Deng and Mansour [1] introduce a rabbit named Hoppy and let him move according to certain rules. At that stage, we don’t need to know the rules. Eventually, the enumeration problem is one about \( k \)-Dyck paths. The up-steps are \((1,k)\) and the down-steps are \((1,-1)\).

The question is about the length of the sequence of down-steps printed in red. Or, phrased differently, how many \( k \)-Dyck paths end on level \( j \), after \( m \) up-steps, the last step being an up-step. The recent paper [4] contains similar computations, although without the restriction that the last step must be an up-step.

Counting the number of up-steps is enough, since in total, there are \( m + km = (k + 1)m \) steps. The original description of Deng and Mansour is a reflection of this picture, with up-steps of size 1 and down-steps of size \(-k\), but we prefer it as given here, since we are going to use the adding-a-new-slice method, see [2, 3]. A slice is here a run of down-steps, followed by an up-step. The first up-step is treated separately, and then \( m - 1 \) new slices are added. We keep track of the level after each slice, using a variable \( u \). The variable \( z \) is used to count the number of up-steps.

Deng and Mansour work out a formula which comprises \( O(m) \) terms. Our method leads only to a sum of \( O(j) \) terms.
The following substitution is essential for adding a new slice:

\[ u^j \longrightarrow z \sum_{0 \leq h \leq j} u^{h+k} = \frac{zu^k}{1-u} (1-u^{j+1}). \]

Now let \( F_m(z, u) \) we the generating function according to \( m \) runs of down-steps. The substitution leads to

\[ F_{m+1}(z, u) = \frac{zu^k}{1-u} F_m(z, 1) - \frac{zu^{k+1}}{1-u} F_m(z, u), \quad F_0(z, u) = zu^k. \]

Let \( F = \sum_{m \geq 0} F_m \), then

\[ F(z, u) = zu^k + \frac{zu^k}{1-u} F(z, 1) - \frac{zu^{k+1}}{1-u} F(z, u), \]

or

\[ F(z, u) \frac{1-u+zu^{k+1}}{1-u} = zu^k + \frac{zu^k}{1-u} F(z, 1). \]

The equation \( 1-u+zu^{k+1} = 0 \) is famous when enumerating \((k+1)\)-ary trees. Its relevant combinatorial solution (also the only one being analytic at the origin) is

\[ \bar{u} = \sum_{\ell \geq 0} \frac{1}{1+\ell(k+1)} \binom{1+\ell(k+1)}{\ell} z^\ell. \]

Since \( u-\bar{u} \) is a factor of the LHS, must also be a factor of the RHS, and we can compute (by dividing out the factor \((u-\bar{u})\)) that

\[ \frac{zu^k(1-u+F(z, 1))}{u-\bar{u}} = -zu^k. \]

Thus

\[ F(z, u) = zu^k \frac{\bar{u}-u}{1-u+zu^{k+1}}. \]

The first factor has even a combinatorial interpretation, as a description of the first step of the path. It is also clear from this that the level reached is \( \geq k \) after each slice. We don’t care about the factor \( zu^k \) anymore, as it produces only a simple shift. The main interest is now how to get to the coefficients of

\[ \frac{\bar{u}-u}{1-u+zu^{k+1}} \]

in an efficient way. There is also the formula

\[ 1-u+zu^{k+1} = (\bar{u}-u) \left( 1 - z \frac{u^{k+1} - \bar{u}^{k+1}}{u-\bar{u}} \right), \]

but it does not seem to be useful here.
First we deal with the denominators

\[ S_j := \frac{[w^j]}{1 - u + zu^{k+1}} = \sum_{0 \leq m \leq j/k} (-1)^m \binom{j - km}{m} z^m. \]

One way to see this formula is to prove by induction that the sums \( S_j \) satisfy the recursion

\[ S_j - S_{j-1} + zS_{j-k-1} = 0 \]

and initial conditions \( S_0 = \cdots = S_k = 1 \). In [4] such expressions also appear as determinants. Summarizing,

\[ \frac{1}{1 - u + zu^{k+1}} = \sum_{m \geq 0} (-1)^m \sum_{j \geq km} \binom{j - km}{m} u^j. \]

Now we read off coefficients:

\[ [u^j] \frac{\bar{\pi}}{1 - u + zu^{k+1}} = \sum_{0 \leq m \leq j/k} (-1)^m \binom{j - km}{m} z^m \sum_{\ell \geq 0} \frac{1}{1 + \ell(k + 1)} \binom{1 + \ell(k + 1)}{\ell} z^\ell \]

and further

\[ [z^n] [w^j] \frac{\bar{\pi}}{1 - u + zu^{k+1}} = \sum_{0 \leq m \leq j/k} (-1)^m \binom{j - km}{m} \frac{1}{1 + (n - m)(k + 1)} \binom{1 + (n - m)(k + 1)}{n - m}. \]

The final answer to the Deng-Mansour enumeration (without the shift) is

\[ \sum_{0 \leq m \leq j/k} (-1)^m \binom{j - km}{m} \frac{1}{1 + (n - m)(k + 1)} \binom{1 + (n - m)(k + 1)}{n - m} - (-1)^n \binom{j - 1 - kn}{n}. \]

If one wants to take care of the factor \( zu^k \) as well, one needs to do the replacements \( n \to n + 1 \) and \( j \to j + k \) in the formula just derived. That enumerates then the \( k \)-Dyck paths ending at level \( j \) after \( n \) up-steps, where the last step is an up-step.

We hope that the methods presented here might be useful for other questions related to ascents/descents.
References

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