MICHEL TRUSS TYPE THEORIES AS A $\Gamma$-LIMIT OF OPTIMAL DESIGN IN LINEAR ELASTICITY

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Abstract. We show how to derive (variants of) Michell truss theory in two and three dimensions rigorously as the vanishing weight limit of optimal design problems in linear elasticity in the sense of $\Gamma$-convergence. We improve our results from [Olb17] in that our treatment here includes the three dimensional case and that we allow for more general boundary conditions and applied forces.

1. Introduction

In the present article we improve our results from [Olb17], where we have derived a certain form of Michell truss theory as the vanishing weight limit of optimal design problems in linear elasticity in a rigorous fashion. The improvement that we present here is twofold: First, we extend the analysis to the three-dimensional case. Second, we allow for more general applied forces, see Remark 1 (v) below.

We briefly explain how the variational problem for finite values of the “weight” parameter that we will present in this introduction can be interpreted as an optimal design problem in linear elasticity in Section A of the appendix. For a short discussion of how our limit problem can be considered as the Michell truss problem (at least for the case of two dimensions), see Section B of the appendix and [BGS08]. Michell trusses, first devised more than a century ago [Mic04], are a very popular model in applied mathematics and engineering, see e.g. [Hem73, Roz12, LSG18].

On a formal level, the relation between these variational models – in both two and three dimensions – had been observed by Allaire and Kohn [AK93]. As in [Olb17], our statements should be viewed as rigorous versions of their formal ones, in the framework of $\Gamma$-convergence.

1.1. Notation. Let $\mathcal{L}^d$ denote the $d$-dimensional Lebesgue measure, and $\mathcal{H}^d$ the $d$-dimensional Hausdorff measure. Let $E \subset \mathbb{R}^n$ be either open or closed. By $\mathcal{M}(E)$ (respectively $\mathcal{M}(E;\mathbb{R}^p)$) we denote the space of Borel signed measures on $E$ (respectively $\mathbb{R}^p$-valued Borel measures). We denote the symmetric $d \times d$ matrices by $\mathbb{R}^{d \times d}_{\text{sym}} = \{ A \in \mathbb{R}^{d \times d} : A^T = A \}$. The space $\mathcal{M}(E;\mathbb{R}^{d \times d}_{\text{sym}})$ is the subspace of $\mu \in \mathcal{M}(E;\mathbb{R}^{d \times d})$ satisfying $\mu_{ij} = \mu_{ji}$ for all $i,j \in \{1,\ldots,d\}$.

The set of non-negative Borel measures is denoted by $\mathcal{M}^+(E)$.

For $\Omega \subset \mathbb{R}^n$ open and bounded with Lipschitz boundary, consider $\mu \in \mathcal{M}(\Omega;\mathbb{R}^n)$ and $g \in \mathcal{M}(\Omega)$. We say that $-\text{div } \mu = g$ if

$$\int_\Omega \sum_i \partial_x \varphi_i d\mu_i = \int_\Omega \varphi d g$$

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for every compactly supported \( \varphi \in C^1(\mathbb{R}^n) \). Put differently, the measures \( \mu \) and \( g \) are being viewed as measures on \( \mathbb{R}^n \) with support on \( \overline{\Omega} \). When \( \mu \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^{n \times n}_{\text{sym}}) \) and \( g \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^n) \), then \( -\text{div} \mu = g \) has to be understood row-wise.

Let \( 1 < p < \infty \), and \( U \subset \mathbb{R}^n \) open. By \( W^{-1,p}(U) \), we denote the dual of \( W_0^{1,p'}(U) \), where \( (p')^{-1} = 1 - p^{-1} \). It is well known that the following norm on \( W^{-1,p}(U) \) is equivalent to the norm as a dual space of \( W_0^{1,p'} \):

\[
\|g\|_{W^{-1,p}(U)} = \inf \left\{ \|\alpha\|_{L^p(U)} + \|\beta\|_{L^p(U)} : g = \alpha + \text{div} \beta \right\},
\]

where the equation \( g = \alpha + \text{div} \beta \) has to be understood in the sense of distributions,

\[
\langle g, \varphi \rangle = \int_U (\varphi \alpha - \nabla \varphi \cdot \beta) \, dx
\]

for all \( \varphi \in C_c^1(U) \).

By slight abuse of notation, we will write

\[
\mathcal{M} \cap W^{-1,p}(\overline{\Omega}) \equiv \mathcal{M}(\overline{\Omega}) \cap W^{-1,p}(\mathbb{R}^n).
\]

For \( \lambda > 0 \), we define \( \tilde{h}_\lambda : \mathbb{R}^{n \times n}_{\text{sym}} \to \mathbb{R} \) by

\[
(1)
\tilde{h}_\lambda(\sigma) = \begin{cases} 
\frac{\log |\sigma|^p}{p} + \sqrt{\lambda} & \text{if } \sigma \neq 0 \\
0 & \text{if } \sigma = 0,
\end{cases}
\]

where \(| \cdot |\) denotes the Frobenius norm defined by \(|A|^2 = \text{Tr} A^T A\). Now let \( g \in \mathcal{M} \cap W^{-1,2}(\overline{\Omega}; \mathbb{R}^n) \), to be thought of as the applied forces and normal component of the stress \( \sigma \) at the boundary respectively (see Remark 1 (iii) below). We define \( G_{\lambda,g} : \mathcal{M}(\overline{\Omega}; \mathbb{R}^{n \times n}_{\text{sym}}) \to \mathbb{R} \) by

\[
G_{\lambda,g}(\mu) := \left\{ \int_{\Omega} \tilde{h}_\lambda \left( \frac{d\mu}{|d\mu|} \right) \, dx : \mu \ll \mathcal{L}^n, \frac{d\mu}{|d\mu|} \in L^2(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \text{ and } -\text{div} \mu = g \text{ in } \overline{\Omega} \right\},
\]

where \( \mu \ll \mathcal{L}^n \) is the notation for \( \mu \) being absolutely continuous with respect to \( \mathcal{L}^n \), and \( \frac{d\mu}{|d\mu|} \) denotes the Radon-Nikodým derivative of \( \mu \) with respect to \( \mathcal{L}^n \).

The variational functional \( G_{\lambda,g} \) defines an optimal design problem in linear elasticity, see Section 3 of the appendix.

For \( \sigma \in \mathbb{R}^{n \times n}_{\text{sym}} \), let \( \sigma_i, i = 1, \ldots, n \) denote the eigenvalues of \( \sigma \), ordered such that \( |\sigma_1| \leq |\sigma_2| \leq \cdots \leq |\sigma_n| \).

From now on, we will only be concerned with the case \( n = \{2,3\} \). For \( n = 2 \) we define

\[
\rho^{(2)}(\sigma) = |\sigma_1| + |\sigma_2|
\]

and for \( n = 3 \), we let

\[
\rho^{(3)}(\sigma) = \begin{cases} 
\frac{1}{2} \sqrt{(|\sigma_1| + |\sigma_2|)^2 + |\sigma_3|^2} & \text{if } |\sigma_1| + |\sigma_2| \leq |\sigma_3| \\
\frac{1}{2} \sqrt{(|\sigma_1| + |\sigma_2|)^2 + |\sigma_3|^2} & \text{else.}
\end{cases}
\]

Note that \( \rho^{(n)} \) is positively one-homogeneous; hence for a \( \mathbb{R}^{n \times n}_{\text{sym}} \)-valued Radon measure \( \mu \), \( \rho^{(n)}(\mu) \) can be defined as a Radon measure via

\[
\rho^{(n)}(\mu)(A) = \int_A \rho^{(n)} \left( \frac{d\mu}{|d\mu|} \right) \, d|\mu|,
\]
Convergence

Assume that $\lambda$ is the Radon-Nikodym derivative of $\mu$ with respect to its total variation measure $|\mu|$. For $g \in M(\Omega; \mathbb{R}^n)$, we define $G_{\infty,g} : M(\Omega; \mathbb{R}^{n \times n}) \rightarrow \mathbb{R}$ by

$$G_{\infty,g}(\mu) = \begin{cases} 2 \int_{\Omega} d\rho^{(2)}(\mu) & \text{if } n = 2, -\text{div} \sigma = g \\ \sqrt{2} \int_{\Omega} d\rho^{(3)}(\mu) & \text{if } n = 3, -\text{div} \sigma = g \\ +\infty & \text{else.} \end{cases}$$

1.2. Statement of results. We are ready to state our main theorem, namely the $\Gamma$-convergence $G_{\lambda,g} \rightharpoonup G_{\lambda,g}$ under the assumption of weak-* convergence of the applied forces, $g_{\lambda} \rightharpoonup g$ with a $\lambda$-dependent control of $\|g_{\lambda}\|_{W^{1,2}}$. We recall that $\Omega \subset \mathbb{R}^n$ is bounded open with Lipschitz boundary, where $n \in \{2, 3\}$.

