ARITHMETICALLY NEF LINE BUNDLES

DENNIS KEELER

Abstract. Let $L$ be a line bundle on a scheme $X$, proper over a field. The property of $L$ being nef can sometimes be “thickened,” allowing reductions to positive characteristic. We call such line bundles arithmetically nef. It is known that a line bundle $L$ may be nef, but not arithmetically nef. We show that $L$ is arithmetically nef if and only if its restriction to its stable base locus is arithmetically nef. Consequently, if $L$ is nef and its stable base locus has dimension 1 or less, then $L$ is arithmetically nef.

1. Introduction

Algebro-geometric theorems over fields of characteristic zero can sometimes be reduced to theorems over positive characteristic fields. Perhaps most famously, the Kodaira Vanishing Theorem can be proved in this manner, as in [H2002, Theorem 6.10]. The main idea of the reduction is to replace the base field $k$ with a finitely generated $\mathbb{Z}$-subalgebra $R$ “sufficiently close” to $k$. Objects such as schemes, morphisms, and sheaves are replaced with models defined over $R$. This process is sometimes called “arithmetic thickening.” Some properties of the original objects will be inherited by their thickened versions, such as ampleness of a line bundle.

However, nefness is not such a property. Langer gave an example of a nef line bundle that does not have a nef thickening [Lan2015, Section 8]. Thus on a scheme $X$ proper over a field, we call a line bundle $L$ arithmetically nef if $L$ has a nef thickening. (See (2.2) for the exact definition.)

Arithmetic nefness of a line bundle was studied briefly in [AK2004], where it was shown, in characteristic zero, that $L$ is arithmetically nef if and only if $L$ is $F$-semipositive (a cohomological vanishing condition). In this paper, we carry out a more thorough examination, with basic properties proven in Section 2.

We review the stable base locus in Section 3 generalizing the concept to the case of a Noetherian scheme. We then prove the following in Section 4.

Theorem 1.1. Let $X$ be a proper scheme over a field $k$ with line bundle $L$. Let $\text{SBs}(L)$ be the stable base locus of $L$. Then $L$ is arithmetically nef if and only if $L|_{\text{SBs}(L)}$ is arithmetically nef.

As a corollary, we show that if $L$ is nef and $\dim \text{SBs}(L) \leq 1$, then $L$ is arithmetically nef (Corollary 1.9). We also show that if $L$ is arithmetically nef, then any numerically equivalent line bundle is as well (Corollary 4.3).

We end in Section 5 by reviewing counterexamples of Langer of line bundles $L$ that are nef, but not arithmetically nef. We verify that his positive characteristic counterexample does fail our definition of arithmetically nef.

Throughout we work over an arbitrary field $k$ unless otherwise specified. While the results are of most interest when $k$ has characteristic zero, this hypothesis will not be necessary for our proofs.
2. Basic properties of arithmetic nefness

We must first review arithmetic thickening. The idea is to approximate a field $k$ by its subalgebras $R_\alpha$ which are finitely generated over $\mathbb{Z}$. Objects over Spec $k$, such as finite type schemes, morphisms, and coherent modules, are replaced with “thickened” versions over Spec $R_\alpha$. As $k$ is the direct limit (that is, a special case of colimit) of the $R_\alpha$, these thickened objects share many properties with the originals. When $R_\alpha$ is thickening $k$, we use the subscript $\alpha$ on other associated objects. Note that the $R_\alpha$ form a directed system, since if $R_\alpha \cup R_\beta \subseteq k$, then there exists $R_\gamma$ such that $R_\alpha \cup R_\beta \subseteq R_\gamma \subseteq k$. When $R_\alpha \subseteq R_\beta$, we write $\alpha \leq \beta$.

The basic theory is covered in [Hi2002, Section 6], with more detail in [EGA IV], Section 8 and some proofs in [SP2018, Tag 00QL]. (These techniques work more generally in the case of finite presentation, but since we are working with Noetherian rings, this is the same as finite type.)

Since localizing at $0 \neq f \in R_\alpha$ yields $(R_\alpha)_f$ with $R_\alpha \subseteq (R_\alpha)_f \subseteq k$, we can also replace Spec $R_\alpha$ with appropriate basic open subsets. The appendices of [GW2010] contain a large list (with references) of properties that hold on the fibers over open subsets of Spec $R_\alpha$.

Let $S_\alpha = \text{Spec } R_\alpha$. We will always choose $R_\alpha$ large enough so that $f_\alpha : X_\alpha \to S_\alpha$ is proper [EGA IV, 8.10.5], and hence closed. Since the inclusion $R_\alpha \hookrightarrow k$ factors through the fraction field of $R_\alpha$, the generic point of $S_\alpha$ must be in the image of $f_\alpha$. Hence $f_\alpha$ is surjective.

Some authors also require that $X_\alpha \to S_\alpha$ be flat. This can always be accomplished for large enough $\alpha$ [EGA IV, 11.2.6]. We can also guarantee that $L_\alpha$ is invertible [EGA IV, 8.5.5].

The following lemma summarizes some properties that can be preserved in a thickening. Since many of our proofs reference [EGA] or [Har1977], we note that their definitions of projective morphism coincide when the target has an ample line bundle [SP2018, Tag 01WB, 01WF], and this will always be the case for this paper.

**Lemma 2.1.** Let $X$ be an integral scheme, smooth and projective over an algebraically closed field $k$. Then there exists a thickening $(X_\alpha, R_\alpha)$ such that for all $\beta \geq \alpha$, we have

1. $f_\beta : X_\beta \to \text{Spec } R_\beta$ is smooth and projective,
2. For every $s \in \text{Spec } R_\beta$, the fiber $X_s$ is geometrically integral over the residue field $k(s)$ and all fibers have the same dimension,
3. For every $s \in \text{Spec } R_\beta$, the induced map $f_s : X_s \to \text{Spec } k(s)$ is smooth and projective.

