Statistical properties of states in QED with unstable vacuum

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Abstract

We study statistical properties of states of quantized charged massive Dirac and Klein-Gordon fields placed in a background that violates the vacuum stability, first in general terms and then considering a specific external electromagnetic background. As a starting point, we use a nonperturbative expression for the density operators of such fields derived in (NPB 795 (2008) 645). We construct reduced density operators for electron and positron subsystems and discuss a decoherence that may occur in course of the evolution due to an intermediate measurement. Calculating the entropy, we study the loss of the information in the QED states due to the partial reductions and the possible decoherence. Then, we consider the Dirac and Klein-Gordon fields placed in the so-called $T$-constant external electric field. Such an exactly solvable example allows us to calculate explicitly all statistical properties of different quantum states of the massive charged fields under consideration.

Keywords: Density operator, entropy, entanglement, particle creation

1 Introduction

It is known that pure quantum states of a physical system provide us with maximum possible information about this system in contrast with mixed quantum states of the same system. A measure of the information loss of a quantum state can be identified with the entropy of such a state. Any unitary evolution does not change the entropy of a quantum state and possible violations of the unitary evolution can be registered as change of the entropy. On the other side, the entanglement is a pure quantum property which is associated with a quantum non-separability of parts of a composite system and also can be evaluated as a specific quantum entropy. Entangled states became a powerful tool in studying both principal questions in quantum theory and quantum computation and information theory \cite{1,2,3,4}. Despite a number of publications devoted to the entropy and entanglement of quantum states, these related characteristics to be fully understandable have to be studied on more examples of different concrete systems not only in nonrelativistic quantum mechanics but in QFT as well. This explains recent interest to study quantum entanglement and entropy of QFT systems with unstable vacuum, i.e., with strong external backgrounds that may create particles from the vacuum, see e.g. \cite{5,6,7,8}. In this article we would like to attract attention to the fact that studying QFT systems with unstable vacuum we get a possibility to see problems with the loss of information, quantum entanglement, and the entropy change in a very close relation. Two points that explain this assertion. Let we place a quantized charged Dirac
or Klein Gordon (K-G) field\textsuperscript{1} in a strong uniform external electric field. Such a system is a QFT with unstable vacuum, the latter means that the electric field creates electron-positron pairs from the vacuum. Particle creation from the vacuum by strong electromagnetic, Yang-Mills, and gravitational fields is a well-known nonlinear quantum phenomenon which has many applications in modern high energy physics. Its theoretical study has a long story that is described in numerous works, see for example Refs. \cite{11, 9, 12, 10, 14, 15}. Creation of charged particles from the vacuum by strong electric-like fields needs superstrong field magnitudes compared with the Schwinger critical field $E_c = m^2 c^3 / e \hbar \simeq 1.3 \times 10^{16} \text{ V} \cdot \text{cm}^{-1} \text{.} \cite{17}$. Nevertheless, recent progress in laser physics allows one to hope that this effect will be experimentally observed in the near future even in laboratory conditions, see Ref. \cite{18} for the review. Electron-hole pair creation from the vacuum (which is an analog of the electron-positron pair creation from the vacuum) was recently observed in graphene by its indirect influence on the graphene conductivity \cite{19} (the graphene conductivity modification due to the particle creation was calculated in \cite{20}, some other relevant effects in \cite{21}). The particle creation effect in a strong uniform external electric field has an additional important feature: the external field not only creates the pairs from the vacuum, but creates two well separated with time subsystems - created electrons and created positrons. States of each subsystem are described by the corresponding density matrices. In this respect it should be noted that for the first time such density matrices were derived in Refs. \cite{22, 15}. And here it is interesting to study the quantum entanglement of both subsystems and its measure calculating the corresponding entropy. A change of the entropy of QFT systems with unstable vacuum and a quantum entanglement of the above mentioned subsystems can occur also due to some decoherence processes. In the concrete case under consideration that could be intermediate measures or collisions with some semiclassical objects (e.g. with well-known impurities in the graphene).

In the present article, we study the above mentioned characteristics, that is, statistical properties of states of quantized charged massive Dirac or K-G fields placed in a background that violates the vacuum stability, first in general terms and then considering a specific external electromagnetic background. As a starting point, we use a general nonperturbative expression for the density operators of such fields derived in Ref. \cite{15}. In Section 2 we discuss such operators with different initial conditions. Reduced density operators for electron and positron subsystems are derived in Section 3. In Section 4 we study a decoherence that may occur in course of the evolution due to an intermediate measurement and the corresponding modifications of the complete and the reduced density operators. In Section 5 we, calculating the entropy, study the loss of the information in the QED states due to the partial reductions and the possible decoherence. In Section 6 we consider quantized Dirac or K-G fields placed in the so-called $T$-constant external electric field. This exactly solvable example allows us to calculate explicitly all statistical properties of different quantum states of the latter massive fields. In the Appendix A we, recall briefly a nonperturbative formulation of QED with strong time-dependent electric-like background that is used in our calculations.

2 General density operator

It is convenient to introduce a generating operator $\hat{R}(J)$ that allows one to construct density operators $\hat{\rho}$ with different initial conditions (different initial states at the initial time instant

\textsuperscript{1}As basic particles in both cases we consider electrons with the charge $q = -e$, $e > 0$, whereas their antiparticles are positrons. In the K-G case both electrons and positrons are spinless.
whereas differential mean numbers \( N_{\alpha,\zeta} \) are equal and have the form

\[
\rho = \text{tr} \rho = \frac{1}{Z} \text{tr} \rho J, \quad \text{and } Z = \text{tr} R(J),
\]

\[
\hat{R}(J) = \exp \left[ \sum_n \left[ a_n^{\dagger}(\text{in}) (J_{n,+} - 1) a_n(\text{in}) + b_n^{\dagger}(\text{in}) (J_{n,-} - 1) b_n(\text{in}) \right] \right] : , \quad (1)
\]

where the variables \( J_{\alpha,\zeta} \) are sources for electron (\( \zeta = + \)) or positron (\( \zeta = - \)) in-operators, \( Z \) is a normalization factor (the partition function), and here and in what follows : \( \cdots \) : means the normal form with respect to those creation and annihilation operators that are situated inside the colon signs.

Using canonical transformation (79), we can express the in-operators in terms of the out-operators and obtain \( \hat{R}(J) \) (and the corresponding \( \hat{\rho} \)) in terms of the out-operators, (15).

Thus,

\[
\hat{R}(J) = Z^{-1}(J) |c_v|^2 \det (1 + \kappa AB)^{\kappa} \hat{R}(J), \quad \hat{R}(J) = : \exp \left[ -a_1^{\dagger} (\text{out}) (1 - D_+) a(\text{out}) - b_1^{\dagger} (\text{out}) (1 - D_-) b(\text{out}) - a_1^{\dagger} (\text{out}) C b_1^{\dagger} (\text{out}) - b(\text{out}) C a(\text{out}) \right] : ,
\]

\[
D_+ = w (+|+) (1 + \kappa AB)^{-1} \mathbb{J}_+ w (+|+)^\dagger, \quad D_- = w (-|-) (1 + \kappa BA)^{-1} w (-|-), \quad A(J) = \mathbb{J}_+ B^\dagger \mathbb{J}_-, \quad C = w (-|-) \mathbb{J}_- B (1 + \kappa AB)^{-1} \mathbb{J}_+ w (+|+) + \kappa w (+ - |0)^\dagger, \quad B = \kappa w (0 - |), \quad \mathbb{J}_{mn,\zeta} = \delta_{mn} J_{n,\zeta},
\]

where \( \kappa = +1 \) for Fermi case and \( \kappa = -1 \) for Bose case. The normalization factor \( Z \) has the form

\[
Z(J) = \exp \left\{ \kappa \sum_{n,\zeta} \left[ \ln (1 + \kappa J_{n,\zeta}) \right] \right\} = \prod_{n,\zeta} [1 + \kappa J_{n,\zeta}]^{\kappa}. \quad (2)
\]

In what follows, we are going to work with two important cases of the general density operator that correspond to the initial pure vacuum state and to the initial thermal state.

a) Setting \( J = 0 \), and using well-known formula (10), we obtain a density operator \( \hat{\rho}(0) \) that corresponds to the initial vacuum state

\[
\hat{\rho}(0) = : \exp \left\{ -\sum_n \left[ a_n^{\dagger}(\text{in}) a_n(\text{in}) + b_n^{\dagger}(\text{in}) b_n(\text{in}) \right] \right\} : = |0,\text{in} \rangle \langle 0,\text{in} |. \quad (3)
\]

From Eqs. (11), we obtain this operator in terms of the out-operators,

\[
\hat{\rho}(0) = |c_v|^2 : \exp \left\{ -\sum_n \left[ a_n^{\dagger}(\text{out}) a_n(\text{out}) + b_n^{\dagger}(\text{out}) b_n(\text{out}) \right] w (+ + |0)^\dagger, \quad + \kappa a_n^{\dagger}(\text{out}) w (+ + |0)^\dagger, \quad + \kappa b_n(\text{out}) w (+ + |0)^\dagger \right\} : . \quad (4)
\]

