Schwinger–Dyson Equation for Quarks in a QCD Inspired Model

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Abstract—We discuss formulation of QCD in Minkowski–spacetime and effect of an operator product expansion by means of normal ordering of fields in the QCD Lagrangian. The formulation of QCD in the Minkowski–spacetime allows us to solve a constraint equation and decompose the gauge field propagator in the sum of an instantaneous part, which forms a bound state, and a retarded part, which contains the relativistic corrections. In Quantum Field Theory, for a Lagrangian with unordered operator fields, one can make normal ordering by means of the operator product expansion, then the gluon condensate appear. This gives us a natural way of obtaining a dimensional parameter in QCD, which is missing in the QCD Lagrangian. We derive a Schwinger–Dyson equation for a quark, which is studied both numerically and analytically. The critical value of the strong coupling constant \( \alpha_s = \frac{\pi}{4} \), above which a nontrivial solution appears and a spontaneous chiral symmetry breaking occurs, is found. For the sake of simplicity, the considered model describes only one flavor massless quark, but the methods can be used in more general case. The Fourier-sine transform of a function with log-power asymptotic was performed.

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1. INTRODUCTION

Strong interaction physics should be described by Quantum Field Theory (QFT) with the Quantum Chromodynamic (QCD) Lagrangian [1–6]. As it shown [5, 6], the running coupling constant \( \alpha_s \) is strong enough at small energies, so that perturbation expansion is not applicable. This is a big problem due to the lack of general methods of non-perturbative calculations.

In order to describe the strong interaction, the phenomenological models were developed that are based directly on experimental data and use partly the QCD knowledge: the QCD sum rules [7–9], the Chiral Perturbation Theory [10–13], the Nambu–Jona-Lasinio model and its generalizations [14–20], bag models [21–23], and others [24, 25]. These models can relatively easily reproduce experimental data. However, they have a number of disadvantages: each of these models works in a certain application area but fail in others, the accuracy of theoretical calculations are limited and often less than the accuracy of modern experimental data. And these models are not true theory of strong interactions. This gives impetus to construct models based directly on QCD, for instance: instanton liquid model [26–29], domain wall network [30–35], various estimations from Schwinger–Dyson equations [36–42].

Thus, there exist various approximations to the theory of strong interactions with their specific simplifications of the QCD. In the QCD researches, one should find answers to the key questions, which are the description of QCD vacuum, spontaneous breaking of chiral symmetry, the absence of color particles (confinement problem), the description of bound states, their masses and decay widths.

We consider the theory of strong interaction at low energy. Our aim is to emphasize the importance of formulation in Minkowski-spacetime and effect of an operator product expansion by means of normal ordering of fields in Lagrangian, and to discuss some consequences of this novel approach.

The formulation of QCD in the Minkowski–spacetime allows us to solve a constraint equation and decompose the gauge field propagator in the sum of an instantaneous part, which forms a bound state, and a retarded part, which contains the relativistic corrections. At the first stage, we should neglect the retarded part and use the instantaneous part to construct the bound state. Then the retarded part gives corrections to the already existing bound state. This idea comes from QED [43] (see also [44–46]), where any attempts of working with the entire propagator do not lead to satisfactory results or the decomposition occurs implicitly. This method enables us to cover both high- and low-energy ranges and find the relation between fundamental QCD parameters and low...
energy constants. Our approach differs from [47, 48], in which the dispersion relations are used to study the fermion propagator in the Minkowski-spacetime.

In QFT, for a Lagrangian with unordered operator fields, one can make normal ordering by means of the operator product expansion. Then the gluon condensate and a low energy effective gluon mass appear. This mechanism gives us a natural and fundamental way of obtaining a dimensional parameter in QCD, which is missing in the QCD Lagrangian. The existence of non-zero condensates directly linked to the conformal anomaly of QCD.

In the next section, we start from QCD Lagrangian and derive an effective action of strong interaction. Then in Section 3, by using this effective action we obtain the Schwinger–Dyson equation for a quark, which is solved both numerically (in Section 4) and analytically (in the subsequent sections). In conclusion, we summarize the obtained results and discuss the prospects of the developed methods. Here, for the sake of simplicity, we intentionally neglect some effects, for example, the considered model describes only one flavor massless quark. While investigating of the Schwinger–Dyson equation, we focus mainly on the question of the spontaneous symmetry breaking. Nevertheless, all the assumptions made to derive the equation are transparent and well-controlled.

2. EFFECTIVE ACTION FOR THE STRONG INTERACTION

Let us start with the Quantum Chromodynamics Lagrangian, with number of colors \( N_c = 3 \) and number of flavors \( N_f = 1 \):

\[
\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu j^\mu + \overline{\psi} i\gamma^\mu \partial_\mu \psi - m \psi,
\]

where \( A_\mu \), \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g f^{abc} A^b_\mu A^c_\nu, \) \( \psi \), \( m \), and \( j^\mu = -g \sigma^\mu A_\mu \psi \) are the gluon field, gluon field strength tensor, quark field, quark current mass, and color current of quark, respectively.

An effective action for the meson-like bound state can be derived from the Lagrangian (1). To this end, some restrictions and assumptions are needed. Below the symbol • is introduced for convenience when we discuss another one assumption or restriction. Some of the restrictions are not principle but imposed in order to not overload the reader by technical calculations. Anyway, in the developed model, we outline main ideas that may be important for correct description of meson-like bound state rather than give a complete description of strong interaction, which certainly remains a tremendous problem.

• First, we choose the frame of reference where the bound state, which we obtain and discuss below, is at rest. Therefore only the static problems are considered. We emphasize that the proper choice of the reference frame should be done in Minkowski-spacetime rather than in Euclidian-spacetime. Note that the generalization of this theory to one bound state moving on mass shell (and thus for an arbitrary frame of reference) can easily be done [45, 46, 49–51], it is sufficient to rewrite various quantities in the comoving frame of reference.

We fix the gauge

\[
\partial_\xi A^\xi_\mu (x) = 0,
\]

where \( k = 1,2,3 \) and \( a = 1,\ldots,8 \) are the space and gluon color indexes, respectively.

The gluon term in the Lagrangian takes the form

\[
-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2} A^a_\mu A_a^\mu - \frac{1}{4} F^a_\mu F^{a\mu} + g f^{abc} (\partial_\mu A^b_\nu) A^c_\nu A^a_\mu
\]

\[
-\frac{1}{2} A^a_\mu \partial_\mu A^b_\nu - g f^{abc} (\partial_\mu A^b_\nu) A^c_\mu A^a_\nu + \frac{1}{2} g^2 f^{abc} f^{ade} A^b_\mu A^d_\nu A^c_\mu A^e_\nu.
\]

The third term on the right hand side contains the time derivative and thus can be neglected, because only the static problems are considered, as is noted above.

After quantization, the gluon field \( A^a_\mu \) becomes an operator field. One can consider the vacuum 2-point correlator

\[
\langle 0 | A^a_\mu (x) A^a_s (x) | 0 \rangle = 2 C_g \delta^a_s \delta^{ab},
\]

• We assume that \( C_g \neq 0 \) and \( C_g < \infty \). Actually \( C_g \) depends on the energy, but for simplicity we suppose \( C_g \) to be a constant. The constant \( C_g \) can be determined from a phenomenology.

The fields in Eq. (4) obeys the condition (2). A question at once arises: “How should the formula (4) be rewritten in other gauges?” The answer is to make a gauge transform to (2), then impose the condition (4), and then make the inverse gauge transform. As is said above, when solving the on-shell bound state problem, we always have a privileged frame of reference, in which this bound state as whole is at rest; therefore, we always have the privileged gauge (2), and thus we define (4) in a gauge-covariant manner in this way. Note that physically privileged reference frame is absent for any scattering problem of quarks and gluons, and one cannot define (4) the same way.

Usually in Quantum Field Theory, a Lagrangian contains only normally ordered operator fields. This is a result of normal ordering of an initial Lagrangian where the above correlator-like terms, arising due to the ordering, are omitted, because they are considered
as (infinite) vacuum energy contributions. Keeping these terms, we have after the normal ordering

\[ -\frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} = \frac{1}{2} A^a_{\nu} A^a_{\mu} : - \frac{1}{4} F^a_{\mu} F^{a \mu} : + \frac{1}{2} A^a_{\nu} (-\Delta + M^2) A^a_{\mu} : \ldots \tag{5} \]

The term with \( M^2 \) comes from the last term of formula (3) and \( M^2_g = 6g^2 C_g N_c \). Here we use the relation

\[ f_{abc} f_{acd} = N_c \delta^{bd} . \]

Thus the appearance of this term is the consequence of non-Abelian nature of QCD. The quantity \( M^2 \) might be interpreted as an effective gluon mass in the gauge (2). This is essentially a model-dependent quantity. Phenomenological models in which gluons have nonzero effective mass at small energies have been considered earlier by some authors (see [42, 52–59] and references therein). In our approach, the gluon mass appears before a perturbation expansion. At the level of the diagrammatic point of view, the normal ordering of fields (4) in the Lagrangian (5) is the nonperturbative consideration of the tadpole-type diagrams of gluons.