**Proof.** Choose a thickening $(X_\alpha, R_\alpha)$ with $f_\alpha : X_\alpha \to S_\alpha = \text{Spec } R_\alpha$ smooth and projective [EGA IV, 17.7.8, 8.10.5]. Smooth and projective are stable under base change [SP2018, Tag 01VB, 01WF], which gives us claims (1) and (2).

Let $k_\alpha$ be the fraction field of $R_\alpha$. Note $k_\alpha \subseteq k$. Then $X_\alpha \times_{R_\alpha} k_\alpha$ is geometrically integral by definition, since taking the fiber product with Spec $k$ yields the integral scheme $X$.

Since $f_\alpha$ is proper and flat, we can replace $R_\alpha$ with a localization by a non-zero element and assume that for every $s \in \text{Spec } R_\beta$, the fibers $X_s$ of $f_\alpha$ are geometrically integral, of the same dimension [EGA IV, 12.2.1].

Now if $R_\alpha \subseteq R_\beta$ and $s' \in \text{Spec } R_\beta$, then $k(s')$ is a field extension of $k(s)$ for some $s \in \text{Spec } R_\alpha$. Thus the fiber of $f_\beta$ over $s'$ is still geometrically integral. The base
change of Spec \( k(s') \to \text{Spec } k(s) \) also preserves the dimension of the fiber \[ \text{SP2018 Tag 02FY} \]. Hence we have \( (2) \). □

We also must review the concept of relative nefness. For a review of nefness and intersection theory in general, see the seminal paper [Kle1966] (working over an algebraically closed field) or [Kol1996 Chapter VI.2] (working over an arbitrary field).

Let \( S \) be a noetherian scheme, let \( f : X \to S \) be a proper morphism, and let \( L \) be a line bundle on \( X \). For each \( s \in S \), let \( L_s \) be the restriction of \( L \) to the fiber \( X_s \). Recall that \( L \) is \( f \)-nef if \( L_s \) is nef for every closed \( s \in S \) (see, for instance, [Kee2003 Definition 2.9]). If \( S \) is affine, then the property of \( L \) being \( f \)-nef does not depend on \( f \), so we may simply say that \( L \) is nef [Kee2003 Proposition 2.15].

We now define the main concept of arithmetically nef. While the concept is most useful in characteristic 0, there is no harm in using an arbitrary field. (In [AK2004, Definition 3.12], when \( k \) had positive characteristic, the line bundle \( L \) was defined to be arithmetically nef if \( L \) was nef, but this was for convenience.) We do not assume that \( k \) is algebraically closed.

**Definition 2.2.** Let \( X \) be a proper scheme over a field \( k \), and let \( L \) be a line bundle on \( X \). Then \( L \) is **arithmetically nef** if there exists a thickening \( (f_\alpha : X_\alpha \to \text{Spec } R_\alpha, L_\alpha) \) such that \( L_\alpha \) is nef.

**Remark 2.3.** We have not insisted that the field \( k \) have characteristic 0, though this is usually the case when applying arithmetic thickening. When \( k \) is algebraic over a finite field, an arithmetic thickening just yields a subfield \( R_\alpha \subseteq k \) will be algebraic over \( \mathbb{Z}/p\mathbb{Z} \) and hence \( R_\alpha \) is a field [SP2018 Tag 00GS]. In this case, nef and arithmetically nef are equivalent. On the other hand, Example 5.1 reviews a nef, but not arithmetically nef, line bundle when the base field is \( \mathbb{F}_2(t) \).

**Remark 2.4.** The property of \( L_\alpha \) being nef is stable under base change of \( R_\alpha \) [Kee2003 Lemma 2.18]. Thus if \( L \) is arithmetically nef, then \( L_\alpha \) will be nef on every fiber of a certain thickening [Kee2003 Lemma 2.18]. In particular, if \( L \) is arithmetically nef, then \( L \) is nef.

Further, if \( L_\alpha \) is nef, then \( L_\beta \) is nef for any \( \beta \geq \alpha \). That is, once one thickening works, all subsequent thickenings will also work.

Like nefness, the property of being arithmetically nef behaves well under pullbacks. If \( i : Y \to X \) is a closed immersion, we write \( L|_Y = i^*L \).

**Lemma 2.5.** Let \( L \) be a line bundle on a proper scheme \( X \) over a field \( k \), and let \( f : X' \to X \) be a proper morphism.

1. If \( L \) is arithmetically nef, then \( f^*L \) is arithmetically nef, and
2. if \( f \) is surjective and \( f^*L \) is arithmetically nef, then \( L \) is arithmetically nef.

In particular, if \( L \) is arithmetically nef, then \( L|_Y \) is arithmetically nef for any closed subscheme \( Y \subseteq X \).

**Proof.** Choose a thickening \( (f_\alpha : X'_\alpha \to X_\alpha, L_\alpha) \) such that \( f_\alpha \) is proper and also surjective [EGA IV.3, 8.10.5]. If \( L \) is arithmetically nef, then upon choosing a larger \( \alpha \), we may assume \( L_\alpha \) is nef, and so \( f_\alpha^*L_\alpha \) is nef [Kee2003 Lemma 2.17]. Hence \( f^*L \) is arithmetically nef. Now if \( f^*L \) is arithmetically nef, then \( L \) is arithmetically nef by a similar argument [Kee2003 loc. cit.]. □
Similar to ampleness, the concept of arithmetically nef depends only on the reduced, irreducible components.

**Lemma 2.6.** Let $L$ be a line bundle on a proper scheme $X$ over a field $k$. Let $X_i$, $i = 1, \ldots, n$ be the irreducible components of $X$ and $L_i$ be the restriction of $L$ to $X_i$. Then

(1) $L$ is arithmetically nef if and only if $L_{\text{red}}$ (the restriction of $L$ to $X_{\text{red}}$) is arithmetically nef.

(2) $L$ is arithmetically nef if and only if $L_i$ is arithmetically nef for all $i$.

