Dynamical maps, quantum detailed balance and Petz recovery map

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Markovian master equations (formally known as quantum dynamical semigroups) can be used to describe the evolution of a quantum state $\rho_S$ when in contact with a memoryless thermal bath. This approach has had much success in describing the dynamics of real-life open quantum systems in the lab. Such dynamics increase the entropy of the state $\rho_S$ and the bath until both systems reach thermal equilibrium, at which point entropy production stops. Our main result is to show that the entropy production at time $t$ is bounded by the relative entropy between the original state and the state at time $2t$. The bound puts strong constraints on how quickly a state can thermalise, and we conjecture that the factor of 2 is tight. The proof makes use of a key physically relevant property of these dynamical semigroups – detailed balance, showing the this property is intimately connected with the field of recovery maps from quantum information theory. We envisage that the connections made here between the two fields will have further applications. We also use this connection to show that a similar relation can be derived when the fixed point is not thermal.

I. INTRODUCTION

It is very often observed in nature that physical systems relax to an equilibrium state. This phenomenon, which has very evident consequences at the macroscopic scales of our everyday experience, ultimately relies on the dynamics of the microscopic components. This fact was understood in the early days of statistical mechanics, and since then a large amount of work has been produced with the aim of trying to understand how exactly physical systems reach thermal equilibrium.

Any such evolution will be ultimately generated through some reversible dynamics on a large composite system, that is effectively irreversible as seen by a smaller part of that composite system. This irreversibility means that, in a coarse-grained sense, entropy will be produced throughout the process. The entropy production can be linked to the fact that correlations between a big thermal object (a heat bath) and one smaller subsystem $S$ are increasingly harder to access, which forces the coarse-graining of the description [1]. Intuitively, the more irreversible a process is, the more entropy is produced, and the closer a particular system will be to equilibrium.

In this work we look at a commonly used family of quantum evolutions that model the dynamics of a system weakly coupled to a thermal bath, and show explicitly how the amount of entropy produced along a particular evolution is related to how much a state changes along that evolution. These maps were first studied by Davies [2] and are a quantum generalization of the classical Glauber dynamics. They are Markovian, and have the thermal state as a fixed point of the evolution.

In the limit of a large thermal bath, the total entropy produced by such a process is given by how much the free energy of a system decreases with time [3]. The free energy for a state $\rho_S(t)$ at time $t$ is defined as

$$F_\beta(\rho_S(t)) = \text{Tr}[\hat{H}_S \rho_S(t)] + \frac{1}{\beta} \text{Tr}[\rho_S(t) \log \rho_S(t)],$$

where $\hat{H}_S$ is the Hamiltonian of the subsystem of interest, and $\beta^{-1}$ is the temperature of the bath. This is a consequence of the fact that i) the dynamics has the thermal state of the system $\tau_S = \frac{e^{-\beta \hat{H}_S}}{Z}$ as a fixed point and ii) the quantum relative entropy is contractive under quantum evolutions. More explicitly, for an evolution from time $t = 0$ to $t$, the total amount of entropy produced is given by $F_\beta(\rho_S(0)) - F_\beta(\rho_S(t))$, which is always positive.

Our main result is Theorem [2] which states that under the condition that the interaction between system and bath is time-independent, we can lower-bound the entropy production at time $t$ by the state at time $2t$.

This sharpens some intuitive notions, namely that if not much entropy is produced during a time interval $\Delta t$, the state will not change very much during the time interval $2\Delta t$, but if it does, then a large amount of entropy must have been produced at an earlier time, namely during the time interval $\Delta t$. This relationship between the entropy production at an earlier time to the state at a later time is not due to any memory effects, since the dynamics is purely Markovian. Moreover, as we will explain, the key ingredient is a general property of these Davies maps, namely quantum detailed balance.

The bound proves a physically relevant particular case of an open conjecture about general quantum maps first formulated in [4]. The strongest possible version of the conjecture is known to not be true in full generality [5], although it has been shown for particular sets such as unital maps [6], classical stochastic matrices [3], catalytic thermal operations [7] and we here show it for Davies maps. All these results relate the decrease of relative entropy with a measure of how well a given pair of states...
can be recovered through a particular recovery map, and are generalizations of an early result by Petz [8]. For the best results up to date on general quantum maps, see [9][12].

This line of research was very much inspired by the study of quantum conditional mutual information and quantum Markov chains [13], where recent results put lower bounds on the quantum conditional mutual information with measures of how well the operation of tracing out a subsystem within a tripartite state can be undone. See [14] for an early breakthrough, and [5][15][17] for further refinements. Recovery maps have found many applications in quantum information theory in general, such as coding theorems [18], approximate error correction [19] or asymmetry [20]. They also appear in the derivation of quantum fluctuation theorems [21][22].

Our results, inspired by findings in quantum information theory, are a consequence of the observation that if a dynamical map satisfies quantum detailed balance, a property of thermodynamical processes, then this implies that the map is its own recovery map. The connection between information theory and thermodynamics goes back a long way, to the seminal work of Landauer [23] and has furthered our understanding of both significantly. Within the current surge of information-theory approaches to quantum thermodynamics (see [24] for a review), our result provides another example of how ideas from one may find definite applications in the other.

We shall first introduce Davies maps, outline their properties, and explain their entropy production. This is followed by the statement of main result and a discussion on the bound it self. We finally conclude with some suggestions for open questions. The technical results have all been placed in the appendix, where we also include a discussion on Davies maps and of the bound in the infinitesimal time limit.

II. DAVIES MAPS AND ENTROPY PRODUCTION

Davies maps are a particular set of quantum dynamical semigroups that describe the evolution of a system on a $d_S$ dimensional Hilbert space that is weakly interacting with a heat bath. The first rigorous derivation of their form was given in [2] (see [25][26] for more modern treatments). As they are time-continuous quantum semigroups, their generator takes the form of a Lindbladian operator, which we define as

$$\frac{d\rho_S(t)}{dt} = \mathcal{L}(\rho_S(t)),$$

where $\mathcal{L}$ is called the Lindbladian. The solution is a one-parametre family of CPTP maps\footnote{\textit{CPTP} stands for \textit{completely positive trace-preserving}.}, $M_\Delta(\cdot)$, $\Delta \geq 0$ which governs the dynamics, $M_\Delta(\rho(t)) = \rho(t + \Delta)$. We will not delve into the full details here, but instead highlight the important properties the canonical form of Davies maps, denoted $T_t(\cdot)$ possess:

1) They arise from the weak system-bath coupling limit
2) They can be written in the form $T_t(\cdot) = e^{t\mathcal{L}(\cdot)}$, with the Lindbladians time independent
3) They have a thermal fixed point: $T_t(\tau_S) = \tau_S$, where $\tau_S$ is the Gibbs state of the system at temperature $T_S$.
4) Their Lindbladians satisfy Quantum detailed balance (QDB):

$$\langle A, \mathcal{L}^1(B) \rangle_\Omega = \langle \mathcal{L}^1(A), B \rangle_\Omega$$

for all $A, B \in \mathbb{C}^{d_S \times d_S}$, where $\mathcal{L}^1$ is the adjoint Lindbladian and in the case of Davies maps, $\Omega = \tau_S$. The scalar product in Eq. (3) is defined as

$$\langle A, B \rangle_\Omega := \text{Tr}[\Omega^{1/2} A^\dagger \Omega^{1/2} B].$$

This is sometimes referred to as reversibility or KMS condition. It is stronger than (2), since it has as a consequence that $\Omega$ is the fixed point, as $\mathcal{L}(\Omega) = 0$.

In Appendix A we give a more detailed account of the microscopic origin of these maps, and of the form of the weak coupling limit, property 1). In the literature, there are various different definitions of QDB which are in general not equivalent. We show in Appendix C that for maps satisfying time translation symmetry, such as Davies maps, definition 4) is equivalent to the definition of QDB in [25][27].

