Adaptive post-Dantzig estimation and prediction for non-sparse “large \( p \) and small \( n \)” models

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Abstract

For consistency (even oracle properties) of estimation and model prediction, almost all existing methods of variable/feature selection critically depend on sparsity of models. However, for “large \( p \) and small \( n \)” models sparsity assumption is hard to check and particularly, when this assumption is violated, the consistency of all existing estimations is usually impossible because working models selected by existing methods such as the LASSO and the Dantzig selector are usually biased. To attack this problem, we in this paper propose adaptive post-Dantzig estimation and model prediction. Here the adaptability means that the consistency based on the newly proposed method is adaptive to non-sparsity of model, choice of shrinkage tuning parameter and dimension of predictor vector. The idea is that after a sub-model as a working model is determined by the Dantzig selector, we construct a globally unbiased sub-model by choosing suitable instrumental variables and nonparametric adjustment. The new estimation of the parameters in the sub-model can be of asymptotic normality. The consistent estimator, together with the selected sub-model and adjusted model, improves model predictions. Simulation studies show that the new approach has the significant improvement of estimation and prediction accuracies over the Gaussian Dantzig selector and other classical methods have.

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Keywords. Adaptability, bias correction, Dantzig selector, instrumental variable, nonparametric adjustment, Ultra high-dimensional regression.

AMS 2001 subject classification: 62C05, 62F10, 62F12, 62G05.

Running head. Adaptive post-Dantzig inference.
1. Introduction

Estimation consistency is a natural criterion for estimation accuracy. In classical settings with small/moderate number of variables in models, this criterion can be adopted. For high-dimensional models, particularly, when the number $p$ of variables involved is even larger than the sample size $n$, are called “large $p$, small $n$” models. However in these paradigm estimation consistency becomes a very challenging issue. This is because what we can work on is only working models rather than full models after active variables are selected into working models. For variable selection, some classical and newly proposed methods are available, such as the LASSO (including the adaptive LASSO) and the SCAD. These methods provide consistent and asymptotically normally distributed estimation for the parameters in working models, but these properties heavily depend on sparse structure, proper choice of shrinkage tuning parameter and the diverging rate of the dimension of parameter vector. For the relevant references see Huber (1973), Portnoy (1988), Bai and Saranadasa (1996), Fan and Peng (2004), Fan, Peng and Huang (2005), Lam and Fan (2008), Huang et al. (2008), and Li, Zhu and Lin (2009), among others. As such, for models without spare structure, how to construct consistent estimation is a great challenge. It is required to develop new or extended statistical methodologies and theories to handle this challenge; see for example Donoho (2000), Kettenring, Lindsay and Siegmund (2003).

To this end, we further review existing methods to get motivation for new methodology development. The following methods were developed also under sparse structure. The Dantzig selector that was proposed by Candés and Tao (2007) and was extended to handle the generalized linear models by James and Radchenko (2009) has received much attention. The connection between the Dantzig selector and the LASSO was investigated by James et al. (2009). Under the uniform uncertainty
principle, the resulting estimator achieves an ideal risk of order $O(\sigma \sqrt{\log p})$ with a large probability. This implies that for large $p$, such a risk can be however large and then even under sparse structure the estimator may also be inconsistent. To reduce the risk and improve the performance of relevant estimation, the Gaussian Dantzig Selector, a two-stage estimation, was suggested in the literature (Candés and Tao 2007). Such an improved estimation is still inconsistent when the shrinkage tuning parameter is chosen to be large (for details see the next section). Another method is the Double Dantzig Selector (James and Radchenko 2009), by which one may choose a more accurate model and, at the same time, get a more accurate estimator. But it critically depends on the choice of shrinkage tuning parameter. Motivated by these problems, Fan and Lv (2008) introduced a sure independent screening method that is based on correlation learning to reduce high dimensionality to a moderate scale below the sample size. Afterwards, variable selection and parameter estimation can be accomplished by sophisticated methods, such as the LASSO, the SCAD or the Dantzig selector. The relevant references include Kosorok and Ma (2007), Van Der Lanin and Bryan (2001), Chen and Qin (2010), James, Radchenko and Lv (2009) and Kuelbs and Anand (2010), among others.

However, for any model with very large $p$, without model sparsity, all existing methods cannot provide estimation consistency for working models, and any further data analysis would be questionable unless we can correct biases later or at most we can obtain an approximation rather than estimation consistency as the sample size goes to infinity. To deal with this problem, we focus our attention on working sub-model that is chosen by the Dantzig selector. In this paper, we suggest a method to construct consistent and asymptotically normal distributed estimation for the parameters in the sub-model. To achieve this, a nonparametric adjustment is recommended to construct a globally unbiased sub-model and to correct the bias in working model. Here the nonparametric adjustment may depend on
a low-dimensional nonparametric estimation via using proper instrument variables. We will show the following properties. The estimator \( \hat{\theta} \) of the parameter vector \( \theta \) in the sub-model satisfies \( \| \hat{\theta} - \theta \|_{\ell_2}^2 = O_p(n^{-1}) \) and the asymptotic normality if the dimension \( q \) of \( \theta \) is fixed. Even for the case where \( q \) tends to infinity, the consistent and asymptotic normality still hold when \( q \) diverges at a certain rate. We will briefly discuss the theoretical results for the case with diverging \( q \). Furthermore, the new consistent estimator, together with the unbiased adjustment sub-model or the original sub-model, can also improve model prediction accuracy. We will prove that our method possesses the adaptability. That is, the above properties always hold whether the sub-model is small or large, the dimension of the parameter in the original model is high or not, and the original model is sparse or not.

The rest of the paper is organized as follows. In Section 2 the properties of the Dantzig estimator for the high-dimensional linear model are re-examined. In Section 3 a bias-corrected sub-model is proposed via introducing instrumental variables and a nonparametric adjustment, and a method about instrumental variable selection is introduced. Estimation and prediction procedures for the new sub-model are suggested and the asymptotic properties of the resulting estimator and prediction are obtained. In Section 4 the algorithms for constructing instrumental variables are proposed. Simulation studies are presented in Section 5 to examine the performance of the new approach when compared with the classical Dantzig selector and other methods. The technical proofs for the theoretical results are postponed to the Appendix.
2. A brief review for the Dantzig selector

Consider the model

\[ Y = \beta' X + \varepsilon, \quad (2.1) \]

where \( Y \) is the scale response, \( X \) is the \( p \)-dimensional covariate and \( \varepsilon \) is the random error satisfying \( E(\varepsilon|X) = 0 \) and \( \text{Cov}(\varepsilon|X) = \sigma^2 \). Here \( p \) will be greater than \( n \) when we can collect a sample of size \( n \). Throughout this paper, our primary interest is to construct consistent estimators for significant components of the parameter vector \( \beta = (\beta_1, \cdots, \beta_p)' \in B \subset \mathbb{R}^p \). These significant components of \( \beta \), together with the corresponding covariates, composes a working model. Then the second interest of our paper is to obtain reasonable model prediction via our estimation.

To introduce the new estimation, we first re-examine the Dantzig selector. Let \( Y = (Y_1, \cdots, Y_n)' \) be the vector of the observed responses and \( X = (X_1, \cdots, X_n)' = (x_1, \cdots, x_p) \) be the \( n \times p \) matrix of the observed covariates. The Dantzig selector of \( \beta \) is defined as

\[ \tilde{\beta}^D = \arg\min_{\beta \in \mathcal{B}} \|\beta\|_{\ell_1} \quad \text{subject to} \quad \sup_{1 \leq j \leq p} |x_j' r| \leq \lambda_p \sigma \quad (2.2) \]

for some \( \lambda_p > 0 \), where \( \|\beta\|_{\ell_1} = \sum_{j=1}^p |\beta_j| \) and \( r = Y - X \beta \). As was shown by Candés and Tao (2007), under some regularity conditions, this estimator satisfies that, with large probability,

\[ \|\tilde{\beta}^D - \beta\|_{\ell_2}^2 \leq C \sigma^2 \log p, \quad (2.3) \]

where \( C \) is free of \( p \) and \( \|\tilde{\beta}^D - \beta\|_{\ell_2}^2 = \sum_{j=1}^p |\tilde{\beta}_j^D - \beta_j|^2 \). In fact this is an ideal risk and thus cannot be improved in a certain sense. However, such a risk can become large and may not be negligible when the dimension \( p > n \).

To reduce the risk and enhance the performance in practical settings, one often uses a two-stage selection procedure (e. g., the Gaussian Dantzig Selector) to con-
struct a risk-reduced estimator for the obtained sub-model (Candés and Tao 2007). For example, we can first estimate $I = \{ j : \beta_j \neq 0 \}$ with $\tilde{I} = \{ j : |\tilde{\beta}_j^D| > \varsigma \sigma \}$ for some $\varsigma \geq 0$ and then construct an estimator

$$\tilde{\beta}_{(\tilde{I})} = ((X(\tilde{I})'X(\tilde{I}))^{-1}(X(\tilde{I})')'Y$$

for $\beta_{(\tilde{I})}$ and set the other components of $\beta$ to be zero, where $\beta_{(\tilde{I})}$ is the restriction of $\beta$ to the set $\tilde{I}$, and $X(\tilde{I})$ is the matrix with the column vectors according to $\tilde{I}$.

