Lower bounds on the squashed entanglement for multi-party system

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Squashed entanglement is a promising entanglement measure that can be generalized to multipartite case, and it has all of the desirable properties for a good entanglement measure. In this paper we present computable lower bounds to evaluate the multipartite squashed entanglement. We also derive some inequalities relating the squashed entanglement to the other entanglement measure.

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Entanglement has been recognized as a key resource and ingredient in the field of quantum information and computation science. As a result, a remarkable research effort has been devoted to characterizing and quantifying it (see, e.g., Ref. [1, 2] and references therein). Despite a large number of profound results obtained in this field, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30], there is still no general solution to the simplest case, namely the two partite case. It is usually accepted that the following two axioms [19] are satisfied for an appropriate entanglement measure:

(i) The function $I(\rho)$ should vanish on the set of separable quantum states. Some other useful but not necessarily properties require the entanglement measures should be convex, additive, and a continuous function in the state. The issue of entanglement measure for multipartite states poses an even greater challenge [31], and most of existing entanglement measures are constructed for bipartite state except that the quantum relative-entropy of entanglement [5] and squashed entanglement [21] can be generalized to multipartite case. Among the existing two-partite entanglement measures, additivity only holds for squashed entanglement and logarithmic negativity [20] and is conjectured to hold for entanglement of formation, but the quantum relative-entropy of entanglement is nonadditive [32]. Squashed entanglement was introduced by [33] and then independently by Christandl and Winter [21], who showed that it is monotone, and proved its additivity.

(ii) It has all of the desirable properties for a good entanglement measure: it is convex, asymptotically continuous, additive on tensor products and superadditive in general. It is upper bounded by entanglement cost, lower bounded by distillable entanglement. Very recently, the squashed entanglement was extended to multipartite case by Yang et al. [34] and similar ideas have also been developed independently in Ref. [35]. Furthermore, in a recent paper [36], the squashed entanglement is given the operational meaning with the aid of conditional mutual information. Thus the squashed entanglement is a promising candidate among the different kinds of entanglement measures. However, it is still very difficult to compute the squashed entanglement and no analytic formula exists even for bipartite states. In fact, it is usually not easy to evaluate entanglement measures. Entanglement of formation is efficiently computable only for two-qubits [6]. Other measures are usually computable for states with high symmetries, such as Werner states, isotropic state, or the family of "iso-Werner" states, and squashed entanglement can only be evaluated for so called special flower states [37].

In this paper our aim is to explore a computable lower bound to evaluate the multipartite squashed entanglement. Firstly we briefly review the definition of multipartite $q$-squashed entanglement introduced in Ref. [34]. Before describing the details of multipartite squashed entanglement, it is necessary to recall the definition of multipartite mutual information. In this paper we will adopt the function $I(A_1 : A_2 : \ldots : A_n) = S(A_1) + S(A_2) + \ldots + S(A_n) - S(A_1 A_2 \ldots A_n)$ as a multipartite mutual information, where $S(X)$ is the von Neumann entropy of system $X$. This version of multipartite mutual information has an interesting feature: it can be represented as a sum of bipartite mutual informations. Analogous to the definition of bipartite conditional mutual informations $I_A := I(A : B|E) = S(\rho_{ABE}) - S(\rho_{AE}) - S(\rho_{BE})$, we can also define the multipartite conditional mutual information $I_{A_1 : A_2 \ldots : A_N} := I(A_1 : A_2 \ldots : A_N|E)$. For the $N$-party state $\rho_{A_1 \ldots A_N}$, the multipartite $q$-squashed entanglement is defined as

$$E_{sq}^q(\rho_{A_1 \ldots A_N}) = \inf I(A_1 : A_2 : \ldots : A_N|E),$$

where the infimum is taken over states $\sigma_{A_1 \ldots A_N, E}$, that are extensions of $\rho_{A_1 \ldots A_N}$, i.e. $Tr_E \sigma = \rho$. If the extension states $\sigma_{A_1 \ldots A_N, E}$ takes the form $\sum_i p_i |\psi_i\rangle_E \langle \psi_i|$, we call it c-squashed entanglement. Here, we denote $q$-squashed entanglement and c-squashed entanglement both as $E_{sq}(\rho_{A_1 \ldots A_N})$ due to our derivation is irrelevant to the form of the extension states. We begin by considering tri-partite state and later generalize the results to the case of multi-party subsystem. Notice that $I(A_1 : A_2 : \ldots : A_N|E)$ can be represented as the sum of the following terms:

$$I_{A_1 : A_2 : \ldots : A_N|E} = I_{A_1 : A_2 | E} + I_{A_2 : A_3 | E} + \ldots + I_{A_{N-1} : A_N | E}.$$
Now we can prove the following:

**Lemma 1.** For any tripartite state $\rho_{A_1A_2A_3}$, we have

$$E_{sq}(\rho_{A_1A_2A_3}) \geq \max \{C - S(A_1A_2), C - S(A_1A_3), C - S(A_2A_3)\}, \quad (3)$$

where $C = \sum_{i=1}^{3} S(A_i) - 2S(A_1A_2A_3)$.

