Modulational instability of dust-ion-acoustic waves and associated first and second-order rogue waves in super-thermal plasma

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Abstract
A proper theoretical research has been carried out to explore the modulational instability (MI) conditions of dust-ion-acoustic (DIA) waves (DIAWs) in a three-component dusty plasma system containing inertial warm \( \kappa \)-distributed electrons, and inertial warm positive ions and negative dust grains. The novel nonlinear Schrödinger equation (NLSE) has been derived by employing reductive perturbation method. The analysis under consideration demonstrates two types of modes, namely, fast and slow DIA modes. The dispersion and nonlinear properties of the plasma medium as well as the MI conditions of DIAWs and the configuration of the energetic rogue waves (RWs) associated with DIAWs in the Modulationally unstable regime have been rigorously changed by the plasma parameters, namely, charge, mass, temperature, and number density of the plasma species. The findings of our investigation will be useful in understanding the criteria for the formation of electrostatic RWs in both astrophysical environments (viz., Jupiter’s magnetosphere, cometary tails, Earth’s mesosphere, Saturn’s rings, etc.) and laboratory experiments (viz., Q-machines and Coulomb-crystal).

Keywords: NLSE, modulational instability, reductive perturbation method, rogue waves.

1. Introduction
The massive dust grains have been observed in astro-physical plasmas, viz., Jupiter’s magnetosphere [10, 11, 12], cometary tails [2], Earth’s mesosphere [3], Saturn’s rings [4], and also laboratory plasmas, viz., Q-machines [5], Coulomb-crystal [6, 7, 8]. Generally, the dust grain is million to billion times heavier than the ion, and the charges which are living onto the dust grain are thousand times more than ion [9]. The size, mass, and charge of the dust grains are considered to be responsible to change the dynamics and the criteria for the formation of nonlinear electrostatic waves [10, 11, 12]. Barkan et al. [11] experimentally observed that the phase speed of the ion-acoustic waves increases in the presence of negatively charged dust grains.

The highly energetic particles exhibit the deviation from the thermally equilibrium state, and can be identified in Saturn’s magnetosphere [13, 14, 15, 16], Earth’s bow shock [17], the solar wind [18], and laboratory experiments [19]. These highly energetic particles are governed by super-thermal (\( \kappa \)) distribution [20], and the parameter \( \kappa \) in the \( \kappa \)-distribution presents the deviation of energetic particles from the thermally equilibrium state. The \( \kappa \)-distribution becomes Maxwellian distribution for large values of \( \kappa \) (i.e., \( \kappa \rightarrow \infty \)) [21, 22, 23, 24]. Emamuddin and Mamun [22] examined the dust-acoustic shock waves in a multi-component plasma medium having super-thermal electrons, and found that the width of the shock profile decreases with the super-thermal effect of the electrons. Atteya et al. [23] examined the dust-ion-acoustic (DIA) solitary waves in a super-thermal plasma, and reported that the amplitude and width of the DIA solitary waves increase with decreasing the super-thermality of the plasma species. Saini and Sethi [24] analysed the characteristics of DIA cnoidal waves in the presence of super-thermal electrons.

The nonlinear Schrödinger equation (NLSE) is considered as one of the most useful equations for describing the modulational instability (MI) conditions of different kinds of waves and associated energy re-localization [25, 26, 27]. Rogue waves (RWs), which are the rational solution of the NLSE [25, 26, 27], can be observed in super-fluid helium [28], stock-market crashes [29], optics [30, 31], and plasma [32, 33, 34, 35]. Koukakis and Shukla [33] studied the MI and localized excitations of DIA Waves (DIAWs). Javidan and Pakzad [34] considered a three-component plasma system having cold inertial ions, super-thermal \( \kappa \)-distributed electrons, and immobile negative dust grains, and studied the MI of DIAWs by using NLSE, and found that the critical wave number for which the DIAWs becomes modulationally unstable decreases with \( \kappa \). Shalini and Saini [35] investigated on DIA RWs (DIA-RWs) in a three-component plasma containing inertial warm ion, inertial warm \( \kappa \)-distributed electrons, and stationary dust grains, and highlighted that the amplitude of the DIA-RWs increases with increasing \( \kappa \). To the best knowledge of the authors, no attempt has been made to study the MI of the DIAWs and associated DIA-RWs in a three-component plasma having inertial warm positive ion and negative dust grain, and inertial warm \( \kappa \)-distributed...
electrons. The aim of the present investigation is, therefore, to develop NLSE and investigate DIARWs in a three-component dusty plasma.

