Studying Critical String Emerging from
Non-Abelian Vortex in Four Dimensions

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Abstract

Recently a special vortex string was found \cite{5} in a class of soliton vortices supported in four-dimensional Yang-Mills theories that under certain conditions can become infinitely thin and can be interpreted as a critical ten-dimensional string. The appropriate bulk Yang-Mills theory has the $U(2)$ gauge group and the Fayet-Iliopoulos term. It supports semilocal non-Abelian vortices with the world-sheet theory for orientational and size moduli described by the weighted $\mathbb{CP}(2,2)$ model. The full target space is $\mathbb{R}^4 \times Y_6$ where $Y_6$ is a non-compact Calabi-Yau space.

We study the above vortex string from the standpoint of string theory, focusing on the massless states in four dimensions. In the generic case all massless modes are non-normalizable, hence, no massless gravitons or vector fields are predicted in the physical spectrum. However, at the selfdual point (at strong coupling) weighted $\mathbb{CP}(2,2)$ admits deformation of the complex structure, resulting in a single massless hypermultiplet in the bulk. We interpret it as a composite “baryon.”
Introduction.—Studies of non-Abelian vortex solitons supported by some four-dimensional Yang-Mills theories with $\mathcal{N} = 2$ supersymmetry, resulted in identification of a bulk theory which under several conditions gives rise to a critical ten-dimensional string. The appropriate four-dimensional Yang-Mills theory has the $U(2)$ gauge group, the Fayet-Iliopoulos term $\xi$ and four matter hypermultiplets. Its non-Abelian sector would be conformal if it were not for the parameter $\xi$. It supports semilocal non-Abelian vortices with the world-sheet theory for orientational and size moduli described by the weighted $CP(2, 2)$ model. The target space is a six-dimensional non-compact Calabi-Yau manifold $Y_6$, namely, the resolved conifold. Including the translational moduli with the $\mathbb{R}^4$ target space one obtains a bona fide critical string, a seemingly promising discovery of.

In this paper we explore the spectrum of the massless modes of the above critical closed string theory. In a hypothesis was formulated regarding parameters of the bulk theory and the corresponding world sheet model necessary to make the thickness of the vortex string vanish at strong coupling, which is required in order to neglect higher derivative corrections in the world sheet theory on the vortex. Here, using duality we derive an exact formula for the relation between the two-dimensional (2D) coupling $\beta$ and four-dimensional (4D) gauge coupling $g^2$ of $\mathcal{N} = 2$ SQCD. Moreover, we identify the critical point $\beta_* = 0$ at which the vortex string can become infinitely thin. At this point the resolved conifold mentioned above becomes a singular conifold. As the only 4D massless mode of the string which emerges at $\beta_* = 0$ we identify a single matter hypermultiplet associated with the deformation of the complex structure of the conifold. Other states arising from the ten-dimensional graviton are not dynamical in four dimensions. In particular 4D graviton and unwanted vector multiplets are absent. This is due to non-compactness of the Calabi-Yau manifold we deal with and non-normalizability of the corresponding modes.

Then we discuss the question of how the states seen in the bulk theory at weak coupling are related to what we obtain from the string theory at strong coupling. In particular we interpret the hypermultiplet associated with the deformation of the complex structure of the conifold as a monopole-monopole

\footnote{The non-Abelian vortex strings are understood as strings carrying non-Abelian moduli on their world sheet, in addition to translational moduli.}

\footnote{A more accurate mathematical term is the two-dimensional sigma model with the target space $\mathcal{O}(-1)^{[2]}_{\mathbb{CP}^1}$. For brevity we will use $WCP(2, 2)$.}
World sheet model.—The basic bulk theory which supports the string under investigation is described in detail in [6]. Let us briefly review the model emerging on its world sheet.

The translational moduli fields (they decouple from other moduli) in the Polyakov formulation [7] are given by the action

$$S_0 = \frac{T}{2} \int d^2\sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha x^\mu \partial_\beta x_\mu + \text{fermions},$$

where $$\sigma^\alpha (\alpha = 1, 2)$$ are the world-sheet coordinates, $$x^\mu (\mu = 1, \ldots, 4)$$ describe the $$\mathbb{R}^4$$ part of the string world sheet and $$h = \text{det} (h_{\alpha\beta})$$, where $$h_{\alpha\beta}$$ is the world-sheet metric which is understood as an independent variable. The parameter $$T$$ stands for the tension which will be discussed below.

