Abstract
Let $T(x) \in k[x]$ be a monic non-constant polynomial and write $R = k[x]/\langle T \rangle$ the quotient ring. Consider two bivariate polynomials $a(x,y), b(x,y) \in R[y]$. In a first part, $T = p^e$ is assumed to be the power of an irreducible polynomial $p$. A new algorithm that computes a minimal lexicographic Gröbner basis of the ideal $\langle a, b, p^e \rangle$, is introduced. A second part extends this algorithm when $T$ is general through the “local/global” principle realized by a generalization of “dynamic evaluation”, restricted so far to a polynomial $T$ that is squarefree. The algorithm produces splittings according to the case distinction “invertible/nilpotent”, extending the usual “invertible/zero” in classic dynamic evaluation. This algorithm belongs to the Euclidean family, the core being a subresultant sequence of $a$ and $b$ modulo $T$. In particular no factorization or Gröbner basis computations are necessary. The theoretical background relies on Lazard’s structural theorem for lexicographic Gröbner bases in two variables. An implementation is realized in Magma. Benchmarks show clearly the benefit, sometimes important, of this approach compared to the Gröbner bases approach.

Keywords: Gröbner basis, lexicographic order, dynamic evaluation, subresultant

1. Introduction

1.1. Context and results
Gröbner bases are a major tool to solve and manipulate systems of polynomial equations, as well as computing in their quotient algebras. Modern and most efficient algorithms rely on linear algebra on variants of Macaulay matrices \cite{Sturmfels}. Another class of methods, the triangular decomposition, rely on some broad generalizations of the Euclidean algorithm, as initiated by Ritt \cite{Ritt}, Wu \cite{Wu}, and pursued by many researchers (see surveys \cite{Cox, Krick, Weispfenning} for details). Starting from an input system $\mathcal{F}$, these methods produce a family of triangular sets $(T_i)_i$ which enjoy the elimination property and satisfies $\bigcap_i \langle T_i \rangle \simeq \langle \mathcal{F} \rangle$. In dimension zero, these sets form a particular subclass of lexicographic Gröbner bases (lexGb thereafter). This representation is well-understood and implemented since already more than two decades, especially for radical ideals. Although several attempts to represent multiplicities have come out \cite{Buchberger, Gianni, Greuel, Dahan}, they are limited in scope, and resort to sophisticated concepts while obtaining partial information only. Triangular sets, hence standard triangular decomposition methods, cannot in general produce an ideal preserving decomposition: for example a mere primary ideal in dimension zero is not triangular in general (think of the primary ideal $\langle y^2 + 3x, xy + 2x, x^2 \rangle$ of associated prime $\langle x, y \rangle$; it is a reduced lexGb...
for $x < y$ and does not generate a radical ideal). However, the underlying key algorithmic concepts may still be relevant, and the present work shows it is the case, yet in a particular situation.

The algorithms proposed in this work consider lexGbs instead of triangular sets. Although it also builds upon the Euclidean algorithm and hence can be compared to triangular decomposition algorithms, it goes only half-way when decomposing. A splitting follows from those of polynomials in $x$ only, by generalizing dynamic evaluation. In particular, no decomposition is produced in the first part, when $T = p^e$ is the power of an irreducible polynomial. One can expect to decompose along the $y$-variable too, but this requires dynamic evaluation in two variables. Nonetheless, the methods introduced here pave the way toward such decompositions. Let us illustrate this by a toy example.

**Example 1 (Computing a lexGb).** From now on p.r.s stands for (subresultant) pseudo-remainder sequence. Below it is computed modulo $t_1$ (Eq. (1)) or modulo $t_1'$ (Eq. (2)), $S_i(a, b)$ refers to the $i$-th subresultant of $a$ mod $t_1$ and $b$ mod $t_1$. See Definition 2 for details.

$$
\begin{align*}
\left\{ 
\begin{array}{l}
  a := (y + x)(y + 1 + x)/(y - 1) \\
  b := (y + x)(y + 1 - x) \\
  t_1 := x^2 
\end{array}
\right. \\
\left| \begin{array}{l}
  \text{prs}(a, b) \text{ mod } t_1 \\
  a \ \rightarrow \ b \ \rightarrow \ S_1(a, b) = -4xy \\
  \text{prs}(\bar{b}, \bar{c}) \text{ mod } t_1' \\
  y^2 + y \rightarrow \bar{S}_1(\bar{b}, \bar{c}) = y \\
\end{array}\right| \\
(1)
\end{align*}
$$

The last non-zero subresultant $S_1(a, b) = -4xy$ is nilpotent modulo $t_1$. We factor out $-4x$ (it is obvious here, in general realized through a removal of nilpotent coefficients, and Weierstrass factorization), and divide $-4xy$ by $-4x$ and $t_1$ by $\frac{-4x}{4} = x$ in order to make them monic. We obtain now a monic polynomial $c := y = \frac{-4xy}{-4x} \text{ mod } t_1' := \frac{b}{x} = x$. We restart then a subresultant p.r.s of $(b \text{ mod } t_1'(x))$ and $(c \text{ mod } t_1'(x))$ modulo $t_1'(x) = x$, while keeping record of the division by $x$.

$$
\begin{align*}
\left\{ 
\begin{array}{l}
  \bar{b} := b \text{ mod } t_1' = y(y + 1) \\
  \bar{c} := c \text{ mod } t_1' = y \\
  t_1'(x) := x 
\end{array}
\right. \\
\left| \begin{array}{l}
  \text{prs}(\bar{b}, \bar{c}) \text{ mod } t_1' \\
  y^2 + y \rightarrow \bar{S}_1(\bar{b}, \bar{c}) = y \\
\end{array}\right| \\
(2)
\end{align*}
$$

The last non-zero subresultant $S_1(\bar{b}, \bar{c}) = y$ is a gcd of $\bar{b}$ and $\bar{c}$ (modulo $t_1'(x) = x$) so that $\langle \bar{b}, \bar{c}, x \rangle = \langle y, x \rangle$. Note that $\bar{S}_1(\bar{b}, \bar{c}, x) = [y, x]$ is a lexGb of $\langle x, \bar{c}, \bar{b} \rangle$. A minimal lexGb of $\langle a, b, x^2 \rangle$ is then obtained by multiplying by $x$ (of which we have kept record) the lexGb $[y, x]$ and concatenating $b$ (this is Line 44 of Algorithm 5).

$$
[x \cdot x, x \cdot y] \text{ cat } [b] = [x^2, xy, y^2 + (3x + 1)y + x].
$$

This lexGb of $\langle a, b, x^2 \rangle$ is minimal but not reduced. If necessary, it suffices to compute adequate normal forms to obtain the reduced one. We will consider lexGbs as output by the algorithms, hence minimal Gröbner bases but not necessarily reduced. \hfill \Box

As this example shows, several new steps come into play to handle the computation of the subresultant p.r.s. modulo $T = p^e = x^2$: removal of nilpotent (Algorithm 1) and Weierstrass factorization (Algorithm 2). These two algorithms transform a non monic polynomial modulo $p^e$ to an equivalent monic one (Algorithm 3). In this way, pseudo divisions can be carried through to retrieve the “first nilpotent” and the “last non nilpotent” polynomials in a modified subresultant p.r.s (Algorithm 4). It suffices then to iterate the above process to compute a minimal lexGb (Algorithm 5).
**Generalizing dynamic evaluation.** In a second part of the article, $T$ can be any monic polynomial. The most interesting cases are when a (monic) large degree factor of $T$ is also a factor of the resultant of $a$ and $b$. If $T$ has no common factor or only a small common factor, the degree of the ideal $(a, b, T)$ is small compared to the degrees of the input polynomials and while the algorithms proposed work in this case, they are slower than Gröbner bases.

This however covers most important situations, for example when the input are two polynomials $a, b$ without a modulus $T$. Then we take $T = \text{Res}_y(a, b)$ the resultant itself. In the experimental section 5 (Columns 4 & 7 of the tables), timings show indeed that computing the resultant (Column 3) and applying the algorithms of this work (Column 4) reveals faster or simply competitive with computing a lexGb of $(a, b)$ (Column 5). Computing the squarefree decomposition of the resultant (Column 6) and applying the algorithms of this work is always much faster.

When $T$ is not the power of an irreducible polynomial, Weierstrass factorization does not apply in general and making polynomials monic in order to restart subresultant computations becomes impossible “globally” (that is modulo $T$, while modulo primary factors $p^e$, it does). Applying the “local/global” philosophy of classic dynamic evaluation fails here too, the polynomial $T$ being not squarefree. We show that it is possible to extend it to a general $T$ with splittings of type “invertible/nilpotent”, instead of the standard “invertible/zero”. The algorithms developed in the first part to treat the local case are then rewritten under this new dynamic evaluation paradigm. The output are families of lexGbs, deduced from the splittings that occur when attempting a division by a non-invertible element.

**Example 2.** (See also Algorithm 7 “invertNil”, Example 10) Consider a non-zero polynomial $f \in k[x]/\langle T \rangle$. If $T$ is irreducible, $f$ is invertible, being non-zero. The Extended Euclidean algorithm permits to find it. If $T$ is squarefree not irreducible and $\gcd(f, T) = g$ then $f$ is invertible modulo $\frac{T}{g}$ and zero modulo $g$ (the “invertible/zero” dichotomy). This leads to an isomorphism $k[x]/\langle T \rangle \simeq k[x]/\langle g \rangle \times k[x]/\langle T / g \rangle$, as in classic dynamic evaluation.

If $T$ is not squarefree like $T = x^3(x + 1)^2$, then $f$ may still not be invertible modulo $\frac{T}{\gcd(f, T)}$. Take for example $f = x^2(x + 2)$. Then $g_0 = \gcd(T, f) = x^2$ and $\frac{T}{g_0} = x(x + 1)^2$. $f$ is not invertible modulo $\frac{T}{g_0}$. Consider next $g_1 = \gcd(\frac{T}{g_0}, g_0) = \gcd(x(x + 1)^2, x^2) = x$. This time $f$ is invertible modulo $\frac{T}{g_0} = (x + 1)^2$, and nilpotent modulo $g := g_0 g_1 = x^3$ (the “invertible/nilpotent” dichotomy). Moreover, $k[x]/\langle T \rangle \simeq k[x]/\langle g \rangle \times k[x]/\langle T / g \rangle$. In the invertible branch, we can pursue computations (typically invert a coefficient of a polynomial in $(k[x]/\langle y \rangle)$, and in the nilpotent branch, work as introduced in the first part when $T = p^e$. 

**Main results.** In the first part, Algorithm 5 “SubresToGB” and Theorem 3 show how to compute a minimal lexGb of an ideal $(a, b, T)$, where $a, b \in k[x, y]$ and $T = p^e$ is the power of an irreducible polynomial $p \in k[x]$.

In the second part, the input are polynomials $a, b \in k[x, y]$ and a monic non-constant polynomial $T \in k[x]$. They verify the assumption

\[
\text{for any primary factor } p^e \text{ of } T, \quad a \text{ and } b \text{ are not nilpotent modulo } p^e. \tag{H}
\]

Algorithm 12 “SubresToGB_D5” and Theorem 4 computes, on input $a, b, T$, a family of minimal lexGbs $(G_i)$, such that $\prod_i \langle G_i \rangle = \langle a, b, T \rangle$. The product is a direct one. More precisely, $\langle G_i \cap k[x] \rangle + \langle G_j \cap k[x] \rangle = \langle 1 \rangle$ for $i \neq j$. 

3
The outcome translates to faster computations of (a direct product of) lexGbs of an ideal \( \langle a, b, T \rangle \) than the Gröbner engine of Magma \([5]\), one of the fastest available, especially when \( T \) is a factor of the resultant of \( a \) and \( b \) having multiplicities.

There is no difference if in Algorithm 5 “SubresToGB” \( p \) is a prime number instead of an irreducible polynomial, and if \( a, b \in \mathbb{Z}[y] \). The obtained family of lexGbs is made of strong Gröbner bases of the ideal \( \langle a, b, p^e \rangle \subset \mathbb{Z}[y] \). Indeed, Hensel lifting, Weierstrass factorization, Lazard’s structural theorem \([1, \text{Section 4.6}]\) all hold in this context too.

In Algorithm 12 it is similarly possible to replace \( T \) by an integer \( n \) and to consider polynomials \( a, b \in \mathbb{Z}[y] \). The output is then a family of coprime lexGbs whose (direct) product equals the ideal \( \langle a, b, n \rangle \subset \mathbb{Z}[y] \). These lexGbs are strong Gröbner bases.

Motivation. There exists a variety of methods to represent the solutions of a system of polynomials. The most widespread are probably Gröbner bases, triangular-decomposition methods, primitive element representations (among which the RUR \([17]\) and the Kronecker representation \([24]\) have received the most attention), homotopy continuation method \([50, 4]\), multi-resultant/eigenvector methods \([3, 18]\). However, when it comes to representing faithfully the ideal generated by the input polynomial, it remains essentially the Gröbner bases only. Some of the aforementioned methods have the ability to represent a multiplicity of a solution, which is just a number, however this is nowhere near to be ideal preserving. This work is somehow affiliated to the triangular-decomposition method, having the Euclidean algorithm at its core. It thus constitutes a first incursion to ideal preserving methods based on the Euclidean algorithm.

Primary decomposition constitutes another clear motivation. Being factorization free, this work nor its generalizations can compute directly a primary decomposition. Rather, it yields a decomposition at a cheap cost, so that a primary decomposition algorithm can then be run on each component. Finding a decomposition efficiently is indeed a well-known speed-up in the realm of primary decomposition \([14, \text{Remark 2}]\). For example, the PrimaryDecomposition command of Magma tries to compute a triangular decomposition from a Gröbner basis (apparently following \([28]\)) in order to reduce the cost of internal subsequent routines such as factorization. It is not known how to do this for non-radical ideals. In addition, the lexicographic monomial order plays a crucial role in GTZ-like primary decomposition algorithms \([23]\). As mentioned above, in this work decompositions follow only from polynomials in \( x \), so comparisons would be premature. We would rather wait that decompositions following the \( y \) variables are developed, which essentially amounts to dynamic evaluation in two variables.

Direct product of lexGbs. While our algorithms naturally compare to Gröbner bases’ ones, the output differs in that it is a family of lexGbs. However, this is not a drawback. Indeed it carries more information, like an intermediate representation toward the primary decomposition. Normal forms, for example to test ideal membership, can be done componentwise. It is also possible to perform other standard ideal operations. Besides, when the coefficients are rational numbers, their size are smaller: this stems from the fact that one lexGb can be reconstructed from the family output by our algorithm, through the Chinese remainder theorem, which induces a growth in the size of the coefficients. See \([11]\) for details. Additionally, decomposing a lexGb for solving has been known to be advantageous since a long time \([28]\).
1.2. Previous and related work

On the general method. There exists a quite dense literature about representations of the solutions of a system of polynomials \( a, b \in k[x, y] \), whether they are simple or not.

One line of research builds upon a subresultant sequence. It started with [25] by González-Vega and El Kahoui, with several forthcoming articles improving the idea. In the background of these works lies the topology of plane (real) curves defined by polynomials over the rationals. The analysis of the bit-complexity is thus a central aspect, and the forthcoming works aim at improving it [15, 6, 31, 34]. From a subresultant sequence of \( a \) and \( b \), it computes a triangular decomposition of the common set of solutions of \( a \) and \( b \). A key step is to take the squarefree part of the resultant. It thus does not consider representation of multiplicities.

Another direction considers shearing of coordinates, for representations in term of primitive element (RUR) [6, 37]. It should be noted that primitive element representations cannot be ideal preserving [47, Remark 3.1].

As for general triangular decomposition methods, some have studied the representation of multiplicities. In most references below the multiplicity is equal to the dimension of a certain local algebra. In the bivariate case, [9] proposes algebraic cycles and primitive p.r.s. to find the multiplicity of each primary components. A preliminary study of deformation techniques is found in [33] that computes a certain multiplicity number. The intersection multiplicity through Fulton’s algorithm was investigated in the bivariate case in [35]. The complexity of this approach remains unclear. It was extended in [2] to multivariate system by reducing to the bivariate ones, through a method dealing with the tangent cone.

The present work distinguishes from the above in that it faithfully produces an ideal preserving representation, which contains far more information than a multiplicity number, yet relying ultimately on the classic routine: the subresultant. To compare with other formal techniques supplied with the ideal preserving feature, I am only aware of Gröbner bases computations.

About the subresultant algorithm. Algorithm 4 “LastNonNil” presents a subresultant algorithm computed modulo the power of an irreducible polynomial \( T = p^e \). If the input is written \( a, b, T \) then the output written \( u, v, T_1, U \) with \( UT_1 = T \), and \( u, v \in k[x, y] \) monic (in \( y \)) polynomials, satisfy \( \langle a, b, T \rangle = \langle u, vT_1, T \rangle \). \( u \) is a monic gcd of \( a \) and \( b \) modulo \( T \) if there exists one, in which case \( v = 0 \) (such a gcd does not necessarily exist see [12]). Otherwise, \( v \neq 0 \) and this algorithm provides a weak version of the gcd, with the ideal preserving feature. The terminology LastNonNil reflects the fact that \( u \) is the last non-nilpotent polynomial “made monic” in the modified p.r.s, while \( v \) if the first nilpotent one “made monic” too (we assume that 0 is a nilpotent element in this article).

Moroz and Schost [41] propose to compute the resultant modulo \( x^e \), that is when \( p(x) = x \) with our notation. It shares some similar tools, Weierstrass factorization and Hensel lifting. However, their purpose is to write a quasi-linear complexity estimate, rather than producing a practical and neat algorithm. The idea amounts to adapt the half-gcd algorithm that runs asymptotically in quasi-linear time, to this particular context — whereas here it is the standard quadratic time subresultant algorithm that is used. The possibility to compute subresultants along with the resultant, required by Algorithm 4 is left open in [41]. The present article aims rather at practical algorithms and implementations. The algorithms presented here being new, they must demonstrate their practical efficiency, and be as simple as possible to check their correctness and to be
implemented. Despite this, the description is already quite technical, and the possibility to incorporate faster routines such as the half-gcd for computing subresultants appears to be an interesting challenge for future work. The algorithms described are all implemented “as such” in Magma [5] (see http://xdahan.sakura.ne.jp/gb.html), and in many cases outperform the Gröbner engine of Magma, yet equipped with one of the best implementation of the F4 and FGLM algorithms.

Recently, faster algorithms for the computation of the resultant of $a, b$ have come out [51, 49]. It does so by bypassing the computation of a p.r.s sequence — whereas here the p.r.s. is essential. These outcomes imply directly an algorithm that computes a lexGb of the ideal $(a, b)$ within the same complexity. However, they require strong generic assumptions: the lexGb must consist of two polynomials, and the unique monic (in $y$) polynomial shall be of degree one. These assumptions lie far away of the present work that deal with lexGs containing many polynomials of arbitrary degree.

The algorithms of this article adapt straightforwardly to univariate polynomials $a$ and $b$ that have coefficients in $\mathbb{Z}$. When $T = p^e$ with $p$ a prime number, one can think of $p$-adic polynomials, $e$ being the precision. A related work [7] in this situation studies the stability of the subresultant algorithm to compute a ($p$-adic) approximate gcd. It should be mentioned that this gcd is not monic, nor have an invertible leading coefficient, hence is of limited practical interest. The subresultant algorithm therein does not hold the crucial functionalities (Hensel lifting, Weierstrass factorization) that make polynomials monic.

About dynamic evaluation. The second part of this article involves dynamic evaluation. In its broad meaning, this technique automatically produces case distinctions depending on the values of some parameters in an equation. More precisely here, the two algebraic equations are the polynomials $a$ and $b$ in the variable $y$, the parameter being $x$. The case distinctions comes from the image of their coefficients (in $x$) modulo each primary factor of $T$. In this context, classic dynamic evaluation considers $T$ squarefree; A primary factor is then of the form $x - \alpha$ (at least in the algebraic closure) for a root $\alpha$ of $T$: this corresponds to the evaluation of the parameter at the value $\alpha$. This work allows Taylor expansions at $\alpha$ at order $e - 1$ if the primary factor of $T$ is of degree $e$ namely is $(x - \alpha)^e$. The case distinction still follows from the different roots of $T$, thus the different Taylor expansions considered should be at different expansion points.

The computational paradigm of dynamic evaluation, introduced in [16] is quite simple. A squarefree polynomial $T$ is split by a gcd computation when attempting to perform an inversion modulo $T$, see Example [2]. It is not surprising that such splittings actually appeared before [16] in the realm of integer factorization, as Pollard did with his rho method [45]. In the algebraic context, dynamic evaluation, also coined “D5 principle” [16], has been quite thoroughly studied, especially toward manipulation and representation of algebraic numbers [17], generalization to multivariate polynomials through the triangular representation [40], becoming a major component of some triangular decomposition algorithms [32]. Another standard reference is [39]. Concerning complexity results, we refer to [10] that brings fast univariate arithmetic to this multivariate context, or more recently to [48]. The splitting can also be performed in an ideal-theoretic way through Gröbner bases computations, see [43], but at a probably too expensive cost to be competitive with what gcd computations afford. Except this latter article, all these previous works manipulate squarefree polynomials only, the present work being to the best of my knowledge, the first addressing general univariate polynomials.
1.3. Organization

The first part (Sections 2-3) considers \( T \) equal to the power of an irreducible polynomial. Section 2 is made of preliminaries: Lazard’s structural theorem, Weierstrass factorization through Hensel lifting, subresultant p.r.s are recalled. The following section 3 introduces the main subresultant routine “LastNonNil”, and shows how to deduce a minimal (not reduced in general) lexGb of \( \langle a, b, p^e \rangle \). The second part considers \( T \) general and is treated in Sections 4-5. This consists essentially in translating the algorithms presented in the first part into a dynamic evaluation that works when \( T \) is not necessarily squarefree. Section 4 introduces the dichotomy “invertible-nilpotent” and applies it to the algorithms that make polynomials monic. It is followed in Section 5 by the generalization of the algorithm “LastNonNil” that computes the “monic form” of the last nilpotent/first nilpotent polynomials in the modified subresultant p.r.s of \( a \) and \( b \), and by the computation of families of lexGbs whose (direct) product is \( \langle a, b, T \rangle \). An experimental section 6 shows the benefits of an implementation realized in Magma of this approach, compared to Gröbner basis computations. A special emphasize is put on the case of an input \( a, b \) without a modulus \( T \) in Section 6.4 due to the importance of this case and the questions it raises. The last section 7 describes some improvements for future work and various generalizations.

As one can see, the second part encompasses the algorithms proposed in the first part, restricted to when \( T = p^e \). Is this first part then really necessary? Well, rewriting known algorithms to the classic dynamic evaluation paradigm is one thing, rewriting new algorithms in a new dynamic evaluation is another one. Without the first part, the algorithms of the second part would become too obscure. Besides, some of these algorithms rely on those restricted to the “local case” \( T = p^e \) dealt with in the first part, in order to alleviate the proofs.

