Closing a Loophole in Factorization Proofs

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Abstract. We address the possibility in factorization proofs that low-energy collinear gluons can couple to soft gluons.

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This talk is based on Ref. [1], in which one can find a more detailed discussion.

The goal in factorization of QCD processes is to separate perturbative processes at the scale of the large momentum transfer $Q$ from nonperturbative processes at the scale of $\Lambda_{\text{QCD}}$ or smaller. In a factorization formula, the perturbative contributions are contained in short-distance coefficients, which are process dependent. The nonperturbative contributions are contained in long-distance quantities, such as parton distribution functions and fragmentation functions. The predictive power of factorization formulas comes from the process independence (universality) of the nonperturbative quantities.

For hard-scattering processes in QCD, the nonperturbative contributions arise from the emission of soft gluons or gluons that are collinear to external particles. These gluons and the associated propagators to which they attach have virtualities that are much less than the hard-scattering scale $Q$. We use the light-cone momentum components $k = (k^+, k^-, k_\perp)$, with $k^\pm = (1/\sqrt{2})(k^0 \pm k^3)$. Then the components of the momentum of a soft gluon have the orders of magnitude

$$k_S \sim Q \epsilon_S (1, 1, 1_\perp),$$

where $\epsilon_S \ll 1$. There is a soft logarithmic singularity that is associated with the limit $\epsilon_S \to 0$. Suppose that the momenta of the external particles are along the $\pm$ light-cone directions. Gluons that are collinear to these external particles have collinear-to-plus ($C^+$) and collinear-to-minus ($C^-$) momenta:

$$k_{C^+} \sim Q e^+ [1, (\eta^+)^2, (\eta^+)_\perp],$$

$$k_{C^-} \sim Q e^- [(\eta^-)^2, 1, (\eta^-)_\perp],$$

where $\eta^\pm \ll 1$. There is a collinear logarithmic singularity that is associated with the limit $\eta^\pm \to 0$. There is also a soft logarithmic singularity that is associated with the limit $\epsilon^\pm \to 0$.

Leading regions are the Feynman-diagram topologies that yield contributions that are leading in powers of $Q$. One can find the leading regions for gauge theories by analyzing pinch singularities in the momentum contours of integration and by making use of power-counting arguments [2, 3, 4, 5, 6]. For definiteness, consider $e^+e^-$ annihilation into two light mesons. In this discussion and in subsequent discussions, we work in the Feynman gauge. The conventional leading regions have the form that is shown in Fig. 1. $J^+$ are jet subdiagrams, which contain the external particles and associated collinear gluons. $S$ is a soft subdiagram, which contains soft gluons. $H$ is a hard subdiagram, which contains only propagators with virtuality of order $Q^2$. In the conventional picture of the leading regions, soft gluons attach to the collinear subdiagrams and collinear gluons attach only to the hard subdiagram. This form is also implicit in the action in soft-collinear effective theory (SCET) [7].

However, low-energy collinear gluons can couple to soft gluons. Consider the two-loop example in Fig. 2 in which a $C^+$ gluon attaches to a soft gluon. The vertex, propagator and phase-space factors give (for $\epsilon^+ \lesssim \epsilon_S$) the factor $\epsilon_S e^+/(\epsilon_S^2 + \epsilon_S e^+).$ This factor is independent of $Q$ and gives a leading contribution if $\epsilon^+ \sim \epsilon_S$. Hence, the leading regions must include couplings of collinear

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig1.pdf}
\caption{Conventional leading regions for $e^+e^-$ annihilation into two light mesons.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig2.pdf}
\caption{Example of a collinear-collinear-collinear coupling.}
\end{figure}
gluons to soft gluons. Power-counting arguments also show that low-energy collinear gluons can couple to each other, as well as to the hard subdiagram. Thus, the leading regions have the form that is shown in Fig. 3.

Because of color confinement, gluons with momentum components less than of order $\Lambda_{\text{QCD}}$ are unphysical. Therefore, one might ask why we need to consider gluons with energy less than $\Lambda_{\text{QCD}}$. However, as we have seen, low-energy gluons can appear in perturbation theory in leading power in $Q$. Perturbative calculations are used to compute short-distance coefficients. In order to establish the consistency of such calculations, it is necessary to prove that the contributions from low-energy gluons can be re-organized into the standard factorized form. Specifically, for $e^+e^-$ annihilation into two light mesons, we need to show that the amplitude factors into (1) a hard function that contains only propagators with virtualities of order $Q^2$, (2) jet functions that contain all of the collinear contributions, and (3) a soft function that contains all of the soft contributions and that cancels when one sums over connections to quark and antiquark in a meson. Our strategy for proving this factorization has the following steps: (1) we show that the soft and collinear singularities decouple from the hard subdiagram and from each other; (2) we show that the soft singularities cancel; (3) in the jet functions, we extend the ranges of integration of the collinear gluon momenta to regions of order $Q$ around the collinear singularities, thereby incorporating all of the collinear contributions into the jet functions; (4), we re-define the hard function to be the amplitude divided by the extended jet functions. The re-defined hard function is free of soft and collinear singularities and depends only on the scale $Q$. Hence, it contains only virtualities of order $Q^2$.

