CHARACTERIZATION OF THE RIESZ EXPONENTIAL FAMILY ON
HOMOGENEOUS CONES

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Abstract. In the paper we present a characterization theorem of Riesz measure on homogeneous cones
through the invariance property of a natural exponential family under the action of the triangular group.

1. Introduction

Following Casalis [2], we consider natural exponential family (NEF), which is invariant under a sub-

group $G$ of the general linear group of finite dimensional linear space. NEF is considered on arbitrary
finite dimensional linear space $E$. Under a weak assumption on $G$ (see Theorem 2.1), generating measure
of such family is of the form

$$e^{-(\theta_0, x)}\mu_0(dx)$$

for some $\theta_0$ in the dual space $E^*$, where $\mu_0$ is a $G$-invariant measure.

We apply Theorem 2.1 to a problem of a characterization of the Riesz measure on homogeneous cone
through the invariance property of NEF under the action of the triangular group. Since NEF generated by
the Riesz measure consists of Wishart distributions, this is also a characterization of a Wishart distribution
on homogeneous cones [1, 10, 25]. We would like to mention that this problem was announced by Letac
in [23, Section 4], who pointed out that the natural framework for such characterizations is indeed
homogeneous cone.

There are essentially two types of characterizations of NEFs. The first one is connected with the
central object of NEF, that is, the variance function. The aim is then to describe a generating measure
of a NEF with given variance function. Much have been done in this direction, but still much more
is unknown. A generating measure is known for $E = \mathbb{R}$, when the variance function is a quadratic
polynomial [27], cubic polynomial [14, 26] and of some more sophisticated forms, like $P\Delta + Q\sqrt{\Delta}$, where
$P$, $\Delta$ and $Q$ are quadratic polynomials [24]. For $E = \mathbb{R}^d$ we have to emphasize [2] with its deep connection
between homogeneous quadratic variance functions and Euclidean Jordan algebras, simple quadratic [3]
and simple cubic variance functions [15], just to name a few.

The second type of characterizations of NEFs are through the invariance properties under some group
action. We quote some results on invariant NEFs under a given subgroup $G$ of the general affine group: one
parameter group [4], group of rotations [11], Moebious group [22], the connected component containing
the identity of the subgroup of the general linear group $GL(Sym(N, \mathbb{R}))$ preserving the cone $Sym_+(N, \mathbb{R})$,
triangular group of a simple Euclidean Jordan algebra [12] and its modification [13]. Here $Sym(N, \mathbb{R})$
stands for the symmetric $N \times N$ matrices with real entries and $Sym_+(N, \mathbb{R})$ is the cone of positive definite
real $N \times N$ matrices. In the last three quoted papers the characterizations were carried out by showing
that the variance function (which uniquely determines NEF) coincides with the variance function of some
Riesz measure (or its image by the involution $x \mapsto -x$) on $Sym_+(N, \mathbb{R})$ and symmetric cones, respectively.

In the present paper we generalize [12] to the homogeneous cones setting. We use a matrix realization
of homogeneous cones, which proves here very useful and is more accessible to the reader who is not

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2. Natural Exponential Families

In the following section we will introduce all the necessary facts on NEFs that will be needed later on. It is however worth mentioning that the standard reference book on exponential families is [2].

Let $\mathbb{E}$ be a finite dimensional real linear space endowed with an inner product $\langle \cdot, \cdot \rangle$ and $\mathbb{E}^*$ its dual space. Let $\mu$ be a positive Radon measure on $\mathbb{E}$. We define its Laplace transform $L_\mu : \mathbb{E}^* \rightarrow (0, \infty]$ by

$$L_\mu(\theta) = \int_\mathbb{E} e^{\langle \theta, x \rangle} \mu(dx).$$

By $\Theta(\mu)$ we will denote the interior of the set $\{ \theta \in \mathbb{E}^* : L_\mu(\theta) < \infty \}$. Hölder’s inequality implies that the set $\Theta(\mu)$ is convex and the cumulant function

$$k_\mu(\theta) = \log L_\mu(\theta)$$

is convex on $\Theta(\mu)$ and it is strictly convex if and only if $\mu$ is not concentrated on some affine hyperplane of $\mathbb{E}$. Let $\mathcal{M}(\mathbb{E})$ be the set of positive Radon measures on $\mathbb{E}$ such that $\Theta(\mu)$ is not empty and $\mu$ is not concentrated on some affine hyperplane of $\mathbb{E}$.

