Lower- and higher-order nonclassicality in a Bose-condensed optomechanical-like system and a Fabry–Perot cavity with one movable mirror: squeezing, antibunching and entanglement

Nasir Alam¹, Kishore Thapliyal¹, Anirban Pathak¹,†, Biswajit Sen¹, Amit Verma*, Swapan Mandal§

¹ Jaypee Institute of Information Technology, A-10, Sector-62, Noida UP-201307, India
* Jaypee Institute of Information Technology, Sector-128, Noida, UP-201304, India
† Department of Physics, Vidyasagar Teachers’ Training College, Midnapore-721101, India
§ Department of Physics, Visva-Bharati, Santiniketan-731235, India

(Dated:)

Various lower- and higher-order nonclassical properties have been studied for two physical systems- (i) an optomechanical system composed of a Fabry–Perot cavity with one nonlinearly oscillating mirror and (ii) an optomechanical-like system formed using a Bose-Einstein condensate (BEC) trapped inside an optical cavity. The investigation is performed using a perturbation method that leads to closed form analytic expressions for the time evolution of the relevant bosonic operators. In the first system, it is observed that the radiation pressure coupling leads to the emergence of lower- and higher-order squeezing, antibunching, entanglement and intermodal squeezing. The effects of the coherent interaction of a nonlinear oscillating mirror with the cavity mode are studied, and it is observed that the optomechanical system studied here becomes more nonclassical (entangled) when the coupling strength is increased. It is also observed that the possibility of observing entanglement depends on the phase of the movable mirror. The Hamiltonian of the trapped BEC system is obtained as a special case of the Hamiltonian of the first system, and the existence of various nonclassicality in the trapped BEC system has been established, and variations of those with various physical parameters have been reported with an aim to understand the underlying physical process that leads to and controls the nonclassicality.

Keywords: Optomechanical system, optomechanics-like system, higher-order nonclassicality, squeezing, antibunching, entanglement

I. INTRODUCTION

Recent success in detecting gravitational wave [1, 2] at the Laser Interferometer Gravitational-Wave Observatory (LIGO), has enhanced the interest in the physical systems that can be used to detect gravitational waves. A particularly important example of such a physical system is an optomechanical system formed by movable mirror of Fabry–Perot cavity pumped by detuned laser [3]. In fact, the field of cavity optomechanics was originated from the Braginsky and Khalili’s pioneering proposal [4] for the gravitational wave detection. Later on, several applications have been reported for cavity optomechanics and related fields, and various optomechanical [5–12], nanomechanical [13–14] and optomechanics-like [15–19] systems have been investigated both theoretically [7–11] and experimentally [5–12, 17]. Interestingly, a majority of the recent investigations on the optomechanical systems are focused on the nonclassical properties of the modes of the optomechanical systems [5–6, 8–10, 15]. To stress on the relevance of these studies, it would be apt to note that a nonclassical state does not have a classical analogue, and it is characterized by the negative values of the Glauber-Sudarshan P function [20, 21]. The existence of nonclassicality in general, and entanglement in particular has already been reported in a nano-mechanical oscillator with a Cooper-pair box [22], ultracold atomic Bose-Einstein condensate (BEC) [23, 24], arrays of nano-mechanical oscillators [25], two mirrors of an optical ring cavity [26], or two mirrors of two different cavities illuminated with entangled light beams [27], etc. The presence of other types of nonclassical states, like squeezed and antibunched states has also been reported in optomechanical systems [7, 28–33]. Specifically, squeezed states are reported in Refs. [7, 29, 31], and antibunched states are reported in Refs. [28, 32, 33]. In many of these studies, Fabry–Perot cavity played a crucial role [7, 29, 30, 32, 33], and that is what motivated us for the present study.

Typically, it is assumed that one of the mirrors constituting the Fabry–Perot cavity oscillates linearly. However, Joshi et al. [7] have recently investigated a more general model that allows nonlinear oscillation of the mirror [7]. The model adopted in [7] appears to be more general, as the previously studied Fabry–Perot cavity having linearly oscillating mirror can be obtained as a special case of it. Further, in the same limit (i.e., when the nonlinear part vanishes), the general Hamiltonian of the Fabry–Perot cavity with nonlinearly oscillating mirror reduces to the Hamiltonian of an optomechanics-like system comprising of a BEC trapped in an optical cavity, which has been recently studied by various groups [5, 6, 15]. In addition, Mancini et al. [10] have shown that the Schroedinger cat states [34, 35] of the cavity field can be generated in a Fabry–Perot cavity with a movable mirror that can be treated as quantum harmonic oscillators. The model of such a Fabry–Perot cavity using two two-level systems in a one dimensional waveguide was proposed by Fratini et al., in Ref. [36]. The opto-mechanical coupling, provided by radiation pressure, in various optomechanical systems has been established as a useful tool for quantum state engineering [37, 38] as it can be used to manipulate the quantum state of light [39, 42].

The nonclassical states (i.e., quantum states having negative
Glauber-Sudarshan $P$ function) have various applications in quantum information processing and other domains. Specifically, squeezed states have been used for continuous variable quantum cryptography [43], teleportation of coherent states [44], reduction of noise in LIGO experiment [45–47], etc.; antibunched states are an essential ingredient of quantum cryptography as it is useful in characterizing single photon sources [28, 48]; also, entangled states are essential for quantum teleportation [49], densecoding [50], entangled state-based quantum cryptography [51], etc. Due to their variety of applications, characterization of nonclassical states is considered as an important task. However, the Glauber-Sudarshan $P$ function is not directly measurable through any experiment. Therefore, a set of moment based criteria for nonclassical states have been introduced. Although only an infinite set of these moment based criteria is both necessary and sufficient as the $P$ function is, here we restrict our task to use some of these criteria to establish the nonclassical behavior of the optomechanical and optomechanics-like systems under consideration. As the criteria used here are only sufficient (not necessary), satisfaction of them would ensure the presence of the corresponding nonclassical state, but the failure would not lead to any conclusion. In the analogy of the lower-order criteria of nonclassicality, there exists some criteria based on moments, which are functions of higher powers of annihilation and creation operators and work as witnesses for higher-order nonclassicality. Some recent experimental studies [52–55] have also established that sometimes it becomes easier to detect weak nonclassicality using higher-order nonclassicality criteria in comparison to their lower-order counterparts. Therefore, here we aim to study both lower- and higher-order nonclassicality.

The importance of Fabry–Perot cavity and the potential applications of nonclassical states discussed above have motivated us to investigate the possibility of observing lower- and higher-order nonclassicality in a Fabry–Perot cavity with specific attention to entanglement. In fact, this investigation is further motivated by the recent works [56, 57] that clearly established the relevance of cavity optomechanics in the interdisciplinary field of quantum information processing. Specifically, in [56], it was shown that preparation, storage and read-out of heralded single phonon Fock state is possible in a cavity optomechanical system, and in [57], it was established that the nonlinearities induced in an optomechanical system (on or near resonance) can be used to realize controlled quantum gates involving optical and phononic qubits. This observation has specially motivated us to investigate the possibility of observing intermodal entanglement in optomechanical systems.

Following independent approaches, nonclassical properties of BECs are also studied in detail ([5, 6, 15, 58, 59] and references therein). Interestingly, in some of these studies, efforts have been made to investigate optomechanics-like properties of BECs, too [5, 6, 15]. Specifically, in Ref. [15], investigations have been performed for a BEC trapped inside an optical resonator and driven by both a classical and a quantized light field; and very interestingly, a role reversal between the matter-wave field and the quantized light field has been observed. This role reversal phenomenon was of particular interest as in this system, the matter-wave field (quantized light field) was observed to play the role of the quantized light field (movable mirror), and it was in sharp contrast to the earlier studied BEC-based cavity optomechanical systems [5, 6]. This interesting feature motivated us to investigate the nonclassical properties of different modes of the BEC optomechanical system described in Ref. [16], too. Specifically, in Ref. [15], the authors studied the nonclassical nature of the system using Wigner function. In the present study, we aim to study a more general system and obtain the optomechanics-like system studied in [15] as a special case, and subsequently extend the results of [15] by showing the existence of lower- and higher-order nonclassicality via other witnesses of nonclassicality. In what follows, we investigate nonclassical properties of a system composed of a Fabry–Perot cavity with one movable mirror [7] as depicted in Fig. 1. At first, we would consider the most general case in which the movable mirror is nonlinear in nature with a nonlinearity proportional to the $x^4$, where $x$ represents the displacement of the mirror from its equilibrium position. The mirror is coupled to the field mode of the cavity through the radiation pressure. Subsequently, we would consider a special case of this system, where the nonlinear coupling constant vanishes and the system reduces to a system which is mathematically equivalent to the BEC trapped optomechanics-like system studied in Ref. [15]. The coherent interaction of the movable mirror with the cavity mode or trapped BEC is responsible for lower- and higher-order nonclassical properties in the cavity resonator. Such an optomechanical system is useful in various interdisciplinary fields in quantum technology and quantum information [56]. Consequently, numerous authors have previously treated the system of a cavity field and a movable mirror, but no one has yet investigated the possibilities of observing higher-order nonclassicalities. In this article, we present the lower- and higher-order nonclassical properties of the above mentioned optomechanical systems.

