STRANGE EXAMPLES OF LOCAL SIGNATURES FOR FIBERED SURFACES OF SMALL GENUS

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Abstract. We give examples of local signatures, completely different from the usual ones, for general fibrations of genus 2 and genus 3.

Introduction

For a closed oriented real 4-manifold $X$, the signature of $X$ is defined to be the signature of the intersection form $H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \to \mathbb{R}$, which is a symmetric bilinear form. We consider the situation that $X$ admits a fibration $f : X \to B$ over a closed oriented real surface $B$. Under some conditions, the signature of $X$ happens to localize around a finite number of fiber germs $F_1, F_2, \ldots, F_m$:

$$\text{Sign}(X) = \sum_{i=1}^{m} \sigma(F_i).$$

We call this phenomenon a localization of the signature and the value $\sigma(F_i)$ a local signature of $F_i$. A first example of local signatures is the one for genus 1 fibrations due to Matsumoto [14]. He also gave a local signature for Lefschetz fibrations of genus 2 in [15], which was generalized by Endo [9] for hyperelliptic fibrations. Later, Kuno [13] defined a local signature for non-hyperelliptic fibrations of genus 3. On the other hand, Horikawa [12] defined a function $\text{Ind}(F)$ on the set of holomorphic fiber germs $F$ of genus 2, which is nowadays called a Horikawa index, in order to study algebraic surfaces of general type near the Noether line. Once a Horikawa index is defined (for a certain type of holomorphic fibrations), we can define a local signature by using it, as shown in [4]. After Horikawa’s work, Xiao [19] and Arakawa-Ashikaga [1] defined a Horikawa index and a local signature for hyperelliptic fibrations. Terasoma [18] showed the coincidence of Endo’s local signature and Arakawa-Ashikaga’s one. For non-hyperelliptic genus 3 fibrations, Reid [16] defined a Horikawa index. Similarly to Terasoma’s proof, Kuno’s local signature and Reid’s one for non-hyperelliptic fibrations of genus 3 also coincide (cf. [3]).

In this short note, in the algebro-geometric category, we construct a local signature associated with an effective divisor $D$ on the moduli space $\mathcal{M}_g$ of smooth curves of genus $g$ and compute some examples of local signatures for general fibrations of genus 2 or 3, which are different from Endo-Arakawa-Ashikaga’s one and Kuno-Reid’s one. The idea of constructions is essentially due to Ashikaga-Yoshikawa [5], who called the divisor $4\lambda - \delta$ on the moduli space of curves. The 2010 Mathematics Subject Classification is 14D06. The Keywords are fibration, local signature.
space $\overline{M}_g$ of stable curves of genus $g$ the signature divisor and gave a local signature by pulling back the signature divisor using a geometric meaningful effective divisor $D$, e.g., the Brill-Noether locus, via the moduli map of a fiber germ. Replacing $D$ by another effective divisor, the associated local signature varies. We compute local signatures in the case that $g = 2$ and $D$ is the bielliptic locus and that $g = 3$ and $D$ is the locus of curves having a hyperflex.

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1. Local signature associated with an effective divisor on $M_g$

Let $M_g$ and $\overline{M}_g$ respectively denote the moduli space of smooth curves of genus $g$ and the moduli space of stable curves of genus $g$. The rational Picard group of $\overline{M}_g$ is generated freely by the Hodge bundle $\lambda$ and the boundary divisors $\delta_0, \delta_1, \ldots, \delta_{g/2}$ for $g \geq 3$, where we use the notation in [11]. Let $D$ be an effective divisor on $M_g$ and $D$ the compactification of $D$ in $\overline{M}_g$. Then we can write $D \sim_\mathbb{Q} a\lambda - \sum b_i\delta_i$ for some rational numbers $a, b_i > 0$, where the symbol $\sim_\mathbb{Q}$ means the $\mathbb{Q}$-linear equivalence.

Let $f: S \to B$ be a surjective morphism from a complex smooth projective surface $S$ to a smooth projective curve $B$ whose general fiber $F_\eta$ is a smooth projective curve of genus $g$, which is called a fibered surface or a global fibration of genus $g$. Let $K_f = K_S - f^*K_B$ denote the relative canonical bundle of $f$ and put

$$\chi_f = \deg f_*O_S(K_f) = \chi(O_S) - (g - 1)(b - 1),$$
$$e_f = e_{\text{top}}(S) - e_{\text{top}}(F_\eta)e_{\text{top}}(B) = e_{\text{top}}(S) - 4(g - 1)(b - 1),$$

where $b$ is the genus of $B$ and $e_{\text{top}}(X)$ the topological Euler number of $X$.

