Open environments for quantum open systems

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The majority of quantum open system models in the literature are simplistic in the sense that they only explicitly account for that part of the environment that directly interacts with the system of interest. A quantum open system with an open environment is examined using the projection operator method in the weak coupling limit. The openness of environment is modelled by nonunitary evolution of the Lindblad form. Under certain conditions, the resulting master equation for the system is insensitive to the initial state of the environment and to initial entanglements between the rest of the universe with which the rest of the universe interacts. A quantum open system with an open environment is examined using the projection operator method in the weak coupling regime on a composite system where the environment relaxes to thermal equilibrium via a nonunitary evolution of the Lindblad form. [5] We illustrate the results for the particular case of linear coupling to an oscillator bath in Section V. We finish in Section VI with a discussion of our results.

I. INTRODUCTION

There has been much interest in quantum open systems (particularly with applications to quantum decoherence [1, 2, 3] over the past several decades. [4]

As discussed by Alicki, [4] models that have been extensively studied in the literature have arisen from one of two approaches. The first starts with an a priori proposal of a mathematical form for the quantum dynamical semigroup (Alicki’s axiomatic approach), and proceeds with a study of the resulting properties of the dynamics. The second is to study the reduced dynamics of the system of interest which is part of a composite system which also includes environmental degrees of freedom (Alicki’s constructive approach). The models primarily explored for the constructive approach have a set of common features. The system of interest evolves unitarily in isolation. The environment (often an oscillator bath) evolves unitarily in isolation. The interaction between system and environment consists of a term in the composite hamiltonian, so that the composite system evolves unitarily. The initial state of the composite system is an important feature of the particular model being studied and is often but not exclusively taken to be a factor state between an initial state of the system of interest and a thermal equilibrium state for the environment.

The study of open systems is an attempt to describe how “the rest of the universe” influences the system of interest. However, most constructive models only include that part of the rest of the universe with which the rest of the universe directly interacts (through an interaction term in the composite hamiltonian). In order to account for that part of the universe with which the system of interest does not directly interact, we explore properties of models for which the environment degrees of freedom are themselves open systems. In the Section II, we motivate the quantum models by examining a classical system and environment where the environment degrees of freedom are damped oscillators relaxing into thermal equilibrium. In Section III we review some important properties of the model we will use to simulate the open nature of the environment. In Section IV we proceed to use the projection operator method in the weak coupling regime on a composite system where the environment relaxes to thermal equilibrium via a nonunitary evolution of the Lindblad form. [5] We illustrate the results for the particular case of linear coupling to an oscillator bath in Section V. We finish in Section VI with a discussion of our results.

II. CLASSICAL OPEN SYSTEM

In this section we examine a classical open system with a damped environment to gain insight and anticipate properties of analogous quantum models. We are essentially following Zwanzig [6] with the addition of dissipation and thermalization to the dynamics of the environment.

The system is described in terms of its coordinate $x$. The environment consists of a bath of independent oscillators with coordinates $q_{\mu}$. The composite system equations of motion are given by

$$m\ddot{x} = -\frac{\partial U(x)}{\partial x} + \sum_{\mu} m_{\mu}\omega_{\mu}^{2}(q_{\mu} - a_{\mu}(x))\frac{\partial a_{\mu}(x)}{\partial x} \quad (2.1)$$

for the system coordinate, and

$$m_{\mu}\ddot{q}_{\mu} = -m_{\mu}\omega_{\mu}^{2}q_{\mu} - 2m_{\mu}\gamma_{\mu}q_{\mu} + m_{\mu}\omega_{\mu}^{2}a_{\mu}(x) + F_{\mu}(t) \quad (2.2)$$

for each environment degree of freedom. These equations of motion correspond to the addition of damping (with friction coefficients $\gamma_{\mu} = 2m_{\mu}\gamma_{\mu}$) and thermal noise terms $F_{\mu}$ to the environment portion of the closed system dynamics obtained from the composite system.
Lagrangian:

\[ L = \frac{1}{2} m \dddot{x} - U(x) + \sum_{\mu} \frac{m_{\mu}}{2} \{ \dot{q}_{\mu}^2 - \omega_{\mu}^2 (q_{\mu} - \dot{q}_{\mu}(x))^2 \} \quad (2.3) \]

Each oscillator is taken to be independent but driven to the same temperature by dissipation and white noise so that the following fluctuation-dissipation relation holds:

\[ \langle F_{\mu}(t) F_{\nu}(s) \rangle = 4 \gamma_{\mu} m_{\mu} k_B T \delta_{\mu,\nu} \delta(t-s). \quad (2.4) \]

The Kronecker delta indicates the oscillators in the bath are driven by independent noise terms. The Dirac delta function indicates that the driving forces are white noise.

\[ q_{\mu}(t) = [\dot{q}_{\mu 0} \cos \Omega_{\mu} t + \frac{1}{\Omega_{\mu}} (\dot{q}_{\mu 0} + \gamma_{\mu} q_{\mu 0}) \sin \Omega_{\mu} t] e^{-\gamma_{\mu} t} + \int_0^t \frac{1}{m_{\mu} \Omega_{\mu}} \sin \Omega_{\mu} (t-s) e^{-\gamma_{\mu}(t-s)} F_{\mu}(s) ds \]

\[ a_{\mu}(x(t)) - \frac{1}{\Omega_{\mu}} q_{\mu 0}(x(0)) \Omega_{\mu} \cos \Omega_{\mu} t + \gamma_{\mu} \sin \Omega_{\mu} t e^{-\gamma_{\mu} t} \]

\[ - \int_0^t \frac{1}{m_{\mu} \Omega_{\mu}} [\Omega_{\mu} \cos \Omega_{\mu} s + \gamma_{\mu} \sin \Omega_{\mu} s] e^{-\gamma_{\mu} s} \frac{\partial a_{\mu}(x(t-s))}{\partial x} \dot{x}(t-s) ds. \quad (2.5) \]