For $\mu \in \mathcal{M}(\mathbb{E})$ we define the natural exponential family (NEF) generated by $\mu$ (denoted by $F(\mu)$) as the set of probability measures of the form

$$P(\theta, \mu)(dx) = e^{\langle \theta, x \rangle - k_\mu(\theta)} \mu(dx), \quad \theta \in \Theta(\mu).$$

Let us note that $F(\mu) = F(\mu')$ if and only if $\mu'(dx) = e^{\langle a, x \rangle + b} \mu(dx)$ for some $a \in \mathbb{E}^*$ and $b \in \mathbb{R}$.

We will now describe the action of elements from the general linear group $GL(\mathbb{E})$ on a NEF. The identity element of $GL(\mathbb{E})$ will be denoted by $\text{Id}$. Let $F = F(\mu)$ be a NEF on $\mathbb{E}$. Then, for any $g \in GL(\mathbb{E})$, we have $g.F(\mu) = F(g_* \mu)$, where $g_* \mu$ denotes the image measure of $\mu$ by $g$ and $g.F(\mu)$ stands for the family of image measures $g_* P(\theta, \mu)$ ($\theta \in \Theta(\mu)$).

We say that a measure $\mu_0$ is invariant under a subgroup $G$ of $GL(\mathbb{E})$ if for all $g \in G$ there exists a constant $c_g > 0$ for which $\mu_0(gA) = c_g \mu_0(A)$ for any measurable set $A \subset \mathbb{E}$. This condition is equivalent to

$$L_{\mu_0}(g^* \theta) = c_g^{-1} L_{\mu_0}(\theta), \quad \theta \in \Theta(\mu_0).$$

Note that the correspondence $G \ni g \mapsto c_g \in \mathbb{R}$ is a character of the group $G$.

Let $\mu \in \mathcal{M}(\mathbb{E})$. Observe that the condition $g.F(\mu) = F(\mu)$ implies that for any $g \in G$ there exists $a(g) \in \Theta(\mu) \subset \mathbb{E}^*$ and $b(g) \in \mathbb{R}$ such that

$$g_* \mu(dx) = e^{\langle a(g), x \rangle + b(g)} \mu(dx).$$

Then, Casalis [3] showed that (see [3], Theorem 2.2) functions $a$ and $b$ satisfy the following system of equations: for any $(g, g') \in G^2$

$$a(gg') = (g^*)^{-1} a(g') + a(g),$$
$$b(gg') = b(g) + b(g').$$
Let us assume that $G$ contains $cld$ for some $c \neq 1$. Then, we obtain for any $g \in G$,
\[
\begin{align*}
a(cld \ g) &= \frac{1}{c} a(g) + a(cld), \\
a(g \ cld) &= (g^*)^{-1} a(cld) + a(g).
\end{align*}
\]

Equating the right hand sides of above formulas, we arrive at
\[
a(g) = \theta_0 - (g^*)^{-1} \theta_0, \quad g \in G,
\]
with $\theta_0 = \frac{c}{1} a(cld)$. Define $\mu_0(dx) = e^{(\theta_0, x)} \mu(dx)$. Then, \[22\] implies that $\mu_0$ is $G$-invariant. Thus, we obtain the following

**Theorem 2.1** Let $G$ be a subgroup of $GL(\mathbb{E})$ and let $F = F(\mu)$ be a $G$-invariant NEF on $\mathbb{E}$, that is, for any $g \in G$ one has $gF = F$. If $G$ contains $cld$ for some $c \neq 1$, then there exist $\theta_0 \in \mathbb{E}^*$ and a $G$-invariant measure $\mu_0$ such that
\[
\mu(dx) = e^{-(\theta_0, x)} \mu_0(dx).
\]

