Evidence for Aharony duality for orthogonal gauge groups

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Abstract: We study the Aharony duality for three dimensional $\mathcal{N} = 2$ supersymmetric gauge theories for orthogonal gauge groups with matters in vector representation. We provide the evidence for the duality by working out the partition function on $S^3$ and the superconformal index, which show perfect agreement.
1. Introduction

We have witnessed the tremendous progress in understanding the 3d SCFTs recently. One of the key observations was put forward by J. Schwarz that such theories could be described by Super Chern-Simons matter theories (SCSM) [1]. This led to rapid progress in $AdS_4/CFT_3$ correspondence [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the understanding the CFT associated with M2 branes [13, 14, 15, 16, 17, 18]. Along with this development, there also have been sophisticated tools developed to probe 3d SCFTs such as the partition function on $S^3$ [19, 20, 21] and the superconformal index [22, 23, 24], which gives the detailed information on the 3d SCFTs. Furthermore in this setting 3d SCFTs do not have to be realized by SCSM type. In the Yang-Mills type theories of 3d, the YM kinetic term is irrelevant in the IR and one can take simply $g_{YM} \to \infty$ in the partition function and the superconformal index in many examples. With such sophisticated tools available one can understand the various dualities in 3d far better than before. Such examples are mirror symmetry and Seiberg-like dualities for SCSM theories [27, 28, 29, 30]. Also one can find the evidences that some YM type theories are flowing to SCSM type SCFTs in the IR [24].

In this paper, we are interested in the Aharony duality [31]. It was shown that Seiberg-like duality for SCSM theories could be derived from Aharony duality. In the available literatures this duality was discussed for U/Sp gauge group where the index computation and the partition function give impressive confirmation of the claimed duality [21, 22]. Curiously the discussion on the Aharony duality for orthogonal groups are lacking. Here we are filling the gap by working out the partition
function (numerically) and the superconformal index to show that the duality works for orthogonal groups with matters in the vector representation.

In section 2, we propose the Aharony duality for orthogonal groups with matters in the vector representation and first work out the partition function on $S^3$ numerically, which exhibits nice agreement between electric theory and magnetic theory. Also we work out the $R$-charge of the various fields by using the $Z$-minimization procedure for simple cases [12]. In section 3, we work out the superconformal index and see the perfect matching in both sides. We also discuss chiral ring structures and enumerate operators of lower dimensions appearing in the index. Then we discuss and conclude. As this work is near the final stage, we are aware of the work [33], which also discusses the Aharony duality for orthogonal groups.

2. Partition function on $S^3$

We consider the following electric and magnetic pair of $\mathcal{N} = 2$ gauge theory in 3-d.

- The electric theory is the $\mathcal{N} = 2\ O(N_c)$ supersymmetric gauge theory with $N_f$ flavors of chiral multiplets $Q^a$ in the vector representation with no superpotential.

- The magnetic theory is the $\mathcal{N} = 2\ O(N_f - N_c + 2)$ supersymmetric gauge theory with $N_f$ flavors of chiral multiplets $q_a$ in the vector representation as well as gauge singlet chiral multiplets $M^{(ab)}$ and $Y$. The magnetic theory has the tree-level superpotential

$$W = M^{(ab)} q^i_a q^i_b + Y y,$$

(2.1)

where $y$ is the monopole operator, parametrizing the Coulomb branch of the magnetic theory.

Here $Y$ is fundamental (non-composite) field while $y$ is the monopole operator, which can be expressed in terms of other fields. In fact $Y$ is generically mapped under the duality to the monopole operator of the electric theory. We use the same $Y$ to denote the monopole operator in the electric side.

The global symmetries and the corresponding charges of the elementary fields and the monopole fields are listed in Table [1]. In the table, $r$ denotes the $R$-charge of the fields $Q^a$ in the IR limit, which is the same as its conformal dimension. In the UV limit, $r = \frac{1}{2}$. There are several cases worthy of mention. For electric $O(1)$, there would be no monopole operator. Nevertheless, when the magnetic theory admits monopole operator $y$, we have to introduce the singlet operator $Y$ whose conformal dimension and other quantum numbers are dictated by the superpotential.
term $W = Yy + \cdots$. This additional singlet is crucial to match the partition function and the superconformal index as we will see later. On the other hand, if the magnetic group is $O(1)$, while electric side has the monopole operator, we have to introduce the singlet $Y$ in the magnetic side, whose conformal dimension is the same as that of the monopole operator in the electric side. In this case the superpotential term is $W = M_{\{ab\}} q^a q^b + Y^2 \det M$. Its existence is motivated by the similar reasoning to eq. (3.12). This term is also discussed in [37].

