Entanglement of Bipartite Gaussian States: a Simple Criterion and its Geometric Interpretation

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Abstract

Werner and Wolf have proven in Phys. Rev. Lett. 86(16) (2001) a very elegant necessary and sufficient condition for a bosonic continuous variable bipartite Gaussian mixed quantum state to be separable. This condition is, however, difficult to implement in practice. In the present Letter we propose a simpler condition which only involves the calculation of the symplectic matrix in the Williamson diagonalization of the covariance matrix of the state under consideration. The main tool in our construction is the observation, proved in previous work, that the Wigner transform is covariant only under symplectic or antisymplectic linear transformations. We also give a geometric interpretation of our condition in terms of the orthogonal projections of “quantum blobs”.

1 Statement of the Problem

By definition a Gaussian state \( \hat{\rho}_\Sigma \) is a density matrix on \( \mathbb{R}^n \) with Wigner distribution

\[
W_{\hat{\rho}_\Sigma}(z) = \left( \frac{1}{2\pi n} \right)^n \int e^{-\frac{i}{\hbar} p \cdot y} \langle x + \frac{1}{2} y | \hat{\rho} | x - \frac{1}{2} y \rangle d^n y
\]

of the type

\[
\rho_\Sigma(z) = \left( \frac{1}{2\pi} \right)^n (\det \Sigma)^{-1/2} e^{-\frac{1}{2} \Sigma^{-1} z \cdot z}
\]
where \( z = (x,p) \) and \( \Sigma \) is a real positive definite \( 2n \times 2n \) matrix (the “covariance matrix”) satisfying the quantum condition

\[
\Sigma + \frac{i}{2} J \geq 0
\]

which ensures the positivity of the corresponding density matrix \( \widehat{\rho}_\Sigma \). Conditions for the separability of such states are well-known [4, 5, 16, 15]; for instance the Peres–Horodecki PPT condition on the positivity of the partial transpose [10, 14] is always a necessary condition for separability.

Werner and Wolf [17] showed that \( \widehat{\rho}_\Sigma \) is separable if and only if there exist partial covariance matrices \( \Sigma_A \) and \( \Sigma_B \) satisfying \( \Sigma_A + \frac{i}{2} J_A \geq 0 \) and \( \Sigma_B + \frac{i}{2} J_B \geq 0 \) such that \( \Sigma \geq \Sigma_A \oplus \Sigma_B \) (we are using here the phase space splitting \( \mathbb{R}^{2n} \equiv \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B} \) corresponding to two subsystems \( A \) and \( B \)).

The Werner and Wolf criterion is very elegant but it is difficult in practice to determine \( \Sigma_A \) and \( \Sigma_B \) in general (however Giedke et al. [5] have constructed an algorithm allowing the determination of these partial covariance matrices; also see Lami et al. [11] for recent developments and a refinement of the Werner and Wolf condition).

## 2 Statement of the Result

Let \( \lambda_1^\sigma, \ldots, \lambda_n^\sigma \) be the symplectic eigenvalues [6] of the matrix \( J\Sigma \) where \( J = J_A \oplus J_B \) is the standard symplectic matrix on \( \mathbb{R}^{2n_A} \oplus \mathbb{R}^{2n_B} \). The condition (3) is equivalent to the statement [7, 9]

\[
\text{We have } \lambda_j^\sigma \geq \frac{1}{2} \hbar \text{ for } 1 \leq j \leq n. \tag{4}
\]

where the \( \lambda_j^\sigma \) are the symplectic eigenvalues of the covariance matrix \( \Sigma \); they appear in the so-called Williamson symplectic diagonalization theorem [6]: there exists \( S \in \text{Sp}(n) \) (the symplectic group) such that

\[
\Sigma = SDS^T, \quad D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}. \tag{5}
\]

where \( \Lambda \) is the diagonal matrix diag(\( \lambda_1^\sigma, \ldots, \lambda_n^\sigma \)). In this Letter we prove the following improvement of Werner and Wolf’s criterion:

