Kollár–Nadel Type Vanishing Theorem*

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Abstract. We prove an analytic generalization of Kollár’s vanishing theorem, which contains the Nadel vanishing theorem as a special case.

Keywords: Injectivity theorem; Nadel vanishing theorem; Kollár vanishing theorem; Multiplier ideal sheaves.

1. Introduction

In this paper, I discussed about the Hodge theoretic aspect of injectivity and vanishing theorems (see [2, 3, 4]). Here, I will explain some analytic generalizations. In [6], Shin-ichi Matsumura and I established the following theorems.

Theorem 1.1. [6, Theorem A] Let $F$ be a holomorphic line bundle on a compact Kähler manifold $X$ and let $h$ be a singular hermitian metric on $F$. Let $M$ be a holomorphic line bundle on $X$ equipped with a smooth hermitian metric $h_M$. We assume that $\sqrt{-1} \Theta_{h_M}(M) \geq 0$ and $\sqrt{-1} \Theta_h(F) - a \sqrt{-1} \Theta_{h_M}(M) \geq 0$ for some $a > 0$. Let $s$ be a nonzero global section of $M$. Then the map

$$\times s : H^i(X, \omega_X \otimes F \otimes J(h)) \to H^i(X, \omega_X \otimes F \otimes J(h) \otimes M)$$

induced by $\otimes s$ is injective for every $i$, where $\omega_X$ is the canonical bundle of $X$ and $J(h)$ is the multiplier ideal sheaf of $h$.

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Theorem 1.1 is a generalization of Enoki’s injectivity theorem (see [1, Theorem 0.2]). Although the formulation of Theorem 1.1 may look artificial, it has many interesting applications (see [6]). Theorem 1.2 below is a Bertini-type theorem for multiplier ideal sheaves.

**Theorem 1.2.** [6, Theorem 1.10] Let $X$ be a compact complex manifold, let $\Lambda$ be a free linear system on $X$ with $\dim \Lambda \geq 1$, and let $\varphi$ be a quasi-plurisubharmonic function on $X$. We put $\mathcal{G} = \{H \in \Lambda | H$ is smooth and $\mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}$. Then $\mathcal{G}$ is dense in $\Lambda$ in the classical topology. Note that $\mathcal{J}(\varphi)$ is the multiplier ideal sheaf of $\varphi$.

The main purpose of this paper is to prove the following theorem, which is a slight generalization of [6, Theorem D], as an application of Theorem 1.1 and Theorem 1.2.

**Theorem 1.3.** (Vanishing Theorem of Kollár–Nadel Type) Let $f: X \to Y$ be a holomorphic map from a compact Kähler manifold $X$ to a projective variety $Y$. Let $F$ be a holomorphic line bundle on $X$ equipped with a singular hermitian metric $h$. Let $H$ be an ample line bundle on $Y$. Assume that there exists a smooth hermitian metric $g$ on $f^*H$ such that

$$\sqrt{-1}\Theta_g(f^*H) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_h(F) - \varepsilon \sqrt{-1}\Theta_g(f^*H) \geq 0$$

for some $\varepsilon > 0$. Then we have $H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$ for every $i > 0$ and $j$, where $\omega_X$ is the canonical bundle of $X$ and $\mathcal{J}(h)$ is the multiplier ideal sheaf associated to the singular hermitian metric $h$.

We can easily see that Theorem 1.3 contains Demailly’s original formulation of the Nadel vanishing theorem (see [6, Theorem 1.4]) and Kollár’s vanishing theorem (see [7, Theorem 2.1 (iii)]) as special cases. Therefore, we call Theorem 1.3 the vanishing theorem of Kollár–Nadel type. For a related vanishing theorem, see [8, Theorem 1.3]. We note that we can find some relative generalizations of Theorems 1.1, 1.2, and 1.3 in [5] and [9].

In this paper, we will freely use the same notation as in [6].

