Scaling Laws in Magnetohydrodynamic Turbulence

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We analyze the decay laws of the kinetic and magnetic energies and the evolution of correlation lengths in freely decaying incompressible magnetohydrodynamic (MHD) turbulence.

Scale invariance of MHD equations assures that, in the case of constant dissipation parameters (i.e., kinematic viscosity and resistivity) and null magnetic helicity, the kinetic and magnetic energies decay in time as \( E \sim t^{-1} \), and the correlation lengths evolve as \( \xi \sim t^{1/2} \).

In the helical case, assuming that the magnetic field evolves towards a force-free state, we show that (in the limit of large magnetic Reynolds number) the magnetic helicity remains constant, the kinetic and magnetic energies decay as \( E_k \sim t^{-1} \) and \( E_B \sim t^{-1/2} \) respectively, while both the kinetic and magnetic correlation lengths grow as \( \xi \sim t^{1/2} \).

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Magnetic fields are observed in all gravitationally bound large-scale structures in the Universe. They have been detected in galaxies, in galaxy clusters, and there are strong hints that they exist in superclusters, and in galaxies at high redshifts. These last astronomical observations support the conjecture that magnetic fields have been generated in the early Universe by microphysics processes (for a full discussion see Ref. \[1, 2, 3, 4, 5, 6, 7\] and references therein). The study of the evolution of primordial magnetic fields has been developed in the framework of the so-called freely decaying magnetohydrodynamic (MHD) turbulence. Numerical and analytical studies show that the relevant integral quantities in MHD turbulence, as for example the magnetic energy and correlation length, evolve in time following simple power laws \[8, 9, 10, 11, 12, 13, 14, 15, 16, 17\]. In this paper we shall give a possible explanation of those laws in the light of some recent high-resolution numerical simulations performed by Biskamp and Muller \[18\], and Christensson et al. \[19\].

We start by writing down the magnetohydrodynamic equations for a incompressible fluid in the case in which the expansion of the Universe can be neglected \[20\]:

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\nabla p + \mathbf{J} \times \mathbf{B} + \nu \nabla^2 \mathbf{v}, \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \\
\nabla \cdot \mathbf{v} &= \nabla \cdot \mathbf{B} = 0,
\end{align*}
\]

(1) (2) (3)

where \( \mathbf{v} \) is the velocity of bulk fluid motion, \( \mathbf{B} \) is the magnetic field, \( \mathbf{J} = \nabla \times \mathbf{B} \) is the magnetic current, and \( p \) is the thermal pressure of the fluid \[1\]. The kinematic viscosity \( \nu \) and the resistivity \( \eta \) are dissipative parameters, and are determined by microscopic physics. It is useful to define the kinematic and magnetic Reynolds numbers, \( \text{Re} = v l / \nu \) and \( \text{Re}_B = v l / \eta \), where \( v \) and \( l \) are the typical velocity and length scale of the fluid motion. We say that the dynamics is turbulent when \( \text{Re} \) and \( \text{Re}_B \) are much greater than unity \[21\].

In the case of the expanding Universe (with zero curvature and in the radiation era), it has been shown that the MHD equations are the same as the Eq. \([11, 13]\) provided that time, coordinates and dynamical variables are replaced by the following quantities \[22\]:

\[
t \to \tilde{t} = \int a^{-1} dt, \quad x \to \tilde{x} = ax,
\]

(4)

\[
\mathbf{B} \to \tilde{\mathbf{B}} = a^2 \mathbf{B}, \quad \nu \to \tilde{\nu} = a^{-1} \nu, \quad \eta \to \tilde{\eta} = a^{-1} \eta,
\]

(5)

where \( a \) is the expansion parameter, and we note that \( \mathbf{v} \) is not scaled. Because of the formal coincidence of the MHD equations in the expanding and non-expanding Universe, we can study the evolution of the integral quantities in MHD turbulence in both cases in a similar way. For definiteness, in this paper we shall consider only the case of non-expanding Universe.

