Nearest-Neighbour-Interactions from a minimal discrete flavour symmetry within SU(5) Grand Unification

D. Emmanuel-Costa and C. Simões
Departamento de Física and Centro de Física Teórica de Partículas, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
(Dated: January 12, 2013)

A flavour symmetry based on $\mathbb{Z}_4$ is analysed in the context of SU(5) Grand Unification with the standard fermionic content plus three right-handed neutrinos. The role of $\mathbb{Z}_4$ is to forbid some Yukawa couplings of up- and down-quarks to Higgs scalars such that the quark mass matrices $M_u$, $M_d$ have Nearest-Neighbour-Interaction (NNI) structure, once they are generated through the electroweak symmetry breaking. It turns out in this framework that $\mathbb{Z}_4$ is indeed the minimal discrete symmetry and its implementation requires the introduction of at least two Higgs quintets, which leads to a two Higgs doublet model at low energy scale. Due to the SU(5) unification, it is shown that the charged lepton mass matrix develops also NNI form. However, the effective neutrino mass matrix exhibits a non parallel pattern, in the framework of the type-I seesaw mechanism. Analysing all possible zero textures allowed by gauge-horizontal symmetry SU(5) $\times$ $\mathbb{Z}_4$, it is seen that only two patterns are in agreement with the leptonic experimental data and they could be further distinguished by the light neutrino mass spectrum hierarchy. It is also demonstrated that $\mathbb{Z}_4$ freezes out the possibility of proton decay through exchange of colour Higgs triplets at tree-level.

PACS numbers: 11.30.Hv, 12.15.Fi, 12.10.Dm, 14.60.Pq
Keywords: Flavour symmetries, Quark and lepton masses and mixing, Unified theories and models of strong and electroweak interactions, Neutrino mass and mixing

I. INTRODUCTION

Grand Unified Theories (GUTs) are beautiful attempts beyond the Standard Model (SM) for understanding the observed quark and lepton masses and their mixings, the so-called “flavour puzzle”. This is indeed corroborated by the fact that the running gauge couplings when evolved to very large energy scales, typically $10^{14} - 10^{16}$ GeV, seem to unify to a unique coupling. The simplest GUT model which accommodates the SM gauge and fermion fields in a few multiplets is based on the group SU(5), proposed in 1974 by Georgi and Glashow [1]. Some small extensions of the Georgi-Glashow model are still viable today [2–6], even in its supersymmetric version [7–10].

Generically in GUT models, not only the SM gauge couplings do unify but also the SM fermions are unified in a small number of large multiplets that lead to new phenomenological signatures. An important signature of most GUTs is the prediction for proton decay [11], which has not yet been observed and then severely constrains the GUT models. The fact that the quarks and leptons are tied together in GUT multiplets is not enough to fully determine the properties of their observed masses and mixings. However, the GUT relations among quark and lepton Yukawa matrices are an excellent starting point for building a flavour symmetry. Thus, if one requires a flavour symmetry to enforce a particular pattern in the up- and down-quark Yukawa couplings, this engenders physical consequences in the leptonic sector.

During the last decades, a huge number of flavour symmetries with different purposes have been extensively presented in the literature (in the context of SM, Grand Unification, etc.). The simplest and attractive possibility is to assume the vanishing of some Yukawa matrix elements (“texture zeroes”) by the requirement of a discrete symmetry [12–14], such that it would naturally lead to the flavour mixing angles be expressed in terms of mass ratios. The converse is not necessarily true, since one can obtain zeroes in the Yukawa matrices [15–20] just by performing some set of transformations (weak basis transformations) leaving the gauge sector diagonal. For instance, it is remarkable in the SM that one can always go to a weak basis where both up- and down-quark mass matrices $M_u$, $M_d$ have simultaneously the form or a “parallel structure”,

$$ M_{u,d} = \begin{pmatrix} 0 & A_{u,d} & 0 \\ A_{u,d} & 0 & B_{u,d} \\ 0 & B_{u,d} & C_{u,d} \end{pmatrix}, $$

known as the Nearest-Neighbour-Interaction (NNI) basis [21]. Being the matrix form in Eq. (1) for both quark sectors just a weak basis, no zero in the NNI matrices has physical meaning. The NNI basis is closely connected to the Fritzsch ansatz [22–24], which further assumes the Hermiticity condition on the NNI quark mass matrices $M_u$, $M_d$. Through a simple choice of weak basis transformation it is always possible to make $M_u$, $M_d$ Hermitian, but their structures are no longer of the NNI form. Assuming that $M_u$, $M_d$ are in the NNI basis, it has been shown in Ref. [25] that the experimental data are still in agreement with relatively small deviations from Hermiticity, at the 20% level. This procedure was also extended to the leptonic sector in Ref. [26].
Furthermore, it was also shown in Ref. [29] that it is possible to attain the up- and down-quark mass matrices $M_u$, $M_d$ with NNI structure through the implementation of an Abelian discrete flavour symmetry in the context of the two Higgs doublet model (2HDM). In that context, the minimal realisation is the group $\mathbb{Z}_4$. In a general 2HDM, a NNI form for each Yukawa coupling matrices cannot be a weak basis choice. Indeed, the requirement of the $\mathbb{Z}_4$-symmetry does imply restrictions on the scalar couplings to the quarks, although one gets no impact on the quark masses and the Cabibbo-Kobayashi-Maskawa (CKM) matrix [27, 28].

The purpose of this article is to study whether it is possible to construct a $\mathbb{Z}_4$ flavour symmetry, similar to the one implemented in Ref. [29], that leads to quark mass matrices $M_u$, $M_d$ in the NNI form in the context of SU(5) Grand Unification with the usual fermionic content. In addition, since SU(5) implies relations among quarks and leptons, we also explore the physical consequences of the $\mathbb{Z}_4$ flavour symmetry on the leptonic sector for the case where three right-handed neutrinos are added, being the type-I seesaw [24, 32] the mechanism responsible for the light neutrinos to acquire Majorana masses.

This article is organised as follows. In Section II we introduce the SU(5) $\times$ $\mathbb{Z}_4$ model. Next, in Section III we analyse the different channels of proton decay as well as the issues of unification in this model. Then in Section IV we discuss in detail the form of the leptonic mass matrix textures provided by the flavour symmetry. In Section V we present our numerical analysis on the leptonic sector for the case where three right-handed neutrinos are added, being the type-I seesawtextures obtained for the effective neutrino mass matrix are confronted with leptonic observable data. Finally, our conclusions are drawn in Section VI.

II. THE MODEL

Following Ref. [29], we build an Abelian discrete flavour symmetry within a SU(5) GUT model which yields at low energies to a NNI form for both up- and down-quark mass matrices. We choose the flavour symmetry to be Abelian, because it is the simplest way to forbid some Yukawa couplings, so that texture zeroes appear naturally in the mass matrices. The flavour symmetry is also chosen to be discrete in order to avoid the presence of Nambu-Goldstone bosons. To simplify our search for a minimal flavour symmetry realisation on the full Lagrangian, we consider only the case where the flavour group belongs to the $\mathbb{Z}_n$ family. Thus, each fermionic or Higgs multiplet, $R$, transforms as

$$R \rightarrow R' = e^{i \frac{2\pi}{n} Q(R)} R,$$

where the charges $Q(R) \in \mathbb{Z}_n$.

The particle content of our GUT model is a small extension of the original SU(5) model proposed by Georgi and Glashow [1] in 1974. It contains three generations of 10, 5* fermionic multiplets, which accommodate the left-handed fermions of the SM, $Q_i, u^c_i, d^c_i, L_i, e^c_i$, as follows

$$10_i = (Q_i, u^c_i, e^c_i), \quad 5^*_i = (L_i, d^c_i), \quad (3)$$

where $i = 1, 2, 3$ stands for the generation index. Furthermore, we introduce three right-handed neutrinos $\nu^c_i$ (in the left-handed picture), singlets under SU(5), as the simplest way to generate the light neutrino masses needed to explain the observed neutrino oscillation data. Being not constrained by any gauge symmetry, the singlets $\nu^c_i$ can have a Majorana mass term and a Dirac Yukawa term requiring that $\eta > 0$ and $\xi > 0$.

The Higgs sector of the model consists of an adjoint Higgs multiplet, $\Sigma(24)$, chargeless under $\mathbb{Z}_n$, and two quintets $H_1(5), H_2(5)$, with different $\mathbb{Z}_4$ charges $\phi_1, \phi_2$, respectively. Under these charge assignments the full scalar potential $V$ reads

$$V = -\frac{1}{2} \mu^2 \text{Tr}(\Sigma^2) + \frac{1}{3} a \text{Tr}(\Sigma^3) + \frac{1}{2} b^2 \left[\text{Tr}(\Sigma^2)\right]^2 + \frac{\lambda}{4} \text{Tr}(\Sigma^4) + H_1^1 \left(\frac{1}{2} \mu^2 + a \Sigma + \lambda_{11} \text{Tr}(\Sigma^2) + \lambda_{12} \Sigma^2\right) H_1^1 + H_2^1 \left(\frac{1}{2} \mu^2 + a \Sigma + \lambda_{21} \text{Tr}(\Sigma^2) + \lambda_{22} \Sigma^2\right) H_2^1 + \lambda_1 |H_1|^4 + \lambda_2 |H_2|^4 + \lambda_3 |H_1|^2 |H_2|^2 + \lambda_4 \left(H_1 H_2^* H_1^* H_2\right), \quad (4)$$

where the parameters $\mu, a$ and $b$ have mass dimensions, while $\lambda$ is dimensionless. Note that the self-potential terms for the adjoint field $\Sigma$ is as general as in the minimal SU(5) model.

