Simpler derivation of bounded pitch inequalities for set covering, and minimum knapsack sets
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Abstract
A valid inequality \(a^T x \geq \alpha_0\) for a set covering problem is said to have pitch \(\leq \pi\) (\(\pi\) a positive integer) if the \(\pi\) smallest positive \(\alpha_j\) sum to at least \(\alpha_0\). This paper presents a new, simple derivation of a relaxation for set covering problems whose solutions satisfy all valid inequalities of pitch \(\leq \pi\) and is of polynomial size, for each fixed \(\pi\). We also consider the minimum knapsack problem, and show that for each fixed integer \(p > 0\) and \(0 < \epsilon < 1\) one can separate, within additive tolerance \(\epsilon\), from the relaxation defined by the valid inequalities with coefficients in \(\{0, 1, \ldots, p\}\) in time polynomial in the number of variables and \(1/\epsilon\).

1 Introduction

In this paper we first consider set-covering problems. Formally, given the \(m \times n\), 0/1 matrix \(A\) we write

\[
\Sigma(A) \doteq \{ x \in \{0,1\}^n : Ax \geq e \}
\]

where \(e\) is the vector of \(m\) 1s. We assume, without loss of generality, that the supports of constraints in \(Ax \geq e\) never contain one another and have cardinality greater than 1. Under these assumptions any undominated inequality \(a^T x \geq \alpha_0\) valid for \(\Sigma(A)\) satisfies \(\alpha_0 \geq 0\) and \(\alpha_j \geq 0\) for \(1 \leq j \leq n\). Given such an inequality, we write \(S(\alpha) = \{1 \leq j \leq n : \alpha_j > 0\}\).

The set \(\Sigma(A)\) can have extreme points with high fractionality that are difficult to cut-off by standard integer programming techniques. Balas and Ng [1] characterize the facets for (1) with coefficients in \(\{0, 1, 2\}\). The work in [3] introduced a general class of combinatorial inequalities generalizing those consider in [1]. This work relies on the concept of pitch:

Definition 1 [3]. Given an inequality \(a^T x \geq \alpha_0\) with \(\alpha \geq 0\) and integer \(0 \geq \pi \leq |S(\alpha)|\), we say that \(a^T x \geq \alpha_0\) is of pitch \(\leq \pi\) if the \(\pi\) smallest positive entries in \(\alpha\) sum to at least \(\alpha_0\). When \(\pi > 0\) the inequality is said to be of pitch \(= \pi\) if it is of pitch \(\leq \pi\) and it is not of pitch \(\leq \pi - 1\).

As a result, an inequality with coefficients in \(\{0, 1, \ldots, \pi\}\) has pitch \(\leq \pi\). Note that when \(\pi = 0\) we must have \(\alpha_0 \leq 0\). The main result in [3] is the following:

Theorem 2 [3]. Consider a set-covering problem given by a matrix \(A\), and let \(\pi \geq 2\) be a fixed integer. There is a polynomial-sized extended formulation whose projection to \(x\)-space satisfies all valid inequalities for \(\Sigma(A)\) of pitch \(\leq \pi\).

Here, an extended formulation is of the form \(\{(x, y) \in \mathbb{R}^{N \times m} : Cx + Dy \geq b\}\) (for some \(N, C, D\) and \(b\)) whose projection to \(x\)-space contains \(\Sigma(A)\), and the theorem states that the size of this description (number of bits) is polynomial in \(n\) and \(m\). The construction in [3] is admittedly complex. Also see [4]. Recent work [10], [6] has provided new constructions related to bounded pitch inequalities for set covering.

A first result in the paper provides a simpler construction for Theorem [2].

Theorem 3. Given a set-covering problem given by a matrix \(A\), and integer \(\pi \geq 2\) there is a disjunctive formulation for (1) with \(O(mn^2 + \pi m^{\pi-1}n)\) variables and constraints.

In the second part of this paper we consider the so-called minimum-knapsack problem, i.e. a problem whose feasible region is of the form

\[
\left\{ x \in \{0,1\}^n : \sum_{j=1}^n w_j x_j \geq w_0 \right\}
\]
with \( w_j \in \mathbb{Z}_+ \) for \( 0 \leq j \leq n \). Obviously by complementing the variables we obtain a standard ("maximum") knapsack problem; however from a polyhedral standpoint there are significant differences.

Without loss of generality, we assume \( w_j \leq w_0 \) for \( 1 \leq j \leq n \). For a vector \( v \in \mathbb{R}^n \) and a subset of indices \( S \subseteq \{1, \ldots, n\} \) we write \( v(S) = \sum_{j \in S} v_j \). As is well-known, a minimum-knapsack problem can be equivalently restated as a set-covering problem, albeit one with an exponential number of rows:

**Remark 4** Any minimum-knapsack set \( \Pi = \{ x \in \{0,1\}^n : \pi^T x \geq \pi_0 \} \) with \( \pi \geq 0 \) is equivalently described by a set covering system, namely

\[
x \in \{0,1\}^n : x(S) \geq 1 \quad \forall S \subseteq \text{supp}(\pi) \quad \text{with } \pi(S) \geq \sum_{j=1}^n \pi_j - \pi_0 + 1,
\]

i.e. the set of cover inequalities for \( \Pi \). As a corollary, an inequality \( \alpha^T x \geq \alpha_0 \) with \( \alpha \geq 0 \) and \( \text{supp}(\alpha) \subseteq \text{supp}(\pi) \) is valid for \( \Pi \) iff, \( \forall S \subseteq \text{supp}(\alpha) \), we have \( \pi(S) \geq \sum_{j=1}^n \pi_j - \pi_0 + 1 \) whenever \( \alpha(S) \geq \sum_{j=1}^n \alpha_j - \alpha_0 + 1 \).

Here, a cover inequality is a valid inequality of the form \( x(S) \geq 1 \) for some \( S \subseteq \{1, \ldots, n\} \). Given remark 4 one wonders if a result similar to Theorem 2 exists for the minimum-knapsack problem. This question has been taken up in recent work [5], where the following result is proved:

**Theorem 5** [5] Given a minimum-knapsack problem \( x^* \in [0,1]^n \) with \( \alpha \geq 0 \) and \( \text{supp}(\alpha) \subseteq \text{supp}(\pi) \) is valid for \( \Pi \) iff, \( \forall S \subseteq \text{supp}(\alpha) \), we have \( \pi(S) \geq \sum_{j=1}^n \pi_j - \pi_0 + 1 \) whenever \( \alpha(S) \geq \sum_{j=1}^n \alpha_j - \alpha_0 + 1 \).

