Abstract: We give a lattice theory treatment of certain one and two dimensional quantum field theories. In one dimension we construct a combinatorial version of a non-trivial field theory on the circle which is of some independent interest in itself while in two dimensions we consider a field theory on a toroidal triangular lattice.

We take a continuous spin Gaussian model on a toroidal triangular lattice with periods $L_0$ and $L_1$ where the spins carry a representation of the fundamental group of the torus labeled by phases $u_0$ and $u_1$. We compute the exact finite size and lattice corrections, to the partition function $Z$, for arbitrary mass $m$ and phases $u_i$. Summing $Z^{-1/2}$ over a specified set of phases gives the corresponding result for the Ising model on a torus. An interesting property of the model is that the limits $m \to 0$ and $u_i \to 0$ do not commute. Also when $m = 0$ the model exhibits a vortex critical phase when at least one of the $u_i$ is non-zero. In the continuum or scaling limit, for arbitrary $m$, the finite size corrections to $-\ln Z$ are modular invariant and for the critical phase are given by elliptic theta functions. In the cylinder limit $L_1 \to \infty$ the “cylinder charge” $c(u_0, m^2L_0^2)$ is a non-monotonic function of $m$ that ranges from $2(1 + 6u_0(u_0 - 1))$ for $m = 0$ to zero for $m \to \infty$ but from which one can determine the central charge $c$. The study of the continuum limit of these field theories provides a kind of quantum theoretic analog of the link between certain combinatorial and analytic topological quantities.

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§ 1. Introduction

In this paper we examine some discretely formulated quantum field theories as well as successfully computing their continuum limits. We shall consider field theories in one dimension, over the circle $S^1$, and in two dimensions, over the torus $T^2$. However a crucial feature of the fields will be that they will not be singly-periodic and doubly-periodic functions over $S^1$ and $T^2$ respectively; instead they will be sections of appropriate bundles over $S^1$ and $T^2$. These sections will then not be periodic but will possess a non-trivial holonomy when rotated round a non-trivial loop on their base space. A brief summary of some of these results has been presented in [1].

In our two dimensional model we shall be able to carry out an extensive calculation of finite size effects which are associated with the onset of continuous phase transition. These effects provide corrections to the bulk system behaviour and play a vital role in our theoretical and experimental knowledge of systems at and near criticality cf. [2]. Such effects have a complex interplay with the other features of these systems giving rise to phenomena such as crossovers from one characteristic behaviour to another [3]. Two dimensional physics has provided a framework where substantial progress has been made in understanding such problems and considerable interest has been focused on them. Some of these calculations use generalised Gaussian models [4,5] and our work can be considered in a similar spirit.

In one dimension we use a zeta function to regularise the theory and also construct the theory discretely on a polygon. We find that when the Van Vleck Morette determinant is included in the measure of the discretely calculated partition function the massless case is independent of the number $N$ of points in the polygon. In addition we find that the combinatorial and continuum partition functions agree precisely and are equal to a power of the Ray-Singer torsion for the appropriate flat bundle of sections over $S^1$.

For our two dimensional theory we obtain the exact finite size corrections to the free energy of our lattice model. These corrections form a modular invariant expression and for the critical phase this expression is expressible, in the continuum limit, in terms of elliptic theta functions and the associated partition function holomorphically factorises in the modular parameter $\tau$. The model possesses two holonomy phases (cf. section 3) $u_0$ and $u_1$ and a mass $m$; there is a critical vortex phase when the mass is zero and one of the phases is non-zero. Modular invariance persists whether $m$ is zero or not.

The principal new results contained in this work are

(i) A demonstration that the combinatorial torsion arises naturally when the Van Vleck-Morette determinant is incorporated in the integration measure.

(ii) The determination of the exact partition function for a massive Gaussian model where the spin variables are sections over a generic triangulated torus.

(iii) The demonstration that the finite size corrections are modular invariant irrespective of the value of the mass and the demonstration that in the massless case they are related to a topological quantity known as the $\bar{\partial}$-torsion.

(iv) The determination of equivalent data for the two dimensional Ising model on a trinangular lattice with three independent nearest neighbour couplings.

The organisation of the material in the rest of this paper is as follows. In section two we discuss our one dimensional theory in its continuum and in its discrete formulation.
Section three moves on to two dimensions where we formulate a gauge theory based on flat bundles over the torus $T^2$ and then construct the associated lattice model. This section contains some of our basic results on modular invariance both at and near the critical phase. In section four we deal with bulk properties the lattice model including those due to finite size and finite lattice spacing. We also examine the conformal properties of the model. Finally section five contains our concluding remarks.

§ 2. A twisted lattice theory in one dimension

We start in one dimension with a gauge theory on a circle $S^1$ of circumference $L$. Any connection $A$ of such a theory necessarily has zero curvature; nevertheless $A$ can still be non-trivial since it can have non-trivial holonomy. The action $S$ of our theory is given by

$$S = \frac{1}{2} \int_{S^1} (\bar{f} d^* E d E f + m^2 \bar{f} f)$$

(2.1)

where (as mentioned in the introduction) $f$ is not a function on $S^1$ but a section of a flat bundle $E$ over $S^1$, finally $d_E$ is the covariant derivative which acts on such sections.

A few details about the flat bundle $E$: The sections of $E$ must have non-trivial holonomy and, for simplicity in the subsequent discretisation of the theory, we take the holonomy group to be $U(1)$. Now a general, rank $n$, vector bundle $E$ over $S^1$ is a sum of appropriate powers of line bundles $L_i$ i.e.

$$E = L_1^{\alpha_1} \oplus L_2^{\alpha_2} \oplus \cdots \oplus L_n^{\alpha_n}, \quad \text{with} \quad \alpha_i \in \mathbb{Z}$$

(2.2)

where $L_i$ is a single line bundle over $S^1$.

Because a completely general $E$ is expressible in terms of line bundles $L_i$ we shall just treat the special case where $E$ is a single line bundle and take $E = \mathcal{L}$ a section $f$ of which has the property that

$$f(x + n L) = e^{2\pi i n u} f(x).$$

(2.3)

The computation of the partition function for a general $E$ (of rank $n$) is an easy extension of the line bundle case.

Now the partition function $Z(S^1, \mathcal{L}, m)$ for the action $S$ above is given by the functional integral

$$Z(S^1, \mathcal{L}, m) = \int \mathcal{D} f \mathcal{D} \bar{f} \exp \left[ -\frac{1}{2} \int_{S^1} \bar{f} (d^* \mathcal{L} d \mathcal{L} + m^2) f \right]$$

(2.4)

This is immediately calculable (cf. [6] and [7] and references therein) and is given by

$$Z(S^1, \mathcal{L}, m) = \left\{ \det \left[ \frac{d^* \mathcal{L} d \mathcal{L} + m^2}{\mu^2} \right] \right\}^{-1}$$

(2.5)

where $\mu$ is a mass scale introduced to render the partition function dimensionless. In the above equation $\det \left[ \frac{d^* \mathcal{L} d \mathcal{L} + m^2}{\mu^2} \right]$ is given a meaning by defining it through the $\zeta$-function regularization procedure. To implement this procedure we need to find the eigenvalues of...
the operator $d^*_L d_L + m^2$. Eigensections replace eigenfunctions and a basis of eigensections is given by
\[ e_n(x) = \frac{1}{\sqrt{L}} \exp^{2\pi i (n+u)L} \] (2.6)

Now if we define the inner product between sections by
\[ <f, g> = \int_0^L dx \bar{f}(x) g(x) \] (2.7)

The relation $<d_L f, g> = <f, d^*_L g>$ defines the action of $d^*_L$ and we find the eigenvalues are
\[ \lambda_n = \left( \frac{2\pi (n+u)}{L} \right)^2 + m^2 \] (2.8)

The $\zeta$-function of interest to us is then
\[ \zeta_{d^*_L d_L + m^2}(s) = \left( \frac{\mu L}{2\pi} \right)^{2s} \sum_{n=-\infty}^{\infty} \frac{1}{((n+u)^2 + (\frac{mL}{2\pi})^2)^s} \] (2.9)

The easiest way to evaluate the sum is to use the Plana sum formula [8] which can be cast in the form
\[ \sum_{n=-\infty}^{\infty} \frac{1}{((n+u)^2 + x)^s} = \frac{\pi^{1/2} \Gamma(s - \frac{1}{2})}{\Gamma(s)} x^{-s+\frac{1}{2}} + \frac{\sin(\pi s)}{\pi} \int_0^\infty dq q^{-2s} \frac{d}{dq} \ln \left| 1 - e^{-2\pi \sqrt{q^2 + x + 2\pi i u}} \right|^2 \] (2.10)