In the bulk theory under consideration $$N_f = 2N = 4$$, implying that in addition to orientational zero modes of the vortex string $$n^P (P = 1, 2)$$, there are size moduli $$\rho^K (K = 1, 2)$$ [8, 1, 4, 9, 10, 11].

The gauged formulation of the non-Abelian part is as follows [12]. One introduces the $$U(1)$$ charges $$\pm 1$$, namely +1 for $$n$$’s and −1 for $$\rho$$’s,

$$S_1 = \int d^2\sigma \sqrt{h} \left\{ h^{\alpha\beta} \left( \tilde{\nabla}_\alpha \bar{n}_P \nabla_\beta n^P + \nabla_\alpha \bar{\rho}_K \tilde{\nabla}_\beta \rho^K \right) \right\} + \frac{e^2}{2} \left( |n^P|^2 - |\rho^K|^2 - \beta \right)^2 + \text{fermions},$$

where

$$\nabla_\alpha = \partial_\alpha - iA_\alpha, \quad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha$$

and $$A_\alpha$$ is an auxiliary gauge field. The limit $$e^2 \to \infty$$ is implied. Equation (2) represents the $$\text{WCP}(2, 2)$$ model.

The total number of real bosonic degrees of freedom in (2) is six, where we take into account constraint imposed by $$D$$-term. Moreover, one $$U(1)$$ phase

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3 Both the orientational and the size moduli have logarithmically divergent norms, see e.g. [9]. After an appropriate infrared regularization, logarithmically divergent norms can be absorbed into the definition of relevant two-dimensional fields [11]. In fact, the worldsheet theory on the semilocal non-Abelian string is not exactly the $$\text{WCP}(N, \tilde{N})$$ model [11], there are minor differences unimportant for our purposes. The actual theory is called the $$\text{zn}$$ model. We can ignore the above differences.
is gauged away. These six internal degrees of freedom are combined with four translational moduli from (1) to form a ten dimensional space needed for a superstring to be critical.

In the semiclassical approximation the coupling constant $\beta$ in (2) is related to the bulk $SU(2)$ gauge coupling $g^2$ via

$$\beta = \frac{4\pi}{g^2}. \quad (4)$$

Note that the first (and the only) coefficient of the beta functions is the same for the bulk and world-sheet theories and equals to zero. This ensures that our world sheet theory is conformal invariant.

The total world-sheet action is

$$S = S_0 + S_1. \quad (5)$$

**Bulk duality vs world sheet duality.**—Since our vortex string is BPS saturated, the tension $T$ in Eq. (1) is given by the exact expression

$$T = 2\pi \xi \quad (6)$$

where $\xi$ is the Fayet-Iliopoulos parameter of the bulk theory.

As we know [13, 14] the bulk theory at hand possesses a strong-weak coupling duality

$$\tau \rightarrow \tau_D = -\frac{1}{\tau}, \quad \tau = \frac{4\pi}{g^2} + \frac{\theta_4D}{2\pi}. \quad (7)$$

The bulk duality implies a similar 2D duality which manifests itself in the world sheet theory as the interchange of the roles of the orientational and size moduli,

$$n^P \leftrightarrow \rho^K, \quad \text{or, equivalently, } \beta \rightarrow \beta_D = -\beta, \quad (8)$$

see Eq. (2). Equation (1) is valid only semiclassically and shows no sign of the strong-weak coupling duality (8). An obvious generalization of (4) which possess duality (8) under (7) is

$$\beta = \frac{4\pi}{g^2} - \frac{g^2}{4\pi}. \quad (9)$$

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4Argyres et al. proved this duality for $\xi = 0$. It should allow one to study the bulk theory at strong coupling in terms of weakly coupled dual theory at $\xi \neq 0$ too.
If \( g^2 \to 16\pi^2/g^2 \) then, obviously, \( \beta \to -\beta \) as required by (8). The 4D selfdual point \( g^2 = 4\pi \) is mapped onto \( \beta_* = 0 \). The selfdual point \( \beta = 0 \) is a critical point at which the target space \( WCP(2, 2) \), which is the resolved conifold, becomes a singular conifold.

