ON THE CONTINUITY OF THE SOLUTION OPERATOR TO
THE WAVE MAP SYSTEM

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Abstract

We investigate the continuity properties of the solution operator to the wave
map system from general nonflat target of arbitrary dimension, and we prove by
an explicit class of counterexamples that this map is not uniformly continuous
in the critical norms on any neighbourhood of 0.

1. Introduction

Let $(N, g)$ be a smooth $d$-dimensional Riemannian manifold with metric $g$, which
with no loss in generality we can isometrically embed in $\mathbb{R}^m$ for some $m > d$. For
the functions $u : \mathbb{R} \times \mathbb{R}^n \to N$ defined on the flat Minkowski space $\mathbb{R}_t \times \mathbb{R}_x^n$ with
values in the target $N$, consider the functional

$$J(u) = \int_{\mathbb{R} \times \mathbb{R}^n} \langle \partial_\alpha u, \partial^\alpha u \rangle_g dt dx$$

where summation over $\alpha = 0, 1, ..., n$ is intended, with

$$(\partial_0, ..., \partial_n) = (\partial_t, \partial_{x_1}, ..., \partial_{x_n}), \quad (\partial^0, ..., \partial^n) = (\partial^t, -\partial_{x_1}, ..., -\partial_{x_n})$$
as usual, while $\langle \cdot, \cdot \rangle_g$ is the product in the metric $g$.

The critical points of the functional $J$ are called wave maps. If we choose a
system of coordinates on $N$, then locally smooth wave maps satisfy the equation

$$\Box u^\ell + \Gamma^\ell_{\beta\gamma}(u) \partial_\beta u^\beta \partial^\gamma u^\gamma = 0, \quad \ell = 1, ..., m,$$

where $\Gamma^\ell_{\beta\gamma}$ denote the Christoffel symbols on $N$ in the chosen coordinates. The
natural problem for this system of wave equations is clearly the Cauchy problem
with data at $t = 0$

$$u(0, x) = u_0, \quad u_t(0, x) = u_1;$$

the usual space for the data are Sobolev spaces

$$(u_0, u_1) \in H^s(\mathbb{R}^n, N) \times H^{s-1}(\mathbb{R}^n, N)$$

for suitable values of $s \in \mathbb{R}$. Here we used the space

$$H^s(\mathbb{R}^n; N) = \{ v \in H^s(\mathbb{R}^n; \mathbb{R}^m), \; v(\mathbb{R}^n) \subseteq N \}, \quad s \in \mathbb{R}$$

with the induced norm; notice that $H^s(\mathbb{R}^n; N) = \emptyset$ if $0 \notin N$, but it causes no loss
in generality to assume that $0 \in N$ after a translation in the ambient space.

An alternative description of the wave map system, which usually gives a simpler
expression in presence of symmetry of the target is the following: a wave map is a
function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ such that

$$u(t, x) \in N, \quad \Box u \perp N \quad \text{for all} \; (t, x).$$

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A good introduction on this subject with comprehensive references can be found in [21].

The Cauchy problem for wave maps has been extensively studied in recent years, starting with the work of Ginibre and Velo [10]. Not many general results for the Cauchy problem are known. Very roughly speaking and with several omissions, one has the following types of results:

- Global existence of weak solutions when the target is compact (several authors, see e.g. [9], [20], [34]).
- Local existence for data in $H^s$, $s > n/2$. This is classical if $s$ is large enough, but for $s$ close to the critical value $s = n/2$ it is a much more difficult result, due to Klainerman, Machedon, Selberg and obtained through careful bilinear estimates (see in particular [15]). See also Tataru [29] for the case of Besov spaces.
- Global existence for small data. Again, this result can be proved by “standard” methods in the smooth case (Y. Choquet-Bruhat), but the recent results of Tao (see in particular [28], see also [22]) show that it is sufficient to assume that the data are in $H^s$ for some $s > n/2$ and that they are small in the homogeneous $\dot{H}^{n/2}$ norm.
- In the presence of symmetry one has of course sharper results; to this class belong the radial case, considered by Christodoulou, Tahvildar-Zadeh and more recently by Struwe (see [7], [24], [25]), and the equivariant case, for which a fairly complete theory exists, due to Shatah, Tahvildar-Zadeh, Struwe, and Grillakis (see e.g. [11], [20], [23], [26]).

The precise behaviour of the wave map system in the critical case $s = n/2$ is an open problem (while in the subcritical case $s < n/2$ one has in general ill-posedness in a sharp sense, i.e. non-uniqueness, see e.g. [27] and [8]). A possible line of attack was suggested by Bourgain (see [2], [3] and also Tzvetkov [33]) who proved that the map data $\mapsto$ solution to some nonlinear evolution equations is not $C^2$ in the subcritical Sobolev spaces. This holds for the cubic NLS, for KdV and mKdV with different critical indices. The result was sharpened by Kenig, Ponce and Vega [3] who proved that the solution map actually is not (locally) uniformly continuous in the subcritical spaces. We also mention [1] and [17] where the case of the supercritical nonlinear wave equation and of the Benjamin-Ono equation are considered.

Our aim here is to prove a similar result for the wave map system, in the critical case $s = n/2$. Our assumption on $N$ will be quite general; essentially we only require that $N$ is not flat. More precisely, we assume that

\begin{equation}
\text{there exists a geodesic curve } \gamma : (-s_0, s_0] \rightarrow N \text{ with } \gamma(0) = 0, \gamma''(0) \neq 0.
\end{equation}

From such generality it will be clear that the ill-posedness in the sense of uniform continuity is a general properties of nonlinear equations like (1.1) more than a geometric property. Our result is the following:

**Theorem 1.1.** Let $N$ be a smooth Riemannian manifold, isometrically embedded in $\mathbb{R}^m$, such that there exists a geodesic curve $\gamma : (-s_0, s_0] \rightarrow N$ with

\begin{equation}
\gamma(0) = 0 \in N, \quad \gamma''(0) \neq 0.
\end{equation}

Assume a solution map $\Phi : (u_0, u_1) \rightarrow u$ for system (1.1) with data (1.2) is defined on some neighbourhood $U$ of 0 in $X \times Y = H^{n/2}(\mathbb{R}^n; N) \times H^{n/2-1}(\mathbb{R}^n; N)$. Then, for any $T > 0$, $\Phi$ is not uniformly continuous between the spaces

$$
\Phi : U \subseteq H^{n/2} \times H^{n/2-1} \rightarrow C([0, T]; \dot{H}^{n/2}(\mathbb{R}^n, \mathbb{R}^m))
$$
or

$$\Phi : U \subseteq H^{n/2} \times H^{n/2-1} \to C^1([0,T]; \dot{H}^{n/2-1}(\mathbb{R}^n, \mathbb{R}^m)).$$

As usual, we say that $\Phi : U \subseteq X \times Y \to L$ is uniformly continuous on $U$ if: for any $\epsilon > 0$ there exists $\delta > 0$ such that, for any $(u_0^{(1)}, u_1^{(1)})$ and $(u_0^{(2)}, u_1^{(2)})$ in $U$

$$\| (u_0^{(1)}, u_1^{(1)}) - (u_0^{(2)}, u_1^{(2)}) \|_{X \times Y} \leq \delta \Rightarrow \| u_0^{(1)} - u_0^{(2)} \|_{L} \leq \epsilon$$

where $u_0^{(1)} = \Phi(u_0^{(1)}, u_1^{(1)})$, $u_0^{(2)} = \Phi(u_0^{(2)}, u_1^{(2)})$. Thus the above result excludes in particular that $\Phi$ is (locally) Lipschitz or Hölder continuous.

**Remark 1.1.** It is not difficult to prove by similar arguments that the solution map is not uniformly continuous also in the subcritical case, i.e., from $H^s \times H^{s-1}$, $1 \leq s < n/2$ with values in $C([0,T], H^s)$ or $C^1([0,T], H^{s-1})$. However, it is already known, at least in the case of a rotationally symmetric target, that a much stronger ill posedness result holds, namely the local non-uniqueness can be proved. This was obtained for $n = 3$ in [9], for $n \geq 4$ in [10] and for $n = 2$ in [11]. Since the arguments in these results have a local nature, it is reasonable to argue that non uniqueness may hold also in the general nonsymmetric case.

**Remark 1.2.** The proof of the Theorem is based on an explicit construction of sequences of data such that the corresponding solutions violate (1.4); such solutions are of geodesic type, i.e., of the form $\gamma \circ v(t,x)$ with $v(t,x)$ a real valued solution of the homogeneous wave equation. We recall that if $\gamma(s) = (\gamma_1, ..., \gamma_m)$ is an arbitrary curve in $\mathbb{R}^m$ with values in $N$, and $v(t,x)$ an arbitrary real valued function, for the composition $u(t,x) = \gamma(v(t,x))$ we can write

$$\Box u^t + \Gamma_{bc}^t(u) \partial_{\alpha} u^c \partial^\alpha u^c = \gamma' \cdot \Box v + \gamma'' \gamma' \cdot \partial_{\alpha} v \partial^\alpha v$$

and this is identically zero as soon as $\Box v = 0$ and $\gamma(s)$ is a geodesic curve.

**Remark 1.3.** The ill posedness for the wave map problem in the case $n = 1$, $s = 1/2$ is proved in [12]. It is interesting to mention also the paper [13], where a scalar wave equation of the form

$$\Box u + f(u) \partial_{\alpha} u \partial^\alpha u = 0$$

is studied in the critical spaces $H^{n/2} \times H^{n/2-1}$. Of course in the scalar case it is possible to prove a much stronger ill posedness result (actually, a blow-up result).

