Non-reciprocal forces and exceptional phase transitions in metric and topological flocks

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Many models of flocking involve alignment rules based on the mean orientation of neighboring particles, which we show introduces microscopic non-reciprocal interactions. In the absence of this microscopic non-reciprocity an exceptional phase transition is predicted at low noise strength within the Toner-Tu framework of polar aligning matter; we demonstrate this transition via large-scale numerical simulations. By coarse-graining the microscopic non-reciprocal forces found in more common models of flocking, we identify additional terms in a hydrodynamic description which lead to a highly ordered clustered phase in metric models and restore the homogeneous flocking phase in topological models.

Flocking in non-equilibrium systems has attracted a great deal of attention in recent decades \cite{menon2018}, as the large-scale dynamical behavior of flocks exhibits fascinating collective phenomena such as spontaneously broken rotational symmetry \cite{vicsek1995, czirok1997} that can be observed experimentally across many length-scales \cite{bird1988, czirok2006}. One natural approach to describing the macroscopic properties of flocks involves hydrodynamic field theories which are typically constructed solely by considering the symmetries and conserved quantities - or lack thereof - of microscopic agent-based flocking models \cite{vicsek2012}. These field theories, which capture the broken Galilean invariance and linear momentum conservation that arises from self-propulsion, make many successful predictions about the phase transitions and dynamical properties of both synthetic and simulated flocks \cite{thurner2015, taylor2015}.

A major theoretical challenge for many flocking systems, particularly those composed of living organisms \cite{arts2016}, is the absence of a Hamiltonian governing the interactions between degrees of freedom, generically leading to a violation of the action-reaction symmetry of Newton’s third law \cite{fabris2016}. The effect this has on allowed flocking phases or other dynamical steady states depends crucially on the specific nature of the microscopic non-reciprocity. For instance, in a flock of birds interacting via a limited cone of vision there is a fore-aft non-reciprocity in the alignment interactions which leads to enhanced density fluctuations in the collective dynamics \cite{fabris2016, thurner2015}. There have been recent efforts to coarse-grain such non-reciprocal interactions to the level of interacting fields in order to better understand flocking dynamics \cite{fabris2016, bray2016}. However, the manner in which flocking dynamics are modified by non-reciprocal forces depends on the nature of the non-reciprocal forces themselves; most work has focused on forces which already break spatial \cite{bray2016} or temporal \cite{fabris2016} symmetries, or which act between species in a multi-component mixture \cite{fabris2016, chuang2016, breuer2016}.

In this work we focus on a subtle source of microscopic non-reciprocity: “neighbor-number-normalized” interactions, such as alignment interactions that depend on the average orientation of nearby particles. This type of interaction has been frequently employed in models ranging from flocking to network synchronization \cite{箅apski2010, toker2017, toker2018}, yet its non-reciprocal nature has often been neglected and is poorly understood \cite{breuer2016, philosophopoulos2016}. Consider the minimal Vicsek-like microscopic flocking dynamics described by

\[
\frac{dr_i}{dt} = v_0 \left( \cos \theta_i \right); \quad \frac{d\theta_i}{dt} = \frac{1}{n_i(t)} \sum_{j \in N_i(t)} f_{ij} + \eta \zeta_i(t). \tag{1}
\]

Here each particle $i$ self-propels at constant speed $v_0$ in a direction $\theta_i$; particle direction is subject a random noise $\zeta_i = \text{Unif.}[\pi, \pi]$ with strength $\eta \in [0, 1]$ and alignment interactions $f_{ij} = \sin(\theta_i - \theta_j)$, with $j$ chosen from a set of neighbors $N_i$. To more deeply understand how microscopic non-reciprocity affects macroscopic collective phenomena, we will consider four essential versions of these evolution equations. First, the set of neighbors $N_i$ can be chosen as all particles with a cutoff radius of particle $i$ (the “metric” model \cite{vicsek1995}) or as all Voronoi neighbors of particle $i$ (the “Voronoi” model \cite{czirok2006, czirok2007}). Second, the turning speed $n_i(t)$ can be chosen to be constant for all particles (the “reciprocal” model, which conserves angular momentum) or one can adopt the much more common Vicsek-like choice \cite{vicsek2012, toker2017, toker2018} $n_i = |N_i|$ (the “non-reciprocal” model), for which the torque on particle $i$ due to particle $j$ is not equal to the negative torque on particle $j$ due to $i$ when $n_i \neq n_j$.

