Nonlinear QM as a fractal Brownian motion with complex diffusion constant

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Abstract

A new nonlinear Schrödinger equation is obtained explicitly from the fractal Brownian motion of a massive particle with a complex-valued diffusion constant. Real-valued energy (momentum) plane wave and soliton solutions are found in the free particle case. The hydro-dynamical model analog yields another (new) nonlinear QM wave equation with physically meaningful soliton solutions. One remarkable feature of this nonlinear Schrödinger equation based on a fractal Brownian motion model, over all the other nonlinear QM models, is that the quantum-mechanical energy functional coincides with the field theory one.

1 Introduction

The theoretical study of quantum chaos has been developed mainly in two areas: The phenomenological characterization of the spacing of the energy levels of bound and quasi-bound quantum physical systems, whose main analytical tool is the random matrix theory [6], and the semi-classical limit of chaotic classical systems [7]. The semi-classical approach pretends to seek solutions of the Schrödinger equation and to read in the wave functions any fingerprints of classical chaos. Due to the linearity of the Schrödinger equation there is no place where the sensibility to the initial conditions can be made manifest, which is present in nonlinear chaotic systems. The Riemann zeta function has been considered as a unifying link between those two approaches [8].

We believe that quantum chaos is truly a new paradigm in physics associated with non-unitary and nonlinear QM processes based on non-Hermitian operators (implementing time symmetry breaking). This chaotic behavior stems directly from the nonlinear Schrödinger equation without any reference to the nonlinear behavior of the classical limit. See [9]. For this reason, the genuine quantum chaos should be exhibited only by systems whose behavior is correctly described by a nonlinear Schrödinger equation.

The nonlinear QM has a practical importance in different fields, like condensed matter, quantum optics and atomic and molecular physics; even quantum gravity may involve nonlinear QM. Another important example is in the modern field of quantum computing. If quantum states exhibit small nonlinearities during their temporal evolution, then quantum computers can be used to solve NP-complete (non polynomial) and \#P problems in polynomial time.
Abrams and Lloyd [26] proposed logical gates based on non-linear Schrödinger equations and suggested that a further step in quantum computing consists in finding physical systems whose evolution is amenable to be described by a NLSE.

On other hand, we consider that Nottale and Ord’s formulation of quantum mechanics [1] from first principles based on the combination of scale relativity and fractal space-time is a very promising field of future research. In this work we extend Nottale and Ord’s ideas to derive the nonlinear Schrödinger equation. This could shed some light on the physical systems which could be appropriately described by the nonlinear Schrödinger equation derived in what follows.

The contents of this work are the following. In section 2 we derive different nonlinear Schrödinger-like equations starting from purely hydro-dynamical considerations. In section 3 a review of the derivation of the Schrödinger equation, based on Nottale and Ord’s [1] model of QM as a fractal Brownian motion of a particle zigzagging back and forth in space-time, is presented. In section 4 we derive the nonlinear Schrödinger equation from an extension of the Nottale’s approach to the case of a fractal Brownian motion with a complex diffusion constant. In section 5 real-valued energy solutions of the nonlinear Schrödinger equation are proposed. In the final section 6 we summarize our conclusions and include some additional comments.

2 Nonlinear Schrödinger equations based on hydrodynamics

In this section we will write down two NLSE (nonlinear Schrödinger equations) using the hydro-dynamical models of QM [5, 10]. The first equation is based by adding a hydrostatic pressure term to the Euler-Lagrange equations [4] and the second equation is obtained by adding, instead, a kinematic pressure term. As far as we know, this second equation has not appeared in the literature before.

The hydrostatic pressure experienced by a fluid element at a point \( \vec{r} \) due to the force of gravity is given by the Euler equation

\[
-\nabla p = \rho \vec{g};
\]

\( \nabla \) is the ordinary gradient, \( p \) the pressure, \( \rho \) the density and \( \vec{g} \) the acceleration of gravity. For example, if the density and acceleration are uniform one can integrate such equation and arrived at

\[
p = \rho g x
\]

giving the pressure at a given depth \( x \).

The author [4] proposed to establish the QM analog of the Euler equation by relating the density \( \rho \) to the quantum mechanical probability density \( \psi^* \psi \) and by integrating the equation. Setting \( \rho = \psi^* \psi \), \( b \) a mass-energy parameter and the particular case that \( p = \rho \), then one has that the hydrostatic potential is given by the integral

\[
b \int \vec{g}(\vec{x}) \cdot d\vec{r} = -b \int \frac{\nabla \rho}{\rho} \cdot d\vec{r} = -b \ln \frac{\rho}{\rho_0} = -b \ln(\psi^* \psi),
\]

setting \( \rho_0 = 1 \). This is the nonlinear potential energy induced from a hydrostatic pressure term.
It is important to normalize the logarithms by a constant which we set to unity for convention. \(-b \ln(\psi^* \psi)\) has energy units. From now these logarithmic terms are normalized that way.

