Some adjunction-theoretic properties
of codimension two nonsingular subvarieties of quadrics

Mark Andrea A. de Cataldo

2/5/1996

Abstract

We make precise the structure of the first two reduction morphisms associated with
codimension two nonsingular subvarieties of quadrics $Q^n$, $n \geq 5$. We give a coarse
classification of the same class of subvarieties when they are assumed to be not of log-
general-type.

0 INTRODUCTION

Because of the Barth-Larsen Theorem and the Double Point Formula, low codimensional
embeddings in projective space are special in many respects. Inspired by the study of the
special adjunction-theoretic properties of threefolds in $P^5$ contained in $Q^5$, in this note we study
the similar properties for codimension two nonsingular subvarieties of quadrics $Q^n$, $n \geq 5$. As
it turns out, by analogy with the results of $Q$, the reduction morphisms associated with these
varieties are almost always isomorphisms; see Theorem 2.2. We give a coarse classification
Theorem for the varieties for which the second reduction morphism is not defined, the so-
called varieties not of log-general-type; see Theorem 3.1, Theorem 3.2 and Theorem 3.4. To
prove the latter one we need to analyze the case of Del Pezzo fibrations and, in the same way
as in the paper $Q$, the case of conic bundles on $Q^5$; see sections 4 and 5, respectively.

Notation and conventions. Our basic reference is [Ha]. We work over any algebraically
closed field of characteristic zero. A quadric $Q^n$, here, is a nonsingular hypersurface of degree
two in the projective space $P^{n+1}$. Little or no distinction is made between line bundles,
associated sheaves of sections and Cartier divisors.

By scroll we mean a variety $X \subseteq P^N$, for which $(X, \mathcal{O}_{P^N}(1)_{|X}) \simeq (P_Y(E), \xi_E$, where $E$ is a
vector bundle on a nonsingular variety $Y$. An adjunction-theoretic scroll (see $Q$) is not, in
general, a scroll; we denote them by a.t. scrolls.

Acknowledgments. This paper is an expanded and completed version of parts of our
dissertation. It is a pleasure to thank our Ph.D. advisor A.J. Sommese, who has suggested to
us that we study threefolds on $Q^5$. We thank the C.N.R. of the Italian Government and The
University of Notre Dame for partial support.

1 PRELIMINARY MATERIAL

Let $\iota : X \hookrightarrow Q^n$ be the embedding of a degree $d$ nonsingular subvariety of codimension two of
$Q^n$; let $L$ denote the line bundle $\iota^*\mathcal{O}_{Q^n}(1)$, $g$ the genus of the curve $C$ obtained by intersecting
$(n-3)$ general elements of $[L]$. Denote by $x_i$ the Chern classes of the tangent bundle of $X$
and by $n_i$ the ones of the normal bundle $N_{X,Q^n}$; by adjunction $K_X = -nL + n_1$ and by the
self-intersection formula $n_2 = (1/2)dL^2$.
The following formulæ which hold in the Chow ring of $X$ for $n \geq 5$, are obtained using the Double Point Formulæ (see \cite{17}) for $\iota$.

\begin{align*}
n_2 &= \frac{1}{2} \left(n^2 - n + 2\right) L^2 - n x_1 \cdot L + x_1^2 - x_2; \\
\frac{1}{6} \left(n^3 - 3n^2 + 8n - 12\right) L^3 + \frac{1}{2} \left(-n^2 + n - 2\right) x_1 L^2 + n(x_1^2 - x_2)L + 2x_1 x_2 - x_1^3 - x_3 &= 0.
\end{align*}

(1)

The following formula for surfaces $X$ on $Q^4$ with balanced cohomology class can be found in \cite{2}.

\begin{align*}
2K_X^2 &= \frac{1}{2} d^2 - 3d - 8(g - 1) + 12\chi(O_X).
\end{align*}

(2)

Proposition 1.1 Let $X$ be a nonsingular threefold on $Q^5$. Then

\begin{align*}
60\chi(O_S) \geq \frac{3}{2} d^2 - 12d + (d - 48)(g - 1) + 24\chi(O_X)
\end{align*}

and

\begin{align*}
\chi(O_S) \leq \frac{2}{3} \left(\frac{g - 1}{d}\right) - \frac{1}{24} d^2 + \frac{5}{12} d.
\end{align*}

Proof. Denote by $s_i$ and $n_i$ the Segre and Chern classes respectively of the normal bundle $N$ of $X$ in $Q^5$. Since $N$ is generated by global sections, we have $s_3 \geq 0$. Since $s_3 = n_1^3 - 2n_1 n_2$, we get

\begin{align*}
0 \leq (K_X + 5L)^3 - 2(K_X + 5L) \frac{1}{2} dL^2 = K^3 + 15K_X^2 L + 75K_X L^2 + 125d - d(K_X + 5L)L^2.
\end{align*}

The first inequality follows from \cite{3}, \cite{5} and \cite{4}. We use the Generalized Hodge Index Theorem of \cite{10} (see also \cite{2}):

\begin{align*}
d(K_X^2 L) \leq (K_X L^2)^2
\end{align*}

and we make explicit the left hand side using \cite{3} and the right hand side using \cite{4}. The second inequality follows.

\hfill \Box

In what follows:

- $(a, b, c, \mathcal{O}(1))$ denotes the polarized pair given by a complete intersection of type $(a, b, c)$ in $\mathbb{P}^{n+1}$ and the restriction of the hyperplane bundle to it;
- $(X, L)$ denotes the polarized pair given by a variety $X \subseteq Q^n$ and $L := \mathcal{O}_{Q^n}(1)|_X$;
- $g$, $q$ and $p_g$ denote the sectional genus of the embedding line bundle, the irregularity and geometric genus of a surface section, respectively.
Remark 1.2 Let $X \subseteq Q^n$, $n \geq 5$, be any subvariety. Then the degree $d$ of $X$ is even. This follows from the fact that the cohomology class of $[X]$ equals the class $(1/2)d[Q^{n-2}]$ in $H^4(Q^n, \mathbb{Z})$.

Proposition 1.3 (Cf. [14]) Let $X \subseteq Q^n$, $n \geq 5$, a codimension two nonsingular subvariety of degree $d \leq 10$. Then the pair $(X, L)$ is one of the ones below.

Type A): $d = 2$, $(\{1, 1, 2\}, O(1)): g = q = p_g = 0$.

Type B): $d = 2$, $(\{1, 2, 2\}, O(1)): g = 1$, $q = p_g = 0$.

Type C): $d = 4$, $\{1, 2, 2\}$, $(\{1, 1\}, (\mathbb{P} \times \mathbb{P}, O(1, 1))): g = q = p_g = 0$.

Type D): $d = 4$, $\{1, 2, 2\}$, $(\mathbb{P}(O_{\mathbb{P}^1}(1)^2 \oplus O_{\mathbb{P}^2}(2)), 5): g = q = p_g = 0$.

Type E): $d = 6$, $(\{1, 2, 3\}, O(1)): g = 4$, $q = 0$, $p_g = 1$.

Type F): $d = 6$, $\{1, 2, 3\}$, $(\mathbb{P}(O_{\mathbb{P}^2}), \xi)$, embedded using a general codimension one linear system $\mathcal{H} \leq \{\xi_{\mathbb{P}^2}\}: g = 1$, $q = p_g = 0$.

Type G): $d = 6$, $\{1, 5\}$, $f: X \rightarrow \mathbb{P}^1 \times \mathbb{P}^2 = Y$ a double cover, branched along a divisor of type $(\mathcal{O}_Y(2, 2)): L \simeq p^*\mathcal{O}_Y(1, 1); g = 2$, $q = p_g = 0$.

Type H): $d = 8$, $(\{1, 2, 4\}, O(1)): g = 9$, $q = 0$, $p_g = 5$.