Figure 1: Fabry–Perot cavity with one movable mirror having mass $m$, equilibrium cavity length $L$ and a maximum amplitude of vibrating mirror $x$.

The rest of the paper is organized as follows. In Section II, we introduce the theoretical model describing the interaction between a movable mirror with the quantized cavity mode in a Fabry–Perot cavity. We also describe how Hamiltonian of this system reduces to that of the BEC trapped optomechanics-like
system. In Section II, we provide a closed form analytic solution of the model Hamiltonian in Heisenberg picture using a perturbative technique known as Sen-Mandal technique. In Section III we investigate the presence of lower- and higher-order squeezing, and intermodal squeezing using the field and oscillating mirror operators obtained in the previous section. Similarly, in Section IV we investigate the possibility of observing lower- and higher-order entanglement using various criteria. Finally, the paper is concluded in Section VII.

II. THE MODEL HAMILTONIAN

The optomechanical system of our interest is composed of a Fabry–Perot cavity with one fixed and one movable mirror. As stated in the previous section, the movable mirror is nonlinear in nature with a nonlinearity proportional to the \(x^3\), where \(x\) represents the displacement of the mirror from its equilibrium position. This system models (imitates) a Kerr-like nonlinear medium illuminated with a coherent light. It’s possible to construct a quantum mechanical Hamiltonian representing this system in a closed analytic form. In order to construct such a Hamiltonian, the retardation effects due to the oscillating mirror in the intracavity field are usually neglected. The correction to the radiation pressure force due to the Doppler frequency shift of the photons is also neglected [60]. Further, the Casimir effect [61], in the cavity, can also be safely neglected. If we assume that the leakage of photons from the cavity is negligible, then the main source of decoherence would be the coupling of mirrors to its surroundings, which can also be neglected up to some extent [62]. Thus, neglecting the dissipation of the system, we restrict ourselves to a situation where we only consider the unitary time evolution of the coupled system of cavity and nonlinearly oscillating mirror.

For further investigation, we express the model Hamiltonian in terms of the creation and annihilation operators. On the application of rotating wave approximation (RWA), i.e., neglecting the fast rotating terms, the analytic form of the quantum mechanical Hamiltonian (see Eq. 19 of [7]) of the system described above takes the following form

\[
H = \hbar \omega_k k^\dagger k + \hbar \omega_m a^\dagger a + \hbar \beta a^2 a^2 - \hbar g (k^\dagger k a^\dagger + \text{H.c.}),
\]

(1)

where \(\omega_m = \omega_0 + 2\beta\) and H.c. stands for the Hermitian conjugate. For simplicity, throughout the article we have used \(\hbar = 1\). The annihilation (creation) operator \(a (a^\dagger)\) corresponds to the mode of the movable mirror with frequency \(\omega_m\) and mass \(m\) and the annihilation (creation) operator \(k (k^\dagger)\) corresponds to the quantized cavity mode with resonant frequency \(\omega_k = n\frac{2\pi}{L}\), where \(n\) is an integer, \(L\) is the equilibrium length of the cavity, and \(c\) is the velocity of light in the free space. The parameters \(\beta\) and \(g = \frac{c}{\sqrt{2m_0}}\) are the anharmonic and coupling constants due to radiation pressure, respectively. The first two terms of the Hamiltonian (1) represent the evolution of harmonic oscillators, the third term is an anharmonic term which is due to the oscillation of the nonlinear mirror, and the last term corresponds to the coherent interaction between the quantized cavity mode and the nonlinear mirror. We assume that the nonlinearity and the coupling constant due to radiation pressure are considerably weak compared to frequency \(\omega_m\), so that \(\beta/\omega_m\) and \(g/\omega_m\) are very small compared to unity. We further assume that the interaction time \(t\) is such that \(\beta t\) and \(g t\) are less than unity and terms involving cubes or higher power in \(\beta t\) and \(g t\) can safely be neglected. In the present work, we have considered equilibrium cavity length \(L \sim 0.04\) m, mass of the movable mirror \(m \sim 10^{-14}\) kg, and frequency of the oscillating mirror \(\omega_0 \sim 10^4\) Hz. This choice of parameters is consistent with the existing literature [10].

The Hamiltonian (1) represents a general form of an optomechanical system which contains nonlinearity. If we consider the nonlinearity constant \(\beta = 0\), we would obtain a special case of the Hamiltonian (1) that will be exactly solvable and would be mathematically equivalent to an optomechanical-like system in which Bose-Einstein condensate (BEC) is trapped inside an optical resonator driven by the quantized light field [15]. As mentioned in the previous section, in [15], the authors have studied the role reversal between matter wave and quantized light, i.e., matter wave behaves like the quantized light field and vice versa. With the condition \(\beta = 0\), the Hamiltonian (1) takes the form

\[
H = \hbar \omega_k k^\dagger k + \hbar \omega_0 a^\dagger a - \hbar g (k^\dagger k a^\dagger + \text{H.c.}),
\]

(2)

where \(\omega_m\) reduces to \(\omega_0\) in Eq. (1) as \(\beta = 0\), and the third term vanishes. The reduced Hamiltonian (2) corresponds to a simple form of a Fabry–Perot cavity with one movable mirror, and it also describes a trapped BEC inside the cavity resonator, where \(\omega_k\) represents the frequency of the matter wave \((k^\dagger k)\) originated from the center of mass of the trapped BEC, and \(\omega_0\) corresponds to the cavity-pump detuning [15]. In the BEC system, the role of the movable mirror (in the Fabry-Perot-type systems) is played by the ultracold gas, which is the trapped BEC in this particular case. The analogue of optomechanical coupling is taken as the coupling between the trapped BEC mode and the cavity field detuning. In what follows, the analytic and numerical results for the BEC system is obtained with cavity pump detuning \(\omega_k = 10^4\) Hz, the renormalized ground-state center-of-mass frequency of the trapped BEC \(\omega_0 = 10^5\) Hz and coupling constant \(g = 0.0002 \times \omega_0\) Hz. A similar situation has been investigated experimentally in various works [3-6]. It may be noted that in Eq. (2), the coupling can be visualized to be originated from the radiation pressure from a massive Schrödinger field driving an optical oscillator. This is in contrast to the usual description that considers that a real radiation pressure from a massless optical field drives a mechanical oscillator [15].
III. THE SOLUTION

Our intention is to solve the Hamiltonian in the Heisenberg picture. Generally, it is difficult to obtain closed form analytic solution in the Heisenberg picture due to the presence of non-commuting operators. Perhaps, a more accurate numerical solution may be possible for the present problem. However, in order to obtain a much more physical insight into the system, we prefer an approximate analytic solution using a perturbative technique known as Sen-Mandal technique (6≤69 and references therein). In order to obtain the closed form analytic expressions for the time evolution of the operators under weak nonlinearity and low radiation pressure coupling, we start with the following Heisenberg’s equations of motion that are obtained from the Hamiltonian

\[ \dot{a} = -i\omega_m a - 2i\beta a^3 a^2 + igk^i k, \]
\[ \dot{k} = -i\omega_k k + igk(a^\dagger + a), \]

(3)

where the over-dots correspond to the differentiation with respect to time t, and the coupling constant g and nonlinearity constant \( \beta \) are assumed to be real. The coupled nonlinear differential equations in Eq. (3) involving the operators \( a \) and \( k \) are not exactly solvable. However, perturbative analytic solution for the time evolution of operators \( a \) and \( k \) can be obtained in terms of \( t \), \( a(0) \), \( a^\dagger(0) \), \( k(0) \), and \( k^\dagger(0) \) and other parameters of the system by using Sen-Mandal approach, which has already been successfully used to study various physical systems that lead to Heisenberg’s equations of motion involving the coupled nonlinear differential equations [68 [63-69]. We have two special situations, where these differential equations (3) and hence the model Hamiltonian offers an exact analytic solution. The first one is for \( g = 0 \) (i.e., when there is no coupling), two differential equations (3) are completely decoupled and the system behaves like two independent oscillators. In this case, for mode \( a(t) \), the initial field frequency \( \omega_0 \) gets modified to \( \omega_{0t} \) due to the presence of a Kerr-type nonlinearity involving the nonlinear constant \( \beta \). The second one is \( \beta = 0 \) (i.e., when the mirror oscillates linearly), the modes \( a(t) \) and \( k(t) \) are the system of coupled harmonic oscillators, and are useful for the investigation of nonclassical properties of the radiation fields. The presence of both the nonlinearity constant \( \beta \) and the coupling constant \( g \) makes the coupled equations unsolvable in closed analytic forms. In the present investigation, we shall keep ourselves confined to the solutions up to the second orders in \( \beta t \) and \( gt \), which in our opinion is reasonable enough to deal with the system described above and all the related physical problems. In what follows, we will also establish the consistency of our analytic solution using a numerical solution of the time dependent Schrödinger equation for the same system. Following Sen-Mandal perturbative approach, the assumed approximated trial solutions for the operators of the oscillating mirror and cavity modes up to second order in \( \beta t \) and \( gt \) can be written as

\[ a(t) \simeq f_1 a(0) + f_2 a^\dagger(0) a^2(0) + f_3 k^i(0) k(0) + f_4 a^\dagger(0) a(0) a^\dagger(0) a^2(0) + f_5 k^i(0) k(0) a^\dagger(0) a(0) + f_6 k^i(0) k(0) a^2(0), \]
\[ k(t) \simeq h_1 k(0) + h_2 k(0) a^\dagger(0) + h_3 k(0) a(0) + h_4 k(0) a^\dagger(0) a(0) + h_5 k(0) a^\dagger(0) a^2(0) + h_6 k(0) a^\dagger(0) a(0) + h_7 k(0) a^2(0) \]
\[ + h_8 k(0) a^\dagger(0) a(0) + h_9 k(0) a(0) a^\dagger(0) + h_{10} k(0) k^i(0) k(0). \]

