Explicit Generators of the Centre of the Quantum Group

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Abstract
A finite generating set of the centre of any quantum group is obtained, where the generators are given by an explicit formulae. For the slightly generalised version of the quantum group which we work with, we show that this set of generators is algebraically independent, thus the centre is isomorphic to a polynomial algebra.

Keywords Quantum groups · Central elements · Harish-Chandra isomorphism

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1 Introduction
We construct explicit generators of the centre of any quantum group in this paper. This involves two separate problems, namely, the construction, in terms of explicit formulae, of a finite set of central elements, and the proof that they generate the centre of the quantum group.

Note that even in the classical case of a semi-simple Lie algebra $\mathfrak{g}$, where the algebraic structure of the centre of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is well understood, it is an important and highly non-trivial problem to construct explicit generators of the centre of $\mathfrak{U}(\mathfrak{g})$, that is, to construct the (high order) Casimir operators.

The explicit generators of the centre of the quantum group to be constructed in this paper are quantum analogues of (higher order) Casimir operators of $\mathfrak{U}(\mathfrak{g})$ arising from “characteristic identities” [2]. The operators, both classical and quantum, are particularly useful for explicit computations in representation theory, e.g., for computing Wigner coefficients and developing Racah–Wigner calculus (see, e.g., [8,22]), and for solving Hamiltonian systems in atomic and molecular physics.
We choose to work with a slightly generalised Jimbo version of the quantum group (see [3,4,12] and Definition 2.1 in particular). Recall that the standard quantum group has among its generators $K_{\pm}^{\beta}$ associated with the simple roots $\alpha_i$, thus contains all $K_{\pm}^{\beta}$ for $\beta$ in the root lattice. The generalised quantum group $U_q(g)$ includes all $K_{\pm}^{\mu}$ for $\mu$ in the weight lattice; see Remark 2.2 for more details.

The main results of the paper are summarised in Theorem 4.5 and Corollary 4.8. Let $rk(g)$ be the rank of $g$. In Theorem 4.5, we give a set of $rk(g)$ central elements of $U_q(g)$ in terms of the explicit formulae (3.20) and show that they generate the centre of $U_q(g)$. In Corollary 4.8, we summarise that the set of generators are algebraically independent, thus the centre of the generalised quantum group $U_q(g)$ is isomorphic to a polynomial algebra of $rk(g)$ variables.

We now briefly describe our approach to the results.

Recall that an infinite family of central elements of $U_q(g)$ were constructed from any given finite-dimensional representation $(V, \zeta)$ (where $V$ is a $U_q(g)$-module and $\zeta$ is the corresponding representation) of the quantum group in [9,24]. The construction makes use of the universal $L$-operators $L_V, L_V^T \in \text{End}(V) \otimes U_q(g)$ (see (3.12)), the existence of which in the current version of $U_q(g)$ is explained in Sect. 3. The operator $\Gamma_V = L_V^T L_V$ (see (3.14)) commutes with $(\zeta \otimes \text{id})\Delta(x)$ for all $x \in U_q(g)$, where $\Delta$ is the comultiplication, thus by Theorem 3.1 (see also [9, Proposition 1]), the $q$-trace $C^{(m)}_V$ of $\Gamma_V^m$ is a central element of $U_q(g)$ for each positive integer $m$. The generating set of the centre of $U_q(g)$ given in Theorem 4.5 consists of the elements $C^{(1)}_V$ for $V$ being the simple $U_q(g)$-module with fundamental highest weights.

We prove that the set of central elements given in Theorem 4.5 generates the centre by using a quantum Harish-Chandra isomorphism (see Theorem 2.5 and [4, §18.3]) for the generalised quantum group. In particular, we prove that the images of those central elements generate the image of the centre of the quantum group under the quantum Harish-Chandra isomorphism. We also show that the images are algebraically independent, thus generate a polynomial algebra, leading to Corollary 4.8.

This quantum Harish-Chandra isomorphism is an analogue of a similar result [11,20] for the standard quantum groups. It was presented in [4, §18.3], where a geometric proof was given following familiar arguments in the theory of semi-simple Lie algebras. (There was also an indication of the result in the introduction section of [17].) Due to its importance to us, we give an elementary algebraic proof of the result in Appendix A by adapting [11, Chapter 6] to our context. The algebraic proof works in much the same way for both the standard and generalised quantum groups; however, it is not clear to us how the geometric method of [4, §18.3] would work in the case of the standard quantum groups.

The authors of [9,24] expected that some finite subset of the central elements $C^{(n)}_V$ for all finite-dimensional simple modules $V$ and all $n = 1, 2, \ldots$ generates the centre of $U_q(g)$, but this was not proved before except for $g = gl_\ell$ [15]. In Conjecture 4.9, we suggest another subset of the elements $C^{(n)}_V$ as a generating set of the centre of $U_q(g)$. For $g = gl_\ell$, the conjecture is implied by the results of [15].

In a future publication, we will give a similar treatment of the centres of the standard quantum groups and quantum supergroups [1]. We point out that complete sets of
generators were constructed for the centres of the standard quantum groups $U_q(\mathfrak{sl}_3)$ and $U_q(\mathfrak{sl}_4)$ [16,23].

The paper is organised as follows. In Sect. 2, we introduce the basics of the generalised Jimbo quantum group, which we still denote by $U_q(\mathfrak{g})$, and give the new Harish-Chandra isomorphism for the generalised quantum group $U_q(\mathfrak{g})$. In Sect. 3, we give an analogue of universal $R$-matrix in finite-dimensional representations of $U_q(\mathfrak{g})$, and use it to construct infinite families of central elements of the generalised quantum group. Finally in Sect. 4, we extract from these central elements a generating set of the centre of the generalised quantum group. This is explained in Theorem 4.5. The appendix contains the algebraic proof of the quantum Harish-Chandra isomorphism for $U_q(\mathfrak{g})$ (see Theorem 2.5).

2 Basics on Quantum Groups

2.1 A Generalisation of the Jimbo Quantum Group

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over the field of complex numbers $\mathbb{C}$. Choose Borel and Cartan subalgebras $\mathfrak{h} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$, and fix a basis $H_1, \ldots, H_n$ for the Cartan subalgebra $\mathfrak{h}$, where $n$ is the rank of $\mathfrak{g}$. Denote by $\Phi$ the root system of $\mathfrak{g}$ relative to this choice, and let $\Phi^+$ be the set of positive roots. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the simple root system of $\Phi$. Let $(\ , \ )$ be the non-degenerate bilinear form on the dual space $\mathfrak{h}^*$ normalised so that the square of the length of short roots is 2. Denote by $W$ the Weyl group of $\mathfrak{g}$. The Cartan matrix $A = (a_{ij})$ is the $n \times n$ matrix with $a_{ij} = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$. Moreover, we set $d_i = (\alpha_i, \alpha_i)/2$ and let $\alpha_i^\vee = d_i^{-1} \alpha_i$ be the simple coroot for $1 \leq i \leq n$.

Denote by $\varpi_i$, $1 \leq i \leq n$ the fundamental weights of $\mathfrak{g}$ such that $\frac{2(\varpi_i, \alpha_j)}{(\alpha_i, \alpha_j)} = \delta_{ij}$ for all $i, j$. We write

$$P = \bigoplus_{i=1}^n \mathbb{Z} \varpi_i, \quad Q = \bigoplus_{i=1}^n \mathbb{Z} \alpha_i$$

for the weight and root lattices of $\mathfrak{g}$, respectively. Clearly $(\lambda, \alpha) \in \mathbb{Z}$ for all $\lambda \in P$ and $\alpha \in \Phi$ for the given normalisation of the bilinear form. Let $P^+ = \bigoplus_{i=1}^n \mathbb{N} \varpi_i$ be the set of dominant weights, and $Q^+ = \bigoplus_{i=1}^n \mathbb{N} \alpha_i$ the set of nonnegative integer combinations of simple roots.

We denote by $\ell_\mathfrak{g}$ the minimal positive integer such that $\ell_\mathfrak{g}(\lambda, \mu) \in \mathbb{Z}$ for all $\lambda, \mu \in P$. Let $q^{\frac{1}{\ell_\mathfrak{g}}}$ be an indeterminate, and denote by $\mathbb{C}(q^{\frac{1}{\ell_\mathfrak{g}}})$ the field of rational functions. We will always write

$$\mathbb{F} = \mathbb{C}(q^{\frac{1}{\ell_\mathfrak{g}}}) \quad \text{and} \quad q = \left(q^{\frac{1}{\ell_\mathfrak{g}}}ight)^{\ell_\mathfrak{g}}.$$
For any $\lambda, \mu \in P$, the expression $q^{(\lambda, \mu)}$ will mean \( q^{\frac{1}{\xi_{\mu}}(\lambda, \mu)} \).