Notations and convention. Below are gathered some notations used in this paper:

- sqfp(\( f \)): squarefree part of a polynomial \( f \in k[x] \).
- a polynomial \( p \) strictly divides another polynomial \( q \) if \( p|q \) and \( p \neq q \).
- \( \mathcal{P} \) denotes the set of irreducible polynomials in \( k[x] \). Given \( p \in \mathcal{P} \) and a polynomial \( T \in k[x] \), \( v_p(T) \in \mathbb{N} \) is the largest integer such that \( p^{v_p(T)} | T \).
- an ideal generated by a family of polynomials \( \mathcal{L} \) is denoted \( \langle \mathcal{L} \rangle \).
- \( k \) will denote a perfect field: irreducible polynomials are squarefree in any algebraic extension of \( k \).
- a product of ideals is direct if the ideals are pairwise coprime. Unless otherwise specified, all products of ideals will be direct.
- all lexicographic Gröbner bases are monic, meaning that their leading coefficient is 1.
- although zero is usually not considered a nilpotent element, it is convenient to assume it is one.
- a value output by an algorithm is ignored by writing “__” (underscore) instead of that value.
- algorithms are written as pseudo-codes. There is no obvious shortcut that would allow to simplify them furthermore. Most routines are indeed elementary, and it essentially amounts to recursive calls, and case distinctions. The proof of correctness follows the lines of the algorithms. They are all self-contained though, copiously commented which should allow to read them without too much effort.
2. Preliminaries

We consider a perfect field \( k \), and a monic non-constant polynomial \( T \in k[x] \). Write \( R = k[x]/\langle T \rangle \) the quotient ring. The other input are two polynomials \( a, b \) in \( k[x, y] \) reduced modulo \( T \). The lexicographic monomial order with \( x < y \) is put on the monomials of \( k[x, y] \). In the first part Sections \( 2.3 \) \( T \) is a power of an irreducible polynomial \( p, T = p^e \).

2.1. Nilpotents

**Definition 1.** Let \( T \) be a non-constant monic polynomial in \( k[x] \). An element \( r \in R = k[x]/\langle T \rangle \) is invertible if and only if \( \gcd(T, r) = 1 \). It is nilpotent if all irreducible factors of \( T \) divide \( r \). Equivalently, if the squarefree part \( \text{sqfp} (T) \) of \( T \) divides \( r \).

The following result is classical and elementary:

**Proposition.** A polynomial \( a \in R[y] \) is nilpotent if and only if all its coefficients are nilpotent in \( R \), or equivalently, if \( \text{sqfp} (T) \) divides \( a \).

**Example 3** (nilpotent element). Let \( r = x^2(x + 1) \). Then \( r \) is nilpotent modulo \( T = x^4 \), modulo \( T = x^3(x + 1)^2 \), but not modulo \( T = x^2(x - 1)^2 \).

**Example 4** (nilpotent polynomial). Let \( T = x^2(x + 1)^2 \) and \( P = x(x + 1)y + 2x(x + 1)(x - 1) \). Then \( P \) is nilpotent since its coefficients \( x(x + 1) \) and \( 2x(x + 1)(x - 1) \) are all nilpotents.

2.2. About lexicographic Gröbner bases

The lexicographic monomial order \( \prec \) with \( x \prec y \) is a total order on the monomials of \( k[x, y] \) defined as \( x^a y^b \prec x^c y^d \) if \( b < d \), or \( b = d \) and \( a < c \). Given \( f \in k[x, y] \) a non-zero polynomial, the notation \( \text{LM}(f) \) will refer to the leading monomial for \( \prec \) among the monomials occuring in \( f \). A lexGb of a polynomial system \( (f_1, \ldots, f_m) \) generating an ideal \( I \) is any family of polynomials \( (g_1, \ldots, g_s) \subset I \) such that for any \( f \in I \) there is a \( g_j \) such that \( \text{LM}(g_j) | \text{LM}(f) \). If \( k \) is simply an Euclidean ring, variations exist, and the one corresponding to this definition is called a strong Gröbner basis. The Gröbner basis is minimal if for all \( 1 \leq i \leq s \), \( \text{LM}(g_i) \) is not divided by \( \text{LM}(g_j) \) for any \( j \neq i \).

In 1985, Lazard [30] completely characterized lexGbs in \( k[x, y] \), result now known as “Lazard’s structural theorem”. It will be used in the following form:

**Theorem 1.** (A) Let monic non-constant polynomials \( h_1, h_2, \ldots, h_{\ell - 1} \in k[x] \) such that \( h_i \) divides \( h_{i - 1} \) for \( i = 2, \ldots, \ell - 1 \). Let also monic (in \( y \)) polynomials \( g_2, \ldots, g_\ell \in (k[x])[y] \) of increasing degree: \( \deg_y(g_2) < \deg_y(g_3) < \cdots < \deg_y(g_\ell) \). The list of polynomials

\[
\mathcal{L} = [h_1, h_2g_2, \ldots, h_{\ell - 1}g_{\ell - 1}, g_{\ell}]
\]

is a lexGb if and only if:

\[
g_i \in \langle g_{i - 1}, \frac{h_{i - 2}}{h_{i - 1}}g_{i - 2}, \ldots, \frac{h_1}{h_{i - 1}} \rangle, \text{ for } 3 \leq i \leq \ell\tag{3}
\]

It is moreover minimal if and only if \( h_i \) strictly divides \( h_{i - 1} \) for \( i = 2, \ldots, \ell - 1 \).

(B) Assume now that \( \mathcal{L} \) above is already a lexGb. Given \( h_0 \in k[x] \) a monic non-constant polynomial, and \( g_{\ell + 1} \in (k[x])[y] \) monic (in \( y \)) of degree larger than that of all the polynomials \( g_1, \ldots, g_\ell \), then

\[
\mathcal{L}' := [h_0 f : f \in \mathcal{L}] \quad \text{cat} \quad [g_{\ell + 1}]
\]

is a lexGb if and only if \( g_{\ell + 1} \in \langle \mathcal{L} \rangle \). It is moreover minimal if and only if \( \mathcal{L} \) is minimal.
Example 5. The system of polynomials \((x^2, xy, y^2 + x)\) is a minimal lexGb. Indeed, with the notation of the theorem we have \(h_1 = x^2\), \(h_2 = x\) and \(f_2 = y\), \(f_3 = y^2 + x\). Moreover \(h_2\) strictly divides \(h_1\). The condition \(f_3 \in (\frac{h_1}{h_2}, f_2)\) is the only one that we must verify. It translates to the condition \(y^2 + x \in (x, y)\) which is clearly true.

However, the system \((x^2, xy, y^2 + 1)\) is not a lexGb. Indeed, \(y^2 + 1 \notin (x, y)\). \(\square\)

Proof. Part (A) is a restatement\(^1\) of Lazard’s theorem \[30\] Thm 1 in our particular context.

Part (B) is straightforward when we translate the conditions of Eq. \((3)\) made on the list \(\mathcal{L}\) to the list \(\mathcal{L}’ = [h_0 f : f \in \mathcal{L}]\) cat \([g_{\ell + 1}]\). Indeed, all these conditions, except the one mentioned that is \((g_{\ell + 1} \in (\mathcal{L}))\), are verified since \(\mathcal{L}\) is assumed to be a lexGb. If \(\mathcal{L}\) is minimal, then the condition for minimality is guaranteed since \(h_0\) is assumed to be non-constant. \(\square\)

2.3. Making polynomials monic

In order to cope with subresultants which do not have an invertible leading coefficient modulo \(T\), a routine that transforms them into equivalent monic polynomials is presented here. If the polynomial is not nilpotent, Weierstrass factorization allows to realize this. Else, we need first to “remove” the nilpotent part. When \(T = p^e\), it suffices to factor out the largest power of \(p\) that divides it. When \(T\) is general however, a polynomial which is not nilpotent may not necessarily have a coefficient that is invertible. We need then dynamic evaluation as introduced in Section 4. The main algorithm \[3\] “MonicForm” is a wrapping of two subroutines, Algorithm \[1\] “WeierstrassForm” and Algorithm \[2\] “MusserQ”. The first one puts the polynomial that we want to “make” monic into a form of application of the second algorithm. This second one realizes Weierstrass factorization through Hensel lifting. In the subsequent algorithms, only the “MonicForm” wrapping algorithm will appear (and no more “WeierstrassForm” nor “MusserQ”).

Overview of Algorithm \[7\] “WeierstrassForm”. We assume in this section and the next one that \(T = p^e\) is the power of an irreducible polynomial. The algorithm “WeierstrassForm” has two effects: find the largest power of \(p\) that divides the input polynomial (case of return at Line 9), or, if there is none, find the coefficient of highest degree that is not divided by \(p\) (case of a return at Line 7); This coefficient is then invertible modulo \(T\). The algorithm scans the coefficients by decreasing degree and updates the computation with a gcd (Line 4). If the coefficient \(c_i(x)\) (of \(y^i\)) is invertible modulo \(T\), the algorithm also outputs the largest power of \(p\) that divides all coefficients of degree higher than \(i\), as well as the inverse of \(c_i\) modulo \(T\). These output are indeed required to perform Weierstrass factorization through Hensel lifting (Algorithm \[2\] “MusserQ”).

It may thus be necessary to call Algorithm \[7\] twice: one to remove the nilpotent part, and one to find the largest coefficient that has become invertible. Then, Weierstrass factorization Algorithm \[2\] can be called. Let us see this through an example:

Example 6 (Algorithm \[1\] “WeierstrassForm”). Let \(f = 3x^2 y^2 + (x^2 + 2x)y + x\) and \(p(x) = x\), \(T = p^3 = x^3\). This polynomial is nilpotent, and to “remove” the nilpotent part, running Algorithm \[1\] “WeierstrassForm” gives (see the specifications):

\[
(x, -1, \_\_) \leftarrow \text{WeierstrassForm}(f, T).
\]  

\(1\) with the notations \(k, f_i, H_i, G_i\) of Lazard’s article: \(k = \ell - 1, f_0 = h_1, f_1 = h_2g_2, \ldots, f_{k-1} = h_{\ell-1}g_{\ell-1}, f_k = g_\ell\), and \(H_i = g_{i+1}\) and \(G_i = h_0/h_i\).
Indeed, \(x \mid f\) but \(x^2 \nmid f\). The second output is \(-1\) meaning that there is no coefficient of \(f\) that is invertible modulo \(x\) (since \(f\) is divided by \(x\)). The third output has no importance in this case, and according to the conventions is replaced by an underscore “\(\_\)”. Then \(f/x\) is not nilpotent, so that running \texttt{WeierstrassForm} again will output:

\[
(x, 1, -\frac{1}{4}x + \frac{1}{2}) \leftarrow \texttt{WeierstrassForm}(f/x, x^{3-1}).
\] (5)

The second output 1 indicates the degree one in \(f/x\). Indeed, the coefficient of \(y\) in \(f/x\) is \(x^2 + 2x = x + 2\), which is indeed invertible modulo \(x^3/x = x^2\). The inverse modulo \(x^2\) of \(x + 2\) is then \(-\frac{1}{4}x + \frac{1}{2}\) which is the third output.

Note that the coefficient of \(y^2\) in \(f/x\) is not invertible hence is divided by \(x\) (this coefficient being 3\(x\)). The first output returns the highest power of \(x\) that divides this coefficient: \(x\) divides \(\frac{3x^2}{x}\).

Algorithm 1: \((g, d, \alpha) \leftarrow \texttt{WeierstrassForm}(f, T)\)

\begin{verbatim}
Input:
1. polynomial \(f = c_0(x) + c_1(x)y + \cdots + c_\delta(x)y^\delta \in k[x,y],\)
2. \(T = p^e \in k[x]\) power of an irreducible polynomial \(p\).

Output:
1. \(g = p^\nu\) the largest power of \(p\) that divides all the coefficients \(c_{d+1}, \ldots, c_\delta\)
   (where \(d\) is defined in 2. below).
2. the largest index \(d\) such that the coefficient \(c_d(x)\) is invertible modulo \(T\). If there is none, then \(d = -1\).
3. \(\alpha\) is the inverse of \(c_d\) if \(d \geq 0\), and an arbitrary value if \(d = -1\).

1 \(d \leftarrow \deg_y(f)\); \(g_{\text{new}} \leftarrow T\)
2 \textbf{while} \(d \geq 0\) and \(g_{\text{new}} \neq 1\) \textbf{do}
3 \hspace{1em} \(g \leftarrow g_{\text{new}}\); \(c_d \leftarrow \text{coeff}(f, d)\)
4 \hspace{1em} \((g_{\text{new}}, \ _, \ \ _, \ \alpha) \leftarrow \text{xgcd}(g, c_d)\) \hspace{1em} // \text{xgcd} = \text{Extended gcd}: \_g + \alpha c_d = g_{\text{new}},\)
5 \hspace{1em} \(d \leftarrow d - 1\) \hspace{1em} \hspace{1em} // the underscore \(_\) replaces an unimportant output
6 \textbf{if} \(g_{\text{new}} = 1\) \textbf{then}
7 \hspace{1em} \textbf{return} \((g, d + 1, \alpha)\) \hspace{1em} // case \(f\) not nilpotent
8 \textbf{else}
9 \hspace{1em} \textbf{return} \((g_{\text{new}}, -1, 0)\) \hspace{1em} // case \(f\) nilpotent
\end{verbatim}

Remark 1. \(f\) is nilpotent if and only if \(d = -1\) where \((\_, d, \_) \leftarrow \texttt{WeierstrassForm}(f, T)\) (again, underscores “\(\_\)” replace unimportant output).

Lemma 1 (Correctness of Algorithm\[1\] “\texttt{WeierstrassForm}”). The three output \((g_{\text{new}}, d, \alpha)\) satisfy the specifications 1-2-3.

Proof. We must prove that the output at Line 7 or 9 verifies the specifications 1.-2.-3. of the output.

Case of return at Line 7. Here \(g_{\text{new}} = 1\) so that the while loop (Lines 3-5) exited on that condition and not on the condition \(d < 0\). Therefore \(d \geq 0\). This means that

\[
g_{\text{new}} = \gcd(T, c_{d+1}, c_{d+2}, \ldots, c_\delta) = 1, \hspace{1em} \text{and} \hspace{1em} g = \gcd(T, c_{d+2}, \ldots, c_\delta) \neq 1.
\]

Moreover the extended gcd “xgcd” computation at Line 4 gives \(\alpha c_{d+1} \equiv 1 \mod T\). We have proved that the output \((g, d + 1, \alpha)\) verifies the specifications 1.-2.-3.
Lemma 2 (Weierstrass). Given $f \in R[y]$, written $f = c_d(x)y^d + \cdots + c_0(x)$, with $c_d$ the coefficient of highest degree not in $(p)$ (that such a coefficient exists is a pre-requisite), there exist two polynomials $q$ and $u$ defined below such that:

1. $f = q \cdot u$.
2. $q$ is monic of degree $n$, and
3. $u = u_0 + u_1 y + \cdots + u_{d-n}y^{d-n}$ with $u_0 \not\in (p)$ and $u_i \in (p)$ for $i \geq 1$. In particular $u$ is a unit of $R[[y]]$, so that: $(f, p^e) = (q, p^e)$.

To compute the monic polynomial $q$ in practice, “special” Euclidean division [27], linear algebra, and Hensel lifting [42, Algo Q] are available. The latter is the most efficient and works in more general situations that are required in the second part of this article. This is what is used in [41, Thm. 1] for their “normalization” (whose purpose is to make polynomials monic too). Since a proof (and more) can be found in [42], it is not reproduced here. That proof relies entirely on Hensel lifting and does not involve Weierstrass preparation theorem, however Algorithm (Q) given by Musser is really a Weierstrass factorization when the modulus is the power of an irreducible polynomial. This implementation has the advantage to extend straightforwardly when the ring of coefficients is not necessarily a local ring, but a direct product of thereof — and that no division by zero occur. This feature helps to generalize the dynamic valuation as undertaken in Section 4, in that we can rely again on the same algorithm 2 when the modulus is not a power of an irreducible polynomial.

This algorithm essentially reduces to quadratic Hensel lifting (QHL). It lifts a factorization, and, as always with QHL, a Bézout identity. A standard form of QHL is described in Algorithm 15.10 of [52] and we refer to it for details. Given $N \in k[x]$ and $f, a, b, \alpha, \beta \in k[x, y]$, the notation:

$$(a^*, b^*) \leftarrow \text{HenselLift}(f, a, b, \alpha, \beta, N \sim N^{2^e})$$

assumes that:

$$f \equiv ab \mod N, \quad \alpha a + \beta b \equiv 1 \mod N, \quad f \equiv a^*b^* \mod N^{2^e}, \quad a^* \equiv a \mod N, \quad b^* \equiv b \mod N.$$ 

Here, the initial two factors of the input polynomial $f$ are the coefficient $c_d \equiv \alpha^{-1} \mod T$ of $f$, and $b \equiv \alpha f \mod N$ (so that $b$ is a monic polynomial of degree $d$ modulo $N$). The lifting produces the same equality modulo $N^{2^e}$. This integer $\epsilon$ is the smallest integer for which $N^{2^\epsilon}$ divides $T$. That
such an integer exists follows from \( \text{sqfp}(N) = \text{sqfp}(T) \). The initial Bézout identity is given for free here: indeed \( c_d \alpha + b \cdot 0 = 0 \).

Algorithm 2: \( b \leftarrow \text{MusserQ}(f, T, d, \alpha, N) \)

**Input:** 1. polynomial \( f = c_0(x) + c_1(x)y + \cdots + c_b(x)y^b \in k[x, y] \), with \( \deg_y(f) = \delta \geq d \).
   Its coefficient \( c_d(x) \) of degree \( d \) is invertible mod \( T \) of inverse \( \alpha \).
   Moreover, \( c_{d+1}, \ldots, c_\delta \) are nilpotent modulo \( T \), that is \( \text{sqfp}(T)|\text{sqfp}(c_i) \) for \( i = d + 1, \ldots, \delta \).
2. A monic non-constant polynomial \( T \in k[x] \).
3. \( d \) a non-negative integer (defined in 1.)
4. \( \alpha \equiv c_d(x)^{-1} \mod T \)
5. A monic polynomial \( N \in k[x] \) that divides \( T \) and such that \( \text{sqfp}(N) = \text{sqfp}(T) \), equal to \( \gcd(T, c_{d+1}, \ldots, c_\delta) \). So that if \( N \neq 1 \) then \( \deg_y(f \mod N) = d \).

**Output:** \( b^* \in k[x, y] \), monic (in \( y \)) of degree \( d \), and verifying \( \langle b^*, T \rangle = \langle f, T \rangle \).

10 \( b \leftarrow \alpha f \mod N \) \hspace{1cm} // \( b \) is monic of degree \( d \)
11 \( c_d \leftarrow \text{coeff}(f, d) \)
12 Let \( \epsilon \) be the smallest integer such that \( T|N^{2^\epsilon} \)
13 \( (\_, \ b^*) \leftarrow \text{HenselLift}(f, \ c_d, \ b, \ \alpha, \ 0, \ N \sim N^{2^\epsilon}) \) \hspace{1cm} // \( f \equiv c_d b \mod N, \ \alpha c_d + 0 \cdot b = 1 \)
14 return \( b^* \mod T \)

Example 7 (Algorithm 2 “MusserQ”). With input \( f = x(x + 1)y^2 + (2x + 1)y + x^2 \) modulo \( T = x^2(x+1)^2, \ d = 1 \) and \( \alpha \equiv (2x+1)^{-1} \equiv -8x^3-12x^2-2x+1 \mod T, \) and \( N = x(x+1) \). All input’ specifications are satisfied: the coefficient \( x(x+1) \) of degree 2 is nilpotent modulo \( T \), the coefficient \( 2x + 1 \) of degree 1 is invertible. As expected, we have \( d = 1 \) and \( N = \gcd(T, x(x+1)) = x(x+1) \); as well as \( \alpha \equiv (2x+1)^{-1} \mod T \).

We have \( b \equiv f x \equiv y + x \mod N \). Now we lift \( f \equiv (2x + 1)b \mod N, \ N^2, \ N^4, \ldots \) until \( T|N^{2^\epsilon} \). The Bézout identity \( \alpha(2x + 1) + 0 \cdot b = 1 \) that is also lifted is given for free. In this example one step of lifting is sufficient since \( N^2 = T \), yielding \( b^* = y - x^3 - 2x^2 \). Then \( \langle b^*, T \rangle = \langle f, T \rangle \).

As already explained, Algorithm 3 “MonicForm” encapsulates the two algorithms “WeierstrassForm” and “MusserQ” described above in one algorithm. On input \( f, T \), it outputs \( U \) the largest polynomial that divides \( f \), and \( b \) monic which is “equivalent” to \( f/U \) in the sense that \( \langle b, T/U \rangle = \langle f/U, T/U \rangle \).

**Lemma 3** (Correctness of Algorithm 3). The output \( \langle b, U \rangle \) satisfies the equality of ideals \( \langle b, T/U \rangle = \langle f/U, T/U \rangle \) with \( b \in k[x, y] \) monic (in \( y \)) and \( U \in k[x] \).

**Proof.** The call to \( \text{WeierstrassForm}(f, T) \) at Line 15 outputs \( (U, d, \alpha) \). We must distinguish two cases:

Case of exit at Line 23: Here \( d \geq 0 \). According to the output’ specifications of “WeierstrassForm”, \( U = \gcd(T, c_{d+1}, \ldots, c_\delta) \) divides \( T \) and the coefficients \( c_{d+1}, \ldots, c_\delta \) of \( f \). Additionally, \( c_d \) is invertible modulo \( T \) of inverse \( \alpha \). Therefore the five entries \( (f, T, d, \alpha, U) \) of “MusserQ” at Line 22 satisfy the input’ specifications of “MusserQ”. Its output’ specifications provide: \( \langle b, T \rangle = \langle f, T \rangle \).

Case of exit at Line 20: Here \( d = -1 \), which means that, according to the output’ specifications of “WeierstrassForm”, \( U = \gcd(T, c_0, \ldots, c_\delta) \) divides all coefficients of \( f \), as well as \( T \). It follows also that \( f' = f/U \) has an invertible coefficient modulo \( T' = T/U \). Therefore the second call to “WeierstrassForm” at Line 18 implies that the output \( (N, d, \alpha) \) satisfy: \( c_d/U \) is invertible.
Algorithm 3: \((b, U) \leftarrow \text{MonicForm}(f, T)\)

**Input:**
1. polynomial \(f \in k[x, y]\)
2. \(T = p^e \in k[x]\) the power of an irreducible polynomial \(p\).

**Output:**
1. \(b \in k[x, y]\) is monic (in \(y\))
2. Monic polynomial \(U \in k[x]\) that divides \(T\) and \(f\), such that: \(\langle b, T/U \rangle = \langle f/U, T/U \rangle\).

