Nonlinear Synchronization on Connected Undirected Networks

S. Orange∗ and N. Verdière∗

October 24, 2011

Abstract

This paper gives sufficient conditions for having complete synchronization of oscillators in connected undirected networks. The considered oscillators are not necessarily identical and the synchronization terms can be nonlinear. An important problem about oscillators networks is to determine conditions for having complete synchronization that is the stability of the synchronous state. The synchronization study requires to take into account the graph topology. In this paper, we extend some results to non linear cases and we give an existence condition of trajectories. Sufficient conditions given in this paper are based on the study of a Lyapunov function and the use of a pseudometric which enables us to link network dynamics and graph theory. Applications of these results are presented.

AMS Subject Classification 2010: 93D20, 93D30, 68R10.

Keywords: Nonlinear systems, Synchronization, Networks, Graph topology, Dynamical Systems

1 Introduction

The study of the dynamics of coupled nonlinear dynamical systems are the subject of a growing interest in various communities like in theoretical physic, in information technology or in neuronal biology. The literature on this topic shows different kinds of synchronization (see [10]). Classically, two coupled limit-cycle are said synchronized when their time evolution is periodic with the same period and perhaps the same phase. From the discover of synchronization of chaotic systems (see [1, 5, 8]), the word synchronization recovered different meanings such as having identical or functional related solutions, eventually with a delay. The definition has also been modulated by considering strong forms like complete, cluster form or weaker forms like phase and lag synchronization (see [11]).

An important question about synchronization of a network of oscillators is to determine the stability of the synchronisation state. This question leads to consider some properties of networks and state vectors of oscillators (see, for example, [4, 13, 14, 15, 17]). For this purpose, two methods are proposed in

∗LMAH (Laboratoire de Mathématiques Appliquées du Havre), Université du Havre, 25 rue Philippe Lebon, BP 540, 76658 Le Havre, France. Sebastien.Orange@univ-lehavre.fr, Nathalie.Verdiere@univ-lehavre.fr
the literature. The first one called master stability function is based on the computation of a Lyapunov exponent and the eigenvalues of the connectivity matrix [9]. However, this method is adapted when the coupling terms are linear and the computation of eigenvalues can become a difficult task. A second proposed method is the connection graph stability method (see [4]). It links the study of a Lyapunov function and the graph topology. This productive method has been extended to unbalance and undirected graph (see [2, 3]).

The results presented in this paper generalize some results of [4] to the non-linear synchronization case. For this, we introduce a notion of pseudometric in the graph. The determination of the sign of the Lyapunov function derivative requires two steps. The first one is to use assumptions allowing comparisons between oscillators and synchronization terms. The second step consists in using pseudometrics which enable us to use some graph properties. For the complete synchronization, we present two results. The first one gives a condition on synchronization strength for having a global synchronization of oscillators. The second result is a local versus of the first one, that is when the oscillators are closed to the synchronization variety. In these two cases, we give sufficient conditions that insure existence of trajectories.

This paper is organized as follows. The problem statements are presented in Section 2. First, we precise the kind of systems and the kind of synchronizations considered. Then, we recall the definition and some properties of pseudometrics defined on a graph. In Section 3, after precising the assumptions on the synchronization term, main results, that is conditions for having complete synchronization of the system of oscillators, are presented. These results are applied in Section 4.

2 Problem statements
Thereafter, $Y^T$ is the transpose of the vector $Y = (Y^1, \ldots, Y^m) \in \mathbb{R}^m$.

2.1 Systems and synchronizations considered
Let $G$ be a connected undirected graph and $n$ its number of vertex. The graph $G$ describes the set of interactions between the oscillators. We denote by $E$ the set of its edges. If $G$ contains an undirected edge from a vertex $i$ to a vertex $j$, we denote it by $(i, j)$.

The considered dynamical systems are defined by the following system of equations:

$$
\begin{cases}
\dot{X}_1 = F_1(X_1, t) - \epsilon \sum_{(1,j) \in E} h(X_1, X_j), \\
\vdots \\
\dot{X}_n = F_n(X_n, t) - \epsilon \sum_{(n,j) \in E} h(X_n, X_j),
\end{cases}
$$

where

- $X_i = (X_1^i, \ldots, X_d^i)^T$ is the vector composed of the $d$ coordinates of the $i$-th oscillator,
- $F_i = (F_1^i, \ldots, F_d^i)^T$ is the vectorial function defining one oscillator,
• $h = (h^1, \ldots, h^d)^T$ is the synchronization function which defines the vector coupling between oscillators,

• the real parameter $\epsilon$ corresponds to the synchronization strength

Recall that, for a given initial state of the set of oscillators $(X_1(0), X_2(0), \cdots X_n(0))^T$, system (1) synchronizes completely if, for all $(i, j) \in [1, n]$,

$$\|X_i(t) - X_j(t)\| \xrightarrow{t \to +\infty} 0.$$ 

This means that the vector $(X_1, \ldots, X_n)$ approaches the synchronization manifold defined by $X_1(t) = X_2(t) = \cdots = X_n(t)$. In particular, this implies that the oscillators have the same asymptotic behavior (such as chaotic trajectories, stable and periodic solutions). The complete synchronization of all oscillators can occur whatever their initial states are, in this case, the synchronization is said global; otherwise it is said local.

