On the Performance Analysis of Streaming Codes over the Gilbert-Elliott Channel

Myna Vajha, Vinayak Ramkumar, Mayank Jhamtani, P. Vijay Kumar
Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore.
email: {mynaramana, vinram93, mayankjhamtani26, pvk1729}@gmail.com

Abstract

The Gilbert-Elliot (GE) channel is a commonly-accepted model for packet erasures in networks. Streaming codes are a class of packet-level erasure codes designed to provide reliable communication over the GE channel. The design of a streaming code may be viewed as a two-step process. In the first, a more tractable, delay-constrained sliding window (DCSW) channel model is considered as a proxy to the GE channel. The streaming code is then designed to reliably recover from all erasures introduced by the DCSW channel model. Simulation is typically used to evaluate the performance of the streaming code over the original GE channel, as analytic performance evaluation is challenging. In the present paper, we take an important first step towards analytical performance evaluation. Recognizing that most, efficient constructions of a streaming code are based on the diagonal embedding or horizontal embedding of scalar block codes within a packet stream, this paper provides upper and lower bounds on the block-erasure probability of the underlying scalar block code when operated over the GE channel.

I. INTRODUCTION

Achieving ultra-reliable, low-latency communication is a key cornerstone of the 5th generation (5G) cellular systems as it enables next-generation applications such as augmented reality, assisted driving and telesurgery. Packet erasure, i.e., packet drops, are commonplace in a communication network and erasure coding represents a resource-efficient means of tackling them. The Gilbert-Elliott (GE) [1] [2] channel is commonly used to model the packet erasures that take place over networks [3]–[5]. This motivates the design of efficient packet-level erasure codes for GE channel that are designed to be decoded under a strict decoding-delay constraint. However, the direct design of decoding-delay-constrained erasure codes with guaranteed performance over the GE channel is challenging. To circumvent this, an approach that has been followed in the literature on streaming codes, is to first identify a delay-constrained, sliding-window (DCSW) channel model [6] that approximates the GE channel at hand, and then design a streaming code that is effective when operated over the DCSW channel. The DCSW channel model is characterized by four parameters. In an \((a, b, w, \tau)\) DCSW channel, within any sliding window of length \(w\) there can be either \(\leq a\) random erasures or else a burst erasure of size \(\leq b\). In addition, a \(\tau\)-packet decoding-delay constraint is imposed. Thus, a streaming code that claims to provide reliable communication over the \((a, b, w, \tau)\) DCSW channel, is required to correctly recover the \(t\)-th packet by the time the \((t+\tau)\)-th packet arrives, provided that the erasure pattern encountered is compatible with the \((a, b, w, \tau)\) DCSW channel model. This channel is non-trivial only for \(a \leq b\). It was shown in [6], [7] that without loss of any generality, the parameter \(w\) can be set equal to \(\tau + 1\). As a result, three parameters \(\{a, b, \tau\}\) suffice to characterize the DCSW channel, and we will speak of an \((a, b, w, \tau)\) DCSW channel model, as an \((a, b, \tau)\) DCSW channel model, with the understanding that \(w = \tau + 1\). The \((a, b, \tau)\) DCSW channel is a generalization of an earlier burst-only channel model considered in [3], [9]. The latter channel model may be viewed as being equivalent to an \((a = 1, b, \tau)\) DCSW channel model. We formally define an \((a, b, \tau)\) streaming code as a packet-level erasure code that can recover with decoding-delay constraint \(\tau\), from all erasure patterns permitted by the \((a, b, \tau)\) DCSW channel model. This was shown in [6], that the rate \(R\) achievable by an \((a, b, \tau)\) streaming code is bounded above as per \(R \leq \frac{\tau + 1 - a}{\tau + 1 - a - b} \triangleq R_{\text{opt}}\). This bound is tight as constructions that achieve this bound for all parameter sets \(\{a, b, \tau\}\) are presented in [10], [11]. Simulation is typically used to evaluate the performance of the streaming code over the GE channel approximated by the DCSW channel model, as analytic performance evaluation over the original GE channel is challenging. We take an important first step here towards analytical performance evaluation.
Notation: We set \([a, b] = \{a, a + 1, \ldots, b\}, [b] = [1, b], x^+ = \max\{0, x\}, \bar{x} = 1 - x\). For \(j \geq i\), \(e_j^i = (e_i, e_{i+1}, \ldots, e_j)\) and \(w(e_j^i) = |\{i' \in [i, j] \mid e_i = 1\}|\) and span\((e_j^i) = i_{\text{max}} - i_{\text{min}} + 1\) where \(i_{\text{max}} = \max\{i' \in [i, j] \mid e_i \neq 0\}\) and \(i_{\text{min}} = \min\{i' \in [i, j] \mid e_i \neq 0\}\). For \(j < i\), \(e_i^j = \phi\), \(w(e_i^j) = 0\). Also, \(e_i^j \cup e_j^i = (e_i \mid i \in [i_1, i_2]) \cup [i_2, j_2)]\) and \(1^T \triangleq [1, 1]\).

Outline: In Section II we first provide background on GE channel and streaming codes, then describe our problem setup and contributions. We present methods to calculate probability of a set of erasure patterns in Section III. In Section IV we present the computation of BEP for MDS codes followed by upper bounds on BEP of cyclic codes. In Section V we present upper and lower bounds on BEP of ESC codes and conclude in Section VI.

