On logarithmic solutions
to the conformal Ward identities

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Abstract

A general discussion of the conformal Ward identities is presented in the context of logarithmic conformal field theory with conformal Jordan cells of rank two. The logarithmic fields are taken to be quasi-primary. No simplifying assumptions are made about the operator-product expansions of the primary or logarithmic fields. Based on a very natural and general ansatz about the form of the two- and three-point functions, their complete solutions are worked out. The results are in accordance with and extend the known results. It is demonstrated, for example, that the correlators exhibit hierarchical structures similar to the ones found in the literature pertaining to certain simplifying assumptions.
1 Introduction

Logarithmic conformal field theory is essentially based on the appearance of conformal Jordan cells in the spectrum of fields. We refer to [1] for the first systematic study of logarithmic conformal field theory, and to [2, 3, 4] for recent reviews on the subject. The number of fields making up a conformal Jordan cell is called the rank of the cell. We will focus on conformal Jordan cells of rank two.

We consider the case where the logarithmic fields in the conformal Jordan cells are quasi-primary, and discuss the conformal Ward identities which follow. Without making any simplifying assumptions about the operator-product expansions of the fields, we find the general solutions for two- and three-point functions. Our results thus cover all the possible cases based on primary fields not belonging to conformal Jordan cells, primary fields belonging to conformal Jordan cells, and the logarithmic partner fields completing the conformal Jordan cells.

We also study the generality of two observations made under certain simplifying assumptions. The first observation concerns the expressibility of the correlators in terms of conformal weights with nilpotent parts [5]. This is a non-trivial point as it a priori presumes that the general solutions to the conformal Ward identities factor accordingly. We demonstrate that they do.

The second observation concerns a hierarchical structure for the set of correlators where the links are based on computing derivatives of the correlators with respect to the conformal weights [6, 7]. Also in this case, we find that the basic idea extends from the simpler set-up to our general situation.

This paper proceeds as follows. After a short introduction to the conformal Ward identities, we work out the general solutions for two- and three-point functions. We then affirm the assertions about conformal weights with nilpotent parts and the hierarchical structure. We conclude with some comments on further extensions.

2 Correlators in logarithmic conformal field theory

A conformal Jordan cell of rank two consists of two fields: a primary field, $\Phi$, of conformal weight $\Delta$ and its non-primary, ‘logarithmic’ partner field, $\Psi$, on which the Virasoro algebra generated by $\{L_n\}$ does not act diagonally. With a conventional relative normalization of the fields, we have

\[
\begin{align*}
[L_n, \Phi(z)] &= \left( z^{n+1} \partial_z + \Delta(n + 1)z^n \right) \Phi(z) \\
[L_n, \Psi(z)] &= \left( z^{n+1} \partial_z + \Delta(n + 1)z^n \right) \Psi(z) + (n + 1)z^n \Phi(z)
\end{align*}
\]

(1)

It has been suggested by Flohr [8] to describe these fields in a unified way by introducing a nilpotent, yet even, parameter $\theta$ satisfying $\theta^2 = 0$. We will follow this idea here, though use an approach closer to the one employed in [5, 9]. We thus define the field or unified
cell
\[ \Upsilon(z, \theta) = \Phi(z) + \theta \Psi(z) \]  
which is seen to be ‘primary’ of conformal weight \( \Delta + \theta \) as the commutators \((\mathbb{1})\) are replaced by
\[ [L_n, \Upsilon(z, \theta)] = \left(z^{n+1} \partial_z + (\Delta + \theta)(n+1)z^n\right) \Upsilon(z, \theta) \]  
A primary field belonging to a conformal Jordan cell is referred to as a ‘cellular’ primary field. A primary field not belonging to a conformal Jordan cell may be represented as \( \Upsilon(z, 0) \), and we will reserve this notation for these non-cellular primary fields. To avoid ambiguities, we will therefore refrain from considering unified cells \( \Upsilon(z, \theta) \), as defined in \((2)\), for vanishing \( \theta \).

2.1 Conformal Ward identities

We will consider quasi-primary fields only, ensuring the projective invariance of their correlators constructed by sandwiching the fields between projectively invariant vacua. That is, insertion of any of the three generators \( L_{-1}, L_0, L_1 \) into a correlator annihilates the correlator. When expressed in terms of the differential operators \((3)\), this is known as the conformal Ward identities which are given here for \( N \)-point functions:

\[ 0 = \sum_{i=1}^{N} \partial_z \langle \Upsilon_1(z_1, \theta_1) \ldots \Upsilon_N(z_N, \theta_N) \rangle \]
\[ 0 = \sum_{i=1}^{N} (z_i \partial_z + \Delta + \theta_i) \langle \Upsilon_1(z_1, \theta_1) \ldots \Upsilon_N(z_N, \theta_N) \rangle \]
\[ 0 = \sum_{i=1}^{N} \left(z_i^2 \partial_z + 2(\Delta + \theta_i)z_i\right) \langle \Upsilon_1(z_1, \theta_1) \ldots \Upsilon_N(z_N, \theta_N) \rangle \]  

