Factorizing the hard and soft spectator scattering contributions for the nucleon form factor $F_1$ at large $Q^2$

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In [1] we suggested the factorization formula for the nucleon form factors which consist of the sum of two contributions describing the hard and soft spectator scattering, and we provided a description of the soft rescattering contribution for the FF $F_1$ in terms of convolution integrals of the hard and hard-collinear coefficient functions with the appropriate soft matrix elements.

In present paper we investigate the soft spectator scattering contribution for the FF $F_1$. We focus our attention on factorization of the hard-collinear scale $\sim Q\Lambda$ corresponding to transition from SCET-I to SCET-II. We compute the leading order jet functions and find that the convolution integrals over the soft fractions are logarithmically divergent. This divergence is the consequence of the boost invariance and does not depend on the model of the soft correlation function describing the soft spectator quarks. Using as example a two-loop diagram we demonstrated that such a divergence corresponds to the overlap of the soft and collinear regions. As a result one obtains large logarithm $\ln Q/\Lambda$ which must be included in the correct factorization formalism.

We conclude that a consistent description of the factorization for $F_1$ implies the end-point collinear divergencies in the hard and soft spectator contributions, i.e. convolution integrals with respect to collinear fractions are not well-defined. Such scenario can only be realized when the twist-3 nucleon distribution amplitude has non-vanishing end-point behavior at a certain low normalization point. Such behavior leads to the violation of the collinear factorization for the hard spectator scattering contribution.

In order to perform factorization of the hard-collinear modes one has to formulate an unambiguous method for the separation of the soft and collinear modes in SCET. We expect that the soft spectator scattering plays important role in such derivation.

We also discuss the physical subtraction scheme for SCET-I factorization which can be used for systematical analysis of the hadronic processes in the range of moderate values of $Q^2 \sim 5-20$ GeV$^2$ where the hard collinear scale $\sim Q\Lambda$ is still not large.

I. INTRODUCTION

A substantial progress of the experimental studies of the nucleon form factors (FFs) has been achieved during the last decade. The polarization transfer method allowed one to measure accurately the proton FFs up to momentum transfer $Q^2 \simeq 8.5$ GeV$^2$ [2-5], for recent reviews see, e.g., Refs. [6,8]. It also opened a possibility for systematic studies of the FFs at large space-like $Q^2$ region in the near future at the Jlab 12GeV upgraded facility see, e.g., [9]. At the same time the PANDA Collaboration at GSI is planning to carry out precise measurements of the proton FFs at large time-like momentum transfers, up to around 20 GeV$^2$, using the annihilation process $p + \bar{p} \rightarrow e^+ + e^-$ [10,11]. These experiments will provide us with new information on the FF behavior in the region of large momentum transfers that also provides a strong motivation for the theoretical studies of the large-$Q$ behavior of the nucleon FFs.

It is known for a long time that in the case of nucleon FFs the soft spectator scattering\footnote{Let us mention that in the literature there are many equivalent names associated with the soft spectator scattering: double flow contribution (or regime), soft-overlap contribution, Feynman mechanism.} is not suppressed by inverse powers of large $Q$ [12,15].

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Moreover, results of different phenomenological approaches \[16–22\] allows one to conclude that in the region of moderate $Q^2 \sim 5 - 15 \text{GeV}^2$ such mechanism plays the crucial role in order to obtain correct description of the FFs data and also other hard processes with baryons, e.g., \[23–25\]. A systematic QCD approach for description of such soft mechanism is needed in order to develop a model-independent formalism for description of existing and future experiments.

In \[1\] the contribution of the soft spectator scattering has been included into factorization scheme using the formalism of the soft collinear effective theory (SCET) \[26–31\]. The important feature in the description of the soft-overlap contribution is the presence of the hard-collinear scale $\sim Q \Lambda$. In order to take into account the soft rescattering mechanism one uses the two-step matching technique developed in SCET. Following from the leading-logarithmic analysis we suggested that the tentative factorization formula for the Dirac FF $F_1$ can be written as a sum of the hard and soft rescattering contributions:

$$F_1 \simeq F_1^{(h)} + F_1^{(s)},$$  \hspace{2cm} (1)

where the hard rescattering part $F_1^{(h)}$ is described by convolution of the hard coefficient function $H$ with the nucleon distribution amplitudes (DAs) $\Psi$:

$$F_1^{(h)} = \int Dx_i \int Dy_i \, \Psi(y_i) \, H(x_i, y_i|Q) \, \Psi(x_i) \equiv \Psi \ast H \ast \Psi,$$  \hspace{2cm} (2)

while the soft contribution $F_1^{(s)}$ has the same scaling behavior $\sim 1/Q^4$ as the hard spectator term $F_1^{(h)}$ and can be presented in the following form:

$$F_1^{(s)} \simeq C_A(Q) \int Dy_i \Psi(y_i) \int_0^\infty d\omega_1 d\omega_2 \, J'(y_i, \omega_i|Q) \int Dx_i \Psi(x_i) \int_0^\infty d\nu_1 d\nu_2 \, J(x_i, \nu_i|Q) S(\omega_i, \nu_i).$$  \hspace{2cm} (3)

This formula can be interpreted graphically as reduced diagram in Fig.1. The hard subprocesses in the soft spectator contribution are described by the hard coefficient function $C_A$ and two hard-collinear jet functions $J, J'$. They describe the parton scattering with the hard and hard-collinear momenta, respectively. The non-perturbative DA $\Psi$ and soft correlation function (SCF) $S$ describe the long distance scattering of collinear and soft modes. The convolution integrals in Eq. (3) are performed with respect to the collinear fractions $x_i$ and $y_i$, and with respect to the soft spectator fractions $\omega_i, \nu_i \sim \Lambda$.

In \[1\] we discussed the matching of electro-magnetic current onto SCET-I operator, calculation hard coefficient function $C_A$ and resummation of Sudakov logarithms. The structure of the factorization formula (3) was considered only qualitatively. In the current publication we present more detailed consideration of this contribution. We shall carefully consider the matching of SCET-I to SCET-II and check the validity of the Eq. (3).

In \[1\] we assumed the existence of the all convolution integrals with respect to the collinear and soft fractions in (3). Such assumption is motivated by the following observation. First, one can easily observe that collinear convolution integrals in both contributions (2) and (3) have a similar structure. Second, if one uses phenomenological models of the nucleon DAs existing in literature then the collinear integrals are well defined. After that we expected that the integral over the soft fractions must be also well defined.
However in this work we show using some general model-independent arguments that the convolution with respect to the soft fractions in (3) is divergent. This divergence can be represented as scaleless logarithmic integral over one of the fractions: $\sim \int_{-\infty}^{\infty} d\omega_1/\omega_1$. Obviously, such integral has problems in both ultraviolet (UV) and infrared (IR) regions. The divergence in UV-limit is a signal that here one faces with the well known problem: overlapping of the soft and collinear modes.

In order to clarify the situation we perform more careful analysis of the factorization (1) in the perturbation theory using perturbative expressions for the DAs and soft correlation function. It turns out that in the theory with massive quarks the collinear convolution integrals in (2) and (3) are also divergent. This divergence arises due to the overlap of the soft and collinear sectors. We demonstrate that there is specific large logarithm $\sim \ln Q/\Lambda$ which can be computed due to the overlap of collinear and soft regions associated with (2) and (3). This is exactly the logarithm which was computed in [12]. In order to compute (2) and (3) unambiguously one has to define a certain prescription which allows one to separate the collinear and soft sectors and to avoid a double counting. In the case of perturbation theory such separation can be carried out using the the idea of subtractions discussed in QCD for the Sudakov FF [32] or, similar technique in SCET known as zero-bin subtraction method [33]. We obtain that the perturbative calculations of the studied diagram is in agreement with the factorization formula (1). However generalization of the perturbative results to realistic case faces with certain difficulties and the description of the hard-collinear factorization even for $F_1$ remains challenging.

The overlap of the collinear and soft regions imposes also a qualitative restriction on the end-point behavior of the twist-3 nucleon DA. We expect that the end-point behavior of nucleon DA at low normalization differs from the behavior of the asymptotic DA $\Psi_2(x_i) \sim x_1 x_2 x_3$. The asymptotic-like models of DA always vanish when one of the fractions $x_i$ tends to zero. But DA at a low normalization point should not vanish in order to produce the end-point singularity in the convolution integrals. We show that such behavior can be obtained even in perturbation theory with massive quarks: for instance, if $x_2 \to 0$ and $x_1$ is fixed, then from the calculation of the Feynman diagrams $\Psi(x_1, 0, 1 - x_1) \neq 0$. We expect that the nonperturbative DA could posses the similar behavior.

Our presentation is organized as follows. In Sec. II we describe in details the soft rescattering contribution $F_1^{(s)}$ defined by Eq. (3), present the analytic results for the leading-order jet functions and discuss the divergency of the soft convolution integrals. Sec. III is dedicated to the analysis of the overlap between the soft and collinear modes. Using two-loop QCD diagram we calculate the “soft-collinear” logarithm and perform the interpretation of the obtained results in terms of hard $F_1^{(h)}$ and soft $F_1^{(s)}$ contributions. Generalizing these observations we suggest that collinear integrals in the factorization formulas for $F_1^{(h)}$ and $F_1^{(s)}$ are also divergent and therefore the nucleon DA must have nonvanishing end-point behavior. In Sec.IV we discuss the application of the SCET factorization for the phenomenological analysis at moderate values of $Q^2$. In Sec.V we summarize our results.

II. SOFT RESCATTERING CONTRIBUTION

We begin our discussion from the description of the soft rescattering contribution. All notations are the same as in [1] and are briefly described in Appendix A. For simplicity we consider proton as the target nucleon. In order to describe $F_1^{(s)}$ we need to define two nonperturbative functions: proton DA $\Psi(x_i)$ and soft correlation function (SCF) $S(\omega_i, \nu_i)$.

The proton DA is a well-known object defined as:

$$4 \left\langle 0 \left| W_{i\lambda_1}^{[\lambda_2 n]} [\xi_{\alpha_1}] W_c^{[\lambda_2 n]} W_{c\lambda_3}^{[\lambda_3 n]} | p \right\rangle \equiv \frac{e^{ijk}}{3!} \int D x_i \ e^{-i p \cdot (\sum x_i \lambda_i)} \Psi_\alpha(x_i), \quad (4)$$

where the measure reads $D x_i = dx_1 dx_2 dx_3 \delta(1 - x_1 - x_2 - x_3)$, the multiindex $\alpha \equiv \alpha_1 \alpha_2 \alpha_3$, and the index “$c$” denotes in SCET-II the collinear fields:

$$W_c[x] \xi_{\alpha} \equiv P \exp \left\{ ig \int_{-\infty}^{0} dt \ (n \cdot A_c)(x + tn) \right\} \left[ \frac{\delta^2}{4} u_c(x) \right]_\alpha, \quad (5)$$
The function \( \Psi_\alpha(x_i) \) can be further represented as

\[
\Psi_\alpha(x_i) = V(x_i) \, p_+ \left[ \frac{1}{2} \not{C} \right]_{\alpha_1 \alpha_2} \left[ \frac{\gamma_5 \not{p}}{2N} \right]_{\alpha_3} + A(x_i) \, p_+ \left[ \frac{1}{2} \not{C} \gamma_5 \right]_{\alpha_1 \alpha_2} \left[ \frac{\gamma_5 \not{p}}{2N} \right]_{\alpha_3} + T(x_i) \, p_+ \left[ \frac{1}{2} \not{C} \gamma_\perp \right]_{\alpha_1 \alpha_2} \left[ \frac{\gamma_5 \not{p}}{2N} \right]_{\alpha_3},
\]

where \( C \) is the charge conjugate matrix \( C : C \gamma_\mu C = \gamma_\mu^T \). The scalar functions \( V, A, T \) depend on the collinear fractions \( x_i \) and from the factorization scale which is not shown for simplicity. Alternatively, these functions can be represented through the one twist-3 DA

\[
V(x_1, x_2, x_3) \equiv V(1, 2, 3) = \frac{1}{2} (\varphi_N(1, 2, 3) + \varphi_N(2, 1, 3)),
\]

\[
A(1, 2, 3) = -\frac{1}{2} (\varphi_N(1, 2, 3) - \varphi_N(2, 1, 3)),
\]

\[
T(1, 2, 3) = \frac{1}{2} (\varphi_N(1, 3, 2) + \varphi_N(2, 3, 1)).
\]

In many applications \( \varphi_N(x_i) \) usually is approximated by a few polynomials in \( x_i \) with unknown coefficients. The polynomials represent the eigenfunctions of the evolution kernels and naturally arise from the solution of the evolution equation (see, e.g., \cite{[35][36]} and references therein). For example, the one of most popular parametrizations for \( \varphi_N(x_i) \) can be represented as follows \cite{[34]}:

\[
\varphi_N(x_i) \approx 120 x_1 x_2 x_3 f_N (1 + c_{10}(x_1 - 2x_2 + x_3) + c_{11}(x_1 - x_3)).
\]

The values of the nonperturbative constants \( f_N, c_i \) are not important for our further discussion and we will not provide their numerical estimates.

