Abstract. We determine the all-genus Hodge–Gromov–Witten theory of a smooth hypersurface in weighted projective space, under a mild condition. In particular, we obtain the first genus-zero computation of Gromov–Witten invariants for non-Gorenstein hypersurfaces, where the convexity property fails.

0. Introduction

Gromov–Witten theory has known a tremendous development in the last thirty years. Originated from theoretical physics, it is mathematically formulated as an intersection theory of complex curves traced on a complex smooth projective variety, and provides invariants that one thinks of as a virtual count of these curves. The most famous example is a full computation of the genus-zero invariants enumerating rational curves on the quintic threefold \([6, 19, 32]\).

Gromov–Witten theory is well understood in all genus for toric varieties, or even toric Deligne–Mumford (DM) stacks \([21, 33]\). Precisely, the moduli space of stable maps inherits a torus action from the target space and the computation essentially reduces to a calculation on the moduli space to the fixed locus. This is the content of the virtual localization formula \([21]\), which is an enhancement of the classical Atiyah–Bott localization formula \([2]\). We also refer to \([13]\) for an algebraic proof.

Smooth hypersurfaces in toric DM stacks are the next class of spaces to consider, but little is known in this situation. The difficulty comes from the non-invariance of the hypersurface by the torus action in general, so that there is no direct way to apply a localization formula to decrease the complexity of the problem. Consider the famous example of the quintic hypersurface in \(\mathbb{P}^4\). As we mention above, the genus-zero theory is fully determined \([6, 19, 32]\). The genus-one case is completely proven by Zinger \([37]\) after a great deal of hard work, and nowadays several approaches are solving it up to genus three \([7, 15, 25]\). It is worth noticing that physicists have predictions up to genus 52 \([27]\) and that Maulik–Pandharipande \([34]\) described a
proposal working in any genus, although it is too hard to implement for practical use. We also mention a recent breakthrough proving the BCOV holomorphic anomaly conjecture [4], see [7, 25].

Even in genus zero the problem of computing Gromov–Witten invariants of smooth hypersurfaces in toric DM stacks is far from being completely solved. Consider the special case of hypersurfaces in weighted projective spaces. The genus-zero theory is only known under a restrictive condition: the degree of the hypersurface is a multiple of every weight. One refers to it as the Gorenstein condition, as it is the condition for the coarse space of the hypersurface to have Gorenstein singularities. We recall that Gromov–Witten theory is invariant under smooth deformations, hence we can choose any defining polynomial of degree \( d \) as long as the associated hypersurface is a smooth DM stack. As a consequence, one can also rephrase the Gorenstein condition as the existence of a Fermat hypersurface of degree \( d \), that is defined by a Fermat polynomial of the form \( x_1^{a_1} + \cdots + x_N^{a_N} \).

There is a substantial simplification for the genus-zero theory of hypersurfaces in weighted projective spaces under Gorenstein condition; it is called the convexity property, see [22, Introduction]. It implies that the virtual cycle of the theory, which is the crucial object to handle, equals the top Chern class of a vector bundle over the moduli space of stable maps to the weighted projective space. It is then calculated by a Grothendieck–Riemann–Roch formula [10, 28, 30, 36] and genus-zero Gromov–Witten theory of the hypersurface is deduced from genus-zero Gromov–Witten theory of the weighted projective space; one calls it Quantum Lefschetz Principle [10, 28, 30, 36]. Without the Gorenstein condition, the convexity property may fail and the virtual cycle is not computable. Although Fan and Lee [15] obtain a version of Quantum Lefschetz Principle in higher genus for projective hypersurfaces, a general statement is false [9].

In this paper, we work on smooth hypersurfaces in weighted projective spaces, under a mild condition (4). Precisely, we relax the existence of a Fermat hypersurface to the existence of a chain hypersurface, that is defined by a chain polynomial of the form \( x_1^{a_1} x_2 + \cdots + x_N^{a_N-1} x_N + x_N^{a_N} \). This is more general than the Gorenstein condition and non-convex cases appear. In genus zero, we relax Condition (4) to Condition (4'), see Remark 2.10 and we give in Proposition 2.13 a more conceptual interpretation of this condition. We prove two results for these hypersurfaces:

- a genus-zero Quantum Lefchetz Principle, Corollary 2.9
- a Hodge Quantum Lefchetz Principle in arbitrary genus, Theorem 2.5.

In the first, we express genus-zero Gromov–Witten theory of the hypersurface in terms of genus-zero Gromov–Witten theory of the weighted projective space. In the second, we do the same in arbitrary genus, once we cap virtual cycles with the Hodge class, that is the top Chern class of the Hodge bundle, see Definition 1.1. As a consequence, this paper gives the first computation of genus-zero Gromov–Witten theory of hypersurfaces in a range of cases where the convexity property fails. It also gives the first comprehensive computation of Hodge integrals, that are Gromov–Witten invariants involving the Hodge class, in arbitrary genus for these hypersurfaces.

