Topological classification of Möbius transformations

Tetiana Rybalkina    Vladimir V. Sergeichuk
rybalkina_t@ukr.net    sergeich@imath.kiev.ua
Institute of Mathematics, Kiev, Ukraine

Abstract

Linear fractional transformations on the extended complex plane are classified up to topological conjugacy. Recall that two transformations \( f \) and \( g \) are called topologically conjugate if there exists a homeomorphism \( h \) such that \( g = h^{-1} \circ f \circ h \), in which \( \circ \) is the composition of mappings.

1 Introduction

Möbius transformations are linear fractional transformations of the form

\[
f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{C}
\]  

(1)

on the extended complex plane \( \hat{\mathbb{C}} := \mathbb{C} \cup \infty \). The foundations of the theory of Möbius transformations are developed in [1, Chapters 3 and 4] and [12, Chapters 8–10].

Since the numbers \( a, b, c, d \) can be simultaneously multiplied by any nonzero number without changing \( f \), the transformation (1) can be given by the matrix

\[
M_f := \frac{1}{\sqrt{ad - bc}} \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]  

(2)

which has determinant 1 and is determined by \( f \) uniquely up to multiplication by \(-1\). The composition of transformations corresponds to the product of their matrices:

\[
M_{fg} = M_f M_g.
\]  

(3)
The trace of a matrix $A$ is denoted by $\text{tr } A$.

Two Möbius transformations $f$ and $g$ are said to be

- **conjugate** if there exists a Möbius transformation $h$ such that the diagram

$$
\begin{array}{ccc}
\hat{\mathbb{C}} & \overset{g}{\longrightarrow} & \hat{\mathbb{C}} \\
\downarrow h & & \downarrow h \\
\hat{\mathbb{C}} & \overset{f}{\longrightarrow} & \hat{\mathbb{C}}
\end{array}
$$

is commutative, i.e., $g = h^{-1}fh$;

- **topologically conjugate** if there exists a homeomorphism $h : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $g = h^{-1}fh$ (a mapping $h$ is a homeomorphism if $h$ and $h^{-1}$ are continuous bijections).

If two Möbius transformations are conjugate, then they are topologically conjugate since each Möbius transformation is a homeomorphism.

The following criterion of conjugacy is easily obtained from [1, Theorem 4.3.1]: nonidentity Möbius transformations $f$ and $g$ are conjugate if and only if $\text{tr } M_f = \pm \text{tr } M_g$.

If $g = h^{-1}fh$, then $M_g = \pm M_h^{-1}M_fM_h$ by (3), and so conjugate Möbius transformations are given by similar matrices determined up to multiplication by $-1$. One can take the transforming matrix $M_h$ such that $M_g$ is the Jordan form of $M_f$. Since $\det M_f = \det M_g = 1$,

$$
M_g = \begin{bmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{bmatrix} \quad (\lambda \neq \pm 1, 0) \quad \text{or} \quad M_g = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \quad (\lambda = \pm 1).
$$

The matrix $M_g$ is uniquely determined by $f$, up to multiplication by $-1$ and up to interchanging the diagonal entries $\lambda$ and $1/\lambda$ in the first matrix (4).

Thus, $f$ is conjugate to $z \mapsto \lambda^2 z$, or $z \mapsto (1/\lambda^2)z$, or $z \mapsto z + 1$, and we obtain a canonical form for conjugacy [1, §4.3]:

Each Möbius transformation is conjugate to exactly one Möbius transformation of the form $m_\mu(z) = \mu z$ ($\mu \neq 0, 1$) or $m_1(z) = z + 1$ (5) ($\mu := 1$), in which $\mu$ is determined up to replacement by $1/\mu$.

