A combination of the trimaximal mixing and $\mu$-$\tau$ reflection symmetry in the minimal seesaw model

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Abstract

In this paper, we consider a combination of the trimaximal (TM1 and TM2) mixings and $\mu$-$\tau$ reflection symmetry in the minimal seesaw model. Such a scenario is highly restrictive and predictive: the trimaximal mixings and $\mu$-$\tau$ reflection symmetry will help us pin down all the neutrino mixing parameters except for $\theta_{13}$, while the minimal seesaw model will help us pin down the neutrino masses. We will study the stability of this scenario against the renormalization group corrections and its implications for leptogenesis.
1 Introduction

As we know, the discovery of neutrino oscillations shows that neutrinos are massive and mixed [1]. The most popular and natural way of generating the small neutrino masses is the type-I seesaw model where three super heavy right-handed neutrino fields \( N_I \) (for \( I = 1, 2, 3 \)) are introduced [2]. They not only can have the Yukawa couplings \( (Y_\nu)_{\alpha I} \) with the left-handed neutrino fields \( \nu_\alpha \) (for \( \alpha = e, \mu, \tau \)), which lead to the Dirac neutrino mass matrix \( M_D = Y_\nu v \) after the Higgs field acquires its vacuum expectation value \( v = 174 \text{ GeV} \), but also can have a Majorana mass matrix \( M_R \) for themselves. Without loss of generality, we will work in the basis of \( M_R \) being diagonal \( D_R = \text{diag}(m_1, m_2, m_3) \) with \( M_I \) being three right-handed neutrino masses. Integrating the right-handed neutrino fields out yields an effective Majorana mass matrix for the light neutrinos: \( M_\nu \approx -M_D D_R^{-1} M_D^T \).

In the basis where the flavor eigenstates of three charged leptons align with their mass eigenstates (i.e., \( M_I = \text{diag}(m_e, m_\mu, m_\tau) \)), the neutrino mixing matrix \( U \) arises as the unitary matrix for diagonalizing \( M_\nu \): \( U^\dagger M_\nu U^* = D_\nu = \text{diag}(m_1, m_2, m_3) \) with \( m_i \) being three light neutrino masses. In the standard parametrization, \( U \) is expressed in terms of three unphysical phases \( \phi_\alpha \), three mixing angles \( \theta_{ij} \) (for \( i = 12, 13, 23 \)), one Dirac CP phase \( \delta \) and two Majorana CP phases \( \rho \) and \( \sigma \):

\[
U = \begin{pmatrix}
e^{i\phi_e} & e^{i\phi_\mu} & e^{i\phi_\tau}
\end{pmatrix}
\begin{pmatrix}
c_{12}c_{13} & s_{12}s_{13}e^{i\delta} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} + s_{12}s_{23}c_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13} \\
s_{12}s_{23} - s_{12}s_{23}s_{13}e^{i\delta} & c_{12}s_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{23}c_{13} \\
\end{pmatrix}
\begin{pmatrix}
e^{i\rho} \\
e^{i\sigma} \\
1
\end{pmatrix}, \tag{1}
\]

where the abbreviations \( c_{ij} = \cos \theta_{ij} \) and \( s_{ij} = \sin \theta_{ij} \) have been employed.

Thanks to the various neutrino oscillation experiments, the mixing angles and neutrino mass squared differences \( \Delta m_{ij}^2 \equiv m_i^2 - m_j^2 \) have been measured to a good degree of accuracy, and there is also a preliminary result for \( \delta \). The global-fit results for these parameters are presented in Table 1 [3, 4]. Note that the sign of \( \Delta m_{31}^2 \) remains undetermined, allowing for two possible neutrino mass orderings: the normal ordering (NO) \( m_1 < m_2 < m_3 \) and inverted ordering (IO) \( m_3 < m_1 < m_2 \). Furthermore, neutrino oscillations are completely insensitive to the neutrino mass themselves and Majorana CP phases. Their values can only be inferred from non-oscillatory experiments such as the neutrinoless double beta decay experiments [5]. Unfortunately, so far there has not been any lower constraint on the lightest neutrino mass, nor any constraint on the Majorana CP phases.

From Table 1 we see that \( \theta_{12} \) and \( \theta_{23} \) are close to some special values: \( \sin^2 \theta_{12} \sim 1/3 \) and \( \sin^2 \theta_{23} \sim 1/2 \). Furthermore, before the value of \( \theta_{13} \) was known, the conjecture that it might be vanishingly small was very popular. For the ideal case of \( \sin \theta_{12} = 1/\sqrt{3} \), \( \sin \theta_{23} = 1/\sqrt{2} \) (i.e., \( \theta_{23} = \pi/4 \)) and \( \theta_{13} = 0 \), referred to as the tribimaximal (TBM) mixing [6], the neutrino mixing matrix takes a very simple form as

\[
U_{\text{TBM}} = \frac{1}{\sqrt{6}} \begin{pmatrix}2 & \sqrt{2} & 0 \\1 & -\sqrt{2} & -\sqrt{3} \\1 & -\sqrt{2} & \sqrt{3} \end{pmatrix}. \tag{2}
\]

However, the observation of a relatively large \( \theta_{13} \) compels us to forsake or modify this particular mixing. An economical and predictive way out is to retain its first or second column while modifying the other two columns within the unitarity constraints, giving the first or second trimaximal (TM1 or
which basically retain the TBM prediction for $\theta_{12}$:

$$
\text{TM1: } s_{12}^2 = \frac{1}{3} - \frac{2s_{13}^2}{3 - 3s_{13}^2}; \quad \text{TM2: } s_{12}^2 = \frac{1}{3} + \frac{s_{13}^2}{3 - 3s_{13}^2}.
$$

(4)

The flavor symmetry associated with $\theta_{23} = \pi/4$ and $\theta_{13} = 0$ is known as the $\mu$-$\tau$ symmetry, under which the neutrino mass matrix is required to keep invariant with respect to the interchange between $\nu_\mu$ and $\nu_\tau$ [8, 9]. After the observation of a relatively large $\theta_{13}$ and a preliminary hint for $\delta \sim -\pi/2$ [10], the $\mu$-$\tau$ reflection symmetry [11, 9] — a generalized $\mu$-$\tau$ symmetry — has become increasingly popular, under which the neutrino mass matrix is required to keep invariant with respect to the following transformations of three left-handed neutrino fields

$$
\nu_e \leftrightarrow \nu_e^c, \quad \nu_\mu \leftrightarrow \nu_\tau^c, \quad \nu_\tau \leftrightarrow \nu_\mu^c,
$$

(5)

with the superscript $c$ denoting the charge conjugation of relevant fields. Such a symmetry leads to the following interesting predictions for the neutrino mixing parameters

$$
\phi_e = 0, \quad \phi_\mu = \pi - \phi_\tau, \quad \theta_{23} = \frac{\pi}{4}, \quad \delta = \pm \frac{\pi}{2}, \quad \rho, \sigma = 0 \text{ or } \frac{\pi}{2}.
$$

(6)

The general type-I seesaw model with three right-handed neutrino fields has the shortcoming that the number of its parameters is much larger than that of the measurable neutrino parameters so that it lacks the predictive power. In the literature, there are two typical approaches to reducing the number of its parameters and consequently improving its predictability. One approach is to constrain its flavor structure by invoking some flavor symmetries such as the above-mentioned trimaximal symmetries (i.e., those associated with the trimaximal mixings) and $\mu$-$\tau$ reflection symmetry. The other approach is to reduce the number of right-handed neutrino fields to two (giving the minimal seesaw model [12, 13]), in which case the lightest neutrino mass remains to be vanishing ($m_1 = 0$ in the NO case or $m_3 = 0$ in the IO case) and only one Majorana CP phase is of physical meaning ($\sigma$ in the NO case or $\sigma - \rho$ in the IO case).

