UNDERSTANDING NEUTRINO MASSES AND MIXINGS IN THE SEESAW FRAMEWORK

R. N. MOHAPATRA
Department of Physics, University of Maryland, College Park, MD 20742, USA
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Abstract

Understanding the disparate mixing patterns between quarks and leptons is one of the major challenges in particle theory today. I discuss some of the ways to understand this difference within the seesaw framework using new symmetries of quarks and leptons. After a brief introduction to the various types of seesaw formulae, proposals for understanding large solar and atmospheric mixings for the three mass hierarchies i.e. normal, inverted and degenerate, are presented and their implications discussed.

I. INTRODUCTION

There is now strong evidence for neutrino oscillations from the solar and atmospheric observations in Super-Kamiokande, Homestake, Gallex, SAGE and SNO experiments. The deficit in the neutrino flux observed in detectors on Earth compared to theoretical expectations in both the solar and the atmospheric cases appear to be well understood if one assumes that neutrinos produced in the source ($\nu_e$'s in the first case and $\nu_{\mu}$'s in the second) oscillate into another species ($\nu_{\mu}, \tau$ in the first case and $\nu_{\tau}$ in the second) which are not observable. Laboratory experiments that use accelerator muon neutrinos as in the K2K experiment and reactor electron anti-neutrinos as in the Kamland experiment have also shown deficits in the number of neutrinos compared to expectations providing not only additional evidence for oscillations but also ruling out alternative explanations for the solar and atmospheric neutrino deficits.

Thus the phenomenon of neutrino oscillations is now well established. The questions now are (i) how well do we understand what is observed in the neutrino experiments and (ii) what does it tell us about the nature of new physics beyond the standard model? In this talk I will discuss several ideas that attempt to answer these questions.

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For inter-species oscillations to take place, the neutrinos must be massive and must mix among themselves, with mass differences and mixing angles that are determined by observations. It seems that all the above data (excluding the LSND observations) can be understood in terms of oscillations of the three known neutrinos i.e. $\nu_e, \nu_\mu, \nu_\tau$ among themselves. Since the standard model predicts that neutrinos are massless, this evidence for neutrino mass regardless of the details about mixings is already of fundamental significance and provides one possible direction for physics beyond the standard model. Most likely possibility is that evidence for neutrino mass proves the existence of a right handed neutrino as I elaborate below. The hope is that by studying the details pattern of neutrino mixings required to understand observations, we will have have a clearer roadmap of physics beyond the standard model.

It is worth mentioning that another accelerator experiment, which has also shown positive evidence for neutrino oscillation i.e. $\bar{\nu}_\mu$ to $\bar{\nu}_e$ is the Los Alamos experiment, LSND. This result has not been confirmed by KARMEN which has also looked for the same process. Currently MiniBoone experiment at Fermilab is searching for this process. If LSND is confirmed, it will require drastic change in our understanding of neutrinos- e.g. it will require the existence of sterile neutrinos that mix with the known neutrinos. We will postpone the discussion of this evidence in the present talk.

Two other searches for $\nu_e$ oscillations that have yielded negative results are CHOOZ and PALO-VERDE reactor experiments but they provide an upper limits on one of the mixing angles (to be called $U_{e3}$ below) that has important implications for theories of neutrino masses. Further experiments are in progress or in planning stages to improve the limits on $U_{e3}$ and will significantly improve our understanding of the theoretical picture.

In this brief overview, I wish to draw attention to some of the theoretical ideas for understanding neutrino mass and mixing patterns in extensions of the standard model. This article will focus specifically on the seesaw mechanism that seems to provide the simplest way to understand small neutrino masses [1] and some attempts to understand large neutrino mixings within the models that rely on the seesaw mechanism to explain the small neutrino masses.

A. Major theoretical issues in neutrino physics:

The major issues of interest in neutrino theory are driven by the following experimental results and conclusions derived from them. We will use the notation, where the flavor or weak eigen states $\nu_\alpha$ (with $\alpha = e, \mu, \tau$) are expressed in terms of the mass eigenstates $\nu_i$ ($i = 1, 2, 3$) as follows: $\nu_\alpha = \sum_i U_{\alpha i} \nu_i$. The $U_{\alpha i}$, the elements of the Pontecorvo-Maki-Nakagawa-Sakata matrix represent the observable mixing angles in the basis where the charged lepton masses are diagonal. In any other basis, one has $U = U_\ell^\dagger U_\nu$, where the matrices $U_\ell$ and $U_\nu$ are the ones that diagonalize the charged lepton and neutrino mass matrices respectively.
1. Solar neutrinos:

Thanks to the SNO results on both charged and neutral currents, and the KAMLAND [2] results, there now appears to be a winner among the various possible oscillation solutions to the solar neutrino puzzle [3]. It seems that the so called LMA MSW solution is preferred over the small angle as well as the low and pure vacuum solution [3]. The KAMLAND results [2] also rules out many of the nonoscillation as well as the magnetic moment solution to the solar neutrino problem [4]. The present range of preferred values of the oscillation parameters are: $2 \times 10^{-5} \leq \Delta m^2_{\odot}/eV^2 \leq 4 \times 10^{-4}$ and $0.62 \leq \sin^2 2\theta \leq 0.99$ at 3$\sigma$ confidence level. All nonoscillation mechanisms could however be present at a sub dominant level and higher precision experiments are necessary to test for their presence.

2. Atmospheric neutrinos:

Here evidence appears very convincing that the explanation of observed muon neutrino deficit in upward going muons as well as the azimuthal angle dependence of this spectrum involves oscillation of $\nu_\mu$ to $\nu_\tau$, with $\Delta m^2_{\nu_\mu-\nu_\tau} \simeq 2.5 \times 10^{-3}$ eV$^2$ and maximal mixing $\sin^2 2\theta_A \geq 0.84$ at 99% c.l.

