On determinants of integrable operators with shifts.

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Abstract

Integrable integral operator can be studied by means of a matrix Riemann–Hilbert problem. However, in the case of so-called integrable operators with shifts, the associated Riemann–Hilbert problem becomes operator valued and this complicates strongly the analysis. In this note, we show how to circumvent, in a very simple way, the use of such a setting while still being able to characterize the large-\(x\) asymptotic behavior of the determinant associated with the operator.

Introduction

The theory of integrable integral operators takes its roots in the works of Jimbo, Miwa, Mori and Sato which ultimately led to the development of a Riemann–Hilbert based setting for studying these operators. The initial motivation for studying these operators stemmed from the theory of quantum integrable models at the free fermion point where the correlation functions are expressed in terms of Fredholm determinants -or minors thereof-. Let us recall certain basic facts. An integrable integral operator is an integral operator on \(L^2(\mathbb{C})\) of the type \(I + \tilde{V}\) where \(\mathcal{C}\) is some contour in \(\mathbb{C}\) and the integral kernel \(\tilde{V}\) takes the form

\[
\tilde{V}(\lambda, \mu) = \sum_{a=1}^{N} \frac{f_a(\lambda)e_a(\mu)}{\lambda - \mu} \quad \text{with} \quad \sum_{a=1}^{N} f_a(\lambda)e_a(\lambda) = 0. \quad (0.1)
\]

There is no restriction on the number \(N\) of functions involved in the definition of the kernel. It has been shown in [4] that the study of such integrable operators (calculation of their resolvent kernel, determinant, etc) boils down to the resolution of an \(N \times N\) Riemann–Hilbert problem. This approach is particularly efficient in the asymptotic regime when the functions \(f_a\) and \(e_a\) entering in the composition of the kernel depend on some large parameter

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\[1\] Several principal aspects of the integrable operator theory, especially the ones concerning the integrable differential systems appearing in random matrix theory, have been developed in [13]. Some of the elements of the theory of integrable operators were already implicitly present in the earlier work [11].

\[2\] the last condition may be avoided for the price of dealing with a principal-value regularization, see [1]
x, say in an oscillatory way. Indeed, then, the jump matrix arising in the associated Riemann–Hilbert problem depends on the large-parameter x in an oscillatory way and its asymptotic solution can be studied within the Deift-Zhou non-linear steepest descent method [3]. This program has been carried out for a large number of kernels in the $N = 2$ case, see eg [2, 8].

In fact, one can even consider more general kernels where the discreet sum in (0.1) is replaced by an integration in respect to an arbitrary measure $d\mu$ on some set $E$:

$$V(\lambda, \mu) = \frac{\int_E f(\lambda, s) \cdot e(\mu, s) \cdot d\mu(s)}{\lambda - \mu} \quad \text{under the condition} \quad \int_E f(\lambda, s) \cdot e(\lambda, s) \cdot d\mu(s) = 0. \quad (0.2)$$

There is however a price to pay for such a generalization: as soon as the measure $d\mu$ is not compactly supported and purely atomic, one deals with an operator valued Riemann–Hilbert problem. In the early days of the analysis of integrable integral operators, there have been a few attempts to extract valuable data out of such Riemann–Hilbert problem. The matter is that, even in the most simple cases, the analysis becomes extremely hard and complicated due to the operator valued setting for the Riemann–Hilbert problem. The authors of [5] were able to argue the leading asymptotics of the solution to the Riemann–Hilbert problem arising in the large-$x$ asymptotic analysis of a very specific integrable integral operator of the form (0.2) for certain classes of functions $f$ and $e$ and in the context of specific measures $d\mu(s)$ supported on $\mathbb{R}^+$ which are absolutely continuous in respect to Lebesgue’s one. For instance, the case considered in [5] does fall into this class. As an application of our technique, we shall consider the operator $I + S$ acting on $L^2([a; b])$ with an integrable integral kernel $S(\lambda, \mu)$ of, so-called, shift type:

$$S(\lambda, \mu) = \frac{icF(\lambda)}{2i\pi(\lambda - \mu)} \left\{ \frac{e(\lambda)e^{-1}(\mu)}{(\lambda - \mu + ic)} + \frac{e(\mu)e^{-1}(\lambda)}{(\lambda - \mu - ic)} \right\} \quad \text{where} \quad e(\lambda) = e^{i\frac{\pi}{2}p(\lambda)}. \quad (0.3)$$

