AdS Asymptotic Symmetries from CFT Mirrors

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ABSTRACT: We study Kac-Moody asymptotic symmetries and memory effects in $\text{AdS}_4^{\text{Poincare}}$ gauge theory and (when accompanied by 4D gravity) in its holographic $\text{CFT}_3$ dual. While such infinite-dimensional symmetries are absent in standard asymptotic analyses of $\text{AdS}_4$, we show how they arise with alternate AdS boundary conditions. In the 3D holographic description, these alternate boundary conditions correspond to a modified $\tilde{\text{CFT}}_3$ obtained by Chern-Simons gauging of the $\text{CFT}_3$ dual defined by standard boundary conditions, so that Kac-Moody symmetries then follow from the familiar Chern-Simons/Wess-Zumino-Witten correspondence. Apart from their own intrinsic interest, in abelian $\text{AdS}_4$ gauge theories these alternate boundary conditions are equivalent to standard boundary conditions imposed on electric-magnetic dual variables. In the holographic description this corresponds to 3D “mirror” symmetries connecting the original and modified CFTs. Further, in both abelian and non-abelian theories we show that the alternative/$\tilde{\text{CFT}}_3$ theory emerges at leading order in large Chern-Simons level from the correlators of the standard theory, upon incorporating large-wavelength limits in the holographically emergent dimension. We point out analogies between 4D AdS and Minkowski gauge theories in their asymptotic symmetries, “soft” limits and memory effects, but with AdS having the advantages of greater holographic insight and technical simplicity.
1 Introduction

In gravitational and gauge theories, asymptotic symmetries (AS) are a global remnant of large diffeomorphisms and gauge transformations which act non-trivially on physical data at spacetime infinity. The classic example of infinite-dimensional AS, and in many ways the best understood and applied, is that of (quantum) General Relativity (GR) in asymptotically 3D Anti-de Sitter (AdS$_3$) spacetime. The analysis of Brown and Henneaux [1] uncovered Virasoro symmetries which presaged, and were ultimately elegantly incorporated into, the AdS$_3$/CFT$_2$ correspondence, translating into the implications of 2D conformal invariance and unitarity. The Virasoro structure and central charges, with modular invariance, led to a precise microscopic account [2] of the Bekenstein-Hawking entropy of AdS$_3$ Schwarzchild black holes, dual to the CFT$_2$ Cardy formula [3]. There is an ongoing program of exploiting this symmetry structure to address more detailed aspects of black hole information puzzles [4, 5]. In a similar vein to these gravitational AS, 3D Chern-Simons (CS) gauge theories display infinite-dimensional Kac-Moody (KM) asymptotic symmetries with central extensions, reflecting 2D Wess-Zumino-Witten (WZW) current algebras via the technically simpler CS/WZW correspondence [6–9].

In higher dimensions the situation is intriguing, but less well understood. The primordial example is provided by the infinite-dimensional BMS “supertranslations” of GR in asymptotically 4D Minkowski spacetime (Mink$_4$) [10, 11], later extended to include Virasoro-type “superrotations” [12, 13], and KM AS from 4D gauge theory [14–17]. However, the symmetry algebras have appeared without central extensions, ordinarily required
by unitarity in lower-dimensional contexts. There are new deep aspects in 4D, unifying AS with soft limits of gravitons and gauge bosons, and with gravitational and gauge “memory” effects (see Ref. [18] for a review and extensive list of references). There are also hopes of applying AS to help understand black hole information [19–23], although this is still under debate [24–29]. The AS can be shown to derive from 2D current algebras “living” on the celestial sphere, but it is unclear what the precise connection is between this structure and some form of holography in Minkowski spacetime. One hint comes from an intermediate step between 4D and 2D: the soft limit of gravitational and gauge fields renders them effectively 3-dimensional, in a more nuanced generalization of the trivial loss of the time dimension in the static limit. In particular, some of the soft fields take the form of 3D GR and CS [30], with close ties to the AdS3/CFT2 and CS/WZW correspondences [6–9].

In order to explore the connection of 4D AS to holography, Ref. [31] turned to the study of AS in (portions of) AdS4, taking advantage of the well-established AdS4/CFT3 correspondence. In this context, there is a natural way to include 3D (conformal) GR and CS, by simply having them gauge the holographic CFT3 at the outset. Applying 3D (conformal) GR and CS (+ CFT3 “matter”) analyses then yields a set of infinite-dimensional AS with central extensions. Even in the limit in which the external 3D GR and CS fields decouple from CFT3, these AS symmetries remain, but losing their central extensions as the price for restricting to CFT correlators with a well-defined decoupling limit. The resulting AS closely parallel the supertranslation, superrotation and KM AS of Mink4.

In this paper, we continue the study of AS in the context of AdS4/CFT3. We restrict our attention to gauge theory in the Poincare patch of AdS4 for technical and conceptual simplicity, with 4D GR only an incidental presence needed for duality with CFT3. Within this framework, we will identify different but interconnected ways in which KM AS arise. Most directly we extend the approach of Ref. [31] to the Poincare patch, with CS-gaugings of the holographic CFT defining new CFTs, and the canonical CS structure leading to KM AS with finite central extensions. The AdS dual of the modified CFT shares the same 4D dynamics as the AdS dual of the original CFT, but with the former having an alternate set of AdS boundary conditions [32](particular to 4D). This is key to evading no-go arguments [33, 34] for infinite-dimensional AS in AdSd>3.

In the case of abelian gauge/global symmetries of AdS4/CFT3, we can make a stronger statement because the original CFT and the CFTs are connected by SL(2, Z) “mirror” symmetry [32]. From the AdS4 viewpoint, this SL(2, Z) is associated to electric-magnetic duality, which relates the standard boundary conditions to alternate boundary conditions. In this sense, KM AS structure already resides in the standard AdS4/CFT3 construction, albeit applied in suitable electric-magnetic/mirror dual variables.

For both abelian and non-abelian theories, there is another way in which we will show that the standard AdS4/CFT3 theory contains the “seeds” of the alternate/CFT theory, namely by taking gauge-boson long-wavelength limits in the holographically emergent dimension within ∂AdS correlators. We show that this “holographic soft limit” of the standard theory yields the correlators and KM AS of the alternate theory to leading order.
in the CS level, closely matching and adding physical significance to the decoupling limit AS analysis of Ref. [31].

We will simplify and extend the analogy between gauge theory AS of AdS$_4$ and Mink$_4$, as summarized in Table 1. In Mink$_4$ gauge theory, massive charges emerging from a scattering event asymptotically approach future timelike infinity. This is a space parametrized by particle boosts, geometrically 3D hyperbolic space or, more suggestively, Euclidean AdS$_3$ [35]. Its boundary is future null infinity, $I^+$, the destination for massless particles, which, while 3-dimensional, has 2D geometry due to the one null direction. In AdS$_4$, there are analogous asymptotic 3D and 2D geometries. The asymptotic infinity of AdS$_4^{\text{Poincare}}$ is of course the boundary $\equiv$ Mink$_3$, the entire spacetime from the holographic perspective. Canonically in CS, AS structure is associated to the wavefunctional at some fixed time, say $t = 0$, with a spacelike 2D geometry. We consider scattering in the bulk of AdS$_4$, with some outgoing particles headed to the boundary and absorbed by local (CFT$_3$) operators there. Charged particles arriving at $\partial$AdS$_4$ at $t = 0$ are the analogs of massless charged particles arriving at $I^+$ in Mink$_4$. In both cases, the soft CS/WZW structure gives rise to 2D holomorphic currents, with poles at the locations of the charges, and with Laurent expansions in terms of KM charges. Charged particles arriving at $\partial$AdS$_4$ at more general $t \neq 0$ are the analogs of massive charges arriving at timelike infinity in Mink$_4$. In both cases, the 2D currents exist but are no longer holomorphic, the above-mentioned poles effectively being “smeared” [17, 18, 36].

