Testing the AdS/CFT correspondence beyond large N

Adel Bilal and Chong-Sun Chu

Institute of Physics, University of Neuchâtel, CH-2000 Neuchâtel, Switzerland
E-mail: adel.Bilal@iph.unine.ch, chong-sun.chu@iph.unine.ch

Abstract: According to the AdS/CFT correspondence, the $d = 4$, $\mathcal{N} = 4$ super Yang-Mills theory is dual to the type IIB string theory compactified on $AdS_5 \times S^5$. Most of the tests performed so far are confined to the leading order at large $N$ or equivalently string tree-level. To probe the correspondence beyond this leading order and obtain $\frac{1}{N^2}$ corrections is difficult since string one-loop computations on an $AdS_5 \times S^5$ background generally are beyond feasibility. However, we will show that the chiral $SU(4)_R$ anomaly of the super YM theory provides an ideal testing ground to go beyond leading order in $N$. In this paper, we review and develop further our previous results [1] that the $1/N^2$ corrections to the chiral anomaly on the super YM side can be exactly accounted for by the supergravity/string effective action induced at one loop.

1. Introduction

We begin by briefly reviewing some relevant basic aspects of the AdS/CFT correspondence \[ \frac{1}{\mathcal{N}} \], see in particular [3].

Consider type IIB string theory with a number $N$ of D3 branes. There are open strings ending on these D3 branes and closed strings in the bulk. The effective low energy action consists of IIB supergravity describing the bulk closed strings, and $\mathcal{N} = 4$ supersymmetric $U(N)$ gauge theory (SYM) describing the open strings ending on the D3 branes. The gauge group of the latter actually is $SU(N)$ since the U(1) decouples. Considering low energies at fixed $\alpha'$ is equivalent to fixed energy and taking the $\alpha' \to 0$ limit. In this limit the gravitational coupling $\kappa \sim g_s \alpha'^2$ vanishes and the interactions between the branes and the bulk can be neglected, as well as all higher derivative terms in the brane action. Only free bulk supergravity and pure $\mathcal{N} = 4$ SYM in $d = 4$ dimensions remain. The latter is a conformal field theory (CFT).

There is a different way to describe the same physics. D3 branes may be viewed as certain solutions of the supergravity field equations, namely

$$ds^2 = \frac{1}{\sqrt{f}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)$$

$$f = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi g_s \alpha'^2 N.$$ (1.1)

Here $t, x_1, x_2, x_3$ are the longitudinal coordinates (on the D3) while $r$ and $\Omega_5$ describe the transverse space. $N$ is the number of (coincident) D3 branes. There is also a five-form field strength $F$ which is proportional to $N$. Again, one wants to study the low energy excitations in this description and compare with the first one. Due to the non-trivial function $f(r)$ in the metric there is a red-shift factor between energies measured at $r$ and energies measured at $r = \infty$:

$$E_\infty = f^{-1/4} E_r = \left(1 + \frac{R^4}{r^4}\right)^{-1/4} E_r.$$ (1.2)

One sees that if $r \to 0$, $E_\infty \to 0$ for any finite $E_r$. So there are two types of low energy excitations: low energy at finite $r$ (bulk) or finite energy near the horizon ($r = 0$), and the two types decouple yielding (free) bulk supergravity and near horizon supergravity. Concentrate on the near horizon limit: as $r \to 0$ one has $f \sim \frac{R^4}{r^4}$. Since also $\alpha' \to 0$, it is convenient to introduce the finite quantity $u = r/\alpha'$ so that the near horizon metric becomes

$$\frac{1}{\alpha'} ds^2 = \lambda^{-1/2} u^2 \left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right)$$
+λ^{1/2} \frac{du^2}{u^2} + \lambda^{1/2}d\Omega_5^2, \quad (1.3)

where we introduced the finite quantity

\[ \lambda = 4\pi g_s N = \frac{R^4}{\alpha^2}. \quad (1.4) \]

The metric (1.3) is the metric of AdS$_5 \times S^5$.

Comparing both descriptions of the same physics one then is led to identify the conformally invariant, $N = 4$ SU($N$) SYM theory in $d = 4$ with the supergravity or small $\alpha'$ limit of IIB string theory on AdS$_5 \times S^5$. We will be more precise shortly.