• Let us consider the dotted terms in (5) as a perturbation and neglect them. This assumption means that bound states are formed by only some of the terms which explicitly written in expression (5), while the other terms merely give some corrections to the already existing bound states. In the basic model developed in this paper, these terms are neglected. The neglected terms can influence on quantitative characteristics of the bound states, but not their presence, and numerical amount of corrections might be not small due to large value of strong coupling constant.

Substituting (5) without dotted terms into Lagrangian (1), we arrive at the generating functional

\[ \mathcal{F} = \int \mathcal{D} A^a_{\mu} \delta(\partial_x A^a_{\mu}) \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \left[ i \int d^4 x \left( \frac{1}{2} A^a_{\nu} A^a_{\mu} - \frac{1}{4} F^a_{\mu} F^{a \mu} + \frac{1}{2} A^a_{\nu} (-\Delta + M^2) A^a_{\mu} \right) \right. \]

\[ - \left. \frac{1}{2} A^a_{\nu} F^a_{\mu} F^{a \mu} + A^a_{\nu} j^a_{\mu} + \bar{\psi}(\gamma^\mu \partial_x - m) \psi \right) \]

\[ + i \int d^4 x (A^a_{\nu} J^{a \nu} + \bar{\psi} \gamma_\mu + \psi \gamma_\mu) \right] . \]

The source \( J^a_{\nu} \) is not involved, since the field \( A^a_{\mu} \) is not dynamical degree of freedom with the gauge (2). This is owing to the fact that the corresponding equation of motion is a constraint [60]. Note that the effective gluon mass is taken into account only for \( A^a_{\nu} \), while \( A^a_{\mu} \) are interpreted as massless. For our purposes, it is enough to consider only explicitly written terms in the generating functional. We do not pretend to describe the confinement. Full description of Coulomb gauge QCD is a much harder problem, see for example [3, 41, 61, 62].

Making integration over \( A^a_{\mu} \) yields

\[ \mathcal{F} = \int \mathcal{D} A^a_{\mu} \delta(\partial_x A^a_{\mu}) \mathcal{D} \bar{\psi} \mathcal{D} \psi \exp \left[ i \int d^4 x \left( \frac{1}{2} A^a_{\nu} A^a_{\mu} \right. \right. \]

\[ - \frac{1}{4} F^a_{\mu} F^{a \mu} + A^a_{\nu} j^a_{\mu} + \bar{\psi}(\gamma^\mu \partial_x - m) \psi \right) \]

\[ - \frac{1}{2} \int d^4 x d^4 y j^a_{\nu}(x) \delta(x - y) \frac{1}{4 \pi} \frac{e^{-M_{g|x-y|}}}{|x - y|} f^a(y) \]

\[ + i \int d^4 x (A^a_{\nu} J^{a \nu} + \bar{\psi} \gamma_\mu + \psi \gamma_\mu) \right] . \]

The term

\[ - \frac{1}{2} \int d^4 x d^4 y j^a_{\nu}(x) \delta(x - y) \frac{1}{4 \pi} \frac{e^{-M_{g|x-y|}}}{|x - y|} f^a(y) \]

includes a combination of the Gell-Mann matrices, which may be rewritten in the form

\[ \frac{\lambda^a s_{r s \nu}}{2} = \frac{1}{3} \lambda^a s_{r s} + \frac{1}{6} e^{r s} e^{s r} . \]

• We restrict ourselves to the colorless mesons and so neglect the second term. This term is the diquark channel, which plays a role when baryons are taken into account (“baryon = diquark + quark”).

Thus within this approximation, the above term can be rewritten in the form

\[ - \frac{1}{2} \int d^4 x d^4 x d^4 y d^4 y \bar{\psi}(x_1) \psi(x_2) \delta(x_1 - x_2) \frac{2}{12 \pi} \frac{e^{-M_{g|x-y|}}}{|x_1 - y_2|} \delta(y_1 - y_2) Y^0_{\beta_1} \]

\[ \times \frac{\gamma^0 \alpha \beta_1}{\beta_2} (x_1; x_2; y_1; y_2) \psi^\alpha_{\beta_2} (x_1, y_1) \]

\[ = - \frac{1}{2} \int d^4 x d^4 x d^4 y d^4 y \bar{\psi}(x_1) \psi(x_2) \delta(x_1 - x_2) \psi^\alpha (x_1, y_1) \]

\[ \times \frac{\gamma^0 \alpha \beta_1}{\beta_2} (x_1; x_2; y_1; y_2) \psi^\alpha (x_1, y_1) \psi^\beta (x_2, y_2) , \]

where the above formula is the definition of the operator \( \gamma^{x \beta_1}_{\beta_2} (x_1; x_2; y_1; y_2) \), and \( \psi^\beta (x_1, y_1) \) was shifted to the left. One can see that color indexes \( r \) and \( s \) have been summed inside pairs \( \bar{\psi} \psi \), so the pair \( \bar{\psi} \psi \) as whole is colorless.

• Let us treat \( \psi^\alpha (x_1, y_1) \bar{\psi} (x_1, y_1) \) as a real bilocal field.

The operator \( \gamma^{x \beta_1}_{\beta_2} (x_1; x_2; y_1; y_2) \) is symmetrical and has an inverse operator \( \gamma^{-1} \) that can be defined by:

\[ \int d^4 x_1 d^4 x_2 d^4 y_1 d^4 y_2 \gamma^{x \beta_1}_{\beta_2} (x_1; x_2; y_1; y_2) \gamma^{-1}_{x \beta_1} (x_2; x_3; y_2; y_3) \]

\[ = \delta^4 (x_1 - x_3) \delta^4 (y_1 - y_3) \delta_{\beta_1 \beta_2} (x_2; x_3; y_2; y_3) . \]
This allows us to introduce new bilocal field $M^\alpha_\beta(x^0, x, y)$ and make a bosonization transform (Habbard–Stratanovich transform) \cite{62}: 

$$\exp\left(\frac{1}{2} \int d^4 x_1 d^4 y_1 d^4 x_2 d^4 y_2 \delta(x_1 - y_1) \delta(x_2 - y_2) \right)$$

$$= \int \mathcal{D} \mathcal{M} \exp \left[ \int \mathcal{D} \psi \psi^\dagger \mathcal{H} \right]$$

$$= \int \mathcal{D} \mathcal{M} \exp \left[ \int \mathcal{D} \psi \psi^\dagger \mathcal{H} \right]$$

Finally the generating functional for effective action of strong interaction takes the form

$$\mathcal{F} = \int \mathcal{D} A^a_k \delta(\partial_k A^a_k) \mathcal{D} \psi \mathcal{D} \bar{\psi} \mathcal{D} \mathcal{M} \exp \left[ \int \mathcal{D} x \left( \frac{1}{2} A^a \cdot A^a - \frac{1}{4} F^{a} F^{a} + \mathcal{L} \right) + \mathcal{L} \right]$$

According to this method, we should integrate out the Fermion variables $\psi$ and $\bar{\psi}$ in (6), thus deriving the functional for the action $S_{\text{eff}}$

$$\mathcal{F} = \int \mathcal{D} A^a_k \delta(\partial_k A^a_k) \mathcal{D} \psi \mathcal{D} \bar{\psi} \mathcal{D} \mathcal{M} \exp \left[ \int \mathcal{D} x \left( \frac{1}{2} A^a \cdot A^a - \frac{1}{4} F^{a} F^{a} + \mathcal{L} \right) + \mathcal{L} \right]$$

Then we arrive at the Schwinger–Dyson equation

$$\delta S_{\text{eff}}(A^a_\mu) = 0, \quad \bar{\eta} = 0, \quad \eta = 0, J = 0 = 0, J = 0$$

which gives us the Fermion spectrum inside the bound state \cite{45, 63, 64, 66, 69–72}.

We introduce the operator $G^{-1}_{\text{m}4(\text{f})}(x, y) = \left( \delta^{a} \delta^{c} \delta^{b} \delta^{d} + g A^a_i A^b_j \right)$

and define its inverse as

$$\int d^4 y G^{-1}_{\text{m}4(\text{f})}(x, y) G_{\text{m}4(\text{f})}(x, y, z) = \delta^{a} \delta^{c} \delta^{b} \delta^{d}(x - z).$$

In this notations, formula (7) reads

$$\mathcal{F} = \int \mathcal{D} A^a_k \delta(\partial_k A^a_k) \mathcal{D} \psi \mathcal{D} \bar{\psi} \mathcal{D} \mathcal{M} \exp \left[ \int \mathcal{D} x \left( \frac{1}{2} A^a \cdot A^a - \frac{1}{4} F^{a} F^{a} \right) \right]$$

Inserting the corresponding $S_{\text{eff}}$ into Eq. (8) we arrive at

$$\int d^4 x d^4 y \mathcal{H}^{1}_{\text{m}4(\text{f})}(x, y) \mathcal{H}^{2}_{\text{m}4(\text{f})}(x, y, z) = \delta^{a} \delta^{c} \delta^{b} \delta^{d}(x - z).$$

\section{3. SCHWINGER–DYSON EQUATION}

In what follows we restrict ourselves only to the question of spontaneous symmetry breaking in the theory described by the functional (6). For this purpose, it is convenient to derive and investigate the Schwinger–Dyson (Gap) equation for the quark.

It is difficult to examine the Schwinger–Dyson equation in the general form.

