Birman-Wenzl-Murakami Algebra, Topological parameter and Berry phase

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In this paper, a 3×3-matrix representation of Birman-Wenzl-Murakami (BWM) algebra has been presented. Based on which, unitary matrices $A(\theta, \varphi_1, \varphi_2)$ and $B(\theta, \varphi_1, \varphi_2)$ are generated via Yang-Baxterization approach. A Hamiltonian is constructed from the unitary $B(\theta, \varphi)$ matrix. Then we study Berry phase of the Yang-Baxter system, and obtain the relationship between topological parameter and Berry phase.

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I. INTRODUCTION

To the best of our knowledge, the Yang-Baxter equation (YBE) was initiated in solving the one-dimensional $\delta$-interacting models [1] and the statistical models [2]. Braid algebra and Temperley-Lieb algebra (TLA) [3] have been widely used in the construction of YBE solutions [4–9] and have been introduced to the field of quantum information, quantum computation and topological computation [10–15]. The Birman-Wenzl-Murakami (BWM) algebra [16] which contain two subalgebra (Braid algebra and TLA) was first defined and independently studied by Birman, Wenzl and Murakami. Very recently [17], S.Abramsky demonstrate the connections from knot theory to logic and computation via quantum mechanics. But, the physical meaning of the topological parameter $d$ (describing the single loop in topology) is still unclear.

The geometrical phase [18], such as Berry phase, is an important concept in quantum mechanics [19–24]. In recent years, numerous works have been attributed to Berry phase [25], because of its possible applications to quantum computation (the so-called geometric quantum computation) [26–29]. Quantum logic gates based on geometric phases have been certified in both nuclear magnetic resonance [26] and ion trap based on quantum information architectures [30]. The Ref. [26] pointed out geometric phases have potential fault tolerance when applied to quantum information processing. In 2007, Leek, P.J. et al. [31] illustrated the controlled accumulation of a geometric phase, Berry phase, in a superconducting qubit.

The Ref. [32] applied TLA as a bridge to recast 4-dimensional YBE into its 2-dimensional counterpart. The 2-dimensional YBE have an important application value in topological quantum computation [33, 34]. To date, few studies have reported 3-dimensional YBE which may have potential application values in topological quantum computation. The motivation of this paper is twofold: one is that we structure 3-dimensional YBE, the other is to study the physical meaning of topological parameter $d$ from Berry phase. This paper is organized as follows: In Sec. 2, we introduce a specialized type BWM algebra, and present a $3 \times 3$-matrix representation of BWM algebra. In Sec. 3, we obtain unitary matrices $A(\theta, \varphi_1, \varphi_2), B(\theta, \varphi_1, \varphi_2)$ via Yang-Baxterization approach. Based on the solution, a Hamiltonian of the Yang-Baxter system is constructed, finally we study the Berry phase of this system. We end with a summary.
II. BWM ALGEBRA

As we know the Braid relations are

\[
\begin{align*}
    b_ib_{i\pm 1} &= b_{i\pm 1}b_i, \\
    b_ib_j &= b_jb_i, \quad |i - j| \geq 2,
\end{align*}
\]

(1)

where \(b_i = I \otimes \cdots \otimes I \otimes b \otimes I \otimes \cdots\). When we just consider three tensor product space, the Braid relations becomes

\[
b_{12}b_{23}b_{12} = b_{23}b_{12}b_{23},
\]

(2)

where \(b_{12} = b \otimes I, b_{23} = I \otimes b\), \(b\)-matrix is a \(N^2 \times N^2\) matrix acted on the tensor product space \(\nu \otimes \nu\), where \(N\) is the dimension of \(\nu\). It is also well known that the braid relation can be reduced to a \(N\)-dimensional braid relation \((b_{12} \to A, b_{23} \to B)\)

\[ABA = BAB.\]

(3)

