Convex polytopes and linear algebra

Jonathan Fine*
1 October 1997

Abstract
This paper defines, for each convex polytope ∆, a family $H_w^\Delta$ of vector spaces. The definition uses a combination of linear algebra and combinatorics. When what is called exact calculation holds, the dimension $h_w^\Delta$ of $H_w^\Delta$ is a linear function of the flag vector $f^\Delta$. It is expected that the $H_w^\Delta$ are examples, for toric varieties, of the new topological invariants introduced by the author in Local-global intersection homology (preprint [alg-geom/9709011]).

1 Introduction

The goal, towards which this paper is directed, is as follows. Suppose ∆ is a convex polytope. One wishes to construct from ∆ vector spaces whose dimension is a combinatorial invariant of ∆. The smaller the dimension of these spaces, the better. The convex polytope ∆ has both a linear structure, due to the ambient affine linear space, and a combinatorial structure, due to the incidence relations among the faces. The construction in the paper uses both structures to produce many ‘vector-weighted inclusion-exclusion formulae’, to each one of which corresponds a complex of vector spaces. When such a complex exactly computes its homology (a concept to be explained later) the result is a vector space of the type that is sought. The proof of exact calculation, which is not attempted in this paper, is expected to be difficult.

In a special case, part of this problem has already been solved. If ∆ has rational vertices then from ∆ a projective algebraic variety $P_\Delta$ can be constructed, and the middle perversity intersection homology (mpih) Betti numbers $h_\Delta$ are combinatorial invariants of ∆, by virtue of the Bernstein-Khovanskii-MacPherson formula $\begin{array}{ll} & 1 \end{array}$, that express the $h_i$ as linear functions of the flag vector $f^\Delta$ of ∆. In this case Braden and MacPherson (personal communication) have proved an exact calculation result. Their proof relies on deep results in algebraic geometry, and in particular on Deligne’s proof $\begin{array}{ll} 2 \end{array}$, $\begin{array}{ll} 3 \end{array}$ of the Weil conjectures. Elsewhere $\begin{array}{ll} 4 \end{array}$, the author has defined local-global intersection homology groups. The construction in this paper corresponds to the extension and unwinding ($\begin{array}{ll} 5 \end{array}$, formulae (8)–(10)) of the extended $h$-vector defined in that paper. This correspondence, which is an exercise in combinatorics, is left to the reader. It might also be presented elsewhere. This paper has been written to be independent of $\begin{array}{ll} 6 \end{array}$.

Central to this paper is the study of flags. They too have linear and combinatorial structure. A flag $\delta$ on a convex polytope ∆ is a sequence

$$\delta = (\delta_1 \subset \delta_2 \subset \ldots \subset \delta_r \subset \Delta)$$

of faces $\delta_i$ of ∆, each strictly contained in the next. The dimension $d = \dim \delta$ is the sequence

$$d = (d_1 < d_2 < \ldots < d_r < n = \dim \Delta)$$

*203 Coldhams Lane, Cambridge, CB1 3HY, England. E-mail: j.fine@pmms.cam.ac.uk
of the dimensions $d_i$ of the terms $\delta_i$ of $\delta$. Altogether there are $2^n$ possible flag dimensions. The number $r$ is the order $\text{ord} \delta$ of the flag $\delta$ (and of the dimension vector $d$). If $\delta$ is a face of $\Delta$, the order one flag whose only term is $\delta$, namely $(\delta \subset \Delta)$, will also be denoted by $\delta$. The empty flag will be denoted by $\emptyset$.

Similarly, if $V$ is a vector space then a flag $U$ on $V$ is a sequence

$$U = (U_1 \subset U_2 \subset \ldots \subset U_r \subset V)$$

of subspaces $U_i$ of $V$, each strictly contained in the next. The dimension $d = \dim V$ is the sequence

$$d = (d_1 < d_2 < \ldots < d_r < n = \dim V)$$

where now the $d_i$ are the dimensions of the $U_i$. Throughout $V$ will be the vector space $\langle \Delta \rangle$ spanned by the vectors that lie on $\Delta$. Each face $\delta_i$ similarly determines a subspace $U_i = \langle \delta_i \rangle$ of $V$. Thus, each flag $\delta$ of faces on $\Delta$ determines a flag $\langle \delta \rangle$ of subspaces of $V = \langle \Delta \rangle$.

The construction of this paper is, in general terms, as follows. Suppose $U$ is a flag on $V$. From $U$ many vector spaces can be constructed. This paper constructs for each $U$, and for each $w$ lying in an as yet unspecified index set, a vector space $U_w$. Now let $U'$ be obtained from $U$ by deleting from $U$ one of its terms $U_i$. It so happens that this deletion operator induces a natural map

$$U_w \to U'_w$$

between the associated $w$-spaces. (In the simplest case, which corresponds to $\text{mph}$, all the $U_w$ are subspaces of a single space $V_w$, and the maps are inclusions. In general there is no such a global space $V_w$.) Suppose $U'$ is obtained from $U$ by deleting two or more terms. By choosing an order for the deletion of these terms, a map $U_w \to U'_w$ can be obtained. The single-deletion map is natural (or geometric) in that the induced multiple deletion maps are independent of the choice of deletion order.

Suppose now that such spaces $U_w$ have been defined for every flag on $V$, and that $\Delta$ is a convex polytope whose vector space $\langle \Delta \rangle$ is $V$. Each flag $\delta$ on $\Delta$ thus determines a flag $\langle \delta \rangle$ on $V$, and thus a vector space $\langle \delta \rangle_w$, or $\delta_w$ for short. These vector spaces will be assembled into a complex, according to the order $r$ of $\delta$. Define the space $\Delta(w,r)$ of $w$-weighted $r$-flags to be the direct sum of the $\delta_w$, where $\delta$ has order $r$. Each vector in $\Delta(w,r)$ can be thought of as a formal sum $\sum v_\delta [\delta]$, where $[\delta]$ is a formal object representing an $r$-term flag $\delta$, and where the coefficient $v_\delta$ is drawn from the vector space $\delta_w$. By the assumptions of the previous paragraph, the deletion operator on flags determines a differential

$$d : \Delta(w,r) \to \Delta(w,r-1)$$

and so induces a complex

$$0 \to \Delta(w,r) \to \Delta(w,r-1) \to \cdots \Delta(w,1) \to \Delta(w,0) \to 0$$

of vector spaces. (As is usual, $d = \sum (-1)^{i+1} \partial_i$, where $\partial_i$ is the operator induced by deletion of the $i$-th term. Because the maps $U_w \to U'_w$ are natural, $d^2$ is zero, and thus one indeed has a complex.)