$F$ and $p$ are certain holomorphic function in a neighborhood of the interval $[a; b]$ and $F$ is "sufficiently" small, in a sense that will be specified below. We call these kernels of "shift-type" in that the denominator not only contains the singular factor $\lambda - \mu$ but also shifts thereof, $\lambda - \mu \pm ic$ in this case. It is readily seen that the kernel (0.3) is an integrable kernel in the generalized sense (0.2) since it admits the representation

$$S(\lambda, \mu) = \frac{-ic}{(\lambda - \mu)B} \int_0^{+\infty} F(\lambda) e^{-cs} \cdot \left\{ e(\lambda) \cdot e^{-1}(\mu) \cdot e^{is(\lambda - \mu)} - e(\mu) \cdot e^{-1}(\lambda) \cdot e^{is(\mu - \lambda)} \right\} \cdot ds. \quad (0.4)$$

Our approach allows us to compute the large-$x$ asymptotic behavior of the Fredholm determinant of the operator $I + S$, what constitutes the main result of the paper.

**Theorem 0.1** Let $\tilde{S}$ correspond to the below generalized sine kernel

$$\tilde{S}(\lambda, \mu) = F(\lambda) \frac{e(\lambda)e^{-1}(\mu) - e(\mu)e^{-1}(\lambda)}{2i\pi(\lambda - \mu)}, \quad (0.5)$$

with $F$, $p$ holomorphic functions on some open neighborhood $U$ of $[a; b]$ and such that $|F| < 1$ on $U$ and $p'[a; b] > 0$. Then, the Fredholm determinant of the integral operator $I + S$ acting on $L^2([a; b])$ with a kernel $S(\lambda, \mu)$ given by (0.3) admits the $x \to +\infty$ asymptotic expansion

$$\frac{\det[I + S]}{\det[I + \tilde{S}]} = \frac{\det[I + U_+] \cdot \det[I + U_-]}{\det[I + U_+] \cdot \{1 + O(x^{-1})\}}. \quad (0.6)$$
The asymptotic expansion of $\det[I + \tilde{S}]$ can be found in [8] and $I + U_\pm$ are integral operators acting on $L^2(\Gamma)$ with kernels

$$U_+(\lambda, \mu) = \frac{\alpha(\mu - ic) \cdot \alpha^{-1}(\lambda)}{2i\pi(\lambda - \mu + ic)} \quad \text{and} \quad U_-(\lambda, \mu) = \frac{\alpha^{-1}(\mu + ic) \cdot \alpha(\lambda)}{2i\pi(\lambda - \mu - ic)},$$

whereas $\Gamma$ is a small counterclockwise loop around $[a; b]$ of index 1. Finally, the function $\alpha$ appearing above is given by

$$\alpha(\lambda) = \exp \left\{ \int_a^b \frac{\ln[1 + F(\mu)]}{\lambda - \mu} \cdot \frac{d\mu}{2i\pi} \right\}.$$ (0.8)

This theorem can be as well obtained by the method of multidimensional Natte series whose premises appeared in [7] and which has been further developed and simplified in [9, 10]. In particular, the method has been brought to a satisfactory level of rigor in [9] upon a hypothesis of convergence of a series of multiple integrals. In the case underlying to the kernel $S$, it is fairly easy to establish this convergence. However, the multidimensional Natte series method is definitely much more complicated than the one being developed in the present note, so that applying it to the analysis of such "simple operators" would be, mildly speaking, highly unreasonable.