**Theorem 1** Let \( \Sigma_0 = \frac{1}{2} \hbar SS^T \) where \( S \in \text{Sp}(n) \) is as in (5). The Gaussian mixed state \( \widehat{\rho}_\Sigma \) is separable if and only if \( \Sigma_0 = \Sigma_A \oplus \Sigma_B \) where \( \Sigma_A \) and \( \Sigma_B \) are positive \( 2n_A \times 2n_A \) and \( 2n_B \times 2n_B \) symmetric positive matrices \( \Sigma_A \) satisfying \( \Sigma_A + \frac{i}{2} J_A \geq 0 \) and \( \Sigma_B + \frac{i}{2} J_B \geq 0 \). Equivalently, the diagonalizing
matrix $S$ in $\mathbb{R}^2$ satisfies $SS^T = S_A S_A^T \oplus S_B S_B^T$ for some $S_A \in \text{Sp}(n_A)$ and $S_B \in \text{Sp}(n_B)$. 

The simplicity of this criterion is obvious since it suffices to determine the symplectic matrix $S$ diagonalizing the covariance matrix, and this can easily be done using elementary methods from linear algebra [13]. 

The essential tool for the proof of this result the lemma below which imposes limitations on the linear changes of variables that takes a Wigner transform to another Wigner transform.

3 A Lemma on Symplectic Covariance

It is well-known that the Wigner distribution (of a pure or mixed state) is symplectically covariant with respect to metaplectic operators. In fact, if $S \in \text{Sp}(n)$ and $\tilde{S} \in \text{Mp}(n)$ is any of the two metaplectic operators covering $S$ then

$$W\psi(S^{-1} z) = W(\tilde{S}\psi)(z)$$

for every tempered distribution $\psi$ on $\mathbb{R}^n$. In [2] we addressed the following question: for which $M \in GL(2n, \mathbb{R})$ does there exist $\psi' \in \mathcal{S}(\mathbb{R}^n)$ such that we have more generally

$$W\psi(M^{-1} z) = W\psi'(z)$$

for every $\psi \in \mathcal{S}(\mathbb{R}^n)$? We showed that the only possibilities were the following:

**Lemma 2** Either $M = S \in \text{Sp}(n)$ (in which case (6) holds) or $M$ is antisymplectic, i.e. of the type $M = \tilde{T} S$ for some $S \in \text{Sp}(n)$ where $\tilde{T}$ is the mirror reflection operator $(x, p) \mapsto (x, -p)$; in this case formula (6) is replaced with

$$W\psi((\tilde{T}S)^{-1} z) = W(\tilde{S}\psi^*)(z).$$

4 Proof of Theorem 1

4.1 Preparatory discussion

We are working here in the usual coordinates $z = (x, p) \in \mathbb{R}^{2n}$. If the symplectic eigenvalues of $\Sigma$ are all equal to $\frac{1}{2}\hbar$ then we have $\Lambda = \frac{1}{2}\hbar I_{n \times n}$, and the covariance matrix $\Sigma$ becomes

$$\Sigma_0 = \frac{1}{2}\hbar SS^T$$

(8)
so that the Wigner distribution \[6, 12\] of the corresponding density matrix 
\[\hat{\rho}_\Sigma \] satisfies 
\[\rho_\Sigma(Sz) = (\pi \hbar)^{-n/4} e^{-|z|^2/2\hbar} \]
where \(S\) is the diagonalizing symplectic matrix in \([5]\) and
\[W \phi_0(z) = (\frac{1}{2\pi \hbar})^{n} \int e^{-\frac{i}{2}p \cdot y} \phi_0(x + \frac{1}{2}y) \phi_0^*(x - \frac{1}{2}y) d^n y \]
is the Wigner transform of the standard Gaussian \(\phi_0(x) = (\pi \hbar)^{-n/4} e^{-|x|^2/2\hbar}\).