To simplify the notation we introduce the differential operator
\[ \mathcal{L}_1 = \sum_{i=1}^{N} \left(z_i^2 \partial_z + 2\Delta z_i\right) \]  
in terms of which the third conformal Ward identity reads
\[ 0 = \left(\mathcal{L}_1 + 2 \sum_{i=1}^{N} \theta_i z_i\right) \langle \Upsilon_1(z_1, \theta_1) \ldots \Upsilon_N(z_N, \theta_N) \rangle \]  
It is noted that a correlator satisfying the first and third Ward identities \((4)\) automatically satisfies the second Ward identity. This follows readily from the commutator \([L_1, L_{-1}] = 2L_0\):

The first conformal Ward identity merely imposes translation invariance on the correlators, allowing us to express them solely in terms of differences, \( z_i - z_j \), between the coordinates.
It is stressed that some solutions for correlators involving non-cellular primary fields $\Upsilon_i(z_i, 0)$ may be lost if one simply sets the corresponding $\theta_i$ equal to zero in the solutions for non-vanishing $\theta_i$. This will be illustrated in the following.

Before proceeding, let us indicate how one extracts information on the individual correlators from solutions to the conformal Ward identities involving unified cells. In the case of

$$\langle \Upsilon_1(z_1, \theta_1)\Upsilon_2(z_2, 0)\Upsilon_3(z_3, \theta_3) \rangle$$

for example, the identity (6) reads

$$0 = \left( \mathcal{L}_1^3 + 2(\theta_1 z_1 + \theta_3 z_3) \right) \langle \Upsilon_1(z_1, \theta_1)\Upsilon_2(z_2, 0)\Upsilon_3(z_3, \theta_3) \rangle$$

A solution to the full set of conformal Ward identities is an expression expandable in $\theta_1$ and $\theta_3$. The term proportional to $\theta_1$ but independent of $\theta_3$, for example, should then be identified with $\langle \Psi_1(z_1)\Upsilon_2(z_2, 0)\Phi_3(z_3) \rangle$.

By construction, and as illustrated by this example, correlators involving unified cells and non-cellular primary fields may thus be regarded as generating-function correlators whose expansions in the nilpotent parameters give the individual correlators involving combinations of cellular primary fields, non-cellular primary fields, and logarithmic fields. Our focus will therefore be on correlators of combinations of unified cells and non-cellular primary fields. To the best of our knowledge, most results found in the literature pertain to correlators involving unified cells only or non-cellular primary fields only, though Ref. [7] does contain a discussion of three-point functions involving so-called twist fields as examples of so-called ‘pre-logarithmic’ fields in the $c = -2$ conformal field theory. Those particular results are in accordance with our general results. Furthermore, studies of three-point functions involving unified cells only are most often based on a simplifying, though physically motivated, assumption to which we will return in due time.

### 2.2 Two-point functions

We have three situations to analyze, distinguished by the number of unified cells appearing in the correlator. The case with non-cellular primary fields only is as in ordinary conformal field theory and we have the well-known result

$$\langle \Upsilon_1(z_1, 0)\Upsilon_2(z_2, 0) \rangle \propto \frac{\delta_{\Delta_1, \Delta_2}}{z_{12}^{\Delta_1+\Delta_2}}$$

To simplify the notation, we have introduced the standard abbreviation $z_{ij} = z_i - z_j$.

We now turn to the situation with at least one unified cell (i.e., one or two) in the two-point function. Motivated by the known results for two-point functions of unified cells only, we consider the following common ansatz

$$\langle \Upsilon_1(z_1, \theta_1)\Upsilon_2(z_2, \theta_2) \rangle = \frac{A(\theta_1, \theta_2) + B(\theta_1, \theta_2) \ln z_{12}}{z_{12}^{2\theta_1}}$$

3
where the dependence of the structure constants $A$ and $B$ on $\theta_1$ or $\theta_2$ vanishes if we consider the non-cellular primary field $\Upsilon_1(z_1,0)$ or $\Upsilon_2(z_2,0)$, respectively. The general expansion of $A$ reads

$$A(\theta_1, \theta_2) = A^0 + A^1\theta_1 + A^2\theta_2 + A^{12}\theta_1\theta_2$$

and similarly for $B$. Imposing (6) results in

$$\langle \Upsilon_1(z_1, \theta_1)\Upsilon_2(z_2, \theta_2) \rangle = \delta_{\Delta_1, \Delta_2} \frac{A^1\theta_1}{z_{12}^{\Delta_1 + \Delta_2}}$$

which in terms of individual two-point functions corresponds to

$$\langle \Phi(z_1)\Upsilon(z_2, 0) \rangle = \langle \Upsilon(z_1, 0)\Phi(z_2) \rangle = \langle \Phi_1(z_1)\Phi_2(z_2) \rangle = 0$$