\footnote{Moreover, in gauge (2) the ghosts can be removed by the certain transformation \cite{62}.}

• For the sake of simplicity, we use the Stationary Phase method (that is the Semiclassical approximation). This method simplify the Schwinger–Dyson equation but retain its main properties.
Below in this article, the solution of this equation is denoted by
\[ M_{\beta}(x^0, x, y) = -\Sigma_\beta(x^0, x, y) + m \delta^3(x - y). \]

It is convenient to introduce the operator
\[ G_{\alpha\beta}(x, y) \equiv \gamma^\alpha \gamma^\beta \delta^4(x - y) - \Sigma_\beta(x, y) \delta^3(x - y), \]
which, as the stationary solutions obeying Eq. (9), coincides with the earlier introduced operator
\[ G^{-1}_{\alpha\beta}(x, y) = G_{\alpha\beta}(x, y) \delta^3(x - y). \]

The inverse operator is defined in the standard manner
\[ \int d^4 y G_{\alpha\beta}(x, y) G_{\gamma\beta}(y, z) = \delta_\alpha^\gamma \delta^4(x - z). \]

Acting with the operator \( \mathcal{K} \) on the both sides of Eq. (9) and using the above notations, we obtain
\[ \Sigma_\beta(x_0^0, x, y_1) = m \delta_{\beta_1}^\alpha \delta^3(x_1 - y_1) + 3 \int d^4 x_2 d^4 y_2 \times \gamma^{\alpha_1} \gamma^{\alpha_2} G_{\alpha_2\beta_2}(x_2, y_2) \delta(x_2^0 - y_2^0). \]

This is the equation on \( \Sigma_\beta(x^0, x, y) \), where in the right-hand side \( \Sigma \) enter into the equation via the operators (10) and (11).

- We are looking for a simplest solution of this equation and adopt the following ansatz
  \[ \Sigma_\beta(x^0, x, y) = \delta_{\beta}^\alpha \frac{1}{(2\pi)^3} M(x - y). \]

Due to the isotropy, \( M \) is radially symmetric and depends only on \( |x - y| \). It follows from the formula (10) that the \( M \) can be interpreted as homogeneous isotropic time-independent quark mass.

Making the Fourier transform of Eq. (12), we have
\[ M(p) \delta^\alpha_{\beta} = m \delta^\alpha_{\beta} - i \frac{g^2}{(2\pi)^3} \int d^4 q \frac{1}{(p - q)^2 + M_s^2} \gamma^{\alpha}_{\beta}\gamma^\beta. \]

In momentum space, the operator \( G^{-1}_{\alpha\beta} \) can easily be reversed
\[ G_{\alpha\beta}(q) = e^{-\frac{\gamma^\alpha_{\beta}(q)}{g_0} \left( \frac{1}{q_0 + E(q) - i\varepsilon} + \frac{1}{q_0 - E(q) + i\varepsilon} \right)} \gamma^\beta \gamma^\alpha \gamma^\beta, \]
where we put by definition \( E(q) \equiv \sqrt{M(q)^2 + q^2} \), and:
\[ \cos 2\phi(q) = \frac{M(q)}{E(q)}. \]

One can introduce instead of quark charge \( g \) a strong coupling constant \( \alpha_s \equiv g^2/(4\pi) \). It is well known in QCD \( \alpha_s \) is a running coupling, whose value strongly dependent of the energy scale. Moreover, at low energies, the dependence of momentum \( \alpha_s(p) \) can not be calculated from the perturbation theory, which is inapplicable due to the large value of \( \alpha_s \). In the literature, there exist various predictions about the shape of \( \alpha_s(p) \) (see, e.g., [25, 37–41, 73] and references therein). Nevertheless in this paper, we assume that \( \alpha_s \) is a constant; which is consistent with neglecting corrections to the bound states; this means in particular neglecting all the loop corrections to \( \alpha_s \), and \( \alpha_s \) is really a constant in the framework of this approach. So in a way, the used in this article constant \( \alpha_s \) can be understood as an average of the strong coupling \( \alpha_s(p) \) over \( p \) within a low-momentum range.

As discussed in detail above, we restrict our consideration to the static problems and only to such terms of (5) which forms a bound state, and we use only Coulomb gauge (2). As a consequence of this, the Schrödinger–Dyson Eq. (14) turned out to be oversimplified: it does not depend on \( p^0 \) and \( q^0 \). Whereas the
solution of the full Schwinger–Dyson equation has a nontrivial dependency on the time variable. This solution can be obtained by the perturbation theory, taking the solution of equation (14) as the zeroth approximation. However one may expect that, as instantaneous approximation leads to potential approach (see for example for similar models \([43, 45, 46, 69]\)), usage of the solution of Eq. (14) in the further calculation of the static properties of the bound states (the spectrum and the wave functions), leads to significant simplification of calculations (for example in comparison with works \([39, 41, 62]\)) at a satisfactory result. And as all important for forming of bound state terms have taken into account, one can expect that perturbation theory corrections make contributions of the order of the coupling constant \(\frac{\alpha}{4\pi}\).

- We solve Eq. (14) only for \(m = 0\), which can be justified by the phenomenology. Indeed, \(m \ll M(0)\), because the current mass of light \(u\) and \(d\) quarks is about 5 MeV. On the other hand, the constituent mass of the same quarks is of order 300 MeV for different models.

- One can demand \(M(q) \to 0\) when \(q \to \infty\). Due to the asymptotic freedom at large momenta, the running quark mass tends to the current mass. Although the existence of the asymptotic freedom in our model is questionable, we do not want to violate it explicitly. In addition, if this restriction is fulfilled then Eq. (14) does not need any regularization and renormalization.

Indeed, at large \(q\) the integrand takes form \(\frac{4p}{q} M(q)\) (factor \(\frac{4p}{q}\) comes from expansion of the logarithm) and the integral converges at the upper limit.

It is convenient to introduce the dimensionless variables \(\bar{p} \equiv p/M_g\), \(\bar{q} \equiv q/M_g\), and \(\bar{M}(\bar{p}) \equiv M(p)/M_g\). In this variables, the Schwinger–Dyson Eq. (14) takes the form

\[
\bar{M}(\bar{p}) = \frac{e^2}{4\pi^2} \frac{1}{\bar{p}} \times \int_0^\infty dq \bar{q} \frac{M(q)}{\sqrt{M^2(q) + \bar{q}^2}} \ln \left( \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right).
\]

(15)

It is obvious that there always exists the solution \(\bar{M}(\bar{p}) = 0\). Of course, we are looking for a nontrivial solution of this equation.

An attractive feature of the Schwinger–Dyson Eq. (15) is that it is controlled only by one external parameter \(g\), which should be fixed from the phenomenology. In particular, it follows from the definition of the dimensionless variables and Eq. (15) that the constituent quark mass is a linear function of \(M_g\) and the coefficient of proportionality \(c(g)\) depends only on \(g\): \(M(0) = c(g)M_g\). That is just a consequence of the fact that in the absence of the current quark mass \(m\) and without regularization the only dimensional parameter is \(M_g\). It should be noted that there are also other dimensional parameters—the scales at which the constants \(g\) and \(M_g\) begin to change significantly (However, since only one dimensional parameter can be expected in the chiral limit of QCD, all these dimensional parameters should be connected to each other.). Therefore, for the model considered in this paper, the relationship between \(M(0)\) and \(M_g\) can be considered only as approximate, assuming that the characteristic scale of change \(M(p)\) is much smaller than the characteristic scales of change \(g\) and \(M_g\).

4. NUMERICAL SOLUTION OF THE SCHWINGER–DYSON EQUATION

To solve Eq. (15) numerically, we use the following algorithm. Let us take a zeroth-order approximation function \(\bar{M}_0(\bar{p})\), it is desirable that \(\bar{M}_0(\bar{p})\) differs from the solution \(\bar{M}(\bar{p})\) not much. Then substituting \(\bar{M}_0(\bar{p})\) into the integral in the right-hand side (15) we get \(\bar{M}_1(\bar{p})\) in the left-hand side. Then \(\bar{M}_1(\bar{p})\) is substituted again, and so on. After a certain number of steps we get, up to the errors of computer calculations, the exact solution \(\bar{M}(\bar{p})\) for which the substitution into the right-hand side (15) gives itself. Strictly speaking, we should prove that this algorithm is convergent. We did not try to prove this because in all cases that we calculated this algorithm turned out to be convergent. Moreover, at different choices of zeroth-order approximation function \(\bar{M}_0(\bar{p}) > 0\), the same final answer \(\bar{M}(\bar{p})\) gets (see below for details).

In some sense, the convergence of the algorithm can be explained by the stability of the solution under small perturbations. Namely, substituting the function \((1 + \epsilon)\bar{M}(\bar{p})\), where \(\epsilon \ll 1\), into integral (15) we have up to \(\epsilon^2\) terms

\[
\frac{e^2}{4\pi^2} \frac{1}{\bar{p}} \times \int_0^\infty dq \frac{\bar{q} M(q)}{\sqrt{M^2(q) + \bar{q}^2}} \ln \left( \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right) \\
= (1 + \epsilon)\bar{M}(\bar{p}) - \epsilon \frac{e^2}{4\pi^2} \frac{1}{\bar{p}} \times \int_0^\infty dq \frac{\bar{q} M(\bar{q})}{\sqrt{M^2(\bar{q}) + \bar{q}^2}} \ln \left( \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right).
\]

The last term is smaller than \(\epsilon M(\bar{p})\) and has a minus sign. That is why the obtained expression is closer to the solution \(\bar{M}(\bar{p})\) than \((1 + \epsilon)\bar{M}(\bar{p})\).

