A modified BCS (MBCS) model has been suggested recently to improve the BCS theory at finite temperature for nuclei. It has been claimed that the MBCS "washes out the sharp superfluid-normal phase transition" and "the fluctuations of particle number also become more suppressed especially at high temperature". In this Comment we argue that the corresponding equations cannot be treated as an extension of the modified BCS approach to finite temperature. Consequently, the "physical effects" reported in this article are only the results of incorrect physical treatment and inconsistencies.

The MBCS equations at \( T \neq 0 \) are obtained applying a temperature-dependent unitary transformation to the Bogoliubov quasiparticles \( \{ \alpha_j^+, \alpha_j \} \), thus transforming them into the new bar quasiparticles \( \{ \bar{\alpha}_j^+, \bar{\alpha}_j \} \):

\[
\bar{\alpha}_j^\dagger = \sqrt{1 - n_j} \alpha_j^\dagger + (-1)^{j-m} \sqrt{n_j} \alpha_{j-m},
\]

where \( n_j \) are the thermal Fermi-Dirac occupation numbers for the Bogoliubov quasiparticles. The authors claim: "The MBCS equations... has exactly the same form as the standard BCS equations \( \Delta_T=0 = G \sum_j \Omega_j u_j v_j \) where the coefficients \( u_j \) and \( v_j \) are replaced with \( \bar{u}_j \) and \( \bar{v}_j \):"

\[
\bar{\Delta}(T) = G \sum_j \Omega_j \bar{u}_j \bar{v}_j = G \sum_j \Omega_j \left[ (1-2n_j)u_j v_j - \sqrt{n_j(1-n_j)(u_j^2 - v_j^2)} \right].
\] (2)

Although formally the conventional BCS \( (T=0) \) and MBCS equations look similar, there is one essential difference between them. As far as \( u_j \) and \( v_j \) coefficients are positively defined, the BCS gap \( \Delta = 0 \) (breaking of the Cooper pairing) corresponds to the trivial solution \( \{ u_j, v_j \} = \{ 0,1 \} \) for all \( j \) when all levels give zero contribution to the gap. It is not difficult to show analytically that the MBCS gap equation (2) does not have such a trivial solution \( \{ \bar{u}_j, \bar{v}_j \} = \{ 0,1 \} \). Indeed, from Eq. (38) in Ref. [1]:

\[
\bar{u}_j = u_j \sqrt{1-n_j} + v_j \sqrt{n_j}, \quad \bar{v}_j = v_j \sqrt{1-n_j} - u_j \sqrt{n_j}
\] (3)

the trivial solution \( \{ \bar{u}_j, \bar{v}_j \} = \{ 0,1 \} \) corresponds to:

\[
\begin{align*}
&u_j = \sqrt{1-n_j}; \quad v_j = \sqrt{n_j} - \text{particles} \\
&u_j = -\sqrt{n_j}; \quad v_j = \sqrt{1-n_j} - \text{holes}
\end{align*}
\]

and contradicts the positive definition of \( u_j \).

One may notice from Eq. (3) that \( \bar{v}_j \) coefficients become negative above a certain temperature for particle levels since \( v_j << u_j \) for them. Then the MBCS pairing gap \( \bar{\Delta} \) receives positive contribution from hole levels and negative contribution from particle levels. This leads to quite a strange thermal behavior of \( \Delta(T) \).

On page 5 of Ref. [1] we find a discussion of the MBCS critical temperature \( T_c \) defined as \( \Delta(T_c) = 0 \). According to the authors this temperature is reached when the "new" (second) term in Eq. (2) is equal to the conventional one. Numeric calculations (see below) show that two terms in Eq. (2) compensate each other around the critical temperature of the conventional BCS \( T_c \approx 0.57 \cdot \Delta_{T=0} \) for particle levels and they never do for hole levels because the second term in Eq. (2) is always negative for holes.

As clear from the above paragraphs, \( \bar{\Delta} \) vanishes at \( T_c \) only when a negative contribution from particles and positive contribution from holes cancel each other (notice the difference with the conventional BCS \( \Delta = 0 \)). However if this happens, at higher \( T \) the balance appears to be broken and \( \bar{\Delta} \) becomes finite again.

