Distributionally Robust Goal-Reaching Optimization in the Presence of Background Risk

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In this article, we examine the effect of background risk on portfolio selection and optimal reinsurance design under the criterion of maximizing the probability of reaching a goal. Following the literature, we adopt dependence uncertainty to model the dependence ambiguity between financial risk (or insurable risk) and background risk. Because the goal-reaching objective function is nonconcave, these two problems bring highly unconventional and challenging issues for which classical optimization techniques often fail. Using a quantile formulation method, we derive the optimal solutions explicitly. The results show that the presence of background risk does not alter the shape of the solution but instead changes the parameter value of the solution. Finally, numerical examples are given to illustrate the results and verify the robustness of our solutions.

1. INTRODUCTION

In financial and insurance markets, many objectives of investment or risk management are related solely to the success of specific goals. For example, the practice of benchmarking in institutional money management is quite widespread. In the evaluation of the portfolio manager’s investment, how his or her performance outperforms that of a given benchmark is an important indicator. A typical example of the benchmark is Standard & Poor’s 500 index. Generally speaking, there are two types of portfolio management: passive and active (see, e.g., Sharpe, Alexander, and Bailey 1999; Maginn et al. 2007). With the former, a manager keeps track of an index, whereas with the latter, the manager tries to beat the return of the predetermined benchmark. Because passive portfolio management can simply invest directly in the benchmark for all intents and purposes, we focus on the active portfolio management decision that attempts to maximize the probability of beating the passive portfolio management. This is often called a “goal-reaching problem” (see, e.g., Browne 1999, 2000). Notably, the goal-reaching problem is also commonly faced by insurance companies. Gajek and Zagrodny (2004) argued that a clear aim for the insurers’ risk protection is to decrease the probability of ruin or, equivalently, increase the survival probability. Bernard and Tian (2009) also emphasized that the insurers often transfer part of risks to reinsurers to minimize the probability of insolvency under the regulation of Solvency II. Therefore, analyzing goal-reaching problems is very useful in finance and insurance.

In the academic literature, the theory of background risk has attracted a great deal of attention for its practical importance. As Pratt (1988) noted,

Most real decision makers, unlike those portrayed in our popular texts and theories, confront several uncertainties simultaneously. They must make decisions about some risks when others have been committed to but not resolved. Even when a decision is to be made about only one risk, the presence of others in the background complicates matters (p. 395).

Specifically, for individuals, such non-hedgeable risks can arise from their uncertain labor income, uncertain tax liabilities, real estate investments, unexpected expenses due to health issues, and so forth. For financial institutions, operational risk and environmental risk represent types of background risk in portfolio management. For insurance companies, background risk can stem from their investment risk, economic risk, operational risk, underwriting risk from other lines, and so on. Thus, it is rational to make decisions by taking background risk into account.

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A large body of research has examined the effect of background risk on decision making in finance and insurance (for a review, see Gollier 2001). Some studies assume an independent background risk and then examine its impact on the economic agent’s behavior. For example, Kimball (1993) showed that decreasing absolute risk aversion and decreasing absolute prudence are sufficient to guarantee that people will be less willing to accept another independent zero-mean background risk to their wealth. Gollier and Pratt (1996) further demonstrated that decreasing absolute risk aversion and decreasing absolute prudence are sufficient for risk vulnerability, which implies that any zero-mean background risk raises the degree of risk aversion to any other independent risk. Moreover, Weng and Zhuang (2017) investigated the optimal reinsurance design with an independent background risk in a goal-reaching model.

In addition, some studies have taken into account a correlated background risk. For example, Doherty and Schlesinger (1983) found that the optimal insurance purchase relies heavily on the dependence between the insurable risk and the background risk. In particular, when a background risk is incorporated, the well-known Arrow’s theorem on the optimality of the deductible and Mossin’s theorem are shown to be held only under a very restricted market and risk conditions. Under the same expected utility framework, Gollier (1996) demonstrated that the stop-loss contract is optimal if the insurable risk and the background risk are independent and that the optimal solution changes to be the disappearing deductible when the background risk is stochastically increasing with respect to the insurable risk in the sense of convex order. Dana and Scarsini (2007) and Chi and Wei (2020) further concluded that different stochastic dependence assumptions between the insurable risk and the background risk can lead to different qualitative properties of the optimal insurance contract. On the other hand, Tsetlin and Winkler (2005) considered a general risk choice model with a correlated background risk and showed that a major factor in a project decision is whether the project risk is positively or negatively related to the background risk. Denuit, Eeckhoudt, and Menegatti (2011) investigated the impact of the correlation between financial risk and background risk on optimal choices.

All of these studies clearly indicate that the dependence structure plays a critical role in aggregating the risks and thus affects decision making. In practice, although there exist accurate and efficient approaches to model the marginal distributions of the risks, their dependence structure is very hard to capture because of many computational and convergence issues with statistical inference of multidiimensional data. Embrechts, Wang, and Wang (2015) argued that modeling a high-dimensional dependence structure is typically data costly, and in many realistic settings there is often not enough joint data to make reliable inference for the dependence structure between individual risks. Due to the lack of joint data, the dependence structure is often chosen arbitrarily in finance and insurance. As emphasized by regulatory expert opinion (see, e.g., Basel Fundamental Review of the Trading Book or Solvency II), risk aggregation formulas used for regulation are all ad hoc and neglect the actual dependence between risks. It is worth noting that an inappropriate dependence assumption can lead to very serious

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**TABLE 1**

The Optimal Reinsurance Design without Background Risk for Different Goal Levels

| Goal Level $\xi$ | Optimal Premium $\pi^*$ | Optimal Objective Value $F_X(q^*)$ | Optimal Attachment Point and Detachment Point $(q^*, q'^*)$ |
|------------------|------------------------|----------------------------------|--------------------------------------------------|
| 15               | 4.4356                 | 0.9690                           | (0.5644, 8.7320)                                 |
| 15.5             | 3.9356                 | 0.9047                           | (0.5644, 6.8680)                                 |
| 16               | 3.4356                 | 0.8048                           | (0.5644, 5.5817)                                 |
| 16.5             | 2.9356                 | 0.7714                           | (0.5644, 4.5439)                                 |
| 17               | 2.4356                 | 0.6946                           | (0.5644, 3.6608)                                 |
| 17.5             | 1.9356                 | 0.6090                           | (0.5644, 2.8882)                                 |
| 18               | 1.4356                 | 0.5135                           | (0.5644, 2.2003)                                 |
| 18.5             | 0.9356                 | 0.4069                           | (0.5644, 1.5803)                                 |
| 19               | 0.4356                 | 0.2881                           | (0.5644, 1.0165)                                 |

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1 Embrechts et al. (2014) gave a very detailed discussion on why estimating the dependence structure between risks is statistically and computationally challenging.

2 Wüthrich (2003) emphasized that usually in actuarial problems it is difficult to have a good intuitive feeling for the dependence structure and one has not enough data to really analyze the dependence structure. McNeil, Frey, and Embrechts (2015) claimed that, in the foreseeable future, the lack of operational loss data is a big concern in modeling the dependence structure.
consequences. For example, using the multivariate Gaussian model can result in extremely underestimating the default probability in a large basket of firms (McNeil, Frey, and Embrechts 2015). In the literature, this scenario with unavailable or unreliable dependence information is referred to as dependence uncertainty. More detailed discussions on this topic can be found in Embrechts, Puccetti, and Rüschendorf (2013), Bernard, Jiang, and Wang (2014), Cai, Liu, and Wang (2018), Wang, Xu, and Zhou (2019), and references therein.

To address the dependence uncertainty, a classical approach is to formulate a minimax/maximin decision problem, which explores the best decision under a worst-case scenario. This approach can be traced back at least to Wald’s minimax criterion (Wald 1950). In a seminal study, Ellsberg (1961) showed that in a situation where probability distributions cannot be completely specified, evaluating the worst-case scenario of all plausible distributions seems to appeal to conservative decision makers. There is ample empirical evidence supporting Ellsberg’s observation. Such an observation is in line with many psychological studies indicating that most decision makers display a very low tolerance toward uncertainty (see, e.g., Furnham and Marks2013).

In this article, we use the same approach to model the dependence uncertainty and attempt to study the optimal distributionally robust decisions that perform best in worst-case situations. In particular, we examine two goal-reaching problems with background risk, focusing respectively on portfolio selection and optimal reinsurance design.3 It is worth noting that the goal-reaching objective function is nonconcave, and therefore these two problems are highly unconventional and challenging, such that traditional optimization techniques generally fail. Furthermore, the presence of background risk poses an additional technical hurdle for solving the robust problems analytically. Considering the similarity of these two problems, we apply the quantile formulation approach in both settings to overcome the difficulty and explicitly derive the optimal solutions. More precisely, we explore the investor’s robust portfolio choice with background risk under dependence uncertainty and compare the solution with the case in the absence of background risk analyzed in He and Zhou (2011). We also compare optimal reinsurance contracts with and without background risk, adopting the dependence uncertainty when background risk is incorporated. By comparison, we find that the presence of background risk does not affect the shape of the solution but instead changes its parameter values (see Remarks 1 and 3).

The numerical study in our article may be of independent interest. In recent years, the success of robustness in distributionally robust optimization has received increasing attention because of the concern that the robust decisions might be very conservative. In practice, how to measure the performance of robustness is a very complicated yet important problem (see, e.g., Huang et al.2010). By borrowing a novel idea from some classic literature on this subject (see, e.g., Zhu and Fukushima 2009), we perform some numerical analyses to demonstrate the worthiness of our optimal contracts. Such a mechanism may provide new insights for actuarial practitioners and academics who seek the robust decisions.

3Notably, the robust (or worst-case scenario) portfolio selection and optimal (re)insurance design have been explored in the literature. For example, Balbás, Balbás, and Balbás (2016) considered a robust portfolio selection problem by minimizing the risk for a given guaranteed expected return. Hou and Xu (2016) analyzed a distributionally robust portfolio selection problem under a mean-variance framework instead of goal-reaching. Asimit et al. (2017) investigated the optimal insurance contract with model risk in the robust optimization sense.
The rest of the article proceeds as follows. Section 2 formulates a robust portfolio selection problem with a background risk and we solve it by the quantile formulation approach. In Section 3, a similar technique is applied to analyze the optimal reinsurance design, and numerical examples are given to illustrate the results and test the robustness of the optimal solutions. Section 4 concludes the article. The Appendix, which supports Section 2 and Section 3, provides the analysis of two distributional robust probabilistic problems and investigates the optimal reinsurance design when the insurable risk and background risk are comonotonic.

**Notation**

Throughout the article, we assume an atom-less probability space \((\Omega, F, P)\). For any random variable \(X\) defined in this probability space, we denote its cumulative distribution function by \(F_X(x) = P(X \leq x)\) for all \(x \in \mathbb{R}\). Using \(F_X(x)\), we can define the quantile function of \(X\) as

\[
F_X^{-1}(t) := \inf\{z : F_X(z) \geq t\}, \quad t \in (0, 1],
\]

with the convention \(\inf \emptyset = +\infty\).\(^4\) Note that all of the quantile functions are increasing\(^5\) and left continuous. By definition, we immediately have

\[
F_X(F_X^{-1}(t)) \geq t, \quad t \in (0, 1]
\]

and

\[
F_X^{-1}(F_X(x)) \leq x, \quad x \in \mathbb{R}.
\]

These two inequalities will be used in the subsequent analysis without claim. Because \(X \geq 0\) almost surely is equivalent to \(F_X^{-1}(0^+) \geq 0\), it is easy to show that the set of quantile functions of nonnegative random variables can be expressed as

\[
\mathcal{G} := \{G(\cdot) : (0, 1] \to [0, \infty] \text{ is increasing and left continuous}\}.
\]

We further use \(A^\top\) to denote the transpose of a matrix or vector \(A\). In particular, when \(A\) is a vector, we set \(|A| = \sqrt{A^\top A}\).

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2. PORTFOLIO SELECTION

In this section, we set up a continuous-time financial market that is correlated with a background risk. Then we investigate the robust portfolio choice by formulating a goal-reaching model.

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\(^4\)It should be emphasized that sometimes we have to extend the domain of \(F_X^{-1}(t)\) to \((0, \infty)\) by setting \(F_X^{-1}(t) = +\infty\) for \(t > 1\).

