A Proof of Fatou’s Interpolation Theorem

Arthur A. Danielyan

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Abstract
For Fatou’s interpolation theorem of 1906 we suggest a new elementary proof.

Keywords Disk algebra · Fatou’s theorem · Interpolation · Closed set

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1 Introduction
Denote by \( \Delta \) and \( T \) the open unit disk and the unit circle in the complex plane, respectively. Recall that the disk algebra \( A \) is the algebra of all continuous functions on the closed unit disk \( \overline{\Delta} \) that are analytic on \( \Delta \). Let \( m \) be the Lebesgue measure on \( T \).

The following theorem of Fatou [4] is a cornerstone in many classical and contemporary investigations in complex and harmonic analysis.

Theorem A Let \( F \) be a closed subset of \( T \) such that \( m(F) = 0 \). Then there exists a function \( \omega_F \) in the disk algebra \( A \) which vanishes precisely on \( F \).

A slightly alternative formulation of Theorem A is the following:

Theorem A’ Let \( F \) be a closed subset of \( T \) such that \( m(F) = 0 \). Then there exists a function \( \lambda_F \) in the disk algebra \( A \) such that \( \lambda_F(z) = 1 \) on \( F \) and \( |\lambda_F(z)| < 1 \) on \( T \setminus F \).

An early applications of Fatou’s theorem is the original proof of the F. and M. Riesz theorem on the analytic measures, which asserts that if \( \mu \) is a complex regular Borel measure on the unit circle orthogonal to all polynomials (in \( z \)), then \( \mu \) is absolutely...
continuous with respect to the Lebesgue measure. In fact the proof of the Riesz brothers is the simplest among the other proofs of their theorem (cf. [7], p. 28).

Fatou’s proof of Theorem A still remains the only proof of the theorem. It has been included in many books (such as [5–9] to list a few). For the proof, Fatou [4] first constructs a summable function which is continuously differentiable in the complement of \( F \) and continuous in the extended sense at the points of \( F \). This construction actually is the crucial step of the proof as the final part of the proof is standard.

In this paper we present a new proof for Theorem A. The proof is based on a summable function which we construct using a new method. To complete the proof of Theorem A, we borrow the classical argument on the existence of the boundary values of the conjugate harmonic functions, but, in general, the present proof is both shorter and simpler than the original proof of Fatou. Because of the application of Fatou’s theorem for the proof of the F. and M. Riesz theorem, our proof also simplifies the self-contained proof of the last theorem.

2 The Proof

The main idea of our proof is the following proposition, the proof of which is obvious.

Lemma 1 Let \( F \) be a closed set of measure zero on \( T \). Then \( F \) can be covered by a finite number of open intervals of \( T \) such that the sum of their lengths is arbitrarily small.

The next analogous lemma does not need in a proof either.

Lemma 2 The set \( F \) can be covered by an open set \( G_n \) consisting of a finite number of disjoint open intervals of \( T \) such that \( m(G_n) < \frac{1}{2^n} \), \( \overline{G}_{n+1} \subset G_n \), and \( F = \bigcap_{n=1}^{\infty} G_n \).

Denote by \( I_m^{(n)} \), \( m = 1, 2, ..., k_n \), the disjoint open intervals of \( T \) constituting the set \( G_n \); thus \( G_n = \bigcup_{m=1}^{k_n} I_m^{(n)} \). Of course, we assume that \( I_m^{(n)} \cap F \neq \emptyset \) for each \( m = 1, 2, ..., k_n \).

Let \( J_m^{(n)} \) be an open subinterval of \( I_m^{(n)} \), such that \( \overline{J_m^{(n)}} \subset I_m^{(n)} \) and \( J_m^{(n)} \cap F = I_m^{(n)} \cap F \) (\( m = 1, 2, ..., k_n \)). Denote

\[
E_n = \bigcup_{m=1}^{k_n} \overline{J_m^{(n)}},
\]

for each natural \( n \). Obviously \( F \subset E_n \subset G_n \) and thus \( F = \bigcap_{n=1}^{\infty} E_n \). (With no loss of generality we may assume \( E_{n+1} \subset E_n \); for example, one can just replace \( E_{n+1} \) by \( E_{n+1} \cap E_n \).