In addition, it is sometimes assumed that:

5) The dynamics associated with Davies maps converge to the fixed point, $\lim_{t \to \infty} T_t(\rho_S(0)) = \tau_S$.

The necessary and sufficient conditions for such convergence are not fully understood [22][28], and we will not need to assume 5) here.

Since we wish to bound the distance from the state at time $t$ to the fixed point, we need a distance measure. For this we use the relative entropy $D(\rho||\sigma) = \text{Tr}[\rho(\log \rho - \log \sigma)]$. This measure is meaningful since it is non-negative, zero iff $\rho = \sigma$, and is contractive under CPTP maps. For the special case that $\sigma$ is a Gibbs state, it has an interpretation in terms of a free energy,

$$D(\rho(t)||\tau_S) = \beta F_\beta(\rho_S(t)) - \log Z_S,$$

where $Z_S = \text{Tr}[e^{-\beta H_S}]$ is the partition function of the system, which we assume is constant. We can thus write

$$D(\rho(0)||\tau_S) - D(\rho(t)||\tau_S) = \beta (F_\beta(\rho_S(0)) - F_\beta(\rho_S(t))) = \beta \Delta E - \Delta S,$$

where $\Delta E$ and $\Delta S$ are the change in energy and change in entropy, respectively.
with $\Delta E, \Delta S$ the changes in mean energy and entropy of the system. Due to the contractivity property of $D$, Eq. (9) is non-negative and non-decreasing with $t \geq 0$. The l.h.s. of Eq. (6) is referred to in the literature as the entropy production. The reason for this name is as follows. For a large thermal reservoir, small changes of energy (that is, heat transferred to the system) are proportional to changes of entropy in it, with proportionality constant $\frac{1}{T}$. Hence, we can identify the change in energy in the system with a change of entropy in the reservoir $\beta \Delta E \approx -\Delta S_{\text{bath}}$, so that the difference in free energy of the system for a time interval $\Delta t$ is equal to the total entropy generated during the interval $\Delta t$ in system and bath. Therefore, Eq. (8) constitutes a natural measure of the irreversibility of the process.

### III. MAIN RESULTS

Our main result is a tight lower bound on the change of free energy, or entropy produced, for a finite time. For Davies maps, the following lemma holds:

**Lemma 1.** All Davies maps $T_t(\cdot)$, satisfy the inequality

$$F_\beta(\rho_S(0)) - F_\beta(\rho_S(t)) \geq \frac{1}{\beta} D(\rho_S(0)\|\tilde{T}_t(\rho_S(t))) ,$$

where $\tilde{T}_t(\cdot)$ is the time-reversed map or Petz recovery map, defined as

$$\tilde{T}_t(\cdot) = \tau_S^{1/2} T_t^\dagger \left( \tau_S^{-1/2} \cdot \tau_S^{-1/2} \right) \tau_S^{1/2} ,$$

with $T_t^\dagger$ denoting the adjoint of $T_t$.  

**Proof.** See Appendix A.2

For Lemma 1 to hold, only properties 1) and 3) are required. In addition, we find that there is a striking connection between property 4) and the Petz recovery map which we will not explain. A quantum dynamical semi-group obeying QDB has a Petz recovery map, which is equal to the map itself $\tilde{T}_t = T_t$ (See Theorem 8 in Appendix). Petz derived his famous recovery map in 1986 while the first appearance of the quantum detailed balance condition goes back at least to the work of Davies in 1974 [2]. To the best of the authors knowledge, this connection between results from the communities of quantum information theory and quantum dynamical semi-groups, although evident, was previously unknown.

The classical definition of detailed balance, in terms of the transition probabilities $p(j|i)$ of a classical Markov equation, implies that, at equilibrium, a particular jump between energy levels $E_i \to E_j$ has the same total probability as the opposite jump $E_j \to E_i$, such that $p(j|i)e^{-\beta E_j} = p(i|j)e^{-\beta E_i}$. The condition in Eq. (9) is the most natural quantum generalization of that (although as shown in [29] different ones are also possible).

In that sense, QDB can be understood as the fact that a particular thermalization process coincides with its own time-reversed map, which is defined as

$$\tilde{\Gamma} (\cdot) = \sigma^{1/2} \Gamma^\dagger \left( \Gamma(\sigma)^{-1/2} \cdot \Gamma(\sigma)^{-1/2} \right) \sigma^{1/2} .$$

This map is such that we have that iff $D(\rho||\sigma) = D(\Gamma(\rho)||\Gamma(\sigma))$ then $\tilde{\Gamma}(\Gamma(\rho)) = \rho$ and $\tilde{\Gamma}(\Gamma(\sigma)) = \sigma$. It appears in quantum information theory when one tries to find the best possible way to recover data after it is processed.

We can hence rewrite Lemma 1 as

**Theorem 2.** All Davies maps $T_t(\cdot)$, satisfy the inequality

$$F_\beta(\rho_S(0)) - F_\beta(\rho_S(t)) \geq \frac{1}{\beta} D(\rho_S(0)\|\rho_S(2t)) .$$

**Proof.** See Appendix A.3

In addition to assuming detailed balance, condition 4), we have also used condition 2). If the Lindbladian $\mathcal{L}$ is time dependent, i.e. 2) is not satisfied, Eq. (10) holds but with $\rho_S(2t)$ replaced with $T_t(\rho(t))$.

In Fig. 1 we show a simple example of the inequality for the case of Davies maps applied on a qubit. Eq. (10) is tight at $t = 0$ and also in the large time limit, as long as condition 5) is satisfied. In this limit, the total entropy that has been produced is equal to $\frac{1}{\beta} D(\rho(0)||\tau_S)$, which both sides of the inequality approach as $\rho_S \to \tau_S$.

On the other hand, for very short times, the lower bound becomes trivial. In particular, in Appendix A.4
we show what both sides of the inequality tend to in the limit of infinitesimal time transformations. The entropy production becomes a rate, and the lower bound to it approaches 0. The latter can be seen in Fig. 1 by the fact that the dashed curve has a stationary point at the origin.

Non-trivial lower bounds on the rate of entropy production, in the form of log-Sobolev inequalities [35] can be used to derive bounds on the time it takes to converge to equilibrium for particular instances of Davies maps. Hence, given that Theorem 2 is completely general, and holds also for Davies maps that do not reach thermal equilibrium efficiently, the fact that the lower bound vanishes for infinitesimal times is not surprising.

Recall that the factor of 2 in Eq. (10) is a consequence of the observation that the Petz recovery map is equal to the map itself. A natural question is then, is the factor 2 fundamental? The following conjecture suggests it is:

**Conjecture 3.** [Tightness of entropy production bound] The largest constant \( k \geq 0 \) such that

\[
F_\beta(\rho_S(0)) - F_\beta(\rho_S(t)) \geq \frac{1}{\beta} D(\rho_S(0)\|\rho_S(kt))
\]  

(11)

holds for all Davies maps is \( k = 2 \).

Violations of Eq. (11) have been found numerically for \( k = 2.0001 \), so the conjecture is highly likely to be true. See Fig. 2 for more details. This means that Eq. (2) is likely to be the strongest constraint of its kind that Davies maps obey, and it hence sets a somewhat strong relation between how much the free energy and the systems state at a later time change during a thermalization process.

**IV. BEYOND DAVIES MAPS**

So far we have addressed the issue of universal bounds for Davies maps which constrain the speed of convergence to the thermal fixed point. We now turn our attention to what recent developments from quantum information theory can say about convergence of dynamical semigroups in general. A recent advancement in quantum information is the development of universal recoverability maps [9, 10, 12]. By universal recoverability, it is meant that given a state \( \sigma \) and a map \( \Gamma \), one can use the recover map to lower bound the relative entropy difference \( D(\rho(\sigma)) - D(\Gamma(\rho))\|\Gamma(\sigma)) \) for all quantum states \( \rho \). In general the lower bound takes on a complicated form (see Appendix B). However, for the case of dynamical semi-groups satisfying QDB and the following property, the bound is more explicit.

