Asynchrony-Resilient and Privacy-Preserving Charging Protocol for Plug-in Electric Vehicles

Yunfan Zhang, Feng Liu, Senior Member, IEEE, Zhaojian Wang, Jianhui Wang, Senior Member, IEEE, Yifan Su, Yue Chen, Cheng Wang, and Qiuwei Wu

Abstract—The proliferation of plug-in electric vehicles (PEVs) advocates a distributed paradigm for the coordination of PEV charging. Distinct from existing primal-dual decomposition or consensus methods, this paper proposes a cutting-plane based distributed algorithm, which enables an asynchronous coordination while well preserving individual’s private information. To this end, an equivalent surrogate model is first constructed by exploiting the duality of the original optimization problem, which masks the private information of individual users by a transformation. Then, a cutting-plane based algorithm is derived to solve the surrogate problem in a distributed manner with intrinsic superiority to cope with various asynchrony. Critical implementation issues, such as the distributed initialization, cutting-plane generation and localized stopping criteria, are discussed in detail. Numerical tests on IEEE 37- and 123-node feeders with real data show that the proposed method is resilient to a variety of asynchrony and admits the plug-and-play operation mode. It is expected the proposed methodology provides an alternative path toward a more practical protocol for PEV charging.

Index Terms—Plug-in electrical vehicles (PEV), charging protocol, distributed optimization, asynchronous privacy preserving.

I. INTRODUCTION

A. Background and Motivation

The past years witnessed the proliferation of plug-in electric vehicles (PEVs). However, their rapid growth inevitably creates new challenges to power system operation. Particularly, as traditional distribution systems were not designed to support simultaneous charging of many PEVs [1], transformer capacity expansion or even reconstruction of the distribution system are needed to meet the growing demand for PEV charging [2]. However, such costly countermeasures could be alleviated or even avoided if the charging behavior of PEVs are well managed [1], in either a centralized or distributed manner.

Traditional centralized management of PEV charging needs to collect all PEVs’ information, such as positions, available charging time and state of charge (SOC), etc. Hence it may raise severe privacy concerns in individual PEV owners [3]. Moreover, the decision center may suffer from a heavy communication burden and high computational complexity. In this regard, distributed management was developed, where the charging patterns of PEVs are decided locally following a certain coordination scheme. It is expected to better protect the privacy of PEV owners and enable a faster response to environmental changes [4], which is crucially important when numerous PEVs disperse across the distribution network. However, there always exist various kinds of asynchrony in practice due to non-ideal communication such as time delay and packet drop. These regards motivate us to address the protocol of PEV distributed charging in this paper, considering asynchrony resilience and privacy-preserving.

B. Related Works

Generally, prior works on distributed PEV charging management can be cast into two branches: non-cooperative strategies and cooperative ones, which are briefly reviewed as follows.

1) Non-cooperative Strategies: Non-cooperative charging strategies oftentimes are partially distributed, where a coordinator is needed to broadcast coordination signals (usually electricity price) and then each PEV reacts to its received signals. A typical distributed non-cooperative strategy relies on a one-way communication broadcast [5]. This open-loop approach appears to be less effective due to the absence of feedback adjustments [6], [7]. To address this problem, iterative strategies emerge to search for the optimal or quasi-optimal charging profiles, where bi-directional communication between the coordinator and individual PEVs is required and each PEV solves a restricted sub-problem in every round. Under this framework, several distributed charging algorithms are derived, based on Lagrangian dual decomposition [8], projected gradient [9], [10], Alternating Direction Method of Multipliers (ADMM) [11]–[14] and theory of non-cooperative games [15], [16], to name a few.

2) Cooperative Strategies: Cooperative strategies are usually investigated under a fully distributed framework, where PEVs collaborate (usually with their immediate neighbors) to achieve a certain optimal target [17]. Such peer-to-peer (P2P) based schemes [18] serve as more flexible, scalable and robust alternatives since individual PEVs can autonomously achieve coordination with the absence of a coordinator. In this regard, distributed solution algorithms based on Karush-Kuhn-Tucker (KKT) conditions and consensus techniques are designed, see, for example, [18]–[20].

The potential of P2P based cooperative schemes, however, has not been well addressed yet when it comes to the following two critical implementation issues: i) User-state-information (USI) privacy. PEV users are reluctant to disclose their USI (such as SOC, positions, and demand profiles, etc.) neither to a center nor to other users. In regard to the information exchange which is necessary for coordination in a P2P network, privacy issue also remains as a concern. i) Imperfect communication. Considering time delays, packet drops, topology changes and non-identical computation capabilities of individual users, the participants have to wait for the slowest one to finish before
executing their local actions in the next iteration. Though prior works [21]–[23] have achieved many successes on distributed algorithms under asynchronous communication, their convergence results and solution quality rely on restrictive assumptions. Moreover, it is not trivial to select appropriate parameters and adjust the step size in the updating process.

3) Distributed Decision-making based on Cutting Planes: The cutting-plane theory [24] has recently drawn increasing attention in the community of distributed decision-making. However, there are few works considering its application to power systems, except [25] and [26] that discuss dynamic economic dispatch and microgrid control, respectively. Different from traditional consensus algorithms, a cutting-plane based consensus algorithm can achieve an agreement on a common query point through iterative constraint exchanges. This salient feature can better decompose the computation into individual agents with the minimal requirements of information synchrony, which inspires a suitable framework for the distributed PEV charging coordination against asynchrony.

C. Contribution

In this paper, a novel distributed optimal scheduling method as well as its implementation are proposed for individual PEV charging coordination under both local and global constraints, considering communication asynchrony and privacy preserving. The main contributions of this paper are threefold.

1) Asynchrony-Resilience. Distinct from the celebrated primal-dual decomposition methods [21]–[23] and ADMM algorithms [11]–[14], [27] where primal/dual variables are exchanged, the proposed method turns to exchange cutting planes among neighboring individuals. The resulting algorithm intrinsically admits asynchronous implementations, favoring a strong resilience to various asynchrony in practice such as time delays, packet drops and communication topology changes.

2) Privacy-Preserving. Regarding privacy concerns, this paper uses a surrogate model to mask the private information of individual users during the iterative coordination process. Different from existing works where the charging profiles or multipliers of the corresponding optimization problem are collected/exchanged [8]–[14], the proposed method only delivers aggregated information of the surrogate model, which can well protect the user-state privacy of individual PEV owners.

3) Convergence-guarantee. The existing cutting-plane based works heuristically take any locally converged consensus result as the optimal solution [25], [26], which lacks of a theoretic convergence guarantee. This paper first unfolds the solution quality of the algorithm and derives a completely localized stopping criteria. It is proved that the solution series obtained by the proposed method must converge to the global optimum. Moreover, the localized stopping criteria can achieve an arbitrarily small error.

D. Organization

The rest of the paper is organized as follows. The problem description with necessary preliminaries and notations is stated in Section II and Section III derivates an equivalent surrogate model. Section IV presents the distributed solution algorithm based on cutting-plane consensus. Case studies are introduced in Section V. Finally, Section VI concludes the paper.