Type I): $d = 8$, $(\{2, 2, 2\}, O(1)): g = 5$, $q = 0$, $p_g = 1$.

Type J): $d = 8$, $\{1, 2, 5\}$, $(\mathbb{P}(E), \xi)$, $E$ a rank two vector bundle on $Q^2$ as in [2]: $g = 4$, $q = p_g = 0$.

Type K): $d = 10$, $(\{1, 1, 5\}, O(1)): g = 16$, $q = 0$, $p_g = 14$.

Type L): $d = 10$, $\{1, 2, 5\}$, $f_{K_X + L}: X \rightarrow \mathbb{P}^1$ is a fibration with general fiber a Del Pezzo surface $\mathcal{F}$, $K_F^2 = 4$, $K_X = -L + f^*\mathcal{O}_{\mathbb{P}^1}(1); g = 8$, $q = 0$, $p_g = 2$.

We say that a nonsingular threefold $X$ on $Q^5$ is of Type O), if it has degree $d = 12$ and it is a scroll over a minimal K3 surface. Such a threefold exists. See [14].

Proposition 1.4 (Cf. [13]) The following is the complete list of nonsingular codimension two subvarieties of quadrics $Q^n$, $n \geq 5$, which are scrolls.

Type C), $n \geq 6$, $d = 4$, scroll over $\mathbb{P}^1$ and over $\mathbb{P}^3$;

Type D), $n \geq 5$, $d = 4$, scroll over $\mathbb{P}^1$;

Type F), $n \geq 5$, $d = 6$, scroll over $\mathbb{P}^2$;

Type L), $n \geq 5$, $d = 8$, scroll over $\mathbb{P}^2$;

Type O), $n \geq 5$, $d = 12$, scroll over a minimal K3 surface.

Proposition 1.5 (Cf. [13], or [3] for the case $d > 2k(k - 1)$.) Let $C \subseteq Q^3$ be an integral curve of degree $d$ and geometric genus $g$. Assume that $C$ is contained in a surface of $Q^3$ of degree $2k$. Then

$$g - 1 \leq \frac{d^2}{4k} + \frac{1}{2}(k - 3)d.$$

Proposition 1.6 (Cf. [2], Proposition 6.4.) Let $C$ be an integral curve in $Q^3$, not contained in any surface of $Q^3$ of degree strictly less than $2k$. Then:

$$g - 1 \leq \frac{d^2}{2k} + \frac{1}{2}(k - 4)d.$$
Lemma 1.7  In the above situation: \( 0 \leq \mu_s \leq s^2 d. \)

Proof. The left hand side inequality is just Proposition 1.5 above. To prove the right hand side we first assume \( s = \sigma. \) Using \( 2, \) Lemma 6.8 we conclude (from here on the hypothesis \( d > 2\sigma^2 \) was not used there) in the case at hand.

Now, for the general case, let \( s = \sigma + t, \) where \( t \) is a non-negative integer. Then, as it is easily checked, \( \mu_s = \mu_\sigma + \sigma td + t(\sigma + t - 3)d - 2t(\sigma - 1) \) we conclude by what proved for \( \mu_\sigma \) and by the obvious \( g \geq 0. \)

\[ \square \]

Remark 1.8 Let \( X \) be a nonsingular codimension two subvariety of \( \mathbb{Q}^n. \) As a consequence of the Barth-Larsen Theorem (see \[ 3 \]), we have that: if \( n \geq 6, \) then the fundamental group \( \pi_1(X) \) is trivial; if \( n \geq 7, \) then \( \text{Pic}(X) \approx \mathbb{Z}, \) generated by the hyperplane bundle, so that \( X \) does not carry any nontrivial morphisms.

The following fact is well known when \( \mathbb{Q}^n \) is replaced by \( \mathbb{P}^n, \) see \[ 10 \] for example. The case of \( \mathbb{P}^4 \) is proved in \[ 3, \] Lemma 6.1. The general case can be proved in the same way. See \[ 13 \], where we prove a more general statement. We used this “lifting” criterion as a tool to prove the finiteness of the number of families of nonsingular threefolds on \( \mathbb{Q}^5 \) not of general type; see Proposition 1.10 below.

Proposition 1.9 (Cf. \[ 3 \]) Let \( X \) be an integral subscheme of degree \( d \) and codimension two on \( \mathbb{Q}^n, \) \( n \geq 4. \) Assume that for the general hyperplane section \( Y \) of \( X \) we have \( h^0(\mathcal{I}_Y, \mathcal{O}_{\mathbb{Q}^n-1}(\sigma)) \neq 0, \) for some positive integer \( \sigma \) such that \( d > 2\sigma^2. \) Then \( h^0(\mathcal{I}_X, \mathcal{O}_{\mathbb{Q}^n}(\sigma)) \neq 0. \)

Proposition 1.10 (Cf. \[ 13 \]) Let \( n = 4, 5 \) or \( n \geq 7. \) There are only finitely many components of the Hilbert scheme of \( \mathbb{Q}^n \) corresponding to nonsingular \( (n-2) \)-folds not of general type.

2 THE STRUCTURE OF THE REDUCTION MORPHISMS

In this section we give, by a systematic use of the double point formulae, a precise description of the reduction morphisms associated with codimension two subvarieties of quadrics \( \mathbb{Q}^n, \) \( n \geq 5. \) We apply these formulae also to the case of divisorial contractions of extremal rays on threefolds on \( \mathbb{Q}^5. \) For the language and results of Adjunction Theory, which we are going to use freely for the rest of this note, we refer the reader to \[ 3 \] and to \[ 5 \].

Let \( \nu := n - 2. \)

Lemma 2.1 Let \( X \) be a codimension two nonsingular subvariety of \( \mathbb{Q}^n, n \geq 5. \)

Let \( D \) be a divisor on \( X \) with \( (D, \mathcal{O}_D(D)) \approx (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) \) and \( (K_X + (\nu - 1)L)_D \approx \mathcal{O}_D; \) then \( n = 5, 6 \) and \( d = 10. \)

Let \( n = 5. \) Then we have the following list of possible degrees according to whether \( X \) contains a divisor of the given form \( (D, \mathcal{O}_D(D)) \) with \( (K_X + (\nu - 2)L)_D \approx \mathcal{O}_D: \)

\begin{enumerate}
  \item[2.1.1] if \( (D, \mathcal{O}_D(D)) \approx (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) \), then \( d = 20; \)
  \item[2.1.2] if \( (D, \mathcal{O}_D(D)) \approx (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) \), then \( d = 14; \)
  \item[2.1.3] if \( (D, \mathcal{O}_D(D)) \approx (\mathbb{P}^2, G), \) where \( 2G = K_D, \) then \( d = 14; \)
  \item[2.1.4] \( (D, \mathcal{O}_D(D)) \approx (\mathbb{F}_0, G), \) where \( 2G = K_D, \) then \( d = 14; \)
  \item[2.1.5] the case in which \( D \) has two components as in \[ 3, \] Theorem 0.2.1, case b5), cannot occur;
  \item[2.1.6] the case \( (D, \mathcal{O}_D(D)) \approx (\mathbb{F}_1, -E - f) \) cannot occur.
\end{enumerate}

Let \( n = 6. \) Assume \( X \) contains a surface \( S \) such that \( S \approx \mathbb{P}^2, L|_S \approx \mathcal{O}_{\mathbb{P}^2}(1) \) and such that the normal bundle \( \mathcal{N}_{S, X} \approx \mathcal{T}_{\mathbb{P}^2}(1). \) Then \( d = 14. \)
Proof. For \( n = 5 \) the proof is the same as the one of [1], Proposition 1.1, using (1) in the place of (0.8) of the quoted paper. For \( n = 6 \) we compute all the relevant Chern classes by using (3), the Euler sequence for \( S \cong \mathbb{F}^2 \) and the exact sequence

\[
0 \rightarrow T_S \rightarrow T_{X|S} \rightarrow N_{S,X} \rightarrow 0.
\]

\[\square\]

