Narayana polynomial and some generalizations

Ricky X. F. Chen\textsuperscript{a}, Christian M. Reidys\textsuperscript{b,*}

Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, DK-5230, Odense M, Denmark

\textsuperscript{arickychen@imada.sdu.dk,\textsuperscript{b}duck@santafe.edu

Abstract

In this note, we present several identities involving binomial coefficients combinatorially. These identities can be viewed as specific evaluations of certain generalizations of the Narayana polynomial. Among the results, we obtain in particular the bijective problem (15) in Stanley’s collection “Bijective Proof Problems”, a new formula for the Narayana polynomial, a new expression for the Harer-Zagier formula, i.e., the generating polynomial for unicellular maps, the Walsh-Lehman formula which counts unicellular maps and finally we identify a class of plane trees, whose enumeration is closely connected to the Schröder numbers.

**Keywords:** Narayana polynomial; colored trees; matches; Schröder number; Stanley’s bijective proof problem; Harer-Zagier formula; Walsh-Lehman formula

Mathematics Subject Classifications: 05A19; 05C05

1 Introduction

This note is firstly motivated by the bijective problem (15) in Stanley’s collection “Bijective Proof Problems” [12, (15)]:

\[
\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (x-1)^j. \tag{1}
\]

We note that in Wilf [13, p.117], there is an “odd kind of a combinatorial proof” of (1) based on the Sieve Method in view of generatingfunctionology and by counting \( n \)-subsets of \([2n]\) subject to some \( n \) properties. Later, the authors find that the new expression

\textsuperscript{*Corresponding author}
of the Narayana polynomial obtained by Tourfic [2] (and independently by Chen [10]) is actually related to (1) and the new expression of the Narayana polynomial is

\[
\sum_{k=1}^{n} N_{n,k} y^k = \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k} \binom{2n-k}{n} (y-1)^k,
\]

(2)

where the Narayana number \( N_{n,k} = \frac{1}{n!} \binom{n}{k} \binom{n}{k-1} \).

In this note, we will prove several identities, combinatorially that imply the identities (1) and (2) as special cases. Furthermore, we present a new expression for the Harer-Zagier formula, i.e., the generating polynomial for unicellular maps [15, 17]. Extracting the corresponding coefficients, we show combinatorially that the latter are equal to the Walsh-Lehman formula [16, 17], which counts unicellular maps for fixed number of edges and topological genus. We also present a (possibly new) class of plane trees the enumeration of which involving Schröder numbers.

\section{Narayana polynomial and certain generalization}

In this section, we will mainly prove an identity implying (1) and (2) as special cases. A colored-labeled plane tree with \( n+1 \) vertices is a plane tree with vertices uniquely labeled by \([n+1] = \{1, 2, \ldots, n+1\}\) and where its leaves are either not colored, or colored \( N \) or \( Y \). In the following, we denote the sets of the internal vertices, \( Y \)-leaves and \( N \)-leaves in a colored-labeled plane tree \( T \) by \( \text{int}(T) \), \( \text{lev}_Y(T) \) and \( \text{lev}_N(T) \), respectively. As usual, the cardinality of a set \( S \) is denoted by \(|S|\).

With foresight, we next recall a bijection between labeled plane trees and sets of matches, called Chen’s bijective algorithm [1]

\textbf{Theorem 1.} (Chen [1]) There is a bijection between labeled plane trees with labels in \([n+1]\) and sets of \( n \) matches with labels in \( \{1, \ldots, n+1, (n+2)^*, \ldots, (2n)^*\} \), where a match is a plane tree with two vertices. In addition, vertices with labels in \([n+1]\) which appear as roots in a set of \( n \) matches appear as internal vertices in its corresponding labeled plane tree, while vertices with labels in \([n+1]\) which are leaves in a set of \( n \) matches appear as leaves in the corresponding labeled plane tree.

For details with respect to Chen’s bijective algorithm, we refer the reader to [1].

We also remark that all proofs in the present note can be formulated directly in the language of matches.

Now, we are ready to present our results.

\textbf{Proposition 2.} The number of colored-labeled plane trees \( T \) with labels in \([n+x+q]\), in which all vertices with labels in \([q]\) are all uncolored leaves, all vertices with labels in \( \{q+1, \ldots, q+x\} \) are internal and \(|\text{int}(T)| + |\text{lev}_Y(T)| = k+x \) is given by

\[
\binom{n}{k} \binom{k+n+x+q-2}{n+q-1} (n+x+q-1)!,
\]

where \( n, x, q, k \geq 0 \).
Proof. The set of trees described in the proposition will be denoted by $\Gamma_{n,x,q}$. Based on Theorem 1, it is not difficult to see that colored-labeled plane trees in $\Gamma_{n,x,q}$ with $|\text{int}(T)| + |\text{lev}_Y(T)| = k + x$ are in bijection with set of pairs $(A, \chi)$ where $A \subseteq [n + x + q] \setminus [x + q]$ with $|A| = k$ and $\chi$ is a set of matches with labels in $\{1, \ldots, n + x + q, (n + x + q + 1)^*, \ldots, 2(n + x + q - 1)^*\}$ where all vertices with labels in $\{q + 1, \ldots, q + x\}$ are roots and other unstarred roots of $\chi$ are in $A$. (Figure 1 shows an example of this bijection. Note, leaves with labels in $A$ will be colored $Y$ and others $N$.) However, there are $\binom{n}{k}$ ways to choose $A$, $\binom{k+n+x+q-2}{n+q-1}$ ways to determine the remaining $n + q - 1$ roots of $\chi$ besides those $x$ prescribed roots and at last $(n + x + q - 1)!$ ways to match up, whence Proposition 2. \hfill \Box

