GEOMETRIC CLASSIFICATION OF
UNITAL GRAPH C*-ALGEBRAS OF REAL RANK ZERO

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Abstract. We generalize the classification result of Restorff (Res06) on
Cuntz-Krieger algebras to cover all unital graph C*-algebras with real rank
zero, showing that Morita equivalence in this case is determined by ordered,
filtered K-theory as conjectured by three of the authors. The classification
result is geometric in the sense that it establishes that any Morita equivalence
between C*(E) and C*(F) in this class can be realized by a sequence of moves
leading from E to F in a way resembling the role of Reidemeister moves on
knots. As a key technical step, we prove that the so-called Cuntz splice leaves
unital graph C*-algebras invariant up to Morita equivalence.

The results of this preprint will be generalized in a forthcoming paper.

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1. Introduction

Ever since the inception of graph $C^*$-algebras, it has been a key ambition to classify these objects by their $K$-theory, either up to isomorphism or stable isomorphism. With the simple case resolved by appeal to the celebrated classification results of Elliott on one hand and Kirchberg and Phillips on the other, the focus has been on the nonsimple $C^*$-algebras, and in fact this endeavour has evolved in parallel with the gradual realization of what invariants may prove to be complete in the case when the number of ideals is finite and the $C^*$-algebras in question are not stably finite. In this sense, the fundamental results obtained on the classification of certain classes of graph $C^*$-algebras are playing a role parallel to the one played by Rørdam’s classification of simple Cuntz-Krieger algebras as a catalyst for the Kirchberg-Phillips classification mentioned above.

The first two results on the classification problem for nonsimple graph $C^*$-algebras were obtained by Rørdam in [Rør97] and by Restorff in [Res06] by very different methods. Rørdam showed the importance of involving the full data contained in six-term exact sequences of the $C^*$-algebras given and proved a very complete classification theorem while restricting the ideal lattice to be as small as possible: only one nontrivial ideal. In Restorff’s work, the ideal lattice was arbitrary among the finite ideal lattices, but as his method was to reduce the problem to classification of shifts of finite type and appeal to deep results by Boyle and Huang from symbolic dynamics ([Boy02], [BH03]), only graph $C^*$-algebras in the Cuntz-Krieger class were covered.

Subsequent progress has mainly followed the approach in [Rør97] (see [BK11], [ET10], [ERR13b]), and hence applies only to restricted kinds of ideal lattices but with few further restrictions on the nature of the underlying graphs. The case of purely infinite graph $C^*$-algebras with finitely many ideals has been resolved (very interestingly, by a different invariant than what was proposed in [ERR10]) in recent work by Bentmann and Meyer ([BM14]), but as summarized in [ERR13a] there is not at present sufficient technology to take this approach much farther in the mixed cases than to $C^*$-algebras with three or four primitive ideals.

In the paper at hand we complete the stable classification of unital graph $C^*$-algebras with real rank zero, following the strategy from [Res06] as generalized by the authors in various constellations over a period of 5 years ([Sør13], [ERR10], [ERS12], [ERS15]). Our method of proof, a substantial elaboration of key ideas from the authors’ earlier work along with key ideas from the papers of Boyle and Huang, leads to a geometric classification, allowing us to conclude from Morita equivalence between a pair of graph $C^*$-algebras $C^*(E)$ and $C^*(F)$ that a sequence of basic moves on the graphs may lead from $E$ to $F$ in a way resembling the role of Reidemeister moves on knots.

These moves are closely related to those defining flow equivalence for shift spaces, apart from the so-called Cuntz splice which has no counterpart in dynamics and also fails to preserve the canonical diagonal Abelian subalgebra of the graph $C^*$-algebras (cf. [MM14], [BCW14]). In all cases when classification has been established, invariance of the Cuntz splice follows immediately from the fact that it will not change the $K$-theory, and in particular it was observed in [BM14] that Cuntz splice is invariant in the class of graph $C^*$-algebras which are purely infinite with finitely many ideals. But since our goal is to use the Cuntz splice to establish classification results in classes outside the scope of these results, we must prove here that in the case under investigation, the Cuntz splice leaves the $C^*$-algebras invariant. In fact, this result covers the full case of unital graph $C^*$-algebras without any reference to real rank zero.
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Although the real rank zero condition is often seen to bear importance in classification theory ([Ell93], [Erl96], [DG97], [ARR12]) its role in our proof is of a substantially different nature than in the papers listed. Indeed, since we gave in [ERS15] an example of two finite graphs yielding Morita equivalent graph C*-algebras of real rank one for which no sequence of moves suffices to lead from one to another, we require real rank zero, through its graph algebraic characterization Condition (K), to ensure that the classification result is indeed geometric in the sense of passing through moves.

After posting the first version of this paper, we realized that it was possible to obtain classification by K-theory in the general unital case by exhibiting a new move which allows us to connect the two examples mentioned above, and we have recently completed the proof that this move leaves the C*-algebra invariant up to Morita equivalence. Consequently, we will present a complete classification result in a forthcoming paper which will contain the results in the present paper as a special case.

2. PRELIMINARIES FOR STATEMENT OF MAIN THEOREM

2.1. Graphs and their matrices. By a graph we mean a directed graph. Formally:

**Definition 2.1.** A graph $E$ is a four tuple $E = (E^0, E^1, r, s)$ where $E^0$ and $E^1$ are sets, and $r$ and $s$ are maps from $E^1$ to $E^0$. The elements of $E^0$ are called vertices, the elements of $E^1$ are called edges, the map $r$ is called the range map, and the map $s$ is called the source map.

All graphs considered will be countable, i.e., there are countably many vertices and edges. We call a graph finite, if there are only finitely many vertices and edges. As usual, two graphs $E_1 = (E^0_1, E^1_1, r_1, s_1)$ and $E_2 = (E^0_2, E^1_2, r_2, s_2)$ are called isomorphic if there exist bijections $\phi^1$ from $E^1_1$ to $E^1_2$ such that $s_2 \circ \phi^1 = \phi^0 \circ s_1$ and $r_2 \circ \phi^1 = \phi^0 \circ r_1$. We will freely identify graphs up to isomorphism.

**Definition 2.2.** A loop is an edge with the same range and source. A path $\mu$ in a graph is a finite sequence $\mu = e_1e_2\cdots e_n$ of edges satisfying $r(e_i) = s(e_{i+1})$, for all $i = 1, 2, \ldots, n - 1$, and we say that the length of $\mu$ is $n$. We extend the range and source maps to paths by letting $s(\mu) = s(e_1)$ and $r(\mu) = r(e_n)$. Vertices in $E$ are regarded as paths of length 0 (also called empty paths).

A cycle is a nonempty path $\mu$ such that $s(\mu) = r(\mu)$. A return path is a cycle $\mu = e_1e_2\cdots e_n$ such that $r(e_i) \neq r(\mu)$ for $i < n$.

For a loop, cycle or return path, we say that it is based at the source vertex of its path. We also say that a vertex supports a certain loop, cycle or return path if it is based at that vertex.

**Definition 2.3.** A vertex $v \in E^0$ in $E$ is called regular if $s^{-1}(v)$ is finite and nonempty.

A vertex $v \in E^0$ in $E$ is called source if $r^{-1}(v) = \emptyset$. A vertex $v \in E^0$ in $E$ is called a sink if $s^{-1}(v) = \emptyset$. Note that an isolated vertex is both a sink and a source.

**Notation 2.4.** If there exists a path from vertex $u$ to vertex $v$, then we write $u \geq v$ — this is a preorder on the vertex set, i.e., it is reflexive and transitive, but need not be antisymmetric.

It is key to our approach to graph C*-algebras to be able to shift between a graph and its adjacency matrix. In what follows, we let $\mathbb{N}$ denote the set of positive integers, while $\mathbb{N}_0$ denotes the set of nonnegative integers.
Definition 2.5. Let $E = (E^0, E^1, r, s)$ be a graph. We define its adjacency matrix $A_E$ as a $E^0 \times E^0$ matrix with the $(u, v)$'th entry being

$$|\{e \in E^1 \mid s(e) = u, r(e) = v\}|.$$

As we only consider countable graphs, $A_E$ will be a finite matrix or a countably infinite matrix, and it will have entries from $\mathbb{N} \sqcup \{\infty\}$.

Let $X$ be a set. If $A$ is an $X \times X$ matrix with entries from $\mathbb{N} \sqcup \{\infty\}$ we let $E_A$ be the graph with vertex set $X$ and between two vertices $x, x' \in X$ we have $A(x, x')$ edges.

It will be convenient for us to alter the adjacency matrix of a graph in two very specific ways, removing singular rows and subtracting the identity, so we introduce notation for this.

Notation 2.6. Let $E$ be a graph and $A_E$ its adjacency matrix. Denote by $A_E^*$ the matrix obtained from $A_E$ by removing all rows corresponding to singular vertices of $E$.

Let $B_E$ denote the matrix $A_E - I$, and let $B_E^*$ be $B_E$ with the rows corresponding to singular vertices of $E$ removed.

2.2. Graph $C^*$-algebras. We follow the notation and definition for graph $C^*$-algebras in [FLR00]; this is not the convention used in Raeburn’s monograph [Rae05].

Definition 2.7. Let $E = (E^0, E^1, r, s)$ be a graph. The graph $C^*$-algebra $C^*(E)$ is defined as the universal $C^*$-algebra generated by a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set $\{s_e \mid e \in E^1\}$ of partial isometries satisfying the relations

- $s_es_f = 0$ if $e, f \in E^1$ and $e \neq f$,
- $s_es_e = p_{r(e)}$ for all $e \in E^1$,
- $s_es_e^{*} \leq p_{s(e)}$ for all $e \in E^1$, and,
- $p_v = \sum_{e \in s^{-1}(v)} s_es_e^{*}$ for all $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.

It is clear from the definition that an isomorphism between graphs induces a canonical isomorphism between the corresponding graph $C^*$-algebras.

Definition 2.8. Let $E$ be a graph. We say that $E$ satisfies Condition (K) if for all vertices $v \in E^0$ in $E$, either there is no return path based at $v$ or there are at least two distinct return paths based at $v$.

Remark 2.9. The graph $C^*$-algebra $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra if and only if the graph $E$ is finite with no sinks, see [AR12] Theorem 3.13. If all vertices in $E$ support two loops, then $C^*(E)$ is purely infinite, see [HS03] Theorem 2.3. In our main result, Theorem 3.1, the graphs are assumed to have finitely many vertices and to satisfy Condition (K) — for all such graphs the associated graph $C^*$-algebras are separable, unital, of real rank zero [HS03] Theorem 2.5] and have finitely many ideals.

2.3. Filtered $K$-theory.

Definition 2.10. Let $\mathfrak{A}$ be a $C^*$-algebra with finitely many ideals, and let $\text{Prim} \mathfrak{A}$ denote the primitive ideal space of $\mathfrak{A}$ equipped with the hull-kernel topology. A subset of $\text{Prim} \mathfrak{A}$ is called locally closed, if it is the set difference between two open subsets of $\text{Prim} \mathfrak{A}$. There is a canonical lattice isomorphism between the open subsets of $\text{Prim} \mathfrak{A}$ and the (closed, two sided) ideals of $\mathfrak{A}$ — let us denote this correspondence with $O \mapsto \mathfrak{A}(O)$. If $V \subseteq \text{Prim} \mathfrak{A}$ is a difference set, then $V = U \setminus O$ for some open subsets $O \subseteq U \subseteq \text{Prim} \mathfrak{A}$. If also $V = U' \setminus O'$ for some other open
subsets $O' \subseteq U' \subseteq \text{Prim} \mathfrak{A}$, then there exists a canonical isomorphism between $\mathfrak{A}(U)/\mathfrak{A}(O)$ and $\mathfrak{A}(U')/\mathfrak{A}(O')$. Thus we can let

$$\mathfrak{A}(V) = \mathfrak{A} \left( \bigcap_{V = U \cap O, O' \subseteq U \subseteq \text{Prim} \mathfrak{A}} U \right) / \mathfrak{A} \left( \bigcap_{V = U \cap O, O' \subseteq U \subseteq \text{Prim} \mathfrak{A}} O \right)$$

with a slight abuse of notation, since we identify $\mathfrak{A}(O)$ with $\mathfrak{A}(O)/\{0\}$ whenever $O$ is open. Note, that all singletons of $\text{Prim} \mathfrak{A}$ are locally closed.

For each $x \in \text{Prim} \mathfrak{A}$ we let $S_x$ denote the smallest open subset that contains $x$, and we let $R_x = S_x \setminus \{x\}$, which is an open subset. Whenever we have two open subsets $O \subseteq U \subseteq \text{Prim} \mathfrak{A}$, we get a cyclic six term exact sequence in $K$-theory:

$$\begin{array}{c}
K_0(\mathfrak{A}(O)) \longrightarrow K_0(\mathfrak{A}(U)) \longrightarrow K_0(\mathfrak{A}(U \setminus O)) \\
K_1(\mathfrak{A}(U \setminus O)) \longrightarrow K_1(\mathfrak{A}(U)) \longrightarrow K_1(\mathfrak{A}(O)).
\end{array}$$

(2.1)

In fact, this holds even if $O$ and $U$ are locally closed. If $\mathfrak{A}$ is a real rank zero algebra, then the map from $K_0$ to $K_1$ will be the zero map.

Let

$$I_0(\mathfrak{A}) = \{R_x \mid x \in \text{Prim} \mathfrak{A}, R_x \neq \emptyset\} \cup \{S_x \mid x \in \text{Prim} \mathfrak{A}\} \cup \\{\{x\} \mid x \in \text{Prim} \mathfrak{A}\},$$

and let $\text{Imm}(x)$ denote the set

$$\{y \in \text{Prim} \mathfrak{A} \mid S_y \subseteq S_x \wedge \not\exists z \in \text{Prim} \mathfrak{A} : S_y \subseteq S_z \subseteq S_x\}.$$ 

The reduced filtered $K$-theory of $\mathfrak{A}$, $\text{FK}_R(\mathfrak{A})$, consists of the families of groups $(K_0(\mathfrak{A}(V)))_{V \in I_0(\mathfrak{A})}$ and $(K_1(\mathfrak{A}(O)))_{O \in I_1(\mathfrak{A})}$ together with the maps in the sequences

$$K_1(\mathfrak{A}([x])) \rightarrow K_0(\mathfrak{A}(R_x)) \rightarrow K_0(\mathfrak{A}(S_x)) \rightarrow K_0(\mathfrak{A}([x]))$$

originating from the sequence (2.1), for all $x \in \text{Prim} \mathfrak{A}$ with $R_x \neq \emptyset$, and the maps in the sequences

$$K_0(\mathfrak{A}(S_y)) \rightarrow K_0(\mathfrak{A}(R_x))$$

originating from the sequence (2.1), for all pairs $(x, y) \in \text{Prim} \mathfrak{A}$ with $y \in \text{Imm}(x)$ and $\text{Imm}(x) \setminus \{y\} \neq \emptyset$.

Let also $\mathfrak{B}$ be a $C^*$-algebra with finitely many ideals. An isomorphism from $\text{FK}_R(\mathfrak{A})$ to $\text{FK}_R(\mathfrak{B})$ consists of a homeomorphism $\rho : \text{Prim} \mathfrak{A} \rightarrow \text{Prim} \mathfrak{B}$ and families of isomorphisms

$$(\phi_V : K_0(\mathfrak{A}(V)) \rightarrow K_0(\mathfrak{B}(\rho(V))))_{V \in I_0(\mathfrak{A})}$$

$$(\psi_O : K_1(\mathfrak{A}(O)) \rightarrow K_1(\mathfrak{B}(\rho(O))))_{O \in I_1(\mathfrak{A})}$$

such that all the ladders coming from the above sequences commute.

Analogously, we define the ordered reduced filtered $K$-theory of $\mathfrak{A}$, $\text{FK}^\text{ord}_R(\mathfrak{A})$, just as $\text{FK}_R(\mathfrak{A})$ where we also consider the order on all the $K_0$-groups — and for an isomorphism, we demand that the isomorphisms between the $K_0$-groups are order isomorphisms.

2.4. Moves on graphs. In this section we describe the moves on graphs used in [Ser13]. We mention that these moves have been considered by other authors, and were previously noted to preserve the Morita equivalence class of the associated graph $C^*$-algebra (see [BP04]).
Definition 2.11 (Move (S): Remove a regular source). Let \( E = (E_0^0, E_1^0, r, s) \) be a graph, and let \( w \in E_0^0 \) be a source that is also a regular vertex. Let \( E_S \) denote the graph \((E_0^S, E_1^S, r_S, s_S)\) defined by
\[
E_0^S := E_0^0 \setminus \{w\} \quad E_1^S := E_1^0 \setminus s^{-1}(w) \quad r_S := r \mid_{E_1^S} \quad s_S := s \mid_{E_1^S}.
\]
We call \( E_S \) the graph obtained by removing the source \( w \) from \( E \), and say \( E_S \) is formed by performing move (S) to \( E \).

Definition 2.12 (Move (R): Reduction at a regular vertex). Suppose that \( E = (E_0^0, E_1^0, r, s) \) is a graph, and let \( w \in E_0^0 \) be a regular vertex with the property that \( s(r^{-1}(w)) = \{x\} \), \( s^{-1}(w) = \{f\} \), and \( r(f) \neq w \). Let \( E_R \) denote the graph \((E_0^R, E_1^R, r_R, s_R)\) defined by
\[
E_0^R := E_0^0 \setminus \{w\} \quad E_1^R := (E_1^0 \setminus \{f\} \cup r^{-1}(w)) \cup \{e_f \mid e \in E_1^0 \text{ and } r(e) = w\}
\]
\[
r_R(e) := r(e) \text{ if } e \in E_1^0 \setminus \{f\} \cup r^{-1}(w) \quad \text{and} \quad r_R(e_f) := r(f)
\]
\[
s_R(e) := s(e) \text{ if } e \in E_1^0 \setminus \{f\} \cup r^{-1}(w) \quad \text{and} \quad s_R(e_f) := s(e) = x.
\]
We call \( E_R \) the graph obtained by reducing \( E \) at \( w \), and say \( E_R \) is a reduction of \( E \) or that \( E_R \) is formed by performing move (R) to \( E \).

Definition 2.13 (Move (O): Outsplit at a non-sink). Let \( E = (E_0^0, E_1^0, r, s) \) be a graph, and let \( w \in E_0^0 \) be vertex that is not a sink. Partition \( s^{-1}(w) \) as a disjoint union of a finite number of nonempty sets
\[
s^{-1}(w) = E_1^1 \sqcup E_2^1 \sqcup \cdots \sqcup E_n^1
\]
with the property that at most one of the \( E_i \) is infinite. Let \( E_O \) denote the graph \((E_0^O, E_1^O, r_O, s_O)\) defined by
\[
E_0^O := \{v^1 \mid v \in E_0^0 \text{ and } v \neq w\} \cup \{w^1, \ldots, w^n\}
\]
\[
E_1^O := \{e^1 \mid e \in E_1^0 \text{ and } r(e) \neq w\} \cup \{e^1, e^2, \ldots, e^n \mid e \in E_1^0 \text{ and } r(e) = w\}
\]
\[
r_{E_O}(e^i) := \begin{cases} r(e^i) \setminus \{w^j\} \quad &\text{if } e \in E_1^0 \text{ and } r(e) \neq w \\ w^j \quad &\text{if } e \in E_1^0 \text{ and } r(e) = w \end{cases}
\]
\[
s_{E_O}(e^i) := \begin{cases} s(e^i) \setminus \{w^j\} \quad &\text{if } e \in E_1^0 \text{ and } s(e) \neq w \\ w^j \quad &\text{if } e \in E_1^0 \text{ and } s(e) = w \end{cases}
\]
We call \( E_O \) the graph obtained by outsplitting \( E \) at \( w \), and say \( E_O \) is formed by performing move (O) to \( E \).

Definition 2.14 (Move (I): Inspliit at a regular non-source). Suppose that \( E = (E_0^0, E_1^0, r, s) \) is a graph, and let \( w \in E_0^0 \) be a regular vertex that is not a source. Partition \( r^{-1}(w) \) as a disjoint union of a finite number of nonempty sets
\[
r^{-1}(w) = E_1^i \sqcup E_2^i \sqcup \cdots \sqcup E_n^i
\]
Let \( E_I \) denote the graph \((E_0^I, E_1^I, r_I, s_I)\) defined by
\[
E_0^I := \{v^1 \mid v \in E_0^0 \text{ and } v \neq w\} \cup \{w^1, \ldots, w^n\}
\]
\[
E_1^I := \{e^1 \mid e \in E_1^0 \text{ and } s(e) \neq w\} \cup \{e^1, e^2, \ldots, e^n \mid e \in E_1^0 \text{ and } s(e) = w\}
\]
\[
r_{E_I}(e^i) := \begin{cases} r(e^i) \setminus \{w^j\} \quad &\text{if } e \in E_1^0 \text{ and } r(e) \neq w \\ r(e^i) \setminus \{w^j\} \quad &\text{if } e \in E_1^0 \text{ and } r(e) = w \end{cases}
\]
\[
s_{E_I}(e^i) := \begin{cases} s(e^i) \setminus \{w^j\} \quad &\text{if } e \in E_1^0 \text{ and } s(e) \neq w \\ w^j \quad &\text{if } e \in E_1^0 \text{ and } s(e) = w.\end{cases}
\]
We call $E_I$ the graph obtained by insplitting $E$ at $w$, and say $E_I$ is formed by performing move (I) to $E$.

**Definition 2.15** (Move (C): Cuntz splicing at a regular vertex supporting two return paths). Let $E = (E^0, E^1, r, s)$ be a graph and let $v \in E^0$ be a regular vertex that supports at least two return paths. Let $E_C$ denote the graph $(E^0_C, E^1_C, r_C, s_C)$ defined by
\[
E^0_C := E^0 \cup \{u_1, u_2\},
\]
\[
E^1_C := E^1 \cup \{e_1, e_2, f_1, f_2, h_1, h_2\},
\]
where $r_C$ and $s_C$ extend $r$ and $s$, respectively, and satisfy
\[
s_C(e_1) = v, \quad s_C(e_2) = u_1, \quad s_C(f_1) = u_1, \quad s_C(h_1) = u_2,
\]
and
\[
r_C(e_1) = u_1, \quad r_C(e_2) = v, \quad r_C(f_1) = u_1, \quad r_C(h_1) = u_1.
\]
We call $E_C$ the graph obtained by Cuntz splicing $E$ at $v$, and say $E_C$ is formed by performing move (C) to $E$.

We also use the notation $E_{v,-}$ for this graph — even in the case where $v$ is not regular or not supporting two return paths. We can also Cuntz splice the vertex $u_1$ in $E_{v,-}$, and the resulting graph we denote $E_{v,-}$ — see also Notation 5.3 and Example 5.4 for illustrations of the Cuntz splice.

**Definition 2.16.** The equivalence relation generated by the moves (Ω), (I), (R), (S) together with graph isomorphism is called move equivalence, and denoted $\sim_M$. The equivalence relation generated by the moves (Ω), (I), (R), (S), (C) together with graph isomorphism is called move prime equivalence, and denoted $\sim_M^*$. The following theorem follows from [Sør13] Propositions 3.1, 3.2 and 3.3 and Theorem 3.5.

**Theorem 2.17** ([Sør13]). Let $E_1$ and $E_2$ be graphs such that $E_1 \sim_M E_2$. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

We also extend the notation of move equivalence to adjacency matrices.

**Definition 2.18.** If $A, A'$ are square matrices with entries in $\mathbb{N}_0 \sqcup \{\infty\}$ we define them to be move equivalent, and write $A \sim_M A'$ if $E_A \sim_M E_{A'}$. We define move prime equivalence similarly.