II. BACKGROUND

1) GE Channel: The GE channel is a 2-state Markov channel, defined by parameters \((\alpha, \beta, \epsilon_0, \epsilon_1)\). The channel has two states, good and bad, indexed by 0, 1 respectively. Here, in the good state, the channel is PEC(\(\epsilon_0\)) and in bad state it is PEC(\(\epsilon_1\)), where PEC(\(\epsilon\)) is a packet erasure channel with packet erasure probability \(\epsilon\) and \(\alpha, \beta\) are transition probabilities as shown in Fig. 1. Most of the literature on GE channel focuses on the version of the channel where errors are introduced in place of erasures. In [12], the authors present an analytical expression for the probability \(P(n, k)\) that \(k\) out of \(n\) symbols transmitted across GE error channel are in error. In [13], the performance of burst-error correcting codes over the GE error channel is studied and recursive expressions are derived for codeword erasure probability. A generating-series approach for analytically calculating \(P(n, k)\) is presented in [14] along with expressions for various burst-error statistics. In [15], a performance comparison between random and burst-error correcting codes over the GE error channel is carried out, by coming up with recursive expressions in both cases for the probability of error. Packet loss probability on the GE erasure channel for a class of burst erasure correcting codes called maximally short codes is analyzed in [8].

2) Streaming Code Construction: The streaming code constructions in the literature have employed a packet-expansion approach where each coded packet is composed of \(n\) symbols of which \(k\) are message symbols and the remaining \(n - k\) are parity symbols. The \(t\)-th packet is represented by \(x(t)^T = [u(t)^T \ p(t)^T] = (x_1(t), x_2(t), \ldots, x_n(t))\) where \(u(t) \in \mathbb{F}_q^n\) represents message and \(p(t) \in \mathbb{F}_q^{n-k}\) represents parity. Streaming codes are obtained by diagonal embedding (DE) of an \([n, k]\) scalar block code \(C\), in such a way that the diagonal stream of coded symbols \((x_1(t), x_2(t+1), \ldots, x_n(t+n-1))\) belongs to the underlying \([n, k]\) scalar code \(C\) (see Fig. 2(a)). DE of \(C\) will result in an \((a, b, \tau)\) streaming code (see [7]), if and only if \(C\) satisfies the following recovery property for every codeword \(c = (c_1, c_2, \ldots, c_n)\) belonging to \(C\):

- for any \(i \in [1, n]\), the code symbol \(c_i\) should be recoverable from \(\{c_j \mid j \in [1, i-1]\} \cup \{c_j \mid j \in [i, \min\{i + \tau, n\}] \setminus E\}\) for any \(E \subseteq [i, n]\) such that \(i \in E\) and either \(|E| \leq a\) or \(\max(E) - \min(E) \leq b - 1\).

We declare an erasure pattern \((e_1, \ldots, e_n) \in \{0, 1\}^n\) as an admissible erasure pattern (AEP) of the \((a, b, \tau)\) DCSW channel if for all \(i \in [1, n-\tau]\) either \(w(e_i^{i+\tau}) \leq a\) or \(\text{span}(e_i^{i+\tau}) \leq b\). It can be seen that the above required recovery property of the scalar block code \(C\) is equivalent to guaranteeing recovery of all code symbols with delay \(\tau\) from all AEP of the \((a, b, \tau)\) DCSW channel. We will refer to a scalar block code satisfying this recovery property as an \((a, b, \tau)\) Embedded Scalar Code (ESC). Note that any \([n = \tau + 1, k = \tau + 1 - a]\) MDS code satisfies this property for the special case \(a = b = 1\) and it’s DE results in a rate-optimal \((a, b = a, \tau)\) streaming code. In [7], [10], [11], [16], [17], \([n = \tau + 1 - a + b = \tau + 1 - a]\) ESCs are presented for all valid \((a, b, \tau)\), giving rise to rate-optimal streaming codes for all parameter sets. While the above description has focused on DE, it is also possible to derive a streaming code through horizontal embedding as discussed below.
A. Problem Setup

In this paper, we study the erasure-recovery performance of ESCs over the GE channel. The performance metric we use here is block-erasure probability (BEP). For a block code, we say that a block-erasure has occurred if at least one erased code symbol is not recoverable from non-erased code symbols. Let $E \subseteq \{0,1\}^n$ be the set of erasure patterns that can be recovered by an $[n,k]$ block code $C$, then the BEP of $C$ over $GE(\alpha, \beta, \epsilon_0, \epsilon_1)$ channel is defined as:

$$BEP(C) = 1 - \sum_{e^n_1 \in E} P(e^n_1),$$

where $P(e^n_1)$ is the probability of seeing erasure pattern $e^n_1$ over $GE(\alpha, \beta, \epsilon_0, \epsilon_1)$ channel. In the context of ESC, block-erasure means that at least one code symbol is not recovered within the delay constraint. We note that an ESC might correct some erasures outside the admissible patterns of the DCSW channel, but this depends on the exact structure of the scalar code. Since our goal here is to see how well DCSW channel approximates GE channel we assume that an $(a,b,\tau)$ ESC recovers only the admissible erasure patterns of $(a,b,\tau)$ DCSW channel, within delay constraint $\tau$. The BEP of $(a,b,\tau)$ ESC $C_s$ over $GE(\alpha, \beta, \epsilon_0, \epsilon_1)$ channel is thus defined as:

$$BEP(C_s) = 1 - \sum_{e^n_1 \in AEP} P(e^n_1),$$

where $AEP \subseteq \{0,1\}^n$ is the set of admissible erasure patterns of $(a,b,\tau)$ DCSW channel.

An Application of BEP Analysis: Streaming codes can also be constructed by introducing separate parity packets. In this method, for every $k$ message packets there will be $(n - k)$ parity packets. Here the idea is to horizontally embed $\ell$ codewords of an $[n,k]$ scalar code to form $n$ packets each containing $\ell$ symbols (See Fig. 2(b)). We will repeat this process for every $k$ message packets. The packet level code obtained by horizontal embedding (HE) of scalar block code $C$ is an $(a,b,\tau)$ streaming code if and only if $C$ is an $(a,b,\tau)$ ESC. Thus, the scalar block codes used in diagonal embedding method to obtain rate-optimal streaming codes can be used here as well to obtain rate-optimal streaming codes. The stream of packets obtained via HE can be divided into blocks each containing $n$ packets, such that that no two blocks share symbols from same codeword. Hence the BEP of streaming codes based on HE is equal to BEP of ESC.