(4)

The notations \( \simeq \) in the above equations and the later part of the paper is used to indicate that the terms beyond the second order in \( \beta t \) and \( gt \) are neglected from the right hand side of the corresponding equations. In the rest of the paper, we will simply write \( a \) and \( k \) instead of \( a(0) \) and \( k(0) \). The similar notation will be adopted for the corresponding creation operators. The assumed solution given in Eq. (4) would not be considered complete unless we evaluate the functional form of the time dependent parameters \( f_i \) and \( h_i \). Therefore, after differentiating Eq. (4) with respect to time \( t \) and substituting that in Eq. (3), we obtain a set of coupled first order differential equations involving \( f_i \) and \( h_i \). The time dependent coefficients \( f_i \) and \( h_i \) are actually obtained from the dynamics involving Eqs. (3) and (4) under a set of initial conditions. Here, we consider \( f_1 = h_1 = 1 \) and \( f_i = h_i = 0 \) (for \( i = 2, 3, 4, \ldots \)) as the initial conditions, i.e., at \( t = 0 \). Under these initial conditions the corresponding solutions for the time dependent coefficient \( f_i \) are obtained as

\[ f_1 = e^{-i\omega_m t}, \]
\[ f_2 = -2i\beta f_1, \]
\[ f_3 = \frac{2\beta}{\omega_m} [1 - f_1], \]
\[ f_4 = -i\beta f_2, \]
\[ f_5 = \frac{4\beta^2}{\omega_m^2} f_1 [-e^{i\omega_m t} + i\omega_m t + 1], \]
\[ f_6 = \frac{8\beta^3}{\omega_m^3} f_1 [e^{-i\omega_m t} + i\omega_m t - 1]. \]

On the other hand, to obtain the solution for the quantized cavity mode \( k(t) \), we need to find out the solutions for the time dependent coefficients \( h_i \). Under the initial boundary condition mentioned previously, these are obtained as
\[ h_1 = e^{-i\omega t}, \]
\[ h_2 = h_1 f_3 / f_1, \]
\[ h_3 = h_1 f_5, \]
\[ h_4 = h_1 f_6 / f_3, \]
\[ h_5 = h_1 f_7 / 2, \]
\[ h_6 = h_1 f_8 / (2 f_5^2), \]
\[ h_7 = h_1 f_9 / 2, \]
\[ h_8 = \frac{v}{2} h_1 f_6 / f_3, \]
\[ h_9 = -\frac{2}{2} h_1 h_8 f_1, \]
\[ h_{10} = f_8 - f_9. \]  

The coefficients \( f_1 \) and \( h_1 \) in Eqs. (5) and (9) are due to the free evolution terms which correspond to the harmonic oscillators, and the rest of the coefficients are due to the nonlinearity and/or coupling constants. The solutions of the coefficients \( f_3 \) and \( h_3 \) complete the operator solutions of \( a(t) \) and \( k(t) \).

The operators \( a(t) \) and \( k(t) \) (i.e., corresponding to the movable mirror mode and quantized cavity mode) are the bosonic operators, and hence they must obey the bosonic commutation relations. It may be noted that the obtained solutions are verified to obey the equal time commutation relations (ETCR), i.e., \([a(t), a^\dagger(t)] = [k(t), k^\dagger(t)] = 1\), while all other possible commutations vanish as required. From the above equations, it can be seen that the operators \( a(t) \) and \( k(t) \) commute with each other. As a consequence of this commutation relation, in the rest of the paper, we can compute various moments of these operators without being worried about the intermodal commutation relation and the corresponding operator ordering. In what follows, we use the closed form analytic expressions of \( a(t) \) and \( k(t) \) obtained here to investigate the temporal evolution of lower and higher-order entanglement and different nonclassical properties of the system, using various moment based criteria. To begin with, we look at the possibility of observing squeezed state in the next section.

**IV. SQUEEZING**

In this section, we investigate the possibilities of observing squeezing, higher-order squeezing and intermodal squeezing of the quadratures. In general, the quadrature operators of the various field modes are defined as

\[ X_j = \frac{1}{2}[j(t) + j^\dagger(t)], \]
\[ Y_j = -\frac{1}{2}[j(t) - j^\dagger(t)], \]  

where \( X_j \) (\( Y_j \)) are the quadrature operators of the corresponding modes with \( j \in \{a, k\} \). The fluctuation of these quadratures obey the famous uncertainty relation, and its minimum value corresponds to the minimum uncertainty state (MUS). The fluctuation of the minimum uncertainty state gives rise to the standard quantum limit (SQL), and hence the zero point fluctuation (ZPF). The quasi-classical states, i.e., coherent states are the examples of the MUS in which uncertainties in both of the quadratures are \( \frac{\Delta}{\sqrt{2}} \) (in dimensionless unit). If one of the quadrature fluctuation goes below the SQL for a quantum state, then the corresponding state is called squeezed state. In such a situation, in order to respect the Heisenberg uncertainty relation the other quadrature fluctuation must be greater than the SQL and hence, the simultaneous squeezing of both of the quadratures is not allowed, i.e., squeezing in one of the quadrature components automatically prohibits the same in the conjugate quadrature. The squeezing of the quadrature \( X_j \) (\( Y_j \)) is obtained if the second order variance \((\Delta X_j)^2 < \frac{1}{4} \) \((\Delta Y_j)^2 < \frac{1}{4} \). In order to calculate \((\Delta X_j)^2\) and \((\Delta Y_j)^2\), we assume that the cavity and oscillating mirror modes are initially in the coherent state. Therefore, the composite initial state is the product of the states \(|\alpha_1\rangle\) and \(|\alpha_2\rangle\), which are the eigenket of the operators \(a\) and \(k\). Thus, the initial composite state is

\[ |\psi(0)\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle. \]  

Explicitly, the initial state is separable, and the operators \(a\) and \(k\) obey the eigenvalue equations

\[ a(0)|\psi(0)\rangle = \alpha_1 |\alpha_1\rangle \otimes |\alpha_2\rangle, \]
\[ k(0)|\psi(0)\rangle = \alpha_2 |\alpha_1\rangle \otimes |\alpha_2\rangle, \]  

where \(\alpha_1 = |\alpha_1|e^{-i\theta}\) and \(\alpha_2 = |\alpha_2|e^{-i\phi}\) are the complex amplitudes. The parameters \(|\alpha_1|^2\) and \(|\alpha_2|^2\) represent the number of excitations present in the modes \(a\) and \(k\), respectively. Also, the quantities \(\theta\) and \(\phi\) are the phase angles of the vibrating mirror and quantized cavity modes, respectively. In what follows, we will show that varying phase angle \(\theta\) we can control the depth (values) of nonclassicality parameters. Similarly, it is expected that by controlling \(\phi\), we would be able to control the values of the nonclassicality parameters. However, in all the plots reported here, \(\alpha_1 = \alpha_2 = 1\), and it is assumed that \(\phi = 0\).