$$\langle \Psi(z_1)\Upsilon(z_2, 0) \rangle \propto \delta_{\Delta_1, \Delta_2} \frac{A^2\theta_2}{z_{12}^{\Delta_1 + \Delta_2}}$$

$$\langle \Upsilon(z_1, 0)\Psi(z_2) \rangle \propto \delta_{\Delta_1, \Delta_2} \frac{A^1\delta_{A^1, A^2}(\theta_1 + \theta_2 - 2\theta_1\theta_2 \ln z_{12})}{z_{12}^{\Delta_1 + \Delta_2}} + A^{12}\theta_1\theta_2$$

$$\langle \Phi_1(z_1)\Psi_2(z_2) \rangle = \langle \Psi_1(z_1)\Phi_2(z_2) \rangle = \delta_{\Delta_1, \Delta_2} \frac{A^1}{z_{12}^{\Delta_1 + \Delta_2}}$$

$$\langle \Psi_1(z_1)\Psi_2(z_2) \rangle = \delta_{\Delta_1, \Delta_2} \frac{A^{12} - 2A^1\ln z_{12}}{z_{12}^{\Delta_1 + \Delta_2}}$$

Explicit relations similar to the one between $A^1$ and $A^2$ represented by the delta function in (12) will be omitted in the following. As indicated above, the solution (9) would have been lost if one were to set $\theta_1 = \theta_2 = 0$ in (12), whereas the first two solutions in (12) neatly follow from the last solution in (12) if one sets $\theta_2 = 0$ or $\theta_1 = 0$, respectively.

### 2.3 Three-point functions

We now have four situations to analyze, again characterized by the number of unified cells appearing in the correlator. As for two-point functions, the case with non-cellular primary fields only is as in ordinary conformal field theory and we have the well-known result

$$\langle \Upsilon_1(z_1, 0)\Upsilon_2(z_2, 0)\Upsilon_3(z_3, 0) \rangle \propto \frac{1}{z_{12}^{\Delta_1 + \Delta_2 - \Delta_3} z_{23}^{\Delta_1 + \Delta_2 + \Delta_3} z_{13}^{\Delta_1 - \Delta_2 + \Delta_3}}$$

For the combined three-point functions, associativity and the results on two-point functions suggest that we consider the following ansatz

$$\langle \Upsilon_1(z_1, \theta_1)\Upsilon_2(z_2, \theta_2)\Upsilon_3(z_3, \theta_3) \rangle$$
\[ A(\theta_1, \theta_2, \theta_3) = A^0 + A^1 \theta_1 + A^2 \theta_2 + A^3 \theta_3 + A^{12} \theta_1 \theta_2 + A^{23} \theta_2 \theta_3 + A^{13} \theta_1 \theta_2 + A^{123} \theta_1 \theta_2 \theta_3 \] (16)

and similarly for \( B_i \) and \( D_{ij} \). Imposing the Ward identities (i.e., on this ansatz, (16) suffices), corresponds to the following conditions, obtained from considering the part independent of logarithms

\[ 0 = (2\Delta_1 - h_1 - h_3 + 2\theta_1)A(\theta_1, \theta_2, \theta_3) + B_1(\theta_1, \theta_2, \theta_3) + B_3(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_2 - h_1 - h_2 + 2\theta_2)A(\theta_1, \theta_2, \theta_3) + B_1(\theta_1, \theta_2, \theta_3) + B_2(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_3 - h_2 - h_3 + 2\theta_3)A(\theta_1, \theta_2, \theta_3) + B_2(\theta_1, \theta_2, \theta_3) + B_3(\theta_1, \theta_2, \theta_3) \] (17)

the part linear in logarithms

\[ 0 = (2\Delta_1 - h_1 - h_3 + 2\theta_1)B_1(\theta_1, \theta_2, \theta_3) + 2D_{11}(\theta_1, \theta_2, \theta_3) + D_{13}(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_2 - h_1 - h_2 + 2\theta_2)B_1(\theta_1, \theta_2, \theta_3) + 2D_{11}(\theta_1, \theta_2, \theta_3) + D_{12}(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_3 - h_2 - h_3 + 2\theta_3)B_1(\theta_1, \theta_2, \theta_3) + D_{13}(\theta_1, \theta_2, \theta_3) + D_{12}(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_1 - h_1 - h_3 + 2\theta_1)B_2(\theta_1, \theta_2, \theta_3) + D_{12}(\theta_1, \theta_2, \theta_3) + D_{23}(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_2 - h_1 - h_2 + 2\theta_2)B_2(\theta_1, \theta_2, \theta_3) + D_{12}(\theta_1, \theta_2, \theta_3) + D_{23}(\theta_1, \theta_2, \theta_3) \]
\[ 0 = (2\Delta_3 - h_2 - h_3 + 2\theta_3)B_2(\theta_1, \theta_2, \theta_3) + D_{23}(\theta_1, \theta_2, \theta_3) + D_{23}(\theta_1, \theta_2, \theta_3) \] (18)

and the part quadratic in logarithms

\[ 0 = (2\Delta_1 - h_1 - h_3 + 2\theta_1)D_{ij}(\theta_1, \theta_2, \theta_3), \quad 1 \leq i \leq j \leq 3 \]
\[ 0 = (2\Delta_2 - h_1 - h_2 + 2\theta_2)D_{ij}(\theta_1, \theta_2, \theta_3), \quad 1 \leq i \leq j \leq 3 \]
\[ 0 = (2\Delta_3 - h_2 - h_3 + 2\theta_3)D_{ij}(\theta_1, \theta_2, \theta_3), \quad 1 \leq i \leq j \leq 3 \] (19)