By the symplectic covariance property \[6, 12\] of the Wigner transform we have
\[\rho_\Sigma(Sz) = W \phi_0(S^{-1}z) = W(\hat{S} \phi_0)(z) \tag{9} \]
where \(\hat{S} \in Mp(n)\) is anyone of the two metaplectic operators covering \(S \in Sp(n)\). Using the convolution properties of normal probability distributions we can write
\[\rho_\Sigma = \rho_{\Sigma - \Sigma_0} \ast \rho_{\Sigma_0} = \rho_{\Sigma - \Sigma_0} \ast \rho_{\Sigma_0} \tag{10} \]
where \(\phi = \hat{S} \phi_0\) is a Gaussian \([6, 12]\) and \(\Sigma - \Sigma_0 \geq 0\) in view of the equivalent conditions \([3]\) and \([4]\). If \(\Sigma - \Sigma_0 > 0\) then \(\rho_{\Sigma - \Sigma_0}\) is a classical normal probability distribution; if some eigenvalues of \(\Sigma - \Sigma_0\) are zero, then \(\rho_{\Sigma - \Sigma_0}\) is a generalized (degenerate) Gaussian. Writing now \(z \in \mathbb{R}^{2n}\) as \(z = (z_A, z_B)\) with \(z_A = (x_A, p_A) \in \mathbb{R}^{2n_A}\) and \(z_B = (x_B, p_B) \in \mathbb{R}^{2n_B}\) the partial transpose \(\hat{\rho}_\Sigma^{TB}\) has Wigner distribution \(\rho_{\Sigma}^{TB}(z_A, z_B) = \rho_\Sigma(z_A, \overline{z_B})\) where \(\overline{z_B} = (x_B, -p_B)\), that is,
\[\rho_{\Sigma}^{TB}(z_A, z_B) = (\frac{1}{2\pi})^n (\det \overline{\Sigma})^{-1/2} e^{-\frac{1}{4}\overline{\Sigma}^{-1}z \cdot z} \tag{11}\]
where the covariance matrix is
\[\overline{\Sigma} = (I_A \oplus T_B) \Sigma (I_A \oplus T_B) \]
with \(I_A z_A = z_A\) and \(T_B z_B = \overline{z_B}\); we will abbreviate this expression to \(\overline{\Sigma} = T_B \Sigma T_B\) since \(I_A\) plays no significant role in this notation. The symplectic diagonalization equality \([3]\) becomes, noting that \(T_B D T_B = D\),
\[\overline{\Sigma} = (T_B S T_B) D (T_B S T_B)^T \text{ with } T_B S T_B \in Sp(n). \tag{12}\]

Notice that we do not have necessarily \(\overline{\Sigma} + \frac{i}{2} \hbar J \geq 0\); this relation only holds if \(\rho_\Sigma^{TB}\) is a density matrix, that is if \(\hat{\rho}_\Sigma\) is separable.
4.2 Necessity of the condition $\Sigma_0 = \Sigma_A \oplus \Sigma_B$

Suppose now that $\hat{\rho}_\Sigma$ is separable and let us show that we must then have $\Sigma_0 = \Sigma_A \oplus \Sigma_B$ with

$$\Sigma_A + \frac{i\hbar}{2} J_A \geq 0 \text{ and } \Sigma_B + \frac{i\hbar}{2} J_B \geq 0.$$  \hfill (13)