In analyzing the soft and collinear singularities, we need to consider the possibility that different loop momenta can approach the soft and collinear limits at different rates. The allowed limiting procedures are governed by power-counting arguments. Along a given line, the momentum components of gluons that attach to the exterior provide lower bounds on the momentum components of gluons that attach to the interior. That is, the exterior divergences “control” the interior divergences.

We now describe the technical tools that we need in order to prove factorization: collinear approximations, the soft approximation, and decoupling relations.

If a gluon carrying $C^\pm$ momentum $k$ attaches to a line that does not carry $C^\pm$ momentum, then the $C^\pm$ approximation can be applied [5, 8]. In the $C^\pm$ approximation one replaces the $g_{\mu\nu}$ in the gluon-propagator numerator with $k_\mu n^{\mp}_\nu$, where $n^{\mp}$ is a light-like vector in the $\mp$ direction. In the $C^\pm$ approximation, the index $\mu$ attaches to the non-$C^\pm$ line. The collinear approximations are exact at the collinear singularities $\eta^{\pm} = 0$. The collinear approximations do not depend on the momentum of the line to which the collinear gluon attaches. As we shall see, a very useful property of the collinear approximations is that they result in a longitudinal gluon polarization.

If a gluon carrying a soft momentum $k$ attaches to a line carrying momentum $p$ and the components of $k$ are much less than the largest component of $p$, then the soft approximation applies [9, 10]. The soft approximation consists of replacing the $g_{\mu\nu}$ in the soft-gluon propagator numerator with $k_\mu p_\nu/k\cdot p$. The index $\mu$ attaches to the line with momentum $p$. The soft approximation is exact at the soft singularity $\varepsilon_5 = 0$. In contrast with the collinear approximation, the soft approximation depends on the momentum of the line to which the singular gluon attaches. The soft approximation also results in a longitudinal gluon polarization.

If the longitudinally polarized gluons that result from one of the above approximations (soft, $C^+$, or $C^-$) attach in all possible ways to a subdiagram, then the graphical Ward-Takahashi identities can be used to show that they decouple [5, 8, 10]. The decoupling relations are depicted in Fig. 4. The hash-marks on the external lines indicate that those lines are truncated. There can also be an arbitrary number of on-shell external lines, which are not shown explicitly. The arrows represent the gluon-propagator replacement factors from the soft, $C^+$, or $C^-$ approximation. The “eikonal” (double) lines have vertices of the form $n_\mu$ and propagators of the form $1/(k\cdot n)$, where $n$ is the vector that appears in the soft, $C^+$, or $C^-$ approximation. These eikonal lines are path-ordered ex-
ponents of path integrals of gauge fields.

In non-Abelian gauge theories, the decoupling relations require that the gluons have momenta that are proportional to each other. This is automatically the case for gluons with momenta at a $C^+$ or $C^-$ singularity. If a soft gluon with momentum $k$ attaches to a subdiagram in which all lines have $C^\pm$ singular momenta, then only the component $k^\pm$ enters into the interactions in the subdiagram. Without loss of accuracy, we can replace $k$ in the subdiagram and in the soft approximation with a vector whose only nonzero component is $k^\pm$. Then, all of the soft gluons that couple to the $C^\pm$ singular subdiagram have momenta that are proportional to each other, as is required by the decoupling relation.

In order to carry out the factorization, we follow an iterative procedure, starting with the singular contributions that are innermost in the Feynman diagrams (those with the largest energy scale) and working to the outside. Each stage in the iteration involves soft gluons with energies of order a nominal scale (NS), collinear gluons with energies of order the NS, and collinear gluons with energy of the large scale (LS). The LS is much larger than the NS, but much smaller than the NS of the next larger (inner) level. We apply the soft and collinear approximations and the decoupling relations at each stage. We also make use of relationships between collinear eikonal lines to combine contributions within each stage and to combine contributions from successive stages. We refer the reader to Ref. [1] for details. The result is that the soft and collinear singular contributions factor, and we arrive at the form that is shown in Fig. 5. In this figure, $\tilde{S}$, $J^\pm$ denote the singular parts of $S$ and $J^\pm$.

It can be seen that the soft eikonal lines that attach to a quark and an antiquark in a given meson cancel. This can be established by making use of order-by-order algebraic relations or, more simply, by noticing that the corresponding path-ordered exponentials run in opposite directions and end on space-time points that are separated by $k_\mu / Q \to 0$. The cancelling contributions have the same color factor by virtue of the color-singlet nature of the meson.

There is also a cancellation of the parts of the quark and antiquark collinear eikonal lines for which the energies of the collinear gluons are much less than $Q$. This cancellation implies that the couplings of the low-energy $C^\pm$ gluons to subdiagrams outside $J^\pm$ do not contribute in the end, but this becomes apparent only when one has carried out the re-organization that we have described.

Now we extend the ranges of integration in $J^\pm$ up to an ultraviolet cutoff $\mu_F \sim Q$, which acts as a factorization scale. We also re-define $\tilde{H}$ to be the complete amplitude divided by $J^+$ and $J^-$. Then, $\tilde{H}$ is free of soft and collinear singularities and depends only on the scale $Q$. Hence, we have arrived at the standard factorized form:

$$A = J^- \otimes \tilde{H} \otimes J^+, \quad (4)$$

where $\tilde{H}$ contains only virtualities of order $Q^2$ and $J^+$ and $J^-$ contain all of the collinear contributions with virtualities of order or less than $\Lambda^2_{QCD}$.

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