Denote $\beta_{(\tilde{I})} = \theta$, a $q$-dimensional vector of interest. Without loss of generality, suppose that $\beta$ can be partitioned as $\beta = (\theta', \gamma)'$ and, correspondingly, $X$ is partitioned as $X = (Z', U')'$. Then the above two-stage procedure implies that we can use the sub-model

$$Y = \theta'Z + \eta \quad (2.4)$$

to replace the full-model (2.1), where $\eta = \gamma'U + \varepsilon$ is regarded as error. Here the dimension $q$ of $\theta$ can be either fixed or diverging with $n$ at certain rate. Since the above sub-model is a replacer of the full model (2.1), we call $\theta$ and $Z$ the main parts of $\beta$ and $X$, respectively. From (2.1) and (2.4) it follows that $E(\eta|Z) = \gamma'E(U|Z)$. When both $\gamma \neq 0$ and $E(U|Z) \neq 0$, the sub-model (2.4) is biased and thus the two-stage estimator $\hat{\theta}_S = \tilde{\beta}_{(\tilde{I})}$ is also biased. It shows that the two-stage estimator $\hat{\theta}_S$ of $\theta$ is also inconsistent. Note that for any non-sparse model, the condition $\gamma \neq 0$ always holds. Then the above method is not possible to obtain consistent estimation.

Another method for improving the Dantzig selector is the Double Dantzig Selector. By which more accurate model and estimation can be expected. In the first step, the Dantzig selector is used with a relatively large shrinkage tuning parameter $\lambda_p$ defined above to get a relatively accurate sub-model in the sense that more significant variables are contained. The Dantzig selector is further used in the selected sub-model to obtain a relatively accurate estimator of $\theta$ via a small $\lambda_p$ and
data \((Y, Z)\). However, such a method cannot handle non-sparse model because the sub-model selected in the first step has already been biased. It is also noted that this method critically depends on twice choices of shrinkage tuning parameter \(\lambda_p\); for details see James and Radchenko (2009). On the other hand, when estimation consistency and normality, rather than variable selection, heavily depend on the choice of \(\lambda_p\), it is practically not convenient, and more seriously, the consistency is in effect not judgeable unless a criterion of tuning parameter selection can be defined to ensure consistency. Then it is desirable to have a new estimation/inference method with which consistency is free of the choice of \(\lambda_p\).

3. Adaptive post-Dantzig estimation and prediction

3.1 Bias-corrected model. As was shown above, the sub-model (2.4) is usually biased. Furthermore, this model is regarded as a non-random model after the variable selection given by the Dantzig selector, i.e., the estimate \(\tilde{I}\) for the index set \(I\) defined in the previous section is fixed after variable selection.

It is clear that a bias correction is needed for the selected sub-model (2.4) when we want to have a consistent estimation of the sub-vector \(\theta = (\theta_1, \cdots, \theta_q)'\). To this end, a new model with an instrumental variable is established. Denote \(Z^* = (Z', U^{(1)}, \cdots, U^{(d)})'\) and \(W = AZ^*\), where \(A\) is \(d \times (q + d)\) matrix satisfying that its row vectors have length 1. Without loss of generality, \(U^{(1)}, \cdots, U^{(d)}\) are supposed to be the first \(d\) components of \(U\), although they may be chosen as another components of \(U\) or pseudo-variables (artificial variables). Denote by \(\lambda_M\) the maximum eigenvalue of \(UU'\) and set \(V = (\alpha'U/\rho, W')'\) for some \(\alpha\) to be chosen later, where \(\rho\) is a nonrandom positive number satisfying the condition \(\rho = O(\|\alpha\|_{\ell_2} \sqrt{\lambda_M})\). Choose \(A\) and \(U^{(1)}, \cdots, U^{(d)}\) such that

\[
E\{(Z - E(Z|V))(Z - E(Z|V))'\} > 0. \tag{3.1}
\]
This condition on the matrix we need can trivially hold because $V$ contains $W$ that is a weighted sum of $Z$ and $U^{(1)}, \cdots, U^{(d)}$. The condition (3.1) can be used to guarantee the identifiability of the following model.

Denote $g(V) = E(\eta|V)$. Now we introduce a bias-corrected version of (2.4) as

$$Y_i = \theta'Z_i + g(V_i) + \xi(V_i), \ i = 1, \cdots, n, \ (3.2)$$

where $\xi(V) = \eta - g(V)$. Obviously, if $\alpha$ in $V$ is identical to $\gamma$ in $\eta$, this model is unbiased, i.e., $E(\xi|Z, V) = 0$; otherwise it may be biased. This model can be regarded as a partially linear model with a linear component $\theta'Z$ and a nonparametric component $g(V)$, and is identifiable because of the condition (3.1). From this structure, we can see that when $V$ does not contain the instrumental variable $W$ and $\alpha = \gamma$, the model goes back to the original model (2.4) as $\xi$ is zero and $g(V)$ becomes the error term $\eta$ (if $\varepsilon$ is ignored). This structure motivates our method. By introducing an instrumental variable $V$ so that $\xi$ has a zero conditional mean, and then we can estimate $g(\cdot)$ to correct the bias occurred in the original model. Although a non-parametric function $g(v)$ is involved, it will be verified that the dimension $d + 1$ of the variable $v$ is low. Note that for $V$, the key is to properly select $\alpha$ and $W$. From the above description, we can see that although $\alpha = \gamma$ should be a natural and good choice, it is unknown and when the dimension is large, is cannot be estimated consistently. Taking this into account, we first consider a general $\alpha$ and construct a bias-corrected model with suitable $W$, or equivalently a suitable matrix $A$.

Denote by $l = p - q$ the dimension of $\gamma$ and let $\lambda = (0, \gamma_2 - \alpha_2/\alpha_1, \cdots, \gamma_l - \alpha_l/\alpha_1)'/\rho$, where $\alpha_1, \cdots, \alpha_l$ are the components of $\alpha$ and $\alpha_1$ is supposed to be nonzero. We can ensure that, when $Z_i$ and $U_i$ satisfy

$$\lambda'E(U_i|Z_i, W_i) = \lambda'E(U_i|W_i), \ (3.3)$$

9
the model (3.2) is unbiased, i.e.,

\[ E(\xi(V_i) \mid Z_i, V_i) = 0. \] 

(3.4)

The proof of (3.4) will be presented in the Appendix.

When \((Z, U)\) is elliptically symmetrically distributed, the condition (3.3) can be rewritten at population level as the following form:

\[
\lambda' \Sigma_{U,Z} A' (\Sigma_{Z^*,Z^*} A')^{-1} A (Z^* - E(Z^*)) \\
= \lambda' \Sigma_{U,Z} B' (\Sigma_{Z^*,Z^*} B')^{-1} B (Z^* - E(Z^*)),
\]

(3.5)

where \(\Sigma_{Z^*,Z^*} = \text{Cov}(Z^*, Z^*)\), \(\Sigma_{U,Z^*} = \text{Cov}(U, Z^*)\) and

\[
B = \begin{pmatrix} I & 0 & \cdots & 0 \\ A_1 & a_{q+2} & \cdots & a_{q+d} \end{pmatrix},
\]

\(A = (A_1, a_{q+1}, \cdots, a_{q+d})\), \(A_1\) is a \(d \times q\) matrix and \(a_j, j = q + 1, \cdots, a_{q+d}\), are \(d\)-dimensional column vectors. Further, the ellipticity condition can be slightly weakened to be the following linearity condition:

\[
E(U \mid C' Z^*) = E(U) + \Sigma_{U,Z^*} C (\Sigma_{Z^*,Z^*} C)^{-1} C' (Z^* - E(Z^*))
\]

for some given matrix \(C\). This linearity condition also results in (3.5). The linearity condition has been widely assumed in the circumstance of high-dimensional models. Hall and Li (1993) showed that it often holds approximately when the dimension \(p\) is high.

Under either the equation (3.3) or (3.5), the bias-corrected model (3.2) is unbiased. Thus, we are now in the position to determine the matrix \(A\) by solving either the equation (3.3) or (3.5). A solution is not difficult to be obtained. For example, if \(\Sigma_{Z^*,Z^*} = I_{q+d}\) and \(B^{-1}\) exists, then we choose \(A\) satisfying

\[
\Sigma_{U,Z} A' (AA')^{-1} A (Z^* - E(Z^*)) = \Sigma_{U,Z^*} (Z^* - E(Z^*)).
\]

(3.6)
It is known that, if we can choose variables $U^{(1)}, \ldots, U^{(d)}$ such that the rank of matrix $\Sigma_{U,Z^*}$ is $d$, then

$$\Sigma_{U,Z^*}^+ \Sigma_{U,Z^*} = Q \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} Q' = Q_1 Q_1', $$

where $\Sigma_{U,Z^*}^+$ is the Moore-Penrose generalized inverse matrix of $\Sigma_{U,Z^*}$, $I_d$ is a $d \times d$ identify matrix, $Q = (Q_1, Q_2)$ an orthogonal matrix satisfying $Q'Q = I_{q+d}$ and $Q_1'Q_1 = I_d$. In this case, we choose

$$A = Q_1'. $$

Such a matrix $A$ is a solution of (3.6) and thus a solution of (3.5). With such a choice of $A$, the model (3.2) is always unbiased whether the model (2.1) is sparse or not, the dimension of $\beta$ is high or low, and the choice of $\lambda_p$ is proper or not.

However, sometimes the matrix $\Sigma_{U,Z^*}^+ \Sigma_{U,Z^*}$ is unknown. Under this situation, we will present a detailed procedure in Section 4 to calculate $\Sigma_{U,Z^*}^+ \Sigma_{U,Z^*}$ and $A$. From the above choice of $A$, we can see that $g(v)$ is a $d + 1$-dimensional nonparametric function. If $d$ is large, we choose a row vector to replace $A$ and will give a method in Section 4 to find an approximate solution. With which, $g(v)$ is a 2-dimensional nonparametric function.