Differential mean numbers \( N_{n,\zeta}(0|\text{in}) \) of in-electrons and positrons in the state \( \hat{\rho}(0) \) are zero,

\[
N_{n,+}(0|\text{in}) = \text{tr} \hat{\rho}(0)a_n^{\dagger}(\text{in}) a_n(\text{in}) = 0, \quad N_{n,-}(0|\text{in}) = \text{tr} \hat{\rho}(0)b_n^{\dagger}(\text{in}) b_n(\text{in}) = 0,
\]

whereas differential mean numbers \( N_{n,\zeta}(0|\text{out}) \) of out-electrons and positrons in the state \( \hat{\rho}(0) \),

\[
N_{n,+}(0|\text{out}) = \text{tr} \hat{\rho}(0) a_n^{\dagger}(\text{out}) a_n(\text{out}) = 0, \quad N_{n,-}(0|\text{out}) = \text{tr} \hat{\rho}(0) b_n^{\dagger}(\text{out}) b_n(\text{out}) = 0,
\]

are equal and have the form

\[
N_{n,+}(0|\text{out}) = N_{n,-}(0|\text{out}) = N_n(0|\text{out}), \quad N_n(0|\text{out}) = \frac{|w (+ - |0)^\dagger|^2}{1 + \kappa |w (+ - |0)^\dagger|^2}. \quad (5)
\]
b) To obtain the density operator \( \tilde{\rho}(\beta) \) that corresponds to the thermal initial state, one has to set \( J_{n,\zeta} = J_{n,\zeta}(\beta) \),

\[
J_{n,\zeta}(\beta) = e^{-E_{n,\zeta}}, \quad E_{n,\zeta} = \beta (\varepsilon_{n,\zeta} - \mu_{\zeta}),
\]

where \( \varepsilon_{n,\zeta} \) are energies of electrons (\( \zeta = + \)) or positrons (\( \zeta = - \)) with quantum numbers \( n \); \( \mu_{\zeta} \) are the corresponding chemical potentials, and \( \beta = \Theta^{-1} \), where \( \Theta \) is the absolute temperature \[15\]. It can be checked that an explicit expression for \( \tilde{\rho}(\beta) \) in terms of the in-operators is

\[
\tilde{\rho}(\beta) = Z_{gr}^{-1} \exp \left[ -\beta \left( \tilde{H} - \sum_{\zeta} \mu_{\zeta} \tilde{N}_{\zeta} \right) \right],
\]

\[
Z_{gr} = \exp \left[ \kappa \sum_{n,\zeta} \ln \left( 1 + \kappa e^{-E_{n,\zeta}} \right) \right].
\]

The quantity \( Z_{gr} \) is the partition function of grand canonical ensemble, \( \tilde{H} \) is the Hamiltonian of the system (written in terms of in-operators),

\[
\tilde{H} = \sum_{n} \left[ a_n^\dagger (\text{in}) \varepsilon_{n,+} a_n (\text{in}) + b_n^\dagger (\text{in}) \varepsilon_{n,-} b_n (\text{in}) \right],
\]

and

\[
\tilde{N}_+ = \sum_{n} \left[ a_n^\dagger (\text{in}) a_n (\text{in}) \right], \quad \tilde{N}_- = \sum_{n} \left[ b_n^\dagger (\text{in}) b_n (\text{in}) \right]
\]

are operators of numbers of in-electrons and in-positrons, respectively.

Let \( \tilde{\rho} \) be the general density matrix for an arbitrary initial state and \( N_{n,\zeta}(\cdots|\text{in}) \) are differential mean numbers of in-electrons or positrons in the state \( \tilde{\rho} \) and \( N_{n,\zeta}(\cdots|\text{out}) \) are differential mean numbers of out-electrons or positrons in the state \( \tilde{\rho} \),

\[
N_{n,+}(\cdots|\text{in}) = \text{tr} \tilde{\rho} a_n^\dagger (\text{in}) a_n (\text{in}), \quad N_{n,-}(\cdots|\text{in}) = \text{tr} \tilde{\rho} b_n^\dagger (\text{in}) b_n (\text{in}),
\]

\[
N_{n,+}(\cdots|\text{out}) = \text{tr} \tilde{\rho} a_n^\dagger (\text{out}) a_n (\text{out}), \quad N_{n,-}(\cdots|\text{out}) = \text{tr} \tilde{\rho} b_n^\dagger (\text{out}) b_n (\text{out}).
\]

Calculating traces in the in-basis, one can see \[15\] that

\[
N_{n,\zeta}(\cdots|\text{out}) = N_{n,\zeta}(\cdots|\text{in}) + N_n(0|\text{out}) \{ 1 - \kappa [N_{n,+}(\cdots|\text{in}) + N_{n,-}(\cdots|\text{in})] \}.
\]

In particular, differential mean numbers \( N_{n,\zeta}(\beta|\text{in}) \) of in-electrons or positrons in the state \( \tilde{\rho}(\beta) \) are well-known Fermi-Dirac (\( \kappa = +1 \)) or Bose-Einstein (\( \kappa = -1 \)) distributions,

\[
N_{n,+}(\beta|\text{in}) = \text{tr} \tilde{\rho}(\beta) a_n^\dagger (\text{in}) a_n (\text{in}) = (e^{E_{n,+}} + \kappa)^{-1},
\]

\[
N_{n,-}(\beta|\text{in}) = \text{tr} \tilde{\rho}(\beta) b_n^\dagger (\text{in}) b_n (\text{in}) = (e^{E_{n,-}} + \kappa)^{-1}.
\]

Differential mean numbers \( N_{n,\zeta}(\beta|\text{out}) \) of out-electrons or positrons in the state \( \tilde{\rho}(\beta) \) follows immediately from \[9\].

3 Reduced density operators for electron and positron subsystems

At any fixed time instant, the complete system of quantum electrons and positrons can be conditionally divided into two subsystems: a system of electrons and a system of positrons. Let us suppose that the external electric field is switched off at some big enough time
instant \( t_2 \) in such a way that at \( t_{out} > t_2 \) no particle creation occurs and both subsystems are spatially separated. Thus, the particle creation effect by the time-dependent uniform electric field provides a real separation of the complete quantum field system into the two subsystems. We can introduce the so-called reduced density operators \( \hat{\rho}_\pm \) of the electron subsystem and of the positron subsystem. These operators are defined as follows:

\[
\begin{align*}
\hat{\rho}_+ &= \text{tr}_- \hat{\rho} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} \sum b(0, \text{out}) |b_{mM}(\text{out}) \ldots b_{m1}(\text{out})| \hat{\rho} |b_{m1}(\text{out}) \ldots b_{mM}(\text{out})|0, \text{out})_b, \\
\hat{\rho}_- &= \text{tr}_+ \hat{\rho} = \sum_{M=0}^{\infty} \sum_{\{m\}} (M!)^{-1} \sum a(0, \text{out}) |a_{mM}(\text{out}) \ldots a_{m1}(\text{out})| \hat{\rho} |a_{m1}(\text{out}) \ldots a_{mM}(\text{out})|0, \text{out})_a,
\end{align*}
\]

where \( \hat{\rho} \) is the density operator of the complete system, \(|0, \text{out})_a \) and \(|0, \text{out})_b \) are electron and positron vacua respectively, \(|a_{mM}(\text{out})0, \text{out})_a \) = 0, \(|b_{mM}(\text{out})0, \text{out})_b \) = 0, \(|0, \text{out})_a \otimes 0, \text{out})_b \) and \( \text{tr}_\pm \) are the so-called reduced traces. Obviously, the reduced density operators \( \hat{\rho}_\pm \) describe mixed states.

The reduced density operators \( \hat{\rho}_\pm \) can be obtained from the reduced generating operators \( \hat{R}_\pm (J) \) which are defined as:

\[
\hat{R}_\pm (J) = \text{tr}_\mp \hat{R} (J).
\]

In terms of the out-operators these operators have the form

\[
\begin{align*}
\hat{R}_+ (J) &= Z_+^{-1} (J) : \exp \left\{ - \sum_n a_n^\dagger (\text{out}) \left[ 1 - K_+ (J) \right]_{nn} a_n (\text{out}) \right\} :,
\\
\hat{R}_- (J) &= Z_-^{-1} (J) : \exp \left\{ - \sum_n b_n^\dagger (\text{out}) \left[ 1 - K_- (J) \right]_{nn} b_n (\text{out}) \right\} :,
\\
K_\pm (J) &= D_\pm + C^\dagger \left( 1 + \kappa (\frac{D_+}{\varepsilon}) \right)^{-\kappa} C,
\\
Z_\pm^{-1} (J) &= \left| c_v \right|^2 \text{det} \left( 1 + \kappa AB \right)^\kappa \text{det} \left( 1 + \kappa D_\mp \right)^\kappa.
\end{align*}
\]

The reduced generating operators \( \hat{R}_\pm (J) \) allow one to obtain the reduced density operators \( \hat{\rho}_\pm \) for different initial states of the system. Consider as before two important cases.

a) By setting \( J = 0 \) in (12), we obtain the reduced density operators \( \hat{\rho}_\xi (0) = \hat{R}_\xi (0) \) for both subsystems in the case when the complete system was in the vacuum state at the initial time instant. Taking into account that

\[
\begin{align*}
K_{\pm} (0) &= |w (\mp |0 \rangle|^2 = P(\mp |0 \rangle P_v^{-1},
\\
Z_{\pm}^{-1} (0) &= |c_v|^2 = P_v, \quad P(\mp |0 \rangle = |0, \text{out} |a_n (\text{out}) b_n (\text{out})|0, \text{in})|^2,
\end{align*}
\]

where \( P(\mp |0 \rangle \) and \( P_v \) are probabilities of pair creation and vacuum-to-vacuum transition, respectively, we obtain explicit expressions for \( \hat{\rho}_\xi (0)\):

\[
\begin{align*}
\hat{\rho}_+ (0) &= \hat{R}_+ (0) = |c_v|^2 : \exp \left\{ - \sum_n a_n^\dagger (\text{out}) \left[ 1 - P(\mp |0 \rangle P_v^{-1} \right]_{nn} a_n (\text{out}) \right\} :,
\\
\hat{\rho}_- (0) &= \hat{R}_- (0) = |c_v|^2 : \exp \left\{ - \sum_n b_n^\dagger (\text{out}) \left[ 1 - P(\mp |0 \rangle P_v^{-1} \right]_{nn} b_n (\text{out}) \right\} :.
\end{align*}
\]

