Caccioppoli–type estimates and Hardy–type inequalities derived from degenerated $p$–harmonic problems

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Abstract

We obtain Caccioppoli–type estimates for nontrivial and nonnegative solutions to the anticoercive partial differential inequalities of elliptic type involving degenerated $p$–Laplacian: $-\Delta_{p,a}u := -\text{div}(a(x)|\nabla u|^{p-2}\nabla u) \geq b(x)\Phi(u)$, where $u$ is defined in a domain $\Omega$. Using Caccioppoli–type estimates, we obtain several variants of Hardy–type inequalities in weighted Sobolev spaces.

Key words and phrases: $p$–harmonic PDEs, $p$–Laplacian, nonlinear eigenvalue problems, degenerated PDEs, quasilinear PDEs

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1 Introduction

In this paper we investigate the nonnegative solutions $u : \Omega \to \mathbb{R}$ to the partial differential inequality (PDI):
\begin{equation}
- \Delta_{p,a} u \geq b(x) \Phi(u),
\end{equation}
where $\Omega \subseteq \mathbb{R}^n$ is an arbitrary open domain, $p > 1$, the operator $\Delta_{p,a} u = \text{div}(a(x)|\nabla u|^{p-2}\nabla u)$ is the degenerated $p$–Laplacian involving a weight function $a(\cdot) : \Omega \to [0, \infty)$, $b(\cdot)$ is a measurable function defined on $\Omega$, and $\Phi : [0, \infty) \to [0, \infty)$ is a given continuous function.

One of our main results, Theorem 4.1, says that if the nonnegative function $u$ solves (1.1), then we can apply it to construct the family of Hardy–type inequalities of the form:
\begin{equation}
\int_{\Omega} |\xi|^p \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_2(dx),
\end{equation}
where the measures $\mu_1$ and $\mu_2$ involve $u$ and the other quantities from (1.1), and $\xi$ is an arbitrary Lipschitz compactly supported function defined on $\Omega$. Those inequalities are constructed as a direct consequence of the Caccioppoli–type estimate for solutions to (1.1) derived in Theorem 3.1.

Our purpose is to investigate the two following issues: the qualitative theory of solutions to nonlinear problems and derivation of precise Hardy–type inequalities. We contribute to the first of them by obtaining Caccioppoli–type estimate for a priori not known solution, which in general is an important tool in the regularity theory. In the second issue we assume that the solution to (1.1) is known and we use it in construction of Hardy–type inequalities. Substituting $a \equiv 1$ in our considerations, we retrieve several results obtained in [40], where she dealt with the partial differential inequality of the form $-\Delta_{p,a} u \geq \Phi$, admitting the function $\Phi$ depending on $u$ and $x$. Some of the inequalities derived in [40], which motivated us to write this work, as well as those obtained here are precise as they hold with the best constants, see Remark 4.1, Theorem 4.3 and Theorem 4.2.

The approach presented here and in the papers [40] and [42] is the modification of methods from [25]. In all of these papers, the investigations start with derivation of Caccioppoli–type estimates for the solutions to nonlinear problem. The method was inspired by the well known nonexistence results by Pohozaev and Mitidieri [37].

In contrast with the results from [25] [40], in this paper we admit the degenerated $p$–Laplacian: $\Delta_{p,a}$ instead of the classical one in (1.1). Our main results are the Caccioppoli–type estimate (Theorem 3.1) and the Hardy–type inequality (Theorem 4.1). Some of the results obtained here are new even in the nondegenerated case $a \equiv 1$, see Remark 4.4 for details.

The discussion linking the eigenvalue problems with Hardy–type inequalities can be found in the paper by Gurka [24], which generalized earlier results by Beesack [8], Kufner
and Triebel [32], Muckenhoupt [36], and Tomaselli [44]. See also related more recent paper by Ghoussoub and Moradifam [23]. Derivation of the Hardy inequalities on the basis of supersolutions to \( p \)-harmonic differential problems can be found in papers by D’Ambrosio [16, 17, 18] and Barbatis, Filippas, and Tertikas [5, 6]. Other interesting results linking the existence of solutions in elliptic and parabolic PDEs with Hardy type inequalities are presented in [2, 4, 22, 45, 46], see also references therein. We refer also to the recent contribution by the third author [42], where, instead of the nondegenerated \( p \)-Laplacian in (1.1), one deals with the \( A \)-Laplacian: 
\[
\Delta_A u = \text{div} \left( \frac{A(|\nabla u|)}{|\nabla u|^2} \nabla u \right),
\]
involving a function \( A \) from the Orlicz class. Similar estimates in the framework of nonlocal operators can be found e.g. in [12].

Let us present several reasons to investigate the partial differential inequality of the form
\[-\Delta_{p,a} u \geq b(x)\Phi(u) \]
rather than a simple one
\[-\Delta_{p} u \geq \Phi(u).\]

The first inspiration comes from the investigation of the Matukuma equation
\[
\Delta u + \frac{1}{1 + |x|^2} u^q = 0, \quad q > 1,
\]
which describes the dynamics of globular clusters of stars [34] and existence results for its generalized version, Matukuma–Dirichlet problems studied in [20] and reading as follows:
\[
\begin{cases}
-\text{div} \left( |x|^a m(|\nabla u|) \nabla u \right) + \frac{|x|^{s-b}}{(1+|x|^b)^{s/b}} g(u) = 0 & \text{in } B(0,R), \\
u = 0 & \text{on } \partial B(0,R).
\end{cases}
\]

Similar PDEs arise often in astrophysics to model several phenomena. For instance, classical models of globular clusters of stars are modeled by Eddington’s equation [21]. Similar structure have models of dynamics of elliptic galaxies [3]. Qualitative properties of solutions to the equations inspired by models and their generalizations, are considered e.g. in [3, 7, 9, 15, 20, 39].

The second motivation comes from functional analysis and it concerns the embeddings of \( W^{1,p}_{a|\cdot|} (\Omega) \) into \( L^s_b(\Omega) \) and its generalizations, when Orlicz spaces are considered instead of \( L^s_b(\Omega) \). In such situation the equation
\[
- \text{div}(a(x)|\nabla u|^{p-2}\nabla u) = \gamma b(x) |u|^{s-2} u \tag{1.2}
\]
is the Euler–Lagrange equation for the Rayleigh energy functional
\[
E(u) = \frac{\left( \int_\Omega |\nabla u(x)|^p a(x)dx \right)^\frac{1}{p}}{\left( \int_\Omega |u(x)|^s b(x)dx \right)^\frac{1}{s}}.
\]
The particular case of the embedding $W^{1,p}_{a,\alpha}(\Omega) \to L^{s}_{b,\beta}(\Omega)$, where the weights are $a = |x|^\alpha, b = |x|^\beta$, is the Caffarelli–Kohn–Nirenberg inequality [14].

The third reason to investigate solutions of degenerated PDEs is that even if we deal with equation like (1.2) in the case $a(x) \equiv 1$, and we know that its solution $u(x) = w(|x|)$ is radial, we can transform equation (1.2) into the related degenerated ODE involving two weights. For example, the equation

$$-\text{div} \left( t^{n-1} v'(t)|^{p-2} v'(t) \right) = \gamma t^{n-1} |v(t)|^{p^*-2} v(t),$$

where $v(t) = w(r(t))$, $r(t)$ is inverse to $t(r) = \int_0^r s^{-\beta/p} ds = \frac{p^{-\beta/p}}{p-\beta/p} t^{(p-\beta)/p}$, $p^* = \frac{n-\beta}{n-p}$ is the Sobolev exponent in the embedding $W^{1,p}_{|x|^\beta}(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$ given by the Caffarelli–Kohn–Nirenberg inequality [14], is related to the transformation of equation

$$-\Delta_p u = \gamma |x|^{-\beta} |u|^{p^*-2} u,$$  \hspace{1cm} (1.3)

see e.g. [39] and the discussion on page 525 in [38].

In many cases the solutions are known and therefore we can use them to construct Hardy–type inequalities. For example, it has been shown in [38, Theorem 5.1], that the function

$$u(x) = c \left( 1 + |x|\right)^{-\frac{(n-p)}{p-\beta}} \left[ \frac{n-\beta}{\gamma} \left( \frac{n-p}{p-1} \right)^{p-1} \right]^\frac{(n-p)}{p-\beta},$$  \hspace{1cm} (1.4)

is the solution of the equation (1.3) in the case of $\beta < p < n$. When $\beta = 0$, we deal with Talenti extremal profile [43]. This fact was the motivation for the analysis presented in the paper [29], reported in Section 4, where the authors, under certain assumptions, obtained the inequality

$$\bar{C}_{\gamma,n,p,r} \int_{\mathbb{R}^n} |\xi|^p \left( 1 + r|x|^{\frac{p}{p-\beta}} \right) \left( 1 + |x|^{\frac{p}{p-\beta}} \right)^{(p-1)-p} d\ell \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left( 1 + |x|^{\frac{p}{p-1}} \right)^{(p-1)\gamma}$$

in some cases with the best constants. Such inequalities in the case $p = 2$ are of interest in the theory of nonlinear diffusions, where one investigates the asymptotic behavior of solutions of equation $u_t = \Delta u^m$, see [10] and the related works [11, 13, 23].