The hydrostatic pressure term \[ \text{[4]} \] has energy units and explains in a straightforward fashion the nonlinear term (nonlinear potential) added to the standard Schrödinger equation by Białyńcki-Birula and Mycielski \[ \text{[3]} \] long ago. The parameter \(b\) has units of mass (energy), so the nonlinear wave equation is given by \[ \text{[3]} \] after adding the nonlinear potential \(-b \ln(\psi^* \psi)\).

The Birula-Mycielski NLSE for a particle is

\[
i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \psi + U \psi - b \ln(\psi^* \psi) \psi. \tag{2}\]

A derivation of this equation from the Nelson stochastic QM was given by Lemos (\[16\] p. 615 and \[17\]). As interesting as this equation may be there are some problems. Such equation does not obey the homogeneity condition (see Weinberg \[12\]) which says that if \(|\psi\rangle\) represents a physical state, the rays \(|\lambda \psi\rangle\) must also represent the same physical state, for any complex constant \(\lambda\). But equation \[\text{[3]}\] is not invariant under \(\psi \rightarrow \lambda \psi\) because the logarithmic nonlinear potential breaks such homogeneity, this NLSE is not scaled by \(\lambda\).

Another problem is that plane wave solutions to equation \[\text{[2]}\] do not seem to have a physical interpretation due to extraneous dispersion relations. Only the soliton solutions were physically meaningful \[3\]. Upper limits on the values of the parameter \(b\) had been found to be \[13\] \(b < 3 \cdot 10^{-15} \text{eV}\), which correspond to an electron soliton width of 3 mm \[4\]. The smallness of \(b\) is itself no reason to disregard equation \[\text{[2]}\] as physically relevant. For the authors, another problem with the logarithmic nonlinear potential term is that the hydrostatic pressure term in the NLSE is given by an explicit function of both \(\psi\) and its complex conjugate \(\psi^*\). It is desirable to write a NLSE solely in terms of the \(\psi\) variable, or \(\psi^*\), but not combined. Thus, another new NLSE can immediately be written and/or modified by adding the kinetic-pressure terms to the Euler-Newton hydro-dynamical equations of motion, i.e. by adding the term \((1/2) \rho \nabla V^2\) and taking \(\rho = a \psi^* \psi\); where \(a\) is a mass parameter, different from \(b\), \(\vec{V} = \vec{p}/m\) and \(\vec{p}\) is the momentum.

Using the relations from the Hamilton-Jacobi theory

\[
\psi \psi^* = e^{2iS(x)/\hbar}, \quad \vec{p} = \vec{\nabla} S(x) = m \vec{V},
\tag{3}
\]

we can express the square of the velocity in terms of \(\psi\) and \(\psi^*\) as follows,

\[
\vec{V} = -i \frac{\hbar}{2m} \vec{\nabla} \ln \frac{\psi^*}{\psi},
\tag{4}
\]

so the energy-density becomes

\[
\frac{1}{2} \rho |\vec{V}|^2 = \frac{a \hbar^2}{8m^2} \psi \psi^* \vec{\nabla} \ln \frac{\psi}{\psi^*} \cdot \vec{\nabla} \ln \frac{\psi^*}{\psi},
\tag{5}
\]
from which we immediately conclude that the corresponding nonlinear potential term associated with the kinematical pressure term is

$$\frac{ah^2}{8m^2} \nabla \ln \frac{\psi}{\psi^*} \cdot \nabla \ln \frac{\psi^*}{\psi}.$$ (6)

Hence a candidate for a NLSE is

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U\psi - b[\ln(\psi^*)]\psi + \frac{ah^2}{8m^2} \left( \nabla \ln \frac{\psi}{\psi^*} \cdot \nabla \ln \frac{\psi^*}{\psi} \right) \psi.$$ (7)

Here the Hamiltonian is Hermitian and $a \neq b$ both are mass-energy parameters to be determined experimentally. As far as we know, this NLSE has not been derived so far.

The new term can be written also in the form

$$\nabla \ln \frac{\psi}{\psi^*} \cdot \nabla \ln \frac{\psi^*}{\psi} = -\left( \nabla \ln \frac{\psi}{\psi^*} \right)^2.$$ (8)

For simplicity purposes, from now on we shall refer to these nonlinear potential kinematic pressure terms as those terms of the type

$$U_{kin} \sim (\nabla \ln \psi)^2.$$ (9)

The reason we choose to impose that notation will become clear in the next sections.

Our goal now is to derive NLSE directly from the fractal space time dynamics of a particle undergoing a Brownian random motion. And such fractal space time interpretation does not require to add \textit{ad-hoc} terms like

1. David Bohm’s quantum potential \[5\] into the Hamilton equation. See also \[9\].

2. To use the hydro-dynamical models discussed so far \[4, 5, 10\] nor to add the hydrostatic pressure and kinematic pressure terms to the Euler-Newton equations of motion as we have shown above, in an \textit{ad-hoc} fashion.

The new NLSE can be obtained from first principles if, and only if, we assume a fractal trajectory of a point particle associated with a Brownian random motion \[1\].

