcup_{E} \mathbb{R}_{\geq 0} \{ e_{ij} : a_{ij} \notin E_i \} + \text{rowspan}(\text{Cay}(A))$$

where $E = (E_1, E_2, \ldots, E_k)$ and each $E_i$ consists of two elements in $A_i$. The multiplicity of the cone associated to $E$ is the index of the lattice spanned by the rows of Cay($E$) in rowspace(Cay($E$)) $\cap \mathbb{Z}^m$.

The cones in (3) do not have to intersect each other transversely. See Example 2.18(b).

Proof. The set theoretic statement follows immediately from Lemmas 2.6 and 2.8.

Let $\sigma$ be the cone corresponding to $E$ in the incidence variety, and $\tau = \pi(\sigma)$. Using the multiplicity formula from tropical elimination theory [ST08], the multiplicity of $\tau$ in the tropical resultant is the lattice index $[\mathbb{L}_r : \pi(\mathbb{L}_\sigma)]$, where $\mathbb{L}_r = \mathbb{R} \tau \cap \mathbb{Z}^m$ and $\mathbb{L}_\sigma = \mathbb{R} \sigma \cap \mathbb{Z}^{m+n}$. The lattice $\mathbb{L}_\sigma$ is defined by the following equations on $(c, x) \in \mathbb{Z}^{m+n}$

$$c \cdot (e_{ij} - e_{ik}) + x \cdot (a_{ij} - a_{ik}) = 0 \text{ for } \{ a_{ij}, a_{ik} \} = E_i$$

and is spanned by the integer points in the lineality space of the tropical incidence variety and the standard basis vectors $e_{ij}$ for $a_{ij} \notin E_i$. The rows of the following matrix span the lattice points in the lineality space of the incidence variety:

$$\begin{bmatrix} \text{Cay}(A) & -I_n \\ 0 & 0 \end{bmatrix}.$$
Hence \( \pi(L_\sigma) \) is spanned by the rows of \( \text{Cay}(A) \) and the \( e_{ij} \)'s for \( a_{ij} \notin E_i \). \( \square \)

In Ref. [DFS07], the tropical discriminant is described as a sum of a tropical linear space and an ordinary linear space. This description carries over to the tropical resultant when \( A \) is essential, and in particular \( R(A) \) is a hypersurface. Our description in Theorem 2.9 is different and also works for non-essential cases and non-hypersurface cases. Moreover, it is simpler, and we do not need to compute a nontrivial tropical linear space.

The first summand in (\ref{first_summand}) plus the linear span of the last \( k \) rows of \( \text{Cay}(A) \) is a tropical linear space obtained as a Cartesian product of tropical hyperplanes. Hence Theorem 2.9 can be rephrased as follows. Let \( C \) be the matrix consisting of the last \( k \) rows of \( \text{Cay}(A) \), so the kernel of \( C \) is defined by equations of the form 
\[
c_i,1 + c_i,2 + \cdots + c_i,m_i = 0 \text{ for } i = 1, 2, \ldots, k.
\]
Then the tropical resultant is the set
\[
\text{TR}(A) = T(\ker(C)) + \text{rowspace}[A_1|A_2|\cdots|A_k].
\]

**Example 2.10.** Consider the tuple \( A = (A_1, A_2, A_3) \) of the following point configurations in \( \mathbb{Z}^2 \):

\[
\begin{aligned}
A_1 &= \{(0,0), (0,1), (1,0)\}, \\
A_2 &= \{(0,0), (1,0), (2,1)\}, \\
A_3 &= \{(0,0), (0,1), (1,2)\}.
\end{aligned}
\]

The Cayley configuration \( \text{Cay}(A_1, A_2, A_3) \) consist of columns of the following matrix, which we also denote \( \text{Cay}(A) \):

\[
\text{Cay}(A) = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 2 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

The corresponding system of polynomials consist of

\[
\begin{aligned}
f_1 &= c_{11} + c_{12}y + c_{13}x, \\
f_2 &= c_{21} + c_{22}x + c_{23}x^2y, \\
f_3 &= c_{31} + c_{32}y + c_{33}xy^2.
\end{aligned}
\]

The point
\[
(0,0,0,0,1,5,0,1,5)
\]
is in the tropical resultant variety because the tropical hypersurfaces of the three tropical polynomials

\[
\begin{aligned}
F_1 &= 0 \oplus X \oplus Y, \\
F_2 &= 0 \oplus (1 \circ X) \oplus (5 \circ X \circ Y^2) \\
F_3 &= 0 \oplus (1 \circ Y) \oplus (5 \circ X \circ Y^2)
\end{aligned}
\]
contain the common intersection points \((-1,-1)\) and \((-2,-2)\). See Figure 1.
Consider the incidence variety defined by the ideal
\[ I = \langle f_1, f_2, f_3 \rangle \subseteq \mathbb{C}[c, x^\pm 1, y^\pm 1]. \]

The resultant variety is obtained by eliminating \( x \) and \( y \) from the system, i.e. it is defined by the ideal \( I \cap \mathbb{C}[c] \). In this case, the resultant variety is a hypersurface defined by the resultant polynomial
\[
\begin{align*}
&c_1^2c_2^2c_3^2 - 2c_1c_2c_3c_1c_2c_3 + c_1^2c_2c_3c_1c_2c_3 - c_1^2c_2c_3c_1c_2c_3 + c_1^2c_2c_3c_1c_2c_3 - \\
&c_1^2c_2c_3c_1c_2c_3 - c_1^2c_2c_3c_1c_2c_3 + c_1^2c_2c_3c_1c_2c_3 - c_1^2c_2c_3c_1c_2c_3 + c_1^2c_2c_3c_1c_2c_3 - \\
&c_1^2c_2c_3c_1c_2c_3 - c_1^2c_2c_3c_1c_2c_3 + c_1^2c_2c_3c_1c_2c_3 - c_1^2c_2c_3c_1c_2c_3 + c_1^2c_2c_3c_1c_2c_3 - \\
&c_1^2c_2c_3c_1c_2c_3.
\end{align*}
\]

It is homogeneous with respect to the rows of \( \text{Cay}(A) \). Its Newton polytope is four dimensional, has f-vector \((15, 40, 38, 13, 1)\) and lies in an affine space parallel to the kernel of \( \text{Cay}(A) \).

The tropical resultant is an eight dimensional fan in \( \mathbb{R}^9 \) with a five dimensional linearity space rowspace(\( \text{Cay}(A) \)). As a subfan of the secondary fan of \( \text{Cay}(A) \), it consists of 89 (out of 338) eight dimensional secondary cones, which can be coarsened to get the 40 normal cones dual to edges of the resultant polytope.

2.2. **Secondary fan structure and links in tropical resultants.** Let \( A \in \mathbb{Z}^{d \times m} \) be an integer matrix with columns \( a_1, a_2, \ldots, a_m \in \mathbb{Z}^d \). We will also denote by \( A \) the point configuration \( \{a_1, a_2, \ldots, a_m\} \). We allow repeated points in \( A \), as we consider the points to be labeled by the set \( \{1, 2, \ldots, m\} \), and every point gets a distinct label.

Following Section 7.2A of Ref. [GKZ94], a subdivision of \( A \) is defined as a family \( \Delta = \{C_i \subseteq A : i \in I\} \) of subsets of \( A \), called **facets**, such that

1. \( \dim(\text{conv}(C_i)) = \dim(\text{conv}(A)) \) for each \( i \in I \),
2. \( \text{conv}(A) = \bigcup_{i \in I} \text{conv}(C_i) \), and
3. for every \( i, j \in I \), the intersection of \( \text{conv}(C_i) \) and \( \text{conv}(C_j) \) is a face of both, and \( \text{Cay}(A) \cap \text{conv}(C_i) = C_j \cap \text{conv}(C_i) \).
This notion is also called a marked subdivision by some authors, as a subdivision depends not only on the polyhedra \( \text{conv}(C_i) \) but also on the labeled sets \( C_i \). The elements in \( \bigcup_{i \in J} C_i \) are called marked. If \( F \) is a face of \( \text{conv}(C_i) \) for some \( C_i \in \Delta \), then \( C_i \cap F \) is called a face of the subdivision.

For two subdivisions \( \Delta \) and \( \Delta' \) of \( A \), we say that \( \Delta \) refines \( \Delta' \) or \( \Delta' \) coarsens \( \Delta \) if every \( C_i \in \Delta \) is contained in some \( C_j' \in \Delta' \). A subdivision is a triangulation if no proper refinement exists, and in particular, every facet contains exactly \( \dim(\text{conv}(A)) + 1 \) elements.

Let \( w : A \to \mathbb{R} \) be an arbitrary real valued function on \( A \), called a weight vector. We can define a subdivision of \( A \) induced by \( w \) as follows. Consider the unbounded polyhedron \( P = \text{conv}\{(a, w(a))\} + \mathbb{R}_{\geq 0}\{e_{d+1}\} \) in \( \mathbb{R}^{d+1} \), and let \( \{F_i : i \in I\} \) be its bounded facets. Then the induced subdivision is \( \{C_i : i \in I\} \) where \( C_i = \{a \in A : (a, w(a)) \in F_i\} \). A subdivision \( A \) is regular or coherent if it is induced by some function \( w \). The partition of \( \mathbb{R}^A \) according to induced subdivisions is a fan, called the secondary fan of \( A \).

Following Section 7.1D of Ref. [GKZ94], we can construct the secondary polytope of \( A \) as follows. For a triangulation \( T \) of a point configuration \( A \), define the GKZ-vector \( \phi_T \in \mathbb{R}^A \) as

\[
\phi_T(a) := \sum_{\sigma \in T, u \in \sigma} \text{vol}(\sigma)
\]

where the summation is over all facets \( \sigma \) of \( T \) of which \( a \) is a vertex.

**Definition 2.11.** The secondary polytope \( \Sigma(A) \) is the convex hull in \( \mathbb{R}^A \) of the vectors \( \phi_T \) where \( T \) runs over all triangulations of \( A \).

**Theorem 2.12.** [GKZ94, § 7.1, Theorem 1.7] The vertices of \( \Sigma(A) \) are precisely the vectors \( \phi_T \) for which \( T \) is a regular triangulation of \( A \). The normal fan of the secondary polytope \( \Sigma(A) \) is the secondary fan of \( A \). The normal cone of \( \Sigma(A) \) at \( \phi_T \) is the closure of the set of all weights \( w \in \mathbb{R}^A \) which induce the triangulation \( T \).

The link of a cone \( C \subseteq \mathbb{R}^m \) at a point \( v \in C \) is

\[
\text{link}_v(C) = \{u \in \mathbb{R}^m \mid 3\delta > 0 : \exists \varepsilon \text{ between 0 and } \delta : v + \varepsilon u \in C\}.
\]

The link of a fan \( \mathcal{F} \) at a point \( v \) in the support of \( \mathcal{F} \) is the fan

\[
\text{link}_v(\mathcal{F}) = \{\text{link}_v(C) \mid v \in C \in \mathcal{F}\}.
\]

For any cone \( \sigma \in \mathcal{F} \), any two points in the relative interior of \( \sigma \) give the same link of the fan, denoted \( \text{link}_\sigma(\mathcal{F}) \). If a maximal cone \( C \in \mathcal{F} \) has an assigned multiplicity, we let \( \text{link}_v(C) \in \text{link}_\sigma(\mathcal{F}) \) inherit it.

