**Quasi-local energy-momentum and energy flux at null infinity**

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The null infinity limit of the gravitational energy-momentum and energy flux determined by the covariant Hamiltonian quasi-local expressions is evaluated using the NP spin coefficients. The reference contribution is considered by three different embedding approaches. All of them give the expected Bondi energy and energy flux.

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I. INTRODUCTION

It is well-known that, as a consequence of the equivalence principle, gravitational energy cannot be localized (see e.g. §20.4). An alternative idea is quasi-local, namely quantities associated with a closed two-surface. During the recent decades there have been numerous intensive efforts made in the search for a better definition of quasi-local energy (as well as momentum and angular momentum) for gravitating systems, with the goal of obtaining quasi-local quantities which can provide a description of the gravitational field more elaborate than that given by the total quantities. A very nice review on the development and applications of quasi-local quantities in general relativity can be found in [3]. We are interested in testing certain quasi-local expressions for energy-momentum and energy flux obtained from the Hamiltonian boundary term using the covariant Hamiltonian formalism applied to gravity [4, 5, 6, 7].

Some basic criteria are usually presumed for a physically reasonable definition of quasi-local gravitational energy [8]. One of the most important is to consider the asymptotic behavior of the quasi-local energy when the two surface approaches null and spatial infinity. The first aim of this work is to check, using the NP spin coefficient formalism [9], the null infinity limit of the quasi-local energy-momentum associated with the boundary of a finite region for a certain covariant Hamiltonian quasi-local energy-momentum expression.

We are interested in gravitational quantities like energy in the quasi-local sense, i.e. within a finite region of the space-time. There is an important issue concerning how much energy flux flows into and out of the considered region. There continues to be considerable interest this topic [10, 11, 12, 13]. In this work we also investigate a natural expression for the energy flux associated with the aforementioned quasi-local energy [5]. In order to test whether the definition is suitable, we again look to the the null infinity limit using the spin coefficient techniques. In this case, the gravitational energy flux is expected to be given by the well known Bondi energy loss formula.

In order to have a reasonable definition of gravitational energy, the choice of reference plays a essential role. It is well known that the technique for choosing the reference is an important unsolved problem for the quasi-local energy issue. The ambiguity comes from how to embed the reference configuration into the physical space-time. Here we consider three different embeddings.

This paper is organized as follows: In the next section, we start from a basic review on the asymptotic behavior of space-time near null infinity including a discussion of the expansion of the Newman-Penrose coefficients which is more complete than the well-known ones in [8]. Our main result is contained in three parts: In Section III, we review the covariant Hamiltonian formalism and the associated boundary term approach to quasi-local energy momentum and energy flux. We then re-write the energy-momentum expression in terms of the Newman-Penrose formalism. In section IV, we find the asymptotic behavior at null infinity under three different embedding methods. From the results, we find that the Brown-Lau-York (BLY) embedding [10] directly gives the standard Bondi mass aspect, while two other embeddings include an additional term which, however, vanishes upon integration. We identify

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the source of this difference. In section V, we look at the direct definition of the energy flux, and consider its null infinity limit to test whether this definition is reasonable. The detailed calculation is divided into several subsections. In the first subsection, we calculate the purely physical part of the energy flux and, in later subsections, we will consider the three different embedding methods which have been mentioned. All three types of embedding give the standard Bondi energy flux. In section VI we test, using the spin coefficient formalism in the null infinity limit, the new Hamiltonian identity based expression for energy flux. It directly yields the expected Bondi energy-flux value. Section VII includes our concluding discussion.

II. ASYMPTOTIC BEHAVIOR OF SPACE-TIME NEAR NULL INFINITY

We first review the asymptotic behavior of space-time near null infinity. Here we are interested in the cases in which the space-time is asymptotically flat. There have been many intensive investigations of this subject in the past decades. We will follow the method initiated by Newman, Penrose, and Tod [13, 14]. Many additional useful results have been discussed in [15, 17, 18, 19, 20].

In accord with Penrose’s conformal compactification method, i.e. the Penrose diagram, we assume that \( (\tilde{M}, \tilde{g}_{\mu\nu}, \Omega) \) is the conformal compactified manifold of a physical space-time \( (M, g_{\mu\nu}) \) via a conformal transformation \( \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \). Suppose \( S \) is a section of future null infinity \( \mathcal{I}^+ \), we choose Bondi coordinates near \( \mathcal{I}^+ \) in the following way: \( (\theta, \phi) \) are spherical coordinates on \( S \) and \( u \) is the affine parameter generating \( \mathcal{I}^+ \) such that \( u = 0 \) on \( S \). If \( k^\mu \) is a null vector on \( \mathcal{I}^+ \) which is not tangential, namely \( k^\mu \notin T(\mathcal{I}^+) \), then the null geodesics generated by \( k^\mu \) will take the coordinates \( (u, \theta, \phi) \) into the physical space-time \( (M, g_{\mu\nu}) \). Moreover, on each null geodesic \( \psi(r) = \{ u = \text{const}, \theta = \text{const}, \phi = \text{const} \} \), its affine parameter \( r \), in the sense of the physical metric \( g_{\mu\nu} \), can serve as the forth coordinate (in the neighborhood of \( \mathcal{I}^+ \), we assume the null geodesics are complete). Thus we have constructed coordinates \( (u, r, \theta, \phi) \) for the physical space-time (they are like Bondi’s coordinates except that \( r \) is an affine parameter rather than a luminosity distance, a small difference asymptotically).

In addition to this coordinate construction, we impose, as has been introduced in [19, 20], certain null frame gauges choices; stated in terms of the spin coefficients they are

\[
\rho - \bar{\rho} = \mu - \bar{\mu} = \kappa = \varepsilon = \tau = - \alpha - \beta = \bar{\pi} = - \bar{\beta} = 0. \quad (1)
\]

The fact that \( \rho \) and \( \mu \) are real functions is insured by the null frame \( m \) and \( \tilde{m} \) being always tangent to the two sphere \( r = \text{const} \) and \( u = \text{const} \). The meaning of the other gauge choices is that \( l \) is a null geodesic and the Lie transport of both \( n \) and \( m \) along \( l \) are tangent to the sphere.

We use the standard notation for derivatives, \( D := l^a \nabla_a, \quad D' := n^a \nabla_a, \quad \delta := m^a \nabla_a, \quad \bar{\delta} := \bar{m}^a \nabla_a, \) and the usual complex coordinate on the two sphere, \( \zeta := \cot \frac{\theta}{2} e^{i\phi} \). The null tetrad can be chosen as

\[
l = \frac{\partial}{\partial r}, \quad (2)
\]
\[
n = \frac{\partial}{\partial u} + U \frac{\partial}{\partial r} + X \frac{\partial}{\partial \zeta} + \bar{X} \frac{\partial}{\partial \bar{\zeta}}, \quad (3)
\]
\[
m = \xi \frac{\partial}{\partial \zeta} + \bar{\eta} \frac{\partial}{\partial \bar{\zeta}}, \quad (4)
\]

where \( U \) (real) and \( X, \xi, \eta \) (complex) are undetermined functions.