**Proof.** The first claim follows immediately from $i : X_{\text{red}} \rightarrow X$ and Lemma 2.5. For the second claim, we may assume $X$ is reduced and use the natural surjection $f : \prod_i X_i \rightarrow X$ from the disjoint union of the $X_i$ to $X$. □

Arithmetically nef is preserved by base change, allowing reduction to the case of an algebraically closed field. We use a form of faithfully flat descent for one direction of the proof.

**Lemma 2.7.** Let $X$ be proper over a field $k$, let $k'$ be a field extension of $k$, and let $L$ be a line bundle on $X$. Then $L$ is arithmetically nef on $X$ if and only if $L \otimes_k k'$ is arithmetically nef on $X \times_k k'$.

**Proof.** Let $(X, R, L)$ be an arithmetic thickening of $(X, k, L)$ with $L_\alpha$ invertible. Suppose $L$ is arithmetically nef. So we may assume that $L_\alpha$ is nef on $X_\alpha$. Since $R_\alpha \subseteq k' \subseteq k$, we have natural isomorphisms

$$X_\alpha \times_{R_\alpha} k' \cong (X_\alpha \times_{R_\alpha} k) \times_k k' \cong X \times_k k'.$$

We see that $(X_\alpha, R_\alpha, L_\alpha)$ is also an arithmetic thickening of $(X \times_k k', k', L \otimes_k k')$. Hence $L \otimes_k k'$ is arithmetically nef.

Now suppose $L \otimes_k k'$ is arithmetically nef. Due to Remark 2.4 and the uniqueness in [H1902, Proposition 6.2], we can choose a finitely generated $\mathbb{Z}$-algebra $R_\beta$ such that $R_\alpha \subseteq R_\beta \subseteq k'$ and $(X_\alpha \times_{R_\alpha} R_\beta, R_\beta, L_\alpha \otimes_{R_\alpha} R_\beta)$ is an arithmetic thickening of $(X \times_k k', k', L \otimes_k k')$, where $L_\alpha \otimes_{R_\alpha} R_\beta$ is nef.

Since $R_\alpha \subseteq R_\beta$ and both $R_\alpha, R_\beta$ are finite type over $\mathbb{Z}$, we have $R_\beta$ of finite type over $R_\alpha$. By Generic Freeness [SP2018, Tag 051R], we can choose $f \in R_\alpha$ such that $(R_\beta)_f$ is a free $(R_\alpha)_f$-module. Thus the map $\text{Spec}(R_\beta)_f \rightarrow \text{Spec}(R_\alpha)_f$ is flat and surjective. Note that $L \otimes_{R_\alpha} (R_\beta)_f$ is still nef and thus $L_\alpha \otimes_{R_\alpha} (R_\alpha)_f$ is nef by [Kee2003, Lemma 2.18]. Since $(X_\alpha \otimes_{R_\alpha} (R_\alpha)_f, (R_\alpha)_f, L_\alpha \otimes_{R_\alpha} (R_\alpha)_f)$ is an arithmetic thickening of $(X, k, L)$, we have that $L$ is arithmetically nef. □

Due to the ability to change fields, we can check arithmetic nefness by just checking nefness over open subsets of a fixed base $\text{Spec} R_\alpha$.

**Corollary 2.8.** Let $X$ be proper over a field $k$ with line bundle $L$. Let $(X_\alpha, R_\alpha, L_\alpha)$ be an arithmetic thickening such that $L_\alpha$ is invertible and $X_\alpha \rightarrow \text{Spec} R_\alpha$ is proper. Then $L$ is arithmetically nef if and only if there exists $0 \neq f \in R_\alpha$ such that $L_\beta$ is nef, where $R_\beta = (R_\alpha)_f$.

**Proof.** Of course if $L_\beta$ is nef, then $L$ is arithmetically nef by definition.

So suppose $L$ is arithmetically nef. Let $k_\alpha$ be the fraction field of $R_\alpha$. Since $R_\alpha \subseteq k$, we have $k_\alpha \subseteq k$. Let $M_\alpha = L_\alpha \otimes_{k_\alpha} k_\alpha$. By Lemma 2.7, we have that $M_\alpha$ is arithmetically nef because $L \cong M_\alpha \otimes_{k_\alpha} k$ is arithmetically nef.
By definition, there exists a finite type \( \mathbb{Z} \)-algebra \( R_\beta \subseteq k_\alpha \) such that there is a nef thickening of \( M_\alpha \) over \( R_\beta \). By Remark 2.4 we can choose \( R_\beta \supseteq R_\alpha \), and so \( R_\beta \subseteq R_\beta \subseteq k_\alpha \). Since \( k_\alpha \) is the fraction field of \( R_\alpha \), we must have \( R_\beta = (R_\alpha) f \) for some non-zero \( f \in R_\alpha \).

The above corollary makes it easier to show that a line bundle is not arithmetically nef. It cannot occur that \( R_\alpha \) far from \( k \) does not give nefness, while a closer approximation does. For example, we have the following.

**Corollary 2.9.** Let \( X \) be proper over a field \( k \) with line bundle \( L \). Let \( (X_\alpha, R_\alpha, L_\alpha) \) be an arithmetic thickening such that \( L_\alpha \) is invertible and \( X_\alpha \to \text{Spec} R_\alpha \) is proper.

Suppose that for all closed \( s \in \text{Spec} R_\alpha \), the line bundle \( L_{\alpha,s} \) is not nef on the fiber \( X_{\alpha,s} \). Then \( L \) is not arithmetically nef.

**Proof.** Let \( S = \text{Spec} R_\alpha \). The ring \( R_\alpha \) is Jacobson because it is finitely generated over \( \mathbb{Z} \) [SP2018, Tag 00GC]. Thus \( S \) is a Jacobson space and the closed points of any open subset \( U \) are closed in \( S \) [SP2018, Tag 00G3, 005X]. So \( L_\alpha|_U \) cannot be relatively nef. Thus \( L \) is not arithmetically nef by Corollary 2.8. \( \square \)

We end the section with a few more simple observations. Being arithmetically nef also behaves well under tensor product.