### 3. Characterization of the Riesz measure on homogeneous cones

#### 3.1. Matrix realization of homogeneous cones

Let $V$ be a real linear space and $\Omega$ a regular open convex set in $V$ containing no line. The cone $\Omega$ is said to be homogeneous if the linear automorphism group $G(\Omega) = \{g \in GL(V) : g\Omega = \Omega\}$ acts transitively on $\Omega$, that is, for any $x$ and $y$ in $\Omega$ there exists $g \in G(\Omega)$ such that $y = gx$.

We will now give a very useful representation of homogeneous cones following \[20\], Section 3. For a symmetric matrix $x \in \text{Sym}(N, \mathbb{R})$, we denote by $x$ the lower triangular matrix of size $N$ defined by

\[
(x)_{ij} = \begin{cases} 
  x_{ij}, & \text{if } i > j, \\
  x_{ii}/2, & \text{if } i = j, \\
  0, & \text{if } i < j.
\end{cases}
\]

Then we have $x = \hat{x} + \hat{x}^\top$, where $\hat{x} = x^\top$ is the transpose of $x$. For $x, y \in \text{Sym}(N, \mathbb{R})$, we define
\[
x \Delta y := xy + y\hat{x} \in \text{Sym}(N, \mathbb{R}).
\]

Then, $(\text{Sym}(N, \mathbb{R}), \Delta)$ forms a non-associative algebra with unit element $I_N$. Let $Z$ be a subalgebra of $(\text{Sym}(N, \mathbb{R}), \Delta)$ and $H_Z$ be the set of lower triangular matrices from $Z$ with positive diagonal entries, that is, $H_Z := \{x : x \in Z \text{ and } x_{ii} > 0\}$. Define $\Omega_Z = \{x \in Z : x \text{ is positive definite}\}$ and consider for any $T \in H_Z$ the linear operators $\rho(T) : Z \rightarrow Z, x \mapsto \rho(T)x = T\hat{x}T^\top$. It can be shown that $\rho(H_Z)$ acts on $\Omega_Z$ transitively, which means that $\Omega_Z$ is a homogeneous cone (\[21\], Theorem 3). Furthermore, we have the following

**Theorem 3.1** (\[19\]) For a homogeneous cone $\Omega \subset V$, there exists a subalgebra $Z \subset \text{Sym}(N, \mathbb{R})$ and a linear isomorphism $\phi : V \rightarrow Z$ such that $\phi(\Omega) = \Omega_Z$.

We say that $Z \subset \text{Sym}(N, \mathbb{R})$ admits a normal block decomposition if there exists a partition $N = n_1 + \ldots + n_r$ and subspaces $V_{lk} \subset \text{Mat}(n_l, n_k, \mathbb{R})$, $1 \leq k < l \leq r$, such that $Z$ is the set of symmetric matrices of the form
\[
\begin{pmatrix}
  X_{11} & X_{12}^\top & \cdots & X_{1r}^\top \\
  X_{21} & X_{22} & \cdots & X_{2r} \\
  \vdots & \vdots & \ddots & \vdots \\
  X_{r1} & X_{r2} & \cdots & X_{rr}
\end{pmatrix}
\]

\[
\begin{cases}
  X_{ll} = x_{ll}I_{n_l}, x_{ll} \in \mathbb{R}, 1 \leq l \leq r \\
  X_{lk} \in V_{lk}, 1 \leq k < l \leq r
\end{cases}
\]

We will write $Z_{\psi}$ for this space.
\textbf{Theorem 3.2} \hspace{1em} (i) \cite[Theorem 2]{IshiKo} Let \( Z \) be a subalgebra of \((\text{Sym}(N,\mathbb{R}),\Delta)\) with \( I_N \in Z \). Then there exists a permutation matrix \( w \), such that \( wZw^\top \) admits a normal block decomposition.

