A RAMSEY THEOREM FOR INDECOMPOSABLE MATCHINGS

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A matching is indecomposable if it does not contain a nontrivial contiguous segment of vertices whose neighbors are entirely contained in the segment. We prove a Ramsey-like result for indecomposable matchings, showing that every sufficiently long indecomposable matching contains a long indecomposable matching of one of three types: interleavings, broken nestings, and proper pin sequences.

1. INTRODUCTION

A (labeled, complete) matching is a graph on the vertex set \([2n] = \{1, 2, \ldots, 2n\}\) in which every vertex is incident to exactly one edge. An interval in a matching is a contiguous segment of vertices \([i, j] = \{i, i + 1, \ldots, j\}\) such that no vertex in \([i, j]\) is adjacent to a vertex outside \([i, j]\). Every matching has two trivial intervals: the empty set and the set of all its vertices. A matching is said to be indecomposable if it has no other intervals (and decomposable if it does have nontrivial intervals, see Figure 1). We prove a Ramsey-like result for indecomposable matchings, showing that every such matching contains a large member of one of three explicit families of indecomposable matchings.

Indecomposable matchings have been studied by Nijenhuis and Wilf [2], who provided a recursive algorithm for constructing all indecomposable matchings. From their recursion, it follows that the number, \(s_n\), of indecomposable matchings of \([2n]\) satisfies the recurrence

\[
s_n = (n - 1) \sum_{i=1}^{n-1} s_i s_{n-k}.
\]

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The contribution of this paper is to show that there are, essentially, only three types of indecomposable matchings.

**Theorem 1.** Every indecomposable matching with at least \((2k)^2k\) edges contains a broken nesting, interleaving or proper pin sequence with \(k\) edges.

This result is the matching analogue of the results of Brignall, Huczynska, and Vatter [1], who proved a similar result for permutations.

In Theorem 1 we say that the matching \(M\) contains the matching \(N\) if \(N\) can be obtained from \(M\) by deleting a collection of edges and the vertices incident with those edges, and then relabeling the remaining vertices. For the remainder of this section we discuss the three types of indecomposable matchings mentioned in Theorem 1. The proof of the theorem follows in the next section.

The interleaving on \([2n]\) is the indecomposable matching defined by \(i \sim i + n\) for all \(i \in [n]\). The interleaving on \([8]\) is depicted in the first matching of Figure 2.

The nesting on \([2n - 2]\) is the matching defined by \(i \sim 2n - 2 - i + 1\) for \(i \in [n]\). This matching is not indecomposable, but can be made indecomposable by adding a new edge which breaks the nesting. This new edge can break the nesting either to the left or the right. The right-broken nesting on \([2n]\) has edges \(n \sim 2n\) and \(i \sim 2n - i\) for \(i \in [n - 1]\), while the left-broken nesting on \([2n]\) has edges \(1 \sim n + 1\) and \(i + 1 \sim 2n - i + 1\) for \(i \in [n - 1]\). The right- and left-broken nestings on \([8]\) are depicted in the second and third matchings of Figure 2.

The most diverse family of indecomposable matchings is the family of pin sequences. In order to define pin sequences, we need a few preliminaries. Given a set of edges in a matching, its shadow is the smallest contiguous segment of vertices containing their endpoints. In an indecomposable matching, every nonempty shadow must either consist of all the vertices, or be split, meaning that there is a vertex in the shadow which is adjacent to a vertex outside of the shadow. We refer to such edges as pins.
A pin sequence is then a sequence of edges \( p_1, p_2, \ldots \) such that each \( p_i \) breaks the shadow of \( \{p_1, \ldots, p_{i-1}\} \). Thus one pin sequences on [8] is \( 3 \sim 5, 4 \sim 7, 6 \sim 1, 2 \sim 8 \), shown in the fourth matching of Figure 2.

It is important to verify that all pin sequences are indecomposable.

**Proposition 2.** Every pin sequence is an indecomposable matching.

**Proof.** Let \( P = p_1, p_2, \ldots, p_n \) be a pin sequence and suppose that \( P \) has an interval. There must be at least one pin in any nontrivial interval and since the \( P \) is finite there is a largest \( i \) such that \( p_i \) is in the interval, this is the last pin in the interval. Suppose \( i \neq 2n \). Because \( P \) is a pin sequence we know that \( p_{i+1} \) crosses \( p_i \) which violates our assertion that \( p_i \) was the last pin in the interval so \( p_i \) must have been \( p_{2n} \). We show that every interval contains \( p_1 \). Suppose \( p_i \neq p_1 \) is the first pin an interval. Since \( P \) is a pin sequence then \( p_i \) crosses \( p_{i-1} \) so \( p_{i-1} \) is in the interval, but this contradicts the minimality of \( i \). So we know that \( p_1 \) is in every interval.

