Perfect Packings in Quasirandom Hypergraphs I

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Abstract

Let $k \geq 2$ and $F$ be a linear $k$-uniform hypergraph with $v$ vertices. We prove that if $n$ is sufficiently large and $v|n$, then every quasirandom $k$-uniform hypergraph on $n$ vertices with constant edge density and minimum degree $\Omega(n^{k-1})$ admits a perfect $F$-packing. The case $k = 2$ follows immediately from the blowup lemma of Komlós, Sárközy, and Szemerédi. We also prove positive results for some nonlinear $F$ but at the same time give counterexamples for rather simple $F$ that are close to being linear. Finally, we address the case when the density tends to zero, and prove (in analogy with the graph case) that sparse quasirandom 3-uniform hypergraphs admit a perfect matching as long as their second largest eigenvalue is sufficiently smaller than the largest eigenvalue.

1 Introduction

A $k$-uniform hypergraph $H$ ($k$-graph for short) is a collection of $k$-element subsets (edges) of a vertex set $V(H)$. For a $k$-graph $H$ and a subset $S$ of vertices of size at most $k - 1$, let $d(S) = d_H(S)$ be the number of subsets of size $k - |S|$ that when added to $S$ form a edge of $H$. The minimum degree of $H$, written $\delta(H)$, is the minimum of $d(\{s\})$ over all vertices $s$. The minimum $l$-degree of $H$, written $\delta_l(H)$, is the minimum of $d(S)$ taken over all $l$-sets of vertices. The minimum codegree of $H$ is the minimum $(k - 1)$-degree. Let $K^k_t$ be the complete $k$-graph on $t$ vertices.

Let $G$ and $F$ be $k$-graphs. We say that $G$ has a perfect $F$-packing if the vertex set of $G$ can be partitioned into copies of $F$. An important result of Hajnal and Szemerédi [9] states that if $r$ divides $n$ and the minimum degree of an $n$-vertex graph $G$ is at least $(1 - 1/r)n$, then $G$ has a perfect $K^k_r$-packing. Later Alon and Yuster [2] conjectured that a similar result holds for any graph $F$ instead of just cliques, with the minimum degree of $G$ depending on

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the chromatic number of $F$. This was proved by Komlós-Sárközy-Szemerédi [18] by using the
Regularity Lemma and Blow-up Lemma. Later, Kühn and Osthus [20] found the minimum
degree threshold for perfect $F$-packings up to a constant; the threshold either comes from
the chromatic number of $F$ or the so-called critical chromatic number of $F$.

In the past decade there has been substantial interest in extending this result to $k$-graphs. Nevertheless, the simplest case of determining the minimum codegree threshold that
 guarantees a perfect matching was settled only recently by Rödl-Ruciński-Szemerédi [30].
Since then, there are a few results for codegree thresholds for packing other small 3-graphs [3, 13, 10, 26, 27, 29, 33, 34]. For $\ell$-degrees with $\ell < k/2$ (in particular the minimum degree),
much less is known. After work by many researchers [10, 14, 15, 22, 21, 28], still only the
degree threshold for $K^3_3$-packings and $K^3_4$-packings are known ($\frac{5}{6}$ and $\frac{37}{64}$ respectively). For
$m \geq 5$ and $k \geq 4$ the packing degree threshold for $K^k_m$ is open ([21] contains the current best
bounds).

A key ingredient in the proofs of most of the above results are specially designed random-
like properties of $k$-graphs that imply the existence of perfect $F$-packings. There is a rather
well-defined notion of quasirandomness for graphs that originated in early work of Thoma-
son [31, 32] and Chung-Graham-Wilson [3] which naturally generalizes to $k$-graphs. Our
main focus in this paper is on understanding when perfect $F$-packings exist in quasirandom
hypergraphs. The basic property that defines quasirandomness is uniform edge-distribution,
and this extends naturally to hypergraphs. Let $\mu = \mu(F) = |V(H)|$.

**Definition.** Let $k \geq 2$, let $0 < \mu, p < 1$, and let $H$ be a $k$-graph. We say that $H$ is
$(p, \mu)$-dense if for all $X_1, \ldots, X_k \subseteq H$,
\[ e(X_1, \ldots, X_k) \geq p|X_1| \cdots |X_k| - \mu n^k, \]
where $e(X_1, \ldots, X_k)$ is the number of $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ such that \{x_1, \ldots, x_k\} $\subseteq H$
(note that if the $X_i$'s overlap an edge might be counted more than once). Say that $H$ is an
$(n, p, \mu)$-graph if $H$ has $n$ vertices and is $(p, \mu)$-dense. Finally, if $0 < \alpha < 1$, then an
$(n, p, \mu)$-$\alpha$-$k$-graph if its minimum degree is at least $\alpha \left(\begin{array}{c} n \\ k \end{array}\right)$.

The $F$-packing problem for quasirandom graphs with constant density has been solved
implicitly by Komlós-Sárközy-Szemerédi [17] in the course of developing the Blow-up Lemma.

**Theorem 1.** (Komlós-Sárközy-Szemerédi [17]) Let $0 < \alpha, p < 1$ be fixed and let $F$ be
any graph. There exists an $n_0$ and $\mu > 0$ such that if $H$ is any $(n, p, \mu, \alpha)$-2-graph where
$n \geq n_0$, $\mu(F)/n$ then $H$ has a perfect $F$-packing.

Note that the condition on minimum degree is required, since if the condition “$\delta(H) \geq \alpha n$” in Theorem 1 is replaced by “$\delta(H) \geq f(n)$” for any choice of $f(n)$ with $f(n) = o(n)$,
then there exists the following counterexample. Take the disjoint union of the random graph
$G(n, p)$ and a clique of size either $\lceil f(n) \rceil + 1$ or $\lceil f(n) \rceil + 2$ depending on which is odd.
The minimum degree is at least $f(n)$, there is no perfect matching, and the graph is still
$(p, \mu)$-dense. Because of the use of the regularity lemma, the constant $n_0$ in Theorem 1 is
an exponential tower in $\mu^{-1}$. We extend Theorem 1 to a variety of $k$-graphs. In the process,
Theorem 4. Let \( n \) if packing if \( 6 \). C in the fact that \( 12 \). F in the random 3-graph (see [23, 24]).

A hypergraph is linear if every two edges share at most one vertex. For a k-graph \( H \), Kohayakawa-Nagle-Rödl-Schacht [16] recently proved an equivalence between \((|H|/ \binom{n}{3}, \mu)\)-dense and the fact that for each linear k-graph \( F \), the number of labeled copies of \( F \) in \( H \) is the same as in the random graph with the same density. This leads naturally to the question of whether Problem 2 has a positive answer for linear k-graphs, and our first result shows that this is the case.

Theorem 3. Let \( k \geq 2 \), \( 0 < \alpha, p < 1 \), and let \( F \) be a linear k-graph. There exists an \( n_0 \) and \( \mu > 0 \) such that if \( H \) is an \((n, p, \mu, \alpha)\) k-graph where \( n \geq n_0 \) and \( v(F)|n \), then \( H \) has a perfect F-packing.

We restrict our attention only to 3-graphs now although the concepts extend naturally to larger \( k \). Define a 3-graph to be \((2 + 1)\)-linear if its edges can be ordered as \( e_1, \ldots, e_q \) such that each \( e_i \) has a partition \( s_i \cup t_i \) with \( |s_i| = 2 \), \( |t_i| = 1 \) and for every \( j < i \) we have \( e_j \cap e_i \subseteq s_i \) or \( e_j \cap e_i \subseteq t_i \). In words, every edge before \( e_i \) intersects \( e_i \) in a subset of \( s_i \) or of \( t_i \). Clearly every linear 3-graph is \((2 + 1)\)-linear, but the converse is false. Keevash’s [11] recent proof of the existence of designs and our recent work on quasirandom properties of hypergraphs [23, 24, 25] use a quasirandom property distinct from \((p, \mu)\)-dense that Keevash calls typical and we call \((2 + 1)\)-quasirandom (although the properties are essentially equivalent). These properties imply that the count of all \((2 + 1)\)-linear 3-graphs in a typical 3-graph is the same as in the random 3-graph (see [23, 24]).

Thus a natural direction in which to extend Theorem 3 is to the family of \((2 + 1)\)-linear 3-graphs and we begin this investigation with some of the smallest such 3-graphs. A cherry is the 3-graph comprising two edges that share precisely two vertices - this is the “simplest” non-linear hypergraph. A more complicated \((2 + 1)\)-linear 3-graph is \( C_4(2 + 1) \) which has vertex set \( \{1, 2, 3, 4, a, b\} \) and edge set \( \{12a, 12b, 34a, 34b\} \). The importance of \( C_4(2 + 1) \) lies in the fact that \( C_4(2 + 1) \) is forcing for the class of all \((2 + 1)\)-linear 3-graphs. This means that if \( F \) is a \((2 + 1)\)-linear 3-graph and \( p, \epsilon > 0 \) are fixed, there is \( n_0 \) and \( \delta > 0 \) so that if \( n \geq n_0 \) and \( H \) is an \( n \)-vertex 3-graph with \( \binom{n}{3} \) edges and \( (1 + \delta)p^n \) labeled copies of \( C_4(2 + 1) \), then the number of labeled copies of \( F \) in \( H \) is \((1 \pm \epsilon)p^{|F|}n^{|v(F)|} \) (see [23, 24]).