Before we begin, we deem it very important to add some more comments about the kinematic pressure terms

$$\frac{1}{2} \rho V^2 \leftrightarrow \frac{\hbar^2}{2m} \frac{a}{m} |\nabla \ln \psi|^2,$$ (10)

versus the hydrostatic pressure terms

$$\int \frac{\nabla p}{\rho} \leftrightarrow -b \ln(\psi^*\psi)$$ (11)

in the new NLSE.
The hydrostatic term \(-b \ln(\psi^*\psi)\psi\) explicitly breaks homogeneity \(\psi \rightarrow \lambda \psi\) of the NLSE. Whereas the kinematic pressure term \((\hbar^2)/(2m)(a/m)|\nabla \ln |\psi|^2\psi\) does preserve the homogeneity condition and the NLSE should scale with a \(\lambda\) factor, fact that can be easily verified.

The hydrostatic pressure term is not compatible with the motion kinematics of a particle executing a fractal Brownian motion. Only in the \(m \rightarrow \infty\) limit (heavy particle) it is reasonable to speak of the static limit. See [4] for a discussion of this limit. There are another deeper reasons to derive the NLSE from an underlying dynamics of a particle in a fractal space time, or more simply, from a fractal trajectory in a fixed space time background. We may, or may not, be switching on the quantum gravitational aspects of space time, so deeply linked to the non-linearity of QM. Perhaps the nonlinearity of QM is deeply intertwined with the quantum gravitational aspects of a Cantorian-fractal-space-time [14, 15].

Our goal is far less ambitious. Returning to our main points, the two NLSE have introduced two additional parameters of mass-dimension \(a\), \(b\) (or one parameter in the special case \(a = b\)). Such parameters need to be found experimentally. The advantage of the fractal formalism is that we will be able to relate the \(a\), \(b\) parameters to the Planck constant \(\hbar\) itself, rather to have new parameters in physics unrelated to \(\hbar\).

### 3 QM as mechanics in non differentiable spaces

We will be following very closely Nottale’s derivation of the ordinary Schrödinger equation [1]. The readers familiar with this work may omit this section. Recently Nottale and Celerier [1] following similar methods were able to derive the Dirac equation using bi-quaternions and after breaking the parity symmetry \(dx^\mu \leftrightarrow -dx^\mu\), see references for details. Also see the Ord’s paper [2] and the Adlers’s book on quaternionic QM [23]. For simplicity the one-particle case is investigated, but the derivation can be extended to many-particle systems.

In this approach particles do not follow smooth trajectories but fractal ones, that can be described by a continuous but non-differentiable fractal function \(\vec{r}(t)\). The time variable is divided into infinitesimal intervals \(dt\) which can be taken as a given scale of the resolution. If \(\Phi(t, t', dt)\) is a smoothing function centered on \(t\), for example a step function of width \(2dt\), a continuous and differentiable approximation to the true fractal \(\vec{r}(t)\) can be constructed as follows,

\[
\vec{r}(t, dt) = \int_{-\infty}^{\infty} \Phi(t, t', dt)\vec{r}(t')dt'.
\]

While \(\vec{r}(t) = \vec{r}(t, 0)\) is non-differentiable, any \(\vec{r}(t, dt)\), called “fractal trajectory”, is differentiable for all \(dt \neq 0\).

Non-differentiability implies a lost of causality. For this reason the fractal trajectories are good candidates to describe the quantum behavior. Feynman in his path integral formulation of QM already found a interesting result involving
the time scale $dt$: When seen at a time scale $dt$, the quantum mechanical mean quadratic velocity of a particle is $\langle v^2 \rangle \propto 1/dt$. This can be easily explained by an argument involving fractals. If the trajectory is a fractal curve of fractal dimension $D$, the space and time resolutions are related by $dt \propto dx^D$, so that $\langle v^2 \rangle \propto (dx/dt)^2 \propto dt^{2(1/D-1)}$. The comparison with Feynman’s result leads to $D = 2$.

Here we omit the details of the arguments leading to $dt \propto dx^D$, which the interested reader can find at [1] and references therein. They have as ingredients the scale dependence of any fractal curve and the renormalization group.

A fractal function $f(x, \epsilon)$ can have, besides the derivative $\partial f(x, \epsilon)/\partial x$, the new derivative with respect to the scale, $\partial f(x, \epsilon)/\partial \epsilon$. It was found useful to use $\ln \epsilon$ instead of $\epsilon$ as the variable for resolution. Renormalization group arguments say that the following relation is valid [1],

$$\frac{\partial f(x, \epsilon)}{\partial \ln \epsilon} = a(x) + bf(x, \epsilon), \quad (13)$$

this means that the variation of $f$ under an infinitesimal scale transformation $d \ln \epsilon$ depends only on $f$ itself. This differential equation can be integrated to give us

$$f(x, \epsilon) = f_0(x) \left[ 1 + \zeta(x) \left( \frac{\lambda}{\epsilon} \right)^{-b} \right]. \quad (14)$$

$\lambda^{-b}\zeta(x)$ is an integration constant and $f_0(x) = -a(x)/b$. This result says that any fractal function can be approximated by the sum of two terms, one independent of the resolution and other resolution dependent. Due to the resolution dependence is associated to the fractal properties, and those are product of the non-differentiability, then is expected that $\zeta(x)$ is a fluctuating function with zero mean.