![Figure 1: A tree of $\Gamma_{11,2,2}$ and its corresponding pair $(A, \chi)$.](image)

As an application of Proposition 2 we immediately obtain a combinatorial proof of the following, well-known identity:

**Theorem 3.** For all $n \geq 0, r \in \mathbb{C}, q \in \mathbb{Z}$, there holds

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k+r}{n+q} = \binom{r}{q}. \tag{3}$$

**Proof.** If we weigh each tree $T$ in $\Gamma_{n,x,q}$ by $(-1)^{|\text{lev}_N(T)|}$, we observe that the number $\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k+n+x+q-2}{n+q-1}(n + x + q - 1)!$ counts the total weight over all trees in $\Gamma_{n,x,q}$ from Proposition 2. However, there is a very simple involution on $\Gamma_{n,x,q}$: we change $Y$ into $N$ if the colored leaf with the smallest label is colored $Y$ and vice versa. From this simple involution, all weights over $\Gamma_{n,x,q}$ cancel out except for those trees, whose leaves are all vertices with labels in $[q]$, i.e., no colored leaves. Therefore, the total weight here also counts the number of trees in $\Gamma_{n,x,q}$ in which the leaves are all vertices in $[q]$. We can obtain this number by enumerating the corresponding sets of matches according to Theorem 1 as follows: since all leaves of those trees are in $[q]$, we only need to choose out $n + x - 1$ vertices from $\{(n + x + q + 1)^*, \ldots, 2(n + x + q - 1)^*\}$ to be leaves of the matches and match up with the roots of the matches. Therefore, the number of trees in $\Gamma_{n,x,q}$, whose leaves are all vertices in $[q]$ is

$$\binom{n+x+q-2}{n+x-1}(n+x+q-1)!.$$
Hence, for \( n, x, q \geq 0 \),
\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{k + n + x + q - 2}{n + q - 1} = \binom{n + x + q - 2}{q - 1}.
\] (4)

Now we need to carefully analyze (4): firstly for any fixed \( q \geq 1 \), both sides of (4) are polynomials in \( x \). Since it holds for all \( x \geq 0 \), it holds for any \( x \in \mathbb{C} \). Secondly, for any \( 1 - n \leq q < 1 \), the right side of (4) equals 0 because \( q - 1 < 0 \). However, the term \( \binom{k + n + x + q - 2}{n + q - 1} \) on the left side is a polynomial in \( k \) with degree less than \( n \), so the left side is also 0 by finite difference argument. At last, for \( q < 1 - n \), both \( q - 1 < 0 \) and \( n + q - 1 < 0 \) hold, which obviously leads to 0 for both sides of (4). Hence, we can state that (4) holds for all \( x \in \mathbb{C} \), \( q \in \mathbb{Z} \). Setting \( n + x + q - 2 = r \) in (4) will complete the proof. \( \square \)

The identity in the above theorem is implied by Vandermonde’s convolution [3, 4, 5, 11]
\[
\sum_{k=0}^{n} \binom{r}{k} \binom{m}{n-k} = \binom{r+m}{n}.
\] (5)

The simplest combinatorial interpretation of (5) for \( m, r \geq 0 \) is the enumeration of the number of \( n \)-subsets in two different ways. For certain generalizations of Vandermonde convolution, we refer to [3, 4, 11].

Note if we also assume that \( \binom{r}{q} = 0 \) if \( q \) is not an integer, then (3) holds for all \( q \in \mathbb{C} \). Similarly, if we weight each tree \( T \) in \( \Gamma_{n,x,q} \) by \( z^{\mid int(T)\mid} \) instead, then we will obtain the following

**Theorem 4.** For \( n, q \geq 0 \), \( x \in \mathbb{C} \), we have
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{2n + x + q - k - 2}{n + q - 1} z^{k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n + x + q - 2}{k + q - 1} (z + 1)^{k}.
\] (6)

**Proof.** Firstly it is obvious from Proposition 2 that the total weight over all trees in \( \Gamma_{n,x,q} \) is
\[
\sum_{k=0}^{n} z^{n-k} \binom{n}{k} \binom{k + n + x + q - 2}{n + q - 1} (n + x + q - 1)!.
\]
However, the total weight over all trees in \( \Gamma_{n,x,q} \) with \( \mid int(T)\mid = x + k \) is
\[
\binom{n}{k} \binom{n + x + q - 2}{n + q - 1 - k} (n + x + q - 1)! (z + 1)^{n-k}.
\]

It follows from Theorem 1 that there are \( \binom{n}{k} \) ways to choose the roots of the matches (internal vertices in the corresponding tree) besides the \( x \) prescribed ones. Furthermore, there are \( \binom{n + x + q - 2}{n + q - 1 - k} \) ways to choose the remaining roots from starred vertices and arrange all the roots in \( (n + x + q - 1)! \) different ways. Finally, all \( n - k \) leaves among all leaves except for those in \( [q] \) can be either colored \( Y \) or \( N \), whence each of them contributes a
weight of \((z + 1)\). Summing over all \(0 \leq k \leq n\), we also obtain the total weight over all
trees in \(\Gamma_{n,x,q}\)
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n+x+q-2}{n+q-1-k} (n+x+q-1)!(z+1)^{n-k}.
\]

