Massive Dilaton and Topological Gravity

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A model in which the massive dilaton is introduced by minimally extending the two
dimensional topological gravity is studied semi-classically. The theory is no longer topo-
logical because of the explicit Weyl scale symmetry breaking. Due to the dilaton the
semiclassical stress-energy tensor gets renormalized and it is shown how the gravitational
background coupled to the the dilaton depends on the dilaton mass as well as the renor-
malization mass scale, but not on the Newton’s constant.

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1. Introduction

The two-dimensional gravity defined by the Einstein-Hilbert action is trivial at the classical level in the sense that any two-dimensional metric satisfies the classical equation of motion. No cosmological constant is allowed at the classical level unless matter is present. Quantization, nevertheless, reveals fairly nontrivial structures of the theory, although it is not yet dynamical\(^1\)\(^2\). There are nontrivial classical observables as topological invariants. Being topological, such a theory in its essence is far from the real nature, even in four dimensions. One simple reason we can think about is that we live in the world in which the scale symmetry is broken, but topological gravity respects the scale symmetry.

In this letter we intend to investigate what might happen to the two-dimensional topological gravity if we begin to break the topological structure by hand, keeping in mind that we can take an analogous approach to the four-dimensional topological gravity in the future. It would be much better if we could find a mechanism to break the topological structure in itself, nevertheless we expect the system should still evolve along a similar path to reach to the broken phase.

The minimal thing we shall do is to break the scale symmetry, but leave the theory still reparametrization invariant. One way to achieve this is to introduce a real massive scalar field coupled to the gravity without self-interactions. The case with self-interactions is in progress and will be reported elsewhere\(^3\). This real scalar field can be identified as the dilaton because its presence breaks the scale symmetry explicitly through the mass term. Note that in two dimensions there is no spontaneous symmetry breaking\(^4\) to identify the dilaton as Goldstone boson so that this naming is nothing illegal. On the contrary to the massless case, there is no classical level conformal symmetry to begin with and the stress-energy tensor is not traceless due to the mass of the dilaton unless the equation of motion is imposed.

As a result, although the graviton is not dynamical yet (i.e. not propagating), the theory has nontrivial dynamical degrees of freedom due to the dilaton. Full quantization will involve a Liouville-like mode (which is not exactly Liouville because of different potential form, but is induced from the conformal factor of the metric), however it is no longer regarded as an anomaly. It is just a quantum correction to the stress-energy tensor. Since the quantization of such a mode is still elusive, under this circumstance we are tempted to investigate this theory using semi-classical method as a first step to see what happens. (Later in the discussion, the rationale behind such an approach will be addressed more.)
other words, a quantum field theory of the dilaton in a classical gravitational background will be studied. In principle, we can work in any gravitational background, but we shall focus on the de Sitter background for an example. Other cases will be reported in [3].

2. Lagrangian

The model we are interested in is described by the following action:

\[ S = S_{\text{top}} + S_d = \int_{\Sigma} d^2 x \left( \mathcal{L}_{\text{top}} + \mathcal{L}_d \right), \]  

where

\[ \mathcal{L}_{\text{top}} = \frac{\sqrt{-g}}{16\pi G_0} R, \]  

\[ \mathcal{L}_d = -\sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} m^2 \phi^2 \right). \]

We have chosen the metric signature (−+), but the Euclidean case can be easily obtained by simply replacing all the minus signs in the above with the plus sign. The first term in eq.(2.1) is a constant (Euler number) to define the classical topological gravity and the second term introduces dynamics to the system. Without the mass term in eq.(2.2b), the action is invariant under Weyl rescaling \( g_{\mu\nu} \rightarrow e^{2\rho} g_{\mu\nu} \) which is the two dimensional analogue of the scale symmetry.

The stress-energy tensor is given by

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_d}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\mu\nu} m^2 \phi^2 \]  

such that the trace is

\[ T^{\mu}_{\mu} = -m^2 \phi^2. \]  

Thus the dilaton mass breaks the Weyl symmetry. Note that in two dimensions the existence of a massless real scalar field does not break the scale symmetry, but the mass term does. This, however, does not necessarily imply that the classical vacuum of eq.(2.1) does not respect the conformal symmetry. The classical equation of motion of eq.(2.1) under the variation of the metric is nothing but \( T_{\mu\nu} = 0 \) using the property of two dimensional geometry \( R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \). Together with eq.(2.4), \( T^{\mu}_{\mu} = 0 \) and then \( \phi = 0 \) is the only classically allowed value so that the dilaton does not appear explicitly. In other words,
classically $S_d = 0$ and $S$ is still a topologically invariant quantity to be the Lagrangian of topological gravity.