The above deduction shows that the above bias-correction procedure is free of the choice of $\alpha$. However, choosing a proper $\alpha$ is of importance. It is clear that, combining (3.2) and (3.3), choosing an $\alpha$ as close to $\gamma$ as possible should be a good way although optimal choice leaves an unsolved and interesting problem. In the estimation procedure, a natural choice is the value $\tilde{\gamma}^D$ for $\gamma$, which is obtained in the Dantzig selection step. The details are presented in Subsection 3.2 below. We will also discuss the asymptotic properties of an estimation when we use a given $\alpha$ in the next subsection.
3.2 Asymptotic normality of estimation. Throughout this subsection we assume that the matrix $A$ satisfying (3.5) or (3.6) has been obtained. Although the obtained $A$ is sometimes an estimator rather than an exact solution, in this section we still regard it as a nonrandom solution of (3.5) or (3.6) because such an estimator is $\sqrt{n}$-consistent (see Section 4 below) and, as a result, when $A$ is thought of a random vector, the theoretical conclusions given below still hold.

Recall that the bias-corrected model (3.2) can be thought of as a partially linear model. We therefore design an estimation procedure as follows. First of all, as mentioned above, for any $\alpha$, the model (3.2) is unbiased. Then we can design the estimation procedure after $\alpha$ is determined by any empirical method. An empirical choice $\alpha$ is designed as the Dantzig selector $\tilde{\gamma}^D$ of $\gamma$ determined by (2.2). Generally, given $\theta$ and for any $\alpha$, the nonparametric function $g(v)$ is estimated by

$$g_\theta(v) = \frac{\sum_{k=1}^{n} (Y_k - \theta'Z_k) L_H(V_k - v)}{\sum_{k=1}^{n} L_H(V_k - v)},$$

where $L_H(\cdot)$ is a $(d+1)$-dimensional kernel function. Then $g_\theta(v)$ is a $(d+1)$-variate nonparametric estimator. As was shown above, the dimension $d+1$ is low. A simple choice of $L_H(\cdot)$ is a product kernel as

$$L_H(V - v) = \frac{1}{h^{d+1}} K\left(\frac{V^{(1)} - v^{(1)}}{h}\right) \cdots K\left(\frac{V^{(d+1)} - v^{(d+1)}}{h}\right),$$

where $V^{(j)}$, $j = 1, \cdots, d+1$, are the components of $V$, $K(\cdot)$ is an 1-dimensional kernel function and $h$ is the bandwidth depending on $n$. Particularly, when $\alpha$ is chosen as $\tilde{\gamma}^D$, we get an estimator of $g(v)$ as

$$\hat{g}_\theta(v) = \frac{\sum_{k=1}^{n} (Y_k - \theta'Z_k) L_H(\hat{V}_k - v)}{\sum_{k=1}^{n} L_H(\hat{V}_k - v)},$$

where $\hat{V} = (U'\tilde{\gamma}^D/\hat{\rho}, W')'$ and $\hat{\rho} = O(\|\tilde{\gamma}^D\|\ell_2\sqrt{\lambda_M})$. 

12
With these two estimations of $g(v)$, the bias-corrected model (3.2) can be approximately expressed by the following two models:

$$Y_i \approx \theta' Z_i + g_\theta(V_i) + \xi(V_i) \quad \text{and} \quad Y_i \approx \theta' Z_i + \hat{g}_\theta(\hat{V}_i) + \xi(\hat{V}_i),$$

equivalently,

$$\hat{Y}_i \approx \theta' \hat{Z}_i + \xi(V_i) \quad \text{and} \quad \hat{Y}_i \approx \theta' \hat{Z}_i + \xi(\hat{V}_i), \quad (3.8)$$

where

$$\hat{Y}_i = Y_i - \frac{\sum_{k=1}^{n} Y_k L_H(V_k - V_i)}{\sum_{k=1}^{n} L_H(V_k - V_i)}, \quad \hat{Z}_i = Z_i - \frac{\sum_{k=1}^{n} Z_k L_H(V_k - V_i)}{\sum_{k=1}^{n} L_H(V_k - V_i)},$$

$$\hat{Y}_i = Y_i - \frac{\sum_{k=1}^{n} Y_k L_H(\hat{V}_k - \hat{V}_i)}{\sum_{k=1}^{n} L_H(\hat{V}_k - \hat{V}_i)}, \quad \hat{Z}_i = Z_i - \frac{\sum_{k=1}^{n} Z_k L_H(\hat{V}_k - \hat{V}_i)}{\sum_{k=1}^{n} L_H(\hat{V}_k - \hat{V}_i)}.$$ 

Thus, the sub-models in (3.8) result in the estimations for $\theta$ as

$$\tilde{\theta} = S_n^{-1} \frac{1}{n} \sum_{i=1}^{n} \hat{Z}_i \hat{Y}_i \quad \text{and} \quad \tilde{\theta} = S_n^{-1} \frac{1}{n} \sum_{i=1}^{n} \hat{Z}_i \hat{Y}_i, \quad (3.9)$$

where $S_n = \frac{1}{n} \sum_{i=1}^{n} \hat{Z}_i \hat{Z}_i'$ or $S_n = \frac{1}{n} \sum_{i=1}^{n} \hat{Z}_i \hat{Z}_i'$, respectively. Here we assume that the bias-corrected model (3.2) is homoscedastic, that is $Var(\xi(V_i)) = \sigma_V^2$ or $Var(\xi(\hat{V}_i)) = \sigma_V^2$ for all $i = 1, \ldots, n$. If the model is heteroscedastic, we respectively modify the above estimators as

$$\tilde{\theta}^* = S_n^*^{-1} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sigma_i^2(V_i)} \hat{Z}_i \hat{Y}_i \quad \text{and} \quad \tilde{\theta}^* = S_n^*^{-1} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sigma_i^2(V_i)} \hat{Z}_i \hat{Y}_i,$$

where $S_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sigma_i^2(V_i)} \hat{Z}_i \hat{Z}_i'$ or $S_n^* = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sigma_i^2(V_i)} \hat{Z}_i \hat{Z}_i'$, respectively, and $\sigma_i^2(V_i) = Var(\xi(V_i))$ and $\sigma_i^2(\hat{V}_i) = Var(\xi(\hat{V}_i))$. Here $\sigma_i^2(V_i)$ and $\sigma_i^2(\hat{V}_i)$ are supposed to be known. If they are unknown, we can use their consistent estimators to replace them; for details about how to estimate them see for example Härdle et al. (2000). In the following we only consider the estimators defined in (3.9). Finally, the estimators of $g(v)$ can be defined as either $g_\theta(v)$ or $\hat{g}_\theta(v)$.
To study the consistency of the estimations, the following conditions for the model (3.2) are assumed:

(C1) The first two derivatives of \( g(v) \) and \( \xi(v) \) are continuous.

(C2) Kernel function \( K(\cdot) \) satisfies
\[
\int K(u)du = 1, \quad \int u^j K(u)du = 0, j = 1, \cdots, k - 1, 0 < \int u^k K(u)du < \infty.
\]

(C3) \( nh^{2(d+1)} \to \infty \).

Obviously, the conditions (C1)-(C3) are commonly used for semiparametric models. Under these conditions, the following theorem provides the consistency of the bias-corrected estimator \( \tilde{\theta} \).

**Theorem 3.1** Assume that the conditions (C1)-(C3) hold, and for given \( \alpha \), (3.1) and (3.3) are satisfied. When \( q \) is fixed, and \( p \) may be larger than \( n \), then, as \( n \to \infty \),
\[
\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{D} N(0, \sigma^2 V^{-1}),
\]
where \( S = E\{(Z - E(Z|V))(Z - E(Z|V))'\} \).

**Remark 3.1** For simplicity of presentation, in this theorem we only give the the asymptotic normality for the case with fixed \( q \). In fact, when \( q \) tends to infinity at a certain rate, the asymptotic normality still holds for every component of \( \theta \) (see for example Lam and Fan, 2008). This is because, after bias-correction, the model (3.2) is indeed a partially linear model and then the proof can be similar with more technical and tedious details. The proof of this theorem is postponed to the Appendix. The results in the theorem show that the new estimator \( \tilde{\theta} \) is \( \sqrt{n} \)-consistent regardless of the choice of the shrinkage tuning parameter \( \lambda_p \) and thus it is convenient to be used in practice. Furthermore, by the theorem and the commonly
used nonparametric techniques, we can prove that $g_{\hat{\theta}}(v)$ is also consistent. In effect, we can obtain the strong consistency and the consistency of the mean squared error under some stronger conditions. The details are omitted in this paper.

To investigate the asymptotic properties for the second estimator $\hat{\theta}$ in (3.9) that is based on the Dantzig selector $\tilde{\gamma}^D$, we need the following more conditions:

(C4) The bandwidth $h$ is optimally chosen, i.e., $h = O(n^{-1/(2(k+d+1))})$.

(C5) Suppose that there exists a vector, say $\alpha$, such that $\|\alpha\|_{\ell_2} \geq c$ for a positive constant $c$ and $\|\tilde{\gamma}^D - \alpha\|_{\ell_2}/\|\tilde{\gamma}^D\|_{\ell_2} = O_p(n^{-\mu})$ for some $\mu$ satisfying

$$1/2 - k/(2(k + d + 1)) \leq \mu \leq 1/2.$$ 

As was stated in the previous sections, $\alpha$ was an arbitrary vector. The vector $\alpha$ in the condition (C5) is then different. But for the simplicity of representation we still use the same notation $\alpha$ in different appearance. The condition (C5) is the key for the following theorem and corollary. This condition does not mean that the Dantzig selector $\tilde{\gamma}^D$ is consistent. Note that $\|\tilde{\gamma}^D\|_{\ell_2}$ is large in non-sparse case, and the accuracy of the solution of linear programm can guarantee that $\|\tilde{\gamma}^D - \alpha\|_{\ell_2}$ is relatively small for the true value of linear programm (2.2) at population level (see for example Malgouyres and Zeng, 2009). These show that the condition (C5) is reasonable. Both (C4) and (C5) can actually be weakened, but for the simplicity of technical proof and presentation, we still use the current conditions in this paper.