The zeroth-order approximation function \(\bar{M}_0(\bar{p})\), as well as kernel of the integral Eq. (15), are both strictly positive functions. Consequently, as we use the
straight iteration algorithm, the final solution $\bar{M}(\bar{p})$ is also positive. There are no finite points $\bar{p}^* < +\infty$ such that the solution becomes zero $\bar{M}(\bar{p}^*) = 0$ at this point.

After some attempts to solve Eq. (15) numerically, we have found that there are some things that should be avoided at numerical computation:

1) The upper limit of integration must be $+\infty$ and cannot be replaced by finite quantity $\Lambda$; otherwise a strong dependence of the solution from $\Lambda$ appears.

2) $\bar{M}(+\infty) = 0$, otherwise the integral diverges.

3) It is better to avoid replacing the continuous function $\bar{M}(\bar{p})$ by a discrete table $\bar{M}(\bar{p})$ with fixed numbers of points $\bar{p}$. That is because the value of $\bar{M}$ at the penultimate point (at the last point $\bar{M} = 0$, as it is noted above) depends mainly on the behavior of $\bar{M}(\bar{p})$ between this point and the end point and has a weak dependence on the values of $\bar{M}$ at the other points; the value of $\bar{M}$ at the next to penultimate point depends on the value of $\bar{M}$ at the penultimate point and the behavior of $\bar{M}(\bar{p})$ between these three points, and so on. One cannot approximate the behavior of the function $\bar{M}(\bar{p})$ between two points by a linear segment, otherwise this leads to very low accuracy of numerical calculations. Preferably, $\bar{M}(\bar{p})$ expands in a series of known functions. One may also add that maybe we have more accurate results than in [74], where a similar equation was considered numerically, and such replacing $\bar{M}(\bar{p})$ by the table $\bar{M}(\bar{p})$ was done.

Put by definition $\bar{M}(-\bar{p}) = \bar{M}(-\bar{p})$, then Eq. (15) can be rewritten in the form

$$\bar{M}(\bar{p}) = \frac{g^2}{2(4\pi)^2} \frac{1}{\bar{p}^2} \times \int_{-\infty}^{+\infty} d\bar{q} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \ln \left( \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} \right).$$

Let us define the new function

$$W(\bar{q}) \equiv \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}}. \tag{17}$$

One can easily see that $W(\bar{q})$ has the properties

$$\bar{q} \to +\infty : W(\bar{q}) = \bar{M}(\bar{q}),$$

$$\bar{q} > 0 : 0 \leq W(\bar{q}) \leq \min(\bar{q}, \bar{M}(\bar{q})), \tag{18}$$

$$W(-\bar{q}) = -W(\bar{q}). \tag{19}$$

New variables can be introduced (where $\lambda$ is some parameter)

$$\bar{p} = \lambda \tan \left( \frac{\phi}{2} \right), \quad \phi \in (-\pi, \pi),$$

$$\bar{q} = \lambda \tan \left( \frac{\theta}{2} \right), \quad \theta \in (-\pi, \pi).$$

In this variables the Schwinger–Dyson equation takes form

$$\bar{M}(\varphi) = \frac{g^2}{2(4\pi)^2} \int_{-\pi}^{+\pi} \frac{d\theta}{\tan \frac{\varphi}{2} \cos^2 \frac{\theta}{2}} \times \ln \left( 1 + \lambda^2 \left( \tan \frac{\varphi}{2} + \tan \frac{\theta}{2} \right) \right) W(\theta). \tag{20}$$

On $[-\pi, \pi]$ there is a convenient system of the Fourier series functions:

\[
\begin{align*}
\bar{M}(\varphi) &= \sum_{k=-\infty}^{+\infty} a_k \cos(k\varphi) \\
W(\theta) &= \sum_{k=-\infty}^{+\infty} b_k \sin(k\theta)
\end{align*}
\]

Using the Fourier series expansion we can avoid all the numerical difficulties which were discussed above. As the Fourier harmonics are periodical functions, it would be better if the area of the fastest change of the function lay closer to the center of the interval. The point $\theta = \frac{\pi}{2}$ corresponds to $\bar{q} = \lambda$, so it dictates the choice of $\lambda$. Of course, before the calculation we do not know what value should be taken; fortunately, the incorrect $\lambda$ leads only to hight inaccuracy and low speed of calculation. Equation (20) now takes the matrix form

$$a_k = A_{kj} b_j, \tag{21}$$

where $A_{kj} \equiv \frac{g^2}{32\pi^3} M_{kj}$, where

$$M_{kj} \equiv \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} d\varphi d\theta \frac{\cos(k\varphi)}{2\tan \frac{\varphi}{2} \cos^2 \frac{\theta}{2}} \times \ln \left( 1 + \lambda^2 \sin \varphi \sin \theta \right) \sin(j\varphi \sin \theta) \cos\left( \cos \frac{\varphi}{2} \cos \frac{\theta}{2} + \lambda \sin \frac{\varphi}{2} \theta \right) \sin(j\theta).$$

The matrix $M_{kj}$ contains only the parameter $\lambda$, and can be calculated separately. Of course in computa-
The zeroth-order approximation function \( W_0(\varphi) \) should obey at least condition (19). We tried various \( W_0(\varphi) \), which obey (19), and in all cases got the same results differing only in evaluation time. That is why one can take \( W_0(\varphi) = \varphi \).

Finally, the algorithm is the following.

This procedure gives the solution that is expanded into the Fourier series. We can check how well enough it is by substitution in the initial exact (not matrix (21)) Eq. (15) and comparison left- and right-hand sides of Eq. (15).

The result of numerical research is the following. There is only the trivial solution \( \bar{M}(\bar{p}) = 0 \) when \( g^2 < 16 \). The number 16 is exact and can be obtained from analytical estimations (see Section 5). At \( g^2 > 16 \) a nonzero solution appears. Examples of such calculation are shown in Fig. 1. One can see that the above-mentioned check is successful.

Unfortunately, due to a low accuracy of the numerical calculations, we can not obtain the precise value \( \bar{M}(0) \). Namely, Eq. (15) in the other dimensionless variables \( \bar{p} = p/M(0), \quad \bar{q} = q/M(0) \) and \( \bar{M}(\bar{p}) = M(p)/M(0) \), takes the form

\[
\bar{M}(\bar{p}) = \frac{g^2}{4\pi} \frac{1}{\bar{p}^2} \int_{\bar{p}}^{\infty} d\bar{q} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}}
\]

\[
\times \ln \left( 1 + \frac{4\bar{p} \bar{q}}{\bar{M}_g^2 + (\bar{p} - \bar{q})^2} \right),
\]

where \( \bar{M}_g = M_g/M(0) = 1/\bar{M}(0) \), and there is the condition \( \bar{M}(0) = 1 \). One can see that right-hand side of this equation depends on \( \bar{M}(0) \) only logarithmically, and when \( |\bar{q} - \bar{p}| \gg \bar{M}_g \) it does not depends on \( \bar{M}(0) \) at all. The solutions of the above equation are shown in Fig. 2 for \( g^2 - 16 = 2 \) and \( g^2 - 16 = 3 \).

The number \( N_h = 13 \) is taken because it is optimal with the computer resources available to the authors. For calculations with \( N_h > 13 \), machine round-off errors appear and, despite the presence of a larger number of harmonics, the accuracy of the calculation may fall uncontrollably, therefore it is necessary to use arbitrary-precision arithmetic. Examples of calculation at different \( N_h \) are shown in Fig. 3.

One can consider modification of Eq. (15) with some finite value \( \bar{\Lambda} \) of upper bound of integration over \( \bar{q} \), where we use the same dimensionless variables \( \bar{\Lambda} = \frac{\Lambda}{M_g} \). Above in this section, before Eq. (16), we mention that such modification of this equation changes the solution and strong dependence on \( \bar{\Lambda} \) appears. For calculation of this task our program can easily be changed: it is enough to recalculate the
matrix $M_{ij}$ (21) with some $\int_{-\theta_{\text{max}}}^{+\theta_{\text{max}}} d\theta$ instead of $\int_{-\pi}^{+\pi} d\theta$, where $\theta_{\text{max}} = 2 \arctan \frac{\Lambda}{\lambda}$. Examples of calculation with finite $\Lambda$ are displayed in Fig. 4. The position of the critical coupling constant now is governed by $\Lambda$ (see also discussion at the end of Section 5). Obviously the additional dependence on $\Lambda$ appears in the formula

$$M(0) = c \left( g, \frac{\Lambda}{M_g} \right) M_g.$$  

Numerical estimation demonstrates that the larger $\Lambda$ (or equivalently smaller $M_g$) is taken the larger $\bar{M}(0)$ is generated; and near the critical point the growth $\bar{M}(0)$ is so fast that $M(0)$ is also growing. $\bar{M}(0)$ tends to some finite limit at $\Lambda \to \infty$, and in this limit dependence on $\Lambda$ disappeared $c \left( g, \frac{\Lambda}{M_g} \right) \to c(g)$. The limit $M_g \to 0$, however, has some difficulties with mathematically correct description since the kernel of Eq. (14) becomes singular.

