Reducible second-class constraints of order $L$: an irreducible approach

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Abstract

An irreducible canonical approach to second-class constraints reducible of an arbitrary order is given. This method generalizes our previous results from [10, 11] for first- and respectively second-order reducible second-class constraints. The general procedure is illustrated on Abelian gauge-fixed $p$-forms.

1 Introduction

The canonical approach to systems with reducible second-class constraints is quite intricate, demanding a modification of the usual rules as the matrix of the Poisson brackets among the constraint functions is not invertible. Thus, it is necessary to isolate a set of independent constraints and then construct the Dirac bracket [1, 2] with respect to this set. The split of constraints may however lead to the loss of important symmetries, so it should be avoided. As shown in [3, 4, 5, 6, 7, 8], it is however possible to construct the Dirac
bracket in terms of a noninvertible matrix without separating the independent constraint functions. A third possibility is to substitute (by an appropriate extension of the phase-space) the reducible second-class constraints with some irreducible ones, defined in the extended phase-space, and further work with the Dirac bracket based on the irreducible constraints. This idea, suggested in [9] mainly in the context of 2- and 3-form gauge fields, has been developed in a general manner only for first- and respectively second-order reducible second-class constraints [10, 11]. Other interesting contributions to reducible second-class constrained systems (including the split involution formalism) can be found in [12, 13, 14, 15, 16]. The idea of extending the phase-space is not new. It has been used previously, for instance in the context of the conversion approach exposed in [17], where some supplementary variables are added in order to convert a set of irreducible second-class constraints into a first-class one.

In this paper we give an irreducible approach to third-order reducible second-class constraints and then generalize the results to an arbitrary order of reducibility, $L$. Our strategy includes three main steps. First, we express the Dirac bracket for the reducible system in terms of an invertible matrix. Second, we construct an intermediate reducible second-class system (of the same reducibility order like the original one) on a larger phase-space and establish the (weak) equality between the original Dirac bracket and that corresponding to the intermediate theory. Third, we prove that there exists an irreducible second-class constraint set equivalent to the intermediate one, such that the corresponding Dirac brackets coincide (weakly). These three steps enforce the fact that the fundamental Dirac brackets derived within the irreducible and original reducible settings coincide (weakly). The equality between the fundamental Dirac brackets associated with the original phase-space variables in the reducible and respectively irreducible formulations has major implications on the relationship between the reducible and irreducible systems: i) the two systems exhibit the same number of physical degrees of freedom, which is precisely the rank of the induced symplectic form (since the Dirac bracket restricted to the constraint surface is determined by the inverse of the induced symplectic form, see Theorem 2.5 from [7]); ii) the physical content of the two theories is the same from the perspective of quantization as they display the same fundamental observables; iii) the original, reducible system can be equivalently replaced with the irreducible one. It is important to remark that the irreducible approach is useful mainly in field theory because it does not spoil the important symmetries of the original system,
such as the spacetime locality of second-class field theories.

The present paper is organized into six sections. In Section 2 we briefly review the procedure for second-class constraints that are reducible of order one and respectively two. Sections 3 and 4 define the ‘hard core’ of the paper. We initially approach second-class constraints reducible of order three in Section 3 by implementing the three main steps mentioned above, and then generalize these results to an arbitrary order of reducibility in Section 4. In Section 5 we exemplify in detail the general procedure from Section 4 on gauge-fixed Abelian $p$-form gauge fields. Section 6 ends the paper with the main conclusions.

2 First- and second-order reducible second-class constraints: a brief review

2.1 Dirac bracket for first- and second-order reducible second-class constraints

We start with a system locally described by $N$ canonical pairs $z^a = (q^i, p_i)$ and subject to the constraints

$$
\chi_{\alpha_0} (z^a) \approx 0, \quad \alpha_0 = 1, M_0.
$$

(1)

For simplicity, we take all the phase-space variables to be bosonic. However, our analysis can be extended to fermionic degrees of freedom modulo including some appropriate phase factors. We choose the scenario of systems with a finite number of degrees of freedom only for notational simplicity, but our approach is equally valid for field theories. In addition, we presume that the functions $\chi_{\alpha_0}$ are not all independent, but there exist some nonvanishing functions $Z_{\alpha_0}^{\alpha_1}$ such that

$$
Z_{\alpha_0}^{\alpha_1} \chi_{\alpha_0} = 0, \quad \alpha_1 = 1, M_1.
$$

(2)

Moreover, we assume that the functions $Z_{\alpha_0}^{\alpha_1}$ are all independent and are the only reducibility relations with respect to the constraints (1). These constraints are purely second class if any maximal, independent set of $M_0 - M_1$ constraint functions $\chi_A$ ($A = 1, M_0 - M_1$) among the $\chi_{\alpha_0}$ is such that the matrix

$$
C_{A,B}^{(1)} = [\chi_A, \chi_B]
$$

(3)
is invertible. Here and in the following the symbol $[,]$ denotes the Poisson bracket. In terms of independent constraints, the Dirac bracket takes the form

$$[F,G]^{(1)*} = [F,G] - [F, \chi_A] M^{(1)AB} [\chi_B, G],$$

where $M^{(1)AB} C^{(1)}_{BC} \approx \delta^A_C$. In the previous relations we introduced an extra index, (1), having the role to emphasize that the Dirac bracket given in (1) is based on a first-order reducible second-class constraint set. We can rewrite the Dirac bracket expressed by (1) without finding a definite subset of independent second-class constraints as follows. We start with the matrix

$$C^{(1)}_{\alpha_0 \beta_0} = [\chi_{\alpha_0}, \chi_{\beta_0}],$$

which clearly is not invertible because

$$Z^{\alpha_0}_{\alpha_1} C^{(1)}_{\alpha_0 \beta_0} \approx 0.$$  

If $\tilde{a}^{\alpha_1}_{\alpha_0}$ is a solution to the equation

$$\tilde{a}^{\alpha_1}_{\alpha_0} Z^{\alpha_0}_{\beta_1} \approx \delta^{\alpha_1}_{\beta_1},$$

then we can introduce a matrix (6) of elements $M^{(1)\alpha_0 \beta_0}$ through the relation

$$M^{(1)\alpha_0 \beta_0} C^{(1)}_{\beta_0 \gamma_0} \approx \delta^{\alpha_0}_{\gamma_0} - Z^{\alpha_0}_{\alpha_1} \tilde{a}^{\alpha_1}_{\gamma_0} = d^{\alpha_0}_{\gamma_0},$$

with $M^{(1)\alpha_0 \beta_0} = -M^{(1)\beta_0 \alpha_0}$. Then, formula (6)

$$[F,G]^{(1)*} = [F,G] - [F, \chi_{\alpha_0}] M^{(1)\alpha_0 \beta_0} [\chi_{\beta_0}, G],$$

defines the same Dirac bracket like (1) on the surface (1). We remark that there exist some ambiguities in defining the matrix of elements $M^{(1)\alpha_0 \beta_0}$ since if we make the transformation

$$M^{(1)\alpha_0 \beta_0} \to M^{(1)\alpha_0 \beta_0} + Z^{\alpha_0}_{\alpha_1} q^{\alpha_1 \beta_1} Z^{\beta_1}_{\beta_0},$$

with $q^{\alpha_1 \beta_1}$ some completely antisymmetric functions, then equation (8) is still satisfied. Relations (6) and (8) show that

$$\text{rank} \left(d^{\alpha_0}_{\gamma_0}\right) \approx M_0 - M_1,$$
which ensures the fact that the rank of the matrix of elements \( M^{(1)\alpha_0\beta_0}C^{(1)}_{\beta_0\gamma_0} \) is equal to the number of independent second-class constraints in the presence of the first-order reducibility.

Let us extend the previous construction to the case of second-order reducible second-class constraints. This means that not all of the first-order reducibility functions \( Z^{\alpha_0}_{\alpha_1} \) are independent. Beside the first-order reducibility relations (2), there appear also the second-order reducibility relations

\[
Z^{\alpha_1}_{\alpha_2}Z^{\alpha_0}_{\alpha_1} \approx 0, \quad \alpha_2 = 1, M_2. \tag{12}
\]

We will assume that the reducibility stops at order two, so all the functions \( Z^{\alpha_1}_{\alpha_2} \) are by hypothesis taken to be independent. It is understood that the functions \( Z^{\alpha_1}_{\alpha_2} \) define a complete set of reducibility functions for \( Z^{\alpha_0}_{\alpha_1} \). In this situation, the number of independent second-class constraints is equal to \( M_0 - M_1 + M_2 \). As a consequence, we can work with a Dirac bracket of the type (11), but in terms of \( M_0 - M_1 + M_2 \) independent functions \( \chi^{A} \)

\[
[F, G]^{(2)*} = [F, G] - [F, \chi^{A}] M^{(2)AB} \chi^{B}, G], \quad A = 1, M_0 - M_1 + M_2, \tag{13}
\]

where \( M^{(2)AB}C^{(2)}_{BC} \approx \delta^{A}_{C} \), with \( C^{(2)}_{AB} = [\chi^{A}, \chi^{B}] \). It is obvious that the matrix of elements

\[
C^{(2)}_{\alpha_0\beta_0} = [\chi^{\alpha_0}, \chi^{\beta_0}] \tag{14}
\]

satisfies the relations

\[
Z^{\alpha_0}_{\alpha_1}C^{(2)}_{\alpha_0\beta_0} \approx 0, \tag{15}
\]

so its rank is equal to \( M_0 - M_1 + M_2 \).

Let \( \bar{A}^{(\alpha_2)} \) be a solution of the equation

\[
Z^{\alpha_1}_{\beta_2} \bar{A}^{(\alpha_2)} \approx \delta^{(\alpha_2)}_{\beta_2} \.tag{16}
\]

and \( \bar{\omega}_{\beta_1\gamma_1} = -\bar{\omega}_{\gamma_1\beta_1} \) a solution to

\[
Z^{\beta_1}_{\beta_2}\bar{\omega}_{\beta_1\gamma_1} \approx 0. \tag{17}
\]

We define an antisymmetric matrix, of elements \( \hat{\omega}^{\alpha_1\beta_1} \), through the relation

\[
\hat{\omega}^{\alpha_1\beta_1} \bar{\omega}_{\beta_1\gamma_1} \approx \delta^{\alpha_1}_{\gamma_1} - Z^{\alpha_1}_{\alpha_2} \bar{A}^{(\alpha_2)} \hat{D}^{\alpha_1}_{\gamma_1}. \tag{18}
\]

Taking (17) into account, it results that \( \hat{\omega}^{\alpha_1\beta_1} \) contains some ambiguities, namely it is defined up to the transformation

\[
\hat{\omega}^{\alpha_1\beta_1} \rightarrow \hat{\omega}^{\alpha_1\beta_1} + Z^{\alpha_1}_{\alpha_2} \hat{\omega}^{\alpha_2\beta_2} Z^{\beta_1}_{\beta_2}. \tag{19}
\]
with $q^{\alpha_2\beta_2}$ some arbitrary, antisymmetric functions. On the other hand, simple computation shows that the matrix of elements $D_{\gamma_1}^{\alpha_1}$ satisfies the properties

\[ \bar{A}_{\alpha_1}^{\alpha_2} D_{\gamma_1}^{\alpha_1} \approx 0, \quad Z_{\gamma_2}^{\gamma_1} D_{\gamma_1}^{\alpha_1} \approx 0, \]  
\[ Z_{\alpha_1}^{\alpha_0} D_{\gamma_1}^{\alpha_1} \approx Z_{\gamma_1}^{\alpha_0}, \quad D_{\gamma_1}^{\alpha_1} D_{\lambda_1}^{\gamma_1} \approx D_{\lambda_1}^{\alpha_1}. \]  

(20)

Based on the latter formula from (20), we infer an alternative expression for $D_{\gamma_1}^{\alpha_1}$, namely

\[ D_{\gamma_1}^{\alpha_1} \approx \bar{A}_{\alpha_1}^{\alpha_0} Z_{\gamma_1}^{\alpha_0}, \]  

(21)

for some functions $\bar{A}_{\alpha_1}^{\alpha_0}$. From the former relation in (21) and (22) we deduce that

\[ Z_{\gamma_1}^{\gamma_0} D_{\gamma_0}^{\alpha_0} \approx 0, \]  

(23)

where

\[ D_{\gamma_0}^{\alpha_0} \approx \delta_{\gamma_0}^{\alpha_0} - Z_{\alpha_1}^{\alpha_0} \bar{A}_{\alpha_1}^{\alpha_0}. \]  

(24)

At this stage, we can rewrite the Dirac bracket given in (13) without separating a specific subset of independent constraints. In view of this, we introduce an antisymmetric matrix, of elements $M_{(2)}^{(2)\alpha_0\beta_0}$, through the relation

\[ M_{(2)}^{(2)\alpha_0\beta_0} C_{\beta_0\gamma_0}^{(2)} \approx D_{\gamma_0}^{\alpha_0}, \]  

(25)

such that formula

\[ [F, G]^{(2)*} = [F, G] - [F, \chi_{\alpha_0}] M_{(2)}^{(2)\alpha_0\beta_0} [\chi_{\beta_0}, G] \]  

(26)

defines the same Dirac bracket like (13) on the surface (1). It is simple to see that $M_{(2)}^{(2)\alpha_0\beta_0}$ also contains some ambiguities, being defined up to the transformation

\[ M_{(2)}^{(2)\alpha_0\beta_0} \rightarrow M_{(2)}^{(2)\alpha_0\beta_0} + Z_{\alpha_1}^{\alpha_0} \tilde{q}^{\alpha_1\beta_1} Z_{\beta_1}^{\beta_0}, \]  

(27)

with $\tilde{q}^{\alpha_1\beta_1}$ some antisymmetric, but otherwise arbitrary functions. Relations (12) and (23) ensure that

\[ \text{rank} (D_{\gamma_0}^{\alpha_0}) \approx M_0 - M_1 + M_2, \]  

(28)

so the rank of the matrix of elements $M_{(2)}^{(2)\alpha_0\beta_0} C_{\beta_0\gamma_0}^{(2)}$ is equal to the number of independent second-class constraints also in the presence of the second-order reducibility.
Direct manipulations emphasize that the Dirac bracket in each case, (9) and (26) respectively, satisfies the relations

$$[\chi_{\alpha_0}, G]^{(1,2)*} \approx 0,$$

(where the index (1) corresponds to (9) and the index (2) to (26) respectively), so the property $$[\chi_{\alpha_0}, G]^{(1,2)*} = 0$$ (for any $$G$$) indeed holds on the surface of first- or second-order reducible second-class constraints respectively. In the meanwhile, each of the Dirac brackets (9) or (26) satisfies the Jacobi identity, but only in the weak sense.

### 2.2 Irreducible analysis of first- and second-order reducible second-class constraints

As it has been shown in [10], first-order reducible second-class constraints can be approached in an irreducible manner. To this end, one starts from the solution to equation (7)

$$\bar{a}_{\alpha_0}^{\alpha_1} = \bar{D}_{\gamma_1}^{\alpha_1} a_{\alpha_0}^{\gamma_1},$$

where $$a_{\alpha_0}^{\gamma_1}$$ are some functions chosen such that

$$\text{rank} \left( Z_{\alpha_0}^{\alpha_1} a_{\alpha_0}^{\gamma_1} \right) = M_1$$

and $$\bar{D}_{\gamma_1}^{\beta_1}$$ stands for the inverse of $$Z_{\alpha_0}^{\alpha_1} a_{\alpha_0}^{\gamma_1}$$. In order to develop an irreducible approach, it is necessary to enlarge the original phase-space with some new variables, $$(Y_{\alpha_1})_{\alpha_1=1}^{M_1}$$, endowed with the Poisson brackets

$$\{Y_{\alpha_1}, Y_{\beta_1}\} = \Gamma_{\alpha_1, \beta_1},$$

where $$\Gamma_{\alpha_1, \beta_1}$$ are the elements of an invertible, antisymmetric matrix that may depend on the newly added variables. Consequently, one constructs the constraints

$$\bar{\chi}_{\alpha_0} = \chi_{\alpha_0} + a_{\alpha_0}^{\alpha_1} Y_{\alpha_1} \approx 0,$$

which are second-class and, essentially, irreducible. Following the line exposed in [10] it can be shown that the Dirac bracket associated with the irreducible constraints (33) takes the form

$$[F, G]_{\text{ired}}^{(1)*} = [F, G] - [F, \bar{\chi}_{\alpha_0}] \mu_{\alpha_0, \beta_0}^{(1)} [\bar{\chi}_{\beta_0}, G]$$

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and it is (weakly) equal to the original Dirac bracket (9)

\[ [F, G]^{(1)*} \approx [F, G]^{(1)*} \bigg|_{\text{ired}}. \]  

(35)

In (34) the quantities \( \mu^{(1)\alpha_0\beta_0} \) are the elements of an invertible, antisymmetric matrix, expressed by

\[ \mu^{(1)\alpha_0\beta_0} \approx M^{(1)\alpha_0\beta_0} + Z_{\lambda_1}^{\alpha_0} \tilde{D}_{\beta_1}^{\lambda_1} \Gamma^{\beta_1\gamma_1} \tilde{D}_{\gamma_1}^{\sigma_1} Z_{\sigma_1}^{\beta_0}, \]  

(36)

with \( \Gamma^{\beta_1\gamma_1} \) the inverse of \( \Gamma_{\alpha_1\beta_1} \). Formula (35) is essential in our context because it proves that one can indeed approach first-order reducible second-class constraints in an irreducible fashion.