The arrangement of the paper is as follows: The basic equations are represented in section 2. The derivation of NLSE is demonstrated in section 3. The MI of DIARWs and rogue waves are provided in section 4. The results and discussion are manifested in section 5. The conclusion is given in section 6.

2. Governing equations

We consider an unmagnetized dusty plasma system consisting of inertial warm positive ions (i.e., mass $m_i$; charge $Z_i$; temperature $T_i$) and negative dust grains (i.e., mass $m_d$; charge $Z_d$; temperature $T_d$), and inertialless super-thermal electrons (i.e., mass $m_e$; charge $e$; temperature $T_e$). The charge neutrality condition at equilibrium for our considered plasma system can be written as $n_{e0} + Z_d n_{d0} = Z_i n_{i0}$; where $n_{e0}$, $n_{d0}$, and $n_{i0}$ are the equilibrium number densities of super-thermal electrons, negative dust grains, and positive ions, respectively. The normalized governing equations for our plasma model can be written as

$$\frac{\partial n_d}{\partial t} + \frac{\partial}{\partial x}(n_d u_d) = 0,$$  \hspace{1cm} (1)

$$\frac{\partial n_d}{\partial t} + u_d \frac{\partial n_d}{\partial x} + \alpha_1 n_d \frac{\partial n_d}{\partial x} = \alpha_2 \frac{\partial \phi}{\partial x},$$  \hspace{1cm} (2)

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0,$$  \hspace{1cm} (3)

$$\frac{\partial n_i}{\partial t} + u_i \frac{\partial n_i}{\partial x} + \alpha n_i \frac{\partial n_i}{\partial x} = -\frac{\partial \phi}{\partial x},$$  \hspace{1cm} (4)

$$\frac{\partial^2 \phi}{\partial x^2} = \alpha_4 n_d + (1 - \alpha_4) n_e - n_i,$$  \hspace{1cm} (5)

where $n_d$, $n_i$, and $n_e$ are normalized by $n_{d0}$, $n_{i0}$, and $n_{e0}$, respectively; the dust and ion fluid speed $u_d$ and $u_i$ are normalized by DIARWs speed $C_i = (Z_i k_B T_e/m_i)^{1/2}$ (where $k_B$ is the Boltzmann constant); the electrostatic wave potential $\phi$ is normalized by $k_B T_e/e$; the time and space are normalized by the $\nu_0^d = (m_i/4\pi e^2 Z_d^2 n_{d0})^{1/2}$ and $\lambda_0^d = (k_B T_e/4\pi e^2 Z_d n_{d0})^{1/2}$, respectively. The pressure term for the dust grains and ion can be written as $P_d = P_{d0}(n_{d0}/n_d)^{\gamma}$ and $P_i = P_{i0}(n_{i0}/n_i)^{\gamma}$, respectively [where $P_{d0} = n_{d0} k_B T_d$ ($P_{i0} = n_{i0} k_B T_i$) represents the equilibrium pressure associated with the warm negative dust grains (positive ions), and $\gamma = (N + 2)/N$ and $\gamma = 3$ (for one dimensional case, i.e., $N = 1$)]. Other relevant physical parameters can be written as $\alpha_1 = 3m_i T_d/Z_d m_d T_e$, $\alpha_2 = \nu_0$ (where $\nu = Z_d Z_i$ and $\mu = m_i/m_d$), $\alpha_3 = 3 T_i/Z_d T_e$, and $\alpha_4 = Z_d Z_i n_{d0}/Z_i n_{i0}$. The normalized form of the number density of electron regarding the $k$-distribution is represented as

$$n_e = \left[1 - \frac{ev\phi}{(k - 3/2)}\right]^{-\kappa}.$$  \hspace{1cm} (6)