Like this reduced method, we easily obtain \(N\)-dimensional reduced BWM-algebra relations from classical BWM-algebra relations. The BWM algebra \[16, 35–37\] is generated by the unit \(I\), the braid operators \(S_i\) and the TLA operators \(E_i\) and depends on two independent parameters \(\omega\) and \(\sigma\). Let us take the BWM relations as follows.

\[
\begin{align*}
    S_i - S_i^{-1} &= \omega(I - E_i), \\
    S_iS_{i+1}S_i &= S_{i+1}S_iS_{i+1}, \quad S_iS_j = S_jS_i, \quad |i - j| \geq 2, \\
    E_iE_{i\pm 1}E_i &= E_{i\pm 1}E_i, \quad E_iE_j = E_jE_i, \quad |i - j| \geq 2, \\
    E_iS_i &= S_iE_i = \sigma E_i, \\
    S_{i\pm 1}E_iS_{i\pm 1} &= E_iS_{i\pm 1}S_i = E_iE_{i\pm 1}, \\
    S_{i\pm 1}E_iS_{i\pm 1}^{-1} &= S_{i\pm 1}^{-1}E_iS_{i\pm 1}^{-1}, \\
    E_{i\pm 1}E_iS_{i\pm 1}^{-1} &= E_{i\pm 1}S_{i\pm 1}^{-1}, \quad S_{i\pm 1}E_iE_{i\pm 1} = S_{i\pm 1}^{-1}E_iE_{i\pm 1}, \\
    E_iS_{i\pm 1}^{-1}E_i &= \sigma^{-1}E_i, \\
    E_i^2 &= \left(1 - \frac{\sigma - \sigma^{-1}}{\omega}\right)E_i,
\end{align*}
\]

(4)

where \(0 \neq d = \left(1 - \frac{\sigma - \sigma^{-1}}{\omega}\right) \in \mathbb{C}\) is a topological parameter in knot theory which does not depend on the sites of the lattices.
By reducing to the $N$-dimensional space ($S_{12} \rightarrow A, S_{23} \rightarrow B, E_{12} \rightarrow E_A, E_{23} \rightarrow E_B$), we have:

\[
\begin{align*}
A - A^{-1} &= \omega(I - E_A), \quad B - B^{-1} = \omega(I - E_B), \\
ABA &= BAB, \\
E_A E_B E_A &= E_A, \quad E_B E_A E_B = E_B, \\
E_A A &= AE_A = \sigma E_A, \quad E_B B = BE_B = \sigma E_B, \\
ABE_A &= E_B AB = E_B E_A, \quad BAE_B = E_A BA = E_A E_B, \\
AE_B A &= B^{-1} E_A B^{-1}, \quad BE_A B = A^{-1} E_B A^{-1}, \\
E_A E_B A &= E_A B^{-1}, \quad E_B E_A B = E_B A^{-1}, \\
AE_B E_A &= B^{-1} E_A, \quad BE_A E_B = A^{-1} E_B, \\
E_A B E_A &= \sigma^{-1} E_A, \quad E_B A E_B = \sigma^{-1} E_B, \\
E_A^2 &= (1 - \frac{\sigma - \sigma^{-1}}{\omega}) E_A, \quad E_B^2 = (1 - \frac{\sigma - \sigma^{-1}}{\omega}) E_B, 
\end{align*}
\]

where $A, B$ satisfy the $N$-dimensional Braid relation (3), $E_A, E_B$ satisfy the $N$-dimensional TLA relations

\[
\begin{align*}
E_A E_B E_A &= E_A, \quad E_B E_A E_B = E_B, \\
E_A^2 &= d E_A, \quad E_B^2 = d E_B, 
\end{align*}
\]

It is interesting that Eq. (4) and Eq. (5) have the same topological parameter $d$.

