GAUSSIAN ZONOIDS AND THE DETERMINANT OF A NON–CENTERED GAUSSIAN MATRIX

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Abstract. We study the Vitale zonoids associated to a non–centered Gaussian vector. We compute two ellipsoids in between which each zonoid is trapped. This allows to estimate from above and below the expectation of the absolute value of the determinant of a non–centered Gaussian matrix with independent columns with the mixed volume of ellipsoid.

1. Introduction

Given a random vector $X \in \mathbb{R}^m$ such that $\mathbb{E}\|X\| < \infty$, Richard A. Vitale introduced in [Vit91] a way to build an associated convex body that we denote $\mathbb{E}X \subset \mathbb{R}^m$, that can be thought as the average of the random convex body $\frac{1}{2}[-X, X]$, see (2.2) below and also [BBLM21, Definition 2.3].

The convex bodies that can be obtained as $\mathbb{E}X$ for some random vector $X \in \mathbb{R}^m$ are called zonoids [Vit91, Theorem 3.1] and play an important role in convex geometry, see for example [Sch14, Section 3.5, 5.3 and 6.4]. Moreover this construction with random vectors appears in works such as [MT21, MN21, BBLM21, Sch22] and has a variant, introduced by Mosler, called the lift zonoid which plays a central role in application to statistics, see [Mos02]. Consistently with [BBLM21], we will call $\mathbb{E}X$ the Vitale zonoid associated to the random vector $X$. Note that it is sometime also called selection expectation, see [He15] and also [Sch22].

In this paper we study the Vitale zonoid associated to a Gaussian vector, we call those Gaussian zonoids. A similar construction can be found in [Mos02, Example 2.10]. In the case where the Gaussian vector $X \in \mathbb{R}^m$ is centered, that is when $\mathbb{E}X = 0$, the Gaussian zonoid $\mathbb{E}X$ is an ellipsoid, see Proposition 5 below. However, if $\mathbb{E}X$ is not zero, $\mathbb{E}X$ is not an ellipsoid. For simplicity we will only consider non degenerate Gaussian vectors in this paper. After reducing to the case where the variance is the identity, we compute its support function in Proposition 6.

The main result is Theorem 8 which states that there is an universal constant $b_\infty \sim 0.989 \ldots$ such that for all Gaussian zonoid $\mathbb{E}X$ there is an ellipsoid $\mathcal{E}$ such that

$$b_\infty \mathcal{E} \subset \mathbb{E}X \subset \mathcal{E}.$$ 

The definition of this constant (defined in Theorem 8) may seem arbitrary, but it can be interpreted as the radius of the biggest ball inside $\tilde{G}(\infty)$ which is the limit of a Gaussian zonoid (suitably renormalized) as the mean of the Gaussian vector goes to infinity. The surprising fact, at least for the author, is that it is so closed to one, this is shown in Figure 2.

When the center of the Gaussian vector $X$ is close to the origin we can build another ellipsoid $\mathcal{E}' \subset \mathbb{E}X$ that is closer to the zonoid $\mathbb{E}X$ see Proposition 11. These two results provide estimates on the volume of Gaussian zonoids, this is the content of Corollary 9, Corollary 12 and Proposition 13.

Finally, Vitale shows in [Vit91, Theorem 3.2] that if $M \in \mathbb{R}^{m \times m}$ is a random matrix with iid columns distributed as $X \in \mathbb{R}^m$, then $\mathbb{E}|\det(M)| = m! \text{vol}_m(\mathbb{E}X)$. In [BBLM21,
Section 5], Vitale’s theorem is generalized to a larger class of random determinants. In particular, [BBLM21, Theorem 5.2] implies that if $M$ has independent columns that are not identically distributed, $E|\det(M)|$ is equal to the mixed volume of the associated Vitale zonoids. We state here a slightly more general form that includes rectangular matrices and that is still a particular case of [BBLM21, Theorem 5.4], see Lemma 14 below.

