Abstract
Subresultant of two univariate polynomials is a fundamental object in computational algebra and geometry with many applications (for instance, parametric GCD and parametric multiplicity of roots). In this paper, we generalize the theory of subresultants of two polynomials to arbitrary number of polynomials, resulting in multi-polynomial subresultants. Specifically,

1. we propose a definition of multi-polynomial subresultants, which is an expression in terms of roots;
2. we illustrate the usefulness of the proposed definition via the following two fundamental applications:
   - parametric GCD of multi-polynomials, and
   - parametric multiplicity of roots of a polynomial;
3. we provide several expressions for the multi-polynomials subresultants in terms of coefficients, for computation.

1 Introduction
Subresultant for two univariate polynomials is a fundamental object in computational algebra and geometry with numerous applications in science and engineering. Due to its importance, there have been extensive research on underlying theories, extensions and applications (just list a few [34, 30, 8, 28] and [35, 27, 32, 33]).

One often needs to consider more than two univariate polynomials, in various applications, for instance, in the following two fundamental applications: parametric GCD of several polynomials and parametric multiplicity of roots of a polynomial. A usual way is to apply subresultants to a pair of input or intermediate polynomials repeatedly, resulting in “nested” subresultants, i.e., subresultants of subresultants of ... and so on. This process, however, often produces extraneous or irrelevant factors, complicating theory and in turn applications.

Hence one wonders whether there is a way to generalize the theory of subresultant to more than two polynomials such that nestings are avoided, which would in turn simplify subsequent theory developments and applications. In this article, we provide such a generalized theory. Specifically,

1. We propose a definition of multi-polynomial subresultants, which is an expression in terms of roots.

Naturally there are two different approaches to generalizing subresultants of two polynomials to arbitrary number of polynomials:

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(A1) generalizing expressions of subresultants in *coefficients*, such as minor of a Sylvester matrix.

(A2) generalizing expressions of subresultants in *roots*, such as Poisson formula.

After trying both approaches, we finally chose (A2) because the resulting definition would encode desired geometric information more explicitly and such explicit geometric information would greatly facilitates the developments of applications.

See Definition 2 for the details.

2. *We illustrate the usefulness of the proposed definition via the following two fundamental applications.*

- Parametric GCD of multi-polynomials.

  Consider several parametric univariate polynomials (such as polynomials with parametric coefficients). The GCD of the polynomials will of course depend on the values of the parameters. Hence the parametric GCD problem asks for several conditions on parameters and corresponding expressions for GCD in terms of the parameters. We show that the conditions and the corresponding GCD can be simply/elegantly written in terms of the proposed multi-polynomial subresultants.

  See Theorem 8 for details.

- Parametric multiplicity of roots of a polynomial.

  Consider a parametric univariate polynomial (such as a polynomial with parametric coefficients). The multiplicity structure (the vector of multiplicities of the distinct roots) of the polynomial will depend on the values of the parameters. Hence the parametric multiplicity problem asks for a condition on parameters for each multiplicity structure. We show that the conditions can be also simply/elegantly written in terms of the proposed multi-polynomial subresultants.

  See Theorem 20 for details.

3. *We provide several expressions for multi-polynomials subresultants in terms of coefficients.*

   In the literature, there are roughly three types of expressions of subresultants in coefficients and their combinations: *Sylvester-type*, *Bézout-type* and *Barnnet-type*. We generalize all the three types to the multi-polynomial case.

   See Theorem 27 for details.

   **Comparison with related works:** There have been numerous works on subresultants. We will focus on efforts in generalizing the notion and theory of the subresultant of two univariate polynomials.

   - In [34, 22, 23, 24, 30, 15, 12, 13, 16, 14, 4], the subresultants are expressed as rational function in roots. We extended it to multi-polynomial case. Difference is that the index for the subresultants are not just a natural number, but a tuple of natural numbers.

   - In [38], the authors generalized the following well-known property of the subresultant of two univariate polynomials: the degree of the GCD of two polynomials is determined by the rank of the corresponding Sylvester matrix. They extended Sylvester matrix to several univariate polynomials so that a similar property holds.

   In this paper, we generalize another well-known property of the subresultant of two univariate polynomials: the degree of the GCD of two polynomials is determined by vanishing of subresultants and the GCD is given by the corresponding subresultant polynomial. We extend subresultant to several polynomials so that a similar property holds.

   - In [2], the author considered the problem of computing the GCD for several univariate polynomials and gave an algorithm to write down the expression of GCD by extending the Barnett matrix from bi-polynomial case to multi-polynomial case.

   In this paper, we further refine the result in the sense that we provide an explicit formula to write down GCD in terms of coefficients of the given polynomials.
In [41], the authors considered the Bézout-type subresultant for two univariate polynomials and established the connection between their pseudo-remainders and the subresultants; in [18], the authors presented more alternative expressions which express subresultants in terms of some minors of matrices different from the Sylvester matrix.

In this paper, we provide three different expressions for subresultant of several univariate polynomials, including Sylvester-type, Bézout-type and Barnett-type.

In [7, 11, 15, 19, 35, 36], the authors generalized the subresultant of two univariate polynomials to that of multivariate polynomials while constraining the number of polynomials to be at most one more than the number of variables. In [31, 23, 24], the authors generalized the subresultant of two polynomials to that of two Ore polynomials.

In this paper, we take another route for generalization. We allow arbitrary number of polynomials while staying univariate. Of course, a natural challenge for future work is to generalize further to arbitrary number of multivariate and/or Ore polynomials.

The paper is structured as follows. In Section 2 we give a “geometric” concept (definition) of subresultant for several univariate polynomials (multi-polynomial subresultant) in terms of roots of one polynomials among them. In Section 3 we show that the parametric GCD of several univariate polynomials can be expressed in terms of the multi-polynomial subresultant. In Section 4 we show that the parametric multiplicity of one univariate polynomial can be expressed in terms of subresultants of the given polynomial and its derivatives. For the sake of computation, we study the expression of subresultant for multi-polynomials in coefficients. In Section 5 we identify three different types of determinantal expressions in coefficients. In Section 6 we summarize the main results and mention a natural extension.

2 Definition of multi-polynomial subresultants in terms of roots

Notation 1.

1. \( F = (F_0, F_1, \ldots, F_t) \subset \mathbb{C}[x] \) where \( t \geq 1 \).
2. \( d_i = \deg F_i \)
3. \( \alpha_1, \ldots, \alpha_{d_0} \) are the complex roots of \( F_0 \).
4. \( \delta = (\delta_1, \ldots, \delta_t) \in \mathbb{N}_{\geq 0}^t \) such that \( |\delta| \leq d_0 \), where \( |\delta| = \delta_1 + \cdots + \delta_t \).

With the above notations, we define the concept of subresultant polynomial in terms of roots.

Definition 2 (Subresultant polynomial in terms of roots). We define the \( \delta \)-th subresultant polynomial \( S_\delta \)
Remark 3. It is very important to note that $s_0(F)$ is a polynomial function in $\alpha_1, \ldots, \alpha_{d_0}$, even though written as a rational function, since the numerator is exactly divisible by the denominator. Hence the above definition should be read as follows:

1. Treating $\alpha_1, \ldots, \alpha_{d_0}$ as distinct indeterminates, carry out the exact division obtaining a polynomial.

2. Treating $\alpha_1, \ldots, \alpha_{d_0}$ as numbers, evaluate the resulting polynomial.

Remark 4. Let $\text{sres}_i(F_0, F_1)$ stand for the “classical” $i$-th subresultant polynomial of $F_0$ and $F_1$. Under the new notion, we have

$$\text{sres}_0(F_0, F_1) = S_{(d_0)}(F_0, F_1)$$
$$\text{sres}_1(F_0, F_1) = S_{(d_0-1)}(F_0, F_1)$$
3.1 Main results of several univariate polynomials.

Let $s_{(d_0-i)}(F_0, F_1)$ and $s_{d_0}(F_0, F_1) = S(0) (F_0, F_1)$.

The index scheme is changed. The reason is the "new" one is better for extension.

**Example 5.** Let $F = (F_0, F_1, F_2)$. Let

\[
F_0 = a_{03} (x - \alpha_1) (x - \alpha_2) (x - \alpha_3)
\]

\[
F_1 = a_{13} x^3 + a_{12} x^2 + a_{11} x + a_{10}
\]

\[
F_2 = a_{21} x + a_{20}
\]

where $a_{03} \neq 0$. We would like to determine $S_{(1,1)}(F)$. Note that

\[
\varepsilon = 1 + 3 - (1 + 1) = 2
\]

\[
\delta_0 = \max(3 + 1 - 3, 1 + 1 - 3, 1 - (1 + 1)) = 1
\]

Thus

\[
S_{(1,1)}(F) = a_{03} \cdot \det \begin{bmatrix}
\alpha_1^0 F_1(\alpha_1) & \alpha_2^0 F_1(\alpha_2) & \alpha_3^0 F_1(\alpha_3) \\
\alpha_1^1 F_2(\alpha_1) & \alpha_2^1 F_2(\alpha_2) & \alpha_3^1 F_2(\alpha_3) \\
\alpha_1^0 & \alpha_2^0 & \alpha_3^0 & x^0 \\
\alpha_1^1 & \alpha_2^1 & \alpha_3^1 & x^1
\end{bmatrix}
\]

\[
= a_{03} \cdot \det \begin{bmatrix}
\alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\
\alpha_1^1 & \alpha_2^1 & \alpha_3^1 \\
\alpha_1^1 (a_{21} \alpha_1 + a_{20}) & \alpha_2^1 (a_{21} \alpha_2 + a_{20}) & \alpha_3^1 (a_{21} \alpha_3 + a_{20}) & x^0 \\
\alpha_1^0 & \alpha_2^0 & \alpha_3^0 & x^1
\end{bmatrix}
\]

\[
= a_{03} \cdot \det \begin{bmatrix}
\alpha_1^0 & \alpha_2^0 & \alpha_3^0 \\
\alpha_1^1 & \alpha_2^1 & \alpha_3^1 \\
(\alpha_2 - \alpha_1) (\alpha_3 - \alpha_1) (\alpha_3 - \alpha_2) (a_{12} + a_{13} \alpha_1 + a_{13} \alpha_2 + a_{13} \alpha_3) (a_{21} x + a_{20}) & (\alpha_2 - \alpha_1) (\alpha_3 - \alpha_1) (\alpha_3 - \alpha_2) & x^0
\end{bmatrix}
\]

\[
= -a_{03} (a_{12} + a_{13} \alpha_1 + a_{13} \alpha_2 + a_{13} \alpha_3) (a_{21} x + a_{20})
\]

3 **Application: Parametric GCD**

In this section, we show an application of multi-polynomial subresultant in representing the parametric GCD of several univariate polynomials.

3.1 **Main results**

**Definition 6** (Incremental Cofactor Degree). Let $F = (F_0, F_1, \ldots, F_t) \subset \mathbb{C}[x]$ where $t \geq 1$. The incremental cofactor degree, $icdeg(F)$, is defined by the following

\[
icdeg(F) = (\deg C_1, \ldots, \deg C_t)
\]
where $C_i$ is the incremental cofactor, that is,

$$C_i = \frac{\gcd(F_0, \ldots, F_{i-1})}{\gcd(F_0, \ldots, F_{i-1}, F_i)}$$

Remark 7.

1. In order to make the gcd unique, we take the usual convention that gcd is monic.
2. The gcd of a single polynomial is defined to be the monic version of the polynomial.

Theorem 8 (Parametric GCD in terms of subresultant polynomials). Let

$$\delta = \max_{s_\gamma(F) \neq 0} \gamma$$

where max is with respect to the ordering $\succ_{glex}$. Then we have

1. $\text{icdeg}(F) = \delta$
2. $\gcd(F) = \frac{S_\delta(F)}{s_\delta(F)}$

Remark 9. The ordering $\succ_{glex}$ is defined for two sequences, e.g., $\gamma$ and $\delta$, in $\mathbb{Z}_{\geq 0}^t$. We say $\delta \succ_{glex} \gamma$ if and only if one of the followings occurs:

1. $|\delta| > |\gamma|$;
2. $|\delta| = |\gamma|$ and there exists $i \leq t$ such that $\delta_i > \gamma_i$ and $\delta_j = \gamma_j$ for $j < i$.

Remark 10. In the above, the max always exists since $s_{(0, \ldots, 0)}(F) = a_{0d_0}^{\max(d_1-d_0, \ldots, d_t-d_0, 1)} \neq 0$ from Definition 2.

