Dynamical Aspects of Analogue Gravity: The Backreaction of Quantum Fluctuations in Dilute Bose-Einstein Condensates

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1 Analogue Gravity: An Overview

1.1 The concept of an effective space-time metric

Curved space-times are familiar from Einstein’s theory of gravitation [1], where the metric tensor $g_{\mu\nu}$, describing distances in a curved space-time with local Lorentz invariance, is determined by the solution of the Einstein equations. A major problem for an experimental investigation of the (kinematical as well as dynamical) properties of curved space-times is that generating a significant curvature, equivalent to a (relatively) small curvature radius, is a close to impossible undertaking in manmade laboratories. For example, the effect of the gravitation of the whole Earth is to cause a deviation from flat space-time on this planet’s surface of only the order of $10^{-8}$ (the ratio of Schwarzschild and Earth radii). The fact that proper gravitational effects are intrinsically small is basically due to the smallness of Newton’s gravitational constant $G = 6.67 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{sec}^{-2}$. Various fundamental classical and quantum effects in strong gravitational fields are thus inaccessible for Earth-based experiments. The realm of strong gravitational fields (or, equivalently, rapidly accelerating a reference frame to simulate gravity according to the equivalence principle), is therefore difficult to reach. However, Earth-based gravity experiments are desirable, because they have the obvious advantage that they can be prepared and, in particular, repeated under possibly different conditions at will.

A possible way out of this dilemma, at least inasmuch the kinematical properties of curved space-times are concerned, is the realization of effective curved space-time geometries to mimic the effects of gravity. Among the most suitable systems are Bose-Einstein condensates, i.e., the dilute matter-wave-coherent gases formed if cooled to ultralow temperatures, where the critical temperatures are of order $T_c \sim 1 \text{nK} \cdots 1 \mu\text{K}$; for reviews of the (relatively)
recent status of this rapidly developing field see [2, 3, 4]. In what follows, it will be of some importance that Bose-Einstein condensates belong to a special class of quantum perfect fluids, so-called superfluids [5].

The curved space-times we have in mind in the following are experienced by sound waves propagating on the background of a spatially and temporally inhomogeneous perfect fluid. Of primary importance is, first of all, to realize that the identification of sound waves propagating on an inhomogeneous background, which is itself determined by a solution of Euler and continuity equations, and photons propagating in a curved space-time, which is determined by a solution of the Einstein equations, is of a kinematical nature. That is, the space-time metric is fixed externally by a background field obeying the laws of hydrodynamics (which is prepared by the experimentalist), and not self-consistently by a solution of the Einstein equations.

As a first introductory step to understand the nature of the kinematical identity, consider the wave equation for the velocity potential of the sound field $\phi$, which in a homogeneous medium at rest reads

$$\left[ \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} - \Delta \right] \phi = 0,$$

where $c_s$ is the sound speed, which is a constant in space and time for such a medium at rest. This equation has Lorentz invariance, that is, if we replace the speed of light by the speed of sound, it retains the form shown above in the new space-time coordinates, obtained after Lorentz-transforming to a frame moving at a constant speed less than the sound speed. Just as the light field in vacuo is a proper relativistic field, sound is a “relativistic” field. The Lorentz invariance can be made more manifest by writing equation (1) in the form $\Box \phi \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = 0$, where $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$ is the (contravariant) flat space-time metric (we choose throughout the signature of the metric as specified here), determining the fundamental light-cone-like structure of Minkowski space [6]; we employ the summation convention over equal greek indices $\mu, \nu, \cdots$.

Assuming, then, the sound speed $c_s = c_s(x, t)$ to be local in space and time, and employing the curved space-time version of the 3+1D Laplacian $\Box$ [7, 8], one can write down the sound wave equation in a spatially and temporally inhomogeneous medium in the generally covariant form [7, 8]

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = 0.$$  \hspace{1cm} (2)

Here, $g = \det[ g_{\mu\nu} ]$ is the determinant of the (covariant) metric tensor. It is to be emphasized at this point that, because the space and time derivatives $\partial_\mu$ are

\footnote{More properly, we should term this form of Lorentz invariance \textit{pseudorelativistic} invariance. We will however use for simplicity “relativistic” as a generic term if no confusion can arise therefrom.}
c covariantly transforming objects in (2), the primary object in the condensed-matter identification of space-time metrics via the wave equation (2) is the contravariant metric tensor \( g^\mu_\nu \) [9]. In the condensed-matter understanding of analogue gravity, the quantities \( g^\mu_\nu \) are material-dependent coefficients. They occur in a dispersion relation of the form \( g^\mu_\nu k_\mu k_\nu = 0 \), where \( k_\mu = (\omega/c_s, \mathbf{k}) \) is the covariant wave vector, with \( \hbar \mathbf{k} \) the ordinary spatial momentum (or quasi-momentum in a crystal).

The contravariant tensor components \( g^\mu_\nu \) for a perfect, irrotational liquid turn out to be [7, 8, 10]

\[
 g^\mu_\nu = \left( \frac{1}{A_c} \right) c_s^2 \left( \frac{1}{v} - c_s^2 1 + v \otimes v \right),
\]

where \( 1 \) is the unit matrix and \( A_c \) a space and time dependent function, to be determined from the equations of motion for the sound field (see below). Inverting this expression according to \( g^\beta_\nu g^\nu_\alpha = \delta^\beta_\alpha \), to obtain the covariant metric \( g^\mu_\nu \), the fundamental tensor of distance reads

\[
 g^\mu_\nu = A_c \left( c_s^2 - v^2 \begin{pmatrix} v \\ v \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right),
\]

where the line element is \( ds^2 = g^\mu_\nu dx^\mu dx^\nu \). This form of the metric has been derived by Unruh for an irrotational perfect fluid described by Euler and continuity equations [7]; its properties were later on explored in more detail in particular by M. Visser [8]. We also mention that an earlier derivation of Unruh’s form of the metric exists, from a somewhat different perspective; it was performed by Trautman [10]. The metric belongs to the Painlevé-Gullstrand class of metrics, historically introduced in Refs. [11].

The conformal factor \( A_c \) in (4) depends on the spatial dimension of the fluid. It may be unambiguously determined by considering the effective action of the velocity potential fluctuations above an inhomogeneous background, identifying this action with the action of a minimally coupled scalar field in \( D + 1 \)-dimensional space-time

\[
 A_{\text{eff}} = \int d^{D+1}x \frac{1}{2g} \left[ \left( \frac{\partial}{\partial t} \phi - \mathbf{v} \cdot \nabla \phi \right)^2 - c_s^2 (\nabla^2 \phi)^2 \right] \equiv \frac{1}{2} \int d^{D+1}x \sqrt{-g g^\mu_\nu \partial_\mu \phi \partial_\nu \phi},
\]

where the prefactor \( 1/g \) in front of the square brackets in the first line is identified with the compressibility of the (barotropic) fluid, \( 1/g = d/(\ln \rho)/dp \), with \( p \) the pressure and \( \rho \) the mass density of the fluid; we assume here and in what follows that \( g \) is a constant independent of space and time so that \( c_s^2 = g \rho \), as valid for a dilute Bose gas (see the following subsection). Using the above identification, it may easily be shown that the conformal factor is given by \( A_c = (c_s/g)^{2/(D-1)} = (\rho/g)^{1/(D-1)} \), while the square root of the
negative determinant is $\sqrt{-g} = c_s (c_s / g)^{D+1/(D-1)} = (\rho^D / g)^{1/(D-1)}$. The case of one spatial dimension ($D = 1$) is special, in that the conformal invariance in two space-time dimensions implies that the classical equations of motion are invariant (take the same form) for any space and time dependent choice of the conformal factor $A_c$, explaining the singular character of the conformal factor at the special value $D = 1$.

