A Relation between
Commutative and Noncommutative
Descriptions of D-branes

Nobuyuki Ishibashi

KEK Theory Group, Tsukuba, Ibaraki 305, Japan

ABSTRACT

In string theory, D-branes can be expressed as a configuration of infinitely many lower dimensional D-branes. Using this relation, the worldvolume theory of D-branes can be regarded as the worldvolume theory of the infinitely many lower dimensional branes. In the description in terms of the lower dimensional branes, some of the worldvolume coordinates become noncommutative. Actually this noncommutative theory can be regarded as noncommutative Yang-Mills theory. Therefore the worldvolume theory of D-branes have two equivalent descriptions, namely the usual static gauge description using ordinary Yang-Mills theory and the noncommutative description using noncommutative Yang-Mills theory. It will be shown that these two descriptions correspond to two different ways of gauge fixing of the reparametrization invariance and its generalization. We will give an explicit relation between commutative gauge field and noncommutative gauge field in semiclassical approximation, when the gauge group is $U(1)$.

\footnote{E-mail:ishibash@theory.kek.jp}
1 Introduction

Many physicists are interested in noncommutative geometry, because they expect that it captures some features of quantum gravity. It is intriguing to see that in string theory and M theory, which is considered to be the most promising model of quantum gravity, we come across several occasions in which space-time coordinate becomes noncommutative\cite{1}-\cite{15}. In this note we would like to discuss an example in which noncommutativity of spacetime coordinates appears in string theory. The example we study here is the worldvolume theory of D-branes. As was pointed out in \cite{16,17}, Dp-branes can be represented as a configuration of infinitely many D(p−2r)-branes. If such a relation hold, the worldvolume theory of the Dp-branes can also be regarded as the worldvolume theory of infinitely many D(p−2r)-branes. We will show that some of the coordinates on the worldvolume of the Dp-branes become noncommutative if one consider it as the worldvolume theory of D(p−2r)-branes. Actually the noncommutative theory we have is noncommutative Yang-Mills theory. Such a noncommutative description of the Dp-branes should be equivalent to the usual commutative descriptions. We will pursue this equivalence and show that these two descriptions correspond to two different way to fix the reparametrization invariance and its generalization on the worldvolume. Therefore we have here an example where a noncommutative theory is equivalent to a commutative theory. Such an equivalence in a similar context was studied in a recent paper\cite{18}. We will comment on the relation between our results and theirs.

In section 2, we explain how Dp-branes can be expressed as a configuration of infinitely many D(p−2r)-branes. In section 3, we study the worldvolume theory of the Dp-branes regarding it as a configuration of the D(p−2r)-branes. Section 4 is devoted to discussions.

This note is based on a talk presented at “Workshop on Noncommutative Differential Geometry and its Application to Physics”, Shonan-Kokusaimura, Japan, May 31-June 4, 1999.

After I completed this note, I was informed that there are papers\cite{19,20} whose results have considerable overlap with ours. Especially, in the first paper, they realized that the commutative and noncommutative descriptions of D-branes correspond to two different ways of gauge fixing, and the explicit relation between the commutative and noncommutative gauge fields, which coincides with ours were given in \cite{20}.

2 Dp-branes from D(p−2r)-branes

In this section we will explain how Dp-branes can be expressed as a configuration of infinitely many D(p−2r)-branes. For simplicity, the special case of expressing one Dp-brane by D-instantons, namely p = 2r + 1 case, will be treated here. We will comment on more general cases at the end of this section.
We study this problem in the Euclidean space $\mathbb{R}^D$. $D$ should be 26 for bosonic string and 10 for superstring. We will first deal with bosonic string theory in which the whole manipulations are simpler. Later we will explain how one can generalize the arguments to the superstring case. The configuration of infinitely many D-instantons can be expressed by the $\infty \times \infty$ hermitian matrices $X^M (M = 1, \cdots, D)$. The one we consider is

\[
X^i = \hat{P}^i, \ (i = 1, \cdots, p + 1) \\
X^M = 0 \ (M = p + 2, \cdots, D),
\]

where $\hat{P}^i (i = 1 \cdots, p + 1)$ satisfy

\[
[\hat{P}^i, \hat{P}^j] = i\theta^{ij}.
\]

In this note, the $(p + 1) \times (p + 1)$ matrix $\theta$ is assumed to be invertible.