3. Main result

**Theorem 3.1.** Let $E_1$ and $E_2$ be graphs with finitely many vertices satisfying Condition (K). Then the following are equivalent:

1. $E_1 \sim_M E_2$,
2. $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$, and,
3. $\text{FK}^+_\mathbb{K}(C^*(E_1)) \cong \text{FK}^+_\mathbb{K}(C^*(E_2))$.

3.1. Strategy of proof and structure of the paper. The proof of the main theorem above, Theorem 3.1, is structured as follows.

Section 5 is devoted to show that the move (C) gives stable isomorphism. Thus, (1) implies (2) follows from Theorem 2.17 and Proposition 5.8 — a variant for finite graphs is in [LARS13].

That (2) implies (3) is clear.

The rest of the paper is devoted to proving that (3) implies (1). Mainly we emulate the previous proofs that go from filtered K-theory data to stable isomorphism or flow equivalence, as in [BH03, Boy02, Res06]. A key component of those proofs is manipulation of the matrix $\mathbb{E}^*_E$, in particular that we can perform basic row and
column operations without changing stable isomorphism class or flow equivalence class, depending on context. We prove in Section 4.2 that these matrix manipulations are allowed. Once we understand matrix manipulation our proof that (3) implies (1) goes through 5 steps.

Step 1 First we find graphs $F_1$ and $F_2$ in a certain standard form such that $F_i \sim_{M'} E_i$. This standard form will ensure that the adjacency matrices $B \cdot F_i$ have the same size and block structure, and that they satisfy certain additional technical conditions. This will be done in Section 7.

Step 2 In Section 8 we generalize a result of Boyle and Huang ([BH03]), to show that the isomorphism $FK_R^+(C^*(F_1)) \cong FK_R^+(C^*(F_2))$ is induced by a GLP-equivalence from $B \cdot F_1$ to $B \cdot F_2$.

Step 3 In Section 9 we find graphs $G_1, G_2$ such that $G_i \sim_{M'} F_i$ and $B \cdot G_1$ and $B \cdot G_2$ are SLP-equivalent.

Step 4 Then, in Section 10, we generalize Boyle’s positive factorization result from [Boy02] to show that there exists a positive SLP-equivalence between $B \cdot G_1$ and $B \cdot G_2$.

Step 5 It now follows from the results of Section 4.2 that $G_1 \sim_{M'} G_2$ and hence that $E_1 \sim_{M'} E_2$.

In Section 6, we introduce some notation and concepts about block matrices needed in the proof. In Section 11, we combine the results of the previous sections to prove the main theorem.

4. Derived moves

4.1. Moves on graphs. Here we introduce the derived moves from [Sør13] Section 5]. These are shown not to change the move equivalence class, but using them simplifies working with $\sim_M$.

Definition 4.1 (Collapse a regular vertex that does not support a loop). Let $E = (E^0, E^1, r, s)$ be a graph and let $v$ be a regular vertex in $E$ that does not support a loop. Define a graph $E_{COL}$ by

\[
E_{COL}^0 = E^0 \setminus \{v\}, \\
E_{COL}^1 = E^1 \setminus (r^{-1}(v) \cup s^{-1}(v)) \cup \{[e,f] | e \in r^{-1}(v) \text{ and } f \in s^{-1}(v)\},
\]

the range and source maps extends those of $E$, and satisfy $r_{E_{COL}}([e,f]) = r(f)$ and $s_{E_{COL}}([e,f]) = s(e)$.

According to [Sør13] Theorem 5.2 $E \sim_M E_{COL}$ — in fact, the collapse move can be obtained using move (O) and move (R).

We also introduce move (T).

Definition 4.2. Let $E = (E^0, E^1, r, s)$ be a graph and let $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ be a path such that $A_E(s(\alpha_1), r(\alpha_1)) = \infty$. Define a graph $E_T$ by

\[
E_T^0 = E^0, \\
E_T^1 = E^1 \cup \{\alpha^m | m \in \mathbb{N}\}
\]

the range and source maps extends those of $E$, and satisfy $r_{E_T}(\alpha^m) = r(\alpha)$ and $s_{E_T}(\alpha^m) = s(\alpha)$.

By [Sør13] Theorem 5.4 $E \sim_M E_T$. 

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4.2. Moves on matrices. Let $E$ be a graph with finitely many vertices. In this section we perform row and column additions on $B_E$ without changing move equivalence class of the associated graphs. Our setup is slightly different from what was considered in [Sør13, Section 7], so we redo the proofs from there in our setting. There are no substantial changes in the proof technique.

Lemma 4.3. Let $E = (E^0, E^1, r, s)$ be a graph with finitely many vertices. Let $u, v \in E^0$ be distinct vertices. Suppose the $(u, v)'$th entry of $B_E$ is nonzero (i.e., there is an edge from $u$ to $v$), and that the sum of the entries in the $u'$th row of $B_E$ is strictly greater than 0 (i.e., $u$ emits at least two edges). If $B'$ is the matrix formed from $B_E$ by adding the $u'$th column into the $v'$th column, then

$$A_E \sim_M B' + I.$$  

Proof. Fix an edge $f$ from $u$ to $v$. Form a graph $G$ from $E$ by removing $f$ but adding for each edge $e \in r^{-1}(u)$ an edge $\tilde{e}$ with $s(\tilde{e}) = s(e)$ and $r(\tilde{e}) = v$. We claim that $B' = B_G$. At any entry other than the $(u, v)'$th entry the two matrices have the same values, since we in both cases add entries into the $v'$th column that are exactly equal to the number of edges in $E$. At the $(u, v)'$th entry of $B_G$ we have

$$|(s_{E'}^{-1}(u) \cap r_{E'}^{-1}(v)) - 1| + |s_{E'}^{-1}(u) \cap r_{E'}^{-1}(u)| = B_E(u, v) + B_E(u, u) = B'(u, v).$$

Thus to prove this lemma it suffices to show $E \sim_M G$.

Partition $s^{-1}(u)$ as $E_1 = \{ f \}$ and $E_2 = s^{-1}(u) \setminus \{ f \}$. By assumption $E_2$ is not empty, so we can use move (D). Doing so yields a graph just as $E$ but where $u$ is replaced by two vertices, $u_1$ and $u_2$. The vertex $u_1$ receives a copy of everything $u$ did and it emits only one edge. That edge has range $v$. The vertex $u_2$ also receives a copy of everything $u$ did, and it emits everything $u$ did, except $f$. Since $u_1$ is regular and not the base of a loop, we can collapse it. The resulting graph is $G$ (after we relabel $u_2$ as $u$), so $G \sim_M E$.  

We can also add columns along a path.

Proposition 4.4. Let $E = (E^0, E^1, r, s)$ be a graph with finitely many vertices. Suppose $u, v \in E^0$ are distinct vertices with a path from $u$ to $v$ going through distinct vertices $u = u_0, u_1, u_2, \ldots, u_n = v$ (labelled so there is an edge from $u_i$ to $u_{i+1}$ for $i = 0, 1, 2, \ldots, n - 1$). Suppose further that for each $i = 0, 1, 2, \ldots, n - 1$ the vertex $u_i$ emits at least two edges. If $B'$ is the matrix formed from $B_E$ by adding the $u'$th column into the $v'$th column, then

$$A_E \sim_M B' + I.$$  

Proof. By repeated applications of Lemma 4.3, we first add the $u_{n-1}'$th column into the $u_n'$th column, which we can since there is an edge from $u_{n-1}$ to $u_n$. Then we add the $u_{n-2}'$th column into the $u_n'$th column, which we can since there now is an edge from $u_{n-2}$ to $u_n$. Continuing this way, we end up with a matrix $C$ which is formed from $B_E$ by adding all the columns $u_i$, for $i = 0, 1, 2, \ldots, n - 1$, into the the $u_n'$th column. We have that $A_E \sim_M C + I$.

Consider the matrix $D$ that is formed from $B_E$ by adding all the columns $u_0$ and $u_i$, for $i = 2, 3, \ldots, n - 1$, into the the $u_n'$th column. Adding the $u_1'$th column in $D$ into the $u_n'$th column yields $C$. So by Lemma 4.3, which applies since in $E_{D+I}$ there is an edge from $u_1$ to $u_n$, we get that $D + I \sim_M C + I \sim_M A_E$. Similarly we see that $D + I$ is move equivalent to the matrix formed from $B_E$ by adding all the columns $u_0$ and $u_i$, for $i = 3, \ldots, n - 1$, into the the $u_n'$th column. Continuing to subtract columns in this fashion, we get that $A_E \sim_M B' + I$.  

Remark 4.5. Similar to how we used Lemma 4.3 in the above proof, we can use Proposition 4.4 “backwards” to subtract columns in $B_E$ as long as the addition that undoes the subtraction would be legal.
We now turn to row additions.

**Lemma 4.6.** Let $E = (E^0, E^1, r, s)$ be a graph with finitely many vertices. Let $u, v \in E^0$ be distinct vertices. Suppose the $(v, u)$’th entry of $B_E$ is nonzero (i.e., there is an edge from $v$ to $u$), that the sum of the entries in the $u$’th column of $B_E$ is strictly greater than 0 (i.e., $u$ receives at least two edges), and that $u$ is a regular vertex. If $B'$ is the matrix formed from $B_E$ by adding the $u$’th row into the $v$’th row, then

$$A_E \sim_M B' + I.$$  

**Proof.** Fix an edge $f$ from $v$ to $u$. Form a graph $G$ from $E$ by removing $f$ but adding for each edge $e \in s^{-1}(u)$ an edge $\bar{e}$ with $s(\bar{e}) = v$ and $r(\bar{e}) = r(e)$. We claim that $E \sim_M G$. Arguing as in the proof of Lemma 4.3 we see that this is equivalent to proving $A_E \sim_M B' + I$.

Partition $r^{-1}(u)$ as $E_1 = \{f\}$ and $E_2 = r^{-1}(u) \setminus \{f\}$. By our assumptions on $u$, $E_2$ is nonempty, and $u$ is regular, so we can use move (I). Doing so replaces $u$ with two new vertices, $u_1$ and $u_2$. The vertex $u_1$ only receives one edge, and that edge comes from $v$, the vertex $u_2$ receives the edges $u$ received except $f$. Since $u_1$ is regular and not the base of a loop of length one we can collapse it. The resulting graph is $G$ (after we relabel $u_2$ as $u$), so $G \sim_M E$.

Naturally we can also add rows along a path.

**Proposition 4.7.** Let $E = (E^0, E^1, r, s)$ be a graph with finitely many vertices. Suppose $u, v \in E^0$ are distinct vertices with a path from $v$ to $u$ going through distinct vertices $v = v_0, v_1, v_2, \ldots, v_n = u$ (labelled so there is an edge from $v_i$ to $v_{i+1}$ for $i = 0, 1, 2, \ldots, n - 1$). Suppose further that for each $i = 1, 2, \ldots, n$ the vertex $v_i$ is regular and receives at least two edges. If $B'$ is the matrix formed from $B_E$ by adding the $u$’th row into the $v$’th row, then

$$A_E \sim_M B' + I.$$  

**Proof.** The proof is completely analogous to the proof of Proposition 4.4. □

**Remark 4.8.** We can also use Proposition 4.7 “backwards” to subtract rows in $B_E$ (cf. Remark 4.5).

## 5. Cuntz splice implies stable isomorphism

In this section we prove that (1) implies (2) in Theorem 5.1. We know that the moves (0), (I), (R), (S) imply stable isomorphism, cf. Theorem 2.17. What is missing is to prove that if $E_1$ and $E_2$ are graphs with finitely many vertices satisfying Condition (K) and $E_1$ is the Cuntz splice of $E_2$ on a vertex that supports at least two distinct return paths then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$, which is what we prove in Proposition 5.8. This is also an important result needed in Section 9.

First we reduce to the case where we perform a Cuntz splice on a regular vertex that supports at least two loops.

**Proposition 5.1.** Let $E$ be a graph with finitely many vertices, and let $u \in E^0$ be a vertex that supports at least two distinct return paths. Then there exists a graph $F$ and a regular vertex $v \in F^0$ such that

1. $E \sim_M F$,
2. $E_{u,-} \sim_M F_{v,-}$ and $E_{u,-} \sim_M F_{v,-}$,
3. $v$ supports at least two loops, and,
4. for all $w \in F^0$ with $v \geq w \geq v$ we have that $w$ supports at least one loop, there is a path from $v$ to $w$ through regular vertices, and there is a path from $w$ to $v$ through regular vertices (we say that a path $e_1 e_2 \cdots e_n$ goes through regular vertices if $s(e_i)$ is regular for all $i = 2, 3, \ldots, n$).
Proof. Let \( w \in E^0 \setminus \{ u \} \) be a regular vertex such that \( u \geq w \geq u \). If \( w \) does not support a loop we can use the collapse move (Definition 4.1) to remove it. The resulting graph will be move equivalent to \( E \) and have fewer regular vertices \( u \geq z \geq u \) that do not support a loop. So by repeatedly collapsing regular vertices that do not support loops we arrive at a graph \( E_1 \) such that \( E \sim_M E_1 \), and since the Cuntz splice has no bearing on the collapse move we also see that \( E_u, - \sim_M (E_1)_{u,-} \) and \( E_u, - \sim_M (E_1)_{u,-} \).

For each infinite emitter in \( w \in E^0 \setminus \{ u \} \) with \( u \geq w \geq u \), we can apply move (T) to assert that there is at least one loop based at \( u \). Call the resulting graph \( E_2 \). Again we have that \( E \sim_M E_2 \) and again the Cuntz splice is irrelevant for our move so \( E_u, - \sim_M (E_2)_{u,-} \) and \( E_u, - \sim_M (E_2)_{u,-} \). Thus we have now found a graph where every vertex \( w \neq u \) with \( w \geq u \) supports at least one loop.

We will now modify \( E_2 \) to get the desired paths to and from \( u \) through regular vertices. Let \( w \in E^0 \setminus \{ u \} \) be a vertex with \( u \geq w \geq u \). Suppose every path from \( u \) to \( w \) goes through an infinite emitter and pick a path \( e_1e_2\cdots e_n \) from \( u \) to \( w \) of minimal length (in particular it does not contain any loops nor does it visit \( u \) again). Let \( l \) be the first index such that \( s(e_l) \) is an infinite emitter, and note that \( e_l \) is not a loop. Partition \( s^{-1}(s(e_l)) \) into two sets, one of them \( \{ e_l \} \), and then outsplit according to this partition. After the outsplit we can collapse the vertex that emits \( e_l \), since \( e_l \) is the only edge it emits. Notice that in the post-collapse graph, the singular vertices are the same, and all the paths that were in the graph are still present, and each vertex \( z \neq u \) with \( u \geq z \geq u \) still supports at least one loop. We now have an edge from \( s(e_{l-1}) \) to \( r(e_l) \), so we can change our path to avoid \( s(e_l) \). Continuing in this fashion we eventually modify \( E_2 \) in such a way that there is a path from \( u \) to \( w \) using only regular vertices. Now we continue to do this for every such vertex \( w \).

Exactly the same strategy lets us assure that there is a path from \( u \) to \( w \) through regular vertices when \( u \geq w \geq u \). Call the graph that emerges after all these moves \( E_3 \).

Since we only did out splits and collapses on vertices in \( E^0 \setminus \{ u \} \), we see that these moves are unaffected by the Cuntz splice. Thus we have \( E \sim_M E_3 \), \( E_u, - \sim_M (E_3)_{u,-} \) and \( E_u, - \sim_M (E_3)_{u,-} \).

Now we want to modify \( E_3 \) such that \( u \) has at least two loops. If not, then since \( u \) supports two distinct return paths there exists some vertex \( w \neq u \) such that \( u \geq w \) and \( |s^{-1}(u) \cap r^{-1}(w)| \geq 1 \). As every vertex \( z \neq u \) with \( u \geq z \geq u \) supports a loop, we can use Proposition 4.4 to add the \( w \)th column of \( B_{E_3} \) into the \( w \)th column twice. Call the resulting matrix \( B' \), and let \( E_4 = E_{B'+I} \). In \( E_4 \) \( u \) will support (at least) two loops and all the other properties are preserved, since \( u \) supports a loop. The column addition is also valid in \( (E_3)_{u,-} \) and \( (E_3)_{u,-} \), so we have \( E \sim_M E_4 \), \( E_u, - \sim_M (E_4)_{u,-} \) and \( E_u, - \sim_M (E_4)_{u,-} \).

We will do the proof in cases.

Case 1: If \( u \) is regular, then we can end Case 1 by letting \( F = E_4 \) and \( v = u \).

Case 2: \( u \) is an infinite emitter and there exists \( w_0 \in E^0 \) such that \( w_0 \geq u \) and \( |s^{-1}(u) \cap r^{-1}(w_0)| = \infty \).

Doing what we did above and using move (T) we can find a graph \( E_5 \) such that

(i) \( E \sim_M E_5 \),
(ii) \( E_u, - \sim_M (E_5)_{u,-} \) and \( E_u, - \sim_M (E_5)_{u,-} \),
(iii) \( u \) supports infinitely many loops,
(iv) if \( u \geq w \geq u \) then there are infinitely many edges from \( u \) to \( w \), and,
(v) for all \( w \in E^0 \) with \( u \geq w \geq u \) we have that \( w \) supports at least one loop, there is a path from \( u \) to \( w \) through regular vertices, and there is a path from \( w \) to \( u \) through regular vertices.
Pick two edges \( e_1, e_2 \in s^{-1}(u) \cap r^{-1}(u) \), and pick for each \( u \geq w \geq u, w \neq u \), one edge \( e_w \in s^{-1}(u) \cap r^{-1}(w) \). Partition \( s^{-1}(u) \) as into two sets, one which is

\[
E_1 = \{e_1, e_2\} \cup \{e_w \mid u \geq w \geq u, w \neq u\}.
\]

Out-splitting according to this partition we get a graph \( F \) with \( E \sim_M F \). We will show that \( F_{u_1, -} \sim_M (E_3)_{u, -} \) and \( F_{u_1, -} \sim_M (E_3)_{u, -} \). Hence putting \( v = u_1 \) will complete the proof of this case.

In \( (E_3)_{u, -} \) we call the two vertices in the Cuntz splice \( v_1 \) and \( v_2 \), and let \( f \) be the edge from \( u \) to \( v_1 \). If we outsplit at \( u \) by partitioning \( s^{-1}(u) \) into two sets, one of which is

\[
F_1 = \{e_1, e_2, f\} \cup \{e_w \mid u \geq w \geq u, w \neq u\}
\]

we get a graph \( F_1 \sim_M (E_3)_{u, -} \), which is just like \( F_{u_1, -} \), except that in \( F_1 \), there is an edge from \( v_1 \) to \( u_2 \), while there is no such edge in \( F_{u_1, -} \). But Proposition 4.4.3 lets us add the \( v_2 \)'th column in \( B_{(E_3)_{u, -}} \) to the \( u_2 \)'th, to show that \( F_1 \sim_M F_{u_1, -} \).

A completely analogue argument shows that \( F_{u_1, -} \sim_M (E_3)_{u, -} \). Letting \( v = u_1 \) finishes Case 2.

Case 3: \( u \) is an infinite emitter and for all \( w \in E_4^0 \) with \(|s^{-1}(u) \cap r^{-1}(w)| = \infty\) we have \( w \not\geq u \).

We will perform an outsplit at \( u \), by partitioning \( s^{-1}(u) \) into two sets, one of which is

\[
E_1 = \{e \in s^{-1}(u) \mid r(e) \geq u\}.
\]

Similarly to Case 2, we see that the only difference between outsplitting according to this partition before or after we perform the Cuntz splice is as edge from \( v_1 \) to \( u_2 \) (notation as above). Hence, we see as above that if we let \( F \) be the outsplit graph coming from \( E_4 \), then \( E_4 \sim_M F, (E_4)_{u, -} \sim_M F_{u_1, -} \) and \( (E_4)_{u, -} \sim_M F_{u_1, -} \). Letting \( v = u_1 \) finishes Case 3. \( \Box \)

We now show that performing the Cuntz splice twice is a legal move.

**Proposition 5.2.** Let \( E \) be a graph with finitely many vertices, and let \( v \) be a vertex that supports at least two distinct return paths. Then \( E \sim_M E_{u, -} \).

**Proof.** According to Proposition 5.1 we can assume that \( E \) satisfies the conditions of that proposition — so we assume that \( v \) is a regular vertex that supports at least two loops. Moreover, for convenience, we let \( n \) be the number of vertices in \( E \) and we label the vertices by the numbers \( 1, 2, \ldots, n \) in such a way that \( v \) gets the label \( n \).

For a given matrix size \( N \) and \( i, j \in \{ 1, 2, \ldots, N \} \), we let \( E_{i, j} \) denote the \( N \times N \) matrix that is equal to the identity matrix everywhere except for the \( (i, j) \)'th entry, that is 1. If \( B \) is a \( N \times N \) matrix, then \( E_{i, j} B \) is the matrix obtained from \( B \) by adding \( j \)'th row into the \( i \)'th row, and \( B E_{i, j} \) is the matrix obtained from \( B \) by adding \( i \)'th column into the \( j \)'th column. Using \( E_{i, j}^{-1} \) instead will yield subtraction. In what follows we will make extensive use of Propositions 4.3 and 4.7 and Remarks 4.5 and 4.8 we feel it will only muddle the exposition if we add all the references in.
Now let \( B_2 = E_{(n+2,n+3)}B_1 \) and \( B_3 = B_2E_{(n+3,n+4)}^{-1} \). Then \( B_1 + I \sim_M B_2 + I \sim_M B_3 + I \). We have that

\[
B_3 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

The \( n+4 \)th vertex in \( E_{B_3 + I} \) does not support a loop, so it can be collapsed yielding

\[
B_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

With \( B_4 + I \sim_M B_3 + I \). Now we let \( B_5 = E_{(n+2,n+1)}^{-1}B_4 \), \( B_6 = E_{(n,n+3)}B_5 \), \( B_7 = E_{(n+1,n+1)}^{-1}E_{(n,n+1)}^{-1}B_6 \), \( B_8 = E_{(n+3,n+2)}B_7 \) and \( B_9 = B_8E_{(n+2,n+1)}^{-1} \). We then have \( B_4 + I \sim_M B_5 + I \sim_M B_6 + I \sim_M B_7 + I \sim_M B_8 + I \sim_M B_9 + I \). We have that

\[
B_9 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

In \( E_{B_9 + I} \) the \( n+1 \)th vertex does not support a loop, so it can be collapsed to yield

\[
B_{10} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]

with \( B_9 + I \sim_M B_{10} + I \).
Now we look at the graph $E$ again, and let $B_E = (b_{ij})$. Since the vertex $v$ (number $n$) has at least two loops, we have $b_{nn} \geq 1$. Now we can split by partitioning $r^{-1}(v)$ into two sets, one with a single set consisting of a loop based at $v$, and the other the rest. In the resulting graph, $v$ is split into two vertices $v^1$ and $v^2$, and let $E'$ denote the rest of the graph. The vertex $v^1$ has the same edges in and out of $E'$ as $v$ had, but it has only $b_{nn}$ loops. There is one edge from $v^1$ to $v^2$ and $v^2$ has one loop and there are $b_{nn}$ edges from $v^2$ to $v^1$ as well as all the same edges going from $v^2$ into $E'$ as originally from $v$. Use the inverse collapse move to add a new vertex $u$ at the middle of the edge from $v^1$ to $v^2$ and call the resulting graph $F$. Label the vertices such that $v^1$, $u$ and $v^2$ are the $n'$th, $n + 1$st and $n + 2$nd vertex, then $B_F$ is:

$$
B_F = \begin{pmatrix}
\tilde{B} & (0 \ 0)
\vdots & \vdots
0 \ 0
\end{pmatrix},
$$

where $\tilde{B}$ is $B_E$ except for on the $(n,n)$'th entry, which is $b_{nn} - 1$. Note that $b_{nn} - 1 \geq 0$, so that there is still a loop based at the $n$'th vertex. This is important since it allows us to do the following matrix manipulations. Let $C_2 = B_E E_{(n+2,n+1)} E_{(n+2,n+1)}$, $C_3 = E_{(n+2,n+1)} C_2$, $C_4 = E_{(n+2,n+1)} C_3$, $C_5 = C_4 E_{(n+1,n)}$ and $C_6 = C_5 E_{(n+2,n+1)}$. We have that $C_1 + I \sim_M C_2 + I \sim_M C_3 + I \sim_M C_4 + I \sim_M C_5 + I \sim_M C_6 + I$. The matrix $C_6$ is in fact equivalent to $B_{10}$ upon relabelling of the last two vertices, thus it follows, that $E \sim_M E_{v, \ldots}$. \hfill $\Box$

We now show that Cuntz splicing once and twice yields isomorphic graph $C^*$-algebras. To do this, we first set up some notation.

