Prediction of exotic magnetic states in the alkali metal quasi-one-dimensional iron selenide compound Na$_3$FeSe$_2$

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The magnetic and electronic phase diagram of a model for the quasi-one-dimensional alkali metal iron selenide compound Na$_3$FeSe$_2$ is presented. The novelty of this material is that the valence of iron is Fe$^{3+}$ contrary to most other iron-chain compounds with valence Fe$^{3+}$. Using first-principles techniques, we developed a three-orbital tight-binding model that reproduces the ab initio band structure near the Fermi level. Including Hubbard and Hund couplings and studying the model via the density matrix renormalization group and Lanczos methods, we constructed the ground state phase diagram. A robust region where the block state $\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow$ is stabilized was unveiled. The analog state in iron ladders, employing 2×2 ferromagnetic blocks, is by now well-established, but in chains a block magnetic order has not been observed yet in real materials. The phase diagram also contains a large region of canonical staggered spin order $\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow\downarrow$ at very large Hubbard repulsion. At the block to staggered transition region, a novel phase is stabilized with a mixture of both states: an inhomogeneous orbital-selective charge density wave with the exotic spin configuration $\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\uparrow\downarrow$. Our predictions for Na$_3$FeSe$_2$ may guide crystal growers and neutron scattering experimentalists towards the realization of block states in one-dimensional iron-selenide chain materials.

I. INTRODUCTION

Iron-based pnictides and selenides are fascinating materials with exotic magnetic and superconducting properties [1–3]. For iron-selenides the low-temperature insulating ground state has robust local magnetic moments [4–6], highlighting the importance of Hubbard and Hund coupling interactions among the electrons occupying the 3d orbitals [2, 3]. The competition between charge, spin, lattice, and orbital degrees of freedom can give rise to various types of exotic magnetic and electronic ordering. In particular, recently the two-leg ladder iron selenide materials have received considerable attention. One reason is their similarity with copper-based ladders, with a spin gap in the undoped limit and superconductivity upon doping by high pressure [7, 8]. Moreover, in the two-leg ladder iron-based compound BaFe$_2$Se$_3$, an exotic block-antiferromagnetic (AFM) order (involving 2×2 ferromagnetically aligned blocks, coupled antiferromagnetically along the legs of the ladder) has been reported using inelastic neutron diffraction methods [9–14], confirming earlier predictions by theory [15, 16]. BaFe$_2$Se$_3$ is an insulator with robust Néel temperature $T_N \sim 250$K into the block phase and large individual magnetic moments $\sim 2.8\mu_B$. In another iron-based ladder material, where K replaces Ba leading to KFe$_2$Se$_3$, the magnetic moments align ferromagnetically along the rungs but antiferromagnetically along the legs forming 2×1 blocks [10].

In addition to these ladder materials, there are some experimentally observed iron-selenide compounds, such as TiFeS$_2$, TiFeSe$_2$ and KFeSe$_2$, which contain weakly coupled quasi-one-dimensional chains [17, 18]. In these compounds iron is in a valence Fe$^{3+}$, corresponding to $n = 5$ electrons in the 3d iron orbitals. Based on magnetic susceptibility, electric resistivity, and electron-spin resonance, TiFeSe$_2$ behaves as a quasi one-dimensional standard spin-staggered antiferromagnet [19]. Furthermore, neutron diffraction experiments on TiFeS$_2$ also indicate [20] staggered spin order below $T_N = 196$K.

The experimental developments described above in quasi-1D iron-based materials provides a playground for theoretical many-body calculations based on multi-orbital Hubbard model [21–25]. Using accurate numerical techniques for low-dimensional systems, such as the density matrix renormalization group method (DMRG) [26, 27], the high-pressure superconducting two-leg ladder compound BaFe$_2$S$_3$ [28–30] was explored with regards to magnetic and pairing properties keeping two orbitals active [31, 32]. Evidence for the correct rung-FM and leg-AFM spin order was found over large portions of interaction parameters [31]. Evidence of metallization under high pressure was also reported [33, 34]. This is considered a precursor of superconductivity, which was also shown to appear in theoretical studies of two-orbital one-dimensional models upon hole doping [21, 22]. Even multiferroicity was unveiled in iron ladders [35], indicating an unexpected rich behavior. Moreover, novel Te-based ladders were predicted to display interesting magnetic properties as well [36, 37].