In this paper, the $A$-matrix, $B$-matrix, $E_a$-matrix, $E_b$-matrix, $A(x)$-matrix and $B(x)$-matrix are $3 \times 3$ matrices acting on the 3-dimensional space. To the TLA relations (6), we assume $E_A$ and $E_B$ possess the same eigenvalues $d$ and 0. We assume $E_A$ is a diagonal matrix as following

\[
E_A = \begin{pmatrix}
0 & 0 & 0 \\
0 & d & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

After tedious calculation, we obtain

\[
E_B = \begin{pmatrix}
\frac{d^2 - d - 1}{d} & \frac{\sqrt{d^2 - d - 1}}{d} e^{i\varphi_1} & -\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} e^{i(\varphi_1 + \varphi_2)} \\
\frac{\sqrt{d^2 - d - 1}}{d} e^{-i\varphi_1} & \frac{1}{d} & -\frac{e^{i\varphi_2}}{\sqrt{d}} \\
-\frac{\sqrt{d^2 - d - 1}}{\sqrt{d}} e^{-i(\varphi_1 + \varphi_2)} & -\frac{e^{-i\varphi_2}}{\sqrt{d}} & 1
\end{pmatrix}.
\]
It is worth to mention that $E_B = U E_A U^{-1}$, and $U$ is a unitary transformation matrix as follows

$$U = \begin{pmatrix}
\frac{1}{(d-1)d} & -\frac{\sqrt{d^2 - d - 1}}{d} e^{i\varphi_1} & -\frac{\sqrt{d^2 - d - 1}}{d} e^{i(\varphi_1 + \varphi_2)} \\
\frac{\sqrt{d^2 - d - 1}}{d} & \frac{1}{d} & \frac{e^{i\varphi_2}}{\sqrt{d}} \\
\frac{\sqrt{d^2 - d - 1}}{d} e^{-i(\varphi_1 + \varphi_2)} & \frac{e^{-i\varphi_2}}{\sqrt{d}} & \frac{d - 2}{d - 1}
\end{pmatrix}, \quad (9)$$

where $d$, $\varphi_1$ and $\varphi_2$ are reals. The parameter $d$ is the so-called topological parameter. For simplicity, we just consider the case of $d > 0$ in this paper.

The Ref. [35] has explored $S_i$ have 3 different eigenvalues $(q, -q^{-1}, q^{-2})$ in the BWM-algebra (i.e. Eq.(11)). The same as $E_A$ and $E_B$, we assume $A$ and $B$ have the same eigenvalues $(q, -q^{-1}, q^{-2})$. The simplest $A$ is

$$A = \begin{pmatrix}
qu & 0 & 0 \\
0 & q^{-2} & 0 \\
0 & 0 & -q^{-1}
\end{pmatrix}, \quad (10)$$

using the unitary transformation matrix $U$, we have

$$B = U A U^{-1} = \begin{pmatrix}
\frac{1}{q^2(d-1)d} & \frac{\sqrt{d^2 - d - 1}}{dq} e^{i\varphi_1} & \frac{\sqrt{d^2 - d - 1}}{q^2(d-1)\sqrt{d}} e^{i(\varphi_1 + \varphi_2)} \\
\frac{\sqrt{d^2 - d - 1}}{dq} e^{-i\varphi_1} & \frac{q^2}{d} & \frac{q}{\sqrt{d}} e^{i\varphi_2} \\
-\frac{\sqrt{d^2 - d - 1}}{q^2(d-1)\sqrt{d}} e^{-i(\varphi_1 + \varphi_2)} & \frac{q}{\sqrt{d}} e^{-i\varphi_2} & \frac{d - 2}{d - 1}
\end{pmatrix}, \quad (11)$$

where $d = q^{-1} + 1 + q$ and the parameter $q$ is real. The matrices $A$ and $B$ satisfy the braid relation (i.e. Eq.(3)). Towards braid relation, in some models $\varphi_i, (i = 1, 2)$, may have a physical significance of magnetic flux. In the paper [13], it has been shown the parameters $\varphi_i$’s are related to Berry phase.