Example 11. Let $F = (F_0, F_1, F_2)$ where $\deg F_0 = 2$. We have the following parametric gcd. For the sake of compactness, we present it using “nested if”.

if $s_{(2,0)}(F) \neq 0$ then $\gcd(F) = \frac{S_{(2,0)}(F)}{s_{(2,0)}(F)}$
else if $s_{(1,1)}(F) \neq 0$ then $\gcd(F) = \frac{S_{(1,1)}(F)}{s_{(1,1)}(F)}$
else if $s_{(0,2)}(F) \neq 0$ then $\gcd(F) = \frac{S_{(0,2)}(F)}{s_{(0,2)}(F)}$
else if $s_{(1,0)}(F) \neq 0$ then $\gcd(F) = \frac{S_{(1,0)}(F)}{s_{(1,0)}(F)}$
else if $s_{(0,1)}(F) \neq 0$ then $\gcd(F) = \frac{S_{(0,1)}(F)}{s_{(0,1)}(F)}$
else then $\gcd(F) = \frac{S_{(0,0)}(F)}{s_{(0,0)}(F)}$

Remark 12. Previous works for generating the condition for parametric gcd computation are mostly based on repeated gcd computation using subresultants for two polynomials [1, 8]. We compare them with the conditions given in this paper (Theorem 8).

- In the previous works, the polynomials in the conditions are principal coefficients of nested subresultants, that is, subresultant of subresultant polynomials of .... and so on. In the current work, the polynomials are just principal multi-polynomial subresultants of the input.
- The number of polynomials in the conditions given by the previous works and that given by the current work are compatible and both of them are one less than the number of possible $\delta$'s, i.e., the number of ordered $t$-weak-partitions of all natural numbers less than $d_0$. However, the polynomials in the conditions given by the previous work often have more extraneous factors.

1 By $t$-weak-partition, we mean a partition with $t$ parts where the part 0 is allowed.
3.2 Proof of Theorem 8 (Parametric GCD)

We will use the following short hand notation.

**Notation 13** (Vandermonde). \( V(x_1, \ldots, x_n) = \det \begin{bmatrix} x_1^0 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^0 & \cdots & x_n^n \end{bmatrix} \)

**Notation 14** (Incremental gcd).

1. \( G_i = \gcd(F_0, \ldots, F_i) \). Note that \( G_0 \) is the monic version of \( F_0 \).
2. \( e_i = \deg G_i \)

**Lemma 15.** Let \( \theta = \text{icdeg}(F) \). Then we have

\[
s_\theta(F) \neq 0 \quad \text{and} \quad \gcd(F) = \frac{S_\theta(F)}{s_\theta(F)}
\]

**Proof.** In order to convey the main underlying ideas effectively, we will show the proof for a particular case first. After that, we will generalize the ideas to arbitrary cases.

**Particular case:** Consider the case \( d = (7, 6, 6) \) and \( e = (7, 4, 2) \).

1. We treat \( \alpha_1, \ldots, \alpha_7 \) as distinct indeterminates.
2. Note \( \theta = (e_0 - e_1, e_1 - e_2) = (7 - 4, 4 - 2) = (3, 2) \). Without loss of generality, we index the roots as follows.

\[
\begin{align*}
G_0 & = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)(x - \alpha_5)(x - \alpha_6)(x - \alpha_7) \\
G_1 & = (x - \alpha_4)(x - \alpha_5)(x - \alpha_6)(x - \alpha_7) \\
G_2 & = (x - \alpha_6)(x - \alpha_7)
\end{align*}
\]

Then

\[
F_1(\alpha_j) = 0 \quad \text{for } j = 4, 5, 6, 7 \\
F_2(\alpha_j) = 0 \quad \text{for } j = 6, 7
\] (1)

3. From the definition of subresultant (Definition 2), we have

\[
\begin{bmatrix}
\alpha_0^2 F_1(\alpha_1) & \alpha_0^2 F_1(\alpha_2) & \alpha_0^2 F_1(\alpha_3) & \alpha_0^2 F_1(\alpha_4) & \alpha_0^2 F_1(\alpha_5) & \alpha_0^2 F_1(\alpha_6) & \alpha_0^2 F_1(\alpha_7) \\
\alpha_2 F_1(\alpha_1) & \alpha_2 F_1(\alpha_2) & \alpha_2 F_1(\alpha_3) & \alpha_2 F_1(\alpha_4) & \alpha_2 F_1(\alpha_5) & \alpha_2 F_1(\alpha_6) & \alpha_2 F_1(\alpha_7) \\
\alpha_0^3 F_1(\alpha_1) & \alpha_0^3 F_1(\alpha_2) & \alpha_0^3 F_1(\alpha_3) & \alpha_0^3 F_1(\alpha_4) & \alpha_0^3 F_1(\alpha_5) & \alpha_0^3 F_1(\alpha_6) & \alpha_0^3 F_1(\alpha_7) \\
\alpha_2 F_2(\alpha_1) & \alpha_2 F_2(\alpha_2) & \alpha_2 F_2(\alpha_3) & \alpha_2 F_2(\alpha_4) & \alpha_2 F_2(\alpha_5) & \alpha_2 F_2(\alpha_6) & \alpha_2 F_2(\alpha_7) \\
\alpha_0^3 F_2(\alpha_1) & \alpha_0^3 F_2(\alpha_2) & \alpha_0^3 F_2(\alpha_3) & \alpha_0^3 F_2(\alpha_4) & \alpha_0^3 F_2(\alpha_5) & \alpha_0^3 F_2(\alpha_6) & \alpha_0^3 F_2(\alpha_7) \\
\end{bmatrix}
\]

\[
S_\theta(F) = a_{07}^2 V(\alpha_1, \ldots, \alpha_7)
\]

where \( a_{07} \) is the leading coefficient of \( F_0 \), and

\[
\begin{align*}
\varepsilon & = 1 + d_0 - |\theta| = 1 + 7 - (3 + 2) = 3 \\
\theta_0 & = \max(d_1 + \theta_1 - d_0, d_1 + \theta_1 - d_0, 1 - |\theta|) \\
& = \max(6 + 3 - 7, 6 + 2 - 7, 1 - (3 + 2)) = 2
\end{align*}
\]
4. Substitution of \( \Pi \) into the above expression yields the following determinant with a block lower-
triangular structure:

\[
\begin{vmatrix}
\alpha_0^0 F_1(\alpha_1) & \alpha_0^0 F_1(\alpha_2) & \alpha_0^0 F_1(\alpha_3) \\
\alpha_1^0 F_1(\alpha_1) & \alpha_1^0 F_1(\alpha_2) & \alpha_1^0 F_1(\alpha_3) \\
\alpha_2^0 F_1(\alpha_1) & \alpha_2^0 F_1(\alpha_2) & \alpha_2^0 F_1(\alpha_3) \\
\alpha_0^0 F_2(\alpha_1) & \alpha_0^0 F_2(\alpha_2) & \alpha_0^0 F_2(\alpha_3) \\
\alpha_1^0 F_2(\alpha_1) & \alpha_1^0 F_2(\alpha_2) & \alpha_1^0 F_2(\alpha_3) \\
\alpha_2^0 F_2(\alpha_1) & \alpha_2^0 F_2(\alpha_2) & \alpha_2^0 F_2(\alpha_3) \\
\alpha_0^0 F_2(\alpha_4) & \alpha_0^0 F_2(\alpha_5) \\
\alpha_1^0 F_2(\alpha_4) & \alpha_1^0 F_2(\alpha_5) \\
\alpha_2^0 F_2(\alpha_4) & \alpha_2^0 F_2(\alpha_5) \\
\alpha_0^0 & \alpha_0^0 & \alpha_0^0 x^0 \\
\alpha_1^0 & \alpha_1^0 & \alpha_1^0 x^1 \\
\alpha_2^0 & \alpha_2^0 & \alpha_2^0 x^2 \\
\end{vmatrix}
\]

\[S_\theta(F) = a_{07}^2 \cdot \det M_1 \det M_2 \det N \]

where

\[M_1 = \begin{bmatrix}
\alpha_0^0 F_1(\alpha_1) & \alpha_0^0 F_1(\alpha_2) & \alpha_0^0 F_1(\alpha_3) \\
\alpha_1^0 F_1(\alpha_1) & \alpha_1^0 F_1(\alpha_2) & \alpha_1^0 F_1(\alpha_3) \\
\alpha_2^0 F_1(\alpha_1) & \alpha_2^0 F_1(\alpha_2) & \alpha_2^0 F_1(\alpha_3)
\end{bmatrix}\]

\[M_2 = \begin{bmatrix}
\alpha_0^0 F_2(\alpha_4) & \alpha_0^0 F_2(\alpha_5) \\
\alpha_1^0 F_2(\alpha_4) & \alpha_1^0 F_2(\alpha_5) \\
\alpha_2^0 F_2(\alpha_4) & \alpha_2^0 F_2(\alpha_5)
\end{bmatrix}\]

\[N = \begin{bmatrix}
\alpha_0^0 & \alpha_0^0 & x^0 \\
\alpha_1^0 & \alpha_1^0 & x^1 \\
\alpha_2^0 & \alpha_2^0 & x^2
\end{bmatrix}\]

5. Note that \( M_1, M_2, N \) are all square matrices.

6. By applying elementary properties of determinants to the above expressions, we have

\[S_\theta(F) = a_{07}^2 \cdot \frac{\det M_1 \det M_2 \det N}{V(\alpha_1, \ldots, \alpha_7)}\]

where

\[\det M_1 = V(\alpha_1, \alpha_2, \alpha_3) \prod_{j=1}^{3} F_1(\alpha_j) = V(\alpha_1, \alpha_2, \alpha_3) F_1(\alpha_1) F_1(\alpha_2) F_1(\alpha_3)\]

\[\det M_2 = V(\alpha_4, \alpha_5) \prod_{j=1}^{3} F_2(\alpha_j) = V(\alpha_4, \alpha_5) F_2(\alpha_4) F_2(\alpha_5)\]

\[\det N = V(\alpha_6, \alpha_7) \gcd(F) \quad \text{since} \quad \gcd(F) = G_2 = (x - \alpha_6)(x - \alpha_7)\]

7. Thus we have

\[S_\theta(F) = s_\theta(F) \gcd(F)\]

where

\[s_\theta(F) = a_{07}^2 \cdot \frac{\gcd(F) V(\alpha_1, \alpha_2, \alpha_3) V(\alpha_4, \alpha_5) V(\alpha_6, \alpha_7)}{V(\alpha_1, \ldots, \alpha_7)} \cdot \frac{F_1(\alpha_1) F_1(\alpha_2) F_1(\alpha_3) F_2(\alpha_4) F_2(\alpha_5)}{V(\alpha_1, \ldots, \alpha_7)}\]

\[8\]
8. By applying elementary properties of Vandermonde determinants, we have
\[
\frac{V(\alpha_1, \alpha_2, \alpha_3)V(\alpha_4, \alpha_5)V(\alpha_6, \alpha_7)}{V(\alpha_1, \ldots, \alpha_7)} = \pm \frac{1}{\prod_{j=1}^{7} \prod_{k=4}^{7} (\alpha_k - \alpha_j) \prod_{j=4}^{7} \prod_{k=6}^{7} (\alpha_j - \alpha_k)}
\]
\[
= \pm \frac{1}{\prod_{j=1}^{7} \prod_{k=4}^{7} (\alpha_j - \alpha_k) \prod_{j=4}^{7} \prod_{k=6}^{7} (\alpha_j - \alpha_k)}
\]
\[
= \pm \frac{1}{G_1(\alpha_1)G_1(\alpha_2)G_1(\alpha_3)G_2(\alpha_4)G_2(\alpha_5)}
\]

9. Thus
\[
s_\theta(F) = \pm a_{d_7}^2 \cdot \frac{F_1(\alpha_1) F_1(\alpha_2) F_1(\alpha_3) F_2(\alpha_4) F_2(\alpha_5)}{G_1(\alpha_1) G_1(\alpha_2) G_1(\alpha_3) G_2(\alpha_4) G_2(\alpha_5)}
\]
\[
= \pm a_{d_7}^2 \cdot H_1(\alpha_1) H_1(\alpha_2) H_1(\alpha_3) H_2(\alpha_4) H_2(\alpha_5)
\]
where \(H_i = F_i/G_i\).

10. From now on, we treat \(\alpha_1, \ldots, \alpha_7\) as numbers.

11. Note that \(s_\theta(F) \neq 0\).

12. Thus we also have \(\gcd(F) = \frac{S_4(F)}{s_\theta(F)}\).

**Arbitrary case.** Now we generalize the above ideas to arbitrary cases.

1. We treat \(\alpha_1, \ldots, \alpha_{d_0}\) as distinct indeterminates.

2. Without loss of generality, we index the roots as follows.
\[
\begin{align*}
F_0 &= a_{d_0} \prod_{k=1}^{d_0} (x - \alpha_k) \\
G_i &= \prod_{k=d_0 - e_i + 1}^{d_0} (x - \alpha_k) \quad \text{for } i = 0, \ldots, t
\end{align*}
\]
where \(a_{d_0} \neq 0\). Then
\[
F_i(\alpha_j) = 0 \quad \text{for } j = d_0 - e_i + 1, \ldots, d_0
\]
3. From the definition of subresultant (Definition 2), we have

\[
S_\theta(F) = a_{0d_0}^{\theta_0} \cdot V(\alpha_1, \ldots, \alpha_{d_0})
\]

where \( \varepsilon = 1 + d_0 - |\theta| \) and \( \theta_0 = \max(d_1 + \theta_1 - d_0, \ldots, d_t + \theta_t - d_0, 1 - |\theta|) \).