**Theorem 1.** Assume that $g_{\lambda} \in M(\Omega; \mathbb{R}^{n \times n})$, $g \in M(\Omega; \mathbb{R}^n)$ such that $g_{\lambda} \rightharpoonup g$ in $M(\Omega; \mathbb{R}^n)$ and

$$\lambda^{-1/4}\|g_{\lambda}\|_{W^{1,2}(\mathbb{R}^n; \mathbb{R}^n)} \rightarrow 0.$$ 

(i) (Compactness) Let $\mu_{\lambda} \in M(\Omega; \mathbb{R}^{n \times n})$ be a sequence such that

$$\limsup_{\lambda} G_{\lambda,g_{\lambda}}(\mu_{\lambda}) < \infty.$$ 

Then there exists $\mu \in M(\Omega; \mathbb{R}^{n \times n})$ such that

$$\mu_{\lambda} \rightharpoonup \mu \text{ in } M(\Omega; \mathbb{R}^{n \times n}).$$

(ii) (Lower bound) Let $\mu_{\lambda} \rightharpoonup \mu$ in $M(\Omega; \mathbb{R}^{n \times n})$. Then we have that

$$\liminf_{\lambda \rightarrow \infty} G_{\lambda,g_{\lambda}}(\mu_{\lambda}) \geq G_{\infty,g}(\mu).$$

(iii) (Upper bound) Let $\mu_{\lambda} \in M(\Omega; \mathbb{R}^{n \times n})$. Then there exists a sequence $\mu_{\lambda} \in L^2(\Omega; \mathbb{R}^{n \times n})$ such that $\mu_{\lambda} \rightharpoonup \mu$ in $M(\Omega; \mathbb{R}^{n \times n})$ and additionally

$$\limsup_{\lambda \rightarrow \infty} G_{\lambda,g_{\lambda}}(\mu_{\lambda}) \leq G_{\infty,g}(\mu).$$

**Remark 1.** (i) The proofs for the compactness and upper bound parts are fairly straightforward; the most interesting part is the lower bound part. Here our proof is inspired by the theorem on lower semicontinuity for linear growth functionals under PDE constraints by Arroyo-Rabasa, De Philippis and Rindler [ARDPR18]. Their work in turn builds on the properties of singular points of $A$-free measures [DPR16], the blow-up technique by Fonseca and Müller [FM93], and properties of the projection operator to $A$-free functions proved by the same authors in [FM99].

(ii) It is the combination of the blow-up technique with the application of the projection operator to $A$-free measures that informs our choice of assumptions for the convergence of the right hand sides, i.e.

$$g_{\lambda} \rightharpoonup g \text{ in } M(\Omega; \mathbb{R}^n) \quad \text{and} \quad \lambda^{-1/4}g_{\lambda} \rightarrow 0 \text{ in } W^{1,2}(\mathbb{R}^n; \mathbb{R}^n).$$

These assumptions (or slightly weaker ones) are necessary in order for this method of proof to work. Also in the proof of the upper bound the assumption on the growth of the $W^{1,2}$ norms is heavily used. It is not clear to us if the statements remain true if this assumption is removed.
(iii) The constraint equation \(-\text{div} \mu = g\) contains boundary conditions and applied forces at the same time. To substantiate this claim, we consider the situation 
\[ \mu = a \mathcal{L}^n \mathbb{I} \Omega \] with \(a \in W^{1,1}(\Omega; \mathbb{R}^n)\) and 
\[ g = b \mathcal{L}^n \mathbb{I} \Omega + c \mathcal{H}^{n-1} \mathbb{I} \partial \Omega \] with \(b \in L^1(\Omega), c \in L^1(\partial \Omega)\). Then the equation \(-\text{div} \mu = g\) translates to
\[
\begin{cases}
  -\text{div} a = b & \text{in } \Omega \\
  a \cdot n = c & \text{on } \partial \Omega,
\end{cases}
\]
where \(n\) denotes the unit outer normal to \(\partial \Omega\).

(iv) As an example for the approximation of applied forces, consider a point force 
\[ g = \sum_i g_i \delta x_i \] (with \(x_i \in \Omega, g_i \in \mathbb{R}^n\), and where \(\delta x_i\) denotes the Dirac measure supported in \(\{x\}\) which is permitted in the Michell truss problem in the sense that there exists a measure \(\mu \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n})\) satisfying 
\[ g = -\text{div} \mu. \] Meanwhile, any such \(\mu\) cannot be absolutely continuous with respect to \(\mathcal{L}^n\) with \(d\mu/d\mathcal{L}^n \in L^2\), which implies that such \(\mu\) is not permissible in the linear elasticity problems in the sense that \(G_{\lambda, \bar{g}}(\mu) = +\infty\). A suitable approximation of the limit problem is given by some sequence \(g_j\) satisfying \(g_j \in W^{-1,2}\) and \(2\): this can be easily achieved, e.g., by mollification.

(v) In [Olb17], we only allowed for right hand sides of a very particular form. Namely, the stresses \(\mu\) for the optimal design problems had to be solutions of boundary value problems
\[
\begin{cases}
  -\text{div} \mu = 0 & \text{in } \Omega \\
  \mu \cdot n = \bar{g}_\lambda & \text{on } \partial \Omega
\end{cases}
\]
where \(\bar{g}_\lambda \in W^{-1/2,2}(\partial \Omega)\), and \(n\) denotes the unit outer normal to \(\partial \Omega\). Additionally, we required that \(\Omega\) be simply connected and piecewise \(C^2\). By (iii) above, this is just a special case of the right hand sides that we are treating here. It was our method of proof that limited us to right hand sides that correspond to \(4\) in [Olb17]. There we reformulated the problem as one for \(BV\) functions, which moreover is only possible in two dimensions.

(vi) One can interpret the penalization parameter \(\lambda\) as a Lagrange multiplier enforcing a constraint on the mass of the elastic body in the minimization problem for the compliance defined by \(G_{\lambda, \bar{g}_\lambda}\) (see Section \(A\) of the appendix). The connection between the constrained problem and the one we are considering is however only formal, see the discussion in [KSS06]. As can be seen straightforwardly seen from an inspection of our proof of the upper bound, recovery sequences for the stresses \(\mu_\lambda\) will typically be non-zero on a set of measure \(O(\lambda^{-1/2})\) for singular limits \(\mu\) that are singular with respect to the Lebesgue measure. In terms of the optimal design problem, the set where the stress is non-zero has to be understood as occupied by the elastic material, while the set where the stress is 0 should be thought of as “holes”.

(vii) In [Bou03], a general formula is proposed for describing the \(\Gamma\)-limit of optimal design problems for vanishing volume fractions, also for non-linear cases. This formula agrees with ours for the case we treat here.

**Notation.** The symbol \(C\) will be used as follows: A statement \(f \leq C(a, b, \ldots)g\) has to be read as “there exists a constant \(C > 0\) only depending on \(a, b, \ldots\) such that \(f \leq Cg\)”. The value of \(C\) may change from one inequality to the next. When it is clear on which quantities the constant depends, we also write \(f \lesssim g\) in this situation.
2. Preliminaries

2.1. \(A\)-free singular measures. Let \(A\) denote a linear partial differential operator of order \(k \in \mathbb{N}\),
\[
A = \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha,
\]
where \(A_\alpha \in \mathbb{R}^{p \times m}\) for every multiindex \(\alpha \in \mathbb{N}^n\) with \(\partial^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}\) and \(|\alpha| = \sum_{i=1}^n \alpha_i\).

We define the principal symbol of \(A\), \(A^k : \mathbb{R}^n \to \mathbb{R}^{p \times m}\), by setting
\[
A^k(\xi) = \sum_{|\alpha| = k} A_\alpha \xi^\alpha,
\]
where \(\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}\). In the following definition, \(S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}\).

**Definition 1.** The wave cone associated to a differential operator \(A\) as above is defined by
\[
\Lambda_A = \bigcup_{\xi \in S^{n-1}} \ker A^k(\xi).
\]
We will only be interested in the case \(A = \text{div}\), acting on measures with values in \(\mathbb{R}^{n \times n}\) (i.e., \(p = n, m = n(n+1)/2\)). In this case we obtain
\[
\Lambda_{\text{div}} = \{A \in \mathbb{R}^{n \times n}_{\text{sym}} : \text{Rk } A \leq n - 1\}
\]
where \(\text{Rk } A\) denotes the rank of \(A\).

**Definition 2.** The operator \(A\) is said to satisfy the constant-rank condition if there exists \(r \in \mathbb{N}\) such that
\[
\text{Rk } A^k(\xi) = r \quad \text{for all } \xi \in S^{n-1}.
\]
One easily verifies that the constant-rank condition is fulfilled for \(A = \text{div}\) with \(r = 1\).

