DUALITY BETWEEN MEASURE AND
CATEGORY IN UNCOUNTABLE LOCALLY
COMPACT ABELIAN POLISH GROUPS

Abstract

We show that there is no addition preserving Erdős-Sierpiński mapping on any uncountable locally compact abelian Polish group. This generalizes results of Bartoszyński and Kysiak.

1 Introduction

Let $G$ be a locally compact abelian (LCA) Polish group. Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and null (with respect to Haar measure) subsets of $G$.

Definition 1.1. A bijection $F: G \to G$ is called an Erdős-Sierpiński mapping if

$$X \in \mathcal{N} \iff F[X] \in \mathcal{M} \quad \text{and} \quad X \in \mathcal{M} \iff F[X] \in \mathcal{N}.$$ 

Theorem 1.2. (Erdős-Sierpiński) Assume the Continuum Hypothesis. Then there exists an Erdős-Sierpiński mapping on $\mathbb{R}$.

The existence of such a function is independent from ZFC. Our main question is the following:

Is it consistent that there is an Erdős-Sierpiński mapping $F$ that preserves addition, namely

$$\forall x, y \in G \quad F(x + y) = F(x) + F(y)?$$

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This question is attributed to Ryll-Nardzewski in the case $G = \mathbb{R}$. Besides intrinsic interest, another motivation was the following: If this statement were consistent then the so called strong measure zero and strongly meager sets would consistently form isomorphic ideals. (For the definitions see [1].)

First Bartoszyński gave a negative answer to the question in the case $G = 2^\omega$, see [2], then Kysiak proved this for $G = \mathbb{R}$ and answered the question of Ryll-Nardzewski, see [3], where he used and improved Bartoszyński’s idea. We answer the general case, the goal of this article is to prove the following theorem:

**Theorem 1.3. (Main Theorem)** There is no addition preserving Erdős-Sierpiński mapping on any uncountable locally compact abelian Polish group.

Let $(\varphi_M)$ denote the following statement (considered by Carlson in [1]): For every $F \in M$ there exists a set $F' \in M$ such that
\[
\forall x_1, x_2 \in G \exists x \in G \quad (F + x_1) \cup (F + x_2) \subseteq F' + x.
\]

Let $(\varphi_N)$ be the dual statement obtained by replacing $M$ by $N$. If there exists an Erdős-Sierpiński mapping preserving addition then $(\varphi_M)$ and $(\varphi_N)$ are equivalent. First we show that $(\varphi_M)$ holds in LCA Polish groups. In the second part we begin to show that $(\varphi_N)$ fails for all uncountable LCA Polish groups by reducing the general case to three special cases. Finally, in part three we settle these three special cases.

2 $(\varphi_M)$ holds for all LCA Polish groups

As the known proofs only work for the reals, we had to come up with a new, topological proof.

**Notation 2.1.** Let $X$ be a metric space, $x \in X$ and $r > 0$. Let $B(x, r)$ denote the closed ball of radius $r$ centered at the point $x$.

**Lemma 2.2.** Let $X$ be a metric space and $K \subseteq X$ a nowhere dense compact set. Then there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x \in X$ and $r > 0$ there exists $y \in X$ such that $B(y, f(r)) \subseteq B(x, r) \setminus K$.

**Proof.** Suppose towards a contradiction that $r > 0$ is such a number that there exists a sequence $r_n \to 0$ $(n \in \mathbb{N})$ and a set $\{x_n \in X : n \in \mathbb{N}\}$ such that for all $y \in X$ and $n \in \mathbb{N}$
\[
B(y, r_n) \not\subseteq B(x_n, r) \setminus K. \tag{2.1}
\]
$r_n < r$ holds for large enough $n$, so in the case $y = x_n$ we get that for large enough $n$ there exist $z_n \in B(x_n, r_n) \cap K$. By the compactness of $K$ there exists a convergent subsequence $\lim_{k \to \infty} z_{n_k} = z \in K$, and so $\lim_{k \to \infty} x_{n_k} = z \in K$. There is an $N \in \mathbb{N}$ such that $x_{n_k} \in B(z, \frac{r}{2})$ holds for all $k > N$. Then $B \left( z, \frac{r}{2} \right) \subseteq B(x_{n_k}, r)$, so by (2.1) for all $k > N$ and $y \in B \left( z, \frac{r}{2} \right)$ we get $B(y, r_{n_k}) \not\subseteq B \left( z, \frac{r}{2} \right) \setminus K$, which contradicts that $K$ is nowhere dense, and we are done. \qed

