Superconvergence and accuracy enhancement of discontinuous Galerkin solutions for Vlasov–Maxwell equations

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Abstract
This paper explores the discontinuous Galerkin (DG) methods for solving the Vlasov–Maxwell (VM) system, a fundamental model for collisionless magnetized plasma. The DG method provides an accurate numerical description with conservation and stability properties. This work studies the applicability of a post-processing technique to the DG solution in order to enhance its accuracy and resolution for the VM system. In particular, superconvergence in the negative-order norm for the probability distribution function and the electromagnetic fields is established for the DG solution. Numerical tests including Landau damping, two-stream instability, and streaming Weibel instabilities are considered showing the performance of the post-processor.

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1 Introduction

In this paper, we consider numerical solutions of the Vlasov–Maxwell (VM) system, a fundamental model for collisionless magnetized plasma. The dimensionless form of the equations that describes the evolution of a single species of non-relativistic electrons under the self-consistent electromagnetic field while the ions are treated as uniform fixed background is given by

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0, \tag{1.1a}
\]

\[
\frac{\partial E}{\partial t} = \nabla_x \times B - J, \quad \frac{\partial B}{\partial t} = -\nabla_x \times E, \tag{1.1b}
\]

\[
\nabla_x \cdot E = \rho - \rho_i, \quad \nabla_x \cdot B = 0, \tag{1.1c}
\]

with

\[
\rho(x, t) = \int_{\Omega_v} f(x, v, t) \, dv, \quad J(x, t) = \int_{\Omega_v} f(x, v, t)v \, dv,
\]

where the equations are defined on \( \Omega = \Omega_x \times \Omega_v \), \( x \in \Omega_x \) denotes the position in physical space, and \( v \in \Omega_v \) in velocity space. Here \( f(x, v, t) \geq 0 \) is the distribution function of electrons at position \( x \) with velocity \( v \) at time \( t \), \( E(x, t) \) is the electric field, \( B(x, t) \) in the magnetic field, \( \rho(x, t) \) is the electron charge density, and \( J(x, t) \) is the current density. The charge density of background ions is denoted by \( \rho_i \), which is chosen to satisfy total charge neutrality, \( \int_{\Omega} (\rho(x, t) - \rho_i) \, dx = 0 \). Periodic boundary conditions in \( \Omega_x \) and compact support in \( \Omega_v \) are assumed. The VM system has wide applications in plasma physics for describing space and laboratory plasmas, with application to fusion devices, high-power microwave generators, and large scale particle accelerators.

Particle-in-cell (PIC) methods [3, 21] have long been prevalent numerical tools in which the particles are advanced in a Lagrangian framework, while the field equations are solved on a mesh. However, PIC methods are intrinsically noisy. Due to the increasing computing power in modern ages, in recent years, much of the work in the literature has been focused on accurate deterministic computations for Vlasov equations. Califano et al. used a semi-Lagrangian approach to compute the streaming Weibel instability [8], current filamentation instability [25], magnetic vortices [7], magnetic reconnection [6]. Also, various methods have been proposed for the relativistic VM system [2, 22, 29, 31]. This work concerns the discontinuous Galerkin (DG) method for solving the VM system. The DG method is a class of finite element method that uses discontinuous polynomial spaces, and they have desirable properties for convection-dominated problems [15]. In particular, DG methods have been used
to simulate the Vlasov–Poisson system in plasmas [12, 19, 20] and for a gravitational infinite homogeneous stellar system [10]. They have also been used to solve the VM system [9, 11] and the relativistic VM system [32]. The DG methods have nice properties such as stability, charge and energy conservation and high order accuracy, which are highly desirable for long time simulations.

The main computational challenge for any grid based solver for the VM system is the high-dimensionality of the Vlasov equation (6d + time). This makes the computation extremely expensive even on modern-day exa-scale supercomputers. Post-processing techniques, which can greatly enhance the resolution of the numerical solution at any given time, are therefore desirable because it is only applied once at the end of the simulation with negligible computational cost. The post-processing technique presented here takes advantage of the information contained in the negative-order norm and was originally developed by Bramble and Schatz [4] in the context of continuous finite element methods for elliptic problems. Negative-order norms can be thought of relating information in the Fourier/Signal space to that in physical space. Specifically, they are weighted dual norms. The post-processor consists of a local averaging operator applied to the finite element solution via convolution. We can then establish the convergence in the negative-order norm which is higher than that one obtained in the usual $L^2$-norm. It is well known that DG solutions have a convergence rate of order $k + 1/2$ using piecewise polynomials of degree $k$, but have higher order dissipation/dispersion errors of orders $2k + 2/2k + 1$. Hence SIAC brings forth the higher-order information seen in Fourier space to physical space. In [14], Cockburn, Luskin, Shu and Süli applied this technique to the DG methods for solving linear hyperbolic equations. This technique was further extended to the DG methods for solving nonlinear conservation laws [24, 27] and nonlinear symmetric systems of hyperbolic conservation laws [28]. This method is currently part of a filtering family known as a Smoothness-Increasing Accuracy-Conserving (SIAC) filters [30]. This paper will demonstrate the performance of post-processing by the SIAC filter for DG solutions to the VM system. In particular, we consider benchmark numerical tests for Vlasov-Ampère (VA) and VM systems, and study the numerical error for short and long time simulations with varying polynomial order.

In order to validate the enhanced accuracy of the post-processed solution, an important step is to establish the superconvergence of the negative-order norm of the error and its divided differences. In [14], Cockburn, Luskin, Shu and Süli established a framework to prove negative-order estimates for the DG solutions to linear conservation laws of order $2k + 1$ using piecewise polynomials of degree $k$. After this, there have been important extensions. $L^2$ and $L^\infty$ superconvergence estimates were established for DG solutions for linear constant coefficient hyperbolic systems with the position-dependent SIAC filter [23]. Ji, Xu and Ryan [24] proved superconvergence for non-linear conservation laws in negative-order Sobolev norm of the DG solution of order $2k + m$, where $m$ is a constant that depends on the chosen numerical flux of the method. Ji, Meng et al [27, 28] extended this superconvergence result for non-linear conservation laws and nonlinear symmetric hyperbolic systems to the $L^2$-norm of the postprocessed DG solution of order at least $(3/2k + 1)$. In summary, it is highly nontrivial to establish superconvergence for nonlinear problems because a suitable dual problem has to be identified, and additionally the divided difference of the solution does not
satisfy the PDE, which makes the proof highly technical [27, 28]. In this work, we aim to prove negative-order estimates of DG solutions to the VM system. Since the VM system is nonlinear, it is nontrivial to extend the proof in [14]. We identify a proper dual problem, which aids the estimates of the consistency term. In the end, we proved superconvergence of order \((2k + \frac{1}{2})\) in the negative-order norm for the probability distribution function and the electromagnetic fields.

The paper is organized as follows. In Sect. 2, we introduce the DG method for the VM system as well as relevant notations that will be required for the negative-order estimates. In Sect. 3 we introduce SIAC filtering. In Sect. 4 we prove the negative-order norm estimates of the DG solutions to the VM system. The superconvergence results are confirmed numerically in Sect. 5. We conclude the paper with remarks and future work in Sect. 6.

2 Discontinuous Galerkin numerical scheme

2.1 Notations, definitions and projections

We begin by introducing the necessary notation used in the paper. Without loss of generality, we assume the spatial and velocity domain to be \(\Omega_x = [-L_x, L_x]^d_x\) and \(\Omega_v = [-L_v, L_v]^d_v\), where \(L_v\) is chosen large enough so that \(f = 0\) at \(\partial \Omega_v\). Throughout the paper, standard notations will be used for the Sobolev spaces. Given a bounded domain \(D \in \mathbb{R}^\ast\) (with \(* = d_x, d_v, or d_x + d_v\) and any nonnegative integer \(m\), \(H^m(D)\) denotes the \(L^2\)-Sobolev space of order \(m\) with the standard Sobolev norm \(\| \cdot \|_{m,D}\), \(W^{m,\infty}(D)\) denotes the \(L^\infty\)-Sobolev space of order \(m\) with the standard Sobolev norm \(\| \cdot \|_{m,\infty,D}\) and the semi-norm \(| \cdot |_{m,\infty,D}\). When \(m = 0\), we also use \(H^0(D) = L^2(D)\) and \(W^{0,\infty}(D) = L^\infty(D)\).

Let \(\mathcal{T}_h^x = \{K_x\}\) and \(\mathcal{T}_h^v = \{K_v\}\) be partitions of \(\Omega_x\) and \(\Omega_v\), respectively, with \(K_x\) and \(K_v\) being Cartesian elements or simplices; then \(\mathcal{T}_h = \{K : K = K_x \times K_v, \forall K_x \in \mathcal{T}_h^x, \forall K_v \in \mathcal{T}_h^v\}\) defines a partition of \(\Omega\). Let \(\mathcal{E}_x^i\) be the set of the edges of \(\mathcal{T}_h^x\) and \(\mathcal{E}_v^i\) the set of the edges of \(\mathcal{T}_h^v\); then the edges of \(\mathcal{T}_h\) will be \(\mathcal{E} = \{K_x \times e_v : \forall K_x \in \mathcal{T}_h^x, \forall e_v \in \mathcal{E}_v\}\). Furthermore, \(\mathcal{E}_v = \mathcal{E}_v^i \cup \mathcal{E}_v^b\) with \(\mathcal{E}_v^i\) and \(\mathcal{E}_v^b\) being the set of interior and boundary edges of \(\mathcal{T}_h^v\) respectively. In addition, we denote the mesh size of \(\mathcal{T}_h\) as \(h = \max(h_x, h_v) = \max_{K \in \mathcal{T}_h} h_K\), where \(h_x = \max_{K_x \in \mathcal{T}_h^x} h_{K_x}\) with \(h_{K_x} = \text{diam}(K_x)\), \(h_v = \max_{K_v \in \mathcal{T}_h^v} h_{K_v}\) with \(h_{K_v} = \text{diam}(K_v)\), and \(h_K = \max(h_{K_x}, h_{K_v})\) for \(K = K_x \times K_v\). When the mesh is refined, we assume both \(\frac{h_x}{\rho_{h_{K_x}}}\) and \(\frac{h_v}{\rho_{h_{K_v}}}\) are uniformly bounded from above by a positive constant \(\sigma_0\). Here \(h_{K_x, \min} = \min_{K_x \in \mathcal{T}_h^x} h_{K_x}\) and \(h_{K_v, \min} = \min_{K_v \in \mathcal{T}_h^v} h_{K_v}\). It is further assumed that \(\{\mathcal{T}_h^x\}\) is shape-regular with \(* = x\ or\ v\). That is, if \(\rho_{K_x}\) denotes the diameter of the largest sphere included in \(K_x\), there is

\[
\frac{h_{K_x}}{\rho_{K_x}} \leq \sigma_*, \quad \forall K_x \in \mathcal{T}_h^x
\]
for a positive constant \( \sigma_\star \) independent of \( h_\star \). Furthermore the inner products are defined as

\[
(g, h)_\Omega = \int_\Omega gh \, dx \, dv = \sum_{K \in \mathcal{T}_h} \int_K gh \, dx \, dv, \tag{2.1}
\]

\[
(U, W)_{\Omega_x} = \int_{\Omega_x} U \cdot W \, dx = \sum_{K_x \in \mathcal{T}_h^x} \int_{K_x} U \cdot W \, dx. \tag{2.2}
\]

Now for \( g \in L^2(\Omega) \), \( U, W \in (L^2(\Omega_x))^{d_x} \), we define the \( L^2 \)-norm of \((g, U, W)\) as

\[
\| (g, U, W) \|_{0, \Omega} = \sqrt{\| g \|_{0, \Omega}^2 + \| U \|_{0, \Omega_x}^2 + \| W \|_{0, \Omega_x}^2} \tag{2.3}
\]

This will be helpful in the error analysis of the negative-order norm. The negative-order norm is defined as: given \( l > 0 \) and domain \( \Omega \),

\[
\| (g, U, W) \|_{-l, \Omega} = \sup_{\phi \in C_0^\infty(\Omega), \mathcal{U}, \mathcal{W} \in [C_0^\infty(\Omega_x)]^{d_x}} \frac{(g, \phi)_\Omega + (U, \mathcal{U})_{\Omega_x} + (W, \mathcal{W})_{\Omega_x}}{\sqrt{\| \phi \|_{l, \Omega}^2 + \| \mathcal{U} \|_{l, \Omega_x}^2 + \| \mathcal{W} \|_{l, \Omega_x}^2}}
\]

Next we define the discrete spaces

\[
\mathcal{P}_h^k = \left\{ g \in L^2(\Omega) : g|_{K=K_x \times K_v} \in P^k(K_x \times K_v), \forall K_x \in \mathcal{T}_h^x, \forall K_v \in \mathcal{T}_h^v \right\}
\]

\[
= \left\{ g \in L^2(\Omega) : g|_{K} \in P^k(K), \forall K \in \mathcal{T}_h \right\}, \tag{2.4}
\]

\[
\mathcal{V}_h^l = \left\{ U \in \left[L^2(\Omega_x)\right]^{d_x} : U|_{K_x} \in \left[P^r(K_x)\right]^{d_x}, \forall K_x \in \mathcal{T}_h^x \right\}, \tag{2.5}
\]

where \( P^r(D) \) denotes the set of polynomials of total degree at most \( r \) on \( D \), and \( k \) and \( r \) are nonnegative integers.