The phase diagram of a three-orbital Hubbard model
for chains, was also studied using DMRG [15, 16], unveiling various types of exotic magnetic and electronic phases. More canonical ferromagnetic and staggered $\uparrow\downarrow\uparrow\downarrow$ states were also stabilized varying the Hund and Hubbard interaction parameters. The spin dynamical properties of exotic orbital-selective Mott phases (displaying the selective localization of electrons on a particular orbital) were also analyzed, revealing unusual co-existing modes of spin excitations [24].

The magnetic phase diagram of the five-orbital Hubbard model for iron-selenide materials was initially studied using real-space Hartree-Fock approximations for chains [38] and ladders [39]. At electronic density $n = 5$, relevant to previously known chain compounds such as TiFeSe$_2$, a simple staggered AFM phase in a large parameter space of the phase diagram was reported, in agreement with existing experiments. Interestingly, a much richer phase diagram was theoretically predicted for chains with the electronic density $n = 6$. More reliable DMRG studies of the three-orbital Hubbard model at $n = 6$ have also consistently reported a similar wide variety of exotic phases for $n = 6$, including the block phase $\uparrow\uparrow\downarrow\downarrow$ at robust Hund coupling [15], as well as generalizations to longer blocks [23] and even spontaneously formed spiral phases [40]. But thus far only two-leg ladder materials, such as BaFe$_2$Se$_3$ and BaFe$_2$S$_3$, have been studied experimentally, confirming the block nature of the spin state – either $2 \times 2$ or $2 \times 1$ blocks – as well as exotic superconductivity upon high pressure. However, finding a truly $n = 6$ one-dimensional version, with only chains instead of ladders, would add another interesting member to the existing group of realizations of the theory predictions, opening a novel avenue for research.

Recently, the possibility of preparing the alkali iron selenide compound Na$_2$FeSe$_2$ has been discussed [41]. In Na$_2$FeSe$_2$ the iron atom is in a valence state Fe$^{2+}$, which correspond to an electronic density $n = 6$ for the 3$d$ Fe orbitals. As already discussed, Hartree-Fock studies of low-dimensional multiorbital models with electronic density $n = 6$ displayed a much richer phase diagram with exotic phases, as compared to the canonical staggered order of the $n = 5$ case. Motivated by the recent experimental efforts [41], in this publication we study theoretically the magnetic and electronic properties of the chain compound Na$_2$FeSe$_2$. Using first principles calculations we obtain the relevant hopping amplitudes. Next, using computationally accurate techniques, such as DMRG and Lanczos methods, we construct the ground-state phase diagram by varying the on-site same-orbital Hubbard $U$ repulsion and the on-site Hund coupling $J_H$. At low values of $J_H/U$, the staggered AFM order with wavevector $\pi$ dominates in a large portion of the phase diagram. However, increasing $J_H/U$ into the realistic regime for iron-based compounds, interesting block phases, particularly $\uparrow\uparrow\downarrow\downarrow$, dominate in a large region of parameter space. In contrast to Hartree-Fock methods, DMRG and Lanczos take into account quantum fluctuations rendering the results more reliable. Finally, albeit in a narrow region of parameter space, a novel phase $\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow$ was also found with a mixture of properties of the dominant block and staggered states.

The organization of the paper is as follow. In Section II, details of the ab initio calculations are described. Section III contains the three-orbital Hubbard model and details of the numerical methods. Section IV presents the DMRG and Lanczos predictions, where we focus on the results at the realistic Hund coupling $J_H/U = 1/4$, and later an extended phase diagram of the model is provided. Finally, conclusions are provided in Section V.

