A Poisson-Jacobi-type transformation for the sum
\[ \sum_{n=1}^{\infty} n^{-2m} \exp(-an^2) \] for positive integer \( m \)

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Abstract
We obtain an asymptotic expansion for the sum
\[ S(a; w) = \sum_{n=1}^{\infty} e^{-an^2/n^w} \]
as \( a \to 0 \) in \( |\arg a| < \frac{1}{2}\pi \) for arbitrary finite \( w > 0 \). The result when \( w = 2m \), where \( m \) is a positive integer, is the analogue of the well-known Poisson-Jacobi transformation for the sum with \( m = 0 \). Numerical results are given to illustrate the accuracy of the expansion.

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1. Introduction

The classical Poisson-Jacobi transformation is given by
\[ \sum_{n=1}^{\infty} e^{-an^2} = \frac{1}{2} \sqrt{\pi/a - \frac{1}{2}} + \sqrt{\frac{\pi}{a}} \sum_{n=1}^{\infty} e^{-\pi^2n^2/a}, \]
where the parameter \( a \) satisfies \( \Re(a) > 0 \). This transformation relates a sum of Gaussian exponentials involving the parameter \( a \) to a similar sum with parameter \( \pi^2/a \). In the case \( a \to 0 \) in \( \Re(a) > 0 \), the convergence of the sum on the left-hand side becomes slow, whereas the sum on the right-hand side converges rapidly in this limit. Various proofs of the well-known result (1.1) exist in the literature; see, for example, [3, p. 120], [4, p. 60] and [5, p. 124].

In this note we consider the sum
\[ S(a; w) = \sum_{n=1}^{\infty} e^{-an^2/n^w} \quad (\Re(a) > 0). \]

This sum converges for any finite value of the parameter \( w \) provided \( \Re(a) > 0 \); when \( a = 0 \) then \( S(0; w) \) reduces to the Riemann zeta function \( \zeta(w) \) when \( \Re(w) > 1 \). Consequently, the series in (1.2) can be viewed as a smoothed Dirichlet series for \( \zeta(w) \). The asymptotic expansion
of $S(a;w)$ as $a \to 0$ in $\Re(a) > 0$ is straightforward. The most interesting case arises when $w = 2m$, where $m$ is a positive integer, for which we establish a transformation for $S(a;2m)$ analogous to that in (1.1) valid as $a \to 0$ in $\Re(a) > 0$. This similarly involves the series in (1.2) with $a$ replaced by $\pi^2/a$, but with each term decorated by an asymptotic series in $a$. A recent application of the series with $w = 2$ and $w = 4$ has arisen in the geological problem of thermochronometry in spherical geometry [6].

2. An expansion for $S(a;w)$ as $a \to 0$ when $w \neq 2, 4, \ldots$

Our starting point is the well-known Cahen-Mellin integral (see, for example, [3, §3.3.1])

$$z^{-\alpha} e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s - \alpha) z^{-s} ds \quad (z \neq 0, \ |\arg z| < \frac{1}{2} \pi), \quad (2.1)$$

where $c > \Re(\alpha)$ so that the integration path passes to the right of all the poles of $\Gamma(s - \alpha)$ situated at $s + \alpha - k$ ($k = 0, 1, 2, \ldots$). For simplicity in presentation we shall assume throughout real values of $w > 0$. Then, it follows that

$$S(a;w) = \sum_{n=1}^{\infty} e^{-an^2} = \sum_{n=1}^{\infty} \frac{n^{-w}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)(an^2)^{-s} ds$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(2s+w) a^{-s} ds,$$

upon reversal of the order of summation and integration, which is justified when $c > \max\{0, \frac{1}{2} - \frac{1}{2}w\}$, and evaluation of the inner sum in terms of the Riemann zeta function. The integrand possesses simple poles at $s = \frac{1}{2} - \frac{1}{2}w$ and $s = -k$ ($k = 0, 1, 2, \ldots$), except if $w = 2m + 1$ is an odd positive integer when the pole at $s = \frac{1}{2} - \frac{1}{2}w$ is double. The case when $w = 2m$ is an even positive integer requires a separate investigation which is discussed in Section 3.