The numbers $\mu_1 := \mu$ and $\mu_2 := 1/\mu$ are called the **multipliers** of $f$. They are defined for a holomorphic map on a Riemann surface in [13, p. 45]; for a
nonidentity Möbius transformation $f$ they can be calculated by formula

$$
\mu_i = \begin{cases} 
    f'(z_i) & \text{if } z_i \neq \infty, \\
    \lim_{z \to \infty} \frac{1}{f'(z)} & \text{if } z_i = \infty,
\end{cases}$$

in which $z_1$ and $z_2$ are fixed points of $f$ (their number is 2 or 1; we take $z_1 = z_2$ in the latter case).

The main result of this paper is the following theorem, in which we give 3 criteria of topological conjugacy; the criterion (iv) was published in [4] in Ukrainian by the first author.

**Theorem 1.1.** The following four statements are equivalent for arbitrary nonidentity Möbius transformations $f, g: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$:  

(i) $f$ and $g$ are topologically conjugate;  

(ii) $\text{tr } M_f, \text{tr } M_g \notin [-2; 2]$ or $\text{tr } M_f = \pm \text{tr } M_g$ (in which $[-2; 2]$ is the set of all $a \in \mathbb{R}$ satisfying $-2 \leq a \leq 2$);  

(iii) if $\lambda$ is any eigenvalue of $M_f$ and $\lambda'$ is any eigenvalue of $M_g$, then $|\lambda|, |\lambda'| \neq 1$, or $\lambda = \pm \lambda'$, or $\lambda = \pm \bar{\lambda}'$;  

(iv) if $\mu$ is any multiplier of $f$ and $\nu$ is any multiplier of $g$, then $|\mu|, |\nu| \neq 1$, or $\mu = \nu$, or $\mu = \bar{\nu}$.

The canonical form (5) is used in the following definition: a nonidentity Möbius transformation is called

- **hyperbolic** if it is conjugate to $z \mapsto \mu z$ with $1 \neq \mu \in \mathbb{R}$;
- **loxodromic** if it is conjugate to $z \mapsto \mu z$ with $\mu \notin \mathbb{R}$ and $|\mu| \neq 1$;
- **elliptic** if it is conjugate to $z \mapsto \mu z$ with $|\mu| = 1$ and $\mu \neq 1$;
- **parabolic** if it is conjugate to $z \mapsto z + 1$.

A canonical form of a Möbius transformation for topological conjugacy is easily obtained from the equivalence of (i) and (iv) in Theorem [1.1].
Corollary 1.1. (a) Each hyperbolic or loxodromic M"obius transformation is topologically conjugate to \( z \mapsto 2z \).

(b) Each elliptic M"obius transformation is topologically conjugate to \( z \mapsto \mu z \) (\(|\mu| = 1\)) which is uniquely determined up to replacement of \( \mu \) by \( \bar{\mu} \).

(c) Each parabolic M"obius transformation is topologically conjugate to \( z \mapsto z + 1 \).

Two linear operators \( A, B : \mathbb{C}^2 \to \mathbb{C}^2 \) are said to be topologically conjugate if \( B = h^{-1}Ah \) for some homeomorphism \( h : \mathbb{C}^2 \to \mathbb{C}^2 \). Assigning to a M"obius transformation \( f \) the linear operator \( x \mapsto M_f x \) \((x \in \mathbb{C}^2)\) determined up to multiplication by \(-1\), we obtain the one-to-one correspondence between M"obius transformations on \( \hat{\mathbb{C}} \) and linear operators on \( \mathbb{C}^2 \) with determinant 1 that are determined up to multiplication by \(-1\). This correspondence preserves the topological conjugacy in virtue of the following corollary, which will be proved in Section 4.

Corollary 1.2. The following two conditions are equivalent for M"obius transformations \( f \) and \( g \):

(i) \( f \) and \( g \) are topologically conjugate;

(ii) the linear operator \( x \mapsto M_f x \) on \( \mathbb{C}^2 \) is topologically conjugate to \( x \mapsto M_g x \) or \( x \mapsto -M_g x \).

2 Topological classification of linear operators

In this section, we recall some results of [11, 14, 5] about topological classification of linear operators (they were extended to affine operators in [3, 5, 6]), which will be used in the next sections.