15 \((U, d, \alpha) \leftarrow \text{WeierstrassForm}(f, T)\)
16 \text{if } d = -1 \text{ then} \quad // \text{Case where } f \text{ is nilpotent}
17 \quad f' \leftarrow f/U; \quad T' \leftarrow T/U \quad // \text{Now } f' \text{ is not nilpotent...}
18 \quad (N, d, \alpha) \leftarrow \text{WeierstrassForm}(f', T') \quad // \ldots \text{so that } d \geq 0
19 \quad b \leftarrow \text{MusserQ}(f', T', d, \alpha, N) \quad // \langle b, T' \rangle = \langle f', T' \rangle
20 \quad \text{return } (b, U)\)
21 \text{else} \quad // \text{Here } f \text{ was not nilpotent}
22 \quad b \leftarrow \text{MusserQ}(f, T, d, \alpha, U) \quad // \langle b, T \rangle = \langle f, T \rangle
23 \quad \text{return } (b, 1)\)

modulo \(T' = T/U\) with \(\alpha c_d/U \equiv 1 \text{ mod } T'\). Additionally, \(N = \gcd(T', c_{d+1}/U, \ldots, c_5/U)\) divides all the coefficients \(c_{d+1}/U, \ldots, c_5/U\) of \(f'\). Consequently, the five entries \((f', T', d, \alpha, N)\) satisfy the input’ specifications of \text{MusserQ} at Line 19. The output \(b\) hence verifies \(\langle b, T' \rangle = \langle f', T' \rangle\), which is what we wanted to prove.

\[\square\]

2.4. Subresultant p.r.s

As explained in the introductory examples [1] the last non nilpotent and the first nilpotent (which may be zero according to our convention) polynomials in the modified subresultant p.r.s. need to be retrieved. The p.r.s is computed modulo \(p^e\) which requires special care since very few results exist when coefficients are in a non-reduced ring. In particular the classic formula (6) may fail because of inverting a leading coefficient which may not be invertible. If such a subresultant is met, either all its coefficients are nilpotent and we have found the first nilpotent, either it has one invertible coefficient and it can be made monic (previous section); Then a subresultant computation is restarted (Line 31 of Algorithm 4). Before giving details of Algorithm 4 in the next section, the remaining of this section of preliminaries recalls the most fundamental results of the theory of subresultants.

**Review.** The subresultant p.r.s is a central topic in computer algebra and as such has been studied extensively; We will only recall the key results. It enjoys many convenient properties both algorithmically and theoretically.

On one hand a subresultant of two polynomials is the determinant of a certain matrix derived from the Sylvester matrix. It can be defined over any ring. But computing them in this way is costly. On the other hand, there is the subresultant p.r.s. computed through the formula [6]. The main and classic result is that both objects are related through the block structure (a.k.a gap structure) theorem. Computing subresultants through a p.r.s is cheaper. The latter assumes traditionally that the input polynomials \(a\) and \(b\) are in a unique factorization domain.

It is possible to address polynomials having coefficients in rings of type \(k[x]/(p^e)\) thanks to the specialization property. This is addressed in Section 3.1 here only standard definitions and results are recalled.
Definition 2. Write \( n_j \) the degrees in \( y \) of any Euclidean p.r.s of \( a \) and \( b \) in \((k[x])[y]\), with \( n_1 := \deg_y(a) \), \( n_2 := \deg_y(b) \).

- For \( 0 \leq j \leq n_2 - 1 \), the \( j \)-th subresultant of \( a \) and \( b \), written \( \text{Sres}_j(a, b) \), is defined as the polynomial of degree at most \( j \) whose coefficients are certain minors of the Sylvester matrix (see e.g. [38] Prop. 7.7.1).

- For \( j = n_2 \), \( \text{Sres}_{n_2}(a, b) := \text{lcm}(b)^{n_1 - n_2 - 1} b \).

- For \( n_2 < j < n_1 - 1 \), \( \text{Sres}_j(a, b) := 0 \).

- Let \( \nu_1 := \deg_y(a \mod T) \) and \( \nu_2 := \deg_y(b \mod T) \) and assume \( \nu_1 \geq \nu_2 \). For \( 0 \leq j < \nu_1 - 1 \) we define the \( j \)-th subresultant of \((a \mod T)\) and \((b \mod T)\), written \( \text{S}_j((a \mod T), (b \mod T)) \), whose coefficients are certain minors of the Sylvester matrix of \((a \mod T)\) and \((b \mod T)\).

Both subresultant families \((\text{Sres}_j(a, b))_{0 \leq j \leq n_2 - 1}\) and \((\text{S}_j((a \mod T), (b \mod T)))_{0 \leq j \leq \nu_2 - 1}\) are related by the following functorial property:

Lemma 4 (Corollary 7.8.2 of [38]). Write \( R = k[x]/(T) \), and \( \phi : (k[x])[y] \to R[y] \) the reduction map. As above let \( n_1 := \deg_y(a) \) and \( n_2 := \deg_y(b) \) with \( n_1 \geq n_2 \), and \( \nu_1 := \deg_y(a \mod T) \) and \( \nu_2 := \deg_y(b \mod T) \). Assume \( \nu_1 \geq \nu_2 \). Then for \( 0 \leq j < n_2 \),

1. If \( \nu_2 = n_2 \) and \( \nu_1 \leq n_1 \), then \( \phi(\text{Sres}_j(a, b)) = \phi(\text{lcm}(b))^{n_1 - \nu_1} \text{S}_j(\phi(a), \phi(b)) \).

2. If \( \nu_2 \leq n_2 \) and \( \nu_1 = n_1 \), then \( \phi(\text{Sres}_j(a, b)) = \phi(\text{lcm}(a))^{n_2 - \nu_2} \text{S}_j(\phi(a), \phi(b)) \).

3. If \( \nu_2 < n_2 \) and \( \nu_1 < n_1 \), then \( \phi(\text{Sres}_j(a, b)) = 0 \).

Writing \( F_1 := a \) and \( F_2 := b \), the subresultant p.r.s \([F_1, F_2, \ldots, F_r, 0]\) over \((k[x])[y]\) is defined through the formula below (sometimes called “First kind subresultant p.r.s” [38] Definition 7.6.4)):

\[
F_3 := (-1)^{n_1 - n_2 + 1} \text{prem}(F_1, F_2) \quad \text{and letting} \quad c_3 := -1,
\]

\[
c_i := \left( \frac{\text{lcm}(F_{i-2})}{c_{i-1}} \right)^{n_1 - i - 1} \quad \text{for} \quad i \geq 4
\]

\[
F_i := \frac{\text{prem}(F_{i-2}, F_{i-1})}{\text{lcm}(F_{i-2})(-c_i)^{n_2 - n_1 - 1}} \quad \text{for} \quad i \geq 4
\]

Theorem 2 (Subresultant’s chain theorem — Thm. 7.9.4 of [38]). The subresultant p.r.s \([F_i]_{i \geq 3}\) and the subresultant chain \((\text{Sres}_j(a, b))_{j=0,\ldots,n_2-1}\) are related as follows:

\[
\text{Sres}_{n_j-1}(a, b) = F_j, \quad \text{for} \quad j = 3, \ldots, r.
\]

\( \text{Sres}_{n_j-1}(a, b) \) is the top subresultant in the block it belongs to.

3. Gröbner basis from modified p.r.s

The algorithm [5] “\text{SubresToGB}”, presented in paragraph 3.2, produces a minimal lexGb of the ideal \((a, b, p)\). It is very simple if we take the modified subresultant algorithm [4] “\text{LastNonNil}” for granted: one call to “\text{LastNonNil}” (Line 37), a recursive call (Line 42) and an update of the output of this recursive call (Line 43).
On the other hand the algorithm $^4$ “LastNonNil” is more technical, as often the case when dealing with subresultants. The rough idea summarizes as: “make subresultants monic” if their leading coefficient is not invertible modulo $T$, then pursue the computations of subresultants through a recursive call, until a nilpotent subresultant is found. This simple description hides however the special care that corner cases and degree conditions require. It thus might be useful to have a look at Algorithm $^5$ “SubresToGB” in prior Algorithm $^4$ “LastNonNil”, as a motivation.

3.1. Subresultant p.r.s. modulo $T$

This section introduces the algorithm $^4$ “LastNonNil”. Running the subresultant p.r.s over $k[x,y]$ yields significant and unnecessary growth in the degree in $x$ of the coefficients. Only the image modulo $T$, that is of degree in $x$ bounded by that of $T$, is needed. Unfortunately, the formula $^6$ formally works only over unique factorization domains. The classic workaround consists in taking the homomorphic image by $\varphi$:

**Proposition 1.** Let polynomials $F_1$ and $F_2$ in $(k[x])[y]$ of respective degrees (in $y$) $n_1$, $n_2$, and let $\varphi$ be the homomorphism $(k[x])[y] \to R[y]$ as defined above. For a given integer $2 \leq i \leq r + 1$, if the $(\lcm(F_{i-1}))_i$ are invertible modulo $T$ for $j = 2, \ldots, i$, then the formula $^6$ specializes well by $\varphi$:

Write $\overline{c_i} = \varphi(c_i)$. For $j = 3$ then, $\overline{c_3} = -1$ and $\varphi(F_3) := (-1)^{n_1 - n_2 + 1} \varphi(F_1), \varphi(F_2))$. And, for $i + 1 \geq j \geq 4$:

$$\overline{c_j} = \left(\frac{\lcm(\varphi(F_{j-2}))}{\varphi(\overline{c_{j-1}})}\right)^{n_j - 3 - n_{j-2}} \frac{\lcm(\varphi(F_{j-2}))}{\varphi(\overline{c_{j-1}})} \varphi(F_j) := \frac{\varphi(F_{j-2})}{\overline{c_{j-1}}} \frac{\varphi(F_{j-1})}{\overline{c_{j-1}}} \left(\frac{\varphi(F_{j-2})}{\overline{c_{j-1}}}, \varphi(F_{j-1})\right).$$

In particular $S_{n_{j-1}}(\varphi(F_1), \varphi(F_2)) = \varphi(F_j)$ for $j = 3, \ldots, i$, and can be computed modulo $T$ using formula $^6$ (index $i$ included although $\lcm(F_i)$ is not assumed invertible modulo $T$).

**Proof.** We need to show that $\varphi$ commutes with all operations and primitives involved in the formula $^6$. First of all, $\lcm(\varphi(F_j)) = \varphi(\lcm(F_j))$ for $j = 2, \ldots, i - 1$ by assumption. Hence the Euclidean degree sequence $\langle n_j \rangle_j$ is also that of the one initiated with $\varphi(F_1), \varphi(F_2)$ too, up to $j < i$. Therefore, for $j = 2, \ldots, i - 1$, the pseudo-division equality $\lcm(F_{j-1})^{n_j - n_{j-1} + 1} F_j = q F_{j-1} + \varphi(F_{j-1})$ specializes by $\varphi$: $\lcm(\varphi(F_{j-1}))^{n_j - n_{j-1} + 1} \varphi(F_j) = \varphi(q) \varphi(F_{j-1}) + \varphi(F_{j-1})$, whence: $\varphi(F_{j-1}) = \varphi(F_{j-1})$, for $j = 2, \ldots, i - 1$. Moreover the pseudo-quotient and pseudo-remainder are uniquely determined. Besides, the formula involves only algebraic operations which commute by definition with the homomorphism $\varphi$. Therefore, $\varphi$ commutes as expected. In particular, although $\lcm(F_1)$ is not assumed to be invertible modulo $T$, we still have

$$\varphi(F_i) = \varphi \left( \frac{\varphi(F_{i-1}), \varphi(F_{i-2})}{\overline{c_{j-1}}} \right) = \frac{\varphi(F_{i-1}), \varphi(F_{i-2})}{\overline{c_{j-1}}} \left(\frac{\varphi(F_{i-1}), \varphi(F_{i-2})}{\overline{c_{j-1}}}\right).$$

Finally, the assumptions made on $F_1$ and $F_2$ fall into Case 1. of Lemma $^4$: $\varphi(S_{\varphi}(F_1, F_2)) = S_j(\varphi(F_1), \varphi(F_2))$. Besides, Theorem $^2$ gives $F_j = S_j_n F_1, F_2$ hence

$$\varphi(F_j) = \varphi(S_{n_{j-1}}(F_1, F_2)) \text{ for } j = 2, \ldots, i.$$

**Corollary 1.** With the notations as in Proposition $^2$ let $f_1 := \varphi(F_1)$ and $f_2 := \varphi(F_2)$. Consider the p.r.s $F_1, F_2, \ldots, F_r, F_{r+1}$ computed in $(k[x])[y]$ with $F_{r+1} = 0$. Let $f_1, f_2, f_3, \ldots, f_i, f_{i+1}$ be the one computed modulo $T$ as explained in Proposition $^2$ under the assumption that the $(\lcm(F_j))_{1 \leq j \leq i}$ are all invertible modulo $T$. Assume that $\lcm(F_{i+1})$ is not invertible modulo $T$ or $F_{i+1} \equiv 0 \mod T$.

We have $\langle F_1, F_2, F \rangle = \langle f_{j-1}, F_j, F \rangle$ for any $2 \leq j \leq i + 1$. 

15
Proof. Over the unique factorization domain \( k[x] \), it is classical that the subresultant p.r.s verifies \( \langle F_1, F_2 \rangle = \langle F_{j-1}, F_j \rangle \) for any \( 2 \leq j \leq r + 1 \). By Proposition 1, \( \phi(F_j) = f_j \) for \( j = 1, \ldots, i + 1 \), hence:

\[
\langle F_1, F_2, T \rangle = \langle \phi(F_1), \phi(F_2), T \rangle = \langle \phi(F_{j-1}), \phi(F_j), T \rangle = \langle f_{j-1}, f_j, T \rangle,
\]

for \( j = 2, \ldots, i + 1 \) \( \square \)

---

Algorithm 4: \((u, v, U, T_1) \leftarrow \text{LastNonNil}(f_1, f_2, T)\)

**Input:** 1. polynomials \( f_1, f_2 \in k[x, y], \deg_y(f_1) \geq \deg_y(f_2) \). The leading coefficient of \( f_1 \) is invertible modulo \( T \), and \( f_2 \neq 0 \).

2. \( T = p^r \in k[x] \), the power of an irreducible polynomial \( p \).

**Output:** 1-2. monic \((y)\) polynomials \( u, v \in k[x, y] \) verifying the degree condition of Proposition 2

3-4. monic polynomials \( U, T_1 \in k[x] \) such that: \( T_1 U = T, \quad \langle f_1, f_2, T \rangle = \langle u, v U, T \rangle \).

24 repeat

25 \[
\text{Compute the subresultant p.r.s } f_1, f_2, \ldots, f_\ell, f_{\ell+1} \text{ of } f_1 \text{ and } f_2 \text{ modulo } T \text{ (with } f_{\ell+1} = 0 \text{, following Proposition 1)}
\]

26 until \( \lc(f_\ell) \) is not invertible modulo \( T \) or \( f_{\ell+1} = 0 \)

27 if \( \lc(f_i) \) is not invertible modulo \( T \) then \quad // \( \lc(f_j) \) is invertible modulo \( T \) for \( j < i \)

28 \[
(b, U) \leftarrow \text{MonicForm}(f_1, T)
\]

29 \quad // \( b \) is monic and \( \langle U b, T \rangle = \langle f_1, T \rangle \)

30 \[
a \leftarrow (\lc(f_{i-1})^{-1} \mod T) f_{i-1} \mod T
\]

31 \quad // \( a \) is monic and \( \langle a, T \rangle = \langle f_{i-1}, T \rangle \)

32 if \( \deg_x(U) > 0 \) then \quad // \( f_i \) is nilpotent

33 \quad return \((a, b, U, T/U)\)

34 else \quad // \( f_i \) is not nilpotent

35 \quad return \LastNonNil(a, b, T) \quad // Recursive call

// Here, all the \( \lc(f_1), \ldots, \lc(f_\ell) \) are invertible modulo \( T \), and \( f_{\ell+1} = 0 \)

34 return \((\lc(f_{\ell}) \mod T)^{-1} f_\ell, 0, 1, T)\)

---

**Proposition 2** (Correctness of Algorithm 4). The output \( u, v, U, T_1 \) of \( \text{LastNonNil}(f_1, f_2, T) \) verifies:

\[
\langle u, v U, UT_1 \rangle = \langle f_1, f_2, T \rangle \quad \text{and} \quad UT_1 = T.
\]

If \( v = 0 \) then \( U = 1 \) and \( T_1 = T \). Moreover, we also have the following degree conditions:

\[
\text{if } v \neq 0 \text{ then } \deg_y(v) < \deg_y(u) \quad \text{and} \quad \deg_y(u) < \deg_y(f_1), \quad (7)
\]

except in the following corner cases:

(i) \( \deg_y(u) = \deg_y(v) \) possibly holds if \( u \) and \( v \) are respectively the monic form of \( f_1 \) and \( f_2 \) with \( f_2 \) nilpotent (implying \( 0 < \deg_x(U) < \deg_x(T) \)) and \( \deg_y(f_2) = \deg_y(f_1) \).

(ii) \( \deg_y(u) = \deg_y(f_1) \) possibly holds under the condition (i) above.

(iii) The only other situation where \( \deg_y(u) = \deg_y(f_1) \) is when \( \lc(f_2) \) is not nilpotent (equivalently is invertible) modulo \( T \), and \( \lc(f_2) f_1 \equiv \lc(f_1) f_2 \mod (p) \).

In this case, \( u \) is the monic form of \( f_2 \) and \( v \) that of \( \pm f_3 \).
Remark that \( f_2 \) is nilpotent and \((y + 1, p) \leftarrow \text{MonicForm}(f_2, T)\). This is thus the first nilpotent subresultant and we have \( v = y + 1 \) and \( U = p \). Besides \( f_1 \) is nilpotent and already in monic form, thus \( u = f_1 \). Note that \( U = p \) verifies \( 0 < \deg_x(U) < \deg_x(T) \). We have \( \deg_y(v) = \deg_y(y + 1) = 1 = \deg_y(y + p) = \deg_y(u) \) (degree equality (i)) and \( \deg_y(u) = 1 = \deg_y(f_1) \) (degree equality (ii)).

An example of corner case (iii) is when

\[
\begin{align*}
  f_1 &= y + p, \\
  f_2 &= y \quad \text{modulo } T = p^2.
\end{align*}
\]

We see directly that \( \text{lcm}(f_2) f_1 \equiv \text{lcm}(f_1) f_2 \equiv y \mod (p) \). Then \( \pm f_3 = \text{prem}(f_1, f_2) = p \) is the first nilpotent subresultant. We thus have \( (v, U) \leftarrow \text{MonicForm}(f_3, T) \) with \( v = 1 \) and \( U = p \). Moreover \( u = f_2 \), being the last non nilpotent, and already in monic form. Finally, note the degree equality (iii) \( \deg_y(f_1) = 1 = \deg_y(u) \).

**Proof.** We start by the proof of correctness before turning to the proof of the degree conditions (7).

**Proof of correctness.** We investigate the three returns in the algorithm separately.

**Case 1:** Exit at Line 34. The algorithm does not enter Lines 27-33 which means that the if-test Line 27 was never passed: the leading coefficients \( \text{lcm}(f_1), \ldots, \text{lcm}(f_l) \) in the p.r.s are all invertible modulo \( T \), and \( f_{l+1} = 0 \). The output at Line 34 is

\[
u = (\text{lcm}(f_l) \mod T)^{-1} f_l \mod T, \quad v = 0, \quad U = 1, \quad T_1 = T.
\]

Therefore \( U T_1 = T \). Moreover, \( \langle u, vU, T \rangle = \langle u, T \rangle \). Note that \( u \) is monic verifying \( \langle u, T \rangle = \langle f_l, T \rangle \). Besides, by Corollary \( 1 \) \( \langle f_1, f_2, T \rangle = \langle f_l, f_{l+1}, T \rangle \), and since \( f_{l+1} = 0 \) we get \( \langle f_1, f_2, T \rangle = \langle f_l, T \rangle \). We obtain: \( \langle f_1, f_2, T \rangle = \langle u, T \rangle \) as expected.

In the remaining two cases, \( v \) is not zero, which proves the assertion that if \( v = 0 \) then \( U = 1 \), and \( T_1 = T \).

**Case 2:** Exit at Line 31. The if-test Line 27 tells that \( \text{lcm}(f_i) \) is not invertible modulo \( T \), while the \( \text{lcm}(f_j) \) are for \( j < i \). At Line 28 \( \langle b, U \rangle \leftarrow \text{MonicForm}(f_i, T) \) we have from the definition of the “MonicForm” algorithm that \( \langle U b, T \rangle = \langle f_i, T \rangle \), with \( U | T \) and \( b \) monic. Moreover \( \deg_x(U) > 0 \) (the if-test at Line 30) implies that \( f_i \) is nilpotent. On the other hand, \( f_{i-1} \) is invertible modulo \( T \): the inversion at Line 29 is correct and \( a \) is monic verifying \( \langle a, T \rangle = \langle f_{i-1}, T \rangle \). By Corollary \( 1 \) we have \( \langle f_1, f_2, T \rangle = \langle f_{i-1}, f_i, T \rangle = \langle a, f_i, T \rangle = \langle a, U b, T \rangle \). This is what we wanted to prove in this case 2.

**Case 3:** Exit at Line 33. There, \( \text{lcm}(f_i) \) is not invertible modulo \( T \), and since \( \deg_x(U) = 0 \), \( f_i \) is not nilpotent. As above, we have \( \langle b, T \rangle = \langle f_i, T \rangle \). Additionally, \( \deg_y(b) < \deg_y(f_i) \): \( f_i \) is not nilpotent but \( \text{lcm}(f_i) \) is, so the Weierstrass factorization of \( f_i \) produces a monic polynomial \( b \) of degree smaller than that of \( f_i \). And since \( \text{lcm}(f_{i-1}) \) is invertible modulo \( T \), \( \langle a, T \rangle = \langle f_{i-1}, T \rangle \). The recursive call is thus made with monic polynomials \( a \) and \( b \) of degree smaller than that of \( f_1 \) and \( f_2 \); Indeed, observe that:

\[
\deg_y(b) < \deg_y(f_i) \leq \deg_y(f_2), \quad \deg_y(a) = \deg_y(f_{i-1}), \quad \text{(hence } \deg_y(b) < \deg_y(a)) \quad (8)
\]
Therefore, the recursive call ultimately boils down to Case 1: its output \((u, v, U, T_1)\) verifies \(\langle a, b, T \rangle = \langle u, uU, T_1U \rangle\) and \(T_1U = T\). Since \(\langle a, b, T \rangle = \langle f_{i-1}, f_i, T \rangle\), and that by Corollary 1 we have \(\langle f_{1}, f_{2}, T \rangle = \langle f_{i-1}, f_i, T \rangle\). We conclude that \(\langle u, vU, T_1U \rangle = \langle f_{1}, f_{2}, T \rangle\).

