ON SUBGROUPS OF THE DIXMIER GROUP AND CALOGERO-MOSER SPACES

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Abstract. We describe the structure of the automorphism groups of algebras Morita equivalent to the first Weyl algebra $A_1(k)$. In particular, we give a geometric presentation for these groups in terms of amalgamated products, using the Bass-Serre theory of groups acting on graphs. A key role in our approach is played by a transitive action of the automorphism group of the free algebra $k\langle x, y \rangle$ on the Calogero-Moser varieties $C_n$ defined in [5]. In the end, we propose a natural extension of the Dixmier Conjecture for $A_1(k)$ to the class of Morita equivalent algebras.

1. Introduction. Let $k$ be an algebraically closed field of characteristic 0. Let $A_1(k) := k\langle x, y \rangle/(xy - yx - 1)$ be the first Weyl algebra over $k$, with canonical generators $x$ and $y$. In his classic paper [11], J. Dixmier described the group $\text{Aut}_k A_1$ of automorphisms of $A_1(k)$; specifically, he proved that $\text{Aut}_k A_1$ is generated by the following transformations:

$$
\Phi_p : (x, y) \mapsto (x, y + p(x)) \quad \text{and} \quad \Psi_q : (x, y) \mapsto (x + q(y), y),
$$

where $p(x) \in k[x]$ and $q(y) \in k[y]$. Using this result of Dixmier, L. Makar-Limanov (see [15, 16]) showed that $\text{Aut}_k A_1$ is isomorphic to the group $G_0 \subset \text{Aut}_k k\langle x, y \rangle$ of ‘symplectic’ (i.e. preserving $[x, y] := xy - yx$) automorphisms of the free algebra $k\langle x, y \rangle$: the corresponding isomorphism

$$
G_0 \cong \text{Aut}_k A_1
$$

is induced by the canonical projection $k\langle x, y \rangle \to A_1(k)$. In [15], Makar-Limanov also gave a presentation for the group $G_0$, from which one can easily deduce (see, e.g., [10]) that $G_0$ is the amalgamated free product

$$
G_0 = A * U \ast B,
$$

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where $A$ is the subgroup of symplectic affine transformations
\[(4) \quad (x, y) \mapsto (ax + by + c, cx + dy + f), \quad a, b, \ldots, f \in k, \quad ad - be = 1,\]
$B$ is the subgroup of triangular (Jonquières) transformations
\[(5) \quad (x, y) \mapsto (ax + q(y), a^{-1}y + h), \quad a \in k^*, \quad h \in k, \quad q(y) \in k[y],\]
and $U$ is the intersection of $A$ and $B$ in $G_0$:
\[(6) \quad (x, y) \mapsto (ax + by + e, a^{-1}y + h), \quad a \in k^*, \quad b, e, h \in k.\]
Combining (2) and (3), we thus have the decomposition $\text{Aut}_k A_1 \cong A \ast_U B$, which completely describes the structure of $\text{Aut}_k A_1$ as a discrete group (cf. [1]).

The aim of the present paper is to generalize the above results to the case when $A_1$ is replaced by a noncommutative domain $D$, which is Morita equivalent to $A_1$ as a $k$-algebra. This question was originally raised by J. T. Stafford in [19] (see op. cit., p. 636). To explain why it is natural we recall that the algebras $D_k$ completely describes the structure of $\text{Aut}_k A_1$ and motivation of our results; detailed proofs and computations will appear elsewhere.

2. Calogero-Moser spaces and the Makar-Limanov isomorphism. Recall that $G_0$ is the automorphism group of the free algebra $k(x, y)$ preserving $[x, y]$. Now, for $n > 0$, we introduce the groups $G_n$ geometrically, using a natural action of $G_0$ on the Calogero-Moser spaces
\[(7) \quad C_n := \{(X, Y) \in M_n(k) : \text{rk}([X, Y] + I_n) = 1) / \text{PGL}_n(k),\]
where $M_n(k)$ denotes the space of $n \times n$ matrices with entries in $k$, and $\text{PGL}_n(k)$ operates on pairs of matrices $(X, Y)$ by simultaneous conjugation. The action of $G_0$ on $C_n$ is defined by
\[(8) \quad (X, Y) \mapsto (\sigma^{-1}(X), \sigma^{-1}(Y)), \quad \sigma \in G_0,\]
where $\sigma^{-1}(X)$ and $\sigma^{-1}(Y)$ are the noncommutative polynomials $\sigma^{-1}(x) \in k(x, y)$ and $\sigma^{-1}(y) \in k(x, y)$ evaluated at $(X, Y)$. In [21], G. Wilson showed that $C_n$ is a smooth irreducible affine variety of dimension $2n$, equipped with a natural symplectic form: $\omega = \text{tr}(dX \wedge dY)$. Knowing that $G_0$ is generated by triangular transformations (1), it is easy to see that the action (8) is symplectic. Much less obvious is the fact that (8) is transitive for all $n \geq 0$. This last fact was proven in [5] (Theorem 1.3), and it plays a crucial role in the present paper.