Consider the integral taken round the rectangular contour with vertices at $c \pm iT$, $-c' \pm iT$, where $c' > 0$. The contribution from the upper and lower sides $s = \sigma \pm iT$, $-c' \leq \sigma \leq c$, vanishes as $T \to \infty$ provided $|\arg a| < \frac{1}{2} \pi$, since from the behaviour

$$\Gamma(\sigma \pm iT) = O(t^{\sigma - \frac{1}{2}e^{-\frac{1}{2}\pi T}}), \quad \zeta(\sigma \pm iT) = O(t^{\mu(\sigma) \log^4 t}), \quad (t \to \infty),$$

where for $\sigma$ and $t$ real

$$\mu(\sigma) = 0 \ (\sigma > 1), \quad \frac{1}{2} - \frac{1}{2} \sigma \ (0 \leq \sigma \leq 1), \quad \frac{1}{2} - \sigma \ (\sigma < 0),$$

$$A = 1 \ (0 \leq \sigma \leq 1), \quad A = 0 \ otherwise,$$

the modulus of the integrand is controlled by $O(T^{\sigma + \mu(\sigma) - \frac{1}{2} \log T e^{-\Delta T}})$, with $\Delta = \frac{1}{2} \pi - |\arg a|$. The residue at the double pole $s = -m$ when $w = 2m + 1$ ($m = 0, 1, 2, \ldots$) is given by

$$\frac{(-a)^m}{m!}(\gamma - \frac{1}{2} \log a + \frac{1}{2} \psi(m + 1),$$

where $\gamma$ is Euler’s constant and $\psi(x)$ is the logarithmic derivative of the gamma function. Displacement of the integration path to the left over the poles then yields (provided $w \neq 2m$)

$$S(a;w) = J(a;w) + \sum_{k=0}^{N-1} \frac{(-1)^k}{k!} \zeta(w - 2k)a^k + R_N, \quad (2.2)$$
where
\[
J(a; w) = \begin{cases} 
\frac{1}{2} \Gamma\left(\frac{1}{2} - \frac{1}{2}w\right)a^{w-1}/2 & (w \neq 2m+1) \\
\frac{(-a)^m}{m!} \left(\gamma - \frac{1}{2} \log a + \frac{1}{2} \psi(m+1)\right) & (w = 2m+1),
\end{cases}
\]

\(N\) is a positive integer such that \(N > \frac{1}{2}w + \frac{1}{2}\) and the prime on the sum over \(k\) denotes the omission of the term corresponding to \(k = m\) when \(w = 2m+1\).

The remainder \(R_N\) is
\[
R_N = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(s)\zeta(w + 2s)a^{-s}ds, \quad c = N - \frac{1}{2}.
\]

It is shown in the appendix, when \(w \neq 2, 4, \ldots\), that \(R_N = O(a^{N-\frac{1}{2}})\) as \(a \to 0\) in \(|\arg a| < \frac{1}{2}\pi\), with the constant implied in the \(O\)-symbol growing at least like \(\Gamma(N+1-\frac{1}{2}w)\). This establishes that the above series over \(k\) diverges as \(N \to \infty\) and that (2.2) is therefore an asymptotic expansion.

We remark that the algebraic expansion (2.2) also contains a subdominant exponentially small component as \(a \to 0\); compare [3, §8.1.5] for the particular case \(w = 0\). We do not consider this further in the present paper.

### 3. An expansion for \(S(a; 2m)\) when \(m = 1, 2, \ldots\)

The case \(w = 2m\), where \(m\) is a positive integer, is more interesting as this leads to the analogue of the Poisson-Jacobi transformation (1.1). There is now only a finite set of poles of the integrand in (2.1) at \(s = \frac{1}{2} - \frac{1}{2}w\) and \(s = 0, -1, -2, \ldots, -m\), since the poles of \(\Gamma(s)\) at \(s = -m - k\) \((k = 1, 2, \ldots)\) are cancelled by the trivial zeros of the zeta function \(\zeta(2m + 2s)\) at \(s = -m - 1, -m - 2, \ldots\). This has the consequence that the integrand is holomorphic in \(\Re(s) < -m\), so that further displacement of the contour can produce no additional algebraic terms in the expansion of \(S(a; 2m)\). Thus, we find when \(w = 2m\)

\[
S(a; 2m) = \frac{1}{2} \Gamma\left(\frac{1}{2} - m\right)a^{m-\frac{1}{2}} + \sum_{k=0}^{m} \frac{(-1)^k}{k!} \zeta(2m - 2k)a^k + I_L,
\]

where, upon making the change of variable \(s \to -s\),

\[
I_L = \frac{1}{2\pi i} \int_L \Gamma(-s)\zeta(2m - 2s)a^{-s}ds
\]

and \(L\) denotes a path parallel to the imaginary axis with \(\Re(s) > m\).