How do the different parameters on the SYM side compare to those of the supergravity/string theory? In the former we have the coupling $g_{YM}$ and the integer $N$ determining the gauge group SU($N$). On the supergravity/string side we have $\alpha'$ and $R$ (with only the dimensionless ratio $R^2/\alpha'$ being a relevant parameter) and the coupling $g_s$. A first relation is already obtained in eq. (1.1), namely $R^4 = 4\pi g_s \alpha'^2$ or equivalently eq. (1.4). A second relation is obtained from the D3 brane action from which one reads the YM coupling in terms of the string coupling. Thus

\[ \frac{1}{g_{YM}^2} = \frac{1}{4\pi g_s} \quad \text{and} \quad N = \frac{1}{4\pi g_s} \frac{R^4}{\alpha'^2} \quad (1.5) \]

expresses the SYM parameters $g_{YM}$ and $N$ in terms of the string/supergravity parameters $g_s$ and $R^2/\alpha'$ and vice versa. In large $N$ gauge theories the relevant loop-counting parameter is the 't Hooft coupling $g_{YM}^2 N$ rather than $g_{YM}^2$. Combining both eqs (1.3) yields $g_{YM}^2 N = \frac{R^4}{\alpha'^2}$ which by eq (1.4) is just the quantity called $\lambda$. It is useful to rewrite the relations between the parameters of the two descriptions as

\[ \lambda = \frac{R^4}{\alpha'^2}, \quad \frac{N}{\lambda} = \frac{1}{4\pi g_s} \quad (1.6) \]

with $\lambda$ now meaning the 't Hooft coupling.

Let us first comment on the first relation: perturbative SYM theory is a good description if the 't Hooft coupling $\lambda$ is small. Supergravity, rather than string theory, should be a good description if the radius of curvature of AdS$_5$ and $S^5$ is large, meaning $R^2 \gg \alpha'$ or $\lambda$ large. The two regimes are opposite as is often the case with dualities. This of course avoids the obvious contradiction that both descriptions look so different.

At fixed 't Hooft coupling $\lambda$, the second relation (1.6) tells us that $\frac{1}{N\lambda}$ corresponds to the string loop-counting parameter $g_s^2$, so that the large $N$ limit of SYM corresponds to classical string theory (or classical supergravity if also $\lambda \gg 1$), and $\frac{1}{N\lambda}$ corrections should correspond to one-loop effects in string theory.

One now has various possible conjectures: 1.) The weakest one is: SYM is dual to AdS supergravity only for $\lambda \to \infty$, but the full string theory is different. This version would not be very useful. 2.) The SYM theory is dual to string theory on AdS for finite $\lambda$ but only as $N \to \infty$ or equivalently $g_s \to 0$. This includes $\alpha'$ corrections beyond the supergravity approximation, but no string loops. 3.) This is the strongest version, generally referred to as the Maldacena conjecture: the SYM theory is dual to string theory for all $\lambda$ and all $N$, i.e. all $R^4/\alpha'^2$ and $g_s$.

While there is now reasonable evidence for version 2.) of the conjecture (see e.g. [3]) the strong version 3.) is hard to test since standard string or supergravity loop computations on an AdS$_5 \times S^5$ background are difficult, to say the least, if not unfeasible, with the present state of the art.

Many successful tests are group theoretic in nature. Some are not restricted to tree-level or even perturbation theory, but on the other hand they do not really provide any real test at one-loop. Examples are global symmetries (disregarding possible anomalies for the moment): AdS$_5$ space-time has an SO(4,2) symmetry which also is the conformal group of the $N = 4$ SYM theory. The “internal” symmetry is SO(6) $\simeq$ SU(4): this is the isometry of the $S^5$ as well as the R-symmetry of the SYM theory. The latter actually is anomalous which will be important for us. Both theories have the same amount of supersymmetry, the full supergroup being SU(2,2)$_{4} \supset$ SO(4,2) $\times$ SU(4)$_R$. Also the duality symmetry SL(2,$\mathbb{Z}$) is the same as it acts on

\[ \tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\chi}{2\pi}. \quad (1.7) \]

A promising arena for performing tests beyond the large $N$ limit or string tree-level is to look at certain anomalies. On the SYM side anomaly coefficients are easily established one-
loop effects in $\lambda$ that are protected against higher order corrections. Typically such an anomaly coefficient will depend on the number of fermion fields in the SYM theory, i.e. on $N$. The goal then is to reproduce the exact $N$-dependence from the string theory including subleading terms $\sim \frac{1}{N^2}$ coming from string loops. If we want to have any chance to be able to do this calculation, the relevant quantity to compute in string theory should be of a topological nature, like a Chern-Simons term, so that the actual metric on $AdS_5$ is irrelevant.

