Trees of manifolds as boundaries of spaces and groups

Jacek Świątkowski *

Abstract. We show that trees of manifolds, the topological spaces introduced by Jakobsche, appear as boundaries at infinity of various spaces and groups. In particular, they appear as Gromov boundaries of some hyperbolic groups, of arbitrary dimension, obtained by the procedure of strict hyperbolization. We also recognize these spaces as boundaries of arbitrary Coxeter groups with manifold nerves, and as Gromov boundaries of the fundamental groups of singular spaces obtained from some finite volume hyperbolic manifolds by cutting off their cusps and collapsing the resulting boundary tori to points.

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Introduction

Not many explicit topological spaces are known to be homeomorphic to Gromov boundaries of hyperbolic groups. The list consists of spheres and Sierpiński compacta of arbitrary dimension, Menger compacta in dimensions $d \in \{1, 2, 3\}$, Pontriagin sphere and Pontriagin surfaces (which are 2-dimensional), and some trees of 3-manifolds (which are certain spaces of topological dimension 3). See [PS] for more detailed comments concerning this list. One of the goals of this paper is to extend this list to trees of $n$-manifolds in arbitrary dimension $n$ (for which the methods used in [PS], in the case $n = 3$, are insufficient).

Trees of manifolds have been formally introduced by Włodzimierz Jakobsche in [J], though their idea goes back to Ancel and Siebenmann [AS]. These are certain explicit homogeneous metric compacta, typically not ANR, appearing in abundance in arbitrary finite topological dimension. We describe them in detail in Section 1, and here we only mention that each closed connected topological $n$-manifold $M$ determines uniquely one such space, denoted $X(M)$, which has topological dimension $n$, and which we call the tree of manifolds $M$. Our first result is the following (compare Theorem 5.1 in the text, which is slightly more general).

**Theorem 1.**

1. For any natural $n$, if a closed connected orientable PL $n$-manifold $M$ bounds a compact orientable PL $(n + 1)$-manifold, then the tree of manifolds $X(M)$ is homeomorphic to the Gromov boundary of some hyperbolic group.
2. For any closed connected non-orientable PL manifold $N$ the tree of manifolds $X(N)$ is homeomorphic to the Gromov boundary of some hyperbolic group.

Appropriate hyperbolic groups as in the assertion of Theorem 1 will be constructed in Section 5, using the procedure of strict hyperbolization due to Charney and Davis [CD]. Theorem 1 follows from a more general result, Main Theorem, formulated in Section 2 (and proved in Sections 3 and 4), which concerns boundaries of $(\mathcal{M}, \kappa)$-complexes, i.e. certain

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CAT($\kappa$) pseudomanifolds, $\kappa \leq 0$. Another application of this theorem, discussed in detail in Section 5, is the following correction and extension of the result of H. Fischer [F] (who has studied only the case of right-angled Coxeter groups with orientable manifold nerves).

**Theorem 2.** Let $(W, S)$ be a Coxeter system (not necessarily right angled) whose nerve is a PL triangulation of a closed connected manifold $M$. Then the boundary at infinity of $(W, S)$ (i.e. the visual boundary of the corresponding Davis-Moussong complex for $(W, S)$) is homeomorphic to the following tree of manifolds:

1. $\mathcal{X}(M \# \overline{M})$, if $M$ is orientable, where $M, \overline{M}$ are two oppositely oriented copies of $M$;
2. $\mathcal{X}(M)$, if $M$ is non-orientable.

Proofs of Theorems 1 and 2 (or even slightly more general results) are presented in Section 5. A variation of Main Theorem presented in Section 6 (Theorem 6.2) yields the following result, which applies to a class of hyperbolic groups (having representatives in arbitrary dimension) constructed by L. Mosher and M. Sageev in [MS] (see also [FM]).

**Theorem 3.** Let $M$ be a finite volume complete non-compact hyperbolic $(n+1)$-manifold with toral cusps. Suppose that after removing some open horoball neighbourhoods of all cusps we get a compact $(n+1)$-manifold $M^\circ$ whose euclidean toral boundary components contain no closed geodesics of length $\leq 2\pi$. Let $\Gamma$ be a hyperbolic group which is the fundamental group of a pseudomanifold obtained from $M^\circ$ by collapsing all its boundary components to points. Then the Gromov boundary $\partial\Gamma$ is homeomorphic to the tree of tori $\mathcal{X}(T^n)$.

The proofs of the results in this paper required some new ideas and tools. One of them is a new and more flexible characterization of trees of manifolds as limits of inverse sequences of manifolds, given in Definition 1.1, and established by the author in a separate paper [Sw]. Remarks after Definition 1.1 clarify the novelty of this characterization. Another new ingredient is a more careful or pushed further (than in [DJ], [F] and [FM]) analysis of geodesic projections between concentric spheres in CAT($\kappa$) pseudomanifolds. This analysis allows to approximate projections as above by more regular maps, which in turn allows to understand the boundaries of the corresponding pseudomanifolds.

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1. **Trees of manifolds.**

Trees of manifolds have been introduced by Wlodzimierz Jakobsche in [J]. These are some homogeneous metric compacta, typically not ANR, appearing in abundance in any finite topological dimension. We briefly recall their description, in the original setting of Jakobsche modified and extended as in [Sw]. Although the description is valid for arbitrary topological manifolds, in this paper we make use only of the subclass related to PL manifolds.

**Definition 1.1.** Let $\mathcal{M}$ be a finite family of closed connected $n$-dimensional topological manifolds, either all oriented, or at least one of which is non-orientable. Let

$$\mathcal{J} = (\{X_i : i \geq 1\}, \{\pi_i : i \geq 1\})$$
be an inverse sequence consisting of closed connected topological $n$-manifolds $X_i$ and maps $\pi_i : X_{i+1} \to X_i$. Assume furthermore that if all the manifolds in $\mathcal{M}$ are oriented then all $X_i$ are also oriented. We say that $\mathcal{J}$ is a weak Jakobsche inverse sequence for $\mathcal{M}$ if for all $i \geq 1$ and all $M \in \mathcal{M}$ one can choose finite families $\mathcal{D}^M$ of collared $n$-disks in $X_i$ such that:

1. for each $i \geq 1$ the families $\mathcal{D}^M : M \in \mathcal{M}$ pairwise do not intersect, and the disks in the union $\mathcal{D}_i := \bigcup_{M \in \mathcal{M}} \mathcal{D}^M_i$ are pairwise disjoint;
2. for each $i \geq 1$ the map $\pi_i$ maps the preimage $\pi_i^{-1}(X_i \setminus \bigcup \{\text{int}(D) : D \in \mathcal{D}_i\})$ homeomorphically onto $X_i \setminus \bigcup \{\text{int}(D) : D \in \mathcal{D}_i\}$;
3(a) $X_1$ is homeomorphic to one of the manifolds from $\mathcal{M}$, and if the manifolds in $\mathcal{M}$ are oriented, we require that this homeomorphism respects orientations;
3(b) for each $i \geq 1$, for each $M \in \mathcal{M}$, and for any $D \in \mathcal{D}^M_i$ the preimage $\pi_i^{-1}(D)$ is homeomorphic to $M \setminus \text{int}(\Delta)$, where $\Delta$ is some collared $n$-disk in $M$; furthermore, if the manifolds in $\mathcal{M}$ are oriented, we require that the above homeomorphism respects the orientations induced from $X_{i+1}$ and from $M$;
4. if $i < j$, $D \in \mathcal{D}_i$, $D' \in \mathcal{D}_j$, then $\pi_{i,j}(D') \cap \partial D = \emptyset$, where $\pi_{i,j} := \pi_i \circ \pi_{i+1} \circ \ldots \circ \pi_{j-1}$;
5. for each $i \geq 1$ the family $\{\pi_{i,j}(D) : j \geq i, D \in \mathcal{D}_j\}$ of subsets of $X_i$ is null, i.e. the diameters of these subsets converge to 0; here $\pi_{i,i}$ denotes the identity map on $X_i$;
6. for any $i \geq 1$ and each $M \in \mathcal{M}$ the set $\bigcup_{j=i}^{\infty} \pi_{i,j}(\bigcup \mathcal{D}^M_j)$ is dense in $X_i$.