**Lemma 2.10.** Let \( X \) be a proper scheme over a field \( k \) with line bundles \( L, M \). Then

1. \( L \) is arithmetically nef, if and only if \( L^n \) is arithmetically nef for all \( n > 0 \), if and only if \( L^n \) is arithmetically nef for some \( n > 0 \), and
2. If \( L \) and \( M \) are arithmetically nef, then \( L \otimes M \) is arithmetically nef.

**Proof.** These statements follow immediately from the definition of nef. \( \square \)

In the projective case, arithmetically nef has a strong connection to ample.

**Proposition 2.11.** Let \( X \) be a projective scheme over a field \( k \), let \( H \) be an ample line bundle, and let \( L \) be a line bundle. Then \( L \) is arithmetically nef if and only if there exists an arithmetic thickening \( X_\alpha \to \text{Spec} R_\alpha \) such that \( H_\alpha \otimes L_\alpha^n \) is ample for all \( n > 0 \).

**Proof.** Choose an arithmetic thickening \( f : X_\alpha \to \text{Spec} R_\alpha \) such that \( f \) is projective [EGA] IV, 8.10.5], \( H_\alpha \) is an ample line bundle [EGA] IV, 9.6.4], and \( L_\alpha \) is a line bundle [EGA] IV, 9.4.7]. The result then follows from [Kee2003] Proposition 2.14]. \( \square \)

### 3. Stable base locus

We now consider some basic properties of the stable base locus of a line bundle \( L \) on a scheme \( X \). While the concept is usually defined when \( X \) is proper over a field, it will be useful to consider any Noetherian scheme. Hence, we generalize the definition via the following proposition. We take supports of coherent sheaves to have their reduced induced structure.

**Proposition 3.1.** Let \( X \) be a Noetherian scheme with line bundle \( L \). Consider exact sequences of the form
\[
\oplus_{i=1}^n \mathcal{O}_X \to L \to F \to 0,
\]
for some \( n \geq 1 \). Then there exists a reduced closed subscheme \( Z \) of \( X \) such that \( Z \) is the minimum support for any such cokernel \( F \). That is, \( Z = \text{Supp}(\text{Coker}(u)) \) for some \( u : \oplus_{i=1}^{n} O_X \to L \), and for any \( v : \oplus_{i=1}^{m} O_X \to L \) (with \( m \geq 1 \)), we have \( Z \subseteq \text{Supp}(\text{Coker}(v)) \).

**Proof.** Note that all sheaves involved are coherent, and hence have closed support [GW2010 Corollary 7.31]. Let

\[
S = \{ Y \mid \exists n \geq 1 \text{ and } u : \oplus_{i=1}^{n} O_X \to L \text{ such that } Y = \text{Supp}(\text{Coker}(u)) \}.
\]

Since \( X \) is Noetherian, \( S \) has minimal elements.

Let \( Z_1, Z_2 \) be two minimal elements of \( S \). So there exists \( n_i \geq 1 \) and \( u_i : \oplus_{i=1}^{n_i} O_X \to L \) with \( Z_i = \text{Supp}(\text{Coker}(u_i)) \). Define \( v : \oplus_{i=1}^{n_1+n_2} O_X \to L \) by \( v(a_1 \oplus a_2) = u_1(a_1) + u_2(a_2) \) for \( a_i \in \oplus_{i=1}^{n_i} O_X \).

Support will not change when tensoring all maps by \( L^{-1} \), so we can abuse notation and consider \( I = \text{Image}(u_i) \) to be a sheaf of ideals. Since \( \text{Image}(v) = I_1 + I_2 \), we have \( \text{Coker}(v) = O_X/(I_1 + I_2) \). Thus \( \text{Supp}(\text{Coker}(v)) \) is the scheme theoretic intersection of \( Z_1 \) and \( Z_2 \) [SP2018 Tag 0C4H] (with reduced induced structure). Since \( Z_1, Z_2 \) are minimal, we must have \( Z_1 = Z_2 \). Hence the minimum \( Z \) exists. \( \Box \)

**Definition 3.2.** Let \( X \) be a Noetherian scheme with line bundle \( L \). The base locus of \( L \), denoted \( \text{Bs}(L) \), is the closed subscheme \( Z \) of Proposition 3.1.

Our definition of base locus is similar to that of [Laz2004 Section 1.1.B], in the case where \( X \) is projective over \( \mathbb{C} \). That is, \( \text{Bs}(L) \) can be interpreted as the support of the cokernel of the natural map \( H^0(X, L) \otimes_k L^{-1} \to O_X \). Another common formulation is that of [Bir2017 Section 2.7]. Working with \( X \) a projective scheme over an arbitrary field \( k \), the paper defines

\[
(3.3) \quad \text{Bs}(L) = \{ x \in X \mid s \text{ vanishes at } x \text{ for all } s \in H^0(X, L) \}.
\]

**Proposition 3.4.** Let \( X \) be a scheme, proper over a field \( k \), with line bundle \( L \). Then the definitions (3.2), (3.3), and [Laz2004 Section 1.1.B] define the same reduced closed subscheme \( \text{Bs}(L) \).

**Proof.** Every \( s \in H^0(X, L) \) defines an \( O_X \)-module homomorphism \( \phi_s : O_X \to L \) by \( \phi_s(1) = s \). Conversely, any homomorphism \( \nu : O_X \to L \) gives a global section \( \nu(1) \in H^0(X, L) \) [SP2018 Tag 01AL]. Thus we can define \( u : H^0(X, L) \otimes_k O_X \to L \) by \( u(s \otimes a) = \phi_s(a) = as \).