For simplicity, in this paper we will only focus on the vacuum case. Due to the asymptotic flatness, the behavior of the Weyl curvature satisfies the “peeling off theorem” [3, 14], i.e.

\[
\Psi_i \sim O(r^{i-5}), \quad i = 0, 1, 2, 3, 4. \quad (5)
\]

Near null infinity, we expand all quantities in Taylor series with respect to \( 1/r \). Using the Newman-Penrose (NP) equations, the asymptotic behavior of the spin-coefficients are [16, 20]

\[
\rho = -\frac{1}{r} \left| \sigma^0 \right|^2 + O(r^{-5}), \quad (6)
\]
\[
\sigma = \frac{\sigma^0}{r^2} + \left| \sigma^0 \sigma^0 - \frac{1}{2} \Psi_0 \right| + O(r^{-5}), \quad (7)
\]
\[
\alpha = \frac{\alpha^0}{r} + \left| \bar{\sigma}^0 \alpha^0 + \bar{\beta}^0 \sigma^0 \right| + O(r^{-3}). \quad (8)
\]
The last equation is needed to determine the value of the leading order term of $\pi$ and the property that the spin weight of $\Psi_0(u,\theta,\phi)$, the leading order terms of $\sigma$ (of order $r^{-2}$) and $\Psi_q$ (of order $r^{-5}$), are free functions. Moreover the variable $\sigma^0$ is an abbreviation for $\sigma^0 := \frac{1}{2\sqrt{2}} \zeta$, the spin-weight operator $\bar{\sigma}_0$ is defined by $\bar{\sigma}_0 := \frac{\sqrt{2}}{2} \frac{\partial}{\partial \theta} + 2s\alpha^0$ acting on a variable with spin-weight $s$. Using the results in Eqs. (6–10) and the asymptotic condition (5), it is straightforward to get
\begin{equation}
\beta = -\frac{\sigma^0}{r} - \frac{\alpha^0\sigma^0}{r^2} + O(r^{-3}),
\end{equation}
\begin{equation}
\tau = \bar{\pi} = \bar{\theta}_0\sigma^0 + O(r^{-3}),
\end{equation}
where $\sigma^0(u,\theta,\phi)$ and $\Psi_0^0(u,\theta,\phi)$, the leading order terms of $\sigma$ (of order $r^{-2}$) and $\Psi_q$ (of order $r^{-5}$), are free functions. Moreover the variable $\sigma^0$ is an abbreviation for $\sigma^0 := \frac{1}{2\sqrt{2}} \zeta$, the spin-weight operator $\bar{\sigma}_0$ is defined by $\bar{\sigma}_0 := \frac{\sqrt{2}}{2} \frac{\partial}{\partial \theta} + 2s\alpha^0$ acting on a variable with spin-weight $s$. Using the results in Eqs. (6–10) and the asymptotic condition (5), it is straightforward to get
\begin{equation}
\gamma = \frac{\gamma_1}{r^2} + O(r^{-3}) = \frac{-\alpha^0\bar{\sigma}_0\sigma^0 + \sigma^0\bar{\theta}_0\sigma^0 - \frac{1}{2}\bar{\Psi}_2}{r^2} + O(r^{-3}).
\end{equation}
In order to get the asymptotic expansion of the remaining NP coefficients we need to have some more control on the null tetrad. From the commutation relations, one can easily derive the null tetrad control equations
\begin{align}
DU &= - (\gamma + \bar{\gamma}), \\
DX &= (\bar{\tau} + \sigma^0)\xi + (\tau + \bar{\pi})\eta, \\
D\xi &= \mu\xi + \sigma^0\eta, \\
D\eta &= \mu\eta + \sigma^0\eta,
\end{align}
which lead to the following asymptotic behavior of the null tetrad
\begin{align}
U &= -\frac{1}{2} + \frac{\gamma_1 + \bar{\gamma}_1}{r} + O(r^{-2}), \\
X &= -\frac{\bar{\sigma}_0\bar{\theta}_0\sigma^0}{r^2} + O(r^{-3}), \\
\xi &= \frac{\xi_0}{r} + \frac{1}{r^3}(\sigma^0)^2 + O(r^{-5}), \\
\eta &= -\frac{\sigma^0\xi_0}{r^2} + O(r^{-4}),
\end{align}
where the numerical factor $-\frac{1}{2}$ in (17) and the value of $\xi_0$ ($\xi^0 := \frac{\sqrt{2}}{2\sqrt{2}} \zeta$) are specified by the result from Minkowski space-time. The asymptotic behavior of $\mu$ and $\lambda$ can now be retrieved from the following NP equations
\begin{align}
D\mu - \delta\pi &= (\bar{\rho}\mu + \sigma\lambda) + |\pi|^2 - (\bar{\alpha} - \beta)\pi + \Psi_2, \\
D\lambda - \delta\bar{\pi} &= (\rho\lambda + \bar{\sigma}\mu) + |\pi|^2 - (\alpha - \bar{\beta})\bar{\pi}, \\
\delta\tau - D'\sigma &= (\mu\sigma + \bar{\lambda}\rho) + \bar{\tau} - (\bar{\alpha} - \beta)\tau - (3\gamma - \bar{\gamma})\sigma.
\end{align}
The last equation is needed to determine the value of the leading order term of $\lambda$. Using the obtained results Eq. (17) and the property that the spin weight of $\pi$ is $-1$, it is straightforward to get
\begin{align}
\mu &= -\frac{1}{2r} \frac{\Psi_2^0 + \sigma_0^0\sigma_0^0 + \bar{\theta}_0\bar{\theta}_0\sigma_0^0}{r^2} + O(r^{-3}), \\
\lambda &= \frac{\bar{\sigma}_0^0}{r} + \frac{1}{2r^2} \bar{\sigma}_0^0 - \frac{\bar{\theta}_0\bar{\theta}_0\sigma_0^0}{r^2} + O(r^{-3}),
\end{align}
where the dot means derivative with respect to $u$ and the numerical factor $-\frac{1}{2}$ in the leading term of $\mu$ is specified by the Minkowski space-time result. (It will turn out that $\mu$ and $\lambda$ will play the key roles in our results.) Finally the expansion for the coefficient $\nu$ can be obtained from the NP equation
\begin{equation}
D\nu - D'\pi = (\pi + \bar{\pi})\mu + (\bar{\tau} + \tau)\lambda + (\gamma - \bar{\gamma})\pi + \Psi_3,
\end{equation}
which gives

$$\nu = -\frac{\partial_\nu \hat{\sigma}^0 + \Psi_0^0}{r} + O(r^{-2}).$$ \hspace{1cm} (27)$$

However, the leading term of the NP equation

$$\delta \lambda - \delta \mu = (\alpha + \beta) \mu + (\bar{\alpha} - 3 \beta) \lambda - \Psi_3,$$ \hspace{1cm} (28)

shows that $\Psi_3^0 = -\partial_\nu \hat{\sigma}^0$ (the spin weight of $\hat{\sigma}^0$ is $-2$). Therefore the leading order of $\nu$ indeed is $r^{-2}$ which does not make a contribution in the later calculation.