Let us assume that we have a one-parameter dynamical semi-group \( M_t(\cdot) \) equipped with a fixed point \( \Omega \) that satisfies a condition we call Time-translation symmetry w.r.t. fixed point (TTSFP),

\[
\mathcal{L}(\cdot) = \Omega^t \mathcal{L} (\Omega^{-t}(\cdot)) \Omega^{-t^t} \forall t \in \mathbb{R}
\]  

(12)

This condition is satisfied for example by dynamical semi-groups which arise naturally in the weak coupling limit or the low-density limit. Davies maps are one such example, but there are others [37].

The properties lead to the following result:

**Theorem 4.** Let the Quantum Dynamical Semigroup \( M_t(\cdot) \) satisfy QDB and TTSFP. Then the following holds

\[
D(\rho(0)\|\Omega) - D(\rho(t)\|\Omega) \geq -2\log F(\rho, M_t(\rho(t)) \}
\]  

(13)

where \( F(\rho, \sigma) = \text{Tr}[\sqrt(\sqrt{\rho} \sigma \sqrt{\rho})] \) is the quantum fidelity. Moreover, if the generators are time-independent we may write \( M_t(\rho(t)) = \rho(t) \).

It is well known that \( D(\rho\|\sigma) \geq -2\log F(\rho\|\sigma) \) with equality only for special instances. Therefore, for Davies maps, Eq. (10) is satisfied but with a weaker bound than Theorem 2.

**V. CONCLUSION**

We have proven a lower bound on the amount of entropy produced for a family of thermalization processes described by quantum dynamical semigroups. In order to do this, we have shown that an inequality which is not true for general CPTP maps holds for these maps. From that perspective, it would be interesting to investigate what are the features of these and other particular maps that make them obey this inequality, and
whether such bounds have more interesting physical consequences. One potential application of this in open quantum systems is to use a tightened monotonicity inequality to find when information backflow occurs in non-Markovian dynamics [35].

One of the important questions regarding Davies maps is how fast they converge to equilibrium. There have been several approaches to this question, mostly inspired by their classical analogues, which include the computation of the spectral gaps [29] or the logarithmic-Sobolev inequalities [30] [31]. In particular we note that the latter take the form of lower bounds to the rate of entropy production of particular Davies maps, from which thermalization times can be derived. It would be interesting to know if the bound of Eq. (10), albeit true for all Davies maps no matter their thermalization times, may contain information about their asymptotic convergence. For instance, one could look at how fast is the inequality saturated in particular cases. We however leave this for future work.

One of the main features in the study of dynamical thermalisation processes, such as Davies maps, is quantum detailed balance; and we have shown that this condition can be written in terms of the Petz recovery map that appears in information theory. It is not completely clear what the physical origin of this coincidence is, and begs for a deeper understanding. The condition of detailed balance is ubiquitous in thermalization processes, and in particular, current algorithms for simulating thermal states on a quantum computer, such as the quantum Metropolis algorithm, [40] obey it, which makes it all the more interesting. As such, the useful connection we establish here between the Petz recovery map and quantum detailed balance, is likely to have further implications for both thermodynamics and information theory.

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Appendix A: Technical results

1. Davies maps and the conditions for Lemma 1

Davies maps are derived from considering the dynamics of a state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$, where $\mathcal{H}_S$ is of finite dimension $d_S$, in contact with a thermal bath on an infinite dimensional Hilbert space $\mathcal{H}_B$. We will here specify the minimal assumptions about the bath and its interaction with the system necessary for the derivation of Lemma 5 and Lemma 4. In order to guarantee other properties, such as the existence of a fixed point or detailed balance, more subtle constraints are also necessary.

Let $\hat{H}_B$ be a self-adjoint Hamiltonian on $\mathcal{H}_B$. Since we want states on $\hat{H}_B$ to be thermodynamically stable, we assume that $Z_B = \text{Tr}[\exp(-\beta \hat{H}_B)] < \infty$ for all $\beta > 0$. $\hat{H}_B$ must therefore have a purely discrete spectrum, which is bounded below and has no finite limit points; that is, there are only a finite number of energy levels in any finite interval $\Delta E$. The quantum state $\rho_S \in \mathcal{S}(\mathcal{H}_S)$ with its free self-adjoint Hamiltonian $\hat{H}_S$ of finite dimension interacts with the system via a bounded interaction term $\hat{I} \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_B)$, with a parameter $\lambda > 0$ determining the interaction strength as follows

$$\hat{H}_{SB} = \hat{H}_S \otimes \mathbb{1}_B + \mathbb{1}_S \otimes \hat{H}_B + \lambda \hat{I}.$$  

The initial state on $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_B)$ is assumed to be product, $\rho_S \otimes \tau_B$, with $\tau_B$ the Gibbs state at inverse temperature $\beta$. The dynamics of the system at time $\hat{t}$ is given by the unitary operator

$$U(\hat{t}) := e^{-i\hat{H}_{SB}}$$

after tracing out the environment. More precisely, by

$$\text{Tr}_B [U(\hat{t}) \rho_S \otimes \tau_B U^\dagger(\hat{t})] \in \mathcal{S}(\mathcal{H}_S),$$

where $U^\dagger$ denotes the adjoint of $U$.

The Davies map $T_1(\cdot)$ is defined by taking the limit that the interaction strength $\lambda$ goes to zero, while the time $\hat{t}$ goes to infinity while maintaining $\hat{t}\lambda^2 := t$ fixed. More concisely,

$$T_1(\cdot) = \lim_{\lambda \to 0^+} \text{Tr}_B [U(\hat{t}) \rho_S \otimes \tau_B U^\dagger(\hat{t})] \in \mathcal{S}(\mathcal{H}_S) \quad \text{subject to } \hat{t}\lambda^2 = t \text{ fixed}. \quad (A4)$$

It is assumed that in this limit $U(\hat{t})$ and its inverse $U^\dagger(\hat{t})$ are still unitary operators mapping states on $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_B)$ to states on $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_B)$. To gain more physical insight into this construction, we refer to [2, 28, 41]. We remind the reader that the conditions described in Section A.1 are not sufficient for the map $T_1(\cdot)$ to satisfy other properties, such as the convergence to a fixed point or detailed balance, more subtle constraints are also necessary. We will not go into the details of these additional conditions, since only sufficient (but perhaps not necessary) conditions are known, e.g. [2]. In other Sections, we will additionally take advantage of the known fact that Davies maps satisfy quantum detailed balance.

2. Main theorem

In order to prove the main theorem we need a lemma about Davies maps first. We show that in the weak coupling limit, correlations between the system and the environment (the bath) are not created if both start as initially uncorrelated thermal states. In order to do this, we will need to introduce a finite dimensional cut-off on $\mathcal{H}_B$ and prove the results for the truncated space, and finally proving uniform convergence in the bath system size by removing the cut-off by taking the infinite dimensional limit. Let $\hat{P}_n$ denote the projection onto a finite dimensional Hilbert space $\mathcal{H}_{B,n} \subset \mathcal{H}_B$. Furthermore, assume that $\mathcal{H}_{B,1} \subset \mathcal{H}_{B,2} \subset \mathcal{H}_{B,3} \ldots \text{ and that } \lim_{n \to \infty} \mathcal{H}_{B,n} = \mathcal{H}_B$. For concreteness (although not strictly necessary), one could let $\hat{P}_n = \sum_{k=0}^n |E_k\rangle\langle E_k|$ where $|E_0\rangle, |E_1\rangle, |E_2\rangle, \ldots$ are the eigenvectors of $\hat{H}_B$ ordered in increasing eigenvalue order.