\[
\partial_t \rho + \nabla \cdot (v_0 \rho) = 0 \tag{2}
\]

\[
\partial_t \rho + \lambda (\rho \cdot \nabla) \rho + g(\nabla \frac{1}{N} \cdot \nabla) \mathbf{m} = -\sigma \nabla \rho - \frac{\delta F}{\delta \rho} \tag{3}
\]

The density field dynamics in Eq. 2 reflect the conservation of mass, and are coupled to the non-conserved momentum field, $\mathbf{p} = \rho \mathbf{m}$, via an ideal-gas pressure with compressibility $\sigma$. The $\lambda$ term in Eq. 1 arises from Galilean symmetry-breaking of the microscopic self-propulsion, and the $g$ term arises from the non-reciprocal forces in the equations of motion. Both of these terms drive the system out of equilibrium \cite{bray2016} and into a dynamic steady-state whose configuration is determined by the effective...
In the absence of non-reciprocity Eqs. 2,3 predict a phase transition from a homogeneous flocking phase to a parity-time (PT) symmetric phase in the limit of large global polar order. Parallelling a result obtained in the context of self-propelled rods [31], we begin by investigating the linear stability of homogeneous flocking states to small fluctuations, and consider \( r(x,t) = \rho_0 + \delta \rho(x,t) \) and \( p(x,t) = \rho_0 + \delta p(x,t) \), with \( \rho_0 = \rho_0 \sqrt{\alpha/\beta} \). In the regime of strong global polar order, fluctuations \( \delta \rho \) relax quickly on non-hydrodynamic time-scales relative to \( \delta p \) and \( \delta m_\perp \), and the linear dynamics of Eqs. 2,3 can be expressed in terms of the coupled Fourier amplitudes \( \delta \rho \) and \( \delta m_\perp \) with modes \( e^{-ik \cdot r - i\omega t} \).

Taking the limit of purely transverse fluctuations \( k = k_\perp e_\perp \), the linearized equations of motion from Ref. [31] simplify to

\[
\hat{H}(k_\perp) \left( \begin{array}{c} \delta \rho \\ \delta m_\perp \end{array} \right) = \omega \left( \begin{array}{c} \delta \rho \\ \delta m_\perp \end{array} \right) ; \quad \hat{H}(k_\perp) = \left( \begin{array}{cc} 0 & 1/m_{\text{eff}} \\ -K_{\text{eff}} & -\mu_{\text{eff}} \end{array} \right).
\]

The pseudo-Hamiltonian here has been written in analogy with the dynamical matrix for the phase space coordinate \((x, m \dot{x})\) of a damped harmonic oscillator with the equation of motion \( m \ddot{x} + \mu \dot{x} + K x = 0 \). This structure maps onto that of phase transitions in non-equilibrium systems controlled by an exception point [21], associated with the effective spring constant changing sign.

Here the effective mass of the fictitious oscillator is \( m_{\text{eff}} \equiv -(v_0 k_\perp)^{-1} \), the friction coefficient is \( \mu_{\text{eff}} \equiv i \rho_0 D k_\perp^2 \), and the spring constant is

\[
K_{\text{eff}} \equiv \left[ \sigma - \kappa \rho_0 m_0^2 \right] k_\perp.
\]