It might happen that the solutions to the partial differential inequality or equation (1.1) are known to exist by some existence theory, but their precise form is not known. In such a situation, under certain assumptions, we are still able to construct the Hardy inequality of the type

$$\int_\Omega |\xi|^p b(x) \, dx \leq \int_\Omega |\nabla \xi|^p a(x) \, dx,$$
which perhaps could be applied to study further properties of solutions. For example, the Hardy–Poincaré inequalities like above, where \( a(\cdot) = b(\cdot) \), are often equivalent to the solvability of degenerated PDEs of the type

\[
\text{div} \left( a(x)|\nabla u(x)|^{p-2}\nabla u(x) \right) = x^*,
\]

where \( x^* \) is an arbitrary functional on weighted Sobolev space \( W^{1,p}_{\rho,0}(\Omega) \) defined as the completion of \( C^\infty_0(\Omega) \) in the norm of Sobolev space \( W^{1,p}_\rho(\Omega) \), see Theorem 7.12 in [19].

We hope that by the investigation of the qualitative properties of supersolutions to degenerated PDEs and by constructions of Hardy–type inequalities, we can get deeper insight into the theory of degenerated elliptic PDEs.

## 2 Preliminaries

### Basic notation

In the sequel we assume that \( p > 1, \Omega \subseteq \mathbb{R}^n \) is an open subset not necessarily bounded. By \( a(\cdot) - p \)-harmonic problems we understand those which involve degenerated \( p \)-Laplace operator:

\[
\Delta_{p,a} u = \text{div} \left( a(x)|\nabla u(x)|^{p-2}\nabla u \right),
\]

with some nonnegative function \( a(\cdot) \). The derivatives which appear in (2.1) are understood in a distributional sense. By \( D'(\Omega) \) we denote the space of distributions defined on \( \Omega \). If \( f \) is defined on \( \Omega \), then by \( f\chi_{\Omega} \) we understand a function defined on \( \mathbb{R}^n \) which is equal to \( f \) on \( \Omega \) and which is extended by 0 outside \( \Omega \). Negative part of \( f \) is denoted by \( f^- := \min\{f,0\} \), while positive one by \( f^+ := \max\{f,0\} \). Moreover, every time when we deal with infimum, we set \( \inf \emptyset = +\infty \).

### Weighted Beppo Levi and Sobolev spaces

**\( B_p \) weights.** We deal with the special class of measures belonging to the class \( B_p(\Omega) \).

**Definition 2.1 (Classes \( W(\Omega) \) and \( B_p(\Omega) \)).** Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and let \( \mathcal{M}(\Omega) \) be the set of all Borel measurable real functions defined on \( \Omega \). Denote \( W(\Omega) := \{ \varrho \in \mathcal{M}(\Omega) : 0 < \varrho(x) < \infty, \text{ for a.e. } x \in \Omega \} \), and let \( p > 1 \). We will say that a weight \( \varrho \in W(\Omega) \) satisfies \( B_p(\Omega) \)-condition (\( \varrho \in B_p(\Omega) \) for short) if \( \varrho^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega) \).

The Hölder inequality leads to the following simple observation based on Theorem 1.5 in [31]. For readers’ convenience we enclose the proof.
Proposition 2.1. Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( p > 1 \) and \( \varrho \in B_p(\Omega) \). Then \( L^p_{\varrho,\text{loc}}(\Omega) \subseteq L^1_{\text{loc}}(\Omega) \) and when \( u_k \to u \) locally in \( L^p_\varrho(\Omega) \) then also \( u_k \to u \) in \( L^1_{\text{loc}}(\Omega) \).

**Proof.** For any \( \Omega' \subseteq \Omega \) such that \( \overline{\Omega'} \subseteq \Omega \) and any \( u \in L^p_{\varrho,\text{loc}}(\Omega) \)

\[
\int_{\Omega'} |u| \, dx = \int_{\Omega'} |u|^\frac{1}{p} \varrho^{-\frac{1}{p}} \, dx \leq \left( \int_{\Omega'} |u|^p \varrho \, dx \right)^\frac{1}{p} \left( \int_{\Omega'} \varrho^{-\frac{1}{p}} \, dx \right)^{1-\frac{1}{p}} < \infty.
\]

The substitution of \( u_k - u \) instead of \( u \) implies second part of the statement. \( \square \)

**Weighted Beppo Levi space.** Assume that \( \varrho(\cdot) \in B_p(\Omega) \). We deal with the weighted Beppo Levi space

\[
\mathcal{L}^{1,p}_\varrho(\Omega) := \{ u \in D'(\Omega) : \frac{\partial u}{\partial x_i} \in L^p_\varrho(\Omega) \text{ for } i = 1, \ldots, n \}.
\]

According to the above proposition and [35, Theorem 1, Section 1.1.2], we have \( \mathcal{L}^{1,p}_\varrho(\Omega) \subseteq W^{1,1}_{\text{loc}}(\Omega) \). We will also consider local variants of Beppo Levi spaces:

\[
\mathcal{L}^{1,p}_{\varrho,\text{loc}}(\Omega) := \{ u \in D'(\Omega) : \int_{\Omega'} |\nabla u(x)|^p \varrho(x) \, dx < \infty \},
\]

whenever \( \overline{\Omega'} \) is a compact subset of \( \Omega \). As it is also a subset in \( W^{1,1}_{\text{loc}}(\Omega) \), integration by parts formula applies to elements of \( \mathcal{L}^{1,p}_{\varrho,\text{loc}}(\Omega) \) in the usual way.

**Two-weighted Sobolev spaces.** Let \( \varrho_1(\cdot) \in W(\Omega), \varrho_2(\cdot) \in B_p(\Omega) \). We consider the space \( W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega) = L^p_{\varrho_1}(\Omega) \cap \mathcal{L}^{1,p}_{\varrho_2}(\Omega) \), i.e.

\[
W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega) := \left\{ f \in L^p_{\varrho_1}(\Omega) \cap D'(\Omega) : \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \in L^p_{\varrho_2}(\Omega) \right\}, \quad (2.2)
\]

with the norm \( \| f \|_{W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega)} := \| f \|_{L^p_{\varrho_1}(\Omega)} + \| \nabla f \|_{L^p_{\varrho_2}(\Omega)} \).

**Proposition 2.2 ([31]).** Let \( p > 1 \), \( \Omega \subseteq \mathbb{R}^n \) be an open set and \( \varrho_1(\cdot) \in W(\Omega), \varrho_2(\cdot) \in B_p(\Omega) \). Then \( W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega) \) defined by (2.2) equipped with the norm \( \| \cdot \|_{W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega)} \) is a Banach space.

When \( \varrho_1 \equiv \varrho_2 \), we deal with the usual weighted Sobolev space \( W^{1,p}_{\varrho}(\Omega) \). By \( W^{1,p}_{(\varrho_1,\varrho_2),0}(\Omega) \) we denote the completion of \( C_0^\infty(\Omega) \) in the space \( W^{1,p}_{(\varrho_1,\varrho_2)}(\Omega) \) and we use the standard notation \( W^{1,p}_{(\varrho_1,\varrho_1),0}(\Omega) = W^{1,p}_{\varrho,0}(\Omega) \) when \( \varrho_1 = \varrho_2 \).

**Some additional facts**

Having an arbitrary function \( u \in W^{1,1}_{\text{loc}}(\Omega) \) (local Sobolev space), we define its value at every point \( x \in \Omega \) by the formula

\[
u(x) := \limsup_{r \to 0} \int_{B(x,r)} u(y) \, dy.
\]

**Lemma 2.1 (e.g. [25], Lemma 3.1).** Let \( u \in W^{1,1}_{\text{loc}}(\Omega) \) be defined everywhere by (2.1) and let \( t \in \mathbb{R} \) be given. Then \( \{ x \in \mathbb{R}^n : u(x) = t \} \subseteq \{ x \in \mathbb{R}^n : \nabla u(x) = 0 \} \cup N \), where \( N \) is a set of Lebesgue’s measure zero.
Degenerated $p$–Laplacian

Assume that $p > 1$, $a \in B_p(\Omega) \cap L^1_{loc}(\Omega)$ (see Definition 2.1), and $u \in L^1_{a,loc}(\Omega)$. Then $a|\nabla u|^{p-1} \in L^1_{loc}(\Omega)$ as we have:

$$
\int_{\Omega'} a|\nabla u|^{p-1}dx \leq \left( \int_{\Omega'} adx \right)^{\frac{1}{p}} \left( \int_{\Omega'} |\nabla u|^p dx \right)^{1-\frac{1}{p}} < \infty,
$$

whenever $\Omega'$ is a compact subset of $\Omega$. In particular, $a|\nabla u|^{p-2}\nabla u \in L^1_{loc}(\Omega, \mathbb{R}^n)$ and so the weak divergence of $a|\nabla u|^{p-2}\nabla u \in L^1_{loc}(\Omega, \mathbb{R}^n)$ denoted by $\Delta_{p,a}u$ is well defined via the formula

$$
\langle \Delta_{p,a}u, w \rangle = \langle \text{div} \ (a|\nabla u|^{p-2}\nabla u), w \rangle := -\int_{\Omega} a|\nabla u|^{p-2}\nabla u \cdot \nabla w dx \quad (2.4)
$$

where $w \in C_0^\infty(\Omega)$. Obviously, in the case $a \equiv 1$ the operator $\Delta_{p,a}u$ reduces to the usual $p$–Laplacian div ($|\nabla u|^{p-2}\nabla u$). It particular, it coincides with the Laplace operator in the case $p = 2$.

