ON THE BIERI-NEUMANN-STREBEL-RENZ Σ-INVARIENTS OF
THE BESTVINA-BRADY GROUPS

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Abstract. We study the Bieri-Neumann-Strebel-Renz invariants and we prove
the following criterion: for groups $H$ and $K$ of type $FP_n$ such that $[H, H] \subseteq K \subseteq H$ and a character $\chi : K \to \mathbb{R}$ with $\chi([H, H]) = 0$ we have $|\chi| \in \Sigma^n(K, \mathbb{Z})$ if and only if $|\mu| \in \Sigma^n(H, \mathbb{Z})$ for every character $\mu : H \to \mathbb{R}$ that extends $\chi$. The same holds for the homotopical invariants $\Sigma^n(-)$ when $K$ and $H$ are groups of type $FP_n$. We use these criteria to complete the description of the $\Sigma$-invariants of the Bieri-Stallings groups $G_m$ and more generally to describe the $\Sigma$-invariants of the Bestvina-Brady groups. We also show that the “only if” direction of such criterion holds if we assume only that $K$ is a subnormal subgroup of $H$, where both groups are of type $FP_n$. We apply this last result to wreath products.

1. Introduction

In [39] Wall defined a group $G$ to be of homotopical type $F^n$ if there is a classifying space $K(G, 1)$ with finite $n$-skeleton. The homotopical type $F_2$ coincides with finite presentability (in terms of generators and relations). A homological version of the homotopical type $F^n$ was defined by Bieri in [6]: a group $G$ is of homological type $FP_n$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution with all modules finitely generated in dimension $\leq n$. Though every group of type $F_n$ is of type $FP_n$ the converse does not hold for $n \geq 2$. There are groups constructed by Bestvina and Brady in [4] that are subgroups of right-angled Artin groups and are of type $FP_\infty$ but are not finitely presented.

The first $\Sigma$-invariant was defined by Bieri and Strebel in [13], where it was used to classify all finitely presented metabelian groups. In [11] Bieri, Neumann and Strebel defined the invariant $\Sigma^1(G)$ for any finitely generated group $G$ and for 3-manifold groups they linked $\Sigma^1(G)$ with the Thurston norm. Higher dimensional homological invariants $\Sigma^n(G, A)$ for a $\mathbb{Z}G$-module $A$ were defined by Bieri and Renz in [12], where they showed that $\Sigma^n(G, \mathbb{Z})$ controls which subgroups of $G$ that contain the commutator are of homological type $FP_n$. In [30] Renz defined the higher dimensional homotopical invariant $\Sigma^n(G)$ for groups $G$ of homotopical type $F_n$ and similar to the homological case $\Sigma^n(G)$ controls the homotopical finiteness properties of the subgroups of $G$ above the commutator. In all cases the $\Sigma$-invariants are open subsets of the character sphere $S(G)$. For a group $G$ of type $F_n$ we have $\Sigma^n(G) = \Sigma^n(G, \mathbb{Z}) \cap \Sigma^2(G)$. The description of the $\Sigma$-invariants of right-angled

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Artin groups by Meier, Meinert and Van Wyk shows that for \( n \geq 2 \) the inclusion \( \Sigma^n(G) \subseteq \Sigma^n(G, \mathbb{Z}) \) is not necessarily an equality [29].

In general it is difficult to explicitly calculate the \( \Sigma \)-invariants of a group \( G \) but for some special classes of groups and sometimes in particular small dimensions \( n \) the \( \Sigma^n(G) \) were studied: Thompson group \( F \) [10, 11], generalized Thompson groups \( F_{n,\infty} \) [11], metabelian groups [9], fundamental groups of compact \( K \)ähler manifolds [15], limit groups [20], free-by-cyclic groups [16, 17], permutational wreath products [29], right-angled Artin groups (RAAGs) [25], some Artin groups that are not RAAGs [1, 2].

For a group \( G \) we call a character a non-zero group homomorphism \( G \to \mathbb{R} \). The main result of this paper is the following theorem.

**Theorem 1.1.** Let \( [H, H] \subseteq K \subseteq H \) be groups such that \( H \) and \( K \) are of type \( FP_n \) (resp. \( F_n \)). Let \( \chi : K \to \mathbb{R} \) be a character such that \( \chi([H, H]) = 0 \). Then \( [\chi] \in \Sigma^n(K) \) (resp. \( [\chi] \in \Sigma^n(K) \)) if and only if \( [\mu] \in \Sigma^n(G, \mathbb{Z}) \) (resp. \( [\mu] \in \Sigma^n(G) \)) for every character \( \mu : H \to \mathbb{R} \) that extends \( \chi \).

Theorem 1.1 can be seen as a monoidal version of the Bieri-Renz result [12, Thm. B] and its homotopical counterpart [30, Satz C]. Furthermore our proof of Theorem 1.1 is very much inspired by the proofs of [12, Thm. B] and [30, Satz C]. As an application of Theorem 1.1 we calculate \( \Sigma^n(G, \mathbb{Z}) \) and \( \Sigma^n(G) \) for the Bestvina-Brady groups, in particular this applies to the Bieri-Stallings groups.

The Bieri-Stallings groups \( G_m \) were studied in [6, Section 2], where Bieri proved that \( G_m \) is of type \( FP_{m-1} \) but not \( FP_m \). The case \( m = 3 \) was previously considered by Stallings in [37] and it was the first known example of a group that is finitely presented but not of type \( FP_3 \). In [23] Kochloukova and Lima suggested the Monoidal Virtual Surjection Conjecture and they showed that it holds for \( G_m \) in dimension \( n \) provided \( n \leq m - 2 \) and \( m \geq 3 \), i.e. they calculated \( \Sigma^n(G_m, \mathbb{Z}) \) and \( \Sigma^n(G_m) \). We will discuss in details the Monoidal Virtual Surjection Conjecture in the preliminary section 3. As an application of Theorem 1.1 we complete in Corollary 1.2 the calculation of \( \Sigma^n(G_m, \mathbb{Z}) \) and \( \Sigma^n(G_m) \) for any \( n \).

**Corollary 1.2.** The Monoidal Virtual Surjection Conjecture holds for the Bieri-Stallings groups \( S = G_m \) where \( m \geq 2 \) and the classical embedding \( S = G_m \subseteq L_1 \times \ldots \times L_m \) with \( L_i \) free of rank 2 for \( 1 \leq i \leq m \).

The Bestvina-Brady groups \( BB_G \) are defined as the kernels of homomorphisms \( \varphi : A_G \to \mathbb{Z} \) that take the usual generators to 1 in \( \mathbb{Z} \), where \( A_G \) is the right-angled Artin group defined by the finite graph \( \Gamma \). This is exactly the class of groups studied in [3], where it is shown that \( BB_G \) can have various sets of exotic finiteness properties depending on the topological properties of the flag complex \( \Delta \) associated to \( \Gamma \), in particular \( BB_G \) is \( FP_n \) (resp. \( BB_G = F_n \)) if and only if \( \Delta \) is \((n - 1)\)-acyclic (resp. \( \Delta \) is \((n - 1)\)-connected). Theorem 1.1 implies that the \( \Sigma \)-invariants of \( BB_G \) are determined by those of \( A_G \) and this completes the description of the \( \Sigma \)-invariants of \( BB_G \) since the \( \Sigma \)-invariants of \( A_G \) are completely described in [25, 26]. Note that the Bieri-Stallings group \( G_m = BB_G \) and \( L_1 \times \ldots \times L_m = A_G \) for the graph \( \Gamma \) with vertices \( x_1, y_1, \ldots, x_m, y_m \), where the only vertices that are not connected by an edge in \( \Delta \) are \( x_i \) and \( y_i \) for \( 1 \leq i \leq m \). Here \( L_i \) is a free non-abelian group generated by \( x_i \) and \( y_i \).

**Corollary 1.3.** Let \( \Gamma \) be a finite connected graph such that \( \Delta \) is \((n - 1)\)-acyclic (resp. \( \Delta \) is \((n - 1)\)-connected). Then \( [\chi] \in \Sigma^n(BB_G, \mathbb{Z}) \) (resp. \( [\chi] \in \Sigma^n(BB_G) \)) if
and only if $[\mu] \in \Sigma^n(A_T, \mathbb{Z})$ (resp. $[\mu] \in \Sigma^n(A_T)$) for every character $\mu : A_T \to \mathbb{R}$ that extends $\chi$.

In [9] using valuation theory Bieri and Groves showed that for a finitely generated metabelian group $G$ the complement $\Sigma^1(G)^c$ of $\Sigma^1(G)$ in the character sphere $S(G)$ is a rationally defined spherical polyhedron i.e. a finite union of finite intersections of rationally defined closed semispheres in $S(G)$, where the rationality of the hemisphere means that it is defined by a rational vector. There are examples of groups $G$ of PL automorphism of the unit interval, where $\Sigma^1(G)^c$ contains precisely two points that need not be discrete [11], hence $\Sigma^1(G)^c$ is not a rationally defined polyhedron. We show that this cannot happen for the Bestvina-Brady group $BB_T$.

**Corollary 1.4.** Let $\Gamma$ be a finite connected graph. Then $\Sigma^1(BB_T)^c = S(BB_T) \setminus \Sigma^1(BB_T)$ is a rationally defined spherical polyhedron. In particular for the Bieri-Stallings groups $G_m$, where $m \geq 2$, we have that $\Sigma^1(G_m)^c$ is a rationally defined spherical polyhedron.

We also strengthen one of the directions of Theorem 1.1.

**Theorem 1.5.** Let $N \leq G$ be groups of type $FP_n$ and let $\chi : G \to \mathbb{R}$ be a character. Suppose that $N$ is a subnormal subgroup of $G$. If $[\chi|_N] \in \Sigma^n(N, \mathbb{Z})$, then $[\chi] \in \Sigma^n(G, \mathbb{Z})$.

This generalizes [35, Proposition C2.25], where Strebel considered the case $n = 1$. Notice that we do not require that the subgroups involved in some subnormal series for $N$ are of type $FP_n$. Such flexibility allows us, for instance, to apply the result to wreath products. Recall that for a group $G$ acting on a set $X$ and a group $H$ the (permutational restricted) wreath product $\Gamma = H \wr_X G$ is $\bigoplus_{x \in X} H_x \rtimes G$, where each $H_x \simeq H$ and the $G$-action (via conjugation) on $\bigoplus_{x \in X} H_x$ permutes the copies of $H_x$ via the original $G$-action on $X$.

**Corollary 1.6.** Let $\Gamma = H \wr_X G$ be a (permutational restricted) wreath product of type $FP_n$ and let $\chi : \Gamma \to \mathbb{R}$ be a character. Let

$$T_\chi = \{ x \in X \mid \chi|_{H_x} \neq 0 \}.$$ 

If $T_\chi$ has at least $n + 1$ elements, then $[\chi] \in \Sigma^n(\Gamma, \mathbb{Z})$.