We apply then our Theorem 8 to estimate the expectation of the absolute determinant of a matrix with independent non-centered Gaussian vector with the mixed volume of ellipsoid, see Theorem 15. In particular we show that if $M \in \mathbb{R}^{m \times m}$ is a random matrix whose columns are independent non-centered Gaussian vectors, then there are ellipsoids $\mathcal{E}_1, \ldots, \mathcal{E}_m$ such that

\[(b_\infty)^m V(\mathcal{E}_1, \ldots, \mathcal{E}_m) \leq E|\det(M)| \leq V(\mathcal{E}_1, \ldots, \mathcal{E}_m)\]

where $V$ denotes the mixed volume and $b_\infty$ is defined in Theorem 8.

By [KZ12] the mixed volume of ellipsoid is precisely equal to the expected absolute random determinant of a matrix with centered Gaussian independents columns. Thus the results Theorem 8 and Theorem 15 can be interpreted as follows: for every non-centered Gaussian vector $X \in \mathbb{R}^m$ there is a centered Gaussian vector $Y \in \mathbb{R}^m$ such that, for random determinants, $X$ is “bounded” by $Y$ from above and by $b_\infty Y$ from below.

**Structure of the paper.** In Section 2 we recall shortly the notions of convex geometry that will be useful to us then state and prove our main result Theorem 8 and its consequences in term of volume estimate. In Section 3 we apply our result to random determinants.

**Acknowledgments.** The author wishes to thank Stefano Baranzini for his helpful comments and Ilya Molchanov for pointing out relevant references that were unknown to the author. He would also like to acknowledge Bernd Sturmfels for being the first person to ask him: “what does the Vitale zonoid of a Gaussian vector look like?”.

## 2. The zonoids

We start by recalling a few fact from convex geometry, the reader can refer to [Sch14] for more details. A *convex body* in $\mathbb{R}^m$ is a non-empty convex compact subset of $\mathbb{R}^m$. If $K \subset \mathbb{R}^m$ is a convex body then its *support function*, denoted $h_K : \mathbb{R}^m \to \mathbb{R}$, is defined for all $u \in \mathbb{R}^m$ by

\[(2.1) \quad h_K(u) := \sup \{ \langle u, x \rangle \mid x \in K \} .\]

It turns out that the support function characterizes the convex body, meaning that $K_1 = K_2$ if and only if $h_{K_1} = h_{K_2}$. Moreover the supremum norm of the support function (restricted to the sphere) defines a distance on the space of convex bodies. This is the so called *Hausdorff distance*: $d(K, L) := \sup \{|h_K(u) - h_L(u)| \mid \|u\| = 1\}$. We will always consider the space of convex bodies endowed with the topology induced by this distance.

In addition to this, the support function has some useful properties.

**Proposition 1.** Let $K, L \subset \mathbb{R}^m$ be convex bodies. The following holds.

(i) $K \subset L$ if and only if $h_K \leq h_L$.

(ii) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map, then $h_{T(K)} = h_K \circ T^t$.

(iii) Let $(K_n)_{n \in \mathbb{N}}$ be a sequence of convex bodies and let $h : \mathbb{R}^n \to \mathbb{R}$ be such that $h_{K_n}$ converges pointwise to $h$. Then $h$ is the support function of a convex body $K$ and $K_n$ converges to $K$ in the Hausdorff distance topology.

**Proof.** Item (i) follows from the fact that $x \in K$ if and only if $\langle u, x \rangle \leq h_K(u)$ for all $u \in \mathbb{R}^m$. Item (ii) is a direct consequence of the definition of the support function in (2.1). Finally, item (iii) is [Sch14, Theorem 1.8.15]. \qed
We say that a random vector \( X \in \mathbb{R}^m \) is integrable if \( \mathbb{E}\|X\| < \infty \). Given an integrable random vector \( X \in \mathbb{R}^m \), we denote by \( \mathbb{E}X \) the convex body with support function

\[
(2.2) \quad h_{\mathbb{E}X}(u) := \frac{1}{2} \mathbb{E}|\langle u, X \rangle|.
\]

This function is the support function of a convex body because it is sublinear, see [Sch14, Theorem 1.7.1]. The convex bodies that can be obtained this way (and their translate) are called zonoids, see [Vit91, Theorem 3.1] and the zonoid \( \mathbb{E}X \) is called the Vitale zonoid associated to the random vector \( X \) (see [BBLM21] where \( \mathbb{E}X \) is denoted \( K(X) \)). Note that \( h_{[-X,X]}(u) = |\langle u, X \rangle| \) and thus \( \mathbb{E}X \) can be thought of as the expectation of the random convex body \( \frac{1}{2}[-X,X] \). The following observation is [BBLM21, Proposition 2.4].