It should be noted that for the first time reduced density operators (12) were obtained in Refs. 22.

b) Setting the sources \( J \) in expression (12) to \( J_{\nu,\xi} (\beta) \) according to eqs. (10), we see that the reduced generating operators (13) become the reduced density operators \( \hat{\rho}_\xi (\beta) \) of the system that has been in thermal equilibrium at the initial time instant, \( \hat{R}_+ (J) = \hat{\rho}_+ (\beta) \) and \( \hat{R}_- (J) = \hat{\rho}_- (\beta) \).
4 Decoherence in course of the evolution

4.1 General

In the previous sections we considered the case when the information loss was due to the averaging over one of the subsystems of electrons or positrons. However, an information loss can also occur due to the interaction of the quantum system with classical (or quasiclassical) objects, or, in other words, due to decoherence. One can imagine two possible scenarios for this: first, it can happen during intermediate measurements by a classical tool and, second, as a result of collisions of particles with some semiclassical objects (for example, well-known impurities in the graphene). For us, there is no difference which of the mechanisms is implemented, so in what follows we talk about an intermediate measurement by a classical tool as a source of the decoherence.

Consider the case when unitary evolution of the system is interrupted by single intermediate measurement. The external field starts to act at the time instant $t_{in}$, at which the system is in vacuum state at initial time instant $t_i$, and then again the unitary evolution from $t_i$ to $t_{out}$, during the time $T_2$. In this case, if we consider the Heisenberg picture, out-set of creation and annihilation operators for electrons and positrons of interval $T_1$ is in-set of interval $T_2$.

Suppose that during time interval $T_1$ the system is described by density operator $\tilde{\rho}(0)$, i.e. the system is in vacuum state at initial time instant $t_{in}$. Differential mean numbers of electrons and positrons at time instant $t_1$ are the numbers of electrons/positrons created by external field from the vacuum $N_n(0|out)$ [5]. The electrons and positrons created in pairs by external field are entangled.

During time interval $T_2$ the system is described by density operator which we denote as $\tilde{\rho}'$. The latter in terms of in-set of creation-annihilation operators for electrons and positrons must describe the system without quantum correlations between the electrons and positrons created (i.e. new “initial” state of the system in time interval $T_2$ is the state without any entanglement).

Such an operator can be obtained using the von Neumann’s reduction principle ([24]). Let a system be in a pure state which is described by a state vector $|\psi\rangle$, or equivalently by a density operator $\tilde{\rho}$ that is in such a case the projector, $\tilde{\rho} = \tilde{P}_\psi = |\psi\rangle \langle \psi|$. And let $\tilde{R}$ be a self-adjoint observable of the system. In the simplest case, when this observable has a nondegenerate discrete spectrum the following spectral decomposition holds $\tilde{R} = \sum_{n=1}^\infty \lambda_n P_{\varphi_n}$, where $\lambda_n$ are possible eigenvalues of the observable, and $P_{\varphi_n}$ are projectors on the corresponding eigenvectors $|\varphi_n\rangle$, $\tilde{P}_{\varphi_n} = |\varphi_n\rangle \langle \varphi_n|$. Measuring the observable $\tilde{R}$, we obtain the eigenvalues $\lambda_n$ with the probabilities $|\langle \varphi_n|\psi\rangle|^2 = \langle \varphi_n|\tilde{P}_\psi|\varphi_n\rangle = \langle \varphi_n|\tilde{\rho}|\varphi_n\rangle$, and just after the measurement the state vector $|\psi\rangle$ is reduced to the one $|\varphi_n\rangle$, or the density operator $\tilde{\rho}$ is reduced to the one $\tilde{\rho}' = \tilde{P}_{\varphi_n}$. In the more general case when a system be in a mixed state, which is described by the density operator $\tilde{\rho}$ with a simple discrete spectrum, $\tilde{\rho} = \sum_{n=1}^\infty \lambda_n P_{\psi_n}$, $P_{\psi_n} = |\psi_n\rangle \langle \psi_n|$, $\lambda_n$ are statistical weights of corresponding states $P_{\psi_n}$, and $\tilde{R}$ be the above mentioned observable, a measurement is as follows. We obtain the eigenvalues $r_n$, with the following probabilities

$$\sum_n \lambda_n |\langle \varphi_n|\psi\rangle|^2 = \langle \varphi_n|\tilde{\rho}|\varphi_n\rangle,$$

and just after the measurement the density operator $\tilde{\rho}$ is reduced to the one $\tilde{\rho}'$,

$$\tilde{\rho}' = \sum_\alpha \langle \varphi_\alpha|\tilde{\rho}|\varphi_\alpha\rangle \tilde{P}_{\varphi_\alpha} .$$

The density operator $\tilde{\rho}(0)$ is

$$\tilde{\rho}(0) = |0, in\rangle \langle 0, in| .$$

(15)
The in-vacuum \(|0,\text{in}\rangle\) is connected to out-vacuum \(|0,\text{out}\rangle\) by relation (80). Then density operator \(\hat{\rho}(0)\) can be presented as

\[
\hat{\rho}(0) = V|0,\text{out}\rangle\langle 0,\text{out}|V^\dagger.
\]

We are interested in case of uniform external field which do not mix different quantum modes. Then amplitudes (78) are diagonal in this case. Thus, it is possible to factorize \(V\) which is defined by (80) as:

\[
V = \prod_n V_n, \quad V_n = v_{4n} v_{3n} v_{2n} v_{1n},
\]

where

\[
v_{1n} = \exp\{-\kappa b_n^\dagger(\text{out})w(0) + \beta\}_n a_n(\text{out})\}, \quad v_{2n} = \exp\{\kappa a_n(\text{out})[\ln w(\pm)]_n a_n(\text{out})\},
\]

\[
v_{3n} = \exp\{-\kappa b_n^\dagger(\text{out})[\ln w(-\pm)]_n b_n^\dagger(\text{out})\}, \quad v_{4n} = \exp\{-\kappa a_n^\dagger(\text{out})w(-0)_n b_n^\dagger(\text{out})\}.
\]

Making use of the explicit form of \(V_n\) and definition (80) we can write

\[
|0,\text{in}\rangle = c_v \prod_n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left[\kappa w(-0)_n a_n^\dagger(\text{out})b_n^\dagger(\text{out})\right]^{\kappa m}|0,\text{out}\rangle,
\]

and then easily calculate \(c_v\),

\[
c_v = \prod_n \left[w(-0)_n\right]^{-\kappa}.
\]

Thus, density operators \(\hat{\rho}(0)\) (16) can be represented as

\[
\hat{\rho}(0) = |c_v|^2 \prod_n \left(\sum_{m=0}^{\infty} \frac{(-\kappa w(-0)_n a_n^\dagger(\text{out})b_n^\dagger(\text{out})^{m}}{m!}\right)\prod_n \left(\sum_{m'=0}^{\infty} \frac{(-\kappa w(-0)_n a_n^\dagger(\text{out})b_n^\dagger(\text{out})^{m'}}{m'!}\right),
\]

where \(\tilde{P}_0 = |0,\text{out}\rangle\langle 0,\text{out}|\).

### 4.2 Measurement of differential mean numbers in the system

Suppose that we are going to measure the physical quantity, which is the number of particles, in the state \(\hat{\rho}(0)\) of the system under consideration. The operator corresponding to such a physical quantity is \(\hat{N}\),

\[
\hat{N}(\text{out}) = \sum_{n,\xi} \hat{N}_{n,\xi}(\text{out}) = \sum_n \left[a_n^\dagger(\text{out})a_n(\text{out}) + b_n^\dagger(\text{out})b_n(\text{out})\right].
\]

Its eigenstates are mutually orthonormal vectors of the following form

\[
|s,\text{out}\rangle = |\{i,\gamma\}_{L_P,\text{out}}\rangle_a \otimes |\{j,\kappa\}_{K_Q,\text{out}}\rangle_b,
\]

\[
|\{i,\gamma\}_{L_P,\text{out}}\rangle_a = \left[\frac{a_i^\dagger(\text{out})}{\sqrt{l_i!}}\right]^{l_i} \cdots \left[\frac{a_{i_P}^\dagger(\text{out})}{\sqrt{l_P!}}\right]^{l_P}|0,\text{out}\rangle_a,
\]

\[
|\{j,\kappa\}_{K_Q,\text{out}}\rangle_b = \left[\frac{b_j^\dagger(\text{out})}{\sqrt{k_j!}}\right]^{k_j} \cdots \left[\frac{b_{j_Q}^\dagger(\text{out})}{\sqrt{k_Q!}}\right]^{k_Q}|0,\text{out}\rangle_b,
\]

\(L = 0, 1, 2, \ldots, P = 1, 2, \ldots L, \quad i = i_1, \ldots, i_P, \quad l_1 + l_2 + \ldots + l_P = L, \quad K = 0, 1, 2, \ldots, Q = 1, 2, \ldots K, \quad j = j_1, \ldots, j_Q, \quad k_1 + k_2 + \ldots + k_Q = K,\)
such that its eigenvalues are

\[ \hat{N}(\text{out})|s, \text{out}\rangle = (L + K)|s, \text{out}\rangle, \]

where \( s \) is full set of quantum numbers \( K, L, \{i\}, \{j\}, P, Q; \{i, l\}_{LP}, \text{out}\rangle \) is a state with \( L \) electrons distributed in \( P \) groups \( i_1, \ldots, i_P \), with \( l_1 \) electrons in the group \( i_1 \), \( l_2 \) electrons in the group \( i_2 \), and so on; \( |\{j, k\}_{KQ}, \text{out}\rangle \) is a state with \( K \) positrons distributed in \( Q \) groups \( j_1, \ldots, j_Q \), with \( k_1 \) positrons in group \( j_1 \), \( k_2 \) positrons in group \( j_2 \), and so on.