Corollary 1.6 generalizes previous work by Mendonça in [29] on the low dimensional (homotopical) invariants of wreath products. This also complements [8, Theorem 8.1], where the set $T_\chi$ is assumed to be empty.

This article is organized as follows: Sections 2 and 3 contain preliminaries about $\Sigma$-invariants and the special classes of groups that we consider. We prove Theorem 1.5 and Corollary 1.6 in Section 4. In Sections 5 and 6 we apply Theorem 1.1 to the particular cases. Finally, the main result is proved in Sections 7 and 8.

### 2. Preliminaries on the $\Sigma$-invariants

Let $G$ be a finitely generated group. By definition a character $\chi : G \to \mathbb{R}$ is a non-zero homomorphism and the character sphere $S(G)$ is the set of equivalence classes $[\chi]$ of characters $\chi : G \to \mathbb{R}$, where two characters $\chi_1$ and $\chi_2$ are equivalent if one is obtained from the other by multiplication with any positive real number. For a fixed character $\chi : G \to \mathbb{R}$ define

$$G_\chi = \{ g \in G \mid \chi(g) \geq 0 \}.$$
If not stated otherwise the modules considered in this paper are left ones. Recall that for an associative ring $R$ and an $R$-module $A$ we say that $A$ is of type $FP_n$ over $R$ if $A$ has a projective resolution over $R$ where all projectives in dimension up to $n$ are finitely generated i.e. there is an exact complex

$$\mathcal{P} : \ldots \to P_i \to P_{i-1} \to \ldots \to P_0 \to A \to 0,$$

where each $P_i$ is a projective $R$-module and for $i \leq n$ we have that $P_i$ is finitely generated.

Let $D$ be an integral domain. By definition for a $DG$-module $A$

$$\Sigma_D^n(G, A) = \{[\chi] \in S(G) \mid A \text{ is of type } FP_n \text{ as } DG_{\chi}\text{-module}\}.$$

When $A$ is the trivial (left) $DG$-module $D$, we denote by $\Sigma^n(G, D)$ the invariant $\Sigma_D^n(G, D)$. When $D = A = \mathbb{Z}$, we recover the original BNS invariant $\Sigma^1(G)$ of $\mathbb{Z}$, i.e. $\Sigma^1(G) = \Sigma^1(G, \mathbb{Z})$.

Later we will need the description of $\Sigma^1(G)$ given by the Cayley graph of a finitely generated group $G$. Let $X$ be a finite generating set of $G$. Consider the Cayley graph $\Gamma$ of $G$ associated with the generating set $X$ i.e. the set of vertices is $V(\Gamma) = G$ and the set of edges is $E(\Gamma) = X \times G$ with the edge $e = (x, g)$ having beginning $g$ and end $gx$. The group $G$ acts on $\Gamma$ via left multiplication on $V(\Gamma)$ and $h.e = (x, hg)$ for any $h \in G$. The letter $x$ is called the label of the edge $e$ and we write $(x^{-1}, gx)$ for the inverse of $e$ and call $x^{-1}$ the label of $e^{-1}$. For a fixed character $\chi : G \to \mathbb{R}$ we write $\Gamma_\chi$ for the subgraph of $\Gamma$ spanned by the vertices in $G_\chi$. By definition

$$\Sigma^1(G) = \{[\chi] \in S(G) \mid \Gamma_\chi \text{ is a connected graph}\}.$$

We can define similarly the invariant $\Sigma^2(G)$. Suppose that $G$ is finitely presented and let $\mathcal{C}$ be the Cayley complex of $G$ associated with some finite presentation $G = \langle X \mid R \rangle$. This is the 2-complex obtained from the Cayley graph, as above, by gluing the set of 2-dimensional cells $R \times G$, where the cell associated to $(r, g)$ is glued along the boundary described by the loop with label $r$ and base point $g$ in $\Gamma$. For any character $\chi : G \to \mathbb{R}$, the subset $G_\chi \subset G$ determines a full subcomplex $\mathcal{C}_\chi$ of $\mathcal{C}$. By definition

$$\Sigma^2(G) = \{[\chi] \in S(G) \mid \mathcal{C}_\chi \text{ is 1-connected for some finite presentation } \langle X \mid R \rangle \text{ of } G\}.$$

Finally, the higher homotopical invariants $\Sigma^n(G)$ can be defined for any group of type $F_n$ via the equality $\Sigma^n(G) = \Sigma^2(G) \cap \Sigma^n(G, \mathbb{Z})$ for all $n \geq 2$.

The first result is folklore, it is an obvious corollary of the fact that $\Sigma^1(G, \mathbb{Z}) = \Sigma^1(G)$ and tensoring is a right exact functor.

**Lemma 2.1.** Let $\pi : G_1 \to G_2$ be an epimorphism of finitely generated groups, $\mu_2 : G_2 \to \mathbb{R}$ be a character (i.e. non-zero homomorphism) and $\mu_1 = \mu_2 \circ \pi$. Suppose that $[\mu_1] \in \Sigma^1(G_1)$. Then $[\mu_2] \in \Sigma^1(G_2)$.

**Theorem 2.2.** [25, Thm. 9.3] Let $H$ be a subgroup of $G$, $A$ be a $DG$-module and $\xi : G \to \mathbb{R}$ be a character. If $[G : H] < \infty$ then

$$[\xi_{\mu}] \in \Sigma_D^0(H, A) \iff [\xi] \in \Sigma_D^0(G, A).$$

In particular, if $n = 0$, then

$A$ is a finitely generated $DG_{\xi}$-module $\iff A$ is a finitely generated $DH_{\xi|H}$-module.
In [8] Bieri and Geoghegan proved a formula for the homological invariants \( \Sigma^n(\cdot, F) \) for a direct product of groups, where \( F \) is the trivial module and \( F \) is a field. If \( F \) is substituted with the trivial module \( \mathbb{Z} \) the result is wrong in both homological and homotopical settings provided that the dimension is sufficiently high, see [25] and [33].

**Theorem 2.3. Direct product formula** [8] Thm. 1.3, and Prop. 5.2 Let \( n \geq 0 \) be an integer, \( G_1, G_2 \) be finitely generated groups and \( F \) be a field. Then,

\[
\Sigma^n(G_1 \times G_2, F)^c = \bigcup_{p=0}^{n} \Sigma^p(G_1, F)^c \ast \Sigma^{n-p}(G_2, F)^c,
\]

where \( \ast \) denotes the join of sets in \( S(G_1 \times G_2) \) and \( ^c \) denotes the set-theoretic complement of subsets of a suitable character sphere.

Note that by definition \( \Sigma^0(G, F) = S(G) \).

The above theorem means that if \( \mu : G_1 \times G_2 \to \mathbb{R} \) is a character with \( \mu_1 = \mu |_{G_1} \) and \( \mu_2 = \mu |_{G_2} \), then \( [\mu] \in \Sigma^n(G_1 \times G_2, F)^c = S(G_1 \times G_2, F) \setminus \Sigma^n(G_1 \times G_2, F) \) precisely when one of the following conditions hold:

1. \( \mu_1 \neq 0, \mu_2 \neq 0 \) and \( [\mu_1] \in \Sigma^p(G_1, F)^c = S(G_1) \setminus \Sigma^p(G_1, F), [\mu_2] \in \Sigma^{n-p}(G_2, F)^c = S(G_2) \setminus \Sigma^{n-p}(G_2, F) \) for some \( 0 \leq p \leq n \);

or

2. one of the characters \( \mu_1, \mu_2 \) is trivial and for the non-trivial one, say \( \mu_i \), we have \( [\mu_i] \in \Sigma^n(G_i, F)^c = S(G_i) \setminus \Sigma^n(G_i, F) \).

**Theorem 2.4.** [17] Let \( n \) be a positive integer and let \( G_1, G_2 \) be finitely generated groups. If \( 1 \leq n \leq 2 \), then

\[
\Sigma^n(G_1 \times G_2, \mathbb{Z})^c = \bigcup_{p=0}^{n} \Sigma^p(G_1, \mathbb{Z})^c \ast \Sigma^{n-p}(G_2, \mathbb{Z})^c,
\]

where \( \ast \) denotes the join of sets of the \( S(G_1 \times G_2) \) and \( ^c \) denotes the set-theoretic complement of subsets of a suitable character sphere. Furthermore, the inclusion “\( \subseteq \)” holds for all \( n \).

This implies for instance that, for a direct product \( P = G_1 \times \cdots \times G_{n+1} \) of type \( FP_n \), any character \( \chi : P \to \mathbb{R} \) such that \( \chi |_{G_i} \neq 0 \) for all \( 1 \leq i \leq n+1 \) represents an element of \( \Sigma^n(P, \mathbb{Z}) \).

**Theorem 2.5.** [12], [30] Let \( G \) be a group of type \( F_n \) (resp. \( FP_n \)) and \( N \) be a subgroup of \( G \) that contains the commutator subgroup \( G' \). Then \( N \) is of type \( F_n \) (resp. \( FP_n \)) if and only if

\[
S(G, N) = \{ [\chi] \in S(G) \mid \chi(N) = 0 \} \subseteq \Sigma^n(G) \quad (\text{resp. } \Sigma^n(G, \mathbb{Z})).
\]

Bieri and Renz proved the homological version of Theorem 2.5 in [25]. The homotopical version for \( n = 2 \) was proved by Renz in [30] and the general homotopical case \( n \geq 3 \) follows from the formula \( \Sigma^n(G) = \Sigma^n(G, \mathbb{Z}) \cap \Sigma^2(G) \).

The following theorem can be traced back to several papers: Gehrke results in [17]; the Meier, Meinert and VanWyk description of the \( \Sigma \)-invariants for right-angled Artin groups [25] or the Meinert result on the \( \Sigma \)-invariants for direct products of virtually free groups [27].

**Theorem 2.6.** [17], [25], [27] If \( \chi : F_2^n = F_2 \times \cdots \times F_2 \to \mathbb{R} \) is a character whose restriction on precisely \( n \) copies of \( F_2 \) is non-zero, then \( [\chi] \in \Sigma^{n-1}(F_2^n) \setminus \Sigma^n(F_2^n) \).
3. Preliminaries on subdirect products, limit groups, the Virtual Surjection Conjecture and the Monoidal Virtual Surjection Conjecture

The class of limit groups contains all finite rank free groups and the orientable surface groups. It coincides with the class of the finitely generated fully residually free groups $G$ i.e. for every finite subset $X$ of $G$ there is a free group $F$ and a homomorphism $\varphi : G \to F$ whose restriction on $X$ is injective. Limit groups are of homotopical type $F$ i.e. they are of type $FP_\infty$, finitely presentable and of finite cohomological dimension.

Limit groups were used in the solution of the Tarski problem on the elementary theory of non-abelian free groups of finite rank obtained independently by Kharlampovich and Myasnikov and by Sela in [13], [35]. By a result of Baumslag, Myasnikov and Remeslennikov in [5] every finitely generated residually free group is a subdirect product of finitely many limit groups.