Fig. 1. The running quark mass versus momentum $\bar{M}(p)$, in units of $M_g$, at different values of the coupling constant $g^2$. The computational parameters are $\lambda = 10$, $N_h = 13$ (see the explanations in the text). The numerical solution of matrix Eq. (21) is shown in red dashed thick line. The blue solid thin line represents the result of substitution of the previous solution into the right-hand side of Schwinger–Dyson Eq. (15).

Fig. 2. The running quark mass versus momentum $\bar{M}(p)$, in units of constituent quark mass $M(0)$. Actually, plots are the same as in Fig. 1 but in the breve variables. All lines represent the results of substitutions of the solutions of matrix Eq. (21) into the right-hand side of Schwinger–Dyson equation (15). The purple dashed thick line and the blue solid thin line correspond to the cases $g^2 = 18$ and $g^2 = 19$, respectively.

Fig. 3. The running quark mass versus momentum $\bar{M}(p)$. The computational parameters are $\lambda = 10$ and $g^2 = 19$. The blue solid thin line and the green dashed thick line correspond to the cases $N_h = 13$ and $N_h = 20$, respectively.
In all cases the resulting numerical solution satisfies $\bar{M}(\bar{p}) \to 0$ at $\bar{p} \to \infty$; for $\Lambda = 4\infty$ it is our demand (see above the end of Section 3), besides this for finite $\Lambda$ it can be shown explicitly (see below Subsection 7.1); and the numerical calculation scheme was constructed so as to satisfy this demand.

5. ANALYTICAL RESTRICTIONS

Using the expression $\bar{M}(\bar{p}) = -\int_0^{+\infty} d\bar{q} \bar{M}^\prime(\bar{q}) = -\int_0^{+\infty} d\bar{q} \bar{M}^\prime(\bar{q} + \bar{p})$ one can rewrite Eq. (15) in the form

\[
\int_0^{+\infty} d\bar{q} \left[ \frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \right] \times \ln \left( \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} + \bar{M}^\prime(\bar{q} + \bar{p}) \right) = 0.
\]

Upon integrating this equation over $\bar{p}$ from zero to infinity, exchange of the order of integration, and direct integration over $\bar{p}$ we have

\[
\int_0^{+\infty} d\bar{q} \left[ \frac{g^2}{(4\pi)^2} \frac{1}{\bar{p}} \frac{\bar{q} \bar{M}(\bar{q})}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \right] \times \ln \left( \frac{1 + (\bar{p} + \bar{q})^2}{1 + (\bar{p} - \bar{q})^2} + \bar{M}^\prime(\bar{q} + \bar{p}) \right) = 0.
\] (22)

The formula is correct even if the integral $\int_0^{+\infty} d\bar{q} \bar{M}(\bar{q})$ diverges. That is because in all steps the right-hand side of the equation is zero.

If $g^2 \leq 16$, then: $\frac{g^2}{(4\pi)^2} 2\pi \arctan(\sqrt{\bar{q}^2 + \bar{q}^2}) < \frac{g^2}{4} \leq 1$. So we get

\[
0 = \int_0^{+\infty} d\bar{q} \left[ \frac{\bar{q}}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \right] \times \bar{M}(\bar{q}) < \int_0^{+\infty} d\bar{q} \left[ \frac{\bar{q}}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \right] \times \bar{M}(\bar{q})
\]

Which can be satisfied only when $\bar{M}(\bar{p}) = 0$ for any $\bar{p}$.

For $g^2 > 16$ we have in the limit $\bar{q} \to \infty$:

\[
\left[ \frac{\bar{q}}{\sqrt{\bar{M}^2(\bar{q}) + \bar{q}^2}} \right] \times \bar{M}(\bar{q}) \rightarrow \frac{g^2}{16} - \frac{1}{\bar{M}(\bar{q})}.
\]

This means that $\int_0^{+\infty} d\bar{q} \bar{M}(\bar{q}) < \infty$. For $g^2 > 16$, the integrand in (22) is below zero at small $\bar{q}$ and above zero at large $\bar{q}$, and so the whole integral (22) can be equal to zero.

Thus, for $g^2 \leq 16$, we have only the trivial solution $\bar{M}(\bar{p}) = 0$ of the massless Schwinger–Dyson Eq. (15).

For $g^2 > 16$, the integral $\int_0^{+\infty} d\bar{q} \bar{M}(\bar{q})$ is convergent.

The threshold value $g^2 = 16$ corresponds to $\alpha_s = \frac{4}{\pi} \approx 1.27$. Also recall that we have taken $N_f = 1$, $

\text{Fig. 4. The running quark mass versus momentum } \bar{M}(\bar{p}), \text{ in units of } M_g, \text{ at different values of hard cutoff } \Lambda. \text{ The red solid thick line, the green dashed line and the blue solid thin line correspond to the cases } \Lambda = 500, \Lambda = 1000 \text{ and } \Lambda = \infty \text{ respectively. The left panel is the plots in standard units, the same plots in log-log variables are shown in the right panel.}
for other values of $N_f$, the critical value of $\alpha_c$ might be different. It is worthwhile to notice that this critical value lies near the maximum value $\alpha_c \approx 1.2$ of the function $\alpha_c(\rho)$ obtained from the lattice calculations [73].

There are several based on Schwinger–Dyson equation models that simulate QCD and chiral symmetry breaking, some of them have the critical coupling. In a model with massive vector boson in Landau gauge (see for example [48] and references therein), the critical coupling $\alpha_c$ depends on the ratio of boson mass to the extra dimensional cutoff parameter (the same situation arises in our model at finite cutoff, see Fig. 4); in more or less realistic cases: $\alpha_c \approx 1.35 \pm 0.30$. In a 4-dimensional massless QED in the quenched rainbow approximation (e.g. see review [75]) the critical value is $\alpha_c = \frac{\pi}{3} = 1.05$. One can also mention a 4-dimensional SU($N_c$) vector-like gauge theory [75, 76] and a QED3 [77], in which there are the phase transitions with respect to the number of flavors $N_f$, with noninteger critical value $N_f^{\text{cr}}$.

6. THE LINEARIZED SCHWINGER–DYSON EQUATION

6.1. The Linearized Schwinger–Dyson Equation and Its Numerical Solution

Let us introduce the notation $\tilde{M}_0 \equiv \tilde{M}(0)$ and the function

$$\tilde{W}(\tilde{q}) \equiv \frac{\tilde{q} \tilde{M}(\tilde{q})}{\sqrt{\tilde{M}_0^2 + \tilde{q}^2}}. \quad (23)$$

Then: $\tilde{M}(\tilde{p}) = \frac{1}{\tilde{p}} \sqrt{\tilde{M}_0^2 + \tilde{p}^2} \tilde{W}(\tilde{p})$. The function $\tilde{W}(\tilde{q})$ has the properties

$$\begin{align*}
\tilde{q} \to \infty : & \tilde{W}(\tilde{q}) \approx \tilde{W}(\tilde{q}), \\
\tilde{q} \to 0 : & \tilde{W}(\tilde{q}) \approx \tilde{W}(\tilde{q}), \\
\tilde{q} > 0 : & \tilde{W}(\tilde{q}) < \tilde{M}(\tilde{q}), \\
\tilde{W}(-\tilde{q}) &= -\tilde{W}(\tilde{q}). \quad (24)
\end{align*}$$

• These properties show that $\tilde{W}(\tilde{p})$ can be determined from an approximate to (15) equation

$$\begin{align*}
\sqrt{\tilde{M}_0^2 + \tilde{p}^2} \tilde{W}(\tilde{p}) &= \frac{\tilde{g}^2}{(4\pi)^2} \int_{-\infty}^{+\infty} d\tilde{q} \ln \left( 1 + \frac{1}{\left( \tilde{p} - \tilde{q} \right)^2} \right) \tilde{W}(\tilde{q}). \quad (25)
\end{align*}$$

As well as Eq. (15) was rewritten in the form (16), one can use (25) and then Eq. (26) takes form

$$\begin{align*}
\sqrt{\tilde{M}_0^2 + \tilde{p}^2} \tilde{W}(\tilde{p}) &= \frac{\tilde{g}^2}{(4\pi)^2} \int_{-\infty}^{+\infty} d\tilde{q} \ln \left( 1 + \frac{1}{\left( \tilde{p} + \tilde{q} \right)^2} \right) \tilde{W}(\tilde{q}). \quad (26)
\end{align*}$$

Unique existence of the parameter $\tilde{M}_0$ and the positive solution of this linear equation are discussed in the next subsection. The last Eq. (27) can be solved numerically using the same algorithm as in Section 4 Eq. (16) was solved, only one need to use (23) instead of (17). This method allows us to find the solution $\tilde{W}(\tilde{p})$ together with the $\tilde{M}_0$. The result of such calculation is shown in Fig. 5 together with the numerical solution of the nonlinear Eq. (16). The plots of the solutions of the nonlinear and linear equations turn out to be close.