**Lemma 2.** Let \( X \in \mathbb{R}^m \) be an integrable random vector and let \( T : \mathbb{R}^m \to \mathbb{R}^n \) be a linear map. Then \( T(X) \in \mathbb{R}^n \) is integrable and we have \( \mathbb{E}T(X) = T(\mathbb{E}X) \).

A particular case of integrable random vector are the Gaussian vectors. Recall that a random vector \( X \in \mathbb{R}^m \) is called Gaussian if for every \( u \in \mathbb{R}^m \), the random variable \( \langle u, X \rangle \) follows a normal distribution. In that case the distribution of \( X \) is determined by its mean \( \mathbb{E}X \) and its variance \( \mathbb{E}[(X - \mathbb{E}X)(X - \mathbb{E}X)^\top] \in \mathbb{R}^{m \times m} \). In this paper we study the Vitale zonoids associated to a Gaussian vector. In the following all Gaussian vectors will be assumed to be non-degenerate, that is to have support on the whole space \( \mathbb{R}^m \).

**Definition 3.** A convex body \( K \subset \mathbb{R}^m \) is called a Gaussian zonoid if there is a Gaussian vector \( X \in \mathbb{R}^m \) such that \( K = \mathbb{E}X \).

A particular case is the standard Gaussian vector \( \xi \in \mathbb{R}^m \) which admits a density given for all \( x \in \mathbb{R}^m \) by \( \rho(x) = (2\pi)^{-\frac{m}{2}} \exp\left(-\frac{\|x\|^2}{2}\right) \). One can prove, using for example the general expression of the density of a Gaussian vector, that for every (non–degenerate) Gaussian vector \( X \in \mathbb{R}^m \) there is a linear map \( M : \mathbb{R}^m \to \mathbb{R}^m \) and a vector \( c \in \mathbb{R}^m \) such that \( X \) is distributed as \( M(c + \xi) \). We use this fact and Lemma 2 to reduce our study to the case where the Gaussian vector is of the form \( c + \xi \), i.e. has variance the identity, hence the following definition.

**Definition 4.** For every \( c \in \mathbb{R}^m \) we define

\[
G(c) := \mathbb{E}c + \xi
\]

where \( \xi \in \mathbb{R}^m \) is a standard Gaussian vector.

Hence, by Lemma 2, a convex body \( K \subset \mathbb{R}^m \) is a Gaussian zonoid if and only if there exists a linear map \( M : \mathbb{R}^m \to \mathbb{R}^m \) and a vector \( c \in \mathbb{R}^m \) such that \( K = M(G(c)) \).

The first example is when \( c = 0 \).

**Proposition 5.** If \( \xi \in \mathbb{R}^m \) is a standard Gaussian vector then we have

\[
\mathbb{E}\xi = G(0) = \frac{1}{\sqrt{2\pi}} B_m
\]

where \( B_m \) is the unit ball of \( \mathbb{R}^m \).

**Proof.** Since the random vector \( \xi \) is invariant under the action of \( O(m) \), by Lemma 2, \( \mathbb{E}\xi \) must be a ball. It is then enough to compute \( h_{\mathbb{E}\xi}(e_1) = \frac{1}{2} \mathbb{E}|\langle e_1, \xi \rangle| \). The random variable \( \langle e_1, \xi \rangle \) is a standard Gaussian variable and thus \( \mathbb{E}|\langle e_1, \xi \rangle| = \sqrt{\frac{2}{\pi}} \) which concludes the proof. \( \square \)
Proposition 6. Let \( c \in \mathbb{R}^m \setminus \{0\} \) and let us write every \( u \in \mathbb{R}^m \) as \( u = (x, y) \in \mathbb{R} \times \mathbb{R}^{m-1} \) with \( x = (u, c/\|c\|) \) and \( y \in c^\perp \). Then the support function of \( G(c) \) is given by

\[
h_{G(c)}(x, y) = \frac{\sqrt{x^2(1 + ||c||^2) + ||y||^2}}{2\pi} e^{\frac{-x^2}{2(1 + ||c||^2 + ||y||^2)}} + \frac{x||c||}{2} \text{erf} \left( \frac{x||c||}{\sqrt{2}\sqrt{x^2(1 + ||c||^2) + ||y||^2}} \right)
\]

where \( \text{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} ds \) is the so called error function.