A subgroup $G \subseteq G_1 \times \ldots \times G_m$ is a subdirect product if the projection map $p_i : G \to G_i$ is surjective for all $1 \leq i \leq m$. Denote by $p_{i_1, \ldots, i_n} : G \to G_{i_1} \times \ldots \times G_{i_n}$ the projection map that sends $(g_1, \ldots, g_m)$ to $(g_{i_1}, \ldots, g_{i_n})$.

**Theorem 3.1.** [20] Let $G \subseteq G_1 \times \ldots \times G_m$ be a subdirect product of non-abelian limit groups $G_1, \ldots, G_m$ such that $G \cap G_i \neq 1$ for every $1 \leq i \leq m$. Then if $G$ is of type $FP_n$ for some $n \leq m$ then $p_{i_1, \ldots, i_n}(G)$ has finite index in $G_{i_1} \times \ldots \times G_{i_n}$ for every $1 \leq i_1 < \ldots < i_n \leq m$.

**Theorem 3.2.** [20] Let $G$ be a non-abelian limit group. Then $\Sigma^1(G) = \emptyset$.

The following conjecture was formulated by Kuckuck in [24].

**The Virtual Surjection Conjecture** [24] Let $G \subseteq G_1 \times \ldots \times G_m$ be a subdirect product of groups $G_1, \ldots, G_m$ such that $G \cap G_i \neq 1$ for every $1 \leq i \leq m$ and each $G_i$ is of homotopical type $F_n$ for a fixed $n \leq m$. Suppose that $p_{i_1, \ldots, i_n}(G)$ has finite index in $G_{i_1} \times \ldots \times G_{i_n}$ for every $1 \leq i_1 < \ldots < i_n \leq m$. Then $G$ is of type $F_n$.

The motivation behind the Virtual Surjection Conjecture is that it holds for $n = 2$ [14] and this particular case was established by Bridson, Howie, Miller and Short as a corollary of the 1-2-3 Theorem. Furthermore the Virtual Surjection Conjecture holds for any $n$ when $G$ contains $G'_1 \times \ldots \times G'_m$ [24]. A homological version of the Virtual Surjection Conjecture was suggested in [22] and proved for $n = 2$.

**Theorem 3.3.** [22] Let $G \subseteq G_1 \times \ldots \times G_m$ be a subdirect product of groups $G_1, \ldots, G_m$ such that $G \cap G_i \neq 1$ for every $1 \leq i \leq m$ and each $G_i$ is of homotopical type $FP_2$. Suppose that $p_{i_1, i_2}(G)$ has finite index in $G_{i_1} \times G_{i_2}$ for every $1 \leq i_1 < i_2 \leq m$. Then $G$ is of type $FP_2$.

The following conjecture, called the Monoidal Virtual Surjection Conjecture, was suggested in [23] and should be viewed as a monoidal version of the Virtual Surjection Conjecture from [24].

**The Monoidal Virtual Surjection Conjecture** [23] Let $n$ and $m$ be positive integers such that $m \geq 2$ and $1 \leq n \leq m$. Let $S \leq L_1 \times \ldots \times L_m$ be a subdirect product of non-abelian limit groups $L_1, \ldots, L_m$ such that $S$ is of type $FP_n$ and finitely presented. Then

$$[\chi] \in \Sigma^n(S, \mathbb{Q}) = \Sigma^n(S, \mathbb{Z}) = \Sigma^n(S)$$
if and only if
\[ p_{j_1,\ldots,j_n}(S_\chi) = p_{j_1,\ldots,j_n}(S) \quad \text{for all } 1 \leq j_1 < \ldots < j_n \leq m, \]
where \( p_{j_1,\ldots,j_n} : S \to L_{j_1} \times \ldots \times L_{j_n} \) is the canonical projection.

Note that by Theorem 3.1 the condition that \( S \) is of type \( FP_n \) implies that \( p_{j_1,\ldots,j_n}(S) \) has finite index in \( L_{j_1} \times \ldots \times L_{j_n} \).

We state several results that were proved recently by Kochloukova and Lima in [23].

**Theorem 3.4.** [23] 1. The forward direction of the Monoidal Virtual Surjection Conjecture holds i.e. if \( [\chi] \in \Sigma^n(S,\mathbb{Q}) \) then \( p_{j_1,\ldots,j_n}(S_\chi) = p_{j_1,\ldots,j_n}(S) \) for all \( 1 \leq j_1 < \ldots < j_n \leq m \);
2. The Monoidal Virtual Surjection Conjecture holds for \( n = 1 \);
3. The Monoidal Virtual Surjection Conjecture holds for \( n = m \);
4. If the Virtual Surjection Conjecture from [24] holds then the Monoidal Virtual Surjection Conjecture holds for all discrete characters \( \chi \);
5. The Monoidal Virtual Surjection Conjecture holds for the Bieri-Stallings groups \( S = G_m \) and the classical embedding \( S = G_m \subseteq L_1 \times \ldots \times L_m \) with \( L_i \) free of rank 2 for \( 1 \leq i \leq m \) in the following two cases:
   a) if \( n \leq m - 2 \);
   b) if \( n = m - 1 \) and \( \chi \) is a discrete character.

## 4. Invariants of subnormal subgroups and wreath products

In this section we prove Theorem 1.5. For this we consider a characterization of the homological \( \Sigma \)-invariants in terms of Novikov rings.

Let \( G \) be a group and \( \chi : G \to \mathbb{R} \) a character. The Novikov ring of \( G \) with respect to \( \chi \) is the ring \( \hat{\mathbb{Z}}G_\chi \) of formal sums \( \lambda = \sum_{g \in G} a_g g \), with \( a_g \in \mathbb{Z} \), such that the set
\[ S_{\lambda,r} = \{ g \in G \mid a_g \neq 0, \chi(g) < r \} \]
is finite for all \( r \in \mathbb{R} \). Addition and multiplication in \( \hat{\mathbb{Z}}G_\chi \) extend the operations of the subring \( \mathbb{Z}G \subseteq \hat{\mathbb{Z}}G_\chi \).

**Theorem 4.1.** [7], [36] Let \( G \) be a group of type \( FP_n \) and \( \chi : G \to \mathbb{R} \) a character. Then \( [\chi] \in \Sigma^n(G,\mathbb{Z}) \) if and only if \( H_k(G,\hat{\mathbb{Z}}G_\chi) = 0 \) for all \( k = 0,1,\ldots,n \).

We will employ the following result obtained by Schütz.

**Lemma 4.2.** [24] Corollary 2.5] Let \( G \) be a group of type \( FP_n \) and \( \chi : G \to \mathbb{R} \) a character. If \( [\chi] \in \Sigma^n(G,\mathbb{Z}) \), then \( H_k(G,M) = 0 \) for all \( 0 \leq k \leq n \) and for all \( \hat{\mathbb{Z}}G_\chi \)-modules \( M \).

**Proof of Theorem 1.5** Suppose that \( N \leq G \) is subnormal and let \( N = N_0 \leq N_1 \leq \cdots \leq N_r = G \) with \( N_i \leq N_{i+1} \) for all \( i \). Suppose that \( \chi : G \to \mathbb{R} \) is a character such that \( [\chi]|_N \in \Sigma^n(N,\mathbb{Z}) \). We will show by induction on \( i \) that \( H_j(N_i,\hat{\mathbb{Z}}G_\chi) = 0 \) for \( 0 \leq j \leq n \). The case \( i = 0 \) is a particular case of Lemma 4.2.

For the induction we assume that \( H_j(N_i,\hat{\mathbb{Z}}G_\chi) = 0 \) for \( 0 \leq j \leq n \) and aim to prove that \( H_j(N_{i+1},\hat{\mathbb{Z}}G_\chi) = 0 \) for \( 0 \leq j \leq n \) if \( i \neq r \). Consider the LHS spectral sequence
\[ E^2_{p,q} = H_p(N_{i+1}/N_i,H_q(N_i,\hat{\mathbb{Z}}G_\chi)) \Rightarrow H_{p+q}(N_{i+1},\hat{\mathbb{Z}}G_\chi). \]
Lemma 4.2 implies that $E^2_{p,q} = 0$ whenever $q \leq n$. By the convergence of the spectral sequence, $H_j(N_{i+1}, \mathbb{Z}G_\chi) = 0$ for all $j \leq n$. This completes the induction.

Finally since $N_r = G$ we deduce that $H_j(G, \mathbb{Z}G_\chi) = 0$ for all $0 \leq j \leq n$, that is, $[\chi] \in \Sigma^n(G, \mathbb{Z})$.

Proof of Corollary 1.6 Let $\Gamma = H \wr X G$ be a wreath product of type $FP_n$ and let $\chi : \Gamma \to \mathbb{R}$ be a character with $|T_\chi| \geq n + 1$. Let $T_1$ be any subset of $T_\chi$ containing exactly $n + 1$ elements and let $N \leq \Gamma$ be the subgroup generated by the copies $H_x$ of $H$ such that $x \in T_1$. Clearly $N = \oplus_{x \in T_1} H_x \simeq \prod_{i=1}^{n+1} H$. Notice that $N$ is of type $FP_n$, since the direct product is finite and, under the conditions of the corollary, $H$ is of type $FP_n$ by [3] Theorem A]. By Theorem 2.3 we conclude that $[\chi|_N] \in \Sigma^n(N, \mathbb{Z})$. Furthermore, $N$ is subnormal in $\Gamma$ via $N \leq \oplus_{x \in X} H_x \leq \Gamma$, so $[\chi] \in \Sigma^n(\Gamma, \mathbb{Z})$ by Theorem 1.3.

5. Applications of Theorem 1.1 for the Bieri-Stallings groups $G_m$ : proof of Corollary 1.2

Let $L_i$ be the free group with a free basis $\{x_i, y_i\}$ and

$$G_m \leq H = L_1 \times L_2 \times \ldots \times L_m$$

be the kernel of the character $H \to \mathbb{R}$ that sends each $x_i$ and $y_i$ to 1. Consider the Monoidal Virtual Surjection Conjecture for $S = G_m$ of type $FP_n$. Note that since $G_m$ is $FP_{m-1}$ but not $FP_m$ we have that $n \leq m - 1$. The case $n \leq m - 2$ was proved in [23] i.e. we know that the Monoidal Virtual Surjection Conjecture holds for $S = G_m$ and $n \leq m - 2$. Thus we can assume from now on that $n = m - 1$. Furthermore as the forward direction of the Monoidal Virtual Surjection Conjecture holds (see Theorem 5.3 part 1), it remains to prove the backward direction, i.e. for $S = G_m$ if $p_{j_1 \ldots j_{m-1}}(S) = p_{j_1 \ldots j_{m-1}}(S_\chi)$ for every $1 \leq j_1 < \ldots < j_{m-1} \leq m$ then $[\chi] \in \Sigma^{m-1}(S)$. Note that this is really sufficient to complete the proof of the Monoidal Virtual Surjection Conjecture since in general we have $\Sigma^{m-1}(S) \subseteq \Sigma^{m-1}(S, Q)$.