According to von Neumann [24], a density operator \( \hat{\rho}(0) \) after such a measurement, is reduced to the operator \( \hat{\rho}_N \) that has the form

\[ \hat{\rho}_N = \sum_s \langle s, \text{out}|\hat{\rho}(0)|s, \text{out}\rangle \hat{P}_s, \quad \hat{P}_s = |s, \text{out}\rangle \langle s, \text{out}|. \quad (22) \]

Due to the structure of density operator \( \hat{\rho}(0) \) given by Eq. (20), means \( \langle s, \text{out}|\hat{\rho}(0)|s, \text{out}\rangle \) are non-zero only when states \( |s, \text{out}\rangle \) are states with integer number of pairs. Thus, we obtain

\[
\hat{\rho}_N = |c_v|^2 \sum_f W_f \hat{P}_f, \quad \sum_f = \sum_{M=0}^{\infty} \sum_{Z=1}^{M} \sum_{\{m,n\}} \hat{P}_f = |f, \text{out}\rangle \langle f, \text{out}|, \\
W_f = |w (+ - |0\rangle_{n_1n_1} |2m_1 \ldots w (+ - |0\rangle_{n_Zn_Z} |2m_Z, \ m_1 + m_2 + \ldots + m_Z = M, \\
|f, \text{out}\rangle = \left[ a_{n_1}^\dagger(\text{out})b_{n_1}^\dagger(\text{out}) \right]^{m_1} \ldots \left[ a_{n_Z}^\dagger(\text{out})b_{n_Z}^\dagger(\text{out}) \right]^{m_Z} \frac{m_1! \ldots m_Z!}{m_1! \ldots m_Z!} |0, \text{out}\rangle. \quad (23) \]

where \( f \) is a complete set of quantum numbers \( M, Z, \{m\}, \{n\}; |f, \text{out}\rangle \) is a state with total number of pairs \( M \) distributed in \( Z \) groups; \( m_1 \) pairs are in the group \( n_1, m_2 \) pairs are in group \( n_2 \) and so on. Unlike (20), the last expression contains only terms diagonal in \( f \). Thus, the measurement destroy nondiagonal terms of density operator (20).

Let us now calculate reduced (in the sense of sect. 3) operators \( \hat{\rho}_N \) by

\[
[\hat{\rho}_N]_c = \operatorname{tr}_- \hat{\rho}_N = |c_v|^2 \sum_f W_f \operatorname{tr}_- \hat{P}_f, \\
\operatorname{tr}_- \hat{P}_f = \frac{[a_{n_1}^\dagger(\text{out})]^{m_1}}{\sqrt{m_1!}}, \ldots \frac{[a_{n_Z}^\dagger(\text{out})]^{m_Z}}{\sqrt{m_Z!}} |0, \text{out}\rangle_{aa} (0, \text{out}) \frac{[a_{n_1}^\dagger(\text{out})]^{m_1}}{\sqrt{m_1!}} \ldots \frac{[a_{n_Z}^\dagger(\text{out})]^{m_Z}}{\sqrt{m_Z!}}, \\
\operatorname{tr}_+ \hat{P}_f = \frac{[b_{n_1}^\dagger(\text{out})]^{m_1}}{\sqrt{m_1!}}, \ldots \frac{[b_{n_Z}^\dagger(\text{out})]^{m_Z}}{\sqrt{m_Z!}} |0, \text{out}\rangle_{bb} (0, \text{out}) \frac{[b_{n_1}^\dagger(\text{out})]^{m_1}}{\sqrt{m_1!}} \ldots \frac{[b_{n_Z}^\dagger(\text{out})]^{m_Z}}{\sqrt{m_Z!}}. \quad (24) \]

On the other hand, one can calculate the reduced density operators \( \hat{\rho}_c(0) \) by taking reduced traces (11) of operator (20) to verify that they have exactly the same form,

\[ [\hat{\rho}_N]_c = \hat{\rho}_c(0). \]

### 4.3 Measurements of differential mean numbers in the subsystems

Suppose now that we measure the number of either electrons or positrons. The corresponding operators of these physical quantities are

\[
\hat{N}_+(\text{out}) = \sum_n \hat{N}_{n,\text{out}}(\text{out}) = \sum_n a_{n}^\dagger(\text{out})a_{n}(\text{out}), \\
\hat{N}_-(\text{out}) = \sum_n \hat{N}_{n,\text{out}}(\text{out}) = \sum_n b_{n}^\dagger(\text{out})b_{n}(\text{out}). \quad (25) \]

8
The spectra of operators (25) are

\[ |s_+, \text{out}\rangle_a = |\{i, l\}_{L, P, \text{out}}\rangle_a = \left[ \frac{a_{i_1}^\dagger(\text{out})}{\sqrt{l_1!}} \right]^{l_1} \cdots \left[ \frac{a_{i_p}^\dagger(\text{out})}{\sqrt{l_p!}} \right]^{l_p} |0, \text{out}\rangle_a, \]

\[ L = 0, 1, 2, \ldots, \ P = 1, 2, \ldots L, \ i = i_1, \ldots, i_p, \ l_1 + l_2 + \ldots + l_p = L; \]

\[ |s_-, \text{out}\rangle_b = |\{j, k\}_{K, Q, \text{out}}\rangle_b = \left[ \frac{b_{j_1}^\dagger(\text{out})}{\sqrt{k_1!}} \right]^{k_1} \cdots \left[ \frac{b_{j_Q}^\dagger(\text{out})}{\sqrt{k_Q!}} \right]^{k_Q} |0, \text{out}\rangle_b, \]

\[ K = 0, 1, 2, \ldots, \ Q = 1, 2, \ldots K, \ j = j_1, \ldots, j_Q, \ k_1 + k_2 + \ldots + k_Q = K; \]

\[ \hat{N}_+(\text{out})|s_+, \text{out}\rangle_a = L|s_+, \text{out}\rangle_a, \ \hat{N}_-(\text{out})|s_-, \text{out}\rangle_b = K|s_-, \text{out}\rangle_b. \]

The states |\{i, l\}_{L, P, \text{out}}\rangle_a and |\{j, k\}_{K, Q, \text{out}}\rangle_b are defined in the same way as in the previous subsection.

The density operators after such measurements, which we denote by \( \tilde{\rho}_{N_+} \) and \( \tilde{\rho}_{N_-} \), correspondingly, have the form

\[
\tilde{\rho}_{N_+} = \sum_{s_+} \langle s_+, \text{out}|\tilde{\rho}(0)|s_+, \text{out}\rangle_a P_{s+}, \quad P_{s+} = |s_+, \text{out}\rangle_{aa} \langle s_+, \text{out}|, \quad \sum_{s_+} = \sum_{L=0}^{\infty} \sum_{P=1}^{L} \sum_{\{i, l\}},
\]

\[
\tilde{\rho}_{N_-} = \sum_{s_-} \langle s_-, \text{out}|\tilde{\rho}(0)|s_-, \text{out}\rangle_b P_{s-}, \quad P_{s-} = |s_-, \text{out}\rangle_{bb} \langle s_-, \text{out}|, \quad \sum_{s_-} = \sum_{K=0}^{\infty} \sum_{Q=1}^{K} \sum_{\{j, k\}}.
\]

(26)

Let us now calculate the quantities \( a\langle s_+, \text{out}|\tilde{\rho}(0)|s_+, \text{out}\rangle_a P_{s+} \) and \( b\langle s_-, \text{out}|\tilde{\rho}(0)|s_-, \text{out}\rangle_b P_{s-}. \)

Due to the structure of \( \tilde{\rho}(0) \) they are equal and have the form

\[
a\langle s_+, \text{out}|\tilde{\rho}(0)|s_+, \text{out}\rangle_a P_{s+} = \langle s_-, \text{out}|\tilde{\rho}(0)|s_-, \text{out}\rangle_a P_{s-} \]

\[
= |c_\gamma|^2 \left[ -\kappa w (+ - |0\rangle_{i_1i_1} a_{i_1}^\dagger(\text{out})b_{i_1}^\dagger(\text{out}) \right]^{l_1} \cdots \left[ -\kappa w (+ - |0\rangle_{i_p,i_p} a_{i_p}^\dagger(\text{out})b_{i_p}^\dagger(\text{out}) \right]^{l_p} \langle 0, \text{out}\rangle \\
\times \langle 0, \text{out}| \left[ -\kappa w (+ - |0\rangle_{i_p,i_p} b_{i_p}(\text{out})a_{i_p}(\text{out}) \right]^{l_p} \cdots \left[ -\kappa w (+ - |0\rangle_{i_1,i_1} b_{i_1}(\text{out})a_{i_1}(\text{out}) \right]^{l_1} \rangle.
\]

(27)

It is not difficult to see that density operators \( \tilde{\rho}_{N_+} \) and \( \tilde{\rho}_{N_-} \) have exactly the same form as (23), namely, they are sums over all possible projectors on states with integer amount of pairs:

\[
\tilde{\rho}_{N_+} = \tilde{\rho}_{N_-} = \tilde{\rho}_N.
\]

(28)

Thus, we stress that measurements of \( N, N_+ \) and \( N_- \) produce the same reductions. The reduced density operators \( [\tilde{\rho}_{N_+}]_\zeta = \text{tr}_- \tilde{\rho}_{N_+} \) and \( [\tilde{\rho}_{N_-}]_\zeta = \text{tr}_- \tilde{\rho}_{N_+} \) are equal to reduced density operators \( \tilde{\rho}_\zeta(0) \) given in (14).