Now, we define the groups $G_n$ to be the stabilizers of points of $C_n$ under the action (8): precisely, for each $n \geq 0$, we fix a basepoint $(X_0, Y_0) \in C_n$, with
\[(9) \quad X_0 = \sum_{k=1}^{n-1} E_{k+1,k}, \quad Y_0 = \sum_{k=1}^{n-1} (k - n) E_{k,k+1},\]
where $E_{i,j}$ stands for the elementary matrix with $(i,j)$-entry 1, and let

$$G_n := \text{Stab}_{G_0}(X_0, Y_0), \quad n \geq 0.$$  

The following result can be viewed as a generalization of the above-mentioned theorem of Makar-Limanov.

**Theorem 1.** There is a natural isomorphism of groups $G_n \simto \text{Aut}_k D_n$.

To construct the isomorphism of Theorem 1, we first note that the groups $\text{Aut}_k D_n$ can be naturally identified with subgroups of $\text{Aut}_k D_0$. To be precise, let $\text{Pic}_k D$ denote the (noncommutative) Picard group of a $k$-algebra $D$. By definition, $\text{Pic}_k D$ is the group of $k$-linear Morita equivalences of the category of $D$-modules; its elements can be represented by the isomorphism classes of invertible $D$-bimodules $[P]$ (see, e.g., [3], Ch. II, Sect. 5). There is a natural group homomorphism $\omega_D : \text{Aut}_k D \to \text{Pic}_k D$, taking $\sigma \in \text{Aut}_k D$ to the class of the bimodule $[\sigma D]$, and if $D'$ is a ring Morita equivalent to $D$, with progenerator $M$, then there is a group isomorphism $\alpha_M : \text{Pic}_k D' \simto \text{Pic}_k D$ given by

$$[P] \mapsto [M^* \otimes_{D'} P \otimes_D M].$$

Thus, in our situation, for each $n \geq 0$, we have the following diagram

$$
\begin{array}{ccc}
\text{Aut}_k D_n & \xrightarrow{\omega_D} & \text{Pic}_k D_n \\
\downarrow{i_n} & & \downarrow{\alpha_{M_n}} \\
\text{Aut}_k D_0 & \xrightarrow{\omega_D} & \text{Pic}_k D_0
\end{array}
$$

where the vertical map $\alpha_{M_n}$ is an isomorphism and the two horizontal maps are injective. Moreover, since $D_0 = A_1$, a theorem of Stafford (see [19], Theorem 4.7) implies that $\omega_{D_0}$ is actually an isomorphism. Inverting this isomorphism, we define the embedding $i_n : \text{Aut}_k D_n \hookto \text{Aut}_k D_0$, which makes (10) a commutative diagram. Now, we have group homomorphisms

$$G_n \hookto G_0 \simto \text{Aut}_k A_1 \xhookto \text{Aut}_k D_n,$$

where the first map is the canonical inclusion and the second is Makar-Limanov’s isomorphism (2). The claim of Theorem 1 is that the image of $i_n$ coincides with the image of $G_n$ in $\text{Aut}_k A_1$; this gives the required isomorphism $G_n \simto \text{Aut}_k D_n$.

Theorem 1 is a consequence of the main results of [5]. In fact, it is shown in [5] that there is a natural $G_0$-equivariant bijection between $\bigsqcup_{n \geq 0} C_n$ and the space of isomorphism classes of right ideals of $A_1$. This bijection can be described explicitly as follows (see [4]). A point of $C_n$ is represented by a pair of linear endomorphisms $(X, Y)$ of $k^n$ satisfying the condition that $[X, Y] + I_n$ has rank 1. Factoring $[X, Y] + I_n = ij$ in $\text{End}(k^n)$, with $i \in k^n$ and $j \in \text{Hom}(k^n, k)$, we define $\chi(X, Y) := 1 + j(X - x I_n)^{-1}(Y - y I_n)^{-1}i$ as an element in the quotient field of $A_1(k)$ and assign to $(X, Y)$ the fractional ideal

$$M(X, Y) = \det(X - x I_n) A_1 + \chi(X, Y) \cdot \det(Y - y I_n) A_1.$$

The assignment $(X, Y) \mapsto M(X, Y)$ induces a map from $C_n$ to the set of isomorphism classes of right ideals of $A_1$; amalgamating such maps for all $n$ yields the required bijection. Notice that $M(X_0, Y_0) = M_n$ for $X_0$ and $Y_0$ given by (9), so our basepoints $(X_0, Y_0) \in C_n$ correspond precisely to the classes of the ideals $M_n$ whose endomorphism rings are the algebras $D_n$. 


