Deep neural network expressivity for optimal stopping problems

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Abstract

This article studies deep neural network expression rates for optimal stopping problems of discrete-time Markov processes on high-dimensional state spaces. A general framework is established in which the value function and continuation value of an optimal stopping problem can be approximated with error at most $\varepsilon$ by a deep ReLU neural network of size at most $\kappa d^q \varepsilon^{-r}$. The constants $\kappa, q, r \geq 0$ do not depend on the dimension $d$ of the state space or the approximation accuracy $\varepsilon$. This proves that deep neural networks do not suffer from the curse of dimensionality when employed to solve optimal stopping problems. The framework covers, for example, exponential Lévy models, discrete diffusion processes and their running minima and maxima. These results mathematically justify the use of deep neural networks for numerically solving optimal stopping problems and pricing American options in high dimensions.

1 Introduction

In the past years, neural network-based methods have been used ubiquitously in all areas of science, technology, economics and finance. In particular, such methods have been applied to various problems in mathematical finance such as pricing, hedging and calibration. We refer, for instance, to the articles [Buehler et al. (2019), Becker et al. (2020), Becker et al. (2021), Cuchiero et al. (2020)] and to the survey papers [Ruf and Wang (2020), Germain et al. (2021), Beck et al. (2020)] for an overview and further references. The striking computational performance of these methods has also raised questions regarding their theoretical foundations. Towards a complete theoretical understanding, there have been recent results in the literature which prove that deep neural networks are able to approximate option prices in various models without the curse of dimensionality. For deep neural network expressivity results for option prices and associated PDEs we refer, for

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instance, to Elbrächter et al. (2022), Grohs et al. (2018) for European options in Black-Scholes models, to Hutzenthaler et al. (2020), Cioica-Licht et al. (2022) for certain semilinear PDEs, to Grohs and Herrmann (2021) for certain Hamilton-Jacobi-Bellman equations, to Reisinger and Zhang (2020), Jentzen et al. (2021), Takahashi and Yamada (2021) for diffusion models and game-type options and to Gonon and Schwab (2021b) for certain path-dependent options in jump-diffusion models. A few works are also concerned with generalization (Berner et al. (2020)) and learning errors (Gonon (2021)).

The goal of this article is to analyse deep neural network expressivity for American option prices and general optimal stopping problems in discrete-time. An optimal stopping problem consists in selecting a stopping time \( \tau \) such that the expected reward \( \mathbb{E}[g_d(\tau, X_d^\tau)] \) is maximized. Here \( X^d \) is a given stochastic process taking values in \( \mathbb{R}^d \) and \( g_d(t, x) \) is the reward obtained if the process is stopped at time \( t \) at state \( x \). Optimal stopping problems arise in a wide range of contexts in statistics, operations research, economics and finance. In mathematical finance, arbitrage-free prices of American and Bermudan options are given as solutions to optimal stopping problems. The solution to an optimal stopping problem can be described by the so-called Snell envelope or, equivalently, by a backward recursion (discrete-time) or a free-boundary PDE (continuous-time) in the case when \( X^d \) is a Markov process.

In recent years, a wide range of computational methods have been developed to numerically solve optimal stopping problems also in high-dimensional situations, i.e., when the dimension \( d \) of the state space is large. For regression-based algorithms we refer, e.g., to Tsitsiklis and Van Roy (2001), Longstaff and Schwartz (2001), for duality-based methods we refer, e.g., to Rogers (2002), Andersen and Broadie (2004), Haugh and Kogan (2004), Belomestny et al. (2009), for stochastic grid methods we refer, e.g., to Broadie and Glasserman (2004), Jain and Oosterlee (2015) and for methods based on approximating the exercise boundary we refer, e.g., to Garcia (2003); see for instance also the overview in Bouchard and Warin (2012). Recently proposed methods include signature-based methods Bayer et al. (2021b) and regression trees Ech-Chafiq et al. (2022). Furthermore, various methods based on deep neural network approximations of the value function, the continuation value or the exercise boundary of the optimal stopping problem have been proposed, see, for instance, Kohler et al. (2010), Becker et al. (2019), Becker et al. (2020), Becker et al. (2021), Herrera et al. (2021), Lapeyre and Lelong (2021), Reppen et al. (2022) and the methods for continuous-time free boundary problems Sirignano and Spiliopoulos (2018), Wang and Perdikaris (2021). For many of these methods also theoretical convergence results or even convergence rates (cf., e.g., Clément et al. (2002), Belomestny (2011), Bayer et al. (2021a)) for a fixed dimension \( d \) have been established.

In this article we are interested in mathematically analysing the high-dimensional situation, i.e., in explicitly controlling the dependence on the dimension \( d \). We analyse deep neural network approximations for the value function of optimal stopping problems. We provide general conditions on the reward functions \( g_d \) and the stochastic processes \( X^d \) which ensure that the value function (and the continuation value) of an optimal stopping problem can
be approximated by deep ReLU neural networks without the curse of dimensionality, i.e., that an approximation error of size at most \( \epsilon \) can be achieved by a deep ReLU neural network of size \( \kappa d^q \varepsilon^{-r} \) for constants \( \kappa, q, r \geq 0 \) which do not depend on the dimension \( d \) or the accuracy \( \varepsilon \). The framework, in particular, provides deep neural network expressivity results for prices of American and Bermudan options. Our conditions cover most practically relevant payoffs and many popular models such as exponential Lévy models and discrete diffusion processes. The constants \( \kappa, q, r \) are explicit and thus the obtained results yield bounds for the approximation error component in any algorithm for optimal stopping and American option pricing in high-dimensions which is based on approximating the value function or the continuation value by deep neural networks.

The remainder of the paper is organized as follows. In Section 2 we formulate the optimal stopping problem, recall its solution by dynamic programming and introduce the notation for deep neural networks. In Section 3 we formulate the assumptions and main results. Specifically, in Section 3.1 we formulate a basic framework, Assumptions 1 and 2 and prove that the value function can be approximated by deep neural networks without the curse of dimensionality, see Theorem 3.4. In Section 3.2 we then provide more refined assumptions on the considered Markov processes and extend the approximation result to this refined framework, see Theorem 3.8 which is the main result of the article. In Sections 3.3, 3.4 and 3.5 we then apply this result to exponential Lévy models, discrete diffusion processes and show that also barrier options can be covered via the running maximum or minimum of such processes. In order to make the presentation more streamlined, most proofs, in particular the proofs of Theorems 3.4 and 3.8 are postponed to Section 4.

1.1 Notation

Throughout, we fix a time horizon \( T \in \mathbb{N} \) and a probability space \((\Omega, \mathcal{F}, P)\) on which all random variables and processes are defined. For \( d \in \mathbb{N}, x \in \mathbb{R}^d, A \in \mathbb{R}^{d \times d}\) we denote by \( \|x\| \) the Euclidean norm of \( x \) and by \( \|A\|_F \) the Frobenius norm of \( A \). For \( f: \mathbb{R}^{d_0} \times \mathbb{R}^{d_1} \to \mathbb{R}^{d_2} \) we let

\[
\text{Lip}(f) = \sup_{\substack{(x_1, y_1), (x_2, y_2) \in \mathbb{R}^{d_0} \times \mathbb{R}^{d_1} \atop x_1 \neq x_2, y_1 \neq y_2}} \frac{\|f(x_1, y_1) - f(x_2, y_2)\|}{\|x_1 - x_2\| + \|y_1 - y_2\|}.
\]

2 Preliminaries

In this section we first formulate the optimal stopping problem and recall its solution in terms of the value function. Then we introduce the required notation for deep neural networks.
2.1 The optimal stopping problem

For each \( d \in \mathbb{N} \) consider a discrete-time \( \mathbb{R}^d \)-valued Markov process \( X^d = (X^d_t)_{t \in \{0, \ldots, T\}} \) and a function \( g_d: \{0, \ldots, T\} \times \mathbb{R}^d \to \mathbb{R} \). Assume for each \( t \in \{0, \ldots, T\} \) that \( \mathbb{E}[|g_d(t, X^d_t)|] < \infty \) and let \( \mathcal{F} = (\mathcal{F}_t)_{t \in \{0, \ldots, T\}} \) be the filtration generated by \( X^d \). Denote by \( \mathcal{T} \) the set of \( \mathbb{F} \)-stopping times \( \tau: \Omega \to \{0, \ldots, T\} \) and by \( \mathcal{T}_t \) the set of \( \tau \in \mathcal{T} \) with \( t \leq \tau \). For notational simplicity we omit the dependence on \( d \) in \( \mathcal{F}, \mathcal{T} \) and \( \mathcal{T}_t \).

The optimal stopping problem consists in computing

\[
\sup_{\tau \in \mathcal{T}} \mathbb{E}[g_d(\tau, X^d_\tau)].
\]  

(2.1)

Consider the value function \( V_d \) defined by the backward recursion \( V_d(T, x) = g_d(T, x) \) and

\[
V_d(t, x) = \max(g_d(t, x), \mathbb{E}[V_d(t+1, X^d_{t+1})|X^d_t = x])
\]

(2.2)

for \( t = T - 1, \ldots, 0 \) and \( \mathbb{P} \circ (X^d)^{-1} \)-a.e. \( x \in \mathbb{R}^d \). Then knowledge of \( V_d \) also allows to compute a stopping time \( \tau^* \in \mathcal{T} \) which is a maximizer in (2.1):

\[
\tau^* = \min\{t \in \{0, \ldots, T\} : V_d(t, X^d_t) = g_d(t, X^d_t)\}
\]

satisfies \( \mathbb{E}[g_d(\tau^*, X^d_{\tau^*})] = \sup_{\tau \in \mathcal{T}} \mathbb{E}[g_d(\tau, X^d_{\tau})] \). Indeed, by backward induction and the Markov property we obtain that \( V_d(t, X^d_t) = U^d_t \), \( \mathbb{P} \)-a.s., where \( U^d \) is the Snell envelope defined by the backward recursion \( U^d = g_d(T, X^d_T) \) and \( U^d_t = \max(g_d(t, X^d_t), \mathbb{E}[U^d_{t+1} | \mathcal{F}_t]) \) for \( t = T - 1, \ldots, 0 \). Then, for instance by (Föllmer and Schied, 2016, Theorem 6.18), for all \( t \in \{0, \ldots, T\} \), \( \mathbb{P} \)-a.s.

\[
V_d(t, X^d_t) = U^d_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \mathbb{E}[g_d(\tau, X^d_{\tau}) | \mathcal{F}_t] = \mathbb{E}[g_d(\tau^{(t)}_{\min}, X^d_{\tau^{(t)}_{\min}}) | \mathcal{F}_t],
\]

(2.3)

where \( \tau^{(t)}_{\min} = \inf\{s \geq t : U^d_s = g_d(s, X^d_s)\} \). In particular, \( \tau^{(0)}_{\min} = \tau^* \) is a maximizer of (2.1) and, in the case when \( X^d_0 \) is constant, \( V_d(0, X^d_0) \) is the optimal value in (2.1).

The idea of our approach is as follows: in many situations the function \( g_d \) is in fact a neural network or can be approximated well by a deep neural network. Then the recursion (2.2) also yields a recursion for a neural network approximation. This argument will be made precise in the proof of Theorem 3.4 below.

Remark 2.1. Alternatively, we could also define

\[
V_d(t, x) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}[g_d(\tau, X^d_{\tau}) | X^d_t = x].
\]

(2.4)

Then under the strong Markov property, for each \( \tau \in \mathcal{T}_t \) it holds that \( \mathbb{E}[g_d(\tau, X^d_{\tau}) | \mathcal{F}_t] = h^d_{\tau^{(t)}_{\min}}(X^d_{\tau^{(t)}_{\min}}) \), \( \mathbb{P} \)-a.s., where \( h^d_{\tau^{(t)}_{\min}}(x) = \mathbb{E}[g_d(\tau, X^d_{\tau}) | X^d_t = x] \). The definition of the essential supremum then implies that for each \( \tau \in \mathcal{T}_t \) it holds \( \mathbb{P} \)-a.s. that \( h^d_{\tau^{(t)}_{\min}}(X^d_{\tau^{(t)}_{\min}}) \geq h^d_{\tau^{(t)}_{\min}}(X^d_{\tau^{(t)}_{\min}}) \). But this
implies that for \( P \circ (X_t^d)^{-1} \)-a.e. \( x \in \mathbb{R}^d \) and all \( \tau \in \mathcal{T} \) it holds that \( h_{\tau_{\min}}^d (x) \geq h_t^d(x) \), hence \( h_{\tau_{\min}}^d (x) \geq \sup_{\tau \in \mathcal{T}} h_t^d(x) \) for each such \( x \). Combining this and (Föllmer and Schied, 2016, Theorem 6.18) yields that \( P \)-a.s.,

\[
U_t^d = h_{\tau_{\min}}^d (X_t^d) = \sup_{\tau \in \mathcal{T}} h_t^d(X_t^d) = V_d(t, X_t^d).
\]  

(2.5)

By definition of the Snell envelope, this then yields the recursion \((2.2)\) for the value function.

**Remark 2.2.** The conditional expectation in \((2.4)\) is defined in terms of the transition kernels \( \mu_{s,t}^d \), \( 0 \leq s < t \leq T \) of the Markov process \( X^d \) (see (Kallenberg, p.143)). In fact, formally we start with transition kernels \( \mu^d \) on \( \mathbb{R}^d \) which then allow us to construct a family of probability measures \( P_x \) on the canonical path space \((\mathbb{R}^d)^{T+1}, \mathcal{B}(\mathbb{R}^d)^{T+1})\) such that, under \( P_x \), the coordinate process is a Markov process starting at \( x \) and with transition kernels \( \mu^d \). We refer to (Kallenberg, Theorem 8.4) or (Peskir and Shiryaev, 2006, Chapter II.4.1); see also (Revuz, 1984, Chapter 1).

### 2.2 Deep neural networks

In this article we will consider neural networks with the ReLU activation function \( \varphi : \mathbb{R} \to \mathbb{R} \) given by \( \varphi(x) = \max(x, 0) \). For each \( d \in \mathbb{N} \) we also denote by \( \varphi : \mathbb{R}^d \to \mathbb{R}^d \) the component-wise application of the ReLU activation function. Let \( L, d, N_0 := d, N_1, \ldots, N_L \in \mathbb{N} \) and \( A^\ell \in \mathbb{R}^{N^\ell \times N^\ell-1}, b^\ell \in \mathbb{R}^{N^\ell} \) for \( \ell = 1, \ldots, L \). A deep neural network with \( L \) layers, \( d \)-dimensional input, activation function \( \varphi \) and parameters \((A^1, b^1), \ldots, (A^L, b^L)\) is the function \( \phi : \mathbb{R}^d \to \mathbb{R}^{N_L} \) given by

\[
\phi(x) = W_L \circ (\varphi \circ W_{L-1}) \circ \cdots \circ (\varphi \circ W_1)(x), \quad x \in \mathbb{R}^d
\]  

(2.6)

where \( W_\ell : \mathbb{R}^{N^{\ell-1}} \to \mathbb{R}^{N^\ell} \) denotes the (affine) function \( W_\ell(z) = A^\ell z + b^\ell \) for \( z \in \mathbb{R}^{N^{\ell-1}} \) and \( \ell = 1, \ldots, L \). We let

\[
\text{size}(\phi) = \sum_{\ell=1}^L \sum_{i=1}^{N^\ell} \left( 1_{\{b^\ell_i \neq 0\}} + \sum_{j=1}^{N^\ell-1} 1_{\{A^\ell_{i,j} \neq 0\}} \right)
\]

denote the total number of non-zero entries in the parameter matrices and vectors of the neural network. In most cases the number of layers, the activation function and the parameters of the network are not mentioned explicitly and we simply speak of a deep neural network \( \phi : \mathbb{R}^d \to \mathbb{R}^{N_L} \). We say that a function \( f : \mathbb{R}^d \to \mathbb{R}^m \) can be realized by a deep neural network if there exists a deep neural network \( \phi : \mathbb{R}^d \to \mathbb{R}^m \) such that \( f(x) = \phi(x) \) for all \( x \in \mathbb{R}^d \). In the literature a deep neural network is often defined as the collection of parameters \( \Phi = ((A^1, b^1), \ldots, (A^L, b^L)) \) and \( \phi \) in \((2.6)\) is called the realization of \( \Phi \), see for instance (Petersen and Voigtlaender, 2018, Opschoor et al., 2020, Gonon and Schwab, 2020).
In order to simplify the notation we do not distinguish between the neural network realization and its parameters here, since the parameters are always (at least implicitly) part of the definition. Note that in general a function $f$ may admit several different realizations by deep neural networks, i.e., several different choices of parameters may result in the same realization. However, in the present article this is not an issue, because pathological cases are excluded by bounds on the size of the networks.

3 DNN Approximations for Optimal Stopping Problems

This section contains the main results of the article, the deep neural network approximation results for optimal stopping problems. We start by formulating in Assumption 1 a general Markovian framework. In Assumption we introduce the hypotheses on the reward functions. We then formulate in Theorem 3.4 the approximation result for this basic framework. Subsequently, we provide a more refined framework, see Assumption below, and prove the main result of the article, Theorem 3.8. This theorem proves that the value function can be approximated by deep neural networks without the curse of dimensionality. Corollary 3.9 shows that an analogous approximation result also holds for the continuation value. Subsequently, in Sections 3.3, 3.4 and 3.5 we specialize the result to the case of exponential Lévy models, discrete diffusion processes and show that also barrier options can be covered by including the running maximum or minimum.

3.1 Basic framework

Let $p \geq 0$ be a fixed rate of growth. For instance, in financial applications typically $p = 1$. We start by formulating in Assumption 1 a collection of hypotheses on the Markov processes $X^d$. These hypotheses will be weakened later on in Assumption 1.

**Assumption 1. [Assumptions on $X^d$]**

(i) For each $d \in \mathbb{N}$, $t \in \{0, \ldots, T-1\}$ there exists a measurable function $f_t^d : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and a random vector $Y^d_t$ such that

$$X^d_{t+1} = f_t^d(X^d_t, Y^d_t). \quad (3.1)$$

(ii) For each $d \in \mathbb{N}$ the random vectors $X^d_0, Y^d_0, \ldots, Y^d_{T-1}$ are independent and $\mathbb{E}[\|X^d_0\|] < \infty$.

Furthermore, there exist constants $c > 0$, $q \geq 0$, $\alpha \geq 0$ such that

(iii) for all $\varepsilon \in (0, 1]$, $d \in \mathbb{N}$, $t \in \{0, \ldots, T - 1\}$ there exists a neural network $\eta_{\varepsilon,d,t} : \mathbb{R}^d \times$
\[ \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ with} \]

\[ \|f_t^d(x, y) - \eta_{\varepsilon,d,t}(x, y)\| \leq \varepsilon cd^d (1 + \|x\|^p + \|y\|^p), \quad \text{for all } x, y \in \mathbb{R}^d, \quad (3.2) \]

\[ \text{size}(\eta_{\varepsilon,d,t}) \leq cd^d \varepsilon^{-\alpha}, \]

\[ \text{Lip}(\eta_{\varepsilon,d,t}) \leq cd^d, \quad (3.3) \]

(iv) for all \( d \in \mathbb{N}, t \in \{0, \ldots, T - 1\} \) it holds that \( \|f_t^d(0, 0)\| \leq cd^d \) and \( \mathbb{E}[\|Y_t^d\|^{2 \max\{2,p}\}] \leq cd^d. \)

Assumption (ii) requires a recursive updating of the Markov processes \( X^d \) according to update functions \( f_t^d \) and noise processes \( Y^d \). Assumption (ii) asks that the noise random variables and the initial condition are independent. Assumption (iii) imposes that the updating functions \( f_t^d \) can be approximated well by deep neural networks. Finally, Assumption (iv) requires that certain moments of the noise random variables and the “constant parts” of the update functions exhibit at most polynomial growth.