Theorem 4. Let \( 0 < \alpha, p < 1 \). There exists an \( n_0 \) and \( \mu > 0 \) such that if \( H \) is an \((n, p, \mu, \alpha)\) 3-graph where \( n \geq n_0 \), then \( H \) has a perfect cherry-packing if \( 4|n \) and a perfect \( C_4(2 + 1) \) packing if \( 6|n \).
One might speculate that Theorem 4 can be extended to the collection of all (2+1)-linear $F$ or to the collection of all 3-partite $F$. However, our next result shows that this is not the case and that solving Problem 2 will be a difficult project. If $x$ is a vertex in a 3-graph $H$, the link of $x$ is the graph with vertex set $V(H) \setminus \{x\}$ and edges those pairs who form an edge with $x$.

Theorem 5. Let $F$ be any 3-graph with an even number of vertices such that there exists a partition of the vertices of $F$ into pairs such that each pair has a common edge in their links. Then for any $\mu > 0$, there exists an $n_0$ such that for all $n \geq n_0$, there exists a 3-graph $H$ such that

- $|H| = \frac{1}{8} \binom{n}{3} \pm \mu n^3$,
- $H$ is $(\frac{1}{8}, \mu)$-dense,
- $\delta(H) \geq (\frac{1}{8} - \mu) \binom{n}{2}$,
- $H$ has no perfect $F$-packing.

Two examples of 3-graphs $F$ that satisfy the conditions of Theorem 5 are the complete 3-partite 3-graph $K_{2, 2, 2}$ with parts of size two and the following (2 + 1)-linear hypergraph. A cherry 4-cycle is the (2 + 1)-linear 3-graph with edge set $\{123, 124, 345, 346, 567, 568, 781, 782\}$.
The spectral gap guarantees a perfect matching in graphs. For hypergraphs, there are several definitions of eigenvalues. We will use the definitions that originated in the work of Friedman and Wigderson \[7, 8\] for regular hypergraphs. The definition for all hypergraphs can be found in \[23, Section 3\] where we specialize to \(\pi = 1 + \cdots + 1\). That is, let \(\lambda_1(H) = \lambda_{1,1+\cdots+1}(H)\) and let \(\lambda_2(H) = \lambda_{2,1+\cdots+1}(H)\), where both \(\lambda_{1,1+\cdots+1}(H)\) and \(\lambda_{2,1+\cdots+1}(H)\) are as defined in Section 3 of \[23\].

**Theorem 6.** For every \(\alpha > 0\), there exists \(n_0\) and \(\gamma > 0\) depending only on \(\alpha\) such that the following holds. Let \(H\) be an \(n\)-vertex 3-graph where \(3|n\) and \(n \geq n_0\). Let \(p = 6|H|/n^3\) and assume that \(\delta_2(H) \geq \alpha pn\) and \(\lambda_2(H) \leq \gamma p^{16}n^{3/2}\). Then \(H\) contains a perfect matching.

Let \(\Delta_2(H)\) be the maximum codegree of a 3-graph \(H\), i.e. the maximum of \(d(S)\) over all 2-sets \(S \subseteq V(H)\). If \(\Delta_2(H) \leq cpn\) then \(\lambda_1(H) \leq c'pn^{3/2}\) where \(c'\) is a constant depending only on \(c\). This implies the following corollary.

**Corollary 7.** For every \(\alpha > 0\), there exists \(n_0\) and \(\gamma > 0\) depending only on \(\alpha\) such that the following holds. Let \(H\) be an \(n\)-vertex 3-graph where \(3|n\) and \(n \geq n_0\). Let \(p = |H|/(\binom{n}{3})\) and assume that \(\delta_2(H) \geq \alpha pn\), \(\Delta_2(H) \leq \frac{1}{2}pn\), and \(\lambda_2(H) \leq \gamma p^{15}\lambda_1(H)\). Then \(H\) contains a perfect matching.

The third largest eigenvalue of a graph is closely related to its matching number (see e.g. \[4\]), but currently we do not know the “correct” definition of \(\lambda_3\) for hypergraphs. It would be interesting to discover a definition of \(\lambda_3\) for \(k\)-graphs which extends the graph definition and for which a bound on \(\lambda_3\) forces a perfect matching.

The remainder of this paper is organized as follows. In Section 2 we will develop the tools neccisary for our proofs, including extensions of the absorbing technique and various embedding lemmas. Then in Section 3 we will use these to prove Theorem 3 (Section 3.3) and Theorem 4 (Sections 3.1 and 3.2). Section 4 contains the construction proving Theorem 5 and Section 5 has the proof of the sparse case, Theorem 6.

### 2 Tools

In this section, we state and prove several lemmas and propositions that we will need; our main tool is the absorbing technique of Rödl-Ruciński-Szemerédi \[30\].

**Definition.** Let \(F\) and \(H\) be \(k\)-graphs and let \(A, B \subseteq V(H)\). We say that \(A\) \(F\)-absorbs \(B\) or that \(A\) is an \(F\)-absorbing set for \(B\) if both \(H[A]\) and \(H[A \cup B]\) have perfect \(F\)-packings. When \(F\) is a single edge, we say that \(A\) edge-absorbs \(B\).
Definition. Let $F$ and $H$ be $k$-graphs, $\epsilon > 0$, $a$ and $b$ be multiples of $v(F)$, $\mathcal{A} \subseteq \binom{V(H)}{a}$, and $\mathcal{B} \subseteq \binom{V(H)}{b}$. We say that $H$ is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$-rich if for all $B \in \mathcal{B}$ there are at least $\epsilon n$ sets in $\mathcal{A}$ which $F$-absorb $B$. If $\mathcal{A} = \binom{V(H)}{a}$, we abbreviate this to $(a, \mathcal{B}, \epsilon, F)$-rich and if both $\mathcal{A} = \binom{V(H)}{a}$ and $\mathcal{B} = \binom{V(H)}{b}$, we abbreviate this to $(a, b, \epsilon, F)$-rich.

The following proposition is one of the main results of this section; the proof appears in Section 2.3.

**Proposition 8.** Fix $0 < p < 1$, let $F$ be a $k$-graph such that $F$ is either linear or $k$-partite, and let $a$ and $b$ be multiples of $v(F)$. For any $\epsilon > 0$, there exists an $n_0$ and $\mu > 0$ such that the following holds. If $H$ is an $(a, b, \epsilon, F)$-rich, $(n, p, \mu)$ $k$-graph where $v(F)|n$, then $H$ has a perfect $F$-packing.

We will actually need a slight extension of Proposition 8 for some of our results and this requires an additional definition.

**Definition.** Let $\zeta > 0$, $t$ be any integer, $H$ be a 3-graph, and $B \subseteq V(H)$ with $|B| = 2t$. We say that $B$ is $\zeta$-separable if there exists a partition of $B$ into $B_1, \ldots, B_t$ such that for all $i$ $|B_i| = 2$ and $d_H(B_i) \geq \zeta n$. Set

$$B_{\zeta,b}(H) := \left\{ B \in \binom{V(H)}{b} : B \text{ is } \zeta\text{-separable} \right\}.$$ 

If $H$ is obvious from context, we will denote this by $B_{\zeta,b}$.

The second main result proved in this section is that the property $(a, b, \epsilon, F)$-rich can be replaced by $(a, B_{\zeta,b}, \epsilon, F)$-rich in Proposition 8; the proof is in Section 2.4.

**Proposition 9.** Fix $0 < p, \alpha < 1$ and let $\zeta = \min\left\{ \frac{p}{4}, \frac{\alpha}{4} \right\}$. Let $F$ be a 3-graph such that $F$ is either linear or $k$-partite, let $v(F)|a$, and let $v(F)|b$ where in addition $b$ is even. For any $\epsilon > 0$, there exists an $n_0$ and $\mu > 0$ such that the following holds. If $H$ is an $(a, B_{\zeta,b}, \epsilon, F)$-rich $(n, p, \mu, \alpha)$ 3-graph where $v(F)|n$, then $H$ has a perfect $F$-packing.
Note that if $b$ is even, $H$ is a 3-graph, and $\delta(H) \geq \alpha\left(\binom{n}{2}\right)$, then Proposition 8 implies Proposition 9. The proofs of Propositions 8 and 9 use the absorbing technique of Rödl-Ruciński-Szemerédi [30]. The two key ingredients are the Absorbing Lemma (Lemma 10) and the Embedding Lemmas (Lemma 11 for linear and Lemma 13 for $k$-partite). The remainder of this section contains the statements and proofs of these lemmas plus the proofs of both propositions.

### 2.1 Absorbing Sets

Rödl-Ruciński-Szemerédi [30, Fact 2.3] have a slightly different definition of edge-absorbing where $B$ has size $k + 1$ and one vertex of $A$ is left out of the perfect matching, but the main idea transfers to our setting in a straightforward way as follows. If $H$ is a $k$-graph, $A \subseteq V(H)$, and $A \subseteq 2^{V(H)}$, then we say that $A$ partitions into sets from $A$ if there exists a partition $A = A_1 \cup \cdots \cup A_t$ such that $A_i \in A$ for all $i$.

**Lemma 10.** (Absorbing Lemma) Let $F$ be a $k$-graph, $\epsilon > 0$, and $a$ and $b$ be multiples of $v(F)$. There exists an $n_0$ and $\omega > 0$ such that for all $n$-vertex $k$-graphs $H$ with $n \geq n_0$, the following holds. If $H$ is $(A,B,\epsilon,F)$-rich for some $A \subseteq \binom{V(H)}{a}$ and $B \subseteq \binom{V(H)}{b}$, then there exists an $A \subseteq V(H)$ such that $A$ partitions into sets from $A$ and $A$-F-absorbs all sets $C$ satisfying the following conditions: $C \subseteq V(H) \setminus A$, $|C| \leq \omega n$, and $C$ partitions into sets from $B$.