Exact calculation is when this complex is exact except at one point, say $\Delta(w,j)$. In that case the homology $H_w \Delta$ at that point is a suitable alternating sum of the dimensions of the $\Delta(w,i)$, and thus of the $\delta_w$. By construction the dimension of $\delta_w$ will depend only on the dimension vector $d$ of $\delta$, and thus (provided exact calculation holds) one has that the dimension $h_w \Delta$ of $H_w \Delta$ is a linear function of the flag vector $f_\Delta$ of $\Delta$, and so is a combinatorial invariant. (This is because the $f_{d\Delta}$ component of $f \Delta$ counts how many $d$-flags there are on $\Delta$. Each $f_{d\Delta}$ contributes, up to an
alternating sign, the quantity $\lambda_d = \dim \delta_w$ to $h_\Delta$, where $\delta_w$ is a coefficient space due to any flag $\delta$
of dimension $d$.

Convex polytopes are not the only combinatorial objects for which the concept of a flag can be defined. In [6, 7], the author defines flag vectors for $i$-graphs, or more generally any object that is a union of cells (or edges), and which can be shelled. There seems to be no reason why the general form of the construction described here cannot also be applied in this new context. This is not to say the the proper choice of the vector spaces $\delta_w$ to be associated to the flags $\delta$ is not expected to be a deep question. Both for convex polytopes and for $i$-graphs there are subtle and presently unknown combinatorial inequalities on the flag vectors. An exact homology theory, using vector spaces such as the $\delta_w$, is the only method the author can envision, that will lead to the proof such inequalities.

The exposition is organised as follows. The next section (§2) give the definition of exact homology. The deletion operator applied to flags yields the flag complex (§3). Next comes a complex (§4) that corresponds to middle perversity intersection homology. The local-global variant is more complicated, and is the substance of the paper. First the coefficient spaces to be used are defined (§5), then the maps between them (§6), and finally the local-global homology (§7).

The purpose of §§3–7 is to present the definition as the result of a study of the geometric resources and constraints. Conversely, in §8 decorated bar diagrams are used to reformulate the definition, and allow the properties to be demonstrated, in a concise manner. Finally, §9 provides a wider discussion of what has been done, and what remains to be done.

The reader may at first find §§5–7 somewhat abstruse. They contain a study of the linear algebra of a segmented flag of vector spaces. Once the problem is understood, the key definitions come out in a fairly natural way. In §8, the same definitions are presented, but this time via coordinates. Here, the definitions are clear, but may appear somewhat arbitrary. Each point of view informs the other. This paper attempts to show how the definitions arise naturally out of the logic of the situation, and thus places §§5–7 before §8. The reader may wish to reverse this order.

The basic devices are the combinatorics of flags, and linear algebra. The exterior algebra on a vector space is widely used. The fibre of the moment map from a projective toric variety to its defining polytope is always a product of circles, and so the homology of the fibre is isomorphic to an exterior algebra. This fact is the beginning of the connection between the linear algebra of flags and the existence of cycles on the toric variety. The reader does not need to know this.

Throughout this paper $\Delta$ will be a convex polytope of some fixed dimension $n$, and $V$ will be the vector space spanned by the vectors lying on $\Delta$. It will do no harm to think of $\Delta$ as lying in $V$.

## 2 Exact Homology

This section explains the concept of exact homology. Suppose that a sequence

$$A = (0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0)$$

of vector spaces is given, together with a map

$$d : A_i \to A_{i-1}$$

between successive terms. If, for each $i$, the composite map

$$d \circ d = d^2 : A_i \to A_{i-2}$$

is exact, then $A$ is called exact.
is the zero map, then \( A \) is called a **complex**, and \( d \) its **boundary map**. If \( A \) is a complex then the statement

\[
\text{im}(d : A_{i+1} \to A_i) \subseteq \ker(d : A_i \to A_{i-1})
\]

restates the condition \( d^2 = 0 \), and the quotient of the above kernel by the image is called the \( i \)-th homology \( H_i A \) of the complex \( A \). This formalism originated in the definition, via chains and cycles, of the homology groups \( H_\ast X \) of a topological space \( X \). There, most of the homology groups were expected to be non-zero. The present use will be different.

Suppose \( A \) is a complex. If all the homology groups \( H_i A \) are zero (i.e. at each \( A_i \) the kernel and image are equal) then \( A \) is called a (long) **exact sequence**, or **exact** for short. If \( A \) is exact then the alternating sum

\[
\sum_{i=0}^n (-1)^i \dim A_i
\]

of the dimensions of the \( A_i \) (assumed finite) will be zero. To prove this, introduce in each \( A_i \) a subspace \( B_i \) that is a complement to \( dA_{i+1} \) in \( A_i \). The dimension of \( A_i \) is the sum of the dimensions of \( dA_{i+1} \) and of \( B_i \). The exactness assumption implies that the restricted form

\[
d : B_i \to dA_i \subseteq A_{i-1}
\]

of the boundary map is an isomorphism. Thus, the contributions of \( B_i \) and \( dA_i \) to the alternating sum are equal but opposite, and so the result follows.

Suppose \( A \) is a complex. If \( H_i A \) is zero then \( A \) is said to be **exact at \( A_i \)**. Now suppose that \( A \) is known to be exact at all its \( A_i \) except perhaps one, say \( A_r \). In this case the alternating sum \((\square)\) of the dimensions gives not zero, but the dimension of the only non-zero homology \( H_r A \) of the complex \( A \), multiplied by \( (-1)^r \). If a complex is exact at all but one location, we shall say that it **exactly computes** the homology at that location.

