Bernstein polynomials are defined as follows:

\[ B_n(f, x) \coloneqq \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x), \]

where

\[ p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, 2, \ldots, n, \quad x \in [0, 1]. \]

Let \( \tilde{w}(x) = |x - \xi|^{\alpha}, \quad 0 < \xi < 1, \ \alpha > 0 \) and \( C_0 \coloneqq \{ f \in C([0,1]) \setminus \{ \xi \} : \lim_{x \to \xi^{-}} (\tilde{w}f)(x) = 0 \} \). The norm in \( C_0 \) is defined by

\[ ||f||_{C_0} := ||\tilde{w}f|| = \sup_{0 < x < 1} |(\tilde{w}f)(x)|. \]

\[ W^0_\phi \coloneqq \{ f \in C_0 : f' \in A.C.((0,1)), \| \tilde{w}\phi f' \| < \infty \}, \]

\[ W^2_{\phi, a,k} : f \in C_0 : f \in A.C.((0,1)), \| \tilde{w}\phi^2 f' \| < \infty \}. \]

For \( f \in C_0 \), the weighted modulus of smoothness is defined by

\[ \omega^2_{\phi}(f,t) \coloneqq \sup_{0 \leq h \leq t} \sup_{0 \leq s \leq 1} |(\tilde{w}f)(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))|, \]

where

\[ \Delta^2_{\phi,h}(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)), \]

and \( \phi(x) = \sqrt{x(1-x)} \), \( \delta_{\phi}(x) = \phi(x) + \frac{1}{\sqrt{x}} \).

Recently Felten showed the following two theorems in [1]:

**Theorem A.** Let \( \phi(x) = \sqrt{x(1-x)} \) and let \( \phi: [0,1] \to R, \phi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik modulus of smoothness [4] such that \( \phi^2 \) and \( \phi^2/\phi \) are concave. Then, for \( f \in C[0,1] \) and \( 0 < x < 2 \),

\[ |B_n(f,x) - f(x)| \leq \omega^2_{\phi}(f,n^{-1/2}\phi(x)/\phi^2(x)). \]

**Theorem B.** Let \( \phi(x) = \sqrt{x(1-x)} \) and let \( \phi: [0,1] \to R, \phi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik

\[ \omega^2_{\phi}(f,t) \coloneqq \sup_{0 \leq h \leq t} \sup_{0 \leq s \leq 1} |(\tilde{w}f)(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))|, \]

where

\[ \Delta^2_{\phi,h}(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)), \]

and \( \phi(x) = \sqrt{x(1-x)} \), \( \delta_{\phi}(x) = \phi(x) + \frac{1}{\sqrt{x}} \).

**Theorem A.** Let \( \phi(x) = \sqrt{x(1-x)} \) and let \( \phi: [0,1] \to R, \phi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik

\[ \omega^2_{\phi}(f,t) \coloneqq \sup_{0 \leq h \leq t} \sup_{0 \leq s \leq 1} |(\tilde{w}f)(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))|, \]

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\[ \Delta^2_{\phi,h}(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)), \]

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**Theorem B.** Let \( \phi(x) = \sqrt{x(1-x)} \) and let \( \phi: [0,1] \to R, \phi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik

\[ \omega^2_{\phi}(f,t) \coloneqq \sup_{0 \leq h \leq t} \sup_{0 \leq s \leq 1} |(\tilde{w}f)(x+h\phi(x)) - 2f(x) + f(x-h\phi(x))|, \]

where

\[ \Delta^2_{\phi,h}(x) = f(x + h\phi(x)) - 2f(x) + f(x - h\phi(x)), \]

and \( \phi(x) = \sqrt{x(1-x)} \), \( \delta_{\phi}(x) = \phi(x) + \frac{1}{\sqrt{x}} \).
modulus of smoothness such that \( \phi^2 \) and \( \psi^2/\phi^2 \) are concave. Then, for \( f \in C[0,1] \) and \( 0 < \alpha < 2 \),
\[
|B_n(f, x) - f(x)| = O\left( n^{-1/2} \frac{\phi(x)}{\phi(1)} \right) ^{\alpha}
\]
implies \( \omega^\alpha(f, t) = O(t^\alpha) \).

Approximation properties of Bernstein polynomials have been studied very well [2–5]. In order to approximate the functions with singularities, Della Vecchia et al. [3] introduced some kinds of modified Bernstein polynomials. Throughout the paper, \( C \) denotes a positive constant independent of \( n \) and \( x \), which may be different in different cases.

Let \( \psi : [0,1] \to \mathbb{R}, \psi \neq 0 \) be an admissible step-weight function of the Ditzian–Totik modulus of smoothness, that is, \( \psi \) satisfies the following conditions:

(I) For every proper subinterval \([a,b] \subseteq [0,1]\) there exists a constant \( C_2 = C(a,b) > 0 \) such that \( C_2^{-1} \leq \psi(x) \leq C_2 \), for \( x \in [a,b] \).