**Theorem 2.2 (Structure of the reduction morphisms)** Let \( X \) be a nonsingular codimension two subvariety of \( \mathbb{Q}^n \), \( n \geq 5 \).
Assume that \( (X,L) \) admits a first reduction \( (X',L') \). Then the first reduction morphism is an isomorphism: \( (X,L) \cong (X',L') \).
Assume that \( (X,L) \) admits, in addition, a second reduction \( (X'',L'') \). We have:
- if \( n = 5 \) and \( d \neq 14, 20 \), then \( (X,L) = (X',L') \) and the blowing up map \( \varphi : X' \rightarrow X'' \) is the blowing up on a nonsingular \( X'' \) of a disjoint union of nonsingular integral curves;
- if \( n = 6 \) and \( d \neq 14 \), then \( (X,L) = (X',L') \) and the second reduction map \( \varphi : X' \rightarrow X'' \) is the blowing up on a nonsingular \( X'' \) of a disjoint union of nonsingular integral curves. If in addition \( d \neq 16, 22 \), then the second reduction morphism is an isomorphism: \( (X,L) \cong (X',L') \cong (X'',L'') \);
- if \( n \geq 7 \), then \( (X,L) \cong (X',L') \cong (X'',L'') \).

Proof.

Once \( K_X + (n-1)L \) is nef and big, i.e. out of the lists of Theorem 3.1 and Theorem 3.2, it fails to be ample only if the first reduction is not an isomorphism; in turn, that happens if and only if \( X \) contains some exceptional divisors of the first kind. By Proposition 2.1 this happens only if \( d = 10 \). By Proposition 1.3 the type is either M) or N); neither of them contains an exceptional divisor of the first kind. It follows that if the first reduction exists, then \( (X,L) \cong (X',L') \).

The statements concerning the second reduction morphism can be proved as follows. For \( n = 5 \), we use Theorem 0.2.1 of [5] coupled with Proposition 2.1.
For \( n = 6 \) we use Theorem 0.2.2 of [5] and then we take a general hyperplane section and reduce to the case \( n = 5 \), with the difference that now case b2) of Theorem 0.2.1 of [5] does not occur. The case of the blowing up of curves yields \( d = 16, 22 \), as we now show. Since \( X \cong X' \) we cut (1) with \( F \cong \mathbb{P}^2 \), a general fiber of the blowing up. Define \( a \) to be the positive integer such that \( L|_F \cong \mathcal{O}_{\mathbb{P}^2}(a) \). Since \( N_{F,X} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \) and \( K_{X|F} \cong \mathcal{O}_{\mathbb{P}^2}(-2) \) we get

\[
(16 - d/2)a^2 = 12a - 4.
\]

Since \( a > 0 \) we see that \( d \leq 30 \). The only integer solutions to the relation above are \( (d,a) = (16,1) \) and \( (22,2) \). This concludes the proof for \( n = 6 \).

Finally, for \( n \geq 7 \) we use Remark 1.3. \( \square \)

**Lemma 2.3** Let \( X \) be a nonsingular threefold in \( \mathbb{Q}^5 \). Let \( D \) be an integral divisor on \( X \). We have:

1. (2.3.1) if \( (D,\mathcal{O}_D(D)) \cong (\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-1)) \), then either \( d = 10 \) and \( L|_D \cong \mathcal{O}_{\mathbb{P}^2}(1) \), or \( d = 14 \) and \( L|_D \cong \mathcal{O}_{\mathbb{P}^2}(2) \);
2. (2.3.2) if \( (D,\mathcal{O}_D(D)) \cong (\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(-2)) \), then either \( d = 8 \) and \( L|_D \cong \mathcal{O}_{\mathbb{P}^2}(1) \), or \( d = 16 \) and \( L|_D \cong \mathcal{O}_{\mathbb{P}^2}(2) \);
3. (2.3.3) if \( (D,\mathcal{O}_D(D)) \cong (\mathbb{F}_0,G) \), then \( d \leq 20 \);
4. (2.3.4) if \( (D,\mathcal{O}_D(D)) \cong (\mathbb{F}_2,G) \), then \( d = 14 \) and \( L_D = -G \).
Proof. The proof is the same as the one of [6], Proposition 1.1, using (1) in the place of (0.8) of the quoted paper.

Proposition 2.4 (Structure of Mori contractions) Let $X$ be a nonsingular threefold embedded in $\mathbb{Q}^5$ with $d \geq 22$ and $K_X$ not nef. Let $\rho : X \to Y$ be the contraction of any extremal ray on $X$. Then $Y$ is nonsingular and either $\rho$ is birational and the blowing up of an integral nonsingular curve on $Y$ or $\rho$ is a conic bundle in the sense of Mori Theory. In particular, if $d \gg 0$, then only the former case can occur.

Proof. The proof is the same as the one of [6], Corollary 1.2, using (1) in the place of (0.8) of the quoted paper. As for the last statement, if $\dim Y \leq 2$, then $X$ is not of general type and we apply Proposition 1.10.

The following conjecture is due to Beltrametti, Schneider and Sommese in the case of 3-folds on $\mathbb{P}^5$. It seems a fairly natural question in view of Proposition 2.4.

Conjecture 2.5 There is an integer $d_0$ such that every threefold on $\mathbb{Q}^5$ of degree $d \geq d_0$ is a minimal model.

3 VARIETIES NOT OF LOG-GENERAL-TYPE

In this section we give a coarse classification of varieties as in the title. We still make free use of the language of Adjunction Theory.

Let $(X, L)$ be a degree $d$, $\nu$-dimensional nonsingular subvariety of $\mathbb{Q}^n$ endowed with its embedding line bundle $L$. The “Types” we shall consider correspond to the ones of Propositions 1.3 and 1.4.

We start by observing that $K_X + (\dim X - 1)L$ is spanned by its global sections (spanned for short) except for three varieties.

Theorem 3.1 Let $(X, L)$ be as above. Then $K_X + (\nu - 1)L$ is spanned unless $(X, L)$ is one of the three pairs A), C) or D). In particular, $d \leq 4$.

Proof. By the list on [8] page 381, and by the fact that there are no codimension two linear subspaces on $\mathbb{Q}^n$, $\forall n \geq 5$, we need to analyze the a.t. scroll over a curve case only. By flatness an a.t. scroll over a curve is a scroll. The result follows from Theorem 1.4.

Now we classify those pairs for which $K_X + (\nu - 1)L$ is spanned, but for which $\kappa(K_X + (\nu - 1)L) < \nu$.

Theorem 3.2 Let $(X, L)$ be as above. Assume that $K_X + (\nu - 1)L$ is spanned, i.e. $(X, L)$ is not as in Theorem 3.1, but that it is not big. Then $(X, L)$ is one of the following pairs:

(3.2.1) (Del Pezzo variety): Type B); Type F);
(3.2.2) (Quadric Bundle over a curve): Type G);
(3.2.3) (A.t. scroll over a surface): Type L); Type O).

In particular, $d \leq 12$.