These apply whether or not the individual \( \theta \)'s vanish, even if \( \theta_1 = \theta_2 = \theta_3 = 0 \) as in \( (14) \). In the further analysis, one should distinguish between the different numbers of unified cells, that is, the numbers of non-vanishing \( \theta \)'s. Also, it is understood that an \( A^1 \), for example, appearing in the study of one set of correlators (related through one or several Jordan-cell structures) a priori is independent of an \( A^1 \) appearing in a different set (not related to the former through a Jordan-cell structure).
Now, it is not hard to show that we in every case have

\[ h_1 = \Delta_1 + \Delta_2 - \Delta_3, \quad h_2 = -\Delta_1 + \Delta_2 + \Delta_3, \quad h_3 = \Delta_1 - \Delta_2 + \Delta_3 \]  

(20)

meaning that these identities apply to all combinations of vanishing or non-vanishing \( \theta \)s.

In the case where \( \theta_1 = \theta_2 = \theta_3 = 0 \), there is only one solution to the conditions \( 1 \) and one recovers \( 14 \) with \( A^0 \) as the proportionality constant.

In the case where \( \theta_1 \neq 0 \) while \( \theta_2 = \theta_3 = 0 \), we find

\[
\langle \Upsilon_1(z_1, \theta_1) \Upsilon_2(z_2, 0) \Upsilon_3(z_3, 0) \rangle = \frac{A^0 + A^1 \theta_1 + A^0 \theta_1 (- \ln z_{12} + \ln z_{23} - \ln z_{13})}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3}
\]  

(21)

which in terms of the individual correlators reads

\[
\langle \Phi_1(z_1) \Upsilon_2(z_2, 0) \Upsilon_3(z_3, 0) \rangle = \frac{A^0}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3}
\]

\[
\langle \Psi_1(z_1) \Upsilon_2(z_2, 0) \Upsilon_3(z_3, 0) \rangle = \frac{A^1 - A^0 \ln \frac{z_{12} z_{23}}{z_{13}}}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3}
\]  

(22)

The other two cases with only one unified cell are treated similarly and the corresponding correlators may be obtained from \( 21 \) and \( 22 \) by appropriately permuting the indices. We note that there in each case are two a priori independent structure constants. Before commenting on the structure of these results, let us complete the analysis of the conditions \( 17 \) and \( 19 \).

In the case where \( \theta_1, \theta_2 \neq 0 \) while \( \theta_3 = 0 \), we find

\[
\langle \Upsilon_1(z_1, \theta_1) \Upsilon_2(z_2, \theta_2) \Upsilon_3(z_3, 0) \rangle
= \left\{ A^0 + A^1 \theta_1 + A^2 \theta_2 + A^{12} \theta_1 \theta_2 + \left( -A^0 \theta_1 - A^0 \theta_2 - (A^1 + A^2) \theta_1 \theta_2 \right) \ln z_{12} \right.
+ \left( A^0 \theta_1 - A^0 \theta_2 + (-A^1 + A^2) \theta_1 \theta_2 \right) \ln z_{23} + \left( -A^0 \theta_1 + A^0 \theta_2 + (A^1 - A^2) \theta_1 \theta_2 \right) \ln z_{13}
+ \left. A^0 \theta_1 \theta_2 \left( \ln^2 z_{12} - \ln^2 z_{23} - \ln^2 z_{13} + 2 \ln z_{23} \ln z_{13} \right) \right\}
\]

\[ \times \frac{A^0}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3} \]  

(23)

which in terms of the individual correlators reads

\[
\langle \Phi_1(z_1) \Phi_2(z_2) \Upsilon_3(z_3, 0) \rangle = \frac{A^0}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3}
\]

\[
\langle \Phi_1(z_1) \Psi_2(z_2) \Upsilon_3(z_3, 0) \rangle = \frac{A^1 - A^0 \ln \frac{z_{12} z_{23}}{z_{13}}}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3}
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \Upsilon_3(z_3, 0) \rangle = \frac{A^2 - A^0 \ln \frac{z_{12} z_{23}}{z_{13}}}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3}
\]