It is also an interesting task to consider the case when the unitary evolution of the system is interrupted by multiple measurements. However, because there are some significant technical difficulties, this problem is not considered in this paper. We plan to examine this issue in our future works.
5 Entropy and entanglement of electron and positron subsystems

As was already said in the Introduction, a measure of the information loss in a quantum state \( \hat{\rho} \) can be identified with the entropy of such a state, namely with the von Neumann information entropy \( S(\hat{\rho}) \),

\[
S(\hat{\rho}) = -k_B \text{tr} \ln \hat{\rho} .
\]  

(29)

Let \( \hat{\rho}(t_{in}) = \hat{\rho}(\beta) \), where \( \hat{\rho}(\beta) \) is given by (7), then

\[
S(\hat{\rho}(\beta)) = k_B \left[ \ln Z_{gr} + \sum_{n} E_{n,\zeta} N_{n,\zeta}(\beta|in) \right] ,
\]  

(30)

Corresponding \( N_{n,\zeta}(\beta|in) \) are Fermi-Dirac or Bose-Einstein distributions, given by (10).

The entropy (30) can be written in terms of the Bose (Fermi) occupation number only, if we take into account that

\[
e^{-E_{n,\zeta}} = \frac{N_{n,\zeta}(\beta|in)}{1 - \kappa N_{n,\zeta}(\beta|in)} .
\]  

(31)

Then

\[
S(\hat{\rho}(\beta)) = -k_B \sum_{n} \{ \kappa [1 - \kappa N_{n,\zeta}(\beta|in)] \ln [1 - \kappa N_{n,\zeta}(\beta|in)] + N_{n,\zeta}(\beta|in) \ln N_{n,\zeta}(\beta|in) \} .
\]  

(32)

This expression has a form similar to expressions for entropy of grand canonical ensemble for Fermi- and Bose-particles [25].

Especially interesting information one obtains calculating von Neumann information entropy of reduced density operators of both the electron subsystem and the positron subsystem \( S(\hat{\rho}_\pm) \),

\[
S(\hat{\rho}_\pm) = -k_B \text{tr}_\pm (\hat{\rho}_\pm \ln \hat{\rho}_\pm) .
\]  

(33)

According to the general theory they coincide \( S(\hat{\rho}_+) = S(\hat{\rho}_-) \) and can be treated as a measure of the quantum entanglement of these subsystems.

It is also known that one can recognize entanglement by evaluating the so-called Schmidt measure, which is the trace of the squared reduced density operators [28]

\[
\tilde{S}(\hat{\rho}_\pm) = -\text{tr} \left[ (\hat{\rho}_\pm)^2 \right] .
\]  

(34)

Let us calculate entropy for both the electron subsystem and the positron subsystem in two important cases of vacuum initial state and thermal initial state that are described by the reduced density operators \( \hat{\rho}_\zeta(0) \) and \( \hat{\rho}_\zeta(\beta) \).

5.1 Vacuum initial state

The entropy for the reduced density operator of the system with initial vacuum state has the form

\[
S(\hat{\rho}_\zeta(0)) = -k_B \text{tr}_\zeta (\hat{\rho}_\zeta(0) \ln \hat{\rho}_\zeta(0)) .
\]  

(35)

The terms \( \ln \hat{\rho}_\zeta(0) \) in the right-hand side of (35) can be written as

\[
\ln \hat{\rho}_+(0) = \ln |c_\zeta|^2 : \exp \left\{ -\sum_{n} a_n^\dagger(\text{out}) (1 - P(+) - |0\rangle P^{-1}_v)_{nn} a_n(\text{out}) \right\} : ,
\]

\[
\ln \hat{\rho}_-(0) = \ln |c_\zeta|^2 : \exp \left\{ -\sum_{n} b_n^\dagger(\text{out}) (1 - P(-) - |0\rangle P^{-1}_v)_{nn} b_n(\text{out}) \right\} : .
\]  

(36)
where $\text{tr} = \sum_n a_n^\dagger(0) b_n(0)$.

Taking into account, that matrices $P(+ - |0)P^{-1}_v$ are diagonal, one can rewrite (35) as

$$S (\hat{\rho}_+(0)) = - k_B \left\{ \ln P_v + \sum_n \text{tr}_+ (\hat{\rho}_+(0) a_n^\dagger(0) a_n(0)) \ln [P(+ - |0)P^{-1}_v]_{nn} \right\},$$

$$S (\hat{\rho}_-(0)) = - k_B \left\{ \ln P_v + \sum_n \text{tr}_- (\hat{\rho}_-(0) b_n^\dagger(0) b_n(0)) \ln [P(+ - |0)P^{-1}_v]_{nn} \right\} ,$$

where $\text{tr}_+ \hat{\rho}_+(0) a_n^\dagger(0) a_n(0) = N_n(0|\text{out})$ and $\text{tr}_- \hat{\rho}_-(0) b_n^\dagger(0) b_n(0) = N_n(0|\text{out})$ are differential mean numbers of out-electrons and out-positrons, correspondingly. They obviously coincide. Thus, the entropy takes the form

$$S (\hat{\rho}_\zeta(0)) = - k_B \left\{ \ln P_v + \sum_n N_n(0|\text{out}) \ln [P(+ - |0)P^{-1}_v]_{nn} \right\} .$$

One can use a pair creation probability and the vacuum-to-vacuum probability, written in terms of differential mean numbers (see, for example, (14)),

$$P(+ - |0)_{n,n'} = \delta_{n,n'} \frac{P_v N_n(0|\text{out})}{1 - \kappa N_n(0|\text{out})}, \quad P_v = \exp \left\{ \kappa \sum_n \ln [1 - \kappa N_n(0|\text{out})] \right\} ,$$

which by the following relation (here $D$ and $\tilde{D}$ are some matrices)

$$\exp [- a^\dagger(0) Da(0)] := \exp [\sum_n b_n(0) (1 - P(+ - |0)P^{-1}_v)_{nn} b_n(0)]$$

Taking into account, that matrices $P(+ - |0)P^{-1}_v$ are diagonal, one can rewrite (35) as

$$S (\hat{\rho}_+(0)) = - k_B \left\{ \ln P_v + \sum_n \text{tr}_+ (\hat{\rho}_+(0) a_n^\dagger(0) a_n(0)) \ln [P(+ - |0)P^{-1}_v]_{nn} \right\} ,$$

$$S (\hat{\rho}_-(0)) = - k_B \left\{ \ln P_v + \sum_n \text{tr}_- (\hat{\rho}_-(0) b_n^\dagger(0) b_n(0)) \ln [P(+ - |0)P^{-1}_v]_{nn} \right\} ,$$

where $\text{tr}_+ \hat{\rho}_+(0) a_n^\dagger(0) a_n(0) = N_n(0|\text{out})$ and $\text{tr}_- \hat{\rho}_-(0) b_n^\dagger(0) b_n(0) = N_n(0|\text{out})$ are differential mean numbers of out-electrons and out-positrons, correspondingly. They obviously coincide. Thus, the entropy takes the form

$$S (\hat{\rho}_\zeta(0)) = - k_B \left\{ \ln P_v + \sum_n N_n(0|\text{out}) \ln [P(+ - |0)P^{-1}_v]_{nn} \right\} .$$

One can use a pair creation probability and the vacuum-to-vacuum probability, written in terms of differential mean numbers (see, for example, (14)),

$$P(+ - |0)_{n,n'} = \delta_{n,n'} \frac{P_v N_n(0|\text{out})}{1 - \kappa N_n(0|\text{out})}, \quad P_v = \exp \left\{ \kappa \sum_n \ln [1 - \kappa N_n(0|\text{out})] \right\} ,$$

which by the following relation (here $D$ and $\tilde{D}$ are some matrices)

$$\exp [- a^\dagger(0) Da(0)] := \exp [\sum_n b_n(0) (1 - P(+ - |0)P^{-1}_v)_{nn} b_n(0)]$$

Using the following relation (here $D$ and $\tilde{D}$ are some matrices)

$$\exp [- a^\dagger(0) Da(0)] := \exp [\sum_n b_n(0) (1 - P(+ - |0)P^{-1}_v)_{nn} b_n(0)]$$

Using the following relation (here $D$ and $\tilde{D}$ are some matrices)

$$\exp [- a^\dagger(0) Da(0)] := \exp [\sum_n b_n(0) (1 - P(+ - |0)P^{-1}_v)_{nn} b_n(0)]$$
we obtain

\[
[\hat{\rho}_+(0)]^2 = P_v^2 \exp \left\{ \sum_n a_n^\dagger(\text{out}) \left[ (P(+-|0)P^{-1}_v)^2 - 1 \right]_{nn} a_n(\text{out}) \right\},
\]

\[
[\hat{\rho}_-(0)]^2 = P_v^2 \exp \left\{ \sum_n b_n^\dagger(\text{out}) \left[ (P(+-|0)P^{-1}_v)^2 - 1 \right]_{nn} b_n(\text{out}) \right\}.
\] (45)