In the case of second-order reducible second-class constraints, one constructs the irreducible constraints

\[ \tilde{\chi}_{\alpha_0} = \chi_{\alpha_0} + A^{\alpha_1}_{\alpha_0} Y_{\alpha_1} \approx 0, \quad \tilde{\chi}_{\alpha_2} = Z^{\alpha_1}_{\alpha_2} Y_{\alpha_1} \approx 0, \]  

(37)

where

\[ A^{\rho_1}_{\sigma_0} = \hat{E}^{\rho_1}_{\alpha_1} A^{\alpha_1}_{\sigma_0}, \]  

(38)

with \( \hat{E}^{\nu_1}_{\alpha_1} \) the elements of an invertible matrix \([11]\). Following the line exposed in \([11]\) it can be shown that the Dirac bracket associated with the irreducible constraints (37) takes the form

\[ \left[ F, G \right]^{(2)*} \bigg|_{\text{ired}} = \left[ F, G \right] - \left[ F, \tilde{\chi}_{\alpha_0} \right] \mu^{(2)\alpha_0\beta_0} \left[ \tilde{\chi}_{\beta_0}, G \right] - \left[ F, \tilde{\chi}_{\alpha_0} \right] Z^{\alpha_0}_{\gamma_1} \hat{e}_{\sigma_1}^{\gamma_1} \Gamma^\sigma_{\lambda_1} A^{\rho_2}_{\lambda_1} \tilde{D}^{\beta_1}_{\rho_2} \left[ \tilde{\chi}_{\beta_2}, G \right] - \left[ F, \tilde{\chi}_{\alpha_2} \right] \tilde{D}^{\sigma_2}_{\lambda_2} A^{\lambda_2}_{\sigma_2} \Gamma^\sigma_{\lambda_1} \hat{e}_{\sigma_1}^{\gamma_1} Z^{\beta_0}_{\gamma_1} \left[ \tilde{\chi}_{\beta_0}, G \right] - \left[ F, \tilde{\chi}_{\alpha_2} \right] \tilde{D}^{\sigma_2}_{\lambda_2} A^{\lambda_2}_{\sigma_2} \Gamma^\sigma_{\lambda_1} A^{\rho_2}_{\lambda_1} \tilde{D}^{\beta_1}_{\rho_2} \left[ \tilde{\chi}_{\beta_2}, G \right], \]  

(39)

where

\[ \mu^{(2)\lambda_0\sigma_0} \approx M^{(2)\lambda_0\sigma_0} + Z^{\lambda_0}_{\lambda_1} \hat{e}_{\lambda_1}^{\sigma_1} Z^{\sigma_0}_{\sigma_1}, \]  

\[ \hat{e}_{\sigma_1}^{\gamma_1} = \hat{e}_{\sigma_1}^{\gamma_1} \Gamma^\sigma_{\lambda_1} \hat{e}_{\lambda_1}^{\beta_1}, \]  

(40)

(41)

and \( \hat{e}_{\sigma_1}^{\gamma_1} \) are the elements of the inverse of the matrix with the elements \( \hat{E}^{\nu_1}_{\alpha_1} \). In (39) the quantities denoted by \( A^{\rho_2}_{\lambda_1} \) are some functions chosen such that

\[ \text{rank } \left( Z^{\lambda_1}_{\alpha_2} A^{\rho_2}_{\lambda_1} \right) = M_2 \]  

(42)
and $\bar{D}^\beta_2$ stand for the elements of the inverse of the matrix with the elements $Z^{\lambda_1}_\alpha A^{\tau_2}_\lambda = D^{\tau_2}_\alpha$. Moreover, according to the general proof from [11], one has

$$[F, G]^{(2)*} \approx [F, G]^{(2)*} \bigg|_{\text{ired}},$$

which shows that second-order reducible second-class constraints can also be approached in an irreducible fashion.

3 Third-order reducible second-class constraints

3.1 Reducible approach

3.1.1 Dirac bracket for third-order reducible second-class constraints

In this section we will consider third-order reducible second-class constraints. This means that, beside the first-order reducibility relations (2), the following relations also hold

$$Z^{\alpha_1}_\alpha Z^{\alpha_0}_\alpha \approx 0, \quad \alpha_2 = \overline{1, M_2},$$

$$Z^{\alpha_2}_\alpha Z^{\alpha_1}_\alpha \approx 0, \quad \alpha_3 = \overline{1, M_3}. \quad (44, 45)$$

They are known as the reducibility relations of order two and three, respectively. In addition, all the third-order reducibility functions $Z^{\alpha_2}_\alpha$ are assumed to be independent. Under these circumstances, the number of independent second-class constraint functions is equal to $M \equiv M_0 - M_1 + M_2 - M_3$. As a consequence, we can work again with a Dirac bracket of the type (3), but written in terms of $M$ independent functions $\chi_A$, i.e.

$$[F, G]^{(3)*} = [F, G] - [F, \chi_A] M^{(3)AB} [\chi_B, G], \quad A = \overline{1, M}, \quad (46)$$

where $C^{(3)}_{AB} M^{(3)BC} \approx \delta^C_A$, with $C^{(3)}_{AB} = [\chi_A, \chi_B]$. It is clear that the matrix of elements

$$C^{(3)}_{\alpha_0\beta_0} = [\chi_{\alpha_0}, \chi_{\beta_0}] \quad (47)$$

also satisfies the relations

$$Z^{\alpha_0}_\alpha C^{(3)}_{\alpha_0\beta_0} \approx 0 \quad (48)$$

and, actually, its rank is equal to $M$. 

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Let \( \bar{A}_\alpha \) be a solution to
\[
Z_{\alpha_3} \bar{A}_{\alpha_2} \approx \delta_{\alpha_3},
\]
and \( \bar{\omega}_{\alpha_2} = -\bar{\omega}_{\beta_2} \) a solution to
\[
Z_{\alpha_3} \bar{\omega}_{\alpha_2} \approx 0.
\]
Then, we can introduce an antisymmetric matrix, of elements \( \hat{\omega}_{\beta_2} \), defined through the relation
\[
\bar{\omega}_{\alpha_2} \hat{\omega}_{\beta_2} \approx \delta_{\alpha_2} - \bar{A}_{\alpha_2} Z_{\gamma_2} \equiv D_{\alpha_2}.
\]
If we take into account equation (50), then it can be checked that \( \hat{\omega}_{\beta_2} \) are defined up to the transformation
\[
\hat{\omega}_{\beta_2} \rightarrow \hat{\omega}_{\beta_2} + Z_{\beta_3} q_{\beta_3} Z_{\gamma_3},
\]
where \( q_{\beta_3} \) are some arbitrary, antisymmetric functions. On the other hand, simple computation shows that the matrix of elements \( D_{\alpha_2} \) satisfies the relations
\[
D_{\alpha_2} \bar{A}_{\gamma_2} \approx 0, \quad Z_{\alpha_3} \approx 0,
\]
(53)
\[
D_{\alpha_2} Z_{\gamma_2} \approx Z_{\alpha_2}, \quad D_{\alpha_2} D_{\beta_2} \approx D_{\beta_2},
\]
(54)
Based on the latter formula from (53), we find that \( D_{\alpha_2} \) can alternatively be expressed as
\[
D_{\alpha_2} \approx Z_{\alpha_2} \bar{A}_{\gamma_2},
\]
(55)
for some functions \( \bar{A}_{\gamma_2} \). We notice that the above mentioned functions are defined up to the transformations
\[
\bar{A}_{\alpha_2} \rightarrow \bar{A}_{\alpha_2} + \mu_{\alpha_2} Z_{\alpha_0},
\]
(56)
with \( \mu_{\alpha_2} \) some arbitrary functions.

Using now the former relation from (54) and (55), we deduce that
\[
Z_{\alpha_2} D_{\alpha_1} \approx 0,
\]
(57)
where
\[
D_{\alpha_1} \approx \delta_{\alpha_1} - \bar{A}_{\alpha_2} Z_{\gamma_1}.
\]
Relations (57) and (58) ensure that $D_{\alpha_1}^{\gamma_1}$ is a ‘projection’ (idempotent) in the weak sense

$$D_{\beta_1}^{\alpha_1} D_{\gamma_1}^{\beta_1} \approx D_{\gamma_1}^{\alpha_1}. \quad (59)$$

With $D_{\alpha_1}^{\gamma_1}$ of the form (58) at hand, from (44) it follows that

$$D_{\alpha_1}^{\gamma_1} Z_{\gamma_1}^{\gamma_0} \approx Z_{\alpha_1}^{\gamma_0}. \quad (60)$$

Formula (57) emphasizes an alternative expression for $D_{\alpha_1}^{\gamma_1}$

$$D_{\alpha_1}^{\gamma_1} \approx Z_{\alpha_1}^{\alpha_0} \bar{A}_{\alpha_0}^{\gamma_1}, \quad (61)$$

for some functions $\bar{A}_{\alpha_0}^{\gamma_1}$. Accordingly, from (60) and (61) we find that

$$Z_{\alpha_0}^{\alpha_1} D_{\gamma_0}^{\gamma_0} \approx 0, \quad (62)$$

where

$$D_{\alpha_0}^{\gamma_0} \approx \delta_{\alpha_0}^{\gamma_0} - \bar{A}_{\alpha_0}^{\alpha_1} Z_{\alpha_1}^{\gamma_0}. \quad (63)$$

Just like before, from relations (62) and (63) we obtain that $D_{\alpha_0}^{\gamma_0}$ is also a ‘projection’ in the weak sense

$$D_{\beta_0}^{\alpha_0} D_{\gamma_0}^{\beta_0} \approx D_{\gamma_0}^{\alpha_0}. \quad (64)$$

At this stage, we can rewrite the Dirac bracket expressed by (46) in terms of all the second-class constraint functions. In view of this, we add an antisymmetric matrix, of elements $M^{(3)}_{\alpha_0 \beta_0}$, through the relation

$$C^{(3)}_{\alpha_0 \beta_0} M^{(3)}_{\alpha_0 \gamma_0} \approx D_{\gamma_0}^{\alpha_0}, \quad (65)$$

such that the formula

$$[F, G]^{(3)*} = [F, G] - [F, \chi_{\alpha_0}] M^{(3)}_{\alpha_0 \beta_0} \chi_{\beta_0}, G \quad (66)$$

defines the same Dirac bracket like (46) on the surface (1). It is simple to see that the elements $M^{(3)}_{\alpha_0 \beta_0}$ are defined up to the transformation

$$M^{(3)}_{\alpha_0 \beta_0} \rightarrow M^{(3)}_{\alpha_0 \beta_0} + Z_{\alpha_1}^{\alpha_0} p^{\alpha_1 \beta_1} Z_{\alpha_1}^{\beta_0}, \quad (67)$$

with $p^{\alpha_1 \beta_1}$ some arbitrary, antisymmetric functions. We notice that relations (44), (45), and (62) ensure that

$$\text{rank } (D_{\gamma_0}^{\alpha_0}) \approx M. \quad (68)$$
and hence the rank of the matrix of elements \( C^{(3)}_{\alpha_0 \beta_0} M^{(3)}_{\beta_0 \gamma_0} \) is equal to the number of independent second-class constraints in the case of the reducibility of order three. Meanwhile, we have that

\[
[\chi_{\alpha_0}, G]^{(3)*} \approx -\bar{A}^{\alpha_1}_{\alpha_0} [Z^{\beta_0}_{\alpha_1}, G] \chi_{\beta_0},
\]

so \([\chi_{\alpha_0}, G]^{(3)*} = 0\), for any \(G\), on the surface of third-order reducible second-class constraints.

### 3.1.2 Expressing the Dirac bracket in terms of an invertible matrix

Initially, we will establish some useful properties of the functions \( \bar{A}^{\alpha_1}_{\alpha_0} \), \( \bar{A}^{\alpha_2}_{\alpha_1} \), and \( \bar{A}^{\alpha_3}_{\alpha_2} \). We introduce (55) in the former relation from (53) and infer

\[
Z^{\alpha_1}_{\alpha_2} \bar{A}^{\gamma_2}_{\alpha_1} \bar{A}^{\gamma_3}_{\gamma_2} \approx 0,
\]

which implies the existence of some smooth functions \( M^{\gamma_3}_{\alpha_0} \) such that

\[
\bar{A}^{\gamma_2}_{\alpha_1} \bar{A}^{\gamma_3}_{\gamma_2} \approx M^{\gamma_3}_{\alpha_0} Z^{\alpha_0}_{\alpha_1}.
\]

Using definition (51) and relations (58) and (72), we obtain

\[
\bar{A}^{\alpha_2}_{\alpha_1} \bar{A}^{\gamma_3}_{\gamma_2} \approx 0.
\]

By inserting now (74) in (61), we deduce the relation

\[
\bar{A}^{\alpha_1}_{\alpha_0} \bar{A}^{\alpha_2}_{\alpha_1} \approx 0,
\]

which enables us, by means of equations (58) and (63), to establish the formulas

\[
D^{\alpha_0}_{\gamma_0} \bar{A}^{\alpha_1}_{\alpha_0} \approx 0, \quad \bar{A}^{\alpha_1}_{\alpha_0} D^{\alpha_1}_{\alpha_1} \approx \bar{A}^{\beta_1}_{\alpha_0}.
\]
Before expressing the Dirac bracket in terms of an invertible matrix, let us analyze equations (49) and (50). The solution to (49) may be set under the form
\[ \tilde{A}_{\alpha_2}^{\alpha_3} \approx A_{\alpha_2}^{\beta_3} \tilde{D}_{\beta_3}^{\alpha_3}, \]  
(78)
where \( A_{\alpha_2}^{\beta_3} \) are some functions taken such that the matrix of elements
\[ D_{\alpha_3}^{\gamma_3} = Z_{\alpha_3}^{\alpha_2} A_{\alpha_2}^{\gamma_3} \]  
(79)
is of maximum rank
\[ \text{rank } (D_{\alpha_3}^{\gamma_3}) = M_3. \]  
(80)
The notations \( \tilde{D}_{\beta_3}^{\alpha_3} \) stand for the elements of the inverse of \( D_{\alpha_3}^{\gamma_3} \).

Using (78) in (51), we have
\[ D_{\alpha_2}^{\gamma_2} \equiv \delta_{\alpha_2}^{\gamma_2} - A_{\alpha_2}^{\beta_3} \tilde{D}_{\beta_3}^{\alpha_3} Z_{\alpha_3}^{\gamma_2}, \]  
(81)
while (78) and the former relation from (53) lead to
\[ D_{\alpha_2}^{\gamma_2} \tilde{A}_{\gamma_2}^{\gamma_3} \approx 0. \]  
(82)

Inserting \( D_{\alpha_2}^{\beta_2} \) given by (81) in (73), we deduce
\[ \tilde{A}_{\gamma_1}^{\gamma_2} A_{\gamma_2}^{\gamma_3} \approx 0. \]  
(83)

Employing now the latter relation from (53), we get that the solution to equation (50) reads as
\[ \tilde{\omega}_{\alpha_2 \beta_2} \approx D_{\alpha_2}^{\gamma_2} \tilde{\omega}_{\gamma_2 \delta_2} D_{\delta_2}^{\beta_2}, \]  
(84)
with \( \tilde{\omega}_{\gamma_2 \delta_2} \) the elements of an antisymmetric matrix. Multiplying (51) with \( A_{\gamma_2}^{\gamma_3} \) and taking into account (82), we infer the equation
\[ \tilde{\omega}_{\alpha_2 \beta_2} \tilde{\omega}_{\beta_2}^{\beta_2} A_{\gamma_2}^{\gamma_3} \approx 0, \]  
(85)
\footnote{Strictly speaking, the solution to equation (49) has the general form \( \tilde{A}_{\alpha_2}^{\alpha_3} \approx A_{\alpha_2}^{\lambda_3} \tilde{D}_{\lambda_3}^{\alpha_3} + Z_{\alpha_2}^{\alpha_3} u_{\alpha_2}^{\lambda_3} + \tilde{\omega}_{\alpha_2 \lambda_3} v_{\lambda_3}^{\alpha_3} \), where \( u_{\alpha_2}^{\lambda_3} \) and \( v_{\lambda_3}^{\alpha_3} \) are some arbitrary functions. If we make the redefinitions \( u_{\alpha_2}^{\lambda_3} \rightarrow \tilde{u}_{\alpha_2}^{\lambda_3} \tilde{D}_{\lambda_3}^{\alpha_3} \) and \( v_{\lambda_3}^{\alpha_3} \rightarrow \tilde{v}_{\lambda_3}^{\alpha_3} \tilde{D}_{\lambda_3}^{\alpha_3} \), with \( \tilde{u}_{\alpha_2}^{\lambda_3} \) and \( \tilde{v}_{\lambda_3}^{\alpha_3} \) some arbitrary functions, then we can bring \( \tilde{A}_{\alpha_2}^{\alpha_3} \) to the form \( \tilde{A}_{\alpha_2}^{\alpha_3} \approx (A_{\alpha_2}^{\lambda_3} + Z_{\alpha_2}^{\alpha_3} \tilde{u}_{\alpha_2}^{\lambda_3} + \tilde{\omega}_{\alpha_2 \lambda_3} \tilde{v}_{\lambda_3}^{\alpha_3}) \tilde{D}_{\lambda_3}^{\alpha_3}. \) On the other hand, the quantities \( A_{\alpha_2}^{\lambda_3} \) taken such that the rank of (78) is maximum are defined up to the transformation \( A_{\alpha_2}^{\lambda_3} \rightarrow \tilde{A}_{\alpha_2}^{\lambda_3} = A_{\alpha_2}^{\lambda_3} + Z_{\alpha_2}^{\lambda_3} \tilde{\omega}_{\lambda_3}^{\alpha_3} + \tilde{\omega}_{\alpha_2 \lambda_3} \tilde{v}_{\lambda_3}^{\alpha_3} \), in the sense that \( Z_{\alpha_2}^{\lambda_3} A_{\alpha_2}^{\lambda_3} \approx Z_{\alpha_2}^{\lambda_3} A_{\alpha_2}^{\lambda_3} \), with \( \tilde{\omega}_{\alpha_2 \lambda_3} \) and \( \lambda_3 \lambda_3 \) also arbitrary. Thus, we can absorb the quantity \( Z_{\alpha_2}^{\lambda_3} \tilde{u}_{\alpha_2}^{\lambda_3} + \tilde{\omega}_{\alpha_2 \lambda_3} \tilde{v}_{\lambda_3}^{\alpha_3} \) from \( A_{\alpha_2}^{\lambda_3} \) through a redefinition of \( A_{\alpha_2}^{\lambda_3} \) and finally obtain solution (78).}
whose solution is

\[ \hat{\omega}^{\beta \gamma_2} A_{\gamma_2}^{\gamma_3} \approx Z_{\beta_3}^{\beta_2} Q_{\beta_3}^{\beta_3 \gamma_3}. \]  

(86)

Since the matrix of elements \( \hat{\omega}^{\beta \gamma_2} \) is defined up to transformation (52), we are free to make the choice

\[ \hat{q}_{\beta_3}^{\beta_3} \approx -Q_{\beta_3}^{\beta_3 \lambda_3} \hat{D}_{\lambda_3}^{\gamma_3}, \]

which brings the solution to equation (85) at the form

\[ \hat{\omega}^{\beta_2 \gamma_2} A_{\gamma_2}^{\gamma_3} \approx 0, \]  

(87)

which further implies

\[ \hat{\omega}^{\alpha_2 \beta_2} \approx D_{\rho_2}^{\alpha_2} \hat{\omega}_{\rho_2}^{\rho_2 \sigma_2} D_{\sigma_2}^{\beta_2}, \]  

(88)

with \( \hat{\omega}_{\rho_2}^{\rho_2 \sigma_2} \) the elements of an antisymmetric matrix.