Hence,
\[
\sum_{k=0}^{n} z^{n-k} \binom{n}{k} \binom{k+n+x+q-2}{n+q-1} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+x+q-2}{n+q-1-k} (z+1)^{n-k}.
\]

Since both sides of (6) are polynomials in \(x\), so it holds for any \(x \in \mathbb{C}\), completing the
proof of Theorem 4. \(\square\)

**Corollary 5.** (Stanley [12, (15)]) For \(n \geq 0\), we have
\[
\sum_{k=0}^{n} \binom{n}{k} z^k = \sum_{j=0}^{n} \binom{n}{j} \binom{2n-j}{n} (z-1)^j.
\]
(7)

**Proof.** Taking \(x = q = 1, z = z - 1\) in (6), we obtain the bijective proof problem (15) of
Stanley [12, (15)] as stated in the corollary. \(\square\)

The following corollary gives the new expression of the Narayana polynomial obtained
by Tourfic [2] and Chen [10].

**Corollary 6.** For \(n \geq 0\), the Narayana number \(N_{n,k}\) satisfies
\[
\sum_{k=1}^{n} N_{n,k} z^k = \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k} \binom{2n-k}{n} (z-1)^k.
\]
(8)

**Proof.** Setting \(q = 0, x = 2, z = z - 1\) completes the proof. \(\square\)

It is well-known that the large Schröder numbers \(S_n\) [18, 19], which count the number
of plane trees having \(n\) edges with leaves colored by one of two colors (say color \(Y\)
and color \(N\)), equal the Narayana polynomial, evaluated at \(z = 2\), i.e.,

\[
S_n = \sum_{k=1}^{n} N_{n,k} 2^k.
\]

For the case \(q = 0, x = 2\), i.e., \(\Gamma_{n,2,0}\), Theorem 4 implies:

**Theorem 7.** Denote \(T_{n+1}\) the number of plane trees of \(n+1\) edges with 2 different internal,
marked vertices and bi-colored leaves. Then, \(T_{n+1}\) is equal to the number of plane trees
having \(n\) edges with 2 different, marked vertices and bi-colored leaves, i.e.,
\(T_{n+1} = \binom{n+1}{2} S_n\).
Proof. From the proof of Theorem 4, we know that the number of trees in \( \Gamma_{n,2,0} \) is

\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n}{k+1} (n+1)!2^k.
\]

Then, “deleting” the labels we obtain the number of (unlabelled) plane trees of \( n+1 \) edges with 2 different internal, marked vertices and bi-colored leaves to be

\[
\frac{1}{2n!} \sum_{k=0}^{n} \binom{n}{k} \binom{n}{k+1} (n+1)!2^k = \frac{(n+1)n}{2} \sum_{k=1}^{n} N_{n,k} 2^k = \binom{n+1}{2} S_n,
\]

which completes the proof.

Since the large (and small) Schröder numbers satisfy the recurrence [18]

\[
3(2n-1)S_n = (n+1)S_{n+1} + (n-2)S_{n-1}, n \geq 2
\]

and \( S_1 = S_2 = 1 \), we have the following corollary.

**Corollary 8.** The sequence \( T_n \) satisfies

\[
3(2n-1)(n-1)(n+2)T_{n+1} = n(n-1)(n+1)T_{n+2} + (n-2)(n+1)(n+2)T_n.
\]

It would be interesting to give direct bijections between the two kinds of trees defined in Theorem 7 and the recurrence in Corollary 8.

Based on Theorem 6, we can also give a new expression for the generating polynomial of unicellular maps, also referred to as the Harer-Zagier formula, in \([15]\). A unicellular map is a triangulation, that is a cell-complex of a closed orientable surface. The number of handles of this surface is called the genus of the map. Denote \( A(n,g) \) the number of unicellular maps of genus \( g \) with \( n \) edges. The Harer-Zagier formula \([15]\) reads

\[
\sum_{g \geq 0} A(n,g) x^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{k \geq 1} 2^{k-1} \binom{n}{k-1} \binom{x}{k}.
\]

**Corollary 9.** The generating polynomial of unicellular maps with \( n \) edges satisfies

\[
\sum_{g \geq 0} A(n,g) x^{n+1-2g} = \frac{(2n)!}{2^n n!} \sum_{k \geq 0} \binom{n}{k} \binom{x+n-k}{n+1}.
\]

Proof. Setting \( q = 2, x = x - n, z = 1 \) in (6), we obtain

\[
\sum_{k \geq 0} \binom{n}{k} \binom{x+n-k}{n+1} = \sum_{k \geq 1} \binom{n}{k-1} \binom{x}{k} 2^{k-1},
\]

whence the corollary.
In fact, there is also an explicit formula for $A(n, g)$ given firstly by Walsh and Lehman [16]. In the Walsh-Lehman formula, $A(n, g)$ is connected to the number of certain permutations with only odd cycles [17, 20]. Specifically, let $O(n + 1, g)$ denote the number of permutations on $[n + 1]$ which consists of $n + 1 - 2g$ odd cycles, then the Walsh-Lehman formula can be expressed as

$$A(n, g) = \frac{(2n)!}{(n + 1)! n! 2^{2g}} O(n + 1, g).$$

(13)

On the other hand, $A(n, g)$ should be equal to the coefficient of the term $x^{n+1-2g}$ on the right side of the eq. (11). To the best of our knowledge, there is no simple way to show that the coefficient is equal to the Walsh-Lehman formula. However, in the following, we will show how to obtain the Walsh-Lehman formula from the new expression in above corollary.

