Generalized canonical approach to gyrokinetic theory

P. Nicolini* 1,2 and M. Tessarotto† 1,3

1 Dipartimento di Scienze Matematiche, Università degli Studi di Trieste (Italy).
2 Istituto Nazionale di Fisica Nucleare, Sezione di Trieste (Italy).
3 Consorzio di Magnetofluidodinamica, Università degli Studi di Trieste (Italy).

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Abstract

We face the well-known gyrokinetic problem, which arises in the description of the dynamics of a charged particle subject to fast gyration for the presence of a strong electromagnetic field. The customary approach to gyrokinetic theory, using canonical variables or identifying them “a posteriori” by means of Darboux theorem, leads to potential complications and ambiguities due to the fact that canonical coordinates are field-related. Here we propose an innovative formulation to construct gyrokinetic canonical variables based on the introduction of a new definition of canonical transformation. The new approach permits to shed light on this often controversial issue.

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I. INTRODUCTION

The “gyrokinetic problem” regards the description of the dynamics of a charged particle in the presence of suitably “intense” electromagnetic (EM) fields realized by means of appropriate perturbative expansions for its equations of motion. The expansions are usually performed with respect to the ratio \( \varepsilon = r_L/L < < 1 \), where \( L \) and \( r_L \) are respectively a characteristic scale length of the EM fields and the velocity-dependent particle Larmor radius \( r_L = \frac{\mu}{18} \), with \( \Omega_s = \frac{qB}{mc} \) the Larmor frequency and \( w \) the orthogonal component of a suitable particle velocity [see equation (47)]. The goal of gyrokinetic theory is to construct with prescribed accuracy in \( \varepsilon \) the so called “gyrokinetic” or “guiding center variables”, by means of a suitable “gyrokinetic” transformation, such that the equations of motion result independent of the gyrophase \( \phi \), being \( \phi \) the angle of fast gyration, which characterizes the motion of a charged particle subject to the presence of a strong magnetic field.

The first author who systematically investigated the gyrokinetic problem was probably Alfven\(^1\) who pointed out the existence of an adiabatic invariant, the magnetic moment \( \mu \), proportional to \( p_\phi \) the conjugate canonical momentum to \( \phi \), in the sense:

\[
\frac{d}{dt} \ln \mu \sim O(\varepsilon).
\] (1)

After subsequent work which dealt with direct construction methods of gyrokinetic variables\(^2\)–\(^\)\(^1\)\(^2\)–\(^\)\(^1\)\(^3\), a significant step forward was made by Kruskal\(^1\)\(^3\) who, first, established the consistency of the Alfven approach by proving, under suitable assumptions on the EM fields, that the magnetic moment can be constructed correct at any order \( n \) in \( \varepsilon \) in such a way that, denoting \( M \) such a dynamical variable, it results an adiabatic invariant of order \( n \), namely in the sense

\[
\frac{d}{dt} \ln M \sim O(\varepsilon^n)
\] (2)

\[M = \mu + \varepsilon \mu_1 + \ldots + \varepsilon^n \mu_n.\] (3)

A modern picture of the Hamiltonian formulation which makes easier the formulation of higher order perturbative theories, was given only later by Littlejohn\(^1\)\(^4\) in terms of a non-
canonical Lie-transform approach, adopting a suitable set of noncanonical variables. As a motivation to his noncanonical approach, Littlejohn$^{14-17}$ pointed out what in his views was a critical point of purely canonical formulations such as previously developed Lie transform approaches$^{18,19}$, namely the ambiguity in the separation of the unperturbed and perturbed contributions in the Hamiltonian due the presence of the vector potential $\mathbf{A}$ in the canonical momenta. He showed that this difficulty can be circumvented by making use of suitable non-canonical variables independent of $\mathbf{A}$ and which include the canonical pair $$(\phi, p_\phi).$$

The possibility of constructing canonical gyrokinetic variables has relied, since, on only two methods due respectively to Littlejohn$^{14-17}$ and Gardner$^{2,3}$. The first approach, and probably the most popular in the literature$^{20-23}$ is the based on the use of Darboux theorem which allows, in principle, the construction of canonical variables for an arbitrary differential 1-form. The canonical 1-form expressed in terms of the canonical variables is then obtained by applying recursively the so-called “Darboux reduction algorithm” as pointed out by Littlejohn, which is obtained by a suitable combination of dynamical gauge and coordinate transformations.