For piecewise functions defined with respect to \( \mathcal{T}_h^x \) or \( \mathcal{T}_h^v \), we further introduce the jumps and averages as follows. For any edge \( e = \{ K^+_x \cap K^-_x \} \in \mathcal{E}_x \), with \( n_x^\pm \) as the outward unit normal to \( K_x \), \( g^\pm = g|_{K_x^\pm} \) and \( U^\pm = U|_{K_x^\pm} \), the jump across \( e \) are defined as

\[
[g]_x = g^+_x n_x^+ + g^-_x n_x^-, \quad [U]_x = U^+ \cdot n_x^+ + U^- \cdot n_x^-,
\]

and the averages are

\[
\{g\}_x = \frac{1}{2}(g^+ + g^-), \quad \{U\}_x = \frac{1}{2}(U^+ + U^-).
\]

By replacing the subscript \( x \) with \( v \), one can define \([g]_v\), \([U]_v\), \([g]_v\), and \([U]_v\) for an interior edge of \( \mathcal{T}_h^v \) in \( \mathcal{E}_v \). For a boundary edge \( e \in \mathcal{E}_v^b \) with \( n_v \) being the outward
unit normal we use

\[ [g]_v = gn_v, \quad \{g\}_v = \frac{1}{2}g, \quad \{U\}_v = \frac{1}{2}U. \quad (2.6) \]

This is consistent with the fact that the exact solution \( f \) is compactly supported in \( v \).

Notice that to simplify notation we dropped the superscript \( \pm \) from the normal vectors \( n_v, \star = x \) or \( v \); if we are evaluating the points on \( \partial K^+_v \), \( n_v = n^+_v \), and \( n_v = n^-_v \) if we are doing it on \( \partial K^-_v \).

For convenience, we introduce some shorthand notation, \( \int_{\Omega^*} = \int_{\mathcal{T}^*_h} \), \( \int_{\Omega} = \int_{\mathcal{T}_h} \), \( \int_{\partial} = \int_{\partial \mathcal{T}_h} \), \( \int_{e} = \int_{e^*} \), where again \( \star \) is \( x \) or \( v \). In addition, \( \| g \|_{0, \mathcal{E}} = (\| g \|_{0, \mathcal{E}_x}^2 + \| g \|_{0, \mathcal{E}_v}^2)^{1/2} \), with \( \| g \|_{0, \mathcal{E}_x} = (\int_{\mathcal{E}_x} g^2 dv ds_x)^{1/2} \), \( \| g \|_{0, \mathcal{E}_v} = (\int_{\mathcal{E}_v} g^2 ds_v dx)^{1/2} \). We will make use of the following equality, which can be easily verified using the definition of averages and jumps.

\[ \frac{1}{2}[g^2]_x = [g]_x [g], \text{ with } \star = x \text{ or } v. \quad (2.7) \]

### 2.2 The DG method for the Vlasov–Maxwell system

Now we review the DG method for the VM system proposed in [11]. The scheme seeks a numerical solution \( f_h \in \mathcal{V}_h^k \) and \( (E_h, B_h) \in \mathcal{V}_h^k \times \mathcal{V}_h^k \) such that for any \( g \in \mathcal{V}_h^k \), \( U, W \in \mathcal{V}_h^k \):

\[
\begin{align*}
\int_{K} \partial_t f_h g \, dx \, dv &\quad - \int_{K} f_h v \cdot \nabla_x g \, dx \, dv - \int_{K} f_h (E_h + v \times B_h) \cdot \nabla_v g \, dx \, dv \\
&\quad + \int_{K_v} \int_{\partial K_v} \left( f_h v \cdot n_v \right)^{\wedge} g \, ds_x \, dv + \int_{K_v} \int_{\partial K_v} \left( (f_h (E_h + v \times B_h) \cdot n_v) \right)^{\wedge} g \, ds_x \, dv = 0, \quad (2.8a) \\
\int_{K} \partial_t E_h \cdot U \, dx &\quad = \int_{K} B_h \cdot \nabla_x \times U \, dx + \int_{\partial K_v} \left( n_v \times B_h \right)^{\wedge} \cdot U \, ds_x - \int_{K} J_h \cdot U \, dx, \quad (2.8b) \\
\int_{K} \partial_t B_h \cdot W \, dx &\quad = - \int_{K} E_h \cdot \nabla_x \times W \, dx - \int_{\partial K_v} \left( n_v \times E_h \right)^{\wedge} \cdot W \, ds_x \quad (2.8c)
\end{align*}
\]

with

\[
J_h(x, t) = \int_{\mathcal{T}_h^v} f_h(x, v, t) v \, dv.
\quad (2.9)
\]

Here \( n_v \) and \( n_v \) are outward unit normals of \( \partial K_x \) and \( \partial K_v \), respectively. All “hat” functions \((\cdot)^{\wedge}\) are numerical fluxes that are determined by upwinding, i.e.,

\[
(f_h v \cdot n_v)^{\wedge} = (f_h v)^{\wedge} \cdot n_v = \left( \{f_h\}_x + \frac{|v \cdot n_v|}{2} [f_h]_x \right) \cdot n_v, \quad (2.10a)
\]
Theorem 1 \([11]\) For \(k \geq 2\) when \(d_x = 3\) and \(k \geq 1\) when \(d_x = 1, 2\). If \((f, \mathbf{E}, \mathbf{B})\) is the exact solution of \((1.1)\), satisfying \(f \in C^1([0, T]; H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega))\) and \(\mathbf{E}, \mathbf{B} \in \mathcal{U}_h^k\), the error estimates of the approximations \(f_h, \mathbf{E}_h, \mathbf{B}_h\), is as follows.

\[ L_2 \text{ error estimates of the approximations } f_h, \mathbf{E}_h, \mathbf{B}_h, \text{ is as follows.} \]

The semi-discrete formulation \((2.8)\) can then be solved by a numerical ODE solver, that will be described below. The \(L^2\) and energy stability of \((2.8)\) are established in \([11]\). The main result in \([11]\) for the semi-discrete \(L^2\) error estimates of the approximations \(f_h, \mathbf{E}_h, \mathbf{B}_h\), is as follows.

\[ \text{Theorem 1} \quad (11) \quad \text{For } k \geq 2 \text{ when } d_x = 3 \text{ and } k \geq 1 \text{ when } d_x = 1, 2. \text{If } (f, \mathbf{E}, \mathbf{B}) \text{ is the exact solution of } (1.1), \text{satisfying } f \in C^1([0, T]; H^{k+1}(\Omega) \cap W^{1,\infty}(\Omega)) \text{ and } \mathbf{E}, \mathbf{B} \in \mathcal{U}_h^k, \]
Then, the semi-discrete DG method of (2.11a)–(2.11b), for the Vlasov–Maxwell equations with the upwind fluxes of (2.10a)–(2.10d), has the following error estimate

\[
\| (f - f_h(t)) \|_{0, \Omega}^2 + \| (E - E_h(t)) \|_{0, \Omega_x}^2 + \| (B - B_h(t)) \|_{0, \Omega_x}^2 \leq C h^{2k+1}, \quad \forall t \in [0, T].
\] (2.12)

Here the constant C is independent of \( h \), but depends on the upper bounds of \( \| \partial_t f \|_{k+1, \Omega}, \| f \|_{k+1, \Omega}, \| E \|_{1, \infty, \Omega_x}, \| B \|_{1, \infty, \Omega_x}, \| E \|_{k+1, \Omega_x}, \| B \|_{k+1, \Omega_x} \) over the time interval \([0, T]\), and it also depends on the polynomial degree \( k \), mesh parameters \( \sigma_0, \sigma_x \) and \( \sigma_v \), and domain parameters \( L_x \) and \( L_v \).

In this work, we also consider (1.1) when there is no magnetic field (i.e. when \( B = 0 \)). This reduced problem is called the V A system, and the DG discretizations would follow a similar discussion by setting \( B_h = 0 \) in (2.8) at all times.

2.3 Temporal discretizations

We used a total variation diminishing (TVD) high-order Runge–Kutta method to solve the method of lines ODE resulting from the semidiscrete DG scheme (2.8), \( \frac{d}{dt} G_h = R(G_h) \). Such time stepping methods are convex combinations of the Euler forward time discretization. The commonly third order TVD Runge-Kutta method is given by

\[
\begin{align*}
G_h^{(1)} &= G_h^n + \Delta t R(G_h^n), \\
G_h^{(2)} &= \frac{3}{4} G_h^n + \frac{1}{4} G_h^{(1)} + \frac{1}{4} \Delta t R(G_h^{(1)}), \\
G_h^{n+1} &= \frac{1}{3} G_h^n + \frac{2}{3} G_h^{(2)} + \frac{2}{3} \Delta t R(G_h^{(2)}),
\end{align*}
\] (2.13)

where \( G_h^n \) represents a numerical approximation of the solution at the discrete time \( t_n \). A detailed description of the TVD Runge–Kutta method can be found in [18]. We will also use the classic fourth order Runge–Kutta method.

3 Smoothness-increasing accuracy-conserving filters

We extract the higher-order accuracy of the DG method solved over a uniform mesh contained in the negative-order norm by using the SIAC filter. This technique can also be applied over nonuniform meshes, however this would force us to compute the post-processing coefficients in each element of the mesh, increasing the computational complexity of the implementation [17]. This filter improves the order of accuracy by reducing the spurious oscillations in the error. This is done by convolving the numerical approximation with a specially chosen kernel,

\[
(f_h^*(x, v), E_h^*(x), B_h^*(x)) = K_h^{2(k+1), k+1} \ast (f_h, E_h, B_h)(x, v),
\] (3.1)
where \( (f_h^*, E_h^*, B_h^*) \) is the filtered solution, \( (f_h, E_h, B_h) \) is an approximated solution computed at the final time, and \( K_h^{2(k+1),k+1} \) is the convolution kernel. The kernel is translation-invariant and composed of a linear combination of B-splines of order \( k + 1 \) obtained by convolving the characteristic function over the interval \( (-\frac{1}{2}, \frac{1}{2}) \) with itself \( k \) times and scaled by the uniform mesh size. Using B-splines makes this kernel computationally efficient, provided the mesh is uniform, as the kernel is locally supported in at most \( 2k + 2 \) elements. The one-dimensional convolution kernel is of the form:

\[
K_h^{2(k+1),k+1}(x) = \frac{1}{h} \sum_{\gamma=-k}^{k} c_{\gamma}(x) \psi^{(k+1)} \left( \frac{x}{h} - \gamma \right). \tag{3.2}
\]

The weights \( c_{\gamma}(x) \), of the B-splines \( \psi^{(k+1)} \), are chosen so that accuracy is not destroyed (the kernel can reproduce polynomials of degree up to \( 2k \)), i.e. \( K_h^{2(k+1),k+1} \) is translation-invariant and \( \phi_p = p \) for \( p = 1, x, \ldots, x^{2k} \). In practice the post-processing is implemented in a simple manner by doing small matrix vector multiplications. This involves negligible computational cost, see [14] for details.

