Abstract. We introduce a new cohomology-theoretic method for classifying generic immersed curves in closed compact surfaces by using Gauss codes. This subsumes a result of J.S. Carter on classifying immersed curves in oriented compact surfaces, and provides a criterion for when an immersion is two-colorable. We note an application to twisted virtual link theory.

1. Introduction

We associate with each generic immersed curve in a closed compact surface an isomorphism class of intersigned Gauss codes, which are multi-component double occurrence sequences of symbols drawn from a fixed alphabet whose symbols alternate with signs, with the ordered signs called a sign sequence. Figure 1 shows an example of such a curve

![Figure 1. An immersed curve with (de)stabilization.](image)

in a non-orientable surface, and a representative of its intersigned Gauss code is

1 + 2 − 3 − 4 + 2 − 3 + 5 − 6 + 7 − 7 + 6 − 5 + 8 + 8 − 4 + 1 −

We have the following.

Theorem 1. Stable geotopy classes of generic immersed curves correspond to isomorphism classes of intersigned Gauss codes.

Date: July 8, 2008.
When the generic immersion of curves in a surface is cellular, it determines an embedded graph which is the one-skeleton of a cellular decomposition of the surface, and then the sign sequence of the intersigned Gauss code is interpreted as a cochain on this one-skeleton. By pulling back the intersection points of the curve to the circles in the domain of the immersion, we can see that the embedded graph is a quotient of the union of cycle graphs, and the sign sequence is a cocycle on this graph. There is an obvious correspondence between the cycle graphs with a cocycle and the intersigned Gauss code. A rotation cochain is a zero-cochain on a cycle graph that assigns opposite signs to the two pullbacks of the intersection points. This situation is illustrated in Figure 2 where a cycle graph with a rotation cochain is shown on the left, and the associated immersion in the plane is shown with the sign sequence on the right. It is clear that the rotation cochain also defines an assignment of opposite signs to the symbols of the associated intersigned Gauss code. Then we have the following:

**Theorem 2.** Let $C$ be a curve cellularly immersed in a closed surface $\Sigma$, $\Pi$ be an intersigned Gauss code for $C$, and $r$ a rotation cochain on $\Pi$. The sign sequence of $\Pi$ added mod 2 to the coboundary of $r$ is a cocycle representing $w_1$ of $\Sigma$.

This has the immediate corollary:

**Corollary 1.** The closed surface $\Sigma$ is orientable if and only if the sign sequence of $\Pi$ plus the coboundary of $r$ is a mod 2 coboundary.

In contrast, for the two-colorability of a cellular immersion, we have the following:

**Theorem 3.** Let $C$ be a curve cellularly immersed in a closed surface $\Sigma$ and $\Pi$ be an intersigned Gauss code for $C$. The mod 2 complement...
of the sign sequence of $\Pi$ is a cocycle representing the mod 2 Poincaré dual of the mod 2 homology class of $C$.

Then the theorem has the immediate corollary:

**Corollary 2.** The immersion $C$ is two-colorable if and only if the mod 2 complement of the sign sequence of $\Pi$ is a mod 2 coboundary.

## 2. Background

### 2.1. Gauss Codes.

Gauss introduced the concept of a Gauss code to describe a normal immersion of a circle in the plane [Gau73, pp. 271–286]. Such a code is obtained from an immersion by labeling its crossings and reading them off starting at an arbitrary non-crossing point on the curve.

A **Gauss code** is a finite collection of (non-empty) finite sequences of symbols drawn from a set such that every symbol of the set occurs exactly twice in the sequences of the collection. Two Gauss codes over the same symbols are equivalent under permutation of the set of symbols, and rotation and reflection of the sequences. Each finite sequence is a **component** of the Gauss code.

Gauss codes with one component are sometimes called Gauss words [Car91], double occurrence words or sequences [Ros76], and cross codes. And when multi-component Gauss codes have been used to classify a normal immersion of a disjoint union of circles in an oriented closed compact surface, they have been called Gauss paragraphs [Car91].

A Gauss code does not uniquely specify a collection of planar curves as there are two ways of realizing each crossing in the immersion surface. If the curves are oriented, then at every crossing, one strand is traversed left to right by the other strand, while the other strand is traversed right to left by the first strand. If the orientation of the components is fixed and opposite signs are associated with each symbol of the code to indicate the direction of traversal, the Gauss code is called a **signed Gauss code**, an **intersection sequence** [Fra69], or a **lacet** [CR01]. Signed Gauss codes were used by J.S. Carter [Car91] to classify stable geotopy classes of generic immersions of curves in closed oriented surfaces of higher genus.

