The Secrecy Capacity Region of the Gaussian MIMO Broadcast Channel

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Abstract

In this paper, we consider a scenario where a source node wishes to broadcast two confidential messages for two respective receivers via a Gaussian MIMO broadcast channel. A wire-tapper also receives the transmitted signal via another MIMO channel. First we assumed that the channels are degraded and the wire-tapper has the worst channel. We establish the capacity region of this scenario. Our achievability scheme is a combination of the superposition of Gaussian codes and randomization within the layers which we will refer to as Secret Superposition Coding. For the outerbound, we use the notion of enhanced channel to show that the secret superposition of Gaussian codes is optimal. We show that we only need to enhance the channels of the legitimate receivers, and the channel of the eavesdropper remains unchanged. Then we extend the result of the degraded case to non-degraded case. We show that the secret superposition of Gaussian codes along with successive decoding cannot work when the channels are not degraded. we develop a Secret Dirty Paper Coding (SDPC) scheme and show that SDPC is optimal for this channel. Finally, we investigate practical characterizations for the specific scenario in which the transmitter and the eavesdropper have multiple antennas, while both intended receivers have a single antenna. We characterize the secrecy capacity region in terms of generalized eigenvalues of the receivers channel and the eavesdropper channel. We refer to this configuration as the MISOME case. In high SNR we show that the capacity region is a convex closure of two rectangular regions.

I. INTRODUCTION

Recently there has been significant research conducted in both theoretical and practical aspects of wireless communication systems with Multiple-Input Multiple-Output (MIMO) antennas. Most works have focused on the role of MIMO in enhancing the throughput and robustness. In this work, however, we focus on the role of such multiple antennas in enhancing wireless security.

The information-theoretic single user secure communication problem was first characterized by Wyner in [1]. Wyner considered a scenario in which a wire-tapper receives the transmitted signal over a degraded channel with respect to the legitimate receiver’s channel. He measured the level of ignorance at the eavesdropper by its equivocation and characterized the capacity-equivocation region. Wyner’s work was then extended to the general broadcast channel with confidential messages by Csiszar and Korner [2]. They considered transmitting confidential information to the legitimate receiver while transmitting

1Financial support provided by Nortel and the corresponding matching funds by the Natural Sciences and Engineering Research Council of Canada (NSERC), and Ontario Centers of Excellence (OCE) are gratefully acknowledged.
a common information to both the legitimate receiver and the wire-tapper. They established a capacity-equivocation region of this channel. The secrecy capacity for the Gaussian wire-tap channel was characterized by Leung-Yan-Cheong in [3].

The Gaussian MIMO wire-tap channel has recently been considered by Khisti and Wornell in [4], [5]. Finding the optimal distribution, which maximizes the secrecy capacity for this channel is a nonconvex problem. Khisti and Wornell, however, followed an indirect approach to evaluate the secrecy capacity of Csiszar and Korner. They used a genie-aided upper bound and characterized the secrecy capacity as the saddle-value of a min-max problem to show that Gaussian distribution is optimal. Motivated by the broadcast nature of the wireless communication systems, we considered the secure broadcast channel with an external eavesdropper in [6], [7] and characterized the secrecy capacity region of the degraded broadcast channel and showed that the secret superposition coding is optimal. Parallel and independent with our work of [6], [7], Ekrem et. al. in [8], [9] established the secrecy capacity region of the degraded broadcast channel with an external eavesdropper. The problem of Gaussian MIMO broadcast channel without an external eavesdropper is also solved by Lui. et. al. in [10]–[12].

The capacity region of the conventional Gaussian MIMO broadcast channel is studied in [13] by Weingarten et al. The notion of an enhanced broadcast channel is introduced in this work and is used jointly with entropy power inequality to characterize the capacity region of the degraded vector Gaussian broadcast channel. They showed that the superposition of Gaussian codes is optimal for the degraded vector Gaussian broadcast channel and that dirty-paper coding is optimal for the nondegraded case.

In the conference version of this paper (see [14]), we established the secrecy capacity region of the degraded vector Gaussian broadcast channel. Our achievability scheme, was a combination of the superposition of Gaussian codes and randomization within the layers which we referred to as Secret Superposition Coding. For the outerbound, we used the notion of enhanced channel to show that the secret superposition of Gaussian codes is optimal. In this paper, we aim to characterize the secrecy capacity region of a general secure Gaussian MIMO broadcast channel. Our achievability scheme is a combination of the dirty paper coding of Gaussian codes and randomization within the layers. To prove the converse, we use the notion of enhanced channel and show that the secret dirty paper coding of Gaussian codes is optimal. We investigate practical characterizations for the specific scenario in which the transmitter and the eavesdropper have multiple antennas, while both intended receivers have a single antenna. This model is motivated when a base station wishes to broadcast secure information for small mobile units. In this scenario small mobile units have single antenna while the base station and the eavesdropper can afford multiple antennas. We characterize the secrecy capacity region in terms of generalized eigenvalues of the receivers channel and the eavesdropper channel. We refer to this configuration as the MISOME case. In high SNR we show that the capacity region is a convex closure of two rectangular regions.

Parallel with our work, Ekrem et. al [15] and Liu et. al. [16], [17], independently considered the secure MIMO broadcast channel and established its capacity region. Ekrem et. al. used the relationships between the minimum-mean-square-error and the mutual information, and equivalently, the relationships between the Fisher information and the differential entropy to provide the converse proof. Liu et. al. considered the vector Gaussian MIMO broadcast channel with and without an external eavesdropper. They presented a vector generalization of Costa’s Entropy Power Inequality to provide their converse proof. In our proof, however, we enhance the channels properly and show that the enhanced channels are proportional. We then use the proportionality characteristic to provide the converse proof. The rest of the paper is organized as follows. In section II we introduce some preliminaries. In section III, we establish the secrecy capacity region of the degraded vector Gaussian broadcast channel. We extend our results to non-degraded and non vector case in section IV. In Section V, we investigate the MISOME case. Section VI concludes the paper.
II. PRELIMINARIES

Consider a Secure Gaussian Multiple-Input Multiple-Output Broadcast Channel (SGMBC) as depicted in Fig. 1. In this confidential setting, the transmitter wishes to send two independent messages \((W_1, W_2)\) to the respective receivers in \(n\) uses of the channel and prevent the eavesdropper from having any information about the messages. At a specific time, the signals received by the destinations and the eavesdropper are given by

\[
\begin{align*}
    y_1 &= H_1 x + n_1, \\
    y_2 &= H_2 x + n_2, \\
    z &= H_3 x + n_3,
\end{align*}
\]

where

- \(x\) is a real input vector of size \(t \times 1\) under an input covariance constraint. We require that \(E[xx^T] \preceq S\) for a positive semi-definite matrix \(S \succeq 0\). Here, \(\prec, \succeq, \succ,\) and \(\preceq\) represent partial ordering between symmetric matrices where \(B \preceq A\) means that \((B - A)\) is a positive semi-definite matrix.
- \(y_1, y_2,\) and \(z\) are real output vectors which are received by the destinations and the eavesdropper respectively. These are vectors of size \(r_1 \times 1, r_2 \times 1,\) and \(r_3 \times 1\), respectively.
- \(H_1, H_2,\) and \(H_3\) are fixed, real gain matrices which model the channel gains between the transmitter and the receivers. These are matrices of size \(r_1 \times t, r_2 \times t,\) and \(r_3 \times t\) respectively. The channel state information is assumed to be known perfectly at the transmitter and at all receivers.
- \(n_1, n_2\) and \(n_3\) are real Gaussian random vectors with zero means and covariance matrices \(N_1 = E[n_1n_1^T] > 0,\)
  \(N_2 = E[n_2n_2^T] > 0,\) and \(N_3 = E[n_3n_3^T] > 0\) respectively.

Let \(W_1\) and \(W_2\) denote the the message indices of user 1 and user 2, respectively. Furthermore, let \(\overline{X}, \overline{Y}_1, \overline{Y}_2,\) and \(\overline{Z}\) denote the random channel input and random channel outputs matrices over a block of \(n\) samples. Let \(\overline{V}_1, \overline{V}_2,\) and \(\overline{V}_3\) denote the additive noises of the channels. Thus,

\[
\begin{align*}
    \overline{Y}_1 &= H_1 \overline{X} + \overline{V}_1, \\
    \overline{Y}_2 &= H_2 \overline{X} + \overline{V}_2, \\
    \overline{Z} &= H_3 \overline{X} + \overline{V}_3.
\end{align*}
\]

Note that \(\overline{V}_i\) is an \(r_i \times n\) random matrix and \(H_i\) is an \(r_i \times t\) deterministic matrix where \(i = 1, 2, 3\). The columns of \(\overline{V}_i\) are independent Gaussian random vectors with covariance matrices \(N_i\) for \(i = 1, 2, 3\). In addition \(\overline{V}_i\) is independent of \(\overline{X}, W_1\)
and $W_2$. A $((2^nR_1, 2^nR_2), n)$ code for the above channel consists of a stochastic encoder
\[ f : \{(1, 2, \ldots, 2^n R_1) \times (1, 2, \ldots, 2^n R_2)\} \rightarrow \mathcal{X}, \] (3)
and two decoders,
\[ g_1 : \mathcal{Y}_1 \rightarrow \{1, 2, \ldots, 2^n R_1\}, \] (4)
and
\[ g_2 : \mathcal{Y}_2 \rightarrow \{1, 2, \ldots, 2^n R_2\}. \] (5)
where a script letter with double overline denotes the finite alphabet of a random vector. The average probability of error is defined as the probability that the decoded messages are not equal to the transmitted messages; that is,
\[ P_e(n) = P(g_1(\mathcal{Y}_1) \neq W_1 \cup g_2(\mathcal{Y}_2) \neq W_2). \] (6)

The secrecy levels of confidential messages $W_1$ and $W_2$ are measured at the eavesdropper in terms of equivocation rates, which are defined as follows.

**Definition 1:** The equivocation rates $R_{e1}$, $R_{e2}$ and $R_{e12}$ for the secure broadcast channel are:
\[ R_{e1} = \frac{1}{n} H(W_1 | \mathcal{Z}), \] (7)
\[ R_{e2} = \frac{1}{n} H(W_2 | \mathcal{Z}), \]
\[ R_{e12} = \frac{1}{n} H(W_1, W_2 | \mathcal{Z}). \]
The perfect secrecy rates $R_1$ and $R_2$ are the amount of information that can be sent to the legitimate receivers both reliably and confidentially.