\(^5\)Throughout the article, the terms “increasing” and “decreasing” mean “nondecreasing” and “nonincreasing,” respectively.
2.1. Model Setting

Let \( W(t) := (\mathcal{W}_1(t), \ldots, \mathcal{W}_m(t)) \), \( t \geq 0 \) be an \( n \)-dimensional standard Brownian motion defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\). It constitutes the risk from the financial market. Let \( (\mathcal{F}_t)_{t \geq 0} \) be the natural filtration generated by the Brownian motion and augmented by all \( \mathbb{P} \)-null sets, namely, \( \mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\} \cup \mathcal{N} \) for any \( t \geq 0 \), where \( \mathcal{N} \) denotes the set of all \( \mathbb{P} \)-null sets. Let \( T > 0 \) denote a fixed terminal time of investment. Because the background risk we consider is from outside the financial market, it is essential to assume \( \mathcal{F}_T \subset \mathcal{F} \). In other words, the background risk is \( \mathbb{P} \)-measurable but may not be \( \mathcal{F}_T \)-measurable.

Following Karatzas and Shreve (1998), we define a continuous-time financial market that involves \( m + 1 \) assets being traded continuously. One of these assets is the bond, whose price \( S_0(t) \) follows an ordinary differential equation

\[
\begin{aligned}
\frac{dS_0(t)}{S_0(t)} &= r(t) \, dt, \quad t \in [0, T], \\
S_0(0) &= s_0 > 0,
\end{aligned}
\]

where \( r(\cdot) \), the appreciation rate of the bond, is an \( \mathcal{F}_t \)-progressively measurable and scalar-valued stochastic process with \( \int_0^T |r(s)| \, ds < +\infty \) almost surely (a.s.). Other \( m \) assets are stocks and their price processes \( \{S_i(t) : i = 1, \ldots, m\} \) are driven by the following stochastic differential equations (SDEs):

\[
\begin{aligned}
\frac{dS_i(t)}{S_i(t)} &= S_i(t) \left[ b_i(t) \, dt + \sum_{j=1}^n \sigma_{ij}(t) \, d\mathcal{W}_j(t) \right], \quad t \in [0, T], \\
S_i(0) &= s_i > 0,
\end{aligned}
\]

where \( b_i(\cdot) \) and \( \sigma_{ij}(\cdot) \), the appreciation rate and volatility rate, respectively, are scalar-valued and \( \mathcal{F}_t \)-progressively measurable stochastic processes with

\[
\int_0^T \left[ \sum_{i=1}^m |b_i(t)| + \sum_{i=1}^m \sum_{j=1}^n \sigma_{ij}(t)^2 \right] \, dt < +\infty, \quad \text{a.s.}
\]

Building upon \( b_i(\cdot) \), \( r(t) \), and \( \sigma_{ij}(\cdot) \), we can define the excess return rate vector process

\[
B(t) := (b_1(t) - r(t), \ldots, b_m(t) - r(t))^\top
\]
and the volatility matrix process \( \sigma(t) := (\sigma_{ij}(t))_{m \times n} \). The following assumption, which ensures that the market is arbitrage free and complete (see, e.g., Karatzas and Shreve 1998; Duffie 2010), is imposed throughout this section.

**Assumption 1.** There exists a unique \( \mathcal{F}_t \)-progressively measurable and \( \mathbb{R}^n \)-valued process \( \theta_0(\cdot) \) such that

\[
\mathbb{E}\left[ e^{\int_0^T |\theta_0(t)|^2 \, dt} \right] < +\infty \quad \text{and} \quad \sigma(t)\theta_0(t) = B(t), \text{a.s.}
\]

for almost everywhere \( t \in [0, T] \).

Consider an investor with an initial endowment \( x_0 > 0 \) and an investment horizon \( [0, T] \). We assume that the share trading takes place continuously in a self-financing manner (i.e., there are no incoming or outgoing cash flows during the investment horizon) and that the market is frictionless (i.e., there are no limitations on the trading size of assets and no transaction costs). We denote by \( x(t) \) the investor’s total wealth at time \( t \in [0, T] \), and then \( x(t) \) evolves according to an SDE (see, e.g., Karatzas and Shreve 1998)

\[
\begin{cases}
\frac{dx(t)}{dt} = \left[ r(t)x(t) + B(t)^\top \pi(t) \right] dt + \pi(t)^\top \sigma(t) dW(t), & t \in [0, T], \\
x(0) = x_0,
\end{cases}
\]

(2.3)

where \( \pi_i(t) \) denotes the amount of the investor’s wealth invested in stock \( i \) at time \( t \) for \( i = 1, 2, \ldots, m \). We call the process

\[
\pi(t) := (\pi_1(t), \ldots, \pi_m(t))^\top, \quad t \in [0, T]
\]

an admissible portfolio if it is \( \mathcal{F}_t \)-progressively measurable with

\[
\int_0^T |\sigma(t)^\top \pi(t)|^2 \, dt < +\infty \quad \text{and} \quad \int_0^T |B(t)^\top \pi(t)| \, dt < +\infty, \text{ a.s.}
\]

and is tame (i.e., the corresponding discounted wealth process, \( S_0(t)^{-1} x(t), t \in [0, T] \), is almost surely bounded from below, though the bound may depend on \( \pi(\cdot) \)). It is standard in the literature on continuous-time portfolio selection to assume that admissible portfolios are tame, so as to exclude the notorious doubling strategy.

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**FIGURE 2.** Comparison of Nominal Performances of the Robust Contract and the Nominal Contract under Different Goal Levels.
Define the pricing kernel $q$:

$$q := \exp \left\{ -\int_0^T \left[ r(s) + \frac{1}{2} \theta_0(s) \right]^2 ds - \int_0^T \theta_0(s)^T dW(s) \right\},$$

(2.4)

where $\theta_0(\cdot)$ is the unique market price of risk. Then, using the standard martingale approach (see, e.g., Karatzas and Shreve, 1998), He and Zhou (2011) investigated the following goal-reaching portfolio selection problem:

$$\sup_X \mathbb{E}_q X \left[ \frac{X}{C1} \right] \leq \xi,$$

s.t. $\mathbb{E}[\rho X] \leq x_0$, $X \geq 0$, $X$ is $\mathcal{F}_T$-measurable,

(2.5)

where $\xi > 0$ is the constant goal (level of wealth) intended to be reached by time $T$, $\mathbb{E}[\rho X] \leq x_0$ is the budget constraint, and $X \geq 0$ is the no-bankruptcy constraint.\(^6\) To the best of our knowledge, the goal-reaching problem was proposed by Kulldorff (1993) and investigated extensively by Browne (1999, 2000).

In this article, we extend Problem (2.5) to incorporate a background risk in the terminal wealth and analyze a robust portfolio selection problem with dependence uncertainty

$$\sup_X \inf_{Y \sim F_0} \mathbb{P}(X - Y \geq \xi),$$

s.t. $\mathbb{E}[\rho X] \leq x_0$, $X \geq 0$, $X$ is $\mathcal{F}_T$-measurable.

(2.6)

Here, $Y$ is $\mathbb{F}$-measurable and represents the background risk, whose distribution function $F_0$ is known but whose relationship with the financial market is unclear because our assumption that $\mathcal{F}_T \subseteq \mathbb{F}$.

Notably, Bernard, Moraux, et al. (2015) also studied a portfolio selection problem under the goal-reaching model with state-dependent preferences in the sense that they sought an optimal payoff $X$ which has a desired dependence with a benchmark asset $A_T$. Our model is different from theirs primarily in two aspects. First, we seek an optimal distributionally robust payoff in Problem (2.6); that is, this payoff is feasible in the worst-case scenario. Second, the objective of Problem (2.6) involves a background risk $Y$, whereas theirs does not.

To simplify the analysis, we further make the following assumption.

\(^6\)The martingale approach can be labelled as a two-step scheme. It first identifies the optimal payoff $X^*$ by solving Problem (2.5) and then derives the optimal portfolio $\pi(\cdot)$ by replicating the optimal final payoff $X^*$, where the theory of backward SDE is applied. See Bielecki et al. (2005) for more details on this approach. Because the second step is rather standard, we focus only on the first step in this article.
Assumption 2. The cumulative distribution function $F_0$ is continuous.

2.2. Optimal Solutions

In this subsection, we aim to solve Problem (2.6) explicitly. By setting $\bar{Y} := Y + \zeta$, this problem is equivalent to

$$
\sup_X \inf_{\bar{Y} \sim F_1} \mathbb{P}(X \geq \bar{Y}) \quad \text{s.t.} \ \mathbb{E}[\rho X] \leq x_0, \ X \geq 0,
$$

(2.7)

where $F_1(z) = \mathbb{P}(\bar{Y} \leq z) = \mathbb{P}(Y \leq z - \zeta) = F_0(z - \zeta)$.

Under Assumption 2, it follows from Corollary B.1 and Corollary B.2 that the inner optimal value of Problem (2.7) can be equivalently expressed as

$$
\inf_{\bar{Y} \sim F_1} \mathbb{P}(X \geq \bar{Y}) = \sup_{z \in \mathbb{R}} (F_1(z) - F_X(z)).
$$

(2.8)

This quantity is neither convex nor concave in $X$. As a result, normal optimization techniques largely fail, and we instead employ the quantile formulation approach to overcome this difficulty. For this reason, we impose the following technical assumption.

Assumption 3. The pricing kernel $\rho$ is atomless; that is, its distribution function is continuous.

This assumption is rather standard in the portfolio selection literature (see, e.g., Jin and Zhou 2008; He and Zhou 2011; He, Jin, and Zhou, 2015). It is worth mentioning that, this assumption is satisfied when the investment opportunity set $(r(\cdot), b(\cdot), \sigma(\cdot))$ is deterministic, in which case $\rho$ follows a lognormal distribution (that is the case with the Black-Scholes market).

Now let us focus on the constraint of Problem (2.7). Note that the quantity in (2.8) is increasing and law-invariant in $X$. Using the same argument as in He and Zhou (2011), Xu (2016), and Rüschendorf and Vanduffel (2020), we can show that the optimal solution should be anti-comonotonic with the pricing kernel $\rho$ so that

$$
X = F_X^{-1}(1 - F_\rho(\rho)).
$$

With the help of this relation, the budget constraint $\mathbb{E}[\rho X] \leq x_0$ reads

$$
\int_0^1 F_X^{-1}(t)F_\rho^{-1}(1-t) \ dt \leq x_0.
$$

In addition, we have $F_X^{-1} \in \mathcal{G}$ because of $X \geq 0$. 

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FIGURE 4. Comparison of Nominal Performances of the Robust Contract and the Nominal Contract under Different Levels of the Safety Loading Coefficient.

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Next regarding the objective function of Problem (2.7), we need to rewrite it in terms of $F_X^{-1}$ as follows.

**Lemma 1.** Under Assumption 2, we have

$$\inf_{Y \sim F_1} \mathbb{P}(Y \geq \bar{Y}) = \sup_{z \in \mathbb{R}} (F_1(z) - F_X(z)) = \sup_{t \in (0, 1]} (F_1(F_X^{-1}(t)) - t).$$  \hspace{1cm} (2.9)

**Proof.** The first equality of (2.9) is just (2.8). Now let us show the second equality. For any $t \in (0, 1]$ and $\epsilon > 0$, we have $F_X(F_X^{-1}(t) - \epsilon) < t$ by definition, so

$$F_1(F_X^{-1}(t) - \epsilon) - F_X(F_X^{-1}(t) - \epsilon) > F_1(F_X^{-1}(t) - \epsilon) - t,$$

which leads to

$$\sup_{z \in \mathbb{R}} (F_1(z) - F_X(z)) > F_1(F_X^{-1}(t) - \epsilon) - t.$$  \hspace{1cm} (2.10)

By Assumption 2, $F_1$ is continuous. Sending $\epsilon \to 0$ in (2.10), we further obtain

$$\sup_{z \in \mathbb{R}} (F_1(z) - F_X(z)) \geq F_1(F_X^{-1}(t)) - t.$$

Because $t \in (0, 1]$ is arbitrarily chosen, it follows that

$$\sup_{z \in \mathbb{R}} (F_1(z) - F_X(z)) \geq \sup_{t \in (0, 1]} (F_1(F_X^{-1}(t)) - t).$$

Conversely, for any $z \in \mathbb{R}$, if $F_X(z) = 1$,

$$F_1(z) - F_X(z) = F_1(z) - 1 \leq 0 \leq (F_1(F_X^{-1}(t)) - t)|_{t=0} \leq \sup_{t \in (0, 1]} (F_1(F_X^{-1}(t)) - t);$$

Otherwise, if $F_X(z) < 1$, let $t = F_X(z) + \epsilon$ and $\epsilon > 0$ be sufficiently small so that $t < 1$; then by definition $F_X^{-1}(t) \geq z$. So

$$F_1(z) - F_X(z) \leq F_1(F_X^{-1}(t)) - t + \epsilon,$$

and
\[
\sup_{z \in \mathbb{R}} (F_1(z) - F_X(z)) \leq \sup_{t \in [0,1]} (F_1(F_X^{-1}(t)) - t) + \epsilon.
\]

The claim is thus proved by sending \( \epsilon \to 0 \).