Now, in terms of the initial composite coherent state (8), we calculate the second order variance of quadratures \(X_j\) and \(Y_j\). Therefore, using Eqs. (4)-(9), we have

\[ \left[ \frac{(\Delta X_a)^2}{(\Delta Y_a)^2} \right] = \frac{1}{4} \left[ 1 + 2(|f_3|^2|\alpha_1|^4 + |f_4|^2|\alpha_2|^2) \right. \]
\[ \left. \pm \left( f_1 f_3 |\alpha_1|^2 |\alpha_1|^2 + f_1 f_4 |\alpha_1|^2 |\alpha_2|^2 + f_2 |\alpha_2|^2 \right) \right] \]  

where \(c.c.\) stands for the complex conjugate. The upper and lower signs (+ and − sign) in the right hand side of Eq. (10) correspond to \((\Delta X_a)^2\) and \((\Delta Y_a)^2\), respectively. It is clear from the above expression that for \(t = 0\), the variances reduce to their minimum values \((\Delta X_a)^2 = (\Delta Y_a)^2 = \frac{1}{4}\), hence belong to the coherent state as expected, which is clearly depicted in Fig. 2. a. In which, one can observe that at \(t = 0\) curves start from 0.25 and with increase of rescaled interaction time, the values of \((\Delta X_a)^2\) or \((\Delta Y_a)^2\) show an oscillatory behavior. From Fig. 2. a, it’s clear that at certain times \((\Delta X_a)^2((\Delta Y_a)^2)\) goes below the SQL and thus indicate the existence of squeezing in \(X_a(\alpha_a)\) quadrature. Additionally, with \(\beta = 0\), the variances will correspond to the BEC trapped optomechanics-like system described by Eq. (2).
and the corresponding result is shown in Fig. 2b, where we failed to obtain squeezing in any quadrature corresponding to mode $a$. For obtaining the figures for the trapped BEC system, we have chosen the cavity pump detuning $\omega_h = 10^5 \text{Hz}$, the renormalized ground-state center-of-mass frequency of the trapped BEC $\omega_0 = 10^5 \text{Hz}$, and coupling constant $g = 0.0002 \times \omega_0 \text{Hz}$. Further, as mentioned previously, we obtained the quadrature squeezing in mode $a$ solving the system Hamiltonian \[ 1 \] in Schrödinger picture using matrix forms of various operators with the same initial state. Interestingly, the quadrature squeezing obtained using perturbative solution is observed to match exactly with that obtained from exact numerical solution (which is shown as circles and squares in Fig. 3a). Similarly, we have verified all the results reported in the present work using numerical solution, which are shown using various plotmarkers.

Figure 2: (Color online) Variation of single mode lower and higher-order quadrature squeezing, intermodal, principal and normal squeezing for optomechanical system (in (a), (c), (d), and (f)), and optomechanics-like system (in (b), (e)). (a) Illustration of quadrature squeezing with rescaled time in mode $a$. The analytic results for $(\Delta X_a)^2$ and $(\Delta Y_a)^2$ correspond to solid (blue) and dashed-dotted (red) curves, respectively. (b) The presence of quadrature squeezing for mode $a$ in optomechanics-like system, where solid (blue) and dashed (red) lines correspond to $(\Delta X_a)^2$ and $(\Delta Y_a)^2$, respectively. (c) Illustration of principal and normal squeezing through dotted (red) and smooth (blue) lines, respectively. (d) Variation of intermodal squeezing with rescaled time via solid (blue) and dashed-dotted (red) curves corresponding to $(\Delta X_{ak})^2$ and $(\Delta Y_{ak})^2$. Also, (e) intermodal squeezing between modes $a$ and $k$ in trapped-BEC system is shown using solid (blue) and dashed (red) lines corresponding to $(\Delta X_{ak})^2$ and $(\Delta Y_{ak})^2$, respectively. (f) Illustration of $l$th powered quadrature squeezing in mode $a$ solving the system Hamiltonian \[ 1 \] in Schrödinger picture using matrix forms of various operators with the same initial state. Interestingly, the quadrature squeezing obtained using perturbative solution is observed to match exactly with that obtained from exact numerical solution (which is shown as circles and squares in Fig. 3a). Similarly, we have verified all the results reported in the present work using numerical solution, which are shown using various plotmarkers.

An alternate definition of quadrature squeezing was given by Luks et al., \cite{Luks} by considering the geometrical (elliptical) representation of variances. Further advancement of the geometrical representation of squeezed state happened in the works of Loudon \cite{Loudon}, who represented the quadrature variance in terms of the Booth’s elliptical lemniscate. According to Luks et al., \cite{Luks}, the standard definition of squeezing (normal squeezing) is

$$ S_n = \langle (\Delta a^\dagger \Delta a) \rangle - \text{Re} \langle (\Delta a)^2 \rangle < 0, \quad (11) $$

and the definition of principal squeezing is given by

$$ S_p = \langle (\Delta a^\dagger \Delta a) \rangle - |\langle (\Delta a)^2 \rangle| < 0. \quad (12) $$

The expectation values are calculated in terms of the initial
completely and the analytic expression of the normal and principal squeezing are obtained using Eqs. 4, 11 and 12 as follows

\[ S_n = |f_2|^2|\alpha_1|^4 + |f_3|^2|\alpha_2|^2 - Re \left[ f_1 f_2 \alpha_1^2 + f_1 f_4(2|\alpha_1|^2\alpha_2^2 + \alpha_1^2) + f_1 f_5|\alpha_1|^2\alpha_1 + 2f_2|^2|\alpha_1|^2\alpha_2^2 + f_3|^2|\alpha_2|^2 \right] \] (13)

and

\[ S_p = |f_2|^2|\alpha_1|^4 + |f_3|^2|\alpha_2|^2 - |(f_1 f_2 \alpha_1^2 + f_1 f_4(2|\alpha_1|^2\alpha_2^2 + \alpha_1^2) + f_1 f_5|\alpha_1|^2\alpha_1 + 2f_2|^2|\alpha_1|^2\alpha_2^2 + f_3|^2|\alpha_2|^2) | \] (14)

respectively. The variations of \( S_n \) and \( S_p \) are depicted in Fig. 2c, where \( S_p \) is always negative, but \( S_n \) is positive in some regions. The negative values of \( S_n \) and \( S_p \) is the clear witness of the presence of quadrature squeezing.

### A. Intermodal squeezing

In this subsection, we investigate the intermodal squeezing between the quantized cavity mode and the phonon mode that correspond to the vibrating mirror. The two-mode quadrature operators involving the modes \( a \) and \( k \) are defined as follows:

\[
\begin{align*}
\Delta X_{ak} &= \frac{1}{2\sqrt{2}} \left\{ (\alpha + \alpha^\dagger) + (k + k^\dagger) \right\}, \\
\Delta Y_{ak} &= -\frac{1}{2\sqrt{2}} \left\{ (\alpha - \alpha^\dagger) + (k - k^\dagger) \right\},
\end{align*}
\]

where the variances of quadratures \( \Delta X_{ak} \) and \( \Delta Y_{ak} \) obey the Heisenberg’s uncertainty relation. Consequently, whenever the variance in one of the quadratures goes below \( \frac{1}{4} \), it demonstrates squeezing in corresponding two-mode quadrature. Using Eqs. 4-6, 8, 9, and 13 we obtain the expressions for the variances of the intermodal quadratures as

\[
\left[ \frac{(\Delta X_{ak})^2}{(\Delta Y_{ak})^2} \right] = \frac{1}{2} \left\{ 2(2|f_2|^2|\alpha_1|^4 + |f_3|^2|\alpha_2|^2) + 2|h_2|^2|\alpha_2|^2 + 2f_2 h_2^* \alpha_2 \alpha_1^2 \right. \\
\left. \pm \left\{ f_1 f_2 \alpha_1^2 + f_1 f_4(2|\alpha_1|^2\alpha_2^2 + \alpha_1^2) + f_1 f_5|\alpha_1|^2\alpha_1 + 2f_2^2|\alpha_1|^2\alpha_2^2 + f_3|^2|\alpha_2|^2 \right\} + h_1 h_2 \alpha_2^2 + h_2 h_3 \alpha_3^2 \right\} \\
\left. \pm 2 \left\{ f_1 h_2 \alpha_2 + 2(f_1 h_4 + f_2 h_2)\alpha_2|\alpha_1|^2 + f_1 h_5 \alpha_2 \alpha_1^2 \right\} \right\} + 2f_1 h_6 \alpha_2 \alpha_1^2 + (f_1 h_8 + f_1 h_9)\alpha_2 \alpha_1^2 \} \right] \] (16)

### B. Higher-order squeezing

The idea of the higher-order squeezing originated from the work of Hong and Mandel [73] who generalized the concept of the normal order squeezing. They introduced higher-order squeezing involving higher-order moments of quadrature variables. On the other hand, Hillery proposed a notion of higher-order squeezing by computing the variance of field quadratures which were higher-order in amplitude [74], i.e., amplitude powered squeezing. In order to investigate the higher-order squeezing, here we follow the idea of Hillery [74]. In which, \( l \)th powered quadrature variables are defined as

\[
Y_{1,a} = \frac{\alpha^{l+a^{l\dagger}}}{{l\choose 2}}, \\
Y_{2,a} = -\frac{\alpha^{l-a^{l\dagger}}}{{l\choose 2}}.
\] (17)

Here, we can see that the quadrature variables \( Y_{1,a} \) and \( Y_{2,a} \) do not commute which gives us an uncertainty relation, and
consequently a criterion for higher-order squeezing as
\[ A_{i,a} = (\Delta Y_{1,a})^2 - \frac{1}{2} |\langle [Y_{1,a}, Y_{2,a}] \rangle| < 0, \]
where \( i \in \{1, 2\} \). In what follows, we have used the Hillery’s criterion for amplitude square squeezing, i.e., in Eq. (17), we choose \( l = 2 \) and reduced the criterion of higher-order squeezing given in Eq. (19) as
\[ A_{i,a} = \langle (\Delta Y_{1,a})^2 \rangle - \langle N_a + \frac{1}{2} \rangle < 0, \]
where \( N_a \) is the number operator for the mode \( a \). We have calculated the second order variances of the quadrature for \( l = 2 \), involving the field operator of mode \( a \). After a few algebraic steps and using Eqs. 4, 6, 8, 9 and 17, we obtain the following compact results.

