Exact Finite-Size Spectra in the Kondo Problem and Boundary Conformal Field Theory

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The exact finite-size spectra for several quantum impurity models related to the Kondo problem are obtained from the Bethe ansatz solutions. Using the finite-size scaling in boundary conformal field theory, we determine various surface critical exponents from the exact spectrum, which accord with those obtained by Affleck and Ludwig with Kac-Moody fusion rules. Some applications to critical phenomena observed in connection with the orthogonality catastrophe are also discussed.

\section{I. INTRODUCTION}

The method of two-dimensional conformal field theory (CFT) is very powerful to study (1+1)-dimensional quantum critical phenomena. One of important applications of CFT is due to Affleck and Ludwig to analyse the Kondo effect. As is well-known, the Kondo effect has been providing many interesting issues such as the multi-channel Kondo problem. A new machinery developed in ref. 2 for studying the Kondo problem is based on the Cardy’s boundary CFT\textsuperscript{7}. A new approach of Affleck and Ludwig, however, is based upon a plausible hypothesis about the low-energy fixed point. This is unavoidable since the formulation of quantum impurity problems in terms of boundary CFT requires not only the macroscopic self-consistency in CFT, but depends on some details of microscopic interactions. They hence proposed the so-called fusion hypothesis to describe the low-energy fixed point, from which the finite-size spectrum (FSS) was obtained to evaluate critical exponents.

On the other hand, the exact Bethe-ansatz solutions to various models related to the Kondo problem have been known. Thus it is desirable to calculate the FSS analytically and compare it with the results obtained by Affleck and Ludwig. Motivated by this, we have recently computed the FSS for the Anderson model, the s-d exchange model, the SU(N) Anderson model, and the multi-channel Kondo model from the Bethe ansatz solutions. All the results obtained are in accordance with those predicted by CFT. Furthermore we can discuss various critical properties from the exact spectrum using finite-size scaling arguments. In comparison with bulk systems, these quantum impurity models show quite different critical behaviors reflecting the effect of boundaries.

In this paper, we give a short review of our recent studies on the exact spectrum for the Kondo problem. The organization of the paper is as follows. In Sec.II, we collect some basic results in boundary CFT which will be needed in the following discussions. In Sec.III, we obtain exactly the FSS of the Anderson model by the Bethe ansatz method, and discuss its critical properties characterizing the local Fermi liquid. In Sec.IV, we show how these results can be applied when considering the Fermi-edge singularity in photoemission (absorption) experiments. We then point out in Sec.V that such anomalous exponents determine the correlation functions of pseudo-particles in the Anderson model. In Sec.VI, we derive a general expression for the FSS of 1D multicomponent chiral systems, and discuss its application to the X-ray edge singularity in the edge state of the fractional quantum Hall effect. In Sec.VII, we briefly discuss the FSS of the multi-channel Kondo model in the over-screening case where a non-Fermi liquid fixed point plays a role. Finally a brief summary is given in Sec.VIII.

\section{II. BOUNDARY CONFORMAL FIELD THEORY}

In this section, we summarize some basic results in boundary CFT necessary for the following discussions. For (1+1)-dimensional critical systems in a finite geometry with open boundaries, we need to consider conformal transformations which preserve the shape of the boundaries. Then the anti-holomorphic part in CFT is not independent of the holomorphic part, and critical properties are determined only by the holomorphic part.

The critical behavior near the boundary is characterized by the surface critical exponent $x_s$ which controls the asymptotic behavior of the correlation function, $\langle \phi(t)\phi(0) \rangle \sim 1/t^{2x_s} \ (t \to \infty)$. The surface exponent $x_s$ is generally different from bulk exponents. We can regard the surface critical exponent $x_s$ as the conformal dimension of some boundary scaling operator. Decomposing a scaling field $\phi$ into the holomorphic part and the anti-holomorphic part which is defined by the analytic continuation of the holomorphic part, $\phi(z, \bar{z}) = \phi_L(z)\phi_R(\bar{z})$, we have $\phi(z, \bar{z}) = \phi_L(x+iy)\phi_L(x-iy) \sim y^{-2\Delta+\Delta_s}\phi_b(x)$.\textsuperscript{8}
Here $\Delta$ is the bulk conformal dimension of $\phi_L$, and $\Delta_b$ is the conformal dimension of a boundary operator $\phi_b(x)$. Then in the vicinity of the boundary, we have the correlation function of the scaling operator,

$$
\langle \phi(z, \bar{z}) \phi(z', \bar{z}') \rangle \sim \langle \phi_b(x) \phi_b(x') \rangle \sim |x - x'|^{-2\Delta_b}.
$$

That is, the conformal dimension of the boundary operator $\Delta_b$ directly determines the surface critical exponent $x_s$.