The second approach, due to Gardner$^{2,3}$, was based on a mixed-variable generating function formulation. This was used to construct the gyrokinetic canonical variables by means of a sequence of canonical transformations. In particular, an attempt to extend this approach to higher orders in $\varepsilon$ was later made by Weitzner$^{24}$.

It should be stressed that in both approaches the canonical coordinates are field-related, namely they depend on the particular geometry of the magnetic field flux lines. As a consequence the construction of canonical variables is achieved only under certain restrictions on the magnetic field (for the Lie transform approach see in particular $^{22,23}$). For example, White$^{23}$ assumed only small deviations from the axis-symmetric toroidal geometry, restricting himself to weakly chaotic magnetic fields (assumption of regularity for the magnetic field), while Weitzner$^{24}$, based on his canonical approach, conjectured that the explicit construction of the gyrokinetic canonical variables might not be always possible. He claimed, indeed, that the magnetic moment might not result single-valued for locally chaotic magnetic
fields, such as those occurring in non-symmetric MHD equilibria, for instance in Stellarators (assumption of “quasi-symmetric” magnetic field\textsuperscript{25}). An implication of his conjecture would be that for closed or ergodic orbits the magnetic moment would not be any more an adiabatic invariant. This conclusion, if proven correct, on one side might indicate a potential breakdown of gyrokinetic theory itself and on the other might point out a difficulty intrinsic to the field-related choice of the gyrokinetic canonical coordinates. On the other hand, since no such assumption of regularity is required for the magnetic field in customary noncanonical formulations, this conjecture raises also a potential contradiction between canonical and non-canonical formulations. An open issue is, therefore, the possibility of constructing more general, field-geometry independent canonical coordinates.

The goal of this paper is to propose a new solution to this problem. Our approach is based on a new definition of canonical transformations, which we call generalized canonical transformations. These transformations make use of a suitable set of superabundant gyrokinetic variables, which include in particular the canonical pair $(\phi, p_\phi)$. Basic feature of the new variables is that they can be defined independently of the magnetic flux lines geometry and do not require the use of the Darboux reduction algorithm. Basic consequences are that, on one hand, no such regularity or quasi-symmetry assumptions are required for the magnetic field, contrary to the conjecture of Weitzner, while, on the other hand, the magnetic moment in these variables results in all cases a single-valued function and therefore a physical observable.

The paper is organized as follows: we preliminary reformulate Hamiltonian mechanics, extending it to the case of superabundant coordinates. Then, we face the gyrokinetic problem, showing how a canonical transformation can be written in terms of superabundant gyrokinetic coordinates, which are shown to obey a suitable form of Hamilton modified variational principle.
II. GENERALIZED CANONICAL TRANSFORMATIONS

The purpose of this section is to extend the concept of canonical transformation. In fact, the customary definition (see for example\textsuperscript{26,27}) may result, in some instances, too restrictive for actual applications.

As is well known, the canonical transformation conventionally concerns Hamiltonian systems \(\{x, H(x,t)\}\), natively represented in terms of canonical variables \(x = (q, p)\) and obeying the Hamilton equations

\[
\dot{x} = [x, H(x,t)]
\]  

(4)

being \(H(x,t)\) the a suitably regular Hamiltonian function. In the phase space \(\Gamma\), of dimension \(2g\) (where \(g\) is the degree of freedom), spanned by the vector \(x = (q, p)\), a canonical transformation is usually defined as a \(C^{(n)}\)-diffeomorphism (with \(n \geq 2\))

\[
\gamma_C : x \rightarrow \bar{x}
\]  

(5)

to a phase space \(\Gamma'\) having the same dimension of the initial space \(\Gamma\) and satisfying the symplectic condition

\[
\mathbf{J} = \mathbf{M} \cdot \mathbf{J} \cdot \mathbf{M}^T
\]  

(6)

where \(\mathbf{M} = \frac{\partial \mathbf{x}}{\partial x}\) is the Jacobian matrix and \(\mathbf{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\) is the symplectic (Poisson) matrix of dimension \(2g \times 2g\).

