Velocity Polytopes of Periodic Graphs

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Abstract. A periodic graph in dimension \(d\) is a directed graph with a free action of \(\mathbb{Z}^d\) with only finitely many orbits. It can conveniently be represented in terms of an associated finite graph with weights in \(\mathbb{Z}^d\). Here we use the weight sums along cycles in this associated graph to construct a certain polytope in \(\mathbb{R}^d\), which we regard as a geometrical invariant associated to the periodic graph. It is the unit ball of a norm on \(\mathbb{R}^d\) describing the large-scale geometry of the graph. It has a physical interpretation as the set of attainable velocities of a particle on the graph which can hop along one edge per timestep. Since a polytope necessarily has distinguished directions, there is no periodic graph for which this velocity set is isotropic. In the context of classical physics, this can be viewed as a no-go theorem for the emergence of an isotropic space from a discrete structure.

1. Introduction

Periodic graphs are abstractions of the atomic structure of crystals. A crystal, by definition, is a material whose structure consists of a finite-size pattern which repeats periodically in all spatial directions. Taking the crystal atoms as the vertices of a graph and the chemical bonds as its edges, one obtains a graph which repeats periodically in all spatial directions: a periodic graph. This graph represents the chemical structure of the crystal. Therefore, the problem of classifying and enumerating all periodic graphs in three dimensions is of fundamental importance for crystallography [2, 8, 9, 19]. Periodic graphs have also been studied in operations research [16], spectral graph theory [5], and computer science [4]. Certain generalizations of periodic graphs also appear in topological graph theory [13, 14, 20].

Besides their natural appearance in all these fields, periodic graphs have recently also been suggested as candidates for the fundamental microscopic building blocks of space [6, 7]. In order to get some idea of how physics on a periodic graph can look like, let us consider a classical point particle moving on a periodic graph \(\Gamma \subseteq \mathbb{R}^d\) as follows: the particle moves along the vertices of \(\Gamma\) in discrete timesteps by hopping along one edge per timestep. More precisely, we define a trajectory to be a sequence \((f_n)_{n \in \mathbb{N}}\) of vertices \(f_n \in \Gamma\) such that for each time \(n \in \mathbb{N}\), the

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positions \( f_n \) and \( f_{n+1} \) are adjacent in \( \Gamma \). A vector \( u \in \mathbb{R}^d \) is then called a velocity vector of \( \Gamma \) if there is a trajectory \( f \) such that

\[
    u = \lim_{n \to \infty} \frac{f_n - f_0}{n}.
\]

Intuitively, this equation means that the trajectory’s apparent velocity on the macroscopic scale is given by \( u \). It is the trajectory’s velocity as seen by a macroscopic observer who is not aware of the fundamental discreteness of \( \Gamma \) and perceives space as a continuum \( \mathbb{R}^d \). Note that (1) only makes sense when the limit on the right-hand side exists, which can be interpreted as requiring the trajectory to have a well-defined constant macroscopic velocity.

Now a natural question is: given the periodic graph \( \Gamma \in \mathbb{R}^d \), what is the set of its velocity vectors? In particular, can this set be a Euclidean ball, thereby making the macroscopic observer perceive an isotropic space, such that the achievable absolute values \( ||u|| \) do not depend on the direction \( u/||u|| \)? This would be a very desirable property for the kind of models discussed e.g. in [6, 7].

Using concepts from the theory of periodic graphs, we will prove in Theorem 16 that the set of velocity vectors of any suitably connected periodic graph \( \Gamma \) is a convex polytope in \( \mathbb{R}^d \). In particular, it never is a Euclidean ball, and the set of achievable velocity vectors cannot be isotropic; see Theorem 25. This is a no-go theorem for the emergence of an isotropic space from a discrete structure within the context of classical physics.

From the mathematical point of view, our velocity polytopes are new invariants of periodic graphs. As witnessed by Proposition 19, they encode the periodic graph’s large-scale geometry. The velocity polytope as an invariant can be applied for example as in Corollary 24, which is a criterion for proving the non-existence of translation-invariant maps between periodic graphs ("morphisms").

2. Preliminaries

In this section, we collect some definitions and simple observations. Although most of the relevant literature is concerned with the case of undirected graphs [2, 3, 4], we work with directed graphs, which is more general and has turned out to be technically more convenient.

If \( A \) is a finite set, we write \( |A| \) for its cardinality.

2.1. Graphs and paths. For us, a graph is a directed graph which may have loops and multiple edges. A graph \( G \) is specified by a vertex set \( V_G \), an edge set \( E_G \), a source function \( s_G : E_G \to V_G \), and a target function \( t_G : E_G \to V_G \). We refrain from identifying an edge \( e \) with the vertex pair \( (s_G(e), t_G(e)) \), since typically there will be several edges between \( s_G(e) \) and \( t_G(e) \). When the graph \( G \) is clear from the context, we frequently omit the subscripts and simply write \( s, t : E \to V \) for the source and target maps in order to avoid unnecessary cluttering.

A path \( p \) in \( G \) is a finite sequence of edges \( p = e_1 \ldots e_n, e_i \in E \), such that \( t(e_i) = s(e_{i+1}) \) for all \( i = 1, \ldots, n - 1 \). The length \( |p| = n \) of \( p \) is its number of edges. The empty path \( \emptyset \) is the unique path of length 0.

A closed path is a non-empty path \( p = e_1 \ldots e_n \) such that \( s(e_1) = t(e_n) \). If no other additional vertex repetitions occur, then \( p \) is also called a cycle. By this definition, a cycle of length \( n \) traverses \( n \) distinct vertices.
If \( p = e_1 \ldots e_n \) and \( p' = e'_1 \ldots e'_m \) are paths such that \( t(e_n) = s(e'_1) \), then \( p \) and \( p' \) can be composed to a new path

\[
pp' \overset{\text{def}}{=} e_1 \ldots e_n e'_1 \ldots e'_m.
\]

It is clear that \(|pp'| = |p| + |p'|\).

A directed graph is said to be strongly connected if there is a path from \( v \) to \( w \) for any pair of vertices \((v, w)\).

**Lemma 1.** Let \( G \) be a finite directed graph.

(a) A path \( p = e_1 \ldots e_n \) in \( G \) of length \( n \geq |V_G| \) contains a cycle: there are indices \( k \) and \( l \) such that \( c = e_k e_{k+1} \ldots e_l \) is a cycle.

(b) There are only a finite number of cycles in \( G \).

**Proof.**

(a) By the pigeonhole principle, there have to be indices \( l > k \) with \( t(e_l) = s(e_k) \). Choosing \( l \) minimal with this property guarantees \( c = e_k e_{k+1} \ldots e_l \) to be a cycle.

(b) Since a cycle is defined as not having any vertex repetitions besides the coincidence between the initial and the final vertex, a cycle can have length at most \(|V_G|\). The conclusion follows since there are only a finite number of paths of length at most \(|V_G|\).