Using the idea of Rödl-Ruciński-Szemerédi [30], Treglown and Zhao [33] Lemma 5.2] proved the above lemma for $F$ a single edge, $a = 2k$, $b = k$, $A = \binom{V(H)}{a}$ and $B = \binom{V(H)}{b}$. For the sparse case (Theorem 6) we require a stronger version of Lemma 10 and so a proof of Lemma 10 appears in Section 5 (as a corollary of Lemma 23).

### 2.2 Embedding Lemmas and Almost Perfect Packings

This section contains embedding lemmas for linear and $k$-partite $k$-graphs and a simple corollary of these lemmas which produces a perfect $F$-packing covering almost all of the vertices.

**Definition.** Let $F$ and $H$ be $k$-graphs with $V(F) = \{w_1, \ldots, w_f\}$. A labeled copy of $F$ in $H$ is an edge-preserving injection from $V(F)$ to $V(H)$. A degenerate labeled copy of $F$ in $H$ is an edge-preserving map from $V(F)$ to $V(H)$ that is not an injection. Let $1 \leq m \leq f$ and let $Z_1, \ldots, Z_m \subseteq V(H)$. Set $\text{inj}[F \to H; w_1 \to Z_1, \ldots, w_m \to Z_m]$ to be the number of edge-preserving injections $\psi : V(F) \to V(H)$ such that $\psi(w_i) \in Z_i$ for all $1 \leq i \leq m$. In other words, $\text{inj}[F \to H; w_1 \to Z_1, \ldots, w_m \to Z_m]$ is the number of labeled copies of $F$ in $H$ where $w_i$ is mapped into $Z_i$ for all $1 \leq i \leq m$. If $Z_i = \{z_i\}$, we abbreviate $w_i \to \{z_i\}$ as $w_i \to z_i$.

**Lemma 11.** Let $0 < p, \alpha < 1$ and let $F$ be a linear $k$-graph where $0 \leq m \leq v(F)$ and $V(F) = \{s_1, \ldots, s_m, t_{m+1}, \ldots, t_f\}$ such that there does not exist $E \subseteq F$ with $|E \cap \{s_1, \ldots, s_m\}| > 1$
and there do not exist \(E_1, E_2 \in F\) with \(|E_1 \cap \{s_1, \ldots, s_m\}| = 1\), \(|E_2 \cap \{s_1, \ldots, s_m\}| = 1\), and \(E_1 \cap E_2 \cap \{t_{m+1}, \ldots, t_f\} \neq \emptyset\).

For every \(\gamma > 0\), there exists an \(n_0\) and \(\mu > 0\) such that the following holds. Let \(H\) be an \(n\)-vertex \(k\)-graph with \(n \geq n_0\) and let \(y_1, \ldots, y_m \in V(H), Z_{m+1} \subseteq V(H), \ldots, Z_f \subseteq V(H)\). Assume that for every \(\{s_i, t_{j_1}, \ldots, t_{j_k}\} \in F\)
\[
\left\{ (z_{j_1}, \ldots, z_{j_k}) \in Z_{j_1} \times \cdots \times Z_{j_k} : \{y_{i_1}, z_{j_1}, \ldots, z_{j_k}\} \in H \right\} \geq \alpha |Z_{j_1}| \cdots |Z_{j_k}|
\]
(1)
and for every \(\{t_{i_1}, \ldots, t_{i_k}\} \in F\) and every \(Z_{i_1}' \subseteq Z_{i_1}, \ldots, Z_{i_k}' \subseteq Z_{i_k}\),
\[
eq p|Z_{i_1}'| \cdots |Z_{i_k}'| - \mu n^k.
\]
(2)

Then
\[
inj[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f]
\]
\[
\geq \alpha^{d_F(s_1)} \cdots \alpha^{d_F(s_m)} p^{|F| - \sum d_F(s_i)} |Z_{m+1}| \cdots |Z_f| - \gamma n^{f-m}.
\]

Proof. Kohayakawa, Nagle, Rödl, and Schacht [16] proved this when \(Z_i = V(H)\) for all \(i\), without the distinguished vertices \(s_1, \ldots, s_m\), and under a stronger condition on \(H\), but it is straightforward to extend their proof to our setup as follows. The lemma is proved by induction on number of edges of \(F\) which do not contain any vertex from among \(s_1, \ldots, s_m\).

Let \(\mu = (1 - p)\gamma\).

First, if every edge of \(F\) contains some \(s_i\) then \(F\) is a vertex disjoint union of stars with centers \(s_1, \ldots, s_m\) plus some isolated vertices. Therefore, we can form a copy of \(F\) of the type we are trying to count by picking an edge of \(H\) containing \(y_i\) (of the right type) for each edge of \(F\). More precisely, using (1), the fact that all edges of \(F\) which use some \(s_1, \ldots, s_m\) (so all edges of \(F\)) do not share any vertices from among \(t_{m+1}, \ldots, t_f\), and the fact that \(F\) is linear, the number of labeled copies of \(F\) with \(s_i \to y_i\) and \(t_j \to Z_j\) is at least
\[
\alpha^{|F|} |Z_{m+1}| \cdots |Z_f| = \alpha^{\sum d_F(s_i)} p^{|Z_{m+1}|} \cdots |Z_f|.
\]
The proof of the base case is complete.

Now assume \(F\) has at least one edge \(E\) which does not contain any \(s_i\), with vertices labeled so that \(E = \{t_{m+1}, \ldots, t_{m+k}\}\). Let \(F_+\) be the hypergraph formed by deleting all vertices of \(E\) from \(F\) and notice that \(s_i \in V(F_+)\) for all \(i\). Let \(F_-\) be the hypergraph formed by removing the edge \(E\) from \(F\) but keeping the same vertex set. Let \(Q_*\) be an injective edge-preserving map \(Q_* : V(F_+) \to V(H)\) where \(Q_*(s_i) = y_i\) for \(1 \leq i \leq m\) and \(Q_*(t_j) \in Z_j\) for \(m+1 \leq j \leq f\). For \(m + 1 \leq j \leq m + k\), define \(S_j(Q_*) \subseteq Z_j\) as follows. For each \(z \in Z_j\), add \(z\) to \(S_j(Q_*)\) if \(z \notin Im(Q_*)\) and there exists an edge-preserving injection \(V(F_+) \cup \{t_j\} \to Im(Q_*) \cup \{z\}\) which when restricted to \(V(F_+)\) matches the map \(Q_*\). More informally, \(S_j(Q_*)\) consists of all vertices which can be used to extend \(Q_*\) to embed a labeled copy of \(F_+ \cup \{t_j\}\).

By definition, every edge counted by \(e(S_{m+1}(Q_*), \ldots, S_{m+k}(Q_*))\) creates a labeled copy of \(F\). Also, every ordered tuple from \(S_{m+1}(Q_*) \times \cdots \times S_{m+k}(Q_*)\) creates a labeled copy of
such that the following holds. Let

$$H \gamma >$$

Lemma 13.

For each \( j \), \( S_j(Q_*) \subseteq Z_j \) so that (2) implies that

$$\text{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f]$$

where the last inequality is because there are at most \( n^{f-m-k} \) maps \( Q_* \), since \( F_* \) has \( f - k \) vertices and \( s_i \in V(F_*) \) must map to \( y_i \). Combining (3) and (4) and then applying induction,

$$\text{inj}[F \to H; s_1 \to y_1, \ldots, s_m \to y_m, t_{m+1} \to Z_{m+1}, \ldots, t_f \to Z_f]$$

Since \( \mu = (1 - p)\gamma \), the proof is complete.

**Corollary 12.** Let \( 0 < p < 1 \) and let \( F \) be a linear \( k \)-graph with \( V(F) = \{ t_1, \ldots, t_f \} \). For every \( \gamma > 0 \), there exists an \( n_0 \) and \( \mu > 0 \) such that the following holds. Let \( H \) be an \( (n, p, \mu) \) \( k \)-graph and let \( Z_1, \ldots, Z_f \subseteq V(H) \). Then

$$\text{inj}[F \to H; t_1 \to Z_1, \ldots, t_f \to Z_f] \geq p^{|F|}|Z_1| \cdot \ldots \cdot |Z_f| - \gamma n^f.$$

**Proof.** Apply Lemma 11 with \( m = 0 \). Since \( H \) is \((n, p, \mu)\)-dense, (2) holds. Also, (1) is vacuous since \( m = 0 \).

**Lemma 13.** Let \( 0 < p < 1 \) and let \( K_{t_1,\ldots,t_k} \) be the complete \( k \)-partite, \( k \)-graph with part sizes \( t_1, \ldots, t_k \) and parts labeled by \( T_1, \ldots, T_k \). For every \( 0 < \mu < \frac{2}{3} \), there exists \( n_0 \) and \( 0 < \xi < 1 \) such that the following holds. Let \( H \) be an \((n, p, \mu)\) \( k \)-graph with \( n \geq n_0 \). Then for any \( X_1, \ldots, X_k \subseteq V(H) \) with \( |X_j| \geq (2\mu/p)^{1/k} n \) for all \( j \), the number of labeled copies of \( K_{t_1,\ldots,t_k} \) in \( H \) with \( T_i \subseteq X_i \) for all \( i \) is at least \( \xi \prod |X_i|^{t_i} \).