Every abelian Polish group admits a compatible invariant complete metric, because it admits a compatible invariant metric by [[8], Thm. 7.3.], and a compatible invariant metric is automatically complete by [[8], Lem. 7.4.]. So we may assume that the metric on our group is invariant.

**Lemma 2.3.** Let $G$ be an abelian Polish group and $K \subseteq G$ a nowhere dense compact set. Then there exists a function $l : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that for all $x, x_1, x_2 \in G$ and $r > 0$ there is a $y \in G$ such that

$$B(y, l(r)) \subseteq B(x, r) \setminus (K + x_1) \cup (K + x_2).$$

**Proof.** It is easy to see, using the invariance of our metric, that $l = f \circ f$ works, where $f$ is defined in the previous lemma. \qed

We may assume that $l(r) < r$ for all $r > 0$. Now we prove $(\varphi_M)$.

**Theorem 2.4.** Let $S$ be a meagre set in an LCA Polish group $G$. Then there is a meagre set $T \subseteq G$ such that for all $s_1, s_2 \in G$ there is a $t \in G$ such that $(S + s_1) \cup (S + s_2) \subseteq T + t$.

**Proof.** We may assume by local compactness (by taking closures of the nowhere dense subsets and decomposing each of them to countably many compact sets) that $S = \cup_{n \in \mathbb{N}} S_n$, where the $S_n$'s $(n \in \mathbb{N})$ are nowhere dense compact sets. The idea is that we construct nowhere dense $T_n$'s for the $S_n$'s simultaneously, and $T$ will be the union of the $T_n$'s. Later we will set $T_n = \cap_{k \in \mathbb{N}} T_n^k$ for some open sets $T_n^k$, and we will simultaneously construct a decreasing sequence of closed balls $B(x_k, r_k)$ such that $t = t_{s_1, s_2}$ will be found as $\cap_{k \in \mathbb{N}} B(x_k, r_k)$.

Let $h : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be an enumeration of $\mathbb{N} \times \mathbb{N}$, let $h_1, h_2 : \mathbb{N} \rightarrow \mathbb{N}$ be the first and second coordinate functions of $h$ and let $\{g_n : n \in \mathbb{N}\}$ be a dense set in $G$. Let $l_n$ be the function in the previous lemma for the compact set $S_n$. We define $T$ by induction, independently from $s_1, s_2$. Let $x_0 \in G$ be arbitrary and $r_0 > 0$ so small that $B(x_0, r_0)$ is compact (we can do this by local compactness of $G$), and $r_k = \frac{l_{h_1(i)}(r_{k-1}) - 1}{2}$ if $k > 0$ and $n \in \mathbb{N}$. Let
Let us define 

\[ T_n^k = \begin{cases} 
G & \text{if } n \neq h_1(k), \\
G \setminus B(g_{h_2(k)}, r_k) & \text{if } n = h_1(k).
\end{cases} \]  

(2.2)

The key step of the proof is the following lemma.