Motivated by the above facts, it will be an interesting attempt to consider a combination of the trimaximal mixings and $\mu$-$\tau$ reflection symmetry in the minimal seesaw model. Such a scenario will be highly restrictive and predictive: the trimaximal mixings and $\mu$-$\tau$ reflection symmetry will help us pin down all the neutrino mixing parameters except for $\theta_{13}$ (see Eqs. (4, 6)) [14], while the minimal seesaw model will help us pin down the neutrino masses (in combination with the measurements for their squared differences). We will study the implications of this scenario for the neutrino parameters and leptogenesis and its stability against the renormalization group (RG) corrections.

2 Trimaximal mixings combined with $\mu$-$\tau$ reflection symmetry in minimal seesaw

2.1 TM1 mixing combined with $\mu$-$\tau$ reflection symmetry in minimal seesaw

In the minimal seesaw model, the TM1 mixing can be naturally realized by taking two columns of $M_D$ to be orthogonal to the first column of $U_{\text{TBM}}$ [15] or by taking one column to be proportional to
Note that the cases of $\nu_1$ and $\nu_2$ where $\delta m_{21}$ is zero are orthogonal to each other. As we will see, these two scenarios can fulfill our purpose in the NO and IO cases, respectively.

### 2.1.1 NO case

For the scenario that two columns of $M_D$ are orthogonal to the first column of $U_{\text{TBM}}$, we parameterize $M_D$ as

$$M_D = \begin{pmatrix}
a \sqrt{M_1} & c \sqrt{M_2} \\
-(a+b) \sqrt{M_1} & -(c+d) \sqrt{M_2} \\
(b-a) \sqrt{M_1} & (d-c) \sqrt{M_2}
\end{pmatrix},$$

where $a, b, c$ and $d$ are generally complex. It is direct to verify that the resulting $M_\nu$ from such an $M_D$ respects the TM1 symmetry associated with the TM1 mixing:

$$M_\nu = R_{\text{TBM}} M_\nu R_{\text{TBM}}^\dagger$$

with

$$R_{\text{TBM}} = \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

On the other hand, the $\mu$-$\tau$ reflection symmetry can be naturally realized by taking $M_D$ to have a form as

$$M_D = \begin{pmatrix}
a \sqrt{M_1} & b \sqrt{M_2} \\
c \sqrt{M_1} & d \sqrt{M_2} \\
c^* \sqrt{M_1} & d^* \sqrt{M_2}
\end{pmatrix} P_N,$$

where $a$ and $b$ are real while $c$ and $d$ are generally complex, and $P_N = \text{diag}(\sqrt{\eta}, 1)$ (for $\eta = \pm 1$). Note that the cases of $P_N = \text{diag}(i, i)$ and $\text{diag}(1, i)$ just differ from the cases of $P_N = \text{diag}(1, 1)$ and $\text{diag}(i, 1)$ by an overall factor, so we have not considered them. It is direct to verify that the resulting $M_\nu$ from such an $M_D$ respects the $\mu$-$\tau$ reflection symmetry:

$$M_\nu = R_{\mu\tau} M_\nu^* R_{\mu\tau}^\dagger$$

with

$$R_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now, we are ready to consider the combination of the TM1 mixing and $\mu$-$\tau$ reflection symmetry.
In order for $M_D$ to assume the patterns in Eqs. (7, 9) simultaneously, it is restricted to a form as

$$
M_D = \begin{pmatrix}
a \sqrt{M_1} & c \sqrt{M_2} \\
-(a+ib) \sqrt{M_1} & -(c+id) \sqrt{M_2} \\
(ib-a) \sqrt{M_1} & (id-c) \sqrt{M_2}
\end{pmatrix} P_N ,
$$

(11)

where $a, b, c$ and $d$ are real now. The resulting $M_\nu$ from such an $M_D$ is given by

$$
M_\nu = -\begin{pmatrix}
T_1 & -T_1 -iT_3 & -T_1 + iT_3 \\
-T_1 -iT_3 & T_1 - T_2 + 2iT_3 & T_1 + T_2 \\
-T_1 + iT_3 & T_1 + T_2 & T_1 - T_2 - 2iT_3
\end{pmatrix},
$$

(12)

with

$$
T_1 = \eta a^2 + c^2 , \quad T_2 = \eta b^2 + d^2 , \quad T_3 = \eta ab + cd .
$$

(13)

It is direct to verify that such an $M_\nu$ respects the TM1 and $\mu-\tau$ reflection symmetries simultaneously. Before proceeding, we would like to point out that, as will be seen soon, $M_D$ in Eq. (11) can be further simplified by taking one of $a$ and $c$ (or one of $b$ and $d$) to be vanishing, leaving us with only three model parameters.

$M_\nu$ in Eq. (12) can be diagonalized by the following unitary matrix

$$
U_0 = \frac{1}{\sqrt{6}} \begin{pmatrix}
2 \sqrt{2} & 0 & 0 \\
1 & -\sqrt{2} & -\sqrt{3} \\
1 & -\sqrt{2} & \sqrt{3}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & i
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix} ,
$$

(14)

with

$$
\tan 2\theta = \frac{2\sqrt{6}T_3}{2T_2 - 3T_1} ,
$$

(15)

giving a NO neutrino mass spectrum as

$$
m'_1 = 0 , \quad m'_{2,3} = \frac{1}{2} \left[ -(3T_1 + 2T_2) \pm \sqrt{(3T_1 + 2T_2)^2 + 24(T_3^2 - T_1T_2)} \right] .
$$

(16)

With the help of Eqs. (15, 16) and $\sin^2 \theta = 3s_{13}^2$, one can determine the values of $T_1$, $T_2$ and $T_3$ by inputting the measured values of $s_{13}^2$, $\Delta m_{21}^2$ and $\Delta m_{31}^2$.