**Definition 2.1** The generalised Jimbo quantum group associated with $g$ is the unital associative algebra over $\mathbb{F}$ generated by $E_i, F_i (i = 1, 2, \ldots, n)$ and $K_\lambda (\lambda \in P)$ subject to the following relations:

\[
K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda+\mu}, \quad (2.1)
\]

\[
K_\lambda E_j K_\mu^{-1} = q^{(\lambda, \alpha_j)} E_j, \quad (2.2)
\]

\[
K_\lambda F_j K_\mu^{-1} = q^{-(\lambda, \alpha_j)} F_j, \quad (2.3)
\]

\[
E_i F_j - F_j E_i = \delta_{ij} K_i - K_i^{-1}, \quad (2.4)
\]

\[
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] q_i^{1-a_{ij}-s} E_i^{1-a_{ij}-s} E_i^s = 0, \quad i \neq j, \quad (2.5)
\]

\[
\sum_{s=0}^{1-a_{ij}} (-1)^s \left[ \frac{1-a_{ij}}{s} \right] q_i^{1-a_{ij}-s} F_i^{1-a_{ij}-s} F_i^s = 0, \quad i \neq j, \quad (2.6)
\]

where $K_i = K_{\alpha_i}$, $q_i = q^{d_i}$ and

\[
[m]_{q_i} = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, \quad [m]_{q_i}! = [1]_{q_i} [2]_{q_i} \cdots [m]_{q_i}, \quad \left[ \begin{array}{c} m \\ k \end{array} \right]_{q_i} = \frac{[m]_{q_i}!}{[m-k]_{q_i} ![k]_{q_i}!}, \quad (2.7)
\]

for any $m \in \mathbb{N}$.

**Remark 2.2** The standard Jimbo quantum groups have generators $E_i, F_i$ and $k_i^{\pm 1}$, where $k_i$ correspond to our $K_{\alpha_i}$ for simple roots $\alpha_i$. The structure and representation theories of these two types of quantum groups are similar. However, for our purpose it is more convenient to work with Definition 2.1.

**Remark 2.3** By a slight abuse of notation and terminology, we shall denote the algebra in Definition 2.1 by $U_q(g)$, and simply refer to it as the Jimbo quantum group.

We will write $U = U_q(g)$. It is well known that $U$ is a Hopf algebra with co-multiplication $\Delta$, co-unit $\varepsilon$ and antipode $S$:

\[
\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \quad \Delta(E_i) = K_i \otimes E_i + E_i \otimes 1, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]

\[
\varepsilon(K_\lambda) = 1, \quad \varepsilon(E_i) = 0, \quad \varepsilon(F_i) = 0,
\]

\[
S(K_\lambda) = K_\lambda^{-1}, \quad S(E_i) = -K_i^{-1} E_i, \quad S(F_i) = -F_i K_i.
\]
We will use Sweedler notation for the co-multiplication: $\Delta(x) = \sum_x x^{(1)} \otimes x^{(2)}$ for any $x \in U$. The adjoint representation $\text{ad}$ of $U$ is defined as follows:

$$\text{ad}(x)(y) = \sum_x x^{(1)} y S(x^{(2)}), \quad \forall x, y \in U. \tag{2.8}$$

We denote by $U^+$ (resp. $U^-$) the subalgebra of $U$ generated by all $E_i$ (resp. $F_i$), and by $U^0$ the subalgebra generated by $K_{\lambda}$ with $\lambda \in P$. Then the multiplication in $U$ induces the isomorphism $U^- \otimes U^0 \otimes U^+ \cong U$.

### 2.2 Representations

We follow [11,18] to discuss the representation theory of $U$. Let $\sigma : P \to \mathbb{C}^\times$ be the group character such that

$$\begin{align*}
\sigma(0) &= 1, \quad \sigma(\lambda + \mu) = \sigma(\lambda)\sigma(\mu), \quad \forall \lambda, \mu \in P, \\
\sigma(\alpha_i)^2 &= 1, \quad i = 1, \ldots n. \tag{2.9}
\end{align*}$$

There exists a one-dimensional $U$-module $\mathbb{F}_\sigma$ associated with $\sigma$ defined by

$$E_i \cdot 1 = F_i \cdot 1 = 0, \quad K_\mu \cdot 1 = \sigma(\mu)1, \quad \forall 1 \leq i \leq n, \mu \in P.$$

We also have the following algebra automorphism $\psi_\sigma : U \to U$

$$\psi_\sigma(E_i) = \sigma(\alpha_i)E_i, \quad \psi_\sigma(F_i) = F_i, \quad \psi_\sigma(K_\mu) = \sigma(\mu)K_\mu, \tag{2.10}$$

for $1 \leq i \leq n$ and $\mu \in P$.

Let $V$ be a (left) module over $U$. For any $\lambda \in P$, we define the weight space

$$V_{\lambda,\sigma} = \{v \in V | K_\mu \cdot v = \sigma(\mu)q^{(\lambda,\mu)}v, \forall \mu \in P\},$$

where $\sigma : P \to \mathbb{C}^\times$ is the group character as defined in (2.9). It is well known that every finite-dimensional $U$-module is a weight module, i.e., a direct sum of its weight spaces. We say $V$ is of type $\sigma$ if $V = V^\sigma = \bigoplus_{\lambda \in P} V_{\lambda,\sigma}$. Clearly, $V^\sigma = V^1 \otimes \mathbb{F}_\sigma$, where $1$ is the trivial group character sending every $\lambda \in P$ to $1$. Therefore, we will restrict our attention to only type $1$ modules.

Of particular interest is the Verma module of $U$. If $\lambda \in \mathfrak{h}^*$, we let $\chi_\lambda : U^0 \to \mathbb{F}$ be the character such that $\chi_\lambda(K_\mu) = q^{(\lambda,\mu)}$ for any $\mu \in P$. Then the Verma module $M(\lambda)$ associated to $\lambda$ is defined as the following induced $U$-module

$$M(\lambda) := U \otimes_{U^0} \mathbb{F}_\lambda,$$

where $U^0$ is the subalgebra generated by all $E_i$ and $K_\mu$, and $\mathbb{F}_\lambda$ denotes the one-dimensional $U^0$-module with the action induced from $\chi_\lambda$ such that all $E_i$ actions
vanish. The vector \( v_\lambda = 1 \otimes 1 \in M(\lambda) \) is called the highest weight vector with the property that

\[
E_i.v_\lambda = 0, \quad K_\mu.v_\lambda = q^{(\lambda,\mu)}v_\lambda, \quad \forall 1 \leq i \leq n, \mu \in P.
\]

It turns out that every finite-dimensional simple module of \( U \) is a quotient of \( M(\lambda) \) by its unique maximal proper submodule \( N(\lambda) \) for some dominant weight \( \lambda \in P^+ \). We will write \( V(\lambda) \) for the simple quotient module \( M(\lambda)/N(\lambda) \).

For any \( \lambda \in P^+ \), denote by \( \Pi(\lambda) \) the set of weights of the simple module \( V(\lambda) \). Let \( m_\lambda(\mu) \) be the multiplicity of \( \mu \) in \( V(\lambda) \), that is, \( m_\lambda(\mu) = \dim V(\lambda)_\mu \). Then \( \Pi(\lambda) \) is \( W \)-invariant and

\[
m_\lambda(\mu) = m_\lambda(\mu w), \quad \forall w \in W, \mu \in \Pi(\lambda)
\]

Furthermore, \( m_\lambda(\lambda) = 1 \).

An important fact is that finite-dimensional modules separate the points of \( U \).

**Proposition 2.4** [11, Chapter 5.11] *If an element \( u \in U \) satisfies \( u.V = 0 \) for all finite-dimensional \( U \)-modules \( V \), then \( u = 0 \).*

### 2.3 The Quantised Harish-Chandra Isomorphism

To prepare for the construction of a generating set for the centre of the generalised quantum group \( U_q(g) \), we first consider a quantised Harish-Chandra isomorphism for \( U_q(g) \) in this section. The discussion here will be brief, and we follow closely [11, Chapter 6].

Let \( Z(U) \) denote the centre of \( U \). Recall that \( U = \bigoplus_{\nu \in Q^+} U_\nu \) is a \( Q \)-graded algebra with

\[
U_\nu = \{ u \in U \mid \text{ad}(K_\lambda)u = q^{(\lambda,\nu)}u, \forall \lambda \in P \}. \quad (2.11)
\]

In particular, \( U_0 = U^0 \oplus \bigoplus_{\nu > 0} U^{-\nu} U^0 U^+ \), where \( U^0 \) is the subalgebra generated by \( K_\lambda^{\pm1}, \lambda \in P \). Clearly we have \( Z(U) \subseteq U_0 \), and the following projection

\[
\pi : U_0 \rightarrow U^0 \quad (2.12)
\]

is an algebra homomorphism.