It was conjectured in [5] that the non-Abelian vortex string become infinitely thin at strong coupling and can be described by the string action (5). The condition necessary for the vortex string in the bulk theory at hand to become infinitely thin is that \( m^2 \), the square of the mass of the bulk Higgsed gauge bosons, is much larger than the string tension. At weak coupling \( m^2 \sim \xi g^2 \). It is natural to assume that mass \( m \) goes to infinity at the selfdual point \( \beta = 0 \). This remains a hypothesis. An example of this behavior is

\[
m^2 = \frac{4\pi}{|\beta|} \xi. \tag{10}
\]

The expansion in derivatives of the action on the string world sheet runs in powers of \( \xi/m^2 \), implying that higher derivatives are irrelevant if \( m^2 \to \infty \) [5]. Note that since the bulk theory has a vacuum manifold, there are massless states in the bulk. Most of them are not localized on the string, and therefore are irrelevant for the string study. The only localized zero modes are translational, orientational and size moduli [5] [16]. Other excitations of the string have excitation energies \( \sim m^2 \).

One can complexify the constant \( \beta \) in a standard way, by adding the topological term in the action (5) \( \beta_{\text{compl}} = \beta + i\frac{\theta_{2D}}{2\pi} \), where \( \theta_{2D} \) is the two-dimensional theta angle which penetrates from the bulk theory [15]. In [20] we will present a complexified version of the relation (9).

**Bulk 4D supersymmetry from the critical non-Abelian string.**— The critical string discovered in [5] must lead to \( \mathcal{N} = 2 \) supersymmetric spectrum in our bulk four-dimensional QCD. This is a priori clear because our starting basic theory is \( \mathcal{N} = 2 \). The question is how this symmetry emerges from the string world-sheet model.

Non-Abelian vortex string is 1/2 BPS; therefore, out of eight bulk supercharges it preserves four. These four supercharges form \( \mathcal{N} = (2, 2) \) on the world sheet which is necessary to have \( \mathcal{N} = 2 \) space-time supersymmetry [17] [18].

**Type IIA vs IIB.**— Another question to ask is whether the string theory (5) belongs to Type IIA or IIB? We started with \( \mathcal{N} = 2 \) supersymmetric QCD

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\[5\] More exactly they have logarithmically divergent norms, a marginal case.
which is a vector-like theory and preserve 4D parity. Therefore we expect that the closed string spectrum in this theory should respect 4D parity.

On the other hand we know that Type IIB string is a chiral theory and breaks parity while Type IIA string theory is left-right symmetric and conserves parity \[^{19}\]. Thus we expect that the string theory of non-Abelian vortex is of Type IIA. In the subsequent paper \[^{20}\] we confirm this expectations studying how parity transformation acts on the 2D fermions.

**Spectrum: 4D graviton and vector fields**—As was mentioned above, our target space is \(\mathbb{R}^4 \times WCP(2,2) = \mathbb{R}^4 \times Y_6\) where \(Y_6\) is a non-compact Calabi-Yau conifold, which in fact, at \(\beta \neq 0\) is a resolved conifold, see \[^{21}\] for a review. We will consider the string theory \(^{(5)}\) at small \(\beta\).

Strictly speaking at small \(\beta\) the sigma model quantum corrections blow up. In other words, we can say that at small \(\beta\) the gravity approximation does not work. However, for the massless states, we can do the computations at large \(\beta\) where supergravity approximation is valid and then extrapolate to strong coupling. This is the reason why now we will limit ourselves only to massless states. The latter in the sigma model language corresponds to chiral primary operators. They are protected by \(\mathcal{N} = (2,2)\) world sheet supersymmetry and their masses are not lifted by quantum corrections. However, kinetic terms (the Kähler potentials) can be corrected.

We will argue that all massless string modes on the resolved conifold \(^{(2)}\) have infinite norm and thus the 4D-graviton and vector field states decouple.

The massless 10D boson fields of type IIA string theory are graviton, dilaton and antisymmetric tensor \(B_{MN}\) in the NS-NS sector. In the R-R sector type IIA string gives rise to one-form and three-form. (Above \(M, N = 1, ..., 10\) are 10D indices). We start from massless 10D graviton and study what states it can produce in four dimensions. In fact, 4D states coming from other massless 10D fields listed above can be recovered from \(\mathcal{N} = 2\) supersymmetry in the bulk, see for example \[^{22}\]. We will follow the standard string theory approach well developed for compact Calabi-Yau spaces. The only novel aspect of the case at hand is that for each 4D state we have to check whether its wave function is normalizable on \(Y_6\) keeping in mind that this space is non-compact.