Now suppose that the \( A_i \) are constructed from the convex polytope \( \Delta \), and that their dimensions depend only on the combinatorial structure of \( \Delta \). The same will then be true for \( H_r A \), provided that for each \( \Delta \) the complex exactly computes its homology at \( A_r \). If this holds we shall say that \( H_r A \) is an **exact homology** group of \( \Delta \). It is of course one thing to define a complex \( A \), as is done in this paper, and quite another to prove its exactness. This is expected to require new concepts and methods.

### 3 The flag complex

The convex polytope \( \Delta \) has both a combinatorial structure (incidence relations among faces) and a linear structure (the vectors lying on a face \( \delta \) span a subspace \( \langle \delta \rangle \) of \( V \)). In general both structures will be used to define the complex \( A \). This section defines a complex that uses the combinatorial structure alone. The general case will arise by allowing vectors to be used instead of numbers in the construction that follows.

Recall the definition, in §1, of a flag \( \delta \) on \( \Delta \), its dimension vector \( d \), and its order \( r \). Now suppose \( \delta \) is a flag on \( \Delta \), of order \( r \). By removing one or more of the terms \( \delta_i \) from \( \delta \), new flags can be obtained, of lower order. Let \( \partial_j \) be the **deletion operator** that removes from a flag \( \delta \) the \( j \)-th term.

The operators \( \partial_j \) do not commute. Removing say the 2nd term from a sequence, and then say the 4th, gives the same final result as does first removing the 5th and then the 2nd. This is because removing the 2nd term will cause the subsequent items to move down one place in the sequence. The equation

\[
\partial_j \partial_k = \partial_k \partial_{j+1} \partial_j \quad \text{for } j < k
\]
is an example of the *commutation law* for these deletion operators.

Now let \( A_i \) consist of all formal weighted sums \( x \) of the form

\[
x = \sum_{\text{ord } \delta = i} x_\delta [\delta]
\]

where the coefficients \( x_\delta \) are numbers and, as indicated, the sum is over all flags \( \delta \) on \( \Delta \), which have \( i \) terms. Here \([\delta]\) denotes \( \delta \) considered as a formal object. Where confusion will not then result, \( \delta \) will be written in its place. Thus, a vector \( x \) in \( A_i \) is a formal sum of \( i \)-term flags, with numeric coefficients \( x_\delta \).

The deletion operation \( \partial_j \) induces a map \( A_i \to A_{i-1} \). (It is zero if \( j \) is larger than \( i \)). Now use the formula

\[
d = \partial_1 - \partial_2 + \cdots + (-1)^{i-1} \partial_i + \cdots
\]

to define a map \( d : A_i \to A_{i-1} \). It follows immediately from the commutation laws, that \( d^2 = 0 \), and so \( A \) is a complex. It will be called the *flag complex* of \( \Delta \). It depends only on the combinatorial structure of \( \Delta \).

4 Global coefficient spaces

Instead of using numeric coefficients for the formal sums that constitute the space \( A_i \), one could instead write

\[
v = \sum_{\text{ord } \delta = i} v_\delta \delta
\]

where now \( v_\delta \) is to lie in a vector space \( \Lambda(\delta) \) that is in some way associated to \( \delta \). Thus, \( A_i \) is again to be formal sums of order \( i \) flags, where now the coefficients are to be vectors rather than numbers. This section will describe the simplest way of constructing such coefficient spaces \( \Lambda(\delta) \).

Suppose \( \delta' \) is obtained from \( \delta \) by the deletion of one or more terms. For such a definition to produce a complex, there must also be a natural map \( \Lambda(\delta) \to \Lambda(\delta') \) between the corresponding coefficient spaces. One way to do this, which will be used in this section, is to have this map be an inclusion. Thus, all the coefficients will lie in the global coefficient space \( \Lambda(\Delta) \) associated to the empty flag. Each flag \( \delta \) will then define a subspace \( \Lambda(\delta) \) of \( \Lambda(\Delta) \).

For this to work, one must have that \( \Lambda(\delta) \) is a subspace of \( \Lambda(\delta') \). One way to do this is to have each individual face \( \delta_j \) in \( \delta \) define a condition (or set of conditions) on \( \Lambda(\Delta) \). Now define \( \Lambda(\delta) \) to be those vectors in \( \Lambda(\Delta) \) that satisfy the condition(s) due to the faces \( \delta_j \) in \( \delta \). Provided the conditions due to \( \delta_j \) depend only on the face \( \delta_j \) (and not on its location in \( \delta \), or whatever), it will follow automatically that \( \delta' \) will provide fewer conditions, and so it will be certain that \( \Lambda(\delta') \) will contain \( \Lambda(\delta) \) as a subspace.

To summarise this section so far, suppose that a vector space \( \Lambda(\Delta) \) is given, and for each face \( \delta \) of \( \Delta \) a subspace \( \Lambda(\delta) \) is given (here \( \delta \) stands for the flag which has \( \delta \) as its only term). From this a complex \( A \) can be constructed. First define \( \Lambda(\delta) \) to be the intersection of the spaces \( \Lambda(\delta_j) \) associated to the terms \( \delta_j \) of \( \delta \). Next define \( A_i \) to be all formal sums \( (2) \), where the coefficients \( v_\delta \) are to lie in \( \Lambda(\delta) \). Finally, the boundary map

\[
d(v_\delta \delta) = v_\delta [\partial_1 \delta] - v_\delta [\partial_2 \delta] + \cdots
\]

is defined just as before. To complete such a definition, one must provide a vector space \( \Lambda(\Delta) \), and derive from each face \( \delta \subset \Delta \) a subspace \( \Lambda(\delta) \) of \( \Lambda(\Delta) \).
Recall that $V$ stands for the span of the vectors lying on $\Delta$. Let $V^*$ be the dual space of linear functions. That such a linear function $\alpha$ is constant on $\delta$ (i.e. zero on the vectors lying on $\delta$) describes a subspace $\delta^\perp$ of $V^*$.