We also stress that our method is quite general and not solely applicable to the above kernel. As will be apparent from the core of the text, it allows one even to obtain new types of representations for Fredholm determinants of non-integrable operators of the type

$$V(\lambda, \mu) = \tilde{V}(\lambda, \mu) - \sum_{a=1}^{N} \gamma_a f_a(\lambda)e_{v_a}(\mu) \text{ where } \tilde{V}(\lambda, \mu) = \sum_{a=1}^{N} f_a(\lambda)e_{a}(\mu)$$

$c_{a} \in \mathbb{R}^*$ and $v_\alpha$ is an arbitrary sequence in $[1 ; N]$. For generic constants $\gamma_{a}$ and sequences $v_\alpha$, $a = 1, \ldots, N$ the operators’ kernels are not of integrable type, but they reduce to the integrable case for the specific choice $\gamma_{a} = 1$ and $v_{a} = a$ for any $a$. In fact, in such an integrable integral operator case, it would be interesting to compare the approach developed in the present note with the results that could be obtained with the help of operator valued Riemann-Hilbert techniques of [5]. We are going to address this issue in a forthcoming publication.

The paper is organized as follows. In section 1 we recall several basic facts about Riemann–Hilbert problems for integrable integral operators. Then, in section 2 we establish a factorization for the determinant $\det[I + V]$ in terms of $\det[I + \tilde{V}]$ and another determinant involving the solution to the Riemann–Hilbert problem associated with the integrable kernel $\tilde{V}$. Finally, in section 3 we demonstrate that in the case of the integrable operator of shift-type (0.3) associated with the generalized sine kernel, such a factorization is already enough so as to access to the asymptotics of the determinant $\det[I + S]$ which take the form stated in theorem 0.1.

1 The Riemann–Hilbert problem setting

1.1 The kernel and initial Riemann–Hilbert problem

Let $I + \tilde{V}$ be an integral operator acting on $L^2(J)$ with $J$ a piecewise smooth curve in $\mathbb{C}$ and whose kernel $\tilde{V}(\lambda, \mu)$ is of integrable integral type:

$$\tilde{V}(\lambda, \mu) = \frac{(E_L(\lambda), E_R(\mu))}{\lambda - \mu}.$$ (1.1)
with vector valued smooth functions on $J$

$$E^T_L(\lambda) = (f_1(\lambda), \ldots, f_N(\lambda)) \quad \text{and} \quad E^T_R(\lambda) = (e_1(\lambda), \ldots, e_N(\lambda))$$

(1.2)
satisfying to the regularity condition $(E_L(\lambda), E_R(\lambda)) = 0$ for all $\lambda \in J$.

This kernel is associated with the Riemann–Hilbert problem for a $N \times N$ matrix $\chi(\lambda)$

- $\chi \in O(C \setminus J)$ and has continuous boundary values on $\partial J$;
- $\chi(z) = \ln|z - a| O \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ \end{array} \right)$ as $z \to a$ with $a \in \partial J$;
- $\chi(z) = I_N + O(z^{-1})$ when $z \to \infty$;
- $\chi^{-1}(z) = \chi^+(z) \cdot G(z)$ for $z \in \partial J$ where $G(\lambda) = I_N + 2i\pi E^T_R(\lambda) \cdot E_L(\lambda)$.

Here, we should explain that relations of the type $M(z) = O(R(z))$ for two matrix functions $M, R$ should be understood entry-wise, i.e. $M_{jk}(z) = O(R_{jk}(z))$. Also, given a function $f$ defined on $C \setminus \gamma$, with $\gamma$ an oriented curve in $C$, we denote by $f_\gamma(s)$ the boundary values of $f(z)$ on $\gamma$ when the argument $z$ approaches the point $s \in \gamma$ non-tangentially and from the left (+) or from the right (−) side of the curve. Again, if one deals with matrix function, then this relation has to be understood entry-wise.