Remark 2.1. We observe that

i) as $|\nabla u|^{p-2}\nabla u \in L^p_{a,loc}(\Omega, \mathbb{R}^n)$, then the right–hand side in (2.4) is well defined for every $w \in L^1_a(\Omega)$ which is compactly supported in $\Omega$;

ii) when $u \in L^1_{a,loc}(\Omega)$, formula (2.4) extends for $w \in W^{1,p}_{(b,a),0}(\Omega)$, whenever $b \in W(\Omega)$.

This follows from the estimates

$$
|\langle \Delta_{p,a}u, w \rangle| \leq \int_{\Omega} a|\nabla u|^{p-1}|\nabla w|dx = \int_{\Omega} (a^{\frac{1}{p'}}|\nabla u|^{p-1})(a^\frac{1}{p}|\nabla w|)dx

\leq \left( \int_{\Omega} |\nabla u|^{p} dx \right)^{1-\frac{1}{p}} \left( \int_{\Omega} |\nabla w|^{p} dx \right)^{\frac{1}{p}} < \infty.
$$

Therefore, in that case $\Delta_{p,a}u$ can be also treated as an element of $(W^{1,p}_{(b,a),0}(\Omega))^*$, the dual to the Banach space $W^{1,p}_{(b,a),0}(\Omega)$. We preserve the same notation $\Delta_{p,a}u$ for this functional extension of formula (2.4).

Differential inequality

Our analysis is based on the following differential inequality.
Definition 2.2. Let \( a \in B_p(\Omega) \cap L_{1,loc}^1(\Omega) \) be a given weight, \( u \in L_{a,loc}^{1,p}(\Omega) \) be nonnegative, \( \Phi : [0, \infty) \to [0, \infty) \) be a continuous function, \( b(\cdot) \) be measurable and \( \Phi b \in L_{loc}^1(\Omega) \). Suppose further that for every nonnegative compactly supported function \( w \in L_{1,loc}^{1,p}(\Omega) \) one has

\[
\int_{\Omega} \Phi(u)b(x)w \, dx > -\infty.
\]

We say that partial differential inequality (PDI for short)

\[
-\Delta_{p,a} u \geq \Phi(u)b(x),
\]

holds if for every nonnegative compactly supported function \( w \in L_{a,loc}^{1,p}(\Omega) \) we have

\[
\langle -\Delta_{p,a} u, w \rangle \geq \int_{\Omega} \Phi(u)b(x)w \, dx,
\]

where \( \langle -\Delta_{p,a} u, w \rangle \) is given by (2.4), see also Remark 2.1.

We have the following observations.

Remark 2.2.

i) Inequality (2.5) can be interpreted as a variant of \( p \)-superharmonicity condition for the degenerated \( p \)-Laplacian defined by (2.1).

ii) In the case of equation in (2.5): \(-\Delta_{p,a} u = \Phi(u)b(x)\), we deal with the solution of the nonlinear eigenvalue problem.

iii) When \( u \equiv D \geq 0 \) is a constant on some subdomain \( \Omega' \subseteq \Omega \), inequality (2.5) implies \( b(x)\Phi(D) \leq 0 \) a.e. in \( \Omega' \), equivalently this means that either \( b \leq 0 \) a.e. in \( \Omega' \) and \( \Phi(D) > 0 \) or else \( \Phi(D) = 0 \). Consequently, when \( \Phi(0) = 0 \) and \( \Phi(t) > 0 \) for \( t > 0 \), inequality (2.5) holds on \( \Omega_1 \) with \( u \equiv D \) if either \( D \neq 0 \) and \( b \leq 0 \) on \( \Omega_1 \) or \( D = 0 \) and \( b \) is arbitrary.

Assumption A

By Assumption A we mean the set of conditions: \((a, b), (\Psi, g), (u), \) and a)–d) below.

\( (a, b) \) \( a \in L_{1,loc}^1(\Omega) \cap B_p(\Omega) \), \( b(\cdot) \) is measurable;

\( (\Psi, g) \) The couple of continuous functions \( (\Psi, g) : (0, \infty) \times (0, \infty) \to (0, \infty) \times (0, \infty) \), where \( \Psi \) is Lipschitz on every closed interval in \( (0, \infty) \), satisfy the following compatibility conditions:
i) the inequality
\[ g(t)\Psi'(t) \leq -C\Psi(t) \quad \text{a.e. in } (0, \infty) \] holds with some constant \( C \in \mathbb{R} \) independent of \( t \) and \( \Psi \) is monotone (not necessarily strictly);

ii) each of the functions
\[ t \mapsto \Theta(t) := \Psi(t)g^{p-1}(t), \quad \text{and} \quad t \mapsto \Psi(t)/g(t) \]
is nonincreasing or bounded in some neighbourhood of 0.

\((u)\) We assume that \( u \in L^{1,p}_{a,loc}(\Omega) \) is nonnegative, \((a, b)\) holds, \( \Phi : [0, \infty) \to [0, \infty) \)
is a continuous function, such that for every nonnegative compactly supported function \( w \in L^{1,p}_{a}(\Omega) \) one has \( \int_{\Omega} \Phi(u)b(x)w \, dx > -\infty \) and \( \Phi(u)b \in L^{1}_{loc}(\Omega) \).

Moreover, let us consider the set \( \mathcal{A} \) of those \( \sigma \in \mathbb{R} \) for which
\[ \Phi(u)b(x) + \sigma a(x) \frac{\nabla u}{g(u)} \geq 0 \quad \text{a.e. in } \Omega \cap \{ u > 0 \}. \] (2.9)

We suppose that
\[ \sigma_0 := \inf \mathcal{A} = \inf \{ \sigma \in \mathbb{R} : \sigma \text{ satisfies (2.9)} \} \in \mathbb{R}. \] (2.10)

Since \( \inf \emptyset = +\infty \), \( \mathcal{A} \) can be neither an empty set nor unbounded from below.

a) We suppose that \((\Psi, g)\) and \((u)\) hold. Parameter \( \sigma \) satisfies \( \sigma_0 \leq \sigma < C \), where \( C \) is given by (2.7) and \( \sigma_0 \) by (2.10).

b) We suppose that \((u)\) and \((\Psi, g)\) hold and we assume that for every \( R > 0 \) we have
\[ b^+(x)(\Phi\Psi)(u)\chi_{0 < u \leq R} \in L^{1}_{loc}(\Omega). \]

c) We suppose that \((u)\) and \((\Psi, g)\) hold. When the set \( \Omega_0 := \{ x : u(x) = 0 \} \) has a positive measure, then we assume that at least one of the following conditions are satisfied
\[ x) \Phi(0) = 0, \quad y) b(x)\chi_{\Omega_0} \geq 0, \quad z) \lim_{\delta \to 0} \Psi(\delta) = 0. \]

d) We suppose that \((u)\) and \((\Psi, g)\) hold. We assume that for any compact subset \( K \subseteq \Omega \) we have
\[ \Psi(R) \int_{K \cap \{ u \geq R/2 \}} |\nabla u(x)|^{p-1}a(x) \, dx \xrightarrow{R \to \infty} 0, \]
\[ \Psi(R) \int_{K \cap \{ u \geq R/2 \}} \Phi(u)b(x) \, dx \xrightarrow{R \to \infty} 0. \]
Comments on assumptions

We have the following observations on Condition $(\Psi, g)$.

Remark 2.3. i) Assume that Condition $(\Psi, g)$, i) holds and, moreover, $g'(t) \geq -C$. Then $(\Psi/g)' \leq 0$ and $\Psi(t)/g(t)$ is nonincreasing.

ii) This condition is satisfied by pairs from Table 1.

iii) For our purposes it suffices to weaken assumption $(\Psi, g)$ in the following way. When $u(x) \in (k_1, k_2) \subseteq [0, \infty)$, we can restrict Condition $(\Psi, g)$ to $(k_1, k_2)$ instead of $(0, \infty)$. This follows from the proofs presented below.

| $\Psi(t)$ | $g(t)$ | $C$ | remarks |
|-----------|--------|-----|---------|
| $t^{-\alpha}$ | $t$ | $\alpha$ | $\alpha \in \mathbb{R}$ |
| $(t \log(a + t))^{-1}$ | $t \log(a + t)$ | $\log a$ | $a > 1$ |
| $e^{-t}$ | bounded by $C$, $g' \geq -C$ | $C$ | $C > 0$ |
| $e^{-t}/t$ | $t/(1 + t)$ | 1 | — |

Table 1: Example couples $(\Psi, g)$ which satisfy Condition $(\Psi, g)$.