\(\square\)

### 2.2. Definition of periodic graphs

We now turn to the formal definition of periodic graphs before discussing their representation by finite weighted graphs. For examples, we refer to figure 1 for a basic example in two dimensions and to [2] for abundant visualizations of three-dimensional periodic graphs within the context of crystallography.

**Definition 2** (periodic graph). A \( d \)-dimensional periodic graph is a graph \( \Gamma \) equipped with a free action of the free abelian group \( \mathbb{Z}^d \) on \( \Gamma \), such that \( \Gamma \) has only finitely many \( \mathbb{Z}^d \)-orbits of vertices as well as edges.

Let us disentangle what this definition means. First of all, the graph \( \Gamma \) comes with an action of the group \( \mathbb{Z}^d \). In additive notation, this means that there are given maps

\[
V_{\Gamma} \times \mathbb{Z}^d \to V_{\Gamma}, \quad (v, x) \mapsto v + x,
\]

\[
E_{\Gamma} \times \mathbb{Z}^d \to E_{\Gamma}, \quad (e, x) \mapsto e + x.
\]

We think of the vertex \( v + x \) as the vertex \( v \) translated by the vector \( x \in \mathbb{Z}^d \), and similarly for \( e + x \). In order for these maps to form a \( \mathbb{Z}^d \)-action on \( \Gamma \), they need to satisfy the group action axioms

\[
(v + x) + y = v + (x + y) \quad \forall v \in V_{\Gamma}, x, y \in \mathbb{Z}^d, \quad v + 0 = v \quad \forall v \in V_{\Gamma},
\]

\[
(e + x) + y = e + (x + y) \quad \forall e \in E_{\Gamma}, x, y \in \mathbb{Z}^d, \quad e + 0 = e \quad \forall e \in E_{\Gamma},
\]

as well as be compatible with each other in the sense that source and target of the translate of an edge are precisely the translates of the source and target of the edge,
\[ s_\Gamma(e + x) = s_\Gamma(e) + x, \]
\[ t_\Gamma(e + x) = t_\Gamma(e) + x. \]  
(3)

Furthermore, the action of \( \mathbb{Z}^d \) on \( \Gamma \) should be free,
\[ v + x = v + x' \Rightarrow x = x', \quad e + x = e + x' \Rightarrow x = x'. \]  
(4)

Finally, there should be only finitely many orbits in \( V_\Gamma \) as well as in \( E_\Gamma \) under the \( \mathbb{Z}^d \)-action. This is an abstraction of the crystallographic property that a unit cell of a crystal contains only finitely atoms and chemical bonds.

**Remark 3.** Definition 2 is of an abstract combinatorial nature in the sense that no embedding of \( \Gamma \) into \( \mathbb{R}^d \) is required. However, suppose that \( \Gamma \subseteq \mathbb{R}^d \) is a Euclidean periodic graph which is translation-invariant in \( d \) linearly independent directions. After applying an appropriate affine transformation to \( \Gamma \), the unit cell of \( \Gamma \) can be taken to be the unit cube \([0,1]^d\), which means that \( \Gamma \) is translation-invariant under the group of integer translations \( \mathbb{Z}^d \subseteq \mathbb{R}^d \). With \( \mathbb{Z}^d \) acting on \( \Gamma \) by these translations, \( \Gamma \) is a \( d \)-dimensional periodic graph in the sense of definition 2.

There are other uses of the term “periodic graph” in the mathematical literature which are not related to the one used here. For example, a plot of a periodic functions is a “periodic graph” in a completely different sense. For another interesting notion of periodic graph which is in no way related to the present one, see [11].

### 2.3. Displacement graphs of periodic graphs.

By the freeness condition (4), a non-empty periodic graph is necessarily infinite. A convenient representation of a periodic graph in terms of a finite amount of data has been developed in a more general context in [12, 13], and probably independently in [3]. Due to the diversity of the literature spanning various fields of science, no universal terminology has been established. Here we partly try to follow the terminology of graph theory [14]. While this subsection contains standard material, we try to offer a slightly different point of view emphasizing the analogy to covering spaces [15, ch. 1.3].

Given a group action on some mathematical object, it is natural to consider the quotient object with respect to the group action. For a periodic graph \( \Gamma \), this means to identify two vertices (or edges) if they can be translated into each other by a group element \( x \in \mathbb{Z}^d \); in other words, if the two vertices (edges) lie in the same \( \mathbb{Z}^d \)-orbit. The resulting collection of vertex orbits \( V_\Gamma/\mathbb{Z}^d \) def = \( V_\Gamma/\mathbb{Z}^d \) and the collection of edge orbits \( E_\Gamma/\mathbb{Z}^d \) def = \( E_\Gamma/\mathbb{Z}^d \) form a quotient graph \( \Gamma/\mathbb{Z}^d \): by (3), the source and target functions \( s_\Gamma \) and \( t_\Gamma \) descend to well-defined maps
\[ s_{\Gamma/\mathbb{Z}^d}, \ t_{\Gamma/\mathbb{Z}^d} : E_{\Gamma/\mathbb{Z}^d} \longrightarrow V_{\Gamma/\mathbb{Z}^d}. \]

By the finiteness assumption of definition 2, the quotient graph \( \Gamma/\mathbb{Z}^d \) is finite.

**Remark 4.** In the particular case that \( \Gamma \subseteq \mathbb{R}^d \) is a translation-invariant Euclidean periodic graph, then one can construct \( \Gamma/\mathbb{Z}^d \) also by taking the vertices and edges in a unit cell of \( \Gamma \). Besides the edges inside the unit cell, each edge of \( \Gamma \) which connects a vertex inside the unit cell to a vertex outside the unit cell defines an
By definition of $\Gamma/\mathbb{Z}^d$, there is a canonical projection map $\phi_T : \Gamma \to \Gamma/\mathbb{Z}^d$ which maps every vertex and every edge to its $\mathbb{Z}^d$-orbit. When $\Gamma$ is clear from the context, we also simply write $\phi$ for $\phi_T$.

The map $\phi$ enjoys the nice property that an edge (or a path) in $\Gamma/\mathbb{Z}^d$ can be uniquely lifted to an edge (a path) in $\Gamma$, given that a starting vertex has been specified:

**Lemma 5.** (a) For every $e \in E_{\Gamma/\mathbb{Z}^d}$ and every $v^* \in \phi^{-1}(s(e))$, there is a unique $e^* \in E_{\Gamma}$ with $\phi(e^*) = e$ and $s(e^*) = v^*$.

(b) For every path $p = e_1 \ldots e_n$ in $\Gamma/\mathbb{Z}^d$ and every $v^* \in \phi^{-1}(s(e_1))$, there is a unique path $p^* = e_1^* \ldots e_n^*$ in $\Gamma$ with $\phi(p^*) = p$ and $s(e_1^*) = v^*$.