Collecting the previous results, Problem (2.7) reduces to

\[
\begin{align*}
\sup_{F_X^{-1}} \ & \sup_{t \in [0,1]} (F_1(F_X^{-1}(t)) - t) \\
\text{s.t. } & \int_0^1 F_X^{-1}(t) F_\rho^{-1}(1-t) dt \leq x_0, \ F_X^{-1} \in \mathcal{G}.
\end{align*}
\tag{2.11}
\]

The following theorem solves Problem (2.11) completely.

**Theorem 1.** Under Assumptions 2 and 3, there exists an optimal solution of Problem (2.11) of the form

\[ F_X^{-1}(t) = \kappa^* 1_{\{t > r^*\}}, \]

and the optimal payoff of Problem (2.7) is of the form

\[ X^* = \kappa^* 1_{\{\rho < F_\rho^{-1}(1-r^*)\}}, \]

where \( 1_A \) is the indicator function of the event \( A \),

\[ r^* = \arg \max_{r \in [0,1]} \left\{ F_1 \left( \frac{x_0}{\int_r^1 F_\rho^{-1}(1-s) ds} \right) - r \right\}, \tag{2.12} \]

and

\[ \kappa^* = \frac{x_0}{\int_0^1 F_\rho^{-1}(1-s) ds}. \tag{2.13} \]

**Proof.** Using the fact that \( F_1(F_X^{-1}(t^*)) \geq F_1(F_X^{-1}(t)) \) and Assumption 2, we can get that, for any feasible solution \( F_X^{-1} \) of Problem (2.11), there exists \( 0 \leq t_0 \leq 1 \) such that
\[ F_1(F_X^{-1}(t_0^+)) - t_0 = \sup_{t \in [0, 1]} (F_1(F_X^{-1}(t)) - t). \]

Notably, we deliberately set \( F_X^{-1}(1^+) = F_X^{-1}(1) \) in the above equation, which is different from Footnote 4. If \( t_0 = 1 \), then it follows that

\[ \sup_{t \in [0, 1]} (F_1(F_X^{-1}(t)) - t) = F_1(F_X^{-1}(t_0^+)) - t_0 \leq 0 \leq F_1(F_X^{-1}(0^+)) - 0. \]

Therefore, from now on, we assume \( 0 \leq t_0 < 1 \). Set

\[ G(t) = F_X^{-1}(t_0^+) \mathbb{1}_{\{t > t_0\}}. \]

Because \( X \geq 0 \text{ a.s.} \), we have \( F_X^{-1}(t) \geq 0 \) for any \( t \in (0, 1] \). Consequently, \( G(\cdot) \) is an increasing and left-continuous function with \( G(t) \leq F_X^{-1}(t) \) for \( t \in (0, 1] \), and hence it satisfies the constraint of Problem (2.11). Moreover,

\[ \max_{t \in [0, 1]} (F_1(G(t)) - t) \geq \sup_{t \in [0, 1]} (F_1(G(t)) - t) = \sup_{t \in [0, 1]} (F_1(F_X^{-1}(t_0^+)) - t) = F_1(F_X^{-1}(t_0^+)) - t_0 = \sup_{t \in [0, 1]} (F_1(F_X^{-1}(t)) - t), \]

which means that \( G \) is at least as good as \( F_X^{-1} \) in Problem (2.11).

As a result, we only need to focus on the candidates of the form

\[ G(t) = \kappa \mathbb{1}_{\{\tau > r\}}, \]

where the parameters \( \kappa \) and \( r \) satisfy the budget constraint of Problem (2.11), namely,

\[ 0 \leq \kappa \leq \frac{x_0}{\int_r^1 F^{-1}_\rho(1 - s)ds}, \quad 0 \leq r < 1. \]

Because, for a given \( r \), the objective function \( F_1(G(t)) - t = F_1(\kappa \mathbb{1}_{\{\tau > r\}}) - t \) is increasing in \( \kappa \), one should choose \( \kappa \) as large as possible. Hence, we set

\[ \kappa = \frac{x_0}{\int_r^1 F^{-1}_\rho(1 - s)ds}, \quad 0 \leq r < 1. \]

Therefore, the remaining problem is to choose \( r^* \in [0, 1) \) that solves the following optimization problem:

\[ \max_{t \in [0, 1]} \left\{ \sup_{t \in [0, 1]} (F_1(G(t)) - t) \right\} = \max_{t \in [0, 1]} \left\{ \frac{x_0}{\int_r^1 F^{-1}_\rho(1 - s)ds} \right\} \]

\[ = \sup_{t \in [0, 1]} \left\{ \frac{x_0}{\int_r^1 F^{-1}_\rho(1 - s)ds} - r \right\} \]

\[ = \sup_{t \in [0, 1]} \left\{ F_1(\frac{x_0}{\int_r^1 F^{-1}_\rho(1 - s)ds} - r) \right\}, \quad (2.14) \]

where the last equation is owing to that
(F_1 \left( \int_0^1 F_\rho^{-1}(1-s)ds \right) - r) \bigg|_{r=0} = F_1 \left( \int_0^1 F_\rho^{-1}(1-s)ds \right) \geq F_1(0).$

Note that $F_1$ is continuous and

$$\limsup_{r \to 1} \left( F_1 \left( \int_0^1 F_\rho^{-1}(1-s)ds \right) - r \right) \leq 0 \leq \left( F_1 \left( \int_0^1 F_\rho^{-1}(1-s)ds \right) - r \right) \bigg|_{r=0},$$

so there exists $r^* \in [0, 1)$ solving the above optimization problem (2.14). Accordingly, the optimal $\kappa^*$ is given in (2.13). This completes the proof. \hfill \square

**Remark 1.** For Problem (2.5) (without background risk), He and Zhou (2011) showed that the optimal payoff is $X^* = \xi 1_{\{\rho \leq c^*\}}$ and the optimal value is $F_\rho(c^*)$, where $\xi$ is the goal level of wealth and $c^*$ is a constant such that $E[\rho X^*] = x_0$. They further pointed out that, in this case, the optimal payoff in the goal-reaching model can be viewed as a digital option. Specifically, the investor either receives a fixed payment $\xi$ upon a “winning event” (i.e., $\rho \leq c^*$) or loses all of the investment on a “losing event” (i.e., $\rho > c^*$).

In contrast, the optimal payoff of Problem (2.6) is

$$X^* = \kappa^* 1_{\{\rho \leq F_\rho^{-1}(1-r^*)\}},$$

where $r^*$ and $\kappa^*$ are given in (2.12) and (2.13), respectively. Moreover, the corresponding optimal value is

$$F_0 \left( \int_0^1 F_\rho^{-1}(1-s)ds - \xi \right) - r^*. $$

Because it is the aggregated risk (i.e., $X - Y$) rather than the sole financial risk (i.e., $X$) that is of major concern, the presence of background risk greatly affects the investor’s decision. Although the optimal payoff profile is still a digital option, the fixed payment $\kappa^*$ and winning event (i.e., $\rho \leq F_\rho^{-1}(1-r^*)$) depend on $F_0$ (i.e., the marginal distribution of background risk $Y$).

### 3. OPTIMAL REINSURANCE DESIGN

In this section, we apply the same quantile formulation method to examine goal-reaching problems in optimal reinsurance design. In particular, we investigate the effect of background risk on the reinsurance demand by comparing optimal contracts with and without background risk.

#### 3.1. Model Setting

In a given time period, an insurer endowed with initial wealth $w_0$ faces two sources of risks: $X$ and $Y$, where $X$ is nonnegative and insurable and $Y$ is a background risk and may be negative. The optimal reinsurance design is concerned with an optimal partition of $X$ into two parts: $I(X)$ and $X - I(X)$, where $I(X)$ represents the portion of the loss that is ceded to a reinsurer and $R(X) := X - I(X)$ is the loss retained by the insurer. Thus, $I(x)$ and $R(x)$ are often called the insurer’s ceded and retained loss functions, respectively.

In practice, the reinsurance contract is often asked to satisfy the principle of indemnity, which ensures that the indemnity is nonnegative and less than the amount of the loss (Tan and Weng 2014). Mathematically, the ceded loss function should satisfy $0 \leq I(x) \leq x$ for all $x$. However, such a principle is insufficient to preclude ex post moral hazard. In addition to the principle of indemnity, Huberman, Mayers, and Smith (1983), Xu, Zhou, and Zhuang (2019), and Chi and Zhuang (2020) further require the ceded loss function to meet the *incentive-compatible condition*, which asks both the ceded and retained loss functions to be increasing. This condition is essentially equivalent to that $I(x)$ is absolutely continuous and
0 ≤ f'(x) ≤ 1 almost everywhere.\(^7\) In this article, we follow Huberman, Mayers, and Smith (1983) to investigate the optimal ceded loss function among the set

$$\mathcal{I} := \{I(x) : I(0) = 0, \ 0 \leq f'(x) \leq 1\}.$$  

Because the reinsurer covers part of the loss for the insurer, the reinsurer will be compensated by collecting the premium from the insurer. We refer to Tan, Weng, and Zhang (2009) for a list of premium principles that have been proposed by actuaries. In this article, we assume that the reinsurance premium is calculated according to the distortion premium principle, which has frequently been used in recent studies.

**Definition 1.** For any nonnegative random variable \(Z\), the distortion premium principle is defined as

$$\pi^g(Z) := (1 + g)\mathbb{E}[Z] = (1 + g)\int_0^\infty g(1 - F_Z(z)) \, dz,$$  

(3.1)

where \(g \geq 0\) is a given constant, and \(g : [0, 1] \rightarrow [0, 1]\) is an increasing and left-continuous function with \(g(0) = 0\) and \(g(1) = 1\).

In this premium principle, \(g\) is typically referred to as the safety loading coefficient and \(g\) as the distortion function. When \(g(x) \equiv x\), the distortion premium principle recovers the expected value premium principle. Furthermore, when the distortion function is concave and \(g = 0\), the distortion premium principle reduces to Wang’s principle.

Under the reinsurance contract \(I(x)\), the insurer’s total risk exposure is the sum of the retained insurable loss, the background risk, and the reinsurance premium. Denoting by \(W_I(X, Y)\) the insurer’s final wealth, we thus immediately have

$$W_I(X, Y) := w_0 - Y - X + I(X) - \pi^g(I(X)).$$

Similar to Gajek and Zagrodny (2004) and Bernard and Tian (2009), we aim to minimize the insolvency risk by choosing the appropriate reinsurance contract. By adopting dependence uncertainty between the insurable risk and the background risk, we formulate a goal-reaching problem as

$$\sup_{I \in \mathcal{I}} \inf_{Y \in \mathcal{Y}} \mathbb{P}(W_I(X, Y) \geq \xi),$$  

(3.2)

where \(\xi\) is a deterministic constant. For example, we can view \(\xi\) as the targeted wealth that is required by regulator to meet some risk management requirement.

Problem (3.2) will be solved via a two-step scheme. More precisely, we first derive the optimal reinsurance contract by fixing a reinsurance premium and then investigate the optimal reinsurance premium.

### 3.2. Optimal Contracts without Background Risk

In this subsection, we analyze the optimal reinsurance design in the absence of background risk; that is, \(Y = 0\). Specifically, we consider the optimization problem

$$\sup_{I \in \mathcal{I}} \mathbb{P}(w_0 - X + I(X) - \pi^g(I(X)) \geq \xi).$$  

(3.3)

We denote by \(M\) the essential supremum of \(X\), which is assumed to be finite in this section. To avoid trivial cases, we make the following assumption.

