Emergent diffeomorphism invariance in a discrete loop quantum gravity model

Rodolfo Gambini, Jorge Pullin

1. Instituto de Física, Facultad de Ciencias, Iguá 4225, esq. Mataojo, Montevideo, Uruguay.
2. Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001

Several approaches to the dynamics of loop quantum gravity involve discretizing the equations of motion. The resulting discrete theories are known to be problematic since the first class algebra of constraints of the continuum theory becomes second class upon discretization. If one treats the second class constraints properly, the resulting theories have very different dynamics and number of degrees of freedom than those of the continuum theory. It is therefore questionable how these theories could be considered a starting point for quantization and the definition of a continuum theory through a continuum limit. We show explicitly in a model that the uniform discretizations approach to the quantization of constrained systems overcomes these difficulties. We consider here a simple diffeomorphism invariant one dimensional model and complete the quantization using uniform discretizations. The model can be viewed as a spherically symmetric reduction of the well known Husain–Kuchař model of diffeomorphism invariant theory. We show that the correct quantum continuum limit can be satisfactorily constructed for this model. This opens the possibility of treating 1+1 dimensional dynamical situations of great interest in quantum gravity taking into account the full dynamics of the theory and preserving the space-time covariance at a quantum level.

I. INTRODUCTION

Lattice techniques have proved remarkably useful in the quantization of usual gauge theories. This raised the hope that they may also prove useful in the quantization of gravity. A major difference however is that most theories of gravity of interest are invariant under diffeomorphisms and the introduction of a discrete structure breaks diffeomorphism invariance. One of the appealing features of lattice gauge theories is therefore lost in this case, one breaks the symmetry of the theory of interest. The situation gets further compounded in the case of canonical general relativity, since there one also breaks four dimensional covariance into a 3+1 dimensional split. Spatial diffeomorphisms get implemented via a constraint that has a natural geometrical action and the usual algebra of diffeomorphisms is implemented via the constraint algebra. But the remaining space-time diffeomorphism gets implemented through the complicated Hamiltonian constraint, that has a challenging space-time diffeomorphism algebra with spatial diffeomorphisms. In particular the algebra of constraints has structure functions. If we call $C(\vec{N})$ the diffeomorphism constraint smeared by a test vector field (shift) $\vec{N}$ and $H(N)$ the Hamiltonian constraint smeared by a scalar lapse $N$, the constraint algebra is,

\[ \{ C(\vec{N}), C(\vec{M}) \} = C([\vec{N}, \vec{M}]) \]  
\[ \{ C(\vec{N}), H(M) \} = H(\mathcal{L}_\vec{N} M) \]  
\[ \{ H(N), H(M) \} = C(\vec{K}(q)), \]

where the vector $K^a = q^{ab}(N\partial_b M - M\partial_b N)$ and $q^{ab}$ is the spatial metric. The last Poisson bracket therefore involves structure functions depending on the canonical variables on the right hand side.

The algebra of constraints poses important complications in the context of loop quantum gravity when one wishes to implement it as an operator algebra at a quantum level (see [1] for a lengthier discussion). In particular, if one chooses spin network states with the usual Ashtekar-Lewandowski [2] measure, they form a non-separable Hilbert space. In it, diffeomorphisms are not implemented in a weakly continuous fashion, i.e. finite diffeomorphisms can be represented but infinitesimal ones cannot. This implies that in loop quantum gravity one treats very asymmetrically the spatial and temporal diffeomorphisms. Whereas invariance under spatial diffeomorphisms is implemented via a group averaging procedure [3], invariance under the remaining space-time diffeomorphisms is to be implemented by solving a quantum operatorial equation corresponding to the Hamiltonian constraint. Since the Poisson bracket of two Hamiltonian constraints involves the infinitesimal generator of diffeomorphisms, which is not well defined as a quantum operator, one cannot expect to implement the Poisson algebra at an operatorial level in the quantum theory, at least in the kinematical Hilbert space.

A symmetric treatment of the diffeomorphism and Hamiltonian constraints requires to develop a technique that allows to implement the generators of spatial diffeomorphisms as operators in the loop representation. One could attempt to treat the diffeomorphism and Hamiltonian constraints on the same footing, for instance by lattice regularizing them. Unfortunately, such discretized versions of the constraints are not first class. If one treats them properly
with the Dirac procedure, the resulting theory is vastly different in symmetries and even in the number of degrees of freedom from what one expects to have in the continuum theory. Therefore there is little chance that one could define a continuum theory as a suitable limit of the constructed lattice theories. These problems have led to the consideration of extensions of the Dirac procedure that could better accommodate this particular problem with the constraint algebra. One such approach is the “master constraint” programme of Thiemann and collaborators [4]. Another approach that we have been studying in the last few years are the “uniform discretizations” [5]. Both approaches have some elements in common.

Uniform discretizations are discrete versions of a constrained theory in which the discretized form of the constraints are quantities whose values are under control throughout the system’s evolution. Notice that this would not be the case, for instance, if one simply takes a constrained theory and discretizes it. Initial data on which the discrete version of the constraints vanishes will evolve into data with non-vanishing values of the discrete constraints, without any control on the final value. This situation is well known, for instance, in numerical relativity. Uniform discretizations are designed in such a way that the discrete constraints are kept under control upon evolution and that one can take appropriate limits in the initial data such that one can satisfy the constraints to an arbitrary (and controlled) degree of accuracy. This therefore guarantees the existence of a well defined continuum limit at the level of the classical theory. It has been shown [6] that the uniform discretization technique is classically equivalent to the Dirac procedure when the constraints are first class. For second class constraints, like the ones that arise when one discretizes continuum systems with first class constraints the uniform discretization technique is radically different from the Dirac procedure, yielding a dynamical evolution that recovers in the continuum limit the continuum theory one started with.

Although the existence of a continuum limit is generically guaranteed at a classical level, it is not obvious that it is at the quantum level. It is known [3] that there are models in which the continuum limit cannot be achieved and one is left with a non-zero minimum value of the expectation value of the sum squared of the constraints. It is therefore of interest to show that in examples of growing complexity and of increasing similarity to general relativity one can indeed define a continuum quantum theory with the desired symmetries by applying the uniform discretization procedure. The purpose of this paper is to discuss one such model. We will consider the quantization via uniform discretizations of a 1 + 1 dimensional model with diffeomorphism symmetry and we will show that the symmetry is recovered at the quantum level correctly. This raises the hopes of having a theory where all the constraints are treated on an equal footing.

