Analytical Solution to the Fokker-Planck Equation with a Bottomless Action

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Abstract

A new Langevin equation with a field-dependent kernel is proposed to deal with bottomless systems within the framework of the stochastic quantization of Parisi and Wu. The corresponding Fokker-Planck equation is shown to be a diffusion-type equation and is solved analytically. An interesting connection between the solution with the ordinary Feynman measure, which in this case is not normalizable, is clarified.
One of the interesting and appealing applications of the stochastic quantization (SQ) of Parisi and Wu [1] is found in its application to systems characterized by actions unbounded from below (bottomless systems). An attempt to stabilize and eventually to quantize bottomless systems is of great interest in several important fields of physics, such as Euclidean Einstein gravity. To deal with bottomless systems on the basis of SQ, Greensite and Halpern [2] made use of the Fokker-Planck equation w.r.t. a fictitious time

$$\frac{\partial}{\partial t} P[\phi; t] = H[\phi] P[\phi; t],$$

with an action $S$ unbounded from below. In this case, the distribution functional $e^{-S}$ (Feynman measure) cannot be an equilibrium solution nor even a stationary solution, for it is not normalizable. Because the negative semi-definiteness of the Fokker-Planck Hamiltonian $H$ can be proved irrespectively of the boundedness of the action, they proposed the interesting idea that a possible way to stabilize and to quantize such bottomless systems in Euclidean space-time is such that the probability distribution in the Euclidean path integral formulation is given by the lowest (normalizable) eigenstate of $H$. This corresponds to using the lowest eigenstate of $H$ of the form $e^{-S_{\text{eff}}}$, instead of $e^{-S}$, as an effective distribution functional in evaluating expectation values. It is also shown that such an effective distribution can indeed be extracted by fixing both initial and final configurations [3]. This procedure avoids runaway solutions which are due to the unboundedness of the action $S$ of the Langevin equation. Notice that their proposal for such bottomless systems corresponds to a departure from the traditional approach that the weight functional in the path integral formulation be given by $e^{-S}$, with $S$ a classical action. Remember also that their Langevin equation does not have an equilibrium distribution because the naive candidate $e^{-S}$ is not normalizable and hence does not belong to the spectrum of the Fokker-Planck Hamiltonian $H$: Every eigenstate belongs to its negative-definite eigenvalue.

Recently Tanaka et al. proposed another interesting way of stochastically quantizing bottomless systems [4]. Unlike the above attempts [2,3], they have pursued the possibility of reproducing the desired probability distribution $e^{-S}$ in the equilibrium limit of the fictitious stochastic process even for bottomless systems. Their starting point is the following Langevin equation

$$\frac{\partial}{\partial t} \phi(x, t) = -K[\phi] \frac{\delta S[\phi]}{\delta \phi(x)} + \frac{\delta K[\phi]}{\delta \phi(x)} + K^{2}[\phi] \eta(x, t)$$

with a field-dependent positive kernel $K[\phi]$. Here $\eta$ is a Gaussian white noise with the statistical properties

$$\langle \eta(x, t) \rangle = 0, \quad \langle \eta(x, t) \eta(x', t') \rangle = 2\delta^D(x - x') \delta(t - t')$$
and we adopt, here and in what follows, the Ito-interpretation [5] for stochastic equations. It is straightforward to derive the corresponding Fokker-Planck equation from the above Langevin equation (2), and the Fokker-Planck Hamiltonian is then given by

$$H[\phi] = \int d^Dx \frac{\delta}{\delta\phi(x)} K[\phi] \left( \frac{\delta}{\delta\phi(x)} + \frac{\delta S[\phi]}{\delta\phi(x)} \right).$$  \hfill (4)

