Type the title of your paper here mathematical properties of $n \times n$ nonnegative matrix: case of irreducible Leslie matrix

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Abstract. Perron-Frobenius theorem describe five properties of irreducible nonnegative matrix. Leslie matrix is one of nonnegative matrix. Leslie matrix that used in this research is limited to the irreducible Leslie matrix. In previous research has been proven that irreducible Leslie matrix satisfies three properties in Perron-Frobenius theorem by spectral radius. This research completed the previous research, proving that irreducible Leslie matrix has a unique Perron vector and satisfies Collatz Wielandt formula. Leslie matrix is a primitive matrix. It is used to calculate the number of populations in the future. Growth of population is interpreted by value spectral radius of Leslie matrix.

1. Introduction

Both pure and applied studies concerning the concept of eigenvalues and eigen vectors is still being developed today, Some example for theoretical studies, such as in paper [1,2]. In paper [1], a new numerical method for the numerical solution of eigenvalues with the largest real part of essentially positive matrices is described, thus the proposed method is compared to Power and QR methods. While, in Paper [2] the comparison principles for the solutions of abstract integral equations, and conditions for point-dissipativity of nonlinear positive maps are derived; Collatz-Wielandt numbers, bounds, and order spectral radii are compared for bounded homogeneous maps; and conditions for a local upper Collatz-Wielandt radius to have a lower positive eigenvector are given. To study the basics of research related to this research, refer to [3-6].

Leslie matrix model is a model used by demographers to predict the number and growth rate of a population. The Leslie matrix can provide an overview of the growth process dynamics of a population, for example long-term population growth, population distribution in the age groups in long-term and its application in policy making of developed population.

The irreducible nature of the non-negative matrices was discovered by George Frobenius in 1912 by developing the Perron theorem. So, this theorem is called the Perron-Frobenius theorem. The Perron-Frobenius theorem explains the nature of non-negative matrices based on their spectral radius. Non-negative matrices are applied in several models, one of which is the Leslie matrix model. Leslie matrix on this paper is limited to the irreducible Leslie matrix.

There are several theorems about the Leslie matrix which prove that the Leslie matrix eigenvalues fulfil three of the five Perron-Frobenius theorem properties. In this paper, the general right and left Perron vectors of the Leslie matrix are obtained, thus proving that the irreducible Leslie matrix full fills all of the Perron-Frobenius theorems.
2. Methods
This research is mainly done by investigate some literatures. In the following, some preliminary concepts are given related to this research.

2.1. Eigen vector and spectral radius
The eigenvector is a right eigenvector which appears as a column vector on the right side of a square matrix. For example, for matrix $A_{n \times n}$, then the right eigenvector $x$ satisfies the equation $Ax = \lambda x$.

2.1.1. Eigen vector. For an $n \times n$ matrix $A$, scalars $\lambda$ and vectors $x_{n \times 1} \neq 0$ satisfying $Ax = \lambda x$ are called eigenvalues and eigenvectors of $A$, respectively, and any such pair, $(\lambda, x)$ is called an eigen pair of $A$. The set of distinct eigenvalues, denoted by $\sigma(A)$, is called the spectrum of $A$.

- $\lambda \in \sigma(A) \Leftrightarrow A - \lambda I$ is singular $\Leftrightarrow \det(A - \lambda I) = 0$.
- Nonzero row vectors $y^T$ such that $y^T(A - \lambda I) = 0$ are called left eigenvectors for $A$.

The characteristic polynomial of $A^T$ is $\det(A^T - \lambda I)$, because the determinant of a matrix and its transpose has the same value, the characteristic polynomial of $A^T$ is the same as characteristic polynomial of $A$. Thus, the left and right eigenvalues of matrix $A$ are the same.

2.1.2. Spectral radius. For square matrix $A$, the number $\rho(A) = \max \{|\lambda|\}$ is called the spectral radius of $A$ denoted by $\rho(A)$ is the smallest circle that contains all the eigenvalues of the matrix $A$.

2.2. Irreducible Matrix (7)

$A_{n \times n} \geq 0$ is irreducible if and only if $(I + A)^{n-1} > 0$.

For any $A \in \mathbb{R}^{n \times n}$, graph of matrix $A = [a_{ij}]$ is graph $\mathcal{G}(A) = (V, E)$ such that $V = \{1, 2, ..., n\}$ and $j \rightarrow i \Leftrightarrow a_{ij} \neq 0$. Graph of square matrix $A$ denoted by $\mathcal{G}(A)$. If $A$ is a $n \times n$ matrix, then $\mathcal{G}(A)$ has $n$ vertices.

