Morse theoretic signal compression and reconstruction on chain complexes

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Abstract
At the intersection of Topological Data Analysis (TDA) and machine learning, the field of cellular signal processing has advanced rapidly in recent years. In this context, each signal on the cells of a complex is processed using the combinatorial Laplacian, and the resultant Hodge decomposition. Meanwhile, discrete Morse theory has been widely used to speed up computations by reducing the size of complexes while preserving their global topological properties. In this paper, we provide an approach to signal compression and reconstruction on chain complexes that leverages the tools of algebraic discrete Morse theory. The main goal is to reduce and reconstruct a based chain complex together with a set of signals on its cells via deformation retracts, preserving as much as possible the global topological structure of both the complex and the signals. We first prove that any deformation retract of real degree-wise finite-dimensional based chain complexes is equivalent to a Morse matching. We will then study how the signal changes under particular types of Morse matchings, showing its reconstruction error is trivial on specific components of the Hodge decomposition. Furthermore, we provide an algorithm to compute Morse matchings with minimal reconstruction error.

Keywords Discrete Morse theory · Combinatorial Hodge theory · Topological signal processing · Signal compression and reconstruction

Mathematics Subject Classification 55N31 · 57Q70

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1 Introduction

The analysis of signals supported on topological objects such as graphs or simplicial complexes is a fast-growing field combining techniques from topological data analysis, machine learning and signal processing (Barbarossa et al. 2018; Ortega et al. 2018; Robinson 2014). The emerging field of simplicial and cellular signal processing falls within this paradigm (Barbarossa and Sardellitti 2020; Schaub et al. 2021; Roddenberry et al. 2022), and here the combinatorial Laplacian $\Delta_n$ plays a pivotal role.

In this context, a signal takes the form of a real-valued chain (or cochain) on a chain complex $(C, \partial)$ endowed with a degree-wise inner product. In particular, the eigenvectors of $\Delta_n$, called the Hodge basis, serve as a 'topological' Fourier basis to transform a signal into a topologically meaningful coordinate system (Ebli et al. 2020; Schaub et al. 2021). Additionally, the combinatorial Laplacian gives rise to the combinatorial Hodge decomposition (Eckmann 1944):

$$C_n = \text{Im} \partial_{n+1} \oplus \text{Ker} \Delta_n \oplus \text{Im} \partial_n^\dagger,$$

where $\partial_n^\dagger$ is the adjoint of $\partial$ with respect to the given inner product on $C_n$. The components of the Hodge decomposition each have their own topological interpretation (Barbarossa and Sardellitti 2020) and respect the eigendecomposition of $\Delta_n$. This Fourier representation has proven to be useful in multiple applications (Barbarossa et al. 2018; Ortega et al. 2018). In graph signal processing, it has been exploited for signal smoothing and denoising (Chen et al. 2014; Zhou and Schölkopf 2004), node embeddings via Laplacian eigenmaps (Belkin and Niyogi 2003; Luxburg 2007), graph neural networks (Bronstein et al. 2017; Defferrard et al. 2016), and signal compression and reconstruction (Li et al. 2021).

The compression and reconstruction of data or signals is a vast field in data science. This typically involves two stages: first, compressing high-dimensional data into a smaller space, and secondly, reconstructing or recovering the data in its original space. One example of this is image compression, which utilizes compression algorithms to reduce the number of pixels while simultaneously mapping their values onto a smaller domain. Further, there exist algorithms to reconstruct high-resolution images from their compressed counterparts.

The goal of the paper is to investigate and define signal compression and reconstruction over cell complexes by combining tools of Hodge theory and discrete Morse theory. In this context, a signal is an element of the (co)chain complex. In analogy to image compression and reconstruction, in the world of cell complexes we wish to compress cell complexes equipped with signals by reducing the size of the complex, while preserving a general structure and mapping the signal of the cells to the new cell complex in an appropriate way. For us the structure to preserve is the homology of the underlying complex, which can be done by Morse collapses and this method also gives a way of mapping the signal value to the smaller cell complex.

We take an entirely algebraic approach to this problem, working at the level of degree-wise finite-dimensional based chain complexes endowed with inner products. The classical example is the chain complex of a cell complex equipped with its canonical cellular basis, but more general constructions such as cellular sheaves fit into this
framework as well. This algebraic perspective not only gives us greater flexibility, but also helps to illuminate connections between Hodge theory and discrete Morse theory that occur only at the level of chain complexes.

Our approach to compressing and reconstructing signals over complexes involves deformation retracts of based chain complexes, which have the advantage of reducing the size of complexes while preserving their homology. A deformation retract \((\Psi, \Phi, h)\) of a chain complex \(C\) onto \(D\) consists of a pair of chain maps \(\Psi\) and \(\Phi\)

\[
\begin{array}{ccc}
D & \xrightarrow{\Psi} & C \\
\Phi \downarrow & & \downarrow h \\
C & \xleftarrow{\Phi} & D
\end{array}
\]

such that \(\Psi \Phi = \text{Id}_D\) and a chain homotopy \(h : C \to C\) between \(\Phi \Psi\) and \(\text{Id}_C\).

In this context, the map \(\Psi\) is used to compress the signal \(s\) onto the reduced complex \(D\), and \(\Phi\) serves to reconstruct it back in \(C\). Thus, for every \(s \in C\) one can compute the difference \(\Phi \Psi s - s\), called the reconstruction error, to understand and evaluate how compression and reconstruction changes the signal.

Among the many topological methods to reduce the size of complexes (Singh et al. 2007; Wood et al. 2011), discrete Morse theory (Forman 1998, 2002) provides the perfect tool to efficiently generate such deformation retracts of chain complexes. This technique has already been used with great success in the compression of 3D images (Wood et al. 2011), persistent homology (Mishaikow 2013) and cellular sheaves (Curry et al. 2016). In this paper we utilise Sköldberg’s algebraic version of discrete Morse theory (Sköldberg 2018, 2006). It takes as input a based chain complex \(C\) and, by reducing its based structure with respect to a Morse matching \(M\), returns a smaller, chain-equivalent complex \(CM\). The first result presented in this article connects the Hodge decomposition of a complex with discrete Morse theory by defining a natural pairing in the Hodge basis. In particular, we show that any deformation retract \((\Psi, \Phi, h)\) of degree-wise finite-dimensional, based chain complexes of real inner product spaces can be obtained from a Morse matching over the Hodge basis of a certain sub-complex. This process, called the Morsification of \((\Psi, \Phi, h)\), is described in Theorem 3.7. In the second part of the paper, we study how the reconstruction error associated to a deformation retract \((\Psi, \Phi, h)\) is distributed amongst the three components of the Hodge decomposition. We define a class of deformation retracts \((\Psi, \Phi, h)\), called \((n, n-1)\)-free, for which the reconstruction error has trivial (co)cycle reconstruction. Specifically, they are characterised by the following properties (Theorem 4.5).

1. (Cocycle Reconstruction) A signal \(s \in C_n\) and its reconstruction \(\Phi \Psi s\) encode the same cocycle information:

\[
\text{Proj}_{\text{Ker} \partial_{n+1}}(\Phi \Psi s - s) = 0 \text{ for all } s \in C_n.
\]

2. (Cycle Reconstruction) A signal \(s \in C_{n-1}\) and the adjoint of the reconstruction \(\Psi^\dagger \Phi^\dagger s\) have the same cycle information:

\[
\text{Proj}_{\text{Ker} \partial_{n-1}}(\Psi^\dagger \Phi^\dagger s - s) = 0 \text{ for all } s \in C_{n-1}.
\]
Moreover, the Morsification concept defined above simplifies many of the proofs and allows them to be extended into a more general framework (Corollary 4.6).

Finally, we study how the reconstruction error of $(n, n-1)$-free deformation retracts can be minimized while maintaining (co)cycle reconstruction. We develop an iterative algorithm to find the retract $(\Psi, \Phi)$ that minimizes the norm of the reconstruction error for a given signal $s \in \mathbb{C}$. Our algorithm is inspired by the reduction pair algorithms in Curry et al. (2016), Kaczynski et al. (1998), Mischaikow (2013) and, like these algorithms, computes a single Morse matching at each step with the additional requirement of minimizing the norm. We show that its computational complexity is linear when the complex is sparse, and discuss bounds on how well the iterative process approximates the optimal deformation retract. Finally, we show computationally that iterating single optimal collapses leads to reconstruction loss that is significantly lower than that arising from performing sequences of random collapses.

The paper is structured as follows. In Sect. 2, we present the necessary background in algebraic topology, discrete Hodge theory, and algebraic discrete Morse theory, giving the definitions and main results that will be used throughout the paper. Section 3 introduces the notion of Hodge matching, which allows us to prove that every deformation retract of a degree-wise finite-dimensional based chain complex $\mathbb{C}$ of real inner product spaces is equivalent to a Morse retract (see Morsification Theorem 3.7). In Sect. 4 we investigate the interaction between deformation retracts and Hodge theory. The main results, Theorem 4.5 and Corollary 4.6, utilise the Morsification theorem to prove that $(n, n-1)$-free (sequential) Morse matchings preserve (co)cycles. Section 4.3 presents an additional result that explains how the reconstruction $\Phi \Psi s$ can be understood as a sparsification of the signal $s$ (see Lemma 4.10). Finally, Sect. 5 is dedicated to presenting algorithms to minimize the reconstruction error in case of iterative single pairings (see Algorithms 1 and 2).

1.1 Related work

Many articles incorporate topology into the loss or reconstruction error function (Carrière et al. 2020; Gabrielsson et al. 2020; Kim et al. 2020; Moor et al. 2020), however, these deal almost exclusively with point cloud data. At the same time discrete Morse theory has been used in conjunction with machine learning in Hu et al. (2021) for image processing, but not in the context of reconstruction error optimisation.

In Forman (2002), the concept of taking duals (over $\mathbb{Z}$) of discrete Morse theoretic constructions is introduced using a dual flow. However, in our approach, we use an adjoint flow over $\mathbb{R}$, which enables us to establish a connection to Hodge theory. This connection is not feasible when working solely over $\mathbb{Z}$, as the use of inner products is necessary.

On the computational side, the articles (Curry et al. 2016; Kaczynski et al. 1998; Kaczynski 2006; Mischaikow 2013) involve algorithms to reduce chain complexes over arbitrary principal ideal domains, including those of cellular sheaves but do not investigate the connection with the combinatorial Laplacian (or sheaf Laplacian). Our algorithms are based on the coreduction algorithms of (Kaczynski 2006; Kaczynski et al. 1998), with the additional requirement of a loss minimization.
Numerous studies have been conducted to explore the relationship between persistent homology and discrete Morse theory (Delgado-Friedrichs et al. 2015; Du et al. 2018; Mischaikow 2013; Wood et al. 2011). As far as we are aware, there are only two other recent studies, namely (Contreras and Tawfeek 2021) and (Contreras 2019), that investigate the connection between the combinatorial Hodge decomposition and discrete Morse theory. The former links the coefficients of the characteristic equation of $\Delta_n$ to the $n$-dimensional paths in an acyclic partial matching, while the latter studies the effect of a discrete Morse function on a deformed version of the Laplacians in dimensions 0 and 1.

2 Background

In this section, for the sake of completeness, we first recall some basic notions in algebraic topology. We refer the reader to Hatcher (2002) for a more detailed exposition.

Then we present the main concepts of algebraic discrete Morse theory and finally, we discuss the foundations of discrete Hodge theory.

2.1 Algebraic discrete Morse theory

For two chain complexes $(C, \partial)$ and $(D, \partial')$, a pair of chain maps $\Psi : C \to D$ and $\Phi : D \to C$ are chain equivalences if $\Phi \circ \Psi : C \to C$ and $\Psi \circ \Phi : D \to D$ are chain homotopic to the identities on $C$ and $D$, respectively. Note that this implies that the maps induced on the homology modules by $\Phi$ and $\Psi$ are isomorphisms. The chain equivalences $\Psi$ and $\Phi$ form a deformation retract of the chain complexes $C$ and $D$ if $\Psi \circ \Phi$ is the identity map on $D$. Deformation retracts will be often depicted as the following diagram.

\[
\begin{array}{c}
D \\
\Phi \\
\Psi \\
\hline \\
C \\
\end{array}
\]

With a slight abuse of notation, we denote such deformation retract by the pair $(\Psi, \Phi)$ instead of $(\Psi, \Phi, h)$. Throughout the paper we will be working with the following notion of based chain complexes, as defined in Sköldberg (2018), which in this context are chain complexes with a graded structure.

**Definition 2.1** Let $R$ be a commutative ring. A **based chain complex** of $R$-modules is a pair $(C, I)$, where $C$ is a chain complex of $R$-modules and $I = \{I_n\}_{n \in \mathbb{N}}$ is a set of mutually disjoint sets such that for all $n$ and all $\alpha \in I_n$ there exist $C_\alpha \subseteq C_n$ such that $C_n = \bigoplus_{\alpha \in I_n} C_\alpha$.

The components of the boundary operator $\partial_n$ are denoted $\partial_{\beta, \alpha} : C_\alpha \to C_\beta$ for all $\alpha \in I_n$ and $\beta \in I_{n-1}$. We will refer to the elements of $I_n$ as the $n$-cells of $(C, I)$, and if $\partial_{\beta, \alpha} \neq 0$, we say that $\beta$ is a face of $\alpha$. If $C$ is endowed with a degree-wise inner product, we say that $I$ is an orthogonal base if $C_\alpha \perp C_\beta$ for all $\alpha \neq \beta \in I$. Similarly, a **based cochain complex** is a cochain complex with an indexing set and graded decomposition as above.
Remark 2.2 We would like to draw attention to the fact that the notion of base in the definition of based chain complexes above, and originally defined in Sköldberg (2018), differs from the notion of basis in the following way. The choice of base is an additional structure providing a grading, or a direct sum decomposition, of each chain group. This does not necessarily coincide with a choice of basis, since base elements $C_\alpha$ need not be one-dimensional (See Example 2.6).

The advantage of introducing the notion of base lies in the fact that this allow us to define algebraic Morse matchings (Definition 2.8) and the subsequent deformation retracts given in Theorem 2.9. The framework of these based chain complexes encapsulate not only standard chain complexes generated from cell complexes, but also cellular sheaves as well as purely algebraic chain complexes when no cell structure is specified.

Remark 2.3 In this paper, working with combinatorial Hodge theory means that, if not specified otherwise, we restrict our study to degree-wise finite-dimensional chain complexes over $\mathbb{R}$ with an inner product on each of the chain module $C_n$.$^1$ Moreover, we will refer to degree-wise finite-dimensional based chain complexes as finite-type based chain complexes.