Moreover, for the later calculation, we still need the asymptotic properties of the induced volume element (2-dimensional) on the considered sphere $S^2$. The induced metric $(2)ds^2$ on the sphere $S = \{u = \text{const}, t = \text{const}\}$ asymptotically should be

$$(2)ds^2 = r^2(d\theta^2 + \sin^2 \theta d\phi^2) + O(r).$$ \hspace{1cm} (29)

The induced volume element on $S^2$ can be obtained from the volume element $\epsilon := i^1 \wedge u \wedge m \wedge \hat{m}$, in our case $\epsilon = \sqrt{-g} du \wedge dr \wedge d\theta \wedge d\phi$, by

$$\epsilon_{cd} = \epsilon_{abcd} \left(\frac{\partial}{\partial u}\right)^a \left(\frac{\partial}{\partial r}\right)^b = m \wedge \hat{m} + O(r),$$

$$= r^2 \left(1 - \frac{\rho_0^0}{r^2}\right) \sin \theta d\theta \wedge d\phi + O(r^{-2}).$$ \hspace{1cm} (30)

In the derivation we have used the relation $\frac{\partial}{\partial \rho} = -\sin^2 \theta \frac{\partial}{\partial \theta} e^{-i\phi}(\frac{\partial}{\partial \theta} + i \frac{\sin \theta}{\sin \theta} \frac{\partial}{\partial \phi}).$

### III. QUASI-LOCAL ENERGY-MOMENTUM AND ITS NULL INFINITY LIMIT

There have been many proposals regarding quasi-local quantities. It should be noted that there is as yet no consensus regarding what approach should be used or even what are the proper criteria. However it has been argued that the Hamiltonian approach, which has been used by many researchers and which we adopt here, has certain merits regarding what approach should be used or even what are the proper criteria. However it has been argued that the Hamiltonian approach, which has been used by many researchers and which we adopt here, has certain merits regarding what approach should be used or even what are the proper criteria (see e.g. [4, 5, 6, 10, 11, 21, 22, 23]). For a general region $\Sigma$ (finite or infinite) the Hamiltonian,

$$H(N, \Sigma) = \int_{\Sigma} N^a H_a + \oint_{\Sigma} B(N),$$ \hspace{1cm} (31)

which displaces the region along a vector field $N$, includes not only an integral of a density over the 3-dimensional region but also an integral over its closed 2-surface boundary $S = \partial \Sigma$. For Einstein’s general relativity (as well as any other geometric gravity theory) the Hamiltonian densities $H_a$ are proportional to certain field equations—the initial value constraints—and so the value of the Hamiltonian, $E(N, S)$, is determined purely by the integral of the boundary term. For appropriate choices of the displacement $N$ on the boundary, this Hamiltonian boundary term, for any gravitating system, determines the quasi-local values: in particular from a suitable time-like translation, the quasi-local energy and from a suitable spacelike translation, the quasi-local linear momentum. The approach is quite general; it can incorporate in particular not only all the Noether charge expressions but also all the traditional pseudotensor expressions (and thereby it rehabilitates this often discredited approach) while taming the notorious ambiguities: the choice of boundary expression is linked, via the boundary term in the variation of the Hamiltonian, with the choice of boundary condition, while the reference frame ambiguity can be associated with a choice of boundary reference values, which determine the choice of vacuum or ground state for the system. In this way the traditional ambiguities can be given a clear physical and geometric significance [3, 21, 22]. Within the covariant Hamiltonian formalism certain covariant symplectic expressions for the conserved gravitational quantities have been proposed [4, 5, 6]. When we look at GR from this perspective one of these expressions stands out as being suitable for most applications, (among other virtues it has an associated positive energy proof [21]). Here we consider only this particular expression, specifically, in geometric units

$$E(N, S) = \frac{1}{16\pi} \oint_S \left( \Delta \omega^{ab} \wedge i_N \eta_{ab} + \sqrt{g} N^a \Delta \eta_{ab} \right);$$ \hspace{1cm} (32)
where $\Delta \omega^{ab} := \omega^{ab} - \tilde{\omega}^{ab}$ is the difference between the orthonormal frame connection one-forms (i.e., the Ricci rotation one-forms) and their reference values, $\tilde{\nabla}$ and $\tilde{N}^a$ are the connection and the displacement vector in the reference spacetime, $\eta_{ab} := (1/2)\epsilon_{abcd} \partial^c \wedge \partial^d$, $\Delta \eta_{ab} := (1/2)\epsilon_{abcd} (\partial^c \tilde{\omega}^d - \tilde{\omega}^c \partial^d)$, with $\partial^a$ and $\tilde{\omega}^a$ being, respectively, the dynamic and reference orthonormal co-frames.

The physical and geometric significance of this particular choice of Hamiltonian boundary term expression is revealed by the resultant boundary term in the variation of the Hamiltonian:

$$\delta H(N, \Sigma) = \int_S \left( \text{field equation terms} - \frac{1}{16\pi} \int_S i_N (\Delta \omega^{ab} \wedge \delta \eta_{ab}) \right).$$

This indicates that we should hold fixed on the boundary $S$ the pullback of $\eta_{ab}$, i.e., certain projected components of the coframe—thus, effectively, certain projected components of the metric (arguably the most natural choice).

In the works already cited this quasilocal expression has been tested in various ways. Here we are concerned with the requirement that the value of the quasi-local energy-momentum and the energy-flux have the correct limit at null infinity. To this end we take $S = \partial \Sigma$, the closed boundary of the three dimensional space-like region $\Sigma$, to be a two sphere which approaches in the limit null infinity. The Hamiltonian formalism with a boundary approaching null infinity has been considered from several perspectives, see e.g. [11, 25]. A nice detailed discussion of the topic, addressing all of the important issues has been given recently [11].

By choosing the vector $N$, one can derive the 10 conserved quasi-local quantities for gravity based on the Poincaré symmetry. In particular the covariant quasi-local energy and momentum associated with the time and space translation asymptotic symmetries are

$$p_\nu = \frac{1}{16\pi} \int_S \left( \Delta \omega^{ab} \wedge i_{N^b} \eta_{ab} + \tilde{\nabla}^a \tilde{N}^b \Delta \eta_{ab} \right),$$

Here the value of $\nu$ labels what quantities are evaluated, 0 for energy and 1, 2, 3 for the three components of momentum. In the asymptotically flat case, $N^a$ should be the translation part of the asymptotic Killing vectors. The translation part of the Bondi-Metzner-Sachs(BMS) group is well-defined: its expansion, in the leading order, is of the form $N^a = N^{(0)} + O(r^{-1})$, with $\nu = 0, k = 1, 2, 3$:

$$N^{(0)}_0 = \frac{\partial}{\partial \phi} = f_0 \left(n + \frac{1}{2} l\right) + O(r^{-1}),$$
$$N^{(0)}_k = f_k \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial r}\right) = f_k \left(n - \frac{1}{2} l\right) + O(r^{-1}),$$

where $f_\nu = (1, -\sin \theta \cos \phi, -\sin \theta \sin \phi, -\cos \theta)$.

The energy-momentum expression [24] includes two parts which will be considered separately: the purely physical part and the part including the reference, i.e.

$$p_\nu = p_\nu^{\text{phy}} + p_\nu^{\text{ref}},$$

where

$$p_\nu^{\text{phy}} := \frac{1}{16\pi} \int_S \left( \omega^{ab} \wedge i_{N^b} \eta_{ab} \right), \quad p_\nu^{\text{ref}} := \frac{1}{16\pi} \int_S \left( \tilde{\omega}^{ab} \wedge i_{N^b} \eta_{ab} + \tilde{\nabla}^a \tilde{N}^b \Delta \eta_{ab} \right).$$

We first evaluate the physical part and leave the reference part and the final results to the next section. It is important to keep in mind that the integral is evaluated on a two sphere with constant $u$ and $r$, therefore the only contributing term is $m \wedge \tilde{m}$. The gauge [11] guarantees that the vector $m$ is always tangent to the 2-sphere. Hereafter, we will only present the coefficient of the 2-form $m \wedge \tilde{m}$, denoting this as $\omega m$.