In both AdS and Minkowski 4D spacetimes, the 2D KM currents are connected to in-principle physical gauge memory effects [37–39] (gravitational memories and related references are reviewed in Ref. [18]). We will discuss a general sense in which canonical CS fields are fundamentally “memory” Wilson loops in 3D, or equivalently in 4D near the AdS boundary, and then give an abelian AdS$_4$/CFT$_3$ analog of Mink$_4$ “magnetic” memory effects along the lines of (the electric-magnetic dual of) Ref. [32]. In this case, we will see that the smearing of pole structure mentioned above is explained holographically by the fact that CFT charge density created by a local operator begins pointlike but then spreads out over time. This is in contrast say to theories of 3D massive charges in other CS/WZW contexts, where a charge can remain pointlike over time.

Of course, the central difference between Mink$_4$ and AdS$_4$ is that in the latter case we understand how holography works, making the coupling and significance of CS fields transparent. Such an understanding is still lacking in Mink$_4$, but given the analogy with AdS$_4$, the AS structure must be providing us important clues.

The paper is organized as follows. In Section 2, we introduce gauge theory in the Poincare patch of AdS$_4$, standard and alternate boundary conditions, and their holographic translations in terms of CFT$_3$ and $\widetilde{\text{CFT}}_3 \equiv \text{CS} + \text{CFT}_3$, respectively. In Section 3 we derive the KM AS of the alternate AdS$_4$/CFT$_3$ theory from its canonical CS structure. In Section 4, we restrict to abelian theories and point out the passage from standard AdS$_4$/CFT$_3$ to alternate AdS$_4$/CFT$_3$, and hence KM AS, via electric-magnetic/mirror duality. In Section 5, we derive another passage from standard AdS$_4$/CFT$_3$ to alternate AdS$_4$/CFT$_3$ in abelian theories, this time by introducing the “holographic soft limit” in its
| Mink\(_4\) | AdS\(_4\) |
|----------------|----------------|
| S-matrix | \(\partial\text{AdS}_4/\text{CFT}_3\) local correlators |
| Timelike infinity \(\equiv\) Euclidean AdS\(_3\) | \(\partial\text{AdS}_4 \equiv \text{Mink}_3\) |
| Null infinity (\(I\)) , 2D geometry | Fixed time \(t = 0\) on \(\partial\text{AdS}_4\), 2D geometry |
| Soft limit, \(m_3 \to 0\), where \(m_3\) is the Casimir invariant of Euclidean AdS\(_3\) \([30]\) | Holographic soft limit, \(m_3 \to 0\), where \(m_3\) is the Casimir invariant of Mink\(_3\) |
| CS structure of soft fields | CS structure of soft fields |
| 2D holomorphic-WZW currents \(j^a\) for (massless) charges hitting \(I\) | 2D holomorphic-WZW currents \(j^a\) for charges hitting \(t = 0\) on \(\partial\text{AdS}_4\) |
| (Non-)abelian KM AS | (Non-)abelian KM AS |
| Electric/Magnetic Memories | Electric/Magnetic Memories |
| Electric flux Memory Kernel | Electric flux/Holographic charge density |
| ? ? | Holographic Duality |
| ? ? | \(\tilde{\text{CFT}}_3\) with fully dynamical CS (finite level) |

**Table 1**: The analogy between Mink\(_4\) and AdS\(_4\) gauge dynamics, their soft limits and associated infinite-dimensional KM asymptotic symmetries. AdS/CFT holography provides more of an explanatory structure in the case of AdS\(_4\).

simplest form. In Section 6 we generalize this soft limit analysis to non-abelian gauge theories in AdS\(_4\), involving more careful treatment of multiple soft external lines. In Section 7 we describe (abelian) magnetic memory effects in standard AdS\(_4)/\text{CFT}_3\) and give their holographic interpretation and connections to KM AS structure and soft limits. Section 8 provides our conclusions.

## 2 AdS\(_4\) Gauge Theory, Boundary Conditions and Holography

We describe the Poincare patch of AdS\(_4\) by coordinates \(X^M \equiv (t, x, y, z)\) and metric,

\[
ds_{\text{AdS}_4}^2 = \frac{dt^2 - dx^2 - dy^2 - dz^2}{z^2}, \quad z > 0,
\]

where we work in units of the AdS radius of curvature. Its boundary, \(\partial\text{AdS}_4 \equiv \text{Mink}_3\), is at \(z = 0\), with 3D coordinates \(x^\mu \equiv (t, x, y)\). We consider AdS dynamics of the form,

\[
\mathcal{L}_{\text{AdS}_4} = -\frac{1}{2g^2} \text{Tr} \mathcal{F}_M \mathcal{F}^M + \frac{\theta}{16\pi^2} \text{Tr} \mathcal{F}_M \tilde{\mathcal{F}}^M + \mathcal{A}_M J^M + \cdots,
\]
where $A_M = A^a_M t^a$ is a 4D gauge field with field strength $F_{MN} = F^a_{MN} t^a$, $J^a_M$ is the 4D current due to gauge-charged matter, $t^a$ are the generators of gauge group, normalized as $\text{Tr} t^a t^b = \delta^{ab}/2$, and the ellipsis includes the 4D matter Lagrangian as well as 4D quantum gravity (QG). We will not explicitly need the details of QG in this paper, but with it the AdS$_4$ theory has a CFT$_3$ holographic dual on Mink$_3$, which we will invoke (see Refs. [40, 41] for a review).

2.1 Standard “Dirichlet” Boundary Conditions

The standard AdS$_4$ boundary condition (b.c.) is

$$A^a_\mu(x, z) \xrightarrow{z \to 0} A^a_\mu(x), \quad (2.3)$$

where $A^a_\mu(x')$ is the source for the dual CFT$_3$ conserved global current, $J^a_\mu(x')$. The 4D $\theta$-term introduces a subtlety, seen by the decomposition,

$$\theta = \bar{\theta} + 2\pi \kappa, \quad \bar{\theta} \in [0, 2\pi), \quad \kappa \in \mathbb{Z}. \quad (2.4)$$

4D bulk physics only depends on the angle $\bar{\theta}$ as usual. For simplicity, in this paper we restrict attention to $\bar{\theta} = 0$. However, given the total derivative nature of the $\theta$-term, $\kappa$ survives as a $\partial \text{AdS}_4$ action for the source $A_\mu$, and

$$\mathcal{L}_{\text{Mink}_3} = \mathcal{L}_{\text{CFT}_3} + A^a_\mu J^a_\mu + \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right). \quad (2.5)$$

This gives extra contact terms, consistent with 3D conformal invariance, in multi-current correlators at coincident points [32]. For example,

$$\left\langle T \left\{ J^a_\mu(x) J^b_\nu(x') \cdots \right\} \rightangle \supset \kappa \epsilon_{\mu\nu\rho} \delta^{ab} \partial^\rho \delta^3(x - x') \langle \cdots \rangle. \quad (2.6)$$

For vanishing source, $A = 0$, the b.c. takes the “Dirichlet” (D) form $A^a_\mu(x', z) \xrightarrow{z \to 0} 0$, or more gauge-invariantly,

$$F^a_{\mu\nu}(x') \xrightarrow{z \to 0} 0, \quad (2.7)$$

since the 3D dual description is also gauge-invariant if we transform the source $A_\mu$ as a background 3D gauge field.

2.2 CS-gauged CFT$_3$ and Alternate Boundary Conditions

We define a modified CFT$_3$ by simply elevating the source $A_\mu$ above to a fully dynamical field with the same action, Eq. (2.5). The $\kappa$ terms no longer represent contact terms for global current correlators of CFT$_3$, but rather a CS action for $A_\mu$, which then gauges the CFT$_3$ current $J_\mu$. Schematically, $\widetilde{\text{CFT}}_3 = \text{CS} + \text{CFT}_3$.