The unique solvability of the above Riemann–Hilbert problem for $\chi$ is equivalent to the condition $\det[I + \tilde{V}] \neq 0$. In such a case, the unique solution takes the form

$$\chi(\lambda) = I_N - \int_J \frac{F_R(\mu) \cdot E^T_L(\mu)}{\mu - \lambda} \, d\mu \quad \text{and} \quad \chi^{-1}(\lambda) = I_N + \int_J \frac{E_R(\mu) \cdot F^T_L(\mu)}{\mu - \lambda} \, d\mu$$

(1.3)

where $I_N$ is the $N \times N$ identity matrix and $F_R(\mu)$ and $F_L(\mu)$ correspond to the solutions to the below linear integral equations

$$F_R(\lambda) + \int_J \tilde{V}(\mu, \lambda) F_R(\mu) \, d\mu = E_R(\lambda) \quad \text{and} \quad F_L(\lambda) + \int_J \tilde{V}(\mu, \lambda) F_L(\mu) \, d\mu = E_L(\lambda).$$

(1.4)

Also, the vector functions $F_R(\mu)$ and $F_L(\mu)$ can be reconstructed in terms of $\chi$ as follows

$$F_R(\mu) = \chi(\lambda) \cdot E_R(\mu) \quad \text{and} \quad F^T_L(\mu) = E^T_L(\mu) \cdot \chi^{-1}(\lambda).$$

(1.5)

## 2 Factorization of determinants

In this section, we consider a trace class integral operator $I + V$ on $L^2(J)$ with a kernel given by (0.9). We establish various alternative representations for its Fredholm determinant.

**Proposition 2.1** Assume that $\det[I + \tilde{V}] \neq 0$. Then, the Fredholm determinant $\det[I + V]$ admits the decomposition

$$\det[I + V] = \det[I + \tilde{V}] \cdot \det[I + W]$$

(2.1)

where the integral kernel $W(\lambda, \mu)$ takes the form

$$W(\lambda, \mu) = -\sum_{n=1}^{N} \gamma_n \frac{(F_L(\lambda), \chi(\mu - ic_n) e_n) \cdot (e_n, E_R(\mu))}{\lambda - \mu + ic_n},$$

(2.2)
in which we have introduced
\[ e^T_p = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^N. \] (2.3)

**Proof —**

It follows that
\[ I + V = (I + \tilde{V}) \cdot (I + W) \] (2.4)

where the kernel
\[ W(\lambda, \mu) = -\sum_{a=1}^{N} \gamma_a \mathcal{U}_a(\lambda, \mu) e_v(\mu) \] (2.5)
is expressed in terms of the unique solutions to the integral equations
\[ \mathcal{U}_n(\lambda, \mu) + \int_{J} \tilde{V}(\lambda, \tau) \mathcal{U}_n(\tau, \mu) \cdot d\tau = \frac{(E_L(\lambda), e_n)}{\lambda - \mu + ic_n}. \] (2.6)

One can solve these equations in terms of the entries of \( \chi \), the unique solution of the RHP associated with the kernel \( \tilde{V}(\lambda, \mu) \). Setting \( \mathcal{U}_n(\lambda, \mu) = (\lambda - \mu + ic_n) \mathcal{U}_n(\lambda, \mu) \), we get that \( \mathcal{U}_n(\lambda, \mu) \) solves the integral equation
\[ \mathcal{U}_n(\lambda, \mu) + \int_{J} \tilde{V}(\lambda, \tau) \mathcal{U}_n(\tau, \mu) \cdot d\tau + \int_{J} (E_L(\lambda), E_R(\tau)) \cdot \mathcal{U}_n(\tau, \mu) \cdot d\tau = (E_L(\lambda), e_n). \] (2.7)

Thus one has that
\[ \mathcal{U}_n(\lambda, \mu) = \frac{(F_L(\lambda), e_n)}{\lambda - \mu + ic_n} - \int_{J} \frac{(F_L(\lambda), E_R(\tau))}{\lambda - \mu + ic_n} \cdot \mathcal{U}_n(\tau, \mu) \cdot d\tau. \] (2.8)

It solely remains to fix the unknown vector coefficient. This can be done by solving the consistency conditions
\[ \int_{J} E_R(\tau) \cdot \mathcal{U}_n(\tau, \mu) d\tau = \left( -\int_{J} \frac{E_R(\tau) \cdot F^T_L(\tau)}{\tau - \mu + ic_n} d\tau \right) \cdot \left( \int_{J} E_R(\tau) \cdot \mathcal{U}_n(\tau, \mu) d\tau \right) + \int_{J} \frac{E_R(\tau) \cdot F^T_L(\tau) \cdot e_n}{\tau - \mu + ic_n}. \] (2.9)