We will first show that the link of a cone in a secondary fan is a common refinement of secondary fans, or equivalently that a face of a secondary polytope is a Minkowski sum of secondary polytopes. For a sub-configuration \( C \subseteq A \), we can consider the the secondary polytope of \( C \) as embedded in \( \mathbb{R}^A \) by setting \( \phi_T(a) = 0 \) for \( a \in A \setminus C \) for every triangulation \( T \) of \( C \). On the other hand, the secondary fan of \( C \) embeds in \( \mathbb{R}^A \) with lineality space containing the coordinate directions corresponding to \( a \in A \setminus C \).

**Lemma 2.13.** Let \( A \) be a configuration of \( m \) points, \( \omega \in \mathbb{R}^A \), and \( \Delta_\omega \) be the regular subdivision of \( A \) induced by \( \omega \). Then the face \( F_\omega \) of the secondary polytope of \( A \) supported by \( \omega \) is the Minkowski sum of secondary polytopes of facets in \( \Delta_\omega \).
Proof. Let \( \omega' \in \mathbb{R}^4 \) be generic and \( p \) be the vertex of the Minkowski sum picked out by \( \omega' \). Then \( p = \sum_i p_i \), where \( p_i \) is the vertex of the secondary polytope of the \( i \)th facet of \( \Delta_\omega \) picked by (a restriction of) \( \omega' \). For each cell in \( \Delta_\omega \), the vector \( \omega + \epsilon \omega' \) induces the same triangulation for all \( \epsilon > 0 \) sufficiently small. These triangulations together give the triangulation of \( A \) induced by \( \omega + \epsilon \omega' \). For each \( i \)th facet of \( \Delta_\omega \), \( p_i \) is the GKZ-vector of the triangulation induced by \( \omega + \epsilon \omega' \), so their sum \( p \) is the GKZ vector of the triangulation of \( A \) induced by \( \omega + \epsilon \omega' \). Hence \( p \) is the vertex of \( F_\omega \) in direction \( \omega' \). We can then conclude that the two polytopes are the same since they have the same vertex in each generic direction. \( \square \)

We now define mixed subdivisions as in Ref. [DLRS10]. For point configurations \( A_1, A_2, \ldots, A_k \) in \( \mathbb{R}^n \), with \( A_i = \{a_{i,j} : 1 \leq j \leq m_i \} \), the Minkowski sum
\[
\sum_{i=1}^k A_i = \{a_{1,j_1} + a_{2,j_2} + \cdots + a_{k,j_k} : 1 \leq j_i \leq m_i \}
\]
is a configuration of \( m_1 m_2 \cdots m_k \) points labeled by \([m_1] \times [m_2] \times \cdots \times [m_k]\). We use the word cell to denote the set of labels of a face of a subdivision.

**Definition 2.14.** A subset of labels is a mixed cell if it is a product of labels \( J_1 \times J_2 \times \cdots \times J_k \) where \( J_i \) is a nonempty subset of \([m_i]\), and it is fully mixed if in addition \( J_i \) contains at least two elements for every \( i = 1, 2, \ldots, k \). A subdivision of the Minkowski sum \( \sum_{i=1}^k A_i \) is mixed if every facet is labeled by a mixed cell.

A mixed subdivision of \( \sum_{i=1}^k A_i \) is also referred to as a mixed subdivision \( A = (A_1, A_2, \ldots, A_k) \). Our definition of fully mixed cell differs from that of Ref. [DFS07] Section 6 where it is required that \( \text{conv}(a_{i,j} : j \in J_i) \) has affine dimension at least one, while we only require that \( J_i \) contains at least two elements. These two definitions coincide if \( J_i \) contains no repeated points.

A mixed subdivision is called regular if it is induced by a weight vector
\[
w : \sum_{i=1}^k A_i \rightarrow \mathbb{R}, \quad \text{where } w : \sum_{i=1}^k a_{i,j_i} \mapsto \sum_{i=1}^k w_{i,j_i}
\]
for some \( (w_1, w_2, \ldots, w_k) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_k} \). In Ref. [Stu94] regular mixed subdivisions (RMS) were also called coherent mixed decompositions.

**Theorem 2.15.** [Stu94] Theorem 5.1] For a subdivision \( \Delta \) of \( \text{Cay}(A) \), the collection of mixed cells of the form \( \sum_{i=1}^k C_i \) such that \( C_i \subseteq A_i \) and \( \bigcup_{i=1}^k C_i \) is a facet of \( \Delta \) forms a mixed subdivision of \( \sum_{i=1}^k A_i \). This is a one-to-one correspondence between the subdivisions of \( \text{Cay}(A) \) and RMSs of \( \sum_{i=1}^k A_i \). Moreover the partition of weight vectors \( (w_1, w_2, \ldots, w_k) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \cdots \times \mathbb{R}^{m_k} \) according to the induced RMS coincides with the secondary fan of \( \text{Cay}(A) \).

From our description of tropical resultants, we get the following result which was proven for the resultant hypersurfaces in Theorem 5.2 of Ref. [Stu94] and stated for the essential configurations with no repeated points in Proposition 6.8 of Ref. [DFS07]. See Remark 2.23 for a characterization of being essential.

**Theorem 2.16.** The tropical resultant is a subfan of the secondary fan of the Cayley configuration \( \text{Cay}(A_1, A_2, \ldots, A_k) \), consisting of the cones dual to subdivisions with fully mixed cells.
The multiplicities of secondary cones in the tropical resultant will be computed in Proposition 2.20 below.

Proof. For a tropical polynomial \( F \in \mathbb{R}^A \) the tropical solution set \( \mathcal{T}(F) \) is dual to the cells with at least two elements in the subdivision of \( A \) induced by the coefficients of \( F \). More precisely, by the definition of tropical solution sets, \( w \in \mathcal{T}(F) \) if and only if \( (1,w) \) is an inner-normal vector for the convex hull of lifted points \( \{(c,a) \in \mathbb{R}^{n+1} : c \circ x^a \text{ is a term in } F\} \) supporting at least two points of \( A \). The two points supported need not have distinct coordinates.

Let \( (F_1, F_2, \ldots, F_k) \in \mathbb{R}^{A_1} \times \mathbb{R}^{A_2} \times \cdots \times \mathbb{R}^{A_k} \). The union of tropical solution sets \( \bigcup_{i=1}^k \mathcal{T}(F_i) \) has a polyhedral complex structure in the common refinement of the completions of \( \mathcal{T}(F_i) \) to \( \mathbb{R}^m \), which is dual to the RMS of \( A \) induced by the coefficients of \( (F_1, F_2, \ldots, F_k) \). The tuple \( (F_1, F_2, \ldots, F_k) \) is in the tropical resultant if and only if the tropical solution sets have a common intersection. In other words, there is a fully mixed cell in the dual RMS.

The tropical resultant is a subfan of the secondary fan. It is pure and connected in codimension one, so we can compute it by traversing. To traverse the resultant fan, we need to know how to find the link of a cone.

Proposition 2.17. Let \( A = (A_1, A_2, \ldots, A_k) \). The support of the link of a point \( \omega \) in the tropical resultant \( TR(A) \) is a union of tropical resultants corresponding to sub-configurations of fully mixed cells in the RMS \( \Delta_\omega \) of \( A \) induced by \( \omega \).

Proof. By definition, a point \( u \) is in the link if and only if \( \omega + \varepsilon u \) induces a RMS with a fully mixed cell for all sufficient small \( \varepsilon > 0 \). This happens if and only if at least one of the fully mixed cells in \( \Delta_\omega \) is subdivided by \( u \) into a RMS with a fully mixed cell, i.e. \( u \) is in the tropical resultant of the sub-configurations of fully mixed cells.

Example 2.18. Let \( A \) be as in Example 2.10

(a) The link of the point \( (0,0,0,0,1,5,0,1,5) \) in the tropical resultant is a union of two hyperplanes whose normal vectors are:

\[
(0,-1,1,-1,1,0,1,-1,0) \quad \text{and} \quad (0,0,0,0,1,-1,0,-1,1)
\]

respectively. They are the resultant varieties of the sub-configurations of the two fully mixed cells. See Figure 1

(b) The link of the point \( (0,0,0,0,-1,-1,0,0,1) \) consists of four rays modulo linearity space, three rays from the resultant of one fully mixed cell and two from the other, and the two resultants overlap along a ray. See Figure 2

The following lemma follows immediately from the definition of induced or regular subdivisions and shows that the description of the tropical resultant as a union of cones in Theorem 2.9 is somewhat compatible with the secondary fan structure. For any tuple \( E = (E_1, E_2, \ldots, E_k) \) of pairs \( E_i \subset A_i \), let \( C_E := \mathbb{R}_{\geq 0}(e_{ij} : a_{ij} \notin E_i) + \text{rowspace}(\text{Cay}(A)) \) be the cone as in Theorem 2.9

Lemma 2.19. For each tuple \( E \) as above, the cone \( C_E \) is a union of secondary cones of \( \text{Cay}(A) \) corresponding to mixed subdivisions of \( \sum_{i=1}^k A_i \) having a mixed cell containing \( \sum_{i=1}^k E_i \).
Figure 2. The tropical solution sets at \((0, 0, 0, 0, -1, -1, 0, 0, 1)\) and the corresponding dual RMS in Example 2.18(b).

Let \(\sigma\) be a secondary cone of \(\text{Cay}(A)\) which is a maximal cone in the tropical resultant \(\mathcal{T}R(A)\), and let \(\Delta_\sigma\) be the corresponding regular mixed subdivision. Then all the fully mixed cells in \(\Delta_\sigma\) are of the form \(\sum_{i=1}^k E_i\) where each \(E\) is a tuple of pairs as above. Otherwise \(\sigma\) would not be maximal in \(\mathcal{T}R(A)\).

**Proposition 2.20.** The multiplicity of the tropical resultant \(\mathcal{T}R(A)\) at a secondary cone \(\sigma\) of \(\text{Cay}(A)\) is the sum of multiplicities of cones \(C_E\) (given in Theorem 2.9) over all tuples \(E\) of pairs forming a mixed cell in the corresponding mixed subdivision \(\Delta_\sigma\).

**Proof.** By Lemma 2.19, for each tuple \(E\) of pairs, the cone \(C_E\) contains \(\sigma\) if and only if \(\sum_{i=1}^k E_i\) is a mixed cell in \(\Delta_\sigma\). Otherwise \(C_E\) is disjoint from the interior of \(\sigma\). The multiplicity of \(\sigma\) is the sum of multiplicities of \(C_E\)'s containing \(\sigma\).

The edges of the resultant polytope are normal to the maximal cones in the tropical resultant, and Proposition 2.20 can be used to find the lengths of the edges. From this description, we can derive Sturmfels' formula \[\text{[Stu94, Theorem 2.1]}\] for the vertices of the resultant polytope.

**2.3. Tropical description of secondary polytopes.** We will give a tropical description of secondary polytopes of arbitrary point configurations and show how tropical resultants fit in.

**Proposition 2.21.** Let \(A\) be a \(d \times m\) integer matrix whose columns affinely span an \(r\)-dimensional space. The tropical hypersurface of the secondary polytope of the columns of \(A\) is the set

\[
\bigcup_{I \subset \{1, \ldots, m\} \atop \mid I\mid = r+2} \mathbb{R}_{\geq 0}\{e_i : i \notin I\} + \text{rowspace}(A) + \mathbb{R}\{1\},
\]

where \(1\) denotes the all one vector in \(\mathbb{R}^m\).

