ON THE CHOW GROUPS OF THE VARIETY OF LINES OF A CUBIC FOURFOLD

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Abstract. Let $X$ be a smooth complex cubic fourfold and let $F$ be the variety of lines of $X$. The variety $F$ is known to be a smooth projective hyperkähler fourfold, which is moreover endowed with a self rational map $\varphi: F \dashrightarrow F$ first constructed by Voisin. Here we define a filtration of Bloch–Beilinson type on $\text{CH}_0(F)$ which canonically splits under the action of $\varphi$, thereby answering in this case a question of Beauville. Moreover, we show that this filtration is of motivic origin, in the sense that it arises from a Chow–Künneth decomposition of the diagonal.

Introduction

To a smooth projective variety $Y$, one can associate its Chow ring $\text{CH}^*(Y)$ of algebraic cycles modulo rational equivalence. Our knowledge about this ring is still very little. The deep conjectures of Beilinson and Bloch predict the existence of a descending filtration on the Chow groups with rational coefficients

$$\text{CH}^p(Y) \cong F_0 \supseteq F_1 \supseteq \cdots \supseteq F_p \supseteq 0$$

which is compatible with the ring structure and functorial with respect to the action of correspondences. Moreover, such a filtration should satisfy $F^1 \text{CH}^p(Y) = \text{CH}^p(Y)_{\text{hom}}$, where the subscript “hom” denotes the homologically trivial cycles, and be such that the action of a correspondence $\gamma$ on the graded parts depends solely on the class of $\gamma$ modulo homological equivalence. In [4], A. Beauville asks if this filtration, assuming it exists, splits canonically for a special class of varieties $Y$. For the case of an abelian variety $Y$, a splitting is constructed in [3] as the eigenspaces of the action of pull back by the “multiplication-by-$n$” endomorphisms. The case of a $K3$ surface is treated in [6]. In [19], C. Voisin established what Beauville called weak splitting, a notion which is concerned only with the splitting of $F^1 \text{CH}^*(Y) \subseteq \text{CH}^*(Y)$, for many projective hyperkähler manifolds, including the variety of lines of a smooth cubic fourfold. In this paper, we focus on the latter case and construct a filtration of Bloch–Beilinson type on $\text{CH}_0$ which splits canonically. The splitting constructed in this paper is thus finer than the weak splitting of Voisin, and will be seen to be an analogue of the splitting in [3].

Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold and let $F$ be its variety of lines. It is known that $F$ is a projective hyperkähler manifold of $K3[^2]$-type; see [5]. There is a
natural polarization $g \in \text{Pic}(F)$ which comes from the Plücker embedding. Crucial to our study is the existence of a natural incidence correspondence

$$I = \{([l_1], [l_2]) \in F \times F : l_1 \cap l_2 \neq \emptyset \}$$

which to a point $[l] \in F$ associates the surface $S_l$ of lines meeting $l$ in $X$. First we define a filtration on $\text{CH}_0(F)$ as

$$(1) \quad F^4 \text{CH}_0(F) = F^3 \text{CH}_0(F) \subset F^2 \text{CH}_0(F) = F^4 \text{CH}_0(F) \subset \text{CH}_0(F),$$

where $F^1 \text{CH}_0(F) = \text{CH}_0(F)_{\text{hom}}$ and $F^3 \text{CH}_0(F) = \ker\{I_* : \text{CH}_0(F)_{\text{hom}} \to \text{CH}_2(F)_{\text{hom}}\}$. We will give many different geometric descriptions of $F^3 \text{CH}_0(F)$. One description is given as follows; see Theorem 2.6. Let $A = I_* \text{CH}_0(F) \subset \text{CH}_2(F)$, then the intersection product induces a natural homomorphism

$$\mathcal{A}_{\text{hom}} \otimes \mathcal{A}_{\text{hom}} \to \text{CH}_0(F)$$

whose image is precisely $F^3 \text{CH}_0(F)$. Note that $H^1(F, \mathbb{Q}) = H^3(F, \mathbb{Q}) = 0$, so that the predicted filtration of Bloch–Beilinson on positive-dimensional cycles only consists of $\text{CH}_{>0}(F)_{\text{hom}, \mathbb{Q}} \subset \text{CH}_{>0}(F)_{\mathbb{Q}}$.

The other crucial feature of the variety of lines of a cubic fourfold is the existence of a degree 16 rational map $\varphi : F \dashrightarrow F$, first defined by Voisin in [18]. It acts contravariantly as multiplication by $-2$ on $H^{2,0}(F)$ and 4 on $H^{4,0}(F)$; see also [2]. The closure of the graph of $\varphi$, denoted by $\Gamma_{\varphi}$, acts via push-forward and pull-back on the Chow groups and also the cohomology groups of $F$. These will be denoted by $\varphi_*$ and $\varphi^*$. We study the action of $\varphi$ on Chow groups and show that the action is diagonalizable and its eigenspaces give interesting splittings of the filtration. To be more precise, the action of $\varphi^*$ on $\text{CH}_0(F)$ satisfies $(\varphi^* - 4)(\varphi^* + 8)(\varphi^* - 16) = 0$, which gives a splitting of

$$\text{CH}_0(F) = V^4_0 \oplus V^{-8}_0 \oplus V^{16}_0$$

as a direct sum of eigenspaces. This decomposition is shown to yield a canonical splitting of the filtration (1) which can be described as

$$F^3 \text{CH}_0(F) = V^4_0, \quad F^1 \text{CH}_0(F) = V^4_0 \oplus V^{-8}_0.$$  

In particular, this shows that if a Bloch–Beilinson filtration exists on $\text{CH}_0(F)$, then it must be the one considered in (1).

The action of $\varphi^*$ on $\text{CH}_1(F)_{\text{hom}}$ satisfies the equation $(\varphi^* - 4)(\varphi^* + 14) = 0$. The associated eigenspace decomposition is

$$\text{CH}_1(F)_{\text{hom}} \otimes \mathbb{Q} = V^4_1 \otimes \mathbb{Q} \oplus V^{-14}_1 \otimes \mathbb{Q}.$$  

The action of $\varphi^*$ on $\text{CH}_2(F)_{\text{hom}}$ satisfies $(\varphi^* - 4)(\varphi^* + 2) = 0$ with associated decomposition

$$\text{CH}_2(F)_{\text{hom}} = V^4_2 \oplus V^{-2}_2.$$  

In the above decomposition, $V^2_2$ is expected to be zero and $V^{-2}_2$ is shown to be the same as $\mathcal{A}_{\text{hom}}$. Furthermore we show that the above eigenspace decompositions are compatible with intersecting the hyperplane class $g$. Actually the intersection with $g$ induces isomorphisms $g : V^{-2}_2 \to V^{-14}_1 \otimes \mathbb{Q}$ and $g : V^{-14}_1 \otimes \mathbb{Q} \to V^{-8}_0$.

Finally, we construct a Chow–Künneth decomposition of the diagonal $\Delta_F \in \text{CH}_3(F \times F)_{\mathbb{Q}}$ whose associated filtration on $\text{CH}_0(F)$ coincides with (1). Moreover, we show that the vanishing of $V^4_2$ is equivalent to Murre’s conjecture (D) being true for the above decomposition.
Here is an outline of the plan of the paper. In section 1, we collect some basics about the geometry of a cubic fourfold and its variety of lines. We first define the filtration \( H \) in section 2 and then give two other characterizations of the group \( F^3 \text{CH}_0(F) \); see Proposition 2.4. The next main result in this section is Theorem 2.6, which in particular gives \( F^3 \text{CH}_0(F) \) as the image of \( A_{\text{hom}} \otimes A_{\text{hom}} \rightarrow \text{CH}_0(F) \). Section 3 is the technical part that gives a key identity involving \( \Gamma_\varphi \); see Proposition 3.4. In section 4, we study in general how a rational map acts on Chow groups; see Proposition 4.2. In section 6, we study the action \( \varphi^* \) on Chow groups and construct the decomposition of Chow groups into eigenspaces. Here we mention that Lemma 1.8 and Proposition 5.2 give the key relations. In section 7, we show some further properties of the eigenspaces obtained above. These include the non-vanishing of \( V_0^{-8} \) and \( V_0^4 \), the compatibility of \( g \) with the above decomposition and some geometric significance of the vanishing of \( V_2^4 \). In section 8, we construct a Chow–Künneth decomposition and show that it gives the filtration \( H \). In the end, we collect some notations that are used throughout the paper.

1. Basics of cubic fourfolds

In this section we set up the notations and collect basic results about the geometry of a cubic fourfolds. We will use \( X \subset \mathbb{P}(V) \) to denote a smooth cubic fourfold, where \( V \) is a 6 dimensional complex vector space. Let \( h \in \text{Pic}(X) \) be the class of a hyperplane section. We fix a basis \( \{e_0, \ldots, e_5\} \) of \( V \) and let \( \{X_0, \ldots, X_5\} \) be the dual basis of \( V^* \). Assume that \( G(X_0, \ldots, X_5) \in \text{Sym}^3(V^*) \) is the degree 3 homogeneous polynomial such that \( X \) is defined by \( G = 0 \). Associated to \( X \), we have its variety of lines \( F \subset \text{Gr}(2, V) \). We use \( g \in \text{Pic}(F) \) to denote the polarization coming from the Plücker embedding of \( \text{Gr}(2, V) \). Let \( c \in \text{CH}_2(F) \) be the second Chern class of the tautological rank 2 vector bundle restricted to \( F \). It is known that \( F \) is an irreducible holomorphic symplectic, or hyperkähler, manifold of \( K3^{[2]} \)-type; see [5]. We will use \( \omega \in H^0(F, \Omega^2_F) \) to denote a nonzero element. Then \( \omega \) is nowhere degenerate and spans \( H^0(F, \Omega^2_F) \). If \( l \subset X \) is a line contained in \( X \), then either

\[ \mathcal{M}_{l/X} \cong \mathcal{O}(1) \oplus \mathcal{O}_2, \]

in which case \( l \) is said to be of first type, or

\[ \mathcal{M}_{l/X} \cong \mathcal{O}(1)^\perp \oplus \mathcal{O}(-1), \]

in which case \( l \) is said to be of second type. It is known that for a general point \( [l] \in F \), the corresponding line \( l \) is of first type. Let \( \Sigma_2 \subset F \) be the closed subscheme of lines of second type. The following lemma was proved in [1].

**Lemma 1.1.** \( \Sigma_2 \) is a surface on \( F \) whose cycle class is \( 5(g^2 - c) \).

If a line \( l \subset X \) is of first type, then there is a unique plane \( \Pi_l \) which contains \( l \) and is tangent to \( X \) along \( l \). If \( \Pi_l \) is not contained in \( X \), then we have

\[ \Pi_l \cdot X = 2l + l' \]

for some line \( l' \). Let \( \Sigma_1 \subset F \) be the subvariety of lines contained in some linear \( \mathbb{P}^2 \subset X \). When \( X \) does not contain any plane, then \( \Sigma_1 = \emptyset \). If \( X \) contains at least a plane, then \( \Sigma_1 \) is a disjoint union of \( \mathbb{P}^2 \)'s.
**Definition 1.2** ([ES]). Let $\varphi : F \setminus (\Sigma_1 \cup \Sigma_2) \to F$ be the morphism defined by the rule $[l] \mapsto [l']$.

**Proposition 1.3** ([2]). The rational map $\varphi : F \dashrightarrow F$ has degree 16 and we have

$$\varphi^* \omega = -2\omega, \quad \varphi^* \omega^2 = 4\omega^2.$$

If $l \subset X$ is a line of second type, then there is a linear $\mathbb{P}^1 \subset \mathbb{P}(V)$ which contains $l$ and is tangent to $X$ along $l$. In particular, there is a family $\{\Pi_t = \mathbb{P}^2 : t \in \mathbb{P}^1\}$ of planes containing $l$ such that each $\Pi_t$ is tangent to $X$ along $l$. A general plane $\Pi_t$ is not contained in $X$ and hence

$$\Pi_t \cdot X = 2l + l'_t,$$

for some line $l'_t$. When $t \in \mathbb{P}^1$ varies, the point $[l'_t]$ traces out a rational curve $R_{[t]} \subset F$. When $[t] \in \Sigma_2$ varies, we get a correspondence

(2)\[ \begin{array}{c}
\Gamma_2 \\
\downarrow \\
\Sigma_2 \\
\end{array} \quad F \]

where $\Gamma_2 \to \Sigma_2$ is a $\mathbb{P}^1$-bundle (meaning all fibers are isomorphic to $\mathbb{P}^1$).

**Lemma 1.4.** Let $l \subset X$ be a line of second type, then $R_{[l]} \cdot g = 3$. In particular, we have $\Gamma_2^* g = 3[\Sigma_2]$.

**Proof.** Let $Y$ be the linear $\mathbb{P}^3$ that is tangent to $X$ along $l$. Let $B \cong \mathbb{P}^1$ be the parameter space of all planes in $M$ that contain $l$. Given $t \in B$, let $\Pi_t$ be the corresponding plane and $l'_t$ be the line such that $\Pi_t \cdot X = 2l + l'_t$. Let $Z = H_1 \cap H_2 \subset X$ be the intersection of two general hyperplanes. Then a representative of $g$ is the space of all lines meeting $Z$. The intersection $M \cap Z$ is a line which meets $X$ in three points $P_i$, $i = 1, 2, 3$. Each of these points together with $l$ determines a plane $\Pi_t$ containing $l$ and $P_i$. Let $t_i \in B$, $i = 1, 2, 3$, be the corresponding points. Then a representative of $R_{[l]} \cdot g$ is $\sum_{i=1}^3 [t'_i]$. \hfill $\square$

For any line $l \subset X$, we define

$$\mathcal{N}_{l/X}^+ = \begin{cases} 
\mathcal{O}(1) \subset \mathcal{N}_{l/X}, & [l] \notin \Sigma_2 \\
\mathcal{O}(1)^2 \subset \mathcal{N}_{l/X}, & [l] \in \Sigma_2
\end{cases}$$

When $[l] \in F \setminus \Sigma_2$ varies, the subbundle $\mathcal{N}_{l/X}^+$ glues into a subbundle of $\mathcal{N}_{P/X}|_{p^{-1}(F \setminus \Sigma_2)}$. Since $\Sigma_2 \subset F$ has codimension 2 and so does $p^{-1}\Sigma_2 \subset P$, we see that the above line bundle extends to a line bundle, denoted $\mathcal{N}_{P/X}^+$, on the whole $P$ together with an injective homomorphism

$$f^+ : \mathcal{N}_{P/X}^+ \to \mathcal{N}_{P/X}$$

which extends the natural inclusion.

