Canonical variables and analysis on so(n,2)

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Abstract. The approach of Berezin to the quantization of so(n,2) via generalized coherent states is considered in detail. A family of $n$ commuting observables is found in which the basis for an associated Fock-type representation space is expressed. An interesting feature is that computations can be done by explicit matrix calculations in a particular basis. The basic technical tool is the Leibniz function, the inner product of coherent states.

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1 Introduction

In the papers [4, 5, 6], Berezin presents an approach to quantization using generalized coherent states, as explained by Perelomov [21], also see [1] and the survey [15].

We first recall the Cartan decomposition and relate it to Berezin’s theory. In the section following, we give the matrix version used for basic computations. The representation space is constructed. Then the Leibniz function is computed. The observables, the natural variables for analysis on $\text{so}(n, 2)$, are found and their joint spectral density is discussed. We conclude showing how the Lie algebra is recovered from the Leibniz function.

Work most closely related to this paper is that of Onofri [20] and Berceanu & Gheorghe [3]. The coherent state methods given by Hecht [13] are closely related to the present article as well.

In addition to the works cited above, in the mathematics literature we have found the book by Hua [14] and the paper of Wolf [22] very useful. An exposition of the present authors’ theory with emphasis on connections with probability is given in [10]. A major aspect of the mathematical point of view is the theory of symmetric cones and Jordan algebras. See [9] for analysis in that context.

The significance of the pseudo-Euclidean group $\text{SO}(n, 2)$ in physics is well-known. It serves as a “linearization” of the conformal group of Minkowski space $\mathbb{R}^{n-1,1}$ (see e.g., [11]), the symmetry group of Maxwell’s equations. Also, the group $\text{SO}(n, 2)$ plays an important rôle in the $n$-dimensional Kepler problem, where the compactified phase space (the Moser phase space) coincides with a coadjoint orbit of the dynamical group $\text{SO}(n+1, 2)$ [8, 16, 19]. In another context, the group $\text{SO}(4, 2)$ serves as the spectrum-generating symmetry group of the hydrogen atom [2, 17].

Remark 1.1 This paper is based on the Ph. D. dissertation of the middle author (M.G.) [12].

Note: matrix computations have been done using Maple V.
2 Cartan decomposition and Berezin theory

Consider a Lie algebra $\mathfrak{g}$. At the heart of our construction is the existence of two abelian subalgebras $\mathcal{R}$ and $\mathcal{L}$ of the same dimension $n$, such that the Lie algebra they generate is $\mathfrak{g}$ itself: $\mathfrak{g} = \text{gen \{L, R\}}$.

An important case of such a structure is the Cartan decomposition for symmetric Lie algebras, where $\mathfrak{g}$ has the form

$$\mathfrak{g} = \mathcal{L} \oplus \mathcal{K} \oplus \mathcal{R}$$

with $\mathcal{L}$ and $\mathcal{R}$ satisfying $[\mathcal{L}, \mathcal{R}] \subseteq \mathcal{K}$, $[\mathcal{K}, \mathcal{R}] \subseteq \mathcal{R}$, and $[\mathcal{K}, \mathcal{L}] \subseteq \mathcal{L}$.

Denote bases for $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{K}$ by $\{R_j\}$, $\{L_j\}$, and $\{\rho_A\}_{1 \leq A \leq m}$, respectively.

Remark 2.1 Later in the paper we will give a Cartan decomposition of $\text{so}(n,2)$. The $\rho$ elements will be taken as generators of rotations in the purely spatial or temporal sectors of $\mathbb{R}^{n,2}$, while the $\mathcal{L}$ and $\mathcal{R}$ elements will be certain combinations of boosts.

A typical element $X \in \mathfrak{g}$ is of the form

$$X = v'_j R_j + u'_A \rho_A + w'_j L_j$$

for some $(2n + m)$-tuple $(v', u', w')$. We can express exponentiation of $X$ to an element of the group either by the standard exponential map, or via factorization into subgroups corresponding to the decomposition of the Lie algebra, thus

$$e^X = \exp(v_i R_i) \left( \prod_A \exp(u_A \rho_A \hat{A} \right)) \exp(w_j L_j)$$

(We use the convention of summation over repeated indices, unless they are dotted; there is no summation over $\hat{A}$ above). Clearly, the coordinates $(v, u, w)$ versus $(v', u', w')$ are mutually dependent as they represent in (3) the same group element.