Let us introduce the magnetic energy density of an anisotropic plasma in a volume \( V = \int_{2\pi/K}^{K} d^3 x \) as

\[
E_B(t) = \frac{1}{2V} \int_V d^3 x \mathbf{B}^2(x,t) = \int_{2\pi/L}^{K} dk \xi_B(k,t),
\]

(6)

where \( \xi_B(k,t) = \frac{2\pi}{k^2} k^2 \mathbf{B}(k) \cdot \mathbf{B}^*(k) \) is the magnetic energy density spectrum, \( \mathbf{B}(k) \) being the magnetic field in Fourier space, and \( k = |k| \). Here, \( 2\pi/L \) and \( K \) are the infrared and ultraviolet cutoffs, respectively. In the following we shall assume that \( L \to \infty \) and \( K \to \infty \).

The expressions for the kinetic energy \( E_k \) and the kinetic energy spectrum \( \xi_k \) are similar to the magnetic ones, with \( \mathbf{B} \) replaced by \( \mathbf{v} \).

\[\text{1}\] The thermal pressure \( p \) is not an independent variable; indeed, taking the divergence of Eq. \([1]\) we can express \( p \) as a function of \( \mathbf{B} \) and \( \mathbf{v} \) as \( \nabla^2 p = \nabla \cdot |\mathbf{J} \times \mathbf{B} - (\mathbf{v} \cdot \nabla) \mathbf{v}| \).
The magnetic helicity density is
\[ H_B(t) = \frac{1}{V} \int_V d^3x \mathbf{A} \cdot \mathbf{B} = \int_0^\infty dk \mathcal{H}_B(k,t), \quad (7) \]
where \( \mathcal{H}_B(k,t) = \frac{2\pi}{k^2} \mathbf{A}(k) \cdot \mathbf{B}^*(k) \) is the magnetic helicity density spectrum \(^2\), and \( \mathbf{A} \) is the vector potential.

The relevant length scale in turbulence theory is the so-called correlation length, which is the characteristic length associated with the large magnetic energy eddies of turbulence. It is defined by
\[ \xi_B(t) = 2\pi \int_0^\infty dk k^{-1} \xi_B(k,t) \frac{dk}{\int_0^\infty dk \xi_B(k,t)}. \quad (8) \]

One can define a kinetic correlation length \( \xi_v \) in the same manner as the magnetic one, with \( \xi_B \) replaced by \( \xi_v \).

Non-helical turbulence

It is well known that the MHD equations, under the scaling transformations \( x \to \ell x \), \( t \to \ell^{1-h} t \), admit solutions of the type
\[ \mathbf{v}(\ell x, \ell^{1-h} t) = \ell^h \mathbf{v}(x,t), \quad (9) \]
\[ \mathbf{B}(\ell x, \ell^{1-h} t) = \ell^h \mathbf{B}(x,t), \quad (10) \]
provided that the dissipative parameters \( \nu \) and \( \eta \) scale as
\[ \nu(\ell^{1-h} t) = \ell^{1+h} \nu(t), \quad (11) \]
\[ \eta(\ell^{1-h} t) = \ell^{1+h} \eta(t). \quad (12) \]

