CONTROLLING A RANDOM POPULATION

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Abstract. Bertrand et al. introduced a model of parameterised systems, where each agent is represented by a finite state system, and studied the following control problem: for any number of agents, does there exist a controller able to bring all agents to a target state? They showed that the problem is decidable and EXPTIME-complete in the adversarial setting, and posed as an open problem the stochastic setting, where the agent is represented by a Markov decision process. In this paper, we show that the stochastic control problem is decidable. Our solution makes significant uses of well quasi orders, of the max-flow min-cut theorem, and of the theory of regular cost functions. We introduce an intermediate problem of independence interest called the sequential flow problem and study its complexity.

1. Introduction

The control problem for populations of identical agents. The model we study was introduced in [BDGG17] (see also the journal version [BDG\textsuperscript{+}19]): a population of agents are controlled uniformly, meaning that the controller applies the same action to every agent. The agents are represented by a finite state system, the same for every agent. The key difficulty is that there is an arbitrary large number of agents: the control problem is whether for every $n \in \mathbb{N}$, there exists a controller able to bring all $n$ agents synchronously to a target state.

The technical contribution of [BDGG17, BDG\textsuperscript{+}19] is to prove that in the adversarial setting where an opponent chooses the evolution of the agents, the (adversarial) control problem is EXPTIME-complete.

In this paper, we study the stochastic setting, where each agent evolves independently according to a probabilistic distribution, \textit{i.e.} the finite state system modelling an agent is a Markov decision process. The control problem becomes whether for every $n \in \mathbb{N}$, there

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exists a controller able to bring all $n$ agents synchronously to a target state with probability one.

Our main technical result is that the stochastic control problem is decidable. In the next paragraphs we discuss four motivations for studying this problem: control of biological systems, parameterised verification and control, distributed computing, and automata theory.

**Modelling biological systems.** The original motivation for studying this model was for controlling population of yeasts ([UMD+15]). In this application, the concentration of some molecule is monitored through fluorescence level. Controlling the frequency and duration of injections of a sorbitol solution influences the concentration of the target molecule, triggering different chemical reactions which can be modelled by a finite state system. The objective is to control the population to reach a predetermined fluorescence state. As discussed in the conclusions of [BDGG17, BDG+19], the stochastic semantics is more satisfactory than the adversarial one for representing the behaviours of chemical reactions, so our decidability result is a step towards a better understanding of the modelling of biological systems as populations of arbitrarily many agents represented by finite state systems.

**From parameterised verification to parameterised control.** Parameterised verification was introduced in [GS92]: the goal is to prove the correctness of a system specification regardless of the number of its components. We refer to [AD16] for a recent survey on the topic. The control problem we study here, which was introduced in [BDGG17, BDG+19], is a step towards parameterised control: the goal is to control a system composed of many identical components in order to ensure a given property.

**Distributed computing.** Our model resembles two models introduced for the study of distributed computing. The first and most widely studied is population protocols, introduced in [AAD+06]: the agents are modelled by finite state systems and interact by pairs drawn at random. The mode of interaction is the key difference with the model we study here: in a time step, all of our agents perform simultaneously and independently the same action. This brings us closer to broadcast protocols as studied for instance in [EFM99], in which one action involves an arbitrary number of agents. As explained in [BDGG17, BDG+19], our model can be seen as a subclass of (stochastic) broadcast protocols, but key differences exist in the semantics, making the two bodies of work technically independent.

The focus of the distributed computing community when studying population or broadcast protocols is to construct the most efficient protocols for a given task, such as (prominently) electing a leader. A growing literature from the verification community focuses on checking the correctness of a given protocol against a given specification; we refer to the recent survey [Esp16] for an overview. We concentrate on the control problem, which can then be seen as a first result in the control of distributed systems in a stochastic setting.

**Alternative semantics for probabilistic automata.** It is very tempting to consider the limit case of infinitely many agents: the parameterised control question becomes the value 1 problem for probabilistic automata, which was proved undecidable in [GO10], and even in very restricted cases, see for instance [FGHO14]. Hence abstracting continuous distributions by a discrete population of arbitrary size can be seen as an approximation technique for probabilistic automata. Using $n$ agents corresponds to using numerical approximation up to $2^{-n}$ with random rounding; in this sense the control problem considers a converging approximation scheme. The plague of undecidability results on probabilistic automata (see
e.g. [Fij17]) is nicely contrasted by our positive result, which is one of the few decidability results on probabilistic automata not making structural assumptions on the underlying graph.

**Our results.** We prove decidability of the stochastic control problem. The first insight is given by the theory of well quasi orders, which motivates the introduction of a new problem called the sequential flow problem. The first step of our solution is to reduce the stochastic control problem to (many instances of) the sequential flow problem. The second insight comes from the theory of regular cost functions, providing us with a set of tools for addressing the key difficulty of the problem, namely the fact that there are arbitrarily many agents. Our key technical contribution is to show the computability of the sequential flow problem by reducing it to a boundedness question expressed with distance and desert automata using the max-flow min-cut theorem.

**Studying infinite-state Markov decision processes.** The technical contribution of this paper is an algorithm for solving an infinite-state Markov decision process with a finite representation. The purpose of the notion of decisive Markov chains [AHM07] is to define a unifying property for studying infinite-state Markov chains with “finite-like” properties. The Markov decision processes we consider in this paper do not yield decisive Markov chains, which suggests that our results are in some sense orthogonal to this notion.

**Organisation of the paper.** We define the stochastic control problem in Section 2, and the sequential flow problem in Section 3, together with its simple variant. We construct a reduction from the former to (many instances of) the latter in Section 4. Our solution of the sequential flow problem (SFP) goes in two steps, first studying the simple variant in Section 5, and then the general case in Section 6. Section 7 proves the PSPACE-hardness of the simple sequential flow problem.

2. **The stochastic control problem**

For a finite set $X$, we let $D(X)$ denote the set of probabilistic distributions over $X$, i.e. functions $\delta : X \to [0,1]$ such that $\sum_{x \in X} \delta(x) = 1$.

**Definition 2.1.** A Markov decision process (MDP for short) consists of

- a finite set of states $Q$,
- a finite set of actions $A$,
- a stochastic transition table $\rho : Q \times A \to D(Q)$.

The interpretation of the transition table is that from the state $p$ under action $a$, the probability to transition to $q$ is $\rho(p,a)(q)$. The transition relation $\Delta$ is defined by

$$\Delta = \{(p,a,q) \in Q \times A \times Q : \rho(p,a)(q) > 0\}.$$ 

We also use $\Delta_a$ given by $\{(p,q) \in Q \times Q : (p,a,q) \in \Delta\}$.

We refer to [Kuc11] for the usual notions related to MDPs; it turns out that very little probability theory will be needed in this paper, so we restrict ourselves to mentioning only the relevant objects. In an MDP $\mathcal{M}$, a strategy is a function $\sigma : Q \to A$; note that we consider only pure and positional strategies, as they will be sufficient for our purposes (see Section 4). Given a source $s \in Q$ and a target $t \in Q$, we say that the strategy $\sigma$ almost surely reaches $t$ if the probability that a path starting from $s$ and consistent with $\sigma$ eventually
leads to \( t \) is 1. As we shall recall in Section 4, whether there exists a strategy ensuring to reach \( t \) almost surely from \( s \), called the \textit{almost sure reachability problem} for MDP can be reduced to solving a non-stochastic two player Büchi game, and in particular does not depend upon the exact probabilities. In other words, the only relevant information for each \((p,a,q) \in \mathcal{Q} \times \mathcal{A} \times \mathcal{Q}\) is whether \( \rho(p,a)(q) > 0 \) or not, equivalently \((p,a,q) \in \Delta\). Since the same will be true for the stochastic control problem we study in this paper, in our examples we do not specify the exact probabilities, and an edge from \( p \) to \( q \) labelled \( a \) means that \( \rho(p,a)(q) > 0 \). When considering a target we implicitly assume that it is a sink: it has a single outgoing transition which is a self-loop.

Let us now fix an MDP \( \mathcal{M} \) and consider a population of \( n \) tokens (we use tokens to represent the agents). Each token evolves in an independent copy of the MDP \( \mathcal{M} \). The controller acts through a strategy \( \sigma : \mathcal{Q}^n \to \mathcal{A} \), meaning that given the state each of the \( n \) tokens is in, the controller chooses one action to be performed by all tokens independently. Formally, we are considering the product MDP \( \mathcal{M}^n \) whose set of states is \( \mathcal{Q}^n \), set of actions is \( \mathcal{A} \), and transition table is \( \rho^n(u,a)(v) = \prod_{i=1}^n \rho(u_i,a)(v_i) \), where \( u,v \in \mathcal{Q}^n \) and \( u_i,v_i \) are the \( i \)th components of \( u \) and \( v \).