For the general case, assume the mesh size is uniform in each direction, given arbitrary an arbitrary point \((x, v) = (x_1, \ldots, x_{d_x}, v_1, \ldots, v_{d_v}) \in \mathbb{R}^{d_x+d_v}, \) we set

\[
\psi^{(k+1)}(x, v) = \prod_{i=1}^{d_x} \psi^{(k+1)}(x_i) \prod_{j=1}^{d_v} \psi^{(k+1)}(v_j). \tag{3.3}
\]

The kernel for our case is of the form

\[
K_h^{2(k+1),k+1}(x, v) = \frac{1}{\left( \prod_{i=1}^{d_x} h_{x_i} \right) \left( \prod_{j=1}^{d_v} h_{v_j} \right)} \sum_{\gamma \in [-k, \ldots, k]} c_{\gamma}^{2(k+1),k+1} \psi^{(k+1)} \left( \frac{x_1}{h_{x_1}}, \ldots, \frac{x_{d_x}}{h_{x_{d_x}}}, \frac{v_1}{h_{v_1}}, \ldots, \frac{v_{d_v}}{h_{v_{d_v}}} \right) - \gamma, \tag{3.4}
\]

where \( h_{x_i} \) and \( h_{v_i} \) denote the mesh size in \( x_i \) and \( v_i \) direction, resp. The success of the filter relies on the following results.

**Theorem 2** (Bramble and Schatz [4]) For \( T > 0 \), let \( u = (f, E, B) \) be the exact solution of problem \( (1.1) \), satisfying \( f \in L^\infty([0, T]; W^{2k+2,\infty}(\Omega)) \cap L^2([0, T]; H^{k+2}(\Omega)) \) and \( E, B \in L^\infty([0, T]; [W^{2k+2,\infty}(\Omega_x)]^{d_x}) \cap L^2([0, T]; [H^{k+2}(\Omega_x)]^{d_x}). \) Let \( \Omega_0 + 2\text{supp}(K_h^{2(k+1),k+1}(x, v)) \subset \Omega \) and \( U = (f_h, E_h, B_h), \) then

\[
\|u(T) - K_h^{2(k+1),k+1} \text{ } \phi \|_{0, \Omega_0} \leq \frac{h^{2k+2}}{(2k + 2)!} |u|_{2k+2,\Omega} + C_P \sum_{|\lambda| \leq k+1} \| \partial_{h}^{\lambda} (u - \phi) \|_{-(k+1),\Omega}. \tag{3.5}
\]
where \( C_P \) depends solely on \( \Omega_0, \Omega, d_x, d_v, k, c_\gamma^{2(k+1),k+1} \), and it is independent of \( h \).

In (3.5), we used the notation of the divided differences, which are defined as

\[
\partial_{h_{x_i}} w(x, v) = \frac{1}{h_{x_i}} (w(x + \frac{1}{2} h_{x_i} e_i, v) - w(x - \frac{1}{2} h_{x_i} e_i, v)),
\]

where \( e_i \) is the unit multi-index whose \( i \)-th component is 1 and all others 0. Analogously, for velocity space variables \( v_j \), the difference quotients are defined as

\[
\partial_{h_{v_j}} w(x, v) = \frac{1}{h_{v_j}} (w(x, v + \frac{1}{2} h_{v_j} e_j) - w(x, v - \frac{1}{2} h_{v_j} e_j)).
\]

For any multi-index \( \lambda = (\alpha_{x_1}, \ldots, \alpha_{d_x}, \beta_{v_1}, \ldots, \beta_{d_v}) \) we set the \( \alpha \)-th order difference quotient to be

\[
\partial_{\lambda} w(x, v) = (\partial_{\alpha_{x_1}}^{\lambda_{x_1}} \ldots \partial_{\alpha_{d_x}}^{\lambda_{d_x}})(\partial_{\beta_{v_1}}^{\lambda_{v_1}} \ldots \partial_{\beta_{d_v}}^{\lambda_{d_v}}) w(x, v).
\]

### 4 Superconvergent error estimates for the DG method

In this section, we prove the superconvergence error estimate in the negative-order norm of the DG solution for the VM system. In Sect. 4.1, we review the basic approximation and regularity properties. In Sect. 4.2 we will construct the dual problem which is the key to our estimates. The main result and the proof will be given in Sect. 4.3.

#### 4.1 Preliminaries

We summarize some of the standard approximation properties of the above discrete spaces, as well as some inverse inequalities [13]. For any nonnegative integer \( k \) that we use to represent the polynomial degree, Let \( \Pi^k \) be the \( L^2 \) projection onto \( \mathcal{V}_h^k \), and \( \Pi^m \) be the \( L^2 \) projection onto \( \mathcal{W}_h^m \). We define \( \zeta^g_h = \Pi^k g - g \) and \( \zeta^U_h = \Pi^k U - U \), as the Projection errors of \( g \) and \( U \) respectively.

**Lemma 3** (Approximation properties) There exist a constant \( C > 0 \), such that for any \( g \in H^{k+1}(\Omega) \) and \( U \in [H^{k+1}(\Omega)]^{d_x} \), the following hold:

\[
\| \zeta^g_h \|_{0,K} + h_K \| \nabla \cdot \zeta^g_h \|_{0,K} + h_K^{1/2} \| \zeta^g_h \|_{0,aK} \leq C h_K^{k+1} \| g \|_{k+1,K}, \quad \forall K \in \mathcal{T}_h
\]

\[
\| \zeta^U_h \|_{0,K_x} + h_{K_x} \| \nabla_x \cdot \zeta^U_h \|_{0,K_x} + h_{K_x}^{1/2} \| \zeta^U_h \|_{0,aK_x} \leq C h_{K_x}^{k+1} \| U \|_{k+1,K_x}, \quad \forall K_x \in \mathcal{T}_{h_x}
\]

\[
\| \zeta^U_h \|_{0,\infty,K_x} \leq C h_{K_x}^{k+1} \| U \|_{k+1,\infty,K_x}, \quad \forall K_x \in \mathcal{T}_{h_x}
\]

where the constant \( C \) is independent of the mesh sizes \( h_K \) and \( h_{K_x} \), but depends on \( k \) and the shape regularity parameters \( \sigma_x \) and \( \sigma_v \) of the mesh.
Lemma 4 (Inverse inequality) There exists a constant $C > 0$, such that for any $g \in P_k(K)$ or $P_k(K_x) \times P_k(K_v)$ with $K = (K_x \times K_v) \in \mathcal{T}_h$, and for any $U \in [P_k(K_x)]^{d_x}$, the following hold:

$$
\| \nabla_x g \|_{0,K} \leq C h_{K_x}^{-1} \| g \|_{0,K}, \quad \| \nabla_v g \|_{0,K} \leq C h_{K_v}^{-1} \| g \|_{0,K},
$$

$$
\| U \|_{0,\infty,K_x} \leq C h_{K_x}^{-d_x/2} \| U \|_{0,K_x}, \quad \| U \|_{0,\partial K_x} \leq C h_{K_x}^{-1/2} \| U \|_{0,K_x},
$$

where the constant $C$ is independent of the mesh sizes $h_{K_x}, h_{K_v}$, but depends on polynomial degree $k$ and the shape regularity parameters $\sigma_x$ and $\sigma_v$ of the mesh.

To assist the proof, we also need a regularity result for a linear PDE system.

Lemma 5 Consider the following system of equations with periodic boundary conditions in $x$ and zero boundary condition in $v$ for all $t \in [0, T]$:

\[
\begin{align*}
\partial_t \varphi + A_1(x, v, t) \cdot \nabla_x \varphi + A_2(x, v, t) \cdot \nabla_v \varphi + A_3(x, v, t) \cdot F &= 0, \quad (4.1a) \\
\partial_t F &= \nabla_x \times D + \int_{\Omega_v} g \nabla_v \varphi \, dv, \quad (4.1b) \\
\partial_t D &= -\nabla_x \times F - \int_{\Omega_v} g (v \times \nabla_v \varphi) \, dv, \quad (4.1c)
\end{align*}
\]

where the given functions $A_1, A_2 \in W^{l+1,\infty}(\Omega)$ satisfy the divergence free constraint $\nabla_x \cdot A_1 = 0$ and $\nabla_v \cdot A_2 = 0$. For any $l \geq 0$ and the fixed time $t$, the solution to (4.1) satisfy the following estimate

\[
\begin{align*}
\| \varphi(\cdot, t) \|_{l,\Omega}^2 + \| F(\cdot, t) \|_{l,\Omega_x}^2 + \| D(\cdot, t) \|_{l,\Omega_x}^2 \\
&\leq C \left[ \| \varphi(\cdot, 0) \|_{l,\Omega}^2 + \| F(\cdot, 0) \|_{l,\Omega_x}^2 + \| D(\cdot, 0) \|_{l,\Omega_x}^2 \right].
\end{align*}
\]

(4.2)

Here $C$ depends on $\| A_3 \|_{L^\infty((0,T); W^{l,\infty}(\Omega))}$ and $\| g \|_{L^\infty((0,T); W^{l+1,\infty}(\Omega))}$.

Proof By using equation (4.1a), the divergence free properties of $A_1, A_2$ and the boundary conditions, we have the following:

\[
\frac{1}{2} \frac{d}{dt} \| \varphi \|^2 = -\int_{\Omega} (A_3 \cdot F) \varphi \, dxdv \leq C (\| \varphi \|^2 + \| F \|^2),
\]

where $C$ depends on $\| A_3 \|_{L^\infty((0,T); L^\infty(\Omega))}$. On the other hand using Eqs. (4.1b) and (4.1c), as well as Gauss theorem on the physical space integrals and Green’s theorem on the velocity space variables, we have

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| F \|^2 + \frac{1}{2} \frac{d}{dt} \| D \|^2 &= \int_{\Omega_x} (\nabla_x \times D \cdot F - \nabla_x \times F \cdot D) \, dx - \int_{\Omega_v} \varphi \nabla_v g \cdot F \, dxdv \\
&\quad + \int_{\Omega_v} \varphi (v \times \nabla g) D \, dxdv
\end{align*}
\]
\[
\begin{align*}
&= -\int_\Omega \nabla \cdot g \cdot F \, dx \, dv + \int_\Omega \varphi(v \times \nabla g) D \, dx \, dv \\
&\leq C \left( \|F\|^2 + \|D\|^2 + \|\varphi\|^2 \right),
\end{align*}
\]

where \(C\) depends on \(\|g\|_{L^\infty((0,T);L^\infty(\Omega))}\).