Calculating the traces in (42) with account taking of (40), we finally obtain

\[
\tilde{S}(\hat{\rho}_\zeta(0)) = -P_v^2 \det \left[ 1 + \kappa (P(+-|0)P^{-1}_v)^2 \right]^\kappa = -\prod_n \left[ 1 - 2\kappa N_n(\text{out}) + (1 + \kappa) (N_n(\text{out}))^2 \right].
\] (46)

5.2 Thermal initial state

Entropy for operators \( \hat{\rho}_\zeta(\beta) \), which describe the system that has been in thermal equilibrium at the initial time instant, has the form

\[
S(\hat{\rho}_\zeta(\beta)) = -k_B \text{tr}_{\zeta} \hat{\rho}_\zeta(\beta) \ln \hat{\rho}_\zeta(\beta).
\] (47)

Transforming expressions \( \ln \hat{\rho}_\zeta(\beta) \) as

\[
\ln \hat{\rho}_+(\beta) = \ln Z_\zeta(J_\beta) + \sum_n a_n^\dagger(\text{out}) \ln [K_+(J_\beta)]_{nn} a_n(\text{out}),
\]

\[
\ln \hat{\rho}_-(\beta) = \ln Z_\zeta(J_\beta) + \sum_n b_n^\dagger(\text{out}) \ln [K_-(J_\beta)]_{nn} b_n(\text{out}),
\] (48)

one can write

\[
S(\hat{\rho}_\zeta(\beta)) = k_B \left\{ \ln Z_\zeta(J_\beta) - \sum_n N_{n,\zeta}(\beta|\text{out}) \ln [K_\zeta(J_\beta)]_{nn} \right\},
\] (49)

where \( N_{n,\zeta}(\beta|\text{out}) \) are given by (41) with \( N_{n,\zeta}(\cdots|\text{in}) = N_{n,\zeta}(\beta|\text{in}) \). One can express diagonal elements of \( K_\zeta(J_\beta) \) in terms of the corresponding occupation numbers \( N_{n,\zeta}(\beta|\text{out}) \)

\[
[K_\zeta(J_\beta)]_{nn} = \frac{N_{n,\zeta}(\beta|\text{out})}{1 - \kappa N_{n,\zeta}(\beta|\text{out})},
\] (50)

and do the same to \( Z_\zeta(J_\beta) \) by means of the normalization condition \( \text{tr}_{\zeta} \hat{\rho}_\beta,\zeta = 1 \)

\[
Z_\zeta(J_\beta) = \exp \left\{ -\kappa \sum_n \ln [1 - \kappa N_{n,\zeta}(\beta|\text{out})] \right\},
\] (51)

to rewrite expression (49) for the entropy in the form

\[
S(\hat{\rho}_\zeta(\beta)) = -k_B \sum_n S(\hat{\rho}_{n,\zeta}(\beta)), \quad S(\hat{\rho}_{n,\zeta}(\beta))
\]

\[
= -k_B \left\{ \kappa [1 - \kappa N_{n,\zeta}(\beta|\text{out})] \ln [1 - \kappa N_{n,\zeta}(\beta|\text{out})] + N_{n,\zeta}(\beta|\text{out}) \ln N_{n,\zeta}(\beta|\text{out}) \right\}.
\] (52)

Considering expressions (52), (41), and (32), one can see that they all have similar forms.

Next, let us find the Schmidt measure for subsystems of positrons and electrons for the system with a thermal state as the initial time instant; subsystems of such state is
described by reduced density operator $\hat{\rho}_\zeta(\beta)$. Entanglement measure of electron and positron subsystem is given by
\[
\tilde{S}(\hat{\rho}_\zeta(\beta)) = -\text{tr} \left[ \hat{\rho}_\zeta(\beta) \right]^2, \tag{53}
\]
where the square of the operators $\hat{\rho}_\zeta(\beta)$ are
\[
[\hat{\rho}_+(\beta)]^2 = Z_+^{-2}(J_\beta); \exp \left[ \sum_n a_n^\dagger(\text{out}) [K_+^2(J_\beta) - 1]_{nn} a_n(\text{out}) \right]: ,
\]
\[
[\hat{\rho}_-(\beta)]^2 = Z_-^{-2}(J_\beta); \exp \left[ \sum_n b_n^\dagger(\text{out}) [K_-^2(J_\beta) - 1]_{nn} b_n(\text{out}) \right]: , \tag{54}
\]
such that
\[
\tilde{S}(\hat{\rho}_\zeta(\beta)) = -\prod_n \left\{ 1 - 2\kappa N_n,\zeta (\beta|\text{out}) + (1 + \kappa) [N_n,\zeta (\beta|\text{out})]^2 \right\}^\kappa. \tag{55}
\]

5.3 Entropy of reduced by measurements density operators

Entropy of a density operator $\hat{\rho} \rho^N (22)$ has the form
\[
S(\hat{\rho} \rho^N) = -k_B \text{tr} \hat{\rho} \rho^N \ln \hat{\rho} \rho^N. \tag{56}
\]
Representation (3) allows one to factorize the complete vacuum in product of single-mode vacua,
\[
|0,\text{out}\rangle \langle 0,\text{out}| = \prod_n |0,\text{out}\rangle_{nn} \langle 0,\text{out}|, \quad a_n(\text{out})|0,\text{out}\rangle_n = 0, \quad b_n(\text{out})|0,\text{out}\rangle_n = 0. \tag{57}
\]
Using this fact and the representation for $|c_v|^2$ from (19), one can rewrite density operator (22) as a product of single-mode density operators:
\[
\hat{\rho} \rho^N = \prod_n \hat{\rho} \rho^N,n, \quad \text{tr} \hat{\rho} \rho^N,n = 1, \quad \hat{\rho} \rho^N,n = |c_v|^2 \sum_{f=0}^{+} W_{f,n} |f,\text{out}\rangle_{nn} \langle f,\text{out}|, \quad |c_v|^2 = |w (-|-n)|^{-2\kappa}, \quad W_{f,n} = |w (+ - |0)_{nn}|^2, \quad |f,\text{out}\rangle_n = \sum_{f=0}^{+} \left[ a_n^\dagger(\text{out}) b_n^\dagger(\text{out}) \right]^f / f! |0,\text{out}\rangle_n. \tag{58}
\]
Quantities $|c_v|^2_n$ and $|w (+ - |0)_{nn}|^2$ can be expressed via differential numbers $N_n(0|\text{out})$ as
\[
|c_v|^2_n = (1 - \kappa N_n(0|\text{out}))^\kappa, \quad |w (+ - |0)_{nn}|^2 = \frac{N_n(0|\text{out})}{1 - \kappa N_n(0|\text{out})}. \tag{59}
\]
Due to expression (58), entropy (56) can be written as
\[
S(\hat{\rho} \rho^N) = -k_B \sum_n \text{tr} \hat{\rho} \rho^N,n \ln \hat{\rho} \rho^N,n. \tag{60}
\]
In order to take the trace of the operator $\hat{\rho} \rho^N,n \ln \hat{\rho} \rho^N,n$, one can use the formal decomposition
\[
\hat{\rho} \rho^N,n \ln \hat{\rho} \rho^N,n = \hat{\rho} \rho^N,n \sum_{k=1}^{\infty} k^{-1} (\hat{\rho} \rho^N,n - 1)^k = \sum_{k=1}^{\infty} k^{-1} \sum_{l=0}^{k} C_k^l (\hat{\rho} \rho^N,n)^{l+1} (-1)^{k-l}, \tag{61}
\]
where $C_k^l$ are binomial coefficients. Due to the orthonormality of the states $|f,\text{out}\rangle_n$ the density operators $(\hat{\rho}_{N,n})^{l+1}$ have the form

$$(\hat{\rho}_{N,n})^{l+1} = |c_{v,n}^{2(l+1)}| \sum_{f=0}^{\infty} (W_{f,n})^{l+1} |f,\text{out}\rangle_n \langle f,\text{out}|.$$  \hfill (62)

Substituting (62) into (61), we obtain

$$\hat{\rho}_{N,n} \ln \hat{\rho}_{N,n} = |c_{v,n}^2| \sum_{f=0}^{\infty} W_{f,n} \ln (|c_{v,n}^2 W_{f,n}|) |f,\text{out}\rangle_n \langle f,\text{out}|.$$  \hfill (63)

Then

$$\text{tr} \hat{\rho}_{N,n} \ln \hat{\rho}_{N,n} = N_n(0|\text{out}) \ln N_n(0|\text{out}) + \kappa [1 - \kappa N_n(0|\text{out})] \ln [1 - \kappa N_n(0|\text{out})].$$  \hfill (64)

Thus, the entropy of density operator (22) reads

$$S(\hat{\rho}_N) = -k_B \sum_n \{\kappa [1 - \kappa N_n(0|\text{out})] \ln [1 - \kappa N_n(0|\text{out})] + N_n(0|\text{out}) \ln N_n(0|\text{out})\}.$$  \hfill (65)

The result has the same form as the entropy $S(\hat{\rho}_N(0))$ given by (11). Thus, we can say that measurement of $N$, $N_+$ or $N_-$ leads to the same information loss as reduction over electrons or positrons.

It was shown in the sect. 4 that reduction of density operator $\hat{\rho}_N$ over electrons and positrons transform it in $[\hat{\rho}_N],_\zeta = \hat{\rho}_\zeta(0)$. This means that if one calculates entropy of density operator $[\hat{\rho}_N],_\zeta$, one obtains the same expression (65) again. The conditional entropy $S_{\text{cond}} = S(\hat{\rho}_N) - S([\hat{\rho}_N],_\zeta)$, which is used as a measure of correlations between subsystems, is zero. This fact means that all quantum correlations between the electrons and positrons are lost due to decoherence, and there is no any entanglement after the measurement.