Let $f: S \to \Delta$ be a relatively minimal degeneration of curves of genus $g$, that is, $f$ is a surjective proper morphism from a complex smooth surface $S$ to a small open disk $\Delta$ such that $f^{-1}(t)$ is a smooth curve of genus $g$ for any $t \neq 0$ and the central fiber $F := f^{-1}(0)$ has no $(-1)$-curves. We take the stable reduction $\tilde{f}: \tilde{S} \to \tilde{\Delta}$ of $f$ via $\Delta \to \Delta; z \mapsto z^N$. Resolving singularities of $\tilde{S}$, we obtain a semi-stable reduction $\hat{f}: \hat{S} \to \hat{\Delta}$. Note that $N$ can be taken as the pseudo-period of the topological monodromy $\mu_f$ of $f$ as a pseudo-periodic class (cf. [2]). Put $F := f^{-1}(0)$ and $\hat{F} := \hat{f}^{-1}(0)$. Let

$$\text{Lsd}(F) := \sigma(f, F; h_{\partial S}) - \frac{1}{N}\sigma(\hat{f}, \hat{F}; h_{\partial \hat{S}})$$

be the local signature defect of $(f, F)$ (more precisely, see [2]) and

$$e_F := (e_{\text{top}}(F) - (2 - 2g)) - \frac{1}{N}\left(e_{\text{top}}(\hat{F}) - (2 - 2g)\right).$$

On the other hand, the local invariants $c_1^g(F), c_2^g(F)$ and $\chi_F$ were defined in [17] for a fiber germ $F$ of a global fibration $f: S \to B$. Indeed,
Proposition 1.1. We have $e_F = c_2(F)$ and

\[ \text{Lsd}(F) = \frac{1}{3}(e_1^2(F) - 2e_F) = 4\chi_F - e_F. \]

Proof. These invariants satisfy the following properties: Let $f : S \to B$ be a fibered surface of genus $g$ and $\hat{f} : \hat{S} \to \hat{B}$ be the semi-stable reduction of $f$ via a cyclic covering $\hat{B} \to B$ of degree $N$. Then we have

\[ \text{Sign}(S) - \frac{1}{N}\text{Sign}(\hat{S}) = \sum_{p \in B} \text{Lsd}(F_p), \]

\[ K_f^2 - \frac{1}{N}K_{\hat{f}}^2 = \sum_{p \in B} c_1^2(F_p), \]

\[ e_f - \frac{1}{N}e_{\hat{f}} = \sum_{p \in B} c_2(F_p) = \sum_{p \in B} e_{F_p}, \]

\[ \chi_f - \frac{1}{N}\chi_{\hat{f}} = \sum_{p \in B} \chi_{F_p}. \]

(1.1)

Let $F$ be an arbitrary fiber germ in a global fibration $f : S \to B$. Taking base change, we may assume that any fiber of $f$ other than $F$ is semi-stable. Thus we get the assertion from Hirzebruch’s signature formula $\text{Sign}(S) = K_f^2 - 8\chi_f$, Noether’s formula $12\chi_f = K_f^2 + e_f$ and (1.1) since $\text{Lsd}(\hat{F}) = c_1^2(\hat{F}) = c_2(\hat{F}) = e_{\hat{F}} = \chi_{\hat{F}} = 0$ for any semi-stable fiber germ $\hat{F}$. \hfill \Box

Let $\rho_{\hat{f}} : \tilde{\Delta} \to \overline{\mathcal{M}}_g$ be the moduli map of the semi-stable reduction $\hat{f} : \hat{S} \to \tilde{\Delta}$. For an effective divisor $E$ on $\overline{\mathcal{M}}_g$ not containing the image $\rho_{\hat{f}}(\tilde{\Delta})$, we can define the pull-back $\rho_{\hat{f}}^*E$. Let $\delta(\hat{F}) := \text{deg}(\rho_{\hat{f}}^*E)$. Note that even when $E \sim E'$ holds for two effective divisors $E$ and $E'$, it is not always true that $E(\hat{F}) = E'(\hat{F})$ because we treat local fibrations here. Given an effective divisor $D$ on $\mathcal{M}_g$ such that $\overline{D}$ does not contain $\rho_{\hat{f}}(\tilde{\Delta})$ with $\overline{D} \sim_Q a\lambda - \sum b_i\delta_i$, we put

\[ \lambda_D(\hat{F}) := \frac{1}{a}\left(\overline{D}(\hat{F}) + \sum b_i\delta_i(\hat{F})\right). \]