Proof. Let $K_X + (\nu - 1)L$ be as in the Theorem, then by [8] page 381 $(X, L)$ is either a Del Pezzo variety, a quadric bundle or an a.t. scroll over a surface. Let us assume that $(X, L)$ is a Del Pezzo variety. By slicing with $(\dim X - 2)$ general hyperplanes we get a surface in $\mathbb{Q}^4$ with $K_S = -L|_S$. Since $S$ is Del Pezzo we get $\chi(O_S) = g(L) = 1$. We plug these values in (3) and get:

$$d^2 - 10d + 24 = 0.$$
It follows that either \( d = 4 \) or \( d = 6 \). The conclusion follows from Proposition \ref{prop:quadric-bundle}. Let us assume that \((X, L)\) is a quadric bundle. Let \( F \cong \mathbb{P}^{n-3} \) be a general fiber of the quadric fibration. Dotting \((\mathbb{P}^{n-3}, \mathcal{O}(3))\) with \( F \) we get \( d = 6 \). We conclude using Proposition \ref{prop:quadric-bundle}.

Let us assume that \((X, L)\) is an a.t. scroll over a surface. By \ref{lem:at-scroll}, Proposition 14.1.3 \((X, L)\) is an ordinary scroll with \( \kappa(K_X + (n - 1)L) = 2 \). We conclude by comparing with Proposition \ref{prop:at-scroll}.

Now we deal with the line bundle \( K_X + (\nu - 2)L \). First we exclude the presence of some special pairs.

**Lemma 3.3** Let \((X, L)\) be as above, then \((X, L)\) cannot be isomorphic to any of the three pairs \((\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))\), \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))\) and \((\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))\). Moreover, there are no Veronese bundles \((X', L')\) associated with a pair \((X, L)\) on \( \mathbb{Q}^5 \).

**Proof.** By contradiction assume that \((X, L) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))\). We intersect two general members of \(|L|\) and get a nonsingular surface section \((S, L_S)\) which is embedded in \( \mathbb{P}^4 \) with \( d = 16 \), \( g = 1 \) and \( \chi(\mathcal{O}_S) = 1 \). This contradicts \ref{lem:at-scroll}. We exclude the case in which \((X, L) \cong (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))\) in a similar way.

The possibility \((X, L) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))\) is ruled out by Remark \ref{rem:at-scroll}. Let us assume that \((X, L)\) is a pair for which \((X', L')\) exists and is a Veronese bundle with associated morphism \( p : X \to Y \); in particular \( n = 5 \). By Theorem 3.3 \((X, L) \cong (X', L')\).

Dotting \((\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))\) with a general fiber \( F \) we get \( d = 10 \). Since for some ample line bundle \( L \) on \( Y \) \( 2K_X + 3L = p^*L \), we have the following relation on a general surface section \( S \) of \( X \):

\[
L_S = -2K_S + L_S,
\]

which “squared” gives \( d = 10 \equiv 0 \mod(4) \), a contradiction.

**Theorem 3.4** Assume that we are not on the lists of Theorems \ref{thm:quadric-bundle} and \ref{thm:at-scroll} so that \((X, L) \cong (X', L')\). If \( K_X + (\nu - 2)L \) is not nef and big then \((X, L)\) is one of the following pairs:

\begin{enumerate}
\item[(3.4.1)] (Mukai variety): Type E; Type I);
\item[(3.4.2)] (Del Pezzo fibration over a curve): either Type N), \( d = 10 \) or as in \ref{lem:at-scroll}, \( d = 12 \);
\item[(3.4.3)] (Quadric bundle over a surface): \( n = 5, 6 \), a flat quadric bundle over a nonsingular surface: if \( n = 6 \), then \( d = 12 \) and if \( n = 5 \), then either \( d \leq 18 \) or \( d = 44 \).
\item[(3.4.4)] (A.t. scroll over a threefold): \( n = 6 \), the scroll map is not flat and \( d \) is either 14 or 20.
\end{enumerate}

**Proof.** Let \( K_X + (\nu - 1)L \) be as in the Theorem, then by \ref{lem:at-scroll} page 381-2 and Lemma 3.3 \((X, L)\) is either a Mukai variety, a Del Pezzo fibration over a curve, a quadric bundle over a surface or an a.t. scroll of dimension \( \nu \geq 4 \) over a normal threefold.

Let us assume that \((X, L)\) is a Mukai variety. By slicing to a surface section \( S \) we find that \( K_S = \mathcal{O}_S \), and since \( X \) is simply connected it follows that \( \pi_1(S) \) is trivial as well; \( S \) is thus a \( K3 \) surface. Using \ref{lem:at-scroll} we get, using \( \chi(\mathcal{O}_S) = 2, 2(g - 1) = d \), that either \( d = 6 \) or \( d = 8 \); accordingly \( g = 4, 5 \), respectively. The conclusion, in this case, follows from Proposition \ref{prop:quadric-bundle}.

We deal with the case of Del Pezzo fibrations over a curve in Lemma 4.1 and Proposition 4.3.

We now deal with quadric bundles over surfaces. Again, \( n = 5, 6 \), by Remark 5.8.

Let \( n = 5 \) and assume, by contradiction, that there is a divisorial fiber \( F \) of the quadric bundle map \( p : X \to Y \). Then \( F \) is as in \ref{lem:at-scroll} Theorem 2.3. This contradicts case (2.1.7) of Lemma 2.1. It follows that all the fibers of \( p \) are equidimensional. By Theorem 5.6 it follows that \( p \) is a quadric fibration in the sense of section 5. The statement follows from Proposition 5.4 and Remark 5.3.

Let \( n = 6 \). \((X, L)\) is a quadric bundle over a surface, \( p : X \to Y \), so is its general hyperplane section. By what proved for the case \( n = 5 \) the base surface \( Y \) is nonsingular and by Corollary 5.7 we deduce that \( p \) is flat. If we cut \( \mathbb{P}^4 \) with a general fiber of \( p \) we get \( d = 12 \).
Case (3.4.3) follows.

Finally case (3.4.4) follows from Proposition 1.4 which ensures us of the absence, on $Q^6$, of adjunction theoretic scrolls over threefolds for which the map $p$ is flat: if $p$ were flat then $Y$ would be nonsingular by [22] Theorem 23.7 and then $X$ would be a projective bundle, a contradiction. If one of these scrolls occurs, since $p$ is not flat and $-K_X$ is $p$-ample, Lemma 5.6 and [22], Theorem 23.1 ensures there must be a fiber $F$ such that either $F$ contains a divisor or, by [7], 14.1.4, $F$ is a surface $S$ as in Proposition 2.1. In the latter case we get $d = 14$. In the former, by slicing with a general hyperplane section, we get a threefold $\tilde{X}$ together with the morphism $\tilde{p} := p|_{\tilde{X}} : \tilde{X} \to Y$, where $Y$ is the base of the scroll. $\tilde{p}$ is the second reduction morphism for $(\tilde{X}, L|_{\tilde{X}})$, so that the result follows by looking at the divisorial fibers of $\tilde{p}$ and Lemma 2.1.

\[\blacksquare\]

4 FIBRATIONS OVER CURVES WITH GENERAL FIBER A DEL PEZZO MANIFOLD

In this section we study codimension two nonsingular subvarieties of $Q^n$, $n \geq 5$, which admit a morphism $f : X \to Y$, with connected fibers, onto a nonsingular curve $Y$, such that the line bundle $K_X + (n-4)L$ is trivial on the general fiber. The general fiber will thus be a nonsingular (adjunction-theoretic) Del Pezzo variety of the appropriate dimension $n-3$. By Remark 1.8 we have $n = 5, 6$.

The following lemma ensures that these fibrations coincide with the Del Pezzo fibrations over curves of Adjunction Theory.

**Lemma 4.1** Let $X$ be a fibration as above. Then $K_X + (n-1)L$ is ample and $\kappa(K_X + (n-2)L) = \kappa(S) = 1$.

**Proof.** Without loss of generality we may assume that $n = 5$, for otherwise we cut with a general hyperplane section to the three dimensional case and it is easy to show that if the statements we want to prove hold for the threefold hyperplane section of $X$, then they also hold for $X$.

