The isotope effect in impure high $T_c$ superconductors.

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The influence of various kinds of impurities on the isotope shift exponent $\alpha$ of high temperature superconductors has been studied. In these materials the dopant impurities, like Sr in $La_{2-x}Sr_xCuO_4$, play different role and usually occupy different sites than impurities like Zn, Fe, Ni etc intentionally introduced into the system to study its superconducting properties. In the paper the in-plane and out-of-plane impurities present in layered superconductors have been considered. They differently affect the superconducting transition temperature $T_c$. The relative change of isotope shift coefficient, however, is an universal function of $T_c/T_{c0}$ ($T_{c0}$ refers to impurity free system) i.e. for angle independent scattering rate and density of states function it does not depend whether the change of $T_c$ is due to in- or out-of-plane impurities. The role of the anisotropic impurity scattering in changing oxygen isotope coefficient of superconductors with various symmetries of the order parameter is elucidated. The comparison of the calculated and experimental dependence of $\alpha/\alpha_0$, where $\alpha_0$ is the clean system isotope shift coefficient, on $T_c/T_{c0}$ is presented for a number of cases studied. The changes of $\alpha$ calculated within stripe model of superconductivity in copper oxides resonably well describe the data on $La_{1.8}Sr_{0.12}CuO_4$, without any fitting parameters.

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I. INTRODUCTION

It is hard to overestimate the role played by the impurities introduced into otherwise clean superconductor. In response to impurities the superconducting properties of the material change. The changes include superconducting transition temperature, slope and jump of the specific heat, upper critical field, superfluid density, and other thermodynamic and electromagnetic characteristics 1,2.

One of the parameters of great experimental and theoretical importance is the isotope coefficient $\alpha$ defined by the power law dependence of superconducting transition temperature $T_c$ on the isotope mass $M$ of the element: $T_c \propto M^{-\alpha}$ or $\alpha = -\partial \ln T_c / \partial \ln M$. Illegal variable name. In the BCS model of superconductivity it has been predicted to take on universal value $\alpha_{BCS} = \frac{3}{8}$ and verified experimentally for a number of superconducting elements and simple compounds. In chemically complex multicomponent systems one usually defines the partial coefficients $\alpha_i = -\partial \ln T_c / \partial \ln M_i$, where $M_i$ is the isotope mass of the $i$-th component. In high temperature superconductors (HTS) typically $O^{18}$ is replaced by $O^{16}$. This defines so called oxygen isotope coefficient $\alpha^O$. Limited data are available on the copper isotope shift $\alpha^{Cu}$ in these materials 3,4.

The effect of impurities introduced into the superconductor on its $T_c$ strongly depends on the symmetry of the order parameter. It is known that non magnetic impurities hardly change $T_c$ of s-wave superconductors (Anderson theorem). On the other hand non magnetic impurities are effective pair breakers in spin singlet d-wave and spin-triplet p- or f- wave superconductors. On the contrary magnetic impurities break time reversal symmetry and strongly affect $T_c$ of all superconductors including s-wave ones. Changing $T_c$ of the material, impurities indirectly influence all its parameters. In particular this is true for isotope coefficient $\alpha$. If one finds the change of the superconducting transition temperature due to impurities as $\frac{T_{c0}}{T_c} = f(T_{c0}, \gamma)$, where $\gamma$ symbolizes all relevant parameters other than $T_{c0}$ itself, than $\frac{\alpha_{c0}}{\alpha} = \frac{\partial \ln T_{c0}}{\partial \ln T_c}$ can easily be calculated from the explicit knowledge of function $f(T_{c0}, \gamma)$.

It is the purpose of this work to systematically study the effect of disorder on the isotope effect of superconductors by assuming that dependence of $\frac{\alpha_{c0}}{\alpha}$ on $\frac{T_{c0}}{T_c}$ can be solely attributed to the effect of impurities. The impurities introduced into the superconductor modify quasi-particle spectrum, interaction parameters and induce pair breaking. This results in a change of the superconducting transition temperature and the isotope coefficient.

The motivation for the present analysis partially comes from the recent experiments which suggest strong effect of electron-phonon coupling on the dynamics of electrons in high-temperature superconductors 5,6.