It is clear from this equation that for any positive definite kernel functional $K[\phi]$ the spectrum of $H$ is still negative semi-definite so that the system relaxes to a unique equilibrium state whose distribution is given by $e^{-S}$, if it is normalizable and if the spectrum has a discrete zero. If the classical action $S$ is not bounded from below, the distribution $e^{-S}$ is not normalizable, and can never be either an equilibrium solution or a stationary solution of the above equation. Nevertheless, a better choice of the kernel $K$ may result in the situation where the stochastic variable $\phi$ is effectively confined to a finite range, so that the distribution $e^{-S}$ can be regarded as a solution of (4) normalizable within such a range. Their choice of the kernel $K$, for example, is

$$K[\phi] = e^{-S_4[\phi]}$$  \hfill (5)

when the action $S$ is given by the difference of two positive monomials

$$S[\phi] = S_2[\phi] - S_4[\phi],$$  \hfill (6a)

$$S_2[\phi] = \frac{1}{2} \int d^Dx \phi(x)[-\partial^2 + m^2]\phi(x), \quad S_4[\phi] = \frac{\lambda}{4} \int d^Dx \phi^4(x), \quad (\lambda > 0).$$  \hfill (6b)

This choice results in the following form of the Langevin equation

$$\frac{\partial}{\partial t}\phi(x, t) = -e^{-S_4[\phi]} \frac{\delta S_2[\phi]}{\delta\phi(x, t)} + e^{-\frac{1}{2}S_4[\phi]} \eta(x, t),$$  \hfill (7)

which has a damping drift term. They have argued [4] that this drift term effectively confines $\phi$ to a finite range so that the desired distribution $e^{-S}$ can be derived in the equilibrium limit of (7). Their numerical simulation of the above kernelized Langevin equation for a simple zero-dimensional model (wrong-sign $x^4$ model) could indeed reproduce the distribution $e^{-S(x)} = e^{-\frac{1}{2}m^2x^2 + \frac{1}{4}x^4}$ at equilibrium [4].

Seeking a possible way of reproducing $e^{-S}$ in the equilibrium limit for bottomless systems is challenging and very appealing. However, their proof is crucially dependent on the plausible argument that the effective range of the stochastic variable may be finite, which makes the apparently nonnormalizable distribution $e^{-S}$ normalizable, but the soundness of which is only verified numerically.

In this short paper, I propose another way of effectively reproducing $e^{-S}$, starting from the above Langevin equation (2) with a slightly different kernel $K$. It will be shown that the resultant Fokker-Planck equation can be solved exactly at any fictitious time $t < \infty$. It turns
out to be a diffusion-type equation, i.e. the spectrum of the Hamiltonian \( H \) is continuous and no equilibrium limit exists. This might be considered an apparent drawback, but it will also be shown that the asymptotic form of the probability distribution is well approximated by the desired form \( e^{-S} \) normalized in the finite range of the stochastic variable, corresponding to a finite fictitious time. These results, though of course not contradictory to it, form an interesting contrast with the previous analysis [4], where a discrete spectrum of \( H \) is implicitly assumed and the equilibrium distribution is numerically found to be well approximated by \( e^{-S} \).

Throughout this paper, we exclusively consider zero-dimensional potential models whose classical action \( S(x) \) is unbounded from below, for notational simplicity.

Let us start our discussion with the simplest possible model, i.e. the wrong-sign Gaussian model

\[
S(x) = -\frac{1}{2}mx^2, \quad (m > 0),
\]

and show that an exact solution to the Fokker-Planck equation (1a) with the Hamiltonian (4) is obtainable if we choose an appropriate kernel \( K(x) \). First we divide the above action into two positive monomials as in (6a),

\[
S(x) = S_0(x) - S_1(x),
\]

with

\[
S_0(x) = \frac{1}{2}mx^2, \quad S_1(x) = mx^2.
\]

Then following Ref. [4], we are naturally led to a kernel

\[
K(x) = e^{-S_1(x)} = e^{-mx^2}.
\]