3. $G_n$ as a fundamental group. We will use Theorem 1 to give a geometric presentation for the groups $\text{Aut}_k D_n$. To this end, we associate to each space $C_n$ a graph $\Gamma_n$ consisting of orbits of certain subgroups of $G_0$ and identify $G_n$ with the fundamental group $\pi_1(\Gamma_n, \ast)$ of a graph of groups $\Gamma_n$ defined by stabilizers of those orbits in $\Gamma_n$. The Bass-Serre theory of groups acting on graphs will then give an explicit formula for $\pi_1(\Gamma_n, \ast)$ in terms of generalized amalgamated products.

To state our results in precise terms we recall the notion of a graph of groups and its fundamental group, following Serre’s classic book [17]. The readers unfamiliar with Bass-Serre theory are recommended to skim §5 of Chapter I of [17].

A graph of groups $\Gamma = (\Gamma, G)$ consists of the following data: (1) a connected oriented graph $\Gamma$ with vertex set $V = V(\Gamma)$, edge set $E = E(\Gamma)$ and incidence maps $i, t : E \to V$, (2) a group $G_a$ assigned to each vertex $a \in V$, (3) a group $G_e$ assigned to each edge $e \in E$, (4) a pair of injective group homomorphisms $\alpha_e : G_e \hookrightarrow G_{i(e)}$ and $\beta_e : G_e \hookrightarrow G_{t(e)}$ defined for each $e \in E$. Associated to $\Gamma$ is the path group $\pi(\Gamma)$, which is given by the presentation

$$\pi(\Gamma) := \left( \ast_{a \in V} G_a \ast \langle E \rangle \right) / \{ e \ast g = \alpha_e(g) \ast e, \forall e \in E, \forall g \in G_e \} ,$$

where ‘$\ast$’ stands for the free product (= coproduct in the category of groups) and $\langle E \rangle$ for the free group with basis set $E = E(\Gamma)$. Now, choosing a maximal tree $T$ in $\Gamma$, we define $\pi_1(\Gamma, T)$, the fundamental group of $\Gamma$ relative to $T$, as a quotient of $\pi(\Gamma)$ by ‘contracting the edges of $T$ to a point’: precisely,

$$\pi_1(\Gamma, T) := \pi(\Gamma)/\{ e = 1 : \forall e \in E(T) \} .$$

For different maximal trees $T \subseteq \Gamma$, the groups $\pi_1(\Gamma, T)$ are isomorphic. Moreover, if $\Gamma$ is trivial (i.e. $G_a = \{ 1 \}$ for all $a \in V$), then $\pi_1(\Gamma, T)$ is isomorphic to the usual fundamental group $\pi_1(\Gamma, a_0)$ of the graph $\Gamma$ viewed as a CW-complex. In general, $\pi_1(\Gamma, T)$ can be also defined in a topological fashion by introducing an appropriate notion of path and homotopy equivalence of paths in $\Gamma$.

When its underlying graph is a tree $(\Gamma = T)$, $\Gamma$ can be viewed as a directed system of groups $\{ G_{i(e)} \xrightarrow{\alpha_e} G_e \xrightarrow{\beta_e} G_{t(e)} \}$ indexed by the edges of $T$. In this case, the fundamental group $\pi_1(\Gamma, T)$ is given by the inductive limit $\varinjlim \Gamma$, which is called the tree product of groups $\{ G_a \}$ amalgamated by $\{ G_e \}$ along $T$. For example, if $T$ is a segment with $V(T) = \{ 0, 1 \}$ and $E(T) = \{ e \}$, the tree product is the usual amalgamated free product $G_0 \ast_{G_e} G_1$. In general, abusing notation, we will denote the tree product by

$$G_{a_1} \ast G_{a_2} \ast G_{a_3} \ast \cdots$$

Now, we return to our situation. To define the graph $\Gamma_n$ we take the subgroups $A, B$ and $U$ of $G_0$ defined by the transformations (4), (5) and (6), respectively. Restricting the action of $G_0$ on $C_n$ to these subgroups, we let $\Gamma_n$ be the oriented bipartite graph, with vertex and edge sets

$$V(\Gamma_n) := (A \setminus C_n) \bigsqcup (B \setminus C_n) , \quad E(\Gamma_n) := U \setminus C_n ,$$

and the incidence maps $E(\Gamma_n) \to V(\Gamma_n)$ given by the canonical projections $i : U \setminus C_n \to A \setminus C_n$ and $t : U \setminus C_n \to B \setminus C_n$. Since the elements of $A$ and $B$ generate $G_0$ and $G_0$ acts transitively on each $C_n$, the graph $\Gamma_n$ is connected.

Now, on each orbit in $A \setminus C_n$ and $B \setminus C_n$, we choose a basepoint and elements $\sigma_A \in G_0$ and $\sigma_B \in G_0$ moving these basepoints to the basepoint $(X_0, Y_0)$ of $C_n$. 