The adjoint field $\Sigma$ breaks spontaneously the SU(5) gauge group to the SM group ($SU(3)_C \times SU(2)_L \times U(1)_Y$), through the vacuum expectation value (VEV),

$$\langle \Sigma \rangle = \sigma \text{diag}(2,2,2,-3,-3), \quad (5)$$

provided that $\sigma$ is the following solution that minimises the scalar potential [33, 34] given in Eq. (1).

$$\sigma = \frac{a}{2\lambda} \left[1 + \sqrt{1 + 4 \xi (60\eta + 7)}\right] \frac{60\eta + 7}{60\eta + 7}, \quad (6)$$

where $\eta \equiv b^2/\lambda$ and $\xi \equiv \lambda \mu^2/a^2$ requiring that $\eta > -7/60$ and $\lambda > 0$. The parameters $\eta$ and $\xi$ are allowed to allow whether the VEV from Eq. (5) corresponds to an absolute minimum of the potential. Thus, the parameters of the potential have to be properly chosen to guarantee the SM group in the broken phase [33] with the natural value for $\sigma$ lying around the unification scale $\Lambda$. The VEV given in Eq. (5) also splits the adjoint Higgs field $\Sigma$.
in its components $\Sigma_3$ (weak isospin triplet), $\Sigma_8$ (colour octet) and $\Sigma_{24}$ (singlet), which become massive.

The Higgs quintets $H_1$, $H_2$ are introduced to break the SM gauge group down to $SU(3)_c \times U(1)_{em}$, and also generate the fermion mass via the Yukawa interactions at the electroweak scale. The quintets $H_1$, $H_2$ are split by the VEV $\langle \Sigma \rangle$ into the Higgs doublets $\Phi_1$, $\Phi_2$ and the Higgs colour-triplets $T_1$, $T_2$, respectively. Thus, the SM group is then broken through the VEVs $v_1$, $v_2$ of the respective Higgs doublets $\Phi_1$, $\Phi_2$, verifying

$$v^2 \equiv |v_1|^2 + |v_2|^2 = \left( \sqrt{2} G_F \right)^{-1} = (246.2 \text{ GeV})^2,$$

where $G_F$ is the Fermi constant.

Furthermore, one has to avoid rapid proton decay mediated by the Higgs colour-triplets $T_1$, $T_2$, which can be solved by fine-tuning the parameters of the Higgs potential, $\mathcal{O} \left( \frac{1}{v^4} \right) \sim 10^{-12 ; +13}$ - the so-called doublet-triplet splitting problem. This fine-tuning can be re-expressed as new constraints on the mass parameters $\mu_1^2$ and $\mu_2^2$,

$$\mu_1^2 = 6 \sigma (a_1 - 10 \sigma \lambda_{11} - 3 \sigma \lambda_{12}), \quad (8a)$$
$$\mu_2^2 = 6 \sigma (a_2 - 10 \sigma \lambda_{21} - 3 \sigma \lambda_{22}), \quad (8b)$$

such that the Higgs colour triplets $T_1$, $T_2$ have masses:

$$m_{T_1}^2 = 5 \sigma (a_1 - \sigma \lambda_{12}), \quad (9a)$$
$$m_{T_2}^2 = 5 \sigma (a_2 - \sigma \lambda_{22}). \quad (9b)$$

Thus, once the heavy Higgs colour-triplets $T_1$, $T_2$ are integrated out, the model is just a two Higgs doublet model (2HDM). Since the adjoint Higgs multiplet carries no $Z_3$ charge, the obtained 2HDM automatically preserves the flavour symmetry in higher orders of perturbation theory, provided that no Nambu-Goldstone boson appears at tree-level due to an accidental global symmetry.

In order to fully determine the $Z_3$ charges for the fermions and the Higgs quintets such that the quark mass matrices $M_u$, $M_d$ have NNI form, one needs to analyse the Yukawa interactions [25]. The most general Yukawa Lagrangian reads as

$$-\mathcal{L}_Y = \frac{1}{4} (\Gamma_u^1)_{ij} 10, 10, H_1 + \frac{1}{4} (\Gamma_u^2)_{ij} 10, 10, H_2 + \sqrt{2} (\Gamma_d^1)_{ij} 10, 5^*_j H_1^* + \sqrt{2} (\Gamma_d^2)_{ij} 10, 5^*_j H_2^* + (\Gamma_d^3)_{ij} 5^*_j \nu^c_j H_1 + (\Gamma_d^4)_{ij} 5^*_j \nu^c_j H_2 + \frac{1}{2} (M_R)_{ij} \nu^c_j \nu^c_j + \text{H.c.}, \quad (10)$$

where $\Gamma^{1,2}_u$ and $M_R$ are symmetric complex matrices, while $\Gamma^{1,2}_d$ are just general complex matrices. The quark mass matrices $M_u$, $M_d$ are then given by

$$M_u = v_1 \Gamma_u^1 + v_2 \Gamma_u^2, \quad (11a)$$
$$M_d = v_1^* \Gamma_d^1 + v_2^* \Gamma_d^2, \quad (11b)$$

and their zeroes are directly settled by the Yukawa matrices $\Gamma^{1,2}_u$. It is clear from Eq. (11a) that the mass matrix $M_u$ is symmetric.

In order to guarantee that the $(33)$-element of up-quark matrix $M_u$ does not vanish we set the charge $\phi_2$ as

$$\phi_2 = -2 q_3, \quad (12)$$

where $q_3 \equiv Q(10_3)$. This choice automatically fixes the $Z_3$ charges of the multiplets $5^*_i$, $10_i$ as a function of $\phi_1$ and $q_3$ as

$$Q(10_i) = (q_3 + \phi_1, -q_3 - \phi_1, q_3),$$
$$Q(5^*_i) = (q_3 + 2 \phi_1, -q_3 - \phi_1, \phi_1). \quad (13)$$

The $Z_3$ charges of the right-handed neutrinos, $\nu^c_i$ (singlets under $SU(5)$) have no restrictions and are taken as free parameters,

$$Q(\nu^c_i) = (\nu_1, \nu_2, \nu_3). \quad (14)$$

It is in fact not enough to have the charge assignments given in Eq. (15) in order to guarantee the NNI structure for the mass matrices $M_u$, $M_d$, one has in addition to preserve the NNI-zero entries, which implies that one should forbid the quark bilinears corresponding to the NNI-zero entries to couple to Higgs doublets. The analysis of the quark bilinears can be made directly from the $10,10_j$ and $10,5^*_j$ bilinears. By taking into account Eq. (12) one can derive the $10,10_j$ bilinear charges as

$$\begin{pmatrix}
6q_3 + 2\phi_1 & 2q_3 & 4q_3 + \phi_1 \\
2q_3 & -2\phi_1 - 2q_3 & -\phi_1 \\
4q_3 + \phi_1 & -\phi_1 & 2q_3
\end{pmatrix} \quad (15)$$

and the $10,5^*_j$ bilinear charges as

$$\begin{pmatrix}
4q_3 + 3\phi_1 & \phi_1 & 2\phi_1 + 2q_3 \\
\phi_1 & -4q_3 - \phi_1 & -2q_3 \\
2\phi_1 + q_3 & -2q_3 & \phi_1
\end{pmatrix}. \quad (16)$$

One sees immediately from Eq. (15), that in the case of $\phi_1 = \phi_2$ one gets that both Higgs doublets $\Phi_1$, $\Phi_2$ can couple to the bilinear $10,10_j$, thus destroying the NNI form for $M_u$. According to the Eqs. (15) and (16), the lowest order group of $Z_3$-type which respects the zero-entries in the NNI form is $Z_4$, as in Ref. [25]. The symmetry group $Z_4$ is indeed the minimal discrete flavour symmetry that gives rise to NNI structure for $M_u$, $M_d$, since the other candidate, $Z_2 \times Z_2$, having all its non-trivial elements of order 2, is not viable.