For each square complex matrix \( A = [a_{ij}] \), we define the matrix \( \bar{A} = [\bar{a}_{ij}] \) whose entries are the complex conjugates of the entries of \( A \), and construct a decomposition of \( A \) into a direct sum of square matrices

\[
S^{-1}AS = A_0 \oplus A_{01} \oplus A_1 \oplus A_{1\infty} \quad (S \text{ is a nonsingular matrix}),
\]

in which all eigenvalues \( \lambda \) of \( A_0 \) (respectively, of \( A_{01}, A_1, \) and \( A_{1\infty} \)) satisfy the condition \( \lambda = 0 \) (respectively, \( 0 < |\lambda| < 1 \), \(|\lambda| = 1\), and \(|\lambda| > 1\)).

The assertion (i) of the following theorem was proved in [11] (see also [14]); the assertion (ii) was proved by the first author in [5] Theorem 2.2.
Theorem 2.1. Let \( f(x) = Ax \) and \( g(x) = Bx \) be linear operators over \( F = \mathbb{R} \) or \( \mathbb{C} \) without eigenvalues that are roots of unity, and let \( A_0, A_{01}, A_1, A_{1}\infty \) and \( B_0, B_{01}, B_1, B_{1}\infty \) be constructed by \( A \) and \( B \) as in (6).

(i) If \( F = \mathbb{R} \), then \( f \) and \( g \) are topologically conjugate if and only if
\[
A_0 \text{ is similar to } B_0, \quad \text{size } A_{01} = \text{size } B_{01}, \quad \det(A_{01}B_{01}) > 0,
A_1 \text{ is similar to } B_1, \quad \text{size } A_{1}\infty = \text{size } B_{1}\infty, \quad \det(A_{1}\infty B_{1}\infty) > 0.
\]

(ii) If \( F = \mathbb{C} \), then \( f \) and \( g \) are topologically conjugate if and only if
\[
A_0 \text{ is similar to } B_0, \quad \text{size } A_{01} = \text{size } B_{01},
A_1 \oplus \bar{A}_1 \text{ is similar to } B_1 \oplus \bar{B}_1, \quad \text{size } A_{1}\infty = \text{size } B_{1}\infty.
\]

Two linear operators \( f, g : \mathbb{R}^n \to \mathbb{R}^n \) are said to be conjugate if there is a linear bijection \( h : \mathbb{R}^n \to \mathbb{R}^n \) such that \( g = h^{-1}fh \).

An operator \( f \) is periodic if \( f^k \) is the identity for some natural \( k \). Kuiper and Robbin [11] proved that if the hypothesis
\[
\text{two periodic linear operators are topologically conjugate if and only if they are conjugate}
\]
is true, then (7) are necessary and sufficient conditions of topological conjugacy for all linear operators. Cappell and Shaneson [7, 8, 9, 10] proved (8) for all linear operators on \( \mathbb{R}^n \) with \( n < 6 \).

These results ensure the assertion (i) of the following theorem. The assertion (ii) is proved as Theorem 2.2 in [5].

Theorem 2.2. Let \( f(x) = Ax \) and \( g(x) = Bx \) be linear operators on \( V = \mathbb{R}^m \) or \( \mathbb{C}^m \); let \( A_0, A_{01}, A_1, A_{1}\infty \) and \( B_0, B_{01}, B_1, B_{1}\infty \) be constructed by \( A \) and \( B \) as in (6).

(i) Suppose \( V = \mathbb{R}^m \) with \( m \leq 5 \); then \( f \) and \( g \) are topologically conjugate if and only if
\[
A_0 \text{ is similar to } B_0, \quad \text{size } A_{01} = \text{size } B_{01}, \quad \det(A_{01}B_{01}) > 0,
A_1 \text{ is similar to } B_1, \quad \text{size } A_{1}\infty = \text{size } B_{1}\infty, \quad \det(A_{1}\infty B_{1}\infty) > 0.
\]

(ii) Suppose \( V = \mathbb{C}^m \) with \( m \leq 2 \); then \( f \) and \( g \) are topologically conjugate if and only if
\[
A_0 \text{ is similar to } B_0, \quad \text{size } A_{01} = \text{size } B_{01},
A_1 \oplus \bar{A}_1 \text{ is similar to } B_1 \oplus \bar{B}_1, \quad \text{size } A_{1}\infty = \text{size } B_{1}\infty.
\]
The following corollary is used in Section 4 for the proof of Corollary 1.2.