Proof. The random variable \((x, y, c + \xi)\) is a Gaussian variable with mean \( x\|c\| \) and variance \( \sqrt{x^2(1 + ||c||^2) + ||y||^2} \). Computing the mean of a folded Gaussian gives the result, see [LNN61, (3)] or [Wik22]. 

Proposition 7. The map \( G : c \mapsto G(c) \) is continuous. Moreover for all \( c \neq 0 \) the map \( t \mapsto G(tc) \) is strictly increasing with respect to inclusion on \( t > 0 \).

Proof. Continuity follows from the fact that the function \( h_{G(c)}(\cdot) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \) given by \( (c, u) \mapsto h_{G(c)}(u) \) is continuous and Proposition 1–(iii).

For the second part, we can assume without loss of generality that \( ||c|| = 1 \). By Proposition 1–(i), it is enough to show that given a fixed non zero point \((x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}\), the function \( t \mapsto h_{G(tc)}(x, y) \) is strictly increasing. We get from Proposition 6:

\[
\frac{d}{dt} h_{G(tc)}(x, y) = \frac{tx^2 e^{2x^2(1+t^2)+||y||^2}}{\sqrt{2\pi} \sqrt{x^2(1+t^2)+||y||^2}} + \frac{x}{2} \text{erf} \left( \frac{xt}{\sqrt{2}\sqrt{x^2(1+t^2)+||y||^2}} \right)
\]

which is positive on \( t > 0 \) and this concludes the proof. 

For \( c \neq 0 \) the Gaussian zonoid \( G(c) \) is not an ellipsoid. However we shall show that it remains close to one in a certain sense that we describe below, see also Figure 2. In order to state the main result, let us first introduce a few definitions.

First we define \( \lambda : \mathbb{R} \to \mathbb{R} \) to be given for all \( s \in \mathbb{R} \) by

\[
\lambda(s) := \sqrt{1 + s^2} e^{\frac{-s^2}{2(1+s^2)}} + s \sqrt{\frac{\pi}{2}} \text{erf} \left( \frac{s}{\sqrt{2\sqrt{1 + s^2}}} \right).
\]

Note that, by Proposition 6 and following the same notation, for all \( c \in \mathbb{R}^m \), we have \( h_{G(c)}(1, 0) = \frac{\lambda(||c||)}{\sqrt{2\pi}} \). Then we define the constant

\[
a := e^{-\frac{1}{2}} + \sqrt{\frac{\pi}{2}} \text{erf} \left( \frac{1}{\sqrt{2}} \right) \sim 1.462 \ldots
\]
and the function $\varphi_{\infty} : \mathbb{R}^2 \to \mathbb{R}$ given for all $(x, z) \in \mathbb{R}^2$ by
\begin{equation}
\varphi_{\infty}(x, z) := \frac{x^2 + a^2 z^2}{a^2} e^{-\frac{x^2}{2a^2}} + \frac{x}{\sqrt{2}} \text{erf} \left( \frac{x}{\sqrt{2} a} \right).
\end{equation}

**Theorem 8.** Let $c \in \mathbb{R}^m$ and consider the linear map $T_c : \mathbb{R}^m \to \mathbb{R}^m$ that is the identity if $c = 0$ and that sends $c \mapsto \lambda(\|c\|)c$ and is the identity on $c^1$ if $c \neq 0$. Then we have

$$b_{\infty} T_c \left( \frac{1}{\sqrt{2\pi}} B_m \right) \subset \mathcal{G}(c) \subset T_c \left( \frac{1}{\sqrt{2\pi}} B_m \right)$$

where $B_m \subset \mathbb{R}^m$ is the unit ball and $b_{\infty} := \min \{ \varphi_{\infty}(\cos(t), \sin(t)) \mid t \in [0, 2\pi] \} \sim 0.989\ldots$

**Proof.** If $c = 0$, $\mathcal{G}(0)$ is equal to the upper bound and there is nothing to prove. Thus we can assume without loss of generality that $c = se_1$ where $e_1$ is the first standard basis vector of $\mathbb{R}^m$ and $s > 0$. Let $\mathcal{G}(s) := \sqrt{2\pi} T_{se_1}^{-1} \mathcal{G}(se_1)$. The idea of the proof is to show that the map $s \mapsto \mathcal{G}(s)$ is strictly decreasing with respect to inclusion for $s > 0$. Once this is established, it is enough to show that the limit object $\mathcal{G}(\infty)$ exists and contains a ball of radius $b_{\infty}$.