Firstly, $A(n, g)$ is equal to the coefficient of the term $x^{n+1-2g}$ on the right side of the eq. (12). We further set $A(n, g) = \frac{(2n)!}{2^m(n+1)!} \bar{A}(n, g)$ and denote the coefficient of the term $x^m$ in the function $f(x)$ as $[x^m]f(x)$. Then, we have

$$\bar{A}(n, g) = \sum_{k=0}^{n} \binom{n}{k} [x^{n+1-2g}](x + n - k)(x + n - k - 1) \cdots x(x - 1) \cdots (x - k)$$

$$= \sum_{k=0}^{n} \binom{n}{k} [x^{n+2-2g}][(x + n - k)(x + n - k - 1) \cdots x][x(x - 1) \cdots (x - k)]$$

(14)

Note there holds

$$x(x + 1)(x + 2) \cdots (x + n - 1) = \sum_{k} C(n, k) x^k,$$

$$x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{k} (-1)^{n-k} C(n, k) x^k,$$

(15)

(16)

where $C(n, k)$ is the unsigned Stirling number of the first kind, i.e., $C(n, k)$ counts the number of permutations on $[n]$ with $k$ cycles. Therefore, from eq. (14) we have

$$\bar{A}(n, g) = \sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=n+2-2g} C(n - k + 1, i)(-1)^{k+1-j} C(k + 1, j).$$

(17)

Inspecting this expression, it is not clear why the right hand side should be always a positive number. However, we shall prove

**Theorem 10.** For $n, g \geq 0$, we have

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=n+2-2g} C(n - k + 1, i)(-1)^{k+1-j} C(k + 1, j) = 2^{n-2g} O(n + 1, g).$$

(18)
Proof. We firstly construct a set of objects which is counted by the left hand side of (18). Let \([n + 1]^* = \{0, 1, 2, \ldots, n + 1\}\) and consider the set \(T\) of all pairs \((\alpha, \beta)\) where \(\alpha\) is a permutation on \(A \subset [n + 1]^*\) with \(0 \in A\) while \(\beta\) is a permutation on \([n + 1]^* \setminus A\) with \(n + 1 \in [n + 1]^* \setminus A\), and where the total number of cycles in \(\alpha\) and \(\beta\) is \(n + 2 - 2g\). Denote the difference between the number of elements in \(\beta\) and the number of cycles in \(\beta\) as \(d(\beta)\), and weigh each pair by \(W[(\alpha, \beta)] = (-1)^{d(\beta)}\). Then, the total weight over all pairs in \(T\) is obtained as follows: consider the set of even cycles contained in \(\beta\) and the set of odd cycles containing \(0\) in \(\alpha\). There is an involution, \(\phi\), between the set \(X\) of pairs \((\alpha, \beta)\) with \(|\beta_{n+1}| = 2i + 1, |\alpha_0| \geq 2\) and the set \(Y\) of pairs \((\alpha', \beta')\) with \(|\beta'_{n+1}| = 2i + 2, |\alpha'_0| \geq 1\), where \(0 \leq i < g\). \(\phi\) is given by

\[
\phi: X \rightarrow Y, \quad (\alpha, \beta) \mapsto (\alpha', \beta'),
\]

where \(\alpha'\) is obtained from \(\alpha\) by removing the element \(\alpha(0)\) while \(\beta'\) is obtained from \(\beta\) by inserting \(\alpha(0)\) between \(n + 1\) and \(\beta(n + 1)\), i.e., \(\beta'(n + 1) = \alpha(0), \beta'(\alpha(0)) = \beta(n + 1)\). Reversing this switch we derive the reverse map \(\phi^{-1}\). It is obvious that \((\alpha, \beta)\) and \((\alpha', \beta') = \phi((\alpha, \beta))\) carry opposite signs, whence their weights will cancel. In particular, the total weight over all pairs \((\alpha, \beta)\) where \(|\beta_{n+1}|\) is even, will cancel.

Since there are \(n + 2 - 2g\) cycles in \(\alpha\) and \(\beta\), \(|\beta_{n+1}| \leq 2g + 1\), equality will be achieved when all other cycles are singletons (which implies \(|\alpha_0| = 1\)). Thus, after applying \(\phi\), the total weight over all pairs \((\alpha, \beta)\) is reduced to the total weight over all pairs \((\alpha, \beta)\) where \(|\alpha_0| = 1, |\beta_{n+1}| = 2i + 1\) for \(0 \leq i < 2g + 1\). We denotes the latter set by \(U\).

There is an involution, \(\varphi\), over all pairs \((\alpha, \beta)\) in \(U\) which have at least one even cycle. \(\varphi\) is obtained as follows: consider the set of even cycles contained in \(\alpha\) and \(\beta\). There is a unique, even cycle containing the smallest element, \(c\). If \(c\) is contained in \(\alpha\) then \(\alpha' = \alpha \setminus c, \beta' = \beta \cup c\) and \(\alpha' = \alpha \cup c\) and \(\beta' = \beta \setminus c\), otherwise.

Note from \(\beta\) to \(\beta'\), the total number of elements changes by an even number while the total number of cycles changes by \(1\) (odd), thus \(\beta\) and \(\beta'\) must carry opposite signs. Therefore, their weight cancels.