II. NOTATION AND PROBLEM FORMULATION

A. Preliminaries and Notations

In this paper, \( \mathbb{R}^n \) (\( \mathbb{R}_+^n \)) depicts the \( n \)-dimensional (non-negative) Euclidean space. Use \( \mathbb{Z}^n \) to denote the set of positive integers. For a column vector \( z \in \mathbb{R}^n \) (matrix \( A \in \mathbb{R}^{m \times n} \)), \( z^T(A^T) \) denotes its transpose. \( \| \cdot \| \) denotes the Euclidean norm. Given a collection of \( y_i \) for \( i \) in a certain set \( N \), define \( \text{col}(y_i) := (y_1, y_2, \ldots, y_n)^T \) and denote its vector form by \( y := \text{col}(y_i) \). We use \( 1 \) (resp. \( 0 \)) to denote vector of ones (resp. zeros). Notations for cutting-plane and cutting-plane set are given in Definition 1 and Definition 2 respectively.

**Definition 1** (Cutting-Plane). Given a convex set \( S \subset \mathbb{R}^n \) and a query point \( z_q \notin S \), a half-space

\[
\begin{align*}
    h_{z_q} := \{ z | a_q^T z \leq b_{z_q} \},
    a_{z_q} \in \mathbb{R}^n, b_{z_q} \in \mathbb{R}^1
\end{align*}
\]

is referred to as the cutting-plane of \( S \) and \( z_q \) if it satisfies the following properties: (i) \( a_{z_q} \neq 0 \), (ii) \( a_{z_q}^T z_q > b_{z_q} \), and (iii) \( a_{z_q}^T z \leq b_{z_q}, \forall z \in S \).

**Definition 2** (Cutting-Plane Set). Cutting-plane set \( H \) is the collection of \( m \) single cutting-planes. Specifically,

\[
H := \bigcup_{k=1}^m h_k
\]

where \( h_k := \{ z | a_k^T z \leq b_k \} \). The induced polyhedron of \( H \) is denoted by \( \mathcal{H} := \{ z | A_H^T z \leq b_H \} \), with \( A_H := [a_1, \ldots, a_m] \) and \( b_H := [b_1, \ldots, b_m]^T \).

Note that in this paper the union symbol (same for intersection symbol) plays an opposite role on cutting-plane set and polyhedron set.

B. Charging of PEVs

To coordinate PEV charging in a fully distributed manner, we adopt the configuration shown in Fig. 1. Each PEV charger is equipped with a local processor with a certain capability of communication and computation. The processor collects data from the user (such as the designated SOC and charging deadline) and data from the PEV (such as the initial SOC and capacity of batteries). It also receives some broadcast information such as the electricity price. The local processor enables a bi-directional information exchange with its neighbors. Once the optimal charging strategy is derived, the control command is generated and sent to the PEV charger.

![Fig. 1. The schematic of the proposed local processor.](image-url)
C. Communication network

We consider a generalized asynchronous communication protocol presented in [22], where each processor has its own concept of time defined by a local clock $k_i \in \mathbb{Z}^+$. $k_i$ triggers when processor $i$ awakes, independently of other processors, to conduct local computations and update information to its neighbors. When $i$ is idle, it listens for messages from neighbors and stores them to its receiving cache. Then let $k \in \mathbb{Z}^+$ denote a virtual global clock that does not exist in reality and is used only for analysis. The relationship between local clocks and the global clock is depicted in fig. 2.

![Diagram of local clocks and a virtual global clock.](image)

Fig. 2. An instance of two local clocks and the virtual global clock.

To consider the change of communication topology, we model the communicating network at the virtual global clock $k$ as a $k$-dependent directed graph $G_k = (\mathcal{N}, \mathcal{E}_k)$, where $\mathcal{N} := \{1, 2, \ldots, n\}$ is the set of local charger processors and $\mathcal{E}_k \subset \mathcal{N} \times \mathcal{N}$ is the edge set. If processor $i$ transmits messages to processor $j$ at $k$, there is an edge from node $i$ to $j$ at $k$, denoted by $(i, j) \in \mathcal{E}_k$. For an edge set $\mathcal{E}_k$, we denote by $\mathcal{N}_{\mathcal{E}_k} := \{j | (i, j) \in \mathcal{E}_k\}$ the out-neighbors of processor $i$ and by $\mathcal{N}_{\mathcal{E}_k} := \{j | (j, i) \in \mathcal{E}_k\}$ the in-neighbors. Also, for the $k$-dependent graph $G_k$, we denote by $d_{G_k}$ its diameter and $d_{G_k}$ equals to the diameter of static graph $(\mathcal{N}, \mathcal{E}_{\infty})$ where $\mathcal{E}_{\infty} := \{(i, j) \in \mathcal{E}_k\}$ for infinitely many indices $k$.

Throughout this paper, we assume a “intermittent” connectivity condition of $G_k$ as in Assumption 1. It is a typical setting in asynchronous multi-agent optimization and control with a changing topology: the information propagation from one part to another is ensured during a certain number of time slots, i.e., every $T$ consecutive global time clocks. I will be used only for localizing stopping criteria design later on.

Assumption 1 ($T$-strongly connected [28]). There exits a $T \in \mathbb{Z}^+$ such that the graph with edge set $\bigcup_{t=T-1}^{T} \mathcal{E}_t$ is strongly connected for every $s \in \mathbb{Z}^+$.

D. Battery Model of PEVs

Consider there are $n$ PEVs to be charged over $T$ discrete time slots. Denote by $\mathcal{N} := \{1, 2, \ldots, n\}$ the set of PEVs, and $\mathcal{T} := \{1, 2, \ldots, T\}$ the set of time slots. For PEV processor $i$, let $p_i^f \in \mathbb{R}$ denote the charging power of PEV $i$ at time $t$ and we use $p_i$ to denote the column vector of $p_i^t$ over the entire time horizon of $T$ for simplicity, i.e., $p_i := (p_1^t, p_2^t, \ldots, p_T^t)^T$.

Each PEV is available for load dispatch once it is plugged in and before the charging is completed. For arbitrary $i \in \mathcal{N}$, limits on the total charging amount should be satisfied according to its SOC, leading to constraints [3].

\[
(SOC_i^b - SOC_i^f) \cdot Cap_i \leq \eta_i \cdot \sum_{t \in \mathcal{T}} p_i^t \quad \text{(3a)}
\]

\[
(SOC_i^f - SOC_i^d) \cdot Cap_i \geq \eta_i \cdot \sum_{t \in \mathcal{T}} p_i^t \quad \text{(3b)}
\]

$SOC_i^b$ and $SOC_i^d$ stand for the initial and final state of charge of PEV $i$, respectively. $Cap_i$ is its battery capacity and $\eta_i$ scales the charging efficiency. Constraint (3a) implies that the charging process stops once the battery reaches its maximum state of charge (denoted by $SOC_i^b$).