- $\mathcal{G}(A)$ is called strongly connected if for each pair of nodes $(i, k)$ there is a sequence of directed edges leading from $i$ to $k$.
- $A$ is an irreducible matrix if and only if $\mathcal{G}(A)$ is strongly connected.

2.3. Positive Matrix

$A_{n \times n}$ is said to be a positive matrix whenever each $a_{ij} > 0$, and this is denoted by writing $A > 0$. There exist several properties of matrix $A_{n \times n} > 0$ which affected by eigenvalue and eigenvector of $A$.

**Theorem 1 (8)** If $A_{n \times n} > 0$ with $\rho(A) = r$, $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ then the following statements are true.

- $r > 0$.
- $r \in \sigma(A)$ ($r$ is called the Perron root).
- $\text{alg mult}_A(r) = 1$.
- There exist an eigenvector $x > 0$ such that $Ax = rx$.
- The Perron vector is the unique vector defined by $A\mathbf{p} = r\mathbf{p}$, $\mathbf{p} > 0$ dan $\|\mathbf{p}\|_1 = 1$.
- And, except for positive multiples of $\mathbf{p}$, there are no other nonnegative eigenvectors for $A$, regardless of the eigenvalue.
- The only eigenvalue on the spectral circle of $A$ is $r = \rho(A)$.
- The Collatz-Wielandt formula says $r = \max_{x \in \mathcal{N}} f(x)$, where $f(x) = \min_{1 \leq i \leq n, x_i \neq 0} \frac{|Ax|_i}{x_i}$ and $\mathcal{N} = \{x| x \geq 0, x \neq 0\}$.
2.4. Perron-Frobenius Theorem

A_{n \times n} is said to be a nonnegative matrix whenever each \( a_{ij} \geq 0 \), and this is denoted by writing \( A \geq 0 \). Frobenius emphasized that the the problem of the Perron theorem on non-negative matrices is not the existence of zero entries, but the zero-entry position in the nonnegative matrix. The zero-entry position determines the reduction of a nonnegative matrix. The reduction of a non-negative matrix affects the Perron theorem which applies to the nonnegative matrix. It is the nature of this irreducible nonnegative matrix that is explained in the Perron-Frobenius theorem.

**Theorem 2 (Perron-Frobenius Theorem)** Suppose that \( A_{n \times n} \geq 0 \) is irreducible nonnegative matrix, then the following things applies

1. \( r = \rho(A) \in \sigma(A) \) and \( r > 0 \)
2. There exists eigenvector \( x > 0 \) such that \( Ax = rx \).
3. \( \text{alg mult}_A(r) = 1 \).
4. There is unique vector \( p \) defined by \( Ap = rp, p > 0 \) and \( \|p\|_1 = 1 \)

   Is called Perron vector or right Perron vector of \( A \). There is no nonnegative eigenvector of \( A \) except the positive multiplication of \( p \).

5. Collatz-Wielandt formula \( r = \max_{x \in N} f(x) \), where \( f(x) = \min_{1 \leq i \leq n, x_i \neq 0} \frac{|Ax_i|}{x_i} \) and \( N = \{x|x \geq 0, x \neq 0\} \)

2.5. Primitive Matrix

For nonnegative square matrix of \( A \), then the following things applies

- Matrix \( A \) is called primitive if the matrix is irreducible and has one diagonal positive element.
- \( A \) is a primitive matrix if and only if \( A^m > 0 \) for some \( m > 0 \).

To facilitate in determine the primitiveness of nonnegative matrix \( A_{m \times m} \), Wielandt found that the matrix \( A_{m \times m} \geq 0 \) is primitive if and only if \( A^{m^2-2m+2} > 0 \). \( m^2 - 2m + 2 \) is the smallest power that proves a primitive \( m \times m \) matrix whose entire diagonal entry is zero.

**Theorem 3** ([9]) Nonnegative irreducible of matrix \( A \) with \( r = \rho(A) \) is primitive if and only if \( \lim_{k \to \infty} (A/\gamma)^k \) exist, where \( \lim_{k \to \infty} (A/\gamma)^k = \frac{p \gamma^k q^*}{q^* p} > 0 \) with \( p \) and \( q \) is Perron vector for \( A \) and \( A^T \).