Remarks.

1. It follows from conditions (1), (2), (3a) and (3b) that each $X_i$ is the connected sum of a family of manifolds each homeomorphic to one of the manifolds in $\mathcal{M}$; moreover, if the manifolds in $\mathcal{M}$ are oriented, the above mentioned homeomorphisms (and the involved operation of connected sum) respect the orientations.
2. In the case when the manifolds in $\mathcal{M}$ are oriented, conditions (1)-(5) in Definition 1.1 basically coincide with conditions (1)-(6) in [J], Section 2, p. 82.
3. Condition (6) in Definition 1.1 implies condition (7) in [J], but it is essentially weaker than the conjunction of conditions (7) and (8) of [J] (except the case when the family $\mathcal{M}$ consists of a single manifold $M$, in which (6) is equivalent to the conjunction of (7) and (8), as it was observed and exploited in [F] and [Z]).
4. Definition 1.1 describes exactly the class of inverse sequences naturally associated to weakly saturated tree systems of manifolds from $\mathcal{M}$, as described in [Sw], Section 3.E. It represents significant, and probably close to optimal, relaxation of the initial set of conditions provided by Jakobsche in [J], for which the inverse limit of the corresponding sequence depends uniquely on $\mathcal{M}$ (see Theorem 1.3 below).

The following result is a reformulation of Corollary 3.E.4 of [Sw].

**Theorem 1.3.** Given $\mathcal{M}$ as in Definition 1.1, any two weak Jakobsche inverse sequences for $\mathcal{M}$ have homeomorphic inverse limits.

**Definition 1.4.** Given $\mathcal{M}$ as in Definition 1.1, denote by $\mathcal{X}(\mathcal{M})$, and call the tree of manifolds from $\mathcal{M}$, the space homeomorphic to the inverse limit of some (and hence any) weak Jakobsche inverse sequence for $\mathcal{M}$.
Remarks.
(1) If $M = \{M\}$, we denote the corresponding space $X(M)$ simply by $X$, and call it the tree of manifolds $M$.
(2) If $M = \{M_1, \ldots, M_k\}$, it is known that the space $X(M)$ is homeomorphic to the tree of manifolds $M_1 \# \ldots \# M_k$, i.e. to the space $X(M_1 \# \ldots \# M_k)$ (see e.g. Corollary 3.E.4 in [Sw]).

As it was shown in [J] and [St], trees of manifolds $X(M)$ are connected homogeneous metric compacta of topological dimension equal to the dimension of the manifolds in $M$. As it was observed in Corollary 3.3 of [PS], trees of manifolds in dimensions $\geq 2$ have no local cut points, and in fact they are Cantor manifolds, i.e. no subsets of topological codimension $\geq 2$ separate them.

Trees of manifolds $X(M)$ of the same topological dimension $n$ can be sometimes distinguished by means of their homotopical or homological invariants, or the shape theoretic invariants. For example, it is known that if $n \geq 3$ then the shape fundamental group $\tilde{\pi}_1(X(M))$ is isomorphic to the inverse limit of the increasing free products $G_1 * G_2 * \ldots * G_k$, where the additional factors in larger products are collapsed to identity while the common factors are mapped to each other through identities, and where groups $G_i$ are the copies of the fundamental groups $\pi_1(M) : M \in M$, each group appearing infinitely often (see [FG]). Obviously, this shape fundamental group is sometimes sufficient to distinguish some trees of manifolds. The general question of classifying trees of manifolds up to homeomorphism remains open.

2. $(M, \kappa)$-pseudomanifolds and Main Theorem.

Let $M$ be a finite collection of (pairwise distinct) closed connected PL manifolds of the same dimension $n$, distinct from the standard PL $n$-sphere, and let $\kappa \in \{0, -1\}$. An $(M, \kappa)$-pseudomanifold is a metric polyhedral complex $X$ of piecewise constant curvature $\kappa$, with finite shapes, and such that

(1) $X$ is $CAT(\kappa)$;

(2) for each point $x \in X$ the link $Lk(x, X)$ (viewed as combinatorial polyhedral complex) is either a PL $n$-sphere, or is PL-homeomorphic to some $M \in M$; furthermore, if all $M \in M$ are oriented, we assume that $X$ is also oriented, and that each homeomorphism $Lk(x, X) \to M$ as above respects the orientations;

(3) for each $M \in M$ the set $\Lambda_M := \{x \in X : Lk(x, X) \cong M\}$ is a net in $X$, i.e. for some $\bar{R} > 0$ each ball of radius $\bar{R}$ in $X$ intersects $\Lambda_M$ (if all $M \in M$ are oriented, the symbol $\cong$ above denotes relation of being PL homeomorphic as oriented manifolds).

Note that, since $X$ above has finite shapes, it is a geodesic metric space, so that it makes sense to speak of the $CAT(\kappa)$ property for it. Moreover, by condition (2), $X$ is automatically an $(n + 1)$-dimensional pseudomanifold. Finite shapes property implies also that each set $\Lambda_M$ as in condition (3) above is discrete. Putting $\Lambda := \bigcup_{M \in M} \Lambda_M$, we get that $\Lambda$, which will be called the singular set of $X$, is also discrete.

We are now ready to state Main Theorem, which is the main result of the paper. Its proof occupies Sections 3 and 4. In Section 5 we show few classes of examples to which this theorem applies, in particular getting the proofs of Theorems 1 and 2 of the introduction.
Main Theorem. The visual boundary $\partial X$ of any $(\mathcal{M}, \kappa)$-pseudomanifold $X$ is homeomorphic to the tree of manifolds $\mathcal{X}(\mathcal{M})$.

3. Proof of Main Theorem.

Fix any point $x_0 \in \Lambda$. Denote the spheres and the balls of radii $R$ in $X$, centered at $x_0$, respectively, by $S_R = \{x \in X : d(x,x_0) = R\}$, $B_R = \{x \in X : d(x,x_0) \leq R\}$ and $B^*_R = \{x \in X : d(x,x_0) < R\}$. By discreteness of $\Lambda$, the set of numbers $\{d(x_0,p) : p \in \Lambda\}$ is also discrete. Order this set into the increasing sequence $R_i : i \geq 0$, with $R_0 = 0$. For each $i \geq 0$ let $p_{i,1}, \ldots, p_{i,k_i}$ be the points of the intersection $S_{R_i} \cap \Lambda$. (In particular, we have $k_0 = 1$ and $p_{0,1} = x_0$.)