**Proof.** (a) Let $\tilde{e} \in E_{\Gamma}$ be some edge with $\phi(\tilde{e}) = e$. Then there is a unique $x \in \mathbb{Z}^d$ such that $s(\tilde{e}) + x = v^*$. Hence, $e^* \overset{\text{def}}{=} \tilde{e} + x$ has the desired properties. For uniqueness, suppose that $e' \in E_{\Gamma}$ would also satisfy $\phi(e') = e$ and $s(e') = v^*$. By definition of $\phi$, the relation $\phi(e^*) = e = \phi(e')$ means that $e^*$ and $e'$ lie in the same $\mathbb{Z}^d$-orbit, so that there exists $y \in \mathbb{Z}^d$ with $e' = e^* + y$. But then, $v^* = s(e') = s(e^*) + y = v^* + y$, which implies $y = 0$ by \eqref{eq:translation}. Hence $e' = e^*$.

(b) This follows from a successive application of part (a) to each edge in the path. $\square$

In the language of graph theory, we have found that the projection $\phi : \Gamma \to \Gamma/\mathbb{Z}^d$ is a covering of graphs [14, ch. 2], [1, ch. 17]. This is completely analogous to the notion of covering space in topology [15, ch. 1.3].

Unfortunately, knowing the quotient graph $\Gamma/\mathbb{Z}^d$ alone is not enough to reconstruct $\Gamma$. For example, there are many Euclidean periodic graphs $\Gamma \subseteq \mathbb{R}^d$ which contain only a single vertex per unit cell, so that $|V_{\Gamma/\mathbb{Z}^d}| = 1$. In this case, all edges of $\Gamma/\mathbb{Z}^d$ are loops. However, knowing $|E_{\Gamma/\mathbb{Z}^d}|$ as the number of these loops is certainly not enough to recover $\Gamma$. For example, it is unclear whether a loop of $\Gamma/\mathbb{Z}^d$ comes from an orbit of loops in $\Gamma$, or whether it represents a class of edges connecting different vertices in $\Gamma$.

The additional piece of data needed in order to recover $\Gamma$ turns out to consist of edge weights on $\Gamma/\mathbb{Z}^d$, which specify, intuitively speaking, the translation required in going from $s_T(e)$ to $t_T(e)$. These edge weights are known under various names — e.g. voltage assignments [12, 13], labels [3, 17], or simply weights [4]. We will prefer the term displacements, which we deem most appropriate given the geometric intuition. The formalism described in the following is closely analogous to gauge theory [18].

Defining displacements requires that one has chosen a vertex representative $t(v) \in V_{\Gamma}$ for every orbit $v \in V_{\Gamma/\mathbb{Z}^d}$. In other words, we fix a map $t : V_{\Gamma/\mathbb{Z}^d} \to V_{\Gamma}$ which is assumed to be a section of $\phi : V_{\Gamma} \to V_{\Gamma/\mathbb{Z}^d}$. For example for a Euclidean periodic graph $\Gamma \subseteq \mathbb{R}^d$, one possibility is to define $t$ by choosing a unit cell and mapping every orbit $v \in V_{\Gamma/\mathbb{Z}^d}$ to its representative in the unit cell.
Every \( v \in V_{\Gamma/Z^d} \) represents a whole \( Z^d \) worth of vertices of \( \Gamma \), namely \( \phi^{-1}(v) \). Given \( \iota \), we can choose a concrete identification of this \( \phi^{-1}(v) \) with \( Z^d \) by defining

\[
\alpha_v : Z^d \xrightarrow{\cong} \phi^{-1}(v), \quad x \mapsto \iota(v) + x.
\]

This map is compatible with translations in the sense that it satisfies the identity \( \alpha_v(x + y) = \alpha_v(x) + y \) for all \( x, y \in Z^d \).

Now for an edge orbit \( e \in E_{\Gamma/Z^d} \), lemma 5 provides an isomorphism \( \phi^{-1}(s(e)) \xrightarrow{\cong} \phi^{-1}(t(e)) \), which is also compatible with translations. In total, we obtain an isomorphism

\[
\gamma_e : \begin{array}{c}
Z^d \\
\xrightarrow{\alpha_{\iota(e)}}
\phi^{-1}(s(e)) \\
\xrightarrow{\cong} \\
\phi^{-1}(t(e)) \\
\xrightarrow{\alpha_{\iota(e)^{-1}}} \\
Z^d
\end{array}
\]

which is again compatible with translations, \( \gamma_e(x + y) = \gamma_e(x) + y \). Therefore, \( \gamma_e(x) = \gamma_e(0) + x \). We now define the displacement along \( e \) to be \( \delta(e) \) defined as \( \gamma_e(0) \). The equation \( \gamma_e(0) = \delta(e) \) expresses the intuition that \( \delta(e) \) is the physical displacement required in going from \( s_\Gamma(e') \) to \( t_\Gamma(e') \) for any \( e' \in \phi^{-1}(e) \).

It is not difficult to see that the quotient graph \( \Gamma/Z^d \) together with the displacement function \( \delta : E_{\Gamma/Z^d} \to Z^d \) is sufficient to recover \( \Gamma \). This works as in the following definition, which can also be regarded as a general scheme for constructing periodic graphs:

**Definition 6 ([13, 3, 4])**. A displacement graph \((G, \delta)\) is a finite graph \( G \) together with edge weights \( \delta : E_G \to Z^d \) (the displacements). Associated to \((G, \delta)\) is a periodic graph \( \Gamma \) given by

\[
V_\Gamma = \{(v, x) \mid v \in V_G, x \in Z^d\}
\]

\[
E_\Gamma = \{(e, x) \mid e \in E_G, x \in Z^d\}
\]

\[
s_\Gamma((e, x)) = (s_G(e), x), \quad t_\Gamma((e, x)) = (t_G(e) + \delta(e), x)
\]

**Remark 7.** Intuitively, \( \Gamma \) is constructed from \( G \) as follows: we start with the lattice \( Z^d \) and place at each point a copy of the vertex set \( V_G \). Each edge \( e \in E_G \) defines a \( Z^d \) worth of edges between the copies of \( s_G(e) \) and \( t_G(e) \), where in adding these edges we have to apply a translation by \( \delta(e) \) in the ambient \( Z^d \).

One needs to keep in mind that determining a displacement graph from a periodic graph \( \Gamma \) requires choosing a representative \( \iota(v) \in V_\Gamma \) for each \( \mathbb{Z}^d \)-orbit \( v \in V_{\Gamma/Z^d} \). How much does the displacement graph depend on the choice of \( \iota \)? Suppose we are given two different choices \( \iota, \iota' : V_{\Gamma/Z^d} \to V_\Gamma \). Then for any \( v \in V_{\Gamma/Z^d} \), there is a unique \( g(v) \in Z^d \) such that

\[
\iota'(v) = \iota(v) + g(v), \forall v \in V_{\Gamma/Z^d}.
\]

By (5), the corresponding displacements \( \delta \) and \( \delta' \) therefore differ by

\[
\delta'(e) = \delta(e) + g(s(e)) - g(t(e)), \forall e \in E_{\Gamma/Z^d}.
\]

**Example 8.** Consider the periodic graph illustrated in figure 1(a). Any one of the elementary parallelograms formed by the dashed lines can be taken as a unit cell. Choosing the two vertices inside such a unit cell defines \( \iota \) in terms of a representative of the “•” vertices which form a \( \mathbb{Z}^d \)-orbit, and a representative of the “■”
vertices which form another $\mathbb{Z}^d$-orbit. The associated displacement graph is shown in figure 1(b). One obtains the displacements of e.g. the edges going from $\bullet$ to $\blacklozenge$ by noting that there are three $\mathbb{Z}^d$-orbits of such edges in figure 1(a): one which stays inside the unit cell, corresponding to the displacement $(0, 0)$; one whose target is one cell away in the positive $y$-direction, having displacement $(0, 1)$; and one whose target is one cell away in the negative $x$-direction with displacement $(-1, 0)$.