A parametrization such as in Eq. (11) always fulfills one important property: it vanishes at the boundary where one of the fractions is zero:

\[
\varphi_N(x_1, x_2 = 0, 1 - x_1) = \varphi_N(x_1 = 0, x_2, 1 - x_2) = \varphi_N(x_1, 1 - x_1, x_3 = 0) = 0.
\]

Such behavior is an important requirement which ensures the existence of the convolution integrals in the factorization formula \cite{[2]} because the hard coefficient functions, as a rule, have the end-point singularities \( \sim 1/x_{1,2} \). The end-point behavior (11) can be associated with the corresponding behavior of the evolution kernel. Performing the expansion of the DA \( \varphi_N \) in terms of the eigenfunctions of the evolution kernel and neglecting by the higher order harmonics one always obtains the model which vanishes at the boundary (11). However such approach is consistent only if one assumes an appropriate convergence of the conformal expansion at a given normalization point.

The second non-perturbative input is the SCF \( S \) introduced in Eq. (2). In SCET it is constructed from the soft quark and gluon fields \( q \) and \( A_s \), respectively. The gluon fields enter only in the form of Wilson lines such as

\[
S_n(x) = P \exp \left\{ i g \int_{-\infty}^{0} ds \ \gamma \cdot A_s(x + sn) \right\},
\]

The definition of \( S \) given in \cite{[1]} implies that it is a tensor with respect to Dirac indices. It is convenient to rewrite it in terms of scalar functions. This can be done with the help of Fierz transformation so that the result reads:

\[
S_{\omega_i, \nu_i} = \frac{1}{8} C \not{\gamma} \otimes \frac{1}{8} \not{\gamma} C S_{V} (\omega_i, \nu_i) + \frac{1}{8} C \not{\gamma} \gamma_5 \otimes \frac{1}{8} \not{\gamma} \gamma_5 C S_{A} (\omega_i, \nu_i) + \frac{1}{4} C \not{\gamma} \gamma_\perp \otimes \frac{1}{4} \not{\gamma} \gamma_\perp C S_{T} (\omega_i, \nu_i).
\]

The scalar functions \( S_{V,A,T} \) defined as

\[
S_X (\omega_i, \nu_i) = \int \frac{d\eta_i}{2\pi} \int \frac{d\eta_j}{2\pi} e^{-i\eta_1 \nu_1 - i\eta_2 \nu_2} \int \frac{d\lambda_1}{2\pi} \int \frac{d\lambda_2}{2\pi} e^{i\lambda_1 \omega_1 + i\lambda_2 \omega_2} \langle 0 | O_X (\eta_i, \lambda_i) | 0 \rangle,
\]

with the operators

\[
O_X (\eta_i, \lambda_i) = \epsilon^{i'j'k'} \left[ S_n^\dagger (0) \right]^{i'i'} \left[ S_n^\dagger \tilde{q} (\lambda_1 n) \right]^{j'j} \left[ S_n^\dagger \tilde{q} (\lambda_2 n) \right]^{k'k} \tilde{\Gamma}_X C [\tilde{q} S_n (\eta_1 \tilde{n})]^{i} \tilde{\Gamma}_X C [\tilde{q} S_n (\eta_2 \tilde{n})]^{k}(15)
\]
The flavor structure of the SCFs $S_X$ is defined by the flavor structure of the proton: it can be described as either $uu$- or $ud$-combinations. Below we will show that the tensor component $S_T$ does not contribute to $F_1$ and hence we can conclude that the matrix $S(\omega, \nu)$ is presented by four scalar functions: $S_{u,u}^{A,V}(\omega, \nu)$. Indeed, SCFs also depend on the factorization scale, which is not shown for simplicity.

One has to keep in mind that the arguments $\omega$ and $\nu$ (soft fractions) are of order $\Lambda$. Obviously, these variables can be associated with the light-cone projections of the soft spectators momenta which are defined to be positive. For instance, the leading order calculation of the SCF $S_{V,A}^{\nu,d}(\omega, \nu)$ in perturbation theory yields (see details in Appendix B)

$$ [S_{V,A}^{\nu,d}(\omega, \nu)]_{LO} = \frac{3m^2}{16\pi^6} \theta(\nu > 0)\theta(\omega > 0) \theta(\omega_1 \nu_1 > m^2)\theta(\omega_2 \nu_2 > m^2), $$

where $m$ denotes the mass of soft quarks. Notice that $S_{V,A}$ is proportional to the square of mass $m^2$ which arises from the numerators of the soft quark propagators and this fact, as we will see later, has an important consequence.

The last elements in Eq. (3) which one has to introduce are jet functions $J$ and $J'$. These are hard-collinear coefficient functions which can be computed in pQCD if the hard-collinear scale $Q_A$ is quite large. They appear in the matching of hard-collinear modes onto collinear and soft fields in SCET-II. Jet functions can be computed from the $T$-products which schematically can be written as $[1]$:

$$ T \left( \bar{\gamma}_{\mu} W^\nu e^{iL^{(n)}_{SCET-I}} \right) \simeq 2 \bar{\gamma}_{\mu} \gamma_{\nu} \gamma_5 \ast J \ast q q, \quad T \left( W \bar{\xi}_{hc} e^{iL^{(n)}_{SCET-I}} \right) \simeq \bar{q} q \ast J \ast \xi_{hc} \xi \xi_e, $$

where the asterisks denote the convolutions with respect to collinear and soft fractions. The matrix elements of the collinear operators in Eq. (18) yield twist-3 DAs $[1]$ describing the initial and final protons, the soft fields $q$ are combined into SCF $S$. Because the collinear operators in the both equations in (18) are the same, the functions $J$ and $J'$ are also the same, therefore one needs to compute only one of it. Corresponding leading order diagrams are shown in Fig. 2. The last two diagrams with the three gluon vertex have vanishing color factors and therefore do not contribute. This is in agreement with the observation made in Ref. $[1]$. The soft-collinear vertices in the diagrams is obtained from the subleading SCET Lagrangian $L^{(1)}_{\xi}$, see details in $[1]$. Computing these diagrams we projected the Dirac indices according to our definitions of the DA (6) and SCF (15).

To present the results for the jet functions let us rewrite Eq. (3) in the following form:

$$ F_1^{(s)}(Q) = C_A(Q) \{ \epsilon_u f_1^{u}(Q) + \epsilon_d f_1^{d}(Q) \}, $$

where $\epsilon_{u,d}$ denote quark charges. The hard coefficient function $C_A(Q)$ in Eq. (19) includes all large logarithms so that SCET-I form factors $f_1^{u,d}$ depend only from the hard-collinear scale $Q_A$ and defined as

$$ \langle p' | \bar{q}_c \gamma_{\mu} W^\nu \gamma_5 W \xi_{hc} | p \rangle_{SCET} = \bar{N}(p) \frac{\bar{q}_c \gamma_{\mu} N(p)}{4} f_1^{u}(Q, \mu_{hc} \sim Q_A) \equiv f_1^{u}(Q), $$

FIG. 2: Leading order SCET diagrams required for the calculation of jet functions. The inner dashed and curly lines denote hard-collinear quarks and gluons, external dashed lines correspond to collinear quarks, fermion lines with crosses denote soft quarks. Black square denotes the vertex of the SCET-I operator.
where the index $g$ describe the flavor of the hard-collinear field (active quark). Computing the diagrams in Fig.2 we obtained analytical expressions for the leading order jet functions. Our results can be presented in the following way

$$f_1^i(Q) = I_A^i(\omega_1, \omega_2) * S_{A}^{ud}(\omega_1, \omega_2, \nu_1, \nu_2) * I_A^u(\nu_1, \nu_2) - I_A^u(\omega_1, \omega_2) * S_{V}^{ud}(\omega_1, \omega_2, \nu_1, \nu_2) * I_V^u(\nu_1, \nu_2),$$

$$f_2^i(Q) = I_A^d(\omega_1, \omega_2) * S_{A}^{uu}(\omega_1, \omega_2, \nu_1, \nu_2) * I_A^u(\nu_1, \nu_2) - I_A^u(\omega_1, \omega_2) * S_{V}^{uu}(\omega_1, \omega_2, \nu_1, \nu_2) * I_V^u(\nu_1, \nu_2),$$

where asterisk denotes the convolution integral with respect to the soft fractions, for instance

$$I_A^d(\omega_1, \omega_2) * S_{A}^{uu}(\omega_1, \omega_2, \nu_1, \nu_2) = \int_0^\infty d\omega_1 d\omega_2 I_A^d(\omega_1, \omega_2) S_{A}^{uu}(\omega_1, \omega_2, \nu_1, \nu_2).$$

The proton DAs and integrations over the collinear fractions enter into the functions $I_{V,A}^{ud}$ in Eqs. (21-22). These functions read:

$$I_V^d(\omega_1, \omega_2) = (V - A - 2T)(y_1, y_2, y_3) * J_a^d(y_1, y_2, y_3, \omega_1, \omega_2) + (A - V - 2T)(y_1, y_2, y_3) * J_b^d(y_1, y_2, y_3, \omega_1, \omega_2),$$

$$I_A^d(\omega_1, \omega_2) = (V - A - 2T)(y_1, y_2, y_3) * J_a^d(y_1, y_2, y_3, \omega_1, \omega_2) + (A - V - 2T)(y_1, y_2, y_3) * J_b^d(y_1, y_2, y_3, \omega_1, \omega_2),$$

$$I_V^d(\omega_1, \omega_2) = V(y_1, y_2, y_3) * J_a^d(y_1, y_2, y_3, \omega_1, \omega_2) - A(y_1, y_2, y_3) * J_b^d(y_1, y_2, y_3, \omega_1, \omega_2),$$

$$I_A^d(\omega_1, \omega_2) = -A(y_1, y_2, y_3) * J_a^d(y_1, y_2, y_3, \omega_1, \omega_2),$$

where the asterisk again denotes the convolution integral over collinear fractions:

$$V(y_1, y_2, y_3) * J_a^d(y_1, y_2, y_3, \ldots) = \int D y_4 V(y_1, y_2, y_3) J_a^d(y_1, y_2, y_3, \ldots).$$

The leading order hard-collinear jet-functions $J_{a,b}^{ud}$ in Eqs. (24-27) read ($\bar{y}_i = 1 - y_i$):

$$J_a^{u}(y_1, y_2, y_3; \omega_1, \omega_2) = \alpha_s^2(\mu_{hc}) \frac{8\pi^2}{27} \frac{1}{(1 - \frac{1}{y_2} + \frac{4(y_1 y_2 + \omega_1 y_3)}{y_2^2 (\omega_1 + \omega_2)})},$$

$$J_b^{u}(y_1, y_2, y_3; \omega_1, \omega_2) = \alpha_s^2(\mu_{hc}) \frac{8\pi^2}{27} \frac{1}{(1 - \frac{1}{y_3} - \frac{4(y_1 y_2 + \omega_1 y_3)}{y_2^2 (\omega_1 + \omega_2)})},$$

and

$$J_a^{d}(y_1, y_2, y_3; \omega_1, \omega_2) = \alpha_s^2(\mu_{hc}) \frac{8\pi^2}{27} \frac{1}{(1 - \frac{1}{y_2} + \frac{4(y_1 y_2 + \omega_1 y_3)}{y_2^2 (\omega_1 + \omega_2)})},$$

$$J_b^{d}(y_1, y_2, y_3; \omega_1, \omega_2) = \alpha_s^2(\mu_{hc}) \frac{8\pi^2}{27} \frac{1}{(1 - \frac{1}{y_3} - \frac{4(y_1 y_2 + \omega_1 y_3)}{y_2^2 (\omega_1 + \omega_2)})}.$$

The argument of the QCD running coupling is defined by the hard-collinear scale: $\mu_{hc} \approx Q A$. From results (21-22) one can deduce that the convolution integrals in (24-27) with the proton DAs vanishing at the boundary as in Eq. (11) are well defined. Therefore one can assume that in the absence of other dominant regions, the convolution integrals with respect to soft fractions in (21-22) should also be finite. One can expect that the SCFIs are concentrated in the region where the soft fractions is of order $A$ and fall quickly in the region where the soft fractions are much larger than $A$.

However the assumption of convergence is not correct. Using the perturbative expression (17) instead of nonperturbative function one can easily obtain, that any convolution integral with respect to the soft fractions is logarithmically divergent and proportional to $\sim \ln[\mu_{UV}/\mu_{IR}]$ where scale $\mu_{UV}$ and $\mu_{IR}$
represent the UV and IR cut-off respectively. This is exactly the logarithmic contribution which was found in \cite{13} and later studied in \cite{14} and called as “nonrenormalization group type logarithmic contribution”.