The generic fiber of $f$ is a nonsingular Del Pezzo surface $F$. Since $K_X + L$ is trivial on the fibers we define

\[\Delta := L^2 \cdot F = L^2_F = K_F^2.\]

Cut (1) with $F$, using the facts that $K_{X|F} = K_F$ and that $x_2 \cdot F = 12 - \Delta$. We get

\[\Delta = \frac{24}{16 - d}.\]

Since $F$ is a Del Pezzo surface and $L$ is very ample, we get $3 \leq \Delta \leq 9$. Since $\Delta$ is an integer we have only the following possibilities:

\[(\Delta, d) = (3, 8), (4, 10), (6, 12).\]

Using the above invariants, and the lists of Adjunction Theory, it is easy to show that $K_X + (n-1)L$ is ample and that $\kappa(K_X + (n-2)L) = 0, 1$. By Theorem 3.4 the case $K_X = -(n-2)L$ cannot occur, since these manifolds do not carry any nontrivial fibration. It follows that $K_X + 2L$ is ample, $\kappa(K_X + L) = 1$ and, by adjunction, $\kappa(S) = 1$.

\[\blacksquare\]

We need the following facts.

**Fact 4.2** Let $f : X \to Y$ be as above. By relative vanishing we have $h^i(O_X) = h^i(O_Y)$, $\forall i$. 8
The proposition is thus proved for manifold so that the above analysis applies. The only difference is that the case hyperplane we get a threefold with a fibration onto a curve whose general fiber is a Del Pezzo

Let

This section is not needed for Theorem 3.4.

### 4.1 More Upper Bounds

We can rule out the cases:

\begin{align}
\chi &= 10 \text{ for curves on } P^2, \text{ see } [19], \text{ page } 72, \text{ either } \kappa(S) = 1 \text{ or only the following systems of invariants is possible:}
\end{align}

\begin{align}
&\text{possible:}
&\begin{align}
1) n &= 5, d = 12, K_F^2 = 6, g = 10, p_g(S) = 2, q(S) = 0, h^i(O_X) = 0, \forall i > 0.
&\end{align}
\end{align}

\begin{proof}
By the proof of Lemma 4.1 and by the knowledge of degree \(d = 8\), 10 varieties stemming from Proposition 4.2, we only need to rule out the case \(d = 8\) and make precise the invariants in the case \(d = 12\). Moreover, by the same lemma, \(\kappa(S) = 1\).

First let \(n = 5\).

Now we determine the invariants in the case \(d = 12\).

We apply formula (3) in the case \(d = 12\). We get

\begin{align}
2(g - 1) - 3\chi(O_S) = 9.
\end{align}

By Fact 4.4 and by Proposition 4.9 we are in the position to apply the Castelnuovo bound for curves on \(P^4\), which gives \(g \leq 13\).

It implies that \(\chi(O_S)\) is not a non-negative integer, unless \((g, \chi(O_S)) = (7, 1), (10, 3), (13, 5)\).

We can rule out the cases: \(d = 12\) and \((g, \chi(O_S)) = (7, 1), (10, 3), (13, 5)\) using Fact 4.3 which gives \(g - 7 = \chi(p_g + q - 1)\); this last equality together with the given values of \(\chi(O_S)\) and \(g\) gives a non-integer value for \(q\), a contradiction. It follows that if \(d = 12\), then \((g, \chi(O_S)) = (10, 3)\).

To compute the values of \(p_g\) and \(q\) we use again Fact 4.3 which gives the number \(p_g + q\). Since we know \(\chi(O_S)\) we get the values of \(p_g\) and \(q\).

Since \(g = q\) we see that \(Y \simeq P^1\). The assertions on \(h^i(O_X)\) follow from Fact 4.2.

The proposition is thus proved for \(n = 5\).

Let \(n = 6\), the only remaining case, by Barth-Larsen Theorem. By slicing with a general hyperplane we get a threefold with a fibration onto a curve whose general fiber is a Del Pezzo manifold so that the above analysis applies. The only difference is that the case \(d = 10\) does not occur by Proposition 4.3.

Now we prove that also the case \(d = 12\) does not occur.

The general fiber of \(f\) is a Del Pezzo threefold with \(K_F = -2L_{|F}\) and \(L^3_{|F} = 6\). By explicit classification, see [16], page 72, either \(F \simeq P^1 \times P^1 \times P^1\) or \(F \simeq P(T_{P^2})\). In both cases formula (2) dotted with \(F\) gives \(x_3 \cdot F = x_3(F) = 24\). But in the former case \(x_3(F) = 8\), in the latter \(x_3(F) = 6\).

\(\square\)

### 4.1 More Upper Bounds

This section is not needed for Theorem 3.4.
We now give an upper bound for the degree of codimension two, nonsingular subvarieties of $\mathbb{Q}^n$, $n \geq 5$, which admit a morphism onto a curve such that the general fiber is a Fano variety. By Barth-Larsen Theorem we need to worry only about the cases $n = 5, 6$.

**Proposition 4.6** Let $X \subseteq \mathbb{Q}^n$ a nonsingular subvariety of codimension two and degree $d$ which admits a morphism onto a curve such that the general fiber is a Fano variety. If $n = 5$, then $d \leq 20$. If $n = 6$, then $d \leq 30$.

*Proof.* Let $n = 5$ and $\mathcal{L} := L_{|F}$. Assume that $d \geq 22$. We cut (1) with a fiber, $F$, and obtain, on $F$:

$$(11 - d/2)\mathcal{L}^2 + 5K_F \mathcal{L} + K_F^3 - c_2(F) = 0.$$  

Since $c_2(F) = 12 - K_F^2$, we get:

$$(d/2 - 11)\mathcal{L}^2 + 12 + 5K_F \mathcal{L} = 0. \quad (9)$$

Now we use $K_F^2 \leq 9$ to get

$$(d/2 - 11)\mathcal{L}^2 \leq 6 + 5K_F \mathcal{L}. \quad (10)$$

Since $K_F \mathcal{L} \leq -1$, we see that either $d = 22$, or $d = 24$ and $\mathcal{L}^2 = -K_F \mathcal{L} = 1$. In the latter case $F \cong \mathbb{P}^2$ and the Hodge Index Theorem on the surface $F$ says that $K_F^2 = 1$, a contradiction. In the former case we use (9):

$$2K_F^2 - 12 + 5K_F \mathcal{L} = 0,$$

which gives a contradiction for each value $K_F^2 = 1, \ldots, 9$. It follows that $d \leq 20$.

The proof of the statement for $n = 6$ is analogous to the proof of Proposition 4.7, where we use (11) with $n = 6$ cut with the cycle $K_X \cdot F$. \hfill $\square$

In the same spirit we give an upper bound on the degree of Fano threefolds on $\mathbb{Q}^5$.

**Proposition 4.7** Let $X \subseteq \mathbb{Q}^5$ be a nonsingular Fano threefold. Then $d \leq 20$.

*Proof.* (Cf. [6], Corollary 1.2.) We cut (1) with $K_X$ and get, using the fact that $x_1x_2 = 24\chi(O_X) = 24$:

$$(11 - d/2)\mathcal{L}^2 + 5LK_X^2 + K_X^3 + 24 = 0.$$  

Let

$$\lambda := LK_X^2, \quad 2\mu := -L^2K_X = -2g + 2 + 2d;$$

clearly $\lambda$ and $\mu$ are positive integers and the above becomes:

$$(d - 22)\mu + 5\mu \lambda + 24 = -K_X^3. \quad (11)$$

By the Generalized Hodge Index Theorem, see [10], we get $(-K_X^3)(-K_X \mathcal{L}^2) \leq (K_X^2 \mathcal{L})^2$, or

$$(-K_X)^3(2\mu) \leq \lambda^2. \quad (12)$$

By combining (11) and (12) we get

$$\lambda^2 - 10\mu \lambda - [2(d - 22)\mu^2 + 48\mu] \geq 0. \quad (13)$$

If we solve the above in $\lambda$ we get either $\lambda < 0$, a contradiction, or $\lambda > 10\mu$. This implies, in turn, that $\lambda \geq 11$. Since, by the classification of Fano threefolds, $-K_X^3 \leq 64$, (13) becomes

$$(d - 22)\mu + 55 + 24 \leq 64,$$

a contradiction for $d \geq 22$. \hfill $\square$

10
5 QUADRIC FIBRATIONS

In this section the term “quadric bundle” is to be intended in the sense of Adjunction Theory. The term “quadric fibration” is introduced below.