In order to tackle non-convexity issues, we develop in this paper a method that we phrase in a very general framework, opening the way to further new results in Gromov–Witten theory. We call it \textit{Regular Specialization} Theorem 1.20 as it consists of deforming a given smooth DM stack into a singular one in a regular way.
It can be understood as an enhancement of the invariance of Gromov–Witten theory under smooth deformations. Precisely, given a regular $\mathbb{A}^1$-family $X$ of DM stacks, that is a flat morphism $X \to \mathbb{A}^1$ with $X$ smooth, the perfect obstruction theory on the moduli space of stable maps to the total space $X$ pulls back to a perfect obstruction theory on every fiber and the associated virtual cycle is independent of the fiber, we call it regularized virtual cycle. Furthermore, on smooth fibers, it equals the cap product of the Gromov–Witten cycle with the Hodge class. Provided we have a global torus action on the $\mathbb{A}^1$-family $X$, the regularized virtual cycle localizes to the fixed locus in the central fiber, see Theorem 1.26.

Genus zero is a special interesting case, as the Hodge class equals the fundamental class and the regularized virtual cycle equals the Gromov–Witten virtual cycle. Let us call a DM stack regularizable, see Definition 2.12, if we can embed it as a fiber of a regular affine family of DM stacks. Although a regularizable DM stack may have bad singularities, we provide it a genus-zero Gromov–Witten theory via the regularized virtual cycle, and we prove invariance of the genus-zero theory under regular deformations, see Proposition 2.14. As a consequence, we can apply the localization formula whenever we have a torus action on the fiber, not necessarily on the total family. One strategy to compute genus-zero Gromov–Witten theory of a DM stack is thus to take a regular specialization to another DM stack admitting a torus action with sufficiently nice fixed locus, see below for more details.

At last, we highlight this paper is the Gromov–Witten counterpart of our previous results [22–24] on the quantum singularity (FJRW, [16, 17, 35]) theory of Landau–Ginzburg orbifolds defined by chain polynomials. It enters the big picture of the Landau–Ginzburg/Calabi–Yau (LG/CY) correspondence [8]. In particular, Theorem 2.9 leads to a computation of the I-function using Givental’s formalism [20] and eventually to a genus-zero mirror symmetry theorem without convexity. Comparing with results in [22], we should then obtain the LG/CY correspondence, extending the work of Chiodo–Iritani–Ruan [8]. We will discuss it in another paper. We also observe that the knowledge of Hodge integrals is crucial for a computation of the hamiltonians of the Double Ramification (DR) hierarchy introduced by Buryak [5] and may lead to new insights on the structure of Gromov–Witten invariants.

**Future works.** This paper is the foundation stone of a strategy aiming at computing all-genus Gromov–Witten invariants of projective hypersurfaces, and possibly other projective varieties. The idea is the following: by Costello’s theorem [11], genus-$g$ Gromov–Witten invariants of a projective variety $X$ are explicitly expressed in terms of genus-0 Gromov–Witten invariants of the symmetric product $S^{g+1}X$.

Let $X$ be a projective variety and assume we have an $\mathbb{A}^1$-family $X$ of DM stacks admitting a torus action and whose fiber at $1 \in \mathbb{A}^1$ is $X$. Taking the symmetric fibered product over $\mathbb{A}^1$, we obtain an $\mathbb{A}^1$-family $X_g$ of DM stacks admitting a torus action and whose fiber at $1 \in \mathbb{A}^1$ is $S^{g+1}X$. Precisely, we have

$$X_g = [X \times_{\mathbb{A}^1} \cdots \times_{\mathbb{A}^1} X/\mathfrak{S}_{g+1}].$$

By Hironaka’s theorem [24] and its equivariant version (see e.g. [29]), there exists a resolution of singularities $X_g'$ of the DM stack $X_g$, which is an isomorphism outside the singular locus of $X_g'$ and which preserves the torus action. In particular, we get a morphism $X_g' \to \mathbb{A}^1$ and the fiber at $1 \in \mathbb{A}^1$ is still $S^{g+1}X$. Moreover, the birational
map \( \widetilde{\mathcal{X}}_g \to \mathcal{X}_g \) is obtained by a sequence of blow-ups and the morphism \( \mathcal{X}_g \to \mathbb{A}^1 \) is flat, hence the morphism \( \widetilde{\mathcal{X}}_g \to \mathbb{A}^1 \) is flat as well, see for instance [18, Appendix B.6.7]. As a consequence, the DM stack \( \widetilde{\mathcal{X}}_g \) is a regular \( \mathbb{A}^1 \)-family admitting a torus action and whose fiber at \( 1 \in \mathbb{A}^1 \) is the symmetric product \( S^{g+1}X \). According to our genus-0 Regular Specialization Theorem, genus-0 Gromov–Witten invariants of \( S^{g+1}X \), and thus genus-\( g \) Gromov–Witten invariants of \( X \), are expressed by the localization formula in terms of genus-0 Gromov–Witten invariants of the torus-fixed loci in (the fiber at \( 0 \in \mathbb{A}^1 \) of) \( \widetilde{\mathcal{X}}_g \).