**Proof.** Let \(\omega \in \mathbb{R}^m\), and let \(\Delta_\omega\) be the regular subdivision of the columns of \(A\) induced by \(\omega\). Then \(\omega\) is not in the tropical hypersurface of the secondary polytope if and only if \(\Delta_\omega\) is not a triangulation, which happens if and only if there exists a facet of \(\Delta_\omega\) containing at least \(r+2\) points of \(A\). For an \(r+2\)-subset \(I\) of \(\{1, \ldots, m\}\),
Figure 3. A projective drawing of the tropical hypersurface of the secondary polytope of the Cayley configuration of two 1-dimensional configurations in Example 2.22. The tropical resultant is shown in bold/color. A vertex labeled $ij$ represents the vector $e_{ij}$ in $\mathbb{R}^6 = \mathbb{R}^{A_1} \times \mathbb{R}^{A_2}$, and an edge between $ij$ and $kl$ represents the cone $\mathbb{R}_{\geq 0}\{e_{ij}, e_{kl}\} + \text{rowspace}(\text{Cay}(A))$. Compare with dual pictures in Figure 2 of Ref. [Stu94] and Figure 3 of Ref. [EFK10].

Comparing with Theorem 2.9, we see that the tropical resultant is the union of the cones in the above tropical description of the secondary polytope of Cay($A$) obtained by dropping an $e_{ij}$ for each $i$.

Example 2.22. Let $A_1 = A_2 = \{0, 1, 2\}$ in $\mathbb{Z}$. For $A = (A_1, A_2)$, the tropical hypersurface of the secondary polytope of Cay($A$) and the tropical resultant of $A$ are depicted in Figure 3. The resultant polytope has $f$-vector $(6, 11, 7, 1)$. The secondary polytope in the case is combinatorially equivalent to the 3-dimensional associahedron and has $f$-vector $(9, 21, 14, 1)$.

2.4. Codimension of the resultant variety. In this section we discuss how to determine the codimension of the tropical resultant variety $\text{TR}(A)$. By the Bieri–Groves Theorem [BGS84] this is also the codimension of $R(A)$.

Theorem 2.23. The codimension of the tropical resultant equals

$$k - \text{Max}_{E}\text{dim}\left(\sum_{i=1}^{k}\text{conv}(E_i)\right)$$

where each $E_i$ runs through all cardinality two subsets of $A_i$.

Proof. The tropical resultant variety is the collection of all lifts of all points in $A$ which give a fully mixed cell in the subdivision. Therefore it is the closure of the collection of lifts which give a zonotope in the mixed subdivision being a sum of convex hull of two points from each $A_i$. Let $P$ be such a zonotope and $E = (E_1, \ldots, E_k)$ the pairs of points. We wish to find the dimension of the (relatively open) cone $C_P$ of lifts which will induce $P$. The height of the first point of each $E_i$ may be chosen freely. The remaining $k$ points of $E$ must be lifted to the
same subspace of dimension \( \dim(P) \), whose lift may be chosen with \( \dim(P) \) degrees of freedom. Finally, the height of the points not in \( E \) may be chosen generically as long as sufficiently large. The codimension of \( CP \) is therefore \( k - \dim(P) \). The theorem follows since there are only finitely many choices for \( E \). \( \square \)

**Lemma 2.24.** Let \( L_i \) denote the linear subspace affinely spanned by \( A_i \). The codimension of \( RA \) only depends on the \( L_i \) and equals

\[
k - \max_{e \in \prod_i L_i} \dim(\text{span}(v_1, \ldots, v_k)).
\]

**Proof.** Since \( \text{conv}(E_i) \subseteq L_i \) the quantity of the lemma is smaller than or equal to that of Theorem 2.23. Conversely, if we have a collection \( v \in \prod_i L_i \) we will now show how we can perform a sequence of changes to \( v \) to make it only consist of vectors \( v_i \) which are each differences between points of \( A_i \) without lowering the dimension of \( \text{span}(v_1, \ldots, v_k) \). Consider a vector \( v_i \). It is a linear combination of some \( u_j \) where each \( u_j \) is of the form \( a_{is} - a_{it} \). If all \( u_j \) belong to \( W := \text{span}(v_1, \ldots, \hat{v}_i, \ldots, v_k) \) then so will \( v_i \) and it may be substituted by an arbitrary \( u_j \) without lowering the dimension. If some \( u_j \) does not belong to \( W \) then substituting \( u_j \) for \( v_i \) will not lower the dimension. \( \square \)

The proof also shows that instead of considering all line segments in Theorem 2.23 it suffices to consider only a basis for the affine span for each polytope. This is useful if computing the codimension using this formula.

**Remark 2.25.** We can define a matroid on a set of polytopes as follows. A set of polytopes is independent if they contain independent line segments. It is straightforward to check that the base exchange axiom holds. The rank of the matroid is the maximal dimension of a fully mixed cell (a zonotope) spanned by two element subsets, one subset from each polytope. The codimension of the tropical resultant equals the corank of the matroid, i.e. the number of polytopes minus the largest dimension of such a zonotope. The (tropical) resultant variety is a hypersurface if and only if the matroid has corank one, which holds if and only if there is a unique circuit in the matroid. The tuple \( A \) is essential if and only if this matroid of \( k \) polytopes is uniform of rank \( k - 1 \), that is, the unique circuit of the matroid consists of the entire ground set.

Using Theorem 2.23 we get a new proof of Sturmfels’ formula for the codimension of \( R(A) \).

**Theorem 2.26.** [Stu94, Theorem 1.1] The codimension of the resultant variety \( R(A) \) in \( \prod_{i=1}^k(C^*)^{m_i} \) is the maximum of the numbers \( |I| - \dim(\sum_{i \in I} Q_i) \) where \( I \) runs over all subsets of \( \{1, \ldots, k\} \).

By the Bieri–Groves Theorem and Theorem 2.24 the codimension of Theorem 2.23 equals that of Theorem 2.26. In the following we explain how the equality of the two combinatorial quantities of Theorems 2.23 and 2.26 can also be seen as a consequence of Perfect’s generalization (Theorem 2.27) of Hall’s marriage theorem and Rado’s theorem on independent transversals.

Let \( S \) be the ground set of a matroid with rank function \( \rho \). Let \( U = \{S_i : 1 \leq i \leq k\} \) be a family of subsets of \( S \). A subset \( S' \) of \( S \) is called an independent partial transversal of \( U \) if \( S' \) is independent and there exists an injection \( \theta : S' \to \{1, 2, \ldots, k\} \) with \( s \in S_{\theta(s)} \) for each \( s \in S' \).
Theorem 2.27. (Perfect’s Theorem [Per69, Theorem 2]) With the notation above, for every positive integer $d$, the family $U$ has an independent partial transversal of cardinality $d$ if and only if

$$d \leq \rho(\cup_{i \in I} S_i) + k - |I|$$

for every $I \subseteq \{1, 2, \ldots, k\}$.

In particular, the maximum cardinality of an independent partial transversal is equal to the minimum of the numbers on the RHS of the inequality.

Proof of Theorem 2.27. Let $S_i = \{a-b : a, b \in A_i\}$, $S = \bigcup_{i=1}^{k} S_i$, and $U = \{S_i : 1 \leq i \leq k\}$. Consider the vector matroid on $S$ given by linear independence. Then the quantity $\max_{E} \dim(\sum_{i=1}^{k} \text{conv}(E_i))$ is the cardinality of the maximal independent partial transversal of $U$. By Perfect’s Theorem,

$$\max_{E} \dim(\sum_{i=1}^{k} \text{conv}(E_i)) = \min_{I \subseteq \{1,2,\ldots,k\}} \dim(\sum_{i \in I} Q_i) + k - |I|.$$ 

Hence the two quantities from Theorems 2.23 and 2.26 are equal. $\square$

Straightforward evaluation of the formulas in Theorems 2.23 and 2.26 will require time complexity exponential in the input. Moreover, the maximal bipartite matching problem is a special case of this codimension problem.

Lemma 2.28. The maximal bipartite matching problem is reducible in polynomial time to the problem of computing codimension of resultants.

Proof. Let $G$ be a bipartite graph with vertices $U \cup V$ and edges $E \subset U \times V$. Let $\{e_u : u \in U\}$ be the standard basis for $\mathbb{R}^U$. For each $v \in V$, let $A_v = \{e_u : (v, u) \in E\}$. Then the maximal cardinality of a bipartite matching in $G$ is equal to the dimension of the resultant variety of $A = \{0\} \cup A_v : v \in V\}$. $\square$

We use Theorem 2.28 to construct an efficient algorithm.

Theorem 2.29. The codimension of the resultant can be computed in polynomial time in the input.

Proof. Let $A = (A_1, A_2, \ldots, A_k)$ where each $A_i$ is a point configuration in $\mathbb{Z}^n$. By Lemma 2.27, the codimension of $\mathcal{R}(A)$ depends only on the linear spaces $L_1, L_2, \ldots, L_k$ affinely spanned by $A_1, A_2, \ldots, A_k$ respectively. Choose a basis $B_i$ for each linear space $L_i$. Let $B = \{B_1, B_2, \ldots, B_k\}$ and $S = \bigcup_{i=1}^{k} B_i$. A subset $S'$ of $S$ is called a partial transversal of $B$ if there is an injection $\theta : S' \rightarrow \{1, 2, \ldots, k\}$ with $s \in B_{\theta(s)}$. The collection of partial transversals form an independent system of a matroid $\mathcal{M}_1$ on ground set $S$, called the transversal matroid of $B$. Let $\mathcal{M}_2$ be the vector matroid on $S$ defined by linear independence. By Theorem 2.28, computing the codimension of the resultant is equivalent to computing the maximal cardinality of a linearly independent partial transversal, i.e. the largest subset of $S$ which is independent in both $\mathcal{M}_1$ and $\mathcal{M}_2$.

We can use the cardinality matroid intersection algorithm [Sch03, Section 41.2] to find the maximum cardinality of a set independent in two matroids with the same ground set. This algorithm is polynomial in the size of $S$ and the time for testing independence in the matroids. Testing independence in $\mathcal{M}_1$ can be reduced to the maximal bipartite matching problem and can be solved in polynomial time.
Testing linear independence in $\mathcal{M}_2$ can be reduced to finding the rank of a matrix, which also takes polynomial time.

The algorithm described in Theorem 2.29 is rather complex, but there is a simpler probabilistic or numerical algorithm. For generic vectors $v_i \in L_i$ for $i = 1, 2, \ldots, k$, the codimension of the resultant is equal to $k - \text{rank}([v_1|v_2|\cdots|v_k])$. The challenge of turning this into a rigorous algorithm lies in making sure that the choices for $v_i$ are generic. Our naive attempts at symbolic perturbations resulted in matrices whose ranks cannot be computed in polynomial time.

2.5. **Traversing tropical resultants.** Tropical resultants are pure and connected in codimension 1. This allows the facets to be enumerated via the well-known adjacency decomposition approach. By this we mean traversing the connected bipartite graph encoding the facet-ridge incidences of the fan. Three operations are essential. We must be able to find some maximal cone in the fan, find the link at a ridge, and compute an adjacent maximal cone given a ray of the link at the ridge. In Ref. [Jen10] these subcomputations were isolated in an oracle, and we discussed a general algorithm for traversing a polyhedral fan (up to symmetry) represented only through oracle calls. In the following paragraphs we will describe how to walk locally in the tropical resultant. More details can be found in the next section for the more general setting of specialized tropical resultant.