The structure of \(A\)-free singular measures by De Philippis and Rindler yields in particular the following result:

**Theorem 2** (See [DPR16]). Let \(\Omega \subset \mathbb{R}^n\) be open and \(\mu \in \mathcal{M}(\Omega; \mathbb{R}^m)\) satisfy \(A\mu = \sigma\), where \(\sigma \in \mathcal{M}(\Omega; \mathbb{R}^p)\). Then for \(|\mu|\) a.e. \(x_0\), we have that
\[
\frac{d\mu}{d|\mu|}(x_0) \in \Lambda_A.
\]

2.2. Generalized Young measures. Generalized Young measures – roughly speaking – are dual objects to functions with linear growth at infinity. They have been introduced by DiPerna and Majda [DMS87]. Here we follow closely the approach by Kristensen and Rindler [KR10], which in turn is based on the work by Alibert and Bouchitté [AB97]. In comparison to [KR10], we drop the dependence of test functions on a variable \(x \in \Omega\), since we will not need this for our purpose.

First we define a suitable set of functions with linear growth at infinity. For \(f \in C(\mathbb{R}^m)\) and \(\xi \in B(0, 1) \subset \mathbb{R}^m\), let
\[
Tf(\xi) = (1 - |\xi|) f \left( \frac{\xi}{1 - |\xi|} \right).
\]
We define
\[
E(\mathbb{R}^m) = \left\{ f \in C(\mathbb{R}^m) : Tf \text{ extends to a continuous function on } \overline{B(0, 1)} \subset \mathbb{R}^m \right\}.
\]
Def. 3. A generalized Young measure $\nu$ parametrized by a set $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{R}^m$ is a triple $(\nu_x, \lambda_\nu, \nu^\infty_x)$, where

- $(\nu_x)_{x \in \Omega}$ is a family of probability measures on $\mathbb{R}^m$,
- $\lambda_\nu \in \mathcal{M}^+(\Omega)$ is a non-negative measure,
- $(\nu^\infty_x)_{x \in \Omega}$ is a family of probability measures on $S^{m-1}$ such that $x \mapsto \nu_x$ is weakly * measurable with respect to $\mathcal{L}^n$, $x \mapsto \nu^\infty_x$ is weakly * measurable with respect to $\lambda_\nu$, and $(x \mapsto |\cdot|, \nu_x) \in L^1(\Omega)$.

In the above definition, weak * measurability means that for every $f \in E(\mathbb{R}^m)$, we have that $x \mapsto \langle f(\cdot), \nu_x \rangle$ is $\mathcal{L}^n$-measurable, and $x \mapsto \langle f(\cdot), \nu^\infty_x \rangle$ is $\lambda_\nu$-measurable. The duality between generalized Young measures and functions $f \in E(\mathbb{R}^m)$ is defined by

$$\langle \langle f, \nu \rangle \rangle = \int_{\Omega} \langle f, \nu_x \rangle dx + \int_{\Omega} \langle f^\infty, \nu^\infty_x \rangle d\lambda_\nu(x),$$

where $f^\infty$ denotes the recession function of $f$,

$$f^\infty(\xi) = \limsup_{t \to \infty} \frac{f(t\xi)}{t}.$$

By Jensen’s inequality, we have for convex $f$ that

$$f(\langle \text{Id}, \nu_x \rangle) \leq \langle f, \nu_x \rangle \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in \Omega$$
$$f^\infty(\langle \text{Id}, \nu^\infty_x \rangle) \leq \langle f^\infty, \nu^\infty_x \rangle \quad \text{for } \lambda_\nu \text{ a.e. } x \in \Omega.$$

By the Radon-Nikodým Theorem, we may decompose any measure $\mu \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^m)$ into two parts, the one regular with respect to $\mathcal{L}^n$, and its singular part:

$$\mu = \frac{d\mu}{d\mathcal{L}^n} \mathcal{L}^n + \mu^s.$$

Such a measure $\mu \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^m)$ can be identified with a Young measure $\delta[\mu]$ via

$$\langle \delta[\mu], x \rangle = \delta \frac{d\mu}{d\mathcal{L}^n}(x), \quad \lambda_\mu = |\mu|^\infty, \quad \langle \delta[\mu], x \rangle^\infty = \delta \frac{d\mu^s}{d|\mu|^\infty}(x).$$

We say that a sequence of Young measures $(\nu_j)_{j \in \mathbb{N}}$ converges weakly * to a Young measure $\nu$ if for every (globally) Lipschitz function $f : \mathbb{R}^m \to \mathbb{R}$, we have that $\langle \langle f, \nu_j \rangle \rangle \to \langle \langle f, \nu \rangle \rangle$. In this case we write $\nu_j \overset{Y}{\rightharpoonup} \nu$.

We say that a sequence of measures $\mu_j$ generates a Young measure $\nu$ if we have $\delta[\mu_j] \overset{Y}{\rightharpoonup} \nu$ weakly * as Young measures.

Finally, we have the following compactness result for generalized Young measures:

**Lemma 1** ([KR10], Corollary 2). Let $(\nu_j)_{j \in \mathbb{N}}$ be a sequence of generalized Young measures such that the functions $x \mapsto |\cdot|, (\nu_j)_x$ are uniformly bounded in $L^1$ and $\lambda_{\nu_j}(\overline{\Omega})$ is uniformly bounded. Then there exists a generalized Young measure $\nu$ such that $\nu_j \overset{Y}{\rightharpoonup} \nu$.
2.3. \( A \)-quasiconvexity. Let \( A = \sum_{|a| \leq k} A_a \partial^a \) be a partial differential operator as in Section 2.1 above. Let \( Q = (-1/2, 1/2)^n \) be the unit cube in \( \mathbb{R}^n \). The smooth \( Q \)-periodic functions with values in \( \mathbb{R}^m \) are denoted by \( C^\infty_{\text{per}}(Q; \mathbb{R}^m) \).

**Definition 4.** (i) A Borel function \( f : \mathbb{R}^m \to \mathbb{R} \) is said to be \( A \)-quasiconvex if for every \( \xi \in \mathbb{R}^m \) and every \( \varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^m) \) satisfying

\[
A \varphi = 0 \quad \text{and} \quad \int_Q \varphi(x)dx = 0
\]

we have that

\[
\int_Q f(\xi + \varphi(x))dx \geq f(\xi).
\]

(ii) For a Borel function \( f : \mathbb{R}^m \to \mathbb{R} \), the \( A \)-quasiconvexification of \( f \), \( Q_A f \), is given by

\[
Q_A f(\xi) = \inf \left\{ \int_Q f(\xi + \varphi(x))dx : \varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^m) \cap \text{Ker} A \text{ with } \int_Q \varphi(x)dx = 0 \right\}.
\]

Since we will only be interested in the case \( A = \text{div} \), we will write \( Q f \equiv Q_{\text{div}} f \).

In the following lemma, functions in \( L^p(Q; \mathbb{R}^m) \) are identified with their \( Q \)-periodic extensions. Furthermore, let \( W^{1,p'}_{\text{per}}(Q) \) denote the \( Q \)-periodic functions in \( W^{1,p'}_{\text{loc}}(\mathbb{R}^n) \), and let \( W^{-1,p}_{\text{per}}(Q) \) denote its dual (where \( (p')^{-1} = 1 - p^{-1} \)). For later usage, we remark that

\[
\|f\|_{W^{-1,p}_{\text{per}}(Q)} \lesssim \|f\chi_Q\|_{W^{-1,p}(Q)}.
\]

**Lemma 2 (FM99, Lemma 2.14).** Let \( A \) be a first order differential operator as above that satisfies the constant rank condition and \( 1 < p < \infty \). There exists an operator \( \mathcal{P}_A : L^p(Q; \mathbb{R}^m) \to L^p(Q; \mathbb{R}^m) \) and a constant \( C = C(p) > 0 \) such that

\[
\mathcal{A} \mathcal{P}_A \varphi = 0, \quad \int_Q \mathcal{P}_A \varphi dx = 0, \quad \|\varphi - \mathcal{P}_A \varphi\|_{L^p(Q; \mathbb{R}^m)} \leq C\|A \varphi\|_{W^{-1,p}_{\text{per}}(Q)}
\]

for every \( \varphi \in L^p(Q; \mathbb{R}^m) \) with \( \int_Q \varphi(x)dx = 0 \).

2.4. Tangent measures. The notion of tangent measures is due to Preiss [Pre87]. We will only need one fact about tangent measures, for which it will not even be necessary to mention the definition. For \( x_0 \in \mathbb{R}^n \), \( r > 0 \), let \( T^{(x_0,r)}(x) = r^{-1}(x - x_0) \). The push-forward of a measure \( \mu \in \mathcal{M}(\mathbb{R}^n) \) by \( T^{(x_0,r)} \) is given by

\[
T^{(x_0,r)}_\# \mu(A) = \mu(x_0 + rA).
\]

The fact that we are going to use is that for \( \mathcal{L}^n \) almost every \( x_0 \in \mathbb{R}^n \), there exists a sequence \( r_j \downarrow 0 \) such that

\[
T^{(x_0,r_j)}_\# \mu \rightharpoonup \frac{d\mu}{d\mathcal{L}^n}(x_0) \mathcal{L}^n.
\]

This follows e.g. from Theorem 2.44 in [AFP00] in combination with the Radon-Nikodým differentiation theorem.
2.5. Quasiconvexification of $\tilde{h}_\lambda$. One of the main ingredients for the derivation of our convergence result are the known relaxations of the functionals $G_{\lambda,g}$ for $\lambda < \infty$. A proof of the following statement can be found in [AK93] (see also [KS86, All02, ABFJ97]).