The following examples motivate such a choice of terminology for based chain complexes.

Example 2.4 In the special case where $(C, I)$ is a finite-type based chain complex over $\mathbb{R}$ and $C_\alpha \cong \mathbb{R}$ for all $\alpha \in I$, we can think of $I$ as a choice of basis, and each $\partial_{\beta, \alpha} \in \text{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}$ as the $(\beta, \alpha)$-entry in the boundary matrix multiplying on the left with respect to such a basis.

Example 2.5 (CW complexes) The chain complex associated to a finite CW complex with a basis given by its cells is an example of a based chain complex (see Hatcher 2002 for a precise definition of CW complex). In this case, the basis and the base coincide. For two cells $\sigma, \tau$ in a CW complex $X$, denote the degree of the attaching map of $\sigma$ to $\tau$ by $[\sigma : \tau]$ and write $\sigma \triangleright \tau$ whenever they are incident.$^2$ For two incident cells, $\partial_{\tau, \sigma}$ is multiplication by $[\sigma : \tau]$.

Example 2.6 (Cellular Sheaves) Here we present the main definitions for cellular sheaves, following the more detailed exposition of sheaf Laplacians found in Hansen and Ghrist (2019). A cellular sheaf of finite dimensional Hilbert spaces over a regular$^3$ CW complex $\mathcal{X}$ consists of an assignment of a finite dimensional vector space $\mathcal{F}(\sigma)$ to each cell $\sigma \in \mathcal{X}$ and a linear map $\mathcal{F}_{\sigma \triangleright \tau} : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ to each pair of incident cells $\sigma \triangleright \tau$. The maps $\mathcal{F}_{\tau \triangleright \sigma} : \mathcal{F}(\tau) \to \mathcal{F}(\sigma)$ must satisfy the two following conditions:

1. identity relation: $\mathcal{F}_{\tau \triangleright \tau} = \text{Id}_{\mathcal{F}(\tau)}$, and
2. if $\tau \triangleright \sigma \triangleright \omega$, then $\mathcal{F}_{\tau \triangleright \omega} = \mathcal{F}_{\sigma \triangleright \omega} \circ \mathcal{F}_{\tau \triangleright \sigma}$.

$^1$ We leave the original definition here to emphasise that algebraic discrete Morse theory works in more generality.

$^2$ Here, incident means that the closure $\bar{\sigma}$ of $\sigma$ contains $\tau$.

$^3$ Regular here indicates that the attaching maps are homeomorphisms.
This defines a cochain complex, with
\[ C_n = \bigoplus_{\tau \in \mathcal{X}_n} \mathcal{F}(\tau), \]
where \( \mathcal{X}_n \) denotes the set of \( n \)-cells of \( \mathcal{X} \), and coboundary maps \( \delta_n : C_n \to C_{n+1} \) defined component-wise by \( \delta_{\sigma, \tau} = [\sigma : \tau] \mathcal{F}_{\tau \sigma} : C_\tau \to C_\sigma \).

Using the inner product on \( C_n \) induced by the inner product on each Hilbert space \( \mathcal{F}(\sigma) \), one can define a boundary map \( \partial_n : C_{n+1} \to C_n \) as the adjoint of the coboundary map \( \delta_n \). This chain complex is an example of a based chain complex, where the \( n \)-cells of the base correspond to the \( n \)-cells of the underlying indexing complex. However, in this case the base does not correspond to a basis since the spaces corresponding to each base element are not one-dimensional.

Discrete Morse theory was originally introduced by Forman in Forman (1998) as a combinatorial version of classical Morse theory. Here we present its fundamental ideas in a purely algebraic setting, following the exposition in Sköldberg (2018).

**Definition 2.7** Let \((C, I)\) be a finite-type based chain complex with base \( I \). We denote by \( \mathcal{G}(C, I) \) the **graph of the complex**, which is the directed graph consisting of vertices \( I \) and edges \( \alpha \to \beta \) whenever \( \partial_\beta, \alpha \) is non-zero. When clear from the context we will denote \( \mathcal{G}(C, I) \) by \( \mathcal{G}(C) \). For a subset of edges \( E \) of \( \mathcal{G}(C) \), denote by \( \mathcal{G}(C)_E \) the graph \( \mathcal{G}(C) \) with the edges of \( E \) reversed.

Using these notions we can define a Morse matching as follows.

**Definition 2.8** An **(algebraic) Morse matching** \( M \) on a based complex \((C, I)\) is a selection of edges \( \alpha \to \beta \) in \( \mathcal{G}(C) \) such that
1. each vertex in \( \mathcal{G}(C) \) is adjacent to at most one edge in \( M \);
2. for each edge \( \alpha \to \beta \) in \( M \), the map \( \partial_\beta, \alpha \) is an isomorphism;
3. the relation on each \( I_n \) given by \( \alpha \succ \beta \) whenever there exists a directed path from \( \alpha \) to \( \beta \) in \( \mathcal{G}(C)_M \) is a partial order.

For context, the third condition corresponds to acyclicity in the classical Morse matching definition, where directed paths akin to gradient flow-lines – which are non-periodic – in the smooth Morse theory setting (Milnor 1969).

When there is an edge \( \alpha \to \beta \) in \( M \), we say that \( \alpha \) and \( \beta \) are **paired** in \( M \), and refer to them as a \((\dim \alpha, \dim \alpha - 1)\)-pairing. We use \( M^0 \) to denote the elements of \( I \) that are not paired by \( M \), and refer to them as critical cells of the pairing. For a directed path \( \gamma = \alpha, \sigma_1, \ldots, \sigma_k, \beta \) in the graph \( \mathcal{G}(C, I)_M \), the index \( \overline{I}(\gamma) \) of \( \gamma \) is then defined as
\[
\overline{I}(\gamma) = \epsilon_n \partial_{\beta, \sigma_n} \circ \cdots \circ \epsilon_1 \partial_{\sigma_2, \sigma_1} \circ \epsilon_0 \partial_{\sigma_1, \alpha} : C_\alpha \to C_\beta
\]
where \( \epsilon_i = -1 \) if \( \sigma_i \to \sigma_{i+1} \) is an element of \( M \), and 1 otherwise. Note that the index is not defined in the case of the trivial path where \( \alpha = \beta \). For any \( \alpha, \beta \in I \), we define
the summed index $\Gamma_{\alpha,\beta}$ to be

$$\Gamma_{\beta,\alpha} = \sum_{\gamma : \alpha \to \beta} I(\gamma) : C_{\alpha} \to C_{\beta},$$

the sum over all possible non-trivial paths from $\alpha$ to $\beta$. In the case that there are no paths from $\alpha \to \beta$ then $\Gamma_{\beta,\alpha} = 0$.

The theorem below is the main theorem of algebraic Morse theory. While this theorem was originally proved in Sköldberg (2006), here we state it in the form presented in Sköldberg (2018) where it is proved as a corollary of the Homological Perturbation Lemma (Sköldberg 2018, Theorem 1, Brown 1965; Gugenheim 1972). This proof provides an explicit description of the chain homotopy $h : C \to C$ that witnesses the fact that the algebraic Morse reduction is a homotopy equivalence.

**Theorem 2.9** (Sköldberg 2018) Let $(C, I)$ be a based chain complex indexed by $I$, and $M$ a Morse matching. For every $n \geq 0$ let

$$C^M_n = \bigoplus_{\alpha \in I_n \cap M^0} C_{\alpha}.$$

The diagram

$$C^M \xleftarrow{\Phi} C \xrightarrow{\Psi} h$$

where for $\alpha \in M^0 \cap I_n$ and $x \in C_{\alpha}$

$$\partial_{C^M}(x) = \sum_{\beta \in M^0 \cap I_{n-1}} \Gamma_{\beta,\alpha}(x) \quad \Phi(x) = \sum_{\beta \in I_n} \Gamma_{\beta,\alpha}(x)$$

and for $\alpha \in I_n$ and $x \in C_{\alpha}$

$$\Psi(x) = \sum_{\beta \in M^0 \cap I_n} \Gamma_{\beta,\alpha}(x) \quad h(x) = \sum_{\beta \in I_{n+1}} \Gamma_{\beta,\alpha}(x)$$

is a deformation retract\(^4\) of chain complexes.

We refer to the finite-type based chain complex $(C^M, \partial_{C^M}, I \cap M^0)$ as the Morse chain complex. Moreover, we call this deformation retract of $C$ into $C^M$ the Morse retract induced by $M$.

**Example 2.10** Given a based chain complex $(C, I)$ and a single $(n+1, n)$-pairing $M = (\alpha \to \beta)$, Lemma 2.9 can be used to get a simple closed form of the Morse chain complex $(C^M, \partial_{C^M})$, as well as the chain equivalences. We write them explicitly here, and will refer to them throughout the paper.

\(^4\) In fact the result is stronger. Specifically the maps form a strong deformation retract.
• For every $\tau, \sigma \in M^0$, the Morse boundary operator is

$$\partial_{\tau,\sigma} = \partial_{\tau,\sigma} - \partial_{\tau,\alpha} \partial_{\beta,\alpha}^{-1} \partial_{\beta,\sigma}.$$ 

• The map $\Psi$ is the identity except at components $C_{\alpha}$ and $C_{\beta}$, where it is

$$\Psi_n^M \bigg|_{C_{\beta}} = \sum_{\tau \in I_n \setminus \beta} -\partial_{\tau,\alpha} \partial_{\beta,\alpha}^{-1} \quad \Psi_n^M \bigg|_{C_{\alpha}} = 0.$$ 

• The map $\Phi$ is the identity except at components $C_{\eta}$ for each $\eta \in M^0 \cap I_{n+1}$, where it is

$$\Phi_{n+1}^M \bigg|_{C_{\eta}} = \text{Id}_{C_{\eta}} - \partial_{\beta,\alpha}^{-1} \partial_{\beta,\eta}.$$ 

Note that these equations are identical to those appearing in Kaczynski et al. (1998), Mischaikow (2013) in the case that each component $C_{\alpha}$ is of dimension 1.

When $(C, I)$ is a finite-type based chain complex of real inner product spaces, the adjoints of the maps in Theorem 2.9 play an important role in later sections. Their discrete Morse theoretic interpretation in terms of flow, however, hinges on the orthogonality of the base of $C$ (see Appendix A.2). We will require the following basic result of linear algebra regarding adjoints throughout the paper.

**Lemma 2.11** Let $V$ be an finite dimensional inner product space and $W \subseteq V$ be a subspace. The adjoint of the inclusion map $i : W \rightarrow V$ is the orthogonal projection $\text{Proj}_W = i^\dag$ onto $W$.

**Example 2.12** Let $(C, I)$ be the canonical based chain complex associated to the cell complex in Fig. 1, (left). Following the standard convention of discrete Morse theory, we visually depict a pairing $\alpha \rightarrow \beta$ by an arrow running from the cell $\beta$ to the cell $\alpha$. We consider the single $(2, 1)$-pairing $M = (\alpha, \beta)$, depicted by the black arrow. Figure 1 illustrates how the maps $\Psi^M$ and $\Phi^M$, made explicit Example 2.10, operate on $s \in C_1$.

**Remark 2.13** Motivated by the emerging field of cellular signal processing, we refer to elements $s \in C_n$ as signals (Barbarossa and Sardellitti 2020; Schaub et al. 2021).
In the next definition we introduce the concept of sequential Morse matching, an iterative sequence of Morse matchings. This type of matching, unlike a Morse matching, has a low computational cost to reduce the chain complex to a minimal number of critical cells. We discuss this in detail in Sect. 5, where we leverage the sequential nature of these matchings to provide efficient algorithms to minimise the reconstruction error.

**Definition 2.14** A sequential Morse matching $M$ on a based chain complex $(C, I)$ is a finite sequence of Morse matchings, $M(1), \ldots, M(n)$ and bases $I_1, \ldots, I_n$ such that the following conditions hold.

1. $M(1)$ is a Morse matching on $(C, I)$.
2. $M(j+1)$ is a Morse matching in $(C^{M(j)}, I_j)$ for every $j \in \{1, \ldots, n-1\}$.
3. $C^{M(j)}$ is a based complex over $I_j \subseteq I_k$ for every $1 \leq j \leq k \leq n$.

**Example 2.15** In Fig. 2, we give an example of a sequential Morse matching with base given by the standard simplicies. Note that this cannot be written as a single Morse matching with the standard base.

We denote by $(C^{M}, \partial_{C^{M}})$ the based chain complex obtained from $C$ by iteratively composing the Morse matchings in the sequential Morse matching $M$, implying that $(C^{M}, \partial_{C^{M}}) = (C^{M(n)}, \partial_{C^{M(n)}})$. Note that in this case, the critical cells of each individual matching in $M$ form a nested sequence $M_{(1)} \supseteq \cdots \supseteq M_{(n)}$. We denote by $M^0$ the set of critical cells of the sequential Morse matching $M$ and define it to be the set of critical cells in the last Morse matching in the sequence, namely $M^0 = M_{(n)}$.

### 2.2 Combinatorial laplacians

For a finite-type based chain complex $C$ over $\mathbb{R}$ with boundary operator $\partial$ and inner products $\langle \cdot, \cdot \rangle_n$ on each $C_n$, define $\partial_n^\dagger : C_n \rightarrow C_{n+1}$ as the adjoint of $\partial_n$, i.e., the map that satisfies $\langle \sigma, \partial_n^\dagger \tau \rangle_n = \langle \partial_n \sigma, \tau \rangle_{n-1}$ for all $\sigma \in C_n$ and $\tau \in C_{n-1}$. The adjoint maps form a cochain complex

$$
\cdots \xleftarrow{\partial_n^\dagger} C_{n+1} \xleftarrow{\partial_n} C_n \xleftarrow{\partial_n} C_{n-1} \xleftarrow{\partial_n} \cdots
$$
where \((\partial^\dagger)^2 = 0\) follows from the adjoint relation.

**Remark 2.16** If \(\partial_n\) is represented as a matrix in a given basis, and the inner products with respect to that basis are represented as \(\langle \sigma, \tau \rangle_n = \sigma^\dagger W_n \tau\) where each \(W_n\) is a positive-definite symmetric matrix, then the matrix form of the adjoint is given by

\[
\partial_n^\dagger = (W_n^{-1})^\dagger \partial_n W_n^{-1}.
\]

Note that in our definition the inner product matrix \(W_n\) does not necessarily preserve the orthogonality of the standard cellular or simplicial basis in case we are working with cell complexes. In practice, other authors require \(W_n\) to be a diagonal matrix to keep the standard basis orthogonal (Horak and Jost 2013). In this way the coefficients of \(W_n\) can be thought as weights on the \(n\)-cells, see Appendix A.1.