**Notation 5.3.** Let $E_*$ and $E_{**}$ denote the graphs:

$$E_* = \begin{array}{ccc}
& e_1 & \\
\bullet & e_2 & \bullet \\
e_3 & \bullet & \\
& e_4 \\
\end{array}$$

$$E_{**} = \begin{array}{ccc}
& f_1 & f_2 & f_3 \\
\circ & f_4 & f_5 & f_6 \\
& f_7 & f_8 & f_9 \\
& f_{10} \\
\end{array}$$

The graph $E_*$ is what we attach when we Cuntz splice, if we instead attach the graph $E_{**}$, it is like we Cuntz spliced twice. Let $E = (E^0, E^1, r_E, s_E)$ be a graph and let $u$ be a vertex of $E$. Then $E_{u,-}$ can be described as follows (up to canonical isomorphism):

$$E_{u,-}^0 = E^0 \sqcup E^0_u,$$

$$E_{u,-}^1 = E^1 \sqcup E^1_u \sqcup \{d_1, d_2\}$$

with $r_{E_{u,-}}|E^1 = r_E$, $s_{E_{u,-}}|E^1 = s_E$, $r_{E_{u,-}}|E^1 = r_{E_*}$, $s_{E_{u,-}}|E^1 = s_{E_*}$, and

$$s_{E_{u,-}}(d_1) = u \quad \quad r_{E_{u,-}}(d_1) = v_1$$

$$s_{E_{u,-}}(d_2) = v_1 \quad \quad r_{E_{u,-}}(d_2) = u.$$
Moreover, $E_{u,-}$ can be described as follows (up to canonical isomorphism):
\[
E_{u,-}^0 = E^0 \sqcup E_{s*}^0, \\
E_{u,-}^1 = E^1 \sqcup E_{s*}^1 \sqcup \{d_1, d_2\}
\]
with $r_{E_{u,-}}(E^1) = r_E$, $s_{E_{u,-}}(E^1) = s_E$, $r_{E_{u,-}}(E_{s*}^1) = r_{E_{s*}}$, $s_{E_{u,-}}(E_{s*}^1) = s_{E_{s*}}$, and
\[
s_{E_{u,-}}(d_1) = u \\
r_{E_{u,-}}(d_1) = w_1 \\
s_{E_{u,-}}(d_2) = w_1 \\
r_{E_{u,-}}(d_2) = u.
\]

**Example 5.4.** Consider the graph
\[
E = \begin{array}{c}
\circ \\
\circ \\
\end{array}
\]
Then
\[
E_{u,-} = \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\]
and
\[
E_{u,-} = \begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array}
\]

By classification of simple purely infinite graph $C^*$-algebras, i.e., by Kirchberg-Phillips classification, the graph $C^*$-algebras $C^*(E_u)$ and $C^*(E_{s*})$ are isomorphic (this important case is actually due to Rørdam, cf. [Rør95]). To show that $C^*(E_{u,-})$ is isomorphic to $C^*(E_{u,-})$ we would like to know that $C^*(E_u)$ and $C^*(E_{s*})$ are still isomorphic if we do not enforce the summation relation at $v_1$ and $w_1$ respectively.

**Proposition 5.5.** The relative graph $C^*$-algebra (in the sense of Muhly-Tomforde [MT04]) $C^*(E_u, \{v_2\})$ and $C^*(E_{s*}, \{w_2, w_3, w_4\})$ are isomorphic.

**Proof.** Following [MT04] Definition 3.6 we define a graph
\[
(E_*)_{\{v_2\}} = \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]
Then by [MT04] Theorem 3.7 we have that $C^*(E_*, \{v_2\}) \cong C^*((E_*)_{\{v_2\}})$. Similarly we define a graph
\[
(E_{**})_{\{w_2, w_3, w_4\}} = \begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]
Using [MT04, Theorem 3.7] again, we have that \(C^*(E_{++, \{w_2, w_3, w_4\}})\) is isomorphic to \(C^*((E_{++, \{w_2, w_3, w_4\}})\).

Both the graphs \((E_+)_v\) and \((E_{++, \{w_2, w_3, w_4\}})\) satisfy Condition (K). Using the well-developed theory of ideal structure and K-theory for graph \(C^\ast\)-algebras, we see that both have exactly one nontrivial ideal, that this ideal is the compact operators, and that their six-term exact sequences are

\[
\begin{array}{ccccccc}
0 & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathbb{Z} & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Furthermore, in \(K_0(C^*((E_+)_v))\) we have

\[
[p_{v_1}] = -[p_{v_1}] = [p_{w_2}],
\]
and in \(K_0(C^*((E_{++, \{w_2, w_3, w_4\}})\))\) we have

\[
[p_{w_1}] = -[p_{v_1}] = [p_{w_2}],
[p_{w_2}] = 0 = [p_{w_4}].
\]

Therefore the class of the unit is \(-[p_{v_1}]\) and \(-[p_{w_1}]\), respectively. It now follows from [BD96, Theorem 2] and [ERR13, Corollary 4.20] that \(C^*(E_+, \{v_2\}) \cong C^*((E_{++}, \{w_2, w_3, w_4\}))\) and hence that \(C^*(E_+, \{v_2\}) \cong C^*(E_{++, \{w_2, w_3, w_4\}})\).

We also need a technical result about the projections in \(E \cong C^*(E_+, \{v_2\})\).

**Lemma 5.6.** Let \(E = C^*(E_+, \{v_2\})\) and choose an isomorphism between \(E\) and \(C^*(E_{++, \{w_2, w_3, w_4\}})\), which exists according to the previous proposition. Let \(p_{v_1}, p_{v_2}, s_{e_1}, s_{e_2}, s_{e_3}, s_{e_4}\) be the canonical generators of \(C^*(E_+, \{v_2\}) = E\) and let \(p_{w_1}, p_{w_2}, p_{w_3}, p_{w_4}, s_{f_1}, s_{f_2}, \ldots, s_{f_{10}}\) denote the image of the canonical generators of \(C^*(E_{++, \{w_2, w_3, w_4\}})\) in \(E\) under the chosen isomorphism. Then

\[
s_{e_1}s_{e_1}^* + s_{e_2}s_{e_2}^* \sim s_{f_1}s_{f_1}^* + s_{f_2}s_{f_2}^* + s_{f_3}s_{f_3}^* + s_{f_4}s_{f_4}^* + p_{v_1} - (s_{e_1}s_{e_1}^* + s_{e_2}s_{e_2}^*) \sim p_{w_1} - (s_{f_1}s_{f_1}^* + s_{f_2}s_{f_2}^* + s_{f_3}s_{f_3}^* + s_{f_4}s_{f_4}^*),
\]
in \(E\), where \(\sim\) denotes Murray-von Neumann equivalence.

**Proof.** By [AMP07, Corollary 7.2], row-finite graph \(C^\ast\)-algebras have stable weak cancellation, so by [MT04, Theorem 3.7], \(E\) has stable weak cancellation. Hence any two projections in \(E\) are Murray-von Neumann equivalent if they generate the same ideal and have the same K-theory class.

As in the proof of Proposition 5.5, we will use [MT04, Theorem 3.7] to realize our relative graph \(C^\ast\)-algebras as graph \(C^\ast\)-algebras of the graphs \((E_+)_v\) and \((E_{++, \{w_2, w_3, w_4\}})\). Denote the image of the vertex projections of \(C^*((E_+)_v)\) inside \(E\) under this isomorphism by \(q_{v_1}, q_{v_2}, q_{v_3}, q_{v_4}\) and denote the image of the vertex projections of \(C^*((E_{++, \{w_2, w_3, w_4\}}))\) inside \(E\) under the isomorphisms \((E_{++, \{w_2, w_3, w_4\}}) \cong C^*((E_{++, \{w_2, w_3, w_4\}}))\) by \(q_{w_1}, q_{w_2}, q_{w_3}, q_{w_4}\). Using the description of the isomorphism in [MT04, Theorem 3.7], we see that we need to show that \(q_{v_1} \sim q_{w_1}\) and \(q_{v_4} \sim q_{w_4}\).

Since \((E_+)^0_v\) satisfies Condition (K) and the smallest hereditary and saturated subset containing \(v_1\) is all of \((E_+)^0_v\), we have that \(q_{v_1}\) is a full projection ([BHR80, Theorem 4.4]). Similarly, \(q_{w_1}\) is full. In \(K_0(E)\) we have, using our calculations from the proof of Proposition 5.5, that

\[
[q_{v_1}] = [1] = [q_{w_1}].
\]

So by weak stable cancellation \(q_{v_1} \sim q_{w_1}\).
Both $q_{e_l}'$ and $q_{e_l}$ generate the only nontrivial ideal $\mathcal{J}$ of $\mathcal{E}$ ([BHRSc02 Theorem 4.4]). Since that ideal is isomorphic to the compact operators and both $[q_{e_l}]$ and $[q_{e_l}']$ are positive generators of $K_0(\mathcal{J}) \cong K_0(\mathcal{E}) \cong \mathbb{Z}$, they must both represent the same class in $K_0(\mathcal{E})$, and thus also in $K_0(\mathcal{E})$. Therefore $q_{e_l} \sim q_{e_l}'$. 

If $E$ is a graph and we have a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set $\{s_e \mid e \in E^1\}$ of partial isometries in a $C^*$-algebra satisfying the relations of Definition 2.7, then we call these elements a Cuntz-Krieger $E$-family. In a graph $E$, we call a cycle $e_1 e_2 \cdots e_n$ a vertex-simple cycle if $r(e_i) \neq r(e_j)$ for all $i \neq j$. A vertex-simple cycle $e_1 e_2 \cdots e_n$ is said to have an exit if there exists an edge $f$ such that $s(f) = s(e_k)$ for some $k = 1, 2, \ldots, n$ with $e_k \neq f$. Note that in [Szy02], the author uses the term loop where we use cycle.

**Theorem 5.7.** Let $E$ be a graph with finitely many vertices and let $u$ be a vertex of $E$. Then $C^*(E_{u,-}) \cong C^*(E_{u,-})$.

**Proof.** As above, we let $\mathcal{E}$ denote the $C^*$-algebra $C^*(\mathcal{E}_*, \{v_2\})$, and we choose an isomorphism between $\mathcal{E}$ and $C^*(\mathcal{E}_*, \{w_2, w_3, w_4\})$, which exists according to Proposition 5.5.

Since $C^*(E_{u,-})$ and $\mathcal{E}$ are unital, separable, nuclear $C^*$-algebras, it follows from Kirchberg’s embedding theorem that there exists a unital embedding

$$C^*(E_{u,-}) \oplus \mathcal{E} \hookrightarrow O_2.$$ 

We will suppress this embedding in our notation. In $O_2$, we denote the vertex projections and the partial isometries coming from $C^*(E_{u,-})$ by $p_v, v \in E^0_{u,-}$ and $s_e, e \in E^1_{u,-}$, respectively, and we denote the vertex projections and the partial isometries coming from $\mathcal{E} = C^*(\mathcal{E}_*, \{v_2\})$ by $p_1, p_2$ and $s_1, s_2, s_3, s_4$, respectively. Since we are dealing with an embedding, it follows from Szymański’s General Cuntz-Krieger Uniqueness Theorem ([Szy02 Theorem 1.2]) that for any vertex-simple cycle $\alpha_1 \alpha_2 \cdots \alpha_n$ in $E_{u,-}$ without any exit, we have that the spectrum of $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$ contains the entire unit circle.

We will define a new Cuntz-Krieger $E_{u,-}$-family. For each vertex $v \in E^0$ we let $q_v = p_v$, we let $q_{v_1} = p_1$ and $q_{v_2} = p_2$. Since any two nonzero projections in $O_2$ are Murray-von Neumann equivalent, we can choose partial isometries $x_1, x_2 \in O_2$ such that

$$x_1 x_1^* = s_{d_1} s_{d_1}, \quad x_1^* x_1 = p_1$$

$$x_2 x_2^* = p_1 - (s_1 s_1^* + s_2 s_2^*), \quad x_2^* x_2 = p_2.$$ 

We let $t_{d_1} = x_1$ and $t_{d_2} = x_2$. Finally we let $t_e = s_e$ for $e \in E^1$ and put $t_{e_i} = s_i$ for $i = 1, 2, 3, 4$.

By construction $\{q_v \mid v \in E^0_{u,-}\}$ is a set of orthogonal projections, and $\{t_e \mid e \in E^1_{u,-}\}$ a set of partial isometries. Furthermore, by choice of $\{t_e \mid e \neq d_1, d_2\}$ the relations are clearly satisfied at all vertices other than $v_1$ and $u$. The choice of $x_1, x_2$ ensures that the relations hold at $u$ and $v_1$ as well. Hence $\{q_v, t_e\}$ does indeed form a Cuntz-Krieger $E_{u,-}$ family. Denote this family by $S$.

Using the universal property of graph $C^*$-algebras, we get a $*$-homomorphism from $C^*(E_{u,-})$ onto $C^*(S) \subseteq O_2$. Let $\alpha_1 \alpha_2 \cdots \alpha_n$ be a vertex-simple cycle in $E_{u,-}$ without any exit. Since $u$ is where the Cuntz splice is glued on, no vertex-simple cycle without any exit uses edges connected to $u, v_1$ or $v_2$. Hence $t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_n} = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$, and so its spectrum contains the entire unit circle. It now follows from Szymański’s General Cuntz-Krieger Uniqueness Theorem ([Szy02 Theorem 1.2]) that $C^*(E_{u,-}) \cong C^*(S)$. 

**GEOMETRIC CLASSIFICATION OF UNITAL GRAPH $C^*$-ALGEBRAS**

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Let $\mathfrak{A}$ be the subalgebra of $\mathcal{O}_2$ generated by $\{p_v \mid v \in E^0\}$ and $\mathcal{E}$. Note that $\mathfrak{A}$ has a unit, and although it does not coincide with the unit of $\mathcal{O}_2$ it does coincide with the unit of $C^*(\mathcal{S})$. In fact $\mathfrak{A}$ is a unital subalgebra of $C^*(\mathcal{S})$. Let us denote by $\{r_{w_i}, y_{f_j} \mid i = 1, 2, 3, 4, j = 1, 2, \ldots, 10\}$ the image of the canonical generators of $C^*(\mathcal{E}_{\alpha,\{w_2, w_3, w_4\}})$ in $\mathcal{O}_2$ under the chosen isomorphism between $C^*(\mathcal{E}_{\alpha,\{w_2, w_3, w_4\}})$ and $\mathcal{E}$ composed with the embedding into $\mathcal{O}_2$. By Lemma 5.6 certain projections in $\mathcal{E}$ are Murray-von Neumann equivalent, hence we can find a unitary $z \in \mathfrak{A}$ such that

$$zq_0^*z = q_v,$$ for all $v \in E^0$,

$$z \left( t_{e_1}t_{e_1}^* + t_{e_2}t_{e_2}^* \right) z^* = y_{f_1}y_{f_1}^* + y_{f_2}y_{f_2}^* + y_{f_3}y_{f_3}^*,$$

$$z \left( q_{w_1} - (t_{e_1}t_{e_1}^* + t_{e_2}t_{e_2}^*) \right) z^* = r_{w_1} - (y_{f_1}y_{f_1}^* + y_{f_2}y_{f_2}^* + y_{f_3}y_{f_3}^*).$$

Note that this implies that $zq_0^*z^* = r_{w_1}$.

We will now define a Cuntz-Krieger $E_{\alpha,-}$-family in $\mathcal{O}_2$. For $v \in E^0$, we let $P_v = q_v$, and we let $P_{w_i} = r_{w_i}$, for $i = 1, 2, 3, 4$. For $e \in E^1 \cup \{d_1, d_2\}$, we let $S_e = zt_ez^*$, and we let $S_{e_1} = y_{f_i}$, for $i = 1, 2, \ldots, 10$. Denote this family by $\mathcal{T}$.

By construction $\{P_v \mid v \in E^0_{\alpha,-}\}$ is a set of orthogonal projections, and $\{S_e \mid e \in E^1_{\alpha,-}\}$ a set of partial isometries. Since $z$ is a unitary in $C^*(\mathcal{S})$ and since $\mathcal{S}$ is a Cuntz-Krieger $E_{\alpha,-}$-family, $\mathcal{T}$ will satisfy the Cuntz-Krieger relations at all vertices in $E^0$. Similarly, we see that since $\{r_{w_i}, y_{f_j} \mid i = 1, 2, 3, 4, j = 1, 2, \ldots, 10\}$ is a Cuntz-Krieger $\{\mathcal{E}_{\alpha,\{w_2, w_3, w_4\}}\}$-family, $\mathcal{T}$ will satisfy the relations at the vertices $w_2, w_3, w_4$. It only remains to check the summation relation at $w_1$, for that we compute

$$\sum_{x_{E_{\alpha,-}(e)=w_1}} S_eS_e^* = S_{f_1}S_{f_1}^* + S_{f_2}S_{f_2}^* + S_{f_3}S_{f_3}^* + S_{d_1}S_{d_2}^* = y_{f_1}y_{f_1}^* + y_{f_2}y_{f_2}^* + y_{f_3}y_{f_3}^* + ztd_2t_d^*z^*$$

$$= z \left( t_{e_1}t_{e_1}^* + t_{e_2}t_{e_2}^* \right) z^* + ztd_2t_d^*z^*$$

$$= z \left( t_{e_1}t_{e_1}^* + t_{e_2}t_{e_2}^* + t_{d_1}t_{d_2}^* \right) z^*$$

$$= q_{w_1}^*z^* = r_{w_1} = P_{w_1}.$$
6. Notation needed for the proof

6.1. Block matrices and equivalences.

Notation 6.1. For \( m, n \in \mathbb{N}_0 \), we let \( \mathcal{M}(m \times n, \mathbb{Z}) \) denote the set of group homomorphisms from \( \mathbb{Z}^n \) to \( \mathbb{Z}^m \). When \( m, n \geq 1 \), we can equivalently view this as the \( m \times n \) matrices over \( \mathbb{Z} \), where composition of group homomorphisms corresponds to matrix multiplication — the (zero) group homomorphisms for \( m = 0 \) or \( n = 0 \) we will also call empty matrices with zero rows or columns, respectively.

For \( m, n \in \mathbb{N} \), we let \( \mathcal{M}^+(m \times n, \mathbb{Z}) \) denote the subset of \( \mathcal{M}(m \times n, \mathbb{Z}) \), where all entries in the corresponding matrix are positive. For a \( m \times n \) matrix, we will also write \( B > 0 \) whenever \( B \in \mathcal{M}^+(m \times n, \mathbb{Z}) \).

For a \( m \times n \) matrix \( B \), where \( m, n \in \mathbb{N} \), we let \( B(i, j) \) denote the \( (i,j) \)'th entry of the corresponding matrix, i.e., the entry in the \( i \)'th row and \( j \)'th column.

Definition 6.2. Let \( m, n \in \mathbb{N} \). For a \( m \times n \) matrix \( B \) over \( \mathbb{Z} \), we let \( \text{gcd}(B) \) be the greatest common divisor of the entries \( B(i, j) \), for \( i = 1, \ldots, m \), \( j = 1, \ldots, n \), if \( B \) is nonzero, and zero otherwise.

Assumption 6.3. Let \( N \in \mathbb{N} \). For the rest of the paper, we let \( \mathcal{P} = \{1, 2, \ldots, N\} \) denote a partially ordered set with order \( \preceq \) satisfying

\[
i \preceq j \Rightarrow i \leq j,
\]

for all \( i, j \in \mathcal{P} \), where \( \preceq \) denotes the usual order on \( \mathbb{N} \). We denote the corresponding irreflexive order by \( \prec \).

Definition 6.4. Let \( \mathbf{m} = (m_i)_{i=1}^N, \mathbf{n} = (n_i)_{i=1}^N \in \mathbb{N}_0^N \) be multiindices. We write \( \mathbf{m} \preceq \mathbf{n} \) if \( m_i \leq n_i \) for all \( i = 1, 2, \ldots, N \), and in that case, we let \( \mathbf{n} - \mathbf{m} \) be \( (n_i - m_i)_{i=1}^N \).

We let \( \mathcal{M}(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \) denote the set of group homomorphisms from \( \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} \oplus \cdots \oplus \mathbb{Z}^{n_N} \) to \( \mathbb{Z}^{m_1} \oplus \mathbb{Z}^{m_2} \oplus \cdots \oplus \mathbb{Z}^{m_N} \), and for such a homomorphism \( B \), we let \( B(i,j) \) denote the component of \( B \) from the \( j \)'th direct summand to the \( i \)'th direct summand. We also use the notation \( B\{i\} \) for \( B(i,i) \). Using composition of homomorphisms we get in a natural way a category \( \mathcal{M}_N \) with objects \( \mathbb{N}_0^N \) and with the morphisms from \( \mathbf{n} \) to \( \mathbf{m} \) being \( \mathcal{M}(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \). Moreover,

\[
(BC)i,j = \sum_{k=1}^N B(i,k)C(k,j),
\]

whenever \( B \in \mathcal{M}(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \) and \( C \in \mathcal{M}(\mathbf{n} \times \mathbf{r}, \mathbb{Z}) \) for a multiindex \( \mathbf{r} \).

A morphism \( B \in \mathcal{M}(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \) is said to be in \( \mathcal{M}_P(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \), if

\[
B(i,j) \neq 0 \Rightarrow i \preceq j,
\]

for all \( i, j \in \mathcal{P} \). It is easy to verify, that this gives a subcategory \( \mathcal{M}_P \) with the same objects but \( \mathcal{M}_P(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \) as morphisms.

Moreover, for a subset \( s \) of \( \mathcal{P} \), we let — with a slight misuse of notation — \( B\{s\} \in \mathcal{M}_s(\mathbf{m}_{i \in s} \times \mathbf{n}_{i \in s}, \mathbb{Z}) \) denote the component of \( B \) from \( \bigoplus_{i \in s} \mathbb{Z}^{m_i} \) to \( \bigoplus_{i \in s} \mathbb{Z}^{n_i} \).

We let \( \mathcal{M}(\mathbf{n}, \mathbb{Z}) \) denote \( \mathcal{M}(\mathbf{n} \times \mathbf{n}, \mathbb{Z}) \), and \( \mathcal{M}_P(\mathbf{n}, \mathbb{Z}) \) denote \( \mathcal{M}_P(\mathbf{n} \times \mathbf{n}, \mathbb{Z}) \).

For \( \mathbf{n} \), we let \( \text{GL}_P(\mathbf{n}, \mathbb{Z}) \) denote the automorphisms in \( \mathcal{M}_P(\mathbf{n}, \mathbb{Z}) \). Then \( U \in \text{GL}_P(\mathbf{n}, \mathbb{Z}) \) if and only if \( U \in \mathcal{M}_P(\mathbf{n}, \mathbb{Z}) \) and \( U\{i\} \) is a group automorphism (meaning that the determinant as a matrix is \( \pm 1 \) whenever \( n_i \neq 0 \), for every \( i \in \mathcal{P} \)).