The shift of $T_c$ with ionic mass was a crucial experiment to confirm the electron-phonon mechanism of superconductivity in BCS superconductors. Similarly the systematic studies of various isotopic substitutions in HTS are important to understand the role of electron-phonon interaction in these materials. After early experiments with contradictory conclusions 5,6,7,8,9 it has later been unequivocally established that the oxygen isotope shift $\alpha$ is non-zero, and takes smallest value for optimally doped materials. It increases when one moves into underdoped region.

Our paper extends the recent work of Openov et al. 10, who consider the effect of magnetic and non-magnetic impurities in s-wave and d-wave superconduc-
itors and that of Kresin and coworkers \cite{12} who study the isotope effects in s-wave superconductors doped with magnetic impurities and those showing the dynamic Jahn-Teller and proximity effects. Our aim here is to elucidate the role (in changing $\alpha$) of the out-of-plane impurities in layered systems and the effect of impurity anisotropy. We also analyze the change of isotope effect due to $Zn$ impurities in the striped phase model of high $T_c$ superconductors. In view of the recent interest in superconductivity and the critical role of impurities played in $Sr_2RuO_4$ possibly spin-triplet superconductor \cite{13} we shall consider isotope coefficient for p-wave order parameter.

In section 2 we explain the approach and apply it to the layered systems in which carriers mainly reside in active layers ($CuO$ in HTS), while impurities are placed either in the active or in the passive layer. It turns out that due to low angle scattering these out-of-plane impurities have small effect on superconducting transition temperature. However their effect on the isotope coefficient is universal in a sense that $\alpha/\alpha_0$ depends only on $T_c/T_{c0}$. In section 3 we analyze the role of in plane anisotropic scatterers in changing transition temperature and the isotope coefficient. Section 4 contains discussion of the results and comparison with experimental data. The predictions of the change of $T_c$ by Zn impurities obtained recently by the stripe theory superconductivity lead to the isotope coefficient, which is calculated in section 5. We end up with conclusions.

II. IN-PLANE AND OUT-OF-PLANE IMPURITIES IN LAYERED SUPERCONDUCTORS

In this section we shall compare the effect of in-plane and out-of-plane impurities in high temperature superconductors on $T_c$ and $\alpha$. The Hamiltonian of the superconductor containing both kinds of impurities takes the form

\[ H = \sum_{k,\sigma} (\varepsilon_k - \mu) c_{k\sigma}^{\dagger} c_{k\sigma} + \sum_{k'k,\sigma} V_{in}(\vec{k},\vec{k}') c_{k\sigma}^{\dagger} c_{k'\sigma} + \sum_{k'k,\sigma} V_{out}(\vec{k},\vec{k}') c_{k\sigma}^{\dagger} c_{k'\sigma} + \sum_{\vec{k}\vec{q}} V_{k\vec{q}} c_{k\uparrow}^{\dagger} c_{\vec{k}\downarrow} + \sum_{\vec{k}\vec{q}} V_{\vec{k}\vec{q}} c_{\vec{k}\downarrow}^{\dagger} c_{\vec{k}\downarrow} c_{\vec{q}\uparrow} c_{\vec{q}\uparrow}, \]

where $\varepsilon_k$ is the single-particle energy, $\mu$ -chemical potential $c_{k\sigma}^{\dagger}$ ($c_{k\sigma}$) denotes creation (annihilation) operator for an spin $\sigma$ electron in a state $\vec{k}$. The second term describes the scattering of carriers by in-plane while third by out-of-plane impurities. The pair potential $V_{k\vec{q}}$ is assumed to take on the separable form $V_{k\vec{q}} = -V_{0}\phi(\vec{k})\phi(\vec{q})$ dependent on the wave vector direction $\vec{k} = \hat{\phi}/|\vec{k}|$. The superconducting order parameter is defined by

\[ \Delta(\vec{k}) = \sum_{\vec{q}} V_{k\vec{q}}^{\dagger} c_{\vec{q}\downarrow} c_{\vec{q}\uparrow}, \]

and the self-consistent equation for it, easily derived by the Green’s function technique, reads \cite{14}

\[ \Delta(q) = \sum_{q} V_{k\vec{q}} \frac{1}{\beta} \sum_{\omega_n} \frac{\Delta(q)}{\omega_n + (\varepsilon_q - \mu)^2 + |\Delta(q)|^2}. \] (3)

where $\omega_n$ and $\Delta(q)$ are renormalised frequency and order parameter in impure system.