To find a starting cone for the traversal, we use the description of the tropical resultant as a union of orthants plus a linear space, as described in Theorem 2.9. Alternatively, a generic vector in a maximal cone of a resultant fan can be found in polynomial time using the algorithms for the codimension described in Section 2.4.

To find the link of a point in the tropical resultant, we use the fact that the link of a point $\omega$ is a union of smaller tropical resultants associated to the fully mixed cells in the mixed subdivision of $\mathcal{A}$ induced by $\omega$, as shown in Proposition 2.17.

In the tropical resultant, as a subfan of the secondary fan of $\text{Cay} (\mathcal{A})$, each cone can be represented by a regular subdivision of $\text{Cay} (\mathcal{A})$. The smallest secondary cone containing a given vector $\omega$ can be constructed from the regular subdivision induced by $\omega$ as explained in Section 5.2 of Ref. [DLRS10].

In our implementation we represent the regular subdivision $\Delta$ induced by $\omega$ by $\omega$ and the triangulation induced by a “placing” or “lexicographic” perturbation of $\omega$. From this triangulation, we can easily recover the subdivision $\Delta$ by comparing the normal vectors of the facets of the triangulation lifted by $\omega$. For this to work, it is important to perturb $\omega$ in such a way that marked points in $\Delta$ remain marked in the refined triangulation. A full triangulation of $\text{Cay} (\mathcal{A})$ is only computed from scratch once at the beginning. To obtain a desired triangulation from a known triangulation, we find a path in the flip graph of regular triangulations and perform flips as in Section 8.3.1 of Ref. [DLRS10]. This is the analogue of a Gröbner walk in the setting of secondary fans.

To find the secondary cone in the link of $u$ given by a ray $v$, we compute the subdivision induced by $u + \varepsilon v$ for sufficiently small $\varepsilon > 0$. Such a vector $u + \varepsilon v$ is represented symbolically in a way similar to a matrix term order in Gröbner basis theory.
3. Resultants with specialized coefficients

For some applications such as implicitization we need to compute resultant varieties while specializing some of the coefficients to constants. This problem was studied in Refs. [EKP07, EKP11] for the case when the resultant variety is a hypersurface. In that case, the Newton polytope of the specialized resultant is the projection of the resultant polytope, and the authors computed the projection of resultant polytopes using Sturmfels' formula for vertices of resultant polytopes [Stu94, Theorem 2.1] and beneath-beyond or gift-wrapping methods for computing convex hulls. In our language, computing a projection of a polytope is equivalent to computing the restriction of the normal fan to a subspace.

In tropical geometry, specialization of certain coefficients amounts to taking stable intersection of the tropical resultant with certain coordinate hyperplanes. In this section we first define the specialized tropical resultants and then present an algorithm for their computation.

A polyhedral complex in \( \mathbb{R}^n \) is called locally balanced if it is pure dimensional and the link of every codimension one face positively spans a linear subspace of \( \mathbb{R}^n \).

**Definition 3.1.** Let \( F_1 \) and \( F_2 \) be locally balanced fans in \( \mathbb{R}^n \). We define the stable intersection as the fan

\[
F_1 \cap_{st} F_2 := \{ C_1 \cap C_2 : (C_1, C_2) \in F_1 \times F_2 \text{ and } \text{supp}(\text{link}_{C_1}(F_1)) - \text{supp}(\text{link}_{C_2}(F_2)) = \mathbb{R}^n \}
\]

with support

\[
\text{supp}(F_1 \cap_{st} F_2) = \{ \omega \in \mathbb{R}^n : \text{supp}(\text{link}_{\omega}(F_1)) - \text{supp}(\text{link}_{\omega}(F_2)) = \mathbb{R}^n \}.
\]

If in addition \( F_1 \) and \( F_2 \) are balanced then the stable intersection inherits multiplicities from \( \text{link}_{C_1}(F_1) \) and \( \text{link}_{C_2}(F_2) \) as follows:

\[
\text{mult}_{C_1}(F_1 \cap_{st} F_2) := \sum_{C_1, C_2} \text{mult}_{C_1}(\text{link}_{C_1}(F_1)) \cdot \text{mult}_{C_2}(\text{link}_{C_2}(F_2)) \cdot [Z^n : (Z^n \cap \mathbb{R}C_1) + (Z^n \cap \mathbb{R}C_2)]
\]

where the sum runs over \( C_1 \in \text{link}_{C_1}(F_1) \) and \( C_2 \in \text{link}_{C_2}(F_2) \) such that \( \omega' \in C_1 - C_2 \) for a fixed generic vector \( \omega' \in \mathbb{R}^n \).

Notice that the support of \( F_1 \cap_{st} F_2 \) depends only on \( \text{supp}(F_1) \) and \( \text{supp}(F_2) \). We will therefore extend the definition of stable intersections to intersections of supports of locally balanced fans and regard them as subsets of \( \mathbb{R}^n \).

Orthogonally projecting a polytope onto a linear space is equivalent to stably intersecting the tropical hypersurface of the polytope with the linear space.

**Theorem 3.2.** Let \( P \subset \mathbb{R}^n \) be a polytope, \( L \subset \mathbb{R}^n \) be a linear subspace, and \( \pi : \mathbb{R}^n \to L \) be the orthogonal projection. Then

\[
T(\pi(P)) = (T(P) \cap_{st} L) + L^\perp.
\]

**Lemma 3.3.** For any locally balanced fans \( F_1, F_2, \) and \( F_3 \), we have

1. \( (F_1 \cap_{st} F_2) \cap_{st} F_3 = F_1 \cap_{st} (F_2 \cap_{st} F_3) \)
2. \( (\text{supp}(F_1) \cup \text{supp}(F_2)) \cap_{st} \text{supp}(F_3) = \text{supp}(F_1 \cap_{st} F_3) \cup \text{supp}(F_2 \cap_{st} F_3) \).
Lemma 3.4. For locally balanced fans \( F_1 \) and \( F_2 \) in \( \mathbb{R}^n \)
\[
\text{supp}(F_1 \cap_{st} F_2) = \bigcup_{C_1 \in F_1, C_2 \in F_2 \text{dim}(C_1+C_2) = n} C_1 \cap C_2.
\]

Corollary 3.5. Let \( A \) and \( B \) be locally balanced polyhedral fans in \( \mathbb{R}^n \). Then
\[
\text{link}_\omega(A) \cap_{st} \text{link}_\omega(B) = \text{link}_\omega(A \cap_{st} B).
\]

Proposition 3.6. The stable intersection of two locally balanced fans is either empty or a locally balanced fan whose codimension is the sum of the codimensions.

Lemma 3.7. Let \( I \) be an ideal in \( k[x_1, x_2, \ldots, x_n] \). Then
\[
\text{supp}(T(I)) \cap_{st} \{x : x_1 = 0\} = \text{supp}(T(\langle I \rangle + (x_1 - \alpha)))
\]
where \( \langle I \rangle \) is the ideal in \( k(\alpha)[x_1, x_2, \ldots, x_n] \) generated by \( I \).

Definition 3.8. Let \( S = (S_1, \ldots, S_k) \) with \( S_i \subseteq \{1, \ldots, m_i\} \) represent a choice of points in the configuration \( A \). The coefficients of the monomials indexed by \( S \) are called specialized. Let \( U_i := \{x \in \mathbb{R}^{m_i} : \forall j \in S_i : x_j = 0\} \) and \( U_S := \prod_{i=1}^k U_i \). We define the specialized tropical resultant variety
\[
\mathcal{T}\mathcal{R}_S(A) := \mathcal{T}\mathcal{R}(A) \cap_{st} U_S.
\]

We will use the following proposition to justify the word “specialized”:

Proposition 3.9. Let \( A \) and \( S \) be as in Definition 3.8. Let \( I \) be the ideal of \( \mathcal{R}(A) \) and add to it, to obtain an ideal \( J \), for each specialized coefficient \( c_j \) the binomial \( c_j - \gamma_j \) where \( \gamma_j \) is a parameter. We define the specialized resultant variety
\[
\mathcal{R}_S(A) := V(J) \subseteq \prod_{i=1}^k (K)^{m_i}, \text{ where } K \text{ is the field of rational functions in the } \gamma_j
\]
with coefficients in \( \mathbb{C} \). Then the tropicalization of \( \mathcal{R}_S(A) \) is \( \mathcal{T}\mathcal{R}_S(A) \).

Proof. The statement follows from Lemmas 3.4 and 3.7. \( \square \)

The computation of the tropicalization of \( \mathcal{R}_S(A) \) can be performed using Buchberger’s algorithm as explained in Ref. [BJS+07] over the field of rational functions in the \( \gamma \)'s. During this computation finitely many polynomials in the \( \gamma \)'s appear as numerators and denominators of the coefficients. Substituting constant values for the \( \gamma \)'s will give the same computation unless one of these polynomials vanish. Hence specializing \( \gamma \)'s to values outside a hypersurface in \( (\mathbb{C}^*)^S \) will lead to a specialized tropical resultant variety. This explains the word “specialized”.

If \( \mathcal{T}\mathcal{R}_S(A) \) is nonempty, then its codimension can be computed using Proposition 3.9 and the codimension formulas from Section 2.4. Thus it remains to give an algorithm for checking if the specialized resultant is empty. Recall that \( m := \sum_i m_i \) is the total number of points in \( A \).

Lemma 3.10. Let \( A \) and \( S \) be as in Definition 3.8. Define the extended tuple \( B = (B_1, \ldots, B_k) \) where \( B_i \) consists of \( b_{i,j} = (a_{i,j}, v_{i,j}) \in \mathbb{Z}^n \times \mathbb{Z}^{m-|S|}, \) with \( v_{i,j} \in \mathbb{Z}^{m-|S|} \) being \( 0 \) if \( j \in S_i \) and a standard basis vector otherwise. If the standard vector is chosen differently for every non-specialized coefficient then
\[
\mathcal{T}\mathcal{R}_S(A) \neq \emptyset \iff \mathcal{T}\mathcal{R}(B) = \mathbb{R}^m.
\]

Proof. According to Lemma 3.3 \( \mathcal{T}\mathcal{R}_S(A) \neq \emptyset \) if and only if there exists a cone \( C \subseteq \mathcal{T}\mathcal{R}(A) \) such that \( U_S + C = \mathbb{R}^m \) where \( U_S \) is as defined in Definition 3.8. According to the simple description of tropical resultants in Theorem 2.4 we may assume that \( C \) has the form \( \mathbb{R}_{\geq 0}[e_{ij} : a_{ij} \notin E_i] \) + rowspace(Cay(A)). Equivalently, the stable
intersection is nonempty if and only if there exists a choice \( E \) such that \( \mathbb{R}_{>0}\{e_{ij} : a_{ij} \not\in E_i\} \) + rowspace(Cay(\(A\))) + \(U_S\) has dimension \(m\). Applying Theorem 2.9 to \( \mathcal{B} \), this is equivalent to \( \mathcal{TR}(\mathcal{B}) \) being full-dimensional, since rowspace(Cay(\(A\))) + \(U_S\) = rowspace(Cay(\(B\))). \[\square\]

Combining Lemma 3.10 and the results from Section 2.4 about codimension computations, we get a polynomial time algorithm for deciding if a specialized result is nonempty. Another consequence of the lemma is the following algorithm for checking membership of a point in a specialized tropical resultant.