An automorphism \( U \in \text{GL}_P(\mathbf{n}, \mathbb{Z}) \) is in \( \text{SL}_P(\mathbf{n}, \mathbb{Z}) \) if the determinant of \( U\{i\} \) is 1 for all \( i \in \mathcal{P} \) with \( n_i \neq 0 \).
Remark 6.5. Let $m, n \in \mathbb{N}^N_0$ be multiindices. Set $k_1 = m_1 + \cdots + m_N$ and $k_2 = n_1 + \cdots + n_N$. If $k_1 \neq 0$ and $k_2 \neq 0$, we can equivalently view the elements $B \in \mathfrak{M}(m \times n, \mathbb{Z})$ as block matrices

$$B = \begin{pmatrix} B\{1,1\} & \cdots & B\{1,N\} \\ \vdots & \ddots & \vdots \\ B\{N,1\} & \cdots & B\{N,N\} \end{pmatrix}$$

where $B\{i,j\} \in \mathfrak{M}(m_i \times n_j, \mathbb{Z})$ with $B\{i,j\}$ the empty matrix if $m_i = 0$ or $n_j = 0$.

Note that from this point of view, the matrices in $\mathfrak{M}_P(m \times n, \mathbb{Z})$ are upper triangular matrices with a certain zero block structure dictated by the order on $P$, and the matrices in $\mathrm{GL}_P(n, \mathbb{Z})$ (respectively $\mathrm{SL}_P(n, \mathbb{Z})$) are matrices in $\mathfrak{M}_P(m \times n, \mathbb{Z})$ with all nonempty diagonal blocks having determinant $\pm 1$ (respectively 1).

Note that if $B \in \mathfrak{M}(m \times n, \mathbb{Z})$ and $C \in \mathfrak{M}(n \times r, \mathbb{Z})$ for a multiindex $r$, then the matrix product makes sense, and — as matrices — we have that

$$(BC)\{i,j\} = \sum_{k \in P, n_k \neq 0} B\{i,k\} C\{k,j\},$$

for all $i, j \in P$ with $n_i \neq 0$ and $r_j \neq 0$.

We will therefore also allow ourselves to talk about matrices with zero rows or columns (by considering it as an element of $\mathfrak{M}(m \times n, \mathbb{Z})$); and then $B\{s\}$ for a subset $s$ of $P$ as defined above is just the principal submatrix corresponding to indices in $s$ (remembering the block structure).

Definition 6.6. Let $m$ and $n$ be multiindices. Two matrices $B$ and $B'$ in $\mathfrak{M}_P(m \times n, \mathbb{Z})$ are said to be $\mathrm{GL}_P$-equivalent (respectively $\mathrm{SL}_P$-equivalent) if there exist $U \in \mathrm{GL}_P(m, \mathbb{Z})$ and $V \in \mathrm{GL}_P(n, \mathbb{Z})$ (respectively $U \in \mathrm{SL}_P(m, \mathbb{Z})$ and $V \in \mathrm{SL}_P(n, \mathbb{Z})$) such that

$$UBV = B'.$$

Note that this is a generalization of the definitions in [Boy02, BH03] (in the finite matrix case) to the cases with rectangular diagonal blocks or vacuous blocks.

6.2. K-web and induced isomorphisms. We define the $K$-web, $K(B)$, of a matrix $B \in \mathfrak{M}_P(m \times n, \mathbb{Z})$ and describe how a $\mathrm{GL}_P$-equivalence $(U, V): B \to B'$ induces an isomorphism $\xi_{(U, V)}: K(B) \to K(B')$.

For an element $B \in \mathfrak{M}(m \times n, \mathbb{Z})$ (i.e., a group homomorphism $B: \mathbb{Z}^n \to \mathbb{Z}^m$), we define as usual $\mathrm{cok} \, B$ to be the abelian group $\mathbb{Z}^m / B\mathbb{Z}^n$ and $\mathrm{ker} \, B$ to be the abelian group $\{x \in \mathbb{Z}^n \mid Bx = 0\}$. Note, that if $m = 0$, then $\mathrm{cok} \, B = \{0\}$ and $\mathrm{ker} \, B = \mathbb{Z}^n$, and if $n = 0$, then $\mathrm{cok} \, B = \mathbb{Z}^m$ and $\mathrm{ker} \, B = \{0\}$.

For $m, n \in \mathbb{N}_0$, $B, B' \in \mathfrak{M}(m \times n, \mathbb{Z})$, $U \in \mathrm{GL}(m, \mathbb{Z})$, and $V \in \mathrm{GL}(n, \mathbb{Z})$ with $UBV = B'$, it is now clear that this equivalence induces isomorphisms

$$\mathrm{cok} \, B \xrightarrow{[x] \mapsto [Ux]} \mathrm{cok} \, B' \quad \text{and} \quad \mathrm{ker} \, B \xrightarrow{[x] \mapsto [V^{-1}x]} \ker \, B'.$$

Lemma 6.7. Let $P = P_2 = \{1, 2\}$ be a partially ordered set and let $B \in \mathfrak{M}_P(m \times n, \mathbb{Z})$. Then the following sequence

$$\begin{array}{c}
\mathrm{cok} \, B\{1\} \xrightarrow{[v] \mapsto [\begin{pmatrix} v \\ u \end{pmatrix}]} \mathrm{cok} \, B \xrightarrow{[\begin{pmatrix} v \\ w \end{pmatrix}] \mapsto [w]} \mathrm{cok} \, B\{2\} \\
\mathrm{ker} \, B\{2\} \xrightarrow{w \mapsto \begin{pmatrix} v \\ w \end{pmatrix}} \mathrm{ker} \, B \xrightarrow{\begin{pmatrix} v \\ w \end{pmatrix} \mapsto \ker \, B\{1\}} \mathrm{ker} \, B\{1\}
\end{array}$$

is exact.
Moreover, if \( B \) and \( B' \) are elements of \( \mathfrak{M}_P(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \) and \( (U, V) : B \rightarrow B' \) is a \( \text{GL}_P \)-equivalence, then \( (U, V) \) induces an isomorphism
\[
(\xi_{(U(1), V(1))}, \xi_{(U(2), V(2))}, \delta_{(U(1), V(1))}, \delta_{(U(2), V(2))})
\]
of (cyclic six-term) exact sequences.

**Proof.** The first part of the lemma follows directly from the Snake lemma applied to the diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{Z}^{n_1} & \rightarrow & \mathbb{Z}^{n_1} \oplus \mathbb{Z}^{n_2} & \rightarrow & \mathbb{Z}^{n_2} & \rightarrow & 0 \\
& & \downarrow{B(1)} & & \downarrow{B} & & \downarrow{B(2)} & & \downarrow{0} \\
0 & \rightarrow & \mathbb{Z}^{m_1} & \rightarrow & \mathbb{Z}^{m_1} \oplus \mathbb{Z}^{m_2} & \rightarrow & \mathbb{Z}^{m_2} & \rightarrow & 0
\end{array}
\]
The second part of the proof is a straightforward verification. \( \square \)

Completely analogous to [BH03], we make the following definitions.

**Definition 6.8.** A subset \( c \) of \( P \) is called convex if \( c \) is nonempty and for all \( k \in P \), \( i, j \subseteq c \) and \( i \preceq k \preceq j \implies k \in c \).

A subset \( d \) of \( P \) is called a difference set if \( d \) is convex and there are convex sets \( r \) and \( s \) in \( P \) with \( r \subseteq s \) such that \( d = s \setminus r \) and
\[
i \in r \text{ and } j \in d \implies j \npreceq i.
\]
Whenever we have such set \( r, s \) and \( d = s \setminus r \), we get a canonical functor from \( \mathfrak{M}_P \) to \( \mathfrak{M}_{P_2} \), where \( P_2 = \{1, 2\} \) with the usual order if there exist \( i \in r \) and \( j \in d \) such that \( i \preceq j \), and the trivial order otherwise. Thus such sets will also give a canonical (cyclic six-term) exact sequence as above.

**Definition 6.9.** Let \( B \in \mathfrak{M}_P(\mathbf{m} \times \mathbf{n}, \mathbb{Z}) \). The (reduced) \( K \)-web of \( B, K(B) \), consists of a family of abelian groups together with families of group homomorphisms between these, as described below.

For each \( i \in P \), let \( r_i = \{ j \in P \mid j \prec i \} \) and \( s_i = \{ j \in P \mid j \preceq i \} \). Note that if \( r_i \) in the above definition is nonempty, then \( \{i\} = s_i \setminus r_i \) is a difference set. We let \( \text{Imm}(i) \) denote the set of immediate predecessors of \( i \) (we say that \( j \) is an immediate predecessor of \( i \) if \( j \prec i \) and there is no \( k \) such that \( j \preceq k \prec i \)).

For each \( i \in P \) with \( r_i \neq \emptyset \), we get an exact sequence from Lemma 6.7
\[
(6.1) \quad \ker B\{i\} \rightarrow \cok B\{r_i\} \rightarrow \cok B\{s_i\} \rightarrow \cok B\{i\}
\]
Moreover, for every pair \( (i, j) \in P \times P \) satisfying \( j \in \text{Imm}(i) \) and \( \text{Imm}(i) \setminus \{j\} \neq \emptyset \) is \( s_j \subseteq r_i \); consequently we have a homomorphism
\[
(6.2) \quad \cok B\{s_j\} \rightarrow \cok B\{r_i\}
\]
originating from the exact sequence above (cf. Lemma 6.7 used on the division into the sets \( r_i, s_j \) and \( r_i \setminus s_j \)).

Set
\[
I^P_0 = \{ r_i \mid i \in P \text{ and } r_i \neq \emptyset \} \cup \{ s_i \mid i \in P \} \cup \{ \{i\} \mid i \in P \},
\]
\[
I^P_1 = \{ i \in P \mid r_i \neq \emptyset \}.
\]
The \( K \)-web of \( B \), denoted by \( K(B) \), consists of the families \( (\cok B\{c\})_{c \in I^P_0} \) and \( (\ker B\{i\})_{c \in I^P_1} \) together with all the homomorphisms from the sequences (6.1) and (6.2). Let \( B' \) be an element of \( \mathfrak{M}_P(\mathbf{m}' \times \mathbf{n}', \mathbb{Z}) \). By a \( K \)-web isomorphism, \( \kappa : K(A) \rightarrow K(B) \), we mean families
\[
(\phi_c : \cok B\{c\} \rightarrow \cok B'\{c\})_{c \in I^P_0}
\]
and

\[(\psi_i: \ker B\{i\} \to \ker B'\{i\})_{i \in I^r}\]

of isomorphisms satisfying that the ladders coming from the sequences in $K(B)$ and $K(B')$ commute.

By Lemma 6.7 any $GL_P$-equivalence $(U,V): B \to B'$ induces a $K$-web isomorphism from $B$ to $B'$. We denote this induced isomorphism by $\kappa_{(U,V)}$.

**Remark 6.10.** The definitions above are completely analogous to the definitions in [BHRS02], and are the same in the case $m_i = n_i \neq 0$ for all $i \in P$. Note that the last homomorphism in (6.1) is really not needed, because commutativity with this map is automatic.

7. Standard form

In this section, we prove that every graph with finitely many vertices is move equivalent to a graph in canonical form (see Definition 7.6). This will allow us to reduce the proof of our classification result to graphs in canonical form. In fact, we will do even better. We will reduce the proof of our classification result to graphs whose adjacency matrices are in the same block form.

The first result of this type is the following that allows us to remove breaking vertices (see [BHRS02] for a definition) and regular vertices that do not support a loop.

**Lemma 7.1.** Let $E$ be a graph with finitely many vertices. Then $E \sim_M E'$, where $E'$ is a graph with finitely many vertices such that every vertex of $E'$ is either a regular vertex that is the base point of a loop or a singular vertex $v$ satisfying the property that if there exists a path of positive length from $v$ to $w$, then $|s^{-1}(v) \cap r^{-1}(w)| = \infty$.

**Proof.** First we show how to modify $E$ to get a graph with the property that if $v$ is an infinite emitter, then $v$ emits infinitely many edges to any vertex it emits any edges to. Let $v \in E^0$ be an infinite emitter. If there exists a vertex $u \in E^0$ such that $v$ emits only finitely many edges to $u$, we partition $s^{-1}(v)$ into two sets, $E_1 = \{e \in s^{-1}(v) | |s^{-1}(v) \cap r^{-1}(r(e))| < \infty\}$ and $E_2 = \{e \in s^{-1}(v) | |s^{-1}(v) \cap r^{-1}(r(e))| = \infty\}$, i.e. $E_1$ consists of the edges out of $v$ that only have finitely many parallel edges. Note that since $E^0$ is finite, $E_1$ is a finite set. Hence we can perform move (Q) according to this partition, resulting in a graph $F'$ that is move equivalent to $E$. Call the vertices $v$ got split into $v_1$ and $v_2$. In $F'$, $v_2$ is an infinite emitter with the property that it emits infinitely many edges to any vertex it emits any edges to, and any infinite emitter in $E$ that already had that property keeps it. On the other hand $v_1$ is a finite emitter.

Since $E^0$ is finite, we can do the above process a finite number of times, ending with a graph $F$ that is move equivalent to $E$, and with the property that if $v$ is an infinite emitter, then $v$ emits infinitely many edges to any vertex it emits any edges to. Now we can use move (T) a finite number of times to get a graph $G$ that is move equivalent to $F$ and satisfies that for every infinite emitter $v \in G^0$ and every $w \in G^0$ for which there exists a path of positive length from $v$ to $w$ we have $|s^{-1}(v) \cap r^{-1}(w)| = \infty$. Finally we use the collapse move (Definition 1.11) on each regular vertex of $F$ that does not support a loop to produce a new graph, $E'$ say, with $E' \sim_M G \sim_M E$ and such that every regular vertex in $E'$ supports a loop. Because of the way the collapse move adds edges this process maintains the property that $|s^{-1}(v) \cap r^{-1}(w)| = \infty$ for any infinite emitter $v$ and any vertex $w$ with a path of positive length from $v$ to $w$. \[\square\]
Assume $E$ satisfies Condition (K) and satisfies the conclusion of Lemma 7.1. Then every hereditary subset of $E^0$ is saturated and $E$ has no breaking vertices. Moreover, every ideal of $C^*(E)$ is gauge invariant. In particular, there is a lattice isomorphism from the ideal lattice of $C^*(E)$ to the lattice of hereditary subsets of $E^0$ with ordering given by set containment.

Therefore, $B^*_E \in \mathfrak{M}_P(\mathfrak{m}_E \times \mathfrak{n}_E, \mathbb{Z})$ (in a canonical way) for a partially ordered set $P = \{\{1, \ldots, N\}, \preceq\}$, where $N$ is the number of points in $\text{Prim}(C^*(E))$, and \( \preceq \) is chosen so that it satisfies Assumption (3). More formally we have:

**Lemma 7.2.** Let $E$ be a graph with finitely many vertices such that every vertex of $E$ is either a regular vertex that is the base point of a loop or a singular vertex $v$ satisfying the property that if there exists a path of positive length from $v$ to $w$, then $|s^{-1}(v) \cap r^{-1}(w)| = \infty$. Suppose $E$ satisfies Condition (K). Then $B^*_E \in \mathfrak{M}_P(\mathfrak{m}_E \times \mathfrak{n}_E, \mathbb{Z})$, where

$$n_{E,i} = |H^E_{i,1} \setminus H^E_{i,0}|$$

with $I_{H^E_{i,0}}$ the prime ideal corresponding to $i$ and $I_{H^E_{i,0}}$ the maximal proper ideal of $I_{H^E_{i,0}}$, and

$$m_{E,i} = n_{E,i} - \{|v \in H^E_{i,1} \setminus H^E_{i,0} : v \text{ is a singular vertex in } H^E_{i,1} \setminus H^E_{i,0}\}|.$$

Note that the hereditary subsets of vertices — as usually defined for graphs, when we consider graph $C^*$-algebras — correspond to subsets $S$ of $P$ satisfying that $i \preceq j$ implies that $j \in S$ whenever $i \in S$. This is due to that fact that we generally do not work with the transposed matrix in this paper, since we find it more convenient to work with the non-transposed matrix (see also the proof of Theorem 11.1).

We now expand on the conditions we can put on graphs. To turn $K$-theory isomorphisms into $\text{GL}_P$-equivalences or $\text{SL}_P$-equivalences, the matrices $B^*_E$ and $B^*_E'$ must have sufficiently big diagonal blocks; this requirement is captured in (3) and (4) below. The positivity condition, (4), is also critical when dealing with matrix manipulations. Condition (2) ensures that we can apply Propositions 4.4 and 4.7 to actually do matrix manipulations.

**Theorem 7.3.** Let $E$ be a graph with finitely many vertices that satisfies Condition (K). Then there exists a graph $E'$ with finitely many vertices such that $E \sim_M E'$ and $E'$ satisfies the following properties:

1. every vertex of $E'$ is either a regular vertex that is the base point of a loop or a singular vertex $v$ satisfying the property that if there exists a path of positive length from $v$ to $w$, then $|s^{-1}(v) \cap r^{-1}(w)| = \infty$;
2. for all regular vertices $v, w$ of $E'$ with $v \geq w$, there exists a path in $E'$ from $v$ to $w$ through regular vertices in $E$;
3. $m_{E', i} \geq 3$ whenever there exists a cycle in the graph

$$\left(H^E_{i,1} \setminus H^E_{i,0}, r^{-1}(H^E_{i,1} \setminus H^E_{i,0}) \cap s^{-1}(H^E_{i,1}), r, s\right);$$
4. if $i \preceq j$ and $B^*_E\{i, j\}$ is not the empty matrix, then $B^*_E\{i, j\} > 0$; and
5. if $B^*_E\{i\}$ is not the empty matrix, then the Smith normal form of $B^*_E\{i\}$ has at least two 1’s.

**Proof.** Lemma 7.3 lets us find a graph $F$ such that $F \sim_M E$ and $F$ satisfies (1). Using the same technique as described in the proof Proposition 5.1, we can guarantee that $F$ also satisfies (2).

Suppose now $i$ is such that $m_{F, i} < 3$ and there exist a cycle in

$$\left(H^F_{i,1} \setminus H^F_{i,0}, r^{-1}(H^F_{i,1} \setminus H^F_{i,0}) \cap s^{-1}(H^F_{i,1}), r, s\right),$$
We want to reduce to the case where \( m_{F,i} = 2 \).

If \( m_{F,i} = 0 \) then all the vertices in \( H_{i,1}^F \setminus H_{i,0}^F \) are infinite emitters. Since the subgraph has a cycle and \( F \) satisfies (1) each of the vertices in \( H_{i,1}^F \setminus H_{i,0}^F \) supports an infinite number of loops. By using move (0) to split two loops of an infinite emitter, we get a graph \( F' \) that is move equivalent to \( F \), satisfies (1) and (2) and where \( m_{F',i} < 3 \).

If \( m_{F,i} = 1 \) there are two cases. Case one is that \( H_{i,1}^F \setminus H_{i,0}^F \) only consists of one vertex. In this case, that vertex must support at least two loops (since \( F \) satisfies Condition (K)) and we can use move (0) to split the vertex in to two, thus giving us a move equivalent graph, \( F' \), that satisfies (1) and (2) and where \( m_{F',i} = 2 \). The other case is that \( H_{i,1}^F \setminus H_{i,0}^F \) also contains an infinite emitter. The regular vertex \( v \) has to emit at least one edge to one such infinite emitter \( w \). By the construction of \( H_{i,1}^F \setminus H_{i,0}^F \) \( w \) must emit an edge to \( v \), and therefore we can use column addition (Proposition 4.4) to add the \( \ast \) th column of \( B_F \) into the \( \ast \) th column. The result will be a graph \( F' \) that satisfies (1) and (2) and where \( v \) supports at least two loops.

Outsplitting, as in case one, we reduce to the case where \( m_{F,i} = 2 \).

Suppose now that \( m_{F,i} = 2 \). Then there are two regular vertices \( u, v \in H_{i,1}^F \setminus H_{i,0}^F \) and there is at least one edge from \( u \) to \( v \) and at least one from \( v \) to \( u \). Hence we can add the \( \ast \) th column of \( B_F \) into the \( \ast \) th, using Proposition 4.4, to ensure that \( u \) supports at least 2 loops. We can now use move (0) to outsplit \( u \), by dividing the outgoing edges into two in such a way that each partition has a loop, to yield a graph move equivalent to \( F \) that satisfies (1), (2) and (3). Hence we can assume that \( F \) also satisfies (4).

By (3) each nonempty diagonal block of \( B_F^* \) will have a nonzero entry. Hence we may use row and column additions (Propositions 4.7 and 4.3), which are legal because of (2), to make sure that all entries in the diagonal blocks are nonzero. Then we can use column addition to guarantee that all offdiagonal blocks (that are not forced to be zero by the block structure) are strictly positive. Since adding rows and columns together will keep conditions (1), (2) and (3), we may assume that \( F \) also satisfies (4).

Note, that by the above reasoning, we can assume that any entry in \( B_F^* \) is not only positive, but greater than or equal to any natural number we see fit. Hence, for each \( i \) with \( B_F^*(i) \) nonempty, we can find a regular vertex, \( v \), say, in \( H_{i,1}^F \) such that \( v \) emits at least 4 edges to each vertex \( v \) reaches. Partition the outgoing edges of \( v \) into two sets in such a way that each partition contains at least one edges to each vertex \( v \) can reach, and at least two loops. Let \( d_1 \) be the number of loops in the first partition, and let \( d_2 \) be the number in the second (then \( d_1 + d_2 = d \)). Outsplitting according to this partition will yield a graph \( F' \) such that \( F \sim_M F' \) and \( F \) satisfies (1), (2), (3) and (4). \( B_F^*(i) \) will contain the following two rows (corresponding to the vertices \( v \) was split into)\

\[
\begin{pmatrix}
(d_1 - 1 & d_1 & \ast & \ast & \cdots \\
(d_2 & d_2 - 1 & \ast & \ast & \cdots)
\end{pmatrix},
\]

where the asterisks can be any positive numbers, with \( m_{F',i} = m_{F,i} + 1 \). Repeating this process we can increase the size of the relevant block so much that the Smith normal form must contain at least two 1’s.

Continuing in this fashion for each diagonal block we can construct \( E' \) such that \( E \sim_M E' \) and \( E' \) satisfies (1), (2), (3) and (4).

\[\square\]

**Remark 7.4.** Suppose that \( E \) is a graph with finitely many vertices that satisfies (1) and (3) of Theorem 7.3 Then \( E \) satisfies Condition (K). Moreover, if \( H_{i,1}^E \setminus H_{i,0}^E = \{v_i\} \), then either \( v_i \) is an infinite emitter that does not support a cycle or is a sink.
Remark 7.5. Let $E$ be a graph with finitely many vertices that satisfies Condition (K). It follows from the proof of Theorem 7.3 that if $E$ satisfies (1), (2) and (3) from the theorem, then there exists a graph $E'$ that is move equivalent to $E$ and satisfies (1), (2), (3), (4) and (5), furthermore $B^*_E$ and $B^*_F$ have the same block form and are $SL_p$-equivalent.

Since the Smith normal form of a matrix is invariant under $SL$-equivalence $E'$ will satisfy condition (6) if $E$ does.

Definition 7.6. A graph $E$ with finitely many vertices is in canonical form if $E$ satisfies the properties (1), (2), (3), (4), and (5) of Theorem 7.3. A pair of graphs $(E, F)$ with finitely many vertices are in standard form if $E$ and $F$ are in canonical form with $m_E = m_F$, and $n_E = n_F$.

The notion of a standard form is of course only useful if we can assume that our graphs have the standard form, the next proposition shows that we can indeed assume that, if the corresponding $C^*$-algebras have isomorphic ordered reduced filtered $K$-theory.