By quadric fibration we mean a nonsingular projective variety $X \subseteq P$, of dimension $x$, together with a fibration $p : X \to Y$ onto a (a fortiori) nonsingular variety $Y$ of positive dimension $y$, all of which fibers are quadrics, not necessarily integral, of the appropriate dimension $(x - y)$. One has non integral fibers only if the relative dimension is one.

The case $\dim Y = 0$ is trivial. In virtue of Remark \[\text{we have:} \]

**Fact 5.1** There are no codimension two quadric fibrations on $Q^n$, for $n \geq 7$ and, for $n = 6$, any such is simply connected.

We restrict ourselves to the case of $n \geq 5$.

We begin by fixing some notation and establishing some simple facts. Let $L$ denote the restriction to $X$ of the hyperplane bundle. The sheaf $E := p_* L$ is locally free on $Y$ of rank $(x - y + 2)$. It is easy to check that $E$ is generated by its global sections. The surjection $p^* p_* L \to L$ defines an embedding: $X \hookrightarrow \mathbb{P}(E)$, where $L = \xi_{E|X}$ and $X$ is defined by a nonzero section of the line bundle $2\xi - \pi^* M$, for some $M \in \text{Pic}(Y)$, where $\pi : \mathbb{P}(E) \to Y$ is the bundle projection.

The following gives a sufficient condition for a general hyperplane section of $X$ to be a quadric fibration over $Y$. It is a well known “counting dimensions” argument.

**Lemma 5.2** Let $X \to Y$ be a quadric fibration as above. Assume $2y < x + 2$. Then a general hyperplane section $X'$ of $X$ is a quadric fibration over $Y$ via $p_{X'} : X' \to Y$.

**Proof.** Since $E$ is generated by global sections and, by assumption $\text{rank}(E) > y$, a general section of it does not vanish on $Y$. Such a section will define, for every $y \in Y$, a hyperplane $\lambda_y$ of the corresponding fiber $\pi^{-1}(y) \subseteq \mathbb{P}(E)$. In the case in which the quadrics $p^{-1}(y)$ were integral $\forall y \in Y$, we would be done. This is, in general, not true. However, the singular quadrics of the fibration are parameterized by a proper closed subset $D$ of $Y$ with $\dim D \leq (y - 1)$. The hyperplanes of $\mathbb{P}$ which contain the reduced part, $\Sigma \subseteq \mathbb{P}^{x - y}$, of one of the components of one non integral quadric of the fibration form a linear space of dimension $(\dim \mathbb{P} - x + y - 1)$ contained in $\mathbb{P}^{\Sigma}$. The space of these bad hyperplanes is of dimension at most $(\dim D + \dim \mathbb{P} - x + y - 1) \leq \dim \mathbb{P} - x + 2y - 2 < \dim \mathbb{P}^{\Sigma}$. It follows that the general section of $E$ gives a hyperplane section of $X$ which cuts every quadric of the fibration in a quadric of dimension one less. $\square$

**Proposition 5.3** There are no quadric fibrations over curves on $Q^n$. The only quadric fibrations over curves on $Q^5$ are of Type G). If there is a quadric fibration over a surface on $Q^6$, then it has degree $d = 12$.

**Proof.** As to quadric fibrations over curves, we cut $[\square]$ with a nonsingular fiber $F \simeq Q^{n-3}$, we get $d = 6$. We conclude by comparing with Proposition $[\square]$.

As to quadric fibrations over a surface we cut $[\square]$ with a nonsingular fiber $F \simeq Q^{n-4}$ and get $d = 12$. $\square$

The following proposition and remark describe the situation for threefolds quadric bundles over surfaces.

**Proposition 5.4** Let $X \subseteq Q^5$ be a threefold quadric fibration (conic bundle) over a surface $Y$. Then either $d \leq 98$ or $X$ is contained in a hypersurface $V \in |\mathcal{O}_{Q^5}(3)|$ and $d \leq 276$.  

11
Proof. We denote the Chern classes of $X$ and $Y$ by $x_i$ and $b_i$, respectively. We omit the symbol "p" for ease of notation. We follow closely the paper [11]. First we introduce the following entities and we report from [11], for the reader’s convenience, the relations among them which are essential to the computations below (one warning: some of the equalities are only numerical equalities):

$\mathcal{M}$ was defined at the beginning of the section; $D \in |2e_1 - 3\mathcal{M}|$, it is called the discriminant divisor; its points correspond to the singular fibers of $p$;

$2R \subseteq Y$ the branching divisor associated with a general hyperplane section, $S$, of $X$, which, in view of Lemma 5.2 is a cyclic double cover of $Y$;

$e_1 = 3R - D$;

$\mathcal{M} = 2R - D$;

$x_1 = L + b_1 - R$;

$x_2 = L^2 + L \cdot (b_1 - 2R + D) + (-2R^2 - R \cdot b_1 + D \cdot R + b_2 + e_2)$;

$x_3 = 2b_2 - D^2 + Db_1$;

$L \cdot W \cdot W' = 2W \cdot W'$, for every pair of divisors $W$ and $W'$ on $Y$;

$L^2 \cdot W = (4R - D) \cdot W$;

$e_2 = \frac{1}{4}(12R^2 + D^2 - 7DR - d)$.

Now we plug the above values of $x_1$ and $x_2$ for $x_1$ and $x_2$ in [11]:

$\begin{align*}
(6 - \frac{d}{2})L^2 - 4Lb_1 + 5LR + b_1^2 - b_1R - LD + 3R^2 - DR - b_2 - e_2 = 0. \quad (14)
\end{align*}$

Next we equate the expression above for $x_3$ to the one of [11]; using again the above expressions for $x_1$ and $x_2$:

$\begin{align*}
-(2d + 10)b_1R + 2dR^2 + \frac{(2d + 4)Db_1 + D^2 - 10b_2 + 2b_1^2 - \left(\frac{d}{2} + 5\right)DR - d(\frac{d}{2} - 13) = 0. \quad (15)
\end{align*}$

Now we set

$x := b_1^2$ and $y := DR$,

we cut [11] with $R, -b_1, D$ and $L$, respectively, so that we obtain four linear equations to which we add [11], after having substituted in $x$ and $y$. The result is the following linear system of equations:

$\begin{align*}
Mv^t = c^t,
\end{align*}$

where

$\begin{align*}
M := \begin{pmatrix}
-8 & 34 - 2d & 0 & 0 & 0 \\
2d - 34 & 0 & -\frac{d}{2} + 8 & 0 & 0 \\
0 & 0 & -8 & \frac{d}{2} - 8 & 0 \\
-18 & 14 & +4 & 0 & -2 \\
-2d - 10 & 2d & \frac{d}{2} + 4 & 1 & -10
\end{pmatrix},
\end{align*}$

$v := (b_1R, R^2, Db_1, D^2, b_2 )$

and

$c := (8 - \frac{d}{2})y, -8x, (2d - 34)y, 2x + 4y + d(\frac{d}{2} - 7), -2x + (\frac{d}{2} + 5)y + d(\frac{d}{2} - 13)$.