Exterior algebra will now be used. Fix a degree $r$, and let $\Lambda(\Delta)$ be the $r$-fold exterior product $\Lambda^r = \bigwedge^r V^*$ of arbitrary linear functions on $V$. The face $\delta$ defines a filtration of $\Lambda(\Delta)$ in the following way. For each decomposition $r = s + t$ one can take the span of expressions of the form

$$\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_s \wedge \beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_t$$

where the $\alpha_i$ are to vanish on $\delta$. No conditions are placed on the $\beta_i$. The result is of course a subspace of $\Lambda(\Delta) = \bigwedge^r V^*$.

To conclude this definition it is enough, for each face $\delta$ of $\Delta$, to choose one of these subspaces of $\Lambda(\Delta)$. One would like the resulting complex to produce an exact homology group, a matter which is presently not well understood, and which involves concepts that lie outside the scope of this paper.

The condition

$$s > \text{codim} \delta - s$$

is satisfied by some smallest value of $s$. (The codimension $\text{codim} \delta$ is defined as usual to be $\dim \Delta - \dim \delta$.) Use this value to define for each face $\delta$ the subspace $\Lambda(\delta)$. If the resulting $s$ is greater than $r$, then $\Lambda(\delta)$ is taken to be zero.

This choice of spaces corresponds to middle perversity intersection homology (for the associated toric variety $P_\Delta$, if it exists). There is little doubt that this gives the correct choice of subspaces, for reasons that will be discussed in the final section.

5 Local coefficient spaces

To produce a complex $A$ one requires a coefficient space $\Lambda(\delta)$ for each flag $\delta$ on $\Delta$, and natural maps $\Lambda(\delta) \rightarrow \Lambda(\delta')$ whenever $\delta'$ is obtained from $\delta$ by the deletion of one or more terms. The previous section assumed the maps were inclusions (and so all the $\Lambda(\delta)$ were subspaces of $\Lambda(\Delta)$). This section will relax this assumption, to obtain the coefficient spaces that will later be used to define further complexes. In the next section, the boundary map will be defined.

Suppose $V_1$ is a subspace of $V$. A basic construction of the previous section was to use $V_1$ to define subspaces (in fact a filtration) of the $r$-fold exterior product $\bigwedge^r V^*$. Such a construction is in fact forced upon us, provided we assume that the deletion operator on flags induces inclusion, and also that $\Lambda(\Delta) = \bigwedge^r V^*$.

Relaxing this assumption allows the following. Given $V_1 \subset V$ one can form the vector spaces $V_1$ and $V/V_1$ and then form the tensor product

$$\bigwedge^{r_1} V_1^* \otimes \bigwedge^{r_2} V_1^\perp$$

of the corresponding exterior products. Here, $V_1^\perp$ consists of the $\alpha$ in $V^*$ that vanish on $V_1$. It is of course naturally isomorphic to $(V/V_1)^*$.

In this way $V$ can be broken into two or more segments, within each of which the construction of the previous section can be applied. For example, each subspace $U$ of $V_1$ will determine a filtration of the first factor $\bigwedge^{r_1} V_1^*$ above. Similarly, if $U$ lies between $V_1$ and $V$, a filtration of the second factor $\bigwedge^{r_2} V_1^\perp$ will arise.

All the coefficient spaces $\Lambda(\delta)$ will be obtained in this way. Given a flag $\delta$ use some (perhaps all or none) of its terms $\delta_i$ to segment $V$. This gives a tensor product of exterior algebras. For each
factor choose a component, i.e. a degree. The resulting space is used in the same way as $\Lambda(\Delta)$ was, in the previous section. Note that this space depends on the flag, or more exactly the subflag used to segment $V$. There is no longer one global space, in which all the coefficient vectors lie.

Now choose a term $\delta_j$ of $\delta$. This can be used to filter $\wedge^{r+1}(V_{k+1}/V_k)^*$, where $\delta_k$ is the largest segmenting face contained in $\delta_j$. (If there is none such, set $k = 0$ and use $\wedge^{r+1} V_1^*$ instead.) Now, as before, use the condition

$$s_j > \text{codim } \delta_j - s_j$$

to select a term in the filtration. Here, however,

$$\text{codim } \delta_j = \dim \delta_{k+1} - \dim \delta_j$$

is to be the codimension of $\delta_j$ not within $\Delta$ but within its segment.

The above filtration is to be applied for all the terms in the flag, including those used to segment. Such terms can produce of course only a trivial filtration. However, when

$$r_k > (\dim \delta_k - \dim \delta_{k-1}) - r_k$$

holds, the whole of $\wedge^r(V_k/V_{k-1})$ is to be used, as the coefficient space for $[\delta]$. If the condition fails, zero is the only coefficient to be used with $[\delta]$.

This concludes the definition of the coefficient spaces $\Lambda(\delta)$. Note that to specify such a space the following is required, in addition to $\delta$. One must select some (or all or none) of the faces of $\delta$, to be used for segmentation. One must also specify a degree $r_k$ for each segment. A more explicit notation might be

$$\Lambda(\delta, -r, s)$$

where $r$ is the sequence $(r_1, r_2, \ldots)$ of degrees, and $s$ gives the subflag $\delta_s$ of $\delta$ that is used to segment $\delta$. The minus sign in $-r$ is to distinguish this notation from $\Lambda(\delta, r, s)$, which will be introduced later. As already noted, each degree must be greater than half the length of the corresponding segment, for the coefficient space to be non-zero.

6 The boundary map

Suppose that $\delta$ is a flag of $\Delta$, and that some segmentation $s$ of $\delta$ is chosen, which breaks $\delta$ (and $V$) into $l$ segments. Suppose also that a multi-degree $r = (r_1, \ldots, r_l)$ is given. The construction of the previous section yields a coefficient space $\Lambda(\delta, -r, s)$. Now suppose that $\delta'$ is obtained from $\delta$ by removing one of the terms $\delta_j$ from $\delta$. Provided $\delta_j$ was not used to segment $\delta$, the space $\Lambda(\delta', -r, s)$ will contain $\Lambda(\delta, -r, s)$.