**Assumption 4.** \(w_0 - \min\{\pi^g(X), M\} < \xi < w_0\).

It is necessary to assume \(\xi < w_0\), because otherwise the optimal objective value is trivially 0 and thus any feasible contract is a solution to Problem (3.3). On the other hand, \(\xi\) should be assumed to be strictly larger than \(w_0 - \min\{\pi^g(X), M\}\) to rule out the trivial solution \(I^*(x) = x\) or \(I^*(x) = 0\).

\(^7\) Xu, Zhou, and Zhuang (2019) provided a detailed discussion on the incentive-compatible condition. It is worth noting that the value change of \(I^*(x)\) on a set with zero Lebesgue measure has no impact on \(R(x)\). Therefore, we do not repeatedly emphasize the term “almost everywhere” when mentioning the marginal ceded loss function \(I^*(x)\).
Given a fixed premium \( p \in [0, \pi^*(X)] \), Problem (3.3) reduces to
\[
\sup_{I \in \mathcal{I}} \mathbb{P}(w_0 - X + I(X) - \pi \geq \zeta) \quad \text{s.t. } \pi^*[I(X)] = \pi.
\] (3.4)

By denoting
\[
\pi_0 := \mathbb{E}^\pi[X] - \frac{\pi}{1 + q} \quad \text{and} \quad \eta := w_0 - \pi - \zeta,
\]
we rewrite Problem (3.4) in terms of the retained loss function \( R(\cdot) \) as
\[
\sup_{R \in \mathcal{R}} \mathbb{P}(R(X) \leq \eta) \quad \text{s.t. } \mathbb{E}^\pi[R(X)] = \pi_0,
\] (3.5)

where
\[
\mathcal{R} := \{ R(x) : R(0) = 0, \ 0 \leq R'(x) \leq 1 \}.
\]

Here we use the comonotonic additive property of the distortion premium principle. Obviously, \( \pi_0 \) should be nonnegative. Moreover, note that when \( \eta < 0 \) (i.e., \( w_0 - \pi - \zeta < 0 \)), the optimal objective value of Problem (3.5) is automatically 0. As a consequence, the feasible range of the reinsurance premium should be \( \pi \in [0, w_0 - \zeta] \).

Motivated by the objective function of Problem (3.5), we define
\[
R_t(x) := \min\{x, t\} = \begin{cases} x, & 0 \leq x \leq t; \\ t, & x > t, \end{cases}
\] (3.6)

for \( t \geq 0 \), then
\[
\mathbb{E}^\pi[R_t(X)] = \int_0^t g(S_X(y))dy,
\]
where \( S_X(x) \) is the survival distribution function of \( X \). The following analysis is divided into two cases based on the comparison between \( \pi_0 \) and \( \mathbb{E}^\pi[R_0(X)] \).

Case (A) \( 0 \leq \pi_0 \leq \mathbb{E}^\pi[R_0(X)] \). By continuity and the monotone convergence theorem, we can find a \( t^* \leq \eta \) such that \( \mathbb{E}^\pi[R_{t^*}(X)] = \pi_0 \). Because \( \mathbb{P}(R_{t^*}(X) \leq \eta) = \mathbb{P}(\min\{t^*, X\} \leq \eta) = 1 \), we have the following proposition.

**Proposition 1.** If \( \pi_0 \leq \mathbb{E}^\pi[R_0(X)] \), then the optimal value of Problem (3.5) is 1 and \( R_{t^*}(x) \) is an optimal solution to it, where \( t^* \leq \eta \) is determined by \( \mathbb{E}^\pi[R_{t^*}(X)] = \pi_0 \).

Case (B) \( \pi_0 > \mathbb{E}^\pi[R_0(X)] \). For any \( q \geq \eta \), we define
\[
R_{x,q}(x) := \begin{cases} x, & 0 \leq x \leq \eta; \\ \eta, & \eta < x \leq q; \\ \eta + x - q, & q < x. \end{cases}
\] (3.7)

This class of retained loss functions has the following interesting properties.

**Lemma 2.** The map \( q \mapsto \mathbb{E}^\pi[R_{x,q}(X)] \) is decreasing and 1-Lipchitz continuous on \( [\eta, M] \). Moreover, if \( R \in \mathcal{R} \) satisfies \( R(q) \leq \eta \) for some \( q \geq \eta \), then \( R(x) \leq R_{x,q}(x) \) for any \( x \in [0, M] \).
Proof. For any \( \eta \leq q < q' \leq M \), we have

\[
R_{n,q}(x) - R_{n,q'}(x) = \begin{cases} 
0, & 0 \leq x \leq q; \\
q - x, & q < x < q'; \\
q' - q, & x \geq q'. 
\end{cases}
\]

which together with the comonotonic additive property of the distortion premium principle implies the first and second claims. If \( R \in \mathcal{R} \) satisfies \( R(q) \leq \eta \) for some \( q \geq \eta \), then \( R(x) \leq \min\{x, R(q)\} \leq \min\{x, \eta\} = R_{n,q}(x) \) for any \( x \leq q \). Because \( R(q) \leq \eta = R_{n,q}(q) \) and \( R'(x) \leq 1 = R_{n,q}'(x) \) for any \( x > q \), we see \( R(x) \leq R_{n,q}(x) \) for any \( x \in [q, M] \).

With the help of this lemma, we can solve Problem (3.5) explicitly.

Proposition 2. If \( \pi_0 > \mathbb{E}^x[R_q(X)] \), then the optimal value of Problem (3.5) is \( F_X(q^*) \) and \( R_{n,q^*}(x) \) is an optimal solution, where

\[
q^* := \max\{q \in [\eta, M] : \mathbb{E}^x[R_{n,q}(X)] = \pi_0\}.
\]

Proof. If \( \pi_0 > \mathbb{E}^x[R_q(X)] = \mathbb{E}^x[R_{n,M}(X)] \), then Lemma 2 together with the continuity and the monotone convergence theorem implies that there exists at least a \( q \in [\eta, M] \) such that \( \mathbb{E}^x[R_{n,q}(X)] = \pi_0 \). Therefore, \( q^* \) is well defined and \( q^* < M \).

We now show that \( R_{n,q^*}(x) \) is an optimal solution to Problem (3.5) with the optimal value \( F_X(q^*) \) by contradiction. Otherwise, there must be a retained loss function \( \bar{R}(\cdot) \in \mathcal{R} \) satisfying

\[
\mathbb{P}(\bar{R}(X) \leq \eta) > \mathbb{P}(R_{n,q^*}(X) \leq \eta) = F_X(q^*)
\]

and \( \mathbb{E}^x[\bar{R}(X)] = \pi_0 \). Let

\[
\bar{q} = \max\{0 \leq q \leq M : \bar{R}(q) \leq \eta\}.
\]

Then, it follows that \( \bar{q} > q^* \) and \( \bar{R}(\bar{q}) = \eta \), which together with Lemma 2 implies

\[
\mathbb{E}^x[\bar{R}(X)] \leq \mathbb{E}^x[R_{n,q}(X)] < \mathbb{E}^x[R_{n,q^*}(X)] = \pi_0,
\]

where the second inequality follows from the definition of \( q^* \). A contradiction is reached, and the proof is thus completed.

To obtain the optimal reinsurance premium \( \pi^* \) in Problem (3.3), we define

\[
\psi(\pi) := \pi_0 - \mathbb{E}^x[R_q(X)] = \mathbb{E}^x[X] - \frac{\pi}{1 + \bar{q}} - \mathbb{E}^x[R_q(X)] = \mathbb{E}^x[X] - \frac{\pi}{1 + \bar{q}} - \int_0^{w_0 - \bar{q}} g(S_X(y)) dy
\]

for \( \pi \in [0, w_0 - \bar{q}] \). It is obvious that \( \psi(0) \geq 0 \) and \( \psi(w_0 - \bar{q}) > 0 \) according to Assumption 4. Taking the derivative of \( \psi(\pi) \) with respect to \( \pi \) yields

\[
\psi'(\pi) = g(S_X(w_0 - \pi - \bar{q})) - \frac{1}{1 + \bar{q}},
\]

(3.8)

Because \( \psi(\pi) \) is a convex function, we can establish the following result.

Theorem 2. Under Assumption 4, we have

i. If there exists \( \hat{\pi} \in [0, w_0 - \bar{q}] \) such that \( \psi(\hat{\pi}) \leq 0 \), then an optimal solution to Problem (3.3) is given as

\[
I_1(x) = \min\{x - \hat{\pi}, 0\},
\]
where \( i \in [0, w_0 - \pi - \xi] \) is such that \( \mathbb{E}[I_i(X)] = \frac{\pi}{1 + \varrho} \). Moreover, the corresponding reinsurance premium is \( \hat{\pi} \), and the objective value is 1.

ii. Otherwise, an optimal solution to Problem (3.3) is given as

\[
I_{q^*, q}(x) = \begin{cases} 
0, & x \leq \eta^*; \\
(x - \eta^*), & \eta^* < x \leq q^*; \\
q^* - \eta^*, & q^* < x,
\end{cases}
\]

where \( \eta^* = w_0 - \pi^* - \xi \), \( \pi^* \) is given in the proof, and \( q^* = \max\{q \in [\eta^*, M] : \mathbb{E}[I_{q^*, q}(X)] = \frac{\pi^*}{1 + \varrho}\} \). Moreover, the corresponding reinsurance premium is \( \pi^* \) and the objective value is \( F_X(q^*) \).

**Proof.** If there exists \( \hat{\pi} \in [0, w_0 - \xi] \) such that \( \psi(\hat{\pi}) \leq 0 \), then \( \hat{\pi}_0 := \mathbb{E}[X] - \frac{\hat{\pi}}{1 + \varrho} \leq \mathbb{E}[R_q(X)] \), where \( \hat{\eta} := w_0 - \hat{\pi} - \xi \).

According to Proposition 1, there exists a \( i \leq \hat{\eta} \) such that \( R_i(x) \) is a solution to Problem (3.5) with parameter \( \hat{\pi}_0 \) and the corresponding optimal value is 1. Obviously, \( I_i(x) \) is also an optimal solution to Problem (3.3).

Otherwise, we must have \( \psi(\pi) > 0 \) for all \( \pi \in [0, w_0 - \xi] \), and thus Case (A) will not take place. Proposition 2 implies that the optimal retained loss function to Problem (3.3) is in the form of \( R_{n,q}(X) \) for some \( q \geq \eta \). Considering that the corresponding optimal objective value is \( F_X(q) \), we should maximize \( q \) over \( \pi \in [0, w_0 - \xi] \) under the constraint of \( \mathbb{E}[I_{n,q}(X)] = \mathbb{E}[X] - \frac{\pi}{1 + \varrho} \) or, equivalently, \( \mathbb{E}[I_{n,q}(X)] = \frac{\pi}{1 + \varrho} \). Recalling that \( \eta = w_0 - \pi - \xi \), we have

\[
\frac{\pi}{1 + \varrho} = \int_{w_0 - \pi - \xi}^{\eta} g(S_X(y)) \, dy = \mathbb{E}[X] - \int_{0}^{w_0 - \pi - \xi} g(S_X(y)) \, dy - \int_{q}^{\pi} g(S_X(y)) \, dy,
\]

which is equivalent to

\[
\psi(\pi) = \int_{q}^{\pi} g(S_X(y)) \, dy.
\]

In order to maximize the corresponding \( q \), we need to find a \( \pi^* \in [0, w_0 - \xi] \) that solves

\[
\min_{\pi \in [0, w_0 - \xi]} \psi(\pi). \tag{3.9}
\]

Define

\[
\bar{\pi} := \inf \left\{ \pi \in \mathbb{R} : \psi'(\pi) = g(S_X(w_0 - \pi - \xi)) - \frac{1}{1 + \varrho} \geq 0 \right\}. \tag{3.10}
\]

Because \( \psi(\pi) \) is convex, an optimal \( \pi^* \) to Problem (3.9) is given by

\[
\pi^* = \max\{0, \min\{\bar{\pi}, w_0 - \xi\}\}.
\]

Note that \( \psi(\pi) \) is strictly positive over \( [0, w_0 - \xi] \), which in turn implies \( q^* \in [w_0 - \pi^* - \xi, M] \). The proof is thus completed. \( \square \)

**Remark 2.** Theorem 2 indicates that the optimal reinsurance in the absence of background risk can be either a stop-loss contract or a stop-loss contract with an upper limit. In particular, if \( \bar{\pi} \) defined in (3.10) is negative, then the optimal contract is no reinsurance. To be specific, for part (i) of Theorem 2, the condition \( \psi(x) \leq 0 \) for some \( x \in [0, w_0 - \xi] \) implies that there exists \( \pi_0 \) such that Case (A) will occur. In this situation, the stop-loss contract \( I_i(x) \) is an optimal contract with reinsurance premium \( \pi^*[I_i(X)] = \pi \) and the corresponding objective value of Problem (3.3) is 1. It should be emphasized that the solution to Problem (3.3) for this case may not be unique, and here we only provide a best strategy for the insurer. For part (ii) of Theorem 2, because \( \psi(x) > 0 \) for all \( x \in [0, w_0 - \xi] \), the optimal objective value of Problem (3.3) cannot be 1 anymore. In this case, the intuition for the optimality of the stop-loss contract with an upper limit (i.e., \( I_{q^*, q}(x) \)) is given as follows. For one thing, due to the goal-reaching objective, when the loss is larger than \( q^* \), it is optimal for the insurer to retain the tail risk because such a portion of risk will not affect the objective value, and it can additionally reduce the insurer’s reinsurance
premium such that the objective value is improved. For another, given \( R(q') = \eta' \), the insurer can further reduce the reinsurance premium by ceding the loss below \( q' \) as little as possible, and hence only the risk over the layer \([\eta', q']\) would be ceded.