**Definition 2:** A secrecy rate pair $(R_1, R_2)$ is said to be achievable if for any $\epsilon > 0, \epsilon_1 > 0, \epsilon_2 > 0, \epsilon_3 > 0$, there exists a sequence of $((2^nR_1, 2^nR_2), n)$ codes, such that for sufficiently large $n$,
\[ P_e^{(n)} \leq \epsilon, \] (8)
\[ R_{e1} \geq R_1 - \epsilon_1, \] (9)
\[ R_{e2} \geq R_2 - \epsilon_2, \] (10)
\[ R_{e12} \geq R_1 + R_2 - \epsilon_3. \] (11)

In the above definition, the first condition concerns the reliability, while the other conditions guarantee perfect secrecy for each individual message and both messages as well. The model presented in (1) is SGMBC. However, we will initially consider two subclasses of this channel and then generalize our results for the SGMBC.

The first subclass that we will consider is the Secure Aligned Degraded MIMO Broadcast Channel (SADBC). The MIMO broadcast channel of (1) is said to be aligned if the number of transmit antennas is equal to the number of receive antennas at each of the users and the eavesdropper ($t = r_1 = r_2 = r_3$) and the gain matrices are all identity matrices ($H_1 = H_2 = H_3 = I$).

Furthermore, if the additive noise vectors’ covariance matrices are ordered such that $0 < N_1 \preceq N_2 \preceq N_3$, then the channel is SADBC.

The second subclass we consider is a generalization of the SADBC. The MIMO broadcast channel of (1) is said to be Secure Aligned MIMO Broadcast Channel (SAMBC) if it is aligned and not necessarily degraded. In other words, the additive noise
vector covariance matrices are not necessarily ordered. A time sample of an SAMBC is given by the following expressions,

\[ y_1 = x + n_1, \]
\[ y_2 = x + n_2, \]
\[ z = x + n_3, \]

where, \( y_1, y_2, z, x \) are real vectors of size \( t \times 1 \) and \( n_1, n_2, \) and \( n_3 \) are independent and real Gaussian noise vectors such that \( N_i = E[n_i n_i^T] \succ 0_{t \times t} \) for \( i = 1, 2, 3. \)

### III. The Capacity Region of the SADBC

In this section, we characterize the capacity region of the SADBC. In [6], we considered the degraded broadcast channel with confidential messages and establish its secrecy capacity region.

**Theorem 1:** The capacity region for transmitting independent secret messages over the degraded broadcast channel is the convex hull of the closure of all \((R_1, R_2)\) satisfying

\[ R_1 \leq I(X; Y_1|U) - I(X; Z|U), \]
\[ R_2 \leq I(U; Y_2) - I(U; Z), \]

for some joint distribution \( P(u)P(x|u)P(y_1, y_2, z|x). \)

**Proof:** Our achievable coding scheme is based on Cover’s superposition scheme and random binning. We refer to this scheme as the Secret Superposition Scheme. In this scheme, randomization in the first layer increases the secrecy rate of the second layer. Our converse proof is based on a combination of the converse proof of the conventional degraded broadcast channel and Csiszar Lemma. Please see [6], [7] for details.

Note that evaluating (13) and (14) involves solving a functional, nonconvex optimization problem. Usually nontrivial techniques and strong inequalities are used to solve optimization problems of this type. Indeed, for the single antenna case, [18], [19] successfully evaluated the capacity expression of (13) and (14). Liu et al. in [20] evaluated the capacity expression of MIMO wire-tap channel by using the channel enhancement method. In the following section, we state and prove our result for the capacity region of SADBC.

First, we define the achievable rate region due to Gaussian codebook under a covariance matrix constraint \( S \succ 0. \) The achievability scheme of Theorem 1 is the secret superposition of Gaussian codes and successive decoding at the first receiver. According to the above theorem, for any covariance matrix input constraint \( S \) and two semi-definite matrices \( B_1 \succ 0 \) and \( B_2 \succ 0 \) such that \( B_1 + B_2 \preceq S, \) it is possible to achieve the following rates,

\[ R_1^G(B_{1,2}, N_{1,2,3}) = \frac{1}{2} \left[ \log |N_1^{-1}(B_1 + N_1)| - \frac{1}{2} \log |N_3^{-1}(B_1 + N_3)| \right]^{+}, \]
\[ R_2^G(B_{1,2}, N_{1,2,3}) = \frac{1}{2} \left[ \log \frac{|B_1 + B_2 + N_2|}{|B_1 + N_2|} - \frac{1}{2} \log \frac{|B_1 + B_2 + N_3|}{|B_1 + N_3|} \right]^{+}. \]

The Gaussian rate region of SADBC is defined as follows.

**Definition 3:** Let \( S \) be a positive semi-definite matrix. Then, the Gaussian rate region of SADBC under a covariance matrix constraint \( S \) is given by

\[ R^G(S, N_{1,2,3}) = \left\{ \left( R_1^G(B_{1,2}, N_{1,2,3}), R_2^G(B_{1,2}, N_{1,2,3}) \right) \mid s.t \ S - (B_1 + B_2) \succeq 0, \ B_k \succeq 0, \ k = 1, 2 \right\}. \]
We will show that $R^G(S, N_{1,2,3})$ is the capacity region of the SADBC. Before that, certain preliminaries need to be addressed. We begin by characterizing the boundary of the Gaussian rate region.

Remark 1: Note that in characterizing the capacity region of the conventional Gaussian MIMO broadcast channel Weingarten et al. [13] proved that on the boundary of the above region we have $B_1 + B_2 = S$ which maximizes the rate $R_2$. In our argument, however, the boundary is not characterized with this equality as rate $R_2$ may decreases by increasing $B_1 + B_2$.

Definition 4: The rate vector $R^* = (R_1, R_2)$ is said to be an optimal Gaussian rate vector under the covariance matrix $S$, if $R^* \in R^G(S, N_{1,2,3})$ and if there is no other rate vector $\tilde{R}^* = (\tilde{R}_1, \tilde{R}_2) \in R^G(S, N_{1,2,3})$ such that $\tilde{R}_1 \geq R_1$ and $\tilde{R}_2 \geq R_2$ where at least one of the inequalities is strict. The set of positive semi-definite matrices $(B_1^*, B_2^*)$ such that $B_1^* + B_2^* \preceq S$ is said to be realizing matrices of an optimal Gaussian rate vector if the rate vector $(R_1^G(B_{1,2}^*, N_{1,2,3}), R_2^G(B_{1,2}^*, N_{1,2,3}))$ is an optimal Gaussian rate vector.

In general, there is no known closed form solution for the realizing matrices of an optimal Gaussian rate vector. Note that finding an optimal Gaussian rate vector once again, involves solving a nonconvex optimization problem. The realizing matrices of an optimal Gaussian rate vector, $B_1^*, B_2^*$ are the solution of the following optimization problem:

$$\max_{(B_1, B_2)} R_1^G(B_{1,2}, N_{1,2,3}) + \mu R_2^G(B_{1,2}, N_{1,2,3}) \quad (15)$$

s.t $B_1 \succeq 0, \quad B_2 \preceq 0, \quad B_1 + B_2 \preceq S$,

where $\mu \geq 1$. Next, we define a class of enhanced channel. The enhanced channel has some fundamental properties which help us to characterize the secrecy capacity region. We will discuss its properties later on.

Definition 5: A SADBC with noise covariance matrices $(N'_1, N'_2, N'_3)$ is an enhanced version of another SADBC with noise covariance matrices $(N_1, N_2, N_3)$ if

$$N'_1 \preceq N_1, \quad N'_2 \preceq N_2, \quad N'_3 = N_3, \quad N'_1 \preceq N_2.'$$

Obviously, the capacity region of the enhanced version contains the capacity region of the original channel. Note that in characterizing the capacity region of the conventional Gaussian MIMO broadcast channel, all channels must be enhanced by reducing the noise covariance matrices. In our scheme, however, we only enhance the channels for the legitimate receivers and the channel of the eavesdropper remains unchanged. This is due to the fact that the capacity region of the enhanced channel must contain the original capacity region. Reducing the noise covariance matrix of the eavesdropper’s channel, however, may reduce the secrecy capacity region. The following theorem connects the definitions of the optimal Gaussian rate vector and the enhanced channel.

Theorem 2: Consider a SADBC with positive definite noise covariance matrices $(N_1, N_2, N_3)$. Let $B_1^*$ and $B_2^*$ be realizing matrices of an optimal Gaussian rate vector under a transmit covariance matrix constraint $S > 0$. There then exists an enhanced SADBC with noise covariance matrices $(N'_1, N'_2, N'_3)$ that the following properties hold.

1) Enhancement:

$$N'_1 \preceq N_1, \quad N'_2 \preceq N_2, \quad N'_3 = N_3, \quad N'_1 \preceq N_2.'$$

2) Proportionality:

There exists an $\alpha \geq 0$ and a matrix $A$ such that

$$(I - A)(B_1 + N'_1) = \alpha A(B_1 + N'_3).$$

3) Rate and optimality preservation:

$$R_k^G(B_{1,2}^*, N_{1,2,3}) = R_k^G(B_{1,2}^*, N_{1,2,3}) \quad \forall k = 1, 2,$$

furthermore, $B_1^*$ and $B_2^*$ are realizing matrices of an optimal Gaussian rate vector in the enhanced channel.
Proof: The realizing matrices $B_1^*$ and $B_2^*$ are the solution of the optimization problem of [15]. Using Lagrange Multiplier method, this constraint optimization problem is equivalent to the following unconditional optimization problem:

$$\max_{(B_1, B_2)} \mathcal{R}_1^C(B_{1,2}, N_{1,2,3}) + \mu \mathcal{R}_2^C(B_{1,2}, N_{1,2,3}) + Tr\{B_1 O_1\}$$

$$+ Tr\{B_2 O_2\} + Tr\{(S - B_1 - B_2)O_3\},$$

where $O_1$, $O_2$, and $O_3$ are positive semi-definite $t \times t$ matrices such that $Tr\{B_1^* O_1\} = 0$, $Tr\{B_2^* O_2\} = 0$, and $Tr\{(S - B_1^* - B_2^*)O_3\} = 0$. As all $B_k^*$, $k = 1, 2$, $O_i$, $i = 1, 2, 3$, and $S - B_1^* - B_2^*$ are positive semi-definite matrices, then we must have $B_k^* O_k = 0$, $k = 1, 2$ and $(S - B_1^* - B_2^*)O_3 = 0$. According to the necessary KKT conditions, and after some manipulations we have:

$$(B_1^* + N_1)^{-1} + (\mu - 1)(B_1^* + N_3)^{-1} + O_1 = \mu(B_1^* + N_2)^{-1} + O_2,$$

