Commensurability of two-multitwist pseudo-Anosovs

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Abstract

This paper analyzes commensurability of the class of surface automorphism generated by two Dehn multitwists. We show pairwise noncommensurability between several classes arising from canonical curve configurations. In addition, we consider the Kenyon–Smillie invariant $J$ of flat surfaces in this setting. We also introduce a general construction of infinite classes of commensurable pseudo-Anosov homeomorphisms.

1 Introduction

Danny Calegari, Hongbin Sun, and Shicheng Wang, in [1], introduced a relation called commensurability among surface automorphisms. An mapping class $\tilde{\phi} \in \text{MCG}(\tilde{S})$ covers $\phi \in \text{MCG}(S)$ if we have a covering $p: \tilde{S} \rightarrow S$ such that $p\tilde{\phi} = \phi p$, and two mapping classes $\phi_1 \in \text{MCG}(S_1)$ and $\phi_2 \in \text{MCG}(S_2)$ are said to be commensurable if there are nonzero numbers $m, n \in \mathbb{Z}$ such that $\phi_1^m$ and $\phi_2^n$ are both covered by some map $\tilde{\phi}$. In Section 2, we deal with generalities about this relation.

In [7], Bill Thurston introduces a construction for a class of pseudo-Anosovs we call two-multiwist pseudo-Anosovs. He associates to each pair of filling multicurves in a surface the subgroup of the mapping class group generated by Dehn twists around the multicurves, and shows that “most of” the elements of this group are pseudo-Anosov. This construction is appealing in its simplicity and allows some invariants of pseudo-Anosovs to be calculated more easily than in the general case. We review this construction in Section 3.

In the following Section 4, we review an invariant $J$ of flat structures introduced by Richard Kenyon and John Smillie in [4]. Since the Thurston construction allows us to determine a flat structure preserved by a collection of pseudo-Anosovs, within this class of pseudo-Anosovs $\mathbb{R}^+ J$ gives a useful commensurability invariant.

Chris Leininger, in [3], studies the Thurston construction to understand for which Teichmüller curves the associated stabilizers contain with finite index a group generated by two positive multi-twists. In Section 5 we look at the invariants of some of these groups determining for some pairs $G, H$ of groups that no element of $G$ is commensurable with an element of $H$.

Finally, in Section 6, we introduce a simple construction that produces many commensurabilities between different elements of different two-multitwist groups. The idea is to find covers that either lift or uniformly break multicurves, and to consider Dehn twists in the covers.

1.1 Acknowledgements

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2 Generalities

If $S$ is a compact surface, we will refer to the isotopy class $\phi = [f]$ of a homeomorphism $f$ of $S$ as an automorphism of $S$. We will also refer to an automorphism by the pair $(S, \phi)$.
**Definition 1.** An automorphism $(\tilde{S}, \tilde{\phi})$ covers $(S, \phi)$ if there are a finite covering $p: \tilde{S} \to S$ and $\tilde{\phi}$ and $\phi$ have representatives $\tilde{f}$ and $f$ such that $p \circ \tilde{f} = f \circ p: \tilde{S} \to S$. We then write $p: \tilde{\phi} \to \phi$ or $p_{\#}(\tilde{\phi}) = \phi$.

**Definition 2.** Automorphisms $(S_1, \phi_1)$ and $(S_2, \phi_2)$ are commensurable if there are an automorphism $(\tilde{S}, \tilde{\phi})$ and nonzero integers $m, n \in \mathbb{Z}$ such that $(\tilde{S}, \tilde{\phi})$ covers $(S_1, \phi_1^m)$ and $(S_2, \phi_2^n)$. If $|m| = |n| = 1$ we say $(S_1, \phi_1)$ and $(S_2, \phi_2)$ are topologically commensurable and if $\tilde{S} = S_1 = S_2$ we say they are dynamically commensurable.

The relation of commensurability is of interest to 3-manifold theorists because if $(S_1, \phi_1)$ and $(S_2, \phi_2)$ are commensurable, the mapping tori associated to these automorphisms are 3-manifolds that admit a common finite-sheeted cover, that is, are commensurable as 3-manifolds [8]. The especial interest of the pseudo-Anosov case is that the interior of the mapping torus of $(S, \phi)$ is a complete hyperbolic 3-manifold that fibers over $S^1$, by work of Thurston [3].

If we have a covering $p: \tilde{S} \to S$, the set of elements of $\text{MCG}(\tilde{S})$ that cover an element of $\text{MCG}(S)$ under $p$ is a subgroup $H_p$, and the covering relation is a homomorphism $p_{\#}: H_{\tilde{p}} \to \text{MCG}(S)$. This is because the covering relation $p_{\#}$ distributes over composition, in the sense that if $p: \tilde{\phi} \to \phi$ and $p: \tilde{\psi} \to \psi$, then

$$p(\tilde{\phi}\tilde{\psi}) = \phi p\tilde{\psi} = \phi \psi,$$

i.e., $\phi \psi = p_{\#}(\tilde{\phi}\tilde{\psi})$. Thus if we have two covering maps $p_1: \tilde{S} \to S_1$ and $p_2: \tilde{S} \to S_2$, the set of elements of $\text{MCG}(\tilde{S})$ that cover both through $p_1$ elements of $S_1$ and through $p_2$ elements of $S_2$ is a subgroup $H$, and elements of $(p_1)_{\#}H$ are each commensurable with some element of $(p_2)_{\#}(H)$.

Associated with each $(S, \phi)$, where $\phi$ is pseudo-Anosov, are a number of invariants that are useful in detecting commensurability, as given in [4]:

1. whether or not $\partial S = \emptyset$;
2. the commensurability class of the mapping torus,
3. the commensurability class in $\mathbb{R}$ of $\log(\lambda)$, where $\lambda$ is the expansion factor of the pseudo-Anosov homeomorphism $f \in \phi$;
4. the set of orders of the singular points of the invariant foliations of $f$.

The set of orders of singular points in the invariant foliations of $f$ can be more helpfully viewed as an infinite vector.

**Definition 3.** Define $\delta_n(f)$ to be the number of $n$-prong singularities in the invariant foliation associated to a pseudo-Anosov $f$, and write $\delta(f) = (\delta_n(f))_{n \in \mathbb{N}^+\setminus\{2\}}$ for the infinite-length vector describing the singularity data for that foliation. (The case $n = 2$ is excluded because a 2-prong is not a singularity.)

In [4] it is shown that the rational commensurability class of $\delta(f)$ is a commensurability invariant of the mapping class $\phi$, so that a pair of pseudo-Anosovs $\phi$ and $\psi$ can be commensurable only if there is some rational number $q \in \mathbb{Q}^+$ such that $q\delta(\phi) = \delta(\psi)$; that is, $q\delta_n(\phi) = \delta_n(\psi)$ for each $n$.

## 3 Thurston’s two-multitwist construction of pseudo-Anosovs

Let $a$ and $b$ be multicurves on a surface $S$, and write $T_a, T_b$ for the mapping classes of the Dehn multitwists around $a$ and $b$. Write $G(a, b)$ for the subgroup $\langle T_a, T_b \rangle$ of the mapping class group $\text{MCG}(S)$. We now, following [7], create a flat structure on $S$ depending on $a$ and $b$. To do this, we first create the dual cell decomposition $\Sigma(a, b)$ of $S$ into rectangular cells, dual to the decomposition of $S$ determined by $a \cup b$; we then assign lengths to each rectangle in such a way that $T_a$ and $T_b$ act by affine transformations.

