Approximate Eigensolutions of the Deformed Woods–Saxon Potential via AIM

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Using the Pekeris approximation, the Schrödinger equation is solved for the nuclear deformed Woods–Saxon potential within the framework of the asymptotic iteration method. The energy levels are worked out and the corresponding normalized eigenfunctions are obtained in terms of hypergeometric function.

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The deformed Woods–Saxon (dWS) potential is a short range potential which is widely used in nuclear, particle, atomic, condensed matter and chemical physics.[1–7] This potential is reasonable for nuclear shell models and is used to represent the distribution of nuclear densities. The dWS and spin-orbit interaction are important and applicable to deformed nuclei[8] and to strongly deformed nuclei.[9] The dWS potential parameterization at large deformations for plutonium 237,239,241Pu odd isotopes was analyzed.[10] The structure of single-particle states in the second minima of 237,239,241Pu has been calculated with the dWS potential. The Nuclear shape was parameterized. The parameterization of the spin-orbit part of the potential was obtained in the region corresponding to large deformations (second minima) depending only on the nuclear surface area. The spin-orbit interaction of a particle in a non-central self consistent field of the WS-type potential was investigated for light nuclei and the scheme of single-particle states has been found for mass numbers $A_0 = 10$ and 25.[8] Two parameters of the spin-orbit part of the dWS potential, namely the strength parameter and radius parameter were adjusted to reproduce the spins for the values of the nuclear deformation parameters.[11]

Badalov et al.[12,13] investigated the Woods–Saxon potential in the framework of Schrödinger and Klein–Gordon equations by means of the Nikiforov–Uvarov method. In our recent works,[14,15] we have studied the relativistic Duffin–Kemmer–Petiau and the Dirac equation for dWS potential. In addition, we have also obtained the bound state solutions of the PT-/non-PT-symmetric and non-Hermitian modified Woods–Saxon potential with the real and complex-valued energy levels.[16]

In this Letter, we study the Schrödinger equation with the dWS potential for any arbitrary orbital quantum number $\ell$. We obtain analytical expressions for the energy levels and wave functions in a closed form via the asymptotic iteration method (AIM).

We briefly outline the AIM here, further details can be found in Refs. [17,18]. The AIM is proposed to solve the homogenous linear second-order differential equation in the form

$$y''(x) = \lambda(x)y'(x) + s(x)y(x),$$

where $\lambda(x) \neq 0$, and the prime denotes the derivative with respect to $x$, the external parameter $n$ is thought of as a radial quantum number. The variables $s_0(x)$ and $\lambda(x)$ are sufficiently differentiable. To find a general solution to this equation, we differentiate Eq. (1) with respect to $x$ as

$$y'''(x) = \lambda_1(x)y'(x) + s_1(x)y(x),$$

where

$$\lambda_1(x) = \lambda_0(x) + s_0(x) + \lambda_0'(x),$$

and the second derivative of Eq. (1) is obtained to be

$$y''''(x) = \lambda_2(x)y'(x) + s_2(x)y(x),$$

where

$$\lambda_2(x) = \lambda_1'(x) + s_1(x) + \lambda_0(x),$$

and the second derivative of Eq. (1) can be iterated up to the $(k + 1)$th and $(k + 2)$th derivatives, $k = 1, 2, 3, \ldots$. Therefore we have

$$y_k^{(k+1)}(x) = \lambda_k(x)y_k'(x) + s_k(x)y_k(x),$$

where

$$\lambda_k(x) = \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_{k-1}(x),$$

$$s_k(x) = s_{k-1}'(x) + s_{k-1}(x)\lambda_{k-1}(x).$$

From the ratio of the $(k + 2)$th and $(k + 1)$th derivatives, we obtain

$$\frac{dy^{(k+2)}(x)}{dy^{(k+1)}(x)} = \frac{\lambda_k(x)y_k'(x) + s_k(x)y_k(x)}{\lambda_{k-1}(x)y_{k-1}'(x) + s_{k-1}(x)y_{k-1}(x)},$$

if $k > 0$, for sufficiently large $k$, we can obtain the $\alpha$ values from

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x),$$

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under the quantization condition

\[ \delta_k(x) = \begin{vmatrix} \lambda_k(x) & s_k(x) \\ \lambda_{k-1}(x) & s_{k-1}(x) \end{vmatrix} = 0, \quad k = 1, 2, 3, \ldots \]  