Let \( Y \) be the base locus defined by Equation (3.3) and let \( W \) be the base locus defined by [Laz2004 Section 1.1.B]. If \( x \in Y \), then \( s(x) = 0 \) for all \( s \in H^0(X, L) \). This means \( s \otimes k(x) = 0 \) where \( k(x) \) is the residue field at \( x \). So \( u \otimes k(x) = 0 \) and thus \( u \) cannot be onto the stalk \( L_x \). So \( \text{Coker}(u)_x \neq 0 \) and hence \( x \in W \). On the other hand, if \( x \notin Y \), then there exists \( s \in H^0(X, L) \) such that \( s \otimes k(x) \neq 0 \). We have \( s_x \) invertible in \( L_x \cong O_{X,x} \). So \( u \) is onto the stalk \( L_x \) and \( x \notin W \). Thus \( Y = W \).

Let \( Z \) be the base locus defined by Definition 3.2 and \( \oplus_{i=1}^{n} O_X \to L \to F \to 0 \) be an exact sequence with \( Z = \text{Supp}(F) \). Let \( \phi_j : O_X \to L \) be the induced homomorphism from the \( j \)th component of \( \oplus_{i=1}^{n} O_X \). If \( x \notin Z \), then \( \oplus_{i=1}^{n} O_{X,x} \) is surjective. Thus there exists \( j \in \{ 1, \ldots, n \} \) such that \( \phi_j(1)_x \) is not an element of the maximal submodule of \( L_x \). Hence \( \phi_j(1) \otimes k(x) \neq 0 \). In other words, the global section \( \phi_j(1) \in H^0(X, L) \) does not vanish at \( x \). Thus \( x \notin Y \).
On the other hand, \( Z \subseteq Y = Z \) by the minimality in the definition of \( Z \). Hence \( Z = Y = W \), as desired. \( \square \)

Proving the existence of the stable base locus is similar to the projective case.

**Proposition 3.5.** Let \( X \) be a Noetherian scheme with line bundle \( L \). For any \( p, q > 0 \), we have \( \text{Bs}(L^{pq}) \subseteq \text{Bs}(L^p) \). Thus there exists \( m > 0 \) such that for all \( n > 0 \), \( \text{Bs}(L^m) \subseteq \text{Bs}(L^n) \) and \( \text{Bs}(L^m) = \text{Bs}(L^{mn}) \).

**Proof.** It suffices to prove the first claim for \( p = 1 \). Let \( Y = \text{Bs}(L) \) and \( j > 0 \), \( u : \oplus_{i=1}^n \mathcal{O}_X \to L \) such that \( Y = \text{Supp}(\text{Coker}(u)) \). Taking \( U = X \setminus Y \), we have that \( u|_U : \oplus_{i=1}^n \mathcal{O}_U \to L|_U \) is surjective. Thus \( (u|_U)^{\otimes q} : (\oplus_{i=1}^n \mathcal{O}_U)^{\otimes q} \to L^q|_U \) is surjective. Therefore \( \text{Bs}(L^q) \subseteq \text{Supp}(\text{Coker}(u^{\otimes q})) \subseteq \text{Bs}(L) \) by the minimality of \( \text{Bs}(L^q) \).

The remaining claims follow by the Noetherian property on closed sets. See for example [Laz2004, Proposition 2.1.21]. \( \square \)

**Definition 3.6.** Let \( X \) be a Noetherian scheme with line bundle \( L \). The **stable base locus** of \( L \), denoted \( \text{SBs}(L) \), is the closed subscheme \( Y \) of \( X \) of Proposition 3.5.

We work over a general Noetherian scheme so that we need not worry about the behavior of \( H^0(X, L) \) when \( X \) is not reduced or not irreducible. Also, right exact sequences can be thickened [EGA IV, 8.5.6] and these sequences behave well under pullbacks, leading to the following lemma.

**Lemma 3.7.** Let \( X, Y \) be Noetherian schemes, \( f : Y \to X \) a morphism, and \( L \) a line bundle on \( X \). Then

\[
\text{Bs}(f^*L) \subseteq f^{-1}(\text{Bs}(L)), \quad \text{SBs}(f^*L) \subseteq f^{-1}(\text{SBs}(L)).
\]

In particular, if \( Y \) is a locally closed subscheme of \( X \), then

\[
\text{Bs}(L|_Y) \subseteq Y \cap \text{Bs}(L), \quad \text{SBs}(L|_Y) \subseteq Y \cap \text{SBs}(L).
\]

**Proof.** Let \( Z = \text{Bs}(L) \) with associated right exact sequence

\[
\oplus_{i=1}^n \mathcal{O}_X \to L \to F \to 0.
\]

Pulling back by \( f \) is right exact [GW2010, Remark 7.9], so we have

\[
\oplus_{i=1}^n \mathcal{O}_Y \to f^*L \to f^*F \to 0.
\]

By [SP2015, Tag 056J], we have \( \text{Supp}(f^*F) = f^{-1}(\text{Supp}(F)) = f^{-1}(Z) \). Then by the minimality in the definition of base locus, \( \text{Bs}(f^*L) \subseteq f^{-1}(Z) \).

The claim regarding stable base locus then follows easily. \( \square \)

**Remark 3.8.** Let \( k \subseteq k' \) be an extension of fields and let \( X \) be a projective scheme over \( k \), and \( L \) a line bundle on \( X \). Then taking \( f : X \times_k k' \to X \), Birkar shows \( \text{Bs}(f^*L) = f^{-1}(\text{Bs}(L)) \) and \( \text{SBs}(f^*L) = f^{-1}(\text{SBs}(L)) \) [Bir2017, Section 2.7]. The proof only uses the finite dimensionality of \( H^0(X, L) \) and \( H^0(X \times_k k', f^*L) \cong H^0(X, L) \otimes_k k' \), so this works for proper schemes as well.

Fujita shows that when \( Y, X \) are proper over an algebraically closed field, with \( X \) normal and integral, \( Y \) irreducible, and \( f : Y \to X \) surjective, then \( \text{SBs}(f^*L) = f^{-1}(\text{SBs}(L)) \) [Fuj1983, Theorem 1.20]. On the other hand, [Fuj1983, 1.21] gives an example of proper containment when \( X \) is a non-normal rational curve and \( Y \) is its normalization.
4. Main results

As stated, a line bundle can be nef, yet not arithmetically nef. However, in this section we examine some properties that guarantee arithmetic nefness. Weaker versions of some of these results were mentioned in [AK2004 Appendix] without proof.

