A Two-Dimensional Map Derived From An Ordinary Difference Equation of mKdV and Its Properties

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Abstract. The discrete modified Korteweg–de Vries (mKdV) is a class of discrete integrable systems that may be distinguished as integrable partial difference equations (P∆E) and integrable ordinary difference equations (O∆E). By considering traveling wave solutions, the O∆E mKdV can be obtained from P∆E mKdV. Meanwhile, a mapping can be constructed from an O∆E mKdV. In this paper, we will focus on producing a new map using a process (replacement), the interchange of a single parameter, and an integral and investigate its properties.

Keywords: O∆E mKdV, P∆E mKdV, Anti measure-preserving.

1. Introduction

The theory of discrete dynamical system and difference equations has been being developed in the last thirty years of the twentieth century. Recently, there is much application of the discrete dynamical system, and difference equations have appeared in the areas of biology, economics, physics, resource management, and others [1]. One of the discrete dynamical system types is the discrete integrable system. In this type, the system can have an integral or invariant (for a two-dimensional case with invariants of high degree [2]. An example of this type is the discrete modified Korteweg–de Vries (mKdV) equation. The mKdV is a partial differential equation which known to has soliton solution; hence it is also called one of the soliton equation [3]. As a class of discrete integrable systems, discrete mKdV may be distinguished integrable partial difference equations (P∆E) and integrable ordinary difference equations (O∆E). Discrete mKdV is a class of QRT (Quispel-Roberts-Thompson) map[4,5].

The discretization of the mKdV equation has been done in various ways. One of them, the method by describing its Lax-pair, can be found in [3,4,5]. There is a connection between the two classes, namely that many integrable maps can be obtained from integrable PÆ by imposing periodic boundary conditions [6]. By using the staircase method, PÆ mKdV can be reduced into O∆E mKdV [3]. To study the dynamics and also the bifurcation, we need to have a parameter in the system. By modifying the parameter, the system can be generalized. In 2019, Zakaria and Tuwankotta
constructed a new map of a 2D double discrete sine-Gordon map by replacing a number of parameters in the original map[7].

The outline of this article is the following. In section 2, a three-dimensional mapping derived from generalized OΔE mKdV equation and its integral function will be described. In section 3, a new map will be constructed by re-parametrizing the parameters of the map. This technique is proposed by [8] and is also used by [7]. The properties of the new map are also explored.

2. Formulation of the Problem

Consider the standard PAEmKdV equation on the 2D lattice \((\mathbb{Z}^2)\) that defined as ([3],[7])

\[
p(V_{l+1,m} V_{l+1,m+1} - V_{l,m} V_{l+1,m+1}) = q(V_{l,m+1} V_{l+1,m} - V_{l,m} V_{l+1,m+1})
\]

(4)

Suppose \(\theta_3 = \theta_4 = p\) and \(\theta_1 = \theta_2 = q\). The map in eq. (1) can be written as follow:

\[\theta_3 V_{l,m} V_{l+1,m} - \theta_2 V_{l+1,m} V_{l,m+1} - \theta_1 V_{l,m} V_{l+1,m+1} + \theta_4 V_{l+1,m} V_{l+1,m+1} = 0\]

(5)

In [9] has studied four parameters family of mappings, which is derived from the generalized PΔE mKdV equation (5). Note that a system of ordinary difference equations, OΔE mKdV, can be derived from Eq. (4) by restriction to traveling wave solution by setting

\[V_{l,m} = V_{a,n} = z_l + z_m,\]

(6)

where \(z_1\) and \(z_2\) are relatively prime integers (see [3,7] for applying on PΔE sine-Gordon).

Substituting Eq. (6) into Eq. (5), the following discrete mapping can be obtained

\[\theta_3 V_{n} V_{n+1} - \theta_2 V_{n+1} V_{n+2} + \theta_1 V_{n} V_{n+2} - \theta_4 V_{n+1} V_{n+2} = 0\]

(7)

The map in equation (7) represents an infinite hierarchy of mapping labeled by \(z_1\) and \(z_2\). For fixed \(z_1\) and \(z_2\), the equation (7) is a mapping from \(\mathbb{R}^{z_1+1}\) to \(\mathbb{R}^{z_2+1}\).

Let \(z_1 = 1\) and \(z_2 = 2\). Therefore we have the following relation from the equations (7).

\[V_{n+1} = V_{n+1}(\theta_3 V_{n+2} - \theta_4 V_{n+3})
\]

(8)

Discrete equation (8) can be written as follow:

\[V_{n+2} = \frac{V_n(\theta_3 V_{n+2} - \theta_4 V_{n+3})}{(\theta_4 V_{n+1} - \theta_2 V_{n+2})}
\]

(9)

If \(\zeta_n = \frac{V_{n+2}}{V_{n+1}}\) and \(\zeta_n = \frac{V_{n+1}}{V_{n}}\) then, Eq. (9) can be written as

\[\zeta_{n+1} = h_\theta(\zeta_n)
\]

(10)

where

\[h_\theta : \mathbb{R}^{z_1+1} \rightarrow \mathbb{R}^{z_2+1},\]

\[(x, y) \mapsto \left(-\frac{1}{xy} \frac{\theta_3 x - \theta_2}{\theta_4 x - \theta_2}, x\right).
\]

