Theory of the Quantum Speed Up

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Abstract

Insofar as quantum computation is faster than classical, it appears to be irreversible. In all quantum algorithms found so far the speed-up depends on the extra-dynamical irreversible projection representing quantum measurement. Quantum measurement performs a computation that dynamical computation cannot accomplish as efficiently.

I. PREMISE

The quantum algorithms are sometimes faster than their classical counterparts. We show that this quantum speed-up results from a succession of entanglement and disentanglement, the former due to dynamical quantum-parallel computation, the latter to the extra-dynamical projection of quantum measurement. Thus the quantum speed-up implies irreversibility.

Some standard notions concerning problem solving must be modified to understand the speed-up. Standard problem solving has three stages:

(i) State the problem. This defines the problem solution, usually implicitly. E.g. consider the problem of finding two primes \(x, y\) (unknown) such that \(x \cdot y = c\) (known). This
equation implicitly defines the values of \( x \) and \( y \) which satisfy it. An implicit definition does not represent the process required to compute the solution.

(ii) *Program the computation.* Change the implicit definition into an explicit, finite step-by-step logical procedure for constructing the solution. This procedure is specified by the solution algorithm.

(iii) *Run the program.* The execution is dynamical in character. By dynamics we mean, here and in the following, deterministic dynamics\(^1\).

The standard assumption is that the solution of a problem must be computed by a dynamical development. Step (ii) changes a definition which does not represent a dynamical process, into one which represents it.

Quantum computation does not fit this scheme, as we will see in the next Section.

**II. THE SPEED-UP IN SHOR’S ALGORITHM**

Consider Shor’s algorithm \([1]\). The problem is to efficiently find the period \( r \) of a hard-to-reverse function \( f(x) \) from \( \{0, 1\}^n \) to \( \{0, 1\}^n \). Fig. 1 gives the algorithm block diagram.

\[ H \text{ is the Hadamard and } F \text{ the digital Fourier transform, } M \text{ denotes measurement of a register content. We need to consider only two steps of the algorithm (see Castagnoli et al., } [2] \).

(I) The process of computing \( f(x) \) for all possible \( x \), in quantum superposition, puts two \n-qubit registers \( X \) and \( F \) into the state \[^{\text{3}}\]"}

\[^{1}\text{Classical nondeterministic computation, at the current fundamental level, will be seen as pseudorandom deterministic computation.}\]

\[^{2}\text{Quantum theory can be formulated praxically or ontically. The former (for example, Finkelstein}\]
\[ |\psi, t_2\rangle_{XF} = \frac{1}{\sqrt{N}} \sum_x |x\rangle_X |f(x)\rangle_F, \]

\( x \) runs over 0, 1, ..., \( N - 1 \), with \( N = 2^n \).

(II) Let \([F]\) be the content of register \( F \), an observable. Measuring \([F]\) in \(|\psi, t_2\rangle_{XF}\) and finding the result \( \mathcal{F} \) yields the state

\[ |\psi, t_3\rangle_{XF} = k (|x\rangle_X + |x + r\rangle_X + |x + 2r\rangle_X + ...) |\mathcal{F}\rangle_F; \]

\( k \) is for normalization, \( f(x) = f(x + r) = ... = \mathcal{F} \).

The transition from \(|\psi, t_2\rangle_{XF}\) to \(|\psi, t_3\rangle_{XF}\) obeys the quantum principle:

(A) measuring an observable is extra-dynamically represented by projection on the eigenspace of one eigenvalue;

(B) this eigenvalue is selected at random according to the square of the probability amplitudes.

If \( Q \) and \( Q' \) are the projection operators on the state just before and after the measurement \( M \), and \( P \) is the projection operator for the observed value of the function, then classically \( Q' = Q \), while quantally \( Q' = PQP \) (up to normalization), depending on both \( P \) and \( Q \).

Thus, selecting \( \mathcal{F} \) projects into the post-measurement state all and only those tensor products of \(|\psi, t_2\rangle_{XF}\) ending with \(|\mathcal{F}\rangle_F\). Point (A) of the quantum principle, by selecting one eigenvalue, imposes a logical constraint on the output of computation. Because of this constraint, quantum measurement filters, out of an exponentially larger superposition, all and only those values of \( x \) whose function is that eigenvalue.