Proposition 7.7. Let there be given graphs $E_1$ and $E_2$ with finitely many vertices.
If $FK_R(C^*(E_1)) \cong FK_R(C^*(E_2))$, then there exists a pair of graphs $(F_1, F_2)$ with finitely many vertices such that the pair $(F_1, F_2)$ is in standard form and $E_1 \sim_M E_2$.

Proof. It follows from Theorem 7.3 that we can find graphs $G_1, G_2$ such that $G_i \sim_M E_i$ and $G_i$ are in canonical form, $i = 1, 2$. The $K$-theory condition gives a specific isomorphism between the primitive ideal spaces of $C^*(E_1)$ and $C^*(E_2)$, hence $B^*_{G_1}$ and $B^*_{G_2}$ can be chosen to have the same same block structure according to this isomorphism. Furthermore, the number of singular vertices in $H_{E_i} \setminus H_{E_0}$ is determined by its $K$-theory, since $C^*(H_{E_i} \setminus H_{E_0})$ is simple (see [Sør13, Lemma 9.2]). The same holds for $E_2$ so $n_{E_1,i} - m_{E_1,i} = n_{E_2,i} - m_{E_2,i}$ for all $i$, and therefore $n_{G_1,i} - m_{G_1,i} = n_{G_2,i} - m_{G_2,i}$ for all $i$.

The only potential problem is now that the we may not have $n_{G_1,i} = n_{G_2,i}$ for all $i$. Since all the entries in $B^*_{G_1}$ are positive, unless forced to be zero by the block structure, we may use row and column additions to ensure that all nonzero entries in $B^*_{G_1}$ are at least 4. Similarly we can assume that all nonzero entries of $B^*_{G_2}$ are at least 4. So if $n_{G_1,i} < n_{G_2,i}$ for some $i$, we can use an outsplit (similar to what is described at the end of the proof of Theorem 7.3) to grow $n_{G_1,i}$ by 1 while keeping it in canonical form. Proceeding this way, we construct graphs $F_1, F_2$ in canonical form such that $F_1 \sim_M E_1, F_2 \sim_M E_2$ and $r_{F_1,i} = r_{F_2,i}$ for all $i$. Since we also have $n_{F_1,i} - m_{F_1,i} = n_{F_2,i} - m_{F_2,i}$ we must have $m_{F_1,i} = m_{F_2,i}$ for all $i$.

When $E$ is in canonical form, the rows of $B_E$ that are removed to form $B^*_E$ either have all entries equal to 0 except on which is $-1$, this is in case the corresponding vertex is a sink, or it only contains 0 and $\infty$, in case it is an infinite emitter. It therefore follows from Proposition 4.4 and Remark 4.8 that adding one column in $B^*_E$ into another will preserve move equivalence, so long as it maintains the block structure and similarly for rows by Proposition 4.4 and Remark 4.8. Hence we have:

Corollary 7.8. Let $E$ be a graph with finitely many vertices and suppose that $E$ is in canonical form. In $B^*_E$ we can add column $l$ into column $k$ without changing the move equivalence class of the associated graph if the diagonal entry of column $l$ is in block $i$, the diagonal entry of column $k$ is in block $j$ and $i \leq j$. Similarly we can add row $l$ into row $k$ without changing move equivalence class if the diagonal entry of row $l$ is in block $i$, the diagonal entry of row $k$ is in block $j$ and $j \leq i$.

For the results in Section 8 we need a final refinement of our standard form, we also need the diagonal blocks of $B^*_E$ to have has greatest common divisor 1. We
Proposition 7.9. Let there be given graphs $E_1$ and $E_2$ with finitely many vertices. If $\text{FK}_K^+(C^*(E_1)) \cong \text{FK}_K^+(C^*(E_2))$, then there exists a pair of graphs $(F_1,F_2)$ with finitely many vertices such that the pair $(F_1,F_2)$ is in standard form, $E_i \sim_M F_i$ and each nonempty diagonal block $B^*_{F_i}$ contains a 1.

Proof. By Proposition 7.7 we can find $F_1, F_2$ satisfying the conclusion of the proposition, except for the last condition. As in the proof of Proposition 7.7 we may use row and column operations to ensure that all nonzero entries of $B^*_{G_i}$ and $B^*_{G_j}$ are at least 4. For each nonzero diagonal block, we will now do an outsplit similar to what is described at the end of the proof of Theorem 7.3 where we find a regular vertex, $v$ say, that supports at least two loops. Partition the outgoing edges of $v$ into two sets in such a way that each partition contains at least one edges to each vertex $v$ can reach, but we also insist that one partition only contains two loops. Let $d_1$ be the number of loops in the first partition, and let $d_2$ be the number in the second (then $d_1 + d_2 = d$). Our assumption forces either $d_1$ or $d_2$ to be 2, for simplicity let us say that $d_1 = 2$. As noted in the proof of Theorem 7.3 in the resulting graph, the diagonal block will contain the rows

$$
\begin{pmatrix}
(d_1 - 1) & d_1 & * & * & \cdots \\
 d_2 & (d_2 - 1) & * & * & \cdots
\end{pmatrix}
= \begin{pmatrix}
1 & 2 & * & * & \cdots \\
 d_2 & d_2 - 1 & * & * & \cdots
\end{pmatrix}.
$$

Hence it contains a 1. Doing this for all nonzero diagonal blocks yields the desired graphs. \qed

8. Generalization of Boyle-Huang’s lifting result

We aim to prove Theorem 8.12 which says that — in certain cases — every $K$-web isomorphism is induced by a $GL_p$-equivalence. This is the main result of this section. To prove Theorem 8.12 we first strengthen [BH03, Theorem 4.5]. The following theorem is a classical well-known theorem, cf. [New72, Section II.15].

Theorem 8.1 (Smith normal form). Suppose $B$ is an $m \times n$ matrix over $\mathbb{Z}$. Then there exist matrices $U \in \text{GL}(m,\mathbb{Z})$ and $V \in \text{GL}(n,\mathbb{Z})$ such that the matrix $D = UBV$ satisfies the following

- $D(i,j) = 0$ for all $i \neq j$,
- the $\min(m,n) \times \min(m,n)$ principal submatrix of $D$ is a diagonal matrix
  $$
  \text{diag}(d_1, d_2, \ldots, d_r, 0, 0, \ldots, 0),
  $$
  where $r \in \{0, 1, \ldots, \min(m, n)\}$ is the rank of $B$ and $d_1, d_2, \ldots, d_r$ are positive integers such that $d_i | d_{i+1}$ for $i = 1, \ldots, r - 1$.

For each matrix $B$, the matrix $D$ is unique and is called the Smith normal form of $B$.

We now recall some terminology that was introduced in [BH03].

Definition 8.2. Let $B$ be an element of $\mathfrak{M}(m \times n, \mathbb{Z})$. A $GL$ self-equivalence of $B$ is a $GL$-equivalence $(U,V): B \rightarrow B$. We say that an automorphism $\phi$ of $\text{cok} B$ is $GL$-allowable if there exists a $GL$ self-equivalence, $(U,V)$, of $B$ such that the isomorphism $\kappa(U,V)$ induces $\phi$.

Lemma 8.3. Let $B$ be an $m \times n$ matrix over $\mathbb{Z}$, and let $U \in \text{GL}(m,\mathbb{Z})$ and $V \in \text{GL}(n,\mathbb{Z})$ be given invertible matrices. Then $\gcd B = \gcd(UBV)$. In particular, if $D$ is the Smith normal form of $B$, then $\gcd B = D(1,1) = d_1$. 

Proof. We may assume that $B \neq 0$. Let $d$ be a positive integer. Then $d$ divides all entries of $B$ if and only if $d | x^TBy$ for all $x, y \in \mathbb{Z}^n$.

Now the lemma follows. \hfill \Box

Remark 8.4. Let $B$ be an $m \times n$ matrix over $\mathbb{Z}$. Then it follows from the above, that $m$ is greater than the number of generators of cok $B$ according to the decomposition from the Smith normal form into direct sums of nonzero cyclic groups if and only if $\gcd B = 1$.

Boyle and Huang show in their paper [BH03] the following fundamental theorem.

Theorem 8.5 (BH03, Theorem 4.4). Let $B$ be a $n \times n$ (square) matrix over a PID $\mathcal{R}$, and let $\delta = \gcd B$. Let $\phi$ be an automorphism of cok $B$, and let $M$ be any $n \times n$ matrix over $\mathcal{R}$ such that $\phi([x]) = [Mx]$ for all $x \in \mathbb{Z}^n$.

Then $\det(M) \equiv 1 \mod \delta$ if and only if there exist $n \times n$ matrices $U$ and $V$ over $\mathcal{R}$ such that $UBV = B$ and $U$ is defining $\phi$. Then $\det(M) \equiv u \mod \delta$ for some unit $u$ in $\mathcal{R}$ if and only if there exist $n \times n$ invertible ($\text{GL}$) matrices $U$ and $V$ over $\mathcal{R}$ such that $UBV = B$ and $U$ is defining $\phi$.

Remark 8.6. As we will see, it is possible to generalize the part about $\text{GL}$-allowance in this theorem to rectangular matrices, the analogous statement to the part about $\text{SL}$-allowance in Theorem 8.5 does not hold in general (for rectangular matrices). If we consider the matrix

$$B = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

and the automorphism $- \text{id}$ on $\text{cok} B \equiv \mathbb{Z}/3 \oplus \mathbb{Z}$ induced by the matrix

$$M = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

it is easy to see that we get a counterexample.

Although it can be done, we do not investigate this further, since for our purposes we do not need to know when automorphisms can be lifted to $\text{SL}$-equivalences.

In [BH03] there is the following useful theorem. Note that all $n_i$’s are assumed to be nonzero in [BH03].

Theorem 8.7 (BH03, Theorem 4.5). Suppose $B$ and $B'$ are matrices in $\mathcal{M}_P(n, \mathbb{Z})$ with corresponding diagonal blocks equal, and $\kappa: K(B) \to K(B')$ is a $K$-web isomorphism. Then there exist matrices $U, V \in \text{GL}(n, \mathbb{Z})$ such that $UBV = B$ and $U$ satisfies $\kappa(U, V) = \kappa$ if and only if each of the automorphisms $d_i: \text{cok} B[i] \to \text{cok} B'[i]$ defined by $\kappa$ is $\text{GL}$-allowable.

Together with [BH03, Theorem 4.4] (see Theorem 8.5 above), this gives us the following useful corollary.

Corollary 8.8 (BH03, Corollary 4.7). Let $B$ and $B'$ be matrices in $\mathcal{M}_P(n, \mathbb{Z})$ with $\gcd B[i] = \gcd B'[i]$ for all $i \in P$. Then for any $K$-web isomorphism $\kappa: K(B) \to K(B')$ there exist matrices $U, V \in \text{GL}(n, \mathbb{Z})$ such that $UBV = B$ satisfying $\kappa(U, V) = \kappa$. 
But even more is true. We can generalize [BH03, Theorem 4.4 (and Proposition 4.1)] (cf. Theorem 8.5) as follows (we here only consider the case $R = \mathbb{Z}$).

**Theorem 8.9.** Let $B$ be a $n \times n$ (square) matrix over $\mathbb{Z}$, and let $\delta = \gcd B$. Let $\phi$ be an automorphism of $\text{cok} B$, let $\psi$ be an automorphism of $\ker B$, and let $M$ be any $n \times n$ matrix over $\mathbb{Z}$ defining $\phi$, i.e., $\phi([x]) = [Mx]$ for all $x \in \mathbb{Z}^n$.

Then $\det(M) \equiv \pm 1 \pmod{\delta}$ if and only if there exist $n \times n$ invertible (GL) matrices $U$ and $V$ over $\mathbb{Z}$ such that $UBV = B$ and $U$ is defining $\phi$ and $V^{-1}$ is defining $\psi$.

**Proof.** The only thing that does not follow from [BH03, Theorem 4.4] is that we can choose the GL-equivalence $(U, V)$ such that it also induces the right automorphism on $\ker B$. For this, it is clear that we may assume that $B$ is its own Smith normal form (just like in the proof of [BH03, Theorem 4.4]). We use [BH03, Theorem 4.4] to get a GL-equivalence $(U, V) : B \rightarrow B$ that induces $\phi$ on $\text{cok} B$. The matrix $V^{-1}$ induces an automorphism $\psi'$ of $\ker B$. Now we will find a GL-equivalence $(I, V') : B \rightarrow B$ that induces $\psi \circ \psi'^{-1}$ on $\ker B$ — then $(U, V'V)$ is a GL-equivalence that induces $\phi$ on $\text{cok} B$ and $\psi$ on $\ker B$. Now, the automorphism $\psi \circ \psi'^{-1}$ on $\ker B$ uniquely determines what $V'^{-1}$ should be on the lower right block matrix (where we write the matrices as $2 \times 2$ block matrices according to the nonzero respectively zero part of the diagonal of $B$). Let $V'^{-1}$ be the block diagonal matrix and the identity as the upper left block matrix. 

Now we let

$$P_{\text{min}} = \{i \in \mathcal{P} : j \prec i \Rightarrow i = j\}.$$  

Using the above result, we get the following stronger version of Theorem 8.7.

**Theorem 8.10** (Strengthening of [BH03, Theorem 4.5]). Let $n = (n_i)_{i \in \mathcal{P}}$ be a multiindex with $n_i \neq 0$, for all $i \in \mathcal{P}$. Suppose $B$ and $B'$ are matrices in $\mathcal{M}_\mathcal{P}(\mathbb{Z})$ with corresponding diagonal blocks equal, and $\kappa : K(B) \rightarrow K(B')$ is a $K$-web isomorphism. Suppose that for each $i \in P_{\text{min}}$, we have an automorphism $\psi_i : \ker B\{i\} \rightarrow \ker B\{i\}$. Then there exist matrices $U, V \in GL(n, \mathbb{Z})$ such that we have a GL-equivalence $(U, V) : B \rightarrow B'$ satisfying $\kappa_{(U, V)} = \kappa$ if and only if each of the automorphisms $d_i : \text{cok} B\{i\} \rightarrow \text{cok} B\{i\}$ defined by $\kappa$ are GL-allowable — moreover, the GL-equivalence can always be chosen such that $V^{-1}\{i\}$ induces $\psi_i$ for each $i \in P_{\text{min}}$.

**Proof.** The only thing that does not follow from [BH03, Theorem 4.4] (cf. Theorem 8.7), is that we can choose the GL-equivalence $(U, V)$ such that it also induces the right automorphisms on $\ker B\{i\}$, $i \in P_{\text{min}}$. We choose a GL-equivalence $(U, V)$ according to Theorem 8.7, so that it induces the given $K$-web isomorphism. For each $i \in P_{\text{min}}$, this gives an automorphism $\psi_i'$ of $\ker B\{i\}$. Now choose GL-equivalences $(I, V'_i)$ of $B\{i\}$ according to (the proof of) Theorem 8.7 so that $V'^{-1}_i$ induces $\psi_i \circ \psi'^{-1}_i$ for each $i \in P_{\text{min}}$. Let $V$ be the block matrix that is the identity matrix everywhere except that $V\{i\} = V'_i$ for every $i \in P_{\text{min}}$. It is straightforward to verify that $(I, V')$ is a GL-equivalence from $B'$ to $B'$, and that $(U, V'V)$ induces exactly what we want. 

Together with [BH03, Theorem 4.4 (and Proposition 4.1)] (see Theorem 8.5 above), this gives us the following stronger version of Corollary 8.8.

**Corollary 8.11** (Strengthening of [BH03, Corollary 4.7]). Let $n = (n_i)_{i \in \mathcal{P}}$ be a multiindex with $n_i \neq 0$, for all $i \in \mathcal{P}$. Suppose $B$ and $B'$ are matrices in $\mathcal{M}_\mathcal{P}(\mathbb{Z})$ with $\gcd B\{i\} = 1 = \gcd B'\{i\}$ for all $i \in \mathcal{P}$. Then for any $K$-web isomorphism $\kappa : K(B) \rightarrow K(B')$ together with automorphisms $\psi_i : \ker B\{i\} \rightarrow \ker B\{i\}$, for
\(i \in \mathcal{P}_{\text{min}},\) there exist matrices \(U, V \in \text{GL}(n, \mathbb{Z})\) such that we have a \(\text{GL}_\mathcal{P}\)-equivalence \((U, V) : B \rightarrow B'\) satisfying \(\kappa_{(U, V)} = \kappa\) and \(V^{-1}\{i\}\) induces \(\psi_i\) for each \(i \in \mathcal{P}_{\text{min}}\).

The following theorem is the main result of this section, and allows us — in certain cases — to lift \(K\)-web isomorphisms to \(\text{GL}_\mathcal{P}\)-equivalences for rectangular cases. Although it is possible to prove this directly, imitating the proof in \([\text{BH}03]\), the present proof is much shorter and reduces the rectangular case to the square case and uses the results from \([\text{BH}03]\).

**Theorem 8.12.** Let \(m = (m_i)_{i \in \mathcal{P}}, n = (n_i)_{i \in \mathcal{P}} \in (\mathbb{N}_0)^\mathcal{P}\) be multiindices. Suppose \(B\) and \(B'\) are matrices in \(\mathfrak{M}_\mathcal{P}(m \times n, \mathbb{Z})\) with \(\gcd B\{i\} = 1 = \gcd B'\{i\}\) for all \(i \in \mathcal{P}\) with \(m_i \neq 0\) and \(n_i \neq 0\).

Then for any \(K\)-web isomorphism \(\kappa : K(B) \rightarrow K(B')\) there exist matrices \(U \in \text{GL}(m, \mathbb{Z})\) and \(V \in \text{GL}(n, \mathbb{Z})\) such that we have a \(\text{GL}_\mathcal{P}\)-equivalence \((U, V) : B \rightarrow B'\) satisfying \(\kappa_{(U, V)} = \kappa\).

If, moreover, we have given an isomorphism \(\psi_i : \ker B\{i\} \rightarrow \ker B'\{i\}\), for every \(i \in \mathcal{P}_{\text{min}},\) then we can choose the above \(\text{GL}_\mathcal{P}\)-equivalence \((U, V)\) such that — in addition to the above — also \(V^{-1}\{i\}\) induces the \(\psi_i\), for all \(i \in \mathcal{P}_{\text{min}}\).

**Proof.** For each \(i \in \mathcal{P},\) choose \(U_i, V_i \in \text{GL}(m_i, \mathbb{Z})\) and \(U_i', V_i' \in \text{GL}(n_i, \mathbb{Z})\) such that \(D_i = U_i B V_i\) and \(D_i' = U_i' B' V_i'\) are the Smith normal forms of \(B\) and \(B'\), respectively (cf. Theorem 8.1). Let \(U, U' \in \text{GL}(m, \mathbb{Z})\) and \(V, V' \in \text{GL}(n, \mathbb{Z})\) be the block diagonal matrices with \(U_i, U_i', V_i\) and \(V_i'\) in the diagonals, respectively.

Then \(UBV\) and \(U'B'V'\) are in \(\mathfrak{M}_\mathcal{P}(m \times n, \mathbb{Z})\) and \((U, V) : B \rightarrow UBV\) and \((U', V') : B' \rightarrow U'B'V'\) are \(\text{GL}_\mathcal{P}\)-equivalences inducing \(K\)-web isomorphisms \(\kappa_{(U, V)}\) from \(K(B)\) to \(K(UBV)\) and \(\kappa_{(U', V')}\) from \(K(B')\) to \(K(U'B'V')\), respectively. Moreover, Lemma 8.2 ensures that we still have \(\gcd(UBV)\{i\} = 1 = \gcd(U'B'V')\{i\}\). Thus we can without loss of generality assume that each diagonal block is equal to its Smith normal form. Also note, that because we have a \(K\)-web isomorphism from \(K(B)\) to \(K(B')\), now the diagonal blocks are necessarily identical.

Let \(r\) be such that \(r_i = \max(m_i, n_i)\) for all \(i \in \mathcal{P}\). Let, moreover, \(C, C' \in \mathfrak{M}_\mathcal{P}(r, \mathbb{Z})\) denote the matrices \(B\) and \(B'\) enlarged by putting zeros outside the original matrices. Define \(\mathcal{r}\) and \(\mathcal{r}'\) by \(\mathcal{r}_i = \max(n_i - m_i, 0)\) and \(\mathcal{r}'_i = \max(m_i - n_i, 0)\) for all \(i \in \mathcal{P}\). In the (reduced) \(K\)-web we are considering the modules \(C_i(B) = \text{cok}(\varphi_i)\) where \(\varphi_i = \{j \in \mathcal{P} : j < i\} \neq \emptyset\) or \(\{j \in \mathcal{P} : j \geq i\} \neq \emptyset\) for \(i \in \mathcal{P}\) — and similarly for \(B'\). It is clear that when we consider \(C\) and \(C'\) we just add onto these cokernels

\[
\bigoplus_{j \in \mathcal{r}} \mathbb{Z}^{r_j},
\]

and that the maps between the modules are the obvious ones. Similarly for the modules \(K_d(B) = \ker B(d)\) where \(d\) is \(\{i\}\) where \(\{j \in \mathcal{P} : j < i\} \neq \emptyset\) — and similarly for \(B'\). It is clear that when we consider \(C\) and \(C'\) we just add onto these kernels

\[
\bigoplus_{j \in \mathcal{r}'} \mathbb{Z}^{r'_j},
\]

where \(d = \{i\}\). And connecting homomorphism will be the zero maps.

Thus we can extend the isomorphism \(\kappa\) to an isomorphism \(\tilde{\kappa} : K(C) \rightarrow K(C')\) by setting it to be the identity on the new groups. By Corollary 8.3 we see that there exist matrices \(U, V \in \text{GL}(r, \mathbb{Z})\) such that we have a \(\text{GL}_\mathcal{P}\)-equivalence \((U, V) : C \rightarrow C'\) satisfying \(\kappa_{(U, V)} = \tilde{\kappa}\). We may (according to Theorem 8.11) actually assume that \((U, V)\) induces \(\psi_i\) plus the identity on the new summands of \(\ker B\{i\}\) for \(i \in \mathcal{P}_{\text{min}}\) as well.

Now let us look at the \(i\)th diagonal block. We now want to cut \(U\) and \(V\) down to match the original structure. Naturally there are three cases to consider. The first one, \(m_i = n_i\) is trivial.
Now consider the case $m_i < n_i$. In this case, $\text{coker} C\{i\} = \text{coker} B\{i\} \oplus \mathbb{Z}^{m_i - n_i}$ and $\ker C\{i\} = \ker B\{i\}$ — and similarly for $B'$ and $C'$. We write $C\{i\} = C'\{i\}$ as

$$\begin{pmatrix} C_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $C_{00}$ is an invertible matrix over $\mathbb{Q}$ and the last diagonal block has size $(n_i - m_i) \times (n_i - m_i)$. We write $U$ and $V$ as

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

according to the block structure of $C\{i\}$ (and $C'\{i\}$).

The condition

$$UCV = C'$$

implies that

$$U_{11}C_{00}V_{11} = C_{00}, \quad U_{i1}C_{00}V_{1j} = 0, \quad \text{for all} \ (i, j) \neq (1, 1).$$

Since $C_{00}$ is invertible as a matrix over $\mathbb{Q}$, we see that also $U_{11}$ and $V_{11}$ have to be invertible over $\mathbb{Q}$. Thus $V_{12} = 0$, $V_{13} = 0$, $U_{21} = 0$, and $U_{31} = 0$. Moreover, since we have to get the identity homomorphism on the new direct summand, we need to have $U_{33} = I$, $U_{23} = 0$ and $U_{32} = 0$. So now let $U_0$ be the block matrix where we erase the rows and columns corresponding to change the size of the $i$'th diagonal block from $r_i \times r_i$ to $m_i \times m_i$ — call the new size $r'$. Moreover, we let $C_0$ and $C_0'$ be the block matrices where we erase the rows corresponding to change the size of the $i$'th diagonal block from $r_i \times r_i$ to $m_i \times n_i$. Note that the $i$'th diagonal block now is the matrix $(U_{11}, U_{12})$. This is a GL matrix that induces the right automorphism of $\text{coker} B\{i\}$. Moreover, clearly $U_0(i)B\{i\}V\{i\} = B\{i\}$. But more is true. We have that $U_0$ is a $\text{GL}(r', \mathbb{Z})$ matrix and that $U_0C_0V = C_0'$ and the induced $K$-web isomorphism agrees with the original on all parts except for the direct summands we cut out.