We now employ the functional relation for \(\zeta(s)\) given by [5, p. 269]

\[
\zeta(s) = 2^s\pi^{s-1}\sin\frac{1}{2}\pi s \Gamma(1-s)\zeta(1-s)\Gamma(1-s)\sin\frac{1}{2}\pi s
\]

and convert the argument of the zeta function in (3.2) into one with real part greater than unity. The integral in (3.2) can then be written in the form

\[
\frac{(-1)^m(2\pi)^m}{2\pi i} \int_L \left(2s - 2m + 1\right) \frac{\Gamma(2s - 2m + 1)}{\Gamma(s + 1)} \left(\frac{a}{4\pi^2}\right)^s ds.
\]

Since on the integration path \(\Re(2s - 2m + 1) > 1\), we can expand the zeta function and reverse the order of summation and integration to obtain

\[
I_L = (-)^m \frac{\pi^{2m-\frac{1}{2}}}{\Gamma(2m-\frac{1}{2})} \sum_{n=1}^{\infty} n^{2m-1} K_n(a; m),
\]

(3.4)
where
\[ K_n(a; m) := \frac{1}{2\pi i} \int_L \frac{\Gamma(s - m + \frac{1}{2})\Gamma(s - m + 1)}{\Gamma(s + 1)} \left( \frac{a}{\pi^2 n^2} \right)^s ds, \]
and we have employed the duplication formula for the gamma function
\[ \Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2}). \]

The quotients of gamma functions may then be expanded by making use of the result given in [3, p. 53]
\[ \frac{\Gamma(s - m + \frac{1}{2})\Gamma(s - m + 1)}{\Gamma(s + 1)^2} = \sum_{j=0}^{M-1} (-)^j c_j \Gamma(s + \vartheta - j) + \rho_M(s) \Gamma(s + \vartheta - M) \tag{3.5} \]
for positive integer \( M \), where \( \vartheta = \frac{1}{2} - 2m \),
\[ c_j = \frac{(m)_j(m + \frac{1}{2})_j}{j!} = \frac{2^{-2j}(2m)_{2j}}{j!} \]
and \( \rho_M(s) = O(1) \) as \( |s| \to \infty \) in \( |\arg s| < \pi \). Substitution of this expansion into the integrals \( K_n(a; m) \) then produces
\[ K_n(a; m) = \sum_{j=0}^{M-1} (-)^j c_j \frac{1}{2\pi i} \int_L \Gamma(s + \vartheta - j) \left( \frac{a}{\pi^2 n^2} \right)^s ds + R_M \]
\[ = \sum_{j=0}^{M-1} (-)^j c_j \left( \frac{a}{\pi^2 n^2} \right)^{2m+j-\frac{1}{2}} e^{-\pi^2 n^2/a} + R_M \tag{3.6} \]
by (2.1), where
\[ R_M = \frac{1}{2\pi i} \int_L \rho_M(s) \Gamma(s + \vartheta - M) \left( \frac{a}{\pi^2 n^2} \right)^s ds. \]

Bounds for the remainder \( R_M \) have been considered in [3, p. 71, Lemma 2.7], where it is shown that
\[ R_M = O \left( \left( \frac{a}{\pi^2 n^2} \right)^{M-\vartheta} e^{-\pi^2 n^2/a} \right) \tag{3.7} \]
as \( a \to 0 \) in the sector \( |\arg a| < \frac{1}{4} \pi \).

Collecting together the results in (3.2), (3.4), (3.6) and (3.7), we obtain
\[ I_L = (-)^m \left( \frac{a}{\pi} \right)^{2m-\frac{1}{2}} \sum_{n=1}^{\infty} e^{-\pi^2 n^2/a} n^{2m} \left\{ \sum_{j=0}^{M-1} c_j \left( \frac{-a}{\pi^2 n^2} \right)^j + O \left( \left( \frac{a}{\pi^2 n^2} \right)^M \right) \right\}. \]

From (3.1) we now have the following theorem:

**Theorem 1.** Let \( m \) and \( M \) be positive integers. Then, when \( w = 2m \), we have the expansion valid as \( a \to 0 \) in \( |\arg a| < \frac{1}{4} \pi \)
\[ S(a; 2m) = \frac{1}{2} \Gamma(\frac{1}{2} - m) a^{m-\frac{1}{2}} + \sum_{k=0}^{m} \frac{(-)^k}{k!} \zeta(2m - 2k) a^k \]
\[ +(-)^m \left( \frac{a}{\pi} \right)^{2m-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\Upsilon_n(a; m)}{n^{2m}} e^{-\pi n^2/a}, \tag{3.8} \]

where \( \Upsilon_n(a; m) \) has the asymptotic expansion

\[ \Upsilon_n(a; m) = \sum_{j=0}^{M-1} \frac{(m+\frac{1}{2})_j}{j!} \left( -\frac{a}{\pi} n^2 \right)^j + O \left( \left( \frac{a}{\pi^2 n^2} \right)^M \right). \]