The correction to the self-energy due to impurity scattering $\Sigma_{k}(\omega_n) = G_{0k}^{-1} - G_{k}^{-1}$, where $\omega_n = (2n + 1)\pi k_B T_c$ is the Matsubara frequency, $G_{0k}$ and $G_k$ are the Nambu-Gorkov (imaginary time) Green’s functions, is calculated here in the Born approximation \cite{14} \cite{15}

\[ \Sigma(\vec{k},i\omega_n) = n_{in} \sum_{\vec{q}} |V_{in}(\vec{k} - \vec{q})|^2 \tau_3 G_{q}(i\omega_n) \tau_3 + n_{out} \sum_{\vec{q}} |V_{out}(\vec{k} - \vec{q})|^2 \tau_3 G_{q}(i\omega_n) \tau_3 \]

where $V_{in}$ and $V_{out}$ represent in-plane and out-of-plane impurity potentials while $n_{in}$ and $n_{out}$ their concentrations. After Kee \cite{16} we assume the out of plane impurity potential to be of short range type $V(\vec{r}) = u_0/(r^2 + d^2)$, where $d$ is the distance of the impurity from conducting plane and $u_0$ its scattering amplitude. Fourier transform of this potential is expressed in terms of modified Bessel function of the second kind $K_0(\tilde{d}|\vec{k} - \vec{k}'|)$, which is strongly decreasing function of its argument. Therefore we approximate it by constant $K_0$ for $|\vec{k} - \vec{k}'| > 1$ and zero otherwise. This means that the momentum transfer $|\vec{k} - \vec{k}'|$ in the scattering process by the out-of-plane impurities is limited to small values. In two-dimensions this translates into small angle scattering. There is no such limitation on the momentum transfer in the scattering by the in-plane impurities. Therefore we take \cite{16} \cite{17}

\[ V_{out}(\vec{q}) = \begin{cases} V_{out} & \text{for } \phi - \phi' < \theta_c \\ 0 & \text{otherwise} \end{cases} \]

and

\[ V_{in}(\vec{q}) = V_{in}. \]

In the following we shall calculate the changes of $T_c$ and $\alpha$ due to both kinds of impurities in superconductors.

A. Impurities in d-wave superconductor

Now we specialise the calculations to specific symmetries of the order parameter starting with d-wave one with the order parameter $\Delta(\vec{k}) = \Delta_0 \cos 2\phi$. To proceed we introduce angle dependent density of states (DOS) function $N_F(\phi)$, normalised to its average value, and assume it to be energy independent in the energy window near the Fermi energy. The integrals over $(\varepsilon - \mu)$ can be easily
FIG. 1: Effect of in-plane (thick solid line) and out-of-plane impurities (thin lines) in d-wave HTS on \( T_c \) for different values of \( \theta_1 \) and for \( \frac{1}{\tau_{in}} = 0 \). The two groups of curves correspond to two values of \( \theta_1 = 0.2, 0.3 \), while different labels refer to different angle dependent density of states functions: A and A’ are calculated with \( N(\phi) = 1 \), B and B’ with \( N(\phi) = \frac{\pi}{2} |\cos 2\phi| \). C and C’ with \( N(\phi) = \frac{\pi}{4} |\cos 4\phi| \). Note much weaker influence of out-of-plane impurities on \( T_c \).

We are interested in the change of the superconducting gap equation (3) is linearised near \( \theta \), transition temperature and simplify the equations by neglecting powers of \( \Delta(\phi) \) higher than the first.