(II) There are two numbers \( \beta(0) \geq 0 \) and \( \beta(1) \geq 0 \) for which
\[
\psi(x) \sim \begin{cases} x^\beta(0), & x \to 0^+, \\ (1-x)^\beta(1), & x \to 1^- . \end{cases}
\]

\((X \sim Y) = C^{-1}X \leq Y \leq CX \) for some \( C \).

Combining conditions (I) and (II) on \( \psi \), we can deduce that
\[
C^{-1} \psi(x) \leq \psi(x) \leq C \psi(x), \quad x \in [0,1],
\]
where \( \psi(x) = x^\beta(0)(1-x)^\beta(1) \).

2. The main results

Let
\[
\psi(x) = \begin{cases} 10x^3 - 15x^4 + 6x^5, & 0 < x < 1, \\ 0, & x \leq 0, \\ 1, & x \geq 1. \end{cases}
\]

Obviously, \( \psi \) is non-decreasing on the real axis, \( \psi \in C^2([-\infty, +\infty]), \psi(0) = 0, \psi(1) = 0, 1, 2, \psi(\beta(0)) = 0, \psi(\beta(1)) = 0, i = 1, 2, \text{and } \psi(1) = 1. \) Further, let
\[
x_1 = \frac{[n_2^2 - 2\sqrt{n}]}{n}, \quad x_2 = \frac{[n_2^2 - \sqrt{n}]}{n}, \quad x_3 = \frac{[n_2^2 + \sqrt{n}]}{n}, \quad x_4 = \frac{[n_2^2 + 2\sqrt{n}]}{n},
\]
and
\[
\psi_1(x) = \psi\left( \frac{x - x_1}{x_2 - x_1} \right), \quad \psi_2(x) = \psi\left( \frac{x - x_3}{x_4 - x_3} \right).
\]

Consider
\[
P(x) := x - x_4 \psi_1(f(x)) + \frac{x_1 - x}{x_1 - x_4} \psi_2(f(x)).
\]
the linear function joining the points \((x_1, f(x_1))\) and \((x_4, f(x_4))\).

And let
\[
\overline{F}_n(f, x) := \overline{F}_n(f)
\]
\[
= f(x)(1 - \psi_1(x) + \psi_2(x)) + \psi_1(x)(1 - \psi_2(x))P(x).
\]

From the above definitions it follows that
\[
\mathcal{T}_n(f, x) = \begin{cases} f(x), & x \in [0, x_1] \cup [x_4, 1], \\ f(x)(1 - \psi_1(x) + \psi_2(x)) + \psi_1(x)(1 - \psi_2(x))P(x), & x \in [x_1, x_2], \\ P(x), & x \in [x_2, x_3], \\ (1 - \psi_2(x)) + \psi_2(x)f(x), & x \in [x_3, x_4]. \end{cases}
\]

Evidently, \( \mathcal{T}_n \) is a positive linear polynomials which depends on the functions values \( f(k/n) \), \( 0 \leq k/n \leq x_2 \) or \( x_3 \leq k/n \leq 1 \). it reproduces linear functions, and \( \mathcal{T}_n \in C^2([0, 1]) \) provided \( f \in W^2 \). Now for every \( f \in C_\psi \) define the Bernstein type polynomials
\[
\overline{B}_n(f, x) := \mathcal{T}_n(\mathcal{T}_n(f, x), x)
\]
\[
= \sum_{k=0}^{n-1} p_{nk}(f_{k/n}) \frac{k}{n} + \sum_{k=0}^{n-1} p_{nk}(f_{k/n}) \frac{k}{n} + \sum_{k=0}^{n-1} p_{nk}(f_{k/n}) \{(1 - \psi_1(k/n)) + \psi_2(k/n)P(k/n)\}
\]
\[
+ \sum_{k=0}^{n-1} p_{nk}(f_{k/n}) \{(1 - \psi_1(k/n)) + \psi_2(k/n)P(k/n)\}.
\]

(2.1)

Obviously, \( \overline{B}_n \) is a positive linear polynomials, \( \overline{B}_n(f) \) is a polynomial of degree at most \( n \), it preserves linear functions, and depends only on the function values \( f(k/n), k/n \in [0, x_2] \cup [x_3, 1] \). Now we state our main results as follows:

Theorem 1. If \( \alpha > 0 \), for any \( f \in C_\psi \), we have
\[
\|w \overline{B}_n(f)\| \leq Cn^2 \|w\|f\|.
\]