The classical equation of motion of the dilaton is

$$(\Delta + m^2) \phi = 0,$$  

(2.5)

where $\Delta \equiv -\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$ is the Laplacian. Note that $\phi = 0$ is a trivial solution and other solutions fail to satisfy $T_{\mu\nu} = 0$.

One can add a cosmological constant $L_c = -\sqrt{-g} \Lambda_0 / 8\pi G_0$ that also breaks the Weyl symmetry. In this case the classical equation of motion modifies to read $8\pi G_0 T_{\mu\nu} = g_{\mu\nu} \Lambda_0$, whose solution $4\pi G_0 m^2 \phi^2 = -\Lambda_0$ still leads to $L_c + L_d = 0$. Thus there is no consequence of the classical dilaton. However the situation becomes much different quantum mechanically.

3. Quantum Physics of Dilaton

As alluded in the introduction, we shall incorporate the gravity classically and investigate the implications of quantum physics of the dilaton. Although the validity of such a semi-classical approach may be always arguable, for our purpose it should be still good enough to learn the quantum effect of the dilaton near the classical topological vacuum. We use the functional integral method to compute the quantum effect of the dilaton. Thus integrating out the dilaton, we obtain

$$W_d \left( \equiv \int d^2 x L_{d,\text{eff}} \right) = \frac{i}{2} \text{tr} \ln(\Delta + m^2).$$  

(3.1)

If the exact result of the right-hand side were known, the semi-classical equation could be derived by simply taking the variation with respect to the metric. Unfortunately, the exact result is unknown so that we have to take a detour. First, we shall compute the divergent contributions to renormalize $S_{\text{top}}$, then the rest will be computed to derive the semi-classical equation.

For a real scalar field this has been computed using the WKB approximation method in [5][6] so that we can just recapture the result in two dimensions. We expect that there would be ultraviolet divergences and it is important to isolate them to renormalize. We follow DeWitt who used Schwinger’s method to handle divergent terms[7][8]. The essence of this method involves representing the propagator as

$$\frac{1}{\Delta + m^2} = i \int_0^\infty d\tau e^{-i\tau(\Delta + m^2)}.$$  

(3.2)
Then we compute $W_d$ for $d = 2 + \epsilon$ and use the dimensional regularization to take $\epsilon \to 0$ limit later:

$$W_d = \frac{i}{2} \int d^d x \sqrt{-g} \langle x | \ln(\Delta + m^2) | x \rangle$$

$$= \frac{i}{2} \int d^d x \sqrt{-g} \lim_{x' \to x} \int_0^\infty d\tau \int_{m^2}^{\infty} dm^2 \langle x | e^{-i\tau(\Delta + m^2)} | x' \rangle$$

$$= \frac{i}{2} \int d^d x \sqrt{-g} \lim_{x' \to x} \int_0^\infty d\tau \int_{m^2}^{\infty} dm^2 \frac{1}{(4\pi i\tau)^{d/2}} \left[ V(x, x') \right]^{1/2} e^{-i(\tau m^2 - \frac{\sigma}{2\tau})} F(x, x'; i\tau).$$

In the above equation, we use $\sigma$ which is one-half of the square of the proper distance between $x$ and $x'$

$$\sigma(x, x') = \frac{1}{2} y_\alpha y^\alpha$$

in terms of the Riemann normal coordinates $y^\alpha$ of $x$ with origin at $x'$, and the Van Vleck determinant is defined by

$$V(x, x') \equiv -\frac{1}{\sqrt{g(x)g(x')}} \det \left( \partial_\mu \partial_\nu \sigma(x, x') \right).$$

Since we are interested in the short distance behavior of $F$, assuming that the high frequency behavior of the massive scalar field is relatively insensitive to the long term time-dependence of the metric we study, we introduce the following asymptotic adiabatic expansion

$$F(x, x'; i\tau) = a_0(x, x') + a_1(x, x')i\tau + a_2(x, x')(i\tau)^2 + \cdots$$