Lemma 3 Let \( F, G_n, I_m^{(n)}, J_m^{(n)}, \) and \( E_n \) be as above. Then there exists a continuously differentiable and periodic function \( \phi_n \) on \( [0, 2\pi] \), \( 0 \leq \phi_n(\theta) \leq 1 \), such that \( \phi_n(\theta) = 0 \) if \( e^{i\theta} \in T \setminus G_n \) and \( \phi_n(\theta) = 1 \) if \( e^{i\theta} \in E_n \).

\(^1\) If \( 1 \notin F \), then we assume \( 1 \notin G_n \) for all \( n \) (this will simplify the proofs).
Proof For a fixed $n$, we have only finite number of (disjoint) intervals $I^{(n)}_m$, and $J^{(n)}_m \subset I^{(n)}_m \ (m = 1, 2, \ldots, k_n)$. Set $\gamma_{n,m}(\theta) = 0$ if $e^{i\theta} \in T \setminus I^{(n)}_m$ and $\gamma_{n,m}(\theta) = 1$ if $e^{i\theta} \in J^{(n)}_m$. The graph of $\gamma_{n,m}$ consists of three horizontal line segments (on the lines $y = 0$ and $y = 1$). Using appropriate pieces of shiftings of cosine function we “smoothly” join these segments, thus, extending $\gamma_{n,m}$ up to a periodic and continuously differentiable on $[0, 2\pi]$ function $\phi_{n,m}$ such that $0 \leq \phi_{n,m}(\theta) \leq 1$. Denote $\phi_{n}(\theta) = \sum_{m=1}^{k_n} \phi_{n,m}(\theta)$. The function $\phi_{n}$ has all the necessary properties and the proof of Lemma 3 is complete. \hfill \Box

Denote
\[ \phi(\theta) = \sum_{n=1}^{\infty} \phi_n(\theta), \] (1)
which is summable on $[0, 2\pi]$ since
\[ \int_0^{2\pi} \phi(\theta) d\theta = \sum_{n=1}^{\infty} \int_0^{2\pi} \phi_n(\theta) d\theta < \sum_{n=1}^{\infty} \frac{1}{2\pi} = 1. \]

Lemma 4 At each $\theta_0$ such that $e^{i\theta_0} \in T \setminus F$ the function $\phi$ is finite and continuously differentiable.

Lemma 4 is obvious since, as Lemma 3 implies, at certain neighborhood of $\theta_0$ all but finite number of (continuously differentiable) functions $\phi_n$ vanish.

Let $u_{n}(z)$ and $u(z)$ be the Poisson integrals of $\phi_{n}(\theta)$ and $\phi(\theta)$, respectively. They are positive harmonic functions in $\Delta$. The function $u_{n}(z)$, as a Poisson integral of a continuous function, is continuous on $\overline{\Delta}$, and $u_{n}(e^{i\theta}) = \phi_{n}(\theta)$ for all $e^{i\theta} \in T$. By Lemma 4 the function $\phi$ is continuous at each $\theta_0$ such that $e^{i\theta_0} \in T \setminus F$ and the elementary property of Poisson integrals immediately implies the following lemma.

Lemma 5 Define the function $u(z)$ at each point $e^{i\theta_0} \in T \setminus F$ by setting $u(e^{i\theta_0}) = \phi(\theta_0)$. Then the (extended) function $u(z)$ is continuous on the set $\overline{\Delta} \setminus F$.

Thus $u(z)$ is defined and continuous on $\overline{\Delta} \setminus F$. We also define $u(z) = \infty$ on $F$. If $l$ is any natural number and $z \in \overline{\Delta}$, we have
\[ u(z) \geq \sum_{n=1}^{l} u_{n}(z). \] (2)

Indeed, $\phi_{n}(\theta) \geq 0$ implies (2) for $z \in \Delta$. If $z \in F$, (2) is evident as $u(z) = \infty$ on $F$. Finally, if $z \in T \setminus F$, then $u(z) = u(e^{i\theta}) = \phi(\theta) \geq \sum_{n=1}^{l} \phi_{n}(\theta) = \sum_{n=1}^{l} u_{n}(e^{i\theta}) = \sum_{n=1}^{l} u_{n}(z)$.