It is worth noting that the optimal reinsurance design problem has been studied extensively in the past few decades. This problem was first formally analyzed by Borch (1960) and Arrow (1963), who both demonstrated that the stop-loss contract is the optimal strategy if the reinsurance premium is calculated by the expected value principle. The objective of Borch (1960) was to minimize the variance of an insurer’s total risk exposure, whereas Arrow (1963) considered maximizing the expected utility of a risk-averse insurer’s terminal wealth. Young (1999) and Chi and Zhuang (2020) extended Arrow’s model by assuming Wang’s premium principle and showed that the optimal solution is partial reinsurance over a number of closed intervals. Cui, Yang, and Wu (2013) instead studied the minimization of Value at Risk (VaR) and Conditional Value at Risk (CVaR) of the insurer’s total risk exposure with a general premium principle that includes Wang’s premium principle as a special case. Using a construction approach, they found that the stop-loss reinsurance with an upper limit is often optimal. Cui, Yang, and Wu (2013) by assuming a distortion risk measure and a distortion premium principle and concluded that the optimal reinsurance strategy is a combination of a stop-loss contract with an upper limit and quota share reinsurance.

3.3. Optimal Contracts with Background Risk

In this subsection, we analyze the effect of background risk in the optimal reinsurance design. More specifically, background risk \( Y \) is assumed to be random instead of a constant. Similar to the previous subsection, we use a two-step approach to solve Problem (3.2). In particular, given a reinsurance premium \( \pi \), we examine the following optimization problem:

\[
\sup_{R \in \mathcal{R}} \inf_{Y \sim F_0} \mathbb{P}(w_0 - Y - R(X) - \pi \geq \xi) \\
\text{s.t. } \mathbb{E}[R(X)] = \pi_0. \tag{3.11}
\]

Recall \( \pi_0 = \mathbb{E}[X] - \frac{\eta}{1-W} \) and \( \eta = w_0 - \pi - \xi \). Define

\[
\Xi := w_0 - \pi - \xi - Y \quad \text{and} \quad F_\pi(z) := 1 - F_0(w_0 - \pi - \xi - z),
\]

then Problem (3.11) is further equivalent to

\[
\sup_{R \in \mathcal{R}} \inf_{\Xi \sim F_\pi} \mathbb{P}(R(X) \leq \Xi) \\
\text{s.t. } \mathbb{E}[R(X)] = \pi_0. \tag{3.12}
\]

Under Assumption 2, Corollaries B.1 and B.2 imply \(^8\)

\[
\inf_{\Xi \sim F_\pi} \mathbb{P}(R(X) \leq \Xi) = \sup_{z \in \mathbb{R}} (F_{R|X}(z) - F_\pi(z)). \tag{3.13}
\]

Because \( F_{R|X}(z) = 0 \) for \( z < 0 \) and \( F_{R|X}(z) = 1 \) for \( z \geq M \), we have

\[
\sup_{z < 0} (F_{R|X}(z) - F_\pi(z)) = \sup_{z < 0} (-F_\pi(z)) \leq 0 \leq 1 - F_\pi(M) = (F_{R|X}(z) - F_\pi(z))|_{z=M}
\]

and

\(^8\)It is worth mentioning that the worst-case dependence structure between the insurable risk \( X \) and the background risk \( Y \) that solves the optimization problem on the left-hand side of Equation (3.13) is not (completely) comonotonic. In fact, the results in Appendix B indicate that the worst-case dependence structure between \( X \) and \( Y \) is associated with a so-called shuffle of min. Such a family of dependence structures shows that \( X \) and \( Y \) are strongly piecewise monotone functions of each other or piecewise comonotonic (see, e.g., Embrechts, Hoing, and Puccetti 2005).
\[ \sup_{z \geq M} (F(z) - F_x(z)) = \sup_{z \geq M} (1 - F_x(z)) = 1 - F_x(M) = (F(z) - F_x(z)) |_{z=M}. \]

Therefore, by (3.13),
\[ \inf_{z \in [0, M]} \mathbb{P}(R(X) \leq z) = \sup_{z \in [0, M]} (F(z) - F_x(z)). \] (3.14)

Consequently, Problem (3.12) becomes
\[ \sup_{R \in \mathcal{R}} \sup_{z \in [0, M]} (F(z) - F_x(z)) \]
\[ \text{s.t. } \mathbb{E}[R(X)] = \pi_0. \] (3.15)

Because two supremums in the objective of Problem (3.15) are exchangeable, we can rewrite this problem as
\[ \sup_{z \in [0, M]} \sup_{R \in \mathcal{R}} (F(z) - F_x(z)) \]
\[ \text{s.t. } \mathbb{E}[R(X)] = \pi_0. \] (3.16)

In what follows, Problem (3.16) will be further analyzed via a two-step scheme. We first fix a \( z \in [0, M] \) and find the optimal retained loss function \( R(\cdot) \). Then, we determine the optimal \( z^* \). Accordingly, for each fixed \( z \in [0, M] \), we consider
\[ \sup_{R \in \mathcal{R}} (F(z) - F_x(z)) \]
\[ \text{s.t. } \mathbb{E}[R(X)] = \pi_0. \] (3.17)

Because \( z \) is fixed, \( F_x(z) \) is a constant. We can remove this term from the objective of Problem (3.17) and investigate the following equivalent problem:
\[ \sup_{R \in \mathcal{R}} F(z) \]
\[ \text{s.t. } \mathbb{E}[R(X)] = \pi_0. \]

Note that the above optimization problem is equivalent to Problem (3.5). Therefore, from Propositions 1 and 2, we obtain the following result.

**Proposition 3.** Under Assumption 2, we have

i. If \( \pi_0 \leq \mathbb{E}[R_z(X)] \), then an optimal solution to Problem (3.17) is
\[ R_{t'}(x) = \begin{cases} x, & x \leq t'; \\ t', & x \geq t', \end{cases} \]

where \( t' \leq z \) is such that \( \mathbb{E}[R_{t'}(X)] = \pi_0 \), and the corresponding optimal value is \( 1 - F_x(z) \).

ii. If \( \pi_0 > \mathbb{E}[R_z(X)] \), then the optimal solution to Problem (3.17) is given as
\[ R_{z,q^*}(x) = \begin{cases} x, & x \leq z; \\ z, & z < x \leq q^*; \\ z + x - q^*, & q^* < x, \end{cases} \]

where \( q^* = \sup \{ q \in [z, M] : \mathbb{E}[R_{z,q}(X)] = \pi_0 \} \) and the corresponding optimal value is \( F_x(q^*) - F_x(z) \).

In order to determine the optimal reinsurance form, it is necessary to compare \( \pi_0 \) and \( \mathbb{E}[R_z(X)] \). Because \( \mathbb{E}[R_z(X)] \) is increasing and continuous in \( z \), we can find a \( z^* \in [0, M] \) such that \( \mathbb{E}[R_{z^*}(X)] = \pi_0 \) and the condition \( \pi_0 \leq \mathbb{E}[R_z(X)] \) (or \( \pi_0 > \mathbb{E}[R_z(X)] \)) is equivalent to \( z^* \leq z \) (or \( z^* > z \)). Therefore, if \( z \geq z^* \), then the optimal value of Problem (3.17) is \( 1 - F_x(z) \),
which is decreasing in \( z \). On the other hand, if \( z < z^* \), then the optimal value of Problem (3.17) is

\[
\sup_{y \in [z, M]} \{ F_X(y) - F_X(z) : \mathbb{E}^x[R_{z,y}(X)] = \pi_0 \}.
\]

According to Proposition 3 and the above discussion, Problem (3.16) (or, equivalently, Problem (3.11)) is equivalent to

\[
\sup_{0 < z \leq y \leq M} \sup_{\mathbb{E}[R_{z,y}(X)] \leq \pi_0} (F_X(y) - F_X(z))
\]

s.t. \( \mathbb{E}^x[R_{z,y}(X)] = \pi_0 \). (3.18)

Therefore, the original problem (3.2) can be rewritten as

\[
\sup_{\pi_0 \in [0, \mathbb{E}[X]]} \sup_{0 < z \leq y \leq M} \sup_{\mathbb{E}[R_{z,y}(X)] \leq \pi_0} (F_X(y) - F_X(z))
\]

s.t. \( \mathbb{E}^x[R_{z,y}(X)] = \pi_0 \). (3.19)

Accordingly, we can exchange two supremums in the objective of Problem (3.19) and consider the following equivalent problem

\[
\sup_{z \in [0, M]} \sup_{0 < z \leq y \leq M} \sup_{\mathbb{E}[R_{z,y}(X)] \leq \pi_0} (F_X(y) - F_X(z))
\]

s.t. \( \mathbb{E}^x[R_{z,y}(X)] = \pi_0 \). (3.20)

Recalling that \( F_X(z) = 1 - F_0(w_0 - \pi - \xi - z) \) and \( \pi_0 = \mathbb{E}[X] - \frac{\pi_0}{1 + g} \), we can express the objective function of Problem (3.20) as

\[
F_X(y) - F_X(z) = F_X(y) + F_0 \left( w_0 - (1 + g) \int_z^y g(S_X(t)) \, dt - \xi - z \right) - 1.
\]

Moreover, it easy to see that the condition \( \mathbb{E}^x[R_{z,y}(X)] \leq \pi_0 \leq \mathbb{E}^x[X] \), together with \( \mathbb{E}^x[R_{z,y}(X)] = \pi_0 \), is equivalent to

\[
0 \leq z \leq y \leq M.
\]

Therefore, we can transform Problem (3.20) into an optimization problem over a triangle area in \( \mathbb{R}^2 \) as follows:

\[
\sup_{0 \leq z \leq y \leq M} K(z,y),
\]

where

\[
K(z,y) := F_X(y) + F_0 \left( w_0 - (1 + g) \int_z^y g(S_X(t)) \, dt - \xi - z \right).
\]

In particular, if the pair \((\bar{z}, \bar{y})\) is optimal to Problem (3.21), then the optimal \( \bar{\pi}_0 \) in Problem (3.20) is \( \bar{\pi}_0 = \mathbb{E}^x[R_{\bar{z},\bar{y}}(X)] \), and thus the optimal premium for the original problem (3.2) is \( \pi^* = (1 + g)(\mathbb{E}^x[X] - \bar{\pi}_0) \).

Noting that, for each fixed \( y \in [0, M] \), maximizing \( K(z,y) \) over \( z \in [0, y] \) is equivalent to minimizing

\[
\int_{x}^{y} (1 + g) g(S_X(t)) \, dt + z.
\]

Because the above function is convex, one minimizer is \( z^* = \min \{ y, z_0 \} \), where

\[
z_0 := \sup \{ z \in \mathbb{R} : (1 + g) g(S_X(z)) \geq 1 \}.
\]
Now, the optimization problem (3.21) reduces to a simple scalar optimization problem over a compact interval

$$\sup_{0 \leq y \leq M} K(\min\{y, z_0\}, y) = \max \left\{ \sup_{0 \leq y \leq z_0} K(y, y), \sup_{z_0 \leq y \leq M} K(z_0, y) \right\}. \quad (3.23)$$

Combining all of the above results, we are able to present the main result of this subsection.