The organization of this paper is as follows. In section II we discuss the model we will consider. In section III we discretize the model. In section IV we review the uniform discretization procedure and how it departs from the Dirac traditional approach. Section VI discusses the quantization using uniform discretizations and how one recovers the correct continuum limit. We conclude with a discussion.

II. THE MODEL

We would like to construct a model by considering spherically symmetric gravity and ignoring the Hamiltonian constraint. This is analogous to building a “Husain–Kuchar” [7] version of spherically symmetric gravity. It is known that these models correspond to degenerate space-times when translated in terms of the metric variables.

We refer the reader to our previous work on spherically symmetric gravity [8] for the setup of the model in terms of Ashtekar’s new variables. Just as a recap, the model has two canonical pairs \( K_x, E^x \) and \( K_\varphi, E^\varphi \). The relation to the more traditional metric canonical variables is,

\[
g_{xx} = \frac{(E^\varphi)^2}{|E^x|}, \quad g_{\theta \theta} = |E^x|, \quad (4)
\]

\[
K_{xx} = -\text{sign}(E^x)(E^\varphi)^2 \sqrt{|E^x|} K_x, \quad K_{\theta \theta} = -\sqrt{|E^x|} K_\varphi \quad (5)
\]

and we have set the Immirzi parameter to one for simplicity, since it does not play a role in this analysis.

The Lagrangian for spherically symmetric gravity ignoring the Hamiltonian constraint is,

\[
L = \int dx E^x \dot{K}_x + E^\varphi \dot{K}_\varphi + N ((E^x)' K_x - E^x (K_\varphi)') \quad (6)
\]

with \( N \) a Lagrange multiplier (the radial component of the shift vector). The equations of motion are

\[
\dot{K}_x - (NK_x)' = 0, \quad (7)
\]

\[
\dot{E}_x - N (E^x)' = 0, \quad (8)
\]
\[
\begin{align*}
\dot{K}_\varphi - NK'_\varphi &= 0, \\
\dot{E}^\varphi - (NE^\varphi)' &= 0.
\end{align*}
\] (9) (10)

The theory has one constraint, which is the remaining diffeomorphism constraint in the radial \((x)\) direction, \(\phi = -(E'^x)K_x + E^xK'_\varphi\), which we will write smeared as \(\phi(N) = \int dxN\phi\). The constraint generates diffeomorphisms of the fields, with \(K_\varphi\) and \(E^x\) behaving as scalars and \(K_x\) and \(E^\varphi\) as densities of weight one, \(\delta K_\varphi = \{K_\varphi, \phi(N)\} = NK'_\varphi\), \(\delta K_x = \{K_x, \phi(N)\} = (NK_x)'\), \(\delta E^\varphi = \{E^\varphi, \phi(N)\} = (NE^\varphi)'\), \(\delta E^x = \{E^x, \phi(N)\} = N(E^x)'\). (11) (12) (13) (14)

The constraint has the usual algebra of diffeomorphisms, \(\{\phi(N), \phi(M)\} = \phi\left(NM' - MN'\right)\). (15)

Observables are integrals of densities of weight one constructed with the fields, for example, \(O = \int dx f(E^x, K_\varphi)K_x\) with \(f\) a function. One then has
\[
\{O, \phi(N)\} = \int dx \left[ \frac{\partial f}{\partial E^x} N(E^x)' + \frac{\partial f}{\partial K_\varphi} NK_\varphi' + (NK_x)' f \right] = \int dx \partial_x (fNK_x) = 0,
\] (16)

if one considers a compact spatial manifold, \(S^1\), which we will do throughout this paper. (This may not make a lot of sense if one is thinking of the model as a reduction of \(3 + 1\) spherical symmetry, but we are just avoiding including boundary terms, which are straightforward to treat in the spherical case, see [8], in order to simplify the discussion of diffeomorphism invariance).

### III. DISCRETIZATION

We now proceed to discretize the model. The spatial direction \(x\) is discretized into points \(x_i\) such that \(x_{i+1} - x_i = \epsilon_i\) and the distances are smaller than a bound \(d(\epsilon_i) < d_*\) when measured in some fiducial metric. To simplify notation, from now on we will assume the points are equally spaced and drop the suffix \(i\) on \(\epsilon\), but the analysis can be straightforwardly extended to the case with variable \(\epsilon_i\). The variables of the model become \(K_{x,i} = K_x(x_i)\), \(K_{\varphi,i} = K_\varphi(x_i)\) and \(E^x_i = \epsilon E^x(x_i)\) and \(E^\varphi_i = \epsilon E^\varphi(x_i)\). The constraint is,
\[
\phi_i = E^\varphi_i \left( K_{\varphi,i+1} - K_{\varphi,i} \right) - K_{x,i} \left( E^x_{i+1} - E^x_i \right).
\] (17)

The constraint algebra is not first class, i.e.,
\[
\{\phi_i, \phi_j\} = -E^\varphi_{i-1} \left( K_{\varphi,i+1} - K_{\varphi,i} \right) \delta_{i,j+1} + E^\varphi_{j-1} \left( K_{\varphi,j+1} - K_{\varphi,j} \right) \delta_{j,i+1}
\]
\[
K_{x,i-1} \left( E^x_{i+1} - E^x_i \right) \delta_{i,j+1} - K_{x,j-1} \left( E^x_{j+1} - E^x_j \right) \delta_{j,i+1}
\] (18)

which does not reproduce the constraint. What one has is a “classical anomaly” of the form \((E^x_{i+1} - E^x_i) (K_{\varphi,i} - K_{\varphi,i-1}) - (E^x_{i+1} - E^x_i - E^x_{i+1} - E^x_i) (K_{x,i} - K_{x,i-1})\). These terms would tend to zero if one takes the separation \(\epsilon\) to zero and the variables behave continuously in such a limit.

So if one were to simply quantize the discrete model, one would run into trouble since one would be quantizing a classical theory with second class constraints. We will expand more on the problems one faces in the next section. In this paper we would like to show that in spite of this problem of the classical theory, which implies that the discrete theory loses diffeomorphism invariance, if one follows the uniform discretization approach to quantization the diffeomorphism invariance is recovered in the limit \(\epsilon \to 0\) both at the classical and quantum level.