Lemma 6 Let $u(z)$ be defined on the closed unit disk $\overline{\Delta}$ as above and let $e^{i\theta_0} \in F$. Then $u(z) \to \infty$ as $z \ (z \in \overline{\Delta})$ tends to $e^{i\theta_0}$ arbitrarily.
Proof Since \( e^{i\theta_0} \in F \), we have \( e^{i\theta_0} \in E_n \) for each \( n \), and, by Lemma 3, \( u_n(e^{i\theta_0}) = \phi_n(\theta_0) = 1 \). Thus \( \sum_{n=1}^{l} u_n(e^{i\theta_0}) = 1 \). Since \( l \) is arbitrarily large and each \( u_n(z) \) is continuous on \( \Delta \), (2) implies that \( u(z) \to \infty \) when \( z \to e^{i\theta_0} \; (z \in \Delta) \). Lemma 6 is proved.

Proof of Theorem A. Let \( v(z) \) be a conjugate harmonic function of \( u(z) \). By Lemma 4 the function \( \phi \) is continuously differentiable at each \( \theta \) such that \( e^{i\theta} \in T \setminus F \). Thus, by the well known property of the conjugate function (see, e.g. [6], p. 79), \( v(z) \) can be extended continuously to the set \( T \setminus F \). Thus, \( v(z) \) is continuous on \( \Delta \setminus F \). Taking into account also Lemma 5 and Lemma 6, we conclude that the function \( \omega_F(z) = \frac{1}{1 + u(z) + iv(z)} \) belongs to the disk algebra, and \( \omega_F(z) = 0 \) if and only if \( z \in F \). Theorem A is proved.

To prove Theorem A', note that the function \( \lambda_F(z) = \frac{u(z) + iv(z)}{1 + u(z) + iv(z)} \) belongs to the disk algebra and has the properties formulated in Theorem A'.

Remark The above proof of Lemma 6 shows that \( u(z) \) approaches to \( \infty \) uniformly as \( z \; (z \in \Delta) \) approaches to any point of the set \( F \).

3 Some Further Remarks

An important ingredient of Fatou’s proof of Theorem A is the following proposition:

If the series with positive terms \( \sum_{k=1}^{\infty} l_k \) converges, then there exists a sequence \( \{A_k\} \), with \( A_k > 0 \), \( A_k \to +\infty \), such that \( \sum_{k=1}^{\infty} l_k A_k \) converges.

The above proof does not need in this proposition. Another advantage of our proof is that it readily implies that the harmonic function \( u(z) \) tends to \( \infty \) as \( z \; (z \in \Delta) \) approaches to the points of \( F \). In Fatou’s proof the same conclusion does not follow so easily and a reference to the theorem on the infinite (unrestricted) boundary values is needed.

Using the above approach, we derive an elementary (self contained) proof for Lemma 5.

By (1) for \( z \in \Delta \) we have

\[
  u(z) = \sum_{k=1}^{\infty} u_k(z) = \sum_{k=1}^{n} u_k(z) + \sum_{k=n+1}^{\infty} u_k(z) = W_n(z) + \tilde{W}_n(z),
\]

where \( W_n(z) \) and \( \tilde{W}_n(z) \) denote the partial sum and the remainder series, respectively.