Using the leading order expressions given in Eqs. (29)-(32) one can show that the similar situation is also takes place for the nonperturbative SCFs. Recall that the soft fractions $\omega_i$ and $\nu_i$ can be associated with plus and minus projections of the soft momenta describing soft spectators. Therefore the boost invariance implies that SCF depends on the products $\omega_i \nu_j$:

$$S_X(\omega_1, \omega_2, \nu_1, \nu_2) \equiv S_X(\omega_1 \nu_1, \omega_1 \nu_2, \omega_2 \nu_2, \omega_2 \nu_1),$$

(33)

Using this observation one can easily obtain that the convolution integrals of jet-functions and SCF in Eqs. (21)-(22) are divergent. Consider, as example, the integral from the Eq. (21):

$$J = I^u_A(\omega_1, \omega_2) \ast S^u_A(\omega_1 \nu_1, \omega_1 \nu_2, \omega_2 \nu_1, \omega_2 \nu_2) \ast I^u_A(\nu_1, \nu_2),$$

(34)

Using substitutions

$$\nu_1 = \nu_1'/\omega_1, \quad \nu_2 = \nu_2'/\omega_1, \quad \omega_2 = \omega_2' \omega_1$$

(35)

one obtains

$$J = \int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_2} \int_0^\infty d\nu'_1 \int_0^\infty d\nu'_2 \, I^u_A(\omega_1, \omega_2 \omega_1) I^u_A(\nu'_1, \nu'_1'/\omega_1) S^u_A(\nu'_1, \nu'_2, \omega_2 \nu'_1, \omega_2 \nu'_2).$$

(36)

Using homogeneity of the leading order jet functions in Eqs. (29)-(32) yields

$$I^u_A(\omega_1, \omega_2 \omega_1) I^u_A(\nu'_1, \nu'_1'/\omega_1) = I^u_A(1, \omega_2^2) I^u_A(\nu'_1, \nu'_2)$$

(37)

and one obtains

$$J = \int_0^\infty \frac{d\omega_1}{\omega_1} \int_0^\infty \frac{d\omega_2}{\omega_2} \int_0^\infty d\nu'_1 \int_0^\infty d\nu'_2 \, I^u_A(1, \omega_2^2) I^u_A(\nu'_1, \nu'_2) S^u_A(\nu'_1, \nu'_2, \omega_2 \nu'_1, \omega_2 \nu'_2).$$

(38)

The first integral in the last Eq. (38) is divergent $\int_0^\infty \frac{d\omega_1}{\omega_1} \sim \ln[\mu_{UV}/\mu_{IR}]$. The similar arguments can be used for the all contributions in Eqs. (21)-(22). Hence we can conclude that the soft convolution integrals are divergent and this singularity is independent on the particular properties of SCF. We performed the complete calculation of the jet functions in order to be convinced that this divergency does not cancel in the sum of all diagrams.

It is natural to assume that dimensionless integral $\int_0^\infty \frac{d\omega_1}{\omega_1}$ indicates that the soft region overlaps with the collinear one as it happens, for instance, in the classical case of Sudakov form factor \cite{32}. Then we may also expect the appearance of the IR-divergencies in the collinear convolution integrals which describe the hard rescattering term and referred often as “end-point” divergencies. However, the corresponding collinear integrals computed with the existing models of DA, like one in Eq. (10), are well defined. In order to clarify this situation we suggest to investigate the factorization properties in pQCD using as example appropriate 2-loop diagram which obtain contributions associated with the soft and collinear sectors. From analysis of the factorization of this diagram we may find the missing element in the discussed factorization picture.

III. SOFT-COLLINEAR OVERLAP AND SEPARATION OF THE HARD AND SOFT SPECTATOR SCATTERING CONTRIBUTIONS

A. Overlap of the soft and collinear regions

In order to perform the required analysis we consider the diagram shown in Fig. 3. This diagram was discussed in \cite{11} in order to show nontrivial contribution arising from the soft region. But we did not investigate the possibility of the overlap of the soft region with other dominant regions. We consider this more accurately now because it will help us to solve the problem with the soft divergent integral and understand the factorization for $F_1$. The contribution of this diagram into the nucleon FF $F_1$ can be
written as

\[ \tilde{N}'\gamma_\mu^\alpha N F_1[D] = \int Dx_i \int Dy_i \Psi_\alpha(y_i) [D^{\mu+}(x_i, y_i)]_{\alpha\beta} \bar{\Psi}_\beta(x_i) \]  

(39)

where DA \( \Psi \) is associated with the blobs in Fig.3. In order to simplify our consideration we use simple observation which follow from Eq. (3):

\[ \Psi(y_i)\hat{\gamma} = 0, \quad \bar{\Psi}(x_i) = 0. \]  

(40)

This allows us in the intermediate calculations to substitute instead of full DA \( \Psi \) the large components of collinear spinors:

\[ \Psi_\beta(x_i) \rightarrow \left[ \frac{\hat{\gamma}_i}{4} u(p_1) \right]_{\beta_1} \left[ \frac{\hat{\gamma}_j}{4} u(p_2) \right]_{\beta_2} \left[ \frac{\hat{\gamma}_k}{4} d(p_3) \right]_{\beta_3} \equiv [\xi_1]_{\beta_1}[\xi_2]_{\beta_2}[\xi_3]_{\beta_3}, \]  

(41)

\[ \Psi_\alpha(y_i) \rightarrow \left[ \bar{u}(p_1) \frac{\hat{\gamma}_i}{4} \right]_{\alpha_1} \left[ \bar{u}(p_2) \frac{\hat{\gamma}_j}{4} \right]_{\alpha_2} \left[ \bar{d}(p_3) \frac{\hat{\gamma}_k}{4} \right]_{\alpha_3} \equiv [\xi'_1]_{\alpha_1}[\xi'_2]_{\alpha_2}[\xi'_3]_{\alpha_3}, \]  

(42)

For simplicity we always assume that the bottom fermion line in the diagram corresponds to \( d \)-quark. We always assume that color indices are properly contracted and don’t show them explicitly. Hence, contribution of the diagram in Fig.3 can be associated with the perturbative diagram in Fig.4 and corresponding analytical expression reads:

\[ D^\mu = C \int d^4k_1 d^4k_2 \left[ \frac{1}{k_1^2 - m^2} \right] \frac{1}{k_2^2 - m^2} \frac{1}{k_3^2 - m^2} \times \frac{\xi_1^\gamma \xi_1^\alpha (k_1 + m) \gamma_1 \xi_1^\beta (p_1' - k_1') - k_1 + m) \gamma_1 (k_2 + m) \gamma_1 (p_1' - k_1') - k_1 + m) \gamma_1 (p_3 - k_1 - k_2)}{(p_1' - k_1')^2 - m^2} \times \frac{\xi_2^\gamma \xi_2^\alpha (p_1' - k_1') - k_1 - k_2 + m) \gamma_2 \xi_3 (p_1' - k_1') - m_1 - k_2 + m) \gamma_2 \xi_3 (p_3 - k_1 - k_2)}{(p_1' - k_1')^2 - m^2} \times \frac{\xi_3^\gamma \xi_3^\alpha (p_1' - k_1') - m_1 - k_2 + m) \gamma_3 \xi_3 (p_1' - k_1') - m_1 - k_2 + m) \gamma_3 \xi_3 (p_3 - k_1 - k_2)}{(p_1' - k_1')^2 - m^2} \]  

(43)

The quark mass \( m \) is used as a soft scale in order to regularize IR-divergencies and to describe the soft contribution, see Eq. (17). The factor \( C \) accumulates color structures and others factors according to Feynman rules.
Our task is to clarify the relation between the collinear and soft regions performing the interpretation of $D^\mu$ in terms of different contributions according to factorization formulas (4) and (3). If the overlap of these sectors is not possible then we may be find the missing elements in the factorization description. Notice that diagram in Fig. does not have UV-divergent subgraphs and therefore we do not need to consider renormalization of QCD parameters like quark mass and running coupling. Taking into account the logarithmic structure of the hard spectator term $F_1^{(h)}$ we may expect that after 2-loop integration the answer can be schematically written as

$$D^{\mu\nu} = \frac{1}{Q^6} [A^{\mu\nu} \ln^2 Q^2/m^2 + B^{\mu\nu} \ln Q^2/m^2 + C^{\mu\nu} + O(1/Q^7)].$$

(44)

where the leading power $Q^{-6}$ is obtained from the dimension reasons. In the absence of the divergent soft contribution we could expect that coefficients $A$ and $B$ can be interpreted in terms of convolutions of the LO and NLO hard coefficient functions with the two- and one-loop evolution kernels, respectively. However, the divergency of the soft convolution integral in (3) allows us to suggest that there is a large logarithm of $Q$ which has a different interpretation.

The exact calculation of the coefficients $A, B, C$ in Eq.(44) is a difficult task involving 2-loop massive integrals and we are not going to do this calculation here. We will focus our attention on the possible overlap of the soft and collinear contributions which can most probably provide the solution of the problem. We will use the technique, known as strategy of regions [38, 39] which allows to find and interpret the contributions originated from the different regions in (43). This formalism also allows one to establish whether the collinear contributions overlap with the soft sectors.

Let us begin our discussion from the soft region where

$$k_{1\mu} \sim k_{2\mu} \sim \Lambda,$$

(45)

Corresponding contribution can be represented as [1]:

$$D^{\mu\nu}_s = \mathcal{C} \left[(\gamma_\perp^\mu)_{\alpha_3\beta_3}\right]_{CA} \int dk_{1,2}^\perp \left[ \frac{[\gamma_1^\gamma]_{\alpha_1} [\gamma_2^\gamma]_{\alpha_2} [\gamma_3^\gamma]_{\alpha_3}}{y_1 y_2 [-Q(k_1^+ + k_2^+)]^2 [-Q k_1^+] \right] J_f,$n

$$\int dk_{1,2} \left[ \frac{[\gamma^\beta_1 \xi_1]_{\beta_1} [\gamma^\beta_2 \xi_2]_{\beta_2} [\xi_3]_{\beta_3}}{x_1 x_3 [-Q(k_1^+ + k_2^+)]^2 [-Q k_1^+] \right] J_f \int dk_{1,2} \left( \hat{k}_1 + m \right)_{\alpha_1\beta_1} \left( \hat{k}_2 + m \right)_{\alpha_2\beta_2} \right].$$

(46)

The subscripts $CA, J, S$ are associated with the appropriate contributions in Eq.(3) so that expressions in the brackets can be associated with these quantities.

Computing the transverse integrals inside the brackets associated with SCF $S$ one obtains the answer given in Eq.(17) which is proportional to $m^2$. Redefining the light-cone variables $k_{1,2}^\perp$ in the convolution integrals one can eliminate the mass $m$ from the consideration. But such redefinition performed in the divergent integrals may lead to a mistake. We will see that calculation of the regularized soft integrals leads to a logarithmic dependence on the mass $m$.

We find that the contributions of the remaining regions in $D^{\mu}$ can be associated only with the hard rescattering term $F_1^{(h)}$ [2]. For brevity we skip the discussion of all the dominant regions and pass directly to those which are relevant to our consideration. These regions are collinear regions, where both spectator momenta $k_{1,2}$ collinear either to initial $p$ or to final $p'$ momentum. Because of the symmetry between in and out states in Eq.(43) it is enough to consider only one of them. Let us choose

$$k_1 \sim k_2 \sim p' : k_1^\perp \sim Q, k_+ \sim \Lambda^2/Q, k_\perp \sim \Lambda.$$

(47)

Then one obtains that corresponding contribution has the leading power suppression $Q^{-6}$ as in Eq.(44) and can be written as a convolution of hard $[\ldots]_H$ and collinear $V$ parts (see details in Appendix C):

$$D^{\mu\nu}_{c-p'} \simeq \mathcal{C} \left[(\gamma_\perp^\mu)_{\alpha_3\beta_3}\right] \int dk_1^\perp dk_2^\perp V(k_1^\perp) \left[ \frac{[\gamma^\gamma_1 \xi_1]_{\beta_1} [\gamma^\gamma_2 \xi_2]_{\beta_2} [\gamma^\gamma_3 \xi_3]_{\beta_3}}{x^2 x_1 [-Q(k_1^+ + k_2^+)]^2 [-Q k_1^+] \right],$$

(48)
where
\[
\mathcal{V}(k_i^-) = \frac{1}{2} \int dk_1^+dk_2^+dk_1^\perp dk_2^\perp \frac{\{\xi^i_1(\vec{k}_1+m)\}}{[k_2^2-m^2][k_1^2-m^2]} \times \frac{\{\xi^\alpha_1(\vec{p}'^\perp-\vec{k}_1+\vec{m})\gamma^\alpha(\vec{k}_2+m)\}}{[(p'-k_1-k_2)^2-m^2]} \frac{\{\xi^\alpha_2(\vec{p}'^\perp-\vec{k}_2+\vec{m})\}}{[(k_1+k_2-p'+p')^2(k_1-p_1'^2)]}.
\]

The collinear part \(\mathcal{V}\) can be clearly associated with the two-loop contribution to the evolution kernel of the DA. Recall, that light-cone components \(k_\pm\) scale according to (17). Computing the integrals over \(k_{1,2}^\perp\) in Eq. (49) we obtain that the integration region for the minus components are restricted \(0 < k_i^- < Q\). Therefore these variables can be rescaled to the dimensionless quantities which, as a rule, used as collinear fractions. The integrals over the transverse momenta in (49) are UV-divergent and these are the logarithmical divergencies associated with the evolution of DA.