Under these conditions, the next theorem can be proved to hold.

**Theorem 1** The matrices of elements \( \hat{\omega}_{\gamma_2 \delta_2} \) and \( \hat{\omega}_{\rho_2 \sigma_2} \) can always be taken to satisfy the following properties:

(a) invertibility;

(b) fulfillment of relation

\[ \hat{\omega}^{\rho_2 \sigma_2} D_{\sigma_2}^{\gamma_2} \hat{\omega}_{\gamma_2 \delta_2} \approx D_{\delta_2}^{\rho_2}. \]  

(89)

**Proof.** (a) Inserting the latter relation from (54) in (84) and (88), we reach the equations

\[ D_{\alpha_2}^{\gamma_2} \hat{\omega}_{\gamma_2 \delta_2} D_{\beta_2}^{\delta_2} \approx D_{\alpha_2}^{\gamma_2} \hat{\omega}_{\gamma_2 \delta_2} \hat{D}_{\beta_2}^{\delta_2}, \]  

(90)

\[ D_{\rho_2}^{\alpha_2} \hat{\omega}_{\rho_2 \sigma_2} D_{\sigma_2}^{\beta_2} \approx D_{\rho_2}^{\alpha_2} \hat{\omega}_{\rho_2 \sigma_2} \hat{D}_{\sigma_2}^{\beta_2}, \]  

(91)

which give

\[ \hat{\omega}_{\gamma_2 \delta_2} \approx \hat{\omega}_{\gamma_2 \delta_2} + A_{\gamma_2}^{\gamma_3} \hat{D}_{\gamma_3}^{\rho_3} \xi_{\rho_3 \sigma_3} \hat{D}_{\sigma_3}^{\delta_3} A_{\delta_2}^{\delta_2}, \]  

(92)

\[ \hat{\omega}_{\rho_2 \sigma_2} \approx \hat{\omega}_{\rho_2 \sigma_2} + Z_{\rho_2}^{\rho_3} \xi_{\rho_3 \sigma_3} Z_{\sigma_3}^{\sigma_2}, \]  

(93)

with \( \xi_{\rho_3 \sigma_3} \) and \( \xi_{\rho_3 \sigma_3} \) the elements of some invertible, antisymmetric matrices. With the help of (50), (51), and (87) and relying on relations (92) and (93) we find

\[ \hat{\omega}_{\gamma_2 \delta_2} \hat{\omega}_{\delta_2 \sigma_2} \approx D_{\gamma_2}^{\sigma_2} + Z_{\gamma_2}^{\rho_3} \hat{D}_{\gamma_3}^{\rho_3} A_{\sigma_2}^{\sigma_2}. \]  

(94)

As \( D_{\gamma_2}^{\sigma_2} \) is of the form (81), we find immediately

\[ \hat{\omega}_{\gamma_2 \delta_2} \hat{\omega}_{\delta_2 \sigma_2} \approx \delta_{\gamma_2}^{\sigma_2}. \]  

(95)
which proves (a).

(b) Simple computation outputs

\[ \hat{\omega}^{\rho_2 \sigma_2} \hat{D}_2^{\beta_2} \approx \hat{\omega}^{\rho_2 \beta_2}, \]

\[ \hat{\omega}^{\rho_2 \beta_2} \hat{\omega}_{\beta_2 \lambda_2} \approx \hat{\omega}^{\rho_2 \beta_2} \hat{\omega}_{\beta_2 \lambda_2} \approx \hat{D}^{\rho_2}_{\lambda_2}, \]

which further imply

\[ \hat{\omega}^{\rho_2 \sigma_2} \hat{D}_2^{\beta_2} \hat{\omega}_{\beta_2 \lambda_2} \approx \hat{D}^{\rho_2}_{\lambda_2}, \]

such that (b) is also proved. \(\Box\)

Let \(\bar{\omega}_{\alpha_1 \beta_1} = -\bar{\omega}_{\beta_1 \alpha_1}\) be a solution to the equation

\[ Z_{\alpha_1}^{\alpha_2} \bar{\omega}_{\alpha_1 \beta_1} \approx 0. \] (99)

Then, one can introduce an antisymmetric matrix, of elements \(\hat{\omega}^{\beta_1 \gamma_1}\), through the relation

\[ \bar{\omega}_{\alpha_1 \beta_1} \hat{\omega}^{\beta_1 \gamma_1} \approx D_{\alpha_1}^{\gamma_1}. \] (100)

Due to (99), we conclude that the elements \(\hat{\omega}^{\beta_1 \gamma_1}\) are defined up to the transformation

\[ \hat{\omega}^{\beta_1 \gamma_1} \rightarrow \hat{\omega}^{\beta_1 \gamma_1} + Z_{\beta_2}^{\beta_1} \hat{q}^{\beta_2 \gamma_2} Z_{\gamma_1}^{\gamma_2}, \]

(101)

with \(\hat{q}^{\beta_2 \gamma_2}\) some antisymmetric, but otherwise arbitrary functions. Recalling relation (57), we obtain that the solution to (99) can be expressed as

\[ \bar{\omega}_{\alpha_1 \beta_1} \approx D_{\alpha_1}^{\gamma_1} \bar{\omega}_{\gamma_1 \delta_1} D_{\beta_1}^{\delta_1}, \]

(102)

with \(\bar{\omega}_{\gamma_1 \delta_1}\) the elements of an antisymmetric matrix. Acting with \(\bar{A}_{\gamma_1}^{\gamma_2}\) on (100) and taking into account the result given by (74), we infer the equation

\[ \bar{\omega}_{\alpha_1 \beta_1} \hat{\omega}^{\beta_1 \gamma_1} \bar{A}_{\gamma_1}^{\gamma_2} \approx 0, \]

(103)

whose solution reads as

\[ \hat{\omega}^{\beta_1 \gamma_1} \bar{A}_{\gamma_1}^{\gamma_2} \approx Z_{\beta_2}^{\beta_1} Q^{\beta_2 \gamma_2}. \]

(104)

Due to the fact that the matrix of elements \(\hat{\omega}^{\beta_1 \gamma_1}\) is defined up to transformation (101), we are free to make the choice \(\hat{q}^{\beta_2 \gamma_2} \approx -Q^{\beta_2 \gamma_2}\), which brings equation (103) at the form

\[ \hat{\omega}^{\beta_1 \gamma_1} \bar{A}_{\gamma_1}^{\gamma_2} \approx 0, \]

(105)
such that its solution can be taken as

\[ \hat{\omega}^{\beta_1 \gamma_1} = D^{\beta_1}_{\lambda_1} \bar{\omega}^{\lambda_1 \rho_1} D^{\gamma_1}_{\rho_1}, \]  

(106)

with \( \bar{\omega}^{\lambda_1 \rho_1} \) the elements of an antisymmetric matrix.

Except from being antisymmetric, the matrices of elements \( \bar{\omega}_{\gamma_1 \delta_1} \) and respectively \( \bar{\omega}^{\lambda_1 \rho_1} \) are arbitrary at this stage. The next theorem shows that they are in fact related.

**Theorem 2** The matrices of elements \( \bar{\omega}_{\gamma_1 \delta_1} \) and \( \bar{\omega}^{\lambda_1 \rho_1} \) can always be taken to satisfy the following properties:

(a) invertibility;

(b) fulfillment of relation

\[ \bar{\omega}^{\lambda_1 \rho_1} D^{\gamma_1}_{\rho_1} \bar{\omega}_{\gamma_1 \delta_1} \approx D^{\lambda_1}_{\delta_1}. \]  

(107)

**Proof.** (a) Substituting the latter relation from (59) in (102) and (106), we obtain the equations

\[ D^{\gamma_1}_{\alpha_1} \bar{\omega}_{\gamma_1 \delta_1} D^{\delta_1}_{\beta_1} \approx D^{\gamma_1}_{\alpha_1} \bar{\omega}_{\gamma_1 \delta_1} D^{\delta_1}_{\beta_1}, \]  

(108)

\[ D^{\alpha_1}_{\rho_1} \bar{\omega}^{\rho_1 \sigma_1} D^{\sigma_1}_{\beta_1} \approx D^{\alpha_1}_{\rho_1} \bar{\omega}^{\rho_1 \sigma_1} D^{\sigma_1}_{\beta_1}, \]  

(109)

which then give

\[ \bar{\omega}_{\gamma_1 \delta_1} \approx \bar{\omega}_{\gamma_1 \delta_1} + \bar{A}_{\gamma_1}^{\xi_2} \xi_{2 \delta_1} \bar{A}_{\delta_1}^{\beta_1}, \]  

(110)

\[ \bar{\omega}^{\rho_1 \sigma_1} \approx \bar{\omega}^{\rho_1 \sigma_1} + Z^{\rho_1 \xi_{2 \sigma_2}} Z^{\sigma_1}, \]  

(111)

with \( \xi_{2 \delta_1} \) and \( \xi_{2 \sigma_2} \) the elements of some antisymmetric matrices, taken to be invertible. Each of the terms from the right-hand sides of relations (110) and (111) possesses null vectors. It is known that the null vectors of \( \bar{\omega}_{\gamma_1 \delta_1} \) and \( \bar{\omega}^{\rho_1 \sigma_1} \) are \( Z^{\gamma_1}_{\alpha_2} \) and \( \bar{A}^{\beta_1}_{\delta_1} \) respectively (see (99) and (105)), while \( \bar{A}^{\xi_{2 \delta_1}}_{\gamma_1} \xi_{2 \sigma_1} \) and \( Z^{\rho_1 \xi_{2 \sigma_2}}_{\rho_2} Z^{\sigma_1}_{\sigma_2} \) display the null vectors \( A^{\gamma_1}_{\gamma_0} \) and \( Z^{\rho_0}_{\sigma_0} \) respectively.\(^2\)

\(^2\)In fact, the general solution to equation (103) has the expression \( \hat{\omega}^{\alpha_1 \beta_1} = D^{\rho_1}_{\alpha_1} \bar{\omega}^{\rho_1 \sigma_1} D^{\sigma_1}_{\beta_1} + Z^{\rho_1 \alpha_{2 \beta_2}} Z^{\beta_2}_{\beta_2} \), for some antisymmetric functions \( u^{\alpha_{2 \beta_2}} \). But the quantities \( \hat{\omega}^{\alpha_1 \beta_1} \) are defined up to transformation (101), so one can absorb the terms \( Z^{\rho_1 \alpha_{2 \beta_2}} Z^{\beta_2}_{\beta_2} \) through a redefinition of \( \hat{\omega}^{\alpha_1 \beta_1} \) and obtain in the end precisely solution (106).

\(^3\)The most general form of the null vectors corresponding to \( \bar{\omega}_{\gamma_1 \delta_1} \) and \( \bar{\omega}^{\rho_1 \sigma_1} \) is of the type \( \nu^{\gamma_1} Z^{\gamma_1}_{\gamma_2} \) and \( A^{\rho_1 \xi_{2 \sigma_2}} \) respectively, with \( \nu^{\gamma_2} \) and \( \xi_{2 \sigma_2} \) arbitrary functions. Along the same line, the functions \( \bar{A}^{\nu^{\xi_{2 \beta_1}}} \xi_{2 \sigma_1} \) and \( Z^{\nu_{\rho_2 \sigma_2} \xi_{2 \sigma_2}} Z^{\sigma_1}_{\sigma_2} \) display (the most general) null vectors \( \tau^{\gamma_0} \bar{A}^{\gamma_0}_{\gamma_0} \) and \( Z^{\rho_0 \kappa_{\sigma_0}} \kappa_{\sigma_0} \) respectively, with \( \tau^{\gamma_0} \) and \( \kappa_{\sigma_0} \) arbitrary functions. However, these observations do not affect the proof in any way.
this reason, the only candidates for null vectors of $\tilde{\omega}_{\gamma_1\delta_1}$ and $\tilde{\omega}^{\rho_1\sigma_1}$ are on the one hand $Z^{\gamma_1}_{\alpha_2}$ and $\tilde{A}^{\beta_2}_{\sigma_1}$ respectively and on the other hand $\tilde{A}^{\gamma_1}_{\sigma_0}$ and $Z^{\sigma_0}_{\sigma_1}$ respectively. We show that none of these candidates are null vectors.

Indeed, from (110) and (111) we find

$$Z^{\gamma_1}_{\alpha_2}\tilde{\omega}_{\gamma_1\delta_1} \approx D^{\gamma_2}_{\alpha_2}\xi_{\gamma_2\delta_2}\tilde{A}^{\beta_2}_{\delta_1},$$

$$\tilde{\omega}^{\rho_1\sigma_1}\tilde{A}^{\beta_2}_{\sigma_1} \approx Z^{\rho_1}_{\rho_2}\xi^{\rho_2\sigma_2}D^{\beta_2}_{\sigma_2}.$$  

(112)

(113)

The right-hand sides of (112) and (113) are (weakly) vanishing for

$$\xi_{\gamma_2\delta_2} = A^{\gamma_2}_{\gamma_2}\theta_{\gamma_2\delta_2}A^{\delta_3}_{\delta_2},$$

$$\xi^{\rho_2\sigma_2} = Z^{\rho_3}_{\rho_2}\theta^{\rho_3\sigma_3}Z^{\sigma_3}_{\sigma_2},$$

(114)

(115)

with $\theta_{\gamma_2\delta_2}$ and $\theta^{\rho_3\sigma_3}$ the elements of some antisymmetric matrices. It is simple to see that the matrices of elements $\xi_{\gamma_2\delta_2}$ and $\xi^{\rho_2\sigma_2}$ given by (114) and (115) respectively are degenerate,

which contradicts the hypothesis on their invertibility. Thus, it follows that the matrices of elements $\tilde{\omega}_{\gamma_1\delta_1}$ and $\tilde{\omega}^{\rho_1\sigma_1}$ do not have the functions $Z^{\gamma_1}_{\alpha_2}$ and $\tilde{A}^{\beta_2}_{\sigma_1}$ as null vectors respectively. Multiplying (110) by $\tilde{A}^{\gamma_1}_{\gamma_0}$ and (111) by $Z^{\sigma_0}_{\sigma_1}$, we deduce that

$$\tilde{A}^{\gamma_1}_{\gamma_0}\tilde{\omega}_{\gamma_1\delta_1} \approx \tilde{A}^{\gamma_1}_{\gamma_0}\tilde{\omega}_{\gamma_1\delta_1},$$

$$\tilde{\omega}^{\rho_1\sigma_1}Z^{\sigma_0}_{\sigma_1} \approx \tilde{\omega}^{\rho_1\sigma_1}Z^{\sigma_0}_{\sigma_1}.$$  

(116)

(117)

The right-hand sides of (116) and (117) vanish for

$$\tilde{\omega}_{\gamma_1\delta_1} = \tilde{A}^{\gamma_1}_{\gamma_1}\theta_{\gamma_2\delta_2}\tilde{A}^{\delta_2}_{\delta_1},$$

$$\tilde{\omega}^{\rho_1\sigma_1} = Z^{\rho_1}_{\rho_2}\theta^{\rho_2\sigma_2}Z^{\sigma_2}_{\sigma_2},$$

(118)

(119)

with $\hat{\theta}_{\gamma_2\delta_2}$ and $\hat{\theta}^{\rho_2\sigma_2}$ the elements of some antisymmetric matrices. It is now easy to see that neither $\tilde{\omega}_{\gamma_1\delta_1}$ nor $\tilde{\omega}^{\rho_1\sigma_1}$, given by (118) and (119) respectively, can be brought to the form expressed by relations (102) and (106) respectively, for any choice of $\hat{\theta}_{\gamma_2\delta_2}$ or $\hat{\theta}^{\rho_2\sigma_2}$. Thus, it follows that neither of relations (118) or (119) can hold, so neither of the quantities $\tilde{A}^{\gamma_1}_{\gamma_0}\tilde{\omega}_{\gamma_1\delta_1}$ or

\footnote{The matrix of elements $\xi_{\gamma_2\delta_2}$ displays the null vectors $u^{\gamma_1}\tilde{A}^{\gamma_2}_{\sigma_2}$ and that of elements $\xi^{\rho_2\sigma_2}$ exhibits the null vectors $v^{\rho_1}Z^{\rho_2}_{\sigma_2}$.}
\( \tilde{\omega}^{\rho_1 \sigma_1} Z_{\sigma_1}^{\rho_1} \) can vanish. This further implies that the matrices of elements \( \tilde{\omega}_{\gamma_1 \delta_1} \) and \( \tilde{\omega}^{\rho_1 \sigma_1} \) do not possess the functions \( \tilde{A}_{\gamma_1}^{\sigma_0} \) and \( Z_{\sigma_1}^{\sigma_0} \) as null vectors respectively, so we conclude that both the matrices of elements \( \tilde{\omega}_{\gamma_1 \delta_1} \) and \( \tilde{\omega}^{\rho_1 \sigma_1} \) (having the expressions (110) and (111) respectively) are invertible.