(17)

$$\mu(B_1^* + B_2^* + N_2)^{-1} + O_2 = \mu(B_1^* + B_2^* + N_3)^{-1} + O_3.$$  

(18)

We choose the noise covariance matrices of the enhanced SADBC as the following:

$$N_1' = (N_1^{-1} + O_1)^{-1},$$

(19)

$$N_2' = \left((B_1^* + N_2)^{-1} + \frac{1}{\mu}O_2\right)^{-1} - B_1^*,$$

$$N_3' = N_3.$$

As $O_1 \geq 0$ and $O_2 \geq 0$, then the above choice has the enhancement property. Note that

$$(B_1^* + N_1)^{-1} + O_1 = (B_1^* + N_1) \quad (I + (B_1^* + N_1)O_1))^{-1}$$

(20)

$$(a) = (I + N_1O_1)^{-1} (B_1^* + N_1) - B_1^*,

= (I + N_1O_1)^{-1} ((B_1^* + N_1) - (I + N_1O_1)B_1^*) + B_1^*

(b) = (I + N_1O_1)^{-1} N_1 + B_1^*$$

$$= (N_1(N_1^{-1} + O_1))^{-1} N_1 + B_1^*$$

$$= (N_1^{-1} + O_1)^{-1} + B_1^*$$

$$= B_1^* + N_1'.$$

where $(a)$ and $(b)$ follows from the fact that $B_1^* O_1 = 0$. Therefore, according to (17) the following property holds for the enhanced channel.

$$(B_1^* + N_1')^{-1} + (\mu - 1)(B_1^* + N_3')^{-1} = \mu(B_1^* + N_2)^{-1}.$$  

Since $N_1' \leq N_2' \leq N_3'$ then, there exists a matrix $A$ such that $N_2' = (I - A)N_1' + AN_3'$ where $A = (N_2' - N_1')(N_3' - N_1')^{-1}$. Therefore, the above equation can be written as.

$$(B_1^* + N_1')^{-1} + (\mu - 1)(B_1^* + N_3')^{-1} = \mu \left[(I - A)(B_1^* + N_1') + A(B_1^* + N_3') \right]^{-1}.$$  

Let $(I - A)(B_1^* + N_1') = \alpha A(B_1^* + N_3')$ then after some manipulations, the above equation becomes

$$\frac{1}{\alpha} + (\mu - 1 - \frac{1}{\alpha})A = \frac{\mu}{\alpha + 1}I,$$

(21)
The above equation is satisfied by $\alpha = \frac{1}{\mu - 1}$ which completes the proportionality property. We can now prove the rate conservation property. The expression $\frac{B_1^* + N_1^*}{|N_1^*|}$ can be written as follow.

\[
\frac{|B_1^* + N_1'|}{|N_1'|} = \frac{|I|}{N_1' (B_1^* + N_1')^{-1}} = \frac{|I|}{(B_1^* + N_1') (B_1^* + N_1')^{-1}} = \frac{|I|}{I - B_1^* (B_1^* + N_1')^{-1}} = \frac{|I - B_1^* ((B_1^* + N_1')^{-1} + O_1)|}{I - B_1^* (B_1^* + N_1')^{-1}} = \frac{B_1^* + N_1'}{|N_1'|},
\]

where (a) once again follows from the fact that $B_1^* O_1 = 0$. To complete the proof of rate conservation, consider the following equalities.

\[
\frac{|B_1^* + B_2^* + N_2|}{|B_1^* + N_2|} = \frac{|B_2^* (B_1^* + N_2)^{-1} + I|}{|I|} = \frac{|B_2^* ((B_1^* + N_2)^{-1} + \frac{1}{\mu} O_2) + I|}{|I|} = \frac{B_1^* + B_2^* + N_2|}{|B_1^* + N_2|},
\]

where (a) follows from the fact $B_2^* O_2 = 0$. Therefore, according to (22), (23), and the fact that $N_3' = N_3$, the rate preservation property holds for the enhanced channel. To prove the optimality preservation, we need to show that $(B_1^*, B_2)$ are also realizing matrices of an optimal Gaussian rate vector in the enhanced channel. For that purpose, we show that the necessary KKT conditions for the enhanced channel coincides with the KKT conditions of the original channel. The expression $\mu (B_1^* + B_2^* + N_2)^{-1}$ can be written as follows.

\[
\mu (B_1^* + B_2^* + N_2)^{-1} = \mu (B_1^* + B_2^* + \left(N_2^{-1} + \frac{1}{\mu} O_2\right)^{-1})^{-1} = \mu (B_1^* + B_2^* + \left(N_2^{-1} + \frac{1}{\mu} O_2\right)^{-1} B_2^*)^{-1} = \mu (B_1^* + B_2^* + \left(N_2^{-1} B_2^*\right)^{-1})^{-1} = \mu (B_1^* + B_2^* + N_2)^{-1},
\]

where (a) follows from the definition of $N_2$ and (b) follows from the fact that $B_2^* O_2 = 0$. Therefore, according to (22), and the above equation, the KKT conditions of (17) and (18) for the original channel can be written as follows for the enhanced
According to Theorem 1, the preceding expression can be rewritten as follows:

\[
(B_1^* + N_1')^{-1} + (\mu - 1)(B_1^* + N_3')^{-1} = \mu (B_1^* + N_2')^{-1},
\]

\[
\mu (B_1^* + B_2^* + N_2')^{-1} = \mu (B_1^* + B_2^* + N_3')^{-1} + O_3 - O_2.
\]

where \(O_3 - O_2 \geq 0\). Therefore, \(R_1^G(B_{1,2}, N_{1,2,3}) + \mu R_2^G(B_{1,2}, N_{1,2,3}')\) is maximized when \(B_k = B_k^*\) for \(k = 1, 2\). We can now use Theorem 2 to prove that \(R_k^G(S, N_{1,2,3})\) is the capacity region of the SADBC. We follow Bergman’s approach [21] to prove a contradiction. Note that since the original channel is not proportional, we cannot apply Bergman’s proof on the original channel directly. Here we apply his proof on the enhanced channel instead.

**Theorem 3:** Consider a SADBC with positive definite noise covariance matrices \((N_1, N_2, N_3)\). Let \(C(S, N_{1,2,3})\) denote the capacity region of the SADBC under a covariance matrix constraint \(S > 0\). Then, \(C(S, N_{1,2,3}) = R^G(S, N_{1,2,3})\).

**Proof:** The achievability scheme is secret superposition coding with Gaussian codebook. For the converse proof, we use a contradiction argument and assume that there exists an achievable rate vector \(\bar{R} = (R_1, R_2)\) which is not in the Gaussian region. We can apply the steps of Bergman’s proof of [21] on the enhanced channel to show that this assumption is impossible. Since \(\bar{R} \notin R^G(S, N_{1,2,3})\), there exist realizing matrices of an optimal Gaussian rate vector \(B_1^*, B_2^*\) such that

\[
R_1 \geq R_1^G(B_{1,2}^*, N_{1,2,3}),
\]

\[
R_2 \geq R_2^G(B_{1,2}^*, N_{1,2,3}) + b,
\]

for some \(b > 0\). We know by Theorem 2 that for every set of realizing matrices of an optimal Gaussian rate vector \(B_1^*, B_2^*\), there exists an enhanced SADBC with noise covariance matrices \(N_1', N_2'\), such that the proportionality and rate preservation properties hold. According to the rate preservation property, we have \(R_k^G(B_{1,2}^*, N_{1,2,3}) = R_k^G(B_{1,2}^*, N_{1,2}^*)\), \(k = 1, 2\). Therefore, the preceding expression can be rewritten as follows:

\[
R_1 \geq R_1^G(B_{1,2}^*, N_{1,2,3}) = R_1^G(B_{1,2}^*, N_{1,2,3}'),
\]

\[
R_2 \geq R_2^G(B_{1,2}^*, N_{1,2,3}) + b = R_2^G(B_{1,2}^*, N_{1,2,3}) + b.
\]

According to the Theorem 1, \(R_1\) and \(R_2\) are bounded as follows:

\[
R_1 \leq h(y_1 | u) - h(z | u) - (h(y_1 | x, u) - h(z | x, u))
\]

\[
R_2 \leq h(y_2) - h(z) - (h(y_2 | u) - h(z | u))
\]

Let \(y_1'\) and \(y_2'\) denote the enhanced channel outputs of each of the receiving users. As \(u \rightarrow y_k' \rightarrow y_k\) forms a Markov chain for \(k = 1, 2\) and \(z' = z\), then we can use the data processing inequality to rewrite the above region as follows:

\[
R_1 \leq h(y_1' | u) - h(z' | u) - \left( h(y_1' | x, u) - h(z' | x, u) \right)
\]

\[
= h(y_1' | u) - h(z' | u) - \frac{1}{2} \left( \log |N_1'|- \log |N_3'| \right)
\]

\[
R_2 \leq h(y_2') - h(z') - \left( h(y_2' | u) - h(z' | u) \right)
\]

Now, the inequalities of (28) and (29) have shifted to the enhanced channel.

Since \(R_1 > R_1^G(B_{1,2}, N_{1,2,3})\), the inequality (29) means that

\[
h(y_1' | u) - h(z' | u) > \frac{1}{2} \left( \log |B_1^* + N_1'| - \log |B_1^* + N_3'| \right)
\]
By the definition of matrix $A$ and since $y'_1 \rightarrow y'_2 \rightarrow z'$ forms a Markov chain, the received signals $z'$ and $y'_2$ can be written as $z' = y'_1 + \tilde{n}$ and $y'_2 = y'_1 + A^\dagger \tilde{n}$ where $\tilde{n}$ is an independent Gaussian noise with covariance matrix $\tilde{N} = N_3 - N_1$.

According to Costa’s Entropy Power Inequality and the previous inequality, we have

$$h(y'_2|u) - h(z'|u) \geq \frac{t}{2} \log \left( |I - A|^\dagger 2^{h(y'_2|u) - h(z'|u)} + |A|^\dagger \right)$$

$$\geq \frac{t}{2} \log \left( \frac{|I - A|^\dagger |B_1^* + N_1^*|^\dagger}{|B_1^* + N_3^*|^\dagger} + |A|^\dagger \right)$$

$$\geq \frac{1}{2} \log |B_1^* + N_2'| - \frac{1}{2} \log |B_1^* + N_3'| \quad (31)$$

where (a) is due to the proportionality property. Using (30) and the fact that $R_2 > R_2^C(B_{1,2}, N_{1,2,3})$, the inequality (30) means that

$$h(y'_2) - h(z') \geq R_2 + h(y'_2|u) - h(z'|u) >$$

$$\frac{1}{2} \log |B_1^* + B_2^* + N_2'| - \frac{1}{2} \log (B_1^* + B_2^* + N_3')$$

On the other hand, Gaussian distribution maximizes $h(x + n_2) - h(x + n_3)$ (See [22]) and $(B_1^*, B_2^*)$ satisfying the KKT conditions of (26). Therefore, the above inequality is a contradiction.

