Algorithm of applying the energy conservation law to study the thermal stress-strain state of a rod of variable cross section with the simultaneous presence of local temperatures, heat exchanges and thermal insulations

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Abstract: The bearing elements of power plants, internal combustion engines of jet engines, nuclear thermal power plants, gas-generating power plants, processing lines, and spacecraft antennas are rods of variable cross-section. They work in a difficult temperature environment. In this paper, we propose an algorithm for applying the fundamental law of energy conservation for a deep study of the resulting thermal-stress-strain state of a rod of limited length and variable cross-section in the simultaneous presence of local temperatures, heat exchange and thermal insulation. At the same time, the obtained solutions in the form of determining the temperature field, displacement, all the components of deformations and stresses, as well as the calculated values of the rod elongation and the resulting axial force are highly accurate at the level of satisfying the energy conservation law.

1. Formulation of the problem

A horizontal rod of limited length is considered L[cm]. The axis Ox is directed from left to right, which coincides with the axis of the considered rod. Radius of section of the left end of the rod r(x=0)=r₀[cm].

Radius of section of the right end of the rod r(x=L)=r_L[cm]. The radius of the rod section along its length varies linearly, i.e.

\[
r(x) = \left(\frac{r_L-r_0}{L}\right)x + r_0, \quad 0 \leq x \leq L
\]  

(1)
The temperature is given at all points of the cross section of the left end of the rod $T(x = 0) = T_1[\degree \text{C}]$. Heat exchange with the environment occurs through the lateral surface of the first (1/3) length of the rod. In this case, the heat exchange coefficient is $h \left[ \text{watt/cm}^2\degree \text{C} \right]$, and $T_{oc}[\degree \text{C}]$ - ambient temperature. The lateral surface of the middle (1/3) part of the rod $\left( \frac{L}{3} \leq x \leq \frac{2L}{3} \right)$ is insulated. As a result, there will be no heat loss through the lateral surface of the section $\left( \frac{L}{3} \leq x \leq \frac{2L}{3} \right)$ of the rod. Now we go to the last third part of the rod section $\left( \frac{2L}{3} \leq x \leq L \right)$. Heat exchange with the environment also takes place through the lateral surface of this section. A similar process occurs through the cross-sectional area of the right end of the rod $\left( x=L \right)$. In these places, $h$ is also the heat exchange coefficient, and the ambient temperature is $T_{oc}$. If the radius of the rod in question changes linearly. According to (1), the cross-sectional area along the length of the rod varies nonlinearly, as follows

$$F(x) = \left( \frac{r_0^2 - 2r_0r_1 + r_1^2}{L^2} \right) x^2 + 2r_0 \left( \frac{r_L - r_0}{L} \right) x + r_0^2, \quad 0 \leq x \leq L \quad (2)$$

Physical and mechanical properties of the rod material are characterized by thermal conductivity coefficients $K_{xx} \left[ \text{watt/cm} \cdot \text{C} \right]$, of thermal expansion $\alpha \left[ 1/\text{C} \right]$ and the elastic modulus $E \left[ \text{kg/cm}^2 \right]$. It is required to determine:

1. The law of temperature distribution along the length of the investigated rod section;
2. The magnitude of the elongation of the rod if one end of the rod is rigidly pinched, and the other is free;
3. The magnitude of the resulting axial force, if both ends of the rod are rigidly pinched;
4. The laws of distribution of thermo-elastic, temperature and elastic components of deformations and stresses along the length of the rod;
5. The law of distribution of displacement along the length of the rod.

2. Building the functional of total thermal energy for a rod of limited length and variable cross-section.

We will consider the first discrete element of the rod in question. Figure – 2.

![Diagram](image-url)
For this element, the total thermal energy functional has the following form [1].

\[ J_1 = \int_{\frac{h}{2} \frac{\partial T}{\partial x} + \frac{d}{dx}\left[\frac{1}{2}\left(T - T_{OC}\right)^2 ds + \int_{V_1} \frac{K_{xx}}{2} \frac{\partial T}{\partial x}^2 dv, \quad (0 \leq x \leq l) \] \]

(3)

where \( V_1 \) – volume of the first discrete element of the rod under study, \( S_{lsa} \) – lateral surface area of the first discrete element.

Now we go to the second discrete element - Figure – 3.

For the second discrete element, the total thermal energy functional has the following form [1].

\[ J_2 = \int_{V_2} \frac{K_{xx}}{2} \left(\frac{\partial T}{\partial x}\right)^2 dv, \quad (0 \leq x \leq l) \]

(4)

where \( V_2 \) – volume of the second discrete element of the rod under study.

