Causal Structure of Vacuum Solutions to Conformal(Weyl) Gravity

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Abstract

Using Penrose diagrams the causal structure of the static spherically symmetric vacuum solution to conformal (Weyl) gravity is investigated. A striking aspect of the solution is an unexpected physical singularity at $r = 0$ caused by a linear term in the metric. We explain how to calculate the deflection of light in coordinates where the metric is manifestly conformal to flat i.e. in coordinates where light moves in straight lines.

KEY WORDS: Conformal gravity; Penrose diagrams

1 INTRODUCTION

Conformal gravity is a higher derivative metric theory of gravity whose action is given by the Weyl tensor squared,

$$I = \alpha \int \sqrt{-g} C_{\lambda\mu\sigma\tau} C^{\lambda\mu\sigma\tau} d^4x$$ (1)

where $\alpha$ is a dimensionless constant. It is the simplest action that can be constructed which is conformally invariant i.e. invariant under the conformal transformation $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$ where $\Omega^2(x)$ is a finite, non-vanishing, continuous real function. It therefore encompasses the largest symmetry

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group which keep the light cones invariant i.e. the 15 parameter conformal group (which includes the 10 parameter Poincaré group). Many important results have been obtained for conformal gravity. It has been shown that Birkhoff’s theorem is valid for conformal gravity \[1\]. The linearized equations about flat space-time have also been obtained \[2\]. An important result, called the zero-energy theorem, was obtained for conformal gravity \[3\]. This theorem states that for the special case of an asymptotically flat space-time the total energy is zero. However, we will see in section 2, that far from a localized source the metric for conformal gravity is conformal to flat and not flat. Hence, the zero-energy theorem only applies to cases where asymptotically flat space-time is imposed as a boundary condition. Interest in conformal gravity was rekindled in the early 90’s after the metric exterior to a static spherically symmetric source was obtained \[4\]. For a metric in the standard form

\[d\tau^2 = B(r) \, dt^2 - A(r) \, dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)\]  

(2)

the static spherically symmetric vacuum solutions are \[1, 4\]

\[B(r) = A^{-1}(r) = 1 - \frac{(2 - 3\gamma\beta)\beta}{r} - 3\beta\gamma + \gamma r - kr^2\]  

(3)

where \(\beta, \gamma\) and \(k\) are constants. The above solution is only valid up to a conformal factor. Constraints from phenomenology imply that \(\gamma, k\) and \(\gamma\beta << 1\) (see \[1, 5, 6\]). The constant \(\gamma\beta\) is usually negligible but for the purposes of the next section we include it here. In subsequent sections it will be dropped.

Mannheim and Kazanas used the above metric solution to fit galactic rotation curves without recourse to dark matter i.e. they wanted to verify whether the linear \(\gamma r\) term could replace dark matter in explaining the rotation curves. They had some success in these fittings but the deflection of light which was later calculated \[4, 7\] was incompatible with the fitting of galactic rotation curves. It was shown \[3\] that at large distances, non-relativistic massive particles and light behaved in opposite ways i.e. if the former was attracted to the source the latter would be repelled and vice versa. Hence, the theory could not simultaneously explain galactic rotation curves and the observed deflection of light in galaxies. In one important respect, the calculation of the deflection of light in conformal gravity is less ambiguous than the calculation of galactic rotation curves. The fitting of galactic rotation
curves requires one to fix the conformal factor because massive geodesics are not conformally invariant. The conformal factor is chosen to fit experiments but there is no theoretical justification for choosing one conformal factor over another. In contrast, there is no need to specify any conformal factor for null geodesics because they are conformally invariant. It follows that causal analysis and the calculation of the deflection of light can be carried out without specifying any conformal factor. It is therefore worthwhile to investigate the causal structure of the metric (2) with solution (3). We find the coordinate transformations that render the metric in a form which is manifestly conformal to flat. The causal structure is then analysed using Penrose diagrams and we identify which space-times allow a calculation of the scattering of light i.e. which space-times allow light to approach the source from infinity. The trajectories and deflection of light for these space-times is then calculated. We begin by investigating some of the geometrical properties of solution (3).