**Theorem 3.** The div-quasiconvexification of $\tilde{h}_\lambda$, $h_\lambda = Q\tilde{h}_\lambda$, is given by the following formulas:

- If $n = 2$ then
  $$h_\lambda(\tau) = \begin{cases} 
  \lambda^{-1/2}|\tau|^2 + \lambda^{1/2} & \text{if } \rho^{(2)}(\tau) \geq \sqrt{\lambda} \\
  2\left(\rho^{(2)}(\tau) - \lambda^{-1/2}|\det \tau|\right) & \text{else.}
  \end{cases}$$

- If $n = 3$, then
  $$h_\lambda(\tau) = \begin{cases} 
  \lambda^{-1/2}|\tau|^2 + \lambda^{1/2} & \text{if } \rho^{(3)}(\tau) \geq \sqrt{\lambda} \\
  2\left(\sqrt{(|\tau_1| + |\tau_2|)^2 + |\tau_3|^2} - \lambda^{-1/2}|\tau_1\tau_2|\right) & \text{if } \rho^{(3)}(\tau) \leq \sqrt{\lambda} \text{ and } |\tau_1| + |\tau_2| \leq |\tau_3| \\
  h_\lambda^{(2)}(\tau) & \text{else},
  \end{cases}$$

where

$$h_\lambda^{(2)}(\tau) = \sqrt{2}(|\tau_1| + |\tau_2| + |\tau_3|) + \lambda^{-1/2}\left(\frac{1}{2}|\tau|^2 - (|\tau_1\tau_2| + |\tau_1\tau_3| + |\tau_2\tau_3|)\right).$$

Obviously we have the following pointwise convergences: If $n = 2$, then
$$\lim_{\lambda \to \infty} h_\lambda(\tau) = 2\rho^{(2)}(\tau) =: h^{(2)}(\tau)$$

and if $n = 3$, then
$$\lim_{\lambda \to \infty} h_\lambda(\tau) = \sqrt{2}\rho^{(3)}(\tau) =: h^{(3)}(\tau).$$

Whenever we make statements that are true for $n \in \{2, 3\}$, we also write $h \equiv h^{(n)}$.

We consider the divergence operator on symmetric matrices (which may be identified with $\mathbb{R}^{n(n+1)/2}$). We have already noted that the wavecone is given by
$$\Lambda_{\text{div}} = \{ A \in \mathbb{R}^{n \times n} : A = A^T, \text{Rk } A \leq n - 1 \}.$$

This readily implies that the restriction of $h^{(n)}$ to $\Lambda_{\text{div}}$ (which is obtained by setting $\tau_1 = 0$) is given by, for $n = 2$,
$$h^{(2)}|_{\Lambda_{\text{div}}}(\tau) = 2|\tau_2|,$$

and for $n = 3$ by
$$h^{(3)}|_{\Lambda_{\text{div}}}(\tau) = 2\sqrt{\tau_2^2 + \tau_3^2}.$$}

Of course, the right hand side of the last two equations is defined on all of $\mathbb{R}_{\text{sym}}^{n \times n}$. We denote it by $H^{(n)}$,
$$H^{(2)}(\tau) = 2|\tau_2|, \quad H^{(3)}(\tau) = 2\sqrt{\tau_2^2 + \tau_3^2}.$$

Again we write $H \equiv H^{(n)}$ whenever statements hold simultaneously for $n = 2$ and $n = 3$.

**Lemma 3.** The function $H$ is convex, and for every $\lambda > 0$, we have that
$$H(\tau) \leq h_\lambda(\tau).$$

**Proof.** The convexity is straightforward from the formulas above. Concerning the inequality, for $n = 2$, this is obvious. For $n = 3$, we insert $\tau_1 = 0$, verify the inequality, and then verify by a direct computation that $\partial_\tau h_\lambda(\tau) \geq 0$ (for a.e. $\tau$).
3. Proof of the lower bound

By the Radon-Nikodým theorem, we have the decomposition of the limit measure \( \mu \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \) into one part that is singular with respect to the Lebesgue measure, and the regular part,

\[
\mu = \frac{d\mu}{d\mathcal{L}^n} \mathcal{L}^n + \mu^s.
\]

Using the blow-up technique, we will prove the lower bound at regular and at singular points separately.

**Lower bound at singular points.**

**Proposition 1.** Let \( u_j \in L^2(\Omega; \mathbb{R}^{n \times n}) \), \( \mu \in \mathcal{M}(\Omega; \mathbb{R}^{n \times n}_{\text{sym}}) \) with \( u_j \mathcal{L}^n \rightharpoonup \mu \) and \( \text{div} \mu \in \mathcal{M}(\Omega; \mathbb{R}^n) \). For \( |\mu^s| \) almost every \( x_0 \), with \( \frac{d\mu}{d|\mu|}(x_0) = \xi \), we have that

\[
\lim_{r \to 0} \liminf_{\lambda \to \infty} \frac{1}{|\mu|(Q(x_0, r))} \int_{Q(x_0, r)} h_\lambda(u_\lambda) \, dx \geq h(\xi).
\]

**Proof.** By Lemma 1 we have – possibly after passing to a subsequence – that \( u_\lambda \mathcal{L}^n \) generates a Young measure \( \nu \), \( u_\lambda \mathcal{L}^n \rightharpoonup \nu \). Now by Lemma 3, we have for \( \lambda \nu \) almost every \( x_0 \),

\[
\lim_{r \to 0} \liminf_{\lambda \to \infty} \frac{1}{\lambda \nu(Q(x_0, r))} \int_{Q(x_0, r)} H(u_\lambda) \, dx \geq \langle H, \nu^\infty \rangle_{x_0} \quad \text{for } \lambda \nu \text{ a.e. } x_0.
\]

Here the limit \( \lim_{r \to 0} \) has to be understood as the choice of a sequence \( (r_i)_{i \in \mathbb{N}} \), \( r_i \downarrow 0 \), such that \( \lambda \nu(\partial Q(x_0, r_i)) = 0 \) for all \( i \in \mathbb{N} \), in order to justify the penultimate equality in (6). In the last equality of (6), we have used that

\[
\frac{d\langle H, \nu^\infty \rangle_{x_0}}{d\lambda \nu}(x_0) = \langle H, \nu^\infty \rangle_{x_0} \quad \text{for } \lambda \nu \text{ a.e. } x_0
\]

Using equation (5) and the convexity of \( H \), we obtain

\[
\lim_{r \to 0} \liminf_{\lambda \to \infty} \frac{1}{|\mu|(Q(x_0, r))} \int_{Q(x_0, r)} h_\lambda(u_\lambda) \, dx \geq \langle H, \nu^\infty \rangle_{x_0} \geq H((\text{Id}, \nu^\infty)) = H(\xi) = h(\xi).
\]

In the last equality, we have used the fact that \( \xi \in \Lambda_{\text{div}} \) for \( \lambda \nu \) almost every \( x_0 \) by Theorem 2. \( \Box \)
Lower bound for regular points. Our proof can be viewed as an adaptation of the proof of Lemma 2.15 in [ARDPR18], which itself is a variation of the proof of Proposition 3.1 in [FLM04]. Even though we will only need the case $A = \text{div}$, we will prove the lower bound at regular points in a slightly more general setting. Namely, let $A$ be a first order linear partial differential operator. Let $P \equiv P_A$ denote the projection operator onto the (mean-free) $A$-free functions from Lemma 2. In the following, let $p > 1$, $q > 0$. Furthermore, let $f_\lambda : \mathbb{R}^m \to \mathbb{R}$ be $A$-quasiconvex and locally Lipschitz with the estimate

$$|\nabla f_\lambda(A)| \lesssim 1 + \frac{|A|^{p-1}}{\lambda^q} \quad \text{for a.e. } A.$$  

This assumption translates into the estimate

$$|f_\lambda(A) - f_\lambda(B)| \lesssim |A - B| \left(1 + \frac{|A|^{p-1} + |B|^{p-1}}{\lambda^q}\right).$$

We write $f(\xi) = \liminf_{\lambda \to \infty} f_\lambda(\xi)$.

One further property that we are going to assume (and that is valid in the case $f_\lambda = h_\lambda$ that we will be interested in later) is

$$f_\lambda(sA) \leq Cs f_\lambda(A)$$

for $s \leq 1$.