Upon substitution of Eq. (2.5) into Eq. (2.1) and rearranging, the equation of motion for the system can be written

\[ m \dddot{x} = -\frac{\partial U(x)}{\partial x} - \int_0^t \eta(x(t),x(s);t,s) \dot{x}(s) ds \]

\[ + F_{E1}(t) + F_{E2}(t) + F_{E3}(t) \quad (2.6) \]

where the noise terms are given by

\[ F_{E1}(t) = \sum_{\mu} m_{\mu} \omega_{\mu}^2 [\dot{q}_{\mu 0} - a_{\mu}(x(0))] [\cos \Omega_{\mu} t + \frac{\Omega_{\mu}}{\gamma_{\mu}} \sin \Omega_{\mu} t] e^{-\gamma_{\mu} t}, \quad (2.7a) \]

\[ F_{E2}(t) = \sum_{\mu} m_{\mu} \omega_{\mu}^2 \dot{q}_{\mu 0} [\cos \Omega_{\mu} t + \frac{\sin \Omega_{\mu} t}{\Omega_{\mu}}] e^{-\gamma_{\mu} t}, \quad (2.7b) \]

\[ F_{E3}(t) = \sum_{\mu} \omega_{\mu}^2 \Omega_{\mu} \int_0^t \sin \Omega_{\mu} (t-s) e^{-\gamma_{\mu}(t-s)} F_{\mu}(s) ds \frac{\partial a_{\mu}(x(t))}{\partial x}, \quad (2.7c) \]

and the dissipation kernel is given by

\[ \eta(x(t),x(s);t,s) = \sum_{\mu} m_{\mu} \omega_{\mu}^2 \{ [\cos \Omega_{\mu} (t-s) + \frac{\Omega_{\mu}}{\gamma_{\mu}} \sin \Omega_{\mu} (t-s)] e^{-\gamma_{\mu}(t-s)} \} \frac{\partial a_{\mu}(x(s))}{\partial x} \frac{\partial a_{\mu}(x(t))}{\partial x}. \quad (2.8) \]

In the past, the properties of noise terms like \( F_{E1}(t) \) and \( F_{E2}(t) \) have generally been extracted from the statistical distributions of the initial conditions. With the introduction of dissipation in the environment, these noise terms can readily be seen to be transient terms on the time scales of the environment (as determined by \( \gamma_{\mu} \)). Thus the details of the initial state of the environment are not important to the long term system dynamics.
The correlations \( \langle F_{E3}(t)F_{E3}(\tau) \rangle \) of the remaining noise term depend upon the correlations of the individual oscillators’ noise terms via Eq. (2.4), and can be written

\[
\langle F_{E3}(x, t)F_{E3}(x', \tau) \rangle = k_B T \eta(x, x'; t, \tau) + \frac{1}{4\gamma_{\mu}\omega_{\mu}^2} \left[ \gamma_{\mu} \cos \Omega_{\mu}(t + \tau) - \omega_{\mu}^2 \cos \Omega_{\mu}(t - \tau) - \omega_{\mu} \gamma_{\mu} \sin \Omega_{\mu}(t + \tau) \right] e^{-\gamma_{\mu}(t+\tau)} \frac{\partial a_{\mu}(x)}{\partial x} \frac{\partial a_{\mu}(x')}{\partial x'}.
\]  

(2.9)

The first term on the right hand side of Eq. (2.9) is the long term correlation function of the effective noise. The second term contains a factor of \( e^{-\gamma_{\mu}(t+\tau)} \), and thus is transient for long timescales.

Since \( \eta(x, x'; t, \tau) \) is a narrow function of \( t - \tau \), we can make a Markov approximation in Eq. (2.6):

\[
m\dot{x} = -\frac{\partial U(x)}{\partial x} - \tilde{\eta}(x, x)\dot{x} + F_s(x, t),
\]  

(2.10)

where

\[
\tilde{\eta}(x, x') = \sum_{\mu} 2m_{\mu}\gamma_{\mu} \frac{\partial a_{\mu}(x)}{\partial x} \frac{\partial a_{\mu}(x')}{\partial x'}.
\]  

(2.11)

The corresponding fluctuation-dissipation relation is

\[
\langle F_s(x, t)F_s(x', \tau) \rangle = 2k_B T \tilde{\eta}(x, x')\delta(t - \tau). \quad (2.12)
\]

The role of the spatial correlations of the noise in quantum decoherence and the lack of importance of those correlations in classical phenomena has been discussed elsewhere.[7, 8]

Thus, adding “fast” thermal relaxation to the environment of the system of interest lead to Markovian equations of motion with the usual fluctuation-dissipation relations. Memory effects due to the details of the initial environment state (including possible correlations with the initial system state) are erased on the environment relaxation time scales. This result excellent motivation for exploring similar models in the quantum mechanical domain.

### III. PROPERTIES OF QUANTUM OPEN ENVIRONMENT

As with the classical model, in order to account for the open nature of the environment we need to incorporate modifications to the dynamics of the environment degrees of freedom. The environment is modelled as a set of independent oscillators whose evolution is governed by a Markovian master equation:

\[
\frac{\partial \rho}{\partial t} = L[\rho]. \quad (3.1)
\]

The generator \( L \) of the evolution is taken to be of the Lindblad form[5]:

\[
\frac{\partial \rho}{\partial t} = L[\rho] = \frac{1}{i\hbar} [H, \rho] + \frac{1}{2\hbar} \sum_{\mu} [V_{\mu} \rho, V_{\mu}^\dagger] + [V_{\mu}, \rho V_{\mu}^\dagger]. \quad (3.2)
\]

Formally, the solution to Eq. (3.2) is

\[
\rho(t) = \Lambda(t)[\rho] = e^{t\Lambda}[\rho], \quad (3.3)
\]

expectation values are defined via the trace operation, allowing the definition of the adjoint representation of the evolution operator \( \Lambda^*(t)[O] \) via

\[
\text{Tr}[\rho \Lambda^*(t)[O]] = \text{Tr}[\Lambda(t)[\rho]O], \quad (3.4)
\]

and for the generator \( L^* \)

\[
\text{Tr}[\rho L^*[O]] = \text{Tr}[L[\rho]O]. \quad (3.5)
\]

The adjoint representation of \( L \) (i.e. Heisenberg picture) is given by

\[
L^*[O] = -\frac{1}{i\hbar}[H, O] + \frac{1}{2\hbar} \sum_{\mu} V_{\mu}^\dagger [O, V_{\mu}] + [V_{\mu}^\dagger, O]V_{\mu}. \quad (3.6)
\]

To model the relaxation of the environment, we will use a subset of a family of master equations that have been studied extensively in the literature [9, 10, 11, 12] where the Lindblad form of the master equation can be rewritten as:

\[
\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar} [H_0, \rho] - \frac{i}{2\hbar} (\lambda + \mu)[p, \{q, \rho\}] + \frac{i}{2\hbar} (\lambda - \mu)[p, \{q, \rho\}] - \frac{D_{qq}}{\hbar^2} [q, [q, \rho]] - \frac{D_{pp}}{\hbar^2} [p, [p, \rho]] + \frac{D_{pq}}{\hbar^2} ([q, [p, \rho]] + [p, [q, \rho]]). \quad (3.7)
\]