3. Neutrinoless double beta decay:

Oscillations involve only mass differences and therefore do not give information on the over all scale of the neutrino masses. One may hope that neutrinoless double beta decay may provide this information. It however turns out that this hope may not be completely justified even if the present limits on lifetime go up by two orders of magnitude as is contemplated in many experiments unless the neutrinos are quasi-degenerate with common mass in the range bigger than 0.05 eV.

Nevertheless, neutrinoless double beta decay is an experiment of fundamental significance since its observation will for the first time give evidence that neutrino is its own antiparticle and signal the breakdown of B-L quantum number. Whether a positive signal will lead to any conclusion about the detailed pattern of masses is not a simple question. The point is that in extensions of physics beyond the standard model, there are several phenomenologically viable mechanisms for $\beta\beta_{0\nu}$ decay that do not involve neutrino mass but rather arise from exchanges of heavy particles such as doubly charged Higgs bosons, right handed W’s or supersymmetric particles. Once a positive signal is observed, one will have to understand which contribution has shown up; for this one not only needs a precise value of the nuclear matrix element but also some way to isolate any possible contribution from heavy particle exchange, before any conclusion regarding the magnitude of the neutrino mass can be deduced.

Searches for $\beta\beta_{0\nu}$ decay has been going on for several years and a new round of higher precision experiments are on the verge of being launched. The most stringent limits on this decay are from the enriched $^{76}$Ge experiment by the Heidelberg-Moscow as well as the IGEX collaborations and can be converted to a constraint on masses and mixing angles as: $\sum_i U^2_{ei}m_i \leq 0.3$ eV, with an uncertainty of a factor of 2 to 3 due to nuclear matrix elements. There appears to be some evidence for a positive signal in the existing Heidelberg-Moscow
data [5], which if confirmed will be a significant discovery. Presently planned experiments such as GENIUS, MAJORANA, CUORE, EXO, XMASS and MOON can not only test this claim but are expected to push the limit down by one order of magnitude, if they fail to substantiate the claim.

4. $U_{e3}$:

The CHOOZ and PALO VERDE reactor experiments mentioned earlier searched for disappearance of reactor anti-neutrinos. Their null result can be translated into an upper limit on the $U_{e3}$ parameter i.e. $U_{e3} \leq 0.16 - 0.2$ for mass differences given by $\Delta m^2 \geq 3 \times 10^{-3}$ eV$^2$.

All this information can be summarized in the following form for the $U$ matrix (ignoring CP violation):

$$
U \simeq \begin{pmatrix}
\frac{c + s \epsilon}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{\epsilon}{\sqrt{2}} \\
\frac{c - s \epsilon}{\sqrt{2}} & \frac{s}{\sqrt{2}} & \frac{-\epsilon}{\sqrt{2}} \\
\frac{s}{\sqrt{2}} & \frac{-c}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
$$

(1)

where $\epsilon \equiv U_{e3}$.

As far as the mass pattern goes however, there are three possibilities all equally viable from experimental point of view:

- (i) normal hierarchy: $m_1 \ll m_2 \ll m_3$;
- (ii) inverted hierarchy : $m_1 \simeq -m_2 \gg m_3$ and
- (iii) approximately degenerate pattern $m_1 \simeq m_2 \simeq m_3$;

where $m_i$ are the eigenvalues of the neutrino mass matrix. In the first case, the atmospheric and the solar neutrino data give direct information on $m_3$ and $m_2$ respectively. On the other hand, in the last case, the mass differences between the first and the second eigenvalues will be chosen to fit the solar neutrino data and the second and the third which then must be close to each other are given the atmospheric neutrino data to be $m_1 \simeq m_2 \simeq \sqrt{\Delta m^2} \simeq 0.05$ eV.

Three of the major theoretical challenges in neutrino physics now are:

- How does one understand the extreme smallness of the neutrino masses?
- How does one understand two large mixing angles among neutrinos given that there is so much similarity between quarks and leptons at the level of interactions and that the quark mixings are small?
- What is the mass pattern among the neutrinos and how does one understand them from a theoretical point of view simultaneously with the near bimaximal mixing pattern? In particular, why is $\Delta m^2 / \Delta m^2_A \ll 1$. 

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II. SEESAW MECHANISM FOR SMALL NEUTRINO MASSES

It is well known that in the standard model the neutrino is massless due to a combination of two reasons: (i) one, its righthanded partner ($\nu_R$) is absent and (ii) the model has exact global $B - L$ symmetry. Clearly, to understand a nonzero neutrino mass, one must give up one of the above assumptions. If one blindly included a $\nu_R$ to the standard model as a singlet, the status of neutrino would be parallel to all other fermions in the model and one would be hard put to understand why its mass is so much smaller than that of other fermions. Clearly there must be some other new ingredient that must be added.

A first hint of this new ingredient came from the observation of Weinberg that if $B$-$L$ symmetry is broken by some high scale physics, in the effective low energy theory, one can have operators of the form $(LH)^2/M$, where $M$ denotes the scale of new physics [6]. This after electroweak symmetry breaking would lead to a neutrino mass $\sim \frac{\alpha^2 \mu^2}{M}$. The key question now is what is the value of $M$?

In the absence of any $B$-$L$ violating physics all the way upto the Planck scale and assuming that nonperturbative Planck scale physics breaks all global symmetries such as the global $B$-$L$ symmetry present in the standard model, a plausible higher dimensional operators takes the form [7] $LHLH/M_{Pl}$ (where $L$ is a lepton doublet and $H$ is the Higgs doublet). This after electroweak symmetry breaking leads to masses for neutrinos of order $10^{-5}$ eV or less and are therefore not adequate for understanding observations. Thus a nontrivial extension of the standard model is called for wherein, the requisite value for $M$ to explain the atmospheric neutrino data (of order $10^{15}$ GeV or so) must be the scale of $B$-$L$ breaking or the breaking of some other symmetry. A concrete example of a particle that will provide the ultraviolet completion of the standard model with the desired neutrino mass operator is to add right handed neutrinos which have a large Majorana mass. This is the seesaw mechanism [1], that I will discuss below.