From which we deduce that
\[ \zeta_{d^*_L d_L + m^2}(0) = 0 \] (2.11)

and
\[ \zeta'_{d^*_L d_L + m^2}(0) = -mL - \ln \left| 1 - e^{-mL + 2\pi i u} \right|^2 \] (2.12)

Notice that the arbitrary undetermined scale $\mu$ has completely disappeared from the answer and we are led to the surprisingly elegant result that the
\[ \det \left[ d^*_L d_L + \frac{m^2}{\mu^2} \right] = \left| 1 - e^{-mL + 2\pi i u} \right|^2 e^{mL} \] (2.13)

and hence $\Gamma = -\ln Z(S^1, L, m)$ is given by
\[ \Gamma = mL + \ln \left| 1 - e^{-mL + 2\pi i u} \right|^2 \] (2.14)

In the massless limit, $m = 0$, we find that $Z(S^1, L, 0)$ is independent of the circumference $L$ of $S^1$, a reflection of metric independence. In fact the resulting expression is a topological invariant of the bundle
\[ Z(S^1, \mathcal{L}, 0) = \frac{1}{T(S^1, \mathcal{L})} \] (2.15)
where $T(S^1, L)$ denotes the Ray-Singer analytic torsion\(^{(a)}\) of the line bundle $\mathcal{L}$ and is given by

$$T(S^1, \mathcal{L}) = \{2 \sin(\pi u)\}^2.$$  

Though the resulting expressions above are elegant many aspects of the procedure that leads to them are mysterious. For example, from general principles one would expect that the partition function would have a term proportional to the number of degrees of freedom, which would give rise to a divergent contribution in the continuum limit. It is therefore illuminating to examine the above theory from a lattice point of view. We therefore turn to its discretisation.

To this end we replace the circle $S^1$ with its natural combinatorial counterpart namely a polygon with $N$ vertices. The discretised basis of eigensections is then take to be

$$\left\{ \frac{1}{\sqrt{Na}} e^{i x_n k} : k = 1, \ldots, N \right\} \text{ where } x_n = \frac{2\pi(n + u)}{N}, \ n = 0, \ldots, N - 1. \quad (2.16)$$

where the lattice spacing $a = \frac{L}{N}$. Since the connection is locally trivial the action of the discretised derivative $d\mathcal{L}$ on sections can be taken to be

$$d\mathcal{L} f(k) = \frac{f(k+1) - f(k)}{a} \quad (2.17)$$

The discretised inner product becomes

$$< f, g > = a \sum_{k=0}^{(N-1)} \bar{f}(k)g(k) \quad (2.18)$$

and hence

$$d'^\ast\mathcal{L} f(k) = \frac{f(k-1) - f(k)}{a}. \quad (2.19)$$

Applying these rules to the basis $e_n(k)$ implies that

$$d\mathcal{L} e_n(k) = \frac{2i}{a} e^{i \frac{2\pi}{N} n} \sin \left( \frac{x_n}{2} \right) e_n(k)$$

and

$$d'^\ast\mathcal{L} e_n(k) = -\frac{2i}{a} e^{-i \frac{2\pi}{N} n} \sin \left( \frac{x_n}{2} \right) e_n(k) \quad (2.20)$$

and so we find that the finite dimensional lattice version of $\det[d\mathcal{L}^\ast d\mathcal{L} + m^2]$, which we shall denote by $\det_N[d\mathcal{L}^\ast d\mathcal{L} + m^2]$, is given by

$$\det_N[d\mathcal{L}^\ast d\mathcal{L} + m^2] = \prod_n \lambda_n, \text{ where } \lambda_n = \frac{1}{a^2} \sin(x_n)^2 + m^2 \quad (2.21)$$

\(^{(a)}\) The equality of the combinatorial and the analytic torsion for $\mathcal{L}$ provides one with a topological derivation of Lerch’s classical formula: $\zeta_u'(0) = -2 \ln 4 - 4 \ln \sin(\pi u)$, where $\zeta_u(s) = \sum_{n=-\infty}^{\infty} 2/(n+u)^2$. In fact this is pointed out in Cheeger’s paper \[9\] though the details are different: Cheeger considers, not a (complex) line bundle $\mathcal{L}$, but a (real) $SO(2)$ bundle $E.$
The corresponding lattice partition function we denote by $Z_N(S^1, \mathcal{L}, m)$, and it is, of course, automatically finite. We have

$$Z_N(S^1, \mathcal{L}) = \int \prod_k d\phi_1(k)d\phi_2(k) \exp[-S] \quad \text{where} \quad S = \frac{1}{2} < d\mathcal{L} \varphi, d\mathcal{L} \varphi > . \quad (2.22)$$

We have taken $\phi = \phi_1 + i\phi_2$ and $\varphi = \sqrt{a}\phi$, then $\varphi$ corresponds to the usual continuum field and $\phi$ is a dimensionless lattice field. This identification renders the partition function dimensionless as we require it to be and in fact we have

$$Z_N(S^1, \mathcal{L}, m) = \left\{ \det_N \left( \frac{d^* d + m^2 a^2}{2\pi} \right) \right\}^{-1} \quad (2.23)$$

Defining $W = -\ln Z_N$ we find that $W$ is given by the sum

$$W = \sum_{n=0}^{N-1} \ln \frac{\lambda_n a^2}{2\pi} \quad (2.24)$$

$$= -N \ln \pi + \sum_{n=0}^{N-1} \ln \left[ 1 + \frac{m^2 a^2}{2} - \cos(x_n) \right]$$

The sum can be performed using the basic identity

$$\sum_{n=0}^{N-1} \ln [z - \cos(x_n)] = N \ln \left[ \frac{z_+}{2} \right] + \ln |1 - z e^{2\pi i u}|^2 \quad (2.25)$$

where $z_\pm = z \pm \sqrt{z^2 - 1}$. Hence we find that $W$ splits in the form

$$W = N W_B + W_F$$

where the extensive term is known as the bulk contribution. The quantity $W_B$ is the contribution to $W$ per lattice site in the thermodynamic limit i.e.

$$W_B = \lim_{N \to \infty} \frac{W}{N}$$

and is given by

$$W_B = \ln \left[ 1 + \frac{m^2 a^2}{2} + \sqrt{\frac{m^2 a^2 + m^4 a^4}{4}} \right] - \ln 2\pi. \quad (2.26)$$

The remaining term $W_F$ captures the effects of a finite lattice and is given by

$$W_F = \ln |1 - \left( 1 + \frac{m^2 a^2}{2} - \sqrt{\frac{m^2 a^2 + m^4 a^4}{4}} \right) e^{2\pi i u}|^2 \quad (2.27)$$
The continuum or scaling limit is a constrained thermodynamic limit and corresponds to taking $N \to \infty$ while keeping $Na = L$ fixed. In this limit we have

$$NW_B = -N \ln 2\pi + \Gamma_B$$

where

$$\Gamma_B = mL$$

and

$$\Gamma_F = \ln |1 - e^{-mL+2\pi i u}|^2$$

We see therefore that the quantity $W = N \ln 2\pi$ tends to a finite continuum limit which in fact coincides with the result provided by the $\zeta$-function prescription for the functional determinant.

It is clear from the above that we could alter the integration measure to absorb the term $-N \ln 2\pi$. This is most naturally done by including the Van Vleck-Morette determinant $J$ in the integration measure so that

$$Z_c^c(S^1, L, m) = \int \prod_k d\phi(k) d\phi(k') J \exp[-S]$$

where

$$J = \left\{ \det \left( \frac{1}{2\pi} \frac{\partial^2 S}{\partial \phi_i(k) \partial \phi_j(k+1)} \right) \right\}^{1/2}.$$

The resulting partition function $Z_c^c(S^1, L, m)$ is given by

$$Z_c^c(S^1, L, m) = z_+^N |1 - z_\pm^N e^{2\pi i u}|^2$$

where

$$z_\pm = 1 + \frac{m^2 a^2}{2} \pm \sqrt{m^2 a^2 + \frac{m^4 a^4}{4}}$$

With $L = Na$ fixed we have

$$\lim_{N \to \infty} Z_c^c(S^1, L, m) = Z(S^1, L, m)$$

But notice that if $m = 0$ then

$$Z_c^c(S^1, L, 0) = \frac{1}{T(S^1, L)}$$

In other words the discrete partition function with the Van Vleck-Morette determinant included in the continuum limit yields exactly the expression obtained by the $\zeta$-function method; in addition in the massless limit this lattice partition function $Z_N^c(S^1, L, 0)$ coincide exactly with $Z(S^1, L, 0)$ and is therefore inversely proportional to the torsion without the need for a limit. The partition function $Z_N^c(S^2, \mathcal{L}, 0)$ is therefore closely related to the combinatorial torsion and can be viewed as providing a realization of this object.