It is well-known that canonical transformations can be defined also in the extended phase-space with dimension \(2g + 2\), spanned by the vector \((q, q_{g+1} = t, p, p_{g+1})\) [see\textsuperscript{26}] and characterized by a set of superabundant variables (canonical extended variables) which satisfy Hamilton equations. Such extended transformations can be regarded as a particular case of what we shall define as \textit{generalized canonical transformation}, i.e., a \(C^{(n)}\) diffeomorphism (with \(n \geq 2\))
\( \gamma_G : \mathbf{x} \rightarrow \mathbf{x}_G = \mathbf{x}_G(\mathbf{x}, t, \alpha), \) 

(7)

(with \( \alpha \) a real parameter) for which the transformed phase-space \( \Gamma_G \) can have larger dimension of the initial phase space \( \Gamma \) and is characterized by superabundant variables, subject either to Hamilton or finite terms constraint equations. More precisely, first we assume 

\[
\dim(\Gamma_G) = 2g' + k > \dim(\Gamma) = 2g,
\]

with \( g' \geq g \) and letting \( \mathbf{x}_G \) of the form \( \mathbf{x}_G = (\mathbf{z}, \mathbf{u}) \), with \( \mathbf{z} = (z_1, \ldots, z_{2g'}) \) and \( \mathbf{u} = (u_1, \ldots, u_k) \). Then, we require that \( \mathbf{z} \) and \( \mathbf{u} \) obey respectively the extended Hamilton equations (for \( i = 1, \ldots, 2g' \))

\[
\frac{d}{dt}z_i(\mathbf{x}, t, \alpha) = \sum_{j=1,2g'} J'_{ij} \cdot \frac{\partial}{\partial z_j} K(\mathbf{x}_G, t, \alpha),
\]

(8)

(with \( J'_{ij} \) the canonical Poisson tensor of rank \( 2g' \) and assuming that the temporal variable \( t \) is left invariant by the transformation) and \( k \) constraint equations of the form

\[
f_s(\mathbf{z}, \mathbf{u}, t) = 0,
\]

(9)

with \( s = 1, k \) and \( f_s(\mathbf{z}, \mathbf{u}, t) \) real \( C^{(2)} \) functions.

The components of the transformed state \( \mathbf{x}_G \) will be here denoted as superabundant canonical variables and the corresponding “equations of motion” (8), (9) generalized canonical equations.

It is immediate to point out that generalized canonical equation can be set in variational form in terms of a suitable constrained form of modified Hamilton variational principle, just as the usual canonical equations. To provide a straightforward example, which will be also used in the sequel, let us consider the case of a charged point particle subject to an EM field and defined the following generalized canonical transformation

\[
\mathbf{x} = (\mathbf{r}, \mathbf{p}) \rightarrow \mathbf{x}_G = (\mathbf{r}, \mathbf{p}, \mathbf{v}).
\]

(10)

One can prove that that the superabundant state \( \mathbf{x}_G \) is an extremal curve of the action functional

\[
S(\mathbf{x}_G) = \int_{t_1}^{t_2} dt \hat{\mathcal{L}}(\mathbf{x}_G, \dot{\mathbf{r}}, t)
\]

(11)
with the fundamental 1-form:

\[ \mathcal{L}(x_G, \dot{r}, t) dt = d\mathbf{r} \cdot \mathbf{p} - dt H(r, p, t) - [d\mathbf{r} - \mathbf{v} dt] \cdot \left[ \mathbf{p} - m\mathbf{v} - q \mathbf{A} \right] \]  