The parameter $\kappa$ denotes the super-thermality of electrons. Now, replacing Eq. (6) into Eq. (5) and expanding up to third order in $\phi$, we get

$$\frac{\partial^2 \phi}{\partial x^2} + n_i = 1 - \alpha_4 + \alpha_3 n_d + T_1 \phi + T_2 \phi^2 + T_3 \phi^3 + \cdots,$$  \hspace{1cm} (7)

where

$$T_1 = \frac{(1 - \alpha_4)(2\kappa - 1)}{2(k - 3)},$$

$$T_2 = \frac{(1 - \alpha_4)(2\kappa - 1)(2\kappa + 1)}{2(2k - 3)^2},$$

$$T_3 = \frac{(1 - \alpha_4)(2\kappa - 1)(2\kappa + 1)(2\kappa + 3)}{6(2k - 3)^3}.$$

It should be noted here that the terms containing $T_1$, $T_2$, and $T_3$ at the right hand side of Eq. (7) are due to the contribution of super-thermal electrons.

3. Derivation of the NLSE

To study the MI of the DIARWs, first we want to derive the NLSE by employing the reductive perturbation method. In that case, the stretched co-ordinates can be written in the following form [36]:

$$\xi = \epsilon(x - v_g t),$$  \hspace{1cm} (8)

$$\tau = \epsilon^2 t,$$  \hspace{1cm} (9)

where $v_g$ is the group speed and $\epsilon$ is a small parameter. After that the dependent variables can be represented as [36]

$$\Upsilon(x, t) = \Upsilon_0 + \sum_{m=1}^{\infty} \sum_{|n| = 0}^{\infty} \Upsilon_m^{(n)}(\xi, \tau) \exp[i(kx - \omega_\tau)].$$  \hspace{1cm} (10)

Here $\Upsilon = [n_d, u_d, n_i, u_i, \phi, \Upsilon_0 = [1, 0, 1, 0, 0]^T$, and $\Upsilon_m^{(n)} = [n_d^{(m)}, u_d^{(m)}, n_i^{(m)}, u_i^{(m)}, \phi^{(m)}]$. The carrier wave number (frequency) is defined as $k (\omega)$. We can write the derivative operators as

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial \tau} - \epsilon v_g \frac{\partial}{\partial \xi} + \epsilon^2 \frac{\partial}{\partial \tau},$$  \hspace{1cm} (11)

$$\frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \xi} + \epsilon \frac{\partial}{\partial \xi}.$$  \hspace{1cm} (12)

Now, by substituting Eqs. (10)–(12) into Eqs. (1)–(4), and (7), and collecting the power terms of $\epsilon$, the first order ($m = 1$ with $l = 1$) reduced equations can be written as

$$n_d^{(1)} = \frac{\alpha_2 k^2}{\epsilon \alpha_1 k^2 - \omega^2} g^{(1)}$$  \hspace{1cm} (13)

$$u_d^{(1)} = \frac{k a_2 \omega}{\epsilon \alpha_1 k^2 - \omega^2} g^{(1)}$$  \hspace{1cm} (14)

$$n_i^{(1)} = \frac{k^2}{\epsilon \omega^2 - \alpha_3 k^2} g^{(1)}$$  \hspace{1cm} (15)

$$u_i^{(1)} = \frac{k \omega}{\epsilon \omega^2 - \alpha_3 k^2} g^{(1)}.$$  \hspace{1cm} (16)

These equations provide the dispersion relation of DIARWs in the following form

$$\omega^2 = \omega_d^2 = \frac{k^2 A + k^2 \sqrt{A^2 - 4(k^2 + T_1)B}}{2(k^2 + T_1)},$$  \hspace{1cm} (17)

$$\omega^2 = \omega_d^2 = \frac{k^2 A - k^2 \sqrt{A^2 - 4(k^2 + T_1)B}}{2(k^2 + T_1)}.$$  \hspace{1cm} (18)
where \( \omega_f \) (\( \omega_s \)) is the fast (slow) mode of the DIAWs, \( A = \alpha_1 k^2 + \alpha_3 k^2 + \alpha_1 T_1 + \alpha_3 T_1 + \alpha_2 + 1 \), and \( B = \alpha_1 + \alpha_3 T_1 + \alpha_1 \alpha_3 k^2 + \alpha_3 \alpha_3 \alpha_4 \). We have graphically observed the variation of the fast mode (left panel) and slow mode (right panel) with \( k \) in Fig. 1. It is clear from this figure that (a) the angular frequency of the DIAWs increases with carrier wave number (left panel); (b) but \( \omega_s \) exponentially increases with \( k \) to a particular value of \( k \), and then it becomes saturated showing no change with the variation of \( k \) (right panel). Now, the second-order (\( m = 2 \) with \( l = 1 \)) equations can be written as