One then faces a “naturalness” problem similar to the Higgs mass problem of the standard model i.e. why the radiative corrections do not send the mass $M$ of the right handed neutrino to the Planck scale.

An associated question is: is there an independent reason for the right handed neutrino other than the neutrino mass and seesaw mechanism? We will see below that there are several candidate symmetries which are compelling from other arguments and provide a reason for the stability of the new scale mass $M$ and naturally bring in the right handed neutrino into the theory. These symmetries are local symmetries.

- (i) local $B - L$ and/or
- (ii) $SU(2)_H$ horizontal symmetry acting on the first two generations [8];
- (iii) $SU(3)_H$ horizontal symmetry [9].

The most widely discussed example is the local $B$-$L$ symmetry but the second case has also very interesting predictions for neutrino masses. The mass $M$ in these examples is the Majorana mass of the right handed neutrinos that break either or both of these symmetries (i.e. in the exact symmetry limit the RH neutrinos have zero mass).
A. Quark lepton symmetry and local B-L symmetry

As the first example of a model with right handed neutrinos ($N_R$), consider making the standard model completely quark lepton symmetric by adding one $N_R$ per generation. This expands the gauge symmetry of the electroweak interactions to $SU(2)_L \times U(1)_{I_3R} \times U(1)_{B-L}$ or to its full left-right symmetric extension $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetry. In the latter case, the fermion doublets $(u,d)_{L,R}$ and $(\nu,e)_{L,R}$ are assigned to the left-right gauge group in a parity symmetric manner. The electric charge formula for the model takes a very interesting form [10]:

$$Q = I_{3L} + I_{3R} + \frac{B-L}{2}.$$ 

It can be concluded from this that below the scale $v_R$ where the $SU(2)_R \times U(1)_{B-L}$ symmetry breaks down to the standard model and above the scale of $M_W$, one has the relation $\Delta I_{3R} = -\frac{B-L}{2}$. This simple looking relation has the profound consequence that neutrinos must be Majorana fermions and that there must be lepton number violating interactions in nature. Furthermore it explains why the right handed neutrino mass is so much smaller than the Planck mass- it is connected with the breaking of local $B-L$ symmetry.

B. Type I vrs type II seesaw

To see how small neutrino masses are explained, note that the $\nu_L - \nu_R$ mass matrix for three generations takes the form:

$$M = \begin{pmatrix} M_{LL} & M_{LR} \\ M_{LR}^T & M_{RR} \end{pmatrix}$$

where $M_{RR} = f v_R$ is the Majorana mass matrix of the right handed neutrinos, ($f$ is the new Yukawa coupling matrix that determines the right handed neutrino masses). The first term $M_{LL} \simeq \frac{f v_R^2}{v_R}$ is the induced triplet vev that leads to a direct Majorana mass matrix for the left handed neutrinos and is characteristic of the existence of asymptotic parity symmetry. (It would for example be absent if the local symmetry is $SU(2)_L \times U(1)_{I_3R} \times U(1)_{B-L}$.) Note that the flavor structure of the induced triplet vev contribution (or the type II seesaw contribution), is same as for the right handed neutrino.

The contribution $M_{LR} \equiv M_D = Y v_{wk}$ is the Dirac mass matrix connecting the left and the right handed neutrinos. The diagonalization of this mass matrix leads to following form for the light neutrino masses:

$$M_\nu \simeq f \frac{v_R^2}{v_R} - \frac{1}{v_R} M_D^T f^{-1} M_D;$$

where $M_D$ is defined as $\mathcal{L}_{mass} = \bar{\nu}_R M_D \nu_L$; $f$, the Yukawa coupling matrix that is responsible for the masses of the heavy right handed neutrinos characterizes the high scale physics, whereas all other parameters denote physics at the weak scale. We have called this generalized formula for neutrino masses, the type II seesaw formula [11] to distinguish it from the type I seesaw formula, the one that is commonly used in literature where the first term of Eq. 3 is absent. Important feature of this formula is that both terms vanish as $v_R \rightarrow \infty$ and since $v_R \gg v_{wk}$, the the neutrino masses are much smaller than the charged fermion masses. As was particularly emphasized in the third paper of ref. [1], the dominance of
V-A interaction in the low energy weak processes is now connected to smallness of neutrino masses.

If in the above seesaw formula, the second term dominates, this leads to the canonical type I seesaw formula and leads to the often discussed hierarchical neutrino masses, which in the approximation of small mixings lead to \( m_{\nu_i} \simeq m_{f_i}^2/v_R \), where \( f_i \) is either a charged lepton or a quark depending on the kind of model for neutrinos.

On the other hand, in models where the first term dominates, the neutrino masses can be almost generation independent unless \( f \) itself has the flavor structure of the charged fermions. For example, if there is indication for neutrinos being degenerate in mass from observations, one will have to resort to type II seesaw mechanism for its understanding, with the first term dominating the neutrino masses.

A further advantage of the right handed neutrino within the B-L symmetry framework and seesaw mechanism is that they fit very nicely into grand unified frameworks based on SO(10) models. The coupling constant unification then provides a theoretical justification for the high seesaw scale and hence the small neutrino masses. Furthermore, the 16-dim. spinor representation of SO(10) has just the right quantum numbers to fit the \( \nu_R \) in addition to the standard model particles of each generation.

