Exceptional points in Fermi liquids with quadrupolar interactions

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We show the existence of non-Hermitian degeneracies, known as exceptional points, in the collective mode spectrum of Fermi liquids with quadrupolar interactions. Through a careful analysis of the analytic properties of the dynamic quadrupolar susceptibility, we show that, in the weak attractive region, two stable collective modes coalesce to an exceptional point. We completely characterize this singularity, explicitly showing its topological properties. Experimental signatures are also discussed.

Introduction– Open quantum systems play a central role in most applications of quantum mechanics\textsuperscript{1}. An important theoretical tool to describe dissipative quantum systems is the modelling of locally non-conservative systems by effective non-Hermitian Hamiltonians\textsuperscript{2,3}. These types of Hamiltonians has several counterintuitive properties. Perhaps, one of the most intriguing ones is the appearance of non-Hermitian degeneracies\textsuperscript{4} known as exceptional points (EP)\textsuperscript{5,6}.

When a non-Hermitian Hamiltonian continuously depends on external parameters, it could happen that, for certain values of the parameters, two or more eigenvalues coalesce to an EP. However, this is not a usual degeneracy, as observed in Hermitian systems. In an EP, not only the eigenvalues coincide but also the eigenvectors become linearly dependent\textsuperscript{7}, reducing in this way the dimension of the subspace associated to the degenerated eigenvalue. This singularity of the Hilbert space has remarkable topological consequences\textsuperscript{8,9}. The relation between EPs and dynamical phase transitions was early recognized in theoretical as well as experimental works\textsuperscript{10–13}. In recent years, exciting findings of EPs are shown up in the condensed matter community because it is the simplest and a variety of Quantum Hall Effects\textsuperscript{24}. The relation between EPs and dynamical phase transitions was early recognized in theoretical as well as experimental works\textsuperscript{10–13}. In recent years, exciting findings of EPs are shown up in the condensed matter community because it is the simplest and a variety of Quantum Hall Effects\textsuperscript{24}.

In recent years, exciting findings of EPs are shown up in the spectrum of collective excitations of Fermi liquids\textsuperscript{22} with higher order Landau parameter interactions. Fermi liquids with quadrupolar interactions began to call the attention of the condensed matter community because it is the simplest and a variety of Quantum Hall Effects\textsuperscript{24}.

Collective excitations of Fermi liquids with quadrupolar interactions have been studied in different regimes\textsuperscript{22,25,29}. Here, we explicitly show the appearance of a non-Hermitian singularity for weak quadrupolar attraction. We completely characterize this exceptional point, by analyzing the Hilbert space structure and its topological properties. Finally, we discuss some possible experimental set ups.

Model– We consider the simplest model of bi-dimensional spinless Fermions with local quadrupolar interactions. The Hamiltonian is

\[ H = \int d^2 r \left\{ \psi^\dagger(r) \epsilon(\nabla) \psi(r) + \frac{F_2}{4} \text{Tr} \left[ Q^2(r) \right] \right\} \]

(1)

where \( \psi(r) \) is a spinless Fermionic field operator. The bare density and quadrupolar susceptibilities, respectively. Eq. (1) has the usual structure of an effective interaction in the Fermi liquid regime. Since the quadrupolar moment has two degrees of freedom, the susceptibility has essentially two independent polarizations, the longitudinal \( \chi_2^x(\omega, q) \) and the transversal polarization \( \chi_2^y(\omega, q) \). These quantities have been computed using different approximation approaches\textsuperscript{23,25–27,32}. In the limit of small momentum \( q << k_F \), where \( k_F \) is the Fermi momentum, the result is\textsuperscript{25}

\[ \chi_2^y(\omega, q) = \frac{\chi_2^0(\omega) \pm \chi_2^1(\omega)}{1 - F_2 (\chi_2^0(\omega) \pm \chi_2^1(\omega))} \]

(2)

where

\[ \chi_2^0 = \frac{1}{2} \left[ -\delta_{\ell,0} + K_0(s) \left( \frac{1 - K_0(s)}{1 + K_0(s)} \right)^\ell \right] \]