**Definition 2.17** The *combinatorial Laplacian* is then defined as the sequence of operators

\[
(\Delta_n = \partial_n^\dagger \partial_n + \partial_{n+1}^\dagger \partial_{n+1} : C_n \rightarrow C_n)_{n \geq 0}.
\]

For each \(n\), the two summands can be further delineated into

1. the \(n\)-th up-Laplacian \(\Delta_n^+ = \partial_{n+1}^\dagger \partial_{n+1} : C_n \rightarrow C_n\) and
2. the \(n\)-th down-Laplacian \(\Delta_n^- = \partial_n^\dagger \partial_n : C_n \rightarrow C_n\).

The fundamental results concerning the combinatorial Laplacian were proved by Eckmann in the 1940s (Eckmann 1944).

**Theorem 2.18** (Eckmann 1944) If \(C\) is a finite-type based chain complex over \(\mathbb{R}\) equipped with an inner product in each degree, then for all \(n \geq 0\)

1. \(H_n(C) \cong \text{Ker} \Delta_n\), and
2. \(C_n\) admits an orthogonal decomposition

\[
C_n \cong \text{Im} \partial_{n+1} \oplus \text{Ker} \Delta_n \oplus \text{Im} \partial_n^\dagger. \tag{1}
\]

The decomposition in the second point, called the *combinatorial Hodge decomposition*, is the finite-dimensional analogue of the Hodge decomposition for smooth differential forms. Two additional orthogonal decompositions associated with adjoints that we will use frequently are

\[
C_n = \text{Ker} \partial_n^\dagger \oplus \text{Im} \partial_{n+1} = \text{Ker} \partial_n \oplus \text{Im} \partial_n^\dagger. \tag{2}
\]

**2.3 Singular value decomposition**

Let \(V, W\) be real finite-dimensional inner-product spaces. Let \(f : V \rightarrow W\) be a linear map and \(f^\dagger : W \rightarrow V\) its adjoint. The Spectral Theorem states that \(f^\dagger f\) and \(f f^\dagger\) have the same set of real eigenvalues \(\Lambda\). Moreover, the singular value decomposition guarantees that there exist orthonormal bases \(\mathcal{R}(f)\) and \(\mathcal{L}(f)\) of \(V\) and \(W\) formed by
eigenvectors of $f^\dagger f$ and $ff^\dagger$ such that for each non-zero $\lambda \in \Lambda$ there exists a unique $v \in R(f)$ and a unique $w \in L(f)$ such that

$$f(v) = \sqrt{\lambda}w.$$ 

We denote by $L_+(f)$ and $R_+(f)$ the subsets of $L(f)$ and $R(f)$ respectively corresponding to non-zero eigenvalues. Consider now $f = \partial_n : C_n \to C_{n-1}, n \geq 0$, the boundary operators associated to a based chain complex. Note that $L_+(\partial_{n+1})$ and $R_+(\partial_n)$, the sets of eigenvectors with positive eigenvalues of $\Delta^+_n = \partial_{n+1} \partial_n^\dagger$ and $\Delta^-_n = \partial_n^\dagger \partial_n$, form orthonormal bases for $\text{Im} \partial_{n+1}$ and $\text{Im} \partial_n^\dagger$, respectively (by Equation (2)). In the next section we will see how these eigenvectors together with the Hodge decomposition will allow us to define a canonical Morse matching.

### 3 Morsification of deformation retracts

The aim of this section is to prove that every deformation retract of a finite-type based chain complex $C$ over $\mathbb{R}$ equipped with degree-wise inner products is equivalent to a Morse retract, with a canonical choice of basis. In doing so, we are able to prove statements about all such deformation retracts using the techniques of algebraic discrete Morse theory.

We first introduce the notion of the Hodge matching on $C$, a Morse matching defined over the eigenbasis of the combinatorial up and down Laplacians $\Delta^+_n$ and $\Delta^-_n$. We can see the matching obtained by Hodge decomposition and the eigenvectors of $\Delta^+_n$ and $\Delta^-_n$ as a canonical Morse matching.

#### 3.1 Hodge matchings

The following concept marries the discrete Morse theoretic notion of pairing to the pairing inherent to the eigendecomposition of $\Delta^+_n$ and $\Delta^-_n$, which is intrinsically connected to the Hodge decomposition of a finite real chain complex.

**Definition 3.1 (Hodge basis)** Let $C$ be a finite-type based chain complex over $\mathbb{R}$. A Hodge basis of $C$ is the basis given by $I^\Delta = \{I^\Delta_n\}_{n \in \mathbb{N}}$, where

$$I^\Delta_n = L_+(\partial_{n+1}) \cup R_+(\partial_n) \cup \mathcal{B}(\text{Ker} \Delta_n),$$

for some choice of bases $L_+(\partial_{n+1})$, $R_+(\partial_n)$ and $\mathcal{B}(\text{Ker} \Delta_n)$.

Observe that in the definition above each set in $I^\Delta_n$ forms a basis for one of the components in the Hodge decomposition (see Eq. 1). Our discussion on the singular value decomposition ensures that Hodge bases always exist.

**Definition 3.2 (Hodge matching)** Let $C$ be a finite-type based chain complex of real inner product spaces, and let $I^\Delta$ be a Hodge basis. The Hodge matching on $(C, I^\Delta)$
is
\[ M^\Delta := \bigcup_i \{ v \in R_+(\partial_i) \to w \in L_+(\partial_i) \mid \partial_i v = \sigma w, \sigma \neq 0 \}. \]

**Lemma 3.3** For a finite-type based chain complex \((C, I^\Delta)\) of real inner product spaces and \(I^\Delta\) be a Hodge basis. The Hodge matching \(M^\Delta\) on \((C, I^\Delta)\) is a Morse matching and satisfies
1. \((M^\Delta)^0_n = \text{Ker} \Delta_n\), where \(\Delta : C \to C\) is the combinatorial Laplacian of \(C\) and
2. \(\partial M^\Delta = 0\).

**Proof** The description of orthonormal bases \(L(\partial_n)\) and \(R(\partial_n)\) described at the end Sect. 2 implies that each cell is adjacent to at most one other cell in \(G(C)\). This means there are no nontrivial paths from any \(n\)-cell to any other \(n\)-cell for all \(n\) in \(G(C)\). Thus, condition (3) in Definition 2.8 is trivially satisfied, and \(M^\Delta\) indeed constitutes a Morse matching. By definition,
\[ \text{Im} \partial_{n+1} = \text{span} L_+(\partial_{n+1}) \text{ and } \text{Im} \partial_n^\perp = \text{span} R_+(\partial_n), \]
and all basis elements are paired. The remaining Hodge basis elements of \(C_n\), i.e. \(B(\text{Ker} \Delta_n)\), are critical, as they are not being paired to any other vectors through the natural pairing defined in 3.2, and constitute \((M^\Delta)^0_n = \text{Ker} \Delta_n\) for all \(n\). Furthermore, \(\text{Ker} \Delta_n \simeq H_n(C)\) the critical elements generate the homology of \(C\), which is consistent with the fact that in an optimal Morse matching there is one critical cell per dimension in \(H_n\).

Since there are no non-trivial paths, \(\partial M^\Delta\) agrees with the boundary operator \(\partial\) of \(C\) on \(\text{Ker} \Delta\), which is indeed the zero map. □

We call the data
\[ \text{Ker} \Delta \xrightarrow{\psi M^\Delta} C \xleftarrow{\phi M^\Delta} h \]
the **Hodge retract** of \((C, I^\Delta)\). Noting that the maps \(\Phi M^\Delta, \Psi M^\Delta\) are chain equivalences reprovess Eckmann’s result that \(\text{Ker} \Delta\) is isomorphic to the homology \(H(C)\) of the original complex.

The same proof also encompasses the case of cellular sheaves discussed in Hansen and Ghrist (2019). Note that here, a Hodge matching will be over a Hodge base \(I^\Delta\) rather than the one specified by the cellular structure of the indexing complex. Nevertheless, since \(\text{Ker} \Delta\) does not depend on the choice of base, the result is the same.

**Example 3.4** In Fig. 3 we depict two different choice of bases – the standard cellular basis and the Hodge basis – for the cellular chain complex of the pictured simplicial complex. Two matchings \(M\) and \(M^\Delta\) are visualized through their corresponding Morse graphs \(\mathcal{G}(C)^M\) and \(\mathcal{G}(C)^{M^\Delta}\). The structure of the singular value decomposition of \(\partial\) and ensuing Hodge matching ‘straightens out’ the connections in the matching graph, as pictured in Fig. 3.
3.2 Morsification theorem

In this section, we say that two deformation retracts

\[ D \xleftarrow{\Phi} C \xrightarrow{h} \quad \text{and} \quad D' \xleftarrow{\Phi'} C' \xrightarrow{h'} \]

are equivalent if there exist isomorphisms of chain complexes, \( f : D \to D' \) and \( g : C \to C' \) such that the diagrams

\[
\begin{array}{ccc}
D & \xrightarrow{\Phi} & C \\
\downarrow f & \cong & \downarrow g \\
D' & \xleftarrow{\Phi'} & C'
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
D & \xrightarrow{\Phi} & C \\
\downarrow f & \cong & \downarrow g \\
D' & \xleftarrow{\Phi'} & C'
\end{array}
\]

commute. Our goal is to show that any deformation retract of finite-type chain complexes of real inner product spaces is equivalent to a Morse retract (Theorem 3.7).

In the special case that \( C = C' \) and \( g \) is the identity, the commutativity of the diagrams above implies that

\[
\Phi' \Psi' = \Phi f^{-1} f \Psi = \Phi \Psi. \tag{3}\]

Thus, to study the reconstruction error of a deformation retract, it is enough to study that of an equivalent deformation retract of the original complex. Two equivalent deformation retracts over a shared domain \( C \) may have different homotopies, however, they are related by

\[
\partial h + h \partial = 1 - \Phi \Psi = 1 - \Phi' \Psi' = \partial h' + h' \partial.\]

The main theorem of this section relies on the observation that deformation retracts share a number of characteristics with projection maps in linear algebra i.e. a linear endomorphism \( P : V \to V \) of a vector space \( V \) satisfying \( P^2 = P \). For any projection map, there exists a decomposition \( V = \text{Im} P \oplus \text{Ker} P \) such that \( P \) can be decomposed as

\[
P = 1_{\text{Im} P} + 0 : \text{Im} P \oplus \text{Ker} P \to \text{Im} P \oplus \text{Ker} P.
\]
The following lemma describes an analogous structure for real chain complexes, where a deformation retract plays the role of a projection.

**Lemma 3.5** For any deformation retract

\[
D \xrightarrow{\Phi} C \xsetminus h
\]

of chain complexes over \(\mathbb{R}\),

\[
C = \text{Ker} \Psi \oplus \text{Im} \Phi
\]

as chain complexes.

**Proof** The deformation retract condition \(\Psi \Phi = \text{Id}_D\) implies that

\[
(\Phi_n \Psi_n)^2 = \Phi_n \Psi_n \Phi_n \Psi_n = \Phi_n \Psi_n,
\]

i.e., each component \(\Phi_n \Psi_n\) of \(\Phi \Psi\) is a projection operator. Thus there is a splitting of vector spaces

\[
C_n = \text{Ker}(\Phi \Psi)_n \oplus \text{Im}(\Phi \Psi)_n
\]

for each \(n\). Since \(\Phi \Psi\) is a chain map, the decomposition above commutes with the boundary operator of \(C\), whence

\[
C = \text{Ker} \Phi \Psi \oplus \text{Im} \Phi \Psi
\]

as chain complexes. Lastly, \(\Psi\) is surjective and \(\Phi\) is injective since \(\Psi \Phi = \text{Id}_D\), implying that \(\text{Im} \Phi \Psi = \text{Im} \Phi\) and \(\text{Ker} \Phi \Psi = \text{Ker} \Psi\).

The decomposition defined in Eq. 4 has an interesting interpretation when passing to homology: all of the non-trivial homology of \(C\) arises from the \(\text{Im} \Phi\) component of the decomposition. One way to think of this decomposition is that \(\text{Ker} \Psi\) is the component of \(C\) that is discarded by the deformation retract, whereas \(\text{Im} \Phi\) is preserved.

**Lemma 3.6** Under the hypotheses of Lemma 3.5

1. \(\text{H}(C) \cong \text{H}(\text{Im} \Phi)\), and
2. \(\text{H}(\text{Ker} \Psi) = 0\).

**Proof** Since \(\Psi\) is a weak equivalence, \(\text{H}(C) \cong \text{H}(D)\). Since \(\Psi \Phi = \text{Id}_D\), \(\Phi\) is injective, so \(D \xrightarrow{\Phi} \text{Im} \Phi\) is an isomorphism of chain complexes, proving point (1). Since \(C = \text{Ker} \Psi \oplus \text{Im} \Phi\) by Eq. 4, it follows that \(\text{H}(\text{Ker} \Psi) = 0\).

**Theorem 3.7** (Morsification) Any deformation retract

\[
D \xleftarrow{\Phi} C \xsetminus h
\]

of finite-type chain complexes of real inner product spaces is equivalent to a Morse retract \((\Psi^M, \Phi^M)\) over \(C\).
Notation 3.8 We refer to the pairing $\mathcal{M}$ in this theorem as the Morsification of a deformation retract.

Proof Define a pairing $\mathcal{M} = \tilde{M}^\Delta \sqcup \hat{M}$ on $C$ as the union of a Hodge pairing $\tilde{M}^\Delta$ on $\text{Ker } \Psi$ (which is given the subspace inner product) and the trivial pairing $\hat{M}$ on $\text{Im } \Phi$. We previously showed that $C = \text{Ker } \Psi \oplus \text{Im } \Phi$ and $H(C) = H(\text{Im } \Phi)$, implying that $H(\text{Ker } \Psi) = 0$. Consequently, all the basis elements in $\text{Ker } \Psi$ are paired by the Hodge pairing, and further, the Morse retract maps

$$H(\text{Ker } \Psi) \cong 0 \xrightarrow{\psi_{\tilde{M}^\Delta}} \text{Ker } \Psi$$

defined by the matching $\tilde{M}^\Delta$ are trivial.