Lemma 2.5. Assume that \( x_0, x_1, \ldots, x_{k-1} \) are already defined. Then there exists \( x_k \in G \) (which depends on \( s_1 \) and \( s_2 \)) such that \( B(x_k, r_k) \subseteq B(x_{k-1}, r_{k-1}) \) and for all \( n \in \mathbb{N} \) and \( t \in B(x_k, r_k) \)

\[ (S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t. \]  

(2.3)

Proof. (of the Lemma) If \( n = h_1(k) \), then \( T_n^k = G \setminus B(g_{h_2(k)}, r_k) \). Using \( 2r_k = l_{h_1(k)}(r_{k-1}) = l_n(r_{k-1}) \) and Lemma 2.3 there exists \( y_k \in B(x_{k-1} + g_{h_2(k)}, r_{k-1}) \) such that

\[ B(y_k, 2r_k) \subseteq B(x_{k-1} + g_{h_2(k)}, r_{k-1}) \setminus ((S_n + s_1) \cup (S_n + s_2)). \]  

(2.4)

Let us define \( x_k := y_k - g_{h_2(k)} \). Using the definition of \( x_k \) and (2.4) we get that

\[ B(x_k, r_k) = B(y_k, r_k) - g_{h_2(k)} \subseteq B(y_k, 2r_k) - g_{h_2(k)} \subseteq B(x_{k-1}, r_{k-1}). \]

We will use the following easy equation, where the first + is the Minkowski sum.

\[ B(g_{h_2(k)}, r_k) + B(x_k, r_k) \subseteq B(g_{h_2(k)} + x_k, 2r_k). \]  

(2.5)

Using (2.4) again, the definition of \( x_k \) and (2.5) in this order we get that for all \( t \in B(x_k, r_k) \)

\[ (S_n + s_1) \cup (S_n + s_2) \subseteq G \setminus B(y_k, 2r_k) = G \setminus B(g_{h_2(k)} + x_k, 2r_k) \]

\[ \subseteq G \setminus B(g_{h_2(k)}, r_k) + t. \]

Hence

\[ (S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t, \]

so (2.3) holds for \( n = h_1(k) \). If \( n \neq h_1(k) \) then \( T_n^k = G \) and (2.3) is obvious, so we are done.

Now we return to the proof of Theorem 2.4. By the compactness of \( B(x_0, r_0) \) the closed sets \( B(x_k, r_k) \) are compact, so the intersection of decreasing sequence of compact sets \( \cap_{k \in \mathbb{N}} B(x_k, r_k) \neq \emptyset \). Let

\[ t_{s_1, s_2} \in \cap_{k \in \mathbb{N}} B(x_k, r_k) \]
be the common shift. By (2.3) and the definition of \( t_{s_1, s_2} \),
\[
(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t_{s_1, s_2}
\]
holds for all \( k, n \in \mathbb{N} \). For every \( n \in \mathbb{N} \) the set \( T_n = \cap_{k \in \mathbb{N}} T_n^k \) is nowhere dense by (2.2). By (2.6) we easily get for every \( n \in \mathbb{N} \)
\[
(S_n + s_1) \cup (S_n + s_2) \subseteq T_n + t_{s_1, s_2}.
\]
The set \( T = \cup_{n \in \mathbb{N}} T_n \) is meagre, and clearly
\[
(S + s_1) \cup (S + s_2) \subseteq T + t_{s_1, s_2},
\]
and the proof is complete. \( \square \)

3 Reduction to \( T, \mathbb{Z}_p \) and \( \prod_{n \in \mathbb{N}} G_n \)

In this section we reduce the general uncountable LCA Polish groups to some special groups. We follow the strategy developed in [5].

**Definition 3.1.** Let us say that an LCA Polish group \( G \) is nice if \((\varphi_N)\) fails in \( G \), that is, there is a nullset \( N \), such that for every nullset \( N' \) there are \( x_1, x_2 \in G \) such that \((N + x_1) \cup (N + x_2) \not\subset (N' + x) \) for all \( x \in G \).