Each PEV can charge only after it plugs in at a certain time slot $T^* \in \mathcal{T}$ and before it leaves at $T^d \in \mathcal{T}$, where $T^* < T^d$. Hence we have constraints [4].

\[
p_i^f \cdot (t - T^*) \geq 0, \quad t \in \mathcal{T}, \quad i \in \mathcal{N} \quad \text{(4a)}
\]

\[
p_i^f \cdot (T^d - t) \leq 0, \quad t \in \mathcal{T}, \quad i \in \mathcal{N} \quad \text{(4b)}
\]

At each time slot, the charging power of a PEV is assumed constant but can vary from 0 to its maximum charging power $p_i$ at different time slots. Then, we have the constraint [5].

\[
0 \leq p_i^f \leq p_i, \quad t \in \mathcal{T}, \quad i \in \mathcal{N} \quad \text{(5)}
\]

Combining (3) to (5), we have individual feasible charging region over the time horizon of $\mathcal{T}$ denoted by $\mathcal{P}_i$, i.e.,

\[
\mathcal{P}_i := \{ p_i \in \mathbb{R}^T | p_i \text{ satisfies (3) - (5)} \} \quad \text{(6)}
\]

Note that $\mathcal{P}_i$ should be kept to PEV processor $i$ itself, as the involved arrival time, departure time and battery state are private information.

E. Coordination of PEV Charging

The charging coordination aims to minimize the total cost while satisfying system operation constraints and individual charging demands. The problem can be formulated as follows.

\[
\text{CoC :} \quad \min_{p_i} \sum_{i \in \mathcal{N}} f_i(p_i) \quad \text{(7a)}
\]

s.t. \[
\sum_{i \in \mathcal{N}} p_i^f \leq L \quad \text{(dual multiplier: } \pi) \quad \text{(7b)}
\]

\[
p_i \in \mathcal{P}_i, \quad i \in \mathcal{N}. \quad \text{(7c)}
\]

where, $p_i$ is a $T$-dimensional decision vector representing the charging power of PEV $i$ in the $T$ time slots. The charging cost of PEV $i$, denoted by $f_i(p_i) : \mathbb{R}^T \to \mathbb{R}$, is a convex function with respect to the charging power $p_i$.

Congestion due to feeder head capacity limit is considered in (7b). The right-hand-side parameter $L$ is a $T$-dimension vector that stand for the maximum available total charging power to avoid overload on feeder head at each time slot. $L$ is determined by the distribution system operator (DSO), and delivered to at least one processor. Note that $L$ can also be properly designed to achieve a valley-filling purpose or other demand response aims. Besides the coupling constraints, individual charging demand is captured by (7c) where $\mathcal{P}_i$ is processor $i$’s feasible region over the time horizon of $\mathcal{T}$ and its specific form is given by [6]. In practice, usually $f_i(\cdot)$, $\mathcal{P}_i$ and $p_i$ are private information of user $i$, which can only be accessed by processor $i$ and should not be disclosed to others.

For the convex problem CoC [7], we also make the following regular assumption to guarantee the strong duality holds.

Assumption 2. The Slater’s condition [29] Chapter 5.2.3] holds for CoC, i.e., there exist $p_i$ in the interior of $\mathcal{P}_i$ such that (7b) holds.

III. A SURROGATE MODEL OF CoC

In this section, first we decompose [7] by constructing an equivalent transformation, then derive a surrogate problem to...
mask the private information of individual PEVs during the iterative charging coordination, as we explain.

First of all, invoking [29, Chapter 5.1.1], the Lagrangian dual of (7) is given by

$$\max_{\pi \geq 0} \min_{p_i \in P_i, \forall i} \left\{ \pi^T \left( -L + \sum_{i \in N} p_i \right) + \sum_{i \in N} f_i(p_i) \right\} \tag{8}$$

where $\pi \in \mathbb{R}^T$ is the dual variable vector corresponding to the global constraint (7b). (8) can be further rewritten as

$$\max_{\pi \geq 0} \min_{p_i, \forall i} \left\{ \pi_i^* (p_i) + \pi^T (p_i - L) \right\}$$

$$+ \sum_{i \in N \setminus \{i^*\}} \min_{p_i \in P_i} \left\{ f_i(p_i) + \pi^T p_i \right\}.$$

(9)

where $i^*$ who is informed of the value of $L$ is uniquely pre-determined. Now define local dual functions $U_i(\pi) : \mathbb{R}^T \mapsto \mathbb{R}$ as in (9). Note that $U_i(\pi)$ are convex functions with respect to $\pi$ [29, Chapter 3.2.3]. Then (9) can be reformulated into (10), a convex problem, for succinctness.

$$\max_{\pi \geq 0} \sum_{i \in N} U_i(\pi) \tag{10}$$

The transformation from (7) to (10) is based on the Lagrangian decomposition [30, Chapter 4.3.1], by dualizing the coupling constraint (7b) to obtain a separate structure as in (10). The equivalence is guaranteed by noting that strong duality holds under Assumption 2. Let $\pi^*$ denote the optimal solution of problem (10). If $f_i(p_i)$ is strictly convex the optimal solution of the primal CoC problem is uniquely determined by

$$p_i^* = \arg \max U_i(\pi^*), \forall i \in N. \tag{11}$$

Note that, without knowing other’s $f_i(\cdot)$ and $P_i$, one cannot infer other PEV’s optimal charging profile, which protects the private information of individual PEVs.

Inspired by Dantzig-Wolfe decomposition [32], we construct a further transformation on (10). New decision variables of the reformulated optimization problem, $z \in \mathbb{R}^{T+n}$, consists of two parts $\pi \in \mathbb{R}^T$ and $u := \text{col}(u_i) \in \mathbb{R}^n$, where $u_i$ is introduced to replace the item $U_i(\pi)$ in (10). The relationship between $\pi$ and $u_i$, which is originally described by $u_i = U_i(\pi)$ as in (9), is now captured by the feasible region of $z$ which is denoted by $Z_i$ and identified in (12b). Let $Z_i$ be convex since $f_i$ is a convex function [29, Chapter 3.2.3]. Then, problem (10) is equivalently converted into the convex problem (12a) where $e := (0^T, 1^T)^T$ is a constant vector.

$$\max e^T z, \text{ s.t. } z \in \cap_{i \in N} Z_i \tag{12a}$$

$$Z_i := \left\{ z \mid \begin{array}{l}
\pi \geq 0 \\
u_i \leq f_i(p_i) + \pi^T p_i, \forall p_i \in P_i, \text{ if } i \in N \setminus \{i^*\} \\
u_i \leq f_i(p_i) + \pi^T (p_i - L), \forall p_i \in P_i, \text{ if } i = i^* \end{array} \right\} \tag{12b}$$

In this way, the original centralized optimization problem is converted into its surrogate dual model, the fully-distributed (12a) with global decision variables $z$ and several isolated feasible regions of $z$. Note that the dual function $e^T z$ as well as the set constraints $Z_i$ are both linear in $z$, providing great benefits for computation efficiency and enabling the cutting-plane exchange. In addition, as we will show in Section IV, this transformation enables processors to reach consensus with respect to the dual variable $z$ (or say $\pi$) under completely asynchronous center-free network without disclosing information about their local objective $f_i(\cdot)$, demand $P_i$ and other sensitive data including $U_i(\cdot)$ and $Z_i$.