2. The chiral SU(4)$_R$ anomaly in the $\mathcal{N} = 4$ SYM

The anomaly we will consider is the chiral SU(4)$_R$ anomaly. As already mentioned, SU(4)$_R$ is a (classical) global symmetry of the $\mathcal{N} = 4$ SYM theory. Due to the presence of chiral fermions transforming in complex conjugate representations of SU(4)$_R$ this symmetry is spoiled at one loop and there is an anomaly: the one-loop effective action in the presence of external SU(4)$_R$ gauge fields is no longer invariant under SU(4)$_R$ and the non-invariance is proportional to the number of fermions. Since they are also in the adjoint representation of the “true” gauge group SU(N) there are $N^2 - 1$ of them, and the anomaly is proportional to $N^2 - 1$. As we recall below, the leading term $\sim N^2$ is accounted for by tree-level supergravity $\mathbb{1}$. It is the $-1$ correction which should originate from a string/supergravity loop effect, and it indeed does as we showed in $\mathbb{1}$ and explain in the remainder of this paper.

Before explaining the string/supergravity loop correction let us review how the leading $N^2$ term is obtained in the string/supergravity description. Here SU(4) $\simeq$ SO(6) acts as an isometry on the $S^5$. As a consequence, the $AdS_5$ supergravity is actually a gauged supergravity $\mathbb{1} \mathbb{2} \mathbb{3}$ and there is an SU(4)$_R$ gauge group with gauge fields $A_a^\mu(x,z), \mu = 0, 1, \ldots, 4$ and $a = 1, \ldots, 15 = \text{dim SU(4)}$. This gauge group is of course not to be confused with the SU(N) of the conformal SYM theory. Note also that in the latter, SU(4)$_R$ is a global symmetry, hence there are SU(4)$_R$ currents $J_\mu^a(x), \mu = 0, 1, 2, 3$, but no associated gauge fields. We can nevertheless couple these currents to external gauge fields $A_a^\mu(x), \mu = 0, 1, 2, 3$ which act as sources for these currents. Then by a standard argument, the non-invariance of the one-loop effective action $\Gamma[A_a]$ under gauge transformations of these external gauge fields is equivalent to the covariant non-conservation of the currents: let $\delta_v$ be such a gauge transformation with parameter $v$ then

$$\delta_v \Gamma[A_a] = \int \delta_v A_a^\mu \frac{\delta \Gamma}{\delta A_a^\mu} = \int \delta_v A_a^\mu J_\mu^a$$

$$= \int (D_\mu v)^a J_\mu^a = \int v^a (D_\mu J^\mu)^a$$

(2.1)

with

$$(D_\mu J^\mu)^a \sim - (N^2 - 1) d^3 \phi e^{\nu \rho \sigma} \partial_\mu A_\rho^a \partial_\sigma A_\nu^a + \ldots$$

(2.2)

the precise numerical coefficient being given below.

There is a standard prescription $\mathbb{1}$ in the AdS/CFT correspondence how to compute correlation functions: we will give this prescription for the case of present interest. For any (SU(N) gauge-invariant) operator $O(x)$ like the currents $J_\mu^a(x)$ of the SYM theory, introduce a source $\phi_0(x)$ like the $A_a^\mu(x)$. Then the generating functional for correlators of $J_\mu^a$ is just

$$e^{-\Gamma[A]} \equiv \langle e^{-\int d^4 x A_a^\mu(x) J_\mu^a(x)} \rangle_{\text{SYM}} .$$

(2.3)