4. Substitution of (2) into the above expression yields the following determinant with a block lower-triangular structure:

\[
\begin{vmatrix}
\alpha_1^0 F_1(\alpha_1) & \cdots & \alpha_{d_0}^0 F_1(\alpha_{d_0}) \\
\vdots & & \vdots \\
\alpha_1^{\theta_1-1} F_1(\alpha_1) & \cdots & \alpha_{d_0}^{\theta_1-1} F_1(\alpha_{d_0}) \\
\vdots & & \vdots \\
\alpha_1^{\theta_t-1} F_t(\alpha_1) & \cdots & \alpha_{d_0}^{\theta_t-1} F_t(\alpha_{d_0}) \\
\alpha_1^1 & \cdots & \alpha_{d_0}^1 \\
\vdots & & \vdots \\
\alpha_1^{\varepsilon-1} & \cdots & \alpha_{d_0}^{\varepsilon-1} \\
\end{vmatrix}
\]

\[
\det V(\alpha_1, \ldots, \alpha_{d_0})
\]

where

\[
M_i = \begin{bmatrix}
\alpha_{d_0-e_i-1+1}^0 F_i(\alpha_{d_0-e_i-1+1}), & \cdots & \alpha_{d_0}^0 F_i(\alpha_{d_0-e_i}) \\
\vdots & & \vdots \\
\alpha_{d_0-e_i-1+1}^{\theta_i-1} F_i(\alpha_{d_0-e_i-1+1}), & \cdots & \alpha_{d_0}^{\theta_i-1} F_i(\alpha_{d_0-e_i}) \\
\end{bmatrix}
\]

for \( i = 1, \ldots, t \)

\[
N = \begin{bmatrix}
\alpha_{d_0-e_t+1}^0 & \cdots & \alpha_{d_0}^0 & x^0 \\
\vdots & & \vdots & \vdots \\
\alpha_{d_0-e_t+1}^{\varepsilon-1} & \cdots & \alpha_{d_0}^{\varepsilon-1} & x^{\varepsilon-1} \\
\end{bmatrix}
\]

5. Note that \( M_1, \ldots, M_t, N \) are all square matrices since

\[
\# \text{ of rows of } M_i = \theta_i = e_i - e_i = \# \text{ of columns of } M_i
\]

\[
\# \text{ of rows of } N = \varepsilon = 1 + d_0 - (\theta_1 + \cdots + \theta_t) = 1 + d_0 - (e_0 - e_1 + e_1 - e_2 + \cdots + e_{t-1} - e_t) = 1 + d_0 - (e_0 - e_t) = 1 + e_t \quad \text{since } d_0 = e_0 = \# \text{ of columns of } N
\]
6. By applying elementary properties of determinants to the above expressions, we have

$$S_\theta(F) = a_{0d_0}^0 \cdot \frac{\prod_{i=1}^{t} \det M_i \cdot \det N}{V(\alpha_1, \ldots, \alpha_d)}$$

where

$$\det M_i = V(\alpha_{d_0-e_{i+1}}, \ldots, \alpha_{d_0-e_1}) \prod_{j=d_0-e_{i+1}}^{d_0-e_1} F_i(\alpha_j) \quad \text{for } i = 1, \ldots, t$$

$$\det N = V(\alpha_{d_0-e_1}, \ldots, \alpha_{d_0}) \gcd(F) \quad \text{since } G_t = \gcd(F)$$

7. Thus we have

$$S_\theta(F) = s_\theta(F) \gcd(F)$$

where

$$s_\theta(F) = a_{0d_0}^0 \cdot \frac{\prod_{i=1}^{t+1} V(\alpha_{d_0-e_{i+1}}, \ldots, \alpha_{d_0-e_1})}{V(\alpha_1, \ldots, \alpha_d)} \prod_{i=1}^{t} \prod_{j=d_0-e_{i+1}}^{d_0-e_1} F_i(\alpha_j)$$

where $e_{t+1} = 0$.

8. By applying elementary properties of Vandermonde matrices, we have

$$\prod_{i=1}^{t+1} V(\alpha_{d_0-e_{i+1}}, \ldots, \alpha_{d_0-e_1}) / V(\alpha_1, \ldots, \alpha_d) = \prod_{i=1}^{t} \prod_{j=d_0-e_{i+1}}^{d_0-e_1} (\alpha_k - \alpha_j)$$

$$= \pm \prod_{i=1}^{t} \prod_{j=d_0-e_{i+1}}^{d_0-e_1} G(\alpha_j)$$

9. Thus

$$s_\theta(F) = \pm a_{0d_0}^0 \cdot \prod_{i=1}^{t} \prod_{j=d_0-e_{i+1}}^{d_0-e_1} \frac{F_i(\alpha_j)}{G_i(\alpha_j)} = \pm a_{0d_0}^0 \cdot \prod_{i=1}^{t} \prod_{j=d_0-e_{i+1}}^{d_0-e_1} H_i(\alpha_j)$$

where $H_i = F_i/G_i$.

10. From now on, we treat $\alpha_1, \ldots, \alpha_d$ as numbers.

11. Note that $s_\theta(F) = \pm a_{0d_0}^0 \cdot \prod_{i=1}^{t} \prod_{j=d_0-e_{i+1}}^{d_0-e_1} H_i(\alpha_j) \neq 0$.

12. Thus we also have $\gcd(F) = \frac{S_\theta(F)}{s_\theta(F)}$.

Lemma 16. Let $\theta = \text{icdeg}(F)$. Let $\delta \in \mathbb{N}_{\geq 0}$ such that $|\delta| \leq d_0$. If $\delta \succ \text{glex} \theta$ then $S_\delta = 0$. 

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Proof. In order to convey the main underlying ideas effectively, we will show the proof for a particular case first. After that, we will generalize the ideas to arbitrary cases.

**Particular case.** We consider the same particular case used in the proof of Lemma 2.4. Recall that \( d = (7, 6, 6), \ e = (7, 4, 2) \) and \( \theta = (3, 2) \).

1. We treat \( \alpha_1, \ldots, \alpha_7 \) as distinct indeterminates.

2. Let \( \delta = (3, 3) \). Note \( \delta \gg \text{lex} \theta \). Note that \( \delta_1 + \delta_2 > \theta_1 + \theta_2 \).

3. From the definition of subresultant (Definition 3), we have

\[
\begin{bmatrix}
\alpha_1^0 F_1 (\alpha_1) & \alpha_1^0 F_1 (\alpha_2) & \alpha_1^0 F_1 (\alpha_3) & \alpha_1^0 F_1 (\alpha_4) & \alpha_1^0 F_1 (\alpha_5) & \alpha_1^0 F_1 (\alpha_6) & \alpha_1^0 F_1 (\alpha_7) \\
\alpha_2^0 F_2 (\alpha_1) & \alpha_2^0 F_2 (\alpha_2) & \alpha_2^0 F_2 (\alpha_3) & \alpha_2^0 F_2 (\alpha_4) & \alpha_2^0 F_2 (\alpha_5) & \alpha_2^0 F_2 (\alpha_6) & \alpha_2^0 F_2 (\alpha_7) \\
\alpha_1 F_1 (\alpha_1) & \alpha_1 F_1 (\alpha_2) & \alpha_1 F_1 (\alpha_3) & \alpha_1 F_1 (\alpha_4) & \alpha_1 F_1 (\alpha_5) & \alpha_1 F_1 (\alpha_6) & \alpha_1 F_1 (\alpha_7) \\
\alpha_2 F_2 (\alpha_1) & \alpha_2 F_2 (\alpha_2) & \alpha_2 F_2 (\alpha_3) & \alpha_2 F_2 (\alpha_4) & \alpha_2 F_2 (\alpha_5) & \alpha_2 F_2 (\alpha_6) & \alpha_2 F_2 (\alpha_7) \\
\alpha_1^2 F_1 (\alpha_1) & \alpha_1^2 F_1 (\alpha_2) & \alpha_1^2 F_1 (\alpha_3) & \alpha_1^2 F_1 (\alpha_4) & \alpha_1^2 F_1 (\alpha_5) & \alpha_1^2 F_1 (\alpha_6) & \alpha_1^2 F_1 (\alpha_7) \\
\alpha_2^2 F_2 (\alpha_1) & \alpha_2^2 F_2 (\alpha_2) & \alpha_2^2 F_2 (\alpha_3) & \alpha_2^2 F_2 (\alpha_4) & \alpha_2^2 F_2 (\alpha_5) & \alpha_2^2 F_2 (\alpha_6) & \alpha_2^2 F_2 (\alpha_7) \\
\end{bmatrix}
\]

\[
de(\delta) (F) = \begin{bmatrix} 2 \alpha_0^2 \end{bmatrix}
\]

where

\[
\begin{bmatrix}
\alpha_1 + d_0 - |\delta| &= 1 + 7 - (3 + 3) = 2 \\
\delta_0 &= \max(d_1 + d_1 - d_0, d_2 + d_2 - d_0, 1 - |\delta|) \\
&= \max(6 + 3 - 7, 6 + 3 - 7, 1 - (3 + 3)) = 2 \\
\end{bmatrix}
\]

4. Substitution of \( \gamma \) into the above expression yields the following determinant with a block lower-triangular structure:

\[
\begin{bmatrix}
\alpha_1^0 F_1 (\alpha_1) & \alpha_1^0 F_1 (\alpha_2) & \alpha_1^0 F_1 (\alpha_3) \\
\alpha_2^0 F_2 (\alpha_1) & \alpha_2^0 F_2 (\alpha_2) & \alpha_2^0 F_2 (\alpha_3) \\
\alpha_1^2 F_1 (\alpha_1) & \alpha_1^2 F_1 (\alpha_2) & \alpha_1^2 F_1 (\alpha_3) \\
\alpha_2^2 F_2 (\alpha_1) & \alpha_2^2 F_2 (\alpha_2) & \alpha_2^2 F_2 (\alpha_3) \\
\alpha_1^4 F_1 (\alpha_1) & \alpha_1^4 F_1 (\alpha_2) & \alpha_1^4 F_1 (\alpha_3) \\
\alpha_2^4 F_2 (\alpha_1) & \alpha_2^4 F_2 (\alpha_2) & \alpha_2^4 F_2 (\alpha_3) \\
\end{bmatrix}
\]

\[
de(\delta) (F) = \begin{bmatrix} 2 \alpha_0^2 \end{bmatrix}
\]

where

\[
M_1 = \begin{bmatrix}
\alpha_1^0 F_1 (\alpha_1) & \alpha_1^0 F_1 (\alpha_2) & \alpha_1^0 F_1 (\alpha_3) \\
\alpha_1^2 F_1 (\alpha_1) & \alpha_1^2 F_1 (\alpha_2) & \alpha_1^2 F_1 (\alpha_3) \\
\alpha_1^4 F_1 (\alpha_1) & \alpha_1^4 F_1 (\alpha_2) & \alpha_1^4 F_1 (\alpha_3) \\
\end{bmatrix}
\]

\[
M_2 = \begin{bmatrix}
\alpha_2^0 F_2 (\alpha_1) & \alpha_2^0 F_2 (\alpha_2) & \alpha_2^0 F_2 (\alpha_3) \\
\alpha_2^2 F_2 (\alpha_1) & \alpha_2^2 F_2 (\alpha_2) & \alpha_2^2 F_2 (\alpha_3) \\
\alpha_2^4 F_2 (\alpha_1) & \alpha_2^4 F_2 (\alpha_2) & \alpha_2^4 F_2 (\alpha_3) \\
\end{bmatrix}
\]

\[
N = \begin{bmatrix}
\alpha_6^0 & \alpha_6^2 & x_0^0 \\
\alpha_6^1 & \alpha_6^3 & x_0^1
\end{bmatrix}
\]

\[
det \begin{bmatrix}
M_1 \\
M_2 \\
N
\end{bmatrix}
\]

\[
= \alpha_0^2 \cdot V (\alpha_1, \ldots, \alpha_7)
\]
5. We repartition the numerator matrix so that the diagonal consists of two square matrices