The statement below shows that under Assumption $A_r(u)$ the function $u$ cannot be constant almost everywhere in $\Omega$. Moreover, in many cases $A$ is not empty and $\inf A$ is a real number.

Lemma 2.2. Suppose $u \in L^{1,p}_{a,loc}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_{p,a} u \geq \Phi(u)b(x)$ in the sense of Definition 2.2, under all assumptions therein. Moreover, let $b \geq 0$ a.e. in $\Omega$. Then $\sigma_0$ given by (2.10) exists and is finite if and only if $u$ is not a constant function a.e. in $\Omega$.

Proof. ($\Leftarrow$) Assume that $u \not\equiv Const$. Then the set $A$ is not empty as it contains zero, in particular $\sigma_0 \leq 0$. If $a(\cdot) > 0$, $b(\cdot) \geq 0$ a.e. in $\Omega$, then the set $A$ cannot be unbounded from below. Indeed, if $A$ was unbounded from below, the inequality:

$$\Phi(u(x))b(x) - \bar{n} \frac{a(x)}{g(u(x))} |\nabla u(x)|^p \geq 0 \quad \text{a.e. in } \Omega \cap \{u > 0\}$$

would hold for every $\bar{n} \in \mathbb{N}$. Consequently we could find $K_1, K_2 > 0$, such that

$$\frac{1}{\bar{n}} \Phi(u(x))b(x) \geq \frac{a(x)}{g(u(x))} |\nabla u(x)|^p \geq K_1 \frac{1}{K_2} > 0$$
a.e. in \( \{ u : \| \nabla u \|^p a(x) \geq K_1, g(u(x)) \leq K_2 \} \), which is the set of positive measure and independent on \( \bar{n} \). Taking the limit for \( \bar{n} \to \infty \), we arrive at the contradiction.

\( \implies \) If \( \sigma_0 \) is a finite number, then \( u \) cannot be constant. Indeed, for \( u \equiv \text{Const} \geq 0 \), condition (2.9) implies \( \mathcal{A} = (-\infty, \infty) \), which violates (2.10).

**Remark 2.4.** Assumption A, d) is satisfied in each of the following cases:

i) When \( u \) is locally bounded.

ii) When \( b \geq 0, u \in L^p_{a,\text{loc}}(\Omega) \) and \( \Psi(R)/R \) is bounded at infinity. Indeed, we have from Hölder’s inequality

\[
Z_1(R) := \Psi(R) \int_{K \cap \{ u \geq R/2 \}} \| \nabla u(x) \|^p a(x) \, dx \leq \Psi(R) \left( \int_{K \cap \{ u \geq R/2 \}} \| \nabla u(x) \|^p a(x) \, dx \right)^{1-\frac{1}{p}} \left( \int_{K \cap \{ u \geq R/2 \}} a(x) \, dx \right)^{\frac{1}{p}}
\]

and \( Z_2(R) := \left( \int_{K \cap \{ u \geq R/2 \}} \| \nabla u(x) \|^p a(x) \, dx \right)^{1-\frac{1}{p}} \to 0 \) as \( R \to \infty \). On the other hand, by Czebyshev’s inequality applied to \( \mu(x) = a(x) \, dx \) on \( K \), we get

\[
\int_{K \cap \{ u \geq R/2 \}} a(x) \, dx = \mu(\{ x \in K : u(x) \geq R/2 \}) \leq \frac{2^p}{R^p} \int_K \| u \|^p a(x) \, dx =: \frac{1}{R^p} Z_3(R).
\]

Therefore, \( Z_1(R) \leq \frac{\Psi(R)}{R} Z_2(R) Z_3(R)^{\frac{1}{p}} \to 0 \) as \( R \to \infty \).

### 3 Caccioppoli–type estimates

Our first goal is to obtain the following estimate. We call it local, because it is stated on a part of the domain where \( u \) is not bigger than a given \( R \).

**Lemma 3.1** (Local estimate). Suppose that Assumption A holds except part d). Assume further that \( 1 < p < \infty \) and \( u \) is a nonnegative solution to PDI

\[
- \Delta_{p,a} u \geq \Phi(u) b(x)
\]

in the sense of Definition 2.2.
Then for any nonnegative Lipschitz function $\phi$ with compact support in $\Omega$ such that the integral $\int_{\{\text{supp}\,\phi\cap\nabla u \neq 0\}} |\nabla \phi|^p \phi^{1-p} a(x)\,dx$ is finite and for any $R > 0$ the inequality
\[
\int_{\{0 < u < R\}} \left( \Phi(u(x)) b(x) + \sigma \frac{a(x)}{g(u(x))} |\nabla u(x)|^p \right) \Psi(u(x)) \phi(x)\,dx 
\leq c \int_{\{\nabla u(x) \neq 0, 0 < u < R\} \cap \text{supp}\,\phi} a(x) \Psi(u(x)) g^{p-1}(u(x)) |\nabla \phi(x)|^p \phi^{1-p}(x)\,dx + \tilde{C}(R),
\]
holds, where $c := \frac{1}{p} \left( \frac{p-1}{p-\sigma} \right)^{p-1}$,
\[
\tilde{C}(R) = \Psi(R) \left[ \int_{\Omega \cap \{u \geq \frac{R}{2}\}} a(x)|\nabla u|^{p-1}|\nabla \phi|\,dx - \int_{\Omega \cap \{u \geq \frac{R}{2}\}} \Phi(u) b(x)\phi\,dx \right].
\]
Moreover, all quantities appearing in (3.2) are finite.

The above result implies the following global estimate (3.3) for solution $s$ to (3.1). It may be used to analyze various qualitative properties of them. We call it Caccioppoli–type inequality, because the right–hand side in (3.3) does not involve $\nabla u$ when we estimate $\chi_{\{\nabla u \neq 0\}}$ by 1, while, on the other hand, the left–hand side does involve $\nabla u$.

**Theorem 3.1** (Caccioppoli–type estimate). Suppose that Assumption A holds, $1 < p < \infty$ and $u \in L^{1,p}_{a,\text{loc}}(\Omega)$ is a nonnegative solution to the PDI
\[-\Delta_{p,a} u \geq \Phi(u)b(x)\]
in the sense of Definition 2.2.

Then for every nonnegative Lipschitz function $\phi$ with compact support in $\Omega$ such that the integral $\int_{\text{supp}\,\phi} |\nabla \phi|^p \phi^{1-p} a(x)\,dx$ is finite, we have
\[
\int_{\Omega \cap \{u > 0\}} \left( \Phi(u(x)) b(x) + \sigma |\nabla u(x)|^p \frac{a(x)}{g(u(x))} \right) \Psi(u(x)) \phi(x)\,dx \leq c \int_{\Omega \cap \{u > 0, \nabla u(x) \neq 0\} \cap \text{supp}\,\phi} a(x) \Psi(u(x)) g^{p-1}(u(x)) |\nabla \phi(x)|^p \phi^{1-p}(x)\,dx,
\]
with $c = \frac{(p-1)^{p-1}}{p^p(C-\sigma)^{p-1}}$.

**Remark 3.1.** Our assumptions do not exclude the case when measure $\left( \Phi(u(\cdot)) b(\cdot) + \sigma |\nabla u(\cdot)|^p \frac{a(\cdot)}{g(u(\cdot))} \right) \Psi(u(\cdot)) \chi_{\{u > 0\}}$ is equal to zero.
Proof of the local estimates

We use the following simple observations (see [40]).

**Lemma 3.2.** Let $p > 1$, $\tau > 0$ and $s_1, s_2 \geq 0$, then

$$s_1 s_2^{p-1} \leq \frac{1}{p^{\tau p - 1}} \cdot s_1^p + \frac{p - 1}{p} \tau \cdot s_2^p.$$ 

**Lemma 3.3.** Let $u, \phi$ be as in the assumptions of Lemma 3.1. We fix $0 < \delta < R$ and denote

$$u_{\delta,R}(x) := \min \{ u(x) + \delta, R \}, \quad G(x) := \Psi(u_{\delta,R}(x)) \phi(x). \quad (3.4)$$

Then $u_{\delta,R} \in \mathcal{L}^{1,p}_{a,\text{loc}}(\Omega)$, $G \in \mathcal{L}^{1,p}_a(\Omega)$ and $G$ is compactly supported in $\Omega$.