Then we can verify that $\{I, A, E_A, B, E_B\}$ satisfy the reduced BWM-algebra (i.e. Eq.(5)), with $d = q^{-1} + 1 + q$. Here we have set $\omega = q - q^{-1}$ and $\sigma = q^{-2}$. It is interesting that $A, B, E_A, E_B$ are Hermitian matrices, and have the same similar transformation $B = U A U^{-1}, E_B = U E_A U^{-1}$, where $U$ is unitary (i.e. $U^\dagger = U^{-1}$).

### III. YANG-BAXTERIZATION, HAMILTONIAN, BERRY PHASE

In this section, A Hamiltonian is constructed from the unitary $B(\theta, \varphi)$ matrix. Then we study the Berry phase of the Yang-Baxter system, and obtain the relationship between the topological parameter and the Berry phase. We
In this paper, we focus on 3-dimensional space. The reduced relativistic YBE reads

$$\hat{R}_{12}(u)\hat{R}_{23}\left(\frac{u + v}{1 + \beta^2 uv}\right)\hat{R}_{12}(v) = \hat{R}_{23}(v)\hat{R}_{12}\left(\frac{u + v}{1 + \beta^2 uv}\right)\hat{R}_{23}(u).$$

(12)

In this paper, we focus on 3-dimensional space. The reduced relativistic YBE reads

$$A(u)B\left(\frac{u + v}{1 + \beta^2 uv}\right)A(v) = B(v)A\left(\frac{u + v}{1 + \beta^2 uv}\right)B(u).$$

(13)

Let the unitary Yang-Baxter matrix take the form

$$\begin{align*}
A(u) &= \rho(u)(I + F(u)E_A) \\
B(u) &= \rho(u)(I + F(u)E_B).
\end{align*}$$

(14)

Following Xue et al. [38], we obtain

$$\begin{align*}
F(u) &= \frac{e^{-2i\theta} - 1}{d}, \\
e^{-2i\theta} &= \frac{\beta^2 u^2 + 2i\beta u\sqrt{d^2/(4 - d^2)} + 1}{\beta^2 u^2 - 2i\beta u\sqrt{d^2/(4 - d^2)} + 1},
\end{align*}$$

(15)

where the new parameter \( \theta \) is real. Let \( \rho(u) = e^{i\theta} \). The Yang-Baxter matrix can be rewritten in the following form

$$\begin{align*}
A(\theta, \varphi_1, \varphi_2) &= e^{i\theta}I - f(\theta)E_A, \\
B(\theta, \varphi_1, \varphi_2) &= e^{i\theta}I - f(\theta)E_B,
\end{align*}$$

(16)

where \( f(\theta) = 2i\sin\theta/d \).

The Yang-Baxter matrix depends on three parameters: the first is \( \theta \) (\( \theta \) is time-independent); the others are \( \varphi_i, (i = 1, 2) \) contained in the matrix \( E \). In physics the parameter \( \varphi_1 \) and \( \varphi_2 \) are flux which depends on time \( t \). Usually take \( \varphi_i = \omega_it, (i = 1, 2) \) and \( \omega_i \) are the frequency. Operators \( A(\theta, \varphi_1, \varphi_2) \) and \( B(\theta, \varphi_1, \varphi_2) \), satisfying \( B(\theta, \varphi_1, \varphi_2) = U A(\theta, \varphi_1, \varphi_2)U^{-1} \), are unitary operators \( A(\theta, \varphi_1, \varphi_2)^\dagger = A(\theta, \varphi_1, \varphi_2)^{-1}, B(\theta, \varphi_1, \varphi_2)^\dagger = B(\theta, \varphi_1, \varphi_2)^{-1} \).