**Lemma 1.5.** The following identity holds in $\text{CH}^1(P)$,

$$c_1(\mathcal{N}_{P/X}^+) = \xi + ap^*g, \quad \xi = q^*h,$$

for some $a \in \mathbb{Z}$. In particular, we have $\mathcal{N}_{P/X}^+ \cong q^*\mathcal{O}_X(1) \otimes p^*(\mathcal{E}_2)^{\otimes (a)}$.

**Proof.** This is because $\text{Pic}(P) = \mathbb{Z}\xi \oplus \mathbb{Z}p^*g$. \hfill $\square$
Remark 1.6. It will be shown in Proposition \([5,2]\) that \(a = -2\).

We recall some definitions from \([14]\). Let \(C_1, C_2 \subset X\) be two curves on \(X\). We say that \(C_1\) and \(C_2\) are well-positioned if there are only finitely many lines that meet both \(C_1\) and \(C_2\), in which case any of such lines is called a secant line of the pair \((C_1, C_2)\). The total number of secant lines (counted with multiplicity) of \((C_1, C_2)\) is equal to \(5d_1d_2\), where \(d_i = \deg(C_i), i = 1, 2\). These concepts can be naturally generalized to the case where \(C_1\) and \(C_2\) are 1-cycles on \(X\). Let \(E_i, i = 1, \ldots, N = 5d_1d_2\) be all the secant lines, then

\[
2d_2C_1 + 2d_1C_2 + \sum_{i=1}^{N} E_i = 3d_1d_2h^3
\]

holds true in \(\text{CH}_1(X)\), where \(d_i = \deg(C_i)\); see \([14]\).

We will need some results and constructions from \([19]\). It is proved in \([19]\) that \(F\) contains a surface \(W\) which represents \(c = c_2(V_2)\) in \(\text{CH}_2(F)\), where \(c_2\) is the pull-back of the tautological rank 2 bundle on \(G(2, 6)\) via the natural embedding \(F \hookrightarrow G(2, 6)\). Furthermore, any two points on \(W\) are rationally equivalent on \(F\). Hence any point on \(W\) defines a special degree 1 element \([x] \in \text{CH}_0(F)\). We summarize some results we need.

Lemma 1.7. (1) Any weighted homogeneous polynomial of degree 4 in \(g\) and \(c\) is a multiple of \([x]\) in \(\text{CH}_0(F)\). To be more precise, we have \(g^4 = 108[x], c^2 = 27[x]\) and \(g^2c = 45[x]\).

(2) Let \(C_x \subset F\) be the curve of all lines passing through a general point \(x \in X\), then \(g \cdot C_x = 6[x]\) in \(\text{CH}_0(F)\).

(3) For any 2-cycle \(\gamma \in \text{CH}_2(F)\), the intersection \(\gamma \cdot c\) is a multiple of \([x]\) in \(\text{CH}_0(F)\).

(4) If \(\Pi = \mathbb{P}^2 \subset X\) is a plane contained in \(X\), then \([l] = [x]\) for any line \(l \subset \Pi\).

(5) Let \(l \subset X\) be a line such that \([l] = [x]\), then \(3l = h^3\) in \(\text{CH}_1(X)\).

Proof. For (1), (2) and (3), we refer to \([19]\). For (4), we take \(g\) to be the lines on \(\Pi\) and then apply (3). To prove (5), we note that the class \(c^2\) can be represented by the sum of all points corresponding to lines on the intersection \(Y = H_1 \cap H_2\) of two general hyperplanes on \(X\). Since \(Y\) is a smooth cubic surface, it contains exactly 27 lines. These 27 lines can be divided into 9 groups, where each group form a “triangle”, i.e. 3 lines meeting each other. Note that each “triangle” is the class \(h^3\) and also \(c^2 = 27[x]\). Then (5) follows naturally. \(\square\)

The next result that we recall is the geometry of the surface \(S_l \subset F\) of all lines meeting a given line \(l \subset X\). Our main reference for this is \([17]\). It is known that for general \(l\) the surface \(S_l\) is smooth. We first assume that \(l\) is general. On \(S_l\) we have an involution \(\iota\) which is defined as follows. Let \([l_1] \in S_l\) which is not \([l]\) then \(\iota([l_1])\) is the residue line of \(l \cup l_1\). If \(l_1 \perp l\), then \(\iota([l]) = \varphi([l])\). There are two natural divisors on \(S_l\). The first one is \(C_x\) which is the curve all the lines passing through a point \(x \in l\). The second one is \(C_x^* = \iota(C_x)\). Then we have

\[
C_x^2 = [l], \quad (C_x^*)^2 = \iota([l]), \quad g|_{S_l} = 2C_x + C_x^*.
\]

Lemma 1.8. For any line \(l \subset X\), the equations \(g^2 \cdot S_l = \varphi([l]) - 4[l] + 24[x]\) and \(c \cdot S_l = 6[x]\) hold in \(\text{CH}_0(F)\).
Let Proposition 2.4. to show that $\Phi : CH_1 \rightarrow CH_1$ equality, we note that the composition $\Phi \circ \Psi$ is equal to $I_*$. To establish the second identity, we may assume that $l$ is general since the special case follows by taking limit. In this case, let $j : S_l \rightarrow F$ be the natural inclusion, then

$$g^2 \cdot S_l = j_* (j^* g^2) = j_* (2C_x + C_x')^2 = j_* (4l + 4C_x \cdot C_x + \epsilon([l]))$$

One notes that $j_* (2l + C_x C_x') = j_* (C_x) \cdot g = 6[\phi]$. The lemma follows easily. □

2. The filtration of $CH_0(F)$

Let $X \subset \mathbb{P}(V)$ be a smooth cubic fourfold and $F = F(X)$ the variety of lines on $X$ as before. Let

$$P \xrightarrow{q} X \xrightarrow{p} F$$

be the universal family of lines on $X$. Then we define $\Phi = p_* q^* : CH_1(X) \rightarrow CH_{i+1}(F)$ and $\Psi = q_* p^* : CH_i(F) \rightarrow CH_{i+1}(X)$ to be the induced homomorphisms of Chow groups.

**Definition 2.1.** We set

$$F^3 CH_0(F) = \ker \{ \Psi : CH_0(F)_{\text{hom}} \rightarrow CH_1(X)_{\text{hom}} \}.$$

If $l \subset X$ is a line, let $S_l \subset F$ be the space of all lines meeting the given line $l$. Then $S_l$ is always a surface and smooth if $l \subset X$ is general. The incidence correspondence $I \subset F \times F$ is defined to be

$$I = \{ (s, t) \in F \times F : s \cap t \neq \emptyset \}.$$

**Definition 2.2.** Three lines $l_1, l_2, l_3 \subset X$ form a triangle if there is a linear $\Pi = \mathbb{P}^2 \subset \mathbb{P}^5$ such that $\Pi \cdot X = l_1 + l_2 + l_3$. Each of the lines in a triangle will be called an edge of the triangle. A line $l \subset X$ is called a triple line if $(l, l, l)$ is a triangle on $X$. Let $\mathcal{R} \subset CH_0(F)$ be the subgroup generated by elements of the form $s_1 + s_2 + s_3$ where $(l_1, l_2, l_3)$ is a triangle.

**Remark 2.3.** A generic triangle $(l_1, l_2, l_3)$ on $X$ has distinct edges. It could happen that two of the lines coincide, in which case the triangle is of the form $(l, l, l')$. Note that in this case, the line $l$ can still be general. Indeed, a general line $l \subset X$ is of first type and determines a linear $\Pi = \mathbb{P}^2 \subset \mathbb{P}^5$ such that $\Pi \cdot X = 2l + l'$ where $[l'] = \varphi([l])$. Hence if $[l'] = \varphi([l])$ then $(l, l, l')$ is a triangle.

The first main result of this section is the following.

**Proposition 2.4.** Let $\mathcal{R}_{\text{hom}} \subset \mathcal{R}$ be the subgroup of all homologically trivial elements, then

$$F^3 CH_0(F) = \mathcal{R}_{\text{hom}} = \ker \{ I_* : CH_0(F)_{\text{hom}} \rightarrow CH_2(F)_{\text{hom}} \}.$$

**Proof.** The first equality follows from (2) of Lemma 2.5. To establish the second equality, we note that the composition $\Phi \circ \Psi$ is equal to $I_*$. Hence we only need to show that $\Phi : CH_1(X)_{\text{hom}} \rightarrow CH_2(F)_{\text{hom}}$ is injective. In [135, Theorem 4.7], it is shown that the composition of $\Phi$ and the restriction $CH_2(F)_{\text{hom}} \rightarrow CH_0(S_l)_{\text{hom}}$
is injective, where $l$ is a general line on $X$. Hence $\Phi$ has to be injective. It follows that

$$\ker(I_*) = \ker(\Phi \circ \Psi) = \ker \Psi,$$

which establishes the second equality. \hfill \qed

**Lemma 2.5.** (1) Let $(l_1, l_2)$ be a pair of well-positioned lines on $X$ and $E_i$, $i = 1, \ldots, 5$, the secant lines of $(l_1, l_2)$. Then we have

$$2[l_1] + 2[l_2] + \sum_{i=1}^{5} [E_i] \in \mathcal{R}.$$

(2) Let $\{\Gamma_t : t \in \mathbb{P}^1\}$ be a family of effective 1-cycles on $X$ such that

$$\Gamma_0 = \sum_{i=1}^{n} l_i + C, \quad \Gamma_\infty = \sum_{i=1}^{n} l'_i + C,$$

where $l_i$ and $l'_i$ are lines on $X$ and $C$ is some 1-dimensional cycle on $X$, then $\sum[l_i] - \sum[l'_i] \in \mathcal{R}$.

**Proof.** To prove (1), we may assume that the pair $(l_1, l_2)$ is general. This is because the special case follows from the generic case by a limit argument. Let $\Pi \cong \mathbb{P}^3$ be the linear span of $l_1$ and $l_2$. Since $(l_1, l_2)$ is general, the intersection $\Sigma = \Pi \cap X \subset \Pi \cong \mathbb{P}^3$ is a smooth cubic surface. Hence $\Sigma \cong \mathbb{B}_1(t_1, \ldots, t_6)(\mathbb{P}^2)$ is the blow-up of $\mathbb{P}^2$ at $6 \geq 27$ points. Let $R_i \subset \Sigma$, $1 \leq i \leq 6$, be the the 6 exceptional curves. Let $L_{ij} \subset \Sigma$ be the strict transform of the line on $\mathbb{P}^2$ connecting $P_i$ and $P_j$, where $1 \leq i < j \leq 6$. Let $C_i \subset \Sigma$, $1 \leq i \leq 6$, be the strict transform of the conic on $\mathbb{P}^2$ passing through all $P_j$ with $j \neq i$. The set $\{R_i, L_{ij}, C_i\}$ gives all the 27 lines on $\Sigma$. Without loss of generality, we may assume that $R_1 = l_1$ and $R_2 = l_2$. Then the set of all secant lines $\{E_i\}_{i=1}^{5}$ is explicitly given by $\{L_{12}, L_{13}, L_{14}, L_{15}, L_{16}\}$. The triangles on $\Sigma$ are always of the form $(R_i, L_{ij}, C_j)$ (here we allow $i > j$ and set $L_{ij} = L_{ji}$) or $(L_{i1j1}, L_{i2j2}, L_{i3j3})$ where $\{i1, i2, i3, j1, j2, j3\} = \{1, \ldots, 6\}$. Then we easily check that

$$2l_1 + 2l_2 + \sum_{i=1}^{5} E_i = 2R_1 + 2R_2 + L_{12} + C_3 + C_4 + C_5 + C_6$$

$$= (R_1 + L_{13} + C_3) + (R_1 + L_{14} + C_4) + (R_2 + L_{25} + C_5)$$

$$+ (R_2 + L_{26} + C_6) + (L_{12} + L_{46} + L_{35})$$

$$- (L_{13} + L_{25} + L_{46}) - (L_{14} + L_{26} + L_{35}).$$

Hence $2[l_1] + 2[l_2] + \sum [E_i] \in \mathcal{R}$.

Now we prove (2). We pick a general line $l \subset X$. For any $t \in \mathbb{P}^1$, let $\gamma_t = \sum_{i=1}^{N} s_{t,i} \in Z_0(F)$, where $\{s_{t,i} : i = 1, \ldots, N\}$ runs through all secant lines of $(l, \Gamma_t)$. Since the parameter $t$ is rational, we know that $\gamma_t$ is constant in $\text{CH}_0(F)$. In particular, $\gamma_0 = \gamma_\infty$ in $\text{CH}_0(F)$. Note that $\gamma_0$ and $\gamma_\infty$ have a common part, namely the cycle corresponding to the secant lines of $(C, l)$. Let $\{E_i : i = 1, \ldots, 5n\}$ be the set of all secant lines of $(l, \sum l_i)$ and $\{E'_i : i = 1, \ldots, 5n\}$ the set of all secant lines of $(l, \sum l'_i)$. The above argument shows that

$$\sum_{i=1}^{5n} [E_i] = \sum_{i=1}^{5n} [E'_i].$$
holds true in $\text{CH}_0(F)$. The result in (a) shows that both of
\[2 \sum_{i=1}^{n} [l_i] + 2n[l] + \sum_{i=1}^{5n} [E_i] \text{ and } 2 \sum_{i=1}^{n} [l'_i] + 2n[l] + \sum_{i=1}^{5n} [E'_i]\]
are in $\mathcal{R}$. Hence
\[2(\sum_{i=1}^{n} [l_i] - \sum_{i=1}^{n} [l'_i]) = (2 \sum_{i=1}^{n} [l_i] + 2n[l] + \sum_{i=1}^{5n} [E_i]) - (2 \sum_{i=1}^{n} [l'_i] + 2n[l] + \sum_{i=1}^{5n} [E'_i]) \in \mathcal{R}_{\text{hom}}.
\]
If we can prove that $\mathcal{R}_{\text{hom}}$ is divisible, then we get $\sum_{i=1}^{n} [l_i] - \sum_{i=1}^{n} [l'_i] \in \mathcal{R}_{\text{hom}}$. Let $\hat{R}$ be the (desingularized and compactified) moduli space of triangles on $X$, then we have a surjection $\text{CH}_0(\hat{R})_{\text{hom}} \to \mathcal{R}_{\text{hom}}$. It is a standard fact that $\text{CH}_0(\hat{R})_{\text{hom}}$ is divisible. Hence so is $\mathcal{R}_{\text{hom}}$. This finishes the proof. \hfill \Box

The second main result of this section is statement (3) in the following theorem.