**Remark 1.4.** If in addition to (1.4) we assume that the geodesic $\gamma(s)$ is defined for all $s \in \mathbb{R}$, i.e., that the target manifold is complete (by the Hopf-Rinow Theorem), and that the dimension of the base space is $n = 2$, then we can modify the proof of Theorem [12] in such a way to use radial solutions exclusively. This is interesting in connection with the recent result of Struwe [24], who proved global existence of smooth radial solutions to the wave map system from $\mathbb{R} \times \mathbb{R}^2$ to the two dimensional sphere. Thus in this case the solution map is well defined for smooth radial data, but not uniformly continuous in $H^1$. (Actually, by a more complex construction, which involves a localization in Fourier space, it is possible to construct a radial counterexample also under assumption (1.4) only). For a more precise statement we refer to the following proposition.

**Proposition 1.2.** Assume $n = 2$ and the target space $N$ is complete. Then the conclusion of Theorem [12] holds also if we restrict the solution map $\Phi$ to the subspace $H^1_{rad} \times L^2_{rad}$ of radial functions in $H^1 \times L^2$.

**Remark 1.5.** It is important to notice that in the proof of Theorem [12] the fact that $\Gamma_{jk}^i$ are Christoffel symbols of some Riemannian manifold is not essential. In
other words, the result holds for any system of the form (1.1), provided the curves locally defined by the system of equations
\[ \gamma'' + \Gamma^{\ell}_{\ell c}(\gamma)\gamma''_c = 0 \]
satisfy an assumption like (1.3) near some point. This means that the ill posedness in the sense of uniform continuity is a general property of systems of the wave map type.

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2. Proof of Theorem 1.1

It is not restrictive to assume that \( \gamma \) is parameterized by arc length; moreover, in the following we shall take \( T = 1 \) for simplicity of notations but the proof is unchanged in general. Assumption (1.5) implies that for some component \( \gamma''_\ell \) of \( \gamma'' \) one has
\[ |\gamma''_\ell(s)| \geq c_1(N) \quad \text{for} \quad |s| \leq c_0(N) \]
for suitable constants \( c_0, c_1 \) depending only on the manifold \( N \).

Let \( v, w \) be two \( C^\infty \) real valued solutions of the homogeneous wave equation
\[ \Box v = \Box w = 0 \]
with data
\[ v(0, x) = w(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x), \quad \partial_t w(0, x) = w_1(x). \]
Notice that \( v(0, x) \equiv w(0, x) \) and only the second datum is different. Moreover, we shall always work with data of compact support, so that \( v(t, \cdot), w(t, \cdot) \) will have support in a fixed ball (say \( B(0, 10) \)) for all \( t \in [-1, 1] \). Then the functions \( u^{(1)} = \gamma \circ v, u^{(2)} = \gamma \circ w \) are solutions of the wave map equation (see Remark 1.2), provided \( v, w \) take their values in the domain of \( \gamma(s) \); more precisely we shall assume that
\[ |v| \leq c_0(N), \quad |w| \leq c_0(N) \]
and these conditions will be verified in the explicit construction of \( v \) and \( w \). The corresponding Cauchy data are given by

\[ u^{(1)}(0) = u^{(2)}(0) = \gamma(v_0), \quad \partial_t u^{(1)}(0) = \gamma'(v_0)v_1, \quad \partial_t u^{(2)}(0) = \gamma'(v_0)w_1. \]

Assume now that the solution map is defined and uniformly continuous from some neighbourhood \( U \) of 0 in \( H^{n/2}(\mathbb{R}^n; N) \times H^{n/2-1}(\mathbb{R}^n; N) \) with values in the space \( C^1([0, 1]; H^{n/2-1}) \) (the case of \( C([0, 1]; H^{n/2}) \) is completely analogous). If we apply this to the data (2.3) we obtain that: for any \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \|\partial_t u^{(1)}(0) - \partial_t u^{(2)}(0)\|_{H^{n/2-1}} \leq \delta \quad \Rightarrow \quad \sup_{t \in [0, 1]} \|\partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot)\|_{H^{n/2-1}} \leq \epsilon \]
for all data \( u^{(1)}(0) = u^{(2)}(0) \) and \( \partial_t u^{(1)}(0), \partial_t u^{(2)}(0) \) in \( U \). We can express this condition in terms of the data for \( v, w \). Indeed, we have
\[ \partial_t u^{(1)}(0) - \partial_t u^{(2)}(0) = \gamma'(v_0)(v_1 - w_1), \]
where \( \gamma'(v_0) \) is a smooth function, equal to a constant outside some compact set in \( \mathbb{R}^n \). Applying Lemma 1.1 in the Appendix for \( s = n/2 - 1 \), we have
\[ \|\gamma(f)g\|_{H^{n/2-1}} \leq c_n \|\gamma(f)\|_{L^\infty \cap H^{n/2}} \cdot \|g\|_{H^{n/2-1}}, \]
where we are using the notation
\[ \| u \|_{X \cap Y} = \| u \|_X + \| u \|_Y; \]
since \( \gamma(0) = 0 \) we can apply the standard Moser type estimate
\begin{equation}
\| \gamma(f) \|_{H^{n/2}} \leq \rho_0(\| f \|_{L^\infty}) \cdot \| f \|_{H^{n/2}}
\end{equation}
for a suitable continuous increasing function \( \rho_0(s) \) (see e.g. \[37], Vol.III, Chapter 13, Proposition 10.2), we obtain an inequality like
\begin{equation}
\| \gamma(f)g \|_{H^{n/2} - 1} \leq \rho_1(\| f \|_{L^\infty \cap H^{n/2}}) \cdot \| g \|_{H^{n/2} - 1}
\end{equation}
for some continuous increasing \( \rho_1(s) \), which is valid provided the range of the real valued function \( f \) is contained in a compact subset of the domain of the smooth function \( \gamma(s) \). Then we have
\begin{equation}
\| \partial_t u^{(1)}(0) - \partial_t u^{(2)}(0) \|_{H^{n/2} - 1} \leq \rho_1(\| v_0 \|_{L^\infty \cap H^{n/2}}) \| v_1 - w_1 \|_{H^{n/2} - 1},
\end{equation}
hence property (2.4) implies the following: for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\begin{equation}
\| v_1 - w_1 \|_{H^{n/2} - 1} \leq \delta \Rightarrow \sup_{t \in [0,1]} \| \partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot) \|_{H^{n/2} - 1} \leq \epsilon
\end{equation}
for all data \((v_0, v_1)\) and \((w_0, w_1)\) belonging to a suitable neighbourhood \( V \) of 0 in \( H^{n/2}(\mathbb{R}^n) \times H^{n/2 - 1}(\mathbb{R}^n) \) and such that \( v_0 = w_0 \) and \( \| v_0 \| \leq c_0 \), where \( c_0 = c_0(N) \) is defined in (2.11).

We now estimate from below the second term in (2.8)
\[ \| \partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot) \|_{H^{n/2} - 1} = \| \gamma'(v) \partial_t v - \gamma'(w) \partial_t w \|_{H^{n/2} - 1}. \]
We have
\begin{equation}
\| \gamma'(v) \partial_t v - \gamma'(w) \partial_t w \|_{H^{n/2} - 1} \geq \| (\gamma'(v) - \gamma'(w)) \partial_t v \|_{H^{n/2} - 1} - \| \gamma'(v) \partial_t (v - w) \|_{H^{n/2} - 1}.
\end{equation}
We apply (2.7) to the last term, obtaining
\[ \| \gamma'(v) \partial_t (v - w) \|_{H^{n/2} - 1} \leq \rho_1(\| v \|_{L^\infty \cap H^{n/2}}) \| \partial_t (v - w) \|_{H^{n/2} - 1}; \]
by the energy identity for \( \Box (v - w) = 0 \), \( v_0 = w_0 \), we know that
\[ \| \partial_t (v - w) \|_{H^{n/2} - 1} \leq \| v_1 - w_1 \|_{H^{n/2} - 1} \]
and in conclusion
\begin{equation}
\| \gamma'(v) \partial_t (v - w) \|_{H^{n/2} - 1} \leq \rho_1(\| v \|_{L^\infty \cap H^{n/2}}) \| v_1 - w_1 \|_{H^{n/2} - 1}.
\end{equation}
To estimate from below the first term in the right side of (2.9) we use the Taylor developments
\[ \gamma'(b) - \gamma'(a) = \gamma''(a)(b - a) + F(a, b)(b - a)^2, \quad \gamma''(a) = \gamma''(0) + G(a) \cdot a \]
where \( F(a, b), G(a) \) are smooth functions of their arguments whose explicit expression is not relevant. Then
\[ \gamma'(v) - \gamma'(w) = \gamma''(0) \cdot (v - w) + R(v, w) \cdot (v - w) \]
where we have written for short
\[ R(u, v) = G(v) \cdot v + F(v, w)(v - w). \]
Recalling that \( |\gamma''(0)| \geq c_1 \) (see (2.1)), we have
\[ \| \gamma''(v - w) \partial_t v \|_{H^{n/2} - 1} \geq c_1 \| (v - w) \partial_t v \|_{H^{n/2} - 1} - \| R(v, w)(v - w) \partial_t v \|_{H^{n/2} - 1}. \]
Now we can apply (1.5) of Lemma 4.1 in the Appendix to obtain
\[ \| R(v, w)(v - w) \partial_t v \|_{H^{n/2} - 1} \leq \| R(v, w) \|_{L^\infty \cap H^{n/2}} \| (v - w) \partial_t v \|_{H^{n/2} - 1}. \]
while using (2.3) it is standard to obtain 
\[ \| R(v, w) \|_{L^\infty \cap H^{n/2}} \leq \rho_2(\|v, w\|_{L^\infty \cap H^{n/2}}) \cdot \|v, w\|_{L^\infty \cap H^{n/2}} \]
for some continuous increasing function \( \rho_2(s) \) whose precise form is not relevant. In conclusion, recalling also (2.9) and (2.10), we have proved the inequality
\[
\| \partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot) \|_{H^{n/2-1}} \geq c_1(\|v-w\|_{H^{n/2}}) \cdot \|v, w\|_{L^\infty \cap H^{n/2}} \cdot \|v-w\|_{H^{n/2}} - \rho_1(\|v, w\|_{L^\infty \cap H^{n/2}}) \cdot \|v-w\|_{H^{n/2-1}}.
\]
To proceed, we must construct explicitly the functions \( v \) and \( w \). This is done with the help of a few lemmas.