Because of results (99), (100), and (105), from relations (110) and (111) one gets

\[
\tilde{\omega}^{\gamma_1}_{\gamma_1} \tilde{\omega}^{\delta_1}_{\sigma_1} \approx D_{\gamma_1}^{\sigma_1} + \tilde{A}_{\gamma_1}^{\sigma_1} \tilde{\omega}_{\gamma_1 \delta_1}^{\rho_2} \tilde{\omega}_{\gamma_1 \delta_1}^{\lambda_2} Z_{\sigma_2}^{\sigma_1}.
\]

(120)

We take the functions \( \xi_{\gamma_2 \rho_2} \) and \( \xi_{\lambda_2 \sigma_2} \) of the form

\[
\xi_{\gamma_2 \rho_2} = \tilde{\omega}_{\gamma_2 \rho_2}, \quad \xi_{\lambda_2 \sigma_2} = \tilde{\omega}_{\lambda_2 \sigma_2}^{\lambda_2},
\]

(121)

which replaced in (120) leads (also due to (89)) to

\[
\tilde{\omega}_{\gamma_1 \delta_1} \approx \delta_{\gamma_1}^{\sigma_1}.
\]

(122)

This proves (a).

(b) By straightforward computation, it results

\[
\tilde{\omega}^{\rho_1 \sigma_1} D_{\sigma_1}^{\lambda_1} \approx \tilde{\omega}^{\rho_1 \lambda_1},
\]

(123)

\[
\tilde{\omega}^{\rho_1 \lambda_1} \tilde{\omega}_{\lambda_1 \delta_1} \approx \tilde{\omega}^{\rho_1 \lambda_1} \tilde{\omega}_{\lambda_1 \delta_1} \approx D_{\delta_1}^{\sigma_1},
\]

(124)

which further yields

\[
\tilde{\omega}^{\rho_1 \sigma_1} D_{\sigma_1}^{\lambda_1} \tilde{\omega}_{\lambda_1 \delta_1} \approx D_{\delta_1}^{\sigma_1},
\]

(125)

and proves (b). \( \Box \)

With these elements at hand, the next theorem is shown to hold.

**Theorem 3** There exists an invertible, antisymmetric matrix of elements \( \mu^{(3)\alpha_0 \beta_0} \) such that Dirac bracket (66) takes the form

\[
[F, G]^{(3)*} = [F, G] - [F, \chi_{\alpha_0}] \mu^{(3)\alpha_0 \beta_0} [\chi_{\beta_0}, G]
\]

(126)

on the surface (1).

**Proof.** First, we observe that \( D_{\gamma_0}^{\alpha_0} \) given in (63) satisfies the relations

\[
D_{\gamma_0}^{\alpha_0} \chi_{\alpha_0} \approx \chi_{\gamma_0}.
\]

(127)

Multiplying (65) by \( \tilde{A}_{\gamma_1}^{\gamma_0} \gamma_1 \) and using (70), we obtain the equation

\[
C_{\alpha_0 \beta_0}^{(3)} M^{(3)\beta_0 \gamma_0} \tilde{A}_{\gamma_1}^{\gamma_0} \approx 0,
\]

(128)
which then leads to
\[ M^{(3)\beta_0\gamma_0} \tilde{A}_{\gamma_0} \approx Z_{\beta_1} f^{\beta_1\gamma_1}, \quad (129) \]
for some functions \( f^{\beta_1\gamma_1} \). Acting with \( D^{\gamma_0}_{\beta_0} \) on (129) and employing (62), we find the relation
\[ M^{(3)\beta_0\gamma_0} \tilde{A}_{\gamma_0} D^{\gamma_0}_{\beta_0} \approx 0, \quad (130) \]
with the help of which (via formula (76)) we can write
\[ M^{(3)\beta_0\gamma_0} D^{\gamma_0}_{\beta_0} \approx D^{\gamma_0}_{\alpha_0} \lambda^{\beta_0\gamma_0}, \quad (131) \]
for some \( \lambda^{\beta_0\gamma_0} \). Acting now with \( D^{\gamma_0}_{\beta_0} \) on (65) and taking into account (131), we deduce
\[ - C^{(3)}_{\alpha_0\beta_0} D^{\gamma_0}_{\beta_0} \lambda^{\beta_0\gamma_0} \approx D^{\gamma_0}_{\alpha_0}. \quad (132) \]
On the other hand, relation (127) implies
\[ D^{\beta_0}_{\alpha_0} C^{(3)}_{\beta_0\gamma_0} \approx C^{(3)}_{\alpha_0\gamma_0}, \quad (133) \]
such that, on behalf of (132) and (133), we have
\[ - C^{(3)}_{\alpha_0\beta_0} \lambda^{\beta_0\gamma_0} \approx D^{\gamma_0}_{\alpha_0}. \quad (134) \]
Comparing (134) with (65) and using the fact that the functions \( M^{(3)\alpha_0\beta_0} \) are defined up to transformation (67), we infer the relation
\[ M^{(3)\beta_0\gamma_0} = - \lambda^{\beta_0\gamma_0}, \quad (135) \]
which substituted in (131) provides the equation
\[ M^{(3)\beta_0\gamma_0} D^{\gamma_0}_{\beta_0} \approx M^{(3)\gamma_0\beta_0} D^{\gamma_0}_{\beta_0}. \quad (136) \]
Using one more time the fact that the elements \( M^{(3)\alpha_0\beta_0} \) are defined up to (67), from (136) we get
\[ M^{(3)\alpha_0\beta_0} \approx \mu^{(3)\gamma_0\sigma_0} D^{\beta_0}_{\sigma_0}, \quad (137) \]
where \( \mu^{(3)\gamma_0\sigma_0} \) is an antisymmetric matrix. Due to formula (76) and relation (137) we can write
\[ M^{(3)\alpha_0\beta_0} \tilde{A}_{\beta_0} \approx 0. \quad (138) \]
Inserting the former relation from (59) in (137), we deduce

\[ D^{\alpha_0}_\lambda M^{(3)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0} \approx D^{\alpha_0}_{\lambda_0} \mu^{(3)\lambda_0\sigma_0} D^{\beta_0}_{\sigma_0}, \]  

which further yields

\[ \mu^{(3)\lambda_0\sigma_0} \approx M^{(3)\lambda_0\sigma_0} + Z^{\lambda_0}_{\lambda_1} \nu^{\lambda_1\sigma_1} Z^{\sigma_0}_{\sigma_1}, \]  

for an antisymmetric matrix, of elements \( \nu^{\lambda_1\sigma_1} \). Now, we show that the matrix of elements \( \mu^{(3)\lambda_0\sigma_0} \) can be taken to be invertible. If we take \( \nu^{\lambda_1\sigma_1} \) under the form \( \tilde{\omega}^{\lambda_1\sigma_1} \), where \( \tilde{\omega}^{\lambda_1\sigma_1} \) are precisely the elements of the invertible matrix given in (111), then we find directly

\[ \mu^{(3)\lambda_0\sigma_0} \approx M^{(3)\lambda_0\sigma_0} + Z^{\lambda_0}_{\lambda_1} \tilde{\omega}^{\lambda_1\sigma_1} Z^{\sigma_0}_{\sigma_1}. \]  

Next, we show that the matrix of elements

\[ \mu^{(3)}_{\rho_0\lambda_0} \approx C^{(3)}_{\rho_0\lambda_0} + \tilde{A}^{\rho_1}_{\rho_0} \tilde{\omega}_{\rho_1\tau_1} \tilde{A}^{\tau_1}_{\lambda_0}, \]  

where \( \tilde{\omega}_{\rho_1\tau_1} \) determines the invertible matrix given in (110), is nothing but the inverse of the matrix of elements \( \mu^{(3)\lambda_0\sigma_0} \) expressed by (111). Indeed, from (48), (61), (65), and (138), direct computation provides

\[ \mu^{(3)\lambda_0\sigma_0} \approx D^{\sigma_0}_{\rho_0} + \tilde{A}^{\rho_1}_{\rho_0} \tilde{\omega}_{\rho_1\tau_1} D^{\tau_1}_{\lambda_1} \tilde{\omega}^{\lambda_1\sigma_1} Z^{\sigma_0}_{\sigma_1}. \]  

Taking into account the results of Theorem 2 (see (107)) and (60), we arrive at the relation

\[ \tilde{A}^{\rho_1}_{\rho_0} \tilde{\omega}_{\rho_1\tau_1} D^{\tau_1}_{\lambda_1} \tilde{\omega}^{\lambda_1\sigma_1} Z^{\sigma_0}_{\sigma_1} \approx \tilde{A}^{\rho_1}_{\rho_0} D^{\sigma_1}_{\rho_1} Z^{\sigma_0}_{\sigma_1} \approx \tilde{A}^{\rho_1}_{\rho_0} Z^{\sigma_0}_{\rho_1}, \]  

which substituted into (143) leads us to the formula

\[ \mu^{(3)\lambda_0\sigma_0} \approx \delta^{\sigma_0}_{\rho_0}, \]  

proving that the matrix of elements \( \mu^{(3)\lambda_0\sigma_0} \) given by (111) is indeed invertible. This proves the theorem. □
3.2 Irreducible approach

3.2.1 Intermediate system

Now, we introduce some new variables, \(y^{\alpha 1}_1\) and \(y^{\alpha 3}_3\), with the Poisson brackets

\[
\{y^{\alpha 1}_1, y^{\alpha 1}_1\} = \omega^{\alpha 1}_1 \beta_1, \quad \{y^{\alpha 3}_3, y^{\alpha 3}_3\} = \omega^{\alpha 3}_3 \beta_3, \quad \{y^{\alpha 1}_1, y^{\alpha 3}_3\} = 0,
\]

where \(\omega^{\alpha 1}_1 \beta_1\) and \(\omega^{\alpha 3}_3 \beta_3\) are the elements of some antisymmetric, invertible matrices, and consider a system subject to the reducible second-class constraints

\[
\chi^{\alpha 0}_0 \approx 0, \quad y^{\alpha 1}_1 \approx 0, \quad y^{\alpha 3}_3 \approx 0.
\]

In what follows we will call the system subject to constraints (147) “intermediate system”. The Dirac bracket on the phase-space locally described by \((z^a, y^{\alpha 1}_1, y^{\alpha 3}_3)\) constructed with respect to the above second-class constraints reads as

\[
[F, G]^{(3)*}_{3, y} = [F, G] - [F, \chi^{\alpha 0}_0] \mu^{(3)\alpha 0 \beta 0} [\chi^{\beta 0}_0, G] - [F, y^{\alpha 1}_1] \omega^{\alpha 1}_1 \beta_1 [y^{\beta 1}_1, G] - [F, y^{\alpha 3}_3] \omega^{\alpha 3}_3 \beta_3 [y^{\beta 3}_3, G],
\]

where the Poisson brackets from the right-hand side of (148) contain derivatives with respect to all the variables \(z^a, y^{\alpha 1}_1,\) and \(y^{\alpha 3}_3\). The notations \(\omega^{\alpha 1}_1 \beta_1\) and \(\omega^{\alpha 3}_3 \beta_3\) denote the elements of the inverses of the matrices of elements \(\omega^{\alpha 1}_1 \beta_1\) and \(\omega^{\alpha 3}_3 \beta_3\) respectively. The most general form of a function defined on the phase-space of coordinates \((z^a, y^{\alpha 1}_1, y^{\alpha 3}_3)\) is given by

\[
F (z^a, y^A) = F_0 (z^a) + \int_0^1 \frac{dF(z^a, \lambda y_A)}{d\lambda} d\lambda = F_0 (z^a) + y_A G^A (z^a, y_B),
\]

where \(y_A = (y^{\alpha 1}_1, y^{\alpha 3}_3)\), \(F_0 (z^a) = F (z^a, 0)\), and

\[
G^A (z^a, y_B) = \int_0^1 \frac{\partial F(z^a, \lambda y_B)}{\partial (\lambda y_A)} d\lambda.
\]

By inserting (149) in (148) we obtain

\[
[F, G]^{(3)*} \approx [F_0, G_0]^{(3)*},
\]

where
where the previous weak equality holds on the surface (147). Moreover, equations (1) and (147) describe the same surface, but embedded in two phase-spaces of different dimensions. In other words, equations (1) and (147) represent equivalent descriptions of one and the same constraint surface. For this reason, we will maintain the symbol of weak equality with respect to both descriptions.\footnote{Obviously, it is understood that we employ description (1) whenever we work with functions defined on the phase-space of local coordinates \( z^a \), but we use representation (147) in relation with the functions defined on the phase-space of local coordinates \( (z^a, y_{a_1}, y_{a_3}) \).}

Substituting (149) in (148) and taking into account (150), we infer

\[
[F,G]^{(3)*}_{z,y} \approx [F,G]^{(3)*}.
\]

(151)

We recall that the Dirac bracket \([F,G]^{(3)*}\) contains only derivatives with respect to the variables \( z^a \).

### 3.2.2 Irreducible system

Let \( \hat{e}^{\alpha_2}_{\sigma_2} \) be the elements of an invertible matrix, taken such that

\[
\bar{A}_{\alpha_1}^{\sigma_2} = A_{\alpha_1}^{\sigma_2} \hat{e}^{\alpha_2}_{\sigma_2},
\]

(152)

with

\[
A_{\alpha_1}^{\sigma_2} = \sigma_{\alpha_1 \lambda_1} Z_{\beta_2}^{\lambda_1} \sigma_{\beta_2 \alpha_2},
\]

(153)

where \( \sigma_{\alpha_1 \lambda_1} \) and \( \sigma_{\alpha_2 \beta_2} \) determine some invertible matrices. From (152) it is easy to see that

\[
A_{\alpha_1}^{\sigma_2} = \bar{A}_{\alpha_1}^{\sigma_2} \hat{E}^{\alpha_2}_{\sigma_2},
\]

(154)

with \( \hat{E}^{\alpha_2}_{\sigma_2} \) the elements of the inverse of the matrix of elements \( \hat{e}^{\alpha_2}_{\sigma_2} \). Substituting (152) in (75) and taking into account the invertibility of the matrix of elements \( \hat{e}^{\alpha_2}_{\sigma_2} \), we obtain

\[
\bar{A}_{\alpha_0}^{\rho_1} A_{\alpha_1}^{\rho_2} \approx 0.
\]

(155)

Next, we add an invertible matrix, whose elements will be denoted by \( \hat{E}^{\gamma_i}_{\rho_1} \), through the relations

\[
\bar{\omega}_{\alpha_1 \beta_1} = \hat{E}^{\gamma_i}_{\alpha_1} \omega_{\gamma_1 \lambda_1} \hat{E}^{\lambda_1}_{\beta_1},
\]

(156)

and define the functions

\[
A_{\sigma_0}^{\rho_1} = \bar{A}_{\sigma_0}^{\rho_1} \hat{E}^{\rho_1}_{\alpha_1}.
\]

(157)
Then, it is clear that
\[ \tilde{\omega}_{\alpha_1 \beta_1} = \hat{\epsilon}_{\sigma_1} \omega_{\sigma_1 \tau_1} \hat{\epsilon}_{\beta_1}, \] (158)
with \( \hat{\epsilon}_{\sigma_1} \) the elements of the inverse of \( \hat{E}^{\gamma_1}_{\alpha_1} \), while (157) produces
\[ \hat{A}^{\alpha_1}_{\sigma_0} = A^{\alpha_1}_{\rho_1} \hat{\epsilon}^{\alpha_1}_{\rho_1}. \] (159)

In this context the next theorem is shown to hold.

