Thermodynamics of Multiple Two-body Systems with Long-range Correlation

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Abstract

We aim to study thermodynamics of multiple two-body systems with long-range correlation using non-extensive statistics. Long-range correlation will cause multiple systems in anomalous diffusion. We consider the influence of long-range correlation as a background noise effect on a two-body system. We solve probability and entropy equations of a two-body system to obtain the temperature and distance dependence of the non-extensive parameter. The result shows the long-range correlation changes the system’s entropy and energy. The more strongly is the system bounded, the less its energy is affected by the long-range correlation. Moreover, the anomalous diffusion approaches Brown motion with increasing temperature. This will help to understand how nonlinear field affects thermodynamics of a system.

Key words: non-extensive statistics, q-entropy, long-range correlation

1 Introduction

Two-body system research is an interesting issue in many areas of physics. In material physics, the critical temperature of two-electron system—cooper pair is used to determine the physical property of superconducting material. In high-energy physics, the thermodynamical character of quark-antiquark

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system–mesons is studied to analyze the physics in heavy-ion collisions. So thermodynamical research of two-body systems is significant in many areas of physics.

The two-body system is not isolated. Normally, research objects are composed of large amount of two-body systems, such as the superconducting material composed of many cooper pairs, quark-gluon plasma containing many mesons. The electric force between electrons is a long-range interaction, so that the electrons in the cooper pair will interact with other electrons in other cooper pairs. The two-body systems are not free and independent. Instead, they are long-range correlated. The diffusion of independent two-body systems is Brown motion. The diffusion of long-range correlated systems is anomalous diffusion\(^1\). Anomalous diffusion was found in many systems including ultra-cold atoms\(^2\), Telomeres in the nucleus of cells\(^3\), single particle movements in cytoplasm\(^4\), worm-like micellar solutions\(^5\). Anomalous diffusion was also found in other biological systems, including heartbeat intervals and in DNA sequences\(^6\). Moreover, anomalous diffusion environment is found to have significant effect on a system. The purpose of this paper is to study the thermodynamics of the two-body system with long-range correlation and analyze the thermodynamical effect.

Long-range interactions will induce system’s non-extensiveness\(^7\). Long-range microscopic interactions exhibit singularities at the origin in Boltzmann-Gibbs (B-G) statistics\(^7\). This made B-G statistics an inefficient theory to describe anomalous thermostatistical behaviour. Then, which method can describe long-range interactions issues a challenge to us. Fractal theory exhibits a repeating self-similar pattern at every scale. It is also known as expanding symmetry or evolving symmetry. Non-extensive statistics is a statistical theory by the light of fractal idea which aims to describe non-extensiveness. In non-extensive statistics, the parameter \(q\) describes the non-extensiveness of the system. In the limit of \(q \to 1\) non-extensive statistics comes to B-G statistics. So the entropic index \(q\) in \(S_q\) can be regarded as physical effect on a standard B-G system which causes the system’s non-extensiveness and non-additive entropy. Here, we will use non-extensive statistics to study how anomalous diffusion, which is caused by long-range correlation, influence the thermodynamics of the two-body system.

This paper is organized as the following. Section 2 constitutes an introduction to the non-extensive statistical mechanics theory. Section 3 is dedicated to solve the probability and entropy equations to obtain the entropic index \(q\), \(q'\) and analyse long-range correlation effect on the system energy. In section 4, we will use electrical dipole system as a case study and analyze the numerical result. Finally, we will come to a conclusion and discuss the thermal environment effect on the system.
Formalism in non-extensive statistics

Non-extensive statistical mechanics has been used to describe phenomena in many physical systems: dusty plasmas [8], trapped ions [9], spin-glasses [10], anomalous diffusion [11], material physics, high-energy physics [12]. Its advantage over B-G statistics is rather than exhibiting singularities, non-extensive statistics can solve the problem in long-range interaction and anomalous diffusion.

Non-extensive statistical mechanics is based on the generalized functional form of the entropy [7] (in natural unit with $k = \hbar = 1$)

$$S_q = \frac{1 - \sum_{i=1}^\Gamma p_i^q}{q - 1} \left( \sum_{i=1}^\Gamma p_i = 1; q \in \mathbb{R} \right), \quad (1)$$

where $\Gamma$ is the number of microscopic states. Here the parameter $q$ describes the non-extensiveness of the system.