For any \( \lambda \in P \), we define the twisting algebra automorphism

\[
\gamma_\lambda : U^0 \rightarrow U^0, \quad \gamma_\lambda(h) = \chi_\lambda(h)h, \quad \forall h \in U^0. \quad (2.13)
\]

Denote \( \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \). Composing (2.12) and (2.13) with \( \lambda = -\rho \), we obtain the Harish-Chandra homomorphism \( \gamma_{-\rho} \circ \pi : U_0 \rightarrow U^0 \), which in particular maps the centre \( Z(U) \) to \( U^0 \).

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We proceed to characterise the image \( \gamma_{-\rho} \circ \pi(Z(U)) \), which will turn out to be the Weyl group invariants in \( U^0 \). For any simple Lie algebra \( g \), the associated Weyl group \( W \) (cf. [10]) is generated by the simple reflections \( s_{\alpha_i} \), \( 1 \leq i \leq n \) with

\[
s_{\alpha_i} \lambda = \lambda - (\lambda, \alpha_i^\vee)\alpha_i, \quad \forall \lambda \in \mathfrak{h}^*.
\]

This induces a natural Weyl group action on \( U^0 \), i.e., \( wK_\lambda = K_{w\lambda} \) for any \( w \in W \), \( \lambda \in P \). Let \( U^0_{ev} = \bigoplus_{\lambda \in P} \mathbb{F}K_{2\lambda} \) be the subalgebra of \( U^0 \) generated by the even elements \( K_{2\lambda} \) for all \( \lambda \in P \), and define

\[
(U^0_{ev})^W := \{ h \in U^0_{ev} \mid wh = h, \forall w \in W \},
\]

which is the subalgebra of \( W \)-invariants in \( U^0 \).

The following result is the quantised Harish-Chandra isomorphism for the generalised Jimbo quantum group \( U \). It is an adaptation to \( U \) of the quantised Harish-Chandra isomorphism for the standard Jimbo quantum groups established in [11,20].

**Theorem 2.5** *The twisted algebra homomorphism*

\[
\gamma_{-\rho} \circ \pi : Z(U) \to (U^0_{ev})^W
\]

is an isomorphism.

Note that Theorem 2.5 has been stated and proven with a geometric method in [4]. However, we have no idea whether the method is applicable to standard quantum groups. Since it is of crucial importance for us, we give an elementary algebraic proof of the result in Appendix A as a reference.

### 3 Construction of Central Elements

#### 3.1 The Construction

The method developed in [9,24] plays a crucial role in the construction of central elements. It enables us to obtain an infinite family of central elements from any finite-dimensional \( U_q(g) \)-modules.

Given any finite-dimensional \( U_q(g) \)-module \( V \), denote by \( \xi : U_q(g) \to \text{End}(V) \) the associated \( U_q(g) \) representation. In this case, we also say that \( (V, \xi) \) is a representation of \( U_q(g) \). We define the partial trace

\[
\begin{align*}
\text{Tr}_1 &: \text{End}(V) \otimes U_q(g) \to U_q(g), \\
\text{Tr}_1(f \otimes x) &= \text{Tr}(f)x, \quad \forall f \in \text{End}(V), \ x \in U_q(g),
\end{align*}
\]

where \( \text{Tr} \) is the usual trace operator on \( \text{End}(V) \).

The following result is proved in [24, Proposition 1].
Theorem 3.1 ([24]) Let $\Gamma_V \in \text{End}(V) \otimes U_q(\mathfrak{g})$. If $\Gamma_V$ commutes with $U_q(\mathfrak{g})$ in the sense that

$$\Gamma_V(\zeta \otimes \text{id})\Delta(x) - (\zeta \otimes \text{id})\Delta(x)\Gamma_V = 0, \ \forall x \in U_q(\mathfrak{g}),$$

then

$$\text{Tr}_1((\zeta(K_{2\rho}) \otimes \text{id})\Gamma_V) \in Z(U_q(\mathfrak{g})). \ (3.1)$$

It immediately follows that

Corollary 3.2 If $\Gamma_V \in \text{End}(V) \otimes U_q(\mathfrak{g})$ commutes with $U_q(\mathfrak{g})$, then for all $m \in \mathbb{N}^+$, the elements $\text{Tr}_1((\zeta(K_{2\rho}) \otimes \text{id})\Gamma_V^m)$ belong to the centre of $U_q(\mathfrak{g})$.

3.2 Constructing $\Gamma_V$

Now the problem is to develop non-trivial elements $\Gamma_V$. We will do this following [9, 24]. We need the explicit expression for the quasi-$R$-matrix $R$ of $U_q(\mathfrak{g})$. Recall the braid group action on the quantum group. The braid group $B_g$ associated with the Weyl group $W$ of $\mathfrak{g}$ is generated by $n$ elements $\sigma_i$ with relations

$$\sigma_i \sigma_j \sigma_i \ldots = \sigma_j \sigma_i \sigma_j \ldots, \ i \neq j,$$

where the number of $\sigma$’s on each side is $m_{ij}$, which is determined by the Cartan matrix. We have $m_{ij} = 2, 3, 4, 6$ for $a_{ij}a_{ji} = 0, 1, 2, 3$. The braid group $B_g$ acts as group of algebra automorphisms of $U_q(\mathfrak{g})$. For the explicit algebra automorphism $T_i$ corresponding to each generator $\sigma_i$, we refer to [14, 18]. These are usually referred to as the Lusztig automorphisms.

Write $s_i$ for the simple reflection $s_{\alpha_i}$ in the Weyl group $W$ of $\mathfrak{g}$. Let $w_0 = s_1s_2\ldots s_N$ be a reduced expression of the longest element of $W$. Then the positive roots of $\mathfrak{g}$ are given by the following successive actions on simple roots $\alpha_i$

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_1(\alpha_{i_2}), \ldots, \beta_N = s_1\ldots s_{i_{N-1}}(\alpha_{i_N}).$$

We define the root vectors of $U_q(\mathfrak{g})$ by

$$E_{\beta_r} = T_1T_2\ldots T_{i_r-1}(E_{i_r}), \quad F_{\beta_r} = T_1T_2\ldots T_{i_r-1}(F_{i_r})$$

for all $1 \leq r \leq N$. Note that $E_{\beta_r} \in U^+$ and $F_{\beta_r} \in U^-$. By [18, 4.1], we have

$$R = \sum_{r_1, \ldots, r_N=0}^{\infty} \prod_{j=1}^{N} q_{\beta_j}^{\frac{1}{2}r_j(r_j+1)} \frac{1 - q_{\beta_j}^{-2}y_j^{r_j}}{[r_j]_{q_{\beta_j}}!} F_{\beta_j}^{r_j} \otimes E_{\beta_j}^{r_j}, \quad (3.2)$$

where $q_{\beta} := q_i$ if $\beta$ and $\alpha_i$ lie in the same orbit under the action of $W$, the factors in the product (3.2) appear in the order $\beta_N, \beta_{n-1}, \ldots, \beta_1$. Now $R$ is an infinite sum, which belongs to some completion of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. 
The quasi $R$-matrix $\mathcal{R}$ has many remarkable properties. In particular, the property discussed below will be important for us. One can easily verify that there is an algebra automorphism $\Psi$ of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ defined by

$$\Psi(K_\lambda \otimes 1) = K_\lambda \otimes 1, \quad \Psi(E_i \otimes 1) = E_i \otimes K_i^{-1}, \quad \Psi(F_i \otimes 1) = F_i \otimes K_i, \quad (3.3)$$

$$\Psi(1 \otimes K_\lambda) = 1 \otimes K_\lambda, \quad \Psi(1 \otimes E_i) = K_i^{-1} \otimes E_i, \quad \Psi(1 \otimes F_i) = K_i \otimes F_i. \quad (3.4)$$

Then it is well-known that $\mathcal{R}$ satisfies the following relation (see, e.g., [21, §4.3]).