**Remark 3.1.** In Assumption (iii)-(iv) we could also put different constants \( c \) and \( q \) in each of the hypotheses. But then Assumption (iii)-(iv) still holds with \( c \) and \( q \) chosen as the respective maximum and so for notational simplicity we choose to directly work with the same constants for all these hypotheses.

**Remark 3.2.** Let \( s \geq t \) and consider \( \tilde{g}_{d,s} : \mathbb{R}^d \rightarrow \mathbb{R} \) defined on all of \( \mathbb{R}^d \) with \( \mathbb{E}[|\tilde{g}_{d,s}(X_s^d)|] < \infty \). Then, under Assumption (i)-(ii),

\[ \mathbb{E}[\tilde{g}_{d,s}(X_s^d)|X_t^d = x] = \mathbb{E}[\tilde{g}_{d,s} \circ f_{s-1}^d(\cdot, Y_{s-1}^d) \circ \cdots \circ f_1^d(\cdot, Y_1^d)](x) \quad (3.5) \]

for \( \mathbb{P} \circ (X_1^d)^{-1} \)-a.e. \( x \in \mathbb{R}^d \). But the right hand side of (3.5) is defined for any \( x \in \mathbb{R}^d \) for which the expectation is finite, and so in what follows we will also consider the conditional expectation \( \mathbb{E}[\tilde{g}_{d,s}(X_s^d)|X_t^d = x] \) to be defined for all such \( x \in \mathbb{R}^d \) (by (3.5)). Note that also \( \mathbb{E}[\max(g_d(t, X_t^d), \mathbb{E}[\tilde{g}_{d,s}(X_s^d)|X_t^d])] \leq \mathbb{E}[|\tilde{g}_{d,s}(X_s^d)|] < \infty \) and so by backward induction this reasoning allows to define in our framework the value function \( V_d(t, \cdot) \) on all of \( \mathbb{R}^d \), for each \( t \).

Next, we formulate a collection of hypotheses on the reward (or payoff) functions \( g_d \).

**Assumption 2.** [Assumptions on \( g_d \)] There exists constants \( c > 0, q \geq 0, \alpha \geq 0 \) such that

(i) for all \( \varepsilon \in (0, 1], d \in \mathbb{N}, t \in \{0, \ldots, T\} \) there exists a neural network \( \phi_{\varepsilon,d,t} : \mathbb{R}^d \rightarrow \mathbb{R} \) with

\[ |g_d(t, x) - \phi_{\varepsilon,d,t}(x)| \leq \varepsilon cd^d (1 + \|x\|^p), \quad \text{for all } x \in \mathbb{R}^d, \quad (3.6) \]

\[ \text{size}(\phi_{\varepsilon,d,t}) \leq cd^d \varepsilon^{-\alpha}, \]

\[ \text{Lip}(\phi_{\varepsilon,d,t}) \leq cd^d, \quad (3.7) \]

\[ (\varepsilon, d, t) \] to be defined for all such \( x \in \mathbb{R}^d \) (by (3.5)). Note that also \( \mathbb{E}[\max(g_d(t, X_t^d), \mathbb{E}[\tilde{g}_{d,s}(X_s^d)|X_t^d])] \leq \mathbb{E}[|\tilde{g}_{d,s}(X_s^d)|] < \infty \) and so by backward induction this reasoning allows to define in our framework the value function \( V_d(t, \cdot) \) on all of \( \mathbb{R}^d \), for each \( t \).

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(i) for all \( \varepsilon \in (0, 1], d \in \mathbb{N}, t \in \{0, \ldots, T\} \) there exists a neural network \( \phi_{\varepsilon,d,t} : \mathbb{R}^d \rightarrow \mathbb{R} \) with

\[ |g_d(t, x) - \phi_{\varepsilon,d,t}(x)| \leq \varepsilon cd^d (1 + \|x\|^p), \quad \text{for all } x \in \mathbb{R}^d, \quad (3.6) \]

\[ \text{size}(\phi_{\varepsilon,d,t}) \leq cd^d \varepsilon^{-\alpha}, \]

\[ \text{Lip}(\phi_{\varepsilon,d,t}) \leq cd^d, \quad (3.7) \]
Example 3.3. Assumption 2(i) means that \( g_d(t, \cdot) : \mathbb{R}^d \rightarrow \mathbb{R} \) can be approximated well by neural networks for any \( d \in \mathbb{N}, t \in \{0, \ldots, T\} \). Assumption 2(ii) imposes that the "constant part" of the payoff grows at most polynomially in \( d \). Lemma 4.7 below shows that the framework indeed ensures that \( \mathbb{E}[|g_d(t, X^d_t)|] < \infty \), as required in Section 2.1.

Theorem 3.4 shows that under Assumptions 1 and 2 the value function \( V_d \) can be approximated well by deep neural networks without the curse of dimensionality: an approximation \( \phi_{\varepsilon,t} \) with size(\( \phi_{\varepsilon,t} \)) \( \leq 6d^3 \). In addition, \( \text{Lip}(g_d(t, \cdot)) = 1 \) and therefore, setting \( \phi_{\varepsilon,d,t} = \phi_{d,t} \) for all \( \varepsilon \in (0,1] \) we get that Assumption 2 is satisfied with \( c = 6, \alpha = 0, q = 3 \). Further examples include basket call options, basket put options, call on min options and, by similar techniques, also put on min options, put on max options and many related payoffs.

We now state the main deep neural network approximation result under the assumptions introduced above.

**Theorem 3.4.** Suppose Assumptions 1 and 2 are satisfied. Let \( c > 0, q \geq 0 \) and assume for all \( d \in \mathbb{N} \) that \( \rho^d \) is a probability measure on \( \mathbb{R}^d \) with \( \int_{\mathbb{R}^d} |x|^2 \max(p,2) \rho^d(dx) \leq cd^q \).

Then there exist constants \( \kappa, \rho, r \in [0, \infty) \) and neural networks \( \psi_{\varepsilon,d,t} \), \( \varepsilon \in (0,1] \), \( d \in \mathbb{N}, t \in \{0, \ldots, T\} \), such that for any \( \varepsilon \in (0,1] \), \( d \in \mathbb{N} \), \( t \in \{0, \ldots, T\} \) the number of neural network weights grows at most polynomially and the approximation error between the neural network \( \psi_{\varepsilon,d,t} \) and the value function is at most \( \varepsilon \), that is, size(\( \psi_{\varepsilon,d,t} \)) \( \leq \kappa d^q \varepsilon^{-r} \) and

\[
\left( \int_{\mathbb{R}^d} |V_d(t,x) - \psi_{\varepsilon,d,t}(x)|^2 \rho^d(dx) \right)^{1/2} \leq \varepsilon. \tag{3.9}
\]

The proof of Theorem 3.4 will be given in Section 4.4 below.

Theorem 3.4 shows that under Assumptions 1 and 2 the value function \( V_d \) can be approximated by deep neural networks without the curse of dimensionality: an approximation error at most \( \varepsilon \) can be achieved by a deep neural network whose size is at most polynomial in \( \varepsilon^{-1} \) and \( d \). The approximation error in Theorem 3.4 is measured in the \( L^2(\rho^d) \)-norm. Theorem 3.4 can also be used to deduce further properties of \( V_d \). In the basic framework we obtain for instance the following corollary, which shows that under Assumptions 1 and 2 for each \( t \) the value function satisfies a certain average Lipschitz property with constant growing at most polynomially in \( d \).

**Corollary 3.5.** Suppose Assumptions 1 and 2 are satisfied. Let \( \nu_0^d \) be the standard Gaussian measure on \( \mathbb{R}^d \). Then for any \( R > 0 \) there exist constants \( \kappa, \rho \in [0, \infty) \) such that for any \( d \in \mathbb{N}, t \in \{0, \ldots, T\}, h \in [-R, R]^d \) the value function satisfies

\[
\left( \int_{\mathbb{R}^d} |V_d(t,x) - V_d(t,x+h)|^2 \nu_0^d(dx) \right)^{1/2} \leq \|h\| \kappa d^q. \tag{3.10}
\]

The proof of Corollary 3.5 will be given at the end of Section 4.4.
3.2 Refined framework

We now introduce a refined framework, in which the approximation hypothesis (3.2) and the Lipschitz condition (3.4) in Assumption [1(iii)] are weakened, see (3.11) and (3.13) below. Due to these weaker hypotheses we need to introduce potentially stronger moment assumptions on the noise variables $Y^d_t$. Note that the additional growth conditions (3.11) and (3.15) are satisfied automatically under Assumption [1(see Lemma 4.5 and Remark 3.6 below).

**Assumption 1’.** *[Weaker assumptions on X^d] Assume that (i), (ii) and (iv) in Assumption [1 are satisfied. Furthermore, assume that there exist constants $c > 0$, $h > 0$, $q,q_0 \geq 0$, $\alpha \geq 0$, $\beta > 0$, $m \in \mathbb{N}$, $\theta \geq 0$ and $\zeta \geq 0$ such that for all $\varepsilon \in (0,1], d \in \mathbb{N}$, $t \in \{0,\ldots,T-1\}$ there exists a neural network $\eta_{\varepsilon,t}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ with

\[
\|f^d_t(x,y) - \eta_{\varepsilon,t}(x,y)\| \leq \varepsilon c d^q (1 + \|x\|^p + \|y\|^p), \quad \text{for all } x,y \in \mathbb{R}^d \quad (3.11)
\]

size($\eta_{\varepsilon,t}$) $\leq$ cd$^q$,$\varepsilon$,$\alpha$, (3.12)

Lip($\eta_{\varepsilon,t}$) $\leq$ cd$^q$,$\varepsilon$,$\zeta$ (3.13)

and for all $x,y \in \mathbb{R}^d$

\[
\mathbb{E}[\|f^d_t(x,Y^d_t)\|^{2m \text{ max}(p,2)}] \leq h d^q (1 + \|x\|^{2m \text{ max}(p,2)}), \quad (3.14)
\]

\[
\|\eta_{\varepsilon,t}(x,y)\| \leq cd^q (2 + \|x\| + \|y\||) \quad (3.15)
\]

and $\mathbb{E}[\|Y^d_t\|^{2m \text{ max}(2,p)}] \leq cd^q$.

**Remark 3.6.** A sufficient condition for (3.14) is that there exist $\tilde{c} > 0$ and $\tilde{q} \geq 0$ such that for all $d \in \mathbb{N}$, $x,y \in \mathbb{R}^d$ we have $\mathbb{E}[\|Y^d_t\|^{2m \text{ max}(2,p)}] \leq \tilde{c} d^q$ and $\|f^d_t(x,y)\| \leq \tilde{c} d^q (1 + \|x\| + \|y\|)$. Then

\[
\mathbb{E}[\|f^d_t(x,Y^d_t)\|^{2m \text{ max}(p,2)}] \leq (\tilde{c} d^q)^{2m \text{ max}(p,2)} \mathbb{E}[(1 + \|x\| + \|Y^d_t\|)^{2m \text{ max}(p,2)}] \leq (\tilde{c} d^q)^{2m \text{ max}(p,2)} (1 + \|x\|^{2m \text{ max}(p,2)} + \tilde{c} d^q).
\]

**Remark 3.7.** While in many relevant applications the number of time steps $T$ is only moderate (e.g. around 10 in [Becker et al., 2019, Sections 4.1–4.2]), it is also important to analyse the situation when $T$ is large. To this end, in Assumption [1] we have introduced the constants $h$ and $\tilde{q}$ instead of using the common upper bounds $c$, $q$. This makes it possible to get first insights about the situation in which $T$ is large from the proofs in Section 4. Indeed, if $h = 1 + \tilde{h}$ and $\tilde{h}$ is sufficiently small (as it is the case for instance in certain discretized diffusion models), then the constants in Lemma 4.6 and Lemma 4.8 are small also for large $T$.

Examples of processes that satisfy Assumption [1] are provided further below. These include, in particular, the Black-Scholes model, more general exponential Lévy processes and discrete diffusions.

We now state the main theorem of the article.
Theorem 3.8. Suppose Assumptions 1 and 2 are satisfied. Let \( c > 0 \), \( q \geq 0 \) and assume for all \( d \in \mathbb{N} \) that \( \rho^d \) is a probability measure on \( \mathbb{R}^d \) with \( \int_{\mathbb{R}^d} \| x \|^{2m \max(p,2)} \rho^d(dx) \leq cd^q \). Furthermore, assume that \( \zeta < \min(1,\beta m^{-\theta}) T^{-1} \), where \( m, \beta, \zeta, \theta \) are the constants appearing in Assumption 1’.

Then there exist constants \( \kappa, q, r \in [0, \infty) \) and neural networks \( \psi_{\varepsilon,d,t}, \varepsilon \in (0,1], d \in \mathbb{N}, t \in \{0,\ldots,T\} \), such that for any \( \varepsilon \in (0,1], d \in \mathbb{N}, t \in \{0,\ldots,T\} \) the number of neural network weights grows at most polynomially and the approximation error between the neural network \( \psi_{\varepsilon,d,t} \) and the value function is at most \( \varepsilon \), that is, \( \text{size}(\psi_{\varepsilon,d,t}) \leq \kappa d^q \varepsilon^{-r} \) and

\[
\left( \int_{\mathbb{R}^d} |V_d(t,x) - \psi_{\varepsilon,d,t}(x)|^2 \rho^d(dx) \right)^{1/2} \leq \varepsilon. \tag{3.16}
\]

The proof of Theorem 3.8 will be given in Section 4.5 below.

Theorem 3.8 shows that for Markov processes satisfying Assumption 1 and for reward functions satisfying Assumption 2 the value function of the associated optimal stopping problem can be approximated by deep neural networks without the curse of dimensionality. In other words, an approximation error at most \( \varepsilon \) can be achieved by a deep neural network whose size is at most polynomial in \( \varepsilon^{-1} \) and \( d \). The condition \( \zeta < \min(1,\beta m^{-\theta}) T^{-1} \) in Theorem 3.8 can be viewed as a condition on \( m \), which needs to be sufficiently large. This means that sufficiently high moments of \( Y^d_t \) need to exist and grow only polynomially in \( d \).

A key step in the proof consists in constructing a deep neural network approximating the continuation value. Therefore, we immediately obtain the following corollary.

Corollary 3.9. Consider the setting of Theorem 3.8. Then for each \( \varepsilon \in (0,1], d \in \mathbb{N}, t \in \{0,\ldots,T\} \) there exists a neural network \( \gamma_{\varepsilon,d,t} \) such that \( \text{size}(\gamma_{\varepsilon,d,t}) \leq \kappa d^q \varepsilon^{-r} \) and

\[
\left( \int_{\mathbb{R}^d} |E[V_d(t+1,X^d_{t+1}|X^d_t = x) - \gamma_{\varepsilon,d,t}(x)|^2 \rho^d(dx) \right)^{1/2} \leq \varepsilon. \tag{3.17}
\]

3.3 Exponential Lévy models

In this subsection we apply Theorem 3.8 to exponential Lévy models. Recall that an \( \mathbb{R}^d \)-valued stochastic process \( L^d = (L^d_t)_{t \geq 0} \) is called a \((d\text{-dimensional})\) Lévy process if it is stochastically continuous, its sample paths are almost surely right continuous with left limits, it has stationary and independent increments and \( \mathbb{P}(L^d_0 = 0) = 1 \). A Lévy process \( L^d \) is fully characterized by its Lévy triplet \((A^d, \gamma^d, \nu^d)\) where \( A^d \in \mathbb{R}^{d \times d} \) is a symmetric, nonnegative definite matrix, \( \gamma^d \in \mathbb{R}^d \) and \( \nu^d \) is a measure on \( \mathbb{R}^d \) describing the jump structure of \( L^d \). We refer, e.g., to \( \text{Sato (1999), Applebaum (2009)} \) for more detailed statements of these definitions, proofs of this characterization and further details on Lévy processes.
A stochastic process $X^d$ is said to follow an exponential Lévy model, if
\begin{equation}
X_t^d = (x_1^d \exp(L_{t,1}^d), \ldots, x_d^d \exp(L_{t,d}^d)), \quad t \in \{0, \ldots, T\}
\end{equation}
for a $d$-dimensional Lévy process $L_t^d = (L_t^{d,i})_{t \geq 0}$ and $x^d \in \mathbb{R}^d$. We refer to Cont and Tankov (2004), Eberlein and Kallsen (2019) for more details on financial modelling using exponential Lévy models.

From Theorem 3.8 we now obtain the following deep neural network approximation result. This result includes the case of a Black-Scholes model ($\nu^d = 0$) as well as pure jump models ($\kappa_{t,j}^d = 0$) with sufficiently integrable tails. In particular, Corollary 3.10 applies to prices of American / Bermudan basket put options, put on min or put on max options in such models (cf. Example 3.3 for payoffs that satisfy Assumption 2).

**Corollary 3.10.** Let $X^d$ follow an exponential Lévy model with Lévy triplet $(\kappa^d, \gamma^d, \nu^d)$ and assume the triplets are bounded in the dimension, that is, there exists $B > 0$ such that for any $d \in \mathbb{N}$, $i, j = 1, \ldots, d$
\begin{equation}
\max \left( \kappa^d, \gamma^d, \int_{\|z\| > 1} e^{2(T+1)\max(p,2)\|z\|} \nu^d(dz), \int_{\{\|z\| \leq 1\}} z^2 \nu^d(dz) \right) \leq B.
\end{equation}

Suppose the payoff functions $g_{\varepsilon}$ satisfy Assumption 2. Let $c > 0$, $q \geq 0$ and assume for all $d \in \mathbb{N}$ that $\rho^d$ is a probability measure on $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} \|x\|^{2(T+1)} \max(p,2) \rho^d(dx) \leq cd^q$. Then there exist constants $\kappa, q, \tau \in [0, \infty)$ and neural networks $\psi_{\varepsilon, d, t}$, $\varepsilon \in (0, 1]$, $d \in \mathbb{N}$, $t \in \{0, \ldots, T\}$, such that for any $\varepsilon \in (0, 1]$, $d \in \mathbb{N}$, $t \in \{0, \ldots, T\}$
\begin{equation}
\text{size}(\psi_{\varepsilon, d, t}) \leq \kappa d^q \varepsilon^{-\tau} \quad \text{and} \quad \left( \int_{\mathbb{R}^d} |V_d(t, x) - \psi_{\varepsilon, d, t}(x)|^2 \rho^d(dx) \right)^{1/2} \leq \varepsilon.
\end{equation}

**Proof.** This follows directly from Theorem 3.8 and Lemma 4.2 with the choice $\zeta = \theta = \frac{1}{T}$, $m = T + 1$, which ensures that $\zeta < \frac{1}{T-1} = \frac{\min(1, 2m-2)}{T-1}$.