The AdS$_4$ dual of CFT$_3$ is given by the same bulk dynamics as for the original CFT$_3$ but with an alternate boundary condition [32]. A large set of gauge-invariant b.c.s respecting the AdS$_4$ isometries (3D conformal invariance) exist because one can replace the “Dirichlet”
vanishing of $\mathcal{F}_{\mu\nu}$ at the boundary by vanishing of a more general linear combination of $\mathcal{F}_{\mu\nu}$ and $\tilde{\mathcal{F}}_{\mu\nu}$. We see that the CS equations of motion corresponding to the action of Eq. (2.5) is matched by alternate b.c. of the form,

$$\frac{\kappa}{2\pi} \mathcal{F}_{\mu\nu} + \frac{1}{g^2} \tilde{\mathcal{F}}_{\mu\nu} \to 0,$$

(2.8)

because of the standard holographic matching

$$2\tilde{\mathcal{F}}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho} \mathcal{F}_{\rho} \to 2g^2 \epsilon_{\mu\nu\rho} J^\rho.$$

(2.9)

In the simplest case, $\kappa = 0$, the alternate b.c. is just a gauge invariant version of "Neumann" (N) b.c.:

$$2\tilde{\mathcal{F}}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho} \mathcal{F}_{\rho} \to 0,$$

(2.10)

as is clear in axial gauge $A^a_z = 0$,

$$\mathcal{F}_{z\rho} = \partial_z A_\rho \to 0.$$

(2.11)

3 Kac-Moody AS from CS Structure

In this section we consider the above AdS$_4$ gauge theory (+ QG) with alternate b.c., or equivalently in 3D, $\widetilde{\text{CFT}}_3 \equiv \text{CS} + \text{CFT}_3$, with level $\kappa$. 3D CS gauge theory coupled to matter (provided here by CFT$_3$) describes relativistic (non)-abelian Aharonov-Bohm (AB) type effects between separated charges (e.g. see Ref. [42] for a review), thereby providing charged matter with quantum “topological hair”. This is manifest already in the CS Gauss Law constraint ($A^a_\rho$ equation of motion),

$$\frac{\kappa}{2\pi} F_{a}^{\mu\nu} = J_0^a,$$

(3.1)

where $F_{a\mu\nu}$ is the field strength of $A$. Outside the support of the charge density $J_0$, $F_{xy} = 0$, but spatial Wilson loops (as seen by test charges) here are non-trivial when enclosing charge $J_0$, as in Fig. 1.

Related to the topological nature of their AB effects, CS structure on 3D spacetimes with a 2D boundary can be mapped to WZW 2D current algebras, exhibiting KM AS at the 2D boundary [6–9]. In the present context however, CS lives on Mink$_3$, with no finite 2D boundary. But from the canonical viewpoint the state wavefunctional, $\Psi$, at some fixed time, say $t = 0$, does exhibit Euclidean signature WZW/KM structure on the spatial $x - y$ plane at that time, the relevant Ward identities supplied by Gauss’ Law [6]. One can think of $\Psi(t = 0)$ as given by a CS + CFT$_3$ path integral on the earlier half of Mink$_3$, $t < 0$, a spacetime with 2D boundary $t = 0$. 

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3.1 Gauss Law constraints on canonical CS fields

To review this, we introduce complex coordinates,
\[ u \equiv x + iy, \quad \bar{u} \equiv x - iy, \]
(3.2)
in which Gauss’ Law (A0 equation of motion) reads
\[ \left( \partial_{\bar{u}} j^a - 2i\kappa \partial_u A^a_u - f^{abc} j^b A^c_{\bar{u}} \right) \Psi[A_{\bar{u}}] = 2\pi J^a_0 \Psi[A_{\bar{u}}]. \]
(3.3)

To explain our notation, from Eq. (2.5) we see from the CS Lagrangian that (after integrating out \( A^a_0 \)) \( A_u \) and \( A_{\bar{u}} \) are canonically conjugate. Here, we choose to work in \( A_{\bar{u}} \) field-space, and denote a (non-canonically normalized, for later convenience) conjugate field-momentum by
\[ j^a(u, \bar{u}) \equiv i\pi \frac{\partial L_{CS}}{\partial \dot{A}^a_{\bar{u}}} = 2i\kappa A^a_u. \]
(3.4)
The wavefunctional \( \Psi \) is taken to depend on \( A_{\bar{u}} \) (coherent state representation) and the CFT fields. At the quantum level the conjugate field-momentum is then given by
\[ j^a(u, \bar{u}) = i\pi \frac{\delta}{\delta A^a_{\bar{u}}(u, \bar{u})}. \]
(3.5)

3.2 Holomorphic 2D WZW current and KM symmetry from CS

We begin by exploring \( \Psi \) at \( A_{\bar{u}} = 0 \) and for the special case of the CFT state consisting only of pointlike disturbances at \( t = 0 \),
\[ \Psi \propto \prod_n O_n(u, \bar{u}) |0\rangle, \]
(3.6)
where the \( O \) are local operators. Gauss’ Law then reduces to
\[ \partial_{\bar{u}} j^a(u, \bar{u}) \Psi[A_{\bar{u}} = 0] = 2\pi \sum_{\alpha=1}^n T^a_{(\alpha)} \delta^2(u - u_\alpha) \Psi[A_{\bar{u}} = 0], \]
(3.7)
where $T^a_{(\alpha)}$ is the representation of the (non-)abelian generator acting on the particular local CFT operator $O_\alpha(u_\alpha, \bar{u}_\alpha)$, giving its charge. This equation can be integrated\(^{1}\) to give

$$ j^a(u, \bar{u})\Psi[A_\alpha = 0] = \sum_{\alpha} \frac{T^a_{(\alpha)}}{u - u_\alpha} \Psi[A_\alpha = 0], \quad (3.8) $$

using the identity $\partial_\bar{u} (1/(u - u_\alpha)) = 2\pi \delta^2(u - u_\alpha)$. From this we can then extract a 2D “OPE”, matching that of a standard holomorphic WZW current with a charged operator in 2D Euclidean field theory (e.g. see Ref. [43] for a review),

$$ j^a(u, \bar{u})O_\alpha(u_\alpha, \bar{u}_\alpha) \xrightarrow{u \to u_\alpha} \frac{T^a_{(\alpha)}}{u - u_\alpha}. \quad (3.9) $$

Next, we act on Gauss’ Law with the operator $j^b(u', \bar{u}') \equiv i\pi \delta/\delta A \bar{u}(u', \bar{u}')$ and then set $A_\bar{u} = 0$:

$$ \left[ \kappa \partial_u \delta^2(u - u')\delta^{ab} + \frac{1}{2\pi} \partial_u j^a j^b - \frac{i}{2} f^{abc} \delta^2(u - u') j^c \right] \Psi = j^b(u', \bar{u}').J^0_a(u, \bar{u})\Psi. \quad (3.10) $$

We consider $u$ away from any CFT local operators at $u_\alpha$ (within $\Psi$), so the right-hand side is non-singular in $u - u'$. The left-hand side can again be integrated, using the identity $-\partial_\bar{u} (1/(u - u')^2) = \partial_u \partial_\bar{u} (1/(u - u')) = 2\pi \partial_u \delta^2(u - u')$, to give the $jj'$ OPE,

$$ j^a(u, \bar{u})j^b(u', \bar{u}') \xrightarrow{u \to u'} \frac{\kappa}{(u - u')^2} \delta^{ab} + \frac{i f^{abc}}{2(u - u')} j^c. \quad (3.11) $$

Choosing $u' = 0$ the 2D holomorphic current can be expanded in a Laurent expansion of KM charges

$$ j^a(u) = \sum_m \frac{Q^a_m}{u^{m+1}}. \quad (3.12) $$

Plugging this into the OPE and interpreting the result in standard 2D Euclidean radial quantization gives the KM symmetry algebra,

$$ [Q^a_m, Q^b_n] = \kappa m \delta^{ab} \delta_{m-n} + i f^{abc} Q^c_{m+n}, \quad (3.13) $$

where the central extension is provided by the CS level $\kappa$.

Via AdS$_4$/CFT$_3$ duality, we then conclude that with alternate b.c., Eq. (2.8), AdS$_4$ gauge theory inherits this KM symmetry algebra as its AS.