In the uniform discretization approach one constructs a “master constraint” \(\mathbb{H}\) by considering the sum of the discretized constraints squared. One then promotes the resulting quantity to a quantum operator and seeks for the eigenstates of \(\mathbb{H}\) with minimum eigenvalue. In the full theory the quantity \(\mathbb{H}\) would be constructed from the diffeomorphism constraints \(\phi_a\) as,
\[
\mathbb{H} = \frac{1}{2} \int dx \phi_a \phi_b \frac{g^{ab}}{\sqrt{g}},
\] (19)
which motivates in our example to choose,

$$\mathcal{H} = \frac{1}{2} \int dx \phi \frac{\sqrt{E^\epsilon}}{(E^\epsilon)^4},$$

(20)

or, in the discretized theory as,

$$\mathcal{H}' = \frac{1}{2} \sum_{i=0}^{N} \phi_i \phi_i \frac{\sqrt{E_i^\epsilon}}{(E_i^\epsilon)^{3/2}}.$$

(21)

To understand better how to promote these quantities to quantum operators, it is best to start with the constraint $E^\epsilon$ that although the operator as we will see later. When one is to promote these quantities to quantum operators, one needs to remember

which again would reproduce the constraint in the continuum limit. Let us rewrite it in terms of the discrete variables,

$$E^\epsilon \ni \frac{1}{2} \sum_{i=0}^{N} \phi_i \phi_i \frac{\sqrt{E_i^\epsilon}}{(E_i^\epsilon)^{3/2}}.$$  

which would reproduce the constraint $\phi(N) = \lim_{\epsilon \to 0} \phi^\epsilon(N)$ though we see that the explicit dependence on $\epsilon$ drops out. We have chosen to regularize $E^\epsilon$ at the midpoint in order to simplify the action of the resulting quantum operator as we will see later. When one is to promote these quantities to quantum operators, one needs to remember that although the $E$ variables promote readily to quantum operators in the loop representation, the $K$’s need to be written in exponentiated form. To this aim, we write, classically,

$$\phi^\epsilon(N) = \sum_{j=0}^{N} N(x_j) \left\{ \frac{[E^\epsilon(x_{j+1}) - E^\epsilon(x_j)]K_x(x_j) + \frac{1}{2}[E^\epsilon(x_j) + E^\epsilon(x_{j+1})](K_{\phi(x_{j+1})} - K_{\phi(x_j)})}{\epsilon} \right\},$$

(22)

which again would reproduce the constraint in the continuum limit. Let us rewrite it in terms of the discrete variables,

$$\phi^\epsilon(N) = \sum_{j=0}^{N} \frac{N(x_j)}{2i\epsilon} \{ \exp \left[ \frac{1}{i} \left( E^\epsilon_{j+1} - E^\epsilon_j \right) K_x + \left( E^\epsilon_j + E^\epsilon_{j+1} \right) (K_{\phi_{j+1}} - K_{\phi_j}) \right] - 1 \}.$$  

(23)

For later use, it is convenient to rewrite $\phi^\epsilon_j = (D_j - 1)/(2i\epsilon)$ and then one has that,

$$H^\epsilon = \sum_{j=0}^{N} (D_j - 1)(D_j - 1)^* \epsilon^{-1/2} \frac{\sqrt{E_j^\epsilon}}{(E_j^\epsilon)^{3/2}}.$$  

(25)

We dropped the $\epsilon$ in $D$ since it does not explicitly depend on it, but it does through the dependence on $E^\epsilon$ and an irrelevant global factor of $1/8$ to simplify future expressions.

**IV. UNIFORM DISCRETIZATIONS**

Before quantizing, we will study the classical theory using uniform discretizations and we will verify that one gets in the continuum limit a theory with diffeomorphism constraints that are first class. The continuum theory can be treated with the Dirac technique and has first class constraints that generate diffeomorphisms on the dynamical variables. However, the discrete theory, when treated with the Dirac technique, has second class constraints and does not have the gauge invariances of the continuum theory. The number of degrees of freedom changes and the continuum limit generically does not recover the theory one started with.

As mentioned before, it has been shown [3] that the uniform discretization technique is equivalent to the Dirac procedure when the constraints are first class. For second class constraints, like the ones that appear when one discretizes continuum systems with first class constraints the uniform discretization technique is radically different from the Dirac procedure, yielding a dynamical evolution that recovers in the continuum limit the continuum theory one started with.

Let us review how this works. We start with a classical canonical system with $N$ configuration variables, parameterized by a continuous parameter $\alpha$ such that $\alpha \to 0$ is the “continuum limit”. We will assume the theory in the
continuum has $M$ constraints $\phi_j = \lim_{\alpha \to 0} \phi_j^\alpha$. In the discrete theory we will assume the constraints generically fail to be first class,

$$\{\phi_j^\alpha, \phi_k^\alpha\} = \sum_{m=1}^M C_{jk}^m \phi_m^\alpha + A_{jk}^\alpha,$$  \hspace{1cm} (26)

where the failure is quantified by $A_{jk}^\alpha$. We assume that in the continuum limit one has $\lim_{\alpha \to 0} A_{jk}^\alpha = 0$ and that the quantities $C_{jk}^m$ become in the limit the structure functions of the (first class) constraint algebra of the continuum theory $C_{jk}^{m^C} = \lim_{\alpha \to 0} C_{jk}^m$, so that,

$$\{\phi_j, \phi_k\} = \sum_{m=1}^M C_{jk}^m \phi_m.$$  \hspace{1cm} (27)

If one were to insist on treating the above discrete theory using the Dirac procedure, that is, taking the constraints $\phi_j^\alpha = 0$ and a total Hamiltonian $H_T = \sum_{j=1}^M C_j \phi_j^\alpha$ with $C_j$ functions of the canonical variables, one immediately finds restrictions on the $C_j$'s of the form $\sum_{j=1}^M C_j A_{jk}^\alpha = 0$ in order to preserve the constraints upon evolution. Only in the continuum $\alpha \to 0$ limit are the $C_j$ free functions and one has in the theory $2N - 2M$ observables. Notice that away from the continuum limit the number of observables is generically larger and could even reach $2N$ if the matrix $A_{jk}^\alpha$ is invertible. Therefore one cannot view the theory in the $\alpha \to 0$ limit as a limit of the theories for finite values of $\alpha$, since they do not even have the same number of observables and have a completely different evolution.

The uniform discretizations, on the other hand, lead to discrete theories that have the same number of observables and an evolution resembling those of the continuum theory. One can then claim that the discrete theories approximate the continuum theory and the latter arises as the continuum limit of them.