### 6 T-constant external electric field

To illustrate some of the above general formulas we consider so-called $T$-constant electric field as an external background. Such a field acts only for a finite time $T$ and is constant within this time interval. Using such a field allows one to avoid problems with the definition of in- and out-states in non-switching external fields at $t \to \pm \infty$. Another important point is that this field produces a finite work in a finite space volume. Let us consider $d = (D+1)$-dimensional space, then the $T$-constant electric field $E$ is acting during the time interval $T = t_{\text{out}} - t_{\text{in}}$,

$$E = (0, E(t), 0, ..., 0), \quad E(t) = \begin{cases} 0, & -\infty < t \leq t_{\text{in}} \\ E > 0, & t_{\text{in}} < t < t_{\text{out}} \\ 0, & t_{\text{out}} \leq t < \infty \end{cases}.$$  \hfill (66)

Processes of pair creation in such field were studied in Refs. [26, 27, 14, 23]. Similar to these works, we consider big enough $T$.

Since there is no particle production after the time instant $t_{\text{out}}$, differential mean numbers of particles $N_{n,\zeta}(\cdots|\text{out})$ created in a given state $n = \mathbf{p}, r$ ($\mathbf{p}$ is a $D$-dimensional vector of momentum and $r$ is spin) depend only on the time interval. Electric field acting during the sufficiently big time $T$ creates a considerable number of pairs only in a finite region in the momentum space. We suppose that $T$ is big enough and $T \gg \max\{1, E_c/E\}$, one needs to consider only the range

$$|p_\perp| \leq \sqrt{eE} \left[\sqrt{eET}\right]^{1/2}, \quad -T/2 \leq p_\parallel /eE \leq T/2$$  \hfill (67)

in the momentum space, see [14] for details. Note that for the case $d = 2$ there are no transversal components of momentum.
6.1 Vacuum initial state

First let us consider the case when the system initially was in the vacuum state. For this case differential mean numbers in the momentum range (77) are

\[ N_n(0|\text{out}) = e^{-\lambda}, \quad \lambda = (p^2_\perp + m^2)/eE. \] (68)

They have the same form as in the case of the constant uniform electric field [22, 11], and are the same for bosons and fermions. Entropy (41) is expressed in terms of \( N_n(0|\text{out}) \) and do not depend on the spin quantum number \( \gamma \), thus, the summation over the latter results in the factor \( \gamma_1(\delta) = 2(\frac{\delta}{2})^{-1} \).

First we consider the Dirac case with \( \kappa = +1 \):

\[ S(\hat{\rho}_{n,\zeta}(0)) = -k_B \left[ (1 - N_n(0|\text{out})) \ln(1 - N_n(0|\text{out})) + N_n(0|\text{out}) \ln N_n(0|\text{out}) \right]. \] (69)

For the case of electric field the mean number of particles created \( N_n(0|\text{out}) \) can change only from 0 to 1 and depend only on strength of the external field. Expression (69) is symmetric with respect to \( N_n,\zeta(0|\text{out}) \). It reaches maximum at \( N_n(0|\text{out}) = 1/2 \) and turn to zero at \( N_n(0|\text{out}) = 1 \) and \( N_n(0|\text{out}) = 0 \). This fact can be interpreted as follows. In case of \( N_n(0|\text{out}) = 0 \) there is no particles created by external field and initial vacuum state in the mode remain unchanged. Case \( N_n(0|\text{out}) = 1 \) correspond to situation when particle is created with certainty. The maximum of (69), corresponding to \( N_n(0|\text{out}) = 1/2 \), is connected to the state with maximal amount of uncertainty.

Representing logarithm in the first term of expression (69) as Taylor series in powers of \( \hat{n}_x,\zeta(0) \), we see that \( S(\hat{\rho}_{n,\zeta}(0)) \) is proportional to \( N_n(0|\text{out}) \). The latter plays the role of the cut-off factor for the integral over \( p_1 \). Thus, the summation over the quantum numbers can be reduced to an integration over momenta that satisfy restrictions (77),

[\[ \sum_n \gamma_1(\delta)V \frac{(2\pi)^{d-1}}{(2\pi)^d} \int dp, \]

where \( V \) is \( D \)-dimensional spatial volume. The mean numbers (68) do not depend on longitudinal component of momentum. Outside of the range (77) contribution to the integral is very small, allowing to extend integration limits of \( p_\perp \) to infinities. Integration over \( p_\perp \) can be done using the Taylor series. The result of the integration is

\[ S(\hat{\rho}_\zeta(0)) = \gamma_1(\delta)k_B \frac{(eE)^{\frac{d}{2}}TV}{(2\pi)^{d-1}} A_{\text{Dirac}}(d, E_c/E), \] (70)

where the factor \( TV \) can be considered as \( d \)-dimensional volume. To get finite and correct expressions, one should use volume normalization. The factor \( A_{\text{Dirac}}(d, E_c/E) \) has the form

\[ A_{\text{Dirac}}(d, E_c/E) = \left[ \sum_{l=1}^{\infty} l^{-d/2} \exp[-\pi l E_c/E] \ight. \]

\[- \left. \sum_{l=1}^{\infty} l^{-1}(l+1)^{\frac{d-2}{2}} \exp[-\pi l (l+1) E_c/E] + \left( \frac{E_c}{E} + \frac{d-2}{2} \right) \exp(-\pi E_c/E) \right]. \]

It is possible to estimate the entropy in strong \( E_c/E \ll 1 \), critical \( E_c/E = 1 \), and weak \( E_c/E \gg 1 \) field limits. For example, for a strong field with \( d = 4 \) we have \( A_{\text{Dirac}}(4,0) = \pi^2/6 \), for the critical field, we have \( A_{\text{Dirac}}(4,1) \approx 0.22 \). In case of a weak field the entropy has a small value of the order \( (\pi E_c/E) \exp[-\pi E_c/E] \) for any \( d \). For \( d = 3 \) the following estimations hold \( A_{\text{Dirac}}(3,0) \approx 0.93 \), \( A_{\text{Dirac}}(3,1) \approx 0.2 \); for \( d = 2 \) the factor \( A(2,0) \) is a value of order 1 and \( A(2,1) = e^{-\pi} \).
Let us consider the K-G case \((\kappa = -1)\),

\[
S(\rho_n,\zeta(0)) = k_B \left\{ [1 + N_n(0|\text{out})] \ln [1 + N_n(0|\text{out})] - N_n(0|\text{out}) \ln N_n(0|\text{out}) \right\}. \tag{71}
\]

Expression \((71)\) just increases with \(N_0\). Let us discuss two cases, one the case of a low temperature and the other, the case of a high temperature. When all the energies of the particles created with a given \(P_{\perp}\) are considerably higher than the temperature, and the case of a high temperature \(\beta eE \ll 1\), when all the energies of the created particles are much lower than the temperature. We assume for simplicity that \(eE \gg \mu\) and \(T\) is big enough so that \((eE)^2 \gg m^2 + p_{\perp}^2\).

In the case of the low temperature, the number of created particles does not depend on the longitudinal momenta:

\[
N_{n,\zeta}(\beta|\text{in}) \approx \exp (-\beta \varepsilon_n) \rightarrow 0, \quad N_{n,\zeta}(\beta|\text{out}) \rightarrow N_{n,\zeta}(0|\text{out}).
\]

In such a limit entropy \(S(\rho_{n,\zeta}(\beta))\) tends to zero temperature case (initial vacuum state). Then integration over transversal momenta can be done exactly as in the initial vacuum case. Formal calculations of \(N_{n,\zeta}(\beta|\text{out})\) and the entropy in the case of high temperature, \(\beta eE \ll 1\), are also quite simple. However, it was shown that in the Dirac case under such a condition the current density due to the work of the external field on the already existing in the initial state particles (which was denoted as \(\text{Re}(\bar{j}_\mu (t))\bar{\gamma}_0\) in \([23]\)) is much greater than

\[16\]
the current density of particles created from vacuum \cite{23}. Therefore in such a case the particle creation effect may be disregarded.

We note that the general form of the reduced density operators $\hat{\rho}_\pm$, given by Eq. (13), allows one to study the change of the entropy and the corresponding entanglement during many consecutive measurements. In this case the density operator $\hat{\rho}_N$ \cite{23} has to be considered as the initial state for the second stage of the evolution and so on. In the general case it is not simple to describe such a decoherence procedure for arbitrary stage. However, as was mentioned above, if the mean numbers $N_n(0|_{\text{out}})$ are not small for large enough range of momenta, already on the second stage the particle creation effect may be disregarded, and the subsequent decoherence is described in usual terms.

7 Summary

Using a general nonperturbative expression for the density operators of quantized Dirac or K-G fields, we derive its specific form corresponding to different initial conditions. Applying a reduction procedure to specific density operators we construct mixed states of both electron and positron subsystems. Calculating the entropy of such states, we obtain the loss of information due to the reduction and, at the same time, entanglement of electron and positron subsystems. We pay attention on the fact that any measurement in the system under consideration implies a decoherence and the corresponding modifications of the complete and the reduced density operators. We study results of such a decoherence and related to it loss of the information calculating the information entropy. To illustrate some of the obtained general results, we consider the slowly varying $T$-constant electric field as an external background. Here we derive the following conclusions. The entropy of any subsystem (of the electrons or positrons) with the vacuum as the initial state is proportional to the factor $(eE)^{2\gamma_d}$ and the number of spin degree of freedom $\gamma_d$. It grows linearly with the time of the field action $T$. The above behavior remains in the thermal case at low temperatures, in fact here the entropy does not depend on the temperature.