Packet Erasure Probability For DE of ESC: Without loss of generality we assume recovery of $k$-th packet. Packet erasure probability (PEP) for streaming code obtained by DE is given by:

$$PEP = P\left( (E_k = 1) \cap \left( \bigcup_{i=1}^{k} D_i (E_{k-i+1}^{k-i+n}) \right) \right);$$

where $D_i$ is the event that we fail to recover $x_i(k)$ and this depends on the erasures seen in the interval $[k-i+1 : k-i+n]$ given by $E_{k-i+1}^{k-i+n}$ as shown in Fig. 3.
Fig. 3: Each column here is a coded packet and row is a symbol within the packet. Say we are interested in recovery of $k$-th packet. The erasures $E^k_1$ impact recovery of message symbol $x_k(k)$ whereas the erasures $E^{k+n-1}_k$ impact recovery of $x_1(k)$.

Note that if $E^{k-i+n}_{k-i+1} \in \text{AEP}$ then it implies successful recovery of $x_i(k)$. Therefore the failure event is a subset of $E^{k-i+n}_{k-i+1} \notin \text{AEP}$. Hence we get:

$$
\text{PEP} \leq P\left( (E_k = 1) \cap \left( \bigcup_{i=1}^{n}(E^{k-i+n}_{k-i+1} \notin \text{AEP}) \right) \right)
$$

Note that this is a loose bound and can be improved further by

1) characterizing the superset of AEP in equation (1) that would still allow for recoverability of $x_i(k)$,
2) by coming up with bounds on $P(E^{k+i+n-1}_{k+i} \notin \text{AEP} | E_k = 1)$ instead of $P(E^{k+i+n-1}_{k+i} \notin \text{AEP})$ and
3) by tightening the bound for $P \left( \left( \bigcup_{i=1}^{n}(E^{k+i+n-1}_{k+i} \notin \text{AEP}) \right) \right) | E_k = 1$ instead of using union bound.

Our Contributions: In the present paper, we provide analytical expressions for the BEP over GE channel of block codes that can recover exactly from either at most $a$ random erasures or a burst erasure of size at most $b$. Direct computation of the BEP for ESCs is computationally difficult as it would require enumerating over all admissible erasure patterns. In order to make this tractable, we come up with a superset and a subset of the AEP set whose probability can be computed in polynomial time. This results in upper and lower bounds for BEP of ESCs. We observe that these bounds closely approximate BEP for various $(a, b, \tau)$ values, GE channel parameters (see an example in Fig. 5). The upper bounds we prove for BEP allows for picking the best possible $a, b$ for a given decoding delay $\tau$ and BEP requirement $P_{rh}$ (see Fig. 6).

Outline: In Section III, we first provide a recap on how to compute probability of seeing $k$ erasures in a window of size $n$ over GE channel. We then follow it up with a window based method to calculate probability of a set of erasure patterns that are characterized by their weights in smaller windows. In Section IV we present the computation of BEP for MDS codes followed by upper bounds on BEP of cyclic codes. In Section V we present upper and lower bounds on BEP of ESC codes and conclude in Section VI.

III. Computing Probabilities in GE Channel

In this section, we first show computation of probability of seeing $k$ erasures in an $n$ length window, $P(n, k)$. The closed form expression for this is presented in [14], we provide a simple proof here for completeness. We will then introduce a window based method to calculate probability of a set of erasure patterns that are characterized by their weights in smaller windows. Let $S_t \in \{0, 1\}$ be the random variable denoting the state of GE channel at time $t$ and let $E_t \in \{0, 1\}$ be the random variable indicating the presence of erasure at time $t$, where $E_t = 1$ indicates an erasure at time $t$. Thus $P(E_t = 1|S_t = 0) = \epsilon_0$, $P(E_t = 1|S_t = 1) = \epsilon_1$. Let $\pi_g = \frac{\beta}{\alpha + \beta}$ denote the steady state probability in good state. For any set $T \subseteq \{0, 1\}^n$, any element $e^n_1 \in \{0, 1\}^n$ we use the notation $P(T) = P(E^n_1 \in T)$ and $P(e^n_1) = P(E^n_1 = e^n_1)$.
We will now show a closed form expression for \( m \)
expression for the probability of seeing \( k \)
given by \( g(n, k, \pi) \), \( b(n, k, \pi) \) respectively. More formally, \( g(n, k, \pi) = P(w(E_{t+1}^{t+n}) = k, S_{t+n} = 0) \) and 
\( b(n, k, \pi) = P(w(E_{t+1}^{t+n}) = k, S_{t+n} = 1) \). This is clearly independent 
of \( t \) given \( \pi \). Probability of seeing \( k \) erasures in an \( n \)-length window \( P(n, k) \) can be computed as \( P(n, k) = g(n, k, \pi_g) + b(n, k, \pi_g) \).

Given the probability of being in bad state for \( r \) out of \( n \) instances and ending in good, bad states, given by 
\( g_B(n, r, \pi) = P(w(S_{t+n}^{t+1}) = r, S_{t+n} = 0) \), \( b_B(n, r, \pi) = P(w(S_{t+n}^{t+1}) = r, S_{t+n} = 1) \) respectively, the probabilities 
\( g(n, k, \pi) \) and \( b(n, k, \pi) \) can be computed as shown below:

\[
\begin{bmatrix}
g(n, k, \pi) \\
b(n, k, \pi)
\end{bmatrix} = \sum_{r=0}^{n} \sum_{b=0}^{\min\{k, r\}+} \binom{r}{b} \binom{n-r}{k-r} \left( \frac{\pi_0 + b - r}{\epsilon_0} \right) \frac{k-b}{\epsilon_1} \left[ g_B(n, r, \pi) b_B(n, r, \pi) \right].
\]

We find the generating functions corresponding to \( g_B(n, r, \pi) \) and \( b_B(n, r, \pi) \) given by:

\[
G_B(L, Z) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} g_B(n, r, \pi) L^n Z^r
\]

and

\[
B_B(L, Z) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} b_B(n, r, \pi) L^n Z^r
\]

respectively.