Applying \(\varphi\), the total weight over all pairs \((\alpha, \beta)\) is further reduced to the total weight over all pairs \((\alpha, \beta)\) which have total number of \(n + 2 - 2g\) odd cycles. It is clear that all pairs in the latter set \((V)\) carry a positive sign, whence the total weight is equal to the total number of elements in \(V\).

\[
\sum_{(\alpha, \beta) \in T} W[(\alpha, \beta)] = \sum_{k=0}^{n} \binom{n}{k} \sum_{i+j=n+2-2g} C(n-k+1,i)(-1)^{k+1-j}C(k+1,j).
\]

Next, we will prove

\[
\sum_{(\alpha, \beta) \in T} W[(\alpha, \beta)] = 2^n \cdot \text{O}(n+1,g).
\]

Denote the length of the cycle containing \(n + 1\) in \(\beta\) as \(|\beta_{n+1}|\) while the length of the cycle containing \(0\) in \(\alpha\) as \(|\alpha_0|\).

The total number of cycles in \(\alpha\) and \(\beta\) changes by an even number while the total number of elements changes by an even number whenever \(|\alpha_0| = 1, \beta_{n+1} = 2i + 1\) for \(0 \leq i < 2g + 1\). We denotes the latter set by \(T'\), and the set \(T'\) of pairs \((\alpha, \beta)\) with \(|\alpha_0| = 1, \beta_{n+1} = 2i + 1\) for \(0 \leq i < 2g + 1\). We denotes the latter set by \(T'\).

The total weight over all pairs \((\alpha, \beta)\) where \(|\beta_{n+1}|\) is even, will cancel.

Since there are \(n + 2 - 2g\) cycles in \(\alpha\) and \(\beta\), \(|\beta_{n+1}| \leq 2g + 1\), equality will be achieved when all other cycles are singletons (which implies \(|\alpha_0| = 1\)). Thus, after applying \(\phi\), the total weight over all pairs \((\alpha, \beta)\) is reduced to the total weight over all pairs \((\alpha, \beta)\) where \(|\alpha_0| = 1, \beta_{n+1} = 2i + 1\) for \(0 \leq i < 2g + 1\). We denotes the latter set by \(U\).

There is an involution, \(\varphi\), over all pairs \((\alpha, \beta)\) in \(U\) which have at least one even cycle. \(\varphi\) is obtained as follows: consider the set of even cycles contained in \(\alpha\) and \(\beta\). There is a unique, even cycle containing the smallest element, \(c\). If \(c\) is contained in \(\alpha\) then \(\alpha' = \alpha \setminus c, \beta' = \beta \cup c\) and \(\alpha' = \alpha \cup c\) and \(\beta' = \beta \setminus c\), otherwise.

Note from \(\beta\) to \(\beta'\), the total number of elements changes by an even number while the total number of cycles changes by \(1\) (odd), thus \(\beta\) and \(\beta'\) must carry opposite signs. Therefore, their weight cancels.

Applying \(\varphi\), the total weight over all pairs \((\alpha, \beta)\) is further reduced to the total weight over all pairs \((\alpha, \beta)\) which have total number of odd cycles. It is clear that all pairs in the latter set \((V)\) carry a positive sign, whence the total weight is equal to the total number of elements in \(V\).
Finally, each pair \((\alpha, \beta) \in V\) can be viewed as a partition of all cycles except the cycle containing \(n + 1\), of a permutation on \([n + 1]\) with \(n + 1 - 2g\) odd cycles, into two parts. Conversely, given a permutation on \([n + 1]\) with \(n + 1 - 2g\) odd cycles, there are \(2^{n-2g}\) different ways to partition all cycles except the one containing \(n + 1\) into two parts. Therefore, we have

\[
\sum_{(\alpha, \beta) \in V} W[(\alpha, \beta)] = |V| = 2^{n-2g}O(n+1, g).
\]

Hence,

\[
\sum_{(\alpha, \beta) \in T} W[(\alpha, \beta)] = 2^{n-2g}O(n+1, g),
\]

completing the proof. \(\square\)

Accordingly, we obtain the Walsh-Lehman formula

\[
A(n, g) = \frac{(2n)!}{(n+1)!n!2^a}A(n, g) = \frac{(2n)!}{(n+1)!n!2^a}O(n+1, g).
\]

### 3 Partial sums and special cases

In this section, we consider further generalizations based on the discussion in Section 2.

**Theorem 11.** For \(n, q \geq 0, x \in \mathbb{C}, 0 \leq n_1 \leq n\), we have

\[
\sum_{k=0}^{n_1} \binom{n}{k} \left( \frac{n + x + q + k - 2}{n + q - 1} \right) z^{n-k} =
\sum_{k=0}^{n_1} \binom{n}{k} \left( \frac{n + x + q - 2}{q + n - 1 - k} \right) \sum_{i=0}^{n_1-k} \binom{n-k}{i} z^{n-k-i}.
\]

**Proof.** Firstly it is obvious from Proposition 2 that the total weight over all trees \(T\) in \(\Gamma_{n,x,q}\) with \(|\text{int}(T)| + |\text{lev}_Y(T)| \leq x + n_1\) is

\[
\sum_{k=0}^{n_1} z^{n-k} \binom{n}{k} \left( \frac{k + n + x + q - 2}{n + q - 1} \right) (n + x + q - 1)!.
\]