On the other hand, since $\hat{M}$ is the trivial pairing, the entirety of $\text{Im } \Phi$ is critical in the pairing $\mathcal{M}$. Further, the Morse boundary operator $\partial \hat{M}$ is the same as the boundary operator on $C$, implying $C^\mathcal{M} = \text{Im } \Phi$ and that the maps

$$C^\mathcal{M} \cong \text{Im } \Phi \xleftarrow{\psi_{\hat{M}}} \text{Im } \Phi$$

are identities. We conclude that $\Phi^\mathcal{M} \Psi^\mathcal{M} = i_{\text{Im } \Phi} \circ \pi_{\text{Im } \Phi}$, where $i_{\text{Im } \Phi} : \text{Im } \Phi \hookrightarrow C$ is the inclusion.

Now we show that this is equivalent to the original deformation retract. To do so, first note that $\Phi : D \to \text{Im } \Phi$ is an isomorphism. We then need to show that the following diagram

$$\begin{array}{ccc}
C & \xrightarrow{\Psi} & C \\
\downarrow{\Psi^\mathcal{M}} & & \downarrow{\Psi^\mathcal{M}} \\
D & \xrightarrow{\Phi} & \text{Im } \Phi
\end{array}$$

commutes. For any $(s, \Phi(t)) \in C = \text{Ker } \Psi \oplus \text{Im } \Phi$, we have

$$\Phi \Psi(s, \Phi(t)) = (\Phi \Psi(s), \Phi(\Phi(t))) = (0, \Phi(t)) = i \circ \pi_{\text{Im } \Phi}(s, \Phi(t)) = \Phi^\mathcal{M} \Psi^\mathcal{M}(s, \Phi(t))$$

as required. Finally, to see that

$$\begin{array}{ccc}
C & \xleftarrow{\Phi^\mathcal{M}} & C \\
\downarrow{\Phi} & & \downarrow{\Phi} \\
D & \xleftarrow{\Phi} & \text{Im } \Phi
\end{array}$$

commutes simply note that $\Phi^\mathcal{M}$ is the inclusion map. \qed

Remark 3.9 When the original deformation retract comes from a Morse matching, the subspace $\text{Im } \Phi = \text{Im } \Phi \Psi = \text{Ker } (1 - \Phi \Psi)$ is the space of flow-invariant chains used by Forman in his foundational articles (Forman 1998, 2002). The difference here is that these chains are linear combinations of genuine critical cells, albeit for a Morse matching in a new base.
It is not difficult to see that the Morsification of a deformation retract is unique up to a choice of bases in the eigenspaces of $\Delta^+$ and $\Delta^-$, and that each such choice produces equivalent deformation retracts. Combining Theorem 3.7 with Eq. 3, we get a simple expression for the reconstruction error of a deformation retract in terms of the paired cells in its Morsification.

**Corollary 3.10** For any deformation retract

$$D \xrightarrow{\Phi} C \xrightarrow{h}$$

of finite-type chain complexes of real inner product spaces and Morsification $M$ we have that

$$1 - \Phi \Psi = \sum_{\alpha \in I^M \setminus M^0} i_\alpha \circ \pi_\alpha$$

**Proof** By Eq. 3 and Theorem 3.7, we have

$$1 - \Phi \Psi = 1 - i_{\text{Im}} \Phi \circ \pi_{\text{Im}} \Phi = i_{\text{Ker}} \Phi \Psi \circ \pi_{\text{Ker}} \Phi \Psi = \sum_{\alpha \in I^M \setminus M^0} i_\alpha \circ \pi_\alpha$$

which proves the statement, noting that the paired cells in $M$ span Ker $\Psi$. \(\square\)

In the case that the deformation retract arises from a Morse matching on a based complex, the Morsification construction will most likely alter the base. However, the number of pairings and critical cells in each dimension are related, as described in the following proposition.

**Notation 3.11** For a sequential Morse matching $M$ on a based chain complex $(C, I)$, let $M_n^-$ and $M_n^+$ denote the elements of $I_n$ that are the union of all start and endpoints respectively of edges in each of the matchings $M_{(i)} \in M$ for all $i$.

This means that

$$I_n = M_n^- \sqcup M_n^0 \sqcup M_n^+.$$ 

Further, let

$$|M_n^*| = \sum_{\alpha \in M_n^*} \dim C_\alpha$$

where $* \in \{+, -, 0\}$, and the subscript $n$ refers to the dimension of the cells.

**Proposition 3.12** Let $M$ be a sequential Morse matching on a finite-type based chain complex $(C, I)$ of real inner product spaces and $M$ be its Morsification. Then

$$|M_n^*| = |M_n^*|$$

for $* \in \{+, -, 0\}$, in each dimension $n \geq 0$. 

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By Theorem 3.7 we know that $C^M \cong C^M$, implying that the dimensions spanned by critical cells

$$\left| M_n^0 \right| = \dim C_n^M = \dim C_n^M = \left| M_n^0 \right|$$

are equal for all $n$. This implies that

$$\left| M_n^+ \right| + \left| M_n^- \right| = \dim C_n - \dim C_n^M = \left| M_n^+ \right| + \left| M_n^- \right|$$

where we have used the identity $\dim C_n = \left| M_n^+ \right| + \left| M_n^- \right| + \left| M_n^0 \right|$. Since the chain complex is concentrated in non-negative degrees, cells in dimension 0 can be paired only with elements in dimension 1, implying that $\left| M_0^+ \right| = \left| M_0^- \right| = 0$. Combining this with Eq. 5 we conclude that $\left| M_0^+ \right| = \left| M_0^- \right|$. The bijection between cells paired up in dimension $i$ with those paired down in dimension $i + 1$ then implies that

$$\left| M_1^- \right| = \left| M_0^+ \right| = \left| M_1^+ \right| = \left| M_1^- \right|,$$

and, again using Eq. 5, that $\left| M_1^+ \right| = \left| M_1^- \right|$. By inductively performing this procedure, we prove the result for all $n$ as required.

It is not difficult to see that two equivalent Morse retracts of $C$ must have the same Morsification. Thus the above proposition then implies that when two sequential Morse retracts $M$ and $M'$ of a complex $C$ under two different bases $I$ and $I'$ are equivalent, there are equalities between the number of dimensions paired up $\left| M_n^+ \right| = \left| M_n'^+ \right|$ and down $\left| M_n^- \right| = \left| M_n'^- \right|$ for all $n$. Notably, this occurs independently of the bases $I$ and $I'$.

4 (Co)cycle preservation and sparsification

Discrete Morse theory aims to reduce the dimension of a chain complex while preserving its homology. Meanwhile, for combinatorial Hodge theory, understanding the effect of deformation on the components of the Hodge decomposition is of equal importance. However, because of the ‘adjointness’ inherent in the Hodge decomposition, neither chain or cochain maps between two complexes usually respect the grading of the Hodge decomposition.

Here, we define a different notion of preservation by examining the effect of applying either $\Phi \Psi$ or $\Psi^\dagger \Phi^\dagger$ to an element $s \in C$. For a pair of chain maps

$$D \xrightarrow{\Psi} \Phi \rightleftharpoons C$$

we define the reconstruction error at $s \in C$ as $\Phi \Psi s - s \in C$. The goal of this section is to examine the projection of $\Phi \Psi s - s$ on the different components of the Hodge decomposition. In particular, we describe which components of the signal are
preserved and discarded by $\Phi \Psi$ when the deformation retract arises from a $(n, n - 1)$-free Morse matching, a special type of (sequential) Morse matchings described in the next section.

Furthermore, we demonstrate that by applying the map $\Phi \Psi$ (or $\Psi^\dagger \Phi^\dagger$) associated to appropriate types of matchings, the initial signal can be sparsified. Sparsification of a signal or vector commonly entails nullifying a considerable number of its entries while maintaining essential features. In the present context, the reconstruction of $\Phi \Psi$ (or $\Psi^\dagger \Phi^\dagger$) is supported only over the critical cells of the matching. In effect, this sparsifies the signal in the original complex while retaining the information regarding (co)cycles.

### 4.1 $(n, n - 1)$-free matchings

**Definition 4.1** A Morse matching $M$ is said to be $(n, n - 1)$-free if $|M_{n-1}^-| = 0$.

An equivalent condition is that $|M_{n-1}^+| = 0$. Put simply, a Morse matching is $(n, n - 1)$-free if no $n$-cells are paired with $(n - 1)$-cells. In what follows, the mantra is that preservation of (co)cycle information in dimension $n - 1$ (or $n$) is equivalent to absence of such pairings. We define an $(n, n - 1)$-free sequential Morse matching $M = (M_1, \ldots, M_k)$ to be a sequential Morse matching where all $M_i$ are $(n, n - 1)$-free Morse matchings.

**Example 4.2** Fig. 4 shows a $(1, 0)$-free and a $(2, 1)$-free matching. The matchings are computed on the cellular chain complex of the depicted cell complex, based with the standard cellular basis. We visually depicted the pairings in the matchings by black arrows. Note that being $(n, n - 1)$-free does not necessarily prohibit all $n$ or $(n - 1)$-cells from appearing in the matching, implying that $(n, n - 1)$-free matchings can still lead to dimension reduction of both $C_n$ and $C_{n-1}$.

**Example 4.3** If $C$ is finite-type chain complex of real inner product spaces such that $\partial_n = 0$, then the Hodge matching $M^\Delta$ is $(n, n - 1)$-free for some choice of Hodge basis $I^\Delta$.

The corollary below, which follows immediately from Proposition 3.12, shows that the property of being $(n, n - 1)$-free is not an artifact of our choice of basis. Namely, if two Morse matchings are equivalent, then either they are both $(n, n - 1)$-free or neither is.
Corollary 4.4 A sequential Morse matching $M$ on a based chain complex $(C, I)$ is $(n, n - 1)$-free if and only if its Morsification $\mathcal{M}$ is $(n, n - 1)$-free.

4.2 (Co)cycle preservation for $(n, n - 1)$-free matchings

The following reconstruction theorem shows that both the reconstruction error of the deformation retract and its adjoint are supported on non-kernel components of the Hodge decomposition.

Theorem 4.5 (Reconstruction) Suppose that $M$ is a Morse matching on a finite-type based chain complex $(C, I)$ of real inner product spaces. Let

$C^M \xleftarrow{\Phi} C \xrightarrow{\Psi} h$

be the deformation retract given by Theorem 2.9. Then

1. for all $s \in C_n$,

$$\operatorname{Proj}_{\operatorname{Ker} \partial_{n+1}} (\Phi \Psi s - s) = 0,$$

and

2. for all $s \in C_{n-1}$,

$$\operatorname{Proj}_{\operatorname{Ker} \partial_{n-1}} (\Psi^\dagger \Phi^\dagger s - s) = 0$$

if and only if $M$ is a $(n, n - 1)$-free matching.

Proof We first prove that if $M$ is a $(n, n - 1)$-free matching, then conditions (1) and (2) hold. If $M_n = \emptyset$, then there are no paths in $\tilde{G}(C)^M$ from an $(n - 1)$-cell to an $n$-cell. Theorem 2.9 then implies that $h_{n-1}(x) = 0$ for all $\alpha \in I_{n-1}$ and $x \in C_\alpha$, whence

$$(\Phi \Psi - 1)_n = \partial_{n+1} h_n + h_{n-1} \partial_n = \partial_{n+1} h_n.$$  \hspace{1cm} (6)

The first claim now follows from the orthogonal decomposition

$$C_n = \operatorname{Ker} \partial_{n+1} \oplus \operatorname{Im} \partial_{n+1}.$$  

The argument above also shows that $h_{n-1}^\dagger = 0$, since the adjoint of the zero map is the zero map. By taking the adjoint of Equation 6 one dimension lower, it then follows that

$$(\Psi^\dagger \Phi^\dagger - 1)_{n-1} = (\Phi \Psi - 1)_{n-1} = \partial_{n-1} h_{n-2}^\dagger + h_{n-1}^\dagger \partial_n^\dagger = \partial_{n-1} h_{n-2}^\dagger.$$  

The second claim is then a consequence of the orthogonal decomposition $C_{n-1} = \operatorname{Ker} \partial_{n-1} \oplus \operatorname{Im} \partial_{n-1}^\dagger$. 

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For the other direction we will prove the contrapositive statement. It is sufficient to show that if the Morse matching is not \((n, n - 1)\)-free, then there exists \(s \in C_n\) such that

\[
\text{Proj}_{\text{Ker} \partial_{n+1}^+} (\Phi \Psi s - s) \neq 0.
\]

The Morse matching \(M\) is \((n, n - 1)\)-free if and only if its Morsification \(\mathcal{M}\) is \((n, n - 1)\)-free (Corollary 4.4) and, further, \(1 - \Phi^M \Psi^M = 1 - \Phi^M \Psi^M\) (Eq. 3). Therefore, it is sufficient to prove the contrapositive statement for the Morsification.

Since the Morsification is not \((n, n - 1)\)-free, there exists an \((n, n - 1)\)-pair \(\alpha \to \beta\) such that \(\partial_{\beta, \alpha}\) is an isomorphism. Recall that by 3.10, we have that \((\Phi^M \Psi^M - 1)x = x\) for \(x \in C_{\alpha}\). The orthogonal decomposition of \(C_n\) implies that

\[
x = \text{Proj}_{\text{Ker} \partial_{n+1}^+} x + \text{Proj}_{\text{Im} \partial_n^+} x.
\]

Applying \(\partial_n\) and using the fact that \(\partial_n(x) \neq 0\), we obtain

\[
0 \neq \partial_n \text{Proj}_{\text{Ker} \partial_{n+1}^+} x + \partial_n \text{Proj}_{\text{Im} \partial_n^+} x = \partial_n \text{Proj}_{\text{Im} \partial_n^+} x.
\]

Since \(\text{Im} \partial_n^+ \subseteq \text{Ker} \partial_{n+1}^+\), this implies that

\[
0 \neq \text{Proj}_{\text{Ker} \partial_{n+1}^+} x = \text{Proj}_{\text{Ker} \partial_{n+1}^+} (\Phi^M \Psi^M - 1)x = \text{Proj}_{\text{Ker} \partial_{n+1}^+} (\Phi^M \Psi^M - 1)x,
\]

which proves our statement. 