(10)

Then Eq. (8) reduces to

\[ \frac{d}{dx} \ln[y_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \]

(11)

which yields the general solution to Eq. (1) as follows:

\[ y_n(x) = \exp \left( - \int^x \alpha(x')dx' \right) \left\{ C_2 + C_1 \cdot \int^x \exp \left( \int^{x'} \left[ \lambda_0(x'') + 2\alpha(x'') \right] dx'' \right) dx' \right\}. \]

(12)

For a given potential, the idea is to convert the radial Schrödinger equation to the form of Eq. (1). Then \( \lambda_0(x) \) and \( s_0(x) \) are determined, \( s_k(x) \) and \( \lambda_k(x) \) parameters are calculated by the recurrence relationship given by Eq. (7). The energy eigenvalues are then obtained by the conditions given by Eq. (10) if the problem is exactly solvable.

Now, suppose that we wish to solve the radial Schrödinger equation for which the homogenous linear second-order differential equation takes the following general form

\[ y''(x) = 2 \left( \frac{tx^{N+1}}{1 - bx^{N+2}} - \frac{m + 1}{x} \right) y'(x) - \frac{Wx^N}{1 - bx^{N+2}}, \]

(13)

where \( t, m \) and \( W \) are arbitrary constant. The exact solution \( y_n(x) \) can be expressed as [17]

\[ y_n(x) = (-1)^n C_2(N+2)^n(\sigma)_n \sigma F_1(-n, \rho; n, \sigma; bx^{N+2}), \]

(14)

where we have used the following notations

\[ (\sigma)_n = \frac{\Gamma(\sigma + n)}{\Gamma(\sigma)}, \quad \sigma = \frac{2m + N + 3}{N + 2}, \]

\[ \rho = \frac{(2m + 1)b + 2t}{(N + 2)b}. \]

(15)

The deformed Woods–Saxon potential we investigate in this study is defined as [14,15,19]

\[ V(r) = -\frac{V_0}{q + \exp \left( \frac{r - R}{aR} \right)}, \quad R = r_0 A_0^{1/3}, \]

\[ V_0 = (40.5 + 0.13 A_0) \text{MeV}, \quad R \gg a, \quad q > 0, \]

(16)

where \( V_0 \) is the depth of the potential, \( q \) is a real parameter which determines the shape (deformation) of the potential, \( a \) is the diffuseness of the nuclear surface, \( R \) is the width of the potential, \( A_0 \) is the atomic mass number of the target nucleus and \( r_0 \) is the radius parameter. By inserting this potential into the Schrödinger equation [20,21] as

\[ \left[ -\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] V(r) \Psi_{nlm}(r) = E \Psi_{nlm}(r), \]

(17)

and setting the wave functions \( \Psi_{nlm}(r) = r^{-1} R_{nl}(r) Y_{\ell m}(\theta, \phi) \), we obtain the radial part of the equation by the separation of variables as

\[ \frac{d^2}{dr^2} + 2\frac{\mu}{\hbar^2} \left( E_{nl} + \frac{V_0}{q + \exp \left( \frac{r - R}{aR} \right)} \right) - \frac{\ell(\ell + 1)}{r^2} R_{nl}(r) = 0. \]

(18)

Because of the total angular momentum centrifugal term, Eq. (18) cannot be solved analytically for \( \ell \neq 0 \). Therefore, we shall use the Pekeris approximation in order to deal with this centrifugal term and so we may express it as follows:[14,22,19]

\[ U_{\text{cent}}(r) = \frac{1}{r^2} = \frac{1}{R^2(1 + \frac{x}{R})^2} \approx \frac{1}{R^2} \left( 1 - 2\frac{x}{R} \right) + 3 \left( \frac{x}{R} \right)^2 + \cdots, \]

(19)

with \( x = r - R \). In addition, we may also approximately express it in the following way

\[ \dot{U} = \frac{1}{r^2} \approx \frac{1}{R^2} \left[ \frac{D_0 - \frac{D_1}{q + \exp(\nu x)} + \frac{D_2}{(q + \exp(\nu x))^2}}{\nu R} \right], \]