Let \( s,t \in \mathcal{Q} \) be the source and target states, we write \( n \cdot s \) and \( n \cdot t \) for the constant \( n \)-tuples where all components are \( s \) and \( t \). This “additive notation” will be more convenient later on than the “product notation” suggested by \( \mathcal{M}^n \). For a fixed value of \( n \), whether there exists a strategy ensuring to reach \( n \cdot t \) almost surely from \( n \cdot s \) can be reduced to solving a two player Büchi game by considering the MDP \( \mathcal{M}^n \) and reducing it to a Büchi game as explained above. The stochastic control problem asks whether this is true for arbitrary values of \( n \):

**Problem 2.2 (Stochastic control problem).**

\textbf{Input:} an MDP \( \mathcal{M} \), a source state \( s \in \mathcal{Q} \), and a target state \( t \in \mathcal{Q} \).

\textbf{Output:} Yes if for all \( n \in \mathbb{N} \), there exists a strategy ensuring to reach \( n \cdot t \) almost surely from \( n \cdot s \), and No otherwise.

Our main result is the following.

**Theorem 2.3.** The stochastic control problem is decidable.

The fact that the problem is co-recursively enumerable is easy to see: if the answer is “no”, there exists \( n \in \mathbb{N} \) such that there exist no strategy ensuring to reach \( n \cdot t \) almost surely from \( n \cdot s \). Enumerating the values of \( n \) and solving the almost sure reachability problem for \( \mathcal{M}^n \) eventually finds this out. However, it is not clear whether one can exhibit an upper bound on such a witness \( n \), which would yield a simple (yet inefficient!) algorithm. As a corollary of our analysis we can indeed derive such an upper bound, but it is non elementary in the size of the MDP.

In the remainder of this section we present a few interesting examples.

**Example 2.4.** Let us consider the MDP represented in Figure 1. We show that for this MDP, for any \( n \in \mathbb{N} \), the controller has a strategy almost surely reaching \( n \cdot t \) from \( n \cdot s \). Starting with \( n \) tokens on \( s \), we iterate the following strategy:

- Repeatedly play action \( a \) until all tokens are in \( q \);
- Play action \( b \).

The first step is eventually successful with probability one, since at each iteration there is a positive probability that the number of tokens in state \( q \) increases. In the second step, with
non zero probability at least one token goes to \( t \), while the rest goes back to \( s \). It follows that each iteration of this strategy increases with non zero probability the number of tokens in \( t \). Hence, all tokens are eventually transferred to \( n \cdot t \) almost surely.

Figure 1: The controller can almost surely reach \( n \cdot t \) from \( n \cdot s \), for any \( n \in \mathbb{N} \).

**Example 2.5.** We now consider the MDP represented in Figure 2. By convention, if from a state some action does not have any outgoing transition (for instance the action \( u \) from \( s \)), then it goes to the sink state \( \perp \).

We show that there exists a controller ensuring to transfer seven tokens from \( s \) to \( t \), but that the same does not hold for eight tokens. For the first assertion, we present the following strategy:

- Play \( a \). One of the states \( q_{i1}^1 \) for \( i_1 \in \{ u, d \} \) receives at least 4 tokens.
- Play \( i_1 \in \{ u, d \} \). At least 4 tokens go to \( t \) while at most 3 go to \( q_1 \).
- Play \( a \). One of the states \( q_{i2}^2 \) for \( i_2 \in \{ u, d \} \) receives at least 2 tokens.
- Play \( i_2 \in \{ u, d \} \). At least 2 tokens go to \( t \) while at most 1 token goes to \( q_2 \).
- Play \( a \). The token (if any) goes to \( q_{i3}^3 \) for \( i_3 \in \{ u, d \} \).
- Play \( i_3 \in \{ u, d \} \). The remaining token (if any) goes to \( t \).

Now assume that there are 8 tokens or more on \( s \). The only choices for a strategy are to play \( u \) or \( d \) on the second, fourth, and sixth moves. First, with non zero probability at least 4 tokens are in each of \( q_{i1}^1 \) for \( i \in \{ u, d \} \). Then, whatever the choice of action \( i \in \{ u, d \} \), there are at least 4 tokens in \( q_1 \) after the next step. Proceeding likewise, there are at least 2 tokens in \( q_2 \) with non zero probability two steps later. Then again two steps later, at least 1 token falls in the sink with non zero probability.

Generalising this example shows that if the answer to the stochastic control problem is “no”, the smallest number of tokens \( n \) for which there exist no almost surely strategy for reaching \( n \cdot t \) from \( n \cdot s \) may be exponential in \(|\mathcal{Q}|\). This can be further extended to show a doubly exponential in \( |\mathcal{Q}| \) lower bound, as done in [BDGG17, BDG+19]; the example produced there holds for both the adversarial and the stochastic setting. Interestingly, for the adversarial setting this doubly exponential lower bound is tight. Our proof for the stochastic setting yields a non-elementary upper bound, leaving a very large gap.

**Example 2.6.** We consider the MDP represented in Figure 3. For any \( n \in \mathbb{N} \), there exists a strategy almost surely reaching \( n \cdot t \) from \( n \cdot s \). We iterate the following strategy:

- Repeatedly play action \( a \) until exactly 1 token is in \( q_1 \).
Figure 2: The controller can synchronize up to 7 tokens on the target state \( t \) almost surely, but not more.

- Play action \( b \). The token goes to \( q_i \) for some \( i \in \{l, r\} \).
- Play action \( i \in \{l, r\} \), which moves the token to \( t \).

Note that the first step may take a very long time (the expectation of the number of \( a \)s to be played until this happens is exponential in the number of tokens), but it is eventually successful with probability one. This very slow strategy is necessary: if \( q_1 \) contains at least two tokens, then action \( b \) should not be played: with non zero probability, at least one token ends up in each of \( q_l, q_r \), so at the next step some token ends up in \( \bot \). It follows that any strategy almost surely reaching \( n \cdot t \) can play \( b \) only when there is at most one token in \( q_1 \).

This is a key example for understanding the difficulty of the stochastic control problem.

3. The Sequential Flow Problem

We introduce first a simple variant of the sequential flow problem. Our motivations are twofold: first it will be a technically convenient step for our solution of the general problem, second we provide a complexity lower bound already for this simpler problem.

We let \( Q \) be a finite set of states.

**Capacities.** A *capacity* is an element \( a \in (\mathbb{N} \cup \{\omega\})^{Q \times Q} \), intuitively it defines for each pair of states \((p, q)\) the maximal value that can be transported from \( p \) to \( q \) in one step. Setting \( a(p, q) = 0 \) means that there is no edge from \( p \) to \( q \). Using \( \omega \) as a value is interpreted as having an edge with unbounded capacity.

**Flows.** Let us call *flow* an element \( f \in \mathbb{N}^{Q \times Q} \). We call *configuration* an element of \( \mathbb{N}^Q \). A flow \( f \) induces two configurations \( \text{pre}(f) \) and \( \text{post}(f) \) defined by

\[
\text{pre}(f)(p) = \sum_{q \in Q} f(p, q) \quad \text{and} \quad \text{post}(f)(q) = \sum_{p \in Q} f(p, q).
\]

Given \( c, c' \) two configurations and \( f \) a flow, we say that \( c \) goes to \( c' \) using \( f \) and write \( c \rightarrow^f c' \), if \( c = \text{pre}(f) \) and \( c' = \text{post}(f) \).
Figure 3: The controller can synchronise any number of tokens almost surely on the target state $t$, but since anytime at most one token can be in $\{q_l, q_r\}$, each token must be sent alone from $q_1$ using $b$.

For a flow $f$, we write $\text{supp}(f) = \{(p, q) \in Q^2 : f(p, q) > 0\}$.

**Flow ideals.** For a capacity $a \in (\mathbb{N} \cup \{\omega\})^{Q \times Q}$ we write
$$a \downarrow = \{f \in \mathbb{N}^{Q \times Q} : f \leq a\},$$
and call $a \downarrow$ the (flow) ideal induced by $a$. This terminology will be justified later when using the framework of well quasi orders.

**Sequential flows.** We introduce now an extension of the classical (max-)flow problem with a sequential aspect: instead of a single capacity $a$ we have a finite set of capacities $\mathcal{C}$, and we go from a configuration $c$ to another configuration $c'$ by applying a sequence of flows $f_1, \ldots, f_\ell$ with each flow belonging to some $a \downarrow$ with $a \in \mathcal{C}$.

Formally, a flow word is $f = f_1 \ldots f_\ell$ where each $f_i$ is a flow. We write $c \rightsquigarrow^f c'$ if there exists a sequence of configurations $c_0, c_1, \ldots, c_\ell$ such that $c = c_0$, $c' = c_\ell$, and $c_{i-1} \rightarrow^{f_i} c_i$ for all $i \in [1, \ell]$. In this case, we say that $c$ goes to $c'$ using the flow word $f$.