$\eta = 1$ case: let us first consider the case of $\eta = 1$. In this case where one has $T_1, T_2 > 0$ and $T_3^2 - T_1T_2 = -(ad-bc)^2 < 0$, both $m'_2$ and $m'_3$ in Eq. (16) are negative, and they should respectively take the “+” and “−” signs of “±” to guarantee the realization of $|m'_2| < |m'_3|$. Taking the best-fit values of $s_{13}^2$, $\Delta m_{21}^2$ and $\Delta m_{31}^2$ as typical inputs, we obtain

$$
a^2 + c^2 \simeq 3.8 \times 10^{-3} \text{ eV} , \quad b^2 + d^2 \simeq 0.024 \text{ eV} , \quad ab + cd \simeq \pm 4.3 \times 10^{-3} \text{ eV} ,
$$

(17)

with the “±” signs respectively corresponding to $\sin \theta = \pm \sqrt{3}s_{13}$. $m'_2$ and $m'_3$ can be converted to positive by adding a diagonal phase matrix as $\text{diag}(1,i,i)$ to the right-hand side of $U_0$ in Eq. (14). In this way the neutrino mixing matrix is finally given by $U = U_0\text{diag}(1,i,i)$. Then one can extract the neutrino mixing parameters by comparing it with the standard form in Eq. (1):

$$
s_{13}^2 = |U_{e3}|^2 , \quad s_{12}^2 = \frac{|U_{e2}|^2}{1 - |U_{e3}|^2} , \quad s_{23}^2 = \frac{|U_{\mu3}|^2}{1 - |U_{e3}|^2} ,
$$

$$
\delta = \arg \left( \frac{U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^* + s_{12}^2 s_{13}^2 s_{23}^2}{c_{12}s_{13}^2 s_{13}^2 c_{23} s_{23}} \right) ,
$$

$$
\rho = \arg (U_{e1}U_{e3}^*) - \delta , \quad \sigma = \arg (U_{e2}U_{e3}^*) - \delta .
$$

(18)
With the help of these expressions, the results in Eqs. (4) can be successfully reproduced. In particular, here we arrive at

\[ \delta = \text{sign}(\sin \theta) \frac{\pi}{2}, \quad \sigma = \frac{\pi}{2}. \]  

(19)

This will become more transparent if we reexpress \( U_0 \) in the following standard form

\[ U_0 = \frac{1}{\sqrt{6}} \left( \begin{array}{cc} 1 & -e^{-i\phi_r} \cos \vartheta \sin \delta \\ e^{i\varphi_r} & e^{i\varphi_r} \sqrt{3} - \sin^2 \theta \end{array} \right) \left( \begin{array}{ccc} 2 & \sqrt{2} \cos \theta & \frac{-\sqrt{2} \sin \theta}{\sqrt{3} - \sin^2 \theta} \\ \frac{-\sqrt{3} \cos \theta - i\sqrt{2} \sin \theta}{\sqrt{3} - \sin^2 \theta} & \frac{\sqrt{3} \cos \theta - i\sqrt{2} \sin \theta}{\sqrt{3} - \sin^2 \theta} & \frac{\sqrt{3} - \sin^2 \theta}{\sqrt{3} - \sin^2 \theta} \\ \frac{\sqrt{3} - \sin^2 \theta}{\sqrt{3} - \sin^2 \theta} & \frac{\sqrt{3} - \sin^2 \theta}{\sqrt{3} - \sin^2 \theta} & \frac{\sqrt{3} - \sin^2 \theta}{\sqrt{3} - \sin^2 \theta} \end{array} \right) \left( \begin{array}{c} 1 \\ 1 \\ i \end{array} \right), \]  

(20)

with \( \phi_r = \text{arg}(\sqrt{3} \cos \theta + \sqrt{2} i \sin \theta) \). For these results, the effective neutrino mass that governs the rate of the neutrinoless double beta decays \([5]\) is led to

\[ |M_{ee}| = \frac{1}{3} \sqrt{\Delta m^2_{21} (1 - 3 s^2_{13}) + \Delta m^2_{31} s^2_{13}} \approx (3.7 - 4.2) \times 10^{-3} \text{ eV}, \]  

(21)

for the 3\( \sigma \) ranges of relevant neutrino parameters.

Since there are only three constraints (given by Eq. (17)) on the four model parameters \( a, b, c \) and \( d \) from the measurements for the neutrino parameters, we are left with one degree of freedom in reconstructing the former from the latter: the combinations \( a^2 + c^2, b^2 + d^2 \) and \( ab + cd \) keep invariant under the following transformations

\[ \begin{pmatrix} a' \\ c' \end{pmatrix} = \begin{pmatrix} \cos \vartheta & -i \sin \vartheta \\ i \sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix}, \quad \begin{pmatrix} b' \\ d' \end{pmatrix} = \begin{pmatrix} \cos \vartheta & -i \sin \vartheta \\ i \sin \vartheta & \cos \vartheta \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix}. \]  

(22)

This can be more easily understood with the help of the Casas-Ibarra parametrization of \( M_D \) \([18]\)

\[ M_D = iU \sqrt{D_\nu} R \sqrt{D_R}, \]  

(23)

where \( R \) is given by

\[ \text{NO: \quad R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \cos z & \sin z & 0 & 0 \\ -\sin z & \cos z & 0 & 0 \end{pmatrix}; \quad \text{IO: \quad R} = \begin{pmatrix} \cos z & \sin z & 0 & 0 \\ -\sin z & \cos z & 0 & 0 \end{pmatrix}, \]  

(24)

with \( z \) being generally complex (for \( \cos^2 z + \sin^2 z = 1 \)). In the NO case, for \( \eta = 1 \) and \( \sigma = \pi/2 \), taking account of the results in Eqs. (4), \( iU \sqrt{D_\nu} \) appears as

\[ iU \sqrt{D_\nu} = -\begin{pmatrix} 0 & \sqrt{m^2_{12} c^2_{13}} & \sqrt{m^2_{3} s_{13}} \\ -\sqrt{m^2_{12}} & s_{12} c_{13} + i \sqrt{\frac{3}{2}} s_{13} & -\sqrt{m^2_{3} s_{13}} \\ \sqrt{m^2_{12}} & -s_{12} c_{13} + i \sqrt{\frac{3}{2}} s_{13} & \sqrt{m^2_{3} s_{13}} \end{pmatrix}, \]  

(25)

with \( \phi_r = \text{arg}(\sqrt{3} s_{12} - i \sqrt{2} s_{13} c_{13}) \) and \( s_{12} c_{13} = \sqrt{(1 - 3 s^2_{13})/3}, \) being of the form in Eq. (11). Here we have taken \( \delta = -\pi/2 \) which is more experimentally favored than \( \pi/2 \) in the \( \mu-\tau \) reflection symmetry limit. It is then easy to see that \( M_D \) will keep the form in Eq. (11) provided that \( z \) is real (just \( \vartheta \)). Taking advantage of the freedom of \( \vartheta \), one can further simplify \( M_D \) in Eq. (11) by transforming one of \( a \) and \( c \) (or one of \( b \) and \( d \)) to vanishing, as mentioned in the above.
\( \eta = -1 \) case: in the case of \( \eta = -1 \) where one has \( T_3^2 - T_1 T_2 = (ad - bc)^2 > 0 \), \( m'_2 \) and \( m'_3 \) in Eq. (16) have opposite signs. For \( 3T_1 + 2T_2 > 0 \) (or \( < 0 \)), \( m'_2 \) and \( m'_3 \) should respectively take the “+” (or “−”) and “−” (or “+”) signs of “±” to guarantee the realization of \( |m'_2| < |m'_3| \). For \( 3T_1 + 2T_2 > 0 \), we obtain

\[
\begin{align*}
    c^2 - a^2 & \simeq -1.6 \times 10^{-3} \text{ eV} , \\
    d^2 - b^2 & \simeq 0.023 \text{ eV} , \\
    cd - ab & \simeq \pm 6.0 \times 10^{-3} \text{ eV} ,
\end{align*}
\]  