### 3.3 General CFT states and non-holomorphicity of WZW current

In later sections we will discuss KM structure and associated memory effects in the context of $\partial$AdS$_4$ correlators with standard D b.c., which are more closely analogous to the Mink$_4$ S-matrix and memories. Nevertheless, in the above derivation of KM structure from the

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\(^{1}\)We are assuming the wavefunctional is a well-behaved function of $A_\bar{u}$ at infinity, so that we do not have to include an analytic function of $u$ as integration constant in RHS of Eq. (3.8).
canonical wavefunctional viewpoint for (the holographic dual of) alternate b.c. we can already recognize key features. The restriction of the CFT “matter” state at $t = 0$ to consist of only point-like excitations, created by CFT local operators then, was needed for $j$ to be holomorphic with simple poles at charge locations. This is analogous to the structure of the 2D holomorphic current construction in Mink$_4$ gauge theory, with poles at angular locations of massless charges arriving at $I^+$. Such a CFT state corresponds in AdS$_4$ to the special case of charged particles all being at the boundary at $t = 0$.

In CS theories with weakly coupled massive 3D charged species, the restriction to states with a few pointlike charged excitations is automatic for finite energy states, yielding simple-pole structure of $j$ more generally. But the CFT consists of massless strongly-coupled constituents, so a typical state is a smeared collection of indefinite numbers of these constituents described by the charged density $J_0$, with $j$ given by a “smeared” integral over poles weighted by $J_0$, rather than a discrete sum of poles. Such a CFT state can be created by local CFT operators acting before $t = 0$, dual to particles that have moved off into the bulk of AdS$_4$ by $t = 0$. The smeared pole structure for $j$ also appears in Mink$_4$ amplitudes [17, 18, 36], associated with massive charged particles heading to timelike infinity. We will study and compare this smeared KM structure more fully in Section 7 in the context of the memory effect.

4 AS from 4D Electric-Magnetic Duality/3D Mirror Symmetry

We have seen that alternate AdS$_4$ b.c., dual to the modified $\tilde{\text{CFT}}_3$, explicitly contains CS and hence CS/WZW-related KM structure. But this analysis seems to exclude the case of standard AdS$_4$ b.c., dual to the isolated original CFT$_3$. The remainder of this paper is devoted to showing different senses in which even this original unmodified theory does connect to KM AS. In this section, we will show that in the case of abelian AdS$_4$ gauge symmetry there is a full CS and KM AS structure arising from standard b.c., when these are imposed on the 4D gauge theory in suitable electric-magnetic dual variables. At the holographic level, this shows how the standard and modified CFTs transform into one another via 3D mirror symmetries.

The most familiar form of electric-magnetic duality arises from the invariance of pure Maxwell theory under

$$\mathcal{F} \to \tilde{\mathcal{F}}, \tilde{\mathcal{F}} \to -\mathcal{F}. \quad (4.1)$$

More precisely, in the presence of charged matter it is described by a discrete duality transformation, $S$, which acts on states with electric charge $n g$ and magnetic charge $2\pi m/g$ (where $n, m$ are integers for Dirac quantization) according to

$$S(n, m) = (m, -n). \quad (4.2)$$

From the viewpoint of the 4D magnetic dual gauge field, $\tilde{A}_M : \tilde{F}_{MN} = \partial_M \tilde{A}_N - \partial_N \tilde{A}_M$, the roles of the “standard” $D$ and “Neumann” $N$ b.c.s are exchanged, as is clear from their gauge-invariant forms, Eq. (2.7), and Eqs. (2.10), (2.11). That is, $D \equiv \tilde{N}, N \equiv \tilde{D}$. 

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Electric-magnetic duality extends to a full SL(2, Z), generated by S and T, where T corresponds to the shift in the CP-violating parameter $\theta \to \theta + 2\pi$, another invariance of the bulk 4D physics. Witten has pointed out that general shifts in $\theta$ induce shifts in the spectrum of electric charges of states with non-zero magnetic charge. For the $(2\pi)$-integer shift of $T$ this Witten effect [44] corresponds to

$$T(n, m) = (n + m, m).$$ (4.3)

In this way, SL(2, Z) duality exchanges ordinary electric charges with more general dyonic charges $(n, m)$.

As we saw for the $S$ transformation above, the AdS boundary conditions are not invariant under the more general SL(2, Z) transformations, since they pick out the particular type of $(n, m)$ charge whose gauge field is given D b.c., thereby defining the global current of the dual CFT. The standard b.c. picks out ordinary electric charges $(1, 0)$ of course. For a general $(n, m)$ the boundary conditions involve an obvious linear combination of the D and N b.c.,

$$gF_{\mu\nu} + \frac{2\pi m}{g} \tilde{F}_{\mu\nu} \xrightarrow{z \to 0} 0.$$ (4.4)

SL(2, Z) thereby incarnates as 3D mirror symmetry, transforming between the different CFTs given by these different b.c.s.

For example, if we first apply the $TS$ transformation to the 4D gauge theory and then impose standard boundary conditions, we get D b.c. applied to the gauge field that couples to $TS(1, 0) = (-1, -1)$ charges,

$$gF_{\mu\nu} + \frac{2\pi}{g} \tilde{F}_{\mu\nu} \xrightarrow{z \to 0} 0.$$ (4.5)

From the discussion of subsection 2.2, we see that this corresponds to a CS gauging of the original CFT$_3$, with level $\kappa = 1$.

In this way, SL(2, Z) equates the standard boundary conditions of AdS$_4$ gauge theory with alternative boundary conditions, which then manifest KM AS as described earlier.

5 Alternate/$\tilde{\text{CFT}}$ Correlators from “Holographic Soft Limit”

We now turn to the sense in which the standard AdS$_4$ D b.c., dual to CFT$_3$ in isolation, has implicit CS structure and AS in the original “electric” variables once we include a natural AdS$^\text{Poincare}$ generalization of the notion of “soft limit”, applying whether the 4D gauge theory is abelian or non-abelian. This form of CS/AS represents our closest analog of the Mink$_4$ AS analysis developed in Ref. [30], and also builds on the AdS$^\text{Poincare}$ discussion of Ref [31]. We begin with abelian gauge theory for simplicity in this section, and extend to non-abelian gauge theory in the next.
5.1 Fixed Helicity $\partial \text{AdS}_4$ Correlators

In Mink$_4$ an S-matrix amplitude with an external photon takes the form,

$$\int_{\text{Mink}_4} d^4X A_M J^M, \quad A_M(X) = \epsilon_M^\pm(q)e^{iq\cdot X},$$  (5.1)

where $J$ represents the on-shell current consisting of the rest of the amplitude with amputated photon leg, and $\epsilon_M^\pm(q)$ is the polarization vector for $\pm$ helicity, satisfying

$$q^2 = q \cdot \epsilon^\pm = e^\pm \cdot e^\pm = 0, e^\pm \cdot e'^\pm = 1.$$  (5.2)

In AdS$_4$ we compute boundary correlators rather than an S-matrix,

$$\int_{\partial \text{AdS}_4} d^3x A_\mu(x) \langle T\{ J^\text{CFT}_\mu(x) \cdots \} \rangle = \int_{\text{AdS}_4} d^4X A_M(X) J^M(X), \quad A_\mu(x, z) \xrightarrow{z \to 0} A_\mu(x),$$  (5.3)

where $A_M$ satisfies the AdS Maxwell’s equations. Given the obvious Weyl invariance of the Maxwell action and the Weyl equivalence of AdS$_4$ to half of Mink$_4$,

$$ds^2_{\text{AdS}_4} \sim \text{Weyl} dt^2 - dx^2 - dy^2 - dz^2, \quad z > 0,$$  (5.4)

Mink$_4$ LSZ wavefunctions for external photons, $A^{\pm}_M(X) = \epsilon^{\pm}_M(q)e^{iq\cdot X}$, are also valid choices for AdS correlators. This corresponds to a CFT$_3$ source,

$$A^{\pm}_\mu(x) = \epsilon^{\pm}_\mu(q)e^{i\hat{q}\cdot x}, \quad \hat{q} \equiv (q_0, q_x, q_y).$$  (5.5)

While $A, A$ are complex, their real and imaginary parts define standard $\partial \text{AdS}/\text{CFT}$ correlators, and we are just considering their complex superposition.