As Gauss pointed out, some Gauss codes do not have an associated planar diagram. For example, the Gauss code 1212 is not planar. Gauss identified a necessary condition for a Gauss code to be that of a planar diagram, namely that there be an even number of symbols between the two occurrences of any single symbol in the code. This condition is sometimes called *evenly intersticed* or *evenly interlaced*. That this condition is not sufficient to characterize planar Gauss codes is shown by
the Gauss code 1234534125. Gauss conjectured that the interlacement structure of a Gauss code would determine whether it has a planar diagram. This conjecture was first proved by Rosenstiehl [Ros76].

2.2. Lacets. One direction of development in the study of Gauss codes came from Lins, Richter, and Shank [LRS87] as an extension of the search for combinatorial conditions for the planarity of Gauss codes. In that paper, the authors developed an algebraic approach to determine the possible surfaces in which a given Gauss code is associated with a two-colorable immersed curve based on a partition of the symbols of the Gauss code. These two-colorable immersions of circles have been called lacets [CR01]. Cohomology-theoretic obstructions have been developed to determine conditions under which a Gauss code can be presented as a lacet in a particular surface. These depend upon the structure of the interlacement graph of a Gauss code. This graph has for vertices the symbols of the Gauss code, and has an edge between two vertices whose symbols are interlaced in the Gauss code. Two symbols 1 and 2 in a Gauss code are interlaced if they appear in a pattern of the form \[ \cdots 1 \cdots 2 \cdots 1 \cdots 2 \cdots \]. Typical results include the following theorem from [LRS87].

**Theorem** (Lins et al. (1987)). *The two-colorable immersions of a Gauss code are in an orientable surface if and only if every vertex of its interlacement graph is even-valent.*

So the Gauss code “1234534125” has 2-colorable immersions in orientable surfaces while the Gauss code “1212” has 2-colorable immersions in non-orientable surfaces.

3. Definitions

A *generic immersion* of curves is the immersion of a disjoint union of circles in a surface such that the only intersections are transverse double points. Two generic immersion of curves are *geotopic* if there is a homeomorphism between the immersion surfaces that takes the image of one immersion onto the image of the other immersion. The immersions are *stably geotopic* if there is a collection of one-handles and crosscaps that can be added to (or removed from) either surface in the complement of the immersions such that the resulting immersions are geotopic. An immersion is *cellular* if its complement is homeomorphic to a collection of open disks. It is *two-colorable* if the components of its complement can be assigned one of two colors such that the arcs between intersections always separate one color from the other.
A *generic immersion* of oriented curves is a generic immersion of curves where the circles are oriented, and then two generic immersions of oriented curves are *geotopic* when the homeomorphism preserves the orientations.

*Signed Gauss codes* are presentations of permutations $P$ of the signed symbols $\{1^\pm, \ldots, n^\pm\}$ over an alphabet $\{1, \ldots, n\}$. The *components* of a signed Gauss code are the orbits of $P$. Two signed Gauss codes are *isomorphic* if one can be obtained from the other by a cyclic permutation of any component, by permuting the order of the components, by permuting the alphabet, by changing all signs to their opposite, or by reversing any component while complementing the signs on all symbols that occur on two components where only one component is reversed. For example, the three component signed Gauss code

$$1^-2^+3^-1^-2^-3^+$$

is isomorphic to

$$1^+2^-3^-1^+2^-3^-2^+.$$ 

*Oriented intersigned Gauss codes* are multi-component double occurrence sequences over an alphabet $\{1, \ldots, n\}$ whose symbols alternate with signs, where the sequence of signs is called the *sign sequence*. Two oriented intersigned Gauss codes are *isomorphic* if one can be obtained from the other by a cyclic permutation of any component, by permuting the order of the components, or by permuting the alphabet. For example, the two component intersigned Gauss code

$$1^-2^+1^-3^-2^-3^-$$

is isomorphic to

$$2^+3^-1^-3^-2^-3^-.$$ 

*Intersigned Gauss codes* are oriented intersigned Gauss codes but have the additional isomorphism of reversing any component while at the same time changing the signs between all pairs of symbols where one symbol occurs on a reversed component and the other does not. For example, the two component intersigned Gauss code

$$1^-2^-3^+1^-2^-3^-$$

is isomorphic to

$$1^-3^-2^-1^-2^-3^-.$$ 

Unless otherwise said, graphs may have multiple edges and loops. An *Euler partition* of a graph is a collection of closed paths that together go over every edge exactly once. A graph admits an Euler partition if and only if it is even-valent. An embedding of a graph is an embedding of the corresponding one-complex, and induces a cyclic order of the edges at every vertex. An embedding of a graph that has an Euler partition is *straight-through* at a vertex with respect to that partition if for all edge pairs determined at that vertex by the closed paths, the two edges have the same number of other edges between them in either direction of the cyclic order of the edges at that vertex induced by the embedding of the graph. An embedding is called just straight-through
if it is straight-through at every vertex. An embedding that is not
straight-through at a vertex is bent at that vertex.