Let us first show that $s \mapsto \mathcal{G}(s)$ is decreasing. Let $(x, z) \in \mathbb{R}^2$ and consider the function

$$\varphi_{x,z}(s) := \frac{sx}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \frac{\mu(s)}{\sqrt{2}} \text{erf} \left( \frac{\tau(s)}{\sqrt{2}} \right)$$

where

$$\mu(s) := \frac{sx}{\lambda(s)}; \quad \sigma(s) := \sqrt{\frac{s^2 + 1}{\lambda^2(s)}} x^2 + 2\pi z^2; \quad \tau(s) := \frac{\mu(s)}{\sigma(s)}.$$

Then $h_{\mathcal{G}(s)}(u) = \varphi_{x,y}||\varphi_x(s)$ where, as in Proposition 6, $x = \langle u, e_1 \rangle$ and $y$ is the orthogonal projection of $u$ onto $e_1$. By Proposition 1–(i), it is enough to show that for all $x \in \mathbb{R}$ and $z \geq 0$, the function $s \mapsto \varphi_{x,z}(s)$ is decreasing on $s > 0$. One can check that $\varphi_{0,z} = |z|$ and $\varphi_{x,0} = |x|$ which are constants in $s$. Moreover since $\varphi_{x,z} = \varphi_{x,z}$, we can assume $x, z > 0$. Thus in the following we fix $x, z > 0$ and omit them in the notation, writing $\varphi := \varphi_{x,z}$.

Consider now the change of variable $\tilde{s} := \frac{s}{\sqrt{s^2 + 1}}$. One can write $\varphi(s)$ as a function of the variable $\tilde{s}$. Since $\tilde{s}$ is strictly increasing on $s > 0$ it is enough to show that $\varphi$ is decreasing in $\tilde{s}$ on $0 < \tilde{s} < 1$. Writing $\varphi'$ for the derivative of $\varphi$ with respect to $\tilde{s}$ at $\tilde{s}$, we obtain for $0 < \tilde{s} < 1$ (we omit the dependence on $s$ in the notation):

\begin{equation}
\frac{(1 - \tilde{s}^2)\tilde{s}^2}{x \text{erf} \left( \frac{\tilde{s}}{\sqrt{2}} \right) \text{erf} \left( \frac{\tau}{\sqrt{2}} \right)} \varphi' = \rho \left( \frac{\tilde{s}}{\sqrt{2}} \right) - \rho \left( \frac{\tau}{\sqrt{2}} \right)
\end{equation}

where

$$\rho(t) := \frac{\text{erf}'(t)}{\text{erf}(t)} = \frac{2t e^{-t^2}}{\sqrt{\pi} \text{erf}(t)}.$$

One can show that $\rho(t)$ is strictly decreasing for $t > 0$, see [Alz10, Lemma 2.1]. Moreover, since $x, z > 0$, we have

$$\tau = \frac{x}{\sqrt{x^2 + 2\pi(1 - s^2)}} \tilde{s} < \tilde{s}.$$

And thus $\rho \left( \frac{x}{\sqrt{2}} \right) < \rho \left( \frac{\tau}{\sqrt{2}} \right)$. The coefficient in front of $\varphi'$ in (2.6) is positive on $0 < \tilde{s} < 1$ and thus this shows that $\varphi' < 0$ on $0 < \tilde{s} < 1$. In definitive we have shown that for all $s > 0$ the map $s \mapsto \mathcal{G}(s)$ is (strictly) decreasing with respect to inclusion.