The treatment of the system in question would start with the construction of the “master constraint”

$$H^\alpha = \frac{1}{2} \sum_{i} (\phi_j^\alpha)^2$$  \hspace{1cm} (28)

and defining a discrete time evolution through $H$. In particular, this implies a discrete time evolution from instant $n$ to $n + 1$ for the constraints of the form,

$$\phi_j^\alpha(n + 1) = \phi_j^\alpha(n) + \{\phi_j^\alpha(n), H^\alpha\} + \frac{1}{2} \{\{\phi_j^\alpha(n), H^\alpha\}, H^\alpha\} + \ldots$$  \hspace{1cm} (29)

$$= \phi_j^\alpha(n) + \sum_{i,k=1}^M C_{ji}^k \phi_k^\alpha(n) \phi_i^\alpha(n) + \sum_{i=1}^M A_{ji}^\alpha \phi_i^\alpha(n) + \ldots$$  \hspace{1cm} (30)

This evolution implies that $H^\alpha$ is a constant of the motion, which for convenience we denote via a parameter $\delta$ such that $H^\alpha = \delta^2/2$. The preservation upon evolution of $H^\alpha$ implies that the constraints remain bounded $|\phi_j^\alpha| \leq \delta$.

If one now divides by $\delta$ and defines the quantities $\lambda^\alpha_i \equiv \phi_i^\alpha / \delta$ one can rewrite $\delta^2$ as,

$$\frac{\phi_j^\alpha(n + 1) - \phi_j^\alpha(n)}{\delta} = \sum_{i,j=1}^M C_{ji}^k \phi_k^\alpha(n) \lambda_i^\alpha(n) + \sum_{i,j=1}^M A_{ji}^\alpha \lambda_i^\alpha(n) + \ldots$$  \hspace{1cm} (31)

Notice that the $\lambda_i^\alpha$ remain finite when one takes the limits $\delta \to 0$ and $\alpha \to 0$.

If one now considers the limit of small $\delta$'s, one notes that the first term on the right is of order $\delta$, the second one goes to zero with $\alpha \to 0$, at least as $\alpha$ and the rest of the terms are of higher orders in $\delta, \alpha$. If one identifies with a continuum variable $\tau$ such that $\tau = n \delta + \tau_0$, then $\phi_j^\alpha(\tau) \equiv \phi_j^\alpha(n)$ and $\phi_j^\alpha(\tau + \delta) \equiv \phi_j^\alpha(n + 1)$ one can take the limits $\alpha \to 0$ and $\delta \to 0$, irrespective of the order of the limits one gets that the evolution equations $\delta^2$ for the constraints become those of the continuum theory, i.e.,

$$\dot{\phi}_j \equiv \lim_{\alpha, \delta \to 0} \frac{\phi_j^\alpha(\tau + \delta) - \phi_j^\alpha(\tau)}{\delta} = \sum_{i,j=1}^M C_{ji}^k \phi_k^\alpha \lambda_i$$  \hspace{1cm} (32)

with $\lambda_i$ become the (freely specifiable) Lagrange multipliers of the continuum theory. At this point the reader may be puzzled, since the $\lambda$'s are defined as limits of those of the discrete theory and therefore do not appear to be free.
However, one has to recall that the \( \lambda \)'s in the discrete theory are determined by the values of the constraints evaluated on the initial data, and these can be chosen arbitrarily by modifying the initial data.

If one considers the limit \( \delta \to 0 \) for a finite value of \( \alpha \) (“continuous in time, discrete in space”) and considers the evolution of a function of phase space \( O \), one has that,

\[
\dot{O} = \{O, H^\alpha\} = \{O, \phi_1^\alpha\} \lambda_j^\alpha + \sum_{j=1}^M \{O, \phi_1^\alpha\} A_{ij}^\alpha \lambda_j^\alpha + \sum_{j,k=1}^M \{O, \phi_1^\alpha\} A_{ij}^\alpha A_{jk}^\alpha \lambda_k^\alpha + \ldots
\]  

(33)

The necessary and sufficient condition for \( O \) to be a constant of the motion (that is, \( \dot{O} = 0 \)) is that

\[
\{O, \phi_1^\alpha\} = \sum_{j=1}^M C_{ij} \phi_j^\alpha + B_i^\alpha
\]  

(34)

with \( B_i^\alpha \) a vector, perhaps vanishing, that is annihilated by the matrix,

\[
A_{ij}^\alpha = \delta_{ij} + A_{ij}^\alpha + \sum_{k=1}^M A_{ik}^\alpha A_{kj}^\alpha + \ldots + \sum_{k_1=1,\ldots,k_s=1}^M A_{i_k}^\alpha \cdots A_{k_j}^\alpha + \ldots
\]  

(35)

Up to now we have assumed \( \lambda_j^\alpha \) arbitrary and not necessarily satisfying that \( \sum_{j=1}^N A_{ij}^\alpha \lambda_j^\alpha = 0 \). It is clear that \( \lim_{\alpha \to 0} A_{ij}^\alpha = \delta_{ij} \) and therefore \( \lim_{\alpha \to 0} B_i^\alpha = 0 \) which implies that conserved quantities in the discrete theory yield in the limit \( \alpha \to 0 \) the observables of the continuum theory.

Since the \( \lambda_j^\alpha \)'s are free the theory with continuous time is not the one that would result naively from applying the Dirac procedure since in the latter the Lagrange multipliers are restricted by \( \sum_{j=1}^M A_{ij}^\alpha \lambda_j^\alpha = 0 \) and therefore the theory admits more observables than the \( 2N - 2M \) of the continuum theory. That is, if one takes the “continuum in time” limit first, the discrete theory has a dynamics that differs from the usual one unless \( A_{ij}^\alpha \phi_j^\alpha (n_i) = 0 \) and one is really treating two different theories.

At this point it would be good to clarify a bit the notation. The above discussion has been for a mechanical system with \( M \) configuration degrees of freedom. When one discretizes a field theory with \( M \) configuration degrees of freedom on a lattice with \( N \) points one ends up with a mechanical system that has \( M \times N \) degrees of freedom. An example of such a system would be the diffeomorphism constraints of general relativity in \( 3 + 1 \) dimensions when discretized on a uniform lattice of spacing \( \alpha \). Of course, it is not clear at this point if such a system could be completely treated with our technique up to the last consequences, we just mention it here as an example of the type of system one would like to treat. The above discussion extends immediately to systems of this kind, only the bookkeeping has to be improved a bit. If we consider a parameter \( \alpha(N) = 1/N \), such that the continuum limit is achieved in \( N \to \infty \) the classical continuum constraints can be thought of as limits

\[
\phi_j(x) = \lim_{N \to \infty} \phi_j^{\alpha(N)}(x, N)
\]  

(36)

where \( i(x, N) \) is such that the point \( x \) in the continuum lies between \( i(x, N) \) and \( i(x, N) + 1 \) on the lattice for every \( N \). We are assuming a one dimensional lattice. Similar bookkeepings can be set up in higher dimensional cases.