To get $\Sigma(a, b)$, draw a small rectangle in the surface at each intersection of an $a$ curve and a $b$ curve, with sides transverse to the curves. Expand these rectangles until the surface is covered. After expansion, a rectangle at an intersection point $p$ of $a_j$ and $b_k$ will share a side with just the rectangles drawn around intersection points on $a_j$ and on $b_k$ that are adjacent to $p$. Thus rectangles are ultimately glued if they lie along the same curve and there are no intervening rectangles between them on the same curve. This is called
the dual cell decomposition $\Sigma = \Sigma(a, b)$ to that determined by $a \cup b$, and if the cell decomposition of $S$ determined by $a \cup b$ has $V$ vertices, $E$ edges, and $F$ faces, the dual decomposition has $F$ vertices, $E$ edges, and $V$ faces. For example, if $a \cup b$ does not separate $S$, the cell decomposition has one face, so the dual cell decomposition will have one vertex. In general, one can show that if a face in the cell decomposition determined by $a \cup b$ is bounded by $2n$ edges, the corresponding vertex in the dual cell decomposition will be an $n$-prong if the squares are foliated by lines parallel to the $a$ curves. The natural product foliations of the Euclidean rectangles induce a pair of transverse singular foliations on $S$; the singularities are those vertices of the dual cell decomposition that are the result of identifying $2n \neq 4$ rectangle corners. Call $\delta(a, b)$ the infinite vector whose $n$th entry is the number of $n$-prongs of either foliation The entries of $\delta(a, b)$ are intimately related with the genus $g$ of $S$, by the Hopf index theorem: if $S$ is closed, we have $4(g - 1) = \sum n(n - 2)\delta_n$. Thus if $a \cup b$ does not separate $S$, we have $4g - 4 = n - 2$, so $\delta = \epsilon_{4g-2}$; that is, the single singularity of the foliation is a $(4g - 2)$-prong.

To illustrate the formation of the dual cell decomposition, we do an example. In part (a) of the figure on the above, we have drawn a multicurve configuration $a, b$ filling a surface $S$. We see that $S \setminus (a \cup b)$ has two components. If we count the number of edges bounding the left region, we see there are 12 (six in front, and another six on the side we can’t see) and 20 edges bounding the right one. In (b), we draw squares, which we will grow to be faces of the dual decomposition $\Sigma(a, b)$, centered at the vertices $a \cap b$. Now there are six squares along the edges bounding the left region of the original cell decomposition. Altogether, they have 24 corners, 12 of which lie in the left region, and 12 in the right. Since the dual decomposition is to have one vertex corresponding to the left region, when we expand the squares to fill the surface, the left vertex will have a total angle of $12(\pi/2) = 6\pi$. Since each separatrix has an angle of $\pi$, then, the left vertex of $\Sigma(a, b)$ will be a 6-prong. Similarly, the right region contains $6 \cdot 2 + 2 \cdot 4 = 20$ square corners, so the right vertex of $\Sigma(a, b)$ will be a 10-prong. In (c), we move the centers of the squares in such a way that half of each square lies along the front of the surface, and half along the back, which we can’t see. Since the picture along the back is symmetrical, we can focus on the front, implicitly doing the same thing in the back. We expand the squares along the red curves until adjacent squares along the red curves have had their sides identified. We color the square corners that are to be identified to the left vertex in $\Sigma(a, b)$ light green, and label them $x$,
and color those to be identified with the right one a darker fuchsia, and label them $y$. In (d), we make two side identifications of squares. There is one green corner left in the front of the picture, and three fuchsia ones, which are yet to be identified. In (e), we identify these three fuchsia endpoints. This involves wrapping the edge with two fuchsia endpoints around the circle on the right with one fuchsia endpoint. Once we’ve done that, the picture has one fuchsia point in front and one in back, and one green point in front and one in back. The only sides of squares left unidentified are two outermost ones in the front and their mirror images in back, all parallel to the thick black border of the picture. In (f), we drag these edges and vertices to the black border and identify them. This finally is the dual cell decomposition.

In order to assign lengths to $a = \{a_1, \ldots, a_m\}$ and $b = \{b_1, \ldots, b_n\}$, associate a bipartite graph $\Gamma(a, b)$ as follows. For each curve $a_j$ and $b_k$ create a vertex, and create one edge between $a_j$ and $b_k$ for each intersection. Associated with this graph is $N = N(\Gamma(a, b))$, an $|a| \times |b|$ incidence matrix whose $(j, k)$ entry is $(N)_{j,k} = i(a_j, b_k)$, the geometric intersection number of $a_j$ and $b_k$. (or the number of edges between $a_j$ and $b_k$ in $\Gamma(a, b)$.)

Assume the graph $\Gamma(a, b)$ is connected; then the square matrix $NN^\top$ has nonnegative integer entries, and some power of $NN^\top$ has strictly positive entries. It therefore has a Perron–Frobenius eigenvalue \([2]\): there is a unique positive real eigenvalue $\mu = \mu(a, b)$ of multiplicity one and such that $\mu > |\mu'|$ for all other eigenvalues $\mu'$ of $NN^\top$. The Perron–Frobenius eigenvalue has a Perron–Frobenius eigenvector $v$ all of whose entries are positive. Let $v' = \mu^{-1/2}NN^\top v$. Then $NN^\top v = \mu^{-1/2}NN^\top (\mu v) = \mu^{-1/2}NN^\top v = \mu v'$.

We complete the flat structure $X = X(a, b)$ by giving each rectangle corresponding to an intersection of $a_j$, considered as running horizontally and $b_k$, considered as running vertically, the metric structure of the Euclidean rectangle $[0, v'_k] \times [0, v_j]$.

To see that the multitwists $T_a$ and $T_b$ act in an affine way on this structure, consider doing a Dehn twist around the curve $a_j$. The squares containing arcs of $a_j$ are the $\sum_k i(a_j, b_k)$ squares induced by intersections with curves $b_k$. These squares are isometric to $[0, v'_k] \times [0, v_j]$, so their union is a cylindrical neighborhood of $a_j$ with height $v_j$ and circumference

$$\sum_k i(a_j, b_k)v'_k = (Nv')_j = (\mu^{-1/2}NN^\top v)_j = \mu^{-1/2}\mu v_j = \mu^{1/2}v_j.$$ 

Putting coordinates in $[0, \mu^{1/2}v_j] \times [0, v_j]$ on this cylinder, doing a right-handed Dehn twist on this cylinder takes $(x, y) \mapsto (x + \frac{y}{\mu^{1/2}}(\mu^{1/2}v_j), y) = (x + y\mu^{1/2}, y)$. Since this is the same on every cylinder, $T_a$ has derivative given everywhere by

$$\begin{pmatrix} 1 & \sqrt{\mu} \\ 0 & 1 \end{pmatrix}.$$ 

Similarly, taking all the squares meeting $b_k$ gives a cylinder neighborhood with height $v'_k$ and circumference

$$\sum j v_j \cdot i(a_j, b_k) = (v^\top N)_k = (\mu^{-1/2}NN^\top v')_k = \mu^{1/2}v'_k,$$

and a Dehn multitwist around $b$ (right-handed!) gives an affine map with derivative

$$\begin{pmatrix} 1 & 0 \\ -\sqrt{\mu} & 1 \end{pmatrix}.$$ 

By the chain rule, then, an element $f$ of $G(a, b)$ always has a derivative $Df$. Since the matrices have determinant 1, the eigenvalues of $Df$ multiply to 1. In fact, the eigenvalues of $Df$ are given by $x^2 - \text{tr}(Df) x + 1$, so the eigenvalues are complex conjugates, both $\pm 1$, or real $> 1$ and inverse depending on whether the absolute value of the trace is less than, equal to, or greater than 2, respectively. If $Df$ has real eigenvalues, then invariant foliations for $f$ are given by the eigenvectors for $Df$, and leaves of these foliations are expanded and contracted by their corresponding eigenvalues $\lambda^{\pm 1}$. Thus if $Df$ has real eigenvalues, $f$ is a pseudo-Anosov homeomorphism. The singularities of the natural foliation on $X(a, b)$ give rise to the singularities of the invariant foliations for $f \in G(a, b)$. This is part of the following theorem.