Since $\Sigma_0 = \frac{1}{2} \hbar S S^T$ this amounts to proving that $S S^T = S_A S_A^T \oplus S_B S_B^T$ for some $S_A \in \text{Sp}(n_A)$ and $S_B \in \text{Sp}(n_B)$; note that $\Sigma_A$ and $\Sigma_B$ will then satisfy de facto the conditions (13). Now, the separability of $\hat{\rho}_\Sigma$ implies, in view of the PPT criterion, that the partial transpose $\hat{\rho}_{\Sigma}^{T_B}$ must also be a density matrix i.e., repeating the argument above leading to (10), there must exist a function $\tilde{\phi}$ such that

$$\rho_{\Sigma}^{T_B} = \rho_{\Sigma - \Sigma_0} \ast \rho_{\Sigma_0} \ast W_{\tilde{\phi}} \tag{14}$$

where the matrix $\tilde{\Sigma}_0$ is given by

$$\tilde{\Sigma}_0 = \frac{1}{2} \hbar (T_B S T_B)(T_B S T_B)^T = \frac{1}{2} \hbar T_B SS^T T_B = T_B \Sigma_0 T_B.$$

Let us now calculate directly $\rho_{\Sigma}^{T_B}(z_A, z_B) = \rho_{\Sigma}(z_A, T_B z_B)$ from formula (10): we have

$$\rho_{\Sigma}^{T_B}(z_A, z_B) = \int \int \rho_{\Sigma - \Sigma_0}(z_A - u_A, T_B z_B - u_B) W_{\phi}(u_A, u_B) du_A du_B$$

$$= \int \int \rho_{\Sigma - \Sigma_0}(z_A - u_A, z_B - u_B) W_{\phi}(u_A, T_B u_B) du_A du_B$$

$$= \int \int \rho_{\Sigma - \Sigma_0}(z_A - u_A, z_B - u_B) W(\phi \circ T_B)(u_A, u_B) du_A du_B$$

that is

$$\rho_{\Sigma}^{T_B} = \rho_{\Sigma - \Sigma_0} \ast W(\phi \circ T_B)$$

(we are using the notation $W_{\phi} \circ T_B(z_A, z_B) = W_{\phi}(z_A, T_B z_B)$). Comparison with (14) thus yields the equality

$$\rho_{\Sigma - \Sigma_0} \ast W_{\tilde{\phi}} = \rho_{\Sigma - \Sigma_0} \ast (W_{\phi} \circ T_B). \tag{15}$$

This equality implies that $W_{\tilde{\phi}} = W_{\phi} \circ T_B$; in fact the Fourier transform of $\rho_{\Sigma - \Sigma_0}$ is a strictly positive function so, taking the Fourier transforms of
both sides of (15) we can simplify, which leads to the equality of the Fourier transforms of $W\tilde{\phi}$ and $W\phi \circ I_B$ and hence of these functions themselves. If now $\phi = \phi_A \otimes \phi_B$ for some $\phi_A \in L^2(\mathbb{R}^{n_A})$ and $\phi_B \in L^2(\mathbb{R}^{n_B})$ then we will have

$$W\phi \circ I_B = W\tilde{\phi} = W_A \phi_A \otimes W_B \phi_B^*$$

so that we can take $\tilde{\phi} = \phi_A \otimes \phi_B^*$ ($W_A \phi_A$ and $W_B \phi_B^*$ are the Wigner transforms on $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$, respectively. Assume in contrario that we cannot write $\phi$ as any tensor product $\phi = \phi_A \otimes \phi_B$. Then $W\phi \circ I_B = W\tilde{\phi}$ implies, by Lemma 2 above, that $I_{A} \oplus I_{B}$ is either symplectic or antisymplectic, but neither is the case. We must thus have $\tilde{\phi} = \phi_A \otimes \phi_B$ so that

$$\rho_{\Sigma_T} = \rho_{\Sigma - \Sigma_0} * (W_A \phi_A \otimes W_B \phi_B^*)$$

Recall now from formulas (9) and (10) that $W\phi = W(\hat{S}\phi_0) = W\phi_0 \circ S^{-1}$ where $\phi_0$ is the standard Gaussian, the equality $\phi_A \otimes \phi_B = \hat{S}\phi_0$. It readily follows that we must have $S = (S_A \oplus S_B)U$ for some $S_A \in \text{Sp}(n_A)$, $S_B \in \text{Sp}(n_B)$, and $U \in \text{Sp}(n) \cap O(2n)$ a symplectic rotation (see the proof of Corollary 4 below) and thus. $SS^T = S_A S_A^T \oplus S_B S_B^T$; setting $\Sigma_A = \frac{1}{2}hS_A S_A^T$ and $\Sigma_B = \frac{1}{2}hS_B S_B^T$ we have $\Sigma_0 = \Sigma_A \oplus \Sigma_B$.