For each $i \geq 1$, let $G_i : X \setminus B^*_R \rightarrow S_{R_i}$ be the geodesic projection towards $x_0$, i.e. the map which to any point $x \in X \setminus B^*_R$ associates the unique point $x'$ in the intersection of the sphere $S_{R_i}$ with the geodesic segment $[x,x_0]$. Denote also by $g_i : S_{R_{i+1}} \rightarrow S_{R_i}$ the restriction of $G_i$ to $S_{R_{i+1}}$. Consider the inverse sequence $S = (\{S_{R_i}\}, \{g_i\})$ and recall that

$$\partial X = \lim \leftarrow S$$

(see II.8.5 in [BH], or the comments after Definition (2b.1) in [DJ]). Before getting to the core of the proof, we need few preparatory result.

The next result follows by the arguments in the proof of Proposition (3d.3) in [DJ] (this proposition explicitly deals with the case where $X$ is an $(\mathcal{M},0)$-pseudomanifold and $\mathcal{M}$ consists of homology spheres, but the extension below follows by literally the same proof). Compare also Theorem I.8.4(i) on p. 526 in [Da].

**Lemma 3.1.** Let $X$ be an $(n+1)$-dimensional $(\mathcal{M}, \kappa)$-pseudomanifold. Then each sphere $S_{R}(x,X)$ in $X$ (with positive radius $R$) is a closed $n$-manifold.

The next two preparatory results concern cellular maps between manifolds. Recall that a compact subset of an $n$-manifold $P$ is cellular if it has arbitrarily close neighbourhoods which are collared $n$-disks in $P$. A proper surjective map $f : P \rightarrow Q$ between $n$-manifolds is *collared* if each point preimage of $f$ is a cellular subset of $P$. The following theorem is due to Siebenmann [Si] for $n \geq 5$, Quinn [Qu] for $n = 4$, Armentrout [Ar] for $n = 3$, and Moore [Mo] for $n \leq 2$.

**Theorem 3.2.** Any cellular map $f : P \rightarrow Q$ between topological $n$-manifolds is a near homeomorphism, i.e. it can be approximated by homeomorphisms. More precisely, if $d$ is any metric in $Q$ then for any positive continuous real function $\delta : P \rightarrow R_+$ there is a homeomorphism $h : P \rightarrow Q$ such that for each $p \in P$ we have $d(f(p), h(p)) \leq \delta(p)$.

The proof of the next result is implicitly given inside the proof of Lemma (3b.1) and Theorem (3b.2) in Section 3 of [DJ] (pp. 371–372). Beware that, in view of Lemma 3.1, all the spheres appearing in the statement below happen to be closed manifolds of the same dimension.

**Lemma 3.3.** Given an $(\mathcal{M}, \kappa)$-pseudomanifold $X$, any $R' > R > 0$, and any $x \in X$, let $g : S_{R'}(x,X) \rightarrow S_R(x,X)$ be the geodesic projection towards $x$. If the set $B^*_{R'}(x,X) \setminus B^*_R(x,X)$...
contains no singular point from \( \Lambda \) then \( g \) is a cellular map. In particular, in the notation established at the beginning of this section, for any \( 0 < \epsilon < R_{i+1} - R_i \) the geodesic projection \( \phi_i : S_{R_{i+1}} \rightarrow S_{R_i + \epsilon} \) towards \( x_0 \) is cellular.

Recall that, by a result of M. Brown [Br] (to which we refer as Brown’s Lemma), if we replace the maps \( g_i \) in the inverse sequence \( S \), successively, by sufficiently close maps \( g'_i : S_{R_{i+1}} \rightarrow S_{R_i} \) then the limit of the resulting inverse sequence \( S' = (\{S_{R_i}\}, \{g'_i\}) \) remains unchanged (up to homeomorphism), i.e. \( \lim_{\rightarrow} S' \cong \lim_{\rightarrow} S \). The next proposition describes particularly nice approximations \( g'_i \), as above, of the maps \( g_i \).

**Proposition 3.4.** For all \( i \geq 1 \) there exist maps \( g'_i : S_{R_{i+1}} \rightarrow S_{R_i} \) satisfying the following:

1. \( g'_i \) are so close to \( g_i \) that, according to Brown’s Lemma, we have

\[
\lim_{\rightarrow} S' \cong \lim_{\rightarrow} S \cong \partial X;
\]

2. for each \( i \geq 1 \), putting \( S_{R_i}^\circ = S_{R_i} \setminus \{p_{i,1}, \ldots, p_{i,k_i}\} \), the restriction

\[
g'_i|_{(g'_i)^{-1}(S_{R_i}^\circ)} : (g'_i)^{-1}(S_{R_i}^\circ) \rightarrow S_{R_i}^\circ
\]

is a homeomorphism;

3. for each \( i \geq 1 \) and each \( 1 \leq j \leq k_i \), if \( D_{i,j} \) is a sufficiently small collared \( n \)-disk in \( S_{R_i} \) such that \( p_{i,j} \in \text{int}(D_{i,j}) \) then

\[
(g'_i)^{-1}(D_{i,j}) \cong M \setminus \text{int}(\Delta),
\]

where \( M = \text{Lk}(p_{i,j}, X) \in \mathcal{M} \), and where \( \Delta \) is any collared \( n \)-disk in \( M \);

4. for each \( M \in \mathcal{M} \) and for all \( i \geq 1 \), the images in \( S_{R_i} \) of the points from the set \( \Lambda_M \cap (X \setminus B_{R_i}) \), through appropriate compositions of the maps \( g'_k : k \geq i \), form a dense subset of \( S_{R_i} \).

We postpone the proof of Proposition 3.4 until the next section, and in the remaining part of this section we complete the proof of Main Theorem, using the proposition. We do this by showing that an inverse sequence \( S' = (\{S_{R_i}\}, \{g'_i\}) \) resulting from Proposition 3.4 is a weak Jakobsche inverse sequence for \( \mathcal{M} \). More precisely, we describe families \( \mathcal{D}^M_i \) of disks as required in Definition 1.1, inductively with respect to \( i \).

We start with checking condition (3a) of Definition 1.1, i.e. showing that \( S_{R_i} \) is homeomorphic to one of the manifolds from \( \mathcal{M} \). Here, and later in Section 4, we will need the concept of a cone neighbourhood of a point in a piecewise constant curvature polyhedral complex. Let \( Y \) be a metric polyhedral complex of constant curvature \( k \) equal 1, -1 or 0, with finite shapes. Then for any point \( y \in Y \), and any sufficiently small \( \epsilon > 0 \), the ball \( B_{\epsilon}(y, X) \) isometrically coincides with the ball of the same radius \( \epsilon \) in the metric cone \( \text{Cone}(\text{Lk}(y, X)) \) centered at the cone vertex (see Theorem I.7.16 on p. 103 of [BH], or [Da], p. 505 for the definition of the metric cone, and the proof of Lemma (2d.1) in [DJ] for justification of the above claim). A cone neighbourhood of \( y \) in \( Y \) is any ball \( B_{\epsilon}(y, Y) \) as above. An obvious (and useful) property is that geodesics started at any point of \( Y \) do
not bifurcate inside a cone neighbourhood of this point. It follows that the natural map from the boundary sphere $S_\epsilon(y,Y)$ of any cone neighbourhood, to the link $\text{Lk}(y,Y)$, is a homeomorphism, which we will view as the natural identification of the sphere with the link.