We now note that for all fixed $(x, z) \in \mathbb{R}^2$ and as $s \to \infty$, $\varphi_{x,z}(s)$ tends to $\varphi_{\infty}(x, z)$ defined in (2.5). Writing as before $u = (x, y) \in \mathbb{R} \times \mathbb{R}^{m-1}$, by Proposition 1–(iii), the function
Figure 2. The boundary of $\tilde{G}(\infty)$ and $\tilde{G}(0) = B_2$ in the positive orthant

$\varphi_\infty(x, \|y\|)$ is the support function of a zonoid that we denote by $\tilde{G}(\infty)$. By what we just proved, for all $s > 0$ we have

$$\tilde{G}(\infty) \subset \tilde{G}(s) \subset \tilde{G}(0).$$

Since $T_0$ is the identity and by Proposition 5, we have that $\tilde{G}(0) = B_m$. Moreover, $\tilde{G}(\infty)$ contains a ball of radius $b_\infty$ since it is the minimum of its support function on the sphere. Mapping everything through $\frac{1}{\sqrt{2\pi}} T_{se_1}$ (which preserves inclusion) gives the result. \qed

From Theorem 8 and the fact that $\det(T_c) = \lambda(\|c\|)$, we get as an immediate corollary an estimate on the volume of the Gaussian zonoids $G(c)$.

**Corollary 9.** For every $c \in \mathbb{R}^m$ we have

$$(b_\infty)^m \frac{\lambda(\|c\|)}{(2\pi)^{\frac{m}{2}}} \kappa_m \leq \text{vol}_m(G(c)) \leq \frac{\lambda(\|c\|)}{(2\pi)^{\frac{m}{2}}} \kappa_m$$

where $\kappa_m := \text{vol}_m(B_m)$ the volume of the unit ball $B_m \subset \mathbb{R}^m$, $\lambda$ is defined by (2.3) and $b_\infty$ is defined in Theorem 8.

The function $\lambda$ is explicit and is expressed in terms of special functions (see its definition in (2.3)). However one can get a simpler expression in the following asymptotic: when $s << 1$ and $s >> 1$. In these cases we have the following expansions:

$$\lambda(s) = 1 + s^2 + O\left(s^4\right); \quad \lambda(s) = as + O\left(\frac{1}{s}\right)$$

where recall the constant $a$ defined in (2.4). From this, we see that when $c$ is close to 0, the volume of $G(c)$ tends to the upper bound in Corollary 9 and the lower bound is not sharp. In fact in that case we have a better estimate from below that comes from the following inequality.

**Lemma 10.** For any $t \geq 0$ we have

$$t \text{erf}(t) \geq \frac{1}{\sqrt{\pi}} \left(1 - e^{-t^2}\right).$$

**Proof.** It is enough to see that $t \text{erf}(t) \geq \frac{1}{\sqrt{\pi}} \int_0^t 2se^{-s^2}ds$. \qed
Proposition 11. For all \( c \in \mathbb{R}^m \) define \( L_c : \mathbb{R}^m \to \mathbb{R}^m \) to be the identity if \( c = 0 \) and to be the map that sends \( c \mapsto \sqrt{1 + ||c||^2}c \) and is the identity on \( c^\perp \) if \( c \neq 0 \). Then for any \( c \in \mathbb{R}^m \), we have
\[
L_c \left( \frac{1}{\sqrt{2\pi}} B_m \right) \subset G(c)
\]
where we recall that \( B_m \) is the unit ball of \( \mathbb{R}^m \).

Proof. Applying Lemma 10 to the support function of \( G(c) \) computed in Proposition 6 we find
\[
h_{G(c)}(x, y) \geq \frac{1}{\sqrt{2\pi}} \sqrt{x^2(1 + ||c||^2) + ||y||^2}.
\]
The right hand side is equal to \( \frac{1}{\sqrt{2\pi}} ||L_c(u)|| \) when, as before, \( x = \langle u, c/||c|| \rangle \) and \( y \) is the orthogonal projection of \( u \) onto \( c^\perp \). Since \( L_c \) is equal to its transpose, by Proposition 1–(ii), this is the support function of \( L_c \left( \frac{1}{\sqrt{2\pi}} B_m \right) \) and the result follows by Proposition 1–(i). \( \square \)

Noting that \( \det(L_c) = \sqrt{1 + ||c||^2} \), we get the following.

Corollary 12. For every \( c \in \mathbb{R}^m \) and recalling the notation \( \kappa_m = \text{vol}_m(B_m) \) we have:
\[
\frac{\kappa_m}{(2\pi)^{m/2}} \sqrt{1 + ||c||^2} \leq \text{vol}_m(G(c)).
\]

Combining this result with Corollary 9 and the expansion (2.7) we find the following.