Next, on each $U$-orbit $O_U \in U \setminus C_n$, we also choose a basepoint and an element $\sigma_U \in G_0$ moving this basepoint to $(X_0, Y_0)$ such that $\sigma_U \in \sigma_A A \cap \sigma_B B$, where $\sigma_A$ and $\sigma_B$ correspond to the (unique) $A$- and $B$-orbits containing $O_U$. Then, we assign to the vertices and edges of $\Gamma_n$ the stabilizers $A_\sigma = G_n \cap \sigma A \sigma^{-1}$, $B_\sigma = G_n \cap \sigma B \sigma^{-1}$, $U_\sigma = G_n \cap \sigma U \sigma^{-1}$ of the corresponding elements $\sigma$ in the graph of right cosets of $G_0$ under the action of $G_n$. These data together with natural group homomorphisms $\alpha_\sigma : U_\sigma \rightarrow A_\sigma$ and $\beta_\sigma : U_\sigma \rightarrow B_\sigma$ define a graph of groups $\Gamma_n$ over $\Gamma_n$, and its fundamental group $\pi_1(\Gamma_n, T)$ relative to a maximal tree $T \subseteq \Gamma_n$ has canonical presentation, cf. (11):

$$\pi_1(\Gamma_n, T) = \frac{(A_\sigma \ast U_\sigma B_\sigma \ast \ldots) \ast \langle E(\Gamma_n \setminus T) \rangle}{(e^{-1} \alpha_\sigma(g) e = \beta_\sigma(g) : \forall e \in E(\Gamma_n \setminus T), \forall g \in U_\sigma)}.$$ 

In (13), the amalgam $(A_\sigma \ast U_\sigma B_\sigma \ast \ldots)$ stands for the tree product taken along the edges of $T$, while $\langle E(\Gamma_n \setminus T) \rangle$ denotes the free group based on the set of edges of $\Gamma_n$ in the complement of $T$.

Our main result is the following:

**Theorem 2.** For each $n \geq 0$, the group $G_n$ is isomorphic to $\pi_1(\Gamma_n, T)$. In particular, $G_n$ has an explicit presentation of the form (13).

Theorems 1 and 2 reduce the problem of describing the groups $\text{Aut}_k D_n$ to a purely geometric problem of describing the structure of the orbit spaces of $A$ and $B$ and $U$ on the Calogero-Moser varieties $C_n$. Using the earlier results of [21] and [5] and some geometric invariant theory, one can obtain much information about these orbits (and hence about the groups $G_n$). In particular, the graphs $\Gamma_n$ can be completely described for small $n$; it turns out that $\Gamma_n$ is a finite tree for $n = 0, 1, 2$ (see examples below), but has infinitely many cycles for $n \geq 3$.

### 4. The graphs $\Gamma_n$ and an adelic Grassmannian

We now explain the origin of the graphs $\Gamma_n$ by realizing them as quotient graphs of a certain ‘universal’ tree $\Gamma$, on which all the groups $\text{Aut}_k D_n$ naturally act. Our construction of $\Gamma$ is motivated by algebraic geometry: specifically, an application of the Bass-Serre theory in the theory of surfaces (see, e.g., [13], [23]). In that approach, the automorphism group of an affine surface $S$ is described via its action on a tree, whose vertices correspond to certain (admissible) projective compactifications of $S$. Following a standard philosophy in noncommutative geometry (see, e.g., [20]), we may think of our algebra $D$ as the coordinate ring of a ‘noncommutative affine surface’; a ‘projective compactification’ of $D$ is then determined by a choice of filtration. Thus, we will define $\Gamma$ by taking as its vertices a certain class of filtrations on the algebra $D$. It turns out that these filtrations can be naturally parametrized by an infinite-dimensional adelic Grassmannian $\text{Gr}^{ad}$ introduced in [22] and studied in [21, 5, 8] (in particular, we rely heavily on results of [8]). Our construction is close in spirit to Serre’s classic application of Bruhat-Tits trees for computing arithmetic subgroups of $\text{SL}_2(\mathbb{K})$ over the function fields of curves (see [17], Chap. II. §2); however, we are not aware of a direct connection.

We begin by briefly recalling the definition of $\text{Gr}^{ad}$. Let $k[z]$ be the polynomial ring in one variable $z$. For each $\lambda \in k$, we choose a $\lambda$-primary subspace in $k[z]$; that is, a $k$-linear subspace $V_\lambda \subseteq k[z]$ containing a power of the maximal ideal $m_\lambda$ at $\lambda$. We suppose that $V_\lambda = k[z]$ for all but finitely many $\lambda$’s. Let $V = \bigcap_\lambda V_\lambda$.
(such a subspace $V$ is called primary decomposable in $k[z]$) and, finally, let

$$W = \prod_{\lambda} (z - \lambda)^{-n_\lambda} V \subset k(z),$$

where $n_\lambda$ is the codimension of $V_\lambda$ in $k[z]$. By definition, $\text{Gr}^{ad}_k$ consists of all subspaces $W \subset k(z)$ obtained in this way. For each $W \in \text{Gr}^{ad}_k$ we set

$$A_W := \{f \in k[z] : fW \subseteq W\}.$$

Taking Spec of $A_W$ then gives a rational curve $X$, the inclusion $A_W \hookrightarrow k[z]$ corresponds to normalization $\pi : \mathbb{A}^1_k \rightarrow X$ (which is set-theoretically a birational map), and the $A_W$-module $W$ defines a rank 1 torsion-free coherent sheaf $\mathcal{L}$ over $X$. In this way, the points of $\text{Gr}^{ad}_k$ correspond bijectively to isomorphism classes of triples $(\pi, X, \mathcal{L})$ (see [22]).