We choose to work in 4D axial gauge, $\epsilon_z = 0$. It is clear that $A$’s of the above form span all possible sources in Mink$_3$ with timelike 3-momentum, $\hat{q}$, given that $J$ is conserved (in momentum space, $\hat{q}.J(\hat{q}) = 0$). We see that 4D helicity for massless photons matches a 3D “helicity” for timelike CFT sources. The different helicity sources satisfy Chern-Simons-Proca (CSP) equations:

$$2\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = \pm m_3 A_\mu, \quad m_3 \equiv q_z,$$  (5.6)

for $\pm$ helicity. Here, $m_3$ is the mass Casimir invariant of Mink$_3$, that is $m_3^2 = q^2 \equiv q_\mu q^\mu$ for momentum eigenstates, so that $m_3 = q_z$ by Eq. 5.2. This is analogous to the 3D CSP form of helicity-cut Mink$_4$ S-matrix amplitudes derived in Ref. [30], where $m_3$ was the Casimir invariant of a Euclidean AdS$_3$ foliation of (a future light cone in) Mink$_4$.

5.2 The “holographic soft limit” of $\partial \text{AdS}_4$ correlators

In Mink$_4$, it was shown that the conventional (leading) soft photon limit of amplitudes captured by the Weinberg Soft Theorems, was equivalent to the limit $m_3 \to 0$. Here, we
simply translate the analogous definition of “soft limit” to the AdS\(_4\) context, as vanishing CSP mass, \(m_3 \to 0\), arriving at the (sourceless) CS equation,
\[
e^{\mu\rho} \partial_{\rho} A_{\mu} = 0, \quad \partial_{\mu} A^{\mu} = 0.
\] (5.7)

We also effectively have a Lorentz-gauge fixing condition as can be seen by taking the divergence of the CSP equation (5.6) for \(m_3 \neq 0\) followed by \(m_3 \to 0\). This gives rise to a “soft” \(\partial\text{AdS/CFT correlator}, \text{Eq. (5.3)}, \) where
\[
A_\mu(x, z) = A_\mu(x), \quad A_z = 0.
\] (5.8)

This follows because \(A\) is pure gauge in Mink\(_3\) since \(F = 0\) by Eq. (5.7), and therefore this \(A_M\) is pure gauge in AdS\(_4\), hence trivially satisfying 4D Maxwell’s equations and \(A_\mu(x, z) \to A_\mu(x)\).

From the 4D viewpoint, unlike the standard notion of “soft” in Minkowski spacetime, it is (only) the holographically emergent direction’s \(z\)-dependence, rather than \(t\)-dependence (overall energy) which is softened.\(^2\) The above 4D pure gauge configurations in the holographic soft limit are the “large” gauge transformations at the root of AS, which we now derive.

It is convenient to focus on CFT\(_3\) correlators of the form,
\[
\left\langle 0 | T \left\{ e^{i \int d^3 x A_\mu(x) J^\mu(x)} \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \right\} | \text{in} \right\rangle,
\] (5.9)
as depicted in Fig. 2a, where \(A_\mu(x)\) is the source for “soft” photons, the \(\mathcal{O}_\alpha\) are arbitrary local CFT operators with \(U(1)\) charges \(Q_\alpha\) (including possibly \(J^\mu\) itself, corresponding to \(\partial\text{AdS correlators for 4D photons which are “hard” in our sense}), and the \(| \text{in} \rangle\) represents a generic initial CFT state.

We write the pure gauge form of \(A\) solving the (Lorentz-gauge) CS equations as
\[
A_\mu(x) = \partial_\mu \lambda(x), \quad \Box_{\text{Mink}3} \lambda(x) = 0.
\] (5.10)

We can specify a particular solution in terms of the “initial” value \((t = 0)\), \(\bar{a}(u, \bar{u}) \equiv A_{\bar{u}}(u, \bar{u}, t = 0)\), first determining
\[
\lambda(u, \bar{u}, t = 0) = \int \frac{d^2 u'}{2\pi} \frac{\bar{a}(u', \bar{u}')}{u - u'},
\] (5.11)
and then uniquely extending to all \(t\) once we impose only positive frequencies (absorbing source) in \(\lambda(u, \bar{u}, t)\),
\[
\lambda(q_u, q_{\bar{u}}, t) = \lambda(q_u, q_{\bar{u}}, t = 0) e^{-2i\sqrt{q_u q_{\bar{u}}}} t,
\] (5.12)

By the CFT current Ward identity,
\[
\partial_\mu J^\mu = - \sum_\alpha Q_\alpha \delta^3(x - x_\alpha),
\] (5.13)
we find
\[
i \int d^3 x A_\mu(x) J^\mu(x) = i \sum_\alpha Q_\alpha \lambda(x_\alpha)
\] (5.14)

\(^2\)In both Mink\(_4\) and AdS\(_4\) it is important that the helicity is fixed as we take the soft limit.
Figure 2: Typical $\partial \text{AdS}_4$ correlators involving 4D photons and matter particles, dual to CFT$_3$ correlators of the form Eq. (5.9) involving the $U(1)$ current and other local operators. (a) corresponds to charged matter lines arriving at general times on the boundary, while (b) corresponds to the special case in which all charged matter arrives at $t = 0$.

5.3 2D Holomorphic Abelian WZW Current from Holographic Soft Limit

Let us focus first on the special case that all the $\mathcal{O}_\alpha$ are simultaneous, $t_\alpha = 0$, as depicted in Fig. 2b, so that by Eqs. (5.14), (5.11),

$$i \int d^3 x A_\mu(x) J^\mu(x) = -i \int \frac{d^2 u}{2\pi} \tilde{a}(u, \bar{u}) \sum_\alpha \frac{Q_\alpha}{u - u_\alpha}. \quad (5.15)$$

Thinking of $\tilde{a}(u, \bar{u})$ as a source defining a 2D current $j \equiv 2\pi i \delta/\delta \tilde{a}(u, \bar{u})$, we arrive at a 2D holomorphic form for $j$,

$$\langle 0|j(u, \bar{u}) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)|\text{in}\rangle = \sum_\alpha \frac{Q_\alpha}{u - u_\alpha} \langle 0|\mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)|\text{in}\rangle. \quad (5.16)$$

The simple pole structure of $j$ is clearly very similar to that observed in soft limits of the Mink$_4$ S-matrix. We can straightforwardly obtain multiple-$j$ correlators since the source is simply exponentiated, but there is no central extension singularity in $jj$ correlators as they coincide, for reasons further discussed in the next section.

In the general case of non-simultaneous $t_\alpha$ (Fig. 2a), Eq. (5.14) gives a 2D current defined by source $\tilde{a}$,

$$j(u, \bar{u}) = -2\pi \sum_\alpha Q_\alpha \frac{\partial \tilde{\lambda}[\tilde{a}, x_\alpha]}{\partial \tilde{a}(u, \bar{u})}, \quad (5.17)$$
but this is no longer holomorphic, reminiscent of the case of massive charges in the Minkowski S-matrix. We explore this non-holomorphic structure more closely in Section 7 in the context of the memory effect.

6 Non-abelian Generalization of Holographic Soft Limit and AS

There is a natural generalization of “soft” to (tree-level) non-Abelian AdS\(_4\) gauge theory. Generalizing Eq. (5.3), we consider a 4D “soft” field \(A^a_\mu\) which is a complex solution to the 4D Yang-Mills equations, coupled to a 4D gauge current \(J^{\mu}_{\mu}^a\) representing other charged matter and “hard” gluons. The boundary limit \(A^a_\mu \overset{z \to 0}{\longrightarrow} A^a_\mu\) of such a complex solution simply corresponds to a complex source \(A^a_\mu\) for \(J^{\mu}_{\mu}^\text{CFT}\) and its associated CFT correlators.

When there are multiple “soft” gluons, we must generalize the fixing of helicity of “soft” photons in the Abelian case in a manner that is compatible with Yang-Mills self-couplings. This is given by requiring the complex \(A^a_\mu\) to be self-dual (or alternatively, anti self-dual):

\[
\frac{1}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu}(x,z) = i F^{\rho\mu}_{\text{axial}} = i \partial_z A^a_{\rho}(x,w),
\]

where \(F\) is the full non-abelian 4D field strength. This is closely analogous to what is seen in 4D Minkowski spacetime, where the non-abelian soft “branches” attached to a hard scattering process are self-dual when all its external soft gluons have positive helicity [30].