**C. Double seesaw with a low scale for \( B-L \) symmetry**

As we saw from the previous discussion, the conventional seesaw mechanism requires rather high scale for the B-L symmetry breaking and the corresponding right handed neutrino mass (of order \( 10^{15} \) GeV). There is however no way at present to know what the scale of B-L symmetry breaking is. There are for example models bases on string compactification [12] where the \( B-L \) scale is quite possibly in the TeV range. In this case small neutrino mass can be implemented by a double seesaw mechanism suggested in Ref. [13]. The idea is to take a right handed neutrino \( N \) whose Majorana mass is forbidden by some symmetry and a singlet neutrino \( S \) which has extra quantum numbers which prevent it from coupling to the left handed neutrino but which is allowed to couple to the right handed neutrino. One can then write a three by three neutrino mass matrix in the basis \( (\nu, N, S) \) of the form:

\[
M = \begin{pmatrix}
0 & m_D & 0 \\
m_D & 0 & M \\
0 & M & \mu
\end{pmatrix}
\]

For the case \( \mu \ll M \approx M_{B-L} \) (where \( M_{B-L} \) is the \( B-L \) breaking scale), this matrix has one light and two heavy states. The lightest eigenvalue is given by \( m_\nu \sim m_D M^{-1} \mu M^{-1} m_D \). There is a double suppression by the heavy mass compared to the usual seesaw mechanism and hence the name double seesaw. A generalization of this mechanism to the case of three generations is straightforward. One important point here is that to keep \( \mu \sim m_D \), one also needs some additional gauge symmetries, which often are a part of the string models. It can also be used in models with high scale B-L breaking where the RH neutrino is forbidden by symmetries [14].
D. $SU(2)_H$ local symmetry and $3 \times 2$ seesaw with two $N_R$'s

A symmetry among the different generation has often been suspected as a possible way to understand the different properties of the quarks and leptons of different generations. This symmetry for the three generation case could be either a $U(1)$, $SU(2)$ or an $SU(3)$ local symmetry. Of these three possibilities, the third one requires that we include additional fermions to cancel anomalies. Of the remaining two, we choose $SU(2)_H$ since it has the following interesting property i.e. if it operate on right handed charged leptons, cancellation of global Witten anomaly requires that we must introduce at least two right handed neutrinos ($N_{eR}, N_{\mu R}$) transforming as a doublet under the group. Thus two right handed neutrinos is the minimal set required theoretically. Clearly, the mass of the right handed neutrinos are connected to the breaking of the $SU(2)_H$ symmetry [8]. An additional feature of these matrices is that they lead to a $3 \times 2$ seesaw as compared to the $3 \times 3$ seesaw in the case of the left-right symmetric (or SO(10)) models. We will see the implications of the $3 \times 2$ seesaw later on in this talk. This could of course be a part of a B-L like model if $v_H \ll M_{B-L}$. A distinct feature of the models with $3 \times 2$ seesaw is that one of the light neutrinos is massless. In this sense, in these models all parameters of a real neutrino mass matrix are determinable by only oscillation experiments.

III. MASS MATRIX ANSATZ AND ATTEMPTS TO UNDERSTAND LARGE MIXINGS

One of the major mysteries of neutrino physics is understanding large mixing angles needed to explain solar and the atmospheric neutrino oscillations. This is because of the simple fact that there is so much similarity in the interactions between the quarks and leptons and yet quarks mixings between different generations are of course well known to be very small unlike the lepton mixings.

In the seesaw framework one may attribute the origin of large mixings to the fact that a central ingredient in understanding the neutrino mass matrix is the mass matrix of the right handed neutrino which reflects high scale physics whereas quark physics is presumably a low scale physics. From this perspective, one should “invert” the seesaw formula and deduce the texture of right handed neutrino masses from our knowledge of neutrino masses andf mixings [15]. One then needs to understand where the relevant right handed neutrino mass matrix comes from and draw clues from it as to the nature of high scale physics. An important point is that if the right handed neutrinos are also responsible for origin of matter via leptogenesis [16], then these conclusions about the RH neutrino mass matrix can in principle be “tested” using this cosmological laboratory.

To proceed further, a good starting point is to search useful mass matrices for light neutrinos that explain observed mixings. We first note that in the absence of CP violation, the symmetric Majorana mass matrix for the light neutrinos $M_\nu$ contains six parameters, whereas observations give only five pieces of observation i.e. $\Delta m^2_{A}$, $\theta_{12}$, $\theta_{23}$, $\theta_{13}$ and $U_{e3} \equiv U_{e3}$. The absence of the sixth piece of information is essentially reflected in the fact that the precise mass pattern (normal, inverted or degenerate) of neutrinos is not known. So to make any progress, one may try to make ansatzes that reduce the number of parameters
in a mass matrix either (A) by making different elements equal or (B) putting them to zero in a basis where the charged leptons are diagonal.

An example of the first strategy is the zeroth order mass matrix discussed in [17]:

\[
M_\nu = \begin{pmatrix}
A + D & F & F \\
F & A & D \\
F & D & A
\end{pmatrix}
\]  

(5)

This leads to an exact bimaximal pattern with the MNS matrix of the form

\[
U_{PMNS} = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]  

(6)

but allows for all different mass patterns depending on the relative values of the parameters \(A, D\) and \(F\). Since the present data implies that there are deviations from the exact bimaximal form, this mass matrix must have additional small corrections.

Three different mass patterns can emerge from this mass matrix in various limits: e.g. (i) for \(F \ll A \simeq -D\), one gets the normal hierarchy; (ii) for \(F \gg A, D\), one has the inverted pattern for masses and (iii) the parameter region \(F, D \ll A\) leads to the degenerate case.