Even though it is possible to continue calculation for general \( N_F(\phi) \), let us for a moment take \( N_F(\phi) = N_F \) independent of \( \phi \), as is appropriate for the circular Fermi surface, to underline the effect of small angle scattering only. Assuming that \( u_n = \frac{\Delta}{\omega_n} \) is independent of the angle \( \phi \) we get

\[
\hat{w}_n - \omega_n = \left( \frac{1}{2\tau_{in}} + \frac{\theta_c}{2\tau_{out}} \right) \text{sign} \hat{w}_n
\]

\[
(\hat{\Delta} - \Delta_0) \cos 2\phi = \frac{\Delta}{|\omega_n|} \left( \frac{1}{\tau_{in}} \int_0^{2\pi} d\phi' \cos 2\phi' \right)
\]

\[
+ \frac{1}{2\tau_{out}} \int_{\phi}^{\phi+\theta_1} d\phi' \cos 2\phi'
\]

(6)

In writing this equation we made use of the fact that \( \Delta(\phi) = \Delta_0 \cos 2\phi \) and \( \Delta_0 \) does not depend on \( \phi \) for the considered DOS. Vanishing of the integral multiplying \( 1/\tau_{in} \) in equation (6) for \( \Delta \) means that the in-plane impurities are strong pair-breakers in d-wave superconductors. By the same token non-zero value of the integral multiplying \( 1/\tau_{out} \) means that the out of plane impurities are much weaker pair breakers. This explains the long standing puzzle while dopant impurities do not kill the d-wave superconductivity in HTS.

Performing the integral over \( \phi' \) in the last equation and projecting the result onto \( \cos 2\phi \) function we get equation

\[
\hat{\Delta}(\theta_c) - \Delta_0 = \frac{\hat{\Delta}(\theta_c) - \Delta_0}{|\omega_n|} \left( \frac{1}{2\tau_{out}} \cos 2\phi \sin^2 \theta_c \right),
\]

(7)

which together with that for \( \hat{w}_n \) allows us to solve for \( u_n(\theta_c) = \frac{\Delta(\phi)}{|w_n|} \). The result reads

\[
u_n = \frac{\Delta_0}{\omega_n + \left( \frac{\theta_1}{2\tau_{out}} + \frac{1}{2\tau_{in}} \right) \text{sign} \omega_n - \frac{1}{2\tau_{out}} \left( \sin 2\theta_1 - tg2\phi \sin^2 \theta_c \right)}
\]

(8)

Gap equation (3) is linearised near \( T_c \) and standard manipulations allow us to find the \( T_c \) changes due to simultaneous presence of both in-plane and out-of-plane impurities

\[
\ln \frac{T_c}{T_{c0}} = -\gamma(\theta_c) \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})(n+\frac{1}{2}+\gamma(\theta_c))},
\]

(9)

where to lowest non-vanishing order in \( \theta_c \)

\[
\gamma(\theta_c) = \frac{1}{2\tau_{in}\pi k_B T_c} + \frac{1}{2\tau_{out}\pi k_B T_c} \left( \frac{2\theta_c^3}{3} \right)
\]

plays a role of pair-breaking parameter. Note, that the first and second order (in \( \theta_c \)) contributions due to out of plane impurities vanish. This has previously been derived in [10], where the change of the \( T_c \) due to the out of plane impurities has been calculated. The equation (6) is valid to all orders in \( 1/\tau_{in} \) and \( 1/\tau_{out} \) but to the lowest non-vanishing order in \( \theta_c \). It slightly generalizes the previous result [10] to simultaneous presence of in-and out-of-plane impurities. The sum on rhs of equation
Figure 1 shows large changes of superconducting transition temperature due to in-plane (thick solid line) and much weaker for out-of-plane impurities for two values of $\theta_c$ (thin lines).

For small values of $\theta_c$ even strong impurities have negligible effect on $T_c$. The change of the isotope shift exponent can be easily calculated and found to read

$$\frac{\alpha_0}{\alpha} = 1 - \gamma(\theta_c)\psi'\left(\frac{1}{2} + \gamma(\theta_c)\right).$$  \hspace{1cm} (11)$$

where $\psi'$ denotes the derivative of the di-gamma function. This solution of the impurity problem has been obtained under the assumption of constant (i.e., angle independent) density of states function $N_F(\phi)$ and scattering rate. The solution of equations (6) with $\phi$ dependent density of states can also be easily obtained by assuming that $u_n(\phi') = \frac{\Delta(\phi')}{\epsilon_{\rm iso}^n}$ is slowly varying function of the angle for angles $\phi' \in \langle \phi, \phi + \theta_c \rangle$ and we obtain

$$\ln \frac{T_e}{T_{co}} = \frac{1}{a_0} \int_0^{\pi/4} d\phi N(\phi) \cos^2 2\phi \left[ \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \rho_c(\phi, \theta_e)\right) \right]$$  \hspace{1cm} (12)$$