2 CURVATURE AND RELATED TENSORS

Curvature scalars are invariant under coordinate transformations and therefore are useful for detecting physical singularities. In contrast, the metric (which is coordinate dependent) may have a coordinate singularity which is not a physical singularity (the classic example is the coordinate singularity at the Schwarzschild radius \( r = 2m \) which is not a physical singularity but a horizon). The metric under study is

\[
 ds^2 = (1 - 3\gamma\beta - \beta(2 - 3\gamma\beta)/r + \gamma r - kr^2)dt^2 \\
 - \frac{dr^2}{1 - 3\gamma\beta - \beta(2 - 3\gamma\beta)/r + \gamma r - kr^2} - r^2 d\Omega^2. \tag{4}
\]

The curvature scalar, \( R \equiv R_{\mu\nu}g^{\mu\nu} \), for the general metric (2) with \( B(r) = A^{-1}(r) \) can readily be calculated and yields

\[
 R = B'' + 4B'/r + 2B/r^2 - 2/r^2
\]

where a prime denotes differentiation with respect to \( r \). The curvature scalar for the metric (4) is equal to

\[
 R = 6\gamma/r - 6\gamma\beta/r^2 - 12k. \tag{5}
\]

The first two terms in the curvature scalar are singular at \( r = 0 \) and therefore the space-time described by the metric (4) has a physical singularity at \( r = \)
0. Since the curvature scalar is a linear inhomogeneous function of $B(r)$ it follows that each term that appears in (5) can be traced back to a term in $B(r)$. Therefore the two singular terms are due to the $\gamma r$ and the constant $3\beta\gamma$ term in the metric respectively. The singularity due to the constant $3\beta\gamma$ term is a conical singularity. This can be shown by considering the metric (4) with only the constant term present i.e.

$$ds^2 = (1 - 3\gamma\beta)dt^2 - \frac{1}{(1 - 3\gamma\beta)}dr^2 - r^2(d\theta^2 - \sin^2\theta d\varphi^2).$$

This metric exhibits a conical singularity, the ratio of the area of a sphere at coordinate radius $r$ to the proper radius squared $r^2/(1 - 3\gamma\beta)$ is the constant $4\pi(1 - 3\gamma\beta) \neq 4\pi$. Correspondingly, the deflection of light is given by the angular defect in the scattering two plane, $3\pi\gamma\beta$ in the limit $\gamma\beta \ll 1$.

The singularity due to the $\gamma r$ term can be analyzed by studying the metric

$$ds^2 = (1 + \gamma r)dt^2 - \frac{1}{(1 + \gamma r)}dr^2 - r^2d\Omega^2$$

obtained by setting $\beta = k = 0$ in the metric (3). It is not apparent that the space-time described by the above metric has a singularity at $r = 0$. In fact, at first glance, it seems that the metric approaches Minkowski space-time as $r$ approaches zero! A singularity at $r = 0$ is made apparent by rewriting the metric (7) for small $r$ i.e. $\gamma r \ll 1$. The metric (7) then takes the form

$$ds^2 = (dt^2 - dr^2 - r^2d\Omega^2) + \gamma r(dt^2 + dr^2)$$

$$= (dt^2 - \sum_{i=1}^{3} dx_i^2) + \gamma \left( \sum_{i=1}^{3} \frac{x_i^2dx_i^2}{r} + \sum_{i<j} \frac{2x_i x_j}{r}dx_idx_j + rdt^2 \right)$$

$$= (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu. \quad (8)$$

The metric has therefore been decomposed into Minkowski space-time plus an additional small term. Although neither term is singular, the derivatives of $h_{\mu\nu}$ are singular at $r = 0$. The connection and the Riemann tensor are constructed out of these derivatives and the inverse metric $g^{\mu\nu}$ (which is not singular) subsequently giving rise to physical singularities at $r = 0$ in the space-time.

Besides the physical singularity at $r = 0$, the curvature scalar reveals another interesting feature of the space-time. As $r$ tends to infinity, the curvature scalar does not vanish but approaches the value $-12k$. The original
metric \((4)\) therefore describes a space-time where the region far from the source i.e. the background, is not flat but of constant four-curvature. It will later be shown that this constant four-curvature background is actually conformal to flat.