Now, following [5], for $W \in \text{Gr}^{ad}_k$ we define¹

$$D(W) := \{D \in k(z)[\partial_z] : DW \subseteq W\},$$

where $k(z)[\partial_z]$ is the ring of rational differential operators in the variable $z$. This last ring carries two natural filtrations: the standard filtration, in which both generators $z$ and $\partial_z$ have degree 1, and the differential filtration, in which $\deg(z) = 0$ and $\deg(\partial_z) = 1$. These filtrations induce two different filtrations on the algebra $D(W)$, which we denote by $\{D^a(W)\}$ and $\{D^b(W)\}$ respectively.

Now, let $D$ be a fixed domain Morita equivalent to $A_1(k)$. Following [8], we consider the set² $\text{Gr}^{ad}_k(D)$ of all algebra isomorphisms $\sigma_W : D(W) \rightarrow D$, where $W \in \text{Gr}^{ad}_k$ (more precisely, $\text{Gr}^{ad}_k(D)$ is the set of all pairs $(W, \sigma_W)$, where $W \in \text{Gr}^{ad}_k$ and $\sigma_W$ is an isomorphism as above). Each $\sigma_W \in \text{Gr}^{ad}_k(D)$ maps the two distinguished filtrations $\{D^a(W)\}$ and $\{D^b(W)\}$ into the algebra $D$: we call their images the admissible filtrations on $D$ of type $A$ and type $B$, respectively. Let $\mathbb{P}_A(D)$ and $\mathbb{P}_B(D)$ denote the sets of all such filtrations coming from various $\sigma_W \in \text{Gr}^{ad}_k(D)$. By definition, we have then two natural projections

$$\mathbb{P}_A(D) \rightarrow \text{Gr}^{ad}_k(D) \rightarrow \mathbb{P}_B(D).$$

We say that $(W, \sigma_W)$ and $(W', \sigma_{W'})$ are equivalent in $\text{Gr}^{ad}_k(D)$ if their images under $\pi_A$ and $\pi_B$ coincide. Writing $\text{Gr}^{ad}_k(D)/\sim$ for the set of equivalence classes under this relation, we define an oriented graph $\Gamma$ by

$$V(\Gamma) := \mathbb{P}_A(D) \bigsqcup \mathbb{P}_B(D), \quad E(\Gamma) := \text{Gr}^{ad}_k(D)/\sim,$$

with incidence maps $E(\Gamma) \rightarrow V(\Gamma)$ induced by the projections (15). Observe that the group $\text{Aut}_k D$ acts naturally on the set $\text{Gr}^{ad}_k(D)$ (by composition), and this action induces an action of $\text{Aut}_k D$ on the graph $\Gamma$ via (15). We write $\text{Aut}_k(D)\backslash \Gamma$ for the corresponding quotient graph.

**Theorem 3.**

(a) $\Gamma$ is a tree, which is independent of $D$ (up to isomorphism).

(b) For each $n \geq 0$, the graph $\text{Aut}_k(D_n)\backslash \Gamma_n$ is naturally isomorphic to $\Gamma_n$.

¹In geometric terms, $D(W)$ can be thought of as the ring $D_L(X)$ of twisted differential operators on $X$ with coefficients in $L$.

²More generally, we may think of $\text{Gr}^{ad}_k$ as a groupoid, in which the objects are the $W$’s and the arrows are given by the algebra isomorphisms $D(W) \rightarrow D(W')$. For $D = D(W)$, the set $\text{Gr}^{ad}_k(D)$ is then a costar in $\text{Gr}^{ad}_k$, consisting of all arrows with target at $W$. In [8], this set was denoted by $\text{Grad} D$.
Theorem 3 can be viewed as a generalization of the main results of [8]. Indeed, this last paper is concerned with a description of the maximal abelian ad-nilpotent (mad) subalgebras of $D_n$: its main theorems (Theorem 1.5 and Theorem 1.6) say that the space $\text{Mad}(D_n)$ of all mad subalgebras of $D_n$ is independent of $D_n$ and its quotient modulo the natural action of $\text{Aut}_k D_n$ is isomorphic to the orbit space $B/C_n$. Now, every admissible filtration of type $B$ determines a mad subalgebra of $D_n$, which is simply the degree zero component of that filtration. (Indeed, by definition, a type $B$ filtration comes through an isomorphism from the usual differential filtration on some $D(W)$, but the degree zero component of the differential filtration is just $AW$, which is certainly a mad subalgebra of $D(W)$.) Thus, we have a well-defined map $\mathbb{P}_B(D_n) \to \text{Mad}(D_n)$. This map is injective, since each type $B$ filtration is maximal ad-nilpotent and hence determined by its degree zero component. On the other hand, Theorem 1.4 of [8] says that every mad subalgebra of $D_n$ arises from some $W \in \text{Gr}^{\text{ad}}$: this implies the surjectivity of the above map. Summing up, we have a natural bijection $\mathbb{P}_B(D_n) \cong \text{Mad}(D_n)$, which is equivariant under the action of $\text{Aut}_k D_n$. This means that $\mathbb{P}_B(D_n)$ does not depend on $D_n$, which is part of Theorem 3(a), and

