SOME REMARKS ON MEROMORPHIC
FIRST INTEGRALS

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ABSTRACT. A scholium on a paper by Cerveau and Lins Neto.

Our starting point is the following result, recently established by Cerveau
and Lins Neto in their paper [CLN]:

**Theorem 1.** Let $\mathcal{F}$ be a germ of holomorphic foliation on $(\mathbb{C}^2, 0)$. Suppose
that there exists a germ of real analytic hypersurface $M \subset (\mathbb{C}^2, 0)$ which is
invariant by $\mathcal{F}$. Then $\mathcal{F}$ admits a meromorphic first integral.

Of course, in this statement the hypersurface $M$ may be singular at 0,
and this singularity may be even non-isolated. To say that $M$ is invariant
by the foliation refers to its smooth part $M_{\text{reg}}$.

The proof given in [CLN] is rather involved. There are two cases: the
dicritical case and the nondicritical one. In the first case, the authors find
a first integral by a quite mysterious computation with power series. In the
second case, they use delicate dynamical considerations (holonomy group).

Our aim is to give an almost straightforward proof of Theorem 1 which
is based only on some general principles of analytic geometry (in the spirit
of our previous paper on a closely related subject [Bru]), together with a
general (and simple) criterion for the existence of a meromorphic first inte-
gral. This relatively new proof will reveal the beautiful geometric structure
behind foliations tangent to real analytic hypersurfaces.

Let us also recall that Theorem 1 generalizes to codimension one foliations
in higher dimensional spaces, by a standard sectional argument [M-M] [CLN]
(alternatively, our arguments also generalize to higher dimensions with no
substantial new difficulty). Another possible generalization concerns folia-
tions defined on singular spaces, instead of $\mathbb{C}^2$.

1. **AN INTEGRABILITY CRITERION**

Let $\mathcal{F}$ be a holomorphic foliation on a domain $U \subset \mathbb{C}^2$ containing the
origin, with $\text{Sing}(\mathcal{F}) = \{0\}$. Set $U^\circ = U \setminus \{0\}$. A **meromorphic first integral**
is a nonconstant meromorphic function on $U$ which is constant along the
leaves of $\mathcal{F}$.
Proposition 2. Suppose that there exists an irreducible analytic hyper-surface
\[ W \subset U^o \times V, \]
\( V \) being a neighbourhood of 0 in \( \mathbb{C}^2 \), such that:
(1) for every \( p \in U^o \), the fiber
\[ W_p = W \cap (\{p\} \times V) \subset V \]
is a proper analytic curve in \( V \), passing through the origin;
(2) if \( p, q \in U^o \) belong to the same leaf of \( F \), then \( W_p = W_q \);
(3) the projection of \( W \) to \( V \) is Zariski-dense (i.e., not contained in a curve).
Then \( F \) admits a meromorphic first integral on \( U \).

The germ-oriented reader should here replace \( V \) with its germ at the origin, and \( W \) with its germ along \( U^o \times \{0\} \).

Proof. It can be resumed as follows. We already have, by assumptions (1) and (2), a “first integral”, but, instead of being a meromorphic function, it is a map which takes values into the “space of curves in \( V \) through 0”. Hence, roughly speaking, we shall give an algebraic structure to such a space of curves, so that the true meromorphic first integral will be obtained by composition of the former “first integral” with a generic meromorphic function on the space of curves. Hypothesis (3) will guarantee that such a first integral is not identically constant. All of this is trivial if, for instance, each \( W_p \) is a line: the space of lines through the origin is the familiar algebraic variety \( \mathbb{CP}^1 \). The general case only requires some additional blow-ups.