Using the integral representation for \( \chi^{-1}(\lambda) \), we are led to the equation
\[ \chi^{-1}(\mu - ic_n) \left( \int_{J} E_R(\tau) \cdot \mathcal{U}_n(\tau, \mu) d\tau \right) = \left[ \chi^{-1}(\mu - ic_n) - I_N \right] \cdot e_n \] (2.10)

which is readily solved for the unknown vector coefficients. Therefore
\[ \mathcal{U}_n(\lambda, \mu) = \frac{(F_L(\lambda), \chi(\mu - ic_n) e_n)}{\lambda - \mu + ic_n}. \] (2.11)

Upon inserting these expression into \( W(\lambda, \mu) \), we are led to the claim. ■

The determinant of \( I + W \) can be recast in terms of the Fredholm determinant of a matrix integral operator acting on a small counterclockwise loop \( \Gamma(J) \) encircling the original interval \( J \).
Proposition 2.2 One has the representation

$$\det[I + W] = \det[I + M]$$  \hspace{1cm} (2.12)

Where $M$ is a $N \times N$ matrix integral operator on $L^2(\Gamma(J)) \otimes \mathbb{C}^N$ whose entries are given by

$$M_{kl}(\lambda, \mu) = \gamma_k \frac{(e_{ik}, \chi^{-1}(\lambda) \cdot \chi(\mu - ic_f) \cdot e_l)}{2\pi i(\lambda - \mu + ic_f)}.$$  \hspace{1cm} (2.13)

Proof —

Let $\kappa \in \mathbb{C}$ and $|\kappa|$ be small enough, then one has the series expansion

$$\ln \det[I + \kappa W] = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{p=1}^{n} \left\{ \gamma_{rp} \cdot f^{(n)}(\tau_p) \right\} \text{ where } \mathcal{E}_N = \left\{ 1 ; N \right\}$$  \hspace{1cm} (2.14)

and

$$I^{(n)}_{\mathcal{E}_N} = \int_{\Gamma(J)} \frac{d^n\lambda}{(2\pi i)^n} \prod_{p=1}^{n} \left\{ \frac{(F_L(\lambda_p), \chi(\lambda_{p+1} - ic_{\tau_p}) \cdot e_{\tau_p}) \cdot (e_{\tau_p}, E_R(\lambda_{p+1}))}{(z_p - z_{p+1} + ic_{\tau_p})} \right\} \cdot d^n\lambda.$$  \hspace{1cm} (2.15)

Above, we agree upon the cyclicity notation $\lambda_{n+1} = \lambda_1$. In order to be able to take the $\lambda$-integrals, we separate the $\lambda$-integration by introducing $z$-contour integrals which leads to

$$I^{(n)}_{\mathcal{E}_N} = \int_{\Gamma(J)} \frac{d^n\lambda}{(2\pi i)^n} \prod_{p=1}^{n} \left\{ \frac{(F_L(\lambda_p), \chi(\lambda_{p+1} - ic_{\tau_p}) \cdot e_{\tau_p}) \cdot (e_{\tau_p}, E_R(\lambda_{p+1}))}{(z_p - z_{p+1} + ic_{\tau_p})} \right\} \cdot d^n\lambda.$$  \hspace{1cm} (2.16)

Focusing on the $z_p$-dependent part of the integration, we see that the part involving $I_N$ is holomorphic inside of $\Gamma(J)$ (the singular part is due to $\chi^{-1}(z_p)$). Hence, by closing the loop we see that this part of the integral does not contribute to the value of the integral. In fact, one can replace $I_N$ by any holomorphic function of $z_p$ inside of the loop $\Gamma(J)$. Hence, we replace $I_N$ by 0. Repeating this procedure for every $p$, we get

$$I^{(n)}_{\mathcal{E}_N} = (-1)^n \int_{\Gamma(J)} \frac{d^n\lambda}{(2\pi i)^n} \prod_{p=1}^{n} \left\{ \frac{(e_{\tau_p}, E_R(\lambda_p)) \cdot (F_L(\lambda_p), \chi(\lambda_{p+1} - ic_{\tau_p}) \cdot e_{\tau_p})}{(z_p - z_{p+1} + ic_{\tau_p})} \right\} \cdot d^n\lambda.$$  \hspace{1cm} (2.17)