![State diagram illustrating the recursion of generating functions](image)

**Fig. 4:** State diagram illustrating the recursion of generating functions \( G_B(L, Z) \) and \( B_B(L, Z) \).

By the state diagram shown in Fig. 4 the generating function satisfies following recursion:

\[
\begin{bmatrix}
G_B(L, Z) \\
B_B(L, Z)
\end{bmatrix} = \begin{bmatrix}
\bar{\alpha}L & \beta L \\
\alpha LZ & \bar{\beta}LZ
\end{bmatrix} \begin{bmatrix}
G_B(L, Z) \\
B_B(L, Z)
\end{bmatrix} + \begin{bmatrix}
\pi \\
\bar{\pi}
\end{bmatrix}
= M(L, Z) \begin{bmatrix}
1 - \bar{\beta}LZ & \beta L \\
\alpha LZ & 1 - \bar{\alpha}L
\end{bmatrix} \begin{bmatrix}
\pi \\
\bar{\pi}
\end{bmatrix}
\]

where \( M(L, Z) = (1 - \bar{\alpha}L - \bar{\beta}LZ + (1 - \alpha - \beta)ZL^2)^{-1} \)

\[
= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} m(n, r) L^n Z^r
\]

From (3), \( g_B \) and \( b_B \) can be determined as shown below:

\[
\begin{bmatrix}
g_B(n, r, \pi) \\
b_B(n, r, \pi)
\end{bmatrix} = \begin{bmatrix}
\frac{\pi}{\bar{\pi}} & \frac{\beta}{-\bar{\alpha}} \\
\frac{\bar{\pi}}{\bar{\alpha}} & \frac{\alpha}{\bar{\pi}}
\end{bmatrix} \begin{bmatrix}
m(n, r) \\
m(n-1, r)
\end{bmatrix}
\]

We will now show a closed form expression for \( m(n, r) \). By the power series expansion for \( M(L, Z) \) we have:

\[
M(L, Z) = \sum_{a=0}^{\infty} (\bar{\alpha} + \bar{\beta}Z - (1 - \alpha - \beta)LZ) a \ L^a
\]

\[
= \sum_{a=0}^{\infty} \sum_{z=0}^{\infty} \sum_{b=0}^{\infty} \binom{a}{z} \binom{z}{b} L^{a+b} Z^z (\alpha - \bar{\beta})^b \ \bar{\beta}^z \ a^a \ z^z
\]
In order to obtain \( m(n, r) \), we set \( z = r, b = n - a \) to get:

\[
m(n, r) = \sum_{a=\max\{r,n-r\}}^{n} \binom{a}{r} \binom{r}{n-a} (\alpha - \beta)^{n-a} \beta^{a-r} \alpha^{a-r}.
\]

From (5), (2) and (4) we have an analytical expression for computation of \( g(n, k, \pi), b(n, k, \pi) \). We will now use this to come up with analytical expressions for BEP of random erasure correcting codes, burst erasure correcting codes and either random or burst erasure correcting codes in the next section.

2) Window based method to calculate probability of erasure patterns: In the following Lemma we will show a method to compute the probability of set of erasure patterns defined by their weight in smaller windows.

**Lemma III.1.** Let the window \([1, n]\) be partitioned into \( \ell \) smaller windows defined by \( 0 = n_0 < n_1 < n_2 < \cdots < n_\ell = n \). Let \((i_1, i_2, \ldots, i_\ell)\) be such that \( i_j \leq n_j - n_{j-1} \) for all \( j \in \ell \). Then the probability of seeing \( i_j \) erasures in window \([n_{j-1} + 1, n_j]\) for all \( j \in \ell \) is given by:

\[
P_e = 1^T Q_\ell Q_{\ell-1} \cdots Q_1 \Pi
\]

where \( Q_j = \begin{bmatrix} g(n_j - n_{j-1}, i_j, 1) & g(n_j - n_{j-1}, i_j, 0) \\ b(n_j - n_{j-1}, i_j, 1) & b(n_j - n_{j-1}, i_j, 0) \end{bmatrix} \) and \( \Pi = \begin{bmatrix} \pi_g & \bar{\pi}_g \end{bmatrix}^T \).

**Proof:** We prove this by induction. For \( \ell = 1 \) it is clear that:

\[
\begin{bmatrix}
P(w(E_{1}^{n_1})) = i_1, S_{n_1} = 0 \\
P(w(E_{1}^{n_1})) = i_1, S_{n_1} = 1
\end{bmatrix} = \begin{bmatrix} g(n_1, i_1, 1) & g(n_1, i_1, 0) \\ b(n_1, i_1, 1) & b(n_1, i_1, 0) \end{bmatrix} \Pi,
\]

by the definition of functions \( g, b \). Now let us assume that for \( \ell = j \):

\[
\begin{bmatrix}
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_j} = 0, \forall j' \in [j] \\
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_j} = 1, \forall j' \in [j]
\end{bmatrix} = Q_j Q_{j-1} \cdots Q_1 \Pi.
\]

Therefore:

\[
\begin{bmatrix}
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_{j+1}} = 0, \forall j' \in [j+1] \\
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_{j+1}} = 1, \forall j' \in [j+1]
\end{bmatrix} = \sum_{s_{n_j}} \begin{bmatrix}
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_{j+1}} = s_{n_j}, S_{n_{j+1}} = 0, \forall j' \in [j+1] \\
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_{j+1}} = s_{n_j}, S_{n_{j+1}} = 1, \forall j' \in [j+1]
\end{bmatrix}
\]

\[
= Q_{j+1} \begin{bmatrix}
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_j} = 0, \forall j' \in [j] \\
P(w(E_{n_{j-1}+1}^{n_j})) = i_{j'}, S_{n_j} = 1, \forall j' \in [j]
\end{bmatrix}
\]