Proof of the degree conditions (7). If at least two pseudo-divisions occur in the algorithm (including in recursive calls), they induce at least two strict degree decreases in the modified p.r.s computed. Both \(u\) and \(v\) being the monic form of the last two polynomials in that modified p.r.s, it follows that:

\[
\text{if } v \neq 0, \quad \deg_y(v) < \deg_y(u) < \deg_y(f_2) \leq \deg_y(f_1)
\]

which proves the degree conditions (7). No corner cases can happen in this situation.

If exactly one pseudo-division occurs then:

-1- either \(\text{lc}(f_2)\) is invertible modulo \(T\) and \(f_3 = \pm \text{prem}(f_1, f_2)\)

-2- or not and then, letting \(b\) be the monic form of \(f_2\) (Line 28), and \(a\) that of \(f_1\) (Line 29), a recursive call \(\langle u, v, U, T \rangle \leftarrow \text{LastNonNil}(a, b, T)\) (Line 33). But then, \(a\) and \(b\) being monic at least a pseudo-division should take place inside this recursive call, a contradiction. This situation -2- cannot happen in case of one pseudo-division only.

Note that \(f_2\) cannot be nilpotent for one pseudo-division to occur. The algorithm then stops, meaning that \(f_3\) is nilpotent (which may be zero according to our convention). Then \(v\) is its monic form, and \(u\) is the monic form of \(f_2\). We have:

\[
\text{if } v \neq 0, \quad \deg_y(v) \leq \deg_y(f_3) < \deg_y(f_2) = \deg_y(u) \leq \deg_y(f_1)
\]

which proves the degree conditions (7). For \(\deg_y(u) = \deg_y(f_1)\) (possible corner cases (i) (ii)) to hold, necessarily \(\deg_y(u) = \deg_y(f_2) = \deg_y(f_1)\). The pseudo-division of \(f_1\) by \(f_2\) writes as:

\[
\text{lc}(f_2)^{\deg_y(f_2)-\deg_y(f_1)+1} f_1 = \text{pquo}(f_1, f_2) f_2 \pm f_3.
\]

The degree equality \(\deg_y(f_1) = \deg_y(f_2)\) implies that \(\text{pquo}(f_1, f_2) = \text{lc}(f_1)\), and since \(f_3 \equiv 0 \mod \langle p \rangle\):

\[
\text{lc}(f_2)f_1 \equiv \text{lc}(f_1)f_2 \mod \langle p \rangle.
\]

This is the corner case (iii). Corner cases (i) (ii) do not occur.

If no pseudo-division occurs at all, then \(f_2\) is already the “first nilpotent”. The polynomials \(u\) and \(v\) are then respectively the monic form of \(f_1\) and \(f_2\). Since \(\text{lc}(f_1)\) is invertible modulo \(T\) by assumption, \(u \equiv \text{lc}(f_1)^{-1} f_1 \mod T\), in particular \(\deg_y(u) = \deg_y(f_1)\): this is corner case (ii). We can then observe that:

\[
\deg_y(v) \leq \deg_y(f_2) \leq \deg_y(f_1) = \deg_y(u).
\]

This proves the degree conditions (7). For \(\deg_y(v) = \deg_y(u)\) to hold, the condition \(\deg_y(f_1) = \deg_y(f_2)\) is necessary, proving the situation of corner case (i).
3.2. Deducing the Gröbner basis

The main algorithm essentially iterates the algorithm “LastNonNil” through recursive calls. We refer to Example 1 in Introduction.

Algorithm 5: $\mathcal{G} \leftarrow \text{SubresToGB}(a, b, T)$

**Input:**
1. $a, b \in k[x, y]$, $\deg_y(a) \geq \deg_y(b)$. The leading coefficient of $a$ is invertible modulo $T = p^e$, $b$ is not nilpotent modulo $T$ (which also means $b \neq 0$ by our convention).
2. The power of an irreducible polynomial $T = p^e \in k[x]

**Output:** A minimal lexGb $\mathcal{G}$ of the ideal $\langle a, b, T \rangle$.

35 if $a$ or $b$ is a non-zero constant then
36 | return [1]
37 $(u, v, U, T_1) \leftarrow \text{LastNonNil}(a, b, T)$  // $(u, vU, UT_1) = \langle a, b, T \rangle$
38 if $u = 1$ then
39 | return [1]
40 if $v = 0$ then
41 | return $[u, T]$
42 $\mathcal{G}_1 \leftarrow \text{SubresToGB}(u, v, T_1)$  // Recursive call
43 $\mathcal{G} \leftarrow [u] \cdot \text{cat} [U \cdot g : g \in \mathcal{G}_1]$  // Update of the output of the recursive call
44 return $\mathcal{G}$

Theorem 3. The output $\mathcal{G}$ is a minimal lexGb of $\langle a, b, T \rangle$.

**Proof.** Among the four returns in the algorithm, the first three ones are base cases treated in Cases 1, 2, 3 below. The last return (Case 4) involves a recursive call and requires more care.

**Case 1: Exit at Line 36** Here $a$ or $b$ is a non-zero constant, thus $\langle a, b, T \rangle = \langle 1 \rangle$ and the output should be [1].

**Case 2: Exit at Line 39** By the input’ Specification 1., $a$ is assumed to have an invertible leading coefficient modulo $T$, and by input’ Specification 2. $b$ is not nilpotent (which implies that $b \neq 0$ according to our convention). This legitimates the call to $\text{LastNonNil}(a, b, T)$ at Line 37, according that $a$ and $b$ should verify these assumptions. By definition of “LastNonNil”, the output $(u, v, U, T_1)$ verifies $(u, vU, UT_1) = \langle a, b, T \rangle$. So that if $u = 1$ then $\langle a, b, T \rangle = \langle 1 \rangle$ and the output should be [1]. This is precisely what returns the algorithm at Line 39.

**Case 3: Exit at Line 41** Here $v = 0$, which implies $U = 1$, $T_1 = T$ and $\langle a, b, T \rangle = \langle u, vT_1, UT_1 \rangle = \langle u, T \rangle$ by Proposition 2. The return precisely outputs $[u, T]$, which is a lexGb (actually a bivariate triangular set).

**Case 4: Exit at Line 44** The recursive call at Line 42 makes sense, its input $u, v$ and $T_1$ satisfying the input’ specifications: $u$ and $v$ are monic and $\deg_y(u) \geq \deg_y(v)$ (this degree inequality follows from the degree condition 7 of the output of “LastNonNil”). Here $v \neq 0$, as the case $v = 0$ is treated in Case 3. By the degree condition 7 of Proposition 2 we have $\deg_y(u) < \deg_y(a)$ unless one of the corner cases $\text{ii}$ or $\text{iii}$ occur. In Case $\text{ii}$, which assumptions are that of Case $\text{i}$, $b$ is nilpotent, excluded by the input’ Specifications. And in Case $\text{iii}$, $v = \pm \text{prem}(a, u)$ which is nilpotent (maybe 0 by our convention, but the case $v = 0$ is already treated in Case 3.). Thus, by Eq. 9:

$\deg_y(v) \leq \deg_y(\text{prem}(a, u)) < \deg_y(u) \leq \deg_y(b) \leq \deg_y(a)$

19
Therefore, the input \((u, v, T)\) of the recursive call Line 42 displays in any case a degree decrease compared to the input \((a, b, T)\) of the main call: always hold the following inequalities

\[
\deg_y(u) \leq \deg_y(a), \quad \deg_y(v) \leq \deg_y(b), \quad \deg_y(T) \leq \deg_y(T),
\]

and at least one of the two strict inequalities holds

\[
\deg_y(u) < \deg_y(a), \quad \deg_y(v) < \deg_y(b).
\]

We can assume by induction that this recursive call is correct: \(G_1\) is a minimal lexGb of \(\langle u, v, T \rangle\).

Moreover, inside the recursive call made at Line 42 the algorithm either went through Line 36 or through Line 37. If it exists at Line 36, then since \(u \neq 1\), this means \(v = 1\). The output is \([1]\) and the final output is \(G_1\) which is a minimal lexGb. Else, at Line 37 let us write the call to “LastNonNil” as:

\[
(u_0, v_0, U_0, T_0) \leftarrow \text{LastNonNil}(u, v, T),
\]

and let us prove that \(\deg_y(u_0) < \deg_y(u)\). The degree conditions \([7]\) of the proposition \([2]\) asserts that it is indeed the case, except maybe for the corner cases \([\text{ii}]\) or \([\text{iii}]\) where possibly \(\deg_y(u_0) = \deg_y(u)\) holds. Both corner cases \([\text{ii}]\) or \([\text{iii}]\) imply that \(\deg_y(u) = \deg_y(v)\), but this equality does not hold. Indeed \(\deg_y(u) = \deg_y(v)\) holds only under Case \([\text{i}]\) (as output of the call LastNonNil\((a, b, T)\) at Line 37 this time) where \(b\) is assumed nilpotent. But the specifications of Algorithm 5 “SubresToGb” prescribes \(b\) to be nilpotent. Hence \(\deg_y(v) < \deg_y(u)\) and finally \(\deg_y(u_0) < \deg_y(u)\).

Therefore, by construction of the lexGb, the polynomial \(u\) has a degree \([y]\) larger than all polynomials constructed in \(G_1\). We can thus apply Theorem 1-(B) \((u \text{ and } G_1)\) which allows to conclude that \(G := [u] \text{ cat } \{Ug : g \in G_1\}\) (as defined at Line 43) is a minimal lexGb if and only if \(u \in G_1\). This is obviously the case, since \(\langle G_1 \rangle = \langle u, v, T \rangle\).

It remains to show that the minimal lexGb \(G\) generates the ideal \(\langle a, b, T \rangle\) to achieve the proof of Case 4. Since \(\langle u, vU, T \rangle = \langle a, b, T \rangle\), it suffices to show that \(\langle u, vU, T \rangle = \langle G \rangle\). From \(\langle u, v, T \rangle = \langle G_1 \rangle\), we obtain \(\langle uU, vU, T \rangle = \langle UG_1 \rangle\). Hence \(\langle G \rangle = \langle u \rangle + \langle UG_1 \rangle = \langle u \rangle + \langle uU, vU, T \rangle = \langle u, vU, T \rangle\).

\[\square\]

4. Generalizing dynamic evaluation

4.1. Splitting “invertible-nilpotent”

Part of a polynomial whose support of irreducible factors are given by another polynomial. Assume that two monic polynomials \(a\) and \(b\) are squarefree. Then \(a / \gcd(a, b)\) and \(\gcd(a, b)\) are pairwise coprime. Moreover \(\gcd(a, b) = \prod_{v_p(b) > 0} p^{v_p(a)}\), according that \(v_p(a) = 1\) or \(0\). We need a similar routine when \(a\) and \(b\) are not squarefree. As seen in Example 2, it suffices to iterate a gcd computation until the two factors become coprime.

**Definition 3.** Let \(P\) be the set of irreducible polynomials of \(k[x]\). Write \(a = \prod_{p \in P} p^{v_p(a)}\) the factorization into irreducibles of \(a \in k[x]\), and \(b = \prod_{p \in P} p^{v_p(b)}\) that of \(b \in k[x]\). The \(b\)-component of \(a\), denoted \(a^{(b)}\) is the polynomial \(a^{(b)} := \prod_{p \in P, v_p(b) > 0} p^{v_p(a)}\).

**Example 9** (Isolating irreducible factors algorithm 3 “IsolFactor”). Let \(a = x^3(x + 1)^4(x + 2)\) and \(b = x^4(x + 1)\). Then \(a^{(b)} = x^3(x + 1)^4\).

There are several ways to take the \(b\)-component, we give a standard and easy one.
With the notations of Definition 3, at initialization we have:

\[ v \] for any irreducible polynomial \( p \) such that \( v_p(b) > 0 \), the sequence of integers \( (v_p(b_i))_{i=0}^{\infty} \) strictly decreases to zero, regarding that \( b_{s+1} = 1 \) and thus \( v_p(b_{s+1}) = 0 \).

Assume first that there exists an index \( i \geq 0 \) such that \( v_p(a_i) < v_p(b_i) \) \((\ast)\), and then take the smallest index \( j \) that verifies this inequality. Then \( v_p(b_{j+1}) = \min(v_p(b_j), v_p(a_j)) = v_p(a_j) \) and thus \( v_p(a_j+1) - v_p(b_j+1) = 0 \). It follows that \( v_p(b_{j+2}) = \min(v_p(b_{j+1}), v_p(a_{j+1})) = 0 \), then \( v_p(b_{j+3}) = v_p(a_{j+3}) = 0 \), and \( v_p(c_{j+2}) = v_p(c_{j+1}) \). By induction, we observe that \( v_p(b_{j+\ell}) = \cdots = v_p(b_{j+1}) = 0 \), that \( v_p(a_{j+1}) = \cdots = v_p(a_{s+1}) = 0 \), and that \( v_p(c_{j+1}) = \cdots = v_p(c_{s+1}) \).

Besides, since \( j \) is the smallest index for which \( v_p(a_j) < v_p(b_j) \), we have that \( v_p(b_j) \leq v_p(a_j) \) for \( \ell < j \). Consequently, \( v_p(b_{j+1}) = \min(v_p(b_{j+1}), v_p(a_{j+1})) = v_p(b_{j+1}) \) and by induction we observe that \( v_p(b_j) = v_p(b_{j-1}) = \cdots = v_p(b_2) = v_p(b_1) = v_p(b_0) = v_p(b) \). We obtain:

\[ v_p(c_s) = jv_p(b) + v_p(a_j), \quad \text{and} \quad v_p(a_j) = v_p(a_{j-1}) - v_p(b_j) = v_p(a_{j-1}) - v_p(b). \]

By induction, \( v_p(a_j) = v_p(a_{j-1}) - v_p(b) = v_p(a_{j-2}) - 2v_p(b) = \cdots = v_p(a_0) - jv_p(b) \). We get:

\[ v_p(c_s) = jv_p(b) + v_p(a_0) - jv_p(b) = v_p(a_0) \].

In conclusion, \( v_p(c_s) = v_p(a) \) when \( v_p(b) > 0 \) and when there is a \( j \) that satisfies \((\ast)\).

If there is no such \( j \), then the first equality in Eq. [10] implies that \( v_p(b_{s+1}) = v_p(b_s) = \cdots = v_p(b_0) = v_p(b) \). Moreover the output \( b_{s+1} \) is equal to 1 hence \( v_p(b_{s+1}) = 0 \) thereby \( v_p(b) = 0 \). Additionally, it follows still from Eq. [10] that \( v_p(c_0) = v_p(c_1) = \cdots = v_p(c_{s+1}) = 0 \). For such an irreducible polynomial \( p \), the definition of the b-component \( a^{(b)} \) says that \( v_p(c_{s+1}) \) must be 0. This concludes the proof of the output’s specification 1. in any case.

Lemma 5 (Correctness of Algorithm 6 “IsolFactor”). The output \( c_{s+1} = a_{s+1} \) are coprime polynomials that verify \( c_{s+1} = c_s \), \( a_{s+1} = a_s \), and \( c_s a_s = a \), with \( a = a^{(b)} = \prod_{p \in \mathcal{P}, v_p(a) > 0} p^{v_p(a)} \).

Proof. Let \( s+1 \) be the last index \( i \) at the exit of the repeat loop, so that \( c_{s+1} = a_{s+1} \) are the output. The exit condition of the repeat/until loop is \( b_1 = 1 \), hence \( b_{s+1} = 1 \) with the notation we have adopted. Thus \( c_{s+1} = c_s b_{s+1} = c_s \) and \( a_{s+1} = a_s/b_{s+1} = a_s \). We must show two things. First, that \( v_p(c_s) = v_p(a) \) and \( v_p(c_s) > 0 \) is equivalent to \( v_p(b) > 0 \). Second, that \( a_s = a/c_s \) is coprime with \( c_s \).

If we rewrite the update line 47 in terms of \( p \)-adic valuations, we get:

\[
\begin{align*}
  v_p(b_{i+1}) &= \min(v_p(a_i), v_p(b_i)) \\
  v_p(a_{i+1}) &= v_p(a_i) - v_p(b_i) \\
  v_p(c_{i+1}) &= v_p(c_i) + v_p(b_i)
\end{align*}
\]

With the notations of Definition 3, at initialization we have \( a_0 = \prod_{p \in \mathcal{P}} p^{v_p(a)} \) and \( b_0 = \prod_{p \in \mathcal{P}} p^{v_p(b)} \).

Input: 1-2. Polynomials \( a, b \in k[x] \).

Output: 1-2. The b-component \( a^{(b)} \) (Definition 3), and \( v_p(c_s/a_s) \). They are coprime.

45 \( a_0 \leftarrow a \); \quad b_0 \leftarrow b \; ; \quad c_0 \leftarrow 1 \; ; \quad i \leftarrow 0 \\
46 \textbf{repeat} \\
47 \quad b_{i+1} \leftarrow \gcd(a_i, b_i) \; ; \quad a_{i+1} \leftarrow \frac{a_i}{b_{i+1}} \; ; \quad c_{i+1} \leftarrow c_i b_{i+1} \; ; \quad i \leftarrow i + 1 \\
48 \textbf{until} \; b_i = 1 \\
49 \textbf{return} \; (c_i, a_i) \quad / / \; \text{Write } s+1 \text{ the index } i \text{ at the exit of the repeat loop}
Finally, let us prove that the output \( c_s, a_s \) are coprime. From \( c_s a_s = a \) and \( v_p(b) > 0 \Rightarrow v_p(c_s) = v_p(a) \), we deduce that \( v_p(a_s) = 0 \) when \( v_p(b) > 0 \). When \( v_p(b) = 0 \) then \( v_p(c_s) = 0 \), and thus \( v_p(a_s) = v_p(a) \). It follows that, either \( (v_p(a_s) = 0 \) and \( v_p(c_s) = v_p(a) \)) or \( (v_p(a_s) = v_p(a) \) and \( v_p(c_s) = 0 \). This means that \( c_s \) and \( a_s \) are coprime. \( \square \)

**Remark 2.** \( b \) is nilpotent modulo \( a^{(b)} \), and \( b \) is invertible modulo \( a^{(b)} \). Indeed all irreducible factors of \( a^{(b)} \) are irreducible factors of \( b \), hence sqfp\( (a^{(b)})\) sqfp\( (b) \). And \( a^{(b)} \) does not have any common irreducible factors with \( b \).

The splitting “invertible/nilpotent”. The algorithm \[ \text{invertNil} \] is the cornerstone for generalizing the dynamic evaluation from a modulus \( T \) that is a squarefree polynomial to a general modulus. The splitting that it induces is “invertible/nilpotent” instead of “invertible/zero” in the standard dynamic evaluation. It is similar to “IsolFactor” algorithm except that the inverse of \( b \) modulo \( a^{(b)} \) is also returned.

**Algorithm 7:** \[ ([f_1, T_1], [f_2, T_2]) \leftarrow \text{invertNil}(a, T) \]

**Input:** 1. Non-zero polynomial \( a \in k[x] \)
2. Non constant monic polynomial \( T \in k[x] \)

**Output:** 1. \([f_1, T_1] \): monic polynomial \( T_1 \) and \( f_1 \) such that \( f_1 f \equiv 1 \bmod T_1 \) if \( T_1 \neq 1 \), or \( f_1 = 0 \) if \( T_1 = 1 \),
2. \([f_2, T_2] \): \( T_2 \) monic polynomial, \( f_2 \) nilpotent \( \bmod T_2 \), and \( f_2 \equiv a \bmod T_2 \).

**Condition:** \( T = T_1 T_2, T_1 \) and \( T_2 \) coprime.

50 \((T_2, T_1) \leftarrow \text{IsolFactor}(T, f)\)
51 \(f_2 \leftarrow f \bmod T_2 \); \( f_1 \equiv f^{-1} \bmod T_1 \)
52 **return** \([f_1, T_1], [f_2, T_2]\)

**Proposition 3.** The algorithm \[ \text{invertNil} \] splits the polynomial \( T \) into two coprime polynomials \( T_1 \) and \( T_2 \), and outputs two polynomials \( f_1 \) and \( f_2 \in k[x] \) such that: \( f_1 \) is invertible \( \bmod T_1 \) and \( f_2 \) is nilpotent (maybe zero according to our convention) \( \bmod T_2 \). Moreover, \( f_1 a \equiv 1 \bmod T_1 \), and \( a \equiv f_2 \bmod T_2 \).

**Remark 3.** The polynomials \( T_1, T_2, f_1, f_2 \) are uniquely determined by \( a \) and \( T \).

**Proof.** The specifications of Algorithm 6 “IsolFactor” implies that \( T_1 T_2 = T \) and that \( T_1 \) is coprime with \( T_2 \). Remark 2 implies that \( a \) is nilpotent \( \bmod T_2 \) and invertible \( \bmod T_1 \). \( \square \)

**Example 10** (Algorithm \[ \text{invertNil} \]). Let \( p, q \in k[x] \) be two monic and distinct irreducible polynomials of \( k[x] \), \( T = p^2 q^2 \). Let \( a := ap' \) with \( ap' \in k[x] \) coprime with \( p \) and with \( q \). Then \( a \) is invertible \( \bmod q^2 \), and \( a \) is nilpotent \( \bmod p^2 \). Then \( \left[ (p a \bmod q^2)^{-1}, q^2 \right], \left[ p a \bmod p^2, p^2 \right] \leftarrow \text{invertNil}(a, T) \).

With \( T = p^2 q^2 \) as above and \( a = pq \), then \([0, 1], [a, T] \leftarrow \text{invertNil}(a, T) \).

Still with \( T = p^2 q^2 \), and \( a \) invertible \( \bmod T \) then \([a \bmod T^{-1}, T], [0, 1] \leftarrow \text{invertNil}(a, T) \).
4.2. Monic form according to dynamic evaluation

The next fundamental operation makes polynomials monic. In the case of polynomials over a field, it suffices to invert the leading coefficient. Modulo a squarefree polynomial $T$, it requires classic dynamic evaluation to handle potential zero-divisors: start with the leading coefficient, and if there is a zero branch, carry on with the next coefficient etc. until there is no zero branch.

The situation is more subtle when $T$ is not squarefree. This stems from the necessity to perform a Weierstrass factorization to make a polynomial monic.

Overview of Algorithm 8 “WeierstrassForm_D5”. It investigates all the coefficients of $f$ by decreasing degree through recursive calls, by splitting into an invertible branch, and a nilpotent one. It returns the inverse of the coefficient in the invertible branches, and continues with the next coefficient in the nilpotent one. That the branches do not overlap is due to the specification $\star$ in Algorithm 4, namely the underlying polynomials $T_1$ and $T_2$ obtained in the splitting are coprime. The output is a compilation of the results obtained at the endpoints of all the branches. See Example 11.