**Theorem 2.6.** (1) The natural homomorphism
\[\text{CH}_2(F) \otimes \text{CH}_2(F)_{\text{hom}} \to \text{CH}_0(F)_{\text{hom}}
\]
is surjective.
(2) The natural homomorphism
\[\text{CH}_2(F) \otimes \text{CH}_2(F) \to \text{CH}_0(F)
\]
is also surjective.
(3) Let $\mathcal{A} \subset \text{CH}_2(F)$ be the subgroup generated by $S_i$, $[l] \in F$. Then the image of the natural homomorphism
\[\mathcal{A}_{\text{hom}} \otimes \mathcal{A}_{\text{hom}} \to \text{CH}_0(F)_{\text{hom}}
\]
is equal to $\mathbb{F}^3 \text{CH}_0(F)$.

**Proposition 2.7.** Let $(l_1, l_2, l_3)$ be a triangle, then the following are true.
(1) The identity $S_i \cdot S_k = 6[\mathcal{O}] + [l_3] - [l_1] - [l_2]$ holds true in $\text{CH}_0(F)$.
(2) If $(l_1', l_2', l_3')$ is another triangle, then
\[6(\sum_{i=1}^{n} [l_i] - 3 [l'_i]) = \sum_{1 \leq i < j \leq 3} (S_i - S_j)^2 - \sum_{1 \leq i < j \leq 3} (S'_i - S'_j)^2
\]
holds true in $\text{CH}_0(F)$.
(3) Let $s \in F$ such that $l_s$ is of first type and let $s' = \varphi(s)$. Then
\[S_{l_s} \cdot S_{l_{s'}} = 6[\mathcal{O}] - s'
\]
holds true in $\text{CH}_0(F)$.
(4) Let $l \subset X$ be a line of first type, then we have
\[(S_l)^2 = 6[\mathcal{O}] + \varphi([l]) - 2[l]
\]
holds true in $\text{CH}_0(F)$.
(5) If $l \subset X$ is a triple line, then $(S_l)^2 = 6[\mathcal{O}] - [l]$.

**Proof:** We first prove (1) for the case where all the edges of the triangle are distinct. Then the general case follows by a simple limit argument. Take a 1-dimensional family of lines $\{l_t : t \in T\}$ such that $l_{t_1} = l_1$. We may assume that $(l_{t_1}, l_2)$ is well-positioned when $t \neq t_1$. Let $\{E_{t,i} : i = 1, \ldots, 5\}$ be the secant lines of the pair $(l_{t_1}, l_2)$ for $t \neq t_1$. The tangent space $T_{t_1}(T)$ determines a section of $\mathbb{H}^0(l_1, \mathcal{N}_{l_1/X})$.
I. Remark 2.8.

Hence the image of the intersection of 2-cycles hits a 0-cycle of degree 1.

\[ S_{i_1} \cdot S_{i_2} = 5 \sum_{i=1}^{5} |E_{i,i}|. \]

Let \( t \to t_1 \) and take the limit, we have

\[ S_{i_1} \cdot S_{i_2} = 4 \sum_{i=1}^{4} |L_i| + |l_3|. \]

We combine this with \([L_1] + \cdots + [L_4] + [l_1] + [l_2] = 6\alpha\] and deduce (1). Let \((l_1, l_2, l_3)\) be a triangle, then we have

\[
\sum_{1 \leq i < j \leq 3} (S_{i_1} - S_{i_2})^2 = 2 \left( \sum_{1 \leq i \leq 3} S_{i_1} \right)^2 - 6 \sum_{1 \leq i < j \leq 3} S_{i_1} \cdot S_{i_2} \\
= 2(\Phi(h^3))^2 + 6 \sum_{1 \leq i < 3} |l_i| - 108\alpha
\]

Then (2) follows easily from this computation.

Statements (3), (4) and (5) are all special cases of (1).

\( \square \)

Proof of Theorem 2.6. To prove (1), we only need to show that \([l_1] - [l_2]\) is in the image of the map. Let \( l' \) be a line meeting both \( l_1 \) and \( l_2 \), then \([l_1] - [l_2] = ([l_1] - [l']) + ([l'] - [l_2])\). Hence we may assume that \( l_1 \) meets \( l_2 \). We may further assume that \((l_1, l_2)\) is general. Since \( \text{CH}_0(F)_{\text{hom}} \) is uniquely divisible (see Lemma 6.5), we only need to show that \( 2([l_1] - [l_2]) \) is in the image. Now let \( l_3 \) be the residue line of \( l_1 \cup l_2 \) and hence \((l_1, l_2, l_3)\) form a triangle. Then

\[
2([l_1] - [l_2]) = (S_{l_2} - S_{l_1}) \cdot S_{l_3},
\]

is in the image of \( \text{CH}_2(F) \otimes \text{CH}_2(F)_{\text{hom}} \to \text{CH}_0(F)_{\text{hom}} \).

To prove (2), we only need to show that the image of \( \text{CH}_2(F) \otimes \text{CH}_2(F) \to \text{CH}_0(F) \) contains a cycle of degree 1. But this is trivial since it follows from Lemma 1.7 and Proposition 2.7 that

\[
g^2 \cdot g^2 = 108\alpha, \quad \text{deg}(S_{i_1} \cdot S_{i_2}) = 5.
\]

Hence the image of the intersection of 2-cycles hits a 0-cycle of degree 1.

(3) is an easy consequence of (2) of Proposition 2.7.

\( \square \)

Remark 2.8. In [19], C. Voisin established the following interesting identity in \( \text{CH}^4(F \times F) \),

\[
I^2 = \alpha \Delta_F + I \cdot \Gamma_1(g_1, g_2) + \Gamma_2(g_1, g_2, c_1, c_2),
\]

where \( \Gamma_1(g_1, g_2) \) is a homogeneous polynomial of degree 2 and \( \Gamma_2(g_1, g_2, c_1, c_2) \) is a weighted homogeneous polynomial of degree 4. It is known that the constant \( \alpha \), which equals to the action of \((I^2)^*\) on \( \text{H}^{4,0}(F) \), is nonzero. Since \((I^2)^*|l| = (S_i)^2\), the statement (d) of the above proposition shows that \((I^2)^* = \varphi^* - 2 = 2\) on \( \text{H}^{4,0}(F) \). This implies that \( \alpha = 2 \). Furthermore, if we carry out the computation in [19] explicitly, we get \( \Gamma_1 = g_1^2 + g_1 g_2 + g_2^2 \).
Remark 2.9. The identity (3) also gives a straightforward proof of statements (1) and (2) of Theorem 2.6. However, our proofs also work for the case of cubic threefolds as follows. If \( X \) is a cubic threefold, then its variety of lines, \( S \), is a smooth surface. Then \( F^2\text{CH}_0(S) \), the Albanese kernel, is the same as \( \ker\{ \Psi : \text{CH}_0(S) \to CH_1(X) \} \). As in Proposition 2.7, \( F^2\text{CH}_0(S) \) is identified with \( R_{\text{hom}} \). Statements (1) and (2) of Proposition 2.7 are true in the following sense. In this case \( S_i \), the space of all lines meeting a given line \( l \), is a curve. The constant class \( 6\sigma \) in statement (1) should be replaced by the class of all lines passing through a general point of \( X \). Then the statement (3) of Theorem 2.6 reads as 

\[
\text{Pic}^0(S) \otimes \text{Pic}^0(S) \to F^2\text{CH}_0(S) \text{ being surjective. This result concerning the Fano scheme of lines on a smooth cubic threefold already appears in [7].}
\]

3. The Graph of \( \varphi \) as a Correspondence

Let \( \varphi : F \to F \) be the rational map introduced in Definition 1.2. Let \( \Gamma_{\varphi} \subset F \times F \) be the closure of the graph of \( \varphi \). Hence a general point \( ([l_1], [l_2]) \in \Gamma_{\varphi} \) is a pair of lines on \( X \) such that there is a linear \( F^2 \subset P^5 \) such that \( F^2 \cdot X = 2l_1 + l_2 \). Our next goal is to study the correspondence \( \Gamma_{\varphi} \) in more details.

Now we introduce more natural cycles on \( F \times F \). Let \( \Gamma_h \subset F \times F \) be the 5-dimensional cycle of all points \( ([l_1], [l_2]) \) such that \( x \in l_1 \cap l_2 \) for some point \( x \) on a given hyperplane \( H \subset X \). Let \( \Gamma_{h_2} \subset F \times F \) be the 4-dimensional cycle of all points \( ([l_1], [l_2]) \) such that \( x \in l_1 \cap l_2 \) for some \( x \) on an intersection \( H_1 \cap H_2 \subset X \) of two general hyperplanes. Let \( I_1 = \{([l_1], [l_2]) \in F \times F \} \) where \( l_1 \cup l_2 \subset \Pi \) for some plane \( \Pi = F^2 \subset X \). Note that for general \( X \), we have \( I_1 = 0 \). When \( I_1 \neq 0 \), it is a disjoint union of \( F^2 \times F^2 \) and the number of such copies equals the number of planes contained in \( X \). Let \( g = -c_1(\mathcal{E}_2) \) and \( c = c_2(\mathcal{E}_2) \) where \( \mathcal{E}_2 \subset V \) is the natural rank two subbundle on \( F \). Let \( g_i = p_i^*g \) and \( c_i = p_i^*c \), where \( p_i : F \times F \to F \) is the projection to the \( i \)-th factor, \( i = 1, 2 \). Now we can state the main result of this section.

Proposition 3.1. There exists a 4-dimensional cycle \( I_2 \) supported on \( \Sigma_2 \times F \cap I \) such that following equation holds true in \( CH^4(F \times F) \),

\[
\Gamma_{\varphi} + I_1 + I_2 = 4\Delta_F + ((a^2 - a + 1)g_1^2 + (1 - a)g_1g_2 + g_2^2) \cdot I \\
+ ((3a - 4)g_1 - 4g_2)\Gamma_h + 8\Gamma_{h_2}
\]

where \( a \in \mathbb{Z} \) as in Lemma 1.3. Furthermore, we can write \( (3a - 4)g_1 - 4g_2)\Gamma_h + 8\Gamma_{h_2} = \Gamma'_2(g_1, g_2, c_1, c_2) \) as a weighted homogeneous polynomial of degree 4.

Our proof uses ideas from Voisin’s proof of [19] Proposition 3.3. The new ingredient we input is a section of some vector bundle whose vanishing gives the correspondence \( \Gamma_{\varphi} \). Meanwhile we are able to compute the Chern classes of that vector bundle and hence get the above equation involving \( \Gamma_{\varphi} \). Consider the following diagram

\[
P \times P \xrightarrow{(q,q)} X \times X \\
\downarrow{(p,p)} \\
F \times F
\]

Let \( \bar{I} = (q,q)^{-1}\Delta_X \subset P \times P \) and \( \sigma : \bar{I} \to I \) be the natural morphism. Let 

\[
\pi = pr_1 \circ ((q,q)|_\bar{I}) = pr_2 \circ ((q,q)|_\bar{I}) : \bar{I} \to X.
\]

We set \( Z = \Delta_F \subset I \) and \( E = \sigma^{-1}Z \)
Lemma 3.2. The subscheme $I_0 \subset F \times F$ is a local complete intersection.

Proof. This is shown in [19] (the proof of Proposition 3.3).

On $\tilde{I}_0$, we have the following short exact sequence of vector bundles,

\[
0 \longrightarrow T_{P \times F / F \times F} \longrightarrow \mathcal{N}_{I_0 / P \times P} = \pi^* T_X \longrightarrow \sigma^* \mathcal{N}_{I_0 / F \times F} \longrightarrow 0,
\]

where each term is understood to be its restriction to $\tilde{I}_0$.

Lemma 3.3. The Chern classes of $\sigma^* \mathcal{N}_{I_0 / F \times F}$ are given by

\[
\begin{align*}
c^1(\mathcal{N}_{I_0 / F \times F}) &= \sigma^*(g_1 + g_2)|_I - \pi^* h, \\
c^2(\mathcal{N}_{I_0 / F \times F}) &= 6\pi^* h^2 + \sigma^*(g_1^2 + g_1 g_2 + g_2^2)|_I - 3\pi^* h \cdot \sigma^*(g_1 + g_2)|_I,
\end{align*}
\]

where everything is restricted to $\tilde{I}_0$.

Proof. Note that on $P$, we have

\[
c_1(T_{P / F}) = 2q^* h - p^* g.
\]

By taking Chern classes, the lemma follows from the exact sequence [4].

When $l$ is of first type, we have $\mathcal{N}_{l/X}^+ \cong \mathcal{O}(1)$. We recall from Section 1 that as $[l]$ varies in $F \setminus \Sigma_2$, the positive part $\mathcal{N}_{l/X}^+$ glues together to give a line bundle on $p^{-1}(F \setminus \Sigma_2)$, which further extends to a line bundle $\mathcal{N}_{P/X}^+$ on the whole space $P$ together with a natural injective homomorphism

\[
f^+ : \mathcal{N}_{P/X}^+ \rightarrow \mathcal{N}_{P/X}.
\]

In the exact sequence [1], the kernel of $\pi^* T_X \rightarrow \sigma^* \mathcal{N}_{I_0 / F \times F}$ contains $\tilde{p}_1^* T_{P / F}$. It follows that there is a natural homomorphism

\[
\beta' : \tilde{p}_1^* \mathcal{M}_{P/X} |_{I_0} \rightarrow \sigma^* \mathcal{N}_{I_0 / F \times F} |_{I_0}.
\]

If we compose $\beta'$ with $f^+$, we get

\[
\beta : \tilde{p}_1^* \mathcal{M}_{P/X} |_{I_0} \rightarrow \sigma^* \mathcal{N}_{I_0 / F \times F} |_{I_0}.
\]

Then $\beta$ can be viewed as an element of $H^0(\tilde{I}_0, F)$ where $F = \tilde{p}_1^* (\mathcal{M}_{P/X}^+)^\vee \otimes \sigma^* \mathcal{N}_{I_0 / F \times F}$. Let $y = \sigma^{-1}([l_1], [l_2]) \in \tilde{I}_0$ be a point, then $\beta_y = 0$ if and only if $T_{l_1} l_2$ is pointing in the direction of $\mathcal{N}_{l_1/X}^+$. This is equivalent to saying that $[l_2] = \varphi([l_1])$ if $[l_2] \not\in \Sigma_1 \cup \Sigma_2$. If $[l_1] \in \Sigma_1$, then the condition becomes that $l_1$ and $l_2$ lie on the same plane contained in $X$. Hence we see that

\[
j_* \sigma_*(\beta = 0) = \Gamma_x + I_1.
\]
where \( j : I \to F \times F \) is the natural inclusion. Meanwhile, the class of \((\beta = 0)\) is equal to

\[
c_2(\mathcal{F}) = (c_1(\mathcal{P}^+_I/\mathcal{X}))^2 - (c_1(\mathcal{P}^+_I/\mathcal{X}))c_1(\sigma^*\mathcal{M}_I/\mathcal{F} \times \mathcal{F}) + c_2(\sigma^*\mathcal{M}_I/\mathcal{F} \times \mathcal{F})
\]

\[
= \sigma^*c_2(\mathcal{M}_I/\mathcal{F} \times \mathcal{F}) + (\mathcal{P}^+_I)(ap^*g + q^*h)|_{\mathcal{I}_0})^2
- (\mathcal{P}^+_I)(ap^*g + q^*h)|_{\mathcal{I}_0}(\sigma^*(g_1 + g_2)|_{\mathcal{I}} - \pi^*h)
= 6\pi^*h^2 - 3h(\tilde{g}_1 + \tilde{g}_2) + \tilde{g}_1^2 + \tilde{g}_2^2 + (a\tilde{g}_1 + \pi^*h)^2
- (a\tilde{g}_1 + \pi^*h)(\tilde{g}_1 + \tilde{g}_2 - \pi^*h),
\]

\[
= (a^2 - a + 1)\tilde{g}_1^2 + (1 - a)\tilde{g}_1\tilde{g}_2 + \tilde{g}_2^2 + \pi^*h((3a - 4)\tilde{g}_1 - 4\tilde{g}_2) + 8\pi^*h^2
\]

where all terms are understood to be the restriction to \( \tilde{I}_0 \) and \( a \) is the integer appearing in Lemma \([1, \alpha] \). We apply \( j_*\sigma_* \) to the above equation and get

\[
(6) \quad \Gamma_\varphi + I_1 = ((a^2 - a + 1)\tilde{g}_1^2 + (1 - a)\tilde{g}_1\tilde{g}_2 + \tilde{g}_2^2) \cdot I + ((3a - 4)\tilde{g}_1 - 4\tilde{g}_2)\Gamma_h + 8\Gamma_h^2
\]

which holds in \( CH^4(F \times F \setminus (\Delta_F \cup I \cap \Sigma_2 \times F)) \).