Finally, we go to the last to the third discrete element of the rod in question – Figure – 4.

For the third discrete element, the total thermal energy functional has the following form [1].

\[ J_3 = \int_{V_3} \frac{K_{xx}}{2} \left(\frac{\partial T}{\partial x}\right)^2 dv + \int_{S_{lsa}} \frac{h}{2} \left(T - T_{OC}\right)^2 ds + \int_{S(x = L)} \frac{h}{2} \left(T - T_{OC}\right)^2 ds, \quad (0 \leq x \leq l) \]

(5)

where \( V_3 \) – volume of the third discrete element of the rod under study, \( S_{lsa} \) – lateral surface area of the third discrete element, \( S(x = L) \) – cross-sectional area of the right end of the rod.

For convenience, we approximate the temperature field within the length of each discrete element by quadratic spline functions [2].

\[ T(x) = \phi_1(x)T_1 + \phi_2(x)T_2 + \phi_k(x)T_k = \left(\frac{2x^2 - 3lx + l^2}{l^2}\right)T_1 + \left(\frac{4lx - 4x^2}{l^2}\right)T_j + \left(\frac{2x^2 - lx}{l^2}\right)T_k, \quad (0 \leq x \leq l) \]

(6)
Then the temperature gradient within the length of each discrete element is defined as follows:

\[
\frac{\partial T}{\partial x} = \frac{\partial \phi_i(x)}{\partial x} T_i + \frac{\partial \phi_j(x)}{\partial x} T_j + \frac{\partial \phi_k(x)}{\partial x} T_k = \frac{4x - 3l}{l^2} T_i + \frac{4l - 8x}{l^2} T_j + \frac{4x - l}{l^2} T_k, \quad (0 \leq x \leq l)
\]  

(7)

In the expressions (6-7) it should be noted that for the first discrete element \( T_i = T(x = 0) = T_1 \) is given; \( T_j = T \left( x = \frac{l}{3} \right) = T_2 \); \( T_k = T(x = l) = T_3 \); For the second discrete element \( T_i = T(x = l) = T_3 \); \( T_j = T \left( x = \frac{l}{3} + \frac{l}{2} \right) = T_4 \); \( T_k = T(x = 2l) = T_5 \) takes place. Similarly, for the third discrete element, we have \( T_i = T(x = 2l) = T_5 \); \( T_j = T \left( x = \frac{l}{3} + \frac{l}{2} \right) = T_6 \); \( T_k = T(x = 3l = L) = T_7 \);

Then the general functional of the total thermal energy for the variable cross section rod under study has the following form:

\[
J = J_1 + J_2 + J_3
\]

(8)

Besides \( J = J(T_1, T_2, T_3, T_4, T_5, T_6, T_7) \). But here \( T_1 \) is the given one. Then the true values of \( T_2, T_3, T_4, T_5, T_6, T_7 \) will be the values that will satisfy the law of conservation of energy, i.e. the following system of linear equations takes place:

\[
\frac{\partial J}{\partial T_r} = 0, \quad r = (2 \div 7)
\]

(9)

Solving this system, the nodal temperatures \( T_r \) are determined. By these, a temperature field within the length of each discrete element is determined using (6).

3. DEFINITION OF THE FIELD OF TENSION AND DEFORMATIONS

Then, in the case of pinching at one end, and when the other end of the rod is free, we can calculate the elongation of the studied rod under the influence of existing types of heat source [3].

\[
\Delta l_T = \int_0^\Delta \alpha T(x) dx
\]

(10)

If both ends of the rod are rigidly pinched, the rod cannot extend. But due to thermal expansion, an axial compressive force occurs in it \( R \) [kG]. It is determined from the condition of compatibility of deformation.

\[
R(kG) = - \frac{E F_{cp} \Delta l_T}{\alpha}, [kG]
\]

(11)

where

\[
F_{cp} = \frac{\int_0^L F(x) dx}{L}
\]

(12)

In this case, the distribution law of the thermoelastic stress component is determined based on the generalized Hooke's law [4].

\[
\sigma(x) = R \frac{R}{F(x)}, \quad 0 \leq x \leq L
\]

(13)
Then, on the basis of the same Hooke's law \[4\], the field of the thermoelastic deformation component is also determined:

\[
\epsilon(x) = \frac{\sigma(x)}{E} = -\frac{d_TF_{cp}}{L\kappa(x)}, \quad 0 \leq x \leq L
\]  

(14)