One could obtain a complex by not allowing deletion to occur only at the faces $\delta_j$ used to segment $\delta$, or rather setting the result of deleting such a face to be zero. The resulting complex and its homology will not however properly speaking be an invariant of the polytope $\Delta$. Rather, for each choice of a segmenting flag $\delta_s$ one will have a complex, and any invariant of $\Delta$ so defined will simply be the direct sum of these flag contributions.

The purpose of this section is to define a map from $\Lambda(\delta)$ to $\Lambda(\delta')$, where $\delta'$ is obtained from $\delta$ by the removal of a segmenting face $\delta_j$. This will have the effect of producing a single global complex, that will in general be indecomposable. It can be thought of as a gluing together of the local complexes of the previous paragraph.

Here is an example. Suppose one has subspaces $V_1 \subset V_2 \subset V$, where $V_i$ is the span $\langle \delta_i \rangle$ of vectors lying on the face $\delta_i$. The segmentation of $V$ due to $V_1$ will be compared to that due to $V_2$. 7
In both cases, the raw materials are firstly linear functions $\alpha$ defined on $V_i$, and secondly linear functions $\beta$ vanishing on $V_i$ (and defined on the whole of $V$).

Now suppose that $\alpha_2$ is defined on $V_2$. Because $V_1 \subset V_2$, the linear function $\alpha_2$ can be restricted to give $\alpha_1$ defined on $V_1$. This is straightforward. Now suppose that $\beta_2$ vanishes on $V_2$ (and is defined on $V$). Again because $V_1 \subset V_2$, the linear function $\beta_2$ also vanishes on $V_1$. Thus there is a linear map (restriction of range $\otimes$ relaxation of condition)

$$\bigwedge^r V_2 \otimes \bigwedge^s V_2^\perp \rightarrow \bigwedge^r V_1^* \otimes \bigwedge^s V_1^\perp$$

(3)

between the basic spaces associated to the two segmentations.

This map has an interesting relation to the conditions used to define the coefficient spaces $\Lambda(\delta)$. When the condition

$$s > \text{codim } V_i - s$$

holds, the coefficient space due to the flag $(\delta_i)$ will be one of the above tensor products. When the condition fails, the coefficient space is zero.

Because $V_1$ is a subspace of $V_2$, the condition for $(\delta_1)$ is more onerous than that for $(\delta_2)$, and so the map (3) is going in the wrong direction, to map the one coefficient space to the other.

Regarding conditions however, the situation is different. The trick is to think of the conditions that define the coefficient spaces $\Lambda(\delta_i)$ to be themselves subspaces of an exterior algebra (or more exactly a tensor product of such). If $U$ is a vector space of dimension $l + m$, then each subspace of $\bigwedge^l U$ determines a subspace of $\bigwedge^m U$, and vice versa. This is via the nondegenerate pairing

$$\bigwedge^l U \otimes \bigwedge^m U \rightarrow \bigwedge^{l+m} U$$

provided by the exterior algebra. By a slight abuse of language, this will be called duality. (The value space $\bigwedge^{l+m} U$ is has dimension one, but is not naturally isomorphic to $\mathbb{R}$.)

It is now necessary to formulate the conditions using this new point of view. Suppose $U \subset V$ is a subspace, and $W \subset \bigwedge^r V^*$ is spanned by

$$\alpha_1 \wedge \ldots \wedge \alpha_l \wedge \beta_1 \wedge \ldots \wedge \beta_m$$

where the $\alpha_i$ are to vanish on $U$. Consider the space $W' \subset \bigwedge^{r'} V^*$ spanned by

$$\alpha_1 \wedge \ldots \wedge \alpha_{l'} \wedge \beta_1 \wedge \ldots \wedge \beta_{m'}$$

where $r + r' = \text{dim } V$, $l + l' = \text{codim } U + 1$, and as before the $\alpha_i$ vanish on $U$; this is the subspace of the complementary component of the exterior algebra, determined mutually by $W$.

The $\beta_i$ are arbitrary, and so by a further application of duality, they can be dropped from the definition of the condition space. Thus, take the span of

$$\alpha_1 \wedge \ldots \wedge \alpha_{l'}$$

for $l + l' = \text{codim } U + 1$, $\alpha_i$ vanishing on $U$, as the condition space for $W$.

We now return to the change of segmentation map (3). As already noted, the conditions for $V_2$ are less onerous than those for $V_1$. The condition space for $V_2$ is either zero (no conditions) or $\bigwedge^{\text{codim } V_2} V_2^\perp$ (zero is the only solution to the conditions), and similarly for $V_1$. The map (3) will in either case respect these conditions. (To make sense of this statement, one should think of the conditions as being an ideal in the exterior algebra, and then the image under (3) of the one ideal is contained in the other.)
The goal of this section is now in sight. It is the conditions that are respected by the natural map that is due to change of segmentation, not the coefficient spaces defined by the conditions. Interpret each coefficient vector \( \nu_\delta \) as a linear function, taking values in a tensor product of top-degree exterior products, that vanishes on the condition space.

Recall that the example \( V_1 \subset V_2 \subset V \) gives rise to a natural map

\[
\bigwedge^r V_2^* \otimes \bigwedge^s V_2^\perp \to \bigwedge^r V_1^* \otimes \bigwedge^s V_1^\perp
\]

which respects conditions. Now let \( \nu_\delta \) be a coefficient vector on \( V_1 \), interpreted as a linear function on the range of the above map.

Using the map, this linear function on the range can be pulled back to give a linear function on the domain. Because \( \nu_\delta \) vanishes on the \( V_1 \) condition space, the pull-back vanishes on the \( V_2 \) condition space. By duality, the pull-back linear function is associated to a unique coefficient vector \( \nu'_\delta \) for the \( V_2 \) segmentation. This is an example of a change of segmentation component of the boundary map. Note that it takes a \( V_1 \)-segmentation coefficient to a \( V_2 \) such. In other words, under boundary the segmenting term(s) may move rightwards, or in other words, increase in dimension.