**Corollary 2.1.** Let \( f(x) = Ax \) and \( g(x) = Bx \) be two nonidentity linear operators on \( \mathbb{C}^2 \) whose matrices \( A \) and \( B \) have determinant 1 and are diagonalizable (i.e., their Jordan forms are diagonal). Let \( \lambda \) and \( \lambda' \) be eigenvalues of \( A \) and \( B \), respectively. Then \( f \) and \( g \) are topologically conjugate if and only if \( |\lambda|, |\lambda'| \neq 1 \), or \( \lambda = \lambda' \), or \( \lambda = \bar{\lambda}' \).

**Proof.** The conjugacy of linear operators implies their topological conjugacy, so we can suppose that the matrices of \( f(x) = Ax \) and \( g(x) = Bx \) are given in their Jordan forms:

\[
A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} \lambda' & 0 \\ 0 & \lambda'^{-1} \end{bmatrix}, \quad \lambda, \lambda' \neq \pm 1, 0.
\]

The following 4 cases are possible:

**Case 1:** \( |\lambda| \neq 1 \) and \( |\lambda'| \neq 1 \). Then in notation (6)

\[
A_{01} \oplus A_{1\infty} = [\lambda] \oplus [\lambda^{-1}], \quad B_{01} \oplus B_{1\infty} = [\lambda'] \oplus [\lambda'^{-1}].
\]

By Theorem 2.2(ii), \( f \) and \( g \) are topologically conjugate.

**Case 2:** \( |\lambda| = |\lambda'| = 1 \). Then

\[
A_{1} \oplus \bar{A}_{1} = [\lambda] \oplus [\bar{\lambda}], \quad B_{1} \oplus \bar{B}_{1} = [\lambda'] \oplus [\bar{\lambda}].
\]

By Theorem 2.2(ii), \( f \) and \( g \) are topologically conjugate if and only if \( \lambda = \lambda' \) or \( \lambda = \bar{\lambda}' \).

**Case 3:** \( |\lambda| = 1 \) and \( |\lambda'| \neq 1 \). Then

\[
A_{1} \oplus \bar{A}_{1} = [\lambda] \oplus [\bar{\lambda}], \quad B_{01} \oplus B_{1\infty} = [\lambda'] \oplus [\lambda'^{-1}].
\]

By Theorem 2.2(ii), \( f \) and \( g \) are not topologically conjugate.

**Case 4:** \( |\lambda| \neq 1 \) and \( |\lambda'| = 1 \). Then \( f \) and \( g \) are not topologically conjugate.

\[\square\]

**Lemma 2.1.** Möbius transformations \( f(z) = az \) and \( g(z) = bz \) are topologically conjugate if and only if

\[
|a|, |b| \neq 1, \quad \text{or} \quad a = b, \quad \text{or} \quad a = \bar{b}.
\]
Proof. ⇐. Suppose that \( f \) and \( g \) satisfy (10).

If \( f \) and \( g \) satisfy
\[
|a|,|b| < 1, \quad \text{or} \quad |a|,|b| > 1, \quad \text{or} \quad a = b, \quad \text{or} \quad a = \bar{b},
\]
then by Theorem 2.2(ii) the linear mappings \( z \mapsto az \) and \( z \mapsto bz \) on \( \mathbb{C} \) are topologically conjugate via some homeomorphism \( \eta : \mathbb{C} \to \mathbb{C} \), and so \( f \) and \( g \) are topologically conjugate via the homeomorphism \( h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) defined as follows: \( h(z) := \eta(z) \) if \( z \in \mathbb{C} \) and \( h(\infty) := \infty \).