Our general goal is to construct a representation space for the enveloping algebra of $\mathfrak{g}$ and then find an abelian subalgebra of self-adjoint operators to
take as our observables of interest.

First, let us construct a Hilbert space $\mathcal{H}$ spanned by a basis

$$|k_1, k_2, \ldots, k_n\rangle = R_{1}^{k_1} \cdots R_{n}^{k_n} \Omega$$

(4)

where $\Omega$ is a vacuum state. Define the action of the algebra elements on the vacuum state thus

(i) $\hat{R}_j \Omega = R_j \Omega$
(ii) $\hat{L}_j \Omega = 0$
(iii) $\hat{\rho}_A \Omega = \tau_A \Omega$

where $\tau_A$ are constants. Next, assume that $\mathcal{H}$ admits a symmetric scalar product (not necessarily hermitian!) in some number field, such that the ladder operators are mutually adjoint with respect to it:

$$\hat{R}_i^* = \hat{L}_i$$

Thus, there is a 1-1 map $\mathcal{R} \leftrightarrow \mathcal{L}$ that admits such a pairing via adjoints. Additionally, we shall always consider the vacuum state normalized, $\langle \Omega, \Omega \rangle = 1$.

For the purpose of this paper, we shall assume that only one element of $\mathcal{K}$, say $\rho_0$, acts on $\Omega$ as a nonzero constant $\tau$, so that the group element specified by equation (3) acts on $\Omega$ as follows

$$e^{X} \Omega = e^{\tau u} \exp(v_j R_j) \Omega$$

(5)

The system possesses two types of lowering and raising operators. The \textit{algebraic lowering} and \textit{raising} operators are defined simply by concatenation within the enveloping algebra (operator algebra generated by the representation) of $\mathfrak{g}$ followed by acting on $\Omega$, that is

$$\hat{R}_j \psi = R_j \psi$$
$$\hat{L}_j \psi = L_j \psi$$

for any linear combination $\psi$ of basis elements (4). The “hat” can be thus omitted without causing confusion. We shall also introduce \textit{combinatorial}
raising operators, $\mathcal{R}_j$, and combinatorial lowering operators, $\mathcal{V}_j$, acting on the basis as follows

$$\mathcal{R}_j |k_1, k_2, \ldots, k_n\rangle = |k_1, k_2, \ldots, k_j+1, \ldots, k_n\rangle$$
$$\mathcal{V}_j |k_1, k_2, \ldots, k_n\rangle = k_j |k_1, k_2, \ldots, k_j-1, \ldots, k_n\rangle$$

Notice that the operator $\mathcal{V}_j$ acts formally as the operator of partial differentiation with respect to the corresponding variable $R_j$.

The algebraic raising operators are represented directly by the $\mathcal{R}'s$, namely $\hat{\mathcal{R}}_j = \mathcal{R}_j$. But the combinatorial lowering operators do not necessarily correspond to elements of $\mathfrak{g}$. The idea will be to express the algebraic lowering operators, $\hat{\mathcal{L}}_j$, (and hence the basis for $\mathfrak{g}$), also in terms of the operators $\{\mathcal{R}_j, \mathcal{V}_j\}$.

Let us introduce the coherent states as the image of the subgroup generated by the (abelian) subalgebra $\mathcal{R} \subset \mathfrak{g}$ in the Hilbert space $\mathcal{H}$ constructed above, namely

$$\psi_v = \exp(v_j \mathcal{R}_j) \Omega$$

Thus, the coherent states are parametrized by the elements $v_j R_j$ of $\mathcal{R}$, or, equivalently, by coordinates $v = (v_1, \ldots, v_n)$. We shall denote the manifold of coherent states as $\mathcal{C}$ with the parametrization $\mathcal{R} \rightarrow \mathcal{C}$.

**Observation 2.2** When restricted to coherent states, $\mathcal{R}_j$ acts as differentiation $\partial/\partial v_j$, while $\mathcal{V}_j$ acts as multiplication by $v_j$. Hence, we can determine the action of any operator defined as a (formal) operator function $f(\mathcal{R}, \mathcal{V})$, with all $\mathcal{V}$’s to the right of any $\mathcal{R}_j$, by (1) moving all $\mathcal{R}$’s to the right of all $\mathcal{V}$’s in the formula $f$, yielding the operator $\hat{f}(\mathcal{R}, \mathcal{V})$, and then (2) replacing $\mathcal{V}_j \rightarrow v_j$ and $\mathcal{R}_j \rightarrow \partial/\partial v_j$, $1 \leq j \leq n$. Note that this is a formal Fourier transform combined with the Wick ordering. The Berezin transform extends this by taking the inner product with a coherent state $\psi_w$.