\[
\times \frac{A^0 + A^1 \theta_1 + A^2 \theta_2 + A^{12} \theta_1 \theta_2 + \left( -A^0 \theta_1 - A^0 \theta_2 - (A^1 + A^2) \theta_1 \theta_2 \right) \ln z_{12} \right.
+ \left. A^1 \theta_1 - A^0 \theta_2 + (-A^1 + A^2) \theta_1 \theta_2 \right) \ln z_{23} + \left( -A^0 \theta_1 + A^0 \theta_2 + (A^1 - A^2) \theta_1 \theta_2 \right) \ln z_{13}
+ \frac{A^0 \theta_1 \theta_2 \left( \ln^2 z_{12} - \ln^2 z_{23} - \ln^2 z_{13} + 2 \ln z_{23} \ln z_{13} \right)}{z_{12}^\Delta_1 + z_{23}^\Delta_2 - z_{13}^\Delta_3 + z_{13}^\Delta_3 + z_{23}^\Delta_2 + z_{13}^\Delta_3 + z_{13}^\Delta_3 - z_{13}^\Delta_2 + z_{13}^\Delta_3} \]  

(24)
The other two cases with two unified cells are treated similarly and the corresponding correlators may be obtained from (23) and (24) by an appropriate permutation of the indices. We note that there in each case are four a priori independent structure constants.

In the case with three unified cells, that is, \( \theta_1, \theta_2, \theta_3 \neq 0 \), we find

\[
\langle \gamma_1(z_1, \theta_1) \gamma_2(z_2, \theta_2) \gamma_3(z_3, \theta_3) \rangle = 
\{ A^1 \theta_1 + A^2 \theta_2 + A^3 \theta_3 + A^{12} \theta_1 \theta_2 + A^{23} \theta_2 \theta_3 + A^{13} \theta_1 \theta_3 + A^{123} \theta_1 \theta_2 \theta_3 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{12} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{13} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{23} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{13} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{23} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{13} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{23} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{13} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{23} 
+ (A^{12} - A^{23} - A^{13}) \theta_1 \theta_2 \theta_3 \} \ln z_{13} \}
\times z_{12}^{-A_1} z_{23}^{-A_2} z_{13}^{-A_3} \}
\times z_{12}^{-A_1} z_{23}^{-A_2} z_{13}^{-A_3} \_23 \times z_{12}^{-A_1} z_{23}^{-A_2} z_{13}^{-A_3} \}
\times (25)
\]

which in terms of the individual correlators reads

\[
\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = 0
\]
\[
\langle \psi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = \frac{A^1}{z_{12}^{A_1} z_{23}^{-A_2} z_{13}^{-A_3}}
\]
\[
\langle \phi_1(z_1) \psi_2(z_2) \phi_3(z_3) \rangle = \frac{A^2}{z_{12}^{A_1} z_{23}^{-A_2} z_{13}^{-A_3}}
\]
\[
\langle \phi_1(z_1) \phi_2(z_2) \psi_3(z_3) \rangle = \frac{A^3}{z_{12}^{A_1} z_{23}^{-A_2} z_{13}^{-A_3}}
\]
\[
\langle \psi_1(z_1) \psi_2(z_2) \phi_3(z_3) \rangle = \frac{A^{12} - A^2 \ln z_{12}^{23} - A^2 \ln z_{23}^{12} - A^2 \ln z_{13}^{23}}{z_{12}^{A_1 + A_2 - A_3} z_{13}^{-A_3} z_{23}^{A_1 + A_3} z_{13}^{-A_3} z_{13}^{A_2 + A_3} z_{13}^{-A_3}}
\]
\[
\langle \phi_1(z_1) \psi_2(z_2) \psi_3(z_3) \rangle = \frac{A^{23} - A^2 \ln z_{12}^{23} - A^2 \ln z_{23}^{12} - A^2 \ln z_{13}^{23}}{z_{12}^{A_1 + A_2 - A_3} z_{13}^{-A_3} z_{23}^{A_1 + A_3} z_{13}^{-A_3} z_{13}^{A_2 + A_3} z_{13}^{-A_3}}
\]
\[
\langle \psi_1(z_1) \phi_2(z_2) \psi_3(z_3) \rangle = \frac{A^{13} - A^1 \ln z_{12}^{23} - A^1 \ln z_{23}^{12} - A^1 \ln z_{13}^{23}}{z_{12}^{A_1 + A_2 - A_3} z_{13}^{-A_3} z_{23}^{A_1 + A_3} z_{13}^{-A_3} z_{13}^{A_2 + A_3} z_{13}^{-A_3}}
\]
\[
\langle \psi_1(z_1) \psi_2(z_2) \psi_3(z_3) \rangle = \frac{A^{123} - A^{12} \ln z_{12}^{23} - A^{12} \ln z_{23}^{12} - A^{12} \ln z_{13}^{23}}{z_{12}^{A_1 + A_2 - A_3} z_{13}^{-A_3} z_{23}^{A_1 + A_3} z_{13}^{-A_3} z_{13}^{A_2 + A_3} z_{13}^{-A_3}}
\]
\[ A^3 \ln \frac{z_{12} z_{23}}{z_{13} z_{23}} \ln \frac{z_{12} z_{13}}{z_{23}} \]

(26)

We note that there are seven a priori independent structure constants. In the literature, on the other hand, one deals with three structure constants only (see [2], for example). This discrepancy is due to an assumption usually made in available studies of three-point functions. It concerns a particular property of the cellular primary fields which we will address presently.