If one would start from the displacement graph 1(b), one would probably draw its associated periodic graph as in figure 1(c), which is a different Euclidean embedding of the same periodic graph as in figure 1(a).

3. Velocity Polytopes

From now on, we assume $\Gamma$ to be a periodic graph equipped with a fixed choice of orbit representatives $\iota: V_{\Gamma/\mathbb{Z}^d} \to V_{\Gamma}$. If $p = e_1 \ldots e_n$ is a path in $\Gamma/\mathbb{Z}^d$, then by abuse of notation we define its displacement to be given by
\[
\delta(p) \overset{\text{def}}{=} \sum_i \delta(e_i),
\]
which is nicely coherent with the lifting properties of lemma 5: if each $e_i$ lifts to an edge which intuitively translates by $\delta(e_i)$, then the path $e_1 \ldots e_n$ should lift to a path which intuitively translates by $\sum_i \delta(e_i)$.

3.1. Velocity. We now formalize the concepts introduced in the introduction. For technical reasons, we formally define a trajectory as a sequence of edges rather than vertices:

**Definition 9.** A trajectory in $\Gamma$ is a sequence $(f_n)_{n \in \mathbb{N}}$ of edges $f_n \in E_{\Gamma}$ such that $s(f_{n+1}) = t(f_n)$ for all $n \in \mathbb{N}$.

Intuitively, a trajectory is nothing but an infinite path in $\Gamma$. Due to lemma 5, up to an overall translation a trajectory in $\Gamma$ is uniquely specified by the sequence of edges $\phi(f_n) \in E_{\Gamma/\mathbb{Z}^d}$, which are the images under the projection $\phi: \Gamma \to \Gamma/\mathbb{Z}^d$. In the following, we will abuse notation by also writing $f_n$ for $\phi(f_n)$.

By the definition (6), the displacement traversed by the trajectory $f$ between $n = n_1$ and $n = n_2$, i.e. along the path $f_{n_1} \ldots f_{n_2-1}$, is given by
\[
\sum_{k=n_1}^{n_2-1} \delta(f_k).
\]
Since this displacement gets traversed in $n_2 - n_1$ timesteps, it makes sense to define the velocity in that time interval to be given by the difference quotient
\[
\frac{\sum_{k=n_1}^{n_2-1} \delta(f_k)}{n_2 - n_1} \quad (7)
\]

The trajectory $f$ has a well-defined velocity if the limit
\[
u_f = \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \delta(f_k)}{n} \in \mathbb{R}^d
\]
exists. In this case, the difference quotient (7) also converges to $u_f$ for $n_2 \to \infty$ for any $n_1 \in \mathbb{N}$.

In the following, $\| \cdot \|$ will be a fixed but arbitrary norm on $\mathbb{R}^d$. 
Figure 1. A periodic graph in two different embeddings (a), (c), its associated displacement graph (b) and its velocity polytope (d). For more detail, see examples 8 and 13.

Lemma 10. The velocity $u_f$ of the trajectory $f$ does not depend on the particular choice of representatives $\iota : V_{\Gamma/\mathbb{Z}^d} \to V_{\Gamma}$ used for constructing the displacement function $\delta$, but only on $\Gamma$ and $f$ themselves.
Proof. Let $\iota, \iota' : V_{\Gamma/\mathbb{Z}^d} \to V_{\Gamma}$ be two choices of orbit representatives. It has been noted in section 2.3 that their displacement functions satisfy

$$\delta'(e) = \delta(e) + g(s(e)) - g(t(e))$$

for some appropriate function $g : V_{\Gamma/\mathbb{Z}^d} \to \mathbb{Z}^d$. Then the displacements associated to the path $f_1 \ldots f_n$ differ by

$$\delta'(f_1 \ldots f_n) = \delta(f_1 \ldots f_n) + \sum_{k=1}^{n} [g(s(f_k)) - g(t(f_k))].$$

(9)

Due to $s(f_{k+1}) = t(f_k)$, the sum is telescoping, so that

$$\delta'(f_1 \ldots f_n) = \delta(f_1 \ldots f_n) + g(s(f_1)) - g(t(f_n)).$$

Writing $C \overset{\text{def}}{=} \max_{v \in V} ||g(v)||$, we conclude that

$$\frac{1}{n} \sum_{k=1}^{n} \delta'(f_k) - \frac{1}{n} \sum_{k=1}^{n} \delta(f_k) \leq 2C$$

from which the assertion immediately follows by taking the limit $n \to \infty$. □

Note that the velocity of a finite path as in (7) is in general not well-defined in the sense of the lemma, but does depend on the choice of $\iota$.

Remark 11. When $\Gamma \subseteq \mathbb{R}^d$ is a Euclidean periodic graph, then we would like this notion of velocity to correspond to the usual one familiar from classical mechanics. As already mentioned in remark 3, we can always apply an affine transformation to $\Gamma$ such that the unit cell becomes the ordinary unit cube $[0,1]^d$. As a matter of bookkeeping, this also changes all velocity vectors by the same affine transformation.

We claim that if the unit cell of $\Gamma$ is the unit cube, then the definition (8) gives precisely the usual concept of velocity. To see this, let us choose the orbit representatives $\iota$ to be those in the unit cell, and write $M$ for the maximal distance between any two vertices in the unit cell. Then, the actual distance vector traversed along the path $f_1 \ldots f_n$ will differ from the displacement $\sum_k \delta(e_k)$ by at most $2M$. In the limit as $n \to \infty$, this is negligible, since all distances get divided by the total elapsed time $n$. This proves the claim.

As a trivial example, a constant trajectory has a velocity of 0. Similarly for any trajectory which stays in a bounded region in $\Gamma \subseteq \mathbb{R}^d$.

More non-trivial examples of velocities for not necessarily Euclidean $\Gamma$ are as follows: for a cycle $c = e_1 \ldots e_n$ in $\Gamma/\mathbb{Z}^d$, we define the basic velocity associated to $c$ to be given by

$$u_c = \frac{\delta(c)}{n} = \frac{\sum_{k=1}^{n} \delta(e_k)}{n}$$

This coincides with (7). Lemma 5 guarantees that $c$ lifts to a unique path in $\Gamma$ upon choosing an arbitrary vertex in $\phi^{-1}(s(e_1))$ as starting point. The trajectory defined by lifting a periodic traversal of $c$ from $\Gamma/\mathbb{Z}^d$ to $\Gamma$ has the basic velocity $u_c$ as its velocity.