with the pair breaking parameter

$$\rho_c(\phi, \theta_e) = \frac{1}{2\pi k_B T_e \tau(\phi, \theta_e)}$$  \hspace{1cm} (13)$$

and angle dependent relaxation time $\tau(\phi, \theta_e)$

$$\frac{1}{\tau(\phi, \theta_e)} = \frac{1}{\tau_{\rm in}} + \frac{1}{\tau_{\rm out}} \int_0^{\phi + \theta_e} d\phi' N(\phi') \left( 1 - \frac{\cos 2\phi'}{\cos 2\phi} \right)$$  \hspace{1cm} (14)$$

The parameter $a_0$ is given by

$$a_0 = \int_0^{\pi/4} d\phi N(\phi) \cos^2 2\phi.$$  \hspace{1cm} (15)$$

The first and second terms contributing to $1/\tau(\phi, \theta_e)$ come from frequency renormalization by the in- and out-of-plane impurities respectively, while the third one is due to gap renormalisation by out-of-plane impurities. In-plane impurities do not renormalise the gap, as is seen from equation (6).

The effect of the angle dependent density of states on the suppression of $T_c$ is also illustrated in the figure (1). To facilitate comparisons we have properly normalized via digamma function $\psi(z)$ as

$$\ln \frac{T_e}{T_{co}} = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \gamma(\theta_c)\right).$$  \hspace{1cm} (10)$$
FIG. 3: Simultaneous effect of in-plane and out-of-plane impurities on the isotope coefficient for various values of \(x\) (from the bottom curve \(x=0.0, 0.01, 0.03, 0.1, 1.0\)) which measure the relative contribution of in-plane impurities to the total scattering rate (\(c.f.\) eq. (17)).

Figure (3) illustrates the changes of the normalised isotope coefficient versus normalised transition temperature calculated for the d-wave impure superconductor with constant \(N_F\). The data points are experimental values for a number of materials. Optimally doped \(La_{1.85}Sr_{0.15}Cu_{1−x}M_xO_4\) with \(M=Co\) (opened triangles), \(Zn\) (filled diamonds) and \(Ni\) (crosses); overdoped \(La_{1.80}Sr_{0.20}Cu_{1−x}M_xO_4\) with \(M= Ni\) (filled triangles), \(Fe\) (filled circles) [18]; \(YBa_2(Cu_{1−x}Zn_x)_3O_7−\delta\) (opened squares) [19].

FIG. 4: Solid curve shows the universal dependence of the normalised isotope coefficient versus normalised transition temperature for the d-wave impure superconductor with constant \(N_F\). The data points are experimental values for a number of materials. Optimally doped \(La_{1.85}Sr_{0.15}Cu_{1−x}M_xO_4\) with \(M=Co\) (opened triangles), \(Zn\) (filled diamonds) and \(Ni\) (crosses); overdoped \(La_{1.80}Sr_{0.20}Cu_{1−x}M_xO_4\) with \(M= Ni\) (filled triangles), \(Fe\) (filled circles) [18]; \(YBa_2(Cu_{1−x}Zn_x)_3O_7−\delta\) (opened squares) [19].

\[
\frac{1}{\tau(\phi, \theta_c)} = \frac{1}{\tau} \left[ x + (1 - x) \int_0^{\phi+\theta_c} d\phi' N(\phi') \left( 1 - \frac{\cos 2\phi'}{\cos 2\phi} \right) \right]
\]

\[
\ln \frac{T_c}{T_{co}} = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \rho_c\right), \quad (18)
\]

where \(\rho_c = \frac{1}{2\pi k_B T_c} \tau_m\), \(\tau_m\) is the corresponding magnetic relaxation time. This leads to the changes of the isotope effect

\[
\frac{\alpha_0}{\alpha} = 1 - \rho_c \psi'\left(\frac{1}{2} + \rho_c\right). \quad (19)
\]

C. Impurity effects of \(p\)-wave layered superconductor

The order parameter of the \(p\)-wave superconductor is taken here as

\[
\Delta(\phi) = \Delta_0 \cos \phi. \quad (20)
\]