Theorem 2. For any \( \alpha > 0 \), \( \min\{\beta(0), \beta(1)\} \geq \frac{1}{2} \), \( 0 < \xi < 1 \), we have
\[
\text{w}(x) \phi^2(x) = \mathcal{B}_n(f, x) \leq \left\{ \begin{array}{ll} Cn^2 \|w\|_x, & f \in C_\psi, \\ C\|w\phi^2\|, & f \in W^2_\phi. \end{array} \right.
\]

Theorem 3. For \( f \in C_\psi \), \( 0 < \xi < 1 \), \( \alpha > 0 \), \( \min\{\beta(0), \beta(1)\} \geq \frac{1}{2} \), \( \alpha \in (0, 2) \), we have
\[
\overline{B}_n(f, x) = \mathcal{B}_n(f, x) \leq Cn^2 \|w\|f\|.
\]

3. Lemmas

Lemma 1. [7] For any non-negative real \( u \) and \( v \), we have
\[
\sum_{k=1}^{n} \left( \frac{k}{n} \right)^{-u} \left( 1 - \frac{k}{n} \right)^{-v} \leq Cx^{-u}(1-x)^{-v}.
\]

Lemma 2. [3] For any \( \alpha > 0 \), \( f \in C_\psi \), we have
\[
\|w \overline{B}_n(f)\| \leq C\|w\|f\|.
\]

Lemma 3. [6] Let \( \min\{\beta(0), \beta(1)\} \geq \frac{1}{2} \), then for \( 0 < t < \frac{1}{2} \) and \( t < x < 1 - t \), we have

\[
\overline{B}_n(f, x) \leq Cn^2 \|w\|f\|.
\]
\[ \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \phi^{-1} \left( x + \sum_{k=1}^{n} \mu_k \right) du_1 \, du_2 \leq C \phi^{-1}(x). \quad (3.3) \]

**Proof.** From the definition of \( \phi(x) \), it is enough to prove (3.3) for \( t < x \leq \frac{1}{2} \) since the proof for \( \frac{1}{2} < x < 1 - t \) is very similar. Obviously, we have

\[ \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{x + \sum_{k=1}^{n} \mu_k} \, du_1 \, du_2 \leq C t^{-1}. \]

Therefore, by the Hölder inequality, we have

\[ \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \phi^{-1} \left( x + \sum_{k=1}^{n} \mu_k \right) du_1 \, du_2 \leq C \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{x + \sum_{k=1}^{n} \mu_k} \, du_1 \, du_2. \]

Then the lemma follows.

**Lemma 4.** If \( \gamma \in R, \) then

\[ \sum_{k=0}^{n} p_{n,k}(x) |k - nx|^{\gamma} \leq C \sigma^{\gamma} \phi^{\gamma}(x). \quad (3.4) \]

**Lemma 5.** Let \( A_{\varepsilon}(x) := \tilde{w}(x) \sum_{|k| < \varepsilon} p_{n,k}(x) \). Then \( A_{\varepsilon}(x) \leq Cn^{1-\frac{1}{4}} \) for \( 0 < \varepsilon < 1 \) and \( x > 0 \).

**Proof.** If \( |x - \xi| \leq \frac{1}{\sqrt{n}} \), then the statement is trivial. Hence assume \( 0 < x < \xi < \frac{1}{\sqrt{n}} \) (the case \( \xi + \frac{1}{\sqrt{n}} < x < 1 \) can be treated similarly). Then for a fixed \( x \) the maximum of \( p_{n,k}(x) \) is attained for \( k = k_n := \lfloor n \xi - \sqrt{n} \rfloor \). By using Stirling’s formula, we get

\[ p_{n,k_n}(x) \leq C \left( \frac{\sqrt{n} x}{n} \right)^{k_n} \left( \frac{1}{n} \right)^{n-k_n} \left( \sqrt{n} - k_n \right)^{-k_n} \]

\[ \leq C \frac{nx}{n} \frac{k_n}{n-k_n} \left( \frac{1}{n} \right)^{n-k_n} \]

\[ = C \frac{nx}{n} \left( 1 - \frac{k_n - nx}{n} \right)^{k_n} \left( 1 + \frac{k_n - nx}{n} \right)^{n-k_n}. \]

Now from the inequalities

\[ k_n - nx = \lfloor n \xi - \sqrt{n} \rfloor - nx > n(\xi - x) - \sqrt{n} - 1 \geq \frac{1}{2} n(\xi - x), \]

and

\[ 1 - u \leq e^{-\frac{u}{\sqrt{n}}}, \quad 1 + u \leq e^u, \quad u \geq 0, \]

it follows that the second inequality is valid. To prove the first one we consider the function \( \lambda(u) = e^{-\frac{u}{\sqrt{n}}} + u - 1 \). Here \( \lambda(0) = 0, \lambda'(u) = -(1 + u)e^{-\frac{u}{\sqrt{n}}} + 1, \lambda''(u) = u(u + 2)e^{-\frac{u}{\sqrt{n}}} \geq 0 \), whence \( \lambda(u) \geq 0 \) for \( u \geq 0 \).