To simplify the following discussion, we will restrict attention to the case \( \varphi_1 = -\varphi_2 = \varphi \). Following Ge et al. [13], we can obtain Yang Baxter Hamiltonian through the Schrödinger evolution of the states

$$\hat{H} = i\hbar\frac{\partial B(\theta, \varphi)}{\partial t}B^\dagger(\theta, \varphi),$$

(17)

where \( \varphi \) be time dependent as \( \varphi = \omega t \) and \( \theta \) be time independent.

For convenience, we introduce the Gell-Mann matrices \( I_\lambda \) [39], a basis for \( su(3) \) algebra. Such matrices satisfy

$$[I_\lambda, I_\mu] = if_{\lambda\mu\nu}I_\nu, (\lambda, \mu, \nu = 1, 2, ..., 8),$$

where \( f_{\lambda\mu\nu} \) are the structure constants of \( su(3) \). We denote \( I_\pm = I_1 \pm iI_2 \),
\( V_\pm = I_4 \mp i I_5, U_\pm = I_6 \pm i I_7 \) and \( Y = \frac{2}{\sqrt{3}} I_8 \). Let

\[
\begin{align*}
S_+ &= \zeta (-i(d^2 - d - 1)^{1/2}(e^{-i\theta} + 2i \sin \theta d^{-2}) I_+ + id^{1/2}(e^{-i\theta} + 2i \sin \theta d^{-2}) U_-), \\
S_- &= \zeta (i(d^2 - d - 1)^{1/2}(e^{i\theta} - 2i \sin \theta d^{-2}) I_- - id^{1/2}(e^{i\theta} - 2i \sin \theta d^{-2}) U_+), \\
S_3 &= \frac{1}{2} [(1 + d - d^2)(1 - d^2)^{-1} (I_3 + \frac{Y}{2} + I_3) - (I_3 + \frac{Y}{2} - I_3) - d(1 - d^2)^{-1} (\frac{I_3}{3} - Y)) + d^{1/2}(d^2 - d - 1)^{1/2} (1 - d^2)^{-1} (V_- + V_+)],
\end{align*}
\]

where \( \zeta = \frac{d^2}{\sqrt{(d^2-1)(d^2-4)(d^2-1)\sin^2 \theta}} \). These operators satisfy the \( su(2) \) algebra relations \([S_+, S_-] = 2S_3, [S_3, S_\pm] = \pm S_\pm, (S_\pm)^2 = 0, S_\pm = S_1 \pm i S_2 \).

In terms of the operators \([18]\), the Hamiltonian Eq. \((17)\) can be recast as following

\[
\hat{H} = -4 \omega \hbar \sin \theta (d^2 - 1)^{1/2} d^{-2} (\sin \alpha \cos \beta S_1 + \sin \alpha \sin \beta S_2 + \cos \alpha S_3).
\]

Its eigenvalues are \( E_0 = 0, E_{\pm} = \mp \omega \hbar \cos \alpha \), where \( \cos \alpha = \frac{2 \sin \theta \sqrt{d^2-1}}{d^2} \), \( \beta = \varphi \), here \( d \geq 1 \). By the way, its Casimir operator is \( \kappa = \frac{1}{2} (S_+ S_- + S_- S_+) + S_3^2 \). It is easy to find the eigenvalues of \( \kappa \) are \( \frac{1}{2} (\frac{1}{2} + 1) = \frac{3}{2} \) and \( 0(0 + 1) = 0 \), which correspond to spin-1/2 and spin-0. According to the definition of Berry phase, when \( \varphi(t) \) evolves adiabatically from 0 to \( 2\pi \), the corresponding Berry phase is

\[
\gamma_\alpha = i \int_0^T \langle \Psi_\alpha, \frac{\partial}{\partial t} | \Psi_\alpha \rangle dt.
\]

Noting that Hamiltonian returns to its original form after the time \( T = 2\pi / \omega \), we easily obtain the corresponding Berry phases of this Yang-baxter system

\[
\begin{align*}
\gamma_0 &= 0, \\
\gamma_\pm &= \pm \pi (1 - \cos \alpha) = \pm \frac{\Omega}{2},
\end{align*}
\]

where \( \Omega = 2 \pi (1 - \cos \alpha) \) is the solid angle enclosed by the loop on the Bloch sphere. The system also equals to spin-1/2 system and spin-0 system. Substituting \( \cos \alpha = \frac{2 \sin \theta \sqrt{d^2-1}}{d^2} \) into Eq. \((21)\), we obtain \( \gamma_\pm = \pm \pi (1 - \frac{2 \sin \theta \sqrt{d^2-1}}{d^2}) \).