**Theorem 1** (Thurston \([7]\)). The derivative of the action of the multi-twists on the flat structure gives rise to a representation $\rho$: $G(a, b) \to \text{PSL}(2, \mathbb{R})$, given by

$$T_a \mapsto \begin{pmatrix} 1 & \sqrt{\mu} \\ 0 & 1 \end{pmatrix}, \quad T_b \mapsto \begin{pmatrix} 1 & 0 \\ -\sqrt{\mu} & 1 \end{pmatrix}.$$ 

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The kernel of $\rho$ is finite and there is some $k$-fold covering group $G_k \to PSL(2, \mathbb{R})$ such that $\rho$ lifts to a faithful representation in $G_k$; if $a \cup b$ fills $S$, then $k$ is finite. The image of $\rho$ is a discrete subgroup of $PSL(2, \mathbb{R})$, and is free just if $\sqrt{k} \geq 2$. For $g \in G(a, b)$, the image $\rho(g)$ is elliptic (or the identity), parabolic, or hyperbolic just if $g$ is (respectively) finite-order, reducible, or pseudo-Anosov. If $\rho(g)$ is reducible, some power of $g$ is a multi-twist, and if $\rho(g)$ is hyperbolic, its larger eigenvalue is the expansion factor $\lambda(g)$ of $g$.

If $a \cup b$ fills, then Thurston showed “most of” the classes in the two-generator subgroup $G(a, b) := \langle T_a, T_b \rangle$ of $MCG(S)$ are pseudo-Anosov — all of them except $\langle T_a \rangle$, $\langle T_b \rangle$, and possibly $T_a^{\pm 1}T_b^{\pm 1}$.

**Lemma 1.** If $g \in G(a, b)$ is pseudo-Anosov, the expansion factor $\lambda(g)$ is an algebraic integer quadratic over $\mathbb{Z}[\mu(a, b)]$.

**Proof.** Note that $\rho(T_a) = \begin{pmatrix} 1 & \sqrt{\mu} \\ 0 & 1 \end{pmatrix}$ and $\rho(T_b) = \begin{pmatrix} 1 & 0 \\ -\sqrt{\mu} & 1 \end{pmatrix}$ have diagonal entries (here 1) in $\mathbb{Z}[\mu]$ and anti-diagonal entries in $\sqrt{\mu} \cdot \mathbb{Z}[\mu]$. The product of two such matrices again has this form:

$$
\begin{pmatrix} p(\mu) & q(\mu) \sqrt{\mu} \\ r(\mu) \sqrt{\mu} & s(\mu) \end{pmatrix} \begin{pmatrix} p'(\mu) & q'(\mu) \sqrt{\mu} \\ r'(\mu) \sqrt{\mu} & s'(\mu) \end{pmatrix} = \begin{pmatrix} (pp') + \mu(qr' + gs') & \sqrt{\mu}(pq' + qs') \\ \sqrt{\mu}(rp' + sq') & (qq') + \mu(rs') \end{pmatrix}.
$$

In particular, the trace of such a matrix is in $\mathbb{Z}[\mu]$, so since the determinant is 1, the larger eigenvalue $\lambda(g)$ satisfies the equation $y^2 - p(\mu)y + 1 = 0$ for some $p(x) \in \mathbb{Z}[x]$. One has $\lambda = \frac{1}{2} \left( p(\mu) + \sqrt{p(\mu)^2 - 4} \right)$.

If two pseudo-Anosovs $g \in G(a, b)$ and $h \in G(a', b')$ are commensurable, then there are $n, m \in \mathbb{Z} \setminus \{0\}$ such that $\lambda(g)^n = \lambda(h)^m$, and these powers are in $\mathbb{Q}(\lambda(g)) \cap \mathbb{Q}(\lambda(h))$. Writing $\mu = \mu(a, b)$ and $\mu' = \mu(a', b')$, then there are polynomials $p, q \in \mathbb{Q}[x]$ and $r, s, t, u \in \mathbb{Q}[x]$ such that

$$r(p(\mu)) + s(p(\mu)) \sqrt{p(\mu)^2 - 4} = t(q(\mu')) + u(q(\mu')) \sqrt{q(\mu')^2 - 4}.$$

This would seem to put rather serious constraints on $\lambda(g), \lambda(h)$. For example, if $g, h \in G(a, b)$, then we have $\mathbb{Q}(\mu, \lambda(g))$ and $\mathbb{Q}(\mu, \lambda(h))$ quadratic over $\mathbb{Q}(\mu)$, so if $g, h$ are commensurable, then $\lambda(g)$ and $\lambda(h)$ have some common power, and if it isn’t in $\mathbb{Q}(\mu)$, then $\mathbb{Q}(\mu, \lambda(g)) = \mathbb{Q}(\mu, \lambda(h))/\mathbb{Q}(\mu)$. But a power of $\lambda$ isn’t in $\mathbb{Q}(\mu)$ unless $\sqrt{p(\mu)^2 - 4} \in \mathbb{Q}(\mu)$, in which case $\lambda$ itself is in $\mathbb{Q}(\mu)$.

We might compare, for example, the discriminants of $\mathbb{Q}(\mu, \lambda(g))$ and $\mathbb{Q}(\mu, \lambda(h))$ over $\mathbb{Q}(\mu)$ or $\mathbb{Q}$ with a number theory program.

## 4 The invariant $J$

Associated to a flat surface $X$ (see [9] for definitions), Kenyon and Smillie [4] associate an invariant $J$ in the rational vector space $\mathbb{R}^2 \otimes \mathbb{Q} \mathbb{R}^2$. To define it, one starts with a decomposition of the flat structure into planar polygons. For each planar polygon $P$, with vertices $v_1, \ldots, v_n$, define

$$J(P) = v_n \wedge v_1 + \sum_{j=1}^{n-1} v_j \wedge v_{j+1}.$$ 

Then if $X$ can be decomposed as a union of planar polygons $X = \bigcup_{j=1}^n P_j$, glued along edges by translations, define $J(X) = \sum_{j} J(P_j)$.

One can show that $J(X)$ is independent of the decomposition and that $J(P)$, $P$ a polygon is independent of translations. $J$ does vary under rotations, though. Suppose we rotate a polygon by an angle of $\theta$ around the origin. Making the canonical identification $\mathbb{C} \cong \mathbb{R}^2$, the new vertices are given by $e^{i\theta} v_j$, and so the new $J(P)$ is $e^{i\theta} v_n \wedge e^{i\theta} v_1 + \sum_{j=1}^{n-1} e^{i\theta} v_j \wedge e^{i\theta} v_{j+1}$. In particular, rotation by $\pi$ leaves $J$ unchanged.

A useful formula for $u$ is the following. Define the **edge vectors** by $e_1 = v_1 - v_n$ and $e_j = v_j - v_{j-1}$ for $j > 1$. Then for a rectangle $R$ with vertices $v_1, v_2, v_3, v_4$ arranged to be horizontal and vertical in the plane, we have $J(R) = 2e_1 \wedge e_2$.

Given a covering of surfaces $p: \tilde{S} \to S$, we say it is a covering of flat surfaces if we can put flat structures $\tilde{X}$ and $X$ on $\tilde{S}$ and $S$ given by quadratic differentials $\tilde{q}$ and $q$ such that $p^*q = \tilde{q}$

**Lemma 2.** Given an $n$-fold covering $p: \tilde{X} \to X$ of flat surfaces, $J(\tilde{X}) = nJ(X)$.  

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Theorem 2. For two commensurable pseudo-Anosovs \( \psi \) and \( \tilde{\psi} \), uniquely defined up to scale. That is, consider the map \( \Psi_{r,s}: \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( (x,y) \mapsto (rx,sy) \) for \( r,s \in \mathbb{R}^+ \), which induces a map \( \Psi_{r,s}: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 \) by \( v \wedge w \mapsto \psi_{r,s}(v) \wedge \psi_{r,s}(w) \); \( J \) is determined up to the map \( \Psi_{r,s} \).