**Proposition 2.** Let $A$ be a first order linear differential operator satisfying the constant rank condition, $\mu_\lambda$ a sequence in $\mathcal{M}(Q; \mathbb{R}^m)$ with $|\mu_\lambda| \ll L_n$, $\frac{d\mu_\lambda}{dL^n} \in L^p(Q; \mathbb{R}^m)$ and $\xi \in \mathbb{R}^m$, $\mu := \xi L^n$ such that

- $\mu_\lambda - \mu \xrightarrow{\lambda} 0$ in $\mathcal{M}(Q; \mathbb{R}^m)$
- $\mathcal{A}(\mu_\lambda - \mu) \xrightarrow{\lambda} 0$ in $\mathcal{M}(Q; \mathbb{R}^m)$
- $\lambda^{-q/p} \mathcal{A}(\mu_\lambda - \mu) \to 0$ in $W^{-1,p}(Q; \mathbb{R}^m)$
- $\lambda^{-q/p} \frac{d\mu_\lambda}{dL^n}$ is bounded in $L^p(Q; \mathbb{R}^m)$.

Then

$$f(\xi) \leq \liminf_{\lambda \to \infty} \int_Q f_\lambda \left(\frac{d\mu_\lambda}{dL^n}\right) \, dx.$$  

**Proof.** After taking subsequences we may assume that the right hand side is a limit, and that it is finite. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \eta \, dx = 1$ and $\eta_\varepsilon = \varepsilon^{-n} \eta(\cdot / \varepsilon)$. For $k \in \mathbb{N}$, choose $\varepsilon(\lambda, k) \downarrow 0$ as $\lambda \to \infty$. Now set for $x \in \mathbb{R}^n$,

$$w_{\lambda,k}(x) := \int_{\mathbb{R}^n} \eta_{\varepsilon(\lambda,k)}(x - \cdot) d(\mu_\lambda - \mu) = \left(\eta_{\varepsilon(\lambda,k)} * (\mu_\lambda - \mu)\right)(x).$$

We have that

$$|w_{\lambda,k} L^n - (\mu_\lambda - \mu)|(Q) \downarrow 0 \quad \text{as } \lambda \to 0.$$
Let $\delta \in (0, 1)$, and set
\[ Q_k := \{ x \in Q : \text{dist}(x, \partial Q) > \frac{\delta}{k} \} \]
and let $\varphi_k$ be associated test functions with $\varphi_k = 1$ on $Q_k$ and $\varphi_k = 0$ on $Q \setminus Q_{k+1}$. We set
\[ \hat{w}_{\lambda,k} = \varphi_k w_{\lambda,k}. \]
Now let
\[ \tilde{w}_{\lambda,k} = \hat{w}_{\lambda,k} - \int_Q \hat{w}_{\lambda,k} \, dx \]
\[ \tilde{w}_{\lambda,k} = P_A \tilde{w}_{\lambda,k}. \]
Note that $\tilde{w}_{\lambda,k} \in C^\infty_{\text{per}}(Q; \mathbb{R}^m)$. For every $\lambda$, $k$ we have by the $A$-quasiconvexity of $f_\lambda$ that
\[ f_\lambda(\xi) \leq \int_Q f_\lambda(\xi + \tilde{w}_{\lambda,k}) \, dx. \]
Using (8) and Hölder’s inequality, we have
\[
\int_Q f_\lambda(\xi + \tilde{w}_{\lambda,k}) \, dx \leq \int_Q f_\lambda(\xi + \hat{w}_{\lambda,k}) \, dx + C \| P \hat{w}_{\lambda,k} - \tilde{w}_{\lambda,k} \|_{L^1(Q)}
+ \frac{C}{\lambda^q} \left( \| \hat{w}_{\lambda,k} \|_{L^p(Q)}^{p-1} + \| P \hat{w}_{\lambda,k} \|_{L^p(Q)}^{p-1} \right) \| P \tilde{w}_{\lambda,k} - \tilde{w}_{\lambda,k} \|_{L^p(Q)},
\]
We claim that for the error terms on the right hand side vanish in the limit $\lambda \to 0$. Indeed, we have for any $\bar{p} \in (1, \frac{p}{n+1})$,
\[
\lim_{\lambda \to \infty} \| P \hat{w}_{\lambda,k} - \tilde{w}_{\lambda,k} \|_{L^1(Q)} \leq \lim_{\lambda \to \infty} \| P \hat{w}_{\lambda,k} - \tilde{w}_{\lambda,k} \|_{L^{\bar{p}}(Q)}
\leq \lim_{\lambda \to \infty} \| A \tilde{w}_{\lambda,k} \|_{W^{-1,\bar{p}}(Q)}
\leq \lim_{\lambda \to \infty} \lambda^{-q/p} \| A \tilde{w}_{\lambda,k} \|_{L^p(Q)}.
\]
Furthermore
\[ A \tilde{w}_{\lambda,k} = A(\varphi_k \eta_{x} \ast (\mu_{\lambda} - \mu))
= \varphi_k \eta_{x} \ast A(\mu_{\lambda} - \mu) + \eta_{x} \ast (\mu_{\lambda} - \mu) A \varphi_k. \]
From this expansion and $\mu_{\lambda} - \mu \xrightarrow{\lambda} 0$, $A(\mu_{\lambda} - \mu) \xrightarrow{\lambda} 0$ we obtain $A \tilde{w}_{\lambda,k} \xrightarrow{\lambda} 0$ in $\mathcal{M}(Q; \mathbb{R}^m)$. By the compact embedding $\mathcal{M}(Q; \mathbb{R}^m) \subset W^{-1,\bar{p}}(Q; \mathbb{R}^m)$, we have that $\| A \tilde{w}_{\lambda,k} \|_{W^{-1,\bar{p}}} \to 0$. From this and the assumption on the vanishing of $\lambda^{-q/p} A(\mu_{\lambda} - \mu)$ in $W^{-1,p}$ we obtain our claim that the right hand sides in (13) vanish too.
Next we have, again using (8),
\[
\int_Q f_\lambda(\xi + \tilde{w}_{\lambda,k}) \, dx \leq \int_Q f_\lambda(\xi + \hat{w}_{\lambda,k}) \, dx
+ C \left| \int \hat{w}_{\lambda,k} \, dx \right|
+ C \lambda^{-q} \left( \| \hat{w}_{\lambda,k} \|_{L^p}^{p-1} + \| \tilde{w}_{\lambda,k} \|_{L^p}^{p-1} \right) \left| \int \tilde{w}_{\lambda,k} \, dx \right|,
\]
By the boundedness assumption on \( \lambda^{-q/p} d(\mu_\lambda - \mu) / d\mathcal{L}^n \) in \( L^p \), the error terms on the right hand side converge to 0 in the limit \( \lambda \to 0 \). Next,

\[
\int_Q f_\lambda (\xi + \hat{w}_{\lambda,k}) \, dx = \int_{Q_k} f_\lambda \left( \frac{d\mu_\lambda}{d\mathcal{L}^n} \right) \, dx + \int_{Q_{k+1} \setminus Q_k} f_\lambda (\xi + \varphi_k \eta_\epsilon * (\mu_\lambda - \mu)) \, dx + \int_{Q \setminus Q_{k+1}} f_\lambda (\xi) \, dx.
\]

Combining (11), (12), (14) and (15), and taking the limit \( \lambda \to \infty \), we obtain

\[
f(\xi) \leq \liminf_{\lambda \to \infty} \left( \int_Q f_\lambda \left( \frac{d\mu_\lambda}{d\mathcal{L}^n} \right) + \int_{Q_{k+1} \setminus Q_k} f_\lambda (\xi + \varphi_k w_{\lambda,k}) \, dx + \int_{Q \setminus Q_{k+1}} f_\lambda (\xi) \, dx \right).
\]

Reordering and using \( f_\lambda \geq 0 \) yields

\[
|Q_{k+1}| f(\xi) \leq \liminf_{\lambda \to \infty} \left( \int_Q f_\lambda \left( \frac{d\mu_\lambda}{d\mathcal{L}^n} \right) + \int_{Q_{k+1} \setminus Q_k} f_\lambda (\xi + \varphi_k w_{\lambda,k}) \, dx \right).
\]

Using (13), we observe that

\[
\int_{Q_{k+1} \setminus Q_k} f_\lambda (\xi + \varphi_k w_{\lambda,k}) \, dx \leq C \int_{Q_{k+1} \setminus Q_k} (f_\lambda (\xi) + \varphi_k f_\lambda (w_{\lambda,k})) \, dx,
\]

Here the second error term on the right hand side, when summed over \( k \), can be estimated as follows,

\[
\limsup_{\lambda \to \infty} \sum_{k=1}^L \int_Q \varphi_k f_\lambda (w_{\lambda,k}) \, dx \leq \limsup_{\lambda \to \infty} \int_Q f_\lambda (w_{\lambda,k}) \, dx \leq \limsup_{\lambda \to \infty} \left( \|w_{\lambda,k}\|_{L^1} + \lambda^{-q} \|w_{\lambda,k}\|_{L^p} \right) \lesssim C.
\]

Summing (16) from \( k = 1 \) to \( L \) and dividing by \( L \) yields

\[
|Q_1| f(\xi) \leq \liminf_{\lambda \to \infty} \left( \int_Q f_\lambda \left( \frac{d\mu_\lambda}{d\mathcal{L}^n} \right) \, dx + CL^{-1} \sum_{k=1}^L \int_{Q_{k+1} \setminus Q_k} f_\lambda (\xi + \varphi_k \eta_\epsilon * (\mu_\lambda - \mu)) \, dx \right)
\]

\[
\leq \liminf_{\lambda \to \infty} \left( \int_Q f_\lambda \left( \frac{d\mu_\lambda}{d\mathcal{L}^n} \right) \, dx + CL^{-1} \right).
\]

Taking first the limit \( L \to \infty \) and then \( \delta \to 0 \) (i.e., \( |Q_1| \to |Q| = 1 \)) we obtain the claim of the proposition.