In order to see the overlap of the soft and collinear regions we consider the soft limit in the collinear contribution \(D_{c-p'}^{\mu-}\). In this case one finds that the collinear-soft limit reproduces the soft contribution in Eq. (48): \((D_{c-p'}^{\mu-})_s = D^{\mu-}\). This allows us to conclude that collinear and soft regions overlap. As a consequence, this also allows us to expect that the convolution integrals with respect to \(k_\perp\) in (48) are singular at the end-point region. We will see this explicitly later computing the collinear integral (48) in Sec. III B.

Let us emphasize one important point. Computing the soft integrals in Eq. (48) we can skip the terms with momenta \(k_{1,2}\) in the numerator of the integrand
\[
D^{\mu-}_s = \ldots \int dk_1^\perp dk_2^\perp \frac{[1_{\beta_1\alpha_1}(1_{\beta_2\alpha_2})]}{[k_1^2-m^2][k_2^2-m^2]},
\]
This is possible because only \(m^2\) contribution yields the final result, see Appendix B. From this observation we can conclude that the part of the collinear contribution \(D_{c-p'}^{\mu-}\) in Eq. (48) relevant for the soft-collinear overlap is UV-finite. Thus we obtain that the soft-collinear overlap in the given example is not connected with the UV-divergent collinear sector which is associated with the evolution of nucleon DA. The importance of the mass squared term in the discussion of the soft spectator scattering was also noted in [13].

Taking into account this fact we can conclude that the convolution integrals of the hard coefficient function \(H\) in the factorization formula (2) with 1- and 2-loop evolution kernels \(\mathcal{V}_{1,2}\) or with the models of DA with the vanishing end-point behavior (11) (for instance, like (10)) are free from the end-point singularities. But the convolution of LO hard coefficient function \(H\) in Eq. (48) with the finite \(m^2\)-term originating from the collinear subdiagram \(\mathcal{V}\) is singular. Below we will demonstrate that this singularity provides a logarithmic contribution which in the sum with the UV-logarithm from the soft term (48) yields a contribution proportional to \(\ln Q/m\).

Such situation turns out to be different from the many practical cases when the end-point singularities appear to be “visible” only after the inspection of the end-point behavior of the evolution kernel, or equivalently, asymptotic DA. Such situation takes place, for instance, in the case of Pauli FF \(F_2\). In this case the incompleteness of the collinear contribution can be associated with the double logarithmic contributions. But in case of \(F_1\) the collinear contribution has weaker singularity which is associated with subleading logarithm in the 2-loop diagram. As a result the hard rescattering mechanism can not describe correctly the coefficient \(B\) in Eq. (44). For its correct interpretation one has to involve the soft rescattering contribution.

In order to clarify the interpretation of the UV-finite contribution arising in \(\mathcal{V}\) in Eq. (48) consider the perturbative analog of the proton DA, i.e. as we would substitute three quark states instead of protons. In such case the 3-quark DA is defined as pQCD matrix element and its perturbative expansion
schematically read:\(^2\):

\[
\Psi_{PT}(x, \mu_F) = \Psi_0(x) + \frac{\alpha_s(\mu_R)}{\pi} \Psi_1(x, \mu_F) + \left(\frac{\alpha_s(\mu_R)}{\pi}\right)^2 \Psi_2(x, \mu_F) + \ldots .
\]  

(51)

where the higher order coefficients \(\Psi_{i>0}(x, \mu_F)\) depend logarithmically on the renormalization (or factorization) scale \(\mu_F\). Schematically this can be represented as:

\[
\Psi_1(x_i, \mu_F) = \ln \frac{\mu_F^2}{m^2} \mathcal{V}_0 + \Psi_0(x_i),
\]  

(52)

\[
\Psi_2(x_i, \mu_F) = \ln^2 \frac{\mu_F^2}{m^2} \mathcal{V}_1 + \mathcal{V}_0 + \ln \frac{\mu_F^2}{m^2} \mathcal{V}_2 + \Psi_0 + \Psi_{20}(x_i) + \ldots ,
\]  

(53)

where asterisk as usually denotes the collinear convolution integrals, \(\mathcal{V}_{1,2}\) denote now 1- or 2-loop evolution kernels and dots represent the other contributions associated with the renormalization of QCD coupling.

If we put \(\mu_F = m\) then all large logarithms vanish and we obtain:

\[
\Psi_{PT}(x_i, m) = \Psi_0(x_i) + \frac{\alpha_s(m)}{\pi} \Psi_1(x_i) + \left(\frac{\alpha_s(m)}{\pi}\right)^2 \Psi_{20}(x_i) + \ldots .
\]  

(54)

Obviously, Eq.\(^2\) is understood as a perturbative expansion of the DA at the low energy scale \(m \ll Q\). Performing transition from perturbative consideration to physical FF one substitutes instead of perturbative DA the realistic one: \(\Psi_{PT}(x_i, m) \to \Psi(x_i, \mu_0)\), where it is natural to assume that \(\mu_0\) is a certain soft scale of order \(\Lambda\). Following these arguments we can perform the interpretation of the UV-finite term \(\sim m^2\) in Eq.\(^2\). This term provides the contribution to \(\Psi_{20}(x_i)\) and therefore can be associated with the DA at low normalization \(\mu_0\). If \(\Psi_{20}(x_i)\) has nonvanishing end-point behavior \(\Psi_{20}(x_i, 0, 1 - x_i) \neq 0\) then only the corresponding collinear convolution integrals are singular. Let us emphasize again that such a behavior is closely related with the presence of soft spectator contribution.

Extrapolating this argument beyond perturbation theory we can not exclude that a similar situation takes place for the realistic DA. The singularity arising in the soft spectator contribution can be considered as strong argument in support of such scenario. In this case the nucleon DA \(\varphi_N(x_i, \mu_0)\) at a low normalization point \(\mu_0 \sim \Lambda\) has nonvanishing end-point behavior. Such a behavior is very important for the consistent description of the large logarithm appearing due to overlap of the hard and soft spectator scattering mechanisms. With such DA one immediately sees that the naive collinear factorization \(^2\) for FF \(F_1\) does not hold anymore.

Finally let us note that a rather similar conclusion about the end-point behavior of the nucleon DA was also obtained from the non-perturbative calculations. In \(^2\) the nucleon DA \(\varphi_N(z_i)\) has been computed in the chiral quark-soliton model \(^2\) using large \(N_C\) expansion. The DA was estimated at \(\mu_0 \lesssim 600\text{MeV}\) and it was found that it does not vanish at the end-point limit. However in \(^2\) it is also noted that obtained results are valid only in the region where \(z_i N_c \sim 1\). Therefore for the realistic value \(N_c = 3\) one can not expect an accurate description of the region with the relatively small collinear fractions.

The nucleon DA computed at leading order in \(1/N_C\) can potentially obtain large corrections at the end-point region.

**B. Calculation of the large logarithmic term arising from the overlap of collinear and soft regions**

Having established the soft-collinear overlap one faces with the problem of unambiguous separation of these regions in the factorization formulas \(^2\) and \(^3\). This is quite a complicate problem, especially for the processes involving composite particles like hadrons. Therefore it is useful as a first step to consider more simple examples with similar features. We will continue the discussion of diagram in Fig.\(^4\) and discuss the separation of the soft and collinear contributions in this particular case.

In order to compute the contributions of the soft \(^1\) and collinear \(^2\) regions explicitly we need a specific regularization for the convolution integrals over longitudinal momenta. Dimensional regularization (DR) can not be used in this situation because the soft contribution is given by scaleless integral and

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\(^2\) For brevity we skip the Dirac and color indices
therefore equals to zero in this case. Therefore one has to introduce a different regularization in order to work with the collinear and soft integrals in (2) and (3). This is not only a technical problem. A careful prescription is required in order to avoid double counting computing the different contributions. These questions have been studied during last years in many publications in connection with the Sudakov form factor, see e.g. [32, 44, 45]. In [32] a systematic subtraction procedure has been suggested in order to separate contributions of collinear and soft modes in non-inclusive processes. The factorization in the context of SCET was discussed in [33] where the idea of the so-called zero-bin subtractions was invented. In our analysis we adopt this technique in order to formulate a prescription for the correct separation of the soft and collinear modes and compute required integrals.

Note that the soft contribution in Eq. (10) and the appropriate part of the collinear term in Eq. (15) do not overlap with the hard region. As a result these integrals can be considered as UV-finite (in a sense that integrals over transverse momenta are finite). This allows us to carry out all calculations using a specific regularization in four dimensions which may be the simplest solution in this case. Such regularization prescription must be formulated uniformly for all collinear and soft divergent integrals. There is also a technical problem about applicability of the method of regions with this specific regularization in four dimensions which may be the simplest solution in this case.

In order to fix these details we suggest to investigate a simple one-loop integral which is close to our situation and can be easily computed.

1. Collinear and soft contributions in D = 4: one-loop case

As example consider following integral:

\[ J = \int dk \frac{m^2}{(k^2 - m^2)^2} \frac{1}{|k^2 - 2(pk)|} \frac{1}{|k^2 - 2(p'k)|} = \frac{i\pi}{Q^2} \ln Q^2/m^2 + O(m^2/Q^2). \] (55)

where we assume that expressions in the square brackets are defined with the \(+i\varepsilon\) prescription: \([X] \equiv [X + i\varepsilon]\), and the momenta \(p\) and \(p'\) are the same as used before. It is easily to see that this integral can be related to a well-known scalar vertex integral:

\[ J = m^2 \frac{d}{dm^2} \int dk \frac{1}{|k^2 - m^2|} \frac{1}{|k^2 - 2(pk)|} \frac{1}{|k^2 - 2(p'k)|} \] (56)

The asymptote of the vertex integral is given by large Sudakov logarithm \(\ln Q^2/m^2\) but the mass differentiation reduces this structure to a simple logarithm. Analysis of the dominant regions yields:

\[ J_{\text{coll}} \sim J_s \sim \frac{1}{Q^2}, \quad J_h \sim \frac{m^2}{Q^4}, \] (57)

where the subscripts “coll”, “s” and “h” denotes collinear to \(p\) or \(p'\), soft and hard regions respectively. The hard region is suppressed and the large logarithm is generated only from the overlap of collinear and soft regions. In order to obtain the leading order result \([55]\) with the help of the method of regions we must introduce appropriate regularization. It is not difficult to see that DR can not help in this case.

Consider the regularization by small off-shell external momenta introducing the small transverse components. This yields:

\[ J_{\text{reg}} = \int dk \frac{m^2}{(k^2 - m^2)^2} \frac{1}{|k^2 - 2(pk) - p_1^2|} \frac{1}{|k^2 - 2(p'k) - p'_1^2|} \] (58)

As usually, one may expect that the exact answer \([55]\) can be reproduced by the sum of the contributions from the dominant regions:

\[ J \simeq J_{e-p'} + J_{e-p} + J_s. \] (59)

But in this case this not true and this can be easily seen from the explicit calculation. The contribution from the soft region reads:

\[ J_s \simeq \int dk \frac{m^2}{(k^2 - m^2)^2} \frac{1}{|p_1^2 k_+ - p_1^2|} \frac{1}{|p^2' k_+ - p'_1^2|} = \frac{1}{Q^2} \int dk \frac{m^2}{(k^2 - m^2)^2} \frac{1}{|k^2 - 2(p'k) - p'_1^2|} \frac{1}{|k^2 - m^2|} \] (60)
where we introduced $\tau_- = p_+^2/p_+$ and $\tau_+ = p_+^2/p_-$. Notice that in the absence of the regulators this soft integral is scaleless and has both UV and IR divergencies as the integral in Eq. (38). Performing integration over $k_-$ by residues and taking the transverse integral one obtains

$$J_s = \frac{i\pi}{Q^2} \int_0^\infty dk_+ \frac{1}{[k_+ + \tau_1 + k_+ \tau'/m^2]} = -\frac{i\pi}{Q^2} \frac{m^2 \ln[\tau\tau'/m^2]}{\tau\tau' - m^2} \sim -\frac{i\pi}{Q^2} \ln[\tau\tau'/m^2]. \quad (61)$$

From Eq. (61) it is clearly seen that the regulators $\tau_+$ serve as IR- and UV-regulators ($k_+ \to 0$ or $k_+ \to \infty$ respectively). Passing to the last equation in (61) we neglected the regular in $\tau\tau'$ contributions assuming $\tau\tau' < m^2$.