\[
\left( \frac{\langle (\Delta Y_{1,a})^2 \rangle}{\langle (\Delta Y_{2,a})^2 \rangle} \right) = |f_1|^4 |\alpha_1|^2 + |f_2|^4 |\alpha_2|^2 (2|\alpha_1|^2 + |\alpha_2|^2 + 1) + 2|f_3|^2 |\alpha_1|^6
+ \left[ |\alpha_2|^2 f_1 f_3 + \frac{1}{2} f_1 f_3^* |\alpha_1|^2 + f_2^2 f_3^2 (2|\alpha_1|^2 + 1) + f_3 f_2^* (3|\alpha_1|^2 + 1) \right]
\pm \left[ |\alpha_3|^2 f_1 f_2^* + |\alpha_2|^2 f_1 f_4 |\alpha_1|^2 + 3 + f_2^2 (4|\alpha_1|^2 + \frac{1}{2}) \right] + f_2^2 (f_1 f_5 + 2 f_1 f_2) |\alpha_2|^2 |\alpha_1|^2,
\]

The average value of the excitation number of the field mode \( a \) is given by
\[
\langle N_a \rangle = \langle a^\dagger(t)a(t) \rangle = |f_1|^2 |\alpha_1|^2 + f_1^* f_3 |\alpha_1|^2 |\alpha_2|^2 + f_2^* f_3 |\alpha_2|^2 |\alpha_1|^2 + f_1 f_6 |\alpha_2|^2 |\alpha_1|^2 |\alpha_1|^2 |\alpha_1|^2
+ f_2^2 |\alpha_2|^2 |\alpha_1|^2 |\alpha_1|^2 + f_3 f_4 |\alpha_2|^2 |\alpha_1|^2 + f_2^2 f_1 |\alpha_2|^2 |\alpha_1|^2 |\alpha_1|^2 + f_2^2 f_1 |\alpha_2|^2 |\alpha_1|^2 |\alpha_1|^2 |\alpha_1|^2.
\]

Substituting the right hand sides of (21) and (20) in (19), and using the functional forms of \( f_s \) given in Eq. (5), we analyzed the results for \( A_{1,a} \) and \( A_{2,a} \), which are not explicitly written here. Variation of the amplitude powered quadratures with rescaled time is illustrated in Fig. 3f, where we can see that either \( A_{1,a} \) or \( A_{2,a} \) can be negative at a particular instant of time, and thus, one of the quadratures of the optomechanical system studied here always shows higher-order squeezing in a mode.

In brief, in the present section, we have established that the present systems can be easily employed to generate squeezed states. Further, the presence of higher-order squeezing in optomechanical system makes it feasible to detect the squeezing, in case corresponding lower-order criterion failed to do so. Additionally, compound modes in both the systems of interest are shown to exhibit intermodal squeezing.

V. ANTIBUNCHING

The phenomenon in which the probability of getting two or more photons simultaneously is less than the probability of getting single photons is called the photon antibunching. Experimentally, correlation of intensity was first time observed by Hanbury-Brown and Twiss [25] in their remarkable experiment using an incandescent light source, which concluded that the photons come together. This phenomenon in which the photons come together is known as the bunching of photon. The natural tendency of light source is to emit a cluster of photons rather than the single one, but there are light sources in which photons do not come in cluster, and hence the antibunching of photon is observed [76]. In order to investigate the photon bunching and antibunching, we generally use the quantum statistical properties of the radiation field. For this, we calculate the second order correlation function for zero time delay which is defined as
\[
g^2(0) = \frac{\langle a^\dagger(t)a^\dagger(0)a(0)a(t) \rangle}{\langle a^\dagger(t)a(t) \rangle \langle a(0)a(0) \rangle}. \tag{22}
\]

The second order correlation function is a useful mathematical tool to deal with the quantum statistical properties of the radiation field. Our interest is to investigate the phonon statistics of the vibrating mirror mode. Interestingly, phonon also obey the second order correlation function [77]. Also, from Eq. (22), we can write
\[
g^2(0) - 1 = \frac{\langle (\Delta N(t))^2 \rangle - \langle N(t) \rangle^2}{\langle N(t) \rangle^2} = \frac{D_{a}(1)}{\langle N(t) \rangle^2}, \tag{23}
\]
where
\[
D_{a}(1) = \langle (\Delta N(t))^2 \rangle - \langle N(t) \rangle. \tag{24}
\]
The denominator of Eq. (23) is always positive. Therefore, quantum behavior can be solely determined by the parameter
$$D_a(1) = 0$$ specifically, for \( D_a(1) = 0 \) (i.e., \( g^2(0) = 1 \)), the corresponding field is coherent in nature. Similarly, \( D_a(1) < 0 \) corresponds to the antibunching. Finally, substituting Eqs. (4)-(6) into Eq. (24), we obtained the analytic expression for \( D_a(1) \), which is given by

$$D_a(1) = 2[|a_0|^2|\alpha_2|^2|\alpha_1|^2 + \langle f_1^* f_5 \rangle f_2^* f_3] |\alpha_2|^2 |\alpha_1|^2 |\alpha_1|^2 \} + 4 f_2^* f_1 + 2 f_2^* f_2 f_3 |\alpha_2|^2 |\alpha_1|^2 |\alpha_1|^2 + c.c. \} \] (25)

To investigate the quantum statistical properties, we plot \( D_a(1) \) with respect to the dimensionless interaction time which is depicted in Fig. 3 using a smooth (blue) line. The negative values of the quantity, shown in the Fig. 3 demonstrates the existence of lower-order antibunching and thus, the presence of a nonclassical character in optomechanical system. Therefore, phonons of the vibrating mirror modes are observed to be antibunched for a small period of interaction time.

VI. ENTANGLEMENT

In this section, our main focus is to check whether the separable initial state described by Eq. (8) evolves into an entangled state due to interaction. To do so, we would investigate the possibility of observing entanglement using a set of inseparability criteria. In fact, there exist several inseparability criteria (in terms of the annihilation and creation operators) that are suitable to investigate the entanglement dynamics in this type of approach. In this article, for simplicity, mostly we have used the Hillery-Zubairy criteria to investigate the lower- and higher-order entanglement and a criterion due to Duan et al. to investigate only lower-order entanglement.

A. Higher-order antibunching

In this section, we investigate the higher-order antibunching phenomena. There have been a numerous proposals for detecting the higher-order nonclassical photon statistics. For example, criteria for higher-order photon statistics are given by Lee, Agarwal and Tara, and the generalized criterion of many photon antibunching is proposed by Lee. In this article, we follow the definition of Pathak and Garcia, which can be viewed as one of variants of various equivalent criteria, to investigate \((l - 1)\)th order antibunching of phonon. Specifically, the criterion for higher-order antibunching is

$$D(l - 1) = \langle a^\dagger a^\dagger \rangle - \langle a^\dagger a \rangle^l < 0 \] (26)

Using the closed form of analytic solution in Eqs. (4)-(9) in the criteria, we obtained the corresponding c-number equation which yields

$$D(l - 1) = \langle a^\dagger a^\dagger \rangle - \langle a^\dagger a \rangle^l < 0 \] (26)$$

which is the analytic expression for the \((l - 1)\)th order antibunching. The variation of the obtained result is depicted in Fig. 3 in dashed (red) line with corresponding lower-order antibunching in smooth (blue) line. One can clearly observe that the higher-order antibunching not only survives for relatively longer period of time, also shows more depth of nonclassicality than corresponding lower-order counterpart. This fact reestablishes the previous experimentally reported result that higher-order nonclassicality criteria may be useful in detecting weaker nonclassicality.

As discussed in the previous section that results for optomechanics-like system can be obtained by taking \( \beta = 0 \) in Eqs. (25) and (27). Incidentally, in the present case, we failed to observe antibunching in case of the trapped BEC system. Therefore, here we have not included the graph that was obtained for the BEC system.
On the other hand, the second criteria due to Hillery and Zubairy [84–86] is
\[
E_{2,a,k}^{1,1} = \langle a^\dagger(t)a(t)\rangle\langle k^\dagger(t)k(t)\rangle - \langle a(t)k(t)\rangle^2 < 0.
\] (29)
In the remaining part of this article we refer to these two criteria as HZ-1 criterion and HZ-2 criterion, respectively. Additionally, the inseparability criteria introduced by Duan et al. [87] is:
\[
da_{a,k} = \langle (\Delta u)^2 \rangle + \langle (\Delta v)^2 \rangle - 2 < 0,
\] (30)
where
\[
u = \frac{-1}{\sqrt{2}}\{(a - a^\dagger) + (k - k^\dagger)\},
\]
\[
u = \frac{1}{\sqrt{2}}\{(a + a^\dagger) + (k + k^\dagger)\}.
\]

The set of inseparability criteria listed in Eqs. (28)–(30) are only sufficient, but not necessary. Thus, if any of these criteria fails to detect the entanglement in a given state, it does not mean that the state is separable, but obeying any of these criteria ensures that the given state is entangled (inseparable). This establishes our choice to use more than one such criteria to detect entanglement in the quantum state.

