No-go theorem for static boson stars

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Abstract

It is proved that self-gravitating static scalar fields whose self-interaction potential $V(\psi^2)$ is a monotonically increasing function of its argument cannot form spherically symmetric asymptotically flat bound matter configurations. Our compact theorem rules out, in particular, the existence of spatially regular static boson stars made of nonlinear massive scalar fields.
I. INTRODUCTION

The physical and mathematical properties of the composed Einstein-scalar system have attracted much attention during the last five decades. Intriguingly, while the elegant no-hair theorems of Chase [1], Bekenstein [2], and Teitelboim [3] (see also [4–8] and references therein) have revealed the important fact that static scalar fields cannot form hairy black-hole configurations with regular event horizons [9], it has recently been proved that stationary spatially regular massive scalar hair can be supported in asymptotically flat spinning black-hole spacetimes [10, 11].

Boson stars, horizonless self-gravitating massive scalar configurations, are also known to exist as stationary solutions of the non-linearly coupled Einstein-scalar field equations (see [12, 13] for excellent reviews). These compact objects have been considered in the physics literature as exotic horizonless black-hole mimickers, as possible sources of dark matter configurations, and as self-gravitating compact objects in binary systems [12, 13].

One naturally wonders whether horizonless compact boson stars can be constructed from static (rather than stationary [12, 13]) self-interacting scalar fields? Derrick’s well-known theorem [14] guarantees that, in (3 + 1)-dimensional flat spacetimes, spatially regular static scalar field configurations with positive definite energy density cannot exist. The main goal of the present paper is to extend this no-go theorem for static boson stars to the regime of self-gravitating scalar field configurations in curved spacetimes.

To this end, in the present paper we shall explore the physical and mathematical properties of the static sector of the nonlinearly coupled Einstein-scalar field equations. In particular, below we shall explicitly prove that, within the framework of classical general relativity, self-gravitating static scalar fields with generic self-interaction potentials cannot form spatially regular horizonless matter configurations (boson stars) [15, 16].

II. THE NO-GO THEOREM FOR SPHERICALLY SYMMETRIC ASYMPOTICALLY FLAT STATIC BOSON STARS

We study the physical and mathematical properties of spherically symmetric static matter configurations made of a self-gravitating real scalar field $\psi$ whose action is given by [6, 17, 18]

$$S = S_{EH} - \frac{1}{2} \int \left[ \partial_\alpha \psi \partial^\alpha \psi + V(\psi^2) \right] \sqrt{-g} d^3x,$$  

(1)
where the nonlinear scalar self-interaction potential \( V(\psi^2) \) is assumed to be a monotonically increasing and a positive semidefinite function of its argument:

\[
V(0) = 0 \quad \text{with} \quad \dot{V} \equiv \frac{d[V(\psi^2)]}{d(\psi^2)} \geq 0 \, .
\] (2)

Note, in particular, that the physically interesting case of an asymptotically flat self-gravitating massive scalar field configuration with \( \dot{V} = \mu^2 \geq 0 \) is covered by this functional form of the nonlinear self-interaction potential.

The spherically symmetric spacetime describing the self-gravitating static scalar field configuration is characterized by the curved line element \[6, 19\]

\[
ds^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) ,
\] (3)

where the metric functions \( \{\nu, \lambda\} \) depend on the areal coordinate \( r \). Spatially regular spacetimes are characterized by the near-origin behavior \[20\]

\[
e^\lambda(r \to 0) = 1 + O(r^2) \quad ; \quad 0 < e^\nu < \infty
\] (4)

with \[21, 22\]

\[
\lambda'(r \to 0) \to 0 \quad ; \quad \nu'(r \to 0) \to 0 .
\] (5)

In addition, asymptotically flat spacetimes describing spatially regular matter configurations of finite mass are characterized by the simple dimensionless functional relations \[6\]

\[
\nu(r \to \infty) = O(M/r) \quad ; \quad \lambda(r \to \infty) = O(M/r)
\] (6)

at spatial infinity, where \( M \) is the total ADM mass (as measured by asymptotic observers) of the spatially regular field configuration.

The action \[\text{(1)}\] of the self-interacting scalar field yields the characteristic nonlinear Klein-Gordon wave equation \[6\]

\[
\partial_\alpha \partial^\alpha \psi - \dot{V} \psi = 0 ,
\] (7)

which, taking cognizance of the curved line element \[3\] that characterizes the spherically symmetric static matter configuration, can be written in the form \[6\]

\[
\psi'' + \frac{1}{2}(\frac{4}{r} + \nu' - \lambda')\psi' - e^\lambda \dot{V} \psi = 0 .
\] (8)

The action \[\text{(1)}\] also yields the functional expression \[6\]

\[
\rho = -T^t_t = \frac{1}{2}[e^{-\lambda}(\psi')^2 + V(\psi^2)]
\] (9)
for the energy density of the self-interacting scalar field. Taking cognizance of Eqs. (2) and (9), and using the fact that physically acceptable spacetimes are characterized by finite energy densities [6], one deduces that the gradient of the scalar eigenfunction is finite [23]:

$$\psi'(r) < \infty.$$  \hspace{1cm} (10)