If \( L_1 \) is a linear form in \( (\tilde{g}_1, \tilde{g}_2, h) \), then by an argument from the proof of \([19, \text{Proposition 3.3}] \), we get \( j_*\sigma_*(\pi^*h \cdot L_1) = \Gamma'_2(g_1, g_2, c_1, c_2) \) is a weighted homogeneous polynomial of degree 4. Hence we see that

\[
(3a - 4)g_1 - 4g_2\Gamma_h + 8\Gamma_h^2 = \Gamma'_2(g_1, g_2, c_1, c_2)
\]

is a weighted homogeneous polynomial of degree 4.

**Proof of Proposition \([\Sigma, \tau] \)** By localization sequence of Chow groups, we get the following equation in \( CH^4(F \times F) \),

\[
\Gamma_\varphi + I_1 + I_2 = \alpha'\Delta_F + ((a^2 - a + 1)g_1^2 + (1 - a)g_1g_2 + g_2^2) \cdot I + ((3a - 4)g_1 - 4g_2)\Gamma_h + 8\Gamma_h^2,
\]

where \( \alpha' \) is a constant and \( I_2 \) is supported on \( I \cap \Sigma_2 \times F \). One also sees that \( \alpha' \) is equal to the action of \( \Gamma_\varphi \) on \( H^4,0(F) \). Hence \( \alpha' = 4 \) by Proposition \([\Sigma, \tau] \). \( \square \)

4. Rational map and Chow groups

The notations of this section are independent of the rest of the paper. We consider \( \varphi : X \dashrightarrow Y \) a rational map between projective varieties defined over a field \( k \) of characteristic zero. Let

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\pi \downarrow & & \downarrow \varphi' \\
\tilde{X} & \cong & Y
\end{array}
\]

be a resolution of \( \varphi \), that is \( \pi : \tilde{X} \to X \) is a projective birational morphism such that \( \varphi \) extends to a morphism \( \varphi' : \tilde{X} \to Y \). Note that no smoothness assumption is required on \( \tilde{X} \).

Assume that \( X \) is smooth. Then we define a map \( \varphi_* : CH^*_I(X) \to CH^*_I(Y) \) by the formula

\[
\varphi_*a := \varphi'_*\pi^*a \text{ for all } a \in CH^*_I(X).
\]

Here, \( \pi^* : CH^*_I(X) \to CH^*_I(\tilde{X}) \) is the pullback map as defined in \([10, \S 8] \).
Lemma 4.1. Assume that $X$ is smooth. The map $\varphi_* : \mathrm{CH}_l(X) \to \mathrm{CH}_l(Y)$ defined above does not depend on a choice of resolution for $\varphi$.

Proof. Let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\varphi_1} & Y \\
\pi_1 \downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & Y \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X_2 & \xrightarrow{\varphi_2} & Y \\
\pi_2 \downarrow & & \downarrow \\
X & \xrightarrow{\varphi} & Y \\
\end{array}
\]

be two resolutions of $\varphi$. There is then a projective variety $\bar{X}$ and birational morphisms $\bar{\pi}_1 : \bar{X} \to X_1$ and $\bar{\pi}_2 : \bar{X} \to X_2$ such that both $\varphi_1$ and $\varphi_2$ extend to a morphism $\bar{\varphi} : \bar{X} \to Y$. It is then enough to see that

\[\bar{\varphi}_*(\pi_1 \circ \bar{\pi}_1)^* = (\varphi_1)_*(\pi_1)^* : \mathrm{CH}_l(X) \to \mathrm{CH}_l(Y).\]

The identity $\varphi_1 \circ \bar{\pi}_1 = \bar{\varphi}$ gives $\bar{\varphi}_*(\pi_1 \circ \bar{\pi}_1)^* = (\varphi_1)_*(\bar{\pi}_1)^* (\pi_1 \circ \bar{\pi}_1)^*$. We may then conclude by the projection formula [10, Proposition 8.1.1] which implies that $(\bar{\pi}_1)^* (\pi_1 \circ \bar{\pi}_1)^* = \pi_1^*$. \hfill $\square$

If $Y$, instead of $X$, is assumed to be smooth, then we define a map $\varphi^* : \mathrm{CH}^l(Y) \to \mathrm{CH}^l(X)$ by the formula

\[\varphi^* b := \pi_* \bar{\varphi}^* b \text{ for all } b \in \mathrm{CH}^l(Y).\]

Likewise, we have

Lemma 4.2. Assume that $Y$ is smooth. The map $\varphi^* : \mathrm{CH}^l(Y) \to \mathrm{CH}^l(X)$ defined above does not depend on a choice of resolution for $\varphi$. \hfill $\square$

Let now $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the natural projections, and let $\Gamma_{\varphi} \subset X \times Y$ be the closure of the graph of $\varphi$, given with the natural projections

\[
\begin{array}{ccc}
\Gamma_{\varphi} & \xrightarrow{\eta} & Y \\
p \downarrow & & \\
X
\end{array}
\]

The morphism $p$ is birational and the morphism $q$ clearly extends $\varphi$. Therefore, as a consequence of Lemmas 4.1 and 4.2 we get

Lemma 4.3. If $X$ is smooth, then $\varphi_* a = q_* p^* a$ for all $a \in \mathrm{CH}_l(X)$. If $Y$ is smooth, then $\varphi^* b = p_* q^* b$ for all $b \in \mathrm{CH}^l(Y)$. If $X$ and $Y$ are both smooth, then

\[\varphi_* a = (\Gamma_{\varphi})_* a := (p_Y)_* (\Gamma_{\varphi} \cdot p_X^* a) \text{ and } \varphi^* b = (\Gamma_{\varphi})^* b := (p_X)_* (\Gamma_{\varphi} \cdot p_Y^* b).\] \hfill $\square$

We now want to understand, when $X$ and $Y$ are both smooth, to which extent $\varphi^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ is compatible with intersection product.

Lemma 4.4. Let $\pi : \bar{X} \to X$ be a dominant morphism between smooth projective varieties and let $x$ and $y$ be two cycles in $\mathrm{CH}_*(\bar{X})$. Then

\[\pi_*(x \cdot y) = \pi_* x \cdot \pi_* y + \pi_* ((x - \pi^* \pi_* x) \cdot (y - \pi^* \pi_* y)).\]

Proof. Let's define $x' := x - \pi^* \pi_* x$ and $y' := y - \pi^* \pi_* y$. 

By the projection formula, $\pi_* \pi^*$ acts as the identity on $\text{CH}_*(\tilde{X})$. Therefore

$$\pi_* x' = 0$$ and $$\pi_* y' = 0.$$

The projection formula gives

$$\pi_*(\pi^* x \cdot y') = \pi_* x \cdot \pi_* y' = 0$$ and $$\pi_*(x' \cdot (\pi^* y)) = \pi_* x' \cdot \pi_* y = 0.$$

This yields

$$\pi_* (x \cdot y) = \pi_*(\pi^* x \cdot \pi^* y) + \pi_*(x' \cdot y')$$

$$= \pi_* x \cdot \pi_* y + \pi_* (x' \cdot y').$$

□

For a smooth projective variety $X$ over $k$, we write

$$T^2(X) := \ker\{\text{AJ}^2 : \text{CH}^2(X)_{\text{hom}} \to J^2(X)\}$$

for the kernel of Griffiths’ second Abel-Jacobi map to the second intermediate Jacobian $J^2(X)$ which is a quotient of $H^3(X, \mathbb{C})$. The group $T^2(X)$ is a birational invariant of smooth projective varieties. Precisely, we have

**Lemma 4.5.** Let $\pi : \tilde{X} \to X$ be a birational map between smooth projective varieties. Then $\pi^* \pi_*$ acts as the identity on $\text{CH}_0(\tilde{X})$, $\text{Griff}_1(\tilde{X})$, $\text{Griff}^2(\tilde{X})$, $T^2(\tilde{X})$, $\text{CH}^1(\tilde{X})_{\text{hom}}$ and $\text{CH}^0(\tilde{X})$.

**Proof.** This is proved in [16] Proposition 5.4 in the case when $\pi$ is a birational morphism. In the general case, by resolution of singularities, there is a smooth projective variety $Y$ and birational morphisms $f : Y \to \tilde{X}$ and $\pi : \tilde{X} \to X$ such that $\pi = \pi \circ f$. By definition of the action of rational maps on Chow groups, we have $\pi^* \pi_* = f_* \pi^* \pi_* f^*$ and the proof of the lemma reduces to the case of birational morphisms. □

**Proposition 4.6.** Let $\varphi : X \to Y$ be a rational map between smooth projective varieties. Let $a \in T^2(Y)$ be an Abel-Jacobi trivial cycle and let $b \in \text{CH}^*(Y)$ be any cycle. Then

$$\varphi^* (a \cdot b) = \varphi^* a \cdot \varphi^* b \in \text{CH}^*(X).$$

**Proof.** Let

$$\xymatrix{ \tilde{X} \ar[dr]_{\tilde{\varphi}} & \\
X \ar[r]_{\varphi} & Y}
$$

be a resolution of $\varphi$. For $x \in T^2(\tilde{X})$, we have by Lemma 4.5 $x = \pi^* \pi_* x$, so that Lemma 4.4 yields

$$\pi_* (x \cdot y) = \pi_* x \cdot \pi_* y. \tag{7}$$

If we now set $x = \tilde{\varphi}^* a$, then $x$ defines a cycle in $T^2(\tilde{X})$ by functoriality of the Abel-Jacobi map with respect to the action of correspondences, and (7) gives:

$$\varphi^* a \cdot \varphi^* b = \pi_* (\tilde{\varphi}^* a \cdot \tilde{\varphi}^* b)$$

$$= \pi_* (\tilde{\varphi}^* a \cdot \tilde{\varphi}^* b)$$

$$= \pi_* \tilde{\varphi}^* (a \cdot b)$$

$$= \varphi^* (a \cdot b).$$
Remark 4.7. The proof of Proposition 4.6 actually shows that $\varphi^*(a \cdot b) = \varphi^*a \cdot \varphi^*b \in \operatorname{CH}^*(X)$ for all $b \in \operatorname{CH}_1(Y)$ and all cycle $a$ that belongs to one of the groups $\operatorname{Griff}^2(Y)$, $T^2(Y), \operatorname{CH}^1(Y)_{\text{hom}}$ and $\operatorname{CH}^0(Y)$.

Proposition 4.8. Let $\varphi : X \dashrightarrow Y$ be a rational map between smooth projective varieties. Let $Z$ be a closed subvariety of $X$ of codimension at most 2 such that $\varphi|_{X-Z}$ defines a morphism. Let $a, b \in \operatorname{CH}^1(Y)$ be two divisors. If $\operatorname{codim}_X Z > 2$, then

$$\varphi^*(a \cdot b) = \varphi^*a \cdot \varphi^*b.$$ 

If $\operatorname{codim}_X Z = 2$ and if $Z_1, \ldots, Z_n$ are the irreducible components of $Z$ of codimension 2, then there are rational numbers $a_1, \ldots, a_i$ that depend on $a$ and $b$ such that

$$\varphi^*(a \cdot b) = \varphi^*a \cdot \varphi^*b + \sum_{i=1}^n a_i[Z_i].$$

Proof. Assume first that a resolution of $\varphi$ is given by blowing up a smooth connected closed subvariety $Z$ of $X$. We thus consider a resolution

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{\varphi}} & Y \\
\pi \downarrow & & \downarrow \varphi \\
X - Z & \xrightarrow{\psi} & Y
\end{array}$$

of $\varphi$, where $\pi : \tilde{X} \to X$ is the blow-up of $X$ along a smooth closed subvariety $Z$. We have the following fiber square

$$\begin{array}{ccc}
D & \xrightarrow{j} & \tilde{X} \\
\pi_2 \downarrow & & \downarrow \pi \\
Z & \xrightarrow{j'} & X
\end{array}$$

where $\pi_2 : D \to Z$ is the projectivization of the normal bundle of $Z$ inside $X$, that is $D = \mathbb{P}(N_{Z/X})$, and where $j : D \to \tilde{X}$ is the inclusion of the exceptional divisor. The blow-up formula for Chow groups ensures the existence of cycles $x'', y'' \in \operatorname{CH}^0(Z)$ such that

$$x' := x - \pi^*\pi_2 x = j_*\pi^*_Z x'' \quad \text{and} \quad y' := y - \pi^*\pi_2 y = j_*\pi^*_Z y''.$$

We may now compute

$$\begin{align*}
\pi_*(x' \cdot y') & = \pi_*(j_*\pi^*_Z x'' \cdot j_*\pi^*_Z y'') \\
& = \pi_*j_*((\pi^*_Z x'') \cdot j_*\pi^*_Z y'') \\
& = j'_*(\pi_Z)_*(\pi^*_Z x'' \cdot j_*\pi^*_Z y'') \\
& = j'_*(x'' \cdot (\pi_Z)_* j_*\pi^*_Z y'') \\
& = \begin{cases}
0 & \text{if } \operatorname{codim}_X Z > 2; \\
-j'_*(x'' \cdot y'') & \text{if } \operatorname{codim}_X Z = 2.
\end{cases}
\end{align*}$$

The second and fourth equalities follow from the projection formula and the third equality follows simply by commutativity of the diagram (8). As for the fifth equality, the map $j_*j^* : \operatorname{CH}_s(D) \to \operatorname{CH}_{s-1}(D)$ is given by intersecting with $-h$, where
Lemma 5.1.\degree 4 and \( \Gamma = (\pi_Z)_* \cdot \) to get explicit descriptions of 

The main goal of this section is to explore how the results in Section 3 can be used 

In particular, we have 

Recall that we have a natural self rational map 

We may then conclude, when \( \Gamma \) is resolved by a smooth blow-up, by noting that 

In the general case, the rational map \( \varphi \) can be resolved by successively blowing up smooth closed subvarieties of \( X \). We may then conclude by induction on the number of blow-ups necessary to resolve \( \varphi \).

\[ x' = (\pi_Z)_* \cdot j^* x \text{ and } y'' = (\pi_Z)_* \cdot j^* y \]

\[ \varphi^*(a \cdot b) = \begin{cases} \varphi^*a \cdot \varphi^*b & \text{if } \text{codim}_X Z > 2; \\ \varphi^*a \cdot \varphi^*b - j^*_1((\pi_Z)_* \cdot j^* \varphi^*a \cdot (\pi_Z)_* \cdot j^* \varphi^*b) & \text{if } \text{codim}_X Z = 2. \end{cases} \]

\[
\begin{align*}
\Gamma_*[l] &= a_{22}(\varphi_*[l] - 4[l] + 24[a]), \\
\Gamma^*[l] &= a_{11}(\varphi_*[l] - 4[l] + 24[a]).
\end{align*}
\]

In particular, we have \( \Gamma_* = a_{22}(\varphi_* - 4) \) and \( \Gamma^* = a_{11}(\varphi_* - 4) \) on \( \text{CH}_0(F)_{\hom} \).