2.6. Leslie Matrix

Leslie model with age structure is a model of the division of individual classes based on chronological age. If the lifespan of a female in a population is \( A \) year and the population is divided into \( n \) age classes, then each age class has an \( A/n \) age range. Age classes and age intervals are presented in the table.

| Age Class | Age Interval |
|-----------|--------------|
| 1         | \left[0, \frac{1}{n} \right] |
| 2         | \left[\frac{1}{n}, \frac{2}{n} \right] |
| 3         | \left[\frac{2}{n}, \frac{3}{n} \right] |
| \vdots    | \vdots |
| \( n-1 \) | \left[\frac{(n-2)}{n}, \frac{(n-1)}{n} \right] |
| \( n \)   | \left[\frac{(n-1)}{n}, \frac{1}{n} \right] |
$n \times n$ Leslie matrix $L$ has the general form as follows

$$L = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & b_{n-1} & 0 \end{pmatrix}$$

with $a_i$ as the female fertility rate (female parent) in the population, and $b_i$ is defined as the survival rate of women, then

$$a_i \geq 0, \text{ for } i = 1, 2, 3, \ldots, n$$

$$0 < b_i \leq 1, \text{ for } i = 1, 2, 3, \ldots, n - 1.$$ 

Known the number of female population in each age class at $t = 0$ and suppose $n_i(t)$ is the number of females in the age class $n$, then the total number of female population is $n(t) = n_1(t) + n_2(t) + n_3(t) + \cdots + n_n(t)$. The number of females in each age class when $t$ can be written

$$n(t) = \begin{pmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \\ \vdots \\ n_n(t) \end{pmatrix}$$

To calculate total population in the $p$ years, we use the following formula

$$n(t + p) = L^p n(t) \text{ or } n(p) = L^p n(0).$$

$L$ is primitive, this is obvious if in addition to $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_{n-1}$ are positive and we have $a_1 > 0$. But even $a_1 = 0$, $L$ is still primitive because $L^{n+2} > 0$.

### 3. Results and Discussion

Based on its general form, Leslie matrix is nonnegative matrix. Note that any $n \times n$ Leslie matrix $L$, with $a_1, a_2, \ldots, a_{n-1} = 0, a_n > 0$ and $b_i > 0; i = 1, 2, \ldots, n - 1$ as follow

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & b_{n-1} & 0 \end{pmatrix}$$

can be depicted in graph $G(L)$

![Figure 1. Graph of Leslie matrix L.](image)

From Figure 1, Leslie matrix $L$ with $a_1, a_2, \ldots, a_{n-1} = 0, a_n > 0$ and $b_i > 0; i = 1, 2, \ldots, n - 1$ is the smallest irreducible matrix of Leslie matrix model. Because graph $G(L)$ has guaranteed that each pair of vertices is strongly connected. Thus, for any Leslie matrix of the size of $n \times n$ that satisfies $a_n > 0$ dan $b_i > 0; i = 1, 2, \ldots, n - 1$, then the Leslie matrix is an irreducible matrix. This irreducible Leslie matrix will be investigated through the Perron-Frobenius theorem.

Leslie matrix $L$ as an irreducible non-negative matrix is primitive if $a_1 > 0$. This is in accordance with the definition of a primitive matrix, namely the Leslie matrix $L$ which has exactly one positive main diagonal entry is a primitive matrix. Even though $a_1 = 0$, $L$ remains primitive because $L^{n+2} > 0$. 


Theorem 4 A Leslie matrix has a single positive eigenvalue $\lambda_1$, this eigen value has multiplicity of one and an $x_1$ eigenvector whose all elements are positive. Based on Theorem 1, Leslie matrix $L$ meets the following

1. Leslie matrix has a positive eigenvalue $\lambda_1$, or $\lambda_1 = \rho(L) \in \sigma(L)$ and $r > 0$
2. $\lambda_1$ has the multiplicity of one or $\text{alg mult}_L(\lambda_1) = 1$
3. There exist a eigenvector $x_1 > 0$ such that $Lx = \lambda_1 x$

The following is an explanation that the irreducible Leslie matrix has a Perron vector and fulfils all the Perron-Frobenius theorems.