Coming back to checking condition (3a), choose $\epsilon$ so small that $B_\epsilon$ is a cone neighbourhood of $x_0$ in $X$. Then the sphere $S_\epsilon$ is homeomorphic to the link $\text{Lk}(x_0,X)$, i.e. to some manifold $M_0 \in \mathcal{M}$. Applying Lemma 3.3 followed by Theorem 3.2, we get that the sphere $S_{R_1}$ is homeomorphic to $S_\epsilon$, and hence also to $M_0$, as required.

We now turn to describing families $\mathcal{D}_i^M$ of disks as required in Definition 1.1. Fix an auxiliary sequence $\epsilon_i$ of positive real numbers, converging to 0. To start the inductive description, choose a family of pairwise disjoint collared disks $D_{1,j} : 1 \leq j \leq k_1$ in $S_{R_1}$ such that for each $j$ the following conditions hold:

- (d1) $D_{1,j}$ contains the point $p_{1,j}$ in its interior;
- (d2) the diameter of $D_{1,j}$ is less than $\epsilon_1$;
- (d3) the preimage $(g_1')^{-1}(D_{1,j})$ is homeomorphic to $M \setminus \text{int}(\Delta)$, where $M \in \mathcal{M}$ is homeomorphic to the link $\text{Lk}(p_{1,j},X)$ and $\Delta$ is a collared $n$-disk in $M$;
- (d4) denoting by $\Lambda(1)$ the set of images in $S_{R_1}$, through appropriate compositions of the maps $g_1' : l \geq 1$, of all points in $\Lambda \cap (X \setminus B_{R_1})$, we have that $\Lambda(1) \cap \partial D_{1,j} = \emptyset$.

A choice as above is possible due to property (3) in Proposition 3.4 (which allows to obtain (d3)), and due to the fact that the set $\Lambda(1)$ is countable (which allows to obtain (d4)). Put $\mathcal{D}_1 = \{D_{1,j} : 1 \leq j \leq k_1\}$, and split this set into subsets $\mathcal{D}_i^M$ according to the rule that $D_{1,j} \in \mathcal{D}_i^M$ iff $\text{Lk}(p_{1,j},X) \cong M$.

Now, suppose that $m \geq 2$ and that for all $1 \leq i < m$, and all $M \in \mathcal{M}$, we have already described the families $\mathcal{D}_i^M$ satisfying conditions (1), (2), (3b) and (4) of Definition 1.1. Suppose also that any disk in a so far described family $\mathcal{D}_i^M$, as well as its image in any $S_{R_k} : k < i$ through the appropriate composition of the maps $g_i'$, has diameter less than $\epsilon_i$ (we demand this to ensure that condition (5) of Definition 1.1 holds, after describing all families of disks). Clearly, if $m = 2$ then these assumptions hold, which can be easily deduced from the first step of the construction given in the previous paragraph.

Choose a family of pairwise disjoint collared disks $D_{m,j} : 1 \leq j \leq k_m$ in $S_{R_m}$ such that for each $j$ the following conditions hold:

- (d1') $D_{m,j}$ contains the point $p_{m,j}$ in its interior;
- (d2') the diameter of $D_{m,j}$, as well as of its image in any $S_{R_k} : k < m$ through the appropriate composition of the maps $g_i'$, is less than $\epsilon_m$;
- (d3') the preimage $(g_m')^{-1}(D_{m,j})$ is homeomorphic to $M \setminus \text{int}(\Delta)$, where $M \in \mathcal{M}$ is homeomorphic to the link $\text{Lk}(p_{m,j},X)$ and $\Delta$ is a collared $n$-disk in $M$;
- (d4') denoting by $\Lambda(m)$ the set of images in $S_{R_m}$, through appropriate compositions of the maps $g_i' : l \geq m$, of all points in $\Lambda \cap (X \setminus B_{R_m})$, we have that $\Lambda(m) \cap \partial D_{m,j} = \emptyset$;
- (d5') for any $k < m$ and any $D \in \mathcal{D}_k$ the image of $D_{m,j}$ in $S_{R_k}$, through the appropriate composition of the maps $g_i'$, is disjoint with $\partial D$.

The fact that choosing $D_{m,j}$ sufficiently small we can fulfill (d5') follows from having condition (d4') satisfied, by inductive hypothesis, for $m$ replaced with any $i < m$. (Clearly, condition (d5') corresponds to condition (4) in Definition 1.1.)
4. Proof of Proposition 3.4.

The proof of Proposition 3.4 is split into a series of auxiliary observations and partial results, and it is completed in the last part of the section. We use the notation established at the beginning of Section 3.

The following observation is an easy consequence of the assumption that for each \( M \in \mathcal{M} \) the singular set \( \Lambda_M \) is a net in \( X \) (we omit the proof).

Claim 4.1. For each \( M \in \mathcal{M} \) and for each \( i \geq 1 \) the image set \( G_i(\Lambda_M \cap (X \setminus B_{R_i})) \) is dense in \( S_{R_i} \).

Next result is a slight extension of Lemma (3b.1) in [DJ], which follows from the arguments in that paper.

Lemma 4.2. Let \( L \) be a CAT(1) piecewise spherical PL manifold of dimension \( n \). Then for any \( p \in L \) and any \( r \in (0, \pi) \) the ball \( B_r(p, L) \) is a collared \( n \)-disk in \( L \).

Proof: By Lemma (3b.1) of [DJ], \( B_r(p, L) \) is an \( n \)-disk with the boundary \( S_r(p, L) \). It thus remains to show that this disk is collared in \( L \). Choose \( \epsilon \) satisfying \( 0 < \epsilon < \pi - r \), put \( A_{r, r+\epsilon} = B_{r+\epsilon}(p, L) \setminus B_r(p, L) \), and note that it is an open subset of \( L \), and hence a manifold. Let \( \varphi : A_{r, r+\epsilon} \to S_r(p, L) \times (r, r + \epsilon) \) be the map given by \( \varphi(y) = (G_r(y), d_L(y, p)) \), where \( G_r \) is the geodesic projection on \( S_r(p, L) \) (towards \( p \)), and where \( d_L \) is the metric in \( L \). It is shown on page 372 in [DJ] (inside the proof of Lemma (3b.1) and Theorem (3b.2)) that \( \varphi \) is a cellular map. Consider a function \( \delta : A_{r, r+\epsilon} \to R_+ \) given by \( \delta(y) = d_L(y, p) - r \). By Theorem 3.2, there is a homeomorphism \( h^* : A_{r, r+\epsilon} \to S_r(p, L) \times (r, r + \epsilon) \) such that for each \( y \in A_{r, r+\epsilon} \) we have \( d(h^*(y), \varphi(y)) < \delta(y) \) for the metric \( d = d_L + |\cdot|_{R_+} \) on the product. Moreover, since \( \delta(y) \to 0 \) uniformly as \( d_L(y, S_r(p, L)) \to 0 \), \( h^* \) extends to a continuous map \( h : B_{r+\epsilon}(p, L) \setminus B_r(p, L) \to S_r(p, L) \times (r, r + \epsilon) \) such that for \( y \in S_r(p, L) \) it holds \( h(y) = (y, r) \). Being a bijection, and since the sphere \( S_r(p, L) \) is compact, \( h \) is easily seen to be a homeomorphism. As this shows that \( B_r(p, L) \) is collared in \( L \), the lemma follows.