A capacity word is a finite sequence of capacities. For a flow word $f = f_1 \ldots f_\ell$ and a capacity word $w = a_1 \ldots a_\ell$ we write $f \leq w$ to mean that $f_i \leq a_i$ for each position $i$.

**Configuration ideals.** Given a state $q$, we write $q \in \mathbb{N}^Q$ for the vector which has value 1 on the $q$ component and 0 elsewhere. We use additive notations: for instance, $2 \cdot q_1 + q_2$ has value 2 in the $q_1$ component, 1 in the $q_2$ component, and 0 elsewhere. For a vector $x \in (\mathbb{N} \cup \{\omega\})^Q$ we write
$$x \downarrow = \{c \in \mathbb{N}^Q : c \leq x\},$$
and call $x \downarrow$ the (configuration) ideal induced by $x$. 
The simple sequential flow problem. Let us first introduce the simple variant of the sequential flow problem and give an example.

**Problem 3.1 (Simple sequential flow problem).**

**Input:** $Q$ a finite set of states, $\mathcal{C}$ a finite set of capacities, $s$ a source state, and $F$ a target ideal of configurations.

**Output:** Yes if for all $n \in \mathbb{N}$, there exists a capacity word $w \in \mathcal{C}^*$, a flow word $f \leq w$ and a final capacity $c \in F$ such that $n \cdot s \leadsto f c$, and No otherwise.

When considering an instance of the simple sequential flow problem we let $\text{Flows} = \bigcup_{a \in \mathcal{C}} a \downarrow$ denote the downward closed set of flows, $I = (\omega \cdot s) \downarrow$, and $F = (\omega \cdot t) \downarrow$. We call $I$ the source ideal and $F$ the target ideal.

In the instance of the simple sequential flow problem represented in Figure 4, we ask the following question: can $F = (\omega \cdot q_4) \downarrow$ be reached from any configuration of $I = (\omega \cdot q_1) \downarrow$ using the capacities $\mathcal{C} = \{a, b, c\}$? The answer is yes: for every $n$, there exists a flow word $f \leq (ac^{n-1}b)^n$ such that $n \cdot q_1 \leadsto^f n \cdot q_4 \in F$. The flow word $f$ is represented in Figure 5.

![Figure 4](image-url)

**Figure 4:** An instance of the simple sequential flow problem. Can $F = (\omega \cdot q_4) \downarrow$ be reached from any configuration of $I = (\omega \cdot q_1) \downarrow$ using the three capacities $a$, $b$, and $c$?

![Figure 5](image-url)

**Figure 5:** A flow word $f = f_1 f_2 \ldots f_{n+1} \leq ac^{n-1}b$ such that $n \cdot q_1$ goes to $(n - 1) \cdot q_1 + q_4$ using $f$. This construction can be iterated to construct $f \leq (ac^{n-1}b)^n$ such that $n \cdot q_1$ goes to $n \cdot q_4$ using $f$.

The general sequential flow problem extends its simple variant by considering more general sets of configurations; in the simple sequential flow problem we ask whether all configurations of $I = (\omega \cdot s) \downarrow$ go to some configuration in $F = (\omega \cdot t) \downarrow$, in the general sequential flow problem we consider an arbitrary downward closed set $F$ and ask to compute the (downward closed) set of all configurations reaching $F$. Towards defining the sequential flow problem let us recall some classical terminology about well quasi orders, which allow us
to manipulate general downward closed sets ([Hig52, Kru72], see [Sch17] for an exposition of recent results).

**Well quasi orders.** Let \((E, \leq)\) be a quasi ordered set \((i.e. \leq \text{ is reflexive and transitive})\), it is a well quasi ordered set (WQO) if any infinite sequence contains a non-decreasing pair. A set \(S\) is downward closed if for any \(x \in S\), if \(y \leq x\) then \(y \in S\). An ideal is a non-empty downward closed set \(I \subseteq E\) such that for all \(x, y \in I\), there exists some \(z \in I\) satisfying both \(x \leq z\) and \(y \leq z\).

We equip the set of configurations \(\mathbb{N}^Q\) and the set of flows \(\mathbb{N}^{Q \times Q}\) with the quasi orders \(\leq\) defined component wise; thanks to Dickson’s Lemma [Dic13] they are both WQOs.

**Lemma 3.2.** [Folklore, see [Sch17]] Let \((E, \leq)\) be a WQO.

- Any infinite sequence of non-increasing downward closed sets is eventually constant.
- A subset is downward closed if and only if it is a finite union of incomparable ideals. We call it its decomposition into ideals (or simply, its decomposition), which is unique (up to permutation).
- An ideal is included in a downward closed set if and only if it is included in one of the ideals of its decomposition.

Let \(X\) be a finite set, we consider the WQO \((\mathbb{N}^X, \leq)\) with \(\leq\) defined component wise. A subset of \(\mathbb{N}^X\) is an ideal if and only if it is of the form

\[ a \downarrow = \{ c \in \mathbb{N}^X \mid c \leq a \}, \]

for some \(a \in (\mathbb{N} \cup \{\omega\})^X\).

Thanks to Lemma 3.2, we represent downward closed sets of configurations and flows using their decomposition into ideals.

The most general definition of the sequential flow problem is the following. Let \(Q\) be a finite set of states, \(C\) a finite set of capacities inducing \(\text{Flows} \subseteq \mathbb{N}^{Q \times Q}\) a downward closed set of flows, \(F \subseteq \mathbb{N}^Q\) a downward closed set of configurations, and \(F \subseteq \mathbb{N}^Q\) a downward closed set of configurations, let us define the following downward closed set \(\text{Pre}^*\):

\[ \text{Pre}^*(\text{Flows}, F) = \{ c \in \mathbb{N}^Q \mid c \sim_f c' \in F, \ f \in \text{Flows}\} , \]

i.e. the configurations from which one may reach \(F\) using only flows from \(\text{Flows}\). The objective is to construct \(\text{Pre}^*(\text{Flows}, F)\).

The following classical result of [VJ85] allows us to further reduce our problem. Informally, it says that the task of constructing a downward closed set can be reduced to the task of determining whether an ideal is included in a downward closed set.

**Lemma 3.3.** Let \(X\) be a downward closed set. Assume there exists an algorithm solving the following problem:

**Input:** \(I\) ideal

**Output:** Yes if \(I \subseteq X\), and No otherwise.

Then there is an algorithm constructing the decomposition of \(X\) into ideals.

Noting that \(\text{Pre}^*(\text{Flows}, F_1 \cup F_2) = \text{Pre}^*(\text{Flows}, F_1) \cup \text{Pre}^*(\text{Flows}, F_2)\), we can without loss of generality assume that \(F\) is a single ideal. We therefore define the sequential flow problem as follows.

**Problem 3.4 (Sequential flow problem)**
**Input:** $Q$ a finite set of states, $\mathcal{C}$ a finite set of capacities, $a$ a source capacity, and $b$ a target capacity. We let $\text{Flows} = \bigcup_{c \in \mathcal{C}} c \downarrow$ denote the downward closed set of flows induced by $\mathcal{C}$, $I = a \downarrow$ the source ideal, and $F = b \downarrow$ the target ideal.

**Output:** Yes if $I \subseteq \text{Pre}^*(\text{Flows}, F)$, meaning whether $F$ can be reached from all configurations of $I$ using only flows from Flows, and No otherwise.

As argued above, an algorithm solving the sequential flow problem actually yields an algorithm constructing the ideal decomposition of $\text{Pre}^*(\text{Flows}, F)$.

### 4. Reduction from the Stochastic Control Problem to the Sequential Flow Problem

Let us consider an MDP $\mathcal{M}$ and a target $t \in Q$. We first recall a folklore result reducing the almost sure reachability question for MDPs to solving a two player Büchi game (we refer to [GTW02] for the definitions and notations of Büchi games). The Büchi game is played between Eve and Adam as follows. From a state $p$:

1. Eve chooses an action $a$ and a transition $(p, q) \in \Delta_a$;
2. Adam can either choose to
   - **agree:** and the game continues from $q$, or
   - **interrupt:** and choose another transition $(p, q') \in \Delta_a$, the game continues from $q'$.

We say that Eve wins if either the target state $t$ is reached or Adam interrupts infinitely many times. This is indeed a Büchi objective where the Büchi transitions are the interruptions of Adam and the self-loop around $t$.

**Lemma 4.1.** There exists a strategy ensuring almost surely to reach $t$ from $s$ if and only if Eve has a winning strategy from $s$ in the above Büchi game.