From the expansion of the vector $N$, [33], we realize that the the significant contribution for $N = f_\nu \left(n \pm \frac{1}{2} l\right)$ is

$$\omega^{ab} \wedge i_{N^b} \eta_{ab} = f_\nu \left\{ -i2(\gamma - \tilde{\gamma}) n \wedge i(\alpha - \tilde{\alpha}) \times [n \wedge (a - \tilde{a}) \mp \pm 2\nu] l \wedge \tilde{m} \right\}$$
$$-i2(\alpha - \tilde{\alpha}) n \wedge m + i2(\alpha - \tilde{\alpha}) n \wedge \tilde{m} + i(\alpha - \tilde{\alpha}) m \wedge \tilde{m}. \right\}.$$ (38)

The imaginary unit $i$ comes from the volume element, i.e. $\epsilon = i l \wedge n \wedge m \wedge \tilde{m}$ [24]. However, if we only focus on the value on the two sphere boundary, the result is

$$\omega^{ab} \wedge i_{N^b} \eta_{ab} \equiv i f_\nu [2(\mu + \tilde{\mu}) \pm (\rho + \tilde{\rho})] m \wedge \tilde{m}. \right\}.$$ (39)
IV. GRAVITATIONAL ENERGY-MOMENTUM IN DIFFERENT EMBEDDINGS

Reference configurations play a crucial role in the expressions for gravitational energy and its energy flux (indeed for all the quasi-local quantities). There are two essential related issues: (i) a “suitable” reference configuration choice and (ii) a proper embedding into the physical space-time.

There are two terms in the reference part. The NP formulation for the first one, \( \hat{\omega}_{\alpha\beta} \wedge i_N \eta_{\alpha\beta} \), can be easily read out from \( \text{(35)} \) by replacing all NP coefficients and the frames within the connection with their reference values.

For an asymptotically flat space-time the choice of reference configuration is more or less unambiguous — the Minkowski space-time. In Eddington-Finkelstein coordinates \((u, r, \theta, \phi)\), which are related to the standard coordinates by \( u = t - r \), the first fundamental form of the Minkowski space-time is \( ds^2 = -du^2 - 2udu + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \).

It is easy to verify that the coordinates are just the Bondi coordinates.

Moreover, it is straightforward to check, for \( \hat{\omega}^{ab} \) by \( \hat{n} \), \( \hat{m} \) of physical space-time into the Minkowski space-time is not fully transparent yet. Various proposals have been made in earlier works. In this subsection, we consider several types of embedding used for calculating the reference part of the energy; later we will also use these embeddings for energy flux. We will see that the embeddings we consider actually give the same result for the gravitational energy flux at future null infinity, but not quite the same expression for energy itself.

A. Holonomic embedding

The simplest embedding technique is to identify the Bondi-coordinates with the Minkowski coordinates, i.e. the embedded surface \( S_0 \) is just the standard coordinate round sphere. In this approach, the embedded null frame is

\[
\hat{\nu} = \frac{\partial}{\partial r}, \quad \hat{\mu} = \frac{1}{2r} \frac{\partial}{\partial u}, \quad \hat{\alpha} = -\hat{\beta} = \frac{\zeta}{2\sqrt{2r}}.
\]

Therefore the non-vanishing N-P reference coefficients are

\[
\hat{\rho} = -\frac{1}{r}, \quad \hat{\mu} = -\frac{1}{2r}, \quad \hat{\alpha} = -\hat{\beta} = \frac{\zeta}{2\sqrt{2r}}.
\]

The embedding of the time direction \( \hat{N} \) is \( \hat{N}_\nu = f_\nu \left( \hat{n} + \frac{1}{2} \hat{l} \right) \) which is a Killing vector of the Minkowski space-time. Properly we should calculate \( \hat{\omega}_{\alpha\beta} = \hat{n}_{\alpha\beta} \hat{\omega}^c_c \). However since the difference of the coframes \( \hat{\nu}^c \) and \( \hat{\alpha}^c \) is \( o(r^{-1}) \) we can make an approximation and take for the reference part

\[
\hat{\omega}_{ab} \wedge i_N \eta_{ab} \cong i f_\nu \left[ 2(\hat{\mu} + \hat{\rho}) \pm (\hat{\rho} + \hat{\mu}) \right] m \wedge \bar{m}.
\]

Moreover, it is straightforward to check, for \( \hat{\omega}_{\alpha\beta} = f_\nu \left( \hat{n} + \frac{1}{2} \hat{l} \right) + O(r^{-1}) \), that

\[
\hat{n} \hat{\omega}_{\alpha\beta} \Delta \eta_{ab} = (\hat{\omega}^a_{\nu} f_\nu) \left( \hat{n} + \frac{1}{2} \hat{l} \right) \Delta \eta_{ab} + f_\nu \hat{\omega}^a_{\nu} \left( \hat{n} + \frac{1}{2} \hat{l} \right) \Delta \eta_{ab} + \hat{\nu} O(r^{-1}) \Delta \eta
\]

\[
= (\hat{\omega}^a_{\nu} f_\nu) \left( \hat{n} + \frac{1}{2} \hat{l} \right) \Delta \eta_{ab} + i f_\nu \left[ (\hat{\nu} + \hat{\omega}) \Delta (m \wedge \bar{m}) + \hat{\nu} \Delta (1 \wedge m) - \hat{\alpha} \Delta (1 \wedge \bar{m}) \right] + O(r^{-1})
\]

\[\cong O(r^{-1}).\]

Finally, from the results \( \text{(38) (39) (40)} \), the energy-momentum in the holonomic embedding is

\[
\lim_{\Delta t^+} \rho_\nu = \lim_{\Delta t^+} \frac{1}{16\pi} \int_S f_\nu \left[ 2(\hat{\mu} + \hat{\rho}) \pm (\hat{\rho} - \hat{\mu}) \right] m \wedge \bar{m}
\]

\[= -\frac{1}{4\pi} \int_S \text{Re} \left( \Psi_2^0 + \sigma^0 \hat{\omega}^0 + \hat{\omega}^2_0 \hat{\omega}^0 \right) f_\nu d\Omega^2.
\]
\[ d\Omega^2 = \sin\theta d\theta d\phi. \] The integrand differs slightly from the usual formula for the Bondi energy-momentum; however the integral of the extra term vanishes at least for the energy—which is our real interest here—simply because \( \partial \bar{\sigma} \) is of spin weight -1. Note that entirely analogous terms show up in equivalent calculations done directly in the Bondi-Sachs metric \[7, 25\].

### B. Ó Murchadha-Szabados-Tod embedding

In Ref. \[27\], Ó Murchadha, Szabados and Tod (OST) introduced another kind of embedding method. Let us consider the space-like region \( \Sigma_0 \) in the physical space-time, \( \partial \Sigma_0 = S \). We suppose that \( S \) is a topological two sphere and that isothermal coordinates globally exist on \( S \). In these coordinates, the induced metric \( (2)ds^2 \) on \( S \) is

\[ (2)ds^2 = \omega^2(\theta', \phi')(d\theta'^2 + \sin^2\theta' d\phi'^2), \]

where \( \omega(\theta', \phi') \) is a positive function on \( S \). This two surface is embedded isometrically into the physical space-time (in Bondi coordinates) with

\[ u = \text{const}, \quad \theta = \theta', \quad \phi = \phi', \quad r = \omega(\theta, \phi). \]

Based on this embedding, we define a coordinate transformation in Minkowski space-time,

\[ u \rightarrow U, \quad r \rightarrow R + \omega, \quad \theta \rightarrow \theta, \quad \phi \rightarrow \phi. \]

Then the Minkowski metric becomes

\[
\gamma = \begin{pmatrix}
-1 & -1 & -\partial_\theta \omega & -\partial_\phi \omega \\
-1 & 0 & 0 & 0 \\
-\partial_\theta \omega & 0 & (R + \omega)^2 & 0 \\
-\partial_\phi \omega & 0 & 0 & (R + \omega)^2 \sin^2\theta
\end{pmatrix}.
\]