**Theorem 4** The elements \( \hat{\epsilon}^{\alpha_1}_{\sigma_1} \) and \( \hat{E}^{\gamma_1}_{\beta_1} \) can be taken such that
\[ \hat{E}^{\alpha_1}_{\sigma_1} D^{\gamma_1}_{\tau_1} \hat{\epsilon}^{\beta_1}_{\rho_1} \approx D^{\alpha_1}_{\beta_1}. \] (160)

**Proof.** We take \( \hat{E}^{\alpha_1}_{\beta_1} \) and \( \hat{\epsilon}^{\alpha_1}_{\beta_1} \) such that the following relations are satisfied:
\[ A^{\alpha_0}_{\alpha_1} = \sigma^{\alpha_0 \beta_0} Z^{\beta_0}_{\beta_1} \sigma^{\beta_1 \alpha_1}, \] (161)
\[ \sigma^{\alpha_1 \gamma_1} \hat{\epsilon}^{\gamma_1}_{\beta_1} \sigma^{\beta_1 \beta_1} = \hat{\epsilon}^{\beta_1}_{\alpha_1}, \] (162)
where the matrix of elements \( \sigma^{\alpha_0 \beta_0} \) is taken to be invertible and \( \sigma^{\beta_1 \alpha_1} \) are the elements of the inverse of the matrix of elements \( \sigma^{\alpha_1 \lambda_1} \). By ‘solving’ (153) and (161) with respect to the reducibility functions of order one and two
\[ Z^{\alpha_1}_{\alpha_0} = \sigma^{\alpha_0 \beta_0} A^{\beta_1}_{\beta_0} \sigma^{\beta_1 \alpha_1}, \quad Z^{\lambda_1}_{\lambda_2} = \sigma^{\lambda_1 \tau_1} A^{\tau_2}_{\tau_1} \sigma^{\tau_2 \lambda_2}, \] (163)
where \( \sigma^{\alpha_0 \beta_0} \) and \( \sigma^{\lambda_2 \tau_2} \) are the elements of the inverses of the matrices of elements \( \sigma^{\alpha_0 \beta_0} \) and \( \sigma^{\alpha_2 \beta_2} \) respectively, we can write
\[ Z^{\alpha_0}_{\alpha_1} \hat{\epsilon}^{\alpha_1}_{\lambda_1} Z^{\lambda_1}_{\lambda_2} = \sigma^{\alpha_0 \beta_0} A^{\beta_1}_{\beta_0} \sigma^{\beta_1 \alpha_1} \hat{\epsilon}^{\alpha_1}_{\lambda_1} \sigma^{\lambda_1 \tau_1} A^{\tau_2}_{\tau_1} \sigma^{\tau_2 \lambda_2}. \] (164)
From (164) and taking into account (159) and (162), we deduce the relation
\[ Z^{\alpha_0}_{\alpha_1} \hat{\epsilon}^{\alpha_1}_{\lambda_1} Z^{\lambda_1}_{\lambda_2} = \sigma^{\alpha_0 \beta_0} \hat{A}^{\beta_1}_{\beta_0} A^{\tau_2}_{\tau_1} \sigma^{\tau_2 \lambda_2}. \] (165)
Inserting now (155) in (165), we arrive at
\[ Z^{\alpha_0}_{\alpha_1} \hat{\epsilon}^{\alpha_1}_{\lambda_1} Z^{\lambda_1}_{\lambda_2} \approx 0. \] (166)
Based on the results expressed by (155) and (166), we are able now to prove the validity of (160). If we make the notation
\[ \hat{D}^{\alpha_1}_{\beta_1} = \hat{\epsilon}^{\alpha_1}_{\lambda_1} D^{\tau_1}_{\tau_1} \hat{E}^{\gamma_1}_{\beta_1}, \] (167)
then it is easy to see that $\hat{D}_{\beta_1}^\alpha \hat{D}_{\lambda_1}^\beta \approx \hat{D}_{\lambda_1}^\alpha$. (168)

On the other hand, with the help of relations (153) and (161), we deduce that $A_{\alpha_0}^\alpha A_{\alpha_1}^\alpha \approx 0$, which further implies

$$A_{\alpha_0}^\alpha \tilde{A}_{\alpha_1}^\alpha \approx 0,$$  
(169)

and hence we find

$$\tilde{A}_{\alpha_0}^\beta \hat{D}_{\alpha_1}^\alpha \approx \tilde{A}_{\alpha_0}^\alpha.$$  
(170)

Applying $Z_{\alpha_1}^{\alpha_0}$ on (167) and relying on (166), we get

$$Z_{\alpha_1}^{\alpha_0} \hat{D}_{\beta_1}^\alpha \approx Z_{\beta_1}^{\alpha_0}.$$  
(171)

Multiplying (170) with $Z_{\rho_1}^{\alpha_0}$ and (171) with $\bar{A}_{\alpha_0}^\beta$, we are led to

$$\hat{D}_{\beta_1}^\alpha D_{\rho_1}^\beta \approx D_{\rho_1}^\alpha, \quad D_{\beta_1}^\alpha \bar{A}_{\alpha_0}^\beta \approx D_{\rho_1}^\alpha.$$  
(172)

The general solution to equations (172) is of the form

$$\hat{D}_{\beta_1}^\alpha \approx D_{\beta_1}^\alpha + \tilde{A}_{\beta_1}^\tau M_{\tau_2}^\lambda Z_{\lambda_2}^{\alpha_1},$$  
(173)

for an arbitrary matrix of elements $M_{\tau_2}^\lambda$. Direct computation yields

$$\hat{D}_{\lambda_1}^\alpha \tilde{D}_{\beta_1}^\lambda \approx D_{\beta_1}^\alpha + \tilde{A}_{\beta_1}^\tau M_{\tau_2}^\lambda D_{\rho_2}^\beta M_{\rho_2}^\beta Z_{\beta_2}^{\alpha_1}.$$  
(174)

Comparing (174) with (168) and employing (173), we obtain that the elements $M_{\tau_2}^\lambda$ are subject to the equations

$$\tilde{A}_{\beta_1}^\tau M_{\tau_1}^\lambda D_{\rho_2}^\beta M_{\rho_2}^\beta Z_{\beta_2}^{\alpha_1} \approx \tilde{A}_{\beta_1}^\tau M_{\tau_2}^\lambda Z_{\lambda_2}^{\alpha_1}.$$  
(175)

It is easy to see that equations (175) possess two types of solutions, namely

$$M_{\tau_2}^\lambda = 0,$$  
(176)

and

$$M_{\tau_2}^\lambda = D_{\tau_2}^\lambda.$$  
(177)
If we employ solution (176), from (173) we infer
\[ \hat{D}_{\beta_1} \approx D_{\beta_1}, \] (178)
such that (160) is valid. This proves the theorem. □

Replacing (156) and (158) in (107) and recalling (160), it is easy to obtain the relation
\[ \omega^{\alpha_1 \gamma_1} D_{\gamma_1}^{\sigma_1} \omega_{\sigma_1 \beta_1} \approx D_{\beta_1}^{\sigma_1}, \] (179)
On the other hand, formulae (156)–(158) imply that \( \mu_{(3)}^{(3)} \) and \( \mu_{(3)}^{(3)} \) given by (141) and (142) respectively can be expressed as
\[ \mu_{(3)}^{(3)} \approx \frac{1}{\lambda_0} \hat{e}^{\lambda_1} \omega^{\sigma_1 \gamma_1} \hat{e}_{\sigma_1} Z_{\gamma_1}, \] (180)
\[ \mu_{(3)}^{(3)} \approx C_{(3)}^{(3)} + A_{(3)}^{(3)} \omega_{\rho_1 \gamma_1} A_{(3)}^{\gamma_1}. \] (181)

At this point, we construct the constraints
\[ \tilde{\chi}_{\alpha_0} \equiv \chi_{\alpha_0} + A_{(3)}^{\alpha_1} y_{\alpha_1} \approx 0, \] (182)
\[ \tilde{\chi}_{\alpha_2} \equiv Z_{(3)}^{\alpha_1} y_{\alpha_1} + A_{(3)}^{\alpha_3} y_{\alpha_3} \approx 0. \] (183)

Under these considerations, we are able to prove the following key theorem.

**Theorem 5** Constraints (182) and (183) satisfy the following properties:
(i) equivalence to (147), i.e.\[^6\]
\[ (\tilde{\chi}_{\alpha_0} \approx 0, \tilde{\chi}_{\alpha_2} \approx 0) \Leftrightarrow (\chi_{\alpha_0} \approx 0, y_{\alpha_1} \approx 0, y_{\alpha_3} \approx 0); \] (184)
(ii) second-class behaviour, i.e. the matrix of elements
\[ C_{\Delta \Delta'} = [\tilde{\chi}_{\Delta}, \tilde{\chi}_{\Delta'}] \] (185)
is invertible, where
\[ \tilde{\chi}_{\Delta} \equiv (\tilde{\chi}_{\alpha_0}, \tilde{\chi}_{\alpha_2}); \] (186)
(iii) irreducibility.

\[^6\] The other solution, (177), produces the equation \( \hat{e}_{\sigma_1} D_{\gamma_1}^{\sigma_1} \hat{E}_{\gamma_1}^{\tau_1} \approx \delta_{\alpha_1}^\tau, \) which further implies the relation \( D_{\beta_1}^{\sigma_1} \approx \delta_{\beta_1}^\sigma, \) contradicting thus (63).

\[^7\] Due to the equivalence expressed by (184), in the following we will use the same symbol of weak equality in relation to both the constraints (147) and (182)–(183) respectively.

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Proof. (i) It is easy to see that if (147) hold, then (182) and (183) also hold
\[(\chi_0 \approx 0, y_{a_1} \approx 0, y_{a_3} \approx 0) \Rightarrow (\tilde{\chi}_0 \approx 0, \tilde{\chi}_{a_2} \approx 0).\] (187)
On the other hand, from (182) and (183) one can express \(\chi_0, y_{a_1}, y_{a_3}\) in terms of \(\tilde{\chi}_0\) and \(\tilde{\chi}_{a_2}\) of the form
\[\chi_0 = D_{\alpha_0}^{\beta_0} \tilde{\chi}_{\beta_0}, \quad y_{a_1} = \hat{e}^{\gamma_1}_{\alpha_1} Z^{\alpha_0}_{\gamma_1} \tilde{\chi}_0 + \hat{A}^{\alpha_2}_{\alpha_1} \tilde{\chi}_{a_2}, \quad y_{a_3} = \hat{D}^{\gamma_3}_{\alpha_3} Z^{\alpha_2}_{\gamma_3} \tilde{\chi}_{a_2}.\] (188)
Using (188), it follows that if (182) and (183) hold, then (182) hold, too
\[(\tilde{\chi}_0 \approx 0, \tilde{\chi}_{a_2} \approx 0) \Rightarrow (\chi_0 \approx 0, y_{a_1} \approx 0, y_{a_3} \approx 0).\] (189)
Relations (187) and (189) prove (i).
(ii) With the help of formulae (182) and (183), we find the expression \(s\) of the Poisson brackets of (187) as: \[\tilde{\chi}_{\alpha}, \tilde{\chi}_{\beta}\] index the line, \(\Delta = (\alpha_0, \alpha_2)\) the column, and \(\phi^{(3)}_{\alpha_2 \beta_2}\) reads as in (181). Then, the matrix of their Poisson brackets, of elements \(C_{\Delta \Delta'}\), takes the concrete form
\[C_{\Delta \Delta'} = \begin{pmatrix} \mu^{(3)}_{\alpha_0 \beta_0} & A^{\alpha_0 \omega_{\alpha_1 \beta_1}}_{\omega_{\beta_2}} Z^{\beta_1}_{\beta_2} \\ Z^{\alpha_0 \omega_{\alpha_1 \beta_1}}_{\alpha_2} A^{\beta_1}_{\beta_2} & \phi^{(3)}_{\alpha_2 \beta_2} \end{pmatrix},\] (192)
where \(\Delta = (\alpha_0, \alpha_2)\) indexes the line, \(\Delta' = (\beta_0, \beta_2)\) the column, and \(\phi^{(3)}_{\alpha_2 \beta_2}\) means
\[\phi^{(3)}_{\alpha_2 \beta_2} = Z^{\alpha_0 \omega_{\alpha_1 \beta_1}}_{\alpha_2} A^{\beta_1}_{\beta_2} + A^{\alpha_3 \omega_{\alpha_3 \beta_3}}_{\alpha_2} A^{\beta_3}_{\beta_2}.\] (193)
In order to prove the invertibility of the matrix (192), we will give its inverse. Direct computation shows that the matrix
\[C^{\Delta' \Delta''} = \begin{pmatrix} \mu^{(3)}_{(3) \beta_0 \alpha_0} & Z^{\beta_0}_{\beta_2} \hat{e}^{\gamma_1}_{\gamma_1} A^{\sigma_1 \lambda_1}_{\lambda_1} \hat{A}^{\rho_2}_{\lambda_1} \\ \hat{A}^{\beta_2}_{\alpha_1} \omega^{\sigma_1 \lambda_1} \hat{e}^{\gamma_1}_{\lambda_1} Z^{\rho_0}_{\gamma_1} & \psi^{(3)\beta_2 \rho_2} \end{pmatrix},\] (194)
with \(\mu^{(3)}_{(3) \beta_0 \alpha_0}\) given by (181) and \(\psi^{(3)\beta_2 \rho_2}\) of the form
\[\psi^{(3)\beta_2 \rho_2} = \hat{A}^{\beta_2}_{\alpha_1} \omega^{\sigma_1 \lambda_1} \hat{A}^{\rho_2}_{\lambda_1} + Z^{\beta_2}_{\beta_3} \hat{D}^{\gamma_3}_{\gamma_3} \omega^{\gamma_3 \lambda_3} \hat{A}^{\rho_2}_{\lambda_3} Z^{\rho_2}_{\sigma_2}.\] (195)
satisfies the relations
\[
C_{\Delta \Delta'} C_{\Delta''} \approx \begin{pmatrix} \delta_{\rho_0} & 0 \\ 0 & \delta_{\rho_2} \end{pmatrix},
\]
so it is indeed the inverse of (194). This proves (ii).

(iii) Since matrix (192) is invertible, it follows that it possesses no non-trivial null vectors and hence the functions \( \tilde{\chi}_\Delta \) are independent, which is equivalent to the fact that the constraint set given by (182) and (183) is irreducible. This proves (iii). □

Taking into account the result (194), the Dirac bracket built with respect to the irreducible second-class constraint set (182) and (183)
\[
[F, G]^{(3)\ast}_{\text{ired}} = [F, G] - [F, \tilde{\chi}_\Delta] C_{\Delta \Delta'} \tilde{\chi}_\Delta', G],
\]
takes the concrete form
\[
[F, G]^{(3)\ast}_{\text{ired}} = [F, G] - [F, \bar{\chi}_\alpha] \mu^{(3)\alpha_0\beta_0} [\bar{\chi}_\beta_0, G]
- [F, \bar{\chi}_\alpha] Z_{\sigma_1}^\alpha \tilde{e}_{\sigma_1}^\gamma \omega^{\sigma_1\lambda_1} A_{\lambda_1}^{\beta_2} [\tilde{\chi}_{\beta_2}, G]
- [F, \bar{\chi}_\alpha] A_{\gamma_1}^{\alpha_2} \omega^{\alpha_2\lambda_1} \tilde{A}_{\lambda_1}^{\beta_2} Z_{\gamma_1}^{\beta_2} [\tilde{\chi}_{\beta_2}, G]
- [F, \bar{\chi}_\alpha] \left( A_{\gamma_1}^{\alpha_2} \omega^{\alpha_2\lambda_1} \tilde{A}_{\lambda_1}^{\beta_2}
+ Z_{\gamma_1}^{\alpha_2} D_{\gamma_1}^{\alpha_2} \omega^{\alpha_2\gamma_3} \tilde{D}_{\gamma_3}^{\alpha_2} A_{\gamma_3}^{\beta_2} [\tilde{\chi}_{\beta_2}, G] \right). \tag{198}
\]

**Theorem 6** The Dirac bracket with respect to the irreducible second-class constraints (198) coincides with that of the intermediate system
\[
[F, G]^{(3)\ast}_{\text{ired}} \approx [F, G]^{(3)\ast}_{\text{ired}} \bigg|_{\text{ired}}, \tag{199}
\]

**Proof.** In order to prove this theorem, we start from the right-hand side of (198) and show that it is (weakly) equal with the right-hand side of (148). Collecting the results expressed by relations (2), (45), (55), (63), (61), (81), (157), (160), (180), (182), and (183), by direct computation we obtain:
\[
[F, \bar{\chi}_\alpha] \mu^{(3)\alpha_0\beta_0} [\bar{\chi}_\beta_0, G] \approx
[F, \chi_\alpha] \mu^{(3)\alpha_0\beta_0} [\chi_\beta_0, G] + [F, y_\alpha] D_{\sigma_1}^\alpha \omega^{\sigma_1\lambda_1} D_{\lambda_1}^{\beta_1} [y_{\beta_1}, G], \tag{200}
\]
\[
[F, \bar{\chi}_\alpha] Z_{\gamma_1}^\alpha \tilde{e}_{\sigma_1}^\gamma \omega^{\sigma_1\lambda_1} \tilde{A}_{\lambda_1}^{\beta_2} [\tilde{\chi}_{\beta_2}, G] \approx
\]

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\[ [F, y_{\alpha 1}] \, D^{\alpha 1}_{\sigma 1} \omega^{\alpha 1 \lambda 1} \left( \delta^\beta_{\lambda 1} - D^\beta_{\lambda 1} \right) [y_{\beta 1}, G], \quad (201) \]
\[ [F, \tilde{\alpha}_{\alpha 2}] \, \tilde{A}^{\alpha 2}_{\sigma 1} \omega^{\alpha 1 \lambda 1} \tilde{\epsilon}^\gamma_{\lambda 1} Z_{\gamma 0}^{\beta 0} [\tilde{\chi}_{\beta 0}, G] \approx \]
\[ [F, y_{\alpha 1}] \, (\delta^\alpha_{\sigma 1} - D^\alpha_{\sigma 1}) \omega^{\alpha 1 \lambda 1} D^\beta_{\lambda 1} [y_{\beta 1}, G], \quad (202) \]
\[ [F, \tilde{\alpha}_{\alpha 2}] \left( \tilde{A}^{\alpha 2}_{\sigma 1} \omega^{\alpha 1 \lambda 1} \tilde{A}^{\beta 2}_{\lambda 1} + Z_{\gamma 3}^{\alpha 2} \tilde{D}^{\gamma 3}_{\lambda 3} \omega^{\gamma 3 \lambda 3} \tilde{D}^{\lambda 3}_{\sigma 3} Z^{\beta 2}_{\sigma 3} \right) [\tilde{\chi}_{\beta 2}, G] \approx \]
\[ [F, y_{\alpha 1}] \, (\delta^\alpha_{\sigma 1} - D^\alpha_{\sigma 1}) \omega^{\alpha 1 \lambda 1} \left( \delta^\beta_{\lambda 1} - D^\beta_{\lambda 1} \right) [y_{\beta 1}, G] \]
\[ + [F, y_{\alpha 3}] \, \omega^{\alpha 3 \beta 3} [y_{\beta 3}, G], \quad (203) \]

Substituting the previous results in (198), we arrive precisely at (199), which proves the theorem. □

### 3.3 Basic result for \( L = 3 \)

Combining (151) and (199), we are led to the result
\[ [F, G]^{(3)*} \approx [F, G]^{(3)*} \bigg|_{\text{ired}}. \quad (204) \]

The last formula proves that we can indeed approach third-order reducible second-class constraints in an irreducible fashion.