The following notion is very useful.

**Definition 2.3** The Leibniz function is a map $\mathcal{C} \times \mathcal{C} \rightarrow \mathbb{C}$ defined as the inner product of the coherent states:

$$\Upsilon_{vw} = \langle \psi_w, \psi_v \rangle$$

for any $v, w$ parametrizing $\mathcal{C}$. 

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**Definition 2.4** The *Berezin transform* (the *coherent state representation*) is defined for an operator $Q$ by,

$$\langle Q \rangle_{wv} = \frac{\langle \psi_w, Q \psi_v \rangle}{\langle \psi_w, \psi_v \rangle}.$$  

The algebraic raising operators can be expressed by the Leibniz function,

$$\langle \hat{R}_j \rangle_{wv} = \Upsilon^{-1} \frac{\partial}{\partial v_j} \Upsilon = \partial(\log \Upsilon) / \partial v_j.$$  

Since $L_j$ is adjoint to $R_j$, we can get $L_j$ by differentiating with respect to $w_j$. Suppose $\Upsilon$ satisfies a system of first-order partial differential equations

$$\frac{\partial \Upsilon}{\partial w} = \hat{f}_j(v, \frac{\partial}{\partial v}) \Upsilon$$

for some operator functions $\hat{f}_j$. Then, from the above discussion, we see that $\hat{L}_j$ is given by

$$\hat{L}_j = f_j(R, V)$$

Note that the converse holds as well. Namely, if we have $\hat{L}_j$ expressed via $R$ and $V$, then $\Upsilon$ satisfies the corresponding partial differential equation. In some cases, this can be used to find $\Upsilon$.

The final step is to find in our representation $n$ commuting, self-adjoint operators $X_j$. They will generate a unitary group, $\exp(i \sum s_j X_j)$, with $s = (s_1, \ldots, s_n) \in \mathbb{R}^n$ and $i = \sqrt{-1}$. The scalar function defined by

$$\phi(s) = \langle \Omega, \exp(i \sum s_j X_j) \Omega \rangle$$

will be required to be positive-definite. Then *Bochner’s Theorem* assures that $\phi(s)$ is the Fourier transform of a positive measure, which gives the joint spectral density of the observables ($X_1, \ldots, X_n$).

For $\text{so}(n, 2)$, we will identify these as a (multivariate) random variable on the Lorentz cone \{ $x_1 > 0, x_1^2 > x_2^2 + \cdots + x_n^2$ \} in Minkowski space $\mathbb{R}^{n,1}$.

There are several ways to proceed with the outlined plan. One way is to study a matrix realization of the Lie algebra. Another is to start from the Leibniz function, which has been calculated in [5] and in [14], and reconstruct the Lie algebra from it. We will show how both of these approaches work.
3 Matrix version of so(\(n,2\))

Consider the “Lorentz” group SO(\(n,2\)) of transformations of the \((n + 2)\)-dimensional real “Minkowski” space \(\mathbb{R}^{n,2}\) of signature \((n,2)\). The two-dimensional “time” leads to two sets of independent boosts. Besides the spatial rotations, the group contains a 1-dimensional subgroup, so(2), of temporal rotations. We shall start with the \((n + 2) \times (n + 2)\) skew-symmetric matrices 
\[
\rho_{kl} = E_{kl} - E_{lk}, \quad \text{for} \quad 1 \leq k, l \leq n + 2,
\]
where \(E_{ij}\) denotes the matrix consisting of zeros except for 1 at the \(ij\)-entry.

**Notation** In the following, indices \(j\) and \(k\) run from 1 to \(n\), referring to “spatial coordinates.” Subscripts \(n + 1, n + 2\) refer to “time coordinates.”

One defines and checks that

**Proposition 3.1** The operators \(R_j, L_j\), for \(1 \leq j \leq n\), and \(\rho_0\) defined by

\[
R_j = \rho_{j,n+2} + i \rho_{j,n+1}, \quad L_j = \rho_{j,n+2} - i \rho_{j,n+1}, \quad \rho_0 = 2i \rho_{n+1,n+2}
\]

along with \(\{\rho_{jk}\}\), \(1 \leq j, k \leq n\), form a basis of so(\(n,2\)) corresponding to a Cartan decomposition as in equations (1) and (2).