The line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ gives us the distances travelled by the phonons in an effective space-time world in which the scalar field $\phi$ “lives”. In particular, quasiclassical (large momentum) phonons will follow light-like, that is, here, sound-like geodesics in that space-time, according to $ds^2 = 0$. Noteworthy is the simple fact that the constant time slices obtained by setting $dt = 0$ in the line element are conformally flat, i.e. the quasiparticle world looks on constant time slices like the ordinary (Newtonian) lab space, with a Euclidean metric in the case of Cartesian spatial co-ordinates we display. All the intrinsic curvature of the effective quasiparticle space-time is therefore encoded in the metric tensor elements $g_{00}$ and $g_{0i}$.

1.2 The metric in Bose-Einstein condensates

We assumed in Eq. (5) that the compressibility $1 / g$ is a constant. This entails that the (barotropic) equation of state reads $p = \frac{1}{2} g \rho^2$. We then have, in the microscopic terms of the interaction between the particles (atoms) constituting the fluid, a contact interaction (pseudo-)potential, $V(x-x') = g \delta(x-x')$. This is indeed the case for the dilute atomic gases forming a Bose-Einstein condensate. Well below the transition temperature, they are described, to lowest order in the gas parameter $(\rho a_s^3)^{1/2} \ll 1$ [where $a_s$ is the $s$-wave scattering length, assumed positive] by the Gross-Pitaevskiı mean-field equation for the order parameter $\Psi \equiv \langle \hat{\Psi} \rangle$, the expectation value of the quantum field operator $\hat{\Psi}$:

$$i \hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ -\frac{h^2}{2m} \Delta + V_{\text{trap}}(x, t) + g |\Psi(x, t)|^2 \right] \Psi(x, t). \tag{6}$$

Here, $V_{\text{trap}}$ denotes the one-particle trapping potential and the coupling constant $g$ is related to the $s$-wave scattering length $a_s$ via $g = 4\pi \hbar^2 a_s / m$ (in three spatial dimensions). The Madelung transformation decomposing the complex field $\Psi$ into modulus and phase reads $\Psi = \sqrt{\rho} \exp[i\phi]$, where $\rho$ yields the condensate density and $\phi$ is the velocity potential. It allows for an interpretation of quantum theory in terms of hydrodynamics. Namely, identifying real
and imaginary parts on left- and right-hand sides of (6), respectively, gives us the two equations

\[-\hbar \frac{\partial}{\partial t} \phi = \frac{1}{2} m v^2 + V_{\text{trap}} + g \rho - \frac{\hbar^2}{2m} \Delta \sqrt{\rho},\]

\[\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho v) = 0.\]

The first of these equations is the Josephson equation for the superfluid phase. This Josephson equation corresponds to the Bernoulli equation of classical hydrodynamics, where the usual velocity potential of irrotational hydrodynamics equals the superfluid phase \(\phi\) times \(\hbar/m\), such that \(v = \hbar \nabla \phi / m\). The latter equation implies that the flow is irrotational save on singular lines, around which the wave function phase \(\phi\) is defined only modulo 2\(\pi\). Therefore, circulation is quantized \(^{14}\), and these singular lines are the center lines of quantized vortices. The usual classical terms in Eq. (7) are augmented by the “quantum pressure” \(p_Q \equiv -\frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}\). The second equation (8) is the continuity equation for conservation of particle number, i.e., atom number in the superfluid gas. The dynamics of the weakly interacting, dilute ensemble of atoms is thus that of a perfect Euler fluid with quantized circulation of singular vortex lines. This is true except for regions in which the density rapidly varies and the quantum pressure term \(p_Q\) becomes relevant, which happens on scales of order the coherence length \(^{3}\) \(\xi = \hbar / \sqrt{gm\rho}\) where \(\rho\) is a constant (asymptotic) density far away from the density-depleted (or possibly density-enhanced) region. This is the case in the density-depleted cores of quantized vortices, or at the low-density boundaries of the system. The quantum pressure is negligible outside these domains of rapidly varying and/or low density. The whole armoury of space-time metric description of excitations, explained in the last section, and based on the Euler and continuity equations, is then valid for phonon excitations of a Bose-Einstein condensate, with the space-time metric \(^{11}\), as long as we are outside the core of quantized vortices and far from the boundaries of the system, where both the flow is irrotational and the quantum pressure is negligible.

We mention here in passing that the form (2) of the wave equation is valid in quite general physical contexts. That is, a generally covariant curved space-time wave equation can be formulated not just for the velocity perturbation potential in an irrotational Euler fluid, for which we have introduced the effective metric concept. If the spectrum of excitations (in the local rest frame) is linear, \(\omega = c_{\text{prop}} k\), where \(c_{\text{prop}}\) is the propagation speed of some collective excitation, the statement that an effective space-time metric exists is true, provided we only consider wave perturbations of a single scalar field \(\Phi\) constituting a fixed classical background: More precisely, given the generic requirement that the action density \(L\) is a functional of \(\Phi\) and its space-time

\(^{3}\) Note that the coherence or “healing” length is also frequently defined in the literature, see e.g. \(^{13}\), with an additional factor of \(1/\sqrt{2}\), i.e., as \(\xi_c = \hbar / \sqrt{2gm\rho}\).
derivatives $\partial_\mu \Phi$, i.e. $L = L[\Phi, \partial_\mu \Phi]$, the fluctuations $\phi \equiv \delta \Phi$ around some classical background solution $\Phi_0$ of the Euler-Lagrange equations always satisfy a wave equation of the form of Eq. (2), with a possible additional scalar potential term [15] comprising, for example, a mass term for the scalar field. As a consequence, the effective metric description also applies, inter alia, to the quasiparticle excitations around the gap nodes in the superfluid $^3$He-A [16], photon propagation in dielectrics [17], and surface waves in shallow water [18].

The analogy between photons propagating on given (curved) space-time backgrounds and phonons in spatially and temporally inhomogeneous superfluids or, more generally, quantized quasiparticles with linear quasiparticle dispersion in some background, allows us to apply many tools and methods developed for quantum fields in curved space-times [19]. We can therefore conclude that the associated phenomena occur (provided the fundamental commutation relations of these quantum fields are fulfilled [20]). Among these phenomena are Hawking radiation [21], the Gibbons-Hawking effect in de Sitter spacetime [22], and cosmological particle production [23, 24]. Furthermore, cosmic inflation and the freezing-in of quantum vacuum fluctuations by the horizon crossing of the corresponding modes may be simulated [25]. A comprehensive recent review of the subject of analogue gravity in its broadest sense is given in [26].