**Lemma 3.2.** If an LCA Polish group \( G \) has a nice open subgroup \( U \) then \( G \) is nice.

**Proof.** Let \( \mu \) be the Haar measure of the LCA Polish subgroup \( U \) and \( N \subseteq U \) be the set that witnesses that \( U \) is nice. Because of \( U \) is open and \( G \) is separable, we can write a disjoint countable decomposition \( G = \cup_{n=0}^\infty (U + g_n) \).
It is easy to see that \( \nu(B) = \sum_{n=0}^\infty \mu((B - g_n) \cap U) \) is a Haar measure on \( G \). We show that the \( \nu \) nullset \( N_G = \cup_{n=0}^\infty (N + g_n) \) is a witness that \( G \) is nice. Let \( \nu(N_G') = 0 \). Since \( N \) witnesses that \( U \) is nice, there are \( u_1, u_2 \in U \) such that
\[
((N + u_1) \cup (N + u_2)) \cap ((N_G' \cap U) + u)^c \neq \emptyset \text{ for all } u \in U.
\]
Suppose towards a contradiction that there exists a \( g \in G \) such that
\[
(N_G + u_1) \cup (N_G + u_2) \subseteq (N_G' + g). \text{ There exists } i \in \mathbb{N} \text{ such that } g - g_i \in U.
\]
Then
\[
(N + u_1) \cup (N + u_2) \subseteq (N_G' + (g - g_i)) \cap U = (N_G' \cap U) + (g - g_i)
\]
where \( g - g_i \in U \), a contradiction. \( \square \)
Lemma 3.3. Assume that $G$ is an LCA Polish group, $H \subseteq G$ is a compact subgroup and $G/H$ is nice. Then $G$ is nice, too.

Proof. If $\mu$ is a Haar measure on $G$, $\pi: G \to G/H$ is the canonical homomorphism, then $\mu_G \circ \pi^{-1}$ is a Haar measure on $G/H$ by [[4], 63. Thm. C.]. So the inverse image of a nullset in $G/H$ under $\pi$ is a nullset in $G$. Hence if $N \subseteq G/H$ is a nullset witnessing that $G/H$ is nice then $\pi^{-1}(N) \subseteq G$ is a nullset witnessing that $G$ is also nice since the translated copies of $\pi^{-1}(N)$ are composed of cosets of $H$, so if we would like to cover it with a translation of a set $N'$, then we may assume without loss of generality, that $N'$ consists of cosets, too. It is easy to see that $\pi(N')$ shows that $N$ is not a witness, a contradiction.

Now we start reducing the problem to simpler groups.

Definition 3.4. Let $G$ be a group and $p$ be a prime number, $G_{p^n} = \{ g \in G : p^n g = 0 \}$ for every $n \in \mathbb{N}$, and also let $G_{p^\infty} = \bigcup_{n \in \mathbb{N}} G_{p^n}$. We say that $G$ is a $p$-group, if $G = G_{p^\infty}$.

Definition 3.5. Let $p$ be a prime number. An abelian group $G$ is called quasicyclic if it is generated by a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ with the property that $g_0 \neq 0$ and $pg_{n+1} = g_n$ for every $n \in \mathbb{N}$. For a fixed prime $p$ the unique (up to isomorphism) quasicyclic group is denoted by $C_{p^\infty}$.

Notation 3.6. We denote by $\mathbb{T}$ the circle group, by $\mathbb{Z}_p$ the group of $p$-adic integers for every prime $p$ and by $\prod_{n \in \mathbb{N}} G_n$ the product of the finite abelian groups $G_n$’s.

Remark 3.7. $\mathbb{Z}_p$ is the topological space $\{0, 1, \ldots , p-1\}^\omega$ equipped with the product topology. Addition is coordinate-wise with carried digits from the $n^{th}$ coordinate to the $n+1^{st}$. Both $\mathbb{Z}_p$ and $\prod_{n \in \mathbb{N}} G_n$ are Polish with the product topology.

We will use the following theorem from [5].