\[
\begin{pmatrix}
\alpha_1^0 F_1(\alpha_1) & \alpha_2^0 F_1(\alpha_2) & \alpha_3^0 F_1(\alpha_3) \\
\alpha_1^1 F_1(\alpha_1) & \alpha_2^1 F_1(\alpha_2) & \alpha_3^1 F_1(\alpha_3) \\
\alpha_1^2 F_1(\alpha_1) & \alpha_2^2 F_1(\alpha_2) & \alpha_3^2 F_1(\alpha_3) \\
\alpha_1^0 F_2(\alpha_1) & \alpha_2^0 F_2(\alpha_2) & \alpha_3^0 F_2(\alpha_3) \\
\alpha_1^1 F_2(\alpha_1) & \alpha_2^1 F_2(\alpha_2) & \alpha_3^1 F_2(\alpha_3) \\
\alpha_1^2 F_2(\alpha_1) & \alpha_2^2 F_2(\alpha_2) & \alpha_3^2 F_2(\alpha_3) \\
\alpha_1^0 F_3(\alpha_1) & \alpha_2^0 F_3(\alpha_2) & \alpha_3^0 F_3(\alpha_3) \\
\alpha_1^1 F_3(\alpha_1) & \alpha_2^1 F_3(\alpha_2) & \alpha_3^1 F_3(\alpha_3) \\
\alpha_1^2 F_3(\alpha_1) & \alpha_2^2 F_3(\alpha_2) & \alpha_3^2 F_3(\alpha_3)
\end{pmatrix}
\]

\[
S_\delta(F) = a_{07}^2 \cdot \frac{\det T \det B}{V(\alpha_1, \ldots, \alpha_7)}
\]

where the size of the square matrix \( T \) is \( \delta_1 + \delta_2 = 3 + 3 = 6 \), namely

\[
T = \begin{bmatrix} M_1 & M_2 \end{bmatrix}
\]

where 0 is the \( \delta_2 \times p \) matrix with zeros, where again \( \delta_2 = 3 \) and \( p = (\delta_1 + \delta_2) - (\theta_1 + \theta_2) = 1 \).

6. By applying elementary properties of determinants to the above expressions, we have

\[
S_\theta(F) = a_{07}^2 \cdot \frac{\det T \det B}{V(\alpha_1, \ldots, \alpha_7)}
\]

7. Since \( p = 1 > 0 \), the last column of \( T \) is all zero. Hence \( \det T = 0 \) and in turn \( S_\theta(F) = 0 \).

8. From now on, we treat \( \alpha_1, \ldots, \alpha_7 \) as numbers.

9. Obviously \( S_\theta(F) = 0 \).

**Arbitrary case.** Now we generalize the above ideas to arbitrary cases. Let \( \delta \in \mathbb{N}_{\geq 0}^\ell \) be such that \( |\delta| \leq d_0 \).

1. We will treat \( \alpha_1, \ldots, \alpha_{d_0} \) as distinct indeterminates.

2. Assume \( \delta >_{\text{glex}} \theta \). Then for some \( \ell \), we have \( \delta_1 + \cdots + \delta_\ell > \theta_1 + \cdots + \theta_\ell \).

3. Recall that

\[
S_\delta(F) = a_{0d_0}^\delta \cdot \frac{\det \begin{bmatrix}
\alpha_1^0 F_1(\alpha_1) & \cdots & \alpha_{d_0}^0 F_1(\alpha_{d_0}) \\
\vdots & \ddots & \vdots \\
\alpha_1^{\delta_1-1} F_1(\alpha_1) & \cdots & \alpha_{d_0}^{\delta_1-1} F_1(\alpha_{d_0}) \\
\vdots & \ddots & \vdots \\
\alpha_1^0 F_\ell(\alpha_1) & \cdots & \alpha_{d_0}^0 F_\ell(\alpha_{d_0}) \\
\vdots & \ddots & \vdots \\
\alpha_1^{\delta_\ell-1} F_\ell(\alpha_1) & \cdots & \alpha_{d_0}^{\delta_\ell-1} F_\ell(\alpha_{d_0}) \\
\alpha_1^\ell & \cdots & \alpha_{d_0}^\ell \\
\vdots & \ddots & \vdots \\
\alpha_1^{\ell-1} & \cdots & \alpha_{d_0}^{\ell-1}
\end{bmatrix}}{V(\alpha_1, \ldots, \alpha_{d_0})}
\]

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where $a_{0d_0}$ is the leading coefficient of $F_0$, and
\[
\begin{align*}
\varepsilon &= 1 + d_0 - |\delta| \\
\delta_0 &= \max(d_1 + \delta_1 - d_0, \ldots, d_t + \delta_t - d_0, 1 - |\delta|)
\end{align*}
\]

4. Substitution of (2) into the above expression yields the following determinant with a block lower-triangular structure:
\[
S_\delta(F) = a_{0d_0}^\delta \cdot \det \begin{bmatrix}
M_1 \\
\vdots \\
\vdots \\
\vdots \\
M_t \\
\vdots \\
N
\end{bmatrix}
\]

where $M_i$ is $\delta_i$ by $\theta_i$.

5. We repartition the numerator matrix so that the diagonal consists of two square matrices $T$ and $B$ as follows
\[
S_\delta(F) = a_{0d_0}^\delta \cdot \det \begin{bmatrix}
T \\
B
\end{bmatrix}
\]

where the size of the square matrix $T$ is $\delta_1 + \cdots + \delta_t$, namely
\[
T = \begin{bmatrix}
M_1 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
M_t \\
0
\end{bmatrix}
\]

where 0 is the $\delta_t \times p$ matrix with zeros, where again $p = (\delta_1 + \cdots + \delta_t) - (\theta_1 + \cdots + \theta_t)$.

6. By applying elementary properties of determinants to the above expressions, we have
\[
S_\theta(F) = a_{0d_0}^\delta \cdot \det T \det B
\]

7. Since $p > 0$, the last column of $T$ is all zero. Hence $\det T = 0$ and in turn $S_\theta(F) = 0$.

8. From now on, we treat $\alpha_1, \ldots, \alpha_{d_0}$ as numbers.

9. Obviously $S_\theta(F) = 0$.

Proof of Theorem 8.

1. Let $\delta = \max_{s_\gamma(F) \neq 0} \gamma$ and let $\theta = \text{icdeg} (F)$.

2. Obviously $s_\delta(F) \neq 0$.

3. Hence from the contra-positive of Lemma 16 we have $\delta \preceq \text{glex} \theta$.

4. From Lemma 15 we have $s_\theta(F) \neq 0$. Recalling $\delta = \max_{s_\gamma(F) \neq 0} \gamma$, we have $\delta \preceq \text{glex} \theta$.

5. Thus $\theta = \delta$, that is, $\text{icdeg} (F) = \delta$.

6. By Lemma 15 we have $\gcd (F) = \frac{S_\delta}{\nu_\theta}$.
4 Application: Parametric multiplicity

In this section, we show an application of the subresultant polynomial in computing the multiplicity of a parametric univariate polynomial.

4.1 Main results

Definition 17 (Multiplicity). Let $H \in \mathbb{C}[x]$ with $m$ distinct complex roots, say $r_1, \ldots, r_m$ with multiplicities $\mu_1, \ldots, \mu_m$ respectively. Without losing generality, we assume that $\mu_1 \geq \cdots \geq \mu_m$. Then the multiplicity of $H$, written as $\text{mult}(F)$, is defined by

$$\text{mult}(H) = (\mu_1, \ldots, \mu_m)$$

Definition 18 (Conjugate). Let $\delta = (\delta_1, \ldots, \delta_t)$. Then the conjugate $\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_s)$ of $\delta$ is defined by

$$s = \max \delta$$

$$\bar{\delta}_i = \# \{ j \in [1, \ldots, t] : \delta_j \geq i \} \quad \text{for } i = 1, \ldots, s$$

Example 19.

1. Let $\delta = (3, 2, 0, 0, 0)$. Note

$$s = \max \delta = 3$$

$$\bar{\delta}_1 = \# \{ j : \delta_j \geq 1 \} = 2$$

$$\bar{\delta}_2 = \# \{ j : \delta_j \geq 2 \} = 2$$

$$\bar{\delta}_3 = \# \{ j : \delta_j \geq 2 \} = 1$$

Thus

$$\bar{\delta} = (2, 2, 1)$$

Theorem 20 (Parametric multiplicity in terms of subresultants). Let $H \in \mathbb{C}[x]$ be of degree $t$ and let

$$\delta = \max_{s, (F) \neq 0} \lambda$$

where

1. $F = (H^{(0)}, H^{(1)}, \ldots, H^{(t)})$ and $H^{(i)}$ is the $i$-th derivative of $H$;
2. $\lambda = (\lambda_1, \ldots, \lambda_t)$ such that $\lambda_1 + \cdots + \lambda_t = t$ and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_t \geq 0$;
3. $\max$ is with respect to the lexicographic ordering $>_\text{lex}$.

Then we have

$$\text{mult}(H) = \bar{\delta}$$

Remark 21. In the above, max always exists since $s_{(1, \ldots, 1)}(F) \neq 0$ from Definition [2].

Example 22. We have the following parametric multiplicity for degree 5.

if $s_{(5,0,0,0,0)}(F) \neq 0$ then $\text{mult}(H) = (1, 1, 1, 1, 1)$
else if $s_{(4,1,0,0,0)}(F) \neq 0$ then $\text{mult}(H) = (2, 1, 1, 1)$
else if $s_{(3,2,0,0,0)}(F) \neq 0$ then $\text{mult}(H) = (2, 2, 1)$
else if $s_{(3,1,1,0,0)}(F) \neq 0$ then $\text{mult}(H) = (3, 1, 1)$
else if $s_{(2,2,1,0,0)}(F) \neq 0$ then $\text{mult}(H) = (3, 2)$
else if $s_{(2,1,1,1,0)}(F) \neq 0$ then $\text{mult}(H) = (4, 1)$
else $\text{then } \text{mult}(H) = (5)$
Remark 23. Previous works for generating the condition for complex root classification are mostly based on repeated gcd computation using subresultants for two polynomials \[^2\] We compare them with the conditions given in this paper (Theorem 20).

- In the previous works, the polynomials in the conditions are principal coefficients of nested subresultant polynomials, that is, subresultant of subresultant polynomials of \( \ldots \) and so on. In the current work, the polynomials are just principal multi-polynomial subresultants of the input.

- In the previous works, the number of polynomials in the conditions are often bigger than the number of multiplicity structures. In the current work, the number of polynomials is exactly one less than that of multiplicity structures.

4.2 Proof of Theorem 20 (Parametric Multiplicity)

Lemma 24. Let \( F = (H^{(0)}, H^{(1)}, \ldots, H^{(t)}) \) and let \( \delta = \text{icdeg}(F) \). Then \( \text{mult}(H) = \bar{\delta} \).

Proof. In order to convey the main underlying ideas effectively, we will show the proof for a particular case first. After that, we will generalize the ideas to arbitrary cases.

**Particular case.** Let \( H = a (x - r_1)^2 (x - r_2)^2 (x - r_3)^1 \) where \( a \neq 0 \) and \( r_1, r_2, r_3 \) are distinct.

1. Note \( \text{deg } H = 5 \) and \( \text{mult}(H) = (2, 1, 1) \).

2. Let \( G_i = \gcd(H^{(0)}, \ldots, H^{(i)}) \) for \( i = 0, \ldots, 5 \). Let \( C_i = \frac{G_i - 1}{G_i} \) for \( i = 1, \ldots, 5 \). Then

   \[
   \begin{align*}
   G_0 &= (x - r_1)^2 (x - r_2)^2 (x - r_3)^1 \\
   G_1 &= (x - r_1)^1 (x - r_2)^1 \\
   G_2 &= 1 \\
   G_3 &= 1 \\
   G_4 &= 1 \\
   G_5 &= 1 \\
   
   C_1 &= (x - r_1) (x - r_2) (x - r_3) \\
   C_2 &= (x - r_1) (x - r_2) \\
   C_3 &= 1 \\
   C_4 &= 1 \\
   C_5 &= 1 
   \end{align*}
   \]

   Thus

   \[ \delta = \text{icdeg}(F) = (3, 2, 0, 0, 0) \]

3. Note

   \[ s = \max \bar{\delta} = 3 \]

   \[ \bar{\delta}_1 = \# \{ j \in [1, \ldots, 5] : \delta_j \geq 1 \} = 2 \]

   \[ \bar{\delta}_2 = \# \{ j \in [1, \ldots, 5] : \delta_j \geq 2 \} = 2 \]

   \[ \bar{\delta}_3 = \# \{ j \in [1, \ldots, 5] : \delta_j \geq 2 \} = 1 \]

   Thus

   \[ \bar{\delta} = (2, 1, 1) \]

4. Thus

   \[ \text{mult}(H) = (2, 2, 1) = \bar{\delta} \]

**Arbitrary case.** Let \( H = a (x - r_1)^{\mu_1} \cdots (x - r_m)^{\mu_m} \) where \( a \neq 0 \) and \( r_1, \ldots, r_m \) are distinct.