This result does not guarantee strict separation; however one can show that an inequality of pitch = 1 is violated iff a cover inequality is violated, and separation of such inequalities is known to be NP-hard. In this sense Theorem 5 is best possible, for pitch \( \leq 2 \).

In this paper we prove the following result:

**Theorem 6** Given a minimum-knapsack problem \( x^* \in [0,1]^n \) with \( \alpha \geq 0 \) and integer \( p \geq 1 \), there is an algorithm that, with input \( x^* \) either finds a valid inequality for \( \Pi \) of pitch = \( p \) that is violated by \( x^* \) or shows that \( x^* \) satisfies all valid inequalities of pitch \( p \) within multiplicative error \( \epsilon \). The complexity of the algorithm is polynomial in \( n \) and \( 1/\epsilon \).

One can show that a valid inequality of pitch \( \leq 2 \) is violated by a given vector \( x^* \) iff \( x^* \) violates a valid inequality with coefficients in \( \{0,1,2\} \); thus Theorem 6 generalizes Theorem 5.

## 2 Set covering

In this section we provide a short construction leading to Theorem 6. First we will motivate our approach by introducing a branching technique for 0/1 integer programming that is of independent interest, and which we term vector branching.

Suppose the inequality

\[
\sum_{j \in S} a_j x_j \geq \alpha_0 (> 0)
\]

is valid for a mixed-integer set \( \mathcal{F} \), where the \( x \) variables are assumed to be binary. Write \( S = \{ j_1, j_2, \ldots, j_{|S|} \} \), and for \( 1 \leq h \leq |S| \) define the set

\[
F_h = \{ x \in \{0,1\}^{|S|} : x_{j_i} = 0 \ \forall \ i < h \ \text{and} \ x_{j_h} = 1 \}.
\]
Then the disjunction
\[ F_1 \lor F_2 \lor \ldots \lor F_{|\mathcal{S}|} \]
is valid for \( \mathcal{F} \) since \( \mathcal{F} \) implies \( \sum_{j \in S} x_j \geq 1 \). This simple observation can be used to drive a branch-and-bound algorithm to solve optimization problems over \( \mathcal{F} \). When processing a node \( v \) of branch-and-bound, this scheme (a) identifies an inequality (3) that is valid for the relaxation to \( \mathcal{F} \) used at \( v \) and (b) creates \( |\mathcal{S}| \) new nodes, corresponding to each of the terms in the disjunction (4). The node corresponding to term \( h \) of (4) imposes the constraints defining \( F_h \) in addition to all those present at \( v \). We call this procedure vector branching (5).

Here we are interested in the implications of this scheme toward set covering problems (1) when the inequalities (3) used to drive vector branching are rows of \( Ax \geq e \). Lemma 8 given below will be used to motivate our proof of Theorem 2, but first, for self-containment, we prove a basic result (6) which will be used in the sequel.

**Lemma 7** Suppose \( \sum_{j \in T} \alpha_j x_j \geq \alpha_0 \) is valid inequality for (1) with \( \alpha \geq 0 \) and \( \alpha_0 > 0 \). (1) There is a row \( \sum_{j \in S} \geq 1 \) of \( Ax \geq e \) such that \( S \subseteq T \). (2) Let \( k \in T \) be such that \( \alpha_k < \alpha_0 \). Then there is a row \( \sum_{j \in S} \geq 1 \) of \( Ax \geq e \) such that \( S \subseteq T - k \).

**Proof.** (1) Otherwise setting \( x_j = 1 \) if \( j \notin T \) and \( x_j = 0 \) otherwise yields a feasible binary solution to \( Ax \geq e \) which violates \( \sum_{j \in T} \alpha_j x_j \geq \alpha_0 \). (2) Otherwise setting \( x_j = 1 \) if \( j \notin T \) or \( j = k \), and \( x_j = 0 \) otherwise, yields a feasible binary solution to \( Ax \geq e \) which violates \( \sum_{j \in T} \alpha_j x_j \geq \alpha_0 \).

**Lemma 8** Suppose we apply vector branching to a set covering problem (1). Consider a node that arises when we vector-branch on one of the rows, \( \sum_{j \in S} x_j \geq 1 \), of \( Ax \geq e \). Let \( y \in \mathbb{R}^n \) be a feasible solution to the relaxation at that node. Then \( y \) satisfies every inequality
\[ \sum_{j \in T} \alpha_j x_j \geq \alpha_0 \]
with \( \alpha_j > 0 \) for all \( j \in T \) which is valid for (1), of pitch \( \leq 2 \), and such that \( S \subseteq T \).

**Proof.** By construction \( y_k = 1 \) for some \( k \in T \), so we may assume \( \alpha_k < \alpha_0 \). By Lemma 7 (2) there is some row \( \sum_{j \in S'} x_j \geq 1 \) of \( Ax \geq e \) with \( S' \subseteq T - k \). As a result
\[ \sum_{j \in T} \alpha_j y_j \geq \sum_{j \in S'} \alpha_j y_j + \alpha_k \geq \alpha_0 \]
since \( \sum_{j \in S'} \alpha_j y_j \geq \min \{ \alpha_j : j \in S' \} \).

This result epitomizes the techniques that we will use below to obtain a proof of Theorem 2, the result relies on the explicit variable-fixing constraints defining the \( F_h \) plus the structural properties of valid inequalities for set covering. Letchford [9] has used a similar idea.