The following relations hold; the root space relations

\[
[\rho_0, R_j] = 2R_j, \quad [L_j, \rho_0] = 2L_j, \quad [\rho_{jk}, \rho_0] = 0
\]

and

\[
[L_j, R_j] = \rho_0, \quad [L_k, R_j] = 2\rho_{jk}, \quad [\rho_{jk}, L_k] = L_j
\]

The involution (adjoint map) given by \(R_j^* = L_j\) is effectively a complex conjugation. The commutation relations determine the involution for the remaining elements of \(\mathfrak{g}\) (since \(\mathcal{L}\) and \(\mathcal{R}\) generate \(\mathfrak{g}\) as a Lie algebra). Hence, \(\rho_0\), is automatically symmetric, \(\rho_0^* = \rho_0\), since it equals a commutator of mutually adjoint elements. Also, the \(\rho_{jk}\) are skew-symmetric with respect to this involution, as follows from relations 2 \(\rho_{jk} = [L_k, R_j]\).

Now, we want to find commuting symmetric operators that will provide \(n\) commuting self-adjoint operators spanning \(\mathfrak{g}\). (Note that even though we have complex numbers in the matrices, we in fact are using a “real form” of \(\mathfrak{g}\), admitting only real coefficients.) It turns out that conjugating by \(\exp(L_1)\) almost “does the job.” More precisely,
Proposition 3.2  The elements

\[
X_1 = 2(\rho_{1,n+2} + i\rho_{n+1,n+2}) = R_1 + L_1 + \rho_0 \\
X_j = 2i(\rho_{1,j} + i\rho_{n+1,j}) = -i(R_j - L_j - 2\rho_{1,j})
\]

for \(2 \leq j \leq n\), form a commuting family of Hermitian-symmetric elements in \(\mathfrak{g}\).

Proof:  Since the \(\mathcal{R}\) is abelian, conjugating it by a fixed element of the group will yield an abelian algebra. Calculating the adjoint group action \(\exp(L_1)R_j \exp(-L_1)\) (with a use of commutation relations) yields the indicated operators. For \(j > 1\), the result is skew-symmetric, thus requiring the factor of \(-i\) for those \(X_j\).

\[\square\]

Here are some explicit matrices for \(n = 3\).

\[
R_1 = \begin{pmatrix}
0 & 0 & 0 & i & 1 \\
0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad L_1 = \begin{pmatrix}
0 & 0 & 0 & -i & 1 \\
0 & 0 & 0 & 0 & 0 \\
i & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0
\end{pmatrix} \\
X_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2i \\
-2 & 0 & 0 & -2i & 0
\end{pmatrix}, \quad X_2 = \begin{pmatrix}
0 & 2i & 0 & 0 & 0 \\
-2i & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and

\[
X_3 = \begin{pmatrix}
0 & 0 & 2i & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-2i & 0 & 0 & 2 & 0 \\
0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Remark 3.3  Note that we are using a Hermitian structure here for the inner product so that multiplication by \(i\) converts a skew operator to a symmetric one.
3.1 Representation space

In the matrix formulation given above, we have a vacuum vector

$$\Omega = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}$$

where $\bar{0}$ stands for a column of $n$ 0’s. This vector satisfies

$$L_j \Omega = 0, \quad \rho_{jk} \Omega = 0, \quad \rho_0 \Omega = -2\Omega$$

Coherent states can be found readily,

$$\exp(v_j R_j) \Omega = \begin{pmatrix} 2iv \\ 1 + v^2 \\ i(1 - v^2) \end{pmatrix}$$

where $v$ is a column vector with components $v_j$ and $v^2 = v_jv_j$.

Recalling equation (5), this leads to

**Proposition 3.4** Let $g$ denote a group element, as in equation (3), then we can recover $v$ and $u$ from $g\Omega = \begin{pmatrix} v_0 \\ a \\ b \end{pmatrix}$ by

$$v = \frac{1}{b + ia} v_0, \quad e^{-2u} = -\frac{1}{2} \frac{v_0^\top v_0}{a + ib}$$

where $\top$ denotes transpose.