Provided than $a \neq 0$ and $b < 0$ two cases can be considered: (i) $\epsilon \ll \lambda$, the scale dependent term is dominant and $f$ is given by a scale-invariant fractal-like power law with fractal dimension $D = b - 1$, namely $f(x, \epsilon) = f_0(x) (\lambda/\epsilon)^{-b}$. (ii) if $\epsilon \gg \lambda$ then $f$ becomes independent of the scale. $\lambda$ is the de Broglie wave length.

A continuous but non-differentiable function $f(t)$ at $t$ has two possible values of the derivative at $t$, for this reason its approximation by a fractal function requires considering “left” and “right” derivatives. For the position vector the following two infinitesimal differences can be considered,

$$\vec{r}(t + dt, dt) - \vec{r}(t, dt) = \vec{b}_+(\vec{r}, t)dt + \vec{\xi}_+(t, dt) \left( \frac{dt}{\tau_0} \right)^\beta,$$

$$\vec{r}(t, dt) - \vec{r}(t - dt, dt) = \vec{b}_-(\vec{r}, t)dt + \vec{\xi}_-(t, dt) \left( \frac{dt}{\tau_0} \right)^\beta, \quad (15)$$

where $\beta = 1/D$, and $\vec{b}_+$ and $\vec{b}_-$ are average forward and backward velocities [1].

Adopting the non standard analysis formulation, $dt$ is also the time scale.
The instantaneous velocities are easily obtained from equation (15),

\[ \vec{v}_\pm(\vec{r}, t, dt) = \vec{b}_\pm(\vec{r}, t) + \vec{\xi}_\pm(t, dt) \left( \frac{dt}{\tau_0} \right)^{\beta - 1}, \]  

In the quantum case, \( D = 2 \), then \( \beta = 1/2 \), so that \( dt^{\beta - 1} \) is a divergent quantity, from which is evident the non-differentiability.

Then, following the definitions given by Nelson in his stochastic QM approach (Lemos in [10] p. 615; see also [17], [18]), Nottale define mean backward and forward derivatives as follows,

\[ \frac{d}\pm \vec{r}(t)}{dt} = \lim_{\Delta t \to \pm 0} \left\langle \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} \right\rangle, \]  

(17)

from which the forward and backward mean velocities are obtained,

\[ \frac{d\pm \vec{r}(t)}{dt} = \vec{b}_\pm. \]  

(18)

For his deduction of Schrödinger equation from this fractal space-time classical mechanics, Nottale starts by defining the complex-time derivative operator

\[ \frac{\delta}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} - iD\nabla^2. \]  

(19)

which after some straightforward definitions and transformations takes the following form,

\[ \frac{\delta}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \vec{\nabla} - iD\nabla^2. \]  

(20)

\( D \) is a real-valued diffusion constant to be related to the Planck constant. Now we are changing the meaning of \( D \), since no longer a symbol for the fractal dimension is needed, it will have the value 2.

The \( D \) comes from considering that the scale dependent part of the velocity is a Gaussian stochastic variable with zero mean, (see de la Peña at [10] p. 428)

\[ \langle d\xi_\pm, d\xi_{\pm j} \rangle = \pm 2D\delta_{ij}dt. \]  

(21)

In other words, the fractal part of the velocity \( \vec{\xi} \), proportional to the \( \vec{\zeta} \), amount to a Wiener process when the fractal dimension is 2.

Afterwards, Nottale defines a set of complex quantities which are generalization of well known classical quantities (Lagrange action, velocity, momentum, etc), in order to be coherent with the introduction of the complex-time derivative operator.

The complex time dependent wave function \( \psi \) is expressed in terms of a Lagrange action \( S \) by \( \psi = e^{iS/(2mD)} \). \( S \) is a complex-valued action but \( D \) is real-valued. The velocity is related to the momentum, which can be expressed as the gradient of \( S \), \( \vec{p} = \vec{\nabla}S \). Then the following known relation is found,

\[ \vec{V} = -2iD\vec{\nabla} \ln \psi. \]  

(22)
The Schrödinger equation is obtained from the Newton’s equation (force = mass times acceleration) by using the expression of $\vec{V}$ in terms of the wave function $\psi$,

$$-\vec{\nabla}U = m\frac{\delta}{\delta t}\vec{V} = -2imD\frac{\delta}{\delta t}\vec{\nabla}\ln\psi. \quad (23)$$

Replacing the complex-time derivation $\frac{23}{2}$ in the Newton’s equation gives us

$$-\vec{\nabla}U = -2im\left(D\frac{\partial}{\partial t}\vec{\nabla}\ln\psi\right) - 2D\vec{\nabla}\left(D\frac{\nabla^2\psi}{\psi}\right). \quad (24)$$

Simple identities involving the $\vec{\nabla}$ operator were used by Nottale. Integrating this equation with respect to the position variables finally yields

$$D^2\nabla^2\psi + iD\frac{\partial\psi}{\partial t} - \frac{U}{2m}\psi = 0, \quad (25)$$

up to an arbitrary phase factor which may set to zero. Now replacing $D$ by $\hbar/(2m)$, we get the Schrödinger equation,

$$ih\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\psi = U\psi. \quad (26)$$

The Hamiltonian operator is Hermitian, this equation is linear and clearly is homogeneous of degree one under the substitution $\psi \rightarrow \lambda\psi$.