We now explain how this reduction can be extended to the stochastic control problem. Let us consider an MDP $\mathcal{M}$ and a target $t \in Q$. We define an infinite Büchi game $G_M$. The set of vertices is the set of configurations $NQ$. The game is played as follows from a configuration $c$:

1. Eve chooses an action $a$ and a flow $f$ such that $\text{pre}(f) = c$ and $\text{supp}(f) \subseteq \Delta_a$.
2. Adam can either choose to
   - **agree:** and the game continues from $c' = \text{post}(f)$
   - **interrupt:** and choose a flow $f'$ such that $\text{pre}(f') = c$ and $\text{supp}(f') \subseteq \Delta_a$, and the game continues from $c'' = \text{post}(f')$.

Note that Eve choosing a flow $f$ is equivalent to choosing for each token a transition $(p, q) \in \Delta_a$, inducing the configuration $c'$, and similarly for Adam should he decide to interrupt.

We now define the winning objective: Eve wins if either all tokens are in the target state, or if Adam interrupts infinitely many times. Again this is a Büchi objective.

Although the game is infinite, it is actually a disjoint union of finite games. Indeed, along a play the number of tokens is fixed, so all visited configurations belong to $Q^n$ for some $n \in \mathbb{N}$.

**Lemma 4.2.** Let $c$ be a configuration with $n$ tokens in total, the following are equivalent:

- There exists a strategy almost surely reaching $n \cdot t$ from $c$,
- Eve has a winning strategy in the Büchi game $G_M$ starting from $c$. 
Lemma 4.2 follows from applying Lemma 4.1 on the product MDP $\mathcal{M}^\alpha$.

We define the game $\mathcal{G}_M^{(i)}$ for $i \in \mathbb{N}$, which is defined just as $\mathcal{G}_M$ except for the winning objective: Eve wins in $\mathcal{G}_M^{(i)}$ if either all tokens are in the target state, or if Adam interrupts more than $i$ times. It is clear that if Eve has a winning strategy in $\mathcal{G}_M$ then she has a winning strategy in $\mathcal{G}_M^{(i)}$. Conversely, the following result states that $\mathcal{G}_M^{(i)}$ is equivalent to $\mathcal{G}_M$ for some $i$. Let $F$ be the set of configurations for which all tokens are in state $t$. We let $X^{(i)} \subseteq \mathbb{N}^Q$ denote the winning region for Eve in the game $\mathcal{G}_M^{(i)}$, and $X^{(\infty)} \subseteq \mathbb{N}^Q$ denote the winning region for Eve in the game $\mathcal{G}_M$.

**Lemma 4.3.**
- If $X^{(i)} = X^{(i+1)}$, then $X^{(i)} = X^{(i+j)}$ for all $j \geq 0$.
- $(X^{(i)})_{i \in \mathbb{N}}$ is a non-increasing sequence of downward closed sets.
- There exists $i \in \mathbb{N}$ such that $X^{(i)} = X^{(\infty)}$.

**Proof.** The first item follows from the definition of $X^{(i)}$.

For the second item, we argue that each $X^{(i)}$ is downward closed: if $c \subseteq c'$ and $c' \in X^{(i)}$, then $c \in X^{(i)}$. Let $\sigma$ be a winning strategy from $c'$, we show that it induces a strategy from $c$: for any play from $c'$ ending in $c_e'$ there exists a play from $c$ ending in $c_e$ consistent with $\sigma$ such that $c_e \subseteq c_e'$ and the two plays are interrupted at the same steps. This is easily proved by induction on the length of the play. Since $\sigma$ is a winning strategy from $c'$ all plays either reach $n \cdot t$ for some $n \in \mathbb{N}$ or Adam interrupts $i$ times, so there exists a length $N$ such that either happens within the first $N$ steps. The invariant above implies that the corresponding strategy is winning from $c$.

For the third item, let $X = \bigcap_{i \in \mathbb{N}} X^{(i)}$, we first argue that $X = X^{(\infty)}$. It is clear that $X^{(\infty)} \subseteq X$: if Eve has a strategy to ensure that either all tokens are in the target state, or that Adam interrupts infinitely many times, then it particular this is true for Adam interrupting more than $i$ times for any $i \in \mathbb{N}$. For the converse inclusion, we show that $\mathbb{N}^Q \setminus X^{(\infty)} \subseteq \mathbb{N}^Q \setminus X$: let $c$ such that Adam has a winning strategy in $\mathcal{G}_M$ from $c$. By positional determinacy of Büchi games, we can choose the strategy to be positional. Let $n$ the number of tokens in $c$, and $i > n \cdot |Q|$, we claim that Adam’s strategy ensures never to interrupt more than $i$ times from $c$. Indeed if there exists a play from $c$ consistent with that strategy where he interrupts more than $i$ times, then a simple pumping argument yields a play from $c$ consistent with the strategy interrupting infinitely many times, a contradiction. Thus $c$ is in $\mathbb{N}^Q \setminus X^{(i)}$, hence a fortiori in $\mathbb{N}^Q \setminus X$.

Thanks to the second item $(X^{(i)})_{i \geq 0}$ is a non-increasing sequence of downward closed sets in $\mathbb{N}^Q$, so it stabilises thanks to Lemma 3.2, i.e. there exists $i_0 \in \mathbb{N}$ such that $X^{(i_0)} = \bigcap_i X^{(i)}$, which concludes the proof of the third item. \hfill $\Box$

Note that Lemma 4.2 and Lemma 4.3 substantiate the claims made in Section 2: pure positional strategies are enough and the answer to the stochastic control problem does not depend upon the exact probabilities in the MDP. Indeed, the construction of the Büchi games does not depend on them, and the answer to the former is equivalent to determining whether Eve has a winning strategy in each of them.

We are now fully equipped to show that a solution to the sequential flow problem yields the decidability of the stochastic control problem. Note first that $X^{(0)} = \text{Pre}^*(\text{Flows}^0, F)$ where

$$\text{Flows}^0 = \{ f \in \mathbb{N}^Q \times \mathcal{Q} \mid \exists a \in A, \text{ supp}(f) \subseteq \Delta_a \}.$$
Indeed, in the game $G_M^{(0)}$ Adam cannot interrupt as this would make him lose immediately. Hence, the winning region for Eve in $G_M^{(0)}$ is $\text{Pre}^*(\text{Flows}^0, F)$. We generalise this by setting $\text{Flows}^i$ for all $i > 0$ to be the set of flows $f \in \mathbb{N}^Q \times \mathbb{Q}$ such that for some action $a \in A$,

- $\text{supp}(f) \subseteq \Delta_a$, and
- for every $f'$, if $\text{pre}(f') = \text{pre}(f)$ and $\text{supp}(f') \subseteq \Delta_a$, then $\text{post}(f') \in X^{(i-1)}$.

Equivalently, this is the set of flows for which, when played in the game $G_M$ by Eve, Adam cannot use an interrupt move and force the configuration outside of $X^{(i-1)}$.

**Lemma 4.4.** For all $i \in \mathbb{N}$,

$X^{(i)} = \text{Pre}^*(\text{Flows}^i, F)$

Before proving this key lemma, let us discuss how it induces an algorithm for solving the stochastic control problem. Let us write $I = (\omega \cdot s) \downarrow$ for the source ideal. Thanks to Lemma 4.2 the answer to the stochastic control problem is Yes if and only if $\text{Pre}^*(\text{Flows}^i, F)$.

**Proof.** We proceed by induction on $i$.

Let $c$ be a configuration in $\text{Pre}^*(\text{Flows}^i, F)$. This means that there exists a flow word $f_1 \cdots f_k$ such that $f_k \in \text{Flows}^i$ for all $k$, and $c \sim_f c' \in F$. Expanding the definition, there exist $c_0 = c, \ldots, c_k = c'$ such that $c_{k-1} \rightarrow f_k c_k$ for all $k$.

Let us now describe a strategy for Eve in $G_M^{(i)}$ starting from $c$. As long as Adam agrees, Eve successively chooses the sequence of flows $f_1, f_2, \ldots$ and the corresponding configurations $c_1, c_2, \ldots$. If Adam never interrupts, then the game reaches the configuration $c' \in F$, and Eve wins. Otherwise, as soon as Adam interrupts, by definition of $\text{Flows}^i$, we reach a configuration $d \in X^{(i-1)}$. By induction hypothesis, Eve has a strategy which ensures from $d$ to either reach $F$ or that Adam interrupts at least $i-1$ times. In the latter case, adding the interrupt move leading to $d$ yields at least $i$ interrupts, so this is a winning strategy for Eve in $G_M^{(i)}$, witnessing that $c \in X^{(i)}$.