3) is closest to Heisenberg’s. It deals with operators dispensing with states. The ontic formulation makes quantum theory seem less time-symmetrical than it really is. In the present theory, it makes the speed-up seem to happen all at once at the end of computation. We use the ontic formulation here, misleading as it is, because it is more familiar to most physicists.

3This intermediate measurement can be skipped, but we will show that this makes no difference.
Therefore quantum measurement performs extra-dynamically a computation crucial for finding $r$, which is “readily” extracted out of $|\psi, t_3\rangle_{XF}$. Measurement time is linear in the number of qubits of register $F$, and is independent of the entanglement between $X$ and $F$, which holds problem complexity. Disentanglement comes for free, as a by-product of quantum measurement.

Filtration, together with function evaluation, is essential to speed-up. This can be better seen by comparing step by step quantum and classical computation times\(^4\): (I) function evaluation: $\text{poly}(n)$ vs $\exp(n)$; (II) filtration: $\text{linear}(n)$ vs $\exp(n)$; (III) extracting $r$ out of $|\psi, t_3\rangle_{XF}$: $\text{linear}(n)$ vs $\text{linear}(n)$. Speed-up is due to steps (I) and (II).

The extra-dynamical character of quantum computation is clarified by showing that Shor’s algorithm does not fit standard, dynamical, problem solving. Quantum dynamics is deterministic: any state in time dynamically determines a unique successor. While function evaluation is dynamical in character, the filtration performed by quantum measurement is not.

Classically the state after measurement $M$ is the same as the state before $M$. In quantum fact the state after measurement is influenced by both the state before measurement and the measurement itself, by the quantum principle: $|\psi, t_3\rangle_{XF} = k \langle \overline{f} |_F \langle \overline{f} |_F |\psi, t_2\rangle_{XF} \cdot |\psi, t_2\rangle_{XF}$ is the prior state; the left-multiplication by $|\overline{f} |_F \langle \overline{f} |_F$ represents the final constraint selecting all tensor products ending with $|\overline{f} |_F$. The determination of $|\psi, t_3\rangle_{XF}$ is jointly influenced by an initial condition and a final condition. It is richer than dynamical determination, insofar as it yields the speed-up. We needed a computational context to realize this.

\(^4\)Classical time is that required to derive the symbolic description of a quantum state (e.g. $|\psi, t_2\rangle_{XF}$, or $|\psi, t_3\rangle_{XF}$, with all $x, f(x), \pi, r$, etc. replaced by the proper numerical values) from the previous one by classical computation. Note that the resulting classical algorithm is reasonably efficient in itself, with times on the order of problem size.
Extra-dynamical computation means much more than nondeterministic computation. For example, point (A) of the quantum principle does not involve randomness and yields Shor’s quantum speed-up in $\sim 70\%$ of the cases, when a single run of the algorithm is sufficient to identify $r$.

Determination with joint influence is extra-dynamical, it cannot be represented by a dynamical propagation of an input into an output. Of course we could go through step (ii), and replace joint influence with a dynamical process that leads to a “filtered” state like $|\psi, t_3\rangle_{XF}$. But this would introduce programming and computation, increasing computation time exponentially in problem size. Joint influence bypasses step (ii) as well as speeding-up the computation. It yields a direct physical determination of the object of an implicit definition.

This can also be seen as follows. $|\psi, t_3\rangle_{XF}$ (selected with joint influence) contains the solutions $x$ of the implicit algebraic equation $f(x) = \overline{f}$, although the reverse of $f$ has not been computed. Thus, $f(x) = \overline{f}$ implicitly defines the solutions $x$ while quantum measurement selects them without going through programming and dynamical computation. The extra-dynamical character of this selection is clear. It takes essentially no time.

It can be seen that the same theory of the speed-up holds for Simon’s algorithm, as modified in [3].

Until now we have assumed the intermediate measurement of $[F]$. However, as is well known, this measurement can be skipped without affecting the result of measuring $[X]$ at time $t_4$ (fig. 1). It was introduced by Ekert and Jozsa to clarify the way Shor’s algorithm operates; it can also clarify the speed-up. In fact skipping it is mathematically equivalent

\footnote{In $\sim 30\%$ of the cases, more than one run is needed. Randomness assures that we do not always obtain the same result.}

\footnote{Since the implicit definition is the problem, we see that, in Shor’s algorithm, computation can be identified with problem solving: points (i), (ii), and (iii) of Section I are both altered and unified.}
to performing it: the filtration performed by the extra-dynamical projection of quantum measurement is induced by measuring only \( X \) at the end.