In this approximation eq.(3.3c) is easily computed to yield

$$\mathcal{L}_{d,\text{eff}} = \frac{\sqrt{-g}}{2(4\pi)^{d/2}} \left( \frac{m}{\mu} \right)^{d-2} \sum_{j=0}^{\infty} a_j(x)(m^2)^{1-j}\Gamma(j - \frac{d}{2}),$$

where $a_0 = 1$, $a_1 = R/6$ and the regularization mass scale $\mu$ is introduced explicitly. Using the infinitesimal property of the Gamma function

$$\Gamma\left(-\frac{\epsilon}{2}\right) = -\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \quad \gamma = \text{Euler constant},$$

the ultraviolet divergences of the effective Lagrangian can be identified as

$$\mathcal{L}_{d,\text{eff}} = \frac{\sqrt{-g}}{8\pi} \left[ m^2 \left( N_\epsilon + \ln \frac{m^2}{\mu^2} - 1 \right) - \frac{R}{6} \left( N_\epsilon + \ln \frac{m^2}{\mu^2} \right) \right] + \mathcal{O}(\epsilon) + \cdots$$
where $N_{\epsilon} \equiv 2/\epsilon + \gamma - \ln 4\pi$. The ellipsis includes higher order finite corrections to the effective Lagrangian, which does not necessarily vanish in $\epsilon \to 0$ limit.

Renormalizing these divergences corresponds to dilatonic corrections to the cosmological constant and the Newton’s constant (the coefficient of the order $R$ term) in the effective action. Thus in this case we obtain renormalized semi-classical equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda_{\text{ren}} = 8\pi G_{\text{ren}} \langle T_{\mu\nu}\rangle_{\text{ren}}$$

where

$$\frac{1}{16\pi G_{\text{ren}}} = \frac{1}{16\pi G_{0}} - \frac{1}{48\pi} \left( \ln \frac{m^2}{\mu^2} + N_{\epsilon} \right),$$

$$\Lambda_{\text{ren}} = G_{\text{ren}} \left( \frac{\Lambda_{0}}{G_{0}} - m^2 \left( \ln \frac{m^2}{\mu^2} + N_{\epsilon} - 1 \right) \right).$$

Note that we have introduced the bare cosmological constant $\Lambda_{0}$ to renormalize the functional integral. $\Lambda_{\text{ren}}$ cannot be made finite without $\Lambda_{0}$.

$\langle T_{\mu\nu}\rangle_{\text{ren}}$ on the right-hand side collects the rest of finite quantum contributions of the massive dilaton to the effective action, which should be computed independently. The adiabatic expansion eq. (3.6) is only useful for computing divergences so that we still need to take care of other quantum corrections. The above are renormalization scheme independent because the ambiguity of the subtraction scheme in introducing counter terms are absorbed into the mass scale $\mu$ by proper redefinition. Such $G_{\text{ren}}$, $\Lambda_{\text{ren}}$ and $\mu$ are usually determined by measurement.

We can immediately observe the follows: a heavy dilaton ($m^2 > \mu^2$) increases the Newton’s constant, whilst a light dilaton ($m^2 < \mu^2$) decreases. In four dimensions the Newton’s constant controls the strength of the gravity, but here, as we shall find out soon, the effective gravitational equation does not depend on the renormalized Newton’s constant explicitly to confirm that the gravity is not dynamical. The curvature generated will depend solely on the dilaton mass and the regularization mass scale. For some $m^2$ the dilaton-corrected cosmological constant $\Lambda_{\text{ren}}$ could vanish. Without loss of generality we assume $\Lambda_{\text{ren}} = 0$ at $\ln(m^2/\mu^2) = 0$. For this we have $\Lambda_{0}/G_{0} = m^2(N_{\epsilon} - 1)$ to obtain

$$\Lambda_{\text{ren}} = -G_{\text{ren}} m^2 \ln \frac{m^2}{\mu^2}. \quad (3.11)$$

Then $\Lambda_{\text{ren}}$ is negative for large $m^2$, whilst for small $m^2$, $\Lambda_{\text{ren}}$ is positive. It may look like that there is no cosmological constant generation by a massless dilaton, but this is
not quite true. Eq.(3.11) only tells us that there is no ultraviolet contribution to the cosmological constant from a massless dilaton. In the massless dilaton case, eq.(3.1) has infrared divergences, which can be a source to generation of a cosmological constant\[8\]. A massive dilaton automatically removes such an infrared divergence.