(ii) \cite[Proposition 2]{IshiKo} \( Z_V \) is a subalgebra of \((\text{Sym}(N,\mathbb{R}),\Delta)\) if and only if the subspaces \( \{V_k\}_{1 \leq k < l \leq r} \) satisfy the following conditions:

(V1) \( A \in V_k, B \in V_l \implies AB \in V_i \) for any \( 1 \leq i < k < l \leq r \),

(V2) \( A \in V_i, B \in V_k \implies AB^\top \in V_l \) for any \( 1 \leq i < k < l \leq r \),

(V3) \( A \in V_k \implies AA^\top \in \mathbb{R}I_{n_l} \) for any \( 1 \leq k < l \leq r \).

If \( Z = Z_V \) we shall write \( \Omega_V, H_V \) for \( \Omega_Z \) and \( H_Z \), respectively.

Condition (V3) allows us to define an inner product on \( V_k, 1 \leq k < l \leq r \), by

\[ AA^\top = (A|A)I_{n_l}, \quad A \in V_k. \]

We then define the \textit{standard inner product} on \( Z_V \) by

\[ (x, y) := \sum_{k=1}^r x_{kk} y_{kk} + 2 \sum_{1 \leq k < l \leq r} (X_{lk}Y_{lk}), \quad x, y \in Z_V. \]

Note that \( \langle \cdot, \cdot \rangle \) coincides with the trace inner product only if \( n_1 = \ldots = n_r = 1 \).

Define a one-dimensional representation of \( H_V \) by

\[ \chi_{\underline{s}}(T) := \prod_{k=1}^r \xi_{kk^2}, \]

where \( \underline{s} = (s_1, \ldots, s_r) \in \mathbb{C}^r \). Note that any one-dimensional representation \( \chi \) of \( H_V \) is of the form \( \chi_{\underline{s}} \).

This fact will be important later on.

For any open convex cone \( \Omega \) we define the \textit{dual cone} of \( \Omega \) by

\[ \Omega^* = \{ \xi \in V^*: \langle x, \xi \rangle > 0 \ \forall x \in \bar{\Omega} \setminus \{0\} \}, \]

where \( V^* \) is the dual space of \( V \). If \( \Omega \) is homogeneous, then so is \( \Omega^* \). Let \( \Omega_V^* \) denote the dual cone of \( \Omega_V \).

For \( T \in H_V \), we denote by \( \rho^*(T) \) the adjoint operator of \( \rho(T) \in GL(Z_V) \) defined in such a way that

\[ \langle x, \rho^*(T)\xi \rangle = \langle \rho(T)x, \xi \rangle \]

for \( x, \xi \in Z_V \). Then we see from \cite[Chapter 1, Proposition 9]{IshiKo} that for any \( \xi \in \Omega_V^* \), there exists a unique \( T \in H_V \) such that \( \xi = \rho^*(T)I_N \). Define

\[ \Delta^*_{\underline{s}}(\xi) = \Delta^*_{\underline{s}}(\rho^*(T)I_N) := \chi_{\underline{s}^*}(T), \]

where \( \underline{s}^* = (s_r, \ldots, s_1) \). We see that for any \( S \in H_V \) and \( \xi = \rho^*(T)I_N \in \Omega_V^* \) we have

\[ \Delta^*_{\underline{s}}(\rho^*(S)\xi) = \chi_{\underline{s}^*}(TS) = \chi_{\underline{s}^*}(T)\chi_{\underline{s}^*}(S) = \Delta^*_{\underline{s}}(\xi)\Delta^*_{\underline{s}}(\rho^*(S)I_N) \]

and, since any character of \( H_V \) is of the form \( \chi_{\underline{s}^*} \), this property characterizes \( \Delta^* \). Function \( \Delta^* \) is sometimes termed a generalized power function. Its importance is emphasized by the following result \cite[10]{IshiKo}.