We define the truncated self-adjoint Hamiltonians on $\mathcal{H}_B$ as $\hat{H}_B^{(n)} = \hat{P}_n \hat{H}_B \hat{P}_n$ with a corresponding Gibbs state denoted by $\tau_{B,n} \in \mathcal{S}(\mathcal{H}_{B,n})$. Similarly, we construct unitaries on $\mathcal{H}_{B,n}$ by

$$U_n = \exp\left(-i\Delta \hat{H}_{SB}^{(n)}\right), \quad \hat{H}_{SB}^{(n)} = \mathbb{1}_S \otimes \hat{P}_n \hat{H}_{SB} \mathbb{1}_S \otimes \hat{P}_n. \quad (A5)$$
We recall the definition of the thermal state of the system $\tau_S \in \mathcal{S}(\mathcal{H}_S)$, which is given by

$$\tau_S = \frac{e^{\beta H_S}}{Z_S}, \quad Z_S > 0 \quad (A6)$$

for some inverse temperature $\beta > 0$

The lemma is the following:

**Lemma 5** (Correlations at the fixed point). Let $\alpha > 0$, $\Delta \in \mathbb{R}$ and the constant $\tilde{Z}_{SB}^{n,\alpha} = \text{Tr}[(\tau_S \otimes \tau_{B,n})^{\alpha}]$. Then, for all $n \in \mathbb{N}^+$, we have the bound

$$\frac{1}{2}\lVert U_n(\tau_S \otimes \tau_{B,n})^{\alpha}U_n^\dagger - (\tau_S \otimes \tau_{B,n})^{\alpha} \rVert_1 \leq \tilde{Z}_{SB}^{n,\alpha}\sqrt{\lambda\lVert I_n \rVert}, \quad (A7)$$

where $\lVert \cdot \rVert_1$, $\lVert \cdot \rVert$ is the one-norm and operator norm respectively.

**Proof.** The result is a consequence of mean energy conservation under the unitary transformation $U_n$ and Pinsker’s inequality.

Define the shorthand notation $\tilde{Z}_{SB}^{n,\alpha} = U_n(\tau_S \otimes \tau_{B,n})^{\alpha}U_n^\dagger$, $\tilde{Z}_{SB}^{n,\alpha} \in \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_{B,n})$ and $\tilde{Z}_{SB}^{n,\alpha} := \tilde{Z}_{SB}^{\alpha}, \tilde{Z}_S := \text{Tr}[\tau_S^{\alpha}]$, $\tilde{Z}_{SB}^{n,\alpha} := \text{Tr}[\tau_{B,n}^{\alpha}]$. By direct evaluation of the relative entropy,

$$D(\tilde{Z}_{SB}^{n,\alpha} \| (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}) / \beta = \text{Tr}[\tilde{H}_S \tilde{Z}_{SB}^{n,\alpha}] + \text{Tr}[\tilde{H}_B(n) \tilde{Z}_{SB}^{n,\alpha}] - (\alpha\beta)^{-1} S(\tau_S^{\alpha} \otimes \tau_{B,n}^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}) + \ln(\tilde{Z}_{SB}^{n,\alpha}), \quad (A8)$$

where we have used unitary in-variance of the von Neumann entropy $S(\cdot)$. Thus since

$$0 = D((\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha} \| (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}) / \beta \quad (A9)$$

we conclude

$$D(\tilde{Z}_{SB}^{n,\alpha} \| (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}) / \beta = \text{Tr}[\tilde{H}_S \tilde{Z}_{SB}^{n,\alpha}] + \text{Tr}[\tilde{H}_B(n) \tilde{Z}_{SB}^{n,\alpha}] - (\alpha\beta)^{-1} S(\tau_S^{\alpha} \otimes \tau_{B,n}^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}) + \ln(\tilde{Z}_{SB}^{n,\alpha}). \quad (A10)$$

Energy conservation implies

$$\text{Tr}[\tilde{H}_B(n) (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}] = \text{Tr}[\tilde{H}_B(n) \tilde{Z}_{SB}^{n,\alpha}], \quad (A12)$$

Combining Eqs. (A12), (A11) we achieve

$$D(\tilde{Z}_{SB}^{n,\alpha} \| (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha}) = \text{Tr}[\lambda \tilde{I}_n (\tilde{Z}_{SB}^{n,\alpha} - (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha})] \beta. \quad (A13)$$

Pinsker inequality states that for any two density matrices $\rho$, $\sigma$,

$$D(\rho \| \sigma) \geq \frac{1}{2} \lVert \rho - \sigma \rVert_1^2. \quad (A14)$$

It follows from it, and from Eq. (A13),

$$\lVert U_n(\tau_S \otimes \tau_{B,n})^{\alpha}U_n^\dagger - (\tau_S \otimes \tau_{B,n})^{\alpha} \rVert_1 \leq \tilde{Z}_{SB}^{n,\alpha}\sqrt{2\text{Tr}[\lambda \tilde{I}_n (\tilde{Z}_{SB}^{n,\alpha} - (\tau_S \otimes \tau_{B,n})^{\alpha} / \tilde{Z}_{SB}^{n,\alpha})]} \quad (A15)$$

$$\leq 2\tilde{Z}_{SB}^{n,\alpha}\sqrt{\sup_{\rho \in \mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_{B,n})} \lVert \text{Tr}[\tilde{I}_n \rho] \rVert \lambda} \quad (A16)$$

$$\leq 2\tilde{Z}_{SB}^{n,\alpha}\sqrt{\lambda\lVert I_n \rVert} \quad (A17)$$

This lemma may be of independent interest, as it makes explicit the idea mentioned in previous work such as [12, 13] of how Davies maps, in the weak coupling limit, can be taken as free operations in the resource theory of athermality [14].

With it at hand, we can prove the central lemma.
Lemma 6 (Lemma \[\text{A1}\] of main text). Assume conditions in Section \[\text{A1}\] hold. Then all maps \(T_t(\cdot)\) satisfy the inequality

\[
D((\cdot)) \tau_S - D(T_t(\cdot)) \tau_S \geq D \left( (\cdot) \left| \hat{T}_t(\cdot) \right| \right), \quad \forall t \geq 0
\]  

(A18)

where \(\hat{T}_t(\cdot)\) is the Petz recovery map corresponding to \(T_t(\cdot)\), (see Eq. (A48)).

Proof. The proof will involve simple manipulations of the relative entropy, and the data processing inequality. We will perform the calculations for the map \(\text{Tr}_B \left[ e^{-iH_B(s)} \otimes \rho_B e^{iH_B(s)} \right]\) rather than \(T_t(\cdot)\) itself, keeping only leading order in \(\lambda\) terms. We will finally take the limit described in Eq. (A4) to conclude the proof.

Noting that the relative entropy between two copies is zero, followed by using its additivity and unitarity invariance properties, we find for \(\rho_S \in \mathcal{S}(\mathcal{H}_S)\),

\[
D(\rho_S \| \tau_S) = D(\rho_S \otimes \tau_{B,n} \| \tau_S \otimes \tau_{B,n}) = D(U_n \rho_S \otimes \tau_{B,n} U_n^\dagger \| U_n \tau_S \otimes \tau_{B,n} U_n^\dagger) \tag{A19}
\]

\[
= D(U_n \rho_S \otimes \tau_{B,n} U_n^\dagger \| \tau_S \otimes \tau_{B,n} + \sqrt{\lambda} \tilde{B}_n(\lambda)) \tag{A20}
\]

where \(\tilde{B}_n(\lambda) := (U_n \tau_S \otimes \tau_{B,n} U_n^\dagger - \tau_S \otimes \tau_{B,n}) / \sqrt{\lambda}\).