3.2 Leibniz function

Here we calculate the Leibniz function from the matrix representation. Since $R$’s are adjoint to $L$’s, we have

$$\Upsilon_{uv} = \langle e^{w R} \Omega, e^{v R} \Omega \rangle = \langle \Omega, e^{w L} e^{v R} \Omega \rangle$$

where, e.g., $w \cdot L = w_j L_j$, and similarly for $v \cdot R$. As a result we get
Theorem 3.5 In the matrix realization of \( \text{so}(n,2) \) given in Proposition 3.1, the Leibniz function is

\[
\Upsilon_{wv} = 1 - 2w^\top v + w^2v^2
\]

Proof: First compute

\[
\exp(w_jL_j)\exp(v_jR_j)\Omega = \begin{pmatrix}
2i(v - v^2w) \\
-2w^\top v + 1 + v^2 + w^2v^2 \\
-2iw^\top v + i(1 - v^2 + w^2v^2)
\end{pmatrix}
\]

(7)

When this is expressed in factored form (cf. equations (3) and (5)), taking inner products with \( \Omega \) eliminates all factors except for \( \langle \Omega, e^{\rho_0 u}\Omega \rangle \). In general, this is \( e^{\tau u} \) with \( u \) a function of \( v \)'s and \( w \)'s. In the matrix realization above, applying \( \rho_0 \) to \( \Omega \) shows that \( \tau = -2 \). The rest follows from equation (7) using the result for \( \exp(-2u) \) in Proposition 3.4.

\[\square\]

Generally, we want \( \rho_0 \) to act on \( \Omega \) as multiplication by \( \tau \). This suggests that for \( e^{\tau u} = (e^{-2u})^{-\tau/2} \) we have in general \( \Upsilon_{wv} = (1 - 2w^\top v + w^2v^2)^{-\tau/2} \).

We can now check agreement with the results in [5, 21], cf. the Bergman kernel function given in [14]. A main feature of the Leibniz function is that expanded in powers of \( v \)'s and \( w \)'s it yields the generating function for the inner products of elements of the basis for the Hilbert space. In general, there are conditions on the values of \( \tau \), the Gindikin set, for which the Hilbert space has a positive-definite inner product. In this regard, in addition to Berezin’s paper, see [9].

3.3 Distribution of the observables

In the matrix representation, the raising operators are nilpotent, \( R_j^3 = 0 \), and hence \( X_j^3 = 0 \) for all \( 1 \leq j \leq n \). Consequently, the exponentials reduce to quadratics and the computations are very fast. With the vacuum vector as above, we find

\[
\exp(z_jX_j)\Omega = \begin{pmatrix}
2i(z_1 - \zeta^2) \\
2z_2 \\
\vdots \\
2z_n \\
1 - 2z_1 + 2\zeta^2 \\
i(1 - 2z_1)
\end{pmatrix}
\]
where \( \zeta^2 = z_1^2 - \sum_{j \geq 2} z_j^2 \). Applying Proposition 3.4, we have

**Proposition 3.6** Let \( h^2 = (1 - z_1)^2 - \sum_{j \geq 2} z_j^2 \). For a group element generated by the \( X_j \) acting on the vacuum, \( \exp(z_j X_j) \Omega \), the \( v \) and \( u \) variables are given according to

\[
v = \frac{1}{h^2} \begin{pmatrix} 1 - z_1 - h^2 \\ -iz_2 \\ \vdots \\ -iz_n \end{pmatrix}, \quad \exp(-2u) = h^2
\]

With \( e^{\tau u} = h^{-\tau} = \left( (1 - z_1)^2 - z_2^2 - \cdots - z_n^2 \right)^{-\tau/2} \) as the Fourier-Laplace transform of the joint spectral density of the \( X_j \), we can identify it as a measure on the Minkowski cone \( \{ x_1 > 0, x_1^2 > x_2^2 + \cdots + x_n^2 \} \) in \( \mathbb{R}^{n,1} \). See, e.g., [18] as well as the references mentioned above for determining positivity. Up to an exponential factor in \( x_1 \) the density is the *Wishart distribution on the Lorentz cone*. See [9, Chapter XVI] and Casalis [7]. The important feature is that the positivity implies (means) that we have the Fourier-Laplace transform of probability measures which are given by a function raised to a power in the Fourier domain. Thus, the measures form a convolution family and with a continuous parameter \( \tau = t/\hbar \), we have the fundamental solution to an evolution equation with generator \( u(D) \), replacing \( (z_1, \ldots, z_n) \) in \( u \) as a function of \( z \) by \( (D_1, \ldots, D_n) \), \( D_j = d/dx_j \). Thus, \( u(D) \) is a ‘natural Hamiltonian’ — generator of time-translations — associated to the Lie algebra.