Though the above separation ((9) and (10)) seems quite arbitrary, it will be shown that this choice is crucial for the ensuing Fokker-Planck equation to be solvable and is just an example of a general scheme \( K = e^{2S} \) which in this case is equal to (11) because \( 2S = -S_1 \) (see (29) and (32) below). The explicit forms of the Langevin and the Fokker-Planck equations are

\[
\dot{x} = -e^{-S_1(x)}S_0'(x) + e^{-\frac{1}{2}S_1(x)}\eta = -e^{-mx^2}mx + e^{-\frac{1}{2}mx^2}\eta,
\]

and

\[
\dot{P}(x;t) = \frac{\partial}{\partial x}e^{-S_1(x)} \left( \frac{\partial}{\partial x} + S'(x) \right) P(x;t) = \frac{\partial}{\partial x}e^{-mx^2} \left( \frac{\partial}{\partial x} - mx \right) P(x;t),
\]

respectively. As we know that the corresponding Fokker-Planck Hamiltonian has negative semi-definite eigenvalues, we put

\[
P(x;t) = e^{-c^2}e^{-S(x)}\tilde{P}(x),
\]
with a real parameter $c$. The above Fokker-Planck equation (13) is then reduced to an ordinary differential equation for $\tilde{P}(x)$

$$-c^2 e^{mx^2} \tilde{P}(x) = \left( \frac{d}{dx} - mx \right) \frac{d}{dx} \tilde{P}(x). \tag{15}$$

Now we change the variable from $x$ to $f(x)$ in the above equation and obtain

$$\left\{ (f'(x))^2 \frac{d^2}{df^2} + \left( f''(x) - mx f'(x) \right) \frac{d}{df} + c^2 e^{mx^2} \right\} \tilde{P}(x) = 0. \tag{16}$$

For the equation to be solvable, we require

$$f'(x) \propto e^{mx^2/2} \quad \text{and} \quad f''(x) - mx f'(x) = 0. \tag{17}$$

It is clear that this condition is identically satisfied for $f'(x) \propto e^{mx^2/2}$, or

$$f(x) = \alpha \int_0^x dy e^{my^2/2} + \beta \tag{18}$$

with two real parameters $\alpha$ and $\beta$. Notice that the transformation between $x$ and the above $f$ is one to one and the domain of $f$ is again $[-\infty, \infty]$. This choice of $f(x)$ drastically simplifies the equation (16) and makes it solvable. In fact, $\tilde{P}$ now satisfies

$$\left\{ \alpha^2 \frac{d^2}{df^2} + c^2 \right\} \tilde{P}(x) = 0, \tag{19}$$

which is easily solved to give

$$\tilde{P}(x) = \text{Re} \left( e^{ic \int_0^x dy e^{my^2/2}} \right) \quad \text{or} \quad \text{Im} \left( e^{ic \int_0^x dy e^{my^2/2}} \right). \tag{20}$$

At this stage it is important to notice that the normalization condition, which is the only condition that the probability distribution $P$ is subject to,

$$\int_{-\infty}^{\infty} dx \ P(x; t) = \int_{-\infty}^{\infty} dx \ e^{-c^2 t} e^{-S(x)} \tilde{P}(x) = 1 \tag{21}$$

seems to be satisfied for any real $c \neq 0$ because at large $|x|$, which corresponds to large $|f|$, rapid oscillations of $\tilde{P}$ (see (20)) may make the integral convergent despite the ever-increasing factor $e^{-S(x)}$. No condition has been imposed on the value of $c$, which implies that the system has a continuous spectrum. This will be verified shortly.

Thus we have arrived at the conclusion that the solution of the Fokker-Planck equation (13) is in general of the form

$$P(x; t) = \int_{-\infty}^{\infty} dc \ e^{-c^2 t} e^{-S(x)} \text{Re} \left[ A(c) e^{icf(x)/\alpha} \right] = \int_{-\infty}^{\infty} dc \ e^{-c^2 t} e^{-S(x)} \text{Re} \left[ A(c) e^{ic \int_0^x dy e^{my^2/2}} \right], \tag{22}$$
where the function $A(c)$ in the second equality has been re-defined to absorb the constant term in $f(x)$. Specification of $A(c)$ certainly corresponds to a specific choice of the initial distribution $P(x; 0)$. It is not difficult to see that the simplest choice of $A(c)$ does correspond to the most fundamental initial condition. In fact, if we choose a constant $A = 1/2\pi$, we have, at the initial time $t = 0$,

$$P(x; 0) = \int_{-\infty}^{\infty} dc e^{-S(x)} \text{Re} \left[ \frac{1}{2\pi} e^{ic \int_{0}^{x} dy e^{my^2/2}} \right] = e^{\frac{1}{4\pi^2} \delta \left( \int_{0}^{x} dy e^{my^2/2} \right)^2} = \delta(x). \quad (23)$$

In the last equality we have made use of the one-to-one correspondence between $x$ and $\int_{0}^{x} dy e^{my^2/2}$ and the monotonically increasing character of the latter w.r.t. $x$.