§ 3. A two dimensional theory at genus one

Now we go to two dimensions where we want to study a field theory on a Riemann surface of genus one i.e. a torus which we denote by $T^2$. We shall then go on to discretise with a
specific triangulation to be described below. We begin by we providing a brief summary of the continuum setting.

First of all let us realise the continuum torus $T^2$ as a complex manifold using the standard quotient

$$T^2 = \mathbb{C}/\Gamma$$

with $\Gamma = \mathbb{Z} \oplus \mathbb{Z} \simeq \pi_1(T^2)$

Next, because we are going to have flat connections, we need a flat bundle $E$ over $T^2$, this is obtained by choosing an element $\chi$ of

$$\text{Hom}(\pi_1(T^2), G)$$

where $G$ is the bundle structure group. We take $E$ to be a line bundle $L$ so that $G = U(1)$. Then $\chi \in \text{Hom}(\pi_1(T^2), G)$ is of the form

$$\chi : \pi_1(T^2) \longrightarrow S^1$$

and $\chi$ is then actually a character of $\pi_1(T^2)$ which is the reason for adopting the present notation.

The action we choose is given by

$$S = \frac{1}{2} \int_{T^2} \bar{\nu}(d^*_\mathcal{L}d_\mathcal{L} + m^2)\nu$$

where $m$ is a mass. Now the torus is a Kähler manifold and so has the property that $d^*_\mathcal{L}d_\mathcal{L} = 2\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}$, where $\bar{\partial} = \partial/\partial \bar{z}$. This means that the partition function $Z(T^2, \mathcal{L}, m)$ for this action gives the $\bar{\partial}$-torsion if we set $m = 0$, i.e.

$$Z(T^2, \mathcal{L}, 0) = \int D\nu D\bar{\nu} \exp \left[ -\frac{1}{2} \int_{T^2} \bar{\nu}d^*_\mathcal{L}d_\mathcal{L}\nu \right]$$

$$= \left\{ \det \left( \frac{d^*_\mathcal{L}d_\mathcal{L}}{\mu^2} \right) \right\}^{-1}$$

$$= \{ T_{\bar{\partial}}(T^2, \mathcal{L}) \}^{-1}$$

where $T_{\bar{\partial}}(T^2, \mathcal{L})$ denotes the $\bar{\partial}$-torsion\(^{(b)}\) and $\det(\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L})$ is calculated (as is the analytic $\bar{\partial}$-torsion) using a zeta function and it is useful to note in the present context that

$$d^*_\mathcal{L}d_\mathcal{L} = 2\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}$$

$$\Rightarrow \zeta_{\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}}(0) = \zeta_{\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}}(0) - \ln(2)\zeta_{\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}}(0)(0)$$

$$= \zeta_{\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}}(0), \text{ since calculation will show that } \zeta_{\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L}}(0)(0) = 0$$

$$\Rightarrow \det(d^*_\mathcal{L}d_\mathcal{L}) = \det(\bar{\partial}^*_\mathcal{L}\bar{\partial}_\mathcal{L})$$

\(^{(b)}\) The $\bar{\partial}$-torsion is the complex analogue of the ordinary Ray-Singer analytic torsion $T(M, E)$. For even dimensional $M$ the ordinary torsion is always unity; however if the manifold $M$ is complex then there exists the analytic $\bar{\partial}$-torsion which is non-trivial cf. [10].
Now the $\bar{\partial}$-torsion $T_{\bar{\partial}}(T^2, \mathcal{L})$ is a rather complicated object and was shown in [10] to be expressible in terms of a ratio containing a Jacobi theta function and the Dedekind eta function. Before giving $T_{\bar{\partial}}(T^2, \mathcal{L})$ we need to give a parametrisation of the homomorphism $\chi$ and to do this we use a notation similar, but not identical, to that of [10]: Let the torus $T^2$ be denoted in standard fashion by the usual parameter $\tau \in \mathbb{C}^+$ ($\mathbb{C}^+$ denotes the upper complex half-plane $Im \ z > 0$), the generators of $\pi_1(T^2)$ are denoted by the pair $\{1, \tau\}$ and an element of $\pi_1(T^2)$ by the pair of integers $(m, n)$. Then if $u_0, u_1 \in [0, 1]$ one has

$$\chi : \pi_1(T^2) \rightarrow S^1$$

$$(m, n) \mapsto \exp[2\pi i(mu_0 + nu_1)] \quad (3.8)$$

While for the $T_{\bar{\partial}}(T^2, \mathcal{L})$ we have

$$T_{\bar{\partial}}(T^2, \mathcal{L}) = \left| \exp[\pi i u_0^2 \tau] \frac{\theta_1(u_1 - \tau u_0, \tau)}{\eta(\tau)} \right|^2 \quad (3.9)$$

We finish this continuum summary with a description of the continuum spectral data for $\bar{\partial} \partial \mathcal{L}$: For simplicity we shall work with the Laplacian $d^*_\mathcal{L} d_\mathcal{L}$ and denote it by $\Delta_\mathcal{L}$; its eigensections by $\mathcal{E}_{mn}(x, y)$ and eigenvalues by $\lambda_{mn}$ can be computed fairly straightforwardly. The details are as follows: let $z = x + iy$ denote the local coordinate in $\mathbb{C}$ then take the torus to have periods $L_0$ and $L_1$ but at an angle $\theta$ to one another. Then a point $z$ and $z + (m\tau + n)L_0$ represents points that are identified under the quotient action (3.1) where

$$\tau = \tau_0 + i\tau_1 \quad with \quad \tau_0 = \frac{L_1}{L_0} \cos \theta \quad and \quad \tau_1 = \frac{L_1}{L_0} \sin \theta. \quad (3.10)$$

It is sensible to change $(x, y)$ to new (real) variables $(x_0, x_1)$ defined by writing

$$z = x_0 + x_1 \cos \theta + ix_1 \sin \theta.$$

In these new co-ordinates the metric $ds^2 = dx^2 + dy^2$ becomes $ds^2 = dx_0^2 + 2 \cos \theta dx_0 dx_1 + dx_1^2$ and the Laplacian $\Delta_\mathcal{L}$ becomes

$$\Delta_\mathcal{L} = -\frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial x_0^2} - \cos \theta \frac{\partial^2}{\partial x_0 \partial x_1} + \frac{\partial^2}{\partial x_1^2} \right) \quad (3.11)$$

It is then easy to find that the eigensections are now given by

$$\mathcal{E}_{mn}(x_0, x_1) = \exp \left[ 2\pi i \left\{ \frac{(n_0 + u_0)}{L_0} x_0 + \frac{(n_1 + u_1)}{L_1} x_1 \right\} \right] \quad (3.12)$$

Note that this means that the $\mathcal{E}_{mn}(x_0, x_1)$ are not periodic, being sections of $\mathcal{L}$, but have the holonomy predictable from the character $\chi$ defined in 3.8 above. The eigenvalues can be easily found and are given by

$$\lambda_{mn} = \left( \frac{2\pi}{L_0 \tau_1} \right)^2 [ |\tau|^2 (n_0 + u_0)^2 - 2(n_0 + u_0)(n_1 + u_1)\tau_0 + (n_1 + u_1)^2 ]$$

$$= \left( \frac{2\pi}{L_0 \tau_1} \right)^2 |(n + u_1) - \tau(m + u_0)|^2 \quad (3.13)$$
A computation of \( \det \Delta_L \) by the \( \zeta \)-function method establishes (3.6) where \( T_{\bar{\partial}}(T^2, L) \) is given by (3.9).

This completes our brief description of the continuum spectral data and we pass on to the matter of discretisation where we shall be able to reproduce exactly the continuum partition function—the inverse of the \( \bar{\partial} \)-torsion—in calculating the finite size effects on the lattice.

To this end we now replace the \( \tau \) parallelogram by a discrete lattice which we then triangulate as follows: To retain complete generality over the geometry of the triangulation we construct a triangular lattice composed of similar triangles, pairs of which form parallelograms. The basic triangles have two sides of lengths \( a_0 \) and \( a_1 \) with an angle \( \theta \) between them. The complete lattice forms our torus, \( T^2 \), and consists of \( K_0 K_1 \) sites and \( 2K_0 K_1 \) triangles, forming a parallelogram of sides \( L_0 = K_0 a_0 \) and \( L_1 = K_1 a_1 \).