Conversely, assume that there is a winning strategy $\sigma$ of Eve in $G_M^{(i)}$ from a configuration $c$. Consider a play consistent with $\sigma$, it either reaches $F$ or Adam interrupts. Let $f = f_1 f_2 \ldots f_k$ denote the sequence of flows until then. We argue that $f_k \in \text{Flows}^i$ for $k \in [1, \ell]$. Let $f = f_k$ for some $k$, by definition of the game $\text{supp}(f) \subseteq \Delta_a$ for some action $a$. Let $f'$ such that $\text{pre}(f') = \text{pre}(f)$ and $\text{supp}(f') \subseteq \Delta_a$. In the game $G_M$ after Eve played $f_k$, Adam has the possibility to interrupt and choose $f'$. From this configuration onward the strategy $\sigma$ is winning in $G_M^{(i-1)}$, implying that $f \in \text{Flows}^i$. Thus $f = f_1 f_2 \ldots f_k$ is a witness that $c \in X^{(i)}$. \hfill $\Box$

5. **Solution to the simple sequential flow problem**

We study the simple sequential flow problem, stated as follows. Let $Q$ be a finite set of states, $\text{Flows} \subseteq \mathbb{N}^Q \times \mathbb{Q}$ a downward closed set of flows represented as the finite union of capacity ideals from $G$, $s$ a source state inducing $I = (\omega \cdot s) \downarrow$ the source ideal, and $F = b \downarrow$ the target ideal. Determine whether for all $n \in \mathbb{N}$, there exists a configuration $c \in F$, a capacity word $w \in G^*$, and a flow word $f \leq w$ such that $n \cdot s \sim_f c$. 

Theorem 5.1. The simple sequential flow problem is decidable in EXPSPACE.

The proof makes use of two ingredients: distance automata and the max-flow min-cut theorem.

Distance automata are weighted automata over the semiring \((\mathbb{N} \cup \{\omega\}, +, \min)\): a weighted automaton over the alphabet \(\Sigma\) is given by a set of states \(Q\), an initial state \(q_0 \in Q\), a vector of final costs \(\eta \in (\mathbb{N} \cup \{\omega\})^Q\), and a transition relation

\[
\Delta \subseteq Q \times \Sigma \times (\mathbb{N} \cup \{\omega\}) \times Q,
\]

with the following interpretation: \((p, a, c, q) \in \Delta\) is a transition from \(p\) to \(q\) reading letter \(a\) with cost \(c\). A run \(\rho\) over \(w = w_1 \ldots w_\ell \in \Sigma^*\) is a sequence of transitions

\[
\rho = (q_0, w_1, c_1, q_1)(q_1, w_2, c_2, q_2) \ldots (q_{\ell-1}, w_\ell, c_\ell, q_\ell).
\]

The value of \(\rho\) is \(\text{val}(\rho) = \sum_{i \in [1, \ell]} c_i + \eta(q_\ell)\). A weighted automaton \(A\) induces the function \([A] : \Sigma^* \rightarrow \mathbb{N} \cup \{\omega\}\) defined by

\[
[A](w) = \min_{\rho \text{ run over } w} \text{val}(\rho).
\]

We say that \(A\) recognises the function \([A]\).

The boundedness problem for distance automata asks whether for a given distance automaton \(A\) the function \([A]\) is bounded, meaning:

\[
\exists n \in \mathbb{N}, \forall w \in \Sigma^*, [A](w) \leq n.
\]

Theorem 5.2 [Kir05]. The boundedness problem for distance automata is decidable in PSPACE.

We reduce the simple sequential flow problem to the boundedness problem for distance automata by considering the following function:

\[
\Phi : C^* \rightarrow \mathbb{N} \cup \{\omega\},
\]

\[
w \mapsto \sup\{n \in \mathbb{N} \mid \exists c \in F, \exists f, f \leq w \text{ and } n \cdot s \leadsto^f c\}.
\]

In words, given as input a word of capacities \(w\), \(\Phi\) computes the largest number of tokens one may synchronize in \(F\) using \(w\).

By construction, whether for all \(n \in \mathbb{N}\), there exists a configuration \(c \in F\), a capacity word \(w \in C^*\) and a flow word \(f \leq w\) such that \(n \cdot s \leadsto^f c\) is equivalent to \(\Phi\) being unbounded. We will prove the following.

Lemma 5.3. The function \(\Phi\) is computed by a distance automaton with \(2^{|Q|}\) states.

Theorem 5.1 follows by combining Theorem 5.2 and Lemma 5.3. The construction of a distance automaton for Lemma 5.3 relies on the standard max-flow min-cut theorem, which we expose now.

Max-flow min-cut theorem. In the classical (max-)flow problem the input is a capacity \(a \in (\mathbb{N} \cup \{\omega\})^{V \times V}\), a source vertex \(s\), and a target vertex \(t\). The capacity \(a\) defines a graph over the set of states \(V\): there is an edge from \(u\) to \(v\) if \(a(u, v) \neq 0\). The objective is to compute the maximal \((s, t)\)-flow respecting the capacity \(a\). An \((s, t)\)-flow is an element \(f \in \mathbb{N}^{V \times V}\) such that for all vertices \(v\) except \(s\) and \(t\) we have a conservation law:

\[
\sum_{v' \in V} f(v', v) = \sum_{v' \in V} f(v, v').
\]
The value of an \((s,t)\)-flow is \(|f| = \sum_{v \in V} f(s,v)\), which thanks to conservation laws is also equal to \(\sum_{v' \in V} f(v,t)\). We say that the flow \(f\) respects the capacity \(a\) if for all \(v,v' \in V\) we have \(f(v,v') \leq a(v,v')\).

An \((s,t)\)-cut is a set of edges \(C \subseteq V \times V\) such that removing them disconnects \(t\) from \(s\). The cost of a cut is the sum of the weight of its edges: \(|C| = \sum_{(v,v') \in C} a(v,v')\). The max-flow min-cut theorem states that the maximal value of an \((s,t)\)-flow is exactly the minimal cost of an \((s,t)\)-cut ([FF56]). With this in hands, we are now ready to prove Lemma 5.3.

**Proof of Lemma 5.3.** Let us fix a capacity word \(w = w_1 \ldots w_\ell \in \mathcal{C}^\ast\). Consider the finite graph \(G_w\) with vertex set \((Q \times [0,\ell]) \cup \{t\}\), where \(t\) is an additional fresh vertex, and for all \(i \in [1,\ell]\), an edge from \((q,i-1)\) to \((q',i)\) labelled by \(w_i(q,q')\), and an edge labelled by \(b(q)\) from \((q,\ell)\) to \(t\). Then \(\Phi(w)\) is the maximal value of an \(((s,0),t)\)-flow in \(G_w\). Thanks to the max-flow min-cut theorem, this is also the minimal cost of an \(((s,0),t)\)-cut in \(G_w\).

The distance automaton we construct guesses an \(((s,0),t)\)-cut and outputs its cost, hence its value on a given word corresponds to the minimal cost cut. To verify that the current run indeed corresponds to a cut, it suffices to remember the set of reachable states from \((s,0)\) using a powerset construction. In the last layer, to obtain a cut, one is forced to cut the edge from \((q,\ell)\) to \(t\), which has cost \(b(q)\), for all states \(q\) such that \((q,\ell)\) is reachable from \((s,0)\).

The set of states of \(A\) is \(\mathcal{P}(Q)\), the initial state is \(\{s\}\), and the vector of final costs \(\eta \in (\mathbb{N} \cup \{\omega\})^{\mathcal{P}(Q)}\) takes value \(\sum_{x \subseteq X} b(x)\) on subset \(X \in \mathcal{P}(Q)\). The transition relation \(\Delta\) is defined as follows: \((X,a,c,Y) \in \Delta\) if \(Y \subseteq Z = \{q' \in Q : \exists q \in X, a(q,q') \neq 0\}\) and \(c = \sum_{q \in X} \sum_{q' \in Z \setminus Y} a(q,q')\). In words: from the set \(X\) representing the set of reachable states from \((s,0)\) without the edges already cut, the new set \(Y\) is a subset of the set of reachable states from \(X\) reading \(a\), obtained by removing a set of edges whose cost is accounted for in the current value of the run.

A run of the automaton \(A\) corresponds to an \(((s,0),t)\)-cut, and its value is the cost of the cut. Minimising over all runs yields the minimal cost of an \(((s,0),t)\)-cut, so \([A] = \Phi\) by the max-flow min-cut theorem.

---

6. Solution to the sequential flow problem

This section builds on Lemma 5.3 in order to obtain decidability of the general sequential flow problem. Let \(Q\) be a finite set of states, \(\mathcal{E}\) a finite set of capacities, \(a\) a source capacity and \(b\) a target capacity. We let \(\text{Flows} = \bigcup_{a \in \mathcal{E}} a \downarrow\) denote the downward closed set of flows, \(I = a \downarrow\) the source ideal, and \(F = b \downarrow\) the target ideal. We ask whether \(I \subseteq \text{Pre}^\ast(\text{Flows},F)\), that is, whether \(F\) can be reached from any configuration of \(I\) using only flows from \(\text{Flows}\).