### 4 Generalization to an arbitrary reducibility order \( L \)

#### 4.1 Reducible approach

In the sequel we generalize the previous results to the case of a system of second-class constraints, reducible of an arbitrary order \( L \)
\[ Z^{\alpha 0}_{\alpha 0} \chi_{\alpha 0} = 0, \quad Z^{\alpha 1}_{\alpha 0} Z^{\alpha 0}_{\alpha 0} \approx 0, \ldots, \quad Z^{\alpha L-1}_{\alpha L-1} Z^{\alpha L-2}_{\alpha L-1} \approx 0, \quad (205) \]

with \( \alpha_k = 1, M_k \) for each \( k = 1, L \). In addition, the reducibility functions of maximum order \( (L) \), \( Z^{\alpha L-1}_{\alpha L-1} \), are assumed to be all independent. Consequently, the number of independent second-class constraints is equal to \( M \equiv \sum_{k=0}^{L} (-)^k M_k \). Therefore, we can work again here with a Dirac bracket of the type (46), but in terms of \( M \) independent functions \( \chi_A \), i.e.
\[ [F, G]^{(L)*} = [F, G] - [F, \chi_A] M^{(L)AB} [\chi_B, G], \quad A = 1, M, \quad (206) \]
where \( C_{AB}^{(L)}M_{BC}^{(L)} ≈ \delta^C_A \), with \( C_{AB}^{(L)} = [\chi_A, \chi_B] \). The matrix of the Poisson brackets among the constraint functions

\[
C_{\alpha_0\beta_0}^{(L)} = [\chi_{\alpha_0}, \chi_{\beta_0}]
\]

is not invertible due to the relations

\[
Z_{\alpha_1}^{\alpha_0} C_{\alpha_0\beta_0}^{(L)} ≈ 0,
\]

but its rank is equal to \( M \).

Just like in the case of order three of reducibility, we introduce some functions \( (\bar{A}_{\alpha k}^{\alpha})_{k=1,L} \), subject to the relations

\[
\text{rank} \left( Z_{\alpha_k}^{\beta_{k-1}} \bar{A}_{\beta_{k-1}}^{\gamma_k} \right) \approx \sum_{i=k}^L (-)^{k+i} M_t,
\]

\[
\bar{A}_{\alpha_{k-2}}^{\alpha_k} \bar{A}_{\alpha_{k-1}}^{\alpha_k} ≈ 0.
\]

The Dirac bracket from (206) can be written, like in the previous situation, in terms of all the second-class constraint functions. Going along a line similar to that from subsection 3.1.1, we introduce an antisymmetric matrix, of elements \( M_{\alpha_0\beta_0}^{(L)} \), through the relation

\[
C_{\alpha_0\beta_0}^{(L)} M_{\alpha_0\beta_0}^{(L)} \approx D_{\gamma_0}^{\alpha_0},
\]

such that

\[
[F, G]^{(L)*} = [F, G] - [F, \chi_{\alpha_0}] M_{\alpha_0\beta_0}^{(L)} [\chi_{\beta_0}, G]
\]

defines the same Dirac bracket like (206) on the surface \( [1] \). Similar to the case of third-order reducible second-class constraints, the Dirac bracket for \( L \)-order reducible constraints can be expressed in terms of a noninvertible matrix.

**Theorem 7** There exists an invertible, antisymmetric matrix \( \mu_{\alpha_0\beta_0}^{(L)} \) such that Dirac bracket (210) takes the form

\[
[F, G]^{(L)*} = [F, G] - [F, \chi_{\alpha_0}] \mu_{\alpha_0\beta_0}^{(L)} [\chi_{\beta_0}, G]
\]

on the surface \( [1] \).

The relationship between the invertible matrix \( \mu^{(L)} \) and the matrix \( M^{(L)} \) is given by a relation similar to that from the third-order reducible case

\[
M_{\alpha_0\beta_0}^{(L)} ≈ D_{\gamma_0}^{\alpha_0} \mu_{\alpha_0\beta_0}^{(L)} D_{\beta_0}^{\gamma_0}.
\]
4.2 Irreducible approach

4.2.1 Intermediate system

Now, we introduce some new variables, \((y_{\alpha 2k+1})_{\alpha 2k+1=1,M2k+1}\), with \(k = 0, \left\lfloor \frac{L-1}{2} \right\rfloor\), exhibiting the Poisson brackets

\[
[y_{\alpha i}, y_{\beta j}] = \omega_{\alpha i\beta j} \delta_{ij},
\]

where \(\omega_{\alpha i\beta j}\) are the elements of an antisymmetric, invertible matrix, and consider the system subject to the reducible second-class constraints

\[
\chi_{\alpha 0} \approx 0, \quad (y_{\alpha 2k+1})_{k=0, \left\lfloor \frac{L-1}{2} \right\rfloor} \approx 0.
\]

The system constrained to satisfy (214) will be called “intermediate system” in what follows. The Dirac bracket on the phase-space locally parameterized by the variables \(\left( z^a, (y_{\alpha 2k+1})_{k=0, \left\lfloor \frac{L-1}{2} \right\rfloor} \right)\), constructed with respect to the above second-class constraints, reads as

\[
[F, G]^{(L)}_{z,y} = [F, G] - [F, \chi_{\alpha 0}] \mu^{(L)\alpha_0\beta_0} [\chi_{\beta 0}, G]
- \sum_{k=0}^{\left\lfloor \frac{L-1}{2} \right\rfloor} [F, y_{\alpha 2k+1}] \omega^{\alpha 2k+1\beta 2k+1} [y_{\beta 2k+1}, G],
\]

where the Poisson brackets from the right-hand side of (215) contain derivatives with respect to all the variables \(z^a\) and \((y_{\alpha 2k+1})_{k=0, \left\lfloor \frac{L-1}{2} \right\rfloor}\) and \(\omega^{\alpha 2k+1\beta 2k+1}\) denote the elements of the inverse of the matrix of elements \(\omega_{\alpha 2k+1\beta 2k+1}\). In this case the most general form of a function defined on the phase-space locally parameterized by \(\left( z^a, (y_{\alpha 2k+1})_{k=0, \left\lfloor \frac{L-1}{2} \right\rfloor} \right)\) is given by

\[
F(z^a, y_A) = F_0(z^a) + \int_0^1 \frac{dF(z^a, \lambda y_A)}{d\lambda} d\lambda = F_0(z^a) + y_A G^A(z^a, y_B),
\]

with \(y_A = (y_{\alpha 2k+1})_{k=0, \left\lfloor \frac{L-1}{2} \right\rfloor}\), \(F_0(z^a) = F_0(z^a, 0)\), and

\[
G^A(z^a, y_B) = \int_0^1 \frac{\partial F(z^a, \lambda y_A)}{\partial (\lambda y_A)} d\lambda.
\]
If we introduce (216) in (215), then we obtain
\[ [F, G]^{(L)*} \approx [F_0, G_0]^{(L)*}, \] (217)
where the previous weak equality takes place on the surface defined by (214). Moreover, equations (1) and (214) describe the same surface, but embedded in phase-spaces of different dimensions, such that (1) and (214) are equivalent descriptions of one and the same constraint surface. This is why we will maintain the same sign of weak equality related to both descriptions.\(^8\)

Replacing (216) in (215) and making use of (217), we infer the result
\[ [F, G]^{(L)*} \Big|_{z,y} \approx [F, G]^{(L)*}. \] (218)
We recall the fact that the Dirac bracket \([F, G]^{(L)*}\) contains only derivatives with respect to the original phase-space variables \(z^a\).

### 4.2.2 Irreducible system

In order to construct the irreducible system in the general case, we act in a manner similar to that exposed in subsection 3.2.2 and start by adding the constraints:
- if \(L\) odd
  \[ \tilde{\chi}_{\alpha_0} \equiv \chi_{\alpha_0} + A_{\alpha_0}^a y_{\alpha_1} \approx 0, \] (219)
  \[ \tilde{\chi}_{\alpha_{2k}} \equiv Z_{\alpha_{2k}}^a y_{\alpha_{2k-1}} + A_{\alpha_{2k}}^{a+1} y_{\alpha_{2k+1}} \approx 0, \quad k = 1, \frac{L}{2}; \] (220)
- if \(L\) even
  \[ \tilde{\chi}_{\alpha_0} \equiv \chi_{\alpha_0} + A_{\alpha_0}^a y_{\alpha_1} \approx 0, \] (221)
  \[ \tilde{\chi}_{\alpha_{2k}} \equiv Z_{\alpha_{2k}}^a y_{\alpha_{2k-1}} + A_{\alpha_{2k}}^{a+1} y_{\alpha_{2k+1}} \approx 0, \quad k = 1, \frac{L}{2} - 1; \] (222)
  \[ \tilde{\chi}_{\alpha_L} \equiv Z_{\alpha_L}^a y_{\alpha_{L-1}} \approx 0. \] (223)

\(^8\)It is understood that for the functions defined on the phase-space locally parameterized by the variables \(z^a\) we use (1) and for those defined on the larger phase-space, of coordinates \((z^a, y_{\alpha_{2k+1}})_{k=0,\frac{L}{2}}\), we employ representation (214).
These constraints are defined on the larger phase-space, locally parameterized by \( \left( z^a, (y_{a2k+1})_{k=0,\left\lfloor \frac{L-1}{2} \right\rfloor} \right) \). The functions \( A_{\alpha2k}^{\alpha2k+1} \) appearing in the above are defined by the relations:

- if \( L \) odd

\[
\tilde{A}_{\alpha2k}^{\alpha2k+1} = A_{\alpha2k}^{\beta2k+1} \tilde{e}_{\beta2k+1}^{\alpha2k+1}, \quad k = 0, \left\lfloor \frac{L}{2} \right\rfloor - 1,
\]

\[
\tilde{A}_{\alpha L-1}^{\alpha L} = A_{\alpha L-1}^{\beta L} \tilde{D}_{\beta L}^{\alpha L},
\]

- if \( L \) even

\[
\tilde{A}_{\alpha2k}^{\alpha2k+1} = A_{\alpha2k}^{\beta2k+1} \tilde{e}_{\beta2k+1}^{\alpha2k+1}, \quad k = 0, \left\lfloor \frac{L}{2} \right\rfloor - 1.
\]

The elements \( \tilde{e}_{\beta2k+1}^{\alpha2k+1} \) determine an invertible matrix and \( \tilde{D}_{\beta L}^{\alpha L} \) are the elements of the inverse of the matrix of elements \( D_{\alpha L}^{\beta L} = Z_{\alpha L}^{\gamma L-1} A_{\alpha L}^{\beta L} \).

In the following we show that (219) and (220) (or (221)–(223)) display all the desired properties: equivalence with the intermediate system (214), second-class behaviour, irreducibility and, most important, the fact that associated Dirac bracket (weakly) coincides with the original one, corresponding to the second-order reducible second-class constraints. The proof of all these properties is contained within the next two theorems.

**Theorem 8** Constraints (219) and (220) (or (221)–(223)) fulfill the following properties:

(i) equivalence to (214), i.e.

\[
(\tilde{\chi}_{a2k})_{k=0,\left\lfloor \frac{L}{2} \right\rfloor} \approx 0 \iff \left(\chi_{a0} \approx 0, (y_{a2k+1})_{k=0,\left\lfloor \frac{L-1}{2} \right\rfloor} \approx 0\right);
\]

(ii) second-class behaviour, i.e. the matrix of elements

\[
C_{\Delta \Delta'} = [\tilde{\chi}_{\Delta}, \tilde{\chi}_{\Delta'}],
\]

is invertible, where

\[
\tilde{\chi}_{\Delta} \equiv (\tilde{\chi}_{a2k})_{k=0,\left\lfloor \frac{L}{2} \right\rfloor};
\]

(iii) irreducibility.
Proof. (i) It is easy to see that if (214) hold, then (219) and (220) (or (221)–(223)) also hold

\[ (\chi_{\alpha_0} \approx 0, (y_{\alpha_{2k+1}})_{k=0,\lfloor L/2 \rfloor} \approx 0) \Rightarrow (\bar{\chi}_{\alpha_{2k}})_{k=0,\lfloor L/2 \rfloor} \approx 0. \]  

(230)

From (219) and (220) (or (221)–(223)) it is simple to express the original constraint functions \( \chi_{\alpha_0} \) and the newly added phase-space variables \( (y_{\alpha_{2k+1}})_{k=0,\lfloor L/2 \rfloor} \approx 0, [L-1/2] \approx 0 \) in terms of \( \bar{\chi}_{\alpha_0} \) and \( (\bar{\chi}_{\alpha_{2k}})_{k=1,\lfloor L/2 \rfloor} \) as follows:
- if \( L \) odd

\[ \chi_{\alpha_0} = D^{\beta_0}_{\alpha_0} \bar{\chi}_{\beta_0}; \quad y_{\alpha_{2k+1}} = \tilde{e}^{\beta_{2k+1}}_{\alpha_{2k+1}} Z^{\beta_{2k}}_{\beta_{2k+1}} \bar{\chi}_{\beta_{2k}} + A^{\alpha_{2k+2}}_{\alpha_{2k+1}} \bar{\chi}_{\alpha_{2k+2}}, \quad k = 0, \lfloor L/2 \rfloor - 1, \]  

(231)

\[ y_{\alpha_{L}} = \tilde{D}^{\beta_{L-1}}_{\alpha_{L}} Z^{\beta_{L}}_{\beta_{L-1}} \bar{\chi}_{\beta_{L-1}}; \]  

(232)

- if \( L \) even

\[ \chi_{\alpha_0} = D^{\beta_0}_{\alpha_0} \bar{\chi}_{\beta_0}; \]  

(233)

\[ y_{\alpha_{2k+1}} = \tilde{e}^{\beta_{2k+1}}_{\alpha_{2k+1}} Z^{\beta_{2k}}_{\beta_{2k+1}} \bar{\chi}_{\beta_{2k}} + A^{\alpha_{2k+2}}_{\alpha_{2k+1}} \bar{\chi}_{\alpha_{2k+2}}, \quad k = 0, L/2 - 1. \]  

(234)

From (231)–(233) (or (234) and (235)) we obtain that if (219) and (220) (or (221)–(223)) hold, then (214) holds, too

\[ (\bar{\chi}_{\alpha_0} \approx 0, (\bar{\chi}_{\alpha_{2k}})_{k=1,\lfloor L/2 \rfloor} \approx 0) \Rightarrow (\chi_{\alpha_0} \approx 0, (y_{\alpha_{2k+1}})_{k=0,\lfloor L/2 \rfloor} \approx 0). \]  

(236)

Relations (230) and (236) prove (i).