Now consider instead the case $m_i > n_i$. In this case, $\text{coker} C\{i\} = \text{coker} B\{i\}$ and $\ker C\{i\} = \ker B\{i\} \oplus \mathbb{Z}^{m_i - n_i}$ — and similarly for $B'$ and $C'$. We write $C\{i\} = C'\{i\}$ as

$$\begin{pmatrix} C_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $C_{00}$ is an invertible matrix over $\mathbb{Q}$ and the last diagonal block has size $(m_i - n_i) \times (m_i - n_i)$. We write $U$ and $V$ as

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix},$$

according to the block structure of $C\{i\}$ (and $C'\{i\}$).

The condition

$$UCV = C'$$

implies that

$$U_{11}C_{00}V_{11} = C_{00}, \quad U_{i1}C_{00}V_{1j} = 0, \quad \text{for all} \ (i, j) \neq (1, 1).$$

Since $C_{00}$ is invertible as a matrix over $\mathbb{Q}$, we see that also $U_{11}$ and $V_{11}$ have to be invertible over $\mathbb{Q}$. Thus $V_{12} = 0$, $V_{13} = 0$, $U_{21} = 0$, and $U_{31} = 0$. Moreover, since we have to get the identity homomorphism on the new direct summand, we need to have $V_{33} = I$, $V_{23} = 0$ and $V_{32} = 0$. So now let $V_0$ be the
block matrix where we erase the rows and columns corresponding to change the size of the \(i\)th diagonal block from \(r_i \times r_i\) to \(n_i \times n_i\) — call the new size \(r_i'\). Moreover, we let \(C_0\) and \(C_0'\) be the block matrices where we erase the rows corresponding to change the size of the \(i\)th diagonal block now is the matrix \(V_{\gamma_i} V_{\alpha_i} \). This is a \(\text{GL}\) matrix that induces the right automorphism of \(\text{ker} \, B\{i\}\). Moreover, clearly \(U\{i\} B\{i\} V_0\{i\} = B\{i\}\). But more is true. We have that \(V_0\) is a \(\text{GL}(r', Z)\) matrix and that \(UC_0 V_0 = C_0'\) and the induced \(K\)-web isomorphism agrees with the original on all parts except for the direct summands we cut out.

Induction finishes the proof. □

9. \(\text{GL}_p\)-equivalence to \(\text{SL}_p\)-equivalence

In this section we are concerned with Step 3 in our proof outline in Section \(\text{8}\) of the proof of \(\text{3}\) implies \(\text{1}\) in Theorem \(\text{8.1}\). It is of course not true in general that any two \(\text{GL}_p\)-equivalent matrices will be \(\text{SL}_p\)-equivalent, so we will need to alter our matrices. Our first step in that direction is to create a little more room.

Lemma 9.1. Let \(E\) be a graph with finitely many vertices and suppose

\[
B_E = \begin{pmatrix}
A & X & Y \\
0 & B & Z \\
0 & 0 & C
\end{pmatrix}
\]

where \(B\) is an \(n \times n\) matrix with entries from \(\mathbb{N}_0 \cup \{\infty\}\) for some \(n \geq 2\) and the entries of rows \(n - 1\) and \(n\) of \(B\) are positive integers and the vertices corresponding to these two rows are regular vertices of \(E\).

Then there exists a graph \(E'\) such that \(E \sim_M E'\), and

\[
\begin{pmatrix}
A & X' & Y' \\
0 & B' & Z' \\
0 & 0 & C'
\end{pmatrix}
\]

with \(B'\) an \((n + 2) \times (n + 2)\) matrix with entries from \(\mathbb{N}_0 \cup \{\infty\}\) and there exists \(V \in \mathfrak{M}(n + 2, \mathbb{Z})\) with \(\det(V) = 1\) such that

\[
\begin{pmatrix}
A_0 & X'_0 & Y'_0 \\
0 & B'_0 & Z'_0 \\
0 & 0 & C_0
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
0 & V & 0 \\
0 & 0 & I
\end{pmatrix}
= \begin{pmatrix}
A_0 & X''_0 & Y_0 \\
0 & B'_0 & Z'_0 \\
0 & 0 & C_0
\end{pmatrix}
\]

where

\[
B_* = \begin{pmatrix}
A_0 & X_0 & Y_0 \\
0 & B_0 & Z_0 \\
0 & 0 & C_0
\end{pmatrix}, \quad B^*_E = \begin{pmatrix}
A_0 & X'_0 & Y'_0 \\
0 & B'_0 & Z'_0 \\
0 & 0 & C_0
\end{pmatrix},
\]

\[
X''_0 = (X_0 \, 0), \quad B''_0 = \begin{pmatrix}
B_0 & 0 & 0 \\
0 & I_2
\end{pmatrix}, \quad \text{and} \quad Z''_0 = \begin{pmatrix}
Z_0 \\
0
\end{pmatrix}.
\]

Moreover, if \(E\) satisfies the property that for all \(v, w \in E^0\) with \(v \geq w\), there exists a path in \(E\) from \(v\) to \(w\) through regular vertices in \(E\), then \(E'\) also satisfies the same property.

Proof. Let \(v\) be the vertex in \(E\) corresponding to the entry \(B(n - 1, n - 1) + 1\) and \(w\) be the vertex in \(E\) corresponding to the entry \(B(n, n) + 1\). Outsplitting the vertices \(v\) and \(w\) with respect to the partitions \(s^{-1}(v) = \{e\} \sqcup (s^{-1}(v) \setminus \{e\})\) where \(r(e) = v\) and \(s^{-1}(w) = \{f\} \sqcup (s^{-1}(w) \setminus \{f\})\) where \(r(f) = w\), we get a graph \(F\) such that \(E \sim_M F\).
Let $X_1$ and $B_1$ be the column vector of $X$ and $B$, respectively, that corresponds to the $v$th-column of $B_E$ and let $X_2$ and $B_2$ be the column vector of $X$ and $B$, respectively, that corresponds to the $w$th-column of $B_E$. Then

$$
B_E = \begin{pmatrix}
A & X' & Y \\
0 & \hat{B} & Z' \\
0 & 0 & C
\end{pmatrix}
$$

where $X' = (X \quad X_1 \quad X_2)$, $Z' = \begin{pmatrix} Z \\ 0 \end{pmatrix}$, and

$$
\hat{B} = \begin{pmatrix}
B - J & B_1 & B_2 \\
0 & \cdots & 0 & 1 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 1 & 0 & 0
\end{pmatrix},
$$

where $J$ is the matrix that is zero in all entries except the last two diagonal entries which are 1. These account for the loops $v$ and $w$ lost when we did the outsplit. We can now use row additions (Corollary 4.6), adding the last row of $\hat{B}$ into the third last and the second last into the fourth last, to get a graph $E'$ such that $E' \sim_M F \sim_M E$ and where

$$
B_{E'} = \begin{pmatrix}
A & X' & Y \\
0 & B' & Z' \\
0 & 0 & C
\end{pmatrix},
$$

where

$$
B' = \begin{pmatrix}
B & B_1 & B_2 \\
0 & \cdots & 0 & 1 & 0 & 0 \\
0 & \cdots & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
$$

Let $\tilde{X}_0$ be the matrix obtained from $X_0'$ by replacing the $(n - 1)'$st column of $X_0'$ and the $n'$th column of $X_0'$ by the zero column and let $\hat{B}_0$ be the matrix obtained from $B_0'$ by replacing the $(n - 1)'$st and the $n'$th column of $B_0$ by the zero column but keeping that last two rows intact. Consider the matrix

$$
\begin{pmatrix}
A_0 & \tilde{X}_0 & Y_0 \\
0 & \hat{B}_0 & Z_0' \\
0 & 0 & C_0
\end{pmatrix}.
$$

Let $V_1 \in \mathfrak{M}(n + 2, \mathbb{Z})$ be the matrix obtained from $I_{n+2}$ by switching the $(n - 1)'$st and the $(n + 1)'$st columns and let $V_2 \in \mathfrak{M}(n + 2, \mathbb{Z})$ be the matrix obtained from $I_{n+2}$ by switching the $n'$th and the $(n + 2)'$nd columns. Then $\det(V_1) = \det(V_2) = -1$ and

$$
\begin{pmatrix}
A_0 & \tilde{X}_0 & Y_0 \\
0 & \hat{B}_0 & Z_0' \\
0 & 0 & C_0
\end{pmatrix} \begin{pmatrix}
I & 0 & 0 \\
0 & V_1 & V_2 \\
0 & 0 & I
\end{pmatrix} = \begin{pmatrix}
A_0 & X_0'' & Y_0 \\
0 & B_0'' & Z_0'' \\
0 & 0 & C_0
\end{pmatrix}.
$$

As in the proof of Proposition 4.2 we let $E_{(i,j)}$ denote the elementary matrix that is equal to the identity matrix everywhere except for the $(i, j)'$th entry, that is 1. Let $V_3 = E_{(n+1,n-1)} \in \mathfrak{M}(n + 2, \mathbb{Z})$, and let $V_4 = E_{(n+2,n)} \in \mathfrak{M}(n + 2, \mathbb{Z})$. Then $\det(V_3) = \det(V_4) = 1$ and

$$
\begin{pmatrix}
A_0 & \tilde{X}_0 & Y_0 \\
0 & \hat{B}_0 & Z_0' \\
0 & 0 & C_0
\end{pmatrix} \begin{pmatrix}
I & 0 & 0 \\
0 & V_3 & V_4 \\
0 & 0 & I
\end{pmatrix} = \begin{pmatrix}
A_0 & X_0' & Y_0 \\
0 & B_0' & Z_0' \\
0 & 0 & C_0
\end{pmatrix}.
$$
Set $V = V_4^{-1}V_3^{-1}V_1V_2$. Then
\[
\begin{pmatrix}
A_0 & X_0' & Y_0 \\
0 & B_0' & Z_0' \\
0 & 0 & C_0
\end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix}
A_0 & X_0'' & Y_0 \\
0 & B_0'' & Z_0'' \\
0 & 0 & C_0
\end{pmatrix}.
\]
Since $\det(V_1) = \det(V_2) = -1$ and $\det(V_3) = \det(V_4) = 1$, we have that $\det(V) = 1$.

For the last part of the lemma, let $v_1$ and $v_2$ be the two additional vertices obtained from the outsplitting. It is clear that if $v, w \in E_{\text{reg}}'$ such that $v \geq w$ in $E'$, then $v \geq w$ in $E$. Thus, there exists a path in $E'$ from $v$ to $w$ through regular vertices of $E'$. Suppose $v \in E_{\text{reg}}'$ and $v \geq v_i$. Then by the definition of outsplitting and the assumption on $E$, there exists an edge $e$ in $E'$ such that $r(e) = v_1$, $s(e) \in E_{\text{reg}}'$, and $v \geq s(e)$. It is now clear that there exists a path in $E'$ from $v_i$ to $v_i$ through regular vertices of $E'$. Suppose $v_i \geq v$ with $v \in E_{\text{reg}}'$. By the definition of the outsplitting, there exists a path $\alpha = \alpha_1 \cdots \alpha_m$ in $E'$ such that $s(\alpha_1) = v_i$, $r(\alpha_m) = v$, $r(\alpha_1) \in E_{\text{reg}}'$, and $r(\alpha_1) \geq v_i$. Since $r(\alpha_1) \geq v_i$, and $v_i \geq v$, we have that $r(\alpha_1) \geq v$. By the previous cases, we have that there exists a path in $E'$ from $r(\alpha_1)$ to $v$ in $E'$ through regular vertices in $E'$. Hence, there exists a path in $E'$ from $v_i$ to $v$ through regular vertices in $E'$. For the pair $(v_1, v_2)$ this is clear by construction.

We now connect the space we have created to our method of changing signs, i.e. the Cuntz splice.

**Lemma 9.2.** Let $E$ be a graph with finitely many vertices such that

\[
B_E = \begin{pmatrix}
A & X \\
0 & B \\
0 & 0 & C
\end{pmatrix}
\]

where $B$ is an $n \times n$ matrix with entries from $\mathbb{N}_0 \cup \{\infty\}$ for some $n \geq 1$ and the entries of row $n$ of $B$ are positive integers and the vertex $v$ corresponding to this row is a regular vertex of $E$. Let $E_{v,-}$ be the Cuntz splice of $E$ at the vertex $v$.

Then $\det(U) = 1$, $\det(V) = -1$, and

\[
B_{E_{v,-}}^* = \begin{pmatrix}
I & 0 & 0 \\
0 & U & 0 \\
0 & 0 & I
\end{pmatrix} \begin{pmatrix} A_0 & (X_{-0})' & Y_0 \\
0 & (B_{-0})' & (Z_{-0})' \\
0 & 0 & C_0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\
0 & V & 0 \\
0 & 0 & I \end{pmatrix} = \begin{pmatrix} A_0 & X_0'' & Y_0 \\
0 & B_0'' & Z_0'' \\
0 & 0 & C_0 \end{pmatrix}
\]

where

\[
B_E^* = \begin{pmatrix}
A_0 & X_0 & Y_0 \\
0 & B_0 & Z_0 \\
0 & 0 & C_0
\end{pmatrix}
\]

$X_0'' = (X_0 \ 0)$, $B_0'' = \begin{pmatrix} B_0 \\
0 \\
I_2
\end{pmatrix}$, and $Z_0'' = \begin{pmatrix} Z_0 \\
0
\end{pmatrix}$.

$V = V_4V_2$ where $V_4$ is the matrix obtained from $I_{n+2}$ by subtracting the $(n+2)\text{nd}$ column from the $n$'th column and $V_2$ is the matrix obtained from $I_{n+2}$ by switching the $(n+1)'\text{st}$ and $(n+2)'\text{nd}$ columns, and $U$ is the matrix obtained from $I_{m+2}$ by subtracting the $(m+2)'\text{nd}$ row from the $m$'th row, with $m$ being the number of rows of $B_0$.

**Proof.** The proof of the lemma is just a simple matrix computation using that the row operations to get $U$ only involve regular vertices of $E_{v,-}$, and is left for the reader. \qed
The next proposition will be our first step in going from a \(\text{GL}_P\)-equivalence to an \(\text{SL}_P\)-equivalence. The idea is to alter our graphs in such a way that their \(B^*\) matrices are \(\text{GL}_P\)-equivalent, say by \((U, V)\), but where all the diagonal blocks of \(U\) have determinant one. Thus moving all our problems to \(V\).

**Proposition 9.3.** Let \(E_1\) and \(E_2\) be graphs with finitely many vertices such that \((E_1, E_2)\) is in standard form. Suppose \((U_1, V_1) : B_{E_1}^* \to B_{E_2}^*\) is a \(\text{GL}_P\)-equivalence, where \(U_1 \in \text{GL}_P(m, \mathbb{Z})\), \(V_1 \in \text{GL}_P(n, \mathbb{Z})\), \(m = (m_1, \ldots, m_N)\) and \(n = (n_1, \ldots, n_N)\).

If, for some \(i, m_i \neq 0\), \(\det(U_1(i)) = -1\), then there exist graphs \(F_1\) and \(F_2\) with finitely many vertices and there exist \(U_2 \in \text{GL}_P(m', \mathbb{Z})\), \(V_2 \in \text{GL}_P(n', \mathbb{Z})\), where \(m' = (m_1, \ldots, m_{i-1}, m_i + 2, m_{i+1}, \ldots, m_N)\) and \(n' = (n_1, \ldots, n_{i-1}, n_i + 2, n_{i+1}, \ldots, n_N)\), such that

- \(E_k \sim_M F_k, k = 1, 2;\)
- \((F_1, F_2)\) is in standard form;
- \(U_2 B_{F_1}^* V_2 = B_{F_2}^*;\)
- \(\det(U_2(i)) = 1, \det(V_2(i)) = -\det(V_1(i));\) and
- \(\det(U_2(j)) = \det(U_1(j))\) and \(\det(V_2(j)) = \det(V_1(j))\) for all \(j \neq i\).

**Proof.** Write \(B_{E_1}^*\) as \(
\begin{pmatrix}
A_1 & X_1 & Y_1 \\
0 & B_1 & Z_1 \\
0 & 0 & C_1
\end{pmatrix}
\)
and write \(B_{E_2}^*\) as \(
\begin{pmatrix}
A_2 & X_2 & Y_2 \\
0 & B_2 & Z_2 \\
0 & 0 & C_2
\end{pmatrix}
\), where \(B_k = B_{E_k}^*\).

Apply Lemma 9.1 to both \(E_k\)'s to yield graphs \(E_k'\) and matrices \(V_k'\).

Define \(\tilde{U} \in \text{GL}_P(m, \mathbb{Z})\) by

\[
\tilde{U}(r, s) = \begin{cases}
(U_1(i) & 0 & 0) & \text{if } (r, s) = (i, i) \\
0 & 0 & 1 & \text{if } (r, s) = (r, i), r \neq i \\
0 & 1 & 0 & \text{if } (r, s) = (i, s), s \neq i \\
0 & 0 & 0 & \text{otherwise}
\end{cases}
\]
and set $\tilde{V} = \begin{pmatrix} I & 0 & 0 \\ 0 & V' & 0 \\ 0 & 0 & I \end{pmatrix} V \begin{pmatrix} I & 0 & 0 \\ 0 & (V_2')^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}$ where $V$ is the matrix defined by

$$\begin{cases} 
V_2 \{i\} & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{cases} \quad \text{if } (r, s) = (i, i) \\
\begin{cases} 
V_1 \{r, i\} & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{cases} \quad \text{if } (r, s) = (r, i), r \neq i \\
\begin{cases} 
V_2 \{i, s\} \\
0
\end{cases} \quad \text{if } (r, s) = (i, s), s \neq i \\
\begin{cases} 
V_1 \{r, s\}
\end{cases} \quad \text{otherwise}
$$

Note that $\tilde{V} \in \text{GL}_P(n, Z)$.

By Lemma 9.1 we have that

$$\tilde{U}B_{E_1}^* \tilde{V} = B_{E_2}^*.$$

Note that $\det(\tilde{V}\{i\}) = -\det(V_i\{i\})$, $\det(\tilde{U}\{i\}) = 1$, and $\det(U\{j\}) = \det(U_j\{j\})$ and $\det(V\{j\}) = \det(V_i\{j\})$ for all $j \neq i$. Since $E_k$ is in canonical form, we have that for every vertex $v, w \in (E_k)^0_{\text{reg}}$ with $v \geq w$, there exists a path in $E_k$ from $v$ to $w$ through regular vertices in $E_k$. Hence, by Lemma 9.1, for every $v, w \in (E_k)^0_{\text{reg}}$ with $v \geq w$, there exists a path in $E_i$ from $v$ to $w$ through regular vertices in $E_i$. Since $E_k$ is in canonical form and by definition of the outsplicing graph, $E_i$ satisfies (1), (2), and (3) of Theorem 7.3. Furthermore, the fact that $E_k$ is in canonical form implies that all the diagonal blocks of $B_{E_k}^*$ have Smith normal form with at least two 1’s, so it follows from the constructions of Lemma 9.1 that the diagonal blocks of $B_{E_i}^*$ also have this property. Therefore $E_i$ also satisfies (5) of Theorem 7.3. By Remark 7.6, we get a graph $F_i$ in canonical form such that $m_{F_i} = m_{E_i} = m'$, $n_{F_i} = n_{E_i} = n'$, and $E_i \sim_{M'} F_i$. Also, we get an SL$^P$-equivalence $(W_i, Z_i): B_{F_i}^* \rightarrow B_{E_i}^*$.

Set $U_2 = W_2^{-1} \tilde{U} W_1$ and $V_2 = Z_1 \tilde{V} Z_2^{-1}$. Since $W_2^{-1}, W_1 \in \text{SL}_P(m', Z)$ and since $Z_1, Z_2^{-1} \in \text{SL}_P(n', Z)$, we have that $\det(U_2\{i\}) = 1$, $\det(V_2\{i\}) = -\det(V_1\{i\})$, and $\det(V_2\{j\}) = \det(U_1\{j\})$ and $\det(U_2\{j\}) = \det(V_1\{j\})$ for all $j \neq i$. By construction, the pair $(F_1, F_2)$ is in standard form with $B_{F_i}^* \in \mathcal{W}_P(m' \times n', Z)$, $U_2 B_{F_i}^* V_2 = B_{E_i}^*$, and $F_i \sim_{M'} F_i$.

We now use the Cuntz splice to fix potential sign problems on $V$.

**Proposition 9.4.** Let $E_1$ and $E_2$ be graphs with finitely many vertices such that $(E_1, E_2)$ is in standard form. Suppose $(U_1, V_1): B_{E_1}^* \rightarrow B_{E_2}^*$ is a GL$^P$-equivalence, where $U_1 \in \text{GL}_P(m, Z)$ and $V_1 \in \text{GL}_P(n, Z)$, and $m = (m_1, \ldots, m_N)$ and $n = (n_1, \ldots, n_N)$. If, for some $i, m_i \neq 0$, $\det(U_1\{i\}) = 1$ and $\det(V_1\{i\}) = -1$, there exist graphs $F_1$ and $F_2$ with finitely many vertices and there exist $U_2 \in \text{GL}_P(m', Z)$ and $V_2 \in \text{GL}_P(n', Z)$, where $m' = (m_1, \ldots, m_{i-1}, m_i + 2, m_{i+1}, \ldots, m_N)$ and $n' = (n_1, \ldots, n_{i-1}, n_i + 2, n_{i+1}, \ldots, n_N)$ such that

- $E_k \sim_{M'} F_k$, for $k = 1, 2$;
- $(F_1, F_2)$ is in standard form;
- $U_2 B_{F_1}^* V_2 = B_{E_2}^*$;
- $\det(U_2\{i\}) = \det(V_2\{i\}) = 1$; and

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\[ \det(U_2(j)) = \det(U_1(j)) \text{ and } \det(V_2(j)) = \det(V_1(j)) \text{ for all } j \neq i. \]

**Proof.** Write \( B_{E_1}^* \) as \( \begin{pmatrix} A_1 & X_1 & Y_1 \\ 0 & B_1 & Z_1 \\ 0 & 0 & C_1 \end{pmatrix} \) and write \( B_{E_2}^* \) as \( \begin{pmatrix} A_2 & X_2 & Y_2 \\ 0 & B_2 & Z_2 \\ 0 & 0 & C_2 \end{pmatrix} \), where \( B_k = B_{E_k}^* \{ i \} \) and the entries of the last two rows of \( B_k \) are positive integers, and the corresponding vertices of \( E_k \) are regular. Apply Lemma 9.1 and Lemma 9.2 to the graph \( E_2 \) to yield a graph \( E_2' \) and a matrix \( V_2' \). Let \( U_- \) and \( V_- \) be the matrices that one obtain when applying Lemma 9.2 to the graph \( E_1 \).

Set \( \tilde{U} = U \begin{pmatrix} I & 0 & 0 \\ 0 & U_- & 0 \\ 0 & 0 & I \end{pmatrix} \) where \( U \) is the matrix defined by

\[
\begin{align*}
\tilde{U}(r,s) &= \left\{ \begin{array}{ll}
(U_1\{i\}) & \text{if } (r,s) = (i,i) \\
(0 & 0) & \text{if } (r,s) = (i,i), r \neq i \\
(1 & 0) & \text{if } (r,s) = (i,s) \\
(0 & 0) & \text{otherwise}
\end{array} \right.
\end{align*}
\]

Note that \( \tilde{U} \in \text{GL}_r(n', Z) \).