Massless 10D graviton is represented by fluctuations of the metric \(\delta G_{MN} = G_{MN} - G_{MN}^{(0)}\), where \(G_{MN}^{(0)}\) is the metric on \(\mathbb{R}^4 \times Y_6\) which has a block form since \(\mathbb{R}^4\) and \(Y_6\) are factorized. Moreover, \(\delta G_{MN}\) should satisfy the Lichnerowicz
The equation
\[ D_A D^A \delta G_{MN} + 2 R_{MANB} \delta G^{AB} = 0. \]  
(11)

Here \( D^A \) and \( R_{MANB} \) are the covariant derivative and the Riemann tensor corresponding to the background metric \( G^{(0)}_{MN} \), where \( D_A \delta G_N^A - \frac{1}{2} D_N \delta G^A_A = 0 \). Given the block form of \( G^{(0)}_{MN} \), only the six-dimensional part \( R_{ijkl} \) of \( R_{MANB} \) is nonzero while the operator \( D_A D^A \) is given by \( D_A D^A = \partial_\mu \partial^\mu + D_i D^i \), where the indices \( \mu, \nu = 1, \ldots, 4 \) and \( i, j = 1, \ldots, 6 \) belong to flat 4D space and \( Y_6 \), respectively, and we use 4D metric with the diagonal entries \((-1, 1, 1, 1)\).

Next, we search for the solutions of (11) subject to the block form of \( \delta G_{MN} \),

\[ \delta G_{\mu\nu} = \delta g_{\mu\nu}(x) \phi_6(y), \quad \delta G_{\mu i} = B_\mu(x) V_i(y), \quad \delta G_{ij} = \phi_4(x) \delta g_{ij}(y), \]  
(12)

where \( x_\mu \) and \( y_i \) are the coordinates in \( R^4 \) and \( Y_6 \), respectively. We see that \( \delta g_{\mu\nu}(x) \), \( B_\mu(x) \) and \( \phi_4(x) \) are the 4D graviton, vector and scalar fields, while \( \phi_6(y) \), \( V_i(y) \) and \( \delta g_{ij}(y) \) are the fields on \( Y_6 \).

For the fields \( \delta g_{\mu\nu}(x) \), \( B_\mu(x) \) and \( \phi_4(x) \) to be dynamical in 4D, the fields \( \phi_6(y) \), \( V_i(y) \) and \( \delta g_{ij}(y) \) should have finite norm over the six-dimensional internal space \( Y_6 \). Otherwise 4D fields will appear with infinite kinetic term coefficients, and, hence, will decouple.

Symbolically the Lichnerowicz equation (11) can be written as

\[ (\partial_\mu \partial^\mu + \Delta_6) g_4(x) g_6(y) = 0, \]  
(13)

where \( \Delta_6 \) is the two-derivative operator from (11) reduced to \( Y_6 \), while \( g_4(x) g_6(y) \) symbolically denotes the factorized form in (12). If we expand \( g_6 \) in eigenfunctions

\[ - \Delta_6 g_6(y) = \lambda_6 g_6(y) \]  
(14)

the eigenvalues \( \lambda_6 \) will play the role of the mass squared of the 4D states. Here we will be only interested in the \( \lambda_6 = 0 \) eigenfunctions. Solutions of the equation \( \Delta_6 g_6(y) = 0 \) for the Calabi-Yau manifolds are given by elements of Dolbeault cohomology group \( H^{(p,q)}(Y_6) \), where \((p, q)\) denotes the numbers of holomorphic and anti-holomorphic indices of the form. The numbers of these forms \( h^{(p,q)} \) are called the Hodge numbers for a given \( Y_6 \). Due to the fact that \( h^{(0,0)} = 1 \), we can easily find (the only!) solution for the 4D graviton: it is a constant on \( Y_6 \). This mode is non-normalizable. Hence, no 4D graviton emerges from our string.
This is a good news. We started from $\mathcal{N} = 2$ QCD in four dimensions without gravity and, therefore, do not expect 4D graviton to appear as a closed string state\(^6\).

The infinite norm of the graviton wave function on $Y_6$ rules out other 4D states of the $\mathcal{N} = 2$ gravitational and tensor multiplets: the vector field, the dilaton, the antisymmetric tensor and two scalars coming from the 10D three-form.

Now, let us pass to the components of the 10D graviton $\delta G_{\mu i}$ which give rise to a vector field in 4D. The very possibility of having vector fields is due to continuous symmetries of $Y_6$, namely for our $Y_6$ we expect seven Killing vectors, which obey the equations

$$D_i V^m_j + D_j V^m_i = 0, \quad m = 1, \ldots, 7.$$  \hspace{1cm} (15)

For Calabi-Yau this implies the equation $\Delta_6 g_6(y) = 0$, which takes the form $D_j D^j V^m_i = 0$ and is equivalent to the statement that that $V_i$ is a covariantly constant vector, $D_j V_i = 0$. This is impossible for compact Calabi-Yau manifolds with the SU(3) holonomy \(^{19}\), but possible in the non-compact case. However it is easy to see that the $V^m_i$ fields produced by rotations of the $y_i$ coordinates do not fall-off at large $y^2_i$ (where the metric tends to flat). Thus, they are non-normalizable, and the associated 4D vector fields $B_\mu(x)$ are absent.