We come back to the setting established at the beginning of Section 3, where \( X \) is an \( (\mathcal{M}, \kappa) \)-pseudomanifold, \( \Lambda \) is its singular set, \( S_{R_i} \) are the spheres in \( X \) centered at a fixed point \( x_0 \in \Lambda \), of appropriately chosen radii \( R_i \), and \( G_i : X \setminus B^*_{R_i} \to S_{R_i} \) are the geodesic projections towards \( x_0 \). Next result is one more observation that follows directly from the proof of Lemma (3b.1) and Theorem (3b.2) in [DJ]. Note that, in view of Lemma 3.1, the sets \( S_{R_i} \setminus \Lambda_{R_i} \) and \( \psi^{-1}_{R_i}(S_{R_i} \setminus \Lambda_{R_i}) \) appearing in the statement below are open subsets in manifolds, and hence are themselves manifolds.

Lemma 4.3. Under notation established at the beginning of Section 3, put \( \Lambda_{R_i} := \Lambda \cap S_{R_i} \). Given some \( \epsilon \in (0, R_{i+1} - R_i) \), denote by \( \psi_i : S_{R_i + \epsilon} \to S_{R_i} \) the geodesic projection.
towards $x_0$, i.e. the appropriate restriction of the map $G_i$. Then the restriction of $\psi_i$ to $\psi_i^{-1}(S_{R_i} \setminus \Lambda_{R_i})$ is a cellular map onto $S_{R_i} \setminus \Lambda_{R_i}$.

Next three lemmas deal with approximations of geodesic projections by the maps of better properties. They prepare the ground for the construction of approximations $g_i^\varepsilon$ as required in Proposition 3.4. More precisely, Lemma 4.4 is related to condition (2) of the proposition, while Lemmas 4.5 and 4.6 concern conditions (3) and (4), respectively.

**Lemma 4.4.** Let $\epsilon \in (0, R_{i+1} - R_i)$, and let $\psi_i : S_{R_i + \epsilon} \to S_{R_i}$ be the geodesic projection towards $x_0$ in $X$, i.e. the appropriate restriction of the map $G_i$. Then $\psi_i$ can be approximated by the maps $\psi'_i$ such that:

1. for each $1 \leq j \leq k_i$ we have $(\psi'_i)^{-1}(p_{i,j}) = \psi_i^{-1}(p_{i,j})$;
2. $\psi'_i$ restricted to the preimage $(\psi'_i)^{-1}(S_{R_i} \setminus \Lambda_{R_i})$ maps this set homeomorphically onto $S_{R_i} \setminus \Lambda_{R_i}$.

**Proof:** Let $\delta : S_{R_i + \epsilon} \to [0, \infty)$ be a continuous function such that $\delta^{-1}(0) = \psi_i^{-1}(\Lambda_{R_i})$, and let $\delta : S_{R_i + \epsilon} \setminus \psi_i^{-1}(\Lambda_{R_i}) \to R_+$ be the restriction of $\delta$. Since, by Theorem 3.2, a cellular map between manifolds is a near-homeomorphism, it follows from Lemma 4.3 that there is a homeomorphism $\psi' : S_{R_i + \epsilon} \setminus \psi_i^{-1}(\Lambda_{R_i}) \to S_{R_i} \setminus \Lambda_{R_i}$ such that $d(\psi'(y), \psi_i(y)) \leq \delta(y)$ for all $y$ in the domain of $\psi'$. By this estimate, and since $\delta$ is the restriction of $\delta'$, $\psi'$ can be extended (uniquely) to a continuous map $\psi'_i : S_{R_i + \epsilon} \to S_{R_i}$. This extension necessarily satisfies properties (1) and (2), and clearly we can get in this way a map as close to $\psi_i$ as we wish. This finishes the proof.

**Lemma 4.5.** Under notation of Lemma 4.4, if $\epsilon$ is sufficiently small then any map $\psi'_i$ fulfilling assertions (1) and (2) of this lemma satisfies also the following:

$(\ast)$ for each $1 \leq j \leq k_i$ and any sufficiently small collared $n$-disk $D$ in $S_{R_i}$ containing $p_{i,j}$ in its interior, the preimage $(\psi'_i)^{-1}(D)$ is homeomorphic to $M \setminus \text{int}(\Delta)$, where $M = \text{Lk}(p_{i,j}, X) \in M$, and where $\Delta$ is a collared $n$-disk in $M$.

**Proof:** Let $\epsilon$ be so small that for each $1 \leq j \leq k_i$ the preimage $C_j := \psi_i^{-1}(p_{i,j})$ is contained in the interior of some cone neighbourhood $B_{R_i}(p_{i,j}, X)$. Consider the canonical identification of each sphere $L_j := S_{R_i}(p_{i,j}, X)$ with the link $L^*_j := \text{Lk}(p_{i,j}, X)$. Put $\{w_j\} = \{p_{i,j}, x_0\} \cap L_j$, and denote by $w_j^*$ the point in $L_j^*$ corresponding to $w_j$ under the above identification. Let $A_j^* := \{y \in L_j^* : d_j(y, w_j^*) \geq \pi\}$, where $d_j$ is the piecewise spherical metric in $L_j^*$, be the shadow of the point $w_j$ in $L_j^*$, and let $A_j$ be the corresponding subset in $L_j$. Put $\Omega_j = B_{R_i} \cap B_{R_i}(p_{i,j}, X)$ and note that it is a convex set in $X$ containing $p_{i,j}$ in its interior. It follows that the natural conical projection $c : L_j \to \partial \Omega_j$ (towards $p_{i,j}$) is a homeomorphism. It is also not hard to realize that $C_j = c(A_j)$.

Now, choose $\rho \in (0, \pi)$ so large that, putting $N_j^* := L_j^* \setminus B^*_p(w_j^*, L_j^*)$, and denoting by $N_j$ the corresponding subset in $L_j$, this subset $N_j$ is disjoint with the ball $B_{R_i + \epsilon}$ and consequently its image $N'_j := c(N_j)$ falls in this part of $\partial \Omega_j$ which is contained in $S_{R_i + \epsilon}$. By Lemma 4.2, $B_{\rho}(w_j^*, L_j^*)$ is a collared $n$-disk in $L_j^*$, and hence the same is true for the corresponding set $B^\rho_p$ in $L_j$. It follows that $N_j$ is homeomorphic to $M \setminus \text{int}(\Delta)$. By applying again Lemma 4.2, together with the Annullus Theorem, we get also that $N_j \setminus A_j$ is homeomorphic with the product $S^{n-1} \times [0, 1)$.
Put $D'_j := \psi_i'(N'_j)$. Since all of $C_j$ is then mapped to $p_{i,j} \in D'_j$, it follows that $D'_j$ is a one-point-compactification of $N'_j \setminus C_j \cong \mathbb{S}^{n-1} \times [0, 1)$, and hence a collared $n$-disk in $S_{R_i}$ containing $p_{i,j}$ in its interior. Clearly, we have $(\psi_i')^{-1}(D'_j) = N'_j \cong M \setminus \text{int}(\Delta)$. By the Annuulus Theorem, the same is true for any collared $n$-disk $D$ contained in the interior of $D'_j$ and containing $p_{i,j}$ in its interior, hence the lemma.