The purpose of the algorithm “WeierstrassForm_D5” is to provide the correct input for “MusserQ” algorithm, like its local counterpart, Algorithm 1 “WeierstrassForm”. It is a recursive algorithm, the main call being $\text{WeierstrassForm}(f, T, \deg_y(f), T)$. Recursive calls take as third input an integer smaller than $\deg_y(f)$, and a polynomial $N$ that divides $T$ as the fourth input. The lemma below addresses the validity of the algorithm throughout recursive calls, and Corollary 2finalizes the proof of correctness of Algorithm 5.

Lemma 6. Assume that the algorithm is correct for a fixed polynomial $f = c_0(x) + c_1(x)y + \cdots + c_d(x)y^d$, and for any polynomial $T$, and integers $-1 \leq d \leq d_0$ for a fixed integer $-1 \leq d_0 < \delta$, and a polynomial $N$, all satisfying the input’ specifications. Then the algorithm is correct with input $f$, any $T$ and $N$ satisfying the input’ specifications, for the integer $d = d_0 + 1$.

Proof. Let $c_d$ be the coefficient of degree $d$ in $f$ (Line 55). There are four cases to investigate:

(a) $c_d = 0$, return at Line 70.
(b) $c_d \neq 0$, $c_d$ is invertible modulo $T$, that is $T_2 = 1$. The return occurs at Line 68.
(c) $c_d \neq 0$, $c_d$ is nilpotent, that is $T_2 = T$ and $T_1 = 1$. The algorithm ends at Line 66.
(d) $c_d \neq 0$, $c_d$ has an invertible part and a nilpotent part, that is $T_1 \neq 1$, $T_2 \neq 1$. Return occurs at Line 63.

Case (a). The algorithm goes directly to Line 70 where a recursive call is done with input $d - 1$ (instead of $d$ in the main call), hence is correct by assumption: the output $[[f_i, T_i, d_i, N_i]]$ verifies the specifications. Since $f$ has coefficient of degree $d$ $c_d = 0$ in this case, the output is the same with input $d$. Moreover these specifications are unchanged with $d$ or $d - 1$.

Case (b). The coefficient $c_d$ is invertible modulo $T$. There is no splitting at Line 57. The output $[[f, T, d, a_1, N]]$ at Line 68 verifies the specification: since $T_1 = T$, $a_1$ is the inverse of the coefficient $c_d$ modulo $T$, $N$ is the gcd with $T$ of the coefficients of degree larger than $d$ by assumption, and sqfp($T$) = sqfp($N$).

Case (c). The coefficient $c_d$ is nilpotent hence $T_2 = T$. The recursive call made at Line 66 outputs $[[f_i, T_i, d_i, N_i]]$ which verifies the specifications with input $f$, $T$, $d - 1$, gcd($N$, $c_d$). One easily verifies that this output also complies with the specifications corresponding to the input $f$, $T$, $d$, $N$.

Case (d). There is a splitting into an invertible branch $[a_1, T_1]$ and a nilpotent one $[a_2, T_2]$ (Line 57). In the invertible branch, the algorithm returns an output similar to Case (b). In the
non-invertible branch, a recursive call is performed and reduces to Case (c). The final step merges these output, which all-in-all verify the required specifications.

**Corollary 2.** Algorithm 8 "WeierstrassForm \(D5\)" is correct.

**Proof.** The main call is made with \(f, T, \deg_y(f), T\). The proof proceeds by induction on \(d_0 = -1, \ldots, \deg_y(f)\). The base case corresponds to \(d_0 = -1\) and more precisely to:

\[
\text{WeierstrassForm}_D5(f, T, -1, \gcd(T, \content(f))).
\]

The output at Line 54 is then \([f, T, -1, 0, \gcd(T, \content(f))]\) and satisfies the required specifications.

The induction hypothesis assumes that the algorithm is correct for a fixed integer \(-1 < d_0 < \deg_y(f)\): with input \(\text{WeierstrassForm}_D5(f, T, d, N)\) where \(N = \gcd(T, c_{d+1}, \ldots, c_\delta), \text{sqfp}(N) = \text{sqfp}(T)\), and \(d\) is any integer \(-1, \ldots, d_0 < \deg_y(f)\), the algorithm is correct. Then Lemma 6 shows that the algorithm is correct for input \(\text{WeierstrassForm}_D5(f, T, d, N)\) where \(d \leq d_0 + 1\) achieving the proof by induction. 

\[\Box\]
Algorithm 8: \([ [f_i, T_i, d_i, a_i, N_i] ] \leftarrow \text{WeierstrassForm}_D5(f, T, d, N)\)

\begin{verbatim}
Input: 1. polynomial \( f \in k[x, y] \), \( f = c_0(x) + c_1(x)y + \cdots + c_5(x)y^5 \)
2. a non constant monic polynomial \( T \in k[x] \),
3. an integer \( d \geq -1 \),
4. if \( d < \delta \) then \( N = \gcd(T, c_{d+1}, \ldots, c_{6}) \). It must also verify \( \text{sqfp}(N) = \text{sqfp}(T) \), that is \( c_{d+1}, \ldots, c_{6} \) are nilpotent modulo \( T \). And \( N = T \) if \( d \geq \delta \).

Output: 1. \( f_i \equiv f \mod T_i \)
2. \( \prod_i T_i = T \) and the \( (T_i)s \) are pairwise coprime.
3. \( d_i \leq \delta \) is the largest integer such that \( c_{d_i} = \text{coeff}(f_i, d_i) \) is invertible modulo \( T_i \). If such an integer does not exist then \( d_i = -1 \).
4. \( a_i \equiv \text{coeff}(f_i, d_i)^{-1} \mod T_i \) if \( d_i \geq 0 \) and \( a_i = 0 \) if \( d_i = -1 \).
5. \( N_i = \gcd(T_i, c_{d_i+1}, \ldots, c_{6}) \) is the gcd with \( T_i \) of all the coefficients of \( f_i \) at a degree larger than \( d_i \). In particular, \( \deg_y(f_i \mod N_i) = d_i \). Moreover, \( c_{d_i+1}, \ldots, c_{6} \) are nilpotent modulo \( T_i \), that is \( \text{sqfp}(N_i) = \text{sqfp}(T_i) \).

if \( d = -1 \) then \quad // Base case of the recursive calls
  return \([ [f, T, -1, 0, N] ] \)

\( c_d \leftarrow \text{coeff}(f, d) \)
if \( c_d \neq 0 \) then
  \([ [a_1, T_1], [a_2, T_2] ] \leftarrow \text{invertNil}(c_d, T) \) \quad // \( a_1c_d \equiv 1 \mod T_1 \), \( a_2 \equiv c_d \mod T_2 \)
  if \( T_2 \neq 1 \) then
    if \( T_1 \neq 1 \) then \quad // Case (d)
      \( N_2 \leftarrow \gcd(a_2, N, T_2) \) \quad // Now \( N_2 = \gcd(T_2, c_d, c_{d+1}, \ldots, c_{6}) \)
      \( f_1 \leftarrow f \mod T_1 \); \( f_2 \leftarrow f \mod T_2 \)
      \( N_1 \leftarrow \gcd(N, T_1) \) \quad // Now \( N_1 = \gcd(T_1, c_d, c_{d+1}, \ldots, c_{6}) \)
      return \([ [f_1, T_1, d, a_1, N_1] ] \leftarrow \text{WeierstrassForm}_D5(f_2, T_2, d - 1, N_2) \)
    else \quad // \( c_d \) is nilpotent (Case (c))
      return \([ [N_1, T_1, d, a_1, N_1] ] \left\langle \text{WeierstrassForm}_D5(f, T, d - 1, N) \quad // T_2 = T \)
    else \quad \text{return} \quad // \( c_d \) is invertible (Case (b))
      return \([ [f, T, d, a_1, N] ] \)
    // \( T_1 = T \)
  else \quad // Case (a)
    return \([ [f, T, d - 1, N] ] \)
end if
end if
end if
end if
end if
end if
end if
end if
end if
end if
\end{verbatim}

Example 11 (Algorithm 8 “WeierstrassForm_D5”). Consider the polynomial \( f = x(x + 1)y^2 + 2xy - x - 1 \) and \( T = x^2(x + 1)^2 \). The main call is

\( \text{WeierstrassForm}_D5(f, T, 2, T) \) the third input is 2 because \( f \) is of degree 2.

First coefficient, that of \( y^2 \): \( c_2(x) = x(x + 1) \). We have

\( \left[ [0, 1], [x(x + 1), x^2(x + 1)^2] \right] \leftarrow \text{invertNil}(c_2, T) \)

since \( c_2 \) is nilpotent modulo \( T \). Thus \( T_1 = 1, T_2 = T \) and \( N = \gcd(T, c_2) = x(x + 1) \).

Recursive call at Line 66 is: \( \text{WeierstrassForm}_D5(f, x^2(x + 1)^2, 1, x(x + 1)) \).
Second coefficient, that of \( y \) is: \( c_1(x) = 2x. \)

\[
\left[\left[-\frac{1}{2}x-1, (x+1)^2\right], \, [2x, \, x^2]\right] \leftarrow \text{invertNil}(2x, \, x^2(x+1)^2),
\]

thus \( T_1 = (x+1)^2 \) and \( T_2 = x^2. \) Thus at Line 60, \( N_1 = \gcd(x(x+1), \, (x+1)^2) = x+1 \) and \( N_2 = \gcd(x(x+1), \, x^2, \, 2x) = x. \) Next consider Line 63. We obtain

\[
\text{OUT} = \left[\left[-(-1 - x)y^2 + 2xy - x - 1, \, (x+1)^2\right], \, 1, \, -1 - \frac{1}{2}x, \, x+1\right]
\]

that is supplemented with the recursive call \texttt{WeierstrassForm}$_{D5}$(\( xy^2 + 2xy - x - 1, \, x^2, \, 0, \, x \)).

Third coefficient is \( c_0 \equiv -x - 1 \mod x^2. \) \([[-1 + x, \, x^2], \, [0, \, 1]] \leftarrow \text{invertNil}(-x - 1, \, x^2), \) thus \( T_1 = x^2 \) and \( T_2 = 1. \) The return occurs at Line 68 with \([xy^2 + 2xy - x - 1, \, x^2, \, 0, \, -1 + x, \, x].\)

Finally, \( \text{OUT} = \left[\left[f_1, \, T_1, \, d_1, \, a_1, \, N_1\right], \, [f_2, \, T_2, \, d_2, \, a_2, \, N_2]\right] \) where:

\[
\begin{align*}
f_1 &= (-1 - x)y^2 + 2xy - x - 1 \\
f_2 &= xy^2 + 2xy - x - 1 \\
T_1 &= (x+1)^2 \\
T_2 &= x^2 \\
d_1 &= 1 \\
d_2 &= 0 \\
a_1 &= -1 - \frac{1}{2}x \\
a_2 &= -1 + x \\
N_1 &= x+1 \\
N_2 &= x
\end{align*}
\]

We have indeed \( T = T_1T_2, \, d_1 = 1, \) and \( d_2 = 0. \) Also, for \( i = 1, \, 2, \) \( \deg_y(f_i \mod N_i) = d_i \), and \( \text{sqfp}(N_i) = \text{sqfp}(T_i). \) Moreover, \( a_i \text{coeff}(f_i, \, d_i) \equiv 1 \mod T_i. \) So that all specifications are satisfied. \( \square \)

**Overview of Algorithm** \( [7] \) “MonicForm$_{D5}$”. Again it translates the algorithm \( [3] \) “MonicForm” to dynamic evaluation. Among the two subroutines involved, only \texttt{WeierstrassForm}$_{D5}$ creates splittings, indeed \texttt{MusserQ} does not perform divisions. Note that the description of “\texttt{MusserQ}” in Algorithm \( [2] \) does not assume \( T \) to be the power of an irreducible polynomial. There is no need to adapt it to dynamic evaluation.

After a first call to \texttt{WeierstrassForm}$_{D5}$ at Line 72, \( W \) is a collection of data like the output of the local version \texttt{WeierstrassForm}. For each component \( W = [f_i, \, T_i, \, d_i, \, \alpha_i, \, N_i] \) of \( W \) two cases must be distinguished: if \( d_i = -1 \) then \( f_i \) is nilpotent and \( N_i/f_i \), or \( d_i \geq 0. \) In the former, a second call to \texttt{WeierstrassForm}$_{D5}$ (Line 75) is performed after dividing the input by \( N_i. \) This second call creates subbranches requiring to “project” some polynomials on these new factors (role of the call to \texttt{IsolFactor} at Line 79).

**Proposition 4** (Correctness of Algorithm \( [9] \) “MonicForm$_{D5}$”). Specifications \([a], \, [b], \, [c] \text{ and } [d] \) are satisfied by the output \text{OUT} of the algorithm.

**Proof.** Consider \( W \in W \) at Line 72 written \( W = [f_i, \, T_i, \, d_i, \, \alpha_i, \, N_i]. \) The specifications 1-5. of Algorithm \( [8] \) “WeierstrassForm$_{D5}$” give:

\[1'. \, \langle a, \, T_i \rangle = \langle f_i, \, T_i \rangle, \]
\[2'. \, \prod_i T_i = T \text{ and the } T_i\text{s are pairwise coprime} \]
\[3'. \, d_i = -1 \text{ (Line 74) or } d_i \geq 0 \text{ (Line 81)} \]
\[4'. \text{ (not important)} \]
\[5'. \, N_i = \gcd(T_i, \text{ coeffs. of } f_i \text{ of degree } > d_i), \text{ and } \text{sqfp}(N_i) = \text{sqfp}(T_i). \]
\textbf{Algorithm 9: } \([b_i, T_i, U_i] \leftarrow \text{MonicFormD5}(a, T)\)

\textbf{Input:} 1. polynomial \(a \in k[x,y]\)
2. monic non constant polynomial \(T \in k[x]\)

\textbf{Output:} 1. \(b_i \in k[x,y]\) is monic (in \(y\))
2. monic polynomials \(T_i \in k[x]\), pairwise coprime dividing \(T\).
3. monic polynomials \(U_i \in k[x]\), pairwise coprime dividing \(T_i\). Furthermore:
   (a) \(\prod_i U_i T_i = T\) and, \(c) U_i\) divides \(a\) modulo \(T_i\)
   (b) if \(U_i \neq 1\), \(\text{sqfp}(U_i) = \text{sqfp}(T_i)\)
   (d) \(\prod_i \langle U_i b_i, U_i T_i \rangle \simeq \langle a, T \rangle\).

\begin{verbatim}
71 \text{OUT} \leftarrow []
72 \text{W} \leftarrow \text{WeierstrassFormD5}(a, T, \deg_y(a), T)
73 \text{for } W \in \text{W} \text{ do} \quad \text{// write } W = [f_i, T_i, d_i, \alpha_i, N_i]
74 \quad \text{if } d_i = -1 \text{ then} \quad \text{// } f_i \text{ is nilpotent and } N_i|T, N_i|f_i
75 \quad \quad Z \leftarrow \text{WeierstrassFormD5}(f_i/N_i, T_i/N_i, \deg_y(f_i), T_i/N_i)
76 \quad \quad U_{i0} \leftarrow N_i
77 \quad \text{for } Z \in Z \text{ do} \quad \text{// write } Z = [f_{ij}, T_{ij}, d_{ij}, \alpha_{ij}, N_{ij}]
78 \quad \quad b_{ij} \leftarrow \text{MusserQ}(f_{ij}, T_{ij}, d_{ij}, \alpha_{ij}, N_{ij}) \quad \text{// } f_{ij} \text{ is not nilpotent, } d_{ij} \geq 0
79 \quad \quad (U_{ij}, U'_{ij}) \leftarrow \text{IsolFactor}(U_{ij-1}, T_{ij}) \quad \text{// } U_{ij}U'_{ij} = U'_{ij-1}, U_{ij} = N_i(T_{ij})
80 \quad \text{OUT} \leftarrow \text{OUT cat } [\langle b_{ij}, T_{ij}, U_{ij} \rangle]
81 \text{else} \quad \text{// } f_i \text{ is not nilpotent}
82 \quad b_{i} \leftarrow \text{MusserQ}(f_{i}, T_{i}, d_{i}, \alpha_{i}, N_{i})
83 \quad \text{OUT} \leftarrow \text{OUT cat } [\langle b_{i}, T_{i}, 1 \rangle]
84 \text{return OUT}
\end{verbatim}

\textit{Case } \(d_i \geq 0\). This concerns Line 71 and onward. This means that \(f_i\) is not nilpotent modulo \(N_i\) and \(\deg_y(f \text{ mod } N_i) = d_i\). The output \([f_i, T_i, d_i, \alpha_i, N_i] = W \in \text{W}\) of the first call to \text{WeierstrassFormD5} at Line 72 is then ready to be used by Algorithm \text{MusserQ} at Line 82 (its specifications being satisfied). We obtain \([b_i, T_i] = \langle f_i, T_i \rangle\) with \(b_i\) monic. And consequently, by the Chinese remainder theorem and the specifications 1'-2'. above:

\[
\prod_{i, \, d_i \geq 0} \langle b_i, T_i \rangle = \prod_{i, \, d_i \geq 0} \langle f_i, T_i \rangle = \prod_{i, \, d_i \geq 0} \langle a, T_i \rangle \simeq \langle a, \prod_{i, \, d_i \geq 0} T_i \rangle.
\]

Besides, note that the third component of the list \([b_i, T_i, 1]\) added to OUT at Line 83 is 1, which means that \(U_i = 1\). Specification (c) is clearly true while Specification (b) is void. Moreover:

\[
\prod_{i, \, d_i \geq 0} \langle U_i b_i, U_i T_i \rangle \simeq \langle a, \prod_{i, \, d_i \geq 0} T_i \rangle. \tag{11}
\]

\textit{Case } \(d_i = -1\). This concerns Lines 74-80. We then have \(N_i|f_i\) and \(N_i = \gcd(T_i, \deg(f_i))\). By specification 5'. above, \(\text{sqfp}(N_i) = \text{sqfp}(T_i)\). Thus \(\gcd(T_i, \deg(f_i/N_i)) = 1\). Consequently,

\[
\text{for any } T_{ij} \text{ dividing } T_i \text{ we have } \gcd \left(T_{ij}, \deg \left(f_i \frac{1}{N_i}\right)\right) = 1. \tag{12}
\]
Consider now Line 75 \( Z \leftarrow \text{WeierstrassFormD5}(f_i/N_i, T_i/N_i, \deg(f_i), T_i/N_i) \) as well as a component \( Z \in \mathbb{Z} \), written \( Z = [f_{ij}, T_{ij}, d_{ij}, \alpha_{ij}, N_{ij}] \). It verifies the specifications 1.-5. of Algorithm 2:

1". \( (f_i/N_i, T_{ij}) = (f_{ij}, T_{ij}) \),

2". \( \prod_i T_{ij} = T_i/N_i \) and the \( (T_{ij}) \)'s are pairwise coprime

3". \( d_{ij} \geq 0 \)

4". (not important)

5". \( N_{ij} = \gcd(T_{ij}, \text{coeffs. of } f_{ij} \text{ of degree } d_{ij}) \), and \( \text{sqfp}(N_{ij}) = \text{sqfp}(T_{ij}) \).

Indeed, \( d_{ij} \geq 0 \) holds. Otherwise \( N_{ij} = \gcd(T_{ij}, \text{content}(\frac{b_{ij}}{N_i})) \neq 1 \) would divide \( f_{ij} \), excluded as seen above in Eq \( (12) \). Therefore, the component \( Z \) is ready to be used by Algorithm 2 "MusserQ" at Line 78, its input' specifications being satisfied by \( f_{ij}, T_{ij}, d_{ij}, \alpha_{ij}, N_{ij} \). Its output \( b_{ij} \) hence verifies \( (b_{ij}, T_{ij}) = (f_{ij}, T_{ij}) \) with \( b_{ij} \) monic. Specifications 1"-2". imply, with the Chinese remainder theorem:

\[
\prod_j (b_{ij}, T_{ij}) = \prod_j (f_{ij}, T_{ij}) = \prod_j (f_i/N_i, T_i/N_i) \simeq (f_i/N_i, T_i/N_i).
\]  

(13)

Eq. (13) then implies with the Chinese remainder theorem according to Specifications 1'-2':

\[
\prod_i (N_i \prod_j (b_{ij}, T_{ij})) \simeq \prod_i (N_i (f_i/N_i, T_i/N_i)) \simeq \prod_i (f_i, T_i) \simeq \prod_i (a, T_i)
\]  

(14)

Line 79 introduces the \( T_{ij} \)-component (Definition 3) of \( U_{ij}' \):

\[
(U_{ij}, U_{ij}') \leftarrow \text{IsolFactor}(U_{ij}'-1, T_{ij}), \quad \text{with } U_{ij} = U_{ij}'(T_{ij}) \text{ and } U_{ij}' = N_i \text{ (Line 76).}
\]

The specifications of \text{IsolFactor} tell that \( U_{ij}U_{ij}' = U_{ij}'-1 \) and \( U_{ij} \) is coprime with \( U_{ij}' \). We see then that

\[
N_i = U_{ij}' = U_{i1}U_{i1}' = U_{i2}U_{i2}' = \cdots = U_{ij}U_{ij}', \quad \text{for all } j \geq 1.
\]  

(15)

Moreover the polynomials in the product are pairwise coprime. By definition, \( U_{ij} = U_{ij}'(T_{ij}) = \prod_{p \in \mathcal{P}, \, v_p(T_{ij}) > 0} p^{v_p(U_{ij}-1)}. \) Since \( U_{ij}' \) is a factor in the factorization of \( N_i \) of Eq. (15), we deduce that \( \{ v_p(T_{ij}) > 0 \Rightarrow v_p(U_{ij}-1) = v_p(N_i) \} \), and thus that \( U_{ij} = N_i^{(T_{ij})} \).

On the other hand, the specification 2". above provides \( N_i \prod_j T_{ij} = T_i \) with the \( (T_{ij}) \)'s coprime, and the specification 5". gives \( \text{sqfp}(T_i) = \text{sqfp}(N_i) \). It follows that

\[
\prod_j U_{ij} = \prod_j \prod_{p \in \mathcal{P}, \, v_p(T_{ij}) > 0} p^{v_p(N_i)} = \prod_{p \in \mathcal{P}, \, v_p(T_i) > 0} p^{v_p(N_i)} = \prod_{p \in \mathcal{P}, \, v_p(N_i) > 0} p^{v_p(N_i)} = N_i,
\]  

(16)

as well as \( \text{sqfp}(U_{ij}) = \text{sqfp}(T_{ij}) \) if \( U_{ij} \neq 1 \). This proves Specifications \( \text{[b]} \) in the case \( d_i = -1 \).