Let us show that this configuration of D-instantons is equivalent to a D$p$-brane. In order to show that two different configurations of D-branes are equivalent, one should prove that the open string theory corresponding to these configurations are equivalent.

A quick way to see the equivalence is to look at the boundary states. The boundary state $|B\rangle$ corresponding to the configuration eq.(1) can be written as follows:

\[
|B\rangle = \text{Tr} e^{-i \int_{0}^{2\pi} d\sigma p_i \hat{P}^i} |B\rangle_{-1}.
\]

Here $p_i(\sigma)$ is the variable conjugate to the string coordinate $x^i(\sigma)$ and is equal to $\frac{1}{2\pi \alpha'} \dot{x}_i$ in the usual flat background. $|B\rangle_{-1}$ denotes the boundary state for a D-instanton at the origin and satisfies $x^i(\sigma) |B\rangle_{-1} = 0$. $|B\rangle_{-1}$ includes also the ghost part which is not relevant to the discussion here. The factor in front of $|B\rangle_{-1}$ is an analogue of the Wilson loop and corresponds to the background eq.(1). Eq.(3) can be rewritten by using path integral as

\[
|B\rangle = \int [dP] e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^i \omega_{ij} - i \int d\sigma p_i P^i |B\rangle_{-1}},
\]

where $\omega_{ij} = (\theta^{-1})_{ij}$.

It is straightforward to perform the path integral in eq.(4). Using the Fock space representation of $|B\rangle_{-1}$, the path integral is Gaussian. The determinant factor can be regularized in the usual way [23], and one can show that $|B\rangle$ coincides with the boundary state for a D$p$-brane with the $U(1)$ gauge field strength $F_{ij} = \omega_{ij}$ on the worldvolume. Knowing that the path integration is Gaussian, it is easy to confirm this fact. Indeed, one can show that the following identity holds:

\[
0 = \int [dP] \frac{\delta}{\delta P_i(\sigma)} e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^i \omega_{ij} - i \int d\sigma p_i P^i |B\rangle_{-1}} \]

\[
= [i\omega_{ij} \partial_\sigma x^j - ip_i(\sigma)] |B\rangle.
\]

\(^2\text{Construction of Dp-branes from D}(p - 2r)\text{-branes was done on torus in [21]. Things discussed here are partially generalized to the space compactified on a torus in [22].}\)
Therefore $|B\rangle$ should coincide with $\exp\left[\frac{i}{2} \int d\sigma x^i \partial_\sigma x^j \omega_{ij}\right]|B\rangle_p$ up to normalization. Here $|B\rangle_p$ denotes the boundary state for a $\text{D}p$-brane satisfying $p_i(\sigma)|B\rangle_p = 0$. Hence $|B\rangle$ is equivalent to the boundary state for a $\text{D}p$-brane with the $U(1)$ gauge field strength $F_{ij} = \omega_{ij}$ on the worldvolume.

In the above analysis using the path integral expression eq.(4), the overall normalization of the boundary state is ambiguous. Actually one can prove the equivalence including the normalization by showing that the open string theory corresponding to the configuration eq.(4) is equivalent to the one corresponding to a $\text{D}p$-brane. We refer to [24] for more details.

It is easy to supersymmetrize the above arguments. In the NSR formalism the boundary state for D-instanton can be written as a sum of four states $|B; \pm\rangle_{-1,I}$ where $I = \text{NS, R}$ indicates the sector it belongs to and

\[
\begin{align*}
 x^M(\sigma)|B; \pm\rangle_{-1,I} &= 0, \\
 (\psi^M(\sigma) \pm i\tilde{\psi}^M(\sigma))|B; \pm\rangle_{-1,I} &= 0.
\end{align*}
\]  

Supersymmetric generalization of eq.(4) can be given for each $|B; \pm\rangle_{-1,I}$ as

\[
|B; \pm\rangle_I = \int [dPd\chi] e^{\frac{i}{2} \int d\sigma (p_i P^i + \chi^i \chi^j) \omega_{ij} - \int d\sigma (ip_i P^i - \pi_i \chi^i)}|B; \pm\rangle_{-1,I},
\]  

where $\pi^M(\sigma) = \frac{1}{2}(\psi^M(\sigma) \mp i\tilde{\psi}^M(\sigma))$. We can show following the same arguments as above that this boundary state coincides with the boundary state for a $\text{D}p$-brane up to normalization. It is also possible to prove the equivalence of the open string theories [25].