**Lemma 4.6.** Let $g_i : S_{R_{i+1}} \to S_{R_i}$ be the geodesic projections, and $\Lambda_{R_i} = \{p_{i,1}, \ldots, p_{i,k_i}\}$ be the sets of singular points in the spheres $S_{R_i}$, as in the proof of Main Theorem. Let $Z$ be arbitrary finite subset of $S_{R_i}$ disjoint with $\Lambda_{R_i}$, and for each $z \in Z$ let $y_z \in S_{R_{i+1}}$ be an arbitrary point in the preimage $(g_i)^{-1}(z)$. Then $g_i$ can be approximated arbitrarily close by a map $g_i^Z : S_{R_{i+1}} \to S_{R_i}$ satisfying conditions (2) and (3) of Proposition 3.4 (with $g_i^Z$ substituted for $g_i'$) and such that

$$(C) \quad g_i^Z(y_z) = z \text{ for all } z \in Z.$$

**Proof:** Choose positive $\epsilon$ as small as required in Lemma 4.5, and consider the corresponding geodesic projections $\psi_i : S_{R_i+\epsilon} \to S_{R_i}$ and $\phi_i : S_{R_{i+1}} \to S_{R_i+\epsilon}$ towards $x_0$. We then clearly have $g_i = \psi_i \phi_i$. Consider a very close approximation $\psi'_i$ of $\psi_i$ satisfying conditions (1) and (2) of Lemma 4.4 and condition $(\star)$ of Lemma 4.5. Moreover, since by Lemma 3.3 the map $\phi_i$ is cellular, consider its very close approximation by a homeomorphism $\phi'_i$, as guaranteed by Theorem 3.2. The composition map $\psi'_i \phi'_i$ satisfies conditions (2) and (3) of Proposition 3.4, but not necessarily the condition $(C)$ of the assertion.

Note that, by the choices of $\psi'_i$ and $\phi'_i$, for each $z \in Z$ the point $\psi'_i \phi'_i(y_z)$ is very close to $\psi_i \phi_i(y_z) = g_i(y_z) = z$. Since $S_{R_i}$ is a manifold, we can choose a correcting homeomorphism $\omega : S_{R_i} \to S_{R_i}$, very close to the identity, fixing all points of $\Lambda_{R_i}$, and such that $\omega(\psi'_i \phi'_i(y_z)) = z$ for all $z \in Z$. The lemma follows by taking $g_i^Z := \omega \psi'_i \phi'_i$, since clearly such a composition can be chosen to approximate the map $g_i$ arbitrarily close.

We now pass to the final phase of the proof of Proposition 3.4. Observe that Lemma 4.6 guarantees existence of approximations of the maps $g_i$ satisfying conditions (2) and (3) of the proposition. It remains to justify that appropriately chosen such approximations satisfy also the last condition (4).

Chose any sequence $i_m$ of natural numbers such that $i_m \leq m$ and each $i \geq 1$ appears in this sequence infinitely often. Choose also a sequence $\epsilon_m$ of positive reals converging to 0. Proceed inductively with respect to $m$, as follows. For $m = 1$ and for each $M \in \mathcal{M}$ choose a finite subset $Y_{M,1} \subset \Lambda_M \cap [X \setminus B_{R_1}]$ such that its image $Z_{M,1} = G_1(Y_{M,1})$ is an $\epsilon_1$-net in $S_{R_1}$. Put $Y_1 = \cup_{M \in \mathcal{M}} Y_{M,1}$ and $Z_1 = \cup_{M \in \mathcal{M}} Z_{M,1}$, and assume, without loss of generality, that $Z_1$ is disjoint with $\Lambda_{R_1}$. (The possibility of such a choice of the above sets follows from Claim 4.1.) Put $v_1$ to be the largest $i$ such that the intersection $Y_1 \cap S_{R_i}$ is nonempty. For $1 \leq i \leq \nu_1 - 1$ put $Y_{1,i} := Y_1 \cap [X \setminus B_{R_i}]$, $Z_{1,i} := G_i(Y_{1,i})$, and for each $z \in Z_{1,i}$ put $y_z = G_{i+1}(y)$, where $y \in Y_{1,i}$ is this point for which $z = G_i(y)$. Again for $1 \leq i \leq \nu_1 - 1$ choose successively approximations $g_i^1 = g_i^{Z_{1,i}}$ as in Lemma 4.6, so close to $g_i$ that the requirements of Brown’s Lemma are fulfilled. Note that, apart from satisfying conditions (2) and (3) of Proposition 3.4, these maps $g_i^1 : 1 \leq i \leq \nu_1 - 1$ have the following property: for each $M \in \mathcal{M}$ the images in $S_{R_i}$ of the points from $\cup_{i=1}^{\nu_1}(\Lambda_M \cap S_{R_i})$, through the appropriate compositions of the maps $g_i^1$, form an $\epsilon_1$-net in $S_{R_i}$.  

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Now, assume that \( m \geq 2, \nu_{m-1} \geq m \), and that we have already defined the approximations \( g'_i \) for all \( i \leq \nu_{m-1} - 1 \). For any \( M \in \mathcal{M} \) choose a finite subset \( Y_{M,m} \subset \Lambda_M \cap [X \setminus B_{R_{m-1}}] \) such that its image \( Z_{M,m} = G_{\nu_{m-1}}(Y_{M,m}) \), further projected by the composition \( g_{i_m}^{i_{m-1}} \circ g_{i_{m-1}}^{i_{m-2}} \circ \ldots \circ g_{i_1}^{i_0} \), is an \( \epsilon_m \)-net in \( S_{R_{i_m}} \). Put \( Y_m = \cup_{M \in \mathcal{M}} Y_{M,m} \) and \( Z_m = \cup_{M \in \mathcal{M}} Z_{M,m} \), and assume, without loss of generality, that the image of \( Z_m \) in \( S_{R_{i_m}} \), through the above mentioned composition of the maps \( g'_i \), omits the points \( p_{i_{m,1}}, \ldots, p_{i_{m,k_m}} \). (The possibility of such a choice of the above sets follows from Claim 4.1, and from the fact that the image of a dense subset through a surjective map is dense.) Put \( \nu_m \) to be the largest \( i \) such that the intersection \( Y_m \cap S_{R_i} \) is nonempty. For \( \nu_{m-1} \leq i \leq \nu_{m-1} - 1 \) put \( Y_{m,i} := Y_m \cap [X \setminus B_{R_i}], Z_{m,i} := G_i(Y_{m,i}), \) and for each \( z \in Z_{m,i} \) put \( y_z = G_{i+1}(y) \) for this \( y \in Y_{m,i} \) for which \( G_i(y) = z \). Then for \( \nu_{m-1} \leq i \leq \nu_{m-1} - 1 \) choose successively approximations \( g'_i \) as in Lemma 4.6, close enough to \( g_i \) to fulfil the requirements of Brown’s Lemma. Note that, apart from satisfying conditions (2) and (3) of Proposition 3.4, the maps \( g'_i : 1 \leq i \leq \nu_{m-1} \) have the following property: for each \( M \in \mathcal{M} \) and for each \( j \in \{i_1, \ldots, i_m\} \), the images in \( S_{R_j} \) of the points from \( \cup_{i=j+1}^{\nu_m} (\Lambda_M \cap S_{R_i}) \), through the appropriate compositions of the maps \( g'_i \), form an \( \epsilon_k \)-net in \( S_{R_j} \), where \( k \leq m \) is the largest number such that \( i_k = j \).