\[
\begin{align*}
\eta_{\alpha}^{(2)} &= \frac{\alpha_1 k^2}{2 \omega - \alpha_3 k^2} \phi_{\alpha}^{(1)} + \frac{i I_1}{(\alpha_1 k^2 - \omega^2)^2} \frac{\partial \phi_{\alpha}^{(1)}}{\partial \xi}, \\
\nu_{\alpha}^{(2)} &= \frac{\alpha_2 k^2}{2 \omega - \alpha_3 k^2} \phi_{\alpha}^{(1)} + \frac{i I_2}{(\alpha_1 k^2 - \omega^2)^2} \frac{\partial \phi_{\alpha}^{(1)}}{\partial \xi}, \\
\eta_{\alpha}^{(2)} &= \frac{\alpha_2 k^2}{2 \omega - \alpha_3 k^2} \phi_{\alpha}^{(1)} + \frac{i I_3}{(\alpha_1 k^2 - \omega^2)^2} \frac{\partial \phi_{\alpha}^{(1)}}{\partial \xi}, \\
\nu_{\alpha}^{(2)} &= \frac{\alpha_1 k^2}{2 \omega - \alpha_3 k^2} \phi_{\alpha}^{(1)} + \frac{i I_4}{(\alpha_1 k^2 - \omega^2)^2} \frac{\partial \phi_{\alpha}^{(1)}}{\partial \xi},
\end{align*}
\]

(19)

(20)

(21)

(22)

where

\[
\begin{align*}
I_1 &= \alpha_3 k^2 - 2 \nu_e \alpha_2 k^2 \omega - \alpha_2 k^2 + \omega^2 + \alpha_1 \alpha_3 k^2, \\
I_2 &= \alpha_3 \omega^2 - 2 \nu_e \alpha_2 k^2 \omega + \nu_e \alpha_2 k \omega - \nu_e \alpha_2 k \omega^2 - \nu_e \alpha_2 k \omega^2, \\
I_3 &= \alpha_3 k^3 - 2 \nu_e \alpha_2 k^2 \omega + \nu_e \alpha_2 k \omega^2 + k(\omega^2 - \alpha_3 k^4), \\
I_4 &= \alpha_3 \omega k^2 - 2 \nu_e \alpha_2 k^2 \omega + \nu_e \alpha_2 k \omega^2 + 2 \nu_e \alpha_2 k \omega^2 - \alpha_3 k^4 + \omega^4.
\end{align*}
\]

The second-order (\( m = 2 \) with \( l = 1 \)) equations and with the compatibility condition, we can write the group velocity \( (v_g) \) of the DIAWs in the following form

\[
v_g = \frac{k^2 \omega^2 (\alpha_3 + \alpha_1 \alpha_2 \alpha_4) + k^2 \omega^2 (\alpha_3 + \alpha_2^2 \alpha_4) + I_5}{2 \omega^2 [(\alpha_1 k^2 - \omega^2)^2 + \alpha_2 \alpha_4 (\omega^2 - \alpha_3 k^2)^2] + I_6},
\]

(23)

where

\[
I_5 = \alpha_1 \alpha_3 k^3 (\alpha_3 + \alpha_2 \alpha_4) - 2 \alpha_1 \alpha_3 k^2 \omega^2 (1 + \alpha_2 \alpha_4) \\
- \frac{1}{2} k^2 \omega^2 (\alpha_3 + \alpha_2 \omega^2) + \alpha_1 (1 + \alpha_2 \alpha_4) \\
+ (\alpha_1 k^2 - \omega^2) \omega^2 (1 + \alpha_2 \alpha_4) + \omega^2 (1 + \alpha_2 \alpha_4).
\]

The coefficients of \( \epsilon \) for \( m = 2 \) and \( l = 2 \) provide the second order harmonic amplitudes which are found to be proportional to \( |\phi_{\alpha}^{(1)}|^2 \)

\[
\begin{align*}
\eta_{\alpha}^{(2)} &= T_4 |\phi_{\alpha}^{(1)}|^2, \\
\nu_{\alpha}^{(2)} &= T_5 |\phi_{\alpha}^{(1)}|^2, \\
\eta_{\alpha}^{(2)} &= T_6 |\phi_{\alpha}^{(1)}|^2, \\
\nu_{\alpha}^{(2)} &= T_7 |\phi_{\alpha}^{(1)}|^2, \\
\phi_{\alpha}^{(2)} &= T_8 |\phi_{\alpha}^{(1)}|^2,
\end{align*}
\]