Theorem 3.8. Every infinite abelian group contains a subgroup isomorphic to one of the following:

- $\mathbb{Z}$,
- $C_{p^\infty}$ for some prime $p$,
- $\bigoplus_{n \in \mathbb{N}} G_n$, where each $G_n$ is a finite abelian group of at least two elements.
Theorem 3.9. \( \mathbb{T}, \mathbb{Z}_p, \) and \( \prod_{n \in \mathbb{N}} G_n \) are nice.

Proof. We postpone the proof to the next section. \( \square \)

Theorem 3.10. Every uncountable LCA Polish group is nice, that is, \((\varphi_N')\) fails for every uncountable LCA Polish group.

Proof. Let \( G \) be an LCA Polish group. From the Principal Structure Theorem of LCA groups \([6], 2.4.1\) follows that \( G \) has an open subgroup \( H \) which is of the form \( H = K \otimes \mathbb{R}^n \), where \( K \) is a compact subgroup and \( n \in \mathbb{N} \). \( G \) is uncountable, so non-discrete and \( H \) is open, therefore \( H \) is a non-discrete, uncountable group. By Lemma 3.2 it is enough to prove that \( H \) is nice, so we can assume \( G = H \).

Assume that \( n \geq 1 \). \( \mathbb{R} \) is nice by \([3]\), let \( N \) be a nullset witnessing this fact. It is obvious that \( K \times N \times \mathbb{R}^{n-1} \) witnesses that \( G = K \otimes \mathbb{R}^n \) is nice. So we may assume \( n = 0 \), and we get that \( G \) is compact. It is enough to find a closed subgroup \( H \subseteq G \) such that \( G/H \) is nice by the previous lemma. Using \([6], 2.1.2\) and the Pontryagin Duality Theorem, see \([6], 1.7.2\), we get that factors of \( G \) are the same as (isomorphically homeomorphic to) dual groups of closed subgroups of \( \hat{G} \). If \( G \) is compact, then \( \hat{G} \) is discrete, see \([6], 1.2.5\). So it is enough to find a subgroup \( \hat{M} \subseteq \hat{G} \) such that \( \hat{M} \) is nice.

From the previous theorem follows (using that \( G = K \) is an infinite group) that \( \hat{G} \) has a subgroup isomorphic either to \( \mathbb{Z} \), or to \( \mathbb{C}_p^\infty \) for some prime \( p \), or to \( \bigoplus_{n \in \mathbb{N}} G_n \) (where each \( G_n \) is a finite abelian group of at least two elements). It is sufficient to show that the duals of these groups are nice. It is well-known that \( \hat{\mathbb{Z}} = \mathbb{T} \), and \( \mathbb{T} \) is nice, see \([3]\).

By \([6], 2.2.3\) the dual of a direct sum (equipped with the discrete topology) is the direct product of the dual groups (equipped with the product topology), so \( \bigoplus_{n \in \mathbb{N}} G_n = \prod_{n \in \mathbb{N}} \hat{G}_n \). If \( G_n \) is finite so is \( \hat{G}_n \), so we are done with the second case.

Finally, \( \mathbb{C}_p^\infty = \mathbb{Z}_p \), see \([7], 25.2\). Hence we are done by the previous theorem. \( \square \)

4 \((\varphi_N')\) Fails For \( \mathbb{T}, \mathbb{Z}_p \) and \( \prod_{n \in \mathbb{N}} G_n \)

We will prove that \( \mathbb{T}, \mathbb{Z}_p \) and \( \prod_{n \in \mathbb{N}} G_n \) are nice. \( \mathbb{T} \) is nice, see \([3]\), so we need to handle the last two cases. Our proofs are very similar to the Main Theorem of Kysiak’s paper, see \([3], \text{Main Thm. 4.3.}\), so we only describe the necessary modifications. Hence in order to follow our proof one has to read Kysiak’s paper parallelly. Unless stated otherwise, the numbered references in
Theorem 4.1. \( \mathbb{Z}_p \) is nice.