(20)

where \( \nu = 1/a \). After expanding Eq. (20) in terms of \( x, x^2, x^3, \cdots \) and next, comparing with Eq. (19), we obtain the expansion coefficients \( D_0, D_1 \) and \( D_2 \) as follows:

\[ D_0 = 1 - \frac{1 + \exp(-aR)}{aR} \left\{ 4aR \right\}^2 \]

\[ D_1 = 2\left[ \exp(aR) + 1 \right] \left\{ \frac{3(1 + \exp(-aR))}{aR} \right\} \]

\[ D_2 = \left[ \exp(aR) + 1 \right]^2 \left\{ \frac{3(1 + \exp(-aR))}{aR} \right\} \]

(21)

and higher order terms are neglected. It is worth noting that the above expansion is valid for low rotational energy states. Now, inserting the approximation expression (20) into Eq. (18) and changing the variables \( r \rightarrow z \) through the mapping function \( z = \exp(-\nu x) \), Eq. (18) turns to

\[ \frac{d^2 R_{nl}(z)}{dz^2} + \frac{d R_{nl}(z)}{dz} + \frac{1}{\nu^2(1 + qz)^2} \left\{ \left( \frac{2\mu(q^2 E_{nl} + qV_0)}{\hbar^2} - \frac{\ell(\ell + 1)}{R^2} \right) \right\}^2 \]

\[ + \frac{2\mu}{\hbar^2} (2qE_{nl} + V_0) - \frac{\ell(\ell + 1)}{R^2} \left( 2q D_0 + D_1 \right) \]

\[ + \frac{2\mu E_{nl}}{\hbar^2} \right\} R_{nl}(z) = 0. \]

(22)
Before applying the AIM to this problem, we must obtain the asymptotic wave functions and then transform Eq. (22) into a suitable form of the AIM. This can be achieved by the analysis of the asymptotic behaviours at the origin and at infinity. As a result the boundary conditions of the wave functions $R_{nl}(z)$ are taken as

$$ R_{nl}(z) \to 0 \quad \text{when} \quad z \to -\frac{1}{q}, $$

$$ R_{nl}(z) \to 0 \quad \text{when} \quad z \to 0. \quad (23) $$

Thus, one can write the wave functions for this problem as

$$ R_{nl}(z) = z^\alpha (1 + qz)^\gamma F_{nl}(z), \quad (24) $$

where we have introduced parameters $\alpha$ and $\gamma$ defined by

$$ \alpha = \frac{1}{\nu} \left[ \frac{\ell(\ell + 1)D_0}{R^2} - \frac{2\nu F_{nl}}{\hbar^2} \right]^{1/2}, $$

$$ \gamma = \frac{1}{2} \left[ 1 + \sqrt{1 + \frac{4\nu(\ell + 1)D_0}{\nu^2 R^4 q^2}} \right]^{1/2}. \quad (25) $$

for simplicity. By substituting Eq. (24) into Eq. (22), we have the second-order homogeneous differential equation of the form

$$ F''_{nl}(z) + \left[ \frac{(2\alpha + 1) + q(2\alpha + 2\gamma + 1)z}{z(1 + qz)} \right] F'_{nl}(z) + \left[ \frac{A}{q} + q(\alpha + \gamma)^2 \right] F_{nl}(z) = 0, \quad (26) $$

where $A = \frac{2\nu(q^2 E_{nl} + q\nu\gamma)}{\hbar^2} - \frac{\ell(\ell + 1)}{R^2}(q^2 D_0 - qD_1 + D_2)$. Equation (26) is now suitable for an AIM solution. By comparing this equation with Eq. (1), we can write the $\lambda_0(z)$ and $s_0(z)$ values and consequently; by means of Eq. (7), we may derive the $\lambda_k(z)$ and $s_k(z)$ as follows:

$$ \lambda_0(z) = -\left( \frac{(2\alpha + 1) + q(2\alpha + 2\gamma + 1)}{z(1 + qz)} \right), $$

$$ s_0(z) = -\left( \frac{A}{q} + q(\alpha + \gamma)^2 \right), $$

$$ \lambda_1(z) = -\left( \frac{2\gamma + 2\alpha + 1}{z(1 + qz)} \right. + (\gamma + \alpha)^2 + A \frac{q}{q} \left. \right) $$