Set \( \tilde{V} = V \begin{pmatrix} I & 0 & 0 \\ 0 & (V_2')^{-1} & 0 \\ 0 & 0 & I \end{pmatrix} \) where \( V \) is the matrix defined by

\[
\begin{align*}
\tilde{V}(r,s) &= \left\{ \begin{array}{ll}
(V_1\{i\}) & \text{if } (r,s) = (i,i) \\
(0 & 0) & \text{if } (r,s) = (i,i), r \neq i \\
(1 & 0) & \text{if } (r,s) = (i,s) \\
(0 & 0) & \text{otherwise}
\end{array} \right.
\end{align*}
\]

Note that \( \tilde{V} \in \text{GL}_r(n', Z) \).

By Lemma 9.1 and Lemma 9.2, we have that

\[ \tilde{U}B_{E_1}^* \tilde{V} = B_{E_2}^*. \]

Note that \( \det(\tilde{U}\{i\}) = 1 \) and \( \det(\tilde{V}\{i\}) = 1 \); moreover, \( \det(\tilde{U}(j)) = \det(U_1(j)) \) and \( \det(\tilde{V}(j)) = \det(V_1(j)) \), for all \( j \neq i \). Since \( E_2 \) is in canonical form, by Lemma 9.1 \( E_2' \) has the property that for every vertex \( v, w \in (E_2')_{\text{reg}} \) with \( v \geq w \), there exists a path in \( E_2' \) from \( v \) to \( w \) through regular vertices in \( E_2' \). It is clear
from the construction of \((E_1)_-\) that for all regular vertices \(v, w\) of \((E_1)_-\) satisfying \(v \geq w\), we have that there exists a path in \((E_1)_-\) from \(v\) to \(w\) through regular vertices of \((E_1)_-\) (since \(E_1\) has this property).

Since \(E_2\) is in canonical form and by the definition of the outsplitting graph, we have that \(E_2\) satisfies \([1], [2]\) and \([3]\) of Theorem 6.8. Similarly, \((E_1)_-\) will satisfying the same properties since \(E_1\) is in canonical form. Furthermore, the canonical form of \(E_k\) and the construction of Lemma [5] implies that the diagonal blocks of \(B_{E_k}^*\) have a Smith normal form with at least two 1’s, so \(E_2\) also satisfies \([5]\) of Theorem 7.3. By Remark 7.5 there exist graphs \(F_1, F_2\) in canonical form such that \(m_{F_1} = m_{(E_1)_-} = m', n_{F_1} = n_{(E_1)_-} = n', F_1 \sim_M (E_1)_-, \) and \(F_2 \sim_M E_2\). Moreover, there exist \(\text{SL}_P\)-equivalences \((W_1, Z_1): B_{E_1}^* \rightarrow B_{(E_1)_-}^*\) and \((W_2, Z_2): B_{E_2}^* \rightarrow B_{E_2}^*\).

Set \(U_2 = W_2^{-1}UW_1\) and \(V_2 = Z_1Z_2^{-1}\). Since \(W_2^{-1}, W_1 \in \text{SL}_P(m', Z)\) and since \(Z_1, Z_2^{-1} \in \text{SL}_P(n', Z)\), we have that \(\det(U_2{[i]} = \det(V_2{[i]}) = 1\) and \(\det(U_2{[j]}) = \det(V_2{[j]}) = \det(V_1{[j]})\) for all \(j \neq i\). By construction, the pair \((F_1, F_2)\) is in standard form with \(B_{F_1}^* \in \text{M}_P(m' \times n', Z)\), \(U_2B_{F_1}^*V_2 = B_{F_2}^*\), and \(F_k \sim_M E_k\).

We now have all we need to modify a \(\text{GL}_P\)-equivalence to an \(\text{SL}_P\)-equivalence.

**Theorem 9.5.** Let \(E_1\) and \(E_2\) be graphs with finitely many vertices such that the pair \((E_1, E_2)\) is in standard form. Suppose \((U, V)\) is a \(\text{GL}_P\)-equivalence from \(B_{E_1}^*\) to \(B_{E_2}^*\) satisfying that \(V{[i]} = 1\) whenever \(n_i = 1\). Then there exist graphs \(F_1\) and \(F_2\) such that \(E_i \sim_M F_i\), the pair \((F_1, F_2)\) is in standard form, and \(B_{F_1}^*\) is \(\text{SL}_P\)-equivalent to \(B_{F_2}^*\).

**Proof.** The theorem follows from an argument similar to the argument in [Res06, Theorem 6.8] with Propositions [9] and [4] in place of [Res06, Lemma 6.7].

Brieﬂy, the idea is that we are given a \(\text{GL}_P\)-equivalence, say \((U, V)\). We go down the diagonal blocks and for each of them use Proposition [9] if necessary to make sure the \(V\) has positive determinant. Then we go down the diagonal blocks again this time using Proposition [4] to ﬁx the determinant of the diagonal blocks of \(V\) when necessary.

10. Generalization of Boyle’s positive factorization method

In [Boy02], Boyle proved several factorization theorems for square matrices. These theorems are the key components to go from \(\text{SL}_P\)-equivalence to ﬂow equivalence. In this section, we prove similar factorization theorems for rectangular matrices. This is our key technical result to go from \(\text{SL}_P\)-equivalence to move equivalence. Although the assumptions might seem restrictive, every unital graph \(C^*\)-algebra is move equivalent to another unital graph \(C^*\)-algebra whose adjacency matrix satisfy the assumptions of the factorization theorem. The proof for rectangular matrices will closely follow the proof in [Boy02] for square matrices.

First we introduce a new equivalence called “positive equivalence” of two matrices in \(\text{M}_P^+(m \times n, Z)\) (see Deﬁnition [10.1]) and show that if \(n_i \neq 0\) for all \(i\), then two matrices in \(\text{M}_P^+(m \times n, Z)\) that are \(\text{SL}_P\)-equivalent are positive equivalent.

**Deﬁnition 10.1.** Deﬁne \(\text{M}_P^+(m \times n, Z)\) to be the set of all \(B \in \text{M}_P(m \times n, Z)\) satisfying the following:

(i) If \(i \geq j\) and \(B{[i, j]}\) is not the empty matrix, then \(B{[i, j]} > 0\).

(ii) If \(B{[i]}\) is not the empty matrix, then \(B{[i]} > 0\), the Smith normal form of \(B{[i]}\) has at least two 1’s, and \(n_i, m_i \geq 3\).

Note that condition (ii) implies that the row rank of every non-empty diagonal block is at least 2. In most of what follows, this will sufﬁce for our purposes, but the stronger condition is needed to apply Theorem [10.7] below.
Let \( B, B' \in \mathfrak{M}_P^\pm(m \times n, \mathbb{Z}) \). An \( SL_P \)-equivalence \((U, V) : B \rightarrow B'\) is said to be a positive equivalence if \( U \) has a factorization of basic elementary matrices in \( SL_P(m, \mathbb{Z}) \) and \( V \) has a factorization of basic elementary matrices in \( SL_P(n, \mathbb{Z}) \) such that when applying these basic elementary matrices at each step we get matrices in \( \mathfrak{M}_P^\pm(m \times n, \mathbb{Z}) \) (recall from [Boy02] that a basic elementary matrix is a matrix that is equal to the identity matrix except for on one offdiagonal entry, where it is either 1 or \(-1\)). We denote a positive equivalence by \( B \xrightarrow{(U, V)} B' \).

Note that every element \( U \in SL_P(n, \mathbb{Z}) \) has a factorization of basic elementary matrices in \( SL_P(n, \mathbb{Z}) \). Therefore, a positive equivalence \((U, V) : B \rightarrow B'\) is an \( SL \)-equivalence that allows one to stay in \( \mathfrak{M}_P^\pm(m \times n, \mathbb{Z}) \) for some factorization of \( U \) and \( V \).

10.1. Factorization: Positive case. In this section, we prove a factorization theorem similar to that of [Boy02, Theorem 5.1] for positive rectangular matrices. The proof is imitating the proof in [Boy02] for square matrices.

**Definition 10.2.** By a signed transposition matrix, we mean a matrix which is the matrix of a transposition, but with one of the offdiagonal ‘s replaced by \(-1\). By a signed permutation matrix we mean a product of signed transposition matrices.

Note that for \( K > 1 \), any \( K \times K \) permutation matrix with determinant 1 is a signed permutation matrix. A \( K \times K \) matrix \( S \) is a signed permutation matrix if and only if \( \det(S) = 1 \) and the matrix \( |S| \) is a permutation matrix (where \( |S|(i, j) := |S(i, j)| \)).

For \( B, B' \in \mathfrak{M}^+(m \times n, \mathbb{Z}) \), we say an equivalence \((U, V) : B \rightarrow B'\) is a positive equivalence through \( \mathfrak{M}^+(m \times n, \mathbb{Z}) \) if it can be given as a chain of positive elementary equivalences

\[
B = B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow B_k = B'
\]

in which every \( B_i \in \mathfrak{M}^+(m \times n, \mathbb{Z}) \) (recall from [Boy02] that an equivalence \((U, V)\) is an elementary equivalence if one of \( U \) and \( V \) is a basic elementary matrix and the other is the identity matrix).

Investigating the proof of [Boy02, Lemma 5.3 and Lemma 5.4] one can see the proof also hold for rectangular matrices. Thus, we have the following lemmas.

**Lemma 10.3** (cf. [Boy02, Lemma 5.3]). Suppose \( B \in \mathfrak{M}^+(m \times n, \mathbb{Z}) \), \( E \) is a basic elementary matrix with nonzero offdiagonal entry \( E(i, j) \), and the \( i \)-th row of \( EB \) is not the zero row. Then there exists \( Q \in SL(n, \mathbb{Z}) \) that is a product of nonnegative basic elementary matrices and there exists a signed permutation matrix \( S \in SL(m, \mathbb{Z}) \) such that \((SE, Q) : B \rightarrow SEBQ \) is a positive equivalence through \( \mathfrak{M}^+(m \times n, \mathbb{Z}) \).

**Lemma 10.4** (cf. [Boy02, Lemma 5.4]). Let \( B \) be an element of \( \mathfrak{M}(K_1 \times K_2, \mathbb{Z}) \) for \( K_1, K_2 \geq 3 \) such that the row rank of \( B \) is at least 2. Suppose \( U \in SL(K_1, \mathbb{Z}) \) such that no row of \( B \) and \( UB \) is the zero row. Then \( U \) is the product of elementary matrices \( U = E_k \cdots E_1 \) such that for \( 1 \leq j \leq k \) the matrix \( E_j E_{j-1} \cdots E_1 B \) has no zero rows.

The following lemma is inspired by the reduction step in the proof of [Boy02, Lemma 5.5]. We give the entire proof for the convenience of the reader.

**Lemma 10.5.** Let \( B \in \mathfrak{M}^+(K_1 \times K_2, \mathbb{Z}) \) with \( K_1, K_2 \geq 3 \). Suppose the row rank of \( B \) is at least 2 and there exists \( U \in SL(K_1, \mathbb{Z}) \) such that \( UB > 0 \). Then the equivalence \((U, I_{K_2}) : B \rightarrow UB \) is a positive equivalence through \( \mathfrak{M}^+(K_1 \times K_2, \mathbb{Z}) \).
Proof. By Lemma [10.4] we can write $U$ as a product of basic elementary matrices $U = E_kE_{k-1} \cdots E_1$, such that for $1 \leq j \leq k$, the matrix $E_j \cdots E_1B$ has no zero row. By Lemma [10.9] given the pair $(E_1, \hat{B})$, there is a nonnegative $Q_1$ which is a product of nonnegative basic elementary matrices and a signed permutation $\hat{S}_1$ such that $(S_1E_1, Q_1): B \rightarrow S_1E_1BQ_1$ is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$. Note that

$$UBQ_1 = S_{i-1}^{-1}[S_1E_kS_{i-1}^{-1}] \cdots [S_1E_2S_{i-1}^{-1}][S_1E_1]BQ_1.$$  

Now, for $2 \leq j \leq k$, the matrix $S_1E_jS_{i-1}^{-1}$ is again a basic elementary matrix $E_j'$. Since $E_1' \cdots E_2'(S_1E_1BQ_1) = S_1E_1' \cdots E_2E_1BQ_1$ for $2 \leq j \leq k$ and since $E_1' \cdots E_2E_1BQ_1$ has no zero rows, and $S_1$ is a signed permutation, we have that $E_1' \cdots E_2'(S_1E_1BQ_1)$ has no zero rows for all $2 \leq j \leq k$.

Using Lemma [10.3] for the pair $(S_1E_2S_{i-1}^{-1}, S_1E_1BQ_1)$, we get a signed permutation matrix $S_2$ and a nonnegative $Q_2$ which is a product of nonnegative basic elementary matrices such that

$$(S_2[S_1E_2S_{i-1}^{-1}], Q_2): S_1E_1BQ_1 \rightarrow S_2[S_1E_2S_{i-1}^{-1}]S_1E_1BQ_1Q_2$$

is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$. Thus, we get a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$

$$([S_2S_1E_2S_{i-1}^{-1}][S_1E_1], Q_1Q_2): B \rightarrow S_2S_1E_2E_1BQ_1Q_2$$

and we observe that

$$UBQ_1Q_2 = S_{i-1}^{-1}S_{i-1}^{-1}S_{i-1}^{-1} \cdots [S_2S_1E_2S_{i-1}^{-1}][S_1E_1]BQ_1Q_2.$$  

Continue this, to obtain a signed permutation matrix $S = S_k \cdots S_1$ and a nonnegative matrix $Q = Q_k \cdots Q_1$ that is a product of nonnegative basic elementary matrices such that

$$UBQ = S^{-1}S_{k-1}^{-1}S_{k-1}^{-1} \cdots [S_2S_1E_2S_{i-1}^{-1}][S_1E_1]BQ = S^{-1}(SUQ)$$

and $(SU, Q): B \rightarrow SUQB$ is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$.

We claim that the equivalence $(S, I_{K_2}): UBQ \rightarrow SUQB$ is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$. Since $S$ is a product of signed transposition matrices, it may be described as a permutation matrix in which some rows have been multiplied by $-1$. Since $UBQ$ and $SUQB$ are strictly positive, it must be that $S$ is a permutation matrix. Also, $\det(S) = 1$, so if $S \neq I_{K_1}$, then $S$ is a permutation matrix which is a product of 3-cycles. So it is enough to realize the positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$ in the case that $S$ is the matrix of a 3-cycle. For this we write the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

as the following product $C_0C_1C_2C_3C_4C_5$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

For $0 \leq i \leq 5$, the matrix $C_iC_{i+1} \cdots C_5$ is nonnegative and has no zero row. Therefore, the equivalence $(C, I): D \rightarrow CD$ is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$ whenever $D \in \mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$. Therefore, $(S, I_{K_2}): UBQ \rightarrow SUQB$ is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$ proving the claim. Therefore, $(S^{-1}, I_{K_2}): SUQB \rightarrow UBQ$ is a positive equivalence through $\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})$. Since $Q$ is the product of nonnegative basic elementary matrices and
Thus, the equivalence \((I_{K_1}, Q) : UB \rightarrow UBQ\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\). Therefore, after replacing equivalences through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\), it is enough to show that the equivalence \((U, I_{K_2}) : B \rightarrow UB\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\).

The proof of the next lemma is similar to the proof of [Boy02, Lemma 5.5]. Since there are some differences between the two proofs we provide the entire argument.

Lemma 10.6 (cf. [Boy02, Lemma 5.5]). Let \(B\) and \(B'\) be elements of \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\) with \(K_1, K_2 \geq 3\), and the rank of \(B\) and \(B'\) at least 2. Suppose \(U \in \text{SL}(K_1, \mathbb{Z})\) and \(W \in \text{SL}(K_2, \mathbb{Z})\) such that \(UB\) has at least one strictly positive entry and \(UB = B'W\). Then the equivalence \((U, W^{-1}) : B \rightarrow B'\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\).

Proof. We will first reduce to the case that \(UB > 0\). By assumption \((UB)(i, j) > 0\) for some \((i, j)\). We can repeatedly add column \(j\) to other columns of until row \(i\) of \(UB\) has all entries strictly positive. This corresponds to multiplying from the right by a nonnegative matrix \(Q\) in \(\text{SL}(K_2, \mathbb{Z})\), where \(Q\) is the product of nonnegative basic elementary matrices, giving \(UBQ = B'WQ\). Then we can repeatedly add row \(i\) of \(UBQ\) to other rows until all entries are positive. This corresponds to multiplying from the left by a nonnegative matrix \(P\) in \(\text{SL}(K_1, \mathbb{Z})\), where \(P\) is the product of nonnegative basic elementary matrices, giving \((PU)(BQ) = (PB')(WQ) > 0\). We also have positive equivalences through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\) given by

\[(I, Q) : B \rightarrow BQ \quad \text{and} \quad (P, I) : B' \rightarrow PB'.\]

Note that the equivalence \((U, W^{-1}) : B \rightarrow B'\) is the composition of equivalences, \((I, Q) : B \rightarrow BQ\) followed by \((P, I) : PB' \rightarrow B'\). Since \((I, Q) : B \rightarrow BQ\) and \((P, I) : PB' \rightarrow B'\) are positive equivalences through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\), it is enough to show that the equivalence \((PU, (WQ)^{-1}) : BQ \rightarrow PB'\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\). Therefore, after replacing \((U, B', B', W)\) with \((PU, BQ, PB', WQ)\), we may assume without loss of generality that \(UB > 0\).

By Lemma 10.5, the equivalence \((U, I_{K_2}) : B \rightarrow UB\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\). Therefore, by Lemma 10.5, \((W)^T, I_{K_1}) : (B')^T \rightarrow W^T(B')^T\) is a positive equivalence through \(\mathfrak{M}^+(K_2 \times K_1, \mathbb{Z})\). Thus, the equivalence \((I_{K_1}, W^{-1}) : B'W \rightarrow B'\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\). Since the equivalence \((U, W^{-1}) : B \rightarrow B'\) is the composition of positive equivalences: \((U, I_{K_2}) : B \rightarrow UB\) followed by \((I_{K_1}, W^{-1}) : B'W \rightarrow B'\), the equivalence \((U, W^{-1}) : B \rightarrow B'\) a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\).

Theorem 10.7 (cf. [Boy02, Theorem 5.1]). Let \(K_1, K_2 \geq 3\) and let \(B \in \mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\). Suppose \(U \in \text{SL}(K_1, \mathbb{Z})\) and \(V \in \text{SL}(K_2, \mathbb{Z})\) such that \(UBV \in \mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\) and suppose that \(X \in \text{SL}(K_1, \mathbb{Z})\) and \(Y \in \text{SL}(K_2, \mathbb{Z})\) such that

\[
XB Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & F \end{pmatrix}.
\]
Then the equivalence \((U, V): B \to UBV\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\).

**Proof.** Note that for any \(H \in \text{SL}(2, \mathbb{Z})\), the \(K_1 \times K_1\) matrix \(G_{H,1}\) and \(K_2 \times K_2\) matrix \(G_{H,2}\) given by

\[
G_{H,1} = \begin{pmatrix} H & 0 \\ 0 & I_{K_1-2} \end{pmatrix} \quad \text{and} \quad G_{H,2} = \begin{pmatrix} H & 0 \\ 0 & I_{K_2-2} \end{pmatrix},
\]

give a self-equivalence \((X^{-1}G_{H,1}X, YG_{H,2}^{-1}Y^{-1}): B \to B\).

For a matrix \(Q\), we let \(Q(12; *)\) denote the submatrix consisting of the first two rows. Since \((XYB)(12; *)\) has rank 2 and \(Y\) is invertible, we have that \((X)(12; *)\) has rank two. Therefore, there exists \(H' \in \text{SL}(2, \mathbb{Z})\) such that the first row \(r = (r_1, \ldots, r_{K_2})\) of \(H'([XB](12; *))\) has both a positive entry and a negative entry.

Let \(c = \begin{pmatrix} c_1 \\ \vdots \\ c_{K_1} \end{pmatrix}\) denote the first column of \(X^{-1}\), and note that it is nonzero. Since \(cr\) is the \(K_1 \times K_2\) matrix with \((i, j)\) entry equal to \(c_ir_j\), we have that \(cr\) has a positive and a negative entry. For each \(m \in \mathbb{N}\), set \(H_m = \begin{pmatrix} m & -1 \\ 1 & 0 \end{pmatrix}\). Choose \(m\) large enough such that the entries of the two matrices \(X^{-1}G_{H_m,1}XB\) and \(mcr\) will have the same sign wherever the entries of \(mcr\) are nonzero. In particular, \(X^{-1}G_{H_m,1}XB\) will have a positive entry. By Lemma 10.6, \((X^{-1}G_{H_m,1}X, YG_{H_m,2}^{-1}Y^{-1}): B \to B\) gives a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\).

Similarly for large enough \(m\), the entries of \(UX^{-1}G_{H_m,1}XB\) will agree in sign with the entries \(UCr\) whenever the entries of the latter matrix are nonzero. Since \(U\) is invertible, the matrix \(UCr\) is nonzero, and thus contains positive and negative entries, because \(r\) does. Therefore, \(UX^{-1}G_{H_m,1}XB\) contains a positive entry. By Lemma 10.6,

\[(UX^{-1}G_{H_m,1}X, YG_{H_m,2}^{-1}Y^{-1}V): B \to B'\]
gives a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\) with \(B' = UBV\). Hence, the equivalence \((U, V): B \to B'\) is a positive equivalence through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\) since it is the composition of positive equivalences through \(\mathfrak{M}^+(K_1 \times K_2, \mathbb{Z})\):

\[(X^{-1}G_{H_m,1}X, YG_{H_m,2}^{-1}Y^{-1}V): B \to B\]

followed by

\[(UX^{-1}G_{H_m,1}X, YG_{H_m,2}^{-1}Y^{-1}V): B \to B'\]

\(\square\)

### 10.2. Factorization: General case.
We now use the results of the previous section to prove a factorization for general \(B, B' \in \mathfrak{M}^+_P(\mathfrak{m} \times \mathfrak{n}, \mathbb{Z})\) with \(n_i \neq 0\) that are \(\text{SL}_P\)-equivalent. Again, many of the arguments follow the arguments of Boyle in [Boy02].

**Lemma 10.8** (cf. [Boy02, Lemma 4.6]). Let \(B, B' \in \mathfrak{M}^+_P(\mathfrak{m} \times \mathfrak{n}, \mathbb{Z})\) with \(n_i \neq 0\) for all \(i\). If \((U, V): B \to B'\) is an \(\text{SL}_P\)-equivalence such that \(U\{i\}\) and \(V\{j\}\) are the identity matrices of the appropriate size whenever they are not the empty matrix, then \((U, V): B \to B'\) is a positive equivalence.

**Proof.** We will first find \(Q\) in \(\text{SL}_P(\mathfrak{n}, \mathbb{Z})\) which is a product of nonnegative basic elementary matrices such that \((U, Q): B \to UBQ\) is a positive equivalence. We may assume that \(U\) is not the identity matrix. Factor \(U = U_n \cdots U_1\) where for each \(U_i\) there is an associated pair \((i_t, j_t)\) such that the following hold

- \(U_t = I\) except in the block \(U_t\{i_t, j_t\}\), where it is nonzero
- if \(s \neq t\), then \((i_s, j_s) \neq (i_t, j_t)\).
Factor $U_1 = U^-_1 U^+_1$, where $U^-_1$ and $U^+_1$ are equal to $I$ outside the block $\{i_1, j_1\}$, $U^-_1 \{i_1, j_1\}$ is the nonpositive part of $U_1 \{i_1, j_1\}$ and $U^+_1 \{i_1, j_1\}$ is the nonnegative part of $U_1 \{i_1, j_1\}$. Note that $U^+_1$ is a product of nonnegative basic elementary matrices in $\text{SL}_P(m, \mathbb{Z})$ and $U^-_1$ is a product of nonpositive basic elementary matrices in $\text{SL}_P(m, \mathbb{Z})$. It is now clear that $(U^+_1, I): B \to U^+_1 B$ is a positive equivalence.