**Remark 3.2.**

i) We know that $\mathcal{L}^{1,p}_{a,\text{loc}}(\Omega) \subseteq W^{1,1}_{\text{loc}}(\Omega)$. This inclusion, together with Nikodym ACL Characterization Theorem [35, Section 1.1.3], implies that we can verify if the function belongs to Sobolev space $\mathcal{L}^{1,p}_{a,\text{loc}}(\Omega)$ by checking that it belongs to $W^{1,1}_{\text{loc}}(\Omega)$ and that its derivatives computed almost everywhere belong to $L^p_{a,\text{loc}}(\Omega)$. The fact that $\Psi$ is locally Lipschitz is used to apply Lemma 3.3 in order to ensure that $\Psi(u_{\delta,R}(x))$ belongs to $W^{1,1}_{\text{loc}}(\Omega)$.

ii) The nonnegativity of function $u$ allows to deduce that $G \in \mathcal{L}^{1,p}_a(\Omega)$. This fact plays the crucial role in the proof of Lemma 3.1.

**Proof of Lemma 3.1**

We present the proof under the assumption that the set $\Omega_0$ in Assumption A, c) has a positive measure. The proof in the case $u > 0$ a.e. follows by the simplification of the presented arguments.

Let the quantities $\Phi, \Psi, g, a, b, u$ be as in (3.1) and Assumption A, while $\phi$ be as in the statement of the lemma.

The proof is performed in four steps:

**Step 1.** We prove that for every $0 < \delta < R$, the inequality

$$\int_{\{\Omega \cap u \leq R - \delta\}} \left( \Phi(u) b(x) + \sigma \frac{g(x)}{g(u + \delta)} |\nabla u|^p \right) \Psi(u + \delta) \phi \, dx$$

$$\leq c \int_{\Omega \cap \text{supp} \phi \cap \{ |\nabla u| \neq 0, 0 < u \leq R - \delta\}} \frac{a(x)}{\phi} |\nabla \phi|^p (u + \delta) \phi \, dx$$

$$+ \tilde{C}(\delta, R) \quad (3.5)$$
holds with $\sigma$ from Assumption A, a) and

$$
\tilde{C}(\delta, R) = \Psi(R) \left[ \int_{\Omega \cap \{\nabla u \neq 0, u > R-\delta\}} a(x)|\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx - \int_{\Omega \cap \{u > R-\delta\}} \Phi(u)b(x)\phi \, dx \right].
$$

**Step 2.** We pass to the limit for $\delta \searrow 0$ and obtain

$$
\limsup_{\delta \searrow 0} c \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} a(x)\Psi(u+\delta)g^{p-1}(u+\delta) \left( \frac{|\nabla \phi|}{\phi} \right)^p \phi \, dx + \tilde{C}(\delta, R)
\leq c \int_{\text{supp } \phi \cap \{\nabla u(x) \neq 0, 0 < u < R\}} a(x)\Psi(u(x))g^{p-1}(u(x)) |\nabla \phi(x)|^p \phi^{1-p}(x) \, dx + \tilde{C}(R).
$$

**Step 3.** For $\delta \geq 0$ we denote

$$
A_\delta(x) := \left( \Phi(u(x))b(x) + \sigma \frac{a(x)}{g(u+\delta)}|\nabla u|^p \right) \Psi(u+\delta) \quad \text{when } \delta > 0, \quad (3.6)
$$

$$
A_0(x) := \left( \Phi(u(x))b(x) + \sigma \frac{a(x)}{g(u)}|\nabla u|^p \right) \Psi(u)\chi_{\{u > 0\}} \quad \text{when } \delta = 0.
$$

We show that

$$
\liminf_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} A_\delta(x)\phi(x) \, dx \geq \int_{\{0 < u < R\}} A_0(x)\phi(x) \, dx.
$$

**Step 4.** We show that

$$
\liminf_{\delta \searrow 0} \int_{\{u = 0\}} A_\delta(x)\phi(x) \, dx \geq 0, \quad (3.7)
$$

which implies the statement.
Proof of Step 1

Let us introduce the following notation for \( J_i = J_i(\delta, R), \ i = 1, \ldots, 6 \):

\[
\begin{align*}
J_1 &= \int_{\Omega \cap \{0 < u \leq R - \delta\}} a(x)|\nabla u|^p \Psi'(u + \delta) \phi \, dx, \\
J_2 &= \int_{\Omega \cap \{0 < u \leq R - \delta\}} a(x)|\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx, \\
J_3 &= \int_{\Omega \cap \{0 < u \leq R - \delta\}} a(x)|\nabla u|^{p-2} \Psi(u + \delta) \nabla u \cdot \nabla \phi \, dx, \\
J_4 &= \Psi(R) \int_{\Omega \cap \{u > R - \delta\}} \Phi(u) b(x) \phi \, dx, \\
J_5 &= \Psi(R) \int_{\Omega \cap \{u > R - \delta\}} a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx, \\
J_6 &= \int_{\text{supp} \phi \cap \{\nabla u \neq 0, 0 < u \leq R - \delta\}} a(x) \left(\frac{|\nabla \phi|}{\phi}\right)^p \Psi(u + \delta) g^{p-1}(u + \delta) \phi \, dx.
\end{align*}
\]

By our assumptions all the above quantities are finite (for \(0 \leq u \leq R - \delta\) we have \(\delta \leq u + \delta \leq R\)). Let \(G\) be given by (3.4). Choose \(w := G\) to be a test function in (2.6). Then the right hand side of (2.6) becomes

\[
\begin{align*}
I &= \int_{\Omega} \Phi(u) b(x) G(x) \, dx = \int_{\Omega} \Phi(u) b(x) \Psi(u_{\delta,R}) \phi \, dx = \\
&= \int_{\Omega \cap \{u \leq R - \delta\}} \Phi(u) b(x) \Psi(u + \delta) \phi \, dx + \Psi(R) \int_{\Omega \cap \{u > R - \delta\}} \Phi(u) b(x) \phi \, dx = \\
&= \int_{\Omega \cap \{u \leq R - \delta\}} \Phi(u) b(x) \Psi(u + \delta) \phi \, dx + J_4,
\end{align*}
\]

so that \(I\) is finite. Thus using (2.6) we get the following estimate

\[
\begin{align*}
I &= \int_{\Omega} \Phi(u(x)) b(x) G(x) \, dx \leq \langle -\text{div}\left(a(x)|\nabla u|^{p-2} \nabla u\right), G \rangle \\
&= \int_{\Omega \cap \{\nabla u \neq 0\}} a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla G \, dx \\
&= \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} a(x)|\nabla u|^{p-2} \Psi'(u + \delta) \phi \, dx + \\
&\quad + \int_{\Omega \cap \{\nabla u \neq 0, u \leq R - \delta\}} a(x)|\nabla u|^{p-2} \Psi(u + \delta) \nabla u \cdot \nabla \phi \, dx + \\
&\quad + \Psi(R) \int_{\Omega \cap \{\nabla u \neq 0, u > R - \delta\}} a(x)|\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = J_1 + J_3 + J_5 \\
&\leq -CJ_2 + J_3 + J_5.
\end{align*}
\]
The last inequality follows from $J_1 \leq -CJ_2$ which holds due to (2.7). Moreover,

$$J_3 \leq \int_{\Omega \cap \{\nabla u \neq 0, u \leq R-\delta\}} a(x)|\nabla u|^{p-1}|\nabla \phi|\Psi(u + \delta) \, dx =$$

$$= \int_{\text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\}} \left(\frac{|\nabla \phi|}{\phi} g(u + \delta)\right) |\nabla u|^{p-1}a(x) \frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx.$$

We apply Lemma 3.2 with $s_1 = \frac{|\nabla \phi|}{\phi} g(u + \delta), s_2 = |\nabla u|$ and arbitrary $\tau > 0$, to get

$$J_3 \leq \frac{p - 1}{p} \int \text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\} a(x)|\nabla u|^{p-1}\frac{\Psi(u + \delta)}{g(u + \delta)} \phi \, dx +$$

$$+ \frac{1}{p\tau^{p-1}} \int \text{supp } \phi \cap \{\nabla u \neq 0, u \leq R-\delta\} a(x) \left(\frac{|\nabla \phi|}{\phi}\right)^p \Psi(u + \delta)g^{p-1}(u + \delta)\phi \, dx.$$

$$\leq \frac{p - 1}{p} \tau J_2 + \frac{1}{p\tau^{p-1}} J_6.$$

Combining these estimates we deduce that for $\tau > 0$ such that $C - \frac{p - 1}{p} \tau = \sigma$ we have

$$I \leq -CJ_2 + J_3 + J_5 \leq$$

$$\leq \left(-C + \frac{p - 1}{p} \tau\right) J_2 + \frac{1}{p\tau^{p-1}} J_6 + J_5 = -\sigma J_2 + \frac{1}{p\tau^{p-1}} J_6 + J_5.$$

The last inequality and (3.8) imply

$$\int_{\Omega \cap \{u \leq R-\delta\}} \Phi(u) b(x) \Psi(u + \delta) \phi \, dx + \sigma J_2 \leq \frac{1}{p\tau^{p-1}} J_6 + J_5 - J_4,$$

which implies (3.5), because $\tilde{C}(\delta, R) \geq J_5 - J_4$ and $\tau = (C - \sigma) \frac{p}{p - 1}$.