The NP reference tetrad is chosen to be

\[
\sigma_l = \frac{\partial}{\partial R},
\sigma_n = \frac{\partial}{\partial U} - \frac{1}{2} \left( 1 + \frac{|\delta_0\omega|^2}{(R + \omega)^2} \right) \frac{\partial}{\partial R} + \frac{\partial_\theta \omega}{(R + \omega)^2} \frac{\partial}{\partial \theta} + \frac{\partial_\phi \omega}{(R + \omega)^2 \sin^2\theta} \frac{\partial}{\partial \phi},
\]

\[
\sigma_m = \frac{e^{-i\phi}}{\sqrt{2}(R + \omega)} \left( \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \phi} \right),
\]

where \( \delta_0 = \frac{\partial}{\partial \theta} + \frac{i}{\sin\theta} \frac{\partial}{\partial \phi} \). The associated dual reference tetrad is

\[
\sigma_1 = -dU,
\sigma_2 = -\frac{1}{2} \left( 1 + \frac{|\delta_0\omega|^2}{(R + \omega)^2} \right) dU - dR,
\]

\[
\sigma_3 = -\frac{e^{-i\phi}}{\sqrt{2}(R + \omega)} dU + \frac{1}{\sqrt{2}} e^{-i\phi}(R + \omega)(d\theta + i \sin\theta d\phi).
\]

From the above metric, a direct calculation gives the non-vanishing NP reference coefficients as

\[
\sigma_\rho = -\frac{1}{R + \omega},
\]

\[
\sigma_\mu = -\frac{1}{2} \left[ \frac{1}{R + \omega} + \frac{|\delta_0\omega|^2}{(R + \omega)^3} \right] - \frac{\partial_\theta \omega}{(R + \omega)^2} \cot\theta \partial_\theta \omega + \frac{1}{\sin^2\theta} \frac{\partial_\phi \omega}{(R + \omega)^2},
\]

\[
\sigma_\alpha = -\frac{e^{i\phi}}{2\sqrt{2}} \left[ \frac{\cot\theta}{R + \omega} + \frac{2\delta_0\omega}{(R + \omega)^2} \right],
\]

\[
\sigma_\beta = \frac{e^{i\phi}}{2\sqrt{2}} \cot\theta \frac{1}{R + \omega}.
\]
Minkowski space-time, i.e. the light cone

We can also do the formal Taylor extension near null infinity as in Eqs. (6–10). The only difference is that we have

\[ \Psi^i_0 \frac{\partial}{\partial \phi} \]

Because the spin-weight of \( \Psi^i \) is non-zero, the above results tell us that \( \partial_\phi \tilde{\sigma}^0 = 0 \). Insert this results into Eqs. (6) and (10), the N-P quantities in Minkowski space-time are then

\[ \rho = -\frac{1}{\sqrt{2(R + \omega)^3}} \frac{|\tilde{\sigma}^0|^2}{R^3} + O(R^{-5}), \]

\[ \gamma = \frac{1}{2} \left( \frac{|\tilde{\sigma}^0|^2}{(R + \omega)^3} + \frac{i \cot \theta \partial \phi}{(R + \omega)^2 \sin \theta} \right), \]

\[ \lambda = \frac{e^{2i\phi}}{2(R + \omega)^2} \left[ \frac{\partial \tilde{\sigma}^0}{\partial \phi} - \cot \theta \partial_\theta \tilde{\sigma}^0 + \frac{2i \cot \theta \partial_\phi \tilde{\sigma}^0}{\sin \theta} - \frac{\partial_\theta \partial \phi \tilde{\sigma}^0}{\sin \theta} - \frac{2}{R + \omega} \left( \partial_\phi \tilde{\sigma}^0 - \frac{i \partial \phi \tilde{\sigma}^0}{\sin \theta} \right)^2 \right], \]

\[ \nu = -\frac{e^{i\phi}}{\sqrt{2(R + \omega)^3}} \left[ \frac{|\tilde{\sigma}^0|^2}{R + \omega} - \partial_\phi \tilde{\sigma}^0 \left( \partial_\phi \tilde{\sigma}^0 + \frac{i \cot \theta \partial_\phi \tilde{\sigma}^0}{\sin \theta} \right) + \frac{i \partial_\phi \tilde{\sigma}^0}{\sin \theta} \left( \partial_\phi \tilde{\sigma}^0 + \cot \theta \partial_\phi \tilde{\sigma}^0 + \frac{1}{\sin^2 \theta} \partial_\theta \tilde{\sigma}^0 \right) \right]. \]

The intrinsic geometry of the two sphere is preserved in the OST embedding, therefore \( \Delta \eta_{ab} \equiv 0 \). Moreover, on the considered two sphere \( S, R = 0 \) by the definition of the embedding and, from Eqs. (29) and (30), we have

\[ \omega(\theta, \phi) = r - \frac{|\sigma^0|^2}{2r} + O(r^{-2}). \]

Therefore, this reference contribution differs from the result of the holonomic embedding only by higher orders of \( 1/r \). Hence the energy-momentum at null infinity in the OST embedding is the same as in the holonomic embedding, i.e.

\[ \lim_{x^+} p_\nu = -\frac{1}{4\pi} \int_S \Re \left( \Psi_2^0 + \sigma^0 \bar{\sigma}^0 + \bar{\sigma}^0 \sigma^0 \right) f_\nu \, d\Omega^2. \]

Again the result differs from the usual Bondi integrand by the same extra term—which makes a vanishing contribution to the energy when integrated over the 2-sphere.

C. Brown-Lau-York embedding

In the above two subsections, the two methods used both embedded the surface \( S \) into a standard light cone in Minkowski space-time, i.e. the light cone \( N \) is the light cone from one point. Brown, Lau and York [10] gave another way to do the embedding near null infinity. This method considers a more general light cone.

Suppose we choose a Bondi coordinate system \((u, R, \theta, \phi)\) in Minkowski space-time. The asymptotic shear is \( \tilde{\sigma}^0 \). We can also do the formal Taylor extension near null infinity as in Eqs. (9) and (10). The only difference is that we have \( \tilde{\Psi}_n = 0, n = 0, 1, 2, 3, 4 \). From the NP equations we have

\[ \Psi_3^{(0)} = -\partial_\theta \bar{\sigma}_n \sigma^0, \quad \Psi_4^{(0)} = -\partial_\phi \sigma^0 \sigma^0. \]