**Proof.** We use the description of \( \mathbb{Z}_p \) that can be found in Remark 3.7. We always write \( \mathbb{Z}_p \) instead of \((0, 1]\). Let \( \mu \) be the Haar measure on \( \mathbb{Z}_p \), and let \( \mu_2 = \mu \times \mu \) be the Haar measure on \( \mathbb{Z}_p^2 \). The proof of Lemma 4.7. in [3] is the same with these modifications.

For the construction of \( I_n \)'s we fix a partition of the set \( \omega \) instead of \( \omega \setminus \{0\} \). For the inequalities we write \( p^{I_n} \) and \( p^{|I_n|} \) instead of \( 2^{I_n} \) and \( 2^{|I_n|} \). With these changes the following inequality holds.

\[
\frac{1}{p^{|I_n|}} \leq \frac{1}{12} \left( \frac{1}{n^2} - \frac{1}{n^5} \right) \text{ for } n > 1. \tag{4.1}
\]

Let \( J_n \subseteq p^{I_n} \) such that

\[
1 - \frac{1}{n^2} + \frac{1}{n^5} > \frac{|J_n|}{p^{|I_n|}} > 1 - \frac{1}{n^2}, \tag{4.2}
\]

and \( J_n \) consist of the first \( |J_n| \) consecutive elements of \( p^{I_n} \) with respect to the antilexicographical ordering (sequences are ordered according to the largeness of the rightmost coordinate where they differ).

Let \( N^* \) be the filter of full measure sets of \( \mathbb{Z}_p \). If \( I \subseteq \omega \) and \( J \subseteq p^I \) then [\( J \)] denote the set \( \{ x \in \mathbb{Z}_p : x \mid I \in J \} \). We write \( p^{<\omega} \) instead of \( 2^{<\omega} \) and for \( s \in p^{<\omega} \) let \( [s] \) be the set \( \{ x \in \mathbb{Z}_p : x \text{ extends } s \} \).

For the definition of \( C \) (at the end of the page 274) we omit the \( C \subseteq \mathbb{R} \setminus \mathbb{Q} \) condition. We may assume about \( C \) the following: for every \( s \in p^{<\omega} \) we have \( [s] \cap C = \emptyset \) or \( \mu([s] \cap C) > 0 \) (if not, consider \( C' = C \setminus \bigcup\{[s] : s \in p^{<\omega} \text{ and } \mu([s] \cap C) = 0\} \) instead).

The inductive definition and its proof (beginning on page 275) is almost the same, too. Write \( p^{I_1 \cup \ldots \cup I_l} \) and \( p^{\max I_l, \infty} \) instead of \( 2^{I_1 \cup \ldots \cup I_l} \) and \( 2^{\max I_l, \infty} \). Let \( 0^n \) be the element of \( p^{(0, \ldots, n-1)} \) with zero coordinates. For the proof of

\[
\mu([r_s] \cap C) > \frac{t-1}{t} \mu([r_s]) \tag{4.3}
\]

(that can be found at the end of the page 275) we use (instead of the Lebesgue density theorem) the following more general density theorem.

**Definition 4.2.** Let \( X \) be a topological group. A **B-sequence** in \( X \) is a non-increasing sequence \( \langle V_n \rangle_{n \in \mathbb{N}} \) of closed neighborhoods of the identity, constituting a base of neighborhoods of the identity, such that there is some \( M \in \mathbb{N} \).
such that for every \( n \in \mathbb{N} \) the set \( V_n - V_n \) can be covered by at most \( M \) translates of \( V_n \).

**Theorem 4.3.** Let \( X \) be a topological group with a left Haar measure \( \mu \), and \( \langle V_n \rangle_{n \in \mathbb{N}} \) a B-sequence in \( X \). Then for any Haar measurable set \( E \subseteq X \),

\[
\lim_{n \to \infty} \frac{\mu(E \cap (x + V_n))}{\mu(V_n)} = \chi_E(x)
\]

for almost every \( x \in X \).