**Deformations of the conifold metric.**—It might seem that our 4D string does not produce massless 4D states at all. This is a wrong impression. At the selfdual value of $\beta = 0$ it does!

Consider the last option in (12), scalar 4D fields. In this case the appropriate Lichnerowicz equation on $Y_6$ reduces to

$$D_k D^k \delta g_{ij} + 2 R_{ikjl} \delta g^{kl} = 0.$$  \hspace{1cm} (16)

The target space of the sigma model (2) is defined by the $D$ term condition

$$|n^P|^2 - |\rho^K|^2 = \beta.$$  \hspace{1cm} (17)

Also, a U(1) phase can be gauged away. We can construct the U(1) gauge-invariant “mesonic” variables

$$w^{PK} = n^P \rho^K.$$  \hspace{1cm} (18)

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\(^6\) The alternative option that massless 4D spin-2 state has no interpretation in terms of 4D gravity is ruled out by Weinberg-Witten theorem \(^{23}\).
In terms of these variables the condition \((17)\) can be written as \(\det w^{PK} = 0\), or
\[
\sum_{\alpha=1}^{4} w_{\alpha}^2 = 0,
\]
where \(w^{PK} = \sigma^{PK}_\alpha w_{\alpha}\), and \(\sigma\)-matrices are \((1, -i\tau^a), a = 1, 2, 3\). Equation \((19)\) define the conifold—a cone with the section \(S_2 \times S_3\). It has the Kähler Ricci-flat metric and represents a non-compact Calabi-Yau manifold [24, 12, 21].

At \(\beta = 0\) the conifold develops a conical singularity, so both \(S_2\) and \(S_3\) shrink to zero. The conifold singularity can be smoothed in two different ways: by a deformation of the Kähler form or by a deformation of the complex structure. The first option is called the resolved conifold and amounts to introducing the non-zero \(\beta\) in \((17)\). This resolution preserves the Kähler structure and Ricci-flatness of the metric. If we put \(\rho^K = 0\) in \((2)\) we get the \(\mathbb{CP}(1)\) model with the \(S_2\) target space (with the radius \(\sqrt{\beta}\)). The explicit metric for the resolved conifold can be found in [24, 25, 26]. The resolved conifold has no normalizable zero modes. In particular, we will demonstrate in [20] that the 4D scalar \(\beta\) associated with deformation of the Kähler form is not normalizable.

If \(\beta = 0\) another option exists, namely a deformation of the complex structure [21]. It preserves the Kähler structure and Ricci-flatness of the conifold and is usually referred to as the deformed conifold. It is defined by deformation of Eq. \((19)\), namely
\[
\sum_{\alpha=1}^{4} w_{\alpha}^2 = b,
\]
where \(b\) is a complex number. Now the \(S_3\) can not shrink to zero, its minimal size is determined by \(b\). The explicit metric on the deformed conifold is written down in [24, 28, 29]. The parameter \(b\) becomes a 4D complex scalar field. The effective action for this field is
\[
S(b) = T \int d^4 x \ h_b |\partial_\mu b|^2,
\]
where the metric \(h_b(b)\) is given by the normalization integral over the conifold \(Y_6\),
\[
h_b = \int d^6 y \sqrt{g} g^{ij} \left( \frac{\partial}{\partial b} g_{ij} \right) g^{jk} \left( \frac{\partial}{\partial \bar{b}} g_{kl} \right),
\]
and $g_{ij}(b)$ is the deformed conifold metric.

We calculated $h_b$ by two different methods, one of them [27] was kindly pointed out to us by Cumrun Vafa. Details of these calculations will be presented in [20]. Here we only state that the norm of the $b$ modulus is marginal (i.e. logarithmically divergent), in full accord with the analysis of [27]. Thus, this scalar mode is localized on the string in the same sense as the orientational and size zero modes are localized on the vortex-string solution.