**Theorem 2.** For two commensurable pseudo-Anosovs \( \phi_1 \) and \( \phi_2 \), with associated flat surfaces \( X_1 \) and \( X_2 \), there are \( r,s \in \mathbb{R}^+ \) with such that \( \Psi_{r,s}(J(X_1)) = J(X_2) \).

**Proof.** If \( \phi_1 \) and \( \phi_2 \) are commensurable, their invariant foliations lift to the same foliations \( \tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u \) under the surface coverings \( p_j: \tilde{S} \to X_j \). Since the common lift \( \tilde{\phi} \) must alter the transverse measures according to \( \tilde{\phi} \mathcal{F}^s = \frac{1}{k} \tilde{\mathcal{F}}^s \) and \( \tilde{\phi} \mathcal{F}^u = \lambda \tilde{\mathcal{F}}^u \), the transverse measures are uniquely determined up to one scaling factor each. Thus we can alter the measures on \( \tilde{\mathcal{F}}^s \) and \( \tilde{\mathcal{F}}^u \) induced by lifting the invariant foliations of \( \phi_1 \) to be equal to those induced by \( \phi_2 \). This induces \( J(X_1) \mapsto \Psi_{r,s}(J(X_2)) \) for some \( r',s' \in \mathbb{R}^+ \).

It may be that one of the covering maps \( p_j \) is orientation-reversing. In this case, this happens, we alter the flat structure of one of our surfaces, say \( X_1 \) by the map \( x \mapsto -x \). This induces the transformation \( J(X_1) \mapsto -J(X_1) \).

Now the two flat surfaces \( X_1 \) and \( X_2 \) have a common covering flat surface, then there is \( q \in \mathbb{Q}^+ \) such that \( qJ(X_1) = J(X_2) \). If we define the stable foliation to run “north-south’’ and the unstable to run “east-west,’’ then the frame is determined up to a rotation by \( \pi \).

Putting this all together, we have \( \pm q \Psi_{r,s}(J(X_1)) = J(X_2) \).

One free variable can be removed by requiring the total area of the surface to be 1, but there is more flexibility in this construction than we would like.

The natural invariant foliations for different pseudo-Anosovs in a group \( G(a,b) \) generally point in different directions, so the associated flat structures for different pseudo-Anosovs differ by \( v \wedge w \mapsto e^{i\theta} v \wedge e^{i\theta} w \). However, once we fix a scaling, this is the only difference between different flat structures for pseudo-Anosovs in a group \( G(a,b) \).

### 5 Some two-multitwist groups and their associated invariants

Leininger\footnote{Leininger [2] found precise conditions on the intersection graph \( \Gamma(a,b) \) of the multicurves \( a \) and \( b \) for the subgroup \( G(a,b) \) to be free: it is free just if the graph has some component that is not among the graphs \( A_j, D_j, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8, j \in \mathbb{N} \). The Teichmüller curves for which the associated stabilizers contain with finite index a group generated by two positive multi-twists are these and others corresponding to graphs \( \mathcal{P}_{2j}, \mathcal{Q}_j, \mathcal{R}_7, \mathcal{R}_8, \mathcal{R}_9, j \in \mathbb{N} \). These graphs are Dynkin diagrams with simple edges, pictures of which will appear in Figures 2–11.}

In this section, we describe the invariants \( \delta(a,b), \mu(a,b), \) and \( J(X(a,b)) \) for certain multicurve configurations \( a,b \). For pairs of multicurve configurations \( a,b \) and \( a',b' \), if \( \delta(G(a,b)) \) and \( \delta(G(a',b')) \) are not rationally commensurable, then no pseudo-Anosov element of \( G(a,b) \) is commensurable with any in \( G(a',b') \). The same is true if the invariants \( J \) differ other than by the action of \( S^1 \times \mathbb{R}^+ \).

In all of our pictures, \( a \) will be the red curves and \( b \) the light blue ones. For a bipartite graph \( \Gamma \) we show

1. a multicurve configuration \( a,b \) on a surface \( S \) such that \( \Gamma(a,b) = \Gamma \);

2. \( \Gamma \) itself;

3. the dual cell decomposition \( \Sigma(a,b) \) corresponding to invariant foliations for pseudo-Anosovs in \( G(a,b) \), with singularity orders written at vertices;
4. the flat structure $X(a, b)$ minus length information—this is basically the picture of $\Sigma(a, b)$ cut along a few edges and straightened out. Note that this picture is only an approximation; the cells are not really squares, but are just notated that way for uniformity of presentation.

We sometimes omit the picture of $\Sigma(a, b)$ on the surface in preference for the square-tile picture. The edge labelling in the former is preserved in the latter when both are presented. In the picture of $\Sigma(a, b)$, only one side of the surface is shown, for clarity, which amounts to an assumption the surface is not transparent. The other side looks the same, except for edges that would otherwise be along the dark black boundary, which I have pushed in into the visible side. The labels in parentheses are for “invisible” edges that lie wholly on the other side of $S$.

We remark that the pictures we have drawn are essentially unique in the following sense.

**Lemma 3** (Leininger [3]). Suppose $a \cup b$ fills $S$ and $a' \cup b'$ fills $S'$, and their incidence graphs $\Gamma(a, b)$ are the same. If $\Gamma(a, b)$ is a tree with all but possibly one vertex of valence $\leq 2$ and the remaining vertex of valence $\leq 3$, then there is a homeomorphism $S \to S'$ taking $a \cup b \to a' \cup b'$, up to adding marked points.
Our first item is the intersection graph \( A_n \). Shown below are cases \( n = 2, 4, 6 \). The groups \( G(A_n) \) are not free, according to Leininger’s result. The graph \( A_n \) determines a filling curve configuration uniquely up to conjugacy and adding punctures, by Lemma 3, so our picture is essentially unique. Since \( a \cup b \) does not separate the surface if \( n \) is even, we get that the singularity data \( \delta(A_{2n}) = e_{4n-2} \), a single \((4n - 2)\)-prong singularity. (If \( n = 1 \), \( \delta = e_2 = 0 \) has no singularities.) Since these vectors are rationally incommensurable for different \( n \), if \( \phi \in G(A_{2n}) \) and \( \psi \in G(A_{2m}) \) are commensurable pseudo-Anosovs, we must have \( m = n \).

![Figure 2: \( A_{2n} \)](image)

The incidence matrix \( N(A_2) = (1) \), so \( \mu = 1 \). The incidence matrix \( N(A_4) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \), so \( NN^\top = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \) and \( \mu = \frac{1}{2}(3 + \sqrt{5}) \). An eigenvalue \( \begin{pmatrix} x \\ y \end{pmatrix} \) then satisfies \( x + y = \mu x \), or \( y = (\mu - 1)x = \frac{1 + \sqrt{5}}{2}x \), where \( \mu - 1 = \sqrt{\mu} = \gamma \) is the golden ratio. Arbitrarily setting \( x = 1 \), we get a Perron–Frobenius eigenvector \( v = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} \). The associated eigenvector for \( N^\top \) is \( \mu^{-1/2}N^\top v = \gamma^{-1} \begin{pmatrix} \gamma^2 \\ \gamma \end{pmatrix} = \begin{pmatrix} \gamma \\ 1 \end{pmatrix} = v \) again. So the flat structure associated to \( A_4 \) consists of three rectangles of proportions \( \gamma \times 1, \gamma \times \gamma \), and \( 1 \times \gamma \). For this reason, this table is called the golden table; see [5]. The \( J \)-invariants for these rectangles are respectively \( 2(\gamma, 0) \land (0, 1), 2(\gamma, 0) \land (0, \gamma) \), and \( 2(1, 0) \land (0, \gamma) \), and so the \( J \)-invariant for this L-shaped table is
$2[(\gamma, 0) \wedge (0, 1) + (\gamma, 0) \wedge (0, \gamma) + (1, 0) \wedge (0, \gamma)]$.