The details of the model are determined by the specification of the diffusion coefficients \( D_{qq}, D_{pp} \) and \( D_{pq} \) and damping constants \( \mu \) and \( \lambda \), subject to the constraints:[9]

\[
D_{qq} > 0 \quad D_{pp} > 0.
\]

\[
D_{qq}D_{pp} - D_{pq}^2 \geq \left( \frac{\lambda\hbar}{2} \right)^2. \quad (3.8)
\]

The nominal Hamiltonian is

\[
H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2. \quad (3.9)
\]

\( \tilde{\rho} \) is taken to be the (unique) asymptotic state of \( \Lambda(t) \), that is

\[
\Lambda_\infty[\rho] \equiv \lim_{t \to \infty} \Lambda(t)[\rho] = \tilde{\rho} \quad (3.10)
\]

for any initial environment state \( \rho \). Since \( \tilde{\rho} \) is a stationary state of \( L \)

\[
L[\tilde{\rho}] = 0. \quad (3.11)
\]

The asymptotic behavior of the environment is encapsulated in \( \Lambda_\infty \), so that

\[
\Lambda_\infty[\rho] = \tilde{\rho}^{(E)}. \quad (3.12)
\]
for any state $\rho$. Although $\Lambda_\infty$ is defined only on the space of density operators (including pure states), we will need to extend its domain. If $\{|e_k\rangle\}$ is a basis for the environment Hilbert space, then

$$\Lambda_\infty[|e_k\rangle\langle e_k|] = \tilde{\rho}$$

(3.13)

for any $k$. Furthermore, if

$$|\psi_{nm}\rangle \equiv \frac{1}{\sqrt{2}}(|e_n\rangle + |e_m\rangle)$$

and

$$|\chi_{nm}\rangle \equiv \frac{1}{\sqrt{2}}(|e_n\rangle + i|e_m\rangle)$$

(3.14)

then

$$\Lambda_\infty[|\psi_{nm}\rangle\langle \psi_{nm}|] = \tilde{\rho}$$

$$= \frac{1}{2}(\Lambda_\infty[|e_n\rangle\langle e_n|] + \Lambda_\infty[|e_m\rangle\langle e_m|]$$

$$+ \Lambda_\infty[|e_n\rangle\langle e_m|] + \Lambda_\infty[|e_m\rangle\langle e_n|])$$

$$= \tilde{\rho} + \frac{1}{2}(\Lambda_\infty[|e_n\rangle\langle e_m|]$$

$$+ \Lambda_\infty[|e_m\rangle\langle e_n|])$$

(3.15)

and

$$\Lambda_\infty[|\chi_{nm}\rangle\langle \chi_{nm}|] = \tilde{\rho}$$

$$= \frac{1}{2}(\Lambda_\infty[|e_n\rangle\langle e_n|] + \Lambda_\infty[|e_m\rangle\langle e_m|]$$

$$+ i\Lambda_\infty[|e_n\rangle\langle e_m|] - i\Lambda_\infty[|e_m\rangle\langle e_n|])$$

$$= \tilde{\rho} + i\frac{1}{2}(\Lambda_\infty[|e_n\rangle\langle e_m|]$$

$$- i\Lambda_\infty[|e_m\rangle\langle e_n|])$$

(3.16)

which implies

$$\Lambda_\infty[|e_n\rangle\langle e_m|] = \delta_{nm}\tilde{\rho}.$$  

(3.17)

For an arbitrary operator $O$

$$\Lambda_\infty[O] = \sum_{nm} O_{nm}\Lambda_\infty[|e_n\rangle\langle e_m|]$$

$$= \sum_{nm} O_{nm}\tilde{\rho}$$

$$= \text{Tr}(O)\tilde{\rho}$$

(3.18)

There is in general no guarantee that an asymptotic state exists for arbitrary $L$, that is, not all choices for the parameters $D_{qq}$, $D_{pp}$, etc. will be appropriate to model the a system dynamically relaxing to equilibrium. For example, if the parameters satisfy

$$D_{pp} = \frac{\lambda + \mu}{2}\hbar m\omega \coth \frac{\hbar \omega}{2k_BT},$$

(3.19a)

$$D_{qq} = \frac{\lambda - \mu}{2}\frac{\hbar}{m\omega}\coth \frac{\hbar \omega}{2k_BT},$$

(3.19b)

then the Gibbs state is the asymptotic state.\cite{9, 10} On the other hand, if

$$D_{qq} = \frac{\hbar \lambda}{2m\Omega},$$

(3.20a)

$$D_{pp} = \frac{\hbar \lambda m\omega^2}{2\Omega},$$

(3.20b)

$$D_{pq} = -\frac{\hbar \lambda \mu}{2\Omega},$$

(3.20c)

(where $\Omega^2 = \omega^2 - \mu^2$) there can be persistent pure states.\cite{11, 12} Since we are considering the primary effect of the openness of the environment to be effectiveness which relax the environment towards an equilibrium state, we would only consider those choices of parameters for which there is a unique asymptotic state.

Sandulescu and Scutaru have determined the time dependence and asymptotic behavior of various moments of $p$ and $q$, which will be useful in later calculations. The evolution of the first order moments is given by

$$\frac{\partial (q)}{\partial t} = -(\lambda - \mu)(q) + \frac{1}{m}(p)$$

(3.21a)

$$\frac{\partial (p)}{\partial t} = -(\lambda + \mu)(p) - m\omega^2(q),$$

(3.21b)

which illustrates the role of the dissipation coefficients $\lambda$ and $\mu$. With

$$S_{pq} = \frac{1}{2}[p, q],$$

(3.22)

the evolution of the second order moments is given by

$$\frac{\partial (p^2)}{\partial t} = -2m\omega^2(S_{pq}) - 2(\lambda + \mu)(p^2) + 2Dpp$$

(3.23a)

$$\frac{\partial (q^2)}{\partial t} = \frac{2}{m}(S_{pq}) - 2(\lambda - \mu)(q^2) + 2Dqq$$

(3.23b)

$$\frac{\partial (S_{pq})}{\partial t} = \frac{1}{m}(p^2) - m\omega^2(q^2) - 2\lambda(S_{pq}) + 2Dpq$$

(3.23c)

which illustrates the role of the diffusion coefficients $D_{pp}$, $D_{qq}$ and $D_{pq}$.