Note that Lemma 8 does not yield a strategy for developing a polynomial-size branch-and-bound tree whose leaf nodes satisfy all valid inequalities of pitch \( \leq 2 \). In order to prove Theorem 2, we instead rely on the well-known equivalence between branching and disjunctive formulations. Specifically, for each integer \( \pi \geq 1 \) we will present a lifted formulation of the form
\[ C^\pi x + D^\pi y^\pi \geq b^\pi, \quad (x, y^\pi) \in [0, 1]^n \times [0, 1]^{N^\pi} \]
where, for some integer \( N^\pi, y^\pi \in \) is a vector of additional variables and \( C^\pi, D^\pi, b^\pi \) are of appropriate dimension, such that
(a) (13) is a relaxation of (1), i.e. any feasible solution to (1) can be lifted so as to satisfy (13).
(b) The \( x \)-component of any vector \((x, y^\pi)\) feasible for (20) satisfies every valid inequality for (1) of pitch \( \leq \pi \), and

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1While this technique may amount to folklore, it was used in [11].
We term (6) the level-$\pi$ formulation. We will first inductively describe the level-$\pi$ formulation and then prove that it satisfies the desired properties. We start with the level-1 formulation, which is simply the original set-covering set of inequalities and variable bounds:

$$Ax \geq e, \quad x \in [0, 1]^n$$

(7)

and then define

$$x_j^t = 1, \quad x_j^h = 0, \quad \forall \ 1 \leq h < t,$$

(8)

and then define

$$D_{\pi+1}^t = \text{conv}\{D_{\pi+1}^t : 1 \leq t \leq |S_i|\}.$$  

(9)

Finally, let

$$M_{\pi+1} = \bigcap_i D_{\pi+1}^i.$$  

(10)

The intuition here is that for each $1 \leq i \leq m$, the collection of all systems (7), (8), plus (9) indeed implements vector branching on the $i^{th}$ constraint of $Ax \geq e$. To this end, let $1 \leq i \leq m$, and suppose the support of the $i^{th}$ row of $A$ is $S_i = \{j_1, \ldots, j_{|S_i|}\}$. Then, for each $1 \leq t \leq |S_i|$, we define the polytope $D_{\pi+1}^i(t) \subseteq [0, 1]^n$ by the system

$$x_{j_i} = 1, \quad x_{j_h} = 0, \quad \forall \ 1 \leq h < t,$$

(7)

and then define

$$D_{\pi+1}^t = \text{conv}\{D_{\pi+1}^t : 1 \leq t \leq |S_i|\}.$$  

(9)

Finally, let

$$M_{\pi+1} = \bigcap_i D_{\pi+1}^i.$$  

(10)

The intuition here is that for each $1 \leq i \leq m$, the collection of all systems (7), (8), plus (9) indeed implements vector branching on the $i^{th}$ constraint of $Ax \geq e$ from a disjunctive perspective, while (10) enforces the simultaneous application of all such disjunctions. When $\pi = 2$, a simple rewording of Lemma 8 shows that every point in $M^2$ satisfies all pitch $\leq 2$ valid inequalities.

Now we will provide our formal description of the level-$\pi$ formulation.

(I) Let $1 \leq i \leq m$. For $1 \leq t \leq |S_i|$, define the polyhedron $D_{\pi+1}^i(t)$ by the system

$$x_{j_i} = 1, \quad x_{j_h} = 0, \quad \forall \ 1 \leq h < t,$$

(7a)

$$C^\pi x + D^\pi y^\pi \geq b^\pi$$

(11b)

$$(x, y^\pi) \in [0, 1]^n \times [0, 1]^{N^\pi}$$

(11c)

Let $D_{\pi+1}^i = \text{conv}\{D_{\pi+1}^i(t) : 1 \leq t \leq |S_i|\}.$

Remark: the projection of $D_{\pi+1}^i(t)$ to $x$-space is precisely $D_{\pi+1}^i(t)$, and likewise the projection of $D_{\pi+1}^i$ to $x$-space is $D_{\pi+1}^i$. 
Formally, $D^{\pi+1}_i$ is described by the system

\begin{equation}
\sum_{t=1}^{\vert S_i \vert} x_{j_t}^{\pi,i,t} = 1, \quad \text{and,} \tag{12a}
\end{equation}

\begin{align}
\forall \ t \text{ with } 1 \leq t \leq \vert S_i \vert:
\end{align}

\begin{align}
x_{j_h}^{\pi,i,t} - x_{j_h}^{\pi,i,h} \leq 0 \quad & \forall h \text{ with } t \leq h \leq \vert S_i \vert, \tag{12b}
x_{j_h}^{\pi,i,t} = 0 \text{ for } 1 \leq h < t \tag{12c}
y_{h}^{\pi,i,t} - x_{j_h}^{\pi,i,t} \leq 0 \quad & \forall h \text{ with } 1 \leq h \leq N^\pi \tag{12d}

C^\pi x_{\pi,i,t} + D^\pi y_{\pi,i,t} - x_{j_h}^{\pi,i,t} b^\pi \geq 0 \tag{12e}
\end{align}

\begin{align}
x = \sum_{t=1}^{\vert S_i \vert} x_{\pi,i,t} \tag{12f}
\end{align}

\begin{align}
x \in [0,1]^n, \ x_{\pi,i,t} \in [0,1]^n, \ y_{\pi,i,t} \in [0,1]^{N^\pi} \tag{12g}
\end{align}

**Remark:** Constraints (12b), (12c), and (12d), together with (12a), enforce the desired vector-branching disjunction (i.e. the disjunction over all the systems (11)), while (12a) is the linearization of (11b).

\[\text{(II)}\] The level-$(\pi+1)$ formulation is the union, over $1 \leq i \leq m$, of all systems (12).

Next we prove the desired facts regarding this formulation.

**Lemma 9** Let $\hat{x}$ be a feasible solution to the set covering problem (1), i.e. $\hat{x} \in \{0,1\}^n$ satisfies $A\hat{x} \geq e$. Then for every integer $\pi \geq 1$, $\hat{x}$ can be lifted to a vector feasible for the level-$\pi$ formulation.

**Proof.** By induction on $\pi$ with the case $\pi = 1$ valid by definition. Assume that the assertion has been proved for $\pi$; we now show it is true for $\pi + 1$. It suffices to prove, for $1 \leq i \leq m$, that $\hat{x}$ can be lifted to a vector contained in $D^{\pi+1}_i$. And in order to prove this fact we need to show that $\hat{x}$ can be lifted to a vector contained in $D^{\pi+1}_i(t)$ for some $t$ with $1 \leq t \leq \vert S_i \vert$. This fact follows by setting $t = \min\{1 \leq h \leq \vert S_i \vert : \hat{x}_h = 1\}$, and induction. \[\square\]

**Lemma 10** Let $\pi \geq 1$ and suppose $(\hat{x}, \hat{y})$ is a feasible solution to the level-$\pi$ formulation. Then $\hat{x}$ satisfies every valid inequality for (11) of pitch $\leq \pi$.