(24)

(25)

(26)

(27)

(28)

where

\[
\begin{align*}
T_4 &= \frac{\alpha_1 \alpha_3 k^3 (2 \alpha_1 T_8 - \alpha_2) - \alpha_2 k^2 \omega^2 (4 \alpha_1 T_8 + 3 \alpha_2) - \alpha_1 \alpha_3 k^6}{2 (\alpha_1 k^2 - \omega^2)^2}, \\
T_5 &= \frac{k^2 \omega(T_4 \alpha_3^3 - \alpha_3^2) - 2 \alpha_1 T_8 k^2 \omega^3 + T_4 \omega^5}{k(\alpha_1 k^2 - \omega^2)^2}, \\
T_6 &= \frac{\alpha_3 (2 \alpha_1 T_8 + 1) - k^2 \omega^2 (4 \alpha_1 T_8 - 3) + 2 T_8 k^2 \omega^4}{2 (\alpha_1 k^2 - \omega^2)^3}, \\
T_7 &= \frac{k^2 \omega(T_6 \alpha_3^3 - 1) - 2 \alpha_1 T_8 k^2 \omega^3 + T_6 \omega^5}{k(\alpha_1 k^2 - \omega^2)^2}, \\
T_8 &= \frac{2 T_2 (\alpha_1 k^2 - \omega^2)^3 (\omega - \alpha_3 k^2)^3 O_1}{2 (\alpha_1 k^2 - \omega^2)^3 (\omega - \alpha_3 k^2)^3 - O_2}.
\end{align*}
\]

(29)

(30)

(31)

(32)

(33)

Now, we consider the expression for \( m = 3 \) with \( l = 0 \) and \( m = 2 \) with \( l = 0 \), which leads the zeroth harmonic modes. Thus, we obtain

\[
\begin{align*}
\eta_{\alpha}^{(2)} &= T_9 |\phi_{\alpha}^{(1)}|^2, \\
\nu_{\alpha}^{(2)} &= T_{10} |\phi_{\alpha}^{(1)}|^2, \\
\eta_{\alpha}^{(2)} &= T_{11} |\phi_{\alpha}^{(1)}|^2, \\
\nu_{\alpha}^{(2)} &= T_{12} |\phi_{\alpha}^{(1)}|^2, \\
\phi_{\alpha}^{(2)} &= T_{13} |\phi_{\alpha}^{(1)}|^2,
\end{align*}
\]

(34)

(35)

(36)

(37)

(38)

where

\[
\begin{align*}
T_9 &= \frac{\alpha_1 \alpha_3 k^3 (\alpha_3 + \alpha_2 \alpha_4) + O_3 - \alpha_2 T_{13} \omega^4}{(v_x^2 - \alpha_3)(\alpha_1 k^2 - \omega^2)^2}, \\
T_{10} &= \frac{v_x T_{10} \alpha_3^3 k^6 - 2 T_{10} \alpha_3^2 k^2 \omega^2 - v_x T_{10} \alpha_3^2 k^2 \omega^2}{(\alpha_1 k^2 - \omega^2)^3}, \\
T_{11} &= \frac{2 \alpha_3 k^2 \alpha_3 k^2 \omega^2 + 2 T_{13} \alpha_3 O_3}{(v_x^2 - \alpha_3)(\omega - \alpha_3 k^2)^2}, \\
T_{12} &= \frac{2 T_{12} (v_x^2 - \alpha_3)(\omega - \alpha_3 k^2)^2 O_3}{(\alpha_1 k^2 - \omega^2)^3 (\omega - \alpha_3 k^2)^2}, \\
T_{13} &= \frac{2 T_{13} (v_x^2 - \alpha_3)(\omega - \alpha_3 k^2)^2 (\omega - \alpha_3 k^2)^2 + O_4}{(\alpha_1 k^2 - \omega^2)^3 (\omega - \alpha_3 k^2)^2},
\end{align*}
\]
where
\[ O_1 = 2\nu_1 \alpha_1^2 k^3 \omega + \alpha_1^2 k^2 \omega^2 (\alpha_2 + 2\kappa_1 \alpha_3), \]
\[ O_4 = (v_0^2 - \alpha_3)(\omega^2 - \alpha_3 k^2) + \alpha_2 \alpha_4^2 k^2 (2\nu_1 k \omega + \alpha_1 k^2 + \omega^2), \]
\[ O_5 = (v_0^2 - \alpha_3) - \kappa_1 (v_0^2 - \alpha_3)(v_0^2 - \alpha_3) + \alpha_2 \alpha_4 (v_0^2 - \alpha_3). \]

