ONE SMOOTHING PROPERTY OF THE SCATTERING MAP OF THE KDV ON $\mathbb{R}$

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Abstract. In this paper we prove that in appropriate weighted Sobolev spaces, in the case of no bound states, the scattering map of the Korteweg-de Vries (KdV) on $\mathbb{R}$ is a perturbation of the Fourier transform by a regularizing operator. As an application of this result, we show that the difference of the KdV flow and the corresponding Airy flow is 1-smoothing.

1. Introduction. In the last decades the problem of a rigorous analysis of the theory of infinite dimensional integrable Hamiltonian systems in 1-space dimension has been widely studied. These systems come up in two setups: (i) on compact intervals (finite volume) and (ii) on infinite intervals (infinite volume). The dynamical behaviour of the systems in the two setups have many similar features, but also distinct ones, mostly due to the different manifestation of dispersion.

The analysis of the finite volume case is now quite well understood. Indeed, Kappeler with collaborators introduced a series of methods in order to construct rigorously Birkhoff coordinates (a cartesian version of action-angle variables) for 1-dimensional integrable Hamiltonian PDE’s on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. The program succeeded in many cases, like Korteweg-de Vries (KdV) [23], defocusing and focusing Nonlinear Schrödinger (NLS) [19, 22]. In each case considered, it has been proved that there exists a real analytic symplectic diffeomorphism, the Birkhoff map, between two scales of Hilbert spaces which conjugate the nonlinear dynamics to a linear one.

An important property of the Birkhoff map $\Phi$ of the KdV on $\mathbb{T}$ and its inverse $\Phi^{-1}$ is the semi-linearity, i.e., the nonlinear part of $\Phi$ respectively $\Phi^{-1}$ is 1-smoothing. A local version of this result was first proved by Kuksin and Perelman [32] and later extended globally by Kappeler, Schaad and Topalov [25]. It plays an important role in the perturbation theory of KdV – see [31] for randomly perturbed KdV equations and [12] for forced and weakly damped problems. The semi-linearity of $\Phi$ and $\Phi^{-1}$

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can be used to prove 1-smoothing properties of the KdV flow in the periodic setup [25].

The analysis of the infinite volume case was developed mostly during the ‘60–’70 of the last century, starting from the pioneering works of Gardner, Greene, Kruskal and Miura [17, 18] on KdV on the line. In these works the authors showed that KdV can be integrated by a scattering transform which maps a function $q$, decaying sufficiently fast at infinity, into the spectral data of the operator $L(q) := -\partial_x^2 + q$. Later, similar results were obtained by Zakharov and Shabat for NLS on $\mathbb{R}$ [43], by Ablowitz, Kaup, Newell and Segur for the Sine-Gordon equation [2], and by Flaschka for the Toda lattice with infinitely many particles [16]. Furthermore, using the spectral data of the corresponding Lax operators, action-angle variables were (formally) constructed for each of the equations above [41, 42, 36, 35]. See also [39, 14, 1] for monographs about the subject. Analytic properties of the scattering transform of the KdV on $\mathbb{R}$ have been studied by Kappeler and Trubowitz [26, 27]. Unfortunately, they developed a theory for data in weighted Sobolev spaces which are not preserved by the nonlinear flow, making it impossible to deduce dynamical consequences. In the present paper we want to fill this gap.

The aim of this paper is to show that for the KdV on the line, the scattering map is an analytic perturbation of the Fourier transform by a 1-smoothing nonlinear operator. With the applications we have in mind, we choose a setup for the scattering map so that the spaces considered are left invariant under the KdV flow. Recall that the KdV equation on $\mathbb{R}$

$$
\begin{align*}
\partial_t u(t, x) &= -\partial_x^2 u(t, x) - 6u(t, x)\partial_x u(t, x), \\
u(0, x) &= q(x),
\end{align*}
$$

is globally in time well-posed in various function spaces such as the Sobolev spaces $H^N_\mathbb{R} \equiv H^N(\mathbb{R}, \mathbb{R})$, $N \in \mathbb{Z}_{>2}$ (e.g. [5, 28, 30]), as well as on the weighted spaces $H^{2N} \cap L^2_M$, with integers $N \geq M \geq 1$ [29], endowed with the norm $\|\cdot\|_{H^{2N}} + \|\cdot\|_{L^2_M}$. Here $L^2_M \equiv L^2_M(\mathbb{R}, \mathbb{C})$ denotes the space of complex valued $L^2$-functions satisfying

$$
\|q\|_{L^2_M} := \left( \int_{-\infty}^{\infty} (1 + |x|^2)^M |q(x)|^2 dx \right)^{1/2} < \infty.
$$

Introduce for $q \in L^2_M$ with $M \geq 3$ the Schrödinger operator $L(q) := -\partial_x^2 + q$ with domain $H^2_\mathbb{C}$, where, for any integer $N \in \mathbb{Z}_{>0}$, $H^N_\mathbb{C} := H^N(\mathbb{R}, \mathbb{C})$. For $k \in \mathbb{R}$ denote by $f_1(q, x, k)$ and $f_2(q, x, k)$ the Jost solutions, i.e. solutions of $L(q)f = k^2f$ with asymptotics $f_1(q, x, k) \sim e^{ikx}$, $x \to \infty$, $f_2(q, x, k) \sim e^{-ikx}$, $x \to -\infty$. As $f_i(q, \cdot, k)$, $f_i(q, \cdot, -k)$, $i = 1, 2$, are linearly independent for $k \in \mathbb{R} \setminus \{0\}$, one can find coefficients $S(q, k)$, $W(q, k)$ such that for $k \in \mathbb{R} \setminus \{0\}$ one has

$$
\begin{align*}
f_2(q, x, k) &= S(q, -k) \frac{2ik}{f_1(q, x, k)} + W(q, k) \frac{2ik}{f_1(q, x, -k)}, \\
f_1(q, x, k) &= S(q, k) \frac{2ik}{f_2(q, x, k)} + W(q, k) \frac{2ik}{f_2(q, x, -k)}.
\end{align*}
$$

It’s easy to verify that the functions $W(q, \cdot)$ and $S(q, \cdot)$ are given by the wronskian identities

$$
W(q, k) := [f_2, f_1](q, k) := f_2(q, x, k)\partial_x f_1(q, x, k) - \partial_x f_2(q, x, k)f_1(q, x, k),
$$

and

$$
S(q, k) := [f_1(q, x, k), f_2(q, x, -k)],
$$
which are independent of $x \in \mathbb{R}$. The functions $S(q,k)$ and $W(q,k)$ are related to the more often used reflection coefficients $\tau_\pm(q,k)$ and transmission coefficient $t(q,k)$ by the formulas

$$r_+(q,k) = \frac{S(q,-k)}{W(q,k)}, \quad r_-(q,k) = \frac{S(q,k)}{W(q,k)}, \quad t(q,k) = \frac{2ik}{W(q,k)} \quad \forall k \in \mathbb{R} \setminus \{0\} \quad (5)$$

It is well known that for $q$ real valued the spectrum of $L(q)$ consists of an absolutely continuous part, given by $[0,\infty)$, and a finite number of eigenvalues referred to as bound states, $-\lambda_1 < \cdots < -\lambda_1 < 0$ (possibly none). Introduce the set

$$Q := \{ q : \mathbb{R} \to \mathbb{R}, \quad q \in L_3^2 : W(q,0) \neq 0, \quad q \text{ without bound states} \} \quad (6)$$

We remark that the property $W(q,0) \neq 0$ is generic. In the sequel we refer to elements in $Q$ as generic potentials without bound states. Finally we define

$$Q^{N,M} := Q \cap H^N \cap L_2^2, \quad N \in \mathbb{Z}_{\geq 0}, \quad M \in \mathbb{Z}_{\geq 3}.$$ 

We will see in Lemma 3.5 that for any integers $N \geq 0$, $M \geq 3$, $Q^{N,M}$ is open in $H^N \cap L_2^2_M$.

Our main theorem analyzes the properties of the scattering map $q \mapsto S(q,\cdot)$ which is known to linearize the KdV flow [18]. To formulate our result on the scattering map in more details let $S$ denote the set of all functions $\sigma : \mathbb{R} \to \mathbb{C}$ satisfying

\begin{enumerate}[(S1)]
    \item $\sigma(-k) = \overline{\sigma(k)}$, \quad $\forall k \in \mathbb{R}$;
    \item $\sigma(0) > 0$.
\end{enumerate}

For $M \in \mathbb{Z}_{\geq 1}$ define the real Banach space

$$H_\zeta^M := \{ f \in H_c^{M-1} : \zeta \partial_k^M f \in L^2 \}, \quad (7)$$

where $\zeta : \mathbb{R} \to \mathbb{R}$ is an odd monotone $C^\infty$ function with

$$\zeta(k) = k \quad \text{for} \quad |k| \leq 1/2 \quad \text{and} \quad \zeta(k) = 1 \quad \text{for} \quad k \geq 1. \quad (8)$$

The norm on $H_\zeta^M$ is given by

$$\| f \|_{H_\zeta^M}^2 := \| f \|^2_{H_c^{M-1}} + \| \zeta \partial_k^M f \|_{L^2}^2.$$ 

For any $N, M \in \mathbb{Z}_{\geq 0}$ let

$$\mathcal{S}^{M,N} := \mathcal{S} \cap H_\zeta^M \cap L_2^N_N. \quad (9)$$

Different choices of $\zeta$, with $\zeta$ satisfying (8), lead to the same Hilbert space with equivalent norms. We will see in Lemma 3.6 that for any integers $N \geq 0$, $M \geq 3$, $\mathcal{S}^{M,N}$ is an open subset of $H_\zeta^M \cap L_2^N_N$. Moreover let $F_{\pm}$ be the Fourier transformations defined by $F_{\pm}(f) = \int_{-\infty}^{\pm\infty} e^{\mp 2ikx} f(x) \, dx$. In this setup, the scattering map $S$ has the following properties – see [37, 23] for a discussion of the notion of real analytic.

**Theorem 1.1.** For any integers $N \geq 0$, $M \geq 3$, the following holds:

\begin{enumerate}[(i)]
    \item The map $S : Q^{N,M} \to \mathcal{S}^{M,N}, \quad q \mapsto S(q,\cdot)$ is a real analytic diffeomorphism.
    \item The maps $A := S - F_-$ and $B := S_{-1} - F_{-1}$ are 1-smoothing, i.e. $A : Q^{N,M} \to H_\zeta^M \cap L_2^N_{N+1}$ and $B : \mathcal{S}^{M,N} \to H^N_{N+1} \cap L_2^{M-1}$. Furthermore they are real analytic maps.
\end{enumerate}
Remark 1. Actually $B$ takes values in the space $\mathcal{F}^{-1}\left(H^M_\zeta \cap L^{2}_{N+1}\right)$, which has not an obvious characterization in terms of Sobolev-weighted spaces. However $H^M_\zeta \cap L^{2}_{N+1} \subset H^{M-1}_C \cap L^2_{N+1}$, thus $\mathcal{F}^{-1}\left(H^M_\zeta \cap L^{2}_{N+1}\right) \subset H^{N+1} \cap L^{2}_{M-1}$.

Before discussing some applications of Theorem 1.1, let us comment about the condition $M \geq 3$. It is standard in direct and inverse scattering theory to work with potentials in the space $L^2_N$ (cf. [10, 13]). We do not try to improve this condition, thus we work in the largest weighted $L^2_M$ space ($M \in \mathbb{Z}_{>0}$) which is contained in $L^2_3$, which clearly is $L^2_3$. Thus here comes our restriction $M \geq 3$.

As a first application of Theorem 1.1 we prove analytic properties of the action variable for the KdV on the line. For a potential $q \in Q$, the action-angle variable were formally defined for $k \neq 0$ by Zakharov and Faddeev [41] as the densities

$$I(q,k) := \frac{k}{\pi} \log \left(1 + \frac{|S(q,k)|^2}{4k^2}\right), \quad \theta(q,k) := \arg(S(q,k)), \quad k \in \mathbb{R} \setminus \{0\}.$$

We can write the action as

$$I(q,k) := -\frac{k}{\pi} \log \left(\frac{4k^2}{4k^2 + S(q,k)S(q,-k)}\right), \quad k \in \mathbb{R} \setminus \{0\}.$$  

By Theorem 1.1, $S(q,\cdot) \in \mathcal{S}$, thus property (S2) implies that $\lim_{k \to 0} I(q,k)$ exists and equals 0. Furthermore, by (S1), the action $I(q,\cdot)$ is an odd function in $k$, and strictly positive for $k > 0$. Thus we will consider just the case $k \in [0, +\infty)$. The properties of $I(q,\cdot)$ for $k$ near 0 and $k$ large are described separately.

Corollary 1. For any integers $N \geq 0$, $M \geq 3$, the maps

$$Q^{N,M} \to L^1_{2N+1}([1, +\infty), \mathbb{R}) \quad q \mapsto I(q,\cdot)|_{[1, +\infty)}$$

and

$$Q^{N,M} \to H^M([0,1], \mathbb{R}) \quad q \mapsto I(q,\cdot)|_{[0,1]} + \frac{k}{\pi} \ln \left(\frac{4k^2}{4k^2 + 1}\right)$$

are real analytic. Here $I(q,\cdot)|_{[1, +\infty)}$ (respectively $I(q,\cdot)|_{[0,1]}$) denotes the restriction of the function $k \mapsto I(q,k)$ to the interval $[1, +\infty)$ (respectively $[0,1]$).

Finally we compare solutions of (1) to solutions of the Cauchy problem for the Airy equation on $\mathbb{R}$,

$$\begin{cases}
\partial_t v(t,x) = -\partial_x^3 v(t,x) \\
v(0,x) = p(x)
\end{cases}$$

(11)

Being a linear equation with constant coefficients, one sees that the Airy equation is globally in time well-posed on $H^N$ and $H^{2N} \cap L^2_M$, with integers $N \geq M \geq 1$ (see Remark 8 below). Denote the flows of (11) and (1) by $U^t_{\text{Airy}}(q) := v(t,\cdot)$ respectively $U^t_{\text{KdV}}(q) := u(t,\cdot)$. Our third result is to show that for $q \in H^{2N} \cap L^2_M$ with no bound states and $W(q,0) \neq 0$, the difference $U^t_{\text{KdV}}(q) - U^t_{\text{Airy}}(q)$ is 1-smoothing, i.e. it takes values in $H^{2N+1}$. More precisely we prove the following theorem.

Theorem 1.2. Let $N$, $M$ be integers with $N \geq 2M \geq 6$. Then the following holds true:

(i) $Q^{N,M}$ is invariant under the KdV flow.
(ii) For any \( q \in \mathcal{Q}^{N,M} \) the difference \( U^t_{KdV}(q) - U^t_{Airy}(q) \) takes values in \( H^{N+1} \cap L^2_M \). Moreover the map
\[
\mathcal{Q}^{N,M} \times \mathbb{R}_{\geq 0} \to H^{N+1} \cap L^2_M, \quad (q,t) \mapsto U^t_{KdV}(q) - U^t_{Airy}(q)
\]
is continuous and for any fixed \( t \) real analytic in \( q \).

Outline of the proof. In Section 2 we study analytic properties of the Jost functions \( f_j(q,x,k) \), \( j = 1, 2 \), in appropriate Banach spaces. We use these results in Section 3 to prove the direct scattering part of Theorem 1.1. The inverse scattering part of Theorem 1.1 is proved in Section 4. Finally in Section 5 we prove Corollary 1 and Theorem 1.2.

Related works: This paper is motivated in part from the study of the 1-smoothing property of the KdV flow in the periodic setup, established recently in [3, 11, 25]. More precisely in [3, 11] the authors showed 1-smoothing properties of the difference of the nonlinear and linear KdV flow for low regularity initial data and proved that the Sobolev norms can grow at most polynomially in time. The techniques employed do not make use of the integrability of the KdV, and have been used to study also non integrable KdV type equations. On the opposite, in [25], the authors exploit the 1-smoothing property of the Birkhoff map of KdV to prove that for \( q \in H^N(\mathbb{T},\mathbb{R}) \), \( N \geq 1 \), the difference \( U^t_{KdV}(q) - U^t_{Airy}(q) \) is bounded in \( H^{N+1}(\mathbb{T},\mathbb{R}) \) with a bound which grows linearly in time.

One smoothing property of the KdV flow on \( \mathbb{R} \) is still not yet completely understood. Indeed Theorem 1.2 deals with decaying initial data of high regularity, but it does not apply to rough data. Furthermore we do not have bounds on the growth of the Sobolev norms of the difference \( U^t_{KdV}(q) - U^t_{Airy}(q) \). Our main contribution concerning 1-smoothing is that we can treat the low frequencies. Indeed smoothing effects of the KdV flow on the line with rough initial data have been proved in [9], but with an additional vanishing weight on the low frequencies.

Results concerning the 1-smoothing property of the inverse scattering map were obtained previously in [40], where it is shown that for a potential \( q \) in the space \( W^{n,1}(\mathbb{R},\mathbb{R}) \) of real valued functions with weak derivatives up to order \( n \) in \( L^1 \)
\[
q(x) - \frac{1}{\pi} \int_\mathbb{R} e^{-2ikx} \chi_c(k) 2ikr_+(q,k) dk \in W^{n+1,1}(\mathbb{R},\mathbb{R}).
\]
Here \( c \) is an arbitrary number with \( c > \|q\|_{L^1} \) and \( \chi_c(k) = 0 \) for \( |k| \leq c \), \( \chi_c(k) = |k| - c \) for \( c \leq |k| \leq c + 1 \), and 1 otherwise. The main difference between the result in [40] and ours concerns the function spaces considered. For the application to the KdV we need to choose function spaces such as \( H^N \cap L^2_M \) for which KdV is well posed. To the best of our knowledge it is not known if KdV is well posed in \( W^{n,1}(\mathbb{R},\mathbb{R}) \). Furthermore in [40] the question of analyticity of the map \( q \mapsto r_+(q) \) and its inverse is not addressed.

Analytic properties of the scattering map \( S \) have been studied by Kappeler and Trubowitz in [26, 27]. To state their result, define the spaces
\[
H^{n,\alpha} := \{ f \in L^2 : x^\beta \partial_x^j f \in L^2, 0 \leq j \leq n, 0 \leq \beta \leq \alpha \},
\]
\[
H^{n,\alpha}_2 := \{ f \in H^{n,\alpha} : x^\beta \partial_x^{n+1} f \in L^2, 1 \leq \beta \leq \alpha \}.
\]
In [26], Kappeler and Trubowitz showed that the map \( q \mapsto S(q,\cdot) \) is a real analytic diffeomorphism from \( \mathcal{Q} \cap H^{N,N} \) to \( \mathcal{S} \cap H^{N-1,N}_2 \), \( N \in \mathbb{Z}_{\geq 3} \). They extend their results
to potentials with finitely many bound states in [27]. Unfortunately, \( Q \cap H^{N,N} \) is not left invariant under the KdV flow.

We remark that Theorem 1.1 treats just the case of regular potentials. In [15, 20] a special class of distributions is considered. In particular the authors study Miura potentials \( q \in H^{-1}_{loc}(\mathbb{R}, \mathbb{R}) \) such that \( q = u' + u^2 \) for some \( u \in L^1(\mathbb{R}, \mathbb{R}) \cap L^2(\mathbb{R}, \mathbb{R}) \), and prove that the map \( q \mapsto r_+ \) is bijective and locally bi-Lipschitz continuous between appropriate spaces. Finally we point out the work of Zhou [44], in which \( L^2 \)-Sobolev space bijectivity for the scattering and inverse scattering transforms associated with the ZS-AKNS system are proved.

The case of potentials with bound states and resonant potentials will be considered in future work. We expect one smoothing effects of the KdV flow also there.

2. Jost solutions. In this section we assume that the potential \( q \) is complex-valued. Often we will assume that \( q \in L^2_{M} \) with \( M \in \mathbb{Z}_{\geq 3} \). Consider the normalized Jost functions \( m_1(q,x,k) := e^{-ikx}f_1(q,x,k) \) and \( m_2(q,x,k) := e^{ikx}f_2(q,x,k) \) which satisfy the following integral equations

\[
\begin{align*}
    m_1(q,x,k) &= 1 + \int_{x}^{+\infty} D_k(t-x)q(t)m_1(q,t,k)dt \\
    m_2(q,x,k) &= 1 + \int_{-\infty}^{x} D_k(x-t)q(t)m_2(q,t,k)dt
\end{align*}
\]

where \( D_k(y) := \int_{0}^{y} e^{i k s} ds \).

The purpose of this section is to analyze the solutions of the integral equations (12) and (13) in spaces needed for our application to KdV. We adapt the corresponding results of [26] to these spaces. As (12) and (13) are analyzed in a similar way we concentrate on (12) only. For simplicity we write \( m(q,x,k) \) for \( m_1(q,x,k) \).

For \( 1 \leq p \leq \infty \), \( M \geq 1 \) and \( a \in \mathbb{R} \), \( 1 \leq \alpha < \infty \), \( 1 \leq \beta \leq \infty \) we introduce the spaces \( L^p_{M} := \{ f : \mathbb{R} \to \mathbb{C} : \langle x \rangle^M f \in L^p \} \) and

\[
L^\alpha_{x \geq a} L^\beta := \left\{ f : [a, +\infty) \times \mathbb{R} \to \mathbb{C} : \| f \|_{L^\alpha_{x \geq a} L^\beta} < +\infty \right\},
\]

where \( \langle x \rangle := (1 + x^2)^{1/2} \), \( L^p \) is the standard \( L^p \)-space, and

\[
\| f \|_{L^\alpha_{x \geq a} L^\beta} := \left( \int_{a}^{+\infty} \| f(x, \cdot) \|_{L^\beta} \, dx \right)^{1/\alpha}
\]

whereas for \( \alpha = \infty \), \( \| f \|_{L^\infty_{x \geq a} L^\beta} := \sup_{x \geq a} \| f(x, \cdot) \|_{L^\beta} \). We consider also the space \( C^0_{x \geq a} L^\beta := C^0([a, +\infty), L^\beta) \) with \( \| f \|_{C^0_{x \geq a} L^\beta} := \sup_{x \geq a} \| f(x, \cdot) \|_{L^\beta} \leq \infty \). We will use also the space \( L^\alpha_{x \leq a} L^\beta \) of functions \( f : (-\infty, a] \times \mathbb{R} \to \mathbb{C} \) with finite norm \( \| f \|_{L^\alpha_{x \leq a} L^\beta} := \left( \int_{-\infty}^{a} \| f(x, \cdot) \|_{L^\beta} \, dx \right)^{1/\alpha} \). Moreover given any Banach spaces \( X \) and \( Y \) we denote by \( \mathcal{L}(X,Y) \) the Banach space of linear bounded operators from \( X \) to \( Y \) endowed with the operator norm. If \( X = Y \), we simply write \( \mathcal{L}(X) \).