**Proposition 3.3** Let $\mathcal{R}^T = T(\mathcal{R}), \Delta' = T \circ \Delta$, where $T$ is the flip $T(x \otimes y) = y \otimes x$ for any $x, y \in U_q(\mathfrak{g})$. We have

$$\mathcal{R}\Delta(x) = \Psi(\Delta'(x))\mathcal{R}, \quad \mathcal{R}^T\Delta'(x) = \Psi(\Delta(x))\mathcal{R}^T, \quad \forall x \in U_q(\mathfrak{g}). \quad (3.5)$$

Now we turn to the construction of $\Gamma_V$. Let $(V, \zeta)$ be a finite-dimensional representation of $U_q(\mathfrak{g})$. Define

$$\mathcal{R}_V := (\zeta \otimes \text{id})(\mathcal{R}), \quad \mathcal{R}_V^T := (\zeta \otimes \text{id})\mathcal{R}^T. \quad (3.6)$$

Then both $\mathcal{R}_V$ and $\mathcal{R}_V^T$ belong to $\text{End}(V) \otimes U_q(\mathfrak{g})$, which are well defined.

Denote by $\Pi(V)$ the set of weights of the $U_q(\mathfrak{g})$-module $V$. For any $\eta \in \Pi(V)$, we denote by $V_\eta \subset V$ the weight space of weight $\eta$. Let $P^V_\eta : V \rightarrow V_\eta$ be the projection from $V$ to $V_\eta$, and define the following element in $\text{End}(V) \otimes U_q(\mathfrak{g})$.

$$K_V := \sum_{\eta \in \Pi(V)} P^V_\eta \otimes K_\eta. \quad (3.7)$$

Regard this as an endomorphism of $V \otimes U_q(\mathfrak{g})$ with the inverse $K_V^{-1} = \sum_{\eta \in \Pi(V)} P^V_\eta \otimes K_\eta^{-1}$.

Recall that the generators of $U_q(\mathfrak{g})$ are $K_\lambda, E_i, F_i$ with $\lambda \in P, 1 \leq i \leq n$, and $\zeta : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$ the associated $U_q(\mathfrak{g})$ representation of $V$, then $K_V$ satisfies the following relations,

$$K_V(\zeta(K_\lambda) \otimes K_\mu) = (\zeta(K_\lambda) \otimes K_\mu)K_V, \quad (3.8)$$

$$K_V(\zeta(E_i) \otimes 1) = (\zeta(E_i) \otimes K_i)K_V, \quad K_V(1 \otimes E_i) = (\zeta(K_i) \otimes E_i)K_V, \quad (3.9)$$

$$K_V(\zeta(F_i) \otimes 1) = (\zeta(F_i) \otimes K_i^{-1})K_V, \quad K_V(1 \otimes F_i) = (\zeta(K_i^{-1}) \otimes F_i)K_V. \quad (3.10)$$

Recall that $\Psi$ is the algebra isomorphism defined in (3.3), (3.4), and so is $(\zeta \otimes \text{id})\Psi$. One can verify the following relations by checking them on the generators of $U_q(\mathfrak{g})$, which are equal to the equations (3.8), (3.9), (3.10),

$$K_V\Psi(\Delta(x)) = \Delta(x)K_V, \quad K_V\Psi(\Delta'(x)) = \Delta'(x)K_V, \quad \forall x \in U_q(\mathfrak{g}), \quad (3.11)$$

where we have ignored $\zeta \otimes \text{id}$ before $\Delta(x), \Psi(\Delta(x))$ and etc. to simplify the notation.
Proposition 3.4 Retain notation above. Let

\[ L_V := K_V R_V, \quad L_V^T := K_V R_V^T. \]  

(3.12)

Then for any finite-dimensional representation \((V, \zeta)\) of \(U_q(g)\),

\[ L_V \Delta(x) = \Delta'(x) L_V, \quad L_V^T \Delta'(x) = \Delta(x) L_V^T, \quad x \in U_q(g). \]  

(3.13)

**Proof** It follows from (3.5) that for all \(x \in U_q(g)\),

\[ R_V \Delta(x) = \Psi(\Delta'(x)) R_V. \]

Combining this with the second relation of (3.11), we obtain

\[ K_V R_V \Delta(x) = \Delta'(x) K_V R_V, \]

i.e., \(L_V \Delta(x) = \Delta'(x) L_V\).

The other relation of 3.13 can be similarly proved by noting that

\[ R^T \Delta'(x) = \Psi(\Delta(x)) R^T, \quad \forall x \in U_q(g). \]

We remark that the universal \(R\)-matrix of the Drinfeld version of the quantum group in any representation of finite rank satisfies relations formally the same as (3.13). □

Now we can construct a \(\Gamma_V\) satisfying the condition of Theorem 3.1 as follows [24].

**Theorem 3.5** Given any finite-dimensional representation \((V, \zeta)\) of \(U_q(g)\), let

\[ \Gamma_V := L_V^T L_V. \]  

(3.14)

Then \(\Gamma_V \in \text{End}(V) \otimes U_q(g)\) commutes with \((\zeta \otimes \text{id}) \Delta(x)\) for all \(x \in U_q(g)\).

**Proof** One of the relations in Proposition 3.4 states that \(L_V \Delta(x) = \Delta'(x) L_V\) for all \(x \in U_q(g)\). Thus,

\[ \Gamma_V \Delta(x) = L_V L_V^T \Delta(x) = L_V^T \Delta'(x) L_V. \]

By using Proposition 3.4 again, we can re-write the right-hand side as

\[ \Delta(x) L_V^T L_V = \Delta(x) \Gamma_V. \]

Hence, \(\Gamma_V \Delta(x) = \Delta(x) \Gamma_V\), proving the theorem. □
3.3 Central Elements of the Generalised Quantum Group

Next we turn to the construction of central elements of the generalised quantum groups. Let \((\zeta_\lambda, V(\lambda))\) be a finite-dimensional irreducible representation of \(U_q(\mathfrak{g})\) with highest weight \(\lambda \in P^+\). Define

\[
C^{(m)}_\lambda := \text{Tr}_1((\zeta_\lambda(K_{2,\rho}) \otimes 1)\Gamma^m_{V(\lambda)}), \quad m \in \mathbb{N}^+.
\]  

(3.15)

Then by Theorem 3.5 and Corollary 3.2, \(C^{(m)}_\lambda\) belong to the centre \(Z(U)\) of \(U_q(\mathfrak{g})\) for all \(\lambda \in P^+\) and \(m \in \mathbb{N}^+\).

**Remark 3.6** We also expect that if a given \(\lambda \in P^+\) satisfies certain conditions, then \(C^{(m)}_\lambda\) for finitely many \(m\) generate the centre of \(U_q(\mathfrak{g})\); see Sect. 4.3 for the precise statement.

Let \(C_\lambda = C^{(1)}_\lambda\) be the central element of \(U_q(\mathfrak{g})\) defined by (3.15) for \(m = 1\). The central elements \(C_\lambda\) are particularly easy to study, since the action of \(L V(\lambda)\), \(L^T V(\lambda)\) can be written specifically. Next we will analyse them and give the formulae of \(C_\lambda\) briefly.

Let

\[
R_N := \{(r_1, r_2, \ldots, r_N) | r_i \in \mathbb{N}\}.
\]

For any \(r = (r_1, r_2, \ldots, r_N) \in R_N\), define

\[
F_r := E_{\beta_N}^{r_N} \cdots E_{\beta_2}^{r_2} E_{\beta_1}^{r_1}, \quad E_r := E_{\beta_N}^{r_N} \cdots E_{\beta_2}^{r_2} E_{\beta_1}^{r_1}, \quad K_r := K_{\beta_N}^{r_N} \cdots K_{\beta_2}^{r_2} K_{\beta_1}^{r_1},
\]

\[
D_r := \prod_{j=1}^N \frac{1}{q_{\beta_j}^{r_j(r_j+1)}(1 - q_{\beta_j}^{-2})^{r_j}} \frac{1}{[r_j]_{q_{\beta_j}}!}, \quad \sum r = \sum_{i=1}^N r_i \beta_i.
\]

Then

\[
L V(\lambda) = K_{V(\lambda)} \sum_{r \in R} D_r (\zeta_\lambda(F_r) \otimes E_r).
\]  

(3.16)

Similarly,

\[
L^T V(\lambda) = K_{V(\lambda)} \sum_{t \in R_N} D_t (\zeta_\lambda(E_t) \otimes F_t) = \sum_{t \in R_N} D_t (\zeta_\lambda(E_t K_t^{-1}) \otimes K_t F_t) K_{V(\lambda)},
\]  

(3.17)

where the last equation follows from (3.9), (3.10).