Equation (6) follows as \( Q_{j+1}(s_{n_{j+1}}, s_{n_j}) = P(w(E_{n_{j+1}}^{n_j}) = i_{j+1}, S_{n_{j+1}} = s_{n_{j+1}} | s_{n_j} = s_{n_j}) \). \( \square \)

The probability of observing an erasure pattern \( e_1^{n} \) can be obtained by setting \( \ell = n, n_j = j \) for all \( j \in [0, n] \) and \((i_1, \ldots, i_n) = (e_1, \ldots, e_n)\) in Lemma III.1

\[
P(E_1^n = e_1^n) = 1^T \Psi_{e_n} \Psi_{e_{n-1}} \cdots \Psi_{e_1} \Pi,
\]

where \( \Pi = [\pi_g, \bar{\pi}_g]^T, \Psi_0 = (I - \Gamma) \Psi \) and \( \Psi_1 = \Gamma \Psi \) given

\[
\Psi = \begin{bmatrix} 1 - \alpha & \beta \\ \alpha & 1 - \beta \end{bmatrix}, \quad \Gamma = \begin{bmatrix} e_0 & 0 \\ 0 & e_1 \end{bmatrix}.
\]

**IV. Block Codes over GE Channel**

The computation of BEP of a block code can be done by characterizing the set \( E \) of correctable erasure patterns and then computing the probability of the set \( \{0, 1\}^n \setminus E \). Characterizing a subset of correctable erasures \( E \) results in an upper bound on BEP whereas characterizing a superset of \( E \) results in a lower bound.
1) Random erasure correcting code: The probability of observing erasure patterns with weight larger than \(a\) in window of length \(n\) over \(GE(\alpha, \beta, \epsilon_0, \epsilon_1)\) channel is given by:

\[
P_{\text{rand}}(n,a) = P(\{e_i^n | w(e_i^n) > a\})
\]

\[
= \sum_{i=0}^{n} g(n,i,\pi_g) + b(n,i,\pi_g).
\]  

Suppose an \([n,k = n-a]\) MDS code is used over GE channel. Then, the BEP of the MDS code is given by probability of observing erasure patterns of weight larger than \(a\). Therefore, BEP of an \([n,k = n-a]\) MDS code is equal to \(P_{\text{rand}}(n,a)\).

2) Burst erasure correcting code: Here we calculate the probability of erasure patterns whose span is greater than \(b\) in a window of length \(n\) given by \(P_{\text{burst}}(n,b)\). It is clear to see that BEP of any \(b\) burst erasure correcting code with block length \(n\) is upper bounded by \(P_{\text{burst}}(n,b)\). Let \(B = \{e_i^n \in \{0,1\}^n | \text{span}(e_i^n) \leq b\}\) be the set of erasures whose span is at most \(b\). The set \(B\) can be partitioned by the index \(i \in [n]\) where the erasure starts, \(B = \{\emptyset\} \cup \bigcup_{i=1}^{n} \{e_i^n | e_i^i-1 = 0, e_i = 1, \text{span}(e_i^n) \leq b\}\). The sum of probability of the erasure patterns where the first erasure is at index \(i\) is given by:

\[
b_i = P(\{e_i^n | e_i^i-1 = 0, e_i = 1, \text{span}(e_i^n) \leq b\})
\]

\[
= P(\{e_i^n | e_i^i-1 = 0, e_i = 1, e_i^i+b' = 0\}), \quad b' = \min(b,n-i+1)
\]

\[
= 1^T \sum_{e_{i+b'-1}=0}^1 \cdots \sum_{e_{i+2}=0}^1 \sum_{e_{i+1}=0}^1 \prod_{\psi_i^0}^{n+1-i-b'} \prod_{\psi_i^{i+1}}^{b-i} \prod_{\psi_i^{i+1}}^{1}
\]

\[
= 1^T \prod_{\psi_i^0}^{n+1-i-b'} \prod_{\psi_i^{i+1}}^{b-i} \prod_{\psi_i^{i+1}}^{1}
\]

\[
= 1^T Q_{n+1-i-b'} \prod_{\psi_i^{i+1}}^{b-i} \prod_{\psi_i^{i+1}}^{1}
\]

\[
\text{where } Q_j = \begin{bmatrix} g(j,0,1) & g(j,0,0) \\ b(j,0,1) & b(j,0,0) \end{bmatrix}
\]

\[
(8)
\]

\[
P_{\text{burst}}(n,b) = 1 - P(B) = 1 - g(n,0,\pi_g) - b(n,0,\pi_g) - \sum_{i=1}^{n} b_i.
\]  

\[
(9)
\]

\[
(10)
\]

**Remark IV.1.** In the computation of \(P_{\text{burst}}(n,b)\), the matrix \(\Psi^i\) can be computed easily by writing it in the following form:

\[
\Psi^i = \begin{bmatrix} 1 - \mu(1 - \rho^i) & (1 - \mu)(1 - \rho^i) \\ \mu(1 - \rho^i) & 1 - (1 - \mu)(1 - \rho^i) \end{bmatrix}
\]

where \(\rho = 1 - \alpha - \beta\) and \(\mu = \frac{\alpha}{\alpha+\beta}\). This computation technique was introduced in [12] to analyze the effect of interleaving.