(12)

where the Hamiltonian is

\[ H(r, p, t) = \frac{1}{2m} \left[ p - q \mathbf{A} \right]^2 + q \Phi \]  

(13)

(with \{A, \Phi\} the EM potentials). It follows that the corresponding Euler-Lagrange (E-L) equations coincide with the canonical equations

\[ \dot{x} = [x, H] \]  

(14)

plus the non-holonomic constraint

\[ p = m\mathbf{v} - \frac{q}{c} \mathbf{A}. \]  

(15)

Therefore \( x_G \) defines a generalized canonical state for the Hamiltonian system.

**III. HAMILTONIAN GYROKINETIC THEORY IN SUPERABUNDANT VARIABLES**

In this section we intend to show that the previous Hamiltonian system, under a suitable assumption of “strong” EM field, can be represented by means of a appropriate set of superabundant canonical variables which are gyrokinetic, i.e. the new Hamiltonian results independent of the gyrophase \( \phi \) when expressed in terms of them. The other basic feature is that the the corresponding canonical coordinates can always be chosen to be field-independent. To construct the new gyrokinetic variables we shall follow a two-step approach.

The first step consists in constructing a particular set of hybrid\(^1\) superabundant gyroki-

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\(^1\)In some cases it is convenient to adopt, for Hamiltonian or Lagrangian systems, a set of variables, which are not Hamiltonian or Lagrangian or Newtonian. In such circumstances we shall speak of hybrid variables.
netic variables, here denoted as pseudo-canonical \( \mathbf{x}' \equiv (r', p_r', \phi', p_{\phi'}) \). For definiteness let us require that the Hamiltonian function takes the form

\[
H(\mathbf{r}, p, t) = \frac{1}{2m} \left[ p - \frac{q}{\varepsilon c} \mathbf{A} \right]^2 + \frac{q}{\varepsilon} \Phi
\]  

(16)

where \( \varepsilon \) is a real infinitesimal and assume that the EM potentials \( \Phi, \mathbf{A} \) are analytic functions of \( \varepsilon \) and can be represented in the form

\[
\Phi = \sum_{i=-1}^{\infty} \varepsilon^i \Phi_i(\mathbf{r}, t)
\]

(17)

\[
\mathbf{A} = \sum_{i=-1}^{\infty} \varepsilon^i \mathbf{A}_i(\mathbf{r}, t).
\]

(18)

In validity of this assumption the construction of hybrid gyrokinetic variables is well known and has been achieved by several authors (see for example\textsuperscript{14–17}). In this case the Lagrangian expressed in terms of gyrokinetic variables (gyrokinetic Lagrangian) reads

\[
\mathcal{L}(y', \mathbf{r}', \phi', t) = \mathbf{r}' \cdot \frac{q}{\varepsilon c} \mathbf{A}^*(\mathbf{r}', u', \mu', t) +
\]

(19)

\[
- \left( \frac{\phi'}{\Omega'_{s}} + 1 \right) \mu' b' - \frac{m}{2} \mathbf{v}'^2 - \frac{q}{\varepsilon} \Phi^*(\mathbf{r}', u', \mu', t),
\]

(20)

where the hybrid state \( y' \) is defined as follows

\[
y' \equiv (r', u', \mu', \phi'),
\]

(21)

and, ignoring for sake of simplicity\textsuperscript{2} corrections of order \( O(\varepsilon^2) \), one can directly prove that there results as a consequence

\[
\mathbf{v}'(\mathbf{r}', u', \mu', t) \equiv u' \mathbf{b}' + \mathbf{v}'_E + \varepsilon \mathbf{v}'_D,
\]

(22)

where \( \mathbf{v}'_E = c \mathbf{E}' \times \mathbf{b}' / B' \) is the electric drift velocity and \( \mathbf{v}'_D = \frac{\mathbf{b}'}{B'} \times \left\{ \frac{u'}{m} \nabla' B' + (u' \mathbf{b}' + \mathbf{v}'_E) \cdot (u' \nabla' \mathbf{b}' + \nabla \mathbf{v}'_E) \right\} \) is the diamagnetic drift velocity, both evaluated