The zero-entries in the Yukawa coupling matrices $\Gamma^{1,2}_{u,d}$ are unequivocally determined by Eq. (13), provided that the bilinear entries given in Eqs. (15) and (16) are taken correctly into account, as follows:

$$\Gamma^1_u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_u & 0 \\ 0 & 0 & c_u \end{pmatrix}, \quad \Gamma^2_u = \begin{pmatrix} a_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_u & 0 \end{pmatrix}, \quad (17a)$$
$$\Gamma^1_d = \begin{pmatrix} 0 & a_d & 0 \\ a_d' & 0 & 0 \\ 0 & 0 & c_d \end{pmatrix}, \quad \Gamma^2_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b_d & 0 \\ 0 & 0 & b_d' \end{pmatrix}. \quad (17b)$$
which when applying Eqs. (11) leads to mass matrices $M_u, M_d$ with a NNI form. We point out that the NNI structures thus obtained have a different nature than in the pure SM case. As we have emphasised in the introduction, in the SM the NNI structure for $M_u, M_d$ is just a choice of weak basis. Instead in our model, the NNI form for $M_u, M_d$ arises as the requirement of a $Z_4$ symmetry, i.e. the form of the Yukawa couplings $\Gamma_{1,2}^{\alpha}$ are dictated by $Z_4$ in Eqs. (17), and therefore it is not a weak basis choice. In fact, the $Z_4$ do imply new restrictions on the scalar couplings to quarks.

In order to complete the construction of the flavour symmetry $Z_4$, one has to ensure in addition that the scalar potential does not acquire an extra accidental global symmetry which, upon spontaneous electroweak symmetry breaking, would lead to a massless Nambu-Goldstone boson at tree level [33]. It is indeed true that the most general scalar potential invariant under $SU(5) \times Z_4$, given in Eq. (4), possesses an extra $U(1)$ global invariant transformation on the quintet fields. Analogously of what was described in [25], this problem can be cured by soft-breaking the $Z_4$ symmetry through the introduction of a term like

$$\mu_{12}^2 H_1^\dagger H_2 + H.c., \quad (18)$$

in the scalar potential in Eq. (4). Notice that the soft-breaking term $\mu_{12}$ is not sufficient to break CP spontaneously in this model, so that CP violation arises simply from complex Yukawa couplings, leading to the Nakayashiki-Maskawa mechanism, see KM in Ref. [28].

An elegant way to avoid the term given in Eq. (18), and at the same time prevent the existence of the global $U(1)$ symmetry is simply by adding a complex $SU(5)$ singlet Higgs field $S$ which transforms non-trivially under $Z_4$. The most general potential involving the singlet field $S$ reads as

$$V_S = \left[H_1^\dagger (\mu_{12}^2 + \lambda_{12}^2) H_2 + H.c.\right] - \frac{\mu_S^2}{2} |S|^2 + \lambda_S |S|^4 + \lambda_S^* (S^4 + H.c.). \quad (19)$$

Note that the last term in the Eq. (19), allowed by $Z_4$, explicitly prevents the appearance of Goldstone bosons that could result from an accidental global $U(1)$. In the case of the VEV of the complex singlet field $S$ is of the order of the GUT scale, it is easy to verify that $\arg((S)) = 0$ or $\pi/4$, but it does not break the CP symmetry spontaneously.

We are now able to derive the texture zeroes in the leptonic sector from the constructed $SU(5) \times Z_4$ model. Since in this model the relation

$$M_e = M_d^\dagger, \quad (20)$$

holds at the GUT scale and the zeroes of the NNI structure are placed symmetrically, one easily sees that the flavour symmetry $Z_4$ also leads to a charged lepton mass matrix $M_e$ with NNI form like for quark mass matrices $M_u, M_d$. On the other hand, for the neutrino sector we do not expect a parallel structure neither for the Dirac mass matrix, $m_D$,

$$m_D = v_1 \Gamma_D^1 + v_2 \Gamma_D^2, \quad (21)$$

nor for the Majorana mass matrix $M_R$, since the $Z_4$ charges of the right-handed neutrino are taken arbitrary.

It is well-known that the GUT relation given by Eq. (20) is not in fact compatible with the down-type quark and the charged lepton mass hierarchies observed at low energies (badly violated for the first and second generations) [39]. In the following subsections, we shall roughly sketch two small extensions of the model that have the aim to modify properly the GUT relation given in Eq. (20) without changing the zeroes in the quark mass matrices $M_u, M_d$ and the charged lepton mass matrix $M_e$.

A. “Consistent” SU(5)

A possible way to modify correctly the relation given by Eq. (20), without adding new representations is by considering some non-renormalisable higher dimensional operators $\mathcal{O}(1/\Lambda')$, due to physics above the GUT scale, $\Lambda' \gg \Lambda$. The natural scale for $\Lambda'$ is to be two or three orders of magnitude greater than the GUT scale. Higher dimensional operators involving the adjoint field $\Sigma$ of the type

$$\sum_{n=1,2} \frac{\sqrt{2}}{\Lambda'} (\Delta_n)_{ij} H_{n,a}^* 10_{ab}^i \Sigma_n^* 5_{jc}^i, \quad (22)$$

with the indices $a, b, c = 1, \cdots, 5$ and $i,j = 1, 2, 3$, would contribute to the relation in Eq. (20) as

$$M_d - M_e^\dagger = \frac{5}{\Lambda'} (v_1^* \Delta_1 + v_2^* \Delta_2). \quad (23)$$

The complex matrices $\Delta_1$ and $\Delta_2$ can account for the discrepancies between $M_d$ and $M_e$. The quark and charged lepton mass matrices $M_u, M_d, M_e$ remain in the NNI form, since the adjoint field $\Sigma$ is trivial under $Z_4$ and contributes to the quark Yukawa matrices $\Gamma_{1,2}^{\alpha}$ through dimension-five operators when it acquires VEV. The only higher dimensional operators that could spoil the NNI structure on the quark mass matrices are of dimension-six, e.g.

$$\frac{\lambda}{\Lambda^2} 10_2 10_2 H_1 H_1^* H_2, \quad (24)$$

and therefore very much suppressed, $\mathcal{O}(v^2/\Lambda^2)$. Moreover, the presence of dimension-five operators in the Higgs potential given in Eq. (4) can contribute to the splitting between the masses of the Higgs multiplets $\Sigma_3$ and $\Sigma_8$ by several orders of magnitude.
B. Adjoint SU(5)

The other alternative consists in maintaining the full Lagrangian renormalisable just by requiring a 45 dimensional Higgs scalar $\mathcal{H}(45)$, instead of the quintet $H_2$. The field representation $\mathcal{H}_a^\alpha$ satisfies the relations: $\mathcal{H}_a^\beta = -\mathcal{H}_a^\alpha$ and $\sum_{a=1}^5 \mathcal{H}_a^\alpha = 0$. Thus, the potential given in Eq. (3) is now modified by the following terms:

$$
\mathcal{H}^i \left( \frac{1}{2} H_i^2 + \lambda_2 \text{Tr}(\Sigma^2) + a_2 \Sigma + \lambda_{22} \Sigma^2 \right) \mathcal{H} + \lambda_2 (\mathcal{H}^i H)^2 + \lambda_3 [H_1]^2 (\mathcal{H}^i H) + \lambda_4 (H_1^i H \mathcal{H}^i H_1),
$$

(25)

that substitute the terms involving the quintet $H_2$. For the sake of simplicity of the notation, the SU(5) invariant contractions involving $\mathcal{H}$ were not explicitly written. The mismatch between $M_d$ and $M_e$ is explained as

$$
M_d - M_e^T = 8 \Gamma_d^2 v_{45},
$$

(26)

where $v_{45}$ is the strength of the vacuum expectation value of the field $\mathcal{H}$, assuming

$$
\langle \mathcal{H}_a^{\delta^5} \rangle = v_{45} (\delta_{a}^{\delta} - 4 \delta_{a}^{4} \delta_{a}^{1}).
$$

(27)

It is clear that if the Higgs multiplet $\mathcal{H}$ has the same $Z_4$ charge as the one assigned for $H_2$ in Eq. (12), one recovers the NNI form for the quark mass matrices $M_u$, $M_d$ and the charged lepton mass matrix $M_e$.

III. PROTON DECAY AND UNIFICATION

In this section we analyse the proton decay in the constructed SU(5) $\times Z_4$ model. In this model, the proton decays through the exchange of the heavy lepto-quark gauge bosons $X, Y$ or the colour Higgs triplets $T_1, T_2$. The experimental limits on the proton decay rate severely constrain the masses of such heavy states that we shall assume of the order of the unification scale $\Lambda$, since in this scenario the proton decay rate is inversely proportional to the mass square of the heavy states. On the other hand, the unification scale is by definition the scale where the running gauge couplings measured at the scale $M_Z = 91.1876 \pm 0.0021$ GeV [40] do unify. Thus, the limits on the proton decay rate have to be confronted with the parameters that govern the evolution of the gauge couplings.