In principle we can add higher order finite correction terms to the semi-classical equation, e.g. a curvature square term, we shall however focus only on the leading correction in this letter. In two dimensions every metric satisfies $R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R$ the semi-classical gravitational equation now becomes nothing but

$$g_{\mu\nu}\Lambda_{\text{ren}} = 8\pi G_{\text{ren}} \langle T_{\mu\nu} \rangle_{\text{ren}}.$$ (3.12)

Using eq.(3.11), we find that this effective equation indeed does not depend on the renormalized Newton’s constant $G_{\text{ren}}$.

We now need to compute $\langle T_{\mu\nu} \rangle_{\text{ren}}$ which is the renormalized piece of $\langle T_{\mu\nu} \rangle$. To be consistent we shall use dimensional regularization, but in general it is not easy to compute this expectation value $\langle T_{\mu\nu} \rangle$ in dimensional regularization. However in the de Sitter space the situation is better because

$$\langle T_{\mu\nu} \rangle = \frac{1}{2}g_{\mu\nu} \langle T^\alpha_{\alpha} \rangle = -\frac{1}{2}g_{\mu\nu}m^2 \langle \phi^2 \rangle.$$ (3.13)

$\langle \phi^2 \rangle$ can be computed using the result in [9] or [8] such that

$$\langle \phi^2 \rangle = \frac{2}{R} \left( \frac{R}{8\pi} \right)^{d/2} \frac{\Gamma \left( \nu(d) - \frac{1}{2} + \frac{d}{2} \right) \Gamma \left( -\nu(d) - \frac{1}{2} + \frac{d}{2} \right)}{\Gamma \left( \frac{1}{2} + \nu(d) \right) \Gamma \left( \frac{1}{2} - \nu(d) \right)} \Gamma \left( 1 - \frac{d}{2} \right)$$

$$= \frac{1}{4\pi} \left( -\frac{2}{\epsilon} - \gamma + \ln 4\pi \right) + \frac{1}{4\pi} \left( \psi(\nu_+) + \psi(\nu_-) + \ln \frac{2\mu^2}{R} \right) + O(\epsilon)$$ (3.14)

where $\nu_\pm \equiv \frac{1}{2} \pm \nu$, $\nu^2 \equiv \frac{1}{4} - \frac{2m^2}{R}$, $(\nu(d))^2 = \frac{1}{4}(d-1)^2 - \frac{d(d-1)m^2}{R}$ and $\psi$ is the Digamma function. Note that the same mass scale $\mu$ is used for $R$. Using the same renormalization scheme we used for $\mathcal{L}_{d,\text{eff}}$, we can obtain

$$\langle \phi^2 \rangle_{\text{ren}} = \langle \phi^2 \rangle - \langle \phi^2 \rangle_{\text{adiabatic}} = \frac{1}{4\pi} \left( \psi(\nu_+) + \psi(\nu_-) + \ln \frac{2\mu^2}{R} - \frac{R}{6m^2} + \ln \frac{m^2}{\mu^2} \right)$$ (3.15)

where $\langle \phi^2 \rangle_{\text{adiabatic}}$ that takes care of the counter term contributions can be easily computed from eq.(3.3c) using the definition $\mathcal{L}_{d,\text{eff}} = \frac{i}{2} \int_{m^2}^{\infty} dm^2 \langle \phi^2 \rangle_{\text{adiabatic}}$. Then we obtain

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = -\frac{1}{8\pi} g_{\mu\nu} \left[ m^2 (\psi(\nu_+) + \psi(\nu_-) + \ln 2) + m^2 \ln \frac{m^2}{\mu^2} - m^2 \ln \frac{R}{\mu^2} - \frac{1}{6} \frac{R}{R} \right].$$ (3.16)
Note that as $m^2 \to 0$, $(T_{\mu\nu})_{\text{ren}} \to \frac{1}{48\pi} g_{\mu\nu} R$ to correctly recover the conformal anomaly in the massless case.