In collinear region $k \sim p'$ one obtains

$$J_{c-p'} = \int dk \frac{m^2}{[k_+ + \tau](k_+ + \tau - \tau_+)} = \frac{1}{p_+} \int_{-\infty}^{p_+} dk_- \frac{1}{[k_- + \tau_-]} \int dk_+ \frac{m^2}{[k_+ + \tau_+][k_+ + \tau - \tau_+][k^2 - 2p', k_+]} \quad (62)$$

We neglected the second regulator $\tau_+$ in (63) because the corresponding integral is finite. Again, computing the integrals over $k_+$ and $k_-$ we obtain

$$J_{c-p'} = \frac{i\pi}{p_+} \int_{-\infty}^{p_+} \frac{1}{[k_- + \tau_-]} = -\frac{i\pi}{Q^2} \frac{\tau_-}{p_+}. \quad (64)$$

Similarly one computes the second collinear integral $k \sim p$:

$$J_{c-p} = -\frac{i\pi}{Q^2} \frac{\tau_+}{p_+}. \quad (65)$$

However the sum of the all terms (59) can not reproduce the exact answer (55). The reason is that collinear integrals $J_{c-p'}$ and $J_{c-p}$ obtain contribution also from the soft region. Computing the soft limit in expression in (63)

$$(J_{c-p'})_s = \int dk \frac{m^2}{[k^2 - m^2][k_+ + \tau_1][k_+ + \tau - \tau_+][k^2 - 2p', k_+]} \quad (66)$$

we reproduce expression for the soft integral in (61). Hence in order to compute the contribution of the collinear regions correctly one has to subtract from the expressions for collinear integrals $J_{c-p'}$ and $J_{c-p}$ appropriate soft contributions (62) (or similarly to perform zero-bin subtractions (63)). With such subtractions one has

$$[J_{c-p'} - J_s] = \frac{i\pi}{Q^2} \frac{\ln[\tau_+]}{m^2}, \quad [J_{c-p} - J_s] = \frac{i\pi}{Q^2} \frac{\ln[\tau_-]}{m^2}. \quad (67)$$

Notice that logarithms in (67) originates from the UV-region because IR singularities cancel in the difference exactly as it was discussed in (33). Now the sum of the all contributions reproduces the correct answer:

$$J = [J_{c-p'} - J_s] + [J_{c-p} - J_s] + J_s = \frac{i\pi}{Q^2} \frac{\ln[\tau_+]}{m^2} + \frac{i\pi}{Q^2} \frac{\ln[\tau_-]}{m^2} - \frac{i\pi}{Q^2} \frac{\ln[\tau_+]}{m^2} = \frac{i\pi}{Q^2} \ln[\frac{Q^2}{m^2}]. \quad (68)$$

One can easily see that zero-bin subtractions (66) can be also taken into account by changing the sign of the soft contribution in Eq. (59) that was noted already in (33, 43).

From considered example we can conclude, that evaluations of integrals using the method of regions in $D = 4$ with specific regularization, which allows to avoid scaleless integrals, must be carried out with the proper IR-subtractions. In the dimensional regularization such subtractions as a rule are performed automatically when one neglects the scaleless integrals, see for detailed discussion (43).
2. Collinear and soft contributions in two-loop integral \([43]\)

Now we return to the calculation of the overlapping soft and collinear contributions for the more complicate two-loop integral \([43]\). From discussion in Sec. II A it is clear that the situation with the soft-collinear overlap in \([43]\) is quite similar to the considered above one-loop example. In the 2-loop case we also expect to obtain simple a logarithm \(\ln Q^2/m^2\) originating from overlap of the collinear and soft regions. This can be written as

\[
D^{\mu\perp}_{s} + \text{UV-finite part} \left[ D^{\mu\perp}_{c-p'} + D^{\mu\perp}_{c-p} \right] = B^{\mu\perp}_{sc} \ln \frac{Q^2}{m^2}, \quad (69)
\]

where \(B^{\mu\perp}_{sc}\) denote appropriate part of the total coefficient \(B^{\mu\perp}\) in \([44]\). Computing \(B^{\mu\perp}_{sc}\) we can follow the same line as in the one-loop case.

We will perform all calculations in \(D = 4\). In order to regularize the divergencies we introduce an infinitesimal gluon mass \(\mu\). Such regularization looks quite natural in this case because the gluon mass plays the role of the virtuality cut-off for hard gluons in the hard subdiagram. Then the regularized soft contribution \([44]\) can be written as

\[
D^{\mu\perp}_{s} = \mathcal{C}(\gamma^\mu_{\perp})_{\alpha_3\beta_3} \int dk_1^{+} \int dk_2^{+} \int dk_3^{+} \frac{[\bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp}]}{y_3[-Q(k_1^+ + k_2^+)]} \left\{ \gamma^j_{\perp} \xi_{1,\beta_1} \left[ \gamma^j_{\perp} \xi_{2,\beta_2} \xi_{3,\beta_3} \right] \right\}
\]

\[
\times \int dk_4^{+} \frac{1}{x_4} \left\{ \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\}, \quad (70)
\]

where we used that in Breit system \(p'_- \approx p_+ \approx Q\), see Appendix A. Calculation of these integrals is a bit tedious but follows a basic line: two integrations performed by residue and the remnant integrals can be further computed keeping the most singular at \(\mu \rightarrow 0\) terms. The details can be found in Appendix D.

The result reads

\[
D^{\mu\perp}_{s} = \mathcal{C}(\gamma^\mu_{\perp})_{\alpha_3\beta_3} \left[ \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right] \left\{ \gamma^j_{\perp} \xi_{1,\beta_1} \left[ \gamma^j_{\perp} \xi_{2,\beta_2} \xi_{3,\beta_3} \right] \right\} \left(1/x_4\right), \quad (71)
\]

\[
J_s = \left(2\pi i\right)^2 \frac{Q^6}{x_1 x_2 y_1 y_2} \frac{1}{y_3} \int \frac{1}{1 - \pi^2/6} \ln \frac{\tau_+ \tau_-}{m^2} + \mathcal{O}(1), \quad (72)
\]

where we again introduced \(\tau_+ = \mu^2/p'_+\) and \(\tau_- = \mu^2/p_+\).

Consider now the collinear term \([48]\). We pick up from \(\mathcal{V}\) only UV-finite part (UV-f.p.) relevant for our calculation. Then the collinear integral reads

\[
\text{UV-f.p. } D^{\mu\perp}_{c-p'} \simeq \mathcal{C}(\gamma^\mu_{\perp})_{\alpha_3\beta_3} \int dk_1 \int dk_2 \int dk_3 \Psi_{20}(k'^{-}) \frac{[\bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\]}{-x_3 Q (k_1^+ + k_2^+)} \left\{ \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\}, \quad (73)
\]

with

\[
\Psi_{20}(k'^{-}) = \frac{[\bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\]}{-2(x'k'_1)} \left\{ \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\}, \quad (74)
\]

The notation \(\Psi_{20}\) is introduced taking into account the structure of DA described in Eq. \([53]\). Recall that according to \([48]\) the integrals in Eq. \([73]\) represent the convolution of the hard coefficient function with the perturbative DA \(\Psi_{20}\). We carried out the calculations of the integrals in Eq. \([74]\) keeping only the most singular contributions at the limit \(k'^{-} \sim 0\). The details are described in Appendix E. The result can be written as

\[
\Psi_{20}(k'^{-}) = \frac{[\bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\]}{-2(x'k'_1)} \left\{ \gamma^\gamma_{\perp} + \bar{\epsilon}^\gamma_{\perp} \gamma^\gamma_{\perp} \right\} \left(1/x_3\right) \int \frac{(k'_1)}{x_3} \ln \frac{1 + (k'^{-}k'^{-})}{k'^{-k'^{-}}} + \ldots, \quad (75)
\]
where dots denote the regular at the limit $k_i^- \sim 0$ contributions which we do not consider for simplicity; the quantities $k_{1,2,\text{max}}^- \sim 1$ denote the upper boundary for the relative light-cone components $k_i^-$. Their explicit value is irrelevant because we compute the integral in Eq. (73) with the leading logarithmic accuracy. Notice that from the expression \((75)\) one can see nonvanishing the boundary behavior of $\Psi_{20}$ at $k_2^- \to 0$ discussed in Sec. III A. Now we can substitute \((75)\) into \((73)\) and compute the regularized convolution integrals. For simplicity, we shall take into account the appropriate zero-bin subtractions changing the sign of soft contribution \((72)\) in the sum \((69)\). Then we have

$$\text{UV-f.p.} D_{-\gamma}^{\mu, -\nu} = C\left(\gamma_{\alpha}^\mu \right)_{\alpha_{\beta_{1}}} \left[\xi_3^\gamma_{1} \right]_{\beta_{1}} \left[\xi_2^j \gamma_{2} \right]_{\alpha_{2}} \left[\xi_\gamma_{\gamma} \right]_{\alpha_{3}} \left[\gamma^j \xi_{1} \right]_{\beta_{1}} \left[\gamma^j \xi_{2} \right]_{\beta_{2}} \left[\xi_{3} \right]_{\beta_{3}} (1)_{\alpha_{1}\beta_{1}} (1)_{\alpha_{2}\beta_{2}} J_{-\gamma},$$

where

$$J_{-\gamma} = \frac{(2\pi i)^2}{Q^6} \frac{1}{x_1 x_2 y_1 y_2} \int_{0}^{k_{1,\text{max}}^-} dk_1^- \int_{0}^{k_{2,\text{max}}^-} dk_2^- \frac{k_1^-}{k_1^- + \tau_+ - x_1 Q}$$

$$\simeq \frac{(2\pi i)^2}{Q^6} \frac{1}{x_1 x_2 y_1 y_2} \int_{0}^{k_{1,\text{max}}^-} dk_1^- \int_{0}^{k_{2,\text{max}}^-} dk_2^- \frac{k_1^-, k_2^-}{(k_1^- + \tau_+ - x_1 Q)^2}$$

$$\simeq \frac{(2\pi i)^2}{Q^6} \frac{1}{x_1 x_2 y_1 y_2} \left(1 - \frac{\pi^2}{6}\right) \ln \tau_- / Q.$$  

The similar calculation for the second collinear integral yields

$$J_{c-p} = \frac{(2\pi i)^2}{Q^6} \frac{1}{x_1 x_2 y_1 y_2} \left(1 - \frac{\pi^2}{6}\right) \ln \tau_+ / Q.$$  

Substituting obtained results into \((69)\) and changing sign in front of the soft term we obtain

$$B_{s-c}^{\mu, \nu} \ln \frac{Q^2}{m^2} = C\left(\gamma_{\alpha}^\mu \right)_{\alpha_{\beta_{1}}} \left[\xi_3^\gamma_{1} \right]_{\beta_{1}} \left[\xi_2^j \gamma_{2} \right]_{\alpha_{2}} \left[\xi_\gamma_{\gamma} \right]_{\alpha_{3}} \left[\gamma^j \xi_{1} \right]_{\beta_{1}} \left[\gamma^j \xi_{2} \right]_{\beta_{2}} \left[\xi_{3} \right]_{\beta_{3}} (1)_{\alpha_{1}\beta_{1}} (1)_{\alpha_{2}\beta_{2}} \times \frac{(2\pi i)^2}{Q^6} \frac{1}{x_1 x_2 y_1 y_2} \left(1 - \frac{\pi^2}{6}\right) \ln \frac{m^2}{Q^2}.  

We see that all regulators cancel as they should and we obtain a simple large logarithm. Recall that the collinear contributions are associated with the hard rescattering term $F^{(h)}_1$. Therefore this calculation explicitly demonstrates that the soft and hard rescattering contributions are related and must be computed consistently.

To summarize this section, we have demonstrated that the 2-loop diagram with massive quarks has a large logarithmic term originating from the overlap of the soft and collinear regions. The appearance of this logarithm does not contradict to our factorization scheme. We have demonstrated that in the perturbation theory one can perform the consistent description of large-$Q$ asymptotic of two-loop diagrams using definitions of SCF \((14)\) and DA \((1)\) with free quarks instead of nucleons. We have also seen that collinear integrals in the hard spectator contribution must be singular and such situation can be realized only with the help of additional assumption about the end-point behavior of the nucleon DA.

The factorization of the hard $F^{(h)}_1$ and soft $F^{(s)}_1$ contributions requires an additional regularization for the separation of collinear and soft sectors. We carried out such separation for the case of 2-loop diagram but a realization of such scheme on a physical FF with nonperturbative DA and SCF is challenging because such matching involves different nonperturbative matrix elements.

IV. DISCUSSION OF SCET FACTORIZATION SCHEME AND ITS PHENOMENOLOGICAL APPLICATIONS

From the consideration of Secs. II and III we can conclude that the practical realization of the factorization scheme discussed in the introduction has some difficulties. Suppose that nucleon DA at low normalization point does not vanish at the end-points, as we discussed in Sec.III. Then the collinear and soft convolution integrals are not defined in the both equations \((2)\) and \((3)\). At a formal level we have established all the dominant regions and provided the definitions for the all nonperturbative quantities
but that is not enough. The complication arises due to the overlap of the collinear and soft regions and, as a result, appearance of the specific “soft-collinear” logarithms $\ln Q/\Lambda$. Technically these logarithms arise from the matching of divergent collinear and soft convolution integrals in the soft $F_i^{(s)}$ and hard $F_i^{(h)}$ rescattering contributions. Therefore in order to describe corresponding logarithmic structure of the nucleon FF we must formulate definite recipe for the separation of the collinear and soft modes. This is well known problem, for many processes where collinear factorization is broken due to singular end-point behavior of the convolution integrals see, e.g., \cite{33, 34, 40}.