$$\text{Aut}_k(D_n) / \mathbb{P}_B(D_n) \cong \text{Aut}_k(D_n) / \text{Mad}(D_n) \cong B/C_n,$$

which is part of Theorem 3(b). The main part of Theorem 3 does not follow directly from the results of [8], although its proof relies on techniques of that paper.

In the end, we should mention that, for $D = A_1(k)$, our construction of the tree $\Gamma$ agrees with the one given in [1].

5. Examples. We now look at the graphs $\Gamma_0$ and groups $G_n$ for small $n$. For $n = 0$, the space $C_0$ is just a point, and so are a fortiori its orbit spaces. The graph $\Gamma_0$ is thus a segment, and the corresponding graph of groups $\Gamma_0$ is given by $[A \xrightarrow{U} B]$. Formula (13) then says that $G_0 = A \ast_U B$, which agrees, of course, with Makar-Limanov’s isomorphism (3), and $G_0$ is generated by its subgroups

$$G_{0,x} := \{ \Phi_p \in G_0 : p \in k[x] \},$$
$$G_{0,y} := \{ \Psi_q \in G_0 : q \in k[y] \},$$

which is the Dixmier result cited in the Introduction.

For $n = 1$, we have $C_1 \cong A_1^2$, with $(X_0, Y_0)$ corresponding to the origin. Since each of the groups $A, B$ and $U$ contains translations $(x, y) \mapsto (x + a, y + b)$, $a, b \in k$, they act transitively on $C_1$. So again $\Gamma_1$ is just the segment, and $\Gamma_1$ is given by $[A_1 \xrightarrow{U_1} B_1]$, where $A_1 := G_1 \cap A$, $B_1 := G_1 \cap B$ and $U_1 := G_1 \cap U$. Since, by definition, $G_1$ consists of all $\sigma \in G_0$ preserving $(0, 0)$, the groups $A_1, B_1$ and $U_1$ are obvious:

$$A_1: (x, y) \mapsto (ax + by, cx + dy), \quad a, b, c, d \in k, \quad ad - bc = 1,$$
$$B_1: (x, y) \mapsto (ax + q(y), a^{-1}y), \quad a \in k^*, \quad q \in k[y], \quad q(0) = 0,$$
$$U_1: (x, y) \mapsto (ax + by, a^{-1}y), \quad a \in k^*, \quad b \in k.$$

It follows from (13) that $G_1 = A_1 \ast_U B_1$. In particular, $G_1$ is generated by its subgroups

$$G_{1,x} := \{ \Phi_p \in G_0 : p \in k[x], \quad p(0) = 0 \},$$
$$G_{1,y} := \{ \Psi_q \in G_0 : q \in k[y], \quad q(0) = 0 \}.$$
Now, for \( n = 2 \), the situation is more interesting. A simple calculation shows that \( U \) has three orbits in \( G_2 \): two closed orbits of dimension 3 and one open orbit of dimension 4. Moreover, the \( B \)-orbits coincide with the \( U \)-orbits. Combinatorially, this means that the group \( A \) acts transitively, and the graph \( \Gamma_2 \) is a tree with one nonterminal and three terminal vertices corresponding to the \( A \)-orbit and the \( B \)-orbits, respectively. In this case, the graph of groups \( \Gamma_2 \) is given by

\[
\begin{array}{c}
G_{2,y} \rtimes \Lambda \\
\downarrow \\
\Lambda \\
\downarrow \\
\Lambda \\
\downarrow \\
G_{2,x} \rtimes \Lambda
\end{array}
\]

where \( \Lambda \subset G_0 \) is the subgroup of scaling transformations \( (x, y) \mapsto (\lambda x, \lambda^{-1} y) \), \( \lambda \in k^* \), and the groups \( G_{2,x}, G_{2,y}, G_{2,y}^{(1)} \) are defined in terms of generators (1) by

\[
\begin{align*}
G_{2,x} &:= \{ \Phi_p \in G_0 : p \in k[x] , p(0) = p'(0) = 0 \} , \\
G_{2,y} &:= \{ \Psi_q \in G_0 : q \in k[y] , q(0) = q'(0) = 0 \} , \\
G_{2,y}^{(1)} &:= \{ \Phi_x \Psi_q \Phi_x \in G_0 : q \in k[y] , q(\pm 1) = 0 \} .
\end{align*}
\]

Formula (13) yields the presentation

(16) \[ G_2 = (G_{2,x} \rtimes \Lambda) \star_\Lambda (G_{2,y} \rtimes \Lambda) \star_{\mathbb{Z}_2} (G_{2,y}^{(1)} \rtimes \mathbb{Z}_2) . \]

In particular, \( G_2 \) is generated by its subgroups \( G_{2,x}, G_{2,y}, G_{2,y}^{(1)} \) and \( \Lambda \).