With the identity \(D(\gamma_{CD} \otimes \zeta_{CD}) - D(\gamma_{CD} \otimes \zeta_D) = D(\gamma_{CD} \| \exp(\ln \gamma_{CD} + \ln \mathbb{1}_C \otimes \zeta_D - \ln \mathbb{1}_C \otimes \zeta_{CD}))\) for bipartite states \(\gamma_{CD}, \zeta_{CD}\), we have that

\[
D(U_n \rho_S \otimes \tau_{B,n} U_n^\dagger \| \tau_S \otimes \tau_{B,n} + \sqrt{\lambda} \tilde{B}_n(\lambda)) \leq D \left( U_n \rho_S \otimes \tau_{B,n} U_n^\dagger \| \exp \left( \ln(\tau_S \otimes \tau_{B,n} + \sqrt{\lambda} \tilde{B}_n(\lambda)) + \ln \sigma_S \otimes \mathbb{1}_B - \ln(\tau_S + \sqrt{\lambda} \text{Tr}_B[\tilde{B}_n(\lambda)] \otimes \mathbb{1}_B) \right) \right) \tag{A21}
\]

\[
\geq D \left( \rho_S \| \text{Tr}_B \left[ U_n^\dagger \exp \left( \ln(\tau_S \otimes \tau_{B,n} + \sqrt{\lambda} \tilde{B}_n(\lambda)) + \ln \sigma_S \otimes \mathbb{1}_B - \ln(\tau_S + \sqrt{\lambda} \text{Tr}_B[\tilde{B}_n(\lambda)] \otimes \mathbb{1}_B) \right) \right] \right) \right) \tag{A22}
\]

where \(\sigma_{S,n} := \text{Tr}_B[U_n \rho_S \otimes \tau_{B,n} U_n^\dagger]\) and in the last line we have used the unitarity invariance of the relative entropy followed by the data processing inequality. Plugging Eq. (A20) into Eq. (A23) followed by taking the \(n \to \infty\) limit, we obtain

\[
D(\rho_S \| \tau_S) - D(\sigma_S \| \tau_S + \sqrt{\lambda} \text{Tr}_B[\tilde{B}(\lambda)]) \tag{A24}
\]

\[
\geq D \left( \rho_S \| \text{Tr}_B \left[ U_n^\dagger \exp \left( \ln(\tau_S \otimes \tau_B + \sqrt{\lambda} \tilde{B}(\lambda)) + \ln \sigma_S \otimes \mathbb{1}_B - \ln(\tau_S + \sqrt{\lambda} \text{Tr}_B[\tilde{B}(\lambda)] \otimes \mathbb{1}_B) \right) \right] \right), \tag{A25}
\]

where we have defined \(\tilde{B}(\lambda) := \lim_{n \to \infty} \tilde{B}_n(\lambda), \sigma_S := \lim_{n \to \infty} \sigma_{S,n}\). Before continuing, we will first note the validity of Eq. (A25). We start by showing that \(\tilde{B}(\lambda)\) is trace class for \(\lambda \in [0,1]\). From Lemma \ref{lem:trace-class}, it follows

\[
\| \tilde{B}_n(\lambda) \|_1 \leq 2 \tilde{Z}_{SB}^{n,1} \beta \sqrt{\| \hat{I}_n \|}, \tag{A26}
\]

for all \(\lambda \in [0,1]\) with the r.h.s. \(\lambda\) independent. By definition of \(\tilde{Z}_{SB}^{n,\alpha}\), it follows that it is the partition function of a tensor product of thermal states on \(\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_{B,n})\) at inverse temperatures \(\alpha \beta_S, \alpha\beta\). Since the Hamiltonians \(\hat{H}_{B,1}, \hat{H}_{B,2}, \hat{H}_{B,3}, \ldots, \hat{H}_B\) by definition have well defined thermal states (finite partition functions) for all positive temperatures, it follows that \(\lim_{n \to \infty} \tilde{Z}_{SB}^{n,\alpha} < \infty\) for all \(\alpha > 0\). Thus noting that by definition, \(\lim_{n \to \infty} \| \hat{I}_n \| = \| \hat{I} \|\) and that \(\hat{I}\) is a bounded operator, it follows that

\[
\| \tilde{B}(\lambda) \|_1 = \lim_{n \to \infty} \| \tilde{B}_n(\lambda) \|_1 = 2 \tilde{Z}_{SB}^{-1} \beta \sqrt{\| \hat{I} \|} < \infty, \tag{A27}
\]

Thus since \(\tau_S + \sqrt{\lambda} \text{Tr}_B[\tilde{B}(\lambda)]\) is finite dimensional and hermitian, and the eigenvalues of finite dimensional hermitian matrices are continuous in their entries \cite{85,86}, it follows, since \(\tau_S\) has full support, that there exists \(0 < \lambda^* \leq 1\) such that for all \(\lambda \in [0,\lambda^*]\), \(\tau_S + \sqrt{\lambda} \text{Tr}_B[\tilde{B}(\lambda)]\) has full support. Thus for all \(\lambda \in [0,\lambda^*]\), the r.h.s. of Eq. (A25) is upper bounded by a finite quantity uniformly in \(n \to \infty\) and thus since relative entropies are non-negative by definition, Eq. (A25) is well defined for all \(\lambda \in [0,\lambda^*]\).
We now set \( \Delta \) appearing in \( U \) to \( \Delta = t/\lambda^2 \) followed by taking the limit \( \lambda \to 0^+ \) while keeping \( t \) fixed in Eq. (A25) thus achieving
\[
D(\rho_S\|\tau_S) - D(T_i(\rho_S)\|\tau_S) \geq D\left(\rho_S\bigg\| \text{Tr}_B[U^\dagger T_i(\rho_S) \otimes \tau_B U]\right),
\]
where we have used that by definition, \( T_i(\cdot) = \lim_{\lambda \to 0^+} \text{Tr}_B[U(\cdot) \otimes \tau_B U^\dagger] \).

We now proceed to calculate the Petz’s recovery map for the map \( T_i(\cdot) \). The adjoint map is \( \text{Tr}_B[\tau_B^1U^1(\cdot) \otimes 1_B]U_B^{1/2} \). Hence from the definition in Eq. (A48) it follows that the Petz recovery map for \( T_i(\cdot) \) is
\[
\tilde{T}_i(\cdot) := \tau_S^{1/2}\text{Tr}_B\left[\tau_B^{1/2}U^1\left(\tau_S^{-1/2}(\cdot)\tau_S^{-1/2} \otimes 1_B\right)U_B^{1/2}\right]\tau_S^{1/2}.
\]

Similarly to before, we define a traceless, self adjoint operator \( \tilde{B} = \tilde{B}(\lambda) := \left(U\tau_S^{1/2} \otimes \tau_B^{1/2}U^\dagger - \tau_S^{1/2} \otimes \tau_B^{1/2}\right)/\sqrt{\lambda} \). In analogy with the reasoning which led to Eq. (A27), it follows from Lemma \( 5 \) that \( \|\tilde{B}(\lambda)\|_1 = \lim_{n \to \infty} \|\tilde{B}_n(\lambda)\|_1 = 2Z_{SB}^{1/2}\beta\sqrt{\|I\|} < \infty \), for all \( \lambda \in [0, 1] \). For general \( U = \exp(-i\Delta \tilde{H}_{SB}) \), we can now write
\[
\tau_S^{1/2}\text{Tr}_B\left[\tau_B^{1/2}U^\dagger\left(\tau_S^{-1/2}(\cdot)\tau_S^{-1/2} \otimes 1_B\right)U\tau_B^{1/2}\right]^{1/2}
= \text{Tr}_B\left[U\tau_S^{1/2} \otimes \tau_B^{1/2} + \sqrt{\lambda}U^\dagger \tilde{B}\right]^{1/2}\left(\tau_S^{-1/2}(\cdot)\tau_S^{-1/2} \otimes 1_B\right)^{1/2}\left(\tau_S^{1/2} \otimes \tau_B^{1/2}U + \sqrt{\lambda}\tilde{B}U\right)
= \text{Tr}_B[U^\dagger((\cdot) \otimes \tau_B)U] + \sqrt{\lambda}\tilde{g}_1(\cdot) + \lambda\tilde{g}_2(\cdot) \in \mathcal{S}(\mathcal{H}_B),
\]
where
\[
\tilde{g}_1(\cdot) = \text{Tr}_B\left[U^\dagger(\tilde{B}(\cdot) \otimes \tau_B^{1/2})U\right] + \text{Tr}_B\left[U^\dagger((\cdot)\tau_S^{-1/2} \otimes \tau_B^{1/2}B)U\right]
\]
\[
\tilde{g}_2(\cdot) = \text{Tr}_B\left[U^\dagger B\left(\tau_S^{-1/2}(\cdot)\tau_S^{-1/2} \otimes 1_B\right)\tilde{B}U\right],
\]
which are well defined since they are comprised of products of bounded operators. Similarly to before, in Eq. (A32) we now set \( \Delta \) appearing in \( U \) to \( \Delta = t/\lambda^2 \) followed by taking the limit \( \lambda \to 0^+ \) while keeping \( t \) fixed achieving
\[
\tilde{T}_i(\cdot) = \text{Tr}_B[U^\dagger((\cdot) \otimes \tau_B)U]
\]
where we have used Eq. (A29). Hence substituting Eq. (A33) in to Eq. (A28) and noting the equations holds for all states \( \rho_S \in \mathcal{S}(\mathcal{H}_B) \), we conclude the proof. \( \square \)