Finally, the third harmonic modes \((m = 3)\) and \((l = 1)\), with the help of Eqs. \((13) - (33)\), give a set of equations, which can be reduced to the following NLSE:
\[
\frac{\partial \Phi}{\partial \tau} + P \frac{\partial^2 \Phi}{\partial \xi^2} + Q(\Phi^2) \Phi = 0,
\]
where \(\Phi = \phi^{(1)}\) has been taken for simplicity. In Eq. \((34)\), \(P\) is the dispersion coefficient which can be written as
\[
P = \frac{O_5}{2\omega(\alpha_1 k^2 - \omega^2)(\omega^2 - \alpha_1 k^2) k^2 \times \Omega_f},
\]
where
\[
O_5 = (\alpha_1 k^2 - \omega^2)^2 [v_1 k \omega - \alpha_3 k^2 - \omega^2 - (\omega^2 - \alpha_3 k^2)] + (v_1 k - \omega)(2\nu_1 k \omega - \nu_3 k^2 - \omega^2 - \alpha_3 k^2 - \omega^2) \]
\[
+ \alpha_3 k^2 (\omega^2 - \alpha_3 k^2) - \alpha_2 \alpha_4 (\omega^2 - \alpha_3 k^2)^2 [(v_1 k \omega - \alpha_3 k^2) + (v_1 k - \omega)(2\nu_1 k \omega + \alpha_1 k^2 - \omega^2)]
\]
\[
+ (v_1 k \omega - \alpha_3 k^2 - \omega^2 + \nu_3 k^2) + (v_1 k \omega - \alpha_3 k^2 - \omega^2 + \nu_3 k^2)
\]
\[
- (\alpha_1 k^2 - \alpha_3 k^2)^3 (\omega^2 - \alpha_3 k^2)^3),
\]
\[O_f = (\alpha_1 k^2 - \omega^2)^2 + \alpha_2 \alpha_4 (\omega^2 - \alpha_3 k^2)^2,
\]
and \(Q\) is the nonlinear coefficient which can be written as
\[
Q = \frac{2T_2 (T_8 + T_{13})(\alpha_1 k^2 - \omega^2) (\omega^2 - \alpha_3 k^2)^2 + O_8}{2\omega k^2 (\alpha_1 k^2 - \omega^2)^2 + \alpha_2 \alpha_4 (\omega^2 - \alpha_3 k^2)^2},
\]
where
\[O_8 = 3T_3 (\alpha_1 k^2 - \omega^2)^2 (\omega^2 - \alpha_3 k^2)^2 - 3k^2 \omega (\alpha_1 k^2 - \omega^2)^2 \]
\[\quad (T_7 + T_{12}) - 2\alpha_3 \alpha_4 k^2 (\omega^2 - \alpha_3 k^2)^2 (T_1 + T_{10})
\]
\[\quad - (\alpha_1 k^2 - \omega^2)^2 (k^2 \omega^2 + \alpha_2 \alpha_4 k^2) (T_6 + T_{11})
\]
\[\quad - (\omega^2 - \alpha_3 k^2)^2 (\alpha_1 \alpha_2 \alpha_4 k^2 + \alpha_2 \alpha_4 \omega^2) (T_4 + T_9).
\]

The space and time evolution of the DIAWs in a dusty plasma are directly governed by the coefficients \(P\) and \(Q\), and indirectly governed by different plasma parameters such as \(\alpha_1, \alpha_3, \alpha_4, \mu, v, \kappa, \) and \(k\), etc. Thus, these plasma parameters can significantly modify the stability conditions of DIAWs in a dusty plasma.