We now exhibit the Riemann tensor, \(R_{\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\tau}}\), in an orthonormal basis for the metric \((4)\). Its non-vanishing components are

\[
R_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{\beta(2 - 3\gamma\beta)}{2r^3} + \frac{\gamma}{2r} - k, \quad R_{\hat{t}\hat{t}\hat{t}\hat{t}} = \frac{\beta(2 - 3\gamma\beta)}{r^3} + k,
\]

where other non-vanishing components related to the above by symmetry are not shown. Clearly, the Riemann tensor and the scalars constructed from it, for example the Riemann tensor squared, diverge at \(r = 0\). We see that terms in the metric containing either \(\beta\), \(\gamma\) or both contribute a physical singularity at \(r = 0\) and represent point-like sources. The \(kr^2\) term contributes a constant \(\pm k\) to the Riemann tensor and therefore the components of the Riemann tensor do not vanish as \(r\) tends to infinity i.e. the space-time is not flat at infinity. The curvature scalar and Riemann tensor have revealed that the metric \((4)\) represents point sources localized at \(r = 0\) which are embedded in a constant four-curvature background.

Let us now compute the Weyl tensor \(C_{\hat{\mu}\hat{\nu}\hat{\sigma}\hat{\tau}}\) for the metric \((4)\). This tensor is useful because the requirement that a space-time be conformal to flat is that the components of the Weyl tensor vanish. The components of the Weyl tensor in an orthonormal basis are

\[
C_{\hat{r}\hat{\theta}\hat{r}\hat{\theta}} = C_{\hat{r}\hat{\phi}\hat{r}\hat{\phi}} = \frac{\beta(2 - 3\gamma\beta)}{2r^3} + \frac{\gamma\beta}{2r^2},
\]

\[
C_{\hat{r}\hat{t}\hat{r}\hat{t}} = \frac{\beta(2 - 3\gamma\beta)}{r^3} + \frac{\gamma\beta}{r^2} - \frac{\gamma\beta}{2r},
\]

\[
C_{\hat{t}\hat{t}\hat{r}\hat{t}} = C_{\hat{t}\hat{t}\hat{\phi}\hat{\phi}} = -\frac{\beta(2 - 3\gamma\beta)}{2r^3} - \frac{\gamma\beta}{2r^2}.
\]

where components related to the above by symmetry are not shown. The Weyl tensor is zero when \(\beta = 0\) or when \(r\) approaches infinity. Under these
conditions the original metric (4) reduces to
\[ ds^2 = (1 + \gamma r - kr^2)dt^2 - 1/(1 + \gamma r - kr^2)dr^2 - r^2 d\Omega^2. \] (11)
The above metric is conformal to flat and it describes the \( \beta = 0 \) or very large \( r \) limit of the original metric (4). We will analyze the conformally flat metric (11) in detail in the next sections, to understand the causal structure of the original metric at very large radii and in particular to verify whether light has scattering trajectories.

3 COORDINATE TRANSFORMATIONS

In the original \( r,t \) coordinates, the components of the metric (11) change sign at the roots of the polynomial \( 1 + \gamma r - kr^2 = 0 \). These coordinates are therefore not the most convenient to analyze the causal structure. Our task in this section will be to rewrite the conformally flat metric (11) in coordinates where the conformal flatness is manifest i.e. in a form where the metric is a conformal factor times the Minkowski metric. The effort spent in obtaining the new coordinates is rewarded by having the metric in a form that has the same causal structure as that of Minkowski space-time i.e. null geodesics do not depend on the conformal factor and therefore the light cones are drawn at 45° to the horizontal axis as in Minkowski space-time. There are constraints on the new coordinates when one transforms from the old to the new coordinates. Therefore, the causal structure of the conformally flat space-time is analyzed in the new coordinates as a patch in Minkowski space-time.

We now perform the coordinate transformation from the \( r,t \) coordinates to a new set of coordinates \( \rho, \tau \) where the metric (11) is written in a form which is manifestly conformal to flat. We write
\[ ds^2 = (1 + \gamma r - kr^2)dt^2 - 1/(1 + \gamma r - kr^2)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]
\[ = \Omega^2(\rho, \tau) [d\tau^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2 \theta d\phi^2)] \] (12)
where \( \tau \) and \( \rho \) are the new coordinates and \( \Omega(\rho, \tau) \) is the conformal factor. We therefore have the following two relations:
\[ r = \rho \Omega \] (13)
\[ \Omega^2(d\tau^2 - dp^2) = (1 + \gamma r - kr^2)dt^2 - 1/(1 + \gamma r - kr^2)dr^2. \]  

(14)

The coordinates \( r \) and \( t \) are now functions of both \( \rho \) and \( \tau \) so that \( dr = r'd\rho + \dot{r}d\tau \) and \( dt = t'd\rho + \dot{t}d\tau \) where a prime and dot on \( r \) and \( t \) represent partial derivatives with respect to \( \rho \) and \( \tau \) respectively. Equations (13) and (14) lead to the following three partial differential equations

\[ (1 + \gamma r - kr^2)t'\dot{t} - \frac{r'\dot{r}}{1 + \gamma r - kr^2} = 0 \]  