In general, for a relatively minimal fiber germ $F$, we define

\[ \lambda_D(F) := \chi_F + \frac{\lambda_D(\hat{F})}{N} \]

and

\[ \delta(F) := e_F + \frac{\delta(\hat{F})}{N} = e_{\text{top}}(F) - (2 - 2g), \]

which are independent of the choice of $N$. 3
Now we consider a global fibration $f: S \to B$, that is, a surjective morphism from a smooth projective surface $S$ to a smooth projective curve $B$ with connected fibers. Assume that the moduli point of the general fiber of $f$ is not contained in $D$. From [11], we have

$$\chi_f = \sum_{p \in B} \lambda_D(F_p), \quad e_f = \sum_{p \in B} \delta(F_p).$$

From Hirzebruch’s signature formula $\text{Sign}(S) = 4\chi_f - e_f$, we can write

$$\text{Sign}(S) = \sum_{p \in B} (4\lambda_D(F_p) - \delta(F_p)).$$

We call $\sigma_D := 4\lambda_D(F) - \delta(F)$ the local signature of a fiber germ $F$ associated with $D$. Note that the divisor $4\lambda - \delta$ is called the signature divisor in [5].

2. Examples

Now we consider two effective divisors $E_{g,-1}$ and $E_{g,1}$ on $\mathcal{M}_g$, which parameterize curves $C$ of genus $g$ having a special Weierstrass point. Let $C$ be a smooth curve of genus $g$. Let $p$ be a Weierstrass point of $C$, i.e., a point on $C$ satisfying $h^0(gp) \geq 2$. Then $p$ is said to be exceptional of type $g-1$ (resp. of type $g+1$) if $h^0((g-1)p) \geq 2$ (resp. $h^0((g+1)p) \geq 3$). The locus $E_{g,-1}$ (resp. $E_{g,1}$) on $\mathcal{M}_g$ is (roughly) defined by the set of curves of genus $g$ with an exceptional Weierstrass point of type $g-1$ (resp. of type $g+1$) with the natural scheme structure, which is of codimension 1 for $g \geq 3$. For more details, see [8]. For $g = 2$, the loci $E_{2,-1}$ and $E_{2,1}$ are empty. For $g = 3$, $E_{3,-1}$ is coincide with the hyperelliptic locus $\mathcal{H}_3$ as a set, but as a divisor, we have $E_{3,-1} = 8\mathcal{H}_3$. Indeed, once a genus 3 curve has one exceptional Weierstrass point of type 2, it becomes hyperelliptic and hence has 8 Weierstrass points of type 2 automatically. Since the hyperelliptic Weierstrass point is exceptional of type $g-1$ and $g+1$, the hyperelliptic locus $\mathcal{H}_g$ is contained in both $E_{g,-1}$ and $E_{g,1}$. In particular, $E_{3,-1} = 8\mathcal{H}_3$ is a subdivisor of $E_{3,1}$. Thus we can define an effective divisor $\mathcal{H}_F := E_{3,1} - E_{3,-1}$. As a different definition, let $\mathcal{H}_F$ be the locus on the moduli space $\mathcal{M}_3 \setminus \mathcal{H}_3$ of smooth plane quartics parameterizing plane quartic curves with a hyperflex, i.e., 4-fold tangent point. Then the above $\mathcal{H}_F$ is just the closure of $\mathcal{H}_F$ in $\mathcal{M}_3$. The locus $\mathcal{H}_F$ has multiplicity 1 around general points. For $g \geq 4$, $E_{g,-1}$ and $E_{g,1}$ also have multiplicity 1 around general points. It is known that the rational divisor classes of $E_{g,-1}$ and $E_{g,1}$ are given by

$$E_{g,-1} = \frac{g^2(g-1)(3g-1)}{2} \lambda - \frac{(g-1)^2g(g+1)}{6} \delta_0 - \sum_{i=1}^{[g/2]} \frac{i(g-i)g(g^2 + g - 4)}{2} \delta_i,$$