**Theorem 3.** Assume that $y^* \in [0, M]$ solves Problem (3.23), then an optimal solution to Problem (3.2) is given by

$$I_{x^*, y^*}(x) = \begin{cases} 0, & 0 \leq x \leq z^*; \\ x-z^*, & z^* < x \leq y^*; \\ y^*-z^*, & y^* < x, \end{cases} \quad (3.24)$$

where $z^* = \min\{y^*, z_0\}$. Moreover, if the optimal value of Problem (3.23) is 1, then all feasible policies are indifferent in the sense that the objective value of Problem (3.2) is 0 for any feasible policy.

**Proof.** The first part of this theorem follows immediately from Proposition 3 and the discussion after Proposition 3. For the second part, if the optimal value of Problem (3.23) is 1, which corresponds to the objective value of Problem (3.2) being 0 for any feasible policy, it is obvious that all feasible contracts are indifferent to the original problem (3.2).

**Remark 3.** By Theorem 3, the analysis of an infinite-dimensional optimal reinsurance problem (i.e., Problem (3.2)) is simplified to solving Problem (3.23). Admittedly, it is very hard to obtain an analytical solution to Problem (3.23) because its objective relies heavily on the cumulative distribution functions of $X$ and $Y$. However, such a one-dimensional optimization problem can be numerically solved quickly. Moreover, if $z_0 = M$, then Theorem 3 indicates that no reinsurance is optimal according to (3.24). If $z_0 < M$, then Theorem 3 further shows that the optimal reinsurance contract can be a stop-loss contract (i.e., $y^* = M$), a stop-loss contract with an upper limit (i.e., $z_0 < y^* < M$), or no reinsurance (i.e., $y^* \leq z_0$). Comparing Theorem 2 and Theorem 3, we can find that these three forms of reinsurance treaties are robust in the sense that they are the same with and without background risk. However, similar to the findings in Remark 1, the presence of background risk affects the optimal reinsurance premium and the optimal reinsurance treaty, which heavily depend on $F_0$.

### 3.4. Numerical Examples

In this subsection, numerical examples are given to illustrate the results obtained in the previous two subsections. Moreover, we will show numerically that the optimal contract obtained in Subsection 3.3 performs robustly in some sense.

#### 3.4.1. Illustration of Optimal Reinsurance Contracts

We assume that the random loss $X$ follows a truncated and shifted Pareto distribution with the probability density function

$$f_X(x) = \frac{24}{7} \frac{10^3}{(x+10)^{3.75}} 1_{\{x \in [0,10]\}}. \quad (3.25)$$

Moreover, we set

$$w_0 = 20, \quad g = 0.1, \quad \text{and} \quad g(x) = x^{0.5}.$$ 

A simple calculation shows that $\pi^*(X) = 5.187$. To avoid trivial cases, we consider nine different levels of the goal

$$\bar{\xi} \in \{15, 15.5, 16, 16.5, 17, 17.5, 18, 18.5, 19\}.$$ 

They all satisfy Assumption 4.

---

9Bahnemann (2015), an actuarial report from the Casualty Actuarial Society, pointed out that one of the most popular probability distributions used to model the size of insurance claims is the truncated and shifted Pareto distribution.
First of all, we consider the case without background risk. With the help of Theorem 2, the optimal premium, the optimal objective value and the optimal reinsurance contract under different levels of the goal are calculated and summarized in Table 1.

Notice that, in this example, the attachment point \( \eta^* \) of the optimal layer reinsurance is not affected by the change of the goal level. This is because, according to (3.8), \( \pi^* \) that solves Problem (3.9) should satisfy

\[
\psi'(\pi^*) = g(Sx(w_0 - \pi^* - \xi)) - \frac{1}{1 + q} = g(Sx(\eta^*)) - \frac{1}{1 + q} = 0,
\]

provided that it is an interior point of \([0, w_0 - \xi]\). The above equation implies that \( \eta^* + \pi^* + \xi \) remain unchanged. In contrast, the optimal detachment point is decreasing in \( \xi \). In other words, for a larger goal level, the insurer will purchase less reinsurance coverage and pay less reinsurance premium.

Next, we analyze the optimal distributionally robust reinsurance design when a background risk is incorporated. For comparison purposes, the background risk is assumed to have zero mean and the probability density function

\[
f_0(x) = \frac{\phi(x)}{\mathcal{N}(5) - \mathcal{N}(-5)} \mathbb{1}_{\{x\in[-5,5]\}},
\]

where \( \phi(x) \) and \( \mathcal{N}(x) \) are the probability density function and cumulative distribution function of the standard normal distribution, respectively. Thanks to Theorem 3, the optimal premium, optimal objective value, and optimal reinsurance contract under different levels of the goal in the worst-case scenario are obtained numerically and are listed in Table 2.

Comparing Tables 1 and 2, we can conclude that incorporating the zero-mean background risk does affect the optimal reinsurance design. Particularly, when \( \xi = 18.5 \) or 19, it becomes impossible to achieve the goal in the presence of background risk, and therefore all feasible contracts are indifferent in Problem (3.2) according to Theorem 3. It is worth mentioning that, similar to Table 1, the optimal attachment point \( z^* \) in Table 2 is not influenced by the change of the goal level within \([15, 15.5, 16, 16.5, 17, 17.5, 18]\). Such a phenomenon can be explained by noting that \( y^* > z_0 = 0.5644 \). In addition, the optimal detachment point \( y^* \) still decreases in the goal level. However, \( \pi^* + \xi \) is no longer a constant.

### 3.4.2. Robustness Analysis

In this part, we will follow the way of Zhu and Fukushima (2009) and Kang et al. (2019) to carry out a robustness analysis of the solution to Problem (3.2). Noting that such a solution is obtained under the worst-case dependence between the retained risk and the background risk, we simply call it “nominal scenario.” In addition to the worst-case scenario, we shall introduce a “nominal scenario.” When facing a risk aggregation problem with an uncertain dependence structure of risks, actuarial researchers typically make a conservative assumption that risks are comonotonic.\(^{10}\) Accordingly, we choose the comonotonic dependence structure as the “nominal scenario” and analyze the optimal reinsurance contract (denoted as “nominal contract”) for this scenario in Appendix C.

We use the same parameter values and the same distributions of \( X \) and \( Y \) as in Subsection 3.4.1. Under this setting, in the nominal scenario, we therefore have \( Y = F_Y^{-1}(F_Y(X)) \) a.s. Thanks to Theorem C.1, we solve the optimal reinsurance problem with the nominal dependence (i.e., Problem (C.1)) numerically, and the corresponding optimal reinsurance premium, optimal objective value, and optimal reinsurance contract for different goal levels are displayed in Table 3. Analogous to Tables 1 and 2, the optimal attachment point is not affected by the change of the goal level, and the insurer will purchase less reinsurance coverage for a larger goal level.

We now compare the worst-case performances (in terms of the objective value in the worst-case scenario) of the robust contract and the nominal contract. The worst-case objective value of the robust contract is reported in Table 2. We only need to derive the worst-case objective value of the nominal contract by substituting the nominal contract into the optimal robust reinsurance design problem (i.e., Problem (3.2)). Therefore, it follows from (3.14) that the worst-case objective value of the nominal contract becomes

\[
\inf_{\gamma \in \mathcal{H}_0} \mathbb{P}(w_0 - Y - X + \pi^*(I(X)) \geq \xi) = \inf_{\pi^*} \mathbb{P}(R(X) \leq \Xi) = \sup_{\pi^*} \left( F_R(X)(\xi) - F_\pi(\xi) \right),
\]

where \( I(x) \) is the nominal contract and \( R(x) \) is the corresponding retained loss function. Figure 1 depicts the worst-case performances of these two contracts.

\(^{10}\)One can refer to Dhaene et al. (2002) for some detailed discussions of comonotonic risk aggregation in actuarial practice.
For the purpose of comparison, we then calculate the nominal objective value of the robust contract by substituting the robust contract into the optimal reinsurance design problem with a comonotonic dependence (i.e., Problem (C.1)). Together with the nominal objective value of the nominal contract in Table 3, we present the nominal performances (in terms of objective values in the nominal scenario) of the robust contract and the nominal contract in Figure 2.

We can observe from Figures 1 and 2 that when the goal level is not extremely high, the worst-case performance gap between the robust contract and the nominal contract is larger than the nominal performance gap between these two contracts. In other words, the performance of the robust contract is very close to that of the nominal contract in the nominal scenario, whereas the performance of the robust contract is much better than that of the nominal contract in the worst-case scenario. In this sense, we numerically demonstrate that the robust contract performs more robustly than the nominal contract. Such robustness can also be explained from another perspective: the objective value of the robust contract drops less than that of the nominal contract when the scenario changes from nominal to worst-case (see, e.g., Gorissen 2015).

To test the sensitivity of this phenomenon, we will tweak the parameter values in the following numerical examples. First, we set the goal level to be 17 and employ 10 different levels of the safety loading coefficient

\[ \varrho \in \{0.02, 0.04, 0.06, 0.08, 0.1, 0.12, 0.14, 0.16, 0.18, 0.2\}, \]

while keeping other parameters. Figures 3 and 4 compare the worst-case and nominal performances of the robust contract and the nominal contract under different levels of the safety loading coefficient, respectively.

The same phenomenon can be discovered from Figures 3 and 4. More specifically, the robust contract performs much better than the nominal contract in the worst-case scenario, and the performance gap becomes larger as the safety loading coefficient increases; however, in the nominal scenario, the performance of the robust contract is very close to that of the nominal contract. Therefore, these observations support the robustness of the robust contract relative to the nominal contract. Furthermore, we can see that the objective values are decreasing in the safety loading coefficient in both figures. In other words, as the reinsurance becomes more costly, it is less likely that the insurer can reach the goal even though the optimal strategy is chosen.

Next, we test whether such a performance gap phenomenon is insensitive to the change of the distribution of the insurable risk \( X \). In the previous examples, the random loss \( X \) is assumed to follow a truncated and shifted Pareto distribution with a scale parameter \( \beta = 10 \) and a shape parameter \( \gamma = 3 \), that is, \( f_X(x) = \frac{\gamma^\beta / (\beta + 1)^{\gamma+1}}{\Gamma(\gamma+1) / (\beta+10)^{\gamma+1}} I_{\{x \in [0, 10]\}} \). By fixing \( \zeta = 17 \) and \( \varrho = 0.1 \), we set a range of the shape parameter to be

\[ \gamma \in \{2, 2.2, 2.4, 2.6, 2.8, 3, 3.2, 3.4, 3.6, 3.8, 4\}, \]

and other parameters remain unchanged. Figures 5 and 6 illustrate the worst-case and nominal performances of the robust contract and the nominal contract under different levels of the shape parameter, respectively.

An interesting finding is that the nominal performance gap between the robust contract and the nominal contract is quite stable with regard to the change of the shape parameter of the Pareto distribution, whereas the worst-case performance gap between these two contracts can change significantly. Moreover, we can see from Figures 5 and 6 that the performance gap phenomenon still exists when the shape parameter varies. This phenomenon is particularly more pronounced with a lower shape parameter. Therefore, we numerically demonstrate that the relative robustness of the robust contract to the nominal contract does not depend on the choice of parameter values.

4. CONCLUSION

In this article, we have examined the effect of background risk on portfolio selection and optimal reinsurance design in goal-reaching models with dependence uncertainty. This is motivated by the fact that the dependence structure between risks or assets is often ambiguous (Embrechts, Puccetti, and Rüschendorf 2013), although their marginal distributions can be estimated with high accuracy. Though the exposure to background risk does not alter the form of the optimal payoff of portfolio selection or the optimal reinsurance contract, it does change the parameter value of the optimal solution. Numerical analyses have also been conducted to study the robustness of our solutions.