Here \( \ell > 0 \) is the “scaling factor” and \( h \) is a arbitrary real parameter. Starting from the scaling relations \( 9-12 \), in the seminal paper \( \cite{8} \), Olesen obtained the following expression for the magnetic energy spectrum:
\[ \xi_B(k,t) = \lambda_B k^p \psi_B(k^{\frac{4-p}{2}}), \quad (13) \]
where \( \lambda_B \) is a constant, \( \psi_B \) is a arbitrary scaling-invariant function, and \( p = -1 - 2h \). For a theory for which \( \nu \) and \( \eta \) are constants, we must take \( h = -1 \), corresponding to \( p = 1 \). Numerical simulations show that the scaling invariance of the solutions of MHD equations is approached only asymptotically for \( t \geq t_s \), where \( t_s \) is an unknown parameter. Hence, the scaling exponent \( p \) is not fixed by initial energy spectrum, the scaling law \( 13 \) being in general not valid at \( t = 0 \). Indeed, if we differentiate Eqs. \( 11 \) and \( 12 \) with respect to \( \ell \), and put \( \ell = 1 \) afterwards, we get \( \nu \sim \eta \sim \ell^{(1-p)/(3+p)} \). We conclude that \( p \) depends only on the scaling properties of dissipation parameters. Now, inserting Eq. \( 13 \) in Eq. \( 10 \) we have
\[ E_B(t) = E_B(t_s) \left( \frac{t}{t_s} \right)^{\frac{2(1+p)}{3+p}}, \quad (14) \]
and it is assumed that, due to the presence of dissipation terms, the integral is convergent. (Here and in the following we tacitly assume that scaling laws are valid only for \( t \geq t_s \). Because \( \nu \) scales the same way as \( B \), Eqs. \( 13 \) and \( 14 \) hold also in the case of kinetic energy. In the case of constant dissipation parameters (i.e. \( p = 1 \)) we get \( E_B(t) \sim t^{-1} \).

Following the same procedure performed in Ref. \( \cite{8} \), we get for the magnetic helicity spectrum the expression:
\[ \mathcal{H}_B = \mu_B p^{-1} \phi_B(k^{(3+p)/2}), \quad (16) \]
where \( \mu_B \) and \( \phi_B \) are arbitrary scaling-invariant function. Integration with respect to \( k \) gives \( H_B(t) = H_B(t_s) t/t_s^{-2(1+p)}/(3+p) \).

Theoretical arguments (see e.g. \( \cite{12, 20} \)) and numerical simulations of MHD equations (see e.g. \( \cite{18, 19} \) have clearly show that the magnetic helicity is an approximately conserved quantity in MHD turbulence (i.e. for large Reynolds numbers \( H_B \approx \text{const} \)). In the light of this, we are led to the conclusion that all the above scaling arguments can correctly describe the decay laws in freely decaying MHD only in the case \( H_B(t_s) = 0 \), or in others words, in the case of null magnetic helicity.

As regarding the correlation length, inserting Eq. \( 13 \) into Eq. \( 5 \) we obtain
\[ \xi_B(t) = \xi_B(t_s) \left( \frac{t}{t_s} \right)^{\frac{2}{3+p}}, \quad (16) \]
where
\[ \xi_B(t_s) = 2\pi t_s^{-\frac{1}{2}} \int_0^{\infty} dx x^{^{-3}+\frac{3}{3+p}} \psi_B(x), \quad (17) \]
In the case of constant dissipation parameters we get \( \xi_B \sim t^{1/2} \). The scaling law \( 16 \) is also valid for the case of kinetic correlation length.

The evolution laws \( 13 \) and \( 14 \) for \( p = 1 \) are exactly the laws found by Biskamp and Müller \( \cite{18} \) in high-resolution numerical simulations of non-helical MHD turbulence.

Helical turbulence

Mechanisms for generating helical magnetic fields in the early Universe has been proposed during the last

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\(^2\) The magnetic energy spectrum and magnetic helicity spectrum are not independent, since any magnetic field configuration satisfies the realizability condition \( \mathcal{H}(k,t) \leq 2k^{-1}E_B(k,t) \) (see e.g. \( \cite{22} \)). The field is said to be “maximally helical” if, for all \( k \), \( \mathcal{H} \) is of the same sign and saturates the above inequality.

\(^3\) The fact that Eq. \( 13 \) cannot be true at \( t = 0 \) it easily understood. Assuming the validity of Eq. \( 13 \) led to the inconsistency that the magnetic energy, \( E_B(t) = (\lambda_B/2t) \int_0^\infty dx \psi_B(x) \), diverges for \( t = 0 \) (here, we have considered, for simplicity, the case \( p = 1 \)).
years \[24, 25, 26, 27\]. In the helical case, the magnetic field is expected to evolve towards a substantially force-free configuration, \( \mathbf{J} \times \mathbf{B} = 0 \). This feature of helical MHD is expected on theoretical ground \[23, 28\], and verified in numerical simulations \[29\].