$$E_{g,1} = \frac{(g+1)(g+2)(3g^2 + 3g + 2)}{2} \lambda - \frac{g(g+1)^2(g+2)}{6} \delta_0 - \sum_{i=1}^{[g/2]} \frac{i(g-i)(g+1)(g+2)^2}{2} \delta_i.$$
an irreducible fiber germ of \( f \) irreducible curves with one node and the number of them is 27. Thus we have \( F_{60} \). Let number of smooth curves in a general Lefschetz pencil of quartic curves with a hyperflex is we obtain a non-hyperelliptic fibration \( f \). Let \( \lambda \) define a (pre-)Horikawa index \( \text{Ind}(F) \) and \( \sigma \) two local signatures \( \lambda \) define \( \lambda(\hat{F}) = (\delta_0(\hat{F}) + 2\delta_1(\hat{F}))/10 \). For a not necessarily semi-stable fiber germ \( F \), we define \( \lambda(\hat{F}) \) by using the semi-stable reduction similarly as in the previous section. We also define a (pre-)Horikawa index \( \text{Ind}(F) := 10\lambda(F) - \delta(F) \) for a relatively minimal genus 2 fiber germ \( F \). It coincides with the original Horikawa index defined by using the double covering data (cf. \[13, 12, 19\]) and hence it is non-negative. A local signature can be defined by \( \sigma(F) := 4\lambda(F) - \delta(F) \) for any fiber germ \( F \) of genus 2.

Example 2.1. Let \{\( C_\lambda \)\} be a general Lefschetz pencil of quartics. The base locus of \{\( C_\lambda \)\} consists of 16 points and they are on smooth members. Blowing up at these 16 points, we obtain a non-hyperelliptic fibration \( f: S \to \mathbb{P}^1 \) of genus 3. By a simple computation, we get \( \chi_f = 3, e_f = 27, K_f^2 = 9 \) and \( \text{Sign}(S) = -15 \). Note that all singular fibers of \( f \) are irreducible curves with one node and the number of them is 27. Thus we have \( \mathcal{H}_3(f) = 0, \lambda(f) = 3, \delta_0(f) = 27 \) and \( \delta_1(f) = 0 \). Hence we have \( \mathcal{H}_3(f) = 60 \). This implies that the number of smooth curves in a general Lefschetz pencil of quartic curves with a hyperflex is 60. Let \( F_{\text{hf}} \) and \( F_{\text{h}} \) respectively be a smooth quartic fiber germ of \( f \) with one hyperflex and an irreducible fiber germ of \( f \) with one node. Then clearly we have

\[
\delta_0(F_{\text{hf}}) = 0, \quad \delta_1(F_{\text{hf}}) = 0, \quad \mathcal{H}_3(F_{\text{hf}}) = 0, \quad \mathcal{H}_3(F_{\text{hf}}) = 1
\]

and

\[
\delta_0(F_{\text{h}}) = 1, \quad \delta_1(F_{\text{h}}) = 0, \quad \mathcal{H}_3(F_{\text{h}}) = 0, \quad \mathcal{H}_3(F_{\text{h}}) = 0.
\]

Thus we get

\[
\lambda(\mathcal{H}_3)(F_{\text{hf}}) = 0, \quad \lambda(\mathcal{H}_3)(F_{\text{h}}) = \frac{1}{9}, \quad \sigma(\mathcal{H}_3)(F_{\text{hf}}) = 0, \quad \sigma(\mathcal{H}_3)(F_{\text{h}}) = -\frac{5}{9}
\]

and

\[
\lambda(\mathcal{H}_3)(F_{\text{hf}}) = \frac{1}{308}, \quad \lambda(\mathcal{H}_3)(F_{\text{h}}) = \frac{8}{11}, \quad \sigma(\mathcal{H}_3)(F_{\text{hf}}) = \frac{1}{11}, \quad \sigma(\mathcal{H}_3)(F_{\text{h}}) = -\frac{45}{11}.
\]

Thus two local signatures \( \sigma(\mathcal{H}_3) \) and \( \sigma(\mathcal{H}_3) \) are different.