II. AB INITIO CALCULATIONS

The crystal structure of Na$_2$FeSe$_2$ is shown in Fig. 1. The most prominent feature is that edge-sharing FeSe$_2$ tetrahedral form one-dimensional chains running along the $c$ axis. Here, first-principles density functional theory (DFT) calculations are used employing the lattice...
constants $a$, $b$, and $c$, and the atomic positions of the Na, Fe, and Se atoms as reported in Ref. 41. These lattice constants are $a = 6.608\AA$, $b = 11.903\AA$, and $c = 5.856\AA$. The space group is Ibam (no. 72), and the atomic positions of Na(8j), Fe(4a), and Se(8j) are (0.1562, 0.35565, 0.0), (0.0, 0.0,0.25), and (0.21638, 0.11435, 0.0), respectively. The band structure and the projected density-of-states for the 3$d$ orbitals are presented in Fig. 2(a,b). The orbital $d_{x^2-y^2}$ contributes primarily near the Fermi level. The contribution of the orbitals $d_{xz}$ and $d_{yz}$ is subdominant, but not negligible.

Considering the one-dimensional character of the atomic structure, all interchain electron hopping amplitudes are neglected and we only focus on the intra-chain hoppings. In other words, only a Na$_2$FeSe$_2$ single chain [shown in Fig. 1(b)] is considered in the DFT procedure. The calculations were performed using the generalized gradient approximation [42] and the projector augmented wave (PAW) pseudopotentials [43], implemented in the Vienna ab initio Simulation Package (VASP) code [44, 45]. Since the magnetic properties will be considered via many-body calculations, magnetism was not included in the derivation of the bands and hopping amplitudes from first principles. Following a self-consistent calculation with total energy convergence of order eV, the maximally localized Wannier functions [46] were constructed using the WANNIER90 code [47] from the ab initio ground-state wave function.

![Figure 2](image.png)

**FIG. 2.** (a) Band structure and (b) projected density-of-states of the Na$_2$FeSe$_2$ single-chain compound obtained using DFT calculations. (c) Tight-binding (TB) band structure in the folded zone. (d) Tight-binding unfolded (TB-unfold) band structure used in the DMRG calculations. The zero in the vertical axis is the position of the Fermi level.

We constructed three Wannier functions involving the orbital basis $d_{xz}$, $d_{yz}$, $d_{x^2-y^2}$ for each iron and deduced the hopping parameters, readjusted to fit properly the band structure after reducing the original five orbitals to three (see Sec. III for details). The corresponding band structure using these hoppings is displayed in Fig. 2(c), which agrees well with the DFT band structure. Note that there are two Fe atoms in the primitive unit cell in the DFT calculation because of the alternating positions of the Se atoms, leading to a unit cell of $2d$ length, where $d$ is the distance between two nearest-neighbor iron atoms. The band structure can be unfolded since we only focus on the iron-only, leading to the bands in Fig. 2(d) that were used in the DMRG calculations.

### III. MODEL AND METHOD

The Hamiltonian for the one-dimensional chain of Na$_2$FeSe$_2$, with three orbitals at each iron site, will be described by the multi-orbital Hubbard $H = H_k + H_{in}$. The kinetic or tight-binding component contains the nearest- and next-nearest neighbors hopping:

$$H_k = \sum_{i,\sigma,\gamma} t_{\gamma,\gamma'} \left( c_i^\dagger \gamma \sigma c_{i+1,\sigma,\gamma'} + H.c. \right) + t'_{\gamma,\gamma'} \left( c_i^\dagger \gamma \sigma c_{i+2,\sigma,\gamma'} + H.c. \right) + \sum_{i,\gamma} \Delta_\gamma n_{i,\gamma},$$

where $t_{\gamma,\gamma'}$ is the nearest-neighbor (NN) $3\times3$ hopping amplitude matrix between sites $i$ and $i+1$ in the orbital space $\gamma = \{d_{xz}, d_{yz}, d_{x^2-y^2}\}$. $n_{i,\gamma}$ stands for the orbital and spin resolved particle number operator. These orbitals will be referred to as $\gamma = \{1, 2, 3\}$, respectively, in the remaining of the paper, for notation simplicity. As explained before, the hopping matrices for Na$_2$FeSe$_2$ were obtained from a tight-binding Wannier function analysis of first-principles results and they are in eV units. Explicitly, the NN $3\times3$ matrix $t_{\gamma,\gamma'}$ between sites $i$ and $i+1$ in orbital space is given by:

$$t_{\gamma,\gamma'} = \begin{bmatrix} -0.177 & 0.171 & 0.000 \\ -0.171 & 0.114 & 0.000 \\ 0.000 & 0.000 & 0.144 \end{bmatrix}$$

where $\gamma$ are the orbitals for site $i$ and $\gamma'$ for site $i+1$. $t'_{\gamma,\gamma'}$ is the NNN hopping matrix between sites $i$ and $i+2$:

$$t'_{\gamma,\gamma'} = \begin{bmatrix} -0.037 & -0.003 & 0.000 \\ 0.003 & -0.053 & 0.000 \\ 0.000 & 0.000 & -0.064 \end{bmatrix}$$

The on-site matrix containing the crystal fields $\Delta_\gamma$ for each orbital is given by:

$$t_{\gamma,\gamma'}^{OnSite} = \begin{bmatrix} -0.068 & 0.000 & 0.000 \\ 0.000 & -0.134 & 0.000 \\ 0.000 & 0.000 & -0.188 \end{bmatrix}$$
Note that we follow the convention that each $3 \times 3$ matrix (both $t_{\gamma,\gamma'}$ and $t'_{\gamma,\gamma'}$) represent the hopping matrix to move from one iron site to another. The full hopping matrix, which includes both the back and forth hopping processes, are of size $6 \times 6$ containing $t_{\gamma,\gamma'}$ in the upper off-diagonal block, the transpose of $t_{\gamma,\gamma'}$ in the lower off-diagonal block, and the on-site matrix $t'_{\gamma,\gamma'}$ on-site in both diagonal blocks [39]. The kinetic energy bandwidth is $W = 0.94$ eV.

The electronic interactions portion of the Hamiltonian is standard:

$$H_{\text{int}} = U \sum_{i,\gamma} n_{i\gamma} n_{i\gamma'} + \left( U' - \frac{J_H}{2} \right) \sum_{i,\gamma<\gamma'} n_{i\gamma} n_{i\gamma'}$$

$$-2J_H \sum_{i,\gamma<\gamma'} S_{i\gamma} \cdot S_{i\gamma'} + J_H \sum_{i,\gamma<\gamma'} \left( P_{i\gamma}^+ P_{i\gamma'} + \text{H.c.} \right)$$

The first term is the Hubbard repulsion between electrons in the same orbital. The second term is the electronic repulsion between electrons at different orbitals where the standard relation $U' = U - 2J_H$ is assumed. The third term represents the Hund’s interaction between electrons occupying the active 3d orbitals. The operator $S_{i\gamma}$ is the total spin at site $i$ and orbital $\gamma$. The fourth term is the pair-hopping between different orbitals at the same site $i$, where $P_{i\gamma} = c_{i\gamma}^+ c_{i\gamma'}$.

To solve numerically this Hamiltonian and obtain the ground state properties of Na$_2$FeSe$_2$, the DMRG and Lanczos methods were used. Open boundary conditions were employed in DMRG and at least 1200 states kept during the calculations. For these DMRG calculation, we used the DMRG++ computer program [48]. We fixed the electronic density per-orbital to be $n = 4/3$ (four electrons per site, i.e. four electrons in three orbitals). Such electronic density is used in the context of iron superconductors where iron is in a valence Fe$^{2+}$, corresponding to 6 electrons in five orbitals. A common simplification is to drop one orbital doubly occupied and one empty, leading to 4 electrons in the remaining three orbitals. Most of the DMRG calculations were performed using chains of length $L = 16$ and $L = 24$ which for our purposes of finding the magnetic properties of the ground state are sufficient. Furthermore, by investigating small lattice sizes ($L = 4$) with exact Lanczos diagonalization we reached the same conclusions.