**Remark 2.**

(i) The blowup technique (in the context of lower semicontinuity of integral functionals) that we use here has been developed by Fonseca and Müller, first for Sobolev functions [FM92], then for \( BV \) functions [FM93], then for \( A \)-free \( L^p \) functions [FM99]. The paper [FLM04] discusses lower semicontinuity for \( A \)-free functions in the weak-* convergence of measures; it is proved there in particular that

\[
\int_{\Omega} f \left( x, \frac{d\mu}{d\mathcal{L}^n}(x) \right) \, dx \leq \liminf_{k \to \infty} \int_{\Omega} f(x, v_k(x)) \, dx
\]

for any sequence \( v_k \in \mathcal{L}^1 \) that converges weakly-* in the sense of measures to some \( \mathbb{R}^d \) valued Radon measure \( \mu_0 \), where \( f : \Omega \times \mathbb{R}^d \to \mathbb{R} \) is \( A \)-quasiconvex in
the second argument with linear growth at infinity, see Theorem 1.4 in [FLM04].
(Some additional regularity is required of \( f \), which we omit here for the sake of brevity.)
For \( f = f_\lambda \), our Proposition 2 is a direct consequence of this theorem.
(ii) Note that we use the \( A \)-quasiconvexity of \( f_\lambda \), and not any convexity properties of
the limit \( f \) to show our claim.

**Lemma 4.** Let \( \mu_\lambda \) be a sequence in \( \mathcal{M}(\overline{Q}; \mathbb{R}^{n \times n}) \) with \( |\mu_\lambda| \ll \mathcal{L}^n \), \( d\mu_\lambda/d\mathcal{L}^n \in L^2(Q; \mathbb{R}^{n \times n}) \),
and \( \xi \in \mathbb{R}^{n \times n} \), \( \mu = \xi \mathcal{L}^n \) such that

\[
\mu_\lambda - \mu \xrightarrow{\lambda} 0 \quad \text{in} \quad \mathcal{M}(\overline{Q}; \mathbb{R}^{n \times n})
\]
\[
\text{div} (\mu_\lambda - \mu) \xrightarrow{\lambda} 0 \quad \text{in} \quad \mathcal{M}(\overline{Q}; \mathbb{R}^{n \times n})
\]
\[
\lambda^{-1/4} \text{div} (\mu_\lambda - \mu) \to 0 \quad \text{in} \quad W^{-1,2}(Q; \mathbb{R}^{n \times n})
\]
\[
\lambda^{-1/4} \frac{d\mu_\lambda}{d\mathcal{L}^n} \quad \text{is bounded in} \quad L^2(Q; \mathbb{R}^{n \times n})
\]

Then

\[
h(\mu) \leq \liminf_{\lambda \to \infty} \int_Q h_\lambda(\mu_\lambda)dx.
\]

**Proof.** We apply Proposition 2 with \( f_\lambda = h_\lambda \), \( q = \frac{1}{2} \) and \( p = 2 \). The estimates (7), (9) for
\( h_\lambda \) are easily verified by an explicit calculation.

**Proof of the lower bound in Theorem 7** After choosing a suitable subsequence, we may
assume that the lim inf is a limit. We recall that \( h_\lambda(\mu_\lambda) = Qh_\lambda(\mu_\lambda) \leq h_\lambda(\mu_\lambda) \).
Hence \( h_\lambda(\mu_\lambda)\mathcal{L}^n \) is a bounded sequence in \( \mathcal{M}(\overline{\Omega}; \mathbb{R}^{n \times n}) \).
After passing to a further subsequence, we have that \( h_\lambda(\mu_\lambda)\mathcal{L}^n \xrightarrow{\ast} \pi \) for some \( \pi \in \mathcal{M}(\overline{\Omega}) \), with

\[
\pi(\overline{\Omega}) = \lim_{\lambda} \int_{\Omega} h_\lambda(\mu_\lambda)dx \leq \lim_{\lambda} \int_{\Omega} \tilde{h}_\lambda(\mu_\lambda)dx.
\]

Since \( \mu^s = \mu - \frac{d\pi}{d\mathcal{L}^n} \mathcal{L}^n \) is singular with respect to \( \mathcal{L}^n \), we have that

\[
\pi \geq \frac{d\pi}{d\mathcal{L}^n} \mathcal{L}^n + \frac{d\pi}{d|\mu^s|}|\mu^s|.
\]

By a well known representation of \( W^{-1,p} \) (see e.g. [Zie89] Theorem 4.3.3), we may write, in the sense of distributions,

\[
g_\lambda = \alpha_\lambda + \text{div} \beta_\lambda
\]

with \( \alpha_\lambda \in L^2(\mathbb{R}^n; \mathbb{R}^n) \) and \( \beta_\lambda \in L^2(\mathbb{R}^n; \mathbb{R}^{n \times n}) \).
We recall that \( \lambda^{-1/4} g_\lambda \to 0 \) in \( W^{-1,2}(\mathbb{R}^n) \) by assumption. This implies that \( \alpha_\lambda, \beta_\lambda \) may be chosen such that

\[
\lambda^{-1/4} \left( ||\alpha_\lambda||_{L^2(\mathbb{R}^n)} + ||\beta_\lambda||_{L^2(\mathbb{R}^n)} \right) \to 0.
\]

We have that for \( \mathcal{L}^n \) almost every \( x_0 \), \( T_{\#}^{x_0, r} \mu \xrightarrow{\ast} d\mu/d\mathcal{L}^n(x_0)\mathcal{L}^n \), see Section 2.4. For fixed \( r \), we have that \( T_{\#}^{x_0, r} \mu \xrightarrow{\ast} T_{\#}^{x_0, r} \mu \). Hence we may choose a sequence \( r_\lambda \downarrow 0 \) such that

\[
\tilde{\mu}_\lambda := T_{\#}^{x_0, r_\lambda} \mu \xrightarrow{\ast} \frac{d\mu}{d\mathcal{L}^n}(x_0)\mathcal{L}^n
\]

In the same way we may assume

\[
\tilde{g}_\lambda := T_{\#}^{x_0, r_\lambda} g_\lambda \xrightarrow{\ast} \frac{dg}{d\mathcal{L}^n}(x_0)\mathcal{L}^n.
\]
By the Radon-Nikodým Theorem, we also have (again, for \( L^n \) almost every \( x_0 \))

\[
\frac{d\pi}{dL^n}(x_0) = \lim_{\lambda \to \infty} \frac{1}{L^n(Q(x_0, r_\lambda))} \int_{Q(x_0, r_\lambda)} h_\lambda \left( \frac{d\mu_\lambda}{dL^n} \right) dx.
\]

Now we verify for an \( x_0 \) that satisfies the above relations that the conditions of Lemma \ref{lemma4} are fulfilled for the sequence \( \bar{\mu}_\lambda := T^{(x_0, r_\lambda)} \mu_\lambda \). The first condition of that lemma is just \( \ref{14} \). Furthermore, note that

\[-\text{div } \bar{\mu}_\lambda = r_\lambda \bar{g}_\lambda,\]

and hence we obtain by \( \ref{19} \) that

\[\text{div } \bar{\mu}_\lambda \rightharpoonup 0,
\]

which is the second condition of Lemma \ref{lemma4}.

Setting

\[\bar{\alpha}_\lambda(x) := \alpha_\lambda(x_0 + r_\lambda x)\]

\[\bar{\beta}_\lambda(x) := \beta_\lambda(x_0 + r_\lambda x)\]

we have by \( \ref{17} \) (assuming that \( r_\lambda^{-1} \lambda^{-1/2} (\|\alpha_\lambda\|_{L^2}^2 + \|\beta_\lambda\|_{L^2}^2) \to 0 \), which may be achieved by possibly modifying the sequence \( r_\lambda \)

\[\|\lambda^{-1/4} \text{div } \bar{\mu}_\lambda\|_{W^{-1,2}(Q)} \lesssim \lambda^{-1/4} \left( \|\bar{\alpha}_\lambda\|_{L^2(Q)} + r_\lambda \|\bar{\beta}_\lambda\|_{L^2(Q)} \right) \to 0
\]

This is just the third condition of Lemma \ref{lemma4}.