For the notion of an analytic map between complex Banach spaces we refer to [37] (see also [23, Appendix A]).

We begin by stating a well known result about the properties of \( m \).

**Theorem 2.1** ([10]). Let \( q \in L^1_1 \). For each \( k, \text{Im} \, k \geq 0 \), the integral equation

\[
m(x,k) = 1 + \int_{x}^{+\infty} D_k(t-x)q(t)m(t,k)dt, \quad x \in \mathbb{R}
\]
has a unique solution $m \in C^2(\mathbb{R}, \mathbb{C})$ which solves the equation $m'' + 2ikm' = q(x)m$ with $m(x, k) \to 1$ as $x \to +\infty$. If in addition $q$ is real valued the function $m$ satisfies the reality condition $m(q, k, \eta) = m(q, -k)$. Moreover, there exists a constant $K > 0$ which can be chosen uniformly on bounded subsets of $L^1_1$ such that the following estimates hold for any $x \in \mathbb{R}$

(i) $|m(x, k) - 1| \leq e^{\eta(x)/|k|} \eta(x)/|k|, \quad k \neq 0$;

(ii) $|m(x, k) - 1| \leq K \left(1 + \max(-x, 0)\right) \int_x^{+\infty} (1 + |t|)|q(t)|dt/(1 + |k|)$;

(iii) $|m'(x, k)| \leq K \left(\int_x^{+\infty} (1 + |t|)|q(t)|dt\right)/(1 + |k|)$

where $\eta(x) = \int_x^{+\infty} |q(t)|dt$. For each $x$, $m(x, k)$ is analytic in $\text{Im} \, k > 0$ and continuous in $\text{Im} \, k \geq 0$. In particular, for every $x$ fixed, $k \mapsto m(x, k) - 1 \in H^{2+}$, where $H^{2+}$ is the Hardy space of functions analytic in the upper half plane such that $\sup_{t>0} \int_{-\infty}^{+\infty} |h(k + iy)|^2 \, dk < \infty$.

Estimates on the Jost functions.

**Proposition 1.** For any $q \in L^2_M$ with $M \geq 2$, $a \in \mathbb{R}$ and $2 \leq \beta \leq +\infty$, the solution $m(q)$ of (12) satisfies $m(q) - 1 \in C^0_{x \geq a}L^\beta \cap L^2_{x \geq a}L^2$. The map $L^2_M \ni q \mapsto m(q) - 1 \in C^0_{x \geq a}L^\beta \cap L^2_{x \geq a}L^2$ is analytic. Moreover there exist constants $C_1, C_2 > 0$, only dependent on $a, \beta$, such that

$$
||m(q) - 1||_{C^0_{x \geq a}L^\beta} \leq C_1 e^{\|q\|_{L^1_1}^\frac{1}{2}} \|q\|_{L^1_1}^\frac{1}{2},
$$

$$
||m(q) - 1||_{L^2_{x \geq a}L^2} \leq C_2 \|q\|_{L^2_1}^\frac{1}{2} \left(1 + \|q\|_{L^1_1} + e^{\|q\|_{L^1_1}}\right).
$$

**Remark 2.** In comparison with [26], the novelty of Proposition 1 consists in the choice of spaces.

To prove Proposition 1 we first need to establish some auxiliary results.

**Lemma 2.2.**

(i) For any $q \in L^1_1$, $a \in \mathbb{R}$ and $1 \leq \beta \leq +\infty$, the linear operator

$$
\mathcal{K}(q) : C^0_{x \geq a}L^\beta \to C^0_{x \geq a}L^\beta, \quad f \mapsto \mathcal{K}(q)[f](x, k) := \int_x^{+\infty} D_k(t - x)q(t)f(t, k)dt \quad (14)
$$

is bounded. Moreover for any $n \geq 1$, the $n^\text{th}$ composition $K(q)^n$ satisfies

$$
||\mathcal{K}(q)^n||_{L^2_{x \geq a}L^\beta} \leq C^n \|q\|_{L^1_1}^n / n!
$$

where $C > 0$ is a constant depending only on $a$.

(ii) The map $\mathcal{K} : L^1_1 \to L^2_{x \geq a}L^\beta$, $q \mapsto \mathcal{K}(q)$, is linear and bounded, and $\text{Id} - \mathcal{K}$ is invertible. More precisely,

$$
(\text{Id} - \mathcal{K})^{-1} : L^1_1 \to L^2_{x \geq a}L^\beta, \quad q \mapsto (\text{Id} - \mathcal{K}(q))^{-1}
$$

is analytic and $|||\text{Id} - \mathcal{K}^{-1}|||_{L^1_1 \to L^1_1} \leq C\|q\|_{L^1_1}^\frac{1}{2}$.
Proof. Let \( h \in L^\alpha \) with \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \). Using \( |D_\nu(t - x)| \leq |t - x| \), one has

\[
\left| \int_{-\infty}^{+\infty} h(k)\mathcal{K}(q)[f](x,k)\,dk \right| \leq \int_{x}^{+\infty} |t - x||q(t)||f(t,\cdot)||_{L^\alpha} \, dt \, ||h||_{L^\alpha} \\
\leq \left( \int_{a}^{+\infty} |t - a||q(t)|\,dt \right) ||f||_{C_{x \geq a}L^\alpha} \, ||h||_{L^\alpha},
\]

and hence \( ||\mathcal{K}(q)||_{\mathcal{L}(C_{x \geq a}L^\beta)} \leq \int_{a}^{+\infty} |t - a||q(t)|\,dt \leq C ||q||_{L^1} \), where \( C > 0 \) is a constant depending just on \( a \). To compute the norm of the iteration of the map \( \mathcal{K}(q) \) it’s enough to proceed as above and exploit the fact that the integration in \( t \) is over a simplex, yielding \( ||\mathcal{K}(q)^n||_{\mathcal{L}(C_{x \geq a}L^\beta)} \leq C^n ||q||_{L^1}^n/n! \) for any \( n \geq 1 \). Therefore the Neumann series of the operator \( (Id - \mathcal{K}(q))^{-1} = \sum_{n \geq 0} \mathcal{K}(q)^n \) converges absolutely in \( \mathcal{L}(C_{x \geq a}L^\beta) \). Since \( \mathcal{K}(q) \) is linear and bounded in \( q \), the analyticity and, by item (i), the claimed estimate for \( (Id - \mathcal{K})^{-1} \) follow. 

\[\square\]

Lemma 2.3. Let \( a \in \mathbb{R} \).

(i) For any \( q \in L^2_{3/2} \), \( \mathcal{K}(q) \) defines a bounded linear operator \( L^2_{3/2} \to L^2_{3/2} \). Moreover the \( n \)th composition \( \mathcal{K}^n \) satisfies

\[
||\mathcal{K}(q)^n||_{\mathcal{L}(L^2_{3/2}L^2)} \leq C^n ||q||_{L^2_{3/2}} ||q||_{L^1}^{n-1}/(n-1)!
\]

where \( C > 0 \) depends only on \( a \).

(ii) The map \( \mathcal{K} : L^2_{3/2} \to \mathcal{L}(L^2_{3/2}L^2), \quad q \mapsto \mathcal{K}(q) \) is linear and bounded; the map

\[
(Id - \mathcal{K})^{-1} : L^2_{3/2} \to \mathcal{L}(L^2_{3/2}L^2), \quad q \mapsto (Id - \mathcal{K}(q))^{-1}
\]

is analytic and \( \left| (Id - \mathcal{K})^{-1} \right|_{\mathcal{L}(L^2_{3/2},L^2_{3/2},L^2)} \leq C \left( 1 + ||q||_{L^2_{3/2}} e^{||q||_{L^1}} \right) \).

Proof. Proceeding as in the proof of the previous lemma, one gets for \( x \geq a \) the estimate

\[
||\mathcal{K}(q)[f](x,\cdot)||_{L^2} \leq \int_{x}^{+\infty} |t - x||q(t)||f(t,\cdot)||_{L^2} \, dt \\
\leq \left( \int_{x}^{+\infty} (t - x)^2|q(t)|^2 \, dt \right)^{1/2} \, ||f||_{L^2_{x \geq a}} L^2,
\]

from which it follows that

\[
||\mathcal{K}(q)[f]||^2_{L^2_{x \geq a}} L^2 \leq \left( \int_{x}^{+\infty} (t - x)^2|q(t)|^2 \, dt \right)^{1/2} \, ||f||^2_{L^2_{x \geq a}} L^2 \leq C ||q||_{L^2_{3/2}} ||f||_{L^2_{x \geq a}} L^2
\]
proving item (i). To estimate the composition $\mathcal{K}(q)^n$ viewed as an operator on $L^2_{x \geq a} L^2_y$, remark that

$$
\|\mathcal{K}(q)^n[f](x,\cdot)\|_{L^2_y} \leq \int_{x \leq t_1 \leq \cdots \leq t_n} |t_1 - x||q(t_1)| \cdots |t_n - t_{n-1}||q(t_n)||f(t_n,\cdot)||_{L^2_y} dt
$$

$$
\leq \int_{x \leq t_1 \leq \cdots \leq t_n} \left( \int_{t_n - t_{n-1}}^{+\infty} dt_n |t_n - t_{n-1}|^2 |q(t_n)|^2 \right)^{1/2} \|f\|_{L^2_{x \geq a} L^2_y} dt
$$

$$
\leq \left( \int_{x}^{+\infty} (t - x)^2|q(t)|^2 dt \right)^{1/2} \|f\|_{L^2_{x \geq a} L^2_y} \left( \int_{x}^{+\infty} |t - x||q(t)| dt \right)^{n-1}/(n-1)!
$$

Therefore

$$
\|\mathcal{K}(q)^n[f]\|_{L^2_{x \geq a} L^2_y} \leq \left\| \begin{array}{c}
\int_{x}^{+\infty} (t - x)^2|q(t)|^2 dt \\
\begin{array}{c} \int_{L^1_{x \geq a}} \|f\|_{L^2_{x \geq a} L^2_y} \\
C^{n-1} \|q\|_{L^1_{x \geq a}}^{n-1} \end{array}
\end{array} \right\|_{L^1_{x \geq a}} \|q\|_{L^2_{x \geq a}}^{n-1} \frac{(n-1)!}{(n-1)!}
$$

from which item (i) follows. Item (ii) is then proved as in the previous Lemma. \(\square\)

Note that for $f \equiv 1$, the expression $\mathcal{K}(q)[1](x,k) = \int_{x}^{+\infty} D_k(t-x)q(t) dt$ is well defined.

**Lemma 2.4.** For any $2 \leq \beta \leq +\infty$ and $a \in \mathbb{R}$, the map $L^2_{x \geq a} \ni q \mapsto \mathcal{K}(q)[1] \in C^0_{x \geq a} L^\beta \cap L^2_{x \geq a} L^2$ is analytic. Furthermore

$$
\|\mathcal{K}(q)[1]\|_{C^0_{x \geq a} L^\beta} \leq C_1 \|q\|_{L^2_y}, \quad \|\mathcal{K}(q)[1]\|_{L^2_{x \geq a} L^2} \leq C_2 \|q\|_{L^2_y},
$$

where $C_1, C_2 > 0$ are constants depending on $a$ and $\beta$.

**Proof.** Since the map $q \mapsto \mathcal{K}(q)[1]$ is linear in $q$, it suffices to prove its continuity in $q$. Moreover, it is enough to prove the result for $\beta = 2$ and $\beta = +\infty$ as the general case then follows by interpolation. For any $k \in \mathbb{R}$, the bound $|D_k(y)| \leq |y|$ shows that the map $k \mapsto D_k(y)$ is in $L^\infty$. Thus

$$
\|\mathcal{K}(q)[1](x,\cdot)\|_{L^\infty} \leq \int_{x}^{+\infty} (t - x)|q(t)| dt \leq \int_{a}^{+\infty} |t - a||q(t)| dt \leq C \|q\|_{L^1_{x \geq a}},
$$

where $C > 0$ is a constant depending only on $a \in \mathbb{R}$. The claimed estimate follows by noting that $\|q\|_{L^1_{x}} \leq C \|q\|_{L^2_y}$.

Using that for $|k| \geq 1$, $|\hat{D}_k(y)| \leq \frac{1}{|k|}$, one sees that $k \mapsto D_k(y)$ is $L^2$-integrable. Hence $k \mapsto D_k(t-x)D_{-k}(s-x)$ is integrable. Actually, since the Fourier transform $\mathcal{F}_+(D_k(y))$ in the $k$-variable of the function $k \mapsto D_k(y)$ is the function $\eta \mapsto \mathbb{R}_{[a,b]}(\eta)$, by Plancherel's Theorem

$$
\int_{-\infty}^{\infty} D_k(t-x)D_k(s-x) dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{R}_{[0,t-x]}(\eta)\mathbb{R}_{[0,s-x]}(\eta) d\eta = \frac{1}{\pi} \min(t-x, s-x).
$$
For any $x \geq a$ one thus has
\[ \|\mathcal{K}(q)[1](x, \cdot)\|_{L^2}^2 = \int_{-\infty}^{\infty} \mathcal{K}(q)[1](x, \cdot) \cdot \overline{\mathcal{K}(q)[1](x, \cdot)} \, dk \]
\[ = \iint_{[x, \infty) \times [x, \infty)} dt \, ds \, q(t) \overline{q(s)} \int_{-\infty}^{+\infty} D_k(t-x)D_{-k}(s-x) \, dk. \]
and hence
\[ \|\mathcal{K}(q)[1](x, \cdot)\|_{L^2}^2 \leq \frac{2}{\pi} \int_{a}^{+\infty} (t-x)|q(t)| \int_{t}^{+\infty} |q(s)|ds \]
\[ \leq \frac{2}{\pi} \int_{a}^{+\infty} ds \, |q(s)| \int_{a}^{t} (t-a) \, |q(t)| \, dt \leq C \|q\|_{L^2}^2, \]
where the last inequality follows from the Hardy-Littlewood inequality. The continuity in $x$ follows from Lebesgue convergence Theorem.

To prove the second inequality, start from the second term in (15) and change the order of integration to obtain
\[ \|\mathcal{K}(q)[1]\|_{L^2_{x \geq a}}^2 \leq \left\| \int_{x}^{+\infty} |t-a||q(t)| \int_{t}^{+\infty} |q(s)|ds \right\|_{L^1_{x \geq a}} \]
\[ \leq \int_{a}^{+\infty} |q(s)| \int_{a}^{s} (s-a)^2 |q(s)|ds \leq C \|q\|_{L^2} \|q\|_{L^2}, \]

\[ \square \]

**Proof of Proposition 1.** Formally, the solution of equation (12) is given by
\[ m(q) - 1 = \left( Id - \mathcal{K}(q) \right)^{-1} \mathcal{K}(q)[1]. \] (16)

By Lemma 2.2, 2.3, 2.4 it follows that the r.h.s. of (16) is an element of $C_{x \geq a}^\beta \cap L^2_{x \geq a} L^2$, $2 \leq \beta \leq \infty$, and analytic as a function of $q$, since it is the composition of analytic maps.

Properties of $\partial_k^n m(q, x, k)$ for $1 \leq n \leq M - 1$. In order to study $\partial_k^n m(q, x, k)$, we deduce from (12) an integral equation for $\partial_k^n m(q, x, \cdot)$ and solve it. Recall that for any $M \in \mathbb{Z}_{\geq 0}$, $H^M_\mathbb{C} = H^M(\mathbb{R}, \mathbb{C})$ denotes the Sobolev space of functions \{ $f \in L^2$ | $\hat{f} \in L^2_M$ \}. The result is summarized in the following

**Proposition 2.** Fix $M \in \mathbb{Z}_{\geq 3}$ and $a \in \mathbb{R}$. For any integer $1 \leq n \leq M - 1$ the following holds:

(i) for $q \in L^M_M$ and $x \geq a$ fixed, the function $k \mapsto m(q, x, k) - 1$ is in $H^{M-1}_\mathbb{C}$;

(ii) the map $L^M_M \ni q \mapsto \partial_k^n m(q) \in C_{x \geq a}^\beta L^2$ is analytic. Moreover $\|\partial_k^n m(q)\|_{C_{x \geq a}^\beta L^2} \leq K \|q\|_{L^M_M}$, where $K$ can be chosen uniformly on bounded subsets of $L^M_M$.

**Remark 3.** In [8] it is proved that if $q \in L^1_{M-1}$ then for every $x \geq a$ fixed the map $k \mapsto m(q, x, k)$ is in $C^{M-2}$; note that since $L^M_M \subset L^1_{M-1}$, we obtain the same regularity result by Sobolev embedding theorem.
To prove Proposition 2 we first need to derive some auxiliary results. Assuming that \( m(q, x, \cdot) - 1 \) has appropriate regularity and decay properties, the \( n^{th} \) derivative \( \partial^n_k m(q, x, k) \) satisfies the following integral equation

\[
\partial^n_k m(q, x, k) = \sum_{j=0}^{n} \binom{n}{j} \int_{x}^{+\infty} \partial_k^j D_k(t - x) q(t) \partial^{n-j}_k m(q, t, k) \, dt. \tag{17}
\]

To write (17) in a more convenient form introduce for \( 1 \leq j \leq n \) and \( q \in L^2_{n+1} \) the operators

\[
K_j(q) : C^0_{x \geq a} L^2 \to C^0_{x \geq a} L^2, \quad f \mapsto K_j(q)[f]
\]

leading to

\[
(Id - K(q)) \partial^n_k m(q) = \sum_{j=1}^{n-1} \binom{n}{j} K_j(q)[\partial^{n-j}_k m(q)] + K_n(q)[m(q) - 1] + K_n(q)[1]. \tag{18}
\]

In order to prove the claimed properties for \( \partial^n_k m(q) \) we must show in particular that the r.h.s. of (18) is in \( C^0_{x \geq a} L^2 \). This is accomplished by the following

**Lemma 2.5.** Fix \( M \in \mathbb{Z}_{\geq 3} \) and \( a \in \mathbb{R} \). Then there exists a constant \( C > 0 \), depending only on \( a, M \), such that the following holds:

(i) for any integers \( 1 \leq n \leq M - 1 \)

1(i) the map \( L^2_M \ni q \mapsto K_n(q)[1] \in C^0_{x \geq a} L^2 \) is analytic, and moreover

\[
\|K_n(q)[1]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M}.
\]

(ii) the map \( L^2_M \ni q \mapsto K_n(q) \in \mathcal{L}(L^2_{x \geq a}, L^2) \) is analytic. Moreover

\[
\|K_n(q)[f]\|_{C^0_{x \geq a} L^2} \leq \|q\|_{L^2_M} \|f\|_{L^2_{x \geq a} L^2}.
\]

(ii) For any \( 1 \leq n \leq M - 2 \), the map \( L^2_M \ni q \mapsto K_n(q) \in \mathcal{L}(C^0_{x \geq a} L^2) \) is analytic. Moreover one has

\[
\|K_n(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{C^0_{x \geq a} L^2}.
\]

(iii) As an application of item (i) and (ii), for any integers \( 1 \leq n \leq M - 1 \) the map \( L^2_M \ni q \mapsto K_n(q)[m(q) - 1] \in C^0_{x \geq a} L^2 \) is analytic, and

\[
\|K_n(q)[m(q) - 1]\|_{C^0_{x \geq a} L^2} \leq K'_n \|q\|_{L^2_M}^2,
\]

where \( K'_n > 0 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

**Proof.** First, remark that all the operators \( q \mapsto K_n(q) \) are linear in \( q \), therefore the continuity in \( q \) implies the analyticity in \( q \). We begin proving item (i).

(i) Let \( \varphi(x, k) := \int_{x}^{+\infty} \partial_k^n D_k(t - x) q(t) \, dt \) and compute the Fourier transform \( \mathcal{F}_+(\varphi(x, \cdot)) \) with respect to the \( k \) variable for \( x \geq a \) fixed, which we denote by \( \hat{\varphi}(x, \xi) \equiv \int_{-\infty}^{+\infty} dk \, e^{ik\xi} \varphi(x, k) \). Explicitly

\[
\hat{\varphi}(x, \xi) = \int_{x}^{+\infty} \, dq(t) \int_{-\infty}^{+\infty} dk \, e^{ik\xi} \partial_k^n D_k(t - x) = \int_{x}^{+\infty} \, dq(t) \xi^n \mathbf{1}_{[0, t-x]}(\xi) \, dt.
\]
Proof of Proposition 2.

By Parseval’s Theorem \( \| \varphi(x, \cdot) \|_{L^2} = \frac{1}{\sqrt{2\pi}} \| \hat{\varphi}(x, \cdot) \|_{L^2} \). By changing the order of integration one has

\[
\| \hat{\varphi}(x, \cdot) \|_{L^2}^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt \, ds \, q(t) \, \overline{q(s)} \int_{|x,\infty \times [x,\infty)} |\xi|^{2n} \, \mathbb{1}_{[0,t-x]}(\xi) \, \mathbb{1}_{[0,s-x]}(\xi) \, d\xi
\]

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\[
\leq 2 \int_{x} dt \, |q(t)||t-x|^{2n+1} \int_{t}^{+\infty} |q(s)| \, ds
\]

\[
\leq \|(t-a)^{n+1}q\|_{L^2_{x \geq a}} \left( \|(t-a)^n \int_{t}^{+\infty} |q(s)| \, ds \right)_{L^2_{x \geq a}} \leq C \|q\|_{L^2_{x \geq a}},
\]

where we used that by Remark 4, \( \|(t-a)^n \int_{t}^{+\infty} |q(s)| \, ds \|_{L^2_{x \geq a}} \leq C \|q\|_{L^2_{x \geq a}} \).

(ii) Let \( f \in L^2_{x \geq a} L^2 \), and using \( |\partial_x^D_k(t-x)| \leq 2^n |t-x|^{n+1} \) it follows that

\[
\|K_n(q)|f|(x, \cdot)\|_{L^2} \leq C \int_{x}^{+\infty} |q(t)||t-x|^{n+1} \|f(t, \cdot)\|_{L^2} \, dt \leq C \|q\|_{L^2_{x \geq a}} \|f\|_{L^2_{x \geq a}} \;
\]

by taking the supremum in the \( x \) variable one has \( K_n(q) \in \mathcal{L}(L^2_{x \geq a} L^2, C^0_{x \geq a} L^2) \), where the continuity in \( x \) follows by Lebesgue’s convergence theorem. The map \( q \mapsto K_n(q) \) is linear and continuous, therefore also analytic.