**Remark 3.7** Note that \(V(\lambda)\) is finite dimensional, there are only a finite number of nonzeros in the summation terms of (3.16), (3.17).
Recall that $\Gamma_V = L^T V$. By (3.16) and (3.17), we can get

$$\Gamma_{V(\lambda)} = \sum_{r,t \in R_N} D_r D_t (\xi_\lambda(E_t K_t^{-1}) \otimes K_t F_t) K_{V(\lambda)}(\xi_\lambda(F_t) \otimes E_r). \quad (3.18)$$

After substituting $\sum_{\mu \in \Pi(\lambda)} P_{\mu}^{V(\lambda)} \otimes K_{\mu}$ for $K_{V(\lambda)}$ and shifting items, we can get

$$\Gamma_V = \sum_{\mu \in \Pi(\lambda)} \sum_{r,t \in R_N} A_{r,t} \xi_\lambda(E_t K_t^{-1} F_r) P_{\mu}^{V(\lambda)} \otimes F_t K_{2\mu-2} - \sum_r \sum_t E_r, \quad (3.19)$$

where $A_{r,t}$ arises from $D_r$, $D_t$, and the exchange of $F_t$ and $K_t$. In particular, $A_{r,t} = 1$ for $r = (0, 0, \ldots, 0)$ and $t = (0, 0, \ldots, 0)$.

Given that $C_\lambda = \text{Tr}_1((\xi_\lambda(K^{2\rho}) \otimes 1) \Gamma_{V(\lambda)})$, we only need to consider the items that contribute to the trace, its necessary condition is $\sum r = \sum t$. In addition, for any $v_\mu \in V(\lambda)$, with $\mu \in \Pi(\lambda)$, $\xi_\lambda(K^{2\rho})v_\mu = q^{(2\rho, \mu)}v_\mu$. Then we can get

$$C_\lambda = \sum_{\mu \in \Pi(\lambda)} q^{(2\rho, \mu)} \sum_{t=\sum r} A_{r,t} \text{Tr}(\xi_\lambda(E_t K_t^{-1} F_r) P_{\mu}^{V(\lambda)} F_t K_{2\mu-2} - \sum_r \sum_t E_r). \quad (3.20)$$

So far, we have a series of central elements $C_\lambda$ obtained from the simple $U_q(g)$-modules $V(\lambda)$ with $\lambda \in P^+$. We will show in Theorem 4.5 that the subset of $C_\lambda$ for $\lambda$ being the fundamental weights generate the entire centre of $U_q(g)$.

Recall from (2.14) the Harish-Chandra isomorphism $\gamma_{-\rho} \circ \pi$ from $Z(U_q(g))$ to $(U_0^0)^W$, where $U = U_q(g)$. By (3.20), we can immediately get the following proposition.

**Proposition 3.8** For any $\lambda \in P^+$, we have

$$\gamma_{-\rho} \circ \pi(C_\lambda) = \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) K_{2\mu} \in (U_0^0)^W. \quad (3.21)$$

**Proof** Recall that $\pi$ is an algebra homomorphism from $U_0$ to $U_0^0$, the subalgebra of $U$ generated by $K_\lambda$’s. By inspecting the formula (3.20), we can easily see that $\pi(C_\lambda)$ is the sum of terms in which $r = (0, 0, \ldots, 0)$ and $t = (0, 0, \ldots, 0)$. In this condition, $A_{r,t} = 1$, and $\text{Tr}(\xi_\lambda(1) P_{\mu}^{V(\lambda)}) = m_\lambda(\mu)$. Then we can get that

$$\pi(C_\lambda) = \sum_{\mu \in \Pi(\lambda)} q^{(2\rho, \mu)} m_\lambda(\mu) K_{2\mu}.$$  

It then follows from the definition of $\gamma_{-\rho}$ in (2.13) that

$$\gamma_{-\rho} \circ \pi(C_\lambda) = \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) K_{2\mu},$$

where the right-hand side clearly belongs to $(U_0^0)^W$. \qed
Explicit Generators of the Centre of the Quantum Group

Remark 3.9 Proposition 3.8 makes $C_\lambda$ with the complicative expression (3.20) more clear to us. Moreover, the images of them under the Harish-Chandra isomorphism is similar with the character of $V(\lambda)$. This fact makes it easily for us to construct the generators of the centre from the generators of $(U_{ev}^0)^W$.

4 Generators of the Centre and Their Relations

4.1 Grothendieck Group of $U_q(g)$

In this section, we will write $U = U_q(g)$.

Let $K(U)$ be the Grothendieck group over $\mathbb{F}$ of the category $\text{Rep}_f(U)$ of finite-dimensional $U$-modules. Since $\text{Rep}_f(U)$ is a tensor category, $K(U)$ is an associative algebra, the multiplication of which is induced by the tensor product of $U$-modules. More explicitly, for any objects $V, V'$ in $\text{Rep}_f(U)$, we write $[V]$ and $[V']$ for the corresponding elements in $K(U)$. Then $[V][V'] = [V \otimes V']$.

We will prove $K(U) \cong (U_{0v}^0)^W$ in Lemma 4.7. Thus, the algebraic structure and the generators of $K(U)$ are of crucial importance to us. In Lie theory, it is well known that the representation ring $R(g)$ for the finite-dimensional simple Lie algebra $g$ is a polynomial algebra generated by the irreducible representations with highest weights $\varpi_1, \varpi_2, \ldots, \varpi_n$, see [7, Theorem 23.24] for details. For $K(U)$, we have the same result.

Theorem 4.1 Let $\varpi_1, \varpi_2, \ldots, \varpi_n$ be the fundamental weights of $g$. Then $K(U)$ is a polynomial algebra over $\mathbb{F}$ in variables $[V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]$, where $n$ is the rank of $g$.

We will take some steps to prove Theorem 4.1. It is easy to verify the following lemma.

Lemma 4.2 The elements $[V(\lambda)]$ with $\lambda \in P^+$ form a basis of $K(U)$.

Endow $P$ with the standard partial order such that $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a nonnegative integral linear combination of positive roots. Next we prove that for any $\lambda \in P^+$, $[V(\lambda)]$ is generated by $[V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]$, and $[V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]$ are algebraically independent.

Lemma 4.3 There exists a polynomial $f_\lambda \in \mathbb{F}[x_1, x_2, \ldots, x_n]$ such that

$$[V(\lambda)] = f_\lambda([V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]), \quad \forall \lambda \in P^+.$$

Proof We use induction on $\lambda$. It is trivial when $\lambda = 0$. Suppose that $\lambda = \sum_{i=1}^n k_i \varpi_i$, then the irreducible representation $V(\lambda)$ is contained in the tensor product $\bigotimes_{i=1}^n V(\varpi_i)^{\otimes k_i}$ with multiplicity 1. This tensor product can be decomposed as

$$\bigotimes_{i=1}^n V(\varpi_i)^{\otimes k_i} = V(\lambda) \oplus \bigoplus_{\mu \in P^+, \mu < \lambda} m_\mu V(\mu),$$

\[\text{Springer}\]
where \( m_\mu \in \mathbb{N} \) denotes the multiplicity of \( V(\mu) \). By induction, there exist \( f_\mu \in \mathbb{F}[x_1, \ldots, x_n] \) such that \( [V(\mu)] = f_\mu([V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]) \). Therefore, we have \( f_\lambda = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} - \sum_{\mu \in P^+, \mu < \lambda} m_\mu f_\mu \). \( \square \)

Recall that any integral dominant weight \( \lambda \) is a nonnegative linear combination of fundamental weights, that is, \( \lambda = \sum_{i=1}^n k_i \varpi_i \) for \( k_i \in \mathbb{N} \). We can define the lexicographic order \( < \) on \( P^+ = \bigoplus_{i=1}^n \mathbb{N}\varpi_i \).

**Lemma 4.4** The elements \([V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]\) are algebraically independent over \( \mathbb{F} \). As a consequence, the polynomial \( f_\lambda \) as defined in Lemma 4.3 is unique.

**Proof** Assume for contradiction that in \( \mathbb{F}[x_1, \ldots, x_n] \) there exists

\[
f = c_{k_1, \ldots, k_n} x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} + \sum_{(a_1, \ldots, a_n) < (k_1, \ldots, k_n)} c_{a_1, \ldots, a_n} x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}
\]

such that \( f([V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]) = 0 \), where all monomials in \( f \) are arranged lexicographically such that \( x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \) is the maximal one and \( c_{k_1, \ldots, k_n} \neq 0 \). Note that \([V(\varpi_1)]^{k_1} [V(\varpi_2)]^{k_2} \ldots [V(\varpi_n)]^{k_n} = [\bigotimes_{i=1}^n V(\varpi_i)]^{k_i} \). We express this as a linear combination of the basis elements \([V(\mu)] \) with \( \mu \in P^+ \), then \([V(\lambda)] \) with \( \lambda = \sum_{i=1}^n k_i \varpi_i \) has coefficient 1. However, \([V(\lambda)] \) never appears in \([V(\varpi_1)]^{a_1} [V(\varpi_2)]^{a_2} \ldots [V(\varpi_n)]^{a_n} \) for any \((a_1, \ldots, a_n) < (k_1, \ldots, k_n)\). Therefore, \([V(\lambda)] \) appears in \( f([V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]) \) with coefficient \( c_{k_1, \ldots, k_n} \neq 0 \), contradicting \( f([V(\varpi_1)], [V(\varpi_2)], \ldots, [V(\varpi_n)]) = 0 \). \( \square \)

**Proof of Theorem 4.1** By Lemmas 4.2, 4.3 and 4.4, we can immediately complete the proof. \( \square \)

### 4.2 Explicit Generators of the Centre

The following theorem is the main result of this paper.