An interesting symmetry of this mass matrix is the \(\nu_\mu \leftrightarrow \nu_\tau\) interchange symmetry, which is obvious from the matrix; but in the limit where \(A = D = 0\), there appears a much more interesting symmetry i.e. the continuous symmetry \(L_e - L_\mu - L_\tau\) [18]. If the inverted mass matrix is confirmed by future experiments, this symmetry will provide an important clue to new neutrino related physics beyond the standard model. Inverted mass pattern is the only case where such an interesting leptonic symmetry appears. We explore the implications of this symmetry further in the next section. But before that, let us explore some other ansatz in the literature.

Another mass matrix of interest [19] has the form

\[
M_\nu = \begin{pmatrix}
A & F & F \\
F & G & D \\
F & D & G
\end{pmatrix}
\]  

(7)

which is a four parameter mass matrix which introduces a new parameter that can be varied to obtain the desired solar neutrino mixing. The MNS matrix that results from this mass matrix is:

\[
U_{PMNS} = \begin{pmatrix}
c & s & 0 \\
-s \sqrt{2} & c \sqrt{2} & \frac{1}{\sqrt{2}} \\
-s \sqrt{2} & c \sqrt{2} & -\frac{1}{\sqrt{2}}
\end{pmatrix}
\]  

(8)

Clearly as in the case of ref. [17], if \(F \gg A, D, G\), we get the inverted pattern and for \(F, A \ll D, G\), we get the normal mass hierarchy.

One can make the parameters in this mass matrix complex [20] as follows:

\[
M_\nu = \begin{pmatrix}
A & F & F^* \\
F & G & D \\
F^* & D & G^*
\end{pmatrix}
\]  

(9)
in which case one gets a $U_{e3}$ imaginary corresponding to maximal Dirac CP violation, a possibility that has important experimental implications. There are many other interesting mass matrix ansatizes \cite{21} which have their characteristic predictions.

One lesson that one may learn from these studies is the possible existence of symmetries, which may shed light on the nature of new physics beyond the standard model and in all the cases discussed, $L_e - L_\mu - L_\tau$ emerges as a possible candidate as noted earlier. We study the implications and tests of this symmetry below.

A. Approximate $L_e - L_\mu - L_\tau$ symmetry and neutrino mixings

In the exact $L_e - L_\mu - L_\tau$ symmetry limit, the model not only leads naturally to large solar and atmospheric mixing angles but it also leads to vanishing $U_{e3}$ as well as $\Delta m^2_{\odot}/\Delta m^2_A = 0$. Therefore the model raises the hope that a small $U_{e3}$ as well as $\Delta m^2_{\odot}/\Delta m^2_A$ can be understood in a natural manner. One must therefore add small symmetry breaking terms to this model and examine the consequences.

This question was studied in two papers \cite{22}. In the second paper of \cite{22}, the following mass matrix for neutrinos was considered that includes small $L_e - L_\mu - L_\tau$ violating terms.

$$M_\nu = m \begin{pmatrix} z c s \\ c y d \\ s d x \end{pmatrix}. \tag{10}$$

where $c = \cos \theta$ and $s = \sin \theta$ and $x, y, z, d \ll 1$ (we allow $x, y, z, d$ to be random and as large as 0.3. The charged lepton mass matrix is chosen to have a diagonal form in this basis and $L_e - L_\mu - L_\tau$ symmetric.

In the presence of the small symmetry breaking terms $(x, y, z, d)$, we find the following sumrules involving the neutrino observables and the elements of the neutrino mass matrix. The nontrivial ones are:

$$\sin^2 2\theta_\odot = 1 - \left( \frac{\Delta m^2_\odot}{4\Delta m^2_A} - z \right)^2 + O(\delta^3)$$

$$\frac{\Delta m^2_\odot}{\Delta m^2_A} = 2(z + \vec{v} \cdot \vec{x}) + O(\delta^2)$$

$$U_{e3} = \vec{A} \cdot (\vec{v} \times \vec{x}) + O(\delta^3) \tag{11}$$

where $\vec{v} = (\cos^2 \theta, \sin^2 \theta, \sqrt{2} \sin \theta \cos \theta)$, $\vec{x} = (x, y, \sqrt{2} d)$ and $\vec{A} = \frac{1}{\sqrt{2}}(1, 1, 0)$. $\delta$ in the preceding equations represents the small parameters in the mass matrix.

One of the major consequences of these relations is that (i) there is a close connection between the measured value of the solar mixing angle and the neutrino mass measured in neutrinoless double beta decay i.e. $z$; (ii) the present values for the solar mixing angle can be used to predict the $m_{\beta\beta}$ for a value of the $\Delta m^2_\odot$. For instance, for $\sin^2 2\theta_\odot = 0.9$, we would predict $(\frac{\Delta m^2_\odot}{4\Delta m^2_A} - z) = 0.3$. For small $\Delta m^2_\odot$, this implies $m_{\beta\beta} \simeq 0.01$ eV. The second relation involving the $\Delta m^2_\odot/\Delta m^2_A$ in terms of $x, y, z, d$ tells us that for this to be the case, we must have strong cancellation between the various small parameters. Given this, the
above \( m_{\beta\beta} \) value is expected to be within the reach of new double beta decay experiments contemplated. Note however that the \( \sin^2 2\theta_\odot \) cannot be smaller than 0.9 in the case of approximate \( L_e - L_\mu - L_\tau \) symmetry.

If the value of \( \sin^2 2\theta_\odot \) is ultimately determined to be less than 0.9, the question one may ask is whether the idea of \( L_e - L_\mu - L_\tau \) symmetry is dead. The answer is in the negative since so far we have explored the breaking of \( L_e - L_\mu - L_\tau \) symmetry only in the neutrino mass matrix. It was shown in the first paper of [22] that if the symmetry is broken in the charged lepton mass, one can lower the \( \sin^2 2\theta_\odot \) to 0.85 or so.