Just like we did in the mechanical system we can define

\[
\{\phi_j^{\alpha(N)}, \phi_k^{\alpha(N)}\} = \sum_{l,m=1}^M C_{i,l,k,m}^{\alpha(N)} \lim_{i,m \to \alpha(N)} \phi_{l,m}^{\alpha(N)} + A_{j,l,k,m}^{\alpha(N)}
\]  

(37)

(where we have assumed that for the sites different from \( i \) the lattice the Poisson bracket vanishes, the generalization to other cases is immediate) and one has that

\[
\lim_{N \to \infty} A_{j,l,k,m}^{\alpha(N)} = 0.
\]  

(38)

If one takes the spatial limit \( \alpha \to 0 \) first, one has a theory with discrete time and continuous space and with first class constraints and we know in that case the uniform discretization procedure agrees with the Dirac quantization.

If one has more than one spatial dimension to discretize, then the situation complicates, since the continuum limit can be achieved with lattices of different topologies and connectivity. Once one has chosen a given topology and connectivity for the lattice, the continuum limit will only produce spin networks of connectivities compatible with such lattices. For instance if one takes a “square” lattice in terms of connectivity in two spatial dimensions, one would
produce at most spin networks in the continuum with four valent vertices. If one takes a lattice that resembles a honeycomb with triangular plaquettes one would produce sextuple vertices, etc. It is clear that this point deserves further study insofar as to how to achieve the continuum limit in theories with more than one spatial dimension.

In addition to this, following the uniform discretization approach one does not need to modify the discrete constraint algebra since it satisfies \( \lim_{N \to \infty} \{ \phi_i, \phi_j \} = 0 \), and all the observables of the continuum theory arise by taking the continuum limit of the constants of the motion of the discrete theory. The encouraging fact that we recover the continuum theory in the limit classically is what raises hopes that a similar technique will also work at the quantum level.

V. QUANTIZATION

To proceed to quantize the model, we need to consider the master constraint given in equation (23),

\[
\mathcal{H}^q = \sum_{j=0}^N (D_j - 1) (D_j - 1)^* e^{-1/2} \frac{E_j^q}{(E_j^q)^3},
\]

and quantize it. The quantization of this expression will require appropriate ordering of the exponential that appears in \( D_j \), putting the \( K \)'s to the left of the \( E \)'s, as in usual normal ordering. One would then have,

\[
\hat{D}_j =: \exp \left( -2 \left[ \hat{E}_{j+1}^x - \hat{E}_j^x \right] \hat{K}_{x,j} + \left[ \hat{E}_j^x + \hat{E}_{j+1}^x \right] \left( \hat{K}_{\varphi,j} + 1 - \hat{K}_{\varphi,j} \right) \right);
\]

Notice that \( \hat{D}_j \) is not self-adjoint and, due to the factor ordering, neither is \( \hat{\phi}_j \), but we will see that one can construct an \( \hat{\mathcal{H}} \) that is self-adjoint.

To write the explicit action, let us recall the nature of the basis of spin network states in one dimension (see [8] for details). One has a lattice of points \( j = 0 \ldots N \). On such lattice one has a graph \( g \) consistent of a collection of links \( e \) connecting the vertices \( v \). It is natural to associate the variable \( K_x \) with links in the graph and the variable \( K_\varphi \) with vertices of the graph. For bookkeeping purposes we will associate each link with the lattice site to its left. One then constructs the “point holonomies” for both variables as,

\[
T_{g,k,\mu}(K_x, K_\varphi) = \langle K_x, K_\varphi \mid \begin{array}{c} \mu_i \mu_i \kappa_i \kappa_i \end{array} \rangle = \exp \left( i \sum_j k_j K_{x,j} \epsilon \right) \exp \left( i \sum_j \mu_j, \varphi K_{\varphi,j} \right)
\]

The summations go through all the points in the lattice and we allow the possibility of using “empty” links to define the graph, i.e. links where \( k_j = 0 \). The vertices of the graph therefore correspond to lattice sites where one of the two following conditions are met: either \( \mu_i \neq 0 \) or \( k_{i-1} \neq k_i \).

In terms of this basis it is straightforward to write the action of the operator defined in (40),

\[
\hat{D}_i \begin{array}{c} \mu_i \mu_i \kappa_i \kappa_i \end{array} = \begin{array}{c} \mu_i \mu_i \kappa_i \kappa_i \kappa_i + \kappa_i + \kappa_i \end{array}
\]

The above expression is easy to obtain, since the \( \hat{E}_j^x \) may be substituted by the corresponding eigenvalues \( \mu_j \) and \( \hat{E}_j^x \) produces \( \lambda K_{\varphi,j} \) adds \( \lambda \) to \( \mu_j \), whereas the exponential of \( c n K_{x,i} \) adds \( n \) to \( k_i \).

An interesting particular case is that of an isolated \( \mu \) populated vertex,

\[
\hat{D}_i \begin{array}{c} \mu_i \kappa_i \kappa_i \end{array} = \begin{array}{c} \mu_i \kappa_i \kappa_i \end{array}
\]

So we see that the operator \( \hat{D} \) moves the line to a new vertex. This clean action is in part due to the choice of “midpoint” regularization we chose for the \( E^x \). This will in the end be important to recover diffeomorphism invariance in the continuum.
Something we will have to study later is the possibility of “coalescing” two vertices, as in the case,

\[
\hat{D}_i \left| \begin{array}{c}
k \\
\mu \\
(\mu + k)/2 \\
i \\
i+1 
\end{array} \right\rangle = \left| \begin{array}{c}
k \\
(\mu + k)/2 \\
(\mu + k)/2 \\
i \\
i+1 
\end{array} \right\rangle.
\]

(45)

or the case in which a new vertex is created,

\[
\hat{D}_i \left| \begin{array}{c}
k \\
\mu \\
2\mu \\
i \\
i+1 
\end{array} \right\rangle = \left| \begin{array}{c}
k \\
2\mu \\
k \\
i \\
i+1 
\end{array} \right\rangle.
\]

(46)

To compute the adjoint of \( \hat{D} \) is easy, since it is a one-to-one operator. We start by noting that,

\[
\left< \begin{array}{c}
-\mu \\
k \\
(\mu + k)/2 \\
i \\
i+1 
\end{array} \right| \hat{D}_i \right| \begin{array}{c}
k \\
\mu \\
2\mu \\
i \\
i+1 
\end{array} \left> = 1,
\]

(47)

and the insertion of any other bra in the left gives zero. Therefore

\[
\hat{D}^\dagger_i \left| \begin{array}{c}
k \\
\mu_1 \\
\mu_2 \\
i \\
i+1 
\end{array} \right\rangle = \left| \begin{array}{c}
k \\
\mu_1 \\
(\mu_1 + \mu_2)/2 \\
i \\
i+1 
\end{array} \right\rangle,
\]