Recall that \( C_\lambda = C^1_\lambda \) is defined in (3.15), which has the explicit expression in (3.20), and \( \varpi_1, \varpi_2, \ldots, \varpi_n \) are the fundamental weights of \( g \).

**Theorem 4.5** The centre \( Z(U) \) of \( U_q(g) \) is the polynomial algebra over \( \mathbb{F} \) in the variables \( C_{\varpi_1}, C_{\varpi_2}, \ldots, C_{\varpi_n} \), i.e., \( Z(U) \cong \mathbb{F}[C_{\varpi_1}, C_{\varpi_2}, \ldots, C_{\varpi_n}] \).

**Proof** By the quantised Harish-Chandra isomorphism in Theorem 2.5, \( Z(U) \) is isomorphic to \((U^0_{ev})^W \) as algebras. Thus, Theorem 4.5 can be proved by showing that \((U^0_{ev})^W \) is a polynomial algebra over \( \mathbb{F} \) in the variables \( \gamma_{-\rho} \circ \pi(C_{\varpi_i}) \) for \( 1 \leq i \leq n \). This is proven in Corollary 4.8. \( \square \)

The remainder of this section is devoted to the proof of Corollary 4.8. This will be done by establishing a series of lemmas. Now the following result is clear.
Lemma 4.6  Set for any $\lambda \in P$,  

$$av(\lambda) = \sum_{\mu \in W\lambda} K_{2\mu},$$

then $av(\lambda)$ for all $\lambda \in P^+$ form a basis of $(U^0_{ev})^W$.

Lemma 4.7  There is an algebra isomorphism  

$$\text{Ch} : K(U) \longrightarrow (U^0_{ev})^W, \quad \text{Ch}([V]) = \sum_{\mu \in \Pi(V)} \dim V_\mu K_{2\mu}, \quad (4.1)$$

where $\Pi(V)$ is the set of weights of $V$ and $\dim V_\mu$ is the dimension of the weight space $V_\mu$.

Proof  It is easy to verify that Ch is an algebra homomorphism. Firstly, we prove that it is injective. For any $[V], [W] \in K(U)$ such that $\text{Ch}([V]) = \text{Ch}([W])$, by the definition of Ch, we have  

$$\sum_{\mu_1 \in \Pi(V)} \dim V_{\mu_1} K_{2\mu_1} = \sum_{\mu_2 \in \Pi(W)} \dim W_{\mu_2} K_{2\mu_2}.$$  

Since $K_{\mu}$ with $\mu \in P$ are linear independent, then we can get that $\Pi(V) = \Pi(W)$, and for any $\mu \in \Pi(V)$, $\dim V_{\mu} = \dim W_{\mu}$, thus $[W] = [V]$.

Next we prove it is surjective. By Lemma 4.6, $av(\lambda)$ for all $\lambda \in P^+$ form a basis of $(U^0_{ev})^W$. We need to show that all these $av(\lambda)$ are in the image of Ch. Note that $P$ is endowed with the standard partial order that $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a nonnegative integral linear combination of positive roots. We use upward induction on the partial ordering of $P^+$. Starting with $\lambda$ minimal, i.e., no other $\mu \in P^+$ can occur as a weight of $V(\lambda)$, then we have $\text{Ch}([V(\lambda)]) = av(\lambda)$. Suppose that $\lambda = \sum_{i=1}^n k_i \varpi_i$, then the irreducible representation $V(\lambda)$ is contained in the tensor product $\bigotimes_{i=1}^n V(\varpi_i)^{\otimes k_i}$ with multiplicity 1. This tensor product can decomposes as the direct sum of $V(\mu)$ with $\mu \leq \lambda$. Recall that $\Pi(\mu)$ is the set of weights of $V(\mu)$, which is $W$-invariant, and $\dim V_v = \dim V_{wV}$ for any $v \in \Pi(\mu)$, $w \in W$. Then we can get  

$$\bigotimes_{i=1}^n V(\varpi_i)^{\otimes k_i} = \left( \bigoplus_{w \in W} V_{w\lambda} \right) \oplus \left( \bigoplus_{w \in W, \eta \in P^+, \eta < \lambda} \dim V_\eta V_{w\eta} \right).$$

Note that $\text{Ch}(V(\varpi_i)) = av(\varpi_i)$. Applying the ring homomorphism Ch to both sides, we obtain  

$$\prod_{i=1}^n av(\varpi_i)^{k_i} = av(\lambda) + \sum_{\eta \in P^+, \eta < \lambda} \dim V_\eta av(\eta).$$

By induction, there exist inverse images for all $av(\eta)$ with $\eta < \lambda$, then so is $av(\lambda)$. □
Combining Proposition 3.8, we can get the following corollary.

**Corollary 4.8** Let $\tilde{C}_\lambda = \gamma_{-\rho} \circ \pi(C_\lambda)$, then $(U^0_{ev})^W \cong \mathbb{F}[\tilde{C}_{\sigma_1}, \tilde{C}_{\sigma_2}, \ldots, \tilde{C}_{\sigma_n}]$, the polynomial algebra in the $n$ variables $\tilde{C}_{\sigma_1}, \tilde{C}_{\sigma_2}, \ldots, \tilde{C}_{\sigma_n}$.

**Proof** By Proposition 3.8 and the definition of Ch, we have $Ch([V(\lambda)]) = \tilde{C}_\lambda$. In particular, $Ch([V(\sigma_i)]) = \tilde{C}_{\sigma_i}$. By Theorem 4.1 and Lemma 4.7, we can immediately complete the proof. □

### 4.3 Some Remarks

We have shown that the subset of central elements $C^{(m)}_\lambda$ defined by (3.15), with $m = 1$ and $\lambda$ being the fundamental weights, generates the centre of $U_q(g)$. We also expect the following to be true.

**Conjecture 4.9** If tensor powers of a finite-dimensional simple $U_q(g)$-module $V(\lambda)$ separate points of $U_q(g)$ (see Proposition 2.4), then there is a finite subset $M_{g,\lambda}$ of $\mathbb{N}^+$ such that $\{C^{(m)}_\lambda | m \in M_{g,\lambda}\}$ generates the centre of $U_q(g)$.

This is the case [15] for $g = gl_n$ and $V(\lambda) = \mathbb{F}^n$ being the natural module; and one can also easily extract from op. cit. such a set of generators for the center of $U_q(sl_n)$.

It will be very interesting to prove this for $U_q(g)$ for the other simple Lie algebras $g$ by identifying such $\lambda$ and the corresponding minimal sets $M_{g,\lambda}$.

### Appendix A. Proof of the Harish-Chandra isomorphism

This part is about the algebraic proof of Theorem 2.5, i.e., the quantised Harish-Chandra isomorphism of $U_q(g)$. Note that it can be proven in much the same way as the proof in [11, Chapter 6]. However, we can hardly find a proof in detail with the method developed in [11]. Hence, we give some pertinent steps in the following.

Write $U = U_q(g)$. We first show that $\gamma_{-\rho} \circ \pi$ indeed maps $Z(U)$ into the invariant subalgebra $(U^0_{ev})^W$.

Observe the following elementary result.

**Lemma A.1** Let $\lambda \in P$. Any $u \in Z(U)$ acts on the Verma module $M(\lambda)$ as a scalar multiplication by $\chi_\lambda(\pi(u))$.

As an immediate consequence, we have

**Lemma A.2** The restriction of $\pi$ to $Z(U)$ is injective, and hence so is $\gamma_{-\rho} \circ \pi$.

**Proof** If $\pi(u) = 0$, then by Lemma A.1, we have $u.M(\lambda) = 0$ and hence $u.V(\lambda) = 0$ for all $\lambda \in P^+$. By Proposition 2.4, $u = 0$. □

We now show that the image $\gamma_{-\rho} \circ \pi(Z(U))$ of the centre is invariant under the Weyl group action.