**Theorem 3.2** Under the conditions (C1)-(C5), (3.1) and (3.3), we have the following asymptotic representation for the second estimator in (3.9):

$$\sqrt{n}(\hat{\theta} - \theta) = S^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \tilde{Z}_i \tilde{g}(V_i) + \tilde{Z}_i \tilde{\xi}(V_i) \right) + o_p(1),$$ 

15
where \( S = E\{ (Z - E(Z|V))(Z - E(Z|V))' \} \) and
\[
\tilde{g}(V_i) = g(V_i) - \frac{\sum_{k=1}^{n} g(V_k)L_H(V_k-V_i)}{\sum_{k=1}^{n} L_H(V_k-V_i)},
\]
\[
\tilde{\xi}(V_i) = \xi(V_i) - \frac{\sum_{k=1}^{n} \xi(V_k)L_H(V_k-V_i)}{\sum_{k=1}^{n} L_H(V_k-V_i)},
\]
\[
\tilde{Z}_i = Z_i - \frac{\sum_{k=1}^{n} Z_kL_H(V_k-V_i)}{\sum_{k=1}^{n} L_H(V_k-V_i)}.
\]

The proof of the theorem is given in the Appendix. From Theorem 3.2, and Theorem 2.1.2 of Härdle et al (2000), the asymptotic normality follows directly. The following corollary states the detail.

**Corollary 3.3** Under the conditions of Theorem 3.1, when \( q \) is fixed but \( p \) may be larger than \( n \), then, as \( n \to \infty \),
\[
\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, \sigma^2_{V,S}^{-1}).
\]

As aforementioned in Remark 3.1, for the sub-model with diverging \( q \), the asymptotic normality can still hold under some stronger conditions, the details are omitted here.

**3.3 Prediction.** Together the estimation consistency with the adjusted sub-model (3.2), we obtain an improved prediction as
\[
\hat{Y} = \hat{\theta}'Z + \hat{g}_\hat{\theta}(V) \quad (3.10)
\]
and the corresponding prediction error is
\[
E(Y - \hat{Y})^2 = E((\hat{\theta} - \theta)'Z)^2 + E(\hat{g}_\theta(V) - g(V))^2 + E(\xi^2(V))
\]
\[
+ 2E((\hat{\theta} - \theta)'Z(\hat{g}_\theta(V) - g(V))) + 2E((\hat{\theta} - \theta)'Z\xi(V))
\]
\[
+ 2E((\hat{g}_\theta(V) - g(V))\xi(V))
\]
\[
= E(\xi^2(V)) + o(1).
\]
Such a prediction is of a smaller prediction error than the one by the classical Dantzig selector, and interestingly it is no need with any high-dimensional nonparametric estimation.

In contrast, if we use the new estimator \(\hat{\theta}\) and the sub-model (2.4), rather than the adjusted sub-model (3.2), to construct prediction, the resulting prediction is defined as
\[
\hat{Y}_S = \hat{\theta}'Z + \bar{\hat{g}}_\theta,
\]
where
\[
\bar{\hat{g}}_\theta = \frac{1}{n} \sum_{i=1}^{n} \hat{g}_\theta(V_i).
\]
For prediction, we need to add \(\bar{\hat{g}}_\theta\) in (3.11) because the sub-model (2.4) has a bias \(E(g(V))\), otherwise, the prediction error would be even larger. In this case, \(\bar{\hat{g}}_\theta\) is free of the predictor \(U\) and the resultant prediction (3.11) only uses the predictor \(Z\) in the sub-model (2.4). This is different from the prediction (3.10) that depends on both the low-dimensional predictor \(Z\) and high-dimensional predictor \(U\). Thus (3.11) is a sub-model based prediction. The corresponding prediction error is
\[
E(Y - \hat{Y}_S)^2 = E((\hat{\theta} - \theta)'Z)^2 + E(\bar{\hat{g}}_\theta - g(V))^2 + E(\xi^2(V))
\]
\[
+ 2E((\hat{\theta} - \theta)'Z(\bar{\hat{g}}_\theta - g(V))) + 2E((\hat{\theta} - \theta)'Z\xi(V))
\]
\[
+ 2E((\bar{\hat{g}}_\theta - g(V))\xi(V))
\]
\[
= E(\xi^2(V)) + Var(g(V)) + 2E(E(g(V)) - g(V))\xi(V)) + o(1).
\]
This error is usually larger than that of the prediction (3.10). But,
\[
|E(E(g(V)) - g(V))\xi(V))| \leq (Var(g(V))Var(\xi(V)))^{1/2}
\]
17
and usually the values of both $\text{Var}(g(V))$ and $\text{Var}(\xi(V))$ are small. Then such a prediction still has a smaller prediction error than the one obtained by the sub-model (2.4) and the common LS estimator $\tilde{\theta}_S = (Z'Z)^{-1}Z'Y$ as:

$$\tilde{Y}_S = \tilde{\theta}_S Z$$

(3.12)

with the corresponding error as

$$E(Y - \tilde{Y}_S)^2 = E((\tilde{\theta}_S - \theta)'Z)^2 + E(\gamma'U)^2 + \sigma^2 + 2E((\tilde{\theta}_S - \theta)'Z\gamma'U).$$

Because $\tilde{\theta}_S$ does not tend to $\theta$, the values of both $E((\tilde{\theta}_S - \theta)'Z)^2$ and $2E((\tilde{\theta}_S - \theta)'Z\gamma'U)$ are large and as a result the prediction error is large.

The above results show that in the scope of prediction, the new estimator can reduce prediction error under both the adjusted sub-model (3.2) and the original sub-model (2.4). We will see that the simulation results in Section 5 coincide with these conclusions.

4. Calculation for $A$

4.1 Calculation of $A$ for the case with unknown $\Sigma_{U,Z}^+, \Sigma_{U,Z}^*$. In the previous section, we suggested a simple choice of $A$ for the case with known $\Sigma_{U,Z}^+, \Sigma_{U,Z}^*$. We now introduce an approach for choosing vector $A$ such that (3.6) holds for the case with unknown $\Sigma_{U,Z}^+, \Sigma_{U,Z}^*$. For the convenience of representation, we here suppose $E(Z) = 0$, $E(U)=0$ and $\text{Cov}(Z^*) = I$. In this case, (3.6) can be rewritten as

$$\Sigma_{U,Z}^* A'(AA')^{-1}AZ^* = \Sigma_{U,Z}^* Z^*.$$  

(4.1)
We denote $\Sigma_{U,Z} = (\omega_{ij})$ with
\begin{align*}
\omega_{ij} &= \sum_{k=1}^{l} E(U^{(k)}Z^{(i)})E(U^{(k)}Z^{(j)}), \ i, j \leq q, \\
\omega_{i,q+s} &= \omega_{q+s,i} = \sum_{k=1}^{l} E(U^{(k)}Z^{(i)})E(U^{(k)}U^{(s)}), \ i = 1, \ldots, q, s = 1, \ldots, d, \\
\omega_{q+r,q+s} &= \sum_{k=1}^{l} E(U^{(k)}U^{(r)})E(U^{(k)}U^{(s)}), \ r, s = 1, \ldots, d,
\end{align*}

where $Z^{(i)}$ and $U^{(k)}$ are the components of $Z$ and $U$, respectively. It is known that $\Omega$ can be decomposed as
\[\Omega = Q \text{ diag}\{\phi_{1}, \ldots, \phi_{d}, 0, \ldots, 0\}Q',\]

where $\phi_{k}, k = 1, \ldots, d$, are the positive eigenvalues of $\Omega$ and $Q$ is the orthogonal matrix. Note that $l$ depends on $n$ and tends to infinity as $n \to \infty$. To get consistent estimator of $Q$, we need the following condition
\[\# \left\{ E(U^{(i)}Z^{(j)}), E(U^{(k)}U^{(s)}) : E(U^{(i)}Z^{(j)}) \neq 0, E(U^{(k)}U^{(s)}) \neq 0, \text{ for all } i, j, k, s \right\} \leq C (4.2)\]

for a positive constant $C$, where $\#\{S\}$ denotes the number of elements in the set $S$. Also we can use some weaker conditions to replace (4.2). In fact the conditions we need are similar to those required for high-dimensional linear models, for example, the weak and strong irrepresentable conditions (Zhao and Yu 2006) and the uniform uncertainty principle (Candés and Tao 2007). Note that $\Omega$ is a low-dimensional matrix. Then, under the condition (4.2), $\Omega$ can be $\sqrt{n}$-consistently estimated; for example, a naive estimator of $\hat{\omega}_{ij}$ for $i, j \leq q$ can be chosen as
\[\hat{\omega}_{ij} = \sum_{k=1}^{l} \frac{1}{n} \sum_{s=1}^{n} U_{s}^{(k)}Z_{s}^{(i)} 1\left\{ \frac{1}{n} \sum_{s=1}^{n} U_{s}^{(k)}Z_{s}^{(i)} > \frac{1}{\sqrt{n}} \right\} \frac{1}{n} \sum_{s=1}^{n} U_{s}^{(k)}Z_{s}^{(j)} 1\left\{ \frac{1}{n} \sum_{s=1}^{n} U_{s}^{(k)}Z_{s}^{(j)} > \frac{1}{\sqrt{n}} \right\},\]

where $1\{S\}$ is the indicator function of the set $S$. As was shown above, we can express $\hat{\Omega}$ as
\[\hat{\Omega} = \hat{Q} \text{ diag}\{\hat{\phi}_{1}, \ldots, \hat{\phi}_{d}, 0 \cdots, 0\}\hat{Q}' (4.3)\]
and \( \hat{Q} \) as \( \hat{Q} = (\hat{Q}_1, \hat{Q}_2) \). Finally, the estimator of \( A \) is obtained by

\[
\hat{A} = \hat{Q}_1.
\]