**Remark 7.** In the above proof, we have taken two independent limits, namely 1st the infinite bath volume limit \( n \to \infty \) followed by the Van Hove limit \( \lambda \to 0^+ \) while keeping \( t \) fixed. This is the order in which Davies performed the limits. From physical reasoning, one would expect the Davies map to be equally valid if the order of the limits is reversed. We note that the proof of Theorem 2 follows also if the order of the these two limits is reversed but now with the new definitions
\[
T_i(\cdot) = \lim_{n \to \infty} \lim_{\lambda \to 0^+} \text{Tr}_{B,n}\left[U_n(\hat{t})(\cdot) \otimes \tau_{B,n} U_n^\dagger(\hat{t})\right] \in \mathcal{S}(\mathcal{H}_S) \quad \text{subject to } \tilde{\lambda}^2 = t \text{ fixed}.
\]
\[
\tilde{T}_i(\cdot) = \lim_{n \to \infty} \lim_{\lambda \to 0^+} \tau_S^{1/2}\text{Tr}_{B,n}\left[\tau_B^{1/2}U_n(\hat{t})\left(\tau_S^{-1/2}(\cdot)\tau_S^{-1/2} \otimes 1_B\right)U_n^\dagger(\hat{t})\tau_{B,n}^{1/2}\right]^{1/2}. \in \mathcal{S}(\mathcal{H}_S) \quad \text{subject to } \tilde{\lambda}^2 = t \text{ fixed}.
\]

An interesting technical question is whether the above limits commute i.e. whether Eqs. (A36), (A37) are identical to Eqs. (A4), (A29).

3. Quantum detailed balance and Petz recovery map

Now we show that all Davies maps have the peculiar property that they are the same as their Petz recovery map. This is because of a crucial property satisfied by their generators: quantum detailed balance. For Theorem 6 in the main text to hold, we require both the conditions of Section A1 and the following Lemma to hold. For the sake of generality, we show the that the results is true for any fixed point \( \Omega \) with full support.
Theorem 8. A quantum dynamical semigroup \( M_t(\cdot) \) with Lindbladian \( \mathcal{L} \) satisfying quantum detailed balance (Eq. (3)) for the state \( \Omega \) with full rank, is equal to its corresponding Petz recovery map, namely,

\[ M_t(\cdot) = \tilde{M}_t(\cdot), \]

where

\[ \tilde{M}_t(\cdot) = \Omega^{1/2} M_t^T (M_t(\Omega)^{-1/2} \cdot M_t(\Omega)^{-1/2}) \Omega^{1/2}. \]

Proof. The property of quantum detailed balance (also sometimes referred to as the reversibility, or KMS condition) reads

\[ \langle A, \mathcal{L}^\dagger (B) \rangle_\Omega = \langle \mathcal{L}^\dagger (A), B \rangle_\Omega \]

for all \( A, B \in \mathbb{C}^{d_{\mathcal{S}} \times d_{\mathcal{S}}} \), where \( \mathcal{L}^\dagger \) is the adjoint Lindbladian, and we define the scalar product

\[ \langle A, B \rangle_\Omega := \text{Tr}[\Omega^{1/2} A^\dagger \Omega^{1/2} B]. \]

Because Eq. (A40) holds for all \( A, B \in \mathbb{C}^{d_{\mathcal{S}} \times d_{\mathcal{S}}} \), Eq. (A40) implies that

\[ \mathcal{L}(\cdot) = \Omega^{1/2} \mathcal{L}^\dagger (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{1/2}. \]

Eq. (A40) automatically implies that any power of the generator also obeys the same relation, that is, \( \forall n \in \mathbb{N}^+ \)

\[ \langle A, \mathcal{L}^{\dagger n} (B) \rangle_\Omega = \langle A, \Omega^{-1/2} \mathcal{L} (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{-1/2} \mathcal{L} (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{-1/2} \rangle_\Omega \]

(\(A43\))

\[ = \langle A, \Omega^{-1/2} \mathcal{L}^n (\Omega^{1/2} B \Omega^{1/2}) \mathcal{L} (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{1/2} \rangle_\Omega \]

(\(A44\))

\[ = \langle \mathcal{L}^\dagger n (A), B \rangle_\Omega, \]

(\(A45\))

where in the first line we use Eq. (A42) \( n \) times and the 2nd line follows from the definition of the adjoint map. Hence we can also write

\[ \mathcal{L}^n(\cdot) = \Omega^{1/2} \mathcal{L}^\dagger n (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{1/2}. \]

(A46)

The semigroup can be written as \( M_t(\cdot) = e^{\mathcal{L}t} \cdot \). Its adjoint semigroup is given by \( e^{\mathcal{L}^\dagger t} \cdot \) and hence the Petz recovery map is (see Eq. (A48))

\[ \tilde{M}_t(\cdot) = \Omega^{1/2} e^{\mathcal{L}^\dagger t} (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{1/2}. \]

(A47)

Since, \( \tilde{M}_t(\cdot) = \Omega^{1/2} (\sum_{n=0}^{\infty} (t \mathcal{L})^n (\Omega^{-1/2} \cdot \Omega^{-1/2})/(n!) \cdot \Omega^{1/2} \), Eq. (A47) together with Eq. (A46), means that \( \tilde{M}_t(\cdot) = M_t(\cdot) \).

We note that the Petz recovery map is defined in terms of a map \( \Gamma(\cdot) \) and a state \( \sigma_S \) as the unique solution to

\[ \langle A, \Gamma^\dagger (B) \rangle_{\sigma_S} = \langle \tilde{T}(A), B \rangle_{\Gamma(\sigma_S)} \]

(A48)

for all \( A, B \in \mathbb{C}^{d_{\mathcal{S}} \times d_{\mathcal{S}}} \) and the scalar product is given by Eq. (A41). The solution takes the form \( \tilde{T}(\cdot) = \sigma_S^{1/2} T^\dagger (T(\sigma_S)^{-1/2} \cdot T(\sigma_S)^{-1/2}) \sigma_S^{1/2} \)

(A49)

such that we always have that \( \tilde{T}(T(\sigma_S)) = \sigma_S \). Here this simplifies by choosing \( \sigma_S = \Omega \) a fixed point of \( M_t(\cdot) \).

When the generator is time-independent, we also have that the combination of a map for a finite time and its recovery map is equivalent to applying the map for a time \( 2t \). That is \( T_t(T_t(\cdot)) = T_{2t}(\cdot) \). This means we can write Eq. (A18) in a particularly simple form.