Primary fields are called *proper primary* if their operator-product expansions with each other cannot produce a logarithmic field. It is argued in [10] (see also [11]) that correlators not involving improper primary fields satisfy

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \ldots \Phi_N(z_N) \rangle = \langle \Phi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \ldots \Phi_N(z_N) \rangle
\]

\[
\vdots
\]

\[
= \langle \Phi_1(z_1) \ldots \Phi_{N-1}(z_{N-1}) \Psi_N(z_N) \rangle
\]

(27)

in particular, and that the general form of the individual three-point functions of logarithmic fields and cellular primary fields hence read

\[
\langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle = 0
\]

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle = \langle \Phi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle = \langle \Phi_1(z_1) \Phi_2(z_2) \Psi_3(z_3) \rangle
\]

\[
= C_{123,1} \frac{z_{12} - \Delta_1 - \Delta_2 + \Delta_3 \ln z_{13}}{z_{12} + \Delta_1 + \Delta_2 + \Delta_3 \ln z_{12} - \Delta_3}
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle = \frac{C_{123,2} - 2C_{123,1} \ln z_{12}}{z_{12} + \Delta_1 - \Delta_2 + \Delta_3 \ln z_{13}}
\]

\[
\langle \Phi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle = \frac{C_{123,2} - 2C_{123,1} \ln z_{23}}{z_{23} + \Delta_1 - \Delta_2 + \Delta_3 \ln z_{13}}
\]

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \Psi_3(z_3) \rangle = \frac{C_{123,2} - 2C_{123,1} \ln z_{13}}{z_{13} + \Delta_2 - \Delta_3 + \Delta_1 \ln z_{12} + \ln z_{13}}
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle = \left\{ C_{123,3} - C_{123,2} (\ln z_{12} + \ln z_{23} + \ln z_{13}) + C_{123,1} (2 \ln z_{12} - \ln z_{13} + 2 \ln z_{23} \ln z_{13} - \ln^2 z_{12} - \ln^2 z_{23} - \ln^2 z_{13}) \right\} \times \frac{z_{12} - \Delta_1 + \Delta_2 + \Delta_3 \ln z_{23} - \Delta_1 + \Delta_2 - \Delta_3 \ln z_{13}}{z_{12} + \Delta_1 - \Delta_2 + \Delta_3 \ln z_{12} - \Delta_3}
\]

(28)

We may recover this result from (26) by relating the structure constants appearing there as

\[
A^1 = A^2 = A^3, \quad A^{12} = A^{23} = A^{13}
\]

(29)

in which case the correlators (26) are seen to reduce to (28) with \(C_{123,1} = A^1\), \(C_{123,2} = A^{12}\), and \(C_{123,3} = A^{13}\).
Regarding the reduction in the number of unified cells in a three-point function, it is observed that setting $\theta_3 = 0$ in (25) does not reproduce the full expression (23) but only the part independent of $A^0$. Setting $\theta_2 = 0$ in (23) or $\theta_1 = 0$ in (21), on the other hand, neatly reproduces the expressions (21) and (14), respectively.

According to the general results above, a logarithmic singularity may appear in a three-point function involving only one logarithmic field as long as at least one of the other two (primary) fields is non-cellular. This is in contrast to the situation based on conformal Jordan cells only, where at least two logarithmic fields are required to have a logarithmic singularity. Likewise, a singularity quadratic in logarithms may appear in a three-point function with two logarithmic fields and one non-cellular primary field, while such a singularity cannot appear if the primary field is cellular.

### 2.4 In terms of weights with nilpotent parts

It has been discussed how the correlators of unified cells only may be represented compactly if one considers the nilpotent parameter $\theta_i$ as part of a generalized conformal weight given by $\Delta_i + \theta_i$ [5]. A general version of this assertion is of course very natural from the point of view of the extended Virasoro action (3). It nevertheless presumes that the general solution to the conformal Ward identities may be factored accordingly. This has been shown to be the case when the simplifying assumption about the cellular primary fields being proper primary is imposed. The extension to our general set-up is discussed in the following and is found to affirm the assertion.