**Proof.** Suppose that $E$ is an optimum extension for system $A_1A_2A_3$ satisfying $E_{sq}(\rho_{A_1A_2A_3}) = I(A_1 : A_2 : A_3|E)$. Then

$$E_{sq}(\rho_{A_1A_2A_3}) = 2E_{sq}(\rho_{A_1A_2}) - 2E_{sq}(\rho_{A_1A_2A_3}) \geq I(A_1 : A_2 : A_3|E) - I(A_1 : A_2|E) - I(A_1 : A_2 : A_3|E) = 0.$$ 

Thus we have $E_{sq}(\rho_{A_1A_2A_3}) \geq 2E_{sq}(\rho_{A_1A_2}) + 2E_{sq}(\rho_{A_1A_2A_3})$. Using a lower bound of the bipartite squashed entanglement presented in Ref. 21, thus we obtain: $E_{sq}(\rho_{A_1A_2A_3}) \geq \sum_{i=1}^{3} S(A_i) - S(A_1A_2) - 2S(A_1A_2A_3)$. If we permute the indices cyclically we get three inequalities and obtain the sharpest bound. This ends the proof.

It should be noted that the constant 2 in Eq. (4) is due to the difference of the definition between bipartite squashed entanglement and multipartite squashed entanglement. The measures we propose in the case of two parties reduces to twice the original squashed entanglement.

**Corollary 1:** For any tripartite state $\rho_{A_1A_2A_3}$, we have

$$E_{sq}(\rho_{A_1A_2A_3}) \geq 2E_{sq}(\rho_{A_1A_2}) + 2E_{sq}(\rho_{A_2A_3}) + 2E_{sq}(\rho_{A_1A_3}) \quad (5)$$

**Proof.** Notice that the monogamy inequality of two-partite squashed entanglement[38], i.e., $E_{sq}(\rho_{A:B:C}) \geq E_{sq}(\rho_{A:B}) + E_{sq}(\rho_{A:C})$, the proof is obtained immediately.

By taking the average over all combinations of two parties in Eq. (3) we get the following corollary:

**Corollary 2:** For any tripartite states $\rho_{A_1A_2A_3}$, we have

$$E_{sq}(\rho_{A_1A_2A_3}) \geq S(A_1) + S(A_2) + S(A_3) - \frac{1}{3}[S(A_1A_2) + S(A_2A_3) + S(A_1A_3)] - 2S(A_1A_2A_3). \quad (6)$$

Eq. (3) and Eq. (6) provide computable lower bounds to evaluate the tripartite squashed entanglement. Using an inequality presented in Ref. 39, we can also relate the relative-entropy of entanglement to the squashed entanglement measure. For tripartite pure state we have $E_{sq}(\rho_{A_1A_2A_3}) = S(A_1) + S(A_2) + S(A_3)$. Employing the inequality (12) in Ref. 39 an immediate corollary is as follows:

**Corollary 3:**

$$\frac{3}{2}E_{RE}(\rho_{A_1A_2A_3}) \leq E_{sq}(\rho_{A_1A_2A_3}) \leq 3E_{RE}(\rho_{A_1A_2A_3}) - E_{RE}(\rho_{A_1A_2}) - E_{RE}(\rho_{A_1A_3}) - E_{RE}(\rho_{A_2A_3}) \quad (7)$$

for any pure tri-partite state $\rho_{A_1A_2A_3}$.

Furthermore, we can derive an inequality relating the conditional entanglement of mutual information with the squashed entanglement. Conditional entanglement of mutual information is a new entanglement measure introduced in Ref. 29. Remarkably, it is additive and has an operational meaning and can straightforwardly be generalized to multipartite cases. Conditional entanglement of mutual information is defined as follows:

**Definition.** Let $\rho_{AB}$ be a mixed state on a bipartite Hilbert space $H_A \otimes H_B$. The conditional entanglement of mutual information for $\rho_{AB}$ is defined as

$$C_{I}(\rho_{AB}) = \inf \frac{1}{2} \left\{ I(AA' : BB') - I(A' : B') \right\}, \quad (8)$$

where the infimum is taken over all extensions of $\rho_{AB}$, i.e., over all states satisfying the equation $Tr_{A'B'}\rho_{AA'B'B'} = \rho_{AB}$, and the factor 1/2 is to make it equal to the entanglement of formation for the pure state case. Yang et al. 29 have proved that $C_I$ satisfied all the desired property of a good entanglement measure and is easy generalized to the multipartite case. For multipartite mixed state $\rho_{A_1\ldots A_n}$, $C_I(\rho_{A_1\ldots A_n}) = \inf \{ I_n(A_1A_1' : \ldots : A_nA_n') - I_n(A_1' : \ldots : A_n') \}$, where $I_n = \sum_i S(A_i) - S(A_1\ldots A_n)$. Now we present our result which is the following lemma:

**Lemma 2.** For any tripartite state $\rho_{A_1A_2A_3}$, we have

$$C_{I}(\rho_{A_1A_2A_3}) \geq \max \{2C_{I}(\rho_{A_1A_2}) + 2E_{sq}(\rho_{A_1A_2A_3}) \}, \quad 2C_{I}(\rho_{A_1A_3}) + 2E_{sq}(\rho_{A_1A_2A_3}), \quad (9)$$

**Proof.** Suppose that $A_1' A_2' A_3'$ is a minimum extension for system $A_1A_2A_3$ satisfying $C_I(\rho_{A_1A_2A_3}) = I_3(A_1A_1' : A_2A_2' : A_3A_3') - I_3(A_1' : A_2' : A_3')$. Then
The general result: the other two inequalities. Analogously we can prove the von Neumann entropy. Finally, we show an inequality of the multipartite state. Using the similar procedure as proving Lemma 1, we obtain the following general result:

**Lemma 3.** For any $N$-partite state $\rho_{A_1, A_2, ... A_N}$, we have

$$E_{sq}(\rho_{A_1, A_2, ... A_N}) \geq \sum_{i=1}^{N} S(A_i) - \sum_{M=2}^{N-1} \frac{1}{M} \sum_{i_1 < ... < i_M = 1, 2, ..., N} S(A_{i_1} ... A_{i_M}) - 2S(A_1 ... A_N).$$

(11)

Finally, we show an inequality of the multipartite squashed entanglement analogous to the monogamy inequality for the two-party case.

**Lemma 4.** For any multipartite state $\rho_{A_1, A_2, ... A_N}$

$$E_{sq}(\rho_{A_1; A_2; ...; (A_{N-1}, A_N)}) \geq E_{sq}(\rho_{A_1; A_2; ...; A_{N-1}}) + E_{sq}(\rho_{A_1, A_2; ...; A_{N-2}; A_N}).$$

(12)

**Proof.** Suppose that $E$ is a minimum extension for state $\rho_{A_1, A_2, ... A_N}$, then

$$E_{sq}(\rho_{A_1; A_2; ...; (A_{N-1}, A_N)}) = I(A_1 : A_2 : ... : (A_{N-1}, A_N) | E) = I(A_1 : A_2 : ... : A_N) - I(A_1, A_2, ... A_N | E) \geq E_{sq}(\rho_{A_1; A_2; ...; A_{N-1}}) + E_{sq}(\rho_{A_1, A_2; ...; A_{N-2}; A_N}).$$

(13)

Below we give some examples to show the application of Eq. (11).

**Example 1.** Consider a family of mixed 4-qubit state $\rho(p) = p|GHZ\rangle \langle GHZ| + (1-p)|W\rangle \langle W|$, where $|GHZ\rangle = \frac{1}{\sqrt{8}}(|0000| + |1111|)$, and $|W\rangle = \frac{1}{\sqrt{3}}(|0010| + |0100| + |1000|)$. In order to evaluate the multipartite entanglement of $\rho(p)$, we plot the lower bound of the squashed entanglement as a function of $p$ in Fig. 1. We find the lower bound for $0 \leq p < 0.113$ and $0.842 < p \leq 1$ is positive, which shows that $\rho(p)$ is an entangled state in these cases. It should be noted that the analytic expression of the 3-tangle for the 3-qubit state $\rho(p)$ has been obtained in Ref. [4] recently, and the 3-tangle can be used as an entanglement measure for the genuine 3-party entanglement. However, their results only restricted to the 3-qubit state and it is not obvious to generalize the 3-tangle to the multipartite case. In contrast, our lower bound can be used to evaluate the squashed entanglement for arbitrary party systems.

**Example 2.** Consider a class of generalized Werner states [11] [12] for $2 \otimes 2 \otimes 2$ systems: $\rho_W(p) = \frac{1}{2}I \otimes I \otimes I + (1-p)|\psi\rangle \langle \psi|$, where $|\psi\rangle = \frac{1}{\sqrt{6}}(|0110| - |101| - |011|)$. The tripartite mixed state $\rho_W(p)$ are invariant under $\rho_W \rightarrow \int dU U \rho_W U^\dagger \otimes U^\dagger \otimes U^\dagger$ and can be regarded as generalized tripartite Werner states. Now we employ the lower bound to evaluate the squashed entanglement of $\rho_W(p)$. The lower bound is plotted in Fig. 2. We can still get a positive lower bound for $0 \leq p < 0.103$.

Our results provide computable lower bounds on the multipartite squashed entanglement for the first time, which allow us to evaluate the multipartite squashed entanglement for a wide class of mixed states. These bounds also help us to judge whether a general mixed multipartite state is entangled or not, and some useful
results can be obtained in some cases. We also relate the squashed entanglement to the other entanglement measure, such as quantum relative-entropy of entanglement, and conditional entanglement of mutual information. An interesting question remained is to derive a tighter lower bound of the multipartite squashed entanglement or the upper bound of the squashed entanglement for the twopartite and multipartite case.

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