**Proof.** By induction on $\pi$. Let $\sum_{j \in T} \alpha_j x_j \geq \alpha_0$ be a valid inequality for (11) with $\alpha \geq 0$ and $\alpha_0 > 0$, $\alpha_j \leq \alpha_0$ for each $j \in T$, of pitch $\leq \pi$. Since, for any $\pi \geq 1$, $M^{\pi+1} \subseteq M^\pi$, i.e. the set of feasible solutions for the level-$(\pi+1)$ formulation is contained in the set of feasible solutions for the level-$\pi$ formulation, we may assume that $\sum_{j \in T} \alpha_j x_j \geq \alpha_0$ has pitch $= \pi$. Further, since $\sum_{j \in T} \alpha_j x_j \geq \alpha_0$ is valid for (11), by Lemma 7(1) there exists $1 \leq i \leq m$ with $S_i \subseteq T$.

Hence, if $\pi = 1$ the result clearly follows. Suppose now that we have proved the assertion for $\pi$ and wish to prove it for $\pi + 1$. Since $(\hat{x}, \hat{y})$ is contained in $D^{\pi+1}_i$ the result will follow if we can prove that, for every $t$ with $1 \leq t \leq \vert S_i \vert$, any point in $D^{\pi+1}_i(t)$ satisfies $\sum_{j \in T} \alpha_j x_j \geq \alpha_0$.

Hence, let $(\hat{x}, \hat{y}) \in D^{\pi+1}_i(t)$. So $\hat{x}_{j_t} = 1$. Note that $j_t \in S_i \subseteq T$. By definition, the inequality

$$\sum_{j \in T - j_t} \alpha_j x_j \geq \alpha_0 - \alpha_{j_t}$$

has pitch $\leq \pi - 1$ and so by induction (and constraint (11f)) it is satisfied by $\hat{x}$. The result now follows. \[\square\]

**Lemma 11** For each fixed $\pi$, the level-$\pi$ formulation has size polynomial in $m$ and $n$.

**Proof.** By construction of the systems $D^{\pi+1}_i$, $N^{\pi+1} \leq mn(N^\pi + n)$, which, together with $N^1 = 0$ implies $N^\pi = O(m^{\pi-1}n^{\pi-1})$. Likewise let $M^\pi$ denote the number of constraints in the level-$\pi$ formulation. Then $M^{\pi+1} \leq m(1 + O(n^2 + nN^\pi + nM^\pi)) = O(mn^2 + m^{\pi-1}n^\pi + nM^\pi) = O(mn^2 + \pi m^{\pi-1}n^\pi)$. \[\square\]
3 Minimum-knapsack

In this section we consider minimum-knapsack problems \((2)\) and prove Theorem 6. We will use Remark 4 together with a structural characterization of valid inequalities for \((2)\), so as to obtain a polynomial-time algorithm for near-separation from the relaxation for the knapsack \((2)\) defined by the valid inequalities with coefficients in \(\{0, 1, \ldots, p\}\) for some fixed \(p > 0\).

3.0.1 Motivation: the cases \(p = 2\) and \(p = 3\)

We first illustrate our approach in the case \(p = 2\) and outline a difficulty that arises when \(p = 3\).

Let \(p = 2\); here we want to check if a vector \(y \in [0, 1]^n\) violates any valid inequality

\[
x(S_1) + 2x(S_2) \geq 2.
\]

(13)

(where \(S_1 \cap S_2 = \emptyset\)) that is not dominated by another inequality of the same kind. Here the situation is simple because by a result to be proven below (Theorem 16) when \(p = 2\) we must have \(w_h < w_k\) for every \(h \in S_1\) and \(k \in S_2\).

Example 12 Consider the minimum-knapsack set \((2)\) given by inequality

\[
\sum_{j=1}^n w_j x_j = 10x_1 + 10x_2 + 5x_3 + 6x_4 + 7x_5 \geq 10, \quad x \in \{0, 1\}^5
\]

The inequality

\[
2(x_1 + x_2 + x_3) + x_4 + x_5 \geq 2
\]

(14)
is valid. However this inequality is not monotone in that \(w_3 = 5 < 6 = w_4\) and yet the coefficients of \(x_3\) and \(x_4\) are 2 and 1, respectively. However, the inequality

\[
2(x_1 + x_2) + x_3 + x_4 + x_5 \geq 2
\]
is also valid, and dominates \((14)\).

As a result, we next argue that checking if \(y\) violates any inequality \((13)\) can be reduced to the solution of the following \(n\) minimum-knapsack problems, one for each index \(1 \leq k \leq n\), where the \(k^{th}\) case checks for violations of those inequalities \((13)\) where \(k = \arg\max\{w_j : j \in S_1\}\), and is formulated as follows:

\[
V(k) = \min \sum_{j : w_j \leq w_k} y_j z_j + 2 \sum_{j : w_j > w_k} y_j z_j
\]

(15a)

s.t. \[
\sum_{j \neq k} w_j z_j \geq \sum_{j=1}^n w_j - w_0 + 1
\]

(15b)

\[z \in \{0, 1\}^n, \quad z_k = 1\]

(15c)

To see that this approach works, suppose \(\hat{z}\) is feasible for \((15)\). Then constraint \((15b)\) guarantees that for every index \(h\) with \(\hat{z}_h = 1\) and \(w_h \leq w_k\) we have

\[
\sum_{j \neq h} w_j \hat{z}_j \geq \sum_{j=1}^n w_j - w_0 + 1
\]
i.e. \(\{j : \hat{z}_j = 1\} \setminus \{h\}\) forms a cover for \((2)\). Thus, by Remark 4 the inequality

\[
\sum_{j : w_j \leq w_k, \hat{z}_j = 1} x_j + 2 \sum_{j : w_j > w_k, \hat{z}_j = 1} x_j \geq 2
\]
is valid for (2) and if it is violated by \( y \) then
\[
\sum_{j : w_j \leq w_k} y_j \hat{z}_j + 2 \sum_{j : w_j > w_k} y_j \hat{z}_j < 2
\]
and therefore \( V(k) < 2 \). The reverse construction is similar. In summary, \( V(k) < 2 \) if and only if \( y \) violates some inequality (13) where \( k = \text{argmax}\{w_j : j \in S_1\} \), as desired. Note that this separation argument requires exact solution of a knapsack problem, however near-separation follows using the usual FPTAS argument.