**Algorithm 3.11.** \((\text{SpecializedResultantContains}(A, S, \omega))\)

**Input:** A tuple \( A \) of point configurations and a choice \( S \) of specialized coefficients. A vector \( \omega \in \mathbb{R}^m \).

**Output:** “True” if \( \omega \in \mathcal{TR}_S(A) \), “False” otherwise.

- Compute the mixed subdivision of \( A \) induced by \( \omega \) by computing the regular subdivision of Cay(\(A\)) induced by \( \omega \).
- For each fully mixed cell:
  - construct a subconfiguration \( A' \) of points involved in the cell.
  - Return “True” if the specialized resultant of \( A' \) is nonempty.
- Return “False”.

**Proof.** By Lemma 3.3(2), Proposition 2.17 and Corollary 3.5, we have that the support of \( \operatorname{link}_\omega(\mathcal{TR}_S(A)) \) is the union of supports of \( \mathcal{TR}_S(A') \), under the appropriate identification of \( \mathcal{TR}_S(A') \) as a subset of \( \mathbb{R}^m \), where \( A' \) runs over all fully mixed cells of the mixed subdivision of \( A \) induced by \( \omega \). Hence \( \omega \in \mathcal{TR}_S(A) \) if and only if one of \( \mathcal{TR}_S(A') \) is nonempty. \[\square\]

**Algorithm 3.12.** \((\text{NonTrivialVectorInSpecializedResultant}(A, S))\)

**Input:** A tuple \( A \) of configurations, a choice \( S \) of specialized coefficients such that \( U_S \cap \) rowspace(Cay(\(A\))) \(\subseteq\) \( \mathcal{TR}_S(A) \).

**Output:** A vector \( \omega \in \mathcal{TR}_S(A) \setminus \) rowspace(Cay(\(A\)))

- For each \( E = (E_1, E_2, \ldots, E_k) : E_i \) is a two-element subset of \( A_i \),
  - Let \( C = \mathbb{R}_{>0}\{e_{ij} : i \not\in E_j\} + \) rowspace(Cay(\(A\))).
  - If \( \dim(C + U_S) = n \) and \( U_S \cap C \neq U_S \cap \) rowspace(Cay(\(A\))) then
    - Find among the generators of \( U_S \cap C \) a vector \( v \) outside the subspace \( U_S \cap \) rowspace(Cay(\(A\))).
    - Return \( v \).

The following recursive algorithm finds a perturbed point in a starting cone for the specialized tropical resultant \( \mathcal{TR}_S(A) \).

**Algorithm 3.13.** \((\text{StartingPoint}(A, S))\)

**Input:** A tuple \( A \) of configurations, a choice \( S \) of specialized coefficients such that \( \mathcal{TR}_S(A) \neq \emptyset \).

**Output:** A vector \( \omega_\varepsilon \in \mathbb{Q}(\varepsilon)^m \) such that for every fan structure of \( \mathcal{TR}_S(A) \) defined over \( \mathbb{Q} \) it holds that for \( \varepsilon > 0 \) sufficiently small, \( \omega_\varepsilon \) is in a facet of \( \mathcal{TR}_S(A) \).

- If \( \dim(\mathcal{TR}_S(A)) = \dim(U_S \cap \) rowspace(Cay(\(A\))) \), then return \( b_1 + \varepsilon b_2 + \cdots + \varepsilon^{t-1}b_t \) where \( b_1, b_2, \ldots, b_t \) is some basis of \( U_S \cap \) rowspace(Cay(\(A\))).

- Compute an \( \omega \in \mathcal{TR}_S(A) \setminus \) rowspace(Cay(\(A\))) using Algorithm 3.12.

- Compute the subdivision \( \Delta_\omega \) of Cay(\(A\)) induced by \( \omega \).

- For every fully mixed cell in \( \Delta_\omega \).
– Let $A'$ be the subconfiguration of the involved points.
– Let $S'$ be the restriction of $S$ to $A'$.
– If $\text{codimension}(\mathcal{T}R_{S'}(A')) = \text{codimension}(\mathcal{T}R_{S}(A))$ then
  * Return $\omega + \varepsilon \cdot \text{StartingPoint}(A', S')$.

Proof. The correctness of the algorithm follows from the facts that the link of $\omega$ in the tropical resultant is the union of tropical resultants corresponding to the fully mixed cells in $\Delta_\omega$ (Proposition 2.17), and that taking links commutes with taking stable intersections (Corollary 3.5), and because the returned value from the recursive call is, after expansion with zeros, a vector outside of the secondary cone of $\omega$. □

We now turn to the problem of enumerating all facets in $\mathcal{T}R_{S}(A)$ considered as a subfan of the restriction of the secondary fan of $\text{Cay}(A)$ to the subspace $U_S$. While stable intersections are not in general connected in codimension 1, this is indeed the case for $\mathcal{T}R_{S}(A)$ since it is defined by a prime ideal. See the argument in Ref. [BJS+07] using Kleinman’s version of Bertini’s Theorem.

The output of Algorithm 3.13 can be converted into a secondary cone in $\mathcal{T}R(A)$ containing $\omega_\epsilon$ in its relative interior, for example by computing a maximal secondary cone containing $\omega_\epsilon$ and taking the face containing $\omega_\epsilon$ in its relative interior.

The polynomial time algorithm of Theorem 2.29 for computing codimension a resultant varieties can be used for finding generic points in $\mathcal{T}R(A)$ in polynomial time. Simply remove points from $A$ as long as possible without dropping the dimension. When done we have exactly two points left from each configuration of $A$. We then compute a generic point in $\mathcal{T}R(A)$ using Theorem 2.9 possibly using a symbolic $\varepsilon$. It is unclear if a polynomial time algorithm exists for finding a generic point in specialized tropical resultants.

Following the approach of Ref. [Jen10] discussed in Section 2.5, we are left with the problem of computing the link of a ridge in $\mathcal{T}R_{S}(A)$. If the subspace $U_S$ had been generic enough to intersect the lineality space of $\mathcal{T}R(A)$ transversely, i.e. $\text{codim}(U_S \cap \text{rowspace}(\text{Cay}(A))) = \text{codim}(U_S) + \text{codim}(\text{rowspace}(\text{Cay}(A)))$, then the link would be combinatorially equivalent to the link in $\mathcal{T}R(A)$ and the support of the link would be a union of resultant fans of subconfigurations (Proposition 2.17) where each fan can be found using Theorem 2.9. If $U_S$ is not generic, then computing a stable intersection with $U_s$ is required for finding the link in $\mathcal{T}R_{S}(A)$ (Corollary 3.5). This is Algorithm 3.14 below. Another approach is to compute the restriction of the secondary fan of each fully mixed subconfiguration to $U_S$. We then get the resultant fan as certain rays of the secondary fan. This is Algorithm 3.15.

Algorithm 3.14. StableLink($A, S, \omega$)
Input: A tuple $A$ of configurations, a choice $S$ of specialized coefficients, a vector $\omega \in \mathbb{R}^n$ in the relative interior of a ridge $R$ of $\mathcal{T}R_{S}(A)$.
Output: A vector in each facet of $\text{link}_\omega(\mathcal{T}R_{S}(A))$.

* Let $d$ be the dimension of $\mathcal{T}R(A) \cap_{st} \{U_S\}$.
* Compute the subdivision $\Delta_\omega$ of $\text{Cay}(A)$ induced by $\omega$.
* $l := \emptyset$.
* For every fully mixed cell in $\Delta_\omega$,
  * Let $A'$ be the subconfiguration of involved points in the cell.
  * For each $E = (E_1, E_2, \ldots, E_k) : E_i$ is a two-element subset of $A'$,
    * Let $C = \mathbb{R}_{\geq 0}\{e_{ij} : i \notin E_j\} + \text{rowspace}(\text{Cay}(A))$. 

Algorithm 3.15. StableLink($A, S, \omega$)
Input: A tuple $A$ of configurations, a choice $S$ of specialized coefficients, a vector $\omega \in \mathbb{R}^n$ in the relative interior of a ridge $R$ of $\mathcal{T}R_{S}(A)$.
Output: A vector in each facet of $\text{link}_\omega(\mathcal{T}R_{S}(A))$.

* Let $d$ be the dimension of $\mathcal{T}R(A) \cap_{st} \{U_S\}$.
* Compute the subdivision $\Delta_\omega$ of $\text{Cay}(A)$ induced by $\omega$.
* $l := \emptyset$.
* For every fully mixed cell in $\Delta_\omega$,
  * Let $A'$ be the subconfiguration of involved points in the cell.
  * For each $E = (E_1, E_2, \ldots, E_k) : E_i$ is a two-element subset of $A'$,
    * Let $C = \mathbb{R}_{\geq 0}\{e_{ij} : i \notin E_j\} + \text{rowspace}(\text{Cay}(A))$. 


If \( \dim(U_S + C) = m \) and \( \dim(U_S \cap C) = d \) then

- Let \( V \) be a set of one or two vectors in \( U_S \cap C \) such that \( (U_S \cap C) + \text{span}(R) \) is positively spanned by \( V \cup \text{span}(R) \).
- \( l := l \cup V \)

- Return \( l \).

Algorithm 3.15. StableLink(\( A, S, \omega \))

Input: A tuple \( A \) of configurations, a choice \( S \) of specialized coefficients, a vector \( \omega \in \mathbb{R}^n \) in the relative interior of a ridge \( R \) of \( \mathcal{T}R_S(A) \).

Output: A vector in each facet of \( \text{link}_\omega(\mathcal{T}R_S(A)) \).

- Let \( d \) be the dimension of \( \mathcal{T}R(A) \cap \{U_S\} \).
- Compute the subdivision \( \Delta_\omega \) of \( \text{Cay}(A) \) induced by \( \omega \).
- \( l := \emptyset \).
- For every fully mixed cell in \( \Delta_\omega \)
  - Let \( \mathcal{A}' \) be the subconfiguration of the involved points of the cell.
  - If the codimension of the lineality space of the restriction \( F \) of the secondary fan \( \Sigma(\text{Cay}(A')) \) to \( U_S \) is \( m - d \), then
    - Choose \( v \) such that \( v \) completes \( \text{span}(R) \cap U_S \) to a generating set of the lineality space of \( F \).
    - If \( \text{SpecializedResultantContains}(\mathcal{A}', S, r) \) then \( l := l \cup \{v, -v\} \).
  - else
    - Compute all maximal cones in \( F \) (by traversal).
    - For each ray \( r \) in \( F \), if \( \text{SpecializedResultantContains}(\mathcal{A}', S, r) \) then \( l := l \cup \{r\} \).
- Return \( l \).

The above algorithm is to be read with proper identifications - when restricting to \( \mathcal{A}' \) the vectors in \( \mathbb{R}^m \) need to be truncated accordingly, and so does the set \( S \), and \( r \) needs to be expanded when adding it to \( l \). When adding vectors to \( l \), it is advantageous to choose the vectors as primitive vectors orthogonal to the span of the ridge so that duplicates can be removed easily.