We investigate the entanglement between movable mirror and the cavity mode using HZ-1 criterion. Using Eqs. (4)–(9) and (28), we obtain
\[
E_{1,a,k}^{1,1} = \langle N_a N_k \rangle - \langle a(t)k(t)\rangle^2
= \{|f_3|^2 |\alpha_2|^4 + (|\alpha_1|^2 - 1) |h_3|^2 |\alpha_2|^2 - \{|h_1 h_2^*|^2 |\alpha_2|^2 |\alpha_1|^2 f_1^* f_2^* |\alpha_1|^4 |\alpha_2|^2 + f_1 h_1^* h_2 |\alpha_2|^4 + (h_1 h_2^* + f_1 f_2^* h_1 h_2^*) |\alpha_2|^2 |\alpha_1|^2 + c.c.\}.
\] (31)

The negative value of the right hand side of Eq. (31) is the witness of entanglement between the oscillating mirror and the cavity modes which is shown in Fig. 4a. The corresponding result for the BEC system is easily reducible and is shown in Fig. 5a. Specifically, we have established possibility of observing entanglement between modes \(a\) and \(k\) in both the systems studied here. However, the presence of entanglement depends on various parameters as shown in Figs. 4b. Specifically, existence of entanglement between both the modes can be controlled by controlling the phase of the oscillating mirror (cavity) mode in optomechanical (optomechanics-like) system.

Similarly, using Eqs. (4)–(9) and (29), we obtained the analytic expression corresponding to HZ-2 criterion, which is given by
\[
E_{2,a,k}^{1,1} = \langle N_a N_k \rangle - \langle a(t)k(t)\rangle^2
= \{|f_3|^2 |\alpha_2|^4 + (|\alpha_1|^2 - 1) |h_3|^2 |\alpha_2|^2 - \{|h_1 h_2^*|^2 |\alpha_2|^2 |\alpha_1|^2 f_1^* f_2^* |\alpha_1|^4 |\alpha_2|^2 + f_1 h_1^* h_2 |\alpha_2|^4 + (h_1 h_2^* + f_1 f_2^* h_1 h_2^*) |\alpha_2|^2 |\alpha_1|^2 + c.c.\}.
\] (32)

Variation of \(E_{2,a,k}^{1,1}\) obtained through Eq. (32) is illustrated in Fig. 4b, and corresponding plot for the BEC system is shown in Fig. 5b. In analogy of the HZ-1 criterion, HZ-2 criterion also shows negative values of the parameter \(E_{2,a,k}^{1,1}\) (resulting in inseparability of the two modes) depending up on the values of phase of the oscillating mirror (cavity) mode in optomechanical (optomechanics-like) system.

Specifically, in Figs. 4a and 5a, we observed entanglement for \(\theta = 0\) and \(\pi\), but not for \(\theta = \pi\), using HZ-1 criterion. On the other hand, HZ-2 criterion of entanglement as shown in Figs. 4b and 5b could establish inseparability of two modes for \(\theta = \pi\), \(\pi\), while failed to detect for \(\theta = 0\). Note that all these inseparability criteria are sufficient in nature, not necessary, therefore, the negative values of the inequalities conclude that the states are definitely entangled but fail to reach to any conclusion for the positive values.

To check the entanglement in the regions where HZ-1 and HZ-2 criteria failed to detect them, we have used Duan et al.’s inseparability criteria. Specifically, using Eqs. (4)–(9) and (30) we obtained the analytic expression as
\[
da_{a,k} = \left[|f_2|^2 |\alpha_1|^4 + (|f_3|^2 + |h_2|^2) |\alpha_2|^2 + \{f_2 h_2^* |\alpha_2|^2 + c.c.\}\right].
\] (33)

The corresponding results for optomechanical and optomechanics-like systems are shown in Fig. 4c and
Apart from the phases of both the modes involved in interaction, one can also control the possibility of entanglement generation by varying the coupling constant $g$. This fact can be clearly established from Fig. 6 a and b. Specifically, we can see that the entanglement detected using HZ-1 criterion (in Fig. 6a) tends to go deeper with higher values of the coupling constant. In contrast, while we use HZ-2 criterion we observed opposite nature (see Fig. 6b) as the plot tends to become more positive.

Figure 4: (Color online) The intermodal entanglement between cavity and mirror modes is illustrated for optomechanical system with $|\alpha_1|^2 = 4$ and $|\alpha_2|^2 = 1$. In all these plots, negative values of the shown quantities establish entanglement. (a) The smooth (blue), dashed (red) and dash-dotted (black) lines correspond to the variation of quantity for $\theta = 0$, $\pi/2$ and $\pi$, respectively, using HZ-1 criterion. (b) Using HZ-2 criterion, the smooth (red), dashed (blue), and dash-dotted (black) lines correspond to $\theta = 0$, $\pi/2$ and $\pi$, respectively. (c) Illustration of the variation of Duan et al.’s entanglement parameter $d_{a,k}$. (d) Higher-order entanglement is illustrated using HZ-1 and HZ-2 criteria $\theta = \phi = 0$, where dotted (black) line ($m = 2$, $l = 2$), dashed-dotted (blue) line ($m = 2$, $l = 3$) correspond to HZ-1 criterion and smooth (red) line ($m = 2$, $l = 3$), dashed (magenta) line ($m = 2$, $l = 2$) correspond to HZ-2 criterion, respectively.

A. Higher-order entanglement

Higher-order entanglement criteria ensure the possibility of detecting the presence of weak nonclassical effects (entanglement) present in the system. Therefore, here we use two criteria to investigate the higher-order two mode entanglement introduced Hillery and Zubairy [84, 85]. According to these criteria, the higher-order entanglement exists if either

\begin{equation}
E_{1,a,k}^{l,m} = \langle a^{(l)}(t) a^{(l)}(t) k^{m}(t) k^{m}(t) \rangle - |\langle a^{(l)}(t) k^{m}(t) \rangle|^2 < 0
\end{equation}

or

\begin{equation}
E_{2,a,k}^{l,m} = \langle a^{(l)}(t) a^{(l)}(t) k^{m}(t) k^{m}(t) \rangle - |\langle a^{(l)}(t) k^{m}(t) \rangle|^2 < 0
\end{equation}

is satisfied, where $l$ and $m$ are non zero integers with $l \geq 1$ and $m \geq 1$. It is clear from Eqs. (34) and (35), for the lowest possible values of these integers, the higher-order entanglement criteria would reduce to the HZ-1 and HZ-2 given in Eqs. (28) and (29), respectively. The higher-order two mode entanglement is present between two modes if $l$ and $m$ satisfy the relation $l + m \geq 3$. Here, from Eq. (34) and using Eqs. (4)-9, we obtain the generalized higher-order entanglement for HZ-1 criterion as
using HZ-2 criterion, respectively. Similarly, (b) entanglement using HZ-2 criterion, dashed (blue) and dash-dotted (black) lines corresponding to \( \theta = 0, \frac{\pi}{2} \) and \( \pi \), respectively. (c) Illustration of the variation of Duan et al.’s entanglement parameter \( d_{\alpha,k} \). (d) Higher-order entanglement is illustrated with smooth (red), dashed (blue) lines corresponding to \( (m = 2, l = 2) \), \( (m = 2, l = 3) \) using HZ-1 criterion, and dashed-dotted (blue), dotted (black) corresponding to \( (m = 2, l = 2) \), \( (m = 2, l = 3) \) using HZ-2 criterion, respectively.