4.1 Calculation of \( A \) for large \( d \). As we mentioned before, when \( d \) is large, the solution \( A \) of (4.1) has \( d \) columns and then \((d + 1)\)-dimensional nonparametric estimation will be involved, which leads an inefficient estimation. Thus, we consider an approximation solution of (4.1), which is a row vector. Without confusion, we still use the notation \( A \) to denote this row vector. That is, we choose a row vector \( A \) such that

\[
A'AZ^* = \Sigma^+_{U,Z} \Sigma_{U,Z} Z^*.
\]  

(4.4)

The approximation solution is identical to the solution of (4.1) in form as when \( A \) is a row vector, recalling that it is normalized to be norm one, \( AA' = 1 \). In this case, to get a low-dimensional nonparametric function \( g(v) \), we choose \( d = 1 \), i.e., \( Z^* \) is a \((q + 1)\)-dimensional vector. Similar to the above determination, when \( A \) is unknown, we can also construct an estimation as follows. Denote \( A = (a_1, \cdots, a_q, a_{q+1}) \), \( A_k = a_k A \) and \( \Sigma^+_{U,Z} \Sigma_{U,Z} = (D'_1, \cdots, D'_q, D'_{q+1})' \), where \( D_k, k = 1, \cdots, q + 1 \), are \((q + 1)\)-dimensional row vectors. Then we estimate \( A \) via solving the following optimization problem:

\[
\inf \left\{ Q(a_1, \cdots, a_{q+1}) : \sum_{k=1}^{q+1} a_k^2 = 1 \right\},
\]  

(4.5)

where \( Q(a_1, \cdots, a_{q+1}) = \frac{1}{n} \sum_{i=1}^{n} \sum_{k=1}^{q+1} \| (A_k - D_k) Z_i^* \|^2 \). By the Lagrange multiplier, we obtain the estimators of \( A_k, k = 1, \cdots, q + 1 \), as

\[
\hat{A}_k = \left( D_k \frac{1}{n} \sum_{i=1}^{n} Z_i^* Z_i'^* + cc_k e_k/2 \right) \left( \frac{1}{n} \sum_{i=1}^{n} Z_i^* Z_i'^* + c_k I \right)^{-1},
\]  

(4.6)

where \( c_k > 0 \), which is similar to a ridge parameter, depends on \( n \) and tends to zero as \( n \to \infty \), and \( e_k \) is a row vector with \( k \)-th component 1 and the others being zero.
Note that the constraint $\|A\| = 1$ implies $\|A_k\| = \pm a_k$. Finally, by combining (4.6) and this constraint we get an estimator of $a_k$ as

$$\hat{a}_k = \pm \|\hat{A}_k\|$$

and consequently the estimator of $A$ is obtained by

$$\hat{A} = (\hat{a}_1, \cdots, \hat{a}_q, \hat{a}_{q+1}).$$

5. Simulation studies

In this section we examine the performance of the new method by simulations. By mean squared error (MSE), model prediction error (PE) and their std MSE and std PE as well, we compare the method with the Gaussian-dantzig selector first. In ultra-high dimensional scenarios, the Dantzig selector cannot work well, we use the sure independent screening (SIS) (Fan and Lv 2008) to bring dimension down to a moderate size and then to make comparison with the Gaussian-dantzig selector. As is well known, there are several factors that are of great impact on the performance of variable selection methods: dimensions $p$ of covariate $X$, correlation structure between the components of covariate $X$, and variation of the error which can be measured by theoretical model R-square defined by $R^2 = (\text{Var}(Y) - \sigma^2_\epsilon)/\text{Var}(Y)$. In order to comprehensively illustrate the theoretical conclusions and performance, we design three experiments. The main goal of the first experiment is to examine the effect of $R^2$ as the smaller $R^2$ is, the more difficult correctly selecting variables is. The second experiment is to investigate the impact from the correlation between the components of covariate $X$, and the third is to check whether the two-step procedure of the SIS and the Dantzig selector works or not.

**Experiment 1.** This experiment is designed mainly for: (1) comparing the new
estimator \( \hat{\theta} \) defined by (3.9) with the Gaussian-dantzig selector \( \tilde{\theta}_S \); (2) examining the effect of different choices of the theoretical model \( R^2 \) of the full model (2.1); (3) checking the effect of the correlation between the components of \( X \) when \( R^2 \) is fixed. To achieve these goals, we compare the MSEs, the PEs and their \( \text{stdMSE} \) and \( \text{stdPE} \) of the two different estimators \( \hat{\theta} \) and \( \tilde{\theta}_S \), and the two models (2.4) and (3.2). In the simulation, to determine the regression coefficients in our simulation, we decompose the coefficient vector \( \beta \) as two parts: \( \beta_I \) and \( \beta_{-I} \), where \( I \) denotes the set of locations of significant components of \( \beta_I \), and let \( S = |I| \) denote the number of elements contained in \( I \). Three types of \( \beta_I \) are considered:

Type (I): \( \beta_I = (1, 0.4, 0.3, 0.5, 0.3, 0.3, 0.3)' \) and \( I = \{1, 2, 3, 4, 5, 6, 7\} \);

Type (II): \( \beta_I = (1, 0.4, 0.3, 0.5, 0.3, 0.3, 0.3)' \) and \( I = \{1, 17, 33, 49, 65, 81, 97\} \);

Type (III): \( \beta_I = (1, 0.4, -0.3, -0.5, 0.3, 0.3, -0.3)' \) and \( I = \{1, 2, 3, 4, 5, 6, 7\} \).

As it is very rare that all other coefficients are exactly zero, non-sparse models are considered. To mimic practical scenarios, we set the values of the components \( \beta_{-I_i} \)'s of \( \beta_{-I} \) as follows. Before performing the variable selection and estimation, we generate \( \beta_{-I_i} \)'s from uniform distribution \( U(-0.5, 0.15) \) and the negative values of them are then set to be zero. After the coefficient vector \( \beta \) is determined, we consider it as a fixed value vector and regard \( \beta_I \) as the main part of the coefficient vector \( \beta \). We use this way to set the values of \( \beta_{-I_i} \)'s because in the simulations below, there are too many insignificant variables with small/zero coefficients and it makes little sense to give a common value for them. As too many values for these insignificant coefficients, we do not list all of them here. We use \( \hat{I} \) to denote the set of subscript of coefficients \( \theta \) in \( \beta \), that is the coefficients’ subscript of variables selected into sub-model. we assume \( X \sim N_p(\mu, \Sigma_X) \), with \( \mu \) the components corresponding to \( I \) are 0 and others are 2 and the \((i, j)\)-th element \( \Sigma_{ij} = (-\rho)^{|i-j|} \), \( 0 < \rho < 1 \). Furthermore, the error term \( \epsilon \) is assumed to be normally distributed as \( \epsilon \sim N(0, \sigma^2) \). In this experiment, we choose different \( \sigma \) to obtain different type of full model with
different $R^2$. In the simulation procedure and the kernel function is chosen to be Gaussian kernel $K(u) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{u^2}{2}\}$. In this experiment, the choice of parameter $\lambda_p$ in the Dantzig selector is just like that given by Candès and Tao (2007), which is the empirical maximum of $|X'z_i|$ over several realizations of $z \sim N(0, I_n)$.

The following Tables 1 and 2 report the MSEs and the corresponding PEs via 200 repetitions. In these tables, $\hat{Y}$ is the prediction via the adjusted model (3.2) that is based on the full dataset, $\hat{Y}_S$ is the prediction via the sub-model (2.4) with the new estimator $\hat{\theta}$ defined in (3.9), $\tilde{Y}_S$ stands for the prediction via the sub-model (2.4) and the Gaussian-dantzig selector $\tilde{\theta}_S$. For the definitions of $\hat{Y}$, $\hat{Y}_S$ and $\tilde{Y}_S$ see (3.10), (3.11) and (3.12), respectively. The purpose of such a comparison is to see whether the adjustment works and whether we should use the sub-model (2.4) when the high-dimensional data are not available (say, too expensive to collect), whether the new estimator $\hat{\theta}$ together with the sub-model (2.4) is helpful for prediction accuracy. The sample size is 50, and for the prediction, we perform the experiment with 200 repetitions to compute the proportion $\tau$ of which the prediction error of $\hat{Y}_S$ is less than that of $\tilde{Y}_S$ in the 200 repetitions. The larger $\tau$ is, the better the new estimator is. We have the following considerations in designing the experiment: a). We will study models with the theoretical model $R^2$ ranging between 0.3 and 1.0, which can be determined by the value of the variance of error term $\sigma^2$, here we choose $\sigma^2=0.2, 0.6, 0.9, 1.3$ and 1.9 respectively; b). The correlation between the components of $X$ should have effect for the estimation, we then consider different correlation coefficients 0.1 and 0.7.

1. Let $n = 50, p = 100, S = 7$ and $\rho = 0.1$. For each type of $\beta$, we choose different $\sigma$ to control the theoretical $R^2$ and consider five cases. For type (I), we have the following results:

Case 1. $R^2 = 0.98$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 2, 3, 4, 6, 7\}$;
Case 2. $R^2 = 0.82$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 2, 4, 6, 7, 55\}$;
Case 3. $R^2 = 0.67$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 2, 3, 4, 15, 22, 28, 81\}$;
Case 4. $R^2 = 0.50$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 2, 4, 27, 29, 49, 53, 84\}$;
Case 5. $R^2 = 0.31$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 4, 5, 24, 25, 42, 43, 62\}$.