3) Burst or random erasure correcting code: Consider an \([n,k]\) block code \(C\) that can recover only from erasure patterns given by \(E = A \cup B\) where \(A\) is the set of erasure patterns that have weight at most \(a\) and \(B\) is the set of erasure patterns that have span at most \(b\). The BEP of this code when used over GE is given by:

\[
P_{\text{rand,burst}}(n,a,b) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B)
\]

\[
= P_{\text{rand}}(n,a) + P_{\text{burst}}(n,b) - 1 + P(A \cap B).
\]  

\[
(11)
\]

where the last equation follows from equations (7) and (10). The \(P(A \cap B)\) can be counted in a way similar to \(P_{\text{burst}}(n,b)\). The erasure patterns in the set \(A \cap B\) can be partitioned into subsets indexed by \(i\), where \(i \in [n]\) is the index where first erasure is observed. The weight of erasure patterns are limited to be \(\leq a\) here. For a given \(i\), the span of erasure can be atmost \(b' = \min(b,n-i+1)\). For a fixed \(i\), the probability of these erasure patterns can be determined by dividing the \(n\)-length window into \(\ell = 4\) small windows such that \(n_0 = 0, n_1 = i-1, n_2 = i, n_3 = i+b'-1, n_4 = n\) with weights of erasures in each of these windows being \((i_1, i_2, i_3, i_4) = (0,1, <\leq a-1,0)\).

Then the probability of erasure patterns in the set indexed by \(i\) can be determined from Lemma III.1 as:

\[
a_i = 1^T Q_{n-i-b'+1} M \Psi_1 Q_{i-1} \Pi,
\]
where

\[ M = \min \{a-1, b'-1\} \sum_{i_3=0}^{\min(a-1, b'-1)} \begin{bmatrix} g(b'-1, i_3, 1) & g(b'-1, i_3, 0) \\ b(b'-1, i_3, 1) & b(b'-1, i_3, 0) \end{bmatrix}, \]

and \( Q_j \) is defined as shown in (9). By also adding the probability of not seeing any erasure in the \( n \) length window we get:

\[ P(A \cap B) = g(n, 0, \pi_g) + b(n, 0, \pi_g) + \sum_{i=1}^{n} a_i. \]  

(12)

By substituting (12) in (11) we get the analytical expression for BEP of a code that can only correct from either a random erasures or a burst erasure of size atmost \( b \).

It follows from definition that

\[ P_{\text{rand, burst}}(n, a = 1, b) = P_{\text{burst}}(n, b) \]

and

\[ P_{\text{rand, burst}}(n, a = b, b) = P_{\text{rand}}(n, b). \]

Hence,

\[ P_{\text{rand}}(n, b) \leq P_{\text{rand, burst}}(n, a, b) \leq P_{\text{burst}}(n, b) \]

for all \( a \in [b] \).

We note that \( P_{\text{rand, burst}}(n, a, b) \) is an upper bound for the BEP of cyclic codes with parameters \( [n, k = n-b, d_{\text{min}} = a + 1] \). This follows as the cyclic code can correct either a random erasures or a burst erasure of size \( b \).

When an \([n, k]\) block code is used, any lost symbol can be recovered, if recoverable, within a decoding delay of \( \tau = n - 1 \). If we compare \([n = \tau + 1, k = n - b]\) block codes which can recover only from erasure patterns that have weight atmost \( a \) or span atmost \( b \), then it can be seen that block code with \( a = b \) (MDS code) gives the smallest BEP. However ESCs offer a family of codes that encompass MDS family of codes and can offer better BEPs in comparison to MDS family under same rate and same delay constraint by picking \( n > \tau + 1 \).

V. ESC OVER GE CHANNEL

Now consider the setting where a ESC that is designed for DCSW channel is used over a GE channel. The BEP can be computed by enumerating all the non admissible erasure patterns and then summing over their probabilities. However this has computational complexity exponential in block length. We therefore propose tractable upper bounds and lower bounds for BEP in this section that can be computed in polynomial time. In the following Lemma we show an upper and lower bound for BEP of \((a, b, \tau)\) ESCs over GE channel using expression for BEP of random or burst erasure correcting block codes. We then improve these bounds in the Lemmas that follow. Let \( P_{\text{ESC}}(n, a, b, \tau) \) be the BEP of \((a, b, \tau)\) ESC, of block length \( n \).

**Lemma V.1.** Let \( n \geq \tau + 1 \). Then, \( P_{\text{rand, burst}}(\tau + 1, a, b) \leq P_{\text{ESC}}(n, a, b, \tau) \leq P_{\text{rand, burst}}(n, a, b) \).

**Proof:** Let \( A_j \) and \( B_j \) be defined as

\[ A_j = \{ e^n_j \mid w(e^{\tau+j}_j) \leq a \}, \quad B_j = \{ e^n_j \mid \text{span}(e^{\tau+j}_j) \leq b \}. \]

The erasure patterns that are admissible by \((a, b, \tau)\) DCSW channel are given by \( \cap_{j=1}^{n-\tau} (A_j \cup B_j) \). Therefore,

\[ P_{\text{ESC}}(n, a, b, \tau) = 1 - P\left( \cap_{j=1}^{n-\tau} (A_j \cup B_j) \right). \]

By definition

\[ P_{\text{rand, burst}}(n, a, b) = 1 - P(R_A \cup R_B) \]

where,

\[ R_A = \{ e^n_i \mid w(e^n_i) \leq a \}, \quad R_B = \{ e^n_i \mid \text{span}(e^n_i) \leq b \}. \]

Clearly \( R_A \cup R_B \) is a subset of admissible erasure patterns. Therefore the upper bound follows. By definition in equation (11),

\[ P_{\text{rand, burst}}(\tau + 1, a, b) = 1 - P(A_1 \cup B_1). \]
Clearly this is a superset of admissible erasure patterns resulting in the lower bound.