6.2. Analytical Restrictions for the Linearized Schwinger–Dyson Equation

Equation (26) is the eigenvalue problem, where $\tilde{g}^2$ plays role the inverse eigenvalue. There is the Perron–Frobenius theorem for a real square matrix with positive entries. It might be supposed that analog of this theorem holds true for Eq. (26). Then we can conclude, Eq. (26) has the unique strictly positive eigenvector, and the corresponding $\tilde{g}^2$ is real positive smallest inverse eigenvalue.
Let us assume that at fixed \( g^2 > 16 \), Eq. (26) has two different nontrivial solutions \( \mathcal{W}(p) \) and \( \mathcal{W}(\bar{p}) \) for different \( \mathcal{M}_0 \) and \( \mathcal{M}_0' \) (let be \( \mathcal{M}_0 > \mathcal{M}_0' \)), respectively,

\[
\left\{ \begin{align*}
\sqrt{\mathcal{M}_0^2 + \bar{p}^2} W(p) &= \frac{g^2}{(4\pi)^2} \int_0^{+\infty} dq \ln\left(\frac{1 + (\bar{p} + q)^2}{1 + (\bar{p} - q)^2}\right) W(q), \\
\sqrt{\mathcal{M}_0'^2 + q^2} W'(q) &= \frac{g^2}{(4\pi)^2} \int_0^{+\infty} dp \ln\left(\frac{1 + (\bar{p} + q)^2}{1 + (\bar{p} - q)^2}\right) W'(p).
\end{align*} \right. \tag{28}
\]

Upon multiplying the first equation by the function \( W'(\bar{p}) \), integrating the result over \( \bar{p} \) from zero to plus infinity and exchange of the order of integration we have

\[
\int_0^{+\infty} d\bar{p} W'(\bar{p}) \sqrt{\mathcal{M}_0^2 + \bar{p}^2} W(\bar{p}) = \frac{g^2}{(4\pi)^2} \int_0^{+\infty} dq W(q) \ln\left(\frac{1 + (\bar{p} + q)^2}{1 + (\bar{p} - q)^2}\right) W'(\bar{p}).
\]

Making integration over \( \bar{p} \) by substituting the second equation of the set (28) we arrive at

\[
\int_0^{+\infty} d\bar{p} \left(\sqrt{\mathcal{M}_0^2 + \bar{p}^2} - \sqrt{\mathcal{M}_0'^2 + \bar{p}^2}\right) W(\bar{p}) = 0.
\]

The last formula is the contradiction. Consequently \( \mathcal{M}_0 = \mathcal{M}_0' \), in other words there is the single \( \mathcal{M}_0 \) for which Eq. (26) admits the positive solution. And vice versa, for any \( g^2 > 16 \) there is the single \( \mathcal{M}_0 > 0 \) for which the unique positive solution exist.

6.3. The Linearized Schwinger–Dyson Equation in the Fourier Space

In Eq. (27) in the numerator of the logarithm one can change the variable \( \bar{q} \rightarrow -\bar{q} \). The Schwinger–Dyson Eq. (27) then takes the form

\[
\sqrt{\mathcal{M}_0^2 + p^2} W(p) = -\frac{g^2}{(4\pi)^2} \int dq \ln(1 + (p - q)^2) W(q).
\]

The right-hand side of the last equation can be simplified by means of the Fourier transform

\[
\mathcal{W}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp e^{ipx} \mathcal{W}(p), \quad \mathcal{W}(\bar{p}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{-ix\bar{p}} \mathcal{W}(x),
\]

\[
\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp e^{ipx} \int dq \ln(1 + (p - q)^2) W(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq e^{iqx} \int_{-\infty}^{+\infty} dp e^{ip\bar{q}} W(\bar{q}) \ln(1 + (p - q)^2)
\]

\[
= \sqrt{2\pi} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq e^{iqx} W(q)\right) \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp e^{ip\bar{q}} \ln(1 + p^2)\right) = -2\pi \frac{e^{-|x|}}{|x|} \mathcal{W}(x).
\]

Using this equation it is convenient to examine the \( \bar{p} \rightarrow \infty \) asymptotics. The left-hand side (29) has a simple form \( \sqrt{\mathcal{M}_0^2 + \bar{p}^2} \rightarrow |\bar{p}| \). The right-hand side of the Schwinger–Dyson equation has a simple form in the Fourier space (31). The limit \( \bar{p} \rightarrow \infty \) corresponds to the limit \( x \rightarrow 0 \), so the Taylor expansion can be used.

\[
\sqrt{\mathcal{M}_0^2 - \partial^2} \mathcal{W}_s(x) = g^2 \frac{e^{-x}}{8\pi x} \mathcal{W}_s(x). \tag{31}
\]
7. ASYMPTOTICS OF SOLUTION
AT HIGH MOMENTUM

7.1. Contribution from Low Momentum

Let \( \overline{p} \) be a point such that at \( \overline{p} > a \) the solution has the asymptotic behavior. If we consider \( \overline{p} \gg a \), the contribution from the right-hand side (15) from a low \( \overline{q} \) is

\[
\frac{g^2}{(4\pi)^2} \int_0^a d\overline{q} \frac{\overline{M}(\overline{q})}{\sqrt{\overline{M}^2(\overline{q}) + \overline{q}^2}} \ln \left( 1 + \frac{4\overline{p}\overline{q}}{1 + (\overline{p} - \overline{q})^2} \right)
\]

\[
\approx \frac{g^2}{4\pi} \int_0^a d\overline{q} \frac{W(\overline{q})}{\overline{q}} \frac{1}{\overline{p}}
\]

Consequently, (remind that from (18) and (24), as \( \overline{q} \to +\infty \) all three functions have the same asymptotic behavior: \( \overline{M}(\overline{q}) \approx W(\overline{q}) \approx \overline{W}(\overline{q}) \)) the asymptotics of \( W(\overline{p}) \) cannot be less than \( \frac{1}{\overline{p}} \):

\[
\lim_{\overline{p} \to +\infty} \frac{1}{\overline{p} W(\overline{p})} < +\infty.
\]  \( \tag{32} \)

Hence, it follows that \( W(\overline{p}) \) cannot decrease exponentially.

7.2. Power Asymptotics

One can examine the following ansatz as \( \overline{p} \to +\infty \):

\[
W(\overline{p}) = C \text{ sgn}(\overline{p}) \frac{1}{|\overline{p}|^\beta},
\]  \( \tag{33} \)

where \( C \) is some constant.

\( \beta \) should be real, otherwise the demand \( M(p) \geq 0 \) is violated. From the requirement of the convergence of the integral \( \int d\overline{q} \overline{M}(\overline{q}) \) on the upper limit (see Section 5) there follows \( \beta > 1 \). It follows from (32) that \( \beta \leq 2 \).

The Fourier-sine transform (30) of the function (33) can easily be calculated, for \( 0 < \beta < 2 \):

\[
\mathcal{W}_s(x) = \frac{1}{\pi} \sqrt{\beta/2} \Gamma(1 - \beta) \frac{1}{x^{1-\beta}}.
\]

Substituting this and (33) in (31) and (29) we get that power asymptotics for \( 1 < \beta < 2 \) is self-consistent if:

\[
\frac{1}{g^2} = \frac{\cot(\beta \pi / 2)}{8\pi (1 - \beta)}.
\]  \( \tag{34} \)

Unfortunately, to obey this formula, one needs \( g^2 \leq 16 \), which is in contradiction with the results of Section 5.

The value \( \beta = 2 \) may easily be examined and it does not suit too (see Subsection 7.4).

Combining all together, we have that power asymptotics (33) is not valid for all \( \beta \).

7.3. Log-Power Asymptotic

As in the previous subsection we can test a log-power asymptotics as \( \overline{p} \to +\infty \):

\[
\mathcal{W}(\overline{p}) = C \frac{\ln(\overline{p})^\gamma}{\overline{p}^\beta}.
\]  \( \tag{35} \)

From the demand \( M(p) \geq 0 \) it follows that \( \beta \in \mathbb{R} \) and \( \gamma \in \mathbb{R} \). From the requirement of the convergence of the integral \( \int d\overline{q} \overline{M}(\overline{q}) \) on the upper limit (see Section 5) there follows \( \beta > 1, \gamma \in \mathbb{R} \) or \( \beta = 1, \gamma < 1 \). To obey (32), we need \( \beta < 2, \gamma \in \mathbb{R} \) or \( \beta = 2, \gamma \geq 0 \).

The Fourier-sine transform (30) of some function with asymptotics (35) can be expressed in terms of elementary functions or relatively simple special functions only in a small number of cases of \( \gamma \). Fortunately, we do not need the whole \( \mathcal{W}_s(x) \), for our purposes just the asymptotic as \( x \to 0 \) is sufficient, and it can be calculated for \( 0 < \beta < 1 \) and \( 1 < \beta < 2 \) and \( \gamma \in \mathbb{R} \) (see (A.5) in Appendix):

\[
\mathcal{W}_s(x) = C \sqrt{\frac{2}{\pi}} \cos\left(\frac{\beta \pi}{2}\right) \Gamma(1 - \beta) \frac{1}{x^{1-\beta}} \left(\ln\frac{1}{x}\right)^\gamma.
\]

This leads to the same constraint (34), so the case \( 1 < \beta < 2, \gamma \in \mathbb{R} \) cannot be.

The cases \( \beta = 1, \gamma < -1 \) and \( \beta = 2, \gamma \geq 0 \) can easily be considered directly (see Subsection 7.4), and they also do not suit.