(3) Let \( a \in \text{CH}_1(F)_{\hom} \), then 

\[ \Gamma_* a = \Gamma^* a = a_{122} \cdot I_*(g \cdot a). \]

(4) Let \( a \in \text{CH}_2(F)_{\hom} \), then 

\[ \Gamma_* a = a_{11} I_*(g^2 \cdot a), \quad \Gamma^* a = a_{22} I_*(g \cdot a). \]

Proof. For any \( a \in \text{CH}_1^*(F)_{\hom} \), \((\Gamma'_0)_* a \) (and also \((\Gamma'_2)_* a \)) is a weighted homogeneous polynomial in \( c \) and \( g \) which is homologically trivial. By the result of [19], the polynomial is already 0 in Chow group. This proves (1).

To prove (2), we note that \( \Gamma_*[l] = a_{22}g^2 \cdot S_i \). Then the result follows from Lemma 1.8.

Let \( a \in \text{CH}_1(F)_{\hom} \). Then \((g^2 \cdot I)_* a = 0 \) for dimension reasons. Note that 

To prove (4), we take \( a \in \text{CH}_2(F)_{\hom} \). Then \((g_1 g_2 \cdot I)_* a = g \cdot I_*(a \cdot g) = 0 \) since \( I_*(a \cdot g) \in \text{CH}_1^*(F)_{\hom} \). Similarly, \((g^2 \cdot I)_* a = 0 \). Hence (4) follows.
Proposition 5.2. There exists an integer $a$ and a correspondence $I_2 \in \text{CH}^4(F \times F)$ which is supported on $\Sigma_2 \times F \cap I$ such that the following are true.

1. The action of $\varphi^*$ on homologically trivial cycles can be described by
   \[
   \varphi^* a = (a^2 - a + 1)\varphi_* a + 4(a - a^2)\alpha - I_2 a, \quad a \in \text{CH}_0(F)_{\text{hom}}
   \]
   \[
   \varphi^* a = 4a + (1 - a)g \cdot I_* (g \cdot a), \quad a \in \text{CH}_1(F)_{\text{hom}}
   \]
   \[
   \varphi^* a = 4a + I_* (g^2 \cdot a), \quad a \in \text{CH}_2(F)_{\text{hom}}
   \]

2. The action of $\varphi_*$ on homologically trivial cycles can be described by
   \[
   \varphi_* a = 4a + (1 - a)g \cdot I_* (g \cdot a), \quad a \in \text{CH}_1(F)_{\text{hom}}
   \]
   \[
   \varphi_* a = 4a + (a^2 - a + 1)I_* (g^2 \cdot a) - (I_2)_a, \quad a \in \text{CH}_2(F)_{\text{hom}}
   \]

3. On the divisor $g$, we have $\varphi^* g = (1 - 3a)g$. In particular, we have $a = -2$.

Proof. In Proposition 5.1 it is easy to see that $I_1$ acts trivially on homologically trivial cycles. Then (1) and (2) are direct applications of Proposition 5.1 Lemma 5.1 and Lemma 5.2. To prove (3), we first note that $I_1^* g = 0$ and $I_2^* g = 0$ since they factor through $\text{CH}_3$ of a surface. We then apply the identity in Proposition 5.1 and Lemma 5.3 and get
   \[
   \varphi^* g = 8g + (1 - a)g \cdot I_* (g^2 + 3a - 4)g \cdot (\Gamma_h)^* g - 4(\Gamma_h)^* g^2 + 8(\Gamma_h^2)^* g
   \]
   \[
   = 4g + 21(1 - a)g + 36g + 3a - 4)(6g - 4 \cdot 21g + 8 \cdot 6g
   \]
   One simplifies the above expression and gets $\varphi^* g = (1 - 3a)g$. Finally, Amerik [11] Theorem 8 showed that $\varphi^* g = 7g$. Thus $a = -2$. □

Remark 5.3. We will see (Remark 6.9 following Theorem 6.8) that $I_2^* a = (a^2 - a + 1)I_* (g^2 \cdot a)$ for all $a \in \text{CH}_2(F)_{\text{hom}}$. Therefore,
   \[
   \varphi_* a = 4a + 2I_* (g^2 \cdot a), \quad a \in \text{CH}_2(F)_{\text{hom}}
   \]

Lemma 5.4. Let $I_2 \in \text{CH}^4(F \times F)$ be a cycle supported on $\Sigma_2 \times F \cap I$, then $I_2^* = 0$ on $\text{CH}_i(F)_{\text{hom}}$, $i = 1, 2$. Furthermore, $(I_2)_i = 0$ on $\text{CH}_i(F)_{\text{hom}}$, $i = 0, 1$.

Proof. The only case we need to prove is $I_2^* a = 0$, for all $a \in \text{CH}_1(F)_{\text{hom}}$. We only do this under the assumption that $\Sigma_2$ is a smooth surface while the general case follows by taking a resolution of $\Sigma_2$. In this case
   \[
   I_2^* : \text{CH}_1(F)_{\text{hom}} \to \text{CH}_1(F)_{\text{hom}}
   \]

factors through $\text{CH}_1(\Sigma_2)_{\text{hom}}$. Note that since $F$ has trivial intermediate jacobian, it follows that $I_2^* a \in \text{CH}_1(\Sigma_2)$ is in the the Abel-Jacobi kernel. Hence $I_2^* a = 0$. □

Lemma 5.5. (1) The cylinder homomorphism $\Psi$ satisfies
   \[
   \Psi(g) = 6[X], \quad \Psi(g^2) = 21h, \quad \Psi(g^3) = 36h^2, \quad \Psi(g^4) = 36h^3.
   \]

(2) The action of $I$ on self-intersections of $g$ is given by
   \[
   I_* (g^2) = 21[F], \quad I_* (g^3) = 36g, \quad I_* (g^4) = 36\Phi(h^3).
   \]

(3) The action of $\Gamma_h$ on the self-intersections of $g$ is given by
   \[
   (\Gamma_h)^* g = 6[F], \quad (\Gamma_h)^* g^2 = 21g, \quad (\Gamma_h)^* g^3 = 36\Phi(h^3), \quad (\Gamma_h)^* g^4 = 108C_x.
   \]

(4) The action of $\Gamma_{h^2}$ on the self-intersections of $g$ is given by
   \[
   (\Gamma_{h^2})^* g = 6g, \quad (\Gamma_{h^2})^* g^2 = 21\Phi(h^3), \quad (\Gamma_{h^2})^* g^3 = 108C_x, \quad (\Gamma_{h^2})^* g^4 = 0.
   \]
Given a smooth projective variety $X$, let $\Gamma: CH_2(X) \rightarrow \Sigma_2$ be the correspondence $\Gamma_f).$ The subschemes $\Gamma_f \circ \Gamma_\sigma$ and $\Gamma_\sigma \circ \Gamma_f$ intersect properly on $(F \times F) \setminus (\Gamma_2 \times \Sigma_2 \Gamma_2)$: their intersection is given by $\{((\varphi(x), x, \varphi(x)) : x \in F \setminus \Sigma_2\}$. Since $\varphi$ has degree 16, it follows that $\Gamma_\varphi \circ \Gamma_\sigma = 16\Delta_F \rightarrow 0 \in CH_4((F \times F) \setminus (p_2, p_2)(\Gamma_2 \times \Sigma_2 \Gamma_2)).$ The subvariety $\Gamma_2 \times \Sigma_2 \Gamma_2$ has dimension 4 and the proposition thus follows from the localization exact sequence for Chow groups.\hfill $\square$

Corollary 6.2. The map $\varphi_* \varphi^*$ acts by multiplication by 16 on $CH_0(F)$, $Griff_1(F)$, $Griff_2(F)$, $CH_2(F}_{\text{hom}}$ and $CH_4(F)$.\hfill $\square$

Proof. Let $\tilde{\Sigma}_2 \rightarrow \Sigma_2$ be a desingularization of $\Sigma_2$. The action of $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ on $CH_0(F)$ (resp. $Griff_1(F)$) factors through $CH_{-1}(\tilde{\Sigma}_2)$ (resp. $Griff_1(\tilde{\Sigma}_2)$). Therefore, $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ acts as zero on $CH_0(F)$, $Griff_1(F)$, $Griff_2(F)$, and on $CH_4(F)$ since the dimension of $\Sigma_2$ is 2; see Lemma 1.1. The group $CH_2(F)_{\text{hom}}$ is actually equal to $T^2(F) := \ker\{A_{J_2} : CH_2(F)_{\text{hom}} \rightarrow J_2(F)\}$ because $H^3(F, \mathbb{C}) = 0$. Thus the action of $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ on $CH_2(F)_{\text{hom}}$ factors through $T^2(\tilde{\Sigma}_2)$, which is zero since $\Sigma_2$ is a surface. \hfill $\square$

Remark 6.3. Given a smooth projective variety $X$ and a rational map $f : X \dashrightarrow X$, it is more generally the case that $f_*f^*$ acts by multiplication $\deg(f)$ on $CH_0(X)$, $Griff_1(X)$, $Griff_2(X)$, $T^2(X)$, $CH_4(X)_{\text{hom}}$ and $CH_4(X)$. This is because the correspondence $\Gamma_f \circ \Gamma_f - \deg(f)\Delta_F$ is supported on $D \times X$ for some divisor $D$. In our case of interest, Proposition 6.1 is somewhat more precise.

6. Splitting of filtration and eigenvalues

Proposition 6.1. There exists an integer $e$ such that the identity

$$\Gamma_\varphi \circ \Gamma_\varphi^e = 16\Delta_F + e[\Gamma_2 \times \Sigma_2 \Gamma_2]$$

holds true in $CH_4(F \times F)$, where by abuse we wrote $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ to mean the image of $\Gamma_2 \times \Sigma_2 \Gamma_2 \subset (F \times F) \times (F \times F)$ in $F \times F$ via $(p_2, p_2)$.

Proof. The subschemes $F \times \Gamma_\varphi$ and $\Gamma_\varphi \times F$ of $F \times F \times F$ intersect properly on $(F \times F \times F) \setminus (\Gamma_2 \times \Sigma_2 \Gamma_2)$: their intersection is given by $\{(\varphi(x), x, \varphi(x)) : x \in F \setminus \Sigma_2\}$. Since $\varphi$ has degree 16, it follows that $\Gamma_\varphi \circ \Gamma_\varphi^e = 16\Delta_F \rightarrow 0 \in CH_4((F \times F) \setminus (p_2, p_2)(\Gamma_2 \times \Sigma_2 \Gamma_2)).$ The subvariety $\Gamma_2 \times \Sigma_2 \Gamma_2$ has dimension 4 and the proposition thus follows from the localization exact sequence for Chow groups.\hfill $\square$

Corollary 6.2. The map $\varphi_* \varphi^*$ acts by multiplication by 16 on $CH_0(F)$, $Griff_1(F)$, $Griff_2(F)$, $CH_2(F)_{\text{hom}}$ and $CH_4(F)$.\hfill $\square$

Proof. Let $\tilde{\Sigma}_2 \rightarrow \Sigma_2$ be a desingularization of $\Sigma_2$. The action of $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ on $CH_0(F)$ (resp. $Griff_1(F)$) factors through $CH_{-1}(\tilde{\Sigma}_2)$ (resp. $Griff_1(\tilde{\Sigma}_2)$). Therefore, $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ acts as zero on $CH_0(F)$, $Griff_1(F)$, $Griff_2(F)$, and on $CH_4(F)$ since the dimension of $\tilde{\Sigma}_2$ is 2; see Lemma 1.1. The group $CH_2(F)_{\text{hom}}$ is actually equal to $T^2(F) := \ker\{A_{J_2} : CH_2(F)_{\text{hom}} \rightarrow J_2(F)\}$ because $H^3(F, \mathbb{C}) = 0$. Thus the action of $[\Gamma_2 \times \Sigma_2 \Gamma_2]$ on $CH_2(F)_{\text{hom}}$ factors through $T^2(\tilde{\Sigma}_2)$, which is zero since $\tilde{\Sigma}_2$ is a surface. \hfill $\square$

Remark 6.3. Given a smooth projective variety $X$ and a rational map $f : X \dashrightarrow X$, it is more generally the case that $f_*f^*$ acts by multiplication $\deg(f)$ on $CH_0(X)$, $Griff_1(X)$, $Griff_2(X)$, $T^2(X)$, $CH_4(X)_{\text{hom}}$ and $CH_4(X)$. This is because the correspondence $\Gamma_f \circ \Gamma_f - \deg(f)\Delta_F$ is supported on $D \times X$ for some divisor $D$. In our case of interest, Proposition 6.1 is somewhat more precise.