Because the spin-weight of \( \partial_\theta \tilde{\sigma}^0 \) is non-zero, the above results tell us that \( \partial_\theta \tilde{\sigma}^0 = 0 \). Insert this results into Eqs. (9) and (10), the N-P quantities in Minkowski space-time are then

\begin{align*}
\rho &= -\frac{1}{R} - \frac{|\tilde{\sigma}^0|^2}{R^3} + O(R^{-5}), \\
\gamma &= \frac{1}{2} \left( \frac{|\tilde{\sigma}^0|^2}{(R + \omega)^3} + \frac{i \cot \theta \partial \phi}{(R + \omega)^2 \sin \theta} \right), \\
\lambda &= \frac{e^{2i\phi}}{2(R + \omega)^2} \left[ \frac{\partial \tilde{\sigma}^0}{\partial \phi} - \cot \theta \partial_\theta \tilde{\sigma}^0 + \frac{2i \cot \theta \partial_\phi \tilde{\sigma}^0}{\sin \theta} - \frac{\partial_\theta \partial \phi \tilde{\sigma}^0}{\sin \theta} - \frac{2}{R + \omega} \left( \partial_\phi \tilde{\sigma}^0 - \frac{i \partial \phi \tilde{\sigma}^0}{\sin \theta} \right)^2 \right], \\
\nu &= -\frac{e^{i\phi}}{\sqrt{2(R + \omega)^3}} \left[ \frac{|\tilde{\sigma}^0|^2}{R + \omega} - \partial_\phi \tilde{\sigma}^0 \left( \partial_\phi \tilde{\sigma}^0 + \frac{i \cot \theta \partial_\phi \tilde{\sigma}^0}{\sin \theta} \right) + \frac{i \partial_\phi \tilde{\sigma}^0}{\sin \theta} \left( \partial_\phi \tilde{\sigma}^0 + \cot \theta \partial_\phi \tilde{\sigma}^0 + \frac{1}{\sin^2 \theta} \partial_\theta \tilde{\sigma}^0 \right) \right].
\end{align*}
\[ \begin{align*}
\mu^\circ &= -\frac{1}{2R} \frac{\theta_0^2 \sigma^0}{R^2} + O(R^{-3}), \\
\lambda^\circ &= \frac{1}{2} \frac{\sigma^0}{R^2} - \frac{\theta_0 \theta_0^0 \sigma^0}{R^2} + O(R^{-3}), \\
\nu^\circ &= O(R^{-3}).
\end{align*} \tag{65} \]

The tetrad part is
\[ \begin{align*}
\mathcal{U}^\circ &= -\frac{1}{2} \frac{\gamma_1 + \tilde{\gamma}_1}{R} + O(R^{-2}), \\
\mathcal{X}^\circ &= -\frac{\theta_0 \sigma^0 \xi^0}{R^2} + O(R^{-3}), \\
\tilde{\xi}^\circ &= \frac{\xi^0}{R} + \frac{|\sigma^0|^2 \xi^0}{R^3} + O(R^{-4}), \\
\tilde{\eta}^\circ &= -\frac{\sigma^0 \xi^0}{R^2} + O(R^{-4}).
\end{align*} \tag{66} \]

The intrinsic geometry of the 2-sphere is preserved in the embedding, i.e.
\[ \mathcal{R} = \tilde{\mathcal{R}}, \tag{67} \]
where the two dimensional Ricci scalar \( \mathcal{R} \) is given by \( \mathcal{R} = -2\rho \mu - 2\tilde{\rho} \tilde{\mu} + 2\sigma \lambda + 2\tilde{\sigma} \tilde{\lambda} + 2\Psi_2 + 2\tilde{\Psi}_2. \) Consequently, this leads to
\[ -\mu \rho + \lambda \sigma + \Psi_2 + \tilde{\Psi}_2 + \tilde{\lambda} \tilde{\sigma} - \tilde{\mu} \tilde{\rho} = -\mu^\circ \rho - \tilde{\mu}^\circ \tilde{\rho} + \lambda^\circ \sigma + \tilde{\lambda}^\circ \tilde{\sigma}. \tag{68} \]

Using the Taylor expansion in Eqs. \( \text{(60)} \) and Eqs. \( \text{(69)} \), we find that the relation between the parameter \( r \) and \( R \) is
\[ R = r + k + O(r^{-1}), \quad \text{with} \quad k = \theta_0^2 \sigma^0 + \tilde{\theta}_0^2 \tilde{\sigma}^0 - \theta_0^2 \tilde{\tau}^0 - \tilde{\theta}_0 \tilde{\tau}^0. \tag{69} \]

We choose \( \sigma^0|_{S_0} = \sigma^0|_{S} \), where \( S \) is the section on \( \mathcal{I}^+ \) and \( S_0 \) is its image under the embedding. We have
\[ R = r + O(r^{-1}) = r[1 + O(r^{-2})]. \tag{70} \]

As for the OST embedding, the two sphere geometry is preserved, therefore \( \Delta \eta_{ab} \cong 0 \). Finally the gravitational energy-momentum in the BLY embedding is
\[ \lim_{\tilde{\mathcal{I}}^+} p_\nu = -\frac{1}{4\pi} \int_S \text{Re} \left( \Psi_2^0 + \sigma^0 \tilde{\sigma}^0 \right) f_\nu \, d\Omega^2. \tag{71} \]

It is worth noting that the BLY embedding is a little neater, in that it directly gives the standard Bondi energy-momentum integrand, whereas there is an additional term in the other two embeddings (which vanishes upon integration). That term is associated with the embedding methods in which the section \( S \) is not embedded into a standard light cone, generated by the null geodesics starting from a single point. This difference again shows us that keeping the inner geometry unchanged under the embedding is not enough to ensure physically reasonable quasi-local quantities; generally, as is especially clear from \[27\], we need more restrictions.

**V. THE ENERGY FLUX AT NULL INFINITY VIA THE DIRECT METHOD**

In this section, we directly calculate the energy flux through a two sphere. For simplicity we choose the time-like translation to be \( N = n + \frac{1}{2} l \). Asymptotically, this agrees with the natural choice, the time translation of the BMS group at null infinity \( \partial_\mu \). The difference between the two vectors is asymptotically negligible.

We consider the energy \( E = H(N, S) \). Suppose \( \Sigma_0 \) is the space-like region that we want to consider, \( \partial \Sigma_0 = S \), and \( \Sigma_{\Delta t} \) is the time evolution of \( \Sigma_0 \). The energy within the region during the time interval changes by the amount \( E(\Sigma_{\Delta t}) - E(\Sigma_0) \), hence a natural direct definition of the rate of energy change is
\[ \dot{E} := \lim_{\Delta t \to 0} \frac{E(\Sigma_{\Delta t}) - E(\Sigma_0)}{\Delta t}. \tag{72} \]
Looking to the value of the Hamiltonian, taking into account the vanishing of the initial value constraints, from (62) we straightforwardly get the quasilocal energy flux relation

$$\dot{E} = \dot{H}(N, \Sigma) := \frac{1}{16\pi} \int_S \nabla^a \nabla^b \Delta \eta_{ab} \dot{N}$$

$$= \frac{1}{16\pi} \int_S \left[ \mathcal{L}_N (\omega^{ab} \wedge i_N \eta_{ab}) - \mathcal{L}_N \left( \dot{\omega}^{ab} \wedge i_N \eta_{ab} \right) - \mathcal{L}_N \left( \dot{N} \mathcal{N}^a \Delta \eta_{ab} \right) \right].$$ (73)

The right hand side defines an energy flux expression \( F \) which includes two parts that will be considered separately: the purely physical part and the other part, which includes the reference, i.e. \( F = F_{\text{phy}} + F_{\text{ref}} \). The reason for making such a separation is that the part including the reference, as we have already seen, depends on the embedding of the reference configuration into physical space-time.