$$ + \frac{2q(\alpha + 1/2) - 1 - 2\alpha}{z^2(1 + qz)} $$

$$ - \frac{2q(\gamma + \alpha + 1/2) - 1 - 2\alpha}{z^2(1 + qz)} $$

$$ - \frac{(2q(\gamma + \alpha + 0.5) + 1 + 2\alpha)^2}{z^2(1 + qz)^2}, $$

$$ s_1(z) = \frac{(q(\gamma + \alpha)^2 + A/q)(2q(\gamma + \alpha + 1/2) + 1 + 2\alpha)}{z^2(1 + qz)^2} $$

$$ - \frac{(q(\gamma + \alpha)^2 + A/q)(2q(\gamma + \alpha + 1/2) + 1 + 2\alpha)}{z^2(1 + qz)^2}. \quad (27) $$

By the substitution of the above equations into Eq. (10), we obtain the first $\delta$ values as

$$ \delta_0(z) = \left\{ \left( q^2(\alpha + \gamma)(4 + \alpha + \gamma) + A + 4q^2 \right) \right. $$

$$ \left. \cdot \left( q^2(\alpha + \gamma)(2 + \alpha + \gamma) + A + q^2 \right) \right. $$

$$ \left. \cdot \left( q^2(\gamma + \alpha + 1) + A \right) \right\} \cdot \left\{ q^2z^2(1 + qz)^3 \right\}^{-1} = 0. \quad (28) $$

From the root of Eq. (28), we obtain the first relationship between $\alpha$ and $\gamma$ as $\alpha_0 = -[\gamma + \frac{1}{2} \sqrt{-1}]$. In a similar fashion, we can obtain other $\delta$ values and consequently establish a relationship between $\alpha_n$ and $\gamma$ ($n = 1, 2, 3, \ldots$) as

$$ \delta_1(z) = \left\{ \left( \frac{\lambda_2(z)}{\lambda_1(z)} \right) \right. $$

$$ \left. \cdot \left( s_2(z) \right) \right\} = 0 $$

$$ \rightarrow \alpha_1 = -[\gamma + 1 + \frac{1}{q} \sqrt{-A}], $$

$$ \delta_2(z) = \left\{ \left( \frac{\lambda_3(z)}{\lambda_2(z)} \right) \right. $$

$$ \left. \cdot \left( s_3(z) \right) \right\} = 0 $$

$$ \rightarrow \alpha_2 = -[\gamma + 2 + \frac{1}{q} \sqrt{-A}], \ldots \quad (29) $$

The $n$th term of the above arithmetic progression is found to be

$$ \alpha_n = -[\gamma + n + \frac{1}{q} \sqrt{-A}]. \quad (30) $$

By substituting for $\alpha$ and $\gamma$, we obtain a more explicit expression for the eigenvalues energy as

$$ E_{nl} = \frac{\hbar^2 \ell(\ell + 1)}{2\mu R^2} \left[ \frac{D_0 - D_1}{q} + \frac{D_2}{q^2} \right] - \frac{V_0}{q} - \frac{\hbar^2 \nu^2}{8\mu} $$

$$ \cdot \left[ \frac{1}{q^2} \left( \frac{2\nu V_0}{\hbar^2} + \frac{\ell(\ell + 1)}{R^2} \left( D_1 - \frac{D_2}{q} \right) \right) - \chi(n, \ell)^2 \right] . \quad (31) $$

where $\chi(n, \ell) = n + \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4\ell(\ell + 1)\frac{D_2}{q^2\nu^2 R^2}}$. Let us now turn to the calculation of the wave functions. By comparing Eq. (26) with Eq. (13) we have

$$ t = -q\gamma, \quad b = -q, \quad N = -1, \quad m = \alpha - \frac{1}{2}, $$

$$ \sigma = 2\alpha + 1, \quad \rho = 2(\alpha + \gamma). \quad (32) $$

Having determined these parameters, we can easily find the wave functions as

$$ F_{nl}(z) = (-1)^n C_2 \Gamma(2\alpha + n + 1) \frac{\Gamma(2\alpha + 1)}{\Gamma(2\alpha + 1)} ^{2F_1} $$

$$ \cdot (-n, 2(\alpha + \gamma) + n; 2\alpha + 1; -qz), \quad (33) $$

where $\Gamma$ and $2F_1$ are the gamma function and hypergeometric function respectively. By using Eqs. (24) and (33), the total radial wave function can be written as