Let $\mu : H \to \mathbb{R}$ be a character and $\mu_i$ be the restriction of $\mu$ on $L_i$. By Theorem 2.6 $[\mu] \in \Sigma^{m-1}(H)$ if and only if $\mu_i \neq 0$ for all $1 \leq i \leq m$. Note that $[G_m, G_m] = [H, H]$; hence we can apply Theorem 1.1 i.e. $[\chi] \in \Sigma^{m-1}(S)$ if and only if for every character $\mu : H \to \mathbb{R}$ that extends $\chi$ we have $[\mu] \in \Sigma^{m-1}(H)$. Thus to finish the proof of the fact that $[\chi] \in \Sigma^{m-1}(S)$ for $S = G_m$ it remains to show that $\mu_i \neq 0$ for all $1 \leq i \leq m$.

By $23$ $p_{j_1 \ldots j_{m-1}}(S_\chi) = p_{j_1 \ldots j_{m-1}}(S)$ is equivalent to $\chi(Ker(p_{j_1 \ldots j_{m-1}})) \neq 0$. Note that for $S = G_m$ we have

$$Ker(p_{j_1 \ldots j_{m-1}}) = G_m \cap L_i,$$

where $i = \{1, \ldots, m\} \setminus \{j_1, \ldots, j_{m-1}\}$. Thus $\chi(G_m \cap L_i) \neq 0$ and hence $\mu_i \neq 0$ for $1 \leq i \leq m$. This completes the proof of Corollary 1.2.

6. Applications of Theorem 1.1 to Bestvina-Brady groups : proof of Corollary 1.2 and Corollary 1.3

Let $A_\Gamma$ be the right-angled Artin group associated to a finite graph $\Gamma$. Let $BB_\Gamma$ be the Bestvina-Brady group associated to $\Gamma$ i.e. $BB_\Gamma$ is the kernel of the homomorphism $\varphi : A_\Gamma \to \mathbb{Z}$ that sends each of the usual generators of $A_\Gamma$ (which
are identified with the vertices of $\Gamma$) to 1 $\in \mathbb{Z}$. By [4] we know that $BB_\Gamma$ is finitely generated if and only if $\Gamma$ is connected.

**Lemma 6.1.** If $\Gamma$ is connected, then $[A_\Gamma, A_\Gamma] = [BB_\Gamma, BB_\Gamma]$.

**Proof.** It suffices to show that for any two vertices $x, y \in V(\Gamma)$, the commutator $[x, y]$ lies in $[BB_\Gamma, BB_\Gamma]$. We prove this by induction on the length $n$ of the shortest path joining $x$ and $y$ in $\Gamma$. If $n = 1$ there is nothing to do. Suppose that $n > 1$ and let $t \in V(\Gamma)$ be a vertex which is connected to $y$ by an edge and which is connected to $x$ through a path of length $n - 1$. By induction we can assume that $[x, t] \in [BB_\Gamma, BB_\Gamma]$. Keeping in mind that $y$ and $t$ commute in $A_\Gamma$, it is not hard to verify that

$$[x, y] = [xy^{-1}, t^{-1}y]^y, [x, t]^{t^{-1}y},$$

thus $[x, y] \in [BB_\Gamma, BB_\Gamma]$. $\square$

6.1. **Proof of Corollary 1.3.** It follows from Lemma 6.1 that Theorem 1.1 applies to any non-zero homomorphism $\chi : BB_\Gamma \to \mathbb{R}$. Thus $[\chi] \in \Sigma^m(BB_\Gamma, \mathbb{Z})$ if and only if $[\mu] \in \Sigma^m(A_\Gamma, \mathbb{Z})$ for any extension $\mu : A_\Gamma \to \mathbb{R}$, and similarly for the homotopical invariants. In this sense, the invariants of $A_\Gamma$ determine the invariants of $BB_\Gamma$.

6.2. **Proof of Corollary 1.4.** Let $\Gamma$ be a connected graph with set of vertices $V(\Gamma)$. Let $\chi : A_\Gamma \to \mathbb{R}$ be a character and $L_\chi$ be the subgraph of $\Gamma$ spanned by the vertices of $\Gamma$ with non-zero $\chi$-value. By [20] $[\chi] \in \Sigma^1(A_\Gamma)$ if and only if $L_\chi$ is connected and dominant in $\Gamma$ i.e. every vertex in $V(\Gamma) \setminus V(L_\chi)$ is linked by an edge from $\Gamma$ with at least one vertex from $L_\chi$. This description easily implies that $\Sigma^1(A_\Gamma)^c$ is a rationally defined spherical polyhedron. Actually there is a conjecture suggested in [2] that predicts the structure of the $\Sigma^1$ for general Artin groups. It was recently shown in [21] that for the Artin groups that satisfy this conjecture $\Sigma^1(G)^c$ is a rationally defined polyhedron. In particular this holds for $G = A_\Gamma$.

By Lemma 6.1 $S(BB_\Gamma)$ is a subsphere of $S(A_\Gamma)$. The fact that $\Sigma^1(A_\Gamma)^c$ is a rationally defined spherical polyhedron together with Theorem 1.1 implies that $\Sigma^1(BB_\Gamma)^c$ is a rationally defined spherical polyhedron.

7. The proof of the homological part of Theorem 1.1

In this section we will prove the homological part of Theorem 1.1. The proof is long and technical and subdivided in several subsections. We use the techniques developed in [12] but at several points subtle modifications are required.

7.1. **Preliminaries from the Bieri-Renz paper [12].** First we recall some notation from [12]. Assume that

$$\ldots \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \ldots \xrightarrow{} F_0 \to A \to 0$$

is an admissible free resolution (over $\mathbb{Z}$) with finitely generated $n$-skeleton i.e. for every $0 \leq i \leq n$ the free $\mathbb{Z}G$-module $F_i$ is endowed with a basis $X_i$ such that for each $x_i \in X_i$ we have $\partial_i(x_i) \neq 0$ and

$$X(n) = \cup_{0 \leq i \leq n} X_i \text{ is finite.}$$

Set

$$X = \cup_{i \geq 0} X_i.$$ We write $F$ for the complex $\ldots \to F_i \to F_{i-1} \to \ldots \to F_0 \to 0$ and write $F \to A$ for the free resolution (7.1). We think of $F$ as $\oplus_{i \geq 0} F_i$. 
Let \( v : X \to \mathbb{R} \) be a set-theoretic map and 
\[
\nu : G \to \mathbb{R}
\]
be a character. We extend it to a valuation (in the sense of Bieri-Renz in \[12\])
\[
v = v_\nu : F \to \mathbb{R} \cup \{\infty\},
\]
given by
\[
v(0) = \infty, \ v(gx) = \nu(g) + v(x) \text{ for } g \in G, x \in X
\]
and
\[
v(\sum_{z_{g,x} \in \mathbb{Z} \setminus \{0\}, g \in G, x \in X} z_{g,x}gx) = \min \{v(gx) \mid z_{g,x} \neq 0\}.
\]
The map \( v \mid_{X_i} = v_i \) is defined inductively on \( i \geq 0 \) by
\[
v_i(x) = 0 \text{ if } i = 0 \text{ and } v_i(x) = v_{i-1}(\partial_i(x)) \text{ for } i > 0.
\]
Thus
\[
v(\partial c) \leq v(c) \text{ for } c \in F.
\]
We define support \( \text{supp}_X(c) \) with respect to \( X \) for \( c = \sum_{g \in G, x \in X} z_{g,x}gx \in F_i \setminus \{0\} \), as a finite subset of \( G \) given by
\[
\text{supp}_X(c) = \bigcup_{g \in G, x \in X, z_{g,x} \neq 0} g \cdot \text{supp}_X(x) \text{ for } i \geq 0,
\]
and
\[
\text{supp}_X(x) = \text{supp}_X(\partial_i(x)) \text{ for } x \in X_i, i \geq 1
\]
and
\[
\text{supp}_X(x) = 1_G \text{ for } x \in X_0.
\]
For completeness we set \( \text{supp}_X(0) = \emptyset \).
Note that for \( c \in F_i \) we have that
\[
\text{supp}_X(\partial(c)) \subseteq \text{supp}_X(c)
\]
and if \( a, b \in F_i \) then
\[
\text{supp}_X(a + b) \subseteq \text{supp}_X(a) \cup \text{supp}_X(b).
\]
If \( a, b, a + b \in F_i \setminus \{0\}, a = \sum_{g \in G, x \in X} z_{g,x}gx, b = \sum_{g \in G, x \in X} z_{g,x}gx \) and
\[
\{(g, x) \in G \times X_i \mid z_{g,x} \neq 0\} \cap \{(g, x) \in G \times X_i \mid z_{g,x} \neq 0\} = \emptyset
\]
then in \((7.2)\) equality is attained.

We state an important result of Bieri and Renz that works for compact subsets of \( S(G) \).

**Theorem 7.1.** \[12\] Let \( F \to A \) be an admissible free resolution with finitely generated \( n \)-skeleton. Then the following three conditions are equivalent for a compact subset \( \Gamma \subseteq S(G) \):

(i) \( \Gamma \subseteq \Sigma^n(G, A) \);

(ii) there is a finite set \( \Phi \) of chain endomorphisms \( \varphi : F \to F \), lifting \( \text{id}_A \), with the property that for each point \( [x] \in \Gamma \) there is some \( \varphi \in \Phi \) with
\[
v_\chi(\varphi(x)) > v_\chi(x) \text{ for every } x \in X(\chi);
\]

(iii) after replacing \( F \) by a suitable admissible free resolution, obtained by performing on \( F \) a finite sequence of elementary expansions, we can find a finite set \( \Phi \) as in (ii) and for each \( \varphi \in \Phi \) a chain homotopy \( \sigma : \varphi \to \text{Id}_F \) with \( \sigma(X_i) \subseteq X_{i+1} \cup \{0\} \) for every \( i \) with \( 0 \leq i \leq n \).
7.2. Proof of the homological part of Theorem 1.1 Notice that Theorem 1.5 already proves one of the directions of Theorem 1.1 in the homological case. In this subsection we prove the other direction.

Theorem 7.2. Let $[H, H] \subseteq K \subseteq H$ be groups such that $H$ and $K$ are of type $FP_n$. Let $\chi : K \to \mathbb{R}$ be a character such that $\chi([H, H]) = 0$. Suppose that $[\mu] \in \Sigma^n(H, \mathbb{Z})$ for every character $\mu : H \to \mathbb{R}$ that extends $\chi$. Then $[\chi] \in \Sigma^n(K, \mathbb{Z})$.

Proof. Let $H_0$ be a subgroup of finite index in $H$ that contains the commutator $[H, H]$ such that $H_0/[H, H]$ is torsion-free and for $K_0 = H_0 \cap K$ we have $H_0/K_0$ is torsion-free. Thus $H_0/[H, H] \cong K_0/[H, H] \times H_0/K_0 \simeq \mathbb{Z}^m \times \mathbb{Z}^{k-m} \simeq \mathbb{Z}^k$.