(3)

with \( K_0(s) = s/\sqrt{s^2 - 1} \). Eq. (3) with \( \ell = 0, 2 \) are the bare density and quadrupolar susceptibilities, respectively. Eq. (2) has the usual structure of an effective interaction in the traditional RPA approximation. Due to the locality of the quadrupolar interaction, the DQS is not a function of \( \omega \) and \( q \) independently. Instead, it depends on the dimensionless variable \( s = \omega/\nu_F \), where \( \omega \) is the frequency and \( \nu_F \) is the maximum energy of a particle-hole excitation with momentum \( q = |q| \) and Fermi velocity \( \nu_F = |v_F| \). It is worth mentioning that in the computation of Eq. (2), rotational invariance and particle-hole symmetry were imposed.
Collective modes— The DQS is an analytic function of $s$, having poles and cuts. It has branch points at $s = \pm 1$; the threshold of Landau damping $\omega = \pm \nu F q$. We will focus on the longitudinal polarization $\chi^+_{1s}(s)$ since, as we will show, this component displays an EP. Collective modes are computed by solving the algebraic equation $F_2 (\chi^0_{1s}(s) + \chi^+_{1s}(s)) = 1$. We have numerically solved it for $F_2$ running from the strong attractive ($F_2 = -1$) to the strong repulsive regime ($F_2 > 1$). We display the result in Fig. 1. In the upper panel, we show the real part of the collective modes as function of $F_2$, while in lower panel, we show the imaginary part. In the repulsive region ($F_2 > 0$), we observe a stable (real) mode that tends to $s = 1$ when $F_2 \to 0$. This is the quadrupolar equivalent of the Landau zero sound. In addition, a damped mode also appears in the same region. The stable mode is continuously extended to the weak attractive region $F_2 \lesssim 1$. However, in this regime, there is another stable mode with divergent behavior, $s \to +\infty$ when $F_2 \to 0^-$. The existence of such a mode was reported in Ref. 27. Interestingly, there is a special point, $F_2^0$, where both stable modes meet together. For $F_2 < F_2^0$, these modes become damped as can be easily seen in the lower panel of Fig. 1. We can also observe an overdamped mode (purely imaginary) in all the attractive region. This mode is the precursor of the isotropic-nematic phase transition that occurs at $F_2 = -1$ and has already been extensively studied $^{23, 25}$.

Exceptional point— In order to analytically characterize the singularity at $F_2 = F_2^0$, we first observe that $s(F^0) \gtrsim 1$, being well separated from the cut $s^2 \lesssim 1$. On the other hand, the singularity is sufficiently close to $s = 1$, allowing us to try a series expansion of $\chi^+_{1s}(s)$ in the neighborhood of $s = 1$. For simplicity, let us work with the inverse of the DQS, $\mathcal{L}^+(s) = (\chi^+_{1s}(s))^{-1}$. We find the following expansion,

$$
\mathcal{L}^+(s) = -F_2 + \sqrt{\frac{s - 1}{2}} + 5 \left(\frac{s - 1}{2}\right) + O\left((s - 1)^{3/2}\right).
$$

(4)

Longitudinal quadrupolar fluctuations $\delta Q^+(s, q)$ are governed by the effective action

$$
S_{\text{eff}} = \int \frac{d\omega d^2q}{(2\pi)^3} \mathcal{L}^+(s)|\delta Q^+(s, q)|^2.
$$

(5)

The collective modes are given by the roots of $\mathcal{L}^+(s) = 0$. Using Eq. (4), we obtain

$$
s_{\pm} = \frac{1}{25}\left((26 + 10F_2) \pm \sqrt{20F_2 + 1}\right).
$$

(6)

$s_{\pm}(F_2)$ have a square root singularity (branch point) at $F_2^0 = -1/20$. At this point, both zeros are degenerated, $s_{\pm}(F_2^0) = 51/50$. We depict the real and imaginary part of $s_{\pm}(F_2)$ in Fig. 2. Thus, the approximation made in Eq. 4 for $|s - 1| \ll 1$, correctly captures the presence of the degeneracy

![FIG. 1. Collective modes from the longitudinal polarized component of the DQS $\chi^+_{1s}(s)$. In the upper panel we plot $\Re[s(F_2)]$ while in the lower panel we depict $\Im[s(F_2)]$.](image1)

![FIG. 2. Solutions of $\mathcal{L}^+(s_{\pm}) = 0$, given by Eq. (6) as a function of the parameter $F_2$. The upper panel shows the real part of $s_{\pm}$, while lower one depicts the imaginary part. The point $F_2 = -1/20$, where both eigenvalues coalesce and the imaginary part emerges, is the exceptional point.](image2)
The square-root singularity is a typical signature of an exceptional point [33].