The twelve lepto-quark gauge bosons $X, Y$ (components of a colour weak isospin doublet) arise from the adjoint 24 representation that also contains the twelve gauge bosons of the SM. The gauge bosons $X, Y$ become massive through the Higgs mechanism with a common mass, $M_V$,

$$
M_V = \frac{25}{8} g_U^2 \sigma^2,
$$

(28)

where $g_U$ is the unified gauge coupling. To suppress the $X, Y$ boson proton decay channels, one has necessarily that $M_V \gg M_p$ (the proton mass) which then leads to an approximate four fermion interaction (dimension-six operators) proportional to $1/M_V^2$ and the unified coupling $\alpha_U \equiv g_U^2/4\pi$. In this approximation, the proton decay width can be estimated as [41]:

$$
\Gamma \approx \alpha_U^2 m_p^5 / M_V^4.
$$

(29)

Making use of the most restrictive constraints on the partial proton lifetime $\tau(p \rightarrow \pi^0 e^+) > 8.2 \times 10^{33}$ years [40], one can derive a rough lower bound for the $X, Y$ mass scale $M_V$,

$$
M_V > (4.0 - 5.1) \times 10^{15} \text{ GeV},
$$

(30)

which corresponds a range of the unified gauge coupling $\alpha_U^{-1} \approx 25 - 40$, as suggested by performing the renormalisation group evolution of the gauge couplings (see details in Appendix A). Since we assume for the unification scale $\Lambda \sim M_V$, the constraint given by Eq. (30) determines the scale where the gauge couplings should unify (for a recent review see [41]).

Usually in non-supersymmetric scenarios, the proton decay through the exchange of Higgs colour triplets $T_1, T_2$ is very suppressed. Being these decay modes also described by dimension-six operators, their suppression is proportional to products of Yukawa couplings, which are much smaller than the gauge couplings in the $X, Y$ boson exchange. In fact, the contribution of these dimension-six operators vanishes at tree-level when the $Z_4$ symmetry is exact. The dimension-six operators contributing to the proton decay via the colour triplet exchange are given at tree-level by:

$$
\sum_{n=1,2} \frac{(\Gamma^a_{\nu})_{ij}(\Gamma^a_\nu)_{kl}}{M_T^2} \left[ \frac{1}{2} (Q_i Q_j)(Q_k L_l) + (u^c_i e^c_j)(u^c_k d^c_l) \right].
$$

(31)

It is then clear from the pattern of the Yukawa coupling matrices $\Gamma^a_{\nu}$ and $\Gamma^a_{\nu}$ given in Eqs. (17) that the only possible non-vanishing contribution of the dimension-six operators given in Eq. (31) involve necessarily fermions of the third generation. One concludes that at tree-level the proton does not decay through the four-fermion interactions described by the operators given in Eq. (31).

If one soft-breaks the $Z_4$ symmetry of the potential with a $\mu_{12}$-term as the one written in Eq. (18), such term mixes already at tree-level the heavy states $T_1, T_2$ and therefore induces proton decay through dimension-six operators proportional to $(\Gamma^a_\nu)_{ij}(\Gamma^a_\nu)_{kl}$, thus involving fermions of the first and the second generations. This can be avoided when a singlet scalar $S$, charged under $Z_4$, is introduced in the potential as shown in Eq. (19) and in this case proton decay via $T_1, T_2$ exchange vanishes once more at tree-level.

In what concerns unification, it is widely established that the running gauge couplings do not unify in the
context of the SM even in the presence of an extra Higgs doublet \[11\]. However, if one considers the splitting between the masses of the Higgs doublets \(\Phi\) for the unification scale \(\Lambda\), \(\Phi\) lying around the electroweak scale. We also assume the masses of the Higgs doublets \(\Phi_1, \Phi_2\) lying around the electroweak scale.

From our numerics, we have found a small variation for the unification scale \(\Lambda\),

\[ 1.3 \times 10^{14} \text{ GeV} \leq \Lambda \leq 2.4 \times 10^{14} \text{ GeV}, \]

and for the masses \(M_{\Sigma_3}\) and \(M_{\Sigma_8}\),

\[ M_Z \leq M_{\Sigma_3} \leq 1.8 \times 10^4 \text{ GeV}, \]
\[ 5.4 \times 10^{11} \text{ GeV} \leq M_{\Sigma_8} \leq 1.3 \times 10^{14} \text{ GeV}, \]

which corresponds to a mass difference of the order \(M_{\Sigma_8}/M_{\Sigma_3} = \mathcal{O}(10^{3-10})\). These results do not get improved even when a large splitting between the two Higgs doublets is considered. The details concerning the equations of gauge coupling evolution are sketched in Appendix [A]. For illustration, in Fig. [1] we plot the gauge coupling evolution at two-loop level for \(M_{\Sigma_3} = 500\) GeV, which fixes \(\Lambda = 1.9 \times 10^{14}\) GeV and \(M_{\Sigma_8} = 3.2 \times 10^{12}\) GeV in order to achieve unification.

The unification of the gauge couplings suffers from two potential problems. First, the scale \(\Lambda\) in Eq. (32) seems to be lower than what is required by the lower bound given in Eq. (59). Second, the mass splitting between \(M_{\Sigma_3}\) and \(M_{\Sigma_8}\) is unnaturally large. This may suggest the necessity of introducing extra multiplets in order to relax such constraints, e.g. see Refs. [2, 6]. It is interesting to remark that the alternative presented in subsection [II B] can successfully adjust the running of SM gauge couplings, since the broken components of the 45 Higgs representations have the correct quantum numbers [8] for the unification of the gauge couplings.

\[ \text{FIG. 1. The plot of the gauge coupling evolution, } \alpha_i, \text{ at two-loop level for the choice of } M_{\Sigma_3} = 500 \text{ GeV. The unification then occurs for } \Lambda = 1.3 \times 10^{14} \text{ GeV and } M_{\Sigma_8} = 3.2 \times 10^{11} \text{ GeV. The dashed line corresponds to the running of the gauge couplings without considering the intermediate scales } M_{\Sigma_3} \text{ and } M_{\Sigma_8}. \]

IV. QUARK AND LEPTON MASS MATRICES

In Section [II] we have derived a \(Z_4\) flavour symmetry that induces the quark mass matrices \(M_u, M_d\) to have a NNI structure in the framework of SU(5) Grand Unification with minimal fermion content. Due to the fact that quarks and leptons are tied together in fermionic multiplets, the flavour symmetry imposed at the Lagrangian also restricts the form of the charged lepton mass matrix \(M_e\), which develops a NNI structure. Moreover, the extensions considered in subsections [II A] and [II B] not only lead to \(M_d \neq M^T_d\) but also \(M_u\) is no longer symmetric \[2, 8, 11\]. Hence, in what follows we shall assume arbitrary NNI mass matrices \(M_u, M_d\) and \(M_e\), where the asymmetry among the elements \(A_{u,d,e}, A'_{u,d,e}, B_{u,d,e}\), \(B'_{u,d,e}\) can be measured by the deviation parameters \(\varepsilon_{u,d,e}^{a,b}\) as

\[ \varepsilon_{u,d,e}^{a} = \left| \frac{A'_{u,d,e} - A_{u,d,e}}{A_{u,d,e} + A'_{u,d,e}} \right|, \]
\[ \varepsilon_{b} = \left| \frac{B'_{u,d,e} - B_{u,d,e}}{B_{u,d,e} + B'_{u,d,e}} \right|, \]

where the parameters \(A_{u,d,e}, A'_{u,d,e}, B_{u,d,e}\), \(B'_{u,d,e}\) follow the convention made in Eq. (1) for the quark sector. Global measurements of the asymmetry in the quark, \(\varepsilon_q\), and charged lepton, \(\varepsilon_e\), sectors are defined as

\[ \varepsilon_q = \frac{1}{2} \sqrt{(\epsilon_{q_a}^2 + (\epsilon_{q_b}^2 + (\epsilon_{q_c}^2 + (\epsilon_{q_d})^2)), \]
\[ \varepsilon_{e} = \sqrt{\left(\epsilon_{e_a}^2 + (\epsilon_{e_b})^2 \right)} \]

It is interesting to see that in the limit when \(\varepsilon_q = \varepsilon_e = 0\) one recovers the Fritzsch ansatz [22, 24].