In AdS$_5$ string theory there is a field $\phi(x,z)$ such that at the boundary $z = 0$ of AdS$_5$ (note that $z = 1/u$) which is just the four-dimensional space of the SYM theory one has $\phi(x,z = 0) = \phi_0(x)$. In our case this is just $A_\mu^a(x,z = 0) = A_\mu^a(x)$ (for $\mu = 0, 1, 2, 3$ only) where the $A_\mu^a$ are the gauge fields of the gauged supergravity. The prescription $\mathbb{1}$ then is

$$e^{-\Gamma[A]} = Z_{\text{string}} \bigg|_{A_\mu^a(x,z=0)=A_\mu^a(x)} ,$$

(2.4)

meaning that the string partition function should be evaluated subject to the boundary condition $A_\mu^a(x,z = 0) = A_\mu^a(x)$ for $\mu = 0, 1, 2, 3$. Writing

$$Z_{\text{string}} = e^{-S_{\text{string}} - S_{\text{1-loop}}_{\text{string}} - \ldots} = e^{-S_{\text{eff}}_{\text{string}}}$$

(2.5)

eq (2.4) together with eq (2.3) implies that, if the AdS/CFT correspondence is correct, we should
have

\[ \delta_v S_{\text{string}}^{\text{eff}} \bigg|_{\tilde{A}_\mu^a(x, z = 0) = A_\mu^a(x)} = \delta_v \Gamma[A] \]

\[ = - \int \sigma^a (D_\mu J^\mu)^a \] (2.6)

which is non-vanishing according to (2.2). Thus the SU(4) gauge variation of \( S_{\text{string}}^{\text{eff}} \) should directly reproduce the SYM chiral SU(4) anomaly. Actually for the purpose of reproducing the leading \( N^2 \) part of the anomaly it is enough to consider the classical supergravity action [10].

Let us now determine the exact anomaly coefficient of the \( N = 4 \) SYM theory in 4 dimensions. This theory has four complex Weyl fermions \( \lambda \) in the fundamental representation of SU(4) with the chirality part \((0,1/2)\) in 4 and \((1/2,0)\) in 4* (see for example [10]). Our conventions here are equivalent to those of [14]. Moreover, all fields are also in the adjoint representation of the “true” gauge group SU(N) which acts as a “flavour” group with respect to the SU(4)_R. Thus there are actually \( N^2 - 1 \) complex Weyl fermions \( \lambda \) in the 4, resp. 4*. The correctly normalised R-symmetry anomaly is given by

\[ \delta_v \Gamma[A] = (N^2 - 1) \int_{S^4} \omega_4^1 (v, A). \] (2.7)

The differential forms

\[ \omega_4^1(v, A) = \frac{1}{24 \pi^2} \text{Tr} [v dA dA + \frac{1}{2} A^3], \]

\[ \omega_5(A) = \frac{1}{24 \pi^2} \text{Tr} [A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5] \] (2.8)

satisfy the descent equations \( d\omega_5 = \frac{1}{24 \pi} \text{Tr} F^3 \), and \( \delta_v \omega_5 = d\omega_1^1 \) with \( F = dA + A^2 \), \( A = A^a T^a \) and \( v = v^a T^a \) as usual, the \( T^a \) being the generators of SU(4) in the fundamental 4 representation. For later use we note that for \( T^a \) in a general representation \( \mathbf{R} \) of SU(4), the corresponding quantities with the trace taken in \( \mathbf{R} \) are

\[ \omega_{2n}^1 \mathbf{R} = A(\mathbf{R}) \omega_{2n}^1, \quad \omega_{2n+1}^R = A(\mathbf{R}) \omega_{2n+1} \] (2.9)

where \( A(\mathbf{R}) \) is the anomaly coefficient defined by the ratio of the \( d \)-symbols taken in the representation \( \mathbf{R} \) and in the fundamental representation. In general 2\( n \) or 2\( n + 1 \) dimensions, since the \( d \)-symbol is given by a symmetrized trace of \( n + 1 \) Lie algebra generators, it is easy to show that the complex conjugate representation \( \mathbf{R}^* \) has an anomaly coefficient

\[ A(\mathbf{R}^*) = (-1)^{n+1} A(\mathbf{R}). \] (2.10)

Due to the connection of anomalies and Chern-Simons actions in one higher dimension, it is natural to expect that the four-dimensional field theory anomaly is dual to a Chern-Simons action in the gauged AdS_5 supergravity. This is indeed the case as was first pointed out in [8]. The tree level supergravity action on AdS_5 contains the following terms [8][8][8][8]

\[ S_{\text{cl}}[A] = \frac{1}{4 g^2_{5G}} \int d^5 x \sqrt{g} F_{\mu\nu}^a F^{\mu\nu a} + k \int_{\text{AdS}_5} \omega_5. \] (2.11)