According to the finite-size scaling analysis in CFT, it is well-known that $\Delta_b$ enters in the finite-size corrections,

$$
E = E_\infty - \frac{\pi v_c}{24l} + \frac{\pi v}{l} (\Delta_b + n)
$$

with $c$ being the central charge of the Virasoro algebra. Here $E_\infty$ is the energy in the thermodynamic limit, $l$ is the system size and $n$ is a non-negative integer which features the conformal tower structure.

Since critical systems with boundaries involve only the holomorphic part of CFT, we can describe the systems only in terms of left-moving (or right-moving) currents. Such systems are referred to as chiral systems. Since quantum impurity systems such as the Kondo model can be mapped to 1D chiral systems, boundary CFT is applicable. In the rest of the paper, we shall obtain the finite-size spectrum from the exact solution of various models related to the Kondo problem, and then discuss their critical properties using boundary CFT.

### III. EXACT FINITE-SIZE SPECTRUM IN THE KONDO PROBLEM

There are several models related to the Kondo problem, which have been exactly solved by the Bethe ansatz method. So far thermodynamic quantities have been mainly studied by the Bethe ansatz solution, and the FSS has not been examined in detail. As mentioned above, the FSS provides us with important information on critical properties. Motivated by this, applying standard methods in the Bethe ansatz solution, we have systematically calculated the FSS of the Anderson model, the SU($N$) Anderson model and the $s$-$d$ exchange model, and compared the obtained results with the predictions by CFT. In this section, we summarize the results for the single-impurity Anderson model as an example, since the manipulations in deriving the FSS are essentially the same for these models. The Hamiltonian of the Anderson model is given by

$$
H = \sum_{k, \sigma} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} + V \sum_{k, \sigma} (c_{k\sigma}^\dagger d_{\sigma} + d_{\sigma}^\dagger c_{k\sigma}) + \epsilon_d \sum_{\sigma} d_{\sigma}^\dagger d_{\sigma} + U d_{\uparrow}^\dagger d_{\uparrow} d_{\downarrow}^\dagger d_{\downarrow},
$$

with standard notations. The model describes free conduction electrons coupled with correlated $d$-electrons at the impurity site via the resonant hybridization $V$. After reducing the model to the one-dimensional system with the use of partial wave representation, we linearize the spectrum of conduction electrons as $\epsilon_k = \nu k$ near the Fermi point. The kinetic energy is then replaced by the operator $-iv\partial/\partial x$ in the coordinate representation. These simplifications enable us to apply the Bethe-ansatz method to diagonalize the Hamiltonian.

#### A. Finite-size spectrum

Exploiting standard techniques in Bethe-ansatz method, we can compute the finite-size corrections to the ground-state energy as,

$$
E_0 = L \xi_0 - \frac{\pi v_c}{12L} - \frac{\pi v_s}{12L},
$$

where $L \xi_0$ is the ground state energy in the thermodynamic limit ($L \to \infty$), and $v_c$ and $v_s$ are the velocities of massless charge and spin excitations. For the present model, these velocities take the same value $v$. To compare the result (3) with the finite-size scaling formula in CFT, one has to replace $L$ by $2l$ since $L$ has been defined as the periodic length of the system. Then we indeed find the scaling behavior predicted by boundary CFT for the charge and spin sectors in (4), i.e. their Virasoro central charges are given by $c = 1$.