In general the incidence matrix $N(A_{2n})$ has 1 on the diagonal and subdiagonal, and 0 elsewhere:

\[
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 1 \\
\end{pmatrix}
\]

Thus $NN^\top = \begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
1 & 2 & 1 & \cdots & 0 & 0 \\
0 & 1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & 1 \\
0 & 0 & 0 & \cdots & 1 & 2 \\
\end{pmatrix}$ has $(1, 2, \ldots, 2)$ along the diagonal, 1 along the super- and subdiagonals, and 0 elsewhere.

Note that up to reflections, the pattern of 1s in $N$ is the same as the pattern of crossings in the flat structure.
Shown below are configurations for the intersection graph $A_{2n+1}$, $n = 1, 2, 3$. The singularity data are $2e_{2n}$; that is, there are two $2n$-prong singularities and no others. (If $n = 1$, $\delta = 2e_2 = 0$.) Again, the different $G(A_{2n+1})$ have no commensurable pseudo-Anosovs. The singularity data doesn’t rule out commensurable pseudo-Anosovs in $G(A_{2m})$ and $G(A_{2n+1})$ if $4m - 2 = 2n$, so $n = 2m - 1$. In this case the surface carrying the configuration corresponding to $A_{2n+1}$ double covers that for $A_{2n}$. Commensurability between different elements of these groups must be ruled out in other ways.

Figure 3: $A_{2n+1}$

The incidence matrix $N(A_3) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so that $N^T N = (2)$ and $\mu = 2$. An eigenvector is $(1)$, and $2^{1/2} N^T (1) = \begin{pmatrix} 2^{-1/2} \\ 2^{-1/2} \end{pmatrix}$, so the flat structure $X(a, b)$ is two rectangles of dimensions $2^{-1/2} \times 1$. The invariant $J$ is then $4(2^{-1/2}, 0) \wedge (0, 1)$. 
The incidence matrix $N(A_5) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$, so that $N^T N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mu = 3$. An eigenvector is $v' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $3^{-1/2} N v = 3^{-1/2} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. The invariant $J$ is then $4(1,0) \wedge (0,1/\sqrt{3}) + 4(1,0) \wedge (0,2/\sqrt{3})$.

In general, $N(A_{2n+1})$ is a $(n+1) \times n$ matrix with 1s on the diagonal and subdiagonal and 0 elsewhere, so that $N^T N$ is an $n \times n$ matrix with 2 along the diagonal, 1 on the super- and subdiagonals, and 0 elsewhere:

$\begin{pmatrix} 2 & 1 & \cdots & 0 \\ 1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{pmatrix}$. In particular $\mu(A_7) = 2 + \sqrt{2}$, so $N^T N$ has eigenvector $v' = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$, and $NN^T$ has eigenvector $v = (2 + \sqrt{2})^{-1/2} \begin{pmatrix} 1 \\ 1 + \sqrt{2} \\ 1 + \sqrt{2} \end{pmatrix}$. The corresponding $J$-invariant is $4(1,0) \wedge (0,(2 + \sqrt{2})^{-1/2}) + 4(1,0) \wedge (0,1/\sqrt{2 + \sqrt{2}}) + 4(\sqrt{2},0) \wedge (0,1/\sqrt{2 + \sqrt{2}})$. Comparing this with the L-shaped table of $A_4$, we see that the covering of $A_4$ by $A_7$ is not a covering of flat surfaces, so pseudo-Anosovs in $G(A_4)$ do not lift to elements of $A_7$. 

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Shown below are configurations for the intersection graph $\mathcal{D}_{2n}$, $n = 2, 3, 4$. The singularity data are $\delta(\mathcal{A}_{2n+1}) = 2e_{2n} = \delta(\mathcal{D}_{2n+2})$, so they don’t rule out commensurability between pseudo-Anosovs in these groups. Again, the $G(\mathcal{D}_{2n})$ for different $n$ have no commensurable pseudo-Anosovs, and they also have no pseudo-Anosovs commensurable with elements of the $G(\mathcal{A}_{2n})$.

Figure 4: $\mathcal{D}_{2n}$

$$N(\mathcal{D}_4) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$ so $N^\top N = (3)$ and $\mu = 3$, and an eigenvector for $N^\top N$ is $v' = (1)$. Then $v = 3^{-1/2}Nv' = \begin{pmatrix} 3^{-1/2} \\ 3^{-1/2} \\ 3^{-1/2} \\ 3^{-1/2} \end{pmatrix}$, so $J = 6(1, 0) \wedge (0, 3^{-1/2})$.

$$N(\mathcal{D}_6) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},$$ so $N^\top N = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ and $\mu = \frac{1}{2}(5 + \sqrt{5})$. In general, $N$ is as predicted by the flat structure, and $N^\top N$ has $(3, 2, \ldots, 2)$ along the diagonal and 1 along the super- and subdiagonals:
The $D_{2n}$ groups can trivially be seen to not have commensurable elements with elements of the $A_j$ groups because the former exist on a punctured surface. If punctures are added to the $A_j$ surfaces, other invariants must be used.
Shown below are configurations for the intersection graph $D_{2n+1}$, $n = 2, 3$. The singularity data are $\delta = e_{4n-2}$, the same as that for $G(A_{2n})$. Again, the $G(D_{2n+1})$ for different $n$ have no commensurable pseudo-Anosovs, and they also have no pseudo-Anosovs commensurable with elements of the $G(A_{2n+1})$.

![Diagram of $D_{2n+1}$](image)

Figure 5: $D_{2n+1}$

$N(D_5)^T = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, so $N^T N = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mu(D_5) = 2 + \sqrt{2} = \mu(A_7)$. $\mu(D_7) = 2 + \sqrt{3}$. $N$ is as predicted by the flat structure, and $N^T N$ has $(3, 2, \ldots, 2, 1)$ along the diagonal and 1 along the sub- and superdiagonals:

$$
\begin{pmatrix}
3 & 1 & 0 & \cdots & 0 & 0 \\
1 & 2 & 1 & \cdots & 0 & 0 \\
0 & 1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & 1 \\
0 & 0 & 0 & \cdots & 1 & 1
\end{pmatrix}
$$
Our next graphs are $E_{2n}$, $n \geq 3$. Shown are the cases $n = 3, 4, 5$. Since $a \cup b$ does not separate, $\delta = e_{4n-2}$, and the different $G(E_{2n})$ have no commensurable pseudo-Anosovs. $G(E_6)$ and $G(E_8)$ are not free; all others are.

$\mu(E_6) = 2 + \sqrt{3}$ and in general $NN^T$ has $(1, 3, 2, \ldots, 2, 1)$ along the diagonal and $1$ along the super- and subdiagonals:

$\begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
1 & 3 & 1 & \cdots & 0 & 0 \\
0 & 1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & 1 \\
0 & 0 & 0 & \cdots & 1 & 1
\end{pmatrix}$. $T_aT_b$ for $E_{10}$ turns out to have the smallest $\lambda$ among all two-multitwist pseudo-Anosovs, and this is Lehmer’s number.
Consider the graphs $E_{2n+1}$, $n \geq 3$. Shown is $E_7$. We have $\delta = e_{2(n-1)} + e_{2(n+1)}$, so and the different $G(E_n)$ have no commensurable pseudo-Anosovs. $G(E_7)$ is not free; higher $G(E_{2n+1})$ are.

$\mu(E_7)$ is a root of $x^3 - 6x^2 + 9x - 3$, and in general $NN^\top$ has $(1, 3, 2, \ldots, 2)$ along the diagonal and 1

$$
\begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
1 & 3 & 1 & \cdots & 0 & 0 \\
0 & 1 & 2 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & 1 \\
0 & 0 & 0 & \cdots & 1 & 2
\end{pmatrix}
$$

along the super- and subdiagonals:
These graphs are called $\mathcal{P}_n$. Shown are the cases $n = 1, 2, 3, 4$. We have $\delta = 0$ for the first two, and $\delta = 4e_n$ for $n \geq 3$. Thus none of the different groups have commensurable pseudo-Anosovs.