### 4 Nonlinear QM as a fractal Brownian motion with a complex diffusion constant

Having reviewed Nottale’s work we can generalize it by relaxing the assumption that the diffusion constant is real; we will be working with a complex-valued diffusion constant; i.e. with a complex-valued $\hbar$. This is our new contribution. The reader may be immediately biased against such approach because the Hamiltonian ceases to be Hermitian and the energy becomes complex-valued. However this is not always the case. We will explicitly find plane wave solutions and soliton solutions to the nonlinear and non-Hermitian wave equations with real energies and momenta.

For a detailed discussion on complex-valued spectral representations in the formulation of quantum chaos and time-symmetry breaking see [13]. Also a complex-valued time and two-times (see [22] and references therein) complex-valued dimensions have been discussed in [20, 21].

Nottale’s derivation of the Schrödinger equation in the previous section required a complex-valued action $S$ stemming from the complex-valued velocities due to the breakdown of symmetry between the forwards and backwards velocities in the fractal zigzagging. If the action $S$ was complex then it is not farfetched to have a complex diffusion constant and consequently a complex-valued $\hbar$ (with same units as the complex-valued action).
Our derivation follows closely that of Nottale, sketched into the previous section, but with some crucial differences in the evaluation of the correlation functions and the definition of the complex-time derivative operator, respectively.

Before the derivation further comments on complex-energies are in order. The energy functional $E_{QM}$ contains imaginary components. Since meaningful physical solutions demand real-valued energies this imposes constraints on the physically acceptable states in these non-linear QM equations, see Puszkarz [11].

Complex energy is not alien in ordinary linear QM. They appear in optical potentials (complex) usually invoked to model the absorption in scattering processes [11] and decay of unstable particles. Complex potentials have also been used to describe decoherence [22]. The accepted way to describe resonant states in atomic and molecular physics is based on the complex scaling approach, which in a natural way deals with complex energies [24]. We will show that real-valued energy solutions exist to the NLSE based on a fractal Brownian motion.

The imaginary part of the linear Schrödinger equation yields the continuity equation
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0. \]

Because, as we shall see, our potential is complex, the imaginary part of such potential acts as a source term in the continuity equation.

Before, Nottale wrote,
\[ \langle d\zeta_+ d\zeta_- \rangle = \pm 2D dt, \] (27)
with $D$ and $2mD = \hbar$ real.

Now we set
\[ \langle d\zeta_+ d\zeta_- \rangle = \pm (D + D^*) dt, \] (28)
with $D$ and $2mD = \hbar = \alpha + i\beta$ complex.

The complex-time derivative operator becomes now
\[ \frac{\delta}{dt} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla - \frac{i}{2} (D + D^*) \nabla^2. \] (29)

In the real case $D = D^*$. It reduces to the complex-time-derivative operator described previously by Nottale.

Writing again the $\psi$ in terms of the complex action $S$,
\[ \psi = e^{iS/(2mD)} = e^{iS/\hbar}, \] (30)
where $S$, $D$ and $\hbar$ are complex-valued, the complex velocity is obtained from the complex momentum $\vec{p} = \nabla S$ as
\[ \vec{V} = -2iD \vec{\nabla} \ln \psi. \] (31)

The NLSE is obtained after we use the generalized Newton’s equation (force = mass times acceleration) in terms of the $\psi$ variable,
\[ -\nabla U = m \frac{\delta}{dt} \vec{V} = -2i m D \frac{\delta}{dt} \vec{\nabla} \ln \psi. \] (32)
Replacing the complex-time derivation [29] in the generalized Newton’s equation gives us

$$\vec{\nabla} U = 2im \left[ \frac{\partial}{\partial t} \vec{\nabla} \ln \psi - 2iD^2(\vec{\nabla} \ln \psi \cdot \vec{\nabla})(\vec{\nabla} \ln \psi) - \frac{i}{2} (D + D^*)D\nabla^2(\vec{\nabla} \ln \psi) \right] .$$

(33)

Now, using the three identities (1) $\vec{\nabla} \nabla^2 = \nabla^2 \vec{\nabla}$, (2) $2(\vec{\nabla} \ln \psi \cdot \vec{\nabla})(\vec{\nabla} \ln \psi) = \vec{\nabla}(\vec{\nabla} \ln \psi)^2$ and (3) $\nabla^2 \ln \psi = \nabla^2 \psi/\psi - (\vec{\nabla} \ln \psi)^2$ allows us to integrate such equation above yielding, after some straightforward algebra, the new NLSE that has the nonlinear (kinematic pressure) potential found before [1],

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m \hbar} \alpha \nabla^2 \psi + U \psi - i\frac{\hbar^2}{2m \hbar} \beta \left( \frac{\vec{\nabla} \ln \psi}{\psi} \right)^2 \psi.$$

(34)

Note the crucial minus sign in front of the kinematic pressure term and that $\hbar = \alpha + i\beta = 2mD$ is complex. When $\beta = 0$ we recover the linear Schrödinger equation.