We prove now item (ii). Let \( g \in C^0_{x \geq a} L^2 \). From the inequality \( \|K_n(q)|g|(x, \cdot)\|_{L^2} \leq \int_{x}^{+\infty} |q(t)||t-x|^{n+1} \|g(t, \cdot)\|_{L^2} \, dt \) it follows that

\[
\sup_{x \geq a} \|K_n(q)|g|(x, \cdot)\|_{L^2} \leq \|g\|_{C^0_{x \geq a} L^2} \int_{a}^{+\infty} |q(t)||t-a|^{n+1} \, dt \leq C \|g\|_{C^0_{x \geq a} L^2} \|q\|_{L^2_{x \geq a}},
\]

which implies the claimed estimate. The analyticity follows from the linearity and continuity of the map \( q \mapsto K_n(q) \).

Finally we prove item (iii). By Proposition 1, the map \( L^2_{n+1} \ni q \mapsto m(q) - 1 \in L^2_{x \geq a} L^2 \) is analytic. By item (ii) above the bilinear map \( L^2_{n+1} \times L^2_{x \geq a} \ni (q, f) \mapsto K_n(q)[f] \in C^0_{x \geq a} L^2 \) is analytic; since the composition of analytic maps is analytic, the map \( L^2_{n+1} \ni q \mapsto K_n(q)[m(q) - 1] \in C^0_{x \geq a} L^2 \) is analytic. By (ii) and Proposition 1 one has

\[
\|K_n(q)[m(q) - 1]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_{n+1}} \|m(q) - 1\|_{L^2_{x \geq a} L^2} \leq K'_0 \|q\|_{L^2_{n+1}}^2,
\]

where \( K'_0 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

Proof of Proposition 2. The proof is carried out by a recursive argument in \( n \). We assume that \( q \mapsto \partial_x^r m(q) \) is analytic as a map from \( L^2_M \) to \( C^0_{x \geq a} L^2 \) for \( 0 \leq r \leq n-1 \), and prove that \( L^2_M \to C^0_{x \geq a} L^2 : q \mapsto \partial_x^n m(q) \) is analytic, provided that \( n \leq M-1 \). The case \( n = 0 \) is proved in Proposition 1.
Lemma 2.5, the fact that the operator norm of $\text{Id}$ maps is analytic. To estimate all the remaining lines, use the induction hypothesis, the estimates of the latter identity, use item (i). In order to estimate the term in the fourth line on the right hand side of the following characterization of $H^1$, we begin by showing that for every $n \geq 0$ fixed, $\|\tau_n f - f\|_{L^2} \leq C|h|$, $\forall h \in \mathbb{R}$, (19) where $(\tau_n f)(k) := f(k + h)$ is the translation operator. Moreover the constant $C$ above can be chosen to be $C = \|\partial_k u\|_{L^2}$. Starting from (18) (with $n - 1$ instead of $n$), an easy computation shows that for every $x \geq a$ fixed $(\tau_n)\partial_k^{n-1}m(q) \equiv \partial_k^{n-1}m(q, x, k + h)$ satisfies the integral equation

$$(1d - K(q))(\tau_n\partial_k^{n-1}m(q) - \partial_k^{n-1}m(q)) = \int_{x}^{+\infty} (\tau_n\partial_k^{n-1}D_k(t - x) - \partial_k^{n-1}D_k(t - x))q(t)(m(q, t, k + h) - 1)\;dt$$

$$+ \int_{x}^{+\infty} (\tau_n\partial_k^{n-1}D_k(t - x) - \partial_k^{n-1}D_k(t - x))q(t)\;dt$$

$$+ \int_{x}^{+\infty} (\partial_k^{n-1}D_k(t - x))q(t)\;dt$$

$$+ \sum_{j=1}^{n-2} \left( \int_{x}^{+\infty} (\tau_n\partial_k^{n-1}D_k(t - x) - \partial_k^{n-1}D_k(t - x))q(t)\;dt \right)$$

$$\int_{x}^{+\infty} (\partial_k^{n-1}D_k(t - x))q(t)\;dt$$

$$+ \int_{x}^{+\infty} \partial_k^{n-1}D_k(t - x)q(t)(\tau_n\partial_k^{n-1-j}m(q, t, k) - \partial_k^{n-1-j}m(q, t, k))\;dt$$

$$+ \int_{x}^{+\infty} (\tau_nD_k(t - x) - D_k(t - x))q(t)\;dt.$$

(20)

In order to estimate the term in the fourth line on the right hand side of the latter identity, use item (i) of Lemma 2.5 and the characterization (19) of $H^1$. To estimate all the remaining lines, use the induction hypothesis, the estimates of Lemma 2.5, the fact that the operator norm of $(1d - K(q))^{-1}$ is bounded uniformly in $k$ and the estimate

$$|\tau_n\partial_k D_k(t - x) - \partial_k D_k(t - x)| \leq C|t - x|^j|\partial^j| H^1,$$

$\forall h \in \mathbb{R}$, to deduce that for every $n \leq M - 1$

$$\|\tau_n\partial_k^{n-1}m(q) - \partial_k^{n-1}m(q)\|_{L^2} \leq C|h|, \quad \forall h \in \mathbb{R},$$

which is exactly condition (19). This shows that $k \mapsto \partial_k^{n-1}m(q, x, k)$ admits a weak derivative in $L^2$. Formula (17) is therefore justified. We prove now that the map $L^2_M \ni q \mapsto \partial_k^{n}m(q) \in C^0_{x \geq a}L^2$ is analytic for $1 \leq n \leq M - 1$. Indeed equation (18) and Lemma 2.5 imply that

$$\|\partial_k^n m(q)\|_{C^0_{x \geq a}L^2} \leq K' \left( \|q\|_{L^2_M}^2 + \left\|\partial_k^{n-j}m(q)\right\|_{C^0_{x \geq a}L^2} \right)$$

where $K'$ can be chosen uniformly on bounded subsets of $q$ in $L^2_M$. Therefore $\partial_k^nm(q) \in C^0_{x \geq a}L^2$ and one gets recursively $\|\partial_k^nm(q)\|_{C^0_{x \geq a}L^2} \leq K\left\|q\right\|_{L^2_M}$, where $K$ can be chosen uniformly on bounded subsets of $q$ in $L^2_M$. The analyticity of the map $q \mapsto \partial_k^nm(q)$ follows by formula (18) and the fact that composition of analytic maps is analytic. \qed
Properties of $k\partial^m_k m(q, x, k)$ for $1 \leq n \leq M$. The analysis of the $M^{th}$ $k$-derivative of $m(q, x, k)$ requires a separate attention. It turns out that the distributional derivative $\partial^M_k m(q, x, \cdot)$ is not necessarily $L^2$-integrable near $k = 0$ but the product $k\partial^M_k m(q, x, \cdot)$ is. This is due to the fact that $\partial^M_k D_k(x)q(x) \sim x^{M+1} q(x)$ which might not be $L^2$-integrable. However, by integration by parts, it’s easy to see that $k\partial^M_k D_k(x)q(x) \sim x^M q(x) \in L^2$. The main result of this section is the following

**Proposition 3.** Fix $M \in \mathbb{Z}_{\geq 3}$ and $a \in \mathbb{R}$. Then for every integer $1 \leq n \leq M$ the following holds:

(i) for every $q \in L^2_M$ and $x \geq a$ fixed, the function $k \mapsto k\partial^m_k m(q, x, k)$ is in $L^2$;
(ii) the map $L^2_M \ni q \mapsto k\partial^m_k m(q) \in C^0_{x \geq a} L^2$ is analytic. Moreover $\|k\partial^m_k m\|_{C^0_{x \geq a} L^2} \leq K_1 \|q\|_{L^2_M}$ where $K_1$ can be chosen uniformly on bounded subsets of $L^2_M$.

Formally, multiplying equation (17) by $k$, the function $k\partial^m_k m(q)$ solves

$$
(Id - K(q)) (k\partial^m_k m(q)) = \left( \sum_{j=1}^{n-1} \binom{n}{j} \hat{K}_j(q)[\partial^{n-j}_k m(q)] + \hat{K}_n(q)[m(q) - 1] + \hat{K}_n(q)[1] \right) \tag{21}
$$

where we have introduced for $0 \leq j \leq M$ and $q \in L^2_M$ the operators

$$
\hat{K}_j(q) : C^0_{x \geq a} L^2 \to C^0_{x \geq a} L^2, \quad f \mapsto \hat{K}_j(q)[f] \quad \text{and} \quad \hat{K}_j(q)[f](x, k) := \int_x^\infty k\partial^j_k D_k(t - x) q(t) f(t, k) \, dt.
$$

We begin by proving that each term of the r.h.s. of (21) is well defined and analytic as a function of $q$. The following lemma is analogous to Lemma 2.5:

**Lemma 2.6.** Fix $M \in \mathbb{Z}_{\geq 3}$ and $a \in \mathbb{R}$. There exists a constant $C > 0$ such that the following holds:

(i) for any integers $1 \leq n \leq M$

(i1) the map $L^2_M \ni q \mapsto \hat{K}_n(q)[1] \in C^0_{x \geq a} L^2$ is analytic, and moreover

$$
\|\hat{K}_n(q)[1]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M};
$$

(i2) the map $L^2_M \ni q \mapsto \hat{K}_n(q) \in \mathcal{L}(L^2_{x \geq a} L^2, C^0_{x \geq a} L^2)$ is analytic. Moreover

$$
\|\hat{K}_n(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{C^0_{x \geq a} L^2};
$$

(ii) for any $1 \leq j \leq M - 1$ the map $L^2_M \ni q \mapsto \hat{K}_j(q) \in \mathcal{L}(C^0_{x \geq a} L^2)$ is analytic, and

$$
\|\hat{K}_j(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{C^0_{x \geq a} L^2};
$$

(iii) As an application of item (i) and (ii) we get

(iii1) for any $1 \leq n \leq M$, the map $L^2_M \ni q \mapsto \hat{K}_n(q)[m(q) - 1] \in C^0_{x \geq a} L^2$ is analytic with

$$
\|\hat{K}_n(q)[m(q) - 1]\|_{C^0_{x \geq a} L^2} \leq K'_1 \|q\|_{L^2_M}^2, \tag{22}
$$

where $K'_1$ can be chosen uniformly on bounded subsets of $L^2_M$.\]
(iii2) For any $1 \leq j \leq n-1$, the map $L^2_M \ni q \mapsto \tilde{K}_j(q)[\partial^n_{k}^j m(q)] \in C^0_{x \geq a}L^2$ is analytic with

$$
\left\| \tilde{K}_j(q)[\partial^n_{k}^j m(q)] \right\|_{C^0_{x \geq a}L^2} \leq K'_2 \|q\|_{L^2_M}^2,
$$

where $K'_2$ can be chosen uniformly on bounded subsets of $L^2_M$.

**Proof.**

(i) Since the maps $q \mapsto \tilde{K}_n(q)$, $0 \leq n \leq M$, are linear, it is enough to prove that these maps are continuous.

(i1) Introduce $\varphi(x, k) := \int_{-\infty}^{+\infty} k \partial^n_k D_k(t - x)q(t) \, dt$. The Fourier transform

$$
\hat{\varphi}(x, \xi) \equiv \mathcal{F}_+(\varphi(x, \cdot))
$$

of $\varphi$ with respect to the $k$-variable is given explicitly by

$$
\int_{-\infty}^{+\infty} dt q(t) \int_{-\infty}^{+\infty} dk \, e^{-2ik\xi} k \partial^n_k D_k(t - x) = -(2i)^{n-1} \int_{-\infty}^{+\infty} dt q(t) \partial_k (\xi^n \mathbb{1}_{[0, t-x]}(\xi)),
$$

where $\partial_k (\xi^n \mathbb{1}_{[0, t-x]}(\xi))$ is to be understood in the distributional sense. By Parseval’s Theorem $\|\varphi(x, \cdot)\|_{L^2} = \frac{1}{\sqrt{\pi}} \|\hat{\varphi}(x, \cdot)\|_{L^2}$. Let $C^\infty_0$ be the space of smooth, compactly supported functions. Since

$$
\|\hat{\varphi}(x, \cdot)\|_{L^2} = \sup_{\chi \in C^\infty_0, \|\chi\|_{L^2} \leq 1} \left| \int_{-\infty}^{+\infty} \chi(\xi) \hat{\varphi}(x, \xi) \, d\xi \right|
$$

one computes

$$
\left| \int_{-\infty}^{+\infty} \chi(\xi) \hat{\varphi}(x, \xi) \, d\xi \right| = \left| \int_{-\infty}^{+\infty} dt q(t) \int_{-\infty}^{+\infty} dk \chi(\xi) \partial_k (\xi^n \mathbb{1}_{[0, t-x]}(\xi)) \, d\xi \right| = \left| \int_{-\infty}^{+\infty} dt q(t) \int_{0}^{t-x} d\xi \, \xi^n \partial_k \chi(\xi) \right|
$$

$$
\leq \left| \int_{-\infty}^{+\infty} dt q(t) \chi(t - x)(t - x)^n \right| + n \left| \int_{-\infty}^{+\infty} dt q(t) d\xi \, \chi(\xi) \xi^n \right|
$$

$$
\leq \|q\|_{L^2_M} \|\chi\|_{L^2}^2 + n \left| \int_{-\infty}^{+\infty} dt |q(t)| |t - x|^{n-1} \int_{0}^{t-x} d\xi \, |\chi(\xi)| \right|
$$

$$
\leq \|q\|_{L^2_M} \|\chi\|_{L^2}^2 + n \left| \int_{-\infty}^{+\infty} dt |q(t)| |t - x|^{n} \frac{\int_{0}^{t-x} d\xi \, |\chi(\xi)|}{|t - x|} \right| \leq C \|q\|_{L^2_M} \|\chi\|_{L^2}^2 + n \left| \int_{-\infty}^{+\infty} dt |q(t)| |f(t, \cdot)|_{L^2} \right|
$$

where the last inequality follows from Cauchy-Schwartz and Hardy inequality, and $C > 0$ is a constant depending on $a$ and $M$.

(ii2) As $|k \partial^n_k D_k(t - x)| \leq 2^n |t - x|^n$ by integration by parts, it follows that for some constant $C > 0$ depending only on $a$ and $M$, $\left\| \hat{K}_n(q)[f(x, \cdot)] \right\|_{L^2} \leq C \int_{-\infty}^{+\infty} dt |q(t)| \|f(t, \cdot)\|_{L^2}$ and now take the
supremum over $x \geq a$ in the expression above and use Lebesgue’s dominated convergence theorem to prove item (i2).

(ii) The claim follows by
\[
\left\| \tilde{K}_j(q)[f](x,\cdot) \right\|_{L^2} \leq C \int \frac{|f(x)|}{|x|} \|f(t,\cdot)\|_{L^2} \, dt \leq C \|q\|_{L^1_j} \|f\|_{C^0_{x\geq a} L^2} \text{ and the remark that } \|q\|_{L^1_j} \leq C \|q\|_{L^2_M} \text{ for } 0 \leq j \leq M - 1.
\]

(iii) By Propositions 1 and 2 the maps $L^2_M \ni q \mapsto m(q) - 1 \in C^0_{x\geq a} L^2$ and $L^2_M \ni q \mapsto \partial_x^{n-1} m(q) \in C^0_{x\geq a} L^2$ are analytic; by item (ii) for any $1 \leq n \leq M - 1$, the bilinear map $(q,f) \mapsto \tilde{K}_n(q)[f]$ is analytic from $L^2_M \times C^0_{x\geq a} L^2$ to $C^0_{x\geq a} L^2$. Since the composition of two analytic maps is again analytic, item (i) follows. Moreover, $\tilde{K}_n(q)[m(q) - 1]$, $\tilde{K}_j(q)[\partial_x^{n-1} m(q)] \in C^0_{x\geq a} L^2$ since $m(q,x,k)$ and $\partial_x^{n} m(q,x,k)$ are continuous in the $x$-variable. The estimate (22) follows from item (i) and Proposition 1, 2.

\[
\square
\]

Proof of Proposition 3. One proceeds in the same way as in the proof of Proposition 2. Given any $1 \leq n \leq M$, we assume that $q \mapsto k\partial_x^{n-1} m(q)$ is analytic as a map from $L^2_M \to C^0_{x\geq a} L^2$ for $1 \leq r \leq n - 1$, and deduce that $q \mapsto k\partial_x^{n-1} m(q)$ is analytic as a map from $L^2_M \to C^0_{x\geq a} L^2$ and satisfies equation (21) (with $r$ instead of $n$).

We begin by showing that for every $x \geq a$ fixed, $k \mapsto k\partial_x^{n-1} m(q,x,k)$ is a function in $H^1$. Our argument uses again the characterization (19) of $H^1$. Arguing as for the derivation of (20) one gets the integral equation
\[
(Id - K(q)) (\tau_n(k\partial_x^{n-1} m(q)) - k\partial_x^{n-1} m(q)) =
\]
\[
\int_0^\infty \tau_n(k\partial_x^{n-1} D_k(t-x)) - k\partial_x^{n-1} D_k(t-x)) q(t)(m(q,t,k+h) - 1) \, dt
\]
\[
+ \int_0^\infty \tau_n(k\partial_x^{n-1} D_k(t-x)) - k\partial_x^{n-1} D_k(t-x)) q(t) \, dt
\]
\[
+ \int_0^\infty (k\partial_x^{n-1} D_k(t-x)) q(t)(m(q,t,k+h) - m(q,t,k)) \, dt
\]
\[
+ \sum_{j=1}^{n-2} \binom{n-1}{j} \left( \int_0^\infty \tau_n(k\partial_x^{j} D_k(t-x)) - k\partial_x^{j} D_k(t-x)) q(t) \partial_x^{n-1-j} m(q,t,k+h) \, dt \right)
\]
\[
+ \int_0^\infty k\partial_x^{j} D_k(t-x) q(t) \left( \tau_n(k\partial_x^{n-1-j} m(q,t,k) - \partial_x^{n-1-j} m(q,t,k)) \right) \, dt
\]
\[
+ \int_0^\infty (\tau_n D_k(t-x) - D_k(t-x)) q(t)(k+h)\partial_x^{n-1} m(q,t,k+h) \, dt.
\]

Using the estimates
\[
|\tau_n D_k(t-x) - D_k(t-x)| \leq C|x-x|^2|h|
\]
and
\[
|\tau_n(k\partial_x^{j} D_k(t-x)) - k\partial_x^{j} D_k(t-x)| \leq C|x-x|^{j+1}|h|, \quad \forall h \in \mathbb{R},
\]
obtained by integration by parts, the characterization (19) of $H^1$, the inductive hypothesis, estimates of Lemma 2.5 and Lemma 2.3 one deduces that for every
\[ n \leq M \]

\[ \| \tau_n(k \partial_k^{n-1}m(q)) - k \partial_k^{n-1}m(q) \|_{L^2} \leq C|h|, \quad \forall h \in \mathbb{R}. \]

This shows that \( k \mapsto k \partial_k^{n-1}m(q, x, k) \) admits a weak derivative in \( L^2 \). Since

\[ k \partial_k m(q, x, k) = \partial_k(k \partial_k^{n-1}m(q, x, k)) - \partial_k^{n-1}m(q, x, k), \]

the estimate above and Proposition 2 show that \( k \mapsto k \partial_k^m m(q, x, k) \) is an \( L^2 \) function. Formula (17) is therefore justified.

The proof of the analyticity of the map \( q \mapsto k \partial_k^m m(q) \) is analogous to the one of Proposition 2 and it is omitted.

\[ \Box \]

Analysis of \( \partial_x m(q, x, k) \). Introduce a odd smooth monotone function \( \zeta : \mathbb{R} \to \mathbb{R} \) with \( \zeta(k) = k \) for \( |k| \leq 1/2 \) and \( \zeta(k) = 1 \) for \( k \geq 1 \). We prove the following

Proposition 4. Fix \( M \in \mathbb{Z}_{\geq 3} \) and \( a \in \mathbb{R} \). Then the following holds:

(i) for any integer \( 0 \leq n \leq M-1 \), the map \( L_M^2 \ni q \mapsto \partial_k^n \partial_x m(q) \in C^0_{x \geq a} L^2 \) is analytic, and \( \| \partial_k^n \partial_x m(q) \|_{C^0_{x \geq a} \rightarrow L^2} \leq K_2 \| q \|_{L^2_M}, \) where \( K_2 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

(ii) the map \( L_M^2 \ni q \mapsto \zeta \partial_k^M \partial_x m(q) \in C^0_{x \geq a} L^2 \) is analytic, and moreover one has \( \| \zeta \partial_k^M \partial_x m(q) \|_{C^0_{x \geq a} \rightarrow L^2} \leq K_3 \| q \|_{L^2_M}, \) where \( K_3 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

The integral equation for \( \partial_x m(q, x, k) \) is obtained by taking the derivative in the \( x \)-variable of (12):

\[ \partial_x m(q, x, k) = - \int x e^{2ik(t-x)} q(t) m(q, t, k) \, dt. \]  

Taking the derivative with respect to the \( k \)-variable one obtains, for \( 0 \leq n \leq M-1 \),

\[ \partial_k^n \partial_x m(q, x, k) = - \sum_{j=0}^n \binom{n}{j} \int x e^{2ik(t-x)} (2i(t-x))^j q(t) \partial_k^{n-j} m(q, t, k) \, dt. \]  

For \( 0 \leq j \leq M \) introduce the integral operators \( G_j(q) : C^0_{x \geq a} L^2 \to C^0_{x \geq a} L^2 \) defined by

\[ f \mapsto G_j(q)[f](x, k) := - \int x e^{2ik(t-x)} (2i(t-x))^j q(t) f(t, k) \, dt \]

and rewrite (24) in the more compact form

\[ \partial_k^n \partial_x m(q) = \sum_{j=0}^{n-1} \binom{n}{j} G_j(q)[\partial_k^{n-j} m(q)] + G_n(q)[m(q) - 1] + G_n(q)[1]. \]

Proposition 4 (i) follows from Lemma 2.7 below.