Also $O(2)$ is interesting. If we consider the $SO(2)$ gauge theory, this has two independent monopole operators $Y_+, Y_-$ since it is isomorphic to $U(1)$. However under nontrivial $Z_2$ action of $O(2)$ $Y_+$ is mapped to $Y_-$ thus we have to consider $Z_2$ invariant combination $Y \equiv \frac{Y_+ + Y_-}{\sqrt{2}}$. Thus it is crucial to consider $O(N)$ gauge theory instead of $SO(N)$ for Aharony duality. This also removes the topological current $J = *dA$ for abelian theory with $A$ being the gauge potential, whose dual in the nonabelian case is not clear.

The above Aharony duality can be motivated by considering the Hanany-Witten setup with $D3 - NS5_\theta - NS5_{-\theta} - O5$ and the brane move passing through the infinite coupling where $NS5_\theta$ and $NS5_{-\theta}$ branes are coincident [24]. Here $D3$ spans (0126), $NS5$ spans (012789) and $O5$ span (012345) and (012789) respectively. $NS5_\theta$ denotes the rotated $NS5$ brane by $\theta$ in (345)-(789) planes. We put the flavor D5 branes parallel to $NS5_\theta$ to avoid the quartic superpotential. For orthogonal case, we can consider D3 in the presence of $O5^+$ plane. Due to the fact that RR charge of $O5^+$ plane is the same as D5 brane, starting from $O(N_c)$ gauge theory one ends up with $O(N_f - N_c + 2)$ theory. By this way one can guess the above form of the Aharony duality. And for symplectic case, one has to consider D3s with $O5^-$ plane.

One can use the partition function on $S^3$ and the superconformal index to give evidence for this conjecture. As a first test, one can work out the partition function on $S^3$ for the theories on both sides. One can also find the $R$-charges of $Q^a$ using the Z-minimization [32]. For other discussions on the partition function on $S^3$, see [33, 36].

| Fields | $U(1)_R$ | $U(1)_A$ | $SU(N_f)$ |
|--------|---------|---------|-----------|
| $Q^a$  | $r$     | 1       | $N_f$     |
| $Y$    | $N_f - N_c + 2 - N_f r$ | $-N_f$ | 1         |
| $q_a$  | $1 - r$ | $-1$    | $N_f$     |
| $M^{ab}$ | $2r$    | 2       | $N_f(N_f + 1)/2$ |
| $Y$    | $N_f - N_c + 2 - N_f r$ | $-N_f$ | 1         |
| $y$    | $N_c - N_f + N_f r$   | $N_f$  | 1         |

**Table 1:** The global symmetry charges of the elementary fields and the monopole fields.
Supersymmetric localization gives the following expression for the partition function of the “electric” theory \[19, 32\]:

\[
Z_{N_c}^{el,N_f}(\Delta_Q) = \frac{1}{|\mathcal{W}|} \int \left( \prod_a du^a \right) F_{N_c}(u)e^{N_f G_{N_c}(u,\Delta_Q)},
\]

(2.2)

where \(\Delta_Q\) denote the conformal dimension of \(Q^a\) in the electric side, the variables \(u^a\) are real, the indices \(a, b\) range from 1 to \([N_c/2]\), and \(|\mathcal{W}|\) is the order of the Weyl group \(\mathcal{W}\). The expressions for functions \(F_{N_c}\) and \(G_{N_c}\) depend on whether \(N_c\) is even or odd. If \(N_c\) is even, say, \(N_c = 2n\), we have

\[
F_{2n}(u) = \prod_{a<b} (4 \sinh(\pi(u_a - u_b)) \sinh(\pi(u_a + u_b)))^2,
\]

(2.3)

\[
G_{2n}(u, \Delta_Q) = \sum_a (l(1 - \Delta_Q + iu_a) + l(1 - \Delta_Q - iu_a)).
\]

(2.4)

Here the function \(l(z)\) is given by

\[
l(z) = -z \log(1 - e^{2\pi iz}) + \frac{i}{2} \left( \pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz}) \right) - \frac{i\pi}{12},
\]

(2.5)

which satisfies \(dl(z)/dz = -\pi z \cot(\pi z)\). If \(N_c\) is odd, \(N_c = 2n + 1\), we have

\[
F_{2n+1}(u) = \prod_c (2 \sinh(\pi u_c))^2 \prod_{a<b} (4 \sinh(\pi(u_a - u_b)) \sinh(\pi(u_a + u_b)))^2,
\]

(2.6)

\[
G_{2n+1}(u, \Delta_Q) = l(1 - \Delta_Q) + \sum_a (l(1 - \Delta_Q + iu_a) + l(1 - \Delta_Q - iu_a)).
\]

(2.7)

The partition function of the “magnetic” theory is similar:

\[
Z_{N_c}^{mag,N_f}(\Delta_Q, \Delta_M, \Delta_Y) = \frac{1}{|\mathcal{W}'|} e^{l(1-\Delta_Y)}e^{l(1-\Delta_M)}N_f(N_f+1)/2 \int \left( \prod_a du^a \right) F_{N_c}(u)e^{N_f G_{N_c}(u,\Delta_q)},
\]

where \(N_c' = N_f - N_c + 2\) and the indices \(a, b\) now range from 1 to \([N_c'/2]\). The pre-factor is the contribution of the gauge singlet chiral multiplets. As argued in \[20, 21\], the partition function should be the same for electric theory and magnetic theory as a function of \(\Delta_Q\). It’s convenient to list the free energy \(F = -\log|Z|\). We check the partition function on \(S^3\) for a few cases and find the agreement up to the accuracy of \(10^{-6}\). Some of the simple cases are listed in Table 2 3 4.