IV. THE CAPACITY REGION OF THE SAMBC

In this section, we characterize the secrecy capacity region of the aligned (but not necessarily degraded) MIMO broadcast channel. Note that since the SAMBC is not degraded, there is no single-letter formula for its capacity region. In addition, the secret superposition of Gaussian codes along with successive decoding cannot work when the channel is not degraded. In [6], we presented an achievable rate region for the general secure Broadcast channel. Our achievable coding scheme is based on a combination of the random binning and the Gelfand-Pinsker binning schemes. We first review this scheme and then based on this result, we develop an achievable secret coding scheme for the SAMBC. After that, based on the Theorem 2 we provide a full characterization of the capacity region of SAMBC.

A. Secret Dirty-Paper Coding Scheme and Achievability Proof

In [6], we established an achievable rate region for the general secure broadcast channel. This scheme enables both joint encoding at the transmitter by using Gelfand-Pinsker binning and preserving confidentiality by using random binning. The following theorem summarizes the encoding strategy. The confidentiality proof is given in Appendix II for completeness.

**Theorem 4**: Let $V_1$ and $V_2$ be auxiliary random variables and $\Omega$ be the class of joint probability densities $P(v_1, v_2, x_1, y_2, z)$ that factors as $P(v_1, v_2)P(x|v_1, v_2)P(y_1, y_2, z|x)$. Let $\mathcal{R}_I(\pi)$ denote the union of all non-negative rate pairs $(R_1, R_2)$ satisfying

$$R_1 \leq I(V_1; Y_1) - I(V_1; Z),$$

$$R_2 \leq I(V_2; Y_2) - I(V_2; Z),$$

$$R_1 + R_2 \leq I(V_1; Y_1) + I(V_2; Y_2) - I(V_1, V_2; Z) - I(V_1; V_2),$$

for a given joint probability density $\pi \in \Omega$. For the general broadcast channel with confidential messages, the following region is achievable.

$$\mathcal{R}_I = \text{conv} \left\{ \bigcup_{\pi \in \Omega} \mathcal{R}_I(\pi) \right\}$$

(32)
Remark 2: If we remove the secrecy constraints by removing the eavesdropper, then the above rate region becomes Marton’s achievable region for the general broadcast channel.

Proof: 1) Codebook Generation: The structure of the encoder is depicted in Fig. 2. Fix $P(v_1), P(v_2)$ and $P(x|v_1, v_2)$. The stochastic encoder generates $2^{n(I(V_1;Y_1)−\epsilon)}$ independent and identically distributed sequences $v_1^n$ according to the distribution $P(v_1^n) = \prod_{i=1}^n P(v_{1,i})$. Next, randomly distribute these sequences into $2^{nR_1}$ bins such that each bin contains $2^{n(I(V_1;Z)−\epsilon)}$ codewords. Similarly, it generates $2^{n(I(V_2;Y_2)−\epsilon)}$ independent and identically distributed sequences $v_2^n$ according to the distribution $P(v_2^n) = \prod_{i=1}^n P(v_{2,i})$. Randomly distribute these sequences into $2^{nR_2}$ bins such that each bin contains $2^{n(I(V_1;Z)−\epsilon)}$ codewords. Index each of the above bins by $w_1 \in \{1, 2, ..., 2^{nR_1}\}$ and $w_2 \in \{1, 2, ..., 2^{nR_2}\}$ respectively.

2) Encoding: To send messages $w_1$ and $w_2$, the transmitter looks for $v_1^n$ in bin $w_1$ of the first bin set and looks for $v_2^n$ in bin $w_2$ of the second bin set, such that $(v_1^n, v_2^n) \in A_e^{(n)}(P_{V_1,V_2})$ where $A_e^{(n)}(P_{V_1,V_2})$ denotes the set of jointly typical sequences $v_1^n$ and $v_2^n$ with respect to $P(v_1, v_2)$. The rates are such that there exist more than one joint typical pair. The transmitter randomly chooses one of them and then generates $x^n$ according to $P(x^n|v_1^n, v_2^n) = \prod_{i=1}^n P(x_i|v_{1,i}, v_{2,i})$. This scheme is equivalent to the scenario in which each bin is divided into subbins and the transmitter randomly chooses one of the subbins of bin $w_1$ and one of the subbins of bin $w_2$. It then looks for a joint typical sequence $(v_1^n, v_2^n)$ in the corresponding subbins and generates $x^n$.

3) Decoding: The received signals at the legitimate receivers, $y_1^n$ and $y_2^n$, are the outputs of the channels $P(y_1^n|x^n) = \prod_{i=1}^n P(y_{1,i}|x_i)$ and $P(y_2^n|x^n) = \prod_{i=1}^n P(y_{2,i}|x_i)$, respectively. The first receiver looks for the unique sequence $v_1^n$ such that $(v_1^n, y_1^n)$ is jointly typical and declares the index of the bin containing $v_1^n$ as the message received. The second receiver uses the same method to extract the message $w_2$.

4) Error Probability Analysis: Since the region of (8) is a subset of Marton region, then the error probability analysis is the same as [3].

5) Equivocation Calculation: Please see Appendix A.

The achievability scheme in Theorem 4 introduces random binning. However, when we want to construct the rate region of (32), it is not clear how to choose the auxiliary random variables $V_1$ and $V_2$. Here, we employ the Dirty-Paper Coding (DPC) technique to develop the secret DPC (SDPC) achievable rate region for the SAMBC. We consider a secret dirty-paper encoder with Gaussian codebooks as follows.
First, we separate the channel input $x$ into two random vectors $b_1$ and $b_2$ such that

$$b_1 + b_2 = x$$ (33)

Here, $b_1$ and $b_2$ and $v_1$ and $v_2$ are chosen as follows:

$$b_1 \sim \mathcal{N}(0, B_1),$$
$$b_2 \sim \mathcal{N}(0, B_2),$$
$$v_2 = b_2,$$
$$v_1 = b_1 + Cb_2.$$ (34)

where $B_1 = E[b_1b_1^T] \succeq 0$ and $B_2 = E[b_2b_2^T] \succeq 0$ are covariance matrices such that $B_1 + B_2 \preceq S$, and the matrix $C$ is given as follows:

$$C = B_1 (N_1 + B_1)^{-1}$$ (35)

By substituting (34) into the Theorem 4, we obtain the following SDPC rate region for the SAMBC.

**Lemma 1:** (SDPC Rate Region): Let $S$ be a positive semi-definite matrix. Then the following SDPC rate region of an SAMBC with a covariance matrix constraint $S$ is achievable.

$$\mathcal{R}^{SDPC}(S, N_{1,2,3}) = \text{conv} \left\{ \bigcup_{\pi \in \Pi} \mathcal{R}^{SDPC}(\pi, S, N_{1,2,3}) \right\}$$ (36)

where $\Pi$ is the collection of all possible permutations of the ordered set $\{1, 2\}$, $\text{conv}$ is the convex closure operator and $\mathcal{R}^{SDPC}(\pi, S, N_{1,2,3})$ is given as follows:

$$\mathcal{R}^{SDPC}(\pi, S, N_{1,2,3}) = \left\{ (R_1, R_2) \mid \begin{aligned} &R_k = \mathcal{R}^{SDPC}_{\pi^{-1}(k)}(\pi, B_{1,2,3}, N_{1,2,3}) \quad k = 1, 2 \\ &\text{s.t. } S - (B_1 + B_2) \succeq 0, \quad B_1 \succeq 0, \quad B_2 \succeq 0 \end{aligned} \right\},$$

where

$$\mathcal{R}^{SDPC}_{\pi^{-1}(k)}(\pi, B_{1,2,3}, N_{1,2,3}) = \frac{1}{2} \log \frac{\sum_{i=1}^{\pi^{-1}(k)} B_{\pi(i)} + N_k}{\sum_{i=1}^{\pi^{-1}(k)-1} B_{\pi(i)} + N_k} - \frac{1}{2} \log \frac{\sum_{i=1}^{\pi^{-1}(k)} B_{\pi(i)} + N_3}{\sum_{i=1}^{\pi^{-1}(k)-1} B_{\pi(i)} + N_3} +$$

Note that for the identity permutation, $\pi_I$, where $\pi_I(k) = k$ we have,

$$\mathcal{R}^{SDPC}(\pi_I, S, N_{1,2,3}) = \mathcal{R}^G(S, N_{1,2,3})$$

**Proof:** We prove the lemma for the case of identity permutation $\pi_I = \{1, 2\}$. This proof can similarly be used for the case that $\pi = \{2, 1\}$. According to the Theorem 4 we have,

$$R_1 \leq \min \left\{ I(V_1; Y_1) - I(V_1; Z), I(V_1; Y_1) + I(V_2; Z) - I(V_1, V_2; Z) - I(V_1; V_2) \right\},$$

(a)

$$\leq \min \left\{ I(V_1; Y_1) - I(V_1; Z), I(V_1; Y_1) - I(V_1; Z|V_2) - I(V_1; V_2) \right\},$$

(b)

$$\leq I(V_1; Y_1) - I(V_1; Z|V_2) - I(V_1; V_2),$$

$$R_2 \leq I(V_2; Y_2) - I(V_2; Z),$$ (37)

where (a) follows from the fact that $I(V_1, V_2; Z) = I(V_2; Z) + I(V_1; Z|V_2)$ and (b) follows from the fact that $I(V_1; Z|V_2) + I(V_1; V_2) = I(Z, V_2; V_1) \geq I(Z; V_1)$. To calculate the upper-bound of $R_1$, we need to review the following lemma which has been noted by several authors [23].
Lemma 2: Let \( y_1 = b_1 + b_2 + n_1 \), where \( b_1, b_2 \) and \( n_1 \) are Gaussian random vectors with covariance matrices \( B_1, B_2 \) and \( N_1 \) respectively. Let \( b_1, b_2 \) and \( n_1 \) be independent, and let \( v_1 = b_1 + Cb_2 \), where \( C \) is an \( t \times t \) matrix. Then an optimal matrix \( C \) which maximizes \( I(v_1; y_1) - I(v_1; b_2) \) is \( C = B_1 (N_1 + B_1)^{-1} \). Further, the maximum value of \( I(v_1; y_1) - I(v_1; b_2) \) is \( I(v_1; y_1|b_2) \).