Because $\text{alg mult}_L(\lambda_1) = 1$ then $\lambda_1$ is a simple eigenvalue. Because simple eigenvalues are semi-simple eigenvalues, $\lambda_1$ is a semisimple eigenvalue, or $\text{alg mult}_L(\lambda_1) = \text{geo mult}_L(\lambda_1) = 1$. Such that dim $N(L - \lambda_1 I) = 1$. $N(L - \lambda_1 I)$ is a one-dimensional eigen space that can be constructed by an $x_1 > 0$, then there is a single eigenvector $p \in N(L - \lambda_1 I) \ni p > 0$ and $\sum_j p_j = 1$ which is formed from $p = \frac{x_1}{||x_1||}$. It is proven that irreducible Leslie matrix has a Perron vector $p$ which is defined as $Lp = \lambda_1 p$ with $p > 0$ and $\sum_j p_j = 1$. This Perron vector $p$ is also called the right Perron vector from the Leslie matrix $L$. That means it corresponds to the fourth properties of the Perron-Frobenius theorems.

Define $\lambda_1 = \max_{x \in N} f(x)$ with $f(x) = \min_{1 \leq i \leq n, x_i \neq 0} \frac{|Lx_i|}{x_i}$ and $N = \{x | x \geq 0, x \neq 0\}$. Because Leslie matrix $L$ is a non-negative matrix, then $L \geq 0 \iff L^T \geq 0$ and $\rho(L) = \rho(L^T)$. It is clear that if $L \geq 0$ then $(\lambda_1, p)$ is the eigen pair for $L$ and $(\lambda_1, q)$ is the eigen pair for $L^T$. Suppose $q^T = (q_1 q_2 q_3 \ldots q_n) > 0$, then $q^T(L_1 I - L) = 0$

We obtain

\[
-a_nq_1 + \lambda_1 q_n = 0 \iff q_n = \frac{a_n q_1}{\lambda_1}
\]

\[
-a_{n-1}q_1 + \lambda_1q_{n-1} - b_{n-1}q_n = 0 \iff q_{n-1} = \frac{a_{n-1}q_1 + b_{n-1}q_n}{\lambda_1}
\]

\[
q_{n-1} = \frac{a_{n-1}q_1 + b_{n-1}q_n}{\lambda_1} \iff q_{n-1} = \left(\frac{\lambda_1 a_{n-1} + \lambda_1 b_{n-1}}{\lambda_1^3}\right) q_1
\]
\[-a_3q_1 + \lambda_1 q_3 - b_3 q_4 = 0 \Leftrightarrow q_3 = \frac{-a_2q_1 + b_2q_4}{\lambda_1}\]

\[-a_2q_1 + \lambda_1 q_2 - b_2 q_3 = 0 \Leftrightarrow q_2 = \frac{-a_2q_1 + b_2q_3}{\lambda_1}\]

and \((\lambda_1 - a_1)q_1 - b_1q_2 = 0\) (5)

Substitute (4) to (5), then

\[
\begin{align*}
(\lambda_1 - a_1)q_1 - b_1 & \left[ (\lambda_1^{n-2}a_2 + \lambda_1^{n-4}a_4b_2 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_2)q_1 \right] = 0 \\
(\lambda_1 - a_1)q_1 & \left[ (\lambda_1^{n-2}a_2 + \lambda_1^{n-4}a_4b_2 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_2)q_1 \right] = 0 \\
q_1 & \left[ (\lambda_1^{n-2}a_2 + \lambda_1^{n-4}a_4b_2 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_2) \right] = 0 \\
q_1 & \left[ (\lambda_1^{n-2}a_2 + \lambda_1^{n-4}a_4b_2 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_2) \right] = 0
\end{align*}
\]

So that \(q_1 = 0\).

It is clear that \(q\) is a left Perron vector corresponding to \(\lambda_1\), so \(q\) is a non-zero vector. If \(q_1 = 0\) then the eigenvector \(q\) corresponding to \(\lambda_1\) is a zero vector. Such that, suppose \(q_1 = t\), then (1), (2), (3) and (4) can be written as

\[
q_n = \frac{a_n t}{\lambda_1} ; \quad q_{n-1} = \frac{(\lambda_1 a_{n-1} + a_n b_{n-1}) t}{\lambda_1^{n-2}}
\]

\[
q_3 = \frac{(\lambda_1^{n-3}a_3 + \lambda_1^{n-4}a_4b_3 + \cdots + \lambda_1 a_nb_{n-1}b_{n-2} \cdots b_3 + a_nb_{n-1}b_{n-2} \cdots b_3) t}{\lambda_1^{n-2}}
\]

\[
q_2 = \frac{(\lambda_1^{n-2}a_2 + \lambda_1^{n-4}a_4b_2 + \cdots + \lambda_1 a_nb_{n-1}b_{n-2} \cdots b_2 + a_nb_{n-1}b_{n-2} \cdots b_2) t}{\lambda_1^{n-1}}
\]