**Lemma 2.1.** Let \( n \geq 2 \). There exists a sequence of real valued functions \( \phi_j \in C_c^\infty(\mathbb{R}^n) \) supported in the ball \( \{|x| \leq 2\} \), with
\[
\phi_j \to 0 \quad \text{in} \quad H^{n/2-1}(\mathbb{R}^n) \quad \text{as} \quad j \to \infty
\]
such that, denoting by \( z_j(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) the solution of the linear problem
\[
\square z = 0, \quad z(0, x) = 0, \quad \partial_t z(0, x) = \phi_j(x)
\]
one has
\[
z_j(1, 0) = 1 \quad \text{for any} \quad j.
\]
The functions \( \phi_j \) and hence \( z_j(t, x) \) can be chosen as radial functions in \( x \), i.e., depending only on \( |x| \).

**Proof.** We begin by the case \( n = 2 \). For \( 0 < p < q < 1 \), we define \( \psi_{p,q}(y) \) on \( \mathbb{R}^2 \) as follows:
\[
\psi_{p,q}(y) = -\frac{I_{\{p \leq |y| \leq q\}}(y)}{\sqrt{1 - |y|^2} \log(1 - |y|^2)}
\]
where \( I_A(y) \) denotes the characteristic function of the set \( A \). An elementary computation gives
\[
2\|\psi_{p,q}(x)\|_{L^2(\mathbb{R}^2)}^2 = \frac{1}{\log(1 - q^2)} - \frac{1}{\log(1 - p^2)}.
\]
Notice that taking any \( 0 < p_1 < p < q < q_1 < 1 \), and an arbitrary smooth radial cutoff function \( \chi_{p,q} \) with
\[
I_{\{p \leq |y| \leq q\}}(y) \leq \chi_{p,q}(y) \leq I_{\{p' \leq |y| \leq q'\}}(y),
\]
we can modify definition (2.15) as follows:
\[
\tilde{\psi}_{p,q}(y) = -\frac{\chi_{p \leq |y| \leq q}(y)}{\sqrt{1 - |y|^2} \log(1 - |y|^2)}
\]
in order to obtain a smooth initial datum with similar norm:
\[
\frac{1}{\log(1 - q^2)} - \frac{1}{\log(1 - p^2)} \leq 2\|\tilde{\psi}_{p,q}(x)\|_{L^2(\mathbb{R}^2)} \leq \frac{1}{\log(1 - q_1^2)} - \frac{1}{\log(1 - p_1^2)}.
\]
On the other hand, the solution \( z_{p,q}(t, x) \) of the problem
\[
\square z = 0, \quad z(0, x) = 0, \quad \partial_t z(0, x) = \psi_{p,q}
\]
is explicitly given by
\[
z_{p,q}(t, x) = \frac{t}{2\pi} \int_{|x-y| \leq t} \frac{\psi_{p,q}(y)}{\sqrt{t^2 - |x-y|^2}} dy.
\]
and in particular at \((t, x) = (1, 0)\) one has
\[
z_{p,q}(1, 0) = -\frac{1}{2\pi} \int_{p \leq |y| \leq q} \frac{1}{(1 - |y|^2) \log(1 - |y|^2)} dy = \frac{1}{4\pi} \log \left| \log(1 - q^2) \right| \log \left( 1 - \frac{q^2}{p^2} \right).
\]
By the positivity of the kernel we have immediately, for the solution \(\tilde{z}_{p,q}\) obtained by replacing \(\psi_{p,q}\) with \(\tilde{\psi}_{p,q}\),
\[
\frac{1}{4\pi} \log \left| \log(1 - q^2) \right| \log \left( 1 - \frac{q^2}{p^2} \right) \leq \tilde{z}_{p,q}(1, 0) \leq \frac{1}{4\pi} \log \left| \log(1 - q^2) \right| \log \left( 1 - \frac{q^2}{p^2} \right).
\]
If we now choose for \(\delta \in ]0, 1[\)
\[
1 - p_1^2 = \delta, \quad 1 - p_2^2 = \delta^2, \quad 1 - q_1^2 = \delta^3, \quad 1 - q_2^2 = \delta^4
\]
and write \(\psi_{\delta} = \tilde{\psi}_{p,q}\), we obtain
\[
\frac{1}{\sqrt{12}} \| \log \delta \|^{-1/2} \leq \| \psi_{\delta} \|_{L^2} \leq \sqrt{\frac{3}{8}} \| \log \delta \|^{-1/2} \to 0 \quad \text{as } \delta \to 0
\]
while \(z_{\delta} = \tilde{z}_{p,q}\) satisfies
\[
\frac{1}{4\pi} \log \frac{3}{2} \leq z_{\delta}(1, 0) \leq \frac{1}{4\pi} \log 4.
\]
Defining \(\phi_{\delta} = z_{\delta}(1, 0)^{-1} \psi_{\delta}\), for any \(\delta_1 \downarrow 0\) we obtain the thesis.

The general case for even \(n\), starting from \(n = 3\) it follows easily by modifying the above example, using the fact that the solution of \((2.13)\) can be represented as
\[
z(t, x) = \sum_{0 \leq |\alpha| \leq (n-2)/2} a_\alpha t^{|\alpha|+1} \int_{|y| \leq 1} y^\alpha D^\alpha \phi(x + ty)(1 - |y|^2)^{-1/2} dy,
\]
which, for a radial function \(\phi\), gives
\[
z(1, 0) = \sum_{\nu = 0}^{(n-2)/2} c_\nu \int_0^1 \partial^\nu x \phi(r)(1 - r^2)^{-1/2} r^{\nu+n-1} dr.
\]
Here of course we shall choose a datum \(\phi\) such that its radial derivative of order \(n/2 - 1\) is of the form \(\tilde{\psi}_{p,q}\) seen above.

Let us now consider the case of odd \(n\), starting from \(n = 3\). In this case it is sufficient to use the well known fact (see e.g., Theorem 11.1 in Volume I of [16]) that, for any bounded \(\Omega \subset \mathbb{R}^n\) with \(C^\infty\) boundary,
\[
C^\infty_0(\Omega) \text{ is dense in } H^{1/2}(\Omega)
\]
and also in \(H^s\) for \(s \leq 1/2\). Since we shall need a special version of this result for radial functions, we shall give here a self-contained proof adapted to our situation.

Indeed, consider the space
\[
(2.19) \quad Z = \{ \phi \in C^\infty_0(\mathbb{R}) : \phi(x) = \phi(-x), \, \phi \equiv 0 \text{ near } 1 \text{ and } -1 \}
\]
(where “near \(\pm 1\)” means “on some neighbourhood of these two points, depending on \(\phi\)”). It is easy to see that \(Z\) is a dense subset of the space of even \(H^{1/2}(\mathbb{R})\) functions
\[
(2.20) \quad H^{1/2}_{\text{even}}(\mathbb{R}) = \{ u \in H^{1/2}(\mathbb{R}) : u(x) = u(-x) \}
\]
by the following argument: in the Hilbert space \(H^{1/2}_{\text{even}}(\mathbb{R})\) we can certainly choose a \(u_0\) orthogonal to \(Z\), and we must only prove that \(u_0 = 0\). The tempered distribution \(T\) whose Fourier transform is given by
\[
\hat{T} = \langle \xi \rangle \hat{u}_0
\]
belongs to $H^{-1/2}(\mathbb{R})$ and by the identity

$$T(\phi) = ((\xi)\hat{u_0}, \tilde{\phi})_{L^2} = (u_0, \overline{\phi})_{H^{1/2}} = 0$$

for any test function in $Z$, we see that the support of $T$ is contained in the set $\{\pm 1\}$, i.e., $T$ is a linear combination of a finite number of derivatives of $\delta_1, \delta_{-1}$. Hence $\hat{T}(\xi)$ is a function of the form

$$\hat{T}(\xi) = \sum_{\ell=0}^{N} (c_{\ell}e^{i\xi} + d_{\ell}e^{-i\xi})\xi^\ell$$

for a suitable $N \geq 0$ and complex numbers $c_{\ell}, d_{\ell}$, and at the same time $\langle \xi \rangle^{-1/2}\hat{T}(\xi)$ must belong to $L^2$. It is trivial to see that the only such function $\hat{T}$ is 0, and this implies $u_0 \equiv 0$ too.