Note that here \( F \) is the field strength associated with the five-dimensional gauge field \( \tilde{A}_\mu \). The exact values of the coefficients \( \frac{1}{4 g^2_{5G}} \) and \( k \) will be important for us. Their ratio is fixed by supersymmetry [8][8][8][8]. They may be obtained by dimensional reduction of the ten-dimensional IIB supergravity on S^5 using the fact that the radius of S^5 is given by eq. (1.4) as \( R^4/\alpha'^2 = 4 g_y N \). Then it is easy to determine the normalization of the gauge kinetic energy term and one finds

\[ g^2_{5G} = \frac{16 \pi^2}{N^2}, \quad k = N^2. \] (2.12)

Note that the action (2.11) with the values (2.12) has been used to compute the 2-point and 3-point correlators of the currents \( J_\mu^a \) in the SYM theory [8]. To leading order in \( N \) this gives the correct result.

In usual considerations of supergravity on AdS, one considers gauge configurations \( \tilde{A}_\mu \) that vanish at the boundary and so the Chern-Simons term is gauge invariant since \( \delta \omega_5 = d\omega_1^1 \) and the integral vanishes. For the considerations of the AdS/CFT correspondence however, we precisely want nonvanishing boundary values for \( \tilde{A}_\mu \) as explained above, cf eq (2.4). Then under a gauge variation \( \delta_\epsilon \tilde{A} \), the variation of the Chern-Simons term is a boundary term

\[ \delta_\epsilon S_{\text{cl}} = k \int_{S^4} \omega_4^1 (v, A). \] (2.13)

(We take the SYM theory to be defined on compactified Euclidean space, i.e. on S^4.) Now by
eq (2.6), approximating $S_{\text{string}}^{\text{eq}} \to S_3$ and using (2.13) one can read off the SU(4)$_R$ anomaly obtained from the supergravity action (2.11). It is
\begin{equation}
\delta_s \Gamma[A] = \delta_s S_3 = N^2 \int_{S^4} \omega_1^s (v, A), \tag{2.14}
\end{equation}
which agrees with the gauge theory computation (2.7) to leading order in $N$.

We thus see that the IIB supergravity action contains a Chern-Simons term at tree level which can account for the chiral anomaly of the gauge theory to leading order in $N$. But there is also a mismatch of “-1” which is of order $1/N^2$ relative to the leading term. As discussed above, this should correspond to a 1-loop effect in IIB string theory. Thus we are lead to examine the string one-loop effective action.

### 3. One-loop induced Chern-Simons action

Loop effects in $AdS_5$ supergravity are technically very difficult to compute due to the complicated propagators in $AdS$ geometry. Here however, this is possible due to the topological character of the Chern-Simons action.

**Fermionic contributions**

Consider a Dirac fermion $\psi$ in odd dimensions (flat) minimally coupled to vector bosons $A_\mu$ of a group $G$. At the quantum level, a regularization needs to be introduced to make sense of the theory and one cannot preserve both the gauge symmetry (small and large) and the parity at the same time $[11, 12]$. If one chooses to preserve the gauge symmetry by doing a Pauli-Villars regularization, then there will be an induced Chern-Simons term generated at one loop. The result is independent of the fermion mass. In our notation, the induced Chern-Simons term is
\begin{equation}
\Delta \Gamma = \pm \frac{1}{2} \int_{S^{2n+1}} \omega_{2n+1} R = \pm \frac{1}{2} \hat{A}(R) \int \omega_{2n+1}, \tag{3.1}
\end{equation}
where $R$ is the representation of the Dirac fermion. The $\pm$ sign depends on the regularization and can often be fixed within a specific context.

This result was originally $[11, 12]$ obtained for fermions coupled to gauge fields in a flat space-time and has been extended to full generality for arbitrary curved backgrounds and any odd dimension $d = 2n + 1$ $[13]$. The induced parity violating terms are given (up to a normalization factor) by the secondary characteristic class $Q(A, \omega)$ satisfying
\begin{equation}
dQ(A, \omega) = \hat{A}(R) ch(F)|_{2n+2}, \tag{3.2}
\end{equation}
where $\omega$ is the gravitational connection. Since $\hat{A}(R) = 1 + O(R^2)$ and $\text{Tr} F = 0$ for SU-groups, it is clear that for $n = 2$ ($d = 5$) there are no mixed gauge/gravitational terms. Also, there can be no purely gravitational term since it would correspond to a gravitational anomaly in four dimensions which is not possible. Hence for the present case of SU(4) with $n = 2$, (3.2) simply reduces to $dQ = ch(F)|_6$ giving rise to the Chern-Simons action upon descent, which does not depend on the geometry of $AdS_5$ at all! Hence the result of (3.1) for a Dirac fermion in flat space(-time) remains valid on $AdS_5$.