Given a sequence of \( \ell \) blow-ups
\[ \pi : \tilde{V} \to V \]
over the origin, denote by \( D = \cup_{j=1}^{\ell} D_j \) the exceptional divisor \( \pi^{-1}(0) \), and set
\[ \Pi = id \times \pi : U^o \times \tilde{V} \to U^o \times V. \]
Denote by \( \tilde{W} \) the strict transform of \( W \), i.e. the closure of the inverse image by \( \Pi \) of \( W \setminus (W \cap U^o \times \{0\}) \). The trace of \( \tilde{W} \) on \( U^o \times D \) is a hypersurface (of dimension 2), and we shall denote by \( Z \) the union of those irreducible components whose projection to \( U^o \) is dominant (the other components project to curves). Thus, for \( p \in U^o \) generic, the fiber
\[ Z_p = Z \cap (\{p\} \times D) \]
is a finite subset of \( D \), which actually coincides with the trace on \( D \) of the strict transform of \( W_p \) (here we have to exclude not only those points \( p \) such that \( Z_p \) contains some component of \( D \), but also those points which belong to the projection of the non-dominant components of the trace of \( \tilde{W} \): these

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1To avoid confusion: “analytic” without the “real” attribute means “complex analytic”
are precisely the conditions ensuring that the fiber of $\tilde{W}$ over $p$ is equal to the strict transform of $W_p$).

Now, hypothesis (3) implies the following: there exists a sequence of blow-ups $\pi: \tilde{V} \to V$ over the origin such that $Z$ is not of the type $U^0 \times \{\text{finite set}\}$. Indeed, in the opposite case the generic curves $W_p$ would be all unseparable by any sequence of blow-ups, i.e. they would be all equal, and this contradicts the Zariski-density of the projection $W \to V$ (here we use the irreducibility of $W$, and also the fact that every $W_p$ pass through the origin).

In this way, we get an irreducible component of $D$ (say, $D_\ell$) such that the part of $Z$ inside $U^0 \times D_\ell$ (call it $Z_\ell$) is dominant over $U^0$ and Zariski-dense over $D_\ell$.

If $k$ is the degree of $Z_\ell \to U^0$, then $Z_\ell$ defines a meromorphic map $I$ from $U^0$ to $D_\ell^{(k)}$, the $k$-fold symmetric product of $D_\ell$. Such a map is not constant, but it is constant along the leaves of $F$, by hypothesis (2). Since $D_\ell^{(k)}$ is an algebraic variety, we can find $F \in \mathcal{M}(D_\ell^{(k)})$ such that $f = F \circ I$ is a nonconstant meromorphic function, constant along the leaves. Finally, $f$ extends from $U^0$ to $U$ by Levi's theorem. □

**Remark 3.** Consider a foliation $F$ on $U \subset \mathbb{C}^2$, $\text{Sing}(F) = \{0\}$, such that every leaf $L$ is a so-called separatrix at $0$: $L \cup \{0\}$ is a proper analytic curve in $U$. This occurs if $F$ has a meromorphic first integral having $0$ as indeterminacy point, but, as is well known, the converse implication is far from being true, see for instance [Mou] and references therein. We have a naturally defined subset $S$ of $U^0 \times U$: its fiber over $p$ is, by definition, the curve $L_p = L_p \cup \{0\}$. However, generally speaking this subset $S$ is not an analytic subset, since it may be not closed.

Of course, we may take the Zariski-closure $\hat{S}$ of $S$, which however could be the full $U^0 \times U$. If it is not the case, i.e. if $\dim \hat{S} = 3$, then by Proposition 2 we get a meromorphic first integral, and the converse is also true by an easy argument. Note, however, that in this special case our Proposition 2 is closely related to old results by B. Kaup and Suzuki [Suz, §5], relating the existence of first integrals with the analyticity of the graph of the foliation.

Let us stress that, even when a first integral exists, the subset $S$ is typically not an analytic subset, that is its Zariski-closure $\hat{S}$ may be much larger than $S$. Indeed, the fiber of $\hat{S}$ over $p$ may contain, besides $T_p$, other components $T_{p_1}, \ldots, T_{p_n}$. These additional separatrices are precisely the ones which cannot be separated from $T_p$ by meromorphic first integrals. In other words, whereas $S$ represents the (nonanalytic) equivalence relation generated by the leaves, $\hat{S}$ represents the (analytic) equivalence relation generated by level sets of meromorphic first integrals.