The asymptotic first order moments are

$$\langle p \rangle_\infty = 0$$

(3.24a)
\[ \langle q \rangle_\infty = 0. \quad (3.24b) \]

For the second order moments we have
\[
\langle p^2 \rangle_\infty = \frac{1}{2m^2 \omega^2 (\lambda^2 + \omega^2 - \mu^2)} \left( m^2 \omega^4 D_{qq} \right. \\
+ \left. 2\mu \omega^2 (\lambda - \mu) D_{pp} \right) - 2m^2 \omega^2 (\lambda - \mu) D_{pq}, \tag{3.25} \]
\[
\langle q^2 \rangle_\infty = \frac{1}{2m \omega^2 (\lambda^2 + \omega^2 - \mu^2)} \left( (m \omega)^2 \omega^2 D_{qq} \right. \\
+ \left. \omega^2 D_{pp} + 2m \omega^2 (\lambda + \mu) D_{pq} \right), \tag{3.26} \]

and
\[
\langle S_{pq} \rangle_\infty = \left\langle \frac{1}{2} \{p, q\} \right\rangle_\infty \\
= \frac{1}{2m \omega^2 (\lambda^2 + \omega^2 - \mu^2)} \left[ - (\lambda + \mu) (m \omega)^2 D_{qq} \right. \\
+ \left. (\lambda - \mu) D_{pp} + 2m (\lambda^2 - \mu^2) D_{pq} \right]. \tag{3.27} \]

We will also need the time dependence of \( q(t) \) in the Heisenberg picture. This can be readily extracted from Sandulescu and Scutaru’s results for the time dependence for the first order moments.\[9\] The result is
\[
q_H(t) = \Lambda(t)^* q \\
= q \left( \cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t \right) e^{-\lambda t} \\
+ \frac{p}{m \Omega} \sin \Omega t e^{-\lambda t}, \tag{3.28} \]
with
\[
\Omega = \sqrt{\omega^2 - \mu^2} \tag{3.29} \]

and \( \Omega \) taken to be real (the oscillators are underdamped).

The specific environment we will employ consists of a set of independent oscillators, each subject to evolution of the form Eq. (3.7), with position operators \( \{ q_i \} \) and associated parameters \( \{ \mu_i \}, \{ \lambda_i \} \), etc. Thus the Hilbert space of the environment is actually the tensor product of the Hilbert space corresponding to each environment degree of freedom. The interaction between the environment and system is accounted for by adding an interaction term to the composite system’s hamiltonian:
\[
U_I = \sum_n V_{n}^{(S)} \otimes q_{n}^{(E)}. \tag{3.30} \]

We will need the correlation functions \( \langle q_n(t)q_m \rangle \) in the next section. Using Eq. (3.28), these correlations can be written in terms of the asymptotic correlations as
\[
\langle q_n(t)q_m \rangle = \text{Tr}[\Lambda^*(t) \langle q_n \rangle_\infty \langle q_m \rangle_\infty] \\
= \delta_{nm} \left[ \langle q_n \rangle_\infty (\cos \Omega t + \frac{\mu}{\Omega} \sin \Omega t) e^{-\lambda t} ight. \\
\left. + \frac{2(S_{pq})_\infty - i\hbar}{2m\Omega} \sin \Omega t e^{-\lambda t} \right], \tag{3.31} \]

where we have made use of the relation
\[
p_n q_n = \frac{1}{2} \{ p_n, q_n \} + [p_n, q_n] \\
= S_{pq} - \frac{i\hbar}{2}. \tag{3.32} \]

IV. QUANTUM MASTER EQUATION FROM AN OPEN ENVIRONMENT

To construct the master equation, we will use the projection operator method in the weak coupling regime, largely following the presentation of Alicki and Lendi.\[13\] The composite system (the system of interest plus its environment) is taken to evolve according to Eq. (3.1). The generator of this evolution is the combination of the nominal dynamics and an interaction:
\[
L = L_0 + L_I, \tag{4.1} \]

where the system and environment dynamics each contribute separately:
\[
L_0 = L_S + L_E. \tag{4.2} \]

The generators for the system and environment nominal evolutions act on on the corresponding subspaces, so that
\[
L_S[A^{(S)} \otimes B^{(E)}] = L_S^{(S)}[A^{(S)}] \otimes B^{(E)}, \tag{4.3} \]
and
\[
L_E[A^{(S)} \otimes B^{(E)}] = A^{(S)} \otimes L_E^{(E)}[B^{(E)}]. \tag{4.4} \]

for all \( A^{(S)} \) and \( B^{(E)} \). \( L_S^{(S)} \) is taken to be unitary, and \( L_E^{(E)} \) is taken to be of the Lindblad form. It is useful to note that necessarily \( L_S \) and \( L_E \) commute.

The projection operator is defined in terms of a partial trace:
\[
P_0[O] = \text{Tr}_E[O] \otimes \tilde{\rho}^{(E)}. \tag{4.5} \]

We wish to establish some important relations between \( P_0 \) and the generators \( L_S \) and \( L_E \).
\[
L_E P_0[O] = L_E[\text{Tr}_E O] \otimes \tilde{\rho}^{(E)} = \text{Tr}_E[O] \otimes L_E^{(E)}[\tilde{\rho}^{(E)}] = 0 \tag{4.6} \]
for any operator \( O \) so that \( L_E P_0 = 0 \). \( L_E^{(E)} \) generates trace preserving evolution so that
\[
\text{Tr}_E[L_E^{(E)}[O^{(E)}]] = 0. \tag{4.7} \]

for all \( O^{(E)} \). Thus
\[
P_0 L_E[A^{(S)} \otimes B^{(E)}] = P_0[A \otimes L_E^{(E)}[B^{(E)}]] \\
= (\text{Tr}_E[L_E^{(E)}[B^{(E)\dagger}]] A^{(S)} \otimes \tilde{\rho}^{(E)} \\
= 0, \tag{4.8} \]
for all $A^{(S)}$ and $B^{(E)}$ so that in general $P_0 L_E = 0$.