The utility of the theorem above is that an \((n, n - 1)\)-free matching \(M\) reduces the dimension of \(C_n\), while perfectly preserving the \(n\)-cocycles of a signal \(s \in C_n\) under the reconstruction \(\Phi_n \Psi_n\). The extent of this reduction depends on the \((n + 1, n)\)-pairs in \(M\). Indeed, the direct sum of the components \(\bigoplus_{\alpha \in M_{n+1}^+} C_{\alpha}\) of \(n\)-cells in such pairs is isomorphic to the subspace \(\text{Ker} \Psi_n\) discarded by the deformation retract. One way to see this is using the fact that the Morsification has the same pair structure as the sequential Morse matching, and the Morsification \(\Phi^M\) is zero on non-critical cells.

If, on the other hand, one is interested in preserving the cycle information of a signal \(s \in C_{n-1}\), then one can use the adjoint maps \(\Phi^+ \Psi^+\) to perform a similar procedure. Namely, an \((n, n - 1)\)-free matching \(M\) will perfectly preserve the \((n - 1)\)-cycle part of \(s\) under the reconstruction \(\Psi_{n-1}^+ \Phi_{n-1}^+\). Analogously to the dual case, the extent of reduction depends on the \((n - 1, n - 2)\)-pairings, where the subspace \(\bigoplus_{\alpha \in M_{n-1}^{-}} C_{\alpha}\) is isomorphic to the discarded subspace \(\text{Ker} \Phi_{n-1}^+\).

Using Morsification, we can extend the (co)cycle reconstruction theorem to \((n, n - 1)\)-free sequential Morse matchings.

**Corollary 4.6** Let \(M\) be a sequential Morse matching on a based chain complex \((C, 1)\). Then the (co)cycle preservation conditions (1) and (2) of Theorem 4.5 hold if and only if \(M\) is \((n, n - 1)\)-free.
By Corollary 4.4 we know that $M$ is $(n, n - 1)$-free if and only if its Morsification $\mathcal{M}$ is $(n, n - 1)$-free. Further, we know that

$$1 - \Phi^M \Psi^M = 1 - \Phi^\mathcal{M} \Psi^\mathcal{M}$$

by Eq. 3. Then the statement follows by applying Theorem 4.5 to $C$ and $\mathcal{M}$. □

One may wonder whether there is a proof by induction that follows directly from Theorem 4.5. The problem with using induction is that each chain complex in the sequential Morse matching has a different Hodge decomposition, and that the maps between them do not necessarily respect the grading. So Theorem 4.5 implies the (co)cyle preservation conditions will be satisfied between the $i$-th and $(i + 1)$-th chain complexes but not necessarily between $C$ and $C^\mathcal{M}$.

In the general case of deformation retracts that do not arise from a Morse matching, combining Theorem 4.5 and Corollary 4.4 yields the following.

**Corollary 4.7** Let $(\Phi, \Psi)$ be a deformation retract of based finite-type chain complexes $(C, I)$ and $(D, I')$ of real inner product spaces. Then the (co)cyle preservation conditions (1) and (2) of Theorem 4.5 hold if and only if the Morsification $\mathcal{M}$ associated to $(\Phi, \Psi)$ is $(n, n - 1)$-free.

### 4.3 Sparsification for $(n, n - 1)$-free matchings

In the previous section, we showed how a signal’s projection onto each Hodge component is related to that of its reconstruction. In addition, one would like to know how the reconstructed signal sits in the complex with respect to the base on which the Morse matching is constructed.

In this section we will show that, for a $(n, n - 1)$-free (sequential) Morse matching, the image of $\Phi_n \Psi_n$ is supported only on the critical cells $M^0_n$ of $I_n$. Intuitively, applying $\Phi_n \Psi_n$ can be thought of as a form of sparsification, a method that reduces the support of the signal, which preserves one of either cycles or cocycles (Theorem 4.5).

**Lemma 4.8** Let $M$ be an $(n, n - 1)$-free matching of an orthogonally based finite-type chain complex $(C, I)$ of real inner product spaces. Then

1. $\Phi_n : C^M_n \rightarrow C_n$ and
2. $\Psi_{n-1}^\dagger : C^M_{n-1} \rightarrow C_{n-1}$

are subspace inclusions and, thus, isometries.

**Proof** For item (1), by Theorem 2.9

$$\Phi_n = \sum_{\alpha \in M^0_n} \sum_{\beta \in I_n} \Gamma_{\beta, \alpha}.$$ 

A path in $\mathcal{G}(C)^M$ starting at an $n$-dimensional critical cell must first step down a dimension. Since $M$ is $(n, n - 1)$-free, it cannot return to dimension $n$. This shows...
that the only paths starting at critical cells in dimension $n$ are trivial and hence

$$\Phi_n(x) = \sum_{\beta \in I_n} \Gamma_{\beta,\alpha}(x) = x$$

for all $x \in C_\alpha$, $\alpha \in M_n^0$, thus proving item (1).

For item (2), recall that

$$\Phi^{M}_n \Psi_n^{M}(s) \in \bigoplus_{\alpha \in M_n^0 \cap I_n} C_\alpha \quad \text{for all} \quad s \in C_n$$

and

$$\Psi_{n-1}^{M} \Phi_{n-1}^{M}(s) = \bigoplus_{\beta \in M_n^1 \cap I_{n-1}} C_\beta \quad \text{for all} \quad s \in C_{n-1}.$$
**Proof** By definition we know that

\[
\Psi^M_n(s) \in \bigoplus_{\alpha \in M^n \cap I_n} C_\alpha = C^M_n \quad \text{and} \quad \Phi^M_{n-1}(s) \in \bigoplus_{\beta \in M^{n-1} \cap I_{n-1}} C_\beta = C^M_{n-1}.
\]

The result then follows from Lemma 4.8, which implies that both \(\Phi^M_n\) and \(\Psi^M_{n-1}\) are compositions of subspace inclusions. \(\square\)

**Example 4.11** In this example we consider the based chain complex \(C\) associated to the cell complex \(X\) in Fig. 5A. We work with the standard basis generated by the \(n\)-cells and the standard boundary operator \(\partial_n\). The signal \(s \in C_1\) is obtained by randomly sampling from \([0, 1]\). We consider the \((1, 0)\)-free matching \(M\) in Fig. 5C, where there are two 1-cells are paired with two 2-cells, denoted by the arrows. All the other cells are critical.

In Fig. 5A we show how the signal \(s\) is transformed by the maps \(\Phi^M\) and \(\Psi^M\) induced by the \((1, 0)\)-free matching \(M\). The absolute value of the reconstruction error \(s - \Phi^M \Psi^M s\) on each 1-cell is shown in Fig. 6B. As proved in Theorem 4.5, we observe in Fig. 6D that the reconstructed signal \(\Phi^M \Psi^M s\) is perfectly preserved on \(\ker \partial_1 = \ker \Delta_1 \oplus \im \partial_1^\dagger\), and all changes in the reconstructed signal are contained in \(\im \partial_2\). Note that \(\Phi^M_1 \Psi^M_1 s\) is supported only on the critical 1-cells as proved in Lemmas 4.10 and 4.8.
5 Algorithms and experiments

The goal of this section is to reduce a based complex \((C, I)\) together with a signal \(s \in C\) (or set of signals \(S \subset C\)) via a sequential Morse matching while trying to minimize the norm of the reconstruction error.

We propose the following procedure to iteratively reduce a based chain complex \((C, I)\) with signal \(s\) via a sequential Morse matching. The method is inspired by the classical reduction pair algorithm described in Kaczynski (2006); Kaczynski et al. (1998) but differs in the optimization step in (1).

1. If \(\partial \neq 0\), select a single pairing \(\alpha \rightarrow \beta\) in \((C, \partial)\) minimizing \(\|s - \Phi \Psi s\|\).
2. Reduce \(C\) to \(C^M\) and repeat with \(C = C^M\) and \(\partial = \partial C^M\).

Note that this procedure differs as well from that of Nanda et al. which, in the context of both persistent homology (Mischaikow 2013) and cellular sheaves (Curry et al. 2016), requires an actual Morse matching. The details of the algorithm are provided in Sect. 5.1 (see Algorithm 1 and Algorithm 2), where we also show that their computational complexity is linear in the number of \((n + 1)\)-cells. In Sect. 5.2 we discuss the behaviour of the norm of the reconstruction error when performing this type of iterated reduction. In Sect. 5.3 we prove that such an algorithm converges to a based chain complex with the minimal number of critical cells. Finally, in Sect. 5.4 we provide experiments on synthetic data.

Remark 5.1 Since in most of the applications \(\dim C_\alpha = 1\) for all \(\alpha \in I\), we will work with this assumption throughout the following sections. Thus, without loss of generality, we will refer to the elements of \(I_n\) as a basis of \(C_n\) and denote \(\partial_{\beta, \alpha} = [\alpha : \beta]\) (see Example 2.4 for more details).

5.1 Algorithms for optimal (sequential) morse matchings

For a pair of chain maps
\[
\mathbf{D} \xleftarrow{\Phi} \mathbf{C} \xrightarrow{\Psi} \mathbf{D}
\]
between based chain complex with inner product on each \(C_n\) and \(D_n\), and a signal \(s \in C_n\), define the loss of the maps \((\Phi, \Psi)\) over \(s\) to be the norm of the reconstruction error
\[
\mathcal{L}_s(\Phi, \Psi) = \langle s - \Phi \Psi s, s - \Phi \Psi s \rangle_{C_n}^{1/2} = \|s - \Phi \Psi s\|_{C_n}.
\] (7)

For a finite subset \(S \subset C_n\), the loss is defined to be the sum
\[
\mathcal{L}_S(\Phi, \Psi) = \sum_{s \in S} \mathcal{L}_s(\Psi, \Phi)
\]
of the individual losses. The loss of a single collapse can be given a closed form by using Theorem 2.9, in the case of a deformation retract associated to a Morse matching.

Specifically, suppose we have a single \((n + 1, n)\)-pairing \(\alpha \rightarrow \beta\). Theorem 2.9 implies that the homotopy \(h\) maps \(\beta\) to \(-\frac{1}{|\alpha:\beta|} \alpha\) and is zero elsewhere. For a signal

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$s \in C_n$, using the equations developed in Example 2.10, we have

$$\mathcal{L}_s(\Psi, \Phi) = \| (1 - \Phi \Psi) s \|_{C_n} = \| \partial_{n+1} h_n s \|_{C_n} = \left\| \frac{s_\beta}{[\alpha : \beta]} \cdot \partial_{n+1} (\alpha) \right\|_{C_n}$$  \hspace{1cm} (8)

where $s_\beta$ is the component of $s$ on basis element $\beta$. Similarly, for a signal $s \in C_{n+1}$ we have a dual loss

$$\mathcal{L}_s(\Phi^\dagger, \Psi^\dagger) = \left\| (1 - \Psi^\dagger \Phi^\dagger) s \right\|_{n+1} = \| \partial_{n+1}^{\dagger} h_n^{\dagger} s \|_{n+1}.$$  \hspace{1cm} (9)

If $I$ is an orthogonal basis for $C$, Theorem A.2 implies that we can write this loss as

$$\mathcal{L}_s(\Phi^\dagger, \Psi^\dagger) = \left\| s_\alpha \partial_{n+1}^\dagger (\beta) \right\|_{C_{n+1}} [\alpha : \beta].$$

Note that to write a compact form for Equation (7), in case $M$ is not a single Morse matching, one needs to sum over all possible non-trivial paths in Theorem 2.9. Therefore finding the matching $M$ minimizing this norm would be computationally expensive, if not infeasible. On the other hand, it is not hard to find the single $(n+1, n)$-pairing $\alpha \rightarrow \beta$ minimizing the loss in Equation (8). Therefore, as a first approach towards finding an approximate solution of the problem, we begin by studying optimal matchings by restricting to iterated single pairings.

**Remark 5.2** Naturally, one can ask the same questions about finding the optimal pairing minimizing the loss for $\Psi^\dagger \Phi^\dagger s - s$. Given the duality of the problem, we will present algorithms and experiments only for $\Phi^\dagger \Psi s - s$. The algorithms and computations for the dual loss can be found by dualizing the chain and boundary maps.

Given a finite-type based chain complex $(C, I)$ of real inner product spaces and a signal $s$ on the $n$-cells, our goal is now to find the the $(n+1, n)$-pairing $\alpha \rightarrow \beta$ minimizing the loss in Equation (8). Computing the minimum and its arguments for a single pair boils down to storing for each $(n+1)$-cell $\tau$ in the basis the face $\sigma$ where the quantity

$$\frac{|s_\sigma|}{||[\tau : \sigma]||} \| \partial_{n+1} \tau \|_n$$

is minimal, and choosing among all the $(n+1)$-cells the one realizing the minimum of $\mathcal{L}_s$.

**Example 5.3** Consider the based chain complex associated to a simplicial complex $X$ with basis induced by its cells and $\partial_*$ the standard boundary operator. Let $s$ be a signal on the $n$-cells. The minimum of the reconstruction loss $\mathcal{L}_s$ in Equation (8) is then realized on the $n$-cell $\beta$, where $|s_\beta|$ is minimum, paired with any of its cofaces $\alpha$. Note that the minimum and its argument might not be unique.
Algorithm 1 Perform a single optimal pairing

**Input** A based chain complex $C$ with basis $I$, a signal on $C_n, \partial_{n+1}$, the non-zero $n + 1$-boundary. **Output** A a single $(n+1, n)$-pairing $\alpha \rightarrow \beta$ which minimize the loss.

1. **function** OPTIMALPAIRING($C, I, \text{signal}, \partial_{n+1}$)

2. for each $n + 1$-cell $\tau$ in $I_{n+1}$ do
3. $\text{OptCol}[] \leftarrow 0$  $\triangleright$ OptCol keeps track of the face which realizes the optimal collapse on $\tau$
4. end for

5. for each $n + 1$-cell $\tau$ in $I_{n+1}$ do
6. $\text{ValOptCol}[] \leftarrow \infty$  $\triangleright$ ValOptCol keeps track of the value of the optimal collapse on $\tau$
7. end for

8. for each $n + 1$-cell $\tau$ in $I_{n+1}$ do
9. for each face $\xi$ of $\tau$ in $F_\tau$ do
10. $x \leftarrow |\text{signal}|[\xi]|/\|\partial_{n+1}(\tau)\|_{C_n}$
11. $\text{ValOptCol}[] \leftarrow \min(\tau, \text{ValOptCol}[])$
12. end for
13. $\sigma \leftarrow \text{random}(\arg \min_{\xi \in F_\tau} (|\text{signal}|[\xi]|/\|\partial_{n+1}(\tau)\|_{C_n} = \text{ValOptCol}[]))$
14. $\text{OptCol}[] \leftarrow \sigma$
15. end for
16. $\text{TotalMin} \leftarrow \min(\text{ValOptCol})$  $\triangleright$ The value TotalMin is the minimum reconstruction loss.
17. $\alpha \leftarrow \text{random}(\arg \min_{\tau \in I_{n+1}} (\text{ValOptCol}=\text{TotalMin}))$  $\triangleright$ The $n + 1$ cell $\alpha$ to collapse is randomly chosen among the $n + 1$ cells where the reconstruction loss is minimal.
18. $D \leftarrow \text{OptCol}[]$  $\triangleright$ The $n$ cell $\beta$ to collapse is the face of $\tau$ obtaining minimal reconstruction loss.
19. return $(\alpha, \beta)$
20. end function

Following the idea above, Algorithm 1 returns a single $(n + 1, n)$-pairing $\alpha \rightarrow \beta$ that minimizes the loss for a given based chain complex $(C, I)$ and signal $s$.