To finish, there are some details to be taken care of. First, although the one-dimensional value spaces for \( V_1 \) and \( V_2 \) are not the same, they are naturally isomorphic. This is good enough. Secondly, because the map is natural, the argument that shows \( d^2 = 0 \) works just as before. Thirdly, the above argument has been applied only to the conditions due to the segmenting faces. The reader may wish to show that it works also for the conditions, as imposed in the previous section. (An alternative way of defining the map and checking the statements made will be outlined, as part of the discussion of bar diagrams.)

The final matter concerns the degree of the coefficient \( \nu_\delta \). The map preserves the degree of (tensor products of) exterior powers. It induces the map on coefficients via duality and pull-back, and so as (tensor products of) exterior powers the coefficients \( \nu_\delta \) on \( V_1 \) and \( \nu'_\delta \) on \( V_2 \) will have different degrees. However, they will by definition have the same co-degree, by which is meant the amount by which they fall short of being of top degree. For this reason, in the rest of the paper the coefficient spaces will be indexed by co-degree. For this the notation \( \Lambda(\delta, r, s) \) will be used. Here, \( \delta \) stands for a flag that has been broken into \( l \) segments by segmentating data \( s \), and \( r = (r_1, r_1, \ldots, r_l) \) provides a co-degree for each segment. The same flag can perhaps be segmented in many ways, even if \( l \) is fixed. The boundary map preserves co-degree.

7 Local-global homology

The main definition of this paper can now be given. Recall that if \( \delta \) is a flag on the convex polytope \( \Delta \), and \( s \) breaks \( \delta \) (and \( V \)) into \( l \) segments, and if a multi-component co-degree \( r = (r_1, \ldots, r_l) \) is given, then from all this a coefficient space \( \Lambda(\delta, r, s) \) has been defined. Recall also that if \( \delta' \) is obtained from \( \delta \) by removal of the \( i \)-th term from \( \delta \), then there is a natural map

\[
\partial_i : \Lambda(\delta, r, s) \to \Lambda(\delta', r, s')
\]

such that the usual definition of \( d \) will produce a boundary map on

\[
A_r = (0 \to A_{r,n} \to A_{r,n-1} \to \ldots \to A_{r,1} \to A_{r,0} \to 0)
\]

where \( A_{r,i} \) is the direct sum of the \( \Lambda(\delta_i) \) for \( \text{ord} \delta = i \).

(The following detail is important. When \( \delta_i \) is removed from \( \delta \) to obtain \( \delta' \), one might as a result have to change the segmentation \( s \). This happens when \( \delta_i \) is used to segment \( \delta \). In this case
the next term $\delta_{i+1}$ is used to segment $\delta'$. If $\delta_{i+1}$ is already used by $s$ to segment $\delta$, or does not exist because $\delta_i$ is the last term in $\delta$, then this component of the boundary map is treated as zero. In this way, one obtains either a satisfactory $s'$, or the zero map.

Thus, for each co-degree $r = (r_1, \ldots, r_l)$, a complex $A_r$ has been defined. The boundary map $d$ of $A_r$ may cause the terms of the segmenting flag $\delta_i$ to move rightwards. This observation leads to the following. One can define subcomplexes of $A_r$ by placing conditions of the segmentation subflags that are to be used. These conditions must of course allow the movement to the right of the segmentation terms.

One way to formulate this is to introduce a multi-dimension $s = (s_1 < \ldots < s_{l-1})$ and allow only those segmentations $\delta$ to be used, for which the dimension $d = (d_1 < \ldots < d_l)$ is term by term at least as large as $s$. In this way one obtains for each $r$ and $s$ a complex $A_{r,s}$. Of course, if $s$ is too great compared to $r$, then the complex will be zero. The complex $A_{r,s}$ is the $A_w$ mentioned in §1.)

We now define the $(r, s)$ homology space $H_{r,s}\Delta$ of $\Delta$ to be the homology of $A_{r,s}$ at the level where $\text{ord} \delta = i$ is equal to the total degree of $\Lambda(\delta, r, s)$. In other words, it is the homology at $A_{r,s;i}$, where $i + r_1 + \ldots + r_l = n$, for $r$ gives the co-degree. If $A_{r,s}$ exactly computes $H_{r,s}\Delta$ then its dimension $h_{r,s}\Delta$ is of course a linear function of the flag vector. The next section clarifies this.

As mentioned in the introduction, these spaces correspond to the local-global intersection homology spaces of $P_\Delta$, introduced in [8].

8 Decorated bar diagrams

The previous discussion made no use of coordinates, and does not give the dimension of the various $\Lambda(\delta, r, s)$ spaces involved in the construction of local-global homology. This section provides another approach. The contribution made by a flag $\delta$ to the homology $H_{r,s}\Delta$, and the maps between these contributions, can be described using coordinates, via the use of decorated bar diagrams. (Part of the theory of bar diagrams was first published in the survey paper [1].)

Suppose, for example, that the dimension $n$ of $\Delta$ is eleven, and that $\delta$ is a flag of dimension $d = (3 < 5 < 9 < 11)$. The (undecorated) bar diagram

\[ \ldots | \ldots | \ldots | \ldots | \ldots | \ldots \]

expresses this situation. There are eleven dots, and a bar ‘|’ is placed after the $d_i$-th dots. Each bar represents a term $\delta_i$ of $\delta$. Now choose a basis $e_1, \ldots, e_n$ of $V$ such that for each $i$, the initial sequence $e_1, \ldots, e_j$ (with $j = d_i$) is a basis for the span $\langle \delta_i \rangle$ of the vectors lying on the $i$-th face. Finally, let each dot represent not $e_i$ but the corresponding linear function $\alpha_i$, that vanishes on each $e_j$ except $e_i$. Call this a system of coordinates for $V$, that is subordinate to $\delta$.

By construction, each dot represent a linear function that vanishes on the faces $\delta_i$ of $\delta$ represented by the bars that lie to its left. So that we can speak more concisely, we shall think of the dots and bars as actually being the linear functions and terms of the flag respectively.