Repeating the calculations for the in-plane and out-of-plane impurities in \(p\)-wave superconductor with constant density of states we get the equations [14] with

\[
\gamma(\theta_c) = \frac{1}{2\pi k_B T_c} + \frac{1}{2\tau_{out} \pi k_B T_c} \left( \frac{\theta_c^2}{6} \right).
\]

It is important to note that for constant density of states function the calculated changes of the isotope coefficient follow the universal curve independendtly what is

B. Impurity effects in \(s\)-wave layered superconductor

In the \(s\)-wave superconductor the order parameter \(\Delta(\vec{k}) = \Delta_0\), and nonmagnetic isotropic impurities do change neither \(T_c\) nor the isotope coefficient \(\alpha\). This is true for both in-plane and out-of-plane impurities.

Contrary, the magnetic (in-plane) impurities are pair breakers in \(s\)-wave superconductors and do change \(T_c\) [14].

rewrite [14] as

\[
\frac{1}{\tau(\phi, \theta_c)} = \frac{1}{\tau} \left[ x + (1 - x) \int_0^{\phi+\theta_c} d\phi' N(\phi') \left( 1 - \frac{\cos 2\phi'}{\cos 2\phi} \right) \right]
\]

\[
\frac{1}{\tau(\phi, \theta_c)} = \frac{1}{\tau} \left[ x + (1 - x) \int_0^{\phi+\theta_c} d\phi' N(\phi') \left( 1 - \frac{\cos 2\phi'}{\cos 2\phi} \right) \right]
\]

\[
\frac{1}{\tau(\phi, \theta_c)} = \frac{1}{\tau} \left[ x + (1 - x) \int_0^{\phi+\theta_c} d\phi' N(\phi') \left( 1 - \frac{\cos 2\phi'}{\cos 2\phi} \right) \right]
\]
the cause changing the superconducting transition temperature. The universal $\alpha/\alpha_0$ vs. $T_c/T_{c0}$ dependence is presented by the curve A in figure 2. Note that numerically this dependence for $d$-wave superconductor is the same as that obtained for magnetic impurities in $s$-wave material.

III. ANISOTROPIC MAGNETIC AND NONMAGNETIC IMPURITY POTENTIALS

High temperature superconductors are strongly anisotropic systems with relatively low carrier concentration and layered structure. This means that the screening is not very effective and the impurity - quasi-particle interaction is anisotropic. This motivates the study of anisotropic impurities in HTS. The effect of anisotropic magnetic and nonmagnetic impurities on $T_c$ of superconductors with the general form of the order parameter $\Delta (\vec{k}) = \Delta e (\vec{k})$ have been considered in Ref. 20. These authors have taken the momentum - dependent impurity potential $u (\vec{k}, \vec{k'}) = v (\vec{k}, \vec{k'}) + J (\vec{k}, \vec{k'}) \vec{S} \cdot \vec{\sigma}$, (where $\vec{S}$ is a classical spin of the impurity and $\vec{\sigma}$ the electron spin density) and assumed a separable form of scattering probabilities

$$v^2 (\vec{k}, \vec{k'}) = v_0^2 + v^2 f (\vec{k}) f (\vec{k'})$$

$$J^2 (\vec{k}, \vec{k'}) = J_0^2 + J_0^2 g (\vec{k}) g (\vec{k'})$$

where $v_0 (v_1)$, $J_0 (J_1)$ are isotropic (anisotropic) scattering amplitudes for non-magnetic and magnetic potentials. $f (\vec{k})$, $g (\vec{k})$ are the momentum - dependent anisotropy functions in the nonmagnetic and magnetic scattering channel, respectively. The averages over the Fermi surface of $f (\vec{k})$ and $g (\vec{k})$ vanish $\langle f (\vec{k}) \rangle_{FS} = 0$ and are normalised as $\langle f^2 (\vec{k}) \rangle_{FS} = 1$. The change of transition temperature calculated in the Born approximation is given in 20 and the isotope effect is found to be

$$\frac{\alpha_0}{\alpha} = 1 - (1 - \langle e f \rangle^2 - \langle e f \rangle^2) \cdot \left( \frac{\Gamma_0 + G_0}{2 \pi k_B T_c} \right) \psi' \left( \frac{1}{2} + \frac{\Gamma_0 + G_0}{2 \pi k_B T_c} \right) - \langle e f \rangle^2 \left( \frac{\Gamma_0 + G_0 + G_1 - \Gamma_1}{2 \pi k_B T_c} \right) \psi' \left( \frac{1}{2} + \frac{\Gamma_0 + G_0 + G_1 - \Gamma_1}{2 \pi k_B T_c} \right) \psi' \left( \frac{1}{2} + \frac{G_0}{2 \pi k_B T_c} \right) - \langle e f \rangle^2 \left( \frac{G_0}{2 \pi k_B T_c} \right) \psi' \left( \frac{1}{2} + \frac{G_0}{2 \pi k_B T_c} \right).$$