(15)

\[ (1 + \gamma r - kr^2)t^2 - \frac{r^2}{1 + \gamma r - kr^2} = \frac{r^2}{\rho^2} \]  

(16)

\[ (1 + \gamma r - kr^2)t'^2 - \frac{r'^2}{1 + \gamma r - kr^2} = -\frac{r^2}{\rho^2}. \]  

(17)

We can eliminate \( t \) from the above three equations to obtain two partial differential equations for \( r \):

\[ \frac{r^2}{\rho^2(\dot{r} + r')} = f(\tau - \rho) \]  

(18)

\[ r'^2 - r^2 = \frac{r^2(1 + \gamma r - kr^2)}{\rho^2} \]  

(19)

where \( f(\tau - \rho) \) is an arbitrary function of \( \tau - \rho \). To solve the above two equations for \( r \), it is convenient to introduce two new coordinates \( u \) and \( v \) related to \( \tau \) and \( \rho \) by

\[ u = \tau - \rho ; \quad v = \tau + \rho. \]  

(20)

In \( u, v \) coordinates (18) reduces to

\[ \frac{2r^2}{(v-u)^2 \partial r/\partial v} = f(u). \]  

(21)

The solution to the above equation is

\[ r = \frac{f(u)(v-u)}{2 + h(u)(v-u)} \]  

(22)

where \( h(u) \) is an arbitrary function of \( u \). Substituting (21) into (19) one obtains

\[ -2 \int \frac{dr}{1 + \gamma r - kr^2} = \int f(u)du = g(u) + p(v) \]  

(23)
where \(dg(u)/du = f(u)\) and \(p(v)\) is an arbitrary function of \(v\). The solution to the above equation depends on whether the polynomial \(1 + \gamma r - kr^2\) has roots or not. If the polynomial has roots the integral of \(1/(1 + \gamma r - kr^2)\) is given by

\[
\frac{-1}{k(r_+ - r_-)} \ln \left| \frac{r - r_+}{r - r_-} \right| ; \quad k > -\frac{\gamma^2}{4}
\]  

(24)

where the two roots \(r_+\) and \(r_-\), which can have negative values, are given by

\[
r_{\pm} = \frac{\gamma}{2k} \pm \sqrt{\frac{\gamma^2}{4k^2} + \frac{1}{k}}.
\]

(25)

If the polynomial has no roots the integral of \(1/(1 + \gamma r - kr^2)\) is given by

\[
\left(-k - \frac{\gamma^2}{4}\right)^{-1/2} \arctan \left(\frac{-kr + \gamma/2}{\sqrt{-k - \gamma^2/4}}\right) ; \quad k < -\frac{\gamma^2}{4}.
\]

(26)

We now solve (23) separately for each of the two cases i.e. case 1: polynomial has roots and case 2: polynomial has no roots.

**Case 1: Roots at \(r_{\pm}\)**

Substituting (24) for the integral in (23) one obtains

\[
r = \frac{r_+ \pm r_- e^{k(r_+ - r_-)(g(u) + p(v))/2}}{1 \pm e^{k(r_+ - r_-)(g(u) + p(v))/2}}
\]

(27)

where the negative sign corresponds to the region where \(\infty > r > r_+\) and \(0 < r < r_-\) whereas the plus sign corresponds to the region \(b\) where \(r_+ > r > r_-\). We now equate \(r\) in (23) to \(r\) in (27). Note that \(f(u)\) in (22) is \(g'(u) \equiv dg(u)/du\). One obtains the following equality

\[
g'(u) e^{-k(r_+ - r_-)(g(u) + p(v))/2} \frac{r_+ e^{-k(r_+ - r_-)(g(u) + p(v))/2}}{r_+ e^{-k(r_+ - r_-)(g(u) + p(v))/2} \pm r_- e^{k(r_+ - r_-)(g(u) + p(v))/2}} = \frac{2}{v - u} + h(u).
\]

(28)

After integrating the above equation and performing some algebraic manipulations we obtain

\[
\ln \left( r_+ e^{-k(r_+ - r_-)(g(u) + p(v))/4} \pm r_- e^{k(r_+ - r_-)(g(u) + p(v))/4} \right) = \ln(v - u) + S(u) + T(v)
\]