As a further example, suppose we want to check if there exists some violated valid inequality of the form
\[
x(\mathbb{T}_1) + 2x(\mathbb{T}_2) \geq 3
\]
with \( \mathbb{T}_1, \mathbb{T}_2 \) nonempty and pairwise disjoint. [This is the \( p = 3 \) case but without a term of therm 3\( x(\mathbb{T}_3) \) in the left-hand side.] Again we aim to reduce this task to a polynomially-large set of knapsack problems. To extend the approach used for \( p = 2 \) we need, to begin with, some way to summarize the structure of covers (for (16)) while guaranteeing that such covers are also covers for the original knapsack (2) (this will be done, in the general case, in Lemma 25 below). The salient point is that there are two critical cases that are needed to guarantee that any cover for (16) is also a cover for (2): first, the case where the indices of the two largest \( w_j \) with \( j \in \mathbb{T}_1 \) are excluded from \( \mathbb{T}_1 \cup \mathbb{T}_2 \) and second, the case where the single largest \( w_j \) with \( j \in \mathbb{T}_2 \) is excluded from \( \mathbb{T}_1 \cup \mathbb{T}_2 \). Thus, we have two cases, rather than one, but we need to be able to represent both using a single constraint similar to (15).

Moreover, the approach used for \( p = 2 \) relied on the “monotonicity” property illustrated by Example 12. In the case \( p = 3 \) the monotonicity does not hold.

**Example 13** Consider now the minimum-knapsack set (2) given by
\[
\sum_{j=1}^{n} w_j x_j = 6x_1 + 6x_2 + 5x_3 + 4x_4 + 4x_5 \geq 13, \quad x \in \{0,1\}^5
\]
The inequality
\[
x_1 + x_2 + 2x_3 + x_4 + x_5 \geq 3
\]
is valid. Again this inequality is not monotone in that \( w_3 = 5 < 6 = w_1 \) and yet the coefficients of \( x_3 \) and \( x_1 \) are 2 and 1, respectively. In this case, the “strengthening” of (17) along the lines of Example 12, namely
\[
x_1 + x_2 + x_3 + x_4 + x_5 \geq 2
\]
is not valid.

As the example shows, the technique outlined for the case \( p = 2 \) is not easily extended due to non-monotonicities in coefficients. However, Theorem 16 will show that for any fixed \( p \) the total number of such non-monotonicities is bounded (i.e. independent of \( n \)).

In our general approach we will handle both aspects outlined above while nevertheless relying on enumeration of a polynomial number of cases: the multiple types of cover inequalities that have to be considered, and the non-monotonicity illustrated by Example 13.

### 3.1 Near-monotonicity

In this section we describe an important, near-monotonicity property of valid inequalities for (2) that will be critical in developing our polynomial-time separation algorithm.
Definition 14 Let \( \sum_{j=1}^{n} \alpha_j x_j \geq \alpha_0 \) be an inequality (valid or not) with \( \alpha_j \in \mathbb{Z}_+ \) for \( 0 \leq j \leq n \). Let \( k \in \mathbb{Z}_+ \) be such that \( \alpha_j = k \) for some \( j \). The drag of \( k \) is defined as

\[
\delta(k) = \left\{ h : w_h \geq \min_{j : \alpha_j = k} \{ w_j \} \text{ and } 0 < \alpha_h < k \right\}.
\]

If there is no \( j \) with \( \alpha_j = k \) then we set \( \delta(k) = \emptyset \).

Example 15 Consider the minimum knapsack set given by

\[
10x_1 + 10x_2 + 80x_3 + 100x_4 + 80x_5 + 20x_6 + 50x_7 + 25x_8 \geq 280, \quad x \in \{0, 1\}^8.
\]

Then

\[
x_1 + x_2 + x_3 + x_4 + 3(x_5 + x_7) + 4x_8 \geq 4
\]

is valid for the knapsack. Applying Definition 14 some selected drag sets are as follows. \( \delta(3) = \{3, 4\} \), because \( w_1 = w_2 = 10 < 20 = w_6 \) and both \( w_3 \geq 20 \) and \( w_4 \geq 20 \). Similarly, \( \delta(4) = \{3, 4, 7\} \).

Using this definition, we obtain a criterion for validity of an inequality for the knapsack set \( \{2\} \).

Theorem 16 Let \( P \in \mathbb{Z}_+ \), and suppose \( \sum_{j=1}^{n} \alpha_j x_j \geq \alpha_0 \) is a valid inequality for \( \{2\} \) with \( \alpha_j \in \{0, 1, \ldots, P\} \) for all \( j \). Then either \( \sum_{h \in \delta(k)} \alpha_h \leq P - 2 \) for every \( k \geq 2 \) or there is another valid inequality with coefficients in \( \{0, 1, \ldots, P\} \) that strictly dominates \( \sum_{j=1}^{n} \alpha_j x_j \geq \alpha_0 \).

Proof. Aiming for a contradiction suppose for some \( k \geq 2 \) we have \( \sum_{h \in \delta(k)} \alpha_h \geq P - 1 \). Choose some index \( i \in \argmin_j : \alpha_j = k \{ w_j \} \). For \( 1 \leq j \leq n \) write

\[
\alpha'_j = \begin{cases} 
\alpha_j & \text{if } j \neq i \\
 k - 1 & \text{if } j = i.
\end{cases}
\]

Let \( S = \suppt(\alpha) = \suppt(\alpha') \). To complete the proof we will show that the inequality

\[
\sum_{j=1}^{n} \alpha'_j x_j \geq \alpha_0,
\]

which dominates \( \sum_{j=1}^{n} \alpha_j x_j \geq \alpha_0 \), is valid for \( \{2\} \). To do so we appeal to Remark 4. In other words, we need to prove that for any \( C \subseteq S \) if \( \alpha'(S \setminus C) < \alpha_0 \) then \( w(S \setminus C) < w_0 \).

Let \( C \subseteq S \) be given. If \( i \notin S \setminus C \) then \( \alpha'(S \setminus C) = \alpha(S \setminus C) \) and we are done. In the remainder of the proof we assume \( i \in S \setminus C \) and \( \alpha'(S \setminus C) < \alpha_0 \). Let \( \delta = \delta(k) \) be defined as in Definition 14. If \( \delta \subseteq S \setminus C \) then

\[
\alpha'(S \setminus C) \geq \alpha'(\delta) + \alpha'_i \geq P - 1 + \alpha'_i \geq P
\]

since \( \alpha' = k - 1 \geq 1 \). But this is a contradiction since we assumed \( \alpha'(S \setminus C) < \alpha_0 \). Hence \( \exists h \in \delta \cap C \).