The resulting lattice is depicted in fig. 1 and we can use this figure to deduce that with this model for the torus a point \( z \in \mathbb{C} \) is identified with points of the form

\[
(z + m L_0 + n L_1 \exp[i \theta]) \quad \text{where} \quad m, n \in \mathbb{Z}.
\]

This geometric information also determines the holonomy of a section \( \varphi \) of the bundle \( L \) over \( T^2 \) which we record in the equation

\[
\varphi(x + m L_1 \cos \theta + n L_0, y + m L_1 \sin \theta) = e^{2 \pi i (m u_1 + n u_0)} \varphi(x, y), \quad m, n \in \mathbb{Z} \quad (3.15)
\]

We shall take a statistical mechanical standpoint and compute the energy of the lattice configuration rather than the, equally possible, approach of computing its action. To do this, we label the lattice sites by \( (k_0, k_1) \equiv k = k_0 + (k_1 - 1) K_0 \) with \( k_i = 1, \ldots, K_i \), and the energy is then given by

\[
\mathcal{E}_L[T^2, \varphi^*, \varphi] = \frac{1}{2} \sum_{kk'} \sqrt{g} \varphi^*(k) \left( \Delta(k, k') + m^2 \delta_{k,k'} \right) \varphi(k') \quad (3.16)
\]

where \( \sqrt{g} = a_0 a_1 \sin \theta \).

The discrete eigensections of \( \Delta \) are \( \mathcal{E}_{n_0 n_1}(k_0, k_1) \) where

\[
\mathcal{E}_{n_0 n_1}(k_0, k_1) = \exp \left[ 2 \pi i \left( (n_0 + u_0) \frac{k_0}{K_0} + (n_1 + u_1) \frac{k_1}{K_1} \right) \right] \quad (3.17)
\]

and the discrete version of the Laplacian \( \Delta \) is a \( K_0 K_1 \times K_0 K_1 \) symmetric matrix \( \Delta(K_0, K_1) \) a general element of which we denote by \( \Delta\{(k_0, k_1), (k_0', k_1')\} \) so that we have

\[
\Delta(K_0, K_1) = [\Delta\{(k_0, k_1), (k_0', k_1')\}]_{K_0 K_1 \times K_0 K_1}
\]

with \( 1 \leq k_0, k_0' \leq K_0 \), and \( 1 \leq k_1, k_1' \leq K_1 \) (3.18)
Using a nearest neighbour interaction all the non-zero entries of $\Delta(K_0, K_1)$ can be deduced by its symmetry and by recording explicitly that

$$
\begin{align*}
\Delta\{(k_0, k_1), (k_0 + 1, k_1)\} &= -\alpha = -\frac{1}{\sin^2 \theta} \left( \frac{1}{a_0^2} - \frac{\cos \theta}{a_0 a_1} \right) \\
\Delta\{(k_0, k_1), (k_0, k_1 + 1)\} &= -\beta = -\frac{1}{\sin^2 \theta} \left( \frac{1}{a_1^2} - \frac{\cos \theta}{a_0 a_1} \right) \\
\Delta\{(k_0 + 1, k_1), (k_0, k_1 + 1)\} &= -\gamma = -\frac{1}{\sin^2 \theta} \left( \frac{\cos \theta}{a_0 a_1} \right) \\
\Delta\{(k_0, k_1), (k_0, k_1)\} &= 2\sigma, \quad \text{where } \sigma = \alpha + \beta + \gamma
\end{align*}
$$

(3.19)

the remaining elements being zero. The lattice eigenvalues are then

$$
\lambda_{n_0, n_1} = 2 \left( \sigma - \alpha \cos (x_{n_0}) - \beta \cos (x_{n_1}) - \gamma \cos (x_{n_0} - x_{n_1}) \right)
$$

where

$$
x_{n_i} = \frac{2\pi(n_i + u_i)}{K_i}, \quad i = 0, 1
$$

(3.20)

The partition function that interests us is

$$
Z_{K_0, K_1}(T^2, \mathcal{L}, m) = \int \left[ \prod d\varphi^* d\varphi \right] e^{-\mathcal{L}[T^2, \varphi^*, \varphi]} = \prod_{(n_0, n_1)} \left\{ \frac{2\pi}{\sqrt{g(\lambda_{n_0, n_1} + m^2)}} \right\}
$$

(3.21)

and $W = -\ln Z$.

We pass now to the lattice determinant in which we are interested namely

$$
\det \left( \frac{\Delta(k, k') + m^2 \delta_{k, k'}}{2} \right) = \prod_{(n_0, n_1)} \left( \frac{\lambda_{n_0, n_1} + m^2}{2} \right)
$$

(3.22)

The computation of this determinant is a non-trivial task but one that we shall accomplish.

Purely for convenience of the reader following the calculation we set $K_0 = K_1 = K$—all our results can be derived without making this specialisation and we shall quote formulae later (in the next section) without this restriction imposed. Now we want to compute $\prod_{(n_0, n_1)} (\lambda_{n_0, n_1} + m^2)/2$ and it suits us to compute the logarithm of this quantity which we shall denote by $\ln \det_K ((\Delta + m^2)/2)$ for short, i.e.

$$
\ln \det_K \left( \frac{\Delta + m^2}{2} \right) = \sum_{n_0, n_1 = 0}^{(K-1)} \ln \left( \frac{\lambda_{n_0, n_1} + m^2}{2} \right)
$$

(3.23)

It turns out that we can actually make sense of this expression and even evaluate its $K \to \infty$ continuum limit where we shall find a modular invariant expression. A key step
in achieving this is that we do one of the sums in 3.23 completely. We now outline how
this latter step comes about.
Let us define \( \delta \), by
\[
\delta = \sigma + m^2 / 2,
\]
so that using 3.20 this implies that
\[
\lambda_{n_0 n_1} + m^2 = 2 (\delta - \alpha \cos (x_{n_0}) - \beta \cos (x_{n_1}) - \gamma \cos (x_{n_0} - x_{n_1})) \tag{3.24}
\]
Define the new quantities, \( b_{n_1}, z_{n_1} \) and \( \theta_{n_1} \) by writing
\[
b_{n_1} = (\alpha + \gamma \exp[-ix_{n_1}]) \equiv |b_{n_1}| \exp[i\theta_{n_1}], \quad z_{n_1} = \frac{(\delta - \beta \cos(x_{n_1}))}{|b_{n_1}|}, \tag{3.25}
\]
\[
\Rightarrow [\delta - \alpha \cos(x_{n_0}) - \beta \cos(x_{n_1}) - \gamma \cos(x_{n_0} - x_{n_1})] = |b_{n_1}| ([z_{n_1} - \cos(x_{n_0} + \theta_{n_1})])
\]
Then we factorise \((z_{n_1} - \cos(x_{n_0} + \theta_{n_1}))\) by observing that
\[
(z_{n_1} - \cos(x_{n_0} + \theta_{n_1})) = \frac{z_+}{2} (1 - z_- \exp[i(x_{n_0} + \theta_{n_1})]) (1 - z_- \exp[-i(x_{n_0} + \theta_{n_1})])
\]
where \(z_\mp = z_{n_1} \mp \sqrt{z_{n_1}^2 - 1}\).\tag{3.26}
Hence the sum over \( n \) in 3.23 requires us to compute
\[
\ln \det K \left( \frac{\Delta + m^2}{2} \right) = \sum_{n_0, n_1=0}^{K-1} \ln \left( \frac{\lambda_{n_0 n_1} + m^2}{2} \right)
\]
\[
\quad = \sum_{n_0, n_1=0}^{K-1} \ln |b_{n_1}| (z_{n_1} - \cos(x_{n_0} + \theta_{n_1})) \tag{3.27}
\]
\[
\quad = K \sum_{n_1=0}^{K-1} \ln \left( \frac{z_+ |b_{n_1}|}{2} \right) + \sum_{n_0, n_1=0}^{K-1} \ln |1 - z_- \exp[i(x_{n_0} + \theta_{n_1})]|^2
\]
But
\[
\sum_{n_0=0}^{K-1} \ln (1 - z_- \exp[i(x_{n_0} + \theta_{n_1})]) = - \sum_{n_0=0}^{K-1} \sum_{r=1}^{\infty} \left(\frac{z_-}{r}\right)^r \exp[i r (x_{n_0} + \theta_{n_1})]
\]
\[
\quad = - \sum_{r=1}^{\infty} \left(\frac{z_- \exp[i \theta_{n_1}]}{r}\right)^r \sum_{n_0=0}^{K-1} \exp[i r x_{n_0}]
\]
\[
\quad = - \sum_{l=1}^{\infty} \left(\frac{z_- \exp[i \theta_{n_1}]}{lK}\right)^l K \exp[2\pi i u_0 l]
\]
\[
\quad = \ln (1 - v_{n_1})^K
\]
where \(v_{n_1} = z_- \exp[i \theta_{n_1} \exp[2\pi i u_0 / K]]\)
Now\textsuperscript{(c)} we note that we have entirely done one of the summations. It turns out that we can do still more: for the first term \( \ln \left( \frac{z_+ |b_{n_1}|}{2} \right) \) we can accomplish the remaining sum over \( n_1 \). We have