For type (II), we have the following results:
Case 1. $R^2 = 0.98$, $I = \{1, 17, 33, 49, 65, 81, 97\}$ and $\hat{I} = \{1, 17, 33, 49, 65, 81, 97\}$;
Case 2. $R^2 = 0.84$, $I = \{1, 17, 33, 49, 65, 81, 97\}$ and $\hat{I} = \{1, 17, 33, 43, 49, 81\}$;
Case 3. $R^2 = 0.71$, $I = \{1, 17, 33, 49, 65, 81, 97\}$ and $\hat{I} = \{1, 15, 17, 33, 49, 62, 72\}$;
Case 4. $R^2 = 0.53$, $I = \{1, 17, 33, 49, 65, 81, 97\}$ and $\hat{I} = \{1, 5, 26, 29, 33, 43, 49, 53, 65, 74\}$;
Case 5. $R^2 = 0.35$, $I = \{1, 17, 33, 49, 65, 81, 97\}$ and $\hat{I} = \{1, 7, 17, 26, 29, 31, 49, 72, 80, 96, 97, 98\}$.

For type (III), we have the following results:
Case 1. $R^2 = 0.98$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 2, 3, 4, 5, 6\}$;
Case 2. $R^2 = 0.83$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 3, 4, 5, 6, 7, 15\}$;
Case 3. $R^2 = 0.69$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 2, 4, 5, 7, 92\}$;
Case 4. $R^2 = 0.51$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 5, 7, 8, 67, 71\}$;
Case 5. $R^2 = 0.33$, $I = \{1, 2, 3, 4, 5, 6, 7\}$ and $\hat{I} = \{1, 4, 6, 7, 21, 23, 38, 50, 75, 83\}$.

**Table 1.** MSE, PE and their standard errors with $n = 50, p = 100, S = 7$ and $\rho = 0.1$
The simulation results are reported in Table 1. The results suggest that the adjustment of (3.2) works very well, the corresponding estimation and prediction are uniformly the best among the competitors. Further, as we mentioned, when the full dataset is not available and we thus use the sub-model of (2.4), the new estimator \( \hat{\theta} \) is also useful for prediction. It can be seen that \( \hat{Y}_S \) is better than \( \tilde{Y}_S \), and the value of \( \tau \) is larger than 0.7 in 13 cases out of 15 cases and in the other 2 cases, it is larger than or about 0.6.

2. To provide more information, we also consider the case with higher correlation \( \rho = 0.7: n = 50, p = 100, S = 7 \). Also different \( \sigma \)'s are chosen to control the theoretical \( R^2 \).

For type (I), we consider the following five cases.

| Case | \( R^2 \) | \( \hat{\theta} \) | \( \tilde{\theta}_S \) | \( \hat{Y} \) | \( \tilde{Y}_S \) | \( \hat{Y}_S \) | \( \tau \) |
|------|---------|-----------|---------|-------|--------|---------|-----|
| 1    | 0.98    | 0.0032(0.0118) | 0.0866(0.3519)| 0.1630(0.0405) | 0.2299(0.0535) | 1.587(0.5549) | 200/200 |
| 2    | 0.82    | 0.0134(0.0544) | 0.1197(0.1654)| 0.6603(0.1497) | 0.7249(0.1564) | 1.4755(0.3475) | 200/200 |
| I    | 0.67    | 0.0273(0.1288) | 0.0430(0.1283)| 1.3038(0.2952) | 1.3438(0.3018) | 1.4821(0.3266) | 166/200 |
| 3    | 0.50    | 0.0543(0.2387) | 0.0694(0.2221)| 2.5371(0.5500) | 2.5919(0.5633) | 2.7176(0.6020) | 142/200 |
| 4    | 0.31    | 0.1028(0.4689) | 0.1131(0.4876)| 4.9199(1.1856) | 4.9960(1.2070) | 5.0708(1.1965) | 126/200 |
| 5    | 0.12    | 0.0052(0.0202) | 0.3540(1.4263)| 0.2584(0.0569) | 0.2744(0.0583) | 1.1324(2.4262) | 200/200 |

The simulation results are reported in Table 1. The results suggest that the adjustment of (3.2) works very well, the corresponding estimation and prediction are uniformly the best among the competitors. Further, as we mentioned, when the full dataset is not available and we thus use the sub-model of (2.4), the new estimator \( \hat{\theta} \) is also useful for prediction. It can be seen that \( \hat{Y}_S \) is better than \( \tilde{Y}_S \), and the value of \( \tau \) is larger than 0.7 in 13 cases out of 15 cases and in the other 2 cases, it is larger than or about 0.6.

2. To provide more information, we also consider the case with higher correlation \( \rho = 0.7: n = 50, p = 100, S = 7 \). Also different \( \sigma \)'s are chosen to control the theoretical \( R^2 \).

For type (I), we consider the following five cases.

Case 1. \( R^2 = 0.96, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 2, 4, 5, 6, 7\} \);

Case 2. \( R^2 = 0.71, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 2, 4, 81\} \);

Case 3. \( R^2 = 0.53, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 4, 8, 9\} \);

Case 4. \( R^2 = 0.35, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 4, 8, 51\} \);

Case 5. \( R^2 = 0.20, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 2, 6, 84\} \).
For type (II), we consider the following five cases.

Case 1. \( R^2 = 0.98, I = \{1, 17, 33, 49, 65, 81, 97\} \) and \( \hat{I} = \{1, 17, 33, 49, 65, 97\} \);

Case 2. \( R^2 = 0.84, I = \{1, 17, 33, 49, 65, 81, 97\} \) and \( \hat{I} = \{1, 18, 49, 65, 97\} \);

Case 3. \( R^2 = 0.69, I = \{1, 17, 33, 49, 65, 81, 97\} \) and \( \hat{I} = \{1, 2, 49, 65\} \);

Case 4. \( R^2 = 0.52, I = \{1, 17, 33, 49, 65, 81, 97\} \) and \( \hat{I} = \{1, 15, 33, 49, 76, 84, 98\} \);

Case 5. \( R^2 = 0.34, I = \{1, 17, 33, 49, 65, 81, 97\} \) and \( \hat{I} = \{1, 2, 24, 48, 49, 55, 87, 97\} \).

For type (III), we consider the following five cases.

Case 1. \( R^2 = 0.96, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 2, 4, 6, 7\} \);

Case 2. \( R^2 = 0.74, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 4, 6, 7\} \);

Case 3. \( R^2 = 0.56, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 6, 7, 33, 56\} \);

Case 4. \( R^2 = 0.38, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 4, 7, 51, 93\} \);

Case 5. \( R^2 = 0.23, I = \{1, 2, 3, 4, 5, 6, 7\} \) and \( \hat{I} = \{1, 2, 7, 31, 45, 80, 85, 88\} \).
Table 2. MSE, PE and their standard errors with \( n = 50, p = 100, S = 7 \) and \( \rho = 0.7 \)

| type | \( R^2 \) | MSE \( (\text{stdMSE}) \) | PE \( (\text{stdPE}) \) | \( \tau \) |
|------|----------|-----------------|-----------------|-----|
|      |          | \( \hat{\theta} \) | \( \hat{\theta}_S \) | \( \hat{Y} \) | \( \hat{Y}_S \) | \( \hat{Y}_S \) |       |
| (I)  | 0.96     | 0.0136(0.0504)  | 0.3285(0.4226)  | 0.2472(0.0517) | 0.2706(0.0599) | 1.7397(0.3804) | 200/200 |
|      | 0.71     | 0.0253(0.1426)  | 0.0709(0.2401)  | 0.6530(0.1463) | 0.6945(0.1557) | 0.9892(0.2070) | 197/200 |
|      | 0.53     | 0.0373(0.1621)  | 0.1108(0.2310)  | 1.2779(0.2744) | 1.3235(0.2861) | 1.5985(0.3736) | 177/200 |
|      | 0.35     | 0.0613(0.3122)  | 0.0999(0.3289)  | 2.3431(0.5342) | 2.3694(0.5395) | 2.6339(0.5799) | 161/200 |
|      | 0.2      | 0.1198(0.6479)  | 0.1292(0.6619)  | 5.1184(1.2643) | 5.1347(1.2729) | 5.1764(1.2420) | 129/200 |
| (II) | 0.98     | 0.0122(0.0484)  | 0.2730(0.3789)  | 0.2648(0.0730) | 0.2809(0.0757) | 1.1952(0.2440) | 200/200 |
|      | 0.84     | 0.0201(0.0924)  | 0.1799(0.2037)  | 0.6567(0.1453) | 0.6580(0.1452) | 1.6477(0.3560) | 200/200 |
|      | 0.69     | 0.0303(0.1338)  | 0.2899(0.4442)  | 1.2955(0.2992) | 1.2996(0.3047) | 2.7125(0.5861) | 200/200 |
|      | 0.52     | 0.0644(0.3395)  | 0.1141(0.4388)  | 2.5572(0.5558) | 2.5633(0.5582) | 3.2790(0.6834) | 191/200 |
|      | 0.34     | 0.1245(0.5615)  | 0.1831(0.6787)  | 5.0731(1.1850) | 5.0818(1.1743) | 5.5988(1.2782) | 161/200 |
| (III)| 0.96     | 0.0239(0.0626)  | 0.6020(2.1653)  | 0.2596(0.0560) | 0.2897(0.0630) | 1.6754(1.4970) | 200/200 |
|      | 0.74     | 0.0315(0.1158)  | 0.4401(0.5248)  | 0.6435(0.1435) | 0.6485(0.1442) | 2.7859(0.6035) | 200/200 |
|      | 0.56     | 0.0749(0.2373)  | 0.1736(0.2679)  | 1.3334(0.2947) | 1.4367(0.3217) | 1.8643(0.3965) | 189/200 |
|      | 0.38     | 0.0687(0.3227)  | 0.1701(0.3809)  | 2.3637(0.4538) | 2.4645(0.4818) | 2.9415(0.5992) | 178/200 |
|      | 0.23     | 0.1740(0.8078)  | 0.2446(0.8718)  | 4.8488(1.1812) | 4.8887(1.1968) | 5.1471(1.1499) | 145/200 |

Table 2 shows that when \( \rho \) is larger, the conclusions about the comparison are almost identical to those presented in Table 1; Thus it concludes that no matter \( \rho \) is larger or not, our new method always works quite well.