The propagation frequencies admitted by these equations are \( \omega_\pm = (-\mu_{\text{eff}} \pm \sqrt{\mu_{\text{eff}}^2 - 4K_{\text{eff}} m_{\text{eff}}^2})/2 \), or in the hydrodynamic limit,

\[
\omega_\pm(k_\perp) = -i D k_\perp^2 \pm k_\perp \sqrt{v_0 (\sigma - \kappa \rho_0 m_0^2)},
\]

with corresponding eigenvectors \( e_\pm = (\omega_\pm/K_{\text{eff}}, 1) \). By varying the macroscopic transport coefficients to change the balance between \( \sigma \) and \( \kappa \rho_0 m_0^2 \), we see that there can be an instability associated with going from a flocking phase \( K_{\text{eff}} > 0 \) to a ‘bound’ phase \( K_{\text{eff}} < 0 \) in which the long wavelength \( k_\perp \) modes become periodic in time due to the negative effective mass of the oscillator. Based on its structure as an exceptional phase transition [21], we expect that the bound phase will possess an additional macroscopic PT symmetry in the \( \delta \rho \) and \( \delta m_\perp \) fields.

We directly confirm this phase transition and its associated striking patterned phases by large-scale simulations of reciprocal versions of both metric and topological models evolving according to Eq. 7. These simulations were carried out by suitably modifying the HOOMD-blue package [32] for metric simulations and adapting the cellGPU package [33] for topological simulations. For the topological model, we use \( \eta \) as a control parameter for tuning \( m_0 \) in Eq. 5. Decreasing \( \eta \), and correspondingly increasing \( m_0 \), we observe an instability of the homogeneous flocking phase as transverse fluctuations cease to be normally distributed about zero and
instead follow a bimodal distribution (Fig. 1a). This transition is accompanied by a large increase in number fluctuations and a decrease in their scaling (Fig. 1b) which is shown qualitatively in Figs. 1–d. The defining macroscopic structure of the low-η phase, though, is in its \( m_\perp \) field. In the homogeneous flocking phase \( m_\perp \) exhibits weak long-wavelength fluctuations about the steady-state value \( \langle m_\perp \rangle = 0 \) (Fig. 1d), while in the low-η phase chiral bands emerge - with chirality referring to the sign of the \( m_\perp \) field - that extend transverse to their direction of propagation (Fig. 1f). This chirality gives the phase a PT symmetry where evolving the system by a time \( \tau = L/m_0 \) is equivalent to evolving it by \( \tau/2 \) and flipping the sign of \( m_\perp \).

The metric model has an extra length scale (the range of the alignment interactions) so, in addition to noise strength, the self-propulsion speed of the particles is a natural control parameter that can be used to tune the effective spring constant of Eq. 5. Boltzmann-equation-inspired coarse-graining suggests that while the ideal-gas compressibility has the proportionality \( \sigma \propto v_0 \), the compressibility arising from the free energy goes as \( \kappa \propto v_0^2 \). Therefore, we would expect to observe a PT symmetry-breaking phase transition in the metric model not only at low noise but also at sufficiently large \( v_0 \). Indeed, as shown in Fig. 2 the homogeneous flocking phase becomes unstable at large \( v_0 \), with macroscopic patterns emerging in the \( \rho \) and \( m_\perp \) fields. However, unlike the Voronoi model, patterning occurs along both the longitudinal and transverse directions, giving rise to mesoscopic chiral clusters.

Why have these striking phases not been observed in simulations of the usual Vicsek model (See Supplemental Material [31]), and what role does microscopic non-reciprocity play on these low-noise or high-\( v_0 \) dynamical patterns? To address this we return to the explicitly non-reciprocal terms we derived in Eqs. [2, 3]. Note that the introduction of the neighbor-number field \( N(r, t) \) requires an additional relation in order to close Eqs. [2, 3]. In the metric model this is straightforward, as we can assume that the number of neighbors at a point is proportional to the local density at that point, \( N \propto \rho \). In the Voronoi model the situation is less straightforward, so to make initial progress we adopt the ansatz \( N \propto \rho^{-1} \). This approximation qualitatively captures the fact that in a disordered point set there is a linear relationship between the average area of a Voronoi cell and its number of sides (i.e. Lewis’ Law [35–37]), although it neglects possible correlations in the emergent structure of the dynamical steady state.

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served in reciprocal state predicted by Toner-Tu hydrodynamics is indeed exceptional phase transition to a patterned PT symmetric simulation in the Supplemental Material [30].