### 2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 as an application of Theorem 1.1 and Theorem 1.2. I hope that the following proof will show the reader how to use Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.3.** We use the induction on $\dim Y$. If $\dim Y = 0$, then the statement is obvious. We take a sufficiently large positive integer $m$ and a general member $B \in |H^{\otimes m}|$ such that $D = f^{-1}(B)$ is smooth, contains no
associated primes of $\mathcal{O}_X / \mathcal{J}(h)$, and satisfies $\mathcal{J}(h|_D) = \mathcal{J}(h)|_D$ by Theorem 1.2. By the Serre vanishing theorem, we may further assume that $$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0$$ (1) for every $i > 0$ and $j$. We have the following big commutative diagram.

\[
\begin{array}{ccccccc}
0 & \to & \mathcal{J}(h) \otimes \mathcal{O}_X(-D) & \to & \mathcal{J}(h) & \to & \text{Coker}\alpha & \to & 0 \\
\downarrow & & \downarrow \alpha & & \downarrow \beta & & \downarrow & \\
0 & \to & \mathcal{O}_X(-D) & \to & \mathcal{O}_X & \to & \mathcal{O}_D & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\mathcal{O}_X / \mathcal{J}(h)) \otimes \mathcal{O}_X(-D) & \to & \mathcal{O}_X / \mathcal{J}(h) & \to & 0 & & 0 & & 0 \\
\end{array}
\]

Since $D$ contains no associated primes of $\mathcal{O}_X / \mathcal{J}(h)$, $\gamma$ is injective. This implies that $\beta$ is injective by the snake lemma and that $\text{Coker}\alpha = \mathcal{J}(h)|_D = \mathcal{J}(h)|_D$. Thus we obtain the following short exact sequence:

$$0 \to \mathcal{J}(h) \otimes \mathcal{O}_X(-D) \to \mathcal{J}(h) \to \mathcal{J}(h)|_D \to 0.$$ 

By taking $\otimes \omega_X \otimes F \otimes \mathcal{O}_X(D)$ and using adjunction, we obtain the short exact sequence:

$$0 \to \omega_X \otimes F \otimes \mathcal{J}(h) \to \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* H^{\otimes m} \to \omega_D \otimes F|_D \otimes \mathcal{J}(h|_D) \to 0.$$

Therefore, we see that

$$0 \to R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \to R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m} \to R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D)) \to 0$$ (2)

is exact for every $j$ since $B$ is a general member of $|H^{\otimes m}|$. By induction on $\dim Y$, we have

$$H^i(B, R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D))) = 0$$ (3)

for every $i > 0$ and $j$. By taking the long exact sequence associated to (2), we obtain

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m})$$

for every $i \geq 2$ and $j$ by (3). Thus we have

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$$ (4)
for every \( i \geq 2 \) and \( j \) by (1). By Leray’s spectral sequence and (1) and (4), we have the following commutative diagram:

\[
\begin{array}{ccc}
H^1(Y, F^j) & \to & H^{j+1}(X, \omega_X \otimes F \otimes J(h)) \\
\downarrow^{a} & & \downarrow^{b} \\
H^1(Y, F^j \otimes H^\otimes m) & \to & H^{j+1}(X, \omega_X \otimes F \otimes J(h) \otimes f^* H^\otimes m)
\end{array}
\]

for every \( j \), where \( F^j = R^j f_*(\omega_X \otimes F \otimes J(h)) \). Note that the horizontal arrows are injective. Since \( b \) is injective by Theorem 1.1, we obtain that \( a \) is also injective. By (1), we have

\[
H^1(Y, R^j f_*(\omega_X \otimes F \otimes J(h)) \otimes H^\otimes m) = 0
\]

for every \( j \). Therefore, we see that \( H^1(Y, R^j f_*(\omega_X \otimes F \otimes J(h))) = 0 \) for every \( j \). Thus we obtain the desired vanishing theorem: Theorem 1.3.

We close this section with a remark on Nakano semipositive vector bundles.

**Remark 2.1.** Let \( E \) be a Nakano semipositive vector bundle on \( X \). We can easily see that Theorem 1.3 holds even when \( \omega_X \) is replaced by \( \omega_X \otimes E \). We leave the details as an exercise for the reader (see [6, Section 6]).

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