In axial-gauge, the holographic soft limit is again that in which \(A^a_\mu\) is \(z\)-independent. Self-duality then implies the vanishing of all of \(F\), so that \(A\) is pure-gauge. The CFT source is simply given by \(A^a_\mu \equiv A^a_\mu(x,z \to 0) = A^a_\mu(x)\), so that it satisfies a (sourceless) non-Abelian CS equation,

\[
\epsilon^{\mu\nu\rho} F^a_{\nu\rho}(x) = 0,
\]

again closely analogous to the Minkowski analysis. More precisely, there will also be an effective 3D gauge-fixing condition that results from the approach to the soft limit, but it will be more complicated than the simple 3D Lorentz gauge of the Abelian case, Eq. (5.7). As for the Abelian case, this condition will not be relevant for the special case of equal-time correlators of CFT local operators, to which we now turn.

6.1 2D Holomorphic Non-abelian WZW Current from Holographic Soft Limit

The vanishing of the non-Abelian field strength of the source in the soft limit has the solution,

\[
i A_\mu(x) = e^{-i\lambda(x)} \partial_\mu e^{i\lambda(x)}, \quad \lambda \equiv \lambda^a t^a, \quad A_\mu \equiv A^a_\mu t^a,
\]

where the \(\lambda^a(x)\) are complex gauge transformation fields, reflecting the complex nature of \(A^a_\mu\) (necessary for Lorentzian self-dual gauge fields). Starting from the general correlator,

\[
\left\langle T \left\{ e^{i \int d^3 x \, A^a_\mu(x) J^{\mu\nu}(x) O_1(x_1) \ldots O_n(x_n) } \right\} \right\rangle,
\]

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we will again consider $\bar{a}^a(u, \bar{u}) \equiv A^a_{\mu}(u, \bar{u}, t = 0)$ as the independent variables behind our soft source $A_\mu(x)$, and define a 2D current

$$j^a(u, \bar{u}) \equiv 2\pi i \frac{\delta}{\delta \bar{a}^a(u, \bar{u})}.$$ \hfill (6.5)

For single $j$ correlators with equal-time “hard” operators, $t_\alpha = 0$, the non-Abelian structure is clearly irrelevant, and we arrive at the analog of Eq. (5.16) again,

$$\langle 0| j^a(u, \bar{u}) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)|\text{in}\rangle = \sum_\alpha \frac{T_{(\alpha)}}{u - u_\alpha} \langle 0| \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n)|\text{in}\rangle. \hfill (6.6)$$

Next we probe correlators $\langle j^a(u, \bar{u}) j^b(u', \bar{u}') \cdots \rangle$, to search for a non-abelian contribution to the $jj'$ 2D “OPE”. This requires us to work to order $\bar{a}^2$. At first order in $\bar{a}$, we obviously have

$$\lambda^{(1)} a(u, \bar{u}, t = 0) = 2 \int \frac{d^2 u'}{2\pi} \bar{a}^a(u', \bar{u}') \frac{\partial}{\partial u'},$$ \hfill (6.7)

as in the Abelian case. To second order, by Eq. (6.3),

$$A^a_\mu(x) \approx \partial_\mu \lambda^{(1)} a(x) - \frac{1}{2} f^{abc} \lambda^{(1)} b(x) \partial_\mu \lambda^{(1)} c(x) + \partial_\mu \lambda^{(2)} a(x). \hfill (6.8)$$

We can use the $\bar{u}$ component of this to solve for $\lambda^{(2)} (t = 0)$,

$$\partial_\bar{u} \lambda^{(2)} a(u, \bar{u}, t = 0) = \frac{1}{2} f^{abc} \lambda^{(1)} b(u, \bar{u}, t = 0) \bar{a}^c(u, \bar{u}), \hfill (6.9)$$

from which we derive

$$\lambda^{(2)} a(u, \bar{u}, t = 0) = \frac{1}{2} f^{abc} \int \frac{d^2 u'}{2\pi} \int \frac{d^2 u''}{2\pi} \frac{\bar{a}^b(u', \bar{u}') \bar{a}^c(u'', \bar{u}'')}{(u - u'')(u'' - u')}. \hfill (6.10)$$

In this way we see two types of non-abelian corrections enter into the typical $\partial \text{AdS}_4/\text{CFT}_3$ correlator compared to the abelian case, as depicted in Fig. 3. Of course there are non-abelian interactions in the 4D bulk, but we also have non-abelian corrections to the CFT “softened” source $A^a_\mu$ when expressed in terms of the independent variables $\bar{a}^a$.

We see that Eq. (6.10) can give rise to a non-trivial “OPE” divergence for coinciding $j$’s, so we drop $\lambda^{(1)}$ contributions to focus on that of $\lambda^{(2)}$:

$$\int d^3 x A^a_\mu(x) J^{\mu a}(x) \supset \int d^3 x \partial_\mu \lambda^{(2)} a(x) J^{\mu a}(x)$$

$$= - \int d^3 x \lambda^{(2)} a(x) \partial_\mu J^{\mu a}(x) = \sum_\alpha \lambda^{(2)} a(x_\alpha) T_{(\alpha)}^a. \hfill (6.11)$$

Specializing to the simultaneous limit, $t_\alpha = 0$,

$$\int d^3 x A^a_\mu(x) J^{\mu a}(x) \supset \sum_\alpha \lambda^{(2)} a(u_\alpha, \bar{u}_\alpha, t_\alpha = 0) T_{(\alpha)}^a$$

$$= \frac{1}{2} f^{abc} \int \frac{d^2 u}{2\pi} \int \frac{d^2 u'}{2\pi} \frac{\bar{a}^b(u, \bar{u}) \bar{a}^c(u', \bar{u}')}{(u_\alpha - u')(u' - u)} T_{(\alpha)}^a. \hfill (6.12)$$
Figure 3: A typical $\partial \text{AdS}_4$ correlator for non-abelian AdS gauge theory, with all hard matter arriving at $t = 0$. Note that there are both non-abelian bulk interactions and non-abelian corrections to the “softened” source in terms of the independent variables $\bar{a}^a$. The leading source term $A^{(1)}$ is similar in form to the abelian case, while the next non-abelian correction $A^{(2)}$ is given by the last two terms in Eq. (6.8).

We thereby derive,

$$\langle 0 | T \left\{ j^a(u, \bar{u}) j^b(u', \bar{u}') \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \right\} | \text{in} \rangle$$

$$\sim \frac{1}{2} f^{abc} \sum_{\alpha} \frac{T^c_{\alpha}}{(u_{\alpha} - u)(u - u')} \left( \frac{1}{u_{\alpha} - u'} - \frac{1}{u_{\alpha} - u} \right) \langle 0 | \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) | \text{in} \rangle$$

$$= \frac{f^{abc}}{u' - u} \langle 0 | T \left\{ j^c(u, \bar{u}) \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \right\} | \text{in} \rangle. \quad (6.13)$$

In this sense, we have arrived at the Euclidean 2D KM “OPE”,

$$j^a(u, \bar{u}) j^b(u', \bar{u}') \sim \frac{f^{abc}}{u - u'} j^c(u, \bar{u}), \quad (6.14)$$

but unlike the canonical Eq. (3.11) we see that we have vanishing central extension here! This absence of a central extension in AS from soft limits matches what is seen in 4D Minkowski spacetime. But as pointed out in Ref. [30], it is closer to the truth to say that we have infinite central extension, as we review below.