**Proof.** Let \( H' \) be the \( k \)-graph on \( \sum |X_i| \) vertices with vertex set \( Y_1 \cup \cdots \cup Y_t \) where the sets \( Y_i \) are disjoint and \( Y_i \cong X_i \) for all \( i \). Note that because the sets \( X_i \) might overlap, a vertex
of $H$ might appear more than once in $H'$. Make $y_1 \in Y_1, \ldots, y_k \in Y_k$ a hyperedge of $H'$ if $y_1, \ldots, y_k$ are distinct vertices of $H$ and \{y_1, \ldots, y_k\} \in H$. Let $t = \sum t_i$. Since $H$ is $(p, \mu)$-dense,

$$e(H') = e_H(X_1, \ldots, X_k) \geq p \prod_i |X_i| - \mu n^k \geq p \left( \frac{2\mu}{p} \right)^{n^k} \mu n^k \geq \frac{\mu}{k^k} v(H)^k.$$ 

Therefore, by supersaturation (see [12, Theorems 2.1 and 2.2]), there exists an $n'_0$ and $\xi' > 0$ such that if $v(H') \geq n'_0$ then $H'$ contains at least $\xi'v(H')^t$ labeled copies of $K_{t_1, \ldots, t_k}$. Each of these labeled copies of $K_{t_1, \ldots, t_k}$ in $H'$ produces a possibly degenerate labeled copy of $K_{t_1, \ldots, t_k}$ in $H$ where $T_i \subseteq X_i$ for all $i$. Pick $\xi = \frac{1}{2}\xi'$, $n_0 \geq n'_0(p/2\mu)^{1/k}$, and $n_0 \geq \frac{1}{2}(p/2\mu)^{t/k}$.

Now assume that $n \geq n_0$. This implies that $v(H') \geq |X_1| \geq (2\mu/p)^{1/k}n \geq n'_0$ so that there are at least $\xi'v(H')^t$ labeled copies of $K_{t_1, \ldots, t_k}$ in $H'$. Therefore, the number of possibly degenerate labeled copies of $K_{t_1, \ldots, t_k}$ in $H$ with $T_i \subseteq X_i$ for all $i$ is at least

$$\xi'v(H')^t = \xi' \prod_i v(H')^{t_i} \geq \xi' \prod_i |X_i|^{t_i} = 2\xi \prod_i |X_i|^{t_i}. \quad (5)$$

Since there are at most $n^{t-1}$ degenerate labeled copies, by the choice of $n_0$ and since $|X_i| \geq (2\mu/p)^{1/k}n$ for all $i$, the number of degenerate labeled copies is at most

$$n^{t-1} = \frac{1}{n} \left( \frac{p}{2\mu} \right)^{t/k} \prod_i \left( \frac{2\mu}{p} \right)^{1/k} n^{t_i} \leq \frac{1}{n} \left( \frac{p}{2\mu} \right)^{t/k} \prod_i |X_i|^{t_i} \leq \xi \prod_i |X_i|^{t_i}. \quad (6)$$

Combining (5) with (6) shows that there are at least $\xi \prod_i |X_i|^{t_i}$ labeled copies of $K_{t_1, \ldots, t_k}$ with $T_i \subseteq X_i$ for all $i$, completing the proof. \qed

With these lemmas in hand, we can prove that if $H$ is $(p, \mu)$-dense and $F$ is linear or $k$-partite, then $H$ has an $F$-packing covering almost all the vertices of $H$.

**Lemma 14.** (Almost Perfect Packing Lemma) Fix $0 < p < 1$ and a $k$-graph $F$ with $f$ vertices such that $F$ is either linear or $k$-partite. Let $v(F)|b$. For any $0 < \omega < 1$, there exists $n_0$ and $\mu > 0$ such that the following holds. Let $H$ be an $(n, p, \mu)$-graph with $n \geq n_0$ and $f|n$. Then there exists $C \subseteq V(H)$ such that $|C| \leq \omega n$, $b|C|$, and $H'[C]$ has a perfect $F$-packing.

**Proof.** First, select $n_0$ large enough and $\mu$ small enough so that any vertex set $C$ of size $\left\lceil \frac{\omega}{2} \right\rceil$ contains a copy of $F$. To prove that this is possible, there are two cases to consider.

If $F$ is linear, let $\gamma = \frac{1}{2^{f(k)}}(\frac{\omega}{2})^{f}$ and select $n_0$ and $\mu > 0$ according to Corollary [12]. Now if $C \subseteq V(H)$ with $|C| \geq \frac{\omega}{2}n$, then Corollary [12] implies there are at least $p^{f} |C|^f - \gamma n^f \geq p^f \left( \frac{\omega}{2} \right)^f n^f - \gamma n^f = \gamma n^f > 0$ copies of $F$ inside $C$.

If $F$ is $k$-partite, then Lemma [13] is used in a similar way as follows. Let $\mu = \frac{p}{2} \left( \frac{\omega}{2} \right)^k$ and select $n_0$ and $\xi$ according to Lemma [13]. Now by the choice of $\mu$, if $|C| \geq \frac{\omega}{2}$ then $|C| \geq (2\mu/p)^{1/k}n$ so that by Lemma [13] $C$ contains at least $\xi(\frac{\omega}{2})^f n^f > 0$ copies of $F$. 

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Now let $F_1, \ldots, F_t$ be a greedily constructed $F$-packing. That is, $F_1, \ldots, F_t$ are disjoint copies of $F$ and $C := V(H) \setminus V(F_1) \setminus \cdots \setminus V(F_t)$ has no copy of $F$. By the previous two paragraphs, $|C| \leq \frac{\omega}{2} n$. Since $f|n$ and $H[\bar{C}]$ has a perfect $F$-packing, $f||C|$. Thus we can let $y \equiv -\frac{|C|}{f} \pmod{b}$ with $0 \leq y < b$ and take $y$ of the copies of $F$ in the $F$-packing of $H[\bar{C}]$ and add their vertices into $C$ so that $b||C|$. \hfill \Box

### 2.3 Proof of Proposition 8

**Proof of Proposition 8.** First, select $\omega > 0$ according to Lemma 10 and $\mu_1 > 0$ according to Lemma 14. Also, make $n_0$ large enough so that both Lemma 10 and 14 can be applied. Let $\mu = \mu_1 \omega^k$. All the parameters have now been chosen.

By Lemma 10 there exists a set $A \subseteq V(H)$ such that $A$ $F$-absorbs $C$ for all $C \subseteq V(H) \setminus A$ with $|C| \leq \omega n$ and $b \mid |C|$. If $|A| \geq (1 - \omega)n$, then $A$ $F$-absorbs $V(H) \setminus A$ so that $H$ has a perfect $F$-packing. Thus $|A| \leq (1 - \omega)n$. Next, let $H' := H[A]$ and notice that $H'$ is $(p, \mu_1)$-dense since $v(H') \geq \omega n$ and

$$\mu n^k \leq \frac{\mu}{\omega^k} v(H')^k = \mu_1 v(H')^k.$$ 

Therefore, by Lemma 14 there exists a vertex set $C \subseteq V(H') = V(H) \setminus A$ such that $|C| \leq \omega n$, $|C|$ is a multiple of $b$, and $H'[\bar{C}]$ has a perfect $F$-packing. Now Lemma 10 implies that $A$ $F$-absorbs $C$. The perfect $F$-packing of $A \cup C$ and the perfect $F$-packing of $H'[\bar{C}]$ produces a perfect $F$-packing of $H$. \hfill \Box

### 2.4 Proof of Proposition 9

This section contains the proof of Proposition 9 but first we need an extension of Lemma 14 that produces a perfect $F$-packing covering almost all the vertices where in addition the unsaturated vertices are $\zeta$-separable. To do so, we need a well-known probability lemma.

**Lemma 15. (Chernoff Bound)** Let $0 < p < 1$, let $X_1, \ldots, X_n$ be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for all $i$, and let $X = \sum X_i$. Then for all $\alpha > 0$,

$$\mathbb{P}[|X - \mathbb{E}[X]| > \alpha] \leq 2e^{-\alpha^2/2n}.$$  

**Lemma 16.** Fix $p, \alpha \in (0, 1)$, $\zeta = \min\{\frac{p}{4}, \frac{\alpha}{2}\}$, and a 3-graph $F$ such that either $F$ is linear or $F$ is 3-partite. Let $v(F)|b$ where in addition $b$ is even. For any $0 < \omega < 1$, there exists $n_0$ and $\mu > 0$ such that the following holds. Let $H$ be an $(n, p, \mu, \alpha)$ 3-graph with $n \geq n_0$ and $v(F)|n$. Then there exists a set $C \subseteq V(H)$ such that $|C| \leq \omega n$, $C$ partitions into sets of $B_{\zeta, b}$, and $H[\bar{C}]$ has a perfect $F$-packing.
Proof. Use Lemma [14] to select \( n_0 \) and \( \mu_1 > 0 \) to produce an \( F \)-packing \( F_1, \ldots, F_t \) where \( W := V(H) \setminus V(F_1) \setminus \cdots \setminus V(F_t) \) is such that \(|W| \leq \omega n\). Let \( f = v(F) \) and let

\[
\phi = \min \left\{ \frac{\omega}{8}, \frac{\alpha}{4}, \frac{p}{4} \right\}, \\
\mu = \min \left\{ \frac{p}{2} \left( \frac{\alpha \phi}{16f} \right)^2, \mu_1 \right\}.
\]

First, form a vertex set \( C' \) by starting with \( W \) and for each 1 \( \leq i \leq t \), add \( V(F_i) \) to \( C' \) with probability \( \phi \) independently. After this, take \( \frac{\phi}{f} - \frac{|C'|}{f} \) (mod \( \frac{\phi}{f} \)) of the unselected copies of \( F \) and add their vertices into \( C' \) to form the vertex set \( C \).