One can regard the set of basic velocities as an invariant of the periodic graph:

Lemma 12. For a given cycle $c$, the basic velocity $u_c$ does not depend on the particular choice of $\iota : V_{\Gamma/\mathbb{Z}^d} \to V_{\Gamma}$ used in constructing the displacement function $\delta$. 
Proof. Applying equation (9) in this case, one finds that all terms in the sum cancel each other, so that \( \delta'(c) = \delta(c) \). \( \square \)

Since by lemma 1.(b) there are only a finite number of cycles in \( \Gamma / \mathbb{Z}^d \), there is only a finite number of basic velocities for fixed \( \Gamma \).

Example 13. We go back to the periodic graph illustrated in figure 1. There are 9 cycles in the displacement graph of figure 1(b) which have “●” as their starting vertex, all of length 2; the other 9 cycles with “●” as their starting vertex have the same basic velocities, so it is sufficient to consider the former. The basic velocities are

\[
\begin{align*}
(0,0) + (0,0) & \quad \frac{2}{2}, & (0,0) + (0,-1) & \quad \frac{2}{2}, & (0,0) + (1,0) & \quad \frac{2}{2}, \\
(0,1) + (0,0) & \quad \frac{2}{2}, & (0,1) + (0,-1) & \quad \frac{2}{2}, & (0,1) + (1,0) & \quad \frac{2}{2}, \\
(-1,0) + (0,0) & \quad \frac{2}{2}, & (-1,0) + (0,-1) & \quad \frac{2}{2}, & (-1,0) + (1,0) & \quad \frac{2}{2}.
\end{align*}
\]

The nonzero ones are depicted in figure 1(d).

Now that we have seen some examples of velocities, a natural question to ask is the following:

Question 14. Given a periodic graph \( \Gamma \), what is the set of its velocities?

3.2. Main theorem. We now proceed to answer question 14 and give an explicit description of the set of velocities of \( \Gamma \) as a subset of \( \mathbb{R}^d \). We still take the periodic graph \( \Gamma \) with its associated displacement graph \( (\Gamma / \mathbb{Z}^d, \delta) \) fixed. As before, \( || \cdot || \) will be a fixed but arbitrary norm on \( \mathbb{R}^d \). The constant

\[ C \overset{\text{def}}{=} \max_{e \in E} ||\delta(e)|| \]

will be of some use. \( |V| \) will always stand for \( |V_{\Gamma / \mathbb{Z}^d}| \).

Lemma 15. For every path \( p \) in \( \Gamma / \mathbb{Z}^d \), there are cycles \( c_1, \ldots, c_k \) in \( \Gamma / \mathbb{Z}^d \) such that

\[ \left| \delta(p) - \sum_{i=1}^{k} \delta(c_i) \right| < C |V| \quad \text{and} \quad 0 \leq |p| - \sum_{i=1}^{k} |c_i| \leq |V|. \]

Proof. For \( |p| < |V| \), there is nothing to prove since one can just take \( k = 0 \), i.e. the sum over cycles to be empty. For \( |p| \geq |V| \), we use induction on \( |p| \). By lemma 1.(a), the path \( p \) can be written in the form

\[ p = p_0 cp_1 \]

where \( c \) is a cycle, so that the paths \( p_0 \) and \( p_1 \) can be composed to \( p' = p_0 p_1 \). Since \( |c| \geq 1 \), we conclude \( |p'| = |p_0| + |p_1| < |p| \), so that an application of the induction assumption to \( p' \) gives cycles \( c_2, \ldots, c_k \) with

\[ \left| \delta(p') - \sum_{i=2}^{k} \delta(c_i) \right| < C |V| \quad \text{and} \quad 0 \leq |p'| - \sum_{i=2}^{k} |c_i| \leq |V|. \]
The conclusion follows by setting $c_1 = c$ and observing $\delta(p) = \delta(c_1) + \delta(p')$ and $|p| = |c_1| + |p'|$. □

**Theorem 16** (Main theorem). Let $\Gamma$ be such that $\Gamma/\mathbb{Z}^d$ is strongly connected. Then the set of velocities of $\Gamma$ coincides with the convex hull

$$P_\Gamma \overset{\text{def}}{=} \text{conv} \{ u_f \mid u_f \text{ basic velocity in } \Gamma/\mathbb{Z}^d \}.$$  \hspace{1cm} (10)

In particular, $P_\Gamma$ is a rational polytope in $\mathbb{R}^d$, the velocity polytope of $\Gamma$.

We now give an outline of the proof before diving into the details. The idea is as follows: if $c_1$ and $c_2$ are cycles in $\Gamma/\mathbb{Z}^d$ with the same initial vertex, then the closed path $c_1c_2$ has a velocity which is a convex combination of the basic velocities associated to $c_1$ and $c_2$. An analogous statement holds for longer combinations of cycles.

So in order to show that the velocity of a trajectory is always a convex combination of basic velocities, one can apply lemma 15 in order to decompose the trajectory into cycles, noting that the right-hand side becomes irrelevant in the limit.

Conversely, for every convex combination of basic velocities one needs to construct a trajectory which has this velocity. By choosing the number of times that each cycle appears in a closed path, one can adjust the coefficients of the convex combination which corresponds to the velocity of (the lift of) that closed path. Therefore, one can try to combine the cycles such that they appear in the trajectory with the appropriate frequencies, while also inserting some auxiliary paths which connect between cycles with different starting vertices.

**Proof of Theorem 16.** Working with the displacement graph $(\Gamma/\mathbb{Z}^d, \delta)$ instead of the periodic graph $\Gamma$, we begin by showing that any velocity lies in the convex hull of the basic velocities. We first claim that for any trajectory $f$ and any $n \in \mathbb{N}$, the quotient

$$w_n = \frac{\sum_{i=1}^{n} \delta(f_i)}{n}$$

has the property that there is a vector $u$ in the convex hull of the basic velocity vectors such that

$$||w_n - u|| \leq \frac{2 |V| C}{n}$$  \hspace{1cm} (11)

To this end, we first approximate the path $f_1 \ldots f_n$ as in lemma 15 by cycles $c_1, \ldots, c_k$, so that

$$\left| \sum_{i=1}^{n} \delta(f_i) - \sum_{j=1}^{k} \delta(c_j) \right| < C |V| \quad \text{and} \quad 0 \leq n - \sum_{j=1}^{k} |c_j| \leq |V|. \hspace{1cm} (12)$$

The first inequality of these two can be rewritten as

$$\left| w_n - \sum_{j} \frac{|c_j|}{n} \cdot \frac{\delta(c_j)}{|c_j|} \right| < \frac{C |V|}{n} \hspace{1cm} (13)$$

Since the fractions $\delta(c_j)/|c_j|$ are basic velocities, the expression

$$u \overset{\text{def}}{=} \sum_{j} \frac{|c_j|}{\sum_{j} |c_j|} \cdot \frac{\delta(c_j)}{|c_j|}$$
is a convex combination of basic velocities; by the second inequality of (12), this approximates well the term \( \sum_j \frac{|c_j|}{n} \frac{\delta(c_j)}{|c_j|} \) appearing in (13). More precisely,

\[
\|w_n - u\| = \left\| w_n - \frac{\sum_i |c_i|}{n} u - \left( 1 - \frac{\sum_i |c_i|}{n} \right) u \right\| < \frac{C|V|}{n} + \left( 1 - \frac{\sum_i |c_i|}{n} \right) u
\]

Since \( \|u\| \leq C \) and \( n - \sum_i |c_i| \leq |V| \), we can also bound the second term on the right-hand side by \( C|V|/n \), which proves the claim (11).