In this paper, we focus naturally on the differences $\Delta_{i,j} = X_i^T - X_j^T$ and therefore on the vector

$\Delta = (\Delta_{1,2}, \cdots, \Delta_{1,n}, \Delta_{2,3}, \cdots, \Delta_{2,n}, \cdots, \Delta_{n-1,n})^T$.

Thus, proving the complete synchronization of system (1) is equivalent to prove that $\|\Delta(t)\| \xrightarrow{t \to +\infty} 0$.

2.2 Quasimetrics defined on a graph

In the following, we consider pseudometric verifying the $\rho$-relaxed triangle inequality for a positive real $\rho$, that is an application $\varphi : D \times D \to \mathbb{R}^+$, where $D$ is an non empty set, satisfying the following three axioms:

• $\varphi(z_1, z_1) = 0$;

• $\varphi(z_1, z_2) = \varphi(z_2, z_1)$ (symmetry property);

• $\varphi(z_1, z_3) \leq \rho(\varphi(z_1, z_2) + \varphi(z_2, z_3))$ ($\rho$-relaxed triangle inequality).

Remark that any classical metric is such a pseudometric with $\rho = 1$.

Let $\varphi$ be a pseudometric on a set $D$. Let’s set, for all $m \in \mathbb{N}^*$, $\rho(m)$ the smallest real such that

$$\varphi(z_1, z_{m+1}) \leq \rho(m) [\varphi(z_1, z_2) + \cdots + \varphi(z_m, z_{m+1})]. \quad (2)$$

Note that $\rho(1) = 1$.

In the following examples, expressions of $\rho(m)$ appearing in inequalities (2) are direct consequences of the convexity of functions $x \to (x^2)\alpha$ and $x \to x^2 e^{1-|x|}$.

**Example 2.1.** 1. The application $\varphi_\alpha : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ defined by

$$\varphi_\alpha (\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = \left((x_1 - x_2)^2\right)^\alpha$$

with $\alpha \geq 1/2$ is a pseudometric for which $\rho(m) = m^{2\alpha-1}$.
2. Let \( D \) be the closed ball of center 0 and radius \( 2 - \sqrt{2} \). The application \( \varphi : D \times D \to \mathbb{R}^+ \) defined by
\[
\varphi \left( \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right) = (x_1 - x_2)^2 e^{1 - |x_1 - x_2|} \]
is a pseudometric for which \( \rho(m) = m \).

We have the following properties.

**Proposition 2.1.**
1. The sequence of reals \( (\rho(m))_{m \geq 1} \) is increasing.
2. For all \( m \in \mathbb{N}^* \), we have \( \rho(m) \leq \rho^{m-1} \) (see [16]).
3. Let \( \varphi_1 \) and \( \varphi_2 \) be two pseudometrics on \( D \) and \( \rho_1(m) \) and \( \rho_2(m) \) be the smallest respective reals verifying (2). For all \( \alpha > 0 \) and \( \beta > 0 \), the application \( \alpha \varphi_1 + \beta \varphi_2 \) is a pseudometric on \( D \) satisfying \( \rho(m) = \) Max\( \{\rho_1(m), \rho_2(m)\} \).

We now apply pseudometrics to networks of oscillators. Recall that a state vector \( z_i \) of an oscillator is associated to \( i \)-th vertex of \( G \). Let’s consider a pseudometric \( \varphi \) defined on the set of state vectors of oscillators. This pseudometric enables one to define the pseudolength \( \varphi(z_i, z_j) \) between vertices \( i \) and \( j \) and also the pseudolength \( \varphi(z_{i_1}, z_{i_2}) + \cdots + \varphi(z_{i_{m-1}}, z_{i_m}) \) of any path \( P_{i,j} = (i = i_1, i_2, \cdots, i_m = j) \) from vertex \( i \) to vertex \( j \).

In the following proposition, we bound, up to a multiplicative constant \( C(G) \), the sum of pseudolengths between any two oscillators by the sum of pseudolengths of paths joining any two oscillators. This constant plays an important role in Theorems 3.1 and 3.2 since the synchronization strenght \( \epsilon \) appearing in these theorems is proportionnal to this constant.