Combining the aforesaid we get that log-power asymptotics (35) is not valid for all \( \beta \) and \( \gamma \).

7.4. Integral Power-Log Asymptotics

Consider asymptotic (35) in the cases \( \beta = 1, \gamma < -1 \) and \( \beta = 2, \gamma \geq 0 \).

Also let \( a \) be a point such that for \( \overline{p} > a \) the solution \( \mathcal{W}(\overline{p}) \) has the asymptotic behavior, and we can take \( a > 1 \). For such \( \overline{p} \) the integral in the right-hand side (15) can be decomposed into the sum:

\[
\int_0^{+\infty} d\overline{q} \overline{W}(\overline{q}) \ln \left( 1 + (\overline{p} + \overline{q})^2 \right)
\]

\[
\approx \int_0^a d\overline{q} \overline{W}(\overline{q}) \ln \left( 1 + \frac{4\overline{p}\overline{q}}{1 + (\overline{p} - \overline{q})^2} \right)
\]

\[
+ \frac{C}{\overline{p}^{(\beta-1)}} \int_0^{\infty} d\overline{y} \frac{\ln(\overline{p}\overline{y})^\gamma}{\overline{y}^\beta} \ln \left( 1 + \frac{\overline{p}^2(1 + y)^2}{1 + \overline{p}^2(1 - y)^2} \right),
\]

where we introduced a new variable \( \overline{y} \) by the formula \( \overline{q} \equiv \overline{p}\overline{y} \). The asymptotics of \( I_s(\overline{p}) \) was considered in subsection 7.1 and it is proportional to \( \frac{1}{\overline{p}} \).
Let us choose \( y_1 \) and \( y_2 \) so that \( 0 < y_1 \ll 1 \) and \( 1 \ll y_2 \). Then the second integral in the right-hand side can be rewritten in the form of the sum:

\[
I_1(\bar{p}) + I_2(\bar{p}) + I_3(\bar{p}),
\]

where

\[
I_1(\bar{p}) = \frac{C}{p^{\beta-1}} \int_{\gamma}^{y_1} dy \frac{(\ln(\bar{p}y))^\gamma}{y^\beta} \ln \left(1 + \frac{4y}{p^2 + (1 - y^2)^2}\right),
\]

\[
I_2(\bar{p}) = \frac{C}{p^{\beta-1}} \int_{y_1}^{y_2} dy \frac{(\ln(\bar{p}y) + \ln(y))^\gamma}{y^\beta} \ln \left(1 + \frac{1}{p^2 + (1 + y^2)^2}\right),
\]

\[
I_3(\bar{p}) = \frac{C}{p^{\beta-1}} \int_{y_2}^{+\infty} dy \frac{(\ln(\bar{p}y))^\gamma}{y^\beta} \ln \left(1 + \frac{4y}{p^2 + (1 - y^2)^2}\right).
\]

Further, we work only with such \( \bar{p} \) that\(a < y_1, \frac{1}{\bar{p}} \ll y_1, y_2 \ll \bar{p} \). For this \( \bar{p} \) the integrands are simplified

\[
I_1(\bar{p}) \approx \frac{4C}{p^{\beta-1}} \int_{\gamma}^{y_1} \frac{dy}{y^\beta} \ln(\bar{p}y)^\gamma, 
\]

\[
I_2(\bar{p}) \approx \frac{C(\ln(\bar{p}))^\gamma}{p^{\beta-1}} \int_{y_1}^{y_2} \frac{dy}{y^\beta} \ln \left(1 + \frac{1}{p^2 + (1 + y^2)^2}\right),
\]

\[
I_3(\bar{p}) \approx \frac{4C}{p^{\beta-1}} \int_{y_2}^{+\infty} \frac{dy}{y^\beta} \ln(\bar{p}y)^\gamma.
\]

This integrals can now be calculated directly. For \( \beta = 1, \gamma < -1 \) this leads to:

\[
I_1(\bar{p}) = 4C y_1(\ln \bar{p})^\gamma, 
\]

\[
I_2(\bar{p}) = C \left(\pi^2 - 4y_1 - \frac{4}{y_2}\right)(\ln \bar{p})^\gamma, 
\]

\[
I_3(\bar{p}) = \frac{4C}{y_2}(\ln \bar{p})^\gamma.
\]

Combining all together the right-hand side of (15) equals: \( C \frac{g^2(\ln \bar{p})^\gamma}{16 \bar{p}} \), which can be consistent with the left-hand side of (15) only if \( g^2 = 16 \), but this value is forbidden by the arguments of Section 5. Thus, the case \( \beta = 1, \gamma < -1 \) does not suit.

For \( \beta = 2, \gamma \gg 0 \) the integration leads to

\[
I_1(\bar{p}) = \frac{4C}{1 + \gamma} \frac{(\ln \bar{p})^{\gamma+1}}{\bar{p}} + 4C \ln(y_1) \frac{(\ln \bar{p})^\gamma}{\bar{p}}, 
\]

\[
I_2(\bar{p}) = C \left(4 - 4 \ln(y_1) - \frac{2}{y_2}\right) \frac{(\ln \bar{p})^\gamma}{\bar{p}},
\]

\[
I_3(\bar{p}) = C \frac{2}{y_2^2} \frac{(\ln \bar{p})^\gamma}{\bar{p}}.
\]

From the aforesaid

\[
I_0(\bar{p}) + I_1(\bar{p}) + I_2(\bar{p}) + I_3(\bar{p}) 
\]

\[
= 4C \frac{\ln(\bar{p})^{\gamma+1}}{\bar{p}} + 4C \frac{(\ln \bar{p})^\gamma}{\bar{p}} + 4 \frac{dW(q)q}{p_0^2}.
\]

We can see that the right- and left-hand sides of (15) here are not self-consistent. So the case \( \beta = 2, \gamma > 0 \) is not valid either.

The \( I_0(\bar{p}) \) always gives the contribution to asymptotics proportional to \( \frac{1}{p^2} \). After substitution this asymptotics into the right-side of (15), according to (36), this should lead to a contribution proportional to \( \frac{\ln \bar{p}}{\bar{p}} \); after substitution the last one we should get \( \frac{(\ln \bar{p})^2}{\bar{p}} \), and so on. Consequently, we can conclude that condition (32) can be generalized to:

\[
\lim_{\bar{p} \to +\infty} \frac{(\ln \bar{p})^\gamma}{\bar{p}} < +\infty
\]

where \( \gamma \in \mathbb{R} \).

Furthermore, the form of (36) suggests that the solution should be searched in the form of a series in powers of the logarithm.

### 7.5. Series Asymptotics

We can suppose that asymptotics of the solution of Eq. (15) as \( \bar{p} \to +\infty \) has the form

\[
W(\bar{p}) = C_0 \frac{1}{\bar{p}^\beta} + C_1 \frac{\ln(\bar{p})}{\bar{p}^\beta} + C_2 \frac{(\ln \bar{p})^2}{\bar{p}^\beta} + \cdots + L(\bar{p}),
\]

where \( 1 < \beta < 2 \), the function \( L(\bar{p}) \) decreases faster than \( \frac{1}{\bar{p}^2 \ln(\bar{p})} \), and the series does not reduce to powers of \( \bar{p} \) or \( \ln(\bar{p}) \).

In the left-hand side of (15), if we neglect \( L(\bar{p}) \), then with the same accuracy \( M(\bar{q}) \approx W(\bar{q}) \), this comes from the formula inverse to (17).

In the right-hand side (15), the integral can be expanded into the sum:

\[
\int_0^\infty = \int_0^a + \int_a^\infty, \quad \text{the integral } \int_0^a
\]

was considered in Subsection 7.1, the other integral (to the accuracy of \( L(\bar{p}) \), which can give not more than \( \frac{1}{\bar{p}} \) asymptotic) is

\[
\int_a^\infty dq W(q) \ln \frac{1 + (\bar{p} + q)^2}{1 + (\bar{p} - q)^2} 
\]

\[
= \left(C_0 + C_1 \left(-\frac{\partial}{\partial \beta}\right) + C_2 \left(-\frac{\partial}{\partial \beta}\right)^2 + \cdots\right)
\]

\[
\times \int_a^\infty dq \frac{1}{q^2} \ln \frac{1 + (\bar{p} + q)^2}{1 + (\bar{p} - q)^2}.
\]
The right-hand side integral can also be expanded into the sum: \( \int_0^\infty = \int_0^0 + \int_0^a \). The \( \int_0^a \) gives the asymptotics 1 for the same reasons as in the low momentum case (see Subsection 7.1). The integral \( \int_0^\infty \) can easily be evaluated by parts

\[
\int dq \frac{1}{\beta+1} \left( \frac{\partial}{\partial q} q^{-\beta+1} \right) \ln \frac{1 + (p + q)^2}{1 + (p - q)^2} = \frac{2\pi}{\beta-1} \int dq \frac{1}{q^{\beta-1}} \left( \frac{p + q}{1 + (p + q)^2} + \frac{p - q}{1 + (p - q)^2} \right)
\]

\[
= 2\pi \left( \frac{1 + p^2}{\beta-1} \right)^{\frac{1}{2}} \left( p^2 - 1 \right) \sin(\beta \cot(\beta)) - 2p \cos(\beta \arccot(\bar{p}))
\]

\[
- p^2 \sin(\beta \pi - (-2 + \beta) \arccot(\bar{p})) + 2p \cos\left(\frac{\beta \pi}{2}\right) \cos(\beta \arctan(\bar{p}))
\]

\[
+ \bar{p} \sin(\beta \arctan(\bar{p})) + \sin\left(\frac{\beta \pi}{2} + (-2 + \beta) \arctan(\bar{p})\right).
\]

As \( \bar{p} \to +\infty \) this leads to:

\[
\int dq \frac{1}{q^\beta} \ln \frac{1 + (p + q)^2}{1 + (p - q)^2} = \frac{2\pi}{\beta-1} \frac{1}{p^\beta} + h(p, \beta) \frac{1}{p^\beta},
\]

where \( \lim_{p \to +\infty} h(p, \beta) < +\infty \), so this term can be neglected.