This shows pin sequences contain only trivial intervals and thus are indecomposable.

In the statement of Theorem [I] we use a particular type of pin sequence. A proper pin sequence satisfies, for each \( 1 < i < 2n \), \( p_{i+1} \) splits the shadow cast by \( \{p_1, \ldots, p_i\} \) but not the shadow cast by \( \{p_1, \ldots, p_{i-1}\} \).

### 2. **Proof of Theorem [I]**

Our proof of Theorem [I] consists of analyzing two possibilities. First, we show that if a single edge is crossed by many different edges, then the matching contains an interleaving or broken nesting. The alternative is that no edge is crossed by many different edges, in which case we show that the matching contains a long proper pin sequence.

**Lemma 3.** If a single edge \( e \) is crossed by \( 2(k-1)^2 + 2 \) edges of a matching, then the matching contains either a broken nesting or an interleaving of length \( k \).

**Proof.** Every edge that crosses \( e \) crosses either to the left or to the right, thus at least \( (k - 1)^2 + 1 \) of the edges must cross to the same side of \( e \). By symmetry call that side left. Now order these \( (k - 1)^2 + 1 \) edges by their left endpoints, preserving the natural order on the integers. Let \( S \) be the unique sequence formed by the right vertices of the the edges when read in order.

By the Erdős-Szekeres theorem, \( S \) has a monotone subsequence of length \( k \). If this subsequence is increasing, the matching contains an interleaving. Otherwise this subsequence is decreasing and the matching contains a nesting that is broken by \( e \).
In order to prove the main theorem, we will need pin sequences that are right-reaching, that is, their final pin is incident with the vertex $2n$.

It is helpful to know that proper right-reaching pin sequences are always available in indecomposable matchings.

**Lemma 4.** Every indecomposable matching has a proper right-reaching pin sequence beginning with any edge.

**Proof.** Let $p_1$ be an arbitrary edge of the indecomposable matching $M$. If the vertex $2n$ is incident with $p_1$, then we are done. Otherwise, by the indecomposability of $M$, there is an edge which crosses $p_1$; label this edge $p_2$. If $2n$ is incident with $p_2$, then we are done. Otherwise, the edges $p_1$ and $p_2$ define a new shadow, which is still not an interval, so there is an edge, $p_3$, which splits this shadow. Since the only interval is $[2n]$, by repeating this process, we can create a pin sequence $p_1, \ldots, p_m$ such that $2n$ is incident with $p_m$.

We now construct from this right-reaching pin sequence a proper right-reaching pin sequence $q_1, q_2, \ldots, q_s$. First, set $q_1 = p_1$. Then we successively extend this sequence by choosing $q_i$ to be the pin $p_j$ of the greatest index which crosses $q_{i-1}$. We stop when $q_i$ is incident with $2n$. Note that by this selection procedure, $q_i$ crosses $q_{i-1}$ but does not cross $q_1, \ldots, q_{i-2}$. Therefore the resulting sequence $q_1, \ldots$ is a proper right-reaching pin sequence, as desired.

In the proof of Theorem 1 we use Lemma 4 to show that every indecomposable matching with $n$ edges contains at least $n$ distinct right-reaching proper pin sequences.

We can now derive the main result.

**Proof of Theorem 1.** Let $M$ be a matching which does not contain a broken nesting, interleaving, or proper pin sequence with at least $k$ edges. We construct a tree of all the proper right-reaching pin sequences of $M$ in the following manner. The parent of the pin sequence $p_1, \ldots, p_m$ ($m \geq 2$) is the sequence $p_2, \ldots, p_m$, so the root of this tree is the edge (thought of as a pin sequence) incident with the vertex of the greatest label.

Since $M$ does not have a pin sequence of length $k$, this tree has height at most $k - 1$. Because $M$ does not contain an interleaving or broken nesting of with $k$ edges, Lemma 3 implies that no node may have $2(k-1)^2 + 2$ children. This bounds the size of the tree with the sum

$$
\sum_{i=0}^{k-1} (2k^2 - 4k + 3)^k - 1 < \sum_{i=0}^{k-1} (2k^2)^i = (2k^2)^{2k}
$$

By Lemma 4 every edge of $M$ begins a proper right-reaching pin sequence. Therefore $M$ can have at most $(2k^2)^{2k}$ edges, proving the theorem. 

\[\square\]
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