In the Voronoi model, the steady-state configuration of the patterned fields (Figs. 3a,c) are well-described by

\[ f(r) = A(r_{\perp}) \cos \left( \frac{r_{\parallel}}{\lambda} + \phi(r_{\perp}) \right) + f_0 , \]

where the amplitude and phase are themselves fit by a similar functional form. Fitting this function to the fields obtained from microscopic simulations (See Supplemental Material [30], the coarse-grained non-reciprocal neighbor-number force fields are calculated and shown in Figs. [3]a,d. We find that the oscillations of the density field give rise to an effective attractive force between high and low density clusters (with opposite signs of \( m_{\perp} \)) due to the equilibrium \( D_N \) term. These forces destabilize the pattern. This is consistent with observations of extremely dense ‘cluster’ phases that have been previously observed in the canonical Vicsek model [38] and its variants [6].

In the Voronoi model, the steady-state fields are inhomogeneous only along the flocking direction and we can consider a simpler 1D representation of the field profiles (Figs. 4a,c). Unlike the metric model, here the equilibrium \( D_N \) term re-enforces the phase’s pattern (Fig. 4a). However, the ‘true’ (strictly non-equilibrium) non-reciprocal \( g \) term penalizes transverse flows near the boundaries of the two chiral bands (Fig. 4b). We write the combined effect of the hydrodynamic coarse graining of the microscopically non-reciprocal forces as

\[ \psi(r_{\parallel}) = -\partial^2_{r_{\perp}}(\rho)(r_{\parallel} m_{\perp})_r_{\perp} - \partial_{r_{\parallel}}(\rho) r_{\perp} \partial_{r_{\parallel}}(m_{\perp})_r_{\perp} , \]

where for simplicity we take \( g = D_N = 1 \). As shown in Figs. [4],f the net effect of \( \psi \) is to render the interface of the bands unstable, and restore the system to an ordinary homogeneous flocking phase (shown explicitly via simulation in the Supplemental Material [30]).

In summary, in this letter we have shown that the exceptional phase transition to a patterned PT symmetric state predicted by Toner-Tu hydrodynamics is indeed observed in reciprocal models of active polar aligning matter. We suggest that this transition is inhibited in more common flocking models by the non-reciprocal forces that stem from alignment rules that depend on the average dynamics of neighboring particles. A natural extension of our work is to investigate whether the exceptional phase transition observed here follows the detailed framework of Fruchart et al. [21], e.g., whether it proceeds discontinuously through a ‘swap’ phase and into a chiral phase. It is also unknown how the transition behaves in three dimensions where there are two Goldstone modes which may couple to density fluctuations in the flocking phase.

We speculate that the non-reciprocal field theory derived here may also be relevant to understanding other inhomogeneous steady-states observed in simulations of the metric Vicsek model [38][59]. Additionally, the distinction between the neighbor-number field of metric and Voronoi models within the non-reciprocal field theory may shed light on the difference between their transitions from disorder to collective motion. Field theories in which the topological nature of interactions in Voronoi-like flocking models are explicitly coarse grained predict that fluctuations should always render the transition from disorder to collective motion discontinuous [40] (as is seen in metric models [5][41]); yet Voronoi model simulations to date find the transition to be continuous [27]. The presence of...
the additional non-reciprocal terms in the present field theory may account for this discrepancy.