We have seen that \( N_i | f_i \), hence \( f_i \in \langle N_i \rangle \). Specification 1'. thus implies that \( a \in \langle N_i, T_i \rangle \). By Eq. (16) \( U_{ij} \) divides \( N_i \) and since \( T_{ij} \) divides \( T_i \), we obtain that \( a \in \langle U_{ij}, T_{ij} \rangle \), or equivalently that \( U_{ij} \) divides \( a \) modulo \( T_{ij} \). This proves Specification \( \text{[e]} \) in the case \( d_i = -1 \).

In Eq. (14), substituting the \( N_i \)'s by \( \prod_j U_{ij} \)'s from Eq. (16), and combining with Eq. (11) yield:

\[
\prod_i \prod_j (U_{ij}b_{ij}, U_{ij}T_{ij}) \cdot (U_{i1}b_{i1}, U_{i2}T_{i1}) \simeq \prod_i \prod_j (a, T_i) \cdot (a, T_i) \simeq (a, T_i).
\]  

28
This proves Specification \([d]\) in all cases.

Moreover Specification \([2']\) above implies that \(\prod_{i, d_i=-1} \prod_{j} U_{ij}T_{ij} = \prod_{i, d_i=-1} T_i\) with the \((T_{ij})j\)’s pairwise coprime. With Specification \([2']\) above \(\prod_{i} T_{i} = T\), we get

\[
\prod_{i, d_i=-1} \prod_{j} U_{ij}T_{ij} \cdot \prod_{i, d_i\geq 0} U_{i}T_{i} = T.
\]

This proves the specification \([a]\) (in all cases). \(\Box\)

5. Computation of lexGb through dynamic evaluation

5.1. Subresultant p.r.s

With the ability to make monic nilpotent polynomials occurring in the subresultant p.r.s. of \(a\) and \(b\) modulo \(T\), we are ready to generalize the algorithm \([4]\) “LastNonNil”. The skeleton is the same as that of Algorithm \([4]\) find the “last non nilpotent” and the “first nilpotent” polynomials in the modified subresultant p.r.s. The difference lies in the management of the splittings arising when making polynomials monic at Line \([28]\) and from the recursive call at Lines \([33]\). The output is a family of objects each similar to the output of the local version Algorithm \([3]\)

**Example 12 (Algorithm \([10]\) “LastNonNil_D5”).** Consider \(T = x^2(x+1)^2\), and \(a = y^3 + (x-1)y^2 + x(x+1)y + x\) and \(b = y^3 - y^2 + y - x\). The subresultant of degree 2 is \(S_2(a, b) = xy^2 + (x^2 + x - 1)y + 2x\). Since \(\text{lcm}(S_2) = x\) is not invertible modulo \(T\), the algorithm calls \(\text{MonicForm}_D5(S_2, T)\) at Line \([91]\). It outputs two branches:

\[
[[y^2 + (2x + 3)y + 2, (x + 1)^2, 1], [y - 2x, x^2, 1]] \leftarrow \text{MonicForm}_D5(xy^2 + (x^2 + x - 1)y + 2x, T)
\]

Recursive calls are then performed on each of these two branches:

1st branch: \([1, 0, 1, (x+1)^2] \leftarrow \text{LastNonNil}_D5(b, y^2 + (2x + 3)y + 2, (x + 1)^2)\).

2nd branch: \([y - 2x, 1, x, x] \leftarrow \text{LastNonNil}_D5(b, y - 2x, (x + 1)^2)\).

Finally: \([1, 0, 1, (x+1)^2, [y - 2x, 1, x, x]] \leftarrow \text{LastNonNil}_D5(a, b, T)\), meaning that \(\langle a, b, T \rangle \simeq \langle 1, 0, (x + 1)^2 \rangle \cdot \langle y - 2x, x, x^2 \rangle\) (actually, the latter product of ideals is equal to \(\langle y, x \rangle\)\).

**Proposition 5 (Correctness of Algorithm \([10]\) “LastNonNil_D5”).** The family \([a_i, b_i, T_i, U_{ij}]\) output by \(\text{LastNonNil}_D5(f_1, f_2, T)\) satisfies the specifications 1-4 and Eq. \([17]\).

We also have the degree conditions: For all \(j\),

\[
\text{deg}_y(a_j) \leq \text{deg}_y(f_1), \quad \text{and if } \quad b_j \neq 0, \quad \text{deg}_y(b_j) < \text{deg}_y(f_2), \quad \text{deg}_y(b_j) < \text{deg}_y(a_j)
\]

Moreover \([\text{deg}_y(a_j) = \text{deg}_y(f_1)] \Rightarrow [\text{deg}_y(f_1) = \text{deg}_y(f_2)]\).

**Proof.** We separate the proof of correctness, from the proof of the degree conditions, treated after.

**Proof of correctness:** We investigate the two returns at Line \([98]\) and at Line \([97]\).

**Case of return at Line \([98]\):** The subresultant p.r.s. was computed modulo \(T\) without failure until to get a zero \(f_{t+1} = 0\). By Corollary \([1]\) \(\langle f_1, f_2, T \rangle = \langle f_{t'}, f_{t+1}, T \rangle\), whence the output \([\langle u, v, 1, T \rangle] \) with \(u \equiv \text{lcm}(f_{t'})^{-1} f_t \mod T, v = f_{t+1} = 0\) verifies \(\langle u, v, T \rangle = \langle f_1, f_2, T \rangle\). This equality is a special easy case of Eq. \([17]\) when the output has one component only.
Algorithm 10: \([a_i, b_i, T_i, U_i] \leftarrow \text{LastNonNil}_D S(f_1, f_2, T)\)

**Input:** 1. polynomial \(f_1 \in k[x,y]\) which has an invertible leading coefficient modulo \(T\).
2. polynomial \(f_2 \in k[x,y]\), satisfies Assumption [\(\Pi\)] (in particular is not zero).
3. \(T\) is a monic polynomial \(T \in k[x]\)

**Output:** 1-2. monic (in \(y\)) polynomials \(a_i, b_i \in k[x,y]\) verifying the degree condition of Proposition [\(5\)]

3.-4. monic polynomials \(U_i, T_i \in k[x]\) such that the family \((T_i U_i)_i\) is pairwise coprime, and if \(T_i \neq 1\) then sqfp\((T_i) = \text{sqfp}(U_i)\).

\[
\prod_i T_i U_i = T, \quad \langle f_1, f_2, T \rangle \simeq \prod_i \langle a_i, T_i b_i, U_i T_i \rangle \tag{17}
\]

85 OUT \(\leftarrow \emptyset\)
86 repeat
87 \hfill Compute the subresultant p.r.s \(f_1, f_2, \ldots, f_{\ell+1}\) of \(f_1\) and \(f_2\) modulo \(T\) (with \(f_{\ell+1} = 0\), following Proposition [\(1\)].
88 until \(\text{lc}(f_1)\) is not invertible modulo \(T\) or \(f_{\ell+1} = 0\)
89 if \(\text{lc}(f_i)\) is not invertible modulo \(T\) then \(// \text{lc}(f_j)\) is invertible modulo \(T\) for \(j < i\)
90 \hfill \(a \leftarrow (\text{lc}(f_{i-1})^{-1} \bmod T) f_{i-1} \bmod T\) \(// a\) is monic and \(\langle a, T \rangle = \langle f_{i-1}, T \rangle\)
91 \hfill \text{Mb} \leftarrow \text{MonicForm}_D S(f_i, T) \(// \text{Write Mb} = \langle [b_j], U_j, T_j \rangle\)
92 \hfill for \([b_j, U_j, T_j] \in \text{Mb}\) do
93 \hfill \text{if } T_j \neq 1 \text{ then} \quad \text{// } f_i \text{ is nilpotent modulo } U_j T_j
94 \hfill \quad \text{OUT} = \text{OUT cat } \langle [a, b_j, T_j, U_j] \rangle
95 \hfill \text{else} \quad \text{// } f_i \text{ is not nilpotent modulo } U_j T_j
96 \hfill \quad \text{OUT} = \text{OUT cat } \text{LastNonNil}_D S(a, b_j, U_j)
97 \hfill \text{return OUT}
98 \hfill \text{// Here, all lc}(f_i), \ldots, \text{lc}(f_{\ell})\) are invertible modulo \(T\), and \(f_{\ell+1} = 0\)
99 \hfill \text{return } \langle ([\text{lc}(f_i) \bmod T]^{-1} f_i, 0, 1, T) \rangle

*Case of return at Line [\(97\)].* The subresultant p.r.s \(f_1, f_2, \ldots\), was correctly computed until \(f_i\), \(\text{lc}(f_i)\) being not invertible modulo \(T\). Since \(f_{i-1}\) passed the if-test at Line [\(89\)] it has an invertible leading coefficient and \(a \equiv \text{lc}(f_{i-1})^{-1} f_{i-1} \bmod T\) makes sense at Line [\(90\)] and \(\langle f_{i-1}, T \rangle = \langle a, T \rangle\).

Algorithm [\(5\)] “\text{MonicForm}_D S” is then called at Line [\(91\)] to “make” \(f_i\) monic. The output \text{Mb} is a list of lists of three polynomials, say \([b_j, T_j, U_j]_j\) verifying \(\langle T_j b_j, U_j T_j \rangle = \langle f_i, U_j T_j \rangle\) and \(\prod_j U_j T_j = T\), with \(b_j\) monic; Also the polynomials \((U_j T_j)_j\)s are pairwise coprime and sqfp\((U_j) = \text{sqfp}(T_j)\) if \(T_j \neq 1\). Next two cases occur: either \(f_i\) is nilpotent modulo \(U_j T_j\) (Line [\(93\)] or not (Line [\(95\)])).

In the first case, \(f_{i-1}\) is the last non-nilpotent and \(f_i\) is the first nilpotent polynomial met in the modified p.r.s. This is the expected result, thus the component \([a, b_j, T_j, U_j]\) is added to the output OUT. With Corollary [\(1\)] we obtain:

\[
\langle f_1, f_2, U_j T_j \rangle = \langle f_{i-1}, f_i, U_j T_j \rangle = \langle a, T_j b_j, T_j U_j \rangle \tag{19}
\]

In the second case, \(T_j = 1, f_i\) is not nilpotent modulo \(U_j T_j\). Since \(\text{lc}(f_i)\) is not invertible modulo \(T\) (Line [\(89\)], all monic forms \((b_j)_j\) verify \(\text{deg}_y(b_j) < \text{deg}_y(f_i)\). Besides, \(\text{deg}_y(a) = \text{deg}_y(f_{i-1}) \leq\)
polynomials in the modified subresultant p.r.s, this implies: a implies at least two strict degree decreases, and since number of pseudo-divisions (including those occurring in recursive calls) is at least two. This

And with these notations, the isomorphism (21) above is precisely Eq. (17). This achieves the proof of correctness of the algorithm in all cases.

Proof of the degree condition. Consider one component \([a_j, b_j, U_j]\). Assume that the total number of pseudo-divisions (including those occurring in recursive calls) is at least two. This implies at least two strict degree decreases, and since \(a_j\) and \(b_j\) are the monic form of the two last polynomials in the modified subresultant p.r.s, this implies:

\[
\deg_y(a_j) < \deg_y(f_1), \quad \text{and if } b_j \neq 0 \quad \deg_y(b_j) < \deg_y(f_2), \quad \text{and } \deg_y(b_j) < \deg_y(a_j),
\]

proving (18) in case of more than one pseudo-division.

If only one pseudo-division occurs in the algorithm, then two possibilities may happen:

-1- either \(\text{lcm}(f_2)\) is invertible modulo \(T\) and \(f_3 = \pm \text{prem}(f_1, f_2)\)

-2- or not and then, let \([b_j, U_j, T_j]_j\) be the monic form of \(f_2\) (Line 91), and \(a\) that of \(f_1\) (Line 90). According to Assumption (1), \(f_2\) is not nilpotent modulo \(U_j T_j\) for any \(j\): the algorithm then never goes through the lines 93-94. At Line 96, a recursive call occurs: “\(\text{LastNonNil}D_5(a, b_j, U_j)\)” , with \(a\) and \(b_j\) monic. Inside this recursive call, another pseudo-division takes place, a contradiction.

Only Case -1- can happen. Let \(a\) be the monic form of \(f_2\) (Line 90). Write the monic form of \(f_3\) at Line 91 as \([b_j, U_j, T_j]_j\).

\[
\text{if } b_j \neq 0, \quad \deg_y(b_j) \leq \deg_y(f_3) < \deg_y(f_2) = \deg_y(a) \leq \deg_y(f_1).
\]
This proves Eq (18). For \( \deg_y(a) = \deg_y(f_1) \) to hold, necessarily \( \deg_y(f_1) = \deg_y(f_2) \). This achieves the proof of the degree condition in the case of one pseudo-division.

If no pseudo-division at all occurs, then \( \text{lc}(f_2) \) is not invertible modulo \( T \). Moreover the recursive call \text{"LastNonNil\_D5}(a, b_j, U_j)\) at Line 96 does not involve pseudo-division neither. But since \( a \) and \( b_j \) are monic, this cannot be the case. Thus the algorithm never reaches Line 96. It cannot reach the lines 93–94 neither: this would mean that \( f_2 \) is nilpotent modulo \( U_i T_i \), excluded by Assumption (H). The situation of no pseudo-division does not occur.

Remark 4. A component \([a_i, b_i, T_i, U_i]\) of the output verifies \( T_i = 1 \) if and only if \( b_i = 0 \). Indeed, this occurs only at an exit Line 98. The other possibility at Line 94 is made under the condition \( T_i \neq 1 \) (Line 93). At Line 94, we have \( b_i \neq 0 \), as an output of \text{MonicForm\_D5} with input a non-zero polynomial.

5.2. Family of minimal lexicographic Gröbner bases from the subresultant p.r.s

We focus now on translating Algorithm 5 \text{"SubresToGB\_D5"} to dynamic evaluation. A direct consequence is that the output is no more one lexGb, but a family of thereof.

Overview of Algorithm 12 \text{"SubresToGB\_D5"}. Naively, it suffices to track divisions in the local version Algorithm 5 \text{"SubresToGB\_D5"} and to create splittings accordingly. A slight difficulty occurs in that the \text{"LastNonNil\_D5}(a, b, T)\) algorithm assumes that \( a \) has an invertible leading coefficient modulo \( T \). This assumption is too restrictive in many cases: there are some local components of \( T \) over which this assumption is verified and others over which it is not. To circumvent this, it suffices to “make monic” \( a \) with a call to \text{"MonicForm\_D5"} (Line 117) and proceed with this output.

To clarify the exposition, an auxiliary algorithm 11 \text{"SubresToGB\_Aux\_D5"} first treats the case of a polynomial \( a \) that has an invertible leading coefficient modulo \( T \), which becomes a subroutine in the main algorithm 12.

However, it is assumed that both \( a \) and \( b \) satisfy Assumption (H). It says that for any primary factor \( p \) of \( T \), \( a \) and \( b \) are not nilpotent modulo \( p \). This assumption can likely be removed; we let if for future work.

Proposition 6 (Correctness of Algorithm 11 \text{"SubresToGB\_Aux\_D5"}). Let \( G = (G_i)_i \) be the output of \text{SubresToGB\_Aux\_D5}(a, b, T). Writing the systems \( (G_i)_i \) as in Eq. (22),

\[
G_i = [ h_{i1}, h_{i2} f_{i2}, \ldots, h_{i\mu-1} f_{i\mu-1}, f_{i\mu} ],
\]

they are minimal lexGbs verifying the specifications:

(i) \( h_{i1} \) and \( h_{j1} \) are coprime. Additionally, \( \prod_i h_{i1} | T \) and \( \prod_i \text{sqfp}(h_{i1}) = \text{sqfp}(T) \).

(ii) \( \text{sqfp}(h_{i1}) = \text{sqfp}(h_{i2}) = \cdots = \text{sqfp}(h_{i\mu-1}) \).

(iii) \( \prod_i \langle G_i \rangle = \langle a, b, T \rangle \) and \( \langle G_i \rangle + \langle G_j \rangle = \langle 1 \rangle \) for \( i \neq j \).

Proof. If the input \( a \) or \( b \) is an invertible constant modulo \( T \) (Line 99) then the ideal \( \langle a, b, T \rangle = \langle 1 \rangle \). The output Line 100 is then logically \([1] \).

The call to \text{LastNonNil\_D5}(a, b, T) at Line 101 outputs a family \(([g_i, h_i, U_i, T_i])_i \) such that

\[
\langle g_i, h_i T_i, T_i U_i \rangle = \langle a, b, U_i T_i \rangle
\] (23)
with the polynomials \( (U_iT_i) \), pairwise coprime and \( \text{sqfp}(U_i) = \text{sqfp}(T_i) \) if \( U_i \neq 1 \).

If \( g_i = 1 \) then the ideal \( \langle g_i, h_iU_i, T_iU_i \rangle = \langle 1 \rangle \) and the corresponding lexGB is \([1]\), which does not need to be added to the output. That is why the if-test at Line 103 discards this case. If \( g_i \neq 1 \) and \( h_i = 0 \) then \( \langle a, b, U_iT_i \rangle = \langle g_i, T_iU_i \rangle \), and \( g_i \), \( T_iU_i \) is a minimal lexGB, which is added to the output (Line 105).

Otherwise, a recursive call at Line 107 is performed. Its validity is proved in the lemma 7 hereafter: we can assume that the output \( G_i \) verifies the output’s specifications.

**Lemma 7.** The recursive call \( G_i \leftarrow \text{SubresToGBAux}(g_i, h_i, U_i) \) at Line 107 is valid.

**Proof.** By the degree condition Eq. (18) of Proposition 5 related to Algorithm LastNonNil_D5, we have \( \deg_y(g_i) > \deg_y(h_i) \) unless \( h_i = 0 \). Regarding that \( h_i \neq 0 \) (else at Line 106) we have \( \deg_y(g_i) > \deg_y(h_i) \). Moreover \( g_i \) is monic so the input \( (g_i, h_i, U_i) \) satisfies the input’s specifications of Algorithm “SubresToGBAux”. Still from that proposition 5 \( \deg_y(g_i) = \deg_y(a) \) possibly holds only if \( \deg_y(a) = \deg_y(b) \). But then \( \deg_y(h_i) < \deg_y(g_i) < \deg_y(b) \). Thus, in any situations the input \( (g_i, h_i, U_i) \) presents a degree decrease compared to that of the main call \( (a, b, T) \). By induction, we can assume that the output is as expected. 

---

Algorithm 11: \( G \leftarrow \text{SubresToGBAux}(a, b, T) \)

**Input:** 1-2. polynomials \( a, b \in k[x,y] \), \( \deg_y(a) \geq \deg_y(b) \). The leading coefficient of \( a \) is invertible modulo \( T \).
3. \( T \in k[x] \) is a monic polynomial.

**Output:** Family of minimal lexGbs \( G = (G_i)_i \) written (following Theorem 1) as in Eq. (22) and verifying the specifications (i)-(ii)-(iii) of Proposition 6

\[
G_i = [h_{i1}, h_{i2}f_{i2}, \ldots, h_{i\mu-1}f_{i\mu-1}, f_{i\mu}], h_{ij} \in k[x], f_{ij} \in k[x,y] \text{ monic in } y
\]

99 if \( a \) or \( b \) is an invertible constant modulo \( T \) then
100 return [1]
101 \( S \leftarrow \text{LastNonNil_D5}(a, b, T) \) \quad // Write \( S = [[g_i, h_i, U_i, T_i]] \)
102 for \( [g_i, h_i, U_i, T_i] \in S \) do
103 \quad if \( g_i \neq 1 \) then \quad // Non-trivial input: pursue computations
104 \quad \quad \quad if \( h_i = 0 \) then \quad // Here \( g_i \) is a gcd mod \( T_i \)
105 \quad \quad \quad \quad \quad \text{OUT} \leftarrow \text{OUT cat } [(g_i, U_iT_i)]
106 \quad \quad \quad \text{else}
107 \quad \quad \quad \quad G_i \leftarrow \text{SubresToGBAux}(g_i, h_i, U_i) \quad // Recursive call
108 \quad \quad \quad \quad V'_{i0} \leftarrow T_i
109 \quad \quad \quad \quad for G_{ij} \in G_i \ do \quad // G_{ij} is a minimal lexGB
110 \quad \quad \quad \quad \quad // Write \( G_{ij} = [h_{i1j}, h_{i2j}f_{i2j}, \ldots, h_{i\mu-1j}f_{i\mu-1j}, f_{i\mu}] \) as in Theorem 1
111 \quad \quad \quad \quad (V'_{ij}, V'_{ij'}) \leftarrow \text{IsolFactor}(V'_{ij-1}, h_{ij1})
112 \quad \quad \quad \quad G_{ij}^{\text{new}} \leftarrow V'_{ij} \cdot G_{ij} \text{ cat } [g_i]
113 \quad \quad \quad \text{OUT} \leftarrow \text{OUT cat } [G_{ij}^{\text{new}}]
114 return OUT

33
Writing $G_i = (G_{ij})_j$, where $(G_{ij})_j$ is a family of minimal lexGs that we denote $G_{ij} = [h_{ij1}, h_{ij2} f_{ij2}, \ldots, h_{ij\lambda-1} f_{ij\lambda-1}, f_{ij\lambda}]$ as in Theorem 1. They verify:

(i') $h_{ij1}$ and $h_{ij\ell}$ are coprime and $\prod_j h_{ij1} | U_i$, as well as $\prod_j \text{sqfp}(h_{ij1}) = \text{sqfp}(U_i)$.

(ii') $\text{sqfp}(h_{ij1}) = \cdots = \text{sqfp}(h_{ij\lambda-1})$.

(iii') $\prod_j (G_{ij}) = \langle g_i, h_i, U_i \rangle$ and $(G_{ij}) + (G_{\ell\ell}) = \langle 1 \rangle$ if $j \neq \ell$.