If \( [F] \) measurement is skipped, the state of registers \( X \) and \( F \) at time \( t_4 \) is entangled. This establishes an equivalence between measuring \( [X] \) or \( [F] \). From a mathematical standpoint, the outcome of measuring \( [F] \) at time \( t_4 \) can be backdated in time along the reversible process, provided that the overall state undergoes the inverse of the usual forward-time evolution. This is equivalent to having measured \( [F] \) at time \( t_2 \).

We should counter the objection that register \( F \) can be annihilated immediately after function evaluation. This would leave register \( X \) in a mixture that is the partial trace over \( F \) of the density matrix of the two registers:

\[
|\psi', t_2\rangle_X = \frac{1}{\sqrt{N}} \sum_h e^{i\delta_h} (|x_h\rangle_X + |x_h + r\rangle_X + |x_h + 2r\rangle_X + ... |f_h\rangle_F ,
\]

where the range of \( h \) is such that \( f_h \) ranges over all the values assumed by \( f(x) \), \( f(x_h) = f(x_h + r) = ... = f_h \), and \( \delta_h \) are random phases independent of each other\(^7\). For the current purposes, annihilating \( F \) is like having performed the intermediate \( [F] \) measurement.

### III. THE SPEED-UP IN QUANTUM ORACLE COMPUTING

Quantum oracle computing can be seen as a competition between two players. One produces the problem, the other is challenged to produce the solution. We shall call the former player Sphinx, the latter Oedipus.

Let us consider Grover’s algorithm\(^7\). The game is as follows. The Sphinx hides an object in drawer number \( k \), among \( n \) drawers. Oedipus must find where it is, in the most efficient way. The chest of drawers is actually a quantum computer that, set in the mode \( k \)

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\(^7\)We are using the random phase representation. Let us exemplify it for a two-state system. The mixture \( \rho = \sin^2 \varphi |0\rangle \langle 0| + \cos^2 \varphi |1\rangle \langle 1| \) becomes \( |\psi\rangle = \sin \varphi |0\rangle + e^{i\delta} \cos \varphi |1\rangle \), where \( \delta \) is a random phase with uniform distribution in \([0, 2\pi]\); \( \rho \) is the average over \( \delta \) of \( |\psi\rangle \langle \psi| \).
and given a drawer number \( x \) as the input, yields the output \( f_k (x) = \delta_{k,x} \) (\( \delta_{k,x} = 1 \) if \( k = x \) and \( \delta_{k,x} = 0 \) if \( k \neq x \)). Fig. 2a gives the usual Grover’s algorithm for \( n = 4 \).

The Sphinx sets the mode \( k \) at random and passes the computer on to Oedipus. Oedipus must find \( k \) in the most efficient way by testing the computer input-output behaviour.

Without entering into detail, we note that the computer has two registers \( X \) and \( F \). Oedipus prepares them in the initial state \( \frac{1}{\sqrt{2}} |00\rangle_X (|0\rangle_F - |1\rangle_F) \), the same for all \( k \), and invokes the algorithm presented in fig. 2a. The state \( \psi \) before final measurement depends on the Sphinx’ choice \( k \):

\[
\begin{align*}
    k = 00 & \leftrightarrow \psi = \frac{1}{\sqrt{2}} |00\rangle_X (|0\rangle_F - |1\rangle_F), \\
    k = 01 & \leftrightarrow \psi = \frac{1}{\sqrt{2}} |01\rangle_X (|0\rangle_F - |1\rangle_F), \\
    k = 10 & \leftrightarrow \psi = \frac{1}{\sqrt{2}} |10\rangle_X (|0\rangle_F - |1\rangle_F), \\
    k = 11 & \leftrightarrow \psi = \frac{1}{\sqrt{2}} |11\rangle_X (|0\rangle_F - |1\rangle_F).
\end{align*}
\]

Measuring \( [X] \) yields Oedipus answer. This is reached in \( O(\sqrt{n}) \) time, versus \( O(n) \) with classical computation. But it is reached in a dynamical way, without any interplay between quantum parallel computation and the extra-dynamical projection of quantum measurement.