We are now in the position to make another comparison. In Experiments 2 and 3 below, we do not use the data-driven approach as given in Experiment 1 to select \( \lambda_p \), while manually select several values to see whether our method works or not. This is because in the two experiments, it is not our goal to study shrinkage tuning parameter, but is our goal to see whether the new method works after we have a sub-model.

**Experiment 2.** In this experiment, our focus is how the correlation between variables affects the estimations. The distribution of \( X \) is the same as that in Experiment 1 except for the dimension. The coefficient vector \( \beta \) is designed as
type (I) in Experiment 1. Furthermore, the error term $\varepsilon$ is assumed to be normally distributed as $\varepsilon \sim N(0, 0.2^2)$.

As different choices of $\lambda_p$ will usually lead to different sub-models, equivalently, to different estimators $\hat{I}$ of $I$, we are then able to examine, when the numbers of significant variables that are included into the submodels are different, the performance of the new estimation by MSE and PE. In this experiment, we consider two cases with two values of $\lambda$. The setting is as follows. For $n = 50, p = 100, S = 7, \rho = 0.1, 0.3, 0.5, 0.7$. We consider two cases for each $\rho$:

**$\rho = 0.1$**
- Case 1. $\lambda_p = 3.97$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 2, 3, 4, 5, 6, 7\}$
- Case 2. $\lambda_p = 6.53$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 3, 4, 6, 95\}$

**$\rho = 0.3$**
- Case 1. $\lambda_p = 3.32$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 2, 3, 4, 5, 6\}$
- Case 2. $\lambda_p = 6.77$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 2, 4, 6, 23\}$

**$\rho = 0.5$**
- Case 1. $\lambda_p = 3.72$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 2, 4, 5, 6, 7\}$
- Case 2. $\lambda_p = 7.29$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 4, 5, 7, 41, 58, 72\}$

**$\rho = 0.7$**
- Case 1. $\lambda_p = 3.50$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 3, 4, 7, 41, 75\}$
- Case 2. $\lambda_p = 7.22$, $I = \{1, 2, 3, 4, 5, 6, 7\}$, $\hat{I} = \{1, 4, 7, 51, 64, 67, 68, 83\}$
Table 3. MSE, PE and their standard errors with $n = 50, p = 100, S = 7$

| $\rho$ | Case | MSE(std MSE) | PE(std PE) | $\tau$ |
|--------|------|--------------|------------|--------|
|        |      | $\hat{\theta}$ | $\hat{\theta}_S$ | $\hat{Y}$ | $\hat{Y}_S$ | $\hat{Y}_S$ |        |
| 0.1    | 1    | 0.0052(0.0242) | 0.2929(0.3877) | 0.2580(0.0528) | 0.2612(0.0527) | 3.0195(0.6691) | 200/200 |
|        | 2    | 0.0104(0.0357) | 0.2347(0.1784) | 0.5135(0.1074) | 0.6430(0.1282) | 5.921(0.4172) | 200/200 |
| 0.3    | 1    | 0.0070(0.0289) | 0.4067(1.6692) | 0.2732(0.0590) | 0.3324(0.0735) | 5.6406(1.8289) | 200/200 |
|        | 2    | 0.0163(0.0458) | 0.5048(0.4107) | 0.4048(0.0881) | 0.5014(0.1078) | 6.4471(0.7697) | 200/200 |
| 0.5    | 1    | 0.0079(0.0336) | 0.4826(1.9425) | 0.2436(0.0551) | 0.3053(0.0674) | 5.8204(1.8152) | 200/200 |
|        | 2    | 0.0136(0.0512) | 0.1532(0.1835) | 0.3655(0.0841) | 0.4245(0.0914) | 6.4357(0.3262) | 200/200 |
| 0.7    | 1    | 0.0157(0.0602) | 0.2296(0.2970) | 0.2688(0.0580) | 0.3198(0.0711) | 6.6313(0.3560) | 200/200 |
|        | 2    | 0.0149(0.0637) | 0.1914(0.1420) | 0.2974(0.0624) | 0.3225(0.0672) | 7.5435(0.1169) | 197/200 |

From Table 3, we can see clearly that the correlation is of impact on the performance of the variable selection methods: the estimation gets worse with larger $\rho$. However, the new method uniformly works much better than the Gaussian Dantzig selector, when we compare the performance of the methods with different values of $\lambda$ and then with different sub-models. We can see that in case I, the sub-models are more accurate than those in case II in the sense that they can contain more significant variables we want to select. Then, the estimation based on the Gaussian Dantzig selector can work better and so can the new method. Note that $\rho$ is about 1 meaning that in all the 200 repetitions, $\hat{Y} \leq \hat{Y}_S$.

In the following, we consider ultra high-dimensional data.

**Experiment 3.** For very large $p$, the Dantzig selector method alone cannot work well. Thus, we use the sure independent screening (SIS, Fan and Lv 2008) to reduce the number of variables to a moderate scale that is below the sample size, and then perform the variable selection and parameter estimation afterwards by the Gaussian Dantzig selector and our adjustment method.

We first consider $n = 100, p = 1000$ and $S = 10$ with $\rho = 0.1, 0.5$ and 0.9 respectively, and for each $\rho$ two $\lambda_p$ are used to obtain two $\hat{I}$ as follows.
For $\rho=0.1$, $\beta_I = (1.0, -1.5, 2.0, 1.1, -3.0, 1.2, 1.8, -2.5, -2.0, 1.0)'$, consider two cases:

Case 1. $\lambda_p=4.50$, $I=\{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{1,3,5,6,7,8,9,318,514,723,760\}$;
Case 2. $\lambda_p=7.30$, $I=\{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{2,3,5,8,515,886\}$.

For $\rho=0.5$, $\beta_I = (1.0, -1.5, 2.0, 1.1, -3.0, 1.2, 1.8, -2.5, -2.0, 1.0)'$, consider two cases:

Case 1. $\lambda_p=3.56$, $I=\{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{1,2,5,7,8,9,846,878,976\}$;
Case 2. $\lambda_p=6.92$, $I=\{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{2,3,5,8,10,882,963\}$.

For $\rho=0.9$, $\beta_I = (1.0, -1.5, 2.0, 1.1, -3.0, 1.2, 1.8, -2.5, -2.0, 1.0)'$, consider two cases:

Case 1. $\lambda_p=1.80$, $I=\{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{3,5,8,10,415,432\}$;
Case 2. $\lambda_p=5.83$, $I=\{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{2,3,5,114,121,839,853,882,984\}$.

With this design, the $\lambda$ in case 1 results in that more significant variables are selected into the sub-model than those in case 2 so that we can see the performance of the adjustment method.

### Table 4. MSE, PE and their standard errors with $n = 100, p = 1000, S = 10$

| $\rho$ | Case | $\text{MSE}$(stdMSE) | $\hat{\theta}$ | $\hat{\theta}_S$ | $\text{PE}$(stdPE) | $\hat{Y}$ | $\hat{Y}_S$ | $S$ | $S$ | $\tau$ |
|-------|------|-----------------------|----------------|------------------|-----------------|---------|-----------|----|----|------|
| 0.1   | 1    | 0.7588(0.3497)        | 71.4031(7.5501)| 6.8104(1.5485)  | 8.0107(1.6574)| 94.7515(19.2968) | 200/200 |     |     |      |
| 2     | 0.8523(0.5343) | 122.8426(15.0952) | 13.1274(2.7772)| 16.0812(3.4160)| 189.7134(34.8081)| 200/200 |     |     |      |
| 0.5   | 1    | 3.6170(1.1823)        | 104.8420(13.5089)| 9.9151(1.9902) | 11.2352(2.2316)| 133.4762(26.5058)| 200/200 |     |     |      |
| 2     | 3.4771(1.2683) | 92.3485(12.5122) | 11.6643(2.6704)| 12.7811(2.8941)| 134.3821(24.4896)| 200/200 |     |     |      |
| 0.9   | 1    | 5.9027(2.7039)        | 107.6118(23.4383)| 8.2842(1.6181) | 11.3518(2.1745)| 148.3143(27.4828)| 200/200 |     |     |      |
| 2     | 3.8963(2.1760) | 59.1525(11.3152) | 10.8033(2.1411)| 12.9395(2.4835)| 68.7272(13.4061)| 200/200 |     |     |      |

From Table 4, we can see that the SIS does work to reduce the dimension so that the Gaussian Dantzig selector and our method can be performed. Whether the correlation coefficient is small or large (the values of $\rho$ change from 0.1 to 0.9), the new method works better than the Gaussian Dantzig selector. The conclusions are almost identical to those when $p$ is much smaller in Experiments 1 and 2. Thus,
we do not give more comments here. Further, when comparing the results of case 1 and case 2, we can see that the adjustment can work better when the submodel is not well selected. The value of $\rho = 1$.