Set $Q_0 = H_0/[H, H]$, $V = Q_0 \otimes \mathbb{R} \simeq \mathbb{R}^k$ and $W = K_0/[H, H] \otimes \mathbb{R} \subseteq V$.

We identify $W$ with a subspace of $V$. We identify $V$ with $\mathbb{R}^k$, $Q_0$ with $\mathbb{Z}^k$ and consider the standard inner product $\langle -, - \rangle$ in $\mathbb{R}^k$.

By Theorem 2.2 for a character $\mu : H \to \mathbb{R}$ we have that $[\mu] \in \Sigma^n(H, \mathbb{Z})$ if and only if $[\mu_0] \in \Sigma^n(H_0, \mathbb{Z})$,

where $\mu_0$ is the restriction of $\mu$ on $H_0$. Now the same holds for the character $\chi : K \to \mathbb{R}$ from the statement and its restriction $\chi_0 : K_0 \to \mathbb{R}$ i.e.

$[\chi] \in \Sigma^n(K, \mathbb{Z})$ if and only if $[\chi_0] \in \Sigma^n(K_0, \mathbb{Z})$.

Thus to prove the result we assume that for each character $\mu_0 : H_0 \to \mathbb{R}$ that extends the character $\chi_0$ we have

\[ [\mu_0] \in \Sigma^n(H_0, \mathbb{Z}) \]

and aim to prove that $[\chi_0] \in \Sigma^n(K_0, \mathbb{Z})$.

Note that there are unique $c_{\mu_0} \in V$, $c_{\chi_0} \in W$ such that

$\mu_0(h) = \langle \pi(h), c_{\mu_0} \rangle$ for $h \in H_0$

and

$\chi_0(g) = \langle \pi(g), c_{\chi_0} \rangle$ for $g \in K_0$,

where

$\pi : H_0 \to Q_0$

is the canonical projection. Thus $c_{\chi_0}$ is the orthogonal projection of $c_{\mu_0}$ to the subspace $W$.

Consider

$W_{\chi_0 \geq 0} = \{ v \in W \mid \langle v, c_{\chi_0} \rangle \geq 0 \}$

and

$W_{\chi_0 = 0} = \{ v \in W \mid \langle v, c_{\chi_0} \rangle = 0 \}$.

Define

$D_r = \cup_{w \in W_{\chi_0 \geq 0}} B(w, r)$,

where $B(w, r)$ is the closed ball in $V$ with center $w$ and radius $r$. Write

$B(w, r)_{\chi_0 \leq 0} = \{ v \in B(w, r) \mid \langle v - w, c_{\chi_0} \rangle \leq 0 \}$.

Let $S(w, r)$ be the boundary of the closed ball $B(w, r)$, i.e. $S(w, r)$ is the sphere with center $w$ and radius $r$. Write

$\delta : V \to W$
for the orthogonal projection. Consider the boundary $B_r$ of $D_r$
\[ B_r = \partial D_r = (\bigcup_{w \in W_{x_0}=0} \{ v \in S(w, r) \mid \delta(w - v) = \lambda c_{x_0} \text{ for } \lambda \in \mathbb{R}_{>0} \}) \]
\[ \cup (\bigcup_{w \in W_{x_0} \geq 0} \{ v \in S(w, r) \mid \delta(v) = w \}). \]

We start by fixing an admissible free resolution with finite $n$-skeleton
\[ \mathbf{F} \to \mathbb{Z} \]
of the trivial $\mathbb{Z}H_0$-module $\mathbb{Z}$. Consider the compact set $\Gamma$
\[ \Gamma = \{ [\mu_0] \in S(H_0) \mid \mu_0|_{K_0} = \chi_0 \} \cup S(H_0, K_0) \subseteq \Sigma^n (H_0, \mathbb{Z}) \subseteq S(H_0), \]
where the first inclusion uses (7.3) and Theorem 2.3. Thus we can apply Theorem 7.1, i.e. there is a finite set $\Phi$ of chain endomorphisms
\[ \varphi : \mathbf{F} \to \mathbf{F} \]
lifting $id_\mathbb{Z}$ such that for every $[\nu] \in \Gamma$ there is some $\varphi \in \Phi$ with
\[ (7.4) \quad v_\nu (\varphi(x)) > v_\nu (x) \text{ for every } x \in X(n) \]
and chain homotopy
\[ (7.5) \quad \sigma_\varphi : \varphi \simeq id_\mathbb{F} \text{ for each } \varphi \in \Phi \]
such that
\[ \sigma_\varphi (X_i) \subseteq X_{i+1} \cup \{ 0 \} \text{ for every } 0 \leq i \leq n. \]

Similarly to [12] we define
\[ (7.6) \quad s = \max \{ ||\pi(\varphi(y)|| \mid \varphi \in \Phi, 1_{H_0} \in supp_X(y), y \in H_0 X(n) \}. \]
Note that $s$ is well-defined since both the sets $X(n)$ and $\{ y \in H_0 X(n) \mid 1_{H_0} \in supp_X(y) \}$ are finite.

Since $\Gamma$ is compact and both $X(n)$ and $\Phi$ are finite there is a real number
\[ (7.7) \quad t = \min_{x \in X(n), \varphi \in \Phi, [\nu] \in \Gamma} (v_\nu (\varphi(x)) - v_\nu (x)) > 0. \]
Then by [12] Lemma 4.3] for $\lambda \in \cup_{j \leq n} F_j$ and $\varphi \in \Phi$, $[\nu] \in \Gamma$ we have
\[ (7.8) \quad supp_X \sigma_\varphi (\lambda) \subseteq supp_X (\lambda) \cup (H_0)_{\nu \geq v_\nu (\lambda) + t}. \]
Set the constants
\[ (7.9) \quad \epsilon_0 = \frac{1}{2} \min \{ s, t \} \text{ and } a = \max \{ \frac{s^2 - \epsilon_0^2}{2 (t - \epsilon_0)}, \epsilon_0 \}. \]
Let $\Delta$ be a finite subset of $H_0$ and recall that $\pi : H_0 \to Q_0 = H_0/[H, H] \subset V$ is the canonical projection. Define
\[ (7.10) \quad \alpha (\Delta) = \min \{ r \mid \pi(\Delta) \subseteq D_r \}. \]
We will prove later the following fact.

Claim For every cycle $z \in F_{j-1} \setminus \{ 0 \}$ with $j \leq n$ such that $\alpha(supp_X(z)) \geq a$ there is $c_0 \in F_j$ with $\partial(c_0) = z$ and $\alpha(supp_X(c_0)) \leq \alpha(supp_X(z))$.

The Claim implies that for every $r \geq a$
\[ F_r := \{ f \in \mathbf{F} \mid \pi(supp_X(f)) \subseteq D_r \} \]
is a subcomplex of $\mathbf{F}$ that is exact in dimensions $1, \ldots, n-1$ and $\mathbf{F}_r \to \mathbb{Z}$ is exact in dimension 0. This will complete the proof modulo the following argument.

Define
\[ W_{x_0 \geq j} = \{ v \in W \mid \langle v, c_{x_0} \rangle \geq j \} \]
The proof of the Claim will show that for every \( r \geq a \) and every \( j \in \mathbb{R} \) we have
\[
\mathbf{F}_{r, \chi_0 \geq j} := \{ f \in \mathbf{F} | \pi(\text{supp}_X(f)) \subseteq D_{r, \chi_0 \geq j} \}
\]
is a subcomplex of \( \mathbf{F} \) that is exact in dimensions 1, \ldots, \( n-1 \).

The original Bieri-Renz argument from [12] gives as well that \( \mathbf{F}_r := \{ f \in \mathbf{F} | \pi_0(\text{supp}_X(f)) \subseteq \hat{B}_r \} \) is a subcomplex of \( \mathbf{F} \) that is exact in dimensions 1, \ldots, \( n-1 \) and \( H_0(\mathbf{F}_r \rightarrow \mathbb{Z}) = 0 \), where
\[
\pi_0 : H_0 \rightarrow H_0/K_0 \simeq \mathbb{Z}^{k-m}
\]
is the canonical projection and \( \hat{B}_r \) is the closed ball in \( H_0/K_0 \otimes \mathbb{R} \simeq \mathbb{R}^{k-m} \) with radius \( r \) and center the origin. Thus \( \mathbf{F}_r \) is a partial free resolution up to dimension \( n \) of the trivial \( \mathbb{Z}K_0 \)-module \( \mathbb{Z} \) with all modules in dimensions up to \( n \) being finitely generated.

Observe that
\[
\mathbf{F}_r = \bigcup_{j \in \mathbb{R}} \mathbf{F}_{r, \chi_0 \geq j}.
\]

Let
\[
(7.12) \quad \pi_1 : H_0 \rightarrow H_0/[H, H] = (K_0/[H, H]) \times (H_0/K_0) \rightarrow K_0/[H_0, H_0],
\]
be the composition where the first map is the canonical projection and the second map is the projection on the first factor. Set
\[
\tilde{\mathbf{F}}_{r, \chi_0 \geq j} := \{ f \in \mathbf{F}_r | \pi_1(\text{supp}_X(f)) \subseteq \chi_0^{-1}([j, \infty)) \}.
\]

Fix some \( r \geq a \). Note that there is a fixed negative integer \( r_0 \) that depends on \( r \) such that for every \( j \in \mathbb{R} \)
\[
\mathbf{F}_{r, \chi_0 \geq j} \subseteq \mathbf{F}_{r, \chi_0 \geq j} \subseteq \mathbf{F}_{r, \chi_0 \geq j+r_0} \subseteq \mathbf{F}_{r, \chi_0 \geq j+r_0}.
\]
The above inclusions induce maps in homology
\[
H_i(\tilde{\mathbf{F}}_{r, \chi_0 \geq j}) \rightarrow H_i(\mathbf{F}_{r, \chi_0 \geq j}) = 0 \rightarrow H_i(\tilde{\mathbf{F}}_{r, \chi_0 \geq j+r_0}) \text{ for } 1 \leq i \leq n-1 \text{ and } j \in \mathbb{R}
\]
and
\[
H_0(\tilde{\mathbf{F}}_{r, \chi_0 \geq j} \rightarrow \mathbb{Z}) \rightarrow H_0(\mathbf{F}_{r, \chi_0 \geq j} \rightarrow \mathbb{Z}) = 0 \rightarrow H_0(\tilde{\mathbf{F}}_{r, \chi_0 \geq j+r_0} \rightarrow \mathbb{Z}) \text{ for } j \in \mathbb{R}.
\]

Hence
\[
(7.13) \quad H_i(\tilde{\mathbf{F}}_{r, \chi_0 \geq j}) \rightarrow H_i(\mathbf{F}_{r, \chi_0 \geq j+r_0}) \text{ is the zero map for } 1 \leq i \leq n-1 \text{ and } j \in \mathbb{R}
\]
and
\[
(7.14) \quad H_0(\tilde{\mathbf{F}}_{r, \chi_0 \geq j} \rightarrow \mathbb{Z}) \rightarrow H_0(\mathbf{F}_{r, \chi_0 \geq j+r_0} \rightarrow \mathbb{Z}) \text{ is the zero map for } j \in \mathbb{R}.
\]

Recall that \( \mathbf{F}_r \) is a partial free resolution up to dimension \( n \) of the trivial \( \mathbb{Z}K_0 \)-module \( \mathbb{Z} \) with all modules in dimensions up to \( n \) being finitely generated. Then (7.13) and (7.14) we can apply [12 Thm. 3.2] to deduce
\[
[\chi_0] \in \Sigma^n(K_0, \mathbb{Z}).
\]
Remark. Notice that $\tilde{F}_{r,\lambda_0 \geq j}$ is a filtration of $\tilde{F}_r$ that comes from the valuation $v$ such that $v(f) = \min(\chi_0(\pi_1(\text{supp}_X(f))))$ for all $f \in \tilde{F}_r$. We could also consider a new basis $\tilde{X}$ of $\tilde{F}_r$ as complex of free $\mathbb{Z}K_0$-modules, which gives rise to a new Bieri-Renz valuation. The valuations are clearly equivalent (in the sense of [12, Section 2.2]), which justifies the application of [12, Thm. 3.2] (see the remark following it).