Since the arguments in this section are essentially about the variables $x^i, \psi^i$ ($i = 1, \cdots, p + 1$) on the worldsheet, it is quite straightforward to apply the arguments here to prove that a $\text{D}p$-brane can be expressed as a configuration of infinitely many $\text{D}(p-2r)$-branes. It is also easy to generalize the argument to the case of $N \text{D}p$-branes. In such a case we should consider the block diagonal background

\[
X^i = \hat{P}^i \otimes I_N,
\]  

where $I_N$ is the $N \times N$ identity matrix which is an element of $U(N)$ Lie algebra. The expression of the boundary state in eq.(4) should be modified to

\[
|B\rangle = \int [dP] \text{TrP} e^{\frac{i}{2} \int d\sigma P^i P^j \omega_{ij} - i \int d\sigma P_i P_i} |B\rangle_{-1},
\]  

where TrP here means the trace of the path-ordered product with respect to the $U(N)$ indices. It is easy to see that this configuration corresponds to $N \text{D}p$-branes following the same arguments as above.
3 Worldvolume Theory

In the previous section, we explained that the open string theory corresponding to the configuration of D-instantons in eq.(I) is equivalent to the one for a Dp-brane with \( F = \omega \). This means that the worldvolume theory of one Dp-brane can also be described as the worldvolume theory of D-instantons. In this section, we investigate the worldvolume theory from two different points of view, i.e. either as the worldvolume theory of one Dp-brane or D-instantons. We call them Dp-brane picture and D-instanton picture respectively. We will show how these two are related with each other. The argument in this section will be done for bosonic string case for simplicity. For superstring case, similar results can be derived starting from the expression eq.(7).

Our argument in this section can be applied to study the worldvolume theory of Dp-branes by regarding it as the worldvolume theory of infinitely many D \((p-2r)\)-branes for \( 2r < p+1 \). It is also straightforward to deal with the case when the number of the Dp-brane is more than one. We will comment on these generalizations at the end of this section.

3.1 Dp-brane Picture

Let us start from the following expression of the boundary state for the Dp-brane:

\[
|B\rangle = \int [dP] e^{\frac{i}{2} \int d\sigma P^i \partial_{\sigma} P^i \omega_{ij} - i \int d\sigma \int d\rho_i P^i |B\rangle_{-1}.
\]

This corresponds to a Dp-brane longitudinal to \( x^i (i = 1, \ldots, p+1) \) directions with the \( U(1) \) gauge field strength \( F = \omega \). The worldvolume theory of a Dp-brane consists of a gauge field \( A_i \) and scalar fields \( \phi^M (M = p+2, \ldots, D) \). \( \phi^M \) correspond to the shape of the worldvolume which can be expressed by the equation \( x^M = \phi^M (x^1, \ldots, x^{p+1}) \). We are considering here the field configurations in the static gauge, so that the coordinates on the worldvolume are taken to be \( x^1, \ldots, x^{p+1} \). The boundary state corresponding to a configuration of \( A_i, \phi^M \) is easily guessed to be

\[
|B\rangle = \int [dP] \exp \left[ i \int d\sigma A_i(P) \partial_{\sigma} P^i - i \int d\sigma (p_i P^i + \sum_{M=p+2}^{D} p_M \phi^M(P)) \right] |B\rangle_{-1}.
\]