The dynamics described by Eq. (4) is non-local in time. However, since the degeneracy is separated from the cut, we can further expand $\mathcal{L}^+(s)$ in the neighborhood of $s = s_\pm$. In addition, we observe that the local character of the interaction imposes that $\mathcal{L}^+(s)$ only depends on the dimensionless variable $s$. Thus, we can consider quadrupolar fluctuations, $\delta Q^+(s)$, ignoring any momentum dependence not scaling with $s$. The consequence is that all collective modes in this approximation have linear dispersion relation $\omega \sim v_F q$.

This is a good approximation for weak interactions. However, it breaks down in the strongly attractive regime ($F_2 \sim -1$), where non-local interactions $F_2(q)$ are essential [35]. With these considerations, we arrive to the effective action

$$ S_{\text{eff}} = \int ds \{ (s - \epsilon_1)(s - \epsilon_2) + w^2 \} |\delta Q^+(s)|^2 \quad (7) $$

where $\epsilon_1 = (1/25)(27 + 10F_2)$, $\epsilon_2 = (1/25)(25 + 10F_2)$ and $w = (1/25)\sqrt{20|F_2|}$ are real positive numbers in the vicinity of the EP. The zeros of the Lagrangian are given of course by Eq. (6).

In order to rewrite the effective action in the Hamiltonian formalism (first order in time), we introduce a two-component vector field $\delta Q^+ = (\delta Q_1, \delta Q_2)$. In terms of this field, the effective action reads

$$ S_{\text{eff}} = \int ds \{ (s - \epsilon_1)(s - \epsilon_2) + w^2 \} (sI - H_{\text{eff}}) \delta Q^+ \quad (8) $$

where $I$ is the $2 \times 2$ identity matrix and the effective Hamiltonian is

$$ H_{\text{eff}} = \begin{pmatrix} \epsilon_1 & iw \\ iw & \epsilon_2 \end{pmatrix} \quad (9) $$

It is straightforward to verify that, integrating out the vector component $\delta Q_2$, we obtain the effective action of Eq. (7) for the field $\delta Q_1$. Therefore, the dynamics near the singularity is driven by a $2 \times 2$ symmetric effective Hamiltonian (non-Hermitian), which determines the properties of the EP [34].

**Hilbert space and topology**—The Hilbert space spanned by the basis $\psi_{\pm}$ and its dual, spanned by $\phi_{\pm}$, are in general different in non-Hermitian Hamiltonian systems. They are defined by

$$ H_{\text{eff}} \psi_{\pm} = s_{\pm} \psi_{\pm} \quad (10) $$

$$ H_{\text{eff}}^\dagger \phi_{\pm} = s_{\pm}^* \phi_{\pm} \quad (11) $$

Bi-orthogonality requires $\langle \phi_i | \psi_j \rangle = \delta_{ij}$ with $i, j = \pm$. Since the effective Hamiltonian is symmetric, the dual space is spanned by $\phi_{\pm} = \psi_{\pm}^*$. Solving Eq. (10), we find

$$ \psi_{\pm} = c_{\pm} \begin{pmatrix} i \omega_- - 1 \\ i \omega_+ - 1 \end{pmatrix} \quad (12) $$