Proof of the Claim Let $c \in F_j$ be any element such that $\partial(c) = z$. If $\alpha(\text{supp}_X(c)) \leq \alpha(\text{supp}_X(z))$ we are done, so assume that

$$\alpha(\text{supp}_X(c)) > \alpha(\text{supp}_X(z)) \geq a.$$ 

Let $g \in \text{supp}_X(c)$ such that

$$\alpha\{g\} = \alpha(\text{supp}_X(c)) = r > a.$$

Thus

$$\pi(g) \in \partial D_r = B_r.$$

Then either

1. $\pi(g) \in S(w, r)$ for some $w \in W_{\lambda_0 \geq 0}$, where $w$ is the orthogonal projection $\delta(\pi(g))$ of $\pi(g)$ to $W$;

or

2. $\pi(g) \in S(w, r)$ for some $w \in W_{\lambda_0 = 0}$, where $w$ is the orthogonal projection of $\pi(g)$ to $W_{\lambda_0 = 0}$ and $\delta(w - \pi(g)) = \lambda \epsilon_{\lambda_0}$ for some $\lambda \in \mathbb{R}_{> 0}$.

Consider the character

$$\nu : H_0 \to \mathbb{R}$$

given by

$$(7.15) \qquad \nu(h) = \langle \pi(h), \frac{w - \pi(g)}{|w - \pi(g)|} \rangle.$$ 

Then in case 1 $\nu(K_0) = 0$ and in case 2 the restriction of $\nu$ to $K_0$ is $\lambda \epsilon_{\lambda_0}$ for some $\lambda \in \mathbb{R}_{> 0}$. In both cases

$$[\nu] \in \Gamma \subseteq \Sigma^n(H_0, \mathbb{Z}).$$ 

For the people familiar with the Bieri-Renz paper [12] this means that we can "push" every element of $H_0 \cap B_r$ inside $D_r$ using the pushing algorithm from [12]. We will explain in details in the rest of the section what this means. The aim of the proof is to find an element $\tilde{c} \in F_j$ such that

$$\partial_j(c) = z = \partial_j(\tilde{c})$$

and

$$\text{supp}_X(\tilde{c}) \subseteq (\text{supp}_X(c) \setminus \{g\}) \cup D_r - \epsilon_0,$$

where $\epsilon_0$ was defined in (7.9). Applying this procedure several times we get the desired element $c_0 \in F_j$ from the Claim.

In order to construct $\tilde{c}$ we decompose $c = \sum z_y y$, where $z_y \in \mathbb{Z} \setminus \{0\}, y \in H_0 X_j$ and write

$$c = c' + c'',$$

where $c'$ contains precisely the terms $z_y y$ for which $g \in \text{supp}_X(y)$. Thus $y$ cannot appear simultaneously in $c'$ and $c''$ and this guarantees that

$$\text{supp}_X(c) = \text{supp}_X(c') \cup \text{supp}_X(c'').$$
Then
\[ g \notin \text{supp}_X(c''), \supp_X(c'') \subseteq \supp_X(c) \text{ and } g \in \supp_X(c') \subseteq \supp_X(c). \]

Fix \( \varphi, \sigma \) and \( v \), as before, for simplicity we denote by \( \varphi, \sigma \) and \( v \). Define
\begin{equation}
(7.16) \quad \bar{c} = c + \partial \sigma(c') = \varphi(c') - \sigma \partial(c') + c''.
\end{equation}
Recall (7.8), i.e. for \( t \) defined in (7.7) and for every element \( \lambda \in \bigcup_{j \leq n} F_j \) we have
\begin{equation}
(7.17) \quad \supp_X(\sigma) \subseteq \supp_X(\lambda) \cup (H_0)_{\nu \geq v(\lambda) + t}.
\end{equation}

Hence
\begin{equation}
(7.18) \quad \supp_X(\partial(c')) \subseteq \supp_X(\partial(c')) \cup (H_0)_{\nu \geq v(\partial(c')) + t} \subseteq \supp_X(\partial(c')) \cup (H_0)_{\nu \geq v(c') + t},
\end{equation}
where the last inclusion follows from the fact that
\[ \supp_X(\partial(c')) \subseteq \supp_X(c'). \]

Since \( \partial(c') = z - \partial(c'') \) we get
\begin{equation}
(7.19) \quad \supp_X(\partial(c')) \subseteq \supp_X(z) \cup \supp_X(\partial(c'')).
\end{equation}

Furthermore
\begin{equation}
(7.20) \quad \supp_X(\partial(c'')) \subseteq \supp_X(c'').
\end{equation}

Thus combining (7.18), (7.19) and (7.20) we obtain
\[ \supp_X(\sigma \partial(c')) \subseteq (H_0)_{\nu \geq v(\partial(c')) + t} \cup \supp_X(\partial(c')) \cup \supp_X(\partial(c'') \cup \supp_X(c''). \]

Since \( v(c') = v(g) \) we have \( g \notin (H_0)_{\nu \geq v(\partial(c')) + t} \). Furthermore \( \alpha(\supp_X(c)) = \alpha(\{g\}) > \alpha(\supp_X(z)) \) implies that \( g \notin \supp_X(z) \) and so
\[ g \notin (H_0)_{\nu \geq v(c') + t} \cup \supp_X(z) \cup \supp_X(c'') \]
and we deduce that
\begin{equation}
(7.21) \quad g \notin \supp_X(\sigma \partial(c')).
\end{equation}

In addition by (7.17)
\[ \supp_X(\bar{c}) \subseteq \supp_X(c) \cup \supp_X(\partial(c')) \subseteq \supp_X(c) \cup \supp_X(\sigma(c')) \subseteq \supp_X(c) \cup \supp_X(c') \cup (H_0)_{\nu \geq v(c') + t} = \supp_X(c) \cup (H_0)_{\nu \geq v(c') + t} \subseteq \supp_X(c) \cup (H_0)_{\nu \geq v(c') + t}. \]

Note that \( v(c) = v(c') = -\alpha(\{g\}) \) and \( t > 0 \). Hence
\begin{equation}
(7.22) \quad \supp_X(\bar{c}) \subseteq \supp_X(c) \cup (H_0)_{\nu \geq -\alpha(\{g\}) + t}. \end{equation}

Note that by (7.4)
\[ v(\varphi(c')) > v(c') = -\alpha(\{g\}), \]

hence
\begin{equation}
(7.23) \quad g \notin \supp_X(\varphi(c')).
\end{equation}

Furthermore by the definition of \( c'' \) we have \( g \notin \supp_X(c'') \). Then by (7.16), (7.21) and (7.23) we obtain
\begin{equation}
(7.24) \quad g \notin \supp_X(\bar{c}). \end{equation}

Now for every \( y \in H_0 X_j \) in the decomposition \( c' = \sum z_{y} y \) we have \( g \in \supp_X(y) \), hence
\[ 1_{H_0} \in g^{-1} \supp_X(y). \]
By the definition of $s$ in (7.6) we have that
\[ ||\pi(b)|| \leq s \text{ for } b \in g^{-1}\text{supp}_X(\sigma(\epsilon')). \]

Then by the definition of $\epsilon$ for every
\[ h \in \text{supp}_X(\epsilon) \setminus \text{supp}_X(c) \subseteq \text{supp}_X(\partial \sigma(\epsilon')), \]
hence
\[ g^{-1}h \in g^{-1}\text{supp}_X(\partial \sigma(\epsilon')) \subseteq g^{-1}\text{supp}_X(\sigma(\epsilon')) \]
and by the above
\[ ||\pi(g^{-1}h)|| \leq s. \]

Moving to additive notation in $V$ we have
\[ (7.25) \quad ||\pi(h) - \pi(g)|| \leq s. \]

Recall that by the definition of $h$ we have $h \not\in \text{supp}_X(c)$ and furthermore by (7.18)
\[ h \in \text{supp}_X(\epsilon) \setminus \text{supp}_X(c) \subseteq \text{supp}_X(\partial \sigma(\epsilon')) \subseteq \text{supp}_X(\sigma(\epsilon')) \subseteq \text{supp}_X(c') \cup (H_0)_{\nu \geq \nu(c')} + t \subseteq \text{supp}_X(c) \cup (H_0)_{\nu \geq \nu(c')} + t, \]

hence
\[ h \in (H_0)_{\nu \geq \nu(c')} + t \]
and
\[ (7.26) \quad \nu(h) \geq \nu(c') + t = -\alpha(\{g\}) + t = -r + t. \]

Recall that $g$ belongs to the boundary of the sphere with radius $r = \alpha(\{g\})$ and center $w$ and $\nu$ is defined by (7.15). The conditions (7.25) and (7.26) imply that $h$ belongs to the sphere with radius $r_0$ and center $w$, where by the Pythagorean theorem applied twice
\[ r_0^2 = s^2 - t^2 + (r - t)^2 = r^2 - 2rt + s^2. \]

Let
\[ \epsilon_0 = \frac{1}{2}\min\{s, t\}. \]

Then $r_0 \leq r - \epsilon_0$ is equivalent to $r \geq \epsilon_0$ and $r^2 - 2rt + s^2 \leq (r - \epsilon_0)^2 = r^2 - 2r\epsilon_0 + \epsilon_0^2$. The last is equivalent to $r \geq \frac{s^2 - \epsilon_0^2}{2(t - \epsilon_0)}$. Thus it suffices that
\[ r \geq a = \max\{\frac{s^2 - \epsilon_0^2}{2(t - \epsilon_0)}, \epsilon_0\}. \]
\[ \square \]

8. Renz’s criteria and the homotopical part of Theorem [1.1]

Recall that for a group $G$ of type $F_n$ we have $\Sigma^n(G) = \Sigma^2(G) \cap \Sigma^n(G, \mathbb{Z})$, thus in order to complete the proof of Theorem [1.1] we only need to consider the homotopical invariant in dimension 2. The result that we wish to prove can be stated as follows.