In the similar manner, using Eqs. (4)-(9) and Eq. (35), we obtain the expression for the generalized higher-order entanglement for HZ-2 criterion

\[
E_{1,a,k}^{l,m} = \left\{ (2m + 1)|\alpha_1|^2 + m^2 \right\} f_3^2|\alpha_2|^{2m}|\alpha_1|^{2(l-1)} + 2m|h_2|^2|\alpha_2|^{2m}|\alpha_1|^{2l} + \frac{m(l-1)}{2} f_3^2 |\alpha_2|^{2m}|\alpha_1|^{2l-1} + \frac{m^2}{2} f_3^4 |\alpha_2|^{2m}|\alpha_1|^{2l-2} - f_3^4 \left\{ 2C_2 A_2 h_3^2 + h_3^2 f_3^2 \right\} + \frac{m}{2} (l+1) f_3^2 h_3^2 \left\{ 2C_2 A_2 |\alpha_2|^{2m}|\alpha_1|^{2l} + f_3^2 \left\{ 2C_2 A_2 |\alpha_2|^{2m}|\alpha_1|^{2l} \right\} \right\}
\]

In the similar manner, using Eqs. (4)-(9) and Eq. (35), we obtain the expression for the generalized higher-order entanglement for HZ-2 criterion

\[
E_{2,a,k}^{l,m} = m^2 |h_2|^2 |\alpha_2|^{2m}|\alpha_1|^{2l} + f_3^2 |\alpha_2|^{2m}|\alpha_1|^{2l-1} + \frac{m^2}{2} f_3^4 |\alpha_2|^{2m}|\alpha_1|^{2l-2} + \frac{m}{2} (l+1) f_3^2 h_3^2 \left\{ 2C_2 A_2 |\alpha_2|^{2m}|\alpha_1|^{2l} + f_3^2 \left\{ 2C_2 A_2 |\alpha_2|^{2m}|\alpha_1|^{2l} \right\} \right\}
\]

It is a tedious job to obtain the compact analytic expressions reported in Eqs. (36)-(37), and also quite difficult to interpret the results directly. However, it allows us to establish the dependence of these quantities on various physical parameters. Therefore, we plot the analytic results with various parameters as shown in Figs. 4a and 5a for the optomechanical and BEC systems, respectively. Specifically, in Fig. 4a, we have shown both HZ-1 and HZ-2 higher-order entanglement criteria together and higher-order entanglement is evidently present for most of the values of rescaled time (except for very high values of rescaled time). Along the same line, the BEC system also exhibits higher-order entanglement for all the val-
Figure 6: (Color online) Variation of entanglement for $|\alpha_1| = 4, |\alpha_2| = 1$ with respect to interaction constant $g$ and rescaled time using (a) HZ-1 and (b) HZ-2 criteria, respectively. Higher-order entanglement varying with respect to the phase of the oscillating mirror using (c) HZ-1 and (b) HZ-2 criteria, respectively. (d) Using HZ-2 criterion. Here, $\theta = \phi = 0$ for (a) and (b), and $\phi = 0$ and $g = 2$ in (c) and (d).

Dependence of higher-order entanglement on various physical parameters can also be established. Here, we show the variation of the HZ-1 and HZ-2 higher-order entanglement parameter with the phase of the oscillating mirror (cavity) mode in optomechanical (optomechanics-like) system in Fig. 6c and d, respectively. The plots illustrate that the possibility of detecting entanglement can be controlled by the value of the phase parameter. It is important to note here that the three dimensional variation of both lower- and higher-order entanglement parameters with the coupling constant and phase parameter is similar. Therefore, we are avoiding repetition and would like to emphasize this point only in the text.

Before we conclude the paper, it becomes imperative to discuss the single mode nonclassicality present in the systems of our interest that are not discussed so far. Specifically, we have not reported squeezing in the cavity (trapped BEC) mode in the optomechanical (optomechanics-like) system due to weak nonclassicality observed in them. Therefore, for the sake of completeness of the present work, we are reporting here only numerical results obtained in that case in Fig. 7. In brief, both the systems the corresponding modes show squeezing phenomena only after an appreciable time evolution. On top of that these modes fail to show antibunching. However, the nonclassical behavior of the concerned mode has already been established with the help of intermodal nonclassicality.

VII. CONCLUSION

In conclusion, we would like to stress on the fact that the work presented here offers prospects for the observation of lower- and higher-order nonclassical features present in the cavity with a nonlinearly movable mirror comprising of an optomechanical system and in a BEC trapped optomechanics-like system. The optomechanical system with nonlinearly movable mirror can be considered as a Kerr-like nonlinear medium. In the absence of nonlinearity, the system Hamiltonian reduces to that of the BEC trapped optomechanics-like system. We first obtain the solution for the generalized nonlinear system, subsequently, the solution for the BEC system is obtained as the limiting case.

In the present study, we have assumed that there is no leakage of photon through the cavity. Therefore, the main source of decoherence is due to the interaction of the movable mirror with environment which can be neglected up to some extent unless the mirror is heavily damped. Moreover, for simplicity, we have assumed that the nonlinearity present in the system is quartic in nature. The model Hamiltonian of the physical system is constructed using the rotating wave approximation to eliminate the non conserving energy terms. In Heisenberg picture, we obtain the equation of motion of the corresponding field operators. Subsequently, we obtain a perturbative analytic solution using Sen-Mandal technique. Finally, we used this perturbative analytic solution to obtain analytic expressions for various nonclassicality parameters and plot those parameters obtain signatures of nonclassicality. Various types of lower- and higher-order nonclassicality have been observed and they are summarized in Table I. The validity of the ob-
Figure 7: (Color online) The presence of quadrature squeezing in mode $k$ for the (a) optomechanical and (b) BEC systems. In both figures, the solid (blue) and dashed (red) lines correspond to the variances in quadratures $X_k$ and $Y_k$, respectively.

Table I: Observation of nonclassicality in cavity with a movable mirror and BEC trapped in a cavity are summarized here.

| Cavity with a movable mirror | Cavity with BEC |
|-----------------------------|------------------|
| Squeezing                   | Observed for mode $a$ | Not observed |
| Higher-order squeezing       | Observed for mode $a$ | Not observed |
| Intermodal squeezing         | Observed between modes $a$ and $k$ | Observed between modes $a$ and $k$ |
| Antibunching                | Observed for mode $a$ | Not observed for mode $a$ |
| Higher-order antibunching    | Observed for mode $a$ | Not observed for mode $a$ |
| Entanglement using HZ-1     | Observed for mode $a$ for $\theta = 0, \pi/2$ | Observed for mode $a$ for $\theta = 0, \pi/2$ |
| Entanglement using HZ-2     | Observed for mode $a$ for $\theta = \pi/2, \pi$ | Observed for mode $a$ for $\theta = \pi/2, \pi$ |
| Entanglement using Duan criterion | Not observed | Not observed |
| Higher-order entanglement using HZ-1 | Observed for $\theta = 0$ | Observed for $\theta = 0$ |
| Higher-order entanglement using HZ-2 | Observed for $\theta = 0$ | Not observed for $\theta = 0$ |

In brief, the present study not only revealed nonclassical features present in both optomechanical and optomechanics-like systems, it also established that the nonclassical features observed here can be controlled by controlling various parameters, such as the phase of the movable mirror and cavity modes. Note that the single mode quadrature squeezing in the cavity and trapped BEC modes was not observed for the smaller values of the rescaled time, but intermodal squeezing was verified by comparing the obtained results with those obtained as numerical solution of time dependent Schrödinger equation. Analytic and numerical results are found to be in good agreement.

As summarized in in Table I in this paper, we have used the obtained perturbative solution to observe the possibility of generation of squeezed, antibunched and entangled states in both the systems of our interest. Specifically, both lower- and higher-order squeezing are observed in the movable mirror mode in the optomechanical system. In contrast, the reduced results for the optomechanics-like system could generate neither lower-order nor higher-order squeezing in the corresponding cavity mode. However, both the systems are found to demonstrate intermodal squeezing. Similarly, lower- and higher-order antibunching is observed to be present (absent) in the movable mirror (cavity) mode in the optomechanical (optomechanics-like) system. It’s particularly noteworthy that so many higher-order nonclassical phenomena have been observed in these two systems. This is important because of the facts that higher order nonclassicality have yet been observed only in a limited number of physical systems and it helps to identify very week nonclassicality.

In case of lower and higher-order entanglement using Hillery and Zubairy’s set of criteria, both the systems are found to show possibility of inseparable states, which can be easily controlled by changing the phase parameter for the movable mirror (cavity) mode in the optomechanical (optomechanics-like) system. Additionally, Duan et al.’s criteria of lower-order entanglement failed to detect nonclassicality in any system.
including these modes has been observed for the same values of the rescaled time. Further, the presence of entanglement and its controllable behavior opens up new doors of possibilities in both optomechanical and optomechanics-like systems. Various types of nonclassicality observed here can be employed in different quantum information processing tasks.

We can conclude the paper with a hope that the rich variety of nonclassical behavior observed in both the cavity systems we have studied here will soon be experimentally verified by growing experimental facilities and escalated interest in the field of optomechanical and optomechanics-like systems.

Acknowledgments

AP and NA thank Department of Science and Technology (DST), India for the support provided through the project number EMR/2015/000393. KT acknowledges support from the Council of Scientific and Industrial Research (CSIR), Government of India.