Thus we have proved that $C_0^\infty([-1,1])$ is dense in $H^{1/2}_{\text{even}}([-1,1])$ since this last space coincides with the space of restrictions of functions in $H^{1/2}_{\text{even}}(\mathbb{R})$ to $[-1,1]$, with the restriction norm (the norm of $u$ is the infimum of the norms of its possible extensions).

It would not be difficult to prove the same result for higher dimensions, but actually here we only need to construct a sequence of radial smooth functions $\psi_j \in C_0^\infty(B_1)$ which converges to 1 in $H^{1/2}(B_1)$, where $B_1$ is the unit ball $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. To this end, it is sufficient to remark that the operator $A : H^{s}_{\text{even}}([-1,1]) \to H^s(B_1)$ defined as

$$A(f)(x) = f(|x|)$$

is bounded for all $0 \leq s \leq 1$: this is proved directly for $s = 0,1$ and follows e.g. by interpolation for the intermediate values of $s$. Hence taken any sequence $f_j(x) \in C_0^\infty([-1,1])$, with $f(x) = f(-x)$, converging to 1 in the $H^{1/2}([-1,1])$ norm, we need only define

$$\psi_j(x) = f_j(|x|)$$

to obtain the desired result.

Now, setting $\phi_j = 1 - \psi_j$ we obtain a sequence of radial smooth functions on $B_1$, converging to 0 in the norm of $H^{1/2}(B_1)$, and identically equal to 1 on some neighbourhood of $\partial B_1$ (depending on $j$). By Kirchhoff’s formula then we obtain

$$z(1,0) = \frac{1}{4\pi} \int_{\partial B_1} \phi_j(y) dS = 1$$

as needed.

In the general case $n \geq 3$ odd, we proceed in a similar way using the general representation of the solution; notice that for radial $\phi$ the following formula holds

$$z(1,0) = \sum_{\nu=0}^{(n-3)/2} b_\nu \partial_\nu^\nu \phi(1)$$

for suitable constants $b_\nu$. $\square$
In the construction of the Lemma we have no control on the $L^\infty$ norm of the functions $z_j$; if we give up the requirement that the $z_j$ be radial, however, it is easy to obtain the following result:

**Corollary 2.2.** Let $n \geq 2$. There exists a sequence of real valued functions $\phi_j \in C_0^\infty(\mathbb{R}^n)$ supported in the ball $\{ |x| \leq 5 \}$ with

$$\phi_j \to 0 \quad \text{in} \quad H^{n/2-1}(\mathbb{R}^n) \quad \text{as} \quad j \to \infty$$

such that, denoting by $z_j(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ the solution of the linear problem

$$\square z = 0, \quad z(0, x) = 0, \quad \partial_t z(0, x) = \phi_j(x)$$

one has

$$z_j(t_j, 0) = 1 \quad \text{for some sequence} \quad t_j \in [0, 1]$$

and

$$|z_j(t, x)| \leq 1 \quad \text{for all} \quad (t, x, j) \in [0, 1] \times \mathbb{R}^n \times \mathbb{N}.$$

**Proof.** The functions $z_j$ constructed in the Lemma are smooth and compactly supported, let $(t_j, x_j)$ be a point where $|z_j|$ attains its maximum value $m_j$ on the strip $[0, 1] \times \mathbb{R}^n$, and define

$$\tilde{z}_j(t, x) = m_j^{-1}z_j(t, x - x_j)$$

(and possibly multiply by the sign of $z_j(t_j, x_j)$). Notice that $t_j > 0$ since $z(0, x) \equiv 0$. This concludes the proof. □

Before passing to the main body of the proof, a last elementary rescaling lemma is necessary.

**Lemma 2.3.** Let $\chi(x)$ with $||\chi||_{H^{n/2-1}} \neq 0$ be a smooth compactly supported (radial) function, vanishing for $|x| \geq 2$, and with the property

$$\int_{\mathbb{R}^n} \chi(x)dx = 0.$$

Let $R \geq 1$, $M \geq 0$ be positive numbers, $0 \leq T \leq 1$, and denote by $v_{R, M, T}(t, x)$ the (radial) solution of the homogeneous wave equation

$$\square v = 0, \quad v(T, x) = 0, \quad \partial_t v(T, x) = \chi_{R, M}(x) \equiv M \cdot \chi(Rx)$$

with data at $t = T > 0$. Denote by $v_0, v_1$ the traces

$$v_0 = v_{R, M, T}(0, x), \quad v_1 = \partial_t v_{R, M, T}(0, x)$$

so that (2.26) is equivalent to a Cauchy problem for the homogeneous wave equation with data $v_0, v_1$ at $t = 0$. Then the following estimates hold, for a constant $c_n$ depending only on the space dimension $n$ and on the function $\chi(x)$:

$$||v_0||_{H^{n/2}} + ||v_1||_{H^{n/2-1}} \leq c_n \frac{M}{R},$$

and, for all $(t, x) \in [0, 1] \times \mathbb{R}^n$,

$$|v_{R, M, T}(t, x)| \leq c_n \frac{M}{R}.$$

Finally, for all $0 \leq s \leq n/2$ and all $t \in \mathbb{R}$

$$||v(t, \cdot)||_{H^s} + ||\partial_t v(t, \cdot)||_{H^{s-1}} \leq c_n \frac{M}{R} \cdot R^{s-n/2}.$$
Proof. Rescale $v(t, x)$ as
\[ v(t, x) = w(Rt, Rx) \]
so that
\[ \Box w = 0, \quad w(RT, x) = 0, \quad \partial_t w(RT, x) = \frac{M}{R} \chi(x). \]
By the energy estimates we have for all real $s$ and all $t \in \mathbb{R}$
\[ (2.31) \quad \|w(t, \cdot)\|_{\dot{H}^s} + \|\partial_t w(t, \cdot)\|_{\dot{H}^{s-1}} \leq 2 \frac{M}{R} \|\chi\|_{\dot{H}^{s-1}}. \]
which scaling back to $v$ gives
\[ \|v(t, \cdot)\|_{\dot{H}^s} + \|\partial_t v(t, \cdot)\|_{\dot{H}^{s-1}} \leq 2 \frac{M}{R} R^{s-n/2} \|\chi\|_{\dot{H}^{s-1}}. \]
Notice that $(2.31)$ gives a finite bound also for $s = 0$; indeed, by assumption $(2.25)$ we have $\hat{\chi}(0) = 0$ and hence $\hat{\chi}/|\xi| \in L^2$, i.e., $\chi \in H^{-1}$. Thus for all $0 \leq s \leq n/2$ we obtain ($R \geq 1$)
\[ \|v(t, \cdot)\|_{\dot{H}^s} + \|\partial_t v(t, \cdot)\|_{\dot{H}^{s-1}} \leq 2 \frac{M}{R} R^{s-n/2}. (\|\chi\|_{\dot{H}^{n/2-1}} + \|\chi\|_{\dot{H}^{-1}}). \]
This proves $(2.30)$; inequality $(2.28)$ is just the special case $s = n/2$ computed at $t = 0$.
To prove $(2.29)$ we use $(2.31)$ again for $s = n/2 + 1$, which gives
\[ \sup_{t \in \mathbb{R}} \|w(t, \cdot)\|_{\dot{H}^{n/2 + 1}} \leq c_n \frac{M}{R} \|\chi\|_{\dot{H}^{-1}}, \]
while for $s = 0$ it gives
\[ \sup_{t \in \mathbb{R}} \|w(t, \cdot)\|_{L^2} \leq c_n \frac{M}{R} \|\chi\|_{\dot{H}^{-1}}, \]
and this is bounded by $(2.23)$ as already remarked. Thus, by Sobolev embedding, we have
\[ \|w\|_{L^\infty(\mathbb{R} \times \mathbb{R}^n)} \leq c_n \sup_{t \in \mathbb{R}} \|w(t, \cdot)\|_{H^{n/2 + 1}} \leq c'_n \frac{M}{R} (\|\chi\|_{\dot{H}^{n/2}} + \|\chi\|_{\dot{H}^{-1}}). \]
Since $||v||_{L^\infty} = \|w\|_{L^\infty}$, this concludes the proof; the constant $c_n$ depends only on $n$ and the quantity $\|\chi\|_{\dot{H}^{n/2}} + \|\chi\|_{\dot{H}^{-1}}$. \qed

We revert now to the main proof. The next step is the explicit construction of sequences of functions $v, w$ appearing in $(2.11)$. As $v$ we shall choose the function $v_{R, M, T}$ constructed in the preceding lemma, with a suitable choice of the parameters. Notice that by $(2.28)$ we can assume that the initial data $v_0, v_1$ belong to the neighbourhood $V$ of 0 in $H^{n/2} \times H^{n/2-1}$ on which property $(2.5)$ holds, as soon as $M/R$ is small enough; e.g., if $V$ contains a ball of radius $r_0(V)$ centered in 0, we may assume that
\[ (2.32) \quad 4c_n \|\chi\|_{H^{n/2-1}} \frac{M}{R} < r_0. \]
Notice also that, in order to define the composition $\gamma \circ v$, we must ensure that $|v| < s_0$ (at least on the strip $[0, 1] \times \mathbb{R}^n$) since the geodesic curve is only defined on the interval $]-s_0, s_0[$, or even better, that $|v| < \gamma_0$ given by $(2.4)$. Using $(2.23)$, we see that it is sufficient to further decrease $M/R$, e.g., to impose the condition
\[ (2.33) \quad c_n \|\chi\|_{H^{n/2}} \frac{M}{R} < \gamma_0/2. \]
In connection with Remark $(1.4)$, we observe that condition $(2.33)$ is not necessary when we assume that $\gamma(s)$ is defined for all $s \in \mathbb{R}$. 