If \( f \) and \( g \) do not satisfy (11) (but satisfy (10)), then either \( |a| < 1 \) and \( |b| > 1 \), or \( |a| > 1 \) and \( |b| < 1 \). Suppose that \( |a| < 1 \) and \( |b| > 1 \). Then \( |1/b| < 1 \) and by (11) \( f \) is topologically conjugate to \( g^{-1}(z) = (1/b)z \), which is topologically conjugate to \( g \) via the homeomorphism \( z \mapsto 1/z \) on \( \hat{\mathbb{C}} \).

⇒. Let Möbius transformations \( f(z) = az \) and \( g(z) = bz \) on \( \hat{\mathbb{C}} \) be topologically conjugate; that is, there exists a homeomorphism \( h : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) such that
\[
hg(z) = fh(z) \quad \text{for all} \quad z \in \hat{\mathbb{C}}.
\]
Since \( h \) transforms all fixed points of \( g \) to all fixed points of \( f \) and their fixed points are 0 and \( \infty \), the following two cases are possible.

Case 1: \( h(\infty) = \infty \) and \( h(0) = 0 \). By (12), the linear operators \( z \mapsto az \) and \( z \mapsto bz \) on \( \mathbb{C} \) (which are the restrictions of \( f \) and \( g \) to \( \mathbb{C} \)) are topologically conjugate via the homeomorphism that is the restriction of \( h \) to \( \mathbb{C} \). Theorem 2.2(ii) ensures that
\[
|a|,|b| < 1, \quad \text{or} \quad |a|,|b| > 1, \quad \text{or} \quad a = b, \quad \text{or} \quad a = \bar{b}.
\]

Case 2: \( h(\infty) = 0 \) and \( h(0) = \infty \). The Möbius transformations \( f^{-1}(z) = (1/a)z \) and \( g(z) = bz \) are topologically conjugate via \( h_1 := \varphi h \) in which \( \varphi(z) := 1/z \). Since \( h_1(\infty) = \infty \), we have (13) in which \( a \) is replaced by \( 1/a \).

In both the cases, \( a \) and \( b \) satisfy (11).

3 Proof of Theorem 1.1

In this section, we denote by \( f \) and \( g \) two nonidentity Möbius transformations, by \( \lambda \) and \( \lambda' \) arbitrary eigenvalues of \( M_f \) and \( M_g \), respectively, and by
\[
n(f) \quad \text{and} \quad n(g) \quad \text{the numbers of fixed points of} \quad f \quad \text{and} \quad g.
\]
Recall that the number of fixed points of any nonidentity Möbius transformation is equal to 1 or 2.

(i) $\iff$ (iv). Suppose (i) holds. Then $n(f) = n(g)$. Since $f$ and $g$ are not the identity, the following two cases are possible.

Case 1: $n(f) = n(g) = 1$. By (5), $f$ and $g$ are conjugate to $m_1(z) = z + 1$, whose multiplier is 1. This ensures (iv).

Case 2: $n(f) = n(g) = 2$. Let $\mu, \nu \notin \{0, 1\}$ be multipliers of $f$ and $g$, respectively. By (5), $f$ and $g$ are conjugate to $m_\mu(z) = \mu z$ and $m_\nu(z) = \nu z$, whose fixed points are 0 and $\infty$. Lemma 2.1 ensures (iv).

Thus (i) $\implies$ (iv). The converse arguments give (i) $\iff$ (iv).

(iii) $\iff$ (iv). By (5) and (2), if $\mu$ is a multiplier of $f$, then $f$ is conjugate to $m_\mu(z) = \mu z$ ($\mu \neq 0, 1$) or $m_1(z) = z + 1$ ($\mu = 1$), and so $M_f$ is similar to

$$M_{m_\mu} = \pm \frac{1}{\sqrt{\mu}} \begin{bmatrix} \mu & 0 \\ 0 & 1 \end{bmatrix} = \pm \begin{bmatrix} \sqrt{\mu} & 0 \\ 0 & 1/\sqrt{\mu} \end{bmatrix}$$

or

$$M_{m_\mu} = \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

($\mu = 1$).