The discrete map in Eq. (10) is well known as a 3-dimensional mapping reduced to 2-dimensional mapping derived from Eq. (8). The mKdV map in Eq. (10) can be used to design text cryptography [10].
3. Results and Discussion

Let us assume that $\theta_2$ not equal to zero. If $\alpha = \frac{\theta_1}{\theta_2}$, $\beta = \frac{\theta_1}{\theta_2}$, and $\lambda = \frac{\theta_4}{\theta_2}$ then the following special mapping can be derived from Eq. (10), namely

$$\zeta_{n+1} = h_{\alpha, \beta, \lambda}(\zeta_n)$$

where

$$h_{\alpha, \beta, \lambda} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto \left(\frac{(\beta - \alpha x)}{x y (x - \lambda)}, x\right).$$

And its integral normal forms is [5,9]

$$H(x, y; \alpha, \beta, \lambda) = \alpha \left(\frac{1}{x} + \frac{1}{y}\right) + (x + y) - \beta \left(\frac{1}{x y}\right) - \lambda (x y)$$

(12)

where $\alpha, \beta, \lambda \in \mathbb{R}$. Thus, for all $n \in \mathbb{N}$, the solution of Eq. (11) lies on a level set of $H(x, y; \alpha, \beta, \lambda)$. Note that a new mapping can be constructed by re-parametrizing the parameter of the original mapping Eq. (13). This technique is introduced in [8] and also used in [7].

Consider the mKdV map in Eq. (11). Let fix the parameters $\alpha=(\mu_0+\mu_1\alpha)$ and $\beta=\lambda=1$. It follows immediately that the map

$$\tilde{\zeta}_{n+1} = h_{\mu_0, \mu_1, \alpha}(\tilde{\zeta}_n)$$

(13)

where

$$h_{\mu_0, \mu_1, \alpha} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$(x, y) \mapsto \left(\frac{1-(\mu_0 + \mu_1\alpha)x}{x y (x - 1)}, x\right).$$

and possesses the following integral

$$\tilde{H}_{(\mu_0, \mu_1, \lambda_0, \beta_0, \alpha)}(x, y) = (\mu_0 + \mu_1\alpha)\left(\frac{1}{y} + \frac{1}{x}\right) + (x + y)$$

$$- \left(\frac{1}{x y}\right)(x y) + (\beta_0 + \beta_1\alpha).$$

(14)

Since $\tilde{H}_{(\mu_0, \mu_1, \lambda_0, \beta_0, \alpha)}(x, y)$ is linear in $\alpha$. And because of

$$H(x, y; \alpha) = 0 \Rightarrow H\left(\frac{1-(\mu_0 + \mu_1\alpha)x}{x y (x - 1)}, x; \alpha\right) = 0,$$

therefore, we have

$$\alpha = \alpha(x, y) = \frac{1-\mu_0(x + y) - \beta_0 x y - (x^2 y + x y^2) + x^2 y^2}{\mu_1 (x + y) + \beta_1 x y}$$

(15)

And it follows that $\tilde{h}_{(\mu_0, \mu_1, \alpha)}$ with the replacement $\alpha = \alpha(x, y)$ satisfies

$$\alpha\left(\frac{1-(\mu_0 + \mu_1\alpha)x}{x y (x - 1)}, x\right) = \alpha(x, y)$$

(16)
Explicitly, \( \hat{h}_{(\mu_0, \mu_1, \alpha)} \) with the replacement \( \alpha = \alpha(x, y) \) yields the map,

\[
\hat{h}_{(\mu_0, \mu_1, \alpha)} : x, y \to \left( \frac{\beta(x(1 - \mu_0 x) + \mu_1 (1 + \beta_0 x^2 + x^2 y - x^2 y)}{(-1 + x)(\beta_0 x + \mu_1 (x + y))}, x \right)
\]

(17)

The mapping Eq. (17) have some properties:

- The mapping \( \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)} \) has an integral Eq. (15).

- \( \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)} \) is anti-measure-preserving, i.e.

\[
\left| \frac{\partial h_{(\mu_0, \mu_1, \alpha)}}{\partial x} \right| = 
\rho(x, y)\frac{\beta(x(1 - \mu_0 x) + \mu_1 (1 + \beta_0 x^2 + x^2 y - x^2 y)}{(-1 + x)(\beta_0 x + \mu_1 (x + y))}, x
\]

where

\[
\rho(x, y) = \frac{1}{xy} \left[ \frac{\partial}{\partial x} H(x, y) \right]^{-1}
\]

\[
= \frac{1}{\mu_1 (x + y) + \beta_1 x y}.
\]

- There exists a reversing symmetry \( L(x, y) = (y, x) \) such that

\[
L \circ \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)} \circ L = \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)}^{-1}.
\]

It means that \( \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)} \) is reversible ( \( \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)} \circ L \circ \hat{h}_{(\mu_0, \mu_1, \beta_0, \beta_1)} = L \)).

4. Conclusions

Based on the results in the previous section, we have described in detail that a mapping derived from an OΔE mKdV has (anti) measure-preserving and reversing symmetry properties.

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