### 6.2 Holographic Soft Limit as Portal from Standard to Alternate Theory

The structure of correlators of $j$ we see in the holographic soft limit with D b.c precisely matches that found in Ref. [31] for alternate b.c in the $\kappa \to \infty$ limit, as shown there by
simple $\kappa$-counting diagrammatic arguments. Here, we just give a heuristic argument for why this is so, based on the path integral for dynamical CS coupled to the CFT (dual to alternate b.c.),

$$\int \mathcal{D}A_\mu \exp \left\{ i \int d^3x \frac{\kappa}{4\pi} \epsilon^{\mu\nu\rho} \text{Tr} \left( A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) + A_\mu^a J^{\mu a}_\text{CFT} \right\}. \quad (6.15)$$

We see that as the CS level $\kappa \to \infty$, there is a wild phase in the path integral, forcing the $\kappa$-dependent part of the action to be extremized, yielding Eq. (6.2), derived here via the “soft” limit. With the $t = 0$ condition on the path integral, $A_\mu^a(t = 0) = \bar{a}^a$ (and gauge-fixing), this leads to a specific $A_\mu^a(x)$. In this way, the alternate b.c. becomes effectively D b.c as $\kappa \to \infty$, in particular matching the holographic soft limit. The one “flaw” with this argument is that the $\kappa \to \infty$ limit for dynamical $A$ is ill-defined for $jj'$ correlators, precisely because of the central term in Eq. (3.11). As pointed out in Ref. [31], this is avoided by only considering connected correlators of the CS fields with the CFT, since the central term arises from connected correlators of CS with only itself. From the D. b.c. viewpoint, this restriction is automatic since we are always considering soft dressing of “hard” CFT correlators. With this restriction, the central extension of KM is absent, as if it vanished, when in fact it is infinite as $\kappa \to \infty$.

The seeds of alternate b.c. correlators are contained in the D. b.c. AdS$_4$ (pure CFT$_3$) correlators via their holographic soft limits. One can then unitarize these leading-in-$\kappa$ correlators by going to finite large $\kappa < \infty$, and including the simple pure-CS correlators, which contain the central extension. In this nuanced sense, AS from soft limits are a remnant of the alternate b.c theory, dual to the CS-gauged CFT$_3$.

## 7 CS Memory Effects and the Holographic Soft Limit

Finally, we point out that AdS$_4$ gauge theory exhibits an analog of the electromagnetic “memory” phenomenon of Mink$_4$ [18, 37–39], closely connected to AS structure. The memory effect compares the parallel transport between two test charges far from a scattering process, long before and after the scattering event, more precisely given by a Wilson loop consisting of spatial transport between the two charges at early and late times, and temporal transport between those times. We focus on the abelian case.

### 7.1 Alternate boundary conditions and electric memory

We begin with alternate b.c., in its dual formulation as $U(1)$ CS + CFT$_3$. Canonically, the CS fields are $A_0, A_u$, effectively in temporal gauge $A_0 = 0$ after deriving the Gauss Law constraint. For simplicity focusing on vanishing electromagnetic field strengths at early times (hence only neutral particles in the initial state), we can choose the further gauge condition $A_u(t = -\infty), A_0(t = -\infty) = 0$. We see that our canonical (can) fields therefore
precisely define “memory” Wilson loops in more general (gen) gauges,

\[ A_i^\text{can}(u, \bar{u}, t = 0) dx^i = A_i^\text{gen}(u, \bar{u}, t = 0) dx^i + \int_0^{\infty} dt' A_0^\text{gen}(u + du, \bar{u} + d\bar{u}, t') \]

\[ - A_i^\text{gen}(u, \bar{u}, t = -\infty) dx^i + \int_{-\infty}^0 dt' A_0^\text{gen}(u, \bar{u}, t') , \quad \text{where } i \equiv u, \bar{u}. \]

(7.1)

The four terms on the right define four sides of a narrow gauge-invariant “memory” Wilson loop, from \( u \) to \( u + du \) at time \( t = 0 \), to time \( -\infty \) at \( u + du \), back from \( u + du \) to \( u \) at time \( -\infty \), and then from time \( -\infty \) to \( t = 0 \) at \( u \). Similarly, a Wilson line of \( A^\text{can} \) along a finite spatial curve \( C \) in the \( x - y \) plane at time \( t = 0 \) is equivalent to a more general memory Wilson loop in a general gauge, completing the curve with time-like lines to \( t = -\infty \) and spatial Wilson line reversing \( C \) at time \( -\infty \). This is depicted in Fig. 4. Because of the Gauss Law constraint, the precise choice of \( C \) does not matter as along as one does not cross 3D charges in deforming the curve.

In the above sense, arbitrary CS gauge theories describe the dynamics of memory effects in 3D. But when the CS charged matter is a CFT\(_3\) with AdS\(_4\) dual, the memory effects “lift” to 4D. The 3D memory Wilson loop above is now seen as a 4D memory Wilson loop in a general gauge, completing the curve with time-like lines to \( t = -\infty \) and spatial Wilson line reversing \( C \) at time \( -\infty \). This is depicted in Fig. 4. Because of the Gauss Law constraint, the precise choice of \( C \) does not matter as along as one does not cross 3D charges in deforming the curve.

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Figure 4: A general CS memory Wilson loop in Mink$_3$, comparing parallel transport along the spatial curve $C$ at early and late times, where for simplicity the early state has vanishing gauge field strength. It can be viewed as composed of many narrow memory Wilson loops, with shared timelike lines canceling due to their opposing orientations. In terms of the canonical CS fields, effectively in temporal gauge, this general Wilson loop is therefore given by just the late Wilson line along $C$. (See Eq. (7.1))

$\partial$AdS$_4$ at $t_\alpha < 0$, before the memory measurement at $t = 0$. More generally, we take the radiation and particles to arrive at $\partial$AdS$_4$ earlier than $t = 0$, and either be reflected away into the bulk or absorbed by boundary/CFT operators. Therefore, radiation from the bulk scattering does not contribute to the boundary $\tilde{A}$ gauge fields at $t = 0$. This set-up is depicted in Fig. 5.

But further radiation can result when the charged particles are absorbed by $O_\alpha$ on $\partial$AdS$_4$, effectively “annihilating” with their images in the Mink$_4$ covering space of Mink$_4$/2 $\sim$ AdS$_4$. This secondary radiation from $z \sim 0$ can spread until $t = 0$ and contribute to the boundary fields $\tilde{A}$ then. In temporal gauge, the transverse radiation satisfies $\partial_x \tilde{A}_x + \partial_y \tilde{A}_y + \partial_z \tilde{A}_z = 0$ as usual. Since the secondary radiation travels in the $x - y$ directions but remains at $z \sim 0$ in order to contribute to the memory measurement there, the $z$-momentum is subdominant, and we have

$$\partial_x \tilde{A}_x + \partial_y \tilde{A}_y \equiv \partial_u \tilde{A}_u(u, \bar{u}, t = 0) + \partial_{\bar{u}} \tilde{A}_{\bar{u}}(u, \bar{u}, t = 0) \approx 0.$$ (7.3)

We can then solve the simultaneous equations, Eqs. (7.2), (7.3), for the memory fields,

$$\tilde{A}_u(u, \bar{u}, t = 0) = -\frac{ig^2}{4} \int \frac{d^2 u'}{2\pi} \frac{J_0(u', \bar{u}', t = 0)}{u' - u},$$

$$\tilde{A}_{\bar{u}}(u, \bar{u}, t = 0) = \frac{ig^2}{4} \int \frac{d^2 u'}{2\pi} \frac{J_0(u', \bar{u}', t = 0)}{\bar{u} - \bar{u}'}.$$ (7.4)

We now show that the above memory effect precisely matches the holographic soft limit we derived in Section 5. First we note that the secondary radiation satisfies the Maxwell
Figure 5: A $\partial \text{AdS}_4$ correlator for radiation and charged matter created by a distant bulk scattering, initiated from an electromagnetically neutral state. We focus on a 't Hooft line at $t = 0$ in temporal gauge, corresponding to a magnetic memory loop, allowed by the standard boundary conditions. It receives contributions from the secondary radiation emitted by charged matter annihilated at the boundary by local operators. Radiation from the bulk scattering is either absorbed by the CFT current $J^\mu$ or reflected by the boundary, and therefore does not contribute to the late-time 't Hooft line.