By construction, \( H[\bar{C}] \) has a perfect \( F \)-packing (the copies of \( F \) which were not selected) and \( b || C \). Since \( b \) is even, \(|C|\) is also even. So to complete the proof, we just need to show that with positive probability, \( C \) is \( \zeta \)-separable and \(|C| \leq \omega n\). (Note that if \( C \) is \( \zeta \)-separable then it can be partitioned into sets from \( B_{\zeta,b} \).

Let \( G \) be the graph where \( V(G) = V(H) \) and for every \( Z \in \binom{V(G)}{2} \), \( Z \) is an edge of \( G \) if \( d_H(Z) \geq \zeta n \), i.e. the codegree of \( Z \) in \( H \) is at least \( \zeta n \). We will now prove that with positive probability, the following two events occur:

- \(|C| \leq \frac{1}{2} \omega n\),
- \( \delta(G[C]) \geq \frac{\alpha \phi}{8f} n \).

First, the expected number of vertices added to \( W \) to form \( C \) is \( \phi ft \leq \frac{\omega}{8} n \) plus potentially a few copies of \( F \) to make \( b || C \). By the second moment method, with probability at least \( \frac{1}{4} \), at most \( \frac{\omega}{4} n \) vertices are added to \( W \) so that \(|C| \leq \frac{1}{2} \omega n\). Secondly, since \( \delta(H) \geq \alpha n^2 \) it is the case that \( \delta(G) \geq \frac{\omega}{2} n \). Indeed, if there was some vertex \( x \) with \( d_G(x) < \frac{\omega}{2} n \), then \( d_H(x) \leq |N_G(x)| \cdot n + n \cdot \zeta n \leq \frac{\omega}{4} n + \frac{\zeta}{2} n \), a contradiction to the fact that \( \delta(H) \geq \alpha n^2 \) and \( \zeta \leq \frac{\omega}{4} \). Since \(|W| \leq \frac{\omega n}{4} \), we have that any vertex \( x \) has at least \( \frac{\omega}{4} n \) neighbors in \( G \) outside \( W \). Since each \( F_i \) has size \( f \), the vertex \( x \) therefore has a neighbor in \( G \) inside at least \( \frac{\omega}{4f} n \) of the copies of \( F \). Therefore, the expected size of \( \{y \in C : xy \in E(G)\} \) is at least \( \frac{\alpha \phi}{4f} n \) and by Chernoff’s Inequality (Lemma [15]),

\[
\mathbb{P} \left[ \left| \{y \in C : xy \in E(G)\} \right| < \frac{\alpha \phi}{8f} n \right] \leq e^{-cn}
\]

for some constant \( c \). Thus \( n_0 \) can be selected large enough so that with probability at most \( \frac{1}{4} \), there is some \( x \in V(G) \) such that \( \left| \{y \in C : xy \in E(G)\} \right| < \frac{\alpha \phi}{8f} n \). This implies that with probability at least \( \frac{1}{2} \), \(|C| \leq \omega n \) and \( \delta(G[C]) \geq \frac{\alpha \phi}{8f} n \).

To complete the proof, we will show that \( \delta(G[C]) \geq \frac{\alpha \phi}{8f} n \) implies that \( G[C] \) has a perfect matching (which is equivalent to \( C \) being \( \zeta \)-separable). Divide \( C \) into two equal sized parts \( C_1 \) and \( C_2 \) (recall that \(|C|\) is even since \( b \) is even and \( b || C \)). Such a partition exists since a random partition has this property with positive probability. Assume towards a contradiction that
Hall’s Condition fails in $G[C_1, C_2]$, i.e. there exists a set $T \subseteq C_1$ such that $|N_G(T) \cap C_2| < |T|$. In a slight abuse of notation, let $\tilde{T} = C_1 \setminus T$. Now $|T| \geq \frac{\alpha \phi}{16f} n$ since $\delta(G[C_1, C_2]) \geq \frac{\alpha \phi}{16f} n$. Similarly, $|\tilde{T}| \geq \frac{\alpha \phi}{16f} n$ since if $z \in C_2 \setminus N_G(T)$ then $N_G(z) \cap C_1 \subseteq \tilde{T}$. This implies that

$$|C_2 \setminus N_G(T)| = |C_2| - |N_G(T)| = |C_1| - |N_G(T)| > |C_1| - |T| = |\tilde{T}| \geq \frac{\alpha \phi n}{16f}.$$ 

Since there are no edges of $G$ between $T$ and $C_2 \setminus N_G(T)$,

$$e_{H}(T, C_2 \setminus N_G(T), V(H)) \leq |T| \cdot |C_2 \setminus N_G(T)| \cdot \zeta n = \zeta |T||C_2 \setminus N_G(T)| n. \quad (7)$$

On the other hand, since $H$ is $(p, \mu)$-dense,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq p|T||C_2 \setminus N_G(T)| n - \mu n^2.$$ 

Since $|T|$ and $|C_2 \setminus N_G(T)|$ are both larger than $\frac{\alpha \phi}{16f} n$,

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq \left( p - \mu \left( \frac{16f}{\alpha \phi} \right)^2 \right) |T||C_2 \setminus N_G(T)| n$$

Since $\mu \leq \frac{p}{2} \left( \frac{\alpha \phi}{16f} \right)^2$, we have

$$e_H(T, C_2 \setminus N_G(T), V(H)) \geq \frac{p}{2} |T||C_2 \setminus N_G(T)| n \quad (8)$$

Since $\zeta < \frac{p}{2}$, (8) contradicts (7). Therefore, $G[C_1, C_2]$ satisfies Hall’s condition so that $G[C]$ has a perfect matching, i.e. $C$ is $\zeta$-separable. \hfill \square

**Proof of Proposition 8.** The proof is similar to the proof of Proposition 8 except Lemma 16 is used instead of Lemma 14. \hfill \square

### 3 Rich hypergraphs

This section contains the proofs of Theorems 3 and 4. By the previous section, these proofs come down to showing that $(p, \mu)$-dense and large minimum degree imply either $(a, b, \epsilon, F)$-rich or $(a, B_{\zeta,b}, \epsilon, F)$-rich, where we get to select $a, b, \epsilon$ but $\zeta = \min\{\frac{L}{T}, \frac{\alpha}{T}\}$. As a warm-up before Theorem 3 (see Section 3.3), we start with the cherry.

#### 3.1 Packing Cherries

Let $K_{1,1,2}$ be the cherry.

**Lemma 17.** Let $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{L}{T}, \frac{\alpha}{T}\}$. There exists an $n_0, \epsilon > 0$, and $\mu > 0$ such that if $H$ is an $(n, p, \mu, \alpha)$ 3-graph with $n \geq n_0$, then $H$ is $(4, B_{\zeta,4}, \epsilon, K_{1,1,2})$-rich.
Proof. Our main task is to come up with an $\epsilon > 0$ such that for large $n$ and all $B \in \mathcal{B}_{\zeta,4}$, there are at least $\epsilon n^4$ vertex sets of size four which $K_{1,1,2}$-absorb $B$; we will define $\epsilon$ and $\mu$ later.

Fix $B = \{b_1, b_2, b_3, b_4\} \in \mathcal{B}_{\zeta,4}$, labeled so that $d_H(b_1, b_2) \geq \zeta n$ and $d_H(b_3, b_4) \geq \zeta n$. Let $X_1 = N(b_1, b_2) = \{x : xb_1b_2 \in E(H)\} \subseteq V(H)$ and $X_2 = N(b_3, b_4)$ and notice that $|X_1|, |X_2| \geq \zeta n$. Arbitrarily divide $X_1$ in half and call the two parts $Y_1$ and $Y_2$. Let $\mu = \frac{\zeta}{2}(\frac{\zeta}{2})^3$.

Since $|Y_1|, |Y_2|, |X_2| \geq \frac{\zeta}{2}n = (2\mu/p)^{1/3}n$, by Lemma 13 there exists a $\xi > 0$ and $n_0$ such that $H[Y_1, Y_2, X_2]$ contains at least $\xi(\frac{\zeta}{2})^4$ copies of $K_{1,1,2}$ with one degree two vertex in each of $Y_1$ and $Y_2$ and the degree one vertices in $X_2$. The proof is now complete, since each of these cherries absorbs $B$. Indeed, let $\epsilon = \xi(\frac{\zeta}{2})^4$ and let $y_1 \in Y_1, y_2 \in Y_2$, and $x_1, x_2 \in X_2$ be such that $y_1y_2x_1, y_1y_2x_2 \in E(H)$. Then $A = \{y_1, y_2, x_1, x_2\} K_{1,1,2}$-absorbs $B$ because $b_1b_2y_1, b_1b_2y_2 \in E(H)$ (recall that $Y_1, Y_2 \subseteq N(b_1, b_2)$) and similarly $b_3b_4x_1, b_3b_4x_2 \in E(H)$. Since there are at least $\epsilon n^4$ choices for $y_1, y_2, x_1, x_2$, the proof is complete. \hfill \square

### 3.2 Packing Cycles

Throughout this section, let $C_4$ denote the hypergraph $C_4(2+1)$. This section completes the proof of Theorem 4.