Finally we observe that

\[
\int_Q \lambda^{-1/2} \left| \frac{d\bar{\mu}_\lambda}{dL^n} \right|^2 dx \lesssim \int_{\{\mu(\mu_\lambda) \leq \sqrt{\lambda}\}} \left| \frac{d\bar{\mu}_\lambda}{dL^n} \right| dx + \int_{\{\mu(\mu_\lambda) \geq \sqrt{\lambda}\}} \left( \lambda^{-1/2} \left| \frac{d\bar{\mu}_\lambda}{dL^n} \right|^2 + \lambda^{1/2} \right) dx
\]

\[\lesssim \int_Q h_\lambda \left( \frac{d\bar{\mu}_\lambda}{dL^n} \right) dx
\]

\[\leq C
\]

which proves boundedness of \( \lambda^{-1/4} d\bar{\mu}_\lambda / dL^n \) in \( L^2 \), the last condition in Lemma \ref{lemma4}.

The application of Lemma \ref{lemma4} yields

\[\frac{d\pi}{dL^n}(x_0) \geq h \left( \frac{d\mu}{dL^n}(x_0) \right).
\]

For \( |\mu^s| \) almost every \( x_0 \in \Omega \), we have that

\[
\frac{d\pi}{d|\mu^s|}(x_0) = \lim_{r \to 0} \frac{1}{\mu(|Q(x_0, r)|)} \lim_{\lambda \to \infty} \int_{Q(x_0, r)} h_\lambda \left( \frac{d\mu_\lambda}{dL^n} \right) dx
\]

\[\geq h \left( \frac{d\mu}{d|\mu|}(x_0) \right),
\]

where the last inequality is obtained by Proposition \ref{prop1}.
Hence we have shown
\[
\pi(\Omega) \geq \int_{\Omega} h \left( \frac{d\mu}{d\mathcal{L}^n} \right) \, dx + \int_{\Omega} h \left( \frac{d\mu}{d|\mu|} \right) \, d|\mu^s| \\
= \int_{\Omega} hd\mu \\
= \mathcal{G}_{\infty,g}(\mu).
\]
This completes the proof of the lower bound. \(\square\)

4. Compactness, upper bound

Proof of compactness in Theorem 1. We have that
\[
|\mu_\lambda| \leq h_\lambda(\mu_\lambda) \leq \tilde{h}_\lambda(\mu_\lambda),
\]
and hence the statement follows from the standard compactness result for sequences in \(M(\overline{\Omega}; \mathbb{R}^{n \times n}_{\text{sym}})\) in the weak * topology. \(\square\)

Proof of the upper bound in Theorem 1. We may assume that
\[
\mathcal{G}_{\infty,g}(\mu) < \infty,
\]
otherwise there is nothing to show. We consider \(g_\lambda, g\) as measures in \(M(\mathbb{R}^n; \mathbb{R}^n)\) with support in \(\Omega\).

We observe that \(M(\mathbb{R}^n; \mathbb{R}^n) \subset W^{-1,p}(\mathbb{R}^n; \mathbb{R}^n)\) for \(p \in (1, \frac{n}{n-1})\) with compact embedding.

Now we apply standard results for strongly elliptic equations with constant coefficients:

Let \(\zeta_\lambda \in W^{1,p}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^n)\) be the solution of
\[
\begin{cases}
-\text{div} \, e(\zeta_\lambda) = g_\lambda & \text{in } \mathbb{R}^n \\
\nabla \zeta_\lambda \in L^p(\mathbb{R}^n; \mathbb{R}^{n \times n})
\end{cases}
\]
where \(e(\zeta_\lambda) = \frac{1}{2}(\nabla \zeta_\lambda + \nabla \zeta_\lambda^T)\). The application of elliptic regularity theory yields
\[
\|\nabla \zeta_\lambda\|_{L^p(\mathbb{R}^n)} \lesssim \|g_\lambda\|_{W^{-1,p}(\mathbb{R}^n)}.
\]
In the same way, we obtain a solution \(\zeta\) of
\[
\begin{cases}
-\text{div} \, e(\zeta) = g & \text{in } \mathbb{R}^n \\
\nabla \zeta \in L^p(\mathbb{R}^n, \mathbb{R}^{n \times n})
\end{cases}
\]
with
\[
\|\nabla \zeta\|_{L^p(\mathbb{R}^n)} \lesssim \|g\|_{W^{-1,p}(\mathbb{R}^n)}.
\]

By the assumption \(g_\lambda \rightharpoonup g\), the compact embedding \(\mathcal{M} \subset W^{-1,p}\) and elliptic regularity, we have that
\[
e(\zeta_\lambda) \to e(\zeta) \quad \text{in } L^p(\mathbb{R}^n; \mathbb{R}^{n \times n}_{\text{sym}}).
\]
By \(\lambda^{-1/4}\|g_\lambda\|_{W^{-1,2}(\mathbb{R}^n; \mathbb{R}^n)} \to 0\) and elliptic regularity, we have that
\[
\lambda^{-1/4}\|e(\zeta_\lambda)\|_{L^2(\mathbb{R}^n; \mathbb{R}^{n \times n}_{\text{sym}})} \to 0.
\]
Set \(\bar{\mu} = \mu - e(\zeta)\). Let \(\eta \in C_c^\infty(\mathbb{R}^n)\) such that \(\int \eta = 1\) and \(\eta_\varepsilon := \varepsilon^{-n} \eta(\cdot/\varepsilon)\). Choose a monotone decreasing sequence \(\varepsilon(\lambda)\) with \(\varepsilon(\lambda) \downarrow 0\) as \(\lambda \to \infty\) and \(\frac{1}{\varepsilon(\lambda)} \sup |\eta| \leq \frac{1}{4} \sqrt{\lambda}\). We set
\[
\bar{\mu}_\lambda := \eta_\varepsilon * \bar{\mu}
\]
and
\[
\mu_\lambda = \bar{\mu}_\lambda + e(\zeta_\lambda).
\]
Note that these definitions imply in particular that $|\bar{\mu}_\lambda| \leq \frac{1}{4} \sqrt{\lambda}$. Furthermore we have that $\mu_\lambda \xrightarrow{\ast} \mu$.

Let
\[
A_\lambda := \{ x \in \Omega : \rho^{(n)}(\mu_\lambda) \geq \sqrt{\lambda} \}
\]
\[
\tilde{A}_\lambda := \{ x \in \Omega : |e(\zeta_\lambda)| \geq \frac{1}{4} \sqrt{\lambda} \}.
\]

By $\rho^{(n)}(\xi) \leq 2|\xi|$ for all $\xi \in \mathbb{R}^{n \times n}_{sym}$, we have that $A_\lambda \subset \tilde{A}_\lambda$. Now we may estimate as follows (not distinguishing between measures and their densities),
\[
\limsup_{\lambda \to \infty} \int_{\Omega} h_\lambda(\mu_\lambda) dx = \limsup_{\lambda \to \infty} \left( \int_{\Omega \setminus A_\lambda} h_\lambda(\mu_\lambda) dx + \int_{A_\lambda} h_\lambda(\mu_\lambda) dx \right)
\leq \limsup_{\lambda \to \infty} \int_{\Omega \setminus A_\lambda} h(\mu_\lambda) dx + \int_{A_\lambda} \left( \frac{\mu_\lambda^2}{\sqrt{\lambda}} + \sqrt{\lambda} \right) dx.
\]

Now we have that $\lambda^{-1/2}||e(\zeta_\lambda)||^2_{L^2} \to 0$ and hence $\lambda^{1/2} L^n(\tilde{A}_\lambda) \to 0$ as $\lambda \to \infty$. This implies
\[
\int_{A_\lambda} \left( \frac{\mu_\lambda^2}{\sqrt{\lambda}} + \sqrt{\lambda} \right) dx \leq \int_{\tilde{A}_\lambda} \left( 2\frac{\mu_\lambda^2}{\sqrt{\lambda}} + |e(\zeta_\lambda)|^2 + \sqrt{\lambda} \right) dx
\leq L^n(\tilde{A}_\lambda) \left( \frac{1}{8} \sqrt{\lambda} + \sqrt{\lambda} \right) + \lambda^{-1/2}||e(\zeta_\lambda)||^2_{L^2}
\to 0 \quad \text{as} \quad \lambda \to \infty.
\]

Hence we get
\[
\limsup_{\lambda \to \infty} \int_{\Omega} h_\lambda(\mu_\lambda) dx \leq \limsup_{\lambda \to \infty} \int_{\Omega} h(\mu_\lambda) dx
= \int_{\Omega} dh(\mu)
= \mathcal{G}_{\infty,g}(\mu).
\]

**Appendix A. Derivation of the variational form of the compliance minimization problem**

Here we repeat basically our presentation from Section 2.4 of [Olb17]. We include this part in order to keep the present article self-contained.