\[
P_0 L_S [A^{(S)} \otimes B^{(E)}] = P_0 [L_S^{(S)} [A^{(S)}] \otimes B^{(E)}]
\]

\[
= L_S^{(S)} [A^{(S)}] \otimes \bar{\rho}^{(E)} T^{(E)} / [B^{(E)}] \\
= L_S [[A^{(S)}] \otimes \bar{\rho}^{(E)} T^{(E)} / [B^{(E)}]] \\
= L_S P_0 [A^{(S)} \otimes B^{(E)}]
\]  \hspace{1cm} (4.9)

for all $A^{(S)}$ and $B^{(E)}$ so that in general $P_0 L_S = L_S P_0$. Defining a second projection $P_1 \equiv 1 - P_0$, it is easy to see that $P_1 L_S = L_S P_1$. In order to focus on the reduced dynamics of the system of interest, we study the reduced dynamics of the system of interest, we study the

\[
\frac{\partial P_0 \rho}{\partial t} = P_0 L_P P_0 \rho + P_0 L_P P_1 \rho,
\]  \hspace{1cm} (4.10a)

\[
\frac{\partial P_1 \rho}{\partial t} = P_1 L_P P_1 \rho + P_1 L_P P_0 \rho.
\]  \hspace{1cm} (4.10b)

Eq. (4.10b) can be formally integrated and substituted into Eq. (4.10a), so that

\[
\frac{\partial P_0 \rho}{\partial t} = P_0 L_P P_0 \rho(t) \\
+ P_0 L_P P_1 e^{P_1 L_P t} P_1 \rho(0) \\
+ \int_0^t e^{P_1 L_P (t-s)} P_1 L_P P_0 \rho(s) ds.
\]  \hspace{1cm} (4.11)

Using the relations between the projectors and generators discussed above, we have

\[
P_0 L_P P_1 = P_0 L_P P_1, \\
P_1 L_P P_0 = P_1 L_P P_0, \\
P_0 L_P P_0 = L_S P_0 + P_0 L_P P_0,
\]  \hspace{1cm} (4.12)

so that Eq. (4.11) can be written

\[
\frac{\partial P_0 \rho}{\partial t} = (L_S + P_0 L_P P_0) P_0 \rho(t) \\
+ \int_0^t P_0 L_P P_1 e^{L_P (t-s)} P_1 L_P P_0 \rho(s) ds.
\]  \hspace{1cm} (4.13)

This result is exact, within the constraints placed on the model so far.

The interaction, as specified by Eq. (3.30), can be written

\[
L_I [O] = \frac{1}{i \hbar} [U_I, O] \\
= \frac{1}{i \hbar} \sum_n [V^{(S)}_n \otimes q^{(E)}_n, O].
\]  \hspace{1cm} (4.14)

This form, along with the first order moments of the environment asymptotic state, simplifies the first term of the right hand side of Eq. (4.13). Specifically, for any operator $O$,

\[
P_0 L_I P_0 [O] = \text{Tr}_E \left[ \frac{1}{i \hbar} \sum_n [V^{(S)}_n \otimes q^{(E)}_n, \text{Tr}_E [O] \otimes \bar{\rho}^{(E)}] \otimes \bar{\rho}^{(E)} \right] \\
= \frac{1}{i \hbar} \sum_n \text{Tr}_E \left[ q^{(E)}_n \bar{\rho}^{(E)} [V^{(S)}_n, \text{Tr}_E [O]] \otimes \bar{\rho}^{(E)} \right] \\
+ \frac{1}{i \hbar} \sum_n \langle q^{(E)}_n \rangle_{\infty} [V^{(S)}_n, \text{Tr}_E [O]] \otimes \bar{\rho}^{(E)}.
\]  \hspace{1cm} (4.15)

so that Eq. (4.13) can be written

\[
\frac{\partial P_0 \rho}{\partial t} = L_S P_0 \rho(t) \\
+ P_0 L_I P_1 e^{L_I t} P_1 \rho(0) \\
+ \int_0^t P_0 L_I e^{L_I (t-s)} L_I P_0 P_0 \rho(s) ds.
\]  \hspace{1cm} (4.19)

The second term on the right hand side of Eq. (4.19)

\[
P_0 L_I P_1 e^{L_I t} P_1 \rho(0)
\]  \hspace{1cm} (4.20)

represents a transient term which depends upon the initial state of the composite system. For most constructive models the initial state is taken to be a factored state
of an arbitrary system state and the environment in its asymptotic state:

$$\rho(0) = \rho^{(S)}(0) \otimes \rho^{(E)}.$$  \hfill (4.21)

For this type of initial condition $\rho(0) = P_0 \rho(0)$ so that

$$P_1 \rho(0) = 0.$$  \hfill (4.22)

However, because we have added relaxation to the dynamics of the environment, this type of factoring assumption is not necessary. We will be taking the weak coupling limit and so we will show that Eq. (4.20) is (approximately) zero to the lowest nonvanishing order of the interaction in the remaining terms of Eq. (4.13), which turns out to be second order. Formally we can write

$$e^{Lt} = e^{L_0 t} + \int_0^t e^{L(t-s)} L_I e^{L_0 s} ds.$$  \hfill (4.23)

Substituting Eq. (4.23) into Eq. (4.20) and keeping only second order, we have the approximation

$$P_0 L_I P_1 e^{L_0 t} P_1 \rho(0) \approx P_0 L_I P_1 e^{L_0 t} P_1 \rho(0) + P_0 L_I P_1 \int_0^t e^{L_0(t-s)} L_I e^{L_0 s} ds P_1 \rho(0).$$  \hfill (4.24)

We are interested in timescales $t$ which are assumed to be much longer than the relaxation timescales of the environment, so that

$$e^{L_0 t} = e^{L_{St} e^{L_0 t}} \approx e^{L_{St} \Lambda_{E \infty}}.$$  \hfill (4.25)

Similarly, for the integral from $t$ to $s$, either $t$ or $t - s$ (or both) is large compared to the relaxation timescales of the environment, so that either

$$e^{L_{0(t-s)}} \approx e^{L_{S(t-s)} \Lambda_{E \infty}}$$  \hfill (4.26)

or

$$e^{L_{0s}} \approx e^{L_{Ss} \Lambda_{E \infty}}.$$  \hfill (4.27)

or both. From Eq. (3.18), for an operator $O$ decomposed in terms of the basis for the environment $\{|e_k\}$ and the basis for the system $\{|\phi_i\}$ we have

$$\Lambda_{E \infty}[O] = O_{am, \beta n} \Lambda_{E \infty}[|\phi_a\rangle \otimes |e_k\rangle \langle e_k| \otimes |\phi_\beta\rangle \langle \phi_\beta|].$$

$$= O_{am, \beta n} |\phi_a\rangle \langle \phi_a| \otimes \Lambda_{E \infty}[|e_m\rangle \langle e_m|].$$

$$= O_{am, \beta n} |\phi_a\rangle \langle \phi_a| \otimes \delta_{m \beta} \rho^{(E)}.$$  \hfill (4.28)