The \( M^{th} \) derivative requires a separate treatment, as \( \partial_k^M m \) might not be well defined at \( k = 0 \). Indeed for \( n = M \) the integral \( \int x e^{2ik(t-x)} q(t) \partial_k^M m(q, t, k) \, dt \) in (24) might not be well defined near \( k = 0 \) since we only know that \( k \partial_k^M m(q, x, \cdot) \in \]
$L^2$. To deal with this issue we use the function $\zeta$ described above. Multiplying (25) with $n = M$ by $\zeta$ we formally obtain that $\zeta \partial_k^M \partial_x m(q)$ equals
\[
\sum_{j=1}^{M-1} \binom{M}{j} \zeta G_j(q)[\partial_k^M \partial_x m(q)] + \zeta G_M(q)[m(q) - 1] + \zeta G_M(q)[1] + G_0(q)[\zeta \partial_k^M m(q)].
\]

Proposition 4 (ii) follows from item (iii) of Lemma 2.7 and the fact that $\zeta \in L^\infty$:

**Lemma 2.7.** Fix $M \in \mathbb{Z}_{\geq 3}$ and $a \in \mathbb{R}$. There exists a constant $C > 0$ such that

(i) for any integer $0 \leq n \leq M$ the following holds:

(i1) the map $L^2_M \ni q \mapsto G_n(q)[1] \in C^0_{x \geq a} L^2$ is analytic.

Moreover $\|G_n(q)[1]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M}$.

(i2) The map $L^2_M \ni q \mapsto G_n(q) \in \mathcal{L}(L^2_{x \geq a} C^0_{x \geq a} L^2)$ is analytic and

\[
\|G_n(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{L^2_{x \geq a} L^2}.
\]

(ii) For any $0 \leq j \leq M - 1$, the map $L^2_M \ni q \mapsto G_j(q) \in \mathcal{L}(C^0_{x \geq a} L^2)$ is analytic, and

\[
\|G_j(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{C^0_{x \geq a} L^2}.
\]

(iii) For any $1 \leq n \leq M - 1$, $0 \leq j \leq n - 1$ and $\zeta : \mathbb{R} \to \mathbb{R}$ odd smooth monotone function with $\zeta(k) = k$ for $|k| \leq 1/2$ and $\zeta(k) = 1$ for $k \geq 1$, the following holds:

(iii1) the maps $L^2_M \ni q \mapsto G_n(q)[\partial_k^{n-j} m(q)] \in C^0_{x \geq a} L^2$ and $L^2_M \ni q \mapsto G_n(q)[m(q) - 1] \in C^0_{x \geq a} L^2$ are analytic. Moreover

\[
\|G_j(q)[\partial_k^{n-j} m(q)]\|_{C^0_{x \geq a} L^2}, \quad \|G_n(q)[m(q) - 1]\|_{C^0_{x \geq a} L^2} \leq K'_2 \|q\|_{L^2_M}^2,
\]

where $K'_2$ can be chosen uniformly on bounded subsets of $L^2_M$.

(iii2) The map $L^2_M \ni q \mapsto G_0(q)[\zeta \partial_k^M m(q)] \in C^0_{x \geq a} L^2$ is analytic and

\[
\|G_0(q)[\zeta \partial_k^M m(q)]\|_{C^0_{x \geq a} L^2} \leq K'_3 \|q\|_{L^2_M}^2
\]

where $K'_3$ can be chosen uniformly on bounded subsets of $L^2_M$.

**Proof.** As before it’s enough to prove the continuity in $q$ of the maps considered to conclude that they are analytic.

(i1) For $x \geq a$ and any $0 \leq n \leq M$, $\|G_n(q)[1](x, \cdot)\|_{L^2}^2 \leq C \int_x^{+\infty} |t - x|^2 |q(t)|^2 dt \leq C \|q\|_{L^2_M}$. The claim follows by taking the supremum over $x \geq a$ in the inequality above.

(i2) For $x \geq a$ and any $0 \leq n \leq M$ one has $\|G_n(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{L^2_{x \geq a} L^2}$, which implies the claimed estimate.

(ii) For $x \geq a$ and $0 \leq j \leq M - 1$ one has the bound

\[
\|G_j(q)[f]\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_{j-1}} \|f\|_{C^0_{x \geq a} L^2} \leq C \|q\|_{L^2_M} \|f\|_{C^0_{x \geq a} L^2}.
\]

(iii1) By Proposition 2 one has that for any $1 \leq n \leq M - 1$ and $0 \leq j \leq n - 1$ the map $L^2_M \ni q \mapsto \partial_k^{n-j} m(q) \in C^0_{x \geq a} L^2$ is analytic. Since composition of analytic maps is again an analytic map, the claim regarding the analyticity follows. The first estimate follows from item (ii). A similar argument can be used to prove the second estimate.
(iii2) By Proposition 3, the map \( L^2_M \ni q \mapsto \zeta \partial_k^M m(q) \in C^0_{x \geq a} L^2 \) is analytic, implying the claim regarding the analyticity. The estimate follows from
\[
||G_0[\zeta \partial_k^M m(q)]||_{C^0_{x \geq a} L^2} \leq ||q||_{L^2_M} ||\zeta \partial_k^M m(q)||_{C^0_{x \geq a} L^2}.
\]

The following corollary follows from the results obtained so far:

**Corollary 2.** Fix \( M \in \mathbb{Z}_{\geq 3} \). Then the normalized Jost functions \( m_j(q, x, k), j = 1, 2 \), satisfy:

(i) the maps \( L^2_M \ni q \mapsto m_j(q, 0, \cdot) - 1 \in L^2 \) and \( L^2_M \ni q \mapsto k^\alpha \partial_k^M m_j(q, 0, \cdot) \in L^2 \)
are analytic for \( 1 \leq n \leq M - 1 \) [\( 1 \leq n \leq M \)] if \( \alpha = 0 \) [\( \alpha = 1 \)]. Moreover
\[
||m_j(q, 0, \cdot) - 1||_{L^2}, ||k^\alpha \partial_k^M m_j(q, 0, \cdot)||_{L^2} \leq K_1 ||q||_{L^2_M},
\]
where \( K_1 > 0 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

(ii) For \( 0 \leq n \leq M - 1 \), the maps \( L^2_M \ni q \mapsto \partial_k^M \partial_x m_j(q, 0, \cdot) \in L^2 \) and \( L^2_M \ni q \mapsto \zeta \partial_k^M \partial_x m_j(q, 0, \cdot) \in L^2 \)
are analytic. Moreover
\[
||\partial_k^M \partial_x m_j(q, 0, \cdot)||_{L^2}, ||\zeta \partial_k^M \partial_x m_j(q, 0, \cdot)||_{L^2} \leq K_2 ||q||_{L^2_M},
\]
where \( K_2 > 0 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

**Proof.** The Corollary follows by evaluating formulas (12), (17), (24) at \( x = 0 \) and using the results of Proposition 1, 2, 3 and 4.

3. One smoothing properties of the scattering map. The aim of this section is to prove the part of Theorem 1.1 related to the direct problem. To begin, note that by Theorem 2.1, for \( q \in L^2_3 \) real valued one has \( \tilde{m}_1(q, x, k) = m_1(q, x, -k) \) and \( \tilde{m}_2(q, x, k) = m_2(q, x, -k) \); hence
\[
S(q, k) = S(q, -k), \quad W(q, k) = W(q, -k).
\]
Moreover one has for any \( q \in L^2_3 \)
\[
W(q, k)W(q, -k) = 4k^2 + S(q, k)S(q, -k) \quad \forall k \in \mathbb{R} \setminus \{0\}
\]
which by continuity holds for \( k = 0 \) as well. In the case where \( q \in \mathcal{Q} \), the latter identity implies that \( S(q, 0) \neq 0 \).

Recall that for \( q \in L^2_3 \) the Jost solutions \( f_1(q, x, k) \) and \( f_2(q, x, k) \) satisfy the following integral equations
\[
f_1(x, k) = e^{ikx} + \int_{-\infty}^{+\infty} \sin k(t-x) \frac{q(t)}{k} f_1(t, k) dt, \quad (27)
\]
\[
f_2(x, k) = e^{-ikx} + \int_{-\infty}^{+\infty} \sin k(x-t) \frac{q(t)}{k} f_2(t, k) dt. \quad (28)
\]
Substituting (27) and (28) into (4), (3), one verifies that \( S(q, k), W(q, k) \) satisfy for \( k \in \mathbb{R} \) and \( q \in L^2_3 \)
\[
S(q, k) = \int_{-\infty}^{+\infty} e^{ikt} q(t) f_1(q, t, k) dt, \quad (29)
\]
Corollary 3. 

Note that the integrals above are well defined thanks to the estimate in item \((ii)\) of Theorem 2.1.

Inserting formula (27) into (29), one gets that

\[ S(q, k) = \mathcal{F}_-(q, k) + O \left( \frac{1}{k} \right). \]

The main result of this section is an estimate of

\[ A(q, k) := S(q, k) - \mathcal{F}_-(q, k), \quad (31) \]

saying that \( A \) is 1-smoothing. To formulate the result in a precise way, we need to introduce the following Banach spaces. For \( M \in \mathbb{Z}_{\geq 1} \) define

\[ H^M_* := \{ f \in H^{M-1}_C : \overline{f(k)} = f(-k), \ k\partial^M_k f \in L^2 \}, \]

endowed with the norm

\[ \| f \|_{H^M_*}^2 := \| f \|_{H^{M-1}_*}^2 + \| k\partial^M_k f \|_{L^2}^2. \]

Note that \( H^M_* \) is a real Banach space. We will use also the complexification of the Banach spaces \( H^M_* \) and \( H^M_C \) (this last defined in (7)), in which the reality condition \( \overline{f(k)} = f(-k) \) is dropped:

\[ H^M_* := \{ f \in H^{M-1}_C : k\partial^M_k f \in L^2 \}, \quad H^M_C := \{ f \in H^{M-1}_C : \zeta \partial^M_k f \in L^2 \}. \]

Note that for any \( M \geq 2 \)

\( (i) H^M_* \subset H^{M-1}_* \) and \( H^M_* \subset H^{M-1}_C \), \( (ii) fg \in H^M_* \quad \forall f \in H^M_* \subset H^{M-1}_C, \ g \in H^M_C. \quad (32) \)

We can now state the main theorem of this section. Let \( L^2_{M, \mathbb{R}} := \{ f \in L^2_M \mid f \text{ real} \} \).

**Theorem 3.1.** Let \( N \in \mathbb{Z}_{\geq 0} \) and \( M \in \mathbb{Z}_{\geq 3} \). Then one has:

- \( (i) \) The map \( q \mapsto A(q, \cdot) \) is analytic as a map from \( L^2_M \) to \( H^M_C \).
- \( (ii) \) The map \( q \mapsto A(q, \cdot) \) is analytic as a map from \( H^N \cap L^2_3 \) to \( L^2_{N+1} \). Moreover

\[ \| A(q, \cdot) \|_{L^2_{N+1}} \leq C_A \| q \|_{H^N \cap L^2_3} \]

where the constant \( C_A > 0 \) can be chosen uniformly on bounded subsets of \( H^N \cap L^2_3 \).

Furthermore for \( q \in L^2_{3, \mathbb{R}} \) the map \( A(q, \cdot) \) satisfies \( \overline{A(q, k)} = A(q, -k) \) for every \( k \in \mathbb{R} \). Thus its restrictions \( A : L^2_{M, \mathbb{R}} \to H^M_C \) and \( A : H^N \cap L^2_3 \to L^2_{N+1} \) are real analytic.

The following corollary follows immediately from identity (31), item \( (ii) \) of Theorem 3.1 and the properties of the Fourier transform:

**Corollary 3.** Let \( N \in \mathbb{Z}_{\geq 0} \). Then the map \( q \mapsto S(q, \cdot) \) is analytic as a map from \( H^N \cap L^2_3 \) to \( L^2_N \). Moreover

\[ \| S(q, \cdot) \|_{L^2_N} \leq C_S \| q \|_{H^N \cap L^2_3} \]

where the constant \( C_S > 0 \) can be chosen uniformly on bounded subsets of \( H^N \cap L^2_3 \).
In [25], it is shown that in the periodic setup, the Birkhoff map of KdV is 1-smoothing. As the map \( q \mapsto S(q, \cdot) \) on the spaces considered can be viewed as a version of the Birkhoff map in the scattering setup of KdV, Theorem 3.1 confirms that a result analogous to the one on the circle holds also on the line.

The proof of Theorem 3.1 consists of several steps. We begin by proving item (i). Since \( F_- : L^2_M \to H^M_M \) is bounded, item (i) will follow from the following proposition:

**Proposition 5.** Let \( M \in \mathbb{Z}_{\geq 3} \), then the map \( L^2_M \ni q \mapsto S(q, \cdot) \in H^M_M \) is analytic and

\[
\|S(q, \cdot)\|_{H^M_M} \leq K_S \|q\|_{L^2_M},
\]

where \( K_S > 0 \) can be chosen uniformly on bounded subsets of \( L^2_M \).

**Proof.** Recall that \( f_1(q, x, k) = e^{ikx} m_1(q, x, k) \) and \( f_2(q, x, k) = e^{-ikx} m_2(q, x, k) \). The \( x \)-independence of \( S(q, k) \) implies that

\[
S(q, k) = [m_1(q, 0, 0), m_2(q, 0, -k)].
\]

As by Corollary 2, \( m_j(q, 0, 1) - 1 \in H^M_M \) and \( \partial_x m_j(q, 0, \cdot) \in H^M_M \), \( j = 1, 2 \), the identity (33) yields

\[
S(q, k) = (m_1(q, 0, k) - 1) \partial_x m_2(q, 0, -k) - (m_2(q, 0, -k) - 1) \partial_x m_1(q, 0, k)
+ \partial_x m_2(q, 0, 0) - \partial_x m_1(q, 0, k),
\]

thus \( S(q, \cdot) \in H^M_M \) by (32). The estimate on the norm \( \|S(q, \cdot)\|_{H^M_M} \) follows by Corollary 2.

**Proof of Theorem 3.1 (i).** The claim is a direct consequence of Proposition 5 and the fact that for any real valued potential \( q \), \( S(q, k) = S(q, -k) \), \( F_-(q, k) = F_-(q, -k) \) and hence \( A(q, k) = A(q, -k) \) for any \( k \in \mathbb{R} \).

In order to prove the second item of Theorem 3.1, we expand the map \( q \mapsto A(q) \) as a power series of \( q \). More precisely, iterate formula (27) and insert the formal expansion obtained in this way in the integral term of (29), to get

\[
S(q, k) = F_-(q, k) + \sum_{n \geq 1} \frac{s_n(q, k)}{k^n}
\]

where, with \( dt = dt_0 \cdots dt_n \),

\[
s_n(q, k) := \int_{\Delta_{n+1}} e^{ikt_0 q(t_0)} \prod_{j=1}^n \left( q(t_j) \sin k(t_j - t_{j-1}) \right) e^{ikt_n} dt
\]

is a polynomial of degree \( n + 1 \) in \( q \) and \( \Delta_{n+1} \) is given by

\[
\Delta_{n+1} := \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} : t_0 \leq \cdots \leq t_n \}.
\]

Since by Proposition 5 \( S(q, \cdot) \) is in \( L^2 \), it remains to control the decay of \( A(q, \cdot) \) in \( k \) at infinity. Introduce a cut off function \( \chi \) with \( \chi(k) = 0 \) for \( |k| \leq 1 \) and \( \chi(k) = 1 \) for \( |k| > 2 \) and consider the series

\[
\chi(k)S(q, k) = \chi(k)F_-(q, k) + \sum_{n \geq 1} \frac{\chi(k)s_n(q, k)}{k^n}. \tag{34}
\]

Item (ii) of Theorem 3.1 follows once we show that each term \( \frac{\chi(k)s_n(q, k)}{k^n} \) of the series is bounded as a map from \( H^N_C \cap L^2_N \) into \( L^2_{N+1} \) and the series has an infinite
radius of convergence in $L^2_{N+1}$. Indeed the analyticity of the map then follows from general properties of analytic maps in complex Banach spaces, see [37].

In order to estimate the terms of the series, we need estimates on the maps $k \mapsto s_n(q,k)$. A first trivial bound is given by

$$\|s_n(q,\cdot)\|_{L^\infty} \leq \frac{1}{(n+1)!} \|q\|^n_{L^1}.$$  \hfill (35)

However, in order to prove convergence of (34), one needs more refined estimates of the norm of $k \mapsto s_n(q,k)$ in $L^2_{N}$. In order to derive such estimates, we begin with a preliminary lemma about oscillatory integrals:

**Lemma 3.2.** Let $f \in L^1(\mathbb{R}^n, \mathbb{C}) \cap L^2(\mathbb{R}^n, \mathbb{C})$. Let $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$ and

$$g : \mathbb{R} \to \mathbb{C}, \quad g(k) := \int_{\mathbb{R}^n} e^{i\alpha \cdot t} f(t) \, dt.$$  

Then $g \in L^2$ and for any component $\alpha_i \neq 0$ one has

$$\|g\|_{L^2}^2 \leq \int_{\mathbb{R}^{n-1}} \left( \int_{-\infty}^{+\infty} |f(t)|^2 \, dt \right)^{1/2} dt_1 \cdots dt_i \cdots dt_n.$$  

**Proof.** The lemma is a variant of Parseval’s theorem for the Fourier transform; indeed

$$\|g\|_{L^2}^2 = \int_{\mathbb{R}} g(k) \overline{g(k)} \, dk = \int_{\mathbb{R} \times \mathbb{R}^n} e^{i\alpha \cdot (t-s)} f(t) \overline{f(s)} \, dt \, ds \, dk.$$  

Integrating first in the $k$ variable and using the distributional identity $\int_{\mathbb{R}} e^{ikx} \, dk = 2\pi \delta_0$, where $\delta_0$ denotes the Dirac delta function, one gets

$$\|g\|_{L^2}^2 = 2\pi \int_{\mathbb{R} \times \mathbb{R}^n} f(t) \overline{f(s)} \delta(\alpha \cdot (t-s)) \, dt \, ds.$$  

Choose an index $i$ such that $\alpha_i \neq 0$; then $\alpha \cdot (t-s) = 0$ implies that $s_i = t_i + c_i/\alpha_i$, where $c_i = \sum_{j \neq i} \alpha_j(t_j - s_j)$. Denoting $d\sigma_i = dt_1 \cdots \hat{dt}_i \cdots dt_n$ and $d\tilde{\sigma}_i = ds_1 \cdots \hat{ds}_i \cdots ds_n$, one has, integrating first in the variables $s_i$ and $t_i$,

$$\|g\|_{L^2}^2 = 2\pi \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} d\sigma_i \, d\tilde{\sigma}_i \int_{\mathbb{R}^n} f(t_1, \ldots, t_i, \ldots, t_n) f(s_1, \ldots, s_i + c_i/\alpha_i, \ldots, s_n) \, dt_i$$

$$\leq 2\pi \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} d\sigma_i \, d\tilde{\sigma}_i \left( \int_{-\infty}^{+\infty} |f(t)|^2 \, dt_i \right)^{1/2} \left( \int_{-\infty}^{+\infty} |f(s)|^2 \, ds_i \right)^{1/2}$$

$$\leq 2\pi \left( \int_{-\infty}^{+\infty} d\tilde{\sigma}_i \left( \int_{-\infty}^{+\infty} |f(s)|^2 \, ds_i \right)^{1/2} \right)^2$$

where in the second line we have used the Cauchy-Schwarz inequality and the invariance of the integral $\int_{-\infty}^{+\infty} |f(s_1, \ldots, t_i + c_i/\alpha_i, \ldots, s_n)|^2$ by translation. \hfill $\square$
To get bounds on the norm of the polynomials $k \mapsto s_n(q, k)$ in $L_N^2$ it is convenient to study the multilinear maps associated with them:

$$\tilde{s}_n : (H^N_t \cap L^1)^{n+1} \to L^2_N, \quad (f_0, \cdots, f_n) \mapsto \tilde{s}_n(f_0, \cdots, f_n),$$

$$\tilde{s}_n(f_0, \cdots, f_n) := \int_{\Delta_{n+1}} e^{ikt_0} f_0(t_0) \prod_{j=1}^n \left( f_j(t_j) \sin(k(t_j - t_{j-1})) \right) e^{ikt_n} \, dt.$$

The boundedness of these multilinear maps is given by the following

**Lemma 3.3.** For each $n \geq 1$ and $N \in \mathbb{Z}_{\geq 0}$, $\tilde{s}_n : (H^N_t \cap L^1)^{n+1} \to L^2_N$ is bounded. In particular there exist constants $C_{n,N} > 0$ such that

$$\|\tilde{s}_n(f_0, \cdots, f_n)\|_{L^2_N} \leq C_{n,N} \|f_0\|_{H^N_t \cap L^1} \cdots \|f_n\|_{H^N_t \cap L^1}. \tag{36}$$

For the proof, introduce the operators $I_j : L^1 \to L^\infty$, $j = 1, 2$, defined by

$$I_1(f)(t) := \int_{-\infty}^t f(s) \, ds \quad I_2(f)(t) := \int_t^{+\infty} f(s) \, ds.$$

It is easy to prove that if $u, v \in H^N_t \cap L^1$, then $u I_j(v) \in H^N_t \cap L^1$ and the estimate $\|u I_j(v)\|_{H^N_t \cap L^1} \leq \|u\|_{H^N_t \cap L^1} \|v\|_{H^N_t \cap L^1}$ holds for $j = 1, 2$.

**Proof of Lemma 3.3.** As $\sin x = (e^{ix} - e^{-ix})/2i$ we can write $e^{ikt_0} \left( \prod_{j=1}^n \sin(k(t_j - t_{j-1})) \right) e^{ikt_n}$ as a sum of complex exponentials. Note that the arguments of the exponentials are obtained by taking all the possible combinations of $\pm$ in the expression $t_0 \pm (t_1 - t_0) \pm \cdots \pm (t_n - t_{n-1}) + t_n$. To handle this combinations, define the set

$$\Lambda_n := \left\{ \sigma = (\sigma_j)_{1 \leq j \leq n} : \sigma_j \in \{\pm 1\} \right\}$$

and introduce

$$\delta_\sigma := \#\{1 \leq j \leq n : \sigma_j = -1\}.$$

For any $\sigma \in \Lambda_n$, define $\alpha_\sigma = (\alpha_j)_{0 \leq j \leq n}$ as

$$\alpha_0 = (1 - \sigma_1), \quad \alpha_j = \sigma_j - \sigma_{j+1} \text{ for } 1 \leq j \leq n-1, \quad \alpha_n = 1 + \sigma_n.$$

Note that for any $t = (t_0, \ldots, t_n)$, one has $\alpha_\sigma \cdot t = t_0 + \sum_{j=1}^n \sigma_j (t_j - t_{j-1}) + t_n$.