Concerning the neutrino sector, since neutrino charges are taken as free parameters in Eq. (14), one has to classify all viable textures for the mass matrices \(M_D, M_R\) by scanning all combinations of the \(Z_4\) neutrino charges. Hence, no NNI form is expected for the mass matrices \(M_D\) and \(M_R\) and their resulting textures have to be confronted with the neutrino experimental data. We start our scanning by deriving the allowed range for the Higgs doublet charges \(\phi_1, \phi_2\). As dictated by Eq. (12), the
charge of the Higgs doublet $\Phi_2$ can only take two values: $\phi_2 = 0$ or 2. On the other hand, the charge of the Higgs doublet $\Phi_1$ has to be odd ($\phi_1 = 1$ or 3), which can be clearly seen, for instance, by noting that the entries (2,2) and (3,3) of the bilinear given in Eq. (15) would be equal when $\phi_1$ is even. Once the $Z_4$ charges of the right-handed neutrino fields, $\nu_i$, are fixed, one can immediately determine the pattern of the effective neutrino mass matrix $m_\nu$. From the structure of the Lagrangian in Eq. (10) one can derive the effective neutrino mass matrix $m_\nu$, which is given by the usual standard type-I seesaw formula [24,52]:

$$m_\nu = -m_D M_R^{-1} m_D^\top,$$  \hspace{1cm} (36)

to an excellent approximation. Since the symmetric Majorana mass matrix $M_R$ is directly introduced at the Lagrangian level, its pattern is determined by the charges $\nu_i$. Thus the $Z_4$ charges of bilinears $\nu_i^\top \nu_j$ are given by

$$\begin{pmatrix} 2\nu_1 & \nu_1 + \nu_2 & \nu_1 + \nu_3 \\ \nu_2 + \nu_1 & 2\nu_2 & \nu_2 + \nu_3 \\ \nu_3 + \nu_1 & \nu_3 + \nu_2 & 2\nu_3 \end{pmatrix}.$$  \hspace{1cm} (37)

The texture zeroes in the Dirac neutrino mass matrix $m_D$ are then obtained thereby computing the charges of the bilinears $\nu_i^\top \nu_j^c$,

$$\begin{pmatrix} q_3 + 2\phi_1 + \nu_1 & q_3 + 2\phi_1 + \nu_2 & q_3 + 2\phi_1 + \nu_3 \\ -3q_3 + \nu_1 & -3q_3 + \nu_2 & -3q_3 + \nu_3 \\ -q_3 + \phi_1 + \nu_1 & -q_3 + \phi_1 + \nu_2 & -q_3 + \phi_1 + \nu_3 \end{pmatrix}.$$  \hspace{1cm} (38)

and verifying their couplings to the Higgs doublets. Finally, one is then able to compute the texture zeroes in the effective neutrino mass matrix, $m_\nu$, just by applying the seesaw formula given in Eq. (36). Thus, by spanning all allowed values for the charges $\phi_1$, $q_3$ and $\nu_i$, one can draw all possible zero textures allowed by the symmetry $Z_4$. We sketched in Table I all possible effective neutrino mass matrices $m_\nu$ obtained as a function of the parameters $\phi_1$, $q_3$ and $\nu_i$. The notation used to represent all textures in Table I reflects the fact that each of them is related to one of the following three classes of textures:

| I | (0  A  0) | II | (0  A  0) | III | (0  A  0) |
| A  0  B | A  B  C | 0  B  C | 0  C  D | 0  C  D |

through a permutation matrix $P_g$ of the form

$$m_\nu^{(g)} = P_g m_\nu P_g^\top,$$  \hspace{1cm} (39)

where $\{P_g\}$ are the six $3 \times 3$ permutation matrices isomorphic to the symmetric group $S_3$. This identification is also useful in simplifying the diagonalisation procedure. It is clear from the seesaw formula given in Eq. (36) that any permutation among the right-handed neutrino $Z_4$ charges does not change the pattern of the effective neutrino mass matrix $m_\nu$. This is indeed the reason why we have denoted the charges $\nu_i$ in Table I just by one ordered 3-tuple.

Unlike what happens on the quark mass matrix pair $M_q$, $M_D$, it is remarkable to verify that the seesaw formula in Eq. (36) together with the allowed $\phi_1$, $q_3$ and $\nu_i$ charges lead necessarily to a non-parallel structure in the leptonic seeaw mass matrix pair $M_e$, $m_\nu$. No NNI form was found for the effective neutrino mass matrix $m_\nu$. Nevertheless, in the case where neutrinos are Dirac fermions, a parallel structure with NNI form in the leptonic sector is indeed possible, if one also requires the flavour symmetry to forbid the appearance of the Majorana right-handed mass matrix, $M_R$. The minimal discrete group realisation is $Z_7$ with, for example, the following charge assignments: $(\phi_1, \phi_2) = (1, 0)$, $Q(10) = (2, 4, 3)$, $Q(5^\ast) = (3, 5, 4)$ and $\nu = (1, 3, 2)$. In this example, the NNI mass matrix $m_D$ encodes all neutrino masses and mixings and it was shown in Ref. [26] that the mass matrix pair $M_e$, $m_D$ with parallel structure can accommodate the leptonic experimental data.

Analysing carefully the zero textures obtained in Table I, some comments are in order. If at least two right-handed neutrinos have the same $Z_4$ charge one realises that one generation decouples from the others (i.e., Texture-III and its permutation (12)), it corresponds to the right-handed neutrino charges having only values $\nu_i = 0$ or 2. Due to the fact that the charged lepton mass matrix is in the NNI form, it is rather easy to see that having a generation which decouples in the neutrino sector is not phenomenologically viable, since it is impossible to obtain large mixing angles. In a similar way Class-I, where the effective neutrino mass matrix is a permutation of the NNI matrix, is not viable too since it leads to small mixing angles. Also by counting the number of independent free parameters of the mass matrix set $M_e$, $m_\nu$ in Class-I one gets a total of ten parameters which have to account for the twelve low energy lepton observables (six lepton masses, three mixing angles and three phases).

### Table I. The obtained textures zeroes for the effective neutrino mass matrices in the context of $SU(5) \times Z_4$ symmetry with two-Higgs doublets.

| $q_3$ | $\nu = (0, 1, 3)$ | $\nu = (1, 2, 3)$ | $\nu_i \in \{0, 2\}$ |
|------|------------------|------------------|------------------|
| 0    | $I_{(132)}$      | $II_{(12)}$      | $III_{(12)}$     |
| $\phi_1 = 1$ | 1 $I_{(13)}$ | II $I_{(132)}$ | III $I_{(132)}$ |
| 2    | $II_{(12)}$      | $I_{(132)}$      | $II_{(12)}$      |
| 3    | $II_{(13)}$      | $II_{(132)}$     | $II_{(132)}$     |

| $\phi_1 = 3$ | 1 $I_{(132)}$ | II $I_{(132)}$ | III $I_{(132)}$ |
| 2    | $II_{(12)}$      | $I_{(132)}$      | $II_{(132)}$      |
| 3    | $I_{(13)}$      | II $I_{(132)}$ | III $I_{(132)}$ |


One is then left with only two zero textures of Class-II, namely Texture-II and -II\textsubscript{(12)}. The confrontation of such textures with the neutrino data is explored in the next section.

V. NUMERICAL ANALYSIS OF NEUTRINO MASS MATRIX WITHIN THE CLASS-II

In this section we analyse the phenomenological consequences of each effective neutrino mass matrix belonging to the Class-II. As discussed in the previous section only two zero textures II and II\textsubscript{(12)} from Table I need to be confronted with the observable neutrino data. Without loss of generality one can write the charged lepton mass matrix, $M_e$, and the effective neutrino mass matrices, $m_{\nu}^{(g)}$ of Class-II as:

$$M_e = K_e^\dagger \begin{pmatrix} 0 & A_e(1 - e_\alpha^2) & 0 \\ A_e(1 + e_\alpha^2) & 0 & B_e(1 - e_\theta^2) \\ 0 & B_e(1 + e_\theta^2) & C_e \end{pmatrix},$$

(41a)

$$m_{\nu}^{(g)} = P_g \begin{pmatrix} 0 & A_\nu & 0 \\ A_\nu & B_\nu & C_\nu \\ 0 & C_\nu & D_\nu e^{i\varphi} \end{pmatrix} P_g^\dagger,$$

(41b)

where $g = e$ or (12) according to Table I the constants $A_e, B_e, A_\nu, B_\nu, C_\nu, D_\nu$ are taken real and positive. The diagonal phase matrix $K_e$ can be parameterised as

$$K_e = \text{diag}(e^{i\kappa_1}, e^{i\kappa_2}, 1).$$

(42)

and the phase $\varphi$ in Eq. (41b) cannot be absorbed by any field redefinition. In the case of Texture-II\textsubscript{(12)} the permutation matrix $P_{(12)}$ is simply given by

$$P_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(43)

Although the number of the parameters encoded in the pair $M_e, m_{\nu}$ is twelve as the number of independent physical parameters experimentally observed at low energy, the zero pattern exhibited in Eqs. (41) does imply new constraints among the independent physical parameters, as it will be shown.