Now we need the particle spectrum of the type IIB string theory on $AdS_5 \times S^5$. The only explicitly known states are the KK states coming from the compactification of the 10 dimensional IIB supergravity multiplet $[14]$. So we will examine them first. We will argue in the discussion section that the other string states are not likely to modify the result.

Particles in $AdS_5$ are classified by unitary irreducible representations of SO(2, 4). SO(2, 4) has the maximal compact subgroup SO(2) $\times$ SU(2) $\times$ SU(2) and so its irreducible representations are labelled by the quantum numbers $(E_0, J_1, J_2)$. The complete KK spectrum of the IIB supergravity on $AdS_5 \times S^5$ was obtained in $[14, 8]$ together with information on the representation content under SU(4)$_R$. We reproduce these results in the table below. Actually, all fermions are symplectic Majorana, giving half the anomaly of a Dirac fermion. But there also is a mirror table with conjugate SU(4)$_R$ representations and SU(2) $\times$ SU(2) quantum numbers exchanged (opposite chiralities). So these “mirror” fermions give the same anomaly as those in the table and the net effect is that we may restrict ourselves to the fermions of the table treating them as if they
of the gravitino. Denote the local coordinates of $AdS \times S^5$ by $(x^\mu, y^a)$. A general spinor $\epsilon$ has the decomposition

$$\epsilon = \sum e^{I, \pm}(x) \Xi^{I, \pm}(y)$$

(3.6)

where $\Xi^{I, \pm}(y)$ are the spinor spherical harmonics on $S^5$ and satisfy ($P_y = \gamma^a D_a, a = 5, \ldots 9$)

$$P_y \Xi^{I, \pm} = \mp i(k + \frac{5}{2}) \frac{1}{R} \Xi^{I, \pm}$$

(3.7)

where $k = I \geq 0$ and $\Xi^{I, \pm}$ can be written in terms of the killing spinors $\eta^{I, \pm}$ on $S^5$. Substituting (3.6) into (3.5) and using (3.7) we get

$$\delta(\gamma^a \psi_\alpha) = \frac{i}{R} \sum \left[ \mp (k + \frac{5}{2}) + \frac{5}{2} \right] e^{I, \pm}(x) \Xi^{I, \pm}(y).$$

(3.8)

So one finds that only the component corresponding to $\Xi^{0, +}$ is gauge invariant while all other components of $\gamma^a \psi_\alpha$ can be gauged away. Thus we arrive at the gravitino gauge fixing condition

$$\gamma^a \psi_\alpha(x, y) \sim \chi^{I_0}(x) \eta^{I_0, +}(y)$$

(3.9)

where $\chi^{I_0}(x)$ are some arbitrary spacetime spinor fields. We refer the reader to [14] for more details. Therefore we see that (3.4) is the closest one can get to the gauge condition $\gamma^a \psi_\alpha = 0$. One can also rewrite this condition as

$$\psi_\alpha = \psi_\alpha^{(\alpha)} + \chi^{I_0}(x) \gamma^a \eta^{I_0, +}(y)$$

(3.10)

where the part $\psi_\alpha^{(\alpha)}$ satisfies $\gamma^a \psi_\alpha^{(\alpha)} = 0$. The other Killing spinors $\eta^{-}$ have been gauged away. The coefficient of $\eta^{-}$ would be a field in the $4^*$ of SU(4) and is precisely the doubleton spinors we are after. Now the constraint can be taken care of in the functional approach by introducing in the path integral the factor

$$\int db \bar{b} \frac{1}{\det M} \int d^4 x \delta M \bar{b}$$

(3.11)

$$\delta(\psi_\alpha - \sum \chi^I \eta^{I, +} - \bar{b}(x) \eta^{I, -}(y)) \cdot \delta(\text{h.c.})$$