Some assumptions are imposed on the targeted wealth and the dependence uncertainty in order to derive the robust decisions explicitly in this article. It is meaningful to revisit these goal-reaching problems by relaxing these assumptions. More specifically, the wealth target \( \zeta \) can be random, and partial dependence uncertainty can be used to model dependence ambiguity if additional dependence information is available (see, e.g., Pflug and Pohl 2018). Furthermore, the optimization criterion
can be changed to maximize the decision maker’s final wealth with other risk preferences, such as rank-dependent expected utility (see, e.g., Bernard, He, et al. 2015; Xu, Zhou, and Zhuang 2019). We leave these problems for future research.

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Discussions on this article can be submitted until April 1, 2023. The authors reserve the right to reply to any discussion. Please see the Instructions for Authors found online at http://www.tandfonline.com/uaaj for submission instructions.

APPENDIX

A. A Technical Result

Lemma A.1. For any continuous probability distribution function $F$, we have

$$
\lim_{\Delta \to 0} \sup_{z \in \mathbb{R}} |F(z + \Delta) - F(z)| = 0.
$$

Proof. For any $\epsilon > 0$, there exists an $N > 0$ such that

$$(1 - F(N)) + F(-N) < \epsilon.$$  

Because $F$ is a continuous function, it is uniformly continuous on $[-3N, 3N]$, so there exists $0 < \delta < N$ such that

$$|F(z_1) - F(z_2)| < \epsilon, \quad \text{for any } z_1, z_2 \in [-3N, 3N] \text{ with } |z_1 - z_2| < \delta.$$  

By the above inequalities, for $|\Delta| < \delta$, we have

$$|F(z + \Delta) - F(z)| \leq F(z + |\Delta|) - F(-N) < \epsilon, \quad \text{if } z \leq -2N;$$  

and

$$|F(z + \Delta) - F(z)| \leq 1 - F(z + \Delta) + 1 - F(z) \leq 1 - F(N) + 1 - F(2N) < 2\epsilon, \quad \text{if } z \geq 2N;$$  

and

$$|F(z + \Delta) - F(z)| < \epsilon, \quad \text{if } -2N \leq z \leq 2N.$$  

Therefore, for any $|\Delta| < \delta$, one has

$$\sup_{z \in \mathbb{R}} |F(z + \Delta) - F(z)| \leq 2\epsilon,$$

which gives the conclusion.  

B. Fréchet Problems

In the literature, the problem of finding extremal joint distributions with given marginals is typically referred to as the Fréchet problem. The study of this problem has a long history. We refer to the survey paper and the book by Rüschendorf (1991, 2013) for the detailed treatments, historical remarks, and many applications in different areas of the Fréchet problem. Specifically, let

$$l_s := \inf \{ P(X_1 + X_2 \geq s) : X_1 \sim F^1, X_2 \sim F^2 \},$$

and
where $F^i$ is the known marginal distribution of $X_i$ for $i = 1, 2$. Here, $l_s$ and $u_s$ are respectively the lower and upper Fréchet bounds of the sum $X_1 + X_2$ in $s$ over all possible dependence structures. Notably, such a problem has been solved independently in Makarov (1981) and Rüschendorf (1982). Though this problem is not the main focus of this article, we revisit it in our context and provide a self-contained and elementary proof for completeness.

For any given distribution function $F_V$ and a random variable $W$, we consider the following two classes of optimization problems:

$$
A : \sup_{Y \sim F_V} \mathbb{P}(W \leq V), \quad \sup_{Y \sim F_V} \mathbb{P}(W > V), \quad \inf_{Y \sim F_V} \mathbb{P}(W < V), \quad \inf_{Y \sim F_V} \mathbb{P}(W > V),
$$

and

$$
B : \sup_{Y \sim F_V} \mathbb{P}(W < V), \quad \sup_{Y \sim F_V} \mathbb{P}(W > V), \quad \inf_{Y \sim F_V} \mathbb{P}(W < V), \quad \inf_{Y \sim F_V} \mathbb{P}(W > V).
$$

By the following identities

$$
\sup_{V \sim F_V} \mathbb{P}(W \leq V) = \sup_{V \sim F_V} \mathbb{P}(-W \leq -V),
$$

$$
\inf_{V \sim F_V} \mathbb{P}(W < V) = 1 - \sup_{V \sim F_V} \mathbb{P}(-W \leq -V),
$$

$$
\inf_{V \sim F_V} \mathbb{P}(W > V) = 1 - \sup_{V \sim F_V} \mathbb{P}(W \leq V),
$$

we can see that four problems in Class A are essentially equivalent. Similarly, one can show that four problems in Class B are equivalent as well. We now endeavor to solve the first problems in Class A and Class B, respectively. Sometimes we need the following technical assumptions.

**Assumption B.1.** $F_W(x)$ is continuous.

**Assumption B.2.** $F_V(x)$ is continuous.

### B.1. Problems in Class A

In this part, we want to study the following problem in Class A:

$$
\sup_{V \sim F_V} \mathbb{P}(W \leq V). \tag{B.1}
$$

For any $z \in \mathbb{R}$ and $V \sim F_V$, we have

$$
\mathbb{P}(W \leq V) = \mathbb{P}(V > z, W \leq V) + \mathbb{P}(V \leq z, W \leq V)
\leq \mathbb{P}(V > z) + \mathbb{P}(W \leq z) = 1 - F_V(z) + F_W(z).
$$

By minimizing the right-hand side of the above equation with respect to $z \in \mathbb{R}$, we deduce an upper bound for the optimal value of Problem (B.1)

$$
\mathbb{P}(W \leq V) \leq \inf_{z \in \mathbb{R}}(1 - (F_V(z) - F_W(z))) = 1 - \sup_{z \in \mathbb{R}}(F_V(z) - F_W(z)) = 1 - \alpha,
$$

where

$$
\alpha := \sup_{z \in \mathbb{R}}(F_V(z) - F_W(z)). \tag{B.2}
$$

It is easily seen that $\alpha \in [0, 1]$. We will show below that $1 - \alpha$ is in fact the optimal value of Problem (B.1) by construction.
The following lemma is the key to construct a solution to Problem (B.1).

**Lemma B.1.** Under Assumption B.2 or Assumption B.1, we have \( F_V^{-1}(t + z) \geq F_W^{-1}(t) \) for any \( t \in (0, 1] \).

**Proof.** Fix any \( t \in (0, 1] \). Let us assume that \( t + z \leq 1 \); otherwise, the claim is trivially true.

Suppose that Assumption B.2 holds. For any \( \epsilon > 0 \), we have \( F_W(F_W^{-1}(t) - \epsilon) < t \). By the definition of \( z \), it follows that

\[
z \geq F_V(F_W^{-1}(t) - \epsilon) - F_W(F_W^{-1}(t) - \epsilon) > F_V(F_W^{-1}(t) - \epsilon) - t.
\]

Letting \( \epsilon \to 0 \), we have \( z \geq F_V(F_W^{-1}(t)) - t \) because \( F_V \) is continuous. The same inequality can be proved when Assumption B.1 holds by letting \( \epsilon \to 0 \) in

\[
z \geq F_V(F_W^{-1}(t) + \epsilon) - F_W(F_W^{-1}(t) + \epsilon) \geq F_V(F_W^{-1}(t)) - F_W(F_W^{-1}(t) + \epsilon).
\]

Note that \( F_V(z) - F_W(z) \leq z \) for any \( z \in \mathbb{R} \). If \( z = F_V(F_W^{-1}(t)) - t \), then for any \( z < F_W^{-1}(t) \), one has

\[
F_V(z) \leq z + F_W(z) = F_V(F_W^{-1}(t)) - t + F_W(z) < F_W^{-1}(t),
\]

which implies

\[
F_V^{-1}(z + t) = F_V^{-1}(F_V(F_W^{-1}(t))) = \inf\{z \in \mathbb{R} : F_V(z) \geq F_V(F_W^{-1}(t))\} = F_W^{-1}(t).
\]

Otherwise, if \( z > F_V(F_W^{-1}(t)) - t \), then

\[
F_V^{-1}(z + t) = \inf\{z \in \mathbb{R} : F_V(z) \geq z + t\} \geq \inf\{z \in \mathbb{R} : F_V(z) > F_V(F_W^{-1}(t))\} \geq F_W^{-1}(t).
\]

The claim is thus proved. \( \square \)

We are now ready to construct a solution to Problem (B.1). When the probability space is atomless, it is well known that there exists a \( Z \sim U(0, 1) \) such that \( W = F_W^{-1}(Z) \) a.s. Let us define

\[
\tilde{V} := \begin{cases} 
F_V^{-1}(Z + z), & \text{if } Z \leq 1 - z; \\
F_V^{-1}(1 - Z), & \text{if } Z > 1 - z.
\end{cases}
\]

Then by Lemma B.1, we have \( \tilde{V} \geq W \) a.s. on the set \( \{Z \leq 1 - z\} \). Hence,

\[
P(W \leq \tilde{V}) \geq P(Z \leq 1 - z) = 1 - z.
\]

This would imply that \( \tilde{V} \) is an optimal solution to Problem (B.1) if we could show that \( \tilde{V} \) follows the distribution \( F_V \). In fact, for any \( z \in \mathbb{R} \),

\[
P(\tilde{V} \leq z) = P(\tilde{V} \leq z, Z \leq 1 - z) + P(\tilde{V} \leq z, Z > 1 - z)
\]

\[
= P(F_V^{-1}(Z + z) \leq z, Z \leq 1 - z) + P(F_V^{-1}(1 - Z) \leq z, Z > 1 - z)
\]

\[
= P(Z + z \leq F_V(z), Z \leq 1 - z) + P(1 - Z \leq F_V(z), 1 - Z < z
\]

\[
= P(Z \leq F_V(z) - z, Z \leq 1 - z) + P(1 - Z \leq F_V(z), 1 - Z < z)
\]

\[
= P(Z \leq F_V(z) - z) + P(1 - Z \leq F_V(z), 1 - Z < z)
\]

\[
= \max\{0, F_V(z) - z\} + \min\{F_V(z), 1\}
\]

\[
= F_V(z).
\]

Collecting the above arguments, we have the following theorem.

**Theorem B.1.** Suppose Assumption B.2 or Assumption B.1 holds, then \( \tilde{V} \) defined in (B.3) solves Problem (B.1) with the optimal value
\[
\sup_{V \sim F_V} \mathbb{P}(W \leq V) = 1 - \sup_{z \in \mathbb{R}} (F_V(z) - F_W(z)).
\]

**Corollary B.1.** Under the same condition in Theorem B.1, we have

\[
\begin{align*}
\sup_{V \sim F_V} \mathbb{P}(W \geq V) &= 1 - \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)); \\
\inf_{V \sim F_V} \mathbb{P}(W < V) &= \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)); \\
\inf_{V \sim F_V} \mathbb{P}(W > V) &= \sup_{z \in \mathbb{R}} (F_V(z) - F_W(z)).
\end{align*}
\]

**Proof.** In fact, we have

\[
\begin{align*}
\sup_{V \sim F_V} \mathbb{P}(W \geq V) &= \sup_{V \sim F_V} \mathbb{P}(-W \leq -V) \\
&= 1 - \sup_{z \in \mathbb{R}} (\mathbb{P}(-V \leq z) - \mathbb{P}(-W \leq z)) \\
&= 1 - \sup_{z \in \mathbb{R}} (\mathbb{P}(W < z) - \mathbb{P}(V < z)) \\
&= 1 - \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)),
\end{align*}
\]

provided that

\[
\sup_{z \in \mathbb{R}} (\mathbb{P}(W < z) - \mathbb{P}(V < z)) = \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)). \tag{B.4}
\]

To show (B.4), suppose that Assumption B.2 holds. Then

\[
\sup_{z \in \mathbb{R}} (\mathbb{P}(W < z) - \mathbb{P}(V < z)) \leq \sup_{z \in \mathbb{R}} (\mathbb{P}(W \leq z) - \mathbb{P}(V < z)) \\
= \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)). \tag{B.5}
\]

On the other hand, for any \( \epsilon > 0 \), we have

\[
\begin{align*}
\sup_{z \in \mathbb{R}} (\mathbb{P}(W < z) - \mathbb{P}(V < z)) &= \sup_{z \in \mathbb{R}} (\mathbb{P}(W < z + \epsilon) - \mathbb{P}(V < z + \epsilon)) \\
&\geq \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z + \epsilon)) \\
&\geq \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)) - \sup_{z \in \mathbb{R}} (F_V(z + \epsilon) - F_V(z)).
\end{align*}
\]

Sending \( \epsilon \to 0^+ \), by Lemma A.1, we deduce

\[
\sup_{z \in \mathbb{R}} (\mathbb{P}(W < z) - \mathbb{P}(V < z)) \geq \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)).
\]

This together with (B.5) implies (B.4). Similarly, one can prove (B.4) when Assumption B.1 holds.