It only remains to identify the sums over $\tau_p \in \mathcal{E}_N$, $p = 1, \ldots, n$ as a matrix trace over the additional matrix indices of the integral operator $M$ so as to obtain the Fredholm determinant $\ln \det[I + \kappa M]$. Taking the exponent, one obtains an equality, valid in some open neighborhood of 0, between two entire functions. They are thus equal everywhere, and, in particular, at $\kappa = 1$.

In fact, one can even derive an alternative representation for $\det[I + M]$ by deforming the loop $\Gamma(J)$. The new matrix integral operator acts on the whole real axis.
Proposition 2.3 One has the representation

$$\det[I + M] = \det[I + N],$$

where $N$ is a $N \times N$ matrix integral operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^N$ whose entries are given by

$$N_{kk}(\lambda, \mu) = \text{sgn}(c_k) \cdot \chi \cdot \frac{(e_{k,n} \cdot [I_N - \chi^{-1}(\lambda + ic_k/2) \cdot \chi(\mu - ic\ell/2)] \cdot e_{k})}{2i\pi(\lambda - \mu + i(c_k + c\ell)/2)}.\quad (2.19)$$

We do stress that the obtained operator is non singular in that, should $c_k + c\ell = 0$ for some $k, \ell$, then the simple zero of the denominator at $\lambda = \mu$ is compensated by a zero of the numerator.

Proof —

The starting point for obtaining the representation (2.19) is to recast (2.16) in a slightly different manner with the help of contour deformations. Hence, starting from (2.16), in the integral relative to $z$, we now replace $I_N$ by $\chi^{-1}(z_{p+1} - ic\tau_p)$ instead of 0. Note that such a replacement also does not affect the analyticity properties of the integrand in $z_{p+1}$ belonging to the inside of the sufficiently small loop $\Gamma(J)$. Repeating this procedure for every $p$, we get

$$I_{(\tau_p)}^{(n)} = \oint_{\Gamma(J)} \prod_{p=1}^{n} \left( e_{v_{p-1}} \cdot \frac{[I_N - \chi^{-1}(z_{p-1}) \cdot \chi(z_{p+1} - ic\tau_p)] \cdot e_{v_p}}{2i\pi(z_p - z_{p+1} + ic\tau_p)} \right). \quad (2.20)$$

We now pass on to deforming the loop $\Gamma(J)$. For this it is enough to consider the model integral

$$I_p = \oint_{\Gamma(J)} f_p(z_p) \cdot dz_p \quad (2.21)$$

where

$$f_p(z_p) = \frac{(e_{v_{p-2}} \cdot [I_N - \chi^{-1}(z_{p-1}) \cdot \chi(z_p - ic\tau_{p-1})] \cdot e_{v_{p-1}}) \cdot (e_{v_{p-1}} \cdot [I_N - \chi^{-1}(z_p) \cdot \chi(z_{p+1} - ic\tau_p)] \cdot e_{v_p})}{(z_{p-1} - z_p + ic\tau_{p-1}) \cdot (z_p - z_{p+1} + ic\tau_p)}. \quad (2.22)$$

and then apply the results recursively. We deform the integration contour in (2.21) as

$$\oint_{\Gamma(J)} \left\{ \sum_{\varepsilon = \pm} \int_{\mathbb{R}_- + i\varepsilon 0^+} \right\} \quad \text{where} \quad \mathbb{R}_+ = [-\varepsilon_0 : \varepsilon_0]. \quad (2.23)$$