**Lemma 18.** Let $0 < p, \alpha < 1$ and let $\zeta = \min\{\frac{p}{10}, \frac{\alpha}{5}\}$. There exists an $n_0, \epsilon > 0, \mu > 0$ such that if $H$ is a $(n, p, \alpha, \mu)$ 3-graph with $n \geq n_0$, then $H$ is $(18, B_{\zeta,6}, \epsilon, C_4)$-rich.

**Proof.** Similar to the proof of Lemma 17 our task is to come up with an $\epsilon > 0$ such that for large $n$ and all $B \in \mathcal{B}_{\zeta,6}$, there are at least $\epsilon n^{18}$ vertex sets of size eighteen which $C_4$-absorb $B$; we will define $\epsilon$ and $\mu$ later.

Consider $B = \{b_1, b'_1, b_2, b'_2, b_3, b'_3\} \in \mathcal{B}_{\zeta,6}$ labeled so that $d_H(b_i, b'_i) \geq \zeta n$ for all $i$. For $1 \leq i \leq 3$, let $X_i = N(b_i, b'_i)$ and note that $|X_i| \geq \zeta n$. Now for each $1 \leq i \leq 3$, define

$$R_i = \left\{ (r_1, r_2) \in \left( \binom{V(H)}{2} \right) : |N(r_1, r_2) \cap X_i| \geq \frac{1}{10}p\zeta n \right\}.$$ 

In other words, $R_i$ is the set of pairs with neighborhood in $X_i$ at least one-tenth the “expected” size. If $|R_i| \leq \frac{1}{10}p\zeta n^2$, then

$$e(X_1, V(H), V(H)) \leq |R_i|n + \binom{n}{2} - |R_i| \leq \frac{1}{10}p\zeta n \leq \frac{1}{5}p\zeta n^3. \quad (9)$$

On the other hand, since $H$ is $(p, \mu)$-dense,

$$e(X_1, V(H), V(H)) \geq p|X_1|n^2 - \mu n^3 \geq (p\zeta - \mu) n^3.$$ 

Let $\mu = \frac{\zeta}{2}(\frac{\zeta}{10})^3 < \frac{4}{5}p\zeta$ so that this contradicts (9). Thus $|R_i| \geq \frac{1}{10}p\zeta n^2$ and similarly for $1 \leq i \leq 3$, $|R_i| \geq \frac{1}{10}p\zeta n^2$.

Now fix $r_1r'_1 \in R_1$, $r_2r'_2 \in R_2$, and $r_3r'_3 \in R_3$. There are at least $\left(\frac{1}{10}p\zeta\right)^3 n^6$ such choices. For $1 \leq i \leq 3$ let $Y_i = N(r_i, r'_i) \cap X_i$, so $|Y_i| \geq \frac{1}{10}p\zeta n = (\frac{2\mu}{p})^{1/3}n$. By Lemma 13 there exists a
$$\xi > 0$$ such that there are at least $$\xi \left( \frac{1}{10} p \zeta \right)^{12} n^{12}$$ copies of $$K_{4,4,4}$$ across $$Y_1, Y_2, Y_3$$. Let $$T_1, T_2, T_3$$ be the three parts of $$K_{4,4,4}$$ with $$T_i \subseteq Y_i$$ and let $$T_i = \{ y_{i1}^1, y_{i2}^1, y_{i3}^1, y_{i4}^1 \}$$.

Let $$\epsilon = \xi \left( \frac{1}{10} p \zeta \right)^{15}$$; we claim that there are at least $$\epsilon n^{18}$$ vertex sets of size 18 which $$C_4$$-absorb $$B$$. Indeed, $$A := \{ r_i, r'_i, y^1_i, y^2_i, y^3_i, y^4_i : 1 \leq i \leq 3, 1 \leq j \leq 4 \}$$ forms a $$C_4$$-absorbing 18-set for $$B$$ as follows. First, $$A$$ has a perfect $$C_4$$-packing: one $$C_4$$ uses vertices $$r_1, r'_1, y^1_1, y^2_1, y^3_1, y^4_1$$, another uses vertices $$r_2, r'_2, y^1_2, y^2_2, y^3_2, y^4_2$$, and the last uses $$r_3, r'_3, y^1_3, y^2_3, y^3_3, y^4_3$$. Secondly, $$A \cup B$$ has a perfect $$C_4$$-packing: one $$C_4$$ using $$b_1, b'_1, r_1, r'_1, y^1_1, y^2_1$$, one using $$b_2, b'_2, r_2, r'_2, y^1_2, y^2_2$$, one using $$b_3, b'_3, r_3, r'_3, y^1_3, y^2_3$$, and one using $$y^1_3, y^2_3, y^3_3, y^4_3$$. Since there are $$\left( \frac{1}{10} p \zeta \right)^3 n^6$$ choices for $$r_1, r'_1, r_2, r'_2, r_3, r'_3$$ and then $$\xi \left( \frac{1}{10} p \zeta \right)^{12} n^{12}$$ choices for $$y^2_i$$, there are a total of at least $$\epsilon n^{18}$$ choices for $$A$$.

**Proof of Theorem 4** Apply Lemmas 17 and 18 and then Proposition 9.

### 3.3 Packing Linear Hypergraphs

In this section, we prove Theorem 8.

**Lemma 19.** Let $$0 < p, \alpha < 1$$ and let $$F$$ be a linear $$k$$-graph on $$f$$ vertices. There exists an $$n_0, \epsilon > 0$$, and $$\mu > 0$$ such that if $$H$$ is a $$(n, p, \mu, \alpha)$$ $$k$$-graph with $$n \geq n_0$$, then $$H$$ is $$(f^2 - f, f, \epsilon, F)$$-rich.

**Proof.** Let $$a = f(f - 1)$$ and $$b = f$$. Similar to the proofs in the previous two sections, our task is to come up with an $$\epsilon > 0$$ such that for large $$n$$ and all $$B \in \binom{V(H)}{b}$$, there are at least $$\epsilon n^a$$ vertex sets of size $$a$$ which $$F$$-absorb $$B$$; we will define $$\epsilon$$ and $$\mu$$ later. Let $$V(F) = \{ w_0, \ldots, w_{f-1} \}$$ and form the following $$k$$-graph $$F'$$. Let

$$V(F') = \{ x_{i,j} : 0 \leq i, j \leq f - 1 \}.$$  

(We think of the vertices of $$F'$$ as arranged in a grid with $$i$$ as the row and $$j$$ as the column.) Form the edges of $$F'$$ as follows: for each fixed $$1 \leq i \leq f - 1$$, let $$\{ x_{i,0}, \ldots, x_{i,f-1} \}$$ induce a copy of $$F$$ where $$x_{i,j}$$ is mapped to $$w_{i+j \mod f}$$. More precisely, if $$\{ w_{\ell_1}, \ldots, w_{\ell_k} \} \in F$$, then $$\{ x_{i,\ell_1-i \mod f}, \ldots, x_{i,\ell_k-i \mod f} \} \in F'$$. Similarly, for each fixed $$0 \leq j \leq f - 1$$, let $$\{ x_{0,j}, \ldots, x_{f-1,j} \}$$ induce a copy of $$F$$ where $$x_{i,j}$$ is mapped to $$w_{i+j \mod f}$$. Note that we therefore have a copy of $$F$$ in each column and a copy of $$F$$ in each row besides the zeroth row.

Now fix $$B = \{ b_0, \ldots, b_{f-1} \} \subseteq V(H)$$; we want to show that $$B$$ is $$F$$-absorbed by many $$a$$-sets. Note that any labeled copy of $$F'$$ in $$H$$ which maps $$x_{0,0} \to b_0, \ldots, x_{0,f-1} \to b_{f-1}$$ produces an $$F$$-absorbing set for $$B$$ as follows. Let $$Q : V(F') \to V(H)$$ be an edge-preserving injection where $$Q(b_j) = x_{0,j}$$ (so $$Q$$ is a labeled copy of $$F'$$ in $$H$$ where the set $$B$$ is the zeroth row of $$F'$$). Let $$A = \{ Q(x_{i,j}) : 1 \leq i \leq f - 1, 0 \leq j \leq f - 1 \}$$ consist of all vertices in rows 1 through $$f - 1$$. Then $$A$$ has a perfect $$F$$-packing consisting of the copies of $$F$$ on the rows, and $$A \cup B$$ has a perfect $$F$$-packing consisting of the copies of $$F$$ on the columns. Therefore, $$A$$ $$F$$-absorbs $$B$$.

To complete the proof, we therefore just need to use Lemma 11 to show there are many copies of $$F'$$ with $$B$$ as the zeroth row. Apply Lemma 11 to $$F'$$ where $$m = f, s_1 =$$
Lemma 20. For every $s_f = x_0, \ldots, s_f = x_{0,f-1}$ and $Z_{m+1} = \cdots = Z_f = V(H)$. Since $\delta(H) \geq \alpha\left(\frac{n}{d-1}\right)$, (11) holds (with $\alpha$ replaced by $\frac{\alpha}{2}$) and since $H$ is $(p, \mu)$-dense (2) holds. Let $\gamma = \frac{1}{2}(\alpha \delta)\sum d(x_0, j)p|E| - \sum d(x_0, j)$ and ensure that $n_0$ is large enough and $\mu$ is small enough apply Lemma 11 to show that

$$\text{inj}[F' \rightarrow H; x_0, 0 \rightarrow b_0, \ldots, x_{0, f-1} \rightarrow b_{f-1}] \geq \gamma n^{f^2-f} = \gamma n^a.$$ 

Each labeled copy of $F'$ produces a labeled $F$-absorbing set for $B$, so there are at least $\frac{\gamma}{\alpha^2} n^a$ $F$-absorbing sets for $B$. The proof is complete by letting $\epsilon = \frac{\gamma}{\alpha^2}$. \hfill \Box

**Proof of Theorem 3.** Apply Lemma 19 and then Proposition 8. \hfill \Box

### 4 Avoiding perfect $F$-packings

In this section we prove Theorem 3 using the following construction.