In this way, we finally obtain the solution of the Fokker-Planck equation (13)

$$P(x; t) = \int_{-\infty}^{\infty} dc e^{-c^2 t} e^{-S(x)} \text{Re} \left[ \frac{1}{2\pi} e^{ic \int_{0}^{x} dy e^{my^2/2}} \right] = \frac{1}{\sqrt{4\pi t}} e^{-S(x)} e^{-\frac{1}{4\pi} \left( \int_{0}^{x} dy e^{my^2/2} \right)^2}, \quad (24)$$

subject to the initial condition $P(x; 0) = \delta(x)$. It is easily confirmed that this solution is normalizable, i.e. that $\int_{-\infty}^{\infty} dx P(x; t) = 1$. As a trivial generalization, the solution subject to the initial condition $P(x; 0) = P_0(x)$ appears to be

$$P(x; t) = \int_{-\infty}^{\infty} dx_0 \frac{1}{\sqrt{4\pi t}} e^{-S(x)} e^{-\frac{1}{4\pi} \left( \int_{0}^{x} dy e^{my^2/2} \right)^2} P_0(x_0). \quad (25)$$

It is instructive to observe that this analytical solution obtained for the simplest case can be expressed in terms of $S$ only

$$P(x; t) = \int_{-\infty}^{\infty} dx_0 \frac{1}{\sqrt{4\pi t}} e^{-S(x)} e^{-\frac{1}{4\pi} \left( \int_{0}^{x} dy e^{-S(y)} \right)^2} P_0(x_0). \quad (26)$$

This suggests that there may in general exist an analytical solution. This is indeed the case, as will be shown explicitly later.

It is remarkable that we can obtain analytical solutions to the Fokker-Planck equation (13), corresponding to the nonlinear Langevin equation (12). It should be stressed again that the solution decays at large $t$ and therefore no equilibrium limit exists and its behaviour is thus quite different from that expected in the ordinary case: It is rather similar to that of diffusion processes, which again implies that the spectrum is continuous. If, for example, we choose a trivial kernel $K = 1$ in (13), no nonlinearity appears and the standard technique is applicable even though the potential has the wrong sign [6]. In this case, the solution $P_{K=1}$ subject to $P_{K=1}(x; 0) = \delta(x-x_0)$ is

$$P_{K=1}(x; t) = \sqrt{\frac{m}{2\pi(1-e^{-2mt})}} \exp \left[ -\frac{m(xe^{-mt}-x_0)^2}{2(1-e^{-2mt})} \right] e^{-mt}. \quad (27)$$
whose asymptotic behaviour at large $t$ shows the discreteness of the spectrum

$$P_{K=1}(x; t) \xrightarrow{t \to \infty} \sqrt{\frac{m}{2\pi}} e^{-mx_0^2/2} e^{-mt}.$$  \hspace{1cm} (28)$$

Now let us turn our attention to more general cases. No specific form of the classical action $S$ is assumed here, except for its unboundedness from below. The problem is to find an appropriate kernel $K$ which makes the Fokker-Planck equation solvable, as in the previous example. If we write a positive kernel $K$ as

$$K(x) = e^{-W(x)}$$  \hspace{1cm} (29)$$

using a real function $W(x)$, the function $\tilde{P}$ already introduced in (14) satisfies

$$\begin{cases} \left( f'(x) \right)^2 \frac{d^2}{df^2} + \left( f''(x) - [W'(x) + S'(x)]f'(x) \right) \frac{d}{df} + e^2 e^W(x) \end{cases} \tilde{P}(x) = 0.$$  \hspace{1cm} (30)$$