We can obtain the conformal dimension of fields by extremizing \(\log|Z|\). Generally, it is hard to find the conformal dimension that extremizing the partition function analytically but for the simple case of \(N_f - N_c + 2 = 1\), the dual magnetic theory becomes theory of \(O(1)\) and it is relatively easy to find. The partition function of magnetic \(O(1)\) theory is

\[
Z_{N_c}^{mag,N_f} = Z_{N_f+1}^{mag,N_f} = e^{l(1-\Delta_Y)}e^{l(1-\Delta_M)}N_f(N_f+1)/2 e^{N_f l(1-\Delta_q)}
\]

\[
= e^{l(N_f r + l(1-2r))N_f(N_f+1)/2 + N_f l(r)}
\]

(2.8)
Table 2: Free energy $F = -\log|Z|$ for $O(2)^{el}_2$ and its dual $O(2)^{mag}_2$ theory.

| $\Delta Q$ | 0.275 | 0.3 | 0.325 | 0.35 | 0.375 | 0.4 | 0.425 | 0.45 |
|------------|-------|-----|-------|-----|-------|----|-------|-----|
| $-\log|Z|$ | 1.66573 | 1.75794 | 1.8338 | 1.88659 | 1.91895 | 1.93301 | 1.93056 | 1.91308 |

Table 3: Free energy $F = -\log|Z|$ for $O(2)^{el}_3$ and its dual $O(2)^{mag}_3$ theory.

| $\Delta Q$ | 0.275 | 0.3 | 0.325 | 0.35 | 0.375 | 0.4 | 0.425 | 0.45 |
|------------|-------|-----|-------|-----|-------|----|-------|-----|
| $-\log|Z|$ | 2.24168 | 2.43027 | 2.5767 | 2.68642 | 2.76383 | 2.81249 | 2.83538 | 2.83499 |

Table 4: Free energy $F = -\log|Z|$ for $O(3)^{el}_4$ and its dual $O(3)^{mag}_4$ theory.

| $\Delta Q$ | 0.275 | 0.3 | 0.325 | 0.35 | 0.375 | 0.4 | 0.425 | 0.45 |
|------------|-------|-----|-------|-----|-------|----|-------|-----|
| $-\log|Z|$ | 4.76985 | 5.05169 | 5.25322 | 5.38042 | 5.45165 | 5.43218 | 5.42046 | 5.33034 |

where $\Delta_Y = N'_c - N_f r$, $\Delta_M = 2r$ and $\Delta_q = 1 - r$. This expression is real positive, so it is effectively the same as extremizing its logarithm

$$0 = \frac{d \log Z}{dr} = -\pi N_f^2 r \cot(\pi N_f r) + \pi N_f (N_f + 1)(1 - 2r) \cot(\pi (1 - 2r)) - \pi N_f r \cot(\pi r) \quad (2.9)$$

We can find the analytic solution of (2.9) when $N_f = 1$, $r = \Delta_Q = 1/3$. The magnetic side is $O(1) = Z_2$ gauge theory with one flavor $q$, two singlets $Y, M$ with the superpotential

$$W = M(q^2 + Y^2). \quad (2.10)$$

Thus the R-charge of $q$ is $\frac{2}{3}$ and that of $Y$ is $\frac{2}{3}$. Note that would-be monopole operator $y$ has conformal dimension $-\frac{4}{3}$ if coupled to $Y$ via the superpotential term $W = Y y + \cdots$. This is in violation of the unitarity and is consistent with fact that for $O(1)$ theory we do not have the monopole operator. Also $Z_2$ acts on $q$ by flipping its sign. Note that for both electric and magnetic side, the moduli space is parametrized by the gauge invariant operators $Y, M$.

Generically we can find the conformal dimension numerically, so when $N_f = 2$, $r = 0.2697$ and when $N_f = 3$, $r = 0.4256$ and they are within the unitarity bound $\Delta_M = 2\Delta_Q \geq 1/2$. We can see that the $\Delta_M$ is irrational. These values we get from the above are coincident with those of the corresponding electric theory obtained numerically as listed at Table 3.