### 4.3 Sufficiency of the condition $\Sigma_0 = \Sigma_A \oplus \Sigma_B$

Assume conversely that we have

$$\Sigma_0 = \frac{1}{2}hSS^T = \Sigma_A \oplus \Sigma_B$$

where $\Sigma_A + \frac{i}{2}J_A \geq 0$ and $\Sigma_B + \frac{i}{2}J_B \geq 0$. Since we have $\Sigma = (\Sigma - \Sigma_0) + \Sigma_0$ and that $\Sigma - \Sigma_0 \geq 0$ by definition of $\Sigma_0$, we have $\Sigma \geq \Sigma_0 = \Sigma_A \oplus \Sigma_B$. Since the partial covariance matrices $\Sigma_A$ and $\Sigma_B$ trivially satisfy the positivity conditions $\Sigma_A + \frac{i}{2}J_A \geq 0$ and $\Sigma_B + \frac{i}{2}J_B \geq 0$ it follows from Werner and Wolf’s result that $\rho_{\Sigma}$ is separable.

### 5 Geometric Interpretation in Terms of Quantum Blobs

By definition [6, 8, 9] a “quantum blob” is the image of a phase space ball with radius $\sqrt{\hbar}$ by a symplectic transformation (it is thus a symplectic ball with radius $\sqrt{\hbar}$ to use the language of symplectic topology [1]). Quantum blobs can be viewed as the smallest phase space units compatible with the uncertainty principle of quantum mechanics.
Let $\hat{\rho}_\Sigma$ be a Gaussian density matrix with Wigner distribution (2); we define as usual its covariance ellipsoid $\Omega_\Sigma$ by being the set of all phase space points $z$ such that $\frac{1}{2} \Sigma^{-1} z \cdot z \leq 1$. Introducing the matrix $M = \frac{1}{2} \hbar \Sigma^{-1}$ we thus have $\Omega_\Sigma = \{ z : Mz \cdot z \leq \hbar \}$. In [7] (also see [9]) we have proven that the condition (3), which reads $\Sigma + \frac{1}{2} i \hbar J \geq 0$, is equivalent to the inequality $c(\Omega_\Sigma) \geq \pi \hbar$ where $c(\Omega_\Sigma)$ is the symplectic capacity of the phase space ellipsoid $\Omega_\Sigma$. This means that the supremum of the set of all numbers $\pi r^2$ where $r$ is the radius of a centered phase space ball $B^{2n}(r)$ which can be embedded in $\Omega_\Sigma$ by some $S \in \text{Sp}(n)$ is precisely $\pi \hbar$ (thus, the radius of the largest ball that can be sent inside $\Omega_\Sigma$ using a linear symplectic transformation is $\sqrt{\hbar}$). As we have shown in [6, 8, 9] this is a geometric restatement of the uncertainty principle in its strong Robertson–Schrödinger form. Consider now the covariance ellipsoid $\Omega_0$ corresponding to $\Sigma_0 = \frac{1}{2} \hbar SS^T$: it is defined by the inequity $(SS^T)^{-1} z \cdot z \leq \hbar$ and is hence the ellipsoid $S(B^{2n}(\sqrt{\hbar}))$. Now, we have seen in the course of our proof above that $\hat{\rho}_\Sigma$ is separable if and only if $S = (S_A \oplus S_B)U$ for some $S_A \in \text{Sp}(n_A)$, $S_B \in \text{Sp}(n_B)$, and $U \in \text{Sp}(n) \cap O(2n)$; an equivalent statement is that

$$\Omega_0 = S(B^{2n}(\sqrt{\hbar})) = (S_A \oplus S_B)B^{2n}(\sqrt{\hbar}).$$

In [3] (also see [1]) we have proven the following geometric result, which we state here as a Lemma:

**Lemma 3** Let $S \in \text{Sp}(n)$. Let $\Pi_A$ and $\Pi_B$ be the orthogonal projections $\mathbb{R}^{2n} \to \mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n} \to \mathbb{R}^{2n_B}$, respectively. (i) There exists $S_A \in \text{Sp}(n_A)$ and $S_B \in \text{Sp}(n_B)$ such that

$$\Pi_A(S(B^{2n}(R))) = S_A(B^{2n_A}(R)) \quad \text{(16)}$$
$$\Pi_B(S(B^{2n}(R))) = S_B(B^{2n_B}(R)) \quad \text{(17)}$$

if and only if $S = (S_A \oplus S_B)U$ for some $U \in \text{Sp}(n) \cap O(2n)$.

From this we easily get the following geometric interpretation of Theorem 1

**Corollary 4** The Gaussian density matrix $\hat{\rho}_\Sigma$ is separable if and only if the orthogonal projections $\Pi_A(\Omega_0)$ and $\Pi_B(\Omega_0)$ of the phase space ellipsoid $\Omega_0 = S(B^{2n}(\sqrt{\hbar}))$ are quantum blobs in $\mathbb{R}^{2n_A}$ and $\mathbb{R}^{2n_B}$.

**Proof.** In view of Theorem 1 $\hat{\rho}_\Sigma$ is separable if and only if

$$\Sigma_0 = \frac{1}{2} \hbar SS^T = \Sigma_A \oplus \Sigma_B.$$
Let us show that if $\Sigma_0 = \Sigma_A \oplus \Sigma_B$ then the diagonalizing symplectic matrix $S$ in (5) must be of the type $S = (S_A \oplus S_B)U$ for some $S_A \in \text{Sp}(n_A)$, $S_B \in \text{Sp}(n_B)$, and $U \in \text{Sp}(n) \cap O(2n)$ a symplectic rotation. In fact, by the polar decomposition theorem for symplectic matrices [6], there exists $U \in \text{Sp}(n) \cap O(2n)$ such that $S = (SS^T)^{1/2}U$, that is, since $SS^T = \frac{2}{\hbar}(\Sigma_A \oplus \Sigma_B)$,

$$S = \left(\frac{2}{\hbar}(\Sigma_A \oplus \Sigma_B)\right)^{1/2}U = (S_A \oplus S_B)U$$

with $S_A = \left(\frac{2}{\hbar}\Sigma_A\right)^{1/2} \in \text{Sp}(n_A)$ and $S_B = \left(\frac{2}{\hbar}\Sigma_B\right)^{1/2} \in \text{Sp}(n_B)$. If conversely $S$ has the form above we have

$$SS^T = S_AS_A^T \oplus S_BS_B^T = \frac{2}{\hbar}(\Sigma_A \oplus \Sigma_B)$$

so that condition (18) is equivalent to the separability of $\hat{\rho}_\Sigma$. Suppose now that we have $S = (S_A \oplus S_B)U$. Then, by rotational symmetry of $B^{2n}(\sqrt{\hbar})$,

$$\Omega_0 = S(B^{2n}(\sqrt{\hbar})) = (S_A \oplus S_B)B^{2n}(\sqrt{\hbar})$$

and hence, by Lemma 3, $\Pi_A(\Omega_0) = S_A(B^{2n_A}(\sqrt{\hbar}))$ and $\Pi_B(\Omega_0) = S_B(B^{2n_B}(\sqrt{\hbar}))$ are quantum blobs. If conversely this is the case, then (again by Lemma 3) we have $\Omega_0 = S(B^{2n}(\sqrt{\hbar}))$. ■

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