Introduction of parameters $\delta$ and $R$ is necessary as we need to move the quantities $J_2, J_4$ in the estimates to the opposite sides of inequalities. For doing this we have to know that they are finite.

**Proof of Step 2**

We show first that under our assumptions, when $\delta \searrow 0$, we have

$$\int \text{supp } \phi \cap \{\nabla u \neq 0, u + \delta \leq R\} a(x) \Psi(u + \delta) g^{p-1}(u + \delta)|\nabla \phi|^{p-1} \phi \, dx \quad (3.9)$$

$$\rightarrow \int \text{supp } \phi \cap \{\nabla u \neq 0, 0 < u \leq R\} a(x) \Psi(u) g^{p-1}(u)|\nabla \phi|^{p-1} \phi \, dx.$$
To verify this we note that for a.e. $x \in \Omega$ we have
\[
\Psi(u(x) + \delta)g^{p-1}(u(x) + \delta) \chi_{\{\nabla u(x) \neq 0, u(x) + \delta \leq R\}} \xrightarrow{\delta \to 0} \Psi(u(x))g^{p-1}(u(x)) \chi_{\{\nabla u(x) \neq 0, 0 \leq u(x) \leq R\}}.
\]
Indeed, when $0 < u(x) < R$ or $u(x) > R$ this follows from the continuity of the involved functions, while according to Lemma 2.1 the set $\{x : u(x) = 0, |\nabla u(x)| \neq 0\} \cup \{x : u(x) = R, |\nabla u(x)| \neq 0\}$ is of measure zero.

For the proof of (3.9) we recall the nonnegative function $\Theta(t) := \Psi(t)g^{p-1}(t)$ given by (2.8), which is nonincreasing or bounded in the neighbourhood of zero.

Let us start with the first case, i.e. there exists $\varepsilon > 0$ such that for $t < \varepsilon$ the function $\Theta(t)$ is nonincreasing. Without loss of generality we may assume $2\delta \leq \varepsilon \leq R$ and
\[
E_{\varepsilon} = \left\{ \nabla u \neq 0, u < \frac{\varepsilon}{2} \right\} \cap \text{supp} \phi, \quad F_{\varepsilon} = \left\{ \nabla u \neq 0, \frac{\varepsilon}{2} \leq u \right\} \cap \text{supp} \phi.
\]
Then we have
\[
\int_{\text{supp} \phi \cap \{\nabla u \neq 0, u + \delta \leq R\}} \Theta(u + \delta)a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx = \\
= \int_{E_{\varepsilon}} \Theta(u + \delta)a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx + \int_{F_{\varepsilon}} \Theta(u + \delta)\chi_{\{u + \delta \leq R\}}a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx.
\]

Let us concentrate on the integral on $E_{\varepsilon}$. We consider $\delta < \varepsilon/2$, so on $E_{\varepsilon}$ we have $u + \delta < \varepsilon$. Note that mapping $t \mapsto \Theta(t)$ is nonincreasing for $t \in (0, \varepsilon)$. For $\delta \searrow 0$ functions $\Theta_{\delta}(x) := \Theta(u(x) + \delta)$ converge to $\Theta(u(x))$ for almost every $x$. Therefore, due to Lebesgue’s Monotone Convergence Theorem we obtain
\[
\lim_{\delta \searrow 0} \int_{E_{\varepsilon}} \Theta(u + \delta)a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx = \int_{E_{\varepsilon}} \Theta(u)a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx.
\]

To deal with integrals over $F_{\varepsilon}$ we note that
\[
\Theta(u + \delta)\chi_{\{u + \delta \leq R\}}a(\cdot)|\nabla \phi|^{p}\phi^{1-p} \leq \chi_{\{\varepsilon/2 \leq u + \delta \leq R\}} \sup_{\text{supp} \phi} \Theta(u + \delta)a(\cdot)|\nabla \phi|^{p}\phi^{1-p} \leq \sup_{t \in [\varepsilon/2, R]} \Theta(t) \chi_{\text{supp} \phi} a(\cdot)|\nabla \phi|^{p}\phi^{1-p} \in L^{1}(\Omega).
\]
Application of Lebesgue’s Dominated Convergence Theorem yields
\[
\lim_{\delta \searrow 0} \int_{F_{\varepsilon}} \Theta(u + \delta)\chi_{\{u + \delta \leq R\}}a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx = \int_{F_{\varepsilon} \cap \{u < R\}} \Theta(u)a(x)|\nabla \phi|^{p}\phi^{1-p} \, dx.
\]
This completes the case of $\Theta$ decreasing in the neighbourhood of 0. The case of bounded $\Theta$ follows from Lebesgue’s Dominated Convergence Theorem (cf. as above for integral over $F_{\varepsilon}$ with $\varepsilon = 0$).

To complete the proof of Step 2 we note that for $\delta \leq R/2$ we have $\tilde{C}(\delta, R) \leq \tilde{C}(R)$.
Proof of Step 3

We note that, when $A_\delta(x)$ is given by (3.6), we have $A_\delta(x) \to A_0(x)$ a.e. in $\Omega_0$ as $\delta \searrow 0$, but we do not have information about the sign of $A_\delta$. Therefore we cannot apply for example Lebesgue’s Dominated Convergence Theorem directly to justify the convergence of the integrals. Thus we distinguish between two cases: when $\sigma \geq 0$ and when $\sigma < 0$. In both cases we prove the statement under each of the restrictions below on $\Psi$ and $\Psi/g$. They cover all the cases in Condition $(\Psi, g)$.

3a) $\Psi$ is nonincreasing and $\Psi/g$ is nonincreasing;
3b) $\Psi$ is increasing and $\Psi/g$ is nonincreasing;
3c) $\Psi$ is nonincreasing and $\Psi/g$ is bounded in some neighbourhood of 0;
3d) $\Psi$ is increasing and $\Psi/g$ is bounded in some neighbourhood of 0.

Case $\sigma \geq 0$. In this case $\Psi$ is decreasing because $0 \leq \sigma < C$ by Assumption A, a). Therefore, we consider restrictions 3a) and 3c) only.

Let us start with restriction 3a). Then $\Psi(u + \delta) \leq \Psi(u)$, $\sigma \frac{\Psi(u + \delta)}{g(u + \delta)} \leq \sigma \frac{\Psi(u)}{g(u)}$. Set

$$B_\delta(x) := \left( b^+(x)\Phi(u) + \sigma \frac{a(x)}{g(u + \delta)} |\nabla u|^p \right) \Psi(u + \delta).$$

(3.10)

Then $B_\delta \geq 0$ and we have

$$A_\delta(x) = \left( b^+(x)\Phi(u) + \sigma \frac{a(x)}{g(u + \delta)} |\nabla u|^p \right) \Psi(u + \delta) + b^-(x)\Phi(u)\Psi(u + \delta)\phi = B_\delta(x) + b^-(x)\Phi(u)\Psi(u + \delta) \geq B_\delta(x) + b^-(x)\Phi(u)\Psi(u).$$

Lebesgue’s Monotone Convergence Theorem yields

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R - \delta\}} B_\delta(x)\phi(x)dx = \int_{\{0 < u < R\}} \left( b^+(x)\Phi(u) + \sigma \frac{a(x)}{g(u)} |\nabla u|^p \right) \Psi(u)\phi(x)dx.$$

For restriction 3c) we verify the convergence of integrals involving $B_\delta$, given by (3.10), by noticing that

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R - \delta\}} b^+(x)\Phi(u)\Psi(u + \delta)\phi(x)dx \to \int_{\{0 < u < R\}} b^+(x)\Phi(u)\Psi(u)\phi(x)dx$$

by Lebesgue’s Monotone Convergence Theorem, while the convergence

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R - \delta\}} a(x)|\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} \phi(x)dx \to \int_{\{0 < u < R\}} a(x)|\nabla u|^p \frac{\Psi(u)}{g(u)} \phi(x)dx$$
follows from Lebesgue’s Dominated Convergence Theorem, as $\Psi/g$ is bounded near 0.