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Then we define \( v_j = v_{R,M,T} \) with the following choices. The parameter \( T \) will be chosen as
\[
T = t_j \quad \Rightarrow \quad v_{R,M,T}(t_j, x) = M\chi(Rx),
\]
where \( t_j \) are given by Corollary \( (2.2) \); the parameter \( R = R_j \) will be chosen such that
\[
z_j(t_j, x) \geq \frac{1}{2} \quad \text{on the ball} \quad \{ ||x| \leq 2R_j^{-1} \};
\]
this is possible in view of \((2.23)\) and of the continuity of \( z_j \); it is not restrictive to assume that \( R_j \uparrow +\infty \). On the parameter \( M = M_j \), besides \((2.32), (2.33)\) further smallness conditions will be imposed in the following.

We now define \( w_j \); let \( \mu > 0 \) be a small parameter, and set \((w_0 \equiv v_0 \) and\)
\[
w_j = v_j + \mu z_j
\]
where \( v_j \) was defined above and \( z_j \) is given by Corollary \((2.2)\). Thus the data for \( w_j \) are
\[
w_0 \equiv v_0, \quad w_1 = v_1 + \mu \phi_j
\]
with \( v_0, v_1 \) the traces of \( v_j \) at \( t = 0 \), studied in Lemma \((2.3)\). Again, in order to define the composition \( \gamma \circ w \), we must ensure that \( |w| < c_0 \), at least for \( 0 \leq t \leq 1 \).

Using \((2.24)\) and recalling \((2.29), (2.33)\), we see that it is sufficient to impose the condition
\[
0 < \mu < c_0/2.
\]
Notice that the data \( w_0, w_1 \) belong to the given neighbourhood \( V \) as soon as \( j \) is large enough, since \( \phi_j \to 0 \) in \( H_{n/2-1}^n \).

Consider inequality \((2.11)\); our aim is to estimate its right side from below. The first term at \( t = t_j \) gives
\[
\|(v_j - w_j)\partial_t v_j\|_{H_{n/2-1}} = \mu \|z_j(t_j, \cdot)\chi_{R_j,M_j}\|_{H_{n/2-1}};
\]
we can apply \((4.12)\) of the Appendix, with \( s = n/2 - 1 \); since \( z_j \geq 1/2 \) on the support of \( \chi_{R,M} \), we have
\[
\|z_j(t_j, \cdot)\chi_{R_j,M_j}\|_{H_{n/2-1}} \geq \frac{c}{2}\|\chi_{R_j,M_j}\|_{H_{n/2-1}} - c'\|z_j\|_{H_{n/2}}\|\chi_{R_j,M_j}\|_{H_{n/2-1}}.
\]
Now we have for \( R \) large enough
\[
\|\chi_{R,M}\|_{H_{n/2-1}} = \frac{M}{R^{\kappa}}, \quad \|\chi_{R,M}\|_{H_{n/2-1}} \leq 2\frac{M}{R^{\kappa}},
\]
where by assumption
\[
\kappa = \|\chi\|_{H_{n/2-1}} \neq 0.
\]
Moreover, by the energy identity we have for all \( t \)
\[
\|z_j(t, \cdot)\|_{H_{n/2}} \leq c\|\phi_j\|_{H_{n/2-1}} \to 0 \quad \text{as} \quad j \to \infty
\]
and also for all \( |t| \leq 1 \)
\[
\|z_j(t, \cdot)\|_{L^2} = \left\| \frac{\sin(t\xi)}{\xi} \phi_j \right\|_{L^2} \leq \|\phi_j\|_{L^2} \to 0 \quad \Rightarrow \quad \|z_j(t, \cdot)\|_{H_{n/2}} \to 0.
\]

Hence we have proved that
\[
\|(v_j - w_j)\partial_t v_j\|_{H_{n/2-1}} \geq c(n, \kappa) \frac{M_j}{R_j}.
\]

In view of the second term in \((2.11)\) we need also a bound from above for the quantity \( (v - w)\partial_t v \); by \((4.9)\) from the Appendix, with \( s = n/2 - 1 \), we have
\[
\|z_j(t_j, \cdot)\chi_{R_j,M_j}\|_{H_{n/2-1}} \leq C\|z_j\|_{L^2 \cap H_{n/2}}\|\chi_{R_j,M_j}\|_{H_{n/2-1}} \leq C \frac{M_j}{R_j} \|z_j\|_{L^2 \cap H_{n/2}},
\]
for $j$ large enough, and recalling that $z_j \to 0$ in $H^{n/2}$ uniformly in $|t| \leq 1$ as remarked above, and $|z_j| \leq 1$ by construction, we finally obtain

\begin{equation}
(2.42) \quad \|(v_j - w_j)\partial_t v_j\|_{H^{n/2-1}} \geq c'(n, \kappa) \frac{M_j}{R_j}
\end{equation}

provided $j$ is large enough.

We notice that, by (2.29), (2.30),

\[ \|v_j\|_{L^\infty \cap H^{n/2}} \leq c \frac{M_j}{R_j} \]

while, recalling that $|z_j| \leq 1$ and that $\|z_j\|_{H^{n/2}} \leq 1$ for $j$ large enough, we have

\[ \|w_j\|_{L^\infty \cap H^{n/2}} = \|v_j + \mu z_j\|_{L^\infty \cap H^{n/2}} \leq \epsilon M_j + c \frac{M_j}{R_j} \]

Together with (2.42) this gives us the following estimate for the second term in (2.11):

\begin{equation}
(2.43) \quad \rho_2(\|v_j, w_j\|_{L^\infty \cap H^{n/2}}) \|v_j, w_j\|_{L^\infty \cap H^{n/2}} \|(v_j - w_j)\partial_t v_j\|_{H^{n/2-1}} \leq \\
\leq \rho_3(\mu + M_j/R_j) \cdot \left( \mu + \frac{M_j}{R_j} \right) \frac{M_j}{R_j}.
\end{equation}

We can impose now the last smallness condition on $\mu$ and $M_j$ (recall that $M_j/R_j$ is bounded):

\begin{equation}
(2.44) \quad \rho_3(\mu + M_j/R_j) \cdot \left( \mu + \frac{M_j}{R_j} \right) \leq \frac{1}{2} c(n, \kappa)
\end{equation}

where $c(n, \kappa)$ is the constant appearing in (2.43). Thus we get

\begin{equation}
(2.45) \quad \rho_2(\|v_j, w_j\|_{L^\infty \cap H^{n/2}}) \|v_j, w_j\|_{L^\infty \cap H^{n/2}} \|(v_j - w_j)\partial_t v_j\|_{H^{n/2-1}} \leq \frac{1}{4} c(n, \kappa) \frac{M_j}{R_j}.
\end{equation}

The last term in (2.11) is quite easy to estimate: we have for $j \to \infty$

\begin{equation}
(2.46) \quad \rho_1(\|v\|_{L^\infty \cap H^{n/2}}) \|v_j - w_j\|_{H^{n/2-1}} \leq \rho_4(\mu + M_j/R_j) \cdot \mu \phi_j\|_{H^{n/2-1}} \to 0.
\end{equation}

We can finally choose $M_j$ and $\mu$; we define $M_j = \lambda \cdot R_j$, and $\lambda, \mu$ are two positive constants so small that conditions (2.32), (2.33), (2.44) are satisfied.

Summing up, by (2.41), (2.43), (2.46), we obtain

\begin{equation}
(2.47) \quad \|\partial_t u^{(1)}(t_j, \cdot) - \partial_t u^{(2)}(t_j, \cdot)\|_{H^{n/2-1}} \geq \frac{1}{4} c(n, \kappa) \frac{M_j}{R_j} = \frac{1}{4} c(n, \kappa) \cdot \lambda.
\end{equation}

provided $j$ is large enough.

We can now conclude the proof. Recalling (2.27), we can choose as data for $v$ the sequences

\[ v_0^{(j)} = v_{R_j, M_j, t_j}(0, x), \quad v_1^{(j)} = \partial_t v_{R_j, M_j, t_j}(0, x) \]

while the data for $w$ are chosen as

\[ w_0^{(j)} = v_0^{(j)}, \quad w_1^{(j)} = v_1^{(j)} + \mu \phi_j \equiv v_1^{(j)} + c_0 \phi_j / 2. \]

By (2.33) the data for $v$ belong to $V$; as a consequence, the data for $w$ belong to $V$ provided $j$ is large enough, since $\phi_j \to 0$ in $H^{n/2-1}$. Thus we are in position to apply the uniform continuity property (2.8); since $w_1 - v_1 = \mu \phi_j / 2$ we have that for all $\epsilon > 0$ there exists $\delta > 0$ such that

\[ \|\phi_j\|_{H^{n/2-1}} < \delta \Rightarrow \sup_{t \in [0,1]} \|\partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot)\|_{H^{n/2-1}} \leq \epsilon; \]

hence in particular at $t = t_j$ we must have

\[ \|\phi_j\|_{H^{n/2-1}} < \delta \Rightarrow \|\partial_t u^{(1)}(t_j, \cdot) - \partial_t u^{(2)}(t_j, \cdot)\|_{H^{n/2-1}} \leq \epsilon \]
3. Proof of Proposition 1.2