Now, using the above Lemma and substituting (34) into (37), we obtain the following achievable rate region when \( \pi = \pi_1 \).

\[
R_1 \leq \frac{1}{2} \left[ \log |N_1^{-1}(B_1 + N_1)| - \frac{1}{2} \log |N_3^{-1}(B_1 + N_3)| \right]^+, \\
R_2 \leq \frac{1}{2} \left[ \log \frac{|B_1 + B_2 + N_2|}{|B_1 + N_2|} - \frac{1}{2} \log \frac{|B_1 + B_2 + N_3|}{|B_1 + N_3|} \right]^+.
\]

\[\blacksquare\]

B. SAMBC- Converse Proof

For the converse part, note that not all points on the boundary of \( \mathcal{R}^{SDPC}(S, N_{1,2,3}) \) can be directly obtained using a single SDPC scheme. Instead, we must use time-sharing between points corresponding to different permutations. Therefore, unlike the SADBC case, we cannot use a similar notion to the optimal Gaussian rate vectors, as not all the boundary points can immediately characterized as a solution of an optimization problem. Instead, as the SDPC region is convex by definition, we use the notion of supporting hyperplanes of \([13]\) to define this region.

In this section, we first define the supporting hyperplane of a closed and bounded set. Then, we present the relation between the ideas of a supporting hyperplane and the enhanced channel in Theorem 5. This theorem is an extension of Theorem 2 to the SAMBC case. Finally, we use Theorem 5 to prove that \( \mathcal{R}^{SDPC}(S, N_{1,2,3}) \) is indeed the capacity region of the SAMBC.

Definition 6: The set \( \overline{\mathcal{R}} = (R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b \), for fixed and given scalars \( \gamma_1, \gamma_2 \) and \( b \), is a supporting hyperplane of a closed and bounded set \( \mathcal{X} \subset \mathbb{R}^m \), if \( \gamma_1 R_1 + \gamma_2 R_2 \leq b \) \( \forall (R_1, R_2) \in \mathcal{X} \), with equality for at least one rate vector \( (R_1, R_2) \in \mathcal{X} \).

Note that as \( \mathcal{X} \) is closed and bounded, \( \max_{(R_1, R_2) \in \mathcal{X}} \gamma_1 R_1 + \gamma_2 R_2 \), exists for any \( \gamma_1, \gamma_2 \). Thus, we always can find a supporting hyperplane for the set \( \mathcal{X} \). As \( \mathcal{R}^{SDPC}(S, N_{1,2,3}) \) is a closed and convex set, for each rate pair of \( \overline{\mathcal{R}} = (R_1, R_2) \notin \mathcal{R}^{SDPC}(S, N_{1,2,3}) \) which lies outside the set, there exists a separating hyperplane \( \{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b \} \) where \( \gamma_1 \geq 0, \gamma_1 \geq 0, b \geq 0 \) and

\[
\gamma_1 R_1 + \gamma_2 R_2 \leq b \quad \forall (R_1, R_2) \in \mathcal{R}^{SDPC}(S, N_{1,2,3}) \\
\gamma_1 R_1^* + \gamma_2 R_2^* > b
\]

The following theorem illustrates the relation between the ideas of enhanced channel and a supporting hyperplane.

Theorem 5: Consider a SAMBC with noise covariance matrices \( (N_1, N_2, N_3) \) and an average transmit covariance matrix constraint \( S \succ 0 \). Assume that \( \{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b \} \) is a supporting hyperplane of the rate region \( \mathcal{R}^{SDPC}(\pi_1, S, N_{1,2,3}) \) such that \( 0 \leq \gamma_1 \leq \gamma_2, \gamma_2 > 0 \) and \( b \geq 0 \). Then, there exists an enhanced SADBC with noise covariance matrices \( (N'_1, N'_2, N'_3) \) such that the following properties hold.

1) Enhancement:

\[
N'_1 \preceq N_1, \quad N'_2 \preceq N_2, \quad N'_3 = N_3, \quad N'_1 \preceq N'_2.
\]

2) Supporting hyperplane preservation:

\[
\{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b \} \text{ is also a supporting hyperplane of the rate region } \mathcal{R}^{G}(S, N'_{1,2,3})
\]
Proof: To prove this theorem, we can follow the steps of the proof of Theorem 2. Assume that the hyperplane \{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b\} touches the region \(\mathcal{R}^{SDPC}(\pi_I, S, N_{1,2,3})\) at the point \((R^*_1, R^*_2)\). Let \(B^*_1, B^*_2\) be two positive semi-definite matrices such that
\[
B^*_1 + B^*_2 \preceq S
\]
and such that
\[
\mathcal{R}^{SDPC}(\pi_I, B^*_1, B^*_2, N_{1,2,3}) = R^*_k, \quad k = 1, 2
\]
By definition of the supporting hyperplane, the scalar \(b\) and the matrices \((B^*_1, B^*_2)\) are the solution of the following optimization problem:
\[
\max_{B_1, B_2} \gamma_1 R_1^{SDPC}(\pi_I, B_1, B_2, N_{1,2,3}) + \gamma_2 R_2^{SDPC}(\pi_I, B_1, B_2, N_{1,2,3})
\]
\[\text{s.t.} \quad B_1 + B_2 \preceq S \quad B_k \succeq 0 \quad k = 1, 2\]
We define the noise covariance matrices of the enhanced SADBC as \(\mathcal{C}(\pi_I, S, N_{1,2,3})\). Since for the permutation \(\pi = \pi_I\) we have \(\mathcal{R}^{SDPC}(\pi_I, S, N_{1,2,3}) = \mathcal{R}^{G}(S, N_{1,2,3})\), the supporting hyperplane \(\{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b\}\) is also a supporting hyperplane of the rate region \(\mathcal{R}^{G}(S, N'_{1,2,3})\). \(\square\)

We can now use Theorem 5 and the capacity result of the SADBC to prove that \(\mathcal{R}^{SDPC}(S, N_{1,2,3})\) is indeed the capacity region of the SAMBC. The following theorem formally states the main result of this section.

Theorem 6: Consider a SAMBC with positive definite noise covariance matrices \((N_1, N_2, N_3)\). Let \(C(S, N_{1,2,3})\) denote the capacity region of the SAMBC under a covariance matrix constraint \(S \succeq 0\). Then, \(C(S, N_{1,2,3}) = \mathcal{R}^{SDPC}(S, N_{1,2,3})\).

Proof: To proof this theorem, we use Theorem 5 to show that for every rate vector \(\overline{R}'\), which lies outside the region \(\mathcal{R}^{SDPC}(S, N_{1,2,3})\), we can find an enhanced SADBC, whose capacity region does not contain \(\overline{R}'\). As the capacity region of the enhanced channel outer bounds that of the original channel, therefore, \(\overline{R}'\) cannot be an achievable rate vector.

Let \(\overline{R}' = (R'_1, R'_2)\) be a rate vector which lies outside the region \(\mathcal{R}^{SDPC}(S, N_{1,2,3})\). There exists a supporting and separating hyperplane \(\{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b\}\) where \(\gamma_1 \geq 0, \gamma_2 \geq 0\), and at least one of the \(\gamma_k\)'s is positive. Without loose of generality, we assume that \(\gamma_2 \geq \gamma_1\). If that is not the case, we can always reorder the indices of the users such that this assumption will hold. By definition of the region \(\mathcal{R}^{SDPC}(S, N_{1,2,3})\), we have,
\[
\mathcal{R}^{SDPC}(\pi_I, S, N_{1,2,3}) \subseteq \mathcal{R}^{SDPC}(S, N_{1,2,3})
\]
Note that, as \(\{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b\}\) is a supporting hyperplane of \(\mathcal{R}^{SDPC}(S, N_{1,2,3})\), we can write,
\[
b' = \max_{(R_1, R_2)\in \mathcal{R}^{SDPC}(\pi_I, S, N_{1,2,3})} \gamma_1 R_1 + \gamma_2 R_2 \leq \max_{(R_1, R_2)\in \mathcal{R}^{SDPC}(S, N_{1,2,3})} \gamma_1 R_1 + \gamma_2 R_2 = b.
\]
Furthermore, we can also write,
\[
\gamma_1 R_1^o + \gamma_2 R_2^o > b \geq b'.
\]
Therefore, the hyperplane of \(\{(R_1, R_2)|\gamma_1 R_1 + \gamma_2 R_2 = b'\}\) is a supporting and separating hyperplane for the rate region \(\mathcal{R}^{SDPC}(\pi_I, S, N_{1,2,3})\). By Theorem 5, we know that there exists an enhanced SADBC whose Gaussian rate region \(\mathcal{R}^{G}(S, N'_{1,2,3})\) lies under the supporting hyperplane and hence \((R^o_1, R^o_2) \notin \mathcal{R}^{G}(S, N'_{1,2,3})\). Therefore, \((R^o_1, R^o_2)\) must lies outside the capacity region of the enhanced SADBC. To complete the proof, note that the capacity region of the enhanced SADBC contains that of the original channel and therefore, \((R^o_1, R^o_2)\) must lies outside the capacity region of the original SAMBC. As this statement is true for all rate vectors which lie outside \(\mathcal{R}^{SDPC}(S, N_{1,2,3})\), therefore we have
\[ \mathcal{C}(S, N_{1,2,3}) \subseteq \mathcal{R}_{SDPC}(S, N_{1,2,3}). \] However, \[ \mathcal{R}_{SDPC}(S, N_{1,2,3}) \] is the set of achievable rates and therefore, \[ \mathcal{C}(S, N_{1,2,3}) = \mathcal{R}_{SDPC}(S, N_{1,2,3}). \]

With the same discussion of [13], the result of SAMBC can extend to the SGMBC and may be omitted here. The results of the secrecy capacity region for two receiver can be extended for \( m \) receivers as follows.