IV. ASYNCHRONOUS DISTRIBUTED ALGORITHM

In this section, we develop a cutting-plane based distributed algorithm to solve the surrogate optimization problem (12) asynchronously, and then derive an error-bounded solution under a completely asynchronous communication protocol.

### A. Cutting-Plane based Asynchronous Distributed Algorithm

Considering each PEV processor runs at its local clock $k_i$, the asynchronous algorithm is derived as in Algorithm 1.

The basic idea of the algorithm 1 is as follows. A set of cutting planes are generated by each processor and is individually updated in every round of iterations. After reading the cutting-plane set from neighbors’ output cache, the local processor collects all received cutting-planes and its own cutting-planes to form a polyhedron, denoted by $\mathcal{H}^{[i]}(k_i)$ for processor $i$ in the $k_{th}$ round of local iteration. The polyhedron can be regarded as an approximation of the feasible region $S := \cap_{i \in N} Z_i$. Moreover, in each round of iteration, additional cutting planes are added to constantly shrink the polyhedron, leading to a more and more accurate estimation. Mathematically, this procedure is almost the same as the outer approximation method. Since $\mathcal{H}^{[i]}(k_i) \subseteq \mathcal{H}^{[i]}(k_i - 1)$ holds for each iteration, $\mathcal{H}^{[i]}(k_i)$ will eventually approach $S$. In this way, each local processor only needs to know its own $Z_i$, and iteratively approaches the feasible region $S$. Once the consensus on $S$ is achieved, the consensus value of $z$ is obtained. Then each PEV processor can extract its optimal charging profile via (11).

For Algorithm 1 we have the following useful remarks.

**Remark 1** (Asynchrony-resilience). Note that no global clock is required in Algorithm 1 implying intrinsic permission for asynchronous computation and updates. As demonstrated in Section V the proposed distributed algorithm is also resilient to other imperfect communication such as packet drops.

**Remark 2** (Privacy-preserving). The cutting-planes exchanged by processors are approximations of the feasible region $Z_i$ of the surrogate problem (12), other than the exact feasible region $P_i$ or objective function $f_i(x)$ of the original CoC problem (7). Therefore, one’s private information will not be disclosed to others. Moreover, since the union process in step 2 aggregates cutting-planes from all processors, one cannot infer private information of any individual.

### B. Distributed Generation of Cutting-planes

The consensus on feasible region depends upon the generation of new cutting-planes. In this subsection, we omit for 1In case $f_i(p_i)$ is not strictly convex, say, it is linear in $p_i$, the difficulty in primal recovery can be avoided by adding a sufficiently small quadratic item to the objective function without revising the optimal solution.

2In the rest of this paper, for a variable $x_{1i}^{[m]}(k)$, $i$ is processor $i$, $k$ is the $k_{th}$ iteration round of processor $i$, $m$ stands for the $m_{th}$ component of $x$.}
Algorithm 1 Asynchronous PEV Charging for processor $i$

**Input:** The local feasible region $\mathcal{P}_i$ of processor $i$

**Iteration at $k_i$:** Suppose processor $i$‘s clock ticks at $k_i$. Then, it is activated to update its cutting-plane set as follows:

**Step 1: Reading Phase**

Get cutting-plane set from its input-neighbors’ output cache. Generate temporary cutting-plane set according to (13).

$$H^{[i]}_{\text{tmp}}(k_i) = \left( \bigcup_{j \in \mathcal{N}_I(i,k_i)} H^{[j]}(k_i) \right) \cup H^{[i]}(k_i)$$

**Step 2: Computation Phase**

Solve a linear programming

$$z^{[i]}(k_i) := \arg\max_z e^T z - \rho \|z\|^2 \text{ s.t. } z \in H^{[i]}_{\text{tmp}}(k_i)$$

where $H^{[i]}_{\text{tmp}}(k_i)$ is the polyhedron induced by $H^{[i]}_{\text{tmp}}(k_i)$ and $\rho$ is a sufficiently small positive number. The penalty item $-\rho \|z\|^2$ is added to derive a unique solution which has minimal Euclidean norm. Then, shrink $H^{[i]}_{\text{tmp}}(k_i)$ by remaining the set of active constraints.

Based on $z^{[i]}(k_i)$ and $\mathcal{Z}_i$, generate a new cutting-plane which is denoted by $h_i(z^{[i]}(k_i))$ and its specific form is given in (16). Then, update local cutting-plane set according to (15).

$$H^{[i]}(k_i + 1) = H^{[i]}_{\text{tmp}}(k_i) \cup h_i(z^{[i]}(k_i))$$

**Step 3: Writing Phase**

Write $H^{[i]}(k_i + 1)$ to its output cache. Update local clock by $k_i = k_i + 1$.

C. Fully Distributed Initialization

To develop the cutting-plane based distributed algorithm, each local processor has to generate an initial cutting-plane set $H^{[i]}(0)$ without knowing the whole picture of the feasible region $\mathcal{S}$. To guarantee convergence, it is required that $\mathcal{S} \subset H^{[i]}(0)$ and $\max_{z \in H^{[i]}(0)} e^T z < \infty$. To this end, we utilize the observation that the objective of (7), which represents a total cost of PEV charging, must have an upper bound in practice. Since strong duality holds for (7) and (12a), there also exists an upper bound of the equivalent maximization problem (12a).

Hence, each processor can individually choose a properly large number $M_i > 0$ according to his historical data, and construct a initial cutting-plane set as

$$H^{[i]}(0) = \left\{ z | e^T z \leq M_i \right\}, i \in \mathcal{N}.$$ (17)

D. Localized Stopping Criteria with Convergence Guarantee

By implementing the proposed distributed algorithm, each local processor will derive a sequence of solutions during the iterations. It is crucial to find an appropriate stopping criteria for consensus. First, we will introduce an empirical and centralized criterion, then extend it to a completely localized form. We will prove that the local criterion is the sufficient condition for the global criterion, deferring its detailed rationale, regarding optimality and feasibility, to subsection IV-E.

Before we start, the temporary objective value of processor $i$ at its $k_i$ round is denoted by

$$J^{[i]}(k_i) := e^T z^{[i]}(k_i) - \rho \|z^{[i]}(k_i)\|^2.$$ (18)

1) **Global Criterion:** Denote the global objective error at $k$ by

$$\max_{i,j \in \mathcal{N}} |J^{[i]}(k_j) - J^{[j]}(k_j)|$$ (19)

where $k_i, k_j$ are local clocks associated with the global clock $k$. Empirically, if (19) is less than a pre-specified convergence tolerance, consensus on solution is regarded as been encountered and the algorithm terminates. The underlying rationale is that strict concavity of $J^{[i]}(\cdot)$ follows that $|J^{[i]}(k_j) - J^{[j]}(k_j)| \geq \sigma \|z^{[i]}(k_i) - z^{[j]}(k_j)\|^2$ for some $\sigma > 0$ [26]. Therefore, the consensus on objective value can be approximated to that on solution $z$. Eq. (19), however, is essentially a global criterion, entailing temporary objective values from all local processors. It implies that individuals cannot implement this criterion by only accessing to local data. To circumvent this issue, a local criterion is proposed below.