(26)

with the “±” signs respectively corresponding to \( \sin \theta = \pm \sqrt{3}s_{13} \). While the results for \( 3T_1 + 2T_2 < 0 \) just differ by a sign, we will only consider the case of \( 3T_1 + 2T_2 > 0 \) in the following discussions. In this case where one has \( m'_2 > 0 \) and \( m'_3 < 0 \), the neutrino mixing matrix is finally given by

\[
U = U_0 \text{diag}(1, 1, i) .
\]

It is easy to see that the predictions for the neutrino mixing parameters are same as in the case of \( \eta = 1 \) except that \( \sigma \) now takes the value of 0 instead of \( \pi/2 \). As a result, \( |M_{ee}| \) becomes

\[
|M_{ee}| = \frac{1}{3} \sqrt{\Delta m^2_{21}(1 - 3s^2_{13})} - \sqrt{\Delta m^2_{31}s^2_{13}} \simeq (1.5 - 1.9) \times 10^{-3} \text{ eV} .
\]

(27)

Since there are also only three constraints (given by Eq. (26)) on the four model parameters \( a, b, c \) and \( d \) from the measurements for the neutrino parameters, we are left with one degree of freedom in reconstructing the former from the latter: the combinations \( c^2 - a^2, d^2 - b^2 \) and \( cd - ab \) keep invariant under the following transformations

\[
\begin{pmatrix}
    ia' \\
    c'
\end{pmatrix} = \begin{pmatrix}
    \cosh \vartheta & -i \sinh \vartheta \\
    i \sinh \vartheta & \cosh \vartheta
\end{pmatrix} \begin{pmatrix}
    ia \\
    c
\end{pmatrix} , \quad \begin{pmatrix}
    ib' \\
    d'
\end{pmatrix} = \begin{pmatrix}
    \cosh \vartheta & -i \sinh \vartheta \\
    i \sinh \vartheta & \cosh \vartheta
\end{pmatrix} \begin{pmatrix}
    ib \\
    d
\end{pmatrix} .
\]

(28)

This can also be understood with the help of the Casas-Ibarra parametrization of \( M_D \): in the NO case, for \( \eta = -1 \) and \( \sigma = 0 \), taking account of the results in Eqs. (4, 6), \( iU \sqrt{D} \) has a form that only differs from that in Eq. (25) by an overall i factor for the second column. It is then easy to see that \( M_D \) in Eq. (23) will be of the form in Eq. (11) provided that \( \cos \vartheta = \cosh \vartheta \) and \( \sin \vartheta = i \sinh \vartheta \) hold. Taking advantage of the freedom of \( \vartheta \), one can also further simplify \( M_D \) in Eq. (11) by transforming one of \( a \) and \( c \) (or one of \( b \) and \( d \)) to vanishing.

2.1.2 IO case

For the scenario that one column of \( M_D \) is proportional to the first column of \( U_{\text{TBM}} \) and the other column is orthogonal to it, we parameterize \( M_D \) as

\[
M_D = \begin{pmatrix}
2a \sqrt{M_1} & b \sqrt{M_2} \\
2a \sqrt{M_1} & -(b + c) \sqrt{M_2} \\
a \sqrt{M_1} & (c - b) \sqrt{M_2}
\end{pmatrix} ,
\]

(29)

where \( a, b \) and \( c \) are generally complex. Then, the combination of the TM1 mixing and \( \mu-\tau \) reflection symmetry will lead \( M_D \) to

\[
M_D = \begin{pmatrix}
2a \sqrt{M_1} & b \sqrt{M_2} \\
2a \sqrt{M_1} & -(b + ic) \sqrt{M_2} \\
a \sqrt{M_1} & (ic - b) \sqrt{M_2}
\end{pmatrix} P_N ,
\]

(30)

where \( a, b \) and \( c \) are real now.
The resulting $M_\nu$ from $M_D$ in Eq. (30) is given by

$$M_\nu = -\begin{pmatrix} 4\eta a^2 + b^2 & 2\eta a^2 - b^2 - ibc & 2\eta a^2 - b^2 + ibc \\ 2\eta a^2 - b^2 - ibc & \eta a^2 + b^2 - c^2 + 2ibc & \eta a^2 + b^2 + c^2 \\ 2\eta a^2 - b^2 + ibc & \eta a^2 + b^2 + c^2 & \eta a^2 + b^2 - c^2 - 2ibc \end{pmatrix}.$$  
(31)

It can also be diagonalized by $U_0$ in Eq. (14) with

$$\tan 2\theta = \frac{2\sqrt{6}\eta c}{2c^2 - 3b^2},$$  
(32)

giving an IO neutrino mass spectrum as

$$m'_1 = -6\eta a^2, \quad m'_2 = -(3b^2 + 2c^2), \quad m'_3 = 0.$$  
(33)

By confronting these results with the experimental results, one obtains

$$a^2 \approx 8.3 \times 10^{-3} \text{ eV}, \quad b^2 \approx 1.0 \times 10^{-3} \text{ eV}, \quad c^2 \approx 0.024 \text{ eV},$$  
(34)

with sign($bc$) = ±1 respectively corresponding to $\sin \theta = \pm \sqrt{3}s_{13}$.