Consider “electric” $O(2)^{el}_2$ theory as an example. Its dual “magnetic” theory is $O(2)^{mag}_2$. The critical value of the dimension is $\Delta_Q = 0.4086$. And “electric” $O(2)^{el}_3$ case, the critical value of $\Delta_Q = 0.4370$. Note that the conformal dimension $\Delta_Q$
| Theory | $O(2)_1$ | $O(2)_2$ | $O(2)_3$ | $O(2)_4$ | $O(2)_5$ | $O(3)_3$ | $O(3)_4$ | $O(3)_5$ |
|--------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\Delta_Q$ | 0.3333 | 0.4086 | 0.4370 | 0.4520 | 0.2697 | 0.3531 | 0.3923 | 0.4149 |

Table 5: Conformal dimensions for various “electric” theories.

become closer to $1/2$ as $N_c/N_f$ decreases. Since the theory is more weakly coupled in this limit, this is an expected result. For some other cases, we list the conformal dimension of $Q$ in Table 5.

3. Superconformal index

We consider the superconformal index for 3-d $\mathcal{N} = 2$ superconformal field theory (SCFT). The bosonic subgroup of the 3-d $\mathcal{N} = 2$ superconformal algebra is $SO(2,3) \times SO(2)$. There are three Cartan elements denoted by $\epsilon, j_3$ and $R$ which come from three factors $SO(2)_{\epsilon} \times SO(3)_{j_3} \times SO(2)_R$ in the bosonic subalgebra. One can define the superconformal index for 3-d $\mathcal{N} = 2$ SCFT as follows [22],

$$I = Tr(-1)^F \exp(-\beta'\{Q,S\})x^{\epsilon+j_3}\prod_j y_j^{F_j}$$

(3.1)

where $Q$ is a special supercharge with quantum numbers $\epsilon = \frac{1}{2}, j_3 = -\frac{1}{2}$ and $R = 1$ and $S = Q^1$. They satisfy the following anti-commutation relation:

$$\{Q,S\} = \epsilon - R - j_3 := \Delta.$$  

(3.2)

In the index formula, the trace is taken over gauge-invariant local operators in the SCFT defined on $\mathbb{R}^{1,2}$ or over states in the SCFT on $\mathbb{R} \times S^2$. As is usual for Witten index, only BPS states satisfying the bound $\Delta = 0$ contributes to the index and the index is independent of $\beta'$. If we have additional conserved charges commuting with chosen supercharges $(Q,S)$, we can turn on the associated chemical potentials and the index counts the number of BPS states with the specified quantum number of the conserved charges denoted by $F_j$ in eq. (3.1).

The superconformal index is exactly calculable using localization technique [23,24]. Following their works, the superconformal index can be written in the following form,

$$I(x) = \sum_m \int da \frac{1}{|W|} e^{-S_{\text{eff}}(\epsilon) - \epsilon \phi_0(a) - \epsilon \phi_0} x^\epsilon \exp \left[ \sum_{n=1}^\infty \frac{1}{n} f_{\text{tot}}(e^{i\alpha}, y_j^n, x^n) \right].$$

(3.3)

To take trace over Hilbert-space on $S^2$, we impose proper periodic boundary conditions on time direction $\mathbb{R}$. As a result, the base manifold become $S^1 \times S^2$. For saddle points in localization procedure, we need to turn on monopole fluxes on $S^2$.
and holonomy along \( S^1 \). These configurations of the gauge fields are denoted by \( \{ m \} \) and \( \{ a \} \) collectively. Both variables take values in the Cartan subalgebra of \( G \). \( S_0 \) denotes the classical action for the (monopole+holonomy) configuration on \( S^1 \times S^2 \). \( \epsilon_0 \) is called the Casimir energy. If we consider the theory without Chern-Simons (CS) term, there would be no contribution from \( S_0 \). Also if we consider non-chiral theories, we have \( b_0(a) = 0 \). In (3.3), \( \sum_m \) is over all integral magnetic monopoles charges, \( f_{\text{tot}} = f_{\text{chiral}} + f_{\text{vector}} \) and \(|W| = (\text{the order of the Weyl group}) \). Each component in (3.3) is given by

\[
\begin{align*}
\epsilon_0 &= \frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_b} |\rho(m)| - \frac{1}{2} \sum_{\alpha \in G} |\rho(m)|, \\
F_{\Phi} &= \sum_{\Phi} \sum_{\rho \in R_b} \left[ e^{i\rho(a)} y_j F_j x^{|\rho(m)|+\Delta_{\Phi}} - e^{-i\rho(a)} y_j^{-1} F_j x^{|\rho(m)|+2-\Delta_{\Phi}} \right], \\
f_{\text{chiral}}(e^{ia}, y_j, x) &= \sum_{\Phi} \sum_{\rho \in R_b} \left[ e^{i\rho(a)} y_j F_j x^{|\rho(m)|+\Delta_{\Phi}} - e^{-i\rho(a)} y_j^{-1} F_j x^{|\rho(m)|+2-\Delta_{\Phi}} \right], \\
f_{\text{vector}}(e^{ia}, y_j, x) &= -\sum_{\alpha \in G} e^{i\alpha(a)} x^{|\alpha(m)|}.
\end{align*}
\]

(3.4)

where \( \sum_{\Phi} \sum_{\rho \in R_b} \) and \( \sum_{\alpha \in G} \) represent the summations over all chiral multiplets, all weights and all roots, respectively. \( F_i \) are the Cartan generators acting only on the \( i \)-th flavor.