When Eq. (4.28) is applied to Eq. (4.24) with either Eq. (4.28) or Eq. (4.28) applying, then all terms in Eq. (4.24) end up with factors of either $P_0 P_1$ or $P_1 P_0$, both of which are 0. Using this result and keeping only up to second order in the interaction term, the master equation Eq. (4.19) becomes

$$\frac{\partial P_0 \rho}{\partial t} = L_S P_0 \rho(t)$$

$$+ \int_0^t P_0 I_{t} e^{L_0(t-s)} L_I P_1 P_0 \rho(s) ds.$$  \hfill (4.29)

We can rewrite Eq. (4.29) using Eq. (4.17) and Eq. (4.18) to get

$$\frac{\partial P_0 \rho}{\partial t} = L_S P_0 \rho(t)$$

$$+ \int_0^t P_0 I_{t} P_1 e^{L_0(t-s)} L_I P_1 P_0 \rho(s) ds.$$  \hfill (4.30)

In the integrand in Eq. (4.30) we see the factor $P_1 e^{L_0(t-s)} P_1$ which (following the discussion above) will be zero for $t - s$ longer than environment relaxation timescales so that the primary contribution to the integral is for $s \approx t$. This yields the simplest Markovian master equation we will extract from the model:

$$\frac{\partial P_0 \rho}{\partial t} = L_S P_0 \rho(t)$$

$$+ \frac{1}{\hbar^2} \sum_n (\langle q_n^2 \rangle_{\infty} |V_n, [V_n, P_0 \rho(t)]|].$$  \hfill (4.33)

Eq. (4.33) represents the added effect of environment induced noise on the system’s dynamics which is responsible for phenomena such as quantum decoherence. However, additional effects such as dissipation are not present and will require a more careful handling of the Markov approximations.

To reconsider the Markov approximations, we return to Eq. (4.29). The unitarity of the isolated system’s evolution implies that

$$e^{L_{S}t}[AB] = e^{L_{S}t}[A]e^{L_{S}t}[B].$$  \hfill (4.34)

The naive Markov approximation is introduced into Eq. (4.29) by examining

$$e^{L_{0(t-s)}} L_I P_0 \rho(s) = e^{L_{E(t-s)}} e^{L_{S(t-s)} L_I P_0 \rho(s)}$$

$$= \frac{1}{i \hbar} e^{L_{E(t-s)}} e^{L_{S(t-s)}} [U_I, P_0 \rho(s)]$$

$$= \frac{1}{i \hbar} e^{L_{E(t-s)}} [(e^{L_{S(t-s)} U_I})(e^{L_{S(t-s)} P_0 \rho(s)})]$$

$$\approx \frac{1}{i \hbar} e^{L_{E(t-s)}} [(e^{L_{S(t-s)} U_I}, P_0 \rho(t)]$$  \hfill (4.35)
The Markovian master equation can be written as

$$\frac{\partial P_0}{\partial t} = L_S P_0(t) + K P_0(t) \quad (4.36)$$

where

$$K[P_0\rho] = \int_0^\infty P_0 L_I e^{L_\omega s} L_I P_0 \rho \, ds, \quad (4.37)$$

which becomes for our particular model:

$$K[P_0\rho(t)] = -\frac{1}{\hbar^2} \int_0^\infty ds \{ \sum_{nm} \text{Tr}_E [V_m^{(S)} \otimes q_n^{(E)}(s), \{ V_m^{(S)}(s) \otimes q_n^{(E)}(s), P_0\rho(t) \}] \} \otimes \hat{\rho}^{(E)}, \quad (4.39)$$

where

$$V_m^{(S)}(s) = e^{L_\omega(s) \tau} V_m^{(S)} \quad (4.40a)$$

$$q_n^{(E)}(s) = A^{(E)*}_n \rho A^{(E)}_n \quad (4.40b)$$

In terms of the correlation functions specified by Eq. (3.31) we can write

$$K[P_0\rho(t)] = \frac{1}{\hbar^2} \int_0^\infty ds \{ \sum_n \left\{ \begin{array}{l} \langle q_n(s)q_n \rangle V_n V_n(s) P_0\rho(t) \\ - \langle q_n(s)q_n \rangle V_n P_0\rho(t) V_n(s) \\ - \langle q_n(s)q_n \rangle V_n(s) P_0\rho(t) V_n \\ + \langle q_n(s)q_n \rangle P_0\rho(t) V_n(s) V_n \end{array} \right. \} \quad (4.41)$$

where $$V_n = V_n^{(S)} \otimes \|^{(E)}$$ and $$\|^{(E)}$$ is simply the identify operator on the environment subspace. We note that here $$V_n(s) = e^{L_\omega s} V_n$$ is not the Heisenberg evolved operator, that is $$V_n(s) \neq e^{L_\omega s} V_n$$.

While Eq. (4.39) is similar to previous results, the main difference comes from the nature of $$L_\omega^{(E)}(t)$$. Previous authors have taken $$L_\omega^{(E)}(t)$$ to necessarily generate unitary evolution in order to use properties analogous to Eq. (4.34) which does not generally apply to nonunitary evolution.

Since Eq. (4.39) has the same mathematical structure as results obtained without dissipative effects in the environment, we expect similar shortcomings. In particular, to insure complete positivity for the reduced dynamics, we will now apply an averaging process (sometimes refereed to as the Rotating Wave Approximation),[13, 14] defined by:

$$\overline{K} = \lim_{a \to \infty} -\frac{1}{a} \int_0^a e^{-L_\omega \tau} K e^{L_\omega \tau} d\tau. \quad (4.42)$$

Since $$L_\omega P_0$$

$$e^{L_\omega \tau} P_0 = P_0, \quad (4.43)$$

so that with Eq. (4.2) and Eq. (4.37) we can write

$$\overline{K} = \lim_{a \to \infty} -\frac{1}{a} \int_0^a e^{-L_\omega \tau} K e^{L_\omega \tau} d\tau. \quad (4.44)$$

Using Eq. (4.34) and Eq. (4.38),Eq. (4.44) becomes

$$\overline{K}[P_0\rho(t)] = -\frac{1}{\hbar^2} \lim_{a \to \infty} -\frac{1}{a} \int_0^a d\tau \int_0^\infty ds \left\{ \begin{array}{l} \langle q_n(s)q_n \rangle V_n(-\tau) V_n(s) P_0\rho(t) \\ - \langle q_n(s)q_n \rangle V_n(-\tau) P_0\rho(t) V_n(s) \\ - \langle q_n(s)q_n \rangle V_n(s) P_0\rho(t) V_n(-\tau) \\ + \langle q_n(s)q_n \rangle P_0\rho(t) V_n(s) V_n(-\tau) \end{array} \right. \} \quad (4.45)$$