In the following we check the effect of model size when the dimension is larger. In doing so, we choose $n = 150, p = 2000, \rho = 0.3$ with $S = 5, 10$. For each $S$ we choose two $\lambda_p$ to obtain two $\hat{I}$.

For $S=5$, $\beta_I = (4.0, -1.5, 6.0, -2.1, -3.0)'$, we consider two cases:
Case 1. $\lambda_p=3.45$, $I = \{1,2,3,4,5\}$, $\hat{I} = \{1, 2, 3, 5, 1099, 1733\}$;
Case 2. $\lambda_p=8.36$, $I = \{1,2,3,4,5\}$, $\hat{I} = \{1, 3, 554, 908\}$.

For $S=10$, $\beta_I = (4.0, -1.5, 6.0, -2.1, -3.0, 1.2, 3.8, -2.5, -2.0, 7.0)'$, consider two cases:
Case 1. $\lambda_p=3.02$, $I = \{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{1, 2, 3, 5, 7, 8, 9, 10, 1701\}$;
Case 2. $\lambda_p=9.08$, $I = \{1,2,3,4,5,6,7,8,9,10\}$, $\hat{I} = \{1, 3, 5, 7, 8\}$.

Table 5. MSE, PE and their standard errors with $n = 150, p = 2000, \rho = 0.3$

| $S$ | Case | $\hat{\theta}$ | $\hat{\theta}_S$ | $\hat{Y}$ | $\hat{Y}_S$ |
|-----|------|----------------|-----------------|-----------|-------------|
| 5   | 1    | 0.4245(0.2102) | 262.6392(21.2109) | 6.4015(1.3038) | 6.3439(1.2879) | 322.9945(62.6228) | 200/200 |
| 10  | 2    | 1.9510(1.0923) | 359.5838(32.4150) | 24.1959(4.8932) | 24.8013(5.1629) | 559.3584(98.1216) | 200/200 |
| 5   | 1    | 0.8799(0.5108) | 498.7862(59.0383) | 10.6009(2.3903) | 12.3505(2.6381) | 946.3400(175.1009) | 200/200 |
| 10  | 2    | 1.8524(0.7599) | 68.1862(43.3612)  | 15.0471(2.8069) | 16.9161 (3.1755) | 1623.4936(111.5972)| 200/200 |

The results in Table 5 show that the SIS is again useful for reducing the dimension for the use of the Gaussian Dantzig selector and our method. When the model size is smaller, estimation accuracy can be better with smaller MSE and PE. In other words, when the model size is smaller, variable selection can perform better and sub-model can be more accurate (case 1 with $S = 5$), the adjustment method does not have much help, and in contrast, it is useful for improving estimation accuracy.
when the sub-model is very different from the full model.

In summary, the results in the five tables above clearly show the superiority of the new estimator $\hat{\theta}$ and the new sub-model (3.2)/the sub-model (2.4) over the others in the sense with smaller MSEs, PEs and standard errors, and large proportion $\tau$. The good performance holds for different combinations of the sizes of selected sub-models (values of $\lambda_p$), $n, p, S, I, R^2$ and the correlation between the components of $X$. The new method is particularly useful when a submodel, as a working model, is very different from underlying true model. Thus, the adjustment method is very worth of recommendation. However, as a trade-off, the adjustment method involves nonparametric estimation, although low-dimensional ones, it might not be that helpful when the sub-model is accurate enough. Thus, we may consider using it after a check whether the submodel is significantly biased. The relevant research is ongoing.

Appendix

Proof of (3.4) Note that

\[
E(\xi(V)|Z,V) = E(Y - \theta'Z - g(V)|Z,V) \\
= E(Y - \theta'Z|Z,V) - E(g(V)|Z,V) \\
= E(\gamma'U + \varepsilon|Z,V) - E(E(\gamma'U + \varepsilon|V)|Z,V) \\
= \gamma'E(U|Z,V) - \gamma'E(U|V) \\
= \gamma'E(U|Z,\alpha'U/\rho,W) - \gamma'E(U|\alpha'U/\rho,W). 
\]
Further,
\[
E(\gamma'U/\rho|Z = z, \alpha'U/\rho = t, W = w) = E(\gamma'U/\rho|Z = z, U^{(1)} = (t\rho - \sum_{j=2}^{l} U^{(j)}\alpha_j)/\alpha_1, W = w) \\
= E(\frac{2\gamma}{\alpha_1}\rho (t\rho - \sum_{j=2}^{l} U^{(j)}\alpha_j) + \sum_{j=2}^{l} U^{(j)}\gamma_j/\rho|Z = z, W = w) \\
= E(\frac{2\gamma}{\alpha_1}t + \sum_{j=2}^{l} U^{(j)}(\gamma_j - \frac{2\gamma}{\alpha_1}\alpha_j)|Z = z, W = w) \\
= \frac{2\gamma}{\alpha_1}t + \lambda' E(U|Z = z, W = w).
\]
Similarly, \(E(\gamma'U/\rho|W = w, U'\gamma/\rho = t) = \frac{2\gamma}{\alpha_1}t + \lambda' E(U|W = w)\). Combining the above results leads to
\[
E(\xi(V)|Z, V) = \rho\lambda'(E(U|Z, W) - E(U|W)),
\]
as required.

**Proof of Theorem 3.1** The proof follows directly from the unbiasedness of the model (3.2) for any \(\alpha\) and Theorem 2.1.2 of Härdle et al (2000).

**Proof of Theorem 3.2** Denote \(V^* = (\gamma'U/\rho^*, W')\) and \(\rho^* = O(\|\gamma^*\|_2\sqrt{\lambda_M})\), where \(\gamma^*\) is a \(l\)-dimensional vector between \(\tilde{\gamma}D\) and \(\alpha\). Then there exists a vector \(\gamma^*\) such that
\[
\frac{1}{n} \sum_{k=1}^{n} L_H(\hat{V}_k - v) = \frac{1}{n} \sum_{k=1}^{n} L_H(V_k - v) + \frac{1}{n} \sum_{k=1}^{n} \hat{L}_H(V^*_k - v)(\hat{V}_k - V_k),
\]
where \(\hat{L}_H(\cdot)\) is the derivative of \(L_H(\cdot)\). By the conditions (C1) and (C5), we have
\[
\hat{V}_k - V_k = O_p(n^{-\mu})
\]
and, consequently,
\[
\frac{1}{n} \sum_{k=1}^{n} \hat{L}_H(V^*_k - v)(\hat{V}_k - V_k) = O_p(n^{-\mu}).
\]
By standard nonparametric technique, see Härdle, et al (2000), it is easy to have

$$\frac{1}{n} \sum_{k=1}^{n} L_H(V_k - v) - f_V(v) = O_p(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}),$$

and then

$$\frac{1}{n} \sum_{k=1}^{n} L_H(\hat{V}_k - v) - f_V(v) = O_p(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}) + O_p(n^{-\mu}),$$

where $f_V$ is the density function of $V$. Similarly, we can prove

$$\frac{1}{n} \sum_{k=1}^{n} Z_k L_H(\hat{V}_k - v) - \int z f_{Z,V}(z,v) dz = O_p(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}) + O_p(n^{-\mu}),$$

where $f_{Z,V}$ is the joint density function of $(Z,V)$. Combining the results above leads to $\hat{Z} = Z - E(Z|V) + O_p(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}) + O_p(n^{-\mu})$ and, consequently,

$$S_n - S = O_p(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}) + O_p(n^{-\mu}).$$

Further, by the definition of $\hat{\theta}$ and the above result, we have

$$\hat{\theta} - \theta = S^{-\frac{1}{n}} \sum_{i=1}^{n} \left( \hat{Z}_i \hat{g}(\hat{V}_i) + \hat{Z}_i \hat{\xi}(\hat{V}_i) \right) \left\{ 1 + O_p(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}) + O_p(n^{-\mu}) \right\},$$

where

$$\hat{g}(\hat{V}_i) = g(\hat{V}_i) - \frac{1}{n} \sum_{k=1}^{n} g(V_k) L_H(V_k - \hat{V}_i),$$

$$\hat{\xi}(\hat{V}_i) = \xi(\hat{V}_i) - \frac{1}{n} \sum_{k=1}^{n} \xi(V_k) L_H(V_k - \hat{V}_i).$$

Again by the conditions (C1) and (C4), we have

$$\hat{g}(\hat{V}_i) = g(V_i) - \frac{1}{n} \sum_{k=1}^{n} g(V_k) L_H(V_k - V_i) + O_p(n^{-\mu}) = \hat{g}(V_i) + O_p(n^{-\mu}),$$

$$\hat{\xi}(\hat{V}_i) = \xi(V_i) - \frac{1}{n} \sum_{k=1}^{n} \xi(V_k) L_H(V_k - V_i) + O_p(n^{-\mu}) = \hat{\xi}(V_i) + O_p(n^{-\mu}),$$

$$\hat{Z}_i = Z_i - \frac{1}{n} \sum_{k=1}^{n} Z_k L_H(V_k - V_i) + O_p(n^{-\mu}) = \hat{Z}_i + O_p(n^{-\mu}).$$
Note that, under the condition (C4),
\[
\tilde{g}(V_i) = O_p\left(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}\right) = O_p\left(n^{-k/(2(2k+d+1))}\right),
\]
\[
\tilde{\xi}(V_i) = O_p\left(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}\right) = O_p\left(n^{-k/(2(2k+d+1))}\right),
\]
\[
\tilde{Z}_i = O_p\left(h^k + \frac{1}{\sqrt{nh^{2(d+1)}}}\right) = O_p\left(n^{-k/(2(2k+d+1))}\right).
\]
Therefore combining the above results can complete the proof of the theorem.

Proof of Corollary 3.3  The proof follows directly from the result of Theorem 3.2 and Theorem 2.12 of Härdle et al (2000).

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