**3.4 Discrete diffusion models**

Let $T > 0$ and let $X^d$ follow a discrete diffusion model with coefficients $\mu^d: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma^d: [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, that is, $X^d$ satisfies $X_0^d = x^d$ and
\begin{equation}
X_{k+1}^d = X_k^d + \mu^d(t_k, X_k^d)(t_{k+1} - t_k) + \sigma^d(t_k, X_k^d)(W_{k+1}^d - W_k^d), \quad k \in \{0, \ldots, T-1\}
\end{equation}
for some $0 \leq t_0 < t_1 < \ldots < t_T \leq T$, $x^d \in \mathbb{R}^d$ and $W^d$ a $d$-dimensional Brownian motion. Consider the following assumption on the drift and diffusion coefficients:

**Assumption 3.** Assume that there exist constants $C > 0$, $q, \bar{\alpha}, \bar{\zeta} \geq 0$ and, for each $d \in \mathbb{N}$, $t \in \{0, \ldots, T-1\}$, $\varepsilon \in (0, 1]$, there exist neural networks $\mu_{\varepsilon, d, t}: \mathbb{R}^d \to \mathbb{R}^d$ and
Corollary 3.11. Let \( \sigma_{\varepsilon,d,t,i} : \mathbb{R}^d \rightarrow \mathbb{R}^d, i = 1, \ldots, d, \) such that for all \( d \in \mathbb{N}, \varepsilon \in (0, 1], t \in \{0, \ldots, T-1\}, x \in \mathbb{R}^d \) it holds that

\[
\| \mu^d(t, x) - \mu_{\varepsilon,d,t}(x) \| + \| \sigma^d(t, x) - \sigma_{\varepsilon,d,t}(x) \|_F \leq \varepsilon C d^q (1 + \| x \|),
\]
\[
\| \mu^d(t, x) \| + \| \sigma^d(t, x) \|_F \leq C d^q (1 + \| x \|),
\]
\[
\text{size}(\mu_{\varepsilon,d,t}) + \sum_{i=1}^d \text{size}(\sigma_{\varepsilon,d,t,i}) \leq C d^d \varepsilon^{-\delta},
\]
\[
\max(\text{Lip}(\mu_{\varepsilon,d,t}), \text{Lip}(\sigma_{\varepsilon,d,t,1}), \ldots, \text{Lip}(\sigma_{\varepsilon,d,t,d})) \leq C d^d \varepsilon^{-\tilde{\zeta}}.
\]

Here we denote by \( \sigma_{\varepsilon,d,t}(x) \in \mathbb{R}^{d \times d} \) the matrix with \( i \)-th row \( \sigma_{\varepsilon,d,t,i}(x) \).

**Corollary 3.11.** Let \( X^d \) follow a discrete diffusion model with coefficients satisfying Assumption 3 with \( \zeta < \frac{1}{T-1} \). Suppose \( p \geq 2 \) and the payoff functions \( g_d \) satisfy Assumption 2. Let \( c > 0, q \geq 0 \) and assume for all \( d \in \mathbb{N} \) that \( \rho^d \) is a probability measure on \( \mathbb{R}^d \) with \( \int_{\mathbb{R}^d} \| x \|^{2m \max(p,2)} \rho^d(dx) \leq c d^n \) for \( m = \lceil \frac{2(1+\tilde{\zeta})}{\beta} + 1 \rceil \).

Then there exist constants \( \kappa, q, r \in [0, \infty) \) and neural networks \( \psi_{\varepsilon,d,t}, \varepsilon \in (0, 1], d \in \mathbb{N}, t \in \{0, \ldots, T\} \), such that for any \( \varepsilon \in (0, 1], d \in \mathbb{N}, t \in \{0, \ldots, T\} \)

\[
\text{size}(\psi_{\varepsilon,d,t}) \leq \kappa d^q \varepsilon^{-r} \quad \text{and} \quad \left( \int_{\mathbb{R}^d} |V_d(t, x) - \psi_{\varepsilon,d,t}(x)|^2 \rho^d(dx) \right)^{1/2} \leq \varepsilon. \tag{3.22}
\]

**Proof.** By Lemma 3.3 it follows that Assumption 1 is satisfied. In addition, the constant \( \beta > 0 \) in Assumption 1 may be chosen arbitrarily and \( \tilde{\zeta} = \theta = \beta + \tilde{\zeta} \). Thus, we may select \( \beta = \frac{1}{T-1} - \tilde{\zeta} - \delta \) for some \( \delta > 0 \) and then \( \beta > 0 \) and \( \zeta = \theta = \frac{1}{T-1} - \delta \). Choosing \( \delta = \frac{1}{2}(\frac{1}{T-1} - \tilde{\zeta}) \), \( m = \lceil \frac{1+\tilde{\zeta}}{\beta} + 1 \rceil \) then ensures that \( \zeta < \frac{1}{T-1} = \min(1, \frac{3m-\theta}{T-1}) \). Theorem 3.8 hence implies (3.22).

### 3.5 Running minimum and maximum

In this subsection we show that our framework can also cover barrier options. This follows from the next proposition, which proves that for processes satisfying Assumption 1 also the processes augmented by their running maximum or minimum satisfy Assumption 1.

**Proposition 3.12.** Suppose Assumption 1 holds. Let \( \bar{X}^1 = X^1 \) and for \( d \in \mathbb{N}, d \geq 2, t \in \{0, \ldots, T\} \) consider the \( \mathbb{R}^d \)-valued process \( \bar{X}^d_t = (X^{d-1}_t, M^d_t) \), where either \( M^d_t = \min_{i=1,\ldots,d-1} \min_{s=0,\ldots,t} X^{d-1}_{s,i} \) or \( M^d_t = \max_{i=1,\ldots,d-1} \max_{s=0,\ldots,t} X^{d-1}_{s,i} \). Then \( \bar{X}^d, d \in \mathbb{N} \), satisfy Assumption 1.

The proof is given at the end of Section 4.2 below.
4 Proofs

This section contains the remaining proofs of the results in Section 3. The section is split in several subsections. In Section 4.1 we provide a refined result on deep neural network approximations of the product function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto xy$. Section 4.2 then contains two lemmas in which this approximation result is applied to verify that suitable exponential Lévy and discrete diffusion models satisfy Assumption 1. Subsequently, Section 4.3 contains auxiliary results needed for the proofs of Theorem 3.4 and Theorem 3.8. The proofs of these two results are then contained in Sections 4.4 and 4.5.

4.1 Deep neural network approximation of the product

Based on (Yarotsky, 2017, Proposition 3) we provide here a refined result regarding deep neural network approximations of the product function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto xy$.

**Lemma 4.1.** There exists $c > 0$ such that for any $\varepsilon \in (0, 1]$, $M \geq 1$ there exists a neural network $n_{\varepsilon, M} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with

$$\sup_{x, y \in [-M, M]} |n_{\varepsilon, M}(x, y) - xy| < \varepsilon,$$

(size($n_{\varepsilon, M}) \leq c(\log(\varepsilon^{-1}) + \log(M) + 1)$ and for all $x, x', y, y' \in \mathbb{R}$

$$|n_{\varepsilon, M}(x, y) - n_{\varepsilon, M}(x', y')| \leq Mc(|x - x'| + |y - y'|).$$

**Proof.** By (Grohs and Herrmann, 2021, Lemma 4.2) or (Opschoor et al., 2020, Proposition 4.1) (based on (Yarotsky, 2017, Proposition 3)), there exists $c > 0$ such that for any $\tilde{\varepsilon} \in (0, \frac{1}{2})$ there exists a neural network $n_{\varepsilon} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with the property that $\sup_{x, y \in [-1, 1]} |n_{\varepsilon}(x, y) - xy| < \tilde{\varepsilon}$, size($n_{\varepsilon}) \leq c(\log(\tilde{\varepsilon}^{-1}) + 1)$ and

$$\sup_{x, x', y, y' \in [-1, 1]} |n_{\varepsilon}(x, y) - n_{\varepsilon}(x', y')| \leq c(|x - x'| + |y - y'|).$$

Consider now the capped neural network $\tilde{n}_{\varepsilon}(x, y) = n_{\varepsilon}(\pi_1(x), \pi_1(y))$, where we set $\pi_1(z) = \max(-1, \min(z, 1))$. Then $\tilde{n}_{\varepsilon}(x, y) = n_{\varepsilon} \circ \text{cap}(x, y)$ and it can be verified that cap is again a neural network and for $x, y \in [-1, 1]$ we have $\tilde{n}_{\varepsilon}(x, y) = n_{\varepsilon}(x, y)$. The fact that the composition of two ReLU neural networks can again be realized by a ReLU neural network with size bounded by twice the sum of the respective sizes (see, e.g. Opschoor et al., 2020, Proposition 2.2)) hence proves that there exists $\delta \geq c$ such that for all $\varepsilon \in (0, \frac{1}{2})$ we have size($\tilde{n}_{\varepsilon}) \leq \delta(\log(\tilde{\varepsilon}^{-1}) + 1)$. Furthermore, $\sup_{x, y \in [-1, 1]} |\tilde{n}_{\varepsilon}(x, y) - xy| < \tilde{\varepsilon}$ and for all $x, x', y, y' \in \mathbb{R}$

$$|\tilde{n}_{\varepsilon}(x, y) - \tilde{n}_{\varepsilon}(x', y')| \leq c(|\pi_1(x) - \pi_1(x')| + |\pi_1(y) - \pi_1(y')|) \leq \delta(|x - x'| + |y - y'|).$$
Now let $\varepsilon \in (0, 1]$, $M \geq 1$ be given, choose $\tilde{\varepsilon} = 3^{-1}M^{-2}\varepsilon$ and define the rescaled network $n_{\varepsilon,M}(x, y) = M^2\tilde{n}_\varepsilon\left(\frac{x}{M}, \frac{y}{M}\right)$. Then
\[
\sup_{x,y \in [-M,M]} |n_{\varepsilon,M}(x, y) - xy| = M^2 \sup_{x,y \in [-M,M]} |\tilde{n}_\varepsilon\left(\frac{x}{M}, \frac{y}{M}\right) - \frac{x}{M}y| < M^2\tilde{\varepsilon},
\]
size$(n_{\varepsilon,M}) \leq \tilde{c}(\log(\tilde{\varepsilon}^{-1}) + 1)$ and for all $x, x', y, y' \in \mathbb{R}$
\[
|n_{\varepsilon,M}(x, y) - n_{\varepsilon,M}(x', y')| = M^2|\tilde{n}_\varepsilon\left(\frac{x}{M}, \frac{y}{M}\right) - \tilde{n}_\varepsilon\left(\frac{x'}{M}, \frac{y'}{M}\right)| \leq M\tilde{c}(|x - x'| + |y - y'|).
\]

\[\square\]

### 4.2 Sufficient conditions

In this subsection we prove Lemma 4.2 and Lemma 4.3 which show that the exponential Lévy and discrete diffusion models considered above satisfy Assumption 1. We also provide a proof of Proposition 3.12.

**Lemma 4.2.** Let $X^d$ follow an exponential Lévy model (cf. (3.18)) for each $d \in \mathbb{N}$ and assume that the Lévy triplets $(\Lambda^d, \gamma^d, \nu^d)$ are bounded in the dimension, that is, there exists $B > 0$ such that for any $d \in \mathbb{N}$, $i, j = 1, \ldots, d$
\[
\max \left(\Lambda^d_{i,j}, \gamma^d_i, \int_{\|z\| > 1} e^{\tilde{c}_i d^\alpha (dz)}, \int_{\{\|z\| \leq 1\}} z^2_i \nu^d (dz)\right) \leq B,
\]
where $\tilde{\rho} = 2m \max (2, p)$. Then Assumption T is satisfied with constant $\beta > 0$ in Assumption II chosen arbitrarily and with $\zeta = \theta = \beta$.

**Proof.** Firstly, (3.18) shows for each $d \in \mathbb{N}$, $t \in \{0, \ldots, T - 1\}$ that $X^{d}_{t+1,i} = X^d_{t,i} \exp(L^d_{t+1,i} - L^d_{t,i})$ for all $i = 1, \ldots, d$. Therefore, $X^d_{t+1} = f^d_t(X^d_t, Y^d_t)$ with $Y^d_{t,i} = \exp(L^d_{t+1,i} - L^d_{t,i})$ and $f^d_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ given by $f^d_t(x, y) = (x_1 y_1, \ldots, x_d y_d)$ for $x, y \in \mathbb{R}^d$, i.e., (3.11) is satisfied. Since $L^d$ has independent increments, it follows that Assumption II is satisfied. Next, we can employ an argument from the proof of (Gonon and Schwab, 2021a, Theorem 5.1) (which uses (Sato, 1999, Theorem 25.17) and (4.2)) to obtain for any $d \in \mathbb{N}$, $i = 1, \ldots, d$ that
\[
\mathbb{E}[e^{\tilde{p}L^d_{i,i}}] = \exp \left(\frac{\tilde{p}^2}{2} A_{i,i}^d + \int_{\mathbb{R}^d} (e^{\tilde{p}y_i} - 1 - \tilde{p}y_i 1\{\|y\| \leq 1\}) \nu^d (dy) + \tilde{p} \gamma^d_i\right)
\]
\[
\leq \exp \left(\frac{5\tilde{p}^2}{2} B + \tilde{p}^2 e^{\tilde{p}B}\right).
\]
Combined with Minkowski’s inequality and the stationarity increments property of $L^d$ this yields

$$\mathbb{E}[\|Y_t^d\|^{2m \max(2,p)}] = \left( \mathbb{E} \left[ \left( \sum_{i=1}^{d} |Y_{t,i}^d|^2 \right)^{\frac{1}{m \max(2,p)}} \right] \right)^{m \max(2,p)}$$

$$\leq \left( \sum_{i=1}^{d} \mathbb{E} \left[ |Y_{t,i}^d|^{2m \max(2,p)} \right] \right)^{\frac{1}{m \max(2,p)}}$$

$$= \left( \sum_{i=1}^{d} \mathbb{E} \left[ e^{2m \max(2,p)L_{1,i}} \right] \right)^{\frac{1}{m \max(2,p)}} \leq d^{m \max(2,p)} \exp \left( \frac{5\bar{p}^2}{2} B + \bar{p}^2 e^{\bar{p} B} \right).$$ (4.4)

Furthermore, $f_t^d(0,0) = 0$ and thus Assumption 1(iv) is satisfied. Next, for $\varepsilon \in (0,1]$, $d \in \mathbb{N}$, $t \in \{0,\ldots,T-1\}$ let $M = \varepsilon^{-\beta}$ and let $\eta_{\varepsilon,d,t} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be the $d$-fold parallelization of $n_{\varepsilon,M}$ from Lemma 4.1. Then for all $x,y \in \mathbb{R}^d$ we obtain

$$\|f_t^d(x,y) - \eta_{\varepsilon,d,t}(x,y)\| = \left( \sum_{i=1}^{d} |x_i y_i - n_{\varepsilon,M}(x_i, y_i)|^2 \right)^{1/2} \leq C d^{\frac{1}{2}},$$

$$\text{size}(\eta_{\varepsilon,d,t}) \leq d \text{size}(n_{\varepsilon,M}) \leq C d((\beta + 1) \log(\varepsilon^{-1}) + 1) \leq C_1 d \varepsilon^{-1}$$

with $C_1 = c(\beta + 2)$ and for all $x,x',y,y' \in \mathbb{R}^d$

$$\|\eta_{\varepsilon,d,t}(x,y) - \eta_{\varepsilon,d,t}(x',y')\| = \left( \sum_{i=1}^{d} |n_{\varepsilon,M}(x_i, y_i) - n_{\varepsilon,M}(x'_i, y'_i)|^2 \right)^{1/2} \leq \left( \sum_{i=1}^{d} |M c(|x_i - x'_i| + |y_i - y'_i|)|^2 \right)^{1/2} \leq \sqrt{2} \varepsilon^{-\beta} c(\|x - x'\| + \|y - y'\|).$$

Finally, for all $x,y \in \mathbb{R}^d$

$$\|\eta_{\varepsilon,d,t}(x,y)\| \leq \|\eta_{\varepsilon,d,t}(x,y) - \eta_{\varepsilon,d,t}(0,0)\| + \|\eta_{\varepsilon,d,t}(0,0) - f_t^d(0,0)\| \leq \varepsilon^{-\beta} c(\|x\| + \|y\|) + \varepsilon d^{\frac{1}{2}} \leq \varepsilon^{-\beta} \max(c,1)d^{\frac{1}{2}}(1 + \|x\| + \|y\|).$$
and Minkowski’s integral inequality and (4.4) imply

\[ \mathbb{E}[\|f^d_t(x, Y^d_t)\|^{2m \max(p, 2)}] = \mathbb{E} \left[ \left( \sum_{i=1}^{d} x_i^2(Y^d_{t,i})^2 \right)^{m \max(p, 2)} \right] \]

\[ \leq \left( \sum_{i=1}^{d} \left( \mathbb{E}[x_i^{2m \max(p, 2)} (Y^d_{t,i})^{2m \max(p, 2)}] \right)^{1/(m \max(p, 2))} \right)^{m \max(p, 2)} \]

\[ \leq \left( \sum_{i=1}^{d} x_i^2 \right)^{m \max(p, 2)} \mathbb{E}[\|Y^d_t\|^{2m \max(p, 2)}] \]

\[ \leq d^{m \max(2, p)} \exp \left( \frac{\tilde{\bar{\rho}}^2}{2} + \tilde{\bar{\rho}}^2 \tilde{\bar{e}}^2 B \right) (1 + \|x\|^{2m \max(p, 2)}). \]

\[ \square \]

**Lemma 4.3.** Assume \( p \geq 2 \), let \( X^d \) follow a discrete diffusion model with coefficients \( \mu^d: [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma^d: [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d} \) and suppose Assumption \( 5 \) holds. Then Assumption \( 1' \) is satisfied with constants \( m \in \mathbb{N}, \beta > 0 \) in Assumption \( 1 \) chosen arbitrarily and with \( \zeta = \theta = \beta + \zeta \).

**Proof.** Firstly, (3.1) holds with \( f^d_t(x, y) = x + \mu^d(t, x)(t_{t+1} - t_t) + \sigma^d(t, x)y \) and \( Y^d_t = W^d_{t_{t+1}} - W^d_t \). Assumption \( 1(ii) \) holds by the independent increment property of Brownian motion.