The presence of two hard scales: the hard $\sim Q^2$ and hard-collinear $\sim \Lambda Q^2$ in the description of nucleon FFs allows one to perform the factorization in two steps. First, one integrates over the hard fluctuations and passes from QCD to SCET-I. This effective theory includes hard-collinear modes which can be further factorized if the virtualities of the hard-collinear particles are quite large. Integrating over hard-collinear modes one passes from SCET-I to SCET-II which includes only collinear and soft particles with the virtualities of order $\Lambda^2$. However if the value of $Q$ is moderate, (for instance, the hard-collinear scale $Q\Lambda \sim m_N \sim 1\text{GeV}^2$ is not large in order to serve as expansion parameter) then one can not perform SCET-II factorization. Phenomenologically such situation is relevant for quite a large range of $Q^2$ such suppression is still weak and the soft mechanism provides essential contribution to the physical FFs. Therefore in order to estimate the relative importance of the two terms it is necessary to include both of them consistently within SCET-I factorization scheme. But the end-point singularities in the hard scattering contribution make such program very complicated. One must introduce a factorization scale and factorize the end-point contribution into $F_i^{(s)}$. In some cases this difficulty can be solved using the universality of the SCET-I matrix elements.

Suppose that we have different scattering processes which are described within SCET-I factorization and depend on the same SCET-I matrix element. Using universality of the SCET-I amplitude one can define the so-called physical subtraction scheme \cite{47, 48} which allows one to perform the systematic calculations of the hard spectator scattering contributions associated with the symmetry breaking corrections. The idea of this approach is very simple: the SCET-I soft-overlap form factor or amplitude can be rewritten as a sum of one of the physical amplitudes and the corresponding hard spectator contribution. Then this combination can be used further for the analysis of physical amplitudes of other processes with the same SCET-I matrix elements. After such redefinition the end-point singularities in the combination of the hard spectator terms must cancel and one obtains the well defined hard correction. This scheme has been successfully used for analysis of different B-meson decay amplitudes and we expect that it can also be used for the analysis of the different hadronic reactions with the soft spectator scattering contributions.

Let us illustrate the above discussion by one concrete example. Consider the following processes: $\gamma^* N \rightarrow N$ describing proton and neutron form factors at large $Q^2$ and $\gamma^* p \rightarrow \pi^0 p$ describing wide-angle hard electroproduction of pion with $s, t, Q^2 \gg \Lambda^2$. We suppose that nucleon form factors are described by following tentative formulae:

\begin{align}
F_i^p(Q) &= C_A(Q) \left\{ e_u f_i^u(Q) + e_d f_i^d(Q) \right\} + \Psi * H_p * \bar{\Psi}, \\
F_i^n(Q) &= C_A(Q) \left\{ e_u f_i^u(Q) + e_d f_i^d(Q) \right\} + \Psi * H_n * \bar{\Psi},
\end{align}  

(82)

where we used definitions \cite{19, 20} and isotopic symmetry, symbols $H_{p,n}$ denote the hard scattering kernel for the proton and nucleon cases, respectively. We also assume that the convolution integrals denoted by asterisk are regularized using some IR-regulator. Solving these equations with respect to SCET FFs one finds ($e_u = 2/3$, $e_d = -1/3$)

\begin{align}
f_i^u &= C_A^{-1} \left\{ 2F_i^p + F_i^n - \Psi * (2H_p + H_n) * \bar{\Psi} \right\}, \\
f_i^d &= C_A^{-1} \left\{ F_i^p + 2F_i^n - \Psi * (H_p + 2H_n) * \bar{\Psi} \right\},
\end{align}  

(84)

The pion production process can also be described as a sum of two contributions as shown in Fig.[5]. So
far we are not going to prove the exact factorization theorem for this process. Most important for us is that this configuration can be considered as one possible contribution to the nucleon helicity conserving amplitudes $A_1^{\alpha\beta}$. The first term in Fig.1 describes the soft overlap nucleon contribution and can be expressed in terms of the same SCET-I form factor $f_1^{\pi q}$. The pion blob is described by pion distribution amplitudes. The second contribution in Fig.1 can be associated with the hard spectator scattering and it is described as usually as a convolution of nucleon and pion DAs with the hard scattering kernel $T_H$. Therefore schematic expression of the contribution of Fig.1 reads

$$A_1^{\alpha\beta} = \varphi_\pi(z) * C_H(z)(e_u f_u^1 - e_d f_d^1) + \Psi(x_i) * \varphi_\pi(z) * T_H(x_i, y_i, z) * \Psi(y_i) \quad (86)$$

where we again assume a some regularization for the divergent convolution integrals. Substitution of expressions for the SCET form factors $(84)$ and $(85)$ in Eq.$(86)$ yields

$$A_1^{\alpha\beta} = \varphi_\pi(z) * C_H(z)/C_A \frac{1}{3}(5F_1^p + 4F_1^n) + \Psi(x_i) * \varphi_\pi(z) * \left( T_H(x_i, y_i, z) - C_H(z)/C_A \frac{1}{3}\{5H_p(x_i, y_i) + 4H_n(x_i, y_i)\}\right) * \Psi(y_i). \quad (87)$$

Thus the soft overlap contribution is represented in terms of the physical FFs $F_1^{p,n}$. The ratio $C_H(z)/C_A$ depends only from the factorization scale associated with the evolution of pion DA. All Sudakov logarithms must cancel in this ratio. On the other hand, the end-point singularities in the hard scattering kernels must be compensated in the combination of $T_H$ and $H_{a,p}$ in rhs of $(88)$. The simple analysis show that the hard spectator correction is of order $\alpha_s^2$. At the same time the ratio $C_H(z)/C_A = \alpha_s C_{LO}(z) + O(\alpha_s^2)$. The next-to-leading order can be computed from the one loop corrections to $C_H(z)$ and $C_A$ and the hard spectator corrections can appear only in the next-next-to-leading order. The more detailed analysis we are going to present in the separate publication.

Therefore with this example we demonstrated how the SCET-I factorization on the soft and hard spectator contributions allows one to analyze the realistic hadronic processes at the intermediate momentum transfer. Of course, such method can be used when the number of observables is larger then the number of unknown SCET amplitudes.

For the asymptotically large values of $Q$ when the hard-collinear scale is a good parameter for the asymptotic approximation one can try to perform the second factorization step and pass from the intermediate SCET-I to the low energy SCET-II. Performing this matching one must provide the solution of the soft-collinear mixing in order to treat correctly corresponding logarithms and avoid double counting between the soft and hard rescattering contributions.

An interesting proposal for the separation of collinear and soft modes for the exclusive processes in the framework of SCET was discussed in [33]. In order to avoid double counting it is suggested to perform zero-been subtractions in the collinear convolution integrals. In our consideration in Sec.III we already used this idea in order to compute the investigated integrals in $D = 4$. The technique developed in [33] allows one to carry out zero-bin subtractions in dimensional regularization. In that case these subtractions remove collinear end-point singularities but generate the UV-poles. These UV-poles have been subtracted with the help of specific counterterms represented by non-local operators built from the collinear fields.

Then, for instance, a divergent collinear convolution integral can be interpreted as following

$$\int_0^1 dx \frac{\phi_\pi(x)}{(x^2)^{\tilde{\sigma}}} = \int_0^1 dx \frac{\phi_\pi(x) - x\phi'_\pi(0)}{x^2} + \phi'_\pi(0) \ln p_+ / \mu_+, \quad (89)$$
In this example we used the pion DA $\phi_\pi(x)$ which, as it is often assumed, at the end point region behaves as $\phi_\pi(x \to 0) \sim x$. The auxiliary scale $\mu_+$ arises after UV-renormalization and is quite similar to scales $\tau_\pm$ introduced by us in the calculations in Sec.III.

Application of this scheme has been illustrated in [33] by a calculation of the transition form factor $F_{0\pi}$. However, we expect that this analysis is not complete because the soft spectator contributions were not considered. The end-point singularities of the collinear integrals can be understood as a clear signal that such mechanism can also contribute in this case. We expect that, as in the examples considered in Sec.III, the soft spectator contributions will play the important role in the full compensation of the $\mu_+$-scale dependence in the physical amplitudes. This means, that the logarithmic term in Eq.(89) with the derivative $\phi_\pi'(0)$ is closely related to the soft rescattering contribution. Then one does not need the specific collinear UV-counterterms which were introduced in [33]. It means that the properly defined zero-bin subtractions must be unambiguously related with the UV-renormalization of the soft rescattering contribution. We expect that in this case one could carry out systematic analysis and avoid double counting.

Formulation of such scheme is a difficult task and at present time we have not yet found a convincing technical realization of this idea. The most challenging problem is the consistent formulation of the zero-bin (or infrared) subtractions for the collinear convolutions integrals with nonpertubative DAs (or other collinear matrix elements) and corresponding UV-renormalization of the soft rescattering contribution involving soft correlation functions, i.e. unambiguous matching of collinear and soft modes and consequently the definition of the hard and soft rescattering contributions. The other difficult moment is the absence of a well-defined regularization. Dimensional regularization can not be used for the calculation of the soft convolution integrals because corresponding integrands are scaleless. Therefore one has to invent the other method, for instance gluon mass, off-shell momenta or analytic regularization. As a rule, such regularization has potential problems beyond the leading order that makes difficult systematic calculations and analysis.\(^3\)

V. CONCLUSIONS

In this paper we have investigated the soft spectator scattering term appearing in the calculation of the large-$Q$ asymptotic form of the proton FF $F_1$. In our previous publication [1] we suggested that the full factorization formula for the nucleon FF $F_1$ is given by the sum of two contributions: hard rescattering term $F_1^{(h)}$ and the second term involving soft spectator quarks which we refer as the soft rescattering (or soft spectator scattering) contribution $F_1^{(s)}$.

Using SCET technique in [1] we provided an explicit definition for the leading power contribution $F_1^{(s)}$ in terms of the hard and hard-collinear coefficient functions and the nonpertubative matrix elements. In this paper we have investigated more closely the proposed formula. We have computed the leading order jet functions $J$ which appear in the matching from SCET-I to SCET-II. Using this result we have demonstrated that the corresponding soft convolution integrals in factorization formula [3] are logarithmically divergent in the ultraviolet and infrared regions simultaneously. This observation follows from the boost invariance of the soft correlation function $S$ in Eq.(3) and therefore does not depend on any model assumptions. It is natural to assume that such divergency arises due to the overlap of the collinear and soft regions.

In order to clarify the situation we studied as example a two-loop QCD diagram with massive quarks and investigated the dominant contributions associated with the different regions. It was demonstrated that the overlap of soft and collinear sectors provides the large logarithm $\sim \ln Q/m$ where $m$ is the quark mass which plays the role of the soft scale. Therefore this confirms our assumption that the logarithmic divergence of the soft convolution integrals can be naturally explained by the overlap with the collinear region.

\(^3\) During the preparation of this text new article [49] has been published where authors suggest a systematic formalism for the separation of soft and collinear regions and resummation of the arising large logarithms. In their terminology these are large rapidity logarithms. Application of this approach to the problems discussed in our presentation might be an interesting subject for future studies.
The obtained results allow us to make important conclusions about factorization FF $F_1$ at large $Q$ limit. We expect that the general scheme as described in Eqs (2)-(9) is valid but the separation of the soft $F_1^{(s)}$ and hard $F_1^{(h)}$ rescattering contributions requires a certain prescription for the separation of the soft and collinear degrees of freedom. From the overlap of the soft and collinear modes it follows that collinear convolution integrals in $F_1^{(h)}$ must be also divergent. This naturally leads to assumption that the nucleon distribution amplitude $\varphi_N$ at some low normalization point has nonvanishing end-point behavior

$$\varphi_N(x_1, x_2 = 0, 1 - x_1) \neq 0, \varphi_N(x_1 = 0, x_2, 1 - x_2) \neq 0.$$  

(90)

Such behavior is opposite to the asymptotic DA $\varphi_N^{as}(x_i) \sim x_1 x_2 x_3$ vanishing at this limit.

If Eq. (90) is realized then the corresponding collinear convolution integrals in factorization formula for $F_1^{(h)}$ also suffer from the logarithmical end-point singularities which reflect the admixture of the soft mode. However, in the sum $F_1^{(s)}$ and $F_1^{(h)}$ the end-point singularities must cancel if computed consistently.

The specifics of the factorization for $F_1$ allows us to suggest that the correct treatment of the soft-collinear mixing becomes important only for a transition from SCET-I to SCET-II. In present paper we do not provide any systematic formalism for the factorization of the soft and collinear degrees of freedom. However, we would like to stress that the soft spectator contribution plays the important role in the correct description of such factorization.