Define \( C' = C + i - h \), so that \( S \setminus C' = S \setminus (C - i + h) \). Thus \( \alpha'(S \setminus C') = \alpha'(S \setminus C) - \alpha'_i + \alpha_h \leq \alpha'(S \setminus C) < \alpha_0 \) and since \( \alpha(S \setminus C') = \alpha'(S \setminus C') \) we conclude that \( x(C') \geq 1 \) is a valid inequality for the minimum-knapsack set defined by \( \sum_{j=1}^{n} \alpha_j x_j \geq \alpha_0 \) and hence it is also valid for \( \{2\} \), i.e.

\[
w(S \setminus C') < w_0.
\]

But \( w(S \setminus C) = w(S \setminus C') - w_h + w_i \leq w(S \setminus C') \) since \( h \in \delta \). So \( w(S \setminus C) < w_0 \) as desired. ■

\[\text{2For simplicity of notation in this proof we use } \text{“+” and “-” for singletons, rather than } \cap \text{ and } \backslash.\]

8
3.2 Separation

Given a fractional vector \( y \in [0, 1]^n \) we consider separation of \( y \) from the relaxation for the knapsack [2] defined by all valid inequalities of the form

\[
x(\mathcal{S}_1) + 2x(\mathcal{S}_2) + \ldots + qx(\mathcal{S}_q) \geq q
\]

(where the \( \mathcal{S}_k \) are assumed pairwise disjoint), in polynomial time. Enumeration of all \( q \in \{2, \ldots, p\} \) will yield Theorem xyzp. As a first step in our procedure, we will present an efficient procedure that succinctly enumerates all possible inequalities [20] that are valid and undominated. The enumeration will be accomplished by first classifying all inequalities [20] using a compact scheme. Throughout, we will rely on Example [15] given above.

**Example 17** (Example [15] continued.) In this example the knapsack set was given by the inequality \( 10x_1 + 10x_2 + 80x_3 + 100x_4 + 80x_5 + 20x_6 + 50x_7 + 25x_8 \geq 280 \) and we consider the valid inequality \( x_1 + x_2 + x_3 + x_4 + 3(x_6 + x_7) + 4x_8 \geq 4 \). Thus \( q = 3 \), and \( \mathcal{S}_1 = \{1, 2, 3, 4\} \), \( \mathcal{S}_2 = \emptyset \), \( \mathcal{S}_3 = \{6, 7\} \) and \( \mathcal{S}_4 = \{8\} \).

Consider an inequality [20] (valid or not). In order to classify this inequality we define

\[
\mathcal{I} = \{1 \leq i \leq q : \mathcal{S}_i \neq \emptyset\}, \quad \text{and} \quad \mathcal{D}_i = \bigcup_{q > k > i} \delta(k) \cap \mathcal{S}_i
\]

The following observation will be useful throughout:

**Remark 18** Let \( i \in \mathcal{I} \) and suppose \( j \in \mathcal{S}_i \setminus \mathcal{D}_i \). For any \( k > i \) with \( k \in \mathcal{I} \) the definition of \( \delta(k) \) implies \( w_j < \min\{w_h : h \in \mathcal{S}_k\} \). Therefore \( w_j < \min\{w_h : h \in \mathcal{D}_i\} \). Thus \( \mathcal{D}_i \) contains the indices of the \( |\mathcal{D}_i| \) largest \( w_j \) with \( j \in \mathcal{S}_i \).

Now we present our classification scheme. We say that inequality [20] (valid or not) has type \( \tau = (\mathcal{I}, \mathcal{L}, m) \), if \( \mathcal{L} = \{\mathcal{L}_i : i \in \mathcal{I}\} \) and \( m = \{m_i : i \in \mathcal{I}\} \) satisfy:

\[ (t.1) \] For each \( i \in \mathcal{I} \), \( \mathcal{L}_i \) is a subset of \( \mathcal{S}_i \) satisfying

- (a) For any \( j \in \mathcal{S}_i \setminus \mathcal{L}_i \) and \( h \in \mathcal{L}_i \) we have \( w_j \leq w_h \).
- (b) \( |\mathcal{L}_i| = \max\{|\mathcal{D}_i|, \min\{q - 1, |\mathcal{S}_i|\}\} \).

**Comment.** Remark [18] implies \( \mathcal{D}_i \subseteq \mathcal{L}_i \). If \( |\mathcal{S}_i| \leq q - 1 \) then \( \mathcal{L}_i = \mathcal{S}_i \) and if \( |\mathcal{D}_i| \geq q - 1 \) then \( \mathcal{L}_i = \mathcal{D}_i \).

In Example [15], \( \mathcal{I} = \{1, 3, 4\} \). Further \( \delta(3) = \{3, 4\} \), so \( \mathcal{D}_{1,3} = \{3, 4\} \). Similarly, \( \delta(4) = \{3, 4, 7\} \), so \( \mathcal{D}_{1,4} = \{3, 4\} \) and \( \mathcal{D}_{3,4} = \{7\} \). Hence \( \mathcal{D}_1 = \{3, 4\} \), \( \mathcal{D}_3 = \{7\} \) and \( \mathcal{D}_4 = \emptyset \).

Thus \( \mathcal{L}_1 = \{2, 3, 4\} \), \( \mathcal{L}_3 = \{6, 7\} \) and \( \mathcal{L}_4 = \{8\} \) are valid choices.

\[ (t.2) \] For each \( k \in \mathcal{I} \), \( m_k \in \text{argmin}\{w_j : j \in \mathcal{S}_k\} \) with ties broken arbitrarily.

In Example [15], \( m_1 = 1, m_3 = 6 \) and \( m_4 = 8 \).

**Remark 19** Let \( i \in \mathcal{I} \). Then \( \forall j \in \mathcal{S}_i \setminus \mathcal{L}_i \), \( w_j \leq \min\{\min_{h \in \mathcal{L}_i} w_h, \min_{k \in \mathcal{I}} w_{m_k} - 1\} \).

In Example [15] we have \( w_1 < w_{m_3} = w_8 = 20 \), \( w_1 < w_{m_4} = w_8 = 25 \). Also, \( \min\{\min_{h \in \mathcal{L}_1} w_h, \min_{k \in \{3,4\}} w_{m_k} - 1\} = \min\{10, 20\} = 10 \).

**Remark 20** A given inequality can have more than one type. However, Lemma [21] given next narrows the choices, without loss of generality, and furthermore Lemma [24] will show a common attribute for all types.