However, the total weight over all trees \(T\) in \(\Gamma_{n,x,q}\) with \(|\text{int}(T)| = x + k\) \((k \leq n_1)\) is

\[
\left( \frac{n}{k} \right) \left( \frac{n + x + q - 2}{n + q - 1 - k} \right) (n + x + q - 1)! \sum_{i=0}^{n_1-k} \binom{n-k}{i} z^{n-k-i}.
\]
since we can color at most \( n_1 - k \) leaves as \( Y \) if there are \( k \) roots (of the matches) from \( \{x + q + 1, \ldots, x + q + n\} \). Summing over all \( 0 \leq k \leq n_1 \), we also obtain the total weight over all trees \( T \) in \( \Gamma_{n,x,q} \) with \(|\text{int}(T)| + |\text{lev}_N(T)| \leq x + n_1\)

\[ \sum_{k=0}^{n_1} \binom{n}{k} \binom{n + x + q - 2}{n + q - 1 - k} (n + x + q - 1)! \sum_{i=0}^{n_1-k} \binom{n - k}{i} z^{n - k - i}. \]

Hence,

\[ \sum_{k=0}^{n_1} z^{n-k} \binom{n}{k} \binom{k + n + x + q - 2}{n + q - 1} = \sum_{k=0}^{n_1} \binom{n}{k} \binom{n + x + q - 2}{n + q - 1 - k} \sum_{i=0}^{n_1-k} \binom{n - k}{i} z^{n - k - i}. \]

Since both sides of (22) are polynomials in \( x \), (22) holds for any \( x \in \mathbb{C} \), completing the proof of (22).

**Corollary 12.** For \( n, q \geq 0, x \in \mathbb{C}, 0 \leq n_1 \leq n \), we have

\[ \sum_{k=0}^{n_1} (-1)^{n-k} \binom{n}{k} \binom{k + x + q + n - 2}{q + n - 1} \]

\[ = \sum_{k=0}^{n_1} (-1)^{n-n_1} \binom{n}{k} \binom{x + q + n - 2}{q + n - 1 - k} \binom{n - k - 1}{n_1 - k}. \] (23)

**Proof.** Note the left hand side is the total weight of all trees where \(|\text{int}(T)| + |\text{lev}_N(T)| \leq n_1\) and each tree \( T \) is weighted by \((-1)^{|\text{lev}_N(T)|}\). (Note both sides have a factor \((n + x + q - 1)!\), they will be canceled out.) Similar as in the proof of Theorem 2.3, we apply the involution. In this case, all weight over trees \( T \) where \(|\text{int}(T)| + |\text{lev}_N(T)| < x + n_1 \) or \(|\text{int}(T)| + |\text{lev}_N(T)| = x + n_1 \) and the colored leave with the smallest label is colored \( Y \) will cancel. Thus, the total weight is equal to the weight over trees \( T \), where \(|\text{int}(T)| + |\text{lev}_N(T)| = x + n_1 \) and the colored leave with the smallest label is colored \( N \). This number is calculated as follows:

1. choose \( k \) labels from \( \{x + q + 1, \ldots, x + q + n\} \) as the roots of the matches in \( \binom{n}{k} \) different ways;

2. choose from the starred labels \( n + q - 1 - k \) labels as other roots of the matches in \( \binom{n+x+q-2}{n+q-1-k} \) ways;

3. color the leave with the smallest label \( N \), and choose another \( n_1 - k \) leaves with labels in \( \{x + q + 1, \ldots, x + q + n\} \) in \( \binom{n-k-1}{n_1-k} \) different ways and color all of them \( Y \) (the rest will be colored \( N \)).

Note that each such tree has weight \((-1)^{n-n_1}\). Summing over \( 0 \leq k \leq n_1 \), we obtain for the total weight

\[ \sum_{k=0}^{n_1} (-1)^{n-n_1} \binom{n}{k} \binom{n + x + q - 2}{n + q - 1 - k} \binom{n - k - 1}{n_1 - k}, \]

which equals the right hand side of (23), whence the corollary.
We remark that it is interesting to observe from the right hand side of (23) that the partial sum on the left hand side alternates in sign as $n_1$ goes from 0 to $n$. Setting $x = q = 1$ and $q = 0, x = 2$ respectively, we obtain

Corollary 13. For $0 \leq n_1 \leq n$, we have

\[
\sum_{k=0}^{n_1} (-1)^{n_1+k} \binom{n}{k} \binom{k+n}{n} = \sum_{k=0}^{n_1} \binom{n}{k}^2 \binom{n-k-1}{n_1-k}, \quad (24)
\]
\[
\sum_{k=0}^{n_1} N_{n,k+1}(n-k-1)_{n_1-k} = \sum_{k=0}^{n_1} \frac{(-1)^k kn}{k+n+1} \binom{n}{k} \binom{k+n+1}{n}. \quad (25)
\]

Corollary 14. The following partial sum holds:

\[
\sum_{k=0}^{n} (-1)^k \binom{x}{k} = (-1)^n \binom{x-1}{n}. \quad (26)
\]

Proof. Comparing the right hand side of (22) when $z = -1$, the right hand side of (23), and noting that $\binom{n}{k}$ form a basis, we have

\[
\sum_{i=0}^{n_1-k} (-1)^i \binom{n-k}{i} = (-1)^{n_1-k} \binom{n-k-1}{n_1-k}.
\]

Since this holds for every $n-k$ as a polynomial of $n-k$, it holds for any $x$. \hfill \Box