In the case of $\eta = 1$ where one has $m'_1 < 0$ and $m'_2 < 0$, the neutrino mixing matrix is finally given by $U = U_0 \text{diag}(i, i, 1)$, which gives $\sigma' \equiv \sigma - \rho = 0$. And $|M_{ee}|$ is found to be

$$|M_{ee}| = \frac{2}{3}\sqrt{|\Delta m^2_{31}|} + \frac{1}{3}\sqrt{|\Delta m^2_{31}| + \Delta m^2_{21}(1 - 3s^2_{13})} \approx 0.048 - 0.050 \text{ eV}.$$  
(35)

In the case of $\eta = -1$ where one has $m'_1 > 0$ and $m'_2 < 0$, the neutrino mixing matrix is finally given by $U = U_0 \text{diag}(1, i, 1)$, which gives $\sigma' = \pi/2$. And $|M_{ee}|$ becomes

$$|M_{ee}| = \frac{2}{3}\sqrt{|\Delta m^2_{31}|} - \frac{1}{3}\sqrt{|\Delta m^2_{31}| + \Delta m^2_{21}(1 - 3s^2_{13})} \approx 0.017 - 0.018 \text{ eV}.$$  
(36)

### 2.2 TM2 mixing combined with $\mu$-$\tau$ reflection symmetry in minimal seesaw

Let us proceed to consider the combination of the TM2 mixing and the $\mu$-$\tau$ reflection symmetry. In the minimal seesaw model, the TM2 mixing can be naturally realized by taking one column of $M_D$ to be proportional to the second column of $U_{\text{TBM}}$ and the other column to be orthogonal to it. Such an $M_D$ can be parameterized as

$$M_D = \begin{pmatrix} a\sqrt{M_1} & 2b\sqrt{M_2} \\ -a\sqrt{M_1} & (b - c)\sqrt{M_2} \\ -a\sqrt{M_1} & (b + c)\sqrt{M_2} \end{pmatrix},$$  
(37)

where $a$, $b$ and $c$ are generally complex. It is direct to verify that the resulting $M_\nu$ from such an $M_D$ respects the TM2 symmetry associated with the TM2 mixing:

$$M_\nu = R_{\text{TM2}}M_\nu R_{\text{TM2}} \quad \text{with} \quad R_{\text{TM2}} = -\frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}. $$  
(38)

Then, the combination of the TM2 mixing and the $\mu$-$\tau$ reflection symmetry leads $M_D$ to

$$M_D = \begin{pmatrix} a\sqrt{M_1} & 2b\sqrt{M_2} \\ -a\sqrt{M_1} & (b - ic)\sqrt{M_2} \\ -a\sqrt{M_1} & (b + ic)\sqrt{M_2} \end{pmatrix} P_N,$$  
(39)
where $a$, $b$ and $c$ are real now.

The resulting $M_\nu$ from $M_D$ in Eq. (39) is given by

$$M_\nu = - \begin{pmatrix} \eta a^2 + 4b^2 & -\eta a^2 + 2b^2 - 2bc & -\eta a^2 + 2b^2 + 2bc \\
-\eta a^2 + 2b^2 - 2bc & \eta a^2 + b^2 - c^2 - 2bic & \eta a^2 + b^2 + c^2 \\
-\eta a^2 + 2b^2 + 2bic & \eta a^2 + b^2 + c^2 & \eta a^2 + b^2 - c^2 + 2bic \end{pmatrix}, \quad (40)$$

which can be diagonalized by the following unitary matrix

$$U_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2} & 0 \\
1 & -\sqrt{2} & -\sqrt{3} \\
1 & -\sqrt{2} & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & i \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta \end{pmatrix}, \quad (41)$$

with

$$\tan 2\theta = \frac{2\sqrt{3}bc}{c^2 - 3b^2}. \quad (42)$$

The resulting neutrino masses can be either of NO or IO

$$\text{NO:} \quad m'_1 = 0, \quad m'_2 = -3\eta a^2, \quad m'_3 = -2(c^2 + 3b^2);$$
$$\text{IO:} \quad m'_1 = -2(c^2 + 3b^2), \quad m'_2 = -3\eta a^2, \quad m'_3 = 0. \quad (43)$$

By confronting these results with the experimental results, with the help of the relation $\sin^2 \theta = 3s_{13}^2/2$, one obtains

$$\text{NO:} \quad a^2 \simeq 2.9 \times 10^{-3} \text{ eV}, \quad b^2 \simeq 2.8 \times 10^{-4} \text{ eV}, \quad c^2 \simeq 0.024 \text{ eV};$$
$$\text{IO:} \quad a^2 \simeq 0.017 \text{ eV}, \quad b^2 \simeq 8.0 \times 10^{-3} \text{ eV}, \quad c^2 \simeq 8.4 \times 10^{-4} \text{ eV}. \quad (44)$$

with $\text{sign}(bc) = \pm 1$ (or $\mp 1$) respectively corresponding to $\sin \theta = \pm \sqrt{3/2}s_{13}$ in the NO (or IO) case.

In the NO case, for $\eta = 1$ (or $-1$) where one has $m'_2 < 0$ (or $> 0$) and $m'_3 < 0$, the neutrino mixing matrix is finally given by $U = U_0 \text{diag}(1, i, i)$ (or $U_0 \text{diag}(1, 1, i)$). With the help of the expressions in Eq. (18), the results in Eqs. (40-42) can also be successfully reproduced. In particular, here we arrive at

$$\delta = \text{sign}(\sin \theta) \frac{\pi}{2}, \quad \sigma = \frac{\pi}{2} \text{ (or 0)} \quad (45)$$

This will become more transparent if we reexpress $U_0$ in the following standard form

$$U_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -e^{-i\phi_r} & e^{i\phi_r} \\
\sqrt{-3 - 2i cos \theta \sin \theta} & \sqrt{3 - 2i sin^2 \theta} & \sqrt{3 - 2i sin^2 \theta} \\
\sqrt{3 - 2i sin \theta sin \theta} & \sqrt{3 - 2i sin^2 \theta} & \sqrt{3 - 2i sin^2 \theta} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{pmatrix}, \quad (46)$$

with $\phi_r = \text{arg}(\sqrt{3} \cos \theta - i \sin \theta)$. For these results, $|M_{ee}|$ is found to be

$$|M_{ee}| = \frac{1}{3} \sqrt{\Delta m^2_{21}} \pm \sqrt{\Delta m^2_{31}s_{13}^2} \simeq (3.8 - 4.2) \times 10^{-3} \text{ eV} \quad \text{or} \quad (1.5 - 2.0) \times 10^{-3} \text{ eV}. \quad (47)$$

In the IO case, for $\eta = 1$ (or $-1$) where one has $m'_1 < 0$ and $m'_2 < 0$ (or $> 0$), the neutrino mixing matrix is finally given by $U = U_0 \text{diag}(i, i, i)$ (or $U_0 \text{diag}(i, 1, 1)$), which gives $\sigma' = 0$ (or $\pi/2$). And $|M_{ee}|$ becomes

$$|M_{ee}| = \frac{2 - 3s_{13}^2}{3} \sqrt{|\Delta m^2_{21}|} \pm \frac{1}{3} \sqrt{|\Delta m^2_{31}| + \Delta m^2_{21}} \simeq 0.048 - 0.050 \text{ eV} \quad \text{or} \quad 0.015 - 0.016 \text{ eV}. \quad (48)$$
3 Stability against RG corrections and implications for leptogenesis

3.1 Stability against RG corrections

In the literature, the flavor symmetries are usually placed at very high energies \[19\]. In this situation, when confronting a flavor-symmetry model with the measurements performed at low energies, one needs to consider the renormalization group (RG) corrections \[20\]. During the RG evolution process, the difference among the Yukawa couplings \(y_{\alpha}\) of three charged leptons can trigger the breakings of the trimaximal and \(\mu-\tau\) reflection symmetries \[21\].