(29)
where \( S(u) \) (related to \( h(u) \)) and \( T(v) \) are arbitrary functions of \( u \) and \( v \) respectively. After exponentiating both sides (29) reduces to

\[
 r_+ e^{-k(r_+-r_-)p(v)/2} \pm r_- e^{k(r_+-r_-)g(u)/2} = (v - u)A(u)B(v). \tag{30}
\]

The functions \( A(u), B(v), g(u), p(v) \) are arbitrary functions of \( u \) and \( v \) and we can therefore write the above equation as

\[
 (v - u)A(u)B(v) = N(v) + M(u) \tag{31}
\]

where all the functions above are arbitrary functions of \( u \) and \( v \). The coordinate \( r \), given by (27), can be expressed in terms of the functions \( M(u) \) and \( N(v) \) i.e.

\[
 r = \frac{r_+ r_- (M(u) + N(v))}{r_- N(v) + r_+ M(u)} \tag{32}
\]

where the above is valid for the entire region \( \infty > r > 0 \). Fortunately, equation (31) can be solved algebraically. We arrive also at (31) in case 2 and therefore postpone finding its solution until case 2 is completed.

**Case 2: Polynomial has no roots**

We proceed in a fashion similar to case 1. Substituting (23) for the integral in (22) one obtains

\[
 r = \frac{1}{kc} \tan \left( \frac{g(u) + p(v)}{2c} \right) + \frac{\gamma}{2k} \tag{33}
\]

where \( c = -1/\sqrt{-k - \gamma^2/4} \). We now equate \( r \) in (22) to \( r \) in (33). After integration one obtains

\[
 \ln \left[ \cos \left( \frac{(g(u) + p(v))}{2c} \right) \frac{\gamma c}{2} + \sin \left( \frac{(g(u) + p(v))}{2c} \right) \right] = \ln(v-u)+S(u)+T(v) \tag{34}
\]

where \( S(u) \) and \( T(v) \) are arbitrary functions. After algebraic manipulations one obtains

\[
 \tan(g(u)/2c) + \frac{\sin(p(v)/2c) + \cos(p(v)/2c) \gamma c/2}{\sin(p(v)/2c) \gamma c/2 + \cos(p(v)/2c)} = (v - u)A(u)B(v). \tag{35}
\]

We therefore obtain the same equation as in case 1 i.e.

\[
 (v - u)A(u)B(v) = N(v) + M(u). \]
In terms of the functions $M(u)$ and $N(v)$, the coordinate $r$, given by (33) is

$$
r = \frac{-c(M(u) + N(v))}{1 + (M(u) + N(v))\gamma c/2 - M(u)N(v)}. 
$$

(36)

Though we arrive at the same equation (31), the coordinate $r$ in case 1 and case 2 are obviously not the same.

We now solve (31) and discuss its physical significance. The right hand side of the equation does not contain any mixed terms of $u$ and $v$ and therefore the mixed terms on the left hand side must vanish. We write $A(u)$ as

$$
A(u) = A_0 + a(u') 
$$

(37)

where $A_0 = A(u_0)$ is a constant and $a(u')$ is a function of $u' \equiv u - u_0$ which vanishes at $u' = 0$. Similarly

$$
B(v) = B_0 + b(v'). 
$$

(38)

With $A(u)$ and $B(v)$ given above, the left hand side of (31) yields

$$
(v' - u' + C_0)(A_0B_0 + A_0b(v') + B_0a(u') + a(u')b(v'))
$$

(39)

where $C_0 = v_0 - u_0$ is a constant. The mixed terms must vanish and we obtain the following equation

$$
v'a(u')B_0 + v'a(u')b(v') - u'b(v')A_0 - u'a(u')b(v') + C_0a(u')b(v') = 0. 
$$

(40)

The solution to (40) is obtained by separating the variables i.e.

$$
b(v') = \frac{-v'B_0}{C_0 + v' - u'(1 + A_0/a(u'))}. 
$$

(41)

The function $b(v')$ is a function of $v'$ only and therefore the term $u' (1 + A_0/a(u'))$ must be a constant (call it $D$). We therefore obtain the following solutions

$$
a(u') = \frac{-u'A_0}{D + u'}; \quad b(v') = \frac{-v'B_0}{C_0 + D + v'}. 
$$

(42)