IV. COMPARISON WITH EXPERIMENTS

Let us start the comparison of the obtained results with existing experimental data with a word of caution. There may exist a number of factors which can affect the value of isotope coefficient of the impure system $\alpha$. In particular electron-phonon interaction and the phonon spectrum may change after the impurity doping. We do not take such effects into account.

We have also implicitly assumed that electron-phonon interaction does play a role in driving the superconducting instability of the system and makes the clean material coefficient $\alpha_0$ non-zero albeit it may take on very small value. This, however, seem to be well established by various experimental techniques 21, 22.

Here we concentrate on the oxygen isotope effect, but it has to be noted that interesting results have also been obtained in the studies of copper isotope substitutions. The negative value of $\alpha_0$, observed in some underdoped $\text{YBa}_2\text{Cu}_3\text{O}_{6-\delta}$ samples 23, 24, have been recently explained 24 as due to scattering of electrons by low-frequency phonons with large momenta. It is to be checked whether the same mechanism is responsible for negative oxygen isotope effect in very clean $\text{Sr}_2\text{Ru}_2\text{O}_6$ $p$-wave superconductor 24.

The oxygen isotope shift in high temperature superconductors does depend on the concentration of carriers. The explanation of this dependence has been a subject of number of papers 25-30. The authors invoke such effects as: anharmonicity, zero point motion, mass dependence of carrier concentration, energy dependence of the density of states, opening of the normal state pseudo-gap and the $T_c$ changes due to impurities.

FIG. 5: The isotope coefficient of $d$-wave superconductor ($\langle e f \rangle = 0.0$), for different values of the normalized anisotropic scattering rate (from top): $\Gamma_1/T_0 = 0.6, 0.5, 0.9, 0.95, 1.0$. We have taken $\langle e f \rangle^2 = 0.2$ and assumed non-magnetic impurities.
In the present work we contribute to the elucidation of the role played by various impurities in changing $T_c$ and $\alpha$ of superconductors. Because the $T_c$ reduction due to dopant impurities like Sr in La$_{2-x}$Sr$_x$CuO$_4$ is small their influence on $\alpha$ follows the standard curve like one presented in the figure (4). The departures would be seen for smaller ratio of $T_c/T_{c0}$ as is evident from figure (4). Small values of $\theta_c$ which reflect weaker pair breaking character of out-of-plane impurities does not influence the slope of the $\alpha/\alpha_0$ vs. $T_c/T_{c0}$ curve. Isotope changes due to impurities in systems with the same bare scattering rate but with differing values of $\theta_c$ follow the same universal curve.

On the other hand the effect of anisotropic impurity scattering changes the universal dependence. Isotope coefficients of the systems with various degree of anisotropy follow slightly differing curves. This additional degree of freedom allows for a better fit to the experimental data. The fit, however, is ambiguous. The same changes can be induced by anisotropy of scattering and the anisotropy of the order parameter.

V. ISOTOPE EFFECT IN IMPURE STRIPED CUPRATES

The strong sensitivity of $T_c$ to in-plane impurities in high temperature superconductors, independently on their magnetic nature, and the difficulties of the existing theories to fully describe the wealth of experimental data has resulted in new approaches to the problem. In a recent work Morais Smith and coworkers [31] have developed the scenario of $T_c$ suppression by Zn impurities, based on the stripe picture of high $T_c$ oxides. Assuming that the doped Zn does not affect the superfluid density the authors considered the increase of local inertia of the stripe due to pinning forces and local slowing down of their dynamics. The calculations within this model give, in agreement with experimental findings, the suppression of $T_c$ which is linear in the concentration of impurities $z$. It also gives the quadratic dependence of critical Zn concentration $z_c$ ($z_c$ is concentration at which superconductivity disappears) on the superconducting transition temperature of Zn free (‘clean’) systems $T_{c0}$.