The index formula for the duality that we are considering is similar to that for theGiveon-Kutasov duality \[25\] except for the absence of the CS term and the contribution of the additional gauge singlet chiral multiplet \( Y \) on the magnetic side. It’s important to take account of nontrivial action of \( Z_2 \) element in \( O(N) \) whose determinant is \(-1\). We review it following \[25\]. Let us first consider \( O(2N) \) case. Since the weights of the fundamental representation are \( \pm \epsilon_i \) where \( i = 1, \ldots, N \) and the roots of \( O(2N) \) are \( \pm \epsilon_i \pm \epsilon_j \) where \( i, j = 1, \ldots, N \) and \( i \neq j \), we obtain for example the contribution from one chiral multiplet with vector representation with turning off the chemical potential \( y_j = 1 \).

\[
f_{\text{chiral}}(e^{ina}, y_j = 1, x^n) = \frac{x_{nr} - x_{(2-r)n}}{1 - x^{2n}} \left[ \sum_{i=1}^{N} x^n |m_i| 2 \cos na_i \right].
\]

(3.5)

This formula holds for \( SO(2N) \) case. We should consider the additional projection for \( Z_2 \) element of \( O(2N) \) not belonging to \( SO(2N) \) group. We choose the specific \( Z_2 \)
action,

\[
Z_2 = \begin{pmatrix}
1 & & \\
-1 & & \\
& \ddots & \\
& & 1
\end{pmatrix}.
\] (3.6)

Under this \(Z_2\) action, the eigenvalues of the holonomy and the monopoles are mapped to

\[e^{\pm ia_1} \to \pm 1, \quad \pm m_1 \to 0.\] (3.7)

The other variables are not affected. Thus, \(f_{\text{chiral}}\) turns into

\[
f_{\text{chiral}}(e^{ia}, y_j = 1, x^n) = \frac{x^{nr} - x^{(2-r)n}}{1 - x^{2n}} \left[ (1 + (-1)^n) + \sum_{i=2}^{N} x^{n|m_i|} 2 \cos na_i \right],
\] (3.8)

Let us turn to \(O(2N+1)\) theory. The weights of the fundamental representation are \(\pm \epsilon_i\) where \(i = 1, \cdots, N\) and the roots of \(O(2N+1)\) are \(\pm \epsilon_i \pm \epsilon_j\) where \(i, j = 1, \cdots, N\) and \(i \neq j\). In this case, we choose \(Z_2\) action,

\[
Z_2 = \begin{pmatrix}
1 & & \\
& \ddots & \\
& & 1 \\
& & -1
\end{pmatrix},
\] (3.9)

where an eigenvalue 1 of the holonomy in the fundamental representation is mapped to

\[1 \to -1\] (3.10)

while the others are not influenced. Furthermore, eigenvalues \(e^{\pm ia_1}\) of the holonomy in the adjoint representation are transformed to

\[e^{\pm ia_1} = e^{\pm ia_1} \cdot 1 \to e^{\pm ia_1} \cdot (-1)\] (3.11)

while the others, which are in the form of \(e^{i(\pm a_i \pm a_j)} = e^{\pm ia_i} \cdot e^{\pm ia_j}\) are not influenced.