Theorem 9. (Theorem 2 of main text) Assume conditions in Section A1 hold and \( T_t(\cdot) \) satisfies quantum detailed balance (Eq. (3)). Then \( T_t(\cdot) \) satisfies the inequality

\[ D(\cdot||\tau_S) - D(T_t(\cdot)||\tau_S) \geq D (\cdot||T_{2t}(\cdot)), \quad t \geq 0. \]

(A50)

Proof. Direct consequence of Theorems 8 and 6.
4. Spohn’s inequality: rate of entropy production

We give an alternative proof of a well-known result which was first shown in [3] that gives the expression for the infinitesimal rate of entropy production of a Davies map. This is stated without a proof in many standard references such as [25, 45]. Then we show in a similar way how in the infinitesimal time limit our lower bound becomes trivial.

First we need the following lemma, which proof can be found in, for instance, [49].

**Lemma 10.** Let $\mathbb{1} \in \mathbb{C}^{n \times n}$ be the identity matrix, and $A, B, C \in \mathbb{C}^{n \times n}$ be matrices such that both $A$ and $A + tB$ are positive with $t \in \mathbb{R}$, we have that

$$\log (A + tB) - \log A = t \int_0^1 \frac{1}{(1 - x)A + xB} \, \frac{1}{(1 - x)A + x\mathbb{1}} \, dx + O(t^2) \quad \text{(A51)}$$

With this, we can show the following:

**Theorem 11.** Let $\mathcal{L}(\rho_S(t))$ be the generator of a dynamical semigroup, with a fixed point $\tau_S$ such that $\mathcal{L}(\tau_S) = 0$. We have that the entropy production rate $\sigma(\rho_S(t))$ is given by

$$\sigma(\rho_S(t)) := -\frac{dD(\rho_S(t)||\tau_S)}{dt} = \text{Tr}[\mathcal{L}(\rho_S(t))(\log \tau_S - \log \rho_S(t))] + \text{Tr}[\mathcal{L}(\rho_S(t))\Pi_{\rho_S(t)}] \geq 0, \quad \text{(A52)}$$

where $\Pi_{\rho_S(t)}$ is the projector onto the support of $\rho_S(t)$. The second term of the sum vanishes at all times for which the rate is finite.

**Proof.** The last inequality (positivity) follows from the data processing inequality for the relative entropy, so we only need to prove the equality. The proof only requires Lemma 10 and some algebraic manipulations. We have that

$$\frac{dD(\rho_S(t)||\tau_S)}{dt} = \lim_{h \to 0} \frac{D(\rho_S(t)||\tau_S) - D(\rho_S(t)||\tau_S)}{h} \quad \text{(A53)}$$

$$= \lim_{h \to 0} \frac{\text{Tr}[(\rho_S(t) + \mathcal{L}(\rho_S(t))h)\log \{\rho_S(t) + \mathcal{L}(\rho_S(t))h\} - \log \tau_S]}{h} - \text{Tr}[\rho_S(t)\log \rho_S(t) - \log \tau_S] \quad \text{(A54)}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[ \text{Tr}[\rho_S(t)\mathcal{L}(\rho_S(t))\log \rho_S(t)] + h \int_0^h \frac{1}{(1 - x)pS(t) + x\mathbb{1}} \mathcal{L}(\rho_S(t)) \frac{1}{(1 - x)pS(t) + x\mathbb{1}} \, dx - \log \tau_S \right]$$

$$- \text{Tr}[\rho_S(t)\log \rho_S(t) - \log \tau_S] \quad \text{(A55)}$$

$$= \text{Tr}[\mathcal{L}(\rho_S(t))\log \rho_S(t) - \log \tau_S] + \text{Tr}[\rho_S(t)\int_0^1 \frac{1}{(1 - x)pS(t) + x\mathbb{1}} \mathcal{L}(\rho_S(t)) \frac{1}{(1 - x)pS(t) + x\mathbb{1}} \, dx] \quad \text{(A56)}$$

$$= \int_0^1 \frac{1}{(1 - x)pS(t) + x\mathbb{1}} \, dx \quad \text{(A57)}$$

Where to go from the 2nd to the third line we used Lemma 10 and from the 4th to the 5th we use the ciclicity and linearity of the trace. Now note the following integral

$$\int_0^1 \frac{1}{(1 - x)p + x} \, dx = \frac{1}{p} \quad \forall p \neq 0. \quad \text{(A58)}$$

This means that, on the support of $\rho_S(t)$,

$$\int_0^1 \frac{1}{(1 - x)pS(t) + x\mathbb{1}} \, dx = \frac{1}{\rho_S(t)}. \quad \text{(A59)}$$

Note that outside the support of $\rho_S(t)$ this integral is not well defined. Given this, we can write

$$\frac{dD(\rho_S(t)||\tau_S)}{dt} = \text{Tr}[\mathcal{L}(\rho_S(t))(\log \rho_S(t) - \log \tau_S)] + \text{Tr}[\mathcal{L}(\rho_S(t))\Pi_{\rho_S(t)}]. \quad \text{(A60)}$$

where $\Pi_{\rho_S(t)}$ is the projector onto the support of $\rho_S(t)$. The Lindbladian is traceless $\text{Tr}[\mathcal{L}(\rho_S(t))] = 0$ and hence second term of this Equation vanishes as long as $\text{supp}(\mathcal{L}(\rho_S(t))) \subseteq \text{supp}(\rho_S(t))$, which we can expect for most times. At instants in time when this is not the case and this term may give a finite contribution (that is, when the rank increases), the first term in Eq. (A60) diverges logarithmically [3], and hence that finite contribution is negligible. \( \square \)
A similar reasoning can be used to show that the instantaneous lower bound on entropy production rate that we can get from our main result in Eq. (10) is trivial for most times. In particular, we can show

**Lemma 12.** The lower bound of Eq. (10) vanishes in the limit of infinitesimal time transformations. More precisely, we have that

\[
\lim_{h \to 0} \frac{D(\rho_S(t)||\rho_S(t+2h))}{h} = -2\text{Tr}[\mathcal{L}(\rho_S(t))\Pi_{\rho_S(t)}],
\]

where \(\Pi_{\rho_S(t)}\) is the projector onto the support of \(\rho_S(t)\). This vanishes as long as \(\text{supp}(\mathcal{L}(\rho_S(t))) \subseteq \text{supp}(\rho_S(t))\).

**Proof.** The proof is similar to the one for Theorem 11 above.

\[
\lim_{h \to 0} \frac{D(\rho_S(t)||\rho(t+2h))}{h} = \lim_{h \to 0} \frac{1}{h} \text{Tr}[\rho_S(t)(\log \rho_S(t) - \log (\rho_S(t) + 2h\mathcal{L}(\rho_S(t)))]
\]

\[
= \text{Tr}[2\rho_S(t) \int_0^1 \frac{1}{(1-x)\rho_S(t) + x\mathbb{1}} \mathcal{L}(\rho_S(t)) \, dx]
\]

\[
= -2\text{Tr}[\mathcal{L}(\rho_S(t))\Pi_{\rho_S(t)}],
\]

where in the second line we applied Lemma 10, and in the third we used Eq. (A59).

Hence for infinitesimal times, the lower bound gives the same condition as the positivity condition in Eq. (A52). It will be nonzero only when \(\text{supp}(\mathcal{L}(\rho_S(t))) \not\subseteq \text{supp}(\rho_S(t))\), in which case the rate of entropy production diverges (at points in time when the rank of the system increases).

**Appendix B: Maps Beyond Davies**

Given that the inequality in Eq. (7) is saturated in some limits, such as when the evolution approaches the fixed point, it is unlikely that a stronger inequality of a similar kind can be derived even in particular cases. However, general results are known for CPTP maps, leading to weaker forms of such bound. In this Section we state the best known general result from [10] and show how they simplify in particular cases of maps with properties similar to Davies maps. This means that we can also bound the entropy production of maps that may not be Davies maps.