$$ R_{nl}(r) = (-1)^n N_{nl} \left( \frac{1 + q \exp(\frac{Rr}{\hbar})}{\exp(\frac{\alpha - R}{\hbar})} \right)^{2F_1} $$

$$ \cdot \left[ -n, 2(\alpha + \gamma) + n; 2\alpha + 1; -q \exp(\frac{R - r}{\hbar}) \right]. \quad (34) $$
where $N_{nl}$ is the normalization constant. For a special case $\ell = 0$, Eq. (31) reduces to
\[
E_n = \frac{V_o}{q} - \frac{\hbar^2 \nu^2}{8\mu} (2\mu V_o) - (n + 1)^2 \left( -\frac{\hbar^2 \nu^2}{2\mu q^2} (n + 1) \right)^2,
\]
and for the $q$-deformed Hulthén potential ($q \rightarrow -q; \nu \rightarrow \delta$) we have
\[
E_n = -\frac{\hbar^2}{2\mu} \left[ \frac{V_o \delta}{q \hbar^2 \delta (n + 1)} - \frac{(n + 1) \delta^2}{2} \right],
\]
which is identical to the ones obtained before using the factorization method,[11] SUSYQM approach,[24–26] quasi-linearization method[27] and Nikiforov–Uvarov method.[28,29]

Table 1. The bound state energy eigenvalues $E_{nl}$ for some values of $n$ and $\ell$ with $R = 1.285 A^{1/3}$, $V_o = 40.5 + 0.13 A_o$, $a = 0.65$, $2a/\hbar^2 = 0.4727 \text{MeV}^{-1}\text{fm}^{-2}$ and $h = 5.8521 \times 10^{-22} \text{MeV}$. 

| $n\ell$ | $E_{nl}$ | $A = 40$ | $E_{nl}$ | $A = 40$ | $E_{nl}$ | $A = 56$ |
|-------|--------|----------|--------|----------|--------|----------|
| 00    | $-38.74580876$ | $-41.69282044$ | $-25.38765097$ | $-26.37123090$ |
| 10    | $-21.08802055$ | $-27.88702375$ | $-22.97221589$ | $-22.97221589$ |
| 20    | $-6.800578910$ | $-9.821051731$ | $-18.34331348$ | $-16.99573336$ |
| 30    | $-0.479194846$ | $0.235908620$ | $-12.94203150$ | $-9.881694792$ |
| 4     | $7.576059522$ | $-3.388054444$ | $-2.951028070$ | $0.600789897$ |
| 5     | $5.77741219$ |                  |                  |                  |
| 6     |                  |                  |                  |                  |

1 0 $-22.94937178$ $-24.07612820$ $33.06875931$ $33.75348842$ $17.82139784$ $19.05824465$ $-31.51835298$ $-30.93614994$ $11.08351873$ $11.63630999$ $-28.21190897$ $-25.78665449$ $-5.012942659$ $-5.329738972$ $-23.52936071$ $-19.02278925$ $0.753601214$ $0.596581631$ $-18.05658773$ $-11.68537314$ $1.057060553$ $1.237118321$ $-4.946616232$ $-6.900359073$ $-6.62098829$ $7$ $7$ $-2.372969819$ $1.676807238$ $8$ $8$ $1.103404777$

![Fig. 1](a) The variation of the ground state ($n = 0$) energy level for various $\ell$ as a function of the deformation parameter. We choose $\mu = 1 \text{fm}^{-1}$, $V_0 = 40.5 + 0.13 A\_o$, $A = 40$, $R = 1.25 A^{1/3}$ and $a = 0.65 \text{fm}$. (b) The variation of the first excited state for various $\ell$ as a function of the deformation parameter.

To examine the behavior of the energy spectrum with the deformation parameter, we plot the ground $n = 0$ and first excited $n = 1$ states for various values of $\ell$ and as a function of deformation parameter $q$ as shown in Figs. 1(a) and 1(b), respectively.

In Figs. 1(a) and 1(b), when the rotational quantum number $\ell$ and deformation parameter $q$ increases, the particle becomes less attractive or the energy is less negative (tends to continuum states). However, when $q < 0.4$, the particle is strongly bound since the increasing $q$ shields the WS field.