Based on the fundamental laws of thermophysics, the law of the distribution of the temperature component of deformation is determined

\[
\epsilon_T(x) = -\alpha T(x), \quad 0 \leq x \leq L
\]  

(15)

Then, on the basis of Hooke's law, the law of distribution of the temperature component of the stress along the length of the studied rod of variable cross-section and limited length is determined.

\[
\sigma_T(x) = E\epsilon_T(x), \quad 0 \leq x \leq L
\]  

(16)

After that, using the basic relations of thermo-elasticity, the law of distribution of the elastic component of deformation and stress along the length of the rod in question is determined.

\[
\epsilon_x(x) = \epsilon(x) - \epsilon_T(x), \quad 0 \leq x \leq L
\]  

(17)

\[
\sigma_x(x) = E\epsilon_x(x), \quad 0 \leq x \leq L
\]  

(16)

Then, on the basis of Hooke's law, the law of distribution of the temperature component of the stress along the length of the studied rod of variable cross-section and limited length is determined.

\[
\epsilon_T(x) = -\alpha T(x), \quad 0 \leq x \leq L
\]  

(15)

After that, using the basic relations of thermo-elasticity, the law of distribution of the elastic component of deformation and stress along the length of the rod in question is determined.

\[
\epsilon_x(x) = \epsilon(x) - \epsilon_T(x), \quad 0 \leq x \leq L
\]  

(17)

\[
\sigma_x(x) = E\epsilon_x(x), \quad 0 \leq x \leq L
\]  

(16)

The law of displacement distribution is determined based on the potential energy functional \[4\].

\[
\Pi = \int_V \frac{\sigma_x}{2} \epsilon_x(x) dV - \int_V \alpha E T(x) \epsilon_x(x) dV, \quad 0 \leq x \leq L
\]  

(18)

Where, according to the Cauchy relation, the relationship between displacement and deformation is defined as follows.

\[
\epsilon_x(x) = \frac{\partial u}{\partial x} = \frac{\partial \phi_i}{\partial x} U_i + \frac{\partial \phi_j}{\partial x} U_j + \frac{\partial \phi_k}{\partial x} U_k, \quad 0 \leq x \leq l
\]  

(19)

By minimizing \(\Pi\) by nodal values of displacement, a resolving system of linear algebraic equations with natural boundary conditions is constructed. By solving which the displacement field is constructed.

### 4. NUMERICAL SOLUTION OF THE TASK

To test the above algorithm, we take the following parameter values as initial data

| \(r_0\) | \(r_L\) | \(L\) | \(n\) | \(\frac{L}{n}\) | \(T_1\) |
|---|---|---|---|---|---|
| 2 cm | 1 cm | 30 cm | 3 | 10 cm | 500°C |

| \(K_{xx}\) | \(\alpha\) | \(E\) | \(h\) | \(T_{oc}\) |
|---|---|---|---|---|
| 100 \(\frac{W}{cm^2C}\) | 125 \(\cdot 10^{-7}\) \(\frac{W}{C}\) | 2 \(\cdot 10^6\) \(\frac{kg}{cm^2}\) | 10 \(\frac{W}{cm^2C}\) | 20°C; |
Figure 5a shows the law of temperature distribution along the length of the rod under study. The distribution pattern is monotonously decreasing. At the right end of the rod, the temperature is equal to $T(x=L=30 \text{cm}) = 20.257 \, ^\circ \text{C}$. Figure 5b shows the distribution law for the three components of deformation. It can be seen from this figure that the thermo-elastic and temperature components of the deformation along the entire length of the rod under study have a compressive character. The elastic component of the deformation $\varepsilon_\varepsilon(x)$ on the interval $0 \leq x < 5 \, \text{cm}$ is tensile and then compressive in nature. The corresponding components of the stress behave similarly, figure 5c. Figure 5d shows that all sections of the rod under study (except the pinched ones) are moved from left to right. In this case the largest amplitude corresponds to the cross section whose coordinate $x=10 \, \text{cm}$, $U(x = 10 \text{cm}) = 0.01553701 \, \text{cm}$. This process is caused by the fact that there is a large local temperature at the left end of the rod.

5. Conclusion
The results show that the developed algorithm based on the fundamental law of energy conservation allows to solve the set problems taking into account natural heterogeneous boundary conditions, as a result of which the obtained solutions are characterized by increased accuracy. For this reason, the above algorithm is applicable to solving complex and priority engineering problems, the results of which can form the basis for creating new technologies. Which in turn give an impulse to increase the country's economy.

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