**Lemma A.3** The images of $Z(U)$ under the Harish-Chandra isomorphism are all in $(U^0)^W$, i.e., $\gamma_{-\rho} \circ \pi(Z(U)) \subseteq (U^0)^W$. □
Proof} Fix any central element \( u \in Z(U) \), we write \( h = \gamma - \rho \circ \pi (u) \).

Given any \( \lambda \in P \) and \( i \in \{1, 2, \ldots, n\} \), we let \( \mu = s_{\alpha_i}(\lambda + \rho) - \rho \).

If \( (\lambda, \alpha_i^\vee) \geq 0 \), there is a nontrivial homomorphism \( M(\mu) \to M(\lambda) \) [11, Chapter 5.9]. By Lemma A.1,

\[
\chi_{\lambda + \rho}(h) = \chi_{\mu + \rho}(h) = \chi_{\lambda + \rho}(s_{\alpha_i}h).
\] (A.1)

If \( (\lambda, \alpha_i^\vee) < -1 \), then \( (\mu, \alpha_i^\vee) \) is nonnegative, thus we may apply the above arguments to \( \mu \) to show that (A.1) still holds.

Then the only other possibility is that \( (\lambda, \alpha_i^\vee) = -1 \). In this case, \( \mu = \lambda \), and (A.1) holds trivially.

Since (A.1) holds for all \( \lambda \) and \( i \), and \( s_{\alpha_i} \) generate \( W \), we have

\[
\chi_{\lambda}(wh - h) = 0, \quad \forall w \in W, \lambda \in P.
\] (A.2)

We can always write \( wh - h = \sum_{\eta} a_{\eta} K_{\eta} \). Then (A.2) leads to

\[
\sum_{\eta} a_{\eta} \chi_{\lambda}(K_{\eta}) = \sum_{\eta} a_{\eta} q^{(\lambda, \eta)} = \sum_{\eta} a_{\eta} \chi_{\eta}(K_{\lambda}) = 0, \quad \forall \lambda \in P.
\]

Thus, \( \sum_{\eta} a_{\eta} \chi_{\eta} = 0 \). The linear independence of characters then implies \( a_{\eta} = 0 \) for all \( \eta \). Hence, \( wh - h = 0 \) for all \( w \in W \), i.e., \( h \in (U^0_w) \) as claimed. \( \square \)

Now the following lemma justifies the range of \( \gamma - \rho \circ \pi \) as defined in (2.14).

**Lemma A.4** The Harish-Chandra homomorphism \( \gamma - \rho \circ \pi \) maps \( Z(U) \) to \( (U^0_e)^W \).

**Proof** Take an arbitrary \( u \in Z(U) \), and write

\[
\gamma - \rho \circ \pi(u) = \sum_{\mu \in P} a_{\mu} K_{\mu}.
\]

By Lemma A.3, \( \gamma - \rho \circ \pi(u) \in (U^0)^W \). Thus, \( a_{w_{\mu}} = a_{\mu} \) for all \( w \in W \) and \( \mu \in P \).

We have to show that \( a_{\mu} \neq 0 \) only if \( \mu \in 2P \).

Recall from (2.10) that there is an automorphism \( \psi_{\sigma} \) of \( U \) associated to each group character \( \sigma \) as defined in (2.9). It can be easily verified that \( \psi_{\sigma} \) commutes with both \( \pi \) and \( \gamma - \rho \). Therefore, we have

\[
\gamma - \rho \circ \pi(\psi_{\sigma}(u)) = \psi_{\sigma}(\sum_{\mu} a_{\mu} K_{\mu}) = \sum_{\mu} a_{\mu} \sigma(\mu) K_{\mu},
\]

which lands in \( (U^0)^W \) since \( \psi_{\sigma}(u) \) is central. It follows that

\[
a_{\mu} \sigma(\mu) = a_{w_{\mu}} \sigma(w_{\mu}) = a_{w_{\mu}} \sigma(w_{\mu}) \quad \forall w \in W, \mu \in P.
\]
Since we have assumed that $a_\mu \neq 0$, this in particular implies $1 = \sigma(\mu - s_\alpha, \mu)$ for $1 \leq i \leq n$. Fixing a group character $\sigma : P \to \mathbb{C}^\times$ such that $\sigma(\alpha_i) = -1$ for all $i$, we have

$$\sigma(\mu - s_\alpha, \mu) = \sigma((\mu, \alpha_i^\vee)\alpha_i) = (-1)^{(\mu, \alpha_i^\vee)} = 1.$$ 

This implies that $\mu_i^\vee$ is even for $1 \leq i \leq n$, i.e., $\mu \in 2P$. \hfill \qed

Now we prove the quantum Harish-Chandra isomorphism following [11, Chapter 6].

### A.1. Proof of the isomorphism

By Lemma A.2, the restriction of $\gamma_\rho \circ \pi$ to $Z(U)$ is injective. Therefore, it suffices to show surjectivity of the map (2.14) in order to prove Theorem 2.5. We do this by showing that each basis element of the invariant subalgebra $(U_0^\alpha)^W$ has a pre-image in $Z(U)$.

We will follow the strategy of [11] to prove the surjectivity. This relies in an essential way on a non-degenerate bilinear form on $U$, which can be constructed in exactly the same way as in [11, Chapter 6]. However, the explicit construction is rather involved and technical. We will merely describe the main properties of the form here, and refer to op. cit. for details.

**Lemma A.5** [11, Chapter 6] There exists a unique bilinear form $(, ) : U^{\leq 0} \times U^{\geq 0} \to \mathbb{F}$ with the following properties:

\[
\begin{align*}
(K_\lambda, K_\mu) & = q^{-\langle \lambda, \mu \rangle}, & (K_\lambda, E_i) & = 0, \\
(F_i, E_j) & = -\delta_{ij}(q_i - q_i^{-1})^{-1}, & (F_i, K_\lambda) & = 0, \\
(x, y_1, y_2) & = (\Delta(x), y_2 \otimes y_1), & (x_1, x_2, y) & = (x_1 \otimes x_2, \Delta(y)),
\end{align*}
\]

for all $x, x_1, x_2 \in U^{\leq 0}$, $y, y_1, y_2 \in U^{\geq 0}$, $\lambda, \mu \in P$ and $1 \leq i, j \leq n$.

**Proposition A.6** [11, Chapter 6] Let $\lambda, \eta \in P$, $\mu, \nu \in Q^+$.  

1. $(x K_\lambda, y K_\eta) = q^{-\langle \lambda, \eta \rangle}(x, y)$ for any $x \in U^-$ and $y \in U^+$.
2. $(U^-_\nu, U^+_\mu) = 0$ for any $\mu \neq \nu$.
3. The restriction $(, )|_{U^-_\mu \times U^+_\mu}$ is non-degenerate.

We now define a bilinear form on $U$ by using Lemma A.5. Recall that $U^+$ (resp. $U^-$) is $Q^+$-graded (resp. $Q^-$-graded) vector space with respect to the $U^0$-action given in (2.11), and the multiplication induces an isomorphism $U^- \otimes U^0 \otimes U^+ \cong U$. Since $K_\mu$ is a unit in $U$, we can rearrange this isomorphism into

$$\bigoplus_{\mu, \nu \in Q^+} U^-_\mu K_\mu \otimes U^0 \otimes U^+_\nu \cong U.$$
Now the bilinear form $\langle \cdot , \cdot \rangle : U \times U \to \mathbb{F}$ is defined on the graded components by

$$\langle yK_{\lambda}x, y'K_{\mu}x' \rangle := (y', x')(y, x')q^{(\lambda, \eta)}(q^{1/2}) - (\lambda, \eta)$$  \hspace{1cm} (A.3)

for all $x \in U^+, x' \in U^{+\prime}, y \in U^{-\prime},$ and $y' \in U^{-\mu},$ with $\lambda, \eta \in P, \mu, \mu', \nu, \nu' \in Q^+.$

It follows immediately from part (2) of Proposition A.6 that

$$\langle U^{-\nu}U^0U^{-\mu}, U^{-\nu}U^0U^{+\mu} \rangle = 0, \quad \text{unless} \quad \mu = \nu', \nu = \mu'.$$

The following proposition gives two significant properties for the bilinear form (A.3), which will be used in the proof of surjectivity of the Harish-Chandra homomorphism.