The solution to $(v - u)A(u)B(v) = M(u) + N(v)$ is therefore

$$
A(u) = \frac{A}{B + u}, \quad B(v) = \frac{C}{B + v}, \quad M(u) = \frac{-AC u}{B(B + u)}, \quad N(v) = \frac{AC v}{B(B + v)}. 
$$

(43)
where the solution (42) was substituted into equations (37) and (38) and the quantities $A, B$ and $C$ are constants related to the constants $A_0, B_0, C_0$ and $D$. With the above solution we can finally obtain the coordinate $r$. For case 1, $r$ is given by (32) and yields

$$r = \frac{r_+ r_-(v - u)}{vr_-(1 + u/B) - ur_+(1 + v/B)}.$$  \hfill (44)

For case 2, $r$ is given by (36) and yields

$$r = \frac{-cAC(v - u)}{(B + u)(B + v) - \gamma cAC(v - u)/2 + (AC/B)^2 uv}. \hfill (45)$$

It is worth mentioning that (31) is invariant under the following coordinate transformation

$$u \rightarrow Au+B, \quad v \rightarrow Av$$

where $A$ and $B$ are arbitrary constants. This is because these transformations form a subgroup of the special conformal transformations i.e. that leave the equation defining null surfaces

$$ds^2 = d\tau^2 - d\rho^2 - \rho^2 d\Omega^2 = dudv - \left(\frac{v - u}{2}\right)^2 d\Omega^2 = 0 \hfill (47)$$

invariant (where $u$ and $v$ are related to $\tau$ and $\rho$ via (21)). The transformations form only a subgroup of the 15 parameter conformal group because two coordinates are not involved in the transformation. We see that equations (31) and (47) share the same symmetries. There are two more transformations that leave equations (31) and (47) invariant. These are

space inversion: $u \rightarrow v, \quad v \rightarrow u$ \ i.e. \ $\rho \rightarrow -\rho$  \hfill (48)

time reversal: $u \rightarrow -v, \quad v \rightarrow -u$ \ i.e. \ $\tau \rightarrow -\tau$. \hfill (49)

Of course, these can be combined with transformations (46).

4 PENROSE DIAGRAMS

The causal structure of the conformally flat metric (11) will now be analyzed for different choices of $\gamma$ and $k$ in the “conformally flat coordinates” $u$ and $v$
(or $\tau$ and $\rho$). The possible choices of $\gamma$ and $k$ are the following

\begin{align}
    a) & \quad k > 0 : r_+ > 0, r_- < 0 \\
    b) & \quad -\frac{\gamma^2}{4} < k < 0 \text{ and } \gamma < 0 : r_+ > 0, r_- > 0 \\
    c) & \quad -\frac{\gamma^2}{4} < k < 0 \text{ and } \gamma > 0 : r_+ < 0, r_- < 0 \\
    d) & \quad k < -\frac{\gamma^2}{4} : \text{ no roots}
\end{align}

Altogether there are four cases to consider and a Penrose diagram has been drawn for each showing the axes of both the $u, v$ and $\rho, \tau$ coordinates (see figure 1). The causal analysis proceeds as in Minkowski space-time except that only a patch of the $u, v$ (or $\rho, \tau$) coordinates are allowed. This is due to the condition that in the original $r, t$ coordinates the radius $r$ must be positive. In every diagram the singularity at $r = 0$ is shown in bold as a vertical dashed line occurring at $u = v$ or $\rho = 0$. We draw a circle at the point $u = v = 0$ in the first three diagrams to show that $r$ is indeterminate at that point i.e. the origin does not correspond to any one specific value of $r$ but depends on the limit with which one approaches it. If a line crosses the origin, the value of $r$ at the origin will depend on the slope of that line. For all the four diagrams, the line at $r = \infty$ is represented by a dashed line. The region where $r$ is positive and runs from the singularity at $r = 0$ to the dashed line at $r = \infty$ is shown by an arc (there is a second arc that is shown that represents an identical patch but with time running the opposite direction). The lines with arrows represent radial null geodesics i.e. the light cones. All the diagrams are drawn for the special case where the constant $B$ in (44) and (45) approaches infinity. Diagrams a), b) and c) represent the case with roots at $r_\pm$ and therefore the coordinate $r$ is given by (44). One obtains the following features for all three diagrams: $r_+$ is a $45^0$ line at $u = 0$ and $r_-$ is a $-45^0$ line at $v = 0$, lines of constant $r$ are simply straight lines that go through the origin and the radius $r$ approaches $r_+$ as $v \to \infty$ (this is not shown on the diagrams to avoid clutter). Diagram d) represents the case with no roots and therefore the coordinate $r$ is given by (45). Lines of constant $r$ are hyperbolas and do not go through the origin. Unlike the first three diagrams, $r$ is zero at the origin and therefore a physical singularity exists at the origin. We therefore do not draw a circle at the origin as with the other three diagrams.