Therefore, the matrix $M_f$ has an eigenvalue $\lambda$ that is equal to $\sqrt{\mu}$ or $-\sqrt{\mu}$ ($\mu \neq 0$).

Analogously, if $\nu$ is a multiplier of $g$, then $M_g$ has an eigenvalue $\lambda'$ that is equal to $\sqrt{\nu}$ or $-\sqrt{\nu}$ ($\nu \neq 0$), which proves the equivalence of (iii) and (iv).

(ii) $\iff$ (iii). Let us prove that

$$|\lambda| = 1 \iff \text{tr } M_f = \pm \text{tr } M_{m_\mu} \in [-2; 2].$$

The equality $\pm \text{tr } M_f = \text{tr } M_{m_\mu}$ follows from the similarity of $M_f$ and $M_{m_\mu}$.

If $|\lambda| = 1$, then by (11)

$$\text{tr } M_{m_\mu} = \lambda + \lambda^{-1} = \lambda + \bar{\lambda} \in [-2; 2].$$

If $|\lambda| \neq 1$, then $\lambda^{-1} \neq \bar{\lambda}$ and $\text{tr } M_{m_\mu} = \lambda + \lambda^{-1} \notin [-2; 2]$, which proves (16). The following 3 cases are possible.

Case 1: $|\lambda| \neq 1$ and $|\lambda'| \neq 1$. Then (iii) holds. By (16), (ii) holds too.
Case 2: $|\lambda| = 1$ and $|\lambda'| \neq 1$, or $|\lambda| \neq 1$ and $|\lambda'| = 1$. Then (ii) and (iii) do not hold.

Case 3: $|\lambda| = |\lambda'| = 1$. The condition $\text{tr} M_f = \pm \text{tr} M_g$ is equivalent to $\text{tr} M_{m_\mu} = \pm \text{tr} M_{m_{\mu'}}$ is equivalent to $\lambda + \lambda^{-1} = \pm (\lambda' + \lambda'^{-1})$ is equivalent to $\lambda + \bar{\lambda} = \pm (\lambda' + \bar{\lambda}')$ is equivalent to $\lambda = \pm \lambda'$ or $\lambda = \pm \bar{\lambda}'$.

4 Proof of Corollary 1.2

The following 4 cases are possible for any Möbius transformations $f$ and $g$.

Case 1: $n(f) \neq n(g)$ (see (14)). Then the assertion (i) of Corollary 1.2 does not hold. Let us prove that (ii) does not hold too. Suppose that $n(f) < n(g)$. If $n(g) = \infty$, then $g$ is the identity, $n(f) \in \{1, 2\}$, and (ii) does not hold. Suppose that $n(g) < \infty$. Then $n(f) = 1$ and $n(g) = 2$. By (5) and (15), $f$ is conjugate to $m_1(z) = z + 1$ and $g$ is conjugate to $m_{\mu}(z) = \lambda^2 z$. The linear operators $x \mapsto M_f x$ and $x \mapsto M_g x$ are conjugate to $x \mapsto \pm M_{m_1} x$ and $x \mapsto \pm M_{m_{\mu}} x$, respectively, in which

$$M_{m_1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_{m_{\mu}} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \quad (\lambda \neq 0, 1).$$

The vector $[0, 0]^T$ is the only fixed point of the linear operator $x \mapsto M_{m_{\mu}} x$. All vectors $[a, 0]^T (a \in \mathbb{C})$ are fixed points of $x \mapsto M_{m_1} x$. Thus, the assertion (ii) does not hold.

Case 2: $n(f) = n(g) = 1$. By (5), $f$ and $g$ are conjugate to $z \mapsto z + 1$. By (4), the matrices $M_f$ and $M_g$ are similar to

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$ 

Hence $M_f$ is similar to $M_g$ or $-M_g$. Therefore, $x \mapsto M_f x$ is conjugate to $x \mapsto \pm M_g x$, and so they are topologically conjugate.