1. Note \( \text{deg } H = t = \mu_1 + \cdots + \mu_m \) and \( \text{mult}(H) = \mu = (\mu_1, \ldots, \mu_m) \).

\[^2\]The authors considered a more general problem, i.e., the real root classification problem. When restricted to the complex case, the methods gave conditions for complex root classification.
2. Let \( G_i = \gcd(H^{(0)}, \ldots, H^{(i)}) \) for \( i = 0, \ldots, t \). Let \( C_i = \frac{G_{i-1}}{G_i} \) for \( i = 1, \ldots, t \). Then

\[
G_i = \prod_{1 \leq k \leq m, \mu_k > i} (x - r_k)^{\mu_k - i}
\]

\[
C_i = \frac{G_{i-1}}{G_i} = \prod_{1 \leq k \leq m, \mu_k > i-1} (x - r_k)^{\mu_k - (i-1)}
\]

\[
\prod_{1 \leq k \leq m, \mu_k \geq i} (x - r_k)^{\mu_k - i} = \prod_{1 \leq k \leq m, \mu_k \geq i} (x - r_k)
\]

Hence

\[
\text{icdeg}(F) = \delta = (\delta_1, \ldots, \delta_t) \quad \text{where} \quad \delta_i = \text{deg}C_i = \# \{ k \in [1, \ldots, m] : \mu_k \geq i \}
\]

3. Note

\[
s = \max \delta = m
\]

\[
\tilde{\delta}_i = \# \{ j : \delta_j \geq i \} = \# \{ j : \# \{ k \in [1, \ldots, m] : \mu_k \geq j \} \geq i \} = \mu_i \quad \text{for} \quad i = 1, \ldots, s
\]

Hence

\[
\tilde{\delta} = \mu
\]

4. Thus

\[
\text{mult}(H) = \tilde{\delta}
\]

Proof of Theorem 27

1. Let \( \delta' \) be such that \( \text{mult}(H) = \delta' \).

Note that \( \delta' \) exists and it is unique.

2. By Lemma 24, we have \( \text{icdeg}(F) = \delta' \).

3. From Lemma 15, we have \( s_{\delta'} \neq 0 \).

4. By hypothesis, \( \delta' \preceq_{\text{lex}} \delta \).

5. Since \( \text{icdeg}(F) = \delta' \), by Lemma 16, \( s_{\gamma} = 0 \) for any \( \gamma \) satisfying \( |\gamma| = t \) and \( \gamma \succ_{\text{lex}} \delta' \).

6. Since \( s_{\delta} \neq 0 \), \( \delta \preceq_{\text{lex}} \delta' \).

7. Combining 4 and 6, we have \( \delta' = \delta \).

8. Therefore, \( \text{mult}(H) = \tilde{\delta} \).

\[\square\]

5 Multi-polynomial subresultants in terms of coefficients

A natural question is how to express \( S_\delta \) in terms of coefficients. In this section, we give three different determinantal expressions of \( S_\delta \) in terms of coefficients whose explicit forms are presented in Subsection 5.1. The three expressions are extensions of Sylvester-type subresultant, Barnett-type subresultant and Bézout subresultant, respectively. The proof ideas and techniques are elaborated in the remaining three subsections.
5.1 Main results

We begin by recalling a few basic notions/notations.

Notation 25.

1. The companion matrix of $A = a_n x^n + \cdots + a_0$ is given by
   
   $$
   \begin{bmatrix}
   0 & -a_0/a_n \\
   1 & -a_1/a_n \\
   \vdots \\
   1 & -a_{n-1}/a_n 
   \end{bmatrix}
   $$

2. The Bézout matrix $M$ of $A, B \in \mathbb{C}[x]$ is such that
   
   $$
   \begin{bmatrix}
   A(x) & A(y) \\
   B(x) & B(y)
   \end{bmatrix}
   \begin{bmatrix}
   x^{-1} \\
   \vdots \\
   x^0
   \end{bmatrix}
   $$
   
   where $\ell = \max(\deg A, \deg B)$.

3. $X_{\delta,h} = \begin{bmatrix} x \\ -1 \\ \vdots \\ \vdots \\ x \\ -1 \end{bmatrix}$ \hspace{1cm} \text{h rows}

   \hspace{1cm} \text{d}_0-(\delta_1+\cdots+\delta_t) \text{ columns}

To give the expression of $S_\delta$ in coefficients, we first extend the three well known subresultant matrices for two polynomials to those for arbitrary number of polynomials.

Definition 26 (Extended subresultant matrices). The extended subresultant matrices of Sylvester/Barnett/Bézout type are given by:

1. $M^{\text{Sylvester}}_\delta = \begin{bmatrix} R_{01} & \cdots & R_{0\delta_0} & \cdots & \cdots & R_{t1} & \cdots & R_{t\delta_t} & X_{\delta,d_0+\delta_0} \end{bmatrix}^T \in \mathbb{R}^{(d_0+\delta_0) \times (d_0+\delta_0)}$
   
   where $R_{ij} \in \mathbb{R}^{(d_0+\delta_0) \times 1}$ such that $R_{ij,k} = \text{coeff } (x^{j-1} F_i, x^{k-1})$ if $\delta_0 = \max((\delta_1+d_1)-d_0, \ldots, (\delta_t+d_t)-d_0, 1-|\delta|) \geq 0$.

2. $M^{\text{Barnett}}_\delta = \begin{bmatrix} R_{11} & \cdots & R_{1\delta_1} & \cdots & \cdots & R_{t1} & \cdots & R_{t\delta_t} & X_{\delta,d_0} \end{bmatrix}^T \in \mathbb{R}^{d_0 \times d_0}$
   
   where $R_{ij} \in \mathbb{R}^{d_0 \times 1}$ is the $j$-th column of $F_i(C_0)$ where $C_0$ is the companion matrix of $F_0$.

3. $M^{\text{Bézout}}_\delta = \begin{bmatrix} R_{11} & \cdots & R_{1\delta_1} & \cdots & \cdots & R_{t1} & \cdots & R_{t\delta_t} & X_{\delta,d_0} \end{bmatrix}^T \in \mathbb{R}^{d_0 \times d_0}$
   
   where $R_{ij} \in \mathbb{R}^{d_0 \times 1}$ is the $j$-th column of the Bézout matrix of $F_0$ and $F_i$. Note that it is defined only when $\deg F_0 \geq \deg F_i$ for every $i$. Otherwise $R_{ij}$ may have different size.

Then we have:

Theorem 27 (Subresultant polynomials in coefficients). Let $\delta \neq (0, \ldots, 0)$. Then

1. $S_\delta(F) = \begin{cases} 
   c \det M^{\text{Sylvester}}_\delta, & \text{if } \delta_0 \geq 0, \\
   0, & \text{otherwise}
   \end{cases}$
   
   where $c = (-1)^{d_0 \delta_0}$.

2. $S_\delta(F) = c \det M^{\text{Barnett}}_\delta$ where $c = a_{0d_0}^{\delta_0}$.
3. \( S_\delta(F) = c \det M^B_\delta \) where \( c = a^\delta_{\delta_0} \).

**Remark 28.** When \( \delta = (0, \ldots, 0) \), Definition 2 immediately implies \( S_\delta(F) = a^\delta_{\delta_0} F_0 \) where \( \delta_0 = \max(d_1 - d_0, \ldots, d_4 - d_0, 1) \).

**Example 29.** Let

\[ F = (F_0, F_1, F_2) = (a_{04}x^4 + a_{03}x^3 + a_{02}x^2 + a_{01}x + a_{00}, a_{13}x^3 + a_{12}x^2 + a_{11}x + a_{10}, a_{22}x^2 + a_{21}x + a_{20}) \]

and \( \delta = (2, 1) \). Then

\[ \delta_0 = \max(\max(\delta_1 + d_1, \delta_2 + d_2) - d_0, 0) = \max(\max(5, 3) - 4, 0) = 1 \]

Thus we have

\[
M^Sylvester_\delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
M^Barnett_\delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
M^B\varepsilonout_\delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

From the above three theorems, we have

\[ S_\delta(F) = \det M^Sylvester_\delta = a^0_{04} \det M^Barnett_\delta = a^{-2}_{04} \det M^B\varepsilonout_\delta \]

### 5.2 Proof of Theorem 27-1 (Sylvester-type)

Here is a high level view of the proof. We start with converting \( S_\delta \) into an equivalent expression which is easier to be connected with the expression in coefficients. The equivalent expression will also be used for proving the Barnett-type and Bézout-type expressions. Then several proof techniques from [12, 25] are adapted. More specifically, we first multiply the extended Sylvester matrix by the following object:

\[
\begin{bmatrix}
\alpha_1^0 & \cdots & \alpha_{d_0}^0 \\
\vdots & & \vdots \\
\alpha_1^{d_0-1} & \cdots & \alpha_{d_0}^{d_0-1} \\
\alpha_1^{d_0} & \cdots & \alpha_{d_0}^{d_0} \\
\vdots & & \vdots \\
\alpha_1^{d_0+\delta_0-1} & \cdots & \alpha_{d_0}^{d_0+\delta_0-1}
\end{bmatrix}
\]

whose determinant is the same as \( V(\alpha_1, \ldots, \alpha_{d_0}) \), the denominator of \( S_\delta \). Next we repeatedly rewrite the determinant by using multi-linearity and anti-symmetry of determinants and eventually achieve the result of Theorem 27-1.
Lemma 30. We have

\[
S_\delta = a_{d_0}^0 \cdot \det \begin{bmatrix}
\alpha_1^0 F_1 (\alpha_1) & \cdots & \alpha_{d_0}^0 F_1 (\alpha_{d_0}) \\
\vdots & & \vdots \\
\alpha_1^{d_1-1} F_1 (\alpha_1) & \cdots & \alpha_{d_0}^{d_1-1} F_1 (\alpha_{d_0}) \\
\vdots & & \vdots \\
\alpha_1^{d} (x - \alpha_1) & \cdots & \alpha_{d_0}^{d} (x - \alpha_{d_0}) \\
\end{bmatrix} / V(\alpha_1, \ldots, \alpha_{d_0})
\]

Proof. The lemma follows from the derivation below:

1. Let \( N \) be the numerator of the fractional part of \( S_\delta \) (see Definition 2).

2. Note \( N = \det \begin{bmatrix} U \\ L_1 \end{bmatrix} \) where

\[
U = \begin{bmatrix}
\alpha_1^0 F_1 (\alpha_1) & \cdots & \alpha_{d_0}^0 F_1 (\alpha_{d_0}) \\
\vdots & & \vdots \\
\alpha_1^{d_1-1} F_1 (\alpha_1) & \cdots & \alpha_{d_0}^{d_1-1} F_1 (\alpha_{d_0}) \\
\vdots & & \vdots \\
\alpha_1^{d} (x - \alpha_1) & \cdots & \alpha_{d_0}^{d} (x - \alpha_{d_0}) \\
\end{bmatrix}
\]

\[
L_1 = \begin{bmatrix} \alpha_1^0 & \cdots & \alpha_{d_0}^0 \\ \vdots & \cdots & \vdots \\ \alpha_1^{d-1} & \cdots & \alpha_{d_0}^{d-1} \end{bmatrix}, \quad L_2 = \begin{bmatrix} x^0 \\ \vdots \\ x^{d-1} \end{bmatrix}
\]

3. We will now cancel \( x^{d-1} \) by carrying out the row operation on the last two rows of \( L = [L_1 \ L_2] \) as follows.

\[
N = \det \begin{bmatrix} U \\ V \end{bmatrix} = \det \begin{bmatrix} \alpha_1^0 & \cdots & \alpha_{d_0}^0 & x^0 \\ \vdots & \cdots & \vdots & \vdots \\ -\alpha_1^{d-2} x + \alpha_1^{d-1} & \cdots & -\alpha_{d_0}^{d-2} x + \alpha_{d_0}^{d-1} & -x^{d-2} x - x^{d-1} \\
\end{bmatrix}
\]
4. By repeating the above operation for the other rows of $L$ except the first row, we get

$$N = \det \begin{bmatrix}
U & x^0 \\
\alpha_1^0 & \cdots & \alpha_{d_0}^0 \\
-\alpha_1^0 (x - \alpha_1) & \cdots & -\alpha_{d_0}^0 (x - \alpha_{d_0}) \\
\vdots & \vdots & \vdots \\
-\alpha_1^{\varepsilon-2} (x - \alpha_1) & \cdots & -\alpha_{d_0}^{\varepsilon-2} (x - \alpha_{d_0})
\end{bmatrix}$$