Next, for all \( d \in \mathbb{N}, t \in \{0, \ldots, T-1\} \) it holds that \( \|f^d_t(0, 0)\| = \|\mu^d(t, 0)(t_{t+1} - t_t)\| \leq C \tilde{T} d^\eta \) and, with \( \bar{\rho} = m \max(2, p) \), \( \mathbb{E}[\|Y^d_t\|^{2\bar{\rho}}] \leq \tilde{T}^{\bar{\rho}} \mathbb{E}[\|Z\|^{2\bar{\rho}}] \) for \( Z \) standard normally distributed in \( \mathbb{R}^d \). The fact that \( \|Z\|^2 \sim \chi^2(d) \) and the upper and lower bounds for the gamma function (see, e.g., Gonon et al., 2021, Lemma 2.4)) thus yield

\[ \mathbb{E}[\|Y^d_t\|^{2\bar{\rho}}] \leq \tilde{T}^{\bar{\rho}} 2^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \leq \tilde{T}^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \left( \frac{e}{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \right)^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \]

\[ \leq \left( 4 \tilde{T} \bar{\rho} \right)^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \left( \frac{e}{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \right)^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \]

\[ \leq \left( \frac{e}{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \right)^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} \]

with \( c_{\bar{\rho}} = \max_{n \in \mathbb{N}} \left( 1 + \frac{2\bar{\rho}}{n} \right)^{\bar{\rho} / \Gamma(\frac{\tilde{T} + 1}{2})} < \infty. \)

Next, for \( \varepsilon \in (0, 1], d \in \mathbb{N}, t \in \{0, \ldots, T-1\} \) let \( M = 4 \max(C, 1) d^{\tilde{T} + 1} e^{-\beta} \) and consider \( \eta_{\varepsilon, d, t}: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \) given by

\[ \eta_{\varepsilon, d, t}(x, y) = x + \mu_{\varepsilon, d, t}(x)(t_{t+1} - t_t) + \sum_{j=1}^{d} n_{\varepsilon, M}(\sigma_{\varepsilon, d, t,i,j}(x), y_j) \]

(4.6)
for $i = 1, \ldots, d$ with $n_{\varepsilon,M}$ from Lemma 4.1. By using the operations of parallelization and concatenation it follows that we can realize $(x, y) \mapsto n_{\varepsilon,M}(\sigma_{\varepsilon,d,t,i,j}(x), y_j)$ by a neural network of size $s_{i,j} := 2(\text{size}(n_{\varepsilon,M}) + 2 + \text{size}(\sigma_{\varepsilon,d,t,i,j}))$, see, e.g., Gonschot et al. (2020, Proposition 2.2). Recall that the identity on $\mathbb{R}$ can be realized by a ReLU deep neural network of arbitrary depth $\ell$ and size $2\ell$ (see (Petersen and Voigtlaender, 2018, Remark 2.4), (Opschoor et al., 2020, Proposition 2.4)). Thus, we may insert identity networks in (4.6) to ensure that all summands can be realized by networks of the same depth, which is at most $\max(\text{size}(\mu_{\varepsilon,d,t,i}), s_{i,1}, \ldots, s_{i,d})$ due to the fact that the depth of a network is bounded by its size. By applying the summing operation for neural networks of equal depth (see, e.g., Gonschot and Schwab, 2021a, Lemma 3.2) it follows that $n_{\varepsilon,d,t,i}$ can be realized by a deep neural network with

$$
\text{size}(n_{\varepsilon,d,t,i}) \leq 2 \left( 2 + \text{size}(\mu_{\varepsilon,d,t,i}) + \sum_{j=1}^{d} s_{i,j} \right) + 4 \max(\text{size}(\sigma_{\varepsilon,d,t,i}), s_{i,1}, \ldots, s_{i,d})
\leq 12 \left( 1 + Cd^{\ell} \varepsilon^{-\tilde{a}} + d(\varepsilon(\log(\varepsilon^{-1}) + \log(M) + 1) + 2) \right).
$$

Next, we use Assumption $\text{B}$ to estimate

$$
\|\sigma_{\varepsilon,d,t}(x)\|_{F} \leq \|\sigma_{\varepsilon,d,t}(x) - \sigma^{d}(t_{i}, x)\|_{F} + \|\sigma^{d}(t_{i}, x)\|_{F} \leq 2Cd^{\beta}(1 + \|x\|)
$$

and thus for $x \in [-\varepsilon^{-\beta}, \varepsilon^{-\beta}]^{d}$ it follows that $\|\sigma_{\varepsilon,d,t}(x)\|_{F} \leq 2Cd^{\beta}(1 + \alpha d \varepsilon^{-\beta}) \leq M$. Hence, Assumption $\text{B}$ and (4.1) imply for $x, y \in [-\varepsilon^{-\beta}, \varepsilon^{-\beta}]^{d}$ that

$$
\|f^{d}(x, y) - n_{\varepsilon,d,t}(x, y)\|
\leq \|\mu^{d}(t_{i}, x) - \mu_{\varepsilon,d,t}(x)(t_{i+1} - t_{i}) + \left( \sum_{i=1}^{d} (\sigma^{d}(t_{i}, x)y_{i} - \sum_{j=1}^{d} n_{\varepsilon,M}(\sigma_{\varepsilon,d,t,i,j}(x), y_{j}) \right)\|^2 \right)^{1/2}
\leq \varepsilon CTd^{\beta}(1 + \|x\|) + \|\sigma^{d}(t_{i}, x)y - \sigma_{\varepsilon,d,t}(x)y\|
+ \left( \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_{\varepsilon,d,t,i,j}(x)y_{j} - n_{\varepsilon,M}(\sigma_{\varepsilon,d,t,i,j}(x), y_{j}) \right)^{2} \right)^{1/2}
\leq \varepsilon CTd^{\beta}(1 + \|x\|) + \|\sigma^{d}(t_{i}, x) - \sigma_{\varepsilon,d,t}(x)\|_{F}\|y\| + d^{\beta} \varepsilon
\leq \varepsilon CTd^{\beta}(1 + \|x\|) + \|\sigma^{d}(t_{i}, x) - \sigma_{\varepsilon,d,t}(x)\|_{F}\|y\| + d^{\beta} \varepsilon
\leq 2\varepsilon(C(T + 1)d^{\beta} + d^{\beta})\|x\|^{2} + \|y\|^{2}.
$$
Furthermore, Assumption 3 and the Lipschitz property of \( n_{\varepsilon,M} \) yield for all \( x, x', y, y' \in \mathbb{R}^d \)

\[
\| n_{\varepsilon,d,t}(x, y) - n_{\varepsilon,d,t}(x', y') \| \leq \| x - x' \| + \| \mu_{\varepsilon,d,t}(x) - \mu_{\varepsilon,d,t}(x') \| (u_{t+1} - u_t) \\
+ \left( \sum_{i=1}^d \sum_{j=1}^d n_{\varepsilon,M}(\sigma_{\varepsilon,d,t,i,j}(x'), y_j) - n_{\varepsilon,M}(\sigma_{\varepsilon,d,t,i,j}(x), y_j) \right)^2 \right)^{\frac{1}{2}} \\
\leq (1 + C d^2 \varepsilon^{-\tilde{\zeta}T}) \| x - x' \| \\
+ \left( \sum_{i=1}^d \sum_{j=1}^d M c(\| \sigma_{\varepsilon,d,t,i,j}(x') - \sigma_{\varepsilon,d,t,i,j}(x) \| + | y_j - y'_j |) \right)^2 \right)^{\frac{1}{2}} \\
\leq 8 \max(C, 1)^2 d^{2q+2} \varepsilon^{-\beta-\tilde{\zeta} c}(1 + C T) (\| x - x' \| + \| y - y' \|).
\]

Finally, for all \( x, y \in \mathbb{R}^d \)

\[
\mathbb{E}[\| f_t^d(x, Y_t^d) \|^{2p}]^{\frac{1}{p}} \leq \| x \| + \| \mu_t^d(x, x) \| T + \mathbb{E}[\| \sigma_t^d(x, x) \| Y_t^d \|^{2p}]^{\frac{1}{p}} \\
\leq (1 + C T d^q)(1 + \| x \|) + C d^q (1 + \| x \|) \mathbb{E}[\| Y_t^d \|^{2p}]^{\frac{1}{p}}
\]

so that (4.3) implies a polynomial growth bound (3.14) and the estimate

\[
\| n_{\varepsilon,d,t}(x, y) \| = \| n_{\varepsilon,d,t}(x, y) - n_{\varepsilon,d,t}(0, 0) \| + \| n_{\varepsilon,d,t}(0, 0) - f_t^d(0, 0) \| + \| f_t^d(0, 0) \|
\]

combined with the Lipschitz, growth and approximation properties that we already established implies a polynomial bound (3.14) with \( \theta = \beta + \tilde{\zeta} \).

Altogether, this proves that Assumption 1 is satisfied with the claimed choices of \( \zeta \) and \( \theta \).

\( \square \)

**Proof of Proposition 7.12.** Consider first the case of the running minimum. For \( z \in \mathbb{R}^d \)

partition \( z = (z_{d-1}, z_d) \) into the first \( d-1 \) and the last component. Define the transition map for the augmented process, \( f_t^d: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \), by

\[
\bar{f}_t^d(x, y) = (f_t^{d-1}(x_{d-1}, 1:1:1), \min_j (\min_{j=1,\ldots,d-1} f_t^{d-1}(x_{1:d-1}, 1:1:1, x_d))
\]

and \( \bar{Y}_t^d = (Y_t^{d-1}, 0) \). Then \( \bar{X}_t^d = (X_0^{d-1}, \min_{j=1,\ldots,d-1} X_0^{d-1}) \) and so the independence and moment conditions on \( Y_t^d \) are satisfied and \( \| \bar{f}_t^d(0, 0) \| \leq 2 \| f_t^{d-1}(0, 0) \| \). Thus, (i), (ii) and (iv) in Assumption 1 are satisfied.

Furthermore, by the identity \( x = x^+ - (-x)^+ \) and (Grohs et al., 2018, Lemma 4.12) the function \( \min_{j=1,\ldots,k} z_j \) can be realized by a deep neural network with size at most \( 12 k^3 \). We now set

\[
\bar{n}_{\varepsilon,d,t}(x, y) = (\eta_{\varepsilon,d-1,t}(x_{1:d-1}, 1:1:1), \min_j (\min_{j=1,\ldots,d-1} \eta_{\varepsilon,d-1,t}(x_{1:d-1}, 1:1:1)))(x_d)).
\]

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Then the 1-Lipschitz property of $\min_k$, which follows from the fact that the pointwise minimum of 1-Lipschitz functions is again 1-Lipschitz, implies that $\text{Lip}(\tilde{\eta}_{e,d,t}) \leq \sqrt{2}\text{Lip}(\eta_{e,d-1,t})$ and $\|\tilde{f}_t^d(x,y) - \tilde{\eta}_{e,d,t}(x,y)\| \leq \sqrt{2}\|f_{t-1}^d(x_{1:d-1}, y_{1:d-1}) - \eta_{e,d-1,t}(x_{1:d-1}, y_{1:d-1})\|$. The bound on size($\tilde{\eta}_{e,d,t}$) follows from the bound on size($\eta_{e,d-1,t}$) and bounds for the operations composition, parallelization and the realization of the identity (which yields a bound for the size of the neural network realizing $x \mapsto x_{1:d}$). Finally, $\|\tilde{f}_t^d(x,y)\| \leq \sqrt{2}\|f_{t-1}^d(x_{1:d-1}, y_{1:d-1})\|$ and $\|\tilde{\eta}_{e,d,t}(x,y)\| \leq \sqrt{2}\|\eta_{e,d-1,t}(x_{1:d-1}, y_{1:d-1})\|$ so that all the required bounds follow from the corresponding properties of $X_{d-1}$.

In the case of the running maximum one proceeds analogously, except that the growth bounds are now a bit different, namely, $\|\tilde{f}_t^d(x,y)\| \leq d\|f_{t-1}^d(x_{1:d-1}, y_{1:d-1})\| + \|x\|$ and $\|\tilde{\eta}_{e,d,t}(x,y)\| \leq d\|\eta_{e,d-1,t}(x_{1:d-1}, y_{1:d-1})\| + \|x\|$, which still allows us to deduce the claimed statement.

\[\square\]

### 4.3 Auxiliary results

This section contains auxiliary results that are needed for the proof of Theorems 3.4 and 3.8. We start with Lemma 4.4, which establishes growth properties of the payoff function and its neural network approximation.

**Lemma 4.4.** Suppose Assumption 2 is satisfied. Then for all $\varepsilon \in (0,1)$, $d \in \mathbb{N}$, $t \in \{0,\ldots,T\}$, $x \in \mathbb{R}^d$ it holds that

\begin{align}
|g_d(t,x)| &\leq cd^\varepsilon(1 + \|x\|), \quad (4.7) \\
|\phi_{e,d,t}(x)| &\leq cd^\varepsilon(2 + \|x\|). \quad (4.8)
\end{align}

**Proof.** First note that from (3.6), (3.8) and the growth assumption on $g_d$ we obtain for every $\bar{\varepsilon} \in (0,1]$ that

\[|g_d(t,x)| \leq |g_d(t,x) - \phi_{e,d,t}(x)| + |\phi_{e,d,t}(x) - \phi_{e,d,t}(0)| + |\phi_{e,d,t}(0) - g_d(t,0)| + |g_d(t,0)| \leq \bar{\varepsilon}cd^\varepsilon(2 + \|x\|^p) + cd^\varepsilon(1 + \|x\|). \quad (4.9)\]

Letting $\bar{\varepsilon}$ tend to 0 we thus obtain (4.7). Next, note that the same properties of $g_d$ and $\phi_{e,d,t}$ imply

\[|\phi_{e,d,t}(x)| \leq |\phi_{e,d,t}(x) - \phi_{e,d,t}(0)| + |\phi_{e,d,t}(0) - g_d(t,0)| + |g_d(t,0)| \leq cd^\varepsilon(2 + \|x\|). \quad (4.10)\]

\[\square\]

The next result, Lemma 4.5, establishes growth properties of the Markov update function and its neural network approximation.
Lemma 4.5. Suppose Assumption 7 is satisfied. Then for all $\varepsilon \in (0, 1]$, $d \in \mathbb{N}$, $t \in \{0, \ldots, T-1\}$, $x, y \in \mathbb{R}^d$ it holds that

\begin{align}
\| f^d_t(x, y) \| &\leq c d^q (1 + \|x\| + \|y\|), \\
\| \eta_{\varepsilon, d, t}(x, y) \| &\leq c d^q (2 + \|x\| + \|y\|).
\end{align}

Proof. The proof is a straightforward consequence of (3.2), Assumption 1(iv) and (3.4). Indeed, these hypotheses imply for every $\bar{\varepsilon} \in (0, 1]$ that

\begin{align}
\| f^d_t(x, y) \| &\leq \| f^d_t(x, y) - \eta_{\varepsilon, d, t}(x, y) \| + \| \eta_{\varepsilon, d, t}(x, y) - \eta_{\varepsilon, d, t}(0, 0) \|
+ \| \eta_{\varepsilon, d, t}(0, 0) - f^d_t(0, 0) \| + \| f^d_t(0, 0) \|
\leq \bar{\varepsilon} c d^q (2 + \|x\|^p + \|y\|^p) + c d^q (1 + \|x\| + \|y\|).
\end{align}

Letting $\bar{\varepsilon}$ tend to 0 we thus obtain (4.11). In addition, the same hypotheses yield

\begin{align}
\| \eta_{\varepsilon, d, t}(x, y) \| &\leq \| \eta_{\varepsilon, d, t}(x, y) - \eta_{\varepsilon, d, t}(0, 0) \| + \| \eta_{\varepsilon, d, t}(0, 0) - f^d_t(0, 0) \| + \| f^d_t(0, 0) \|
\leq c d^q (2 + \|x\| + \|y\|).
\end{align}

In the next lemma we establish a bound on the conditional moments of $X^d$. The proof and $\mathbb{E}[\|X^d_0\|] < \infty$ also yield $\mathbb{E}[\|X^d_t\|] < \infty$ for all $t$ and so we may consider the conditional expectation in (4.15) to be defined for all $x \in \mathbb{R}^d$, cf. Remark 3.2 above.

Lemma 4.6. Suppose Assumption 1 or 7 is satisfied. Then for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$, $s, t \in \{0, \ldots, T\}$ with $s \geq t$ it holds that

\begin{equation}
\mathbb{E}[\|X^d_s\| | X^d_t = x] \leq \tilde{c}_1 d^{\tilde{q}_1} (1 + \|x\|) \tag{4.15}
\end{equation}

with $\tilde{c}_1 = 2 \max(c, 1)^{T+1} T$, $\tilde{q}_1 = q(T + 1)$ in the case of Assumption 7 and with $\tilde{c}_1 = T \max(h, 1) \frac{q}{2 m \max(p, 2)}$, $\tilde{q}_1 = \frac{q T}{2 m \max(p, 2)}$ in the case of Assumption 1.

Proof. Assume w.l.o.g. that $c \geq 1$. Consider first the case when Assumption 1 holds. Then (4.11) can be used to prove inductively that for all $s \geq t$,

\begin{equation}
\mathbb{E}[\|X^d_s\| | X^d_t = x] \leq (c d^q)^{s-t} \|x\| + \sum_{k=1}^{s-t} (c d^q)^k (1 + \mathbb{E}[\|Y^d_{s-k}\|]). \tag{4.16}
\end{equation}

Indeed, for $s = t$ this directly follows from the definition. Assume now $s > t$ and (4.16)
holds for \( s - 1, s - 2, \ldots, t \), then (4.11), the induction hypothesis and independence yield
\[
\mathbb{E}[\|X^d_s\| | X^d_t = x] = \mathbb{E}[\|f^d_{s-1}(X^d_{s-1}, Y^d_{s-1})\| | X^d_t = x]
\]
\[
\leq cd^q (1 + \mathbb{E}[\|X^d_{s-1}\| | X^d_t = x] + \mathbb{E}[\|Y^d_{s-1}\| | X^d_t = x])
\]
\[
\leq cd^q \left( 1 + (cd^q)^{s-1-t} \|x\| + \sum_{k=1}^{s-1-t} (cd^q)^k (1 + \mathbb{E}[\|Y^d_{s-1-k}\|]) + \mathbb{E}[\|Y^d_{s-1}\|]\right)
\]
\[
= cd^q (1 + \mathbb{E}[\|Y^d_{s-1}\|]) + (cd^q)^{s-t} \|x\| + \sum_{k=2}^{s-t} (cd^q)^k (1 + \mathbb{E}[\|Y^d_{s-k}\|]),
\]
(4.17)
as claimed. This shows that (4.16) holds for all \( s \geq t \). From (4.16) and Assumption 1(iv) we obtain
\[
\mathbb{E}[\|X^d_s\| | X^d_t = x] \leq c^T d^T q \|x\| + s - t \sum_{k=1}^{s-t} (cd^q)^k (1 + \mathbb{E}[\|Y^d_{s-k}\|]),
\]
(4.18)

In the case when Assumption 1' holds we first note that independence, Jensen’s inequality and (3.14) yield
\[
\mathbb{E}[\|f^d_{s-1}(X^d_{s-1}, Y^d_{s-1})\| | X^d_t = x] = \int_{\mathbb{R}^d} \mathbb{E}[\|f^d_{s-1}(z, Y^d_{s-1})\|] \mu^d_{t,s-1}(x, dz)
\]
\[
\leq \int_{\mathbb{R}^d} (hd^q (1 + \|z\|^{2m \max(p,2)}) \frac{1}{2m \max(p,2)} \mu^d_{t,s-1}(x, dz)
\]
\[
\leq (hd^q) \frac{1}{2m \max(p,2)} \int_{\mathbb{R}^d} (1 + \|z\|) \mu^d_{t,s-1}(x, dz)
\]
\[
= (hd^q) \frac{1}{2m \max(p,2)} (1 + \mathbb{E}[\|X^d_{s-1}\| | X^d_t = x]).
\]
(4.19)
We can now apply this estimate instead of (4.11) to get from the first to the second line in (4.17) and arrive at
\[
\mathbb{E}[\|X^d_s\| | X^d_t = x] \leq \left( (hd^q) \frac{1}{2m \max(p,2)} \right)^{s-t} \|x\| + \sum_{k=1}^{s-t} \left( (hd^q) \frac{1}{2m \max(p,2)} \right)^k,
\]
(4.20)
hence the conclusion follows.

The next result ensures that the optimal value (2.1) is finite in our setting.

**Lemma 4.7.** Suppose Assumption 2 holds and Assumption 1 or 1' is satisfied. Then \( \mathbb{E}[g_d(t, X^d_t)] < \infty \) for all \( d \in \mathbb{N}, t \in \{0, \ldots, T\} \).

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Proof. Let \(d \in \mathbb{N}, t \in \{0, \ldots, T\}\). Then Lemma 4.4, Lemma 4.6 and Assumption \(I\) or \(I'\) ensure that

\[
\mathbb{E}[|g_d(t, X^d_t)|] \leq \hat{c}d^q(1 + \mathbb{E}[||X^d_t||]) = \hat{c}d^q(1 + \mathbb{E}[||X^d_t |||X^d_0||]) \\
\leq \hat{c}d^q(1 + \hat{c}_d d^{\hat{q}_1}(1 + \mathbb{E}[||X^d_0||])) < \infty.
\]

The next lemma proves that the value function grows at most linearly. Recall from Remark 3.2 that Lemma 4.7 allows us to recursively define the value function for all \(x \in \mathbb{R}^d\) as the right hand side of (2.2).

**Lemma 4.8.** Suppose Assumption \(\mathcal{Z}\) holds and Assumption \(I\) or \(I'\) is satisfied. Then for all \(d \in \mathbb{N}, t \in \{0, \ldots, T\}\), \(x \in \mathbb{R}^d\) it holds that

\[
|V_d(t, x)| \leq \hat{c}_t d^{\hat{q}_1}(1 + ||x||), \quad (4.21)
\]

where \(\hat{c}_t = \max(c, 1)(3 \max(c, 1)^2)^{T-t}, \hat{q}_t = q + 2q(T-t)\) in the case of Assumption \(I\) and \(\hat{c}_t = c(T+1) \max(h, 1) \frac{2m \max(p, 2)}{\max(h, 1)^2} \hat{q}_t = q + \frac{2q(T-t)}{2m \max(p, 2)}\) in the case of Assumption \(I'\).

