A PROOF OF QUARK CONFINEMENT IN QCD *

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ABSTRACT

I propose to reformulate the gauge field theory as the perturbative deformation of a novel topological quantum field theory. It is shown that this reformulation leads to quark confinement in \( \text{QCD}_4 \). Similarly, the fractional charge confinement is also derived in the strong coupling phase of \( \text{QED}_4 \). As a confinement criterion, we use the area decay of the expectation value of the Wilson loop.

1. Introduction

An idea of magnetic monopole and its condensation was used in the analytical proofs of ”quark” confinement so far:

1975: Polyakov\(^1\) for 3D compact \( \text{U}(1) \) gauge theory

1977: Polyakov\(^2\) for 3D Georgi-Glashow model

1994: Seiberg-Witten\(^3\) for 4D \( \text{N}=2 \) SUSY YM/QCD

How to prove quark confinement in \( \text{QCD}_4 \) analytically? Recent numerical simulations have established that also in \( \text{QCD}_4 \) the magnetic monopole plays the dominant role in quark confinement (monopole dominance) under the Abelian projection.\(^4\) To explain this fact analytically and then to prove quark confinement, we propose to use a novel type of topological quantum field theory (TQFT)\(^5\) which describes 4D magnetic monopole, and to reformulate QCD as a perturbative deformation of the TQFT.

We can show:\(^6–9\)

1. The TQFT is derived from QCD (without any approximation) in the maximal Abelian gauge (MAG).

2. The calculation of the Wilson loop in 4D YM theory is reduced to that in 2D nonlinear sigma model (NLSM) due to the dimensional reduction. This is an exact result.

3. Area decay of the Wilson loop is derived from 2D Instanton calculus in NLSM.

\(^*\)Report-no: CHIBA-EP-107 (hep-th/9808186); Talk given at the 3rd International conference on Quark Confinement and the Hadron Spectrum, 7-12 June 1998, Jefferson Lab., Newport News, VA, USA.
2. Basic Idea

We take into account the topological nontrivial background field $\Omega_\mu$ which corresponds to 4D magnetic monopole current $k_\mu$. The current $k_\mu$ obeys the topological conservation law $\partial_\mu k_\mu = 0$ and hence $k_\mu$ denotes a loop in 4D spacetime. Usually, $\Omega_\mu$ denotes an arbitrary, but fixed background (as a solution of classical field equation). In our case, we sum up all topological nontrivial background $\Omega_\mu$ after integrating out quantum fluctuations $Q_\mu$,

$$Z_{QCD} = \sum_\Omega \int [dQ_\mu] \exp\{-S_{QCD}[\Omega_\mu + Q_\mu]\},$$  \hspace{1cm} (1)

where

$$S_{QCD} = \int d^D x \left[ -\frac{1}{2g^2} \text{tr}_G(F_{\mu\nu} F_{\mu\nu}) + \bar{\psi}(i\gamma^\mu \slashed{D}_\mu [A] - m)\psi \right].$$  \hspace{1cm} (2)

We must regard this background to be different from the 4D instanton discussed in the topological YM theory (Witten) and also from 3D magnetic monopole (Polyakov).

3. Quantization of YM theory based on BRST formalism

Using the nilpotent BRST transformation $\delta_B$, it is well known that the gauge fixing and the Faddeev-Popov (FP) ghost part is written in the form,

$$\mathcal{L}_{gf} = -i\delta_B G_{gf}, \quad G_{gf} : = \text{tr}_G \left\{ \bar{C} \left( [A] + \frac{\alpha}{2} \phi \right) \right\},$$  \hspace{1cm} (3)

where $\alpha$ is the gauge-fixing parameter and $\phi$ is the Lagrange-multiplier field. For the manifestly covariant Lorentz gauge, we choose $F[A] = \partial^\mu A_\mu$.

In the maximal Abelian gauge (MAG), a suitable choice of gauge fixing parameter $\alpha = -2$ leads to

$$G_{gf} : = \bar{\delta}_B \text{tr}_{G/H} \left( \frac{1}{2} A_\mu(x) A_\mu(x) + iC(x) \bar{C}(x) \right).$$  \hspace{1cm} (4)

where $\bar{\delta}_B$ is the (nilpotent) anti-BRST transformation. Using the decomposition,

$$A_\mu(x) = \Omega_\mu(x) + Q_\mu(x) = \frac{i}{g} U(x) \partial_\mu U(x) \dagger + U(x) \gamma_\mu(x) U \dagger(x),$$  \hspace{1cm} (5)

the action of the TQFT is given by

$$S_{TQFT} : = \int d^D x \ i\delta_B \bar{\delta}_B \text{tr}_{G/H} \left( \frac{1}{2} \Omega_\mu(x) \Omega_\mu(x) + iC(x) \bar{C}(x) \right),$$  \hspace{1cm} (6)

and the partition function is given by

$$Z_{TQFT} = \int [dU] [dC] [d\bar{C}] [d\phi] \exp\left\{-S_{TQFT}[\Omega_\mu]\right\},$$  \hspace{1cm} (7)

where $U$ is an element of the gauge group $G$. 
4. Strategy of a proof

We consider D-dim. QCD (with a gauge group G) for $D > 2$.