There is a variant of Proposition 2 in which the hypothesis that every $W_p$ is a curve passing through the origin of $V$ is replaced by a similar asymptotic hypothesis over the singular point of the foliation. Firstly observe that if
W ⊂ U° × V is as in Proposition 2, then, by standard extension theorems, W can be prolonged to an irreducible analytic hypersurface in U × V. However, it may happen that the fiber over 0 of this extension is not a curve, but the full V; this is precisely the case in which the meromorphic first integral has an indeterminacy point at 0.

Proposition 4. Suppose that there exists an irreducible analytic hypersurface W ⊂ U × V, V being a neighbourhood of 0 in C², such that:

(1) for every p ∈ U, the fiber W_p = W ∩ ({p} × V) ⊂ V is a proper analytic curve in V, passing through the origin when p = 0;
(2) if p, q ∈ U° belong to the same leaf of F, then W_p = W_q;
(3) the projection of W to V is Zariski-dense.

Then F admits a holomorphic first integral on some (possibly smaller) neighbourhood of 0.

Proof. It is even simpler than the previous one; in some sense, it is the “no blow-up case”.

Take a (possibly singular) disk D ⊂ V passing through 0 and intersecting W₀ only at 0. Take the trace Z of W on U × D. Then, up to shrinking U, Z is a hypersurface in U × D and the projection Z → U is proper, say of degree k. We thus obtain, as before, a first integral with values in D^(k). This last space admits a lot of holomorphic functions, and so we get a holomorphic first integral. Thanks to hypothesis (3), and by a suitable choice of D, this first integral will be not identically constant (it is sufficient to choose D highly tangent to a branch of W₀).

□

Remark 5. Consider a foliation F on U ⊂ C², Sing(F) = {0}, such that there is a finite number of separatrices and any other leaf is a proper analytic curve in U. Then, on a possibly smaller U' ⊂ U, the foliation admits a holomorphic first integral [M-M]. This result can be recast into Proposition 4 but one needs some further work. The idea is to look again at the subset S ⊂ U° × U of Remark 3, and to show that the topological closure S in U × U is an analytic hypersurface, which cuts the fiber over 0 along a curve passing through 0. This last curve will be the union of the separatrices (plus the origin).

This indispensable further work can be found in [Mou]. Let Σ ⊂ U be the union of the separatrices and the origin. We may assume that the closure of each separatrix is a (singular) disk passing through 0 and transverse to the boundary of U. According to [Mou, Lemme 1], if p is sufficiently close to 0, and outside Σ, then L_p is a curve transverse to the boundary of U. Using the finiteness of the holonomy of L_p (which is an elementary fact) and Reeb stability, it is then easy to see that the restriction of S to (U' \ Σ') × U is an analytic hypersurface, where U' is a sufficiently small neighbourhood of 0 and Σ' = Σ ∩ U'. Take now the topological closure S of S in U' × U. By standard results (Remmert-Stein), if S is not an analytic hypersurface, then it must contain an irreducible component of Σ' × U; this is however
impossible, again by \[\text{Mou}, \text{Lem} 1\] (which implies that the \(F\)-saturation of \(U'\) cannot be the full \(U\)). Hence, \(\overline{S}\) is an analytic hypersurface in \(U' \times U\). By the same reason, its fiber over 0 cannot be the full \(U\), and therefore it must coincide with \(\Sigma\). We can now apply Proposition 4.

It is also worth observing that the fiber of \(\overline{S}\) over a point \(p \in U' \setminus \Sigma'\) is the single leaf \(L_p\). This corresponds to the fact that the leaves outside the separatrices can be separated by holomorphic first integrals.