Equations,

\[ 0 = \partial_i B_i + \partial_z B_z \approx \partial_i B_i \]
\[ 0 = \partial_0 B_i + \epsilon_{ij} \partial_j E_z - \epsilon_{ij} \partial_z E_j \approx \partial_0 B_i + \epsilon_{ij} \partial_j E_z, \quad \text{where } i \equiv x, y, \]

(7.5)

and where again the $z$-momentum is subdominant so that we drop the $\partial_z$ terms. Since we are near the boundary, we can translate $B_i \rightarrow g^2 \epsilon_{ij} J_j$ and $E_z \rightarrow g^2 J_0$, so that the above relations become

\[ \epsilon^{\mu\nu\rho} \partial_\nu J_\rho \approx 0. \]

(7.6)

Therefore $J_\mu \approx \partial_\mu \Phi$ is a total gradient. The current Ward identity, Eq. (5.13), then reads

\[ \partial_\mu \partial^\mu \Phi = -\sum_\alpha Q_\alpha \delta^3(x - x_\alpha), \]

(7.7)

with solution

\[ \Phi(x) = -i \sum_\alpha Q_\alpha G_S(x - x_\alpha), \]

(7.8)
where $G_S$ is the Mink$_3$ scalar $\Phi$ propagator. Therefore, Eq. (7.4), reads

$$\tilde{A}_u(u, \bar{u}, t = 0) = -\frac{g^2}{4} \sum_\alpha Q_\alpha \int \frac{d^2 u' }{2\pi} \frac{\partial_0 G_S(u' - u_\alpha, \bar{u}' - \bar{u}_\alpha, -t_\alpha)}{u - u'} . \quad (7.9)$$

Let us compare this result with the holographic soft limit for non-simultaneous $\mathcal{O}_\alpha$, as given by

$$j(u, \bar{u}) \equiv 2\pi i \frac{\delta}{\delta \bar{u}(u, \bar{u})} \int d^3 x' A_\mu(x') J^\mu(x')$$

$$= -2\pi \sum_\alpha Q_\alpha \frac{\delta \lambda(x_\alpha)}{\delta \bar{u}(u, \bar{u})}$$

$$= \sum_\alpha Q_\alpha \int \frac{d^2 u' }{u - u'} \int \frac{dq_u}{2\pi} \frac{dq_\bar{u}}{2\pi} e^{iq_u(u_\alpha - u')} e^{iq_\bar{u}(\bar{u}_\alpha - \bar{u}')} e^{-2i\sqrt{q_u q_\bar{u}} t_\alpha} \quad (7.10)$$

following from Eqs. (5.14), (5.12), (5.11). This precisely matches the form of memory, Eq. (7.9), since the time-ordering in $G_S$ is fixed because all $t_\alpha < 0$.

The special case of $t_\alpha \to 0$ in AdS$_4$ is analogous to the case of massless charges in Mink$_4$ reaching lightlike infinity, in each case leading to holomorphic $j$ with simple poles. We see this explicitly at $t_\alpha = 0$ in Eq. (7.10), where the Fourier transforms give $\delta^2(u' - u_\alpha)$. General $t_\alpha \neq 0$ in AdS$_4$ is analogous to the case of massive charges in Mink$_4$ which approach timelike infinity, in which case $j$ is not holomorphic. See Ref. [17, 18, 36] for the same smeared structure of poles in Mink$_4$ memory for massive charges as our Eq. (7.4). However, we see that in AdS$_4$ we have a clear holographic interpretation for this smearing in terms of the spreading of holographic charge density over time starting from $\delta$-function localization, $J_0 \propto \partial_0 G_S$, because the 3D charges are “blobs” of massless CFT fields. This is in contrast to a 3D theory with only 3D-massive point-particle charges (without 4D dual), where $J_0$ would retain the form of $\delta$-functions at particle locations over time, and the analogous construction of $j$ would have simple poles in $u$ without smearing over time.

8 Conclusion

In this paper we have studied infinite-dimensional Kac-Moody (KM) asymptotic symmetries arising in AdS$_4^{\text{Poincare}}$ gauge theories. The standard asymptotic analysis, famously admitting only the finite-dimensional global symmetries of a holographically dual CFT$_3$, was evaded in two steps, identified in Ref. [31] but taking their simplest form here. In the present context, the first step was to consider alternate AdS boundary conditions peculiar to four dimensions, holographically dual to a modified CFT$_3$ obtained by an external Chern-Simons (CS) gauging of the original CFT$_3$. The second step was to restrict attention to boundary/CFT correlators (or wavefunctional) at a fixed time, say $t = 0$, where the canonical CS structure yields KM symmetries.

We showed several ways in which the resulting KM structure bears a striking resemblance to that of gauge theories in 4D Minkowski spacetime, as summarized in Table 1. This is particularly clear when comparing with the Minkowski analysis of Ref. [30] where
an underlying CS structure to KM asymptotic symmetries was also found. Yet, naively, a close resemblance would have seemed unlikely. In 4D Minkowski spacetime KM asymptotic symmetries reflect gauge-boson soft limits, whereas standard AdS\textsuperscript{global} lacks soft limits. Nevertheless, we showed a simple generalization of “soft limits” in AdS\textsuperscript{Poincare} which underlies its KM asymptotic symmetries. In both Mink\textsubscript{4} and AdS\textsubscript{4}\textsuperscript{Poincare}, the limiting soft 4D fields take the form of 3D CS fields (in AdS, implying alternate boundary conditions) which then lead to KM symmetries on an effectively 2D boundary of the CS spacetime, via the CS/WZW correspondence and a 2D holomorphic WZW current.

In Mink\textsubscript{4}, this “2D boundary” effectively corresponds to lightlike infinity \( I \), whereas for AdS\textsubscript{4} it is provided by \( t = 0 \) on \( \partial \text{AdS} \). In both 4D spacetimes, restriction of charged particles to arrive at the 2D subspace is not generic. In Mink\textsubscript{4}, only exactly massless 4D charges arrive at \( I \), whereas massive particles head towards timelike infinity where the KM structure becomes “smeared” and there is a loss of holomorphicity of the 2D WZW current. In AdS\textsubscript{4}, generic boundary correlators with unequal times have similarly smeared KM structure and non-holomorphic WZW current. In Mink\textsubscript{4} there is also a close connection between KM symmetries and the memory effect, given by a large asymptotic Wilson loop, but in AdS\textsubscript{4} the analogous Wilson loop at the AdS boundary must vanish by standard boundary conditions. Nevertheless, we demonstrated that non-trivial “magnetic” memory effects exist even with standard boundary conditions in AdS\textsubscript{4}, associated with non-vanishing ’t Hooft loops on the boundary, and that these are closely related to soft limits and KM structure. Furthermore, the smearing effect in general non-simultaneous correlators finds a natural holographic explanation in the tendency of 3D charge density to spread in a CFT\textsubscript{3} even if initially created in point-like form by a local operator. The smearing of KM structure in Mink\textsubscript{4} for massive charges is so similar to this that it may provide some clue as to the form Minkowski holography might take.

While soft limits yield the alternate/CFT\textsubscript{3} theory to leading order in the associated CS level, in the sense of Ref. [31], it is interesting to see if the all-orders theory (finite CS level) can naturally emerge from the standard/CFT\textsubscript{3} construction. We showed this for the case of abelian gauge/global symmetry, where the standard construction imposed on electric-magnetic/mirror dual variables assumes the alternate/CFT\textsubscript{3} form in the original variables, with finite CS level in the holographic description! The KM symmetries were thereby seen to be generalizations of dyonic charge conservation rather than simple electric charge conservation. It is less clear whether there is a non-abelian generalization, given the key role played by the S-duality transformation exchanging electric and magnetic charges. Perhaps a good theoretical laboratory is provided by those special supersymmetric non-abelian theories in which S-duality persists [45].

Given the rich structure of asymptotic symmetries and memories in AdS\textsubscript{4}, and their deep connection with holography, it is obviously of interest to know whether they imply a new form of “hair” for complex 4D states such as black holes, and can algebraically encode information that might seem lost according to 4D effective field theory analysis. We hope that the simple form and derivation of asymptotic symmetry presented here will help to decide this issue in the future.
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