Each subset of the dots represents an element of the exterior algebra generated by the linear functions on $V$. To represent the selection of a subset, promote the chosen dots ‘.’ into circles ‘o’. Thus

\[ \ldots o|o.|o.o.|o.o.|o.o.|o.o.| \]

represents (or more concisely is) a degree six element of the exterior algebra. It is an example of a decorated bar diagram.

(Each circle represents an element in the homology of the torus, that is the generic or central fibre of the moment map. As the fibre is moved towards a face that lies to the left of the circle,
so the circle shrinks to a point. Thus, circles represent 1-cycles with specified vanishing properties. Similarly, the promotion of say 6 dots to circles produces a diagram that represent a 6-cycle in the generic fibre, with specific shrinking properties, as the fibre is moved to the faces of the flag. This will be important, in the topological interpretation of exact homology.)

The exterior form (4) has certain vanishing properties, with respect to the faces of $\delta$. The condition of §4 is equivalent to the following: That between each bar and the right hand end of the diagram, there should be strictly more ‘o’s than ‘.’s. Our example satisfies this condition, and so is an admissible decorated bar diagram.

Now fix an unsegmented degree $r = (r_1)$ and define the coefficient space $\Lambda(\delta, r)$ to be the span of the admissible $r$-circle bar diagrams (considered as exterior forms). (Because there is no segmentation, $s$ is trivial, and will be omitted.) For the $\Lambda(\delta, r)$ to come together to produce a complex, the following must hold. First, $\Lambda(\delta, r)$ as a subspace of the degree $r$ forms should not depend on the choice of a basis subordinate to $\delta$. Second, the boundary map should not depend on the basis (or in other words should be covariant for such change). Third, when a bar is removed from an admissible diagram, the result should also be admissible. The last two conditions are immediately seen to be true.

There are two ways to see that the first requirement (that $\Lambda(\delta, r)$ not move when the subordinate basis is changed) is true. One method is to show that the span $\Lambda(\delta, s)$ of the admissible diagrams is the solution set to a problem that can be formulated without recourse to use of a basis. This is the approach taken in §§3–7. The other method is to show directly that $\Lambda(\delta, r)$ does not move, under change of subordinate basis.

Any change of subordinate basis can be obtained as a result of applying the following moves. First, one can multiply basis elements by non-zero scalars. Second, one can permute the basis elements (dots and circles), provided so doing does not cause a dot or circle to pass over a bar. Thirdly, one can increase a basis element by some multiple of another basis element, that lies to its right.

It is clear that applying either of the first two moves to the basis will not change $\Lambda(\delta, r)$. Regarding the third move, if a diagram such as (4) is admissible, then the result of moving one or more ‘o’s to the right, perhaps over bars, will also be admissible. This is obvious, from the nature of the conditions defining admissibility. The third type of move makes such changes. Thus, the span $\Lambda(\delta, r)$ of the admissible decorated diagrams does not move.

It has now been shown that the admissible bar diagrams, such as (4), define coefficient spaces $\Lambda(\delta, r)$ that can be assembled to produce a complex. It is left to the reader, to check that it is exactly the same complex, as was defined in §4.

The remainder of this section is devoted to the description of the segmented form of the above construction. As before, let 

\[ \ldots | \ldots | \ldots | \ldots \]

denote a flag of dimension $d = (3 < 5 < 9 < 11)$, and now choose some of the bars, say just the second, to segment the diagram. The result is

\[ \ldots | \ldots | \ldots \]

or more concisely

\[ \ldots | \ldots | \ldots \]

where the promotion of a ‘|’ to a ‘!’ indicates that is is being used for segmentation. Note that each segmenting face is also used as the first face in the following segment.
As before, one can promote some of the ‘. ’s to ‘o ’s to obtain a decorated bar diagram, which will be admissible if between any bar (either ‘|’ or ‘!’) and the end of the segment, there are more ‘o ’s than ‘. ’s.

To be able to assemble the span of the admissible diagrams into a complex, certain requirements must be met. They have already been formulated. The first is that the span should not depend on the choice of a subordinate basis. Scalar and permutation moves on the basis clearly leave the span unchanged, as in the single segment case. Adding to one basis vector another, lying to its right, has no effect on the span, provided the second lies in the same segment as the first.

Now consider the situation, where one of the basis linear forms is changed, by adding to it another basis linear form, as before lying to the right, but this time in a different segment. The way to have this have no effect on the span is to have the dots and circles in a segmented diagram represent not linear functions on $V$, but rather linear functions on the (span of vectors lying on) the face at the right end of the segment (which is $V$ for the last segment).

Thus, by having each decorated diagram (possibly segmented) represent an element in a tensor product of exterior algebras, the span of all the admissible diagrams becomes a space that does not move, when the subordinate basis is changed. This agrees with §5.

For these spaces to be assembled into a complex, the boundary map must be well defined. As in the single segment situation, all is well when a non-segmenting term (a ‘|’ rather than a ‘!’) is removed from a decorated bar diagram.

The removal of a ‘!’ terms has a more subtle effect, for the segmentation will have to change. Whatever rule is used, it must respect admissibility of decorated diagrams, and it must respect change of subordinate basis.

The rule, which we state without prior justification, is this: whenever something such as

```
|ooooo|
```

occurs in a decorated diagram, one can obtain a component of the boundary by replacing it with

```
ooooo!
```

while the result of removing any other ‘!’ terms is zero. (The number of ‘o ’s is not relevant, that one has ‘! ’ followed by some circles, and then a ‘|’ is.)

It is clear that this rule will, from admissible diagrams, generate only admissible diagrams. This is because it can but only cause some extra ‘o ’s to appear, whenever the ‘.’ and ‘o ’ counts are to be compared. The same is of course not true, for the replacement of ‘! . o . |’ by ‘. o . ! ’ and similar situations.