[1] B. P. Abbott, et al., Phys. Rev. Lett., 116, 061102 (2016).
[2] B. P. Abbott, et al., Phys. Rev. Lett., 116, 241103 (2016).
[3] A. A. Rakhabovsky and S. P. Vyatchanin, Phys. Lett. A 377, 1317 (2013).
[4] V. B. Braginsky and F. Y. Khalili, Quantum Measurements (Cambridge University Press, Cambridge, England, 1995).
[5] F. Brennecke, S. Ritter, T. Donner and T. Esslinger, Science 322, 235 (2008).
[6] K. W. Murch, K. L. Moore, S. Gupta, and D. M. Stamper-Kurn, Nature Phys. 4, 561 (2008).
[7] C. Joshi, M. Jonson, E. Anderson and P. Ohberg, J. Phys. B: At. Mol. Opt. Phys. 4, 245503 (2011).
[8] C. Joshi, J. Larson, M. Jonson, E. Anderson and P. Ohberg, Phys. Rev. A 85, 033805 (2012).
[9] S. Bose, K. Jacobs and P. L. Knight, Phys. Rev. A 56, 4175 (1997).
[10] S. Mancini, V. I. Manko and P. Tombesi, Phys. Rev. A 55, 3042 (1997).
[11] N. Lambert, R. Johansson and F. Nori, Phys. Rev. B 84, 245421 (2011).
[12] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, R. W. Simmons, Nature 475, 359 (2011).
[13] K. C. Schwab and M. L. Roukes, Phys. Today 58, 108 (2005).
[14] C. Joshi, A. Hutter, F. E. Zimmer, M. Jonson, E. Anderson and P. Ohberg, Phys. Rev. A 82, 043846 (2010).
[15] K. Zhang, P. Meystre and W. Zhang, Phys. Rev. Lett., 108, 240405 (2012).
[16] K. Zhang, P. Meystre and W. Zhang, In APS Division of Atomic, Molecular and Optical Physics Meeting Abstracts, 1, 4005 (2011).
[17] J. D. Teufel, D. Li, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker and R. W. Simmons, Nature 471, 204 (2011).
[18] Z.-L. Xiang, S. Ashhab, J.Q. You, F. Nori, Rev. Mod. Phys. 85, 623 (2013).
[19] J.R. Johansson, G. Johansson and F. Nori, Phys. Rev. A 90, 053833 (2014).
[20] R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[21] E. C. G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963).
[22] A. D. Armour, M. P. Blencowe and K. C. Schwab, Phys. Rev. Lett. 88, 148301 (2002).
[23] T. Byrnes, K. Wen and Y. Yamamoto, Phys. Rev. A 85, 040306(R) (2012).
[24] Z. B. Chen and Y. D. Zhang, Phys. Rev. A 65, 022318 (2002).
[25] J. Eisert, M. B. Plenio, S. Bose and J. Hartley, Phys. Rev. Lett. 93, 190402 (2004).
[26] S. Mancini, V. Giovannetti, D. Vitali and P. Tombesi, Phys. Rev. Lett. 88, 120401 (2002).
[27] J. Zhang, K. Peng and S. L. Braunstein, Phys. Rev. A 68, 013808 (2003).
[28] Z. Yuan, B. E. Kardynal, R. M. Stevenson, A. J. Shields, C. J. Lobo, K. Cooper, N. S. Beattie, D. A. Ritchie and M. Pepper, Science 295, 102 (2002).
[29] X.-Y. Lu, J.-Q. Liao, L. Tian and F. Nori, Phys. Rev. A 91, 013834 (2015).
[30] T. P. Purdy, P.-L. Yu, R. W. Peterson, N. S. Kampel and C. A. Regal, Phys. Rev. X 3, 031012 (2013).
[31] M. Asjad, G. S. Agarwal, M. S. Kim, P. Tombesi, G. Di Giuseppe and D. Vitali Phys. Rev. A 89, 023849 (2014).
[32] S. Huang and G. S. Agarwal, Phys. Rev. A 81, 033830 (2010).
[33] X.-W. Xu and Y.-J. Li. J. Phys. B: Atom. Mol. Opt. Phys. 46, 035502 (2013).
[34] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 77, 4887 (1996).
[35] C. Monroe, D. M. Meekhof, B. E. King and D. J. Wineland, Science 272, 1131 (1996).
[36] F. Fratini, E. Mascalzhas, L. Safari, J.-P. Poizat, D. Valente, A. Auffeves, D. Gerace and M. F. Santos, Phys. Rev. Lett. 113, 243601 (2014).
[37] J. Sperling, W. Vogel and G. S. Agarwal, Phys. Rev. A 89, 043829 (2014).
[38] K. Vogel, V. M. Akulin and W. P. Schleich, Phys. Rev. Lett. 71, 1816 (1993).
[39] A. Miranowicz and W. Leonski (2004), J. Opt. B: Quantum and Classical Optics, 6, S43.
[40] D. Kleckner, W. Marshall, M. J. Dood, K. N. Dinyari, B.-J. Pors, W. T. Irvine, and D. Bouwmeester et al., Phys. Rev. Lett. 96, 173901 (2006).
[41] S. Gigan, H. R. Bohm, M. Paternostro, F. Blaser, G. Langer, J. B. Hertzberg, K. Schwab, D. Bauerle, M. Aspelmeyer, A. Zeilinger, Nature 444, 67 (2006).
[42] O. Arcizet, P.-F. Cohadon, T. Briant, M. Pinard, and A. Heidmann, Nature 444, 71 (2006).
[43] M. Hillery, Phys. Rev. A 61, 022309 (2000).
[44] A. Furusawa, J. L. Sorensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble and E. S. Polzik, Science 282, 706 (1998).
[45] J. Aasi, et al., Nature Photonics, 7, 613 (2013).
[46] M. M. Harry and LIGO Scientific Collaboration, 27, 084006 (2010).
[47] H. Grote, K. Danzmann, K. L. Dooley, R. Schnabel, J. Slutsky and H. Vahlbruch, Phys. Rev. Lett. 110, 181101 (2013).
[48] A. Pathak, Ind. J. Phys. 80, (2006) 495.
[49] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and
[50] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[51] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[52] A. Allevi, S. Olivares and M. Bondani, Phys. Rev. A 85, 063835 (2012).
[53] A. Allevi, S. Olivares and M. Bondani, Int. J. Quant. Info. 8, 1241003 (2012).
[54] M. Avenhaus, K. Laiho, M. V. Chekhova and C. Silberhorn, Phys. Rev. Lett. 104, 063602 (2010).
[55] M. Hamar, V. Michalek, and A. Pathak, Measurement Science Review 14, 227 (2014).
[56] C. Galland, N. Sangouard, N. Piro, N. Gisin, and T. J. Kippenberg, Phys. Rev. Lett. 112, 143602 (2014).
[57] K. Stannigel, P. Komar, S. J. M. Habraken, S. D. Bennett, M. D. Lukin, P. Zoller, and P. Rabl, Phys. Rev. Lett. 109, 013603 (2012).
[58] S. K. Giri, B. Sen, C. H. R. Ooi, and A. Pathak, Phys. Rev. A 89, 033628 (2014).
[59] S. K. Giri, K. Thapliyal, B. Sen, A. Pathak, Physica A 466, 140 (2017).
[60] W. Unruh, in Quantum Optics, Experimental Gravitation and Measurements Theory, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983).
[61] C. K. Law, Phys. Rev. A 51, 2537 (1995).
[62] S. Singh, G. A. Phelps, D. S. Goldbaum, E. M. Wright and P. Meystre, Phys. Rev. Lett. 105, 213603 (2010).
[63] B. Sen and S. Mandal, J. Mod. Phys. 52, 1798 (2005).
[64] K. Thapliyal, A. Pathak, B. Sen, and J. Perina, Phys. Rev. A 90, 013808 (2014).
[65] K. Thapliyal, A. Pathak, B. Sen, and J. Perina, Phys. Lett. A 378, 3431 (2014).
[66] K. Thapliyal, A. Pathak, and J. Perina, Phys. Rev. A 93, 022107 (2016).
[67] N. Alam and S. Mandal, J. Phys. B: At. Mol. Opt. Phys. 48, 045503 (2015).
[68] N. Alam and S. Mandal, Optics Comm. 359, 221 (2016).
[69] N. Alam and S. Mandal, Opt. Lett. 366, 340 (2016).
[70] A. Luks, V. Perinova, and J. Perina, 67, 149 (1988).
[71] R. Loudon, Opt. Comm. 70, 109 (1989).
[72] R. Loudon and P. L. Knight, J. Mod. Opt. 34, 709 (1987).
[73] C. K. Hong and L. Mandel, Phys. Rev. Lett. 54, 323 (1985).
[74] M. Hillery, Phys. Rev. A 36, 3796 (1987).
[75] R. Hanbury-Brown and R. Q. Twiss, Nature 177, 27 (1956).
[76] D. F. Walls, Nature, 280, 451 (1979).
[77] N. Lorch and K. Hammerer, Phys. Rev. A 91, 061803 (2015).
[78] F. Diedrich and H. Walther, Phys. Rev. Lett. 58, 203 (1987).
[79] C. T. Lee, Phys. Rev. A 41, 1721 (1990).
[80] G. S. Agarwal and K. Tara, Phys. Rev. A 46, 485 (1992).
[81] C. T. Lee, Phys. Rev. A 41, 1569 (1990).
[82] A. Pathak and M. Garcia, App. Phys. B 84, 484 (2006).
[83] G. S. Agarwal and A. Biswas, New J. Phys. 7, 211 (2005).
[84] M. Hillery and M. S. Zubairy, Phys. Rev. Lett. 96, 050503 (2006).
[85] M. Hillery and M. S. Zubairy, Phys. Rev. A 74, 032333 (2006).
[86] M. Hillery, H. T. Dung, and H. Zheng, Phys. Rev. A 81, 062322 (2010).
[87] L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000).