Thus \( A_{\varepsilon}(x) \leq C(\xi - x)^2 e^{-C \xi/z^2} \). An easy calculation shows that here the maximum is attained when \( \xi = \frac{\xi}{n} \) and the lemma follows.

**Lemma 6.** For \( 0 < \xi < 1, \alpha, \beta > 0, \) we have

\[ \tilde{w}(x) \sum_{|k| < \alpha} |k - nx|^\beta p_{n,k}(x) \leq C n^{\alpha^2/\beta} \phi^{\beta}(x). \quad (3.5) \]

**Proof.** By (3.4) and the Lemma 5, we have

\[ \tilde{w}(x)^{\frac{1}{2}} \left( \tilde{w}(x) \sum_{|k| < \alpha} p_{n,k}(x) \right)^{\frac{1}{2}} \leq C n^{\alpha^2/\beta} \phi^{\beta}(x). \]

**Lemma 7.** For any \( \alpha > 0, f \in W_2^\alpha, \min\{\beta(0), \beta(1)\} \geq 1/2, \) we have

\[ \tilde{w}(x) |f(x) - P(f,x)|_{[x_1, x_2]} \leq C \left( \frac{\delta(x)}{\sqrt{n} \phi(x)} \right)^2 \|\tilde{w} f\|^2. \quad (3.6) \]

**Proof.** If \( x \in [x_1, x_2] \), for any \( f \in W_2^\alpha \), we have

\[ f(x_1) = f(x) + f'(x)(x_1 - x) + \int_{x_1}^{x} (t - x) f''(t) dt, \]

\[ f(x_2) = f(x) + f'(x)(x_4 - x) + \int_{x_4}^{x} (t - x) f''(t) dt, \]

\[ \delta(x) \sim \frac{1}{\sqrt{n}}, \quad n = 1, 2, \ldots. \]

So

\[ \tilde{w}(x) |f(x) - P(f,x)| = \tilde{w}(x) \frac{X - x}{x_1 - x_2} \int_{x_1}^{x} |(t - x) f''(t)| dt + \tilde{w}(x) \frac{x_1 - x}{x_1 - x_4} \int_{x_1}^{x} |(t - x) f''(t)| dt \]

\[ = I_1 + I_2. \]

Whence \( t \) between \( x_1 \) and \( x_2 \), we have \( \frac{|x_1 - x_2|}{x_1 - x_2} \leq \frac{|x_1 - x_2|}{|x_1 - x_2|} \), then

\[ I_1 \leq C \tilde{w} \|\tilde{w} f\|^2 |(x - x_1)(x - x_2)| \int_{x_1}^{x} f''(t) dt \]

\[ \leq C \left( \frac{\phi(x)}{\sqrt{n} \phi(x)} \right)^2 \|\tilde{w} f\|^2 \|\tilde{w} f''\|. \]

Analogously, we have

\[ I_2 \leq C \left( \frac{\delta(x)}{\sqrt{n} \phi(x)} \right)^2 \|\tilde{w} f''\|. \]
Now the lemma follows from combining these results together. □

**Lemma 8.** If $f \in W^2_{n}$, $\min\{\beta(0), \beta(1)\} \geq \frac{1}{2}$, then
\[
||\bar{w}\varphi^2T_n\| = O(||\bar{w}\varphi^2f''||).
\]  

**(Proof)** Again, it is sufficient to estimate $(\bar{w}\varphi^2T_n^i(x))$ for $x \in [x_3, x_4]$, and the same as $x \in [x_1, x_2]$. For $x \in [x_2, x_3]$, $T_n^i(x) = 0$, while for $x \in [0, x_1] \cup [x_4, 1]$, $T_n^i(x) = f(x)$. Thus for $x \in [x_3, x_4]$, then $T_n^i(x) = P(x) + \psi_2(x)(f(x) - P(x))$ and
\[
T_n^i(x) = n\psi^2\left[n\psi(x - x_3)\right](f(x) - P(x))
+ \frac{n\psi^2\left[n\psi(x - x_1)\right]}{n}\left(f(x) - P(x)\right)'
+ \frac{n\psi\left[n\psi(x - x_3)\right]}{n}\left(f(x) - P(x)\right)''
:= I_1(x) + I_2(x) + I_3(x).
\]

From the proof of Lemma 7, we have
\[
|\bar{w}(x)\varphi^2(x)I_1(x)| = O\left(\|\bar{w}\varphi^2f''\|\right).
\]