A direct verification shows that the whole sequence of maps \( g'_i : i \geq 1 \) described above meets all the requirements of Proposition 3.4, which finishes the proof.

5. Applications of Main Theorem.

In this section we describe vast classes of examples to which Main Theorem (as presented in Section 2) applies. Among others, we provide proofs of Theorems 1 and 2 from the introduction.

Hyperbolizations of \( \mathcal{M} \)-pseudomanifolds and the proof of Theorem 1.

Given a finite family \( \mathcal{M} \) of PL manifolds as in Section 2 (all of the same dimension \( n \)), a closed \( \mathcal{M} \)-pseudomanifold is a compact connected polyhedral cell complex \( P \) such that

1. for each point \( x \in P \) the link \( \text{Lk}(x, P) \) is either a PL \( n \)-sphere, or is PL-homeomorphic to some \( M \in \mathcal{M} \); furthermore, if all \( M \in \mathcal{M} \) are oriented, we assume that \( P \) is also oriented, and that each homeomorphism \( \text{Lk}(x, P) \to M \) as above respects the orientations;

2. for each \( M \in \mathcal{M} \) the set \( \Lambda_P^M := \{x \in P : \text{Lk}(x, P) \cong M\} \) is nonempty (if all \( M \in \mathcal{M} \) are oriented, the symbol \( \cong \) above denotes relation of being PL homeomorphic as oriented manifolds).

Recall that hyperbolization, as described e.g. in [DJ], is a procedure that turns an arbitrary compact simplicial complex into a compact nonpositively curved piecewise euclidean polyhedral complex with the same local PL topology. In particular, if we apply this procedure to a simplicial closed \( \mathcal{M} \)-pseudomanifold \( P \), we get a nonpositively curved piecewise euclidean complex \( P_h \) which is also a closed \( \mathcal{M} \)-pseudomanifold. (If all manifolds in \( \mathcal{M} \) are oriented, we consider the procedure which respects all requirements concerning orientations, e.g. the procedure described in Subsection (4c) in [DJ], for which each hyperbolized simplex is an oriented manifold, and the associated hyperbolization map is of degree 1.) Observe that the universal cover \( \tilde{P}_h \) is then an \( (\mathcal{M}, 0) \)-pseudomanifold.
There is also an analogous procedure, called strict hyperbolization and described in [CD], which turns any simplicial closed $\mathcal{M}$-pseudomanifold $P$ into a piecewise hyperbolic locally $\text{CAT}(-1)$ closed $\mathcal{M}$-pseudomanifold $P^s_h$. The universal cover $\tilde{P}^s_h$ is then an $(\mathcal{M},-1)$-pseudomanifold.

Theorem 1 of the introduction is an easy consequence of the following.

5.1 Theorem. Let $\mathcal{M} = \{M_1, \ldots, M_k\}$ be a finite family of closed connected PL manifolds of the same dimension $n$, either all oriented, or at least one of which is non-orientable. Suppose that for some positive integers $m_1, \ldots, m_k$ the disjoint union $\bigsqcup_{i=1}^k m_i M_i$ bounds a compact $(n+1)$-dimensional PL manifold $W$ (in oriented sense, if all manifolds from $\mathcal{M}$ are oriented). Then there is a hyperbolic group $G$ such that its Gromov boundary $\partial G$ is homeomorphic to the tree of manifolds $X(\mathcal{M})$.

Proof: Consider any PL triangulation of any manifold $W$ as in the assumptions. For each boundary component $M$ of $W$ consider the simplicial cone over $M$ and glue it to $W$ via the identity of $M$. This gives a simplicial closed $\mathcal{M}$-pseudomanifold which we denote by $P$. Consider its strict hyperbolization $P^s_h$, and its universal cover $\tilde{P}^s_h$, which is a $(\mathcal{M},-1)$-pseudomanifold. The group $G = \pi_1(P^s_h)$ is then a word hyperbolic group, and its Gromov boundary $\partial G$ coincides with the visual boundary $\partial \tilde{P}^s_h$. Since by Main Theorem the latter boundary is homeomorphic to the tree of manifolds $X(\mathcal{M})$, this completes the proof.

Proof of Theorem 1: Part (1) follows from Theorem 5.1 by taking $\mathcal{M} = \{M\}$, while part (2) follows by taking $\mathcal{M} = \{N\}$ and $W = N \times [0,1]$.

5.2 Remark. Note that Theorem 1 provides new examples of Gromov boundaries already in dimension 3. More precisely, since each closed connected PL 3-manifold $M$ bounds a compact PL 4-manifold, each tree of 3-manifolds $M$, $X(\mathcal{M})$, is homeomorphic to the Gromov boundary of some hyperbolic group. On the other hand, in the case of orientable 3-manifolds the arguments of [PS] (after appropriate correction of the main result of [F] used in [PS]) justify this statement only for manifolds of form $M \# \overline{M}$, where $M$ and $\overline{M}$ are the oppositely oriented copies of any orientable 3-manifold.

5.3 Question. Theorem 5.1 leaves open the following question: is there a hyperbolic group $G$ whose Gromov boundary $\partial G$ is homeomorphic to the tree of complex projective planes $X(CP^2)$? Recall that neither $CP^2$ nor any positive number of its copies bounds a compact oriented 5-manifold, so $G$ cannot be obtained by referring to Theorem 5.1. On the other hand, this theorem easily implies that the space $X(\{CP^2, \overline{CP^2}\})$ is homeomorphic to the Gromov boundary of some hyperbolic group. Since one can use properties of Čech cohomology rings to show that the spaces $X(CP^2)$ and $X(\{CP^2, \overline{CP^2}\})$ are not homeomorphic, this also does not help to answer the question. My guess is that the answer is negative. Of course, a similar question can be asked for other than $CP^2$ manifolds which represent the elements of infinite order in the corresponding oriented cobordism additive semi-group.

Coxeter groups with manifold nerves and the proof of Theorem 2.

Our main reference concerning Coxeter groups is the book [Da] by Mike Davis. We start with explaining the terms appearing in the statement of Theorem 2, following the terminology and notation from [Da].
Given a finite set $S$, a Coxeter matrix on $S$ is a matrix $m = (m_{st})_{s,t \in S}$ such that

1. for each $s \in S$ we have $m_{ss} = 1$;
2. for all $s, t \in S$, $s \neq t$, we have that $m_{st}$ is an integer $\geq 2$, or $\infty$.