Proposition 6.4. $I_* \varphi^* I_* : CH_0(F)_{\text{hom}} \rightarrow CH_2(F)_{\text{hom}}.$

Proof. Let $\sigma \in CH_2(F)_{\text{hom}}$. Then, by Proposition 6.2 we have

$$\varphi^* \sigma = 4\sigma + I_*(g^2 \cdot \sigma).$$
By Lemma 1.8 we have \( \varphi_*x = 4x + g^2 \cdot I_\ast x \) for all \( x \in \text{CH}_0(F)_{\text{hom}} \). Therefore, if \( \sigma = I_\ast x \), we find that

\[
\varphi^* I_\ast x = 4I_\ast x + I_\ast (\varphi_*x - 4x) = I_\ast \varphi_*x.
\]

Lemma 6.5. The group \( \text{CH}_0(F)_{\text{hom}} \) is uniquely divisible.

Proof. Indeed, on the one hand, Roitman [13] showed that the albanese map \( \text{CH}_0(Z)_{\text{hom}} \to \text{Alb}(Z) \) is an isomorphism on the torsion for all smooth projective complex varieties \( Z \). Here \( H^1(F, \mathbb{C}) = 0 \), so that \( \text{Alb}(F) = 0 \) and thus \( \text{CH}_0(F)_{\text{hom}} \) has no torsion. On the other hand, it is well-known that \( \text{CH}_0(Z)_{\text{hom}} \) is divisible for all smooth projective complex varieties \( Z \).

Lemma 6.6. The group \( \text{CH}_2(F)_{\text{hom}} \) is torsion-free.

Proof. A famous theorem of Colliot-Thélène, Sansuc and Soulé [9] says that, for all smooth projective varieties \( X \), defined over the complex numbers say, the Abel-Jacobi map \( \text{AJ} : \text{CH}^2(X)_{\text{hom}} \to \text{J}^2(X) \) is injective when restricted to the torsion. Here \( H^3(F, \mathbb{Z}) = 0 \). Hence \( \text{CH}_2(F)_{\text{hom}} \) is torsion-free.

Proposition 6.7. Let \( V^2 := \ker \{ \varphi^* + 2 : \text{CH}_2(F)_{\text{hom}} \to \text{CH}_2(F)_{\text{hom}} \} \). Then

\[
\mathcal{A}_{\text{hom}} = V^2.
\]

Proof. The group \( \mathcal{A}_{\text{hom}} \) is spanned by cycles of the form \([S_1] - [S_2]\]. Let’s compute \( \varphi^* \sigma \) for \( \sigma = [S_1] - [S_2] = I_\ast x \) where \( x = [l_1] - [l_2] \in \text{CH}_0(F) \).

\[
\varphi^* \sigma = 4\sigma + I_\ast (\varphi_*x - 4x) = 4\sigma + I_\ast (\varphi_*x + 2x) - I_\ast (6x) = 4\sigma - 6\sigma = -2\sigma.
\]

Thus \( \mathcal{A}_{\text{hom}} \subseteq V^2 \). Here the first equality is Proposition 6.4. The third equality follows simply from the fact that \( \varphi_*x + 2x \) satisfies the triangle relation, i.e. belongs to \( R_{\text{hom}} \), and that \( I_\ast \) acts as zero on \( R_{\text{hom}} \) by Proposition 2.4.

Consider now \( \sigma \in \text{CH}_2(F)_{\text{hom}} \) such that \( \varphi^* \sigma = -2\sigma \). By Proposition 6.2 we have \( \varphi^* \sigma = 4\sigma + I_\ast (g^2 \cdot \sigma) \) so that \( I_\ast (g^2 \cdot \sigma) = -6\sigma \). Thus \( 6V^2 \subseteq I_\ast \text{CH}_0(F)_{\text{hom}} = \mathcal{A}_{\text{hom}} \). We may then conclude that \( V^2 \subseteq \mathcal{A}_{\text{hom}} \) because the group \( \text{CH}_0(F)_{\text{hom}} \) is uniquely divisible (Lemma 6.5) and because the group \( V^2 \subseteq \text{CH}_2(F)_{\text{hom}} \) is torsion-free (Lemma 6.6).

Theorem 6.8. The action \( \varphi^* \) on \( \text{CH}_2(F)_{\text{hom}} \) satisfies the following quadratic equation

\[
(\varphi^* - 4)(\varphi^* + 2).
\]

Hence there is a canonical splitting

\[
\text{CH}_2(F)_{\text{hom}} = V^2 \oplus V^4,
\]

where \( V^2 \) is the eigenspace corresponding to the eigenvalue \( n \) for the action of \( \varphi^* \) on \( \text{CH}_2(F)_{\text{hom}} \).
Proof. By Proposition 6.7 for all $\sigma \in \text{CH}_2(F)_{\text{hom}}$, there is a 2-cycle $\sigma' := I_*(g^2 \cdot \sigma) \in A_{\text{hom}}$ such that

$$\varphi^* \sigma = 4\sigma + \sigma'.$$

By Proposition 6.7 and Corollary 6.2 we have that $\varphi_* \sigma' = -8\sigma'$ and $\varphi_* \varphi^* \sigma = 16\sigma$. Therefore, after applying $\varphi_*$ to (9), we get

$$4\sigma = \varphi_* \sigma - 2\sigma'.$$

Combining (9) and (10) we get

$$\sigma' = \varphi_* \sigma - \varphi^* \sigma.$$

Plugging in this identity in (9) and applying $\varphi_*$ again gives

$$\varphi_* \varphi_\sigma + 4\varphi_* \varphi^* \sigma = 0.$$

In view of Corollary 6.2 we find that $(\varphi_* - 4)(\varphi_* + 8)$ acts as zero on $\text{CH}_2(F)_{\text{hom}}$. Thus, thanks to Corollary 6.2, $(\varphi_* - 4)(\varphi_* + 2)$ acts as zero on $\text{CH}_2(F)_{\text{hom}}$.

That $V_2^{-2} \cap V_4^2 = \{0\}$ follows from the fact that $\text{CH}_2(F)_{\text{hom}}$ is torsion-free (Lemma 6.10) and, that $V_2^{-2} + V_4^2 = \text{CH}_2(F)_{\text{hom}}$ follows from the fact that $V_2^{-2}$ is divisible (Proposition 6.7).

Remark 6.9. The identity $I_*(g^2 \cdot \sigma) = \varphi_* \sigma - \varphi^* \sigma$, combined with the expressions of $\varphi^* \sigma$ and $\varphi_* \sigma$ of Proposition 5.2 gives

$$(I_2)_* \sigma = (a^2 - a - 1)I_*(g^2 \cdot \sigma), \quad \sigma \in \text{CH}_2(F)_{\text{hom}}.$$  

Theorem 6.10. The action $\varphi^*$ on $\text{CH}_1(F)_{\text{hom}}$ satisfies the following quadratic equation

$$(\varphi^* - 4)(\varphi^* - 6a + 2) = 0.$$  

Hence there is a canonical splitting

$$\text{CH}_1(F)_{\text{hom}} \otimes \mathbb{Q} = V_1^4 \otimes \mathbb{Q} \oplus V_1^{6a-2} \otimes \mathbb{Q}$$

where $V_1^n = \ker(\varphi^* - n)$ is the eigenspace corresponding to the eigenvalue $n$ for the action of $\varphi^*$ on $\text{CH}_1(F)_{\text{hom}}$.

Remark 6.11. In Proposition 7.2 we show that if we replace $V_1^{6a-2}$ by $g \cdot V_2^{-2}$ then we get a decomposition of $\text{CH}_1(F)_{\text{hom}}$ without tensoring with $\mathbb{Q}$.

Proof. Let $x \in \text{CH}_1(F)_{\text{hom}}$, then by Proposition 5.2 we get

$$(\varphi^* - 4)x = (1 - a)g \cdot I_*(g \cdot x).$$

Since $I_*(g \cdot x) \in V_2^{-2}$ and $\varphi^* g = (1 - 3a)g$, Proposition 4.6 implies

$$\varphi^*(g \cdot I_*(g \cdot x)) = -2(1 - 3a)g \cdot I_*(g \cdot x),$$

or equivalently,

$$(\varphi^* - 6a + 2)g \cdot I_*(g \cdot x) = 0.$$  

Combining the above identities, we obtain the required quadratic identity. The eigenspace splitting follows because $a = -2$ and thus $6a - 2 = 14 \neq 4$; see Proposition 5.2.

□
Theorem 6.12. The action of $\varphi^*$ on $\text{CH}_0(F)$ is diagonalizable and induces a canonical splitting of the filtration $F^\bullet$ on $\text{CH}_0(F)$. Precisely, there is a canonical splitting

$$\text{CH}_0(F) = V_0^{16} \oplus V_0^{-8} \oplus V_0^4,$$

where $V_0^t$ is the eigenspace corresponding to the eigenvalue $t$ for the action of $\varphi^*$ on $\text{CH}_0(F)$, and

- $\text{Gr}_0^0 \text{CH}_0(F) := \text{CH}_0(F)/\text{CH}_0(F)_{\text{hom}} = V_0^{16} = \mathbb{Z}[\alpha]$;
- $\text{Gr}_0^2 \text{CH}_0(F) := \text{CH}_0(F)_{\text{hom}}/F^3 \text{CH}_0(F) = V_0^{-8}$;
- $\text{Gr}_0^4 \text{CH}_0(F) := F^3 \text{CH}_0(F) = V_0^4$.

Proof. The rational map $\varphi$ has degree 16. Thus, for all $x \in \text{CH}_0(F)$, we have $\deg(\varphi^* x) = 16 \deg x$. Therefore, $\varphi^*$ acts by multiplication by 16 on $\text{Gr}_0^0 \text{CH}_0(F)$. Moreover, the zero-cycle $[\alpha]$ is a canonical choice of a degree-1 cycle in $\text{CH}_0(F)$. We need to see that $\varphi^*[\alpha] = 16[\alpha]$, or equivalently $\varphi_*[\alpha] = \alpha$. By Lemma 1.7, we have

$$I_*[\alpha] = \frac{1}{3} \Phi(h^3) = \frac{1}{3}(g^2 - c).$$

At the same time by Proposition 2.22, we have $(I_*[\alpha])^2 = \varphi_*[\alpha] + 4[\alpha]$. Hence using the identities in (1) of Lemma 1.7, we get

$$\varphi_*[\alpha] + 4[\alpha] = \frac{1}{9}(g^4 - 2g^2c + c^2) = 5[\alpha].$$

It follows that $\varphi_*[\alpha] = [\alpha]$. There is therefore a canonical splitting

$$\text{CH}_0(F) = \text{Gr}_0^0 \text{CH}_0(F) \oplus \text{CH}_0(F)_{\text{hom}}.$$

The group $\text{CH}_0(F)_{\text{hom}}$ is uniquely divisible by Lemma 6.3 and hence has the structure of a $\mathbb{Q}$-vector space. Let now $x \in \text{CH}_0(F)_{\text{hom}}$ be a homologically trivial cycle. By the very definition of $\varphi$ (Definition 1.2), the cycle $\varphi_* x + 2x$ satisfies the triangle relation of Definition 2.22. Therefore $\varphi_*$ acts by multiplication by $-2$ on $\text{Gr}_0^2 \text{CH}_0(F)$. By Corollary 6.2, we get that $\varphi^*$ acts by multiplication by $-8$ on $\text{Gr}_0^2 \text{CH}_0(F)$. By Theorem 2.6, the subgroup $F^3 \text{CH}_0(F)$ is spanned by $A_{\text{hom}} \otimes A_{\text{hom}}$. By Proposition 6.7, $\varphi^*$ acts as multiplication by $-2$ on $A_{\text{hom}}$. Proposition 4.6, together with the fact that $H^3(F, \mathbb{Z}) = 0$ and hence $J^2(F) = 0$, then implies that $\varphi^*$ acts by multiplication by 4 on $F^3 \text{CH}_0(F)$. \qed

7. Further properties of the eigenspaces $V_i^n$

In this section we study further properties of the eigenspaces $V_i^n$. In particular, we show that the intersection with hyperplane class $g$ respects the decomposition of $\text{CH}_1(F)_{\text{hom}}$.

Lemma 7.1. (1) If $\sigma \in \text{CH}_1(F)_{\text{hom}}$ is torsion, then $\sigma \in V_1^4$.
(2) $g \cdot V_2^{-2} \subset V_1^{6a - 2}$.

Proof. If $\sigma \in \text{CH}_1(F)_{\text{hom}}$ is torsion, then $g \cdot \sigma = 0$ since $\text{CH}_0(F)_{\text{hom}}$ is torsion free. Hence by Proposition 6.2, we have

$$\varphi^* \sigma = 4\sigma + (1 - a)g \cdot I_*(g \cdot \sigma) = 4\sigma.$$

This proves (1). Now we prove (2). Let $\sigma \in V_2^{-2}$, then Proposition 4.4 implies $\varphi^*(g \cdot \sigma) = \varphi^* g \cdot \varphi^* \sigma = (1 - 3a)g \cdot (-2)\sigma = (6a - 2)g \cdot \sigma$. \qed
Proposition 7.2. (1) The space $V_2^4$ can also be described as follows
\[ V_2^4 = \ker\{g^2 : CH_2(F)_{\text{hom}} \to CH_0(F)_{\text{hom}}\} = \ker\{g : CH_2(F)_{\text{hom}} \to CH_1(F)_{\text{hom}} \otimes Q\} \]
(2) $g^2 : V_2^{-2} \to V_0^{-8}$ is an isomorphism.
(3) There is a canonical decomposition $V_1^{6a-2} = g \cdot V_2^{-2} \oplus T_{18}$, where $T_{18} = \{a \in CH_1(F)_{\text{hom}} : 18a = 0\}$ is the subgroup of all elements of 18-torsion. Furthermore, the group $CH_1(F)_{\text{hom}}$ has the following canonical decomposition
\[ CH_1(F)_{\text{hom}} = g \cdot V_2^{-2} \oplus V_1^4. \]
(4) $g : V_1^{6a-2} \to V_0^{-8}$ induces an isomorphism $g \cdot V_2^{-2} \cong V_0^{-8}$.