The nonlinear potential is now complex-valued in general. Defining

$$W = -\frac{\hbar^2}{2m \hbar} \beta \left( \frac{\vec{\nabla} \ln \psi}{\psi} \right)^2,$$

(35)

and $U$ the ordinary potential, then the NLSE can be rewritten as

$$i\hbar \frac{\partial \psi}{\partial t} = \left( \frac{\hbar^2}{2m \hbar} \alpha \nabla^2 + U + iW \right) \psi.$$

(36)

This is the fundamental nonlinear wave equation of this work. It has the form of the ordinary Schrödinger equation with the complex potential $U + iW$ and the complex $\hbar$. The Hamiltonian is no longer Hermitian and the potential itself depends on $\psi$. Nevertheless, one could have meaningful physical solutions with real valued energies and momenta. Like the plane-wave and soliton solutions. Notice that the new NLSE obeys the homogeneity condition $\psi \to \lambda \psi$ for any constant $\lambda$. All the terms in the NLSE are scaled respectively by a factor $\lambda$. We did not obtain the hydrostatic pressure term $-b(\ln \psi^* \psi) \psi$ which breaks the homogeneity condition for a simple reason: We are studying the true kinematics and dynamics of a particle of mass $m$ undergoing a fractal Brownian motion. It would be meaningless to have a hydrostatic pressure term in such a model. Moreover, our two parameters $\alpha, \beta$ are intrinsically connected to a complex Planck constant $\hbar = \alpha + i\beta$ rather that being *ad-hoc* constants to be determined experimentally. Thus, the nonlinear QM equation derived from the fractal Brownian motion with complex-valued diffusion coefficient is intrinsically tied up with a non-Hermitian Hamiltonian and with complex-valued energy spectra [13]. To be more precise, the nonlinear $\beta$ term in (36) is really the nonlinear partner of the kinetic energy term.

We will show that despite having a non-Hermitian Hamiltonian we still could have eigenfunctions with real valued energies and momenta. When $\hbar$ is real ($\beta = 0$) and the NLSE is linearized back to the ordinary one.
The reader may ask why not simply propose as a valid NLSE the following,

\[ i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + U \psi + \frac{\hbar^2}{2m} a \nabla \ln |\psi|^2. \] (37)

Such equation (with the ordinary real Planck constant) is based on a real Hamiltonian that satisfies the homogeneity condition. It also admits soliton solutions of the type

\[ \psi = C A(x - Vt)e^{i(kx - \omega t)}, \] (38)

\(A(x - Vt)\) is a function to be determined by solving the NLSE. Galilean invariance imposes that the soliton is a traveling wave, a function of \(x - Vt\). We will present explicit expressions for the function \(A(x - Vt)\) afterwards. Therefore, in principle, this NLSE is a more suitable candidate than the Bialynicki-Birula and Mycielski NLSE with a nonlinear potential term \(-b \ln(\psi^*\psi)\) that breaks homogeneity and introduces also a \(\psi^*\) dependence into the wave equation.

The only problem (perhaps there are others) with the NLSE above is that it suffers also from an extraneous dispersion relation. Plugging-in the plane-wave solution \(\psi \sim e^{-i(Et - px)/\hbar}\) one gets an extraneous energy-momentum relation, after setting \(U = 0\),

\[ E = \frac{\vec{p}^2}{2m} \left(1 + \frac{a}{m}\right), \] (39)

not the usual \(E = \vec{p}^2/(2m)\). So in this case we have that \(E_{QM} \neq E_{FT}\) (FT means field theory).

It has been known for some time, see Puskarz [3], that the expression for the energy functional in nonlinear QM does not coincide with the QM energy functional, nor it is unique. The simplest way to see this is, for example, writing down the Birula and Mycielski NLSE (2) in the Weinberg form [4]

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{\partial H(\psi, \psi^*)}{\partial \psi^*}, \] (40)

where \(\psi\) and \(\psi^*\) are a pair of canonically-conjugate variables. The real-valued Hamiltonian density is given by

\[ H(\psi, \psi^*) = -\frac{\hbar^2}{2m} \psi^* \nabla^2 \psi + U \psi^* \psi - b \psi^* \ln(\psi^*\psi)\psi + b \psi^* \psi, \] (41)

using \(E_{FT} = \int d^3r H\), so we can see it is different from \(<\hat{H}\rangle_{QM}\). Notice the last term in [3]. Hence, one can immediately see that the \(H(\psi, \psi^*)\) stemming from the field theory approach does not coincide with the Birula-Mycielski Hamiltonian. They differ by a constant, \(E_{FT} - E_{QM} = \int d^3r \psi^* \psi = b\). Exactly like it occurs when we plug in the plane wave solution into the NLSE with the nonlinear potential, real valued,

\[ \frac{\hbar^2}{2m} \frac{a}{m} |\nabla \ln |\psi|^2|^2. \] (42)
Notice that this problem does not occur in the fractal-based NLSE, because such NLSE is written entirely in terms of the \( \psi \) variable and does not contain the \( \psi^* \) variable explicit or implicitly, like it occurs in the Birula-Mycielski NLSE.