Let $s_p = \text{sgn}(c\tau_p)$. The integrand is analytic for $z_p \in \mathbb{H}_{-s_{p-1}}$ since the apparent pole at $z_p = z_{p+1} - ic\tau_p$ - which may or may not belong to $\mathbb{H}_{-s_{p-1}}$ - is canceled by a zero of the numerator. Furthermore, $f_p(z_p) = O(z_p^{-2})$ when $z_p \to \infty$. Hence, by deforming the contour $\mathbb{R}_{s_{p-1}} - is_{p-1} 0^+$ up to $\mathbb{R}_{s_{p-1}} - is_{p-1} \times +\infty$, we can drop the corresponding contribution, leading to

$$I_p = \int_{\mathbb{R}_{-s_{p-1}} + is_{p-1} 0^+} f_p(z_p) \cdot dz_p. \quad (2.24)$$
Repeating these manipulations in every integral arising in \( I_{(r_p)}^{(a)} \), we get

\[
I_{(r_p)}^{(a)} = \prod_{p=1}^{n} \left\{ \int_{\mathbb{R}^{n-1} + it_{p-1}0^+} \frac{(e_{\tau_{p-1}},(I_N - \chi^{-1}(z_p + ic_{r_{p-1}}/2)\cdot\chi(z_{p+1} - ic_{r_p})/2)e_{\tau_{p-1}})}{-\text{sgn}(c_{r_p}) \cdot (z_p - z_{p+1} + ic_{r_{p-1}} + c_{r_p})/2} \right\} \frac{dnz}{(2\pi)^n}. \tag{2.28}
\]

The integrand is holomorphic in the multidimensional strip in \( \mathbb{C}^n \)

\[ S = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : 0 < s_{p-1} \Im(z_p) < s_{p-1} c_{r_{p-1}}, p = 1, \ldots, n\} \tag{2.26} \]

in that the numerator is holomorphic on this strip whereas the apparent simple poles due to the denominator are, in fact, canceled by the numerator’s zeros. Hence, we can deform the integration contour in \( z_p \) from \( \mathbb{R}^{-s_{p-1}} + is_{p-1}0^+ \) to \( \mathbb{R}^{-s_{p-1}} + ic_{r_{p-1}}/2 \). \tag{2.27}

This leads to

\[
I_{(r_p)}^{(a)} = \int_{\mathbb{R}^n} \frac{(e_{\tau_{p-1}},(I_N - \chi^{-1}(z_p + ic_{r_{p-1}}/2)\cdot\chi(z_{p+1} - ic_{r_p})/2)e_{\tau_{p-1}})}{-\text{sgn}(c_{r_p}) \cdot (z_p - z_{p+1} + ic_{r_{p-1}} + c_{r_p})/2} \frac{dnz}{(2\pi)^n}. \tag{2.28}
\]

It then only remains to recognize the sum over \( r_p \in E_N, p = 1, \ldots, N \) as the trace associated with the products over the matrix indices. The result then follows by an analytic continuation in \( \kappa \), just as in the previous proposition.

3 Application to the two-shifted generalized sine kernel

In this section, we apply our general formalism to the case of the so-called two-shifted generalized sine kernel whose kernel has been defined in (0.3). In such a way, we prove theorem (0.1). The kernel (0.3) is of the form (0.9) with \( N = 2 \), \( v_a = a \), \( \gamma_1 = \gamma_2 = 1 \) and \( c_1 = -c \), \( c_2 = c \) and the two-dimensional vectors \( E_R(\lambda) \) and \( E_L(\lambda) \) being expressed as:

\[
E_L^T(\lambda) = \frac{F(\lambda)}{2\pi}(-e^{-\lambda} \cdot e(\lambda)) \quad \text{and} \quad E_R^T(\lambda) = (e(\lambda) \cdot e^{-\lambda}(\lambda)). \tag{3.1}
\]

It has been established in [8] that, when \( x \rightarrow +\infty \), given any open and relatively compact neighborhood \( O \) of \([a,b]\), the unique solution \( \chi \) to the Riemann–Hilbert problem associated with the operator \( \hat{S} \) (0.5) takes, on \( \mathbb{C} \setminus \overline{O} \), the form

\[
\chi(\lambda) = \Pi(\lambda) \cdot \sigma^{-\alpha}(\lambda) \quad \text{with} \quad ||\Pi - I_2||_{L^{\infty}(\mathbb{C})} = O\left(\frac{1}{\lambda}\right). \tag{3.2}
\]

Here, \( \Pi \) is some holomorphic matrix in \( \mathbb{C} \setminus \overline{O} \) and the function \( \sigma \) is as defined by (0.8).