Acknowledgement. The author is extremely grateful to his former PhD advisor Alessandro Chiodo for many interesting discussions on this topic. Results in this paper have been presented during the Conference “New perspectives in Gromov–Witten theory” in Paris. The author is also partially funded by the ANR project Catag, ANR-17-CE40-0014.

1. Hodge–Gromov–Witten theory

In this section, we prove a general theorem on Hodge–Gromov–Witten theory, that we call ‘Regular Specialization Theorem’. The context is the following.

**Definition 1.1.** Given a smooth DM stack \( \mathcal{Y} \), Gromov–Witten theory provides a virtual fundamental cycle for the moduli space \( \mathcal{M}_\mathcal{Y} \) of stable maps to \( \mathcal{Y} \). We call Hodge virtual cycle the cup product of the virtual fundamental cycle with the top Chern class of the Hodge bundle. Hodge–Gromov–Witten theory is then intersection theory on \( \mathcal{M}_\mathcal{Y} \) against this cycle.

**Definition 1.2.** A morphism \( f: \mathcal{X} \to \mathcal{Y} \) between two DM stacks is called a family when it is flat. We also say that \( \mathcal{X} \) is a \( \mathcal{Y} \)-family. Inverse images of geometric points \( y \in \mathcal{Y} \) are called fibers. A regular family is a family for which the DM stack \( \mathcal{X} \) is smooth.

Let \( p: \mathcal{X} \to \mathbb{A}^1 \) be a regular \( \mathbb{A}^1 \)-family of DM stacks, and denote by \( X_0 \) and \( X_1 \) its fibers at \( 0 \in \mathbb{A}^1 \) and at \( 1 \in \mathbb{A}^1 \). We assume \( X_0 \) and \( X_1 \) to be proper, and \( X_1 \) to be smooth, but we do not impose any restriction on singularities of \( X_0 \). Depending on the purpose, we may also assume the \( \mathbb{A}^1 \)-family \( \mathcal{X} \) is equipped with a torus action leaving \( X_0 \) invariant.

Let \( \mathcal{M}_{X_0} \) and \( \mathcal{M}_{X_1} \) be the moduli spaces of stable maps to \( X_0 \) and to \( X_1 \), with arbitrary genus, degree, number of markings, and isotropy type at markings. Gromov–Witten theory for smooth DM stacks provides a perfect obstruction theory and a virtual fundamental cycle for the moduli space \( \mathcal{M}_{X_1} \), but not for \( \mathcal{M}_{X_0} \).

In the first subsection, we construct perfect obstruction theories on the moduli spaces \( \mathcal{M}_{X_0} \) and \( \mathcal{M}_{X_1} \), and we call the associated virtual fundamental cycles ‘regularized virtual cycles’.

The Regular Specialization Theorem can be phrased as an equality between regularized virtual cycles of \( \mathcal{M}_{X_0} \) and \( \mathcal{M}_{X_1} \). Moreover, whenever the target space is smooth, e.g. for \( X_1 \), we show the regularized virtual cycle equals the Hodge–Gromov–Witten virtual cycle, up to a sign.

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1For a family \( \pi: \mathcal{C} \to S \) of genus-\( g \) curves, the Hodge bundle is a rank-\( g \) vector bundle on \( S \) defined by the push-forward \( \pi_* \omega_{\mathcal{C}/S} \) of the relative canonical sheaf.
Graber–Pandharipande’s virtual localization formula [21] applies to regularized virtual cycles. Therefore, provided we have a torus action on the $\mathbb{A}^1$-family preserving the central fiber $X_0$ and since the fixed locus in $X_0$ is smooth, we can decompose the Hodge–Gromov–Witten cycle of $M_{X_1}$ into Hodge–Gromov–Witten cycles of the fixed loci in $M_{X_0}$.