2. Complexification of real hypersurfaces

Consider now the setting of Theorem 1: \(F\) is a foliation on \(U \subset \mathbb{C}^2\), singular at \(0 \in U\), and \(M\) is a real analytic hypersurface passing through the origin and invariant by \(F\). We denote by \(M_{\text{reg}} \subset M\) the (open) subset of regular points, i.e. the points where \(M\) is a real analytic submanifold of dimension 3. We assume that \(0 \in \overline{M_{\text{reg}}}\) (otherwise, the germ of \(M\) at 0 would not be a germ of hypersurface, as prescribed by Theorem 1). Without loss of generality, we may also assume that \(M\) is irreducible, and even that the germ of \(M\) at 0 is irreducible.

Let us recall few facts concerning complexification, see also \([\text{Bru}, \S 3]\).

Denote by \(U^*\) the complex manifold conjugate to \(U\): it is the same differentiable manifold, but with the opposite complex structure; equivalently, holomorphic functions on \(U^*\) are the same as antiholomorphic functions on \(U\). Remark that if \(A\) is an analytic subset of \(U\), then it is analytic also as a subset of \(U^*\). As such, it will be denoted by \(A^*\). In particular, every point \(p \in U\) has a “mirror” point \(p^* \in U^*\). Similarly, if \(F\) is a holomorphic foliation on \(U\), then it is holomorphic also as a foliation on \(U^*\), and as such it will be denoted by \(F^*\). Remark that, generally speaking, the two foliations \(F\) and \(F^*\) are different as holomorphic foliations: the identity map \(U \rightarrow U^*\) obviously conjugate \(F\) to \(F^*\), but such a map is antiholomorphic, and not holomorphic. For example, if \(\gamma \subset L \in F\) is a loop with linear holonomy \(\lambda\), then the same loop \(\gamma \subset L^* \in F^*\) has linear holonomy \(\overline{\lambda}\).

In the product space \(U \times U^*\) (with the product complex structure) we have the involution

\[j : U \times U^* \rightarrow U \times U^*\]

\[j(p, q^*) = (q, p^*).\]

It is antiholomorphic. Its fixed point set is the diagonal \(\Delta\), and it is a totally real submanifold.

It is convenient to look at our real analytic hypersurface \(M\) in \(U\) as a subset of the diagonal:

\[M \subset \Delta \subset U \times U^*\]

Then, \(M\) can be complexified: there exists a neighbourhood \(\mathring{U} \subset U \times U^*\) of the diagonal and an irreducible complex analytic hypersurface \(M^C\) in \(\mathring{U}\) such that

\[M^C \cap \Delta = M.\]
Up to restricting $U$ around the origin, we may assume $\hat{U} = U \times U^*$. Remark that
\[ j(M^C) = M^C \quad \text{and} \quad \text{Fix}(j|_{M^C}) = M. \]

Actually, this complexification can be done on any real analytic subset. In particular, we can start with a complex analytic subset $A \subset U$, and look at it as a subset of $\Delta$, thus forgetting its complex analytic structure and retaining only its real analytic one. Its complexification is then simply the product $A \times A^*$ (which could be pompously called “complexification of the decomplexification of $A$”).

Consider now the projection
\[ pr : M^C \rightarrow U \]
to the first factor, and for every $p \in U$ set
\[ M^C_p = pr^{-1}(p). \]
It is an analytic subset of $U^*$.

**Lemma 6.** Up to shrinking $U$ around the origin, we have: for every $p \in U^o$, $M^C_p$ is a (nonempty) curve in $U^*$. 

**Proof.** The irreducibility of $M^C$ implies that the set of points of $U$ over which the fiber is two-dimensional (i.e., the full $U^*$) is discrete. Hence, up to shrinking $U$, we get that $M^C_p$ is at most one-dimensional for every $p \in U^o$ (note that a shrinking of $U$ implies a simultaneous shrinking of $U^*$, but this is not a problem).