**Corollary 1:** Consider a SGMBC with \( m \) receivers and one external eavesdropper. Let \( S \) be a positive semi-definite matrix. Then the SDPC rate region of \[ \mathcal{R}_{SDPC}(S, N_{1,...,m}, H_{1,...,m}), \] which is defined by the following convex closure is indeed the secrecy capacity region of the SGMBC under a covariance constraint \( S \).

\[
\mathcal{R}_{SDPC}(S, N_{1,...,m}, H_{1,...,m}) = \text{conv} \left\{ \bigcup_{\pi \in \Pi} \mathcal{R}_{SDPC}(\pi, S, N_{1,...,m}, H_{1,...,m}) \right\}
\]

where \( \Pi \) is the collection of all possible permutations of the ordered set \( \{1, ..., m\} \), \( \text{conv} \) is the convex closure operator and \( \mathcal{R}_{SDPC}(\pi, S, N_{1,...,m}, H_{1,...,m}) \) is given as follows:

\[
\mathcal{R}_{SDPC}(\pi, S, N_{1,...,m}, H_{1,...,m}) = \left\{ (R_1, R_2) \mid R_k = R_{\pi^{-1}(k)}(\pi, B_{1,...,m}, N_{1,...,m}, H_{1,...,m}) \quad k = 1, ..., m \right\}
\]

where

\[
R_{\pi^{-1}(k)}(\pi, B_{1,...,m}, N_{1,...,m}, H_{1,...,m}) = \frac{1}{2} \log \left| \frac{H_k \left( \sum_{i=1}^{\pi^{-1}(k)} B_{\pi(i)} \right) H_k^T + N_k}{H_k \left( \sum_{i=1}^{\pi^{-1}(k)-1} B_{\pi(i)} \right) H_k^T + N_k} \right| - \frac{1}{2} \log \left| \frac{H_3 \left( \sum_{i=1}^{\pi^{-1}(k)-1} B_{\pi(i)} \right) H_3^T + N_3}{H_3 \left( \sum_{i=1}^{\pi^{-1}(k)-1} B_{\pi(i)} \right) H_3^T + N_3} \right|^+ 
\]

**V. MULTIPLE-INPUT SINGLE-OUTPUTS MULTIPLE EAVESDROPPER (MISOME) CHANNEL**

In this section we investigate practical characterizations for the specific scenario in which the transmitter and the eavesdropper have multiple antennas, while both intended receivers have a single antenna. We refer to this configuration as the MISOME case. The significance of this model is when a base station wishes to broadcast secure information for small mobile units. In this scenario small mobile units have single antenna while the base station and the eavesdropper can afford multiple antennas. We can rewrite the signals received by the destination and the eavesdropper for the MISOME channel as follows.

\[
y_1 = h_1^T x + n_1, \\
y_2 = h_2^T x + n_2, \\
z = H_3 x + n_3,
\]

where \( h_1 \) and \( h_2 \) are fixed, real gain matrices which model the channel gains between the transmitter and the legitimate receivers. These are matrices of size \( t \times 1 \). The channel state information again is assumed to be known perfectly at the transmitter and at all receivers. Here, the superscript \( \dagger \) denotes the Hermitian transpose of a vector. Without lost of generality, we assume that \( n_1 \) and \( n_2 \) are i.i.d real Gaussian random variables with zero means unit covariances, i.e., \( n_1, n_2 \sim \mathcal{N}(0, I) \). Furthermore, we assume that \( n_3 \) is a Gaussian random vector with zero mean and covariance matrix \( I \). In this section, we assume that the input \( x \) satisfies a total power constraint of \( P \), i.e.,

\[
\text{Tr}\{E(xx^T)\} \leq P
\]

Before we state our results for the MISOME channel, we need to review some properties of generalized eigenvalues and eigenvectors. For more details of this topic, see, e.g., [24].
Definition 7: (Generalized eigenvalue-eigenvector) Let $A$ be a Hermitian matrix and $B$ be a positive definite matrix. Then, $(\lambda, \psi)$ is a generalized eigenvalue-eigenvector pair if it satisfy the following equation.

$$A\psi = \lambda B\psi$$

Note that as $B$ is invertible, the generalized eigenvalues and eigenvectors of the pair $(A, B)$ are the regular eigenvalues and eigenvectors of the matrix $B^{-1}A$. The following Lemma, describes the variational characterization of the generalized eigenvalue-eigenvector pair.

Lemma 3: (Variational Characterization) Let $r(\psi)$ be the Rayleigh quotient defined as the following.

$$r(\psi) = \frac{\psi^\dagger A\psi}{\psi^\dagger B\psi}$$

Then, the generalized eigenvectors of $(A, B)$ are the stationary point solution of the Rayleigh quotient $r(\psi)$. Specifically, the largest generalized eigenvalue $\lambda_{\text{max}}$ is the maximum of the Rayleigh quotient $r(\psi)$ and the optimum is attained by the eigenvector $\psi_{\text{max}}$ which is corresponded to $\lambda_{\text{max}}$, i.e.,

$$\max_{\psi} r(\psi) = \frac{\psi_{\text{max}}^\dagger A\psi_{\text{max}}}{\psi_{\text{max}}^\dagger B\psi_{\text{max}}} = \lambda_{\text{max}}$$

Now consider the MISOME channel of (40). Assume that $0 \leq \alpha \leq 1$ and $P$ are fixed. Let define the following matrices for this channel.

$$A_{1,1} = I + \alpha P h_1 h_1^\dagger,$$
$$B_{1,1} = I + \alpha P H_3 H_3$$

Suppose that $(\lambda_{(1,1)} \text{max}, \psi_{1 \text{max}})$ is the largest generalized eigenvalue and the corresponding eigenvector pair of the pencil $(A_{1,1}, B_{1,1})$. We furthermore define the following matrices for the MISOME channel.

$$A_{2,2} = I + \frac{(1 - \alpha) P}{1 + \alpha P |h_2^\dagger H_2|^2} h_2 h_2^\dagger,$$
$$B_{2,2} = I + (1 - \alpha) P H_3^\dagger \left( I + \alpha P H_3 \psi_{1 \text{max}} H_3 \right)^{-1} H_3$$

Assume that $(\lambda_{(2,2)} \text{max}, \psi_{2 \text{max}})$ is the largest generalized eigenvalue and the corresponding eigenvector pair of the pencil $(A_{2,2}, B_{2,2})$. Moreover, consider the following matrices for this channel.

$$A_{2,1} = I + (1 - \alpha) P h_2 h_2^\dagger,$$
$$B_{2,1} = I + (1 - \alpha) P H_3 H_3,$$
$$A_{1,2} = I + \frac{\alpha P}{1 + (1 - \alpha) P |h_1^\dagger H_1|^2} h_1 h_1^\dagger,$$
$$B_{1,2} = I + \alpha P H_3^\dagger \left( I + (1 - \alpha) P H_3 \psi_{2 \text{max}} H_3 \right)^{-1} H_3,$$

where we assume that $(\lambda_{(2,1)} \text{max}, \psi_{3 \text{max}})$ and $(\lambda_{(1,2)} \text{max}, \psi_{4 \text{max}})$ are the largest generalized eigenvalue and the corresponding eigenvector pair of the pencils $(A_{2,1}, B_{2,1})$, and $(A_{1,2}, B_{1,2})$ respectively. The following theorem then characterizes the capacity region of the MISOME channel under a total power constraint $P$ based on the above parameters.

Theorem 7: Let $C^{\text{MISOME}}$ denote the secrecy capacity region of the the MISOME channel under an average total power constraint $P$. Let $\prod$ be the collection of all possible permutations of the ordered set $\{1, 2\}$ and $\text{conv}$ be the convex closure operator, then $C^{\text{MISOME}}$ is given as follows.

$$C^{\text{MISOME}} = \text{conv} \left\{ \bigcup_{\pi \in \prod} \mathcal{R}^{\text{MISOME}}(\pi) \right\}$$
where $R^{MISOME}(\pi)$ is given as follows.

$$R^{MISOME}(\pi) = \bigcup_{0 \leq \alpha \leq 1} R^{MISOME}(\pi, \alpha)$$

where $R^{MISOME}(\pi, \alpha)$ is the set of all $(R_1, R_2)$ satisfying the following condition.

$$R_k \leq \frac{1}{2} \left[ \log \lambda_{(k, \pi^{-1}(k))} \right]^+, \quad k = 1, 2.$$

Proof: This theorem is a special case of Theorem $\square$ and corollary $\lozenge$. First assume that the permutation $\pi = \pi_f = \{1, 2\}$.

In the SDPC achievable rate region of (39), we choose the covariance matrices $B_1$ and $B_2$ as follows.

$$B_1 = \alpha P \psi_{1\max} \psi_{1\max}^\dagger,$$
$$B_2 = (1 - \alpha)P \psi_{2\max} \psi_{2\max}^\dagger.$$

In other words, the channel input $x$ is separated into two vectors $b_1$ and $b_2$ such that

$$x = b_1 + b_2,$$
$$b_1 = u_1 \psi_{1\max},$$
$$b_2 = u_2 \psi_{2\max},$$

where $u_1 \sim \mathcal{N}(0, \alpha P)$, $u_2 \sim \mathcal{N}(0, (1 - \alpha)P)$, and $0 \leq \alpha \leq 1$. Using these parameters, the region of $R^{SDPC}(\pi_f, S, N_{1,2,3})$ becomes as follows.

$$R_1 \leq \frac{1}{2} \left[ \log \left| 1 + h_1^\dagger B_1 h_1 \right| - \frac{1}{2} \log \left| I + H_3 B_1 H_3^\dagger \right| \right]^+, \quad (41)$$

where (a) is due to the fact that $|I + AB| = |I + BA|$ and the fact that $\psi_{1\max} \psi_{1\max}^\dagger = 1$ Similarly for the $R_2$ we have,

$$R_2 \leq \frac{1}{2} \left[ \log \left| 1 + h_2^\dagger (B_1 + B_2) h_2 \right| - \frac{1}{2} \log \left| I + H_3 (B_1 + B_2) H_3^\dagger \right| \right]^+$$

$$= \frac{1}{2} \left[ \log \left| 1 + \frac{h_2^\dagger B_2 h_2}{1 + h_2^\dagger B_1 h_2} \right| - \frac{1}{2} \log \left| I + \frac{H_3 B_2 H_3^\dagger}{I + H_3 B_1 H_3^\dagger} \right| \right]^+$$

$$= \frac{1}{2} \left[ \log \left| \psi_{2\max} \left( I + (1 - \alpha)P h_2^\dagger \psi_{2\max} \psi_{2\max}^\dagger \right)^{-1} H_3 \right| \right]^+ = \frac{1}{2} \left[ \log \lambda_{(2, \pi^{-1}(2))} \right]^+. \quad (42)$$
Similarly, when \( \pi = \{2, 1\} \), in the SDPC region, we choose \( b_1 = u_1 \psi_{3, \max} \) and \( b_2 = u_2 \psi_{3, \max} \). Then the SDPC region is given as follows.