Proposition 13. When \( ||c|| << 1 \) we have
\[
1 + \frac{1}{2} ||c||^2 + O \left( ||c||^4 \right) \leq \frac{(2\pi)^m}{\kappa_m} \text{vol}_m(G(c)) \leq 1 + ||c||^2 + O \left( ||c||^4 \right).
\]

3. Application to random determinants

Vitale in [Vit91] shows that if \( M \in \mathbb{R}^{m \times m} \) is a random matrix with iid columns distributed as \( X \in \mathbb{R}^m \) integrable, then \( \mathbb{E} |\det(M)| = m! \text{vol}_m(\mathbb{E}X) \) (recall the definition of \( \mathbb{E}X \) in (2.2)). In [BBLM21, Section 5] this is generalized to a larger class of random determinants where the columns are not necessarily identically distributed. We state below the case of a rectangular matrix with independent columns which is a particular case of [BBLM21, Theorem 5.4] reformulated in a language more suitable for our context.

Let us first recall the notion of mixed volume. Given two convex bodies \( K, L \subset \mathbb{R}^m \) one can define their Minkowski sum: \( K + L := \{x + y \mid x \in K, y \in L\} \). A fundamental result by Minkowski (see [Sch14, Theorem 5.1.7]) says that given convex bodies \( K_1, \ldots, K_m \subset \mathbb{R}^m \) the function \( (t_1, \ldots, t_m) \mapsto \text{vol}_m(t_1 K_1 + \cdots + t_m K_m) \) is a polynomial in \( t_1, \ldots, t_m \in \mathbb{R} \). The coefficient of \( t_1 \cdots t_m \) is called the mixed volume of \( K_1, \ldots, K_m \) and is denoted \( \text{V}(K_1, \ldots, K_m) \). In some sense this is a polarization of the function \( \text{vol}_m \) on convex bodies of \( \mathbb{R}^m \). In particular, for any convex body \( K \subset \mathbb{R}^m \), we have \( \text{V}(K, \ldots, K) = \text{vol}_m(K) \).

Lemma 14. Let \( 0 < k \leq m \) and let \( X_1, \ldots, X_k \in \mathbb{R}^m \) be independent integrable random vectors. Consider the random matrix \( \Gamma := (X_1, \ldots, X_k) \in \mathbb{R}^{m \times k} \) whose columns are the vectors \( X_i \) and let \( K_i := \mathbb{E}X_i \) be the zonoid defined in (2.2). Then we have
\[
\mathbb{E} \sqrt{\det(\Gamma^T \Gamma)} = \frac{m!}{(m-k)! \kappa_{m-k}} \text{V}(K_1, \ldots, K_k, B_m[m-k])
\]
where \( B_m[m-k] \) denotes the unit ball \( B_m \subset \mathbb{R}^m \) repeated \( m-k \) times in the argument and \( \kappa_{m-k} = \text{vol}_{m-k}(B_{m-k}) \).
Proof. First note that \( \sqrt{\det(\Gamma\Gamma')} \) is equal to the \( k \)-th dimensional volume of the cube spanned by the vectors \( X_1, \ldots, X_k \). Writing \( X \) to denote the segment \( \frac{1}{2}[-X, X] \), we have:

\[
\sqrt{\det(\Gamma\Gamma')} = \binom{m}{k} V(X_1 + \cdots + X_k, B_m[m-k]) = \frac{m!}{(m-k)!k_{m-k}} V(X_1, \ldots, X_k, B_m[m-k])
\]

where the first equality follows from [Sch14, (5.31)] and the second is a consequence of the symmetries of the mixed volume. Finally by [BBLM21, Theorem 5.4] we obtain, taking the expectation, \( \mathbb{E} V(X_1, \ldots, X_k, B_m[m-k]) = V(K_1, \ldots, K_k, B_m[m-k]) \) which concludes the proof.

\[\square\]

In the case where \( X_i \) is a (non–degenerate) Gaussian vector, the zonoid \( K_i \) is, by definition, a Gaussian zonoid and thus is a linear image of some \( G(c_i) \) (see the previous section). Since the mixed volume is increasing with respect to inclusion (see [Sch14, (5.25)]), Lemma 14 and Theorem 8 imply the following.