Obviously $M^C_p$ cannot contain isolated points, because $M^C$ is a hypersurface. Therefore, it remains to show that it is not empty. Of course $M^C_0$ is not empty, for any choice of $U$, and we can distinguish two cases:

(a) $M^C_0 = U^*$, i.e. $M^C$ contains $\{0\} \times U^*$. Because $M^C$ is $\gamma$-invariant, this means that also the horizontal fiber $U \times \{0^*\}$ is fully contained in $M^C$. As a consequence, every $M^C_p, p \in U^o$, is a curve which, moreover, passes through the origin.

(b) $M^C_0$ is a curve in $U^*$. Then, by a standard result (Remmert’s Rank Theorem), the map $pr$ is open, and hence surjective for a suitable choice of $U$. \qed

We shall see that case (a) corresponds to the dicritical case, and case (b) to the nondicritical one.

Recall now that we have a holomorphic foliation $\mathcal{F}$ on $U$, leaving $M$ invariant.

**Lemma 7.** For every $p \in U^o$, the curve $M^C_p \subset U^*$ is invariant by $\mathcal{F}^*$. Moreover, if $p$ and $q$ belong to the same leaf, then $M^C_p = M^C_q$.

**Proof.** This is basically [Bru, Lemma 3.1], but let us explain it in a slightly different manner.
On $U \times U^*$ we have the foliation (of dimension 2) $\mathcal{F} \times \mathcal{F}^*$. It is nonsingular on $U^\circ \times U^{\circ *}$, and its leaf through $(p, q^*)$ is

$$L_{p,q^*} = L_p \times L_{q^*}$$

where the first factor is the leaf of $\mathcal{F}$ through $p$ and the second factor is the leaf of $\mathcal{F}^*$ through $q^*$. In particular, if $(p, p^*) \in \Delta$, then $L_{p,p^*} = L_p \times L_{p^*}$, and this is also the complexification of $L_p \subset \Delta$ (here $L_p$ is not yet properly embedded, but it doesn't matter for the next arguments). It follows that if we take a leaf $L_p$ contained in $M$, then the leaf $L_{p,p^*}$ is contained in $M^\mathbb{C}$.

As a consequence of this, if $L$ is any leaf of $\mathcal{F}$, its preimage $pr^{-1}(L) \subset M^\mathbb{C}$ is a union of leaves of $\mathcal{F} \times \mathcal{F}^*$ (plus possibly some singular point on $U^\circ \times \{0^*\}$), i.e. it is of the form $L \times (L_1^* \cup \ldots \cup L_n^*)$ for suitable leaves $L_i^*$ of $\mathcal{F}^*$ (plus possibly some singular point). But this is precisely the assertion of the lemma.

**Remark 8.** Without assuming the existence of $\mathcal{F}$, the same argument shows the following: if $M \subset U$ is any real analytic Levi-flat hypersurface, then on $M^\mathbb{C}$ we have a two-dimensional foliation whose leaves are products of horizontal and vertical fibers of $M^\mathbb{C}$. Here the essential point is that if we take a horizontal fiber and a vertical fiber of $M^\mathbb{C}$, passing through the same point of $M^\mathbb{C}$, then their product is still contained in $M^\mathbb{C}$. This is a remarkable symmetry property of $M^\mathbb{C}$, and of course it is a manifestation of the Levi-flatness of $M$. This foliation appears also in [CLN], as complexification of the Levi foliation, but the authors obtain the properness of leaves only after a long tour.

**Remark 9.** The fact that $M_p^\mathbb{C}$ may contain several leaves of $\mathcal{F}^*$ should be compared with the phenomenon described in Remark 3. Note also that on a neighbourhood of $M_{\text{reg}}$ we have a Schwarz reflection at the level of the leaf space [Bru p. 669]. If $p$ is close to $M_{\text{reg}}$, then $M_p^\mathbb{C}$ contains the Schwarz reflection of $L_p$ (which must be understood as a leaf of $\mathcal{F}^*$). The fact that $M_p^\mathbb{C}$ is defined for every $p$ could be interpreted as a sort of “globalization” of that Schwarz reflection, and the fact that $M_p^\mathbb{C}$ contains several leaves suggests that “reflection” should be replaced by “correspondence”.