Then a call to $\text{IsolFactor}$ has the effect of “projecting” $T_i$ along the $(h_{ij1})_j$'s. We have $V_{ij} = V_{ij-1}'(h_{ij1})$ where $(V_{ij}, V_{ij}') \leftarrow \text{IsolFactor}(V_{ij-1}'$, $h_{ij1})$, with $V_{i0}' = T_i$ (Line 108). The specifications of Algorithm 6 “IsolFactor” provides $V_{ij}V_{ij}' = V_{ij-1}'$, the product being that of coprime polynomials. By induction we get:

$$T_i = V_{i0}' = V_{i1}V_{i1}' = V_{i1}V_{i2}V_{i2}' = \cdots = V_{i1}V_{i2} \cdots V_{i\lambda}V_{ij}, \quad \text{for all } j \geq 1. \quad (24)$$

Moreover the factors in the product are pairwise coprime. It follows that

$$V_{ij} = V_{ij-1}'(h_{ij1}) = \prod_{p \in P, v_p(h_{ij1}) > 0} p_{v_p(V_{ij-1})} = \prod_{p \in P, v_p(h_{ij1}) > 0} p_{v_p(T_i)} = T_i(h_{ij1}).$$

Since the $(h_{ij1})_j$'s are pairwise coprime, and that $\prod_j \text{sqfp}(h_{ij1}) = \text{sqfp}(U_i)$, each irreducible factor of $U_i$ appears in those of one (and only one) of $h_{ij1}$. Moreover, $U_i \neq 1$ by Remark 4, hence $\text{sqfp}(U_i) = \text{sqfp}(T_i)$ by the output’ specifications of Algorithm 10 “LastNonNil.DS”. Therefore

$$\prod_j V_{ij} = \prod_j T_i(h_{ij1}) = \prod_j \prod_{p \in P, v_p(h_{ij1}) > 0} p_{v_p(T_i)} = \prod_{p \in P, v_p(U_i) > 0} p_{v_p(T_i)} = \prod_{p \in P, v_p(T_i) > 0} p_{v_p(T_i)} = T_i. \quad (25)$$

At Line 111 $G_{ij}^{new} := V_{ij}G_{ij} \text{ cat } [g_i]$, which rewrites using the notation above:

$$G_{ij}^{new} := [h_{ij1}V_{ij}, h_{ij2}V_{ij}f_{ij2}, \ldots, h_{ij\lambda-1}V_{ij}f_{ij\lambda-1}, V_{ij}f_{ij\lambda}, g_i].$$

Since $g_i \in \langle G_{ij} \rangle$, Theorem 1(B) implies that $G_{ij}^{new}$ is a lexGb. The following lemma shows that it is minimal.

**Lemma 8.** The lexGb $G_{ij}^{new}$ is a minimal one.

*Proof.* According to Theorem 1(B), it suffices to prove that deg$_y(g_i)$ is larger than the degree (in $y$) of all the polynomials in $G_{ij}$. Consider the first recursive call at Line 107. Recall that $g_i \neq 1$ and $h_i \neq 0$. If $h_i$ is a constant (Line 100) then the recursive call outputs $[[1]]$ and the minimality of $G_{ij}$ is obvious. Else it goes to another call to $\text{LastNonNil.DS}$ at Line 101

$$[[g_{\ell\ell}, h_{\ell\ell}, U_{\ell\ell}, T_{\ell\ell}]] \leftarrow \text{LastNonNil.DS}(g_i, h_i, U_i).$$

According to the degree condition of Proposition 5, an equality deg$_y(g_{\ell\ell}) = \text{deg}_y(g_i)$ can only occur if $\text{deg}_y(h_i) = \text{deg}_y(g_i)$, which does not hold as seen above.

This proves one output’s specification. According to the specification (ii') above, we obtain $\text{sqfp}(h_{ij1}V_{ij}) = \cdots = \text{sqfp}(h_{ij\lambda-1}V_{ij})$. Moreover, all irreducible factors of $V_{ij} = T_i^{(h_{ij1})}$ are irreducible factors of $h_{ij1}$ by definition. We obtain that $\text{sqfp}(h_{ij1}V_{ij}) = \text{sqfp}(V_{ij})$ for $1 \leq \ell \leq \lambda - 1$. This proves Specification (ii).
Additionally, $\prod_i \prod_j h_{ij} V_{ij} \prod_i U_i T_i | T$ proving the second statement of Specification (i). The first statement is that the $(h_{ij} V_{ij})_s$ are pairwise coprime. This follows from Eq. (24) and the specification (i').

Besides, $\prod_j (G_{ij}^\text{new}) = \prod_j (V_i G_{ij} + \langle g_i \rangle)$. Lemma 9 in Appendix, proves that $\prod_j (G_{ij}^\text{new}) = (\prod_j (V_i G_{ij})) + \langle g_i \rangle$. This ideal is equal to $(\prod_j V_i) (\prod_j (G_{ij})) + \langle g_i \rangle$. By the specification (iii') and Eq. (25) above, we get: $\prod_j (G_{ij}^\text{new}) = T_i (g_i, h_i, U_i) + \langle g_i \rangle = (T_i g_i, T_i h_i, T_i U_i, g_i, g_i, T_i h_i, T_i U_i)$. Finally, with Eq. (23) and the Chinese remainder theorem we obtain:

$$\langle a, b, T \rangle \simeq \prod_i \langle a, b, U_i T_i \rangle = \prod_i \langle g_i, T_i h_i, U_i T_i \rangle = \prod_{i, g_i \neq 1, h_i = 0} \langle g_i, U_i T_i \rangle \cdot \prod_{i, g_i \neq 1, h_i \neq 0} \prod_j \langle G_{ij}^\text{new} \rangle$$

which is output’s specification (iii). That they are pairwise coprime follows from the fact that the ideals $(\langle g_i, T_i h_i, T_i U_i \rangle)_i$ are pairwise coprime.

Next the assumption that $a$ has an invertible leading coefficient modulo $T$ is lifted. It suffices to run the algorithm [9] “MonicFormD5($a, T$)” (Line 117), and naively to call the previous algorithm [11] “SubresToGB_Aux” on each component of that monic form. But this induces a minor issue, in that the monic forms $(a_j)_j$ of $a$ may not all satisfy $\deg_y (b) \leq \deg_y (a_j)$, a requirement for calling “SubresToGB_Aux”. The if-test Line 120 distinguishes the $j$’s for which $\deg_y (a_j) < \deg_y (b)$ from those $j$’s for which $\deg_y (b) \leq \deg_y (a_j)$. In the former case (Lines 122, 129), calling directly SubresToGB_Aux($b_j$, $a_j$, $T_j$) does not necessarily work because $b_j \equiv b \mod T_j$ may not have an invertible leading coefficient modulo $T_j$ — another requirement of “SubresToGB_Aux”. Thus we make $b_j$ monic first (Line 123). Then a last if-test (Line 126) distinguishes the cases $\deg_y (a_j) < \deg_y (b_j)$ from the cases $\deg_y (a_j) \geq \deg_y (b_j)$, before this time calling SubresToGB_Aux with all its specifications guaranteed.

**Theorem 4** (Correctness of Algorithm 12 “SubresToGB_D5”). The output $\text{OUT}$ of SubresToGB_D5($a, b, T$) verifies the specifications below.

(i) $h_{i1}$ and $h_{j1}$ are coprime. Additionally, $\prod_i h_{i1} | T$ and $\prod_i \text{sqfp}(h_{i1}) = \text{sqfp}(T)$.

(ii) $\text{sqfp}(h_{i1}) = \text{sqfp}(h_{i2}) = \cdots = \text{sqfp}(h_{i\mu-1})$.

(iii) $\prod_i \langle G_i \rangle = \langle a, b, T \rangle$ and $\langle G_i \rangle + \langle G_j \rangle = \langle 1 \rangle$ for $i \neq j$.

**Proof.** If $a$ or $b$ is a non-zero constant, then $\langle a, b, T \rangle = \langle 1 \rangle$ so that the output [[1]] at Line 115 is correct.

By Assumption [11], for each primary factor $p^e$ of $T$, a mod $p^e$ is not nilpotent. Therefore, when “making” a monic with MonicFormD5($a, T$), the collection of output is always of the form $[a_j, T_j, 1]$. The third component is indeed a polynomial $U_i \neq 1$ if and only if $U_i$ divides a modulo $T_i$. Since $\text{sqfp}(U_i) = \text{sqfp}(T_i)$, this would imply that $a$ is nilpotent modulo $T_i$, in contradiction with Assumption [11].

The output $\text{Ma} = [[a_j, T_j, 1]]$ of MonicFormD5($a, T$) at Line 117 verify $\langle a_j, T_j \rangle = \langle a, T \rangle$ with $a_j$ monic and $\prod_j T_j = T$, the $(T_j)_s$ being pairwise coprime. The algorithm treats each component $T_j$ of $T$ separately in the for loop Line 118, 129.

Naively, a call to “SubresToGB_Aux” shall be performed with $(a_j, b_j, T_j)$, but this input does not necessarily satisfy the specifications of that algorithm if $\deg_y (a_j) < \deg_y (b_j)$. In that case (Lines 122, 129), we cannot simply switch $a_j$ with $b_j$ by calling directly SubresToGB_Aux($b_j$, $a_j$, $T_j$)
Algorithm 12: \( G \leftarrow \text{SubresToGB}_D(a, b, T) \)

**Input:** 1-2. polynomials \( a, b \in k[x, y] \), \( \deg_y(a) \geq \deg_y(b) \) satisfying Assumption [II].
3. \( T \in k[x] \) is a monic non-constant polynomial.

**Output:** Family of minimal lexGbs \( G = (G) \) written following Theorem [I] as in Eq. (26), and verifying the specifications (i)-(ii)-(iii) of Theorem [II].

\[
G_i = [h_{i1}, h_{i2}f_{i2}, \ldots, h_{iy-1}f_{iy-1}, f_{iy}], \quad h_{ij} \in k[x], \; f_{ij} \in k[x, y] \text{ monic in } y, \quad (26)
\]

114 if \( a \) or \( b \) is a non-zero constant then
115 \( \text{return } [[1]] \)
116 \( \text{OUT} \leftarrow [] \)
117 \( \text{Ma} \leftarrow \text{MonicForm}_D(a, T) \) \quad // Write \( \text{Ma} = [[a_j, T_j, 1]] \)
118 for \([a_j, T_j, 1] \in \text{Ma} \) do
119 \( b_j \leftarrow b \mod T_j \)
120 if \( \deg_y(a_j) \geq \deg_y(b_j) \) then
121 \( \text{OUT} \leftarrow \text{OUT} \; \text{cat} \; \text{SubresToGB}_A(a_j, b_j, T_j) \)
122 else
123 \( \text{Mb}_j \leftarrow \text{MonicForm}_D(b_j, T_j) \) \quad // Write \( \text{Mb}_j = [[b_{ij}, T_{ij}, 1]] \)
124 for \([b_{ij}, T_{ij}, 1] \in \text{Mb}_j \) do
125 \( a_{ij} \leftarrow a_j \mod T_{ij} \)
126 if \( \deg_y(a_{ij}) \geq \deg_y(b_{ij}) \) then
127 \( \text{OUT} \leftarrow \text{OUT} \; \text{cat} \; \text{SubresToGB}_A(a_{ij}, b_{ij}, T_{ij}) \)
128 else
129 \( \text{OUT} \leftarrow \text{OUT} \; \text{cat} \; \text{SubresToGB}_A(b_{ij}, a_{ij}, T_{ij}) \)
130 if \( \text{OUT} = [] \) then
131 \( \text{return } [[1]] \)
132 return \text{OUT}

neither since \( b_j \) may not have a leading coefficient invertible modulo \( T_j \) (a requirement of that algorithm). Therefore we “make” \( b_j \) monic with a call to \( \text{MonicForm}_D(b_j, T_j) \) at Line 123. Since \( b \) satisfies Assumption [II], \( b \mod p^e \) is not nilpotent for all primary factor \( p^e \) of \( T \). Therefore, \( \text{Mb}_j = [[b_{ij}, T_{ij}, 1]] \) (third component is always 1).

\( b_{ij} \) monic \( \quad \langle b_{ij}, T_{ij} \rangle = \langle b_j, T_{ij} \rangle, \quad \prod_i T_{ij} = T_j, \quad (T_{ij}) \text{s pairwise coprime.} \)

Now both families \( \langle a_{ij} \rangle \) and \( \langle b_{ij} \rangle \) are made of polynomials having an invertible coefficient modulo \( T_{ij} \), hence calls to \( \text{SubresToGB}_A \) at Line 127 or Line 129 are correct. \( \text{OUT} \) concatenates all these minimal lexGbs. This leads to the following equalities, where the second one is deduced from Lemma [II] in Appendix.

\[
\prod_{G \in \text{OUT}} \langle G \rangle \simeq \prod_j \prod_i \langle a_j, b_{ij}, T_{ij} \rangle \simeq \prod_j \langle a_j \rangle + \prod_i \langle b_{ij}, T_{ij} \rangle \simeq \prod_j \langle a_j, b_j, T_j \rangle = \langle a, b, T \rangle.
\]

This proves the first statement of the specification (iii), the second statement being that these lexGbs are pairwise coprime. Let \( G_1 \) and \( G_2 \) be two different lexGbs in \( \text{OUT} \). If \( G_1 \) and \( G_2 \) are in the
same output of a call to “SubresToGB_Aux” at Line 121 [127 or 129] then the output’ specifications of “SubresToGB_Aux” guarantee that $G_1$ and $G_2$ are coprime.

If $G_1$ and $G_2$ are obtained from different calls to “SubresToGB_Aux”, then the $(T_j)_j$’s (Line 121) are pairwise coprime, or the $(T_{ij})_{ij}$’s (Lines 127 or 129) are pairwise coprime, so that $G_1$ and $G_2$ are coprime. This ends the proof of the specification (iii).

Finally, the specifications (i)-(ii) are also satisfied according that the lexGbs are computed by calls to “SubresToGB_Aux” whose output satisfy these two specifications. $\square$

6. Implementation and experimental section

All algorithms are implemented in Magma v2.25-1 and can be found at `http://xdahan.sakura.ne.jp/gb.html`. Experiments were realized on an Intel processor Core i7-6700K clocked at 4GHz. The timings naturally compare with that of the `GröbnerBasis` command. We recall that a common strategy to compute a lexGb divides in two steps: first compute a Gröbner basis for the degree reverse lexicographic order (`grevlex` for short) then applies a change of monomial order algorithm, typically FGLM since we are constrained to systems of dimension zero. Magma is equipped with one of the best implementation of Faugère algorithm F4 [20], which is called in the first step. We report on the timing F4+FGLM, with the timing of FGLM appended inside parentheses when not negligible.

We consider only polynomials over a finite field. Rational coefficients induce large coefficients swell, and without modular methods become difficult to compare. We could have turned the `Modular` option off in Magma when calling a Gröbner basis computation, but for now we would rather stick to finite fields. The comparison with small size (16bits) and medium size (64bits) finite fields bring enough evidences of the advantage of the proposed algorithms.

6.1. Testing suite

We consider two families of examples, each tested over a 16-bits finite field and a 64-bits finite field. The polynomials $a, b$ are defined as follows:

- First family (Tables 1-4): 16 polynomials $a, b$ are constructed with respect to a modulus $T$ whose factors are defined as follows:

  - Consider $T = p_1^{\ell_1}p_2^{\ell_2}$ (two factors where $p_1 = x$ and $p_2 = (x+1)$), $T = p_1^{\ell_1}p_2^{\ell_2}p_3^{\ell_3}$ (three factors, where $p_1 = (x+10)$, $p_2 = (x+20)$, $p_3 = (x+30)$), and finally $T = p_1^{\ell_1}p_2^{\ell_2}p_3^{\ell_3}p_4^{\ell_4}$ (four factors, where $p_1 = x$, $p_2 = (x+5)$, $p_3 = (x+10)$, $p_4 = (x+15)$).

  - The exponents are respectively

    - (examples 1-4) $(e_1, e_2) = (5i, 5(i+1))_{i=1,2,3,4}$ when there are two factors,
    - (examples 5-11) $(e_1, e_2, e_3) = (3i-1, 3i, 3i+1)_{i=3,4,5,6,7,8,9}$, when there are three factors,
    - (examples 12-16) $(e_1, e_2, e_3, e_4) = (4i, 4i+1, 4i+2, 4i+3)_{i=1,2,3,4,5}$, when there are four factors.

  - For each $p_i^{\ell_i}$ polynomials $a_i, b_i$ are built as follows:

    - $a_i := (y + p_i) \cdot \prod_{\ell=1}^{\ell_i-1} (y + p_i + p_i^{\ell_1} + \cdots + p_i^{\ell + \ell_2 + \cdots + p_i^{\ell_i-1}})$ mod $p_i^{\ell_i}$
    - $b_i := (y + 2p_i) \cdot \prod_{\ell=1}^{\ell_i-1} (y + p_i + p_i^{\ell_1} + \cdots + p_i^{\ell + \ell_2 + \cdots + p_i^{\ell_i-1}})$ mod $p_i^{\ell_i}$

  - Then the CRT is applied to construct polynomials $a, b$ modulo $T$ from their local images $a_i, b_i$. 37
• Second family (Tables 2-5-6): 6 polynomials $a, b$ constructed according to a modulus $T$ whose factors are defined as follows:

- Consider $T = q_0^{e_0} \cdots q_6^{e_6}$ where each polynomial $q_i = r_{i1} \cdots r_{i(8-i)}$ and $r_{ij} = x + i(i + 1)/2 + j - 2$. For example $r_{71} = x + 27$ and $r_{11} = x + 1 + j - 2 = x + j - 1$.
- $e = 7, 8, 9, 10, 11, 12$.
- For each $r_{ij}^{e-i}$ polynomials $a_{ij}, b_{ij}$ are built as follows:

\[
\begin{align*}
    a_{ij} &:= (y + r_{ij}) \cdot \prod_{\ell=1}^{e-i}(y + r_{ij} + r_{ij}^2 + \cdots + r_{ij}^\ell + \ell + 2r_{ij}^{\ell+1}) \mod r_{ij}^{e-i} \\
    b_{ij} &:= (y + 2r_{ij}) \cdot \prod_{\ell=1}^{e-i}(y + r_{ij} + r_{ij}^2 + \cdots + r_{ij}^\ell + \ell + r_{ij}^{\ell+1}) \mod r_{ij}^{e-i}
\end{align*}
\]

- Then the CRT is applied to construct polynomials $a, b$ modulo $T$ from their local images $a_{ij}, b_{ij}$.

The first family concerns a modulus $T$ with few factors of moderate exponent degrees, each exponent degree being moderately distant. The second family targets a modulus $T$ having 27 factors, and a squarefree decomposition made of seven factors. The exponent degrees are distant of one, and are relatively small.

The correctness of the output of Algorithm 12 “SubresToGB" was checked as follows: Take $\mathcal{G}$ the lexGb output by the Gröbner engine of Magma, and $(\mathcal{G}_i)$, the family of lexGbs output by Algorithm 12. We checked that each polynomial in $\mathcal{G}$ reduces to zero modulo each $\mathcal{G}_i$. In this way $\mathcal{G} \subset \prod_i \langle \mathcal{G}_i \rangle$. And compared the dimension of the two vector spaces $k[x, y]/\langle \mathcal{G} \rangle$ and $\prod_i k[x, y]/\langle \mathcal{G}_i \rangle$. The two tables summarize data attached to the system $(a, b)$ (without a modulus $T$, which will turn out to be $\text{Res}_y(a, b)$, see next Section 6.2). This kind of input is of practical importance and the timings are not always favorable to Algorithm 12, which require a special analysis, undertaken in Section 6.4.

Description of Tables 1-2

Column 1: refers to the example’s number. (4 + 7 + 5 = 16 examples for the 1st family, and 6 examples for the 2nd family).

Column 2: degree $\text{DEG}$ of the ideal generated by $\langle a, b \rangle$, equal to $\dim_k(k[x, y]/\langle a, b \rangle)$.

Column 3: degrees $\deg_y(a)$ and $\deg_y(b)$ of the input polynomials $a$ and $b$ (always equal).

Column 4: total degrees $t\deg(a)$, $t\deg(b)$ of the input polynomials $a$ and $b$ (always equal).

Column 5: $\deg(\text{Res}_y(a, b)) \in k[x]$ degree of the resultant of $a$ and $b$.

Column 6: sum of the degrees of the factors having multiplicity one in the factorization of $\text{Res}_y(a, b)$.

Column 7: sum of the degrees of the factors having multiplicity $> 1$ in the factorization of $\text{Res}_y(a, b)$.

Column 8: average multiplicity of an irreducible factor of $\text{Res}_y(a, b)$ having multiplicity $> 1$ (do not count the factor of multiplicity one).

Column 9: number of lexGbs output by Algorithm 12 “SubresToGB". Inside parentheses, the average number of polynomials in these lexGbs.
### Table 1: Data attached to the polynomials $a$, $b$ in the 1st family of examples

| ex. nbr. | DEG | $\deg_y(a)$ | $\deg_y(b)$ | $\tdeg(a)$ | $\tdeg(b)$ | degree resultant | deg. multiplicity 1 | deg. multiplicity >1 | avg. multiplicity >1 | #lexGbs (avg. #poly.) |
|----------|-----|--------------|--------------|------------|------------|-----------------|---------------------|---------------------|---------------------|----------------------|
| 1        | 241 | 10           | 24           | 266        | 171        | 95              | 47.5                |                    |                     | 2 (8.5)              |
| 2        | 646 | 15           | 39           | 696        | 471        | 225             | 112.5               |                    |                     | 2 (14.5)             |
| 3        | 1251| 20           | 54           | 1326       | 919        | 407             | 135.67              |                    |                     | 2 (18.5)             |
| 4        | 2056| 25           | 69           | 2165       | 1521       | 635             | 317.5               |                    |                     | 2 (23.5)             |
| 5        | 281 | 8            | 28           | 300        | 196        | 104             | 34.67               |                    |                     | 3 (8)                |
| 6        | 581 | 11           | 40           | 609        | 415        | 194             | 64.67               |                    |                     | 3 (11)               |
| 7        | 989 | 14           | 52           | 1026       | 715        | 311             | 103.67              |                    |                     | 3 (14)               |
| 8        | 1505| 17           | 64           | 1551       | 1096       | 455             | 151.67              |                    |                     | 3 (17)               |
| 9        | 2129| 20           | 76           | 2184       | 1558       | 626             | 208.67              |                    |                     | 3 (20)               |
| 10       | 2861| 23           | 88           | 2925       | 2101       | 824             | 274.67              |                    |                     | 3 (23)               |
| 11       | 3701| 26           | 100          | 3774       | 2725       | 1049            | 349.67              |                    |                     | 3 (26)               |

### Table 2: Data attached to the polynomials $a$, $b$ in the 2nd family of examples

| ex. nbr. | DEG | $\deg_y(a)$ | $\deg_y(b)$ | $\tdeg(a)$ | $\tdeg(b)$ | degree resultant | deg. multiplicity 1 | deg. multiplicity >1 | avg. multiplicity >1 | #lexGbs (avg. #poly.) |
|----------|-----|--------------|--------------|------------|------------|-----------------|---------------------|---------------------|---------------------|----------------------|
| 1        | 827 | 7            | 90           | 1079       | 617        | 462             | 6.5                 |                    |                     | 7 (5)                |
| 2        | 1301| 8            | 119          | 1665       | 979        | 686             | 24.5                |                    |                     | 7 (6)                |
| 3        | 1887| 9            | 148          | 2363       | 1425       | 938             | 33.5                |                    |                     | 7 (7)                |
| 4        | 2585| 10           | 177          | 3173       | 1955       | 1218            | 43.5                |                    |                     | 7 (8)                |
| 5        | 3395| 11           | 206          | 4095       | 2569       | 1526            | 54.5                |                    |                     | 7 (9)                |
| 6        | 4317| 12           | 235          | 5129       | 3267       | 1862            | 66.5                |                    |                     | 7 (10)               |
6.2. Timings

Tables 3–4 display timings for the first family of 16 polynomials described in Section 6.1 (see also Table 1). The double horizontal lines separate the four polynomials \( T \) which have two primary factors, from the seven polynomials \( T \) that have three, and from the last five polynomials \( T \) which have four primary factors. Tables 5–6 show the timings for the second family of 6 polynomials described in Section 6.1 (see also Table 2). We remark that the polynomials \( a \) and \( b \) do not have solutions at infinity: \( \gcd(\lc(a), \lc(b)) = 1 \). The four tables all have eight columns. They represent:

**Description of Tables 3–4–5–6**

**Column 1**: timing of the algorithm 12 “SubresToGB\_D5” with input \( a, b, T \).

**Column 2**: timing of GroebnerBasis([\( a, b, T \)]) command in Magma: F4+FGLM with FGLM in parenthesis when not negligible.