**Theorem 6.1.** The sequential flow problem is decidable.

To prove Theorem 6.1, we will use so-called desert automata. Dually to distance automata, these non-deterministic automata output the maximal value over all runs. Formally, desert automata are weighted automata over the semi-ring \((\mathbb{N} \cup \{\omega\}, +, \max)\). They are defined as tuples \((Q, q_0, \delta, \eta)\), just as distance automata (see Section 5), except for their semantics which is now given by

\[
[A](w) = \max_{\rho \text{ run over } w} \text{val}(\rho).
\]

We say that \(A\) recognises the function \([A]\).
The theory of regular cost functions. The following duality theorem is crucial to our proof and is a central result of the theory of regular cost functions [Col13], which is a set of tools for solving boundedness questions. Since in the theory of regular cost functions, when considering functions we are only interested in whether they are bounded or not, we consider functions “up to boundedness properties”. Concretely, this means that a cost function is an equivalence class of functions \( \Sigma^* \to \mathbb{N} \cup \{\omega\} \), with the equivalence being \( f \approx g \) if there exists \( \alpha : \mathbb{N} \to \mathbb{N} \) such that \( f(w) \) is finite if and only if \( g(w) \) is finite, and in this case, \( f(w) \leq \alpha(g(w)) \) and \( g(w) \leq \alpha(f(w)) \). This is equivalent to stating that for all \( X \subseteq \Sigma^* \), \( f \) is bounded over \( X \) if and only if \( g \) is bounded over \( X \).

**Theorem 6.2** [Col13]. Let \( A' \) be a distance automaton. There exists a desert automaton \( A \) recognising the cost function \([A']\).

The boundedness problem for desert automata is decidable.

**Theorem 6.3** [Kir05]. The boundedness problem for desert automata is decidable in PSPACE.

Our goal is to determine the boundedness of the function

\[
\Phi : \mathcal{C}^* \to \mathbb{N} \cup \{\omega\}
\]

\[
w \mapsto \sup\{n \in \mathbb{N} \mid \exists f, f \leq w \text{ and } a_n \leadsto f F\}.
\]

**Lemma 6.4.** The function \( \Phi \) is computed by a desert automaton.

**Proof.** We let \( S = \{s \in Q \mid a(s) = \omega\} \), and for \( n \in \mathbb{N} \cup \{\omega\} \), let \( a_n \) denote the vector obtained from \( a \in (\mathbb{N} \cup \{\omega\})^Q \) by replacing occurrences of \( \omega \) by \( n \). Note that \( a = a_0 + \sum_{s \in S} \omega \cdot s \).

The automaton recognising \( \Phi \) will non-deterministically decompose a flow into synchronous flows:

- a finitary part bringing \( a_0 \) to the target, and
- for each \( s \in S \), an instance of the simple sequential flow problem with source \( s \).

We now formalise the notion of decomposition. Recall that \( \mathcal{C} \) is a finite set of capacities, let \( k = \max\{n \in \mathbb{N} : \exists a \in C, q \in Q, a(q) = n\} \) the largest integer appearing in \( \mathcal{C} \): then \( \mathcal{C} \subseteq ([0, k] \cup \{\omega\})^Q \).

Let \( J = S \cup \{\text{fin}\} \), we consider the alphabets

\[
\Sigma_2 = \left( ([0, k] \cup \{\omega\})^Q \right)^J ; \quad \Sigma_1 = \left( ([0, k] \cup \{\omega\})^Q \right)^J ; \quad \Sigma = \Sigma_2 \cup \Sigma_1.
\]

For a letter \( a^J = (a^j)_{j \in J} \) in \( \Sigma_2 \) we say that \( a^J \) is a decomposition of \( a \in \mathcal{C} \) if \( \sum_{j \in J} a^j = a \), and similarly a letter \( b^J \in \Sigma_1 \) is a decomposition of the target ideal \( b \) if \( \sum_{j \in J} b^j = b \).

We extend decompositions to words by considering each position: a word \( w^J \in \Sigma_w^* \) is a decomposition of \( w \in C^* \) if for each position \( i \), the letter \( w_i^J \) is a decomposition of \( w_i \).

We start with constructing the following automata.

- the distance automaton \( A_{\text{fin}} \) has semantics

\[
[A_{\text{fin}}](w_{\text{fin}}, b_{\text{fin}}) = \begin{cases} \omega \text{ if } \exists f, \ f \leq w_{\text{fin}} \text{ and } a_0 \leadsto f b_{\text{fin}}, \\ 0 \text{ otherwise}. \end{cases}
\]

Every transition has cost 0 or \( \omega \); the automaton non-deterministically guesses the path of each of the finitely many tokens. The set of states is \( \mathbb{N}^p \) where \( p = \sum_{q \in Q} a_0(q) \).
For each \( s \in S \), the distance automaton \( A_s \) computes

\[
[A_s](w^s b^s) = \sup \left\{ n \in \mathbb{N} \mid \exists f, f \leq w^s \text{ and } n \cdot s \rightsquigarrow f b^s \right\},
\]

and its existence is guaranteed by Lemma 5.3.

Note that \( \Phi(w) = \max_{w^J, b^J} \min_{j \in J} [A_j](w^j b^j) \).

To obtain a desert automaton recognising the cost function \( \Phi \), we need three more steps. The first is to construct a distance automaton \( A' \) computing

\[
[A'](w^J b^J) = \min_{j \in J} [A_j](w^j b^j),
\]

which is obtained as a direct product of the automata \( (A_j)_{j \in J} \). The second step is to apply Theorem 6.2 to obtain a desert automaton \( A_d \) recognising the cost function \( [A'] \). The third step is to construct a desert automaton \( A \) guessing the decomposition:

\[
[A](w) = \max_{w^J, b^J} \min_{j \in J} [A_d](w^j b^J).
\]

By construction, \( A \) recognises the cost function \( \Phi \).

Theorem 6.1 follows by combining Theorem 6.3 and Lemma 6.4.

7. Lower bound for the simple sequential flow problem

We now provide a PSPACE lower bound for simple SFP, by reduction from NFA intersection emptiness [Koz77].

**Problem 7.1 (NFA intersection emptiness).** Given \( k \) non-deterministic finite automata \( A_1, \ldots, A_k \), decide whether \( \bigcap_i L(A_i) = \emptyset \).

Recall that the instance discussed in Figure 4 is an instance of simple SFP. It can be used as a gadget to encode intersection of two NFAs, essentially by encoding in action \( c \) a synchronous run in both NFA, one from \( q_2 \) to \( q_1 \) and one from \( q_3 \) to \( q_3 \).

**The gadget \( S_k \).** We now introduce and study a gadget \( S_k \), which generalizes the one from Figure 4 and is tailored to encode synchronous runs of \( k \) NFA. It is illustrated in Figure 6.

- There are \( 2k + 1 \) states, \( s, r_1, t_1, r_2, t_2, \ldots, r_k, t_k \). The source is \( s \) and the target is \( t_k \). Intuitively, to move a token from \( s \) to \( t_i \), one must first move it into \( r_i \), then recursively move all tokens to \( t_{i-1} \), before the token in \( r_i \) is allowed to move to \( t_i \), which drives all other tokens back to the source \( s \).
- There are \( 2k \) capacities \( a_1, b_1, \ldots, a_k, b_k \). The role of \( a_i \) is to allow to move a token to \( r_i \), provided all tokens are in \( \{s, t_i, t_{i+1}, t_{i+1}, \ldots, r_k, t_k\} \). The role of \( b_i \) is to allow a token to move from \( r_i \) to \( t_i \), provided all tokens are in \( \{t_{i-1}, r_i, t_i, \ldots, r_k, t_k\} \). Additionally, \( b_i \) resets all tokens from \( t_{i-1} \) back to \( s \). Formally, we have

\[
a_i(q, q') = \begin{cases} 
\omega & \text{if } q = q' \in \{s, t_i, t_{i+1}, \ldots, t_k\} \\
1 & \text{if } q = q' \in \{r_{i+1}, \ldots, r_k\} \text{ or } (q, q') = (s, r_i) \\
0 & \text{otherwise,}
\end{cases}
\]

\(^1\text{For this definition when } i = 1, \text{ we identify } s \text{ and } t_{i-1}.\)
Figure 6: The capacities $a_i$ and $b_i$ which define the gadget $S_k$

and

$$b_i(q, q') = \begin{cases} 
\omega & \text{if } q = q' \in \{t_i, t_{i+1}, \ldots, t_k\} \text{ or } (q, q') = (t_{i-1}, s) \\
1 & \text{if } q = q' \in \{r_{i+1}, \ldots, r_k\} \text{ or } (q, q') = (r_i, t_i) \\
0 & \text{otherwise}
\end{cases}$$

Towards introducing our reduction from NFA intersection, we study the behaviour of $S_k$. Let us first see that it is a positive instance of simple SFP.