\textbf{Theorem 3.3} \textit{There exists a positive measure} \( R_{\underline{s}} \) \textit{on} \( Z_V \) \textit{with the Laplace transform} \( L_{R_{\underline{s}}}(\theta) = \Delta^*_{\underline{s}^*}(\theta) \) \textit{for} \( \theta \in \Omega_V^* \) \textit{if and only if} \( \underline{s} \in \Xi := \bigsqcup_{\xi \in \{0,1\}^r} \Xi(\xi) \), \textit{where}

\[ \Xi(\xi) := \left\{ \underline{s} \in \mathbb{R}^r; \begin{array}{ll} s_k > p_k(\xi)/2 \text{ if } \varepsilon_k = 1, \\ s_k = p_k(\xi)/2 \text{ if } \varepsilon_k = 0 \end{array} \right. \]

\textit{and} \( p_k(\xi) = \sum_{i<k} \varepsilon_i \dim V_{ki} \).
The measure $R_{\xi}$ with $\xi \in \Xi$ has the support in $\Omega_V$ and is called the Riesz measure, while the set $\Xi$ is called the Gindikin-Wallach set.

In order to define NEF generated by $R_{\xi}$, we have to know if $R_{\xi} \in \mathcal{M}(Z_V)$, that is, if $R_{\xi}$ is not concentrated on some affine hyperplane of $Z_V$. The following result is a generalization of \cite[Theorem 3.1]{12}. Let us define for $\xi \in \{-1,0,1\}^r$, 

$$E_\xi := \begin{pmatrix} \varepsilon_1 I_{n_1} & \cdots & \varepsilon_r I_{n_r} \end{pmatrix} \in Z_V$$

Theorem 3.4 The support of $R_{\xi}$ is not concentrated on any affine hyperplane in $Z_V$ if and only if $s_k > 0$ for all $k = 1, \ldots, r$.

Proof. We write $O_\xi$ for the $\rho(H_V)$-orbit in $Z_V$ through $E_\xi$. Note that $O_{\{1, \ldots, 1\}} = \Omega_V$. It is shown in \cite[Theorem 6.2]{16} that, if $\xi \in \Xi(\xi)$, then $R_{\xi}$ is a positive measure on $O_\xi$, so that the support of $R_{\xi}$ coincides with the closure $\overline{O_\xi}$ of $O_\xi$. In particular, if for any $k = 1, \ldots, r$,

$$s_k > p_k(1, \ldots, 1)/2 = \frac{1}{2} \sum_{i < k} \dim \mathcal{V}_{ki},$$

then $\xi \in \Xi(1, \ldots, 1)$ and $R_{\xi}$ is a regular measure on the cone $\Omega_V = \rho(H_V)I_N$.

Now we show the ‘if’ part of the statement. Assume that $\xi \in \Xi \cap \mathbb{R}_{>0}$. In view of \cite[11], we see that there exists a positive integer $m$ such that $m\xi \in \Xi(1, \ldots, 1)$. Then $R_{m\xi}$ is a regular measure, while $R_{m\xi}$ equals the convolution measure $R_{\xi} \ast R_{\xi} \ast \cdots \ast R_{\xi}$ ($m$-times). It follows that the support of $R_{\xi}$ is not concentrated on any affine hyperplane in $Z_V$.

Next we show the ‘only if’ part. It suffices to show that, if $\xi \in \Xi(\xi)$ with $s_k = 0$ for some $k$, then $\supp R_{\xi} = \overline{O_\xi}$ is contained in the subspace $(\mathbb{R}E_k)^\perp := \{x \in Z_V : x_{kk} = 0\}$ of $Z_V$. Recalling the definition of $\Xi(\xi)$, we see that

$$\varepsilon_k = 0 \quad \text{and} \quad p_k(\xi) = \sum_{i < k} \varepsilon_i \dim \mathcal{V}_{ki} = 0,$$

and the latter equality implies

$$\mathcal{V}_{ki} = 0 \text{ if } \varepsilon_i = 1.$$

Therefore, for any $x = \rho(T)E_\xi = TE_\xi T^T \in \mathcal{O}_\xi$ with $T \in H_V$, we have

$$x_{kk} = \varepsilon_k(t_{kk})^2 + \sum_{i < k} \varepsilon_i \|T_{ki}\|^2 = 0,$$

which means that $\mathcal{O}_\xi \subset (\mathbb{R}E_k)^\perp$. Hence $\supp R_{\xi} \subset (\mathbb{R}E_k)^\perp$ and the proof is completed. \hfill \Box