Next, let us consider the genus 2 case. The rational Picard group of \( \overline{\mathcal{M}}_2 \) is generated by \( \lambda, \delta_0 \) and \( \delta_1 \) with one relation \( 10\lambda = \delta_0 + 2\delta_1 \). For a semi-stable fiber germ \( \hat{F} \) of genus 2, we put \( \lambda(\hat{F}) := (\delta_0(\hat{F}) + 2\delta_1(\hat{F}))/10 \). For a not necessarily semi-stable fiber germ \( F \), we define \( \lambda(\hat{F}) \) by using the semi-stable reduction similarly as in the previous section. We also define a (pre-)Horikawa index \( \text{Ind}(F) := 10\lambda(F) - \delta(F) \) for a relatively minimal genus 2 fiber germ \( F \). It coincides with the original Horikawa index defined by using the double covering data (cf. \[13, 12, 19\]) and hence it is non-negative. A local signature can be defined by \( \sigma(F) := 4\lambda(F) - \delta(F) \) for any fiber germ \( F \) of genus 2.

Now, we define another local signature for non-bielliptic genus 2 fiber germs. Let \( \mathcal{B}_2 \) be the bielliptic locus on \( \mathcal{M}_2 \) and \( \overline{\mathcal{B}}_2 \) its closure in \( \overline{\mathcal{M}}_2 \). They are irreducible codimension 1 loci. From \[10\], the rational linearly equivalence class of \( \overline{\mathcal{B}}_2 \) is

\[
\overline{\mathcal{B}}_2 = \frac{3}{2}\delta_0 + 6\delta_1 = 30\lambda - \frac{3}{2}\delta_0 = 15\lambda + 3\delta_1.
\]
Thus, for non-bielliptic genus 2 fiber germs, two localizations of the Hodge bundle \( \lambda \) can be realized as follows. We put
\[
\lambda_{B_2,0}(\hat{F}) := \frac{1}{30} \tilde{B}_2(\hat{F}) + \frac{1}{20} \delta_0(\hat{F})
\]
and
\[
\lambda_{B_2,1}(\hat{F}) := \frac{1}{15} \tilde{B}_2(\hat{F}) - \frac{1}{5} \delta_1(\hat{F})
\]
for a semi-stable non-bielliptic fiber germ \( \hat{F} \) of genus 2. By using semi-stable reduction, we define \( \lambda_{B_2,i}(F) \) for any non-bielliptic fiber germ \( F \) of genus 2. Then \( \sigma_{B_2,i}(F) := 4\lambda_{B_2,i}(F) - \delta(F) \), \( i = 1, 2 \) are local signatures for genus 2 non-bielliptic fibrations.

**Example 2.2.** Let \( F_0 \) and \( F_b \) respectively be non-bielliptic genus 2 fiber germs the image of whose moduli map meets \( \Delta_0 \), \( \Delta_1 \) and \( B_2 \) transversally (and does not meet other loci among them) at the moduli point of the central fiber. Then we have
\[
\sigma(F_0) = -\frac{3}{5}, \quad \sigma(F_1) = -\frac{1}{5}, \quad \sigma(F_b) = 0,
\]
\[
\sigma_{B_2,0}(F_0) = -\frac{4}{5}, \quad \sigma_{B_2,0}(F_1) = -1, \quad \sigma_{B_2,0}(F_b) = \frac{2}{5},
\]
\[
\sigma_{B_2,1}(F_0) = -1, \quad \sigma_{B_2,1}(F_1) = -\frac{9}{5}, \quad \sigma_{B_2,0}(F_b) = \frac{4}{15}.
\]
For example, take a general member \( R \) in the complete linear system \( |pr_1^*O_{\mathbb{P}^1}(N) \otimes pr_2^*O_{\mathbb{P}^1}(6)| \), \( N \in 2\mathbb{Z}_{>0} \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) and construct the double covering \( S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) branched over \( R \). Then the composite \( f : S \rightarrow \mathbb{P}^1 \) of the double covering and the first projection \( pr_1 \) is a non-bielliptic fibration of genus 2. By a simple computation, we have
\[
\chi_f = N, \quad K_S^2 = 2N, \quad e_f = 10N, \quad \text{Sign}(S) = -6N.
\]
Since \( R \) is general, we may assume that any singular fiber germ of \( f \) is of type \( F_0 \) as above. Thus the number of fiber germs of type \( F_0 \), \( F_1 \) and \( F_b \) is \( 10N \), 0 and \( 15N \), respectively.

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