**Proposition 2.2.** Let \( G \) be a connected graph, \( E \) be the set of its edges and \( \varphi \) be a pseudometric on a set \( D \). For any vertex \( i \), let \( z_i \in D \) be a vector associated to vertex \( i \). There exists a constant \( C \) depending only on \( G \) so that we have
\[
\sum_{i,j} \varphi(z_i, z_j) \leq C \sum_{(i,j) \in E} \varphi(z_i, z_j). \tag{3}
\]
Moreover, the smallest real \( C \) satisfying (3), \( C(G) \), is bounded by
\[
\frac{n(n - 1)}{2} \delta(G) \rho(\delta(G)), \tag{4}
\]
where \( \delta(G) \) is the diameter of \( G \).

**Proof.** Let \( i \) and \( j \) be two vertices of \( G \) and let’s denote
\[
P_{i,j} = (i = i_1, i_2, \cdots, i_{s+1} = j)
\]
a path of \( G \) from the vertex \( i \) to vertex \( j \) (recall that \( G \) is connected). Since \( \varphi \) is a pseudometric on \( D \), we have \( \varphi(z_i, z_j) \leq \rho(s) \sum_{\ell=1}^s \varphi(z_{i_\ell}, z_{i_{\ell+1}}) \).

The path \( P_{i,j} \) can be chosen so that \( s \leq \delta(G) \). Suppose that this choice is done for any vertices \( i \) and \( j \); since the sequence \( (\rho(n))_{n \in \mathbb{N}^*} \) is increasing, we have \( \rho(s) \leq \rho(\delta(G)) \). Consequently, for any vertices \( i \) and \( j \), we have \( \varphi(z_i, z_j) \leq \rho(\delta(G)) \delta(G) \ Max \{\varphi(z_i, z_j) | (i, j) \in E\} \) which implies the result.

\[ \square \]
In Theorem 3.1, we need to determine the lowest bound \( C(G) \) of the set of reals \( C \) satisfying inequality (3). The bound (4) of \( C(G) \) may not lead to a good estimation of \( C(G) \) for a particular graph; nevertheless, this bound is valid for any graph with \( n \) vertices.

In the case of a pseudometric satisfying the classical triangle inequality, i.e. when \( \rho(n) = n \) for all \( n \in \mathbb{N}^* \), a method taking \( G \) as input and returning a bound of \( C(G) \) is proposed in [3]. Its two main steps are:

1. for all \((i, j)\) with \( i > j \), choose a path \( P_{i,j} \); this path is usually chosen with minimal length (number of edges in the path);
2. for each edge \( e \) of the connection graph, determine the sum \( B(e) \) of the lengths of all chosen paths \( P_{i,j} \) containing \( e \). A bound for \( C(G) \) is then \( \text{Max}\{ B(e) : e \in E \} \).

For each choice of paths, these two steps return a bound for \( C(G) \). Clearly, the number of possible paths is huge but computations of bounds for \( C(G) \) are possible since most of these choices are suboptimal. Up to a slight modification of the first step, this method can be applied here: its consists in considering, for all path \( P_{i,j} \), the pseudolength \( \rho(|P_{i,j}|) \) instead of its length \(|P_{i,j}|\).

**Remark 2.1.** In the case of pseudometrics \( \varphi \) satisfying \( \rho(m) = m \), explicit bounds of \( C(G) \) for specific graphs and the method proposed in [4, 3] for computing \( C(G) \) from \( G \) can be directly used. This is the case of the second function in Example 2.1.

### 3 Complete synchronizations

#### 3.1 Hypothesis

Afterwards, two cases are considered. The first one is the global complete synchronization for which oscillators \( X_1, \ldots, X_n \) lies in \( D = \mathbb{R}^d \). The second one is the complete synchronization for which oscillators are in a neighborhood \( D \) of the variety \( X_1 = X_2 = \cdots = X_n \).

Thereafter, we will suppose the following assumptions on system (1).

- For all \((i, j)\) \( \in E \), there exist some non negative reals \( a_1, \ldots, a_d \) such that
  \[
  \forall (X_i, X_j) \in D, \quad \varphi(X_i, X_j) = \sum_{k=1}^{d} a_k (X_i^k - X_j^k) h^k(X_i, X_j) \quad (5)
  \]
  are pseudometrics where \( h = (h^1, \ldots, h^d)^T \) is the synchronization function.

- For all \((i, j)\) \( \in [1, n]^2 \) and, for all \( t \geq t_0 \) where \( t_0 \in \mathbb{R} \),
  \[
  \forall (X_i, X_j) \in D, \sum_{k=1}^{d} a_k (X_i^k - X_j^k) (F_i^k(X_i, t) - F_j^k(X_j, t)) \leq \varphi(X_i, X_j). \quad (6)
  \]
• For all \((i, j) \in \llbracket 1, n \rrbracket^2\), \(\forall (X_i, X_j) \in D\),
\[
\varphi(X_i, X_j) = 0 \text{ and/or } \sum_{k=1}^{d} a_k(X_i^k - X_j^k)(F_i^k(X_i, t) - F_j^k(X_j, t)) = 0
\implies (X_i = X_j).
\]

(7)

**Remark 3.1.**

1. Notice that hypothesis (5) implies that,
\[
\forall (i, j) \in E, \forall (X_i, X_j) \in D, h(X_i, X_j) = -h(X_j, X_i) \text{ (antisymmetry)}.
\]

(8)

2. The assumption (7) is necessary for proving the complete synchronisation of system (1) in Theorems 3.1 and 3.2. The condition \(\varphi(X_i, X_j) = 0\) in this assumption is not always sufficient when it does not imply equalities of all the components of oscillators. In this case, the second condition is necessary for proving the complete synchronisation.