An embedding scheme for a graph consists of a pair \((\pi, \lambda)\) of a rotation
system \(\pi\) which is a set of cyclic permutations \(\pi_v\) of the edges incident
with a vertex \(v\), and a signature \(\lambda\) from the set of edges \(e\) to \(\pm 1\) [MT01].
Each embedding scheme for a graph encodes a cellular immersion for
the graph, where the complement of the image of the graph is home-
omorphic to a collection of open disks. The map between embedding
schemes and cellular immersions is as follows. Disjoint neighborhoods
of each vertex in the graph are chosen. The neighborhood of each vertex
is embedded in an oriented disk with boundary such that the counter-
clockwise order of the vertex’s incident edges on the boundary is given
by its permutation. For each disk, disjoint half-disk neighborhoods of
the intersection of an edge with the boundary of the disk are chosen,
and obtain their orientation from that of the disk. Each pair of neigh-
borhoods intersecting an edge are identified by orientation-preserving
(resp. orientation-reversing) homeomorphisms if the edge’s signature is
+ (resp. –). This extends the embedding of the vertices in disks to the
graph in a compact surface. The resulting surface is closed by attaching
disks to its boundary’s components. Conversely, given an embedded
graph in a closed compact surface, choose a collection of disjoint ori-
ented closed neighborhoods homeomorphic to disks about the vertices
to determine a rotation system. The signature is determined by propagat-
ing the orientations of the disks along tubular neighborhoods of
the edges, and is + when the orientations agree and – otherwise. An
edge is twisted with respect to an embedding scheme if its signature
is –. Every cellular embedding of a graph in a surface is uniquely
determined, up to homeomorphism, by an embedding scheme. Two
embedding schemes for a graph are equivalent if one can be obtained
from the other by reversing some of the cyclic permutations along with
inverting the signatures on their incident edges. Equivalence classes of
embedding schemes for two graphs are isomorphic if there is an iso-
morphism between the graphs that respects the rotation system and
signature of representative embedding schemes.

4. The Classification of Immersed Curves

We begin by classifying cellular immersions of oriented curves, then
we classify cellular immersions of curves, and finally we use surgeries
to obtain a classification for all generic immersed curves.

Lemma 1. Geotopy classes of generic cellularly immersed curves cor-
respond to isomorphism classes of straight-through embedding schemes.
Proof. Generic immersions of unoriented curves can be associated with four-regular embedded graphs in an obvious way, so all generic cellu-
larly immersed curves can be determined from the equivalence classes of embedding schemes of their graphs. A graph that comes from a generic immersion of curves has associated with it an Euler partition. Since our im-
mersions are transverse, we need only consider embedding schemes that are straight-through with respect to the Euler partition. □

Since the embedding schemes are straight-through, we have two choices of permutation for each intersection and two choices of sign for each edge. If we fix the choices of rotations for all intersections of generic immersions of curves that have isomorphic graphs, we can determine a cellular generic immersion of curves by the choice of edge signs. The problem is then to find a presentation of the class of embedding schemes for generic immersions of curves that is independent of the choice of rotation system.

Lemma 2. Geotopy classes of generic cellularly immersed oriented curves correspond to isomorphism classes of oriented intersigned Gauss codes.

Proof. Given an oriented intersigned Gauss code, define a union of cycle graphs with two distinct vertices for each symbol in the alphabet, and an edge for each pair of cyclically adjacent symbols in some component, where the edge goes between the distinct vertices associated with the symbols. Define a four-regular graph as the quotient of the cycle graphs that identifies each pair of distinct vertices that correspond to a symbol. Since the edges of the cycle graphs are in one-to-one correspondence with those of the four-regular graph, by abuse of notation we will identify the edges of the cycle graphs and those of the four-regular graph.

We define an embedding scheme for the four-regular graph by first choosing a straight-through rotation system. This defines a rotation cochain on the cycle graphs that assigns a + to the pullback of a vertex whose outgoing edge is sent by the rotation system to the vertex’s other outgoing edge, and a − to the other pullback of the vertex. Since the sign sequence of the Gauss code is a one-cocycle of the cycle graphs, we add it mod 2 to the coboundary of the rotation cochain and define it to be the embedding scheme’s signature. Since the mod 2 sum of two rotation cochains is the pullback of a zero-cochain on the embedded graph, any another choice of rotation cochain would produce an equivalent embedding scheme. Any other isomorphic oriented intersigned Gauss code will have an isomorphic graph, and since the above map...
does not depend upon the presentation of the Gauss code, it will have an isomorphic embedding scheme.