The two-point functions may thus be represented as

\[
\langle \Upsilon_1(z_1, 0) \Upsilon_2(z_2, 0) \rangle = \delta_{\Delta_1, \Delta_2} \frac{A^0}{z_{12}^{\Delta_1 + \Delta_2}}
\]

\[
\langle \Upsilon_1(z_1, \theta_1) \Upsilon_2(z_2, 0) \rangle = \delta_{\Delta_1, \Delta_2} \frac{\theta_1 A^1}{z_{12}^{(\Delta_1 + \theta_1) + \Delta_2}}
\]

\[
\langle \Upsilon_1(z_1, \theta_1) \Upsilon_2(z_2, \theta_2) \rangle = \delta_{\Delta_1, \Delta_2} \frac{A^1 (\theta_1 + \theta_2) + A^{12} \theta_1 \theta_2}{z_{12}^{(\Delta_1 + \theta_1) + (\Delta_2 + \theta_2)}}
\]

The similar expression for the correlator $\langle \Upsilon_1(z_1, 0) \Upsilon_2(z_2, \theta_2) \rangle$ is obtained from the second one by interchanging the indices.

It is straightforward to verify that the three-point functions may be represented as

\[
\langle \Upsilon_1(z_1, 0) \Upsilon_2(z_2, 0) \Upsilon_3(z_3, 0) \rangle = \frac{A^0}{z_{12}^{\Delta_1 + \Delta_2 - \Delta_3} z_{23}^{\Delta_1 - \Delta_2 + \Delta_3} z_{13}^{\Delta_1 - \Delta_2 + \Delta_3}}
\]

\[
\langle \Upsilon_1(z_1, \theta_1) \Upsilon_2(z_2, 0) \Upsilon_3(z_3, 0) \rangle = \frac{A^0 + A^1 \theta_1}{z_{12}^{(\Delta_1 + \theta_1) + \Delta_2 - \Delta_3} z_{23}^{\Delta_2 - \Delta_1 + \theta_1} z_{13}^{(\Delta_1 + \theta_1) - \Delta_2 + \Delta_3}}
\]

and

\[
\langle \Upsilon_1(z_1, \theta_1) \Upsilon_2(z_2, \theta_2) \Upsilon_3(z_3, 0) \rangle
\]
Logarithmic fields as follows:

\[
\langle Y_1(z_1, \theta_1) Y_2(z_2, \theta_2) Y_3(z_3, \theta_3) \rangle
= A^0 + A^1 \theta_1 + A^2 \theta_2 + A^{12} \theta_1 \theta_2
\]

\[
\frac{z_{12}}{(\Delta_1^2 + \theta_1^2) + (\Delta_2 + \theta_2)^2 - \Delta_1 - \Delta_2 + \Delta_3 + \Delta_4}
\]

\[
\frac{z_{23}}{(\Delta_1^2 + \theta_1^2) + (\Delta_2 + \theta_2)^2 - \Delta_1 - \Delta_2 + \Delta_3 + \Delta_4}
\]

\[
\frac{z_{13}}{(\Delta_1^2 + \theta_1^2) + (\Delta_2 + \theta_2)^2 - \Delta_1 - \Delta_2 + \Delta_3 + \Delta_4}
\]

\[
(30), (31) \text{ and (32)}
\]

(32)

The remaining four combinations are obtained by appropriate permutations in the indices.

As already indicated, it is not clear a priori that the general solutions to the conformal Ward identities (4) based on the ansätze (10) and (15) reduce to expressions which may be factored as in (30), (31) and (32). Our analysis has demonstrated that this is indeed the case.

### 2.5 Derivatives with respect to the conformal weights

Acting on either

\[
W_2 = \frac{\delta_{\Delta_1, \Delta_2}}{z_{12}}
\]

or

\[
W_3 = \frac{1}{z_{12} + \Delta_2 - \Delta_3 - \Delta_1 + \Delta_2 + \Delta_3 - \Delta_2 + \Delta_3}
\]

we may substitute derivatives with respect to the conformal weights by multiplicative factors according to

\[
\partial_{\Delta_1} = \partial_{\Delta_2} \rightarrow -2 \ln z_{12}
\]

or

\[
\partial_{\Delta_1} \rightarrow - \ln \frac{z_{12} z_{13}}{z_{23}}, \quad \partial_{\Delta_2} \rightarrow - \ln \frac{z_{12} z_{23}}{z_{13}}, \quad \partial_{\Delta_3} \rightarrow - \ln \frac{z_{23} z_{13}}{z_{12}}
\]

respectively. This simple observation allows us to represent the correlators involving logarithmic fields as follows:

\[
\langle \Psi_1(z_1) Y_2(z_2, 0) \rangle = A^1 W_2
\]

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \rangle = A^1 W_2
\]

\[
\langle \Psi_1(z_1) Y_2(z_2, 0) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_2
\]