The resulting formula for $T_c$ suppression in the regime of incompressible stripes is [31]

$$\frac{T_c}{T_{c0}} = 1 - \gamma z \frac{T_c}{T_{c0}},$$

(24)

where $z$ is concentration of Zn (or other in-plane impurities), $\gamma$ is a factor depending inter alia on the stripe distance and lattice constant $a$. In the optimally doped and overdoped region, where charged stripes behave as an compressible quantum fluid the $T_c$ reduction has been found to be universal and given by

$$\frac{T_c}{T_{c0}} = 1 - z/z_c,$$

(25)

with constant $z_c$.

In the stripe scenario we find that the change of isotope coefficient due to impurities reads

$$\frac{\alpha}{\alpha_0} = \frac{2}{(T_c/T_{c0})} - 1$$

(26)

in the incompressible region and

$$\frac{\alpha}{\alpha_0} = 1$$

(27)

in the compressible region.

Roughly inverse dependence of $\alpha/\alpha_0$ on $T_c/T_{c0}$ at low $T_c$ is in good qualitative agreement with experimental data (cf. figure (3)) and gives credit to the stripe picture of superconductivity in this material.

It is, however, hard to explain the difference between Fe and Ni substitution to the otherwise overdoped La$_{1.85}$Sr$_{0.15}$CuO$_4$ sample. According to the theory [31] one expects in this material incommensurate stripes and thus no influence of impurities on both $T_c$ and $\alpha$. The difficulty arises from the fact that both impurities (Ni and Fe) do change the transition temperature but the isotope shift is nearly constant for Ni substitution. One explanation is that concentration of carriers may change in the doping process. The divalent Ni does not change the concentration and the position of the Fermi level remains roughly constant while trivalent Fe ions change it driving the system effectively into underdoped or optimally doped regime. If this is true the Fe doped system is thus expected, within stripe scenario, to change both $T_c$ and $\alpha$. It is obvious that the isotope coefficient $\alpha$ of impure system can take non-zero values only for those systems for which $T_{c0}$ depends on isotope mass and $\alpha_0 \neq 0$. The question thus arises whether the formation of stripes and

![FIG. 6: The isotope effect in impure striped cuprates: the continuous line is given by Eq. (26). The data points are experimental values for overdoped La$_{1.80}$Sr$_{0.20}$Cu$_{1-x}$M$_x$O$_4$, with Ni (filled triangles), Fe (circles) [13].](image)
the driving mechanism of superconductivity in striped metal are of purely electronic origin or electron-phonon interaction plays the role to make $\alpha_0 \neq 0$. These issues have been recently addressed in \[32\]. The XANES experiments revealed large oxygen isotope effect on the stripe formation temperature $T^*$, as also on effective supercarrier mass. This shows that electron phonon interaction does play a role in a stripe model of superconductivity and validates the above analysis and conclusions.

VI. CONCLUSIONS

We have studied the effect of various impurities on the isotope coefficient $\alpha$. While the in-plane and out-of-plane impurities affect the superconducting transition temperature quite differently (cf. Fig. 1), their influence on $\alpha$ is universal for angle independent scattering relaxation rate and density of states function.

Experimental changes of $\alpha/\alpha_0$ with $T_c/T_{c0}$ for a number of high temperature copper oxides can be well described solely by the effect of impurities on $d$-wave superconductors. The angle dependence of relaxation rate and $N_F(\phi)$ or the simultaneous presence of anisotropic scatterers of magnetic and non-magnetic nature adds new degrees of freedom which can be used to quantitatively describe the data.

These mechanisms of $T_c$ and $\alpha$ changes operate for both dopant (like Sr) and extra impurities (like Ni or Zn). However, they do not differentiate between underdoped and overdoped systems. On the other hand the stripe model of superconductivity describes the detrimental effect of Ni or Zn on $T_c$. The stripe theory predicts increase of $\alpha$ with impurity concentration on the underdoped side of the phase diagram and no change of it in the overdoped region. This agrees with data on Ni and Fe doped La$_{1-x}$Sr$_{x}$CuO$_{4}$.

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