For $n = 1$ we have $N = (2)$, so $NN^\top = (4)$ and $\mu = 4$. $v = (1)$ is an eigenvector, so $v' = \mu^{-1/2}N^\top v = \frac{1}{2}(2)(1) = (1)$ is an eigenvector for $N^\top N$, and $J = 4(1, 0) \wedge (0, 1)$.

For $n = 2$ we have $N = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, so $NN^\top = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. Thus again $\mu = 4$. $v = v' = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector, so $J = 8(1, 0) \wedge (0, 1)$.

For higher $n$ we have $N = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$, so that $NN^\top = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$. In all
cases, \( \mu = 4 \), with eigenvector \( v = v' = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \), so \( J = 4n(1, 0) \wedge (0, 1) \).

These graphs are called \( Q_{2n+1} \), \( n \geq 2 \). Shown are the cases \( n = 2, 3, 4 \). They have \( \delta = 2e_{2n-2} \) and \( \mu = 4 \).

Figure 9: \( Q_{2n+1} \)
These graphs are $Q_{2n}$, $n \geq 3$. Shown are the cases $n = 3, 4$. They have $\delta = 2e_{2n-3}$ and $\mu = 4$. 

Figure 10: $Q_{2n}$
Finally, the three graphs $\mathcal{R}_7, \mathcal{R}_8, \mathcal{R}_9$ are below. They are the only other connected graphs with $\mu = 4$. We have $\delta = 2e_6, 2e_4 + e_6,$ and $e_6 + e_{10}$ respectively in the three cases.

Figure 11: $\mathcal{R}_7, \mathcal{R}_8, \mathcal{R}_9$
Here are some more multicurve configurations, which have no canonical names.

Figure 12: Miscellaneous curve configurations

The first has $\delta = 2e_4$ and $\mu = 6$. The second has $\delta = 2e_4$ and $\mu = 4$. The third has $\delta = 2e_6$ and
\( \mu = 3 + \sqrt{5} \). It double covers a genus-two surface with an \( A_4 \) configuration, and we shall meet (a configuration homeomorphic to) it again below. The fourth has \( \delta = 2e_6 \) and \( \mu = \frac{1}{2}(5 + \sqrt{17}) \), and again double covers a genus-two surface with an \( A_4 \) configuration. The last has \( \delta = 4e_4 \) and \( \mu = \frac{1}{2}(7 + \sqrt{17}) \). It double covers an \( A_5 \) configuration on a genus-two surface.
These miscellaneous graphs have no names. Their $\delta$s are respectively $e_{14}$, $e_{14}$, $e_6 + e_{10}$, and $3e_6$.

Their $\mu$ are $\frac{1}{2}(5 + \sqrt{21})$, the greatest root of $x^3 - 6x^2 + 8x - 1$, the greatest root of $x^3 - 6x^2 + 8x - 2$, and $\frac{1}{2}(5 + \sqrt{13})$.  

Figure 13: More miscellaneous curve configurations
Here are some more miscellaneous graphs that don’t have specific names. Their singularity data are respectively $2e_3 + 2e_5$, $2e_4 + 2e_6$, $4e_3 + 2e_6$, $e_{14}$.

![Graphs and diagrams illustrating various curve configurations](image)

Figure 14: Even more miscellaneous curve configurations

Their $\mu$ are $\frac{1}{4}(5 + \sqrt{17})$, the greatest root of $x^3 - 8x^2 + 18x - 10$, $6 + \sqrt{20}$, and the largest root of $x^3 - 9x^2 + 11x - 2$. 
A nice class of examples is given by pairs of curves that fill a surface. The first three are part of an infinite sequence, and the last is the beginning of a different infinite family.

Figure 15: pairs of filling curves

Their \( N \) are 1-by-1 matrices \((n)\) with \( n \) the number of intersection points: \( n = 4(\text{genus} - 1) \) for the infinite series, and \( n = 6 \) for the last one. Thus for the sequence, \( NN^T = (n^2) = ([4(\text{genus} - 1)]^2) \), which of course has eigenvalue \( \mu = n^2 \), and for the last one, \( \mu = 36 \). They all have eigenvector \( v = (1) \), so that \( v' = \mu^{-1/2}N^Tv = \frac{1}{n}(n)(1) = 1 \), and \( J = 2n(1, 0) \wedge (0, 1) \).

One can calculate their \( \delta \) by rotating around vertices in the flat structure while counting angles, or by reasoning as follows. Each square in the flat structure has \( 2\pi \) of angle inside it, so there is \( 4 \cdot 2\pi = 8\pi \) of total angle around vertices in the first one. A rotation of \( \pi \) exchanges the components of the cut surface, so there are two vertices each with angle \( 4\pi \), and \( \delta = 2e_4 \). For the rest, one can notice that cutting along the curves results in two pieces at the ends, homeomorphic to the ones before, and \( 2(g - 2) \) pairwise equivalent components in between. Since there is \( 4(g - 1) \cdot 2\pi = 8(g - 1)\pi \) total angle around all vertices, removing the two end pieces leaves \( 8(g - 2)\pi \) of angle to be equally divided among \( 2(g - 2) \) vertices, meaning each has angle \( 4\pi \), and \( \delta = 2(g - 1)e_4 \). The last example, on cutting, falls into four components, one, front and center,
visibly a square (giving rise to a vertex of angle $2\pi$), and another, approximately behind it, a rectangle. Thus the remaining two vertices have combined angle $6 \cdot 2\pi - 2 \cdot 2\pi = 4 \cdot 2\pi$, so by symmetry, each has angle $4\pi$, and $\delta = 2e_2 + 2e_4 = 2e_4$. 
Here is a different construction of two curves filling a surface, for closed surfaces of genus $\geq 3$, generalizing the six-intersection pair on the last page. One takes genus-many of the one handled objects below and glues them to the singular surface next to them to produce a surface with two curves, illustrated to the right in the genus-three case. They have $\delta = \text{genus} \cdot e_4 + 2e_{\text{genus}}$ and $\mu = (3 \cdot \text{genus})^2$.

![Figure 16: More pairs of filling curves](image)

Consider the one-handled surface. On cutting along the blue arc, the red arc is divided into three segments, whose endpoints along what was the blue arc can be moved independently. Thus, on cutting along the red arcs, one gets three surfaces, shown below the one handled surface, all of them topological disks. The middle one becomes a cylinder on cutting the blue curve, and cutting the remaining red arc makes it a disc.

To see that the glued together surface with the two curves is filled by the curves, again one first cuts along the blue curve. The red arcs’ endpoints on the boundary of the picture are now free to move around, so one can move them to the configuration below.
Now cutting along the red arc gives two triangular discs in the middle and genus-many “handle”-shaped objects, which we saw before are actually disks.

6 A construction of some commensurable pseudo-Anosovs

The techniques of the last section allow us to conclude that several accessible groups contain no commensurability classes of pseudo-Anosovs in common. We are able, however, to construct some commensurable pseudo-Anosovs in different $G(a, b)$. Basically, the strategy is to lift Dehn twists and use that $p_\#$ is a homomorphism. We first adopt a strategy that yields compositions of twists covering other compositions of twists; but these will not lie in two-multitwist groups $G(a, b)$. On seeing why this strategy does not yield a covering of elements of one $G(a, b)$ group by another, a modification that does will become apparent.