Let $g \in W^{-1,2}(\Omega; \mathbb{R}^n)$. The aim of this appendix is to give a derivation of the compliance minimization problem in its variational form,
\[
(21) \quad \inf \mathcal{G}_{\lambda,g}(\sigma),
\]
where the infimum is taken over the set
\[
S_g(\Omega) = \{ \sigma \in L^2(\Omega; \mathbb{R}^{n \times n}_{sym}) : -\text{div} \sigma = g \}
\]
Here the equation $-\text{div} \sigma = g$ is to be understood as an equation in the distributional sense in a neighborhood of $\Omega$, with $\sigma$ extended by 0 on the complement of $\Omega$. In this way we incorporate boundary conditions in the equation, see our discussion in Section 1.1. We
want to derive this variational problem starting from the standard formulation of a linear elasticity problem. More details can be found in [All02].

Consider $\Omega \subset \mathbb{R}^n$ as an elastic body, characterized by its elasticity tensor $A_0 \in \text{Lin}(\mathbb{R}_{\text{sym}}^{n \times n}, \mathbb{R}_{\text{sym}}^{n \times n})$, where for simplicity we assume here that $A_0 = \text{Id}_{\mathbb{R}_{\text{sym}}^{n \times n}}$ is the identity. We remove a subset $H \subset \Omega$ from the elastic body and the new boundaries from that process shall be traction-free. The resulting linear elasticity problem is to find $u : \Omega \setminus H \to \mathbb{R}^n$ such that

$$
\begin{aligned}
\sigma &= A_0 e(u) \\
-\text{div} \sigma &= g \quad \text{in} \quad \Omega \setminus H \\
\sigma \cdot n &= 0 \quad \text{on} \quad \partial H,
\end{aligned}
$$

where $e(u) = \frac{1}{2}(\nabla u + \nabla u)^T$. The compliance (work done by the load) is given by

$$
c(H) = \int_{\partial \Omega} g \cdot u d\mathcal{H}^1 = \int_{\Omega \setminus H} (A_0 e(u)) : e(u) dx,
$$

where $u : \Omega \setminus H \to \mathbb{R}^2$ is the unique solution to the linear elasticity system above. We want to minimize the compliance under a constraint on the “weight” $L^2(\Omega \setminus H)$. We do so by the introduction of a Lagrange multiplier $\lambda$, and are interested in the minimization problem

$$
\min_{H} \left( c(H) + \lambda L^2(\Omega \setminus H) \right).
$$

The “equivalence” between the mass constrained problem and the problem including a Lagrange multiplier only holds on a heuristic level, see [KSS90] for a discussion of this point. Accepting this step, taking the limit of vanishing weight corresponds to the limit $\lambda \to \infty$. We now rewrite the problem by considering space-dependent elasticity tensors of the form $A(x) = \chi(x)A_0$, where $\chi \in L^\infty(\Omega; \{0, 1\})$. The equations from above turn into the system

$$
\begin{aligned}
\sigma &= A(x) e(u) \\
-\text{div} \sigma &= g \quad \text{in} \quad \Omega.
\end{aligned}
$$

The compliance turns into a functional on the set of permissible elasticity tensors, and is given by

$$
c(A) = \int_{\Omega} (A(x) e(u)) : e(u) dx,
$$

where $u$ is the solution of (22). By the principle of minimum complementary energy, the compliance can be written as

$$
c(A) = \int_{\Omega} G(A(x), \sigma(x)) dx,
$$

where

$$
G(\bar{A}, \xi) = \begin{cases} 
+\infty & \text{if } \xi \neq 0 \text{ and } \bar{A} = 0 \\
0 & \text{if } \xi = 0 \text{ and } \bar{A} = 0 \\
(\bar{A}^{-1} \xi) : \xi & \text{else,}
\end{cases}
$$

and $\sigma \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{n \times n})$ is a solution of the PDE

$$
-\text{div} \sigma = g \quad \text{in} \quad \Omega.
$$
i.e., $\sigma \in S_g(\Omega)$. We see that the compliance minimization problem can be understood as
the variational problem of finding the infimum
\[
\inf \left\{ \int_{\Omega} (G(\chi(x),A_0,\sigma(x)) + \lambda\chi(x)) \, dx : \chi \in L^\infty(\Omega;\{0,1\}), \sigma \in S_g(\Omega) \right\}.
\]
Of course, the compliance of a pair $(\chi, \sigma)$ is infinite if there exists a set of positive measure $U$ such that $\chi = 0$ and $\sigma \neq 0$ on $U$. Hence the above variational problem is equivalent with
\[
\inf \left\{ \int_{\Omega} F^{A_0}_\lambda(\sigma) \, dx : \sigma \in S_g(\Omega) \right\},
\]
where
\[
F^{A_0}_\lambda(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\
(A_0^{-1}\xi) + \lambda & \text{else},
\end{cases}
\]
Up to a factor $\lambda^{-1/2}$, this is just the integrand (11), and hence (23) is just the variational problem (21), with $A_0 = \text{Id}_{\mathbb{R}^{n \times n}}$. As is well known, this problem does not possess a solution in general and requires relaxation.

**Appendix B. A very brief presentation of Michell trusses**

In this section, we want to sketch very briefly how the limit integral functional $G_{\infty,g}$ is linked to Michell truss theory for the case $n = 2$. What we say here is mainly taken from [BGS08].

A *truss* is a finite union of *bars* (line segments that can resist compression or tension parallel to them) between points $x_i \in \mathbb{R}^2$, $i = 1, \ldots, M$. We write $(x_1, \ldots, x_m) = x \in \mathbb{R}^{2 \times M}$, and let $w \in \mathbb{R}^{M \times M}$. To every bar $[x_i, x_j] = \{tx_i + (1-t)x_j : t \in [0,1]\}$, we associate $w_{ij}$, where $|w_{ij}|$ is the *strength* of the bar, and the sign of $w_{ij}$ is chosen according to whether the bar has to withstand compression or tension.

The force provided the bar $[x_i, x_j]$ is given by
\[
f_{ij}(x,w) = (\delta_{x_i} - \delta_{x_j}) w_{ij} \frac{x_i - x_j}{|x_i - x_j|}.
\]
The set of all bars shall counterbalance a given force
\[
g = \sum_{i=1}^N g_i \delta_{y_i},
\]
where $y_i, g_i \in \mathbb{R}^2$, $i = 1, \ldots, N$, are given. The *truss* $(\{x_i\}_{i=1,\ldots,M}, \{w_{ij}\}_{i,j=1,\ldots,M})$ withstands $g$ if
\[
g + \sum_{i,j=1,\ldots,M} f_{ij}(x,w) = 0.
\]
The *weight* of the truss $(x,w)$ is given by
\[
W(x,w) = \sum_{i,j=1}^M |w_{ij}| |x_i - x_j|.
\]
The task is now the minimization of the weight, given the external forces, as a function of $x, w$. To express how this variational problem relates to $G_{\infty,g}$, we note that the force
supplied by the bars can be written as the divergence of a stress, \( f_{ij}(x, w) = \text{div} \, \sigma_{ij}(x, w) \) with

\[
\sigma_{ij}(x, w) = w_{ij} \frac{x_i - x_j}{|x_i - x_j|} \otimes \frac{x_i - x_j}{|x_i - x_j|} \mathcal{H}^1([x_i, x_j])
\]

\[
\sigma(x, w) = \sum_{i,j=1}^{M} \sigma_{ij}(x, w) \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^{2 \times 2}_{\text{sym}}).
\]

With this notation, the balance of forces becomes the equation

\[-\text{div} \, \sigma = g,
\]

and the weight of the truss is given by \( \mathcal{W}(x, w) = |\sigma(x, w)| \) (the total variation of the measure \( \sigma \)).

Summarizing, we are dealing with the variational problem

\[
\inf \{|\sigma| : \sigma = \sigma(x, w) \text{ for some truss } (x, w), \ -\text{div} \, \sigma = g\}.
\]

To guarantee the existence of a minimizer, this variational problem requires relaxation, as has already been remarked by Michell in 1904 [Mic04]. We will not discuss the derivation of the relaxation here and refer the interested reader to [BGS08]. We only state the result: Namely, that it becomes the variational problem defined by the \( \Gamma \)-limit in the main text:

\[
\inf \left\{ \rho^{(2)}(\sigma)(\mathbb{R}^2) : \sigma \in \mathcal{M}(\mathbb{R}^2; \mathbb{R}^{n \times n}_{\text{sym}}), \ -\text{div} \, \sigma = g \right\}.
\]

Requiring additionally \( \text{supp} \, \sigma \subset \overline{\Omega} \) leads to

\[
\inf \left\{ \rho^{(2)}(\sigma)(\overline{\Omega}) : \sigma \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^{n \times n}_{\text{sym}}), \ -\text{div} \, \sigma = g \right\}.
\]

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