We now investigate the causal structure of all four diagrams. In diagram a), the case of $k > 0$, $r_+$ is a horizon because light between $r = 0$ and $r = r_+$
Figure 1: Penrose diagrams for four different space-times: a) horizon at $r_+$ b) horizon at $r_-$ and $r_+$ c) roots $r_\pm$ are negative, no horizon d) no roots, no horizon
either ends at the singularity or at \( r_+ \) i.e. the \( r_+ \) at \( v \to \infty \) which is not shown on the diagram. Note that there exists no point from which light can reach infinity. Clearly, there are no scattering states for the space-time described by diagram a). In diagram b), both \( r_- \) and \( r_+ \) are positive and act as horizons. Light at a radius greater than \( r_- \) cannot cross the \( r_- \) line. Light between \( r_+ \) and infinity cannot cross the \( r_+ \) line and is trapped between these two values. No relevant scattering can therefore take place i.e. the radius of closest approach is greater than \( r_+ \) (which is a radius on cosmological scales since \( \gamma/k \) is of that magnitude by definition). In diagram c), there are no horizons i.e. both \( r_+ \) and \( r_- \) are in the negative \( r \) region and are outside the patch shown by the arc. In this space-time, light at infinity can reach any radius \( r_0 \) and return back to infinity (see fig.2b). Hence, scattering takes place in diagram c). In diagram d), the case with no roots, there are no horizons and again light at infinity can reach any radius and return back to infinity. Scattering therefore occurs in diagram d). Therefore, of the four possible space-times, only those described by diagram c) and d) have scattering.

5 TRAJECTORIES AND DEFLECTION OF LIGHT

We saw in the previous section that light has scattering states only for the space-times described by diagram c) and d) i.e. when \( 0 > k > -\gamma^2/4 \) with \( \gamma > 0 \) or \( k < -\gamma^2/4 \) respectively (note that no scattering states exist for a positive value of \( k \)). We can therefore calculate the deflection of light for the two cases above. The deflection of light has already been calculated for the original metric (4). The result obtained is \( \beta \)

\[
\frac{4\beta}{r_0} - \gamma r_0
\]

(51)

where \( r_0 \) is the point of closest approach. The calculation was done with the approximation that both terms in (51) are much smaller than one. At large \( r_0 \), however, the \( \gamma r_0 \) term is not small and the approximation is therefore no longer valid. It is therefore worthwhile to perform a separate calculation for the deflection of light at large \( r_0 \). If \( r_0 \) is large enough, we can neglect the \( \beta \) term and light will therefore be scattering in a conformally flat space-time.
Before calculating the deflection of light, it is worthwhile to understand how a deflection is possible in a conformally flat space-time.

In the “conformally flat” coordinates $\rho$ and $\tau$, light moves in a straight line as in Minkowski space-time. How can one then have scattering? In a scattering process light starts far away from the source, approaches the source, and then ends up far away from the source. “Far away” means that the sources no longer have any influence on the trajectory of the light. By looking at the curvature scalar and Riemann tensor we know that the sources no longer have influence as $r$ approaches infinity. However, if the coordinate $r$ approaches infinity this does not imply that the coordinate $\rho$ approaches infinity. As can be seen in all 4 diagrams in figure 1, there are finite values of $\rho$ and $\tau$ that correspond to $r = \infty$. Hence, when light moves from $r = \infty$ to $r = \infty$, it moves from one finite value of $\rho$ and $\tau$, say $\rho_1, \tau_1$, to another finite value of $\rho$ and $\tau$, say $\rho_2, \tau_2$. Hence, in $\rho, \tau$ coordinates light moves in a straight line but it does not cover the entire line i.e. it covers an angle $\delta$ less than $\pi$ (see figure 2a). The deflection angle is therefore equal to $\delta - \pi$. A straight line in polar coordinates $\rho$ and $\tau$ is described by the equation

$$\tau = \pm \sqrt{\rho^2 - \rho_0^2 + b}$$

(52)

where $\rho_0$ and $b$ are constants. For the space-time described by the Penrose diagram figure 1c, we use the straight line equation (52) to draw the path of light as it moves from $\rho_1, \tau_1$ ($r = \infty$), to $r_0$ and then to $\rho_2, \tau_2$ ($r = \infty$). We have therefore seen how light can deflect in coordinates where the space-time is manifestly conformal to flat.