In the picture below, we have a cyclic double cover $p: \tilde{S}_2 \rightarrow S$; the covering symmetry rotates $\tilde{S}_2$ counterclockwise by an angle of $\pi$ around the axis extending vertically through the middle handle. Put another way, to get $\tilde{S}_2$ from $S$, cut along $z$, clone the resulting doubly-punctured torus, and glue the two cut objects together along their boundaries in such a way that the resulting action of $\mathbb{Z}/2\mathbb{Z}$ is free..

We draw multicurves $a, b$ in an $A_4$ configuration on $S$. Note that the curves $a_1, a_2$, and $b_1$ lift to two pairs of curves each on $\tilde{S}_2$, but $b_2$ lifts to a pair of paths since it is cut by $z$. Let $\tilde{c} := b_2 b'_2$ be the concatenation
of these two paths, and let the other lifts be as labeled. Write \( \vec{a} = \{\vec{a}_1, \vec{a}_2, \vec{a}_2'\} \) and \( \vec{b} = \{\vec{b}_1, \vec{b}_1', \vec{c}\} \). This yields an \( \mathcal{A}_7 \) configuration on \( \vec{S}_2 \). \( z \) is the same curve that in the earlier pictures of the \( \mathcal{A}_7 \) cell decomposition, was labeled \( \tilde{z} \). If one cuts that picture along \( z \), clones, and glues, one gets the earlier picture of the \( \mathcal{A}_7 \) decomposition, the curves corresponding to the cut curve \( z \) there being labeled \( \tilde{z} \) and \( \tilde{z}' \).

The lengths of curves the corresponding flat structures, however, are not the same, (this shows up in the noncommensurability of the different \( J \) and we have seen therefore that none of the pseudo-Anosovs in \( G(a, b) \) lift to elements of \( G(\vec{a}, \vec{b}) \). We can find some compositions of Dehn twists in curves of \( a \) and \( b \) that do lift to \( G(\vec{a}, \vec{b}) \), but these will not be elements of \( G(a, b) \). Later, we will find a different cover and multicurves \( \vec{a}, \vec{b} \) such that elements of \( G(a, b) \) do lift to elements of \( G(\vec{a}, \vec{b}) \). As far as the present cover is concerned, we claim the Dehn multitwist \( T_{\vec{a}_1} \) \( T_{\vec{a}_1} \) covers the twist \( T_{a_1} \).

**Lemma 5.** Given an \( n \)-fold covering of surfaces \( p: \vec{S} \rightarrow S \), if a curve \( a \) in \( S \) lifts to \( n \) curves \( \vec{a}(1), \ldots, \vec{a}(n) \) in \( \vec{S} \), then \( p: \prod_{j=1}^{n} T_{\vec{a}(j)} \rightarrow T_{a} \).

**Proof.** Consider a small annular neighborhood \( A(a) \) of \( a \), and give \( A(a) \equiv [0, 1] \times S^1 \) coordinates \((t, \theta) \pmod{2\pi}\). The inverse image \( p^{-1} A \) is a disjoint union of \( n \) annuli \( A(\vec{a}(j)) \), each homeomorphic to \( A(a) \), around \( \vec{a}(j) \). We can assume the twist \( T_{a} \) to be supported on \( A(a) \), where can be written in local coordinates as \((t, \theta) \mapsto (t, \theta + 2\pi t)\). We can pull back these coordinates to coordinates \((\vec{t}(j), \vec{\theta}(j)) \) on \( A(\vec{a}(j)) \) and \( T_{\vec{a}(j)} \) can be taken to be supported on these annuli with local formulas \((\vec{t}(j), \vec{\theta}(j)) \mapsto (\vec{t}(j), \vec{\theta}(j) + 2\pi \vec{t}(j))\). Now if \((\vec{t}(j), \vec{\theta}(j)) \in A(\vec{a}(j)) \) we have

\[
p \prod_{k} T_{\vec{a}(k)}(\vec{t}(j), \vec{\theta}(j)) = p T_{\vec{a}(j)}(\vec{t}(j), \vec{\theta}(j)) = p(\vec{t}(j), \vec{\theta}(j) + 2\pi \vec{t}(j)) = (t, \theta + 2\pi t) = T_{a}(t, \theta) = T_{a} p(\vec{t}(j), \vec{\theta}(j)),
\]

so \( p: \prod_{k} T_{\vec{a}(k)} \big| A(\vec{a}(j)) \rightarrow T_{a} \big| A(a) \). On the complement \( p^{-1}(S \setminus A(a)) \) of the lifted annuli, the multitwist \( \prod_{k} T_{\vec{a}(j)} \) is the identity, as is \( T_{a} \) on \( S \setminus A(a) \), so we have \( p \prod_{k} T_{\vec{a}(j)} = p = T_{a} p \) there as well. Thus \( \prod_{k} T_{\vec{a}(j)} \) covers \( T_{a} \).

The same argument shows \( T_{\vec{a}_2} T_{\vec{a}_2} \) covers \( T_{a_2} \) and \( T_{b_1} T_{b_1} \) covers \( T_{b_1} \). Only a little subtler is that \( T_{\vec{c}} \) covers \( T_{b_2} \).

**Lemma 4.** Given an \( n \)-fold covering of surfaces \( p: \vec{S} \rightarrow S \), if a curve \( a \) in \( S \) lifts to \( n \) curves \( \vec{a}(1), \ldots, \vec{a}(m) \) in \( \vec{S} \), with \( p|_{\vec{a}(j)} \) an \( n_j \)-to-1 map. Let \( L = \text{lcm}\{n_j: j = 1, \ldots, m\} \). Then \( p: \prod_{j=1}^{n} T_{\vec{a}(j)}^{L/n_j} \rightarrow T_{a}^{2} \).

**Proof.** As before, we can take the maps to be the identity off of preselected annuli. Take \( A(a) \) to be an annulus around \( a \) and \( A(\vec{a}(j)) \) to be the component of \( p^{-1} A(a) \) containing \( \vec{a}(j) \). Restricting to these annuli, it will be enough to show that \( p T_{\vec{a}(j)} = T_{a}^{n_j} p \) on \( A(\vec{a}(j)) \). If \( a \) has coordinates \((t, \theta) \in [0, 1] \times [0, 2\pi) \) as before, the lift \( A(a(j)) \) is \( n_j \) rectangular neighborhoods with these coordinates, attached end to end, so they naturally carry coordinates \((\vec{t}, \vec{\theta}) \in [0, 1] \times [0, 2n_j \pi) \). The point \((t, \theta) \in A(b_2) \) has the \( n_j \) lifts \((t + k\pi, \theta + k\pi) \) upstairs, \( 0 \leq k < n_j \) so the \( n_j \)-fold covering \( p|_{A(\vec{a}(j))} \) can be represented by taking the second coordinate modulo \( 2\pi \). Now we can take \( T_{\vec{a}(j)}(\vec{t}, \vec{\theta}) = (t + n_j \pi \vec{t}, \mod{2\pi}) = T_{a}^{n_j} p(\vec{t}, \vec{\theta}) \).

Now using that \( p|_{\#} \) is a homomorphism, we see \( p: T_{a} \rightarrow T_{a} \) and \( p: T_{b} \rightarrow T_{b} \). Thus we have an element-by-element covering \( p: G(\vec{a}, \vec{b}) \rightarrow \langle T_{a}, T_{b}, T_{a}^{2} \rangle \). However, the latter group intersects \( G(a, b) \) in \( \langle T_{a} \rangle \), so they share no pseudo-Anosovs.