**Proposition 4.1.** Let $X$ be a proper scheme over a field $k$ with line bundle $L$. If $L$ is numerically trivial (that is, $L$ and $L^{-1}$ are nef), then $L$ and $L^{-1}$ are arithmetically nef (and hence arithmetically numerically trivial).

**Proof.** We need only show that $L$ is arithmetically nef.

First, nefness is stable under both field extension and descent by [Kee2003, Lemma 2.18], as is arithmetic nefness by Lemma 2.7. So we may assume that $k$ is algebraically closed. Second, we may similarly assume that $X$ is integral by definition of nef and Lemma 2.6. Finally, any pullback of $L$ by a proper morphism is also numerically trivial [Kee2003, Lemma 2.17]. Thus by Lemma 2.5 we may replace $X$ with a Chow cover and thus assume that $X$ is projective. Via Alteration of Singularities, we can even assume that $X$ is smooth over $k$ [deJ1996, Theorem 4.1].

Let $S_\alpha = \text{Spec } R_\alpha$. By Lemma 2.1, we can assume all thickenings $f_\alpha : X_\alpha \to S_\alpha$ are smooth and projective, with geometrically integral, constant dimensional fibers. Let $H$ be an ample line bundle on $X$. There exists a thickening where $H_\alpha$ is ample [EGA III1, 4.7.1], and so any subsequent $H_\beta$ is also ample [SP2018, Tag 0893]. For any $s \in S_\alpha$, let $H_s, L_s$ be the pullbacks of $H_\alpha, L_\alpha$ to the fiber $X_s$ of $f_\alpha$ over $s$. (We omit the $\alpha$ since the thickening is not changing.) Since $f_\alpha$ is smooth, and hence flat [SP2018, Tag 01VF], and $S_\alpha$ is irreducible, we have that the intersection numbers $(L^r_s H^{\dim X-r}_s)$ are constant as $s$ varies, and $r = 0, \ldots, \dim X$ is fixed [Kol1996, VI.2.9]. (Here the exponent means self-intersection, not tensor product.)

Let $L_\tau = L_s \otimes \overline{k(s)}, H_\tau = H_s \otimes \overline{k(s)}$ on the smooth, (geometrically) integral scheme $X_\tau = X_s \times_{k(\alpha)} \overline{k(\alpha)}$. Tensoring cohomology of $H^n_\alpha \otimes L^n_\alpha$ (for fixed $m, n \in \mathbb{Z}$) with the algebraic closure of $k(s)$ will not change the value of the Euler characteristic [SP2018 Tag 02KH]. The intersection numbers $(L^r_\tau H^{\dim X-r}_\tau)$, being derived from the Euler characteristic [Kle1966, Section I.2], are thus also constant with respect to $s$.

If $\dim X = 0$, then $X$ is just a point and $L \cong \mathcal{O}_X$ thickens to a relatively numerically trivial line bundle. If $\dim X = 1$, then $(L_s, X_s) = 0$ for all $s$. So again, $L_s$ is nef on every fiber and hence arithmetically nef.

Note that $L_\tau$ is numerically trivial if and only if the same holds for $L_s$ [Kee2003 Lemma 2.18]. If $\dim X \geq 2$, then $L_\tau$ is numerically trivial if and only if

$$(L_\tau^r H^{\dim X-r}_\tau) = (L^r_s H^{\dim X-r}_s) = 0$$

by [Kle1966 p. 305, Corollary 3], as a consequence of the Hodge Index Theorem. The intersection numbers equal 0 at the generic point and hence for all $s$. (See [Kle1966 p. 335] for a similar argument.) Thus $L$ is arithmetically nef, as is $L^{-1}$.

**Remark 4.2.** Instead of changing to an algebraically closed field, one could use a relativized version of Alteration of Singularities, as in [Kee2018 Lemma E1.3], but replacing $R_\alpha$ with a finite extension might be required.
We now have that arithmetic nefness is preserved under numerical equivalence. We write \( L \equiv L' \) if \( L, L' \) are numerically equivalent. That is, \((L.C) = (L'.C)\) for all integral curves \( C \).

**Corollary 4.3.** Let \( X \) be a proper scheme over a field \( k \) with line bundles \( L, L' \). Suppose \( L \equiv L' \). Then \( L \) is arithmetically nef if and only if \( L' \) is arithmetically nef.

**Proof.** This follows from Lemma 2.10 and Proposition 4.1. \( \square \)

At least when \( X \) is a curve, nef and arithmetically nef are always the same.

**Corollary 4.4.** Let \( X \) be a proper scheme over a field \( k \) with line bundles \( L, L' \). Suppose \( L \equiv L' \). Then \( L \) is arithmetically nef if and only if \( L' \) is arithmetically nef.

**Proof.** The case of \( \dim X = 0 \) is trivial since any line bundle is ample. Using the same argument as in the proof of Proposition 4.1, we may assume that \( X \) is a smooth, integral projective curve over an algebraically closed field. Then \( L \) is either numerically trivial or ample, and hence arithmetically nef by Proposition 4.1 or [EGA III, 4.7.1]. \( \square \)

We now examine connections between nefness of \( L \) and the (stable) base locus \( B = \text{SBs}(L) \). Theorems in this vein already exist. For example, when \( X \) is proper over an algebraically closed field, [Fuj1983, Theorem 1.10] shows that if \( L|_{\text{SBs}(L)} \) is ample, then \( L \) is semiample. On the other hand, [Fuj1983, Example 1.16] shows \( L|_{\text{SBs}(L)} \) can be semiample, yet \( L \) is not semiample.