\[
\ln \left( \frac{z_+ |b_{n_1}|}{2} \right) = \ln \left[ \frac{1}{2} \left( \frac{A_{n_1}}{2} + \sqrt{\frac{A_{n_1}^2}{4} - |b_{n_1}|^2} \right) \right] \tag{3.30}
\]

where \( A_{n_1} = 2\delta - 2\beta \cos(x_{n_1}) \)

But using the integral identity

\[
\int_0^{\pi/2} \frac{d\omega}{\pi} \ln(z^2 - \lambda^2 \cos^2(\omega)) = \ln \left( \frac{z + \sqrt{z^2 - \lambda^2}}{2} \right) \tag{3.31}
\]

we find that

\[
\ln \left( \frac{z_+ |b_{n_1}|}{2} \right) = \int_0^{\pi/2} \frac{d\omega}{\pi} \ln \left[ \left( \frac{A_{n_1}}{2} - |b_{n_1}|^2 \cos^2(\omega) \right) \right] \tag{3.32}
\]

Now we have to do the summation and evaluate

\[
\sum_{n_1=0}^{(K-1)} K \ln \left( \frac{z_+ |b_{n_1}|}{2} \right) = \sum_{n_1=0}^{(K-1)} K \int_0^{\pi/2} \frac{d\omega}{\pi} \ln \left[ \left( \frac{A_{n_1}}{2} - |b_{n_1}|^2 \cos^2(\omega) \right) \right] \tag{3.33}
\]

Next we reuse the two properties summarised in 3.26 and 3.28 to do the summation. But first we must rewrite 3.33 in a suitable factorised form: noting that \( A_{n_1} \) is quadratic in \( \cos(x_{n_1}) \) we get

\[
A_{n_1}^2 / 4 - |b_{n_1}|^2 \cos^2(\omega) = \beta^2 (\cos^2(x_{n_1}) - 2q \cos(x_{n_1}) + 2a) = \beta^2 (\cos(x_{n_1}) - x_0)(\cos(x_{n_1}) - x_1)
\]

with

\[
\begin{align*}
x_0 &= q + \sqrt{q^2 - a^2} \\
x_1 &= q - \sqrt{q^2 - a^2}
\end{align*}
\]

So 3.26 \( \Rightarrow \) \( \frac{A_{n_1}^2}{4} - |b_{n_1}|^2 \cos^2(\omega) = \beta^2 \frac{x_0^2 + x_1^2}{2} |1 - x_0^2|^2 |1 - x_1|^2 \)

where

\[
\begin{align*}
x_0^\pm &= x_0 \mp \sqrt{(x_0^2)^2 - 1} \\
x_1^\pm &= x_1 \mp \sqrt{(x_1)^2 - 1}
\end{align*}
\]

\textsuperscript{(c)} In 3.28 above we use the fact that \( \exp[2\pi ir/K] \) are the \( K \)th roots of unity so that

\[
\sum_{r=0}^{(K-1)} \exp[2\pi ir/K] = \begin{cases} K, & \text{if } n = 0 \mod K \\ 0, & \text{otherwise} \end{cases} \tag{3.29}
\]

a similar algebraic trick was used in [11].
If we now employ 3.28 we find that we can do the sum over \( n_1 \) leaving us with a trivial two term sum originating in the roots of the quadratic. We get

\[
\sum_{n_1=0}^{(K-1)} K \ln \left( \frac{z_+|b_{n_1}|}{2} \right) = K \sum_{i=0}^{1} \left\{ \int_0^{\pi/2} \frac{d\omega}{\pi} K \ln \left( \frac{\beta x_+^i}{2} \right) + \ln |1 - (s_+^i)K|^2 \right\}
\]

(3.35)

where \( s_+^i = x_+^i \exp \left[ \frac{2\pi i u_1}{K} \right] \), \( i = 0, 1 \)

Thus we have accomplished our summation goal and have found for \( \ln \det K \left( \frac{\Delta + m^2}{2} \right) \) the expression

\[
\ln \det K \left( \frac{\Delta + m^2}{2} \right) = K^2 \int_0^{\pi/2} \frac{d\omega}{\pi} \ln \left( \frac{\beta^2 x_+^0 x_+^1}{4} \right) + K \sum_{i=0}^{1} \int_0^{\pi/2} \frac{d\omega}{\pi} \ln |1 - (s_+^i)K|^2
\]

\[\quad + \sum_{n_1=0}^{(K-1)} \ln |1 - v_{n_1}K|^2\]

(3.36)

Actually the coefficient of \( K^2 \) above (the bulk coefficient), is easily checked to be given by

\[
\int_0^{\pi/2} \frac{d\omega}{\pi} \ln \left( \frac{\beta^2 x_+^0 x_+^1}{4} \right) = \int_{-\pi}^{\pi} \frac{d\nu_1 d\nu_2}{(2\pi)^2} \ln \left[ \{\delta - \alpha \cos(\nu_1) - \beta \cos(\nu_2) - \gamma \cos(\nu_1 - \nu_2)\} \right]
\]

(3.37)

We can now begin to calculate what happens when \( K \) is large. Note that we shall now (temporarily) specialise to the case where \( m = 0 \) since our answer in this case should be checkable against the \( \bar{\partial} \)-torsion \( T_{\bar{\partial}}(T^2, \mathcal{L}) \). However after our successful check against \( T_{\bar{\partial}}(T^2, \mathcal{L}) \) we shall give the result for \( m \neq 0 \) also. We shall start with the two terms \( \ln |1 - (s_+^i)K|^2, i = 0, 1 \) it turns out that

\[
\lim_{K \to \infty} K \int_0^{\pi/2} \frac{d\omega}{\pi} \ln |1 - (s_+^i)K|^2 = \left\{ \begin{array}{ll} 0, & \text{for } i = 0 \text{ because } |s_+^0| < 1 \text{ for } \omega \in [0, \pi/2] \\ \neq 0, & \text{for } i = 1 \text{ because } |s_+^1| \leq 1 \text{ for } \omega \in [0, \pi/2] \end{array} \right\}
\]

(3.38)

We briefly describe the calculation of the surviving limit in 3.38 above. The entire contribution to the limit comes from the \( \omega = 0 \) end of the integral since it is only at \( \omega = 0 \) that \( |s_+| \) attains the value 1. Taking this as our starting point we just need the (laboriously obtained) fact that, near \( \omega = 0 \),

\[
x_+^1 = 1 - \frac{|\tau_1^2|}{\tau_1} \omega + \cdots
\]

(3.39)

which is relevant since \( s_+^1 = x_+^1 \exp[2\pi i u_1/K] \). This found we write

\[
\int_{0}^{\pi/2} \frac{d\omega}{\pi} \ln |1 - (s_+^1)K|^2 = \int_{0}^{\epsilon} \frac{d\omega}{\pi} \ln |1 - (s_+^1)K|^2 + \int_{\epsilon}^{\pi/2} \frac{d\omega}{\pi} \ln |1 - (s_+^1)K|^2, \epsilon > 0
\]

(3.40)
The second integral dies exponentially as $K \to \infty$ but the first integral gives, for small $\epsilon$,

$$
\int_0^\epsilon \frac{d\omega}{\pi} \ln |1 - (s^1_\omega)|^{K^2} = -\sum_{l=1}^{\infty} \int_0^\epsilon \frac{d\omega}{\pi} (s^1_\omega)^{K^l}
$$

$$
\Rightarrow K \int_0^\epsilon \frac{d\omega}{\pi} \ln |1 - (s^1_\omega)|^{K^2} \to -K \sum_{l=1}^{\infty} \int_0^\epsilon \frac{d\omega}{\pi} \frac{\exp[2\pi i u_{1l}]}{l} \left(1 - \frac{|\tau|^2}{\tau_1}\omega\right)^{K^l}
$$

$$
= -K \sum_{l=1}^{\infty} \frac{\exp[2\pi i u_{1l}]}{l\pi} \left(\frac{-\tau_1}{|\tau|^2(K^l + 1)}\right) \left(1 - \frac{|\tau|^2}{\tau_1}\omega\right)^{K^l+1} [0, \epsilon]
$$

(3.41)

From which we readily compute that

$$
\lim_{K \to \infty} K \int_0^{\pi/2} \frac{d\omega}{\pi} \ln |1 - (s^1_\omega)|^{K^2} = -\frac{2\tau_1}{\pi|\tau|^2} \sum_{l=1}^{\infty} \frac{\cos(2\pi u_{1l})}{l^2}
$$

$$
= -\frac{2\pi \tau_1}{|\tau|^2} \left(\frac{u_1^2 - u_1 + \frac{1}{6}}{1 - i\tau_0} \right)
$$

(3.42)

Having taken care of the second term in 3.36 we move on.