We now see why in this example elements of \( G(a, b) \) don’t lift to elements of \( G(\vec{a}, \vec{b}) \): the problem is exactly that the exponents of \( T_{b_1} \) and \( T_{b_2} \) are not the same in the image. In order to fix this, we should arrange that \( b_1 \) doesn’t lift. So consider the two-fold covering below.
The $A_4$ curve pattern downstairs is the same, but now we are cutting along a multicurve $z$ that meets both $b_1$ and $b_2$, so each is lifted to a pair of paths. Join the lifts of $b_1$ to form a curve $\tilde{c}_1$ double-covering $b_1$ and similarly form a curve $\tilde{c}_2$ double-covering $b_2$. Now $p_\#(T_j) = T_6^j$ for $j = 1, 2$, so if we let $\tilde{a}$ be the set of lifts of $a_1, a_2$ and $\tilde{c} = \{\tilde{c}_1, \tilde{c}_2\}$, then $p_\#(T_\tilde{a}) = T_a$ and $p_\#(T_\tilde{c}) = T_6^2$, so $p_\#: \Gamma(\tilde{a}, \tilde{c}) \to \langle T_a, T_6^2 \rangle < \Gamma(a, b)$.

There is nothing special about double covers in this example; forming an $n$-fold cover of $S$ by cutting along $z$ and gluing $n$ copies cyclically gives a configuration $\tilde{a}^{(n)}, \tilde{c}^{(n)}$ of curves in the cover $\tilde{S}_n$ such that $p_\#: \Gamma(\tilde{a}^{(n)}, \tilde{c}^{(n)}) \to \langle T_a, T_6^2 \rangle < \Gamma(a, b)$.

In this way we get an infinite family of cyclic covers $p_n: \tilde{S}_n \to S$; moreover, the cover $\tilde{S}_mn \to S$ always factors both through $\tilde{S}_m$ and $\tilde{S}_n$.

The curve systems are compatible with the coverings, so that the group $\Gamma(\tilde{a}^{(mn)}, \tilde{c}^{(mn)})$ in $\text{MCG}(\tilde{S}_mn)$ covers the subgroup $\langle T_{\tilde{a}^{(mn)}}, T_{\tilde{c}^{(mn)}} \rangle$ of $\Gamma(\tilde{a}^{(n)}, \tilde{c}^{(n)}) < \text{MCG}(\tilde{S}_n)$ under the covering map $\tilde{S}_mn \to \tilde{S}_n$. This shows that the elements in the group $\langle T_{\tilde{a}^{(n)}}, T_{\tilde{c}^{(n)}} \rangle < \text{MCG}(\tilde{S}_n)$ are all commensurable with elements of the group $\langle T_{\tilde{a}^{(m)}}, T_{\tilde{c}^{(m)}} \rangle$ of $\text{MCG}(\tilde{S}_m)$.

The transpose of the incidence matrix for the upstairs configuration is $N^T = \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}$, so that $N^TN = \begin{pmatrix} 2n & n \\ n & n \end{pmatrix} = n \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has characteristic polynomial $x^2 - 3nx + n^2$ and Perron–Frobenius eigenvalue $n\frac{1 + \sqrt{5}}{2} = n\mu$. Thus in the derivative representation $G(\tilde{a}^{(n)}, \tilde{c}^{(n)}) \to \text{PSL}(2, \mathbb{R})$ we have $T_{\tilde{a}^{(n)}} \mapsto \begin{pmatrix} 1 & 0 \\ -\sqrt{n\mu} & 1 \end{pmatrix}$ and $T_{\tilde{c}^{(n)}} \mapsto \begin{pmatrix} 1 & \sqrt{n\mu} \\ 0 & 1 \end{pmatrix}$.

The “natural choice” of width assignment for rectangles upstairs is to glue together $n$ copies of the flat structure we’ve chosen for the surface below: namely, to give lifts of $a_1$ the width 1 and lifts of $a_2$ the
width $\gamma$, the golden ratio, $c_1^{(n)}$ the width $\gamma$, and $c_2^{(n)}$ the width 1. If we take derivatives with respect to this flat structure, though, we get $T_{\alpha}^{(n)} \mapsto \begin{pmatrix} 1 & 0 \\ -\sqrt{n} & 1 \end{pmatrix}$ and $T_{\beta}^{(n)} \mapsto \begin{pmatrix} 1 & n\sqrt{\mu} \\ 0 & 1 \end{pmatrix}$, which factors through the representation $G(a, b) \to \text{PSL}(2, \mathbb{R})$, but is a different representation from the one in the last paragraph. This difference demonstrates the ambiguity in choosing flat structures we discussed in Section 4. While the choice of eigenvectors for $NN^\top$ and $N^\top N$ we have made in Section 3 gives the most natural representation, it does not factor through covering maps $p\#$.

It is now clear how to generalize this example.

**Theorem 3.** Let an $n$-fold covering of surfaces $p: \tilde{S} \to S$ be given. Suppose $a$ and $b$ are multicurves filling $S$, with components $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$. Suppose $a_j$ has lifts $\tilde{a}_j^{(k)}$ with $p(\tilde{a}_j^{(k)})$ an $n_{j,k}$-to-1 map and $b_j$ has lifts $\tilde{b}_j^{(k)}$ with $p(\tilde{b}_j^{(k)})$ an $n'_{j,k}$-to-1 map. Let $L_j = \text{lcm}\{n_{j,k} : k\}$ and $L'_j = \text{lcm}\{n'_{j,k} : k\}$ for each $j$, and further suppose $L = L_j$ and $L' = L'_j$ are independent of $j$. Then $p: \prod_{j,k} T_{\alpha_j^{(k)}}^{L_{j,k}} \to T_{\alpha}^{L}$ and $p: \prod_{j,k} T_{\beta_j^{(k)}}^{L'_{j,k}} \to T_{\beta}^{L'}$.

In particular, if $L = n_{j,k}$ and $L' = n'_{j,k}$ are independent of $j, k$, then letting $\tilde{a} = \{\tilde{a}_j^{(k)}\}_{j,k}$ and $\tilde{b} = \{\tilde{b}_j^{(k)}\}_{j,k}$ we have a covering $p\# : \langle T_{\alpha}, T_{\beta} \rangle \to \langle T_{\alpha}^{L}, T_{\beta}^{L'} \rangle$.

**Proof.** This follows from restricting to annuli and applying the preceding two lemmas.

**Example.** In the picture below are two covers of the $A_4$ configuration. The one on the right is from before, but the one on the left is new; one can get it by cutting along the curves indicated. They have a common double cover, a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ cover of $A_4$.

![Figure 20: A pair of commensurable groups](image)
The red curves lift on both sides. The blue curves on the left lift, while those on the right are cut when we produce the double cover. If \( a, b \) are the red and blue multicurves on the left and \( a', b' \) are the red and blue multicurves on the right, respectively, then we have that elements of \( \langle T_a, T_b \rangle \) are commensurable with elements of \( \langle T_{a'}, T_{b'}^2 \rangle \), in the sense that the homomorphism defined by \( T_a \mapsto T_{a'} \) and \( T_b \mapsto T_{b'}^2 \) takes elements to commensurable elements.

A little reflection shows that we never have a full element-by-element group-covering \( G(\tilde{a}, \tilde{b}) \to G(a, b) \) when \( G(a, b) \) contains pseudo-Anosovs and the covering surface is connected; the process of forming an \( n \)-fold covering of a surface \( S \) involves cutting along some curves (and/or arcs between boundary components) \( z \), cloning the cut surface \( n \) times, and gluing the copies of the boundary components together. Since the covering surface is assumed connected, the permutation of \( \{1, \ldots, n\} \) induced by the gluing corresponding to some component \( z_1 \) of \( z \) must not be the identity. If \( G(a, b) \) contains pseudo-Anosovs, \( a \cup b \) fills \( S \), and so at least one curve, say \( a_1 \), must intersect \( z_1 \). Then \( a_1 \) lifts to a collection of paths, which concatenate together to a collection \( \tilde{a}_1 \) of curves, at least one of which multiply covers \( a_1 \). If these multiplicities are the same number \( m \) for all the curves covering \( a_1 \), then \( T_{\tilde{a}_1} \) covers \( T_{a}^m \); if the multiplicities are different, no automorphism of the covering surface covers a power of \( T_{\tilde{a}_1} \). Either way, not all of \( G(a, b) \) can be covered.
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