![Fig. 2](a) The variation of the ground state energy for various values of $\ell$ as a function of the deformation parameter. We choose $\mu = 1 \text{fm}^{-1}$, $V_0 = 40.5 + 0.13 A\_o$, $A = 40$, $R = 1.25 A^{1/3}$ and $q = 1.5 \text{fm}$. (b) The variation of the first excited energy state for various $\ell$ as a function of the deformation parameter.

In Figs. 2(a) and 2(b), we plot the ground and first excited states, respectively, for various $\ell$ and as function of $a$. We see that for the case $q = 1.5$, the energy curve is strongly bound for a wide range of $a$ when $\ell = 0$ and then becomes less attractive. The depth of the attractive energy is much higher for small $\ell$. However, when $\ell = 10$, the depth is small and the attractive energy has a short range of $a$, which then becomes purely repulsive. We conclude that the increase in the rotational quantum number $\ell$ leads to the attractive energy which is sensitive to the parameter $a$ and thus the selection of $a$ is in the short range.

In Figs. 3(a) and 3(b), the variation of the potential depth $V_0$, we choose $\mu = 1 \text{fm}^{-1}$, $a = 0.65 \text{fm}$, $A = 40$, $R = 1.25 A^{1/3}$ and $q = 1.5 \text{fm}$. (b) The variation of the first excited energy state for various $\ell$ as a function of the potential depth $V_0$.

In Figs. 2(a) and 2(b), we plot the ground and first excited states, respectively, for various $\ell$ and as function of $a$. We see that for the case $q = 1.5$, the energy curve is strongly bound for a wide range of $a$ when $\ell = 0$ and then becomes less attractive. The depth of the attractive energy is much higher for small $\ell$. However, when $\ell = 10$, the depth is small and the attractive energy has a short range of $a$, which then becomes purely repulsive. We conclude that the increase in the rotational quantum number $\ell$ leads to the attractive energy which is sensitive to the parameter $a$ and thus the selection of $a$ is in the short range.
Figs. 3(a) and 3(b), we plot energy versus $V_q$. We remark that for the selected orbital state, the energy becomes more attractive with increasing $V_q$. In Figs. 4(a) and 4(b), the energy is strongly attractive when the particle mass increases. As the rotational quantum number $\ell$ increases, the attractive energy speeds up towards the positive energy (i.e., the energy becomes less attractive with the increasing $\ell$).

We have seen that the approximate analytical bound states solutions of the $\ell$–wave Schrödinger equation for the nuclear deformed Woods–Saxon potential can be solved by proper approximation to the centrifugal term within the framework of the AIM. Closed analytical forms for the energy eigenvalues are obtained and the corresponding wave functions have been presented in terms of hypergeometric functions.

To test the accuracy of the present potential model, in Table 1 we present the approximate non-relativistic energy states for various vibrational $n = 0, 1, 2, 3$ and rotational $\ell$ states. As in nuclear physics, we used the parameter values $q = 1$, $2\mu\hbar^2 = 0.4727\text{MeV}^{-1}\text{fm}^{-2}$, $R = 1.285A^{1/3}$, $\alpha = 0.65$ and $V_0 = 40.5 + 0.13A_0$. For example, the energy states are calculated by means of the energy formula (31) using $V_0 = 45.7000$ and $V_0 = 47.7800$ for two nuclei of atomic masses $A_0 = 40$ and $A_0 = 56$, respectively. It is seen that when the vibrational quantum number $n$ increases then the energy increases toward the positive energy, i.e., it is becoming weakly bound (less negative). The entire bound states are calculated up to the continuum states where the number of negative bound states increases. The numerical results obtained in this work are found to be identical to the ones obtained before via the Nikiforov–Uvarov method in Ref. [14]. Furthermore, the energy expression (31) provides the solution for the Hulthen potential if $q$ is being replaced by $-q$. We should also remark that the analytical energy formulae (31) and (58) in Ref. [14] are identical and both provide identical numerical values in the MATLAB code. It is also found that the present approximation to the centrifugal term $1/r^2$ is valid only for the lowest energy states.

Finally, the method presented is an elegant and powerful technique. If there are analytically solvable potentials, it provides the closed forms for the eigenvalues and the corresponding eigenfunctions. However, in the case that the solution is not available, the eigenvalues are obtained by using an iterative approach.

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