Let us now consider the excitation energy. In the present system, there exist particle-hole type excitations as well as excitations which change the number of electrons $N_h$ as well as the $z$-component of spin $S_h$,

$$
N_h \to N_h + \Delta N_h, \quad S_h \to S_h + \Delta S_h.
$$

All these excitations give $1/L$-corrections to the energy spectrum. An important point in the Kondo problem is that the first-order corrections to the energy due to the change of $N_h$ appear in $1/L$-corrections as an effect due to the impurity. This correction is given by

$$
\Delta E^{(1)} = -\frac{2\delta_F}{L} \Delta N_h,
$$

where $\delta_F$ is the phase shift caused by the impurity scattering. Combining this term with the second-order corrections in $\Delta N_h$ and $\Delta S_h$, we obtain the excitation spectrum,

$$
E = E_0 + \frac{1}{L} E_1 + \frac{1}{L^2} E_2 + O(1/L^3),
$$

$$
\frac{1}{L} E_1 = \frac{2\pi v}{L} \left[ \frac{1}{4} \left( \Delta N_h - 2 \frac{\delta_F}{\pi} \right)^2 + n^+_c \right]
$$

$$
+ \frac{2\pi v}{L} \left( (\Delta S_h)^2 + n^+_s \right),
$$

2
\[
\frac{1}{L^2} E_2 = \frac{2\pi v}{L^2} \frac{\chi^{imp}_c}{\chi^h_c} \left[ \frac{(\Delta N_h)^2}{4} + n^+_c \right] \\
+ \frac{2\pi v}{L^2} \frac{\chi^{imp}_s}{\chi^h_s} \left[ (\Delta S_h)^2 + n^+_s \right],
\]

where \(\chi^{imp}_c (\chi^h_c)\) and \(\chi^{imp}_s (\chi^h_s)\) are the charge (spin) susceptibilities of impurity and host electrons, respectively. \(n^+_c, n^+_s\) are non-negative integers which label particle-hole excitations. One can see that the expression for the \(1/L\) correction is consistent with the fusion hypothesis. Using the above formulas for the FSS, we will discuss various critical properties of the Kondo model in the following.

B. Canonical exponents

Let us now discuss the results (6)~(8) by using the finite-size scaling in CFT. It was already found that low-energy critical properties of the Kondo effect can be described by boundary CFT in which we have only the right (or left) moving sector of CFT. The \(1/L\) correction term (8) indeed shows the scaling behavior predicted by boundary CFT. It is seen from eqs.(4) and (8) that the charge sector is described by \(c = 1\) Gaussian CFT and the spin sector by \(c = 1\) SU(2) Kac-Moody CFT in the low-energy regime. Note that the finite-size spectrum for bulk electrons in eq.(8) involves a non-universal phase shift \(\delta_F\). We recall here that this phase shift \(\delta_F\) is regarded as the chemical-potential change due to the impurity (see eq.(6)). This means that the effect of the phase shift amounts to merely imposing boundary conditions on conduction electrons. Therefore, when we derive the dimension of the scaling operator associated with conduction electrons, we should discard the \(\delta_F\) dependence in eq.(8) by redefining \(\Delta N_h - 2\delta_F/\pi \to \Delta N_h\). Hence the scaling dimension \(x\) of the conduction electron field is obtained by taking the quantum numbers

\[
\Delta N_h = 1, \quad \Delta S_h = 1/2,
\]

resulting in \(x = 1/2\). Thus we can see that the single-electron Green function \(\langle c_{s}(t)c^{\dagger}_{s}(0)\rangle \sim 1/t^{\eta}\) has the canonical exponent \(\eta = 1\). This result is consistent with the fact that the system is described by the strong-coupling fixed point of the local Fermi liquid.