Proof. Let $\sigma \in V_2^4$, then we have $\varphi^*\sigma = 4\sigma$. Then by the fact that $\varphi^*g = 7g$ and Proposition 5.6 we see that $\varphi^*(g \cdot \sigma) = \varphi^*g \cdot \varphi^*\sigma = 7g \cdot 4\sigma = 28g \cdot \sigma$. Since 28 is not an eigenvalue of $\varphi^*$ on $CH_1(F)_{\text{hom}} \otimes Q$, we know that $g \cdot \sigma$ is torsion. Hence $g^2 \cdot \sigma = 0$ since $CH_0(F)_{\text{hom}}$ is torsion free. This shows that $V_2^4 \subset \ker\{g^2 : CH_2(F)_{\text{hom}} \to CH_1(F)_{\text{hom}} \otimes Q\}$ and also $V_2^4 \subset \ker\{g : CH_2(F)_{\text{hom}} \to CH_1(F)_{\text{hom}} \otimes Q\}$. Now let $\sigma \in CH_2(F)_{\text{hom}}$ be an element such that $g^2 \cdot \sigma = 0$, then by Proposition 5.2 we have
\[ \varphi^*\sigma = 4\sigma + I_x(g^2 \cdot \sigma) = 4\sigma. \]
Hence $\sigma \in V_2^4$. If $\sigma \in CH_2(F)_{\text{hom}}$ such that $g \cdot \sigma = 0$ in $CH_1(F)_{\text{hom}} \otimes Q$, then again by Proposition 5.2 we have $\varphi^*\sigma = 4\sigma$. Hence we have the inclusion in the other direction, which completes the proof of (1).

Given the characterization of $V_2^4$ in (1) and the decomposition of $CH_2(F)_{\text{hom}}$ in Theorem 6.8, to prove (2) it suffices to show that the image of $g^2 : V_2^{-2} \to CH_0(F)$ is $V_0^{-8}$. Note that by Proposition 5.2 we have $V_2^{-2} = A_{\text{hom}} = I_x(V_0^{-8})$. For any $\sigma \in V_2^{-2}$, we write $\sigma = I_x x$ for some $x \in V_0^{-8}$. Hence by Lemma 5.1 we have
\[ g^2 \cdot \sigma = (g^2 \cdot I_x)x = (\varphi^* - 4)x = 6x \in V_0^{-8}. \]
Thus $g^2 \cdot V_2^{-2} \subseteq V_0^{-8}$. Conversely, if $x \in V_0^{-8}$ and write $\sigma = -\frac{1}{6}I_x x$, then the above equations also shows that $g^2 \cdot \sigma = x$. Hence we conclude that $g^2 \cdot V_2^{-2} = V_0^{-8}$.

Now we prove (3). Let $\sigma \in CH_1(F)_{\text{hom}}$. Since $V_2^{-2} = A_{\text{hom}}$ is uniquely divisible, we have a well-defined element $\sigma_1 = -g \cdot (\frac{1}{6}I_x(g \cdot \sigma)) \in g \cdot V_2^{-2}$. By Lemma 5.1 we have $\varphi^*\sigma_1 = (6a - 2)\sigma_1$. We set $\sigma_2 = \sigma - \sigma_1$. Then
\[ \varphi^*(\sigma_2) = \varphi^*\sigma - \varphi^*\sigma_1 = 4\sigma + (1 - a)g \cdot I_x(g \cdot \sigma) - (6a - 2)\sigma_1 = 4\sigma - (1 - a)6\sigma_1 - (6a - 2)\sigma_1 = 4(\sigma - \sigma_1) = 4\sigma_2. \]
Namely, we have $\sigma_2 \in V_1^4$. It follows that $\sigma = \sigma_1 + \sigma_2 \in g \cdot V_2^{-2} + V_1^4$, which shows that $CH_1(F)_{\text{hom}} = g \cdot V_2^{-2} + V_1^4$. Hence we see that $CH_1(F)_{\text{hom}} = g \cdot V_2^{-2} \oplus V_1^4$. One also easily sees that $V_1^{6a-2} \cap V_1^4 = T_{18}$, since $a = -2$; see Proposition 5.2. This gives the decomposition of $V_1^{6a-2}$.

Proposition 7.3.
\[ [\Sigma_2] \cdot \sigma = 5g^2 \cdot \sigma, \quad \sigma \in CH_2(F)_{\text{hom}}. \]

Proof. Let $\sigma \in CH_2(F)_{\text{hom}}$, then by (3) of Lemma 1.1 we have $c \cdot \sigma = 0$. Then the proposition follows from Lemma 1.1.
Proposition 7.4. The group $V_0^{-8}$ is supported on a surface. To be more precise, we have
\[
V_0^{-8} \subseteq \text{im}\{\text{CH}_0(H_2)_{\text{hom}} \to \text{CH}_0(F)\}
\]
\[
V_0^{-8} \subseteq \text{im}\{\text{CH}_0(\Sigma_2)_{\text{hom}} \to \text{CH}_0(F)\}.
\]
where $H_2$ is the intersection of two general hyperplanes.

Proof. We have $V_0^{-8} = g^2 \cdot \text{CH}_2(F)_{\text{hom}}$. Therefore $V_0^{-8}$ is supported on $H_2$. It is also supported on $\Sigma_2$ thanks to Proposition 6.6. □

Proposition 7.5. $V_0^{-8} \neq 0$ and $V_0^4 \neq 0$.

Proof. $V_0^{-8}$ is isomorphic to $A_{\text{hom}}$ and hence to $\text{CH}_1(X)_{\text{hom}}$. This latter group is known to be non-zero.

Assume that $V_0^4 = 0$. Then, by Theorem 6.12 and Proposition 7.4, $\text{CH}_0(F)_{\text{hom}} = \text{im}\{\text{CH}_0(H_2)_{\text{hom}} \to \text{CH}_0(F)\}$. By Bloch and Srinivas [8], it follows that $H^{4,0}(F) = 0$. This is a contradiction. □

Proposition 7.6. The group $g \cdot V_2^{-2}$ is spanned by rational curves.

Proof. Let $\sigma \in g \cdot V_2^{-2}$. Then $\varphi^{*}\sigma = \varphi_{*}\sigma = (6a - 2)\sigma = -14\sigma$. Hence we have $\varphi \cdot \varphi^{*}\sigma = (6a - 2)^2\sigma = 196\sigma$. By Proposition 6.1, we have $\varphi \cdot \varphi^{*}\sigma = 16\sigma + e(\Gamma_2)^2 \sigma$. Hence $180\sigma = e(\Gamma_2)^2 x$, where $x = (\Gamma_2)^2 \sigma \in \text{CH}_0(\Sigma_2)_{\text{hom}}$. Since $\text{CH}_0(\Sigma_2)_{\text{hom}}$ is divisible and $g \cdot V_2^{-2}$ is uniquely divisible, we conclude that there exists $y \in \text{CH}_0(\Sigma_2)_{\text{hom}}$ such that $\sigma = (\Gamma_2)^2 \gamma$. Note that all fibers of $\Gamma_2 \to \Sigma_2$ are rational curves, we see that $\sigma$ can be written as a linear combination of the rational curves $R_{\mid \sigma \rangle}$. □

Proposition 7.7. Consider the following statements:

1. $g^2 : \text{CH}_2(F)_{\text{hom}} \to \text{CH}_0(F)_{\text{hom}}$ is injective.
2. $V_2^4 = 0$.
3. $\text{CH}_2(F)_{\text{hom}} = A_{\text{hom}} := I_{\text{hom}} \text{CH}_0(F)_{\text{hom}}$.
4. $V_2^{-2} \otimes V_2^4 \to \text{CH}_0(F)$ is zero.
5. $\text{im}\{\text{CH}_2(F)_{\text{hom}} \otimes \text{CH}_2(F)_{\text{hom}} \to \text{CH}_0(F)\} = F^3 \text{CH}_0(F)$.
6. $\text{Griff}_2(F) = 0$.

Then (1) $\iff$ (2) $\iff$ (3) $\rightarrow$ (4) $\iff$ (5) and (1) $\Rightarrow$ (6).

Moreover, all of these statements are true if the Bloch-Beilinson conjectures are true.

Proof. The implications relating the six statements of the proposition are clear in view of the results established above.

Let $G^\bullet$ be a filtration on $H^1(F)_{\mathbb{Q}}$ of Bloch-Beilinson type. Because $H^3(F, \mathbb{Q}) = 0$, we have $G^2 \text{CH}_2(F)_{\mathbb{Q}} = G^1 \text{CH}_2(F)_{\mathbb{Q}} = \text{CH}_2(F)_{\text{hom}, \mathbb{Q}}$. Likewise, because $H^\cdot(F, \mathbb{Q}) = 0$, we have $G^2 \text{CH}_0(F)_{\mathbb{Q}} = G^1 \text{CH}_0(F)_{\mathbb{Q}} = \text{CH}_0(F)_{\text{hom}, \mathbb{Q}}$. The Lefschetz hyperplane theorem gives an isomorphism $g^2 : H^2(F, \mathbb{Q}) \to H^6(F, \mathbb{Q})$. Thus $g^2 : \text{Gr}_2^2 \text{CH}_2(F)_{\mathbb{Q}} = G^2 \text{CH}_2(F)_{\mathbb{Q}} \to G^2 \text{CH}_0(F)_{\mathbb{Q}} \to \text{Gr}_2^0 \text{CH}_0(F)_{\mathbb{Q}}$ is an isomorphism. It follows that $g^2 : \text{CH}_2(F)_{\text{hom}, \mathbb{Q}} \to \text{CH}_0(F)_{\text{hom}, \mathbb{Q}}$ is injective. We may then conclude that $g^2 : \text{CH}_2(F)_{\text{hom}} \to \text{CH}_0(F)_{\text{hom}}$ is injective because $\text{CH}_2(F)_{\text{hom}}$ is torsion-free; see Lemma 6.6. □
8. Chow–Küneth decomposition

Let $X$ be a smooth projective variety of dimension $d$ defined over a field $k$. Murre [12] conjectured the existence of mutually orthogonal idempotents $\pi^0, \ldots, \pi^{2d}$ in the ring (for the composition law) of correspondences $CH_d(X \times X)_{\mathbb{Q}}$ such that $\Delta_X = \pi^0 + \ldots + \pi^{2d} \in CH_d(X \times X)_{\mathbb{Q}}$ and such that the cohomology class of $\pi^i \in H^{2d}(X \times X) \subseteq \text{End}(H^*(X))$ is the projector on $H^i(X)$. Here $H^i(X)$ denotes the $i$-adic cohomology group $H^i(X_{\mathbb{Z}}, \mathbb{Q}_l)$. Such a decomposition of the diagonal is called a Chow–Küneth decomposition; it is a lift of the Künneth decomposition of $id \in \text{End}(H^*(X))$ via the cycle class map $CH_d(X \times X) \rightarrow H^{2d}(X \times X)$. Furthermore, Murre conjectured that any Chow–Küneth decomposition satisfies the following properties:

(B) $\langle \pi^i \rangle^* CH^i(X)_{\mathbb{Q}} = 0$ for $i > 2d$ and for $i < d$;
(D) $\ker\{\langle \pi^2 \rangle^* : CH^1(X)_{\mathbb{Q}} \rightarrow CH^1(X)_{\text{hom,} \mathbb{Q}}\}$.

If such a Chow–Küneth decomposition exists, we may define a descending filtration $G^i$ on $CH^i(X)_{\mathbb{Q}}$ as follows:

$$G^i CH^i(X)_{\mathbb{Q}} := \ker\{\pi^{2d} + \ldots + \pi^{2d-r+1} : CH^i(X)_{\mathbb{Q}} \rightarrow CH^i(X)_{\mathbb{Q}}\}.$$  

Thus $G^0 CH^i(X)_{\mathbb{Q}} = CH^i(X)_{\mathbb{Q}}$, $G^1 CH^i(X)_{\mathbb{Q}} = CH^i(X)_{\text{hom,} \mathbb{Q}}$, and $G^{i+1} CH^i(X)_{\mathbb{Q}} = 0$. Murre conjectured

(C) The filtration $G^i$ does not depend on a choice of Chow-Küneth decomposition.

It is a theorem of Jannsen that Murre’s conjectures for all smooth projective varieties are equivalent to the conjectures of Bloch and Beilinson. Here we show that the Fano scheme of lines on a smooth cubic fourfold has a Chow–Küneth decomposition that satisfies (B) and for which the induced filtration on $CH_0(F)$ is the one constructed in Section 2.

The following lemma relies on a technique initiated by Bloch and Srinivas [8].

Lemma 8.1. Let $X$ be a smooth projective variety over a field $k$ and let $\Omega$ be a universal domain containing $k$. If $f \in CH_{\dim X}(X \times X)_{\mathbb{Q}}$ is a correspondence such that $(f_\Omega)_* CH_0(X\Omega)_{\mathbb{Q}} = 0$, then there is a smooth projective variety $Y$ of dimension $\dim X - 1$ and correspondences $g \in CH^{\dim X - 1}(Y \times X)_{\mathbb{Q}}$ and $h \in CH_{\dim X}(X \times Y)_{\mathbb{Q}}$ such that $f = g \circ h$. If $\dim X = 0$, then $f = 0$.

Proof. The lemma is clear when $\dim X = 0$. Let $k(X)$ be the function field of $X$. The assumption that $(f_\Omega)_* CH_0(X\Omega)_{\mathbb{Q}} = 0$ implies that $(f_{k(X)})_* CH_0(k(X)\times X)_{\mathbb{Q}} = 0$. Let $\eta$ be the generic point of $X$ seen as a $k(X)$-rational point. In particular, $(f_{k(X)})_* [\eta] = 0$. The 0-cycle $(f_{k(X)})_* [\eta] \in CH_0(k(X) \times X)_{\mathbb{Q}}$ coincides with the restriction of $f$ along the map $CH_d(X \times X)_{\mathbb{Q}} \rightarrow CH_0(k(X) \times X)_{\mathbb{Q}}$ obtained as the direct limit, indexed by the non-empty open subsets $U$ of $X$, of the flat pullback maps $CH_d(X \times X)_{\mathbb{Q}} \rightarrow CH_d(U \times X)_{\mathbb{Q}}$. Therefore, by the localization exact sequence for Chow groups, $f$ is supported on $D \times X$ for some divisor $D$ inside $X$. If $Y \rightarrow D$ is an alteration of $D$, then $f$ factors through $Y$.