For $\xi \in \{-1,1\}^r$, consider $\mathcal{O}_\xi^* := \rho^*(H_V)E_\xi$. The set

$$\bigcup_{\xi \in \{-1,1\}^r} \mathcal{O}_\xi^*$$

is dense in $Z_V$ and $\mathcal{O}_\xi^*$ are the only open orbits of $\rho^*(H_V)$ (see \cite[17]{8}).

3.2. Characterization of the Riesz measure on homogeneous cone. In the following section we will give an application of Theorem 2.1 to a characterization of the Riesz measure on homogeneous cone. We generalize the results of \cite{12}, where the characterization of Riesz measure through invariance property of NEF on simple Euclidean algebras was considered.

We say that the subalgebra $Z$ of $(\text{Sym}(N, \mathbb{R}), \triangle)$ is irreducible if $Z$ is not equal to a direct sum of two non-trivial ideals.
Theorem 3.5 Let $\Xi = Z_V$ be an irreducible subalgebra of $(\text{Sym}(N, \mathbb{R}), \triangledown)$ that admits a normal decomposition and let $F(\mu)$ be a NEF invariant by $G = \rho(H_V)$. Then there exists $\theta_0 \in Z_V$, $a_0 \in \mathbb{R}$ and $\xi \in \Xi \cap \mathbb{R}^\geq_0$ such that

$$\mu(dx) = e^{a_0 - \langle \theta_0, x \rangle} R_\xi(dx)$$

or

$$\mu(dx) = e^{a_0 - \langle \theta_0, x \rangle} R_\xi(-dx)$$

Proof. Theorem 2.4 implies that there exists $\theta_0$ such that

$$\mu(dx) = e^{-\langle \theta_0, x \rangle} \mu_0(dx),$$

where

$$L_{\mu_0}(\rho^*(T)\theta) = \chi(T)L_{\mu_0}(\theta), \quad (\theta, T) \in \Theta(\mu_0) \times H_V$$

with a certain character $\chi$ of $H_V$. We will determine $L_{\mu_0}$ and $\Theta(\mu_0)$. The proof is split into four steps:

(i) There exists $\xi \in \{-1, 1\}^r$ such that $E_\xi \in \Theta(\mu_0)$. Moreover, if $E_\xi, E_{\xi'} \in \Theta(\mu_0)$, then $\xi = \xi'$.

(ii) $\Theta(\mu_0) = O^*_\xi$ for some $\xi \in \{-1, 1\}^r$.

(iii) If $\Theta(\mu_0) = O^*_\xi$ then $\xi = (-1, \ldots, -1)$ or $\xi = (1, \ldots, 1)$.

(iiii) $\mu$ is of the postulated form.

First step. By definition, the set $\Theta(\mu_0)$ is open and non-empty. Since $\rho^*(T)\Theta(\mu_0) = \Theta(\mu_0)$ for any $T \in H_V$ and the set $\Theta(\mu_0)$ is dense in $Z_V$, we have $E_\xi \in O^*_\xi \subset \Theta(\mu_0)$ for some $\xi \in \{-1, 1\}^r$.

Assume now that $E_{\xi}, E_{\xi'} \in \Theta(\mu_0)$ and $\xi \neq \xi'$. Since $\Theta(\mu_0)$ is convex, we know that $\sigma := \frac{1}{2}(E_{\xi} + E_{\xi'}) \in \Theta(\mu_0)$. Define $I_0 := \{i \in \{1, \ldots, r\}: \frac{\xi_i + \xi'_i}{2} = 0\}$. The set $I_0$ is not empty. Define $H_0 := \{T \in H_V: t_{ii} = 1 \text{ for any } i \notin I_0\}$