\textsuperscript{2}In the sequel we shall omit higher order correction in \( \varepsilon \).
at the guiding center position. Here the notations are standard. Thus $b' = \mathbf{B}(\mathbf{r}', t)/\mathbf{B}(\mathbf{r}', t)$ while the primes denote quantities evaluated at the guiding center position $\mathbf{r}'$. In particular, $\mu'$ is the magnetic moment evaluated at the guiding center position. Moreover, $\{A^*, \Phi^*\}$ are the effective EM potentials, which at first order in $\varepsilon$ read

$$A^*(\mathbf{r}', u', w', t) = A' + \frac{\varepsilon mc}{q} v' [1 + O(\varepsilon)],$$

$$\Phi^*(\mathbf{r}', u', w', t) = \Phi'[1 + O(\varepsilon)].$$

(23)

(24)

To construct a set of superabundant variables, let us introduce the conjugate momenta

$$p_{\mathbf{r}'} = \frac{\partial \mathcal{L}}{\partial \left( \frac{d}{dt} \mathbf{r}' \right)} = \frac{q}{\varepsilon c} A^* \equiv m v' + \frac{q}{\varepsilon c} A',$$

$$p_{\phi'} = \frac{\partial \mathcal{L}}{\partial \left( \frac{d}{dt} \phi' \right)} = -\frac{1}{\Omega_s} \mu' B' = -\frac{mc}{q} \mu',$$

(25)

(26)

in terms of which the gyrokinetic Lagrangian becomes

$$\mathcal{L}(\mathbf{x}', \mathbf{r}', \phi', t) = \mathbf{r} \cdot p_{\mathbf{r}'} + \phi' p_{\phi'} - K(\mathbf{x}', t).$$

(27)

The superabundant state $\mathbf{x}' \equiv (\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'})$ is here denoted as pseudo-canonical and $K(\mathbf{x}', t)$ is the corresponding Hamiltonian

$$K(\mathbf{x}', t) = -p_{\phi'} \Omega + \frac{1}{2m} \left[ p_{\mathbf{r}'} - \frac{q}{\varepsilon c} A' \right]^2 + \frac{q}{\varepsilon} \Phi^*.$$  

(28)

Manifestly, the transformation

$$\mathbf{x} = (\mathbf{r}, p_{\mathbf{r}}) \rightarrow \mathbf{x}' = (\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'})$$

(29)

is not canonical, even in the generalized sense previously indicated.

The second step concerns the introduction of a further transformation to a new set of superabundant gyrokinetic variables. In particular let us consider the transformation

$$\gamma : \mathbf{x} = (\mathbf{r}, p_{\mathbf{r}}) \rightarrow \mathbf{X}' = (\mathbf{r}', p_{\mathbf{r}'}, \phi', p_{\phi'}, \mathbf{v}')$$

(30)

where $\mathbf{v}' \equiv u' b' + \mathbf{v'}_E + \varepsilon \mathbf{v}'_D$ is here considered as an independent variable. We intend to prove that the gyrokinetic state $\mathbf{X}'$ is canonical in the generalized sense defined above, namely
its components satisfy either Hamilton equations with respect to a suitable Hamiltonian function or finite-terms constraint conditions. To reach the proof, we initially notice that, by construction [see Eq.(25)], the following finite-terms constraint is satisfied by the vector $\mathbf{v}'$

$$\frac{1}{m} \left[ p_{r'} - \frac{q}{\varepsilon c} A' \right] = \mathbf{v}'. \quad (31)$$