The result, which proof involves techniques from complex interpolation theory, is the following:

**Theorem 13.** (Main result of [10]) Let \(\Gamma(\cdot)\) be a CPTP map, and \(\rho, \sigma\) any two quantum states. We have that

\[
D(\rho||\sigma) - D(\Gamma(\rho)||\Gamma(\sigma)) \geq -2 \int_{\mathbb{R}} dt \; p(t) \log F(\rho, \tilde{\Gamma}_t(\Gamma(\rho))),
\]

where \(F(\rho, \sigma) = \text{Tr}[\sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}}]\) is the quantum fidelity, the map \(\tilde{\Gamma}_t\) is the rotated recovery map

\[
\tilde{\Gamma}_t(\cdot) = \sigma^{it}\tilde{\Gamma}(\Gamma(\sigma)^{-it} \cdot \Gamma(\sigma)^{it}\sigma^{-it})
\]

and \(p(t)\) is the probability density function \(p(t) = \frac{\pi}{2} (\cosh(\pi t) + 1)^{-1}\).

**Proof.** See [10].

We now observe that the rotated map can be simplified given the following conditions:

- If the map has a fixed point \(\Gamma(\Omega) = \Omega\), the Petz recovery map simplifies to become

\[
\tilde{\Gamma}_t(\cdot) = \Omega^{it}\tilde{\Gamma}(\Omega^{-it} \cdot \Omega^{it}\Omega^{-it}) \quad \forall t \in \mathbb{R}
\]

This by itself implies that \(\tilde{\Gamma}_t(\Omega) = \Omega\).

- The map may also obey the property of *time-translation symmetry*, where this is given by

\[
\Gamma(\cdot) = \Omega^{it}\Gamma(\Omega^{-it} \cdot \Omega^{it}\Omega^{-it}).
\]

If a map obeys this symmetry, the adjoint map \(\Gamma^\dagger(\cdot)\) also will. This can be seen through the definition of the adjoint, which is that for any two matrices \(A, B\),

\[
\text{Tr}[A\Gamma(B)] = \text{Tr}[\Gamma^\dagger(A)B],
\]
and in particular, it holds for the matrices $A' = \Omega^H A \Omega^{-H}$, $B' = \Omega^{-H} B \Omega^H$. This, together with Eq. \eqref{eq:B4}, means that

$$\text{Tr}[\Gamma^\dagger (A) B] = \text{Tr}[\Omega^H \Gamma^\dagger (\Omega^{-H} \cdot \Omega^H) \Omega^{-H} (A) B]$$ \hspace{2cm} \text{(B6)}$$

Hence this property, together with the fixed-point property, means that the rotated recovery map becomes equal to the Petz map, and the integral in Eq. \eqref{eq:B1} gets averaged out.

It may be the case, however, that the symmetry exists, but that the fixed point is not the thermal state, and hence the simplification does not occur. This may be the case for instance when there is weak coupling to a non-thermal environment.

• If on top of these two the map has the property of detailed balance, the Petz recovery map and the original one are the same $\Gamma(\cdot) = \Gamma(\cdot)$. Examples of maps which satisfy detailed balance which are not Davies maps do exist. See \cite{37} for general characterization of quantum dynamical semigroups.

When all these hold we have that Eq. \eqref{eq:B1} becomes

$$D(\rho|\Omega) - D(\Gamma(\rho)||\Omega) \geq -2 \log F(\rho, \Gamma(\Gamma(\rho))).$$ \hspace{2cm} \text{(B7)}$$

If the map is in a dynamical semigroup with time-independent generator $\Gamma = M_t$, we may also write $M_t(M_t(\cdot)) = M_{2t}(\cdot)$.

Davies maps have all these properties. Further examples where all these appear are semigroups derived from the low-density limit (which models a system immersed in an ideal gas at low density, see \cite{25} for details), or the so-called heat bath generators \cite{50}.

We note however that $D(\rho||\sigma) \geq -2 \log F(\rho, \sigma)$, and hence Eq. \eqref{eq:B7} is a weaker bound than Eq. \eqref{eq:7}, and in particular is not tight as the fixed point is approached.

**Appendix C: Equivalence of definitions of quantum detailed balance**

In the literature, different nonequivalent definitions of the property of quantum detailed balance have been given. While in many places the one given is that of Eq. \eqref{eq:3}, an alternative definition, which can be found for instance in \cite{25, 27} is that the Lindbladian is self-adjoint with respect to the inner product

$$\langle A, L^\dagger (B) \rangle = \langle L^\dagger (A), B \rangle,$$ \hspace{2cm} \text{(C1)}$$

for all $A, B \in \mathbb{C}^{d_S \times d_S}$, where the inner product is defined as

$$\langle A, B \rangle = \text{Tr}[\Omega A^\dagger B].$$ \hspace{2cm} \text{(C2)}$$

Eq. \eqref{eq:C2} is different from that of Eq. \eqref{eq:A41} due to the noncommutativity of the operators. The solution to Eq. \eqref{eq:C1} is $L(\cdot) = \Omega L^\dagger (\Omega^{-1} \cdot)$, \hspace{2cm} \text{(C3)}$$

while the solution to Eq. \eqref{eq:3} is $L(\cdot) = \Omega^{1/2} L^\dagger (\Omega^{-1/2} \cdot \Omega^{-1/2}) \Omega^{1/2}$. \hspace{2cm} \text{(C4)}$$

We now give a simple proof of the fact that, under the condition that the map is time-translation invariant w.r.t. fixed point, the two conditions are the same.

**Theorem 14.** For a Lindbladian operator $L(\cdot)$ which obeys the property of time-translation symmetry w.r.t. fixed point $\Omega$ of full Rank (Eq. \eqref{eq:12}), the quantum detailed balance conditions of Eq. \eqref{eq:C3} and Eq. \eqref{eq:C4} are equivalent.

**Proof.** We rewrite both Eq. \eqref{eq:C3} and Eq. \eqref{eq:C4} in terms of their individual matrix elements in the orthonormal basis $\{ |i\rangle \}$ in which $\Omega = \sum_i p_i |i\rangle \langle i|$ is diagonal. Eq. \eqref{eq:C3} can be written in the form

$$\langle i | L(|n\rangle \langle m|) | j \rangle = \frac{p_n}{p_i} \langle i | L^\dagger (|n\rangle \langle m|) | j \rangle$$ \hspace{2cm} \text{(C5)}$$
and Eq. (C4) is

\[
(i|\mathcal{L}(|n\rangle\langle m|)|j) = \sqrt{\frac{p_i p_j}{p_n p_m}} (i|\mathcal{L}^\dagger(|n\rangle\langle m|)|j).
\]

We can see that for each matrix element the conditions only change by the factors multiplying in front, which are different unless \( \frac{p_n}{p_m} = \frac{p_i}{p_j} \).

Let us now introduce the following decomposition of operators in \( \mathbb{C}^{dS \times dS} \) in terms of their *modes of rotation*

\[
A = \sum_\omega A_\omega, \tag{C7}
\]

where \( A_\omega \) is defined as

\[
A_\omega = \sum_{k,l \text{ s.t. } \omega = \log \frac{p_k}{p_l}} |k\rangle \langle k| A |l\rangle \langle l|. \tag{C8}
\]

The name of *modes of rotation* is due to the fact that under the action of the unitary \( \Omega^{-it} \cdot \Omega^it \) they rotate with a different Bohr frequency, that is

\[
\Omega^{-it} A_\omega \Omega^it = A_\omega e^{-i\omega t}. \tag{C9}
\]

If the Lindbladian has the property of time-translational invariance w.r.t. the fixed point (Eq. (12)), it can be shown [52, 53] that each input mode is mapped to its corresponding output mode of the same Bohr frequency \( \omega \). We can write this fact as

\[
\mathcal{L}(A_\omega) = \mathcal{L}(A)_\omega. \tag{C10}
\]

This means that in Eq. (C5) and (C6), \( (i|\mathcal{L}(|n\rangle\langle m|)|j) = 0 \) unless the Bohr frequencies coincide at the input and the output, that is, when \( \log \frac{p_n}{p_m} = \log \frac{p_i}{p_j} \). That is, the two conditions are nontrivial only in those particular matrix elements in which both are equivalent.