Thus we can choose some relatively compact neighborhood \( O \) of \([a,b]\) that is small enough and take \( \Gamma \) to be a small counterclockwise loop around \([a;b]\), lying entirely in \( \mathbb{C} \setminus \overline{O} \) \( \cap ||\Im(z)|| < c/2 \). Then, we introduce two integral operators \( I + M^{(0)} \) and \( I + M \) on \( L^2(\Gamma) \otimes \mathbb{C}^2 \) having kernels

\[
M(\lambda, \mu) = \begin{pmatrix}
\frac{(e_1 \cdot \chi^{-1}(\lambda) \cdot \chi(\mu + ic) \cdot e_1)}{2\pi(\lambda - \mu + ic)} & \frac{(e_1 \cdot \chi^{-1}(\lambda) \cdot \chi(\mu - ic) \cdot e_2)}{2\pi(\lambda + \mu + ic)} \\
\frac{(e_2 \cdot \chi^{-1}(\lambda) \cdot \chi(\mu + ic) \cdot e_2)}{2\pi(\lambda - \mu + ic)} & \frac{(e_2 \cdot \chi^{-1}(\lambda) \cdot \chi(\mu - ic) \cdot e_2)}{2\pi(\lambda + \mu + ic)}
\end{pmatrix}. \tag{3.3}
\]
and
\[
\mathcal{M}^{(0)}(\lambda, \mu) = \begin{pmatrix}
\frac{\alpha(\lambda) \cdot \alpha^{-1}(\mu + ic)}{2i\pi \cdot (\lambda - \mu - ic)} & 0 \\
0 & \frac{\alpha^{-1}(\lambda) \cdot \alpha(\mu - ic)}{2i\pi \cdot (\lambda - \mu + ic)}
\end{pmatrix}.
\]

(3.4)

\(\mathcal{M}(\lambda, \mu)\) and \(\mathcal{M}^{(0)}(\lambda, \mu)\) being bounded on \(\Gamma^2\), and \(\Gamma\) being compact, the integral operators \(\mathcal{M}\) and \(\mathcal{M}^{(0)}\) are trace class. It thus follows from standard estimates for Fredholm determinants (cf. [12]) that, for some universal constant \(C\),

\[
\left| \det [I + \mathcal{M}^{(0)}] - \det [I + \mathcal{M}] \right| \leq C \|\mathcal{M}^{(0)} - \mathcal{M}\|_1,
\]

(3.5)

with \(\|\cdot\|_1\) being the trace class norm. Hence, the uniform bounds for \(\Pi - I_2\) on \(\Gamma\) ensure that

\[
\det [I + \mathcal{M}] = \det [I + \mathcal{M}^{(0)}] \cdot \left(1 + O\left(\frac{1}{\lambda}\right)\right).
\]

(3.6)

It is then enough to apply the propositions 2.1 and 2.2 specialized to the setting associated with the kernel \(S\) (0.3) and choose the loop arising in proposition 2.2 to coincide with \(\Gamma\) (what is indeed possible).

\[\blacksquare\]

**Conclusion**

In this paper, we have developed an effective technique for dealing with certain classes of integrable integral operators with kernels of shift-type. This allowed us to compute, in the case of the simple example of the generalized sine kernel issued two-shift integrable integral operator, the large-\(x\) asymptotic behavior of its Fredholm determinant. The main achievement of this work is a technique that allows one to circumvent, in a very simple way, the handling of operator valued Riemann–Hilbert problems. It is still interesting to compare the approach we have developed in this paper with the operator valued Riemann-Hilbert technique of [5]. We are going to address this issue in a forthcoming publication.

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