5. Note that there is only one non-zero entry $x^0$ in the last column of $N$ whose row and column indices are $1 + \sum_{i=1}^t \delta_i$ and $d_0 + 1$ respectively. Thus we expand $N$ by the last column and get

$$N = (-1)^{(1 + \sum_{i=1}^t \delta_i) + (d_0 + 1)} \cdot x^0 \cdot \det \begin{bmatrix}
U & x^0 \\
-\alpha_1^0 (x - \alpha_1) & \cdots & -\alpha_{d_0}^0 (x - \alpha_{d_0}) \\
\vdots & \vdots & \vdots \\
-\alpha_1^{\varepsilon-2} (x - \alpha_1) & \cdots & -\alpha_{d_0}^{\varepsilon-2} (x - \alpha_{d_0})
\end{bmatrix}$$

6. Next we extract the factor $-1$ from each of the last $\varepsilon - 1$ row in the resulting determinant and achieve

$$N = (-1)^{(1 + \sum_{i=1}^t \delta_i) + (d_0 + 1)} \cdot (-1)^{\varepsilon - 1} \cdot x^0 \cdot \det \begin{bmatrix}
U & x^0 \\
\alpha_1^0 (x - \alpha_1) & \cdots & \alpha_{d_0}^0 (x - \alpha_{d_0}) \\
\vdots & \vdots & \vdots \\
\alpha_1^{\varepsilon-2} (x - \alpha_1) & \cdots & \alpha_{d_0}^{\varepsilon-2} (x - \alpha_{d_0})
\end{bmatrix}$$

7. Noting that

$$\left(1 + \sum_{i=1}^t \delta_i\right) + (d_0 + 1) + (\varepsilon - 1) = \left(\sum_{i=1}^t \delta_i + \epsilon\right) + (d_0 + 1) = 2(d_0 + 1) \equiv 0 \mod 2$$

and combining $x^0 = 1$, we have

$$N = \det \begin{bmatrix}
U & x^0 \\
\alpha_1^0 (x - \alpha_1) & \cdots & \alpha_{d_0}^0 (x - \alpha_{d_0}) \\
\vdots & \vdots & \vdots \\
\alpha_1^{\varepsilon-2} (x - \alpha_1) & \cdots & \alpha_{d_0}^{\varepsilon-2} (x - \alpha_{d_0})
\end{bmatrix}$$

8. Thus $S_\delta = a_{d_0}^0 \cdot N/V(\alpha_1, \ldots, \alpha_{d_0})$. 

Now combining Lemma 30 and proof techniques borrowed from [12] and [25], we may prove Theorem 27.1.

Proof of Theorem 27.1 (Sylvester-type).

1. Recall

$$M_\delta^{\text{Sylvester}} = \begin{bmatrix}
\Psi_q (x^0 F_0) & \cdots & \Psi_q (x^{\delta_0 - 1} F_0) & \cdots & \cdots & \Psi_q (x^0 F_t) & \cdots & \Psi_q (x^{\delta_t - 1} F_t) & X_{\delta,h}
\end{bmatrix}^T$$

2. Let $d_i = \deg F_i$. Note that $\delta_0 + d_0 \geq \max_{0 \leq i \leq t} (\delta_i + d_i)$. Let $\tilde{d}_i = \delta_0 + d_0 - d_i$. Then $\tilde{d}_0 = d_0$. 

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3. Assume \( F_i = \sum_{0 \leq j \leq d_i} a_{ij} x^j \). Then we may write \( \det M^\text{Sylvester}_\delta \) in the following explicit form.

\[
\begin{vmatrix}
  a_{00} & \cdots & a_{0d_0} \\
  \vdots & \ddots & \vdots \\
  a_{t\delta} & \cdots & a_{t\delta} \\
  x & -1 \\
  \vdots & \ddots & \vdots \\
  x & -1
\end{vmatrix}
\]

\( \delta_0 \text{ rows} \)

\[
\begin{vmatrix}
  \alpha_1^\delta & \cdots & \alpha_d^\delta \\
  \vdots & \ddots & \vdots \\
  \alpha_1^{d_\delta-1} & \cdots & \alpha_d^{d_\delta-1} \\
  \alpha_1^{d_\delta} & \cdots & \alpha_d^{d_\delta} \\
  \alpha_1^{d_\delta+\delta-1} & \cdots & \alpha_d^{d_\delta+\delta-1} \\
\end{vmatrix}
\]

\( \delta_1 \text{ rows} \)

\( \epsilon - 1 \text{ rows} \)

4. Note that \( F_0 = a_{d_0} \prod_{i=1}^{d_0} (x - \alpha_i) \). Here we view \( \alpha_1, \ldots, \alpha_{d_0} \) as distinct indeterminates. Thus

\[ V = V(\alpha_1, \ldots, \alpha_{d_0}) \neq 0 \]

5. In what follows, we show that the product of \( \det M^\text{Sylvester}_\delta \) and \( V \) is proportional to the numerator of the equivalent expression of \( S_\delta \) in Lemma \[10\].

(a) We enlarge the size of \( V \) so that it can match the size of \( M^\text{Sylvester}_\delta \) while the determinant keeps unchanged.

\[
\begin{vmatrix}
  a_{00} & \cdots & a_{0d_0} \\
  \vdots & \ddots & \vdots \\
  a_{t\delta} & \cdots & a_{t\delta} \\
  x & -1 \\
  \vdots & \ddots & \vdots \\
  x & -1
\end{vmatrix}
\]

\( \det M^\text{Sylvester}_\delta \cdot V = \det
\begin{vmatrix}
  \alpha_1^\delta & \cdots & \alpha_d^\delta \\
  \vdots & \ddots & \vdots \\
  \alpha_1^{d_\delta-1} & \cdots & \alpha_d^{d_\delta-1} \\
  \alpha_1^{d_\delta} & \cdots & \alpha_d^{d_\delta} \\
  \alpha_1^{d_\delta+\delta-1} & \cdots & \alpha_d^{d_\delta+\delta-1} \\
\end{vmatrix}
\]

(b) Since \( \det A \cdot \det B = \det AB \) for square matrices \( A \) and \( B \), we have

\[
\begin{vmatrix}
  a_{00} & \cdots & a_{0d_0} \\
  \vdots & \ddots & \vdots \\
  a_{t\delta} & \cdots & a_{t\delta} \\
  x & -1 \\
  \vdots & \ddots & \vdots \\
  x & -1
\end{vmatrix}
\]

\( \det M^\text{Sylvester}_\delta \cdot V = \det
\begin{vmatrix}
  \alpha_1^\delta & \cdots & \alpha_d^\delta \\
  \vdots & \ddots & \vdots \\
  \alpha_1^{d_\delta-1} & \cdots & \alpha_d^{d_\delta-1} \\
  \alpha_1^{d_\delta} & \cdots & \alpha_d^{d_\delta} \\
  \alpha_1^{d_\delta+\delta-1} & \cdots & \alpha_d^{d_\delta+\delta-1} \\
\end{vmatrix}
\]

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(c) Carrying out matrix multiplication, we get
\[
\begin{pmatrix}
\alpha_1^0 F_0 (\alpha_1) & \ldots & \alpha_{d_0}^0 F'_0 (\alpha_{d_0}) & a_{0d_0} \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{\delta_0-1} F_0 (\alpha_1) & \ldots & \alpha_{d_0}^{\delta_0-1} F_0 (\alpha_{d_0}) & \ldots & a_{0d_0} \\
\alpha_1^0 F_1 (\alpha_1) & \ldots & \alpha_{d_0}^0 F_1 (\alpha_{d_0}) \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{\delta_1-1} F_1 (\alpha_1) & \ldots & \alpha_{d_0}^{\delta_1-1} F_1 (\alpha_{d_0}) & \ldots & a_{1d_1} \\
\alpha_1^0 (x - \alpha_1) & \ldots & \alpha_{d_0}^0 (x - \alpha_{d_0}) \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{\delta_0-2} (x - \alpha_1) & \ldots & \alpha_{d_0}^{\delta_0-2} (x - \alpha_{d_0})
\end{pmatrix}
\]

\[\det M_{\delta}^{\text{Sylvester}} \cdot V = \det \]

(d) Since \(\alpha_1, \ldots, \alpha_{d_0}\) are roots of \(F_0\), the above determinant is simplified:

\[
\begin{pmatrix}
\alpha_1^0 F_0 (\alpha_1) & \ldots & \alpha_{d_0}^0 F'_0 (\alpha_{d_0}) & a_{0d_0} \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{\delta_0-1} F_0 (\alpha_1) & \ldots & \alpha_{d_0}^{\delta_0-1} F_0 (\alpha_{d_0}) & \ldots & a_{0d_0} \\
\alpha_1^0 F_1 (\alpha_1) & \ldots & \alpha_{d_0}^0 F_1 (\alpha_{d_0}) \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{\delta_1-1} F_1 (\alpha_1) & \ldots & \alpha_{d_0}^{\delta_1-1} F_1 (\alpha_{d_0}) & \ldots & a_{1d_1} \\
\alpha_1^0 (x - \alpha_1) & \ldots & \alpha_{d_0}^0 (x - \alpha_{d_0}) \\
\vdots & \ddots & \vdots & \vdots \\
\alpha_1^{\delta_0-2} (x - \alpha_1) & \ldots & \alpha_{d_0}^{\delta_0-2} (x - \alpha_{d_0})
\end{pmatrix}
\]

\[\det M_{\delta}^{\text{Sylvester}} \cdot V = \det \]

(e) Note that in the first \(\delta_0\) rows of the righthand side determinant, there is only one nonzero minor of order \(\delta_0\). The entries of this minor lie in the intersections of the first \(\delta_0\) rows and the last \(\delta_0\) columns (i.e., \((d_0 + 1)\)th, \((d_0 + 2)\)th, \ldots, \((d_0 + \delta_0)\)th columns) and obviously the minor has a diagonal structure. Thus after taking Laplace expansion for the determinant, we get

\[\det M_{\delta}^{\text{Sylvester}} \cdot V\]
Here is a high level view of the proof. The main technical challenge is how to connect the extended Barnett matrix with the roots of the first polynomial. For this purpose, we multiply the extended Barnett matrix with the Vandermonde matrix of $\alpha_1, \ldots, \alpha_{d_0}$. Observing that the companion matrix of a univariate polynomial $A(x)$ of degree $n$ represents the endomorphism of $\mathbb{C}_{<n}[x]$ defined by the multiplication of $x$ (see Lemma 31) and generalizing it to the multiplication of an arbitrary univariate polynomial (see Lemma 32), we may convert the product into an expression in terms of roots, which is exactly the numerator of $S_\delta$ in Lemma 30. We start the proof with recalling the following well-known lemma.

**Lemma 31.** Let $A = \sum_{i=0}^{n-1} a_i x^i$ and $C_A$ be as in Notation 22. Then

\[
\bar{x} x \equiv_A \bar{x} C_A \quad \text{where} \quad \bar{x} = \begin{bmatrix} x^0 & \cdots & x^{n-1} \end{bmatrix}
\]
When $x$ is generalized to a univariate polynomial $H$, Lemma \ref{lem:1} can be generalized to the following Lemma \ref{lem:2} which is the essential ingredient for building up the connection between the extended Barnett matrix and the roots of $A$.

**Lemma 32.** $\bar{x}H(C_A) \equiv_A \bar{x}H$.

**Proof.** It is easy to verify that

$$\bar{x}H(C_A) = \bar{x} \sum_{j \geq 0} h_j C_A^j = \sum_{j \geq 0} h_j \bar{x} C_A^j \equiv_A \sum_{j \geq 0} h_j \bar{x} x^j = \bar{x} \left( \sum_{j \geq 0} h_j x^j \right) = \bar{x}H$$

\hfill $\Box$

**Remark 33.** When $x$ is specialized to a root of $A$, say $\alpha_i$, we get $\bar{\alpha}_i H(C_A) = \bar{\alpha}_i H(\alpha_i)$ where $\bar{\alpha}_i = (\alpha_i^0, \ldots, \alpha_i^{n-1})$.

Now we are ready to prove Theorem 27-2.