The first term in 3.36 already has its $K$-dependence displayed as an explicit factor of $K^2$ and so the only other term to consider is the last term, namely $\sum_{n_1=0}^{(K-1)} \ln |1 - v_{n_1}^{K}|^2$. We note that, apart from an explicit factor of $K$, the dependence on $K$ in the sum is via the variable

$$
x_{n_1} = \frac{2\pi(n_1 + u_1)}{K}
$$

(3.43)

and for $K \to \infty$ we distinguish the two cases

$$
K \to \infty \begin{cases} 
n_1/K \to 0, & \text{case 0} 
n_1/K \not\to 0, & \text{case 1}
\end{cases}
$$

(3.44)

Since the procedure for both cases is lengthy but reasonably straightforward we describe it fairly briefly: first we deal with case 0 and then summarise the differences relevant for case 1.

**Case 0:**

We consider first the term $\ln |1 - v_{n_1}^{K}|^2$ and simply quote the fact that, after slightly tedious algebra, one finds that, with $\lambda = \sqrt{(\alpha\beta + \beta\gamma + \gamma\alpha)/(\gamma + \alpha)}$, we have

$$
v_{n_1} \to \exp \left[ -\frac{2\pi \lambda(n_1 + u_1)}{K} \right] \left(1 - \frac{i\gamma}{\alpha + \gamma} \frac{2\pi(n_1 + u_1)}{K} \right) \exp \left[ \frac{2\pi i u_0}{K} \right]
$$

$$
= \exp \left[ -\frac{2\pi \tau_1}{K|\tau|^2} \right] \left(1 - \frac{i\tau_0}{\tau_1} \frac{2\pi(n_1 + u_1)}{K} \right) \exp \left[ \frac{2\pi i u_0}{K} \right]
$$

(3.45)
Hence we now obtain
\[
\lim_{n_1/K \to 0} v_{n_1}^K = \exp \left[ -2\pi i \left\{ \left( n_1 + u_1 \right) \frac{\bar{\tau}}{|\tau|^2} - u_0 \right\} \right]
\]
\[
= \exp \left[ -2\pi i \left\{ \left( n_1 + u_1 \right) \frac{\tau}{\bar{\tau}} - u_0 \right\} \right]
\]
(3.46)

This expression 3.46 will take care of the \( K \to \infty \) limit of the term \( \ln |1 - v_{n_1}^K|^2 \). We must now round off the treatment of the \( K \to \infty \) limit by describing what happens for case 1.

**Case 1:**

Recall that case 1 means that \( n_1/K \not\to 0 \) as \( K \to \infty \). We can achieve this by writing
\[
n_1 = (K - l), \quad l = 1, \ldots, (K - 1)
\]
(3.47)

But since \( v_{n_1} \) only depends on \( n_1 \) via the functions \( \cos(x_{n_1}) \) and \( \sin(x_{n_1}) \) it is enough to note that with \( n_1 \mapsto (K - l) \) we obtain
\[
\cos(2\pi(n_1 + u_1)/K) \equiv \cos(2\pi(l - u_1)/K)
\]
\[
\sin(2\pi(n_1 + u_1)/K) \equiv \sin(2\pi(l - u_1)/K)
\]
(3.48)

The discussions of case 0 and case 1 are now complete and this allows us to deduce that
\[
\lim_{K \to \infty, n_1/K \not\to 0} v_{n_1}^K = \exp \left[ 2\pi i \left\{ \frac{(l - u_1)}{\bar{\tau}} + u_0 \right\} \right]
\]
(3.49)

We now have the entire \( K \to \infty \) limit of \( \ln \det K (\Delta/2) \) and it is that, as \( K \to \infty \)
\[
\ln \det K \left( \frac{\Delta}{2} \right) \longrightarrow K^2 \int_{-\pi}^{\pi} \frac{d\nu_1 d\nu_2}{(2\pi)^2} \ln \left\{ \left[ \sigma - \alpha \cos(\nu_1) - \beta \cos(\nu_2) - \gamma \cos(\nu_1 - \nu_2) \right] \right\}
\]
\[
- \frac{2\pi \tau_1}{|\tau|^2} \left( u_1^2 - u_1 + \frac{1}{6} \right) + \sum_{n=-\infty}^{\infty} \ln \left| 1 - \exp \left[ -2\pi i \left\{ \frac{|n|}{\tau} + \epsilon_n \left( \frac{u_1}{\tau} - u_0 \right) \right\} \right] \right|^2
\]
where \( \epsilon_n = \begin{cases} 1, & n \geq 0 \\ -1, & n < 0 \end{cases} \)
(3.50)

Hence the finite part of \( \ln \det K \Delta_L/2 \) is the function \( F(u_0, u_1, \tau) \) given by
\[
F(u_0, u_1, \tau) = - \frac{2\pi \tau_1}{|\tau|^2} \left( u_1^2 - u_1 + \frac{1}{6} \right) + \sum_{n=-\infty}^{\infty} \ln \left| 1 - \exp \left[ -2\pi i \left\{ \frac{|n|}{\tau} + \epsilon_n \left( \frac{u_1}{\tau} - u_0 \right) \right\} \right] \right|^2
\]
(3.51)

Now to agree with the continuum calculation \( F'(u_0, u_1, \tau) \) should give the \( \tilde{\partial} \)-torsion via the equation
\[
\ln T_{\tilde{\partial}}(T^2, \mathcal{L}) = F(u_0, u_1, \tau)
\]
(3.52)
But $T_{\bar{\partial}}(T^2, \mathcal{L})$ was given in 3.9 from which we obtain

$$
\ln T_{\bar{\partial}}(T^2, \mathcal{L}) = \ln \left| \exp \left[ \pi i u_1/2 \frac{\theta_1(u_1 - \tau u_0, \tau)}{\eta(\tau)} \right] \right|^2
= -2\pi \tau_1 \left( u_0^2 - u_0 + \frac{1}{6} \right) + \sum_{n = -\infty}^{\infty} \ln \left| 1 - \exp \left\{ 2\pi i \left( |n| \tau + \epsilon_n (u_1 - u_0 \tau) \right) \right\} \right|^2,
$$

(3.53)

(The last line of 3.53 follows from the series representations of $\eta(\tau)$ and $\theta_1(u_1 - \tau u_0, \tau)$.) However the apparent disagreement between 3.52 and 3.53 is illusory these two expressions are both modular invariants and if we perform the summations leading to 3.52 in the opposite order then we get precisely 3.53. Hence we do indeed find that the continuum limit of our lattice model has reproduced the $\bar{\partial}$-torsion $T_{\bar{\partial}}(T^2, \mathcal{L})$ in a fashion quite analogous to the one dimensional example of the previous section.