**Theorem 8.1.** Let $[H, H] \subseteq K \subseteq H$ be groups such that $H$ and $K$ are finitely presentable. Let $\chi : K \to \mathbb{R}$ be a character such that $\chi([H, H]) = 0$. Then $[\chi] \in \Sigma^2(K)$ if and only if $[\mu] \in \Sigma^2(H)$ for every character $\mu : H \to \mathbb{R}$ that extends $\chi$.

Note that Theorem [1.5] is not applicable here, so we need to consider both implications of the statement above.
8.1. Extension of characters.

**Proposition 8.2.** Let \([H, K] \subseteq K \subseteq H\) be groups such that \(H\) and \(K\) are finitely presentable. Let \(\chi: K \to \mathbb{R}\) be a character such that \(\chi([H, K]) = 0\) and \([\chi] \in \Sigma^2(K)\). Then \([\mu] \in \Sigma^2(H)\) for every character \(\mu: H \to \mathbb{R}\) that extends \(\chi\).

**Proof.** Notice that \(H/K\) is a finitely generated abelian group. By the homotopical version of Theorem 2.2 (see [28, Cor. 2.7]) we can pass to a finite-index subgroup of \(H\) and assume that \(H/K\) is free-abelian, so \(H\) is built from \(K\) by some extensions by \(\mathbb{Z}\). By induction we may actually assume that \(H = K \times \mathbb{Z}\), and the proposition will follow.

By [28, Thm. B] if \(G\) is an HNN extension with base group \(B\) that is finitely presented and associated subgroup \(A\) that is finitely presented too and \(\mu: G \to \mathbb{R}\) is a character such that \(\mu|_A \neq 0\), \([\mu|_A] \in \Sigma^1(A), [\mu|_B] \in \Sigma^2(B)\) then \([\mu] \in \Sigma^2(G)\). We can apply this for \(G = H = K \times \mathbb{Z}, B = A = K\) to complete the proof. \(\square\)

8.2. Renz's criteria. Let \(G\) be a finitely generated group and let \(X\) be a finite generating set. Suppose that \([\chi] \in S(G)\) is a non-zero character. If \(w = x_1 \cdots x_n\) is a word on the generators \(X^{\pm 1}\), the valuation of \(w\) with respect to \(\chi\) is

\[v_\chi(w) = \min\{\chi(x_1 \cdots x_j) \mid 0 \leq j \leq n\}.\]

**Theorem 8.3.** [31] Let \(G = \langle X \rangle\) with \(X\) finite. Then \([\chi] \in \Sigma^1(G)\) if and only if there is an element \(t \in X^{\pm 1}\) such that \(\chi(t) > 0\) and such that for each \(x \in X^{\pm 1}\) there is a word \(w_x\) that represents \(t^{-1}xt\) in \(G\) and satisfies

\[v_\chi(w_x) > v_\chi(t^{-1}xt).\]

Suppose now that \(G\) is a finitely presentable group and let \(\langle X|R\rangle\) be a finite presentation. Let \(M\) be a (van Kampen) diagram over \(\langle X|R\rangle\). As usual, \(M\) comes equipped with a base point \(x_0\) (a vertex) on its boundary. In the original Renz definition any vertex of \(M\) is labeled by an element of \(G\) as follows: \(x_0\) has label \(1_G\) and if \(x\) is another vertex, then its label is the image \(g \in G\) of the label of any edge-path connecting \(x_0\) and \(x\) inside \(M\).

Now, for any \([\chi] \in S(G)\) and for any diagram \(M\), we can define:

\[v_\chi(M) = \min\{\chi(g) \mid g\ \text{labels a vertex of } M\}.\]

Suppose that \([\chi] \in \Sigma^1(G)\). Let \(t\) and \(\{w_x\}_x\) be as in Theorem 8.3. For a word \(r = x_1 \cdots x_n\), with \(x_i \in X^{\pm 1}\), we put:

\[\tilde{r}_i = w_{x_1} \cdots w_{x_n}.\]

**Theorem 8.4.** [31] Let \(G\) be a finitely presented group and let \(G = \langle X|R\rangle\) be a finite presentation. Let \([\chi] \in \Sigma^1(G)\). With \(t\) and \(\{w_x\}_x\) as above, suppose further that \(R \supseteq \{t^{-1}xtw_x^{-1} \mid x \in X^{\pm 1}\}\). Then \([\chi] \in \Sigma^2(G)\) if and only if for each \(r \in R\), there is a diagram \(M\) such that \(\partial M = \tilde{r}_i\) and \(v_\chi(M) + \chi(t) > v_\chi(r)\).

The boundary \(\partial M\) of \(M\) above is read from the base point in any direction. What we really need is the version of Theorem 8.4 for \(\Sigma^2\), which is the following.

**Theorem 8.5.** [31] Let \(G\) be a finitely presentable group and let \(\Gamma \subseteq S(G)\) be a non-empty compact set. Then \(\Gamma \subseteq \Sigma^2(G)\) if and only if there exist a finite presentation \(G = \langle X|R\rangle\), a finite set \(W\) of words with letters in \(X^{\pm 1}\) and a finite set \(M\) of diagrams over \(\langle X|R\rangle\) such that for each \([\chi] \in \Gamma\), there exists \(t \in X^{\pm 1}\) such that \(\chi(t) > 0\) and
(1) For any \( x \in X^\pm \) there is some \( w_x \in W \) that represents \( t^{-1}xt \) in \( G \) and 
\[ v_\chi(w_x) > v_\chi(t^{-1}xt). \]

(2) For any \( r \in R^\pm \) there is some \( M_{\tilde{r}_t} \in M \) such that \( \partial M_{\tilde{r}_t} = \tilde{r}_t \) and 
\[ v_\chi(M_{\tilde{r}_t}) + \chi(t) > v_\chi(r), \] 
where \( \tilde{r}_t \) is computed as above.

Remark: The element \( w_x \) actually depends on both \( x \) and \( \chi \).

8.3. The converse.

**Proposition 8.6.** Let \([H, H] \subseteq K \subseteq H\) be groups such that \( H \) and \( K \) are finitely presentable. Let \( \chi : K \to \mathbb{R} \) be a character such that \( \chi([H, H]) = 0 \). Suppose that \([\mu] \in \Sigma^2(H)\) for every character \( \mu : H \to \mathbb{R} \) that extends \( \chi \). Then \([\chi] \in \Sigma^2(K)\).

We will use here the same notations as in Section 7.2 except that \( t \) here denotes element of \( X^\pm \) and not some positive real number. In particular, \( H_0 \subseteq H \) is a finite-index subgroup such that \([H, H] \subseteq H_0\) and both \( H_0/[H, H] \) and \( H_0/K_0\) are torsion-free, where \( K_0 = K \cap H_0 \). We will also keep the notations for \( V, W, D, \alpha \) and so on. We denote \( \chi_0 = \chi|_{K_0} \) and \( \mu_0 = \mu|_{H_0} \) for any extension \( \mu : H \to \mathbb{R} \) of \( \chi \).

Again by \cite[Cor. 2.7]{8} we have that \([\mu_0] \in \Sigma^2(H_0)\) for all extensions \( \mu : H \to \mathbb{R} \) of \( \chi \) and, once we show that \([\chi_0] \in \Sigma^2(K_0)\), then also \([\chi] \in \Sigma^2(K)\).

Let \( H_0 = \langle X|R \rangle\) be a finite presentation of \( H_0 \) satisfying the conditions of Theorem 8.5 with respect to
\[ \Gamma = S(H_0, K_0) \cup \{ [\mu_0] \mid \mu_0|_{K_0} = \chi_0 \} \subseteq \Sigma^2(H_0). \]

Thus we are given the finite sets \( W \) and \( M \) of words and diagrams satisfying the required properties (1) and (2) from Theorem 8.5.

We define now some constants that will appear in the sequence. First, let
\[ a_1 = \inf \{ v_\varphi(w_x) - v_\varphi(t^{-1}xt) \}, \]
where the infimum is taken over all \( x \in X^\pm \), \( \varphi \) such that \([\varphi] \in \Gamma\) and \( ||\varphi|| = 1 \), all \( t \in X^\pm \) such that \( \varphi(t) > 0 \), and all \( w_x \in W \) such that \( t^{-1}xt = w_x \) in \( H_0 \) and \( v_\varphi(w_x) > v_\varphi(t^{-1}xt) \). Similarly, we put
\[ a_2 = \inf \{ v_\varphi(M_{\tilde{r}_t}) + \varphi(t) - v_\varphi(r) \}, \]
where the infimum is taken over all \( r \in R^\pm \), \( \varphi \in \Gamma \) with \( ||\varphi|| = 1 \), all \( t \in X^\pm \) such that \( \varphi(t) > 0 \) and all \( M_{\tilde{r}_t} \in M \) such that \( v_\varphi(M_{\tilde{r}_t}) + \varphi(t) > v_\varphi(r) \). Notice that since \( \Gamma \) is a compact set and by the conditions of Theorem 8.5 both \( a_1 \) and \( a_2 \) are positive real numbers. We define
\[ a = \inf \{ a_1, a_2 \}. \]  
(8.1)

We also define
\[ b_1 = \sup \{ \|\pi(tw)\|, \|\pi(x^{-1}tw)\| \}, \]
where the supremum runs over all \( t, x \in X^\pm \) and all initial subwords \( w \) of all \( w' \in W \), and
\[ \pi : H_0 \to Q_0 = H_0/[H, H] \]
is the canonical projection. We define
\[ b_2 = \sup \{ \|\pi(tgh^{-1})\| \}, \]
where the supremum runs over all \( t \in X^\pm \) and all \( g \) and \( h \) are vertices of some \( M \in M \), for all possible \( M \). Finally, we set
\[ b = \sup \{ b_1, b_2, 2a \}. \]  
(8.2)
Let $C$ be the Cayley graph of $H_0$ with respect to the generating set $X$. Define $C_d$ as the full subgraph of $C$ spanned by the vertices $h \in H_0$ such that $\alpha(h) \leq d$, where $\alpha$ was defined in (7.10).

Fix some $0 < \epsilon < a$.

Claim For $d > \max\{\frac{a^2 - a^2}{2(a - 1)}, a\}$, the subgraph $C_d$ is connected.

Proof of the Claim Let $h_0$ be a vertex of $C_d$ and let $\gamma$ be an edge-path in $C$ connecting $1 = 1_{h_0}$ and $h_0$. Let $p$ be the label of $\gamma$ and let $Vert(\gamma)$ be the set of vertices of $\gamma$. If $\alpha(Vert(\gamma)) \leq d$, then $\gamma$ runs inside $C_d$. Otherwise, let $g \in Vert(\gamma) \setminus \{1, h_0\}$ such that

$$\alpha(Vert(\gamma)) = \alpha(\{g\}) = s > d.$$  

As in the proof of Theorem 7.2 there must be some $w \in W_{X \geq 0}$ such that $\|w - \pi(g)\| = s$ and such that the equivalence class of the character $\nu : H_0 \to \mathbb{R}$ defined by

$$\nu(h) = \langle \pi(h), \frac{w - \pi(g)}{\|w - \pi(g)\|} \rangle$$

lies in $\Gamma$. Notice that $\nu$ attains its minimum on $Vert(\gamma)$ at $g$.