C. Local Fermi liquid

The local Fermi-liquid properties are confirmed further by observing the \(1/L^2\) corrections in eq.(6). For this, let us rewrite the \(1/L^2\)-term in eq.(6) by using the velocities for the charge and spin excitations at the impurity site,

\[
v^{imp}_c = \frac{\pi}{3\gamma_c^{imp}}, \quad v^{imp}_s = \frac{\pi}{3\gamma_s^{imp}}.
\]

We then have

\[
E_2 = \pi v_c^{imp} (\Delta n_d)^2/4 + 2\pi v_s^{imp} (\Delta s_z)^2,
\]

where we set \(n^+_c = n^+_s = 0\) for simplicity. Here \(n_d\) is the number of impurity electrons, and \(s_z\) is the \(z\)-component of the impurity spin. It should be noted that this expression for the impurity spectrum directly reflects local Fermi-liquid properties. To see this, we first recall that in the ordinary Tomonaga-Luttinger liquid in 1D, there is a dimensionless coupling parameter \(K_r\), which enters in the FSS as \(v_c K_r\), in addition to the velocities \(v_c\) and \(v_s\). In the expression (12), however, it is seen that \(K_r = 1\), i.e. it is given by the value of free electrons. Moreover, \(K_r = 1\) always holds independent of the strength of interaction \(U\), which suggests that the local Fermi liquid is stable at the impurity site (13). Although the spin-charge separation, which is characterized by the different spin and charge velocities, occurs even in the Kondo system, the local Fermi liquid is always stabilized due to the locality of correlations.

IV. ANOMALOUS EXPONENTS RELATED TO THE X-RAY EDGE PROBLEM

In the previous section, we neglected the effect of the phase shift when obtaining the canonical exponents for the local Fermi liquid. Keeping the phase-shift dependence, on the other hand, we can extract another interesting information from the finite-size spectrum (6), i.e. the critical behavior related to the orthogonalization catastrophe. To see this explicitly, let us consider the time-dependent Anderson model in which the hybridization \(V = 0\) for \(t < t_0\), and then \(V\) is switched on at \(t = t_0\). Note that the orthogonalization catastrophe related to the Fermi edge singularity appears in the long-time behavior of this model. In this case, the phase shift in (6) becomes a key quantity which controls the critical exponent. Actually, critical exponents related to the X-ray problem can be read off from the above FSS with keeping the dependence on the phase shift intact. This is because a sudden potential change occurs in X-ray photoemission (or absorption) experiments. For example, the critical exponent \(\eta\) of the single-particle Green function \(\langle c_{s}(t)c^{\dagger}_{s}(0)\rangle \sim 1/t^{\eta}\) for the model with a sudden potential change is obtained as

\[
\eta = 1 - \frac{2\delta_F}{\pi} + 2 \left(\frac{\delta_F}{\pi}\right)^2.
\]

Note that this is just the exponent which governs the long-time behavior of the overlap integral between the initial and final states in the X-ray absorption problem. It is thus seen that the FSS (6) contains the information about orthogonalization catastrophe in addition to the local Fermi-liquid properties. In the next section, we
shall discuss in more detail the physical meaning of such anomalous exponents in terms of pseudo-particles in the Anderson model.

V. CRITICAL EXPONENTS OF PSEUDO-PARTICLES IN THE KONDO PROBLEM

In the strong-coupling limit \(U \to \infty\) of the Anderson model, the double occupancy of electrons is forbidden, and thus the Fock space of the impurity electron can be mapped to that spanned by the slave-boson field \(b\) and the pseudo-fermion field \(f_m (m = 1, 2, \ldots, N)\) which represent an empty site and a singly occupied site, respectively. In this section, we discuss the long-time asymptotic behavior of dynamical correlation functions of these pseudo-particles, and show that the orthogonality catastrophe mentioned in the previous section manifests itself in these correlation functions. The critical exponents of pseudo-particles can be thus derived exactly from a boundary CFT analysis.

Let us consider the \(U \to \infty\) SU(\(N\)) Anderson model. The Hamiltonian is given by

\[
H = \sum_{m=1}^{N} \int dx c_m^\dag(x) \left( -i \frac{\partial}{\partial x} \right) c_m(x) + \epsilon_f \sum_{m=1}^{N} f_m^\dag f_m + V \sum_{m=1}^{N} \int dx \delta(x) \left( f_m^\dag b c_m(x) + c_m(x) b^\dag f_m \right),
\]