This is the analogue of the Poisson-Jacobi transformation in (1.1). In the case \( m = 0 \), the quotient of gamma functions in (3.5) is replaced by the single gamma function \( \Gamma(s+\frac{1}{2}) \), with the result that \( c_0 = 1, c_j = 0 \) \((j \geq 1)\) and \( \Upsilon_n(a; m) = 1 \) for all \( n \geq 1 \). Then (3.8) reduces to (1.1) and is valid for all values of the parameter \( a \) (not just \( a \to 0 \)) satisfying \(|\arg a| < \frac{1}{4} \pi\).

**Remark 1.** We note that the values of the zeta function appearing in (3.8) can be expressed alternatively in terms of Bernoulli numbers by the result [2, p. 605]

\[ \zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}|. \]

4. Numerical results and concluding remarks

From the well-known values [2, p. 605]

\[ \zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \]

we obtain from Theorem 1 the expansions in the cases \( m = 1 \) and \( m = 2 \) given by

\[ S(a; 2) = \frac{\pi^2}{6} + \frac{a}{2} - (\pi a)^{\frac{3}{2}} - \frac{a^2}{4} \sum_{n=1}^{\infty} \frac{e^{-\pi^2 n^2/a}}{n^2} \left\{ \sum_{j=0}^{M-1} \frac{(\frac{5}{2})_j}{j!} \left( -\frac{a}{\pi^2 n^2} \right)^j + O(a^M) \right\} \tag{4.1} \]

and

\[ S(a; 4) = \frac{\pi^4}{90} - \frac{\pi^2 a}{6} - \frac{a^2}{4} + \frac{2}{5} \pi^2 a^2 \]

\[ + \left( \frac{a}{\pi} \right)^{\frac{3}{2}} \sum_{n=1}^{\infty} \frac{e^{-\pi^2 n^2/a}}{n^4} \left\{ \sum_{j=0}^{M-1} \frac{(\frac{5}{2})_j (2)_j}{j!} \left( -\frac{a}{\pi^2 n^2} \right)^j + O(a^M) \right\} \tag{4.2} \]

valid as \( a \to 0 \) in \(|\arg a| < \frac{1}{4} \pi\).

In Table 1 we show the results of numerical calculations for the case \( m = 2 \). For different values of the parameter \( a \) we present the value of the absolute error in the computation of \( S(a; 4) \) from (4.2). In the computations, we have used only the \( n = 1 \) term (since the order of \( \frac{1}{4} \exp(-4\pi^2/a) \) was found to be less than the error), with the expansion for \( \Upsilon_1(a; 2) \) optimally truncated (corresponding to truncation at, or near, the least term in modulus) at index \( j_0 \approx \left( \pi^2/a \right)^{-\frac{1}{2}} \). It is seen that the error when \( a = 0.1 \) is extremely small and that, only when \( a \approx 2 \) does the relative error start to become significant.

To conclude, we mention that a similar treatment can be carried out for the sum

\[ S_p(a; w) \equiv \sum_{n=1}^{\infty} \frac{e^{-an^p}}{n^w} \quad (a \to 0, \Re(a) > 0) \]
for positive even integer $w$ and $p$. The case $w = 0$ and $p > 0$, corresponding to the Euler-Jacobi series, has been considered in [3, §8.1]; see also [1] for a hypergeometric approach when $p$ is a rational fraction. The details of the small-$a$ expansion of $S_p(a; w)$ will be presented elsewhere.

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**Appendix: A bound for the remainder $R_N$**

Let $\psi = \arg a$ and integer $N > \frac{1}{2} w + \frac{1}{2}$. Upon replacement of $s$ by $-s$ followed by use of (3.3), the remainder $R_N$ in (2.3) becomes

$$R_N = \frac{(2\pi)^w}{2\pi i} \int_{N - \frac{1}{2} - i\infty}^{N - \frac{1}{2} + i\infty} \zeta(1 - w + 2s) \frac{\Gamma(1 - w + 2s)}{\Gamma(1 + s)} \sin \pi s \left( \frac{a}{4\pi^2} \right)^s ds.$$