It follows from (11) that when the trajectory has a well-defined velocity \( \lim_n w_n \), then the distance from this velocity to the convex hull of basic velocities is smaller than \( 2|V|C/n \) for any \( n \). Since that convex hull is a polytope and therefore closed, the limit \( \lim_n w_n \) is itself in the convex hull of basic velocities.

Conversely, it has to be shown that any convex combination of basic velocities

\[
\sum_{i=1}^r \lambda_i \frac{\delta(c_i)}{|c_i|}
\]

for weights \( \lambda_1, \ldots, \lambda_r \geq 0 \) with \( \sum_i \lambda_i = 1 \) and cycles \( c_i \) can be realized by a trajectory. In order to construct such a trajectory, let us choose natural numbers \( \alpha_{ik} \) giving rational approximations to the numbers \( \lambda_i/|c_i| \) as

\[
\alpha_{ik} \overset{\text{def}}{=} \left\lfloor k \cdot \frac{\lambda_i}{|c_i|} \right\rfloor \quad \text{so that} \quad \left| \alpha_{ik} \frac{\lambda_i}{|c_i|} - \frac{\lambda_i}{|c_i|} \right| < \frac{1}{k} \quad \forall i, k \in \mathbb{N}.
\]

Let us also choose paths \( p_1, \ldots, p_r \) in \( \Gamma/\mathbb{Z}^d \) such that \( |p_i| < V \) and \( p_i \) connects the ending vertex of the cycle \( c_i \) to the starting vertex of the cycle \( c_{i+1} \) (with \( c_{r+1} = c_1 \), such that \( p_r \) connects \( c_r \) to \( c_1 \)). Such a choice of paths exists due to the assumption of strong connectivity. Then the building blocks of the trajectory are going to be the paths

\[
q_k \overset{\text{def}}{=} c_1 \cdots c_1 p_1 c_2 \cdots c_2 p_2 \cdots c_r \cdots c_r p_r.
\]

Each \( q_k \) is a closed path in \( \Gamma/\mathbb{Z}^d \) in the sense that its starting vertex coincides with its ending vertex. Its length can be estimated as

\[
|q_k| = \sum_{i=1}^r \alpha_{ik} |c_i| + \sum_{i=1}^r |p_i| = \sum_{i=1}^r \left( k \cdot \frac{\lambda_i}{|c_i|} \cdot |c_i| + O(1) \right) = k + O(1),
\]

where \( O(1) \) refers to a term which does not depend on \( k \). This implies the similar estimate \( ||\delta(q_k)|| \leq O(k) \).

The trajectory \( f \) is defined to be (the lift to \( \Gamma \)) of the infinite path

\[
f : q_1q_2q_3q_4q_1 \ldots q_k \ldots
\]

It needs to be shown that this trajectory has a velocity which equals (14). We consider this trajectory up to a timestep \( n \), i.e. the path \( f_1 \ldots f_n \). We choose \( k \) (as a function of \( n \)) such that the path \( f_1 \ldots f_n \) has already concluded the whole \( q_k \)-segment from (18), but not yet the whole \( q_{k+1} \)-segment. Then due to (17), the \( q_{k+1} \)-segment will contribute to the time \( n \) and the displacement \( \delta \) by at most \( O(k^2) \). Hence the total displacement up to \( n \) timesteps is given by

\[
\sum_{i=1}^n \delta(f_i) = \sum_{m=1}^k m \cdot \delta(q_m) + O(k^2) \quad (16) = \sum_{m=1}^k \sum_{i=1}^r m (\alpha_{im} \delta(c_i) + \delta(p_i)) + O(k^2)
\]
Using condition (15) in the form \( |\alpha_{im} - \lambda_i m | c_i | \leq O(1) \) evaluates this to
\[
\sum_{i=1}^{n} \delta(f_i) = \sum_{m=1}^{k} \sum_{i=1}^{r} m \left( \frac{\lambda_i m c_i}{|c_i|} \delta(c_i) + O(1) \right) + O(k^2) = \frac{k^3}{3} \sum_{i=1}^{r} \lambda_i \frac{\delta(c_i)}{|c_i|} + O(k^2)
\]
On the other hand, the number \( n \) of edges traversed, which equals the time elapsed, can be evaluated very similarly as
\[
n = \sum_{m=1}^{k} m \cdot |q_m| + O(k^2) = \sum_{m=1}^{k} \sum_{i=1}^{r} m (\alpha_{im} c_i + |p_i|) + O(k^2)
\]
\[
= \sum_{m=1}^{k} \sum_{i=1}^{r} m (\lambda_i m + O(1)) + O(k^2) = \frac{k^3}{3} \sum_{i=1}^{r} \lambda_i + O(k^2) = \frac{k^3}{3} + O(k^2).
\]
Hence the velocity of the trajectory is given by
\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{\delta(f_i)}{n} = \lim_{k \to \infty} \frac{k^3}{3} \sum_{i=1}^{r} \lambda_i \frac{\delta(c_i)}{|c_i|} + O(k^2) = \sum_{i=1}^{r} \lambda_i \frac{\delta(c_i)}{|c_i|},
\]
as desired. \( \square \)

**Example 17.** For the periodic graph of figure 1(a) and 1(c), the velocity polytope is the hexagon in figure 1(d).

We now consider what happens when the connectedness assumption of theorem 16 is dropped.

**Proposition 18.** Let \( \Gamma \) be any periodic graph. Let \( \Gamma_1/\mathbb{Z}^d, \ldots, \Gamma_c/\mathbb{Z}^d \) be the strongly connected components of \( \Gamma/\mathbb{Z}^d \) which have preimages \( \Gamma_1, \ldots, \Gamma_c \subseteq \Gamma \). In this case, the set of velocities of \( \Gamma \) is the union of polytopes
\[
P_{\Gamma} = P_{\Gamma_1} \cup \ldots \cup P_{\Gamma_c}.
\]  

**Proof.** The \( \Gamma_i \) are defined as the preimages under \( \phi : \Gamma \to \Gamma/\mathbb{Z}^d \) of the strongly connected components of \( \Gamma/\mathbb{Z}^d \). Then for every trajectory \( (f_n)_{n \in \mathbb{N}} \), there is \( n_0 \in \mathbb{N} \) such that all \( f_n \) for \( n \geq n_0 \) lie in the same \( \Gamma_i \). The velocity of \( \Gamma \) therefore lies in the corresponding velocity polytope \( P_{\Gamma_i} \). This proves the “\( \subseteq \)” inclusion of (19). The “\( \supseteq \)” inclusion is clear since any trajectory in \( \Gamma_i \) is also a trajectory in \( \Gamma \). \( \square \)