**Proof.** Consider first the case of Assumption \(I\). The proof proceeds by backward induction. For \(t = T\) the statement directly follows from (4.7). Assume now the statement holds for \(t + 1\). Then (2.2), (4.7), the induction hypothesis and (4.16) yield

\[
|V_d(t, x)| \leq \max(|g_d(t, x)|, \mathbb{E}[|V_d(t+1, X^d_{t+1})| |X^d_t = x|]) \\
\leq \max(\hat{c}_t d^{\hat{q}_1}(1 + ||x||), \hat{c}_{t+1} d^{\hat{q}_1+1}(1 + \mathbb{E}[||X^d_{t+1}|| |X^d_t = x|])) \\
\leq \max(\hat{c}_t d^{\hat{q}_1}(1 + ||x||), \hat{c}_{t+1} d^{\hat{q}_1+1}(1 + \hat{c}_t d^{\hat{q}_1}(1 + ||x|| + cd^{\hat{q}_1}))) \\
\leq \hat{c}_{t+1} d^{\hat{q}_1+3} \max(c, 1)^2 d^{2\hat{q}_1}(1 + ||x||).
\]

Hence, (4.21) also holds at \(t\) and so by induction the statement follows.

In the case of Assumption \(I'\) we aim to provide a tighter estimate and instead inductively prove that \(|V_d(t, x)| \leq \hat{a}_t + \hat{b}_t ||x||\) with \(\hat{a}_t = \hat{a}_{t+1} + \hat{b}_{t+1}(h d^\hat{q}_1) \frac{1}{2m \max(p, 2)}, \hat{a}_T = cd^\hat{q}_1, \hat{b}_t = \hat{b}_{t+1}(max(h, 1)d^\hat{q}_1) \frac{1}{2m \max(p, 2)}, \hat{b}_T = cd^\hat{q}_1\). Indeed, using (4.20) instead of (4.16) we analogously obtain

\[
|V_d(t, x)| \leq \max(\hat{c}_t d^{\hat{q}_1} + cd^\hat{q}_1 ||x||, \hat{a}_{t+1} + \hat{b}_{t+1}(hd^\hat{q}_1) \frac{1}{2m \max(p, 2)}(1 + ||x||)) \leq \hat{a}_t + \hat{b}_t ||x|| \quad (4.23)
\]

from which the statement follows by noting that \(\hat{b}_t = cd^\hat{q}_1(max(h, 1)d^\hat{q}_1) \frac{1}{2m \max(p, 2)}\),

\[
\hat{a}_t = cd^\hat{q}_1 + \sum_{s=t+1}^{T} \hat{b}_s (hd^\hat{q}_1) \frac{1}{2m \max(p, 2)} \leq cd^\hat{q}_1(T+1)(max(h, 1)d^\hat{q}_1) \frac{1}{2m \max(p, 2)}.
\]

}\]
Lemma [4.9] mathematically proves the intuitively obvious fact that a neural network in which some input arguments are held at fixed values is still a neural network with at most as many non-zero parameters as the original neural network.

**Lemma 4.9.** Let \( d_0, d_1, m \in \mathbb{N} \) and let \( \phi: \mathbb{R}^{d_0+d_1} \to \mathbb{R}^m \) be a neural network. Let \( y \in \mathbb{R}^{d_1} \). Then \( \Phi_y: \mathbb{R}^{d_0} \to \mathbb{R}^m, x \mapsto \phi((x,y)) \) can again be realized by a neural network \( \phi_y \) with size(\( \phi_y \)) \leq \text{size}(\phi).

**Proof.** Denote by \( ((A^1, b^1), \ldots, (A^L, b^L)) \) the parameters of \( \phi \) with \( L \in \mathbb{N}, \ N_0 = d_0 + d_1, N_1, \ldots, N_{L-1} \in \mathbb{N}, \ N_L = m, A^\ell \in \mathbb{R}^{N^\ell \times N_{\ell-1}}, b^\ell \in \mathbb{R}^{N^\ell}, \ell = 1, \ldots, L \). Denote by \( A^{1,0} \in \mathbb{R}^{N_1 \times d_0} \) and \( A^{1,1} \in \mathbb{R}^{N_1 \times d_1} \) the first \( d_0 \) and the remaining \( d_1 \) columns of \( A^1 \), respectively. Consider the neural network \( \phi_y \) with parameters \( ((A^{1,0}, A^{1,1} y + b^1), (A^2, b^2), \ldots, (A^L, b^L)) \).

Then
\[
\phi_y(x) = (W_L \circ (\rho \circ W_{L-1}) \circ \cdots \circ (\rho \circ W_2) \circ \rho)(A^{1,0} x + A^{1,1} y + b^1) \\
= (W_L \circ (\rho \circ W_{L-1}) \circ \cdots \circ (\rho \circ W_1))((x,y)) \\
= \Phi_y(x)
\]
for all \( x \in \mathbb{R}^{d_0} \) and \( \text{size} (\phi_y) \leq \text{size} (\phi) \), as claimed.

The next lemma will allow us to construct a realization of a random neural network and at the same time obtain a bound on the neural network weights.

**Lemma 4.10.** Let \( U \) be a nonnegative random variable, let \( d, N \in \mathbb{N} \), let \( M_1, M_2 > 0 \) and let \( Y_1, \ldots, Y_N \) be i.i.d. \( \mathbb{R}^d \)-valued random variables. Suppose \( \mathbb{E}[U] \leq M_1 \) and \( \mathbb{E}[\|Y_i\|] \leq M_2 \). Then
\[
\mathbb{P} \left( U \leq 3M_1, \max_{i=1,\ldots,N} \|Y_i\| \leq 3NM_2 \right) > 0.
\]

**Proof.** Firstly, by the i.i.d. assumption it follows that
\[
\mathbb{P} \left( \max_{i=1,\ldots,N} \|Y_i\| > 3NM_2 \right) = 1 - \left( \mathbb{P} (\|Y_1\| \leq 3NM_2) \right)^N = 1 - (1 - \mathbb{P} (\|Y_1\| > 3NM_2))^N. \tag{4.25}
\]
Next, note that Bernoulli’s inequality implies \( \left( \frac{2}{3} \right)^{1/N} \leq 1 - \frac{1}{3N} \) and therefore, by Markov’s inequality,
\[
\mathbb{P} (\|Y_1\| > 3NM_2) \leq \frac{\mathbb{E}[\|Y_1\|]}{3NM_2} \leq \frac{1}{3N} \leq 1 - \left( \frac{2}{3} \right)^{\frac{1}{N}}.
\]
Thus, we obtain \( (1 - \mathbb{P}(\|Y_1\| > 3NM_2))^N \geq \frac{2}{3} \) and inserting this in (4.25) yields
\[
\mathbb{P} \left( \max_{i=1,\ldots,N} \|Y_i\| > 3NM_2 \right) \leq \frac{1}{3}. \tag{4.26}
\]
Furthermore, Markov’s inequality implies that
\[ P(U > 3M_1) \leq \frac{\mathbb{E}[U]}{3M_1} \leq \frac{1}{3}. \]  
(4.27)

Combining (4.26) and (4.27) with the elementary fact that \( P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 \) for \( A, B \in \mathcal{F} \) then shows that
\[ P \left( U \leq 3M_1, \max_{i=1,\ldots,N} \|Y_i\| \leq 3NM_2 \right) \geq \frac{2}{3} + \frac{2}{3} - 1 > 0, \]
as claimed.

4.4 Proof of Theorem 3.4 and Corollary 3.5

With these preparations we are now ready to prove Theorem 3.4. The proof is divided into several steps, which are highlighted in bold in order to facilitate reading. Let us first provide a brief sketch of the proof. The proof proceeds by backward induction. This entails some subtleties regarding the probability measure \( \rho^d \), which we will not discuss here. We refer to the proof below for details. Here we rather provide an easy-to-follow overview.

The starting point is the backward recursion (2.2). Our goal is to provide a neural network approximation of the right hand side in (2.2). At time \( t \) we first aim to bound the \( L^2(\rho^d) \)-approximation error \( E^d_t \) between the continuation value \( \mathbb{E}[V_d(t+1,X^d_{t+1})|X^d_t = x] \) and the random function \( \Gamma_{\varepsilon,d,t}(x) = \frac{1}{N} \sum_{i=1}^{N} \hat{v}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x,Y^d_{i,t})) \), where \( \hat{v}_{\varepsilon,d,t+1} \) is a neural network approximating the value function at time \( t+1 \) and \( Y^d_{1,t}, \ldots, Y^d_{N,t} \) are i.i.d. copies of \( Y^d_t \). The existence and suitable properties of \( \hat{v}_{\varepsilon,d,t+1} \) follow from the induction hypothesis.

We derive a bound on \( \mathbb{E}[E^d_t] \), which we can then use to obtain existence of a realization \( \gamma_{\varepsilon,d,t} \) of \( \Gamma_{\varepsilon,d,t} \) satisfying a slightly worse bound and such that the realization of \( \max_{i=1,\ldots,N} \|Y^d_{i,t}\| \) can also be bounded suitably. This last point is necessary to control the growth of \( \gamma_{\varepsilon,d,t}(x) \). Then \( \gamma_{\varepsilon,d,t}(x) \) is an approximation of the continuation value and so we naturally define the approximate value function at time \( t \) by
\[ \hat{v}_{\varepsilon,d,t}(x) = \max (\phi_{\varepsilon,d,t}(x) - \delta, \gamma_{\varepsilon,d,t}(x)) \]  
(4.28)

for a suitably chosen \( \delta \) (depending on \( \varepsilon \)). We then consider the continuation region
\[ C_t = \{ x \in \mathbb{R}^d : g_{d,t}(x) < \mathbb{E}[V_d(t+1,X_{t+1}^d)|X_t^d = x] \} \]  
(4.29)

and the approximate continuation region
\[ \hat{C}_t = \{ x \in \mathbb{R}^d : \phi_{\varepsilon,d,t}(x) - \delta < \gamma_{\varepsilon,d,t}(x) \}. \]  
(4.30)
Then we may decompose
\[
|V_d(t,x) - \hat{v}_{\varepsilon,d,t}(x)|
= |g_d(t,x) - \phi_{\varepsilon,d,t}(x) + \delta 1_{C^\varepsilon_t \cap \hat{C}^\varepsilon_t}(x) + |E[V_d(t+1,X_{t+1}^d)]|X_t^d = x| - \phi_{\varepsilon,d,t}(x) + \delta 1_{C^\varepsilon_t \cap \hat{C}^\varepsilon_t}(x)
+ |g_d(t,x) - \gamma_{\varepsilon,d,t}(x)|1_{C^\varepsilon_t \cap \hat{C}^\varepsilon_t}(x) + |E[V_d(t+1,X_{t+1}^d)]|X_t^d = x| - \gamma_{\varepsilon,d,t}(x)|1_{C^\varepsilon_t \cap \hat{C}^\varepsilon_t}(x).
\]

The \(L^2(\rho^d)\)-error of the last term has already been analysed, and it remains to analyse the remaining terms. The first term is small due to Assumption 2. The second and third term may not necessarily be small, but we will be able to show that \(\rho^d(C_t \cap \hat{C}^\varepsilon_t)\) and \(\rho^d(C^\varepsilon_t \cap \hat{C}_t)\) are small. Hence, the overall \(L^2(\rho^d)\)-error can be controlled. The proof is then completed by showing that the neural network (4.28) satisfies the growth, size and Lipschitz properties required to carry out the induction argument.

**Proof of Theorem 3.4.**

1. **Preliminaries:** Without loss of generality we may assume that the constants \(c > 0, q, \alpha \geq 0\) in the statement of the theorem and in Assumptions 1 and 2 coincide; otherwise we replace each of them by the respective maximum and all the assumptions are still satisfied. We may also assume that \(c \geq 1\).

Furthermore, if for each fixed \(t \in \{0, \ldots, T\}\) there exist constants \(\kappa_t, q_t, r_t \in [0, \infty)\) and a neural network \(\psi_{\varepsilon,d,t}\) such that size(\(\psi_{\varepsilon,d,t}\)) \leq \kappa_t d^{q_t} e^{-r_t} \) and (3.9) holds for all \(\varepsilon \in (0,1], d \in \mathbb{N}\), then also the statement of the theorem follows by choosing \(\kappa, q, r\) as the respective maximum over \(t \in \{0, \ldots, T\}\).

Next, let \(c_0, \ldots, c_T\) satisfy \(c_0 = c, c_{t+1} = \max(3c,1)^{2 \max(p,2)}(1 + c_t + c)\) and set \(q_t = (2 \max(p,2) + 1)q + q\). Then

\[
c_t = c(\max(3c,1)^{2 \max(p,2)})^t + \sum_{k=0}^{t-1} (\max(3c,1)^{2 \max(p,2)})^{k+1}(1 + c)
\]

for all \(t \in \{0, \ldots, T\}\) and \(c_t\) does not depend on \(d\).

2. **Stronger statement:** We will now proceed to prove the following stronger statement, which shows that the constants \(\kappa_t, q_t, r_t\) can be chosen essentially independently of the probability measure \(\rho^d\) and, in addition, \(\rho^d\) may be allowed to depend on \(t\). Specifically, we will prove that for any \(t \in \{0, \ldots, T\}\) there exist constants \(\kappa_t, q_t, r_t \in [0, \infty)\) such that for any family of probability measures \(\rho^d_t\) on \(\mathbb{R}^d\), \(d \in \mathbb{N}\), satisfying

\[
\int_{\mathbb{R}^d} ||x||^{2 \max(p,2)} \rho^d_t(dx) \leq c_t d^{q_t}
\]

and for all \(d \in \mathbb{N}, \varepsilon \in (0,1]\) there exists a neural network \(\psi_{\varepsilon,d,t}\) such that

\[
\left( \int_{\mathbb{R}^d} |V_d(t,x) - \psi_{\varepsilon,d,t}(x)|^2 \rho^d_t(dx) \right)^{1/2} \leq \varepsilon
\]

(4.33)
and
\begin{align}
|\psi_{\varepsilon,d,t}(x)| & \leq \kappa_t d^{d_t} \varepsilon^{-t_\varepsilon} (1 + \|x\|), \quad \text{for all } x \in \mathbb{R}^d, \quad (4.34) \\
\text{size}(\psi_{\varepsilon,d,t}) & \leq \kappa_t d^{d_t} \varepsilon^{-t_\varepsilon}, \quad (4.35) \\
\text{Lip}(\psi_{\varepsilon,d,t}) & \leq \kappa_t d^{d_t}. \quad (4.36)
\end{align}

Choosing $\rho^t_t = \rho^d$ for all $t$ and noting that (4.32) is satisfied due to $q \leq q_t, c \leq c_t$ the statement of Theorem 3.4 then follows.

In order to prove the stronger statement for each fixed $t$, we now proceed by backward induction.

3. **Base case of backward induction:** In the case $t = T$ we have $V_d(T,x) = g_d(T,x)$ and therefore we may choose $\psi_{\varepsilon,d,T} = \phi_{\varepsilon,d,T}$ with $\tilde{\varepsilon} = \varepsilon \left[ \cd \left( 1 + (\int_{\mathbb{R}^d} \|x\|^2 \rho^d_{T+1}(dx)^\frac{p}{2 \max(2,2)}) \right) \right]^{-1}$. Then (3.6), Jensen’s inequality and (4.32) imply
\begin{align}
\left( \int_{\mathbb{R}^d} |g_d(T,x) - \psi_{\varepsilon,d,T}(x)|^2 \rho^d_{T+1}(dx) \right)^{1/2} & \leq \tilde{\varepsilon} \cd \left( 1 + \left( \int_{\mathbb{R}^d} \|x\|^2 \rho^d_{T+1}(dx)^\frac{p}{2 \max(2,2)} \right) \right) \\
& \leq \tilde{\varepsilon} \cd \left( 1 + \left( \int_{\mathbb{R}^d} \|x\|^2 \rho^d_{T+1}(dx)^\frac{p}{2 \max(2,2)} \right) \right) \\
& \leq \tilde{\varepsilon} \cd \left( 1 + (c_T d^{d_T})^{\frac{p}{2 \max(2,2)}} \right) = \varepsilon.
\end{align}
Furthermore, (3.7) implies $\text{size}(\psi_{\varepsilon,d,T}) \leq \cd \varepsilon^{-\alpha} \left[ \cd \left( 1 + (c_T d^{d_T})^{\frac{p}{2 \max(2,2)}} \right) \right]^{-\alpha}$ and so, recalling (3.8) and (4.8), the statement follows in the case $t = T$.

4. **Start of the induction step:** The remainder of the proof will now be dedicated to the induction step. To improve readability we will again divide it into several steps. For the induction step we now assume that the stronger statement formulated in Step 2 above holds for time $t+1$ and aim to prove it for time $t$. To this end, let $\rho^t_t$ be a probability measure satisfying (4.32) and denote by $\nu^t_t$ the distribution of $Y^d_t$.

5. **Induction hypothesis:** Let $\kappa_{t+1}, q_{t+1}, r_{t+1} \in [0, \infty)$ denote the constants with which the stronger statement formulated in Step 2 above holds for time $t+1$.

Consider the probability measure $\rho^t_{t+1} = (\rho^t_t \otimes \nu^t_t) \circ (f^d_t)^{-1}$ given as the pushforward measure of $\rho^t_t \otimes \nu^t_t$ under $f^d_t$. Then, using the change-of-variables formula, (4.11), (4.32)
and Assumption IV iv) and writing \( p = 2 \max(p, 2) \), this measure satisfies

\[
\int_{\mathbb{R}^d} \| z \|^2 \max(p, 2) \rho_{t+1}^d (dz) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \| f_t^d(x, y) \|^p \nu_t^d(dy) \rho_t^d(dx) \\
\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (cd^p(1 + \| x \| + \| y \|)) \nu_t^d(dy) \rho_t^d(dx) \\
\leq (3cd^p)^p (1 + \int_{\mathbb{R}^d} \| x \|^p \nu_t^d(dx) + \int_{\mathbb{R}^d} \| y \|^p \nu_t^d(dy)) \\
\leq (3cd^p)^p (1 + ct q_{t+1} + cd^q) \\
\leq \max(3c, 1)^p d^{pq+q} (1 + ct + c) \\
= \kappa_{t+1} d^{pq+q+1}.
\]

Hence, by induction hypothesis, for any \( \varepsilon \in (0, 1] \), \( d \in \mathbb{N} \) there exists a neural network \( \psi_{\varepsilon, d, t+1} \) such that

\[
\left( \int_{\mathbb{R}^d} |V_d(t + 1, x) - \psi_{\varepsilon, d, t+1}(x)|^2 \rho_{t+1}^d(dx) \right)^{1/2} \leq \varepsilon \quad (4.38)
\]

and

\[
|\psi_{\varepsilon, d, t+1}(x)| \leq \kappa_{t+1} d^{pq+q+1} (1 + \| x \|), \quad \text{for all } x \in \mathbb{R}^d, \quad \text{(4.39)}
\]

\[
\text{size}(\psi_{\varepsilon, d, t+1}) \leq \kappa_{t+1} d^{pq+q+1}, \quad \text{(4.40)}
\]

\[
\text{Lip}(\psi_{\varepsilon, d, t+1}) \leq \kappa_{t+1} d^{pq+q+1}. \quad \text{(4.41)}
\]

Now let \( \varepsilon \in (0, 1] \), \( d \in \mathbb{N} \) be given. The remainder of the proof consists in selecting \( \kappa_t, q_t, t_t \) (only depending on \( c, \alpha, p, q, t, T, \kappa_{t+1}, q_{t+1}, t_{t+1} \)) and constructing a neural network \( \psi_{\varepsilon, d, t} \) such that (4.33)–(4.36) are satisfied. This will complete the proof.