Now, we have two expressions, hence coordinate systems, for a coherent state,

\[
\exp(z_j X_j) \Omega = e^{\tau u} \exp(v_j R_j) \Omega
\]

(8)

where \( u = u(z) \) and \( v_j = v_j(z) \) are functions of \( z = (z_1, \ldots, z_n) \). In order to express the basis of the Hilbert space in terms of the \( X \)'s rather than \( R \)'s, we must solve (8) for the \( z_j \) in terms of the \( v \)'s. This will give the *generating function* for the basis expressed in terms of the \( X_j \), written in spectral form as variables \( x_j \).
In general,
\[ e^{v_j R_j \Omega} = \exp \left[ x_j z_j(v) - \tau u(z(v)) \right] = \sum_{k_1, \ldots, k_n} \frac{v_1^{k_1} \cdots v_n^{k_n}}{k_1! \cdots k_n!} |k_1, \ldots, k_n\rangle \]
where \( z_j = z_j(v) \) are the components of the (functional) inverse to \( v(z) \).

**Theorem 3.7** The generating function for the basis \(|k_1, \ldots, k_n\rangle\) is
\[ \exp \left( \frac{x_1 (v_1 + v^2) + i (\mathbf{x} \cdot \mathbf{v} - x_1 v_1)}{1 + 2v_1 + v^2} \right) (1 + 2v_1 + v^2)^{-\tau/2} \]
where \( v^2 = v_j v_j \) and \( \mathbf{x} \cdot \mathbf{v} = x_j v_j \).

**Proof:** To start, note that \( e^{-2u} = h^2 \) entails \( e^{-\tau u} = h^{\tau/2} \). Now we must solve for the \( v \)'s in Proposition 3.6. First,
\[ v_1 = (h^2 - 1 - z_1)/h^2 \] implies \( 1 + z_1 = h^2(1 - v_1) \)
And with \( z_k = i h^2 v_k \), we square and re-sum on the left-hand side to yield
\[ h^2 = h^4 \left( (1 - v_1)^2 + \sum_{k \geq 2} v_k^2 \right) \]
from which \( h^2 = (1 + 2v_1 + v^2)^{-1} \). Expressing \( h^2 \) in terms of \( v \)'s in the above expressions for the \( z_j \) yields the result. \( \square \)

For \( n = 1 \), the terms for \( k \geq 2 \) drop out, reducing to a generating function for Laguerre polynomials. Hence, in general, the basis in the \( x \)-variables offers a generalization of the classical Laguerre polynomials.

### 4 Leibniz function and the Lie algebra

Now we shall show how the Lie algebra structure expressed in terms of the combinatorial raising and lowering operators \( R_j \) and \( V_j \), the hat-representation, can be constructed from the Leibniz function. (Recall Remark 2.2.)
Theorem 4.1  The hat-representation has the form

\[ \hat{R}_j = R_j, \quad \hat{L}_j = \tau V_j + 2 (R_l V_l) V_j - R_j V^2 \]

for the algebraic raising and lowering operators, while the rotation operators are given by

\[ \hat{\rho}_0 = \tau + 2 R_l V_l, \quad \hat{\rho}_{jk} = R_j V_k - R_k V_j \]

Proof:  Theorem 3.5 provides us the Leibniz function for our representation of so(n,2)

\[ \Upsilon = \left( 1 - 2 w^T v + w^2 v^2 \right)^{-\tau/2} \]

Differentiating, we obtain

\[ \frac{1}{\Upsilon} \frac{\partial \Upsilon}{\partial w_j} = \tau \frac{v_j - w_j v^2}{1 - 2 w^T v + w^2 v^2} \quad \text{and} \quad \frac{1}{\Upsilon} \frac{\partial \Upsilon}{\partial v_j} = \tau \frac{w_j - v_j w^2}{1 - 2 w^T v + w^2 v^2} \]

Combining these, we find the system of partial differential equations

\[ \frac{\partial \Upsilon}{\partial w_j} = \tau v_j \Upsilon + 2 v_j \frac{\partial \Upsilon}{\partial v_l} v_l - \frac{\partial \Upsilon}{\partial v_j} v^2 \]

(implied summation over \( l \) in the middle term) from which we can read off the result stated for \( \hat{L}_j \). Since \( \hat{R}_j = R_j \), taking the commutator with \( \hat{L}_j \), the first relation in equation (6) yields \( \hat{\rho}_0 = \tau + 2 R_l V_l \). And from the middle relation in equation (6),

\[ [\hat{L}_k, \hat{R}_j] = 2 (R_j V_k - R_k V_j) \]

yields \( \hat{\rho}_{jk} \).