(14)

with the constraint \(b^\dag b + \sum_m f_m^\dag f_m = 1\), where the impurity electrons have \(N\)-fold spin states. The Hamiltonian \(H\) can be diagonalized by the Bethe-ansatz method after reducing it to the one-dimensional one as mentioned before. The finite-size spectrum is then computed by standard techniques. The result is succinctly expressed in terms of the \(N \times N\) matrix \(C_f\)

\[
\frac{1}{L} E_1 = \frac{2\pi v}{L} \frac{1}{2} \Delta M^T C_f \Delta M - \frac{\pi v}{L} N \left( \frac{\delta_F}{\pi} \right)^2,
\]

(15)

where \(\Delta M^{(i)} \equiv \Delta M_h^{(i)} - \frac{\delta_F}{\pi} (N - l)\) for \(1 \leq l \leq N - 1\), and \(\Delta M^{(0)} = \Delta N_h - N \delta_F / \pi\) with \(\delta_F\) being the phase shift at the Fermi level. Here \(\Delta N_h\) is the quantum number for charge excitations, whereas \(\Delta M_h^{(i)}\)’s are quantum numbers for spin excitations. The \(N \times N\) matrix \(C_f\) is given as

\[
C_f = \begin{pmatrix}
1 & -1 & \cdots & 0 \\
-1 & 2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & -1 \\
0 & \cdots & -1 & 2
\end{pmatrix}.
\]

(16)

It can be checked that the last term in eq. (15), which is evaluated from the excited states, is equal to the shift of the ground-state energy due to the presence of the impurity. Therefore the increase of the ground-state energy exactly cancels the last term of eq. (15), which is not necessary for the following discussions of critical exponents. We will drop it in the following.

Let us now study the long-time behavior of the Green functions for pseudo-particles; \(\langle f_m^\dag(t) f_m(0) \rangle \sim t^{-\alpha_f}\) and \(\langle b^\dag(t)b(0) \rangle \sim t^{-\alpha_b}\). As explained in Sec. III, when determining canonical exponents for the local Fermi liquid, we can neglect the phase shift in eq. (14). In order to derive critical exponents for pseudo-particles, however, we must regard the number of impurity electrons (or phase shift) \(n_l = \delta_l / \pi\) as a quantum number. Therefore the phase shift plays an essential role to determine the critical exponents. For example, in order to obtain the Green function of pseudo-fermions, we take \(\Delta N_h = 1\) and \(\Delta M_h^{(l)} = 0\) as quantum numbers. We thus obtain the corresponding critical exponent as,

\[
\alpha_f = 1 - 2 \frac{\delta_F}{\pi} + N \left( \frac{\delta_F}{\pi} \right)^2.
\]

(17)

In a similar way, the critical exponent \(\alpha_b\) for the slave-boson Green function can be obtained. Since the slave-boson expresses a vacancy, it carries neither charge nor spin. Putting \(\Delta N_h = \Delta M_h^{(l)} = 0\), one gets

\[
\alpha_b = N \left( \frac{\delta_F}{\pi} \right)^2.
\]

(18)

These expressions for \(\alpha_f\) and \(\alpha_b\) agree with those obtained for the \(N = 1\) and \(N = 2\) cases \[4\] and take the same form as those in the X-ray problem: the exponent of pseudo-fermion corresponds to the X-ray absorption exponent, and that of slave-boson to the X-ray photoemission exponent. In the Kondo effect, the Fermi edge singularity shows up in the intermediate state as pointed out in the Anderson-Yuval approach. The anomalous exponents discussed here reflect this singularity.

VI. SURFACE CRITICAL EXPONENTS IN CHIRAL 1D SYSTEMS

The expression for the FSS obtained for the Kondo problem is essentially common to all 1D chiral quantum systems \[4\]. A number of 1D quantum systems with open boundaries belong to this class. For instance, let us list several examples of exactly solvable models with open boundaries; the Heisenberg model \[21\], the boson model \[22\], the continuum electron model \[23\], the Hubbard model \[24\], and the \(1/r^2\) quantum model \[25\], etc. In contrast to the Kondo problem, however, there is a different point in these 1D critical systems with boundaries. Namely, in 1D quantum models, a continuously varying parameter \(K_f\) enters in the theory, reflecting \(U(1)\) symmetry of the charge sector. For example, the critical exponent of the
SU(\(N\)) interacting electron model with open boundaries is given by:

\[
x_{b} = \Delta M^{T} \left( \begin{array}{ccc}
\frac{1}{N K_{\rho}} + \frac{N-1}{N} & -1 & 0 \\
-1 & 2 & \ddots \\
0 & \ddots & -1 & 2 \\
\end{array} \right) \Delta M, \quad (19)
\]

from which one can see, by comparing it with (16), that the only the charge sector of the matrix is changed. Here the phase shift \(\delta_{l}\) implicitly involved in \(\Delta M^{(l)}\) (see eq. (13)) can be determined by the number of particles with spin \(l\) localized at boundaries, \(n_{l} = \delta_{l}/\pi\). From this spectrum, we can read critical exponents of 1D quantum systems with open boundary conditions. The formula (19) is typical for 1D chiral electron systems with SU(\(N\)) spin symmetry.

Following the arguments given in the previous section, we can apply the above results to the X-ray problem in 1D chiral electron systems. Suppose that electrons move only in one direction and the backward scattering due to the impurity is irrelevant. Such a situation may be realized in the edge state of the fractional quantum Hall effect (FQHE). Let us discuss the X-ray photoemission (or absorption) problem in such chiral systems by applying the formula (19). For example, the critical exponent for the X-ray absorption is given by:

\[
\alpha_{f} = \frac{1}{N K_{\rho}} \left( 1 - \frac{N \delta}{\pi} \right)^{2} + \frac{N - 1}{N}, \quad (20)
\]

whereas the exponent for the photoemission is:

\[
\alpha_{b} = \frac{N}{K_{\rho}} \left( \frac{\delta}{\pi} \right)^{2}. \quad (21)
\]

These expressions should be compared with (17) and (18). Note that for the edge state of the FQHE with filling \(\nu = N/(Nm + 1)\) with even \(m\), \(K_{\rho}\) is solely determined by the filling factor \(\nu\) as \(K_{\rho} = \nu/N\). We expect such anomalous exponents to be observed in the X-ray problem in the edge state of the FQHE.

**VII. EXACT FINITE-SIZE SPECTRUM OF THE MULTI-CHANNEL KONDO MODEL**

We have seen so far that the FSS for the ordinary Kondo effect is in complete accordance with the fusion hypothesis. In this section, we extend our discussions to the multi-channel Kondo model. The Hamiltonian for the multi-channel model is:

\[
H = \sum_{k,l,\sigma} \epsilon_{k} c_{k\sigma}^{\dagger} c_{k\sigma} + J \sum_{k,k',l,\sigma,\sigma'} c_{k\sigma}^{\dagger} \sigma \cdot (\sigma \cdot \sigma') c_{k'\sigma'}^{\dagger} (l = 1, 2, \cdots, n) \quad (22)
\]

with \(J > 0\), where conduction electrons with \(n\)-channels \((l = 1, 2, \cdots, n)\) screen the impurity spin \(S\). Although the multi-channel model is Bethe-ansatz solvable, it is quite difficult to calculate the FSS analytically, in particular, for the overscreening case \(n > 2S\). The difficulty arises from the fact that the Bethe-ansatz solution to the multi-channel model takes the form of the so-called string solution even for the ground state. It has been known that the string solution is valid only in the thermodynamic limit. Thus, a naive application of finite-size techniques fails, giving only the Gaussian part of the spectrum for the spin sector. Namely, the total spectrum has a pathological form,

\[
E = \frac{2 \pi v}{L} \left( \frac{(Q - n)^{2}}{4n} + \frac{S^{2}}{n} + (\text{flavor part}) \right) + nQ + n_{s} + n_{f}, \quad (23)
\]

for the finite system with linear size \(L\), where \(v\) is the Fermi velocity, and \(nQ\), \(n_{s}\), and \(n_{f}\) are non-negative integers. Here the first term expresses charge excitations with the quantum number \(Q\), whereas the second term labels the spin excitations with the magnetization \(S\). Since the spin sector of the multi-channel Kondo model is described by the level-\(n\) SU(2) Kac Moody theory with the central charge \(c_{WZW} = 3n/(2 + n)\), it is seen from (23) that the \(Z_{n}\) parafermion sector with \(c = c_{WZW} - 1 = (n-1)/(n+2)\) is missing. It is thus necessary to exploit alternative methods other than the coordinate Bethe ansatz to obtain the correct spectrum corresponding to the missing \(Z_{n}\) parafermions.