1.3 Pseudo-Energy-Momentum Tensor

An important quantity characterizing the dynamics of the field $\phi$ is the pseudo-energy-momentum tensor, cf. [27]. Since the equation of motion for the scalar mode, $\nabla_\mu g^{\mu\nu} \nabla_\nu \phi = 0$, where $\nabla_\mu$ denotes the space-time-covariant derivative, is equivalent to covariant energy-momentum balance, expressed by $\nabla_\mu T^{\mu\nu} = 0$, the classical pseudo-energy-momentum tensor reads [27]

$$T^{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} g^{\mu\nu} (\partial_\rho \phi)(\partial_\sigma \phi) g^{\rho\sigma}.$$  

We already stressed that the identification of field theoretical effects in curved space-time by analogy (including the existence of the pseudo-energy-momentum tensor), is of a kinematical nature. An important question concerns the dynamics, that is, the backreaction of the quantum fluctuations of the scalar field onto the classical background. Extending the analogy to curved spacetimes a bit further, one is tempted to apply the effective-action method (see, e.g., [19] and [28]). In the effective-action method, one integrates out fluctuations of the quantum fields to one-loop order, and then determines the expectation value of the energy-momentum-tensor by the canonical identification $\delta A_{\text{eff}}/\delta g^{\mu\nu} \equiv \frac{1}{2} \sqrt{-g} \langle T_{\mu\nu} \rangle$. Since the dependence of the effective action $A_{\text{eff}}$ on the degrees of freedom of the background $\eta$ enters via the effective metric $g^{\mu\nu} = g^{\mu\nu}(\eta)$, one finds the backreaction contribution to the equations of motion of the $\eta$ by differentiation of the effective action according to
\[
\frac{\delta A_{\text{eff}}}{\delta \eta} = \frac{\delta A_{\text{eff}}}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta \eta} = \frac{1}{2} \sqrt{-g} \left\langle \hat{T}_{\mu\nu} \right\rangle \frac{\delta g^{\mu\nu}}{\delta \eta} \quad (10)
\]

The precise meaning of the expectation value of the pseudo-energy-momentum tensor, \(\langle \hat{T}_{\mu\nu} \rangle\), is difficult to grasp in general, due to the non-uniqueness of the vacuum state in a complicated curved space-time background and the ultra-violet (UV) renormalization procedure. Adopting a covariant renormalization scheme, the results for \(\langle \hat{T}_{\mu\nu} \rangle\) can be classified in terms of geometrical quantities (cf. the trace anomaly [19]).

However, in calculating the quantum backreaction using the effective-action method, one is implicitly making two essential assumptions: first, that the leading contributions to the backreaction are completely determined by the effective action in Eq. (5), and, second, that deviations from the low-energy effective action at high energies do not affect the (renormalized) expectation value of the pseudo-energy-momentum tensor, \(\langle \hat{T}_{\mu\nu} \rangle\). Since the effective covariance in Eq. (5) is only a low-energy property, the applicability of a covariant renormalization scheme is not obvious in general. In the following, we critically examine the question of whether the two assumptions mentioned above are justified, e.g., whether \(\langle \hat{T}_{\mu\nu} \rangle\) completely determines the backreaction of the linearized quantum fluctuations.

The remainder of these lecture notes, which is based on the results of the publication [29], is organized as follows. In section 2 we give a brief introduction to Bose-Einstein condensates, and introduce a particle-number-conserving ansatz for the field operator separated into condensate, single-particle and multi-particle excitation parts. In the subsequent section 3 based on this ansatz, the backreaction onto the motion of the fluid using the full current will be calculated, and it is shown that this yields a different result than that obtained by the effective-action method. Afterwards, the failure of the effective-action technique is discussed in more detail in section 4. The cutoff dependence of the pseudo-energy-momentum tensor (9) is addressed in section 5. As a simple example, we consider the influence of the backreaction contribution on a static quasi-1D condensate in section 6.

2 Excitations in Bose-Einstein Condensates

2.1 Particle-number-conserving mean-field expansion

In the \(s\)-wave scattering approximation, a dilute many-particle system of interacting bosons is described, on a “microscopic” level corresponding to distance scales much larger than the true range of the interaction potential, by the field operator equation of motion in the Heisenberg picture (we set from now on \(\hbar = m = 1\))

\[
i \frac{\partial}{\partial t} \hat{\Psi} = \left( -\frac{1}{2} \nabla^2 + V_{\text{trap}} + g \hat{\Psi}^\dagger \hat{\Psi} \right) \hat{\Psi} \quad (11)
\]
In the limit of many particles $N \gg 1$, in a finite trap at zero temperature with almost complete condensation, the full field operator $\hat{\Psi}$ can be represented in terms of the particle-number-conserving mean-field ansatz \[30, 31\]

\[
\hat{\Psi} = \left( \psi_c + \hat{\chi} + \hat{\zeta} \right) \hat{A} N^{-1/2}.
\] \tag{12}

Here, the order parameter $\psi_c = \mathcal{O}(\sqrt{N})$ [note that $\psi_c \neq \langle \hat{\Psi} \rangle$, as opposed to the $\Psi$ in Eq. (6)]. The one-particle excitations are denoted $\hat{\chi} = \mathcal{O}(N^0)$, where one-particle here means that the Fourier components of $\hat{\chi}$ are linear superpositions of annihilation and creation operators of quasiparticles $\hat{a}_k$ and $\hat{a}^\dagger_k$, cf. Eq. (38) below. The remaining higher-order, multi-particle corrections are described by $\hat{\zeta} = \mathcal{O}(1/\sqrt{N})$. The above mean-field ansatz can be derived in the dilute-gas limit by formally setting $g = \mathcal{O}(1/N)$ [31, 32, 33, 34]; we shall use this formal definition of the dilute-gas limit in what follows. The dilute-gas limit should be compared and contrasted with the usual thermodynamic limit, in which the density and particle interaction remains constant, while the size of the (trapped) system increases with $N \to \infty$, adjusting the harmonic trapping potential $V_{\text{trap}}$ correspondingly (in $D$ spatial dimensions, the thermodynamic limit corresponds to keeping $N \omega^D$ constant for $N \to \infty$, where $\omega$ is the geometric mean of the trapping frequencies [4]). In the presently used dilute-gas limit, on the other hand, the trapping potential remains constant, but the interaction and the density change. The advantage of the limit $gN$ constant is that in this limit we have a well-defined prescription to implement the mean-field approximation, keeping one power of $g$ for each factor of $N$, cf. [36].