where $c_{\pm}$ are complex normalization constants. We have introduced the variable $z = 1 + 20F_2$, in order to have the EP at $z = 0$. As anticipated, not only $s_+ = s_-$ at the EP, but the eigenvectors collapse to $\psi_{\text{EP}} = c_{\pm}(1, -i)$. This fact produces that $\langle \phi_{\text{EP}} | \psi_{\text{EP}} \rangle = 0$, which is evidently in conflict with bi-orthogonality. In this way, the EP is a singularity in the structure of the Hilbert space [7]. This singularity induces remarkable topological properties. To show this, let us compute the geometric phase that the wave function picks up when the EP is wound in parameter space. For this, we analytically continue $z$ to the complex plane and define the Berry phase as $\gamma = i \frac{\pi}{2} \int d\mathbf{A} \cdot \mathbf{A}$, where the one-form $\mathbf{A} = \langle \phi_+ | \nabla \psi_+ \rangle/\langle \phi_+ | \psi_+ \rangle$ [35]. $C$ is a closed path and $\nabla$ is the gradient in parameter space $z$. The equivalent definition with $\phi_-$ and $\psi_-$ eigenvectors provides the same result. Notice that $\mathbf{A}$ is ill-defined at the EP since, at this point, the denominator is zero. The particular structure of the Hilbert space and its dual allows us to rewrite the vector form as a total derivative (locally a pure gauge), $\mathbf{A} = (1/2)\nabla \ln(\langle \phi_+ | \psi_+ \rangle)$. Thus, the EP is a branch point of the logarithm. Each time the phase $\langle \phi_+ | \psi_+ \rangle$ winds the branch point, the logarithm picks up a $2\pi i$ term. This property does not depend on the specific path, provided the path encircles the EP. Thus, we can compute $\gamma$ considering a very small circumference around the EP. Using Eq. (12), we find for $|z| < 1$, $\langle \phi_+ | \psi_+ \rangle \sim \zeta^{1/2}$. Due to the square-root singularity, the phase of $\langle \phi_+ | \psi_+ \rangle$ is half the phase of $z$. Therefore, taken the path $C$ winding two times the EP, the Berry phase $\gamma = \pi$, in agreement with results obtained for general symmetric non-Hermitian Hamiltonians [36]. In this way, in encircling the EP, it is necessary to wind four times the singularity to return to the original state [37]. Recently, this unique topology of EPs was experimentally confirmed in metamaterials set ups [20, 21].

**Experimental signatures**—Information about collective excitations of strongly correlated systems can be obtained by measuring momentum-resolved dynamic susceptibility in the meV scale [38]. The detection of a stable mode near the meV scale could be an indication of the presence of an EP. Moreover, pump-probe spectroscopy [39–41] yield important information on the dynamic response in the time domain. An experimental signature can be obtained from $\chi_2^+(q, t)$, by Fourier transforming the DQS in the neighborhood of the EP. For $F_2 > F_2^*$, the retarded susceptibility is

$$ \Re \left[ \chi_2^+(q, t) \right] = 2v_Fq \left[ \sin(\omega_- t)/\omega_- \right] \cos(\omega_+ t) \Theta(t) \quad (13) $$

where $\omega_{\pm} = (s_+ \pm s_-) v_F q/2$ and $\Theta(t)$ is the Heaviside distribution. We clearly observe two well separated time scales since $\omega_+ / \omega_- \gg 1$. At the EP, $\omega_- = 0$ and $\sin(\omega_- t)/\omega_- \rightarrow t$. Thus, the signature of the EP is a growing linear modulating function of time, $\chi_2^+(q, t) \sim t \cos(\omega_+ t)$. An approximate linear modulation can be observed on a large range of intermediate times, even when the coupling is not fine tuned at $F_2 = F_2^*$. On the other hand, for $F_2 < F_2^*$, the dynamic response dramatically changes since the modulation is exponentially damped $\chi_2^+(q, t) \sim \exp\{-|\omega_-| t\} \cos(\omega_+ t)$. We depict these different regimes in Fig. 3. The abrupt change in the dynamical response at the EP should also be captured in
quantum quench set ups [42].

Summary and discussion– We have shown the existence of an EP in the collective mode spectrum of a Fermi liquid with weak attractive quadrupolar interactions. We completely characterize this singularity in terms of the Hilbert space structure as well as through its topological properties. We have also provided experimental signatures in the dynamical response. More complex models of Fermi liquids could lead to higher dimensional singularities, such as exceptional lines or surfaces [43, 44]. For instance, if we consider isotropic density interactions \( F_0 \) in addition to the quadrupolar ones [27], we still find square-root singularities which, in the limit of small \( F_0 \), take the form \( s_+ - s_- = \sqrt{1 + 20F_2 + 4F_0} \). In this way, the spectrum has an exceptional line parametrized by \( F_2 + F_0/5 = -1/20 \).

Concluding, non-Hermitian singularities appear in the spectrum of collective modes of Fermi liquids with higher angular momentum attractive interactions. Specific properties, such us the singularity location and dimensionality, are model dependent. However, its existence, its topological properties and experimental signatures are robust results. It could be important to investigate the influence of these singularities in the single quasi-particle spectrum and its effect on charge transport and other out-of-equilibrium properties.

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