If \( U_S \) is high dimensional, a typical situation is that each subconfiguration is a number of edges and a triangle. In this case there are only few choices \( E \) to run through in Algorithms 3.14. For lower dimensional \( U_S \) there can be many choices of \( E \) but with many of the contributions to the stable intersection being the same. See Example 3.16. In such a case Algorithm 3.15 performs better. In general it is difficult to predict which algorithms is best. In our implementation we use Algorithm 3.13 and Algorithm 3.14 only when there is no specialization.

Example 3.16. Let \( A = (A_1, A_2, A_3) \) with

\[
A_1 = \{(0,0), (0,1), (0,3), (1,0), (3,0)\}
\]

\[
A_2 = \{(0,0), (0,1), (0,3), (1,0), (3,0)\}
\]

\[
A_3 = \{(0,0), (0,1), (0,2), (1,0), (1,3), (2,0), (3,1), (3,3)\}
\]

Choosing the specialization \( S \) of every coefficient except the coefficient of the point \( (0,0) \) in each configuration, we get that \( \mathcal{T}R_S(A) \) is a two dimensional fan with f-vector \( (1,13,17) \) living inside \( \mathbb{R}^3 \subset \mathbb{R}^{18} \). The link at \( e_{11} \in \mathbb{R}^{18} \) consists of 4 rays. The traversal of \( \mathcal{T}R_S(A) \) takes 79 seconds if Algorithm 3.14 is used but only 5 seconds if Algorithm 3.15 is used for computing the links. Algorithm 3.14 needs
to iterate through 2100 vertex pair choices at $e_{11}$, but much fewer for many of the other links.

3.1. Implicitization using specialized resultants. In this section we will show that the tropicalization of a variety parameterized by polynomials with generic coefficients can be computed using specialized tropical resultants. Let $f_1, f_2, \ldots, f_k \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ be polynomials parameterizing a variety $V$ in $\mathbb{C}^k$. Let $\Gamma$ be the graph of the parameterizing map, defined by $(y_1 - f_1, y_2 - f_2, \ldots, y_k - f_k)$ in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_k]$. When $f_1, f_2, \ldots, f_k$ have generic coefficients, the tropical variety of $\Gamma$ is the stable intersection of the tropical hypersurfaces of the polynomials $y_1 - f_1, y_2 - f_2, \ldots, y_k - f_k$. Since $V$ is the closure of the projection of $\Gamma \subset \mathbb{C}^n \times \mathbb{C}^k$ onto $\mathbb{C}^k$, by tropical elimination theory, we can compute the tropical variety of $V$ as a projection of $\mathcal{T}(\Gamma)$. This approach was used in Refs. [STY07, SY08].

Another way to compute the tropical variety of $V$ is by using specialized resultants. Let $A = (A_1, A_2, \ldots, A_k)$ where $A_i = \text{supp}(f_i) \cup \{0\}$ for each $i = 1, 2, \ldots, k$. Let $S = (\text{supp}(f_1), \text{supp}(f_2), \ldots, \text{supp}(f_k))$ be a choice of points to specialize, and let $U_S$ be the subspace of $\prod_{i=1}^k \mathbb{R}^{A_i} \times \mathbb{R}^n$ defined by setting the coordinates in $S$ to 0.

**Proposition 3.17.** With the notation above, $\mathcal{T}(V) = \mathcal{T} \mathcal{R}_S(A)$, i.e. the tropicalization of a variety parameterized by polynomials with generic coefficients coincides with a specialized resultant.

**Proof.** Let $W$ be the incidence variety in $\prod_{i=1}^k (\mathbb{C}^{\ast})^{A_i} \times (\mathbb{C}^{\ast})^n$ as seen in (1), defined by equations of the form $y_i - g_i$, where $g_i$ is a polynomial with the same support as $f_i$ but with indeterminate coefficients. Then the graph $\Gamma$ is obtained by specializing the coefficients of $g_i$ to those of $f_i$. Since the coefficients of $f_i$ were assumed to be generic, we get $\mathcal{T}(\Gamma) = \mathcal{T}(W) \cap \text{st} U_S$. By tropical elimination, $\mathcal{T}(V) = \mathcal{T}(\Gamma) + \mathbb{R}^n$, and by the following lemma, this coincides with $(\mathcal{T}(W) + \mathbb{R}^n) \cap \text{st} U_S$, which is the specialized tropical resultant. Here we are using the fact that a linear projection of a fan is combinatorially equivalent to the Minkowski sum with the kernel of the projection. □

**Lemma 3.18.** Let $\mathcal{F}$ be a locally balanced fan in $\mathbb{R}^n$. Let $L$ and $L'$ be linear subspaces of $\mathbb{R}^n$ such that $L' \subset L$. Then

$$(\mathcal{F} \cap \text{st} L) + L' = (\mathcal{F} + L') \cap \text{st} L$$

In other words, stable intersection with a linear space commutes with Minkowski sum with a smaller linear space.

**Proof.** Both $(\mathcal{F} \cap \text{st} L) + L'$ and $(\mathcal{F} + L') \cap \text{st} L$ are empty if $\mathcal{F} + L$ has dimension less than $n$. Suppose this is not the case. Then both sets contain $L'$ in their linearity space and consist of points of the form $u + v \in \mathbb{R}^n$ where $u \in L'$ and $v \in \mathcal{F} \cap L$ are such that $\dim(\text{link}_u(\mathcal{F}) + L) = n$. □

Since the tropical variety of the graph $\Gamma$ only depends on the extreme monomials of the parameterizing polynomials, the following result follows immediately.

**Corollary 3.19.** When using specialized resultants for implicitization, the extreme monomials of the input polynomials determine the tropical variety of the parameterized variety, so we can safely disregard the non-extreme terms.
An advantage of using specialized resultant is that our description of specialized tropical resultant is easier to compute than the stable intersection directly. Moreover, experiments show that the resultant description may speed up the reconstruction of the Newton polytope in some cases. See Section 5 for examples.

Moreover, when the variety \( V \) is not a hypersurface, our resultant description gives a fan structure of \( T(V) \) derived from the restriction of a secondary fan to a linear subspace, which is the normal fan of a fiber polytope. Tropical elimination does not give a fan structure for varieties of codimension more than one.

3.2. Tropical elimination for specialized tropical resultants. As before, let \( A \) be a tuple point configurations in \( \mathbb{Z}^n \) and \( S \) be the tuple of subsets to be specialized. Let \( W \) be the incidence variety and \( TW \) be is tropicalization as in Section 2.1. Let \( W_S \) be a variety cut out by polynomials \( f_i \) where the the coefficients of monomials in \( S \) have been specialized. Then \( f_1, f_2, \ldots, f_k \) may no longer form a tropical basis, but the tropicalization of \( W_S \) can be computed as the stable intersection of tropical hypersurfaces of \( f_1, f_2, \ldots, f_k \) because the coefficients are assumed to be generic (or indeterminate).

The specialized resultant is the projection of \( W_S \) onto the non-specialized coefficient variables, and we can compute this using tropical elimination theory, which gives the tropical variety as a union of cones. When the specialized tropical resultant is a tropical hypersurface, then we can reconstruct the normal fan of the dual Newton polytope using the methods in the next section.

The tropical hypersurface of \( f_i \) only depends on the Newton polytope \( P_i \) of \( f_i \). The non-specialized points in \( A_i \) always contribute as vertices of \( P_i \), but some specialized points of \( A \) may not. From this observation, we obtain the following result, which is not obvious from the resultant point of view.

**Lemma 3.20.** If \( a_{ij} \in A_i \) is a specialized point lying in the convex hull of other specialized points in \( A_i \), then removing \( a_{ij} \) from \( A_i \) does not change the specialized tropical resultant because the Newton polytope and the tropical hypersurface of \( f_i \) remain the same.

In other words, we may disregard the non-vertices among the specialized points. Using this lemma, we may be able to reduce the amount of work for computing the specialized tropical resultant \( TR_S(A) \).

4. Polytope reconstruction

In this section we describe an algorithm for finding a fan structure on a tropical hypersurface \( T \subseteq \mathbb{R}^n \). Recall that the tropical hypersurface of a polytope \( P \subseteq \mathbb{R}^n \) is the set of \( \omega \in \mathbb{R}^n \) for which there exist distinct \( p, q \in P \) such that for any \( r \in P \), \( \omega \cdot p = \omega \cdot q \leq \omega \cdot r \). In other words, the tropical hypersurface of a polytope is the union of the normal cones to the polytope at the edges. The multiplicity of a point in the relative interior of such a normal cone is the (lattice) length of the edge. The tropical hypersurface of a polynomial is the tropical hypersurface of its Newton polytope.

The tropical hypersurface \( T \) will be presented to us as a finite collection of codimension 1 cones which may overlap badly but whose union will be \( T \). What we wish to compute is a collection of codimension 1 cones such that the collection of all their faces is a polyhedral fan with support \( T \). This fan is not unique unless we require it to be the coarsest – that is, that it is the normal fan of the polytope.
defining $T$ with its maximal cones removed. If the codimension 1 cones come with a multiplicity then an advantage of having the fan structure is that it is relatively easy to reconstruct the 1-skeleton of the polytope defining $T$ up to translation. Therefore we will consider the computations of a polytope, its normal fan, and its tropical hypersurface to be equivalent in the following.

One way to perform the polytope reconstruction is to use the beneath-beyond method for computing convex hulls. The key observation is that for any generic $\omega \in \mathbb{R}^n$ the vertex face $\omega(\text{New}(f))$ can be computed using “ray shooting”. See the papers Refs. [DFS07] and [CTY10]. The method we present in this paper uses the adjacency decomposition approach (see Section 2.5) and the following algorithm for computing normal cones, at vertices, of the polytope defining $T$.

**Algorithm 4.1** (Region($S, \omega$)).

**Input:** A collection $S$ of codimension 1 cones in $\mathbb{R}^n$ such that $T := \cup_{C \in S} C$ is the support of a tropical hypersurface. A vector $\omega \in \mathbb{R}^n \setminus T$.

**Output:** The (open) connected component of $\mathbb{R} \setminus T$ containing $\omega$.

- $R := \mathbb{R}^n$.
- For each $C \in S$:
  - While $R \cap C \neq \emptyset$:
    * Find $p \in R \cap C$.
    * Let $h$ be the generic open half line from $\omega$ through $p + \sum_{i=1}^n \varepsilon_i e_i$.
    * Let $\Delta \subseteq S$ be the collection of cones with intersection point with $h$ closest to $\omega$, for $\varepsilon > 0$ sufficiently small.
    * Let for some $D \in \Delta$ the halfspace $H \subset \mathbb{R}^n$ be the connected component of $\mathbb{R}^n \setminus \text{span}(D)$ containing $\omega$.
  * $R := R \cap H$.
- Return $R$.

**Proof.** The set $R$ stays open and convex throughout the computation. At the end $R \cap T = \emptyset$. Each added constraint $H$ for $R$ is necessarily satisfied by the connected component because of its convexity. The symbolic perturbation of $p$ and the convexity of $R$ ensures that $H$ is independent of the choice of $D$ in $\Delta$, as all cones in $\Delta$ must be parallel. In fact the set of constraints give an irredudant inequality description of the returned cone. 

**Proposition 4.2.** Let $a$ be the number of facets of the closure of the returned cone of Algorithm 4.1. The number of checks “$R \cap C \neq \emptyset$” performed in algorithm is $|S| + a$ while the number of interior point computations “$p \in R \cap C$” is $a$.