(ii) Now, we employ formulae (219) and (220) (or (221)–(223)) and find the concrete form of the Poisson brackets among the constraint functions \( \bar{\chi}_{\Delta} \) as:
- if \( L \) odd

\[ [\bar{\chi}_{\alpha_0}, \bar{\chi}_{\beta_0}] \approx \mu(L)_{\alpha_0 \beta_0}; \]  

(237)

\[ [\bar{\chi}_{\alpha_{2k}}, \bar{\chi}_{\beta_{2k}}] \approx Z^{\alpha_{2k-1}}_{\alpha_{2k}} \omega^{\alpha_{2k-1}}_{\alpha_{2k-1}} \beta_{2k-1} \beta_{2k} + A^{\alpha_{2k+1}}_{\alpha_{2k+1}} \omega^{\alpha_{2k+1}}_{\alpha_{2k+1}} \beta_{2k+1} \beta_{2k} \]  

(238)

\[ [\bar{\chi}_{\alpha_{2k-2}}, \bar{\chi}_{\beta_{2k}}] \approx A^{\alpha_{2k-2}}_{\alpha_{2k-2}} \omega^{\alpha_{2k-2}}_{\alpha_{2k-2}} \beta_{2k-1} \beta_{2k} \]  

(239)

\[ [\bar{\chi}_{\alpha_{2k-1}}, \bar{\chi}_{\beta_{2k}}] \approx A^{\alpha_{2k-1}}_{\alpha_{2k-1}} \omega^{\alpha_{2k-1}}_{\alpha_{2k-1}} \beta_{2k-1} \beta_{2k} \]  

(240)
with \( k = 1, \left\lceil \frac{L}{2} \right\rceil \):

-if \( L \) even

\[
\begin{align*}
[\tilde{\chi}_o, \tilde{\chi}_0] & \approx \mu^{(L)}_{\alpha_0 \beta_0}, \\
[\tilde{\chi}_{2k}, \tilde{\chi}_{2k}] & \approx Z_{\alpha_{2k}}^{\alpha_{2k-1} \omega_{2k-1} \beta_{2k-1}} Z_{\beta_{2k}}^{\beta_{2k-1}} + A_{\alpha_{2k}}^{\alpha_{2k+1} \omega_{2k+1} \beta_{2k+1}} A_{\beta_{2k}}^{\beta_{2k+1}} \tag{241} \\
[\tilde{\chi}_{2k-2}, \tilde{\chi}_{2k}] & \approx A_{\alpha_{2k-2}}^{\alpha_{2k-1} \omega_{2k-1} \beta_{2k-1}} Z_{\beta_{2k}}^{\beta_{2k-1}}, \tag{242} \\
[\tilde{\chi}_L, \tilde{\chi}_L] & \approx Z_{\alpha_L}^{\alpha_L-1 \omega_L \beta_L-1} Z_{\beta_L}^{\beta_L-1}, \tag{243}
\end{align*}
\]

with \( k = 1, \left\lceil \frac{L}{2} \right\rceil - 1 \) in (241) and \( k = 1, \left\lceil \frac{L}{2} \right\rceil \) in (242).

Accordingly, the matrix of elements given in (228) reads as

\[
C_{\Delta \Delta'} = \begin{pmatrix}
\mu^{(L)}_{\alpha_0 \beta_0} & A_{\alpha_0}^{\alpha_1 \omega_1 \beta_1} Z_{\beta_2}^{\beta_1} & 0 \\
Z_{\alpha_2}^{\alpha_2 \omega_1 \beta_1} A_{\beta_0}^{\beta_1} & \phi_{\alpha_2 \beta_2} & A_{\alpha_2}^{\alpha_3 \omega_3 \beta_3} Z_{\beta_4}^{\beta_3} \\
0 & Z_{\alpha_4}^{\alpha_4 \omega_3 \beta_3} A_{\beta_2}^{\beta_3} & \phi_{\alpha_4 \beta_4} \\
\end{pmatrix}, \tag{244}
\]

where

\[
\phi_{\alpha_{2k} \beta_{2k}} = Z_{\alpha_{2k}}^{\alpha_{2k-1} \omega_{2k-1} \beta_{2k-1}} Z_{\beta_{2k}}^{\beta_{2k-1}} + A_{\alpha_{2k}}^{\alpha_{2k+1} \omega_{2k+1} \beta_{2k+1}} A_{\beta_{2k}}^{\beta_{2k+1}}. \tag{245}
\]

The last block on the main diagonal of (244) is of the type (245), with \( k = \left\lceil \frac{L}{2} \right\rceil \) for \( L \) odd or respectively of the form

\[
\phi_{\alpha_L \beta_L} = Z_{\alpha_L}^{\alpha_L-1 \omega_L \beta_L-1} Z_{\beta_L}^{\beta_L-1}, \tag{246}
\]

for \( L \) even. The invertibility of \( C_{\Delta \Delta'} \) is obtained by constructing its inverse, which can be checked to have the expression

\[
C_{\Delta' \Delta''} = \begin{pmatrix}
A_{\sigma_1}^{\beta_2} \omega_1 \gamma_1 \sigma_1 \gamma_1 Z_{\gamma_1}^{\gamma_1} & 0 \\
Z_{\gamma_1}^{\gamma_1} & A_{\lambda_1}^{\rho_1} & 0 \\
0 & Z_{\gamma_3}^{\gamma_3} & A_{\lambda_3}^{\rho_3} \\
\end{pmatrix}, \tag{247}
\]

with

\[
\psi_{\beta_{2k} \beta_{2k}} = A_{\sigma_{2k-1}}^{\beta_{2k}} \omega_{2k-1} \lambda_{2k-1} A_{\lambda_{2k-1}}^{\rho_{2k}} + Z_{\gamma_{2k+1}}^{\beta_{2k}} \omega_{2k+1} \lambda_{2k+1} \gamma_{2k+1} \sigma_{2k+1} Z_{\gamma_{2k+1}}^{\rho_{2k+1}}. \tag{248}
\]
The last block on the main diagonal of (247) is given by (248), with \( k = \left[ \frac{L}{2} \right] \) for \( L \) odd or respectively

\[
\psi^\beta_L \rho_L = \bar{A}_\rho_L \omega_{\rho L}^\sigma \lambda_{L-1} \bar{A}_\lambda_L
\]

for \( L \) even. Indeed, simple computation yields

\[
C_{\Delta \Delta'} C_{\Delta' \Delta''} \approx \begin{pmatrix}
\delta_{\rho_0 \alpha_0} & 0 & 0 \\
0 & \delta_{\rho_2 \alpha_2} & 0 \\
0 & 0 & \delta_{\rho_4 \alpha_4} \\
& & \ddots
\end{pmatrix},
\]

such that (244) is indeed invertible and its inverse is expressed by (247). This proves (ii).

(iii) As (244) is invertible, it follows that it displays no null vectors and hence the functions \( \tilde{\chi}_\Delta \) are all independent or, in other words, the constraint set (219) and (220) (or (221)–(223)) is irreducible. This proves (iii). □

Taking into account the result given by (247), it follows that the Dirac bracket built with respect to the irreducible second-class constraints (219) and (220) (or (221)–(223))

\[
\{ F, G \}^{(L)*}_{ired} = \{ F, G \} - \{ F, \tilde{\chi}_\Delta \} C_{\Delta \Delta'} \{ \tilde{\chi}_{\Delta'}, G \}
\]

takes the particular form

\[
\{ F, G \}^{(L)*}_{ired} = \{ F, G \} - \{ F, \tilde{\chi}_\alpha_0 \} \mu^{(L)\alpha_0 \beta_0} [ \tilde{\chi}_{\beta_0}, G ]
\]

\[
- \sum_{k=0}^{\left[ \frac{L}{2} \right] - 1} \left\{ [ F, \tilde{\chi}_{\alpha_{2k}} ] Z_{\alpha_{2k+1}}^{\alpha_{2k+2}} c_{\gamma_{2k+1}}^{\gamma_{2k+1} + 1} \lambda_{\gamma_{2k+1}}^{\beta_{2k+1} \beta_{2k+1} + 1} \bar{A}_{\beta_{2k+1}}^{\beta_{2k+1} + 2} [ \tilde{\chi}_{\beta_{2k+2}}, G ] \\
+ [ F, \tilde{\chi}_{\alpha_{2k+2}} ] \bar{A}_{\alpha_{2k+1}}^{\alpha_{2k+2} + \alpha_{2k+1} \gamma_{2k+1} + 1} c_{\gamma_{2k+1}}^{\gamma_{2k+1} + 1} \lambda_{\gamma_{2k+1}}^{\beta_{2k+1} \beta_{2k+1} + 1} Z_{\beta_{2k+1}}^{\beta_{2k+1} + 2} [ \tilde{\chi}_{\beta_{2k+2}}, G ] \\
+ [ F, \tilde{\chi}_{\alpha_{2k+2}} ] \psi_{\alpha_{2k+2} \beta_{2k+2}} [ \tilde{\chi}_{\beta_{2k+2}}, G ] \right\}.
\]

**Theorem 9** The Dirac bracket with respect to the irreducible second-class constraints (252) coincides with that of the intermediate system

\[
\{ F, G \}^{(L)*}_{ired} \approx \{ F, G \}^{(L)*}_{z,y}.
\]
Proof. We start from the right-hand side of (252) and show that it is (weakly) equal to the right-hand side of (215). By direct computation, we obtain that:

\[
[F, \tilde{\chi}_{\alpha_0}] \mu^{(L)\alpha_0\beta_0} [\tilde{\chi}_{\beta_0}, G] \approx [F, \chi_{\alpha_0}] \mu^{(L)\alpha_0\beta_0} [\chi_{\beta_0}, G] + [F, y_{\alpha_1}] \mathcal{D}_{\alpha_1}^\omega \sigma^1 \lambda_1 \mathcal{D}_{\lambda_1}^\beta_1 [y_{\beta_1}, G], \tag{254}
\]

\[
[F, \tilde{\chi}_{\alpha_{2k}}] \mu^{\alpha_{2k}+\beta_{2k}} [\tilde{\chi}_{\beta_{2k}}, G] \approx [F, \chi_{\alpha_{2k}}] \mu^{\alpha_{2k}+\beta_{2k}} [\chi_{\beta_{2k}}, G] + [F, y_{\alpha_{2k+1}}] \mathcal{D}_{\alpha_{2k+1}}^\omega \sigma_{2k}^1 \lambda_{2k+1} \mathcal{D}_{\lambda_{2k+1}}^\beta_{2k+1} [y_{\beta_{2k+1}}, G], \tag{255}
\]

\[
[F, \tilde{\chi}_{\alpha_{2k+2}}] \mu^{\alpha_{2k+2}+\beta_{2k+2}} [\tilde{\chi}_{\beta_{2k+2}}, G] \approx [F, \chi_{\alpha_{2k+2}}] \mu^{\alpha_{2k+2}+\beta_{2k+2}} [\chi_{\beta_{2k+2}}, G] + [F, y_{\alpha_{2k+3}}] \mathcal{D}_{\alpha_{2k+3}}^\omega \sigma_{2k+3}^1 \lambda_{2k+3} \mathcal{D}_{\lambda_{2k+3}}^\beta_{2k+3} [y_{\beta_{2k+3}}, G], \tag{256}
\]

with \( k = \frac{L}{2} - 1 \). Also direct computation provides:

-if \( L \) odd

\[
[F, \tilde{\chi}_{\alpha_{2k+2}}] \mu^{\alpha_{2k+2}+\beta_{2k+2}} [\tilde{\chi}_{\beta_{2k+2}}, G] \approx
[F, \chi_{\alpha_{2k+2}}] \mu^{\alpha_{2k+2}+\beta_{2k+2}} [\chi_{\beta_{2k+2}}, G] + [F, y_{\alpha_{2k+3}}] \mathcal{D}_{\alpha_{2k+3}}^\omega \sigma_{2k+3}^1 \lambda_{2k+3} \mathcal{D}_{\lambda_{2k+3}}^\beta_{2k+3} [y_{\beta_{2k+3}}, G], \tag{257}
\]

with \( k = \frac{L}{2} - 1 \);

-if \( L \) even

\[
[F, \tilde{\chi}_{\alpha_{2k+2}}] \mu^{\alpha_{2k+2}+\beta_{2k+2}} [\tilde{\chi}_{\beta_{2k+2}}, G] \approx
[F, \chi_{\alpha_{2k+2}}] \mu^{\alpha_{2k+2}+\beta_{2k+2}} [\chi_{\beta_{2k+2}}, G] + [F, y_{\alpha_{2k+3}}] \mathcal{D}_{\alpha_{2k+3}}^\omega \sigma_{2k+3}^1 \lambda_{2k+3} \mathcal{D}_{\lambda_{2k+3}}^\beta_{2k+3} [y_{\beta_{2k+3}}, G], \tag{258}
\]

with \( k = \frac{L}{2} - 2 \).

Further computation finally gives:

\[
[F, \tilde{\chi}_{\alpha_L}] \mathcal{A}_{\gamma_{L-1}}^\alpha \mathcal{A}_{\lambda_{L-1}}^\beta \mathcal{A}_{\beta_{L-1}}^\gamma [\tilde{\chi}_{\beta_{L}}, G] \approx
[F, \chi_{\alpha_L}] \mathcal{A}_{\gamma_{L-1}}^\alpha \mathcal{A}_{\lambda_{L-1}}^\beta \mathcal{A}_{\beta_{L-1}}^\gamma [\chi_{\beta_{L}}, G]. \tag{259}
\]

Inserting the last formulae in (252) we arrive at (253), which proves the theorem. \( \square \)
4.3 Main result

Based on (218) and (253), we are led to the relation

$$[F, G]^{(L)*} \approx [F, G]^{(L)*}_{\text{ired}},$$

which expresses the fact that second-class constraints reducible of an arbitrary order $L$ can be systematically approached in an irreducible manner. This is the key result of the present paper.

4.4 Geometrical interpretation of the irreducible approach

Let us denote by $P$ the original phase-space and by $P'$ the phase-space of the intermediate system, and hence also of the irreducible theory. Both are symplectic manifolds endowed with symplectic two-forms whose coefficients are in each case the elements of the inverse of the matrix having as elements the fundamental Poisson brackets. We denote by $\Sigma$ and respectively $\Sigma'$ the second-class constraint surface for the original system and respectively for the intermediate theory. By Theorem 8 it follows that the second-class constraint surface of the irreducible system, given by equations (219) and (220) (or (221)–(223)), is nothing but an equivalent representation of $\Sigma'$. Let $j$ and respectively $j'$ be the injective immersions of $\Sigma$ in $P$ and respectively of $\Sigma'$ in $P'$. The second-class property of $\Sigma$ and respectively of $\Sigma'$ is equivalent to the fact that the induced symplectic two-forms $j^*\omega$ and respectively $j'^*\omega'$ are non-degenerate [7], which is the same with [18]

$$j_*(T\Sigma) \cap T\Sigma^\perp = \{0\}, \quad j'_*(T\Sigma') \cap T\Sigma'^\perp = \{0\}. \quad (261)$$

It is easy to argue now the preservation of the original number of physical degrees of freedom with respect to the intermediate and irreducible systems. The dimensions of the original and respectively of the intermediate or irreducible phase-space are valued as

$$\dim P = 2N, \quad \dim P' = 2N + \sum_{k=0}^{[L/2]-1} M_{2k+1}, \quad (262)$$
while the dimensions of the corresponding submanifolds $\Sigma$ and respectively $\Sigma'$ are equal by construction

$$2N = \dim \Sigma = \dim \Sigma' = 2N - \sum_{l=0}^{L} (-1)^l M_l.$$ (263)

Because the induced symplectic two-forms $j^*\omega$ and respectively $j'^*\omega'$ are non-degenerate, from (263) we deduce that

$$\text{rank } (j^*\omega) = \text{rank } (j'^*\omega') = 2N,$$ (264)

and therefore all the three systems, original, intermediate, and irreducible, possess the same number of physical degrees of freedom, $N$, defined as half of the rank of the induced two-forms.

Moreover, the induced symplectic two-forms $j^*\omega$ and $j'^*\omega'$ can be brought to exactly the same form in some conveniently chosen charts. For instance, if we (locally) parameterize the submanifolds $\Sigma$ and $\Sigma'$ (having the same dimension $2N$) by the coordinates $(\xi^a)_{a=1,2N}$, then the local expressions of the immersions $j$ and respectively $j'$ read as

$$z^a = z^a(\xi), \quad a = 1,2N$$ (265)

and respectively

$$\begin{cases} 
  z^a = z^a(\xi), & a = 1,2N, \\
  y^{A_{2k+1}} = 0, & A_{2k+1} = 1,M_{2k+1}, \quad k = 0,\left[\frac{L-1}{2}\right].
\end{cases}$$ (266)

Obviously, related to the local expressions of (265) and (266) we have that

$$(j^*\omega)_{\alpha\beta} = (j'^*\omega')_{\alpha\beta}, \quad \forall \alpha, \beta = 1,2N.$$ (267)

One of the main benefits enabled by our irreducible construction is the computation in a standard manner of the coefficients of the induced symplectic two-form (267) as the elements of the inverse of the matrix having as elements the fundamental Dirac brackets (see Theorem 2.5 from [7]). By ‘standard’ we mean without need to take any specific parametrization of the second-class constraint surface and, implicitly, to perform any separation into dependent and independent constraint functions.
We have seen that the matrices $D^\gamma_{\alpha k}$ (with $k > 0$) are some intermediate steps required by the irreducible procedure, which serve to the construction of the projection $D^\gamma_{\alpha 0}$, which projects the system of local generators

$$X_{\alpha 0} = \sigma^{ab} \frac{\partial X_{\alpha 0}}{\partial z^a} \frac{\partial}{\partial z^b}$$

(268)

of the space $T\Sigma^\perp$ into a local basis of the same space.