**For** $I_2(x)$, it is obvious that
\[
|\bar{w}(x)\varphi^2(x)I_2(x)| = O\left(\|\bar{w}\varphi^2f''\|\right).
\]

Finally
\[
|\bar{w}(x)\varphi^2(x)I_3(x)| = O\left(\|\bar{w}\varphi^2f''\|\right).
\]

4. **Proof of theorem**

4.1 **Proof of Theorem 1**

**Case 1.** If $f \in C_{\bar{w}}$, when $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$, by [2], we have
\[
|\bar{w}(x)\overline{B}_n^i(f, x)| \leq n\overline{w}^2(x)\overline{w}(x)\overline{B}_n^i(f, x)
+ \bar{w}(x)\varphi^4(x)\sum_{k=0}^n p_{n,k}(x)k
- nx\left|\overline{B}_n^i\left(\frac{k}{n}\right)\right| + \bar{w}(x)\varphi^4(x)\sum_{k=0}^n p_{n,k}(x)
\times (k - nx)^2\left|\overline{B}_n^i\left(\frac{k}{n}\right)\right|
:= A_1 + A_2 + A_3.
\]  

By (3.2), we have
\[
A_1(x) = n\overline{w}^2(x)\overline{w}(x)\overline{B}_n^i(f, x) \leq Cn^2\|\bar{w}\|f\|.
\]  

and
\[
A_2 = \bar{w}(x)\varphi^4\left(\sum_{k=0}^n k - nx\right)\left|\overline{B}_n^i\left(\frac{k}{n}\right)\right|p_{n,k}(x)
+ \sum_{k=0}^n \left|k - nx\right|\left|\overline{B}_n^i\left(\frac{k}{n}\right)\right|p_{n,k}(x) := \sigma_1 + \sigma_2.
\]  

thereof $A := [0, x_1] \cup [x_3, 1]$. If $\frac{x}{2} \in A$, when $\frac{\bar{w}(x)}{\varphi(x)} \leq C(1 + n^{-2}k - nx)^{\frac{1}{2}}$, we have $|k - n\xi| \geq \frac{\sqrt{2}}{2}$, by (3.4), then
\[
\sigma_1 \leq C\|\bar{w}\|\varphi^4(x)\sum_{k=0}^n p_{n,k}(x)\|k - nx\|\|1 + n^{-2}k - nx\|^2
= C\|\bar{w}\|\varphi^4(x)\sum_{k=0}^n p_{n,k}(x)\|k - nx\|
+ Cn^{-2}\|\bar{w}\|\|\varphi^4(x)\sum_{k=0}^n p_{n,k}(x)\|k - nx\|^{1+\varepsilon}
\leq Cn^2\varphi^{-1}(x)\|\bar{w}\| + Cn^2\varphi^{-3+\varepsilon}(x)\|\bar{w}\| \leq Cn^2\|\bar{w}\|.
\]  

For $\sigma_2$, $\bar{w}$ is a linear function. We note $|P(\frac{3}{2})| \leq \max(\|P(x_1)\|, |P(x_2)|) := \bar{w}(a)$. If $x \in [x_1, x_2]$, we have $\bar{w}(x) \leq \bar{w}(a)$. So, if $x \in [x_1, x_2]$, by (3.4), then
\[
\sigma_2 \leq C\bar{w}(a)\varphi^4(x)\sum_{k=0}^n p_{n,k}(x)\|k - nx\| \leq Cn^2\|\bar{w}\|.
\]  

If $x \notin [x_1, x_2]$, then $\bar{w}(a) > n^{-\frac{1}{2}}$, by (3.5), we have
\[
\sigma_2 \leq C\bar{w}(a)\varphi^4(x)\sum_{x_1 \leq k/n \leq x_2} \left|P(a)\right|\|k - nx\|p_{n,k}(x)
\leq Cn^2\|\bar{w}\|\|\varphi^4(x)\sum_{x_1 \leq k/n \leq x_2} \left|k - nx\right|p_{n,k}(x)
\leq Cn^2\|\bar{w}\|.
\]  

So
\[
A_2 \leq Cn^2\|\bar{w}\|.
\]  

Similarly
\[
A_3 \leq Cn^2\|\bar{w}\|.
\]  

It follows from combining with (4.1)-(4.4) that the inequality is proved.

**Case 2.** When $x \in [0, \frac{1}{n}]$ (The same as $x \in [1 - \frac{1}{n}, 1]$, by [4], then
\[
\overline{B}_n^i(f, x) = n(n - 1)\sum_{k=0}^{n-2} \Delta^2_1\overline{B}_n(k/n)p_{n-2,k}(x).
\]

We have
\[
|\bar{w}(x)\overline{B}_n^i(f, x)| \leq Cn\bar{w}(x)\sum_{k=0}^{n-2} \Delta^2_1\overline{B}_n(k/n)p_{n-2,k}(x)
= Cn\bar{w}(x)\sum_{k=0}^{n-2} \left|\overline{B}_n^i\left(\frac{k}{n}\right)\right|\Delta^2_1\overline{B}_n(k/n)
+ \sum_{x_2 \leq k/n \leq x_3} \left|\overline{B}_n^i\left(\frac{k}{n}\right)\right|\overline{B}_n^i(k/n).
\]
We can deal with it in accordance with Case 1, and prove it immediately, then the theorem is done. □

4.2. Proof of Theorem 2

(1) We prove the first inequality of Theorem 2.