The optimal rate \((a,b,\tau)\) ESCs given in \([7, 10, 11, 16]\) have \(n = \tau + 1 + b - a\). Here, we present improved bounds for \(P_{\text{ESC}}(n,a,b,\tau)\), for \(\tau + 1 \leq n \leq \tau + b\). In order to obtain improved upper bound, we first define \(\hat{U}\) and then show that \(\hat{U}\) is a subset of correctable erasure patterns.

\[
\hat{U} \triangleq U_A \cup U_B
\]

\[
U_B = \bigcup_{i=1}^{n-a} \cup_{b'=a+1}^{\min(n-i+1,b)} U_{B,i,b'}
\]

\[
U_A = \{ e^n_i | w(e^n_i) \leq a \} \cup (\bigcup_{i=1}^{n-\tau-1} \cup_{b'=0}^{\min(\tau-a,n)} U_{A,i,b'})
\]

\[
U_{A,i,b'} = \{ e^n_i | w(e^n_i) = 0, w(e^n_i+1) = 1, w(e^n_{i+b'}) = 0, w(e^n_{i+b'+1}) = x \}
\]

It can be verified that \(U_A\) is disjoint from \(U_B\), any two sets \(U_{A,i_1,b_1}, U_{B,i_2,b_2}\) are disjoint given that either \(i_1 \neq i_2\) or \(b_1 \neq b_2\) and any two sets \(U_{A,i_1,b_1}, U_{A,i_2,b_2}\) are disjoint given that either \(i_1 \neq i_2\) or \(b_1 \neq b_2\).

**Lemma V.2 (Improved Upper Bound).** Let \(\tau + 1 \leq n \leq \tau + b\). Then,

\[
P_{\text{ESC}}(n,a,b,\tau) \leq P_{\text{ESC,}\hat{U}}(n,a,b,\tau) \leq P_{\text{rand,bar,at}}(n,a,b),
\]

where \(P_{\text{ESC,}\hat{U}}(n,a,b,\tau) = 1 - P(\hat{U})\). \(\hat{U}\) is defined in \((13)\).

**Proof:** Let \(e^n_i \in U_A\). If \(w(e^n_i) \leq a\), then \(e^n_i\) is an admissible erasure pattern. Let \(e^n_i \in U_{A,i,b'}\) for some \(i \leq n - \tau\), then \(w(e^{\tau+i}_i) \leq a\) and it is also true that \(w(e^{\tau+i}_i) \leq a\). This implies \(e^n_i \in \cap_{j=1}^{i} A_j\). Now looking at window \([i + i_0 - 1, i + i_0 - 1 + \tau]\) for \(i_0 \in [b']\):

\[
w(e^{i+b'-1}_i) = w(e^{i+b'-1}_i) + w(e^{i+b'+1}_i) 
\]

Therefore it is clear that if \(e^n_i \in U_{A,i,b'}\) then \(e^n_i \in \cap_{j=1}^{\tau+1} A_j\). Therefore \(e^n_i \in \cap_{j=1}^{n-\tau} (A_j \cup B_j)\). Now let \(e^n_i \in U_{B,i,b'}\), then when \(i > n - \tau - b' + a\), it implies that span\((e^n_i) = b' \leq b\). Therefore it is an acceptable erasure pattern. For the case when \(i \leq n - \tau - b' + a\) in the window \([j, \tau + j]\) for \(j \in [i, i + b' - a]\); span\((e^{\tau+j}_j) \leq b' \leq b\). Therefore \(e^n_i \in \cap_{j=1}^{i+b'-a} B_j\). For \(j \in [i + b' - a + 1, i + b' - 1]\):

\[
w(e^{\tau+j}_j) = w(e^{i+b'-1}_i) + w(e^{i+b'+1}_i) 
\]

Therefore \(e^n_i \in \cap_{j=1}^{i+b'-a} B_j \cap \cap_{j=i+b'+1}^{i+b' - a + 1} A_j\). For \(j \in [i + b', n - \tau]\) we have:

\[
\text{span}(e^{\tau+j}_j) \leq \tau + j - (i + b' + \tau - a) 
\]

\[
\leq (n - \tau) - b' + a \leq b.
\]

Therefore \(e^n_i \in \cap_{j=1}^{n-\tau} (A_j \cup B_j)\). Hence \(\hat{U} \subseteq \cap_{j=1}^{n-\tau} (A_j \cup B_j)\) and

\[
P_{\text{ESC}}(n,a,b,\tau) = 1 - P(\cap_{j=1}^{n-\tau} (A_j \cup B_j)) \leq 1 - P(\hat{U}) = P_{\text{ESC,}\hat{U}}(n,a,b,\tau).
\]

It can be verified that \(R_A \subseteq U_A\) and \(R_B \setminus R_A \subseteq U_B\) and therefore \(R_A \cup R_B \subseteq \hat{U}\). The improvement from the upper bound seen in Lemma \([V.1]\) therefore follows.
For obtaining an improved lower bound we come up with a super set of correctable erasure patterns given by,

\[
\hat{L} \triangleq L_0 \cup L_A \cup L_B
\]

\[
L_0 = \{e_1^n \mid w(e_1^{i+1}) = 0\},
\]

\[
L_A = \cup_{i=1}^{\tau+1} L_{A,i}, \quad L_B = \cup_{i=1}^{\tau+1-a} L_{B,i}, \quad L_{A,i} = \{e_1^n \mid w(e_1^{i+1}) = 0, e_i = 1, w(e_1^{i+1} \cup e_{i+b}^{\min\{i+n, n\}}) \leq a - 1\}, \quad L_{B,i} = \{e_1^n \mid w(e_1^{i+b-1}) > a, w(e_1^{i+1}) = w(e_1^{i+b}) = w(e_1^{\min\{i+b, n\}}) = 0\}
\]

where \(b_m = \min\{b, \tau + 2 - i\}\). It can be verified that \(L_0, L_A, L_B\) are mutually disjoint, any two sets \(L_{B,i}, L_{B,i'}\) are disjoint given that \(i_1 \neq i_2\) and any two sets \(L_{A,i}, L_{A,i'}\) are disjoint given that \(i_1 \neq i_2\).