4.1. Step 1

In MAG, D-dim. QCD (QCD$_D$) is reformulated as a perturbative deformation of TQFT$_D$. MAG is a partial gauge fixing which fixes the coset part $G/H$ and leaves the maximal torus group $H$ invariant.

4.2. Step 2

It is shown that TQFT$_D$ is equivalent to the coset $G/H$ nonlinear sigma model (NLSM) in $(D-2)$ dimensions. This is a consequence of Parisi-Sourlas dimensional reduction due to the hidden supersymmetry in TQFT (8). As extensively discussed more than 20 years ago, QCD$_4$ and NLSM$_2$ have common properties: renormalizability, asymptotic freedom ($\beta(g) < 0$), dynamical mass generation, existence of instanton solution, no phase transition for any value of coupling constant (one phase), etc. This similarity between two theories can be understood from this correspondence,

$$QCD_4 \supset TQFT_4 \iff G/H \text{ NLSM}_2$$

(8)

For $G = SU(2)$, $G/H$ NLSM is equivalent to $O(3)$ NLSM. Existence of 2D instanton is guaranteed for any $N$, because $\Pi_2(SU(N)/U(1)^{N-1}) = \Pi_1(U(1)^{N-1}) = \mathbb{Z}^{N-1}$.

4.3. Step 3

We define the full string tension $\sigma$ by

$$\sigma := -\lim_{A(C) \to \infty} \frac{1}{A(C)} \ln \langle W^C[A] \rangle_{YM_4},$$

(9)

where $W^C[A]$ is the non-Abelian Wilson loop with the area $A(C)$,

$$W^C[A] := \text{tr} \left[ \mathcal{P} \exp \left( iq \oint_C A_\mu^a(x) T^a dx^\mu \right) \right].$$

(10)

On the other hand, the diagonal string tension $\sigma_{Abel}$ is defined by

$$\sigma_{Abel} := -\lim_{A(C) \to \infty} \frac{1}{A(C)} \ln \langle W^C[a^a] \rangle_{TFT_4},$$

(11)

using the diagonal Wilson loop,

$$W^C[a^a] = \exp \left( iq \oint_C dx^\mu a_\mu^a(x) \right), \quad a_\mu^a(x) := \Omega_\mu^a(x) := \text{tr}(T^3 \Omega_\mu(x)).$$

(12)

From the dimensional reduction, we find

$$\langle W^C[a^a] \rangle_{TFT_4} = \langle W^C[a^a] \rangle_{NLSM_2}.$$

(13)

Then it is shown that in the large Wilson loop limit

$$|\sigma - \sigma_{Abel}| \downarrow 0 \quad (A(C) \uparrow \infty).$$

(14)

This is derived from the non-Abelian Stokes theorem. For the large (non-intersecting planar) Wilson loop, the full string tension $\sigma$ is saturated by the diagonal string tension $\sigma_{Abel}$. This explains the Abelian dominance and magnetic monopole dominance.
4.4. Step 4

The whole problem is reduced to calculating

\[
\langle W^C[a^\Omega] \rangle_{\text{NLSM}_2} = \langle e^{i\frac{4\pi}{g}Q_S} \rangle_{\text{NLSM}_2}, \quad Q_S = \frac{1}{8\pi} \int_S d^2z \epsilon_{\mu\nu} n \cdot (\partial_\mu n \times \partial_\nu n).
\]

Note that the integrand of \(Q_S\) is the instanton density in \(\text{NLSM}_2\). Therefore, \(Q_S\) counts the number of instantons minus that of anti-instantons inside the area \(S\) bounded by the Wilson loop \(C\). In order to perform the actual calculation, we adopt the naive instanton calculus (dilute gas approximation). This leads to the area law of the diagonal Wilson loop and the non-zero diagonal string tension \(\sigma_{\text{Abel}}\) for the fractional charge \(q\) (i.e., when \(q/g\) is not an integer). More systematic instanton calculations enable us to identify the instanton solution with the Coulomb gas of vortices.

Note that all the above steps are exact except for the instanton calculus of the Wilson loop in NLSM. The above results imply that the non-zero string tension \(\sigma\) in QCD\(_4\) follows from the non-zero diagonal string tension \(\sigma_A\) in \(\text{NLSM}_2\). The problem of proving area law in QCD\(_4\) is reduced to the equivalent problem in \(\text{NLSM}_2\).

D-dim. QCD with a gauge group \(G\)

\[
\text{MAG}
\]

D-dim. Perturbative QCD \(\otimes\) deform

D-dim. TQFT

\[
\text{Dimensional reduction}
\]

D-dim. Perturbative QCD \(\otimes\) deform

(D-2)-dim. G/H NLSM

Similar strategy is also applied to QED\(_4\) to prove the existence of strong coupling confinement phase. This follows from the existence of Berezinski-Kosterlitz-Thouless transition of the O(2) \(\text{NLSM}_2\).

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