For every $\sigma \in \Lambda_n$, $\alpha_\sigma$ satisfies the following properties:

(i) $\alpha_0, \alpha_n \in \{2, 0\}$, $\alpha_j \in \{0, \pm 2\}$ $\forall 1 \leq j \leq n-1$; (ii) $\#\{j | \alpha_j \neq 0\}$ is odd. \tag{37}

Property (i) is obviously true; we prove now (ii) by induction. For $n = 1$, property (ii) is trivial. To prove the induction step $n \mapsto n + 1$, let $\alpha_0 = 1 - \sigma_1, \ldots, \alpha_n = \sigma_n - \sigma_{n+1}, \alpha_{n+1} = 1 + \sigma_{n+1}$, and define $\tilde{\alpha}_n := 1 + \sigma_n \in \{0, 2\}$. By the induction hypothesis the vector $\tilde{\alpha}_\sigma = (\alpha_0, \ldots, \alpha_{n-1}, \tilde{\alpha}_n)$ has an odd number of elements non zero. Case $\tilde{\alpha}_n = 0$: in this case the vector $(\alpha_0, \ldots, \alpha_{n-1})$ has an odd number of non zero elements. Then, since $\alpha_n = \sigma_n - \sigma_{n+1} = \tilde{\alpha}_n - \alpha_{n+1} = -\alpha_{n+1}$, one has that $(\alpha_n, \alpha_{n+1}) \in \{(0,0), (0,2)\}$. Therefore the vector $\alpha_\sigma$ has an odd number of non zero elements. Case $\tilde{\alpha}_n = 2$: in this case the vector $(\alpha_0, \ldots, \alpha_{n-1})$ has an even number of non zero elements. As $\alpha_n = 2 - \alpha_{n+1}$, it follows that $(\alpha_n, \alpha_{n+1}) \in \{(2,0), (0,2)\}$. Therefore the vector $\alpha_\sigma$ has an odd number of non zero elements. This proves (37).
As

\[ e^{ikt_0} \left( \prod_{j=1}^n \sin k(t_j - t_{j-1}) \right) e^{ikt_n} = \sum_{\sigma \in \Lambda_n} \frac{(-1)^{\delta_{n}}}{(2i)^n} e^{ik\alpha_t} \]

\( \tilde{s}_n \) can be written as a sum of complex exponentials,

\[ \tilde{s}_n(f_0, \ldots, f_n)(k) = \sum_{\sigma \in \Lambda_n} \frac{(-1)^{\delta_{n}}}{(2i)^n} \tilde{s}_{n,\sigma}(f_0, \ldots, f_n)(k) \]

where

\[ \tilde{s}_{n,\sigma}(f_0, \ldots, f_n)(k) = \int_{\Delta_{n+1}} e^{ik\alpha_t} f_0(t_0) \cdots f_n(t_n) dt. \]

The case \( N = 0 \) follows directly from Lemma 3.2, since for each \( \sigma \in \Lambda_n \) one has by (37) that there exists \( m \) with \( \alpha_m \neq 0 \) implying \( \| \tilde{s}_{n,\sigma}(f_0, \ldots, f_n) \|_{L^2} \leq C \| f_m \|_{L^2} \| f_j \|_{L^1} \), which leads to (36).

We now prove by induction that \( \tilde{s}_n : (H^{N}_{C} \cap L^1)^{n+1} \to L^2_{N} \) for any \( N \geq 1 \). We start with \( n = 1 \). Since we have already proved that \( \tilde{s}_1 \) is a bounded map from \((L^2 \cap L^1)^2 \) to \( L^2 \), it is enough to establish the stated decay at \( \infty \). One verifies that

\[ \tilde{s}_1(f_0, f_1) = \frac{1}{2i} \int_{-\infty}^{+\infty} e^{2ikt} f_0(t) I_1(f_1)(t) dt - \frac{1}{2i} \int_{-\infty}^{+\infty} e^{2ikt} f_1(t) I_2(f_0)(t) dt \]

\[ = \frac{1}{2i} F_-(f_0 I_1(f_1)) - \frac{1}{2i} F_-(f_1 I_2(f_0)). \]

Hence, for each \( N \in \mathbb{Z}_{\geq 0} \), \( (f_0, f_1) \mapsto \tilde{s}_1(f_0, f_1) \) is bounded as a map from \((H^{N}_{C} \cap L^1)^2 \) to \( L^2_{N} \). Moreover

\[ \| \tilde{s}_1(f_0, f_1) \|_{L^2_{N}} \leq C_1 \left( \| f_0 I_1(f_1) \|_{H^N} + \| f_1 I_2(f_0) \|_{H^N} \right) \leq C_{1,N} \| f_0 \|_{H^N_C \cap L^1} \| f_1 \|_{H^N_C \cap L^1}. \]

We prove the induction step \( n \mapsto n + 1 \) with \( n \geq 1 \) for any \( N \geq 1 \) (the case \( N = 0 \) has been already treated). The term \( \tilde{s}_{n+1}(f_0, \ldots, f_{n+1}) \) equals

\[ \int_{\Delta_{n+2}} e^{ikt_0} f_0(t_0) \prod_{j=1}^n \left( \sin k(t_j - t_{j-1}) f_j(t_j) \right) e^{ikt_n} \sin k(t_{n+1} - t_n) \]

\[ \cdot e^{ik(t_{n+1} - t_n)} f_{n+1}(t_{n+1}) dt \]

where we multiplied and divided by the factor \( e^{ikt_n} \). Writing

\[ \sin k(t_{n+1} - t_n) = \frac{(e^{ikt_{n+1} - t_n} - e^{-ikt_{n+1} - t_n})}{2i}, \]

the integral term \( \int_{t_n}^{+\infty} e^{ikt_{n+1} - t_n} \sin k(t_{n+1} - t_n) f_{n+1}(t_{n+1}) dt_{n+1} \) equals

\[ \frac{1}{2i} \int_{t_n}^{+\infty} e^{2ikt_{n+1} - t_n} f_{n+1}(t_{n+1}) dt_{n+1} - \frac{1}{2i} I_1(f_{n+1})(t_n). \]

Since \( f_{n+1} \in H^{N}_{C} \), for \( 0 \leq j \leq N - 1 \) one gets \( f^{(j)}_{n+1} \to 0 \) when \( x \to \infty \), where we wrote \( f^{(j)}_{n+1} = \partial_{k_j}^j f_{n+1} \). Integrating by parts \( N \)-times in the integral expression
By the inductive assumption it follows that \( \tilde{\rho} \). For the second term in (38) it is enough to note that \( \leq \)

We analyze the first term in the r.h.s. of (38). For \( 0 \leq j \leq N - 1 \), the function \( f^{(j)} + 1 \in H^{N-j}_c \) is in \( L^\infty \) by the Sobolev embedding theorem. Therefore \( f_n \cdot f^{(j)} + 1 \in H^{N-j}_c \cap L^1 \). By the inductive assumption applied to \( N - j \), \( \tilde{s}_n(f_0, \ldots, f_n \cdot f^{(j)} + 1) \in L^2 \). Therefore \( \chi \sum (2ik)^{N+j} \tilde{s}_n(f_0, \ldots, f_n \cdot f^{(j)} + 1) \in L^2 \), where \( \chi \) is chosen as in (34).

For the second term in (38) it is enough to note that \( f_n \cdot I(f^{(j)} + 1) \in H^{N}_c \cap L^1 \) and by the inductive assumption it follows that \( \tilde{s}_n(f_0, \ldots, f_n \cdot I(f^{(j)} + 1)) \in L^2 \).

We are left with (39). Due to the factor \((2ik)^{N} \) in the denominator, we need just to prove that the integral term is \( L^2 \) integrable in the \( k \)-variable. Since the oscillatory factor \( e^{2ikt_n + 1} \) doesn’t get canceled when we express the sine functions with exponentials, we can apply Lemma 3.2, integrating first in \( L^2 \) w.r. to the variable \( t_n + 1 \), getting

\[
\| \chi \cdot (39) \|_{L^2} \leq C_{n+1,N} \left\| f^{(N)} + 1 \right\|_{L^2} \prod_{j=0}^{n} \| f_j \|_{L^1} .
\]

Putting all together, it follows that \( \tilde{s}_n + 1 \) is bounded as a map from \( (H^{N}_c \cap L^1)_{n+2} \) to \( L^2 \) for each \( N \in \mathbb{Z}_{\geq 0} \) and the estimate (36) holds.

By evaluating the multilinear map \( \tilde{s}_n \) on the diagonal, Lemma 3.3 says that for any \( N \geq 0 \),

\[
\| s_n(q, \cdot) \|_{L^2} \leq C_{n,N} \| q \|_{H^{N}_c \cap L^1}^{n+1} , \quad \forall n \geq 1. 
\]

(40)

Combining the \( L^\infty \) estimate (35) with (40) we can now prove item (ii) of Theorem 3.1:

**Proof of Theorem 3.1 (ii).** Let \( \chi \) be the cut off function introduced in (34) and set

\[
\Delta(q, k) := \sum_{n=1}^{\infty} \frac{\chi(k)s_n(q, k)}{k^n} .
\]

(41)

We now show that for any \( \rho > 0 \), \( \Delta(q, \cdot) \) is an absolutely and uniformly convergent series in \( L^2 \) for \( q \in B_{\rho}(0) \), where \( B_{\rho}(0) \) is the ball in \( H^{N}_c \cap L^1 \) with center 0 and radius \( \rho \). By (40) the map \( q \mapsto \sum_{n=1}^{N+1} \frac{\chi(k)s_n(q, k)}{k^n} \) is analytic as a map from \( H^{N}_c \cap L^1 \).
to $L^2_{N+1}$, being a finite sum of polynomials - cf. [37]. It remains to estimate the sum
\[ \hat{A}_{N+2}(q,k) := \hat{A}(q,k) - \sum_{n=1}^{N+1} \frac{\chi(k)s_n(q,k)}{k^n}. \]
It is absolutely convergent since by the $L^\infty$ estimate (35)
\[
\left\| \sum_{n \geq N+2} \frac{\chi s_n(q,\cdot)}{k^n} \right\|_{L^2_{N+1}} \leq C \sum_{n \geq N+2} \left\| \frac{\chi(k)}{k^n} \right\|_{L^2_{N+1}} \|s_n(q,\cdot)\|_{L^\infty} \leq C \sum_{n \geq N+2} \|q\|_{L^1}^{n+1} (n+1)!
\]
for an absolute constant $C > 0$. Therefore the series in (41) converges absolutely and uniformly in $B_\rho(0)$ for every $\rho > 0$. The absolute and uniform convergence implies that for any $N \geq 0$, $q \mapsto \hat{A}(q,\cdot)$ is analytic as a map from $H^N_0 \cap L^1$ to $L^2_{N+1}$.

It remains to show that identity (34) holds, i.e., for every $q \in H^N_0 \cap L^1$ one has $\chi A(q,\cdot) = \hat{A}(q,\cdot)$ in $L^2_{N+1}$. Indeed, fix $q \in H^N_0 \cap L^1$ and choose $\rho$ such that $\|q\|_{H^N_0 \cap L^1} \leq \rho$. Iterate formula (27) $N' \geq 1$ times and insert the result in (29) to get for any $k \in \mathbb{R} \setminus \{0\}$,
\[
S(q,k) = \mathcal{F}_- (q,k) + \sum_{n=1}^{N'} \frac{s_n(q,k)}{k^n} + S_{N'+1}(q,k),
\]
where
\[
S_{N'+1}(q,k) := \frac{1}{k^{N'+1}} \int_{\Delta_{N'+2}} e^{ikt_0} q(t_0) \prod_{j=1}^{N'+1} \left( q(t_j) \sin k(t_j - t_{j-1}) \right) f_1(q,t_{N'+1},k) \, dt.
\]
By the definition (31) of $A(q,k)$ and the expression of $S_{N'+1}$ displayed above
\[
\chi(k) A(q,k) - \sum_{n=1}^{N'} \frac{\chi(k)s_n(q,k)}{k^n} = \chi(k) S_{N'+1}(q,k), \quad \forall N' \geq 1.
\]
Let now $N' \geq N$, then by Theorem 2.1 (ii) there exists a constant $K_\rho$, which can be chosen uniformly on $B_\rho(0)$ such that
\[
\|\chi S_{N'+1}(q,\cdot)\|_{L^2_{N+1}} \leq K_\rho \|q\|_{L^1}^{N'+2} (N' + 2)! \leq K_\rho \rho^{N'+2} (N' + 2)! \to 0, \quad \text{when } N' \to \infty,
\]
where for the last inequality we used that $\|q\|_{L^1} \leq C \|q\|_{L^2}$ for some absolute constant $C > 0$. Since $\lim_{N' \to 0} \sum_{n=1}^{N'} \frac{\chi(k)s_n(q,k)}{k^n} = \hat{A}(q,k)$ in $L^2_{N+1}$, it follows that $\chi(k) A(q,k) = \hat{A}(q,k)$ in $L^2_{N+1}$.

For later use we study regularity and decay properties of the map $k \mapsto W(q,k)$. For $q \in L^2_3$ real valued with no bound states it follows that $W(q,k) \neq 0$, $\forall \Im k \geq 0$ by classical results in scattering theory. We define
\[
\mathcal{Q}_\mathcal{C} := \{ q \in L^2_3 : W(q,k) \neq 0, \forall \Im k \geq 0 \}, \quad \mathcal{Q}^N_M := \mathcal{Q}_\mathcal{C} \cap H^N_0 \cap L^2_M.
\]
We will prove in Lemma 3.5 below that $Q_{C}^{N,M}$ is open in $H_{C}^{N} \cap L_{M}^{2}$. Finally consider the Banach space $W_{C}^{M}$ defined for $M \geq 1$ by

$$W_{C}^{M} := \{ f \in L^{\infty} : \partial_{h} f \in H_{C}^{M-1} \} ,$$

endowed with the norm $\|f\|_{W_{C}^{M}}^{2} = \|f\|_{L^{\infty}}^{2} + \|\partial_{h} f\|_{H_{C}^{M-1}}^{2}$.

Note that $H_{C}^{M} \subseteq W_{C}^{M}$ for any $M \geq 1$ and

$$gh \in H_{C}^{M} \iff g \in H_{C}^{M}, \forall h \in W_{C}^{M} .$$

The properties of the map $W$ are summarized in the following Proposition:

**Proposition 6.** For $M \in \mathbb{Z}_{\geq 3}$ the following holds:

(i) The map $L_{M}^{2} \ni q \mapsto W(q, \cdot) - 2ik + F_{-}(q, 0) \in H_{C}^{M}$ is analytic and

$$\|W(q, \cdot) - 2ik + F_{-}(q, 0)\|_{H_{C}^{M}} \leq C_{W} \|q\|_{L_{M}^{2}} ,$$

where the constant $C_{W} > 0$ can be chosen uniformly on bounded subsets of $L_{M}^{2}$.

(ii) The map $Q_{C}^{0,M} \ni q \mapsto 1/W(q, \cdot) \in L^{\infty}$ is analytic.

(iii) The maps $Q_{C}^{0,M} \ni q \mapsto \partial q/W(q, \cdot) \in L^{2}$ for $0 \leq j \leq M - 1$ and $Q_{C}^{0,M} \ni q \mapsto \frac{\partial q/W(q, \cdot)}{W(q, \cdot)} \in L^{2}$ are analytic. Here $\zeta$ is a function as in (8).

**Proof.** The $x$-independence of the Wronskian function (3) implies that

$$W(q, k) = 2ik m_{2}(q, 0, k) m_{1}(q, 0, k) + [m_{2}(q, 0, k), m_{1}(q, 0, k)].$$

(43)

Introduce for $j = 1, 2$ the functions $\hat{m}_{j}(q, k) := 2ik (m_{j}(q, 0, k) - 1)$. By the integral formula (12) one verifies that

$$\hat{m}_{1}(q, k) = \int_{0}^{+\infty} (e^{2ikt} - 1) q(t) (m_{1}(q, t, k) - 1) dt + \int_{0}^{+\infty} e^{2ikt} q(t) dt - \int_{0}^{+\infty} q(t) dt ;$$

$$\hat{m}_{2}(q, k) = \int_{-\infty}^{0} (e^{-2ikt} - 1) q(t) (m_{2}(q, t, k) - 1) dt + \int_{0}^{-\infty} e^{-2ikt} q(t) dt - \int_{0}^{-\infty} q(t) dt .$$

(44)

A simple computation using (43) shows that $W(q, k) - 2ik + F_{-}(q, 0) = I + II + III$ where

$I := \hat{m}_{1}(q, k) + \hat{m}_{2}(q, k) + F_{-}(q, 0),$

$II := \hat{m}_{1}(q, k)(m_{2}(q, 0, k) - 1)$ and $III := [m_{2}(q, 0, k), m_{1}(q, 0, k)].$

(45)

We prove now that each of the terms $I, II$ and $III$ displayed above is an element of $H_{C}^{M}$. We begin by discussing the smoothness of the functions $k \mapsto \hat{m}_{j}(q, k), j = 1, 2$. For any $1 \leq n \leq M,$

$$\partial_{k}^{n} \hat{m}_{j}(q, k) = 2m \partial_{k}^{n-1}(m_{j}(q, 0, k) - 1) + 2ik \partial_{k}^{n} m_{j}(q, 0, k) .$$

Thus by Corollary 2 (i), $\hat{m}_{j}(q, \cdot) \in W_{C}^{M}$ and $q \mapsto \hat{m}_{j}(q, \cdot), j = 1, 2,$ are analytic as maps from $L_{M}^{2}$ to $W_{C}^{M}$. Consider first the term $III$ in (45). By Corollary 2, $\|III(q, \cdot)\|_{H_{C}^{M}} \leq K_{III} \|q\|_{L_{M}^{2}},$ where $K_{III} > 0$ can be chosen uniformly on bounded subsets of $L_{M}^{2}$. Arguing as in the proof of Proposition 5, one shows that it is an element of $H_{C}^{M}$ and it is analytic as a map $L_{M}^{2} \to H_{C}^{M}$.

II. Next consider the term $II$. Since $\hat{m}_{1}(q, \cdot)$ is in $W_{C}^{M}$ and $m_{2}(q, 0, \cdot) - 1$ is in $H_{C}^{M}$, it follows by (42) that their
product is in $H^M_{\xi,C}$. It is left to the reader to show that $L^2_M \to H^M_{\xi,C}$, $q \mapsto II(q)$ is analytic and furthermore $\|II(q,\cdot)\|_{H^M_{\xi,C}} \leq K_{II} \|q\|_{L^2_M}$, where $K_{II} > 0$ can be chosen uniformly on bounded subsets of $L^2_M$.

Finally let us consider term $I$. By summing the identities for $\hat{m}_1$ and $\hat{m}_2$ in equation (44), one gets that $\hat{m}_1(q,k) + \hat{m}_2(q,k) + F_-(q,0)$ equals

$$\int_0^{+\infty} e^{2ikt} q(t) m_1(q,t,k) \, dt - \int_0^{+\infty} q(t) (m_1(q,t,k) - 1) \, dt$$
$$+ \int_{-\infty}^{0} e^{-2ikt} q(t) m_2(q,t,k) - \int_{-\infty}^{0} q(t) (m_2(q,t,k) - 1) \, dt. $$

We study just the first line displayed above, the second being treated analogously. By equation (23) one has that $\int_0^{+\infty} e^{2ikt} q(t) m_1(q,t,k) \, dt = \partial_t m(q,0,k)$, which by Corollary 2 is an element of $H^M_{\xi,C}$ and analytic as a function $L^2_M \to H^M_{\xi,C}$. Furthermore, by Proposition 2 and Proposition 3 it follows that $W(q,k) = 2ik + L^\infty$; therefore the map $W(q,k)$ is analytic.

Lemma 3.4. For any $q \in Q_{0,3}$, $W(q,0) < 0$.}

Proof. Let $q \in Q_{0,3}$ and $\kappa \geq 0$. By formulas (27) and (28) with $k = ik$, it follows that $f_j(q,x,ik)$ ($j = 1,2$) is real valued (recall that $q$ is real valued). By the definition $W(q,ik) = [f_2,f_1] (q,ik)$ it follows that for $\kappa \geq 0$, $W(q,ik)$ is real valued. As $q$ is generic, $W(q,ik)$ has no zeroes for $\kappa \geq 0$. Furthermore for large $\kappa$ we have $W(q,ik) \sim 2i(ik) = -2\kappa$. Thus $W(q,ik) < 0$ for $\kappa \geq 0$.}

We are now able to prove the direct scattering part of Theorem 1.1.

Proof of Theorem 1.1. direct scattering part. Let $N \geq 0$, $M \geq 3$ be fixed integers. First we remark that $S(q,\cdot)$ is an element of $\mathcal{S}^{M,N}$ if $q \in Q^{N,M}$. By (26), $S(q,\cdot)$ satisfies (S1). To see that $S(q,0) > 0$ recall that $S(q,0) = -W(q,0)$, and by Lemma 3.4 $W(q,0) < 0$. Thus $S(q,\cdot)$ satisfies (S2). Finally by Corollary 3 and Proposition 5 it follows that $S(q,\cdot) \in \mathcal{S}^{M,N}$. The analyticity properties of the map $q \mapsto S(q,\cdot)$ and $q \mapsto A(q,\cdot)$ follow by Corollary 3, Proposition 5 and Theorem 3.1.}

We conclude this section with two results about the openness of $Q^{N,M}$ and $\mathcal{S}^{M,N}$.

Lemma 3.5. For any integers $N \geq 0$, $M \geq 3$, $Q^{N,M} [Q^{N,M}_C]$, is open in $H^N \cap L^2_M [H^2_C \cap L^2_M]$. 

The proof of this lemma can be found in [34, Lemma 3.8].

Denote by \( H^M_{\zeta,C} \) the complexification of the Banach space \( H^M_{\zeta} \), in which the reality condition \( \overline{f(k)} = f(-k) \) is dropped:

\[
H^M_{\zeta,C} := \{ f \in H^{M-1}_{\zeta}: \zeta \partial^M_k f \in L^2 \}.
\]

Furthermore denote by \( \mathcal{F}^{M,N}_C \) the complexification of \( \mathcal{F}^{M,N} \). It consists of functions \( \sigma: \mathbb{R} \to \mathbb{C} \) with \( \text{Re}(\sigma(0)) > 0 \) and \( \sigma \in H^M_{\zeta,C} \cap L^2_N \).