In order to extract the mixings angles and the CP phases encoded in the charged lepton and the light neutrino mass matrices in Eqs. (41), one needs to evaluate the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $U$ \cite{42, 44}. $U$. It is then useful to introduce the Hermitian mass matrix $H_e$ defined by:

$$H_e = M_e M_e^\dagger.$$

(44)

From Eq. (41a) one deduces that $H_e$ is a real Hermitian mass matrix, which can be diagonalised by a real and orthogonal matrix, $O_e$, in the following way:

$$O_e^\dagger H_e O_e = \text{diag}(m_{\nu}^2_1, m_{\nu}^2_2, m_{\nu}^2_3).$$

(45)

To diagonalise the effective neutrino mass matrix $m_{\nu}^{(g)}$ it is practical to define the complex matrix $m_{\nu}^0$ as

$$m_{\nu}^0 = P_g^\dagger m_{\nu}^{(g)} P_g,$$

(46)

where $m_{\nu}^0$ is diagonalised by $U_\nu$ as,

$$U_{\nu}^\dagger m_{\nu}^0 U_\nu = \text{diag}(m_1, m_2, m_3),$$

(47)

and $m_i$ are the positive light neutrino masses. Finally, the PMNS matrix $U$ is given by

$$U = O_e^\dagger K_e^\dagger P_{(12)} U_\nu.$$

(48)

It is useful to re-express the unitary matrix $U$ in terms of the standard parameterisation, which has the property to better express the neutrino observed data in a more clear and uniform way,

$$U = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & e^{i\alpha_1/2} c_{23} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & 0 \\ s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & s_{23} c_{13} \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(49)

the effective Majorana mass, $m_{ee}$, that is proportional to the neutrinoless double beta decay amplitude \cite{46, 48}.
Eqs. (51) and (52) can determine the lightest neutrino mass squared differences, the constraints given by on the sign of the mass squared difference $\Delta$ phenomenologically viable is due to the indetermination hierarchy (IH). The fact that these two hierarchies are still charged lepton masses or constraint given by Eq. (51) implies that light neutrino masses [50–52], with the assumption of a hierarchical spectrum for the (IH) should be less than $2T$ if one takes the upper bound of quantity experiments.

Eq. (52), one gets

$$\nu$$

tering [49].

COSmic Microwave Background data of the WMAP experiment with supernovae data and data on galaxy clustering [49].

The cosmological bound is much more severe and even if one refers to a more restrictive bound on $T \leq 0.28$ eV at 95% C.L. [53], one then gets for the lightest neutrino mass the following restrictions:

$$m_1 < 0.089 \text{ eV (NH)} , \quad m_3 < 0.084 \text{ eV (IH)} . \quad (55)$$

The real and orthogonal matrix $O_e$ can be expressed in terms of the parameters $\epsilon_{a,b}$ defined in Eq. (56), and the charged lepton masses are

$$m_e = 0.486661305 \pm 0.000000056 \text{ MeV} , \quad (56a)$$

$$m_\mu = 102.728989 \pm 0.000013 \text{ MeV} , \quad (56b)$$

$$m_\tau = 1746.28 \pm 0.16 \text{ MeV} , \quad (56c)$$

evaluated at $M_Z$ scale through the renormalisation group equations for QED in the $\overline{MS}$ scheme at 1-loop level [54, 55]. In the limit where the parameters $\epsilon_{a,b}$ are small, the orthogonal matrix $O_e$ that diagonalises $M_e$ takes approximately [22, 56] the following form:

$$O_e(12) \approx -\sqrt{\frac{m_e}{m_\mu}} \left(1 - \epsilon_a^e - \frac{m_\mu}{m_\tau} \epsilon_\tau^e \right) , \quad (57a)$$

$$O_e(13) \approx \sqrt{\frac{m_e m_\mu}{m_\tau}} (1 + \epsilon_\tau^e - \epsilon_a^e) , \quad (57b)$$

$$O_e(21) \approx \sqrt{\frac{m_e}{m_\mu}} \left(1 - \epsilon_\tau^e - \frac{m_e}{m_\tau} \epsilon_a^e \right) , \quad (57c)$$

$$O_e(23) \approx \sqrt{\frac{m_\mu}{m_\tau}} (1 - \epsilon_\tau^e) , \quad (57d)$$

$$O_e(31) \approx -\sqrt{\frac{m_\mu}{m_\tau}} (1 - \epsilon_a^e - \epsilon_\tau^e) , \quad (57e)$$

$$O_e(32) \approx -\sqrt{\frac{m_\mu}{m_\tau}} \left(1 - \epsilon_\tau^e + \frac{m_e}{m_\mu} \epsilon_a^e \right) . \quad (57f)$$

The unitary matrix, $U_\nu$, that diagonalises the neutrino mass matrix $m^0_{\nu}$ given in Eq. (46) can be written in terms of the neutrino masses, the real parameter $D_\nu$ and the phase $\varphi$ from Eq. (11). If one takes the phase $\varphi = 0$ the matrix $U_\nu$ becomes an exact form of the remaining

### Table II. The three-flavour oscillation parameters within 1σ error from Ref. [48].

| Parameter | 1σ |
|-----------|----|
| $\Delta m^2_{21}$ | $(7.59^{+0.23}_{-0.18}) \times 10^{-5}$ eV$^2$ |
| $|\Delta m^2_{31}|$ | $(2.40^{+0.12}_{-0.11}) \times 10^{-3}$ eV$^2$ |
| $\sin^2 \theta_{12}$ | 0.318$^{+0.019}_{-0.016}$ |
| $\sin^2 \theta_{23}$ | 0.50$^{+0.07}_{-0.06}$ |
| $\sin^2 \theta_{13}$ | $< 0.035$ at 90% C.L. |

The constraint from Tritium $\beta$ decay [40], $m_\nu$,

$$m_\nu^2 \equiv \sum_{i=1}^3 m_i^2 |U_{1i}|^2 < (2.3 \text{ eV})^2 \quad \text{at 95% C.L.} , \quad (51)$$

and the bound on the sum of light neutrino masses, $\mathcal{T}$ from cosmological and astrophysical data:

$$\mathcal{T} \equiv \sum_{i=1}^3 m_i < 0.68 \text{ eV} \quad \text{at 95% C.L.} . \quad (52)$$

This upper limit on $\mathcal{T}$ results from the combination of the Cosmic Microwave Background data of the WMAP experiment with supernovae data and data on galaxy clustering [49].

Bounds on $|m_{ee}|$ can be estimated by taking into account the best fit values of neutrino oscillation parameters and the upper limit of $\sin^2 \theta_{13}$ from Table II together with the assumption of a hierarchical spectrum for the light neutrino masses [50, 52],

$$|m_{ee}| \lesssim 0.005 \text{ eV (NH)} , \quad (53a)$$

$$10^{-2} \text{ eV} \lesssim |m_{ee}| \lesssim 0.05 \text{ eV (IH)} . \quad (53b)$$

Since neutrino oscillations are only sensitive to the mass squared differences, the constraints given by Eqs. (51) and (52) can determine the lightest neutrino mass: $m_1$ in the case of normal hierarchy (NH) as in the charged lepton masses or $m_3$ in the case of inverted hierarchy (IH). The fact that these two hierarchies are still phenomenologically viable is due to the indetermination on the sign of the mass squared difference $\Delta m^2_{31}$. The constraint given by Eq. (51) implies that $m_1$ (NH) or $m_3$ (IH) should be less than $2.3 \text{ eV}$, which is a rather poor constraint and needs to get an improvement in future experiments.

The cosmological bound is much more severe and even if one takes the upper bound of quantity $\mathcal{T}$ given in Eq. (52), one gets

$$m_1 < 0.22 \text{ eV (NH)} , \quad m_3 < 0.22 \text{ eV (IH)} . \quad (54)$$
of the lightest neutrino (and neutrino mass differences within their allowed range we have varied all experimental charged lepton masses and the phases \( \kappa_1, \kappa_2, \varphi \), defined in Eq. (11). The remaining six parameters, \( A_e, B_e, A_\mu, B_\mu, C_{e\nu}, e_e \), are determined from the values of lepton masses and \( \epsilon_{a,b}^e, D_\nu, \varphi \). The restriction in this

\[
\begin{align*}
|U_{11}| &= \sqrt{\frac{m_2m_3(D_\nu - m_1)}{D_\nu (m_2 - m_1)(m_3 - m_1)}}, \\
|U_{12}| &= \sqrt{\frac{m_1m_3(m_2 - D_\nu)}{D_\nu (m_2 - m_1)(m_3 - m_2)}}, \\
|U_{13}| &= \sqrt{\frac{m_1m_2(D_\nu - m_3)}{D_\nu (m_3 - m_1)(m_3 - m_2)}}, \\
|U_{21}| &= \sqrt{\frac{m_1(m_1 - D_\nu)}{(m_2 - m_1)(m_3 - m_1)}}, \\
|U_{22}| &= \sqrt{\frac{(D_\nu - m_2)m_2}{(m_2 - m_1)(m_3 - m_2)}}, \\
|U_{23}| &= \sqrt{\frac{m_3(m_3 - D_\nu)}{(m_3 - m_1)(m_3 - m_2)}}, \\
|U_{31}| &= \sqrt{\frac{m_1(D_\nu - m_2)(D_\nu - m_3)}{D_\nu (m_2 - m_1)(m_3 - m_1)}}, \\
|U_{32}| &= \sqrt{\frac{m_2(D_\nu - m_1)(m_3 - D_\nu)}{D_\nu (m_2 - m_1)(m_3 - m_2)}}, \\
|U_{33}| &= \sqrt{\frac{m_3(D_\nu - m_1)(D_\nu - m_2)}{D_\nu (m_3 - m_1)(m_3 - m_2)}},
\end{align*}
\]

and the signs for the matrix elements \( (U_\nu)_{ij} \) are in the case of normal hierarchy given by

\[
\begin{pmatrix}
+ & + & + \\
\text{sign}(m_1) & \text{sign}(m_2) & \text{sign}(m_3) \\
\text{sign}(m_2m_3) & \text{sign}(m_1) & +
\end{pmatrix},
\]

and in the case of inverted hierarchy by

\[
\begin{pmatrix}
+ & + & + \\
\text{sign}(m_1) & \text{sign}(m_2) & \text{sign}(m_3) \\
\text{sign}(m_3) & + & \text{sign}(m_1m_2)
\end{pmatrix}.
\]

In our numerics we have performed the full diagonalisation with all possible values for \( \varphi \) by just applying the Eq. (17).