Eq. (3.12) now becomes

$$0 = \ln \frac{R}{\mu^2} - \psi(\nu_+) - \psi(\nu_-) + \frac{1}{6m^2} R$$

(3.17)

where $\ln \hat{\mu}^2 \equiv \ln 2\mu^2$. This is the key equation of this letter. Solutions to this equation can be obtained as the intersection points of the following two curves:

$$f_1(\hat{R}) = \psi(\nu_+) + \psi(\nu_-) - \frac{1}{6} \hat{R}$$

(3.18a)

$$f_2(\hat{R}; \hat{\mu}^2) = \ln \frac{\hat{R}}{\hat{\mu}^2}$$

(3.18b)

where $\hat{R} \equiv R/m^2$ and $\hat{\mu} \equiv \tilde{\mu}/m$. Since $f_1$ is a decreasing function from $\lim_{R \to 0} f_1 \to 0$ to $\lim_{R \to \infty} f_1 \to -\infty$, there is always an intersection with $f_2$. Note that the intersection points depend only on $m^2$ and $\mu^2$.

Let us call the curvature at the intersection point $R_d(m^2, \tilde{\mu}^2)$, whose exact analytic form is not really important at this moment. $\hat{R}_d$ increases as $\hat{\mu}^2$ increases and $f_1 < 0$ implies that $\hat{R}_d < \hat{\mu}^2$. The above relations work regardlessly whether $\nu$ is real or imaginary because $\text{Im}(\psi(\nu_+) + \psi(\nu_-))$ always vanishes. It nevertheless constrains $\hat{R}_d \geq (<) 4/3 + 2\gamma + 7 \ln 2$ respectively. In principle, $\hat{\mu}^2$ is independent from $m^2$ so that $\hat{R}$ can take any value according to the value of $\hat{\mu}$. However, $\hat{\mu}^2$ should be of order $m^2$ because $m^2$ is the only scale of our model to start with and there is no reason to have a completely different mass scale to generate any hierarchical mystery. Thus $\hat{\mu} \sim \mathcal{O}(1)$ for reasonable solutions. Thus $R \sim \mathcal{O}(m^2)$ and in particular $R$ vanishes for the massless dilaton in this limit (this can be easily shown from eq. (3.10b) and eq. (3.12)).

4. Conclusion

We have seen so far that the massive dilaton explicitly breaks the Weyl symmetry so that the curvature of the gravitational background coupled to the dilaton is no longer arbitrary but constrained by the mass of the dilaton at least in the de Sitter space. In particular we have seen that the curvature does not depend on the Newton’s constant. As a result, the two-dimensional gravity is not trivial any more.
It is certainly arguable if eq.(3.17) is a consistent equation to address the current issue. One should first worry about that quantization of dilaton without quantizing gravity is indeed a reasonable thing to do. The dilaton is usually regarded as part of the gravity rather than a matter field, but in two dimensions we could treat it as a matter field. This is possible because $L_{\text{top}}$ in eq.(2.24) is invariant under the Weyl rescaling so that we cannot generate the dilaton field using the conformal property of the curvature as in the four dimensions. Due to the absence of Goldstone boson we could simply identify a real massive scalar field that breaks the scale symmetry as the dilaton. We avoided the quantization of the gravity simply because we do not know how to quantize the Liouville-like mode. Even though we did that, but, then, it would be difficult to separate and investigate the quantum effect of the dilaton to the system. Thus the semi-classical approach we took in this letter certainly serves our purpose. After knowing the fact that the dilaton constrains the scale of the curvature through its mass, we can fully quantize to investigate what kind of dynamical two-dimensional gravity can be obtained as the broken phase of the two-dimensional topological gravity.

We certainly do not know how dilaton becomes massive, needless to say, neither do we know how two-dimensional topological gravity generates the dilaton. Presumably it would be better off to address these questions in a theory where an extra symmetry related to the dilaton is present, e.g. $N = 2$ supergravity that becomes topological by twisting. Anyhow, once a massive dilaton is obtained, its quantum effect should not be too much different from the result given here.

It would be interesting to see a similar effect in the four-dimensional topological gravity by properly introducing a massive dilaton. In such a way, perhaps we could unlock the mystery of the relationship between the broken phase and the unbroken phase of topological gravity.

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1 Some issues on the validity of the semi-classical approximation in quantum gravity is discussed in [10].
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