Assuming force-freedom (that is taking \( \mathbf{J} \times \mathbf{B} = 0 \) in Eq. (3)), and assuming that \( \nu \) and \( \eta \) scale as in Eqs. (11) and (12), we find that the helical MHD equations admit solutions of the type

\[
\mathbf{v}(\ell \mathbf{x}, \ell^{1-h} t) = \ell^h \mathbf{v}(\mathbf{x}, t),
\]

\[
\mathbf{B}(\ell \mathbf{x}, \ell^{1-h} t) = \ell^m \mathbf{B}(\mathbf{x}, t),
\]

where \( m \) is a new scaling exponents. Proceeding as in Ref. \[3\], we obtain the scaling relations

\[
\mathcal{E}_B(k, t) = \lambda_B k^q \psi_B(k^{3+p} t),
\]

\[
\mathcal{E}_v(k, t) = \lambda_v k^p \psi_v(k^{3+p} t),
\]

\[
\mathcal{H}_B(k, t) = \mu_B k^{q-1} \phi_B(k^{3+p} t),
\]

where \( \lambda_B, \lambda_v, \mu_B \) are constants, \( \psi_B, \psi_v, \phi_B \) arbitrary scaling-invariant functions, \( p = -1 - 2h \), and \( q = -1 - 2m \). Equations (20) and (22) imply the following scaling laws, respectively:

\[
E_B(t) = E_B(t_s) \left( \frac{t}{t_s} \right)^{-\frac{2r}{3+p}},
\]

\[
H_B(t) = H_B(t_s) \left( \frac{t}{t_s} \right)^{-2r},
\]

where we have defined \( r = q/(3+p) \) for notational convenience, and

\[
E_B(t_s) = \frac{2\lambda_B}{3+p} t_s^r \int_0^\infty dx x^{2r+q+2r-1} \psi_B(x),
\]

\[
H_B(t_s) = \frac{2\mu_B}{3+p} t_s^{2r} \int_0^\infty dx x^{2r-1} \phi_B(x).
\]

The kinetic energy scales in time as in Eq. (14), while the magnetic and kinetic correlation lengths follow the same law as in Eq. (16). Since the magnetic helicity is a quasi-conserved quantity in MHD turbulence, we expect that \( r \to 0 \) for large Reynolds numbers. In order to find a relation between these quantities, we shall apply the so-called “Taylor variational principle” \[20, 28\]. This principle states that in magnetohydrodynamic turbulent systems, the asymptotic state is expected to be a minimum-energy state under the constraint \( H_B = \text{const} \). This means that the asymptotic state satisfies the variational equation \( \delta[E - \beta H_B] = 0 \), where \( E = E_v + E_B \) is the total energy, and \( \beta \) is a Lagrangian multiplier. Variation with respect to \( \mathbf{A} \) gives

\[
\mathbf{J} = 2\beta \mathbf{B},
\]

while variation with respect to \( \mathbf{v} \) gives \( \mathbf{v} = 0 \). Thus, the asymptotic state corresponds to a force-free magnetic field configuration \[4\] with a vanishing kinetic energy \[5\].

It is worthwhile to stress that the force-free hypothesis \[27\] is only valid in the large-scale range (i.e. small \( k \)-scales). At small scales (large \( k \)-scales), i.e. in the dissipative range, turbulence is not efficient, and then the force-free hypothesis does not longer hold. Therefore, one has to consider the presence of small-scale structures of the magnetic field when investigating the dissipation of energy and helicity. Theoretical arguments (see e.g. \[12, 21, 30\]) seem to indicate that the magnetic energy dissipation is finite at scale \( k_{\text{diff}} \sim \eta^{-1/2} \), while at this scale the dissipation rate of the magnetic helicity goes to zero. We shall see in the Appendix that this means that the presence of small-scale structures of the magnetic field does not affect significantly the decaying of helicity, while the dissipation of energy is completely ruled by the decay of these small-scale modes. Since in the following we shall analyze the large-scale structure of the magnetic field, and in particular the decay of the magnetic helicity, we shall work in the force-free approximation.