5 Example

Let us exemplify the general theory on gauge-fixed Abelian $p$-form gauge fields. Abelian $p$-forms are described by the Lagrangian action

$$S^L_0 [A_{\mu_1 \ldots \mu_p}] = \int d^D x \left( -\frac{1}{2 \cdot (p + 1)!} F_{\mu_1 \ldots \mu_{p+1}} F^{\mu_1 \ldots \mu_{p+1}} \right),$$

(269)

where the field strength of $A_{\mu_1 \ldots \mu_p}$ is defined in the standard manner by $F_{\mu_1 \ldots \mu_{p+1}} \equiv \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]}$. Furthermore, we take the spacetime dimension $D$ to satisfy $D \geq p + 1$, since otherwise the number of physical degrees of freedom would be strictly negative. Everywhere in the sequel the notation $[\mu \ldots \nu]$ signifies antisymmetry with respect to all the indices between brackets without normalization factors (i.e., the independent terms appear only once and are not multiplied by overall numerical factors). We will briefly expose the canonical analysis of Abelian $p$-forms. For more details, see [19] and [20].

From the definitions of canonical momenta\footnote{We work with a ‘mostly negative’ metric tensor (+ $- - \ldots -$), such that no confusion arises in the notation $A^{\mu_1 \ldots \mu_p}$ for the time derivative of $A^\mu$.}

$$\pi_{\mu_1 \ldots \mu_p} = \frac{\partial L_0}{\partial \dot{A}^{\mu_1 \ldots \mu_p}}$$

(270)

on the one hand one obtains the primary constraints

$$G^{(1)}_{i_1 \ldots i_{p-1}} \equiv \pi_{0i_1 \ldots i_{p-1}} \approx 0,$$

(271)

and, on the other hand, one expresses the time derivatives of $A_{i_1 \ldots i_p}$ as

$$\dot{A}_{i_1 \ldots i_p} = -p! \pi_{i_1 \ldots i_p} - (-)^p \partial_{[i_1} A_{i_2 \ldots i_p]}.$$

(272)
The canonical Hamiltonian in defined in the standard manner for constrained systems \[7\] and reduces to

\[
H = \int d^{D-1}x \left( -pA^{\alpha_1\ldots\alpha_{p-1}} \partial_{\alpha_1} \pi_{\alpha_1\ldots\alpha_{p-1}} \right. \\
\left. - \frac{p!}{2} \pi_{\alpha_1\ldots\alpha_p} \pi^{\alpha_1\ldots\alpha_p} + \frac{1}{2} \frac{1}{(p+1)!} F_{\alpha_1\ldots\alpha_{p+1}} F^{\alpha_1\ldots\alpha_{p+1}} \right),
\]

(273)

where we made the notation \( x = (x^0, \mathbf{x}) \).

Dirac’s algorithm (the consistency conditions for the primary constraints (271)) provides the secondary constraints

\[
\chi^{(1)}_{\alpha_1\ldots\alpha_{p-1}} \equiv -p \partial_{\alpha} \pi_{\alpha_1\ldots\alpha_{p-1}} \approx 0
\]

(274)

and stops after the first step. Therefore, Abelian \( p \)-form gauge fields are subject to the constraints (271) and (274), which are first-class and, moreover, Abelian (the Poisson brackets among the constraint functions vanish strongly). It is easy to check the relations

\[
\left[ H, G^{(1)}_{\alpha_1\ldots\alpha_{p-1}} \right] = \chi^{(1)}_{\alpha_1\ldots\alpha_{p-1}},
\]

(275)

\[
\left[ H, \chi^{(1)}_{\alpha_1\ldots\alpha_{p-1}} \right] = 0,
\]

(276)

which show that (273) is also a first-class Hamiltonian for Abelian \( p \)-form gauge fields. The primary first-class constraints are irreducible, while the secondary first-class ones are off-shell reducible (meaning that the null eigenvector equations for the constraint functions and for all the higher-order reducibility functions hold strongly, everywhere on the phase-space, and not only on the first-class surface) of order \( p - 1 \). The associated reducibility functions are given below.

It is known that the first-class constraints produce some local transformations of the canonical variables, which do not affect the physical state of the system. They are called Hamiltonian gauge transformations. Although only the primary first-class constraints can be shown to generate gauge transformations, we accept Dirac’s conjecture, according to which all first-class constraint generate Hamiltonian gauge transformations. The dynamics of first-class systems is thus not fixed in the sense that for some fixed initial set of canonical variables, the solution to the Hamiltonian equations of motion in the presence of first-class constraints is not unique. In other words,
a given physical state of a first-class system is expressed by more than one set of canonical variables (any two such sets are related by a Hamiltonian gauge transformation). In practice, it is useful to eliminate this ambiguity and restore a one-to-one correspondence between physical states and values of the independent canonical variables. This is realized via the so-called ‘gauge-fixing procedure’ by means of imposing further restrictions on the canonical variables, known as ‘canonical gauge conditions’. These must be ‘good’ canonical gauge conditions in the sense of [7], subsection 1.4.1. It is easy to see that a set of good canonical gauge conditions with respect to the first-class constraints (271) and (274) reads as

\[ \chi^{(1)i_1...i_{p-1}} \equiv A^{0i_1...i_{p-1}} \approx 0, \]
\[ \chi^{(2)j_1...j_{p-1}} \equiv -\partial_m A^{mj_1...j_{p-1}} \approx 0. \]  

The overall constraint set formed with the first-class constraints (271) and (274) together with the chosen canonical gauge conditions (277) and (278) is a second-class constraint set, off-shell reducible of order \((p-1)\). In fact, only (274) and (278) are reducible, each of order \((p-1)\), while (271) and (277) are irreducible.

Due to the fact that the second-class constraints (271) and (277) are independent, we will eliminate them from the theory by means of the Dirac bracket built with respect to themselves and will treat along the irreducible approach exposed in the main body of this paper only the reducible second-class constraints (274) and (278). It is useful to organize these second-class constraints in a column vector

\[ \chi_{\alpha 0} = \left( \begin{array}{c} \chi^{(1)i_1...i_{p-1}} \\ \chi^{(2)j_1...j_{p-1}} \end{array} \right) \approx 0. \]  

Constraints (279) are \((p-1)\)-order reducible, with the reducibility functions of the form

\[ Z_{\alpha k+1} = \left( \begin{array}{cc} \frac{1}{(p-k-2)!} \delta_{m_1}^{i_1} \cdots \delta_{m_{p-k-2}}^{i_{p-k-2}} \partial^{i_{p-k-1}} & 0 \\ 0 & \frac{1}{(p-k-1)!} \delta_{j_1}^{n_1} \cdots \delta_{j_{p-k-2}}^{n_{p-k-2}} \partial_{j_{p-k-1}} \end{array} \right), \]  

for \( k = 0, p-2 \). The matrix of the Poisson brackets among the constraint functions from (279) is expressed by

\[ C_{\alpha 0\beta 0} = \left( \begin{array}{cc} 0 & \Delta D_{i_1...i_{p-1}}^{j_1...j_{p-1}} \\ -\Delta D_{j_1...j_{p-1}}^{i_1...i_{p-1}} & 0 \end{array} \right), \]  

41
and $\Delta = \partial_j \partial^j$.

Formula (212) together with (284) and (285) provides

$$D^j_{i_1 \ldots i_{p-1}} = \frac{1}{(p-1)!} \left( \delta^j_{i_1} \cdots \delta^j_{i_{p-1}} - \frac{\delta^{m_1}_{i_1} \cdots \delta^{m_{p-2}}_{i_{p-2}} \delta^{j_1}_{m_1} \cdots \delta^{j_{p-1}}_{m_{p-2}} \partial^{j_{p-1}}}{(p-2)! \Delta} \right)$$

and $\Delta = \partial_j \partial^j$.

In this particular case the functions $(\tilde{A}_{\alpha_k}^{\alpha_k+1})_{k=0}^{p-2}$ read as

$$\tilde{A}_{\alpha_k}^{\alpha_k+1} = \left( \begin{array}{ccc} \frac{1}{(p-k-1)!} \partial^{m_1}_{j_1} \cdots \delta^{m_{p-k-2}}_{j_{p-k-2}} \partial^{j_{p-k-1}} & 0 & 0 \\ 0 & \frac{1}{(p-k-2)!} \partial^{n_1}_{j_1} \cdots \delta^{n_{p-k-2}}_{j_{p-k-2}} \partial^{j_{p-k-1}} \end{array} \right).$$

If we take $\tilde{A}_{\alpha_0}^{\alpha_1}$ as in (283) for $k = 0$, then we find that $D^{\beta_0}_{\alpha_0}$ is given by

$$D^{\beta_0}_{\alpha_0} = \left( \begin{array}{c} D^{j_1 \ldots j_{p-1}}_{i_1 \ldots i_{p-1}} \\ 0 \\ D^{j_1 \ldots j_{p-1}}_{i_1 \ldots i_{p-1}} \end{array} \right).$$

From equation (209) we obtain $M^{(p-1)\alpha_0 \beta_0}$ under the form

$$M^{(p-1)\alpha_0 \beta_0} = \left( \begin{array}{c} 0 \\ -\frac{1}{\Delta} D^{j_1 \ldots j_{p-1}}_{i_1 \ldots i_{p-1}} \\ 0 \end{array} \right).$$

With $M^{(p-1)\alpha_0 \beta_0}$ at hand, we are able to construct the Dirac bracket given by (210). After some computation, we determine the fundamental Dirac brackets as:

$$[A^{i_1 \ldots i_p}(x), \pi_{j_1 \ldots j_p}(y)]^s_{x_0=y_0} = D^{i_1 \ldots i_p}_{j_1 \ldots j_p} \delta^{D-1}(x-y),$$

$$[A^{i_1 \ldots i_p}(x), A^{j_1 \ldots j_p}(y)]^s_{x_0=y_0} = 0 = [\pi_{i_1 \ldots i_p}(x), \pi_{j_1 \ldots j_p}(y)]^s_{x_0=y_0},$$

where

$$D^{j_1 \ldots j_p}_{i_1 \ldots i_p} = \frac{1}{p!} \left( \delta^j_{i_1} \cdots \delta^j_{i_p} - \frac{\delta^{m_1}_{i_1} \cdots \delta^{m_{p-1}}_{i_{p-1}} \partial^{j_{p-1}}_{m_{p-1}} \delta^{j_1}_{m_1} \cdots \delta^{j_{p-1}}_{m_{p-2}} \partial^{j_{p-1}}}{(p-1)! \Delta} \right).$$

Formula (212) together with (284) and (285) provides

$$\mu^{(p-1)\alpha_0 \beta_0} = \left( \begin{array}{c} 0 \\ -\frac{1}{(p-1)! \Delta} \delta^j_{i_1} \cdots \delta^j_{i_{p-1}} \partial^{j_{p-1}} \delta^{i_1}_{j_1} \cdots \delta^{i_{p-1}}_{j_{p-1}} \\ 0 \end{array} \right).$$
which clearly exhibits the invertibility of $\mu^{(p-1)\alpha_0\beta_0}$. By computing the fundamental Dirac brackets with the help of (2111) (with $\mu^{(p-1)\alpha_0\beta_0}$ given by (289)), we reobtain precisely (286) and (287).

In order to construct the irreducible system of second-class constraints that is equivalent to the original one (like in subsection 4.2.2), we need to enlarge the phase-space by the independent variables $(y_{\alpha_{2k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$ and to know the exact form of the functions $(A_{\alpha_k}^{\alpha_{k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$. For gauge-fixed $p$-forms, it is necessary to add the supplementary variables

$$y_{\alpha_{2k+1}} = \left( \begin{array}{c} P_{i_1,\ldots,i_{p-2k-2}} \\ B_{i_1,\ldots,i_{p-2k-2}} \end{array} \right),$$

with the Poisson brackets

$$\omega_{\alpha_{2k+1},\beta_{2k+1}} = \left( \begin{array}{cc} 0 & \frac{1}{(p-2k-2)!} \delta_{j_1,\ldots,j_{p-2k-2}}^i \delta_{j_{p-2k-2}} \delta_{j_{p-2k-2}} \\ \frac{1}{(p-2k-2)!} \delta_{j_1,\ldots,j_{p-2k-2}}^i & 0 \end{array} \right).$$

The functions $(A_{\alpha_k}^{\alpha_{k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$ can be chosen for instance of the form

$$A_{\alpha_{2k}} = \left( \begin{array}{cc} \frac{(-2k+1)}{(p-2k-1)!} \delta_{m_1,\ldots,m_{p-2k-2}}^{i_1,\ldots,i_{p-2k-2}} \delta_{i_{p-2k-2}} \delta_{i_{p-2k-2}} \\ 0 & 0 \end{array} \right).$$

The link between the function sets $(A_{\alpha_{2k+1}}^{\alpha_{k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$ and $(A_{\alpha_{2k}}^{\alpha_{k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$ is expressed in the case of the model under study by:

- if $p$ is odd, by relations (226), with $(\hat{c}_{\alpha_{2k+1}}^{\beta_{2k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$ taken as

$$\hat{c}_{\alpha_{2k+1}}^{\beta_{2k+1}} = \left( \begin{array}{cc} \frac{(-2k+1)}{(p-2k-2)!} \delta_{m_1,\ldots,m_{p-2k-2}}^{i_1,\ldots,i_{p-2k-2}} \\ 0 & \frac{(-2k+1)}{(p-2k-2)!} \delta_{i_{p-2k-2}} \delta_{i_{p-2k-2}} \end{array} \right);$$

- if $p$ is even, by relations (224) and (225), with $(\hat{c}_{\alpha_{2k+1}}^{\beta_{2k+1}})_{k=0,\frac{p}{2}+1}^{\frac{p}{2}}$ given by (293) and $\bar{D}_{\beta_{p-1}}^{\alpha_{p-1}}$ of the form

$$\bar{D}_{\beta_{p-1}}^{\alpha_{p-1}} = \left( \begin{array}{cc} \frac{1}{\Delta} & 0 \\ 0 & \frac{1}{\Delta} \end{array} \right).$$
The set of irreducible second-class constraints equivalent with (274) and (278) follows from formulae (221)–(223) for \( p \) odd or (219) and (220) for \( p \) even and is given by

\[
\tilde{\chi}^{(1)}_{i_1...i_{p-1}} \equiv -p \partial^i \pi_{i_1...i_{p-1}} + \frac{(-)^{p-1}}{p-1} \partial_{[i_1} P_{i_2...i_{p-1}]} \approx 0, \quad (295)
\]

\[
\tilde{\chi}^{(2)}_{j_1...j_{p-1}} \equiv -\partial_m A^{m{j_1...j_{p-1}}} + (-)^{p-1} \partial^{[j_1} B^{j_2...j_{p-1}]} \approx 0, \quad (296)
\]

together with:

- if \( p \) odd

\[
\tilde{\chi}^{(1)}_{i_1...i_{p-2k-1}} \equiv (-)^{p-1} (p - 2k) \partial^i P_{i_1...i_{p-2k-1}} + \frac{(-)^{p-1}}{p - 2k - 1} \partial_{[i_1} P_{i_2...i_{p-2k-1}]} \approx 0, \quad (297)
\]

\[
\tilde{\chi}^{(2)}_{j_1...j_{p-2k-1}} \equiv (-)^{p-1} \partial_m B^{m{j_1...j_{p-2k-1}}} + (-)^{p-1} \partial^{[j_1} B^{j_2...j_{p-2k-1}]} \approx 0, \quad (298)
\]

\[
\chi^{(1)} \equiv (-)^{p-1} \partial^i P_i, \quad (299)
\]

\[
\chi^{(2)} \equiv (-)^{p-1} \partial_m B^m, \quad (300)
\]

with \( k = 1, \lfloor \frac{p}{2} \rfloor - 1 \);

- if \( p \) even

\[
\tilde{\chi}^{(1)}_{i_1...i_{p-2k-1}} \equiv (-)^{p-1} (p - 2k) \partial^i P_{i_1...i_{p-2k-1}} + \frac{(-)^{p-1}}{p - 2k - 1} \partial_{[i_1} P_{i_2...i_{p-2k-1}]} \approx 0, \quad (301)
\]

\[
\chi^{(2)}_{j_1...j_{p-2k-1}} \equiv (-)^{p-1} \partial_m B^{m{j_1...j_{p-2k-1}}} + (-)^{p-1} \partial^{[j_1} B^{j_2...j_{p-2k-1}]} \approx 0, \quad (302)
\]

with \( k = 1, \frac{p}{2} - 1 \).

At this stage, we have constructed all the objects entering the structure of the irreducible Dirac bracket (252). It is essential to remark that the irreducible second-class constraints are local. If we construct the irreducible Dirac bracket and evaluate the fundamental Dirac brackets among the original variables, then we finally obtain that they are expressed by relations (286) and (287). This completes the irreducible analysis of gauge-fixed \( p \)-form gauge fields.

\section{Conclusion}

To conclude with, in this paper we have exposed an irreducible procedure for approaching systems with second-class constraints reducible of order \( L \). Our
strategy includes three main steps. First, we express the Dirac bracket for the reducible system in terms of an invertible matrix. Second, we establish the equality between this Dirac bracket and that corresponding to the intermediate theory, based on the constraints (214). Third, we prove that there exists an irreducible second-class constraint set equivalent with (214) such that the corresponding Dirac brackets coincide. These three steps enforce the fact that the fundamental Dirac brackets with respect to the original variables derived within the irreducible and original reducible settings coincide. Moreover, the newly added variables do not affect the Dirac bracket, so the canonical approach to the initial reducible system can be developed in terms of the Dirac bracket corresponding to the irreducible theory. The general procedure was exemplified on Abelian gauge-fixed $p$-form gauge fields. It is important to mention that our procedure does not spoil other important symmetries of the original system, such as spacetime locality of second-class field theories.

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