As we have seen, this interplay is associated with an isomorphism between the problem that implicitly defines its solution and the solution determination. This obviously requires that the problem is physically represented in a complete way. Here it is not: the above possible choices of the Sphinx and the related implications are not physically represented.

This is easily altered by introducing an ancillary two-qubit register \( K \) which contains the computer mode \( k \). Given the input \( k \) and \( x \), the output of computation is now \( F(k,x) = f_k(x) \) (fig. 2b). The preparation becomes

\[
\frac{1}{2\sqrt{2}} \left( |00\rangle_K + e^{i\delta_1} |01\rangle_K + e^{i\delta_2} |10\rangle_K + e^{i\delta_3} |11\rangle_K \right) |00\rangle_X (|0\rangle_F - |1\rangle_F),
\]
where \( \delta_1, \delta_2 \) and \( \delta_3 \) are independent random phases. To Oedipus, the Sphinx’ random choice is indistinguishable from a mixture where \( k \) is a random variable with uniform distribution over 00, 01, 10, 11. Fig. 2b includes the physical representation of the problem; we can go directly to the final state before measurement:

\[
\left( |00\rangle_K |00\rangle_X + e^{i\delta_1} |01\rangle_K |01\rangle_X + e^{i\delta_2} |10\rangle_K |10\rangle_X + e^{i\delta_3} |11\rangle_K |11\rangle_X \right) (|0\rangle_F - |1\rangle_F),
\]

\( \delta_1, \delta_2 \) and \( \delta_3 \) are independent random phases. Measuring \( [K] \) gives the Sphinx’ choice, measuring \( [X] \) gives Oedipus answer, or vice-versa. Now that the game has been physically represented, we find again that there is the above “interplay”.

In this game context, joint influence becomes the joint determination of the drawer number on the part of the two players, imposed by the quantum principle.

Why does extra-dynamical joint influence produce a speed-up? We suggest the following argument. The quantum game – yielding joint determination of the drawer number as a special quantum feature – should be as efficient as a classical game where there were joint determination of the drawer number on the part of the two players.

This cannot mean that Oedipus dictates the Sphinx’ choice, or the Sphinx suggests to Oedipus the right answer: this would be unilateral determination. Joint determination is symmetrical. The classical game must be defined as follows, with reference to the square-shaped chest of drawers herebelow\(^8\)

\[
\begin{array}{ccc}
0 & 1 & \\
0 & |00\rangle & |01\rangle \\
1 & |10\rangle & |11\rangle \\
\end{array}
\]

The Sphinx chooses the row number, say 1. Oedipus chooses the column number, say 0. Clearly, the drawer number 10 has been jointly determined by the Sphinx and Oedipus. Now the cost of Oedipus search is \( O(\sqrt{n}) \) rather than \( O(n) \), since he must search only the row. This is in agreement with theory.

\(^8\)If the number of drawers is \( O(n) \), the number of rows or columns is \( O(\sqrt{n}) \).
A similar analysis applies to Deutsch’s algorithm [8] as modified in [5].

IV. CONCLUSIONS

We have shown that quantum computation speed-up depends essentially on the extra-dynamical, irreversible projection of quantum measurement. To be sure, the entropy increase associated with speed-up is proportional only to the size of the output register, not the computation.

Extra-dynamical computation is more efficient than dynamical computation, as it yields the speed-up. It is a high level quantum feature, as it comes from a special interplay between a plurality of lower level ones (entanglement, disentanglement ...).

Earlier, attention was paid only to reversible quantum computation. The seminal well known works of Bennett, Fredkin and Toffoli, Benioff, and Feynmann demonstrated that computation can be reversible both in the classical and quantum framework. With Deutsch and others, quantum computation becomes quantum problem-solving, yields a speed-up and, we point out, ceases to be reversible. The current quantum algorithms ingeniously exploit extra-dynamical computation.

It is natural to ask whether other extra-dynamical projections than the one inherent in quantum measurement can be useful. Two come to mind at once: a statistics symmetry can be seen as an extra-dynamical projection on the “symmetric” subspace; and annealing is a projection on the ground state resulting from gradual cooling by a succession of extra-dynamical interventions. Exploiting these forms of projection might result in further speed-ups, and further reductions in the programming process.

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