Recently, Birkar showed that for any projective scheme over a field \( k \), if \( L \) is nef, then there exists a subscheme \( Z \) with \( Z_{\text{red}} \) equal to the augmented base locus \( B_+(L) \), such that \( L \) is semiample if and only if \( L|_Z \) is semiample [Bir2017, Theorems 1.4, 1.5]. See [Bir2017, Section 1] for a history of similar results.

Also, the diminished base locus or non-nef locus \( B_-(L) \) was studied in [ELMMP, Mus2013]. One has that \( B_-(L) \subseteq \text{SBs}(L) \) and \( L \) is nef if and only if \( B_-(L) = \emptyset \). (Note that it is possible for \( B_-(L) \) to not be closed [Les2014].) The following theorem fits with these properties of \( B_-(L) \) since the non-nef locus is contained in the stable base locus.

**Theorem 4.5.** Let \( X \) be a proper scheme over a field \( k \) with line bundle \( L \). Then \( L \) is nef if and only if \( L|_{\text{SBs}(L)} \) is nef.

**Proof.** Since nefness and the stable base locus are stable under field extension and descent, by [Kee2003, Lemma 2.18] and Remark 3.5, we can assume \( k \) is algebraically closed. This will match the hypotheses of [Fuj1983, Kle1966].

If \( L \) is nef, then clearly \( L|_{\text{SBs}(L)} \) is nef. So suppose \( L|_{\text{SBs}(L)} \) is nef. Let \( C \subseteq X \) be an integral curve. If \( C \subseteq \text{SBs}(L) \), then \((L.C) = (L|_{\text{SBs}(L)}.C) \geq 0\) by Proposition 5, and \((L|_{\text{SBs}(L)}.C) \geq 0\) by hypothesis.

If \( C \not\subseteq \text{SBs}(L) \), then \( C \cap \text{SBs}(L) \) is a finite set. Thus \( \text{SBs}(L|_C) \) is a finite set since \( \text{SBs}(L|_C) \subseteq C \cap \text{SBs}(L) \) by Lemma 3.7. But then \( \text{SBs}(L|_C) = \emptyset \) [Fuj1983, Corollary 1.14]. So \( L|_C \) is semiample and thus we may replace \( L \) with a multiple and assume \( L|_C \) is generated by global sections. Thus \( L|_C \) is the pullback of a very ample line bundle on some projective space. So we have \((L.C) = (L|_C.C) \geq 0\) by [Kle1966, p. 303, Proposition 1]. This shows that \( L \) is nef. \( \square \)
Remark 4.6. The above theorem makes the nef hypothesis unnecessary in [Bir2017 Theorem 1.5]. We have $\text{SBs}(L) \subseteq B_{\alpha}(L)$ by [ELMMP Example 1.16], [CMM2014 Section 2], [Bir2017 Section 2.7]. If $L|_{Z}$ (with $Z = B_{\alpha}(L)$) is semiample, then $L|_{\text{SBs}(L)}$ is semiample and hence nef. So by Theorem 4.5 $L$ is nef.

We now show that Theorem 4.5 also holds for arithmetically nef. Note that all properties discussed in the proof are stable under a base change $\text{Spec} \rightarrow \text{Spec} R_{\alpha}$, so for simplicity we simply speak of choosing $\alpha$ such that a property holds over $\text{Spec} R_{\alpha}$.

Proof of Theorem 4.5. Let $Z = \text{SBs}(L)$. Suppose $L$ is arithmetically nef. Then $L|_{Z}$ is arithmetically nef by Lemma 2.5.

Now assume $L|_{Z}$ is arithmetically nef. Replacing $L$ with a positive power, we may assume $Z = \text{Bs}(L)$. By Definition 3.2 of base locus, there exists $m > 0$ and a right exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{m} O_{X} \rightarrow L \rightarrow F \rightarrow 0$$

with $\text{Supp}(F) = Z$.

As described in Section 2, we can choose a thickening with $X_{\alpha} \rightarrow \text{Spec} R_{\alpha}$ proper and $L_{\alpha}$ invertible. We can also thicken the sequence (4.7) to the right exact

$$0 \rightarrow \bigoplus_{i=1}^{m} O_{X_{\alpha}} \rightarrow L_{\alpha} \rightarrow F_{\alpha} \rightarrow 0$$

by [EGA IV, 8.5.6]. Let $Z_{\alpha} = \text{Supp}(F_{\alpha})$ and $i : X \rightarrow X_{\alpha}$ be the morphism induced by the base change $\text{Spec} k \rightarrow \text{Spec} R_{\alpha}$. Then $F = i^{*}F_{\alpha}$ and $Z = \text{Supp}(F) = i^{-1}Z_{\alpha}$. By hypothesis and the definition of arithmetically nef, we can also choose $\alpha$ so that $L_{\alpha}|_{Z_{\alpha}}$ is nef. That is, if $s \in \text{Spec} R_{\alpha}$ is a closed point, and $Z_{s}, X_{s}$ are the fibers (respectively) of $Z_{\alpha}, X_{\alpha}$ over $s$, then $L_{\alpha}|_{Z_{s}}$ is nef.

Let $i_{s} : X_{s} \rightarrow X_{\alpha}$ be the natural closed immersion. We can pullback (4.8) to the fiber $X_{s}$. Write $L_{s} = i_{s}^{*}L_{\alpha}$ and $F_{s} = i_{s}^{*}F_{\alpha}$. Then $\text{Supp}(F_{s}) = i_{s}^{-1}Z_{\alpha} = Z_{s}$. By definition of (stable) base locus, $\text{SBs}(L_{s}) \subseteq \text{Bs}(L_{s}) \subseteq Z_{s}$. Since $L_{s}|_{Z_{s}} \cong L_{\alpha}|_{Z_{s}}$ is nef, we have $L_{s}|_{\text{SBs}(L_{s})}$ nef. Hence $L_{s}$ is nef by Theorem 4.5. Since this holds for all closed $s \in \text{Spec} R_{\alpha}$, we have that $L_{\alpha}$ is nef by definition, and hence $L$ is arithmetically nef by definition.