Multiplying Eq. (27) by \( \mathbf{A} \) and then integrating in \( d^3 x \), we get \( \beta = E_B/H_B \). Now, using the equation for the dissipation of the magnetic helicity (see e.g. Ref. \[21\]),

\[
\frac{dH_B}{dt} = -2\eta \int d^3 x \mathbf{J} \cdot \mathbf{B},
\]

and taking into account Eq. (24), we obtain

\[
\frac{dH_B}{dt} = -8\eta \frac{E_B^2}{H_B}. \tag{29}
\]

Inserting Eqs. (24) and (22) into the above equation we get \( r = 4\eta/(15) t_s \beta^2(t_s) \). In order to give a physical meaning to the scaling exponent \( r \), let us introduce the magnetic Taylor micro-scale, the diffusion scale, the magnetic Reynolds number evaluated using the magnetic Taylor micro-scale, and the “eddy turnover time”:

\[
\xi_T = \frac{2\pi B_{\text{rms}}}{J_{\text{rms}}}, \quad \xi_{\text{diff}} = 2\pi \sqrt{\eta T}, \quad \mathcal{R}_B = \frac{v_{\text{rms}} \xi_T}{\eta}, \quad \tau_{\text{eddy}} = \frac{\xi_v}{v_{\text{rms}}} \tag{31}
\]

where \( B_{\text{rms}}, J_{\text{rms}} \) and \( v_{\text{rms}} \) are the RMS magnetic field, magnetic current and velocity field respectively. Taking

\[ As \] shown by Field and Carroll \[23\], a maximally helical magnetic field, whose energy spectrum is strongly peaked at some wavenumber \( k_0 \), will be substantially force-free: \( \mathbf{J} \approx k_0 \mathbf{B} \).

\[ We \] shall see that the kinetic energy decays faster than the magnetic energy (for the case of constant dissipative parameters and high Reynolds numbers we shall get \( E_v \sim t^{-4} \) and \( E_B \sim t^{-1/2} \)) and this means that the kinetic energy is asymptotically negligible compared to the magnetic energy.
into account Eqs. [23–27], we get \( \xi_T \sim \xi_{\text{diff}} \sim t^{2/(3+p)} \), and \( \beta = \pi/\xi_T \). Moreover, observing that \( v_{\text{rms}} \propto E_0^{1/2} \), and using the scaling laws for \( E_v, \xi_T, \eta \) and \( \xi_{\text{diff}} \), we find that \( \text{Re}_B \) is constant, while \( \tau_{\text{eddy}} \) scales as \( t \). Finally, we can write \( r \) as

\[
 r = \left( \frac{\xi_{\text{diff}}}{\xi_T} \right)^2 = \frac{t_s}{\tau_{\text{eddy}}} \frac{\xi_v}{\xi_T} \frac{(2\pi)^2}{\text{Re}_B},
\]

where all quantities are evaluated at \( t = t_s \).

For the case of constant dissipation terms we get \( E_B \sim t^{-1/2-2r} \), and \( H_B \sim t^{-2r} \). These scaling laws are in agreement with the laws found by Christensson et al. using different scaling arguments, and are in good agreement with numerical simulation performed by the same authors [19]. For very large magnetic Reynolds number and for initial condition such that \( t_s \xi_v/\tau_{\text{eddy}} \xi_T \) is of order of unity (as in Ref. [18, 19]), we have \( E_v \sim t^{-1}, E_B \sim t^{-1/2}, H_B \sim \text{const} \). These are exactly the evolution laws that Biskamp and Müller have found by direct integration of MHD equations [18].

It is important to stress that in Ref. [19] the initial spectra are taken to be \( E_v(k,0) \propto k^2 \) and \( E_B(k,0) \propto k^4 \), while in Ref. [18] are \( E_v(k,0) \propto k^2 \) and \( E_B(k,0) \propto k^2 \). Then, the interesting fact is that different power-law initial conditions led to the same scaling laws. This suggest that the scaling exponents we found do not depend on the particular choice of initial energy spectra, although the functional form of the scaling functions \( \psi_B, \psi_v, \phi_B \), and the characteristic time \( t_s \) are expected to depend on the initial conditions.