With $s = N - \frac{1}{2} + it, t \in (-\infty, \infty)$ we have

$$|R_N| \leq (2\pi)^{w-1} \left( \frac{a}{4\pi^2} \right)^{N - \frac{1}{2}} \zeta(2N - w) \int_{-\infty}^{\infty} e^{-\psi t} \left| \frac{\Gamma(2N - w + 2it)}{\Gamma(N + \frac{1}{2} + it)} \right| dt,$$

since $|\zeta(x + it)| \leq \zeta(x)$ ($x > 1$) and

$$\left| \frac{\sin \pi(N - \frac{1}{2} - \frac{1}{2} w + it)}{\sin \pi(N + \frac{1}{2} + it)} \right| = \left| \frac{\cos \pi(\frac{x}{2} w - it)}{\cosh \pi t} \right| = \left( \frac{\cos^2 \frac{1}{2} \pi \sinh \pi t}{\cosh \pi t} \right)^{\frac{1}{2}} \leq 1.$$

It then follows that

$$|R_N| = O \left( \left( \frac{a}{\pi^2} \right)^{N - \frac{1}{2}} \int_{-\infty}^{\infty} e^{-\psi t} \left| \frac{\Gamma(N - \frac{1}{2} w + it)}{\Gamma(N + \frac{1}{2} + it)} \right| dt \right). \quad (A.1)$$

Using the argument presented in [3, p. 126], we set $N - \frac{1}{2} w - \frac{1}{2} = M + \delta$, with $-\frac{1}{2} < \delta \leq \frac{1}{2}$ so that $M \leq N - 1$, to find

$$\left| \frac{\Gamma(N - \frac{1}{2} w + \frac{1}{2} + it)}{\Gamma(N + \frac{1}{2} + it)} \right| = P(t)g(t), \quad g(t) := \frac{\left| \Gamma(1 + \delta + it) \right|}{\left| \Gamma(\frac{1}{2} + it) \right|},$$

| $a$ | $S(a; 4)$ | Error | $j_0$ |
|-----|------------|-------|------|
| 0.10 | 0.952696 | $9.662 \times 10^{-86}$ | 96 |
| 0.20 | 0.849025 | $9.768 \times 10^{-43}$ | 46 |
| 0.25 | 0.803169 | $4.045 \times 10^{-34}$ | 36 |
| 0.50 | 0.615128 | $7.769 \times 10^{-17}$ | 17 |
| 0.75 | 0.475493 | $4.656 \times 10^{-11}$ | 10 |
| 1.00 | 0.369026 | $3.642 \times 10^{-8}$ | 6 |
| 1.50 | 0.223285 | $2.856 \times 10^{-5}$ | 3 |
| 2.00 | 0.135356 | $7.500 \times 10^{-4}$ | 1 |
where
\[ P(t) = \left( \frac{1}{4} + t^2 \right)^{-\frac{1}{2}} \prod_{r=1}^{M} \left( \left( r + \delta \right)^2 + t^2 \right)^{-\frac{1}{2}} \leq \left( \frac{1}{4} + t^2 \right)^{-\frac{1}{2}} \leq 2. \]

From the upper bound for the gamma function \( \Gamma(z) \) with \( z = x + it, \ x > 0 \) [3, p. 35]
\[
|\Gamma(z)| \leq |\Gamma(x)| \exp \left[ x \left\{ \frac{1}{2} \pi \pm \psi \right\} e^{-\frac{1}{2} \pi |t|} \right],
\]
where we have put \( \tau = t/x \), defined \( \omega(\tau) = |\tau| \arctan(1/|\tau|) \) and used the fact that \( 0 \leq \omega(\tau) < 1 \) for \( \tau \in [0, \infty) \), with the limit 1 being approached as \( \tau \to \infty \). Substituting the above bounds into (A.1), we see on setting \( x = N - \frac{1}{2} w \) that
\[
|R_N| = e^N \Gamma(N-\frac{1}{2}w+1) \times \left( \frac{a}{\pi} \right)^{N-\frac{1}{2}} \int_0^\infty (1 + \tau^2)^{N/2} g(\tau) \left\{ e^{-\Delta^+} + e^{-\Delta^-} \right\} d\tau,
\]
where \( \Delta_{\pm} = (N - \frac{1}{2} w) (\frac{1}{2} \pi \pm \psi) \). Since \( g(\tau) = O(\tau^{d+\frac{1}{2}}) \) as \( \tau \to \infty \), the integral is convergent provided \( |\psi| < \frac{1}{2} \pi \) and is manifestly an increasing function of \( N \).

Hence
\[
R_N = O(a^{N-\frac{1}{2}}) \quad (a \to 0, \ |\arg a| < \frac{1}{2} \pi), \tag{A.2}
\]
with the constant implied in the \( O \)-symbol growing at least like \( \Gamma(N+1-\frac{1}{2}w) \) as \( N \) increases.

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