The classic Gross-Pitaevskii NLSE (of the 1960’s), based on a quartic interaction potential energy, relevant to Bose-Einstein condensation, contains the nonlinear cubic terms in the Schrödinger equation, after differentiation, \((\psi^* \psi) \psi\). This equation does not satisfy the Weinberg homogeneity condition and also the \( E_{FT} \) differs from the \( E_{QM} \) by factors of two.

In the fractal-based NLSE there is no discrepancy between the quantum-mechanical energy functional and the field theory energy functional. Both are given by

\[
H_{\text{fractal}}^{\text{NLSE}} = -\frac{\hbar^2 \alpha}{2m} \psi^* \nabla^2 \psi + U \psi^* \psi - i \frac{\hbar^2 \beta}{2m} \psi^* (\nabla \ln \psi)^2 \psi. \tag{43}
\]

The NLSE is then unambiguously given by equation (40), \( H(\psi, \psi^*) \) is homogeneous of degree 1 in \( \lambda \) respect to \( \psi \). This is why we push forward the NLSE derived from the fractal Brownian motion with a complex-valued diffusion coefficient. Such equation does admit plane-wave solutions with the dispersion relation \( E = \vec{p}^2/(2m) \). It is not hard to see that after inserting the plane wave solution into the fractal-based NLSE we get (after setting \( U = 0 \)),

\[
E = \frac{\hbar^2 \alpha}{2m} \frac{\vec{p}^2}{\hbar^2} + i \frac{\hbar^2 \beta}{2m} \frac{\vec{p}^2}{\hbar} = \frac{\vec{p}^2}{2m} \frac{\alpha + i\beta}{\hbar} = \frac{\vec{p}^2}{2m}, \tag{44}
\]

since \( \hbar = \alpha + i\beta \). So the plane-wave is a solution to the fractal-based NLSE (when \( U = 0 \)) with a real-valued energy and which has the correct energy-momentum dispersion relation.

5 Soliton solutions to the fractal based NLSE.

One dimensional case

Let us find soliton solutions to the fractal-based NLSE given by (34), in the free particle case. We set the ansatz (one-dimensional for simplicity)

\[
\psi = CA(x - Vt) e^{-(Et - px)/\hbar}. \tag{45}
\]

The function \( A \) must be complex-valued, otherwise no real-valued energy solutions exist. Then we set

\[
A(x - Vt) = F(x - Vt) + iG(x - Vt), \tag{46}
\]

and plugging-in \( \psi \) with this \( A \) into the fractal-based NLSE (34) yields 2 coupled differential equations, after separating the real and imaginary parts, respectively, which yield, in principle, the functions \( F \) and \( G \).

For example, the soliton solution to the NLSE with the \(-b \ln(\psi^* \psi)\) is of the form [1],

\[
\psi(x, t) = Ce^{\alpha \sqrt{B}(x - Vt + d)^2} e^{i(kx - \omega t)}, \tag{47}
\]
where \( c, a, B, d \) are numerical constants which can depend on \( \hbar, m \) and \( b \).

As mentioned before, plane wave solutions to the NLSE based on the \(-b \ln(\psi^*\psi)\) potential exist but they have extraneous dispersion relations. For example, the energy-momentum relation turns out to be \( E = \hbar \omega = \frac{\vec{p}^2}{2m} + b \ln(2\pi) \). Thus, plane-wave solutions do not seem to have physically meaningful interpretation. This was another reason why we believe that this NLSE has problems. Besides, we remarked already that this NLSE breaks the homogeneity condition as well.

To finalize we will find the soliton solutions to the NLSE based on the kinematic pressure potential \(|\vec{\nabla} \ln \psi|^2\) terms, given by equation (37), in the free particle case \( U = 0 \).

Earlier on we have shown that it admits plane wave solutions with the extraneous dispersion relation \( E = \frac{\vec{p}^2}{(2m)(1 + a/m)} \). It obeys the homogeneity condition: Under scaling of \( \psi \) by \( \lambda \) the NLSE scales with an overall factor of \( \lambda \) as expected.

Notice that if we wish to have a Hermitian Hamiltonian we must take the absolute value \(|\vec{\nabla} \ln \psi|^2\) instead of \((\vec{\nabla} \ln \psi)^2\) for our potential. Notice this important difference between these non-linear potentials in the fractal-based NLSE versus the kinematic pressure based one.

Plugging-in the ansatz \( \psi = CF(x - Vt)e^{-(Et - px)/\hbar} \) into the kinematic pressure NLSE for the free particle case \( U = 0 \) yield for the imaginary parts \(-i\hbar VF' = -i(\hbar/m)F'p\). Then, for any \( F \) we have \( V = p/m \). Therefore the ansatz is consistent with the de Broglie relations \( p = \hbar k \) and \( E = \hbar \omega \), as expected from this NLSE soliton solution.