For practical cases, a first problem is to prove the existence of trajectories of system (1) for a sufficient large \(t\). For this goal, the following proposition enables us to link existence of trajectories between synchronized and non synchronized systems.

**Proposition 3.1.** For all \((i, j) \in \llbracket 1, n \rrbracket^2\), suppose that assumptions (5), (6) and (7) are satisfied and that, for all \(t \geq t_0\),
\[
X_i^T F_i(X_i, t) \leq \Psi(||X_i||)
\]
where \(\Psi\) satisfies the conditions
\[
\int_{s=s_0}^{+\infty} ds \frac{\Psi(t)}{\Psi(s)} = +\infty \text{ and } \Psi(s) > 0 \text{ for all } s \geq s_0 \geq 0.
\]

Then, the Cauchy’s problem defined by system (1) and an initial condition
\[
\begin{pmatrix}
X_1(t_0) \\
\vdots \\
X_n(t_0)
\end{pmatrix} \in \mathbb{R}^{nd}
\]
has a solution on the complete semi-axis \([t_0; +\infty)\).

**Proof.** Let’s set \(X = \begin{pmatrix} X_1 \\
\vdots \\
X_n \end{pmatrix} \in \mathbb{R}^{nd}\) and \(F(X, t) = \begin{pmatrix} F_1(X_1, t) \\
\vdots \\
F_n(X_n, t) \end{pmatrix} \in \mathbb{R}^{nd}\). In a first step, we prove that there exists a real \(\beta\) such that the following inequality between the scalar products holds:
\[
X^T \dot{X} \leq \beta X^T F(X, t).
\]

(9)

For this, we consider the \(dn \times dn\) diagonal matrix \(M = \text{Diag}(a_1, \ldots, a_d, \ldots, a_1, \ldots, a_d)\). We have:
\[
X^T M \dot{X} = \sum_{i=1}^{n} \sum_{k=1}^{d} a_k X_i^k F_i^k(X_i, t) - \epsilon \sum_{i=1}^{n} \sum_{k=1}^{d} a_k \sum_{j \in E} X_i^k h^k(X_i, X_j)
= X^T MF(X, t) - \epsilon \sum_{k=1}^{d} \sum_{(i, j) \in E} a_k X_i^k h^k(X_i, X_j)
\]
and, since to any edge \((i, j) \in E\) corresponds the edge \((j, i) \in E\), we obtain

\[
X^T M \dot{X} = X^T MF(X, t) - \frac{\epsilon}{2} \sum_{k=1}^{d} \sum_{(i,j) \in E} X^k_i h^k(X_i, X_j) + X^k_j h^k(X_j, X_i)
\]

\[
= X^T MF(X, t) - \frac{\epsilon}{2} \sum_{k=1}^{d} \sum_{(i,j) \in E} (X^k_i - X^k_j) h^k(X_i, X_j) \quad \text{(see equality (8))}
\]

\[
\leq X^T MF(X, t) \quad \text{(see assumption (5))}
\]

Inequality (9) is then a direct consequence of the fact that the reals \(a_i\) are non negative.

If the conditions of the proposition are verified, inequality (9) shows that we have, for all \(t \geq t_0\),

\[
X^T \dot{X} \leq \tilde{\Psi}(\|X\|)
\]

where \(\tilde{\Psi}\) is a application satisfying the conditions

\[
\int_{s=s_0}^{+\infty} ds = +\infty \quad \text{and} \quad \tilde{\Psi}(s) > 0 \quad \text{for all} \quad s \geq s_0 \geq 0.
\]

Thus, system (1) satisfies the conditions of Wintner’s theorem ([12]) and, consequently, solutions of system (1) are defined for any \(t \geq t_0\).

3.2 Global synchronization

**Theorem 3.1.** Suppose that the assumptions done in Section 3.1 are satisfied for \(D = (\mathbb{R}^d)^2\). If \(\epsilon > \frac{C_G}{2n}\), where \(C_G\) is the optimal bound such that inequality (3) holds, then system (1) synchronizes completely.