For example, suppose that $\mathcal{F}$ is the radial foliation $(zdw - wdz = 0)$, so that $M$ corresponds to a real algebraic curve $\gamma \subset \mathbb{C}P^1$ (= the space of leaves of $\mathcal{F}$). The complexification of $\gamma$ is a complex algebraic curve $\gamma^\mathbb{C} \subset \mathbb{C}P^1 \times \mathbb{C}P^1^*$, which gives an antiholomorphic correspondence of $\mathbb{C}P^1$ with itself.\footnote{We say that a real analytic curve $\gamma \subset \mathbb{C}P^1$ is real algebraic if its complexification $\gamma^\mathbb{C}$, which is in principle defined only on a neighbourhood of the diagonal, extends to the full $\mathbb{C}P^1 \times \mathbb{C}P^1^*$. With this definition, it is easy to see that a radial Levi-flat hypersurface $M$...}
We can now immediately complete the proof of Theorem 1. Firstly note that, by Lemmata 6 and 7, every leaf of $F$ is properly embedded in $U^\circ$. If $M_0^C$ is the full $U^*$ then, as observed in the proof of Lemma 6, every $M_p^C$, $p \neq 0$, is a curve through the origin, and so we can apply Proposition 2 to get a meromorphic first integral. If $M_0^C$ is a curve, then it is a curve through the origin, by symmetry, and so we can apply Proposition 4 (actually, in that Proposition the requirement that $W_0$ passes through 0 can be obviously replaced by $W_0 \neq \emptyset$).

Let us conclude with a question. In the setting of Theorem 1, consider firstly the case where we have a (primitive) holomorphic first integral $f$, with $f(0) = 0$. It is then easy to see that $M = \hat{f}^{-1}(\hat{\gamma})$ where $\hat{f}$ is the projection to the space of leaves $\Sigma$ (a non-Hausdorff Riemann surface [Mon, Suz]) and $\hat{\gamma} \subset \Sigma$ is a real analytic curve. Moreover, $f = e \circ \hat{f}$, where $e : \Sigma \to V \subset \mathbb{C}$ is the map which collapses nonseparated points. However, due to the special structure of $\Sigma$ in this case (there is only a finite set of nonseparated points, all sent to 0 by $e$) we certainly have $\hat{\gamma} = e^{-1}(\gamma)$ for some $\gamma \subset V$, and so $M = f^{-1}(\gamma)$. Now, consider the dicritical case, where the first integral $f$ is only meromorphic, and 0 is an indeterminacy point. Can we find a real algebraic curve $\gamma \subset \mathbb{C}P^1$ such that $M = f^{-1}(\gamma)$? The problem here is that the collapsing map $e : \Sigma \to \mathbb{C}P^1$ is much more complicated, and in principle the curve $\hat{\gamma} \subset \Sigma$ could be not of the form $e^{-1}(\gamma)$, i.e. there could exist two unseparable leaves $L, L'$ with $L$ in $M$ but not $L'$. Of course, we can set $\gamma = e(\hat{\gamma})$ and take $f^{-1}(\gamma)$, but this last one could be reducible, and our initial $M$ could be only one irreducible component of it.

![Figure 1](image.png)

**Figure 1.** Can $M = \hat{f}^{-1}(\hat{\gamma})$ and $M' = \hat{f}^{-1}(\hat{\gamma}')$ be irreducible components of $f^{-1}(\gamma)$?

is analytic at 0 if and only if the corresponding $\gamma$ is real algebraic; the complex curve $\gamma^C$ is then the trace of $\tilde{M}^C$ on $\mathbb{C}P^1 \times \mathbb{C}P^1^* \subset \tilde{U} \times \tilde{U}^*$, where $\tilde{M}$ is the strict transform of $M$ in $\tilde{U} = \text{the blow-up of } U$ at 0.
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