Case 1. If $0 \leq \varphi(x) \leq \frac{1}{n}$, by (2.2), we have

$$|ar{w}(x)(\varphi^2(x)\bar{B}_n(f, x))|=\varphi^2(x)\cdot\frac{\varphi'(x)}{\varphi(x)}|ar{w}(x)\bar{B}_n(f, x)| \leq Cn||\bar{w}||.$$

Case 2. If $\varphi(x) > \frac{1}{n}$, by [4], we have

$$\bar{B}_n(f, x) = B_n(T_n, x) = (\varphi^2(x))^{-1}\sum_{k=0}^{n}Q(x, n)n'$$

$$\times \sum_{k=0}^{n}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x), \quad (\varphi^2(x))^{-1}Q(x, n)n' \leq Cn/\varphi^2(x)^{1+1/2}.$$

So

$$\bar{w}(x)(\varphi^2(x)\bar{B}_n(f, x)) \leq C\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$= C\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$+ C\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$\times \sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x) := \sigma_1 + \sigma_2.$$

where $A := [0, x_1] \cup [x_3, 1]$. Working as in the proof of Theorem 1, we can get $\sigma_1 \leq Cn||\bar{w}||, \sigma_2 \leq Cn||\bar{w}||$. By bringing these facts together, we can immediately get the first inequality of Theorem B.

(2) If $f \in W^{2}_{\alpha, 2}$, by (2.1), then

$$\bar{w}(x)(\varphi^2(x)\bar{B}_n(f, x)) \leq n^2\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n-2}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$= n^2\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n-2}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$+ n^2\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n-2}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$+ n^2\bar{w}(x)(\varphi^2(x))^{-1}\sum_{k=0}^{n-2}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) p_{nk}(x),$$

$$:= I_1 + I_2 + I_3.$$

By [4], if $0 < k < n - 2$, we have

$$\left|\frac{\Delta_{2}^{2}T_{n}}{n}\right| \leq Cn^{-1}\int_{0}^{1/2} \bar{T}_n\left(\frac{k}{n}\right) + u) du.$$ (4.6)

If $k = 0$, we have

$$\left|\frac{\Delta_{2}^{2}T_{n}}{n}\right| \leq C\int_{0}^{1/2} u\bar{T}_n(u) du.$$ (4.7)

Similarly

$$\int_{1/2}^{1} (1 - u)\bar{T}_n(u) du.$$ (4.8)

By (4.6), then

$$I_1 \leq Cn\bar{w}(x)\varphi^2(x)\sum_{k=0}^{n-2}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du + \int_{0}^{1/2} u\bar{T}_n(u) du,$$

$$\leq Cn\bar{w}(x)\varphi^2(x)\sum_{k=0}^{n-2}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du + \int_{0}^{1/2} u\bar{T}_n(u) du,$$

$$+ Cn\bar{w}(x)\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du$$

$$:= Cn\bar{w}(x)\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du,$$

where $A := [0, x_1] \cup [x_3, 1]$. When $k/n \in A$, we have $|k - n\zeta| \geq \sqrt{n}$, by (3.1), (3.4) and (3.7), then

$$I_1 \leq Cn\bar{w}(x)\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du,$$

$$\leq C\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du,$$

$$\leq Cn\bar{w}(x)\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du.$$ (4.9)

So, we can get

$$I_1 \leq Cn\bar{w}(x)\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du,$$

$$\geq C\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du,$$

$$\geq C\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du.$$ (4.10)

Similarly

$$I_2 \leq Cn\bar{w}(x)\varphi^2(x)(1 - x)^{n-2} \int_{0}^{1/2} u\bar{T}_n(u) du,$$

$$\leq Cn\bar{w}(x)\varphi^2(x)(1 - x)^{n-2} \int_{0}^{1/2} u\bar{T}_n(u) du,$$

$$\leq C\varphi^2(x)\sum_{x_k \leq k/n \leq x_k}x - \frac{k}{n} \left(\frac{k}{n}\right) \bar{T}_n\left(\frac{k}{n}\right) + u) du.$$ (4.11)

By bringing (4.5), (4.9)–(4.11) together, we can get the second inequality of Theorem B. □