We first evaluate the physical part of the flux,

$$\mathcal{L}_N (\omega^{ab} \wedge i_N \eta_{ab}) = \frac{1}{2} \mathcal{L}_1 (\omega^{ab} \wedge i_N \eta_{ab}) + \mathcal{L}_n (\omega^{ab} \wedge i_N \eta_{ab}).$$ (74)

We see that it is necessary to know the Lie derivative of all two form elements. However, we are only interested in the final results that can contribute to the integral: the terms proportional to \( \mathbf{m} \wedge \bar{\mathbf{m}} \). After a straightforward verification the contributing terms are

$$\mathcal{L}_1 (\mathbf{m} \wedge \bar{\mathbf{m}}) \cong - (\rho + \bar{\rho}) \mathbf{m} \wedge \bar{\mathbf{m}}, \quad \mathcal{L}_n (\mathbf{m} \wedge \bar{\mathbf{m}}) \cong (\mu + \bar{\mu}) \mathbf{m} \wedge \bar{\mathbf{m}}, \quad \mathcal{L}_n (\mathbf{m} \wedge \bar{\mathbf{m}}) \cong - \bar{\nu} \mathbf{m} \wedge \bar{\mathbf{m}}. \quad (75)$$

Therefore, following the result of \( \omega^{ab} \wedge i_N \eta_{ab} \) in Eq. (35), the first term is

$$\mathcal{L}_1 (\omega^{ab} \wedge i_N \eta_{ab}) \cong i D' [(\rho + \bar{\rho}) + 2(\mu + \bar{\mu})] \mathbf{m} \wedge \bar{\mathbf{m}} + i [(\rho + \bar{\rho}) + 2(\mu + \bar{\mu})] \mathcal{L}_1 (\mathbf{m} \wedge \bar{\mathbf{m}})$$

$$\cong i \left[ D' [(\rho + \bar{\rho}) + 2(\mu + \bar{\mu})] + (\alpha - \beta - \bar{\pi}) \right] \mathbf{m} \wedge \bar{\mathbf{m}}$$

$$= i \left[ 2\bar{\delta} + 2\bar{\delta} + 2\bar{\nu} - \frac{1}{2} R + 4\mu^2 - 4|\lambda|^2 + 2(\gamma + \bar{\gamma})(\rho - 2\mu) + 2|\pi|^2 + (\alpha - \beta)\bar{\pi} + \bar{\nu} (\alpha + 3\beta) + 2(\alpha - \bar{\beta} - \pi) \bar{\nu} - 2(\alpha - \beta - \bar{\pi}) \nu \right] \mathbf{m} \wedge \bar{\mathbf{m}},$$ (76)

where \( \mathcal{R} \) is the two dimensional Ricci scalar on the enclosed surface. Similarly, the second term is

$$\mathcal{L}_n (\omega^{ab} \wedge i_N \eta_{ab}) \cong i D' [(\rho + \bar{\rho}) + 2(\mu + \bar{\mu})] \mathbf{m} \wedge \bar{\mathbf{m}} + i [(\rho + \bar{\rho}) + 2(\mu + \bar{\mu})] \mathcal{L}_n (\mathbf{m} \wedge \bar{\mathbf{m}})$$

$$\cong i \left[ D' [(\rho + \bar{\rho}) + 2(\mu + \bar{\mu})] + (\alpha - \beta - \bar{\pi}) \right] \mathbf{m} \wedge \bar{\mathbf{m}}$$

$$= i \left[ 2\bar{\delta} + 2\bar{\delta} + 2\bar{\nu} - \frac{1}{2} R + 4\mu^2 - 4|\lambda|^2 + 2(\gamma + \bar{\gamma})(\rho - 2\mu) + 2|\pi|^2 + (\alpha - \beta)\bar{\pi} + \bar{\nu} (\alpha + 3\beta) + 2(\alpha - \bar{\beta} - \pi) \bar{\nu} - 2(\alpha - \beta - \bar{\pi}) \nu \right] \mathbf{m} \wedge \bar{\mathbf{m}},$$ (77)

Finally, the purely physical contribution to the energy flux is

$$F_{\text{phy}} = \frac{1}{16\pi} \int_S \mathcal{L}_N \left( \omega^{ab} \wedge i_N \eta_{ab} \right)$$

$$= i \int S \left[ \bar{\delta} + 2\bar{\delta} + 2\bar{\nu} \right] \mathbf{m} \wedge \bar{\mathbf{m}}$$

$$= -\frac{1}{4\pi} \int S |\tilde{\sigma}|^2 d\Omega^2. \quad (78)$$

From the asymptotic expansion of all the NP coefficients (61 14 24 25 27) we find that all except one of the terms fall off as \( O(1/r^3) \) or faster, only the \(-2|\lambda|^2\) term contributes asymptotically. Hence the null infinity limit of the purely physical flux is

$$\lim_{r \to \infty} F_{\text{phy}} = \lim_{r \to \infty} \frac{1}{8\pi} \int_S \left( -2|\tilde{\sigma}|^2 + O(r^{-3}) \right) i \mathbf{m} \wedge \bar{\mathbf{m}}$$

$$= -\frac{1}{4\pi} \int S |\tilde{\sigma}|^2 d\Omega^2. \quad (79)$$
A. Holonomic embedding

A straightforward calculation gives

\[ \mathcal{L}_N \left( \tilde{\omega}^{ab} \wedge i_N \eta_{ab} \right) = \mathcal{L}_N \left[ \left( \tilde{\rho} + \dot{\tilde{\rho}} + 2 \tilde{\mu} + 2 \dot{\tilde{\mu}} \right) i \mathbf{m} \wedge \tilde{\mathbf{m}} + \left( \tilde{\alpha} - \tilde{\beta} \right) i \mathbf{n} \wedge \tilde{\mathbf{m}} - \left( \tilde{\alpha} - \tilde{\beta} \right) i \mathbf{n} \wedge \mathbf{m} + O(r^{-3}) \right] \]

\[ \cong \left[ \left( \tilde{\rho} + \dot{\tilde{\rho}} + 2 \tilde{\mu} + 2 \dot{\tilde{\mu}} \right) \left( \frac{1}{2} \tilde{\rho} - \frac{1}{2} \dot{\tilde{\rho}} + \mu + \tilde{\mu} \right) + O(r^{-3}) \right] i \mathbf{m} \wedge \tilde{\mathbf{m}}, \]

\[ \mathcal{L}_N \left( \tilde{\nabla}^a \tilde{N}^b \Delta \eta_{ab} \right) = \mathcal{L}_N \left[ \left( -\frac{1}{2} \tilde{\gamma} + \tilde{\gamma} \right) i \left( \mathbf{m} \wedge \tilde{\mathbf{m}} - \tilde{\mathbf{m}} \wedge \mathbf{m} \right) + \frac{1}{2} \dot{\tilde{\nu}} i \left( 1 \wedge \tilde{\mathbf{m}} - 1 \wedge \mathbf{m} \right) - \frac{1}{2} \tilde{\nu} i \left( 1 \wedge \mathbf{m} - 1 \wedge \tilde{\mathbf{m}} \right) \right] \cong 0. \] (80)

Therefore, the reference part in the holonomic embedding is

\[ F_{\text{ref}} = -\frac{1}{10\pi} \int_S \left( \tilde{\rho} + \dot{\tilde{\rho}} + 2 \tilde{\mu} + 2 \dot{\tilde{\mu}} \right) \left( \frac{1}{2} \tilde{\rho} - \frac{1}{2} \dot{\tilde{\rho}} + \mu + \tilde{\mu} \right) + O(r^{-3}) \] \[ i \mathbf{m} \wedge \tilde{\mathbf{m}} = \frac{1}{16\pi} \int_S O(r^{-3}) i \mathbf{m} \wedge \tilde{\mathbf{m}}. \] (81)

Finally, we have

\[ \lim_{I^+} F = -\frac{1}{4\pi} \int_S |\dot{\phi}|^2 d\Omega^2. \] (82)

This is just the standard expression of the Bondi energy flux at \( I^+ \). \[ \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2}. \]