We propose an analytic way to investigate the nontrivial \(Z_{n}\) parafermion part. To this end, we first recall the following properties of the S-matrix for “physical particles” in the overscreening Kondo model. The bulk S-matrix in the spin sector is decomposed into two parts, the S-matrix for the Gaussian model and that for the \(Z_{n}\) model. Since the \(Z_{n}\) model can be described by the restricted solid-on-solid (RSOS) model in the regime \(1/11\), the corresponding S-matrix is given by the face weight in the RSOS model. A remarkable point for the overscreening model is that the interaction between “physical particles” and the impurity is described by the S-matrix of multi-kinks, which is given by the fusion of the face weights of the RSOS model with the fusion level \(p = n - 2S\). Therefore, nontrivial properties in the overscreening model are essentially determined by the RSOS model coupled with the impurity. Thus the exact FSS can be derived by combining the spectrum (23) with that of the RSOS model coupled with the impurity.
$$E_{\text{RSOS}} = \frac{2\pi v}{L} \left( \frac{j(j+1)}{n+2} - \frac{(m+p)^2}{4n} \right) + \text{const}, \quad (24)$$

where $m = 2j \mod 2$, and $j = 0, 1/2, 1, \ldots, n/2$. It is seen that the spectrum fits in with $\mathbb{Z}_n$ parafermion theory, and only the selection rule for quantum numbers is changed by the impurity effect, $m \to (m+p)$.

In order to compare our results with those of Affleck and Ludwig, let us consider the total finite-size spectrum, which is given by the sum of eqs.(23) and (24). It is easily seen that the term $-(m+p)^2/4n$ in (24) is canceled by $S_z^2/n$ in (23) by suitably choosing the quantum number for $S_z$. This in turn modifies the selection rule for quantum numbers in SU(2)$_n$ Kac-Moody algebra. Consequently, we find the total spectrum as

$$E = \frac{2\pi v}{L} \left( \frac{(Q-n)^2}{4n} + \frac{j(j+1)}{n+2} + \text{(flavor part)} \right) + nQ + n_s + n_f \right), \quad (25)$$

with new quantum numbers $\tilde{j} = |j - p/2|$ where $j = 0, 1/2, 1, \ldots, n/2$. Here $Q$ and $j$ are the charge and spin quantum numbers for free electrons without the Kondo impurity. We can say that the effect due to the Kondo impurity is merely to modify the selection rule for quantum numbers of spin excitations by $j \to \tilde{j}$, which indeed results in non-Fermi liquid properties. This is the essence of the Kac-Moody fusion hypothesis proposed by Affleck and Ludwig. We think that the above result may be a microscopic description of the fusion hypothesis for the multi-channel Kondo model.

VIII. SUMMARY

We have investigated critical properties of the Kondo problem by using the Bethe ansatz method and boundary CFT. We have obtained analytically the exact FSS of several models related to the Kondo problem, such as the Anderson model, the SU($N$) Anderson model, the $s$-$d$ exchange model, and the multi-channel Kondo model. The Kac-Moody fusion hypothesis proposed by Affleck and Ludwig for the FSS of the Kondo problem has been shown to be consistent with the exact solution. By applying the finite-size scaling of boundary CFT, boundary critical phenomena in the Kondo problem have been investigated.

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30 G. E. Andrews, R. J. Baxter, and P. J. Forrester, J. Stat. Phys. 35, 193 (1984).
31 A. B. Zamolodchikov and V. V. Fateev, Sov. Phys. JETP 62, 215 (1985).
32 A. M. Tsvelick, Nucl. Phys. B305, 675 (1988).
33 M. J. Martins, Nucl. Phys. B426, 661 (1994).
34 E. Date, M. Jimbo, T. Miwa, and M. Okado, Lett. Math. Phys. 12, 209 (1986).
35 N. Yu. Reshetikhin, Lett. Math. Phys. 7, 205 (1983).
36 A. Klümper and P. A. Pearce, Physica A183, 304 (1992).