The operators $$\{V_n\}$$ can be decomposed in terms of the energy eigenstates of the system hamiltonian $$H(s)$$:

$$V_n = \sum_{\mu} |\mu\rangle \langle \mu| V_n |\nu\rangle \langle \nu| \quad (4.46)$$
It is clear that with this decomposition
\[ V_{n,-\Delta \omega} = V_{n,\Delta \omega}^\dagger. \] (4.47)

The time dependence of the operators \( \{V_n(s)\} \) becomes
\[ V_n(s) = \sum_{\Delta \omega} e^{-i\Delta \omega s} V_{n,\Delta \omega}. \] (4.48)

When Eq. (4.48) is substituted into Eq. (4.45), there will be oscillating terms which the integral over \( \tau \) will cancel, via
\[ \lim_{a \to \infty} \frac{1}{a} \int_0^a d\tau e^{i(\Delta \omega' + \Delta \omega)} = \delta_{\Delta \omega',-\Delta \omega} \] (4.49)
so that Eq. (4.45) becomes
\[ \overline{K}[P_0 \rho(t)] = -\frac{1}{\hbar^2} \int_0^\infty ds \sum_{n,\Delta \omega} e^{-i\Delta \omega s} \{ \langle q_n(s)q_n \rangle V_{n,\Delta \omega} V_{n,\Delta \omega}^\dagger P_0 \rho(t) \]
\[ - \langle q_n q_n \rangle V_{n,\Delta \omega} P_0 \rho(t) V_{n,\Delta \omega}^\dagger \]
\[ - \langle q_n q_n \rangle V_{n,\Delta \omega} V_{n,\Delta \omega}^\dagger P_0 \rho(t) V_{n,\Delta \omega} \]
\[ + \langle q_n q_n \rangle P_0 \rho(t) V_{n,\Delta \omega} V_{n,\Delta \omega}^\dagger \}. \] (4.50)

With the definition
\[ h_{n,\Delta \omega} + iS_{n,\Delta \omega} = \int_0^\infty ds e^{-i\Delta \omega s} \langle q_n(s)q_n \rangle, \] (4.51)
and using Eq. (4.47) we can write Eq. (4.50) as
\[ \overline{K}[P_0 \rho(t)] = -\frac{i}{\hbar} \left[ \sum_{n,\Delta \omega} \frac{1}{\hbar} S_{n,\Delta \omega} V_{n,\Delta \omega} V_{n,\Delta \omega}^\dagger P_0 \rho(t) \right] \]
\[ + \frac{1}{\hbar^2} \sum_{n,\Delta \omega} h_{n,\Delta \omega} ([V_{n,\Delta \omega} P_0 \rho(t), V_{n,\Delta \omega}] \]
\[ + [V_{n,\Delta \omega}^\dagger, P_0 \rho(t) V_{n,\Delta \omega}]). \] (4.52)

The first term in the right hand side of Eq. (4.52) is simply an additional Hamiltonian term. The remaining terms (those with \( h_{n,\Delta \omega} \)) are in the Lindblad form if \( h_{n,\Delta \omega} \) is positive. Using Eq. (3.31) and the moments given in Eq. (3.26) and Eq. (3.27) in Eq. (4.51) we find

\[ h_{n,\Delta \omega} = \frac{[(\lambda_n + \mu_n)^2 + \Delta \omega^2]m_n^2D_{qgn} + D_{qpn} + m_n\lambda_n h\Delta \omega + 2(\lambda_n + \mu_n)m_nD_{qpn}}{[\lambda_n^2 + (\Omega_n + \Delta \omega)^2][\lambda_n^2 + (\Omega_n - \Delta \omega)^2]} \] (4.53)
and
\[ S_{n,\Delta \omega} = \frac{C_{0n} + C_{1n} \Delta \omega + C_{2n} \Delta \omega^2 + C_{3n} \Delta \omega^3}{2\lambda m^2[\lambda_n^2 + (\Omega_n + \Delta \omega)^2][\lambda_n^2 + (\Omega_n - \Delta \omega)^2]} \] (4.54)

where
\[ C_{0n} = \hbar m_n \lambda_n (\lambda_n^2 + \Omega_n^2)^2 \] (4.55a)
\[ C_{1n} = [(\Omega_n^2 - 3\lambda_n^2)(\mu + \lambda)^2 + (\lambda_n^2 + \Omega_n^2)^2]m_n^2D_{q gn} + (\Omega_n^2 - 3\lambda_n^2)(2(\mu + \lambda)m_nD_{qpn} + D_{q nn}), \] (4.55b)
\[ C_{2n} = -\hbar \lambda m_n (\lambda_n^2 + \Omega_n^2), \] (4.55c)
\[ C_{3n} = -m_n^2D_{q gn} [\lambda_n^2 + \Omega_n^2 + (\mu + \lambda)^2] - 2D_{q pn} m_n (\mu + \lambda) m_n - D_{q nn}, \] (4.55d)

The denominator in the right hand side of Eq. (4.53) is the product of two sums of squares, and hence is guaranteed to be positive. The numerator is quadratic in \( \Delta \omega \) of the form
\[ y = a\Delta \omega^2 + b\Delta \omega + c \]
\[ = (m_n^2D_{q gn})\Delta \omega^2 - m_n\lambda_n h\Delta \omega \]
\[ + m_n^2(\lambda_n + \mu_n)^2D_{q gn} + D_{q pn} + 2(\lambda_n + \mu_n)m_nD_{q pn}. \] (4.56)
If $y$ is positive, then $h_{n,\Delta\omega}$ is positive as well. The coefficient of the quadratic term is positive, thus $y$ has positive concavity. If $y = 0$ has no real roots, then $y$ must be positive, which can be tested by the condition $4ac - b^2 > 0$. Upon some rearrangement, we can write

$$4ac - b^2 = 4m_n^2 [m_n D_{qnm}(\lambda_n + \mu_n) + D_{qpm}]^2 + D_{qpn} D_{pqn} - D_{qpn}^2 - (\frac{\lambda_n h}{2})^2. \quad (4.57)$$

The first term inside the braces is a square and hence positive, while the remaining terms satisfy Eq. (3.8) and so the resulting expression is necessarily positive. Thus it is sufficient that our open system model for the oscillators is of the Lindblad form (as discussed in the previous section) to guarantee that the rotating wave approximation of the weak coupling limit generates a Lindblad form for the evolution of the system.