One can easily check the adjoint action of \( \hat{\rho}_0 \) as well as the remaining commutation relations corresponding to equation (6). Finally, note that \( \hat{L}_j \) can be written in the form

\[ \hat{L}_j = \hat{\rho}_0 V_j - R_j V^2 \]

which is a variation on the Bessel operator.
5 Conclusion

An important feature of our approach is the identification of an interesting abelian subalgebra that provides a family of commuting observables from the viewpoint of quantization. The special element $\rho_0$ turns out to be dual to a natural Hamiltonian generating a convolution semigroup of measures yielding the joint spectral density of the observables of interest. It is important to note that in Berezin, e.g., [5], what is considered as Planck’s constant should be in fact the ratio $t/h$, namely the ratio between a time variable and a fixed constant. In this paper, the time in so($n,2$) is represented by (imaginary) so(2) and in the representation space by a positive real variable, the corresponding weight. The physical interpretation of this aspect remains to be explored.

References

[1] Barut A O and Girardello L 1971 Commun. Math. Phys. 21 41–55
[2] Barut A O and Kleinert H 1967 Phys. Rev. 156 1541
[3] Berceanu S and Gheorghe A 1992 On equations of motion on compact Hermitian symmetric spaces J. Math Phys. 33 3 998–1007.
[4] Berezin F A 1974 Quantization Izv. Akad. Nauk. SSSR, Ser. Mat. 38 5 1109–1165.
[5] Berezin F A 1975 Quantization in complex symmetric spaces Izv. Akad. Nauk. SSSR, Ser. Mat. 39 2 363–402.
[6] Berezin F A 1975 General concept of quantization Comm. Math. Phys. 40 153–174
[7] Casalis M 1991 Les familles exponentielles à variance quadratique homogène sont des lois de Wishart sur un cône symétrique C.R.Acad.Sci.Paris 312 537-540
[8] Cordani B 1986 Conformal regularization of the Kepler problem Comm. Math. Phys 103 403–413
[9] Faraut J & Koranyi A 1994 Analysis on symmetric cones (Oxford)
[10] Feinsilver P and Schott R 1996 Algebraic structures and operator calculus, Volume 3: Representations of Lie groups (Kluwer Academic Publishers)

[11] Felsager B 1981 Geometry, Particles and Fields (Odense Univ. Press)

[12] Giering M 1995 Representations and differential equations on the classical domains of type IV, using coherent state methods and a new formulation of Berezin quantization for $so(n,2)/so(n) \times so(2)$, (Ph.D. Dissertation, Southern Illinois University, )

[13] Hecht K T 1987 The vector coherent state method and its application to problems of higher symmetries (Springer Lect. Notes in Physics 290)

[14] Hua L K 1963 Harmonic analysis of functions of several complex variables in the classical domains (Transl. Math. Mono. 6, A.M.S.) (Providence)

[15] Klauder J R and Skagerstam B S (Eds.) 1985 Coherent States (Singapore: World Scientific)

[16] Kummer M 1982 On the regularization of the Kepler problem Comm. Math. Phys. 84 133–152

[17] Malkin I A and Man’ko V I 1970 Dynamical symmetries and Coherent States of Quantum systems (in Russian) (Moscow: Nauka Publishers)

[18] Letac G 1994 Les familles exponentielles statistiques invariantes par les groupes du cône et du parabolôde de révolution, J. Appl. Prob. 31A 71–95

[19] Moser J 1970 Comm. Pure Appl Math 23 609–636.

[20] Onofri E 1975 A note on coherent state representations of Lie groups J. Math. Phys. 16 5 1087-1089

[21] Perelomov A 1986 Generalized coherent states and applications (Springer-Verlag)

[22] Wolf J A 1972 Fine structure of Hermitian symmetric spaces, in Symmetric Spaces, W.M. Boothby & G.L. Weiss (eds.) (New York: Marcel Dekker) 271–357