**Proof.** The check is done for every cone in $C \in S$. In addition, whenever the algorithm enters the body of the while loop, a facet constraint $H$ is added to $R$, and an additional check “$R \cap C \neq \emptyset$” and a computation of $p$ is performed.

The condition that the generic $h$ intersects a given polyhedral cone $C$ can be phrased as a condition on the ordering in which $h$ intersects the defining hyperplanes of $C$. We can imagine moving a point starting from $\omega$ and along the half-line $h$, keeping track of which equations and inequalities defining $C$ are satisfied and updating when a defining hyperplane of $C$ is crossed. Hence the implementation reduces to a check of the order in which $h$ intersects two given hyperplanes. The perturbation in such a check is not difficult to handle symbolically. The check can
be used again to actually find a $D$ in the algorithm with intersection point closest to $\omega$.

To apply the adjacency decomposition approach we must be able to compute a starting cone and move across codimension 1 faces to find neighboring cones, while computing links of ridges is trivial for complete fans. To find a starting cone we guess a vector outside $T$ and apply Algorithm 4.1. Suppose now that $C$ is a full dimensional cone in the normal fan and $u$ is a relative interior point on a facet of $C$ with outer normal vector $v$. For $\varepsilon > 0$ sufficiently small, calling Algorithm 4.1 with argument $u + \varepsilon v$ will give us the desired neighboring cone. In our implementation we again use comparison of intersection points on line segments to find an $\varepsilon$ sufficiently small to avoid all hyperplanes appearing in the description of $T$.

If we precompute generators for the cones in $S$ then most of the checks for empty intersection with $R$ can done without using linear programming, but rather for each defining hyperplane of $R$ checking if the cone generators are completely contained on the wrong side. In our current implementation the time spent on finding first intersection along the half-lines is comparable to the time spent on linear programming. We present two examples to illustrate the usability of the algorithm.

**Example 4.3.** In Ref. [HSYY08] we computed the f-vector of the tropical hypersurface of the $2 \times 2 \times 2 \times 2$ hyperdeterminant. Its support is the sum of a tropical linear space and a classical linear space in $\mathbb{R}^{16}$ and is to easy write as a union of cones. We reconstruct the 25448 normal cones of the Newton polytope of the defining equation in 163 minutes. Exploiting the 384 order symmetry as explained in Ref. [Jen10] we reduce the running time to 7 minutes for computing the 111 orbits of maximal cones. With suitable input files the following Gfan command [Jen] will compute the f-vector. Also see Section 5.

```
anders@gureko:~$ gfan_tropicalhypersurfacereconstruction -i tropolinspc.fan
--sum --symmetry <claslinspc_and_symmetry.txt | grep -A1 F_VECTOR
```

```
F_VECTOR
1 268 5012 39680 176604 495936 927244 1176976 1005946 55528017870 25448
```

**Example 4.4.** The implicitization challenge solved in Ref. [CTY10] is to reconstruct the Newton polytope of the defining equation of a tropical variety given as a union of 6865824 cones. This 11-dimensional polytope lives in $\mathbb{R}^{16}$ and has a symmetry group of order 384. In Ref. [CTY10], a modified version of the ray-shooting method was used to produce coordinates of vertices at a rate of a few (2-5) minutes per vertex. Each round took about 45 minutes found 10-20 vertices typically. However a lot more computation, with some human interaction and parallelization, over a period of a few months was required to make sure that all the vertices were discovered, and this was done by computing the tangent cone at each found vertex, up to symmetry. During the process most vertices were re-discovered multiple times.

On this example our new implementation in Gfan spends approximately 1 minute for each call of Algorithm 4.1. We estimate that the enumeration of the 44938 orbits would finish after 30 days of computation. With the new method, we do not need to process a vertex more than once, and we obtain all the facet directions as the rays in the normal fan and all the tangent cones as duals of the normal cones. Moreover, there is no post-processing needed to certify that all vertices have been found.
The method we just described does not make use of multiplicities. In fact, it is not necessary that the fan is polytopal, or even locally balanced. We only require that each connected component of the complement of $T$ is convex.

Before settling with Algorithm 4.1 we also experimented with storing the codimension one cones in a binary space partitioning tree (BSP tree). The tree would be built at initialization, and the connected components of the complement could be computed by gathering convex regions stored in the tree. This method worked as well as Algorithm 4.1 in small dimensions, but in higher dimensions, like the examples above, Algorithm 4.1 would always perform better. In Example 4.3 the difference would be a factor five without exploiting symmetry. But in Example 4.4 the number of required nodes of the tree would grow too large to have any chance of fitting in memory. The intuition behind the explosion in complexity is that cones (for example, simplicial cones of codimension one) in a higher dimensional space have larger chances of intersecting a fixed hyperplane. Therefore in the process of building the BSP tree, a codimension one cone from the input will meet many other hyperplanes coming from other cones, causing an explosion in the number of nodes in the BSP tree.

5. Comparison of algorithms

In this section, we consider various algorithms and compare the combinatorial complexity of the output (e.g. f-vector) and running time (recorded on a laptop computer with a 2.66 GHz Intel Core i5 processor and 8GB of memory). All implementations are single threaded, done in C++ using cddlib [Fuk05] and SoPlex [Wun96], and will be part of Gfan in its next release, unless otherwise noted. The combinatorial complexity of the output is essential for a fair comparison since not the same amount of effort went into making each of the implementations fast. A lot of effort went into the implementation of Algorithm 4.1 and the secondary fan computation because of their broad range of applications, while less optimization effort has gone into algorithms specific to tropical resultants.

In general, the software Gfan uses the $\max$ convention for tropical varieties and Gröbner fans. However, for the fact that the secondary fan of a point configuration is a coarsening of the Gröbner fan of the associated binomial (lattice) ideal, we need the subdivisions to be defined with respect to $\min$ if the initial ideals are defined with respect to $\max$. Therefore Gfan uses $\min$ for secondary fans. As tropical resultants are subfans of secondary fans, we chose to use $\min$ in this paper for tropical addition.

Hypersurfaces. Let us first consider the case where the resultant variety $\mathcal{R}(A)$ is a hypersurface. Following is a list of different methods for computing the resultant polytope (or its tropical hypersurface or its normal fan).

1. Enumerating the vertices of the secondary polytope of Cay($A$), and then using Sturmfels’ formula [Stu94, Theorem 2.1] to obtain the vertices of the resultant polytope. For our timings we only do the first part of the computation using the Gfan command

   \texttt{gfan\_secondaryfan \textless{}cayley.txt}

2. Computing the tropical hypersurface of the resultant as a subfan of the secondary fan by fan traversal using the methods described in Section 4.4

   \texttt{gfan\_resultantfan \--vectorinput \textless{}tuple.txt}
(3) Constructing the normal fan of the resultant polytope from the simple description of the tropical resultant as a union of cones as in Theorem 2.9. Our implementation in Gfan uses Algorithm 4.1 for this.

\texttt{gfan\_resultantfan --vectorinput --projection <tuple.txt}

(4) For a generic direction using Sturmfels’ formula \cite[Theorem 2.1]{Stu94} for finding the optimal vertex of the resultant polytope in that direction and combining this approach with the beneath-beyond convex hull algorithm for recovering the whole polytope. This method has recently been implemented. See Ref. \cite{EFKP11}. Unfortunately the current interface for the implementation only handles implicitization type problems and cannot handle Examples (a) through (f) below. We intend to make comparisons with it for these examples in the near future.

For the third method, one can also use other methods for reconstructing a polytope from its tropical hypersurface, such as ray-shooting/beneath-beyond and BSP trees, as discussed in Section 4, although we found Algorithm 4.1 to perform better, especially for polytopes of dimension 5 or more (compared to beneath-beyond in iB4e \cite{Hug06} and BSP).

For Example 2.10 above, each of the first three methods finished in under one second in Gfan. We will present more challenging examples below.

\textbf{Example (a).}

\[ A = \begin{pmatrix}
0 & 1 & 3 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 1 \\
0 & 2 & 1 \\
2 & 1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
0 & 2 & 2 \\
2 & 1 & 2 \\
2 & 0 & 3
\end{pmatrix} \]

| Method/fan | F-vector of output | Timing |
|------------|--------------------|--------|
| (1) secondary fan | 1 10432 55577 106216 88509 27140 | 467 s |
| (2) traversing tropical resultant | 1 5152 21406 28777 12614 | 733 s |
| (3) normal fan from simple description | 1 78 348 570 391 93 | 1.4 s |

\textbf{Example (b).}

\[ A = \begin{pmatrix}
0 & 1 & 3 \\
0 & 0 & 1 \\
0 & 1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 2 \\
1 & 2 & 0 \\
1 & 2 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 1 & 2 \\
0 & 1 & 2 \\
1 & 1 & 0
\end{pmatrix} \]

| Method/fan | F-vector of output | Timing |
|------------|--------------------|--------|
| (1) secondary fan | 1 3048 38348 178426 407991 494017 304433 75283 | 506 s |
| (2) tropical resultant | 1 2324 26316 106083 197576 173689 58451 | 1238 s |
| (3) normal fan | 1 56 497 1779 3191 3018 1412 249 | 6 s |

\textbf{Example (c).}

\[ A = \begin{pmatrix}
1 & 2 & 2 \\
1 & 2 & 3 \\
3 & 1 & 2
\end{pmatrix}, \quad \begin{pmatrix}
1 & 3 & 3 \\
1 & 2 & 2 \\
0 & 1 & 3
\end{pmatrix}, \quad \begin{pmatrix}
0 & 2 & 2 \\
0 & 2 & 1 \\
0 & 1 & 0
\end{pmatrix} \]

| Method/fan | F-vector of output | Timing |
|------------|--------------------|--------|
| (1) secondary fan | 1 3048 38348 178426 407991 494017 304433 75283 | 506 s |
| (2) tropical resultant | 1 2324 26316 106083 197576 173689 58451 | 1238 s |
| (3) normal fan from simple descr. | 1 937 5257 11288 11572 5589 985 | 55 s |

In Example (c) we were not able to compute the secondary fan and the resultant fan with the secondary fan structure due to integer overflow in intermediate polyhedral computations. Gfan has been designed to work well for Gröbner fans, where the degrees of the polynomials is never very large, since that would prevent us from computing a single Gröbner basis anyway (except for binomial ideals). In Example
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(c), a primitive normal vector of a codimension 1 cone of the normal fan of the resultant is $(-32, 0, 32, 27, 0, -27, 25, -25, 0, -51, -51, -87, 0, 87)$, showing that the resultant has degree at least $32+27+25+51+87$. On such examples overflows typically arise when trying to convert an exactly computed rational generator of a ray to a primitive vector of 32 bit integers. Algorithm 4.1 will show similar behavior on other examples, for example when converting “$p$” to a vector of 32 bit integers. We intend to fix these implementation problems in the future.

**Hypersurfaces with Specialization.** If the specialized resultant is a hypersurface, then we can compute its tropical variety using the following methods.

1. Compute $TR_S(A)$ as a subfan of the restriction secondary fan to a subspace $U_S$ by fan traversal using the algorithms in Section 3.
   
   ```
gfan_resultantfan --vectorinput --special <tuple_and_spvec.txt
   ```

2. Compute the stable intersection $TR_S(A) = TR(A) \cap_{st} \{U_S\}$ as a union of cones, using the simple description from Theorem 2.9 and the characterization of stable intersections from Lemma 3.4. Then reconstruct the normal fan of the dual polytope using Algorithm 4.1.
   