To study the new constraints that arise from a charged lepton mass matrix \( M_e \) in the NNI form and the effective neutrino mass matrix \( m_\nu \) belonging to Class-II, we have varied all experimental charged lepton masses and neutrino mass differences within their allowed range given in Eq. (59) and Table III respectively. The mass of the lightest neutrino (\( m_1 \) in NH or \( m_3 \) in IH) was scanned for different magnitudes below 2 eV. In order to fully reconstruct the PMNS matrix, we have also varied the free parameters \( \epsilon_{a,b}^e, D_\nu \) and the phases \( \kappa_1, \kappa_2, \varphi \), defined in Eq. (11). The remaining six parameters, \( A_e, B_e, A_\mu, B_\mu, C_{e\nu}, e_e \), are determined from the values of lepton masses and \( \epsilon_{a,b}^e, D_\nu, \varphi \). The restriction in this scan was to accept only the input values which correspond to a reconstructed PMNS matrix \( U \) that naturally leads to the mixing angles \( \theta_{12}, \theta_{23} \) and \( \theta_{13} \) within their experimental bounds presented in Table III.

![Plot of \( \sin^2 \theta_{13} \) as a function of \( m_1 \) in the case of Texture-II and normal hierarchy. The dashed lines correspond to \( \sin^2 \theta_{13} = 0.016 \pm 0.010 \) at 1σ C.L. from the global analysis 57. The plot of \( \sin^2 \theta_{23} \) vs \( m_1 \) is not present here, since it reveals no correlation.](image)

From our search, we have found that the neutrino mass hierarchy can distinguish the two zero textures, II and II(12), since Texture-II is only compatible with normal hierarchy, while Texture-II(12) requires inverted hierarchy. In Fig. 2 we plot \( \sin^2 \theta_{13} \) against the lightest neutrino mass \( m_1 \) for the Texture-II where normal hierarchy applies. For the mixing angles \( \theta_{12} \) and \( \theta_{23} \) we found no correlation at all. While in Fig. 2 we plot \( \sin^2 \theta_{23} \), \( \sin^2 \theta_{13} \) over the lightest neutrino mass, which is \( m_3 \) for the Texture-II(12) since it is only compatible with inverted hierarchy. In the plots of \( \sin^2 \theta_{13} \) over the lightest neutrino mass shown in Fig. 2 and 3 we have also drawn the new hint of non-zero \( \sin^2 \theta_{13} \):

\[
\sin^2 \theta_{13} = 0.016 \pm 0.010,
\]

at 1σ C.L. from the global analysis 57 of all available neutrino oscillation data. A direct determination of \( \sin^2 \theta_{13} \) can affect significantly the validity of our model. It is indeed clear in Fig. 4 that neither Texture-II nor Texture-II(12) are compatible with inverted or normal hierarchy, respectively. In both situations, the value of \( |U_{13}| \) is two orders of magnitude outside the required bound for \( \sin^2 \theta_{13} \) given in Table III.

One can clearly see from Fig. 2 and Fig. 3 that for both textures the lightest neutrino mass, \( m_1 \) (NH) or \( m_3 \) (IH), is bounded. In the case of Texture-II one has,

\[
0.0013 \text{eV} \leq m_1 \leq 0.016 \text{eV},
\]

whereas for Texture-II(12) one has,

\[
0.0042 \text{eV} \leq m_3 \leq 0.011 \text{eV},
\]
which let us to conclude that the light neutrino mass spectrum cannot be quasi-degenerated and there is no room to account for a massless neutrino state (as it is still allowed by general neutrino oscillation analyses). The upper bound on the constraint from Tritium $\beta$ decay given in Eq. (51) does not have any impact in these results. Although the cosmological and astrophysical upper bound on the sum of light neutrino masses seems rather severe, the upper limit on the lightest neutrino mass given in Eq. (51) or even the most restrictive value in Eq. (54) are indeed above the bounds reported in Eqs. (62) and (63).

Concerning the effective Majorana mass $|m_{ee}|$, we present in Fig. 5 the value of $|m_{ee}|$ as function of the lightest neutrino mass for both Texture-II and -II(12). From our scan, we have obtained the following limits for $|m_{ee}|$:

$$6.4 \times 10^{-4} \text{eV} < |m_{ee}| < 2.2 \times 10^{-3} \text{eV},$$

in the case of Texture-II and,

$$0.015 \text{eV} < |m_{ee}| < 0.022 \text{eV},$$

for Texture-II(12). These bounds obtained for $|m_{ee}|$ are in full agreement with those given in Eq. (53). One expect that future improvements on the experimental value of $|m_{ee}|$ may have impact on this textures (II and II(12)) or on the $Z_4$ symmetry itself, since a change on the magnitude of the experimental value of $|m_{ee}|$ may drastically constrains our model.

Before closing the section one may address the question which could be the smallest deviations to the Fritzsch ansatz, $\varepsilon_c$, for the charged lepton mass matrix $M_e$ acceptable by the experimental data as it was done for the quark sector in Ref. [25]. In the quark sector, the lower bound for $\varepsilon_q$ consistent with electroweak data can be evaluated by taking into account the values of the quark masses, CKM elements $(V_{us}, V_{cb}, V_{ub})$ and the angle $\beta \equiv \arg(-V_{ud}V_{cb}^*V_{ub})$ of the unitarity triangle listed in Table III. We have performed an update of the lower bound of the parameter $\varepsilon_q$ defined in Eq. (35),

$$\varepsilon_q \geq 0.188,$$

which has increased more than 10% compared with the value reported in [25].

The deviation from Hermiticity of the charged lepton mass matrix $\varepsilon_c$ was computed for the Texture-II and -II(12) and the lower bounds $\varepsilon_c > 0.0011$ and $\varepsilon_c > 0.0013$
TABLE III. Values of the quark masses, the CKM element moduli $|V_{us}|$, $|V_{cb}|$, $|V_{ub}|$ and the angle $\beta$ of the unitarity triangle at the scale $M_Z$. The quark masses are calculated at $M_Z$ scale by taking properly into account the 4-loop renormalisation group equations for QCD in the MS scheme [54, 55, 58–60] and using as input the masses given in [40].

\[
\begin{align*}
m_u &= 1.4 \pm 0.5 \text{ MeV} & m_d &= 2.9 \pm 0.5 \text{ MeV} \\
m_c &= 0.62^{+0.06}_{-0.05} \text{ GeV} & m_s &= 58^{+16}_{-12} \text{ MeV} \\
m_t &= 170.2 \pm 1.0 \text{ GeV} & m_b &= 2.86^{+0.16}_{-0.12} \text{ GeV} \\
|V_{us}| &= 0.2253 \pm 0.0007 & |V_{ub}| &= (3.47^{+0.16}_{-0.12}) \times 10^{-3} \\
|V_{cb}| &= (41.0^{+1.1}_{-0.7}) \times 10^{-3} & \sin 2\beta &= 0.673 \pm 0.023
\end{align*}
\]

were found, respectively. These bounds imply that the neutrino data allow for the charged lepton mass matrices $M_{e}$ to be much closer to Hermiticity than the quark mass matrices, by two orders of magnitude higher.

VI. CONCLUSIONS

In this work we extended the flavour symmetry proposed in Ref. [25] in the context of SU(5) Grand Unification. Such symmetry was imposed in the Lagrangian in order to force the quark mass matrices $M_{u}$, $M_{d}$ to have NNI form, after spontaneous gauge symmetry breaking. In this GUT model, beyond the standard field content, three right-handed neutrinos and two Higgs quintets were added. It turns out that the light neutrino masses are generated through type-I seesaw mechanism and the low energy theory below the GUT scale is just a two Higgs doublet model. In this context, the minimal realisation of such discrete symmetry is $Z_{4}$.