The real parts give the differential equation
\[
-F''F - \frac{a}{m}(F')^2 + \frac{1}{\hbar^2} \left[ 2mE - p^2(1 + \frac{a}{m}) \right] F^2 = 0.
\]
(48)

The solutions to this nonlinear differential equation yield \( F(x - Vt) \). This differential equation involves the derivatives \( F', F'' \) and is much harder to solve than the differential equation given in (37) that involves \( F'' \) but not \( F' \).

For example a nonlinear differential equation which involves \( F'' \) but not \( F' \) is \( F'' - F^3 = 0 \). Such equation, after multiplying both sides by \( F' \), can then be integrated by quadratures, \( \int dF/F^2 = \int dy/y^{3/2} \).

6 Concluding remarks

Based on Nottale and Ord’s formulation of QM from first principles; i.e. from the fractal Brownian motion of a massive particle we have derived explicitly a nonlinear Schrödinger equation. Despite the fact that the Hamiltonian is not Hermitian real-valued energy solution exist like the plane wave and soliton solutions in the free particle case. The hydro-dynamical model analog of this fractal-based NLSE yields another new NLSE with Hermitian (real) Hamiltonian. The remarkable feature of the fractal approach versus all the nonlinear
QM equation considered so far is that the quantum mechanical energy functional coincides precisely with the field theory one.

The hydro-dynamical-based NLSE has a nonlinear (real) potential term

$$\frac{a\hbar^2}{8m^2} \nabla \ln \frac{\psi}{\psi^*} \cdot \nabla \ln \frac{\psi^*}{\psi},$$

with $a$ the mass-energy parameter, bears a very rough similarity to the Starusz-kiewicz imaginary potential term in three dimensions

$$-\frac{\gamma}{8} \nabla^2 \ln \frac{\psi^*}{\psi},$$

see [11], this potential is imaginary, $\gamma$ is a constant.

The fractal model based NLSE admits plane wave (soliton solutions also) with the correct dispersion relation

$$E = \frac{\vec{p}^2}{2m},$$

real. Soliton solutions, with real-valued energy (momentum) are of the form

$$\psi \sim [F(x - Vt) + iG(x - Vt)] e^{ipx/\hbar - iEt/\hbar},$$

with $F, G$ two functions of the argument $x - Vt$ obeying a coupled set of two nonlinear differential equations.

It would be interesting to study solutions when one turns-on an external potential $U \neq 0$.

The reader may ask why concentrate on a complex diffusion constant to generate a nonlinear Schrödinger equation with a non-Hermitian Hamiltonian when one could have written from the start a NLSE with a Hermitian Hamiltonian that obeys the Weinberg homogeneity conditions and also with the correct energy dispersion relations.

Starting from the fundamental equations: $\psi = e^{iS}/S_0 = e^{iS}/\hbar_0$ and the generalized Newtonian law, written in terms of the Nottale complex derivative operator and the $\psi$:

$$\nabla U = iS_0 \frac{\partial \nabla \ln \psi}{\partial t} - i \left( \frac{S_0}{m} (\nabla \ln \psi \cdot \nabla) (\nabla \ln \psi) + D \nabla^2 (\nabla \ln \psi) \right),$$

after adding and subtracting the quantity $D_0 \nabla^2 (\nabla \ln \psi)$, and using the 3 vector-calculus identities used in (36), we get a nonlinear correction to the Schrödinger equation

$$\frac{\hbar_0}{2m} (\hbar - \hbar_0) (\nabla^2 \ln \psi) \psi,$$

where $S_0 = \hbar_0 = 2mD_0$ and $\hbar = 2mD \neq 2mD_0$.

As desirable as this NLSE may look a close inspection reveals that the nonlinearity is just an artifact of the definition of $\psi$. It looks nonlinear from the $\psi$ perspective. It is not difficult to see that under a re-definition of the wavefunction

$$\psi' = e^{iS/\hbar} = e^{iS/(2mD)}$$

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the generalized Newtonian law of motion of a particle undergoing a fractal Brownian motion given by (52) yields now the standard linear Schrödinger equation for the $\psi'$ wavefunction. Where we have chosen for a new value of $S'_{0}=\hbar=2mD$.

It is important to emphasize that the diffusion constant is always chosen to be related to Planck constant as follows: $2mD = \hbar$ which is just the transition length from a fractal to a scale-independence non-fractal regime discussed by Nottale in numerous occasions. In the relativistic scale it is the Compton wavelength of the particle (say an electron): $\lambda_{e} = \hbar/(mc)$. In the nonrelativistic case it is the de Broglie wavelength of the electron.

Therefore, the NLSE based on a fractal Brownian motion with a complex valued diffusion constant $2mD = \hbar = \alpha + i\beta$ represents truly a new physical phenomenon in so far as the small imaginary correction to the Planck constant (unobserved in present day experiments) is the hallmark of nonlinearity in QM. For other generalizations of QM see experimental tests of quaternionic QM (in the book by Adler [23]). Equation (36) is the fundamental NLSE of this work, where the $\beta$ term is essentially the nonlinear partner of the linear kinetic energy term in comparison to all other approaches which focused on nonlinear modifications of the potential.

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