Associated to a Coxeter matrix $m$ on $S$, there is a group $W$ given by the presentation

$$\langle S \mid \{(st)^{m_{st}} : s,t \in S\}\rangle,$$

where the symbol $(xy)^\infty$ denotes absence of any relation of the form $(ab)^k$ in the set of relations. $W$ is called the Coxeter group associated to $m$, and it is known that the set $S$ canonically injects in $W$. The pair $(W, S)$ is called the Coxeter system associated to $m$.

For any subset $T \subset S$ the special subgroup $W_T < W$ is the subgroup generated by $T$. It is known that $(W_T, T)$ can be canonically identified with the Coxeter system associated to the restricted matrix $m_T$, i.e. the matrix $(m_{st})_{s,t \in T}$.

The nerve of a Coxeter system $(W, S)$ is the simplicial complex $L = L(W, S)$ whose vertex set coincides with $S$, and such that $T \subset S$ spans a simplex of $L$ iff the special subgroup $W_T$ is finite.

As it is described in Chapter 7 of [Da], to any Coxeter system $(W, S)$ there is associated a polyhedral cell complex $\Sigma = \Sigma(W, S)$, called the Davis-Moussong complex of $(W, S)$, which satisfies the following properties:

- $(\Sigma 1)$ each vertex link of $\Sigma$ is a simplicial complex isomorphic with the nerve $L$;
- $(\Sigma 2)$ $\Sigma$ carries a natural piecewise euclidean metric with respect to which it is a $CAT(0)$ space, see Theorem 12.3.3 on p. 235 in [Da];
- $(\Sigma 3)$ the group $W$ acts on $\Sigma$ by isometries, properly discontinuously and cocompactly, so that the generators from $S$ correspond to certain geometric reflections in $\Sigma$, and so that the action is simply transitive on the vertex set of $\Sigma$.

**Proof of Theorem 2:** Suppose that the nerve $L(W, S)$ is a PL triangulation of a closed connected manifold $M$. If $M$ is orientable, it follows from conditions $(\Sigma 1)$-$(\Sigma 3)$ above that $\Sigma(W, S)$ is an $(\mathcal{M}, 0)$-pseudomanifold with $\mathcal{M} = \{ M, \overline{M} \}$, where $M$ and $\overline{M}$ are the two oppositely oriented copies of $M$. Similarly, if $M$ is non-orientable, $\Sigma(W, S)$ is an $(\mathcal{M}, 0)$-pseudomanifold with $\mathcal{M} = \{ M \}$. Thus, applying Main Theorem to the Davis-Moussong complex $\Sigma(W, S)$, we immediately get Theorem 2.

**5.4 Remark.** It is known that, when a Coxeter group $W$ is word hyperbolic, its Gromov boundary $\partial W$ coincides with the visual boundary $\partial \Sigma(W, S)$ (see e.g. Remark I.8.5 on p. 527 in [Da]). In such a case, if the nerve $L(W, S)$ is a PL triangulation of a closed connected manifold $M$, the Gromov boundary $\partial W$ is homeomorphic to the space $X(\mathcal{M})$ as in the statement of Theorem 2.

**6. Riemannian $(\mathcal{M}, 0)$-pseudomanifolds with log-injective singularities.**

In this section we explain how to adapt our proof of Main Theorem to a class of smooth $CAT(0)$ pseudomanifolds with Riemannian metrics on their regular part. Using this, we deduce Theorem 3. We start with describing the relevant class of pseudomanifolds. We refer the reader to Subsection 3.2 of [FM] for the definition of the space of directions $\Sigma_p(\mathcal{X})$ at a point $p$ of a $CAT(0)$ space $\mathcal{X}$ (or to Definition II.3.18 on p. 190 of [BH], where the same object is denoted $S_p(\mathcal{X})$).
**Definition 6.1.** Given a finite collection $\mathcal{M}$ of closed connected manifolds of the same dimension $n$, a $\text{CAT}(0)$ complete geodesic metric space $(X,d)$ is a Riemannian $(\mathcal{M},0)$-pseudomanifold with log-injective singularities if the following conditions are satisfied:

1. there is a subset $\Lambda \subset X$, called the singular set of $X$, which is discrete and the complement $X \setminus \Lambda$ is a smooth manifold;
2. the set $\Lambda$ is partitioned into subsets $\Lambda_M : M \in \mathcal{M}$ such that for each $M \in \mathcal{M}$ and any $p \in \Lambda_M$ the space of directions $\Sigma_p(X)$ is homeomorphic to $M$;
3. each of the subsets $\Lambda_M$ is a net in $X$;
4. the regular part $X \setminus \Lambda$ is equipped with a Riemannian metric $g$ of nonpositive sectional curvature, such that $d$ restricted to $X \setminus \Lambda$ coincides with the path metric induced by $g$, and the completion of $(X \setminus \Lambda, d)$ coincides with $(X,d)$;
5. each $p \in \Lambda$ has a neighborhood $U$ such that the logarithmic map $\log_p : U \setminus \{p\} \to \Sigma_p(X) \times \mathbb{R}_+$ (which to each $x \in U \setminus \{p\}$ associates the pair $(a,r)$ such that $a \in \Sigma_p(X)$ is the direction of the geodesic $[p,x]$ and $r = d(p,x)$) is injective;
6. the space $\Sigma_p(X)$ at any singular point $p$ has the property that every ball of radius $r \in (0,\pi)$ in it (with respect to the angle metric) is a collared $n$-disk in $\Sigma_p(X)$.

**Theorem 6.2.** Let $X$ be a Riemannian $(\mathcal{M},0)$-pseudomanifold with log-injective singularities. Then the visual boundary $\partial X$ is homeomorphic to the tree of manifolds $X(\mathcal{M})$.

Theorem 6.2 follows by the same arguments as in the proof of Main Theorem, slightly adapted and simplified according to the following features:

1. whenever in the proof of Main Theorem we use cone neighbourhoods of singular points, we need to use instead small balls which are log-injective neighbourhoods of singular points; existence of the latter balls is justified by condition (5) in Definition 6.1;
2. references to Lemma 4.2 in the proof of Main Theorem, when applied to links at singular points, need to be replaced by references to condition (6) of Definition 6.1; moreover, references to properties of links at non-singular points also need to be replaced by references to the properties of the corresponding spaces of directions, which are just the standard round $n$-spheres of constant curvature 1;
3. since geodesics in Riemannian $(\mathcal{M},0)$-pseudomanifolds do not bifurcate outside the singular set, while proving Theorem 6.2 we never need to approximate various geodesic projections between the spheres (or their restrictions) by homeomorphisms cellular maps appearing in the proof by homeomorphisms, since the corresponding maps are automatically homeomorphisms.

**Proof of Theorem 3:** In the paper [FM] by K. Fujiwara and J. Manning it is explained how to put a Riemannian metric on a regular part of any space appearing in the statement of Theorem 3, so that its lift to the universal cover of this space (and the induced path metric and its completion) satisfies all the requirements of Definition 6.1. In fact, the metrics constructed in [FM] are even $\text{CAT}(-1)$. In view of this, Theorem 3 follows fairly directly from Theorem 6.2.
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Instytut Matematyczny, Uniwersytet Wrocławski,
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail: Jacek.Swiatkowski@math.uni.wroc.pl