Now we add the two inequalities above, to obtain
\[
\frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \frac{1}{2} \frac{d}{dt} \|F\|^2 + \frac{1}{2} \frac{d}{dt} \|D\|^2 \leq C \left( \|F\|^2 + \|D\|^2 + \|\varphi\|^2 \right),
\]

where \(C\) depends on \(\|A_3\|_{L^\infty((0,T);L^\infty(\Omega))}\) and \(\|g\|_{L^\infty((0,T);L^\infty(\Omega))}\). An application of Gronwall’s inequality is used to obtain the conclusion. Now since we are considering the full Sobolev norm, we still need to estimate the higher-order derivatives \(\partial^\alpha_x \partial^\gamma_t\), to do so we apply \(\partial^\alpha_x \partial^\gamma_t\) to the system (4.1) and then we repeat the same steps taken above. \(\square\)

### 4.2 The dual problem

In order to prove negative-order estimates for the system, the key is to find the dual problem associated to (1.1). We note that, for the nonlinear problem, the dual problem is not unique, see [26]. We construct the dual problem as follows: find functions \(\varphi(\cdot, \cdot, t), F(\cdot, t)\) and \(D(\cdot, t)\) such that \(\varphi(\cdot, v, t)\) is periodic in all dimensions in space and \(\varphi(x, \cdot, t)\) vanishes in the boundary of the velocity region for all \(t \in [0, T]\) and

\[
\begin{align*}
\partial_t \varphi + v \cdot \nabla \varphi + (E + v \times B) \cdot \nabla \varphi - v \cdot F &= 0, \quad (4.4a) \\
\partial_t F &= \nabla \times D - \int_{\Omega_v} f \nabla \varphi \, dv, \quad (4.4b) \\
\partial_t D &= -\nabla \times F + \int_{\Omega_v} f (v \times \nabla \varphi) \, dv, \quad (4.4c)
\end{align*}
\]

with final time conditions \(\varphi(x, v, T) = \Phi(x), F(x, T) = \Phi(x)\) and \(D(x, T) = \Phi(x)\), \(\Phi \in C_0^\infty(\Omega)\) and \(D, \Phi \in \left[C_0^\infty(\Omega_x)\right]^{d_x}\).

Notice that by multiplying \((\varphi, F, D)\) on both sides of (1.1a)–(1.1b), and multiplying by \((f, E, B)\) on both sides of (4.4a)–(4.4c), and then summing up and integrating over velocity and physical space, we obtain

\[
\begin{align*}
\int_\Omega \partial_t (f \varphi) \, dx \, dv + \int_\Omega \nabla \cdot (f \varphi v) \, dx \, dv + \int_\Omega \nabla \cdot (f \varphi(E + v \times B)) \, dx \, dv \\
- \int_\Omega f v \cdot F \, dx \, dv &= 0, \\
\int_{\Omega_v} \partial_t (E \cdot F + B \cdot D) \, dx + \int_\Omega f v \cdot F \, dx \, dv \\
= \int_{\Omega_v} \nabla \cdot (B \times F) \, dx + \int_{\Omega_v} \nabla \cdot (E \times D) \, dx - \int_\Omega f (E + v \times B) \cdot \nabla \varphi \, dx \, dv,
\end{align*}
\]
where we used the identities
\[
\nabla_\ast (\phi U) = \phi \nabla_\ast U + U \cdot \nabla_\ast \phi,
\n\nabla_\ast (U \times W) = W \cdot (\nabla_\ast U) - U \cdot (\nabla_\ast W),
\]
for scalar functions \( \phi \) and vector functions \( U \) and \( W \) and the fact that \( \nabla_\ast (E + v \times B) = 0 \).

By adding all equations above and using boundary conditions, we arrive at
\[
\frac{d}{dt} [(f, \varphi)_{\Omega} + (E, F)_{\Omega_x} + (B, D)_{\Omega_x}] + \mathcal{P}(f, E, B; \varphi) = 0, \tag{4.5}
\]
where
\[
\mathcal{P}(f, E, B; \varphi) = \int_\Omega f(E + v \times B) \cdot \nabla \varphi \, dx \, dv.
\tag{4.6}
\]

### 4.3 The main result

In this subsection, we give our main theorem on the negative-order norm of the error for the DG solution. Note that superconvergence of the negative-order norm of the solution itself is not sufficient to prove high order convergence of the post-processed solution according to Theorem 2. However, it is a necessary first step. As shown in [27], it is highly nontrivial to prove superconvergence of the divided difference of the solution for nonlinear problems, we will leave this to explore in our future work.

**Theorem 6** Let \( (f, E, B) \) be the exact solution to (1.1), and let us assume it satisfies \( f \in C^1([0, T]; H^{k+2}(\Omega) \cap W^{1,\infty}(\Omega)) \cap L^2([0, T]; H^{k+2}(\Omega)) \) and \( E, B \in C^0([0, T]; [H^{k+2}(\Omega_x)]^{d_x} \cap [W^{1,\infty}(\Omega_x)]^{d_x}) \cap L^2([0, T]; [H^{k+2}(\Omega_x)]^{d_x}) \). If \( (f_h, E_h, B_h) \) is a solution to (2.11a)–(2.11b) with the numerical initial condition \( f_h = \Pi_x^k f \) and \( E_h = \Pi_x^k E, B_h = \Pi_x^k B \) and \( k \geq (d_x + d_v)/2 \), then
\[
\| (f - f_h, E - E_h, B - B_h) \| - (k+1), \Omega \leq C h^{2k+1/2},
\]
where \( C \) is a constant independent of \( h \) and depends on the upper bounds of \( \| \partial_t f \|_{k+2, \Omega}, \| f \|_{k+2, \Omega}, \| f \|_{1, \infty, \Omega_x}, \| E \|_{1, \infty, \Omega_x}, \| B \|_{1, \infty, \Omega_x}, \| B \|_{k+2, \Omega_x}, \| B \|_{k+2, \Omega_x} \) over the time interval \([0, T]\). It also depends on the polynomial degree \( k \), mesh parameters \( \sigma_0, \sigma_x \), and \( \sigma_v \), and domain parameters \( L_x \) and \( L_v \).

**Proof** We define \( e_h^f = f - f_h = e_h^E - e_h^B \), where \( e_h^E = \Pi_x^k f - f_h \) and \( e_h^B \) is defined just as in Sect. 4.1. Notice that \( e_h^f \) is not an exponential. Analogously \( e_h^E = \Pi_x^k E - E_h, e_h^B = \Pi_x^k B - B_h, \) then \( e_h^E = E - E_h = e_h^E - e_h^B \) and \( e_h^B = B - B_h = e_h^B - e_h^E \). We follow the ideas in [14]. For any \( \Phi \in C_0^\infty(\Omega), \Psi, \Omega \in [C_0^\infty(\Omega_x)]^{d_x} \), we estimate the term
\[
(e_h^f(T), \Phi)_{\Omega} + (e_h^E(T), \Psi)_{\Omega_x} + (e_h^B(T), \Omega)_{\Omega_x}
\]
where for the first equality we used (4.5), and the numerical initial condition is used in the last equality. Notice that for any \( \chi \in \mathcal{G}_h, \xi, \eta \in \mathcal{Y}_h \)

\[
\begin{align*}
&\int_0^T ((f_h)_t, \varphi)_{\Omega} + ((E_h)_t, F)_{\Omega_x} + ((B_h)_t, D)_{\Omega_x} \, d\tau \\
&= \int_0^T ((f_h)_t, \varphi - \chi)_{\Omega} \, d\tau \\
&\quad + \int_0^T ((f_h)_t, \chi)_{\Omega} \, d\tau + \int_0^T ((E_h)_t, F - \xi)_{\Omega_x} + ((B_h)_t, D - \eta)_{\Omega_x} \, d\tau \\
&\quad + \int_0^T ((E_h)_t, \xi)_{\Omega_x} + ((B_h)_t, \eta)_{\Omega_x} \, d\tau \\
&= \int_0^T ((f_h)_t, \varphi - \chi)_{\Omega} \, d\tau - \int_0^T a_h(f_h, E_h, B_h; \chi) \, d\tau \\
&\quad + \int_0^T ((E_h)_t, F - \xi)_{\Omega_x} + ((B_h)_t, D - \eta)_{\Omega_x} \, d\tau \\
&\quad - \int_0^T b_h(E_h, B_h; \xi, \eta) - l_h(J_h, \xi) \, d\tau \\
&= \int_0^T ((f_h)_t, \varphi - \chi)_{\Omega} + a_h(f_h, E_h, B_h; \varphi - \chi) \, d\tau \\
&\quad + \int_0^T ((E_h)_t, F - \xi)_{\Omega_x} + ((B_h)_t, D - \eta)_{\Omega_x} \, d\tau \\
&\quad + \int_0^T b_h(E_h, B_h; F - \xi, D - \eta) - l_h(J_h, F - \xi) \, d\tau - \int_0^T a_h(f_h, E_h, B_h; \varphi) \, d\tau
\end{align*}
\]
\[- \int_0^T b_h(E_h, B_h; F, D) - l_h(J_h, F) \, d\tau.\]

After this calculation we can conclude that
\[
(e^f_h(T), \Phi)_{\Omega} + (e^E_h(T), \xi)_{\Omega_x} + (e^B_h(T), \Omega)_{\Omega_x} = \Theta_M + \Theta_N + \Theta_C, \quad (4.7)
\]

where
\[
\Theta_M = - \left[ (\xi^f_{h0}, \varphi(0))_{\Omega} + (\xi^E_{h0}, F(0))_{\Omega_x} + (\xi^B_{h0}, D(0))_{\Omega_x} \right],
\]
\[
\Theta_N = - \int_0^T ((f_h), \varphi - \chi)_{\Omega} + a_h(f_h, E_h, B_h; \varphi - \chi) \, d\tau
- \int_0^T ((E_h)_t, F - \xi)_{\Omega_x} + ((B_h)_t, D - \eta)_{\Omega_x} + b_h(E_h, B_h; F - \xi, D - \eta)
- l_h(J_h, F) \, d\tau,
\]
\[
\Theta_C = - \int_0^T (f_h, \varphi_t)_{\Omega} - a_h(f_h, E_h, B_h; \varphi) \, d\tau
- \int_0^T (E_h, F)_{\Omega_x} + (B_h, D)_{\Omega_x} - b_h(E_h, B_h; F, D) + l_h(J_h, F) \, d\tau
- \int_0^T \mathcal{F}(f, E, B; \varphi) \, d\tau.
\]

In the following we will estimate \( \Theta_M, \Theta_N \) and \( \Theta_C \).

**Lemma 7** (Projection Estimate) Assume that the same assumptions hold as in Theorem 6, then \( \Theta_M \) satisfies
\[
|\Theta_M| \leq C h^{k+2} \sqrt{\|\varphi(0)\|^2_{k+1, \Omega} + \|F(0)\|^2_{k+1, \Omega_x} + \|D(0)\|^2_{k+1, \Omega_x}}, \quad (4.9)
\]
where \( C \) depends on \( \|f_0\|_{k+1, \Omega}, \|E_0\|_{k+1, \Omega_x} \) and \( \|B_0\|_{k+1, \Omega_x} \).

**Proof** By the definition of \( \Pi^k \),
\[
(f_0 - \Pi^k f_0, \varphi(0))_{\Omega} = (f_0 - \Pi^k f_0, \varphi(0) - \Pi^k \varphi(0))_{\Omega}
\leq \|f_0 - \Pi^k f_0\| \|\varphi(0) - \Pi^k \varphi(0)\|
\leq C h^{k+1} \|f_0\|_{k+1, \Omega} h^{k+1} \|\varphi(0)\|_{k+1, \Omega}.
\]

The last line was obtained by applying the first part of Lemma 3. Using the same reasoning, we obtain analogous results for the \( \mathbf{E} \) and \( \mathbf{B} \). The conclusion follows by grouping them all together and an application of Cauchy-Schwarz inequality. \( \square \)

For the second term, we have the following result:
Lemma 8 (Residual) Let $\chi = \Pi^k f$, $\xi = \Pi^k F$, $\eta = \Pi^k D$ and assume that the same assumptions hold as in Theorem 6, then $\Theta_N$ satisfies

$$|\Theta_N| \leq C h^{2k+1/2} \left[ \int_0^T \varphi \|_{k+1,\Omega}^2 + \|F\|_{k+1,\Omega_x}^2 + \|D\|_{k+1,\Omega_x}^2 \, \tau \right]^{1/2},$$

where $C$ depends on the upper bounds of $\|f\|_{k+2,\Omega}$, $\|f\|_{1,\infty,\Omega}$, $\|\mathbf{E}\|_{0,\infty,\Omega_x}$, $\|\mathbf{B}\|_{0,\infty,\Omega_x}$, $\|\mathbf{E}\|_{k+2,\Omega_x}$, $\|\mathbf{B}\|_{k+2,\Omega_x}$ over the time interval $[0, T]$, and it also depends on the polynomial degree $k$, mesh parameters $\sigma_0$, $\sigma_x$ and $\sigma_v$, and domain parameters $L_x$ and $L_v$.