The picture with the soft spectator scattering mechanism naturally introduces a concept of two large scales: hard $\sim Q^2$ and hard-collinear $\sim \Lambda Q$. This allows one to define the transition region or the region of moderate values of large momentum transfer $Q$. We suggest that it corresponds to the situation when the inverse power of hard scale $1/Q^2$ is a good expansion parameter but hard-collinear scale is still not too large. Taking $\Lambda \approx 300 - 400 \text{MeV}$ and $Q^2 = 25 \text{GeV}^2$ one easily obtains that $Q \Lambda \simeq 1.5 - 2 \text{GeV}^2$. Hence in the situation when $Q^2 \approx 10 - 20 \text{GeV}^2$ which overlaps with the majority of existed and upcoming experiments, one can perform consistently only the SCET-I factorization. Within such factorization scheme it is useful to take into account both soft and hard spectator scattering contributions because their relative contribution might be comparable if the suppression from the Sudakov form factor for the soft-overlap term is not sufficiently large. In order to perform a systematic consideration in this case we suggest to use the physical subtraction scheme which allows to express the SCET-I amplitudes in terms of physical ones. In such situation one can also to get rid of the end-point singularities from the hard spectator corrections. Applications to specific processes will be presented in subsequent publications.

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Appendix A: Brief summary of used notations

Through the paper we imply Breit frame

$$q = p' - p = Q \left( \frac{n}{2} - \frac{\bar{n}}{2} \right), \quad n = (1, 0, 0, -1), \quad \bar{n} = (1, 0, 0, 1), \quad (n \cdot \bar{n}) = 2,$$

(91)

and define the external momenta as

$$p = Q \frac{\bar{n}}{2} + m_N^2 \frac{n}{Q} \frac{2}{2}, \quad p' = Q \frac{n}{2} + m_N^2 \frac{\bar{n}}{Q} \frac{2}{2}, \quad Q = Q^2 \left[ 1 + \sqrt{1 + \frac{4 m_N^2}{Q^2}} \right] = Q + O(m_N^2/Q^2),$$

(92)

$$2(\not{p} \not{p}') = Q^2 + \frac{m_N^4}{Q^2} \approx Q^2,$$

(93)

where $m_N$ is the nucleon mass. For the incoming and outgoing collinear quarks we always imply

$$p_i = x_i Q \frac{\bar{n}}{2} + p_{\perp i} + \left( x_i m_N^2 \frac{n}{Q} \right) \frac{n}{2}, \quad p'_i = y_i Q \frac{n}{2} + p'_{\perp i} + \left( y_i m_N^2 \frac{\bar{n}}{Q} \right) \frac{\bar{n}}{2},$$

(94)
with the transverse momenta
\[ p_\perp^2 \sim p'_\perp^2 \sim \Lambda^2, \tag{95} \]
and where \( x_i \) and \( x'_i \) denote fractions of the corresponding momentum-component. Computing the Feynman diagrams in Sec.III we neglect by power suppressed components and assume
\[ p \simeq Q \frac{n}{2}, \quad p_i \simeq x_i p, \quad p' \simeq Q \frac{n}{2}, \quad p'_i \simeq y_i p', \tag{96} \]
In many formulas we use convenient notation \( \tilde{x}_i = 1 - x_i \). We also use the following notation for scalar products
\[ (a \cdot n) \equiv a_+, \quad (a \cdot \bar{n}) \equiv a_- \tag{97} \]
and Dirac contractions
\[ p_{\mu} \gamma^\mu \equiv \bar{p} \equiv \bar{p}. \tag{98} \]
Nucleon FFs are defined as the matrix elements of the e.m. current between the nucleon states:
\[ \langle p' | J_{e.m.}^\mu (0) | p \rangle = \bar{N}(p') \left[ \gamma^\mu (F_1 + F_2) - \frac{(p + p')^\mu}{2m_N} F_2 \right] N(p), \tag{99} \]
with nucleon spinors normalized as \( \bar{N}N = 2m_N \).

### Appendix B: Soft correlation function in perturbation theory

The leading order perturbative expression for the \( S_{V}^{ud} \) reads:
\[ \left( S_{V}^{ud} \right)_{LO} = \frac{3}{16\pi^2} \int dk_1 dk_2 \delta (k_1^+ - \omega_1) \delta (k_2^+ - \omega_2) \delta (k_1^- - \nu_1) \delta (k_2^- - \nu_2) \]
\[ \frac{1}{8} \text{Tr} \left[ \left( k_1 + m \right) \gamma_{\perp} C \left( k_2 + m \right) \right]^T \]
\[ \left[ \omega_1 \nu_1 - k_{1\perp}^2 - m^2 + i\varepsilon \right] \left[ \omega_2 \nu_2 - k_{2\perp}^2 - m^2 + i\varepsilon \right]. \tag{100} \]
The factor 1/8 in front of trace is chosen for convenience. Calculation of the trace in the numerator yields:
\[ \frac{1}{8} \text{Tr} \left[ \left( k_1 + m \right) \gamma_{\perp} C \left( k_2 + m \right) \right]^T = -m^2 + (k_{1\perp} \cdot k_{2\perp}) \tag{102} \]
This allows us to write:
\[ \left( S_{V}^{ud} \right)_{LO} = -\frac{3m^2}{16\pi^2} \frac{1}{4} \int dk_{1\perp} dk_{2\perp} \frac{1}{\left[ \omega_1 \nu_1 - k_{1\perp}^2 - m^2 + i\varepsilon \right] \left[ \omega_2 \nu_2 - k_{2\perp}^2 - m^2 + i\varepsilon \right]} \tag{103} \]
In order to proceed further we must take into account the specific properties of the jet functions. We always assume that the soft fractions \( \omega_1 \) and \( \nu_1 \) are positive. Mathematically it comes out from the analytical properties of the diagrams and imposes specific restrictions on the integrand in Eq.(103). In order to see this assume that \( -\infty < \omega_1 < \infty \) but then we keep the Feynman \( i\varepsilon \)-prescription in the jet functions. From calculations of the diagrams in Fig.2 one can easily obtain that all denominators of the jet functions in Eqs.\(29-32\) are defined with \( -i\varepsilon \), for instance
\[ \frac{1}{\omega_1 + \omega_2} \frac{1}{\omega_1} \frac{1}{\omega_2} \rightarrow \frac{1}{\left[ \omega_1 + \omega_2 - i\varepsilon \right] \left[ \omega_1 - i\varepsilon \right] \left[ \omega_2 - i\varepsilon \right]}, \tag{104} \]
The same arguments also true for \( \nu_i \). Consider now the convolution integrals
\[ \int d\omega_1 d\omega_2 \frac{1}{\left[ \omega_1 + \omega_2 - i\varepsilon \right] \left[ \omega_1 - i\varepsilon \right] \left[ \omega_2 - i\varepsilon \right]} S_V(\omega_i), \tag{105} \]
where \( S_V(\omega_i) \) is represented by expression (103). Computing \( d\omega_i \) by residues we obtain that nontrivial results originates only from the poles of the propagators in \( S_V(\omega_i) \) in (103). Therefore this allows us to represent the propagators in (103) as \( \delta \)-functions:

\[
\frac{1}{\omega_1 \nu_1 - k_{1\perp}^2 - m^2 + i\varepsilon |\omega_2 \nu_2 - k_{2\perp}^2 - m^2 + i\varepsilon|} = (2\pi i)^2 \theta(\omega_i > 0) \theta(\nu_i > 0) \times \delta (\omega_1 \nu_1 - k_{1\perp}^2 - m^2) \delta (\omega_2 \nu_2 - k_{2\perp}^2 - m^2).
\]

Then we obtain

\[
\left( S_{\nu d}^{\text{ud}} \right)_{\text{LO}} = \frac{3m^2}{16\pi^6} \theta(\nu_i > 0) \int dk_{1\perp} \int dk_{2\perp} \delta (\omega_1 \nu_1 - k_{1\perp}^2 - m^2) \delta (\omega_2 \nu_2 - k_{2\perp}^2 - m^2)
\]

\[
= \frac{3m^2}{16\pi^6} \theta(\nu_i > 0) \theta(\omega_i > 0) \theta(\omega_1 \eta_1 > m^2) \theta(\omega_2 \eta_2 > m^2).
\]

**Appendix C: Derivation of the collinear contribution** 

Consider first denominator (43). In the collinear region (47) we obtain:

\[
[(p - k_1 - k_2)^2 - m^2] (p - p_3 - k_1 - k_2)^2 (p - p_3 - k_1)^2 [(p_1 - k_1)^2 - m^2]
\]

\[
\simeq [-Q (k_1 + k_2)] [-Q x_3 (k_1 + k_3)] [-Q x_3 k_1] [-Q x_3 k_3]
\]

\[
\simeq Q^4 (x_3^2 x_1) \left( - (k_1 + k_3) \right)^2 \left[ -k_3 \right]^2.
\]

The remaining propagators are soft, of order \( \Lambda^2 \). Hence for the denominator we obtain

\[
\text{Den} \sim Q^8 \Lambda^{12}
\]

In the numerator we have

\[
\text{Num} = \xi_1^2 (k_1 + m) [\gamma^\beta \xi_1] \xi_2^2 \gamma^i (p' - \hat{p}_3 - \hat{k}_1 + m) \gamma^\alpha (\hat{k}_2 + m) \left[ \gamma^\beta (p - \hat{p}_3 - \hat{k}_1) \gamma^j \xi_2 \right]
\]

\[
\xi_3 \gamma^\alpha (p' - \hat{k}_1 - \hat{k}_2 + m) \left[ \gamma^\mu \xi_1 (p - \hat{k}_1 - \hat{k}_2 + m) \gamma^\beta \xi_3 \right]
\]

where we single out by brackets \( [...] \) the numerators of the hard propagators and hard gluon vertices. Using that \( \hat{p} \xi \simeq 0 \) we can rewrite this piece as:

\[
[\gamma^\beta \xi_1] \otimes [\gamma^\beta (p - \hat{p}_3 - \hat{k}_1) \gamma^j \xi_2] \otimes [\gamma^\mu \xi_1 (p - \hat{k}_1 - \hat{k}_2 + m) \gamma^\beta \xi_3]
\]

\[
\simeq (-4)(pk_1) [\gamma^3 \xi_1] \otimes [\gamma^j \xi_2] \otimes [\gamma^\mu \xi_3].
\]

Then we obtain

\[
\text{Num} = 2Q (-k_1^\perp) \xi_1^2 \gamma^\beta \hat{k}_1 + m) [\gamma^\beta \xi_1] \xi_2^2 \gamma^i (p' - \hat{p}_3 - \hat{k}_1 + m) \gamma^\alpha (\hat{k}_2 + m) \left[ \gamma^\beta \xi_2 \right]
\]

\[
\times \xi_3 \gamma^\alpha (p' - \hat{k}_1 - \hat{k}_2 + m) \left[ \gamma^\mu \xi_3 \right].
\]

Recall that \( k_1^\perp \sim Q \), then for the remaining terms one finds

\[
\xi_1^2 \gamma^\beta \hat{k}_1 + m) [\gamma^\beta \xi_1] \xi_3 \gamma^\alpha (-\hat{k}_1^\perp - \hat{k}_2^\perp) \left[ \gamma^\mu \xi_3 \right] \xi_2^2 \gamma^i (-\hat{k}_1^\perp) \gamma^\alpha (\hat{k}_2^\perp) \left[ \gamma^\beta \xi_2 \right]
\]

\[
\sim A^4 \xi_1 \Gamma_1 \xi_1 \xi_2 \Gamma_2 \xi_2 \xi_3 \Gamma_3 \xi_3,
\]

where \( \Gamma_i \) denote certain Dirac matrices. Hence one obtains:

\[
\text{Num} \sim Q^2 A^4.
\]
and for the whole diagram one finds:

\[
D_{c-p'} \sim \frac{1}{Q^6} \xi_1^\gamma \xi_1^\alpha \xi_2^\beta \xi_2^\gamma \xi_3^\alpha \xi_3^\beta.
\]  
(120)

The collinear contribution can be written as

\[
D_{c-p'} \approx \frac{1}{x_3^2 x_1} \int dk_1^- dk_2^- \frac{[\gamma^\gamma x_1, \gamma^\gamma x_2, \gamma^\gamma x_3]}{[\gamma^\gamma x_1 + k_2^2] [\gamma^\gamma x_3 + k_2^2]} \mathcal{V}(k_1^-),
\]  
(121)

with

\[
\mathcal{V}(k_1^-) = \frac{1}{2} \int dk_1^+ dk_2^+ \int dk_1^- dk_2^- \left\{ \xi_1^\gamma \left( \bar{k}_1 + m \right) \right\}_{\alpha_1} \left\{ \xi_2^\gamma \left( \bar{k}_2 + m \right) \right\}_{\alpha_2} \left\{ \xi_3^\gamma \left( \bar{k}_3 + m \right) \right\}_{\alpha_3} \left( \bar{k}_1 - \bar{k}_2 \right)^2 - m^2 \right\}_{\alpha_3}.
\]  
(122)

Eq. (122) yields the contribution to the evolution kernel at 2-loop approximation. The integral with respect to $k_{1,2}^-$ in Eq. (121) can be interpreted as a convolution integral of leading order hard coefficient function with the given part of evolution kernel.