In addition, taking cognizance of Eqs. (2), (6), and (9), and using the fact that finite mass matter configurations in asymptotically flat spacetimes are characterized by the simple asymptotic behavior [24]

$$r^3 \rho(r) \to 0 \quad \text{for} \quad r \to \infty,$$

one deduces that the self-gravitating bound-state scalar configurations are characterized by the asymptotic functional behavior

$$\psi(r \to \infty) \to 0.$$  \hspace{1cm} (12)

We shall now prove that the eigenfunction $\psi(r)$, which characterizes the spatial behavior of the spherically symmetric self-gravitating static scalar fields, cannot be a monotonic function of the radial coordinate $r$. Taking cognizance of Eqs. (4), (5), and (8), one finds the near-origin radial scalar equation

$$\psi'' + \frac{2}{r} \psi' - \dot{V} \psi = 0.$$  \hspace{1cm} (13)

The physically acceptable solution of (13) which respects the relation (10) is characterized by the small-$r$ (near-origin) functional behavior

$$\psi(r \to 0) = a \left[1 + \frac{1}{6} \dot{V}(a^2) \cdot r^2\right] + O(r^3),$$

where $a$ is a constant. Using Eqs. (2) and (14), one obtains the near-origin functional relations [25, 26]

$$\psi \psi'(r \to 0) = 0; \quad \psi \psi''(r \to 0) > 0.$$  \hspace{1cm} (15)

From the small-$r$ and large-$r$ functional behaviors (12) and (15) of the radial scalar eigenfunction $\psi(r)$, which characterizes the spatially regular static matter configurations, one deduces that $\psi(r)$ must have (at least) one extremum point, $r = r_{\text{peak}}$, with the characteristic functional relations

$$\{\psi \neq 0; \quad \psi' = 0; \quad \psi \cdot \psi'' < 0\} \quad \text{for} \quad r = r_{\text{peak}}.$$  \hspace{1cm} (16)
In particular, at the extremum point (16), the radial scalar equation (8) yields the remarkably simple relation

$$\psi\psi'' = e^\lambda V\psi^2 \quad \text{for} \quad r = r_{\text{peak}}.$$  \hspace{1cm} (17)

Taking cognizance of Eqs. (2) and (16), one learns that the functional expression on the l.h.s of (17) is negative whereas the functional expression on the r.h.s of (17) is non-negative. Thus, the characteristic differential relation (8) for the self-gravitating static matter configurations cannot be respected at the extremum point (16) of the scalar eigenfunction.

We therefore conclude that spherically symmetric asymptotically flat bound-state matter configurations (boson stars) cannot be constructed from spatially regular static scalar fields whose self-interaction potential $V(\psi^2)$ is a monotonically increasing function of its argument.

III. SUMMARY

In this compact analysis of the static sector of the nonlinearly coupled Einstein-scalar field equations, we have explicitly proved that if a spherically symmetric spatially regular boson star can be constructed from a static (rather than a stationary) scalar field, then the corresponding radial scalar eigenfunction $\psi(r)$ cannot be a monotonic function of the areal coordinate $r$. In particular, at its extremum point (16), the scalar field eigenfunction is characterized by the simple functional relation (17). However, one immediately realizes that this relation is in contradiction with the functional identity (8) [see Eqs. (2) and (16)] which characterizes the radial behavior of the nonlinear self-gravitating static scalar field in the spatially regular curved spacetime (3).

Our compact theorem therefore reveals the interesting fact that, as opposed to stationary scalar fields [12, 13, 27, 31], static self-gravitating scalar fields whose nonlinear self-interaction potential $V(\psi^2)$ is a monotonically increasing function of its argument cannot form spatially regular asymptotically flat boson stars.
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[21] Here a prime $'$ denotes a spatial derivative with respect to the radial coordinate $r$.

[22] The near-origin relations [5] can be deduced from the Einstein field equations, $G_{\mu}^{\nu} = 8\pi T_{\mu}^{\nu}$, which for spherically symmetric static spacetimes are given by the functional relations [6, 20]

$$\lambda' = 8\pi r e^\lambda \rho - (e^\lambda - 1)/r$$

and

$$\nu' = (e^\lambda - 1)/r + 8\pi r e^\lambda p,$$

where $(\rho, p) \equiv (-T_t^t, T_r^r)$ are the energy
density and the radial pressure of the matter fields, respectively. As explicitly proved in [6], physically acceptable spacetimes are characterized by finite values of the energy-momentum components. Thus, taking cognizance of the above stated Einstein equations and the relation (4), one obtains the characteristic near-origin behavior (5) of the metric functions.

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[26] If $\dot{V} = 0$ then Eq. (13) yields the near-origin functional behavior $\psi(r \to 0) = a/r$, which violates the characteristic relation (10) of physically acceptable systems.

[27] It is worth emphasizing the fact that four-dimensional field solitons which violate the assumptions made in the present theorem do exist as spatially regular mathematical solutions of the non-linearly coupled Einstein-scalar field equations. In particular, the composed Einstein-scalar system is characterized by the presence of stationary (rather than static) spatially regular solutions [12, 13, 28], as well as by the presence of non-asymptotically flat field configurations [29, 30]. In addition, solitonic field configurations exist for the case of scalar self-interaction potentials which are not positive semidefinite [31].

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