2) **Local Criterion:** Given a pre-set tolerance $\epsilon > 0$, two conditions constituting the local criterion are given:

**Condition 1.** For processor $i \in \mathcal{N}$, $J^{[i]}(k_i - K) - J^{[i]}(k_i) < \epsilon$ where $K := d_{\mathcal{G}_i} \mathcal{T}$ is a constant with $d_{\mathcal{G}_i}$ being the diameter of the communication topology $\mathcal{G}_k$ and $\mathcal{T}$ a parameter of $\mathcal{G}_k$ stated in Assumption [1].

**Condition 2.** For processor $i \in \mathcal{N}$, $u_i^{[i]}(k_i - K) - U_i^{[i]}(k_i) < \epsilon$.

Both Condition 1 and 2 are stated in localized form. Condition 1 claims to have stagnation on local objective updating within $K$ iterative steps. Condition 2 guarantees a bounded distance from the $z^{[i]}(k_i)$ in hand to $\mathcal{Z}_i$. The local criterion is designed: processor $i$ stops at its local clock $k_i$ when Condition 1 and 2 are fulfilled with a pre-set tolerance $\epsilon > 0$. The local criterion is justified by theorem [1].

**Theorem 1.** If all processors have reached the local criterion with Conditions 1 and 2 fulfilled, then the global criterion is satisfied with $\max_{i,j \in \mathcal{N}} |J^{[i]}(k_j) - J^{[j]}(k_j)| < \epsilon$.

The proof of Theorem 1 is given in Appendix A. Though the local criterion may be more conservative than the global one, it only requires local information and enables a fully distributed and asynchronous implementation.

E. Convergence and Optimality

Let $z^*$ and $J^*$ denote the optimal solution and optimal value of (12) respectively. Convergence of Algorithm [1] is warranted by the monotonically nonincreasing objective value sequence $\{J^{[i]}(k_i)\}_{k_i}$, the lower bound of which is $J^*$. The optimality of Algorithm [1] is guaranteed by classic cutting-plane theory [24], [26]. Specifically, as the feasible region $\mathcal{S}$ is closed

This makes sense since $J^{[i]}(k_i)$ is monotonically nonincreasing with respect to $k_i$ as more constraints are added to the maximization problem while inactive constraints are pruned in every round of communication.
and compact, when Assumptions [12] hold, the limit point of sequence \( \{z[i](k_i)\}_{k_i} \) lies in \( \mathcal{S} \), implying \( J^* \) is greater than or equal to the limit of \( \{J[i](k_i)\}_{k_i} \). Such being the case, the convergence and optimality of Algorithm [1] is ensured by
\[
\lim_{k_i \to \infty} J[i](k_i) = J^*, \forall i \in \mathcal{N}. \tag{20}
\]

In practice, however, we are more concerned with the quality of the solutions obtained within finite rounds of iteration. Moreover, a consensus on \( J \) may not necessarily imply that the optimal \( J^* \) is achieved since \( J^* \) is non-convex, a priori for any individual processor (we will show by case studies in section \[\Box\] how the global criterion may fail at times). These points highlight the need for a measure to estimate the distance between a truncated solution \( J[i](k_i) \) (or \( z[i](k_i) \)) and the exact solution \( J^* \) (or \( z^* \)). Next we will show that, though \( J^* \) (either \( z^* \)) does not appear in the Conditions [12], the two conditions together guarantee bounded error of a truncated solution, with respect to optimality and feasibility.

1) Optimality: The optimality of the obtained solution can be characterized by the following theorem.

**Theorem 2 (Optimality).** Assume Conditions [12] hold for all processors in \( \mathcal{N} \), then for any local processor \( i \in \mathcal{N}, J[i](k_i) - J^* \geq 0 \) and
\[
J[i](k_i) - J^* \in O(\sqrt{\epsilon}). \tag{21}
\]

2) Feasibility: There is no guaranteed feasibility of solution sequence in cutting-plane based algorithms. In other words, the obtainable consensus result within finite rounds may be very close to \( \mathcal{S} \) but still outside \( \mathcal{S} \). To address this problem, we start with the situation that \( z[i](k_i) \) is feasible, i.e., \( z[i](k_i) \in \mathcal{S} \). Then we can easily infer from Theorem 3 that \( J[i](k_i) = J^* \). This implies \( z[i](k_i) \) must be the optimal solution to (12). If unfortunately \( z[i](k_i) \) is infeasible, it is revealed in Theorem 3 that \( z[i](k_i) \) can be close enough to a feasible and quasi- optimal solution.

**Theorem 3 (Feasibility).** Assume Conditions [12] hold for all processors in \( \mathcal{N} \). Then for each local processor \( i \in \mathcal{N} \), there exists a feasible solution \( \bar{z} \in \mathcal{S} \) which is close enough to \( z[i](k_i) \) with \( \|z[i](k_i) - \bar{z}\|^2 \leq B\epsilon \) where \( B \) is a positive constant, and nearly optimal with \( |J(z) - J^*| \in O(\sqrt{\epsilon}) \).

The proof of Theorem 2 and 3 is given in Appendix B.

V. CASE STUDIES

In this section, we test the performance of the proposed method and compare it with the celebrated ADMM algorithm. Simulations are carried out on the IEEE 37- and IEEE 123-node feeders [33], with MATLAB on a laptop with Intel(R) Core(TM) i5-5200U 2.20GHz CPU and 4GB of RAM.

A. Setup

1) Physical and Communication Networks: We consider two typical radial residential distribution networks: the IEEE 37-bus and 123-bus test feeders. The topology of the former system is depicted in Fig. 3, while the latter is omitted for space limitation. For both the systems, suppose that load of one household and one PEV is located at each bus. The feeder head (bus 4) is the only power supplier and its maximal capacity is set as the peak load without PEVs. We use the proposed method to coordinate PEVs’ charging to avoid overload. The communication topology of charging PEVs is chosen similar to the power network topology for simplicity [34]. We use real data from the hourly residential load profile of Los Angeles [34] as the baseline load and scale it to match the household numbers. The information on hourly day-ahead electricity prices comes from California ISO [35].

2) PEV Specifications: The parameters of PEVs are given: Battery capacities lie in a uniform distribution between 18 kWh to 20 kWh [36]. The scheduling horizon is from 5:00 pm to 9:00 am in the next day and is divided into 16 time slots hour-by-hour. Accordingly, we assume that the arrival and departure time of PEVs in the test cases lie in 5:00 pm-9:00 am and their hourly probability distributions are determined according to [37]. Initial and designated SOC are uniformly distributed in [0.3, 0.5] and [0.7, 0.9] respectively [9]. The maximum charge power is set as 3.3 kW for Level II charger. A charging efficiency of 0.9 is considered.