The computational complexity of Algorithm 1 is $O(pc^2) + O(p)$, where $p = \dim C_{n+1}$ and $c = \max_{\tau \in I_{n+1}} |\partial_{n+1}(\tau)|$. The first term follows from the fact that we need to iterate through all the $(n + 1)$-cells and their faces, computing the minimum of lists of size at most $c$. The second summand follows from the fact that the final step of the algorithm requires computations of the minimum of a list of size at most $p$. Since the first summand dominates the second one, the computational complexity of Algorithm 1 is $O(pc^2)$. We assume that in most of the computations we are dealing with sparse based chain complexes, i.e. based chain complexes in which the number of $n$-cells in the boundary of an $(n + 1)$-cell is at most a constant $c \ll p$. In this case the computational complexity of Algorithm 1 is $O(p)$.

In practice, one would like to further reduce the size of a based chain complex. In Algorithm 2 we provide a way to perform a sequence of single optimal collapses. For a based chain complex $C$ and a signal $s$, the algorithm computes at each iteration a single optimal pairing $(\alpha, \beta)$ and it updates $(C, \partial)$ to $(C^M, \partial_{C^M})$ and the signal $s$ to $\Psi^M s$. 

\(@ Springe@\)
**Algorithm 2** Perform $k$ single optimal pairings

**Input** A based chain complex $C$ with basis $I$, a signal on $C_n, \partial_{n+1}$ the non-zero $(n+1)$-boundary and parameter $k$ of the number of single optimal collapses to perform.

**Output** A based chain complex $C^M$ with basis $I^M \subseteq I$ and its boundary $\partial_{C^M}$ obtained by iteratively computing $k$ optimal pairings starting from $C$.

1: function $k$-OPTIMALPAIRINGS($C, I, \text{signal}, \partial_{n+1}, k$)
2:   $i \leftarrow 1$
3:   while $i \leq k$
4:     $(\alpha, \beta) \leftarrow$ OPTIMALUPCOLLAPSE($C, I, \text{signal}, \partial_{n+1}$)
5:     $(C, \partial, I) \leftarrow (C^M, \partial_{C^M}, I^M)$
6:     signal $\leftarrow \Psi_1(\text{signal})$
7:     $i \leftarrow i + 1$
8:   end while
9: return $C, \partial$
10: end function

In fact, Algorithm 2 consists of the classical reduction pair algorithm proposed in Kaczynski (2006); Kaczynski et al. (1998) with the additional step of the loss minimization. If applied only to a $(n, n-1)$-free sequential Morse matching, Algorithm 2 will converge to a based chain complex with given dimensions, as we prove in Proposition 5.10. Otherwise, if applied to cells of every size, it allows us to reduce a chain complex up to a minimal number of critical $n$-cells, as proved in Kaczynski et al. (1998). We state again this result in Sect. 5.3. At the same time, the algorithm constructs a $(n, n-1)$-free sequential Morse matching, therefore the original signal is perfectly reconstructed on part of the Hodge decomposition, as proved in Theorem 4.5. Finally, a further justification for the choice of this iterative algorithm, is that the loss on the original complex is bounded by the sum of the losses in the iterative step. We further discuss this in the next section.

### 5.2 Conditional loss

The computational advantages outlined above are dictated by the fact that Algorithm 2 iteratively searches for optimal pairings. One important detail to understand is then how the loss function interacts with such iterated reductions. For a diagram of chain maps

$$
E \xrightarrow{\Psi'} D \xrightarrow{\Phi} C
$$

and $s \in C_n$, define the conditional loss to be

$$
\mathcal{L}_s(\Psi', \Phi' | \Psi, \Phi) = \mathcal{L}_{\Psi(s)}(\Psi', \Phi') = \|\Psi_s - \Phi' \Psi \Psi_s\|_{D_n}.
$$

In practice, we will generate a sequential Morse matching by taking a series of collapses and optimising the conditional loss at each step.

**Lemma 5.4** Let $C$, $D$, and $E$ be inner product spaces and suppose we have a diagram of linear maps

$$
\begin{array}{c}
\end{array}
$$

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\[ E \xleftarrow{\psi'} D \xrightarrow{\phi'} C \]

where \( \phi \) is an isometry. Then for all \( s \in C \) we have

\[ \| (1 - \phi \phi' \psi' \psi) s \|_C \leq \| (1 - \phi \psi) s \|_C + \| (1 - \phi' \psi') \psi(s) \|_D. \]

**Proof** Using the triangle inequality and the fact that \( \phi \) is an isometry, we have

\[ \| (1 - \phi \phi' \psi' \psi) s \|_C = \| (1 - \phi \psi) s + \phi (1 - \phi' \psi') \psi(s) \|_C \]
\[ \leq \| (1 - \phi \psi) s \|_C + \| (1 - \phi' \psi') \psi(s) \|_C \]
\[ = \| (1 - \phi \psi) s \|_C + \| (1 - \phi' \psi') \psi(s) \|_D \]

as required. \( \square \)

The following corollary justifies the approach of minimizing the conditional loss at each step. It states that the loss on the original complex will be bounded by the sum of the conditional losses. Note that the same result and proof also work for the adjoint case where \( s \in C_{n-1} \), as long as the complex is orthogonally based.

**Corollary 5.5** Suppose we have a diagram of chain maps

\[ E \xleftarrow{\psi'} D \xrightarrow{\phi'} C \]

where each step arises from an \((n, n-1)\)-free Morse matching. Then for all \( s \in C_n \)

\[ \mathcal{L}_s(\psi' \psi, \Phi \Phi') \leq \mathcal{L}_s(\psi, \psi) + \mathcal{L}_s(\psi', \Phi' \psi, \psi, \Phi). \]

**Proof** In the Sparsification Lemma 4.10, we showed that taking \((n, n-1)\)-matchings implied that \( \Phi_n, \Phi'_n \) are isometries. The result then follows from applying the lemma above. \( \square \)

### 5.3 Reduction pairings and convergence

The following proposition ensures that the reduction pair algorithm proposed in Kaczyński et al. (1998), which is the foundation of Algorithm 2, converges in a finite (and pre-determined) number of steps to the homology of \( C \). This advantage of being able to maximally reduce a based complex is in contrast with the well-studied NP-hard problem (Joswig and Pfetsch 2006) of finding Morse matchings. In this section, we will prove an analogous result for \((n, n-1)\)-free matchings.

**Theorem 5.6** (Kaczyński et al. 1998) Let \((C, I)\) be a finite-type based chain complex over \( \mathbb{R} \), where \( \dim C_\alpha = 1 \) for all \( \alpha \in I \). The iteration of the following procedure

1. If \( \partial \neq 0 \), select a single pairing \( \alpha \rightarrow \beta \) in \((C, \partial)\).
2. Reduce \( C \) to \( C^M \) and repeat with \( C = C^M \) and \( \partial = \partial_{C^M} \).
converges to the complex $H(C)$ with $\partial = 0$ after

$$N = \frac{1}{2} \sum_n (\dim C_n - \dim H_n(C))$$

steps.

**Remark 5.7** The above is not necessarily true over $\mathbb{Z}$. Indeed, the fact that $\partial_{\beta,\alpha}$ may be multiplication by a non-unital element may obstruct the convergence of the process.

To prove a similar result for $(n, n - 1)$-free matchings, we first prove two lemmas describing how the dimensions of the summands in the Hodge decomposition of $C^M$ relate to those of $C$ when $M$ is a single pairing.

**Lemma 5.8** Let $M = (\alpha \rightarrow \beta)$ be an $(n + 1, n)$-pairing of a based complex $(C, I)$. Then

$$\dim \text{Im} \partial^M_n = \dim \partial_n.$$

**Proof** Since no $(n - 1)$-cells are deleted by $M$, $C_{n-1} = C^M_{n-1}$. The formulas in the background section in Example 2.10 show that $\partial^M_n = \partial_n|_{C^M}$, implying that $\dim \text{Im} \partial^M_n + \dim (C_\beta) = \dim \partial_n$. To prove the statement it then suffices to show that $\partial_n(C_\beta)$ is contained in $\text{Im} \partial^M_n$. Using $\partial_n \partial_{n+1} = 0$ and the fact that $\partial_{\alpha,\beta}$ is an isomorphism, we then have that

$$0 = \partial_n(\partial_{n+1}(C_\alpha)) = \partial_n(\partial_{\beta,\alpha}(C_\alpha) + \sum_{\tau \in I_n \setminus \beta} \partial_{\tau,\alpha}(C_\alpha))$$

$$\Rightarrow \partial_n(C_\beta) = -\partial_n(\sum_{\tau \in I_n \setminus \beta} \partial_{\tau,\alpha}(C_\alpha)) \subseteq \text{Im} \partial^M_n.$$

which proves the result. $\square$

Note that while the images of both $\partial^M_n$ and $\partial_n$ agree, the eigendecomposition of their correspondent up- and down-Laplacians may not be related in a straightforward way. In other words, the combinatorial Laplacian eigenbases for $C_{n-1}$ and $C^M_{n-1}$ can be rather different, even though the corresponding summands of their Hodge decompositions have the same dimensions.

**Lemma 5.9** Let $M = (\alpha \rightarrow \beta)$ be an $(n + 1, n)$-pairing of a finite-type based complex $(C, I)$ of real inner product spaces. Then

$$\dim \text{Im} (\partial^M_i)^\dagger = \dim \text{Im} \partial^M_i = \begin{cases} \dim \text{Im} \partial_i - \dim C_\beta & i = n + 1 \\ \dim \text{Im} \partial_i & \text{else}. \end{cases}$$

(10)
Proof The left equality is a basic property of adjoints. For the right equality, note that

(1) \( C \cong C^M \) implies \( \text{dim Ker } \Delta_i^M = \text{dim Ker } \Delta_i \) for all \( i \) and (2) Lemma 5.8 implies that \( \text{dim Im } (\partial_n^M)^\dagger = \text{dim Im } \partial_n^\dagger \). Together these imply that

\[
\text{dim } C_n - \text{dim } C_n^M = \text{dim Im } \partial_{n+1}^\dagger - \text{dim Im } (\partial_{n+1}^M)^\dagger = \text{dim } C^\beta.
\]

Equivalently, this says that \( \text{dim Im } (\partial_{n+1}^M)^\dagger - \text{dim Im } (\partial_n^M)^\dagger = \text{dim } C^\alpha \), and now all of the change in dimension from \( C \) to \( C^M \) has been accounted for.

We can now state the convergence theorem for the \((n, n-1)\)-sequential Morse matchings over \( \mathbb{R} \) in Algorithm 2. Along with homology, \( \text{dim Im } \partial_n \) and \( \text{dim Im } \partial_n^\dagger \) provide a (strict) upper bound on how many pairings we can make in an \((n, n-1)\)-free sequential Morse matching.

Proposition 5.10 (Convergence) Let \((C, I)\) be a finite-type based chain complex over \( \mathbb{R} \) with inner products. Then Algorithm 2 for \((n, n-1)\)-free Morse matchings converges to a based chain complex \( D \) such that

\[
D_i \cong \begin{cases} 
H(C_i) \oplus \text{Im } \partial_i^\dagger & i = n \\
H(C_i) \oplus \text{Im } \partial_{i-1} & i = n - 1 \\
H(C_i) & \text{else}
\end{cases}
\]

where \( \partial_i^D = 0 \) for all \( i \neq n \).

Proof Given the conditions on the basis assumed at the beginning of the section, \( \partial_{\alpha, \beta} \) is an isomorphism if and only if it is a multiplication by a non-zero element of \( \mathbb{R} \). Hence, \( \partial_i = 0 \) if and only if we are not able to make any more \((i, i-1)\)-pairings, implying the process must converge to some complex \( D \) with \( \partial_i^D = 0 \) for all \( i \neq n \). Since \( D \) is weakly equivalent to \( C \), this proves that \( D_i = H_i(D) = H_i(C) \) for all \( i \neq \{n, n-1\} \).

By Lemma 5.9, each \((n+1, n)\)-pairing reduces the dimension of \( \text{Im } \partial_{n+1} \) by 1, and each \((n-1, n-2)\)-pairing reduces the dimension of \( \text{Im } (\partial_{n-1}^M)^\dagger \) by 1. One can iterate the process of either \((n+1, n)\)-pairing or \((n-1, n-2)\)-pairing, until \( \text{dim Im } \partial_{n+1} = 0 \) or \( \text{dim Im } (\partial_{n-1}^M)^\dagger = 0 \) respectively. Thus, the isomorphism in the lemma follows from this iterative process and from the Hodge decomposition of \( D_i \).

5.4 Experiments

In this section we provide examples of how Algorithms 1 and 2 can be applied to compress and reconstruct signals on synthetic complexes. Moreover, we show computationally that the reconstruction loss of a sequence of optimal pairings given by Algorithm 2 is significantly lower than the loss when performing sequences of random collapses (see Fig. 6 and Fig. 9). Our main goal is to provide a proof of concept for the theoretical results and algorithms of this article rather than an exhaustive selection of experiments. The code for the experiments can be found in Stefania et al. (2022).
A Sequence of optimal pairings and reconstruction of the signal $s$

$|s - \Phi^M \Psi^M s|$

$\Psi^M s$

$\Phi^M \Psi^M s$

B Reconstruction error

C Projection of $s$ and $\Phi^M \Psi^M s$ on the Hodge basis

Fig. 6 Optimal $(1, 0)$-free sequential Morse matching $M$ obtained by iterating Algorithm 2 for $k = 120$ on $(2, 1)$-pairs. The signal $s$ on the 1-cells is given by the height function.