B. Approximate \( L_e - L_\mu - L_\tau \) symmetry from \( SU(2)_H \) horizontal symmetry

It can be shown that an \( SU(2)_H \) model for leptons leads quite generally to an approximate \( L_e - L_\mu - L_\tau \) symmetry for neutrinos. As already noted, a distinct feature of \( SU(2)_H \) symmetry is that there are two right handed neutrinos instead of three and therefore one has a \( 3 \times 2 \) seesaw rather than the usual \( 3 \times 3 \) one.

To see this in detail, first note that the gauge interactions have the symmetry \( SU(2)_H \times U(1)_{e+\mu+\tau} \) global symmetry. The diagonal generator of \( SU(2)_H \) is given by \( L_e - L_\mu \). If we break horizontal symmetry by an \( SU(2)_H \) triplet Higgs \( \Delta_H \), then \( L_e - L_\mu \) survives as a gauge symmetry of leptons. We further break the symmetry by a doublet Higgs \( \chi_H \), then the allowed Yukawa couplings that contribute to neutrino masses are of the form \( N^c \Delta_H N^c, L_\tau H_a \chi_H N^c \). Note that these two terms reduce the above global symmetry to \( SU(2)_H \times U(1)_{\tau} \). The vevs of these Higgs fields i.e. \( < \Delta_H, 3 > \neq 0 \) and \( \chi_H, 2 \neq 0 \) reduces this symmetry down to \( L_e - L_\mu - L_\tau \). This is the major reason why this model leads to an inverted hierarchy and also two large mixings in zeroth order as desired. Thus if experiments confirm the inverted hierarchy and a possible \( L_e - L_\mu - L_\tau \) symmetry for leptons, it may be signal for the local \( SU(2)_H \) symmetry at a high scale.

The charged lepton masses arise from the couplings of the form \( LH_d \chi_H \tau^c \) and \( L_\tau H_d \chi_H E^c \). The second term breaks \( L_e - L_\mu - L_\tau \) symmetry and is responsible departure from exact maximal mixing angle in the 12 sector as well can contribute to solar mass splittings.

Using the discussion of the above paragraph, the Dirac mass of the neutrino as well as the righthanded neutrino mass matrix can be seen to lead [8] to \( 5 \times 5 \) mass matrix for heavy and light neutrinos of the form:

\[
M_{\nu_L, \nu_R} = \begin{pmatrix}
0 & 0 & 0 & h_0 k_0 & 0 \\
0 & 0 & 0 & 0 & h_0 k_0 \\
0 & 0 & 0 & h_1 k_1 & h_1 k_2 \\
h_0 k_0 & 0 & h_1 k_1 & 0 & f v'_H \\
0 & h_0 k_0 & h_1 k_2 & f v'_H & 0 \\
\end{pmatrix}
\]  \hspace{1cm} (12)

After seesaw diagonalization, it leads to the light neutrino mass matrix of the form:

\[
M_{\nu} = - M_D M_R^{-1} M_D^T
\]  \hspace{1cm} (13)

where \( M_D = \begin{pmatrix} h_0 k_0 & 0 & 0 \\
0 & h_0 k_0 & 0 \\
h_1 k_1 & h_1 k_2 & \end{pmatrix} \); \( M_R^{-1} = \frac{1}{f v'_H} \begin{pmatrix} 0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix} \). The resulting light Majorana neutrino mass matrix \( M_{\nu} \) is given by:
First of all as discussed before, this leads to one neutrino which is massless. Also note that as \( \kappa_1 \rightarrow 0 \), the mass matrix acquires \( L_e - L_\mu - L_\tau \) symmetry and of course has \( \Delta m^2_{\odot} = 0 \).

The smallness of \( \Delta m^2_{\odot} \) thus implies that \( \kappa_1 \ll \kappa_{0.2} \). Also in the limit the solar mixing angle is \( \pi/4 \).

To get the physical neutrino mixings, we also need the charged lepton mass matrix defined by \( \tilde{\psi}_L M_\ell \psi_R \). There are two possibilities for \( M_\ell \) in our model:

**Case (i):**

\[
M_\ell = \begin{pmatrix}
 h'_2 \kappa_0 & 0 & -h'_1 \kappa_2 \\
 0 & h'_2 \kappa_0 & h'_1 \kappa_1 \\
 h'_1 \kappa_1 & h'_4 \kappa_2 & h'_3 \kappa_0
\end{pmatrix}
\]  

(15)

**Case (ii):**

\[
M_\ell = \begin{pmatrix}
 h'_2 \kappa_0 & 0 & h'_1 \kappa_1 \\
 0 & h'_2 \kappa_0 & h'_1 \kappa_2 \\
 -h'_4 \kappa_2 & h'_4 \kappa_1 & h'_3 \kappa_0
\end{pmatrix}
\]  

(16)

There are contributions to neutrino mixings coming from the charged lepton sector, which will help us to get a lower value for \( \sin^22\theta_\odot \). Generically, we get on including the charged lepton contributions a correlation between the solar mixing angle and \( U_{e3} \) as follows: \( \theta_\odot \approx \frac{\pi}{4} - U_{e3} \) [23].

### IV. LARGE MIXINGS IN MODELS WITH QUARK-LEPTON UNIFICATION

The \( L_e - L_\mu - L_\tau \) model discussed above treats the quarks and leptons on a fundamentally different footing. On the other hand it could be that at very short distances there is quark lepton unification [24]. I give below two of a number of ideas, where models with quark lepton symmetry can lead to large neutrino mixings. In the models discussed below large mixings arise dynamically and without need for any extra symmetries.

#### A. Radiative magnification of mixing angles

In this class of models dynamics of radiative corrections plays an essential role in understanding the maximal mixings. The basic idea is that at the seesaw scale, all mixings angles are small, a situation quite natural if the pattern of \( f \) Yukawa coupling is similar to the quark sector. Since the observed neutrino mixings are weak scale observables, one must extrapolate [25] the seesaw scale mass matrices to the weak scale and recalculate the mixing angles.