Corollary 1. For any $\varphi > 0$, $0 \leq \alpha \leq 1$, we have

$$|\bar{w}(x)(\varphi^2(x)\bar{B}_n(f, x))| \leq \left\{ Cn\left(\max\{n^{1-\alpha}, \varphi^{2(\alpha - 1)}\}\right)||\bar{w}||, \quad f \in C_{\alpha}, \right.$$ (4.12)

$$\left. f \in W_{\alpha, 2}^{1} \right\}.$$ (4.13)

4.3. Proof of Theorem 3

4.3.1. The direct theorem

We know

$$\bar{T}_n(t) + \bar{T}_n(t - x) + \int_{x}^{t} (t - u)\bar{T}_n(u) du,$$ (4.14)

$$B_n(t - x, x) = 0.$$ (4.15)
According to the definition of $W^2_\phi$, by (4.13) and (4.14), for any $g \in W^2_\phi$, we have $B_n(g, x) = B_n(\overline{G}_n(x), x)$, then

$$\bar{w}(x) |\overline{G}_n(x) - B_n(\overline{G}_n, x)| = \bar{w}(x) |B_n(R_1(\overline{G}_n, t, x), x)|,$$

thereof $R_1(\overline{G}_n, t, x) = f_1^{\prime}(t - u)G_n(u)du$.

$$\bar{w}(x) |\overline{G}_n(x) - B_n(\overline{G}_n, x)|$$

$$\leq C\bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} \left( \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\bar{w} \right) \left( \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\phi^2 \right)$$

$$= C\bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} \left( \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\bar{w} \right) \left( \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\phi^2 \right)$$

$$\leq Cn^{-2} \bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) (k - nx)^2$$

$$\leq Cn^{-2} \bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) (k - nx)^2$$

$$\leq C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|.$$  

(4.16)

If $u$ between $\frac{1}{2}$ and $x$, we have

$$\left| \frac{\bar{w}(x)}{\bar{w}(u)} \right| \leq \left| \frac{\bar{w}(x)}{\bar{w}(u)} \right| \leq \left| \frac{\bar{w}(x)}{\bar{w}(u)} \right|.$$

(4.17)

By (3.4) and (4.17), then

$$I_1 \leq C\|\bar{w} \phi^2 \overline{G}_n\| \|\bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\bar{w} \| \|\phi^2 \overline{G}_n\|$$

$$\leq C\|\bar{w} \phi^2 \overline{G}_n\| \|\bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\bar{w} \| \|\phi^2 \overline{G}_n\|$$

$$\leq Cn^{-2} \|\bar{w} \phi^2 \overline{G}_n\| \|\phi^2 \overline{G}_n\|$$

$$\leq Cn^{-2} \|\phi^2 \overline{G}_n\| \|\phi^2 \overline{G}_n\|$$

$$= C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|.$$  

(4.18)

For $I_2$, when $u$ between $\frac{1}{2}$ and $x$, we let $k = 0$, then $\frac{\bar{w}(x)}{\bar{w}(u)} \leq \frac{\bar{w}(x)}{\bar{w}(u)}$, and

$$I_2 \leq C\|\bar{w} \phi^2 \overline{G}_n\| \|\bar{w}(x) \sum_{k=0}^{n-1} p_{n,k}(x) \int_x^{x + \frac{|l - u|}{\bar{w}(u)\phi^2(u)}} d\bar{w} \| \|\phi^2 \overline{G}_n\|$$

$$\leq C(\phi(x)(1 - x)\phi^2(x) \|\bar{w} \phi^2 \overline{G}_n\|$$

$$\leq C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|.$$  

(4.19)

Similarly, we have

$$I_3 \leq C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|. $$

(4.20)

By bringing (4.18)-(4.20), we have

$$\bar{w}(x) |\overline{G}_n(x) - B_n(\overline{G}_n, x)| \leq C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|.$$  

(4.21)

By (3.6) and (4.21), when $g \in W^2_\phi$, then

$$\bar{w}(x) |g(x) - \overline{B}_n(g, x)|$$

$$\leq \bar{w}(x) |g(x) - \overline{G}_n(g, x)| + \bar{w}(x) |\overline{G}_n(g, x) - \overline{B}_n(g, x)|$$

$$\leq \bar{w}(x) |g(x) - P_0(g, x)| \|\phi^2 \overline{G}_n\|$$

$$\leq C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|.$$  

(4.22)

For $f \in C_n$, we choose proper $g \in W^2_\phi$, by (3.2) and (4.22), then

$$\bar{w}(x) |f(x) - \overline{B}_n(f, x)| \leq \bar{w}(x) |f(x) - g(x)| + \bar{w}(x) |\overline{B}_n(f, g, x)|$$

$$+ \bar{w}(x) |g(x) - \overline{B}_n(g, x)|$$

$$\leq C\left( \|\bar{w} \phi^2 \overline{G}_n\| \right) + \left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|$$

$$\leq C\left( \frac{\delta_n(x)}{\sqrt{\phi(x)}} \right)^2 \|\bar{w} \phi^2 \overline{G}_n\|.$$  