Since $P := -\frac{1}{4}detM = 3d^3 - 27d^2 - 1520d + 18976 > 0, \forall d > 0$, we can solve the above system [11] and obtain the unique solution:

$\begin{align*}
b_1R &= -\frac{1}{4}((-128d^3 + 4480d - 39168)x + (2d^3 - 111d^2 + 2020d - 12096)y + (2d^3 - 10d^4 + 2678d^3 - 26304d^2 + 95744d)]/P,
R^2 &= \frac{1}{4}((-1024d + 18432)x + (3d^3 - 8d^2 - 2112d + 23552)y + (16d^4 - 688d^3 + 9728d^2 - 45056d)]/P,
\end{align*}$

12
\[ b_1D = -2\left[(-152d^2 + 4440 - 32128)x + (2d^3 - 113d^2 + 2099d - 12852)y + (2d^5 - 122d^4 + 2766d^3 - 27574d^2 + 101728d)/P, \right. \]
\[ D^2 = -4\left[(-1216d + 16064)x + (-3d^3 + 46d^2 + 893d - 13736)y + (16d^4 - 720d^3 + 10608d^2 - 50864d)/P, \right. \]
\[ b_2 = \frac{1}{2}\left[(12d^2 + 20d^2 - 3648d + 13952)x + (d^3 - 30d^2 + 152d + 960)y + (d^5 - 27d^4 + 274d^3 - 4448d^2 + 46016d)/P. \right] \]

Since \( E \) is generated by global sections and \( D \) is effective we see that \( e_2 \geq 0, e_1D \geq 0 \). Also, \([11], \text{Lemma 2.9 gives } y = DR \geq 0 \). We can make explicit \( e_2 \) and \( e_1 \) by the formulæ given at the beginning of this proof and deduce:

\[ \begin{align*}
DR &= y \geq 0, \\
e_2 \cdot P &= (896d - 4480)x - \left(\frac{19}{2}d^2 - 366d + 3616\right)y - \left(\frac{19}{2}d^4 - \frac{843}{2}d^3 + 5864d^2 - 24656d\right) \geq 0, \\
e_1D \cdot P &= -(4864d - 64256)x - (3d^3 - 103d^2 + 988d - 1984)y + (64d^4 - 2880d^3 + 42432d^2 - 203456d) \geq 0.
\end{align*} \]

These three inequalities define a region of the plane \((x, y)\). It is straightforward to check that the two lines \( e_2 = 0 \) and \( e_1D = 0 \) have slopes \( a \) and \( b \) whose sign does not change with \( d \) if \( d \geq 20 \). One can check easily that \( a > 0 \) and \( b < 0 \). The intersection of the first line above with the \( x \)-axis is

\[ (x_1, 0)_{e_2} = \left(\frac{(19/2)d^4 - (843/2)d^3 + 5864d^2 - 24656d}{896d - 4480}, 0\right); \]

the intersection of the second line with the \( x \)-axis is

\[ (x_2, 0)_{e_1D} = \left(\frac{64d^4 - 2880d^3 + 42432d^2 - 203456d}{4864d - 64256}, 0\right). \]

One can check, that, since \( d \geq 20, x_1 < x_2 \). The region we are interested in is a triangle with vertices \((x_1, 0)_{e_2}, (x_2, 0)_{e_1D} \) and \((x_3, y_3)_{(e_2=0)/(e_1D=0)} \).

Now we compute the genus of a general curve section, \( C \), of \( X \). By adjunction \( x_1 \cdot L^2 = 2d + 2 - 2g \), so that by what above:

\[ g - 1 = \frac{d}{2} - 2b_1R + \frac{Dd_1}{2} + 2R^2 - \frac{DR}{2} = -2b_1R + \frac{Dd_1}{2} + 2R^2 - \frac{y}{2} + \frac{d}{2} = \left((24d^2 - 472d + 2176)x + ((23/2)d^2 - 375d + 3044)y + ((23/2)d^4 - (891/2)d^3 + 5374d^2 - 19024d)\right)/P. \]

Again it is not difficult to check that the absolute value of the slope of the above line is bigger than \( |b| \). It follows easily that the maximum possible value for \( g - 1 \) in our region is achieved at \((x_2, 0)_{e_1D} \), while the minimum is at \((x_1, 0)_{e_2} \). We thus get

\[ \frac{19d^3 - 187d^2 + 416d}{22d - 1120} \leq g - 1 \leq \frac{4d^3 - 77d^2 + 321d}{38d - 502}. \]

Assume that \( C \) is not contained in any surface of \( \mathbb{Q}^3 \) of degree strictly less than \( 2 \cdot 11 \). Then by \([13]\) and by the left hand side inequality of \([17]\), we get

\[ \frac{19d^3 - 187d^2 + 416d}{22d - 1120} \leq \frac{d^2}{2} + \frac{7}{2}d, \]

13
which, remembering that \( d \) is even and that we are assuming \( d \geq 20 \), implies \( d \leq 98 \).
Assume that \( C \) is contained in a surface of degree \( 2k \), with \( k = 10, 9, \ldots, 3 \). By Corollary \([1.5]\) we infer:
\[
\frac{19d^3 - 187d^2 + 416d}{224d - 1120} \leq \frac{d^2}{4k} + \frac{k - 3}{2}d,
\]
which implies, as above, that for \( k = 10, 9, \ldots, 3 \), \( d \leq 64, 58, 54, 48, 44, 40, 40 \) and 276, respectively.
Finally, assume that \( C \) is contained in a surface of degree four or two. Using the right hand side inequality of \([17]\) and Lemma \([1.7]\) we get \( d \leq 42 \) and \( d \leq 16 \), respectively. Actually in the last case we get a contradiction, since we are assuming \( d \geq 20 \).
Finally if \( C \) is in a surface of degree six, then \( X \) is in a hypersurface of degree six in \( Q^5 \), provided, \( d > 18 \) (cf. Proposition \([1.9]\))

**Remark 5.5** We have checked with a Maple routine which are the possible degrees of a threefold on \( Q^5 \) which is a quadric fibration over a surface. For \( d \geq 20 \) we have imposed the following restrictions on the triples \((d, x, y)\):
1) \( 20 \leq d \leq 276 \);
2) for every fixed \( d \) as above \((x, y)\) must belong to the triangle of the proof of Proposition \([5.4]\);
3) \( b_1 R, R^2, b_1 D, D^2, b_2, g - 1, \chi(O(Y)) \) and \( \chi(O(S)) \) must be integers;
4) \( (g - 1) \) must satisfy inequality \([17]\) and the bound of Theorem 2.3 in [18];
5) \( \chi(O(S)) \) must satisfy the two inequalities of Proposition \([1.4]\);
6) various inequalities stemming from the Hodge Index Theorem on \( Y \) as, for example, \((K_Y R)^2 \geq K_Y^2 R^2\);
7) if \( d > 98 \) then \( g - 1 \leq (1/12)d^2 \), see Proposition \([1.5]\).

The result is that the only possible degree, for \( d \geq 20 \) is \( d = 44 \).
By taking double covers of the four scrolls of \([24]\), we see that there are flat conic bundles over surfaces for \( d = 6, 12, 14, 18 \). We do not know whether the case \( d = 44 \) occurs.

### 5.1 DIGRESSION

In the course of the proof of Theorem \([3.4]\) we used the fact, due to Besana \([9]\), that the base of an adjunction theoretic quadric bundle over a surface is nonsingular. The following is a result with a similar flavor. It is probably well known.