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \rangle = \left( A^1 + A^0 \partial_{\Delta_1} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \rangle = \left( A^1 + A^0 \partial_{\Delta_1} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \rangle = \left( A^{12} + A^1 \partial_{\Delta_2} + A^2 \partial_{\Delta_1} + A^0 \partial_{\Delta_1} \partial_{\Delta_2} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Phi_2(z_2) \rangle = \left( A^1 + A^0 \partial_{\Delta_1} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_3
\]

\[
\langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle = \left( A^{12} + A^2 \partial_{\Delta_1} + A^1 \partial_{\Delta_2} \right) W_3
\]
in addition to expressions obtained by appropriately permuting the indices. One may therefore represent the correlators hierarchically as

\[
\langle \Psi_1(z_1) Y_2(z_2, 0) \rangle = A^1 W_2 + \partial_{\Delta_1} \langle \Phi_1(z_1) Y_2(z_2, 0) \rangle \\
\langle \Psi_1(z_1) \Phi_2(z_2) \rangle = A^1 W_2 + \partial_{\Delta_1} \langle \Phi_1(z_1) \Phi_2(z_2) \rangle \\
\langle \Psi_1(z_1) \Psi_2(z_2) \rangle = A^{12} W_2 + \partial_{\Delta_1} \langle \Phi_1(z_1) \Psi_2(z_2) \rangle + \partial_{\Delta_2} \langle \Psi_1(z_1) \Phi_2(z_2) \rangle \\
- \partial_{\Delta_1} \partial_{\Delta_2} \langle \Phi_1(z_1) \Phi_2(z_2) \rangle
\]

(38)

in the case of two-point functions, and

\[
\langle \Psi_1(z_1) Y_2(z_2, 0) Y_3(z_3, 0) \rangle = A^1 W_3 + \partial_{\Delta_1} \langle \Phi_1(z_1) Y_2(z_2, 0) Y_3(z_3, 0) \rangle \\
\langle \Psi_1(z_1) \Phi_2(z_2) Y_3(z_3, 0) \rangle = A^1 W_3 + \partial_{\Delta_1} \langle \Phi_1(z_1) \Phi_2(z_2) Y_3(z_3, 0) \rangle \\
\langle \Psi_1(z_1) \Psi_2(z_2) Y_3(z_3, 0) \rangle = A^{12} W_3 + \partial_{\Delta_1} \langle \Phi_1(z_1) \Psi_2(z_2) Y_3(z_3, 0) \rangle \\
+ \partial_{\Delta_2} \langle \Psi_1(z_1) \Phi_2(z_2) Y_3(z_3, 0) \rangle \\
- \partial_{\Delta_1} \partial_{\Delta_2} \langle \Phi_1(z_1) \Phi_2(z_2) Y_3(z_3, 0) \rangle
\]

(39)

in the case of three-point functions. As above, the remaining correlators may be obtained by appropriately permuting the indices. Similar results in the particular case of proper primary fields \(2\) have already appeared in the literature \(\mathfrak{W}\), see also \(\mathfrak{Q}\).

We finally wish to re-address the conformal Ward identities in the realm of these hierarchical structures. Since the latter are the same in all the cases, we will focus on the most complex scenario, the one involving the three-point function of three logarithmic fields. The conformal Ward identity following from inserting \(L_1\) into such a correlator may be written

\[
0 = \left( L_1^3 + 2 z_1 \delta_1 + 2 z_2 \delta_2 + 2 z_3 \delta_3 \right) \langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle
\]

(40)

As is conventional in logarithmic conformal field theory, we have introduced here the operator \(\delta_i\) acting (in the case of a conformal Jordan cell of rank two) on the fields in a correlator as

\[
\delta_i \Psi_j(z_j) = \delta_{ij} \Phi_j(z_j), \quad \delta_i \Phi_j(z_j) = 0
\]

(41)
in addition to $\delta_i \hat{Y}_j(z_j,0) = 0$. This means that the conformal Ward identity (10) reads

$$0 = L^3_i \langle \Psi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle + 2z_1 \langle \Phi_1(z_1) \Psi_2(z_2) \Psi_3(z_3) \rangle + 2z_2 \langle \Psi_1(z_1) \Phi_2(z_2) \Psi_3(z_3) \rangle + 2z_3 \langle \Psi_1(z_1) \Psi_2(z_2) \Phi_3(z_3) \rangle$$

(42)

This condition is easily verified using (37). It is stressed, though, that it is only with hindsight that these structures appear natural.

3 Conclusion

We have studied the conformal Ward identities for quasi-primary fields appearing in logarithmic conformal field theory based on conformal Jordan cells of rank two. Even though our results are based on an ansatz, it appears natural to suspect that they constitute the general solution for two- and three-point functions.

We anticipate that one, in a straightforward manner, may extract general information about the operator-product expansions underlying the correlators we have found. This is an interesting enterprise we intend to undertake.

As already mentioned, our results pertain to conformal Jordan cells of rank two. We hope to study the case of general rank elsewhere. Partial results in this direction may be found in [6, 7]. Conformal Jordan cells of infinite rank have been introduced in [12].

We have found that the results presented in this paper may be extended to affine Jordan cells appearing in certain logarithmic extensions of Wess-Zumino-Witten models [13]. The general solutions in these models also satisfy the Knizhnik-Zamolodchikov equations and are found to reduce, by Hamiltonian reduction, to the solutions provided in the present paper.

Another natural extension of the present work which would be interesting to pursue, is the general solution to the superconformal Ward identities appearing in logarithmic superconformal field theory. Results in this direction may be found in [14].

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