**Lemma 7.2.** For all $n$, there exists a word of capacities $w(n) \in C^*$ and a flow word $f(n) \leq w(n)$ such that $n \cdot s \leadsto_f n \cdot t_k$.

**Proof.** We prove by induction over $i \in \{1, \ldots, k\}$ that from any configuration $c$ such that $c(s) = n, c(r_1) = c(t_1) = \cdots = c(r_{i-1}) = c(t_{i-1}) = c(r_i) = 0, c(r_{i+1}) = c(r_{i+2}) = \cdots = c(r_k) = 1$, and $c(t_i), c(t_{i+1}) \ldots c(t_k)$ are arbitrary, there exists a capacity word $w \in \{a_1, b_1, \ldots, a_i, b_i\}^*$ and a flow word $f \leq w$ such that $c \leadsto_f c'$, where $c'$ is identical to $c$ except for a token which has moved from $s$ to $t_i$. For $i = 0$, the result is easily obtained with $w = a_1 b_1$. For the induction step, note that capacities from $\{a_1, b_1, \ldots, a_{i-1}, b_{i-1}\}$ allow for flows which leave the positions $t_i, r_{i+1}, t_{i+1}, r_{i+2}, \ldots, r_k, t_k$ unchanged (with value 1 at states $r_j$ and arbitrary at states $t_j$). We first use capacity $a_i$ to move one token from $s$ to $r_i$, then apply the induction hypothesis $n - 1$ times to move all remaining tokens to $t_{i-1}$, and finally use capacity $b_k$ to reach the wanted configuration $c'$. This concludes the proof of the induction.

To prove the lemma, it suffices to apply this inductive statement $n$ times for $i = k$, to move each token from $s$ to $t$. Unravelling the inductive structure of the proof yields that the word of capacities $a_k(a_{k-1}(\ldots(a_1 b_1)^{n-k+1} \ldots)^{n-2} b_{k-1})^{n-1} b_k$ goes from $n \cdot s$ to $(n - 1) \cdot s + t_k$.

Note that the capacity word from the proof above has $a_k a_{k-1} \ldots a_1$ as a prefix: the strategy we use starts by moving one token in each $r_i$, and then applies capacity $b_i$ which is able to *synchronously* move each of these single tokens from $r_i$ back to $r_i$. This behaviour in $S_k$ is crucial in our reduction, and formalized now.
Lemma 7.3. Let $n \geq k$, and $w \in \{a_1, b_1, \ldots, a_k, b_k\}^*$ be a word of capacities of size $N$, and $f = f_1 \ldots f_N$ a flow word $\leq w$ such that $n \cdot s \sim f nt_k$. Let $c_l = \text{pre}(f_l)$ for $l \in \{1, \ldots, N\}$. Then there exists an index $l$ such that for all $i \in \{1, \ldots, k\}$, $c_l(r_i) = 1$.

Proof. We prove by induction over $i \in \{1, \ldots, k\}$ that for any $n \geq i$, a flow word $f = f_1 \ldots f_N$ smaller than some $w \in \{a_1, b_1, \ldots, a_k, b_k\}^*$ going from $n \cdot s + r_i+1 + \cdots + r_k$ to $nt_i + r_i+1 + \cdots + r_k$ must have some $c_l = \text{pre}(f_l)$ satisfying for all $j$, $c_l(r_j) = 1$. For $i = 1$, this is true for $c_l = n \cdot s + r_1 + \cdots + r_k$. Let $i > 0$ and assume the result proved for smaller $i$’s. For all $l \in \{1, 2, \ldots, N - 1\}$, let $c_l = \text{pre}(f_l)$ (which is also equal to $\text{post}(f_{l-1})$ for $l \geq 2$). Note that since $f_l \leq w_l \in \{a_1, b_1, \ldots, a_k, b_k\}$ by hypothesis, for all $j \geq i+1$, since $c_l(r_j) = 1$ and since the $r_j$ is the unique state $q$ such that $w_l(q, r_j) = 1$, an easy induction concludes that $c_l(r_j) = 1$ and $f_l(r_j, r_j) = 1$. Let $l_{\text{right}} = \min_j(c_l(r_i) > 0)$. Since $c_{l_{\text{right}}-1}(r_i) = 0$, and $w_l \in \{a_1, b_1, \ldots, a_i, b_i\}$, it must be that $w_{l_{\text{right}}-1} = b_i$ and hence $c_{l_{\text{right}}-1} = (n - 1) \cdot t_i - 1 + r_i + r_{i+1} + \cdots + r_k$. Let $l_{\text{left}} = \min_j(c_l(r_i) > 0) \in \{2, 3, \ldots, l_{\text{right}}\}$. Since $c_{l_{\text{left}}-1}(r_i) = 0$ and $w_{l_{\text{left}}-1} \in \{a_1, b_1, \ldots, a_i, b_i\}$, it must be that $w_{l_{\text{left}}-1} = a_i$ and hence $c_{l_{\text{left}}-1} = n \cdot s + r_i + r_{i+1} + r_{i+2} + \cdots + r_k$. Now, an easy induction concludes that for all $l \in \{l_{\text{left}}, l_{\text{left}} + 1, \ldots, l_{\text{right}} - 1\}$, we have $c_l(r_i) = 1$, so it must be that $w_l \in \{a_1, b_1, \ldots, a_i, b_i\}$. We conclude by invoking the induction hypothesis between $l_{\text{left}}$ and $l_{\text{right}} - 1$.

We now have a sufficiently good understanding of the gadget $S_k$ to present our lower bound.

We fix $k$ finite non-deterministic automata $A_1 = (Q_1, I_1, F_1, \Delta_1), \ldots, A_k = (Q_k, I_k, F_k, \Delta_k)$ on a common alphabet $\Sigma$. Without loss of generality $Q_1, \ldots, Q_k$ are disjoint sets and $\Delta_i \subseteq Q_i \times \Sigma \times Q_i$. To encode intersection of $A_1, \ldots, A_k$ in simple SFP, we add to $S_k$ a copy of each automaton, an additional state $r_1'$, capacities $c_\sigma$ for each $\sigma \in \Sigma$ and two capacities init and fin. The idea is as follows: to proceed from $r_1 + r_2 + \cdots + r_k$ towards $t_1 + r_2 + \cdots + r_k$, one now has to first go to $r_1' + r_2 + \cdots + r_k$, which requires each of the $k$ tokens in $r_1, \ldots, r_k$ to synchronously follow a run in $A_1, \ldots, A_k$ respectively. Formally, we let $Q = \{s, r_1, t_1, \ldots, r_k, t_k, r_1'\} \cup Q_1 \cup \cdots \cup Q_k, \sigma = \{a_1, b_1, \ldots, a_k, b_k, \text{init, fin}\} \cup \{c_\sigma, \sigma \in \Sigma\}$. We let $a_i$ and $b_i$, for $i \in \{1, \ldots, n\}$ be defined just as previously, except for $b_1$ which is now defined as

$$b_1(q, q') = \begin{cases} 
\omega & \text{if } q = q' \in \{s, t_1, \ldots, t_k\}, \\
1 & \text{if } q = q' \in \{r_2, \ldots, r_k\} \text{ or } (q, q') = (r_1', t_1), \\
0 & \text{otherwise.}
\end{cases}$$

Moreover, the capacity $c_\sigma$ is given by

$$c_\sigma(q, q') = \begin{cases} 
\omega & \text{if } q = q' \in \{s, t_1, \ldots, t_k\}, \\
1 & \text{if for some } i \in \{1, \ldots, k\}, (q, \sigma, q') \in \Delta_i, \\
0 & \text{otherwise,}
\end{cases}$$

the capacity init is given by

$$\text{init}(q, q') = \begin{cases} 
\omega & \text{if } q = q' \in \{s, t_1, \ldots, t_k\}, \\
1 & \text{if for some } i \in \{1, \ldots, k\}, q = r_i \text{ and } q' \in I_i, \\
0 & \text{otherwise,}
\end{cases}$$
and the capacity fin is given by
\[
\text{fin}(q, q') = \begin{cases} 
\omega & \text{if } q = q' \in \{s, t_1, \ldots, t_k\}, \\
1 & \text{if } q \in F_1 \text{ and } q' = r_1', \\
1 & \text{if for some } i \in \{2, \ldots, k\}, q \in F_i \text{ and } q' = r_i, \\
0 & \text{otherwise.} 
\end{cases}
\]

We now prove that this construction indeed gives a reduction from NFA intersection to simple SFP.