For the proof of Theorem 4.3 and for Definition 4.2 see [[9], 447D Thm.] and [[9], 446L Def.]. It is easy to see that \( V_n = [0^n + 1] \) (\( n \in \mathbb{N} \)) form a B-sequence, and (4.3) follows by applying Theorem 4.3 for \( \langle V_n \rangle_{n \in \mathbb{N}} \).

**Remark 4.4.** For every countable product space of discrete groups \( \prod_{n \in \mathbb{N}} X_n \) the sets \( V_n = \{e_0, \ldots, e_n\} \) (\( n \in \mathbb{N} \)) form a B-sequence (where \( e_i \) is the neutral element of \( X_i \)), so (4.3) is valid, too.

We follow the proof till the definition of \( U \) (at the end of the page 277), then we deviate from it. We jump to the formula of the following lemma that is at the end of the proof in [3] (in the middle of the page 279). For the following lemma we need a new idea, Kysiak’s arguments are not applicable here, he uses a specific relation between the metric and the addition in \( T \).

**Lemma 4.5.** For every \( k \in U \) there is a \( v_k \in [J'_n] \) such that

\[
z \upharpoonright I_{n_k} = v_k \upharpoonright I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}].
\]

**Proof.** (of the Lemma) Assume \( k \in U \). Remember that \( \varepsilon_{n_k} = \frac{1}{2}\lambda_{n_k} \), and as a consequence of (4.2) \( \lambda_{n_k} = 1 - \mu(J_{n_k}) \geq \frac{1}{n_k^2} - \frac{1}{n_k^3} \). Using the definition of \( U \), the previous facts and (4.1) in this order we get

\[
\mu \left( [J'_{n_k}] \setminus ((x - y) + [J_{n_k}]) \right) = \mu \left( (y + [J'_{n_k}]) \setminus (x + [J_{n_k}]) \right) \geq \frac{\lambda_{n_k}}{12} \geq \frac{1}{12} \left( \frac{1}{n_k^2} - \frac{1}{n_k^3} \right) \geq \frac{1}{p|J_{n_k}|}.
\]

Hence

\[
\mu \left( [J'_{n_k}] \setminus ((x - y) + [J_{n_k}]) \right) > \frac{1}{p|J_{n_k}|}. \tag{4.4}
\]

Let us define

\[
L_k := \left\{ ((x - y) + [J_{n_k}]) \upharpoonright I_{n_k} \right\} \subseteq p^n_{n_k}. \tag{4.5}
\]
Since the elements of $J_{n_k}$ are consecutive (with respect to the antilexicographical ordering), we get $|L_k| \leq |J_{n_k}| + 1$. Using this inequality and the translation invariance of $\mu$

$$\mu ((x - y) + [J_{n_k}]) = \mu ([J_{n_k}]) \geq \mu ([L_k]) - \frac{1}{p|L_{n_k}|}, \quad (4.6)$$

Using $(x - y) + [J_{n_k}] \subseteq [L_k]$, (4.6) and finally (4.4) we get

$$\mu ([J_{n_k}'] \setminus [L_k]) \geq \mu ([J_{n_k}'] \setminus ((x - y) + [J_{n_k}])) - \frac{1}{p|L_{n_k}|} > 0.$$ 

Hence there is a $v_k \in [J_{n_k}'] \setminus [L_k] \neq \emptyset$, and it means by the definition of $L_k$ that

$$z \upharpoonright I_{n_k} = v_k \upharpoonright I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}],$$

and Lemma 4.5 follows.

\[ \Box \]

We can simply follow the last few lines of Kysiak’s proof after the analogous formula.

**Theorem 4.6.** \( \prod_{n \in \mathbb{N}} G_n \) is nice.

**Proof.** We write \( \mathbb{N} \) and \( \prod_{n \in \mathbb{N}} G_n \) instead of \( \omega \) and \( (0, 1] \). Let \( \mu \) be the Haar measure on \( \prod_{n \in \mathbb{N}} G_n \), and let \( \mu_2 = \mu \times \mu \) be the Haar measure on \( (\prod_{n \in \mathbb{N}} G_n)^2 \). The proof of Lemma 4.7, in [3] is the same.