**Lemma 5.6** Let \( X \) a nonsingular projective variety of dimension \( n \), \( p : X \rightarrow Y \) a morphism onto a normal projective variety \( Y \) of dimension \( n - 1 \) such that all fibers have the same dimension, the general scheme theoretic fiber over a closed point is isomorphic to a conic and \( -K_X \) is \( p \)-ample. Then all the scheme-theoretic fibers are isomorphic to conics, \( p \) is flat and \( Y \) is nonsingular.

**Proof.** The proof is the same as the one of \([23]\) Lemma 3.25. The only necessary changes are the following: a) replace the line bundle \( H \) of \([23]\), by a pull-back \( p^* A \) of any ample line bundle \( A \) on \( Y \) and use Kleiman criterion of ampleness to obtain the result analogue to the last assertion of \([23]\) Lemma; b) replace \([23]\) Lemma 3.12 by \([1]\) Lemma 1.5.

**Corollary 5.7** Let \( X \) be a nonsingular projective variety together with a morphism \( p : X \rightarrow Y \), where \( Y \) is a normal variety of dimension \( m \). Let \( D_i, i = 1, \ldots, n - m - 1 \) be divisors on \( X \) such that they intersect transversally; denote by \( X' \) their intersection. Assume that \( p|_{X'} : X' \rightarrow Y \) satisfies the hypothesis of Lemma \([5.6]\). Then \( p \) is flat and \( Y \) is nonsingular.

**Proof.** By the lemma, \( p|_{X'} \) is flat. We can “lift” this flatness to \( p \) by virtue of \([23]\), Corollary to Theorem 22.5. As above the flatness of \( p|_{X'} \) (or of \( p \)) implies the nonsingularity of \( Y \).
Corollary 5.8 Let $X$ a nonsingular projective variety of dimension $n$, $p : X \to Y$ a morphism onto a normal projective variety $Y$ of dimension $n - 1$ such that all fibers have the same dimension. If the general fiber of $p$ is actually embeddable as conics with respect to an embedding of $X$, then all scheme theoretic fibers are actually embedded conics, $p$ is flat, $Y$ is nonsingular and $-K_X$ is $p$-ample.

Proof. We argue as in the proof of the lemma with the simplifications due to the fact that a flat deformation of a conic in projective space is still a conic. The assertion on $-K_X$ follows by observing that, if $L$ denotes the line bundle with which we embed $X$, $K_X + L$ is a pull-back from $Y$. □

Remark 5.9 The assumption $-K_X$ is $p$-ample is essential in the lemma, as the blow up of a $\mathbb{P}^1$ bundle over a curve at two distinct points on a fiber shows. Moreover, the above Lemma does not follow directly from [23] or [1], since there are conic bundles which structural morphism is not a Mori contraction. Finally, the above theorem is certainly false if one has $\dim X = \dim Y$. It is a purely local question: consider the quotient of $\mathbb{A}^2$ by the involution $(x, y) \to (-x, -y)$.

References

[1] T. Ando, “On extremal rays of the higher dimensional varieties,” Invent. Math. 81 (1985), 347-357.

[2] E. Arrondo, I. Sols, “On congruences of lines in the Projective Space,” Soc. Mat. de France, Mémoire n° 50, Suppl. au Bull. de la S.M.F., Tome 120, 1992, fascicule 3.

[3] W. Barth, Submanifolds of low codimension in projective space,” in Proc. Intern. Cong. Math., Vancouver (1974), 409-413.

[4] M.C. Beltrametti, A. Biancofiore, A.J. Sommese, “Projective n-folds of log-general type. I,” Trans. Amer. Math. Soc. 314 (1989), 825-849.

[5] M. C. Beltrametti, M.L Fania, A.J. Sommese, “On the adjunction theoretic classification of projective varieties,” Math. Ann. 290 (1991), 31-62.

[6] M. C. Beltrametti, M. Schneider, A. J. Sommese, “Special properties of the adjunction theory for threefolds in $\mathbb{P}^5$,” to appear in Mem. of the Amer. Math. Soc.

[7] M. Beltrametti and A.J. Sommese, The adjunction theory of complex projective varieties, Expositions in Mathematics, 16 (1995), 398+xxi pages, Walter De Gruyter, Berlin.

[8] M.C. Beltrametti, A.J. Sommese, “New properties of special varieties arising from adjunction theory,” J. Math. Soc. Japan, Vol. 43, No.2, 1991.

[9] G.M. Besana, “On the geometry of conic bundles arising in adjunction theory,” Math. Nachr. 160 (1993), 223-251.

[10] R. Braun, G. Ottaviani, M. Schneider, F.-O. Schreyer, “Boundedness of nongeneral type 3-folds in $\mathbb{P}^5$,” in Complex Analysis and Geometry, ed. by V. Ancona and A. Silva, 311-338 (1993), Plenum Press, New York.

[11] R. Braun, G. Ottaviani, M. Schneider, F.-O. Schreyer, “Classification of log-special 3-folds in $\mathbb{P}^5$, ” preprint Bayreuth, 1992.

[12] M.A.A. de Cataldo, “The genus of curves on the three dimensional quadric,” preprint (1994).
[13] M.A.A. de Cataldo, “A finiteness theorem for low-codimensional nonsingular subvarieties of quadrics,” Preprint (1995).

[14] M.A.A. de Cataldo, “Codimension two subvarieties of quadrics: scrolls and classification in degree $d \leq 10$,” preprint, 1996.

[15] M.L. Fania, L.E. Livorni, “Degree ten manifolds of dimension $n \geq 3$,” preprint (1994).

[16] T. Fujita, Classification Theories of Polarized Varieties, London Math. Soc. Lecture Note Ser. 155, Cambridge University Press, (1990).

[17] W. Fulton, Intersection Theory, Ergeb. Math. Grenzgeb. (3) 2, Springer-Verlag, Berlin, (1984).

[18] M. Gross, “Surfaces of degree 10 in the Grassmannian of lines in 3-space,” J. Reine Angew. Math. 436 (1993), 87-127.

[19] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math. 52, Springer-Verlag, New York, (1978).

[20] P. Ionescu, “Embedded projective varieties of small invariants, III,” in Algebraic Geometry, Proceedings of Conference on Hyperplane Sections, L’Aquila, Italy, 1988, ed. by A.J. Sommese, A. Biancofiore, L.E. Livorni, Lecture Notes in Math., 1417 (1990), 138-154, Springer-Verlag, New York.

[21] E.L. Livorni, A.J. Sommese, “Threefolds of non-negative Kodaira dimension with sectional genus less than or equal to 15,” Ann. Sc. Norm. Sup. Pisa Cl. Sci. Ser (4) 13 (1986), 537-558.

[22] H. Matsumura, Commutative Ring Theory, Cambridge studies in advanced mathematics 8 (1992), Cambridge University Press.

[23] S. Mori, “Threefolds whose canonical bundle are not numerically effective,” Ann. of Math. 116 (1982), 132-176.

[24] G. Ottaviani “On threefolds in $\mathbb{P}^5$ which are scrolls,” Ann. Scuola. Norm. Sup. Pisa Cl. Sci. Ser. (4) 19 (1992), 451-471.

[25] A.J. Sommese, “On the nonemptyness of the adjoint linear system of a hyperplane section of a hyperplane section of a threefold,” J. Reine Angew. Math. 402 (1989), 211-220; “Erratum,” J. Reine Angew. Math. 411 (1990), 122-123.

1991 Mathematics Subject Classification. 14C05, 14E05, 14E25, 14E30, 14E35, 14J10, 14J30, 14J35, 14J40, 14J45, 14M07.

Key words and phrases. Adjunction Theory, classification, codimension two, conic bundles, low codimension, non log-general-type, quadric, reduction, special variety.

AUTHOR’S ADDRESS

Mark Andrea de Cataldo, Department of Mathematics, Washington University in St. Louis, Campus Box 1146, St. Louis, Missouri 63130-4899.

e-mail: mde@math.wustl.edu