It is useful at this point to spell out some of the details of the modular properties of our expressions in a short discussion. We recall some basic facts. If $M$ denotes an element of the modular group then $M$ acts on the modular parameter $\tau$ according to the well known rule $\tau \mapsto (a\tau + b)/(c\tau + d)$ with integral $a,b,c$ and $d$ satisfying $ad - bc = 1$. Hence we usually write something like

$$
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2, \quad a, b, c, d \in \mathbb{Z}, \quad ad - cb = 1 \quad (3.54)
$$

However when such transformations $M$ act on the torus they act on cycles and thus on the generators of the fundamental group $\pi_1(T^2)$; or equivalently on the holonomy phases $u_0$ and $u_1$ which specify the bundle $\mathcal{L}$. Hence we must work out this action on the phases. But the modular group $SL(2, \mathbb{Z})/\mathbb{Z}_2$ is generated by the celebrated $S$ and $T$ elements which are

$$
S : \tau \mapsto -1/\tau, \quad T : \tau \mapsto \tau + 1 \quad (3.55)
$$

Therefore it is sufficient to give the action of $S$ and $T$ on the holonomy phases and this is

$$
\begin{align*}
\text{Under } S &: \begin{array}{ll} u_0 &\mapsto u_0, \\ u_1 &\mapsto u_0 + u_1 \end{array} \\
\text{Under } T &: \begin{array}{ll} u_0 &\mapsto u_1, \\ u_1 &\mapsto -u_0 \end{array}
\end{align*} \quad (3.56)
$$

In addition $S$ and $T$ are isometries of the flat torus so they leave the volume $V$ invariant.

With this information on $S$ and $T$ it is possible to check directly the modular invariance of 3.52 and 3.53. For completeness we give the corresponding information for a general element $M$ of the modular group. It is that, regarding $L_0$ and $L_1$ as complex numbers in an obvious way, we have

$$
\begin{align*}
\text{Under } M &: \begin{array}{ll} u_0 &\mapsto cu_1 + du_0, \\ u_1 &\mapsto au_1 + bu_0 \end{array} \\
\text{and } \tau &\mapsto \frac{a\tau + b}{c\tau + d}
\end{align*}
$$

with $M \in SL(2, \mathbb{Z})/\mathbb{Z}_2$.
For the $\bar{\partial}$-torsion this modular invariance is the assertion that $T_{\bar{j}}(T^2, \mathcal{L})$ only depends on the conformal structure on $T^2$ i.e. on its complex structure. In fact as observed in [10] modular invariance in this case can be shown to follow from the classical functional equations obeyed by $\theta_1$ and $\eta$.

In fact in this work the modular invariance of the continuum theory is guaranteed by, and has its origin in, the independence of the limit defining $F(u_0, u_1, \tau)$ on the order in which its double sums are carried out.

Finally we reinstate the mass and give the corresponding formulae when $m \neq 0$. We find that

$$\ln \det_{\mathcal{K}} \left( \frac{\Delta + m^2}{2} \right) \to K^2 \int_{-\pi}^{\pi} \frac{d\nu_1 d\nu_2}{(2\pi)^2} \ln \left[ \left\{ \delta - \alpha \cos(\nu_1) - \beta \cos(\nu_2) - \gamma \cos(\nu_1 - \nu_2) \right\} \right]$$

$$- \frac{\pi \tau_1}{6} c(u_0, \frac{m^2 V}{\tau_1}) + \sum_{n = -\infty}^{\infty} \ln \left| 1 - e^{-2\pi \tau_1 \sqrt{(n+u_0)^2 + \frac{m^2 V}{4\pi \tau_1} + 2\pi i (u_1 - \tau_0 (n + u_0))}} \right|^2$$

where $V = L_0 L_1 \sin \theta$ and the function $c(u, x)$ that appears in 3.58 above is defined by the equation

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi} \ln \left| 1 - e^{-\sqrt{p^2 + x^2 + 2\pi i u}} \right|^2 = -\frac{c(u, x) \pi}{6}$$

One can verify that 3.58 reduces to 3.53 when $m = 0$—for example this would require that $c(u, 0) = 12(u^2 - u + 1/6)$ which is true.

Of course, for this case where $m \neq 0$ we can use a zeta function method to give the continuum partition function. Repeated use of (2.10) enables one to calculate $\zeta_{d, \mathcal{L} + m^2}(0)$ which is the required object. The result is that we obtain $W_F$ in agreement with 3.58 above; we also find a bulk term of the form $-\frac{V m^2}{4\pi} \ln [(m/2\pi \mu)^2 - 1]$ where $\mu$ is an arbitrary undetermined scale.

We would like to emphasize that, in the continuum or scaling limit, we have found the exact finite size corrections to $\ln Z(T^2, \mathcal{L}, m^2)$ and these are modular invariant whether $m = 0$ or not. But when $m$ is zero we have a critical phase and then the finite size corrections to $Z$ exhibit holomorphic factorisation and are given by elliptic theta functions, i.e.

$$\exp[-W_F] = F F$$

with $F = \exp[\pi i u_0^2 \tau] \frac{\theta_1(u_1 - \tau u_0, \tau)}{\eta(\tau)}$

(3.60)

Concerning holomorphic factorisation note that if the phase factor $\exp[\pi i u_0^2 \tau]$ were absent from $F$ above then $Z$ would also exhibit holomorphic factorisation in the Picard variety label $w = u_1 - \tau u_0$. Actually the presence of the phase $\exp[\pi i u_0^2 \tau]$ is a consequence of the central charge $c$ being non-zero—lack of holomorphic factorisation in this sense is the existence of a holomorphic anomaly [12] of the $\bar{\partial}_\mathcal{L}$ determinant bundle over the Picard variety. If we observe that

$$\partial_w \partial_{\bar{w}} \ln T_{\bar{j}}(w, \bar{w}, \tau) = 4\pi \delta(w, \bar{w}) - \frac{\pi}{\tau_1}$$

(3.61)
we see that the torsion is proportional to the Greens function, where the lowest mode (the zero mode) has been projected out, of the Laplacian $\partial_w \partial_{\bar{w}}$ on the Piccard variety. It is the absence of this lowest mode that is the barrier to holomorphic factorization.

In the next section we turn to the bulk term and the cylinder limit $L_1 \to \infty$.

§ 4. **Conformal properties and effects due to finite size and finite lattice spacing.**

In this section we work with $K_0$, $K_1$ and the mass $m$ at perfectly general values.

We recall that $W$ and $Z$ are related by

$$ W = -\ln Z \quad (4.1) $$

If we now perform the lattice sums, doing first the sum over $n_0$, followed by that over $n_1$ we obtain the result that

$$ W = K_0 K_1 W_B + W_F \quad (4.2) $$

where

$$ W_B = \int_{-\pi}^{\pi} \frac{d\nu_1 d\nu_2}{(2\pi)^2} \ln \left[ \frac{\sqrt{g}}{\pi} \left\{ \delta - \alpha \cos(\nu_1) - \beta \cos(\nu_2) - \gamma \cos(\nu_1 - \nu_2) \right\} \right] \quad (4.3) $$

and

$$ W_F = K_1 \sum_{i=0}^{1} \int_{0}^{\pi/2} \frac{d\omega}{\pi} \ln |1 - (s_i)_{K_0}|^2 + \sum_{n_1=0}^{(K_1-1)} \ln |1 - \nu_{n_1}|^2 \quad (4.4) $$

We note that $W_B$ gives the free energy per lattice site in the thermodynamic limit, i.e. we have

$$ \lim_{K_0, K_1 \to \infty} \frac{W}{K_0 K_1} = W_B \quad (4.5) $$

This in turn means that $W_F$ gives the **complete finite size corrections** to the bulk lattice behaviour. We now examine in more generality and more detail the limits studied in the preceding section. We now no longer set $K_0 = K_1 = K$ we simply take $K_0, K_1 \to \infty$ while keeping fixed their ratio $k = K_0/K_1$, we also keep fixed the quantities $m^2$, $\theta$ and $L_i = K_0 a_i$, we shall refer to this as the scaling limit. We still find the same result for the finite size corrections $W_F$ namely

$$ \lim_{\text{scaling}} W_F = -\frac{\pi \tau_1}{6} c(u_0, \frac{m^2 V}{\tau_1}) + \sum_{n=-\infty}^{\infty} \ln \left| 1 - e^{-2\pi \tau_1 \sqrt{(n+u_0)^2 + \frac{m^2 V}{4\pi^2}} + 2\pi i (u_1 - \tau_0 (n+u_0))} \right|^2 \quad (4.6) $$

while for the bulk term we find the mass dependence

$$ \lim_{\text{scaling}} K_0 K_1 W_B = K_0 K_1 \Lambda_B - \frac{V m^2}{4\pi} \left\{ \ln[K_0 K_1] - 2\rho \right\} - \frac{V m^2}{4\pi} \left( \ln\left[ \frac{m^2 V}{4\pi^2} \right] - 1 \right) + \cdots $$