(48)

with special particular cases that “translate” a \( \mu \) insertion,

\[
\hat{D}^\dagger_i \left| \begin{array}{c}
k \\
\mu_1 \\
\mu_2 \\
i \\
i+1 
\end{array} \right\rangle = \left| \begin{array}{c}
k \\
\mu_1 \\
2\mu_1 \\
i \\
i+1 
\end{array} \right\rangle,
\]

(49)

or create a vertex,

\[
\hat{D}^\dagger_i \left| \begin{array}{c}
k \\
\mu_1 \\
\mu_2 \\
i \\
i+1 
\end{array} \right\rangle = \left| \begin{array}{c}
k \\
-\mu_1 \\
2\mu_1 \\
i \\
i+1 
\end{array} \right\rangle.
\]

(50)

In addition there is a third particular case of interest in which a vertex is annihilated, it happens if \( \mu_{i-2} = -2\mu_1 \) and \( k = (k_1 + k_2)/2 \).

We now need to turn our attention to the other terms in the construction of \( \hat{H} \) in order to have a complete quantum version of (25). The discretization we will propose is, as,

\[
\hat{H} = \sum_{j=0}^{N} (O_{j+1}D_j - O_j)\hat{D}^\dagger_j (O_{j+1}D_j - O_j)
\]

(51)

where \( O_j = \sqrt{\frac{E_j}{E_j^{3/2}}} \), and we have chosen to localize \( O_j \) and \( D_j \) at different points. Intuitively this can be seen in the fact that \( \hat{D} \) “shifts” links in the spin nets to the next neighbor whereas \( \hat{O} \) just acts as a prefactor, as we will discuss in the next paragraph. Therefore if one wishes to find cancellations between both terms in (51) one needs to delocalize the action of both \( \hat{O} \)’s.

The quantization of \( O_j \) has been studied in the literature before [10]. Since these operators only act multiplicatively, it is better to revert to a simpler notation for the states \( |\vec{\mu}, \vec{k}\rangle \). The action of the operator is,

\[
\sqrt{\frac{E_j}{(E_j^{3/2})^{3/2}}} e^{1/4} |\vec{\mu}, \vec{k}\rangle = \left( \frac{4}{3\rho} \right)^6 \sqrt{\frac{k_{j-1} + k_{j+1}}{2}} \left[ |\mu_j + \rho^{3/4}/2 - \mu_j - \rho^{3/4}/2| \right]^6 |\vec{\mu}, \vec{k}\rangle,
\]

(52)
where $\rho$ is the minimum allowable value of $\mu$ as is customary in loop quantum cosmology. Since this operator has a simple action through a prefactor, we will call such prefactor $f(\vec{\mu}, \vec{k}, j)$. One therefore has, for example,

$$\hat{O}_{i+1} \hat{D}_i \left| \begin{array}{c} \mu_i \\ k \\ \bar{i} \\ i+j \end{array} \right\rangle = f(\vec{\mu}, \vec{k}, i+1) \left| \begin{array}{c} \mu_i \\ k \\ \bar{i} \\ i+j \end{array} \right\rangle,$$

or,

$$\hat{O}_{i+1} \hat{D}_i \left| \begin{array}{c} \mu_i \\ k \\ \bar{i} \\ i+j \end{array} \right\rangle = f(\vec{\mu}, \vec{k}, i+1) \left| \begin{array}{c} \mu_i \\ k \\ \bar{i} \\ i+j \end{array} \right\rangle,$$

where the $\vec{\mu}, \vec{k}$ that appear in the prefactor are the ones that appear in the state to the right of the prefactor.

It is worthwhile noticing that if $\mu_2 = 0$ the map is from a diagram with one insertion to another with one insertion, if $\mu_1 = 0$ it goes from one insertion to two and if both $\mu_1$ and $\mu_2$ are non-vanishing it maps two insertions to two insertions. It is not possible to go from a state with two consecutive insertions into one with only one insertion, since $2\mu_2 + \mu_1 = 0$ then $f = 0$. This is a key property one seeks in the regularization. If the regularization were able to fuse two insertions it would be problematic, as we will discuss later on.

This allows us to evaluate the action of the quadratic Hamiltonian $H$ explicitly on a set of states that capture in the discrete theory the flavor of diffeomorphism invariance. For instance, consider a normalized state obtained by superposing all possible states with a given insertion

$$|\psi_1\rangle = \frac{1}{\sqrt{N}} \sum_{i=0}^{N} \left| \begin{array}{c} \mu_i \\ k \\ \bar{i} \\ i+j \end{array} \right\rangle.$$

(55)

Such a state would be the analogue in the discrete theory of a “group averaged” state. If we now consider the action of $\hat{O}_{i+1} \hat{D}_i - \hat{O}_i$ on such a state we get,

$$\left\langle \psi_1 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i \right| \psi_1 \rangle = 0,$$

(56)

since both terms in the difference produce the same prefactor when acting on the state on the right. If one were to consider on the right a state with multiple insertions, then the result will also be zero since the operators do not convert two consecutive insertions at $i, i+1$ into one and the inner product would vanish. As a consequence, we therefore have that,

$$\langle \psi_1 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i \rangle = 0.$$

(57)

Let us now consider states with two insertions, again “group averaged” in the sense that we sum over all possible locations of the two insertions respecting a relative order within the lattice (in this case this is irrelevant due to cyclicity in a compact manifold),

$$|\psi_2\rangle = \frac{1}{\sqrt{N(N-1)}} \sum_{i=0}^{N} \sum_{j \neq i}^{N} \left| \begin{array}{c} \mu_i \\ k' \\ \bar{i} \\ i+j \end{array} \right\rangle.$$

(58)

If one considers a state $|\nu\rangle$, with three or more insertions of $\mu$ one has that,

$$\langle \psi_2 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i | \nu \rangle = 0,$$