In this case, solvability requires that

$$f'(x) \propto e^{W(x)/2} \text{ and } f''(x) - [W'(x) + S'(x)]f'(x) = 0.$$  \hspace{1cm} (31)$$

These conditions are clearly satisfied by the following $f$ and $W$

$$W(x) = -2S(x),$$  \hspace{1cm} (32)$$

$$f(x) = \alpha \int_0^x dy e^{-S(y)} + \beta'. $$  \hspace{1cm} (33)$$

Having found the appropriate kernel and the transformation function, we are therefore able to solve the Fokker-Planck equation, which in this case is written

$$\dot{P}(x; t) = \frac{\partial}{\partial x} e^{2S(x)} \left( \frac{\partial}{\partial x} + S'(x) \right) P(x; t),$$  \hspace{1cm} (34)$$

just as in the previous example. The solution $P(x; t)$, satisfying the initial condition $P(x; 0) = P_0(x)$, is straightforwardly found to be

$$P(x; t) = \int_{-\infty}^{\infty} dx_0 \frac{1}{\sqrt{4\pi t}} e^{-S(x)} e^{-\frac{1}{4t} \left( \int_{x_0}^{x} dy e^{-S(y)} \right)^2} P_0(x_0),$$  \hspace{1cm} (35)$$

as expected from (26).

Finally, let us discuss possible implications of these results in numerical simulations. One usually keeps track of the stochastic variable, whose time development is governed by the Langevin equation

$$\dot{x} = e^{2S(x)} S'(x) + e^{S(x)} \eta$$  \hspace{1cm} (36)$$
when the kernel is given by $K = e^{2S}$. Let the total action $S$ be given by the difference between two terms

$$S(x) = S_0(x) - S_i(x), \quad (37)$$

where $S_i$ is a positive monomial of the highest order, assumed even, and $S_0(x)$ the remaining terms. The above Langevin equation is approximated for large $x$ by

$$\dot{x} \sim -e^{-2S_i(x)} S_i'(x) + e^{-S(x)} \eta, \quad (38)$$

and for small $x$ by

$$\dot{x} \sim e^{2S_0(x)} S_0'(x) + e^{S(x)} \eta, \quad (39)$$

Thus the kernel $K = e^{2S}$ supplies a restoring force for large $x$, but a diverging force for small $x$. To understand the behaviour of the stochastic variable $x(t)$ starting from a finite value $x(0) = x_0$, and the approximate form of the probability distribution $P$, consider only the finite range $|x(t) - x_0| \lesssim R$. This range may be viewed as an actual domain of the variable in numerical simulations; (no infinite range can be realized in practical simulations.) Next we define $t_R$ as a time scale satisfying

$$\frac{1}{4t_R} \left( \int_{x_0-R}^{x_0+R} dy e^{-S(y)} \right)^2 = 1. \quad (40)$$

Then at $t = t_R$, the second exponential factor of the exact solution (35) is expressed as

$$e^{-\frac{1}{4t_R} \left( \int_{x_0}^{x} dy e^{-S(y)} \right)^2} = e^{-\left( \int_{x_0}^{x} dy e^{-S(y)} \right)^2 / \left( \int_{x_0-R}^{x_0+R} dy e^{-S(y)} \right)^2} \quad (41)$$

and is considered to be of the order of unity for any $x$ in the range $|x - x_0| \lesssim R$. Under this estimation, the probability distribution at time $t_R$ is well approximated by a naive distribution $e^{-S}$ normalized in this range

$$P(x; t_R) \sim \frac{e^{-S(x)}}{Z_R}, \quad Z_R = \int_{x_0-R}^{x_0+R} dy e^{-S(y)}. \quad (42)$$

This implies the possibility in the actual simulation of the Langevin equation (36) of reproducing the desired probability distribution $e^{-S}$ at a certain finite time $t_R$, normalized in a finite domain determined by the details of the simulation. Further investigation is necessary to draw more definite conclusions.

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