In what follows we fix \( \bar{\varepsilon} \in (0, 1) \) and choose

\[
N = \left\lceil \bar{\varepsilon}^{-2q_{t+1}+2} \right\rceil \quad \text{and} \quad \delta = \bar{\varepsilon}^{\frac{1}{t}}. \quad (4.42)
\]

The value of \( \bar{\varepsilon} \) will be chosen later (depending on \( \varepsilon \) and \( d \)).

6. Approximation of the continuation value: Let \( Y_t^d, i \in \mathbb{N}, \) be i.i.d. copies of \( Y_t^d \) and set \( \hat{v}_{\varepsilon, d, t+1} = \psi_{\varepsilon, d, t+1} \). Define the (random) function

\[
\Gamma_{\varepsilon, d, t}(x) = \frac{1}{N} \sum_{i=1}^{N} \hat{v}_{\varepsilon, d, t+1}(\eta_{\varepsilon, d, t}(x, Y_t^d)).
\]

Note that \( \Gamma_{\varepsilon, d, t} \) is a random function, since \( Y_t^d,i \) is random.
We now estimate the expected $L^2(\rho_t^d)$-error that arises when $\Gamma_{\bar{t},d,t}$ is used to approximate the continuation value. Denote $Z_{\bar{t},d,i}(x) = \hat{v}_{\bar{t},d,t+1}(\eta_{\bar{t},d,t}(x,Y_{\bar{t}}^{d,i}))$ and recall that Assumption \( \text{I(iii)} \)–(ii) implies that $Y_{\bar{t}}^d$ is independent of $X_{\bar{t}}^d$. Then $\Gamma_{\bar{t},d,t}(x) = \frac{1}{N} \sum_{i=1}^N Z_{\bar{t},d,i}(x)$ and thus the bias-variance decomposition and independence show

\[
\int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, X_{t+1}^d)|X_t^d = x] - \Gamma_{\bar{t},d,t}(x)|^2] \rho_t^d(dx)
\]

\[
= \int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, X_{t+1}^d)|X_t^d = x] - \mathbb{E}[Z_{\bar{t},d,1}(x)]|^2 + \mathbb{E}[|\mathbb{E}[Z_{\bar{t},d,1}(x)] - \Gamma_{\bar{t},d,t}(x)|^2] \rho_t^d(dx)
\]

\[
= \int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, X_{t+1}^d)|X_t^d = x] - \mathbb{E}[Z_{\bar{t},d,1}(x)]|^2 + \frac{1}{N} \mathbb{E}[|\mathbb{E}[Z_{\bar{t},d,1}(x)] - Z_{\bar{t},d,1}(x)|^2] \rho_t^d(dx).
\]

(4.43)

The term corresponding to the first integral in the last line of (4.43) can be estimated as

\[
\int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, X_{t+1}^d)|X_t^d = x] - \mathbb{E}[\hat{v}_{\bar{t},d,t+1}(\eta_{\bar{t},d,t}(x,Y_{\bar{t}}^{d,i}))]|^2 \rho_t^d(dx)
\]

\[
\leq 2 \int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, X_{t+1}^d)|X_t^d = x] - \mathbb{E}[\hat{v}_{\bar{t},d,t+1}(X_{t+1}^d)|X_t^d = x]|^2 \rho_t^d(dx)
\]

\[
+ 2 \int_{\mathbb{R}^d} \mathbb{E}[|\hat{v}_{\bar{t},d,t+1}(X_{t+1}^d)|X_t^d = x] - \mathbb{E}[\hat{v}_{\bar{t},d,t+1}(\eta_{\bar{t},d,t}(x,Y_{\bar{t}}^{d,i}))]|^2 \rho_t^d(dx).
\]

(4.44)

6.a) Applying the error estimate from $t + 1$: Now consider the first term in the right hand side of (4.44) and recall that $\rho_{t+1}^d = (\rho_t^d \otimes \nu_t^d) \circ (f_t^d)^{-1}$ is the pushforward measure of $\rho_t^d \otimes \nu_t^d$ under $f_t^d$. Then Jensen’s inequality, (3.1), Assumption II(ii) and (2.38) yield

\[
\int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, X_{t+1}^d)|X_t^d = x] - \mathbb{E}[\hat{v}_{\bar{t},d,t+1}(X_{t+1}^d)|X_t^d = x]|^2 \rho_t^d(dx)
\]

\[
\leq \int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t + 1, f_t^d(x,Y_t^d)) - \hat{v}_{\bar{t},d,t+1}(f_t^d(x,Y_t^d))]|^2 \rho_t^d(dx)
\]

\[
= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathbb{E}[V_d(t + 1, f_t^d(x,y)) - \hat{v}_{\bar{t},d,t+1}(f_t^d(x,y))]|^2 \nu_t^d(dy) \rho_t^d(dx)
\]

\[
= \int_{\mathbb{R}^d} |\mathbb{E}[V_d(t + 1, z) - \hat{v}_{\bar{t},d,t+1}(z)]|^2 \rho_{t+1}^d(dz)
\]

\[
\leq \bar{\varepsilon}^2.
\]

6.b) Applying the Lipschitz property of the network at $t + 1$: For the second term in the right hand side of (4.44), note that by induction hypothesis (4.41) we have
Lip(\hat{v}_{\xi,d,t+1}) \leq \kappa_{t+1}d^{\tau_{t+1}} + \text{and hence (3.2), the assumption } E[\|Y^d_t\|] \leq cd^t \text{ and (1.32) imply}

\int_{\mathbb{R}^d} |E[\hat{v}_{\xi,d,t+1}(f^d_t(x,Y^d_t))] - E[\hat{v}_{\xi,d,t+1}(\eta_{\xi,d,t}(x,Y^d_t))]|^2 \rho_t^d(dx) 
\leq (\kappa_{t+1}d^{\tau_{t+1}})^2 \int_{\mathbb{R}^d} |E[\|f^d_t(x,Y^d_t) - \eta_{\xi,d,t}(x,Y^d_t)\|]|^2 \rho_t^d(dx) 
\leq (\bar{\varepsilon}\kappa_{t+1}d^{\tau_{t+1}+q+q_{t+1}})^2 \int_{\mathbb{R}^d} [1 + \|x\|^p + E[\|Y^d_t\|^p]]^2 \rho_t^d(dx) 
\leq 3(\bar{\varepsilon}\kappa_{t+1}d^{\tau_{t+1}+q+q_{t+1}})^2 \left(1 + c^2d^{2q} + \left(\int_{\mathbb{R}^d} \|x\|^{2\max(p,2)} \rho_t^d(dx)\right)^{\frac{p}{2\max(p,2)}}\right) 
\leq 3(\bar{\varepsilon}\kappa_{t+1})^2d^{2(q+q_{t+1})+\max(2q,2\max(p,2))} \left(1 + c^2 + c_{t+1}^2\right)^{\frac{p}{2\max(p,2)}}. \tag{4.46}

6.c) Applying the growth property of the network at \(t + 1\): For the last term in \(4.43\), note that the induction hypothesis \(|\hat{v}_{\xi,d,t+1}(x)| \leq \kappa_{t+1}d^{\tau_{t+1}}\bar{\varepsilon}^{-\tau_{t+1}}(1 + \|x\|), \tag{4.12}, \) \(E[\|Y^d_t\|^2] \leq cd^t\), H"older’s inequality and (1.32) yield

\int_{\mathbb{R}^d} E[|E[Z^{d,1}_t(x) - Z^{d,1}_t(x)]|^2] \rho_t^d(dx) 
\leq \int_{\mathbb{R}^d} E[|Z^{d,1}_t(x)|^2] \rho_t^d(dx) 
= \int_{\mathbb{R}^d} E[|\hat{v}_{\xi,d,t+1}(\eta_{\xi,d,t}(x,Y^d_t))|^2] \rho_t^d(dx) 
\leq (\kappa_{t+1}d^{\tau_{t+1}}\bar{\varepsilon}^{-\tau_{t+1}})^2 \int_{\mathbb{R}^d} E[(1 + \|\eta_{\xi,d,t}(x,Y^d_t)\|)]^2 \rho_t^d(dx) 
\leq 2(\kappa_{t+1}d^{\tau_{t+1}}\bar{\varepsilon}^{-\tau_{t+1}})^2 \left(1 + \int_{\mathbb{R}^d} E[(cd^q(2 + \|x\| + \|Y^d_t\|))]^2 \rho_t^d(dx)\right) 
\leq 2(\kappa_{t+1}d^{\tau_{t+1}+q_2\bar{\varepsilon}^{-\tau_{t+1}}})^2 \left(1 + 3c^2 \left[4 + cd^q + \left(\int_{\mathbb{R}^d} \|x\|^{2\max(p,2)} \rho_t^d(dx)\right)^{\frac{p}{\max(p,2)}}\right]\right) 
\leq 2(\kappa_{t+1}d^{\tau_{t+1}+2q+\max(2q,2\max(p,2))}\bar{\varepsilon}^{-2\tau_{t+1}} \left(1 + 3c^2 \left[4 + c_{t+1}^2\right]\right). \tag{4.47}

6.d) Bounding the overall error and constructing a realization: We can now insert the estimates from (4.45) and (4.46) into (4.44) and subsequently insert the resulting bound
and (4.47) into (4.43). We obtain

\[ \int_{\mathbb{R}^d} \mathbb{E}[\mathbb{E}[V_d(t+1, X_{t+1}^d)|X_t^d = x] - \Gamma_{\varepsilon,d,t}(x)]^2 \rho_t^d(dx) \]

\[ \leq 2 \int_{\mathbb{R}^d} |\mathbb{E}[V_d(t+1, X_{t+1}^d)|X_t^d = x] - \mathbb{E}[\hat{v}_{\varepsilon,d,t+1}(X_{t+1}^d)|X_t^d = x]|^2 \rho_t^d(dx) \]

\[ + 2 \int_{\mathbb{R}^d} |\mathbb{E}[\hat{v}_{\varepsilon,d,t+1}(X_{t+1}^d)|X_t^d = x] - \mathbb{E}[\hat{v}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x, Y_t^d))]|^2 \rho_t^d(dx) \]

\[ + \frac{1}{N} \int_{\mathbb{R}^d} \mathbb{E}[\mathbb{E}[Z_{\varepsilon,d,1}(x)] - Z_{\varepsilon,d,1}(x)]^2 \rho_t^d(dx) \] (4.48)

\[ \leq 2 \varepsilon^2 + 6(\bar{c}\varepsilon \kappa_1 + 1)^2 d^{2(q+q_{t+1})+\max(2q, \frac{4p}{\max(p,2)})} \left( 1 + c^2 + \frac{c}{\max(p,2)} \right) \]

\[ + \frac{2}{N} \varepsilon^2 \kappa_1^2 + 2q + \max(q, \frac{4p}{\max(p,2)}) \varepsilon^{-2q_{t+1}} \left( 1 + 3c^2 \left[ 4 + c + \frac{1}{\max(p,2)} \right] \right) \]

\[ < \tilde{c}_2 \tilde{d}^t \varepsilon^2 + N^{-1} \varepsilon^{-2q_{t+1}} \]

with \( \tilde{c}_2 = 2+8 \max(c,1)^2 \kappa_1^2 (4+\max(c,1)^2 + \max(c_1,1)) \) and \( \tilde{q}_2 = 2(q+q_{t+1})+\max(2q, q_t) \). But (4.48) implies

\[ \mathbb{E} \left[ \int_{\mathbb{R}^d} |\mathbb{E}[V_d(t+1, X_{t+1}^d)|X_t^d = x] - \Gamma_{\varepsilon,d,t}(x)|^2 \rho_t^d(dx) \right] < \tilde{c}_2 \tilde{d}^t \varepsilon^2 + N^{-1} \varepsilon^{-2q_{t+1}}. \] (4.49)

Hence, Assumption IV (iv), the fact that \( Y_{t,1}^d, \ldots, Y_{t,N}^d \) are i.i.d. copies of \( Y_t^d \) and Lemma 4.10 show that there exists \( \omega \in \Omega \) such that \( \gamma_{\varepsilon,d,t}(x) = \frac{1}{N} \sum_{i=1}^N \hat{v}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x, Y_{t,i}^d(\omega))) \) (i.e. the realization of \( \Gamma_{\varepsilon,d,t} \) at \( \omega \)) satisfies

\[ \int_{\mathbb{R}^d} |\mathbb{E}[V_d(t+1, X_{t+1}^d)|X_t^d = x] - \gamma_{\varepsilon,d,t}(x)|^2 \rho_t^d(dx) \leq 3\tilde{c}_2 \tilde{d}^t \varepsilon^2 + \frac{N^{-1} \varepsilon^{-2q_{t+1}}}{2} \] (4.50)

and

\[ \max_{i=1, \ldots, N} \| Y_{t,i}^d(\omega) \| \leq 3Nd^q. \] (4.51)

We now define

\[ \hat{v}_{\varepsilon,d,t}(x) = \max (\phi_{\varepsilon,d,t}(x) - \delta, \gamma_{\varepsilon,d,t}(x)) \] (4.52)

and claim that \( \psi_{\varepsilon,d,t} = \hat{v}_{\varepsilon,d,t} \) satisfies all the properties required in (4.33)–(4.36).

7. Growth bound on the constructed network: Let us first verify (4.34). Indeed, the growth bound on \( \phi_{\varepsilon,d,t} \) in (4.8), the induction hypothesis (4.39), the growth bound (4.12) on \( \eta_{\varepsilon,d,t} \), the bound (4.51) on \( \| Y_{t,i}^d(\omega) \| \) and the choice of \( N \) in (4.42) imply for all \( x \in \mathbb{R}^d \)}
that

\[ |\psi_{\varepsilon,d,t}(x)| \leq |\phi_{\varepsilon,d,t}(x)| + \delta + |\gamma_{\varepsilon,d,t}(x)| \]

\[ \leq c d^q (2 + \|x\|) + \delta + \frac{1}{N} \sum_{i=1}^{N} |\tilde{\psi}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x, Y^d,i(\omega)))| \]

\[ \leq c d^q (2 + \|x\|) + \delta + \frac{1}{N} \sum_{i=1}^{N} \kappa_{t+1} d^{q+1} \varepsilon^{-r_{t+1}} (1 + \|\eta_{\varepsilon,d,t}(x, Y^d,i(\omega))\|) \]

\[ \leq c d^q (2 + \|x\|) + \delta + \frac{1}{N} \sum_{i=1}^{N} \kappa_{t+1} d^{q+1} \varepsilon^{-r_{t+1}} (1 + c d^q (2 + \|x\| + \|Y^d,i(\omega)\|)) \]

\[ \leq c d^q (2 + \|x\|) + \delta + \kappa_{t+1} d^{q+1} \varepsilon^{-r_{t+1}} (1 + cd^q (2 + \|x\| + 3N cd^q)) \]

\[ \leq \tilde{c}_3 d^{q+1} \varepsilon^{-r_3} (1 + \|x\|) \]

(4.53)

with \( \tilde{c}_3 = 18 \max(c, 1, \kappa_{t+1}) \max(c, 1)^2, \tilde{q}_3 = q_{t+1} + 2q \) and \( \tilde{r}_3 = 3r_{t+1} + 2 \).

8. **Bounding the size of the constructed network:** Next, we verify (4.35). To achieve this, first note that for each \( i \), Lemma 4.9 shows that the map \( x \mapsto \eta_{\varepsilon,d,t}(x, Y^d,i(\omega)) \) can be realized as a neural network with size at most \( \text{size}(\eta_{\varepsilon,d,t}) \). Next, the composition of two ReLU neural networks \( \phi_1, \phi_2 \) can again be realized by a ReLU neural network with size at most \( 2(\text{size}(\phi_1) + \text{size}(\phi_2)) \) (see, e.g., [Opschoor et al. 2020, Proposition 2.2]). Finally, [Gonon and Schwartz 2021a, Lemma 3.2] shows that the weighted sum of deep neural networks \( \phi_1, \ldots, \phi_N \) with the same number of layers, the same input dimension and the same output dimension can be realized by another deep neural network with size at most \( \sum_{i=1}^{N} \text{size}(\phi_i) \). Therefore, \( \gamma_{\varepsilon,d,t} \) can be realized as a deep neural network with

\[ \text{size}(\gamma_{\varepsilon,d,t}) \leq \sum_{i=1}^{N} 2(\text{size}(\tilde{\psi}_{\varepsilon,d,t+1}) + \text{size}(\eta_{\varepsilon,d,t})) \]

\[ \leq 2N (\kappa_{t+1} d^{q+1} \varepsilon^{-r_{t+1}} + c d^q \varepsilon^{-\alpha}), \]

where the last step follows from the induction hypothesis (4.40) and the bound (3.3) on the size of \( \eta_{\varepsilon,d,t} \). Next, subtracting a constant corresponds to a change of the “bias” \( b^t \) in the last layer and so \( \phi_{\varepsilon,d,t} - \delta \) is a neural network with size \( \text{size}(\phi_{\varepsilon,d,t} - \delta) = \text{size}(\phi_{\varepsilon,d,t}) \). Define the neural network \( m: \mathbb{R}^2 \to \mathbb{R}, m(x, y) = A^2 g(A^1(x, y)^\top) \) with

\[ A^1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A^2 = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \]

then \( m(x, y) = \max(x - y, 0) + \max(y, 0) - \max(-y, 0) = \max(x, y) \). Thus, \( \tilde{\psi}_{\varepsilon,d,t} \) in (4.52) can be realized as a neural network by the composition of \( m \) with the parallelization of \( \phi_{\varepsilon,d,t} - \delta \) and \( \gamma_{\varepsilon,d,t} \) (see, e.g., [Opschoor et al. 2020, Proposition 2.3]) and the size of the
parallelization is bounded by \( \text{size}(\phi_{\varepsilon,d,t}) + \text{size}(\gamma_{\varepsilon,d,t}) \). This and the bound \( 3.7 \) on the size of \( \phi_{\varepsilon,d,t} \), the choice of \( N \) in \( 4.12 \) and the bound \( 4.5.1 \) on the size of \( \gamma_{\varepsilon,d,t} \) imply

\[
\text{size}(\psi_{\varepsilon,d,t}) \leq 2(\text{size}(m) + \text{size}(\phi_{\varepsilon,d,t}) + \text{size}(\gamma_{\varepsilon,d,t}))
\leq 2(7 + cd_d^q\varepsilon^{-\alpha} + 2N(\kappa_{t+1}d^{q+1}\varepsilon^{-\tau_{t+1}} + cd_d^q\varepsilon^{-\alpha}))
\leq \tilde{c}_4 q^4 \varepsilon^{-\tilde{r}_4}
\]

with \( \tilde{c}_4 = 2(7 + 5c + 4\kappa_{t+1}) \), \( \tilde{q}_4 = \max(q, q_{t+1}) \) and \( \tilde{r}_4 = 2\tau_{t+1} + 2 + \max(\alpha, \tau_{t+1}) \).