Conversely, given a generic cellular immersion of oriented curves in a surface, label the \( n \) intersections with symbols from an alphabet \( \{1, \ldots, n\} \). By Lemma 4, we can obtain a representative of an equivalence classes of embedding schemes. The choice of rotation system along with the orientation of the curves determines a + sign for the occurrence of the symbol whose outgoing edge is sent to the other occurrence’s outgoing edge, and a − sign for the other occurrence. This is a rotation cochain. We can then read off an intersigned Gauss code whose sign sequence is the mod 2 sum of the embedding scheme’s signature and the coboundary of the rotation cochain. This is illustrated in Figure 3. Changing the choices of assignment of symbols to intersections, representative embedding scheme, starting point on the curves, and order of curves in reading the intersigned Gauss code will produce an isomorphic oriented intersigned Gauss code.

We characterize the differences between the intersigned Gauss codes for generic cellular immersions of oriented curves that differ only in the orientations of their curves.

**Lemma 3.** Two generic cellular immersions of oriented curves that differ only in the orientation of one component have oriented intersigned Gauss codes that differ in the reversal of the corresponding component while changing the signs between all pairs of symbols where one symbol occurs on a reversed component and the other does not.

**Proof.** Reversing only the direction on a component while keeping fixed the other choices will assign the opposite straight-through rotation
to the intercomponent intersections on that component, and this will change the signs between those intersections and other intersections.

It immediately follows that geotopy classes of generic cellularly immersed curves correspond to isomorphism classes of intersigned Gauss codes.

We consider general generic immersions of curves.

Lemma 4. Stable geotopy classes of generic immersions of curves have a unique cellular immersion.

Proof. The complement of the image of a generic immersion of oriented curves in a surface is homeomorphic to a set of open surfaces, and the individual closure of each component has a collection of circles for a boundary. Then, a collection of the same number of open disks may be substituted for the surface components giving the unique cellular immersion.

Proof of Theorem 1. By Lemma 4 any generic immersion of curves has a unique cellular immersion from which we may obtain an isomorphism class of intersigned Gauss codes. Since the stabilizations occur on the complement of the immersed curve, they do not change the isomorphism class of intersigned Gauss codes.

5. Orientable Surfaces

One lemma is needed:

Lemma 5. The signature of an embedding scheme is a representative of $w_1$ of the associated surface.

Proof. Fix an embedding scheme for a given surface. The signature of the embedding scheme is a one-cochain in the cellular cohomology of the surface that, along with the rotation scheme, determines the face cycles. A one-cochain is a cocycle of the surface if and only if it has an even number of negative edges on every face cycle. Since the face cycles are the boundaries of disks, they must be orientation-preserving, and so have an even number of twisted edges along their path. Obtain another embedding scheme from the surface by choosing the same orientation for every disk containing a vertex, so that the twisting of the edges determines the signature of the scheme. For this scheme, there is an even number of negative edges along each face cycle determined by the signature. The two embedding schemes differ by the choice of the permutation that represents the order of the edges about the vertex, and where they differ defines a zero-cochain of the
surface. Then, the signatures of the two embedding schemes differ by the coboundary of this cochain, so the original signature also had an even number of negative edges along each face cycle.

By cellularity, any cycle in the surface is homologous to a cycle on the immersed curve, and a cycle in the immersed curve is orientation-preserving if and only if it has an even number of twisted edges along its path. But this is true if and only if the path has an even number of negative edges determined by the signature. □

The proof of the characterization of the orientability of the immersion surface is now straightforward.

Proof of Theorem 2. Since the signature of an embedding scheme is a representative of $\omega_1$ of the surface, then the surface is orientable if and only if the signature is a coboundary. This is true if and only if the mod 2 sum of the sign sequence of the intersigned Gauss code with the coboundary of a rotation cochain is a coboundary of a pullback cochain, or, in other words, if the sign sequence of the intersigned Gauss code is a coboundary of the mod 2 sum of a rotation cochain and a pullback cochain. Since this sum is itself a rotation cochain, we are done. □

We now provide a map from the signed Gauss codes used by Carter and intersigned Gauss codes. Clearly, the signs of signed Gauss codes define a rotation cochain on the associated cycle graph. Then, the sign sequence of the corresponding intersigned Gauss code is the coboundary of that rotation cochain. By Theorem 2, the associated surface is orientable, and by construction, the associated cellular immersion is geotopic.