The extrapolation formula is
Note that since $h_\alpha = \sqrt{2}m_\alpha/v_{wk}$ ($\alpha$ being the charged lepton index), in the extrapolation only the $\tau$-lepton makes a difference. In the MSSM, this increases the $M_{\tau\tau}$ entry of the neutrino mass matrix and essentially leaves the others unchanged. It was shown in ref. [26] that if the muon and the tau neutrinos are nearly degenerate in mass at the seesaw scale, in supersymmetric theories, the $\tan\beta \geq 5$, the radiative corrections can become large enough so that at the weak scale the two diagonal elements of $M_\nu$ which were nearly equal but different at the seesaw scale become much more degenerate. This leads to an enhancement of the mixing angle to become almost maximal and a solution to the atmospheric neutrino deficit emerges even though at the seesaw scale, the mixing angles were small.

This can also be seen from the renormalization group equations when they are written in the mass basis [27]. Denoting the mixing angles as $\theta_{ij}$ where $i,j$ stand for generations, the equations are:

$$\frac{ds_{23}}{dt} = -F_\tau c_{23}^2 (-s_{12}U_{\tau 1}D_{31} + c_{12}U_{\tau 2}D_{32}),$$  

(19)

$$\frac{ds_{13}}{dt} = -F_\tau c_{23}c_{13}^2 (c_{12}U_{\tau 1}D_{31} + s_{12}U_{\tau 2}D_{32}),$$  

(20)

$$\frac{ds_{12}}{dt} = -F_\tau c_{12} (c_{23}s_{13}s_{12}U_{\tau 1}D_{31} - c_{23}s_{13}c_{12}U_{\tau 2}D_{32} + U_{\tau 1}U_{\tau 2}D_{21}).$$  

(21)

where $D_{ij} = (m_i + m_j) / (m_i - m_j)$ and $U_{\tau 1,2,3}$ are functions of the neutrino mixings angles. The presence of $(m_i - m_j)$ in the denominator makes it clear that as $m_i \approx m_j$, that particular coefficient becomes large and as we extrapolate from the GUT scale to the weak scale, small mixing angles at GUT scale become large at the weak scale.

Furthermore, this happens only if the experimental observable $\Delta m_{23}^2 \leq 0$ a possibility can be tested in contemplated long base line experiments. Also for this mechanism to work, the overall scale of neutrino masses must be in the range of 0.1 eV or so making the idea testable in forthcoming double beta decay experiments.

Several comment are in order: (i) to get a near degenerate mass spectrum without additional assumptions, one must use the type II seesaw mechanism as in Eq. (3); (ii) An interesting question is whether this mechanism can be extended to the case of three generations and whether it can explain the bimaximal pattern also. It has been shown recently that indeed this can work for three generations [28], where identifying the seesaw scale neutrino mixing angles with the corresponding quark mixings and assuming quasi-degenerate neutrinos, it is found [28] that weak scale solar and atmospheric angles get magnified to the desired level while due to the extreme smallness of $V_{ub}$, the magnified value of $U_{e3}$ remains within its present upper limit. In figure 1, we show the evolution of the mixing angles to the weak scale.

A second recent work [29] has used the techniques of ref. [26] to study radiative magnification of solar angle in texture zero neutrino mass matrices. In this example, the atmospheric neutrino mixing is an input but solar angle is dynamically magnified.
B. A minimal SO(10) model

Another suggestion for understanding large atmospheric mixing has been made within a class of SO(10) models, which are strongly suggested by local B-L symmetry, large seesaw scale and grand unification ideas. The basic ingredients of this suggestion are the following properties of the SO(10) model: (i) that one can construct a minimal SO(10) model with only two multiplets that couple to fermions i.e. 10 and 126 and another that breaks SO(10) down to the left-right model. The second breaks the B-L symmetry and the first the electroweak symmetry. (ii) A second property of SO(10) models [30] is that 126 contains submultiplets that not only contribute to charged fermion but also to the left and right handed Majorana masses ($M_{LL}, M_{RR}$ respectively in Eq. (2)) for the neutrinos. This leads to a tremendous reduction of the number of arbitrary parameters in the model, as we will see below.

There are only two Yukawa coupling matrices in this model: (i) $h$ for the 10 Higgs and (ii) $f$ for the 126 Higgs. SO(10) has the property that the Yukawa couplings involving the 10 and 126 Higgs representations are symmetric. Therefore if we ignore CP violation and work in a basis where one of these two sets of Yukawa coupling matrices is diagonal, then it will have only nine parameters. Noting the fact that the (2,2,15) submultiplet of 126 has a standard model doublet that contributes to charged fermion masses, one can write the quark and lepton mass matrices as follows [30]:

$$
M_u = h\kappa_u + fv_u \\
M_d = h\kappa_d + fv_d \\
M_\ell = h\kappa_d - 3fv_d \\
M_{\nu_D} = h\kappa_u - 3fv_u
$$

(22) (23)

where $\kappa_{u,d}$ are the vev’s of the up and down Higgs vevs of the standard model doublets in 10 Higgs and $v_{u,d}$ are the corresponding vevs for the same doublets in 126. Note that there are 13 parameters in the above equations and there are 13 inputs (six quark masses, three
lepton masses and three quark mixing angles and weak scale). Thus all parameters of the model that go into fermion masses are determined.

To determine the light neutrino masses, we use the seesaw formula in Eq. (3), where the $f$ is nothing but the 126 Yukawa coupling. Thus all parameters that give neutrino mixings except an overall scale are determined. These models were extensively discussed in the last decade [31]. Initially CP phases were ignored and more recently CP phases have been included in the analysis.