2.2 Gross-Pitaevskii and Bogoliubov-de Gennes equations

Insertion of Eq. (12) into (11) yields to $\mathcal{O}(N)$ the Gross-Pitaevskii equation \[35\] for the order parameter $\psi_c$

\[
i \frac{\partial}{\partial t} \psi_c = \left( -\frac{1}{2} \nabla^2 + V_{\text{trap}} + 2g|\psi_c|^2 + 2g \langle \hat{\chi}^\dagger \hat{\chi} \rangle \right) \psi_c + g \langle \hat{\chi}^2 \rangle \psi_c^*.
\] \tag{13}

The Bogoliubov-de Gennes equations \[37\] for the one-particle fluctuations $\hat{\chi}$ are obtained to $\mathcal{O}(N^0)$

\[
i \frac{\partial}{\partial t} \hat{\chi} = \left( -\frac{1}{2} \nabla^2 + V_{\text{trap}} + 2g|\psi_c|^2 \right) \hat{\chi} + g\psi_c^2 \hat{\chi}^\dagger,
\] \tag{14}

Finally, the time evolution of the remaining higher-order corrections in the expansion (12), $\hat{\zeta} = \mathcal{O}(1/\sqrt{N})$, neglecting the $\mathcal{O}(1/N)$ terms, is given by:

\[
i \frac{\partial}{\partial t} \hat{\zeta} \approx \left( -\frac{1}{2} \nabla^2 + V_{\text{trap}} + 2g|\psi_c|^2 \right) \hat{\zeta} + g\psi_c^2 \hat{\zeta}^\dagger
\]

\[
+ 2g(\hat{\chi}^\dagger \hat{\chi} - \langle \hat{\chi}^\dagger \hat{\chi} \rangle) \psi_c + g(\hat{\chi}^2 - \langle \hat{\chi}^2 \rangle) \psi_c^*.
\] \tag{15}

The Gross-Pitaevskii equation in the form (13) ought to be compared with the simple-minded form of (6). The additional terms $2g\langle \hat{\chi}^\dagger \hat{\chi} \rangle$ and $g\langle \hat{\chi}^2 \rangle$ in the
Gross-Pitaevskii equation in the form of Eq. (13) above ensure that the expectation value of the multi-particle operator, $\zeta = \mathcal{O}(1/\sqrt{N})$, vanishes in leading order, $\langle \zeta \rangle = \mathcal{O}(1/N)$. Without these additional terms, the mean-field expansion (12) would still be valid with $\zeta = \mathcal{O}(1/\sqrt{N})$, but without $\langle \zeta \rangle = \mathcal{O}(1/N)$. The proper incorporation of the so-called “anomalous” fluctuation average $\langle \hat{\chi}^2 \rangle$ (the “normal” fluctuation average is $\langle \hat{\chi}^\dagger \hat{\chi} \rangle$) into the description of Bose-Einstein condensates has also been discussed from various points of view in [31, 38, 39].

3 Quantum Backreaction

3.1 Calculation of backreaction force from microscopic physics

The observation that the Gross-Pitaevskii equation (13) yields an equation correct to leading order $\mathcal{O}(\sqrt{N})$, using either $|\psi_c|^2$ or $|\psi_c|^2 + 2 \langle \hat{\chi}^\dagger \hat{\chi} \rangle$ in the first line of (13), hints at the fact that quantum backreaction effects correspond to next-to-leading, i.e. quadratic order terms in the fluctuations and cannot be derived ab initio in the above manner without additional assumptions. Therefore, we shall employ an alternative method: In terms of the exact density and current given by

$$\rho = \langle \hat{\psi}^\dagger \hat{\psi} \rangle, \quad j = \frac{1}{2i} \langle \hat{\psi}^\dagger \nabla \hat{\psi} - \text{H.c.} \rangle,$$

(16)

the time-evolution is governed by the equation of continuity for $\rho$ and an Euler type equation for the current $j$. After insertion of Eq. (11), we find that the equation of continuity is not modified by the quantum fluctuations but satisfied exactly (i.e., to all orders in $1/N$ or $\hbar$)

$$\frac{\partial}{\partial t} \rho + \nabla \cdot j = 0,$$

(17)

in accordance with the $U(1)$ invariance of the Hamiltonian and the Noether theorem, cf. [28]. However, if we insert the mean-field expansion (12) and write the full density as a sum of condensed and non-condensed parts

$$\rho = \rho_c + \langle \hat{\chi}^\dagger \hat{\chi} \rangle + \mathcal{O}(1/\sqrt{N}),$$

(18)

with $\rho_c = |\psi_c|^2$, we find that neither part is conserved separately in general. Note that this split requires $\langle \zeta \rangle = \mathcal{O}(1/N)$, i.e., the modifications to the Gross-Pitaevskii equation (13) discussed above. Similarly, we may split up the full current [with $\rho_c \nu_c = \Im(\psi_c^* \nabla \psi_c)$]

$$j = \rho_c \nu_c + \frac{1}{2i} \langle \hat{\chi}^\dagger \nabla \hat{\chi} - \text{H.c.} \rangle + \mathcal{O}(1/\sqrt{N}),$$

(19)

and introduce an average velocity $\nu$ via $j = \rho \nu$. This enables us to unambiguously define the quantum backreaction $Q$ as the following additional contribution in an equation of motion for $j$ analogous to the Euler equation:
\[ \frac{\partial}{\partial t} j = f_{cl}(j, \varrho) + Q + \mathcal{O}(1/\sqrt{N}), \tag{20} \]

where the classical force density term
\[ f_{cl}(j, \varrho) = -v [\nabla \cdot (\varrho v)] - \varrho (v \cdot \nabla) v + \varrho \nabla \left( \frac{1}{2} \frac{\nabla^2 \sqrt{\varrho}}{\sqrt{\varrho}} \right) - V_{\text{trap}} - g \varrho \tag{21} \]

Here, “classical” means that the force density contains no explicit quantum fluctuation terms (i.e., only those absorbed in the full density and full current), and in addition just the “quantum pressure”, which already occurs on the mean-field level. Formulation in terms of the conventional Euler equation, i.e., using a convective derivative of the velocity defined by \( j = \rho \mathbf{v} \) giving the acceleration, yields
\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\nabla \left( V_{\text{trap}} + g \varrho - \frac{1}{2} \frac{\nabla^2 \sqrt{\varrho}}{\sqrt{\varrho}} \right) + \frac{Q}{\varrho} + \mathcal{O}(N^{-3/2}). \tag{22} \]

The quantum backreaction force density \( Q \) can now be calculated by comparing the two equations above and expressing \( \partial j / \partial t \) in terms of the field operators via Eqs. (11) and (16)
\[ \frac{\partial}{\partial t} j = \frac{1}{4} \left\langle \hat{\psi}^\dagger \nabla^3 \hat{\psi} - (\nabla^2 \hat{\psi})^\dagger \nabla \hat{\psi} + \text{H.c.} \right\rangle 
- \left\langle \hat{\psi}^\dagger \hat{\psi} \right\rangle \nabla V_{\text{trap}} - \frac{1}{2g} \nabla \left\langle g^2 (\hat{\psi})^2 \hat{\psi}^2 \right\rangle. \tag{23} \]

After insertion of the mean-field expansion (12), we obtain the leading contributions in the Thomas-Fermi limit
\[ Q = \nabla \cdot (\mathbf{v} \otimes \hat{j}_\chi + \hat{j}_\chi \otimes \mathbf{v} - \varrho_\chi \mathbf{v} \otimes \mathbf{v}) 
- \frac{1}{2g} \nabla \left\langle 2g^2 (\hat{\psi}_c^2 \hat{\chi}_1^\dagger \hat{\chi}_1 + \psi_2^2 (\hat{\chi}_1^\dagger)^2 + (\psi_2^* \hat{\chi}_1^\dagger)^2) \right\rangle 
- \frac{1}{2} \nabla \cdot \left\langle (\nabla \hat{\chi}_1^\dagger) \otimes \nabla \hat{\chi}_1 + \text{H.c.} \right\rangle, \tag{24} \]

with \( \varrho_\chi = \langle \hat{\chi}_1^\dagger \hat{\chi}_1 \rangle \) and \( \hat{j}_\chi = \Im \langle \hat{\chi}_1^\dagger \nabla \hat{\chi}_1 \rangle \). Under the assumption that the relevant length scales \( \lambda \) for variations of, e.g., \( \varrho \) and \( g \), are much larger than the healing length \( \xi = (gg)^{-1/2} \), we have neglected terms containing quantum pressure contributions \( \nabla^2 \varrho \) and \( [\nabla \varrho]^2 \), which amounts to the Thomas-Fermi or local-density approximation. These contributions would, in particular, spoil the effective (local) geometry in Eq. (3) (the inclusion of the quantum pressure to derive a “nonlocal metric” has been discussed in [10]).