In classical version, the non-extensive entropy can be expressed as

$$S_q = \frac{1 - \int dx [p(x)]^q}{q - 1} \left( \int dx p(x) = 1 \right). \quad (2)$$

The B-G entropy can be obtained in the limit of $q \to 1$.

Shown in Fig.1 is the non-extensive entropy as a function of probability $p$ at fixed states number $\Gamma = 2$ for typical values of $q$ [7]. We can find that the
non-extensive entropy increases with decreasing $q$ at fixed $p$. In other words, $S_q$ is larger than B-G entropy when $q < 1$, less than B-G entropy when $q > 1$.

It can be straightforwardly verified that for system $A_1$ and $A_2$ if the joint probability satisfies $p_{ij}^{A_1 + A_2} = p_i^{A_1} p_j^{A_2}$, then

$$S_q(A_1 + A_2) = S_q(A_1) + S_q(A_2) + (1 - q) S_q(A_1) S_q(A_2). \quad (3)$$

Due to this property, for $q \neq 1$, $S_q$ is said to be nonadditive.

In classical version, the mean value of a variable (referred to as the q-mean value) in non-extensive statistical mechanics is:

$$\langle x \rangle_q \equiv \int_0^\infty dx x P(x), \quad (4)$$

where $P(x)$ is the escort distribution, which is defined as an arbitrary, possibly fractal, probability distribution. In thermostatistics of multifractals, the probability distribution $p(x)$ is sought on the basis of incomplete knowledge, an occurrent event will induce a set of further probability distribution \[13\]

$$P(x) \equiv \int_0^\infty dx' \left[ p(x') \right]^q / \int_0^\infty dx' \left[ p(x') \right]^q. \quad (5)$$

We can immediately verify that $P(x)$ is also normalized, i.e.,

$$\int_0^\infty dx P(x) = 1. \quad (6)$$

It can be understood in this manner that under some physical effects, the probability, entropy and energy of the system changes from $p(x)$, $S_{B-G}$, and $\langle E \rangle$ with $q = 1$ to $P(x)$, $S_q$, and $\langle E \rangle_q$ with $q \neq 1$, respectively.

If the system is in equilibrium, with principle of maximum entropy, the Fermi-Dirac and Bose-Einstein(escort) mean occupation number distributions could be generalizable as follows\[14\]

$$n_i = \frac{1}{[1 + (q - 1) \beta (\varepsilon_i - \mu)]^{\frac{1}{q - 1}} \pm 1}, \quad (7)$$

where $i$ corresponds for microscopic state, $\beta$ and $\mu$ are effective inverse temperature and chemical potential respectively, and $\pm$ correspond to fermions and bosons respectively. In the favor of the distribution function, it has an
impressively good fitting of high temperature experimental data obtained in electron-positron collisions [15].

3 Thermodynamics of two-body system with long-range correlation

Here the research object is supposed to consist of a large amount of two-body systems. The interaction between the two bodies inside a two-body system is a long-range interaction. This will induce one body not only interacts with the other body inside a two-body system, but also interacts with others in other systems. So the systems are long-range correlated. The long-range correlation among different systems will cause anomalous diffusion[1]. The long-range interaction will make other systems perform work on a system, change its energy and entropy.

If a two-body system is isolated and free, the probability of this system can be written as

$$P = \frac{1}{(2\pi)^6} \int e^{-\beta \varepsilon} d^3 p_1 d^3 p_2 d^3 r_1 d^3 r_2,$$

(8)

where $p_1, p_2, r_1, r_2$ are momentum and position of the two bodies. The energy of the system can be written as $\varepsilon = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V_{\text{pot}}$, where $V_{\text{pot}} = V_{\text{pot}}(r)$ is the long range interaction potential between the two bodies inside the bound system, $r$ is the distance between the two bodies.

Meanwhile, since this system is isolated, according to equal probabilities of the fundamental hypothesis of equilibrium statistical mechanics, the probability in any microscopic state can be expressed as

$$P = \frac{1}{\Gamma},$$

(9)

where $\Gamma$ is the total number of microscopic states of the two-body system.

The mean occupation number distribution of this free two-body system as an explicit function is

$$n = \frac{1}{e^{\beta \varepsilon} - 1},$$

(10)

where $\varepsilon$ is the system’s energy.