Proof Due to the definition of the projection operators, $((\mathbf{f}_h)^t, \varphi - \chi)_{\Omega} = 0$, $((\mathbf{E}_h)^t, \mathbf{F} - \xi)_{\Omega_x} = 0$, and $((\mathbf{B}_h)^t, \mathbf{D} - \eta)_{\Omega_x} = 0$, and $l_h (\mathbf{J}_h; \mathbf{F} - \xi) = -(\mathbf{J}_h, \mathbf{F} - \xi)_{\Omega_x} = 0$, we have

$$\Theta_N = \int_0^T -a_h (\mathbf{f}_h, \mathbf{E}_h, \mathbf{B}_h; \xi_h^{\varphi}) - b_h (\mathbf{E}_h, \mathbf{B}_h; \xi_h^{\mathbf{F}} - \xi_h^{\mathbf{D}}) \, d\tau.$$ 

From its definition,

$$b_h (\mathbf{E}_h, \mathbf{B}_h; \xi_h^{\mathbf{F}} - \xi_h^{\mathbf{D}}) = \int_{\mathcal{T}_h} \mathbf{E}_h \cdot \nabla \chi \times \xi_h^{\mathbf{D}} \, dx - \int_{\mathcal{T}_h} \mathbf{B}_h \cdot \nabla \chi \times \xi_h^{\mathbf{F}} \, dx$$

$$+ \int_{\mathcal{E}_x} (\mathbf{E}_h)^{\mathbf{F}} \cdot [\xi_h^{\mathbf{D}}]_\tau \, ds_x - \int_{\mathcal{E}_x} (\mathbf{B}_h)^{\mathbf{D}} \cdot [\xi_h^{\mathbf{F}}]_\tau \, ds_x$$

$$= -\int_{\mathcal{T}_h} e_h^{\mathbf{F}} \cdot \nabla \chi \times \xi_h^{\mathbf{D}} \, dx + \int_{\mathcal{T}_h} e_h^{\mathbf{B}} \cdot \nabla \chi \times \xi_h^{\mathbf{F}} \, dx$$

$$- \int_{\mathcal{E}_x} (e_h^{\mathbf{E}})^{\mathbf{F}} \cdot [\xi_h^{\mathbf{D}}]_\tau \, ds_x + \int_{\mathcal{E}_x} (e_h^{\mathbf{B}})^{\mathbf{D}} \cdot [\xi_h^{\mathbf{F}}]_\tau \, ds_x$$

$$+ \int_{\mathcal{T}_h} (\nabla \chi \times \mathbf{E}) \cdot \xi_h^{\mathbf{D}} \, dx - \int_{\mathcal{T}_h} (\nabla \chi \times \mathbf{B}) \cdot \xi_h^{\mathbf{F}} \, dx.$$ 

By Lemma 3,

$$\left| \int_{\mathcal{T}_h} (e_h^{\mathbf{E}})^{\mathbf{F}} \cdot \nabla \chi \times \xi_h^{\mathbf{D}} \, dx \right| \leq C h^k \|e_h^{\mathbf{E}}\|_{0,\Omega_x} \|\mathbf{D}\|_{k+1,\Omega_x},$$

$$\left| \int_{\mathcal{T}_h} e_h^{\mathbf{B}} \cdot \nabla \chi \times \xi_h^{\mathbf{F}} \, dx \right| \leq C h^k \|e_h^{\mathbf{B}}\|_{0,\Omega_x} \|\mathbf{F}\|_{k+1,\Omega_x},$$

$$\left| \int_{\mathcal{E}_x} (e_h^{\mathbf{E}})^{\mathbf{F}} \cdot [\xi_h^{\mathbf{D}}]_\tau - (e_h^{\mathbf{B}})^{\mathbf{D}} \cdot [\xi_h^{\mathbf{F}}]_\tau \, ds_x \right|$$

$$\leq C h^{k+1/2} \left( \|\mathbf{D}\|_{k+1,\Omega_x} + \|\mathbf{F}\|_{k+1,\Omega_x} \right) \left( \|e_h^{\mathbf{E}}\|_{0,\mathcal{E}_x} + \|e_h^{\mathbf{B}}\|_{0,\mathcal{E}_x} \right).$$
Now notice that
\[
\|e_h^E\|_{0, \Omega_x} \leq \|e_h^E\|_{0, \Omega_x} + \|s_h^E\|_{0, \Omega_x} \\
\leq C [h^{-1/2} \|e_h\|_{0, \Omega_x} + h^{k+1/2}] \\
\leq C h^{-1/2} [\|e_h\|_{0, \Omega_x} + h^{k+1}].
\]

Analogously
\[
\|e_h^B\|_{0, \Omega_x} \leq C h^{-1/2} [\|e_h^B\|_{0, \Omega_x} + h^{k+1}].
\]

Therefore,
\[
\left| \int_{\Omega_x} (e_h^E)^\sim \cdot [\zeta_h^D] - (e_h^B)^\sim \cdot [\zeta_h^F] \, ds_x \right| \\
\leq C h^k \left( \|D\|_{k+1, \Omega_x} + \|F\|_{k+1, \Omega_x} \right) \left( \|e_h^E\|_{0, \Omega_x} + \|e_h^B\|_{0, \Omega_x} + h^{k+1} \right).
\]

Now by the properties of the orthogonal projection $\Pi_h^k$
\[
\left| \int_{\Omega_x} (\nabla_x \times E) \cdot \zeta_h^D \, dx \right| = \left| \int_{\Omega_x} (\nabla_x \times E - \Pi_h^k(\nabla_x \times E)) \cdot \zeta_h^D \, dx \right| \leq C h^{2k+2} \|D\|_{k+1, \Omega_x},
\]
where $C$ depends on $\|E\|_{k+2, \Omega_x}$. By an analogous procedure
\[
\left| \int_{\Omega_x} (\nabla_x \times B) \cdot \zeta_h^F \, dx \right| \leq C h^{2k+2} \|F\|_{k+1, \Omega_x},
\]
where $C$ depends on $\|B\|_{k+2, \Omega_x}$. Putting all the above calculations together, we arrive at,
\[
|b_h(E_h, B_h; \zeta_h^F, \zeta_h^D)| \leq C h^k \left( \|D\|_{k+1, \Omega_x} + \|F\|_{k+1, \Omega_x} \right) \left( \|e_h^E\|_{0, \Omega_x} + \|e_h^B\|_{0, \Omega_x} + h^{k+1} \right),
\]
where $C$ depends on $\|E\|_{k+2, \Omega_x}, \|B\|_{k+2, \Omega_x}$.

We will deal now with the term $a_h$, which is
\[
a_h(f_h, E_h, B_h, \zeta_h^P) = a_{h,1}(f_h, \zeta_h^P) + a_{h,2}(f_h, E_h, B_h; \zeta_h^P).
\]

First, we have
\[
a_{h,1}(f_h; \zeta_h^P) = \int_{\Omega_x} e_h^f \nabla_x \zeta_h^P \, dx \, dv + \int_{\Omega_x} (e_h^f \nabla_x \zeta_h^P) \, ds_x \, dv - \int_{\Omega_x} \nabla_x f \cdot \zeta_h^P \, dx \, dv
\]
The first term can be easily bounded, by using Lemma 3.

\[
\left| \int_{\mathcal{T}_h} e_h f \cdot \nabla x \xi_h^p \, dx d\mathbf{v} \right| \leq Ch^k ||e_h f||_{0,\Omega} ||\varphi||_{k+1,\Omega}.
\]

Similarly,

\[
\left| \int_{\mathcal{T}_h} \int_{\mathcal{E}_x} (e_h f) \sim [\xi_h^p]_x \, dx d\mathbf{v} \right| \leq C ||e_h f||_{\mathcal{T}_h} \|\xi_h^p\|_{\mathcal{T}_h} \|\varphi\|_{k+1,\Omega}
\]

\[
\leq Ch^{k+1/2} ||e_h f||_{\mathcal{T}_h} \|\xi_h^p\|_{\mathcal{T}_h} \|\varphi\|_{k+1,\Omega}
\]

\[
\leq Ch^{k+1} (||e_h f||_{0,\Omega} + h^{k+1}) \|\varphi\|_{k+1,\Omega}.
\]

For the last term notice that by the properties of the projection \(\Pi^k\) and the fact that \(\Pi^k(\nabla x f \cdot \mathbf{v})\) is a polynomial of degree \(k\),

\[
\int_{\mathcal{T}_h} \nabla x f \cdot \mathbf{v} \xi_h^p \, dx d\mathbf{v} = \int_{\mathcal{T}_h} (\nabla x f \cdot \mathbf{v} - \Pi^k(\nabla x f \cdot \mathbf{v})) \xi_h^p \, dx d\mathbf{v}
\]

\[
\leq Ch^{2k+2} \|\varphi\|_{k+1,\Omega},
\]

where \(C\) depends on \(||f||_{k+2,\Omega}\). By using all the calculations above, we can conclude that

\[
|a_{h,1}(f_h; \xi_h^p)| \leq Ch^k ||e_h f||_{0,\Omega} \|\varphi\|_{k+1,\Omega} + Ch^{2k+1} \|\varphi\|_{k+1,\Omega}, \tag{4.12}
\]

where \(C\) depends on \(||f||_{k+2,\Omega}\). To conclude our proof, we only need to bound \(a_{h,2}\). Notice that

\[
a_{h,2}(f_h, E_h, B_h, \xi_h^p) = a_{h,2}(f, E_h, B_h, \xi_h^p) - a_{h,2}(e_h f, E_h, B_h, \xi_h^p).
\]

We will get started by noting that \((f(E_h + \mathbf{v} \times B_h))') = f(E_h + \mathbf{v} \times B_h)_v = f(E_h + \mathbf{v} \times B_h), then

\[
a_{h,2}(f, E_h, B_h, \xi_h^p) = \int_{\mathcal{T}_h} f(E_h + \mathbf{v} \times B_h) \cdot \nabla \xi_h^p \, dx d\mathbf{v} + \int_{\mathcal{T}_h} \int_{\mathcal{E}_x} f(E_h + \mathbf{v} \times B_h) \cdot [\xi_h^p]_x \, dx d\mathbf{v}
\]

\[
\int_{\mathcal{T}_h} f(e_h E + \mathbf{v} \times e_h B) \cdot \nabla \xi_h^p \, dx d\mathbf{v} - \int_{\mathcal{T}_h} \int_{\mathcal{E}_x} f(e_h E + \mathbf{v} \times e_h B) \cdot [\xi_h^p]_x \, dx d\mathbf{v}
\]

\[
+ \int_{\mathcal{T}_h} \nabla \mathbf{v} \cdot (E + \mathbf{v} \times B) \xi_h^p \, dx d\mathbf{v}.
\]
We obtained the last inequality by adding and subtracting $\int_{\mathcal{B}_h} f (E + v \times B) \cdot \nabla v \xi^\varphi_h \, dxdv$, integration by parts, and the fact that $\nabla v \cdot (E + v \times B) = 0$. In this way

$$\left| \int_{\mathcal{B}_h} f (e^E_h + v \times e^B_h) \cdot \nabla v \xi^\varphi_h \, dxdv \right| \leq C h^k (\| e^E_h \|_{0, \Omega_x} + \| e^B_h \|_{0, \Omega_x}) \| \varphi \|_{k+1, \Omega}.$$ 

and

$$\left| \int_{\mathcal{B}_h} \int_{\mathcal{E}_v} f (e^E_h + v \times e^B_h) \cdot [\xi^\varphi_h]_v \, dv \, dx \right| \leq C h^{k+1/2} (\| e^E_h \|_{0, \Omega_x} + \| e^B_h \|_{0, \Omega_x}) \| \varphi \|_{k+1, \Omega}.$$ 