4. **The large-scale geometry of a periodic graph**

What we mean by the large-scale geometry of a strongly connected periodic graph \( \Gamma \) is the following. There is a natural notion of distance between two vertices defined to be the length of the shortest path connecting the two vertices. This defines a metric \( d(\cdot, \cdot) \) on \( \Gamma \) invariant under the action of \( \mathbb{Z}^d \). Given any \( x \in \mathbb{Z}^d \) and any vertex \( v \in V_{\Gamma} \), we can now define the \( \Gamma \)-norm \( ||x||_{\Gamma} \) to be given by
\[
||x||_{\Gamma} \overset{\text{def}}{=} \lim_{n \to \infty} \frac{d(v + nx, v)}{n}.
\]  

The existence of the limit is guaranteed by the triangle inequality and Fekete’s lemma. Taking the limit instead of defining \( ||x||_{\Gamma} \) to be \( d(v, v+x) \) itself is necessary in order guarantee that \( ||x||_{\Gamma} \) does not depend on \( v \). This well-definedness of (20) easily follows from arguments very similar to those made in the proof of lemma 10.
Moreover, it is simple to show that \( \|mx\|_\Gamma = m\|x\|_\Gamma \) for any \( m \in \mathbb{N} \). The norm \( \|x\|_\Gamma \) can be interpreted as follows: for any \( v \in \Gamma \), one needs to traverse \( n \cdot \|x\|_\Gamma + O(1) \) edges in order to get from \( v \) to \( v + xn \).

From the triangle inequality \( \|x + y\|_\Gamma \leq \|x\|_\Gamma + \|y\|_\Gamma \) and \( \|mx\|_\Gamma = m\|x\|_\Gamma \) for \( m \in \mathbb{N} \), one deduces that \( \| \cdot \|_\Gamma \) extends to a unique norm on \( \mathbb{R}^d \).

**Proposition 19.** Let \( \Gamma \) be strongly connected. Then \( P_\Gamma \) is the unit ball of \( \| \cdot \|_\Gamma \).

**Proof.** We first show that \( P_\Gamma \) is contained in the unit ball. To this end, it is enough to prove \( \|u_c\|_\Gamma \leq 1 \) for a basic velocity \( u_c \) associated to a cycle \( c = e_1 \ldots e_k \) in \( \Gamma/\mathbb{Z}^d \). Choosing any \( v \in \phi^{-1}(s(e_1)) \), the cycle \( c \) lifts to a path in \( \Gamma \) from \( v \) to \( v + \delta(c) \). Since this path has length \( k \), we get

\[
\|u_c\|_\Gamma = \left\| \frac{\delta(c)}{k} \right\|_\Gamma \leq \frac{1}{k} \lim_{n \to \infty} \frac{d(v, v + n\delta(c))}{n} \leq \frac{1}{k} d(v, v + \delta(c)).
\]

There is a path from \( v \) to \( v + \delta(c) \) of length \( k \), which implies \( d(v, v + \delta(c)) \leq k \). This results in the desired inequality \( \|u_c\|_\Gamma \leq 1 \).

Conversely, it has to be shown that \( \|x\|_\Gamma \leq k \) for \( x \in \mathbb{Z}^d \) and \( k \in \mathbb{N} \) implies that \( \frac{x}{k} \in P_\Gamma \). Fix \( \varepsilon > 0 \). By assumption, there is \( v \in \Gamma \) and \( n \in \mathbb{N} \) such that

\[
k - \varepsilon \leq \frac{d(v, v + nx)}{n} \leq k + \varepsilon.
\]

This means that there exists a path \( p \) from \( v \) to \( v + nx \) of length between \( nk - n\varepsilon \) and \( nk + n\varepsilon \). Since the final vertex of \( p \) is a translate of its starting vertex, \( p \) can be concatenated with its own translates in order to form a trajectory which periodically traverses translates of \( p \). Since \( \delta(p) = nx \), the velocity of this trajectory is given by \( nx/|p| \), so that

\[
\frac{x}{k} \cdot \frac{kn}{|p|} \in P_\Gamma.
\]

In terms of an arbitrary norm \( \| \cdot \| \) on \( \mathbb{R}^d \), the distance of \( \frac{x}{k} \) to \( P_\Gamma \) can therefore be bounded by

\[
\left\| \frac{x}{k} - \frac{x}{k} \cdot \frac{kn}{|p|} \right\| = \frac{|x|}{k} \cdot \frac{1}{|p|} \cdot \|p| - nk\| \leq \frac{|x|}{k} \cdot \frac{\varepsilon}{k - \varepsilon}.
\]

Since this vanishes as \( \varepsilon \to 0 \), we conclude that \( \frac{x}{k} \in P_\Gamma \) from closedness of \( P_\Gamma \). \( \square \)

**5. Properties of velocity polytopes**

We now study some basic properties of velocity polytopes.

In section 3, we have assigned a velocity polytope to each (reasonably connected) periodic graph. By remark 11, for a Euclidean periodic graph \( \Gamma \subseteq \mathbb{R}^d \), this polytope describes the set of achievable velocities of a classical particle hopping along the edges of \( \Gamma \). In order to get interesting models for physics, the velocity polytope should be not too different from a Euclidean ball, thereby making the set of achievable velocities close to isotropic. This raises the question, which rational polytopes can be realized as velocity polytopes of periodic graphs? This is not difficult to answer:

**Proposition 20.** Every rational polytope arises as the velocity polytope of an appropriate periodic graph \( \Gamma \).
VELOCITY POLYTOPES

Theorem 16. Let $P \subseteq \mathbb{R}^d$ be a non-empty polytope with rational vertices $w_1, \ldots, w_m \in \mathbb{Q}^d$. Let $\gamma$ be the least common multiple of the denominators of the $w_i$, so that $\gamma w_i \in \mathbb{Z}^d$ for all $i$. Then $P$ can be realized as a velocity polytope as follows: let us construct a displacement graph on vertices $u_1, \ldots, u_\gamma$ such that there is a single edge each with displacement $\delta = 0$ from the vertex $u_j$ to the vertex $u_{j+1}$ for each $j = 1, \ldots, \gamma - 1$, and additional edges $e_1, \ldots, e_m$ from $u_\gamma$ to $u_1$ with displacements $\delta(e_i) \overset{\text{def}}{=} w_i$.

This defines a strongly connected displacement graph. Its basic velocities are precisely all the $\frac{w_i}{\gamma}$. Therefore, the associated periodic graph has $P$ as its velocity polytope.

In this paper, we have been considering the general case of directed graphs; since every undirected graph can be made into a directed graph by replacing an undirected edge by a pair of oppositely oriented directed edges, the theory also applies to undirected graphs. In an undirected graph, every path can be reversed, which reverses the sign of its velocity. Therefore, it should not be surprising that the following holds:

Proposition 21. When $\Gamma$ is undirected, then $P_\Gamma$ is symmetric around the origin.