   ```
gfan_resultantfan --vectorinput --special --projection <tuple_and_spvec.txt
   ```

3. Compute the specialized tropical resultant as a union of cones using stable intersection of hypersurfaces and tropical elimination theory as in Section 3.2 and reconstruct the normal fan of the dual polytope using Algorithm 4.1. We combine the commands (see also Ref. SY08):
   
   ```
gfan_tropicalstartingcone --stable >startingcone.txt
gfan_tropicaltraverse --stable <startingcone.txt >stable.fan
gfan_tropicalhypersurfacereconstruction --sum -i stable.fan <lnspc.txt
   ```

4. For a generic direction, Sturmfels’ formula [Stu94, Theorem 2.1] gives the optimal vertex of the resultant polytope in that direction, which can then be projected to get a point in the Newton polytope of the specialized resultant polynomial. This can be combined with the beneath-beyond convex hull algorithm for recovering the whole polytope. This method has recently been implemented as described in Ref. EFKP11. This is the implementation used in the timings below.

In Ref. EKP07, the authors proposed computing a silhouette or a projection of the secondary polytope. This is dual to computing the restriction of the secondary fan to a subspace. We provide the results and timings of this dual computation for comparison.

In the following examples the non-black columns are specialized.

**Example (d).**

\[
\mathcal{A} = \left( \begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{array} \right), \left( \begin{array}{ccc}
0 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & 2 \\
\end{array} \right), \left( \begin{array}{ccc}
0 & 1 & 1 \\
0 & 1 & 2 \\
1 & 0 & 1 \\
\end{array} \right)
\]

| Method/fan | F-vector | Timing |
|------------|----------|--------|
| Restriction of secondary fan | 1372 2514 5829 5661 1976 | 26 s |
| (1) traversing tropical resultant | 1126 476 561 212 | 14 s |
| (2) normal fan from stable intersection | 125 227 250 211 65 | 0.7 s |
| (3) normal fan from tropical elimination | 125 127 250 211 65 | 1.4 s |

In Ref. EKP07, the authors proposed computing a silhouette or a projection of the secondary polytope. This is dual to computing the restriction of the secondary fan to a subspace. We provide the results and timings of this dual computation for comparison.
Example (e).

\[ A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{pmatrix} \]

| Method/fan                  | F-vector | Timing |
|----------------------------|----------|--------|
| Restriction of secondary fan| 1 709 6955 24554 39464 30226 8870 | 116 s |
| (1) traversing tropical resultant | 1 469 3993 11296 12853 5040 | 320 s |
| (2) normal fan from stbl. inters. | 1 29 209 597 792 485 110 | 1.3 s |
| (3) normal fan from trop. elim. | 1 29 209 597 792 485 110 | 3.2 s |

Example (f).

\[ A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 2 \\ 0 & 2 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{pmatrix} \]

| Method      | F-vector | Timing |
|-------------|----------|--------|
| (2)         | 1 1566 19510 98143 265202 424620 413455 238425 73741 9156 | 798 s |
| (3)         | 1 1566 19510 98143 265202 424620 413455 238425 73741 9156 | 974 s |

Implicitization of hypersurfaces. Implicitization is a special case of the specialized resultants, and we compare the three methods as before.

Example (g). (Implicitization of a bicubic surface. Example 3.4 of Ref. [EK05])

\[ A = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 0 & 0 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix} \]

| Method/fan                  | F-vector | Timing |
|----------------------------|----------|--------|
| Restriction of secondary fan| 1 26 66 42 | 5 s |
| (1) traversing tropical resultant | 1 13 17 | 16 s |
| (2) normal fan from stable inters. | 1 5 9 6 | 171 s |
| (3) normal fan from tropical elim. | 1 5 9 6 | 0.4 s |
| (4) beneath-beyond | 1 5 9 6 | < 0.1 s |

As we saw in Corollary 3.19, removing the non-extreme monomials from the parameterizing polynomials does not change the resultant polytope, and in this example, this also does not change the restriction of the secondary fan. However, doing so speeds up the computations, as seen on the right most column.

Example (h). (Implicitization of a hypersurface in four dimensions)

\[ A = \begin{pmatrix} 0 & 0 & 2 & 4 \\ 0 & 2 & 4 & 1 \\ 0 & 2 & 4 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 4 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 2 & 3 & 4 \\ 0 & 4 & 0 & 1 \\ 0 & 2 & 4 & 2 \end{pmatrix} \times \begin{pmatrix} 0 & 0 & 4 & 4 \\ 0 & 2 & 2 & 3 \\ 0 & 4 & 2 & 3 \end{pmatrix} \]

| Method/fan                  | F-vector | Timing |
|----------------------------|----------|--------|
| (2) normal fan from stable intersection | 1 111 358 368 121 | 9 s |
| (3) normal fan from tropical elimination | 1 111 358 368 121 | 2.6 s |
| (4) beneath-beyond | 1 111 358 368 121 | 1.3 s |

For (3), computing the polytope from the tropical hypersurface using ray-shooting and beneath-beyond took 47 s in the TrIm implementation [SY08] using the library iB4e [Hug06] on a slightly slower machine.

Example (i). (Implicitization of a hypersurface in five dimensions)
\[ A = \left( \begin{array}{cccc} 0 & 1 & 3 & 4 \\ 0 & 1 & 4 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 4 & 0 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 0 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 3 & 1 \end{array} \right) \right) \]

| Method/fan | F-vector | Timing |
|------------|----------|--------|
| (2) normal fan from stable inters. | 1 5932 23850 35116 22289 5093 | 351 s |
| (3) normal fan from tropical elim. | 1 5932 23850 35116 22289 5093 | 184 s |
| (4) beneath-beyond | 1 5932 23850 35116 22289 5093 | 1241 s |

For (3), timing includes 17 seconds for computing the specialized tropical incidence variety. The normal fan reconstruction computation in TrIm with iB4e took 3375 seconds on a slightly slower machine.

**Non-hypersurfaces.** When \( R(A) \) is not a hypersurface, the only method we know for computing \( TR(A) \) with a fan structure without knowing the defining ideal is to traverse the secondary fan of Cay(\( A \)) and enumerating just the secondary cones whose RMS contains a fully mixed cell. There are other descriptions of tropical resultants as a set, such as Theorem 2.9, but none gives a fan structure.

**Example (j).**

\[ A = \left( \begin{array}{cccc} 0 & 1 & 3 & 4 \\ 0 & 1 & 4 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 2 & 4 & 0 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 0 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 3 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 3 & 1 \end{array} \right) \right) \]

| Method/fan | F-vector | Timing |
|------------|----------|--------|
| Secondary fan | 1 8876 72744 222108 322303 225040 60977 | 478 s |
| Traversing tropical result. | 1 968 4495 6523 3000 | 81 s |

We used, respectively, the commands:
gfan_secondaryfan <cayley.txt
gfan_resultantfan --vectorinput <tuple.txt

**Non-hypersurfaces with Specialization.** The only method here is to traverse \( TR_{S}(A) \) as a subfan of a restriction of the secondary fan using the algorithms in Section 3.

**Example (k).**

\[ A = \left( \begin{array}{cccc} 0 & 2 & 4 & 0 \\ 4 & 1 & 1 & 0 \\ 1 & 0 & 4 & 3 \\ 3 & 5 & 5 & 1 \end{array} \right), \quad \left( \begin{array}{cccc} 3 & 5 & 5 & 3 \\ 1 & 0 & 4 & 3 \\ 4 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \end{array} \right), \quad \left( \begin{array}{cccc} 3 & 4 & 5 & 0 \\ 1 & 5 & 2 & 4 \\ 0 & 1 & 2 & 3 \end{array} \right), \quad \left( \begin{array}{cccc} 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 1 & 4 & 1 \\ 0 & 1 & 3 & 1 \end{array} \right) \right) \]

| Method/fan | F-vector | Timing |
|------------|----------|--------|
| Restriction of secondary fan | 1 4257 23969 48507 42269 13467 1236 | 256 s |
| Traversing spec. tropical result. | 1 310 831 533 | 81 s |

We used, respectively, the commands:
gfan_secondaryfan --restrictingfan subspace.fan <cayley.txt
gfan_resultantfan --vectorinput --special <tuple_and_sv.txt

**5.1. Conclusion.** The new method of using adjacency decomposition with Algorithm 4.1 for constructing the normal fan of a polytope from it tropical hypersurface works very well in practice. Our implementation of it is much faster than any existing implementation of the beneath-beyond method with ray-shooting, and we think the gap will widen even more in higher dimension since this new method scales well – multi-linearly with respect to the number of cones in input and the number of vertices and edges of the output polytope, as shown in Proposition 4.2.
The normal fan reconstruction method can be used together with either the simple description of tropical resultants (Theorem 2.9) or tropical elimination (Section 3.2) for computing resultant polytopes efficiently. Traversing the (specialized) tropical resultant as a subfan of (a restriction of) the secondary fan of the Cayley configuration is combinatorially interesting but not computationally competitive.

For implicitization, the beneath-beyond method from Ref. [EFKP11] works faster than any of our “tropical” methods when output polytope is low dimensional, while our methods seem to have an advantage in higher dimension (5 or more). However, the method of Ref. [EFKP11] may have an advantage when there are many specialized points in the input configurations, as the number of cones in the tropical description increases rapidly. See the last problem in Section 6 below.

For resultant varieties of codimension higher than one, whether specialized or not, we only know of one method for computing the tropicalization as a fan, without knowing the defining polynomials, which is to traverse the secondary fan of the Cayley configuration or a restriction of it to a subspace.

6. Open problems

Combinatorial classification of resultant polytopes: For 1-dimensional point configurations, the combinatorics of the resultant polytope only depend on the (partial) order of the (not necessarily distinct) points in each \( A_i \) [GKZ94], so a combinatorial classification is easy to obtain. No such classification is known even for point configurations in \( \mathbb{Z}^2 \). A concrete problem is to classify 4 dimensional resultant polytopes combinatorially. This was done for 3 dimensional resultant polytopes by Sturmfels [Stu94], and only one dimensional point configurations were needed for this case. To understand the 4 dimensional resultant polytopes, we need to work with the case \( \mathcal{A} = (A_1, A_2, A_3) \) where each \( A_i \) consists of three points in \( \mathbb{Z}^2 \) that are not necessarily distinct. How can we stratify the space of tuples \( \mathcal{A} \)'s according to the combinatorial type of the resultant polytope?

Finding a point in the specialized tropical resultant: Is there a polynomial time algorithm for finding a generic vector \( \omega \in \mathbb{Q}(\varepsilon)^m \) in the specialized tropical resultant? For non-specialized tropical resultants, the polynomial time algorithm for codimension from Section 2.4 can also be used to find a generic point, by Theorem 2.9.

Improved description of specialized tropical resultants: Combining the descriptions of tropical resultants in Theorem 2.9 and stable intersections in Lemma 3.4, we get a specialized tropical resultant as a union of cones. In computations, we need to go through a list of \( \prod_{i=1}^{k} \binom{m_i}{2} \) choices of tuples of pairs from \( A_i \), many of which do not contribute to a facet of specialized tropical resultant. Give a combinatorial characterization for the choices of the tuples of pairs that contribute to a facet. Corollary 3.19 and Lemma 3.20 are results in this direction.

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