B. Optimality

In this case, Algorithm 1 is applied to the IEEE 37-node test feeder. Results are presented in Fig. 4 compared with two common uncoordinated charging modes:

- **mode (i):** PEVs start charging immediately when they arrive and until the designated SOC is reached.
- **mode (ii):** PEVs optimize charging cost on their own without coordinating with others.

A total charging cost of \$109.12, \$130.60 and \$105.06 is achieved under coordinated charging, uncoordinated mode (i) and (ii) respectively. As shown in Fig. 4 under mode (i), a new peak load is imposed on the baseload profile around evening rush hours, which gives rise to a great burden on the feeder head. Moreover, mode (i) costs the most because of charging during high price periods, which is uneconomical. As for mode (ii), a minimal charging cost is achieved, however, with there being a new peak load around 2 am to 3 am. The aggregation charging behavior during low price period around midnight also threatens the system security, although it is in off-peak time for non-EV baseload. The proposed coordinated charging meets system constraints at minimum cost, striving a balance between security and economic efficiency. It tries to schedule PEV load to off-peak time as well as avoids overload on the feeder head. An observation is made that the strategy plays a role in ‘valley-filling’ of total load profile as a consequence when PEV load and baseload are roughly on the same scale. The outcomes of the distributed algorithm are coincident with that of the centralized method, which is solved by CPLEX.

*This setting is made solely for the clarity of presentation. Theoretically, the communication topology can be arbitrary provided Assumption [1] holds.*
Fig. 4. Total load profile under coordinated/uncoordinated charging modes.

**C. Convergence**

Fig. 5(left) shows the iterative process of several selected nodes, converging to the optimal $J^*$. An observation is made on the different initial values of the optimization objective for individual processors. This is because they independently choose their $N_i$ uniformly in $[150, 200]$ to generate their initial cutting plane set. Also, it is observed that $J^i(k)$ of each processor is monotonously nonincreasing along with the communication round $k$, according with theoretical analysis.

To demonstrate performance of the different criterion, we show the evolution of four alternative errors: (i) $e_I(k) = \max_i J^i(k) - J^*$; (ii) $e_{II}(k)$ which is exactly the global criterion in (19); (iii) $e_{III}(k) = \max_i \{J^i(k) - K\}$, where $K = 15$ in this case, and (iv) $e_{IV}(k) = \max_i \{u^i(k) - U_i(n^i(k))\}$, as in Fig. 5(right). $e_I(k)$ essentially characterizes the convergence performance of the algorithm, however, entails the optimal $J^*$ being known a priori. Note that the $e_{III}(k)$ associated with Condition[1] and $e_{IV}(k)$ associated with Condition[2] together give a sketch of the proposed local criterion. As shown in Fig. 5(right), the algorithm converges after about 18 rounds of iteration, with $e_I(k)$ reduced to less than $10^{-3}$. Notice that the global criterion fails in this case, since the $e_{II}(k)$ reduces almost to 0 (the shaded box in Fig. 5(right)) whereas achieving consensus on a non-optimal objective value. The local criterion with $\epsilon = 10^{-3}$ is met at round 33 with $e_{III}(k) < \epsilon$ (namely, Condition[1] satisfied by all processors) and $e_{IV}(k) < \epsilon$ (namely, Condition[2] fulfilled by all). Though the local criterion may be conservative due to Condition[1], it provides guaranteed optimality of the consensus outturn, as opposed to the empirical global criterion.

We adopt the celebrated ADMM algorithm in [27], which is also center-free, for comparison. Also, case of IEEE 123-node feeder is tested to showcase the scalability to large problems. The error tolerance is set as $\epsilon = 10^{-3}$. Table II compares the convergence performances of the two methods. It is observed that the proposed algorithm needs only half of the communication rounds of ADMM to converge, showing a better convergence performance.

**D. Performance of Asynchronous Charging**

In this subsection, the robustness of the algorithm to communication delay, packet loss and communication topology changes are tested in IEEE 37-node feeder.

1) Communication Delay and Packet Losses: We assume that in each time step, each communication link has a delay of one time step with a probability of 10%. Also we suppose that the packet loss probability is 10% for the transmitted cutting-planes. Results given in Fig. 6(left) evidently illustrate a satisfactory performance of the proposed algorithm even with communication delay and package losses.

The proposed algorithm is tested on the IEEE 37-node feeder topology has a diameter of 15. Consider another topology with a diameter of 10. Cases with different communication topologies are tested and the comparison of convergence performances are shown in Fig. 6(right). It is observed that a static graph with bigger diameter calls for more iteration rounds to converge. The rationale for this observation is that the diameter of the communication topology determines the longest time needed for passing cutting-planes from one node to another indirectly. Surprisingly, when alternating the two topologies in coordinated charging (with Assumption[1] hold always), the algorithm converges even faster than the cases with static topologies. This fact indicates that, with the proposed algorithm, topology varying may even accelerate the cutting-plane passing process in the network, which could facilitate the convergence.

**E. Plug-and-Play Operation**

The proposed algorithm is tested on the IEEE 37-node system in a plug-and-play operation. Classify the 36 nodes into two parts $N_1 = \{1, 2, \ldots, 20\}$ and $N_2 = \{21, 22, \ldots, 36\}$. At the beginning, only PEVs in $N_1$ coordinate on charging. Nodes in $N_2$ participate at the 16 round. Results are shown in Fig. 7 showing that new players can join in at any time, which well supports the plug-and-play operation.
VI. CONCLUSION

In this paper, we have derived a cutting-plane based method to fulfill an optimal distributed coordination of PEV charging under local and global constraints. The proposed method strives the minimal overall charging cost without violating feeder head capacity, which is in accordance to the result of centralized global optimization. During the PEV charging, private information of individual PEVs can be well protected. It performs resiliently under various kinds of asynchrony in practice such as time delays, packet losses, and topology changes. We hope this work can promote an alternative path toward a more practical protocol for PEV charging and other distributed coordination problems.

As an initial study, uncertainties of renewable resources and inaccuracy of baseload forecast have not been taken into account in this work. Extending the proposed cutting-plane based distributed coordination framework to incorporate uncertainties are among our ongoing works.