**Construction.** For $n \in \mathbb{N}$, define a probability distribution $H(n)$ on 3-uniform, $n$-vertex hypergraphs as follows. Let $G = G^{(2)}(n, \frac{1}{2})$ be the random graph on $n$ vertices. Let $X$ and $Y$ be a partition of $V(G)$ where

- if $n \equiv 0 \pmod{4}$, then $|X| = \frac{n}{2} - 1$ and $|Y| = \frac{n}{2} + 1$,
- if $n \equiv 1 \pmod{4}$, then $|X| = \frac{n}{2} - \frac{1}{2}$ and $|Y| = \frac{n}{2} + \frac{1}{2}$,
- if $n \equiv 2 \pmod{4}$, then $|X| = |Y| = \frac{n}{2}$,
- if $n \equiv 3 \pmod{4}$, then $|X| = \frac{n}{2} - \frac{1}{2}$ and $|Y| = \frac{n}{2} + \frac{1}{2}$.

Let the vertex set of $H(n)$ be $V(G)$ and make a set $E \in \binom{V(G)}{3}$ into a hyperedge of $H(n)$ as follows. If $|E \cap X|$ is even, then make $E$ into a hyperedge of $H(n)$ if $G[E]$ is a clique. If $|E \cap X|$ is odd, then make $E$ into a hyperedge of $H(n)$ if $E$ is an independent set in $G$.

**Lemma 20.** For every $\epsilon > 0$, with probability going to 1 as $n$ goes to infinity,

$$\left| |E(H(n))| - \frac{1}{8}\binom{n}{3}\right| \leq \epsilon n^3.$$ 

**Proof.** For each $E \in \binom{V(H(n))}{3}$, $E$ is a clique or independent set in $G(n, \frac{1}{2})$ with probability $\frac{1}{8}$. Thus the expected number of edges in $H(n)$ is $\frac{1}{8}\binom{n}{3}$ so the second moment method shows that with probability going to one as $n$ goes to infinity, $|E(H(n)) - \frac{1}{8}\binom{n}{3}| \leq \epsilon n^3$. See [11] or the proof of Lemma 15 in [25] for details about the second moment method. \hfill \Box

**Lemma 21.** For every $\epsilon > 0$, with probability going to 1 as $n$ goes to infinity the following holds. Let $X_1, X_2, X_3 \subseteq V(H(n))$. Then

$$\left| e(X_1, X_2, X_3) - \frac{1}{8}|X_1||X_2||X_3| \right| < \epsilon n^3.$$
Proof. Let $S_1, \ldots, S_n$ be Steiner triple systems that partition $\binom{V(H(n))}{3}$. That is, view $V(H(n)) \cong \mathbb{Z}_n$ and for $1 \leq i \leq n$ let $S_{i+1}$ consist of the triples $\{a, b, c\}$ such that $a + b + c = i \pmod{n}$. Each triple in $\binom{V(H(n))}{3}$ appears in exactly one $S_i$ and two triples from the same $S_i$ share at most one vertex.

Let $1 \leq i \leq n$ and let $X_1, X_2, X_3 \subseteq V(H(n))$. Let $e_H(X_1, X_2, X_3; S_i)$ be the number of ordered tuples $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that $\{x_1, x_2, x_3\} \in E(H(n)) \cap S_i$. Let $e_{K_n}(X_1, X_2, X_3; S_i)$ be the number of ordered tuples $(x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$ such that $\{x_1, x_2, x_3\} \in S_i$.

The expected value of $e_H(X_1, X_2, X_3; S_i)$ is clearly $\frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i)$. If $E_1, E_2 \in S_i$ then since $E_1$ and $E_2$ share at most one vertex the events $E_1 \in E(H(n))$ and $E_2 \in E(H(n))$ are independent. By Chernoff’s Bound (Lemma 15),

$$\mathbb{P}\left[\left|e_H(X_1, X_2, X_3; S_i) - \frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i)\right| > \epsilon |S_i|\right] < e^{-c_n^2}.$$  

for some constant $c$ since $|S_i| = \frac{n}{3} \binom{n}{3}$ and the number of events is $e_{K_n}(X_1, X_2, X_3; S_i) < |S_i|$. By the union bound,

$$\mathbb{P}\left[\exists i, \exists X_1, X_2, X_3, \left|e_H(X_1, X_2, X_3; S_i) - \frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i)\right| > \epsilon |S_i|\right] < e^{-\frac{c_n^2}{2}}.$$  

Therefore, with high probability, for all $i$ and all $X_1, X_2, X_3$,

$$\left|e_H(X_1, X_2, X_3; S_i) - \frac{1}{8} e_{K_n}(X_1, X_2, X_3; S_i)\right| < \epsilon |S_i|. \tag{10}$$

Summing (10) over $i$ completes the proof. \hfill \Box

Lemma 22. Let $F$ be a 3-graph with an even number of vertices such that there exists a partition of the vertices of $F$ into pairs such that every pair has a common pair in their links. Then $H(n)$ does not have a perfect $F$-packing for any $n$.

Proof. If $n \nmid v(F)$, then obviously $H(n)$ does not have a perfect $F$-packing. Therefore assume that $n | v(F)$ so that $n$ is even. Let $X$ and $Y$ be the partition of $V(H(n))$ in the definition of $H(n)$. Since $n$ is even, by definition both $|X|$ and $|Y|$ are odd. Let $\{w_1, z_1\}, \{w_2, z_2\}, \ldots, \{w_{v(F)/2}, z_{v(F)/2}\}$ be the partition of $V(F)$ into pairs so that $w_i$ and $z_i$ have a common pair in their link for all $i$. By construction, if $x \in X$ and $y \in Y$ then there is no pair of vertices $u, v \in V(H(n))$ such that $xuv, yuv \in E(H(n))$ since the parities of $\{x, u, v\} \cap X$ and $\{y, u, v\} \cap X$ are different. This implies that for each $i$, $w_i$ and $z_i$ must either both appear in $X$ or both appear in $Y$ so that any copy of $F$ in $H(n)$ uses an even number of vertices in $X$ and an even number of vertices in $Y$. Since $|X|$ is odd, $H(n)$ does not have a perfect $F$-packing. \hfill \Box

Proof of Theorem 5. By Lemmas 20, 21 and 22 with high probability $H(n)$ has the required properties. \hfill \Box
5 Perfect Matchings in Sparse Hypergraphs

In this section, we prove Theorem 6. We follow the same outline as Section 3.

Lemma 23. Let \( k \geq 2, c > 0, \) and \( a, b \) be multiples of \( k \). There exists an \( n_0 \) depending only on \( k, a, b, \) and \( c \) such that the following holds for all \( n \geq n_0 \). Let \( H \) be an \( n \)-vertex \( k \)-graph, let \( A \subseteq \binom{V(H)}{a} \), and let \( B \subseteq \binom{V(H)}{b} \). Suppose that \( \ell \geq cn^{a-1/2} \log n \) is an integer such that for every \( B \in B \) there are at least \( \ell \) sets in \( A \) which edge-absorb \( B \). Then there exists set \( A \subseteq V(H) \) such that \( A \) partitions into sets from \( A \) and \( A \) edge-absorbs any set \( C \) satisfying the following conditions: \( C \subseteq V(H) \setminus A, |C| \leq \frac{1}{64} \ell^2 n^{-2a+1} \), and \( C \) partitions into sets from \( B \).

Proof. The proof is similar to Treglown-Zhao [33, Lemma 5.2] which in turn is similar to Rödl-Ruciński-Szemerédi [30, Fact 2.3]. Let \( q = \frac{1}{8} \ell n^{-2a+1} \) and let \( A \subset A \) be the family obtained by selecting each element of \( A \) with probability \( q \) independently. The expected number of intersecting pairs of elements from \( A \) is at most \( q^2 \binom{n}{a} \binom{n}{a-1} \leq \frac{1}{16} \ell q \). By Markov’s inequality, with probability at least \( \frac{1}{2} \) there are at most \( \frac{1}{8} q \ell \) intersecting pairs of elements from \( A \).

Now fix \( B \in B \) and let \( \Gamma_B = \{ A \in A : A \text{ edge-absorbs } B \} \) be such that \( |\Gamma_B| = \ell \). For each \( A \in \Gamma_B \), let \( X_A \) be the event that \( A \in A \). By Chernoff’s Bound (Lemma 15),

\[
\mathbb{P}\left[ |\Gamma_B \cap A| - q \ell > \frac{1}{2} q \ell \right] \leq 2e^{-q \ell/6}.
\]

Using that \( \ell \geq cn^{a-1/2} \log n \), we have that \( q \ell = \frac{1}{8} \ell^2 n^{-2a+1} \geq \frac{c^2}{8} \log^2 n \). By the union bound,

\[
\mathbb{P}\left[ \exists B, |\Gamma_B \cap A| < \frac{1}{2} q \ell \right] \leq \left( \binom{n}{b} \right)^2 \mathbb{P}\left[ \left| \Gamma_B \cap A \right| < \frac{1}{2} q \ell \right] \leq 2 \mathbb{P}\left[ \left| \Gamma_B \cap A \right| < \frac{1}{2} q \ell \right] < \frac{1}{2}
\]

for large \( n \). Thus with probability at least \( \frac{1}{2} \), \( A \) is such that for all \( B \in B \), there exist at least \( \frac{1}{8} q \ell \) \( a \)-sets in \( A \) which edge-absorb \( B \). Also, with probability at least \( \frac{1}{2} \) there are at most \( \frac{1}{8} q \ell \) intersecting pairs of elements from \( A \).