Theorem 15. For $n \geq 0, k \geq 1, 0 \leq t < k, q \in \mathbb{Z}, x \in \mathbb{C}, \omega_k = e^{\frac{j\pi n}{k}}, j^2 = -1$, we have

\[
\sum_{l=0}^{n} \binom{kn+t}{kl+t} \binom{kl+x+q+kn+2t-2}{q+kn+t-1} z^{kl} =
\]
\[
\frac{1}{k} \sum_{i=0}^{kn+t} \binom{kn+t}{i} \binom{x+q+kn+t-2}{q+kn+t-1-i} z^{i-t} \sum_{l=1}^{k} (\omega_k^i)^{i-t}(1+z\omega_k^i)^{kn+t-i}. \quad (27)
\]

Proof. Considering the trees contained in $\Gamma_{kn,x,q}$ with $|\text{int}(T)|+|\text{lev}(T)| = x+kl+t, l \geq 0$, we obtain, along the lines of Theorem 4

\[
\sum_{l=0}^{n} \binom{kn+t}{kl+t} \binom{kl+kn+x+q+2t-2}{kn+t+q-1} z^{kn-kl} =
\]
\[
\sum_{i=0}^{kn+t} \binom{kn+t}{i} \binom{kn+t+q-2}{kn+t+q-i-1} \sum_{kl+t \geq i} \binom{kn+t-i}{kl+t-i} z^{kn-kl}.
\]

Canceling out the term $z^{kn}$ and setting $z = z^{-1}$ in the above identity, the second summation on the right hand side becomes

\[
z^{-t} \sum_{l \geq 0} \binom{kn+t-i}{kl+t-i} z^{kl+t-i}.
\]
From the identity (1.53) in [14], we have
\[
\sum_{i \geq 0} \binom{n}{a + ki} x^{a + ki} = \frac{1}{k} \sum_{i=1}^{\infty} (\omega_k^i)^{-a} (1 + x \omega_k^i)^n, \quad k \geq 1, a, a \in \mathbb{Z}.
\]

Therefore, setting \(a = t - i, n = kn + t - i, k = k\) we obtain
\[
\sum_{i \geq 0} \binom{kn + t - i}{kl + t - i} t^{kl + t - i} = \frac{1}{k} \sum_{i=1}^{\infty} (\omega_k^i)^{-i-t} (1 + z \omega_k^i)^{kn + t - i}
\]
and the proof follows.

Finally, we summarize some particular evaluations of (27) in the following three corollaries.

**Corollary 16.** For \(n \geq 0, k \geq 1, 0 \leq t < 2, 0 \leq s < 4, \omega_k = e^{\frac{i2\pi}{k}}, j^2 = -1\), we have
\[
\sum_{l=0}^{n} \binom{n}{l} \binom{n + x + q + l - 2}{n + q - 1} z^l = \sum_{l=0}^{n} \binom{n}{l} \binom{x + q + n - 2}{q + n - 1 - l} z^l (1 + z)^{n-l}, \quad (28)
\]
\[
\sum_{l=0}^{n} 2 \binom{2n + t}{2l + t} \binom{2l + x + q + 2n + 2t - 2}{q + 2n + t - 1} z^{2l} = \sum_{l=0}^{2n+t} \binom{2n + t}{l} \binom{x + q + 2n + t - 2}{q + 2n + t - 1 - l} z^{l-t} [(z - 1)^{2n+t-l} + (z + 1)^{2n+t-l}], \quad (29)
\]
\[
\sum_{l=0}^{n} 4 \binom{4n + s}{4l + s} \binom{4l + x + q + 4n + 2s - 2}{q + 4n + s - 1} z^{4l} = \sum_{l=0}^{4n+s} \binom{4n + s}{l} \binom{x + q + 4n + s - 2}{q + 4n + s - 1 - l} z^{l-s} [(z - 1)^{4n+s-l} + (z + 1)^{4n+s-l} + (z + j)^{4n+s-l} + (z - j)^{4n+s-l}], \quad (30)
\]

**Proof.** Setting \(k = 1, 2, 4\) resp. in (27), we obtain (29) – (30), respectively.

**Corollary 17.** For \(n \geq 0, k \geq 1, 0 \leq t < k, 0 \leq s_1 < 2, 0 \leq s_2 < 4, \omega_k = e^{\frac{i2\pi}{k}}, j^2 = -1\), there holds
\[
\sum_{l=0}^{n} \binom{kn + t}{kl + t} \binom{kn + kl + 2t}{kn + t} = \sum_{i=0}^{kn+t} \binom{kn + t}{i} \sum_{i=1}^{k} (\omega_k^i)^{-i-t} (1 + \omega_k^i)^{kn + t - i}, \quad (31)
\]
\[
\sum_{l=0}^{n} N_{kn+t,i+1} \sum_{i=1}^{k} (\omega_k^i)^{-i-t} (1 + \omega_k^i)^{kn + t - i} = \sum_{l=0}^{n} \frac{1}{n} \binom{kn + t}{kl + t} \binom{kn + kl + 2t}{kl + t + 1}, \quad (32)
\]
\[
\sum_{l=0}^{n} 2 \binom{2n + s_1}{2l + s_1} \binom{2l + 2n + 2s_1}{2n + s_1} = \sum_{l=0}^{2n+s_1} \binom{2n + s_1}{l} 2^{2n+s_1-l} + 1, \quad (33)
\]
\[
\sum_{l=0}^{n} 4 \left( \frac{4n + s_2}{4l + s_2} \right) \left( \frac{4l + 4n + 2s_2}{4n + s_2} \right) = 4n + s_2 \sum_{l=0}^{4n+s_2} \binom{4n+s_2}{l} \left[ 2^{4n+s_2-l} + (1+j^{l-s_2}) \left( 1 + j \right)^{4n+s_2-l} \right] + 1 \quad (34)
\]