\[
R_1 \leq \frac{1}{2} \left[ \log \left| \frac{1 + h_1^\dagger (B_1 + B_2) h_1}{1 + h_1^\dagger B_2 h_1} \right| - \frac{1}{2} \log \left| \frac{I + H_3 (B_1 + B_2) H_3^\dagger}{I + H_3 B_2 H_3} \right| \right]^+ \\
= \frac{1}{2} \left[ \log \left| \frac{1 + h_1^\dagger B_1 h_1}{1 + h_1^\dagger B_2 h_1} \right| - \frac{1}{2} \log \left| \frac{I + \psi_{3, \max} \left( I + (1 - \alpha) P H_3^\dagger \psi_{3, \max} \right) \psi_{3, \max} H_3^\dagger}{I + H_3 B_2 H_3} \right| \right]^+ \\
= \frac{1}{2} \left[ \log \left| \frac{1 + h_2^\dagger B_2 h_2}{1 + h_2^\dagger B_2 h_2} \right| - \frac{1}{2} \log \left| \frac{I + \psi_{3, \max} \left( I + (1 - \alpha) P H_3^\dagger \psi_{3, \max} \right) \psi_{3, \max} H_3^\dagger}{I + H_3^\dagger B_2 H_3} \right| \right]^+ \\
= \frac{1}{2} \left[ \log \left| \frac{1 + \psi_{3, \max} \left( I + (1 - \alpha) P H_3^\dagger \psi_{3, \max} \right) \psi_{3, \max} H_3^\dagger}{I + H_3^\dagger B_2 H_3} \right| \right]^+ \\
= \frac{1}{2} \left[ \log \lambda_{(2, 1)} \right]^+. \\
\]

and \( R_2 \) is bounded as follows

\[
R_2 \leq \frac{1}{2} \left[ \log \left| \frac{1 + h_2^\dagger B_2 h_2}{1 + h_2^\dagger B_2 h_2} \right| - \frac{1}{2} \log \left| \frac{I + \psi_{3, \max} \left( I + (1 - \alpha) P H_3^\dagger \psi_{3, \max} \right) \psi_{3, \max} H_3^\dagger}{I + H_3^\dagger B_2 H_3} \right| \right]^+ \\
= \frac{1}{2} \left[ \log \left| \frac{1 + \psi_{3, \max} \left( I + (1 - \alpha) P H_3^\dagger \psi_{3, \max} \right) \psi_{3, \max} H_3^\dagger}{I + H_3^\dagger B_2 H_3} \right| \right]^+. \\
\]

Note that the eigenvalues \( \lambda_{(l, k)} = \lambda_{(l, k)}(\alpha, P) \) and the eigenvector \( \psi_{k, \max} = \psi_{k, \max}(\alpha, P) \), \( l, k = 1, 2 \) are the functions of \( \alpha \) and \( P \). The following corollary characterizes the secrecy capacity region of the MISOME channel in high SNR regime.

**Corollary 2:** In the high SNR regime, the secrecy capacity region of the MISOME channel is given as follows.

\[
\lim_{P \to \infty} C_{\text{MISOME}} = \text{conv} \left\{ \bigcup_{\pi \in \Pi} R_{\text{MISOME}}^\infty(\pi) \right\} 
\]

where

\[
R_{\text{MISOME}}^\infty(\pi = \{1, 2\}) = \\
\left\{ (R_1, R_2), R_1 \leq \frac{1}{2} \left[ \log \lambda_{\max} \left( h_1^\dagger H_1^\dagger H_3 \right) \right]^+, R_2 \leq \frac{1}{2} \left[ \log \lambda_{\max} \left( h_2^\dagger H_2^\dagger H_3 \right) \right]^+ \right\}
\]

\[
R_{\text{MISOME}}^\infty(\pi = \{2, 1\}) = \\
\left\{ (R_1, R_2), R_1 \leq \frac{1}{2} \left[ \log \lambda_{\max} \left( h_1^\dagger H_1^\dagger H_3 \right) \right]^+, R_2 \leq \frac{1}{2} \left[ \log \lambda_{\max} \left( h_2^\dagger H_2^\dagger H_3 \right) \right]^+ \right\}
\]

where \((\lambda_{\max}(A_i, B), \psi_{i, \max})\) denotes the largest eigenvalue and corresponding eigenvector of the pencil \((A_i, B)\) and \( b = \frac{|h_i^\dagger \psi_{i, \max}|^2}{\|H_3 \psi_{i, \max}\|^2}, \quad a = \frac{|h_i^\dagger \psi_{i, \max}|^2}{\|H_3 \psi_{i, \max}\|^2} \).

Note that the above secrecy rate region is independent of \( \alpha \) and therefore is a convex closure of two rectangular regions.

**Proof:** We restrict our attention to the case that \( \lambda_{(l, k)}(\alpha, P) > 1 \) for \( l, k = 1, 2 \) where the rates \( R_1 \) and \( R_2 \) are nonzero.
First suppose that \( \pi = \pi_I = \{1, 2\} \). We show that

\[
\lim_{P \to \infty} \lambda_{(1,1)_{\max}}(\alpha, P) = \lambda_{\max}\left( h_1 h_1^\dagger, H_3^\dagger H_3 \right)
\]  
\[
\lim_{P \to \infty} \lambda_{(2,2)_{\max}}(\alpha, P) = \frac{\lambda_{\max}\left( h_2 h_2^\dagger, H_3^\dagger H_3 \right)}{b}.
\]

Note that since

\[
\lambda_{(1,1)_{\max}}(\alpha, P) = \frac{1 + \alpha P|h_1^\dagger \psi_{1_{\max}}(\alpha, P)|^2}{1 + \alpha P\|H_3 \psi_{1_{\max}}(\alpha, P)\|^2} > 1
\]

where

\[
\psi_{1_{\max}}(\alpha, P) = \arg \max_{\{\psi_1 : \|\psi_1\|^2 = 1\}} \frac{1 + \alpha P|h_1^\dagger \psi_1(\alpha, P)|^2}{1 + \alpha P\|H_3 \psi_1(\alpha, P)\|^2}
\]

for all \( P > 0 \) we have,

\[
|h_1^\dagger \psi_{1_{\max}}(\alpha, P)|^2 > |H_3 \psi_{1_{\max}}(\alpha, P)|^2
\]

Therefore, \( \lambda_{(1,1)_{\max}} \) is an increasing function of \( P \). Thus,

\[
\lambda_{(1,1)_{\max}}(\alpha, P) \leq \frac{|h_1^\dagger \psi_{1_{\max}}(\alpha, P)|^2}{\|H_3 \psi_{1_{\max}}(\alpha, P)\|^2} \leq \lambda_{\max}\left( h_1 h_1^\dagger, H_3^\dagger H_3 \right)
\]

Since \( \lambda_{\max}\left( h_1 h_1^\dagger, H_3^\dagger H_3 \right) \) is independent of \( P \) we have

\[
\lim_{P \to \infty} \lambda_{(1,1)_{\max}} \leq \lambda_{\max}\left( h_1 h_1^\dagger, H_3^\dagger H_3 \right)
\]

Next, defining

\[
\psi_1(\infty) = \arg \max_{\{\psi_1 : \|\psi_1\|^2 = 1\}} \frac{|h_1^\dagger \psi_1|^2}{\|H_3 \psi_1\|^2}
\]

we have the following lower bound

\[
\lim_{P \to \infty} \lambda_{(1,1)_{\max}}(\alpha, P) \geq \lim_{P \to \infty} \frac{1}{P} + \frac{\alpha |h_1^\dagger \psi_{1_{\max}}(\infty)|^2}{P + \alpha \|H_3 \psi_{1_{\max}}(\infty)\|^2}
\]

\[
= \lambda_{\max}\left( h_1 h_1^\dagger, H_3^\dagger H_3 \right)
\]

As the lower bound and upper bound coincide then we obtain (43). Similarly to obtain (44) note that since

\[
\lambda_{(2,2)_{\max}}(\alpha, P) = \frac{1 + (1 - \alpha)P|h_2^\dagger \psi_{2_{\max}}(\alpha, P)|^2}{1 + \alpha P\|H_3 \psi_{2_{\max}}(\alpha, P)\|^2} > 1
\]

where

\[
\psi_{2_{\max}}(\alpha, P) = \arg \max_{\{\psi_2 : \|\psi_2\|^2 = 1\}} \frac{1 + (1 - \alpha)P|h_2^\dagger \psi_2_{\max}(\alpha, P)|^2}{1 + \alpha P\|H_3 \psi_{2_{\max}}(\alpha, P)\|^2}
\]

for all \( P > 0 \) we have,

\[
\frac{(1 - \alpha)P|h_2^\dagger \psi_{2_{\max}}(\alpha, P)|^2}{1 + \alpha P\|h_2^\dagger \psi_{1_{\max}}(\alpha, P)\|^2} > \frac{(1 - \alpha)P\|H_3 \psi_{2_{\max}}(\alpha, P)\|^2}{1 + \alpha P\|H_3 \psi_{1_{\max}}(\alpha, P)\|^2}
\]

Therefore, we have

\[
\lim_{P \to \infty} \lambda_{(2,2)_{\max}}(\alpha, P) \leq \frac{|h_2^\dagger \psi_{2_{\max}}(\infty)|^2}{|h_2^\dagger \psi_{1_{\max}}(\infty)|^2} \frac{\|H_3 \psi_{2_{\max}}(\infty)\|^2}{\|H_3 \psi_{1_{\max}}(\infty)\|^2} \leq \frac{\lambda_{\max}\left( h_2 h_2^\dagger, H_3^\dagger H_3 \right)}{b}
\]  
\[
\text{(45)}
\]
where

\[ b = \frac{|b_2 \psi_{1 \max}(\infty)|^2}{||H_3 \psi_{1 \max}(\infty)||^2} \]

\[ \psi_2(\infty) = \arg \max_{\{\psi_j : ||\psi_j|| = 1\}} \frac{|h_2^j \psi_{2 \max}|^2}{||H_3 \psi_{2 \max}||^2} \]

On the other hand we have the following lower bound

\[ \lim_{P \to \infty} \lambda_{(2,2) \max}(\alpha, P) \geq \frac{1 + \frac{(1-\alpha)||h_2^1 \psi_{2 \max}(\infty)||^2}{\alpha ||H_3 \psi_{1 \max}(\infty)||^2}}{1 + \frac{(1-\alpha)||h_2^1 \psi_{2 \max}(\infty)||^2}{\alpha ||H_3 \psi_{1 \max}(\infty)||^2}} \]

(46)

Note that \( 0 \leq \alpha \leq 1 \). It is easy to show that the right side of equation of (46) is a decreasing function of \( \alpha \) and therefore the maximum value of this function is when \( \alpha = 0 \). Thus we have,

\[ \lim_{P \to \infty} \lambda_{(2,2) \max}(\alpha, P) \geq \frac{\lambda_{\max}(b_2 h_2^i, H_3 H_3^\dagger)}{b} \]

As the lower bound and upper bound coincide then we obtain (44). When \( \pi = 2, 1 \), the proof is similar and may be omitted here.