However, this two-body system is not free, it long-range interacts with other two-body systems. Then other systems will collide with it, perform work on it and finally reaches equilibrium. The long-range correlation among the systems will induce anomalous diffusion[1]. The long-range correlation will influence
the system and induce probability changes to $P'$, energy changes to $\varepsilon'$, distribution changes to $n'$ and its entropy changes to $S'$.

The energy content of a system consists of the heat which is put into it and the sum of the work performed on it

$$d\varepsilon = dQ + dW. \tag{11}$$

The variation of the system’s energy $d\varepsilon = d(\sum_i P_i \varepsilon_i)$ in Eq. (11) can also be considered as a result of the variation of energy level and the variation of the corresponding possibility.

$$d\varepsilon = \sum_i \varepsilon_i dP_i + \sum_i P_i d\varepsilon_i. \tag{12}$$

The quantity of heat transferred is given by $dQ = \sum_i \varepsilon_i dP_i$\[16\]. A transfer of heat gives rise to a redistribution of the occupation probabilities. Heating increases the populations of the states at higher energies.

Energy change by an input of work is

$$dW = Y dy = dy \sum_i \frac{\partial \varepsilon_i}{\partial y} P_i = \sum_i P_i d\varepsilon_i \tag{13}$$

where $Y$ is the generalized force, $y$ is the external parameter. This means the work input produces a change in the energy eigenvalues but doesn’t change the corresponding probability. So the probability $P_i$, the number of microscopic states $\Gamma$ and mean occupation number $n_i$ remains the same. Long-range correlation can be considered as outside particles performing work on the system, so that it does not change the probability and mean occupation number distribution of the system, but the energy eigenvalue is changed to $\varepsilon'_i$. So the probability and mean occupation number distribution of the two-body system with long-range correlation are\[7\]

$$P' = \frac{(\frac{1}{\Gamma})^{q'}}{\sum_{i=1}^{(\frac{1}{\Gamma})^{q'}}} = \frac{1}{\Gamma} = P = \frac{e^{-\beta \varepsilon}}{(2\pi)^3} \int e^{-\beta \varepsilon d3p_1 d3p_2 d3r_1 d3r_2}; \tag{14}$$

$$n' = \frac{1}{[1 + (q' - 1)\beta \varepsilon']^{q' - 1}} = n = \frac{1}{e^{\beta \varepsilon} - 1}, \tag{15}$$

where $q'$ is the non-extensive parameter of the two-body system in anomalous diffusion, its departure from $q = 1$ reflects the anomalous diffusion effect on the two-body system. If $q' = 1$, this system is in Brown motion.
Using Eq. (1), the entropy of the two-body system with long-range correlation is
\[ S' = 1 - \sum_{i=1}^{\Gamma} P'^{q_i} \] (16)

We expand the function \( P'^{q_i} \) around the initial free state \( q' = 1 \) and terminate the expansion after the quadratic term if \( q' \) is around 1, we obtain
\[
P'^{q_i} = P'^{q_i}|_{q' = 1} + \frac{\partial P'^{q_i}}{\partial q'}|_{q' = 1} (q' - 1) + \frac{1}{2} \frac{\partial^2 P'^{q_i}}{\partial q'^2}|_{q' = 1} (q' - 1)^2 + \cdots
\]
\[ = P + \ln P \cdot P(q' - 1) + \frac{(\ln P)^2 \cdot P(q' - 1)^2}{2} + \cdots. \] (17)

Then we can rewrite the entropy of the two-body system with long-range correlation as
\[ S' = -\ln P - \frac{1}{2} \ln^2 P \cdot (q' - 1) \] (18)

So that,
\[ q' = \frac{-\ln P - S'}{\frac{1}{2} \ln^2 P} + 1. \] (19)

Next, we come to analyse the physics inside the two-body system. If the two bodies (we name them \( a_1 \) and \( a_2 \)) are free, the two bodies should obey B-G statistics. The probability of body \( a_1 \) is,
\[
P_{a_1} = \frac{1}{(2\pi)^3} \int \int \int e^{-\beta \varepsilon_1} d^3p_1 d^3r_1.
\] (20)

where \( \varepsilon_1 \) is body \( a_1 \)'s energy.

For non-relativistic particles, we have \( \varepsilon_1 = p_1^2/2m \). Then we can get
\[
P_{a_1} = \frac{8\pi^3 e^{-\beta p_1^2/2m}}{V(\frac{2\pi m \beta}{\beta})^{3/2}},
\] (21)

where \( V \) is the system's volume, \( m \) is the mass of the particle, and \( \beta \) is the inverse temperature of the system \( 1/T \).