**Proof.** In order to show this result, we will apply the second method of Lyapunov. Let’s consider the Lyapunov candidate function:

\[
V = \frac{1}{2} \sum_{k=1}^{d} \sum_{i \leq j} a_k (X^k_i - X^k_j)^2.
\]

Clearly, this function is non negative if \(\Delta \neq \overrightarrow{0}\) and equal to 0 iff \(\Delta = \overrightarrow{0}\) that is when the system (1) is synchronized.
The derivative of $V$ gives:

$$
\dot{V} = \sum_{k=1}^{d} \sum_{i=1}^{d} \frac{\partial V}{\partial X_i^k} \dot{X}_i^k
$$

$$
= \sum_{k=1}^{d} \sum_{i=1}^{d} (nX_i^k - nX_j^k) \dot{X}_i^k
$$

$$
= \sum_{k=1}^{d} \left[ \sum_{i=1}^{d} nX_i^k \dot{X}_i^k - \sum_{j=1}^{d} X_j^k \sum_{i=1}^{d} \dot{X}_i^k \right]
$$

$$
= \sum_{k=1}^{d} \left[ \sum_{i=1}^{d} nX_i^k - \sum_{j=1}^{d} X_j^k \right] F_i^k(X_i, t)
$$

$$
- n\epsilon \sum_{(i,j) \in E} X_j^k h^k(X_i, X_j) + \epsilon \left( \sum_{j=1}^{n} X_j^k \right) \sum_{(i,j) \in E} h^k(X_i, X_j)
$$

$$
= \sum_{k=1}^{d} \left[ \sum_{(i,j) \in E} X_j^k - X_j^k \right] F_i^k(X_i, t)
$$

$$
- n\epsilon \sum_{(i,j) \in E} X_j^k h^k(X_i, X_j) + \epsilon \left( \sum_{j=1}^{n} X_j^k \right) \sum_{(i,j) \in E} h^k(X_i, X_j)
$$

Since each edge $(i, j) \in E$ corresponds to an edge $(j, i)$ and using equality (8), we have, for all $k \in [1, n]$,

$$
2 \sum_{(i,j) \in E} h^k(X_i, X_j) = \sum_{(i,j) \in E} h^k(X_i, X_j) + \sum_{(i,j) \in E} h^k(X_j, X_i)
$$

$$
= \sum_{(i,j) \in E} h^k(X_i, X_j) + \sum_{(i,j) \in E} -h^k(X_i, X_j)
$$

$$
= 0
$$

and

$$
2 \sum_{k=1}^{d} \sum_{(i,j) \in E} X_j^k h^k(X_i, X_j) = \sum_{k=1}^{d} \left[ \sum_{(i,j) \in E} X_j^k h^k(X_i, X_j) + \sum_{(i,j) \in E} X_j^k h^k(X_j, X_i) \right]
$$

$$
= \sum_{k=1}^{d} \left[ \sum_{(i,j) \in E} X_j^k h^k(X_i, X_j) + \sum_{(i,j) \in E} -X_j^k h^k(X_i, X_j) \right]
$$

$$
= \sum_{(i,j) \in E} \varphi(X_i, X_j) \text{ (see 5).}
$$
Moreover, we have

\[ 2 \sum_{i,j} (X^k_i - X^k_j) F^k_i(X_i, t) = \sum_{i,j} (X^k_i - X^k_j) F^k_i(X_i, t) + \sum_{i,j} (X^k_j - X^k_i) F^k_j(X_j, t) = \sum_{i,j} (X^k_i - X^k_j) (F^k_i(X_i, t) - F^k_j(X_j, t)). \]

These three equalities gives

\[ \dot{V} = \sum_{i,j} \sum_{k=1}^d \frac{a_k}{2} (X^k_i - X^k_j) (F^k_i(X_i, t) - F^k_j(X_j, t)) - n\epsilon \sum_{(i,j) \in E} \varphi(X_i, X_j) \quad (10) \]

With assumption (6) and inequality (3), we obtain

\[ \dot{V} \leq \frac{1}{2} \sum_{i,j} \varphi(X_i, X_j) - n\epsilon \sum_{(i,j) \in E} \varphi(X_i, X_j) \leq \left( \frac{C_G}{2} - n\epsilon \right) \sum_{(i,j) \in E} \varphi(X_i, X_j) \]

Since \( \varphi \) is a pseudometric the right factor of this last expression is non negative. Therefore, if \( \epsilon > \frac{C_G}{2n} \) then \( \dot{V} \leq 0 \). To prove that \( \dot{V} \) is negative definite, it remains to show that if \( \dot{V} = 0 \) then \( X_1 = X_2 = \cdots = X_n \). Suppose that \( \dot{V} = 0 \). Since \( \left( \frac{C_G}{2} - n\epsilon \right) < 0 \), the last inequality implies that we have \( \varphi(X_i, X_j) = 0 \) for all \( (i, j) \in E \). From equality (10), we obtain

\[ \sum_{i,j} \sum_{k=1}^d a_k (X^k_i - X^k_j) (F^k_i(X_i, t) - F^k_j(X_j, t)) = 0. \]

Consequently, assumption (7) is satisfied and system (1) synchronizes.