**Proposition A.7** [11, Chapter 6]

1. If $\langle v, u \rangle = 0$ for all $v \in U,$ then $u = 0$;
2. $\langle \text{ad}(x)u, v \rangle = \langle u, \text{ad}(S(x))v \rangle$ for all $x, u, v \in U.$

Let $M$ be a finite-dimensional $U$-module. For any $m \in M$ and $f \in M^*,$ let $c_{f,m} \in U^*$ be the linear form with $c_{f,m}(v) = f(vm)$ for any $v \in U.$ The following lemma follows from the non-degeneracy of the form $\langle \cdot , \cdot \rangle$ [11, Chapter 6.22].

**Lemma A.8** Retain notation above. There exists a unique element $u \in U,$ depending on $f \in M^*, m \in M$ such that

$$c_{f,m}(v) = \langle v, u \rangle, \quad \forall v \in U.$$

This leads to the following key lemma.

**Lemma A.9** Fix $\lambda \in P^+,$ and let $V(\lambda)$ be the finite-dimensional simple $U$-module with highest weight $\lambda.$ Then there exists a unique central element $z_{\lambda} \in Z(U)$ such that

$$\langle u, z_\lambda \rangle = \text{Tr}(uK^{-1}_{2\rho}), \quad \forall u \in U,$$  \hspace{1cm} (A.4)

where $\text{Tr}(x)$ denotes the trace of $x \in U$ over $V(\lambda).$

**Proof** Let $m_1, m_2, \ldots, m_r$ be a basis of $V(\lambda)$ and $f_1, f_2, \ldots, f_r$ the dual basis of $V(\lambda)^*,$ i.e., $f_i(m_j) = \delta_{ij}.$ Then the trace of $uK^{-1}_{2\rho}$ over $V(\lambda)$ is equal to $\sum_{i=1}^r c_{f_i,K^{-1}_{2\rho}m_i}(u).$ By Lemma A.8, there is a unique $v_i \in U$ such that $\langle u, v_i \rangle = c_{f_i,K^{-1}_{2\rho}m_i}(u)$ for all $u \in U.$ Let $z_\lambda = v_1 + v_2 + \cdots + v_r,$ then we have $\langle u, z_\lambda \rangle = \sum_{i=1}^r c_{f_i,K^{-1}_{2\rho}m_i}(u),$ which is the trace of $uK^{-1}_{2\rho}$ over $V(\lambda)$.

It remains to show that $z_\lambda$ is central in $U,$ which is equivalent to showing that $\text{ad}(u)z_\lambda = \varepsilon(u)z_\lambda$ for any $u \in U.$ Then the linear representation $\xi_\lambda : U \to \text{End}(V(\lambda))$ is a homomorphism of $U$-modules, where $U$ acts on itself by the adjoint action, that is, $u.v := \text{ad}(u)v$ for any $u, v \in U.$ The quantum trace $\text{Tr}_q : \text{End}(V(\lambda)) \to \mathbb{F}$ that takes

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\[ \varphi \mapsto \operatorname{Tr}(\varphi \circ K^{-1}_{2\rho}) \] is also a U-module homomorphism, where \( \mathbb{F} \) is the trivial module such that \( u.a = \varepsilon(u)a \) for any \( a \in \mathbb{F} \). Let \( \theta = \operatorname{Tr}_q \circ \varsigma_\lambda \). Then by definition

\[ \theta(u) = \operatorname{Tr}_q \circ \varsigma_\lambda(u) = \operatorname{Tr}(u K^{-1}_{2\rho}) = \langle u, z_\lambda \rangle, \quad \forall u \in U. \]

Since \( \theta \) is a U-module homomorphism, we have

\[ \theta(u.v) = u.\theta(v) = \varepsilon(u)\theta(v) = \varepsilon(u)\langle v, z_\lambda \rangle, \]

On the other hand, using the adjoint structure of \( U \) we have

\[ \theta(u.v) = \operatorname{Tr}_q \circ \varsigma_\lambda(\operatorname{ad}(u)v) = \langle \operatorname{ad}(u)v, z_\lambda \rangle = \langle v, \operatorname{ad}(S(u))z_\lambda \rangle, \]

where the last equation follows from part(2) of Proposition A.7. Since the bilinear form is non-degenerate, we have \( \operatorname{ad}(S(u))z_\lambda = \varepsilon(u)z_\lambda \) for all \( u \in U \). Recalling that the antipode \( S \) satisfies \( \varepsilon \circ S = \varepsilon \), we obtain \( \operatorname{ad}(u)z_\lambda = \varepsilon(u)z_\lambda \) for all \( u \in U \). Therefore, \( z_\lambda \in Z(U) \). \( \square \)

Lemma A.10 Let \( \lambda \in P^+ \), and \( V(\lambda) \) the finite-dimensional simple module of \( U \). Let \( z_\lambda \in Z(U) \) be the central element defined in (A.4). Then

\[ \gamma_{-\rho} \circ \pi(z_\lambda) = \sum_{\eta \in \Pi(\lambda)} m_\lambda(\eta) K_{-2\eta}, \]

where \( \Pi(\lambda) \) is the set of weights of \( V(\lambda) \) and \( m_\lambda(\eta) \) denotes the dimension of the weight space \( V(\lambda)_\eta \).

Proof Since \( z_\lambda \) is central and \( Z(U) \subseteq U_0 = U^0 \oplus \bigoplus_{\nu > 0} U^-\nu U^+\nu \), we may write

\[ z_\lambda = z_{\lambda,0} + \sum_{\nu > 0} z_{\lambda,\nu}, \quad \text{with } z_{\lambda,0} \in U^0, \quad z_{\lambda,\nu} \in U^-\nu U^+\nu. \]

It follows that \( \pi(z_\lambda) = z_{\lambda,0} \). By (A.3), we have

\[ \langle K_\mu, z_\lambda \rangle = \langle K_\mu, z_{\lambda,0} \rangle = \langle K_\mu, \pi(z_\lambda) \rangle, \quad \forall \mu \in P. \quad (A.5) \]

On the other hand, using Lemma A.9 we obtain

\[ \langle K_\mu, z_\lambda \rangle = \operatorname{Tr}(K_{\mu-2\rho}) = \sum_{\eta \in \Pi(\lambda)} m_\lambda(\eta)q^{(\eta,\mu-2\rho)} \]

\[ = \sum_{\eta \in \Pi(\lambda)} m_\lambda(\eta)q^{-(2\eta,\rho)} q^{(\mu,\eta)} \quad (A.6) \]

\[ = \sum_{\eta \in \Pi(\lambda)} m_\lambda(\eta)q^{-(2\eta,\rho)} \langle K_\mu, K_{-2\eta} \rangle. \]
Comparing (A.5) and (A.6) and using the non-degeneracy of the bilinear form, we have

$$\gamma_{-\rho} \circ \pi(z_\lambda) = \sum_{\eta \in \Pi(\lambda)} m_\lambda(\eta) K_{-2\eta}. $$

This completes the proof. \(\square\)

Now we are ready to prove Theorem 2.5.

**Proof of Theorem 2.5** We know that \(\gamma_{-\rho} \circ \pi\) is injective from Lemma A.2. It remains to show that \(\gamma_{-\rho} \circ \pi\) is surjective. By Lemma 4.6, the elements \(av(-\mu) = \sum_{\eta \in W_\mu} K_{-2\eta}\) with \(\mu \in P^+\) form a basis for \((U^0_{ev})^W\), since each group orbit \(W_\mu\) in \(P\) contains exactly one \(-\mu\) such that \(\mu\) is dominant.

We use induction on \(\mu\) to show that the basis elements \(av(-\mu)\) are in the image of \(\gamma_{-\rho} \circ \pi\). Endow \(P\) with the standard partial order such that \(\mu \leq \lambda\) if and only if \(\lambda - \mu\) is a nonnegative integral linear combination of positive roots. For the base case \(\nu = 0\), we have \(av(0) = 1 = \gamma_{-\rho} \circ \pi(1)\). For any \(\lambda \in P^+\), we may apply Lemma A.9 and then obtain the element \(z_\lambda \in Z(U)\), which by Lemma A.10 has the image

$$\gamma_{-\rho} \circ \pi(z_\lambda) = \sum_{\eta \in \Pi(\lambda)} m_\lambda(\eta) K_{-2\eta} = av(-\lambda) + \sum_{\mu < \lambda, \mu \in P^+} m_\lambda(\mu) av(-\mu),$$

where the second equality follows from the fact that \(m_\lambda(\lambda) = m_\lambda(w\lambda) = 1\) for any \(w \in W\). The left-hand side of the above equation belongs to \(\gamma_{-\rho} \circ \pi(Z(U))\). By induction hypothesis, all \(av(-\mu)\) with \(\mu < \lambda\) are in the image of \(\gamma_{-\rho} \circ \pi\), hence so is \(av(-\lambda)\). \(\square\)

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