Lemma 8.2. Let $\omega \in H^{2,0}(F)$ be a generator. Then

$$I_1(g^2 \cdot \omega) = -6 \omega.$$  

Proof. This essentially follows from Proposition 3.1. First note that both $I_1$ and $I_2$ are supported on a closed subset of the form $\Sigma \times F$ where $\Sigma \subseteq F$ is a surface.
Then for dimension reasons, we have $I_1^* \omega = I_2^* \omega = 0$. Furthermore, on $\text{CH}_0(F)$, we have

$$(\Gamma_\varphi - 4\Delta_F - g_2^2 I + n[F \times \{\emptyset\}]_\ast)[l] = 0, \forall[l] \in F,$$

where $n$ is an integer ($n = 20$ for degree reasons). This means, by Lemma VIII.1 that some multiple of $\Gamma_\varphi - 4\Delta_F - g_2^2 I + n[F \times \{\emptyset\}]$ is supported on $D \times F$ where $D \subset F$ is a divisor. Hence it follows that

$$(\Gamma_\varphi - 4\Delta_F - g_2^2 I + n[F \times \{\emptyset\}])^* \omega = 0.$$ 

One spells this out and gets

$$I_\ast (g^2 \cdot \omega) = (g_2^2 I)^* \omega = \varphi^* \omega - 4\omega.$$

Since $\varphi^* \omega = -2\omega$, we see that $I_\ast (g^2 \cdot \omega) = -6\omega$. \hfill \qed

**Proposition 8.3.** Let $X$ be a smooth projective variety over a field $k$ and let $\Omega$ be a universal domain containing $k$. If $f \in \text{CH}_{\dim X} (X \times X)_\Omega$ is a correspondence such that $(f_\Omega)_\ast \text{CH}_0 (X_{\Omega})_\Omega = 0$, then $f$ is nilpotent.

**Proof.** We proceed by induction on $\dim X$. If $\dim X = 0$, then by Lemma VIII.1 $f = 0$. Let's now assume that $\dim X > 0$. By Lemma VIII.1, there is a smooth projective variety $Y$ of dimension $\dim X - 1$ and correspondences $g \in \text{CH}_{\dim X} (Y \times X)_\Omega$ and $h \in \text{CH}_{\dim X} (X \times Y)_\Omega$ such that $f = g \circ h$. The correspondence $(h \circ g \circ h \circ g)_{\Omega} \in \text{CH}_{\dim Y} (Y_{\Omega} \times Y_{\Omega})_\Omega$ then acts as zero on $\text{CH}_0 (Y_{\Omega})$. By induction, $h \circ g \circ h \circ g$ is nilpotent. It immediately follows that $f$ is nilpotent. \hfill \qed

**Theorem 8.4.** The variety $F$ has a Chow–Künneth decomposition $\{\pi^0, \pi^2, \pi^4, \pi^6, \pi^8\}$ that satisfies (B) and such that

- $\text{Gr}^0_2 \text{CH}_0 (F) = (\pi^0)^* \text{CH}_0 (F)$;
- $\text{Gr}^2_2 \text{CH}_0 (F) = (\pi^2)^* \text{CH}_0 (F)$;
- $\text{Gr}^4_2 \text{CH}_0 (F) = (\pi^4)^* \text{CH}_0 (F)$.

Moreover, $\{\pi^0, \pi^2, \pi^4, \pi^6, \pi^8\}$ satisfies (D) if and only if $V_2^1 = 0$.

**Proof.** The notations are those of [12]. By Jannsen’s semi-simplicity theorem [11] and by a standard lifting argument, we may decompose the Chow motive of $F$ as

$$\mathfrak{h}(F) := M \oplus N,$$

where $N$ is isomorphic to $\mathbf{1} \oplus \mathbf{1}(-1)^{\oplus \rho} \oplus \mathbf{1}(-2)^{\oplus \rho} \oplus \mathbf{1}(-3)^{\oplus \rho} \oplus \mathbf{1}(-4)$ and where $\text{CH}_0 (M)_\Omega = \text{CH}_0 (F)_\text{hom,\Omega}$. Let’s write $M = (F, r)$ for some $r \in \text{CH}_4 (F \times F)_\Omega$ and let’s write $N = (F, \pi^0) \oplus (F, \pi^2_{\text{alg}}) \oplus (F, \pi^4_{\text{alg}}) \oplus (F, \pi^6_{\text{alg}}) \oplus (F, \pi^8)$ for the decomposition above. We may assume that $\pi^0 = [\emptyset \times F]$ and $\pi^8 = [F \times \emptyset]$. Thus, if $s \in \text{End}(M)$, then $s^* [\emptyset] = 0$ and since $\text{CH}_0 (F)_\text{hom}$ is uniquely divisible, it makes sense to speak of the action of $s^*$ on $\text{CH}_0 (F)$ (rather than $\text{CH}_0 (F)_\Omega$).

By Bertini, let $H_2 \hookrightarrow F$ be a smooth linear section of dimension 2 of $F$: its class is $g^2 \in \text{CH}_2 (F)$. Let $G \in \text{CH}_2 (F \times F)$ be the correspondence given by the class of $\Delta_{H_2}$ inside $F \times F$. The action of $G$ on $\text{CH}_4 (F)$ is given by intersecting with $g^2$, i.e. $G^* x = g^2 \cdot x$ for all $x \in \text{CH}_4 (F)$. We define

$$p := -\frac{1}{r \circ I^\ast \circ G \circ r} \in \text{End}(M) := r \circ \text{CH}_4 (F \times F)_\Omega \circ r.$$ 

The correspondence $p$ defines an idempotent in $H^6 (F \times F, \mathbb{Q})$. Indeed, $p^\ast$ acts as zero in $H^i (M) := r^\ast H^i (F, \mathbb{Q})$ for $i \neq 6$ because $H^i (M)$ is zero for $i$ odd and does not contain any Hodge classes for $i$ even, and acts as the identity on $H^6 (M)$.
thanks to Lemma 5.2 Moreover, the action of \( p^* \) on \( CH_0(F) \) is the identity on \( V_0^{-8} \) and is zero on \( V_0^{16} \oplus V_0^3 \), and is zero on \( CH_i(F) \mathbb{Q} \) for all \( i > 0 \).

Therefore, the correspondence \( f := p \circ p - p \in \text{End}(M) \) acts as zero on \( CH_i(F) \mathbb{Q} \). By Proposition 5.3, \( f \) is nilpotent. We now use a trick due to Beilinson. The correspondence \( f \) is such that \( f \circ p = p \circ f \) and hence

\[
(p + (1 - 2p) \circ f)^{\circ 2} = p + (1 - 2p) \circ f + f^{\circ 2} \circ (-2p - 3).
\]

A straightforward descending induction on the nilpotency index of \( f \) shows that there is no nilpotent correspondence \( n \), of the form \( f \circ P(p) \) for some polynomial \( P \), such that \( q := p + n \) is an idempotent. In particular, since a nilpotent correspondence is homologically trivial, we see that \( p \) and \( q \) are homologically equivalent. It is also apparent that \( p^* \) and \( q^* \) act the same on \( CH_0(F) \mathbb{Q} \). Thus, \( q \) is an idempotent whose cohomology class is the projector on \( H^6(M) \) and whose action on \( CH_0(F) \) is the projector with image \( V_0^{-8} \) and kernel \( V_0^{16} \oplus V_0^3 \).

The idempotent \( q \in \text{End}(M) \) factors through \( \mathfrak{h}(H_2) \) because \( p \) does. Thus \( q^l \in \text{End}(M) \) factors through \( \mathfrak{h}(H_2)(-2) \). Therefore, \( q \circ q^l \) factors through a morphism \( \gamma \in \text{Hom}(\mathfrak{h}(H_2)(-2), \mathfrak{h}(H_2)) = CH^0(H_2 \times H_2) \mathbb{Q} \). We see that \( q \circ q^l = 0 \). Thus \( q \circ q^l = 0 \).

We then define

\[
\pi^{2, tr} := q^l \circ (1 - \frac{1}{2} q) \quad \text{and} \quad \pi^{6, tr} := (1 - \frac{1}{2} q^l) \circ q.
\]

The correspondences \( \pi^{2, tr} \) and \( \pi^{6, tr} \) are mutually orthogonal idempotents in \( \text{End}(M) \) which factor through \( \mathfrak{h}(H_2) \) and \( \mathfrak{h}(H_2)(-2) \) respectively. Therefore, we see that \( \pi^{2, tr} \) is zero on \( CH^0(F) \mathbb{Q} \) for \( l > 2 \) (resp. \( l < 3 \)). The correspondences \( \pi^0, \pi^1, \pi^2, \pi^3, \pi^4, \pi^5, \pi^6, \pi^{7, g}, \pi^{6, alg} \) factor through points and thus only act as non-zero on \( CH^0(F) \mathbb{Q}, CH^1(F) \mathbb{Q}, CH^2(F) \mathbb{Q}, CH^3(F) \mathbb{Q}, CH^4(F) \mathbb{Q} \), respectively. Moreover we have \( H^2(M) = (\pi^{2, tr})^* H^*(F) \) (By Poincaré duality \( (q^l)^* H^*(F) \subseteq H^8(F) \)).

We now define

\[
\pi^2 := \pi^{2, alg} + \pi^{2, tr}, \quad \pi^6 := \pi^{6, alg} + \pi^{6, tr} \quad \text{and} \quad \pi^4 := \Delta_F - (\pi^0 + \pi^2 + \pi^6 + \pi^8).
\]

It is then clear from the above that \( \{ \pi^0, \pi^2, \pi^4, \pi^6, \pi^8 \} \) defines a Chow–Künneth decomposition for \( F \) which satisfies Murre’s conjecture (B) and whose induced filtration on \( CH_0(F) \) coincides with the one of Section 2.

Finally, note that the correspondences \( p^l \) and \( q^l \) act the same on \( CH_2(F)_{\text{hom}, \mathbb{Q}} \): they project onto \( V_2^{-2} \otimes \mathbb{Q} \) along \( V_2^4 \otimes \mathbb{Q} \). Moreover, \( q \) acts as zero on \( CH_2(F)_{\text{hom}} \). Thus \( \pi^2 \) acts like \( p^l \) on \( CH_2(F)_{\text{hom}, \mathbb{Q}} \). By the above, we also have a decomposition \( CH_2(F)_{\mathbb{Q}} = \text{im}\{ (\pi_4)^* \} \oplus \text{im}\{ (\pi_2)^* \} = \text{im}\{ (\pi_4)^* \} \oplus \ker\{ (\pi_4)^* \} \), where \( (\pi^l)^* \) and \( (\pi^2)^* \) are acting on \( CH_2(F) \mathbb{Q} \). Therefore, Murre’s conjecture (D) holds if and only if \( \ker\{ (\pi_4)^* \} = CH_2(F)_{\text{hom}, \mathbb{Q}} \) if and only if \( \text{im}\{ (\pi_2)^* \} = CH_2(F)_{\text{hom}, \mathbb{Q}} \) if and only if \( V_2^4 \otimes \mathbb{Q} = 0 \). As \( V_2^4 \) is torsion-free, this finishes the proof. \( \square \)

GLOSSARY

**V**: A fixed 6 dimensional complex vector space.

\( \{e_0, \ldots, e_5\} \): a basis of \( V \).

\( \{X_0, \ldots, X_5\} \): the dual basis of \( V^* \).

\( \mathbb{P}(V) \): the projectivization of \( V \) that parameterizes 1-dimensional subspaces of \( V \).

\( V_1 \subset V \): the tautological line bundle on \( \mathbb{P}(V) \).
Gr(2, V): the Grassmannian of 2-dimensional subspaces of V.

\( V_2 \subset V \): the tautological rank 2 bundle on Gr(2, V).

\( G = G(X_0, \ldots, X_5) \in \text{Sym}^3(V^*) \): a homogeneous polynomial of degree 3.

\( X \subset \mathbb{P}(V) \): smooth cubic fourfold define by \( G = 0 \).

\( h \in \text{Pic}(X) \): the class of a hyperplane section.

\( F \subset \text{Gr}(2, V) \): the variety of lines on \( X \), with the total family

\[
\begin{array}{c}
\text{P} & \xrightarrow{q} & X \\
\downarrow p & & \downarrow \Psi \\
F & &
\end{array}
\]

\( \Psi = q_\ast p^* : \text{CH}_i(F) \to \text{CH}_{i+1}(X) \): the cylinder homomorphism.

\( \Phi = p_\ast q^* : \text{CH}_i(X) \to \text{CH}_{i+1}(F) \): the Abel-Jacobi homomorphism.

\( \mathcal{E}_2 = V_2^\ast |_F = (p_\ast q^\ast \mathcal{O}_X(1))^\ast \): the restriction of tautological bundle.

\( g \in \text{Pic}(F) \): the Plücker polarization.

\( C_x \subset F \): the curve of all lines passing through a general point \( x \in X \).

\( l \subset X \): a line on \( X \).

\( [l] \in F \): the corresponding point on \( F \).

\( S_l \subset X \): the surface of all lines meeting \( l \).

\( l_s \subset F \): the line corresponding to a point \( s \in F \).

\( \mathcal{N}_{l/X} \): the normal bundle of \( l \) in \( X \).

\( T_{P/F} \): the relative tangent bundle of \( p : P \to F \).

\( \mathcal{N}_{P/X} = \text{coker}(T_{P/F} \to q^\ast T_X) \): the total normal bundle of lines on \( X \).

\( \mathcal{N}_{l/X}^+ \): the positive part of the normal bundle.

\( p_i : F \times F \to F \): the projection to the \( i \)-th factor, \( i = 1, 2 \).

\( \tilde{p}_i : P \times P \to P \): the projection to the \( i \)-th factor, \( i = 1, 2 \).

\( g_i = p_i^\ast g \in \text{Pic}(F \times F) \): pull-back of the polarization to the product.

\( c = c_2(\mathcal{E}_2) \in \text{CH}^2(F) \): second Chern class of the tautological bundle.

\( c_i = p_i^\ast c \in \text{CH}^2(F \times F) \): its pull-back on the product.

\( I \subset F \times F \): the incidence correspondence.

\( \Sigma_1 \subset F \): all lines \( [l] \in F \) such that \( l \) lies on a plane contained in \( X \).

\( \Sigma_2 \subset F \): all lines of second type.

\( H_2 \subset F \): a smooth linear section of \( F \) of dimension 2.

\( \mathcal{A} \subset \text{CH}_2(F) \): the subgroup generated by cycles of the form \( S_l \).

\( \mathcal{A}_{\text{hom}} \subset \mathcal{A} \): the group of homologically trivial cycles in \( \mathcal{A} \).

consider

\[
\begin{array}{c}
P \times X \xrightarrow{p \times p} F \times F \\
\downarrow q^{(2)} & \downarrow \Gamma_h \\
X &
\end{array}
\]

\( \Gamma_h = (p \times p)_\ast (q^{(2)})^\ast h \in \text{CH}^3(F \times F) \)

\( \Gamma_{h^2} = (p \times p)_\ast (q^{(2)})^\ast h^2 \in \text{CH}^4(F \times F) \)

\( V_{i}^n = \ker(\varphi^\ast - n : \text{CH}_{i}(F)_{\text{hom}} \to \text{CH}_{i}(F)_{\text{hom}}) \): the eigenspace of the action \( \varphi^\ast \) on \( \text{CH}_{i}(F)_{\text{hom}} \) corresponding to the eigenvalue \( n \).
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