**3.3 Local synchronization**

Let \( H \) be the diagonal matrix \( \text{Diag}(a_1, \ldots, a_d) \) and \( \mathcal{H} = \begin{pmatrix} H & 0 & \cdots & 0 \\ 0 & H & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & H \end{pmatrix} \), the matrix composed with \( \frac{n(n-1)}{2} \) matrices \( H \). The application

\[ \| \cdot \| \mathcal{V} : \mathbb{R}^{\frac{n(n-1)}{2} \cdot d} \to \mathbb{R}^+, \quad X \to \sqrt{\frac{d}{2} X^T \mathcal{H} X} \]

is a norm since \( a_1, \ldots, a_d \) are non negative. Let’s set

\[ V(t) = \| \Delta(t) \|^2 = \frac{1}{2} \sum_{k=1}^d \sum_{i<j \leq n} a_k (X^k_i(t) - X^k_j(t))^2. \]
Theorem 3.2. Let $B$ the closed ball $\{ X \in \mathbb{R}^{\frac{n(n+1)}{2}} \ | \ \| X \|_V \leq r \}$ where $r$ is a non negative real. Suppose that assumptions of Section 3.1 are satisfied when $\Delta$ belongs to the inner $\overset{\circ}{B}$ of $B$ and suppose that, for an instant $t_0, \Delta(t_0) \in \overset{\circ}{B}$. If $\epsilon > \frac{C_G}{2n}$, where $C_G$ is the optimal bound such that inequality (3) holds, then system (1) synchronizes.

Proof. Let’s show that if $\Delta(t_0) \in \overset{\circ}{B}$ then $\forall t > t_0, \Delta(t) \in B$. If $\Delta(t_0) \in \overset{\circ}{B}$, by definition of $B$, we have $V(t_0) < r^2$. Suppose that there exists $t_1 > t_0$ such that $\Delta(t_1) \notin B$; by definition of $B$, we have $V(t_1) > r^2$. Since $t \to V(t)$ is continuous, there exists a real $t_2 = \inf \{ t \in [t_0, t_1] \ | \ V(t) = r^2 \}$. The mean value theorem shows that there exists $t_3 \in (t_0, t_2)$ such that $V'(t_3) = \frac{V(t_0) - V(t_2)}{t_0 - t_2} > 0$. On the other side, since $t_3 < t_2 = \inf \{ t \in [t_0, t_1] \ | \ V(t) = r^2 \}$, we have $V(t_3) < r^2$ and $\Delta(t_3) \in \overset{\circ}{B}$. Consequently, the hypothesis of Section 3.1 are satisfied by $\Delta(t_3)$ and we can proceed like in the proof of Theorem 3.1 to show that $V'(t_3) \leq 0$. This brings to a contradiction.

Finally, we have $\forall t \geq t_0, \Delta(t) \in B$ and the assumptions of Section 3.1 are satisfied for any $t \geq t_0$. Now, we can proceed like in the proof of Theorem 3.1 to conclude. \hfill \Box

4 Applications

In this section, we focus on applications of Theorems 3.1 and 3.2 in order to have a sufficient condition for global synchronization of two systems. The fact that solutions of these two systems are defined on $\mathbb{R}$ is a direct consequence of Proposition 3.1.

4.1 Global synchronization of a network of neurons

In this section, we apply Theorem 3.1 to a network of neurons satisfying the FitzHugh-Nagumo model (See [6]). Recall that the dynamic of a single neuron is modelised by the equation $\dot{X} = F(X)$ where

- $X = \left( \begin{array}{c} x \\ y \end{array} \right)$;
- $F(X) = \left( \begin{array}{c} -x^3 + x - y + a \\ bx - cy - d \end{array} \right)$ for some real parameters $a, b, c$ and $d$.

In the following, we suppose that $b$ is positive. Let’s set $G$ the connected graph describing the interaction between the oscillators, $n$ its number of vertices and $\mathcal{E}$ the set of its edges. For the synchronization terms, we consider the function $h$ defined by

$$\forall (i, j) \in [1, n]^2, h(X_i, X_j) = \left( \frac{\alpha(x_i - x_j) + \beta \sqrt{(x_i - x_j)^5}}{\gamma(y_i - y_j)} \right)$$
with $\alpha \geq 1$, $\beta \geq 0$ and $\gamma \geq Max\{0,-c\}$. The system of equations for the network of oscillators is then

$$
\begin{align*}
X_1 &= F_1(X_1) - \epsilon \sum_{(i,j) \in E} h(X_1, X_j), \\
& \quad \vdots \\
X_n &= F_n(X_n) - \epsilon \sum_{(n,j) \in E} h(X_n, X_j).
\end{align*}
$$

(12)

The three hypothesis of Section 3.1 are satisfied with $a_1 = 1$ and $a_2 = 1/b$. Indeed,

1. assumption (7) is obvious;