**Column 3**: timing for the resultant computation \( r(x) := \text{Res}_y(a, b) \)

**Column 4**: timing for Algorithm 12 “SubresToGB\_D5” with input \( a, b, r \).

**Column 5**: timing of GroebnerBasis([\( a, b \)]) command in Magma: F4+FGLM with FGLM in parenthesis when not negligible.

**Column 6**: timing for the squarefree decomposition of \( r = R_1^{e_1} \cdots R_s^{e_s} \). Here, \( R_i \) is the product of those irreducible factors of the resultant \( R \) that come with the same multiplicity \( e_i \).

**Column 7**: total time taken for calling Algorithm 12 “SubresToGB\_D5” on input \( (a, b, R_i^{e_i}) \) for \( i = 1, \ldots, s \).

**Column 8**: total time taken for calling GroebnerBasis([\( a, b, R_i^{e_i} \)]) for \( i = 1, \ldots, s \).
| 1   | 2   | 3          | 4          | 5          | 6         | 7         | 8         |
|-----|-----|------------|------------|------------|-----------|-----------|-----------|
| Input: $a, b, T$ | Input: $a, b, 	ext{Res}_p(a, b)$ | Input: $a, b$ | Several input: $a, b, R_i$ | sqf dec | this | Gb(fglm) |
| this | Gb(fglm) | res | this | Gb(fglm) | sqf dec | this | Gb(fglm) |
| 0.01 | 0.02 | 0.04 | 2.28 | 0.25 (0.22) | 0 | 0.28 | 0.44 (0.13) |
| 0.02 | 0.22 (0.02) | 0.21 | 36.13 | 3.83 (3.649) | 0.02 | 3 | 5.21 (2.23) |
| 0.04 | 1.23 (0.09) | 0.74 | 174.69 | 27.56 (26.65) | 0.05 | 17.36 | 32.49 (18) |
| 0.09 | 6.17 (0.33) | 1.92 | 592.18 | 121.1 (118.009) | 0.08 | 73.91 | 143.42 (78.74) |
| 0.01 | 0.05 (0.009) | 0.05 | 1.28 | 0.39 (0.339) | 0 | 0.21 | 0.87 (0.2) |
| 0.01 | 0.27 (0.019) | 0.17 | 10.16 | 3.87 (3.6) | 0.02 | 1.06 | 5.55 (1.79) |
| 0.03 | 1.2 (0.109) | 0.47 | 36.48 | 20.48 (19.39) | 0.04 | 4.19 | 23.41 (8.9) |
| 0.05 | 3.7 (0.339) | 1 | 107.99 | 60.39 (57.279) | 0.07 | 12.22 | 73.15 (32.88) |
| 0.09 | 6.76 (0.74) | 2.06 | 277.45 | 169.11 (162) | 0.13 | 33.37 | 205.93 (102) |
| 0.15 | 27.07 (2.039) | 3.45 | 717.37 | 386.3 (370.8) | 0.18 | 77.88 | 508.07 (245) |
| 0.22 | 60.91 (4.5) | 5.72 | 1403.83 | 793.41 (763.609) | 0.27 | 166.48 | 939.93 (513) |
| 0.01 | 0.04 (0.01) | 0.04 | 0.65 | 0.29 (0.239) | 0.01 | 0.13 | 0.71 (0.15) |
| 0.02 | 0.66 (0.059) | 0.26 | 14.41 | 7.54 (6.88) | 0.03 | 1.34 | 15.18 (4.7) |
| 0.04 | 5.42 (0.399) | 0.98 | 83.5 | 62.34 (58.05) | 0.06 | 7.78 | 97.33 (33.5) |
| 0.09 | 25.44 (1.489) | 2.59 | 296.32 | 273.9 (257.47) | 0.14 | 30.87 | 429.38 (165) |
| 0.17 | 91.02 (6.909) | 7 | 943.93 | 907.33 (856.36) | 0.23 | 101.05 | 1383.59 (624) |

Table 4: First family over a finite field $GF(p)$, $p$ a 64bits prime

| 1   | 2   | 3          | 4          | 5          | 6         | 7         | 8         |
|-----|-----|------------|------------|------------|-----------|-----------|-----------|
| Input: $a, b, T$ | Input: $a, b, 	ext{Res}_p(a, b)$ | Input: $a, b$ | Several input: $a, b, R_i$ | sqf dec | this | Gb(fglm) |
| this | Gb(fglm) | res | this | Gb(fglm) | sqf dec | this | Gb(fglm) |
| 0 | 0.77 (0.03) | 0.03 | 0.23 | 1.4 (0.66) | 0 | 0.06 | 5.09 (0.36) |
| 0 | 3.6 (0.07) | 0.07 | 0.63 | 5.1 (2.39) | 0.01 | 0.15 | 19.36 (1.289) |
| 0.01 | 9.84 (0.119) | 0.13 | 1.53 | 13.08 (5.969) | 0.01 | 0.34 | 50.39 (3.289) |
| 0.01 | 23.91 (0.25) | 0.23 | 3.16 | 30.47 (13.699) | 0.07 | 7.78 | 97.33 (33.5) |
| 0.01 | 60.21 (0.509) | 0.37 | 6.32 | 64.96 (28.7) | 0.02 | 1.28 | 268 (15.9) |
| 0.02 | 154.72 (0.97) | 0.57 | 11.37 | 134.31 (54.509) | 0.03 | 2.28 | 527.95 (27.8) |

Table 5: Second family over a finite field $GF(p)$, $p$ a 16bits prime

| 1   | 2   | 3          | 4          | 5          | 6         | 7         | 8         |
|-----|-----|------------|------------|------------|-----------|-----------|-----------|
| Input: $a, b, T$ | Input: $a, b, 	ext{Res}_p(a, b)$ | Input: $a, b$ | Several input: $a, b, R_i$ | sqf dec | this | Gb(fglm) |
| this | Gb(fglm) | res | this | Gb(fglm) | sqf dec | this | Gb(fglm) |
| 0.01 | 11.79 (0.149) | 0.54 | 3.04 | 18.05 (10.05) | 0.04 | 0.86 | 87.28 (5.5) |
| 0.02 | 90.69 (0.549) | 1.29 | 8.27 | 91.21 (43.519) | 0.08 | 1.9 | 338.78 (20) |
| 0.04 | 297.74 (1.23) | 2.52 | 18.73 | 243.46 (135) | 0.13 | 3.94 | 808.97 (60) |
| 0.06 | 681.95 (3.09) | 4.44 | 37.28 | 560.26 (326.289) | 0.19 | 8.06 | 2186.52 (170) |
| 0.09 | 2402 (7.8) | 7.76 | 68.63 | 1375.74 (899.62) | 0.27 | 15.39 | 5101.74 (300) |
| 0.13 | 5512.63 (16.739) | 12.67 | 134.43 | 2789 (1909.229) | 0.36 | 28.41 | 11789.98 (825) |

Table 6: Second family over a finite field $GF(p)$, $p$ a 64bits prime
6.3. Comments on Tables 3-4-5-6

Columns 1-2. The Gröbner engine is unable to take full advantage of the addition of $T$ in the system. The comparison between the timings of the first two columns is telling: up to several thousands times faster and always more than one hundred times faster.

Columns 6-7-8. Taking the squarefree decomposition $R_{e_1}^e \cdots R_{e_s}^e$ of the resultant $r = \text{Res}_{y}(a, b)$ is cheap and natural. As one can expect, running Algorithm 12 on each factor found brings a clear advantage in any case, even in comparison with Column 5 F4+FGLM. This last experiment supports the conclusion that decomposing helps to alleviate the growth of internal computations.

Columns 3-4-5: preliminary comments. This is related to the computation of a lexGb of a bivariate system of two polynomials: the input is the system $[a, b]$. Column 3 displays timings (always negligible) for computing the resultant $r$, and then Column 4 displays the timing taken by Algorithm 12 to find a product of lexGbs of $[a, b, r]$. This timing compares with that of $\text{GroebnerBasis}([a, b])$ of Column 5. One question arises: is it faster to compute $\text{GroebnerBasis}([a, b])$ or to compute $\text{GroebnerBasis}([a, b, r])$? Experiments show that almost always the first is faster. That is why we have shown the timings of $\text{GroebnerBasis}([a, b])$, rather than those $\text{GroebnerBasis}([a, b, r])$ which are slower.

We observe that the situation is more balanced in this case. First the computation of $r$ is most often negligible. This suggests for future work to take advantage of its computation through a subresultant p.r.s of $a, b$ (not modulo a polynomial $T$) and work directly on it. Second, the timings of the first family of polynomials (Tables 1-3-4 and rows 1-2-3 of Table 7) are slower in general, whereas those of the second family (Tables 2-5-6 and row 4 of Table 7) are better. To get an educated knowledge of this situation, which is of practical importance, let us provide more details.

6.4. Elements of complexity in case of input $a$ and $b$

This section extends the above comment on the timings of Columns 3-4-5 of Tables 3-4-5-6 that concern an input $a$ and $b$ without a modulus $T$. For clarity, Table 7 displays these data into 8 plots. Magma computes with F4 a grevlex Gröbner basis and then applies FGLM to transform it to a lexGb. The complexity of the latter is well-known, it is $O(\text{DEG}^3)$, where $\text{DEG} = \dim_k(R[y]/\langle a, b \rangle)$ is the degree of the ideal generated by $a$ and $b$. The complexity of the F4 algorithm is notoriously more difficult to analyze. Only the less efficient Lazard’s algorithm relying on Macaulay matrices has a known complexity upper bound, written in Eq. (27) below. The complexity results given hereafter can be found in [29] (see also [44, Section 1.5] for a nice overview and the references therein). The standard assumption in the context of [29] is:

$a$ and $b$ form a homogeneous regular sequence

(R)

The examples of polynomials $a$ and $b$ in the testing suite are not homogeneous, but form a regular sequence. The complexity of Lazard’s method in the affine case is not well-understood. Nonetheless, the bounds for the homogenous case provide some indications for affine polynomials. We define the degree of regularity $d_{\text{reg}}$ of the ideal $<a, b>$, equal under (R) to be the smallest integer such that the Hilbert function of $k[x, y]/\langle a, b \rangle$ becomes equal to the corresponding Hilbert polynomial. Then the number of arithmetic operations over $k$ to compute a grevlex Gröbner basis of $<a, b>$ is lower than:

$$O \left( \left( \frac{2 + d_{\text{reg}}}{2} \right)^{\omega} \right) = O(d_{\text{reg}}^{\omega})$$

(27)
We refer to [52] for details about the exponent of linear algebra $\omega$. It is possibly overestimated in many situations. Considering that the Macaulay bound $d_{\text{reg}} = \text{tdeg}(a) + \text{tdeg}(b) - 1$ holds under (R), we obtain:

$$O(\text{tdeg}(a)^{2\omega} + \text{tdeg}(a)^{\omega}\text{tdeg}(b)^{\omega} + \text{tdeg}(b)^{2\omega}) \leq O(\text{tdeg}(a)^{2\omega})$$

The latter inequality follows from $\deg_Y(b) \leq \deg_y(a)$. Adding the cost of FGLM in $\text{DEG}^3$ yields:

$$O(\text{tdeg}(a)^{2\omega} + \text{tdeg}(a)^{\omega}\text{tdeg}(b)^{\omega} + \text{tdeg}(b)^{2\omega} + \text{DEG}^3) \leq O(\text{tdeg}(a)^{2\omega} + \text{DEG}^3). \quad (28)$$

Even in the affine case, we can think that the time taken by F4 depends more on $\text{tdeg}(a)$ and $\text{tdeg}(b)$, whereas the one taken by FGLM depends exclusively on $\text{DEG}$. The comparison of the timings of the examples of the 1st family and of the 2nd one supports this claim:

- Column 5 of Tables 3-4 as well as the plots in rows 1-2-3 of Table 7 in the 1st family show that FGLM occupies most of the computations. We observe that the ratio $\text{DEG}/\text{tdeg}(a)$ ranges from 10 to at least 36 (Table 1), putting a higher cost on FGLM.
- Column 5 of Tables 5-6 as well as the plots in row 4 of Table 7 in the 2nd family shows that FGLM takes more or less half of the total time. We observe that the ratio $\text{DEG}/\text{tdeg}(a)$ ranges from 9 to around 18 (Table 1). The cost of FGLM is then less important.

In comparison, Algorithm 12 “SubresToGB_D5” certainly has the subresultant as a central routine. The number of arithmetic operations in $R = k[x]/(T)$ to compute naively the subresultant p.r.s of $a$ and $b$ modulo $T$, when divisions are possible, is well-known, upper-bounded by $\deg_y(a) \deg_y(b)$. The number of arithmetic operations in $k$ for peforming naively any operation in $R$ is $O(\deg(T)^2)$. Thus the cost of computing the subresultant p.r.s modulo $T$ is within:

$$O(\deg_y(a) \deg_y(b) \deg(T)^2). \quad (29)$$

Without entering into details, other functions to make polynomials monic and to split the computations do increase the cost, but unlikely of a much higher magnitude compared to Eq (29). When $T = \text{Res}_y(a, b)$, the classical upper-bound [52, Thm. 6.22] $\deg_x(\text{Res}_y(a, b)) \leq d_x(\deg_y(a) + \deg_y(b))$ where $d_x = \max(\deg_x(a), \deg_y(b))$ yields:

$$O(d_x^2(\deg_y(a)^3 \deg_y(b) + \deg_y(a)^2 \deg_y(b)^2 + \deg_y(a) \deg_y(b)^3)) \leq O(d_x^2 \deg_y(a)^4). \quad (30)$$

The latter inequality is deduced from $\deg_y(b) \leq \deg_y(a)$. Thus the parameter $\deg_y(a)$ dominates the total cost.

- Looking at Columns 3-4 of Table 1 (1st family), we observe that the ratio $\text{tdeg}(a)/\deg_y(a)$ ranges from around 2 to 4, which is quite small. The estimate (28) involving $\text{tdeg}(a)$ and the estimate (30) involving $\deg_y(a)$ suggest that F4 will spend less time than Algorithm 12.
- Whereas this ratio in Columns 3-4 of Table 2 (2nd family) ranges from around 12 to 19, which is quite bigger. This indicates that Algorithm 12 will spend less time than F4.

This is certainly one main explanation behind the difference in the timings of “SubresToGB_D5” and F4+FGLM observed in Columns 4-5 of Tables 3-4-5-6 and in Table 7.
Table 7: Input a, b (Columns 3-4-5 of Tables 3-4-5-6). Timing against the degree DEG of the ideal. Left: 16bits field. Right: 64bits field. Row 1: examples 1-4 of the 1st family. Row 2: examples 5-11 of the 1st family. Row 3: examples 12-16 of the 1st family. Row 4: examples of the 2nd family.
7. Concluding Remarks

The new algorithms brought in this work open the way to various generalizations and some improvements.

*Subresultant p.r.s of $a$ and $b*. The line of works $[25, 15, 6, 31, 34]$ that produces a triangular decomposition of the set of solutions (not ideal preserving) of $a$ and $b$, works directly on the subresultant p.r.s of $a$ and $b$. The present article requires a modulus $T$ and works on the p.r.s modulo $T$. Although this kind of situations is necessary for more general incremental Euclidean algorithm-based decomposition methods, starting with the p.r.s of $a$ and $b$ directly is desirable. Besides the discussion of Section 6.4 suggests enhanced performances.

*Decomposition along the $y$-variable.* In the case of a radical ideal, decomposing it along the $y$-variable is well-known. This is a special instance of dynamic evaluation in two variables. It amounts to compute a “gcd” in $y$ of two bivariate polynomials. The inversions occurring in the coefficients ring $k[x]$ required by the Euclidean algorithm are managed by dynamic evaluation in one variable, namely the $x$-variable. In this way, dynamic evaluation in two variables boils down to that of one variable. To do the same for non-radical lexGbs, two complications occur: first, Hensel lifting comes into play to compensate the division required to make nilpotent polynomials non-nilpotent, second, the output may be a lexGb not necessarily a triangular set. Computing a certain decomposition (as introduced in $[12]$) of the ideal $\langle a, b, p^e \rangle$ and decomposing a minimal lexGb of $\langle a, b, p^e \rangle$, necessarily along the $y$-variable, does not differ much. Note that Lazard’s structural theorem applies in this situation too. This constitutes a necessary step towards the generalization of this work to more than two variables.

*More than two variables.* Besides a dynamic evaluation in two variables, a form of structure theorem for lexGbs of three variables or more becomes necessary. For now, it is known to hold in the radical case $[22, 30]$, not sufficient for our purpose since we target non-radical ideals. Evidences that a broad class of non-radical lexGbs also holds a factorization pattern are coming out $[54]$. A key step in that direction lies in the development of CRT-based algorithms that reconstruct a lexGb from its primary components $[13]$.

Beyond this first step, the idea would be a recursive algorithm on the number of variables in order to ultimately reduce to the bivariate case treated here.

*Complexity.* Instead of standard quadratic-time subresultant p.r.s, can half-gcd based algorithm be adapted? This would motivate to analyze the complexity carefully. The work of $[41]$ paves the way to this direction. However, the additional algorithms appearing here compared to $[11]$, which is already quite sophisticated, complicate it furthermore, making the complexity analysis an interesting challenge.

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Appendix

In below, is proposed a version of the Chinese Remainder Theorem. Although probably not new, finding a precise reference appears to be not obvious.

Lemma 9. Consider a family of ideals $(I_\ell)_{\ell \in L}$ of $k[x, y]$ and a polynomial $g \in k[x, y]$. Take a monic generator $h_\ell$ of $(I_\ell \cap k[x])$ and assume that the polynomials $h_\ell$’s are pairwise coprime. Then \(\prod_{\ell \in L}(I_\ell + \langle g \rangle) = \langle \prod_{\ell \in L_1} I_\ell \rangle + \langle g \rangle\).

Proof. Write $J = \prod_{\ell \in L}(I_\ell + \langle g \rangle)$. Let $L_1 \cup L_2 = L$ be a partition of the set of indices $L$ and associate to it the ideal $(\prod_{\ell \in L_1} I_\ell)\langle g \rangle^{\#L_2}$ where $\#L_2$ stands for the cardinal of $L_2$. We then have the equality of ideals:

\[
J = \sum_{\text{partitions } (L_1, L_2) \text{ of } L} (\prod_{\ell \in L_1} I_\ell)\langle g \rangle^{\#L_2}. \tag{31}
\]

When $L_2 = \emptyset$, the ideal associated to the partition $(L, \emptyset)$ of the set of indices $L$ is then $A := \prod_{\ell \in L} I_\ell$. As soon as $L_2 \neq \emptyset$, the ideal associated to the partition $(L_1, L_2)$ is $(\prod_{\ell \in L_1} I_\ell)\langle g \rangle^{\#L_2} \subset \langle g \rangle$. Therefore we have:

\[ J \subset \langle g \rangle + A. \]

Since $A \subset J$, it suffices to prove that $g \in J$. To this end, consider the monic polynomial $h_\ell$ generating the ideal $I_\ell \cap k[x]$. Let

\[ H_{\ell'} = \prod_{\ell \in L, \ell \neq \ell'} h_\ell, \quad \text{for all } \ell' \in L. \]
The polynomials $H_{\ell'}$s are product of polynomials coprime with $h_{\ell'}$, hence is coprime with $h_{\ell'}$ hence invertible modulo $h_{\ell'}$. Write $F_{\ell'} := H_{\ell'}^{-1} \mod h_{\ell'}$ in $k[x]$ of degree smaller than that of $h_{\ell'}$. In particular $\deg_x(H_{\ell'}F_{\ell'}) < \deg_x(\prod_{\ell \in L} h_{\ell}) := D$ for all $\ell' \in L$. We have:

$$\sum_{\ell \in L} H_{\ell}F_{\ell} = 1. \tag{32}$$

Indeed, the polynomial $H = \sum_{\ell \in L} H_{\ell}F_{\ell}$ is of degree in $x$ smaller than $D := \deg_x(\prod_{\ell \in L} h_{\ell})$. Because $H_{\ell'} \equiv 0 \mod h_{\ell}$ as soon as $\ell' \neq \ell$, we have $H \equiv H_{\ell}F_{\ell} \mod h_{\ell}$ and since $F_{\ell} \equiv H_{\ell}^{-1} \mod h_{\ell}$ we have $H_{\ell}F_{\ell} \equiv 1 \mod h_{\ell}$, whence $H \equiv 1 \mod h_{\ell}$. Since the $h_{\ell}$ are pairwise coprime, the Chinese remainder theorem implies that $H \equiv 1 \mod (\prod_{\ell \in L} h_{\ell})$. But since $\deg_x(H) < D$ this means $H = 1$. This ends the proof of Eq. (32).

Consider the partitions $[(L \setminus \{\ell\}, \{\ell\})]_{\ell \in L}$ of $L$. Their associated ideals are $(\prod_{\ell' \in L, \ell' \neq \ell} I_{\ell'}) \langle g \rangle$, for $\ell \in L$. Observe that $H_{\ell}g \in (\prod_{\ell' \in L, \ell' \neq \ell} I_{\ell'}) \langle g \rangle$, hence $H_{\ell}F_{\ell}g \in (\prod_{\ell' \in L, \ell' \neq \ell} I_{\ell'}) \langle g \rangle$. Therefore $\sum_{\ell \in L} H_{\ell}F_{\ell}g \in \sum_{\ell \in L}(\prod_{\ell' \in L, \ell' \neq \ell} I_{\ell'}) \langle g \rangle$. By Eq. (32), we obtain $g \in \sum_{\ell \in L}(\prod_{\ell' \in L, \ell' \neq \ell} I_{\ell'}) \langle g \rangle$. Finally, Eq. (31) shows that $g \in J$. \qed