Indeed this coincides with eq.(10) when \( F = \omega, \phi^M = 0 \). For small deformations \( \delta A_i, \delta \phi^M \) from the background \( F = \omega, \phi^M = 0 \), we expect that the boundary state \( |B\rangle \) in eq.(10) is modified by the vertex operator as

\[
(1 + i \int d\sigma \frac{\delta A_i(x) \partial_{\sigma} x^i - \delta \phi^M(x) p_M}{\phi^M} \int [dP] e^{\frac{i}{2} \int d\sigma P^i \partial_{\sigma} P^i \omega_{ij} - i \int d\sigma P^i |B\rangle_{-1}},
\]

which is consistent with eq.(11). Moreover since \( x^M(\sigma)|B\rangle_{-1} = 0 \), this boundary state exactly describes the emission of a closed string from the worldvolume \( x^M = \).
The contribution of the gauge field is in the form of the Wilson loop. This state will be BRS invariant for only those $A_i, \phi^M$ satisfying the equations of motion. $Q_B|B\rangle = 0$ implies that the path integral measure is invariant under the reparametrization $\sigma \rightarrow \sigma'(\sigma)$. Imposing such a condition, one may be able to deduce the equations of motion after calculations similar to [23].

Thus in this picture, the worldvolume theory is a $U(1)$ gauge theory with scalars $\phi^M$. In this note, we always assume that Pauli-Villars regularization on the worldsheet is taken so that the noncommutativity because of the regularization discussed in [18] does not occur.

### 3.2 D-instanton Picture

Now let us consider the worldvolume theory as the worldvolume theory of D-instantons. The boundary state eq.(10) corresponds to the configuration eq.(1) of D-instantons. General configuration of D-instantons can be described as

$$X^M = \phi^M(\hat{P}) \quad (M = 1, \cdots, D),$$

if one assumes that the operators $\hat{P}^i$ ($i = 1, \cdots, p$) generate all the possible operators acting on the Chan-Paton indices of D-instantons. Here in defining the functions $\phi^M$, we need to specify the ordering of the operators $\hat{P}^i$, which will be given shortly. What will be the form of the boundary state corresponding to the configuration eq.(13)? A natural guess is

$$|B\rangle = \int [dP] e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^j \omega_{ij} - i \int d\sigma P = \phi^M(\hat{P}) |B\rangle_{-1}. \quad (14)$$

For small deformations $\delta \phi^M$ from the background eq.(1), we expect that the boundary state in eq.(10) is modified as

$$\int [dP] (1 - i \int d\sigma \delta \phi^M(\hat{P}) p_M) e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^j \omega_{ij} - i \int d\sigma P_i |B\rangle_{-1}. \quad (15)$$

Since the vertex operators corresponding to the transverse deformations $\phi^M (M = p + 2, \cdots, D)$ coincides with those in eq.(12), we expect that the transverse $\phi^M$ appear in the same way as in eq.(11). Eq.(14) is unique choice satisfying this condition and the $D$-dimensional rotational invariance. The boundary state eq.(14) describes the emission of a closed string from the hypersurface $X^M = \phi^M(\hat{P})$. Therefore the fields $\phi^M(P)$ correspond to the shape of the D-branes and $\hat{P}^i$ play the role of the coordinates of the $p$-brane.

In order to be consistent with the path integral form eq.(14), the ordering in eq.(13) should be chosen to be Weyl-ordering [26]. To be more explicit, for each $c$-number function $f(P)$, let us define the Weyl-ordered function $f(\hat{P})$ to be

$$f(\hat{P}) = \int d^{p+1}k e^{ik_i \hat{P}^i} \tilde{f}(k), \quad (16)$$
where
\[ \tilde{f}(k) = \int \frac{dp^{p+1}}{(2\pi)^{p+1}} e^{-ik\cdot P} f(P). \] (17)

Then \( \phi^M(\hat{P}) \) on the right hand side of eq. (13) should be understood to be the Weyl-ordered function corresponding to the c-number function \( \phi^M(P) \) in eq. (14).

The action and other physical quantities in the worldvolume theory of D-instantons are written as a trace of a function of the Weyl-ordered operators in eq. (16). However it is more convenient for us to rewrite everything in terms of the c-number functions \( \phi^M(P) \) in eq. (17). We can do so by using the following formula [27]:
\[ \text{Tr}(f_1(\hat{P}) f_2(\hat{P}) \cdots f_n(\hat{P})) = \int \frac{dp^{p+1}}{(2\pi)^{(p+1)/2}} \sqrt{|\det \theta|} f_1(P) * f_2(P) * \cdots * f_n(P). \] (18)

Here the \(*\)-product is defined as
\[ f(P) * g(P) = e^{\frac{i}{2} \theta^{ij} \frac{\partial}{\partial P^i} \frac{\partial}{\partial P^j} f(P + \xi) g(P + \zeta)|_{\xi=\zeta=0}. \]

Hence, a trace of a function of Weyl-ordered operators can be rewritten in terms of the corresponding c-number functions by replacing product of operators by \(*\)-product of the corresponding c-number functions and trace by integral.