In summary, we have studied the evolution laws of the relevant integral quantities in freely decaying incompressible MHD turbulence. Using scale invariance of MHD equations and force-free hypothesis we have shown that, for constant dissipation parameters and very large Reynolds numbers, the kinetic and magnetic energies and correlation lengths evolve in time according to the following laws:

i) Non-helical case: \( E_v \sim E_B \sim t^{-1}, \xi_v \sim \xi_B \sim t^{1/2} \);

ii) Helical case: \( H_B \sim \text{const}, E_v \sim t^{-1}, E_B \sim t^{-1/2}, \xi_v \sim \xi_B \sim t^{1/2} \).

Finally, taking into account the results of different numerical simulations of MHD equations (in particular see Ref. [18] and Ref. [19]) we are led to suspect that this scaling laws are universal, in the sense that they do not depend on the initial conditions.

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Appendix

We start by noting that Eq. [23] together with Eq. [32] implies that the decay of the magnetic energy is ruled by a dissipative term:

\[
 \frac{dE_B}{dt} = -2\eta k_T^2 E_B - \frac{2}{3+p} \eta k_{\text{diff}}^2 E_B,
\]

where \( k_T = 2\pi/\xi_T \), and \( k_{\text{diff}} = 2\pi/\xi_{\text{diff}} \) is the k-scale at which the diffusion is efficient. On the other hand, taking into account the evolution equation for the magnetic energy (see e.g. Ref. [20]),

\[
 \frac{dE_B}{dt} = -\int d^3x \mathbf{v} \cdot \mathbf{J} \times \mathbf{B} - \eta \int d^3x \mathbf{J}^2,
\]

we get that, in force-free hypothesis, the decaying law of the magnetic energy becomes \( \frac{dE_B}{dt} = -2\eta k_T^2 E_B \), which gives \( E_B \sim t^{-2r} \). This means that the use of the force free-hypothesis correctly reproduce the diffusive part of the power in Eq. (23), but cannot give the other part of the power, \(-2/(3+p)\), which must come from the small-scale modes of \( \mathbf{B} \). In order to get the right decay law for the magnetic energy, we have to take into account the small-scale modes of the magnetic field in the expression of the current. So we write:

\[
 \mathbf{J} \simeq k_T(t) \mathbf{b} + k_{\text{diff}}(t) \mathbf{b},
\]

where \( \mathbf{b}(x) \) is the magnetic field on dissipation scale. (It should be noted that in this case the magnetic field is no longer force-free.) Because \( \mathbf{b} \) is a tangled field defined only in the dissipative range, we expect that \( \int d^3x \mathbf{C} \cdot \mathbf{b} \simeq 0 \), where \( \mathbf{C}(x) \) is an independent function of \( \mathbf{b}(x) \) (i.e. \( \mathbf{C} \) is any combination of \( \mathbf{B} \) and \( \mathbf{v} \) only). It is straightforward to show that the presence of small-scale structures of the magnetic field does not affect the decay of helicity, while the dissipation of energy is completely ruled by the decay of these small-scale modes. Indeed, inserting Eq. (35) into Eq. (28) and Eq. (24) we get, respectively:

\[
 \frac{dH_B}{dt} \simeq -4\eta k_T E_b,
\]

(we remember that \( k_T = 2\beta = 2E_B/H_B \)), and

\[
 \frac{dE_B}{dt} \simeq -2\eta k_T^2 E_B - 2\eta k_{\text{diff}}^2 E_b.
\]

Comparing Eq. (33) and (37), we conclude that the presence of small-scale modes of \( \mathbf{B} \) explains the \(-2/(3+p)\) part of the power in Eq. (23) provided that

\[
 E_b \simeq \frac{2}{3+p} E_B.
\]

The interesting and perhaps unexpected fact is that the energy associated to the small-scale modes, \( E_b \), is comparable to the total energy, \( E_B \), of the magnetic field.

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