Last but not least by the same arguments as previous estimates

$$\int_{\mathcal{B}_h} \nabla v \cdot (E + v \times B) \xi^\varphi_h \, dxdv = \int_{\mathcal{B}_h} (\nabla v \cdot (E + v \times B) - \Pi^k v \cdot (E + v \times B)) \xi^\varphi_h \, dxdv \leq C h^{2k+2} \| \varphi \|_{k+1, \Omega},$$

where $C$ depends on $\| f \|_{k+2, \Omega}$, $\| E \|_{k+1, \Omega_x}$, $\| B \|_{k+1, \Omega_x}$. We can conclude that

$$|a_{h,2} (f, E_h, B_h; \xi^\varphi_h)| \leq C h^k (\| e^E_h \|_{0, \Omega_x} + \| e^B_h \|_{0, \Omega_x}) \| \varphi \|_{k+1, \Omega} + C h^{2k+2} \| \varphi \|_{k+(d+1)3}$$

Finally we just need to estimate

$$a_{h,2} (e^f_h, E_h, B_h; \xi^\varphi_h) = - \int_{\mathcal{B}_h} e^f_h (E_h + v \times B_h) \cdot \nabla v \xi^\varphi_h \, dxdv + \int_{\mathcal{B}_h} \int_{\mathcal{E}_v} (e^f_h (E_h + v \times B_h)) \cdot (\xi^\varphi_h) v \, dsv \, dx$$

We have

$$\left| \int_{\mathcal{B}_h} e^f_h (E_h + v \times B_h) \cdot \nabla v \xi^\varphi_h \, dxdv \right| \leq C h^k \| e^f_h \|_{0, \Omega} (\| E_h \|_{0, \Omega_x}$$

\begin{align*}
+ \| B_h \|_{0, \Omega_x} \| \varphi \|_{k+1, \Omega} & \leq C h^k \| e^f_h \|_{0, \Omega} (\| E_h \|_{0, \Omega_x} + \| B_h \|_{0, \Omega_x} + \| \Pi^k E \|_{0, \Omega_x} + \| \Pi^k B \|_{0, \Omega_x}) \| \varphi \|_{k+1, \Omega} \leq C h^{k-d_x/2} \| e^f_h \|_{0, \Omega} (\| E_h \|_{0, \Omega_x} + \| B_h \|_{0, \Omega_x}) \| \varphi \|_{k+1, \Omega} \\
+ C h^k \| e^f_h \|_{0, \Omega} \| \varphi \|_{k+1, \Omega} & \leq C h^k \| e^f_h \|_{0, \Omega} (h^{-d_x/2} \| e^f_h \|_{0, \Omega_x} + h^{k+1}) \| \varphi \|_{k+1, \Omega} \\
+ h^{d_x/2} \| B_h \|_{0, \Omega_x} + 1 \| \varphi \|_{k+1, \Omega}. 
\end{align*}$$

Here we used the fact whenever $d_x = 1, 2, 3, k+1-d_x/2 > 0$, Lemma 4 and the fact that $\Pi_x$ is bounded in any $L^p$-norm ($1 \leq p \leq \infty$) [1, 16],

$$\| \Pi_x E \|_{0, \Omega_x} \leq C \| E \|_{0, \Omega_x}, \| \Pi_x B \|_{0, \Omega_x} \leq C \| B \|_{0, \Omega_x}.$$
Finally
\[
\int_{\mathcal{T}_h} \int_{\mathcal{T}_h^c} (e^h_f (E_h + v \times B_h))^\top \cdot [e^0_h] \, dv \, dx \\
\leq Ch^{k+1/2} (\|E_h\|_{0, \infty, \Omega_\varepsilon} + \|B_h\|_{0, \infty, \Omega_\varepsilon}) \|e^f_h\|_{0, \mathcal{T}_h^c} \|v\|_{k+1, \Omega}
\leq Ch^{k+1/2} (\|E_h\|_{0, \infty, \Omega_\varepsilon} + \|B_h\|_{0, \infty, \Omega_\varepsilon}) h^{-1/2} \|e^f_h\|_{0, \mathcal{T}_h} + h^{k+1} \|f\|_{k+1, \Omega} \|v\|_{k+1, \Omega}
\leq Ch^k (\|E_h\|_{0, \infty, \Omega_\varepsilon} + \|B_h\|_{0, \infty, \Omega_\varepsilon}) (\|e^f_h\|_{0, \mathcal{T}_h} + h^{k+1} \|f\|_{k+1, \Omega} \|v\|_{k+1, \Omega})
\leq Ch^k (\|e^f_h\|_{0, \mathcal{T}_h} + h^{k+1}) (h^{-d_x/2} \|e^f_h\|_{0, \mathcal{T}_h} + h^{-d_x/2} \|e^0_h\|_{0, \mathcal{T}_h} + 1) \|v\|_{k+1, \Omega}.
\]

In this way we conclude that
\[
|a_{h, 2} (e^f_h, E_h, B_h; \xi^0_h)| \leq Ch^k (\|e^f_h\|_{0, \mathcal{T}_h} + h^{k+1}) (h^{-d_x/2} \|e^0_h\|_{0, \mathcal{T}_h} + 1) \|v\|_{k+1, \Omega}
\] (4.14)

Then by putting together (4.10), (4.12), (4.13), (4.14), and using Theorem 1, we have
\[
|a_h (f_h, E_h, B_h; \xi^0_h)| + b_h (E_h, B_h; \xi^F_h, \xi^D_h) |
\leq Ch^k (\|D\|_{k+1, \Omega} + \|F\|_{k+1, \Omega} + \|v\|_{k+1, \Omega}) \left( \|e^0_h\|_{0, \mathcal{T}_h} + \|e^0_h\|_{0, \mathcal{T}_h} + 1 \right)
+ C h^{k+1} (\|e^f_h\|_{0, \mathcal{T}_h} + h^{k+1}) (h^{-d_x/2} \|e^0_h\|_{0, \mathcal{T}_h} + h^{-d_x/2} \|e^0_h\|_{0, \mathcal{T}_h} + 1) \|v\|_{k+1, \Omega}
\leq Ch^{2k+1/2} (\|D\|_{k+1, \Omega} + \|F\|_{k+1, \Omega} + \|v\|_{k+1, \Omega}).
\]

where we have used \( k + 1/2 - d_x/2 > 0 \). An application of Cauchy-Schwarz inequality concludes the proof. \( \square \)

Lastly, we need to estimate the third term, \( \Theta_C \).

**Lemma 9** (Consistency) Assume that the same assumptions hold as in Theorem 6, then \( \Theta_C \) satisfies
\[
|\Theta_C| \leq Ch^{2k+1} \left[ \int_0^T \|\varphi\|^2_{k+1, \Omega} \, d\tau \right]^{1/2},
\] (4.15)

where \( C \) depends on the upper bounds of \( \|\partial_t f\|_{k+1, \Omega}, \|f\|_{k+1, \Omega}, |f|_{1, \infty, \Omega}, \|E\|_{1, \infty, \Omega}, \|B\|_{1, \infty, \Omega}, \|E\|_{k+1, \Omega}, \|B\|_{k+1, \Omega} \) over the time interval \([0, T]\), and it also depends on the polynomial degree \( k \), mesh parameters \( \sigma_0, \sigma_x \) and \( \sigma_v \), and domain parameters \( L_x \) and \( L_v \).

**Proof** The terms inside the integral of \( \Theta_C \) can be split in \( I + II \), where
\[
I = (f_h, \varphi)_{\Omega} - a_h (f_h, E_h, B_h; \varphi) + l_h (J_h, F)
\]
\[
II = (E_h, F_t)_{\Omega_x} + (B_h, D_t)_{\Omega_x} - b_h (E_h, B_h; F, D) + \mathcal{P} (f, E, B; \varphi)
\]
since \( \varphi \) is a smooth function, \([\varphi]_t = 0\) and \([\varphi]_v = 0\), in this way, by using (4.4a), and the definition of \( l_h \), we conclude that,

\[
I = (f_h, -v \cdot \nabla_x \varphi - (E + v \times B) \cdot \nabla_v \varphi + v \cdot F)_{\Omega} - a_h(f_h, E_h, B_h; \varphi) + l_h(J_h; F) \\
= -\int_{\partial_h} f_h v \cdot \nabla_x \varphi \, dx \, dv - \int_{\Omega} f_h (E + v \times B) \cdot \nabla_v \varphi \, dx \, dv - l_h(J_h; F) \\
+ \int_{\partial_h} f_h v \cdot \nabla_x \varphi \, dx + \int_{\Omega} f_h (E_h + v \times B_h) \cdot \nabla_v \varphi \, dx \, dv + l_h(J_h; F) \\
= -\int_{\Omega} f_h (e^E_h + v \times e^B_h) \cdot \nabla_v \varphi \, dx \, dv.
\]

On the other hand, by using (4.4b) and (4.4c), since \( F \) and \( D \) are smooth functions, \([F]_t = [D]_t = 0\), we have that

\[
II = (E_h, \nabla_x \cdot D)_{\partial_h^\times} - (B_h, \nabla_x \times F)_{\partial_h^\times} - b_h(E_h, B_h; F, D) + \mathcal{F}(f, E, B; \varphi) \\
= \int_{\Omega} f(E_h + v \times B_h) \cdot \nabla_v \varphi \, dx \, dv + \int_{\Omega} f(E + v \times B) \cdot \nabla_v \varphi \, dx \, dv \\
= \int_{\Omega} f(e^E_h + v \times e^B_h) \cdot \nabla_v \varphi \, dx \, dv.
\]

We obtain

\[
I + II = \int_{\Omega} e^f_h (e^E_h + v \times e^B_h) \cdot \nabla_v \varphi \, dx \, dv \\
\leq C \| e^f_h \|_{\Omega} (\| e^E_h \|_{\Omega}, + \| e^B_h \|_{\Omega}) \| \nabla_v \varphi \|_{\infty, \Omega} \\
\leq C \| e^f_h \|_{\Omega} (\| e^E_h \|_{\Omega} + \| e^B_h \|_{\Omega}) \| \varphi \|_{k+1, \Omega},
\]

where we used the Sobolev inequality [5], \( \| \nabla_v \varphi \|_{\infty, \Omega} \leq C \| \varphi \|_{k+1, \Omega} \), which requires \( k > (d_1 + d_2)/2 \). Using Theorem 1, we conclude the proof. \( \square \)

It is easy to transform the dual problem (4.4) to an initial value problem (4.1) by changing time \( t' = T - t \). Then, using Lemma 5, where \( A_1(x, v, t) = -v, A_2(x, v, t) = -(E + v \times B), A_3(x, v, t) = v, g = f \) and \( l = k + 1 \),

\[
\| \varphi \|_{k+1, \Omega}^2 + \| F \|_{k+1, \Omega}^2 + \| D \|_{k+1, \Omega}^2 \leq C[\| \Phi \|_{k+1, \Omega}^2 + \| \mathcal{F} \|_{k+1, \Omega}^2 + \| \mathcal{D} \|_{k+1, \Omega}^2, \]