Thus if one regards the worldvolume theory as a theory of D-instantons, the description should be noncommutative. \( P^i \) can be considered as the coordinates on the \( p \)-brane and they are noncommutative under the \(*\)-product reflecting the commutation relation eq. (2). Now let us discuss what kind of theory this noncommutative field theory is. The Lagrangian of the worldvolume theory of D-instantons can be written in terms of the commutators of the matrices \( \phi^M(\hat{P}) \). Since we started from the background in eq. (1), let us express \( \phi^i \) \( (i = 1, \cdots, p+1) \) in the form of the background and the fluctuations around it:
\[ \phi^i(\hat{P}) = \hat{P}^i + \theta^{ij} a_j(\hat{P}). \] (20)

The c-number expression corresponding to the commutators of \( \phi^i(\hat{P}) \) are easily calculated to be
\[ [\phi^i(\hat{P}), \phi^j(\hat{P})] \to i\theta^{ij} - i(\theta f)_{ij}, \] (21)
where
\[ f_{ij} = \partial_i a_j(P) - \partial_j a_i(P) - ia_i * a_j(P) + ia_j * a_i(P). \] (22)

\( f_{ij} \) can be considered as the field strength of a noncommutative Yang-Mills field \( a_i \). \( \phi^i(\hat{P}) \) essentially corresponds to the covariant derivative \( \partial^i + ia \). Thus the commutators of \( \phi^i \) with other fields give the covariant derivative of these fields. Other commutators are interpreted as the gauge covariant commutators in the noncommutative Yang-Mills theory. Since the Lagrangian is written in terms of these commutators, the noncommutative theory we have here can be considered as noncommutative Yang-Mills theory [28]. The gauge invariance of the theory stems from the transformation
\[ \delta \phi^M(\hat{P}) = i[\epsilon, \phi^M(\hat{P})]. \] (23)
As a theory of D-instantons this is the $U(\infty)$ transformation under which the theory should be invariant. In the c-number formulation such a transformation corresponds to the coordinate transformation

$$\delta P = \theta^{ij} \partial_j \epsilon(P). \quad (24)$$

This is the coordinate transformation which preserves $\omega = \theta^{-1}$. If one regards $\omega$ as a symplectic form, such transformations are the canonical transformations. The invariance under the canonical transformation will be discussed in the next subsection.

### 3.3 Relation between the Two Pictures

In the previous subsections we obtain two points of view about the worldvolume theory. Since they are supposed to describe the same thing, there should be correspondence between the two. In the D$p$-brane picture, the worldvolume fields are the gauge field $A_i (i = 1, \cdots, p + 1)$ and scalars $\phi^M (M = p + 2, \cdots, D)$, where the coordinates on the worldvolume is taken to be $x^i (i = 1, \cdots, p + 1)$. On the other hand, the worldvolume fields in the D-instanton picture are $\phi^M (P) (M = 1, \cdots, D)$ and $P^i$ are the coordinates on the worldvolume.

As we noticed in the previous section, the fields $\phi^M (M = p + 2, \cdots, D)$ common to both correspond to each other. Therefore we should find how the fields $A_i$ and $\phi^i$ are related. Let us first consider small deformations $\delta A_i, \delta \phi^i$ from the background eq.(1). From eq.(12), one can see that $\delta A_i$ changes the boundary state as

$$\delta \langle B \rangle = i \int [dP] \int d\sigma \delta A_i(P) \partial_\sigma P^i e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^j \omega_{ij} - i \int d\sigma p_i P^i |B\rangle_{-1}. \quad (25)$$

which should be compared with the variation corresponding to $\delta \phi^i$ from eq.(15):

$$\delta \langle B \rangle = -i \int [dP] \int d\sigma \delta \phi^i(P) p_i e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^j \omega_{ij} - i \int d\sigma p_i P^i |B\rangle_{-1}. \quad (26)$$

The relation between these two variations can be derived from the following identity

$$0 = \int [dP] \frac{\delta}{\delta P^i(\sigma)} e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^j \omega_{ij} - i \int d\sigma p_i P^i |B\rangle_{-1}}$$

$$= \int [dP] [i \omega_{ij} \partial_\sigma P^j - i p_i] e^{\frac{i}{2} \int d\sigma P^i \partial_\sigma P^j \omega_{ij} - i \int d\sigma p_i P^i |B\rangle_{-1}. \quad (27)$$

which implies that $\delta \langle B \rangle$ in eqs.(25)-(26) coincide with each other when $\delta A_i = \omega_{ij} \delta \phi^j$. Such a relation was given in (29).