**Proof of Theorem 27-2 (Barnett-type).**

1. Specialize $A$, $H$ and $x$ in Lemma \ref{lem:2} with $F_0$, $F_k$ and $\alpha_i$, respectively, where $\alpha_i$'s are the roots of $F_0$. Again let $\bar{\alpha}_i = (\alpha_i^0, \ldots, \alpha_i^{d_0 - 1})$. Then by Lemma \ref{lem:2} and Remark \ref{rem:33} we get

$$\bar{\alpha}_i F_k(C_{F_0}) = \bar{\alpha}_i F_k(\alpha_i)$$

2. Recall that $R_{kj}$ is the $j$-th column of $F_k(C_{F_0})$. Thus selecting the $j$-th column on both sides in the above equation, we have

$$\bar{\alpha}_i R_{kj} = \alpha_i^{j-1} F_k(\alpha_i). \tag{6}$$

3. Assembling $\bar{\alpha}_i$ ($1 \leq i \leq d_0$) vertically and $R_{kj}$ ($1 \leq j \leq \delta_k$) horizontally, we obtain two matrices, denoted by $\bar{\alpha}$ and $R_k$, i.e,

$$\bar{\alpha} = \begin{bmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_{d_0} \end{bmatrix}, \quad R_k = \begin{bmatrix} R_{k1} & \ldots & R_{k\delta_k} \end{bmatrix}$$

4. Multiplying $\bar{\alpha}$ and $R_k$, we obtain

$$\bar{\alpha} R_k = \begin{bmatrix} \bar{\alpha}_1 \\ \vdots \\ \bar{\alpha}_{d_0} \end{bmatrix} \begin{bmatrix} R_{k1} & \ldots & R_{k\delta_k} \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_1 R_{k1} & \ldots & \bar{\alpha}_1 R_{k\delta_k} \\ \vdots & \ddots & \vdots \\ \bar{\alpha}_{d_0} R_{k1} & \ldots & \bar{\alpha}_{d_0} R_{k\delta_k} \end{bmatrix}$$

5. Substituting (6) into the resulting matrix yields

$$\bar{\alpha} R_k = \begin{bmatrix} \alpha_1^0 F_k(\alpha_1) & \ldots & \alpha_1^{\delta_k-1} F_k(\alpha_1) \\ \vdots & \ddots & \vdots \\ \alpha_{d_0}^0 F_k(\alpha_{d_0}) & \ldots & \alpha_{d_0}^{\delta_k-1} F_k(\alpha_{d_0}) \end{bmatrix}$$
6. Next we simplify $\bar{\alpha}X_{\delta,d_0}$. Recall that

$$X_{\delta,d_0} = \begin{bmatrix} x \\ -1 \\ \vdots \\ \vdots \\ \vdots \\ x \\ -1 \end{bmatrix}_{d_0 \text{ rows}}$$

$$d_0-(\delta_1+\cdots+\delta_t)=\varepsilon-1 \text{ columns}$$

By calculation, we have

$$\bar{\alpha}X_{\delta,d_0} = \begin{bmatrix} 0 \alpha_1 & \cdots & \alpha_{d_0-1} \\ \vdots & \vdots & \vdots \\ 0 \alpha_{d_0} & \cdots & \alpha_{d_0-1} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ \vdots \\ \vdots \\ \vdots \\ x \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 x - \alpha_1 & \cdots & \alpha_{d_0-2} x - \alpha_{d_0-1} \\ \vdots & \vdots & \vdots \\ \alpha_{d_0-1} x - \alpha_1 & \cdots & \alpha_{d_0-1} x - \alpha_{d_0-1} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 (x - \alpha_1) & \cdots & \alpha_{d_0-2} (x - \alpha_1) \\ \vdots & \vdots & \vdots \\ \alpha_{d_0-1} (x - \alpha_1) & \cdots & \alpha_{d_0-2} (x - \alpha_1) \end{bmatrix} \quad (7)$$

7. Now assembling $R_k \ (1 \leq k \leq t)$ and $X_{\delta,d_0}$ horizontally and pre-multiplying the resulting matrix by $\bar{\alpha}$, we get

$$\text{LHS} = \bar{\alpha} \begin{bmatrix} R_1 & \cdots & R_t & X_{\delta,d_0} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\alpha} R_1 & \cdots & \bar{\alpha} R_t & \bar{\alpha} X_{\delta,d_0} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1^0 F_1(\alpha_1) & \cdots & \alpha_1^{d_1-1} F_1(\alpha_1) \\ \vdots & \vdots & \vdots \\ \alpha_{d_0}^0 F_1(\alpha_{d_0}) & \cdots & \alpha_{d_0}^{d_0-1} F_1(\alpha_{d_0}) \end{bmatrix} \begin{bmatrix} \alpha_1^0 F_1(\alpha_1) & \cdots & \alpha_1^{d_1-1} F_1(\alpha_1) \\ \vdots & \vdots & \vdots \\ \alpha_{d_0}^0 F_1(\alpha_{d_0}) & \cdots & \alpha_{d_0}^{d_0-1} F_1(\alpha_{d_0}) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1^0 (x - \alpha_1) & \cdots & \alpha_1^{d_1-2} (x - \alpha_1) \\ \vdots & \vdots & \vdots \\ \alpha_{d_0}^0 (x - \alpha_{d_0}) & \cdots & \alpha_{d_0}^{d_0-2} (x - \alpha_{d_0}) \end{bmatrix}$$

$$= \text{RHS}$$

8. Taking transpose and computing the determinants for the lefthand and righthand side expressions yield

$$\det \text{LHS} = \det \left( \begin{bmatrix} R_1 & \cdots & R_t & X_{\delta,d_0} \end{bmatrix}^T \cdot \bar{\alpha}^T \right) = \det (M_j^{\text{Barnett}}) \cdot V$$

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Lemma 35. Then we may establish the following lemmas, which will be needed for the proof of Theorem 27-3.

The substitution of $y = \alpha_i$ into the left and right sides of the above expression yields

$$\det \text{RHS} = \det \begin{bmatrix} F_1(\alpha_1)\alpha_1^0 & \cdots & F_1(\alpha_{d_0})\alpha_{d_0}^0 \\ \vdots & \ddots & \vdots \\ F_1(\alpha_1)\alpha_1^{d_1-1} & \cdots & F_1(\alpha_{d_0})\alpha_{d_0}^{d_1-1} \\ \vdots & \ddots & \vdots \\ F_1(\alpha_1)\alpha_1^{d_2-1} & \cdots & F_1(\alpha_{d_0})\alpha_{d_0}^{d_2-1} \\ \alpha_1^n(x - \alpha_1) & \cdots & \alpha_{d_0}^n(x - \alpha_{d_0}) \\ \vdots & \ddots & \vdots \\ \alpha_1^{\varepsilon-2}(x - \alpha_1) & \cdots & \alpha_{d_0}^{\varepsilon-2}(x - \alpha_{d_0}) \end{bmatrix} = a_{\delta_{0d_0}}^{-d_0} \cdot V S_5$$

9. Finally we have $S_5 = a_{\delta_{0d_0}}^{-d_0} \cdot \det M_0^\text{Barnett}$.

5.4 Proof of Theorem 27-3 (Bézout type)

Here is a high level view of the proof. First, by multiplying the extended Bézout matrix with the Vandermonde matrix of $\alpha_1, \ldots, \alpha_{d_0}$ and exploring the connection between the Bézout matrix and roots of the involved polynomials, we convert the product into an expression in terms of roots. Then we repeatedly rewrite the determinant of the obtained product matrix by using multi-linearity and anti-symmetry of determinants and achieve the result. We start with introducing some notations for constructing Bézout matrices.

Notation 34.

- $A = \sum_{i=0}^n a_i x^i = a_n \prod_{i=1}^n (x - \alpha_i)$ and $B \in \mathbb{C}[x]$ such that $\deg A \geq \deg B$
- $\bar{\alpha}_i = (\alpha_i^0, \ldots, \alpha_i^{n-1})$
- $M$ is the Bézout matrix of $A$ and $B$
- $M_j$ is the $j$-th column of $M$
- $e_j$ is the $j$-th elementary symmetric polynomials in $\alpha_1, \ldots, \alpha_n$ (with $e_0 := 1$)
- $e_j^{(i)}$ is the $j$-th elementary symmetric polynomials in $\bar{\alpha}_i, \ldots, \alpha_n$ (with $e_0^{(i)} := 1$).

Then we may establish the following lemmas, which will be needed for the proof of Theorem 27-3.

Lemma 35. $\bar{\alpha}_i M_j$ is the coefficient of $a_n B(\alpha_i) \prod_{k \neq i} (x - \alpha_k)$ for the term $x^{n-j}$, i.e.,

$$\bar{\alpha}_i M_j = a_n B(\alpha_i) \cdot (-1)^{j-1} e_j^{(i)}$$

Proof. By the definition of Bézout matrix,

$$\det \begin{bmatrix} A(x) & A(y) \\ B(x) & B(y) \end{bmatrix} = \det \begin{bmatrix} A(x) - A(y) & A(y) \\ B(x) - B(y) & B(y) \end{bmatrix} = \begin{bmatrix} y^0 & \cdots & y^{n-1} \end{bmatrix} \begin{bmatrix} x^{n-1} \\ \vdots \\ x^0 \end{bmatrix}$$

The substitution of $y = \alpha_i$ into the left and right sides of the above expression yields
1. LHS = \[ \begin{bmatrix} \frac{A(x) - A(\alpha_i)}{x - \alpha_i} & 0 \\ \frac{B(x) - B(\alpha_i)}{x - \alpha_i} & B(\alpha_i) \end{bmatrix} \begin{bmatrix} x - \alpha_i \\ A(x) - A(\alpha_i) \\ B(x) - B(\alpha_i) \end{bmatrix} = B(\alpha_i) \cdot \frac{A(x) - A(\alpha_i)}{x - \alpha_i} = B(\alpha_i) \cdot \frac{A(x)}{x - \alpha_i} = a_n B(\alpha_i) \prod_{k \neq i} (x - \alpha_k) \]

2. RHS = \[ \bar{\alpha}_i [ M_1 \cdots M_n ] \begin{bmatrix} x^{n-1} \\ \vdots \\ x^0 \end{bmatrix} = [ \bar{\alpha}_i M_1 \cdots \bar{\alpha}_i M_n ] \begin{bmatrix} x^{n-1} \\ \vdots \\ x^0 \end{bmatrix} = \sum_{j=0}^{n-1} \bar{\alpha}_i M_{n-j} x^j \]

Comparing the coefficients of \( x^{n-j} \) in the two expressions, we get
\[
\bar{\alpha}_i M_j = a_n B(\alpha_i) \cdot (-1)^{j-1} e_{j-1}^{(i)}
\]

Next we explore the connection between \( e_j^{(i)} \) and \( e_k \) (1 ≤ k ≤ n).

**Lemma 36.** We have
\[
e_j^{(i)} = \sum_{k=0}^{j} (-1)^k e_{j-k} \alpha_i^k, \quad i = 1, \ldots, n, \quad j = 0, \ldots, n - 1
\]

**Proof.** The proof is given in an inductive way for \( j \) based on the observation \( e_j^{(i)} = e_j - \alpha_i e_{j-1}^{(i)} \) when \( j > 0 \).

- When \( j = 0 \), it is easy to verify that \( LHS = e_0^{(i)} = e_0 = (-1)^0 e_0 \alpha_i^0 = RHS \).
- Assume that the equation holds for \( j < 1 \). Then
\[
e_j^{(i)} = e_j - \alpha_i e_{j-1}^{(i)}
\]
\[
= e_j - \alpha_i \sum_{k=0}^{j-1} (-1)^k e_{j-1-k} \alpha_i^k
\]
\[
= e_j + \sum_{k=0}^{j-1} (-1)^{k+1} e_{j-(k+1)} \alpha_i^{k+1}
\]
\[
= e_j + \sum_{k=1}^{j} (-1)^k e_{j-k} \alpha_i^k
\]
\[
= (-1)^0 e_{j-0} \alpha_i^0 + \sum_{k=1}^{j} (-1)^k e_{j-k} \alpha_i^k
\]
\[
= \sum_{k=0}^{j} (-1)^k e_{j-k} \alpha_i^k
\]

Now we are ready to prove Theorem 27-3.