**Theorem 7.4.** The intersection of \( L(A_i) \) is non-empty if and only if for all \( n \), there is a capacity word \( w \in \mathcal{C}^* \) and a flow word \( f \leq w \) such that \( n \cdot s \sim f \cdot n \cdot t_k \).

**Proof.** Let us first assume that \( \bigcap_i L(A_i) \neq \emptyset \), and let \( u \in \bigcap_i L(A_i) \). Then it is straightforward to adapt the proof of Lemma 7.2 to show that the word of capacities
\[
w_n = a_k(a_{k-1}(\ldots(a_1 \cdot \text{init} \cdot u \cdot \text{fin} \cdot b_1)^{n-1}) \ldots)^{n-2} b_{k-1})^{n-1} b_k
\]
go from \( n \cdot s \) to \((n-1) \cdot s + t_k \). Consequently, the word of capacities\(^2\) \( w = w_n w_{n-1} \ldots w_1 \) goes from \( n \cdot s \) to \( n \cdot t_k \).

Conversely, assume that for all \( n \) there is a word of capacities \( w \in \mathcal{C}^* \) and a flow word \( f \leq w \) going from \( n \cdot s \) to \( n \cdot t_k \), and let \( n = k \). Let \( N = |w| \) and let \( c_l = \text{pre}(f_l) \) for \( l \in \{1, \ldots, N\} \). Again, by a straightforward adaptation of Lemma 7.3 there exists an index \( l_0 \) such that for all \( i \in \{1, \ldots, k\}, c_{l_0}(r_i) = 1 \). This implies that \( w_{l_0+1} = \text{init} \), since for any other capacity \( c \) we have \( \text{pre}(c) \nleq c_{l_0} \). Hence, there are initial states \( q^{(1)} \in I_1, q^{(k)} \in I_k \) in each automaton such that \( c_{l_0+1} = q^{(1)} + \cdots + q^{(k)} + p \), with \( p \) a configuration of support \( \subseteq \{s, t_1, t_2, \ldots, t_k\} \). Let \( L_1 = \min\{l \geq l_0 + 1 \mid w_l \notin \{c_{\sigma}, \sigma \in \Sigma\}\} \). Let \( L = L_1 - l_0 - 1 \), and \( u = u_1 \ldots u_L \) be such that for all \( l \in \{1, \ldots, L\}, w_{l_0+l} = c_{u_l} \). An easy induction shows that for all \( l \in \{1, \ldots, L\} \), there are states \( q^{(1)}_l, \ldots, q^{(k)}_l \in Q_l \) such that \( c_{u_{l+1}} = q^{(1)}_l + \cdots + q^{(k)}_l + p \), and that for each \( i \in \{1, \ldots, k\}, q^{(i)}_1 \xrightarrow{u_1} q^{(i)}_2 \xrightarrow{u_2} \ldots \xrightarrow{u_L} q^{(i)}_L \) is a run in \( A_i \). Since it is the only capacity, other than the \( c_{\sigma}'s \), with arrows leaving from \( \bigcup Q_i \), it must be that \( w_{l_0+L} = \text{fin} \). This implies that for all \( i \in \{1, \ldots, k\}, q^{(i)}_L \in F_i \). We conclude that \( u \) belongs to each of the \( L(A_i) \).

\[\square\]

**8. Conclusions**

We showed the decidability of the stochastic control problem. Our approach uses well quasi orders and the sequential flow problem, which is then solved using the theory of regular cost functions. As an intermediate we also introduce the simple sequential flow problem as a problem of independent interest, and provide an EXPSPACE complexity upper bound as well as a PSPACE complexity lower bound.

Together with the original result of [BDGG17, BDG+19] in the adversarial setting, our result contributes to the theoretical foundations of parameterised control. We return to the first application of this model, control of biological systems. As we discussed the stochastic setting is perhaps more satisfactory than the adversarial one, although in our examples complicated behaviours also exist in the stochastic setting, which are arguably not relevant for modelling biological systems.

\(^2\)With the convention that \( v^n \) is the empty word when \( n \) is a negative integer
We thus pose two open questions. The first is to settle the complexity status of the stochastic control problem. Mascle, Shirmohammadi, and Totzke [MST19] proved the EXPTIME-hardness of the problem, which is interesting because the underlying phenomena involved in this hardness result are specific to the stochastic setting (and do not apply to the adversarial setting). Our algorithm does not even yield elementary upper bounds, leaving a very large complexity gap. We believe that a good understanding of the sequential flow problem and the structure of its positive instances can be useful to make progress in this direction. Closing the complexity gap by either finding a better complexity lower bound or another algorithmic technique appears to be an interesting and challenging problem.

The second question is towards more accurately modelling biological systems: can we refine the stochastic control problem by taking into account the synchronising time of the controller, and restrict it to reasonable bounds?

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References

[AAD+06] Dana Angluin, James Aspnes, Zoë Diamadi, Michael J. Fischer, and René Peralta. Computation in networks of passively mobile finite-state sensors. Distributed Computing, 18(4):235–253, 2006.

[AD16] Parosh Aziz Abdulla and Giorgio Delzanno. Parameterized verification. International Journal on Software Tools for Technology Transfer, 18(5):469–473, 2016.

[AHM07] Parosh Aziz Abdulla, Noomene Ben Henda, and Richard Mayr. Decisive Markov chains. Logical Methods in Computer Science, 3(4), 2007.

[BDG+19] Nathalie Bertrand, Miheer Dewaskar, Blaise Genest, Hugo Gimbert, and Adwait Amit Godbole. Controlling a population. Logical Methods in Computer Science, 15(3), 2019.

[BDGG17] Nathalie Bertrand, Miheer Dewaskar, Blaise Genest, and Hugo Gimbert. Controlling a population. In CONCUR, pages 12:1–12:16, 2017.

[Col13] Thomas Colcombet. Regular cost functions, part I: logic and algebra over words. Logical Methods in Computer Science, 9(3), 2013.

[Dic13] Leonard Eugene Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. American Journal of Mathematics, 35(4):413–422, 1913.

[EFM99] Javier Esparza, Alain Finkel, and Richard Mayr. On the verification of broadcast protocols. In LICS, pages 352–359, 1999.

[Esp16] Javier Esparza. Parameterized verification of crowds of anonymous processes. In Dependable Software Systems Engineering, pages 59–71. IOS Press, 2016.

[FF56] Lester R. Ford and Delbert R. Fulkerson. Maximal flow through a network. Canadian Journal of Mathematics, 8:399–404, 1956.

[FGH14] Nathanaël Fijalkow, Hugo Gimbert, Florian Horn, and Youssouf Oualhadj. Two recursively inseparable problems for probabilistic automata. In MFCS, pages 267–278, 2014.

[Fij17] Nathanaël Fijalkow. Undecidability results for probabilistic automata. SIGLOG News, 4(4):10–17, 2017.

[GO10] Hugo Gimbert and Youssouf Oualhadj. Probabilistic automata on finite words: Decidable and undecidable problems. In ICALP, pages 527–538, 2010.

[GS92] Steven M. German and A. Prasad Sistla. Reasoning about systems with many processes. Journal of the ACM, 39(3):675–735, 1992.
[GTW02] Erich Grädel, Wolfgang Thomas, and Thomas Wilke, editors. *Automata, Logics, and Infinite Games*, volume 2500 of LNCS. Springer, 2002.

[Hig52] Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, s3-2(1):326–336, 1952.

[Kir05] Daniel Kirsten. Distance desert automata and the star height problem. *Theoretical Informatics and Applications*, 39(3):455–509, 2005.

[Koz77] Dexter Kozen. Lower bounds for natural proof systems. In *Foundations of Computer Science*, pages 254–266. IEEE Computer Society, 1977.

[Kru72] Joseph B. Kruskal. The theory of well-quasi-ordering: A frequently discovered concept. *J. Comb. Theory, Ser. A*, 13(3):297–305, 1972.

[Kuč11] Antonín Kučera. *Turn-Based Stochastic Games*. Lectures in Game Theory for Computer Scientists. Cambridge University Press, 2011.

[MST19] Corto Mascle, Mahsa Shirmohammadi, and Patrick Totzke. Controlling a random population is exptime-hard. *CoRR*, 2019.

[Sch17] Sylvain Schmitz. *Algorithmic Complexity of Well-Quasi-Orders*. Habilitation à diriger des recherches, École normale supérieure Paris-Saclay, November 2017.

[UMD+15] Jannis Uhlendorf, Agnès Miermont, Thierry Delaveau, Gilles Charvin, François Fages, Samuel Bottani, Pascal Hersen, and Gregory Batt. In silico control of biomolecular processes. *Computational Methods in Synthetic Biology*, 13:277–285, 2015.

[VJ85] Rüdiger Valk and Matthias Jantzen. The residue of vector sets with applications to decidability problems in Petri nets. *Acta Informatica*, 21:643–674, 03 1985.