If $T \in H_0$ is diagonal, then

$$\rho^*(T)\sigma = \sigma.$$  

On the other hand, by (3.3) we obtain

$$L_{\mu_0}(\rho^*(T)\sigma) = \chi(T)L_{\mu_0}(\sigma),$$

which implies that $\chi(T) = 1$ for any $T \in H_0$ (note that $\chi(T)$ depends on $T$ only through diagonal elements). Further, this implies that for any $T \in H_0$,

$$L_{\mu_0}(\rho^*(T)E_\xi) = L_{\mu_0}(E_\xi),$$

but this contradicts the assumption that $\mu_0$ is not supported on any affine hyperplane of $Z_V$. Indeed, in such case the support of $\mu_0$ is contained in the subspace of $Z_V$ whose $(i, i)$-components are zero for any $i \in I_0$.

Second step. Since $\Theta(\mu_0)$ contains only one open orbit, we have

$$\text{int}\{\Theta(\mu_0) \setminus O^*_\xi\} = \emptyset.$$  

This implies that

$$\Theta(\mu_0) \setminus O^*_\xi = \emptyset,$$

thus

$$O^*_\xi \subset \Theta(\mu_0) \subset \overline{O^*_\xi}$$

which proves the claim, since $\Theta(\mu_0)$ is open.

Third step. We will show that $O^*_\xi$ is not convex unless all $\xi_k$, $1 \leq k \leq r$, are simultaneously 1 or −1. Define

$$I^+(\xi) := \{i \in \{1, \ldots, r\}: \xi_i = 1\} \text{ and } I^-(\xi) := \{1, \ldots, r\} \setminus I^+(\xi)$$

and
Suppose that $I^+(\mathbb{Z})$ and $I^-(\mathbb{Z})$ are non-empty. Then there exists $k \in I^+(\mathbb{Z})$ and $l \in I^-(\mathbb{Z})$ such that the space $\mathcal{V}_k$ is not empty (without loss of generality we may assume that $l > k$). If not, $\mathcal{Z}_\mathcal{V}$ has the block form
\[
w \begin{pmatrix} Z_+ & 0 \\ 0 & Z_- \end{pmatrix} w^T,
\]
for some permutation matrix $w$. This contradicts the assumption that $\mathcal{Z}_\mathcal{V}$ is irreducible.

Thus there exists $\mathcal{V}_k \neq \emptyset$. We have $\varepsilon_k = 1$ and $\varepsilon_l = -1$. For $v \in \mathcal{V}_k$ let $T(v)$ be the element of $H_\mathcal{V}$ such that $T_{ik} = v$, $t_{ii} = 1$ for $i = 1, \ldots, r$ and $T_{ji} = 0$ for all $(i, j) \neq (k, l)$, $1 \leq i < j \leq r$. Take any $v \in \mathcal{V}_k$ with $(v|v) = 2$. Then
\[
1/2 (\rho^*(T(v)) E_+^l + \rho^*(T(-v)) E_-^l) = E_+^l,
\]
where $\varepsilon_i' = \varepsilon_i$ for $i \neq k$ and $\varepsilon_k' = -1$. To show (3.4) we could switch to a matrix realization of the dual cone $\Omega^*_\mathcal{V}$, where (under suitable linear isomorphism) $\rho^*(T)$ is just a multiplication by some upper triangular matrix on the left and its transpose on the right. We will however use only the definition of $\rho^*(T)$. For any $x \in \mathcal{Z}_\mathcal{V}$ consider the matrix
\[
x_v := 1/2 (\rho(T(v)) x + \rho(T(-v)) x) = 1/2 (T(v)xT(v)^\top + T(-v)xT(-v)^\top).
\]
It may be verified by direct calculation that matrices $x$ and $x_v$ differ only on their $(l, l)$-components, which in the latter case equals
\[
(vv^\top x_{kk} + x_{ll}) I_{n_l} = (2x_{kk} + x_{ll}) I_{n_l}.
\]
Thus,
\[
\left\langle x, 1/2 (\rho^*(T(v)) E_+^l + \rho^*(T(-v)) E_-^l) \right\rangle = \left\langle 1/2 (\rho(T(v)) x + \rho(T(-v)) x), E_+^l \right\rangle
\]
\[
= \sum_{i \notin \{k, l\}} x_{ii} \varepsilon_i + x_{kk} + (2x_{kk} + x_{ll})(-1) = \left\langle x, E_+^l \right\rangle.
\]
But $\Theta(\mu_0)$ is convex, thus $E_+^l \in \Theta(\mu_0)$. This contradicts the point (i), that $\Theta(\mu_0)$ contains only one open orbit of $\rho^*(H_\mathcal{V})$. This means that $-I_N$ or $I_N$ belongs to $\Theta(\mu_0)$ and finally,
\[
\Theta(\mu_0) = -\Omega^*_\mathcal{V} \quad \text{or} \quad \Theta(\mu_0) = \Omega^*_\mathcal{V}.
\]