Since in the SU(5) $\times$ $Z_{4}$ model the charged lepton mass matrix, $M_{e}$, are related to the down-quark mass matrix $M_{d}$, it was not surprising to verify that the mass matrix $M_{e}$ acquired a NNI form, too. Instead, the right-handed neutrinos being singlets under SU(5), their $Z_{4}$ charges are free parameters and the neutrino mass matrix, $m_{\nu}$ has no NNI form. There are in fact only six zero textures allowed for the effective neutrino mass matrix, $m_{\nu}$. However, the neutrino oscillation data select just two textures among the six possibilities: texture-II, which is compatible only with normal hierarchy for the neutrino mass spectrum and the Texture-II(12), which demands neutrino mass spectrum to have inverted hierarchy.

In Table IV we summarise the main predictions for Texture-II and -II(12). In both textures, the lightest neutrino mass is predicted to be bounded and the neutrino mass spectrum to be hierarchical, therefore not compatible with a massless neutrino. Our results are also in agreement with three other relevant constraints, namely the effective Majorana mass $m_{ee}$, the constraint from Tritium $\beta$ decay and the cosmological bound on the sum of light neutrino masses. We have also obtained that the deviation of Hermiticity on the charged lepton mass matrices is two orders of magnitude lower than the one estimated for the quark sector.

Future improvements on the knowledge of neutrino oscillations, neutrinoless double beta decay, tritium beta decay and cosmological astrophysics measurements may be decisive for testing the viability of the SU(5) $\times$ $Z_{4}$ model.

TABLE IV. Summary of Texture-II and -II(12) results.

| Texture-II (NH) | Texture-II(12) (IH) |
|-----------------|---------------------|
| (0 * 0)         | ( * * * )           |
| ( * * * )       | (0 * 0)             |
| (0 * * )        | (0 * 0 * )          |
| $\sin^{2}\theta_{13} > 0.010$ | $0.0013 \text{eV} \leq m_{1} \leq 0.016 \text{eV}$ |
| $0.0042 \text{eV} \leq m_{3} \leq 0.011 \text{eV}$ | $0.00064 \text{eV} < |m_{ee}| < 0.0022 \text{eV}$ |
| $0.015 \text{eV} < |m_{ee}| < 0.022 \text{eV}$ | $\varepsilon_{e} > 0.0011$ |
| $\varepsilon_{e} > 0.0013$ | $\varepsilon_{e} > 0.0013$ |

ACKNOWLEDGMENTS

We would like to thank Gustavo C. Branco and M. N. Rebelo for fruitful discussions and reading carefully the manuscript. We would also like to thank Sergio Palomares-Ruiz for pointing out some useful remarks. This work was partially supported by Fundação para a Ciência e a Tecnologia (FCT, Portugal) through the projects CERN/FP/109305/2009, CERN/FP/116328/2010, PTDC/FIS/098188/2008 and CFTP-FCT Unit 777 which are partially funded through POCTI (FEDER), by Marie Curie Initial Training Network UNILHC PITN-GA-2009-237920, by Accion Complementaria Luso-Espanhola FCT and MICYT with project number 20NSML3700. The work of C. Simões is also supported by FCT under the contract SFRH/BD/61623/2009.

Appendix A: Two-loop evolution of the running gauge couplings

In this appendix we collect the two-loop renormalisation group equations for the gauge coupling constants
\[ \frac{d}{d\alpha_i} = -\frac{b_i}{2\pi} - \frac{1}{8\pi^2} \sum_j b_{ij} \alpha_j \]
\[ + \frac{1}{32\pi^4} \sum_{f=a,d,e, k=1,2} C_{if} \text{Tr} \left( \Gamma^k_f \Gamma^k_f \right), \]
(A1)

\[ \alpha_1 (i = 1, 2, 3), \text{which can be written in the form } \]
\[ \frac{d}{d\alpha_i} = -\frac{b_i}{2\pi} - \frac{1}{8\pi^2} \sum_j b_{ij} \alpha_j \]
\[ + \frac{1}{32\pi^4} \sum_{f=a,d,e, k=1,2} C_{if} \text{Tr} \left( \Gamma^k_f \Gamma^k_f \right), \]
(A1)

where \( \alpha_1 = 5/3 \alpha_2, b_i \text{ are the usual one-loop beta co-}

\[ \alpha_i (\Lambda) = \alpha_1 (\Lambda) = \alpha_3 (\Lambda). \]
(A2)

In the region of energy scales where one has only the

\[ b_i = \left( \begin{array}{cc} 41 & 19 \\ -10 & -7 \end{array} \right), \quad b_{ij} = \left( \begin{array}{ccc} 199 & 27 & 44 \\ 9 & 35 & 12 \\ 11 & 6 & -26 \end{array} \right). \]
(A3)

The two-loop coefficients \( C_{if} \) are given by,

\[ C_{if} = \left( \begin{array}{ccc} 17 & 2 & 2 \\ 2 & 10 & 3 \\ 2 & 3 & -7 \end{array} \right), \]
(A4)

and they are neglected in our Runge-Kutta integration.

Concerning the \( \beta \)-function coefficients for the relevant particle content of the \( SU(5) \) theory, absent in the SM,

\[ b_{ij}^{\Phi_{1,2}} = \left( \begin{array}{cc} \frac{1}{10} & 0 \\ 0 & 0 \end{array} \right), \quad b_{ij}^{\Phi_{1,2}} = \left( \begin{array}{cc} \frac{9}{50} & \frac{9}{10} \\ \frac{9}{10} & 0 \end{array} \right), \]
(A5)

\[ b_{ij}^{T_{1,2}} = \left( \begin{array}{cc} \frac{1}{15} & 0 \\ 0 & 0 \end{array} \right), \quad b_{ij}^{T_{1,2}} = \left( \begin{array}{cc} \frac{2}{15} & \frac{1}{3} \\ \frac{1}{3} & 0 \end{array} \right), \]
(A6)

\[ b_{ij}^{S_2} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \quad b_{ij}^{S_2} = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \]
(A7)

which are introduced at the appropriate intermediate scales.

[1] H. Georgi and S. Glashow, Phys.Rev.Lett., 32, 438 (1974).
[2] L. Dorsner and P. Fileviez Perez, Nucl. Phys. B, 723, 53 (2005), arXiv:hep-ph/0504276.
[3] L. Dorsner, P. Fileviez Perez, and G. Rodrigo, Phys. Rev. D, 75, 125007 (2007), arXiv:hep-ph/0607208.
[4] L. Dorsner and P. Fileviez Perez, JHEP, 06, 029 (2007), arXiv:hep-ph/0612216.
[5] B. Bajc and G. Senjanovic, JHEP, 08, 014 (2007), arXiv:hep-ph/0612029.
[6] P. Fileviez Perez, Phys. Lett. B, 654, 189 (2007), arXiv:hep-ph/0702287.
[7] B. Bajc, P. Fileviez Perez, and G. Senjanovic, Phys. Rev. D, 66, 075005 (2002), arXiv:hep-ph/0204311.
[8] B. Bajc, P. Fileviez Perez, and G. Senjanovic, (2002), arXiv:hep-ph/0210374.
[9] D. Emmanuel-Costa and S. Wiesenfeldt, Nucl. Phys. B, 661, 62 (2003), arXiv:hep-ph/0302272.
[10] P. Fileviez Perez, Phys.Rev. D, 76, 071701 (2007), arXiv:0705.3589 [hep-ph].
[11] P. Nath and P. Fileviez Perez, Phys.Rept., 441, 191 (2007), arXiv:hep-ph/0601023 [hep-ph].
[12] G. C. Branco, C. Q. Geng, R. E. Marshak, and P. Y. Xue, Phys. Rev. D, 36, 928 (1987).
[13] K. S. Babu and J. Kubo, Phys.Rev. D, 71, 056006 (2005), arXiv:hep-ph/0411226 [hep-ph].
[14] W. Grimus, A. S. Joshipura, L. Lavoura, and M. Tanimoto, Eur. Phys. J. C, 36, 227 (2004), arXiv:hep-ph/0312218.
[15] C. I. Low, Phys. Rev. D, 71, 073007 (2005), arXiv:hep-ph/0501251.
[16] P. M. Ferreira and J. P. Silva, (2010), arXiv:1012.2874 [hep-ph].
[17] F. G. Canales and A. Mondragon, (2010), arXiv:1101.6932 [hep-ph].
[18] G. C. Branco, D. Emmanuel-Costa, and R. Gonzalez Felipe, Phys. Lett. B, 477, 147 (2000), arXiv:hep-ph/9911418.
[19] G. C. Branco, D. Emmanuel-Costa, R. Gonzalez Felipe, and H. Serôdio, Phys. Lett. B, 670, 340 (2009), arXiv:0711.1613 [hep-ph].
[20] D. Emmanuel-Costa and C. Simões, Phys. Rev. D, 79, 125005 (2009), arXiv:0903.0564 [hep-ph].
[21] G. C. Branco, L. Lavoura, and P. Mota, Phys. Rev. D, 66, 073006 (2002), arXiv:hep-ph/0210374.
[22] G. C. Branco, C. Q. Geng, R. E. Marshak, and P. Y. Xue, Phys. Rev. D, 36, 928 (1987).