We now calculate the deflection of light. The angle $\varphi$ as a function of $r$ for the metric (11) is given by (see 5)

$$\varphi(r) = \int \left[ 1 + \frac{\gamma r_0}{1 + \sin \theta} \right]^{-1/2} d\theta$$

(53)

where $\sin \theta = r_0/r$. We therefore obtain the condition that

$$1 + \frac{\gamma r_0}{1 + \sin \theta} \geq 0.$$  

(54)

The above condition is automatically satisfied if $\gamma$ is positive and this implies that for positive $\gamma$ light will reach infinity($\theta = 0$) for any value of $r_0$. If $\gamma$ is
negative, then condition (54) implies that
\[ r_0 \leq \frac{1 + \sin \theta}{|\gamma|} ; \quad \gamma \text{ negative}. \] (55)

With the above condition, light can reach infinity (\(\theta = 0\)) only if \(r_0 \leq 1/|\gamma|\). If \(r_0\) is in the range \(2/|\gamma| \geq r_0 > 1/|\gamma|\) then light moves in a closed orbit i.e. a bound state. Let us now calculate the integral (53). This yields
\[ \varphi(r) = \arcsin \left( \frac{r_0/r + \gamma r_0/2}{1 + \gamma r_0/2} \right). \] (56)

The deflection from infinity to \(r_0\) and back to infinity is
\[ \Delta \varphi = 2(\varphi(r_0) - \varphi(\infty)) - \pi \]
\[ = -2 \arcsin \left( \frac{\gamma r_0}{2 + \gamma r_0} \right). \] (57)

For small deflections (i.e. \(\gamma r_0 << 1\)) equation (57) reduces to \(-\gamma r_0\) in agreement with (51). The deflection is repulsive for a positive \(\gamma\) and attractive for a negative \(\gamma\). For positive \(\gamma\) the deflection ranges from 0 at \(r_0 = 0\) to \(-\pi\) at \(r_0 = \infty\) and for a negative \(\gamma\) it ranges from 0 at \(r_0 = 0\) to \(\pi\) at \(r_0 = 1/|\gamma|\).
(there are no scattering states for negative $\gamma$ when $r_0 > 1/|\gamma|$). Let us now obtain the shape of the orbits. One can obtain $r$ as a function of $\varphi$ from (56). This yields

$$r = \frac{-2/\gamma}{1 - \left(\frac{2 + \gamma r_0}{\gamma r_0}\right) \sin \varphi}. \quad (58)$$

This is of course the equation for a conic section in polar coordinates with eccentricity

$$e = \left|\frac{2 + \gamma r_0}{\gamma r_0}\right|. \quad (59)$$

The shapes are determined by the value of $e$. The orbits we obtain are

**positive $\gamma$:** hyperbola ($e > 1$) \hfill (60)

\[\begin{cases}
   r_0 < \frac{1}{|\gamma|}, & \text{hyperbola ($e > 1$)} \\
   r_0 = \frac{1}{|\gamma|}, & \text{parabola ($e = 1$)} \\
   \frac{2}{|\gamma|} > r_0 > \frac{1}{|\gamma|}, & \text{ellipse ($0 < e < 1$)} \\
   r_0 = 2/|\gamma|, & \text{circle ($e = 0$)}
\end{cases}\]

**negative $\gamma$:**

For a positive $\gamma$, the shapes of all orbits are hyperbolas and these describe scattering states. For a negative $\gamma$, the shapes of the orbits depend on the value of $r_0$ and bound states as well as scattering states can exist. The ellipse has a minimum value of $r$ which is $r_{\text{min}} = r_0$ and has a maximum value of $r_{\text{max}} = r_0/(|\gamma|r_0 - 1)$. The semi-latus rectum $L$, defined to be the point which occurs at an angle of $\pm \pi/2$ away from $r_{\text{min}}$ is equal to $2/|\gamma|$. All ellipses therefore have the same value for $L$ i.e. independent of $r_0$. The bound states therefore occur on cosmological scales where they can never be traced to any one particular source.
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