(59)

since in the first term $\hat{D}_i$ could produce a two insertion diagram, but then the action of $\hat{O}$ at site $i+1$ would vanish, and the term on the right does not produce a two insertion diagram, as seen in (53). If one considers a state $|\nu\rangle$ with two non-consecutive vertices, the operator also vanishes, for the same reasons as before. Finally, if $|\nu\rangle$ has two
consecutive insertions then we will have a non-trivial contribution. We will see, however, that such a contribution vanishes in the continuum limit. To see this we evaluate,

\[
\langle \psi_2 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i | \psi_2 \rangle = f(\vec{v}, \vec{m}, i+1) \langle \psi_2 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i | \psi_2 \rangle
\]

\[
- f(\vec{v}, \vec{m}, i) \langle \psi_2 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i | \psi_2 \rangle
\]

\[
= [f(\vec{v}, \vec{m}, i+1)] \delta_{\mu, 2m_{i+1} + \nu} \delta_{\mu_1, -v_{i+1}} \delta_{\nu_1, m_{i+1} - m_{i+1} - m_{i+1} - m_{i+1}} \\
- f(\vec{v}, \vec{m}, i) \delta_{\mu_1, \nu_1} \delta_{\mu_2, \nu_1} \delta_{\nu_1, m_{i+1}} \delta_{\nu_1, m_{i+1}} f(\vec{v}, \vec{m}, i+1)] \frac{1}{\sqrt{N(N-1)}}
\]

(60)

If \(|\nu\rangle\) has one \(\mu\) insertion then there is another contribution,

\[
\langle \psi_2 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i | \psi_2 \rangle = f(\vec{v}, \vec{m}, i+1) \langle \psi_2 | \hat{O}_{i+1} \hat{D}_i - \hat{O}_i | \psi_2 \rangle
\]

\[
= \frac{1}{\sqrt{N(N-1)}} [\delta_{k,m}, \delta_{k', 2m_{i+1} - m_{i+1} - m_{i+1}} \delta_{\mu_1, \nu_1} \delta_{\nu_1, m_{i+1}} f(\vec{v}, \vec{m}, i+1)]
\]

(61)

We are now in a position to evaluate the expectation value of \(\hat{H}\). To do that we compute,

\[
\langle \psi_2 | \hat{H} | \psi_2 \rangle = \sum_{j=0}^{N} \langle \psi_2 | \left( \hat{O}_{j+1} \hat{D}_j - \hat{O}_j \right) \left( \hat{O}_{j+1} \hat{D}_j - \hat{O}_j \right)^\dagger | \psi_2 \rangle.
\]

(62)

and we insert a complete basis of states between the two parentheses. Then we can apply all the results we have just worked out. The final result is that only three finite contributions appear for every \(j\) and therefore

\[
\langle \psi_2 | \hat{H} | \psi_2 \rangle = O \left( \frac{1}{N} \right),
\]

(63)

and we see that in the limit \(N \to \infty\) one shows that the spectrum of \(\hat{H}\) contains zero and therefore no anomalies appear and the constraints are enforced exactly.

Analogously, one can show that for spin networks with \(m\) vertices \(\langle \psi_m | \hat{H} | \psi_m \rangle = O(1/N)\), and therefore the states that minimize \(\langle \hat{H} \rangle\) include in the limit \(N \to \infty\) the diffeomorphism invariant states obtained via the group averaging procedure. To see this more clearly we note that the state with \(m\) vertices we are considering is of the form,

\[
|\psi_m\rangle = \frac{1}{\sqrt{NC_m}} \sum_{i_1 < \ldots < i_v} \left[ \begin{array}{c}
\nu_1 \\
\nu_2 \\
\vdots \\
\nu_v
\end{array} \right]
\]

(64)

where the sum is over all the spin nets with the only condition that the cyclic order of the vertices is preserved, that is \(\nu_1\) is always between \(\nu_m\) and \(\nu_2\), etc. The quantities \(C_m^{N-1}\) are the combinatorial numbers of \(N - 1\) elements taken in groups of \(m\) for normalization purposes. This sum is the discrete version of the sum on the group that is performed in the continuum group averaging procedure. The sum preserves the cyclic order placing the vertices in all the positions compatible with such order.

We have shown that the expectation value of \(\hat{H}\) vanishes in the continuum limit. Since \(\hat{H}\) is a positive definite operator this also implies that \(\hat{H} |\psi_n\rangle = 0\), which is the condition one seeks in the uniform discretization approach. This can be explicitly checked by computing, for instance for a state \(|\psi_2\rangle\),

\[
\sum_s \langle \psi_2 | \hat{H} | s \rangle \langle s | = \frac{1}{\sqrt{N(N-1)}} \sum_{i=1}^{N} f_i(s_i)
\]

(65)
where the sum over \(s\) means a sum over a basis of spin networks \(|s\rangle\) and the \(|s_{i}\rangle\) are spin network states that have vertices at consecutive sites \(i\) and \(i+1\). Given that the \(f_{j}\)'s are finite coefficients independent of \(N\) one immediately sees that the right hand side has zero norm when \(N \to \infty\).

There is a rather important difference with the continuum case, however. The states constructed here as limits of discrete states are normalizable with the kinematical inner product and therefore the calculation suggests that in a problem with a Hamiltonian constraint in addition to diffeomorphism constraints one could work all constraints in the discrete theory on an equal footing.

VI. DISCUSSION

We have seen in a 1 + 1 dimensional model with diffeomorphism invariance that one can discretize it, therefore breaking the invariance, and treat it using the “uniform discretizations” approach yielding a diffeomorphism invariant theory in the continuum limit. We have argued that this would have been close to impossible if one had naively discretized the constraints and quantized the resulting theory.

An important point to realize is that the the kinematical Hilbert space has been changed, by considering spin networks on “lattices” with a countable number of points. There exist infinitely many possible such lattices built by considering different spacings between the points. However, in 1 + 1 dimensions the choice of lattice does not influence the diffeomorphism invariant quantum theory, whose observables can be written in terms of the canonical variables and invariant combinations of their derivatives that can be entirely framed in terms of \(\vec{k}\) and \(\vec{\mu}\) without reference to details of the lattice. For instance, the total volume of a slice evaluated on a diffeomorphism invariant spin network \(|\psi_{1}\rangle\) is given by

\[
\hat{V}|\psi_{1}\rangle = 4\pi\ell_{\text{Planck}}^{2} \sum_{v} |\mu_{v}| \sqrt{\frac{k_{e+}(v) + k_{e-}(v)}{2}} |\psi_{1}\rangle
\]

where the sum is over all vertices of the continuum spin network and \(k_{e±}\) are the values of \(k\) emanating to the right and left of vertex \(v\).