To be more specific, we plot in Fig. 1 the MBCS pairing gap as function of temperature in \(^{76}\text{Ni}\) [2]. One notices that it reaches zero at \( T_c \approx 2.1 \) MeV and continues to decrease. The authors of Ref. [1] define an unphysical negative
FIG. 1: (a) the MBCS pairing gap; (b) $\bar{v}_j$ and $v_j$ coefficients for particle levels near the Fermi surface: $2d_{5/2}$ - solid lines, $3s_{1/2}$ - dashed lines; (c) two functions for particle-number fluctuations $\sqrt{\delta N^2}$ in Eq. (4). Critical temperates $T_c$ and $\bar{T}_c$ are indicated by vertical lines. See text for details.

gap of the MBCS as the results “no longer reliable”. Since the authors never discuss any physical limitations of their model, it is not clear upon what basis this conclusion is made. In Fig. 1b we present the $\bar{v}_j$ and $v_j$ coefficients for two particle ($2d_{5/2}$, $3s_{1/2}$) levels near the Fermi surface. They become negative around $T_c$. However, one may notice that nothing special occurs at $\bar{T}_c$, thus confirming the “unreliability” of the method above the “new” critical temperature.

The problem with a negative gap can be removed by increasing the role of hole levels. It may be achieved, e.g., by including the levels from the shell N=0-28 in a single-particle spectrum in addition to the levels near the Fermi surface accounted for in Ref. [1] (with a proper renormalization of the pairing strength). Then, another MBCS crack appears: the gap starts to continuously increase above a certain temperature remaining always positive. Similar effect in $^{84}$Ni is shown in Figs. 2h and 3 of Ref. [1]. But we find it impossible for the MBCS gap to reach zero and remain zero at higher temperature, thus, indicating the superfluid-normal phase transition.

In view of all the above, the author’s conclusions about superfluid-normal phase transition within the MBCS makes no sense from our point of view.

The other main goal of the MBCS is to reduce the particle number fluctuations in the ground state. The authors show in Fig. 5 (solid line) of Ref. [1] that this quantity

$$\delta \bar{N}^2 = 4 \sum_j \Omega_j \bar{u}_j^2 \bar{v}_j^2 = \bar{\Delta}^2 \sum_j \Omega_j / \bar{E}_j^2.$$  \hspace{1cm} (4)

is reducing with the temperature and apparently vanishes. The $\delta \bar{N}^2$ has been calculated from the r.h.s. part of Eq. (11) (or Eq.(41) in Ref. [1]). It remains unclear how transformation in Eq. (4) has been obtained within the MBCS.
at $T \neq 0$. We have calculated both the left and right functions. Numeric results presented in Fig. 1 reveal their rather different behavior. It means that the conclusion on suppression of the particle number fluctuations with a temperature within the MBCS in Ref. [1] is based on calculations from the equation which is not valid.

A legitimate question is what is wrong with the MBCS. In Sec. II.D and Appendix A of Ref. [1] the well-known statistical approach in a many-body theory at finite temperature is sketched. However, deriving the MBCS equations the authors of Ref. [1] do not use this method. Instead, they formally follow the scheme formulated by them for zero temperature. Namely, they find a minimum of the expectation value of the BCS Hamiltonian over the temperature dependent vacuum state $|\bar{0}\rangle$. They imply for reasons unclear that selecting $n_j(T)$ in the form of the quasiparticle thermal occupation numbers is equivalent to averaging over the grand canonical ensemble (as one reads on page 5 of Ref. [1]).

We find it difficult to understand: the authors are aware that a quantum mechanical ground state, for which expectation value of the Hamiltonian equals the statistical average over the grand canonical ensemble, “does not exist in the physical space spanned by eigen vectors” of the pairing Hamiltonian and why it is so (see footnote on page 3). Nevertheless, the basic starting point of the MBCS at $T \neq 0$ is certainly to introduce such a vacuum state $|\bar{0}\rangle$ in Eq. (33).

This means that the MBCS cannot be treated as a theory with the thermal behavior of the pairing correlations under control. It is not surprising that the MBCS equations (Eqs. (39-40) and (43-44) in Ref. [1]) differ from the conventional ones by extra terms but their origin is only inconsistency in treatment of the physical object.

To conclude, we doubt that the modified BCS in Ref. [1] with the above discussed flaws is a step forward in the developing of the conventional BCS at finite temperature.

[1] N. Dinh Dang and A. Arima, Phys. Rev. C 67, 014304 (2003).
[2] Our code reproduces excellently all the results in Ref. [1] for all isotopes. We present them for $^{76}$Ni only (from the resonant continuum MBCS calculations). The ones in $^{68-76}$Ni and $^{80-82}$Ni are qualitatively very similar and lead to the same conclusions.