Now, note that $U^-_1 U^+_1 B = U^+_1 B$ outside the blocks $\{i_1, k\}$ such that $i_1 < j_1 \leq k$. Also note that $m_{i_1} \neq 0$ (since $U_1 \neq I$). Since $n_i \neq 0$ for all $i$, we have that $B \{i_1\}$ is not the empty matrix. Therefore, $B \{i_1\} > 0$ since $B \in \mathbb{M}_n^P(m \times m, \mathbb{Z})$. Hence, $(U^+_1 B) \{i_1\} > 0$ since $(U^+_1, I): B \to U^+_1 B$ is a positive equivalence. We can now add columns of $(U^+_1 B) \{i_1\}$ to columns of $(U^+_1 B) \{i_1, k\}$ for all $i_1 < j_1 \leq k$ enough times to obtain a $Q_1$ which is a product of nonnegative basic elementary matrices in $\text{SL}_P(n, \mathbb{Z})$ such that $(U^-_1, Q_1): U^-_1 B \to U^-_1 U^+_1 B Q_1$ is a positive equivalence. Since $(U_1, Q_1): B \to U_1 B$ is the composition of positive equivalences

$$B \xrightarrow{(U^+_1, I)} U^+_1 B \xrightarrow{(U^-_1, Q_1)} U_1 B Q_1,$$

we get that the equivalence $(U_1, Q_1): B \to U_1 B Q_1$ is a positive equivalence.

Repeat the process for the matrices $U_1 B Q_1$ and $U_2 U_1 B Q_1$, we get $Q_2$ which is the product of nonnegative elementary matrices in $\text{SL}_P(m, \mathbb{Z})$ such that the equivalence $(U_2, Q_2): U_2 B Q_1 \to U_2 U_1 B Q_1 Q_2$ is a positive equivalence. We continue this process to get $Q_i$ that is the product of nonnegative elementary matrices in $\text{SL}_P(n, \mathbb{Z})$ for $1 \leq i \leq n$ such that $(U, Q): B \to U B Q$ is a positive equivalence, where $Q = Q_1 \cdots Q_n$.

We now show that there exists $P$ that is a product of nonnegative basic elementary matrices in $\text{SL}_P(m, \mathbb{Z})$ such that $(P, V^{-1}): B' \to P B' V^{-1}$ is a positive equivalence. Throughout the rest of the proof, if $M \in \mathbb{M}_n^P(m \times m, \mathbb{Z})$, then $M \{1, 2, \ldots, i\}$ will denote the block matrix whose $\{s, r\}$ block is $M \{s, r\}$ for all $1 \leq s, r \leq i$. First note that there are matrices $V_2, \ldots, V_N$ in $\text{SL}_P(n, \mathbb{Z})$ such that $V^{-1} = V_2 V_3 \cdots V_N$, each $V_i$ is the identity matrix except for the blocks $V_i \{l, i\}$, and

$$V_2 \cdots V_i = \begin{pmatrix} V^{-1} \{1, \ldots, i\} & 0 \\ 0 & I \end{pmatrix}.$$ 

Let $V^-_i$ be the matrix in $\text{SL}_P(n, \mathbb{Z})$ that is the identity matrix except for the blocks $V_i \{l, i\}$ and $V^-_i \{l, i\}$ is the nonpositive part of $V_i \{l, i\}$ and let $V^+_i$ be the matrix in $\text{SL}_P(n, \mathbb{Z})$ that is the identity matrix except for the blocks $V_i \{l, i\}$ and $V^+_i \{l, i\}$ is the nonnegative part of $V_i \{l, i\}$. Note that $V^+_i V^-_i$ is equal to the identity matrix except for the blocks $V_i \{l, i\}$ and $(V^+_i V^-_i) \{l, i\} = V^+_i \{l, i\} + V^-_i \{l, i\} = V_i \{l, i\}$. Therefore, $V_i = V^+_i V^-_i$.

We will inductively construct matrices $P_2, P_3, \ldots, P_N$ in $\text{SL}_P(m, \mathbb{Z})$ such that each $P_i$ is the product of nonnegative basic matrices such that each $P_i$ is the identity outside of the blocks $\{i\}$ for $1 < i$ and for each $2 \leq i \leq N$, we have that $(P_i, V_i): P_{i-1} P_i B' V_2 \cdots V_{i-1} \to P_i P_{i-1} B' V_2 \cdots V_i$ is a positive equivalence. Note that if we have constructed $P_i$, then the composition of these positive equivalences gives a positive equivalence $(P, V^{-1}): B' \to P B' V^{-1}$, where $P = P_k \cdots P_2$. Thus, the lemma holds.

We now prove the claim. We first construct $P_2$. Note that if $l$ is not a predecessor of $2$, then $V^+_2 = V^-_2 = I$. Therefore, $(I, V_2): B' \to B' V_2$ is a positive equivalence. Suppose $1 \leq 2$. Suppose $m_1 = 0$. Then $B' V^+_2 V^-_2 = B' V^-_2 = B'$ which implies that $(I, V_2): B' V^+_2 \to B' V_2$ is a positive equivalence. So, $(I, V_2): B' \to B' V_2$ is a positive equivalence since it is the composition of the positive equivalences $(I, V^+_2)$ and $(I, V^-_2)$. Suppose $m_1 \neq 0$. In this situation, we have two cases, $m_2 \neq 0$ and $m_2 = 0$. 
Suppose $m_2 \neq 0$. Note that $B'V_2^+V_2^-$ is equal to $B'$ except for the $\{1,2\}$ block. We have that $B'V_2^+ > 0$ since $B' > 0$ and $(I, V_2^+) : B' \to B'V_2^+$ is a positive equivalence. Hence, we may add rows of $(B'V_2^+\{2\})$ to rows of $(B'V_2^-\{1\})$ to get a matrix $P_2$ in $\text{SL}_P(\mathbf{m}, \mathbb{Z})$ that is the product of nonnegative basic matrices and is the identity outside of the block $\{1,2\}$ such that $(P_2, V_2^-) : B'V_2^+ \to B'V_2^-$ is a positive equivalence. Composing the positive equivalences $(I, V_2^+)$ and $(P_2, V_2^-)$, we get a positive equivalence $(P_2, V_2) : B' \to P_2BV_2$.

Suppose $m_2 = 0$. Then

\[
(B'V_2\{1,2\}) = B'\{1\}V_2\{1,2\} + B'\{1,2\}V_2\{2\}
= B'\{1\}V_2\{1,2\} + B'\{2\},
\]

since $V_2\{1,2\} = V^{-1}\{1,2\}$ and $V_2\{2\} = V^{-1}\{2\} = I$. Therefore,

\[
(B'V_2^+V_2^-)\{1,2\} = (B'V_2^-)\{1,2\} = (UB)\{1,2\}
= U\{1\}B\{1,2\} + U\{1,2\}B\{2\}
= B\{1,2\} > 0,
\]

since $U\{1,2\}$ is the empty matrix and $U\{1\} = I$. Therefore $(I, V_2^-) : B'V_2^+ \to B'V_2$ is a positive equivalence and by composing the positive equivalences $(I, V_2^+)$ and $(I, V_2^-)$, we get a positive equivalence $(I, V_2) : B' \to B'V_2$.

So, in all cases, we have found a matrix $P_2$ in $\text{SL}_P(\mathbf{m}, \mathbb{Z})$ that is the product of nonnegative basic elementary matrices and is the identity outside of the block $\{1,2\}$ such that $(P_2, V_2) : B' \to P_2BV_2$ is a positive equivalence.

Let $2 \leq n \leq N - 1$ and suppose we have constructed $P_2, P_3, \ldots, P_n$ in $\text{SL}_P(\mathbf{m}, \mathbb{Z})$ such that each $P_i$ is the product of nonnegative basic matrices and $P_i$ is the identity outside of the blocks $\{l, i\}$ with $l < i$ and for each $2 \leq i \leq n$, we have that $(P_i, V_i) : P_{i-1} \cdots P_2P'V_2 \cdots V_{i-1} \to P_i \cdots P_2P'V_2 \cdots V_i$ is a positive equivalence.

To simplify the notation, we set $B'_n = P_i \cdots P_2P'V_2 \cdots V_i$. Since $B'_n > 0$, we get a positive equivalence $(I, V_{n+1}^+) : B'_n \to B'V_{n+1}$. Note that $B'V_{n+1} V_n$ is equal to $B_n$ except for the blocks $\{i, n+1\}$ with $i \leq n + 1$.

Suppose $m_{n+1} \neq 0$. Then $(B'_n V_{n+1})\{n+1\} > 0$. Hence, we may add rows of $(B'_n V_{n+1}^+)\{i+1\}$ to rows of $(B'_n V_{n+1}^-)\{i, n+1\}$ for all $i < n + 1$, to obtain a matrix $P_{n+1}$ in $\text{SL}_P(\mathbf{m}, \mathbb{Z})$ which is the product of nonnegative basic matrices and is the identity outside of the blocks $\{i, n+1\}$ for $i < n + 1$ such that $(P_{n+1}, V_{n+1}^-) : B'_n V_{n+1}^- \to P_{n+1}B'V_{n+1}$ is a positive equivalence. Composing the positive equivalences $(I, V_{n+1}^+)$ and $(P_{n+1}, V_{n+1}^-)$, we get that $(P_{n+1}, V_{n+1}^-) : B' \to P_{n+1}B'V_{n+1}$ is a positive equivalence.

Suppose $m_{n+1} = 0$. Let $I = \{i_0, \ldots, i_l\}$ be the set of elements $i_s \in \mathcal{P}$ that satisfy $i_s \leq n + 1$, $m_{i_s} \neq 0$, and if $i_s < l \leq n + 1$, then $m_l = 0$. Note that for all distinct $i_s, i_t \in I$, $i_s$ is not a predecessor of $i_t$. Note that if $I = \emptyset$, then $B'_n V_{n+1}^+ V_{n+1}^- = B'_n V_{n+1}^- = B'_n$. This would imply that $(I, V_{n+1}^-) : B'_n V_{n+1}^+ \to B'_n V_{n+1}^-$ is a positive equivalence and hence $(I, V_{n+1}) : B'_n \to B'_n V_{n+1}^-$ is a positive equivalence.

Suppose $I \neq \emptyset$. Note that for each $i_s \in I$,

\[
(B'_n V_{n+1}^+)\{i_s, n+1\} = \sum_{i_s \leq l \leq n+1} (P_n \cdots P_2B')\{i_s, l\}V_2 \cdots V_n V_{n+1}\{l, n+1\}
= \sum_{i_s \leq l \leq n+1} (P_n \cdots P_2B')\{i_s, l\}V^{-1}\{l, n+1\}
= ((P_n \cdots P_2B')V^{-1})\{i_s, n+1\}
\]
since $(V_2 \ldots V_n V_{n+1})\{(1, \ldots n + 1)\} = V^{-1}\{(1, \ldots n + 1)\}$. Since $P_n \cdots P_2 UB = P_n \cdots P_2 B' V^{-1}$,

$$((P_n \cdots P_2 B') V^{-1})\{i_s, n + 1\} = ((P_n \cdots P_2 U) B)\{i_s, n + 1\} = \sum_{i_s \leq l \leq n + 1} (P_n \cdots P_2 U)\{i_s, l\} B\{l, n + 1\}.$$  

Using the fact that $m_l = 0$ for all $i_s < l \leq n + 1$ and $(P_n \cdots P_2 U)\{i_s\} = I$, we get that

$$(B'_n V_{n+1})\{i_s, n + 1\} = B\{i_s, n + 1\}.$$

Moreover, $(B'_n V_{n+1})\{i_s, n + 1\} = B\{i_s, n + 1\} > 0$ because $m_{i_s} \neq 0$ and $B \in \mathcal{M}_n^p(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$.

For each $l < n + 1$, there exists an $s$ such that $l \leq i_s$. Recall that $(B'_n V_{n+1})\{i_s, n + 1\} > 0$, so if $l < i_s$, we may add rows of $(B'_n V_{n+1})\{i_s, n + 1\}$ to rows of $(B'_n V_{n+1})\{l, n + 1\}$, to get a matrix $P_{n+1}$ in $\text{SL}_p(\mathbf{m}, \mathbb{Z})$ that is the product of nonnegative basic elementary matrices and is the identity outside of the block $\{l, n + 1\}$ such that $(P_{n+1} B'_n V_{n+1})\{l, n + 1\} > 0$. Doing this for all $l < n + 1$, we get a matrix $P_{n+1}$ in $\text{SL}_p(\mathbf{m}, \mathbb{Z})$ that is the product of nonnegative basic elementary matrices and is the identity outside of the blocks $\{l, n + 1\}$ for $l < n + 1$ such that $(P_{n+1} V_{n+1}^-): B'_n V_{n+1}^+ \to P_{n+1} B'_n V_{n+1}$ is a positive equivalence. Composing the positive equivalences $(I, V_{n+1}^-)$ and $(P_{n+1}, V_{n+1})$, we get that $(P_{n+1}, V_{n+1})$: $B'_n \to P_{n+1} B'_n V_{n+1}$ is a positive equivalence.

In all cases, we get a matrix $P_{n+1}$ in $\text{SL}_p(\mathbf{m}, \mathbb{Z})$ that is the product of nonnegative basic elementary matrices and is the identity outside of the blocks $\{l, n + 1\}$ for $l < n + 1$ such that $(P_{n+1}, V_{n+1})$: $B'_n \to P_{n+1} B'_n V_{n+1}$ is a positive equivalence. The claim now follows by induction. □

The next lemma allows us to reduce the general case to the case that the diagonal blocks $U\{i\}$ and $V\{j\}$ are the identity matrices of the appropriate sizes when they are not the empty matrices. This will allow us to use Lemma [10.8] to get the desired positive equivalence.

**Lemma 10.9** ([Boy02] Lemma 4.9). Let $B, B' \in \mathcal{M}_m^p(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$ with $m_i \neq 0$ for all $l$. Fix $i$ with $m_i \neq 0$.

1. Suppose $E$ is a basic elementary matrix in $\text{SL}_p(\mathbf{m}, \mathbb{Z})$ such that $E\{j, k\} = I\{j, k\}$ when $(j, k) \neq (i, i)$ and

$$(E\{i\}, I): B\{i\} \to B'\{i\}$$

is a positive equivalence. Then there exists $V \in \text{SL}_p(\mathbf{n}, \mathbb{Z})$ which is the product of nonnegative basic elementary matrices in $\text{SL}_p(\mathbf{n}, \mathbb{Z})$ such that $V\{k\} = I$ for all $k$ and

$$(E, V): B \to EBV$$

is a positive equivalence.

2. Suppose $E$ is a basic elementary matrix in $\text{SL}_p(\mathbf{n}, \mathbb{Z})$ such that $E\{j, k\} = I\{j, k\}$ when $(j, k) \neq (i, i)$ and

$$(I, E\{i\}): B\{i\} \to B'\{i\}$$

is a positive equivalence. Then there exists $U \in \text{SL}_p(\mathbf{n}, \mathbb{Z})$ which is the product of nonnegative basic elementary matrices in $\text{SL}_p(\mathbf{n}, \mathbb{Z})$ such that $U\{k\} = I$ for all $k$ and

$$(U, E): B \to UB' E$$

is a positive equivalence.
Proof. We prove (1). The proof of (2) is similar. Let $E(s,t)$ be the nonzero offdiagonal entry of $E$. If $E(s,t) = 1$, then set $V = I$. Suppose $E(s,t) = -1$. So, $E$ acts from the left to subtract row $t$ from row $s$. Since $m_i \neq 0$ and $$(E\{i\}, I) : B\{i\} \rightarrow B'\{i\},$$
is a positive equivalence, we have that $(EB)\{i\} > 0$. Thus, there exists $r$ an index for a column through the $\{i, i\}$ block such that $B'(s, r) > B'(t, r)$. Let $V$ be the matrix in $\text{SL}_P(n, \mathbb{Z})$ which acts from the right to add column $r$ to column $q$, $M$ times, for every $q$ indexing a column through an $\{i, j\}$ block for which $i < j$. Choosing $M$ large enough, we have that $(E, I) : BV \rightarrow EB'V$ is a positive equivalence. Therefore, $(E, V) : B \rightarrow EBV$ is a positive equivalence since it is the composition of two positive equivalences: $(I, V) : B \rightarrow BV$ followed by $(E, I) : BV \rightarrow EB'V$. □

We are now ready to prove the main result of this section. This result will be used to show that if the adjacency matrices of $E$ and $F$ are $\text{SL}_P$-equivalent, then $E$ is move equivalent to $F$. Consequently, $C^*(E)$ is Morita equivalent to $C^*(F)$.

**Theorem 10.10** (Boy02, Theorem 4.4). Let $B, B' \in \mathcal{M}_P^n(m \times n, \mathbb{Z})$ with $i_i \neq 0$ for all $i$. Suppose there exist $U \in \text{SL}_P(m, \mathbb{Z})$ and $V \in \text{SL}_P(n, \mathbb{Z})$ such that $UBV = B'$. Then $(U, V) : B \rightarrow B'$ is a positive equivalence.

**Proof.** By Theorem 10.7 for each $i$ with $m_i \neq 0$, we have that $(U\{i\}, V\{i\}) : B\{i\} \rightarrow B'\{i\}$ is a positive equivalence since by (ii) of Definition 10.1 a $2 \times 2$-submatrix can be extracted as stipulated. So, we may find a string of elementary equivalences say $(E_1, F_1), \ldots, (E_t, F_t)$, with every $E_t\{i, j\} = I$, $F_t\{i, j\} = I$ unless $i = j$ with $m_i \neq 0$, which accomplishes the elementary positive equivalences decomposition inside the diagonal blocks. By Lemma 10.9, we may find $(U_1, V_1), \ldots, (U_t, V_t)$ such that $U_1 \in \text{SL}_P(m, \mathbb{Z})$, $V_1 \in \text{SL}_P(n, \mathbb{Z})$, $U_n\{k\} = I$, $V_n\{k\} = I$, and such that we have the following positive equivalences

$$B_{(U_1, F_1)} \cdot (E_t, V_t) \cdot \cdots \cdot (E_2, V_2) \cdot (E_1, V_1) \rightarrow B''.$$ 

Let $X = E_t U_t \cdots E_2 U_2 E_1 U_1$ and $Y = F_t V_t \cdots F_2 V_2 F_1 V_1$. Then for all $i$, we have that $X\{i\} = U\{i\}$ and $Y\{i\} = V\{i\}$. Therefore, $(UX^{-1})\{i\} = I$ and $(Y^{-1}V)\{i\} = I$ for all $i$. Then by Lemma 10.8

$$B'' \xrightarrow{(UX^{-1}, Y^{-1}V)} B'$$
is a positive equivalence. Thus, $(U, V) : B \rightarrow B'$ is a positive equivalence since it is the composition of two positive equivalences

$$B \xrightarrow{(X, Y)} B'' \xrightarrow{(UX^{-1}, Y^{-1}V)} B'.$$

□

11. Putting it all together/Proof of main theorem

**Theorem 11.1.** Let $E_1$ and $E_2$ be graphs with finitely many vertices satisfying Condition (K) and assume that $\text{FK}_{p}^{\ast}(C^*(E_1)) \cong \text{FK}_{p}^{\ast}(C^*(E_2))$. Let $F_1$ and $F_2$ be chosen according to Proposition 7.9. Then there exists a $\text{GL}_P$-equivalence $(U, V)$ from $B_{F_1}^{\ast}$ to $B_{F_2}^{\ast}$ that satisfies that $V\{i\}$ is the identity matrix whenever $n_i = 1$.

**Proof.** As usual, we define $\mathcal{P}$, $\mathbf{m}$ and $\mathbf{n}$ according to the matrices $B_{F_1}^{\ast}$ to $B_{F_2}^{\ast}$ so that it reflects the ideal structure of the associated $C^*$-algebras. Here, $B_{F_1}^{\ast}, B_{F_2}^{\ast} \in \mathcal{M}_P(m \times n, \mathbb{Z})$. We let $\mathcal{P}^\top$ denote the set $\mathcal{P}$ with the opposite order defined by $i \preceq j \in \mathcal{P}^\top$ if and only if $N + 1 - j \preceq N + 1 - i$ in $\mathcal{P}$, for $i = 1, 2, \ldots, N$. Moreover, we let $m_1 = (m_1, \ldots, m_2, m_1)$ and $n_1 = (n_1, \ldots, n_2, n_1)$, we let $m = m_1 + m_2 + \cdots + m_N$ and $n = n_1 + n_2 + \cdots + n_N$, and we let $J_m$ and
$J_n$ be the $m \times m$ respectively $n \times n$ permutation matrix that reverses the order. Then $J_n(B_{F_1}^* \ T \ J_m, J_n(B_{F_2}^* \ T \ J_m \in \mathcal{M}_P(n^T \times m^T, \mathbb{Z})$. So in a similar way as in the proof of [Res06 Proposition 8.3], where we use [CET12 Theorem 4.1 and Remark 4.2] in the place of [Res06 Proposition 3.4], we see that this ordered filtered $K$-theory isomorphism induces a $K$-web isomorphism from $K(J_n(B_{F_1}^* \ T \ J_m)$ to $K(J_n(B_{F_2}^* \ T \ J_m)$. When $n_1 = 1$, positivity implies that the isomorphism from $\text{cok}(J_n(B_{F_1}^* \ T \ J_m)$ to $\text{cok}(J_n(B_{F_2}^* \ T \ J_m)$ is the identity map.

Now we use Theorem 8.12 to get a $GL_P$-equivalence $(U, V)$ from $J_n(B_{F_1}^* \ T \ J_m$ to $J_n(B_{F_2}^* \ T \ J_m$ that induces exactly this $K$-web isomorphism. Note that $U\{i\}$ is the identity matrix whenever $n_{N+1-i} = 1$. As in [Res06 Remark 8.2], we see that $(J_m^V \ T \ J_m, J_m^U \ T \ J_m)$ is a $GL_P$-equivalence from $B_{F_1}^*$ to $B_{F_2}^*$ that satisfies that $(J_n^U \ T \ J_n)\{i\}$ is the identity matrix whenever $n_i = 1$. □

Proof of Theorem 7.7. $\text{(i) } \implies \text{(ii)}$. It follows from Theorem 2.17 that the moves $(\mathcal{O}), (\mathcal{I}), (\mathcal{R}), (\mathcal{S})$ preserve stable isomorphism. By Proposition 5.2 (C) also preserves stable isomorphism so $\sim_{M^*}$ preserves stable isomorphism.

$\text{(ii) } \implies \text{(i)}$. Suppose we are given graphs $E_1, E_2$ that satisfy Condition (K) and have finitely many vertices. By Theorem 11.1 we can find graphs $F_1$ and $F_2$ with finally many vertices such that $E_1 \sim_M F_1, E_2 \sim_M F_2$ and $(F_1, F_2)$ are in standard form and there exists a $GL_P$-equivalence $(U, V)$ from $B_{F_1}^*$ to $B_{F_2}^*$ that satisfies that $V\{i\}$ is the identity matrix whenever $n_i = 1$. Theorem 7.5 lets us find graphs $G_1, G_2$ in standard form such that $G_1 \sim_{M'} F_1, G_2 \sim_{M'} F_2$ and $B_{F_1}^*$ and $B_{F_2}^*$ are $SL_P$-equivalent. By Theorem 10.10 this equivalence is a positive equivalence and so by Corollary 7.8, $G_1 \sim_M G_2$. Thus we have $E_1 \sim_M F_1 \sim_{M'} G_1 \sim_M G_2 \sim_{M'} F_2 \sim_M E_2$. □

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