The energy dependence of the neutrino mass matrix is described by \[22\]

\[
16\pi^2 \frac{dM_\nu}{dt} = C \left( Y^\dagger_i Y_i \right)^T M_\nu + CM_\nu \left( Y^\dagger_i Y_i \right) + \alpha M_\nu, \tag{49}
\]

where \(t\) is defined as \(\ln(\mu/\mu_0)\) with \(\mu\) denoting the renormalization scale, and \(C\) and \(\alpha\) read

\[
\text{SM:} \quad C = -\frac{3}{2}, \quad \alpha \simeq -3g_2^2 + 6y_t^2 + \lambda; \\
\text{MSSM:} \quad C = 1, \quad \alpha \simeq -\frac{6}{5}g_1^2 - 6g_2^2 + 6y_t^2. \tag{50}
\]

An integration of Eq. (49) enables us to obtain the neutrino mass matrix \(M_\nu(\Lambda_{EW})\) at the electroweak scale \(\Lambda_{EW}\) from its counterpart \(M_\nu(\Lambda_{FS})\) at the flavor-symmetry scale \(\Lambda_{FS}\) as \[23\]

\[
M_\nu(\Lambda_{EW}) = I_0 \left( \begin{array}{ccc} 1 - \Delta_e & 1 - \Delta_\mu & 1 - \Delta_\tau \\ 1 - \Delta_\mu & 1 - \Delta_\tau & 1 - \Delta_e \\ 1 - \Delta_\tau & 1 - \Delta_e & 1 - \Delta_\mu \end{array} \right) M_\nu(\Lambda_{FS}) \left( \begin{array}{ccc} 1 - \Delta_e & 1 - \Delta_\mu & 1 - \Delta_\tau \\ 1 - \Delta_\mu & 1 - \Delta_\tau & 1 - \Delta_e \\ 1 - \Delta_\tau & 1 - \Delta_e & 1 - \Delta_\mu \end{array} \right), \tag{51}
\]

with

\[
I_0 = \exp \left( -\frac{1}{16\pi^2} \int_0^{\ln(\Lambda_{FS}/\Lambda_{EW})} \alpha dt \right), \quad \Delta_\alpha = \frac{C}{16\pi^2} \int_0^{\ln(\Lambda_{FS}/\Lambda_{EW})} y_\alpha^2 dt. \tag{52}
\]

We see that \(I_0\) is just an overall rescaling factor, while \(\Delta_\alpha\) can modify the structure of \(M_\nu\) due to the differences among \(y_\alpha\). Because of \(y_e \ll y_\mu \ll y_\tau \ll 1\) which give \(\Delta_e \ll \Delta_\mu \ll \Delta_\tau \ll 1\), it is an excellent approximation for us to only keep \(\Delta_\tau\) in the following discussions. In the SM framework, the RG corrections are negligibly small due to the smallness of \(y_\tau\) which gives \(\Delta_\tau \simeq O(10^{-5})\). But in the MSSM framework where \(y_\tau^2 = (1 + \tan^2 \beta) m_\nu^2 / v^2\) can be greatly enhanced by a large \(\tan \beta\), the RG corrections may become considerable. For the typical value of \(\Lambda_{FS} \sim 10^{14}\) GeV, \(\Delta_\tau\) ranges from 0.001 to 0.05 for \(\tan \beta\) from 10 to 50.

Then, given an \(M_\nu(\Lambda_{FS})\) respecting the trimaximal and \(\mu-\tau\) reflection symmetries, one can diagonalize the resulting \(M_\nu(\Lambda_{EW})\) to derive the deviations of the neutrino mixing parameters from the predictions for them in the flavor-symmetry limit. To a good approximation, we obtain

\[
\Delta \theta_{12} \simeq \frac{\Delta_\tau}{2} c_{12}s_{12}, \quad \Delta \theta_{23} \simeq \frac{\Delta_\tau}{2} \left( 1 + 2s_{12}^2 \cos 2\sigma \right), \quad \Delta \delta \simeq \frac{\Delta_\tau}{2} c_{12}s_{12} \left( \xi s_{12}^2 \cos 2\sigma - s_{13}^2 \right), \quad \Delta \sigma \simeq -2\Delta_\tau \xi c_{12}s_{12}s_{13} \cos 2\sigma, \tag{53}
\]

in the NO case; or

\[
\Delta \theta_{12} \simeq \frac{\Delta_\tau}{2} \xi^2 c_{12}s_{12}(1 + \cos 2\sigma'), \quad \Delta \theta_{23} \simeq -\frac{\Delta_\tau}{2}, \quad \Delta \delta \simeq \frac{\Delta_\tau}{2} s_{13}^2 \left[ c_{12}^2 - s_{12}^2 - \frac{2}{\xi^2} (1 - \cos 2\sigma') \right], \quad \Delta \sigma' \simeq \frac{\Delta_\tau}{2} s_{13}^2 \left[ 1 - \frac{2}{\xi^2} (c_{12}^2 - s_{12}^2)(1 - \cos 2\sigma') \right], \tag{54}
\]
in the NO case, where \( \xi \equiv \sqrt{\Delta m_{21}^2/|\Delta m_{31}^2|} \approx 0.17 \) has been defined. In obtaining these results, we have taken \( \theta_{23} = \pi/4 \) and \( \delta = -\pi/2 \). For a benchmark value of \( \Delta_\tau = 0.01 \) (corresponding to \( \tan \beta \simeq 32 \) in the MSSM framework), we arrive at

\[
\begin{align*}
\sigma = 0 : & \quad \Delta \theta_{12} \simeq 0.1^\circ, \quad \Delta \theta_{23} \simeq 0.4^\circ, \quad \Delta \delta \simeq 0.2^\circ, \quad \Delta \sigma \simeq -0.01^\circ; \\
\sigma = \frac{\pi}{2} : & \quad \Delta \theta_{12} \simeq 0.1^\circ, \quad \Delta \theta_{23} \simeq 0.2^\circ, \quad \Delta \delta \simeq -0.4^\circ, \quad \Delta \sigma \simeq 0.01^\circ,
\end{align*}
\]

(55)
in the NO case; or

\[
\begin{align*}
\sigma' = 0 : & \quad \Delta \theta_{12} \simeq 18^\circ, \quad \Delta \theta_{23} \simeq -0.3^\circ, \quad \Delta \delta \simeq 0.03^\circ, \quad \Delta \sigma \simeq 0.1^\circ; \\
\sigma' = \frac{\pi}{2} : & \quad \Delta \theta_{12} \simeq 0, \quad \Delta \theta_{23} \simeq -0.3^\circ, \quad \Delta \delta \simeq -13^\circ, \quad \Delta \sigma \simeq -5^\circ,
\end{align*}
\]

(56)
in the IO case. For other values of \( \Delta_\tau \), the results can be obtained by simply rescaling these results proportionally. We see that in the NO case \( \theta_{12}, \theta_{23}, \delta \) and \( \sigma \) are stable against the RG corrections, while in the IO case \( \theta_{12} \) (\( \delta \) and \( \sigma \)) may receive considerable RG corrections for \( \sigma' = 0 \) (\( \pi/2 \)).