6. Two-Colorable Immersions

In this section, unless otherwise indicated, all graphs are four-valent embedded graphs, except that the lifts of such graphs are lifts to cycle graphs whose components map to straight-through closed paths in the graphs.

Proof of Theorem 5. To show that the mod 2 complement of the signature is a cocycle representing the mod 2 Poincaré dual of the mod 2 homology class of $C$, we use the fact that if a cohomology class represented by a cocycle $Z$ is the dual of the homology class represented by a cycle $z'$, then

$$Z(z) = z' \cdot z,$$

for all cycles $z$. 
By cellularity, any cycle in the surface is homologous to a cycle on the immersed curve. We associate a cycle on the immersed curve with a homologous cycle chosen such that:

- it is an $\epsilon > 0$ push-off from the curve in the direction of a vector translated along the curve,
- intersects with the immersed curve once at every straight-through crossing,
- intersects with the immersed curve an even number of times at a bent crossing, and
- intersects the immersed curve one final time if the cycle is orientation-reversing.

This is illustrated in Figure 4. Then for $C$ being the immersed curve by

![Figure 4. Choosing a homologous cycle.](image)

a map $L$ and $z$ a cycle that meets the above conditions, we have that:

$$C \cdot z = \#\text{straight-throughs}(z) + w_1(z) \pmod{2}.$$  

Let $r$ be a rotation cochain and $S(\Pi)$ be the one-cochain determined by the sign sequence of $\Pi$. We have:

$$S(\Pi) + 1 = (S(\Pi) + \delta r) + (1 + \delta r) \pmod{2},$$

and we know from the proof of Theorem 2 that $[S(\Pi) + \delta r] = w_1$. It is clear that:

$$1(z) = \#\text{arcs}(z) \pmod{2}.$$  

By identifying the arcs of the embedded curve with those of the pullback by $L$ to the domain circles, we have:

$$\delta r(L^{-1}(z)) = r(\partial L^{-1}(z)) = \#\text{bends}(z) \pmod{2}$$
because $r$ has opposite values on the two lifts of a double point. Then:

$$1 + \delta r = \# \text{straight-throughs} \pmod{2}.$$  

Since the faces of an immersion are orientation-preserving paths that bend at every crossing, then $S(\Pi) + 1$ is a cocycle, and it represents the mod 2 Poincaré dual of $C$. □

7. Related Work

As shown in [Bou08], when the immersion surface of a curve is thickened by its orientation $I$-bundle, it becomes possible to resolve the intersection points of the curve to obtain a link in an orientable three-manifold, and then the immersed curve becomes a diagram of this link.

We have classified these diagrams using intersigned link codes, which are complemented intersigned Gauss codes with the addition of a writhe at every crossing. A Reidemeister move on an intersigned link code is one of the abstractly-defined moves:

R-1 : $ab \sim a1^\varepsilon + 1^\varepsilon b$

R-2 : $a1^\varepsilon + 2\varepsilon b1^\varepsilon - 2\varepsilon c - 3\varepsilon d \sim abc \sim a1^\varepsilon + 2\varepsilon b2^\varepsilon + 1^\varepsilon c$

R-3 : $a1^\varepsilon + 2\varepsilon b3^\varepsilon - 2\varepsilon c1^\varepsilon - 3\varepsilon d \sim a1^\varepsilon + 2\varepsilon b1^\varepsilon - 3\varepsilon c3^\varepsilon - 2\varepsilon d$

$$a1^{-\varepsilon} + 2\varepsilon b3^\varepsilon - 2\varepsilon c1^{-\varepsilon} - 3\varepsilon d \sim a1^\varepsilon + 2\varepsilon b1^\varepsilon - 3\varepsilon c3^\varepsilon - 2\varepsilon d$$

In a move, intersigned link codes are presented left-right, the crossing numbers are assigned to reflect the order in which crossings are encountered, the writhe mark is given as an exponent $\varepsilon = +$ or $-$ such that $-\varepsilon =$, respectively, $-$ or $+$, and the lowercase letters $a, b, c, \ldots$ represent segments of the code in between which the fragments are embedded. Two intersigned link codes are Reidemeister equivalent if there is a sequence of Reidemeister moves or intersigned Gauss code isomorphisms that takes one code to the other. The resulting theory of twisted virtual links is a proper extension of both Lou Kauffman’s virtual link theory [Kau99] and Yu. V. Drobotukhina’s projective link theory [Dro90].

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