In order to see the relation between the two pictures for finite deformations, the most convenient way is to consider a description involving fields $A_i$ and $\phi^M (M = 1, \cdots, D)$. From eqs.(11)-(14), the boundary state involving all these fields should be

$$\langle B \rangle = \int [dP] e^{i \int d\sigma A_i(P) \partial_\sigma P^i - i \int d\sigma p_i \phi^M(P) |B\rangle_{-1}. \quad (28)$$
However there are too many fields in such a description and there should be symmetry to reduce the number of them. The boundary state eq.(28) is invariant under gauge transformation $\delta A_i = \partial_i \lambda$. Moreover, in the D$p$-brane picture, considering $A_i$ and $\phi^M$ just means considering the theory before the gauge fixing of reparametrization invariance. Therefore the theory should have reparametrization invariance.

Indeed we can argue that the boundary state eq.(28) is invariant under the following transformation

$$
\delta A_i(P) = -\epsilon^j(P) F_{ji}(P), \\
\delta \phi^M(P) = -\epsilon^i(P) \partial_i \phi^M(P),
$$

because the variation is proportional to a sum of the equations of motion for $P^i$. This transformation coincides with the coordinate transformation up to field-dependent gauge transformation because

$$
\delta A_i(P) = -\epsilon^j \partial_j A_i - \partial_i \epsilon^j A_j + \partial_i (\epsilon^j A_j).
$$

Since the boundary state eq.(28) is gauge invariant, it is reparametrization invariant.

Therefore the description involving $A_i$ and $\phi^M$ ($M = 1, \cdots, D$) is invariant under the reparametrization on the worldvolume. The D$p$-brane picture obviously corresponds to the static gauge. On the other hand, one can see from eq.(14) that the D-instanton picture apparently corresponds to the gauge $F_{ij} = \omega_{ij}$. We are not sure if such a gauge can be taken for arbitrary configuration of the gauge field $F$, but at least when we are thinking about the fluctuation from the background $F = \omega$ perturbatively, it seems all right. Such a gauge does not fix the whole reparametrization invariance on the worldvolume. The residual invariance consists of the coordinate transformation preserving $\omega_{ij}$, i.e. the canonical transformation. We saw such an invariance as the c-number counterpart of the $U(\infty)$ invariance in the previous subsection.

Since the difference is from the gauge choice, we can give the explicit relation between the static gauge variables $A_i(x), \phi^M_{st}(x) (M = p + 2, \cdots, D)$ and the variables $\phi^M_{nc}(P) (M = 1, \cdots, D)$ in the noncommutative description at least classically, i.e. for small $\theta$:

$$
\phi^M_{nc}(P) = \phi^M_{st}(\phi^1_{nc}(P), \cdots, \phi^{p+1}_{nc}(P)) (M = p + 2, \cdots, D),
$$

$$
\omega_{ij} = F_{kl}(\phi^1_{nc}(P), \cdots, \phi^{p+1}_{nc}(P)) \frac{\partial \phi^k_{nc}}{\partial P^i} \frac{\partial \phi^l_{nc}}{\partial P^j}.
$$

Eq.(32) can be rewritten in terms of the noncommutative Yang-Mills field $a_i$ using eq.(20) as

$$
(\theta^{-1})_{ij} = (M^T F(P + \theta a) M)_{ij},
$$

where

$$
M^i_j = \delta^i_j + \theta^{ik} \partial_j a_k(P).
$$

\(^3\)Here the subscript st and nc are for distinguishing $\phi^M$ in different gauges.
This gives an explicit relation between commutative gauge field $A_i(x)$ and noncommutative gauge field $a_i(P)$.