**Lemma V.3 (Improved Lower Bound).** Let \(\tau + 1 \leq n \leq \tau + b\). Then,

\[
P_{\text{rand,burst}}(\tau + 1, a, b) \leq P_{\hat{E}_\text{SC},\hat{L}}(n, a, b, \tau) \leq P_{\text{ESC}}(n, a, b, \tau)
\]

where \(P_{\text{ESC},\hat{L}}(n, a, b, \tau) = 1 - P(\hat{L})\), \(\hat{L}\) is defined in (14).

**Proof:** We will now show that the set \(\hat{L}\) is a super set of the admissible erasure patterns given by the set \(\cap_{j=1}^{\tau} (A_j \cup B_j)\). Let \(e_1^n \in \cap_{j=1}^{\tau} (A_j \cup B_j)\), suppose \(w(e_1^{i+1}) = 0\) then clearly \(e_1^n \in L_0\). Otherwise, let \(i\) be the first index in \([1, \tau + 1]\) where an erasure is seen. Then \(w(e_1^{i+1}) = 0, e_i = 1\). But we know that \(e_1^n\) is admissible and therefore \(e_1^n \in A_1 \cup B_1\). Hence it satisfies one of the following conditions:

1) \(w(e_1^{i+1}) \leq a\),
2) \(\text{span}(e_1^{i+1}) \leq b_m, w(e_1^{i+1}) > a\) where \(b_m = \min\{b, \tau + 2 - i\}\).

We first look at case 1, where \(w(e_1^{i+1}) \leq a\). For the case when \(i > n - b, e_1^n\) clearly belongs to the set \(L_{A,i}\). For the case when \(i \leq n - b\), it is necessary that \(w(e_1^{i+1} \cup e_{i+b}^{\min\{i+n, n\}}) \leq a\). Suppose \(w(e_1^{i+1} \cup e_{i+b}^{\min\{i+n, n\}}) > a\), then in the window \([j, \tau + j]\) where \(j = \min\{i, n - \tau\}\) we can show that:

\[
w(e_1^{j+\tau}) > a, \quad \text{span}(e_1^{j+\tau}) > b.
\]

As we know that \(w(e_1^{i+1}) \leq a\) it implies that \(w(e_1^{\min\{i+n\}}) \geq 1\) implying that there is an erasure at index \(i + b\) and at \(i\). Therefore, \(\text{span}(e_1^{j+\tau}) > b\). This contradicts the condition that \(e_1^n \in A_j \cup B_j\). Therefore when \(e_1^n\) satisfies case 1, it belongs to \(L_{A,i}\).

In case 2, \(w(e_1^{i+1}) > a\) and \(\text{span}(e_1^{i+1}) \leq b_m\). Therefore \(w(e_1^{i+1} \cup e_{i+b}^{\min\{i+b, n\}}) = w(e_1^{i+b} \cup e_{i+b}^{\min\{i+b, n\}}) > a\). It is also necessary that \(w(e_1^{\min\{i+n\}}) = 0\), otherwise the window \([j, j + \tau]\) where \(j = \min\{i, n - \tau\}\), will have \(\text{span}(e_1^{j+\tau}) > b\) and \(w(e_1^{j+\tau}) > a\) contradicting that \(e_1^n \in A_j \cup B_j\). Therefore it follows that \(\cap_{j=1}^{\tau} (A_j \cup B_j) \subseteq \hat{L}\) and:

\[
P_{\text{ESC},\hat{L}}(n, a, b, \tau) = 1 - P(\hat{L}) = 1 - P(\cap_{j=1}^{\tau} (A_j \cup B_j)) = P_{\text{ESC}}(n, a, b, \tau).
\]

It is clear to see that \(L_0 \cup L_A \subseteq A_1\) and \(L_B \subseteq B_1 \setminus A_1\). Therefore \(\hat{L} \subseteq A_1 \cup B_1\) and it follows that:

\[
P_{\text{rand,burst}}(\tau + 1, a, b) = 1 - P(A_1 \cup B_1) \leq 1 - P(\hat{L}) = P_{\text{ESC},\hat{L}}(n, a, b, \tau).
\]

We note here that the analytical expression for the improved bounds can be obtained by using the techniques developed in Section IV and these expressions are computable in polynomial time.
Fig. 5: Bounds on BEP for \((a = 3, b = 6, \tau = 10)\) ESC with parameters \((n = 14, k = 8)\) over GE\((\alpha = 10^{-4}, \beta = 0.5, \epsilon_0 = \epsilon, \epsilon_1 = 1)\). Exact BEP plot shown above is obtained by enumerating all the AEP and then computing their probabilities.

Choosing an \(a, b\) for a given delay \(\tau\) and BEP threshold \(P_{Th}\): The parameters \((a, b)\) of \((a, b, \tau)\) ESC can be chosen such that it has highest rate under the constraint \(P_{ESC,E}(n, a, b, \tau) \leq P_{Th}\). This guarantees \(\text{BEP} \leq P_{Th}\) and decoding delay of at most \(\tau\). The figure shows rate gain of ESCs over MDS codes over a GE channel, for the same decoding delay constraint \(\tau\) and BEP threshold \(P_{Th}\).

Fig. 6: The figure depicts rate achievable by MDS code \((b = a)\), burst only ESC \((a = 1)\) and ESC when used over GE\((\alpha = 10^{-4}, \beta = 0.5, \epsilon_0 = \epsilon, \epsilon_1 = 1)\). Here, decoding delay constraint \(\tau = 10\), \(\text{BEP} \leq P_{Th}\) and \(n = \tau + 1 + b - a\). The figure also shows code parameters which give the rates shown.

VI. Conclusion

In this paper we derived computable upper and lower bounds on BEP of ESCs over GE channel, by characterizing tractable subset and superset of correctable erasure patterns. This leads to upper and lower bound on BEP of HE based streaming codes. Extending this result to DE based streaming codes by taking into account error propagation remains an open problem.
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