### 3.2 Implications for leptogenesis

Finally, we study the implications of the combination of the trimaximal mixings and \( \mu-\tau \) reflection symmetry in the minimal seesaw model for leptogenesis. As we know, the seesaw model also offers an attractive explanation (the leptogenesis mechanism) for the baryon asymmetry of the Universe \[24\]

\[
Y_B \equiv \frac{n_B - n_{\bar{B}}}{s} = (8.67 \pm 0.15) \times 10^{-11},
\]

(57)
where \( n_B \) (\( n_{\bar{B}} \)) denotes the baryon (anti-baryon) number density and \( s \) the entropy density. The leptogenesis mechanism proceeds in the following manner \[25,26\]: a lepton asymmetry \( Y_L \equiv (n_L - n_{\bar{L}})/s \) is firstly generated from the decays of the right-handed neutrinos, and subsequently converted into the baryon asymmetry through the sphaleron process. Depending on the temperature where leptogenesis takes place (about the right-handed neutrino mass scale), there are three distinct leptogenesis regimes \[27\].

1. **Unflavored regime:** in the temperature range above \( 10^{12} \) GeV where \( y_\alpha \)-related interactions have not yet entered thermal equilibrium, three lepton flavors are indistinguishable and should be treated in a universal way.
2. **Two-flavor regime:** in the temperature range \( 10^9 - 10^{12} \) GeV where \( y_\tau \)-related interactions are in thermal equilibrium, the \( \tau \) flavor becomes distinguishable from the other two flavors which remain indistinguishable. In this regime, the \( \tau \) flavor should be treated separately from the other two flavors.
3. **Three-flavor regime:** in the temperature range below \( 10^9 \) GeV where \( y_\mu \)-related interactions are also in thermal equilibrium, all the three flavors are distinguishable and should be treated separately on their own. It is known that there exists a lower bound \( \sim 10^9 \) GeV for the right-handed neutrino masses in order for leptogenesis to be viable \[28\]. Hence we just need to consider the unflavored and two-flavor regimes.

In the unflavored regime, the final baryon asymmetry contributed by \( N_I \) (for \( I = 1,2 \)) can be calculated according to

\[
Y_{IB} = c r \varepsilon_I \kappa (\bar{m}_I),
\]

(58)
where \( c = -28/79 \) measures the conversion efficiency from the lepton asymmetry to the baryon asymmetry, and \( r \simeq 3.9 \times 10^{-3} \) is the ratio of the equilibrium number density of \( N_I \) to the entropy density. \( \varepsilon_I \) is the unflavored CP asymmetry for the decays of \( N_I \)

\[
\varepsilon_I = \frac{1}{8\pi (M_D^I M_D)^{II} v^2} \text{Im} \left[ (M_D^I M_D^I)_{IJ}^2 \right] \mathcal{F} \left( \frac{M_D^I}{M_J^I} \right),
\]

(59)
with $J \neq I$, which is a sum of the flavored CP asymmetries

$$
\varepsilon_{Ia} = \frac{1}{8\pi(M_D^aM_D^a)_{IJ}} \left\{ \text{Im} \left[ (M_D^a)_{aI}(M_D^a)_{aJ}(M_D^aM_D^a)_{IJ} \right] \mathcal{F} \left( \frac{M_2^a}{M_I^a} \right) \right. \\
+ \left. \text{Im} \left[ (M_D^a)_{aI}(M_D^a)_{aJ}(M_D^aM_D^a)^*_{IJ} \right] \mathcal{G} \left( \frac{M_2^a}{M_I^a} \right) \right\},
$$

with $\mathcal{F}(x) = \sqrt{x((2-x)/(1-x) + (1+x)\ln[x/(1+x)])}$ and $\mathcal{G}(x) = 1/(1-x)$. Finally, $\kappa(\tilde{m}_I)$ is the efficiency factor accounting for the washout effects, and its value is determined by the washout mass parameter $\tilde{m}_I$ [29], which is a sum of the flavored washout mass parameters

$$
\tilde{m}_{Ia} = \frac{|(M_D^a)_{aI}|^2}{M_I^a}.
$$

In the two-flavor regime, the final baryon asymmetry receives two contributions from $\varepsilon_{I\tau}$ and $\varepsilon_{I\gamma} = \varepsilon_{Ie} + \varepsilon_{I\mu}$, which are subject to different washout effects determined by $\tilde{m}_{I\tau}$ and $\tilde{m}_{I\gamma} = \tilde{m}_{Ie} + \tilde{m}_{I\mu}$

$$
Y_{FB} = cr \left[ \varepsilon_{I\tau} \kappa \left( \frac{390}{589} \tilde{m}_{I\tau} \right) + \varepsilon_{I\gamma} \kappa \left( \frac{417}{589} \tilde{m}_{I\gamma} \right) \right].
$$

For $M_D$ in Eqs. [30, 39], due to the orthogonality relation between their two columns, the CP asymmetries for the decays of the right-handed neutrinos vanish, prohibiting leptogenesis to work. Hence we just need to consider $M_D$ in Eq. [11] which realizes the TM1 mixing and $\mu-\tau$ reflection symmetry in the NO case. For such an $M_D$, it is direct to see that $\varepsilon_{Ie} = 0$ and $\varepsilon_{I\tau} = -\varepsilon_{I\mu}$ hold. This would give $\varepsilon_I = 0$, prohibiting the unflavored regime to work. Hence we just need to consider the two-flavor regime. In this regime, Eq. (62) becomes

$$
Y_{FB} = cr\varepsilon_{I\mu} \left[ \kappa \left( \frac{417}{589} \tilde{m}_{I\tau} \right) - \kappa \left( \frac{390}{589} \tilde{m}_{I\gamma} \right) \right].
$$

Provided that $417\tilde{m}_{I\tau}$ does not happen to be exactly equal to $390\tilde{m}_{I\gamma}$, a successful leptogenesis will be possible [30]. To be explicit, one has

$$
\varepsilon_{I\mu} = \frac{M_2}{8\pi v^2} \frac{(ad - bc)(3ac + 2bd)}{3a^2 + 2b^2} \left[ \mathcal{G} \left( \frac{M_2^2}{M_I^2} \right) + \eta \mathcal{F} \left( \frac{M_2^2}{M_I^2} \right) \right],
$$

$$
\tilde{m}_{I\gamma} = 2a^2 + b^2, \quad \tilde{m}_{I\tau} = a^2 + b^2,
$$

while the results for $\varepsilon_{I\mu}$, $\tilde{m}_{I\gamma}$ and $\tilde{m}_{I\tau}$ can be obtained by simply making the interchanges $a \leftrightarrow c$, $b \leftrightarrow d$ and $M_1 \leftrightarrow M_2$. In terms of the Casas-Ibarra parametrization of $M_D$ (with $\sin z = \sin \vartheta$ or $i \sinh \vartheta$ in the case of $\eta = 1$ or $-1$), one arrives at