A very interesting point regarding these models has been noted in Ref. [32], where it is pointed out that if the direct triplet term in type II seesaw dominates, then it provides a very natural understanding of the large atmospheric mixing angle without invoking any symmetries. A simple way to see this is to note that when the triplet term dominates the seesaw formula, we have the neutrino mass matrix $M_\nu \propto f$, where $f$ matrix is the 126 coupling to fermions discussed earlier. Using the above equations, one can derive the following sumrule (sumrule was already noted in the second reference of [31]):

$$M_\nu = c(M_d - M_\ell)$$

(24)

Now quark lepton symmetry implies that for the second and third generation, the $M_{d,\ell}$ have the following general form:

$$M_d = \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_2 & m_b \end{pmatrix}$$

(25)

and

$$M_\ell = \begin{pmatrix} \epsilon'_1 & \epsilon'_2 \\ \epsilon'_2 & m_\tau \end{pmatrix}$$

(26)

where $\epsilon_i \ll m_{b,\tau}$ as is required by low energy observations. It is well known that in supersymmetric theories, when low energy quark and lepton masses are extrapolated to the GUT scale, one gets approximately that $m_b \simeq m_\tau$. One then sees from the above sumrule for neutrino masses that all entries for the neutrino mass matrix are of the same order leading very naturally to the atmospheric mixing angle to be large. Thus one has a natural understanding of the large atmospheric neutrino mixing angle. No extra symmetries are assumed for this purpose.

For this model to be a viable one for three generations, one must show that the minimal SO(10) model with triplet vev dominated seesaw formula indeed can give a large $\theta_{12}$ and a small $\theta_{13}$. This has been shown in a recent paper [33]. It was shown that this is indeed the case. To see roughly how this comes about, let us work in the basis where the down quark mass matrix is diagonal. All the quark mixing effects are then in the up quark mass matrix i.e. $M_u = U_{CKM}^T M_u^d U_{CKM}$. Note further that the minimality of the Higgs content leads to the following sumrule among the mass matrices:

$$k\tilde{M}_\ell = r\tilde{M}_d + \tilde{M}_u$$

(27)

where the tilde denotes the fact that we have made the mass matrices dimensionless by dividing them by the heaviest mass of the species i.e. up quark mass matrix by $m_t$, down quark mass matrix by $m_b$ etc. $k, r$ are functions of the symmetry breaking parameters of
FIG. 2. The figure shows the predictions of the minimal SO(10) model for $\sin^2 \theta^\odot$ and $\sin^2 \theta_A$ for the presently range of quark masses. Note that $\sin^2 \theta^\odot \geq 0.9$ and $\sin^2 \theta_A \leq 0.9$.

the model. Using the Wolfenstein parameterization for quark mixings, we can conclude that that we have

$$M_{d,t} \approx m_{b,\tau} \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

where $\lambda \sim 0.22$ and the matrix elements are supposed to give only the approximate order of magnitude. As we extrapolate the quark masses to the GUT scale, due to the fact that $m_b - m_\tau \approx m_\tau \lambda^2$ for some value of $\tan \beta$, the neutrino mass matrix $M_\nu = c(M_d - M_\ell)$ takes roughly the form

$$M_\nu = c(M_d - M_\ell) \approx m_0 \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda^2 \end{pmatrix}$$

It is then easy to see from this mass matrix that both the $\theta_{12}$ (solar angle) and $\theta_{23}$ (the atmospheric angle) are now large. The detailed magnitudes of these angles of course depend on the details of the quark masses at the GUT scale. Using the extrapolated values of the quark masses and mixing angles to the GUT scale, the predictions of this model for various oscillation parameters are given in Fig. 1, 2 and 3 in a self expalanatory notation. Note specifically the prediction in Fig. 4 for $U_{e3}$ which can be tested in MINOS as well as other planned Long Base Line neutrino experiments such as Numi-Off-Axis, JHF etc. There is a simple explanation of why the $U_{e3}$ comes out to be large. This can also be seen from the mass sumrule in Eq.24. Roughly, for a matrix with hierarchical eigen values as is the case here, the mixing angle $\tan 2\theta_{13} \sim \frac{M_{e,13}}{M_{e,33}} \approx \frac{\lambda^2 m_\tau}{m_\nu(M_U) - m_\tau(M_U)}$. Since to get large mixings, we need $m_b(M_U) - m_\tau(M_U) \simeq m_\tau \lambda^2$, we see that $U_{e3} \simeq \lambda$ upto a factor of order one. Indeed the detailed calculations lead to 0.16 which is not far from this value.

So far in this work, we assumed that all Yukawa couplings are real. Our procedure can be easily generalized to the case when they are all complex. The range of predictions for solar and atmospheric mixing angles become larger although the prediction for $U_{e3}$ remains around 0.14 to 0.16.
FIG. 3. The figure shows the predictions of the minimal SO(10) model for $\sin^2 2\theta_A$ and $\Delta m^2_\odot / \Delta m^2_A$ for the range of quark masses and mixings that fit charged lepton masses.

FIG. 4. The figure shows the predictions of the minimal SO(10) model for $\sin^2 2\theta_A$ and $U_{e3}$ for the allowed range of parameters in the model. Note that $U_{e3}$ is very close to the upper limit allowed by the existing reactor experiments.
V. CONCLUSION

In conclusion, the seesaw mechanism appears by far to be the simplest way to understand the small neutrino masses. The large right handed neutrino mass implied by this also helps in understanding origin of matter in the universe. Our understanding of mixings on the other hand, is at a very preliminary level. A particular challenge to theorists is to understand the so called bimaximal mixing pattern, which is emerging as the favorite. Several symmetry and dynamical approaches to understanding large mixings are noted. Also a minimal SO(10) model whose predictions are currently in accord but testable in near future is also presented. On the experimental side, high precision search for $U_{e3}$ and neutrinoless double beta decay will provide important ways to distinguish between the different models.

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