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A QED with strong electric-like background

In this Appendix, we consider briefly a special case of QFT with unstable vacuum of quantized Dirac or K-G field with time dependent electric-like background that switches on and off at $t \rightarrow \pm \infty$. Quantization of such a theory in terms of in- and out-electrons and positrons was elaborated in Refs. \cite{13}. Some necessary for us results of such a quantization are presented below.

We denote operators in the Schrödinger representation with the hat $\hat{A}$ above, while operators in the Heisenberg representation by the inverted hat $\check{A}$ above. In the Schrödinger picture one can define: at the initial time instant $t_{\text{in}}$, a set of creation and annihilation operators $a_{n_{\text{in}}}(t_{\text{in}})$, $a_{n_{\text{in}}}(t_{\text{in}})$ of electrons, as well as similar operators $b_{n_{\text{in}}}(t_{\text{in}})$, $b_{n_{\text{in}}}(t_{\text{in}})$ of positrons, such that a corresponding vacuum at $t_{\text{in}}$ is $\ket{0,t_{\text{in}}}$; at the final time instant $t_{\text{out}}$, a set of creation and annihilation operators $a_{n_{\text{out}}}(t_{\text{out}})$, $a_{n_{\text{out}}}(t_{\text{out}})$, of electrons and similar operators $b_{n_{\text{out}}}(t_{\text{out}})$, $b_{n_{\text{out}}}(t_{\text{out}})$ of positrons, such that the corresponding vacuum at $t_{\text{out}}$ is $\ket{0,t_{\text{out}}}$.

$$
a_{n_{\text{in}}}(t_{\text{in}})\ket{0,t_{\text{in}}} = b_{n_{\text{in}}}(t_{\text{in}})\ket{0,t_{\text{in}}} = 0, \quad a_{n_{\text{in}}}(t_{\text{in}})\ket{0,t_{\text{out}}} = b_{n_{\text{in}}}(t_{\text{out}})\ket{0,t_{\text{out}}} = 0 \quad \forall n.$$
The probability amplitude for transition from an initial to a final state $M_{\text{in} \rightarrow \text{out}}$ has the following form in the Schrödinger picture:

$$M_{\text{in} \rightarrow \text{out}} = \langle \text{out} | U(t_{\text{out}}, t_{\text{in}}) | \text{in} \rangle,$$

where $U(t, t')$ is a unitary evolution operator of the system. The density operator of an initial state $\hat{\rho}(t_{\text{in}})$ is given as an operatorial function of the creation and annihilation operators of electrons (positrons) at the initial time instant,

$$\hat{\rho}(t_{\text{in}}) = \rho_{\text{in}} (a^\dagger(t_{\text{in}}), a(t_{\text{in}}), b^\dagger(t_{\text{in}}), b(t_{\text{in}})).$$

A mean value of a physical quantity $F$ at the final time instant reads

$$\langle F(t_{\text{out}}) \rangle = \text{tr} \hat{\rho}(t_{\text{out}}) \hat{F}(t_{\text{out}}),$$

where $\hat{\rho}(t)$ is the density operator in Schrödinger representation at time instant $t$, and the $\text{tr}$ stands for the complete trace,

$$\hat{\rho}(t_{\text{out}}) = U (t_{\text{out}}, t_{\text{in}}) \hat{\rho}(t_{\text{in}}) U^\dagger (t_{\text{out}}, t_{\text{in}}).$$

To pass to the Heisenberg picture, we define the finite-time evolution operators $\Omega_{\pm}$,

$$\Omega_{(+)} = U (0, t_{\text{in}}), \quad \Omega_{(-)} = U (0, t_{\text{out}}), \quad U (t_{\text{out}}, t_{\text{in}}) = \Omega_{(-)}^\dagger \Omega_{(+)},$$

$$\hat{\rho} = \hat{\rho}(0) = \Omega_{(+)} \hat{\rho}(t_{\text{in}}) \Omega_{(+)}^\dagger = \Omega_{(-)} \hat{\rho}(t_{\text{out}}) \Omega_{(-)}^\dagger,$$

and a set of creation and annihilation operators $a^\dagger_{\text{in}}(\text{in})$, $a_{\text{in}}(\text{in})$ of in-electrons, as well as similar operators $b^\dagger_{\text{in}}(\text{in})$, $b_{\text{in}}(\text{in})$ of in-positrons, a corresponding in-vacuum $|0, \text{in}\rangle$, and a set of creation and annihilation operators $a^\dagger_{\text{out}}(\text{out})$, $a_{\text{out}}(\text{out})$ of out-electrons and similar operators $b^\dagger_{\text{out}}(\text{out})$, $b_{\text{out}}(\text{out})$ of out-positrons, and a corresponding out-vacuum $|0, \text{out}\rangle$,

$$\{a(\text{in}), \cdots \} = \Omega_{(+)} \{a(t_{\text{in}}), \cdots \} \Omega_{(+)}^\dagger, \quad |0, \text{in}\rangle = \Omega_{(+)} |0, t_{\text{in}}\rangle,$$

$$\{a(\text{out}), \cdots \} = \Omega_{(-)} \{a(t_{\text{out}}), \cdots \} \Omega_{(-)}^\dagger, \quad |0, \text{out}\rangle = \Omega_{(-)} |0, t_{\text{out}}\rangle,$$

$$M_{\text{in} \rightarrow \text{out}} = \langle 0, t_{\text{out}}| \cdots a(t_{\text{in}}) \Omega_{(-)}^\dagger \Omega_{(+)} a^\dagger_{\text{in}}(\text{in}) \cdots |0, t_{\text{in}}\rangle = (0, \text{out}) \cdots a(\text{out}) a^\dagger_{\text{in}}(\text{in}) \cdots |0, \text{in}\rangle,$$

$$c_v = \langle 0, t_{\text{out}}| U(t_{\text{out}}, t_{\text{in}}) | 0, t_{\text{in}}\rangle = (0, \text{out}) |0, \text{in}\rangle.$$

The entire information concerning the processes of particle creation, annihilation and scattering is contained in the elementary probability amplitudes

$$w (+|+)_{mn} = c_v^{-1} \langle 0, \text{out} | a_{\text{out}}(\text{out}) a^\dagger_{\text{in}}(\text{in}) | 0, \text{in} \rangle,$$

$$w (-|-)_{nm} = c_v^{-1} \langle 0, \text{out} | b_{\text{out}}(\text{out}) b^\dagger_{\text{in}}(\text{in}) | 0, \text{in} \rangle,$$

$$w (0|+)_{nm} = c_v^{-1} \langle 0, \text{out} | b^\dagger_{\text{in}}(\text{in}) a^\dagger_{\text{in}}(\text{in}) | 0, \text{in} \rangle,$$

$$w (+|0)_{mn} = c_v^{-1} \langle 0, \text{out} | a_{\text{out}}(\text{out}) b_{\text{out}}(\text{out}) | 0, \text{in} \rangle.$$

The amplitudes (78) can be calculated with the help of some appropriate sets of solutions of the corresponding relativistic wave equation with an external field (Klein–Gordon, Dirac, and so on), see [13]. We are interested in case of uniform external field, that do not mix different quantum modes. Thus, in this paper the amplitudes (78) are diagonal in quantum numbers,

$$w (\zeta | \zeta)_{mn} = \delta_{mn} w (\zeta | \zeta)_{nn}, \quad w (0|+)_{nm} = \delta_{mn} w (0|+)_{nn}, \quad w (+|0)_{nm} = \delta_{mn} w (+|0)_{nn}.$$
The sets of in- and out-operators are related to each other by a linear canonical transformation \[16\], which can be written in terms of the amplitudes \( (78) \) as follows
\[
\begin{align*}
    a^\text{(out)} &= \left[w^{(+|+)^\dagger}\right]^{-1} a^\text{(in)} - \kappa w^{(+|-0)} w^{(-|-)^{-1}} b^\dagger^\text{(in)}, \\
    b^\dagger^\text{(out)} &= \left[w^{(+|+)^\dagger}\right]^{-1} w^{(+|-0)} a^\text{(in)} + [w^{(-|-)^{-1}} b^\dagger^\text{(in)},
\end{align*}
\]
and their Hermitian conjugated. As has been demonstrated, (see \[13\]) such a relation is given by an unitary operator \( V \),
\[
V \{a^\text{(out)}, \cdots \} V^\dagger = \{a^\text{(in)}, \cdots \}, \quad |0,\text{in}\rangle = V|0,\text{out}\rangle,
\]
where a unitary operator \( V \) has the form
\[
\begin{align*}
    v_1 &= \exp \{-\kappa b^\text{(out)} w^{(0|-+)} a^\text{(out)}\}, \quad v_2 = \exp \{a^\dagger^\text{(out)} \ln w^{(+|+)} a^\text{(out)}\}, \\
    v_3 &= \exp \{-\kappa b^\text{(out)} \ln w^{(-|0)} b^\dagger^\text{(out)}\}, \quad v_4 = \exp \{-\kappa a^\dagger^\text{(out)} w^{(+|-0)} b^\dagger^\text{(out)}\}.
\end{align*}
\]
Using this expression for \( V \), one can find
\[
c_v = \langle 0,\text{out}|V|0,\text{out}\rangle = \exp \{-\kappa \text{tr} \ln w^{(-|-)}\}.
\]

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\[2\] We use condensed notations, for example,
\[
ba w (0|-+) a = \sum_{n,m} b_n w (0|-+) a_m a_n.
\]
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