Proof. The edges of $\Gamma$ come in parallel pairs of opposite orientation. Therefore, the edges of $\Gamma/\mathbb{Z}^d$ also come in parallel pairs with opposite orientation and displacement of the opposite sign. Hence for every cycle $c$ in $\Gamma/\mathbb{Z}^d$, there is a cycle $c'$ which corresponds to traversing $c$ backwards by using all the "partner" edges. Therefore, if $u_c$ is a basic velocity, then so is $-u_c$. Now the statement follows from theorem 16 and proposition 18.

Proposition 22 (Connectedness). If $\Gamma$ itself is strongly connected, then $P_\Gamma \subseteq \mathbb{R}^d$ is full-dimensional and $0 \in P_\Gamma$ is an interior point.

Proof. This is best proven without appealing to theorem 16. Let $\epsilon_1, \ldots, \epsilon_d \in \mathbb{Z}^d$ be the standard unit vectors, and $v \in \Gamma$ a fixed starting vertex. Then by the assumption of strong connectedness of $\Gamma$, there is a path $p_1^+$ in $\Gamma$ from $v$ to $v + \epsilon_i$. In $\Gamma/\mathbb{Z}^d$, this is a closed path of displacement $\epsilon_i$ and velocity $\frac{\epsilon_i}{|p_1^+|}$. We can use translates of $p_1^+$ to connect $v + n \epsilon_i$ to $v + (n + 1) \epsilon_i$ for any $n \in \mathbb{N}$. Sequentially traversing these translates of $p_1^+$ defines a trajectory with velocity $\frac{\epsilon_i}{|p_1^+|}$. Similarly, there exists a path $p_1^-$ from $v$ to $v - \epsilon_i$, which gives rise to a trajectory with velocity $-\frac{\epsilon_i}{|p_1^-|}$. The convex hull of these $2d$ velocities is a subset of $P_\Gamma$. By construction, this subset is full-dimensional and includes the origin as an interior point, and so the same also holds for $P_\Gamma$.

The converse is not true: for example, in dimension $d = 1$ we can take $\Gamma/\mathbb{Z}^d$ to consist of a single vertex with a loop $\epsilon_+$ of displacement $\delta(\epsilon_+) = 2$ and a loop $\epsilon_-$ of displacement $\delta(\epsilon_-) = -2$, which makes $\Gamma/\mathbb{Z}^d$ strongly connected and gives $P_\Gamma = [-2, 2]$, although the associated periodic graph $\Gamma$ has two connected components.

So far, we have been talking about the velocity polytope of a single periodic graph $\Gamma$. But given two periodic graphs $\Gamma$ and $\Gamma'$, how do their velocity polytopes relate? In order to find some relation between $P_\Gamma$ and $P_{\Gamma'}$, one needs to assume
a relation between $\Gamma$ and $\Gamma'$. One such notion is that of a morphism $h : \Gamma \to \Gamma'$ between periodic graphs $\Gamma$ and $\Gamma'$ of the same dimension $d$, by which we mean maps
\[ h_V : V_\Gamma \to V_{\Gamma'}, \quad h_E : E_\Gamma \to E_{\Gamma'}, \]
which are compatible with the graph structures,
\[ s_{\Gamma'} \circ h_E = h_V \circ s_\Gamma, \quad t_{\Gamma'} \circ h_E = h_V \circ t_\Gamma, \]
and the $\mathbb{Z}^d$-action,
\[ h_V(v + x) = h_V(v) + x, \quad h_E(e + x) = h_E(e) + x \quad \forall v \in V_\Gamma, e \in E_\Gamma, x \in \mathbb{Z}^d. \quad (21) \]
A morphism of periodic graphs induces a simple relationship between the velocity polytopes:

**Proposition 23** (Functoriality). Let $h : \Gamma \to \Gamma'$ be a morphism of periodic graphs. Then
\[ P_\Gamma \subseteq P_{\Gamma'}. \]

**Proof.** We consider how $h$ operates on the quotient graphs $\Gamma/\mathbb{Z}^d$ and $\Gamma'/\mathbb{Z}^d$. Since by (21) the assignment $h_V$ operates on the quotient graphs $\Gamma/\mathbb{Z}^d$ and $\Gamma'/\mathbb{Z}^d$, we get induced maps $\hat{h}_V : V_{\Gamma/\mathbb{Z}^d} \to V_{\Gamma'/\mathbb{Z}^d}$ and $\hat{h}_E : E_{\Gamma/\mathbb{Z}^d} \to E_{\Gamma'/\mathbb{Z}^d}$. Moreover, the orbit representatives $\iota : V_{\Gamma/\mathbb{Z}^d} \to V_\Gamma$ and $\iota' : V_{\Gamma'/\mathbb{Z}^d} \to V_{\Gamma'}$ can be chosen such that $h_V$ maps representatives to representatives in the sense that $h_V \circ \iota = \iota' \circ \hat{h}_V$. In total, there is a commutative diagram
\[
\begin{array}{ccc}
\Gamma/\mathbb{Z}^d & \xrightarrow{\hat{h}_V} & \Gamma'/\mathbb{Z}^d \\
\iota \downarrow & & \iota' \downarrow \\
\Gamma/\mathbb{Z}^d & \xrightarrow{h_V} & \Gamma/\mathbb{Z}^d \\
\phi \downarrow & & \phi' \downarrow \\
\Gamma/\mathbb{Z}^d & \xrightarrow{\hat{h}_V} & \Gamma'/\mathbb{Z}^d
\end{array}
\]
such that the vertical compositions are identities. Then it follows that the induced map $\hat{h}_E : E_{\Gamma/\mathbb{Z}^d} \to E_{\Gamma'/\mathbb{Z}^d}$ is compatible with the displacements, $\delta'(\hat{h}(e)) = \delta(e)$ for all edges $e \in E_{\Gamma'/\mathbb{Z}^d}$.

To prove the assertion, we now show that every velocity of $\Gamma$, associated to a trajectory $(f_n)_{n \in \mathbb{N}}$, is also a velocity in $\Gamma'$. But this follows from
\[ \lim_{n \to \infty} \frac{\sum_{k=1}^n \delta(f_k)}{n} = \frac{\sum_{k=1}^n \delta'(\hat{h}(f_k))}{n}, \]
so that $(\hat{h}(f_n))_{n \in \mathbb{N}}$ is a trajectory in $\Gamma'$ with the same velocity. \qed

This result can be applied for example as follows:

**Corollary 24.** If $P_\Gamma \nsubseteq P_{\Gamma'}$, then there is no morphism $h : \Gamma \to \Gamma'$.

**5.1. A no-go theorem for digital physics.** We end the paper by concluding from theorem 16 and proposition 18 a no-go theorem for the construction of models in “digital physics” [10].

**Theorem 25.** There is no periodic graph $\Gamma$ for which the set of macroscopic velocities achievable by a classical point particle hopping along the edges is isotropic.
In order to be isotropic, the set of velocities would have to be a Euclidean ball. However, a Euclidean ball is not a polytope or a union of a finite number of polytopes.

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