More generally, consider an observable \(\hat{O}_{\text{Diff}}\), that is an operator invariant under diffeomorphisms. Let us study in the space of lattices with a countable number of points its expectation value on diffeomorphism invariant states \(|\psi_{m,\vec{k},\vec{\mu}}\rangle \hat{O}_{\text{Diff}} |\psi_{m,\vec{k},\vec{\mu}}\rangle\), with \(|\psi_{m,\vec{k},\vec{\mu}}\rangle\) the cyclic state we considered in the previous section. In the continuum the vectors of the Hilbert space of diffeomorphism invariant states \(|\{s\}\rangle\) where \(\{s\}\) is the knot class of a spin network \(s\) belong to the dual of the space of kinematic spin network states \(|s\rangle\). The expectation value of the observable in the continuum is \(|\{s\}| \hat{O}_{\text{Diff}} |\{s\}\rangle\) and the result of both expectation values in the continuum and in the discrete theory coincide. The reason for this is that the action of \(\hat{O}_{\text{Diff}}\) on one of the terms in \(|\psi_{m}\rangle\) coincides with \(\hat{O}_{\text{Diff}}|s\rangle\) except when \(s\) has vertices that occupy consecutive positions on the lattice. In this case, depending on the specific form of \(\hat{O}_{\text{Diff}}\) the results could differ. Due to the normalization factor, however, such exceptional contributions contribute a factor \(1/N\) in the \(N \to \infty\) limit, so we have that in the continuum limit the expectation values in the continuum and the discrete always agree.

An issue of importance in loop quantum gravity is the problem of ambiguities in the definition of the quantum theory. Apart from the usual factor ordering ambiguities in a discrete theory one adds the ambiguities of the discretization process. In this example we have made several careful choices in this process to ensure that the operator \(\hat{H}\) has a non-trivial kernel in the continuum limit. This requirement proved in practice quite onerous to satisfy and it took quite a bit of effort to satisfy the requirement. Though in no way we claim that the results are unique, it hints at the fact that requiring that \(\hat{H}\) has a non-trivial kernel in the continuum significantly reduces the level of ambiguities in the definition of a quantum discrete theory. We have not been able to find another regularization satisfying the requirement an leading to a different non-trivial kernel.

Another point to note is that the quantum diffeomorphism constraints \(\phi^{j}(M) = \sum_{j=0}^{N} M_{j} \left( D_{j} - 1 \right)\) with \(M_{j}\) stemming from discretizing a smooth shift function do not reproduce the continuum algebra of constraints when they act on generic spin networks on the lattice that belong to the kinematical Hilbert space. The algebra almost works, but there appear anomalous contributions for spin networks with vertices in two consecutive sites of the lattice. In spite of this the constraints can be imposed at a quantum level through the condition \(\langle \psi | H = 0 \) and imply, as we showed, that the solutions correspond to a discrete version of the sum in the group that is performed in the group averaging procedure. The difference is that these states are normalizable with the inner product of the kinematical space itself. In this construction the Hilbert space \(\mathcal{H}_{\text{Diff}}\) is a subspace of \(\mathcal{H}_{\text{Kin}}\), unlike the situation in the ordinary group averaging procedure. This property opens interesting possibilities, particularly if it holds in more elaborate models. If such a property were to hold in more complex models, for instance involving a Hamiltonian constraint, it would be very important since it would provide immediate access to a physical inner product.
All of the above suggests that in more realistic models than the one we studied, for instance when there is a Hamiltonian constraint (with structure functions in the constraint algebra) one will also be able to define the diffeomorphism and the Hamiltonian constraints as quantum operators and impose them as constraints (or equivalently, to impose the “master constraint” \( H \)). They would act on the kinematic Hilbert space of the discrete theory, and one would hope that a suitable continuum limit can be defined. We would therefore have a way of defining a continuum quantum theory via discretization and taking the continuum limit even in systems where the discretization changes the nature of the constraints from first to second class. In 1 + 1 dimensions the procedure appears quite promising. It should be noted that this is a quite rich arena in physical phenomena, including Gowdy cosmologies, the Choptuik phenomena and several models of black hole formation. The fact that we could envision treating these problems in detail in the quantum theory in the near future is quite attractive. In higher dimensions the viability of the approach will require further study, in particular since the discretization scheme chosen could constrain importantly the types of spin networks that one can construct in the continuum theory.

Summarizing, we have presented the first example of a model with infinitely many degrees of freedom where the uniform discretization procedure works out to the last consequences, providing a continuum theory with diffeomorphism invariance and where the master constraint has a non-trivial kernel. It also leads to an explicit construction of the physical Hilbert space that is different from the usual one, allowing the introduction of the kinematical inner product as the physical one.

VII. ACKNOWLEDGEMENTS

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[1] K. Giesel and T. Thiemann, Class. Quant. Grav. 24, 2465 (2007) [arXiv:gr-qc/0607099].
[2] A. Ashtekar and C.J. Isham Class. Quantum Grav. 9 (1992), 1433. [hep-th/9202053]; A. Ashtekar and J. Lewandowski. “Representation theory of analytic holonomy \( C^* \) algebras” In Knots and Quantum Gravity, J. Baez (ed.), (Oxford University Press, Oxford 1994); A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourão and T. Thiemann, Journ. Math. Phys. 36 (1995), 6456-6493. [gr-qc/9504018]
[3] D. Marolf, in “Proceedings of the Eleventh Marcel Grossmann Meeting on General Relativity”, V. Gurzadyan, R. T. Jantzen, R. Ruffini (editors), World Scientific, Singapore (2002) [arXiv:gr-qc/0011112].
[4] T. Thiemann. Class. Quant. Grav. 23 (2006), 2211-2248 [gr-qc/0305080]; Class. Quant.Grav. 23 (2006), 2249-2266 [gr-qc/0510011]; B. Dittrich and T. Thiemann. Class. Quant. Grav. 23 (2006), 1025-1066 [gr-qc/0411138]; Class. Quant. Grav. 23 (2006), 1067-1088 [gr-qc/0411139]; Class. Quant. Grav. 23 (2006), 1089-1120 [gr-qc/0411140]; Class. Quant. Grav. 23 (2006), 1121-1142 [gr-qc/0411141]; Class. Quant. Grav. 23 (2006), 1143-1162 [gr-qc/0411142].
[5] M. Campiglia, C. Di Bartolo, R. Gambini and J. Pullin, J. Phys. Conf. Ser. 67, 012020 (2007) [arXiv:gr-qc/0606121].
[6] M. Campiglia, C. Di Bartolo, R. Gambini and J. Pullin, Phys. Rev. D 74, 124012 (2006).
[7] V. Husain and k. V. Kuchar, Phys. Rev. D 42, 4070 (1990).
[8] M. Campiglia, R. Gambini and J. Pullin, Class. Quant. Grav. 24, 3649 (2007) [arXiv:gr-qc/0703135].
[9] P. A. Renteln and L. Smolin, Class. Quant. Grav. 6 (1989) 275.
[10] A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. D 73, 124038 (2006) [arXiv:gr-qc/0604013].