### 3.2 Comparison with effective-action technique

To compare the expression for the backreaction force density derived from the full dynamics of \( \hat{\Psi} \) with the force obtained from Eq. (10), we have to identify the scalar field \( \phi \) and the contravariant metric \( g^{\mu \nu} \). We already
know that phonon modes with wavelength $\lambda \gg \xi$ are described by the action in Eq. (5) in terms of the phase fluctuations $\phi$ provided that $g_{\mu\nu}$ is given by (3), where $A_c = c_s/g$ and $\sqrt{-g} = c_s^3/g^2 = \rho^2/c_s$ in three spatial dimensions. The density fluctuations $\delta \rho$ are related to the phase fluctuations $\phi$ via

$$
\delta \rho = -g^{-1}(\partial / \partial t + v \cdot \nabla )\phi.
$$

The variables $\eta = \{ \rho_b, \phi_b \}$ or alternatively $\eta = \{ \rho_b, \nabla \phi_b \}$ in (10) are then defined by the expectation values of density and phase operators according to

$$
\hat{\Phi} = \langle \hat{\Phi} \rangle + \hat{\phi} = \phi_b + \hat{\phi},
$$

$$
\hat{\rho} = \langle \hat{\rho} \rangle + \delta \hat{\rho} = \rho_b + \delta \hat{\rho}.
$$

The phase operator can formally be introduced via the following ansatz for the full field operator

$$
\hat{\Psi} = e^{i\hat{\Phi} \sqrt{\hat{\rho}}}.
$$

Since $\hat{\phi}$ and $\hat{\rho}$ do not commute, other forms such as $\hat{\Psi} = \sqrt{\hat{\rho}} e^{i\hat{\Phi}}$ would not generate a self-adjoint $\hat{\Phi}$ (and simultaneously satisfy $\hat{\Psi}^\dagger \hat{\Psi} = \hat{\rho}$). Note that, in contrast to the full density which is a well-defined and measurable quantity, the velocity potential, $\hat{\Phi}$, is not \[41\]. This can be seen as follows. The commutator between density and phase operators

$$
[\hat{\rho}(r), \hat{\phi}(r')] = i\delta(r - r'),
$$

yields, if one takes, on both sides, matrix elements in the number basis of the space integral over $r$ for a given volume $V$ in which $r'$ is situated,

$$
(N - N')\langle N|\hat{\phi}(r')|N'\rangle = i\delta_{NN'},
$$

where $N$ and $N'$ are two possible values for the number of particles in $V$. This is an inconsistent relation for $N, N'$ positive semidefinite and discrete (which the very existence of particles requires), most obviously for $N = N'$. The commutator therefore makes sense only if it is understood to be effectively coarse-grained over a sub-volume $V$ with large enough number of particles $N \gg 1$, such that the inconsistency inherent in (28) becomes asymptotically irrelevant. It cannot be defined consistently locally, i.e., in arbitrarily small volumina, where there is just one particle or even none, or when the number fluctuations in larger volumina $V$ are large, such that the probability to have a very small number of particles in $V$ is not negligible \[42\].

The action in terms of the total density $\varrho$ and the variable $\Phi$ reads (neglecting the quantum pressure term, i.e., in the Thomas-Fermi limit)

$$
\mathcal{L} = -\varrho \left( \frac{\partial}{\partial t} \Phi + \frac{1}{2} (\nabla \Phi)^2 \right) - \epsilon [\varrho] - V_{\text{trap}} \varrho,
$$

with $\epsilon [\varrho]$ denoting the internal energy density. The quantum corrections to the Bernoulli equation up to second order in the fluctuations, using the effective-action method in Eq. (10) are then incorporated by writing (the background density here equals the full density, $\varrho = \varrho_b$)
\[
\frac{\partial}{\partial t}\phi_b + \frac{1}{2}(\nabla\phi_b)^2 + h[\phi] - \frac{\delta\mathcal{A}_{\text{eff}}}{\delta\phi} = 0, \tag{30}
\]

where \(h[\phi] = dc/d\phi + V_{\text{trap}}\). We obtain, using (32),

\[
\frac{\delta\mathcal{A}_{\text{eff}}}{\delta\phi} = \frac{\delta\mathcal{A}_{\text{eff}}}{\delta g_{\mu\nu}} \frac{\delta g_{\mu\nu}}{\delta\phi} = \frac{\sqrt{-g}}{2} \langle \hat{T}_{\mu\nu} \rangle \frac{\delta g_{\mu\nu}}{\delta\phi} = -\frac{1}{2} \langle (\nabla\hat{\phi})^2 \rangle. \tag{31}
\]

Clearly, taking the gradient of this result we obtain a backreaction force which markedly differs from the expression (24) derived in the previous subsection. Moreover, it turns out that the backreaction force density in Eq. (24) contains contributions which are not part of the expectation value of the pseudo-energy-momentum tensor, \(\langle \hat{T}_{\mu\nu} \rangle\). For example, the phonon density \(\rho_\chi\) contains \(\langle (\delta\hat{\rho})^2 \rangle_{\text{ren}}\) (where \(\langle \ldots \rangle_{\text{ren}}\) means that the divergent c-number \(\hat{\chi}\hat{\chi}^\dagger - \hat{\chi}^\dagger\hat{\chi} = \delta(0)\) has been subtracted already) which is part of \(\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}\), but also \(\langle \hat{\phi}^2 \rangle_{\text{ren}}\) which is not. [Note that \(\langle \hat{\phi}^2 \rangle_{\text{ren}}\) cannot be cancelled by the other contributions.] The expression \(\langle (\nabla\hat{\chi})^\dagger \otimes \nabla\hat{\chi} + \text{H.c.} \rangle\) in the last line of Eq. (24) contains \(\langle \nabla\hat{\phi} \otimes \nabla\hat{\phi} \rangle_{\text{ren}}\) which does occur in \(\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}}\), but also \(\langle \nabla\delta\hat{\rho} \otimes \nabla\delta\hat{\rho} \rangle_{\text{ren}}\), which does not. One could argue that the latter term ought to be neglected in the Thomas-Fermi or local-density approximation since it is on the same footing as the quantum pressure contributions containing \(\nabla^2\hat{\rho}\) and \(\nabla\hat{\rho}^2\) [which have been neglected in (24)], but it turns out that this expectation value yields cutoff dependent contributions of the same order of magnitude as the other terms, see section 5 below.