**Lemma 3.6.** For any integers \( M \geq 3, N \geq 0 \) the subset \( \mathcal{F}^{M,N}_C \) is open in \( H^M_{\zeta} \cap L^2_N \). The proof of this simple lemma can be found in [34, Lemma 3.9].

4. **Inverse scattering map.** The aim of this section is to prove the inverse scattering part of Theorem 1.1. More precisely we prove the following theorem.

**Theorem 4.1.** Let \( N \in \mathbb{Z}_{\geq 0} \) and \( M \in \mathbb{Z}_{\geq 3} \) be fixed. Then the scattering map \( S: \mathcal{Q}^{N,M} \to \mathcal{F}^{M,N} \) is bijective. Its inverse \( S^{-1}: \mathcal{F}^{M,N} \to \mathcal{Q}^{N,M} \) is real analytic.

The smoothing and analytic properties of \( B := S^{-1} - F_-^{-1} \) claimed in Theorem 1.1 follow now in a straightforward way from Theorem 4.1 and 3.1.

**Proof of Theorem 1.1. inverse scattering part.** By Theorem 4.1, \( S^{-1}: \mathcal{F}^{M,N} \to \mathcal{Q}^{N,M} \) is well defined and real analytic. As by definition \( B = S^{-1} - F_-^{-1} \) and \( S = F_+ + A \) one has \( B \circ S = I - F_-^{-1} \circ S = -F_-^{-1} \circ A \) or

\[
B = -F_-^{-1} \circ A \circ S^{-1}.
\]

Hence, by Theorem 3.1 and Theorem 4.1, for any \( M \in \mathbb{Z}_{\geq 3} \) and \( N \in \mathbb{Z}_{\geq 0} \) the restriction \( B: \mathcal{F}^{M,N} \to \mathcal{F}^{-1} \left( H^M_{\zeta} \cap L^2_{N+1} \right) \) is a real analytic map.

Since \( \mathcal{F}^{-1} \left( H^M_{\zeta} \cap L^2_{N+1} \right) \subset H^{N+1} \cap L^2_{M-1} \), the claim follows. \( \Box \)

The rest of the section is devoted to the proof of Theorem 4.1. By the direct scattering part of Theorem 1.1 proved in Section 3, \( S(\mathcal{Q}^{N,M}) \subset \mathcal{F}^{M,N} \). Furthermore, the map \( S: \mathcal{Q} \to \mathcal{F} \) is 1-1, see [26, Section 4]. Thus also its restriction \( S|_{\mathcal{Q}^{N,M}}: \mathcal{Q}^{N,M} \to \mathcal{F}^{M,N} \) is 1-1.

In order to prove that \( S: \mathcal{Q}^{N,M} \to \mathcal{F}^{M,N} \) is onto, we need some preparation. Firstly let us denote by \( \mathcal{H}: L^2 \to L^2 \) the Hilbert transform

\[
\mathcal{H}(v)(k) := \frac{-1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{v(k')}{k'-k} dk'.
\]

Following [26] define for \( \sigma \in \mathcal{F}^{M,N} \) and \( k \in \mathbb{R} \)

\[
\omega(\sigma, k) := \exp \left( \frac{1}{2} l(\sigma, k) + i \mathcal{H}(l(\sigma, \cdot))(k) \right),
\]

\[
l(\sigma, k) := \log \left( \frac{4(k^2 + 1)}{4k^2 + \sigma(k)\sigma(-k)} \right),
\]

and

\[
\frac{1}{w(\sigma, k)} := \frac{\omega(\sigma, k)}{2i(k + i)}, \quad \tau(\sigma, k) := \frac{2ik}{w(\sigma, k)},
\]

\[
\rho_+(\sigma, k) := \frac{\sigma(-k)}{w(\sigma, k)}, \quad \rho_- (\sigma, k) := \frac{\sigma(k)}{w(\sigma, k)}.
\]
The aim is to show that $\rho_+(\sigma, \cdot), \rho_-(\sigma, \cdot)$ and $\tau(\sigma, \cdot)$ are the scattering data $r_+, r_-$ and $t$ of a potential $q \in Q^{N, M}.$

In the next proposition we discuss the properties of the map $\sigma \to l(\sigma, \cdot).$ To this aim we introduce, for $M \in \mathbb{Z}_{\geq 2}$ and $\zeta$ as in (8), the auxiliary Banach space

$$W^M_\zeta := \{ f \in L^\infty : \overline{f(k)} = f(-k), \partial_k^\zeta f \in L^2 \text{ for } 1 \leq n \leq M-1, \zeta \partial_2^\zeta f \in L^2 \}$$

and its complexification

$$W^M_{\zeta, C} := \{ f \in L^\infty : \partial_k^\zeta f \in L^2 \text{ for } 1 \leq n \leq M-1, \zeta \partial_2^\zeta f \in L^2 \},$$

both endowed with the norm $\| f \|^2_{W^M_{\zeta, C}} := \| f \|^2_{L^\infty} + \| \partial_k f \|^2_{H^{M-\zeta, 2}} + \| \zeta \partial_k^\zeta f \|^2_{L^2}.$ Note that $W^M_{\zeta}$ differs from $H^M_{\zeta}$ since we require that $f$ lies just in $L^\infty$ (and not in $L^2$ as in $H^M_{\zeta}$).

**Proposition 7.** Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 3}$ be fixed. The map $\mathcal{S}^{M, N} \to H^M_{\zeta},$ $\sigma \to l(\sigma, \cdot)$ is real analytic.

**Proof.** Denote by

$$h(\sigma, k) := \frac{4(k^2 + 1)}{4k^2 + \sigma(k)\sigma(-k)}.$$  

We show that the map $\mathcal{S}^{M, N} \to W^M_{\zeta, C}, \sigma \to h(\sigma, \cdot)$ is real analytic. First note that the map $\mathcal{S}^{M, N} \to L^\infty,$ assigning to $\sigma$ the function $\sigma(k)\sigma(-k)$ is analytic by the Sobolev embedding theorem. For $\sigma \in \mathcal{S}^{M, N}$ write $\sigma = \sigma_1 + i\sigma_2,$ where $\sigma_1 := \text{Re} \sigma,$ $\sigma_2 := \text{Im} \sigma.$ Then

$$\text{Re}(\sigma(k)\sigma(-k)) = \sigma_1(k)\sigma_1(-k) - \sigma_2(k)\sigma_2(-k).$$

(49)

Now fix $\sigma^0 \in \mathcal{S}^{M, N}$ and recall that $\mathcal{S}^{M, N} = \mathcal{S} \cap H^M_{\zeta} \cap L^N_{\zeta}.$ Remark that $\sigma_2^0 := \text{Im} \sigma^0 = 0,$ while $\sigma_1^0 := \text{Re} \sigma^0$ satisfies $\sigma_1^0(k)\sigma_1(k) \geq 0$ and $\sigma_1^0(0)^2 > 0.$ Thus, by formula (49) and the Sobolev embedding theorem, there exists $V_{\sigma^0} \subset \mathcal{S}^{M, N}_{\zeta}$ small complex neighborhood of $\sigma^0$ and a constant $C_{\sigma^0} > 0$ such that

$$\text{Re}(4k^2 + \sigma(k)\sigma(-k)) > C_{\sigma^0}, \quad \forall \sigma \in V_{\sigma^0}.$$

It follows that there exist constants $C_1, C_2 > 0$ such that

$$\text{Re} h(\sigma, k) \geq C_1, \quad |h(\sigma, k)| \leq C_2, \quad \forall k \in \mathbb{R}, \forall \sigma \in V_{\sigma^0},$$

(50)

implying that the map $V_{\sigma^0} \to L^\infty, \sigma \to h(\sigma, \cdot)$ is analytic. In a similar way one proves that $V_{\sigma^0} \to W^M_{\zeta, C}, \sigma \to h(\sigma, \cdot)$ is analytic (we omit the details). If $\sigma(k) = \sigma(-k),$ the function $h(\sigma, \cdot)$ is real valued. Thus it follows that $\mathcal{S}^{M, N} \to W^M_{\zeta}, \sigma \to h(\sigma, \cdot)$ is real analytic.

We consider now the map $\sigma \to l(\sigma, \cdot).$ By (50), $l(\sigma, k) = \text{log}(h(\sigma, k))$ is well defined for every $k \in \mathbb{R}.$ Since the logarithm is a real analytic function on the right half plane, the map $\mathcal{S}^{M, N} \to L^\infty, \sigma \to l(\sigma, \cdot)$ is real analytic as well. Furthermore for $|k| > 1$ one finds a constant $C_3 > 0$ such that $|l(\sigma, k)| \leq C_3/|k|^2, \forall \sigma \in V_{\sigma^0}.$ Thus $\sigma \to l(\sigma, \cdot)$ is real analytic as a map from $\mathcal{S}^{M, N}$ to $L^2.$ One verifies that $\partial_k \text{log}(h(\sigma, \cdot)) = \partial_k h(\sigma, \cdot)/h(\sigma, \cdot)$ is in $L^2$ and one shows by induction that the map $\mathcal{S}^{M, N} \to H^M_{\zeta}, \sigma \to l(\sigma, \cdot)$ is real analytic.

In the next proposition we discuss the properties of the map $\sigma \to \omega(\sigma, \cdot).$

**Proposition 8.** Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 3}$ be fixed. The map $\mathcal{S}^{M, N} \to W^M_{\zeta}, \sigma \to \omega(\sigma, \cdot)$ is real analytic. Furthermore $\omega(\sigma, \cdot)$ has the following properties:
(i) \( \omega(\sigma,k) \) extends analytically in the upper half plane \( \text{Im} k > 0 \), and it has no zeroes in \( \text{Im} k \geq 0 \).

(ii) \( \omega(\sigma,k) = \omega(\sigma,-k) \) \( \forall k \in \mathbb{R} \).

(iii) For every \( k \in \mathbb{R} \)

\[
\omega(\sigma,k)\omega(\sigma,-k) = \frac{4(k^2 + 1)}{4k^2 + \sigma(k)\sigma(-k)}.
\]

**Proof.** We begin to show that the Hilbert transform is a bounded linear operator from \( H^M_{\zeta,\mathbb{C}} \) to \( H^M_{\zeta,\mathbb{C}} \). Let \( f \in H^M_{\zeta,\mathbb{C}} \). As the Hilbert transform commutes with the derivatives, we have that \( \mathcal{H}(f) \in H^M_{\zeta,\mathbb{C}} \). Next we show that if \( \zeta \partial_k^M f \in L^2 \), then \( \zeta \partial_k^M \mathcal{H}(f) = \mathcal{H}(\zeta \partial_k^M f) \in L^2 \). To this end we apply Calderon first commutator estimate \([6]\), which states that given \( b : \mathbb{R} \to \mathbb{R} \) with first-order derivative in \( L^\infty \), then for any \( p \in (1, \infty) \) there exists \( C > 0 \) such that

\[
\| [\mathcal{H}, b] \partial_k g \|_{L^p} \leq C \| g \|_{L^p}.
\]

Applying this estimate with \( p = 2 \), \( g = \partial_k^M f \) and \( b = \zeta \), we have that

\[
\| \zeta \mathcal{H}(\partial_k^M f) \|_{L^2} \leq \| \mathcal{H}(\zeta \partial_k^M f) \|_{L^2} + \| [\mathcal{H}, \zeta] \partial_k^M f \|_{L^2} \leq \| f \|_{H^M_{\zeta,\mathbb{C}}} + C \| \partial_k^M f \|_{L^2} < \infty.
\]

Next consider the map

\[ \mathcal{S}^{M,N} \to H^M_{\zeta,\mathbb{C}}, \quad \sigma \mapsto \mathcal{H}(l(\sigma,\cdot)) \].

By Proposition 7 it follows that it is real analytic as well. Since the exponential function is real analytic and \( \partial_k \omega(\sigma,\cdot) = \frac{1}{2} \partial_k(l(\sigma,\cdot) + i\mathcal{H}(l(\sigma,\cdot)))\omega(\sigma,\cdot) \), one proves by induction that \( \mathcal{S}^{M,N} \to W^M_{\zeta,\mathbb{C}}, \sigma \mapsto \omega(\sigma,\cdot) \) is real analytic. Properties (i)–(iii) are proved in [26, Section 4].

Next we consider the map \( \sigma \to \frac{1}{\omega(\sigma,k)} \). The following proposition follows immediately from Proposition 8 and the definition \( \frac{1}{\omega(\sigma,k)} = \frac{\omega(\sigma,k)}{2\pi(k+\sigma)} \).

**Proposition 9.** Let \( N \in \mathbb{Z}_{\geq 0} \) and \( M \in \mathbb{Z}_{\geq 3} \) be fixed. The map \( \mathcal{S}^{M,N} \to H^M_{\zeta,\mathbb{C}} \), \( \sigma \to \frac{1}{\omega(\sigma,k)} \) is real analytic. Furthermore the maps

\[ \mathcal{S}^{M,N} \to L^2, \quad \sigma \to \partial_k^n \frac{2ik}{\omega(\sigma,k)}, \quad 1 \leq n \leq M \]

are real analytic. The function \( \frac{1}{\omega(\sigma,k)} \) fulfills

(i) \( \left( \frac{1}{\omega(\sigma,k)} \right)' = \frac{1}{\omega(\sigma,-k)} \) for every \( k \in \mathbb{R} \).

(ii) \( \frac{2ik}{\omega(\sigma,k)} \leq 1 \) for every \( k \in \mathbb{R} \).

(iii) For every \( k \in \mathbb{R} \)

\[
w(\sigma,k)w(\sigma,-k) = 4k^2 + \sigma(k)\sigma(-k).
\]

In particular \( |w(\sigma,k)| > 0 \) for every \( k \in \mathbb{R} \) and \( \sigma \in \mathcal{S}^{M,N} \).

Now we study the properties of \( \rho_+(\sigma,\cdot) \) and \( \rho_-(\sigma,\cdot) \) defined in formulas (46).

**Proposition 10.** Let \( N \in \mathbb{Z}_{\geq 0} \) and \( M \in \mathbb{Z}_{\geq 3} \) be fixed. Then the maps \( \mathcal{S}^{M,N} \to H^M_{\zeta,\mathbb{C}} \cap L^2_N, \sigma \to \rho_+(\sigma,\cdot) \) are real analytic. There exists \( C > 0 \) so that one has

\[
\| \rho_+(\sigma,\cdot) \|_{H^M_{\zeta,\mathbb{C}} \cap L^2_N} \leq C \| \sigma \|_{H^M_{\zeta,\mathbb{C}} \cap L^2_N},
\]

where \( C \) depends locally uniformly on \( \sigma \in \mathcal{S}^{M,N} \). Furthermore the following holds:
In order to prove the statements, we will use that
$$
\rho_+(\sigma, k) = 1 + \rho_+(\sigma, k)\rho_+(\sigma, -k) = 1
$$
and
$$
\rho_+(\sigma, k)\overline{\rho_+(\sigma, -k)} + \overline{\rho_+(\sigma, k)}\rho_+(\sigma, -k) = 0.
$$
Proof. The real analyticity of the maps $\mathcal{M}_N \to H^M_{\xi} \cap L^2_N$, $\sigma \to \rho_+(\sigma, \cdot)$ follows from Proposition 9 and the definition $\rho_+(\sigma, k) = \sigma(\overline{\tau}, k)/w(\sigma, k)$ (see also the proof of Proposition 11). Since $\sigma \to \frac{1}{w(\sigma, k)}$ is real analytic, it is bounded in $C$, i.e., there exists $C > 0$ so that $\|\rho_+(\sigma, \cdot)\|_{H^M_{\xi} \cap L^2_N} \leq C\|\sigma\|_{H^M_{\xi} \cap L^2_N}$, where $C$ depends locally uniformly on $\sigma \in \mathcal{M}_N$. Properties (i), (ii), (v) follow by simple computations. Property (iii) – (iv) are proved in [26, Lemma 4.1].

Finally define the functions
$$
R_{\pm}(\sigma, k) := 2ik\rho_\pm(\sigma, k).
$$

**Proposition 11.** Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 3}$ be fixed. Then the maps $\mathcal{M}_N \to H^M_{\xi} \cap L^2_N$, $\sigma \to R_{\pm}(\sigma, \cdot)$ are real analytic. There exists $C > 0$ so that one has $\|R_{\pm}(\sigma, \cdot)\|_{H^M_{\xi} \cap L^2_N} \leq C\|\sigma\|_{H^M_{\xi} \cap L^2_N}$, where $C$ depends locally uniformly on $\sigma \in \mathcal{M}_N$. Furthermore the following holds:

(i) $R_{\pm}(\sigma, k) = R_{\pm}(\sigma, -k)$ for every $k \in \mathbb{R}$.
(ii) $|R_{\pm}(\sigma, k)| < 2|k|$ for any $k \in \mathbb{R} \setminus \{0\}$.

Proof. In order to prove the statements, we will use that $R_{\pm}(\sigma, k) = 2ik\frac{\sigma(\overline{\tau}, k)}{w(\sigma, k)}$.

We will consider just $R_-$, since the analysis for $R_+$ is identical. To simplify the notation, we will denote $R_-(\sigma, \cdot) \equiv R(\sigma, \cdot)$.

By Proposition 9(i), $|R(\sigma, k)| \leq |\sigma(\cdot)|$, thus $R(\sigma, \cdot) \in L^2_N$. In order to prove that $R(\sigma, \cdot) \in H^M_{\xi}$, take $n$ derivatives ($1 \leq n \leq M$) of $R(\sigma, \cdot)$ to get the identity
$$
\partial_k^n R(\sigma, k) = \frac{2ik}{w(\sigma, k)} \partial_k^n \sigma(\cdot) + \sum_{j=1}^{n-1} \binom{n}{j} \left( \partial_k^j \frac{2ik}{w(\sigma, k)} \right) \partial_k^{n-j} \sigma(\cdot) + \left( \frac{2ik}{w(\sigma, k)} \right)^n \sigma(\cdot).
$$

We show now that each term of the r.h.s. of the identity above is in $L^2$. Consider first the term $I_1 := \frac{2ik}{w(\sigma, k)} \partial_k^n \sigma(\cdot)$. If $1 \leq n < M$, then $\partial_k^n \sigma \in L^2$ and $|2ik/w(\sigma, k)| \leq 1$, thus proving that $I_1 \in L^2$. If $n = M$, let $\chi$ be a smooth cut-off function with $\chi(k) \equiv 1$ in $[-1, 1]$ and $\chi(k) \equiv 0$ in $\mathbb{R} \setminus [-2, 2]$. Then one has
$$
I_1 = \frac{1}{w(\sigma, k)} \chi(k) 2ik \partial_k^M \sigma(\cdot) + \frac{2ik}{w(\sigma, k)} (1 - \chi(k)) \partial_k^M \sigma(\cdot).
$$

As $\sigma \in \mathcal{M}_N$ it follows that $k \to \chi(k) 2ik \partial_k^M \sigma(k)$ and $k \to (1 - \chi(k)) \partial_k^M \sigma(k)$ are in $L^2$. By Proposition 9, $\frac{1}{w(\sigma, \cdot)}$ and $\frac{2ik}{w(\sigma, \cdot)}$ are in $L^\infty$. Altogether it follows that $I_1 \in L^2$ for any $1 \leq n \leq M$. 


Consider now $I_2 := \sum_{j=1}^{n-1} \binom{n}{j} \left( \frac{\partial^j_k 2ik}{w(\sigma,k)} \right) \partial_n^{n-j} \sigma(k)$. By Proposition 9, one has 
\[
\left( \frac{\partial^j_k 2ik}{w(\sigma,k)} \right) \in H^1_k \text{ for every } 1 \leq j \leq M - 1,
\]
thus by the Sobolev embedding theorem 
\[
\left( \frac{\partial^j_k 2ik}{w(\sigma,k)} \right) \in L^\infty \text{ for every } 1 \leq j \leq M - 1.
\]
As $\partial_n^{n-j} \sigma \in L^2$ for $1 \leq j \leq n - 1 < M$, it follows that $I_2 \in L^2$ for any $1 \leq n \leq M$.

Finally consider $I_3 := \left( \frac{\partial^j_k 2ik}{w(\sigma,k)} \right) \sigma(k)$. By Proposition 9, $\left( \frac{\partial^j_k 2ik}{w(\sigma,k)} \right) \in L^2$ for any $1 \leq n \leq M$.

Altogether we proved that $R(\sigma, \cdot) \in H^M \cap L^2_N$. The claimed estimate on the norm $\| R(\sigma, \cdot) \|_{H^M \cap L^2_N}$, item (i) and (ii) follow in a straightforward way. The real analyticity of the map $\mathcal{M} \rightarrow H^M \cap L^2_N$, $\sigma \rightarrow R(\sigma, \cdot)$ follows by Proposition 9.

For $\sigma \in \mathcal{M} \cap \mathcal{N}$, define the Fourier transforms
\[
F_\pm(\sigma, y) := \mathcal{F}_\pm^{-1}(\rho_{\pm}(\sigma, \cdot))(y) = \frac{1}{\pi} \int_{\mathbb{R}} \rho_{\pm}(\sigma, k) e^{\pm 2iky} dk.
\]  
(52)

Then
\[
\pm \partial_y F_\pm(\sigma, y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} 2ik \rho_{\pm}(\sigma, k) e^{\pm 2iky} dk = \mathcal{F}_\pm^{-1}(R_{\pm}(\sigma, \cdot))(y).
\]  
(53)

In the next proposition we analyze the properties of the maps $\sigma \mapsto F_{\pm}(\sigma, \cdot)$.

**Proposition 12.** Let $N \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}_{\geq 3}$ be fixed. Then the following holds true:

(i) $\sigma \mapsto F_{\pm}(\sigma, \cdot)$ are real analytic as maps from $\mathcal{M} \cap \mathcal{N}$ to $H^1 \cap L^2_2$. Moreover there exists $C > 0$ so that $\| F_{\pm}(\sigma, \cdot) \|_{H^1 \cap L^2_2} \leq C \| \sigma \|_{H^M \cap \mathcal{N}}$, where $C$ depends locally uniformly on $\sigma \in \mathcal{M} \cap \mathcal{N}$.

(ii) $\sigma \mapsto F_{\pm}(\sigma, \cdot)$ are real analytic as maps from $\mathcal{M} \cap \mathcal{N}$ to $H^N \cap L^2_M$. Moreover there exists $C' > 0$ so that $\| F_{\pm}(\sigma, \cdot) \|_{H^N \cap L^2_M} \leq C' \| \sigma \|_{H^M \cap \mathcal{N}}$, where $C'$ depends locally uniformly on $\sigma \in \mathcal{M} \cap \mathcal{N}$.