Fix an arbitrary point \( e^{i\theta_0} \in T \setminus F \). By Lemma 3, there exists a \( \delta > 0 \) such that all but finite number of functions \( \phi_n \) vanish on the neighborhood \( B_\delta(\theta_0) = (\theta_0 - \delta, \theta_0 + \delta) \). Fix \( N \) so large that if \( k > N \) then \( \phi_k \) vanishes on \( B_\delta(\theta_0) \), and denote \( \tilde{\phi}_N(\theta) = \sum_{k=N+1}^{\infty} \phi_k(\theta) \). Then \( \tilde{W}_N(z) \) is the Poisson integral of the summable function \( \tilde{\phi}_N(\theta) \), which vanishes on \( B_\delta(\theta_0) \). Thus

\[
  \tilde{W}_N(z) = \tilde{W}_N(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}_N(t) P_r(\theta - t) dt = \frac{1}{2\pi} \int_{A} \tilde{\phi}_N(t) P_r(\theta - t) dt,
\]

\( \tilde{\phi}_N(t) \) is the Poisson integral of the summable function \( \tilde{\phi}_N(\theta) \), which vanishes on \( B_\delta(\theta_0) \). Thus

\[
  \tilde{W}_N(z) = \tilde{W}_N(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}_N(t) P_r(\theta - t) dt = \frac{1}{2\pi} \int_{A} \tilde{\phi}_N(t) P_r(\theta - t) dt.
\]

\( \tilde{\phi}_N(t) \) is the Poisson integral of the summable function \( \tilde{\phi}_N(\theta) \), which vanishes on \( B_\delta(\theta_0) \). Thus

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  \tilde{W}_N(z) = \tilde{W}_N(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{\phi}_N(t) P_r(\theta - t) dt = \frac{1}{2\pi} \int_{A} \tilde{\phi}_N(t) P_r(\theta - t) dt.
\]
where \( A = [-\pi, \pi] \setminus B_\delta(\theta_0) \). If \( t \in A \) and \( \theta \in B_{\frac{1}{2}\delta}(\theta_0) = (\theta_0 - \frac{1}{2}\delta, \theta_0 + \frac{1}{2}\delta) \), then \(|\theta - t| \geq \frac{\delta}{2}\). Thus, for \( \theta \in B_{\frac{1}{2}\delta}(\theta_0) \),

\[
|\tilde{W}_N(re^{i\theta})| = \frac{1}{2\pi} \int_A \tilde{\phi}_N(t) P_r(\theta - t) dt \leq \max_{\frac{\delta}{2} \leq |\theta - t|} \{ P_r(\theta - t) \} \frac{1}{2\pi} \int_A \tilde{\phi}_N(t)
\]

\[
= \max_{\frac{\delta}{2} \leq |t|} \{ P_r(t) \} \frac{1}{2\pi} \int_A \tilde{\phi}_N(t).
\]

The last quantity tends to zero as \( r \to 1 \). Thus, \( \tilde{W}_N(re^{i\theta}) \) can be extended continuously at the points of the set \( B_{\frac{1}{2}\delta}(\theta_0) \), where its values are zero. In particular, \( \tilde{W}_N(z) \) is continuous at \( e^{i\theta_0} \). Since \( W_N(z) \) is continuous on \(|z| \leq 1\), \( u(z) \) is continuous at \( e^{i\theta_0} \).

Lemma 5 is proved.

In closing we note that some recent applications of Fatou’s interpolation theorem are given in [1–3].

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References

1. Danielyan, A.A.: A theorem of Lohwater and Piranian. Proc. AMS 144, 3919–3920 (2016)
2. Danielyan, A.A.: Fatou’s interpolation theorem implies the Rudin-Carleson theorem. J. Fourier Anal. Appl. 23, 656–659 (2017)
3. Danielyan, A.A.: On Fatou’s theorem. Anal. Math. Phys. 10, 28 (2020)
4. Fatou, P.: Séries trigonométriques et séries de Taylor. Acta Math. 30, 335–400 (1906)
5. Garnett, J.B.: Bounded Analytic Functions. Academic Press, Cambridge (1981)
6. Hoffman, K.: Banach Spaces of Analytic Functions. Prentice Hall, Englewood Cliffs (1962)
7. Koosis, P.: Introduction to \( H^p \) Spaces. Cambridge University Press, Cambridge (1998)
8. Privalov, I.I.: Boundary Properties of Analytic Functions, 2nd edn. GITTL, Moscow-Leningrad (1950). (in Russian)
9. Zygmund, A.: Trigonometric Series, vol. 1. Cambridge University Press, Cambridge (1959)

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