**Case $\sigma < 0$.** Let us consider first restriction 3a). Then we have

$$\frac{\sigma \Psi(u + \delta)}{g(u + \delta)} \geq \frac{\sigma \Psi(u(x))}{g(u(x))}, \quad b^-(x)\Psi(u(x) + \delta) \geq b^-(x)\Psi(u(x))$$

when $\delta > 0$ and $u(x) > 0$, and

$$A_\delta(x) \geq \Phi(u)b^+(x)\Psi(u + \delta) + \sigma \frac{a(x)}{g(u)}|\nabla u|^p\Psi(u) + \Phi(u)b^-(x)\Psi(u)$$

$$= \Phi(u)\Phi(u)\Psi(u) + \sigma \frac{a(x)}{g(u)}|\nabla u|^p\Psi(u) + \Phi(u)b^+(x)(\Psi(u + \delta) - \Psi(u))$$

$$= A_0(u) - \Phi(u)b^+(x)(\Psi(u) - \Psi(u + \delta)).$$

Let us consider the integral over $\Omega$ from the last expression and let $\delta$ converge to 0. Note that $\Phi(u)b^+(x)(\Psi(u) - \Psi(u + \delta))$ is nonnegative and decreasing to 0 a.e. in $\Omega$ as $\delta \searrow 0$. Moreover, according to Assumption A, b), we have

$$\Phi(u)b^+(x)(\Psi(u) - \Psi(u + \delta))\chi_{0 < u \leq R} \phi(x) \leq b^+(x)\Phi(u)\Psi(u)\chi_{0 < u \leq R} \phi(x) \in L^1(\Omega).$$

Lebesgue’s Dominated Convergence Theorem gives

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} \Phi(u)b^+(x)(\Psi(u) - \Psi(u + \delta)) \phi(x) dx = 0.$$

If restriction 3b) applies we have $\sigma \frac{\Psi(u + \delta)}{g(u + \delta)} \geq \sigma \frac{\Psi(u)}{g(u)}$, $b^+(x)\Psi(u + \delta) \geq b^+(x)\Psi(u)$ when $u > 0$, and then

$$A_\delta(x) \geq \Phi(u)b^+(x)\Psi(u) + \sigma \frac{a(x)}{g(u)}|\nabla u|^p\Psi(u) + b^-(x)\Phi(u)\Psi(u + \delta).$$

Now the fact that

$$\lim_{\delta \searrow 0} \int_{\{0 < u \leq R-\delta\}} (-b^-(x))\Phi(u)\Psi(u + \delta)\phi(x) dx \to \int_{\{0 < u < R\}} (-b^-(x))\Phi(u)\Psi(u)\phi(x) dx$$

follows from Lebesgue’s Dominated Convergence Theorem because inequality $\Psi(u + \delta) \leq \Psi(R)$ holds on this domain of integration and by Assumption A, (u).
In case of restriction 3c) we have $b^-(x)\Psi(u + \delta) \geq b^-(x)\Psi(u)$, therefore

$$A_\delta(x) \geq \Phi(u)b^+(x)\Psi(u + \delta) + \sigma a(x)|\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} + b^-(x)\Phi(u)\Psi(u).$$

The convergence of integrals involving $\Phi(u)b^+(x)\Psi(u + \delta)$ follows from Lebesgue’s Monotone Convergence Theorem, and the convergence of integrals involving $a(x)|\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)}$ follows from Lebesgue’s Dominated Convergence Theorem, because we can estimate $(\Psi/g)(u + \delta) \leq \sup\{(\Psi/g)(\lambda) : \lambda \in (0, R)\}$ on domains of integration.

For restriction 3d) we use the following estimate for $u > 0$:

$$A_\delta(x) \geq b^+(x)\Phi(u)\Psi(u) + \sigma a(x)|\nabla u|^p \frac{\Psi(u + \delta)}{g(u + \delta)} + b^-(x)\Phi(u)\Psi(u + \delta).$$

We justify the convergence of integrals from the expression on the right-hand side by Lebesgue’s Dominated Convergence Theorem using the fact that $\Psi(u + \delta) \leq \Psi(R)$, $(\Psi/g)(u + \delta) \leq \sup\{(\Psi/g)(\lambda) : \lambda \in (0, R)\}$ on the domain of integration, and taking into account Assumption A, b).

**Proof of Step 4**

For almost every $x \in \Omega_0$ we have

$$\left(\Phi(u(x))b(x) + \sigma \frac{a(x)}{g(u(x) + \delta)}|\nabla u(x)|^p\right)\Psi(u(x) + \delta)\phi(x) = \Phi(0)\Psi(\delta)b(x)\phi(x)$$

and $(b(x)\Phi(u)\chi_{\Omega_0})\cdot\phi(x)$ is integrable over $\Omega$ by Assumption A, (u). Since Assumption A, c) holds we have either: $\Phi(0)\Psi(\delta)b(x)\chi_{\Omega_0}\phi(x) \geq 0$ when $x$ or $y$ holds, or $\lim_{\delta \to 0} \Psi(\delta) = 0$ in case z). In all cases [3.7] holds.

This completes the proof of Lemma 3.2. □

**Proof of Theorem 3.1 (Caccioppoli estimates)**

Assume at first that $\Psi$ is nonincreasing. It suffices to let $R \to \infty$ in Lemma 3.1. Without loss of generality we may assume that the integral on the right-hand side of (3.3) is finite, as otherwise the inequality follows trivially. Since $a|\nabla u|^{p-1}|\nabla \phi|$ and $\Phi(u)b\phi$ are integrable, we have $\lim_{R \to \infty} \tilde{C}(R) = 0$. Therefore, (3.3) follows from (3.2) by Lebesgue’s Monotone Convergence Theorem.

When $\Psi$ is increasing we apply Assumption A, d) and proceed similarly. □
Remark 3.3. We can weaken the assumption of Lemma 3.1 and thus in Theorem 3.1 if we have more information about $u$. We suppose that $u \geq 0$ a.e. and $\Phi$ is continuous up to zero. In particular, we admit $u$ to be equal to zero on a set of positive measure. If $u > 0$ a.e. the assumption on $\Phi$ can be weakened. It suffices to consider continuous $\Phi : (0, \infty) \to (0, \infty)$ in Condition $(u)$ and omit Assumption A, c). See Step 4 in the proof of Lemma 3.1.

4 Hardy–type inequality

As a direct consequence of Caccioppoli–type estimates for solutions to PDI, we obtain Hardy–type inequality for rather general class of test functions, i.e. Lipschitz and compactly supported functions. The following theorem implies several Hardy–type inequalities with the optimal constants, see Remark 4.1 below.

Theorem 4.1 (Hardy–type inequality). Suppose $a \in L^1_{\text{loc}}(\Omega) \cap B_p(\Omega)$, $b \in L^1_{\text{loc}}(\Omega)$. Assume that $1 < p < \infty$ and $u \in L^{1,p}_{a,\text{loc}}(\Omega)$ is a nonnegative solution to the PDI $-\Delta_{p,a}u \geq \Phi(u)b(x)$ in the sense of Definition 2.2. Moreover, let Assumption A hold.

Then for every Lipschitz function $\xi \in L^{1,p}_{a,\text{loc}}(\Omega)$ with compact support in $\Omega$ we have

$$\int_{\Omega} |\xi|^p \mu_1(dx) \leq \int_{\Omega} |\nabla \xi|^p \mu_2(dx),$$

(4.1)

where

$$\mu_1(dx) = \left( \Phi(u)b(x) + \sigma |\nabla u|^p g(u) \chi_{\{u \neq 0\}} \right) \Psi(u) \chi_{u > 0} dx,$$

$$\mu_2(dx) = \left( \frac{p-1}{C-\sigma} \right)^{p-1} a(x) \Psi(u) g^{p-1}(u) \chi_{\{u > 0, \nabla u \neq 0\}} dx.$$

Proof. We apply of Theorem 3.1 with $\phi = \xi^p$, where $\xi$ is nonnegative Lipschitz function with compact support. Then $\phi$ is Lipschitz and

$$|\nabla \xi|^p = \left( \frac{1}{p} \phi^{\frac{1}{p}-1} |\nabla \phi| \right)^p = \frac{1}{p^p} \left( \frac{|\nabla \phi|}{\phi} \right)^p \phi.$$ 

Therefore (3.3) becomes (4.1). Note that for every Lipschitz function $\xi$ with compact support in $\Omega$ we have $\int_{\Omega} |\nabla \xi|^p a(x) dx < \infty$, equivalently $\int_{\text{supp} \phi} |\nabla \phi|^p \phi^{1-p} a(x) dx < \infty$. As the absolute value of a Lipschitz function is a Lipschitz function as well, we write $|\xi|$ instead of $\xi$ on the left–hand side and do not require its nonnegativeness.
| Inequality          | Optimality | Comment                                                                 |
|---------------------|------------|-------------------------------------------------------------------------|
| classical Hardy     | 30         | proven in [40]                                                           |
| Hardy–Poincaré      | 41         | via [40]; improved constants from [10, 23];                            |
|                     | 29         | Theorem 4.3 and Remark 4.1 here                                         |
| Poincaré            | Remark 7.6 in [19] | concluded from [40]                                                    |
| exponential–weighted| expected   | [40, Theorem 5.5] vs. [26]                                              |
| Hardy               | expected   | [40, Theorem 5.8] vs. [27, Proposition 5.2]                            |

**Remark 4.1.** Let us point out that some of the inequalities derived previously in [40], which motivated us to write this work, are sharp as they hold with the best constants. Namely, they are achieved in the classical Hardy inequality (Section 5.1 in [40]); the Hardy–Poincaré inequality obtained in [41] due to [40], confirming some constants from [23] and [10] and establishing the optimal constants in further cases; the Poincaré inequality concluded from [40], confirmed to hold with best constant in Remark 7.6 in [19]. Moreover, the inequality in Theorem 5.5 in [40] can also be retrieved by the methods from [26] with the same constant, while some inequalities from Proposition 5.2 in [27] are comparable with Theorem 5.8 in [40]. In Theorem 4.3, we provide some extensions of Hardy–Poincaré inequalities from [41], which are proven in [29] by applying the results obtained in this paper. Some of them hold with the optimal constants.