Case 3: $n(f) = n(g) = 2$. By (5) and (15), $f$ is conjugate to $z \mapsto \lambda^2 z$ and $g$ is conjugate to $z \mapsto \lambda'^2 z$, in which $\lambda$ and $\lambda'$ are eigenvalues of $M_f$ and $M_g$, respectively. The Jordan forms of $M_f$ and $M_g$ are $\pm J_f$ and $\pm J_g$, where

$$J_f := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}, \quad J_g := \begin{bmatrix} \lambda' & 0 \\ 0 & \lambda'^{-1} \end{bmatrix}, \quad \lambda, \lambda' \notin \{0, \pm 1\}.$$
By Theorem 1.1(iii), \( f \) and \( g \) are topologically conjugate if and only if
\[
|\lambda|, |\lambda'| \neq 1, \quad \text{or} \quad \lambda = \pm \lambda', \quad \text{or} \quad \lambda = \pm \bar{\lambda}',
\]
if and only if (see Corollary 2.1) the linear operators \( x \mapsto J_fx \) and \( x \mapsto \pm J_gx \) are topologically conjugate, if and only if the linear operators \( x \mapsto M_fx \) and \( x \mapsto \pm M_gx \) are topologically conjugate.

Case 4: \( n(f) = n(g) > 2 \). The assertions (i) and (ii) hold since \( f \) and \( g \) are the identity mappings and \( M_f = \pm M_g = \pm I_2 \), in which \( I_2 \) is the identity matrix.

References

[1] A. F. Beardon, *The geometry of discrete groups*, Springer-Verlag, New York (1983).

[2] J. Blanc, “Conjugacy classes of affine automorphisms of \( \mathbb{K}^n \) and linear automorphisms of \( \mathbb{P}^n \) in the Cremona groups,” *Manuscripta Math.* 119, No. 2, 225–241 (2006).

[3] T. V. Budnitska, “Classification of topological conjugate affine mappings,” *Ukrainian Math. J.* 61, No. 1, 164–170 (2009).

[4] T. V. Budnitska “Topological classification of Möbius transformations,” *Zb. Pr. Inst. Mat. NAN Ukr.*, 6, No. 2, 349–358 (2009) (in Ukrainian, Zbl:1199.37031).

[5] T. V. Budnitska, “Topological classification of affine operators on unitary and Euclidean spaces,” *Linear Algebra Appl.*, 434, 582–592 (2011).

[6] T. Budnitska, N. Budnitska, “Classification of affine operators up to biregular conjugacy,” *Linear Algebra Appl.*, 434, 1195–1199 (2011).

[7] S. E. Cappell, J. L. Shaneson, “Nonlinear similarity of matrices,” *Bull. Amer. Math. Soc. (N.S.)*, 1, 899–902 (1979).

[8] S. E. Cappell, J. L. Shaneson, “Non-linear similarity,” *Ann. of Math.*, 113, No. 2, 315–355 (1981).
[9] S. E. Cappell, J. L. Shaneson, “Non-linear similarity and linear similarity are equivariant below dimension 6,” *Contemp. Math.*, 231, 59–66 (1999).

[10] S. E. Cappell, J. L. Shaneson, M. Steinberger, J. E. West, “Nonlinear similarity begins in dimension six,” *Amer. J. Math.*, 111, 717–752 (1989).

[11] N. H. Kuiper, J. W. Robbin, “Topological classification of linear endomorphisms,” *Invent. Math.*, 19, No. 2, 83–106 (1973).

[12] A. I. Markushevich, *Theory of functions of a complex variable*, Vol. I, Prentice Hall, Inc., Englewood Cliffs, N.J. (1965).

[13] J. Milnor, *Dynamics in one complex variable*, Princeton University Press, Princeton, NJ (2006).

[14] J. W. Robbin, “Topological conjugacy and structural stability for discrete dynamical systems,” *Bull. Amer. Math. Soc.*, 78, 923–952 (1972).