$$
\varepsilon_{I\mu} = \frac{M_2}{8\sqrt{6}\pi v^2} \sqrt{m_2 m_3 (m_3 - m_2)} \cos \vartheta \sin \vartheta \left[ \mathcal{G} \left( \frac{M_2^2}{M_I^2} \right) + \mathcal{F} \left( \frac{M_2^2}{M_I^2} \right) \right],
$$

$$
\tilde{m}_{I\gamma} \simeq \frac{2}{3} m_2 \cos^2 \vartheta + \frac{1}{2} m_3 \sin^2 \vartheta, \quad \tilde{m}_{I\tau} \simeq \frac{1}{3} m_2 \cos^2 \vartheta + \frac{1}{2} m_3 \sin^2 \vartheta,
$$

in the case of $\eta = 1$, and

$$
\varepsilon_{I\mu} = \frac{M_2}{8\sqrt{6}\pi v^2} \sqrt{m_2 m_3 (m_3 + m_2)} \cosh \vartheta \sinh \vartheta \left[ \mathcal{G} \left( \frac{M_2^2}{M_I^2} \right) - \mathcal{F} \left( \frac{M_2^2}{M_I^2} \right) \right],
$$

$$
\tilde{m}_{I\gamma} \simeq \frac{2}{3} m_2 \cosh^2 \vartheta + \frac{1}{2} m_3 \sinh^2 \vartheta, \quad \tilde{m}_{I\tau} \simeq \frac{1}{3} m_2 \cosh^2 \vartheta + \frac{1}{2} m_3 \sinh^2 \vartheta,
$$

in the case of $\eta = 1$. 
Figure 1: The values of $M_1$ versus $\vartheta$ for leptogenesis to be successful in the cases of $\eta = 1$ (left) and $-1$ (right).

in the case of $\eta = -1$, while the results for $\varepsilon_{2\mu}$, $\tilde{m}_{2\gamma}$ and $\tilde{m}_{2\tau}$ can be obtained by simply making the replacements $\cos \vartheta \rightarrow -\sin \vartheta$ (or $\cosh \vartheta \rightarrow -\sinh \vartheta$), $\sin \vartheta \rightarrow \cos \vartheta$ (or $\sinh \vartheta \rightarrow \cosh \vartheta$) and $M_1 \leftrightarrow M_2$.

With the above results, we now can calculate the final baryon asymmetry. Let us first consider the case of $M_1 < M_2$. In this case the contribution to leptogenesis mainly comes from $N_1$ while that from $N_2$ suffers from its washout effects. In Figure 1 (left panel for the case of $\eta = 1$ and right panel for the case of $\eta = -1$) we have shown the values of $M_1$ versus $\vartheta$ for leptogenesis to be successful. These results are obtained by allowing the neutrino parameters to vary in their $3\sigma$ ranges and $M_2$ to vary in the range $(2, 10)M_1$. We see that the minimum value of $M_1$ that can accommodate a successful leptogenesis is about $5 \times 10^{10}$ (or $8 \times 10^{10}$) GeV in the case of $\eta = 1$ (or $-1$). And leptogenesis can not work successfully for a very small or large $|\vartheta|$, which can be easily understood from Eqs. (65, 66): in the case of $\eta = 1$, $\varepsilon_{1\mu}$ is proportional to $\cos \vartheta \sin \vartheta$ which will get highly suppressed for $|\vartheta| \rightarrow 0$ or $\pi/2$; in the case of $\eta = -1$, $\varepsilon_{1\mu}$ is proportional to $\sinh \vartheta$ which will get highly suppressed for $|\vartheta| \rightarrow 0$, and $\tilde{m}_{1\gamma}$ and $\tilde{m}_{1\tau}$ would increase rapidly for large values of $|\vartheta|$ and consequently highly suppress the corresponding efficiency factors. As for the case of $M_2 < M_1$ where the contribution to leptogenesis mainly comes from $N_2$, the results are as follows: in the case of $\eta = 1$, the values of $M_2$ versus $\vartheta$ for leptogenesis to be successful (obtained by allowing $M_1$ to vary in the range $(2, 10)M_2$ instead) are same as those in the left panel of Figure 1 expect for the translation $\vartheta \rightarrow \vartheta + \pi/2$. This can be understood from the observation made below Eq. (66). But in the case of $\eta = -1$ there exists no parameter space of $M_2$ versus $\vartheta$ for leptogenesis to be successful.

4 Summary

As we know, the most popular and natural way of generating the small neutrino masses is the type-I seesaw mechanism, which also provides an attractive explanation for the baryon asymmetry of the Universe via the leptogenesis mechanism. But the general type-I seesaw model with three right-handed neutrinos has the shortcoming that the number of its parameters is much larger than that of the measurable neutrino parameters so that it lacks predictive power. In the literature, a typical approach to reducing the number of its parameters and consequently improving its predictability is to reduce the number of right-handed neutrinos to two (giving the minimal seesaw model).

On the other hand, the experimental results that the neutrino mixing angles are close to some
special values and $\delta$ is likely to be around $-\pi/2$ suggest that there may be some underlying flavor symmetry in the lepton sector. Along this direction, the trimaximal mixings and $\mu$-$\tau$ reflection symmetry are two typical examples and have attracted a lot of attention due to their interesting phenomenological consequences.

In this paper, we have studied a combination of the trimaximal (TM1 and TM2) mixings and $\mu$-$\tau$ reflection symmetry in the minimal seesaw model. Such a scenario is highly restrictive and predictive: as shown in Eqs. (11, 30, 39), $M_D$ has a very simple form which only contains three or four real parameters. And the trimaximal mixings and $\mu$-$\tau$ reflection symmetry will help us pin down all the neutrino mixing parameters except for $\theta_{13}$, while the minimal seesaw model will help us pin down the neutrino masses. These results lead to some definite predictions for the effective neutrino mass that governs the rate of the neutrinoless double beta decays, which can be tested by future measurements.

We then have studied the stability of this scenario against the RG corrections. From the results in Eqs. (55-56) we see that in the NO case $\theta_{12}$, $\theta_{23}$, $\delta$ and $\sigma$ are stable against the RG corrections, while in the IO case $\theta_{12}$ ($\delta$ and $\sigma$) may receive considerable RG corrections for $\sigma' = 0$ ($\pi/2$) in the MSSM framework with large $\tan \beta$ values.

We finally have studied the implications of this scenario for leptogenesis. It turns out that only $M_D$ in Eq. (11) which realizes the TM1 mixing and $\mu$-$\tau$ reflection symmetry in the NO case has chance to give a successful leptogenesis in the two-flavor regime. The values of $M_1$ versus $\theta$ for leptogenesis to be successful are shown in Figure 1. It is found that the minimum value of $M_1$ that can accommodate a successful leptogenesis is about $5 \times 10^{10}$ (or $8 \times 10^{10}$) GeV in the case of $\eta = 1$ (or $-1$).

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