**Lemma 21** Suppose [20] is valid for [2] and not dominated by another valid inequality with coefficients in \( \{0, 1, \ldots, q\} \). Then for any type \( \tau = (\mathcal{I}, \mathcal{L}, m) \) for [20], \( |\mathcal{L}_i| \leq q^2 \) for all \( i \in \mathcal{I} \).
Proof. Considering requirements (t.1)(a,b) for a type \( \tau \), we see that for any \( i \in \mathbb{I} \), \( |L_i| \leq \max\{|D_i|, q-1\} \). But \( |D_i| \leq q, \) by Theorem 10; ■

We will next see how the type of an inequality encodes its validity. This will be done in Lemma 25 below after we introduce some notation.

**Definition 22** Given an inequality (20) of type \( \tau = (I, L, m) \), its signature is defined as

\[
\sigma(\tau) = \max \sum_{i \in I} w(T_i) \quad (22a)
\]

\[
s.t. \quad \sum_{i \in I} |T_i| < q, \text{ and } T_i \subseteq L_i, \text{ for all } i \in I \quad (22b)
\]

**Example 23** Example 15 continued. In constraint (22a) we must have \( T_4 = \emptyset \), i.e. the constraint reads \( |T_1| + 3|T_3| \leq 3 \) and so either \( |T_1| = 0 \) and \( |T_3| = 1 \) or \( |T_1| \leq 3 \) and \( |T_3| = 0 \). Clearly we obtain \( \sigma = 190 \).

**Lemma 24** Suppose an inequality (20) is of type \( \tau \), and that for each \( i \in \mathbb{I} \) we have a subset \( X_i \) such that \( L_i \subseteq X_i \subseteq S_i \). Then we can rewrite

\[
\sigma(\tau) = \max \sum_{i \in I} w(T_i) \quad (23a)
\]

\[
s.t. \quad \sum_{i \in I} |T_i| < q, \text{ and } T_i \subseteq X_i, \text{ for all } i \in I \quad (23b)
\]

As a corollary, all types for a given inequality (20) have the same signature.

**Proof.** Identity (23) follows because as noted in Remark 14 for all \( i \in \mathbb{I}, L_i \) contains the indices of the \( |L_i| \)-largest \( w_h \) with \( h \in S_i \), and by (t.1)(b) \( |L_i| \geq \min\{q-1, |S_i|\} \geq \min\{q-1, |X_i|\} \). The corollary follows (for example) by setting \( X_i = S_i \) for all \( i \in \mathbb{I} \). ■

We now present the characterization of validity that we will useful below.

**Lemma 25** An inequality (20) of type \( \tau \) is valid for (2) iff

\[
\sum_{i=1}^{q} w(S_i) \geq \sigma(\tau) + \sum_{j=1}^{n} w_j - w_0 + 1 \quad (24)
\]

**Proof.** Remark 11 implies that inequality (20) is valid for (2) iff for each family of subsets \( T_i \subseteq S_i \) \((1 \leq i \leq q)\) such that \( \sum_{i=1}^{q} |T_i| < q \) we have

\[
\sum_{i=1}^{q} w(S_i \setminus T_i) \geq \sum_{j=1}^{n} w_j - w_0 + 1
\]

from which the result follows. ■

**Example 26** Consider Example 16. Inequality (18a) is valid because \( w(S_1 \cup S_2 \cup S_3 \cup S_4) = 295 \) while \( \sigma(\tau) + \sum_{j=1}^{8} w_j - w_0 + 1 = 190 + 375 - 280 + 1 = 286 \), i.e. condition (24) is verified.

### 3.2.1 Separation through type enumeration

Our separation procedure will enumerate a set of candidate triple \((I, L, m)\) that includes all possible types \( \tau \) arising from valid inequalities, and for each enumerated candidate perform a polynomial-time test, given in Section 3.3. In this section we describe the enumeration. Let us consider an arbitrary triple \((I, L, m)\) where \( L \) is a collection of \( q \) subsets of \( \{1, \ldots, n\} \) and each \( m_i \in \{0,1,\ldots,n\} \). In order for the triple to arise as the type of an inequality it must satisfy a number of conditions given next:
(r.1) \( \mathbb{I} \subseteq \{1, \ldots, q\} \). The sets \( \{m_i\} \cup \mathbb{I}_i \) (\( i \in \mathbb{I} \)) are pairwise disjoint. For any \( i \in \mathbb{I} \), if \( |\mathbb{I}_i| < q - 1 \) then \( m_i \in \mathbb{I}_i \), and if \( m_i \notin \mathbb{I}_i \) then \( w_{m_i} < w_{m_k} \) for all \( k > i \) with \( k \in \mathbb{I} \).

(r.2) We require that \( |\mathbb{I}_i| < q^2 \) for all \( i \in \mathbb{I} \). In terms of separation from undominated, valid inequalities requirement is valid in light of Theorem 10 and Lemma 21.

Stronger conditions can be imposed, however these assumptions suffice to prove:

**Lemma 27** For given \( q \), the set of pairs satisfying (r.1)-(r.2) includes all types arising from undominated, valid inequalities (20). The total number of tuples that satisfy (r.1)-(r.4) is at most \( O(q^22^n n^3) \).

*Proof.* Follows from the above discussion and the fact that there are at most \( 2^q \) choices for sets of indices \( i \) with \( m_i > 0 \).

### 3.3 Separation using a given type

Assume again a given \( y \in [0,1]^n \). Consider a fixed triple \( \tau = (\mathbb{I}, \mathbb{L}, m) \) that has been enumerated as indicated above, i.e, it satisfies (r.1)-(r.2). Here we will first describe an optimization problem whose solution either:

(a) Proves that \( y \) satisfies all inequalities (20) of type \( (\mathbb{I}, \mathbb{L}, m) \) that are valid for (22) (if any such inequalities exist), or

(b) Finds a valid inequality (20) for (22) that is violated by \( y \).