**9. Lipschitz constant of the constructed network:** Next, we verify \( 4.36 \). To do this, we note that the induction hypothesis \( 4.41 \) and the Lipschitz property \( 3.4 \) of \( \eta_{\varepsilon,d,t} \) imply for all \( x, y \in \mathbb{R}^d \) that

\[
|\gamma_{\varepsilon,d,t}(x) - \gamma_{\varepsilon,d,t}(y)| = \left| \frac{1}{N} \sum_{i=1}^{N} (\hat{v}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x), Y_{t}^{d,i}(\omega))) - \hat{v}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(y), Y_{t}^{d,i}(\omega))) \right|
\leq \frac{1}{N} \sum_{i=1}^{N} \kappa_{t+1}d^{q+1}||\eta_{\varepsilon,d,t}(x), Y_{t}^{d,i}(\omega)) - \eta_{\varepsilon,d,t}(y), Y_{t}^{d,i}(\omega)||
\leq \kappa_{t+1}d^{q+1}cd_d^q \| x - y \|.
\]

In addition, \( 3.8 \) implies \( \text{Lip}(\phi_{\varepsilon,d,t} - \delta) = \text{Lip}(\phi_{\varepsilon,d,t}) \leq cd_d^q \) and the pointwise maximum of two Lipschitz continuous functions is again Lipschitz continuous with Lipschitz constant given by the maximum of the two Lipschitz constants. Combining this with \( 4.56 \) yields

\[
\text{Lip}(\psi_{\varepsilon,d,t}) \leq \max(cd_d^q, \kappa_{t+1}d^{q+1+q}c) \leq c \max(\kappa_{t+1}, 1)d^{q+1+q}.
\]

**10. Bounding the overall approximation error:** We now work towards verifying the approximation error bound \( 4.33 \). To achieve this, let \( C_t = \{ x \in \mathbb{R}^d : g_d(t, x) < E[V_{d}(t+1, X_{t+1}^d)|X_t^d = x] \} \) be the continuation region and let \( \hat{C}_t = \{ x \in \mathbb{R}^d : \phi_{\varepsilon,d,t}(x) - \delta < \gamma_{\varepsilon,d,t}(x) \} \) be the approximate continuation region. Then

\[
|V_{d}(t, x) - \hat{v}_{\varepsilon,d,t}(x)|
= |g_d(t, x) - \phi_{\varepsilon,d,t}(x) + \delta [1_{C_t^c \cap \hat{C}_t^c}(x)] + |E[V_{d}(t+1, X_{t+1}^d)|X_t^d = x] - \phi_{\varepsilon,d,t}(x)| + \delta [1_{C_t \cap \hat{C}_t^c}(x)]
+ |g_d(t, x) - \gamma_{\varepsilon,d,t}(x)| [1_{C_t^c \cap \hat{C}_t^c}(x)] + |E[V_{d}(t+1, X_{t+1}^d)|X_t^d = x] - \gamma_{\varepsilon,d,t}(x)| [1_{C_t \cap \hat{C}_t^c}(x)].
\]

We now estimate (the integral of) each of these four terms separately. For the first term, from \( 3.6 \) we directly get

\[
|g_d(t, x) - \phi_{\varepsilon,d,t}(x)| + \delta [1_{C_t^c \cap \hat{C}_t^c}(x)] \leq \delta + \varepsilon cd_d^q (1 + \|x\|^p)
\]

and so we proceed with analysing the second term.
10.a) Bounding the approximation error on \(C_t \cap \hat{C}_t^c\): From Lemma 4.8 we have the growth bound \(|V_d(t + 1, x)| \leq \hat{c}_{t+1} d^{\hat{q}_{t+1}} (1 + \|x\|)\) and so, using (4.58), the second term in (4.58) can be estimated as

\[
\left[ \hat{c}_{t+1} d^{\hat{q}_{t+1}} (1 + \mathbb{E}[\|X^d_{t+1}\|\|X^d_t\| = x]) + c d^q (2 + \|x\|) + \delta \right] 1_{C_t \cap \hat{C}_t^c}(x)
\]

Combining this with (4.32), (4.15) in Lemma 4.6 and Hölder’s inequality we estimate

\[
\left( \int_{\mathbb{R}^d} [\mathbb{E}[V_d(t + 1, X^d_{t+1})|X^d_t = x] - \phi_{\varepsilon, d, t}(x) + \delta ||1_{C_t \cap \hat{C}_t^c}(x)\rho_t^d(dx)] \right)^{1/2}
\]

\[
\leq 4 \max(\hat{c}_{t+1}, c) d^{\hat{q}_{t+1}} \left( \int_{\mathbb{R}^d} \left[ 1 + \sum_{s=t}^{t+1} \mathbb{E}[\|X^d_s\|\|X^d_s = x\right] \right] 1_{C_t \cap \hat{C}_t^c}(x)\rho_t^d(dx) \right)^{1/2}
\]

\[
\leq 4 \max(\hat{c}_{t+1}, c) d^{\hat{q}_{t+1}} \left[ 1 + \sum_{s=t}^{t+1} \left( \int_{\mathbb{R}^d} \mathbb{E}[\|X^d_s\|\|X^d_s = x]\right] 4 \rho_t^d(dx) \right)^{1/2} \left( \rho_t^d(C_t \cap \hat{C}_t^c) \right)^{1/4}
\]

\[
\leq 4 \max(\hat{c}_{t+1}, c) d^{\hat{q}_{t+1}} \left[ 1 + \frac{1}{2} \left( \int_{\mathbb{R}^d} \mathbb{E}[\|X^d_s\|\|X^d_s = x]\right] \rho_t^d(dx) \right)^{1/2} \left( \rho_t^d(C_t \cap \hat{C}_t^c) \right)^{1/4}.
\]

(4.61)

10.b) Estimating \(\rho_t^d(C_t \cap \hat{C}_t^c)\): Next, we aim to estimate \(\rho_t^d(C_t \cap \hat{C}_t^c)\). To do this, set \(A = \{x \in \mathbb{R}^d : |g_d(t, x) - \phi_{\varepsilon, d, t}(x)| > \frac{\delta}{2}\}\), \(B = \{x \in \mathbb{R}^d : |\mathbb{E}[V_d(t + 1, X^d_{t+1})|X^d_t = x] - \gamma_{\varepsilon, d, t}(x)| > \frac{\delta}{2}\}\) and note that

\[
C_t \cap \hat{C}_t^c = \{x \in \mathbb{R}^d : g_d(t, x) < \mathbb{E}[V_d(t + 1, X^d_{t+1})|X^d_t = x], \phi_{\varepsilon, d, t}(x) - \delta \geq \gamma_{\varepsilon, d, t}(x)\}
\]

\[
\subset A \cup B,
\]

since \(|g_d(t, x) - \phi_{\varepsilon, d, t}(x)| \leq \frac{\delta}{2}, |\mathbb{E}[V_d(t + 1, X^d_{t+1})|X^d_t = x] - \gamma_{\varepsilon, d, t}(x)| \leq \frac{\delta}{2}\) and \(g_d(t, x) < \mathbb{E}[V_d(t + 1, X^d_{t+1})|X^d_t = x]\) implies

\[
\phi_{\varepsilon, d, t}(x) - \delta \leq g_d(t, x) - \frac{\delta}{2} < \mathbb{E}[V_d(t + 1, X^d_{t+1})|X^d_t = x] - \frac{\delta}{2} \leq \gamma_{\varepsilon, d, t}(x).
\]

(4.62)
Furthermore, Markov’s inequality, (4.32) and (3.6) imply that
\[
\rho^d_t(A) = \rho^d_t \left( \left\{ x \in \mathbb{R}^d : |g_d(t,x) - \phi_{\varepsilon,d,t}(x)| > \frac{\delta}{2} \right\} \right)
\leq \frac{4}{\delta^2} \int_{\mathbb{R}^d} |g_d(t,x) - \phi_{\varepsilon,d,t}(x)|^2 \rho^d_t(dx) 
\leq \frac{4}{\delta^2} \int_{\mathbb{R}^d} \varepsilon cd^q (1 + \|x\|^p)^2 \rho^d_t(dx)
\leq \frac{8(\varepsilon cd^q)^2}{\delta^2} \left( 1 + \int_{\mathbb{R}^d} \|x\|^{2p} \rho^d_t(dx) \right)
\leq \frac{8(1 + c_t)\varepsilon^2 d^{2q+q}}{\delta^2}.
\] (4.63)

Similarly, Markov’s inequality and (4.50) yield
\[
\rho^d_t(B) = \rho^d_t \left( \left\{ x \in \mathbb{R}^d : |E[V_d(t+1,X_{t+1}^d)|X_t^d = x] - \gamma_{\varepsilon,d,t}(x)| > \frac{\delta}{2} \right\} \right)
\leq \frac{4}{\delta^2} \int_{\mathbb{R}^d} \left| E[V_d(t+1,X_{t+1}^d)|X_t^d = x] - \gamma_{\varepsilon,d,t}(x) \right|^2 \rho^d_t(dx) 
\leq \frac{12}{\delta^2} \tilde{c}_2 d^{\tilde{q}^2}[\varepsilon^2 + N^{-1}\varepsilon^{-2\tau+1}].
\] (4.64)

Putting together (4.61), (4.62), (4.63) and (4.64) and inserting the choices (4.42) we obtain
\[
\left( \left( \int_{\mathbb{R}^d} \left| E[V_d(t+1,X_{t+1}^d)|X_t^d = x] - \phi_{\varepsilon,d,t}(x) + \delta \right|^2 \mathbb{1}_{C_t \cap C_t^c}(x) \rho^d_t(dx) \right)^{1/2}
\leq 4 \max(\tilde{c}_{t+1},c)d^{\tilde{q}^2+1} \left( 1 + \tilde{c}_1 d^{\tilde{q}^2+1} \right) \left( \rho^d_t(A) + \rho^d_t(B) \right)^{1/4}
\leq 4 \max(\tilde{c}_{t+1},c)d^{\tilde{q}^2+1} \left( 1 + 2\tilde{c}_1 \left( 1 + \frac{c_t}{8} \right) \right)
\cdot \left( \frac{8(1 + c_t)\varepsilon^2 d^{2q+q}}{\delta^2} + \frac{12}{\delta^2} \tilde{c}_2 d^{\tilde{q}^2}[\varepsilon^2 + N^{-1}\varepsilon^{-2\tau+1}] \right)^{1/4}
\leq \tilde{c}_5 \varepsilon^2 d^{\tilde{q}^2+1}
\] (4.65)

with \( \tilde{c}_5 = 4 \max(\tilde{c}_{t+1},c)(1 + 2\tilde{c}_1(1 + c_t^{1/4}))(8(1 + c_t)\varepsilon^2 + 24\tilde{c}_2) \left( 1 + \frac{c_t}{8} \right) \) and \( \tilde{q}_5 = \tilde{q}_{t+1} + \tilde{q}_1 + \frac{2q}{4} + \frac{1}{4} \max(2q + q_t, \tilde{q}_2) \).

10.c) Bounding the approximation error on \( C_t^c \cap C_t \): We are now concerned with the third term in (4.55). Observe that
\[
|g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| \mathbb{1}_{C_t^c \cap C_t}(x) \leq |g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| \mathbb{1}_{A}(x) + \mathbb{1}_{B}(x) + \mathbb{1}_{A \cap B \cap C_t^c \cap C_t}(x).
\] (4.66)
For \( x \in A^c \cap \hat{C}_t \) we have
\[
\gamma_{\varepsilon,d,t}(x) + \frac{3}{2}\delta > \phi_{\varepsilon,d,t}(x) + \frac{\delta}{2} \geq g_d(t,x)
\] (4.67)
and therefore for \( x \in A^c \cap B^c \cap C^c_t \cap \hat{C}_t \) it follows that
\[
|g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| \leq |g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| - \frac{3}{2}\delta + \frac{3}{2}\delta \\
= \gamma_{\varepsilon,d,t}(x) + 3\delta - g_d(t,x) \\
\leq \gamma_{\varepsilon,d,t}(x) + 3\delta - \mathbb{E}[V_d(t+1, X^{d}_{t+1})|X^{d}_t = x] \\
\leq \frac{7}{2}\delta.
\] (4.68)
Combining this with (4.66) we obtain
\[
|g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| \mathbb{1}_{C^c_t \cap \hat{C}_t}(x) \leq |g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| (\mathbb{1}_A(x) + \mathbb{1}_B(x)) + \frac{7}{2}\delta
\] (4.69)
and consequently the growth bounds (4.7), (4.15) and (4.21) on \( g_d \), on the conditional moments and on \( V_d \), Hölder’s inequality and the approximation error bound (4.50) for the continuation value yield
\[
\left( \int_{\mathbb{R}^d} |g_d(t,x) - \gamma_{\varepsilon,d,t}(x)|^2 \mathbb{1}_{C^c_t \cap \hat{C}_t}(x) \rho^d_t(dx) \right)^{1/2} \\
\leq \left( \int_{\mathbb{R}^d} |g_d(t,x) - \gamma_{\varepsilon,d,t}(x)| (\mathbb{1}_A(x) + \mathbb{1}_B(x))^2 \rho^d_t(dx) \right)^{1/2} + \frac{7}{2}\delta \\
\leq \left( \int_{\mathbb{R}^d} |g_d(t,x) - \mathbb{E}[V_d(t+1, X^{d}_{t+1})|X^{d}_t = x]|(\mathbb{1}_A(x) + \mathbb{1}_B(x)) \rho^d_t(dx) \right)^{1/2} + \frac{7}{2}\delta \\
+ \left( \int_{\mathbb{R}^d} \|\mathbb{E}[V_d(t+1, X^{d}_{t+1})|X^{d}_t = x] - \gamma_{\varepsilon,d,t}(x)|(\mathbb{1}_A(x) + \mathbb{1}_B(x))^2 \rho^d_t(dx) \right)^{1/2} + \frac{7}{2}\delta \\
\leq 2\delta \tilde{c}_{t+1} d^{\beta t+1} \left( \int_{\mathbb{R}^d} \left( 1 + \sum_{s=t}^{t+1} \mathbb{E}[||X^{d}_s|||X^{d}_t = x] \right)^2 (\mathbb{1}_A(x) + \mathbb{1}_B(x)^2 \rho^d_t(dx) \right)^{1/2} \\
+ 2 \left( \int_{\mathbb{R}^d} \mathbb{E}[V_d(t+1, X^{d}_{t+1})|X^{d}_t = x] - \gamma_{\varepsilon,d,t}(x)|^2 \rho^d_t(dx) \right)^{1/2} + \frac{7}{2}\delta \\
\leq 2\delta \tilde{c}_{t+1} d^{\beta t+1} \left( \int_{\mathbb{R}^d} \left[ 1 + 2\tilde{c}_1 d^{\beta t+1} (1 + ||x||) \right]^4 \rho^d_t(dx) \right)^{1/4} \left[ (\rho^d_t(A))^{1/4} + (\rho^d_t(B))^{1/4} \right]^4 \\
+ 2 \left( 3\tilde{c}_2 d^{\beta t+2} \mathbb{E}^2 + N^{-1}\mathbb{E}^{-2\varepsilon t+1} \right)^{1/2} + \frac{7}{2}\delta.
\] (4.70)
Inserting the bound (4.32) on the moments of $\rho_t^d$, the bounds (4.63), (4.64) on $\rho_t^d(A)$, $\rho_t^d(B)$ and the choices (4.42) for $N$ and $\delta$ thus shows that

$$
\left( \int_{\mathbb{R}^d} |g_{\delta}(t, x) - \gamma_{\varepsilon,d,t}(x)|^2 \mathbb{1}_{C_{\gamma} \cap C_t}(x) \rho_t^d(dx) \right)^{1/2} \\
\leq 2\tilde{c}_{t+1}d^{\tilde{q}+1}(1 + 2\tilde{c}_1d^{\tilde{q}+1}(1 + c_t^d)} + (8(1 + c_t)c^2\varepsilon^2d^{2q+\alpha}) \right)^{1/2} + \left( \frac{24}{\delta^2}\tilde{c}_2d^{\tilde{q}+2}\varepsilon^2 \right)^{1/2} \\
+ 2(6\tilde{c}_2d^{\tilde{q}+2}\varepsilon^2)^{1/2} + \frac{7}{2}\delta \\
\leq \tilde{c}_6d^{\tilde{p}\tilde{q}\varepsilon^2 + 1}.
$$

(4.71)

with $\tilde{c}_6 = 2\tilde{c}_{t+1}(1 + 2\tilde{c}_1(1 + c_t^d))(8(1 + c_t)c^2\varepsilon^2d^{2q+\alpha}) + 2(6\tilde{c}_2)^{1/2} + \frac{7}{2}$ and $\tilde{q}_6 = \tilde{q}_{t+1} + \frac{1}{2}\rho + \tilde{q}_1 + \tilde{q} + \tilde{q}^2$.

10.d) Combining the individual error estimates: Finally, note that the second and last line of (4.63) yield

$$
\int_{\mathbb{R}^d} |g_{\delta}(t, x) - \phi_{\varepsilon,d,t}(x)|^2 \rho_t^d(dx) \leq 2(1 + c_t)c^2\varepsilon^2d^{2q+\alpha}.
$$

(4.72)

Consequently, combining the decomposition (4.58) with the individual estimates (4.50), (4.65), (4.71) and (4.72) we obtain

$$
\left( \int_{\mathbb{R}^d} |V_{\delta}(t, x) - \psi_{\varepsilon,d,t}(x)|^2 \rho_t^d(dx) \right)^{1/2} \\
\leq \left( \int_{\mathbb{R}^d} |g_{\delta}(t, x) - \phi_{\varepsilon,d,t}(x) + \delta^2 \mathbb{1}_{C_{\gamma} \cap \tilde{C}_t}(x) \rho_t^d(dx) \right)^{1/2} \\
+ \left( \int_{\mathbb{R}^d} |\mathbb{E}[V_{\delta}(t + 1, X_{t+1}^d)|X_{t}^d = x] - \phi_{\varepsilon,d,t}(x) + \delta^2 \mathbb{1}_{C_{\gamma} \cap \tilde{C}_t}(x) \rho_t^d(dx) \right)^{1/2} \\
+ \left( \int_{\mathbb{R}^d} |g_{\delta}(t, x) - \gamma_{\varepsilon,d,t}(x)|^2 \mathbb{1}_{C_{\gamma} \cap \tilde{C}_t}(x) \rho_t^d(dx) \right)^{1/2} \\
+ \left( \int_{\mathbb{R}^d} |\mathbb{E}[V_{\delta}(t + 1, X_{t+1}^d)|X_{t}^d = x] - \gamma_{\varepsilon,d,t}(x)|^2 \mathbb{1}_{C_{\gamma} \cap \tilde{C}_t}(x) \rho_t^d(dx) \right)^{1/2} \\
\leq (2(1 + c_t))^\frac{1}{2}\varepsilon^2d^{\tilde{q}+2}\varepsilon + \delta + \tilde{c}_5d^{\tilde{q}\tilde{p}\varepsilon^2 + 1} + \tilde{c}_6d^{\tilde{p}\varepsilon^2 + 1} + \tilde{c}_7d^{\tilde{q}+2} (6\tilde{c}_2)^{1/2} \\
\leq \tilde{c}_7d^{\tilde{p}\tilde{q}\varepsilon^2 + 1}.
$$

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with \( \tilde{c}_7 = (2(1 + c_\ell))^\frac{1}{2}c + 1 + \tilde{c}_5 + \tilde{c}_6 + (6\tilde{c}_2)^\frac{1}{2} \) and \( \tilde{q}_7 = \max(q + \frac{2\tilde{c}}{\tilde{c}_7}, \tilde{q}_5, \tilde{q}_6, \frac{2\tilde{c}}{\tilde{c}_7}) \). Now choose

\[
\bar{\varepsilon} = \varepsilon_4^A \left[ \frac{1}{\tilde{c}_7} \right]^{-\frac{1}{2}}.
\]  

(4.74)

Inserting (4.74) in (4.73) proves that (4.33) is satisfied. Furthermore, choosing

\[
\kappa_t = \max(\tilde{c}_3(\tilde{c}_7)^{4\tilde{c}_3}, \tilde{c}_4(\tilde{c}_7)^{4\tilde{c}_4}, c \max(\kappa_{t+1}, 1)),
\]

(4.75)

\[
q_t = \max(\tilde{q}_3 + 4\tilde{r}_3 \tilde{q}_7, \tilde{q}_4 + 4\tilde{r}_4 \tilde{q}_7, q_{t+1} + q),
\]

(4.76)

\[
\tau_t = \max(4\tilde{r}_3, 4\tilde{r}_4),
\]

(4.77)

we obtain from (4.53), (4.55) and (4.57) that (4.34), (4.35) and (4.36) are satisfied. This completes the induction step. Hence, the statement follows.