Let \( A' \) be the subfamily of \( A \) consisting only of those \( a \)-sets \( A \) where \( A \) is not in any intersecting pair and also there is at least one \( B \subseteq V(H) \) (of any size) such that \( A \) edge-absorbs \( B \). Thus by the union bound, with positive probability \( A' \) is such that for all \( B \in B \), there exist at least \( \frac{1}{8} q \ell \) \( a \)-sets in \( A' \) which edge-absorb \( B \). Let \( A' \) be such a family of \( a \)-sets and let \( A' = \cup A' \). First, \( H[A'] \) has a perfect matching. Indeed, each \( A \in A' \) edge-absorbs some set so \( H[A] \) has a perfect matching, and the sets in \( A' \) are disjoint so that these perfect matchings combine to form a perfect matching of \( H[A'] \). Second, \( A' \) partitions into sets from \( A \) since the sets in \( A' \subseteq A \) are disjoint. Now let \( C \subseteq V(H) \setminus A' \) with \( |C| \leq \frac{1}{64} \ell^2 n^{-2a+1} = \frac{1}{8} q \ell \) and \( C = B_1 \cup \cdots \cup B_t \) with \( B_i \in B \). Using the bound on the size of \( C \), we have that \( t < \frac{1}{8} q \ell \). Since each \( B_i \) is edge-absorbed by at least \( \frac{1}{8} q \ell \) sets in \( A' \), each \( B_i \) can be edge-absorbed by a different \( a \)-set in \( A' \). Therefore, \( H[A' \cup B'] \) has a perfect matching so the proof is complete. \( \square \)
Proof of Lemma 11. Let $H'$ be the $v(F)$-uniform hypergraph on the same vertex set as $H$, where $X \in [V(H)]$ is a hyperedge of $H'$ if $H[X]$ is a copy of $F$. Let $\ell = \lceil en^{2} \rceil$ and notice since $H$ is $(\mathcal{A}, \mathcal{B}, \epsilon, F)$-rich, for every $B \in \mathcal{B}$ there are at least $\ell$ sets in $\mathcal{A}$ which edge-absorb $B$ in $H'$. Also, since $a, b$ are multiples of $v(F)$ they are multiples of the uniformity of $H'$. Lastly, for large $n$ we have that $\ell \geq n^{\alpha-1/2} \log n$. Therefore, applying Lemma 23 (with $c = 1$) to $H'$ shows that there exists a set $A \subseteq V(H') = V(H)$ such that $A$ partitions into sets from $\mathcal{A}$ and for any $C \subseteq V(H') \setminus A = V(H) \setminus A$ with $|C| \leq \frac{\alpha}{64} n$ and $C$ partitions into sets from $\mathcal{B}$, $A$ edge-absorbs $C$ in $H'$. Because each edge of $H'$ is a copy of $F$, this implies that $A$ $F$-absorbs $C$ in $H$ so the proof is complete by setting $\omega = \frac{\alpha}{64}$. □

Next, similar to the proofs in Sections 3.1, 3.2, and 3.3 we show that a bound on $\lambda_2(H)$ implies that each 3-set is edge-absorbed by many 6-sets. To do so, we need the hypergraph expander mixing lemma, first proved by Friedman and Wigderson [7, 8] (using a slightly different definition of $\lambda_2(H)$) and then extended to our definition of $\lambda_2(H)$ in [23].

Proposition 24. (Hypergraph Expander Mixing Lemma [23, Theorem 4]). Let $H$ be an $n$-vertex $k$-graph and let $S_1, \ldots, S_k \subseteq V(H)$. Then

$$e(S_1, \ldots, S_k) - \frac{k!|E(H)|}{n^k} \prod_{i=1}^{k} |S_i| \leq \lambda_2(H) \sqrt{|S_1| \cdots |S_k|}.$$

Lemma 25. Let $\alpha > 0$. Let $H$ be a 3-graph and let $p = 6|E(H)|/n^3$. Assume $\delta_2(H) \geq \alpha p n$ and $\lambda_2(H) \leq \frac{1}{2} \alpha^2 p^{5/2} n^{-3/2}$. Then for every $B \subseteq V(H)$ with $|B| = 3$, there are at least $\frac{1}{16} \alpha^4 p^3 n^6$ sets $A \subseteq V(H)$ with $|A| = 6$ such that $A$ edge-absorbs $B$.

Proof. Let $B = \{b_1, b_2, b_3\} \subseteq V(H)$. First, there are at least $\frac{1}{8} \alpha p n^3$ edges disjoint from $B$; let $\{x_1, x_2, x_3\}$ be such an edge. For $1 \leq i \leq 3$, let $Y_i \subseteq N(b_i, x_i) = \{y : yx_i b_i \in H\}$ with $|Y_i| = \alpha p n$. Such a $Y_i$ exists since the minimum codegree is at least $\alpha p n$. By the expander mixing lemma (Proposition 24),

$$e(Y_1, Y_2, Y_3) \geq p|Y_1||Y_2||Y_3| - \lambda_2(H) \sqrt{|Y_1||Y_2||Y_3|} \geq \alpha^3 p^4 n^3 - \lambda_2(H) \alpha^{3/2} p^{3/2} n^{3/2}.$$

Since $\lambda_2(H) \leq \frac{1}{2} \alpha^2 p^{5/2} n^{-3/2}$,

$$e(Y_1, Y_2, Y_3) \geq \alpha^3 p^4 n^3 - \frac{1}{2} \alpha^7/2 p^4 n^3 \geq \frac{1}{2} \alpha^3 p^4 n^3.$$

Let $\{y_1, y_2, y_3\}$ be an edge with $y_1 \in Y_1$, $y_2 \in Y_2$, and $y_3 \in Y_3$. Then $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is a six-set that edge-absorbs $B$ and there are at least $\frac{1}{8} \alpha p n^3 \cdot \frac{1}{2} \alpha^3 p^4 n^3 = \frac{1}{16} \alpha^4 p^5 n^6$ such sets. □

Proof of Theorem 8. We are given $\alpha > 0$ such that $\delta_2(H) \geq \alpha p n$. Let $\gamma = 2^{-22} \alpha^{12}$.

First, we can assume that $p \geq \gamma n^{-1/10} \log^{1/5} n$. Indeed, by averaging there exists vertices $s_1, s_2$ such that the codegree of $s_1$ and $s_2$ is at most $2p$. Then taking $S_1 = \{s_1\}$, $S_2 = \{s_2\}$, and $S_3$ as the non-coneighbors of $s_1$ and $s_2$, Proposition 24 shows that

$$\lambda_2(H) \geq p \sqrt{|S_3|} \geq p \sqrt{(1 - 2p)n}.$$
But by assumption, $\lambda_2(H) \leq \gamma p^{16} n^{3/2}$. Therefore,

$$p\sqrt{(1-2p)n} \leq \gamma p^{16} n^{3/2}$$

which implies that $p \geq \gamma n^{-1/10} \log^{1/5} n$ (by a large margin).

By Lemma 25 for every $B \subseteq V(H)$ with $|B| = 3$ there are at least $\frac{1}{16} \alpha^4 p^5 n^6$ 6-sets $A \subseteq V(H)$ which edge-absorb $B$. Let $\ell = \frac{1}{16} \alpha^4 p^5 n^6$. If $n$ is sufficiently large, then since $p \geq \gamma n^{-1/10} \log^{1/5} n$, we have that $\ell = \frac{1}{16} \alpha^4 p^5 n^6 \geq \frac{1}{16} \alpha^4 \gamma^5 n^{5.5} \log n$. Let $c = \frac{1}{16} \alpha^4 \gamma^5$ so that $\ell \geq cn^{5.5} \log n$. Now by Lemma 23 if $n$ is sufficiently large there exists $A \subseteq V(H)$ such that $A$ edge-absorbs all sets of size a multiple of three and at most

$$\frac{1}{64} \ell^2 n^{-11} = \frac{1}{214} \alpha^8 p^{10} n.$$ (11)

We now show how to construct a perfect matching in $H$. First, greedily construct a matching in $H[V(H) \setminus A]$. Say the greedy procedure halts with $B \subseteq V(H) \setminus A$ as the unmatched vertices. Since $3|v(H)$ and $3||A|$ (since $A$ is an edge-absorbing set), $3||B|$. By Proposition 24 (recall that $\gamma = 2^{-22} \alpha^{12}$),

$$e(B, B, B) = p|B|^3 \pm \lambda_2(H)|B|^{3/2} \geq p|B|^3 - \frac{1}{222} \alpha^{12} p^{16} n^{3/2} |B|^{3/2}. \quad (12)$$

If $|B| \geq 2^{-14} \alpha^8 p^{10} n$, then

$$p|B|^3 \geq p|B|^{3/2} \left( \frac{1}{214} \alpha^8 p^{10} n \right)^{3/2} = \frac{1}{221} \alpha^{12} p^{16} n^{3/2} |B|^{3/2}.$$

Combining this with (12) shows that $e(B, B, B) > 0$. This contradicts that the greedy procedure halted with $B$ as the unmatched vertices. Thus $|B| \leq 2^{-14} \alpha^8 p^{10} n$ and then (11) shows that $A$ edge-absorbs $B$, producing a perfect matching of $H$.

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