4. Modulational instability and rogue waves

The stable and unstable parametric regimes of the DIAWs have been determined by the sign of the dispersion \((P)\) and nonlinear \((Q)\) coefficients of the standard NLSE \((34) - (39)\). When \(P\) and \(Q\) have same sign (i.e., \(P/Q > 0\)), the evolution of the DIAWs amplitude is modulational unstable. On the other hand, when \(P\) and \(Q\) have opposite sign (i.e., \(P/Q < 0\)), the DIAWs are modulationally stable in the presence of external perturbations. The plot of \(P/Q\) against \(k\) yields stable and unstable parametric regimes of DIAWs. The point, at which transition of \(P/Q\) curve intersects with \(k\)-axis, is known as threshold or critical wave number \(k = k_c\) \((36) - (39)\). When \(P/Q > 0\) and \(k < k_c\), the MI growth rate \((\Gamma)\) is given by \((36) - (39)\)
\[
\Gamma = |P| \frac{k^2}{k_c^2} - 1,
\]
where \(\tilde{k}\) is the modulated wave number. The first-order rational solution of NLSE in the unstable parametric regime (i.e., \(P/Q > 0\)) is given by \((25) - (27)\)
\[
\Phi_1(\xi, \tau) = \frac{2P}{Q} \sqrt{\frac{4 + 16\tau P}{1 + 4\kappa^2 + 16\tau^2 P^2} - 1} \exp(2i\tau P).
\]
The second-order RWs, which are the super-position of two or more first-order RWs, solution can be written as \((25) - (27)\)
\[
\Phi_2(\xi, \tau) = \frac{P}{Q} \sqrt{1 + \frac{G_2(\xi, \tau) + iM_2(\xi, \tau)}{D_2(\xi, \tau)}} \exp(i\tau P),
\]
where \( G_2, M_2, \) and \( D_2 \) are some polynomials associated with the variables \( \xi \) and \( \tau \), which can be written as

\[
G_2(\xi, \tau) = \frac{\xi^4}{2} - 6(\xi \tau)^2 - 10(\tau^3) - \frac{3\xi^2}{2} - 9(\tau^2) + \frac{3}{8}.
\]

\[
M_2(\xi, \tau) = -P\{\xi^4 + 4(\xi \tau)^2 + 4(\tau^3) - 3\xi^2 + 2(\tau^2) - \frac{15}{4}\}.
\]

\[
D_2(\xi, \tau) = \frac{\xi^6}{12} + \frac{\xi^4(\tau^3)}{2} + \frac{\xi^2(\tau^4)}{8} + \frac{\xi^4}{8} + \frac{9(\tau^2)^2}{2} - \frac{3(\xi \tau)^2}{2} + \frac{9\xi^2}{16} + \frac{33(\tau^2)^2}{8} + \frac{3}{32}.
\]

Eq. (36) and (37) represent the solution of the first and second-order DIARWs associated with DIAWs, respectively.

5. Results and discussion

Generally, in dust-ion-acoustic waves, the positive ion mass provides the moment of inertia and the thermal pressure of the electrons provides the restoring force in the presence of immobile negative dust grains. On the other hand, in dust-acoustic waves, the moment of inertia is provided by the massive negative dust grains and the restoring force is provided by the thermal pressure of electrons and positive ions. But the consideration of thermal effects of the positive ions can contribute substantively to the moment of inertia along with negative dust grains in the formation of the DIAWs. In our present plasma model, we consider thermal effects of the ions along with negative dust grains (both ion and dust grains are inertial) and would like to examine the contribution of inertial ions in the formation of the DIAWs. We have also considered that \( m_d = 10^6m_i \), \( Z_d = 10^3Z_i \), and \( T_d = T_i > T_d \).

We have graphically observed the variation of the \( P/Q \) with respect to \( k \) for different values of \( \kappa \) in Figs. 2 and 3 corresponding to the DIA fast (\( \omega_f \)) and slow (\( \omega_s \)) modes when other plasma parameters are \( \alpha_1 = 3 \times 10^{-8}, \alpha_2 = 0.3, \alpha_3 = 0.5, \nu = 2 \times 10^3, \mu = 3 \times 10^{-6}, \) and \( \omega \equiv \omega_f \).