So eigenvector \(q\) corresponding to \(\lambda_4\) is

\[
q = \begin{pmatrix}
\frac{1}{\lambda_1^{n-2}} (\lambda_1^{n-2}a_2 + \lambda_1^{n-4}a_4b_2 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_2) \\
\frac{1}{\lambda_1^{n-1}} (\lambda_1^{n-3}a_3 + \lambda_1^{n-4}a_4b_3 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_3) \\
\vdots \\
\frac{1}{\lambda_1} (\lambda_1^{n-3}a_3 + \lambda_1^{n-4}a_4b_3 + \cdots + a_nb_{n-1}b_{n-2} \cdots b_3) \\
\frac{1}{\lambda_1} (\lambda_1 a_{n-1} + a_nb_{n-1}) \\
a_n \\
\lambda_1
\end{pmatrix} t ; \ t \in \mathbb{R} \setminus \{0\}
\]

So it proved that \(q > 0\). Because \(L^T q = \lambda_1 q\), then \(q\) is the right Perron vector for \(L^T\) or \(q^T\) is the left Perron vector for \(L\). Because
\[
\mathcal{N} = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} \mid x \geq 0, x \neq 0 \right\} \quad \text{then} \quad [Lx] = \begin{pmatrix} Lx_1 \\ Lx_2 \\ Lx_3 \\ \vdots \\ Lx_n \end{pmatrix}.
\]

If \( \xi = f(x) \) for \( x \in \mathcal{N} \), then \( 0 \leq \xi x \leq Lx \). It is known that \( p \) and \( q^T \) is the right and left Perron vector of \( L \) corresponding to Perron root \( \lambda_1 \), because \( q^T x > 0 \) and \( \xi x \leq Lx \) then

\[
\xi q^T x \leq q^T Lx = \lambda_1 q^T x \Rightarrow \xi \leq \lambda_1 \Rightarrow f(x) \leq \lambda_1 \forall x \in \mathcal{N}.
\]

Because \( f(p) = \lambda_1 \) and \( p \in \mathcal{N} \), then \( \lambda_1 = \max_{x \in \mathcal{N}} f(x) \). Therefore, Collatz Wielandt Formula applies to irreducible Leslie matrix.

4. Conclusion
Leslie matrix model is a non-negative matrix which is divided into reducible Leslie matrix and irreducible Leslie. In addition, the Leslie matrix is a primitive matrix. Leslie matrix fulfills all the properties of the Perron-Frobenius theorem.

5. References
[1] Oepomo T S 2016 An Alternating Sequence Iteration’s Method for Computing Largest Real Part Eigenvalue of Essentially Positive Matrices: Collatz and Perron-Frobenius’ Approach. J Appl Computat Math 5 334
[2] Thiem, H.R.: Comparison of spectral radii and Collatz–Wielandt numbers for homogeneous maps, and other applications of the monotone companion norm on ordered normed vector spaces. arXiv:1406.6657
[3] Friedland, S 2010 Matrices. (USA: Department of Mathematics, Statistics, and Computer Science, University of Illinois, Chicago)
[4] Li, C K and Schneider, H 2002 Applications of Perron-Frobenius Theory to Population Dynamics. Jurnal Matematical Biology 44 450-462
[5] Muchlis, A 2012 Analisis Matriks. (Bandung: Program Studi Magister Matematika, Fakultas Matematika dan Ilmu Pengetahuan Alam, Institut Teknologi Bandung)
[6] Pratama, Y, Prihandono, B and Kusumastuti, N 2013 Aplikasi Matriks Leslie Untuk Memprediksi Jumlah dan Laju Pertumbuhan Suatu Populasi. Buletin Ilmiah Math. Stat dan Terapannya (Bimaster), 2 3 163-172
[7] Fasino, D 2014 Nonnegative and Spectral Matrix Theory with Applications to Network Analysis. (Italia: University of Udine, Rome-Moscow)
[8] Horn, A R and Johnson, C R 1985 Matrix Analysis. (Cambridge University, Cambridge)
[9] Gentle, J E 2007 Matrix Algebra Theory, Computations, and Applications in Statistics. (USA: George Mason University, Fairfax County)

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