Choose $t \in X^\pm$ such that $\nu(t) > 0$ as in Theorem 8.3. Write $p = x_1 \cdots x_n$ for some $x_1, \ldots, x_n \in X^\pm$, so $g = x_1 \cdots x_j$ for some $j < n$. We can find words $w_j, w_{j+1} \in W$ such that $t^{-1} x_i t = w_i$ and $v_\nu(w_i) > v_\nu(t^{-1} x_i t)$ for $i = j, j+1$. Thus the path $\gamma'$ starting at 1 and having label

$$p' = x_1 \cdots x_{j-1} t w_j w_{j+1} t^{-1} x_{j+2} \cdots x_n$$

also ends at $h_0$.

Let $y$ be a vertex in $Vert(\gamma') \setminus Vert(\gamma)$. Thus either $y = gtw'$, where $w'$ is an initial subword of $w_{j+1}$, or $y = gx_j^{-1} tw'$, where $w'$ is an initial subword $w_j$. In any case we have

$$\|\pi(y) - \pi(g)\| \leq b,$$

where $b$ is defined as in (8.2).

On the other hand, we have:

$$\nu(y) = \nu(g) + \nu(t w'),$$

in case $y = gtw'$ and $w'$ is an initial subword of $w_{j+1}$, or

$$\nu(y) = \nu(g) + \nu(x_j^{-1} tw')$$

if $y = gx_j^{-1} tw'$, where $w'$ is an initial subword of $w_j$. Using that $\nu$ attains its minimum on $Vert(\gamma)$ at $g$, we have in any case

$$\nu(y) \geq \nu(g) + a_1 \geq \nu(g) + a,$$

where $a$ is defined in (8.1).

The argument with Pythagoras’ Theorem applied to restrictions (8.3) and (8.4) implies that $\|\pi(y) - w\| < s - \epsilon$, so $\alpha(\{y\}) < \alpha(\{g\}) - \epsilon$. It follows that either $\alpha(Vert(\gamma')) < s - \epsilon$, or $s - \epsilon \leq \alpha(Vert(\gamma')) \leq s$, but $\gamma'$ has less vertices $g'$ with $\alpha(\{g'\}) \geq s - \epsilon$.  


All of this implies that $C_d$ is connected for $d > \max\{\frac{1}{2} - \varepsilon, a\}$. Indeed, if $h \in C_d$ is a vertex and $\gamma$ is any path connecting $1$ and $h$ in $C$, then it can be transformed by some applications of the procedure described above to another path $\gamma_2$ such that 

$$\alpha(\text{Vert}(\gamma_2)) \leq \max\{\alpha(\text{Vert}(\gamma)) - \varepsilon, d\}.$$ 

Then we iterate this until we find $\gamma_n$ such that $\alpha(\text{Vert}(\gamma_n)) \leq d$, so that $1$ and $h$ are connected inside $C_d$. This proves the claim.

**Remark.** This argument actually shows that the full subgraphs $C_{d, \chi_0 \geq j}$ spanned by the vertices $h \in C_d$ such that $\pi(h) \in D_{d, \chi_0 \geq j}$ are connected for any $j$ and for sufficiently large $d$, where $D_{d, \chi_0 \geq j}$ is defined in (7.11). By definition $C_d = C_{d, \chi_0 \geq 0}$ and the proof for $C_d$ works for $C_{d, \chi_0 \geq j}$ with the only difference that the path $\gamma$ starts at a fixed vertex in $C_{d, \chi_0 \geq j}$ instead of $1 = 1_{H_0}$.

Now we consider the Cayley complex $\mathcal{C}$ of $H_0$ associated to the presentation $\langle X|R \rangle$ and the full subcomplex $C_d$, spanned by all $h \in H_0$ such that $\alpha\{\{h\}\} \leq d$, for $d > \max\{\frac{1}{2} - \varepsilon, a\}$. We already know that it is connected since $C_d$ is the 1-skeleton of $C_d$. We want to show that $C_d$ is also simply-connected.

Let $\gamma \subseteq C_d$ be a closed path with basepoint $1$ and label $p = x_1 \cdots x_n$. Since $\mathcal{C}$ is simply-connected, there is a diagram $D$ over $\langle X|R \rangle$ such that $\partial D = \gamma$. If $\alpha(\text{Vert}(D)) \leq d$, we are done. Otherwise there is an internal vertex $g$ such that 

$$\alpha(\text{Vert}(D)) = \alpha\{\{g\}\} = s > d.$$

We consider again the character $\nu : H_0 \to \mathbb{R}$, defined by $\nu(h) = \langle \pi(h), \frac{w - \pi(g)}{\|w - \pi(g)\|} \rangle$, where $w \in W_{\chi_0 \geq 0}$ and $\|w - \pi(g)\| = s$. Recall that $[\nu] \in \Gamma$.

Choose $t \in X^{\pm 1}$ such that $\nu(t) > 0$. Then for each $r \in R^{\pm 1}$ we can find a diagram $M_{\hat{r}} \in \mathcal{M}$ satisfying the conditions of the $\Sigma^2$-criterion, i.e. (2) from Theorem 8.5. This, together with the cells associated to $t^{-1}txw^{-1}$ for the letters $x$ of $r$, gives a new diagram $M'_r$ such that $\partial M'_r = r$ and such that $\nu(h) > v_r(r)$ for all vertices of $M'_r$ that do not lie on the boundary of the diagram.

We transform $D$ as follows: we substitute the cells $e_r$, associated to relators $r$, having $g$ as a vertex, with the diagrams $M'_r$. By canceling the pairs of adjacent cells $e_p$ and $e_{p^{-1}}$ for some relators $p$ of type $t^{-1}txw^{-1}$, we obtain a new diagram $D'$ that has the same boundary but does not contain $g$. This is the same argument as in the proof of the $\Sigma^2$-criterion (Theorem 8.5) in [31]. Any vertex $y$ of $D'$ that was not a vertex of $D$ can be written as $y = gtw'$, where $w'$ is the image in $G$ of the label of a path from a vertex $v_0 = gt$ from the boundary of some $M_{\hat{r}}$ to some other vertex of $M_{\hat{r}}$.

By the definition of $b_2$ we have

$$\|\pi(y) - \pi(g)\| \leq b_2 \leq b.$$

Recall that $r$ is a relation that defines a 2-cell $e_r$, i.e. the closed path $\partial e_r$ has label $r$ and that $g$ is a vertex of $e_r$. Without loss of generality we can assume that $g$ is the base point of this cell (a similar argument works when the base point is different from $g$). Thus if $r = x_1 \cdots x_m$ for $x_1, \ldots, x_m \in X^{\pm 1}$ the consecutive vertices of $\partial(e_r)$ are $g\overline{x_1}, \ldots, g\overline{x_1} \ldots \overline{x_i}, \ldots, g\overline{x_1} \ldots \overline{x_{m-1}}$, where $\overline{x_i}$ is the image of $x_i$ in $G$. Since $\nu(g)$ is minimal among $\{\nu(v) \mid v \in \text{Vert}(D)\}$ we conclude that $v_r(r) = 0$.

Recall that $M_{\hat{r}}$ is attached at the vertex $gt$, that is actually the base point (as $t$ is the base point of $\partial(e_r) = r$). In the original Renz definition a base point is
labeled by $1_G$ but in the Cayley complex it is labeled by $gt$, hence we obtain that
\[ \nu_\nu(M_{r_\nu}) = \min\{\nu(h) - \nu(gt) \mid h \in \text{Vert}(M_{r_\nu})\} \leq \nu(gt'w) - \nu(gt). \]

Thus we have
\[ (8.6) \quad \nu(y) = \nu(gtw') \geq \nu(gt) + \nu_\nu(M_{r_\nu}) \geq \nu(gt) + \nu_\nu(r) - \nu(t) + a_2 = \nu(gt) - \nu(t) + a_2 \geq \nu(gt) - \nu(t) + a = \nu(g) + a. \]

Again by Pythagoras’ Theorem we find that the new vertices $y$ of $D'$ satisfy $\alpha(\{y\}) < \alpha(\{g\}) - \epsilon$. This procedure defines the transformation $D \mapsto D_2$ such that
\[ \alpha(\text{Vert}(D_2)) \leq \max\{\alpha(\text{Vert}(D)) - \epsilon, d\}. \]

We can iterate this to produce a diagram $D_n$ with $\partial D_n = \partial D$ and $\alpha(\text{Vert}(D_n)) \leq d$. This completes the proof of the fact that $C_d$ is simply connected for sufficiently big $d$.

**Remark.** Again, we can use the argument to show that the full subcomplex $C_{d,x \geq j}$ spanned by the vertices of $C_{d,x \geq j}$ is simply-connected for any $j$ and for $d$ big enough.

**Proof of Proposition 8.6** Fix $d$ such that $C_d$ is 1-connected. Let $\tilde{C}_d$ be the full subcomplex of $C$ spanned by the vertices $h$ such that $\pi_0(h) \in \tilde{B}_d$, where $\pi_0 : H_0 \to H_0/K_0 \cong \mathbb{Z}^{k-m}$ is the canonical projection and as defined before in the proof of Theorem 8.2. $\tilde{B}_d$ is the closed ball with center 0 and radius $d$ in $H_0/K_0 \otimes \mathbb{R}$. Notice that $\tilde{C}_d$ is a free $K_0$-complex with finitely many orbits of cells, and it is also connected by the original argument in [30].

Let $\pi_1 : H_0 \to K_0/[H, H]$ be the projection defined in [7.12]. Let $\beta : H_0 \to \mathbb{R}$ be the composite of $\pi_1$ with the induced character $\tilde{\chi}_0 : K_0/[H, H] \to \mathbb{R}$. Then $\beta$ extends to a $\chi_0$-equivariant regular height function (in the sense of [28 Section 2]) defined on $\tilde{C}_d$. We can then consider the valuation subcomplexes $\langle \tilde{C}_d \rangle_{y \geq j}$: this is simply the full subcomplex of $\tilde{C}_d$ spanned by the vertices $h$ such that $\beta(h) \geq j$.

Finally, we can find some fixed negative numbers $j_1 > j_2$, depending only on $d$, such that
\[ \langle \tilde{C}_d \rangle_{y \geq j} \subseteq \langle \tilde{C}_d \rangle_{y \geq j_1} \subseteq \langle \tilde{C}_d \rangle_{y \geq j_2} \]
for all $j$. Since $\langle \tilde{C}_d \rangle_{y \geq j}$ is 1-connected for all $j$, the induced maps
\[ \pi_i(\langle \tilde{C}_d \rangle_{y \geq j}) \to \pi_i(\langle \tilde{C}_d \rangle_{y \geq j_1}) \]
are trivial for $i = 0, 1$. Thus $[\chi_0] \in \Sigma^2(K_0)$ by [28 Theorem A].

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