**Proof.** By Proposition 10 and Proposition 11, the map $\mathcal{M} \cap \mathcal{N} \rightarrow H^1 \cap L^2_2 \rightarrow H^2_{\pm} \cap L^2_1$, $\sigma \mapsto \rho_{\pm}(\sigma, \cdot)$ is real analytic. Thus item (i) follows by the properties of the Fourier transform. By Proposition 10 (ii), $F_{\pm}(\sigma, \cdot) = F_{\pm}^{-1}(\rho_{\pm})$ is real valued. Item (ii) follows from (53) and the characterizations
\[
R_{\pm} \in H^M_{\mathcal{H}} \iff F_{\pm}^{-1}(R_{\pm}) \in L^2_M \text{ and } R_{\pm} \in L^2_N \iff F_{\pm}^{-1}(R_{\pm}) \in H^N_C.
\]  
(54)

The claimed estimates follow from the properties of the Fourier transform, Proposition 10 and Proposition 11.

We are finally able to prove that there exists a potential $q \in \mathcal{Q}$ with prescribed scattering coefficient $\sigma \in \mathcal{M} \cap \mathcal{N}$. More precisely the following theorem holds.

**Theorem 4.2.** Let $N \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 3}$ and $\sigma \in \mathcal{M} \cap \mathcal{N}$ be fixed. Then there exists a potential $q \in \mathcal{Q}$ such that $S(q, \cdot) = \sigma$.

**Proof.** Let $\rho_{\pm} := \rho_{\pm}(\sigma, \cdot)$ and $\tau := \tau(\sigma, \cdot)$ be given by formula (46). Let $F_{\pm}(\sigma, \cdot)$ be defined as in (52). By Proposition 12 it follows that $F_{\pm}(\sigma, \cdot)$ are absolutely
continuous and $F'_\pm(\sigma, \cdot) \in H^N \cap L^2_M$. As $M \geq 3$ it follows that
\[
\int_{-\infty}^{\infty} (1 + x^2)|F'_\pm(\sigma, x)| \, dx < \infty.
\] (55)

The main theorem in inverse scattering [13] assures that if (55) and item (i)–(v) of Proposition 11 hold, then there exists a potential $q \in \mathcal{Q}$ such that $r_\pm(q, \cdot) = \rho_\pm$ and $t(q, \cdot) = \tau$, where $r_\pm$ and $t$ are the reflection respectively transmission coefficients defined in (5). From the formulas (46) it follows that $S(q, \cdot) = \sigma$. □

4.1. Gelfand-Levitan-Marchenko equation. The aim of this subsection is to show that the potential $q$ of Theorem 4.2 is of class $\mathcal{Q}^{N, M}$ and that $S^{-1} : \mathcal{H}^{M, N} \rightarrow \mathcal{Q}^{N, M}$ is a real analytic map. We take here a different approach than [26]. In [26] the authors show that the map $S$ is complex differentiable and its differential $d_q S$ is bounded invertible. Here instead we reconstruct $q$ by solving the Gelfand-Levitan-Marchenko equations and we show that the inverse map $\mathcal{H}^{M, N} \rightarrow \mathcal{Q}^{N, M}$, $\sigma \mapsto q$ is real analytic.

We outline briefly the procedure. Given $\sigma \in \mathcal{H}^{M, N}$, Proposition 10 allowed us to construct two reflection coefficients $\rho_\pm$. Given two arbitrary real numbers $c_+ \leq c_-$, the inverse scattering methods will allow us to construct a potential $q_+$ on $[c_+, \infty)$ using $\rho_+$ and a potential $q_-$ on $(-\infty, c_-]$ using $\rho_-$, such that $q_+$ and $q_-$ coincide on the intersection of their domains, i.e., $q_+[c_+, c_-] = q_-|[c_+, c_-]$. Hence $q$ defined on the whole line by $q|[c_+, \infty) = q_+$ and $q|(-\infty, c_-] = q_-$ is well defined, and furthermore the standard theory of inverse scattering [13, 33] assures that $q \in \mathcal{Q}$ and $r_\pm(q, \cdot) = \rho_\pm$, i.e., the potential $q$ just constructed has the prescribed reflection coefficients $\rho_+$ and $\rho_-$. Before proceeding, we need to introduce some notations. Indeed we will work with functions defined on half-lines of the form $[c, \infty)$ or $(-\infty, c)$, thus we need to introduce the appropriate Lebesgue and Sobolev spaces. For any $c \in \mathbb{R}$ and $1 \leq p \leq \infty$ we define
\[
L^p_{x \geq c} := \left\{ f : [c, \infty) \rightarrow \mathbb{C} : \| f \|_{L^p_{x \geq c}} < \infty \right\},
\]
where $\| f \|_{L^p_{x \geq c}} := \left( \int_c^{+\infty} |f(x)|^p \, dx \right)^{1/p}$ for $1 \leq p < \infty$, while for $p = \infty$ we define $\| f \|_{L^\infty_{x \geq c}} := \text{esssup}_{x \geq c} |f(x)|$. For any integer $N \geq 1$ define
\[
H^N_{x \geq c} := \left\{ f : [c, \infty) \rightarrow \mathbb{R} : \| f \|_{H^N_{x \geq c}} < \infty \right\}, \quad \| f \|_{H^N_{x \geq c}}^2 := \sum_{j=0}^{N} \| \partial_x^j f \|_{L^2_{x \geq c}}^2,
\]
and for any real number $M \geq 1$ define
\[
L^2_{M, x \geq c} := \left\{ f : [c, \infty) \rightarrow \mathbb{C} : \| f \|_{L^2_{M, x \geq c}} < \infty \right\}, \quad \| f \|_{L^2_{M, x \geq c}} := \| (x)^M f \|_{L^2_{x \geq c}},
\]
where $(x) := (1 + x^2)^{1/2}$. We will write $H^N_{x, x \geq c}$ for the complexification of $H^N_{x \geq c}$. For $1 \leq \alpha, \beta \leq \infty$, we define
\[
L^\alpha_{x \geq c, y \geq 0} := \left\{ f : [c, \infty) \times [0, +\infty) \rightarrow \mathbb{C} : \| f \|_{L^\alpha_{x \geq c, y \geq 0}} < \infty \right\}, \quad \| f \|_{L^\alpha_{x \geq c, y \geq 0}} := \left( \int_c^{+\infty} \int_y^{+\infty} |f(x, \cdot)|^\alpha \, dy \, dx \right)^{1/\alpha}.
\]
Analogously one defines the spaces $L^p_{x \leq c}$, $H^N_{x \leq c}$, $L^2_{M, x \leq c}$ and $L^\alpha_{x \leq c, y \leq 0}$, mutatis mutandis.
Let us denote by $C_0^0 := C^0([0, \infty), \mathbb{C})$ and by $C_{x \geq c, y \geq 0}^0 := C^0([c, \infty) \times [0, \infty), \mathbb{C})$. Finally we denote by $C_{x \geq c}^0 L_y^2 \geq 0 := C^0([c, \infty), L_y^2 \geq 0)$ the set of continuous functions on $[c, \infty)$ taking value in $L_y^2 \geq 0$.

The potentials $q_+$ and $q_-$ mentioned at the beginning of this section are constructed by solving two integral equations, known in literature as the Gelfand-Levitan-Marchenko equations, which we are now going to describe in more details.

Given $\sigma \in \mathcal{S}$, define the functions $F_{\pm}(\sigma, \cdot)$ as in (52). See Proposition 12 for the analytical properties of the maps $\sigma \rightarrow F_{\pm}(\sigma, \cdot)$. To have a more compact notation, in the following we will denote $F_{\pm, \sigma} := F_{\pm}(\sigma, \cdot)$.

The Gelfand-Levitan-Marchenko equations are the integral equations given by

\[
F_{+, \sigma}(x + y) + E_{+, \sigma}(x, y) + \int_0^{+\infty} F_{+, \sigma}(x + y + z)E_{+, \sigma}(x, z)dz = 0, \quad y \geq 0 \tag{56}
\]

\[
F_{-, \sigma}(x + y) + E_{-, \sigma}(x, y) + \int_{-\infty}^{0} F_{-, \sigma}(x + y + z)E_{-, \sigma}(x, z)dz = 0, \quad y \leq 0 \tag{57}
\]

where $E_{\pm}(x, y)$ are the unknown functions and $F_{\pm, \sigma}$ are given and uniquely determined by $\sigma$ through formula (52). If (56) and (57) have solutions with enough regularity, then one defines the potentials $q_+$ and $q_-$ through the well-known formula - [13]

\[
q_+(x) = -\partial_x E_{+, \sigma}(x, 0), \quad \forall c_+ \leq x < \infty,
\]

\[
q_-(x) = \partial_x E_{-, \sigma}(x, 0), \quad \forall -\infty < x \leq c_. \tag{58}
\]

The main purpose of this section is to study the maps $R_{\pm, c}$ defined by

\[
\sigma \mapsto R_{\pm, c}(\sigma), \quad R_{\pm, c}(\sigma)(x) := \mp \partial_x E_{\pm, \sigma}(x, 0), \quad x \in [c, \pm \infty). \tag{59}
\]

**Theorem 4.3.** Fix $N \in \mathbb{Z}_{\geq 0}$, $M \in \mathbb{Z}_{\geq 1}$, and $c \in \mathbb{R}$. Then the maps $R_{+, c}$ and $R_{-, c}$ are well defined on $\mathcal{S}^{M, N}$ and take values in $H_{x \geq c}^N \cap L_{M, x \geq c}^2$ and $H_{x \leq c}^N \cap L_{M, x \leq c}^2$. As such they are real analytic.

In order to prove Theorem 4.3 we look for solutions of (56) and (57) of the form

\[
E_{\pm, \sigma}(x, y) = -F_{\pm, \sigma}(x + y) + B_{\pm, \sigma}(x, y) \tag{60}
\]

where $B_{\pm, \sigma}(x, y)$ are to be determined. Inserting the ansatz (60) into the Gelfand-Levitan-Marchenko equations (56), (57), one gets that for $y \geq 0$,

\[
B_{+, \sigma}(x, y) + \int_0^{+\infty} F_{+, \sigma}(x + y + z)B_{+, \sigma}(x, z)dz = \int_0^{+\infty} F_{+, \sigma}(x + y + z)F_{+, \sigma}(x + z)dz, \tag{61}
\]

while for $y \leq 0$

\[
B_{-, \sigma}(x, y) + \int_{-\infty}^{0} F_{-, \sigma}(x + y + z)B_{-, \sigma}(x, z)dz = \int_{-\infty}^{0} F_{-, \sigma}(x + y + z)F_{-, \sigma}(x + z)dz. \tag{62}
\]

We will prove in Lemma 4.8 below that there exists a solution $B_{+, \sigma}$ of (61) and a solution $B_{-, \sigma}$ of (62) with $\partial_x B_{+, \sigma}(\cdot, 0) \in H_{x \geq c}^1$ respectively $\partial_x B_{-, \sigma}(\cdot, 0) \in H_{x \leq c}^1$. 


By (58) we get therefore
\[ q_+ = \partial_x F_{+, \sigma} - \partial_x B_{+, \sigma}(\cdot, 0) \quad \forall c \leq x < \infty, \]
\[ q_- = -\partial_x F_{-, \sigma} + \partial_x B_{-, \sigma}(\cdot, 0) \quad \forall -\infty < x \leq c. \]

Define the maps
\[ B_{\pm, c} : \sigma \mapsto B_{\pm, c}(\sigma) \]
as \[ B_{+, c}(\sigma)(x) := -\partial_x B_{+, \sigma}(x, 0) \quad \forall x \geq c \quad \text{and} \quad B_{-, c}(\sigma)(x) := \partial_x B_{-, \sigma}(x, 0) \quad \forall x \leq c. \]

Now we study analytic properties of the maps \( B_{\pm, c} \) in case the scattering coefficient \( \sigma \) belongs to \( \mathcal{H}^{3, N} \) with arbitrary \( N \in \mathbb{Z}_{\geq 0} \). Later we will treat the case where \( \sigma \in \mathcal{M}^{\infty, 0} \), \( M \in \mathbb{Z}_{\geq 3} \).

**Proposition 13.** Fix \( N \in \mathbb{Z}_{\geq 0} \) and \( c \in \mathbb{R} \). Then \( B_{+, c} [B_{-, c}] \) is real analytic as a map from \( \mathcal{H}^{3, N} \) to \( H^N_{x \geq c} [H^N_{x \leq c}] \). Moreover
\[
\|B_{+, c}(\sigma)\|_{H^N_{x \geq c}}, \quad \|B_{-, c}(\sigma)\|_{H^N_{x \leq c}} \leq K \|\sigma\|_{H^N_{\xi \leq c}} \]
where \( K > 0 \) is a constant which can be chosen locally uniformly in \( \sigma \in \mathcal{H}^{3, N} \).

The main ingredient of the proof of Proposition 13 is a detailed analysis of the solutions of the integral equations (61)-(62), which we rewrite as
\[
(Id + K^+_{x, \sigma}) [B_{+, \sigma}(x, \cdot)](y) = f_{\pm, \sigma}(x, y) \quad (63)
\]
where for every \( x \in \mathbb{R} \) fixed, the two operators \( K^+_{x, \sigma} : L^2_{y \geq 0} \to L^2_{y \geq 0} \) and \( K^-_{x, \sigma} : L^2_{y \leq 0} \to L^2_{y \leq 0} \) are defined by
\[
K^+_{x, \sigma} [f](y) := \int_0^{+\infty} F_{+, \sigma}(x + y + z) f(z) \, dz, \quad f \in L^2_{y \geq 0}, \quad (64)
\]
\[
K^-_{x, \sigma} [f](y) := \int_{-\infty}^{-y} F_{-, \sigma}(x + y + z) f(z) \, dz, \quad f \in L^2_{y \leq 0}, \quad (65)
\]
and the functions \( f_{\pm, \sigma} \) are defined by
\[
f_{\pm, \sigma}(x, y) := \pm \int_0^{\pm\infty} F_{\pm, \sigma}(x + y + z) F_{\pm, \sigma}(x + z) \, dz. \quad (66)
\]

As the claimed statements for \( B_{+, c} \) and \( B_{-, c} \) can be proved in a similar way we consider \( B_{+, c} \) only. To simplify notation, in the following we will omit the subscript " + ". In particular we write \( B_{\sigma} \equiv B_{+, \sigma}, F_{\sigma} \equiv F_{+, \sigma}, f_{\sigma} \equiv f_{+, \sigma} \) and \( K_{x, \sigma} \equiv K^+_{x, \sigma} \).

In order to solve the integral equations (63) we need the operator \( Id + K^-_{x, \sigma} \) to be invertible on \( L^2_{y \geq 0} \) (respectively \( Id + K^+_{x, \sigma} \) to be invertible on \( L^2_{y \leq 0} \)). The following result is well known:

**Lemma 4.4** ([10, 7]). Let \( \sigma \in \mathcal{H}^{3, 0} \) and fix \( c \in \mathbb{R} \). Then the following holds:

(i) For every \( x \geq c \), \( K^+_{x, \sigma} : L^2_{y \geq 0} \rightarrow L^2_{y \geq 0} \) is a bounded linear operator; moreover
\[
\sup_{x \geq c} \|K^+_{x, \sigma}\|_{L(L^2_{y \geq 0})} < 1 \quad \text{and}
\]
\[
\|K^+_{x, \sigma}\|_{L(L^2_{y \geq 0})} \leq \int_{x}^{+\infty} |F_{+, \sigma}(\xi)| \, d\xi \rightarrow 0 \quad \text{if} \quad x \rightarrow +\infty.
\]
Consider the integral equation (63):

\[ \text{Lemma 4.7.} \]

The next step is to solve the Gelfand-Levitan-Marchenko equation. It will be more convenient to state a general result about the properties of the solution of the integral equation (63):

**Lemma 4.7.** Consider the integral equation

\[ (I + K_{\sigma}^+) [g(x,\cdot)](y) = h_{\sigma}(x, y) \]

and assume that \( h_{\sigma}, \partial_x h_{\sigma}, \partial_y h_{\sigma} \) satisfy (P). Then there exists a unique solution \( g_{\sigma} \) to the integral equation which satisfies (P). Its derivatives \( \partial_x g_{\sigma} \) and \( \partial_y g_{\sigma} \) satisfy (P) and solve the equations

\[ (I + K_{\sigma}) [\partial_x g_{\sigma}] = \partial_x h_{\sigma} - K_{\sigma}^+ [g_{\sigma}] \],

\[ \partial_y g_{\sigma} = \partial_y h_{\sigma} - K_{\sigma}^+ [g_{\sigma}] \].

The proof of this lemma, being quite long but elementary, can be found in [34, Appendix C].

We are now in the position to study the properties of the solution of the integral equation (61):

**Lemma 4.8.** Fix \( N \geq 0 \) and \( c \in \mathbb{R} \). For every \( \sigma \in \mathcal{S}^{3, N} \) equation (61) has a unique solution \( B_{\sigma} \in C^0_{\sigma} \cap L^2_{\sigma} \cap L^2_{\sigma} \). Moreover for all integers \( n_1, n_2 \geq 0 \) with \( n_1 + n_2 \leq N + 1 \), the function \( \partial_x^{n_1} \partial_y^{n_2} B_{\sigma} \) satisfies (P).
Remark 4. (Hardy inequality) Fix an arbitrary real number \( K \geq 1 \). Then the linear map \( \sigma \to B_\sigma \) satisfies assumption \((P)\). Thus by Lemma 4.7 (i) it follows that \( B_\sigma = (Id + K_\sigma)^{-1} f_\sigma \) and its derivatives \( \partial_x B_\sigma, \partial_y B_\sigma \) satisfy \((P)\).

Note that if \( N = 0 \) the lemma is proved. Thus in the following we assume \( N \geq 1 \).

Case \( n = 0 \). Then \( j_1 = j_2 = 0 \). We need to prove existence and uniqueness of the solution of equation \((63)\). By Lemma 4.6 the function \( f_\sigma \) and its derivatives \( \partial_x f_\sigma, \partial_y f_\sigma \) satisfy assumption \((P)\). Thus by Lemma 4.7 (i) it follows that \( B_\sigma = (Id + K_\sigma)^{-1} f_\sigma \) and its derivatives \( \partial_x B_\sigma, \partial_y B_\sigma \) satisfy \((P)\).

Proof of Proposition 13. Let \( \{ f_\sigma \}_{\sigma \in 0} \) be a family of functions \( f_\sigma \). By Lemma 4.7 it follows that \( \partial_x ^{j_1} \partial_y ^{j_2} B_\sigma \) satisfies

\[
\begin{align*}
(\partial_x ^{j_1} \partial_y ^{j_2} B_\sigma)(x,y) &= f_\sigma ^{j_1,j_2}(x,y) \quad \text{if } j_2 > 0, \\
(\partial_x ^{j_1} \partial_y ^{j_2} B_\sigma)(x,y) &= f_\sigma ^{j_1,j_2}(x,y) \quad \text{if } j_2 > 0,
\end{align*}
\]

for arbitrary \( \sigma \). Thus the claim follows.

Lemma 4.8 implies in a straightforward way Proposition 13.

Proof of Proposition 13. By Lemma 4.8, \( \partial_x ^{j_1} B_\sigma \) satisfies \((P)\) for every \( 1 \leq n \leq N + 1 \). Thus for every \( 1 \leq n \leq N + 1 \), \( \sigma \in \mathcal{S}^{3,N} \) to \( L^\infty_{x,z} \) and \( \| \partial_x ^{j_1} B_\sigma \|_{L^\infty_{x,z}} \leq K_\sigma \| \sigma \|_{H^3_{x,z} \cap L^1_{x,z}} \). Thus the map \( \sigma \to -\partial_x B_\sigma \) is real analytic as a map from \( \mathcal{S}^{3,N} \) to \( H^N_{x,z,c} \). The claimed estimate follows in a straightforward way.

The next step is to study the case \( \sigma \in \mathcal{S}^{M,0} \) for arbitrary \( M \geq 3 \). We start with some remarks which will be used in our analysis.

Remark 4. (Hardy inequality) Fix an arbitrary real number \( c \) and \( m \geq 0 \). The linear map \( L^2_{m+1,x; \geq c} \to L^2_{m,x; \geq c} \) defined by \( g \to \tilde{g}(x) := \int_x ^{+\infty} g(z) \, dz \) is continuous, and there exists a constant \( K_c > 0 \), depending on \( c \), such that \( \| \tilde{g} \|_{L^2_{m,x; \geq c}} \leq K_c \| g \|_{L^2_{m+1,x; \geq c}} \).

Remark 5. Fix an arbitrary real number \( c \). Then the linear map \( L^2_{1/2,x; \geq c} \to L^2_{x; \geq c} \) defined by \( g \to \tilde{g}(x,y) := g(x+y) \) and the bilinear map \( L^2_{x; \geq c} \times L^2_{x; \geq c} \to L^2_{x; \geq c} \) defined by \( (g,h) \to \tilde{G}(x,y) := g(x+y)h(x+) \)
are continuous, and there exist constants $K_c, K'_c \geq 0$, depending on $c$, such that
\[
\|\tilde{g}\|_{L^2_{x \geq c} L^2_{y \geq 0}} \leq K_c \|g\|_{L^2_{x \geq c} L^2_{y \geq 0}}, \quad \|\tilde{G}\|_{L^2_{x \geq c} L^2_{y \geq 0}} \leq K'_c \|g\|_{L^2_{x \geq c} L^2_{y \geq 0}}.
\]

**Remark 6.** Fix an arbitrary real number $c$. The bilinear map $L^2_{x \geq c} \times L^2_{x \geq c} L^2_{y \geq 0} \rightarrow L^2_{x \geq c}$, $(g, h) \mapsto \tilde{G}(x) := \int_0^{+\infty} g(x + z) h(x, z) \, dz$ is continuous, and there exists a constant $K_c > 0$, depending on $c$, such that $\|\tilde{G}\|_{L^2_{x \geq c}} \leq K_c \|g\|_{L^2_{x \geq c}} \|h\|_{L^2_{x \geq c} L^2_{y \geq 0}}$.