The proof follows exactly the same lines as for Theorem 1.1, and actually it is simpler from a technical point of view. Indeed, when $n = 2$ we must violate the following uniform continuity condition: for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$
\|\partial_t u^{(1)}(0) - \partial_t u^{(2)}(0)\|_{L^2} \leq \delta \quad \Rightarrow \quad \sup_{t \in [0,1]} \|\partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot)\|_{L^2} \leq \epsilon
$$

for all data $u^{(1)}(0) = u^{(2)}(0)$ and $\partial_t u^{(1)}(0)$, $\partial_t u^{(2)}(0)$ in $U$. We choose as above two $C^\infty$ real valued solutions $v, w$ of the homogeneous wave equation

$$
\Box v = \Box w = 0
$$

with data

$$
v(0, x) = w(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x), \quad \partial_t w(0, x) = w_1(x),
$$

and we set $u^{(1)} = \gamma \circ v$, $u^{(2)} = \gamma \circ w$. Since $|\gamma'(s)| = 1$ by the choice of the length parameter, the uniform continuity for our choice of data becomes simply

$$
\|v_1 - w_1\|_{L^2} \leq \delta \quad \Rightarrow \quad \sup_{t \in [0,1]} \|\partial_t u^{(1)}(t, \cdot) - \partial_t u^{(2)}(t, \cdot)\|_{L^2} \leq \epsilon
$$

In order to violate this property, we estimate from below the second term in (3.1).

We can write

$$
\partial_t u^{(1)} - \partial_t u^{(2)} = \gamma'(v)v_t - \gamma'(w)w_t = (\gamma'(v) - \gamma'(w))w_t + \gamma'(w) \cdot \partial_t (v-w)
$$

whence

$$
|\partial_t u^{(1)} - \partial_t u^{(2)}| \geq |\gamma'(v) - \gamma'(w)| \cdot |w_t| - |\partial_t (v-w)|
$$

using the identity $|\gamma'| \equiv 1$. This implies easily

$$
|\partial_t u^{(1)} - \partial_t u^{(2)}|^2 + |\partial_t (v-w)|^2 \geq \frac{1}{2} |\gamma'(v) - \gamma'(w)|^2 \cdot |v_t|^2,
$$

which can be written

$$
|\partial_t u^{(1)} - \partial_t u^{(2)}|^2 + |\partial_t (v-w)|^2 \geq \frac{1}{2} \int_v^w \gamma''(\sigma)d\sigma \cdot |v_t|^2.
$$

As $v$ we shall choose the radial function $v_{R,M,T}$ constructed in Lemma 2.3, with a suitable choice of the parameters. By (2.28) for $n = 2$ we can assume that the initial data $v_0, v_1$ belong to the neighbourhood $V$ of $0$ in $H^1 \times L^2$ on which property (2.8) holds, as soon as $M/R$ is small enough; if $V$ contains a ball of radius $r_0(V)$ centered in $0$, we may assume that

$$
4\|\chi\|_{L^2} \frac{M}{R} < r_0.
$$

Notice that, thanks to the assumption that the geodesic curve is globally defined, it is not necessary to impose any restriction to the $L^\infty$ norm of $v$.

We now choose the data for $w$: let $\mu > 0$ be a small parameter, and set ($w_0 \equiv v_0$ and)

$$
w_1 = v_1 + \mu \phi_j
$$

where $\phi_j$ are the smooth radial functions constructed in Lemma 2.1. Then we have $w = v + \mu z_j$, with $v \equiv v_{R,M,T}$; again, no condition on the $L^\infty$ norm of $w$ is necessary since $\gamma(s)$ is defined for all $s$.

Recall now (3.2) which gives

$$
|\partial_t u^{(1)} - \partial_t u^{(2)}|^2 + |\mu \partial_z z_j|^2 \geq \frac{1}{2} \int_v^{v+\mu z_j} \gamma''(\sigma)d\sigma \cdot |v_t|^2.
$$
Notice that this is a pointwise inequality, valid at any \((t,x)\). We can fix now \(t = 1\), choose \(T = 1\) in the definition of \(v_{R,M,T}\) while leaving \(R, M\) free for the moment (apart from (3.3)), and we get

\[
|\partial_t(u^{(1)} - u^{(2)})(1,x)|^2 + |\mu \partial_t z_j(1,x)|^2 \geq \frac{1}{2} \int_0^{\mu z_j(1,x)} \gamma''(\sigma) d\sigma \cdot |\chi_{R,M}|^2.
\]

Integrating on \(\mathbb{R}^2\), and using the energy inequality \(\|\partial_t z_j(t, \cdot)\|_{L^2} \leq \|\phi_j\|_{L^2}\) we get

\[
(\|\cdot\| = \|\cdot\|_{L^2})
\]

\[
\|\partial_t(u^{(1)} - u^{(2)})(1,\cdot)\|^2 + \|\mu \phi_j\|^2 \geq \frac{1}{2} \int_0^{\mu z_j(1,\cdot)} \gamma''(\sigma) d\sigma \cdot \chi_{R,M}^2.
\]

Since \(z_j\) is smooth and satisfies (2.14), we also have

\[
2 > z_j(t_j, x) > 1/2 \quad \text{for} \quad |x| \leq \frac{1}{R_j}
\]

for some \(R_j\) large enough; this is our choice for the parameter \(R = R_j\) in the definition of \(v_{R,M,T}\). Moreover, we shall choose \(M = M_j\) proportional to \(R_j\), in such a way that (3.3) is satisfied, i.e.,

\[
M_j = \frac{M_j}{R_j} = \frac{\lambda_0}{s|\chi|_{L^2}}.
\]

Recalling (2.3), we can write for \(|s| < c_0(N)\)

\[
\int_0^s \gamma''(\sigma) d\sigma \geq |s| \cdot c_1(N)
\]

with \(c_1(N) > 0\). If we choose

\[
\mu = c_0(N)/2,
\]

by (3.8) we obtain

\[
\int_0^{\mu z_j(1,x)} \gamma''(\sigma) \cdot \chi_{R_j,M_j}(x) \geq \frac{1}{4} c_1(N) c_0(\lambda)|\chi_{R_j,M_j}(x)| \quad \text{for} \quad |x| \leq \frac{1}{R_j}.
\]

By (3.12) and (3.7) we thus get, recalling (3.9),

\[
\|\partial_t(u^{(1)} - u^{(2)})(1,\cdot)\|^2 + \frac{c_2}{4} \|\mu \phi_j\|^2 \geq \frac{1}{8} (c_0 c_1)^2 \|\chi_{R_j,M_j}\|_{L^2}^2 (|x| < 1/R_j) = c_3
\]

where the constant \(c_3\) is given by

\[
c_3 = \frac{1}{16} (c_0 c_1 \lambda_0 \|\chi\|_{L^2})^2
\]

and is independent of \(j\) (\(|\chi_{M_j,R_j}\|_{L^2} = M_j R_j^{-1} \|\chi\|_{L^2} = \lambda_0 \|\chi\|_{L^2}\)).

The conclusion of the proof is now quite similar to the general case \(n \geq 2\); like before, we choose as data for \(v\) the sequences

\[
v_0^{(j)}(x) = v_{R_j,M_j,1}(0,x), \quad v_1^{(j)} = \partial_t v_{R_j,M_j,1}(0,x)
\]

while the data for \(w\) are

\[
w_0^{(j)} = w_0^{(j)}, \quad w_1^{(j)} = w_1^{(j)} + \mu \phi_j = v_1^{(j)} + c_0 \phi_j/2.
\]

By (3.5) the data for \(v\) belong to \(V\); on the other hand, the data for \(w\) belong to \(V\) provided \(j\) is large enough, since \(\phi_j \to 0\) in \(L^2\). The uniform continuity property implies that for all \(\epsilon > 0\) there exists \(\delta > 0\) such that

\[
\|\phi_j\|_{L^2} < \delta \quad \Rightarrow \quad \|\partial_t u^{(1)}(1,\cdot) - \partial_t u^{(2)}(1,\cdot)\|_{L^2} \leq \epsilon
\]

and this contradicts (3.13).
4. Appendix

The aim of this Appendix is to prove two multiplicative estimates needed in the proof of Theorem 4.1. The first one has the following form:

\[
\|fg\|_{H^s(\mathbb{R}^n)} \leq C\|f\|_{H^s} \cdot \|g\|_{L^\infty; H^{s/2}}, \quad s < n/2.
\]

Notice that this estimate is asymmetric in \(f, g\). We can obtain this estimate from the Kato-Ponce estimate (see Lemma 2.2 in [12])

\[
\|fg\|_{H^s} \leq C\|f\|_{L^{p_1}} \cdot \|J^s g\|_{L^{p_2}} + C\|J^s f\|_{L^{p_3}} \cdot \|g\|_{L^{p_4}}
\]

which is valid for all \(s \geq 0\), for all \(p_2, p_3 \in ]1, \infty[\), and \(1/2 = p_1^{-1} + p_2^{-1} = p_3^{-1} + p_4^{-1}\); here \(J^s = (1 - \Delta)^{s/2}\). Then (4.1) follows taking \(p_3 = 2\), \(p_4 = \infty\),

\[
p_1 = \frac{2n}{n - 2s}, \quad p_2 = \frac{n}{s}
\]

and using the Sobolev embeddings

\[
\|f\|_{L^{p_1}} \leq C\|f\|_{H^s}, \quad \|J^s g\|_{L^{p_2}} \leq C\|g\|_{H^{s/2}}.
\]