In type IIA superstring the complex scalar associated with deformations of the complex structure of the Calabi-Yau space enters as a 4D hypermultiplet. Thus our 4D scalar $b$ is a part of a hypermultiplet. Another complex scalar $\tilde{b}$ comes from 10D three-form, see [22] for a review. Together they form the bosonic content of a 4D $\mathcal{N} = 2$ hypermultiplet. The fields $b$ and $\tilde{b}$ being massless can develop VEVs. Thus, we have a new Higgs branch in the bulk which is developed only at the self-dual value of coupling constant $g^2 = 4\pi$. The bosonic part of the full effective action for the $b$ hypermultiplet to be presented in [20] takes the form

$$S(b) = T \int d^4x \left\{ |\partial_\mu b|^2 + |\partial_\mu \tilde{b}|^2 \right\} \log \frac{T^4L^8}{|b|^2 + |\tilde{b}|^2},$$

where $L$ is the size of $R^4$ introduced as an infrared regularization of logarithmically divergent norm of $b$-field.

The logarithmic metric in (23) in principle can receive both perturbative and non-perturbative quantum corrections. However, for $\mathcal{N} = 2$ theory the non-renormalization theorem of [14] forbids the dependence of the Higgs branch metric on the 4D coupling constant $g^2$. Since the 2D coupling $\beta$ is related to $g^2$ we expect that the logarithmic metric in (23) will stay intact.

The presence of the “non-perturbatively emergent” Higgs branch at the self-dual point $g^2 = 4\pi$ at strong coupling is a successful test of our picture. A hypermultiplet is a BPS state. If it were present in a continuous region of $\tau$ at strong coupling it could be continued all the way to the weak coupling domain where its presence would contradict the quasiclassical analysis of bulk QCD [6, 20].

String states interpretation in the bulk.—To find the place for the massless scalar hypermultiplet we obtained as the only massless string excitation at the critical point $\beta = 0$ let us first examine weak coupling domain $g^2 \ll 1$.

As well-known from the previous studies of the vortex-strings at weak coupling, non-Abelian vortices confine monopoles. The elementary monopoles
are junctions of two distinct elementary non-Abelian strings \[30, 3, 4\]. As a result in the bulk theory we have monopole-antimonopole mesons in which monopole and antimonopole are connected by two confining strings. For the U(2) gauge group we have also “baryons” consisting of two monopoles, rather than of monopole-antimonopole pair. The monopoles acquire quantum numbers with respect to the global group \(SU(2) \times SU(2) \times U(1)\) of the bulk theory. Indeed, in the world sheet model on the vortex-string confined monopole are seen as kinks interpolating between two different vacua \[30, 3, 4\]. These kinks are described at strong coupling by the \(n^R\) and \(\rho^K\) fields \[31, 32\] (for \(WCP(N, \tilde{N})\) models see \[33\]) and therefore transform in the fundamental representations of \(SU(2) \times SU(2)\) for \(WCP(2, 2)\)\[7\].

As a result, the monopole-antimonopole mesons and baryons can be either singlets or triplets of both \(SU(2)\) global groups, as well as in the bi-fundamental representations. With respect to baryonic \(U(1)_B\) symmetry which we define as a \(U(1)\) factor in the global \(SU(2) \times SU(2) \times U(1)\), the mesons have charges \(Q_B(\text{meson}) = 0\) while the baryons can have charges

\[
Q_B(\text{baryon}) = 0, 1, 2. \tag{24}
\]

All the above non-perturbative stringy states are heavy, with mass of the order of \(\sqrt{\xi}\), and can decay into screened quarks, which are lighter, and, eventually, into massless bi-fundamental screened quarks.

Now we return from weak to strong coupling and go to the self-dual point \(\beta = 0\). We claim that at this point a new ‘exotic’ Higgs branch opens up which is parameterized by the massless hypermultiplet – the \(b\) state, associated with the deformation of the complex structure of the conifold. It can be interpreted as a baryon constructed of two monopoles. To this end note that the complex parameter \(b\) (promoted to a 4D scalar field) is singlet with respect to two \(SU(2)\) factors of the global world-sheet group while its baryonic charge is \(Q_B = 2\) \[20\].

Since the world sheet and the bulk global groups are identical we can conclude that our massless \(b\) hypermultiplet is a monopole-monopole baryon.

Being massless it is marginally stable at \(\beta = 0\) and can decay into pair of massless bi-fundamental quarks in the singlet channel with the same baryon charge \(Q_B = 2\). The \(b\) hypermultiplet does not exist at non-zero \(\beta\).

\[7\]Here we note that global group \(SU(2) \times SU(2) \times U(1)\) is the same for both bulk and world sheet theories \[6\]
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