### 4 Failure of Effective-Action Technique

After having demonstrated the failure of the effective-action method for deducing the quantum backreaction, let us study the reasons for this failure in more detail. The full action governing the dynamics of the fundamental field \(\Psi\) reads

\[
\mathcal{L}^\Psi = i\Psi^* \frac{\partial}{\partial t}\Psi - \Psi^* \left( -\frac{1}{2} \nabla^2 + V_{\text{trap}} + \frac{g}{2} \Psi^*\Psi \right) \Psi. \tag{32}
\]

Linearization according to \(\Psi = \psi_c + \chi\) yields the effective second-order action generating the Bogoliubov-de Gennes equations (14)

\[
\mathcal{L}_{\text{eff}}^\chi = i\chi^* \frac{\partial}{\partial t}\chi - \chi^* \left( -\frac{1}{2} \nabla^2 + V_{\text{trap}} + 2g|\psi_c|^2 \right) \chi - \frac{1}{2} \left[ g(\psi_c^*)^2 \chi^2 \right. + \text{H.c.} \right]. \tag{33}
\]

If we start with the action (29) in terms of the total density \(\rho\) and the nonfundamental variable \(\Phi\), the quantum corrections to the equation of continuity \(\delta\mathcal{A}_{\text{eff}}/\delta\phi_b\) are reproduced correctly but the derived quantum backreaction contribution to the Bernoulli equation, \(\delta\mathcal{A}_{\text{eff}}/\delta\phi_b\), and therefore the backreaction force, is wrong.
One now is led to the question why the effective-action method works for the fundamental field \( \Psi \) and gives the correct expression for the backreaction force, but fails for the non-fundamental variable \( \Phi \). The quantized fundamental field \( \hat{\Psi} \) satisfies the equation of motion (11) as derived from the above action and possesses a well-defined linearization via the mean-field expansion (12). One of the main assumptions of the effective-action method is a similar procedure for the variable \( \Phi \), i.e., the existence of a well-defined and linearizable full quantum operator \( \hat{\Phi} \) satisfying the quantum Bernoulli equation (for large length scales), cf. Eq. (30)

\[
\frac{\partial}{\partial t} \hat{\Phi} + \frac{1}{2} (\nabla \hat{\Phi})^2 + \hbar [\hat{\Phi}] = 0.
\]  

(34)

The problem is that the commutator of \( \hat{\rho} \) and \( \hat{\Phi} \) at the same position diverges, cf. Eq. (27), and hence the quantum Madelung ansatz in Eq. (26) is singular. As a result, the above quantum Bernoulli equation is not well-defined (in contrast to the equation of continuity), i.e., insertion of the quantum Madelung ansatz in Eq. (26) into Eq. (11) generates divergences [41].

In order to study these divergences by means of a simple example, let us consider a generalized Bose-Hubbard Hamiltonian [43], which considers bosons sitting on a lattice with sites \( i \), which can hop between nearest neighbor sites and interact if at the same site:

\[
\hat{H} = \frac{-\alpha}{2} \sum_{<ij>} (\hat{\Psi}_i^\dagger \hat{\Psi}_j + H.c.) + \sum_i \left( \beta_i \hat{n}_i + \frac{\gamma}{2} \hat{n}_i^2 \right),
\]

(35)

where \( \hat{\Psi}_i \) is the annihilation operator for bosons at a given lattice site \( i \), and \( <ij> \) denotes summation over nearest neighbors; \( \hat{n}_i = \hat{\Psi}_i^\dagger \hat{\Psi}_i \) is the so-called filling factor (operator), equal to the number of bosons at the lattice site \( i \). The quantities \( \beta_i \) multiplying the filling factor depend on the site index. In the continuum limit, the lattice Hamiltonian (35) generates a version of Eq. (11). Setting \( a^{D/2} \hat{\Psi}_i(x_i) = \hat{\Psi}_i \), where the \( \hat{\Psi}_i(x_i) \) are the continuum field operators and \( a \) is the lattice spacing taken to zero (we consider for simplicity a simple cubic lattice in \( D \) spatial dimensions), the effective mass is given by \( 1/m^* = aa^2 \). The bosons moving through the lattice obviously become the heavier the smaller the hopping amplitude \( \alpha \) becomes at given \( a \). The coupling constant is determined by \( g = \gamma a^D \), and the trap potential is governed by \( V_{\text{trap}}(x) = g/2 + \beta(x) - \alpha \).

On the other hand, inserting the quantum Madelung ansatz employing a phase operator, Eq. (26) in its lattice version, the problem of operator ordering arises and the (for the Bernoulli equation) relevant kinetic energy term reads

\[
\hat{H}_\Phi = \frac{1}{4} \sum_i \sqrt{\hat{n}_i(\hat{n}_i+1)} (\nabla \hat{\Phi})_i^2 + \text{H.c.},
\]

(36)

with the replacement \( \hat{n}_i + 1 \) instead of \( \hat{n}_i \) being one effect of the non-commutativity. In the superfluid phase with large filling \( n \gg 1 \), we therefore obtain the following leading correction to the equation of motion
\[
\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} (\nabla \hat{\phi})^2 + h [\partial] + \frac{1}{n} \left( \frac{\nabla \hat{\phi}}{16} \right)^2 \frac{1}{n} = O \left( \frac{1}{n^3} \right). \tag{37}
\]

The Bernoulli equation in its quantum version thus receives corrections depending on microscopic details like the filling of a particular site and is not of the (conjectured) form (34).

By means of this simple example, we already see that the various limiting procedures such as the quantization and subsequent mean-field expansion, the variable transformation \((\Psi^*, \Psi) \leftrightarrow (\hat{\rho}, \Phi)\), and the linearization for small fluctuations, as well as continuum limit do not commute in general – which explains the failure of the effective-action method for deducing the quantum backreaction. The variable transformation \((\Psi^*, \Psi) \leftrightarrow (\hat{\rho}, \Phi)\) is applicable to the zeroth-order equations of motion for the classical background as well as to the first-order dynamics of the linearized fluctuations – but the quantum backreaction is a second-order effect, where the aforementioned difficulties, such as the question of the choice of fundamental variables and their operator ordering, arise.

### 5 Cutoff Dependence of Effective Action

Another critical issue for the applicability of the effective-action method is the UV divergence of \(\langle \hat{T}_{\mu \nu} \rangle\). Extrapolating the low-energy effective action in Eq. (5) to large momenta \(k\), the expectation values \(\langle \hat{\rho}^2 \rangle\) and \(\langle \hat{\phi}^2 \rangle\) entering \(\hat{\rho}_\chi\) would diverge. For Bose-Einstein condensates, we may infer the deviations from Eq. (5) at large \(k\) from the Bogoliubov-de Gennes equations (14). Assuming a static and homogeneous background (which should be a good approximation at large \(k\)), a normal-mode expansion yields a Bogoliubov transformation between the bare bosonic operators \(\hat{\chi}_\mathbf{k}\) and the quasiparticle operators \(\hat{a}_\mathbf{k}, \hat{a}^\dagger_\mathbf{k}\):

\[
\hat{\chi}_\mathbf{k} = \sqrt{\frac{k^2}{2\omega_k}} \left[ \left( \frac{\omega_k}{k^2} - \frac{1}{2} \right) \hat{a}^\dagger_\mathbf{k} + \left( \frac{\omega_k}{k^2} + \frac{1}{2} \right) \hat{a}_\mathbf{k} \right], \tag{38}
\]

where the frequency \(\omega_k\) is determined by the Bogoliubov dispersion relation for the dilute Bose gas, \(\omega^2_k = g\rho k^2 + k^4 / 4\). The above form of the Bogoliubov transformation results, after inversion, in the usual phonon quasiparticle operators at low momenta, and gives \(\hat{\chi}_\mathbf{k} = \hat{a}_\mathbf{k}\) at \(k \to \infty\), i.e., the quasiparticles and the bare bosons become, as required, identical at large momenta.