Also the second commutator estimate we need, i.e.,

\[
\|J^s(fg) - gJ^s f\|_{L^2} \leq C\|f\|_{H^s} \cdot \|g\|_{H^{s/2}}, \quad s < n/2, \quad n \geq 3,
\]

can be proved by a similar argument based on the the Kato-Ponce commutator estimate (see Lemma 2.2 in [12])

\[
\|J^s(fg) - gJ^s f\|_{L^2} \leq C\|\nabla g\|_{L^{p_1}} \cdot \|J^{s-1} f\|_{L^{p_2}} + C\|J^s g\|_{L^{p_3}} \cdot \|f\|_{L^{p_4}}
\]

which is valid for all \(s \geq 0\), for all \(p_2, p_3 \in ]1, \infty[\), and \(1/2 = p_1^{-1} + p_2^{-1} = p_3^{-1} + p_4^{-1}\). Now (4.2) follows taking \(p_1 = n, p_3 = s/n,\)

\[
p_2 = \frac{2n}{n - 2}, \quad p_4 = \frac{n}{n - 2s}
\]

and using the Sobolev embeddings

\[
\|\nabla g\|_{L^{p_1}} \leq C\|g\|_{H^{s/2}}, \quad \|J^{s-1} f\|_{L^{p_2}} \leq C\|f\|_{H^s},
\]

\[
\|J^s g\|_{L^{p_3}} \leq C\|g\|_{H^{s/2}}, \quad \|f\|_{L^{p_4}} \leq C\|f\|_{H^s}.
\]

For completeness, we give a self-contained proof of (4.1), (4.2) and a refined version of (4.2) involving homogeneous Sobolev norms; we hope that our method is of independent interest.

To this end, we must introduce some basic tools from the theory of Sobolev and Besov spaces.

1) Difference operators. Given \(h \in \mathbb{R}^n\) and a function \(f : \mathbb{R}^n \to \mathbb{C}\), we denote by \(f_j(x)\) the \(j\)-th translate of \(f\) in the direction \(h\):

\[
f_j(x) = f(x + j \cdot h), \quad j \in \mathbb{Z}
\]

and the difference operator \(\Delta_h = \Delta\) defined as

\[
\Delta f = f_1 - f, \quad \text{i.e.,} \quad \Delta f(x) = f(x+h) - f(x)
\]

We denote by \(\Delta^j\) the iterates of \(\Delta\). Trivial properties are \(f_0 \equiv f\), \((f_i)_j = f_{i+j}\),

\[
\Delta^1(\Delta^j f) = \Delta^{j+1} f, \quad \Delta(f_j) = (\Delta f)_j \equiv \Delta f_j.
\]

Of special interest here will be the behaviour of the difference operator with respect to products. We have immediately

\[
\Delta(fg) = f_1g_1 - fg = f_1(g_1 - g) + (f_1 - f)g
\]

which can be written shortly

\[
\Delta(fg) = \Delta f \cdot g + f_1 \cdot \Delta g.
\]
By induction one proves easily the Leibnitz rule

\[(4.3) \quad \Delta^k(fg) = \sum_{\ell+m=k} \binom{k}{\ell} \Delta^\ell f \Delta^m g.\]

2) Sobolev spaces with fractional index. All the functions (and the spaces) considered here are defined on the whole \( \mathbb{R}^n \). The \textit{homogeneous Sobolev seminorms} \( W^{k,p} \) with \( k \geq 0 \) integer, \( 1 < p < \infty \) are defined as

\[\|u\|_{W^{k,p}} = \sum_{\alpha = k} \|D^\alpha u\|_{L^p};\]

we write \( \dot{H}^k \) for \( W^{k,2} \). Thus the standard Sobolev norms can be written

\[\|u\|_{W^{k,p}} = \|u\|_{L^p} + \|u\|_{\dot{W}^{k,p}}.\]

For our purposes it is not necessary to enter into the topological details of the definition of the corresponding spaces; only the norms are sufficient, and we shall always apply them to smooth functions. The \( \dot{W}^{s,p}, W^{s,p} \) (semi)norms with noninteger \( s > 0 \) are more troublesome; the usual definition by interpolation is not well suited to prove multiplicative estimates. A handier equivalent characterization can be given using the fractional integrals

\[I_{s,p}(u) = \left( \int \frac{[\Delta_{\alpha}^n + 1]u(x)^p}{|h|^{n+sp}} dxdh \right)^{1/p}\]

where \( [s] \) is the integer part of the noninteger \( s > 0 \), \( 1 < p < \infty \), and integration is performed over \( \mathbb{R}^{2n} \); we shall write \( I_{s,2} = I_s \). Then we have

\[(4.4) \quad \|u\|_{\dot{W}^{s,p}} \simeq \|u\|_{L^p} + I_{s,p}(u)\]

(see e.g. 2.3.1 and Theorem 2.5.1 in [31]). The integral \( I_{s,p}(u) \) plays the role of the homogeneous norm; this can be seen by a simple rescaling argument. For the following application it will be sufficient to consider the \( L^2 \) case, in which we have a simple definition using the Fourier transform

\[\|u\|_{\dot{H}^s} = \|\langle \xi \rangle^s \hat{u}\|_{L^2}, \quad \|u\|_{H^s} = \|(1 + |\xi|^2)^{s/2} \hat{u}\|_{L^2} \simeq \|u\|_{L^2} + \|u\|_{\dot{H}^s}.\]

Indeed, let \( S_\lambda \) be the scaling operators for \( \lambda > 0 \)

\[(S_\lambda u)(x) = u(\lambda x);\]

it is easy to check the scaling properties

\[\|S_\lambda u\|_{L^2} = \lambda^{-n/2}\|u\|_{L^2}, \quad \|S_\lambda u\|_{\dot{H}^s} = \lambda^{s-n/2}\|u\|_{\dot{H}^s} \quad I_s(S_\lambda u) = \lambda^{2s-n}I_s(u).\]

Thus, fixed \( u \in H^s \), if we apply the two equivalent definitions to \( S_\lambda u \) we obtain

\[\lambda^{s-n/2}\|u\|_{\dot{H}^s} + \lambda^{-n/2}\|u\|_{L^2} \simeq \lambda^{s-n/2}I_s(u) + \lambda^{-n/2}\|u\|_{L^2}.\]

Letting \( \lambda \to \infty \), we obtain immediately

\[(4.5) \quad \|u\|_{\dot{H}^s} \simeq I_s(u).\]

3) Besov spaces. With the same type of norms it is possible to define the \textit{Besov spaces} \( B^{s}_{p,q} \) as follows (see Theorem 2.5.1 in [31]): for any \( s > 0 \), \( 1 < p < \infty \), \( 1 < q < \infty \) set

\[(4.6) \quad \|f\|_{B^{s}_{p,q}} = \|f\|_{L^p} + \left( \int \frac{[\Delta_{\alpha}^n + 1]^q}{|h|^{n+sq}} dxdh \right)^{1/q}\]

and define the spaces accordingly. From this definition in particular it is evident that \( B^{s}_{p,p} = W^{s,p} \) for noninteger \( s \). We shall use the fact that

\[B^{s}_{2,2} \equiv H^s\]
for all values of $s$ (including integers).

We finally recall the continuous embedding (see e.g., Theorem 7.58 in [1] and Theorem 2.8.1 in [31]): for $s,t \geq 0$

$$1 < p \leq q < \infty, \quad s - \frac{n}{p} = t - \frac{n}{q} \quad \Rightarrow \quad W^{s,p} \subseteq W^{t,q}$$

and, more generally, the Besov version

$$r \in [1,\infty], \quad 1 < p \leq q < \infty, \quad s - \frac{n}{p} = t - \frac{n}{q} \quad \Rightarrow \quad B^{s,p}_{r} \subseteq B^{t,q}_{r}.$$

We are ready to prove our lemma. We use the notation

$$\|u\|_{X \cap Y} = \|u\|_{X} + \|u\|_{Y}$$

for any two Banach spaces $X,Y$ and $u \in X \cap Y$. We state the following Lemma for smooth functions, the extension to $f,g$ belonging to the appropriate spaces being obvious.

**Lemma 4.1.** For all real $0 \leq s < n/2$ and any smooth functions $f,g$, the following inequality holds:

$$\|fg\|_{H^{s}(\mathbb{R}^{n})} \leq C \|f\|_{H^{s}} \cdot \|g\|_{L^{\infty}}$$

and, for all $\lambda$ with $s < \lambda < n/2$,

$$\|fg\|_{H^{\lambda}(\mathbb{R}^{n})} \leq C \|f\|_{H^{n/2 + \lambda}} \cdot \|g\|_{H^{\lambda}}.$$  

Moreover, assume that

$$|g(x)| \geq C_{1} > 0 \quad \text{on the support of } f;$$

then we have also

$$\|fg\|_{H^{s}} \geq cC_{1} \|f\|_{H^{s}} - c' \|f\|_{H^{s}} \cdot \|g\|_{H^{n/2}}$$

for some constants $c,c' > 0$ depending only on $s,n$.

**Remark 4.1.** Estimate (4.9) can be regarded as the limit case of (4.10) as $\lambda \to n/2$; when $\lambda \to s$ we obtain (4.9) with $f$ and $g$ exchanged.