REFERENCES

[1] L. Gan, U. Topcu, and S. H. Low, “Optimal decentralized protocol for electric vehicle charging,” IEEE Transactions on Power Systems, vol. 29, no. 2, pp. 940–951, 2013.
[2] J. A. P. Lopes, F. J. Soares, and P. M. R. Almeida, “Integration of electric vehicles in the electric power system,” Proceedings of the IEEE, vol. 99, no. 1, pp. 168–183, 2011.
[3] D. Engel, “Privacy and security challenges in the smart grid user domain,” in Proceedings of the First ACM Workshop on Information Hiding & Multimedia Security, 2013.
[4] N. G. Omran and S. Filizadeh, “A semi-cooperative decentralized scheduling scheme for plug-in electric vehicle charging demand,” International Journal of Electrical Power & Energy Systems, vol. 88, no. Complete, pp. 119–132, 2017.
[5] K. Turitsyn, S. Sinitryn, S. Backhaus, and M. Cherktov, “Robust broadcast-communication control of electric vehicle charging,” in First IEEE International Conference on Smart Grid Communications, 2010.
[6] J. A. P. Lopes, F. J. Soares, and P. M. R. Almeida, “Identifying management procedures to deal with connection of electric vehicles in the grid,” in IEEE Bucharest PowerTech, 2009.
[7] M. G. Vaya and G. Andersson, “Centralized and decentralized approaches to smart charging of plug-in vehicles,” in Power & Energy Society General Meeting, 2012.
[8] B. Jiang and Y. Fei, “Decentralized scheduling of pev on-street parking and charging for smart grid reactive power compensation,” in IEEE PES ISGT, 2013.
[9] M. Liu, P. K. Phanivong, S. Yang, and D. S. Callaway, “Decentralized charging control of electric vehicles in residential distribution networks,” IEEE Transactions on Smart Grid Systems Technology, vol. 27, no. 1, pp. 266–281, 2019.
[10] O. Ardakanian, C. Rosenberg, and S. Keshav, “Distributed control of electric vehicle charging,” in International Conference on Future Energy Systems, 2013.
[11] L. Zhang, V. Kekatos, and G. B. Giannakis, “Scalable electric vehicle charging protocols,” IEEE Transactions on Power Systems, vol. 32, no. 2, pp. 1451–1462, 2017.
[12] M. G. Vaya, G. Andersson, and S. Boyd, “Decentralized control of plug-in electric vehicles under driving uncertainty,” in IEEE PES ISGT Europe, 2014.
[13] M. Kraning, “Dynamic network energy management via proximal message passing,” Foundations & Trends in Optimization, vol. 1, no. 2, pp. 73–126, 2014.
[14] J. Rivera, P. Wolfrum, S. Hirche, C. Goebel, and H. A. Jacobsen, “Alternating direction method of multipliers for decentralized electric vehicle charging control,” in 52nd IEEE Conference on Decision & Control, 2013.
[15] Z. Ma, D. S. Callaway, and I. A. Hiskens, “Decentralized charging control of large populations of plug-in electric vehicles,” IEEE Transactions on Control Systems Technology, vol. 21, no. 1, pp. 67–78, 2012.
[16] F. Parise, M. Colombino, S. Grammatico, and J. Lygeros, “Mean field constrained charging policy for large populations of plug-in electric vehicles,” in 53rd IEEE Conference on Decision & Control, 2014.
[17] T. Logenthiran, D. Srinivisan, A. M. Khambashkone, and H. N. Aung, “Multi-agent system for real-time operation of a microgrid in real-time digital simulator,” IEEE Transactions on Smart Grid, vol. 3, no. 2, pp. 925–933, 2012.
[18] J. Mohammadi, G. Hug, and S. Kar, “A fully distributed cooperative charging approach for plug-in electric vehicles,” IEEE Transactions on Smart Grid, vol. 9, no. 4, pp. 3507–3518, 2018.
[19] N. Rabhari-Asr and M. Y. Chow, “Cooperative distributed demand management for community charging of phev/pevs based on kkt conditions and consensus networks,” IEEE Transactions on Industrial Informatics, vol. 10, no. 3, pp. 1907–1916, 2014.
[20] Y. Xu, “Optimal distributed charging rate control of plug-in electric vehicles for demand management,” IEEE Transactions on Power Systems, vol. 30, no. 3, pp. 1536–1545, 2015.
[21] S. H. Low and D. E. Lapsley, “Optimization flow control. i. basic algorithm and convergence,” IEEE/ACM Transactions on Networking, vol. 7, no. 6, pp. 861–874, 1999.
[22] Z. Wang, S. Mei, L. Feng, P. Yi, and M. Cao, “Asynchronous distributed power control of multi-microgrid systems,” arXiv preprint arXiv:1810.11998, 2018.
[23] M. T. Hale, A. Nedich, and M. Egerstedt, “Asynchronous multi-agent primal-dual optimization,” IEEE Transactions on Automatic Control, vol. 62, no. 9, pp. 4431–4435, 2017.
[24] B. Eaves and W. Zangwill, “Generalized cutting plane algorithms,” SIAM Journal on Control, vol. 9, no. 4, pp. 529–542, 1971.
[25] M. Liu, J. Zhu, L. Li, and W. Zhao, “Fully decentralized multi-area dynamic economic dispatch for large-scale power systems via cutting plane consensus,” IET Generation Transmission & Distribution, vol. 10, no. 10, pp. 2486–2495, 2016.
[26] M. Bürger, G. Notarstefano, and F. Allgöwer, “A polyhedral approximation framework for convex and robust distributed optimization,” IEEE Transactions on Automatic Control, vol. 59, no. 2, pp. 384–395, 2014.
[27] E. Wei and A. Ozdaglar, “Distributed alternating direction method of multipliers,” in 51st IEEE Conference on Decision and Control, 2012.
[28] A. Nedic and A. Olshevsky, “Distributed optimization over time-varying directed graphs,” IEEE Transactions on Automatic Control, vol. 60, no. 3, pp. 601–615, 2014.
[29] S. Boyd and L. Vandenberghe, Convex optimization. Cambridge university press, 2004.
[30] Y. Bo and M. Johansson, “Distributed optimization and games: A tutorial overview,” Networked Control Systems, vol. 406, pp. 109–148, 2010.
[31] O. L. Mangasarian and R. R. Meyer, “Nonlinear perturbation of linear programs,” Siam Journal on Control & Optimization, vol. 17, no. 6, pp. 745–752, 1978.
[32] G. B. Dantzig and P. Wolfe, “The decomposition algorithm for linear programs,” Econometrica, vol. 29, no. 4, pp. 767–778, 1961.
[33] IEEE PES AMPS DSAS Test Feeder Working Group. [Online]. Available: http://sites.ieee.org/pes-testfeeders/resources/
[34] OpenEI Datasets. [Online]. Available: https://openiei.org/datasets/dataset7
[35] Energy Online. [Online]. Available: http://www.energionline.com/Data/evsbs.shtml
[36] UEP Agency. [Online]. Available: http://www.fueleconomy.gov/leg
[37] N. G. Omran and S. Filizadeh, “A semi-cooperative decentralized scheduling scheme for plug-in electric vehicle charging demand,” International Journal of Electrical Power & Energy Systems, vol. 88, no. Complete, pp. 119–132, 2017.
APPENDIX A
PROOF OF THEOREM

Let $t_{ij}(ki)$ denote the mapping from processor $i$’s local clock $ki$ to its corresponding unique universal time and $t(k)$ denote the mapping from global clock $k$ to its corresponding unique universal time. ∀i, k, we define

$$Q[i](k) := J[i](ki)$$

(22)

where $t_{ij}(ki) \leq t(k) < t_{ij}(ki + 1)$. Since $J[i](ki)$ is monotonically non-increasing with respect to $k_i$, $Q[i](k)$ is also monotonically non-increasing with respect to $k$.