**Remark 4.2.** It is known [28] that Hardy inequalities can imply Gagliardo–Nirenberg interpolation inequalities for intermediate derivatives:

\[
\|\nabla u\|_{L^q(\Omega, \mu)}^2 \leq C \|u\|_{L^r(\Omega, \mu)} \|\nabla^{(2)} u\|_{L^p(\Omega, \mu)}, \text{ where } \frac{2}{q} = \frac{1}{r} + \frac{1}{p},
\]

if one has Hardy inequality: \(\|u\|_{L^p(\Omega, \varrho \cdot \mu)} \leq C \|\nabla u\|_{L^p(\Omega, \mu)}\) under certain assumptions on the measure \(\mu\) and the weight function \(\varrho\).

**Remark 4.3.** When we know that \(u\) is strictly positive almost everywhere, due to Remark 3.3, the statement of Theorem 4.1 holds under the assumption that \(\Phi : (0, \infty) \to (0, \infty)\) is continuous in Condition (u) and we can omit Assumption A, c).

**Remark 4.4.** In the nondegenerated case, i.e. when \(a(\cdot) = b(\cdot) \equiv 1\), Theorem 4.1 as well as Theorem 3.1 retrieves the results of [40]. In contrast with [40] our function \(\Psi\) need not be increasing here. Hence, broader class of measures \(\mu_1\) and \(\mu_2\) may appear in [41]. Therefore our result generalizes that of [40] even in nondegenerated case.

**Hardy inequalities resulted from existence theorems**

We are going to derive sharp Hardy type inequality, not knowing \(u\) explicitly but only its existence. We assume now that \(b\) is nonnegative and that there exists a nonnegative
nontrivial locally bounded solution of PDE $-\Delta_{p,a} u \geq b(x) u^{p-1}$ i.e.,

$$\langle -\Delta_{p,a} u, w \rangle \geq \int_{\Omega} b(x) u^{p-1} w \, dx,$$

holds for every nonnegative compactly supported function $w \in L^1_a(\Omega)$.

This is the special case of inequality (2.5) for $\Phi(u) = u^{p-1}$. Our result reads as follows.

**Theorem 4.2 (Sharp Hardy–Poincaré inequality).** Assume that $1 < p < \infty$, $a, b \in W(\Omega)$, $a \in L^{1,1}_\text{loc}(\Omega) \cap L^1_b(\Omega)$, and $u \in L^{1,p}_a(\Omega), b u^{p-1} \in L^{1,1}_\text{loc}(\Omega)$, then for every Lipschitz function $\xi \in L^{1,p}_a(\Omega)$ with compact support in $\Omega$ we have

$$\int_{\Omega} |\xi|^p b(x) \, dx \leq \int_{\Omega} |\nabla \xi|^p a(x) \, dx. \quad (4.3)$$

Moreover, if there exists nontrivial, nonnegative, $u_0 \in W^{1,p}_{(b,a),0}(\Omega)$ which is the solution to $-\Delta_{p,a} u_0 = b(x) u_0^{p-1} \in L^{1,1}_\text{loc}(\Omega)$ then inequality (4.3) is sharp, i.e. $C = 1$ is the optimal constant in the inequality $C \int_{\Omega} |\xi|^p b(x) \, dx \leq \int_{\Omega} |\nabla \xi|^p a(x) \, dx$.

**Proof.** We apply Theorem 4.1 with $\Psi(t) = \frac{1}{t^{p-1}}$, $g(t) = t$, $\Phi(t) = t^{p-1}$, $\sigma = 0$ and verify that under our conditions Assumption A is satisfied. This gives (4.3). Suppose now that there exists $u_0$ satisfying all the requirements of the theorem. Let us consider the sequence $(w_k)_{k \in \mathbb{N}}$ of smooth compactly supported functions, such that $w_k \to u_0$ in $W^{1,p}_{(b,a)}(\Omega)$. Since each $w_k$ has a compact support and belongs to $L^{1,p}_a(\Omega)$, we have the equality

$$\langle -\Delta_{p,a} u_0, w_k \rangle = \int_{\Omega} |\nabla u_0|^{p-2} \nabla u_0 \cdot \nabla w_k a(x) \, dx = \int_{\Omega} b(x) u_0^{p-1} w_k \, dx.$$ 

When we let $k \to \infty$, we get $\int_{\Omega} |\nabla u_0|^p a(x) \, dx = \int_{\Omega} b(x) u_0^p \, dx$ which proves sharpness. \qed

**Remark 4.5.** Theorem 4.2 is known in the case $a \equiv 1, b \equiv 1$, see [1] or Remark 1 on page 163 in [33].

**Remark 4.6.** We substitute the special value of $\sigma = 0$, in the proof of the above statement. Therefore, we do not expect that the inequality (1.3) holds with the best constant in general.

**Sharp Hardy–Poincaré inequalities with best constants**

Using the Talenti extremal profile given by (1.4) where $\beta = 0$ in our approach, one obtains the following theorem, cf. [29] for details.
Theorem 4.3. Assume that $1 < p < \infty$, $\gamma > 1 - \frac{n}{p}$, $0 < r < 1 - \frac{p}{n} + \frac{p}{r}$ and $v_1(x) := \left(1 + r|x|^{\frac{p}{p+1}}\right)\left(1 + |x|^{\frac{p}{p+1}}\right)^{(p-1)\gamma}$, $v_2(x) := \left(1 + |x|^{\frac{p}{p+1}}\right)^{(p-1)\gamma}$. Then for every $\xi \in W_{v_1,v_2}^{1,p}(\mathbb{R}^n)$ we have
\[
\mathcal{C}_{\gamma,n,p,r} \int_{\mathbb{R}^n} |\xi|^p \left(1 + r|x|^{\frac{p}{p+1}}\right)\left(1 + |x|^{\frac{p}{p+1}}\right)^{(p-1)\gamma} \, dx \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p+1}}\right)^{(p-1)\gamma} \, dx,
\]
where $\mathcal{C}_{\gamma,n,p,r} = n \left(\frac{p}{p-1}\right)^{(p-1)\gamma} \left(1 + \frac{n}{p}(1-r)\right)^{(p-1)\gamma}$. Moreover, constant $\mathcal{C}_{\gamma,n,p,r}$ is optimal when $\gamma > nr + 1 - \frac{n}{p}$ and when $\gamma = 1 + n(1 - \frac{1}{p})$, $r = 1$.

Remark 4.7. Such inequalities in the case $p = 2$ are very much of interest in the theory of nonlinear diffusions, where one investigates the asymptotic behavior of solutions of the equation $u_t = \Delta u^m$, see [10]. To our best knowledge our inequalities are new if $r \neq 1$ in general. However, as an example dealing with $r \neq 1$ and $p = 2$ we refer to the fourth line on page 434 in [10], which is our case with $r = \gamma/n$. Proof of that inequality in [10] requires knowledge about the best constants in Sobolev inequality, which we do not need. We can also prove Proposition 3 from [10] by our methods and generalize it for an arbitrary $p$.

Remark 4.8. There is a particular interest in the Hardy–Poincaré inequalities with decreasing weights (involving negative power $\gamma < 0$), which are not covered in [11]. In our Theorem 4.3 we do allow some of such inequalities with optimal constants.

The above statement can be compared with the following one obtained in [11], which follows as the special case of Theorem 4.2 when one substitutes $r = 1$. Consequently one has to assume that $\gamma > 1$.

Theorem 4.4 (cf. [11]). Suppose $p > 1$ and $\gamma > 1$. Then, for every function $\xi \in W_{v_1,v_2}^{1,p}(\mathbb{R}^n)$, where $v_1(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)(\gamma-1)}$, $v_2(x) = \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma}$, we have
\[
\mathcal{C}_{\gamma,n,p} \int_{\mathbb{R}^n} |\xi|^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma} \, dx \leq \int_{\mathbb{R}^n} |\nabla \xi|^p \left(1 + |x|^{\frac{p}{p-1}}\right)^{(p-1)\gamma} \, dx,
\]
with $\mathcal{C}_{\gamma,n,p} = n \left(\frac{p(\gamma-1)}{p-1}\right)^{(p-1)\gamma}$. Moreover, for $\gamma > n + 1 - \frac{n}{p}$, the constant $\mathcal{C}_{\gamma,n,p}$ is optimal.

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