**Proof.** Notice that in order to prove (4.9), (4.10) it is sufficient to prove the estimate with the $H^{s}$ norm on the left hand side replaced by the homogeneous $H^{s}$ norm, since the estimates are trivially true for the term $\|fg\|_{L^{2}}$. We need two different (but parallel) proofs in the cases $s$ integer or noninteger, since we have two different representations of the norm in these cases.

The proof for integer $s$ is simple. Indeed, by the Sobolev embedding

$$H^{s}(\mathbb{R}^{n}) \subseteq L^{\frac{2n}{n-2s}}(\mathbb{R}^{n}), \quad \forall\ 0 \leq s < \frac{n}{2}$$

(see (4.7)) and by Hölder’s inequality we have

$$\|uv\|_{L^{2}} \leq \|u\|_{L^{n/\mu}} \|v\|_{L^{2n/(n-2\mu)}} \leq C \|u\|_{H^{n/2 - \mu}} \|v\|_{H^{\mu}}$$

for any real number

$$0 < \mu < \frac{n}{2}.$$

We can apply (4.13) to the product of two derivatives (here and in the following we shall use the shorthand notation $D^{\ell}$ to denote any derivative of order $\ell$):

$$\|D^{\ell}f D^{m}g\|_{L^{2}} \leq C \|f\|_{H^{n/2 + \ell - \mu}} \|g\|_{H^{m + \mu}}.$$

For any integer $s < n/2$ we can write

$$\|fg\|_{H^{s}} \simeq \sum_{\ell + m = s} \|D^{\ell}f D^{m}g\|_{L^{2}}.$$
Now, (4.10) follows directly by applying (4.14) to each term $0 \leq \ell < n/2$ with the choice $\mu = \ell + \lambda - s$, since in this case we have $0 < \mu < n/2$ for all $\ell = 0, \ldots, s$. To prove the limit case (4.9), i.e., with $\lambda = n/2$, the same methods works if we choose for $\ell = 0, \ldots, s - 1$

$$\mu = \ell + n/2 - s$$
and (4.14) gives

$$\|D^\ell f D^m g\|_{L^2} \leq \|f\|_{H^\mu} \|g\|_{H^{n/2}};$$
but we must consider the term with $\ell = s$ separately since $\mu = n/2$ in that case, and we have

$$\|D^s f g\|_{L^2} \leq \|g\|_{L^\infty} \|f\|_{H^\mu}$$
and this concludes the proof.

Consider now (4.12) for $s < n/2$ integer; by (4.14) we have

$$\|f g\|_{H^s} \geq \|g D^s f\|_{L^2} - c \sum_{\ell + m = s + 1} \|D^\ell f D^m g\|_{L^2}$$
and applying (4.14) to each term in the sum, with $\mu = n/2 - \ell + s$ as above (so that $0 < \mu < n/2$), we obtain

$$\|f g\|_{H^s} \geq \|g D^s f\|_{L^2} - c \|g\|_{H^{n/2}} \|f\|_{H^s}.$$ 

Recalling (4.11), we obtain (4.12).

From now on, assume $0 < s < n/2$ is not an integer. To estimate from above $\|f g\|_{H^s}$, we use the characterization (4.15) and the Leibnitz rule (4.13):

$$\|f g\|_{H^s} \simeq I_s(f g) \leq C \sum_{\ell + m = [s]+1} \left(\int \frac{\Delta^\ell f_m \Delta^m g}{|x|^{n+2s}} |dx| \right)^{1/2}.$$ 

Thus we need an analogue of (4.14) for fractional integrals. Consider first the terms with both $\ell \geq 1$ and $m \geq 1$. By Hölder’s inequality we can write for any $p^{-1} + q^{-1} = 1$ and any $\rho + \sigma = s$

$$\int \frac{\Delta^\ell f_m \Delta^m g}{|x|^{n+2s}} |dx| \leq \left(\int \frac{|\Delta^\ell f|^2}{|x|^{n+2\rho}} |dx| \right)^{1/p} \left(\int \frac{|\Delta^m g|^{2q}}{|x|^{n+2\sigma}} |dx| \right)^{1/q}$$

where we replaced $f_m$ with $f$ after a translation in the variable $x$. The parameters $p, q, \rho, \sigma$ must be chosen in an appropriate way. First of all we can set (since $\ell, m \geq 1$)

$$\rho = \ell - \frac{1}{2} + \frac{\{s\}}{2}, \quad \sigma = m - \frac{1}{2} + \frac{\{s\}}{2}$$
where $\{s\} = s - [s]$ is the fractional part of $s$, so that

$$\rho + \sigma = s, \quad \ell = [\rho] + 1, \quad m = [\sigma] + 1.$$ 

Recalling the definition of $I_{\rho, p}$, from (4.17) we thus obtain

$$\left(\int \frac{\Delta^\ell f_m \Delta^m g}{|x|^{n+2s}} |dx| \right)^{1/2} \leq I_{\rho, 2p}(f) \cdot I_{\sigma, 2q}(g) \leq c\|f\|_{W^{\rho, 2p}} \cdot \|g\|_{W^{\sigma, 2q}}.$$ 

Now let $\lambda$ be such that $s \leq \lambda \leq n/2$ (extreme cases included) and choose $p, q \in ]1, \infty[\)$ as follows

$$2p = \frac{n}{\lambda - \sigma}, \quad 2q = \frac{2n}{n - 2(\lambda - \sigma)};$$
notice that $\lambda - \sigma \geq s - \sigma > 0$ and $n - 2(\lambda - \sigma) \geq 2\sigma > 0$. Thus by (4.7) we have the embeddings

$$H^{n/2+s-\lambda} \subseteq W^{\rho, 2p}, \quad H^\lambda \subseteq W^{\sigma, 2q}. $$
In conclusion we have proved for all \( \ell, m \geq 1 \) with \( \ell + m = [s] + 1 \), and any \( s \leq \lambda \leq n/2 \), the inequality

\[
\left( \int \left| \frac{\Delta \Delta f_m g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} \leq c \| f \|_{H^{n/2-\lambda+s}} \cdot \| g \|_{H^\lambda}.
\]

Two terms are left. The term with \( m = 0, \ell = [s] + 1 \) is bounded simply by writing

\[
\left( \int \left| \frac{g \Delta [s+1] g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} \leq \| g \|_{L^\infty} \cdot I_s(f) \leq c \| g \|_{L^\infty} \| f \|_{H^{s}}.
\]

On the other hand, the term with \( \ell = 0 \) and \( m = [s] + 1 \) is more delicate since we can not use the \( L^\infty \) norm of \( f \). We proceed as follows: we apply Hölder inequality in \( dx \) to obtain

\[
\left( \int \left| \frac{f_m \Delta [s+1] g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} \leq \left( \int \left( \frac{\| f_m \|_{L^{n/2}} \| \Delta [s+1] g \|_{L^{2(n/2-s)}} \} dh \right) \right)^{1/2}
\]

and we notice that the norm \( \| f_m \|_{L^{n/2}} = \| f \|_{L^{n/2}} \) is independent of \( h \) and can be drawn out of the integral. What remains is exactly a Besov norm (see (4.6)) and we conclude

\[
\left( \int \left| \frac{f_m \Delta [s+1] g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} \leq \| f \|_{L^{2n/(n-2s)}} \cdot \| g \|_{B^s_{2,2}} \leq c \| f \|_{H^s} \cdot \| g \|_{H^{n/2}}
\]

by the continuous embeddings

\[
H^s \subset L^{2n/(n-2s)}, \quad B^s_{2,2} \subset B^{n/2}_{2,2} = H^{n/2}
\]

(see [7], [8]). By (4.22) for \( \lambda = n/2 \), (4.23) and (4.24) we obtain (4.9).

By the same method we can write

\[
\left( \int \left| \frac{f \Delta g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} \leq \left( \int \left( \frac{\| f \|_{L^{n/2}} \| \Delta g \|_{L^{2(n/2-s)}} \} dh \right) \right)^{1/2}
\]

where, for an arbitrary \( \lambda \) with \( s < \lambda < n/2 \), \( p \) and \( q \) are chosen as

\[
2p = \frac{n}{\lambda - s}, \quad 2q = \frac{2n}{n - 2(\lambda - s)};
\]

proceeding exactly as in the proof of (4.24) we obtain

\[
\left( \int \left| \frac{f_m \Delta [s+1] g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} \leq \| f \|_{L^{n/(\lambda-s)}} \cdot \| g \|_{B^s_{2,2}} \leq c \| f \|_{H^{n/2+s-\lambda}} \| g \|_{H^\lambda}.
\]

By (4.22) and (4.23) we obtain immediately (4.10) for noninteger \( s \).

The proof of (4.12) for noninteger \( s \) proceeds in a similar way. Using again the Leibnitz rule (4.3) we can write

\[
I_s(fg) \geq \left( \int \left| \frac{g \Delta [s+1] f}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} - c \sum_{\ell + m = [s] + 1} \left( \int \left| \frac{\Delta \Delta f_m g}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2}
\]

which by (4.22) for \( \lambda = n/2 \) and (4.24) implies

\[
I_s(fg) \geq \left( \int \left| \frac{g \Delta [s+1] f}{|h|^{n+2s}} \right|^2 \, dxdh \right)^{1/2} - c \| f \|_{H^s} \cdot \| g \|_{H^{n/2}}.
\]

Using now assumption (1.11) we have

\[
I_s(fg) \geq C_1 I_s(f) - c \| f \|_{H^s} \| g \|_{H^{n/2}}
\]

and recalling that \( I_s(u) \simeq \| u \|_{H^s} \), we conclude the proof.
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