To construct the formulation, we write, each \( i \in \mathbb{I} \),

\[
M_i = \min \left\{ \min_{h \in \mathbb{L}_i} w_h, \min_{k > i} w_{m_k} - 1 \right\}, \quad \mathbb{F}_i = \cup_{k \neq i} \{ m_k \} \cup \mathbb{L}_k \quad (25a)
\]

\[
\mathbb{R}_i = \begin{cases} \{ j \notin \mathbb{F}_i : w_{m_j} \leq w_j \leq M_i \}, & \text{if } |\mathbb{I}_i| \geq q - 1 \\ \emptyset, & \text{otherwise.} \end{cases} \quad (25b)
\]

\[
\mathbb{V}_i = \mathbb{R}_i \cup \mathbb{L}_i. \quad (25c)
\]

Now we describe our formulation. Let \( \sigma(\tau) \) be the value of problem (22) (computed exactly as in that formulation). Then we solve the problem:

\[
\Omega(y, \tau) = \min \sum_{i=1}^q \sum_{j \in \mathbb{V}_i} y_j z_j \quad (26a)
\]

\[
\text{s.t.} \quad z_j = 1 \quad \forall j \in \cup_{i \in \mathbb{I}} \{ m_i \} \cup \mathbb{L}_i \quad (26b)
\]

\[
z_j = 0 \quad \forall j \notin \cup_{i \in \mathbb{I}} \mathbb{V}_i, \quad (26c)
\]

\[
\sum_{j=1}^n w_j z_j \geq \sigma(\tau) + \sum_{j=1}^n w_j - w_0 + 1, \quad (26d)
\]

\[
z \in \{0,1\}^n
\]

**Lemma 28** Suppose there is an inequality \( \sum_{i=1}^q x(S_i) \geq q \) of type \( \tau \) valid for the knapsack (2). Then, setting \( \tilde{z}_j = 1 \) iff \( j \in \cup_{i \in \mathbb{I}} S_i \) yields a feasible solution to (20).

*Proof.* By definition of the type \( \tau \), \( \tilde{z} \) satisfies (26d). To show (26c) holds at \( \tilde{z} \), we will show that \( S_i \subseteq \mathbb{V}_i \) for all \( i \in \mathbb{I} \). Let \( j \in S_i \); if \( j \in \{m_i\} \cup \mathbb{L}_i \) then by construction \( j \in \mathbb{V}_i \), so assume \( j \notin \{m_i\} \cup \mathbb{L}_i \).

In this case we prove \( j \in \mathbb{R}_i \). To do so, note that by definition of \( m_i \) in (t.2), \( w_{m_i} \leq w_j \). Moreover \( j \in S_i \setminus \mathbb{L}_i \) and hence (Remark 19) \( w_j \leq M_i \). Finally, the \( S_k \) are disjoint, and so \( j \notin \cup_{k \neq i} S_k \) and hence \( j \notin \mathbb{F}_i \). Thus indeed \( j \in \mathbb{R}_i \), as desired. To complete the proof we need to show that \( \tilde{z} \) satisfies (26d), but this follows from Lemma 25.

11
Lemma 29 Suppose \( \tilde{z} \) is a feasible solution for (26). For \( i \in I \) define \( S_i \equiv V_i \cap \{ j : \tilde{z}_j = 1 \} \). Then
\[
\sum_{i \in I} i x(S_i) \geq q
\] is valid for (2).

Proof. First note that for any \( i \in I \), \( S_i \supseteq \mathbb{L}_i \) (by (26b)). Moreover, for each \( i \in I \), either (a) \( |\mathbb{L}_i| < q - 1 \) in which case \( S_i = \mathbb{L}_i \) or (b) \( |\mathbb{L}_i| \geq q - 1 \) and \( w_j \geq w_h \) for each \( j \in \mathbb{L}_i \) and \( h \in S_i \setminus \mathbb{L}_i \). Thus it follows (Lemma 24) that if (27) has a certain type \( \tau' \), then \( \sigma(\tau') = \sigma(\tau) \). As a result, constraint (26d) and Lemma 25 imply that (27) is valid for (2).

We can now prove our key separation theorem.

Theorem 30 The vector \( y \in [0, 1]^n \) violates an inequality of type \( \tau \) valid for (2) iff \( \Omega(y, \tau) < q \).

Proof. Suppose first that \( \sum_{i \in I} i x(S_i) \geq q \) is an inequality of type \( \tau \), valid for (20) and violated by \( y \). By Lemma 28 by setting \( \hat{z}_j = 1 \) iff \( j \in \bigcup_{i=1}^n S_{e(i)} \) we obtain a feasible solution for problem 26. But since the objective value attained by \( \hat{z} \) in this problem equals \( \sum_{i \in I} i y(S_i) < q \) we conclude as desired.

Now assume \( \Omega(y, \tau) < q \). Let \( \tilde{z} \) be an optimal solution for (26). By Lemma 29 the inequality
\[
\sum_{i \in I} i x(S_i) \geq q
\] is valid for (2) where for \( i \in I \) we define \( S_i \equiv V_i \cap \{ j : \tilde{z}_j = 1 \} \). But since
\[
q > \Omega(y, \tau) = \sum_{i \in I} i y(S_i)
\] we conclude \( y \) violates (28). ■

3.3.1 Near separation in polynomial time

In order to prove Theorem 6 there remains the issue of the complexity of solving problems of the form (26). These are min-knapsack problems, for which an FPTAS exists, based on that for the standard knapsack problem [8], [7]. Relying on such an FPTAS would yield a proof of Theorem 6 (though, technically, the complexity would depend polynomially on \( p/\epsilon \)). However this route would yield an algorithm that relies on the traditional techniques: dynamic programming and coefficient scaling.

Here we indicate a simpler technique that applies in this case. Consider, again, a given value of \( q \) and an inequality of type \( \tau \) as in the sections above. For \( 1 \leq j \leq n \) define \( \hat{y}_j \equiv \frac{1}{qn^2} \lfloor qn^2y_j \rfloor \), i.e. the “round-up” of \( y_j \) to the nearest integer multiple of \( \frac{1}{qn^2} \). Then for any type \( \tau \)
\[
V(y, \tau) \leq V(\hat{y}, \tau) \leq V(y, \tau) + \frac{1}{n}
\] and so \( V(\hat{y}, \tau) < q \) implies that \( y \) violates an inequality of type \( \tau \) whereas if \( V(\hat{y}, \tau) \geq q \) then \( y \) satisfies every inequality of type \( \tau \) within additive error at most \( 1/n \), which is less than \( \epsilon \) for \( n \) large enough.

Moreover \( V(\hat{y}, \tau) \) can be computed in polynomial time, since it can be restated as a min-knapsack problem with nonnegative, integral objective coefficients bounded above by \( qn^2 \). Such a min-knapsack problem can be solved using dynamic-programming (no need for coefficient scaling). We have thus proved Theorem 6.

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\[ ^3 \] We estimate that this is a folklore trick

\[ ^4 \] In fact even the dynamic-programming step can be eliminated.
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