(4.16)

where \( C \) depends on \( \| f \|_{L^\infty((0, T); W^{k+2,\infty}(\Omega))} \). Then apply Lemmas 7, 8 and 9 to obtain

\[
|(e^f_h(T), \Phi)_{\Omega} + (e^E_h(T), \mathcal{F})_{\Omega} + (e^B_h(T), \mathcal{D})_{\Omega}|.
\]
Table 1  $L^2$-errors for the numerical solution and the post-processed solution for Landau Damping using the discontinuous Galerkin method using a $P_k$ polynomial approximation space

| Mesh   | Before post-processing | After post-processing |
|--------|-------------------------|------------------------|
|        | Error $f$ | Order | Error $E$ | Order | Error $f^*$ | Order | Error $E^*$ | Order |
| $P^1$  |           |       |           |       |           |       |           |       |
| $16 \times 16$ | 1.42E-02 | – | 1.19E-02 | – | 2.28E-02 | – | 1.04E-02 | – |
| $32 \times 32$ | 6.22E-03 | 1.19 | 3.16E-03 | 1.91 | 6.16E-03 | 1.89 | 2.84E-03 | 1.88 |
| $64 \times 64$ | 1.59E-03 | 1.97 | 5.65E-04 | 2.48 | 8.74E-04 | 2.82 | 4.36E-04 | 2.70 |
| $128 \times 128$ | 4.08E-04 | 1.96 | 1.12E-04 | 2.33 | 1.10E-04 | 2.99 | 6.31E-05 | 2.79 |
| $256 \times 256$ | 1.03E-04 | 1.98 | 2.51E-05 | 2.16 | 1.37E-05 | 3.00 | 9.01E-06 | 2.81 |
| $512 \times 512$ | 2.60E-05 | 1.99 | 6.14E-06 | 2.03 | 1.71E-06 | 3.00 | 1.71E-06 | 2.39 |
| $P^2$  |           |       |           |       |           |       |           |       |
| $16 \times 16$ | 7.08E-03 | – | 1.97E-03 | – | 2.09E-03 | – | 1.88E-03 | – |
| $32 \times 32$ | 1.08E-03 | 2.71 | 1.13E-04 | 4.12 | 2.87E-03 | 2.87 | 1.08E-04 | 4.12 |
| $64 \times 64$ | 1.35E-04 | 3.00 | 6.62E-06 | 4.11 | 1.20E-04 | 4.58 | 5.15E-06 | 4.39 |
| $128 \times 128$ | 1.63E-05 | 3.04 | 6.62E-07 | 3.57 | 2.70E-06 | 5.47 | 2.04E-07 | 4.66 |
| $256 \times 256$ | 2.01E-06 | 3.03 | 6.57E-08 | 3.09 | 5.29E-06 | 5.47 | 2.04E-07 | 4.66 |
| $P^3$  |           |       |           |       |           |       |           |       |
| $16 \times 16$ | 1.73E-03 | – | 2.19E-04 | – | 2.16E-02 | – | 9.71E-05 | – |
| $32 \times 32$ | 1.52E-04 | 3.51 | 7.18E-06 | 4.93 | 2.60E-03 | 3.05 | 3.09E-06 | 4.97 |
| $64 \times 64$ | 1.06E-05 | 3.84 | 1.30E-07 | 5.79 | 5.65E-05 | 5.52 | 7.52E-08 | 5.36 |
| $128 \times 128$ | 6.45E-07 | 4.04 | 3.42E-09 | 5.25 | 3.95E-07 | 7.16 | 8.24E-10 | 6.51 |

\[
\| (f - f_h, E - E_h, B - B_h) \|_{-(k+1), \Omega} \leq C h^{2k+1/2} \sqrt{\| \Phi \|_{k+1, \Omega}^2 + \| \delta \|_{k+1, \Omega_x}^2 + \| D \|_{k+1, \Omega_x}^2}. \quad (4.17)
\]

Therefore the estimate for the zero-divided difference negative-order norm is given by

\[
\| (f - f_h, E - E_h, B - B_h) \|_{-(k+1), \Omega} \leq C h^{2k+1/2}.
\]

5 Numerical experiments

In this section, we validate our theoretical results using several numerical tests. In particular, we want to demonstrate the performance of the post-processing technique for the VA system and the VM system. We heavily use the fact that the VM (VA) system is time reversible to provide quantitative measurements of the errors. In particular, let $f(x, v, 0)$, $E(x, 0)$, $B(x, 0)$ denote the initial conditions and $f(x, v, T)$, $E(x, T)$,
We consider two classical benchmark examples.

5.1 Vlasov-Ampère examples

We consider two classical benchmark examples.

- Landau damping:

\[
f(x, v, 0) = f_M(v)(1 + A \cos(kx)), \quad x \in [0, L], v \in [-V_c, V_c],
\]

where \( A = 0.5, k = 0.5, L = 4\pi, V_c = 6\pi, \) and \( f_M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}. \)

- Two-stream instability:

\[
f(x, v, 0) = f_{TS}(v)(1 + A \cos(kx)), \quad x \in [0, L], v \in [-V_c, V_c],
\]

where \( A = 0.05, k = 0.5, L = 4\pi, V_c = 6\pi, \) and \( f_{TS}(v) = \frac{1}{\sqrt{2\pi}} v^2 e^{-v^2/2}. \)

Notice that in both examples we have taken \( V_c \) to be larger than the usual values in the literature in order to completely eliminate the boundary effects and accurately reflect the accuracy enhancement property.

In Table 1, we run the VA system with the initial condition for Landau damping to \( T = 1 \) and then back to \( T = 0 \), then we apply the SIAC filter, and compare it with the initial conditions. We use the third order TVD-RK method (2.13) as the time integrator. To have a clean presentation of the numerical experiments we make the following change of notation for the mesh sizes: \( h_x = \Delta x \) and \( h_v = \Delta v. \) To ensure the spatial error dominates, we take \( \Delta t = CFL/(V_c/\Delta x + E_{\max}/\Delta v) \) for \( \mathbb{P}^1, \) \( E_{\max} \) denotes the maximum value of \( E(\cdot, T) \) in \( \Omega_x, \) for \( \mathbb{P}^2 \) we take \( \Delta t = CFL/(V_c/(\Delta x)^{5/3} + E_{\max}/(\Delta v)^{3/3}), \) and \( \Delta t = CFL/(V_c/(\Delta x)^{7/3} + E_{\max}/(\Delta v)^{7/3}) \) for \( \mathbb{P}^3. \) For \( \mathbb{P}^1 \) and \( \mathbb{P}^2 \) we take the CFL = 0.1, and we take the CFL = 0.2 for \( \mathbb{P}^2. \) From the table, we observe \((k+1)\)-th order of convergence for the DG solution before post-processing for both \( f \) and \( E. \) We can clearly see that we improve the order of the error to at least \( O(h^{2k+1/2}) \) after post-processing.

In Fig. 1 we plot the errors of the numerical solution before and after post-processing for \( \mathbb{P}^1 \) using 128 \times 128 elements. We can see that the errors before post-processing are highly oscillatory, and that the post-processing smooths out the error surface and greatly reduces its magnitude. In Fig. 2, we plot the errors of the approximations for \( E \) obtained when solving using a 128 \times 128 mesh with \( \mathbb{P}^1 \) and 32 \times 32 mesh with \( \mathbb{P}^3. \) We can clearly see that the errors before post-processing are highly oscillatory, and the post-processing gets rid of the oscillations and dramatically reduces the magnitude of the error. Another point that we want to make is the following: if we look at Table 1, for \( k = 2 \) and a mesh of 64 \times 64, the \( L^2 \)-errors before and after post-processing are similar in magnitude. However, if we look at Fig. 3 which plots the absolute value of the error in \( f \) in this case, we can clearly see that the \( L^\infty \)-norm of the error of the filtered solution is much smaller than the unfiltered solution. Therefore, by removing
Fig. 1 Errors for $f$ before (on the left) and after post-processing (on the right) for $128 \times 128$ elements and $P^1$. Landau damping.

Fig. 2 Errors before (solid line) and after post-processing (dashed line) for $E$ for different mesh sizes and $P^k$. Landau damping. $T = 2$.

Fig. 3 Absolute value of errors for $f$ before (on the left) and after post-processing (on the right) for $64 \times 64$ elements and $P^2$. Landau damping.

the spurious oscillations, even if the $L^2$-error is comparable, the $L^\infty$ error is further reduced by the post-processor. This is probably due to the high oscillatory nature of the solution.

Now we provide plots comparing the solution profile before and after post-processing for a longer computational time. To compute those plots, we use a
third-order Runge–Kutta method with $\Delta t = \text{CFL}/(V_c/\Delta x + E_{\text{max}}/\Delta v)$ and CFL = 0.1. In Figs. 4, 5, 6 and 7, we show a comparison of contour plots of the numerical solution for $f$ before and after post-processing with different mesh sizes and $k = 1, 2$. There is visible improvement of the resolution of the solution, particularly for $k = 1$.

5.2 Vlasov–Maxwell example

In this section, we will test our post-processor for the VM system. Specifically we will use the streaming Weibel (SW) instability as an example. This is a reduced version of the VM equations with one spatial variable, $x_2$, and two velocity variables $v_1$ and $v_2$. The variables under consideration are the distribution function $f(x_2, v_1, v_2, t)$, a 2D electric field $E = (E_1(x_2, t), E_2(x_2, t), 0)$, and a 1D magnetic field $B = (0, 0, B_3(x_2, t))$. The reduced VM system reads as

$$\frac{\partial f}{\partial t} + v_2 \frac{\partial f}{\partial x_2} + (E_1 + v_2 B_3) \frac{\partial f}{\partial v_1} + (E_2 - v_1 B_3) \frac{\partial f}{\partial v_2} = 0, \quad (5.3a)$$

$$\frac{\partial B_3}{\partial t} = \frac{\partial E_1}{\partial x_2}, \quad \frac{\partial E_1}{\partial t} = \frac{\partial B_3}{\partial x_2} - j_1, \quad \frac{\partial E_2}{\partial t} = -j_2, \quad (5.3b)$$

where

$$j_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_2, v_1, v_2, t) v_1 d v_1 d v_2, \quad j_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_2, v_1, v_2, t) v_2 d v_1 d v_2. \quad (5.4)$$

The initial conditions are given by

$$f(x_2, v_1, v_2, 0) = \frac{1}{\pi \beta} e^{-v_2^2/\beta} \left[ \delta e^{-(v_1 - \omega_{0,1})^2/\beta} + (1 - \delta) e^{-(v_1 + \omega_{0,2})^2/\beta} \right], \quad (5.5a)$$

$$E_1(x_2, v_1, v_2, 0) = E_2(x_2, v_1, v_2, 0) = 0, \quad B_3(x_2, v_1, v_2, 0) = b \sin(\kappa_0 x_2), \quad (5.5b)$$

which for $b = 0$ is an equilibrium state composed of counter-streaming beams propagating perpendicular to the direction of inhomogeneity. Following [8, 11], we trigger the instability by taking $\beta = 0.01$, $b = 0.001$ (the amplitude of the initial perturbation of the magnetic field). Here, $\Omega_x = [0, L_y]$, where $L_y = 2\pi/\kappa_0$, and we set $\Omega_v = [-1.8, 1.8]^2$. We consider the following set of parameters,

$$\delta = 0.5, \quad \omega_{0,1} = \omega_{0,2} = 0.3, \quad \kappa_0 = 0.2.$$

In Table 2, we run the VM system with the initial condition for SW instability to $T = 5$ and then back to $T = 0$. We then apply the SIAC filter and compare it with the initial condition. We use a third order TVD-RK method (2.13) as the time integrator. To ensure the spatial error dominates, we take $\Delta t = \text{CFL} \cdot \Delta x$ for $P^1$ and $\Delta t = \text{CFL} \cdot \Delta x^{5/3}$ for $P^2$, in both cases we used CFL = 0.1. From the table we can observe $(k + 1)$-th order of convergence for the DG solution before post-processing for $f$, $E_1$, $E_2$ and $B_3$. After post-processing we can see overall the order of convergence improves to $O(h^{2k+1/2})$. 