Now let us comment on two generalizations of our results in this section. First one is to consider more than one Dp-branes. Starting from the background eq.(8), we can follow the arguments of $N = 1$ case. This time all the fields $A_i$ and $\phi^M$ are in the adjoint representation of $U(N)$. We should put $\text{Tr} P$ in front of the right hand sides of eqs.(10),(11),(14),(28) and then we can follow the same arguments as $N = 1$ case. The reparametrization invariance of the boundary state in this case can be derived as follows. The path-ordered trace version of eq.(28) can be rewritten by introducing fermions $\psi$ in the fundamental representation $U(N)$ as

$$|B\rangle = \int [dPd\psi] \exp[\int d\sigma \psi^\dagger \partial_\sigma \psi - i \int d\sigma A_i^a(P) \partial_\sigma P^i \psi^\dagger t^a \psi - i \int d\sigma p_M \phi^M a(P) \psi^\dagger t^a \psi] |B\rangle_{-1}. \tag{35}$$

Here $t^a$ are the generators of $U(N)$ in the fundamental representation. In this form, we can see that the boundary state is invariant under the transformation

$$\delta A_i(P) = -e^i(P) F_{ji}(P),$$
$$\delta \phi^M(P) = -e^i(P) D_i \phi^M(P), \tag{36}$$

because the variation is proportional to the equations of motion for $P^i, \psi$. This transformation is again equivalent to the coordinate transformation up to field-dependent gauge transformation. Moreover we can argue that the boundary state is invariant under

$$\delta A_i(P) = -e^i(P) t^a F_{ji}(P),$$
$$\delta \phi^M(P) = -e^i(P) t^a \partial_i \phi^M(P), \tag{37}$$

because the variation is proportional to $\lim_{\sigma' \to \sigma} \psi^\dagger t^a \psi(\sigma') \times \text{ (equations of motion)(}\sigma\text{)}$. This transformation can be considered to be a nonabelian generalization of coordinate transformation up to field-dependent gauge transformation. Fixing such invariance by taking static gauge or $F = \omega$ and we get commutative or noncommutative descriptions respectively.

Secondly it is also possible to study the worldvolume theory of Dp-branes regarding it as the worldvolume theory of infinitely many D$(p - 2r)$-branes. We obtain a description in which some of the coordinates on the worldvolume become noncommutative.

4 Discussions

In this note, we have shown that Dp-branes with constant field strength $F_{ij}$ can be represented as a configuration of infinitely many D$(p - 2r)$-branes. The worldvolume theory of the Dp-branes can be analysed by regarding it as the worldvolume theory of the D$(p - 2r)$-branes and we obtain a noncommutative description of the worldvolume.
theory. The system we studied here is gauge equivalent to the one studied in [18]. In that paper, D-branes in constant $B_{ij}$ background was considered and the authors get commutative and noncommutative descriptions of the worldvolume theory depending on the regularization. Moreover they propose descriptions with continuously varying $\theta$ which connect the commutative and noncommutative descriptions. Actually we can realize such descriptions also in our formalism by considering our system in constant $B_{ij}$ background [30]. Therefore we suspect that our noncommutative description is equivalent to a choice of $\theta$ in [18]. Since we have an explicit relation between the commutative and noncommutative descriptions which is valid for small $\theta$, it may be interesting to compare our relation and theirs. Here we just comment on one crucial difference. In [18], the gauge transformations of the commutative and noncommutative descriptions are related but in the relation we obtained the gauge invariance of non-commutative theory is the residual reparametrization invariance which is not related to the commutative gauge invariance. Therefore the relation we obtained in eq.(32) is for gauge invariant quantities.

Since we have an example in which there is a relation between commutative and noncommutative theory, it may be generalized and be used in studying other non-commutative theories. One example is the noncommutative geometric formulation of open string field theory [31]. We think that the relation we studied in this note may be relevant in revealing symmetries hidden in the string field theory. We hope that we can come back to this problem in the future.

Acknowledgements

We would like to thank the organizers of the workshop for the wonderful workshop. I am grateful to H. Aoki, S. Iso, H. Kawai, Y. Kitazawa and T. Tada for collaborations and K. Okuyama for discussions. This work was supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Culture of Japan.

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