Lemma 4. For any $i, k_i$ and $k$, if $t_{ij}(ki) \leq t(k) < t_{ij}(ki + 1)$, then $J[i](k_i - K) \geq Q[i](k - K)$ where $K \in \mathbb{Z}^+$ is an arbitrary constant.

Proof. Suppose $Q[i](k - K) = J[i](k_i)$ where

$$t_{ij}(k_i) \leq t(k - K) < t_{ij}(k_i + 1).$$

(23)

Due to the relationship of global clock and local clocks, if $t_{ij}(k_i) \leq t(k) < t_{ij}(k_i + 1)$, there must be

$$t_{ij}(k_i - K) \leq t(k - K).$$

(24)

Combining (24) and (23), we have

$$t_{ij}(k_i - K) < t_{ij}(k_i + 1).$$

(25)

which is equivalent to

$$k_i - K \leq k_i.$$

(26)

Since $J[i](k_i)$ is monotonically non-increasing with respect to $k_i$, we have

$$J[i](k_i - K) \geq J[i](k_i) = Q[i](k - K)$$

(27)

which completes the proof of lemma 4.

Now proof of theorem 1 is given as follows by contradiction:

Proof. We have assumed that ∀i, j

$$J[i](k_i - K) - J[j](k_j) < \epsilon,$$

(28a)

$$J[j](k_j - K) - J[i](k_i) < \epsilon.$$  

(28b)

Now suppose there exists i, j and k such that:

$$Q[i](k) - Q[j](k) > \epsilon,$$

(29a)

$$t_{ij}(k_i) \leq t(k) < t_{ij}(k_i + 1),$$

(29b)

$$t_{ij}(k_j) \leq t(k) < t_{ij}(k_j + 1).$$

(29c)

According to (29b) and the definition of $Q[i](\cdot)$, we have

$$Q[i](k) = J[i](k_i).$$

(30)

According to (29b) and lemma 4 we have

$$Q[i](k - K) \leq Q[i](k_i - K).$$

(31)

Then combining (29a), (30) and (31), we have

$$Q[i](k - K) < Q[j](k) + \epsilon.$$  

(32)

Combining (29a) and (32) we derive:

$$Q[i](k) - Q[j](k) < \epsilon.$$  

(33)

Denote $k_0 := k - K$ and index set $I_0 = \{i\}$. For any $s > 0$, $I_s := \{m: \exists l \in I_{s-1}, s.t. (m, l) \in E_{k_0+s}\}$ (34)

As the communication graph is strongly connected with limited intercommunication interval, $I_K$ must contains all the nodes in the network. Therefore there exists $1 \leq s^* \leq K$ such that $j \in I_{s^*}$. The algorithm guarantees that

$$\forall l \in I_s, Q[l](k_0 + s) \leq Q[l](k_0).$$

(35)

Thus we have

$$Q[j](k_0 + K) \leq Q[j](k_0 + s^*) \leq Q[i](k_0)$$

(36)

which contradicts (33).

□

APPENDIX B
PROOF OF THEOREMS 2 AND 3

According to the model in Appendix B-D for any $i \in \mathcal{N}$, $p_i$ is bounded in set $P_i$ with $||p_i|| \leq \sqrt{TP_i}$. Before giving the proof of theorem 2 and 3 two lemmas are provided as follows.

Lemma 5. If Conditions 1-2 are satisfied by all processors in $\mathcal{N}$ at global clock $k$, then $\forall t, m \in \mathcal{N}$ and $m \neq i$,

$$u^m_i(k_i) - U_m(\pi_i(k_i)) \in O(\sqrt{\epsilon}).$$

(37)

Proof. Strict convexity of $J(\cdot)$ follows that $|J[i](k_i) - J[j](k_j)| \geq \sigma ||z[i](k_i) - z[j](k_j)||^2$ for some $\sigma > 0$. Take $\pi$ as row vector, so the transposition symbol on $\pi$ is omitted, and for simplicity reason we omit $k_i$ and $k_m$ in the following proof.

$$u^m_i - U_m(\pi_i) = u^m_i - \min\{f_m(p_m) + \pi_i p_m\}$$

$$= u^m_i - \min\{f_m(p_m) + \pi_i - \pi^m p_m\}$$

$$\leq u^m_i - U_m(\pi_i) + \max\{\pi_i - \pi^m\}$$

$$\leq \epsilon + \sqrt{\epsilon/\sigma} = \max\{\pi_i - \pi^m\}$$

$$\leq \epsilon + \sqrt{\epsilon/\sigma} = \max\{\pi_i - \pi^m\}$$

$$\leq \epsilon + \sqrt{\epsilon/\sigma} = \max_j \{p_j\} \in O(\sqrt{\epsilon})$$

(38)

where (1) comes from $|u^m_i - u^m_i| \leq \sqrt{\epsilon/\sigma}$, (2) comes from Condition 2 and (3) comes from $|\pi_i - \pi^m| \leq \sqrt{\epsilon/\sigma}$ and $||p_m|| \leq \sqrt{TP_m} \leq \sqrt{TP_m} \leq \epsilon$.

Now, proof of theorem 2 and 3 is given:

Proof. Skip the $k_i$ for conciseness. According to Lemma 5 there exists a positive constant $c$ such that $u^m_i - U_m(\pi_i) \leq c\sqrt{\epsilon}$ for any $m \neq i$. Then define

$$\delta := [0, \ldots, 0, c\sqrt{\epsilon}, \ldots, c\sqrt{\epsilon}]^T$$

(38a)

$$\bar{z} := [z[i] - \delta z]$$

(38b)

Note that $||\bar{z} - z[i]|| = c\sqrt{m_\epsilon}$, so $\bar{z}$ lies in the $c\sqrt{m_\epsilon}$ neighborhood of point $z[i]$. According to Lemma 5 we know that for any $m \in \mathcal{N}$ there are $u^m_i - c\sqrt{\epsilon} \leq U_m(\pi_i)$ for any $m$, so $\bar{z} \in \cap_i \mathcal{N} \mathcal{E} \bar{z}$. This implies that the intersection of $\mathcal{S}$ and the $c\sqrt{m_\epsilon}$ neighborhood of $z[i]$ which is denoted by $N_{c\sqrt{m_\epsilon}}(z[i])$, is not empty.

Suppose that $J^*$ is the maximizer of $J(\cdot)$ in $\mathcal{S}$. Since $\bar{z} \in \cap_i \mathcal{N} \mathcal{E} \bar{z}$, $\bar{z}$ is in the $c\sqrt{m_\epsilon}$ neighborhood of $z[i]$. Hence $J[z[i]]$ is the maximizer of $J(\cdot)$ in $\mathcal{H}^{\pi_i}[\mathcal{N}]$, $S \in \mathcal{H}^{\pi_i}[\mathcal{N}]$: we have $J[z[i]] \geq J^*$. So we get the following relationship:

$$J[z[i]] \geq J^* \geq J(\bar{z})$$

(39)
According to squeeze theorem, we get $0 \leq J(z^{[i]}) - J^* \leq$ and $J(z^{[i]}) - J^* \in O(\sqrt{\epsilon})$, which completes the proof of theorem 2 and 3. $\square$