Using the above presentations, it is easy to see that the groups \( G_0, G_1 \) and \( G_2 \) are pairwise non-isomorphic. First of all, \( G_0 \) and \( G_1 \) are perfect groups, since they are generated by the triangular subgroups \( G_{0,x}, G_{0,y} \) and \( G_{1,x}, G_{1,y} \) respectively, while \( G_{i,x} = [\Lambda, G_{i,z}] \) and \( G_{i,y} = [\Lambda, G_{i,y}] \) for \( i = 0, 1 \). Hence, neither \( G_0 \) nor \( G_1 \) have nontrivial one-dimensional linear representations (characters) over \( k \). On the other hand, from our presentation (16) it follows that the natural isomorphism \( \Lambda \tilde{\rightarrow} k^* \) can be extended to a homomorphism \( G_2 \to k^* \) (by simply mapping \( G_{2,x}, G_{2,y}, G_{2,y}^{(1)} \) to the identity). Hence \( G_2 \) has at least one nontrivial character, and thus \( G_2 \not\cong G_0 \) and \( G_2 \not\cong G_1 \) as abstract groups. Now, the fact that \( G_0 \not\cong G_1 \) is probably known. One way to see it is to notice that \( G_0 \) has no nontrivial two-dimensional representations over \( k \), while \( G_1 \) does. Indeed, in every linear representation of \( G_0 \) on \( k^2 \) the translations \( \Phi_a \) and \( \Psi_b \) \( (a, b \in k) \) must act trivially, but then the obvious relations \( \Phi_a \Psi_q \Phi_a = \Psi_{q(y)} \Phi_a \) and \( \Psi_b \Phi_p(x) = \Phi_{p(x-b)} \Psi_b \) between the generators (1) imply that the whole \( G_0 \) acts trivially. On the other hand, the group \( G_1 \) is the stabilizer of a point \( (X_0, Y_0) \) under the action of \( G_0 \) on \( C_1 \); as explained in [5], Sect. 11, this action is algebraic, hence \( G_1 \) acts linearly on the tangent space at \( (X_0, Y_0) \). Since \( C_1 \) is smooth of dimension 2, we get a two-dimensional linear representation of \( G_1 \), which is certainly nontrivial.
6. Questions. We end this paper with some questions and conjectures.

1. It is proved in [15] that $G_0$ is isomorphic to the group $\text{SAut} \mathbb{A}_k^2$ of symplectic automorphisms of the affine plane $\mathbb{A}_k^2$ (as in the case of the Weyl algebra, the isomorphism $G_0 \cong \text{SAut} \mathbb{A}_k^2$ is induced by the canonical projection $k\langle x, y \rangle \to k[x, y]$). Thus, the groups $G_n$ can be naturally identified with subgroups of $\text{Aut} \mathbb{A}_k^2$. Do these last subgroups have a geometric interpretation?

2. In this paper, we have described the structure of $G_n$ and $\text{Aut}_k D_n$ as discrete groups. However, these two groups carry natural algebraic structures and can be viewed as infinite-dimensional algebraic groups (in the sense of Shafarevich [18]). Despite being isomorphic to each other as discrete groups, they are not isomorphic as algebraic groups (for $n = 0$, this phenomenon was observed in [5].) A natural question is to explicitly describe the algebraic structures on $G_n$ and $\text{Aut}_k D_n$; in particular, to compute the corresponding (infinite-dimensional) Lie algebras. The last question was an original motivation for our work. For $G_0$, the answer is known (see [12]).

3. Compute the homology of the groups $G_n$ for all $n$. Again, for $n = 0$, the answer is known (see [2]): $H_*(G_0, \mathbb{Z}) \cong H_*(\text{SL}_2(k), \mathbb{Z})$. One may wonder whether the groups $H_*(G_n, \mathbb{Z})$ are strong enough invariants to distinguish the algebras $D_n$ up to isomorphism. Unfortunately, the answer is ‘no’: in fact, it follows from our description of $G_1$ that $H_*(G_1, \mathbb{Z}) \cong H_*(\text{SL}_2(k), \mathbb{Z})$. However, for $n \geq 2$, it seems that the groups $H_*(G_n, \mathbb{Z})$ are neither isomorphic to $H_*(\text{SL}_2(k), \mathbb{Z})$ nor to each other, so they may provide interesting invariants.