**Appendix D: Calculation of $D_{c-p'}^\gamma$**

The expression for the soft integral $J_s$ in Eq. (72) reads

\[
J_s = \frac{1}{Q^6} \frac{1}{x_1 x_3 y_1 y_3} I_s,
\]  
(123)

with

\[
I_s = \int dk_2^- dk_2^+ \int dk_1^+ \frac{1}{[\bar{k}_1^- - \bar{x}_3] [\bar{k}_1^+ + \bar{x}_3] [\bar{k}_2^+ - \bar{x}_3]} \left( \bar{k}_1^- - \bar{k}_2^- \right)^2 - m^2
\]  
(124)

\[
\int dk_2^- dk_1^- \frac{1}{[\bar{k}_1^- - \bar{x}_3] [\bar{k}_1^+ + \bar{x}_3]} \left( \bar{k}_1^- - \bar{k}_2^- \right)^2 - m^2
\]  
(125)

Let us redefine the notation as

\[
\bar{k}_1^- \rightarrow \beta_1, \quad \bar{k}_1^+ \rightarrow \alpha_1, \quad dk_1^+ \rightarrow dk_i, \quad \tau_1 \equiv \bar{x}_3, \quad \tau_3 \equiv \bar{x}_3, \quad \tau_3 \equiv \bar{x}_3, \quad \tau_3' \equiv \bar{x}_3 + y_1, \quad \tau_3' \equiv \bar{x}_3 + y_3.
\]  
(126)

So that

\[
I_s = \int dk_2^- dk_1^+ \int d\beta_1 d\beta_2 \frac{1}{[\bar{\beta}_1 - \tau_1] [\bar{\beta}_2 + \tau_3] [\bar{\beta}_1 + \tau_3]} \left( \bar{\beta}_1 - \bar{\beta}_2 \right)^2 - m^2
\]  
(127)

Expressions in square brackets implies $[\ldots] \equiv [\ldots + i \varepsilon]$. Next we use the same trick as in Appendix A: we integrate over $d\alpha_1$ by residues, rewrite the poles as $\delta$-functions:

\[
\int d\alpha_1 d\alpha_2 \frac{1}{[\bar{\alpha}_1 - \tau_1] [\bar{\alpha}_1 + \tau_3]} \frac{1}{[\bar{\alpha}_2 - \bar{\tau}_3] [\bar{\alpha}_2 + \tau_3]} \left( \bar{\alpha}_1 - \bar{\alpha}_2 \right)^2 - m^2 \left( \bar{\alpha}_2 - \bar{\alpha}_1 \right)^2 - m^2)
\]  
(128)

and integrate over transverse momenta. This yields

\[
I_s = \frac{(2\pi)^2 m^2}{[\alpha_1 \beta_2 - k_2^2 - m^2]} \int_0^\infty d\beta_1 d\beta_2 \frac{1}{[\beta_1 + \tau_1] [\beta_1 + \tau_3] [\beta_1 + \beta_2]} \theta(\alpha_1 \beta_2 - k_2^2 - m^2) \delta(\alpha_1 \beta_1 - k_1^2 - m^2) \delta(\alpha_2 \beta_2 - k_2^2 - m^2)
\]  
(129)

and integrate over transverse momenta. This yields

\[
I_s = \frac{(2\pi)^2 m^2}{[\alpha_1 \beta_2 - k_2^2 - m^2]} \int_0^\infty d\beta_1 d\beta_2 \frac{1}{[\beta_1 + \tau_1] [\beta_1 + \beta_2 - \tau_3] [\beta_1 + \beta_2]} \theta(\alpha_1 \beta_2 - k_2^2 - m^2) \delta(\alpha_1 \beta_1 - k_1^2 - m^2) \delta(\alpha_2 \beta_2 - k_2^2 - m^2)
\]  
(130)

and integrate over transverse momenta. This yields

\[
I_s = \frac{(2\pi)^2 m^2}{[\alpha_1 \beta_2 - k_2^2 - m^2]} \int_0^\infty d\beta_1 d\beta_2 \frac{1}{[\beta_1 + \tau_1] [\beta_1 + \beta_2 - \tau_3] [\beta_1 + \beta_2]} \theta(\alpha_1 \beta_2 - k_2^2 - m^2) \delta(\alpha_1 \beta_1 - k_1^2 - m^2) \delta(\alpha_2 \beta_2 - k_2^2 - m^2)
\]  
(131)
After simple substitutions this integral can be written as

\[ I_s = (2\pi i)^2 m^2 \int_0^\infty d\beta_2 \frac{1}{(1 + \beta_2)} \int_0^\infty d\alpha_1 d\alpha_2 \frac{\theta(\alpha_2 > m^2) \theta(\alpha_1 > m^2)}{(\alpha_1 + \alpha_2)} \int_0^\infty d\beta_1 \frac{1}{(\beta_1 + \tau_1) [\beta_1 (1 + \beta_2) + \tau_3] (\alpha_1 + \alpha_2 + \beta_1 \tau_3^f)} . \]

Using that

\[ \frac{\beta_1}{[\beta_1 (1 + \beta_2) + \tau_3]} = \frac{1}{(1 + \beta_2)} - \frac{\tau_3}{(1 + \beta_2) [\beta_1 (1 + \beta_2) + \tau_3]} \approx \frac{1}{(1 + \beta_2)}, \]  

\[ \frac{1}{(\alpha_1 + \beta_1 \tau_1^f)} \frac{1}{[\alpha_1 + \alpha_2 + \beta_1 \tau_3^f]} = \frac{1}{(\alpha_1 + \alpha_2)} \left[ \frac{1}{(\alpha_1 + \beta_1 \tau_1^f)} - \frac{\tau_3^f}{[\alpha_1 + \alpha_2 + \beta_1 \tau_3^f]} \right] \]

\[ \approx \frac{1}{(\alpha_1 + \alpha_2)} \frac{1}{(\alpha_1 + \beta_1 \tau_1^f)}, \]  

where we neglected small contributions proportional to infinitesimal mass. Therefore we obtain

\[ I_s = (2\pi i)^2 m^2 \int_0^\infty d\beta_2 \frac{1}{(1 + \beta_2)^2} \int_0^\infty d\alpha_1 d\alpha_2 \frac{\theta(\alpha_1 > m^2)}{(\alpha_1 + \alpha_2)^2} \int_0^\infty d\beta_1 \frac{\theta(\alpha_2 > m^2)}{(\beta_1 + \tau_1^f)} \frac{1}{(\alpha_1 + \beta_1)} . \]

\[ = (2\pi i)^2 m^2 \int_{m^2}^\infty d\alpha_1 \frac{\ln [\alpha_1 / \tau_1^f]}{\alpha_1 - \tau_1^f} \int_0^\infty d\alpha_2 \frac{1}{(\alpha_1 + \alpha_2)^2(1 + m^2/\alpha_2)} \]

\[ \approx (2\pi i)^2 \int_1^\infty \frac{d\alpha_1}{\alpha_1} \left( \ln \alpha_1 + \ln [m^2 / \tau_1^f] \right) \frac{\alpha_1 - \ln \alpha_1 - 1}{(1 - \alpha_1)^2} \]

\[ = (2\pi i)^2 \ln \left[ \tau_1^f / m^2 \right] \left( 1 - \frac{\pi^2}{6} \right) + O(1). \]

Therefore we finally arrive at:

\[ I_s = (2\pi i)^2 \ln \left[ \tau_1^f / m^2 \right] \left( 1 - \frac{\pi^2}{6} \right) + O(1). \]

**Appendix E: Calculation of \( \Psi_{20}(k_i^-) \)**

Let us rewrite the Eq.[74] as

\[ \Psi_{20}(k_i^-) = [\xi_i^i \gamma^i]_{\alpha_1} [\xi_i^2 \gamma^2]_{\alpha_2} [\xi_i^3 \gamma^3]_{\alpha_3} J_{20}, \]

with

\[ J_{20} = \int dk_1^+ dk_2^+ dk_{1\perp} dk_{2\perp} \frac{m^2}{k_2^2 - m^2} \frac{m^2}{k_1^2 - m^2} \]

\[ \times \frac{-2(p', k_1)}{(p' - k_1 - k_2)^2 - m^2} \left[ (p' - p_3 - k_1)^2 - m^2 \right] (k_1 + k_2 + p' + p_3)^2 (k_1 - p_1')^2. \]

The exact answer for \( J_{20} \) is very complicated. We shall compute this expression only in the region of small \( k_i^- \rightarrow 0 \) assuming that their ratio is fixed \( k_i^- / k_i^+ \sim O(1) \). Again, redefine for simplicity light-cone decomposition:

\[ k_i = k_i ^n + k_i ^+ \tilde{n} + k_i ^\perp \equiv \alpha_i \frac{n}{2} + \beta_i \frac{\tilde{n}}{2} + k_i, \]

\[ m^2 \rightarrow \frac{1}{4} \left( k_i ^n - \alpha_i \right)^2 + \frac{1}{2} \left( k_i ^\perp - \beta_i \right)^2 + m^2. \]
The first two propagators in Eq. (139) (in the first line). The result can be written as

\[
\beta \propto \int d\beta_1 \frac{m^2}{(\alpha_1 - y_3 Q) \beta_1 - k_1^2} \frac{1}{((\alpha_1 - y_1 Q) \beta_1 - k_1^2)} \frac{1}{((\alpha_2 - Q) - k_2^2) \frac{1}{((\beta_1 + \beta_2) (\alpha_1 + \alpha_2) - (k_1 + k_2)^2} \frac{1}{((\beta_1 + \beta_2) (\alpha_1 + \alpha_2 - y_3 Q) - (k_1 + k_2)^2}.
\]

Performing integrations over \( \beta_1 \) and \( \beta_2 \) by residues we can neglect by all poles for which

\[
\alpha_i > Q - \alpha_j \text{ or } \alpha_i > Qy_i - \alpha_j,
\]

because in this case \( \alpha_i \) can not be small. Then we have contribution only from the poles associated with the first two propagators in Eq. (139) (in the first line). The result can be written as

\[
J_{20} \approx (2\pi i)^2 \theta(0 < \alpha_1 < Q\alpha_1^\text{max}) \theta(0 < \alpha_2 < Q\alpha_2^\text{max})
\]

\[
\times \int d\beta_1 d\beta_2 dk_2 dk_1 \frac{m^2}{(\alpha_2 - k_2^2 - m^2) \frac{1}{(\alpha_1 - \alpha_1 - k_1^2 - m^2)} \frac{1}{(\beta_1 + \beta_2) (\alpha_1 + \alpha_2) - (k_1 + k_2)^2} \frac{1}{(\beta_1 + \beta_2) (\alpha_1 + \alpha_2 - y_3 Q) - (k_1 + k_2)^2}.
\]

The maximal values \( \alpha_1^\text{max} \sim \alpha_2^\text{max} \sim 1 \). Their explicit values are not important. The integrand can be further simplified. We can neglect the small \( \alpha_i \ll Q \) in the denominator (142):

\[
\text{Den} \approx (\frac{\alpha_1 - y_3 Q) \beta_1 - k_1^2}{(\alpha_1 - y_1 Q) \beta_1 - k_1^2} \frac{1}{((\beta_1 + \beta_2) (\alpha_1 + \alpha_2 - Q) - (k_1 + k_2)^2} \frac{1}{((\beta_1 + \beta_2) (\alpha_1 + \alpha_2 - y_3 Q) - (k_1 + k_2)^2}.
\]

Then we take into account that the dominant contribution arises from the region where

\[
k_1^2 = \alpha_i \beta_i - m^2 \ll y_i Q,
\]

as it follows from the \( \delta \)-functions in Eq. (141). Therefore we can also neglect the transverse momenta \( k_i \) in the propagators (143):

\[
\text{Den} \approx (\frac{-y_3 Q) \beta_1 - k_1^2}{(\alpha_1 - y_1 Q) \beta_1} \frac{1}{((\beta_1 + \beta_2) (\alpha_1 + \alpha_2) - Q) [((\beta_1 + \beta_2) (\alpha_1 + \alpha_2 - y_3 Q) - (k_1 + k_2)^2}.
\]

Finally we obtain

\[
J_{20} \approx (2\pi i)^2 \theta(0 < \alpha_1 < Q\alpha_1^\text{max}) \theta(0 < \alpha_2 < Q\alpha_2^\text{max}) \frac{m^2}{y_1 y_3 Q^3} \frac{1}{\beta_1 (\beta_1 + \beta_2)^2}.
\]

A simple calculation yields:

\[
J_{20} \approx (2\pi i)^2 \theta(0 < \alpha_1 < Q\alpha_1^\text{max}) \theta(0 < \alpha_2 < Q\alpha_2^\text{max}) \frac{m^2}{y_1 y_3 Q^3} \frac{1}{\alpha_2 \ln(1 + \alpha_1/\alpha_2)}
\]

\[
\equiv (2\pi i)^2 \theta(0 < k_1^-/Q < k_{1}^\text{max}) \theta(0 < k_2^-/Q < k_{2}^\text{max}) \frac{m^2}{y_1 y_3 Q^3} k_2^- \ln(1 + k_1^-/k_2^-).
\]

Substituting this into Eq. (136) we obtain the required result.

---

[1] N. Kivel and M. Vanderhaeghen, Phys. Rev. D 83 (2011) 093005 [arXiv:1010.5314].