The next task is to show that this part of the boundary respects change of subordinate basis. Given a fragment such as ‘! . o . |’ that contributes zero to this part of the boundary, whatever change of basis is made, the contribution will still be zero. Consider now a fragment such as ‘|oooo|’. Scalar and permutation moves will have no effect on the boundary. This is obvious. The only way that adding something that lies to the right can have any effect is if that something lies to the right of the whole fragment ‘|oooo|’. This is because the exterior algebra is antisymmetric. Consider now the boundary contribution ‘oooo!’. As already described, the ‘! ’ induces a restriction of linear functions, and so the just considered change of basis will after all have no effect on the boundary map.

This concludes the presentation of the complex $A$ via decorated bar diagrams. The details of co-degree (number of ‘. ’s in each segment) and so forth are as in the earlier exposition (§5–7). It is left to the reader to verify that the two approaches lead to exactly the same family $A_{r,s}$ of complexes.
9 Summary and conclusions

The flag vector of a convex polytope satisfies subtle linear inequalities, and also non-linear inequalities, that are at present not known. Exact homology seems to be the only general method available to us, to prove such results. The difficulty with purely combinatorial means is firstly that there are no natural maps, other than the deletion operators, between the flags on a polytope, and secondly that the convexity of the polytope has to be allowed to enter into the discussion in a significant way. This said, very few results relating to exact homology are known at present.

For the usual middle perversity intersection homology theory, the three approaches mentioned in the introduction are known to have significant areas of agreement. The recursive formula in §4 for \( h_{\Delta} \) can be unwound to express each Betti number as a sum of contributions due to flags. These contributions are exactly the same as those implicit in the definitions of §4. The method of decorated bar diagrams provides a reformulation of the recursive formula for \( h_{\Delta} \), where now one simply counts the number of valid diagrams. As was indicated in §8, once the valid diagrams are known, it is not then hard to determine what the corresponding space of coefficients should be. Thus, the derivation from §4 is without difficulty. Finally, via the identification of the exterior algebra \( \bigwedge^* V^* \) with the homology of the generic fibre of the moment map, one can interpret the complex in §4 in terms of the construction of intersection homology cycles on \( P_\Delta \), and relations between them. This last will be presented elsewhere.

The important property of intersection homology (for middle perversity only) is that it seems to produce exact homology groups. That there are formula for such Betti numbers, and moreover as somewhat geometric alternating sums, supports this view. The local-global construction defined in §§5–7 can, via the moment map, be translated into a class of cycles on \( P_\Delta \), and relations between them. These cycles and relations have special properties with respect to the strata of \( P_\Delta \). In this way one can translate the definition of local-global homology in this paper into a topological one. Indeed, if intersection homology were not already known, it could have been discovered via its similarity to the \( H_\Delta \) theory presented here.

There is a subtlety connected to the concept of a linear function of the flag vector. The flag vectors of convex polytopes span a proper subspace of the space of all possible flag vectors. Now, a linear function on a subspace is not the same as a linear function on the whole space. The latter contains more information. The constructions of this paper provide linear functions of arbitrary (not necessarily polytope) flag vectors. As noted, they agree with the formulae (8)–(10) of §4.

The construction presented here of \( H_\Delta \) is probably not the only one. In particular, it ought to be possible to define a theory, with the same expected Betti numbers, but where \( H_\Delta \) is built out of the \( H_\delta \), for all proper faces \( \delta \subset \Delta \), together with perhaps a little gluing information. This can certainly be done in the simple case, where it corresponds to the natural formula in that context for \( h_\Delta \) in terms of the face vector. For general polytopes, such a theory would correspond to a linear function on arbitrary flag vectors, which agrees on polytope flag vectors with the \( h \)-vector presented in this paper. Such a theory may be part of a geometric proof of exactness of homology.

One, of course, prove that which is not true; and exact homology cannot hold for a complex if the expected Betti number turns out to be negative for some special polytope. Bayer (personal communication) has an example of a 5 dimensional polytope (the bipyramid on the cylinder on a 3-simplex) where this in fact happens. This is discussed further in §3. All this indicates that there are as yet unrealised subtleties in the concept and proof of exact calculation.

The central definitions of this paper, namely of the coefficient spaces \( \Lambda(\delta, r, s) \) and the maps between them, use only linear algebra and the deletion operator on flags. Put another way, the
largest part of this paper has been the study of the flag

\[ V_1 \subset V_2 \subset \ldots \subset V_m \subset V \]

of vector spaces associated to a flag \( \delta \) of faces on \( \Delta \), and the vector spaces that can be defined from it. Given the requirements of exact homology, there is little else available that one could study. However, these definitions are very closely linked to the usual intersection homology theory and also its local-global variant. These connections indicate that there may be unremarked subtleties in the linear algebra of a flag, and undiscovered simplicity in intersection homology. A better understanding of the intersection homology of Schubert varieties would be very useful.

Finally, as mentioned in the Introduction, one could wish for a similar theory that applies to \( i \)-graphs, and similar combinatorial objects. This will probably require a different sort of linear algebra.

References

[1] M.M. Bayer, Face numbers and subdivisions of convex polytopes, in POLYTOPES: Abstract, Convex and Computational (ed. T. Bisztriczky et al), (1994) 155–171. Kluwer.

[2] P. Deligne, La conjecture de Weil, I, *Publ. Math. IHES* 43 (1974), 273–307

[3] ______. La conjecture de Weil, II, *Publ. Math. IHES* 52 (1980), 137–252

[4] J. Denef and F. Loeser, Weights of exponential sums, intersection homology and Newton polyhedra, *Invent. Math.* 106 (1991), 275–294

[5] K. Fieseler, Rational intersection homology of projective toric varieties, *J. reine angew. Math.* 413 (1991), 88–98

[6] J. Fine, Quantum topology, hypergraphs and flag vectors, preprint [q-alg/9708001] (August 1997)

[7] ______. Shelling and flag vectors, preprint [q-alg/9710002] (October 1997)

[8] ______. Local-global intersection homology, preprint [alg-geom/9709011] (September 1997)

[9] R. Stanley, Generalized \( h \)-vectors, intersection cohomology of toric varieties and related results, *Adv. Stud. Pure Math.* 11 (1987), 187–213