4. Finally, we would like to propose an extension of the well-known Dixmier Conjecture for $A_1(k)$ (see [11], Problème 11.1) to the class of Morita equivalent algebras. We recall that if $D$ is a domain Morita equivalent to $A_1$, then there is a unique integer $n \geq 0$ such that $D \cong D_n$, where $D_n$ is the endomorphism ring of the right ideal $M_n = x^n A_1 + (y + nx^{-1}) A_1$. For two unital $k$-algebras $A$ and $B$, we denote by $\text{Hom}_k(A, B)$ the set of all unital $k$-algebra homomorphisms $A \to B$.

Conjecture 1. For all $n, m \geq 0$, we have
\[
\text{Hom}_k(D_n, D_m) = \begin{cases} 
0 & \text{if } n \neq m \\
\text{Aut}_k D_n & \text{if } n = m
\end{cases}
\]

Formally, Conjecture 1 is a strengthening of the Dixmier Conjecture for $A_1$: in fact, in our notation, the latter says that $\text{Hom}_k(D_0, D_0) = \text{Aut}_k D_0$. Does the Dixmier Conjecture actually imply Conjecture 1?

References

[1] J. Alev, Action de groupes sur $A_1(\mathbb{C})$, Lecture Notes in Math. 1197, Springer, Berlin, 1986, pp. 1–9. MR 0859378
[2] R. C. Alperin, Homology of the group of automorphisms of $k[x, y]$, J. Pure Appl. Algebra, 15 (1979), 109–115. MR 0535479
[3] H. Bass, Algebraic $K$-theory, W. A. Benjamin Inc., New York-Amsterdam, 1968. MR 0249491
[4] Yu. Berest and O. Chalykh, $\mathbb{A}_\infty$-modules and Calogero-Moser spaces, J. reine angew Math., 607 (2007), 69–112. MR 2338121
[5] Yu. Berest and G. Wilson, Automorphisms and ideals of the Weyl algebra, Math. Ann., 318 (2000), 127–147. MR 1785579
[6] Yu. Berest and G. Wilson, Classification of rings of differential operators on affine curves, Internat. Math. Res. Notices, 2 (1999), 105–109. MR 170188
Yu. Berest and G. Wilson, *Ideal classes of the Weyl algebra and noncommutative projective geometry* (with an Appendix by M. Van den Bergh), Internat. Math. Res. Notices, 26 (2002), 1347–1396. MR 1904791

Yu. Berest and G. Wilson, *Mad subalgebras of rings of differential operators on curves*, Adv. Math., 212 (2007), 163–190. MR 2319766

Yu. Berest and G. Wilson, *Differential isomorphism and equivalence of algebraic varieties* in “Topology, Geometry and Quantum Field Theory” (Ed. U. Tillmann), London Math. Soc. Lecture Note Ser. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 98–126. MR 2079372

P. M. Cohn, *The automorphism group of the free algebras of rank two*, Serdica Math. J., 28 (2002), 255–266. MR 1952011

J. Dixmier, *Sur les algèbres de Weyl*, Bull. Soc. Math. France, 96 (1968), 299–242. MR 2142897

V. Ginzburg, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett., 8 (2001), 377–400. MR 1839485

M. H. Gizatullin and V. I. Danilov, *Automorphisms of affine surfaces, I, II*, Math. USSR Izv., 9 (1975), 493–534; Math. USSR Izv., 11 (1977), 51–98. MR 0376701

K. M. Kouakou, “Isomorphismes Entre Algèbres d’opérateurs Différentielles sur les Courbes Algébriques Affines,” Thèse de Doctorat, Université Claude Bernard-Lyon I, 1994.

L. Makar-Limanov, *Automorphisms of a free algebra with two generators*, Funct. Anal. Appl., 4 (1970), 262–264. MR 0271161

L. Makar-Limanov, *On automorphisms of the Weyl algebra*, Bull. Soc. Math. France, 112 (1984), 359–363. MR 0794737

J.-P. Serre, “Trees,” Springer-Verlag, Berlin, 1980. MR 0607504

I. R. Shafarevich, “Collected Mathematical Papers,” Springer, Berlin, 1989, pp. 430, 607. MR 0977275

J. T. Stafford, *Endomorphisms of right ideals of the Weyl algebra*, Trans. Amer. Math. Soc., 299 (1987), 623–639. MR 0869225

J. T. Stafford and M. Van den Bergh, *Noncommutative curves and noncommutative surfaces*, Bull. Amer. Math. Soc., 38 (2001), 171–216. MR 1816070

G. Wilson, *Collisions of Calogero-Moser particles and an adelic Grassmannian* (with an Appendix by I. G. Macdonald), Invent. Math., 133 (1998), 1–41. MR 1626461

G. Wilson, *Bispectral commutative ordinary differential operators*, J. reine angew. Math., 442 (1993), 177–204. MR 1234841

D. Wright, *Two-dimensional Cremona groups acting on simplicial complexes*, Trans. Amer. Math. Soc., 331 (1992), 281–300. MR 1038019

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