Now consider the MISOME channel with \( m \) single antenna receivers and an external eavesdropper. Let \( x = \sum_{k=1}^{m} b_k \), where \( b_k = u_k \psi_{k \max} \), \( u_k \sim N(0, \alpha_k P) \), and \( \sum_{k=1}^{m} \alpha_k = 1 \). Assume that \( (\lambda_{k \max}, \psi_{k \max}) \) is the largest generalized eigenvalue and the corresponding eigenvector pair of the pencil

\[ \left( I + \frac{\alpha_k P h_k h_k^\dagger}{1 + \alpha_k P H_3^\dagger (I + H_3 A H_3^\dagger)^{-1} H_3} \right) \]

where \( A = (\sum_{i=1}^{\pi^{-1}(k)} \alpha_{\pi(i)} P \psi_{\pi(i) \max} \psi_{\pi(i) \max}^\dagger) \). The following corollary then characterizes the capacity region of the MISOME channel with \( m \) receivers under a total power constraint \( P \).

**Corollary 3:** Let \( \prod \) be the collection of all possible permutations of the ordered set \( \{1, \ldots, m\} \) and \( \text{conv} \) be the convex closure operator, then \( C_{\text{MISOME}} \) is given as follows.

\[ C_{\text{MISOME}} = \text{conv} \left\{ \bigcup_{\pi \in \prod} R_{\text{MISOME}}(\pi) \right\} \]

where \( R_{\text{MISOME}}(\pi) \) is given as follows.

\[ R_{\text{MISOME}}(\pi) = \bigcup_{0 \leq \alpha_k \leq 1, \sum_{k=1}^{m} \alpha_k = 1} R_{\text{MISOME}}(\pi, \alpha_1, \ldots, \alpha_m) \]

where \( R_{\text{MISOME}}(\pi, \alpha_1, \ldots, \alpha_m) \) is the set of all \( (R_1, \ldots, R_m) \) satisfying the following condition.

\[ R_k \leq \frac{1}{2} \left[ \log \lambda_{k \max} \right]^+, \quad k = 1, \ldots, m. \]

**VI. Conclusion**

A scenario where a source node wishes to broadcast two confidential messages for two respective receivers via a Gaussian MIMO broadcast channel, while a wire-tapper also receives the transmitted signal via another MIMO channel is considered. We considered the secure vector Gaussian degraded broadcast channel and established its capacity region. Our achievability scheme was the secret superposition of Gaussian codes. Instead of solving a nonconvex problem, we used the notion of an enhanced channel to show that secret superposition of Gaussian codes is optimal. To characterize the secrecy capacity region of the vector Gaussian degraded broadcast channel, we only enhanced the channels for the legitimate receivers, and the
channel of the eavesdropper remained unchanged. Then we extended the result of the degraded case to non-degraded case. We showed that the secret superposition of Gaussian codes along with successsive decoding cannot work when the channels are not degraded. we developed a Secret Dirty Paper Coding (SDPC) scheme and showed that SDPC is optimal for this channel. Finally, We investigated practical characterizations for the specific scenario in which the transmitter and the eavesdropper can afford multiple antennas, while both intended receivers have a single antenna. We characterized the secrecy capacity region in terms of generalized eigenvalues of the receivers’ channels and the eavesdropper channel. In high SNR we showed that the capacity region is a convex closure of two rectangular regions.

APPENDIX

A. Equivocation Calculation

The proof of secrecy requirement for each individual message (9) and (10) is straightforward and may therefore be omitted. To prove the requirement of (11) from $H(W_1, W_2 | Z^n)$, we have

$$nR_{e12} = H(W_1, W_2 | Z^n)$$

$$= H(W_1, W_2, Z^n) - H(Z^n)$$

$$= H(W_1, W_2, V_1^n, V_2^n, Z^n) - H(V_1^n, V_2^n | W_1, W_2, Z^n) - H(Z^n)$$

$$= H(W_1, W_2, V_1^n, V_2^n) + H(Z^n | W_1, W_2, V_1^n, V_2^n) - H(V_1^n, V_2^n | W_1, W_2, Z^n) - H(Z^n)$$

$$(a) \geq H(W_1, W_2, V_1^n, V_2^n) + H(Z^n | W_1, W_2, V_1^n, V_2^n) - n\epsilon_n - H(Z^n)$$

$$(b) \leq H(W_1, W_2, V_1^n, V_2^n) + H(Z^n | V_1^n, V_2^n) - n\epsilon_n - H(Z^n)$$

$$(c) \geq H(V_1^n, V_2^n) + H(Z^n | V_1^n, V_2^n) - n\epsilon_n - H(Z^n)$$

$$= H(V_2^n) + H(V_1^n) - I(V_1^n; V_2^n) - I(V_1^n, V_2^n; Z^n) - n\epsilon_n$$

$$(d) \geq I(V_1^n; Y_1^n) + I(V_2^n; Y_2^n) - I(V_1^n, V_2^n; Z^n) - n\epsilon_n$$

$$(e) \geq nR_1 + nR_2 - n\epsilon_n,$$

where (a) follows from Fano’s inequality, which states that for sufficiently large $n$, $H(V_1^n, V_2^n | W_1, W_2, Z^n) \leq h(P_{we}^{(n)}) + nP_{we}^{(n)}R_w \leq n\epsilon_n$. Here $P_{we}^{(n)}$ denotes the wiretapper’s error probability of decoding $(v_1^n, v_2^n)$ in the case that the bin numbers $w_1$ and $w_2$ are known to the eavesdropper. Since the sum rate is small enough, then $P_{we}^{(n)} \to 0$ for sufficiently large $n$. (b) follows from the following Markov chain: $(W_1, W_2) \to (V_1^n, V_2^n) \to Z^n$. Hence, we have $H(Z^n | W_1, W_2, V_1^n, V_2^n) = H(Z^n | V_1^n, V_2^n)$. (c) follows from the fact that $H(W_1, W_2, V_1^n, V_2^n) \geq H(V_1^n, V_2^n)$. (d) follows from that fact that $H(V_1^n) \geq I(V_1^n; V_2^n)$ and $H(V_2^n) \geq I(V_2^n; Y_2^n)$. (e) follows from the following lemmas.

**Lemma 4:** Assume $V_1^n, V_2^n$ and $Z^n$ are generated according to the achievability scheme of Theorem 4, then we have,

$$I(V_1^n, V_2^n; Z^n) \leq nI(V_1, V_2; Z) + n\delta_{1n},$$

$$I(V_1^n, V_2^n; Z^n) \leq nI(V_1, V_2; Z) + n\delta_{2n}.$$

**Proof:** Let $A_n^0(P_{V_1, V_2, Z})$ denote the set of typical sequences $(V_1^n, V_2^n, Z^n)$ with respect to $P_{V_1, V_2, Z}$, and

$$\zeta = \begin{cases} 
1, & (V_1^n, V_2^n, Z^n) \notin A_n^0(P_{V_1, V_2, Z}); \\
0, & \text{otherwise},
\end{cases}$$

...
be the corresponding indicator function. We expand \( I(V^n_1, V^n_2; Z^n) \) as follow,

\[
I(V^n_1, V^n_2; Z^n) \leq I(V^n_1, V^n_2; \zeta; Z^n)
\]

\[
= I(V^n_1, V^n_2; Z^n; \zeta) + I(\zeta; Z^n)
\]

\[
= \sum_{j=0}^1 P(\zeta = j)I(V^n_1, V^n_2; Z^n; \zeta = j) + I(\zeta; Z^n).
\]

According to the joint typicality property, we have

\[
P(\zeta = 1)I(V^n_1, V^n_2; Z^n | \zeta = 1) \leq nP((V^n_1, V^n_2, Z^n) \notin A^n(\mathcal{P}(V_1, V_2, Z))) \log \|Z\| \tag{47}
\]

\[
= nP(V^n_1, V^n_2, Z^n) \log \|Z\|.
\]

Note that,

\[
I(\zeta; Z^n) \leq H(\zeta) \leq 1 \tag{49}
\]

Now consider the term \( P(\zeta = 0)I(V^n_1, V^n_2; Z^n | \zeta = 0) \). Following the sequence joint typicality properties, we have

\[
P(\zeta = 0)I(V^n_1, V^n_2; Z^n | \zeta = 0) \leq I(V^n_1, V^n_2; Z^n | \zeta = 0)
\]

\[
= \sum_{(V^n_1, V^n_2, Z^n) \in A^n} P(V^n_1, V^n_2, Z^n)(\log P(V^n_1, V^n_2, Z^n) - \log P(V^n_1, V^n_2)
\]

\[
- \log P(Z^n)),
\]

\[
\leq n[-H(V_1, V_2, Z) + H(V_1, V_2) + H(Z) + 3\epsilon_n],
\]

\[
= n[I(V_1, V_2; Z) + 3\epsilon_n].
\]

By substituting (48), (49), and (50) into (47), we get the desired result,

\[
I(V^n_1, V^n_2; Z^n) \leq nI(V_1, V_2; Z) + n\epsilon_n \log \|Z\| + 3\epsilon_n + \frac{1}{n}, \tag{51}
\]

where,

\[
\delta_1 = \epsilon_n \log \|Z\| + 3\epsilon_n + \frac{1}{n}.
\]

Following the same steps, one can prove that

\[
I(V^n_1; V^n_2) \leq nI(V_1; V_2) + n\delta_2.
\]

Using the same approach as in Lemma 4, we can prove the following lemmas.

**Lemma 5:** Assume \( V^n_1, Y^n_1 \) and \( Y^n_2 \) are generated according to the achievability scheme of Theorem 4, then we have,

\[
I(V^n_1; Y^n_1) \leq nI(V_1; Y_1) + n\delta_3,
\]

\[
I(V^n_2; Y^n_2) \leq nI(V_1; Z) + n\delta_4.
\]

**Proof:** The steps of the proof are very similar to the steps of proof of Lemma 4 and may be omitted here.
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