where $b(x)$ is a complex fermionic field in the $4^*$ of SU(4) and $M = P_x$. Integrating over $b, \bar{b}$ results in the gauged fixed lagrangian. The factor $(\det M)^{-1}$ can be handled by introducing ghost fields $c, \bar{c}$, which are bosonic spinor fields on $AdS_5$ and are in the $4^*$ of SU(4). Thus

$$\frac{1}{\det M} = \int dc \bar{c} e^{-\int d^4 x \bar{c} M c .}$$

(3.12)
and so give rise to another -1/2 contribution to the induced Chern-Simons action. So altogether we get a total induced Chern-Simons term of -1,

\[ \Delta \Gamma = - \int_{AdS_5} \omega_5, \quad (3.13) \]

which is exactly the desired result. Notice that the induced Chern-Simons action (coming with a constant integer coefficient) is independent of the radius \( R \) and this is consistent with the AdS/CFT proposal since the anomaly and its corrections are independent of \( \lambda \).

**Bosonic contributions**

There is another interesting effect related to the Chern-Simons action. It is known that in three dimensions, the gluons at one loop can modify the coefficient of the Chern-Simons action by an integer shift. It has been argued that there is no such shift for the present case. Therefore only spinor loops contribute to the induced Chern-Simons action and we find that at finite \( N \), the coefficient \( k \) is shifted by

\[ k \rightarrow k - 1 \quad \text{or} \quad N^2 \rightarrow N^2 - 1 \quad (3.14) \]

due to the quantum effects of the full set of Kaluza-Klein states.

A few comments about the absence of a shift due to gluon loops are in order. The bosonic shift in pure Chern-Simons theory was first computed in using a saddle point approximation. Later calculations trying to reproduce this results from the perturbative point of view revealed that the precise shift depends on the choice of regularization scheme. In the present case of 5-dimensions, one may try to employ a regularization scheme and do a 1-loop perturbative calculation to determine the possible shift. However, like in the 3-dimensional case, it can be expected that the result will depend on the choice of regularization and a better way to determine the shift is called for. One possibility might be to do a string theory calculation by embedding the Chern-Simons action in a string setting and to determine the quantum loop effects from the string loop effects. Since string theory is free from divergences, no regularization related ambiguities should occur and a definite answer can be expected.

**4. Discussion**

We have reproduced the correct shift \( N^2 \rightarrow N^2 - 1 \) of the anomaly coefficient as a one-loop effect in IIB supergravity/string theory on \( AdS_5 \times S^5 \). This shift is entirely due to the towers of Kaluza-Klein states. No massive string states need to be invoked. It is indeed likely that the latter play no role at all since anomalies are usually due to massless fields only. Note however that we need the full towers of Kaluza-Klein states to get the correct result. A truncation to five-dimensional \( AdS \) supergravity alone would not give the desired result. Also the \( AdS_5 \) supergravity Chern-Simons term originates from compactifying the full IIB supergravity. This is another indication that string states beyond the Kaluza-Klein towers are unlikely to modify our result.

We have been able to obtain a non-trivial one-loop result within a particularly favourable case. In general, one-loop calculations in \( AdS_5 \) are very difficult - already tree computations are quite non-trivial! Of course, the anomaly coefficient \( \sim N^2 - 1 \) should be exact and there cannot be any further higher-loop corrections \( \sim N^2 \frac{1}{N^2} \). Again, since the induced Chern-Simons term in 5 dimensions is closely related to anomalies in 4 dimensions, we expect some sort of non-renormalisation theorem to be at work, although it would be nice to have a proof of this statement.

Finally, we would like to make some comments on more or less related situations. There are other dualities like those involving \( AdS_7 \times S^4 \) where one can expect similar Chern-Simons terms and doubleton multiplets. The issue of the trace-anomaly in \( AdS_5 \) should be closely related to the present study. Already the leading \( N \) behaviour of this conformal anomaly is non-trivial to establish and to explicitly obtain the sub-leading terms might well turn out to be more difficult than for the chiral anomaly studied here. Effects that are of lower order than \( N^2 \) have also been considered in which essentially studies situations where the leading effect corresponds to open strings at tree level and hence comes with

\[ ^1 \text{We thank R. Stora for a useful discussion about the issues of regularization.} \]
just one power of $N$. A somewhat related discussion is [19].