2. the fact that the application $\varphi$ corresponding to $h$, explicitly defined by

$$
\varphi(X_i, X_j) = \alpha(x_i - x_j)^2 + \beta \sqrt{(x_i - x_j)^4 + \gamma/b(y_i - y_j)^2},
$$

is a pseudometric satisfying $\rho(m) = m^{5/3}$ is a consequence of Example 2.1 and Proposition 2.1. Therefore, assumption (5) is satisfied;

3. the following inequalities shows assumption (6), for all $(X_i, X_j) \in D$,

$$
\begin{align*}
\sum_{k=1}^{2} a_k(X^k - X^k_j) (F^k_i(X_i) - F^k_j(X_j)) \\
&= \left( \begin{array}{c}
(x_i - x_j) \\
y_i - y_j
\end{array} \right) \cdot \left( \begin{array}{c}
-(x_i^4 - x_j^4) + (x_i - x_j) - (y_i - y_j) \\
(x_i - x_j) - c/b(y_i - y_j)
\end{array} \right) \\
&= -(x_i - x_j)(x_i^4 - x_j^4) + (x_i - x_j)^2 - c/b(y_i - y_j)^2 \\
&\leq \varphi(X_i, X_j).
\end{align*}
$$

For any connected graph $G$ with $n$ vertex, inequality (3) is verified for the bound of $C(G)$ given by $C = \frac{n(n-1)}{2} \delta(G) \rho(\delta(G))$. Theorem 3.1 shows then that, for any connected graph $G$ with $n$ vertex, if $\epsilon > \frac{(n-1) \delta(G)^{5/3}}{4}$ then system (12) synchronizes.

### 4.2 Local synchronization of a network of oscillators

In this section, we apply Theorem 3.2 to a network of Chua oscillators. We consider the simplified version suggested by Chua for these oscillators (see [7]): if we set $X = (x,y,z)^T$, the state equation for a single oscillator is given by $\dot{X} = F(X)$ where

$$
F(x, y, z) = \left( \begin{array}{c}
a[y - x - f(x)] \\
x - y + z \\
-b y - c z
\end{array} \right),
$$

$a > 0$, $b > 0$, $c > 0$ and $f$ is a piece-wise function $f(x) = dx + 1/2(d-e)(|x+1|-|x-1|)$ with $2d < e$.

Since $f$ is a piece-wise function, a real $\delta \geq 0$ bounds the set of slopes

$$
\left\{ \frac{f(x) - f(y)}{x-y} \mid 0 < |x-y| \leq 1 \right\}.
$$

In the following, we suppose that:
1. the set of vertex of $G$ is $\mathcal{E} = \{(1; 2), (1; 3), \ldots, (1; n)\}$. In other words, we consider a star configuration of oscillators;

2. the synchronization function $h$ is given by

$$h((x_i, y_i, z_i), (x_j, y_j, z_j)) = \begin{pmatrix} a\delta(x_i - x_j)e^{1-|x_i-x_j|} \\ 0 \\ 0 \end{pmatrix}.$$ 

The equation for the $i$-th oscillator of the network is then

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{pmatrix} = \begin{pmatrix} a[y_i - x_i - f(x_i)] \\ x_i - y_i + z_i \\ -by_i - cz_i \end{pmatrix} + \epsilon \sum_{j \in \mathcal{E}} \begin{pmatrix} a\delta(x_i - x_j)e^{1-|x_i-x_j|} \\ 0 \\ 0 \end{pmatrix}.$$

Assumptions of Section 3.1 have to be verified in order to apply Theorem 3.2. The first one is obvious. For the second and the third one, let’s set $a_1 = 1/a$, $a_2 = 1$ and $a_3 = 1/b$.

Let’s consider a closed ball $B = \left\{X \in \mathbb{R}^d : \|X\|_V \leq (\sqrt{2} - 1)\sqrt{a}\right\}$ where $\|\cdot\|_V$ is defined by (11) and the norm $\|\cdot\|_{\tilde{V}}$ given by

$$\|\cdot\|_{\tilde{V}} : \mathbb{R}^d \rightarrow \mathbb{R}^+ \quad Y \rightarrow \sqrt{\frac{1}{2}YTHTY}$$

where $H$ is the diagonal matrix $\text{Diag}(a_1, \ldots, a_d)$. If we have $\Delta \in B$ then $\|\Delta_{i,j}\|_{\tilde{V}} < (\sqrt{2} - 1)\sqrt{a}$. This implies that $|x_i - x_j| < 2 - \sqrt{2}$ and, according to Example 2.1, the application $\varphi$ corresponding to $h$ satisfies assumption (5).