**IV. RESULTS**

In Fig. 3, we show the phase diagram of the three-orbitals Hubbard model. We use realistic $ab$ initio hopping amplitudes for Na$_2$FeSe$_2$ and vary $U/W$ at fixed Hund coupling $J_H/U = 1/4$ [49]. This phase diagram was constructed based on DMRG calculations measuring several observables: the site average electronic density at each orbital $n_{\gamma} = \frac{1}{L} \sum_{i,\sigma} (n_{i\sigma\gamma})$, the spin-spin correlation $S(r) = \langle S_m \cdot S_l \rangle$ (where $r = |m - l|$; $m$ and $l$ are sites), and the spin structure factor $S(q) = \frac{1}{L^2} \sum_{m,l} e^{-iq(m-l)} \langle S_m \cdot S_l \rangle$ using primarily a system size $L = 16$. The global electronic density is $n = 4/3$ (4 electrons in three orbitals at each site in average).

Four different phases were found: (i) a paramagnetic phase (PM) at small $U/W$, followed by (ii) an unexpected block phase (AF2) where ferromagnetic clusters of two spins are coupled antiferromagnetically in a ↑↓↑↓ pattern. Then (iii) an intermediate electronically inhomogeneous and spin exotic state (AF3) was found, with ferro and antiferro magnetic ordering ↑↑↑↑↓↓. Finally, (iv) a canonical staggered antiferromagnetic phase (AF1) ↑↓↑↓ becomes stable. To distinguish among these magnetic phases and to obtain the approximate phase boundary location, we studied $S(q_p)$ vs $U/W$, where $q = q_p$ is defined as the wavevector that displays a sharp peak for each value of $U/W$ studied.

**Results at Hund coupling $J_H/U = 1/4$**

(a) **AF2 and AF1 phases**

At small Hubbard interaction $U/W$ the system displays metallic behavior without any dominant magnetic order, as expected. In this PM regime, the spin correlation $S(r)$ decays rapidly with distance in the range...
$U/W < 0.8$, as exemplified in Fig. 4(a). Increasing the Hubbard interaction $U/W$, the system enters into the block phase with AF2 magnetic ordering. In Fig. 4(b), the spin-correlations $S(r)$ at $U/W = 4.0$ are shown, clearly showing the formation of antiferromagnetically coupled ferromagnetic spins clusters in a $\uparrow\uparrow\downarrow\downarrow$ pattern. Because of this block order, the spin structure factor $S(q)$ in the AF2 phase displays a sharp peak at $q = \pi/2$, shown in Fig. 4(d). The peak value increases with the system size $L$ providing evidence of a stable exotic $\pi/2$-block magnetic state in the system. Note that the canonical power-law decaying real-space correlations in one dimension prevents $S(q)$ from diverging with increasing $L$, but in a real material it is expected that weak interchain couplings will stabilize the several phases we have observed.

As shown in Fig. 5(a), $S(q_0) = S(\pi/2)$ dominates in the range $0.8 \lesssim U/W \lesssim 8.5$, signalling a stable block-phase in a broad region of parameter space, at $J_H/U = 0.25$. Similar block-AF2 spin patterns, albeit extended in two dimensions into $2 \times 2$ ferromagnetic blocks, have been also experimentally observed in two dimensional iron-selenium based compounds with vacancies, such as Rb$_3$Fe$_{18}$Se$_2$ and K$_8$Fe$_{16}$Se$_2$ and more importantly for our purposes also in the two-leg ladder BaFe$_2$Se$_3$ which is a close “relative” of the Na$_3$Fe$_2$Se$_2$ compound due to the common one-dimensionality and iron valence Fe$^{2+}$. Although it is difficult to establish with clarity what induces this block state, previous work suggests that this phase is a result of competition between the Hund coupling $J_H$, favoring ferromagnetic alignment of spins as in double-exchange manganites, and the standard superexchange Hubbard spin-spin interaction that aligns the spins antiferromagnetically. One surprising aspect is that in the block-AF2 phase the population of orbital $\gamma = 3$ appears locked to 1.5 in all the range of $U/W$ investigated [Fig. 5(b)]. On the other hand, the occupancies of the other orbitals $\gamma = 1$ and $\gamma = 2$ change with varying $U/W$ in the same range.