B. Ó Murchadha-Szabados-Tod embedding

The reference part in the OST embedding is

\[ \mathcal{L}_N \left( \tilde{\omega}^{ab} \wedge i_N \eta_{ab} \right) \cong \left[ (-\tilde{\alpha} + \tilde{\beta} - \tilde{\beta}) \nu + (-\tilde{\alpha} + \tilde{\beta} - \tilde{\beta}) \nu + (\tilde{\rho} + \dot{\tilde{\rho}} + \mu + \tilde{\mu}) \left( \frac{1}{2} \tilde{\rho} + \dot{\tilde{\rho}} + \mu + \tilde{\mu} \right) \right] i \mathbf{m} \wedge \tilde{\mathbf{m}}, \]

\[ \mathcal{L}_N \left( \tilde{\nabla}^a \tilde{N}^b \Delta \eta_{ab} \right) \cong \tilde{\nabla}^a \tilde{N}^b \mathcal{L}_N \left( \Delta \eta_{ab} \right) \equiv - \left[ (\tilde{\gamma} + \tilde{\gamma}) \mathcal{L}_N \Delta (\mathbf{m} \wedge \tilde{\mathbf{m}}) + \tilde{\nu} \mathcal{L}_N \Delta (\mathbf{m} \wedge \tilde{\mathbf{m}}) - \tilde{\nu} \mathcal{L}_N \Delta (\mathbf{m} \wedge \tilde{\mathbf{m}}) \right]. \] (83)

The image of \( S \) is the two sphere in Minkowski space-time such that \( U = \text{const.} \) and \( R = 0 \). From the above calculation, we get the same result as before:

\[ F_{\text{ref}} = \int_S O(r^{-3}) i \mathbf{m} \wedge \tilde{\mathbf{m}}. \] (84)

Consequently, for the energy flux we get the same result as Eq. (84).

C. Brown-Lau-York embedding

Following up on the result in the case of the energy-momentum calculation, the reference part of the energy flux is

\[ F_{\text{ref}} = \int_S O(r^{-3}) i \mathbf{m} \wedge \tilde{\mathbf{m}}. \] (85)

Hence for the total energy flux we again we get the same result as Eq. (85).

VI. THE ENERGY FLUX AT NULL INFINITY VIA AN IDENTITY

In this section, we calculate the energy flux through a two sphere using an interesting formal Hamiltonian identity \[ \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \] derived in detail in \[ \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \frac{\mathcal{L}}{2} \]. The identity is simply the analogue of the classical mechanics identity \( \dot{H} = 0 \), which follows from \( \delta H = \dot{q}^k \delta p_k - \dot{p}_k \delta q_k \) by simply replacing \( \delta \rightarrow a/dt \). We can obtain it directly in essentially the same
way from asymptotic results using that well developed technique. For our calculation here we take coefficients identically (just as they did in the classical mechanics case) leaving way from (33) simply by substituting the time derivative operator £\_n from (33) simply by substituting the time derivative operator £\_n

This is a general quasi-local formula for energy flux, applicable to the boundary of any region. In it was tested in the null infinity limit using the Bondi-Sachs metric.

Here we wish to transcribe this expression into the NP spin coefficient form and confirm that it gives the desired asymptotic results using that well developed technique. For our calculation here we take N = \partial_n = \vec{n} + 1/2 \vec{l} = n + 1/2 \vec{l} + O(1/r), the reference geometry Killing field.

Let us consider Eq. term by term. The first term is

\[ i_N \Delta \omega^{ab} \mathcal{L}_N \eta_{ab} = \Delta \omega^{ab} \mathcal{L}_n \eta_{ab} + \frac{1}{2} \Delta \omega^{ab} \mathcal{L}_i \eta_{ab} + \frac{1}{2} \Delta \omega^{ab} \mathcal{L}_n \eta_{ab} + \frac{1}{4} \Delta \omega^{ab} \mathcal{L}_n \eta_{ab} \]

Based on the asymptotic estimations given in the previous sections and the three considered embedding methods, all of these terms are of higher order than O(1/r), hence they make no contribution to the flux asymptotically. Basically this comes about because (i) in our gauge \( \epsilon = \kappa = 0 \), (ii) the Lie derivative introduces a factor of 1/r, (iii) the spin coefficients \( \sigma, \pi, \tau, \gamma, \nu \) are O(1/r^2), (iii) for \( \rho, \alpha, \beta, \mu \) the \( \Delta \) operation removes their O(1/r) part.

Similarly, for the second term we find

\[ \Delta \omega^{ab} \mathcal{L}_N i_N \eta_{ab} \approx (-\Delta \omega^{ab} i_m \mathcal{L}_N i_N \eta_{ab} + \Delta \omega^{ab} i_m \mathcal{L}_N i_N \eta_{ab}) \mathbf{m} \wedge \bar{\mathbf{m}} \]

Again we used the asymptotic estimations and the three considered embedding methods. In this case we get a non-vanishing asymptotic contribution from \( \lambda = O(1/r) \). Submitting these results into Eq. we find that the null infinity limit of the energy flux calculated from that Hamiltonian identity relation is

\[ \lim_{\mathcal{I}^+} F = -\frac{1}{4\pi} \int_S |\sigma|^2 d\Omega^2, \]

which is, just as was found from the direct calculation, the standard flux loss due to the Bondi news.

VII. DISCUSSION

We have tested certain expressions for the quasilocal energy-momentum and energy flux of gravitating systems. The expressions were obtained from the covariant Hamiltonian formalism. In this formalism the quasilocal quantities are determined by the value of the boundary term in the Hamiltonian. The variation of the Hamiltonian associates the choice of boundary term with specific boundary conditions. Thus the definition of the quasilocal energy-momentum of a gravitating system is linked to the choice of boundary conditions. The boundary term that corresponds to holding
certain projected components of the orthonormal frame fixed seems to be the best choice for most purposes. We have considered only that choice here (the values for certain other choices are given in [7]).

Energy flux can be computed in more than one way. On the one hand it can be obtained directly from the change in the energy expression. On the other hand one can use an interesting identity associated with the specific role of the Hamiltonian and its variation. Here we have evaluated the energy flux by both techniques.

In strong field regions we do not have any sharp test as to what values we should find for energy-momentum and energy flux. Proposed expressions necessarily are first tested in the weak field linearized theory limits. Getting good values at spatial infinity is not the strongest test. The Bondi limit at future null infinity is more delicate. Here we tested our selected expression for energy-momentum and its associate energy flux in this limit.

Technically we used a well known and well developed technique: the Newman-Penrose spin coefficients. We selected a suitable gauge and found that we needed certain quantities expanded in more detail than is usual [30]. In the quasilocal expressions it is necessary to select reference values which determine the “vacuum” or “ground state”. The natural choice is, of course, Minkowski space, but it is not so obvious how to embed the Minkowski space into the asymptotic part of the dynamic space. We considered three types of embeddings which have been used: holonomic, one due to Ó Murchadha, Szabados and Tod [27], and one due to Brown, York and Lau [10]. We found some interesting technical differences between the embeddings but in the end they all gave the same answer: namely the expected Bondi energy and the Bondi energy flux determined by the Bondi news.

In the detailed calculation we noted that, at least in the selected gauge, the quasilocal energy was asymptotically determined by the deviation of the spin coefficient $\mu$ from its asymptotic Minkowski value and that the energy flux was determined by the spin coefficient $\lambda$ (in the notation of [17] these coefficients are $-\rho'$ and $-\sigma'$, respectively).

We have shown that the values of these expressions can be practically calculated in terms of the NP spin coefficient technique; the expressions were found to have the desired asymptotic values. Thus they satisfy an important criterion for quasi-local energy and energy flux expressions.

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