where $\Lambda_B = W_B|_{m=0}$, $V = L_0 L_1 \sin \theta$

and $\rho \equiv \rho(\alpha, \beta, \gamma) = \int_{0}^{\pi} d\nu \left[ \frac{1}{\sin \nu \sqrt{1 + g \alpha^2 \sin^2 \nu}} - \frac{1}{\nu} \right] - \frac{1}{2} \ln \left[ \sqrt{g} (\beta + \gamma) \right] \quad (4.7)$
Our results can be applied to the Ising model on a torus—the special case corresponding, in our language, to \( \tau_0 = 0 \) was studied in [13]. This arises because of the equivalence of the Ising model and a dimer model on a decorated lattice cf. [14,15]. To see what happens we denote \( W_F \) by \( W_F(0, 0) \) in order to display its phase dependence and use [16] for \( \alpha, \beta, \gamma \) and \( \delta \). Let us do this: denoting nearest neighbour spin-spin couplings by \( J_1, J_2 \) and \( J_3 \) then

\[
\alpha = \sinh \left( \frac{2J_1}{k_BT} \right), \quad \beta = \sinh \left( \frac{2J_2}{k_BT} \right), \quad \gamma = \sinh \left( \frac{2J_3}{k_BT} \right),
\]

\[
\delta = \cosh \left( \frac{2J_1}{k_BT} \right) \cosh \left( \frac{2J_2}{k_BT} \right) \cosh \left( \frac{2J_3}{k_BT} \right) + \sinh \left( \frac{2J_1}{k_BT} \right) \sinh \left( \frac{2J_2}{k_BT} \right) \sinh \left( \frac{2J_3}{k_BT} \right)
\]

and the Ising partition function is given by

\[
Z^{Ising} = \frac{1}{2} e^{-W_B^{Ising}} \left\{ \pm e^{\frac{1}{2} W_F(0, 0)} + e^{\frac{1}{2} W_F(0, \frac{1}{2})} + e^{\frac{1}{2} W_F(\frac{1}{2}, 0)} + e^{\frac{1}{2} W_F(\frac{1}{2}, \frac{1}{2})} \right\}
\]

(4.9)

This result is for ferromagnetic couplings, with + referring to \( T < T_c \) and - to \( T > T_c \). If we take the the scaling limit \( W_F(u_0, u_1) \mapsto \Gamma_F(u_0, u_1) \) and we obtain a modular invariant expression for the finite size contributions in the scaling limit for this model also. Our results (4.9) incorporates the complete lattice and finite size corrections for the Ising model on a triangular lattice. With a similar equivalence our results can easily be translated to give the general result for other models.

We point out that the expression of \( Z^{Ising} \) as the four term sum 4.9 above constructs a modular invariant function of \( \tau \) and \( m^2 V \); this construction makes it easy to understand that it is the summing over the \( u_i \) which form an orbit of the \( SL(2, \mathbb{Z})/\mathbb{Z}_2 \)-action on the space of \( u_i \)’s that guarantees the invariance. Mathematically we are working with the action of \( SL(2, \mathbb{Z})/\mathbb{Z}_2 \) on the space of flat bundles \( \mathcal{L} \)—the so called Picard variety—and many other orbits exist, this allows us to construct many additional (phase independent) modular invariant partition functions such as those of the other conformally invariant field theories cf. [17] and references therein. Rational conformal theories will be obtained when the \( u_i \) orbits contain elements corresponding to roots of unity.

Next we would like to consider an interesting geometric limit of the model where \( L_1 \to \infty \): the cylinder limit. This will enable us to access the central charge of the model. For large \( L_1 \) the quantity \( W_F/V \) tends to the finite value\(^{(d)}\) \( \gamma_{cylinder} \) where

\[
\gamma_{cylinder} = -\frac{\pi}{6L_0^2} c(u_0, m^2 L_0^2)
\]

(4.10)

We refer to \( c(u_0, m^2 L_0^2) \) as the ‘cylinder charge’, the central charge \( c \) of the model is obtained by setting \( m = u_0 = 0 \) and we see that this gives us

\[
c = 2
\]

(4.11)

\(^{(d)}\) We could equally take \( L_0 \) large if we interchange \( (u_0, L_0) \) with \( (u_1, L_1) \).
Because the cylinder charge determines the central charge at appropriate values of its variables it might appear reasonable to expect that the cylinder charge $\gamma_{\text{cylinder}}$ be equal to the Zamolodchikov $c$-function [18]—this is not so. To see this note that the $c$-function must be monotonic in $m$ and the cylinder charge is not cf. fig. 2 where we plot $c(u, x)$ versus $x$ for various $u$. In fig. 2 we see that $c(u, x)$ exhibits a universal crossover phenomenon as $x \to \infty$ with $c(u, x)$ having the property that $\lim_{x \to \infty} c(u, x) = 0$ whatever the value of $u$.

The cylinder charge for the Ising model, can be obtained from 4.9 with $L_1 \to \infty$, and it works out to be $\gamma_{\text{Ising}}$ where

$$\gamma_{\text{Ising}} = \frac{\pi}{12 L_0^2} c(\frac{1}{2}, x)$$

This means that comparison with 4.10 above determines that the cylinder charge for the Ising model is

$$-\frac{1}{2} c(\frac{1}{2}, x)$$

Now if we set $x = 0$ we should get the Ising central charge and doing this we find $c = \frac{1}{2}$ (cf. fig. 2) the conventional value [19,20].

A further rich feature of this model is that it posses non-commuting limits namely the limits $u_0, u_1 \to 0$ and $m \to 0$. To establish this we can expand the finite size $W_F$ contribution to 3.58 for small $u_i$ and $m$ obtaining

$$W_F = \ln [(2\pi)^2 |u_1 - \tau u_0|^2 + \tau_1 m^2 V] + 2 \ln |\eta(\tau)|^2 + \cdots$$

But the RHS of 4.14 can tend to the two distinct (logarithmically singular) expressions

$$\ln |u_1 - \tau u_0|^2 + 2 \ln |\eta(\tau)|^2$$

and $\ln |\tau_1 m^2 V| + 2 \ln |\eta(\tau)|^2$

depending on the order in which the limits are taken. Nevertheless both limits and indeed 4.14 itself are modular invariant.

This model is also ideal for studying the Kosterlitz-Thouless phase [21]. One can add on a $|\phi|^4$ interaction and in this way one can study the critical point of the $XY$ model approaching it from the disordered phase. We have holonomy conditions (3.15) that describe the essential features of a vortex phase: To make this more transparent one should perform the cylinder limit of 3.15 and then map the cylinder to the plane using the conformal map

$$y + i \frac{2\pi x}{L_0} = \ln z$$

This gives a vortex at the origin whose charge is given by $u_0$ and whose presence is detected by the existence of an Aharanov-Bohm effect for transport around the origin.

§ 5. Conclusion

We have considered lattice models and their continuum limits in one and two dimensions. Our one dimensional model is that of a gauge theory for a flat bundle $L$ over the circle and
it is non-trivial its partition function being a power of the Ray-Singer torsion $T(S^1, \mathcal{L})$ of $\mathcal{L}$. We find that the combinatorial and continuum partition functions agree precisely and are equal to $T(S^1, \mathcal{L})^{-1}$. This result emphasizes the topological nature of $T(S^1, \mathcal{L})^{-1}$ which is of course expressible purely combinatorially, as it was originally [22], or analytically as it was later [23].

In summary, the finite size corrections to the free energy are modular invariant. This conclusion extends to the entire scaling neighborhood of the critical phase. We use our results to give expressions for the complete lattice and finite size corrections for the two dimensional Ising model on a triangular lattice via its equivalence to a sum over Pfaffians. Modular invariance also extends to models in more than two dimensions when the geometry giving rise to finite size effects contains a flat torus. For a three dimensional cylindrical geometry with toroidal cross-section the result can be obtained from $W_F$ by replacing $m^2$ with $m^2 + q^2$ and integrating the resulting expression over $q$. One can understand the origin of modular invariance in general as the residual freedom to reparametrise coordinates, in the continuum limit, while retaining flat toroidal geometry.

In the two dimensional case the limiting finite size corrections at the critical phase are expressible in terms of classical elliptic functions. Infinitesimally small values of the phases $u_i$ lead to logarithmically divergent contributions to the free energy. This means that the free energy needed to create a vortex becomes infinite for an infinitely large lattice. In general the model has a surprisingly rich structure of non-commuting limits. For example the limits of approaching the critical phase ($m \to 0$) and that of sending the $u_i$ to zero do not commute.
The triangulated torus

![Triangulated Torus Diagram]

Fig. 1.

The ‘cylinder charge’ function $c(u, x)$ for various $u$

![Cylinder Charge Function Graph]

Fig. 2.
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