In the inset of Fig. 5(b), the mean value of the local spin-squared averaged over all sites $\langle S^2 \rangle = \frac{1}{L} \sum_i S_i^2$ is shown vs $U/W$. For $U/W > 1.0$, strong local magnetic moments are fully developed at every site with spin magnitude $S \approx 1$, as expected for four electrons in three orbitals and a robust Hund coupling. In experiments, alkali metal iron selenide compounds generally show large magnetic moments, particularly when compared to iron pnictide compounds.

In Fig. 5(b), the site average occupancy of orbitals $n_{\gamma}$ vs. $U/W$ is shown, and for $U/W > 9.5$ the population of orbital $\gamma = 3$ reaches 2, thus decoupling from the system, while the other two orbitals $\gamma = 1, 2$ reach population 1. This arrangement minimizes the double occupancy at large $U/W$. In this Mott AF1 phase, the spin correlations show a canonical staggered AFM ordering, see Fig. 4(e), due to the dominating effect of the superexchange mechanism in the system, now involving only two active orbitals. The structure factor displays a sharp peak at $q = \pi$, see Fig. 4(e).
FIG. 6. (a) Electronic occupancy $<n_{\gamma,i}>$ for the three orbitals $(\gamma = 1, 2, 3)$ vs site index $i$, at $U/W = 9.1$, $L = 24$, in the AF3 regime showing an orbital-selective charge density wave. (b) Spin correlation $S(r) = \langle S_m \cdot S_i \rangle$ at $U/W = 9.1$ using a $L = 24$ cluster, displaying the AF3 magnetic ordering $\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow$. (c) The spin structure factor for three different values of $L = 8, 16, 24$ and at $U/W = 9.1$. Clear peaks at $q = 3\pi/4$ are shown.

At interaction $8.5 < U/W < 9.5$, a novel orbital-selective charge density wave phase was observed, with an exotic AF3 spin ordering. This phase exists for all the lattice sizes analyzed, and moreover it appears both using DMRG and Lanczos, as shown below, thus we believe it is a real regime of the present model. Figure 6(a) displays the population of the three orbitals $<n_{\gamma,i}>$ vs. the site index $i$, at $U/W = 9.1$. The results show an orbital-selective charge density wave phase. The pattern that develops has two sites with integer fillings, such as 1.0 and 2.0, followed by two sites with a fractional filling for all the three orbitals. Orbital 3 jumps from population 2.0 as in the phase AF1, to population 1.5, as in the phase AF2, as compared with Fig. 5(b). The other two orbitals 1 and 2 display similar characteristics, namely a mixture of AF1 and AF2 features.

Interestingly, in parallel to an inhomogeneous charge density arrangement, a novel spin pattern AF3 $\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\downarrow$, develops in the system for this range of $U/W$, see Fig. 6(b). The structure factor $S(q)$ shows a peak at $q = 3\pi/4$, which grows with increasing the system size, see Fig. 6(c). The phase boundary of this exotic AF3 phase is determined by comparing the peaks of the spin structure factors. As shown in Fig. 5(b), the peak at $q = 3\pi/4$ clearly dominates over other peaks of $S(q)$ in the range $8.5 < U/W < 9.4$. Similar spin configurations have also been reported in the study of the one-dimensional two-orbital Hubbard model [23] at density $n = 2.33$. We believe that this exotic phase stabilizes in the phase diagram mainly due to the NNN hopping $t'_{\gamma,\gamma'}$, since it generates frustration in the system. Eventually, for large enough values of the Hubbard interaction $U/W > 9.5$, the system enters into the insulating Mott phase with staggered AF1 magnetic ordering.

(e) Density of states and charge fluctuations

FIG. 7. DOS of different orbitals corresponding to different phases at $J_H/U = 0.25$ on a 4-site three-orbital system using Lanczos diagonalization. (a) Corresponds to the PM phase at $U/W = 0.4$, (b) is for the AF2 phase at $U/W = 4.0$, while (c) is for the AF1 phase at $U/W = 10.0$.