**Theorem 15.** Let \( 0 < k \leq m \) and let \( X_1, \ldots, X_k \in \mathbb{R}^m \) be independent Gaussian vectors such that \( X_i = M_i(c_i + \xi_i) \) with \( M_i : \mathbb{R}^m \to \mathbb{R}^m \) a linear map, \( c_i \in \mathbb{R}^m \) fixed vectors and \( \xi_i \) iid standard Gaussian vectors of \( \mathbb{R}^m \). Consider the random matrix \( \Gamma := (X_1, \ldots, X_k) \) whose columns are the vectors \( X_i \) and define the ellipsoids \( \mathcal{E}_i := M_i \circ T_{c_i}(B_m) \) for \( i = 1, \ldots, k \) where \( B_m \subset \mathbb{R}^m \) is the unit ball and recall the definition of \( T_{c_i} \) in Theorem 8. We have

\[
(b_\infty)^k \alpha_{m,k} V(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k]) \leq \mathbb{E} \sqrt{\det(\Gamma\Gamma')} \leq \alpha_{m,k} V(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k])
\]

where \( B_m[m-k] \) denotes the unit ball \( B_m \subset \mathbb{R}^m \) repeated \( m-k \) times in the argument of the mixed volume \( V \), \( \alpha_{m,k} := \frac{m!}{(2\pi)^{k/2}(m-k)!k_{m-k}} \) and \( b_\infty \) is defined in Theorem 8.

This result is to be compared with [KZ12, Theorem 1.1] where it is proved that the centered case (i.e. when all the \( c_i \) are equal to 0) is equal to the mixed volume of ellipsoids, that is to the upper bound. More precisely, if \( Y_i \) is a centered Gaussian vector with variance \( \Sigma_i := M_i T_{c_i} T_i^t M_i^t \) and \( Y_1, \ldots, Y_k \) are independent and if \( \bar{\Gamma} := (Y_1, \ldots, Y_k) \), then by [KZ12, Theorem 1.1] we have \( \mathbb{E} \sqrt{\det(\bar{\Gamma}\bar{\Gamma}')} = \alpha_{m,k} V(\mathcal{E}_1, \ldots, \mathcal{E}_k, B_m[m-k]) \). In some sense one can interpret Theorem 8 by saying that, for random determinants, the non–centered Gaussian vector \( X_i \) is "trapped" between the centered Gaussian vectors \( b_\infty Y_i \) and \( Y_i \). Note that in that case we also have \( \mathbb{E} Y_i = \frac{1}{\sqrt{2\pi}} \mathcal{E}_i \) and thus Lemma 14 gives an alternative proof of [KZ12, Theorem 1.1].

**Remark 16.** Note that Lemma 14 is still valid if the random vectors are degenerate (i.e almost surely contained in an hyperplane). Hence, Theorem 15 is still valid if the map \( M_i \) has a non trivial kernel. However one cannot obtain all degenerate Gaussian vectors this way since such vectors must have the mean in the image of the variance.

As before, when some \( c_i \) is close to zero we can have a better estimate from below.

**Proposition 17.** Let \( 0 < l \leq k \leq m \) and let \( X_1, \ldots, X_k \in \mathbb{R}^m \) be independent Gaussian vectors such that \( X_l = M_l(c_l + \xi_l) \) with \( M_l : \mathbb{R}^m \to \mathbb{R}^m \) a linear map, \( c_l \in \mathbb{R}^m \) fixed vectors and \( \xi_l \) iid standard Gaussian vectors of \( \mathbb{R}^m \). Consider the random matrix \( \Gamma := (X_1, \ldots, X_k) \) whose columns are the vectors \( X_i \) and define the ellipsoids \( \mathcal{E}_i := M_i \circ L_{c_i}(B_m) \) for \( i = l+1, \ldots, k \). We have

\[
(b_\infty)^{k-l} \alpha_{m,k} V(\mathcal{E}_1', \ldots, \mathcal{E}_l', \mathcal{E}_{l+1}, \ldots, \mathcal{E}_k, B_m[m-k]) \leq \mathbb{E} \sqrt{\det(\Gamma\Gamma')}
\]

where \( \alpha_{m,k} := \frac{m!}{(2\pi)^{k/2}(m-k)!k_{m-k}} \) and \( b_\infty \) is defined in Theorem 8.
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