Using a linear dispersion \(\omega^2_k \propto k^2\) instead of the full Bogoliubov dispersion, expectation values such as \(\langle \hat{\chi}^\dagger_\mathbf{k} \hat{\chi} \rangle\) would be UV divergent, but the correct dispersion relation implies \(\hat{\chi}_\mathbf{k} \sim \hat{a}_\mathbf{k} \rho / k^2 + \hat{a}_\mathbf{k}\) for large \(k^2\), and hence \(\langle \hat{\chi}^\dagger_\mathbf{k} \hat{\chi} \rangle\) is UV finite in three and lower spatial dimensions. Thus the healing length \(\xi\) acts as an effective UV cutoff, \(k^\text{cut} \equiv 1 / \xi\).

Unfortunately, the quadratic decrease for large \(k\) in Eq. (38), giving asymptotically \(\hat{\chi}_\mathbf{k} \sim \hat{a}_\mathbf{k} \rho / k^2 + \hat{a}_\mathbf{k}\), is not sufficient for rendering the other expectation values (i.e., apart from \(\hat{\rho}_\chi\) and \(\hat{j}_\chi\)) in Eq. (24) UV finite in three spatial
dimensions. This UV divergence indicates a failure of the s-wave pseudo-potential $g\delta^3(r - r')$ in Eq. (11) at large wavenumbers $k$ and can be eliminated by replacing $g\delta^3(r - r')$ by a more appropriate two-particle interaction potential $V_{\text{int}}(r - r')$, see [36]. Introducing another UV cutoff wavenumber $k_{\text{cut}}^s$ related to the breakdown of the s-wave pseudo-potential, we obtain

$$\langle (\nabla \hat{\chi} \hat{\chi}^\dagger) \otimes \nabla \hat{\chi} + \text{H.c.} \rangle \sim g^2 \varrho^2 k_{\text{cut}}^s$$

and

$$\langle \hat{\chi}^2 \rangle \sim g \varrho k_{\text{cut}}^s.$$

In summary, there are two different cutoff wavenumbers: The first one, $k_{\text{cut}}^s$, is associated to the breakdown of the effective Lorentz invariance (change of dispersion relation from linear to quadratic) and renders some – but not all – of the naively divergent expectation values finite. The second wavenumber, $k_{\text{cut}}^s$, describes the cutoff for all (remaining) UV divergences. In dilute Bose-Einstein condensates, these two scales are vastly different by definition. Because the system is dilute, the inverse range of the true potential $k_{\text{cut}}^s \equiv 1/r_0$, must be much larger than the inverse healing length. Thus the following condition of scale separation must hold:

$$k_{\text{cut}}^s \gg k_{\text{cut}}^\xi = k_{\text{cut}}^\text{Lorentz} \equiv \sqrt{4\pi a_s \varrho}.$$

In terms of the length scales governing the system, using that $k_{\text{cut}}^\xi = \sqrt{4\pi a_s \varrho}$, in the dilute gas we must have the following condition fulfilled, $4\pi a_s r_0^2 \ll d^3$, where $d = \varrho^{-1/3}$ is the interparticle separation. Note that the opposite scale separation relation, $k_{\text{cut}}^\text{Lorentz} \gg k_{\text{cut}}^\text{UV}$ is very unnatural in the sense that every quantum field theory which has the usual properties such as locality and Lorentz invariance must have UV divergences (e.g., in the two-point function).

The renormalization of the cutoff-dependent terms is different for the two cases: The $k_{\text{cut}}^s$-contributions can be absorbed by a $\varrho$-independent renormalization of the coupling $g$ [34, 36], whereas the $k_{\text{cut}}^\xi$-contributions depend on the density in a nontrivial way and thus lead to a quantum renormalization of the effective equation of state. We supply an example for this renormalization in the section to follow.

### 6 Static Example for the Backreaction Force

In order to provide an explicit example for the quantum backreaction term in Eq. (24), without facing the above discussed UV problem, let us consider a quasi-one-dimensional (quasi-1D) condensate [44, 45], where all the involved quantities are UV finite. In a quasi-1D condensate the perpendicular harmonic trapping $\omega_\perp$ is much larger than the axial trapping $\omega_z$ such that the condensate assumes the shape of a strongly elongated cigar with all atomic motion in the perpendicular direction frozen out, $\omega_\perp$ being much larger than the mean energy per particle.

In accordance with general considerations [46], the phonon density $\varrho_\chi$ is infrared (IR) divergent in one spatial dimension, therefore inducing finite-size effects, i.e., a dependence of the various quantities of interest on the system size. Nevertheless, in certain situations, we are able to derive a closed local
expression for the quantum backreaction term $Q$: Let us assume a completely static condensate $\nu = 0$ in effectively one spatial dimension, still allowing for a spatially varying density $\rho$ and possibly also coupling $g$. Furthermore, since we require that spatial variations of $\rho$ and $g$ occur on length scales $\lambda$ much larger than the healing length (Thomas-Fermi approximation), we keep only the leading terms in $\xi/\lambda \ll 1$, i.e., the variations of $\rho$ and $g$ will be neglected in the calculation of the expectation values. In this case, the quantum backreaction term $Q$ simplifies considerably and yields (in effectively one spatial dimension, where $g \equiv g_{1D}$ and $\varrho \equiv \varrho_{1D}$ now both refer to the 1D quantities)

$$Q = -\nabla \langle (\nabla \chi) \rangle - \frac{1}{2g} \nabla \left( g^2 \langle 2 \chi^3 \rangle + \langle \chi^2 \rangle \right)$$

$$= -\nabla \left( \frac{1}{2\pi} (g \rho)^{3/2} \right) + \frac{1}{2\pi g} \nabla \left( g^{5/2} \rho^{3/2} \right) + O(\xi^2/\lambda^2)$$

$$= \frac{\varrho}{2\pi} \nabla \sqrt{g^2 \rho} + O(\xi^2/\lambda^2). \quad (40)$$

It turns out that the IR divergences of $2(\chi^3)$ and $\langle (\chi^2) \rangle$ cancel each other such that the resulting expression is not only UV but also IR finite. Note that the sign of $Q$ is positive and hence opposite to the contribution of the pure phonon density $\langle \chi^2 \rangle$, which again illustrates the importance of the “anomalous” term $\langle (\chi^2) \rangle$.

A possible experimental signature of the quantum backreaction term $Q$ calculated above, is the change incurred on the static Thomas-Fermi solution of the Euler equation \cite{22} for the density distribution (cf. \cite{35} 36)

$$\varrho_{1D} = \frac{\mu - V_{\text{trap}}}{g_{1D}} + \frac{\sqrt{\mu - V_{\text{trap}}}}{2\pi} + O(1/\sqrt{N}), \quad (41)$$

with $\mu$ denoting the (constant) chemical potential. The classical [$O(N)$] density profile $\varrho_{cl} = (\mu - V_{\text{trap}})/g_{1D}$ acquires nontrivial quantum [$O(N^0)$] corrections $\varrho_Q = \sqrt{\mu - V_{\text{trap}}}/2\pi$, where the small parameter is the ratio of the interparticle distance $1/\rho = O(1/N)$ over the healing length $\xi = O(N^0)$. Note that the quantum backreaction term $\varrho_Q$ in the above split $\varrho = \varrho_{cl} + \varrho_Q$ should neither be confused with the phonon density $\varrho_{\chi}$ in $\varrho = \varrho_{cl} + \varrho_{\chi}$ (remember that $\varrho_{\chi}$ is IR divergent and hence contains finite-size effects) nor with the quantum pressure contribution $\propto \nabla^2 \sqrt{\varrho}$ in the Euler type Eq. \cite{22}.