Body \( a_1 \) interacts with \( a_2 \) and is affected by long-range correlations from other systems. These two physical aspects could be considered as an occurrent event, that will induce the \( a_1 \)'s probability change to
\[
P_{a_{1\text{q}}} = \frac{[P_{a_1}(p_1, r_1)]^q}{(2\pi)^3} \int \int [P_{a_1}(p_1, r_1)]^q d^3p_1 d^3r_1 = \frac{8\pi^3 e^{-\beta q p_1^2/2m}}{V(\frac{2\pi m \beta}{\beta})^{3/2}}.
\] (22)
according to Eq. (5) in thermostatistics of multifractals. Then with Eq. (2), the changed entropy \( S_{a_1q} \) can also be obtained.

The calculation of body \( a_2 \) should obey the same rule as \( a_1 \).

The probability of the two-body system with long-range correlation \( P' \) is the joint probability of body \( a_1 \) and \( a_2 \), so

\[
P' = P = P_{a_1q} \cdot P_{a_2q}.
\] (23)

The entropy of the two-body system is the sum of the two bodies’ entropy and the non-extensive part, according to Eq. (3),

\[
S' = S_{a_1q} + S_{a_2q} + (1 - q)S_{a_1q}S_{a_2q}.
\] (24)

For convenience, we consider that the bodies inside the two-body bound system are static (\( p_1 = p_2 = 0 \)). Solving the equation (23), \( q \) can be obtained,

\[
q = \left( \frac{Ve^{-\beta V_{pot}}}{\int e^{-\beta V_{pot}} d^3r} \right)^{1/3}.
\] (25)

With the obtained \( q \) in Eq. (25), one can get body \( a_1 \) and \( a_2 \)’s entropy and the entropy of the two-body system with long-range correlation. Substituting the entropy the two-body system into the Eq. (19) gives the value of \( q' \). With the obtained \( q' \), from Eq. (15), the energy of two-particle system is changed to

\[
\varepsilon_{q'} = \frac{1}{(q' - 1)\beta} \left( e^{\frac{q' - 1}{q'} \beta \varepsilon} - 1 \right)
\approx \sum_{n=1}^{\infty} [(q' - 1)\beta]^{n-1} \left( \frac{\varepsilon}{q'} \right)^n,
\] (26)

due to long-range correlation among the two-body systems.

4 Numerical Results

We take electrical dipole bound system with the point charge \( Q = 20e \) and mass \( M = 1MeV \) as a case study. The interaction potential between the electrical dipole is \( V_{pot} = -\alpha_s/r \) with \( \alpha_s = 1/137 \) in natural unit. Because of long-range electromagnetic force, other electrical dipole bound systems will have weak interactions on the dipole bound system and cause anomalous diffusion.
Fig. 2. The Temperature and the two-body distance dependence of the non-extensive parameter $q$ of the body $a_1$ in the two-body system.

The non-extensive parameter $q$ describes both the attraction and environmental effect on the body inside the two-body system. We obtained the temperature and the two body distance dependence of the non-extensive parameter $q$ of the body $a_1$ in the bound system, which is shown in Fig. 2. Normally, binding decreases a body’s entropy but long-range correlation outside the system will increase it. Here, our result is the competition of the two effects. Mostly the interaction inside the system affects the body in the system more than the environmental noise outside. So it is found that here mostly attraction induces $q > 1$. Considering Eq. (1) and Fig. 1, the non-extensive entropy decreases with increasing $q$, so that it shows the bound state decreases the body’s entropy if compared to the free state $q = 1$. However, when the distance is large and the interaction between the bodies is weak, the environmental noise outside affects more. This induces that $q$ is a bit less than unit and the body $a_1$’s entropy is a little larger than that when $q = 1$. Moreover, the departure of non-extensive parameter $q$ from $q = 1$ (B-G statistics) decreases when increasing the temperature.