Now combining Corollary 4.4 and Theorem 1.1 we immediately have the following.

Corollary 4.9. Let $X$ be a proper scheme over a field $k$ with line bundle $L$. If $L$ is nef and $\dim \text{SBs}(L) \leq 1$, then $L$ is arithmetically nef.

In particular, if $X$ is an integral surface and $L$ is nef and $L^{m}$ is effective for some $m > 0$, then $L$ is arithmetically nef.

5. Counterexamples

In this section, we consider examples of line bundles $L$ which are nef, but not arithmetically nef. In [Lan2013 Section 8], Langer gave an example of a nef, but not arithmetically nef, line bundle on a smooth projective scheme over a field of characteristic 0. As he works over $\mathbb{Z}[\frac{1}{N}]$ for some natural number $N$, it is clear that his example is not arithmetically nef.

We now consider a characteristic $p > 0$ example and verify that the line bundle is nef, but not arithmetically nef.
Example 5.1. In [Lan2013, Example 5.3], pulling together the results of a few authors, Langer gives an example of a smooth, projective morphism $\pi : X \to S$ with $S = \mathbb{A}^1_k$ and $k = \mathbb{F}_2$. There exists a line bundle $L$ on $X$ such that $L$ is nef on the generic fiber, but not on any closed fiber. This will also be a counterexample to arithmetic nefness as follows.

Let $\{k_\alpha\}$ be the directed system of finite subfields of $k$ and $R_\alpha = k_\alpha[t]$. By Remark 2.3, the $k_\alpha$ are all the finite type $\mathbb{Z}$-subalgebras of $k$. Since all schemes are finite type over $k$, we have arithmetic thickenings $X_\alpha$, $L_\alpha$ over $k_\alpha$. For $\alpha$ sufficiently large, $\pi_\alpha : X_\alpha \to S_\alpha = \text{Spec } R_\alpha$ is smooth and projective [EGA, IV 4.3, 17.7.8, 8.10.5].

Since $k$ is algebraic over $k_\alpha$, we have $k[t]$ integral over $k_\alpha[t]$. Thus $\text{Spec } k[t] \to \text{Spec } k_\alpha[t]$ is surjective [SP2018, Tag 00GQ] and closed points map to closed points [SP2018, Tag 00GT]. Thus $L_\alpha$ is nef on the generic fiber of $\pi_\alpha$, but not nef on any closed fiber [Kee2003, Lemma 2.18]. The $R_\alpha$ are part of the directed system of finitely generated $\mathbb{Z}$-subalgebras of $K = k(t)$. By Corollary 2.9 $L$ restricted to the generic fiber of $\pi$ is an example of a nef, but not arithmetically nef, line bundle over a field $K$ of positive characteristic.

Acknowledgements

We thank D. Arapura, A. Langer, and D. Litt for discussions on this topic.

References

[AK2004] D. Arapura, Frobenius amplitude and strong vanishing theorems for vector bundles. With an appendix by Dennis S. Keeler. Duke Math. J. 121 (2004), no. 2, 231–267.

[Bir2017] C. Birkar, The augmented base locus of real divisors over arbitrary fields. Math. Ann. 368 (2017), no. 3-4, 905–921.

[CMM2014] P. Cascini, J. McKernan, M. Mustaţă, The augmented base locus in positive characteristic, Proc. Edinb. Math. Soc. (2) 57 (2014), no. 1, 79–87.

[deJ1996] A. J. de Jong, Smoothness, semi-stability and alterations, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 51–93.

[EELMP] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, Asymptotic invariants of base loci, Ann. Inst. Fourier (Grenoble) 56 (2006), 1701–1734.

[Fuj1983] T. Fujita, Semipositive line bundles, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 30 (1983), no. 2, 353–378.

[GW2010] U. Görtz, T. Wedhorn, Algebraic geometry I. Schemes with examples and exercises, Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010.

[EGA] A. Grothendieck, Éléments de géométrie algébrique, Inst. Hautes Études Sci. Publ. Math. (1961, 1963, 1966, 1967), no. 11, 14, 28, 32.

[Har1977] R. Hartshorne, Algebraic geometry, Graduate Texts in Math., no. 52, Springer-Verlag, New York, 1977.

[ILL2002] L. Illusie, Frobenius and Hodge degeneracy. Introduction to Hodge theory, 99–150, SMF/AMS Texts and Monographs, 8. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2002.

[Kee2003] D. S. Keeler, Ample filters of invertible sheaves, J. Algebra 259 (1) (2003) 243–283.

[Kee2018] Corrigendum to “Ample filters of invertible sheaves” [J. Algebra 259 (1) (2003) 243–283], J. Algebra 507 (2018) 592–598.

[Kle1966] S. L. Kleiman, Toward a numerical theory of ampleness, Ann. of Math. (2) 84 (1966), 293–344.

[Kol1996] J. Kollár, Rational curves on algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 32, Springer-Verlag, Berlin, 1996.

[Lan2013] A. Langer, On positivity and semistability of vector bundles in finite and mixed characteristics. J. Ramanujan Math. Soc. 28A (2013), 287–309.
[Lan2015] ______., Generic positivity and foliations in positive characteristic, Adv. Math. 277 (2015), 1–23.

[Laz2004] R. Lazarsfeld, Positivity in algebraic geometry. I, Springer-Verlag, Berlin, 2004.

[Les2014] J. Lesieutre, The diminished base locus is not always closed, Compos. Math. 150 (2014), no. 10, 1729–1741.

[Mus2013] M. Mustaţă, The non-nef locus in positive characteristic, A celebration of algebraic geometry, 535–551, Clay Math. Proc., 18, Amer. Math. Soc., Providence, RI, 2013.

[SP2018] The Stacks Project Authors, Stacks Project. http://stacks.math.columbia.edu (2018).

Department of Mathematics, Miami University, Oxford, OH 45056

E-mail address: keelerds@miamioh.edu

URL: http://www.users.miamioh.edu/keelerds