To characterize, at least qualitatively, the metallic vs. insulating nature of the different phases, we have calculated the orbital-resolved density-of-states, using the Lanczos method for small $L = 4$ three-orbital Hubbard model clusters. While these clusters are small, the results are exact. Figure 7 contains the orbital-resolved density-of-states (DOS) vs $\omega - \mu$ ($\omega$ is the frequency and $\mu$ the chemical potential), for different values of the interaction parameter $U/W$. In the paramagnetic phase, all the three orbitals have a robust weight at the Fermi level, Fig. 7(a), indicating metallic behavior. For the block phase at $U/W = 4$, we observe considerably lower weight at the Fermi level for all the three orbitals, Fig. 7(b), signaling a possible pseudogap and bad metallic behavior in the system. As expected, in the Mott phase Fig. 7(c)
and (ii) below the chemical potential the LDOS is
\[\rho_{i,\gamma}(\omega) = \frac{1}{\pi} Im \left[ \left\langle \psi_0 \left| c_{i,\gamma}^\dagger \omega - H - E_g - i\eta c_{i,\gamma} \right| \psi_0 \right\rangle \right],\]
where \(c_{i,\gamma}\) is the fermionic annihilation operator while \(c_{i,\gamma}^\dagger\) is the creation operator, \(E_g\) is the ground state energy, and \(\psi_0\) is the ground-state wave function of the system. We set the broadening parameter as \(\eta = 0.1\) for the DDMRG calculations. To avoid edge effects, for the LDOS we chose a central site \(i = L/2 + 1\) for the system size \(L = 16\). For the block phase at \(U/W = 4.0\), [Fig. 8(a)], a pseudogap with suppressed weight near the Fermi-energy appears, which is in accord with the Lanczos DOS, suggesting a bad metallic behavior for the AF2 phase. In Fig. 8(b), results for the LDOS at the AF3 phase using \(U/W = 9.1\) are shown. Here, due to the appearance of orbital-selective density order, we calculate results for two sites (one of each kind, i.e. with \(\gamma = 3\) equal to 2.0 and 1.5) and then average to obtain a net LDOS. The resulting LDOS at \(U/W = 9.1\) in Fig. 8(b) indicates insulating behaviour of the system.

In addition to the DOS we have also investigated the charge fluctuations \(\delta N\), to distinguish between a metal and an insulator. Figure 8(c) displays the \(\delta N\) charge fluctuations defined as \(\delta N = 1/L \sum_i (n_i^2 - \langle n_i \rangle^2)\) (where \(n_i = \sum_\gamma n_{i,\gamma}\)) and also the orbital-resolved charge fluctuation \(\delta N_{\gamma,i} = \frac{1}{L} \sum_i (n_{i,\gamma}^2) - \langle n_{i,\gamma} \rangle^2\) varying \(U/W\). For \(U/W \leq 0.8\), the large local charge fluctuations indicate strong metallic behavior in the PM phase as expected. Increasing \(U/W\), the charge fluctuations \(\delta N\) decrease substantially but remain finite for \(U/W \lesssim 8.5\), hinting towards a (bad) metallic behavior of the system in the block AF2-phase. Moving beyond \(U/W > 9.5\), the charge fluctuations approach zero, providing further evidence of insulating behavior in the AF1-phase. The AF3 phase is difficult to judge because of its narrow range nature, but it also seems insulating. These results are in agreement with the Lanczos DOS analysis in Fig. 7.