Thus, from various sources the term \( A \frac{1}{\beta} \) appears in the right-hand side of the equation. We rewrite this term in the form:

\[
\begin{bmatrix}
A \\
\frac{A(\beta - 2)^2}{\beta} \\
\vdots
\end{bmatrix}
= \begin{pmatrix}
C_0 \\
C_1 \\
C_2 \\
\vdots
\end{pmatrix}
\]

\[
\begin{pmatrix}
(4\pi)^2 \\
ge^2 \\
\vdots
\end{pmatrix}
\]

where \( nC_k \). Likely, one cannot reduce the infinite matrix and columns to finite ones, because the elements in the rows in the matrix do not decrease. This is one of the reasons why asymptotics (37) is permitted while asymptotics (33) and (35) are not. The
other reason is that in the evaluation of (37) we do not neglect the term $A \frac{1}{\beta^2}$.

8. CONCLUSIONS

The natural method of obtaining a dimensional parameter in QCD is suggested by means of normal ordering of fields in the Lagrangian. Within our model, the dimensional parameter QCD is nothing else but the effective gluon mass.

Based on QCD the effective action of strong interaction (6) was constructed.

In the framework of the constructed model, the Schwinger–Dyson equation with the effective gluon mass (15) is investigated both analytically and numerically. It is shown that spontaneous chiral symmetry breaking occurs and a nontrivial dependence of the quark mass on momenta appears. The critical value of the strong coupling constant (equals to $\alpha_s = 4/\pi$), above which the spontaneous breaking occurs, is found in the semiclassical approximation. It is proved strictly that below this critical value, the Schwinger–Dyson equation has only trivial non-negative solution $M(p) = 0$. The critical coupling $\alpha_s$ turns out to be independent of any dimensional parameters. Comparison with some other models based on Schwinger–Dyson equations have made.

The Fourier-sine transform of function (A.1) with log-power asymptotic was performed (A.4), and the leading asymptotic was found.

Although the derivation of the effective action of strong interaction (6) from the QCD Lagrangian (1) is clear and well-controlled, numerous assumptions are done during the derivation. Thus, the obtained results are qualitative rather than quantitative. Taking into account the neglected terms could amend the model and make it quite quantitative. For better understanding of the solution of the Schwinger–Dyson equation in the region of the large coupling constant, it would be better to improve the numerical computation scheme. For instance, the accuracy of the numerical simulations is likely to be insufficient to obtain the precise value of $M(p)$ at $p = 0$.

The developed analytical methods of solving and analyzing as well as created programs for numerical calculation the Schwinger–Dyson equation can be used not only in the considered specific kernel but for various other kernels; and not only in the QCD but for other gauge theories, including technicolor.

Detailed derivation and discussion of the Coulomb gauge quark Schwinger–Dyson equation can be found in a series of [62] (see references therein). Our research develops this article; the main difference is in [62], as phenomenological input, the other (and more suitable for heavy quarks) kernel is used in numerical calculations. Due to simplicity, the analytical investigation of Schwinger–Dyson Eq. (15) was done in more detail.

In [43, 49] (see also [50, 66, 69, 72] and references therein) it is shown how the Bethe–Salpeter equation, which describe the spectrum and wave functions of the bound states, can be derived from the generating functional (6) in the framework of the Stationary Phase method. Knowledge of the solutions of the Bethe–Salpeter equation allows us to calculate masses of light mesons ($\pi$, $K$ etc. and respective excited states) and their decays, similarly to Nambu–Jona-Lasinio model [17] but with higher precision. To cope with this Bethe–Salpeter equation, one should already have the solution of the corresponding Schwinger–Dyson Eq. (8) as the “input function”. The investigation of the Bethe–Salpeter equation is beyond the scope of this paper and will be done later.

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APPENDIX

FOURIER–SINE TRANSFORM OF A LOG-POWER FUNCTION

Consider the function

$$F_x(x, \gamma, \beta) \equiv \int_0^{+\infty} d\bar{p} \frac{(\ln(a + \bar{p}))^\gamma}{(a + \bar{p})^{\beta}} \sin(x\bar{p}),$$

(A.1)

where: $a > 1$, $\gamma \in \mathbb{R}$, $\beta \in \mathbb{R}$. As: $F_x(-x, \gamma, \beta) = -F_x(x, \gamma, \beta)$, below we will take: $x > 0$.

Our aim is to find the asymptotic behavior of $F_x(x, \gamma, \beta)$ at $x \to 0$.

This function has the property

$$F_x(x, \gamma, \beta) = -\frac{d}{d\beta} F_x(x, \gamma - 1, \beta),$$

(A.2)

so the practically interesting case is: $-1 < \gamma \leq 0$. Let us introduce the new notation: $\gamma' \equiv -\gamma$, where: $0 \leq \gamma' < 1$.

Using the formula

$$\frac{1}{(\ln(a + \bar{p}))^2} = \int_0^{+\infty} dt \frac{\sin(t\ln(a + \bar{p}))}{t^{\gamma'}} \frac{1}{\sin \left(\frac{\pi \gamma'}{2}\right) \Gamma(\gamma')}$$

and exchanging the order of integrations we get

$$F_x(x, -\gamma, \beta) = \frac{1}{\sin \left(\frac{\pi \gamma'}{2}\right) \Gamma(\gamma')}$$

(A.3)

$$\times \int_0^{+\infty} dt \frac{1}{t^{\gamma'}} \int_0^{+\infty} d\bar{p} \frac{\sin(x\bar{p})}{(a + \bar{p})^{\beta}} \sin(t\ln(a + \bar{p})).$$
One can expand the function: \( \sin(t \ln(a + p)) = \sin\left(t \ln(x(a + p)) + t \ln \frac{1}{x}\right) \) into the series around the point \( t \ln \frac{1}{x} \). Put by definition:

\[
H_a(x, \beta) = \int_0^\infty \frac{\sin(x \beta)}{(a + p)\beta} \, dx
\]

we arrive at

\[
\int_0^\infty \frac{\sin(x \beta) \sin(t \ln(a + p))}{(a + p)\beta} \, dx = x^\beta \sin(t \ln \frac{1}{x})
\]

\[
\times \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} \left( \frac{d^{2k}}{dp^{2k}} \frac{H_a(x, \beta)}{x^\beta} \right) t^{2k} - x^\beta \cos(t \ln \frac{1}{x})
\]

Substituting the latter into (A.3) and changing the variable: \( \bar{t} \equiv t \ln \frac{1}{x} \), we finally have

\[
F_a(x, -\bar{t}, \beta) = \frac{1}{\sin \left( \frac{\pi \bar{t}}{2} \right)} \frac{1}{\Gamma(\bar{t}) \left( \ln \frac{1}{x} \right)^\gamma} \prod_{k=0}^\infty \frac{(-1)^k}{(2k+1)!} \left( \frac{d^{2k+1}}{dp^{2k+1}} \frac{H_a(x, \beta)}{x^\beta} \right) \left( \ln \frac{1}{x} \right)^{2k+1} \cos \bar{t}
\]

Further steps strongly depend on which \( \beta \) we want to consider and how many terms of the series we are interested in. In our case, we are interested in only leading asymptotics and values: \( 0 < \beta < 1 \) or \( 1 < \beta < 2 \). For these \( \beta \):

\[
H_a(x, \beta) \approx \frac{\pi \cos \left( \frac{\beta \pi}{2} \right)}{\Gamma(\beta) \sin(\beta \pi)} \left( \frac{1}{x^{1-\beta}} \right)
\]

and leading asymptotics comes from the \( k = 0 \) term in the first sums (A.4). Thus, we get

\[
F_a(x, -\bar{t}, \beta) \approx \frac{\pi \cos \left( \frac{\beta \pi}{2} \right)}{\Gamma(\beta) \sin(\beta \pi)} \left( \frac{1}{\ln \frac{1}{x}} \right)^\gamma.
\]

Recollecting (A.2) we finally have

\[
F_a(x, -\bar{t}, \beta) = \Gamma(1 - \beta) \cos \left( \frac{\beta \pi}{2} \right) x^{\beta - 1} \left( \ln \frac{1}{x} \right)^\gamma.
\]

where: \( x > 0 \), \( 0 < \beta < 1 \) or \( 1 < \beta < 2 \), \( \gamma \in \mathbb{R} \). The coefficient does not depend on \( \gamma \), \( a > 1 \) and \( a \) is absent in the right-hand side, as it should be.

**CONFLICT OF INTEREST**

The authors declare that they have no conflicts of interest.

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