Evaluating the change $\Delta R$ of the Thomas-Fermi size (half the full length), where $\mu = V_{\text{trap}}$, of a quasi-1D Bose-Einstein condensate induced by backreaction, from Eq. \cite{11} we get $\Delta R = -2^{-5/2}(\omega_{\perp}/\omega_z) a_s$. Here, the quasi-1D coupling constant $g_{1D}$ is related to the 3D s-wave scattering length $a_s$ and the perpendicular harmonic trapping $\omega_{\perp}$ by $g_{1D} = 2a_s \omega_{\perp}$ (provided $a_s \ll a_{\perp} = 1/\sqrt{\omega_{\perp}}$ \cite{14}). In units of the classical size $R_{cl} = (3a_s N \omega_{\perp}/\omega_z^2)^{1/3}$, we have

$$\frac{\Delta R}{R_{cl}} = -\frac{1}{4\sqrt{2}} \left( \frac{1}{3N} \right)^{1/3} \left( \frac{\omega_{\perp}}{\omega_z} \frac{a_s}{a_{\perp}} \right)^{2/3}, \quad (42)$$
where $a_z = 1/\sqrt{\omega_z}$ describes the longitudinal harmonic trapping length. In quasi-1D condensates, backreaction thus leads to a shrinking of the cloud relative to the classical expectation – whereas in three spatial dimensions we have the opposite effect \[4, 36\]. In one dimension, we thus obtain a softening of the quantum renormalized equation of state of the gas; conversely, in three spatial dimensions the effective equation of state becomes stiffer due to quantum fluctuations.

For reasonably realistic experimental parameters, the effect of quantum backreaction on the equation of state should be measurable; for $N \simeq 10^3$, $\omega_\perp/\omega_z \simeq 10^3$, and $a_s/a_z \simeq 10^{-3}$, we obtain $|\Delta R/R_{cl}| \simeq 1\%$.

### 7 Conclusion

By explicit analysis of the analytically tractable case of a dilute Bose gas in the mean-field approximation, we have demonstrated the following. Even given that the explicit form of the quantum backreaction terms depends on the definition of the classical background, the effective-action method does not yield the correct result in the general case (which is, in particular, independent of the choice of variables). The knowledge of the classical (macroscopic) equation of motion – such as the Bernoulli equation – may be sufficient for deriving the first-order dynamics of the linearized quantum fluctuations (phonons), but the quantum backreaction as a second-order effect cannot be obtained without further knowledge of the microscopic structure (which reflects itself, for example, in the operator ordering imposed). It is tempting to compare these findings to gravity, where we also know the classical equations of motion only

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu},$$

which – in analogy to the Bernoulli equation – might yield the correct first-order equations of motion for the linearized gravitons, but perhaps not their (second-order) quantum backreaction. Another potentially interesting point of comparison is the existence of two different high-energy scales – one associated to the breakdown of Lorentz invariance $k_{\xi}^{\text{cut}} = k_{\text{Lorentz}}^{\text{cut}}$ and the other one, $k_{\text{UV}}^{\text{cut}} = k_s^{\text{cut}}$, to the UV cutoff introduced by the true interaction potential range. The question then poses itself whether one of the two cutoff scales (or in some sense both of them) correspond to the Planck scale in gravity.

The dominant $O(\xi/\lambda)$ quantum backreaction contributions like those in Eq. (41) depend on the healing length as the lower UV cutoff and hence cannot be derived from the low-energy effective action in Eq. (5) using a covariant (i.e., cutoff independent) regularization scheme, which does not take into account details of microscopic physics (represented, for example, in the quasiparticle dispersion relation). Note that the leading $O(\xi/\lambda)$ quantum correction to the pressure could be identified with a cosmological term, $\langle \hat{T}_{\mu\nu} \rangle = \Lambda g_{\mu\nu}$ in Eq. (10), provided that the cosmological “constant” $\Lambda$ is not constant but
depends on $g$ and $\rho$. Note that in general relativity, the Einstein equations demand $\Lambda$ to be constant, due to the equivalence principle and the resulting requirement that the metric be parallel-transported, $\nabla^\mu g_{\mu\nu} = 0$.

As became evident, the knowledge of the expectation value of the pseudo-energy-momentum tensor $\langle \hat{T}_{\mu\nu} \rangle$ is not sufficient for determining the quantum backreaction effects in general. Even though $\langle \hat{T}_{\mu\nu} \rangle$ is a useful concept for describing the phonon kinematics (at low energies), we have seen that it does not represent the full dynamics of the fluid dynamical variables defined in terms of the fundamental quantum field $\hat{\Psi}$. Related limitations of the classical pseudo-energy-momentum tensor, in particular the background choice dependence of the description of the second-order effect of the exchange of energy and momentum between excitations and that background, and the resulting form of the conservation laws, have been discussed in [27].

In general, the quantum backreaction corrections to the Euler equation in Eq. (22) cannot be represented as the gradient of some local potential, cf. Eq. (24). Hence they may effectively generate vorticity and might serve as the seeds for vortex nucleation from the vortex vacuum.

In contrast to the three-dimensional case (see, e.g., [35, 36]), the quantum backreaction corrections given by Eq. (40) diminish the pressure in condensates that can be described by Eq. (11) in one spatial dimension (quasi-1D case). This is a direct consequence of the so-called “anomalous” term $\langle (\hat{\chi}^\dagger)^2 + \hat{\chi}^2 \rangle$ in Eq. (10), which – together with the cancellation of the IR divergence – clearly demonstrates that it cannot be neglected in general. We emphasize that even though Eqs. (10)–(12) describe the static quantum backreaction corrections to the ground state, which can be calculated by an alternative method [36] as well, the expression in Eq. (24) is valid for more general dynamical situations, such as rapidly expanding condensates. Quantum backreaction can thus generally not be incorporated by rewriting the Euler equation in terms of a renormalized chemical potential. While the static quantum backreaction corrections to the ground state can be absorbed by a redefinition of the chemical potential $\mu(\rho)$ determining a quantum renormalized (barotropic) equation of state $p(\rho)$, this is not possible for the other terms in Eq. (24), like the quantum friction-type terms depending on $j \otimes v$.

We have derived, from the microscopic physics of dilute Bose-Einstein condensates, the backreaction of quantum fluctuations onto the motion of the full fluid and found a quantum backreaction force that is potentially experimentally observable in existing condensates. We observed a failure of the effective-action technique to fully describe the backreaction force in Eq. (24), and a cutoff dependence of backreaction due to the breakdown of covariance at high energies. Whether similar problems, in particular the question of the correct choice of the fundamental variables and the related operator ordering issues, beset the formulation of a theory of “real” (quantum) gravity remains an interesting open question.
Quantum Backreaction in Bose-Einstein Condensates

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