Furthermore, using $(r', u', p_{\phi'}, \phi')$ the E-L equations for $r'$ reads, omitting higher order terms in $\varepsilon$

$$-\frac{d}{dt} p_{r'} + m \left( u' \nabla'b' + \nabla'v_E' + \frac{q}{\varepsilon mc} \nabla'A' \right) \cdot r' +$$

$$+ p_{\phi'} \nabla'\Omega - m (u' \nabla'b' + \nabla'v_E') \cdot [u'b' + v_E'] - \frac{q}{\varepsilon} \nabla'\Phi^* = 0, \quad (33)$$

from one can prove that it follows

$$r' = \mathbf{v}'. \quad (34)$$

As a consequence, there results

$$-\frac{d}{dt} p_{r'} + \frac{q}{\varepsilon c} \nabla'A' \cdot \mathbf{v}' + p_{\phi'} \nabla'\Omega - \frac{q}{\varepsilon} \nabla'\Phi^* = 0, \quad (35)$$

which can be cast in Hamiltonian form with respect to $K(x', t)$

$$\frac{d}{dt} p_{r'} = -\frac{\partial}{\partial r'} K(x', t), \quad (36)$$

where the partial derivative $\partial/\partial r'$ is defined keeping $p_{r'}$ as a constant. Analogously, from (31) and (34) one obtains the Hamilton equations

$$\frac{d}{dt} r' = \frac{\partial}{\partial p_{r'}} K(x', t) = \frac{1}{m} \left[ p_{r'} - \frac{q}{\varepsilon c} A' \right]. \quad (37)$$

Finally, in a similar way, it is immediate to prove that also $p_{\phi'}$ and $\phi'$ obey Hamilton equations

$$\frac{d}{dt} p_{\phi'} = -\frac{\partial}{\partial \phi'} K(x', t) = 0, \quad (38)$$
\[
\frac{d}{dt} \phi' = \frac{\partial}{\partial p' \phi} K(x', t) = -\Omega_s'.
\]

Thus, given the constraint condition (31), which establishes a finite-terms equation for \(v'\), the remaining variables \(x' = (r', p_r', \phi', p_{\phi'})\) satisfy the Hamilton equations, with respect to the Hamiltonian function \(K(x', t)\). As a consequence, the superabundant state \(X' = (r', p_r', \phi', p_{\phi'}, v')\) establishes a generalized canonical transformation [see for example (7)] in the transformed phase-space \(\Gamma_G\). Furthermore \(X'\) results, by construction, a gyrokinetic state and therefore the gyrophase \(\phi\) is ignorable for the generalized canonical equations (31, 36, 37, 38, 39).

The crucial feature of these variables is that the canonical coordinates \(r'\) are manifestly independent of any particular magnetic field geometry, which implies that the vector \(r'\) can be represented in the form \(r' = r'(q', t)\), being \(q' = (q'_1, q'_2, q'_3)\) arbitrary, field-independent, curvilinear coordinates, such as for instance orthogonal Cartesian coordinates. As a consequence, no restriction is placed on the magnetic field geometry for the definition of these canonical variables, contrary to previous formulations\(^{20,23}\). A fundamental consequence is that, by construction, the magnetic moment \(\mu\) can always be defined in such a way to be a single-valued function with respect with to any angle-like coordinates \(q' = (q'_1, q'_2, q'_3)\) and therefore results, in a suitable gauge, a physical observable. In particular, contrary to the conjecture of Weitzner\(^{24}\), the definition of the magnetic moment results independent of the magnetic field topology and does not require the existence of a single family of nested magnetic surfaces (quasi-symmetric magnetic field)\(^{25}\).

IV. CONSTRAINED HAMILTON MODIFIED VARIATIONAL PRINCIPLE

A further key aspect of the present approach is that the previous generalized canonical equations (namely 31, 36, 37, 38, 39) are necessarily variational, as pointed out in section II. In fact, it is easy to show that they follow from a constrained form of Hamilton modified variational principle. One can prove that this is provided in terms of the following gyrokinetic Lagrangian:
\[ L'(X', r', \phi', t) = \mathbf{r}' \cdot p + \phi' \cdot p_{\phi'} - \mathcal{K}(x', t) - \]

\[ - \left[ \mathbf{r}' - \mathbf{v}' \right] \cdot \left[ p_{r'} - m \mathbf{v}' - \frac{q}{e \varepsilon} \mathbf{A}' \right], \]

where \( \mathcal{K}(x', t) \) is the corresponding Hamiltonian

\[ \mathcal{K}(x', t) = \frac{1}{2m} \left[ p_{r'} - \frac{q}{e} \mathbf{A}' \right]^2 - \frac{1}{e} \Omega_s p_{\phi'} + \frac{q}{e} \Phi' \]

and by definition \( \mathbf{v}' \equiv u'b' + v'E + \varepsilon v'D \).