The result of the index computation is given in Table 6. The indices on both sides agree perfectly.
| (3,1) | O(1) | O(4) | \(1 - 9x^2 + 3x^{1-2r} + 28x^{6r} + x^{4r} (15 - 33x^2) + x^{2r} (6 - 21x^2) + \cdots\) |
|-------|------|------|--------------------------------------------------|
| (1,2) | O(2) | O(1) | \(1 - x^2 - 2x^4 + x^{5-5r} + x^{4-4r} + x^{3-3r} + x^{2-2r} + x^{1-r} + x^{2r} + x^{4r} + x^{3+3r} + \cdots\) |
| (2,2) | O(2) | O(2) | \(1 - 4x^2 + 5x^4 + x^{4-4r} + 6x^{4r} + x^{2r} (3 - 8x^2) + \cdots\) |
| (3,2) | O(2) | O(3) | \(1 - 9x^2 + 36x^4 + x^{3-3r} + 3x^{4-2r} + 21x^{4r} + x^{2r} (6 - 45x^2) + \cdots\) |
| (4,2) | O(2) | O(4) | \(1 - 16x^2 + 148x^4 + x^{4-4r} + 6x^{4-2r} + 55x^{4r} + x^{2r} (10 - 144x^2) + \cdots\) |
| (2,3) | O(3) | O(1) | \(1 + 3x + x^2 + x^{4-4r} + x^{3-6r} + 3x^{2r} + x^{-2r} (x + 3x^2) + x^{-4r} (x^2 + 3x^3) + \cdots\) |
| (3,3) | O(3) | O(2) | \(1 - 9x^2 + x^{1-6r} + 6x^{4-4r} + x^{2-3r} + 6x^{2-2r} + 6x^{2r} + 21x^{4r} + 15x^{2-4r} + \cdots\) |
| (4,3) | O(3) | O(3) | \(1 - 16x^2 + 35x^3 + x^{3-4r} + 10x^{3-2r} + 10x^{2r} + 55x^{4r} + \cdots\) |
| (5,3) | O(3) | O(4) | \(1 - 25x^2 + x^{4-5r} + 15x^{3-4r} + 10x^{2r} + x^{1-4r} (15 - 350x^2) + \cdots\) |
| (3,4) | O(4) | O(1) | \(1 + 46x^2 + 21x^{4-8r} + 21x^{3-5r} + 6x^{2-4r} + x^{1-3r} + 21x^{2-2r} + 6x^{1-r} + 6x^{2r} + 21x^{1+r} + \cdots\) |
| (4,4) | O(4) | O(2) | \(1 + 39x^2 + x^{4-8r} + 10x^{4-6r} + x^{2-4r} + 10x^{2-2r} + 10x^{2r} + \cdots\) |
| (5,4) | O(4) | O(3) | \(1 - 25x^2 + x^{3-5r} + 15x^{3-3r} + 120x^{3-2r} + 15x^{2r} + 120x^{4r} + \cdots\) |
| (6,4) | O(4) | O(4) | \(1 - 36x^2 + x^{8-12r} + 21x^{4-4r} + 246x^{4-2r} + 231x^{4r} + 21x^{2r} - 720x^{2+2r} + x^{4-6r} + \cdots\) |

Table 6: The result of the superconformal index computation.

We can examine the chiral ring structure of the theory. Let us first consider the \(N_c > 1\) cases. In these cases, the chiral primaries on the electric side are the meson operators \(M^{(ab)} = Q_i^a Q_i^b\) and the monopole operator \(Y\). One can show by following closely [23], there are no other monopole operators which are chiral primary. The baryon operators are projected out due to the nontrivial \(Z_2\) element of \(O(N)\) whose determinant is \(-1\). Their counterparts on the magnetic side are the lowest components of the gauge singlet chiral multiplets \(M^{(ab)}\) and \(Y\). The composite meson operators \(m^{(ab)} = q^a_i q^b_i\) and the monopole operator \(y\) are all \(Q\)-exact due to the superpotential (2.4). Thus, the chiral ring structures on both side are exactly the same. Their contribution to the superconformal index can be easily checked.
There are $N_f(N_f+1)/2$ meson operators $M^{(ab)}$ of energy $\epsilon = 2r$, whose contributions to the index is thus $N_f(N_f+1)/2 x^{2r}$. The monopole operator $Y$, which has energy $\epsilon = N_f - N_c + 2 - N_f r$, makes the contribution $x^{N_f - N_c + 2 - N_f r}$ to the index.

For $N_c = 1$ and $N_f - N_c + 2 \neq 1$, on the other hand, there is no monopole operator on the electric side because the gauge group is just $O(1) = \mathbb{Z}_2$. Thus, the meson operators $M^{(ab)}$ are the only chiral primaries. On the magnetic side, there is still the chiral operator $Y$, which seems to be one of the chiral primaries. We propose that it becomes $Q$-exact due to the superpotential

$$W \sim Y y + y^2 \det m \sim \frac{Y^2}{\det m}.$$  \hspace{1cm} (3.12)

The form of the superpotential is similar to the Affleck-Dine-Seiberg (ADS) superpotential \cite{ADS}. Here simple $R$ charge counting shows that such superpotential is possible due to the additional singlet $Y$ to soak up the additional fermion zero modes. It would be interesting to derive this by explicit computation. In the index computation, there is no $x^{N_f + 1 - N_f r}$ term in the index because the contribution $-x^{N_f + 1 - N_f r}$ of the fermionic operator $\psi_Y^\dagger \det m$ exactly cancels out the contribution $x^{N_f + 1 - N_f r}$ of $Y$ as we expected. Indeed, the contribution of the fermionic operators $m^{N_f} \psi_Y^\dagger$ is in general canceled by the contribution of the scalar operators $m^{N_f} y$. Here the contracted gauge indices of $m^{N_f} y = (q^a_i q^b_i) y$ run over $N_f - 1 = N_c' - 2$ values corresponding to the unbroken gauge group $O(N_c' - 2)$ in the presence of the monopole flux associated with $y$. Here $N_c'$ denotes the magnetic gauge group. Schematically we have the block-diagonal structure