Moreover, we have

\[
\begin{align*}
\inf_{V \sim F_V} \mathbb{P}(W < V) &= 1 - \sup_{V \sim F_V} \mathbb{P}(W \geq V) = \sup_{z \in \mathbb{R}} (F_W(z) - F_V(z)),
\end{align*}
\]
and
\[
\inf_{V \sim F_V} \mathbb{P}(W > V) = 1 - \sup_{V \sim F_V} \mathbb{P}(W \leq V) = \sup_{z \in \mathbb{R}} (F_Y(z) - F_W(z)).
\]
The proof is thus completed. \hfill \Box

### B.2. Problems in Class B

**Theorem B.2.** Under Assumption B.2 or Assumption B.1, we have
\[
\sup_{V \sim F_V} \mathbb{P}(W \leq V) = \sup_{V \sim F_V} \mathbb{P}(W < V). \tag{B.6}
\]

**Proof.** By the evident inequalities
\[
\sup_{V \sim F_V} \mathbb{P}(W \leq V) \geq \sup_{V \sim F_V} \mathbb{P}(W < V) \geq \lim_{\epsilon \to 0^+} \sup_{V \sim F_V} \mathbb{P}(W \leq V - \epsilon),
\]
it suffices to show
\[
\lim_{\epsilon \to 0^+} \sup_{V \sim F_V} \mathbb{P}(W \leq V - \epsilon) \geq \sup_{V \sim F_V} \mathbb{P}(W \leq V). \tag{B.7}
\]
Suppose that Assumption B.2 holds, then by Theorem B.1, we obtain
\[
\lim_{\epsilon \to 0^+} \sup_{V \sim F_V} \mathbb{P}(W \leq V - \epsilon) = \lim_{\epsilon \to 0^+} \left( 1 - \sup_{z \in \mathbb{R}} (F_Y(z + \epsilon) - F_W(z)) \right)
\]
\[
= 1 - \lim_{\epsilon \to 0^+} \sup_{z \in \mathbb{R}} (F_Y(z + \epsilon) - F_W(z))
\]
\[
\geq 1 - \lim_{\epsilon \to 0^+} \sup_{z \in \mathbb{R}} (F_Y(z) - F_W(z)) + \sup_{z \in \mathbb{R}} (F_Y(z + \epsilon) - F_Y(z))
\]
\[
\geq 1 - \sup_{z \in \mathbb{R}} (F_Y(z) - F_W(z)) - \lim_{\epsilon \to 0^+} \sup_{z \in \mathbb{R}} (F_Y(z + \epsilon) - F_Y(z))
\]
\[
= 1 - \sup_{V \sim F_V} \mathbb{P}(W \leq V - \epsilon)
\]
\[
= \sup_{V \sim F_V} \mathbb{P}(W \leq V),
\]
where the third equality follows from Lemma A.1. The same inequality (B.7) can be similarly proved under Assumption B.1. The claim is thus proved. \hfill \Box

By using similar arguments as above, we can establish the following corollary.

**Corollary B.2.** Under Assumption B.2 or Assumption B.1, we have
\[
\sup_{V \sim F_V} \mathbb{P}(W \geq V) = \sup_{V \sim F_V} \mathbb{P}(W > V);
\]
\[
\inf_{V \sim F_V} \mathbb{P}(W < V) = \inf_{V \sim F_V} \mathbb{P}(W \leq V);
\]
\[
\inf_{V \sim F_V} \mathbb{P}(W > V) = \inf_{V \sim F_V} \mathbb{P}(W \geq V).
\]

### C. Optimal Contracts with a Comonotonic Background Risk

In this subsection, we consider the case in which the background risk $Y$ is comonotonic with the insurable risk $X$. More precisely, we assume that $Y = h(X)$, where $h$ is a continuous and increasing function. For simplicity, we further assume that $F_Y(x)$ is continuous and strictly increasing on $[0, M]$ and that the distortion function $g$ is continuous and strictly increasing. Now the optimal reinsurance design problem with a comonotonic background risk $Y = h(X)$ becomes...
\[
\sup_{i \in I} \mathbb{P}(w_0 - h(X) - X + I(X) - \pi^X(I(X)) \geq \zeta).
\]

(C.1)

It is obvious that the objective of Problem (C.1) can be rewritten as

\[
\mathbb{P}(h(X) + R(X) + \pi^X(I(X)) \leq w_0 - \zeta).
\]

If \(h(M) + z_0 + \pi^X((X-z_0)_+) \leq w_0 - \zeta\); where \(z_0\) is defined in (3.22), then it can be easily shown that an optimal solution to Problem (C.1) is \((x-z_0)_+\) and the corresponding objective value is 1. Else, if \(w_0 - \zeta < h(0)\), then the objective value is zero for any admissible reinsurance strategy. To avoid these trivial cases in the subsequent analysis, we make the following assumption.

**Assumption C.1.** \(h(M) + z_0 + \pi^X((X-z_0)_+) > w_0 - \zeta \geq h(0)\).

For any given \(I \in I\) (or, equivalently, \(R \in \mathcal{R}\)), we will show that there exists a stop-loss contract with an upper limit that is better than \(I\). More specifically, if \(h(0) > w_0 - \zeta - \pi^X(I(X))\), then the increasing property of \(h\) and \(R\) implies that the objective value for \(I\) is 0, and thus \(I\) is dominated by any feasible contract.

Otherwise, if \(h(0) \leq w_0 - \zeta - \pi^X(I(X))\), we denote by

\[
\bar{x} := \sup\{t \in [0,M] : h(t) + R(t) \leq w_0 - \zeta - \pi^X(I(X))\}.
\]

Notice that \(\bar{x}\) is well defined and should be strictly less than \(M\). Otherwise, if \(\bar{x} = M\), then we have

\[
h(M) + z_0 + \pi^X((X-z_0)_+) \leq h(M) + R(M) + \pi^X((X-R(M))_+) \leq h(M) + R(M) + \pi^X(I(X)) \leq w_0 - \zeta,
\]

where the first inequality follows from the fact that

\[
z + \pi^X((X-z)_+) = z + (1 + g) \int_z^M g(S_X(t)) \ dt
\]

achieves the minimal value at \(z = z_0\) based on the definition of \(z_0\), and the second inequality is owing to \((X-R(M))_+ \leq X-R(X) = I(X)\). This leads to a contradiction with Assumption C.1. Furthermore, recalling that both \(h\) and \(R\) are increasing and continuous and that \(F_X(x)\) is continuous and strictly increasing, we should have

\[
\mathbb{P}(h(X) + R(X) \leq w_0 - \zeta - \pi^X(I(X))) = \mathbb{P}(X \leq \bar{x}).
\]

Denote

\[
\tilde{R}(x) := \begin{cases} 
\min\{x, R(\bar{x})\}, & 0 \leq x \leq \bar{x}; \\
R(\bar{x}) + (x-\bar{x}), & \bar{x} < x \leq M.
\end{cases}
\]

(C.2)

It is easy to see that \(\tilde{R} \in \mathcal{R}\) and

\[
\tilde{R}(x) \geq R(x) \quad \text{and} \quad \tilde{R}(\bar{x}) = R(\bar{x}),
\]

which in turn implies that \(\pi^X(\tilde{I}(X)) \leq \pi^X(I(X))\), where \(\tilde{I}(x) = x - \tilde{R}(x)\). Moreover,

\[
\mathbb{P}\left( w_0 - h(X) - X + \tilde{I}(X) - \pi^X(\tilde{I}(X)) \geq \zeta \right) = \mathbb{P}\left( h(X) + \tilde{R}(X) + \pi^X(\tilde{I}(X)) \leq w_0 - \zeta \right) \\
\geq \mathbb{P}\left( h(X) + \tilde{R}(X) + \pi^X(I(X)) \leq w_0 - \zeta \right) \\
\geq \mathbb{P}\left( h(X) + R(X) + \pi^X(I(X)) \leq w_0 - \zeta \right) \\
= \mathbb{P}(X \leq \bar{x}).
\]

Therefore, the retained loss function \(\tilde{R}(x)\) in (C.2) dominates \(R\) in terms of objective value.
The above analysis demonstrates that an optimal retained loss function $R^*$ can be in the form of $R_{a,b}(x)$, where $R_{a,b}(x)$ is defined in (3.7) and $a$ and $b$ satisfy

$$L(a, b) := h(b) + a + (1 + g) \int_a^b g(S_x(t))dt \preceq w_0 - \zeta.$$  \hfill (C.3)

Note that $L(0, 0) = h(0) \preceq w_0 - \zeta$ under Assumption C.1.

**Lemma C.1.** At least for an optimal contract $R_{a,b}(x)$, the attachment point $a$ and the detachment point $b$ can satisfy

$$L(a, b) = w_0 - \zeta.$$  \hfill (C.4)

**Proof.** Denote

$$c := \sup \{ t \in [0, M] : h(t) + R_{a,b}(t) \preceq w_0 - \zeta - \pi^e(I_{a,b}(X)) \},$$

then we have $c \geq b$ according to (C.3). Moreover, $c$ cannot be equal to $M$; otherwise, we would have

$$w_0 - \zeta \geq h(M) + a + M - b + \pi^e(I_{a,b}(X)) \geq h(M) + z_0 + \pi^e((X - z_0)^+),$$

where the second inequality follows from the fact that

$$a + M - b + (1 + g) \int_a^b g(S_x(t)) dt \geq M \geq z_0 + (1 + g) \int_{z_0}^M g(S_x(t)) dt$$

when $b \leq z_0$ and that

$$a + M - b + (1 + g) \int_a^b g(S_x(t)) dt \geq z_0 + M - b + (1 + g) \int_{z_0}^M g(S_x(t)) dt \geq z_0 + (1 + g) \int_{z_0}^M g(S_x(t)) dt$$

when $b \geq z_0$. This contradicts Assumption C.1. Furthermore, $c$ should satisfy

$$P(h(X) + R_{a,b}(X) + \pi^e(I_{a,b}(X)) \preceq w_0 - \zeta) = P(X \leq c)$$

and

$$h(c) + R_{a,b}(c) + (1 + g) \int_a^b g(S_x(t)) dt = w_0 - \zeta.$$  

If $c = b$, then (C.4) automatically holds. Otherwise, if $c \in (b, M)$, then $R_{a,b}(c) = a + c - b$. We can show that $R_{a+c-b, c}(x)$ is strictly better than $R_{a,b}(x)$. This is because

$$h(c) + R_{a+c-b, c}(c) + (1 + g) \int_{a+c-b}^c g(S_x(t)) dt < w_0 - \zeta,$$

where the inequality is derived by the fact that $\int_{a+c-b}^c g(S_x(t)) dt < \int_a^b g(S_x(t)) dt$. Thus, the case of $c \in (b, M)$ cannot occur.

**Lemma C.1** indicates that there must exist an optimal solution $I_{a,b}(x)$ (or, equivalently, $R_{a,b}(x)$) to Problem (C.1), where $a$ and $b$ satisfy $L(a, b) = w_0 - \zeta$. Moreover, the corresponding objective value is $F_X(b)$. Therefore, to solve Problem (C.1), we only need to find a maximal $b$ such that $b \geq a$ and $L(a, b) = w_0 - \zeta$. Notice that $L(a, b)$ is increasing in $b$ and
Combining all of the above results, we are able to obtain the following result.

**Theorem C.1.** *Under Assumption C.1, we have*

i. *If* \( h(z_0) + z_0 > w_0 - \xi \), *then no reinsurance is optimal to Problem (C.1).*

ii. *Otherwise, if* \( h(z_0) + z_0 < w_0 - \xi \), *then a stop-loss contract with an upper limit*

\[
I_{z_0,b^*}(x) = \begin{cases} 
0, & 0 \leq x \leq z_0; \\
 x - z_0, & z_0 < x \leq b^*; \\
 b^* - z_0, & b^* < x,
\end{cases}
\]

*is optimal to Problem (C.1), where* \( b^* \) *is determined by* \( L(z_0, b^*) = w_0 - \xi \).