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[1] T. Vicsek and A. Zafeiris, Physics reports 517, 71 (2012).
[2] E. Méhes and T. Vicsek, Integrative biology 6, 831 (2014).
[3] F. Ginelli, The European Physical Journal Special Topics 225, 2099 (2016).
[4] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, Physical review letters 75, 1226 (1995).
[5] H. Chaté, F. Ginelli, G. Grégoire, and F. Raynaud, Physical Review E 77, 046113 (2008).
[6] Y. Zhao, T. Ihle, Z. Han, C. Huepe, and P. Romanczuk, Physical Review E 104, 044605 (2021).
[7] B. Szabo, G. Szöllösi, B. Gönci, Z. Jurányi, D. Selmeczi, and T. Vicsek, Physical Review E 74, 061908 (2006).
[8] J. Buhl, D. J. Sumpter, I. D. Couzin, J. J. Hale, E. Despland, E. R. Miller, and S. J. Simpson, Science 312, 1402 (2006).
[9] H.-P. Zhang, A. Be’er, E.-L. Florin, and H. L. Swinney, Proceedings of the National Academy of Sciences 107, 13626 (2010).
[10] A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, and M. Viale, Proceedings of the National Academy of Sciences 107, 11865 (2010).
[11] J. Toner and Y. Tu, Physical review E 58, 4828 (1998).
[12] A. Bricard, J.-B. Caussin, N. Desreumaux, O. Dauchot, and D. Bartolo, Nature 503, 95 (2013).
[13] D. Geyer, A. Morin, and D. Bartolo, Nature materials 17, 789 (2018).
[14] B. Mahault, F. Ginelli, and H. Chaté, Physical review letters 123, 218001 (2019).
[15] V. Guttal and I. D. Couzin, Proceedings of the national academy of sciences 107, 16172 (2010).
[16] A. V. Ilyev, J. Bartnick, M. Heinen, C.-R. Du, V. Nosenco, and H. Löwen, Physical Review X 5, 011035 (2015).
[17] Q.-s. Chen, A. Patelli, H. Chaté, Y.-q. Ma, and X.-q. Shi, Physical Review E 96, 020601 (2017).
[18] T. Bagarti and S. N. Menon, Physical Review E 100, 012609 (2019).
[19] M. Durve, A. Saha, and A. Sayeed, The European Physical Journal E 41, 1 (2018).
[20] L. P. Dadhichi, J. Kethapelli, R. Chajwa, S. Ramaswamy, and A. Maitra, Physical Review E 101, 052601 (2020).
[21] M. Fruchart, R. Hanai, P. B. Littlewood, and V. Vitelli, Nature 592, 363 (2021).
[22] Z. You, A. Baskaran, and M. C. Marchetti, Proceedings of the National Academy of Sciences 117, 19767 (2020).
[23] Y. Sumino, K. H. Nagai, Y. Shitaka, D. Tanaka, K. Yoshikawa, H. Chaté, and K. Oiwa, Nature 483, 448 (2012).
[24] C. Chen, S. Liu, X.-q. Shi, H. Chaté, and Y. Wu, Nature 542, 210 (2017).
[25] T. Sugi, H. Ito, M. Nishimura, and K. H. Nagai, Nature communications 10, 1 (2019).
[26] O. Chepizhko, D. Saintillan, and F. Peruani, Soft Matter 17, 3113 (2021).
[27] F. Ginelli and H. Chaté, Physical Review Letters 105, 168103 (2010).
[28] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, et al., Proceedings of the national academy of sciences 105, 1232 (2008).
[29] D. M. Sussman, arXiv preprint arXiv:2103.10239 (2021).
[30] [url to be inserted by publisher] (2022).
[31] S. Mishra, A. Baskaran, and M. C. Marchetti, Physical Review E 81, 061916 (2010).
[32] J. A. Anderson, J. Glaser, and S. C. Glotzer, Computational Materials Science 173, 109363 (2020).
[33] D. M. Sussman, Computer Physics Communications 219, 400 (2017).
[34] E. Bertin, M. Droz, and G. Grégoire, Journal of Physics A: Mathematical and Theoretical 42, 445001 (2009).
[35] N. Rivier and A. Lissowski, Journal of Physics A: Mathematical and General 15, L143 (1982).
[36] W. Korneta, S. Mendiratta, and J. Menteiro, Physical Review E 57, 3142 (1998).
[37] T. R. Faisal, N. Hristozov, A. D. Rey, T. L. Western, and D. Pasini, Physical Review E 86, 032607 (2018).
[38] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, et al. Proceedings of the national academy of sciences 105, 1232 (2008).
[39] D. M. Sussman, arXiv preprint arXiv:2003.10239 (2021).
[40] [url to be inserted by publisher] (2022).
[41] S. Mishra, A. Baskaran, and M. C. Marchetti, Physical Review E 81, 061916 (2010).