$$
\begin{pmatrix}
  m & 0 \\
  0 & y
\end{pmatrix}.
\hspace{1cm} (3.13)
$$

The mesons $m_{(ab)}$ do not couple to the magnetic flux excited by $y$ and remain as scalars. It is obvious that Gauss constraint is satisfied if we view such operator as a state defined on $S^2 \times \mathbb{R}$. However, one combination of such scalar operators, $y \det m$, is vanishing since $N_f \times N_f$ matrix $m$ has rank $N_f - 1$. As a result, $\psi_Y^\dagger \det m$ can survive and cancel the contribution of $Y$. Therefore, meson operators $M^{(ab)}$ are the only chiral primaries for $N_c = 1$ on either side.

For $N_c = 1$ and $N_f - N_c + 2 = 1$, we have $N_f = 0$ so that both sides have abelian gauge group. In this case, there would be no monopole operators, no meson operators and the theory is trivial. Thus, the superconformal index is just 1 on either sides.

Furthermore, we can trace not only the contributions of the chiral primaries but also those of BPS operators with nonzero angular momentum. For example, there are $N_f^2$ fermionic operators $Q^a_i \psi^\dagger_b$ on the electric side of energy $\epsilon = \frac{3}{2}$ and the angular momentum $j = \frac{1}{2}$. Correspondingly, there are $N_f^2$ fermionic operators
$q^i_n \psi^j_{qi}$ on the magnetic side of same energy and the same angular momentum. On either side, their contribution to the index is $-N_f^2 x^2$. The results listed in Table except for $(N_f, N_c) = (2, 3), (3, 4)$ and $(4, 4)$ cases, confirm this argument. The three exceptional cases are due to the fact that there are additional BPS operators whose $\epsilon + j$ is 2. Let us examine these exceptional cases in detail.

At first, we consider the $(N_f, N_c) = (2, 3)$ case. On the electric side, we can find the operators $M^2 Y^2$, which are scalar BPS operators, of energy $\epsilon = 2$. The contracted gauge indices of $M^2 Y^2 = Q^i_n Q^j_{qi} Q^j_{qj} Q^{i'}_{qk} Y^2$ run over the values corresponding to the unbroken gauge group in the presence of monopole flux associated with $Y$ such that $M^2$ do not couple to the magnetic flux. If the operators $M^2$ couple to the magnetic flux, then they get an effective spin such that their energy are no longer 2. Since the unbroken gauge group is in this case just $O(1)$, the gauge indices are fixed to the one value corresponding to the unbroken $O(1)$. Thus, the operators $M^2 Y^2 = Q^i_n Q^j_{qi} Q^j_{qj} Y^2$ with the fixed gauge index make the contribution $2 H_1 x^2 = 5x^2$ to the index. Therefore, the total $x^2$ term is $(-4 + 5)x^2 = x^2$. Here $-4$ comes from $-N_f x^2$ discussed in the previous paragraph. On the magnetic side, as opposed to the electric side, there is no gauge index to be contracted and no issue of the coupling to the flux because $M^{(ab)}$ and $Y$ are just elementary chiral fields. Thus, the number of $M^2 Y^2$ is $3 \cdot 4/2 = 6$ where we have $2 \cdot 3/2 = 3 M^{(ab)}$. However, some of its contribution is canceled by that of the fermionic operator $Y \psi^+_Y$, whose contribution is $-x^2$. Therefore, the total $x^2$ term in the index is again $(-4 + 6 - 1)x^2 = x^2$ on the magnetic side.

The $(N_f, N_c) = (3, 4)$ case is exactly the same. On the electric side, the scalar BPS operators $M^3 Y^2 = Q^i_n Q^j_{qi} Q^j_{qj} Q^j_{qk} Y^2$ have energy $\epsilon = 2$ where the contracted gauge indices $i, j$ and $k$ run over 2 values corresponding to the unbroken gauge group $O(2)$. Since the number of $M^{(ab)}$ is $3 \cdot 4/2 = 6$, there are naively $6 H_2 = 56 M^3 Y^2$. However, we could check that $\det M = 0$ if the gauge indices run over only 2 values, which means that the number of independent $M^3 Y^2$ is $56 - 1 = 55$. On the magnetic side, there are $56 M^3 Y^2$ and one $Y \psi^+_{Y}$. They make an additional contribution $(55 - 1)x^2 = 55x^2$. Thus, on either electric and magnetic side, the total $x^2$ term is $(-9 + 55)x^2 = 46x^2$.