**Proof of Theorem 27-3 (Bézout-type).**

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1. Recall that $\bar{a}_i = (a^0_i, \ldots, a^{d_0-1}_i)$ and $R_{kj}$ is the $j$th column of the Bézout matrix of $F_0$ and $F_k$. Again we assemble $\bar{a}_i$ ($1 \leq i \leq d_0$) vertically and $R_{kj}$ ($1 \leq j \leq \delta_k$) horizontally and obtain two matrices as follows:

\[
\bar{a} = \begin{bmatrix}
\bar{a}_1 \\
\vdots \\
\bar{a}_{d_0}
\end{bmatrix}, \quad R_k = \begin{bmatrix}
R_{k1} & \ldots & R_{k\delta_k}
\end{bmatrix}
\]

2. Now multiplying $\bar{a}$ and $R_k$ and simplifying the resulting expression by Lemma 35, we get

\[
\bar{a}R_k = \begin{bmatrix}
\bar{a}_1 \\
\vdots \\
\bar{a}_{d_0}
\end{bmatrix} \begin{bmatrix}
R_{k1} & \ldots & R_{k\delta_k}
\end{bmatrix}
= \begin{bmatrix}
\bar{a}_1 R_{k1} & \ldots & \bar{a}_1 R_{k\delta_k} \\
\vdots & \ddots & \vdots \\
\bar{a}_{d_0} R_{k1} & \ldots & \bar{a}_{d_0} R_{k\delta_k}
\end{bmatrix}
= \begin{bmatrix}
a_0\delta_0 F_k(\alpha_1) e_0^{(1)} & \ldots & (-1)^{\delta_k-1} a_0\delta_0 F_k(\alpha_1) e_0^{(1)} \\
\vdots & \ddots & \vdots \\
a_0\delta_0 F_k(\alpha_{d_0}) e_0^{(d_0)} & \ldots & (-1)^{\delta_k-1} a_0\delta_0 F_k(\alpha_{d_0}) e_0^{(d_0)}
\end{bmatrix}
\]

3. Recall (7), i.e.,

\[
\bar{a}X_{\delta,d_0} = \begin{bmatrix}
\alpha^0_0 (x-\alpha_1) & \ldots & \alpha^{\varepsilon-2}_1 (x-\alpha_1) \\
\vdots & \ddots & \vdots \\
\alpha^0_{d_0} (x-\alpha_{d_0}) & \ldots & \alpha^{\varepsilon-2}_{d_0} (x-\alpha_{d_0})
\end{bmatrix}
\]

4. Now assembling $R_k$ ($1 \leq k \leq t$) and $X_{\delta,d_0}$ horizontally and pre-multiplying the resulting matrix by $\bar{a}$, we get

\[
\text{LHS} = \bar{a} \begin{bmatrix}
R_1 & \ldots & R_t & X_{\delta,d_0}
\end{bmatrix}
= \begin{bmatrix}
\bar{a}R_1 & \ldots & \bar{a}R_t & \bar{a}X_{\delta,d_0}
\end{bmatrix}
= \begin{bmatrix}
(-1)^0 a_0\delta_0 F_l(\alpha_1) e_0^{(1)} & \ldots & (-1)^{\delta_1-1} a_0\delta_0 F_l(\alpha_1) e_0^{(1)} & \ldots & \ldots \\
\vdots & \ddots & \vdots & \ddots & \ddots \\
(-1)^0 a_0\delta_0 F_l(\alpha_{d_0}) e_0^{(d_0)} & \ldots & (-1)^{\delta_1-1} a_0\delta_0 F_l(\alpha_{d_0}) e_0^{(d_0)} & \ldots & \ldots \\
\vdots & \ddots & \vdots & \ddots & \ddots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(-1)^0 a_0\delta_0 F_l(\alpha_{d_0}) e_0^{(d_0)} & \ldots & (-1)^{\delta_1-1} a_0\delta_0 F_l(\alpha_{d_0}) e_0^{(d_0)} & \ldots & \ldots \\
\end{bmatrix}
= \text{RHS}
\]

5. Taking transpose and computing the determinants for the left-hand and right-hand side expressions yield

\[
\det \text{LHS} = \det \left( \begin{bmatrix}
R_1 & \ldots & R_t & X_{\delta,d_0}
\end{bmatrix}^T \cdot \bar{a}^T \right) = \det M_{\delta}^{\text{Bézout}} . V
\]

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6. In what follows, we rewrite \( \text{det} \, \text{RHS} \) by using the multi-linearity and anti-symmetry properties of determinant.

(a) Extract the factor \((-1)^{k_i} a_{0d_0}\) from the \(i\)th row \((1 \leq i \leq \delta_1 + \cdots + \delta_t)\) where \(k_i\) is some natural number depending on \(i\).

\[
\text{det} \, \text{RHS} = \sum_{i=1}^{t} \sum_{j=1}^{\delta_i} \alpha_i^{(1)} \cdot \ldots \cdot \alpha_i^{(d_0)} \cdot \det
\]

(b) Substitute \(\Box\) into the above determinant and we get

\[
\text{det} \, \text{RHS} = (-1)^{\sum_{i=1}^{t} (0+\cdots+\delta_i-1)} \cdot a_{0d_0}^{\sum_{i=1}^{t} \delta_i}.
\]
For each block of the form

\[
\begin{array}{cccc}
F_1(\alpha_1)(-1)^0e_0\alpha_1^0 & \cdots & F_1(\alpha_d)(-1)^0e_0\alpha_d^0 \\
\vdots & & \vdots \\
F_1(\alpha_1)\sum_{k=0}^{\delta_1-1}(-1)^ke_{\delta_1-1-k}\alpha_1^k & \cdots & F_1(\alpha_d)\sum_{k=0}^{\delta_d-1}(-1)^ke_{\delta_d-1-k}\alpha_d^k \\
\vdots & & \vdots \\
(-1)^dF_1(\alpha_1)\alpha_1^d & \cdots & (-1)^dF_1(\alpha_d)\alpha_d^d
\end{array}
\]

we subtract the \(\ell\)th row multiplied by \(e_{j-\ell}\) where \(1 \leq \ell \leq j - 1\) successively from the \(j\)th row for \(j = 2, \ldots, \delta_i\).

\[
T_i := \begin{bmatrix}
F_1(\alpha_1)(-1)^0e_0\alpha_1^0 & \cdots & F_1(\alpha_d)(-1)^0e_0\alpha_d^0 \\
F_1(\alpha_1)\sum_{k=0}^{\delta_1-1}(-1)^ke_{\delta_1-1-k}\alpha_1^k & \cdots & F_1(\alpha_d)\sum_{k=0}^{\delta_d-1}(-1)^ke_{\delta_d-1-k}\alpha_d^k \\
\vdots & & \vdots \\
F_1(\alpha_1)\sum_{k=0}^{\delta_1-1}(-1)^ke_{\delta_1-1-k}\alpha_1^k & \cdots & F_1(\alpha_d)\sum_{k=0}^{\delta_d-1}(-1)^ke_{\delta_d-1-k}\alpha_d^k
\end{bmatrix}
\]

substituting \(e_0 = 1\) into the first row

\[
\begin{array}{cccc}
(-1)^0F_1(\alpha_1)\alpha_1^0 & \cdots & (-1)^0F_1(\alpha_d)\alpha_d^0 \\
F_1(\alpha_1)((-1)^0e_1\alpha_1^0 + (-1)^1e_0\alpha_1^1) & \cdots & F_1(\alpha_1)((-1)^0e_1\alpha_1^0 + (-1)^1e_0\alpha_1^1) \\
\vdots & & \vdots \\
F_1(\alpha_1)\sum_{k=0}^{\delta_1-1}(-1)^ke_{\delta_1-1-k}\alpha_1^k & \cdots & F_1(\alpha_d)\sum_{k=0}^{\delta_d-1}(-1)^ke_{\delta_d-1-k}\alpha_d^k
\end{array}
\]

subtracting the first row multiplied by \(e_1\) from the second row and using \(e_0 = 1\)

\[
\begin{array}{cccc}
(-1)^0F_1(\alpha_1)\alpha_1^0 & \cdots & (-1)^0F_1(\alpha_d)\alpha_d^0 \\
(-1)^1F_1(\alpha_1)\alpha_1^1 & \cdots & (-1)^1F_1(\alpha_d)\alpha_d^1 \\
\vdots & & \vdots \\
F_1(\alpha_1)\sum_{k=0}^{\delta_1-1}(-1)^ke_{\delta_1-1-k}\alpha_1^k & \cdots & F_1(\alpha_d)\sum_{k=0}^{\delta_d-1}(-1)^ke_{\delta_d-1-k}\alpha_d^k
\end{array}
\]

now considering the third row and expanding it
8. Next extract the factor \((-1)^{k_i}\) from the \(i\)th row \((1 \leq i \leq \delta_1 + \cdots + \delta_t)\) where \(k_i\) is some natural number.

7. Due to the multi-linear and anti-symmetry properties of determinant, \(\det \text{RHS}\) in Step (6b) keeps unchanged after the transformations. That is,

\[
\det \text{RHS} = (-1)^{\sum_{i=1}^{t} (\alpha_0 + \cdots + \alpha_{\delta_i - 1})} \cdot \sum_{\alpha_{\delta_{0d}} = 0}^{\delta_{1d}} \cdots \cdot \det \begin{bmatrix}
(-1)^0 F_i(\alpha_1) \alpha^0_1 & \cdots & (-1)^0 F_i(\alpha_{\delta_{0}}) \alpha^0_{\delta_{0}} \\
\vdots & \ddots & \vdots \\
(-1)^{\delta_i - 1} F_i(\alpha_1) \alpha^{\delta_i - 1}_1 & \cdots & (-1)^{\delta_i - 1} F_i(\alpha_{\delta_{0d}}) \alpha^{\delta_i - 1}_{\delta_{0d}} \\
\end{bmatrix}
\]

repeating the process and eventually obtaining the following matrix block:

\[
\begin{bmatrix}
(-1)^0 F_i(\alpha_1) \alpha^0_1 & \cdots & (-1)^0 F_i(\alpha_{\delta_{0}}) \alpha^0_{\delta_{0}} \\
\vdots & \ddots & \vdots \\
(-1)^{\delta_i - 1} F_i(\alpha_1) \alpha^{\delta_i - 1}_1 & \cdots & (-1)^{\delta_i - 1} F_i(\alpha_{\delta_{0d}}) \alpha^{\delta_i - 1}_{\delta_{0d}} \\
\end{bmatrix}
\]

subtracting the first two rows multiplied by \(e_2\) and \(e_1\) respectively from the third row and using \(e_0 = 1\)

\[
\begin{bmatrix}
(-1)^0 F_i(\alpha_1) \alpha^0_1 & \cdots & (-1)^0 F_i(\alpha_{\delta_{0}}) \alpha^0_{\delta_{0}} \\
(-1)^1 F_i(\alpha_1) \alpha^1_1 & \cdots & (-1)^1 F_i(\alpha_{\delta_{0}}) \alpha^1_{\delta_{0}} \\
(-1)^2 F_i(\alpha_1) \alpha^2_1 & \cdots & (-1)^2 F_i(\alpha_{\delta_{0}}) \alpha^2_{\delta_{0}} \\
\vdots & \ddots & \vdots \\
F_i(\alpha_1) \sum_{k=0}^{\delta_i - 1} (-1)^k e_{(\delta_i - 1) - k} \alpha^k_1 & \cdots & F_i(\alpha_{\delta_{0d}}) \sum_{k=0}^{\delta_i - 1} (-1)^k e_{(\delta_i - 1) - k} \alpha^k_{\delta_{0d}} \\
\end{bmatrix}
\]

due to the multi-linear and anti-symmetry properties of determinant, \(\det \text{RHS}\) in Step (6b) keeps unchanged after the transformations. That is,
depending on $i$:

$$\det \text{RHS} = (-1)^2 \sum_{i=1}^{t} (0+\cdots+\delta_i-1) \cdot a_{0d_0}^{\delta_i} \cdot \det \begin{bmatrix} F_1(\alpha_1) \alpha_1^0 & \cdots & F_1(\alpha_{d_0}) \alpha_{d_0}^0 \\ \vdots & & \vdots \\ F_1(\alpha_1) \alpha_1^{\delta_i-1} & \cdots & F_1(\alpha_{d_0}) \alpha_{d_0}^{\delta_i-1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ F_t(\alpha_1) \alpha_1^0 & \cdots & F_t(\alpha_{d_0}) \alpha_{d_0}^0 \\ \alpha_1^t (x-\alpha_1) & \cdots & \alpha_{d_0}^t (x-\alpha_{d_0}) \\ \vdots & & \vdots \\ \alpha_1^{\varepsilon-2} (x-\alpha_1) & \cdots & \alpha_{d_0}^{\varepsilon-2} (x-\alpha_{d_0}) \end{bmatrix}$$

9. By Lemma 30

$$\det \text{RHS} = (-1)^2 \sum_{i=1}^{t} (0+\cdots+\delta_i-1) \cdot a_{0d_0}^{\delta_i} \cdot a_{0d_0}^{-\delta_0} \cdot S_\delta \cdot V = a_{0d_0}^{\delta_0+|\delta|} \cdot S_\delta \cdot V$$

Recall that in Step 5, we have shown that

$$\det \text{LHS} = \det M_\delta^{\text{Bézout}} \cdot V$$

10. Finally, comparing the two expressions, we have

$$S_\delta = a_{0d_0}^{\delta_0-|\delta|} \cdot \det M_\delta^{\text{Bézout}}$$

\[33\]

6 Conclusion

In this paper, we proposed a definition of a subresultant for several univariate polynomials in terms of roots. To show that the definition is meaningful and useful, we presented two fundamental applications: parametric gcd and parametric multiplicity. For computation, we gave three different expressions of the subresultants in terms of coefficients. A natural challenge for future work is to generalize further to arbitrary number of multivariate and/or Ore polynomials.

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