Example 5.11 In this example we consider the cell complex $X$ in Fig. 6A-left, constructed as the alpha complex of points sampled uniformly at random in the cube $[0, 1] \times [0, 1]$. We work with the basis given the cells of $X$ and the standard boundary operator $\partial$. The signal $s$ on the 1-cells is given by the height function on the 1-cells. The example illustrates a $(1, 0)$-free sequential Morse matching $M$ obtained by iterating Algorithm 2 for $k = 120$. Note that the optimal matchings correspond to 1-cells where the signal is lower (see Fig. 6A-center). This can be explained by Remark 5.3 and the fact that Equation (8) favors collapsing cells with lower signal even when $X$ is not a simplicial complex.

The absolute value of the reconstruction error after the sequential Morse matching $M$ is shown in Fig. 6B. As expected from Equation (8), the error is mainly concentrated on the 1-cells that are in the boundaries of the collapsed 2-cells. Further, the map $\Phi^M$ is an inclusion as showed in Lemma 4.10. In panel C of Fig. 6 we show the projection of the signal $s$ and the reconstructed signal $\Phi^M \Psi^M s$ on the Hodge decomposition. By Theorem 4.5 the signal is perfectly reconstructed on $\text{Ker } \partial_1 = \text{Ker } \Delta_1 \oplus \text{Im } \partial_1^\dagger$, and only $\text{Im } \partial_2$ contains non-trivial reconstruction error. Due to formatting constraints, we show the projection onto only 30 (randomly chosen) vectors of the Hodge basis in $\text{Im } \partial_1^\dagger$ and $\text{Im } \partial_2$.

In Fig. 7 we propose the same example as above with a non-geometric function on the 1-cells. Specifically, the signal $s$ on the 1-cells is given by sampling uniform at random in $[0, 1]$ and the $(1, 0)$-free sequential Morse matching $M$ is obtained by iterating Algorithm 2 until all 2-cells were removed.
A Sequence of optimal pairings and reconstruction of the signal \( s \)

\[
\begin{align*}
  s &= \text{random uniform} \\
  \Psi^M s &\quad \Phi^M \Psi^M s
\end{align*}
\]

B Reconstruction error

\[
|s - \Phi^M \Psi^M s|
\]

C Projection of \( s \) and \( \Phi^M \Psi^M s \) on the Hodge basis

Fig. 7 Optimal \((1, 0)\)-free sequential Morse matching \( M \) obtained by iterating Algorithm 2 until all 2-cells were removed. The signal \( s \) on the 1-cells is given by sampling uniform at random in \([0, 1]\).

To quantify how low the reconstruction loss is after performing a sequential Morse matching with optimal pairings, we compare the reconstruction loss after a sequence of \( k \) optimal matchings with the reconstruction loss after a sequence of \( k \) random matchings.

Example 5.12 In this example we compare the sequence of optimal collapses presented in Example 5.11 in Fig. 6 and in Fig. 7 respectively with sequence of random collapses. In particular, we consider the complex \( \mathcal{X} \) of Example 5.11 with signal on the 1-cells \( s \) given by the height function as in Fig. 8 and signal \( s \) given by sampling uniformly at random in \([0, 1]\) as in Fig. 7. Instead of finding a sequence of \((2, 1)\)-pairings minimizing the reconstruction loss, at each step of Algorithm 2 we will randomly remove a \((2, 1)\)-pair. We apply this procedure for \( k = 120 \) iterations in case \( s \) is the height function of the 1-cells and until all 2-cells are removed when the signal \( s \) is sampled uniform at random in \([0, 1]\).

Figure 8A shows the projection on the Hodge basis of \( s \) and \( \Phi^M \Psi^M s \) when \( s \) is the height function and Fig. 8B shows the same result for \( s \) sampled uniform at random. Due to formatting constraints, we show the projection onto only 30 (randomly chosen) vectors of the Hodge basis in \( \text{Im} \partial_1^\dagger \) and \( \text{Im} \partial_2^\dagger \). Note that, for both types of signal, the projection of the reconstructed signal \( \Phi^M \Psi^M s \) and \( s \) on \( \text{Im} \partial_2 \) differ significantly more than the the projection on \( \text{Im} \partial_2 \) of the reconstructed error and the signal in the case of optimal sequential Morse matching presented in Example 5.11 (see Figs. 6D and 7D).
Projection of the signal and the reconstructed signal on the Hodge basis after a sequence of random pairings

\[ \text{A } s = \text{ height function} \]
\[ \text{B } s = \text{ random uniform} \]

**Fig. 8** Projection of the signal and the reconstructed signal on the Hodge basis after a sequence of random pairings

The quantitative results shown in the previous examples can be strengthened by comparing the value of the reconstruction loss for random and optimal sequence of pairings. In the next example we show that, for different types of both geometric and random signals, the reconstruction loss is significantly lower in sequentially optimal matchings than in random matchings.

**Example 5.13** We consider again the same complex \( X \) as in Example 5.11. Figure 9 shows the value of the reconstruction loss after a sequence optimal and random pairings. We took sequences of length \( k = 1, 2, \ldots, 244 \), terminating when all 2-cells were reduced. In panel A we consider a signal on the 1-cells sampled from a uniform distribution in \([0, 1] \), in panel B the signal is the height function on the 1-cells, in panel C the signal is sampled from a normal distribution (mean 0.5 and standard deviation 0.1), and in panel D the signal is given by the distance of the middle point of the 1-cells from the center of the cube \([0, 1] \times [0, 1] \). The blue curve is the average over 10 instantiations of optimal pairings while the green curve is the average over 10 instantiations of random pairings. The filled opaque bars show the respective mean square errors. Note that for all type of functions, the loss for the optimal pairings is significantly lower than the loss of random pairings.

**6 Discussion**

**6.1 Contributions**

The contributions of this paper are threefold. First we demonstrated that any deformation retract \( (\Phi, \Psi) \) of finite-type based chain complexes over \( \mathbb{R} \) is equivalent to a deformation retract \( (\Phi^M, \Psi^M) \) associated to a Morse matching \( \mathcal{M} \) in a given basis. Second, we proved that the reconstruction error \( s - \Phi \Psi s \), associated to any signal \( s \in C_n \) and deformation retract \( (\Phi^M, \Psi^M) \), is contained in specific components of the Hodge decomposition if and only if \( \mathcal{M} \) is a \((n, n-1)\)-free (sequential) Morse matching. In the more general case, we showed that the reconstruction error associated to a deformation retract of a based chain complex is contained in specific parts of the
Hodge decomposition if and only if its Morsification $\mathcal{M}$ is $(n, n-1)$-free. Moreover, we proved that the composition $\Phi^M \Psi^M s$ can be thought as a sparsification of the signal $s$ in the $(n, n-1)$-free case. Finally, on the computational side, we designed and implemented algorithms that calculate (sequential) matchings that minimize the norm of the reconstruction error. Further, we demonstrated computationally that finding a sequence of optimal matchings with our algorithm performs significantly better than randomly collapsing.

6.2 Limitations

The type of collapses that preserve cocycles involve chain maps, and those that preserve cycles involve the adjoints of these maps. This has two main limitations. The first one is that one can pick only one of the two features to be encoded at a time. The second limitation is the fact that chain maps do not necessarily send cocycles in $\mathcal{C}$ to cocycles in $\mathcal{D}$, and dually for cochain maps.

The proof of Theorem 4.5 hints at the difficulties of trying to define chain maps that preserve cocycles and dually cochain maps that preserve cycles. Namely, to preserve cocycles with chain maps in dimension $n$, Morsification and Corollary 3.10 yield some insight, saying that this will occur only when the paired $n$-cells of Morsification lie in $\partial_n^\dagger$. A sufficient condition for this is that $\ker \Psi \perp \text{Im} \Phi$, in which case $\partial_n^\dagger \big|_{\ker \Psi} = (\partial_n |_{\ker \Phi \Psi})^\dagger$ (See Appendix A.2). This rarely occurs in the standard CW or sheaf bases.
6.3 Applications and future work

6.3.1 Algorithms for optimal collapses

In this paper we minimize the reconstruction error by considering only single collapses. It would be desirable to find algorithms either for the optimal \((n,n-1)\)-free Morse matchings, with no restriction on the length of the sequence, or for optimal \((n,n-1)\)-free Morse matchings of given length \(k\). We speculate that this task is likely to be NP-hard, given that the simpler task of finding a matching that minimises the number of critical cells is already known to be NP-hard (Joswig and Pfetsch 2006; Martinez-Figueroa 2021). In this case, it would be useful to develop algorithms to approximate optimal matchings. These could be then used to compare how far away the reconstruction error of a sequence of \(k\) optimal pairings (Algorithm 2) is from the reconstruction error of an optimal collapse of size \(k\).

6.3.2 Applications with inner products

In this paper, we have chosen examples that are helpful to visually illustrate the key results. However, the theory is built to accommodate a far larger class of applications. Examples where our theory may be useful for performing reductions that respect the inner product structure include the following

- **Markov-based heat diffusion.** The foundational work of Coifman and Lafon (2006) introduces a graph-theoretic model of heat diffusion on a point cloud, and can be framed in terms of combinatorial (graph) Laplacians. Here, distance kernel functions induce a weighting function on the nodes and edges of fully connected graph over the points. This weighting function is equivalent to specifying an inner product on \(\mathbb{C}\) where the standard basis vectors are orthogonal (Horak and Jost 2013).

- **Triangulated manifolds.** If \(M\) is a Riemannian manifold with smooth triangulation \(K\), then \(\mathbb{C}(K; \mathbb{R})\) has an inner product structure that converges to the canonical inner product on the de Rham complex \(\Omega^1(M)\) under a certain type of subdivision (Dodziuk 1976). This inner product on \(\mathbb{C}(K; \mathbb{R})\) – and variations thereof – are useful in discrete Exterior calculus and its applications (Hiptmair 2002; Hirani 2003).

The main theorems of this paper will hold in any of the circumstances described above, and provide a discrete Morse theoretic procedure for signal compression that is aware of the geometric information contained in the inner product structure.

**Pooling in cell neural networks.** Complementary to theoretical ideas, this research direction may have potential applications in pooling layers in neural networks for data structured on complexes or sheaves, such as in (Bodnar et al. 2021; Ebli et al. 2020; Hansen and Gebhart 2019). One could use Algorithm 2 to reduce the complex for a fixed sized \(k\) and then the map \(\Phi\) to send the signal onto the reduced complex. We also envision that in pooling layers one could learn the \((n,n-1)\)-free Morse matchings.
A adjoints and discrete morse theory

A.1 Matrix representation of adjoints and weights

In this appendix we include a lengthier discussion about inner products and weight functions. To begin, we state a basic result about the matrix representation of the adjoint in finite-dimensional inner product spaces.

**Proposition A.1** Let $V$ and $W$ be finite-dimensional inner product spaces where

$$\langle v_1, v_2 \rangle_V = v_1^\top A v_2$$

and

$$\langle w_1, w_2 \rangle_W = w_1^\top B w_2$$

for some fixed bases of $V$ and $W$, where $A$, $B$ are positive definite symmetric matrices. If $T : V \to W$, then the adjoint $T^\dagger : W \to V$ of $T$ satisfies

$$T^\dagger = (A^{-1})^\top T^\top B^\top.$$

The idea is that inner products are a vehicle to incorporate data with weights on the simplices into the linear algebraic world of combinatorial Laplacians. In particular, as mentioned in Remark 2.16, there is a one-to-one correspondence between inner products where elementary simplicial (co)chains form an orthogonal basis and weight matrices on the simplices. In the literature there are two approaches to associate weights to the simplices.

Firstly, the work of (Mémoli et al. 2022) begins by letting $\partial_n : C_n(X) \to C_{n-1}(X)$ be the standard cellular boundary operator on a simplicial complex $X$, and defines an inner product structure with respect to a basis given by the simplices via

$$\langle \sigma, \tau \rangle_n = \sigma^\top W_n \tau,$$

where where each $W_n$ is a diagonal matrix. The diagonal entries of $W_n$ can be thought as weights on the $n$-cells. Then the coboundary operator $\partial_n^\dagger : C_{n-1}(X) \to C_n(X)$, is given by

$$\partial_n^\dagger = W_n^{-1} \partial_n^\top W_{n-1}$$

following the proposition above.

The second approach, exemplified by the work of Horak and Jost (2013), starts instead with the standard coboundary operator on a simplicial complex $X$, $\delta_n = \partial_n^\top : C_{n-1}(X) \to C_n(X)$. Here the inner product structure on $C_n(X)$ with respect to a basis given by the simplices is defined instead to be

$$\langle \sigma, \tau \rangle_n = \sigma^\top W_n \tau,$$
where each $W_n$ is a diagonal matrix, the entries of which can be thought as weights on the $n$-cells. In this approach, the boundary operator is then written as

$$
\delta_n^\dagger = W_{n-1}^{-1} \delta_n^\top W_n.
$$

(11)

Because we are working with discrete Morse theory, which conventionally is built for homology, we take the approach of always beginning with a boundary operator before constructing its adjoint operator. If one starts by defining a weighted boundary operator

$$
\tilde{\partial}_n = W_{n-1}^{-1} \partial_n W_n,
$$

then the adjoint operator induced by the weighted inner product yields

$$
\tilde{\partial}_n^\dagger = W_{n-1}^{-1} W_n \partial_n^\top W_{n-1}^{-1} W_n W_{n-1}^{-1} = \partial_n^\top.
$$

In other words, the adjoint of this weighted boundary operator is the standard coboundary operator, recovering the method of Horak and Jost (2013).

### A.2 The adjoint of a morse retract

In this section, we explain why the orthogonality condition on the base $I$ of a based chain complex $C$ is important for establishing a discrete Morse theoretic interpretation when taking adjoints in Theorem 2.9. One can of course take the adjoint of the maps in this theorem to construct a deformation retract of the adjoint cochain complex, along with a coboundary operator, cochain weak-equivalences, and a cochain homotopy between them. However, only in the special case of an orthogonal base can these maps be decomposed in terms of adjoint flow backwards along paths in the original matching graph $\mathcal{G}(C)^M$.