**Remark 7.** From the decay properties of $F^\pm_{\sigma,c}$ one deduces corresponding decay properties of $F_{\sigma,c}$. Indeed one has
\[
\langle x \rangle^m F^\pm_{\sigma,c} \in L^2_{x \geq c} \Rightarrow \langle x \rangle^{m-1} F^\pm_{\sigma,c} \in L^2_{x \geq c} \Rightarrow \langle x \rangle^{m-2} F^\pm_{\sigma,c} \in L^2_{x \geq c}, \quad \forall m \geq 2.
\]

The interested reader can find the proofs of these statements in [34, Appendix A].

**Proposition 14.** Fix $M \in \mathbb{Z}_{\geq 3}$ and $c \in \mathbb{R}$. For any $\sigma \in \mathcal{S}^{M,0}$ the equations (56) and (57) admit solutions $E_{\sigma,x}$. The maps $R_{+,c}$ [$R_{-,c}$], defined by (59), are real analytic as maps from $\mathcal{S}^{M,0}$ to $L^2_{M,x \geq c} [L^2_{M,x \leq c}]$. Moreover $\|R_{+,c}(\sigma)\|_{L^2_{M,x \geq c}}$, $\|R_{-,c}(\sigma)\|_{L^2_{M,x \leq c}} \leq K_c \|\sigma\|_{H^M_{\sigma,c}}$, where $K_c > 0$ can be chosen locally uniformly in $\sigma \in \mathcal{S}^{M,0}$.

**Proof.** We prove the result just for $R_{+,c}$, since for $R_{-,c}$ the proof is analogous. As before, we suppress the subscript "+" from the various objects.

Consider the Gelfand-Levitan-Marchenko equation (56). Multiply it by $\langle x \rangle^{M-3/2}$ to obtain
\[
(Id + K_{x,c}) \left[ \langle x \rangle^{M-3/2} E_{\sigma}(x, y) \right] = -\langle x \rangle^{M-3/2} F_{\sigma}(x + y).
\]

The function
\[
h_{\sigma}(x, y) := -\langle x \rangle^{M-3/2} F_{\sigma}(x + y),
\]
satisfies $h_{\sigma}(x, \cdot) \in L^2_{y \geq 0}$ and one checks that $h_{\sigma} \in C^0_{x \geq c} L^2_{y \geq 0} \cap C^0_{x \geq 0} y \geq 0$. We show now that $h_{\sigma} \in L^2_{x \geq c} L^2_{y \geq 0}$. One has that, for any $M \geq 3$,
\[
\|h_{\sigma}\|_{L^2_{x \geq c} L^2_{y \geq 0}} \leq K_c \int_0^{+\infty} \langle x \rangle^{2M-2} \|E_{\sigma}(x)\|^2 \, dx \leq K_c \|E_{\sigma}\|_{L^2_{M-1}}^2 \leq K_c \|\sigma\|_{H^M_{\sigma,c}}^2,
\]
where in the first inequality we exchanged the order of integration, in the second one we used Remark 4 and in the last one Proposition 12. Consider now $h_{\sigma}(x, 0) = -\langle x \rangle^{M-3/2} F_{\sigma}(x)$. One verifies easily that $h_{\sigma}(\cdot, 0) \in L^2_{y \geq c}$. Finally the map $\sigma \mapsto h_{\sigma}$ [$\sigma \mapsto h_{\sigma}(\cdot, 0)$] is real analytic as a map from $\mathcal{S}^{M,0}$ to $L^2_{x \geq c} L^2_{y \geq 0} [L^2_{M-3/2,x \geq c}]$. Then it is easy to show (cf. [34, Appendix C]) that there exists a solution $E_{\sigma}$ of equation (56) which satisfies (i) $\langle x \rangle^{M-3/2} E_{\sigma} \in C^0_{x \geq c} L^2_{y \geq 0} \cap L^2_{x \geq c} L^2_{y \geq 0}$, (ii) $\langle x \rangle^{M-3/2} E_{\sigma}(\cdot, 0) \in L^2_{y \geq c}$, (iii) $\|E_{\sigma}\|_{L^2_{M-3/2,x \geq c}} \leq K_c \|\sigma\|_{H^M_{\sigma,c}}$, (iv) $\sigma \mapsto \langle x \rangle^{M-3/2} E_{\sigma}$ [or $E_{\sigma}(\cdot, 0)$] is real analytic as a map from $\mathcal{S}^{M,0}$ to $L^2_{x \geq c} L^2_{y \geq 0} [L^2_{M-3/2,x \geq c}]$. Furthermore its derivative $\partial_x E_{\sigma}$ satisfies the integral equation
\[
(Id + K_{x,c}) (\partial_x E_{\sigma}(x, y)) = -F'_{\sigma}(x + y) - \int_0^{+\infty} F'_{\sigma}(x + y + z) E_{\sigma}(x, z) \, dz.
\]
Multiply the equation above by \( \langle x \rangle^{M-3/2} \), to obtain \((Id + K_\sigma) \big( \langle x \rangle^{M-3/2} \partial_x E_\sigma \big) = \tilde{h}_{\sigma} \), where

\[
\tilde{h}_{\sigma}(x, y) := -\langle x \rangle^{M-3/2} h'_\sigma(x, y) - \int_{0}^{+\infty} F'_\sigma(x + y + z) \langle x \rangle^{M-3/2} E_\sigma(x, z) \, dz. \tag{68}
\]

where \( h'_\sigma(x, y) := F'_\sigma(x + y) \). We claim that \( \tilde{h}_{\sigma} \in L^2_{x \geq c, y \geq 0} \) and \( \sigma \mapsto \tilde{h}_{\sigma} \) is real analytic as a map \( \mathscr{M}_{0} \to L^2_{x \geq c, y \geq 0} \). By Remark 5 the first term of (68) satisfies

\[
\Big\| \langle x \rangle^{M-3/2} h'_\sigma \Big\|_{L^2_{x \geq c, y \geq 0}} \leq K_c \| \langle x \rangle^{M-1} F'_\sigma \|_{L^2_{x \geq c, y \geq 0}} \leq K_c \| \sigma \|_{H^{M}_{c, \mathbb{C}}},
\]

and the second term of (68) fulfills

\[
\int_{0}^{+\infty} F'_\sigma \Big( x + y + z \Big) \langle x \rangle^{M-3/2} E_\sigma(x, z) \, dz \leq \| F'_\sigma \|_{L^1} \| \langle x \rangle^{M-3/2} E_\sigma \|_{L^2_{x \geq c, y \geq 0}} \leq K_c \| \sigma \|_{H^{M}_{c, \mathbb{C}}}. \]

Moreover \( \sigma \mapsto \tilde{h}_{\sigma} \) is real analytic as a map from \( \mathscr{M}_{0} \) to \( L^2_{x \geq c, y \geq 0} \), being composition of real analytic maps.

By Lemma 4.7, \( \langle x \rangle^{M-3/2} \partial_x E_\sigma \in L^2_{x \geq c, y \geq 0} \), the estimate

\[
\| \langle x \rangle^{M-3/2} \partial_x E_\sigma \|_{L^2_{x \geq c, y \geq 0}} \leq K_c \| \sigma \|_{H^{M}_{c, \mathbb{C}}}
\]

holds and \( \sigma \mapsto \langle \cdot \rangle^{M-3/2} \partial_x E_\sigma \) is real analytic as a map from \( \mathscr{M}_{0} \) to \( L^2_{x \geq c, y \geq 0} \).

Consider now equation (56). Evaluate it at \( y = 0 \) to get \( E_\sigma(x, 0) = -F'_\sigma(x) - \int_{0}^{+\infty} F'_\sigma(x + z) E_\sigma(x, z) \, dz \). Take the \( x \)-derivative of the last equation and multiply it by \( \langle x \rangle^{M} \) to obtain

\[
\langle x \rangle^{M} \partial_x E_\sigma(x, 0) = -\langle x \rangle^{M} F'_\sigma(x) - \int_{0}^{+\infty} \langle x \rangle^{3/2} F'_\sigma(x + z) \langle x \rangle^{M-3/2} E_\sigma(x, z) \, dz \]

\[
- \int_{0}^{+\infty} \langle x \rangle^{3/2} F'_\sigma(x + z) \langle x \rangle^{M-3/2} \partial_x E_\sigma(x, z) \, dz.
\]

We prove now that \( \partial_x E_\sigma(\cdot, 0) \in L^2_{M, x \geq c} \) and \( \sigma \mapsto \partial_x E_\sigma(\cdot, 0) \) is real analytic as a map from \( \mathscr{M}_{0} \) to \( L^2_{M, x \geq c} \). Clearly the map \( \sigma \mapsto F'_\sigma \) is real analytic as a map from \( \mathscr{M}_{0} \) to \( L^2_{M} \). Consider now the third term in the r.h.s. of the equality above (the second one being analogous). One has that

\[
\int_{0}^{+\infty} \langle x \rangle^{3/2} F'_\sigma(x + z) \langle x \rangle^{M-3/2} \partial_x E_\sigma(x, z) \, dz \leq \sup_{x \geq c} \left( \langle x \rangle^{3} \int_{x}^{+\infty} |F'_\sigma(z)|^2 \, dz \right) \| \langle \cdot \rangle^{M-3/2} \partial_x E_\sigma \|_{L^2_{x \geq c, y \geq 0}}
\]
\[ \leq K_c \left( \int_{-\infty}^{+\infty} (z)^3 |F_\sigma(z)|^2 \, dz \right) \left\| \langle \cdot \rangle^{M-3/2} \partial_x E_\sigma \right\|_{L^2_x L^2_y \geq 0} \]

\[ \leq K_c \| \sigma \|^2_{H^M_{0 \leq c}}. \]

The real analyticity follows from the fact that \( \sigma \mapsto F_\sigma \) \( [\sigma \mapsto F_\sigma] \) is real analytic as a map from \( \mathcal{S}^{M,0} \) to \( L^2_M \left[ L^2_{3/2} \right] \), and that \( \sigma \mapsto \langle \cdot \rangle^{M-3/2} E_\sigma \) and \( \sigma \mapsto \langle \cdot \rangle^{M-3/2} \partial_x E_\sigma \) are real analytic as maps from \( \mathcal{S}^{M,0} \) to \( L^2_{x \geq c} L^2_y \geq 0 \).

Combining the results of Proposition 13 and Proposition 14, we can prove Theorem 4.3.

**Proof of Theorem 4.3.** It follows from Proposition 12, Proposition 13 and Proposition 14 by restricting the scattering maps \( \mathcal{R}^{M,N} \) to the spaces \( \mathcal{S}^{M,N} = \mathcal{S}^{3,N} \cap \mathcal{S}^{M,0} \).

Using the results of Theorem 4.3 and Theorem 4.2 we can prove Theorem 4.1, showing that \( S^{-1} : \mathcal{S}^{N,M} \to \mathcal{Q}^{N,M} \) is real analytic.

**Proof of Theorem 4.1.** Let \( \sigma \in \mathcal{S}^{M,N} \). By Theorem 4.2 there exists \( q \in \mathcal{Q} \) with \( S(q, \cdot) = \sigma \). Now let \( c_+ \leq c \leq c_- \) be arbitrary real numbers and consider \( \mathcal{R}_{+,-c+}(\sigma) \) and \( \mathcal{R}_{-,-c-}(\sigma) \), where \( \mathcal{R}_{+,-c+} \) and \( \mathcal{R}_{-,-c-} \) are defined in (59). By classical inverse scattering theory [13], [33] the following holds:

(i) \( \mathcal{R}_{+,-c+} \mid_{x \in [c_+, c]} = \mathcal{R}_{-,-c-} \mid_{x \in [c, c_-]} \),

(ii) the potential \( q_c \) defined by

\[ q_c := \mathcal{R}_{+,-c+}(\sigma) I_{[c, +\infty)} + \mathcal{R}_{-,-c-}(\sigma) I_{(-\infty, c]} \]

is in \( \mathcal{Q} \) and satisfies \( r_+(q_c, \cdot) = r_+(\sigma, \cdot) \), \( r_-(q_c, \cdot) = r_-(\sigma, \cdot) \) and \( t(q_c, \cdot) = t(\sigma, \cdot) \). Thus by formulas (5) and (46) it follows that \( S(q_c, \cdot) = \sigma \).

Since \( S \) is 1-1 it follows that \( q_c = q \). Finally, by Theorem 4.3, \( \mathcal{S}^{M,N} \to H^N_{x \geq c+} \cap L^2_M \), \( \sigma \mapsto \mathcal{R}_{+,-c+}(\sigma) \) and \( \mathcal{S}^{M,N} \to H^N_{x \leq c-} \cap L^2_M \), \( \sigma \mapsto \mathcal{R}_{-,-c-}(\sigma) \) are real analytic. It follows that \( q \in H^N \cap L^2_M \) and the map \( S^{-1} : \sigma \mapsto q \) is real analytic.

5. **Proof of Corollary 1 and Theorem 1.2.** This section is devoted to the proof of Corollary 1 and Theorem 1.2. Both results are easy applications of Theorem 1.1.

**Proof of Corollary 1.** Let \( N \geq 0 \), \( M \geq 1 \) be fixed integers. Fix \( q \in \mathcal{Q}^{N,M} \). By Theorem 1.1 the scattering map \( S(q, \cdot) \) is in \( \mathcal{S}^{M,N} \). Furthermore by the definition (10) of \( I(q, k) \) there exists a constant \( C > 0 \) such that for any \( |k| \geq 1 \)

\[ |I(q, k)| \leq C |S(q, k)|^2 \]

In particular \( I(q, \cdot) \in L^2_{2N+1}([1, \infty), \mathbb{R}) \). By the real analyticity of the map \( q \mapsto S(q, \cdot) \), it follows that \( \mathcal{Q}^{N,M} \to L^2_{2N+1}([1, \infty), \mathbb{R}) \), \( q \mapsto I(q, \cdot) \mid_{[1, \infty)} \) is real analytic.

Now let us analyze \( I(q, k) \) for \( 0 \leq k \leq 1 \). By the definition (10) of \( I(q, k) \) one has

\[ I(q, k) + \frac{k}{\pi} \log \left( \frac{4k^2 + 1}{4(k^2 + 1)} \right) = -\frac{k}{\pi} \log \left( \frac{4k^2 + 1}{4k^2 + S(q, k)S(q, -k)} \right). \]

By Proposition 7, the map

\[ \mathcal{S}^{M,N} \to H^M_0(0, 1), \quad \sigma \mapsto \log \left( \frac{4k^2 + 1}{4k^2 + \sigma(k)\sigma(-k)} \right) \]
is real analytic.

Thus also the map \( Q^{N,M} \to H^M(\mathbb{R}) \), \( q \to l(S(q), \cdot) \) is real analytic, being composition of real analytic maps. Since the interval \([0, 1]\) is bounded, the map \( f \mapsto kf \), which multiplies a function by \( k \), is analytic as a map \( H^M_\zeta([0, 1], \mathbb{R}) \to H^M_\zeta([0, 1], \mathbb{R}) \). It follows that the map \( q \mapsto -\frac{i}{2}l(S(q), k) \) is real analytic as a map from \( Q^{N,M} \) to \( H^M([0, 1], \mathbb{R}) \).

For \( t \in \mathbb{R} \) and \( \sigma \in H^1_\zeta \), let us denote by

\[
\Omega^t(\sigma)(k) := e^{-i8kt^3} \sigma(k) .
\]  

(69)

We prove the following lemma.

**Lemma 5.1.** Let \( N, M \) be integers with \( N \geq 2M \geq 2 \). Let \( \sigma \in \mathcal{S}^{M,N} \). Then \( \Omega^t(\sigma) \in \mathcal{S}^{M,N} \), \( \forall t \geq 0 \).

**Proof.** As a first step we show that \( \Omega^t(\sigma) \in \mathcal{S} \) for every \( t \geq 0 \). Since \( \Omega^t(\sigma)(0) = \sigma(0) > 0 \) and \( \Omega^t(\sigma)(k) = \Omega^t(\sigma)(-k) \), \( \Omega^t(\sigma) \) satisfies (S1) and (S2) for every \( t \geq 0 \). Thus \( \Omega^t(\sigma) \in \mathcal{S} \), \( \forall t \geq 0 \). Next we show that \( \Omega^t(\sigma) \in H^M_{\zeta,k} \cap L^2_N \). Clearly \( |\Omega^t(\sigma)(k)| \leq |\sigma(k)| \), thus \( \Omega^t(\sigma) \in L^2_N \), \( \forall t \geq 0 \). Now we show that \( \Omega^t(\sigma) \in H^M_{\zeta,k} \), \( \forall t \geq 0 \). In particular we prove that \( \zeta \partial_k^M \Omega^t(\sigma) \in L^2 \), the other cases being analogous. Let \( M \geq 2 \) (the case \( M = 1 \) being simpler). Using the expression (69) one gets that \( \zeta(k) \partial_k^M \Omega^t(\sigma)(k) \) equals

\[
e^{-i8kt^3} \left( \zeta(k) \partial_k^M \sigma(k) + \sum_{j=1}^{M-1} \binom{M}{j} (-i24tk^2)^j \zeta(k) \partial_k^{M-j} \sigma(k) + (-i24tk^2)^M \zeta(k) \sigma(k) \right) .
\]

As \( \sigma \in \mathcal{S}^{M,N} \), the first and last term above are in \( L^2 \). Now we show that for \( 1 \leq j \leq M-1, |k|^{2j} \zeta \partial_k^{M-j} \sigma \in L^2 \). We will use the following interpolating estimate, proved in [38, Lemma 4]. Assume that \( J^a f := (1 - \partial^2_l)^a/2 f \in L^2 \) and \( \langle k \rangle^b f := (1 + |k|^2)^b/2 f \in L^2 \). Then for any \( \theta \in (0,1) \)

\[
\left\| \langle k \rangle^{\theta b} J^{(1-\theta)a} f \right\|_{L^2} \leq c \|f\|_{L^2} \|f\|_{H^0}^{1-\theta} . \quad (70)
\]

Note that \( \zeta \sigma \in H^M_\zeta \cap L^2_N \), thus we can apply estimate (70) with \( f = \zeta \sigma, b = N, a = M, \theta = \frac{J}{M} \), to obtain that \( \langle k \rangle^{\frac{M}{2}} \partial_k^{M-j} (\zeta \sigma) \in L^2 \). Since \( N \geq 2M \), it follows that \( \langle k \rangle^{2j} \partial_k^{M-j} (\zeta \sigma) \in L^2 \). By integration by parts

\[
\langle k \rangle^{2j} \left[ \zeta(k) \partial_k^{M-j} \sigma(k) \right] = \langle k \rangle^{2j} \left[ \partial_k^{M-j} (\zeta \sigma) - \sum_{l=1}^{M-j} \left( \binom{M-j}{l} \partial_k^{l} \zeta(k) \partial_k^{M-j-l} \sigma(k) \right) \right] .
\]

Since \( \forall l \geq 1, \partial_k^l \zeta \) has compact support, it follows that \( \langle k \rangle^{2j} \partial_k^l \zeta \partial_k^{M-j-l} \sigma \in L^2 \), thus all the terms in the r.h.s. above are in \( L^2 \). Thus for every \( 1 \leq j \leq M-1 \) we have \( \langle k \rangle^{2j} \partial_k^l \zeta \partial_k^{M-j-l} \sigma \in L^2 \) and it follows that \( \zeta \partial_k^M \Omega^t(\sigma) \in L^2 \) for every \( t \geq 0 \).

**Remark 8.** One can adapt the proof above, putting \( \zeta(k) \equiv 1 \), to shows that the spaces \( H^N \cap L^2_M \), with integers \( N \geq 2M \geq 2 \), are invariant by the Airy flow. Indeed the Fourier transform \( F_- \) conjugates the Airy flow with the linear flow \( \Omega^t \), i.e., \( U_{\text{Airy}}^t = F_- \circ \Omega^t \circ F_- \).

**Proof of Theorem 1.2.** Recall that by \([18]\) the scattering map \( S \) conjugate the KdV flow with the linear flow \( \Omega^t(\sigma)(k) := e^{-i8kt^3} \sigma(k) \), i.e.,

\[
U_{\text{KdV}}^t = S^{-1} \circ \Omega^t \circ S ,
\]
where \( U^t_{\text{Airy}} = \mathcal{F}^{-1} \circ \Omega^t \circ \mathcal{F} \). Take now \( q \in Q^{N,M} \), where \( N, M \) are integers with \( N \geq 2M \geq 6 \). By Theorem 1.1, \( S(q) \equiv S(q, \cdot) \in \mathcal{S}^{M,N} \). By Lemma 5.1 the flow \( \Omega^t \) preserves the space \( \mathcal{S}^{M,N} \) for every \( t \geq 0 \). Thus \( \Omega^t \circ S(q) \in \mathcal{S}^{M,N} \), \( \forall t \geq 0 \). By the bijectivity of \( S \) it follows that \( S^{-1} \circ \Omega^t \circ S(q) \in Q^{N,M} \), \( \forall t \geq 0 \). Thus item (i) is proved.

We prove now item (ii). Remark that by item (i), \( U^t_{KdV}(q) \in L^2 \), for any \( t \geq 0 \). Since \( U^t_{\text{Airy}} \) preserves the space \( H^N \cap L^2_M (N \geq 2M \geq 6) \), it follows that for \( q \in Q^{N,M} \) the difference \( U^t_{KdV}(q) - U^t_{\text{Airy}}(q) \in H^N \cap L^2_M \), \( \forall t \geq 0 \). We prove now the smoothing property of the difference \( U^t_{KdV}(q) - U^t_{\text{Airy}}(q) \). Since \( S^{-1} = \mathcal{F}^{-1} + B \),

\[
U^t_{KdV}(q) = U^t_{\text{Airy}}(q) + \mathcal{F}^{-1} \circ \Omega^t \circ A(q) + B \circ \Omega^t \circ S(q)
\]

and since \( S = \mathcal{F}^{-1} + A \),

\[
\mathcal{F}^{-1} \circ \Omega^t \circ S(q) = \mathcal{F}^{-1} \circ \Omega^t \circ \mathcal{F}(q) + \mathcal{F}^{-1} \circ \Omega^t \circ A(q) .
\]

Hence

\[
U^t_{KdV}(q) = U^t_{\text{Airy}}(q) + \mathcal{F}^{-1} \circ \Omega^t \circ A(q) + B \circ \Omega^t \circ S(q).
\]

The 1-smoothing property of the difference \( U^t_{KdV}(q) - U^t_{\text{Airy}}(q) \) follows now from the smoothing properties of \( A \) and \( B \) described in item (ii) of Theorem 1.1. The real analyticity of the map \( q \mapsto U^t_{KdV}(q) - U^t_{\text{Airy}}(q) \) follows from formula (71) and the real analyticity of the maps \( A, B \) and \( S \).

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