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Fig. 4 Comparison of contour plots before (left) and after post-processing (right) for different mesh-sizes. Landau damping, $k = 1$ and $T = 10$
Fig. 5 Comparison of the definition of the contour plots before (left) and after post-processing (right) for different mesh-sizes. Landau damping, $k = 2$ and $T = 10$. We can see that the definition is improved after post-processing.
Fig. 6 Comparison of contour plots before (left) and after post-processing (right) for different mesh-sizes. Two stream instability, $k = 1$ and $T = 20$
Fig. 7 Comparison of the contour plots before (left) and after post-processing (right) for different mesh-sizes. Two stream instability, \( k = 2 \) and \( T = 20 \). We can see that the definition is improved after post-processing.
Table 2 $L^2$ errors for the numerical solution (top) and the post-processed solution (bottom) for streaming Weibel instability using the discontinuous Galerkin method using a $P^k$ polynomial approximation space

| Mesh | Error $f$ | Order | Error $B_3$ | Order | Error $E_1$ | Order | Error $E_2$ | Order |
|------|-----------|-------|-------------|-------|-------------|-------|-------------|-------|
| $p^1$ | 20 $\times$ 20 $\times$ 20 | 2.20E-01 | – | 2.61E-06 | – | 2.12E-06 | – | 5.31E-06 | – |
|      | 40 $\times$ 40 $\times$ 40 | 7.17E-02 | 1.61 | 6.54E-07 | 2.00 | 7.06E-07 | 1.58 | 5.46E-07 | 3.28 |
|      | 80 $\times$ 80 $\times$ 80 | 1.92E-02 | 1.90 | 1.63E-07 | 2.00 | 1.96E-07 | 1.85 | 7.05E-08 | 2.95 |
|      | 160 $\times$ 160 $\times$ 160 | 4.89E-03 | 1.98 | 4.07E-08 | 2.00 | 5.13E-08 | 1.94 | 6.40E-09 | 3.46 |
| $p^2$ | 20 $\times$ 20 $\times$ 20 | 1.07E-01 | – | 2.56E-07 | – | 2.49E-07 | – | 1.02E-06 | – |
|      | 40 $\times$ 40 $\times$ 40 | 1.64E-02 | 2.70 | 3.14E-08 | 3.03 | 2.93E-08 | 3.09 | 9.72E-08 | 3.40 |
|      | 80 $\times$ 80 $\times$ 80 | 2.23E-03 | 2.88 | 1.63E-09 | 4.27 | 1.90E-09 | 3.95 | 6.93E-09 | 3.81 |
|      | 160 $\times$ 160 $\times$ 160 | 2.92E-04 | 2.93 | 1.41E-10 | 3.52 | 1.72E-10 | 3.46 | 2.46E-10 | 4.81 |

| Mesh | Error $f^*$ | Order | Error $B^*_3$ | Order | Error $E^*_1$ | Order | Error $E^*_2$ | Order |
|------|-------------|-------|-------------|-------|-------------|-------|-------------|-------|
| $p^1$ | 20 $\times$ 20 $\times$ 20 | 2.95E-01 | – | 3.17E-07 | – | 1.08E-07 | – | 5.08E-06 | – |
|      | 40 $\times$ 40 $\times$ 40 | 6.13E-02 | 2.27 | 7.16E-08 | 2.14 | 1.49E-08 | 2.87 | 4.38E-07 | 3.54 |
|      | 80 $\times$ 80 $\times$ 80 | 5.87E-03 | 3.38 | 1.12E-08 | 2.68 | 3.11E-09 | 2.26 | 6.33E-08 | 2.79 |
|      | 160 $\times$ 160 $\times$ 160 | 4.19E-04 | 3.81 | 2.01E-09 | 2.48 | 7.47E-10 | 2.06 | 6.22E-09 | 3.35 |
| $p^2$ | 20 $\times$ 20 $\times$ 20 | 2.89E-01 | – | 1.24E-08 | – | 9.06E-09 | – | 4.41E-07 | – |
|      | 40 $\times$ 40 $\times$ 40 | 4.58E-02 | 2.66 | 5.61E-10 | 4.46 | 2.97E-10 | 4.93 | 2.63E-08 | 4.07 |
|      | 80 $\times$ 80 $\times$ 80 | 2.03E-03 | 4.49 | 2.94E-11 | 4.25 | 1.31E-11 | 4.50 | 2.57E-09 | 3.36 |
|      | 160 $\times$ 160 $\times$ 160 | 4.43E-05 | 5.52 | 1.65E-12 | 4.15 | 5.55E-13 | 4.56 | 1.12E-10 | 4.53 |
Fig. 8 Cross-sectional plot of the error for $f$ at $x_2 \approx 0.15\pi$, before (on the left) and after post-processing (on the right) for $80^3$ elements and $P^1$. SW instability

In Fig. 8 we plot a cross-section of the errors of the numerical solution at $x_2 \approx 0.15\pi$ before and after post-processing for $P^1$ using $80 \times 80 \times 80$ elements. We can see that before post-processing that the errors are highly oscillatory, and after post-processing the error surface is smooth and the error is much smaller in magnitude. In Fig. 9 we plot the errors of $E_1$, $E_2$ and $B_3$. We used the same number of elements as in Fig. 8 and obtained similar conclusions.

6 Concluding remarks

In this paper, we proved theoretically and demonstrated computationally the effectiveness of the SIAC filter for DG solutions of the nonlinear VM system. We proved superconvergence of order $(2k + 1/2)$ in the negative-order norm of the DG solution. This is nontrivial for nonlinear systems, and is achieved by identifying a suitable dual problem. Numerical experiments verify the performance of the filter in reducing spurious oscillations in the numerical errors. For low order $k$, the resolution of the numerical solution is greatly enhanced, which is highly desirable for long time kinetic simulations. In the future, we plan to prove superconvergence for the divided difference of the numerical solution to fully justify the enhanced resolution of the post-processed solution.
Fig. 9 Errors before (solid line) and after post-processing (dashed line) for the different fields using mesh size of $80 \times 80 \times 80$ and $p^1$, $T = 10$. SW instability.
Declarations

Conflict of interest The author has no conflicts of interest.

References

1. Ayuso de Dios, B., Carrillo de la Plata, J.A., Shu, C.-W.: Discontinuous Galerkin methods for the one-dimensional Vlasov–Poisson system. (2009)
2. Besse, N., Latu, G., Ghizzo, A., Sonnendrüker, E., Bertrand, P.: A wavelet-MRA-based adaptive semi-Lagrangian method for the relativistic Vlasov–Maxwell system. J. Comp. Phys. 227(16), 7889–7916 (2008)
3. Birdsall, C.K., Langdon, A.B.: Plasma Physics Via Computer Simulation. Institute of Physics Publishing, Bristol (1991)
4. Bramble, J.H., Schatz, A.H.: Higher order local accuracy by averaging in the finite element method. Math. Comput. 31(137), 94–111 (1977)
5. Brenner, S.C., Scott, L.R., Scott, L.R.: The Mathematical Theory of Finite Element Methods, vol. 3. Springer, New York (2008)
6. Califano, F., Attico, N., Pegoraro, F., Bertin, G., Bulanov, S.V.: Fast formation of magnetic islands in a plasma in the presence of counterstreaming electrons. Phys. Rev. Lett. 86(23), 5293–5296 (2001)
7. Califano, F., Pegoraro, F., Bulanov, S.V.: Impact of kinetic processes on the macroscopic nonlinear evolution of the electromagnetic-beam-plasma instability. Phys. Rev. Lett. 84, 3602–3605 (2000)
8. Califano, F., Pegoraro, F., Bulanov, S.V., Mangeney, A.: Kinetic saturation of the Weibel instability in a collisionless plasma. Phys. Rev. E 57(6), 7048–7059 (1998)
9. Cheng, Y., Christlieb, A.J., Zhong, X.: Energy-conserving discontinuous Galerkin methods for the Vlasov–Ampere system. J. Comput. Phys. 256, 630–655 (2014)
10. Cheng, Y., Gamba, I.M.: Numerical study of Vlasov-Poisson equations for infinite homogeneous stellar systems. Commun. Nonlinear Sci. Numer. Simul. 17, 2052–2061 (2012)
11. Cheng, Y., Gamba, I.M., Li, F., Morrison, P.J.: Discontinuous Galerkin methods for the Vlasov–Maxwell equations. SIAM J. Numer. Anal. 52(2), 1017–1049 (2014)
12. Cheng, Y., Gamba, I.M., Morrison, P.J.: Study of conservation and recurrence of Runge–Kutta discontinuous Galerkin schemes for Vlasov–Poisson systems. J. Sci. Comput. 2012. arXiv:1209.6413v2 [math.NA]
13. Ciarlet, P.G.: The finite element method for elliptic problems. Bull. Amer. Math. Soc 1, 800–802 (1979)
14. Cockburn, B., Luskin, M., Shu, C.-W., Süli, E.: Enhanced accuracy by post-processing for finite element methods for hyperbolic equations. Math. Comput. 72(242), 577–606 (2003)
15. Cockburn, B., Shu, C.-W.: Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. J. Sci. Comput. 16, 173–261 (2001)
16. Crouzeix, M., Thomée, V.: The stability in $L_p$ and $W^1_p$ of the $L_2$-projection onto finite element function spaces. Math. Comput. 48(178), 521–532 (1987)
17. Curtis, S., Kirby, R.M., Ryan, J.K., Shu, C.-W.: Postprocessing for the discontinuous Galerkin method over nonuniform meshes. SIAM J. Sci. Comput. 30(1), 272–289 (2008)
18. Gottlieb, S., Shu, C.-W.: Total variation diminishing Runge–Kutta schemes. Math. Comput. 67(221), 73–85 (1998)
19. Heath, R.E.: Numerical analysis of the discontinuous Galerkin method applied to plasma physics. 2007. Ph.D. dissertation, the University of Texas at Austin
20. Heath, R.E., Gamba, I.M., Morrison, P.J., Michler, C.: A discontinuous Galerkin method for the Vlasov–Poisson system. J. Comput. Phys. 231, 1140–1174 (2012)
21. Hockney, R.W., Eastwood, J.W.: Computer Simulation Using Particles. McGraw-Hill, New York (1981)
22. Huot, F., Ghizzo, A., Bertrand, P., Sonnendrüker, E., Coulard, O.: Instability of the time splitting scheme for the one-dimensional and relativistic Vlasov–Maxwell system. J. Comp. Phys. 185(2), 512–531 (2003)
23. Ji, L., Van Slingerland, P., Ryan, J.K., Vuik, K.: Superconvergent error estimates for position-dependent smoothness-increasing accuracy-conserving (SIAC) post-processing of discontinuous Galerkin solutions. Math. Comput. pp 2239–2262 (2014)
24. Ji, L., Xu, Y., Ryan, J.K.: Negative-order norm estimates for nonlinear hyperbolic conservation laws. J. Sci. Comput. 54(2), 531–548 (2013)
25. Mangeney, A., Califano, F., Cavazzoni, C., Travnicek, P.: A numerical scheme for the integration of the Vlasov–Maxwell system of equations. J. Comput. Phys. 179(2), 495–538 (2002)
26. Marchuk, G.I.: Construction of adjoint operators in non-linear problems of mathematical physics. Sbornik: Math. 189(10), 1505 (1998)
27. Meng, X., Ryan, J.K.: Discontinuous Galerkin methods for nonlinear scalar hyperbolic conservation laws: divided difference estimates and accuracy enhancement. Numer. Math. 136(1), 27–73 (2017)
28. Meng, X., Ryan, J.K.: Divided difference estimates and accuracy enhancement of discontinuous Galerkin methods for nonlinear symmetric systems of hyperbolic conservation laws. IMA J. Numer. Anal. 38(1), 125–155 (2018)
29. Sircombe, N., Arber, T.: VALIS: a split-conservative scheme for the relativistic 2d Vlasov–Maxwell system. J. Comput. Phys. 228(13), 4773–4788 (2009)
30. Steffan, M., Curtis, S., Kirby, R.M., Ryan, J.: Investigation of smoothness enhancing accuracy-conserving filters for improving streamline integration through discontinuous fields. IEEE Trans. Visual Comput. Graph. 14(3), 680–692 (2008)
31. Suzuki, A., Shigeyama, T.: A conservative scheme for the relativistic Vlasov–Maxwell system. J. Comput. Phys. 229(5), 1643–1660 (2010)
32. Yang, H., Li, F.: Discontinuous Galerkin methods for relativistic Vlasov–Maxwell system. J. Sci. Comput. 73(2), 1216–1248 (2017)

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