It will also be interesting to investigate these anomaly issues within the non-commutative version of the AdS/CFT correspondence [20] to see the origin of the higher derivative Chern-Simons terms [21] on the supergravity side.

Acknowledgments

This work was partially supported by the Swiss National Science Foundation, by the European Union under TMR contract ERBFMRX-CT96-0045.

References

[1] A. Bilal, C.-S. Chu, A Note on the Chiral Anomaly in the AdS/CFT Correspondence and $1/N^2$ Correction, Nucl. Phys. B562 (1999) 181, hep-th/9907106.

[2] J. Maldacena, The Large $N$ Limit of Superconformal Field Theories and Supergravity, Adv. Theor. Math. Phys. 2 (1998) 231, hep-th/9711200.

[3] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge Theory Correlators from Non-Critical String Theory, Phys. Lett. B428 (1998) 105, hep-th/9802109.

[4] E. Witten, Anti de Sitter Space and Holography, Adv. Theor. Math. Phys. 2 (1998) 253, hep-th/9802150.

[5] For a review, see O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Large $N$ Field Theories, String Theory and Gravity, hep-th/9905111.

[6] D. Z. Freedman, S. D. Mathur, A. Matusis, L. Rastelli, Correlation Functions in the CFT(d)/AdS(d+1) Correspondence, hep-th/9804058.

[7] M. Gunaydin, L.J. Romans, N.P. Warner, Gauged $N = 8$ Supergravity in Five Dimensions, Phys. Lett. B154 (1985) 268.

[8] M. Gunaydin, L.J. Romans, N.P. Warner, Compact and Non-compact Gauged Supergravity Theories in Five Dimensions, Nucl. Phys. B272 (1986) 508.

[9] M. Pernici, K. Pilch, P. van Nieuwenhuizen, Gauged $N = 8$, $d = 5$ Supergravity, Nucl. Phys. B259 (1985) 460.

[10] S. Ferrara, C. Fronsdal, A. Zaffaroni, On $\mathcal{N}=8$ Supergravity on $AdS_5$ and $\mathcal{N}=4$ Superconformal Yang-Mills theory, Nucl. Phys. B532 (1998) 153, hep-th/9802203.

[11] A.N. Redlich, Gauge Noninvariance and Parity Nonconservation of Three-dimensional Fermions, Phys. Rev. Lett. 52 (1984) 18; Parity Violation and Gauge Noninvariance of the Effective Gauge Field Action in Three Dimensions, Phys. Rev. D29 (1984) 2366.

[12] A.J. Niemi, G.W. Semenoff, Axial-Anomaly-Induced Fermion Fractionization and Effective Gauge Theory Actions in Odd-Dimensional Spacetimes, Phys. Rev. Lett. 51 (1983) 2077.

[13] L. Alvarez-Gaume, S. Della Pietra, G. Moore, Anomalies and Odd Dimensions, Ann. Phys. 163 (1985) 288.

[14] H.J. Kim, L.J. Romans, P. van Nieuwenhuizen, Mass Spectrum of Chiral Ten-dimensional $\mathcal{N} = 2$ Supergravity on $S^5$, Phys. Rev. D32 (1985) 389.

[15] M. Gunaydin, P. van Nieuwenhuizen, N.P. Warner, Nucl. Phys. B255 (1985) 63.

[16] E. Witten, Quantum Field Theory and Jones Polynomial, Adv. Theor. Math. Phys. 2 (1998) 203, hep-th/9806087.

[17] O. Aharony, J. Pawelczyk, S. Theisen, S. Yankielowicz, A Note on Anomalies in the AdS/CFT Correspondence, hep-th/9901134.

[18] M. Blau, E. Gava, K.S. Narain, On Subleading Contributions to the AdS/CFT Trace Anomaly, hep-th/9904179.

[19] S. Nojiri, S. D. Odintsov, On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence, hep-th/9903033.

[20] A. Hashimoto, N. Itzhaki, Non-Commutative Yang-Mills and the AdS/CFT Correspondence, hep-th/9907166.

[21] J. M. Maldacena and J. G. Russo, Large $N$ Limit of Non-Commutative Gauge Theories, hep-th/9908134.

[22] C.-S. Chu, Induced Chern-Simons and WZW action in noncommutative spacetime, hep-th/0003007.