Let’s verify assumption (6). We have

$$\sum_{k=1}^3 a_k (X^k_i - X^k_j)(P^k_i(X_i) - P^k_j(X_j))$$

$$= \begin{pmatrix} x_i - x_j \\ y_i - y_j \\ z_i - z_j \end{pmatrix} \begin{pmatrix} a((y_i - y_j) - (x_i - x_j) - (f(x_i) - f(x_j))) \\ (x_i - x_j) - (y_i - y_j) + (z_i - z_j) \\ -b(y_i - y_j) - c(z_i - z_j) \end{pmatrix}$$

$$= (x_i - x_j)(f(x_i) - f(x_j)) - (x_i - x_j)^2 - (y_i - y_j)^2 - c(b(z_i - z_j)^2).$$

By definition of $\delta$, we have $\varphi(x_i - x_j)(f(x_i) - f(x_j)) \leq \delta(x_i - x_j)^2e^{1-|x_i-x_j|}$. This shows inequality (6).

Moreover, if $\varphi(x_i, x_j) = 0$ and $\sum_{k=1}^3 a_k (X^k_i - X^k_j)(P^k_i(X_i) - P^k_j(X_j)) = 0$ then we have $X_i = X_j$. Consequently, assumption (7) holds.

Since the induced pseudometric $\varphi$ satisfies $\forall m \in \mathbb{N}^*$, $\rho(m) = m$ (see Example 2.1), the bound $C_G$ is given explicitly by $2n - 3$ (See Remark 2.1 and [4]).

Theorem 3.2 can now be applied : if $\Delta(t_0) \in \mathcal{B}$ for an instant $t_0$ and if $\epsilon > \frac{2n - 3}{2n}$ then system (1) synchronizes.
5 Conclusion

In this paper, sufficient conditions for proving complete synchronization of oscillators in a connected undirected network are presented. The contribution of this paper lies in the extension of results established in the case of linear synchronization to the non-linear case. For this, we have introduced pseudometrics which enable us to link graph topology and minimal synchronization strength between oscillators. Under our assumptions, a criterion proving the existence of trajectories is given. Two results for proving the complete synchronization are then proposed: the first one gives a global criterion and the second one deals with local synchronization, that is when the trajectories lie in a neighborhood of the synchronization variety. To illustrate these results, two applications are treated.

References

[1] VS Afraimovich, NN Verichev, and MI Rabinovich. Stochastically synchronized oscillators in dissipative systems. Radiophys. Quant. Electron, 29:795–803, 1986.

[2] I. Belykh, V. Belykh, and M. Hasler. Synchronization in asymmetrically coupled networks with node balance. Chaos: An Interdisciplinary Journal of Nonlinear Science, 16:015102, 2006.

[3] I. Belykh, M. Hasler, M. Lauret, and H. Nijmeijer. Synchronization and graph topology. Int. J. Bifurcation and Chaos, 15(11):3423–3433, 2005.

[4] V.N. Belykh, I.V. Belykh, and M. Hasler. Connection graph stability method for synchronized coupled chaotic systems. Physica D: nonlinear phenomena, 195(1-2):159–187, 2004.

[5] H. Fujisaka and T. Yamada. Stability theory of synchronized motion in coupled dynamical systems. Prog. Theor. Phys, 69(1):32–47, 1983.

[6] JL Hindmarsh and RM Rose. A model of the nerve impulse using two first-order differential equations. 1982.

[7] T. Matsumoto. A chaotic attractor from chua’s circuit. Circuits and Systems, IEEE Transactions on, 31(12):1055–1058, 1984.

[8] L.M. Pecora and T.L. Carroll. Synchronization in chaotic systems. Physical review letters, 64(8):821–824, 1990.

[9] L.M. Pecora and T.L. Carroll. Master stability functions for synchronized coupled systems. Physical Review Letters, 80(10):2109–2112, 1998.

[10] A. Pikovsky, M. Rosenblum, and J. Kurths. Synchronization: A universal concept in nonlinear sciences, volume 12. Cambridge Univ Pr, 2003.

[11] M.G. Rosenblum, A.S. Pikovsky, and J. Kurths. From phase to lag synchronization in coupled chaotic oscillators. Physical Review Letters, 78(22):4193–4196, 1997.
[12] A. Wintner. The non-local existence problem of ordinary differential equations. *American Journal of Mathematics*, 67(2):277–284, 1945.

[13] C.W. Wu. *Synchronization in coupled chaotic circuits and systems*, volume 41. World Scientific Pub Co Inc, 2002.

[14] C.W. Wu. Synchronization in networks of nonlinear dynamical systems coupled via a directed graph. *Nonlinearity*, 18:1057, 2005.

[15] C.W. Wu and L.O. Chua. Synchronization in an array of linearly coupled dynamical systems. *Circuits and Systems I: Fundamental Theory and Applications, IEEE Transactions on*, 42(8):430–447, 1995.

[16] Q. Xia. The geodesic problem in quasimetric spaces. *J. Geom. Anal.*, 19(2):452–479, 2009.

[17] J. Zhou, J. Lu, and J. Lü. Pinning adaptive synchronization of a general complex dynamical network. *Automatica*, 44(4):996–1003, 2008.