V. EXAMPLE: OSCILLATOR LINEARLY COUPLED TO BATH

In this section we illustrate our results with a test model consisting of an oscillator linearly coupled to a damped oscillator bath. To simplify notation we take $\rho$ to be the reduced density operator for the system. The system has a nominal hamiltonian given by

$$H_S = \frac{1}{2} m_S \omega_S^2 Q^2 + \frac{P^2}{2 m_S}, \quad (5.1)$$

and the interaction with the environment is given by

$$U_I = \sum_n C_n Q \otimes q_n. \quad (5.2)$$

With this choice, we can write the operators $V_{n,\Delta\omega}$ in terms of the system creation and annihilation operators $a^\dagger$ and $a$

$$V_{n,\Delta\omega} = C_n \sqrt{\frac{\hbar}{2m_S \omega_S}} (a^\dagger \delta_{\Delta\omega,\omega_S} + a \delta_{\Delta\omega,-\omega_S}). \quad (5.3)$$

The contribution to the hamiltonian through $K$, as it appears in Eq. (4.52), becomes

$$\Delta H = \frac{1}{\hbar} \sum_{n,\Delta\omega} (S_{n,\Delta\omega} V_{n,\Delta\omega} V_{n,\Delta\omega})^\dagger
= h \delta \omega_S (aa^\dagger - \frac{1}{2}) + \Delta E. \quad (5.4)$$

where

$$\delta \omega_S = \sum_n \frac{C_n^2}{2 m_S \omega_S h} (S_{n,\omega_S} + S_{n,-\omega_S}) \quad (5.5)$$

and

$$\delta E = \sum_n \frac{C_n^2}{2 m_S \omega_S} (S_{n,\omega_S} - S_{n,-\omega_S}). \quad (5.6)$$

Thus $\delta \omega_S$ is simply a frequency shift for the system and $\delta E$ is a C-number shift in the energy. The remaining contributions to $K$ essentially are of the form of Eq. (3.7) with $\mu = 0$, that is

$$K[\rho] = \frac{i}{\hbar} [h \delta \omega_S (aa^\dagger - \frac{1}{2}) + \Delta E, \rho]
- \frac{i}{2h} \lambda ([q, \{p, \rho\}] - [p, \{q, \rho\}])
- \frac{D_{pp}}{\hbar^2} [q, \{q, \rho\}] - \frac{D_{qq}}{\hbar^2} [p, \{p, \rho\}]. \quad (5.7)$$

The dissipation and diffusion coefficients are given by

$$\lambda = \sum_n \frac{C_n^2}{2 m_S \omega_S} (h_{n,\omega_S} - h_{n,-\omega_S})$$
$$D_{pp} = (m_S \omega_S)^2 D_{qq}$$
$$= \sum_n \frac{C_n^2}{4} (h_{n,\omega_S} + h_{n,-\omega_S}) \quad (5.8)$$

If the environment degrees of freedom are thermalized (i.e. driven asymptotically to a Gibbs state) then Eq. (3.19) holds and the dissipation and diffusion coefficients are given by
\[
\lambda = \sum_n \frac{C_n^2 \lambda_n}{m_S^2 (\lambda_n^2 + (\Omega_n - \omega_S)^2)(\lambda_n^2 + (\Omega_n + \omega_S)^2)}
\]

\[
D_{pp} = \sum_n \frac{C_n^2 \hbar \coth \left( \frac{\hbar \omega_S}{2k_B T} \right) \left( (\lambda_n + \mu_n)(\lambda_n^2 + \Omega_n^2) + (\lambda_n - \mu_n)\omega_S^2 \right)}{m_S \omega_n (\lambda_n^2 + (\Omega_n - \omega_S)^2)(\lambda_n^2 + (\Omega_n + \omega_S)^2)}
\]  

(5.9)

and the frequency shift is given by

\[
\delta \omega_S = \sum_n \frac{C_n^2 (\lambda_n^2 + \Omega_n^2 - \omega_S^2)}{2m_S^2 \omega_S (\lambda_n^2 + (\Omega_n - \omega_S)^2)(\lambda_n^2 + (\Omega_n + \omega_S)^2)}
\]  

(5.10)

From Eq. (3.21), the dissipative time scales of the environment are determined by \(\lambda_n - \mu_n\) and \(\lambda_n + \mu_n\). If dissipation is weak, then

\[
\lambda_n - \mu_n, \lambda_n + \mu_n \ll \omega_n
\]

and

\[
\Omega_n \approx \omega_n.
\]

The dissipation and diffusion coefficients each have a factor of

\[
\frac{1}{(\lambda_n^2 + (\Omega_n - \omega_S)^2)} \approx \frac{1}{(\omega_n - \omega_S)^2}
\]

(5.13)

from which we see that for a weakly damped environment, the greatest effect is from the oscillators in the environment with frequencies close to the system’s frequency. With this approximation, we can further simplify the dissipation and diffusion coefficients with:

\[
\lambda \approx \sum_{\omega_n \approx \omega_S} \frac{C_n^2}{4m_S^2 \omega_S^2 \lambda_n}
\]

\[
D_{pp} \approx \frac{\hbar m_S \omega_S}{2} \coth \left( \frac{\hbar \omega_S}{2k_B T} \right) \sum_{\omega_n \approx \omega_S} \frac{C_n^2}{4m_S^2 \omega_S^2 \lambda_n}
\]

(5.14)

Thus the parameters of \(K\) satisfy Eq. (3.19) and the system is driven towards thermal equilibrium by an effective evolution of the Lindblad form.

**VI. CONCLUSIONS**

We have constructed open environment models to account for the environment’s interaction with the “rest of the universe”. In Section II, our brief investigation of a classical model provides some insight and expectations for the development of a quantum mechanical model. The review the open system model in Section III served to establish many properties used in our derivation of the effective master equation in Section IV. Although Section IV largely follows previous work, the introduction of nonunitary evolution for the environment provided some novel aspects to the master equation derivation. We were able to show that an effective master equation of the Lindblad form could be obtained for a rich family of dissipative environment models. Our illustration of the resulting master equation with a bilinear environment-system interaction provided an demonstration in which an environment dynamically driven toward thermal equilibrium can naturally result in the dynamical thermalization of the system of interest.

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