For the construction of \( I_n \)’s we fix a partition of the set \( \mathbb{N} \) instead of \( \omega \setminus \{0\} \). Write \( \prod_{i \in I_n} G_i \) and \( \prod_{i \in I_n} G_i \) instead of \( 2^{|I_n|} \) and \( 2^{|I_n|} \). At the choice of the sets \( J_n \) we omit the ordering condition.

Let \( \mathcal{N}^\ast \) be the filter of full measure sets of \( \prod_{n \in \mathbb{N}} G_n \). If \( I \subseteq \mathbb{N} \) and \( J \subseteq \prod_{n \in I} G_n \) then \( [J] \) denote the set \( \{ x \in \prod_{n \in \mathbb{N}} G_n : x \upharpoonright I \in J \} \). We write \( \{ \prod_{i \in I} G_i : |I| < \infty \} \) instead of \( \mathcal{N}^\ast \) and for \( s \in \{ \prod_{i \in I} G_i : |I| < \infty \} \) let \( \{ s \} \) be the set \( \{ x \in \prod_{n \in \mathbb{N}} G_n : x \upharpoonright I = s \} \).

For the definition of \( C \) we omit the \( C \subseteq \mathbb{R} \setminus \mathbb{Q} \) condition, and we may assume that for every \( s \in \{ \prod_{i \in I} G_i : |I| < \infty \} \) we have \( [s] \cap C = \emptyset \) or \( \mu ([s] \cap C) > 0 \). (4.3) follows from Remark 4.4 instead of the Lebesgue density theorem.

The inductive definition and its proof need the following modifications. Use \( \prod_{i \in I_n} \bigcup_{i \leq n} G_i \) and \( \prod_{i=\max I_n} \bigcup_{i \geq \infty} G_i \) instead of \( 2^I \cup \cdots \cup 1 \) and \( 2^{(\max I_n, \infty)} \).

We follow the proof till the definition of \( U \), then we deviate from it. Note that by the definition of \( U \), for every \( k \in U \)

$$\mu ([J_{n_k}' \setminus ((x - y) + [J_{n_k}])) = \mu (y + [J_{n_k}'] \setminus ((x + [J_{n_k}])) > 0. \quad (4.7)$$
We jump to the formula of the following lemma that is at the end of the proof in [3]. The following lemma is analogous to Lemma 4.5, but the proof is much easier than in the case $\mathbb{Z}_p$.

**Lemma 4.7.** For every $k \in U$ there is a $v_k \in [J'_{n_k}]$ such that

$$z \mid I_{n_k} = v_k \mid I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}].$$

**Proof.** (of the Lemma) Let us define

$$L_k := \{(x - y) + [J_{n_k}) \mid I_{n_k} \subseteq \prod_{i \in I_{n_k}} G_i, \}
\text{then obviously } (x - y) + [J_{n_k}] = [L_k].$$

Using (4.7) we get

$$\mu \left( [J'_{n_k}] \setminus [L_k] \right) = \mu \left( (x - y) + [J_{n_k}] \right) > 0.$$

So there is a $v_k \in [J'_{n_k}] \setminus [L_k] \neq \emptyset$, and it means by the definition of $L_k$ that

$$z \mid I_{n_k} = v_k \mid I_{n_k} \Rightarrow z \notin (x - y) + [J_{n_k}],$$

and Lemma 4.7 follows.

The last few lines of the proof is the same as in Kysiak’s paper after the analogous formula.

**Remark 4.8.** Theorem 4.6 could be proved easier based on Bartoszyński’s paper [2], but we do not know to generalize Bartoszyński’s method to the case $\mathbb{Z}_p$ in Theorem 4.1.

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