**Fourth step.** Let us first consider the case $\Theta(\mu_0) = -\Omega^*_\mathcal{V}$. Putting $\theta = \rho^*(S)(-I_N)$ for $S \in H_\mathcal{V}$, we obtain
\[
L_{\mu_0}(-\rho^*(ST)I_N) = \chi(T)L_{\mu_0}(-\rho^*(S)I_N), \quad (S, T) \in H_\mathcal{V}^2,
\]
which implies that for $a_0 = \log L_{\mu_0}(-I_N)$,
\[
L_{\mu_0}(-\rho^*(T)I_N) = e^{a_0}\chi(T)
\]
and $\chi(T)$ is a one-dimensional representation of $H_\mathcal{V}$. Thus there exists $\underline{\chi} \in \Xi$ such that $\chi = \chi_{-\underline{\chi}}$ and then
\[
L_{\mu_0}(\theta) = e^{a_0}\Delta^*_\underline{\chi}(\theta), \quad \theta \in \Omega^*_\mathcal{V}.
\]
By Theorems 3.3 and 3.4 we see that $\underline{\chi} \in \Xi \cap \mathbb{R}_{<0}$. This means that $\mu_0(dx) = e^{a_0}R_{-\underline{\chi}}(dx)$ and $\Theta(\mu_0) = -\Omega^*_\mathcal{V}$.

In the second case, when $\Theta(\mu_0) = \Omega^*_\mathcal{V}$, one can show that $\mu_0(dx) = e^{a_0}R_{\underline{\chi}}(-dx)$. \qed
3.3. Comments.

(1) In [13], the authors considered characterization of the Riesz measure $\mathcal{R}_x$ through the invariance property of NEF on a simple Euclidean algebra $\mathbb{E}$ by some subgroup $G$ of $GL(\mathbb{E})$. This subgroup was carefully chosen in order to ensure that some components of vector $x \in \Xi$ are equal. Taking ordinary triangular group $\rho(H_V)$ imposes no additional conditions on these components. On the other hand if one considers the invariance property of NEF by the connected component containing identity of $G(\Omega)$ (this is in principle what Letac did on symmetric matrices in [23], but it is true on homogeneous cones also), then all $s_i$ has to be equal, $s_i = p$ for $1 \leq i \leq r$. Then $\Delta_x^* = \det \rho$ and $p$ belongs to the set $\Lambda$ called the Jorgensen set (see [6]).

(2) Elements of $F(\mathcal{R}_x)$ are actually the Wishart distributions on homogeneous cones introduced in [1] (the subcase of $s_i = p$ for $1 \leq i \leq r$) and in [10].

(3) It should be stressed that our approach is very different than that of [12, 23], where the characterization of NEF was proved by showing that the variance function of $\rho(H_V)$-invariant NEF coincides with the one of $F(\mathcal{R}_x)$ for some $x \in \Xi$. Here we didn’t even need to know what is the variance function of the Riesz measure on homogeneous cones. We perceive our approach as less technical and more natural.

(4) The formula for variance function of $F(\mathcal{R}_x)$ on homogeneous cones is the topic of our joint paper with Piotr Graczyk [11].

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