For $(N_f, N_c) = (4, 4)$, the scalar BPS operators $M^2 Y = Q^i_n Q^j_{qi} Q^j_{qj} Y$ on the electric side have energy $\epsilon = 2$ where the contracted gauge indices $i$ and $j$ run over 2 values corresponding to the unbroken $O(2)$ as before. Since there are $4 \cdot 5/2 = 10 M^{(ab)}$ and $10 H_2 = 55 M^2$, the contribution of $M^2 Y$ to the index is $55x^2$. On the magnetic side, there are $55 M^2 Y$ as on the electric side. Therefore, the total $x^2$ term is $(-16 + 55)x^2 = 39x^2$ on either sides.

We can check the indices in more detail by turning on the chemical potentials of the global flavor symmetry. There are few differences in index formula that there

\[^1 n H_m = n(m-1) C_m = \frac{(n+m-1)!}{m!(n-1)!} \] is the combination with repetition.
are flavor charge terms $y_j^{\text{fl}}$ and chemical potentials $y_j$ are in letter index $q$. The resulting indices can be rewritten in terms of $U(1) \times SU(N_f)$ characters. All of the above results correspond to the cases with setting $y_j = 1$. For example, for the case of $N_c = 2$, $N_f = 2$, either indices of electric and magnetic theory is

$$I = 1 - 2x^{2-2r} \frac{1}{y_1 y_2} - x^2 \left( \frac{y_1}{y_2} + \frac{y_2}{y_1} + 2 \right) + x^{2r} \left( y_1^2 + y_1 y_2 + y_2^2 \right) + \cdots$$

$$= 1 - 2x^{2-2r} y_0^{-2} - x^2 (\chi_1 (u) + 1) + x^{2r} y_0^2 \chi_1 (u) + \cdots$$

(3.14)

where $y_0 = (y_1 y_2)^{1/2}$ and $u = y_1 / y_2$ correspond to the chemical potentials for the global symmetry $U(1)_A \times SU(2)$ and $\chi_n (u) = u^{-n} + u^{-n+1} + \cdots + u^n$ are the characters of $SU(2)$.

As a final remark, Seiberg-like duality for the orthogonal gauge group considered in [24, 25] can be derived from the duality we have considered here. It is well known that the CS term $\pm \frac{1}{8\pi} \int \text{tr} A \wedge F$, which is one unit of the CS level for the orthogonal groups, is generated by integrating out a charged fermion by giving it axial mass. This mass term can be understood as arising from weakly gauging the axial symmetry $U(1)_A$ by a background vector field $V_{\text{mass}} = -i \bar{\theta} \mu$. Thus the mass term for a chiral multiplet $Q$ on the electric side with $U(1)_A$ charge $+1$ is given by

$$\mathcal{L}_{\text{mass}} = \int d^4 \theta Q^+ e^{V_{\text{mass}}} Q.$$  

(3.15)

Note that on the magnetic theory under Aharony duality, $q$ picks up axial mass term of negative sign since it has $U(1)_A$ charge $-1$. Thus on the electric side, we flow from $O(N_c)$ with $N_f$ flavors into $O(N_c)$ with $N_f - 1$ flavors with Chern-Simons level 1 while in the magnetic side, we are flowing to $O(N_f - N_c + 2)$ with $N_f - 1$ flavors with Chern-Simons level $-1$. Thus from Aharony duality, one can obtain Seiberg-like duality for Chern-Simons matter theories. By repeatedly integrating out charged chiral multiplets one can obtain Seiberg-like duality for Chern-Simons matter theories with higher Chern-Simons level. Note also that by weakly gauging $U(1)_A$ symmetry, one also gives mass to the monopole operators $Y, y$.

4. Conclusions

In this paper we provide evidences for Aharony duality for orthogonal groups. By using available tools of the partition function and the superconformal index, we give sufficient evidences for Aharony duality for orthogonal gauge groups with matters in the vector representation. Along the investigation, we come up with the proposal that for $O(N_f + 1)_{N_f}$ theory in the magnetic side, it should develop the superpotential

$$W = \frac{Y^2}{\det m}$$

(4.1)
reminiscent of the ADS superpotential in 4d. Recall that without the additional singlet \( Y \) we could not have such superpotential so it would be interesting to work out the proposed supertpotential explicitly. Further we expect that for \( N_f - N_c + 2 = 0 \) there would be no SCFT dual in the magnetic side. Rather, we will have the quantum moduli space to be modified
\[
Y^2 \det M = 1. \tag{4.2}
\]
This is another interesting exercise for instanton calculus. Analogous 4d computation was done in \cite{39, 40}.

**Acknowledgements**

We are grateful to A. Kapustin for the discussion on the Aharony duality. J.P. is supported by the KOSEF Grant R01-2008-000-20370-0, the National Research Foundation of Korea (NRF) Grants No. 2009-0085995 and 2005-0049409 through the Center for Quantum Spacetime (CQUEST) of Sogang University. J. P. appreciates APCTP for its stimulating environment for research and acknowledges Simons summer workshop on mathematics and physics 2011 for hospitality while a part of the current work was carried out.

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