The effects of mass and charge state of the positive ion and negative dust grain on the MI of DIAWs in the modulationally unstable parametric regime can be seen in Figs. 4 and 5. Figure 4 describes the modification of the MI growth rate due to existence of heavy negative dust grains and light positive ions. It is obvious from this figure that the \( \Gamma \), initially, increases with \( \tilde{k} \), and becomes maximum for a particular value of \( \tilde{k} \), then decreases to zero. The maximum value of the growth rate increases with the mass of the positive ion but decreases with mass of the negative dust grain. On the other hand, it is clear...
from Fig. 5 that the maximum value of the MI growth rate increases (decreases) with the increase in the value of the negative dust (positive ion) charge state. Physically, the nonlinearity as well as the maximum value of the growth rate increases (decreases) with the increasing charge state of the negative dust grains (positive ions). So, the mass and charge state of the positive ions and negative dust grains have to play an opposite role in the dynamics of the plasma medium.

In our present investigation, we have considered the thermal effect of ions as well as the moment of inertia of positive ions along with negative dust grains. So, it is important to examine how the dynamics of the plasma system changes due to the consideration of thermal effect as well as the moment of inertia of positive ions along with negative dust grains. We have numerically analyzed Eq. (35) in Fig. 6 to observe the effects of ion temperature (via $\alpha_d$) in the formation of first-order DIARWs. The amplitude and width of the first-order DIARWs increase with the increase in the value of the ion temperature ($T_i$) for a fixed value of ion charge state ($Z_i$) and electron temperature ($T_e$). The physics behind this result is that the nonlinearity as well as the amplitude of the first-order DIARWs increases with ion temperature. So, the consideration of the thermal effect of the inertial positive ion significantly changes the dynamics of the plasma medium. Figure 7 illustrates the variation of the first-order DIARWs with different values of $\alpha_d$, and it is clear from this figure that the amplitude and width of the first-order DIARWs increases with $\alpha_d$, and this means that the charge state of the positive ion ($Z_i$) minimizes the nonlinearity as well as the amplitude of the first-order DIARWs while the charge state of the negative dust grain ($Z_d$) maximizes the nonlinearity as well as the amplitude of the first-order DIARWs for a constant value of $n_d$ and $n_0$.

The second-order DIARWs has been depicted by using Eq. (37) in Fig. 8 and it is clear from this figure that the negative dust population ($n_d$) enhances the amplitude of the second-order DIARWs while the positive ion population ($n_0$) reduces the amplitude of the second-order DIARWs for their constant charge states (via $\alpha_d$). Similarly, the increasing charge state of the negative dust grains (positive ions) enhances (reduces) the nonlinearity as well as amplitude of the second-order DIARWs when their number density remains constant.

6. Conclusion

We have studied the stability conditions of the DIARWs in a three-component dusty plasma by considering the thermal effect of the positive ions. In our present analysis the moment of inertia for the formation of the DIARWs is provided by the positively charged warm ions and negatively charged dust grains, and the restoring force is provided by the thermal pressure of the electrons. The evolution of the DIARWs associated with DIARWs is governed by the standard NLSE, and it is interesting that the nonlinear and dispersive coefficients of the NLSE can easily predict the modulationally stable and unstable parametric regimes of DIARWs. The results that have been found from our investigation can be summarized as follows:

- Both modulationally stable (i.e., $P/Q < 0$) and unstable (i.e., $P/Q > 0$) parametric regimes of the DIARWs can exist for both fast and slow modes.
- The nonlinearity as well as the amplitude of the first-order DIARWs increases with ion temperature.
- The negative dust population ($n_d$) enhances the amplitude of the second-order DIARWs while the positive ion population ($n_0$) reduces the amplitude of the second-order DIARWs for their constant charge state.

It may be noted here that the gravitational effect is very important but beyond the scope of our present work. In future and for better understanding, someone can investigate the nonlinear propagation in a three-component plasma by considering the gravitational effect. Hopefully, we can emphasize that the results obtained from the investigation would be helpful to realize various nonlinear phenomenon, where we can expect to have some possibilities for the occurrence of MI as well as the construction of RWs in astrophysical ambience, space, viz., Jupiter’s magnetosphere [10, 11], cometary tails [12], Earth’s mesosphere [13], Saturn’s rings [14, 15], and also laboratory plasmas, viz., Q-machines [16], Coulomb-Crystal [17, 18, 19].

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