The E-L equations corresponding to (40) coincide with the previous equations of motion (31, 36, 37, 38, 39) and hence provide a variational formulation for them.

Finally, we emphasize that the constrained Lagrangian (40) defined above can also be obtained directly from the Lagrangian of a charged point particle in the presence of a strong EM field previously defined [see equation (12)]. This is achieved by means of the gyrokinetic transformation defined in terms of generalized canonical variables, i.e. of the form

\[ r = r' + \varepsilon r_1 + ... \]

\[ p_r = p_{r'} + \varepsilon p_{r_1} + ... \]

\[ \mathbf{v} = \mathbf{v}' + \varepsilon \mathbf{v}_1 + ... \]

with perturbations \( \varepsilon r_1, \varepsilon p_{r_1}, \varepsilon \mathbf{v}_1 \) to be suitably defined. This transformation can be constructed using standard techniques (see for example\(^{14-17}\)) and can be identified at any order in \( \varepsilon \) with a generalized canonical transformation. In particular, to the leading order in \( \varepsilon \) one can prove

\[ \begin{cases} 
\mathbf{r} \\
p_r \\
\mathbf{v}
\end{cases} \rightarrow \begin{cases} 
\mathbf{r} = \mathbf{r}' + \varepsilon \rho' , \\
p_r = p_{r'} + m \mathbf{w}' + \frac{q}{e \varepsilon} \mathbf{\rho}' \cdot \nabla \mathbf{A}' , \\
\mathbf{v} = \mathbf{v}' + \mathbf{w}' ,
\end{cases} \]

where \( \varepsilon \rho \) denotes the Larmor radius

\[ \varepsilon \rho = -\varepsilon \frac{\mathbf{w}' \times \mathbf{b}'}{\Omega'} . \]
The remaining notation is standard. Thus we require

\[
\begin{align*}
\mathbf{v}' & = u'b' + v'_E + \varepsilon v'_D, \\
\mathbf{w}' & = w'(e'_1 \cos \phi' + e'_2 \sin \phi'), \\
\phi' & = \arctg \left( \frac{(v-v'_E) e'_2}{(v-v'_E) e'_1} \right), \\
\end{align*}
\]

\[
w' \equiv \sqrt{2B'\mu'}, \\
p_{\phi'} = -\frac{mc}{q} \mu',
\]

where \(\mathbf{w}'\) is a vector in the plane orthogonal to the magnetic flux line. Therefore, by means of the generalized canonical variables \(\mathbf{X}'=(r',p_r',\phi',p_{\phi'},\mathbf{v}')\) gyrokinetic theory can be directly represented in canonical form without recurring to the use of hybrid gyrokinetic variables as considered by most of previous authors\(^{14-24}\)

V. FINAL REMARKS

Based on the concept of generalized canonical transformation a new set of superabundant canonical variables has been defined for the set of gyrokinetic variables, which can be used to represent the Lagrangian of a charged point particle immersed in a strong EM field. These canonical variables, contrary to the customary ones previously considered in the literature\(^{2,3,23,24}\), do not depend on the magnetic field geometry and do not require subsidiary restrictions for the magnetic field. As a consequence, gyrokinetic theory can be developed in principle at any order in \(\varepsilon\), by means of gyrokinetic transformation expressed in terms of suitable generalized canonical variables. The present formalism for its straightforward simplicity and handy construction appears susceptible of interesting new applications both in plasma theory and mathematical physics.

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