(4.23)

**4.3.2. The inverse theorem**

The main-part $K$-functional is given by

$$K_{2,0}(f, t^2)_w = \sup_{0 \leq h < t} \inf_{g \in C_n} \left\{ \|\bar{w} \phi^2 \overline{G}_n\|, g \in A.C. \right\}.$$  

(4.24)

**Proof** Let $\delta > 0$, by (4.23), we choose proper $g$ so that

$$\|\bar{w} \phi^2 \overline{G}_n\| \leq C\left( \delta \phi(x) \right)_w, \|\bar{w} \phi^2 \overline{G}_n\| \leq C\delta^{-2} \phi(x) \delta_w.$$  

(4.25)

then

$$\left| \bar{w}(x) \Delta_n^2 f(x) \right| \leq \left| \bar{w}(x) \Delta_n^2 f(x) - \overline{B}_n(f, x) \right|$$

$$+ \left| \bar{w}(x) \Delta_n^2 \overline{B}_n(f, g, x) + \bar{w}(x) \Delta_n^2 \overline{B}_n(g, x) \right|$$

$$\leq \sum_{j=0}^{n-1} C_j \left| \frac{n \phi(x)}{\phi(x + (1 - j) \phi(x))} \right| \bar{w}(x) \overline{B}_n(f, g, x)$$

$$+ \int_{\frac{\phi(x)}{n}}^{\frac{\phi(x)}{n+1}} \int_{\frac{\phi(x)}{n+1}}^{\frac{\phi(x)}{n+2}} \bar{w}(x) \overline{B}_n(f, g, x + \sum_{k=1}^{j} u_k) \|d_1, d_2 \|$$

$$+ \int_{\frac{\phi(x)}{n+1}}^{\frac{\phi(x)}{n+2}} \int_{\frac{\phi(x)}{n+2}}^{\frac{\phi(x)}{n+3}} \bar{w}(x) \overline{B}_n(g, x + \sum_{k=1}^{j} u_k) \|d_1, d_2 \|$$

$$:= J_1 + J_2 + J_3.$$  

(4.26)

Obviously

$$J_1 \leq C \left( \left( \frac{n \phi(x)}{\phi(x + (1 - j) \phi(x))} \right) \bar{w}(x) \right)^{\delta_w}.$$  

(4.27)

By (2.2) and (4.24), we have

$$J_2 \leq Cn^2 \|\bar{w} \phi^2 \overline{G}_n\| \|d_1, d_2 \|$$

$$\leq Cn^2 \|\bar{w} \phi^2 \overline{G}_n\| \|d_1, d_2 \|.$$  

(4.28)

By the second inequality of (4.12) and (4.24), we have
\[ J_2 \leq C n^2 \left( \frac{\sum_{k=1}^{n} (x + \frac{2}{k})}{\sum_{k=1}^{n} (x + \frac{2}{k})} \right) \left( x + \frac{2}{k} \right) \left( \sum_{k=1}^{n} (x + \frac{2}{k}) \right) \]

By the second inequality of (2.3), (3.3) and (4.24), we have

\[ J_3 \leq C \left( \frac{\sum_{k=1}^{n} (x + \frac{2}{k})}{\sum_{k=1}^{n} (x + \frac{2}{k})} \right) \left( x + \frac{2}{k} \right) \left( \sum_{k=1}^{n} (x + \frac{2}{k}) \right) \]

Now, by (4.25)–(4.29), there exists a constant \( M > 0 \) so that

\[ \left| \tilde{w}(x) \Delta^2_{\mbox{ap}} f(x) \right| \leq C \left( n \frac{n^2 \delta_0(x)}{\phi(x)} + \min \left\{ \frac{n \phi^2(x)}{\phi^2(x)} n^2 \phi^2(x) \right\} \right) \times h^2 \delta^2_{\phi}(f, \delta) \]

Therefore

\[ \left| \tilde{w}(x) \Delta^2_{\mbox{ap}} f(x) \right| \leq C \left\{ \delta^0 + h^2 \delta^2_{\phi}(f, \delta) \right\} \]

Which implies

\[ \omega^0_{\phi}(f; t)_{\delta} \leq C \left\{ \delta^0 + h^2 \delta^2_{\phi}(f, \delta) \right\} \]

So, by Berens–Lorentz lemma in [4], we get

\[ \omega^2_{\phi}(f; t)_{\delta} \leq C \delta^0. \quad \square \]

We can obtain the similar results when the Bernstein polynomials have no singularities. Now, we can consider the combinations of Bernstein Polynomials with inner singularities as Theorem 3 with countable or uncountable singularities.

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