\[ \square \]

**Proof of Corollary 4.3** Fix \( d \in \mathbb{N}, h \in [-R, R]^d \) and set \( \rho^d = \frac{1}{2} \nu_0^d + \frac{1}{2} \nu_h^d \), where \( \nu_x^d \) denotes a multivariate normal distribution on \( \mathbb{R}^d \) with mean \( x \) and identity covariance. Then we estimate

\[
\int_{\mathbb{R}^d} \|x\|^{2\max(p, 2)} \rho^d(dx) \leq \frac{1}{2}(1 + 2^{\max(p, 2) - 1}) \int_{\mathbb{R}^d} \|x\|^{2\max(p, 2)} \nu_0^d(dx) + \frac{1}{4}(2\|h\|)^{2\max(p, 2)}
\]

and thus \( \|h\| \leq Rd^{1/2} \) and (4.2) show that there exist \( c > 0, q \geq 0 \) only depending on \( p \) and \( R \) such that the bound \( \int_{\mathbb{R}^d} \|x\|^{2\max(p, 2)} \rho^d(dx) \leq cd^q \) holds. Hence, we can apply Theorem 3.4 and obtain for all \( \varepsilon \in (0, 1], t \in \{0, \ldots, T\} \) the existence of a neural network \( \psi_{\varepsilon, d, t} \) such that (3.9) holds. From the proof of Theorem 3.4 we obtain that these networks satisfy the Lipschitz condition (4.36). Therefore, for any \( \varepsilon > 0 \) we may use Minkowski’s inequality, the bound \( \| \cdot \|_{L^2(\nu^d)} \leq \sqrt{2} \| \cdot \|_{L^2(\nu^d)} \) for \( \psi_{\varepsilon, d, t} \) and the Lipschitz property (4.36) to estimate

\[
\|V_d(t) - V_d(t, \cdot + h)\|_{L^2(\nu^d)} \\
\leq \|V_d(t) - \psi_{\varepsilon, d, t}\|_{L^2(\nu^d)} + \|\psi_{\varepsilon, d, t} - \psi_{\varepsilon, d, t}(\cdot + h)\|_{L^2(\nu^d)} + \|\psi_{\varepsilon, d, t}(\cdot + h) - V_d(t, \cdot + h)\|_{L^2(\nu^d)} \\
\leq \sqrt{2}\|V_d(t) - \psi_{\varepsilon, d, t}\|_{L^2(\nu^d)} + \|\psi_{\varepsilon, d, t} - \psi_{\varepsilon, d, t}(\cdot + h)\|_{L^2(\nu^d)} + \|\psi_{\varepsilon, d, t} - V_d(t, \cdot)\|_{L^2(\nu^d)} \\
\leq \sqrt{2} \varepsilon + \|h\| \kappa_t d^q.
\]

(4.78)

This holds for any \( \varepsilon > 0 \) and from the statement in Step 2 of the proof of Theorem 3.4 the constants \( \kappa_t, q_t \) do not depend on \( d, \varepsilon \) and \( h \) (but they depend on \( R \)); letting \( \varepsilon \) tend to 0 therefore yields the claimed statement.

\[ \square \]

**4.5 Proof of Theorem 3.8**

This subsection is devoted to the proof of Theorem 3.8. It is based on the proof of Theorem 3.4 given in the previous subsection.
Proof of Theorem 3.8

Proving this result just requires slight modifications in the proof of Theorem 3.4. W.l.o.g. we may assume $c \geq h$, $q \geq \bar{q}$. In Step 1 we only need to choose $c_0, \ldots, c_T$ differently. Indeed, let $c_0 = c$, $c_{t+1} = h(1 + c_t)$ and set $q_t = \bar{q}(t + 1)$. In Step 2, the stronger statement is modified accordingly: we will prove that for any $t \in \{0, \ldots, T\}$ there exist constants $\kappa_t, q_t, r_t \in [0, \infty)$ such that for any family of probability measures $\rho^d_t$ on $\mathbb{R}^d$, $d \in \mathbb{N}$, with

\[ \int_{\mathbb{R}^d} \|x\|^{2m \max(p, 2)} \rho^d_t(dx) \leq c_t d q^t \]  (4.79)

and for all $d \in \mathbb{N}$, $\varepsilon \in (0, 1]$ there exists a neural network $\psi_{\varepsilon, d, t}$ such that the approximation error estimate (4.33) holds and $\psi_{\varepsilon, d, t}$ satisfies the growth (4.34) and size conditions (4.35) and the modified Lipschitz condition

\[ \text{Lip}(\psi_{\varepsilon, d, t}) \leq \kappa_t d^{q^t} \varepsilon^{-\zeta(T-t)}. \]  (4.80)

For $t = T$ condition (4.80) coincides with (4.36) and so the base case (Step 3) remains the same as in the proof of Theorem 3.4. Due to the assumption (3.14) also Steps 4 and 5 only require slight modifications: (4.37) becomes

\[ \int_{\mathbb{R}^d} \|z\|^{2m \max(p, 2)} \rho^d_{t+1}(dz) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \|f^d_t(x, y)\|^{2m \max(p, 2)} \nu^d_t(dy) \rho^d_t(dx) \\
= \int_{\mathbb{R}^d} \mathbb{E}[\|f^d_t(x, Y^d_t)\|^{2m \max(p, 2)}] \rho^d_t(dx) \\
\leq h d^{\bar{q}} \left( 1 + \int_{\mathbb{R}^d} \|x\|^{2m \max(p, 2)} \rho^d_t(dx) \right) \\
\leq h d^{\bar{q}+\bar{q}} (1 + c_t) \\
= c_{t+1} d^{\bar{q}+1}, \]  (4.81)

the Lipschitz condition (4.41) is replaced by

\[ \text{Lip}(\psi_{\varepsilon, d, t+1}) \leq \kappa_{t+1} d^{\bar{q}+1} \varepsilon^{-\zeta(T-t-1)} \]  (4.82)

and we modify the choice of $N$ in (4.42) to $N = \lceil \varepsilon^{-2} - 2^{2-2\theta} \rceil$.

For the beginning of Step 6 and for Step 6.a) we proceed precisely as above and obtain the error estimates (4.43), (4.44) and (4.45). In 6.b) the Lipschitz property (1.82) of the network now yields the additional factor $\varepsilon^{-2\zeta(T-t-1)}$ and the approximation property (3.2)
only holds on \([-((\varepsilon^{-\beta}), \varepsilon^{-\beta})^d]\), see \((3.11)\). Hence, we estimate

\[
\int_{\mathbb{R}^d} |\mathbb{E}[\hat{v}_{\varepsilon,d,t+1}(f_t^d(x, Y_t^d))] - \mathbb{E}[\hat{v}_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x, Y_t^d))]|^2 \rho_t^d(dx)
\]

\[
\leq (\kappa_{t+1} d^{m+1} \varepsilon^{-\zeta(T-t-1)})^2 \int_{\mathbb{R}^d} |\mathbb{E}[||f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)||]|^2 \rho_t^d(dx)
\]

\[
\leq 2(\kappa_{t+1} d^{m+1} \varepsilon^{-\zeta(T-t-1)})^2 \left[ (\varepsilon cd^2)^2 \int_{[-(\varepsilon^{-\beta}), \varepsilon^{-\beta})^d} |1 + ||x||^p + \mathbb{E}[||Y_t^d||^p]|^2 \rho_t^d(dx) \right. \]

\[
+ \int_{\mathbb{R}^d \setminus [-\varepsilon^{-\beta}, \varepsilon^{-\beta})^d} |\mathbb{E}[||f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)||]|^2 \rho_t^d(dx)
\]

\[
\left. + \int_{\mathbb{R}^d} |\mathbb{E}[||f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)|| I_{\{Y_t^d \in \mathbb{R}^d \setminus [-\varepsilon^{-\beta}, \varepsilon^{-\beta})^d\}}|^2 \rho_t^d(dx) \right].
\]

The first term can be bounded as before. For the second term, note that Jensen’s inequality and \((3.14)\) imply that \(\mathbb{E}[||f_t^d(x, Y_t^d)||] \leq cd^q(1 + ||x||)\). Hence, we may apply Hölder’s inequality, the growth bound \((3.15)\) on \(\eta_{\varepsilon,d,t}\), Markov’s inequality and \((4.79)\) to obtain

\[
\int_{\mathbb{R}^d \setminus [-\varepsilon^{-\beta}, \varepsilon^{-\beta})^d} |\mathbb{E}[||f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)||]|^2 \rho_t^d(dx)
\]

\[
\leq \left( \int_{\mathbb{R}^d} |\mathbb{E}[||f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)||]|^4 \rho_t^d(dx) \right)^{\frac{1}{2}} \left( \rho_t^d(\mathbb{R}^d \setminus [-\varepsilon^{-\beta}, \varepsilon^{-\beta})^d) \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_{\mathbb{R}^d} |\mathbb{E}[2cd^q \varepsilon^{-\theta} (2 + ||x|| + ||Y_t^d||)]|^4 \rho_t^d(dx) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} ||x||^2 m \max(2,p) \rho_t^d(dx) \right)^{\frac{1}{2}}
\]

\[
\leq 3(2cd^q)^2 \varepsilon^{-2\theta} \left( 4 + \mathbb{E}[||Y_t^d||^2] + \left( \int_{\mathbb{R}^d} ||x||^4 \rho_t^d(dx) \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \varepsilon^{m \max(2,p)}
\]

\[
\cdot \left( \int_{\mathbb{R}^d} ||x||^2 m \max(2,p) \rho_t^d(dx) \right)^{\frac{1}{2}}
\]

\[
\leq 3\varepsilon^{m \max(2,p)-2\theta} (2cd^q)^2 \left( 4 + (cd^p)^2 + (c_t d^p)^{\frac{1}{2}} \right) (c_t d^p)^{\frac{1}{2}}.
\]

For the last term in \((4.83)\) we note that \((3.14)\) and Jensen’s inequality imply

\[
\mathbb{E}[||f_t^d(x, Y_t^d)||^2] \leq (cd^q(1 + ||x||^{2m \max(p,2)}))^{\frac{1}{m \max(p,2)}} \leq cd^q(1 + ||x||)^2.
\]

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Using this, Hölder’s inequality, (3.15) and Markov’s inequality we obtain

\[
\int_{\mathbb{R}^d} \left| \mathbb{E}[f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)] \right|^2 \rho_t^d(dx)
\leq \int_{\mathbb{R}^d} \mathbb{E}[|f_t^d(x, Y_t^d) - \eta_{\varepsilon,d,t}(x, Y_t^d)|^2] \rho_t^d(dx) \mathbb{P}(Y_t^d \in \mathbb{R}^d \setminus [-\bar{\varepsilon}^{-\beta}, \bar{\varepsilon}^{-\beta}])
\leq 2 \int_{\mathbb{R}^d} \mathbb{E}[2(\bar{c}d^q \bar{\varepsilon}^{-\beta})^2 (2 + \|x\| + \|Y_t^d\|)^2] \rho_t^d(dx) \left( \frac{1}{(\bar{\varepsilon}^{-\beta})^{2m \max(2,p)}} \mathbb{E}[\|Y_t^d\|^{2m \max(2,p)}] \right)
\leq 2(2\bar{c}d^q)^2 \left( 4 + \mathbb{E}[\|Y_t^d\|^2] + \left( \int_{\mathbb{R}^d} \|x\|^4 \rho_t^d(dx) \right)^{\frac{1}{2}} \right) \bar{\varepsilon}^{2m \beta \max(2,p) - 2\theta} \mathbb{E}[\|Y_t^d\|^{2m \max(2,p)}]
\leq 3 \bar{\varepsilon}^{\beta m \max(2,p) - 2\theta} (2\bar{c}d^q)^2 \left( 4 + cd^q \left( c_t d^q \right)^{\frac{1}{2}} \right) cd^q.
\]

(4.85)

Together this yields

\[
\int_{\mathbb{R}^d} \left| \mathbb{E}[v_{\varepsilon,d,t+1}(f_t^d(x, Y_t^d))] - \mathbb{E}[v_{\varepsilon,d,t+1}(\eta_{\varepsilon,d,t}(x, Y_t^d))] \right|^2 \rho_t^d(dx)
\leq 6(\bar{\varepsilon}^{1-\zeta(T-t-1)} \kappa_{t+1})^2 d^{2(q+4q_{t+1})+\max(2q, \frac{p}{\max(2,\beta)})} \left( 1 + c^2 + c_t \right)
\]

\[
+ 48 \bar{\varepsilon}^{\beta m \max(2,p) - 2\theta - 2\zeta(T-t-1)} (\kappa_{t+1})^2 d^{4q+4q_{t+1}+4q_{t}} \left( 4 + c^2 + c_t^{\frac{1}{2}} \right) \epsilon_t.
\]

(4.86)

Similarly, in Step 6.c) the factor \(\bar{\varepsilon}^{-2t_{t+1}}\) is replaced by \(\bar{\varepsilon}^{-2t_{t+1}-2\theta}\) due to the growth bound (3.15). This and the modified bound (4.86) (as opposed to (4.46)) then also leads to a different estimate in Step 6.d), where we obtain

\[
\int_{\mathbb{R}^d} \mathbb{E}[|\mathbb{E}[V_d(t+1, X_{t+1}^d)|X_t^d = x] - \Gamma_{\varepsilon,d,t}(x)|^2] \rho_t^d(dx)
\leq 3\bar{c}_2 \tilde{q}^2 \left( \bar{\varepsilon}^{2-2\zeta(T-t-1) + \bar{\varepsilon}^{\beta m \max(2,p) - 2\theta - 2\zeta(T-t-1)} + N^{-1} \bar{\varepsilon}^{-2t_{t+1} - 2\theta}} \right)
\]

with slightly modified constants \(\bar{c}_2 = 96c^2\kappa_{t+1}^2 (4+c^2+c_t) c_t\) and \(\tilde{q}^2 = 4q+4q_{t+1}+\max(2q, q_t)\). Thus, the same argument as before yields the existence of \(\omega \in \Omega\) such that \(\gamma_{\varepsilon,d,t}\), the realization of \(\Gamma_{\varepsilon,d,t}\) at \(\omega\), satisfies

\[
\int_{\mathbb{R}^d} \mathbb{E}[|V_d(t+1, X_{t+1}^d)|X_t^d = x] - \gamma_{\varepsilon,d,t}(x)|^2] \rho_t^d(dx)
\leq 3\bar{c}_2 \tilde{q}^2 \left( \bar{\varepsilon}^{2-2\zeta(T-t-1) + \bar{\varepsilon}^{\beta m \max(2,p) - 2\theta - 2\zeta(T-t-1)} + N^{-1} \bar{\varepsilon}^{-2t_{t+1} - 2\theta}} \right)
\]

(4.87)

and (4.51) holds. In Step 7 the modified growth bound (3.15) and the modified choice \(N = \bar{\varepsilon}^{-2t_{t+1} - 2\theta}\) lead to an additional factor \(\bar{\varepsilon}^{-2\theta}\) in (4.53) and to a modified choice \(\bar{r}_3 = 3t_{t+1} + 2 + 3\theta\). Similarly, in Step 8 the modified choice of \(N\) leads to an additional
Thus, we obtain
\[ \kappa_{t+1} d^{q_{t+1}+1} \varepsilon^{-\zeta(T-t)} \leq c \max(\kappa_{t+1}, 1) d^{q_{t+1}+1} \varepsilon^{-\zeta(T-t)}. \]  

Step 10 requires a different choice of \( \delta \) and otherwise only minor modifications. We choose \( \delta = \varepsilon^{1/2}(\min(1, \beta m - \theta) - \zeta(T-1)) \). Then in Step 10.b) the new bound (4.88) leads to a slightly different bound than in (4.64) and (4.65). We obtain

\[ \left( \int_{\mathbb{R}^d} |E[V_d(t+1, X^d_{t+1})] - X^d_{t} - \delta \rho^d_t(x) \rho^d_t(x) | \delta^2 \right)^{1/2} \]
\[ \leq 4 \max(\hat{c}_{t+1}, c) d^{q_{t+1}+\tilde{q}_1+\frac{q}{2}} \left[ 1 + 2\hat{c}_1 (1 + \frac{1}{c_t}) \right] \left( \frac{8(1 + c_t)\varepsilon^2 d^{2q+q_t}}{\delta^2} \right) \]
\[ + \frac{12}{\delta^2} \hat{c}_2 d^{q_{t+1}+\tilde{q}_1+\frac{q}{2}} \left[ \varepsilon^{-2\zeta(T-t-1)} + \varepsilon^{-\zeta(T-t-1)} + N^{-1} \varepsilon^{-2r_{t+1}-2\theta} \right]^{1/4} \]
\[ \leq \tilde{c}_5 d^{q_{t+1}+\tilde{q}_1+\frac{q}{2}} (\min(1, \beta m - \theta) - \zeta(T-1)) \]  

with slightly modified constant \( \tilde{c}_5 = 4\hat{c}_{t+1}(1 + 2\hat{c}_1 (1 + \frac{1}{c_t})) \) \( (8(1 + c_t)\varepsilon^2 d^{2q+q_t})^{1/2} \) and \( \tilde{q}_5 \) as before. In Step 10.c), (4.88) leads to analogous modifications in (4.70) and (4.71), yielding

\[ \left( \int_{\mathbb{R}^d} |g_d(t, x) - \gamma_{x, t+d}(x) |^2 \right)^{1/2} \leq \tilde{c}_6 d^{q_{t+1}+\tilde{q}_1+\frac{q}{2}} (\min(1, \beta m - \theta) - \zeta(T-1)) \]

with \( \tilde{c}_6 = 2\hat{c}_{t+1}(1 + 2\hat{c}_1 (1 + \frac{1}{c_t})) \) \( (8(1 + c_t)\varepsilon^2 d^{2q+q_t})^{1/2} + (36\hat{c}_2)^{1/2} | + 2(9\hat{c}_2)^{1/2} + \frac{7}{2} \) and \( \tilde{q}_6 \) as before. Combining (4.88), (4.91), (4.92) and (4.72) we thus obtain for Step 10.d) the bound

\[ \left( \int_{\mathbb{R}^d} |V_d(t, x) - \psi_{x, t+d}(x) |^2 \right)^{1/2} \leq \tilde{c}_7 d^{q_{t+1}+\tilde{q}_1+\frac{q}{2}} (\min(1, \beta m - \theta) - \zeta(T-1)) \]

with \( \tilde{c}_7 = 2\hat{c}_{t+1}(1 + 2\hat{c}_1 (1 + \frac{1}{c_t})) \) \( (8(1 + c_t)\varepsilon^2 d^{2q+q_t})^{1/2} + (36\hat{c}_2)^{1/2} | + 2(9\hat{c}_2)^{1/2} + \frac{7}{2} \) and \( \tilde{q}_7 \) as before.
with \( \tilde{c}_7 = (2(1 + \epsilon_t))^\frac{1}{2} c + 1 + \tilde{c}_5 + \tilde{c}_6 + (9\tilde{c}_2)^\frac{1}{2} \) and \( \tilde{q}_7 \) as before. Choose

\[
\tilde{\varepsilon} = \left( e^{-1}\tilde{c}_7 d\tilde{q}_7 \right)^{-\min(1, \beta_m - \theta)} - \zeta(T - 1)
\]

and note that \( \tilde{\varepsilon} \in (0, 1) \) because \( \tilde{c}_7 > 1 \) and \( \frac{\min(1, \beta_m - \theta)}{T - 1} > \zeta \). By inserting this choice of \( \tilde{\varepsilon} \) in the bounds for the growth, size and Lipschitz constant of \( \psi_{\varepsilon, d, t} \) we may then appropriately choose \( \kappa_t, q_t, r_t \) (analogously to (4.75), (4.76), (4.77)) and complete the proof.

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