Substituting \( \theta \) with \( \frac{\pi}{2} - \theta \), we rewrite Berry phase as follows

\[
\gamma_\pm = \pm \pi (1 - \frac{2 \cos \theta \sqrt{d^2-1}}{d^2}).
\]

It is worth mentioning that in some papers \([13]\), the Berry phase \( \gamma_\pm = \pm \pi (1 - \cos \theta) \) of Yang-Baxter system only depends on the spectral parameter \( \theta \). It is interesting that in our paper, the Berry phases \([22]\) not only depends on the spectral parameter \( \theta \), but also depends on the topological parameter \( d \). The Berry phase (Eq. \((22)\)) reduce to \( \gamma_\pm = \pm \pi (1 - \cos \theta) \) if \( d = \sqrt{2} \). The FIG. \([1]\) which corresponds to the Berry phase \( \gamma_+ \). The FIG. \([1(a)]\) illustrate the
Berry phases $\gamma_\pm = \pi (1 - \frac{2 \cos \theta \sqrt{d^2 - 1}}{d})$ versus the topological parameter $d$ and the spectral parameter $\theta$. The right figure, the sectional drawings have also provided with the same values of parameters. $d = 1$ (solid line), $d = \sqrt{2}$ (dot-dashed line), $d = 2$ (dotted line), $d = 3$ (dashed line), $d = 5$ (star line).

Berry phases Eq.(22) versus the spectral parameter $\theta$ and the topological parameter $d$. The FIG. 1(b) illustrate the Berry phase $\gamma_+$ versus the spectral parameter $\theta$, when $d$ choice specific values. It is demonstrated that the Berry phase $\gamma_+$ is maximum (minimum) at parameter $\theta = (2n + 1)\pi \ (\theta = 2n\pi)$, for a given definit value of $d$. Where $n$ is integer. The maximum (minimum) of $\gamma_+$ is $\pi (1 + \frac{2\sqrt{d^2 - 1}}{d}) \ (\pi (1 - \frac{2\sqrt{d^2 - 1}}{d}))$. As $d$ increase, the maximum of $\gamma_+$ decreases, the minimum of $\gamma_+$ increases. The Berry phase $\gamma_+$ tend to a constant value $\pi$, with $d$ approaches infinity.

IV. SUMMARY

In this paper we presented BWM-algebra ($A$, $B$, $E_A$, $E_B$) and solution of YBE ($A(\theta, \varphi_1, \varphi_2)$, $B(\theta, \varphi_1, \varphi_2)$) in 3-dimensional representation which satisfy $B = UAU^{-1}$, $E_B = UE_AU^{-1}$ and $B(\theta, \varphi_1, \varphi_2) = U A(\theta, \varphi_1, \varphi_2)U^{-1}$. The evolution of the Yang-Baxter system is explored by constructing a Hamiltonian from the unitary $B(\theta, \varphi)$ matrix. We study the Berry phase of the Yang-Baxter system, and obtain the relationship between topological parameter and Berry phase $\gamma_\pm = \pm \pi (1 - \frac{2 \cos \theta \sqrt{d^2 - 1}}{d})$. Then we compare the Berry phase of Yang-Baxter system in Ref.[13] with us, and find the topological parameter $d$ plays a deformation role in the Berry phase. We have been discussing in this paper is still an open problem that will require a deal of further investigations.
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