\[
\sum_{l=0}^{2n+s_1} N_{2n+s_1, l+1} 2^{2n+s_1-l} = \sum_{l=0}^{n} \frac{1}{n} \left( \frac{2n + s_1}{2l + s_1} \right) \left( \frac{2n + 2l + 2s_1}{2l + s_1 + 1} \right) \quad (35)
\]

\[
\sum_{l=0}^{4n+s_2} N_{4n+s_2, l+1} \left[ 2^{4n+s_2-l} + (1+j^{l-s_2}) \left( 1 + j \right)^{4n+s_2-l} \right] = \sum_{l=0}^{n} \frac{1}{n} \left( \frac{4n + s_2}{4l + s_2} \right) \left( \frac{4n + 4l + 2s_2}{4l + s_2 + 1} \right) \quad (36)
\]

Proof. Setting \( x = q = 1, z = 1 \) in (27) leads to eq. (31); Setting \( q = 0, x = 2, z = 1 \) in (27) leads to eq. (32); Setting \( k = 2, 4 \) in eq. (31) leads to eq. (33) and resp. eq. (34); Setting \( k = 2, 4 \) in eq. (32) leads to eq. (35) and resp. eq. (36); This completes the proof. \( \square \)

Corollary 18. For \( n \geq 0 \), we have

\[
\sum_{l=0}^{n} \binom{2n}{2l} \binom{2l + 2n}{2n} = \sum_{l=0}^{n-1} \binom{2n}{2l + 1} \binom{2n + 2l + 1}{2n} + 1 \quad (37)
\]

\[
\sum_{l=0}^{n} \binom{2n + 1}{2l} \binom{2l + 2n + 1}{2n + 1} = \sum_{l=0}^{n} \binom{2n + 1}{2l + 1} \binom{2n + 2l + 2}{2n + 1} - 1 \quad (38)
\]

Proof. Employing eq. (7) and eq. (34) completes the proof. \( \square \)

Acknowledgements

We acknowledge the financial support of the Future and Emerging Technologies (FET) programme within the Seventh Framework Programme (FP7) for Research of the European Commission, under the FET-Proactive grant agreement TOPDRIM, number FP7-ICT-318121.

References

[1] W. Y. C. Chen, A general bijective algorithm for trees, Proc. Nat. Acad. Sci. U.S.A., 1990, 87:9635–9639.

[2] T. Mansour and Y. Sun, Identities involving Narayana polynomials and Catalan numbers, Discrete Math. v309, 4079-4088.

[3] H. W. Gould, Final analysis of Vandermonde’s convolution, Amer. Math. Monthly, Vol. 64 (1957) pp. 409–415.
[4] H. W. Gould, Some Generalization of Vandermonde’s Convolution, The American Mathematical Monthly, Vol. 63, No. 2. (Feb., 1956), pp. 84–91.

[5] J. H. van Lint, R. M. Wilson, A Course in Combinatorics, Second Edition, Cambridge University Press, 2001.

[6] J. W. Moon, Counting Labelled Trees, Canadian Mathematical Congress, Canadian Mathematical Monographs, No. 1, 1970.

[7] A. Postnikov, Intransitive Trees, J. Combin. Theory Ser. A 79 (1997), 360–366.

[8] H. Prüfer, Never beweis eines satzesuber permutationen, Arch. Math. Phys. Sci., 27:742–744, 1918.

[9] R. Sprugnoli, Riordan arrays and the Abel–Gould identity, Discrete Math. 142, 213–233 (1995).

[10] William Y. C. Chen and Sabrina X. M. Pang, On the combinatorics of the Pfaff identity, Discrete Mathematics 309(8):2190-2196 (2009).

[11] Ricky X. F. Chen, A Refinement of the Formula for k-ary Trees and the Gould-Vandermonde’s Convolution, Electr. J. Comb. 15(1) (2008).

[12] R. P. Stanley, http://www-math.mit.edu/~rstan/. Bijective Proof Problems, version of 18 August 2009,

[13] H. S. Wilf, Generatingfunctionology, 2nd ed., Academic Press, London, 1994.

[14] R. Sprugnoli, Riordan array proofs of identities in Gould’s book, 2006.

[15] J. Harer and D. Zagier, The Euler characteristic of the moduli space of curves, Invent. Math., 85(3):457-485, 1986.

[16] T. R. S. Walsh and A. B. Lehman, Counting rooted maps by genus I, J. Combinatorial Theory Ser. B13 (1972) 192-218.

[17] G. Chapuy, V. Féray, É. Fusy: A simple model of trees for unicellular maps. J. Comb. Theory, Ser. A 120(8): 2064-2092 (2013).

[18] D. Foata, D. Zeilberger: A Classic Proof of a Recurrence for a Very Classical Sequence. J. Comb. Theory, Ser. A 80(2): 380-384 (1997).

[19] Louis W. Shapiro, Robert A. Sulanke, Bijections for the Schröder numbers, Mathematics Magazine 73 (5): 369-376 (2000).

[20] Thomas J. X. Li, Christian M. Reidys, A combinatorial interpretation of the $\kappa_n^*(n)$ coefficients, arXiv:1406.3162.