\section*{Phase Diagram varying \(J_H/U\) and \(U/W\)}

Figure 9 contains the phase diagrams of our three-orbital Hubbard model using realistic hopping parameters for Na\(_2\)FeSe\(_2\) and varying \(J_H/U\) from 0.15 to 0.30 and \(U/W\) from 0 to 10. The phase diagram shown in Fig. 9(a) is based on the DMRG calculations (\(L = 16\)), while Fig. 9(b) is based on Lanczos calculations using \(L = 4\) sites. To obtain the phase boundaries among the different phases, we have used the peak values of the spin-structure factor \(S(q_p)\) and the site-average occupancies of each of the orbitals \(n_i = \frac{1}{L} \sum_{i,\gamma} \langle n_{i,\gamma} \rangle\). For lower values of \(U/W\), as expected the metallic PM phase dominates in the phase diagram for any values of \(J_H/U\). The phase boundary of the PM phase clearly is very similar be-
between the DMRG and Lanczos results. Further increasing the Hubbard interaction $U/W$, in the lower range of Hund couplings $J_H/U$ shown, the block AF2 phase stabilizes in a small region of the phase diagram, while the staggered AF1 phase dominates over a larger portion. At not too large $J_H/U$, the superexchange mechanism dominates and promotes primarily staggered AF1 magnetic ordering, as expected. For these moderate values of $J_H/U$, a rapid cascade of transitions (PM $\rightarrow$ AF2 $\rightarrow$ AF3 $\rightarrow$ AF1) is observed. For $J_H/U < 0.19$, the narrow region in between AF2 $\rightarrow$ AF1 shows incommensurate behavior (not shown), while for $J_H/U > 0.19$ this intermediate region displays the exotic AF3-spin order with peak at $q = 3\pi/4$.

Interestingly, by increasing $J_H/U$ the block AF2 phase with spin configuration $\uparrow\downarrow\uparrow\downarrow$ stabilizes over a large portion of the phase diagram. This magnetic block state (AF2) is the same as found before in the context of orbital-selective Mott phases [15, 23, 24], although here the three orbitals remain itinerant, i.e. none has a population locked to one. As in those previous efforts, we believe the block spin order AF2 arises from competing superexchange order at small $J_H$ and double-exchange ferromagnetism at large $J_H$. While in our phase diagram there is no ferromagnetic phase in the range studied, we found that removing the NNN hopping leads to a stable ferromagnetic region, as in previous efforts [15, 23, 24]. Thus, the ferromagnetic state is certainly close in energy.

Also note the good agreement between the DMRG and Lanczos results found for the phase diagrams, see Fig. 9(a) vs Fig. 9(b), except for small $J_H/U$ where the AF3 phase is broader with Lanczos than DMRG, with opposite effects for the AF2 region. This small difference may be due to size effects. However, at moderate $J_H/U$ between 0.19 and 0.25 – a region considered realistic for iron-based compounds – the AF2 phase, which represents our main prediction for the physics of Na$_2$FeSe$_2$ if ever synthesized, is large and robust as Fig. 5(a) shows using DMRG and Fig. 10(b) using Lanczos.
V. CONCLUSIONS

In this publication, the phase diagram of the one-dimensional chain compound Na$_2$FeSe$_2$ has been investigated. We used a realistic three-orbital Hubbard model with the hopping amplitudes derived from ab initio calculations. The phase diagram presented here was constructed at electronic density $n = 4$ per site (the analog of $n = 6$ in a five-orbital system). To our best knowledge, this is the first material of the family of iron superconductors that has both a dominant chain geometry in the structure (not ladder) and valence Fe$^{2+}$. Our phase diagram is based primarily on DMRG measurements of the orbital occupancy and spin structure factor, supplemented by Lanczos techniques. In comparison to previously studied $n = 5$ one-dimensional three-orbital models for iron based compounds such as TlFeSe$_2$, which display a trivial staggered spin order, we find a much richer phase diagram for the alkali metal iron selenide compound Na$_2$FeSe$_2$. In particular, at low $J_H/U$ the staggered spin order dominates, but increasing $J_H/U$ the block AF2 phase $\uparrow\uparrow\downarrow\downarrow$ is stabilized over a large region of the phase diagram. We also observed a narrow region of a new phase AF3, with charge density wave properties and a combination of features of the AF1 and AF2 dominant phases, leading to a net $\uparrow\uparrow\downarrow\downarrow\uparrow\uparrow\downarrow$ magnetic order. Previous results with iron ladders suggest that high pressure probes may also bring surprises, such as metallicity and even superconductivity. As a consequence, we encourage experimentalists to synthesize Na$_2$FeSe$_2$ and investigate its magnetic properties via neutron scattering experiments.

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