**Adjoint paths and flow.** Suppose we have a Morse matching $M$ on any based finite-type chain complex $C$ over $\mathbb{R}$ with inner products. One can always define a notion of adjoint flow. First, observe that

$$
\partial_{\beta,\alpha} = 0 \iff \partial_{\beta,\alpha}^\dagger = 0
$$

and further

$$
\partial_{\beta,\alpha}^\dagger \text{ isomorphism } \iff \partial_{\beta,\alpha} \text{ isomorphism}.
$$

The opposite digraph $\mathcal{G}^{op}(C)^M$ (same vertices with edges reversed) of the directed graph $\mathcal{G}(C)^M$ then has an analogous relationship with the adjoint of the boundary operator. Namely, there is an edge $\beta \rightarrow \alpha$ whenever $\partial_{\beta,\alpha}^\dagger$ is non-zero, and a reversed edge $\beta \rightarrow \alpha$ in $\mathcal{G}^{op}(C)^M$ whenever $\alpha \rightarrow \beta$ is in $M$ and $\partial_{\beta,\alpha}^\dagger$ is an isomorphism. The same cells are unpaired in the adjoint world as the original one, and thus the critical cells of both are the same.
For a directed path \(\gamma = \alpha, \sigma_1, \ldots, \sigma_k, \beta\) in the graph \(\mathcal{G}(C)^M\), the adjoint index \(I^\dagger(\gamma)\) of \(\gamma\) is written as

\[
I^\dagger(\gamma) = e_0^\dagger \partial e_0^\dagger \circ \cdots \circ e_{n-1}^\dagger \partial e_{n-2}^\dagger, e_{n-1}^\dagger \circ e_n^\dagger \partial e_n^\dagger : C_\beta \to C_\alpha
\]

where \(k_i = -1\) if \(\sigma_i \to \sigma_{i+1}\) is an element of \(M\), and 1 otherwise. For any \(\alpha, \beta \in I\), we can interpret this as following the path backwards and taking the adjoint of each map. The adjoint of the summed index also has a similar structure:

\[
\Gamma^\dagger_{\beta, \alpha} = \sum_{\gamma: \alpha \to \beta} I^\dagger(\gamma) : C_\beta \to C_\alpha
\]

where the sum runs over all paths \(\gamma\) from \(\alpha \to \beta\) in \(\mathcal{G}(C)^M\) or, equivalently, over all paths \(\beta \to \alpha\) in \(\mathcal{G}^\text{op}(C)^M\).

**Main theorem for adjoint matching.** To see what can go wrong, we need to be careful to distinguish categorical projections – those that simply delete components of a direct sum – from orthogonal projections that arise from the inner product structure.

Let \(f : C = \oplus_\alpha C_\alpha \to D = \oplus_\beta D_\beta\) be a map of finite-type graded Hilbert spaces, based by \(I\) and \(J\) respectively. Each component \(f_{\beta, \alpha}\) can be thought of as the composition of maps

\[
f_{\beta, \alpha} : C_\alpha \xrightarrow{i_\alpha} C \xrightarrow{f} D \xrightarrow{\pi_\beta} D_\beta
\]

such that we recover the total map \(f\) via sums

\[
f = \sum_{\alpha, \beta} f_{\beta, \alpha}.
\]

In a Hilbert space, the the inclusion \(i_\alpha\) is adjoint to the orthogonal projection \(\text{Proj}_{C_\alpha}\) onto \(C_\alpha\) (Lemma 2.11), which not necessarily the categorical projection \(\pi_\alpha\). The categorical projection map \(\pi_\alpha\) agrees with \(\text{Proj}_{C_\alpha}\) if and only if

\[
C_\alpha \perp C_{\alpha'}
\]

for all \(\alpha' \in I \setminus \alpha\). If this equation holds for both \(\alpha \in I\) and \(\beta \in J\), then the adjoint of the component map

\[
(f_{\beta, \alpha})^\dagger : D_\beta \xrightarrow{\pi_\beta^\dagger} D \xrightarrow{f^\dagger} D \xrightarrow{i_\alpha^\dagger} D_\alpha
\]

agrees with the component maps of the adjoint

\[
(f^\dagger)_{\alpha, \beta} : D_\beta \xrightarrow{i_\beta} D \xrightarrow{f^\dagger} D \xrightarrow{\pi_\alpha} D_\alpha.
\]

If Eq. 13 holds for all \(\alpha \in I\) and \(\beta \in J\), then

\[
f^\dagger = \bigoplus_{\alpha, \beta} (f_{\beta, \alpha})^\dagger.
\]
In other words, the adjoint commutes with the direct sum.

The reasoning above underpins why orthogonal components lead to a natural interpretation of the adjoint maps of 2.9 in terms of the adjoint flow. If this is the case, we can take the adjoint of 2.9 everywhere to prove the following important result.

**Theorem A.2** (Sköldberg 2018) Let $C$ be a finite-dimensional chain complex indexed by an orthogonal base $I$, $M$ a Morse matching, and

$$
C_n^M = \bigoplus_{\alpha \in I_n \cap M^0} C_{\alpha}.
$$

The diagram

$$
\begin{array}{ccc}
C_n^M & \xrightarrow{\Phi^\dagger} & C \\
\xrightarrow{\Psi^\dagger} & & \xrightarrow{h^\dagger}
\end{array}
$$

is a deformation retract of cochain complexes, where for $x \in C_\beta$ with $\beta \in I_n$,

- $(\partial_{C_n^M}^\dagger)_n(x) = \sum_{\alpha \in M^0 \cap I_n} \Gamma_{\beta,\alpha}^\dagger(x)$
- $\Phi_n^\dagger(x) = \sum_{\alpha \in I_n} \Gamma_{\beta,\alpha}^\dagger(x)$
- $\Psi_n^\dagger(x) = \sum_{\alpha \in M^0 \cap I_n} \Gamma_{\beta,\alpha}^\dagger(x)$
- $h_n^\dagger(x) = \sum_{\alpha \in I_{n-1}} \Gamma_{\beta,\alpha}^\dagger(x)$

In most circumstances – weighted Laplacians, cellular sheaves, etc. – there is indeed an orthogonal basis. However, in the Morsification Lemma 3.7, we perform a reduction on the left component of $\text{Ker } \Psi \oplus \text{Im } \Phi$

which, in general, is not orthogonal to $\text{Im } \Phi$. One needs to be careful in such situations not to utilise the adjoint flow decompositions given in Theorem A.2.

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**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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References

Barbarossa, S., Sardellitti, S.: Topological signal processing over simplicial complexes. IEEE Trans. Signal Process. 68, 2992–3007 (2020)
Barbarossa, S., Sardellitti, S., Ceci, E.: Learning from signals defined over simplicial complexes. In: IEEE 2018 IEEE Data Science Workshop (DSW), 51–55 (2018)
Belkin, M., Niyogi, P.: Laplacian eigenmaps for dimensionality reduction and data representation. Neural Comput. 15(6), 1373–1396 (2003)
Bodnar, Cristian, Frasca, Fabrizio, Wang, Yu Guang, Otter, Nina, Montúfar, Guido, Liò, P., Bronstein, M.: Weisfeiler and Lehman go topological: Message passing simplicial networks. Proceedings of the 38th International Conference on Machine Learning PMLR 139, 1026–1037 (2021)
Bronstein, M.M., Bruna, J., LeCun, Y., Vandergheynst, P.: Geometric deep learning: going beyond euclidean data. IEEE Signal Process. Mag. 34(4), 18–42 (2017)
Brown, R.: The twisted Eilenberg-Zilber theorem. In: Simposio di Topologia (Messina, 1964). Edizioni Oderisi, Gubbio (1965)
Carrière, M., Chazal, F., Glisse, M., Ike, Y., Kannan, H.: A note on stochastic subgradient descent for persistence-based functionals: convergence and practical aspects, CoRR (2020) available at arXiv:2010.08356
Chen, S., Sandryhaila, A., Moura, J.M.F., Kovacevic, J.: Signal denoising on graphs via graph filtering. In: IEEE 2014 IEEE Global Conference on Signal and Information Processing (GlobalSIP), pp. 872–876 (2014)
Coifman, R.R.: Lafon, Stéphane: diffusion maps. Appl. Comput. Harmonic Anal. 21(1), 5–30 (2006)
Contreras, I., Tawfeek, A. R.: On discrete gradient vector fields and laplacians of simplicial complexes (2021), available at arXiv:2105.05388
Contreras, I., Xu, B.: The graph Laplacian and Morse inequalities. Pac. J. Math. 300(2), 331–345 (2019)
Curry, J., Ghrist, R., Nanda, V.: Discrete Morse theory for computing cellular sheaf cohomology. Found. Comput. Math. 16(4), 957–97 (2016)
Defferrard, M., Bresson, X., Vandergheynst, P.: Convolutional neural networks on graphs with fast localized spectral filtering. Adv. Neural. Inf. Process. Syst. 29, 3844–3852 (2016)
Delgado-Friedrichs, O., Robins, V., Sheppard, A.: Skeletonization and partitioning of digital images using discrete Morse theory. IEEE Trans. Pattern Anal. Mach. Intell. 37, 654–666 (2015)
Dodziuk, J.: Finite-difference approach to the Hodge theory of harmonic forms. Am. J. Math. 98(1), 79–104 (1976)
Du, C., Szul, C., Manawa, A., Rasekh, N., Guzman, R., Davidson, R.: RGB image-based data analysis via discrete morse theory and persistent homology, CoRR, abs/1801.09530 (2018). available at arXiv:1801.09530
Ebli, S., Defferrard, M., Spreemann, G.: Simplicial neural networks. Topological Data Analysis and Beyond workshop at NeurIPS (2020)
Eckmann, B.: Harmonische Funktionen und Randwertaufgaben in einem Komplex. Commentarii Mathematici Helvetici 17(1), 240–255 (1944)
Forman, R.: Morse theory for cell complexes. Adv. Math. 134(1), 90–145 (1998)
Forman, R.: Discrete Morse Theory and the Cohomology Ring. Trans. Am. Math. Soc. 354(12), 5063–5085 (2002)
Gabrielsonn, R.B., Nelson, B. J., Dwaraknath, A., Skraba, P.: A Topology Layer for Machine Learning. In: Proceedings of the Twenty Third International Conference on Artificial Intelligence and Statistics, 202026 pp. 1553–1563
Gugenheim, V.K.A.M.: On the chain-complex of a fibration. Ill. J. Math. 16, 398–414 (1972)
Hansen, J., Gebhart, T.: Sheaf neural networks (2019), available at arXiv:2012.06333
Hansen, J., Ghrist, R.: Toward a spectral theory of cellular sheaves. J. Appl. Comput. Topol. 3(4), 315–358 (2019)
Hatcher, A.: Algebraic topology. Cambridge University Press (2002)
Hiptmair, R.: Finite elements in computational electromagnetism. Acta Numer. 11, 237–339 (2002)
Hirani, A.N.: Discrete exterior calculus. California Institute of Technology (2003)
Horak, D., Jost, J.: Spectra of combinatorial Laplace operators on simplicial complexes. Adv. Math. 244, 303–336 (2013)
Hu, X., Wang, Y., Fuxin, L., Samaras, D., Chen, C.: Topology-aware segmentation using discrete Morse theory. in: International conference on learning representations (2021)
Joswig, M., Pfeitsch, M.E.: Computing optimal Morse matchings. SIAM J. Discret. Math. 20, 11–25 (2006)
Kaczynski, T., Mischaikow, K., Mrozek, M.: Computational homology. Springer, Berlin (2006)
Kaczynski, T., Mrozek, M., Slusarek, M.: Homology computation by reduction of chain complexes. Comput. Math. Appl. 35(4), 59–70 (1998)
Kim, K., Kim, J., Zaheer, M., Kim, J., Chazal, F., Wasserman, L.: Pllay: Efficient topological layer based on persistent landscapes. Adv. Neural Inf. Process. Syst. 33, 15965–77 (2020)
Li, P., Shleizinger, N., Zhang, H., Wang, B., Eldar, Y.C.: Graph signal compression via task-based quantization, ICassp 2021 - 2021 In: IEEE international conference on acoustics, speech and signal processing (ICASSP), 5514–5518 (2021)
Martinez-Figueroa, F.: Optimal Discrete Morse Theory Simplification (Expository Survey), (2021), available at arXiv:2111.05774
Mémoli, F., Wan, Z., Wang, Y.: Persistent Laplacians: properties, algorithms and implications. SIAM J. Math. Data Sci. 4(2), 858–884 (2022)
Milnor, J.: Morse theory. Princeton University Press (1969)
Mischaikow, K., Nanda, V.: Morse theory for filtrations and efficient computation of persistent homology. Discrete & Computational Geometry 50(2), 330–353 (2013)
Moor, M., Horn, M., Rieck, B., Borgwardt, K.: Topological autoencoders. In: Proceedings of the 37th international conference on machine learning, pp. 7045–7054 (2020)
Ortega, A., Frossard, P., Kovačević, J., Moura, J.M.F., Vandergheynst, P.: Graph signal processing: Overview, challenges, and applications. Proc. IEEE 106(5), 808–828 (2018)
Robinson, M.: Topological signal processing, vol. 81. Springer (2014)
Roddenberry, T.M., Schaub, M.T., Hajij, M.: Signal processing on cell complexes. In: IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP), pp. 8852–8856 (2022)
Schaub, M.T., Zhu, Yu., Seby, J.-B., Roddenberry, T Mitchell, Segarra, S.: Signal processing on higher-order networks: Livin’ on the edge... and beyond. Signal Process. 187, 108149 (2021)
Singh, G., Memoli, F., Carlsson, G.: Topological Methods for the Analysis of High Dimensional Data Sets and 3D Object Recognition. In: Eurographics symposium on point-based graphics (2007)
Sköldberg, E.: Algebraic Morse theory and homological perturbation theory. Algebra Discrete Math. 26, 124–129 (2018)
Sköldberg, E.: Morse theory from an algebraic viewpoint. Trans. Am. Math. Soc. 358(01), 115–129 (2006)
Stefania, E., Celia, H., Kelly, M.: Morse theoretic signal compression and reconstruction on chain complexes, GitHub (2022), note Available at https://github.com/stefaniaebli/dmt-signal-processing
Von UlrikeL, L.: A tutorial on spectral clustering. Stat. Comput. 17(4), 395–416 (2007)
Wood, P., Sheppard, A.P., Robins, V.: Theory and algorithms for constructing discrete Morse complexes from grayscale digital images. IEEE Trans. Pattern Anal. Mach. Intell. 33(08), 1646–1658 (2011)
Zhou, D., Schölkopf, B.: A regularization framework for learning from graph data. In: ICML 2004 workshop on statistical relational learning and its connections to other fields (SRL 2004), pp. 132–137 (2004)

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