The non-extensive parameter $q'$ describes the environmental noise effect (long-range correlation effect with other systems) on the two-body system. Its departure from unit comes from long-range interaction by other similar systems and work performed on it. We neglect the approximation in Eq. (17) and solve Eq. (16), Eq. (24) numerically, then we obtain the temperature and two-body distance dependence of $q'$, which is shown in Fig. 3. It indicates that the inter-
Fig. 3. The Temperature and the two-body distance dependence of the non-extensive parameter $q'$ of the two-body system with long-range correlation.

action inside the system affect the environment. When the distance between the two bodies is small and the binding energy is large, it is found $q' > 1$. This means this environment is a supper-diffusion environment [17]. When the distance becomes larger and the binding energy is smaller, there is strongly overlapping among the similar two-body systems, $q'$ turns to be less than unit, which means the environment is a sub-diffusion one. Secondly, although $q'$ is different at different circumstances, we find that its departure from $q = 1$ (B-G statistics) decreases with the increase of the temperature. That indicates with increasing temperature, thermal motion of the multiple two-body systems rises while the influence of the long-range correlation declines, so that the anomalous diffusion approaches Brown motion.

Fig. 4 illustrates the ratio of the system energy in the anomalous diffusion to the isolated system energy $\varepsilon'/\varepsilon$ as a function of temperature $T$ and the isolated energy $\varepsilon$. It is found that the ratio decreases with the increase of absorption between the two particles inside the two-body system. This indicates that the more strongly the system is bounded, the less it is affected by the anomalous diffusion environment. In order to analyse the validness of our result. We compare it with BCS-BEC crossover picture, which also describes the two-body state with an absorption between the two bodies. It is found that the result is consistent with BCS-BEC crossover [20]. In the BCS phase, the binding inside the cooper pair is weak, the pair size is large, so that there is strongly overlapping among the cooper pairs, the environment affects the cooper pair a lot. In
Fig. 4. The ratio of the system energy in the anomalous diffusion to the isolated system energy $\varepsilon'/\varepsilon$ as a function of temperature $T$ and the isolated energy $\varepsilon$.

In the BEC phase, the binding is strong, the pair size is small, so that preformed pair nearly compose ideal gas, the environment affect the pair very little. We also find that the ratio’s departure from unit decreases with the increase of the temperature, this again indicates that the anomalous diffusion approaches Brown motion with increasing temperature.

5 Conclusion

We use non-extensive statistical mechanics to study the thermodynamics of multiple two-body systems. We solve probability and entropy equations and obtain the non-extensive parameter. We also calculate the ratio of the system energy in the anomalous diffusion to the isolated system energy $\varepsilon'/\varepsilon$ as a function of temperature $T$ and the isolated energy $\varepsilon$. We take an electrical dipole bound system with long-range correlation with the point charge $Q = 20e$, mass $M = 1\text{MeV}$ and the interaction potential $V_{pot} = -\alpha_s/r$ as a case study.

We consider the non-extensive parameter $q$ of body $a_1$ in the two-body system. The result is the competition of two effects, absorption between $a_1, a_2$ and long-range correlation effect outside system, which is considered as environmental noise. Mostly the binding inside the system affects the body more than the environmental noise outside so that it decreases the body’s entropy if compared to the free state. However, when the distance is large and the inter-
action is weak, the environmental noise outside affects more. This induces the particle’s entropy is a little larger than free state. Moreover, the departure of $q$ from $q = 1$ (B-G statistics) decreases when increasing the temperature. The non-extensive parameter $q'$ describes the environmental noise effect (collision effect from other systems which induces anomalous diffusion) on the system. Both the interaction inside the system and environmental temperature affect $q'$. When the system’s binding energy is large and two-body distance is small, $q'$ is larger than unit, the environment is a supper-diffusion environment. In the opposite condition, there is strongly overlapping among the two-body systems, $q' < 1$ and the environment is a sub-diffusion one. Although $q'$ is different at different circumstances, its departure from $q = 1$ (B-G statistics) decreases with the increase of temperature. That indicates with increasing temperature, thermal motion of the multiple two-body systems rises while the influence of the long-range correlation declines, so that the anomalous diffusion approaches Brown motion. We also find that the ratio of the system energy with long-range correlation to the isolated system energy $\varepsilon'/\varepsilon$ decreases with the increase of absorption between the two bodies inside the two-body system. The more strongly the system is bounded, the less it is affected by the long-range correlation outside the two-body system. The ratio’s departure from unit also decreases with increasing temperature, this again indicates that the anomalous diffusion approaches Brown motion with the increase of temperature. The result is consistent with BCS-BEC crossover picture. This will help to understand how nonlinear field affect the system energy.

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