1.1. Perfect obstruction theories.

**Notation 1.3.** For a DM stack $\mathcal{Y}$, we denote by $M_{\mathcal{Y}}$ the moduli space of stable maps to $\mathcal{Y}$, by $\pi_{\mathcal{Y}}: C_{\mathcal{Y}} \to M_{\mathcal{Y}}$ the universal curve, by $f: C_{\mathcal{Y}} \to \mathcal{Y}$ the universal map, and by $\omega_{\pi_{\mathcal{Y}}}$ the relative dualizing sheaf. In the special cases of $\mathcal{X}$, $X_0$, and $X_1$, we simplify notations of the maps as

$$\pi = \pi_{\mathcal{X}}, \quad \pi_0 = \pi_{X_0}, \quad \pi_1 = \pi_{X_1}, \quad f = f_{\mathcal{X}}, \quad f_0 = f_{X_0}, \quad f_1 = f_{X_1}.$$  

The flat morphism $p: X \to \mathbb{A}^1$ induces a flat morphism

$$q: M_{\mathcal{X}} \to M_{\mathbb{A}^1} \simeq \mathbb{A}^1 \times M_{g,n} \to \mathbb{A}^1.$$  

Furthermore, we have fiber diagrams

$$\begin{array}{ccc}
M_{X_0} & \xrightarrow{j_0} & M_{\mathcal{X}} \\
q_0 \downarrow & & \downarrow q \\
0 & \xrightarrow{j_1} & \mathbb{A}^1
\end{array}$$

In particular, the maps $j_0$ and $j_1$ are closed immersions, hence proper. We also introduce notations for maps in the following fiber diagrams

$$\begin{array}{ccc}
X_0 & \xrightarrow{i_0} & \mathcal{X} \\
p_0 \downarrow & & \downarrow p \\
0 & \xrightarrow{i_1} & \mathbb{A}^1
\end{array}$$

$$\begin{array}{ccc}
X_1 & \xrightarrow{i_1} & \mathcal{X} \\
p_1 \downarrow & & \downarrow p \\
1 & \xrightarrow{i_0} & \mathbb{A}^1
\end{array}$$

The map $i_0$ yields an exact triangle of cotangent complexes

$$i_0^* L_{\mathcal{X}} \to L_{X_0} \to L_{X_0/\mathcal{X}} \to i_0^* L_{\mathcal{X}}[1].$$

From the construction of obstruction theories on moduli spaces of maps, we obtain a commutative diagram

$$\begin{array}{cccc}
j_0^* E_{\mathcal{X}} & \to & E_{X_0} & \to & E_{X_0/\mathcal{X}} & \to & j_0^* E_{\mathcal{X}}[1] \\
| & \downarrow & | & \downarrow & | & \downarrow & \\
j_0^* L_{M_{\mathcal{X}}} & \to & L_{M_{X_0}} & \to & L_{M_{X_0}/M_{\mathcal{X}}} & \to & j_0^* L_{M_{\mathcal{X}}}[1]
\end{array}$$  

(1)

where each row is an exact triangle and where obstruction theories are defined as

$$E_{\mathcal{X}} := R\pi_* (f^* L_{\mathcal{X}} \otimes \omega_{\pi_{\mathcal{X}}}) \simeq (R\pi_* f^* T_\mathcal{X})^\vee,$$

$$E_{X_0} := R\pi_{X_0} (f_0^* L_{X_0} \otimes \omega_{\pi_{X_0}}),$$

$$E_{X_0/\mathcal{X}} := R\pi_{X_0} (f_0^* L_{X_0/\mathcal{X}} \otimes \omega_{\pi_{X_0}}) \simeq E[2] \otimes O[1].$$

Note that for the second equality of the third line, we use that $\mathcal{X}$ is smooth. For the second equality of the first line, we use that $\mathcal{X}$ is smooth. For the second equality of the third line, we use that $p$ is flat to compute $L_{X_0/\mathcal{X}} \simeq p_0^* L_{0/\mathbb{A}^1} = O[1]$ and then $E := \pi_{X_0} (\omega_{\pi_{X_0}})$ is the Hodge bundle$^2$.

$^2$We do not specify the subscript $X_0$ for the Hodge bundle because it is a pull-back from the moduli space of stable curves $\overline{M}_{g,n}$.
In the exact same way, we use the map $i_1$ to obtain a commutative diagram

$$
\begin{array}{cccccc}
 j^*_1 E_X & \longrightarrow & E_{X_1} & \longrightarrow & E_{X_1/X} & \longrightarrow & j^*_1 E_X[1] \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 j^*_1 L_{M_X} & \longrightarrow & L_{M_{X_1}} & \longrightarrow & L_{M_{X_1}/M_X} & \longrightarrow & j^*_1 L_{M_X}[1]
\end{array}
$$

where each row is an exact triangle and where obstruction theories are defined as

$$
E_{X_1} := R\pi_{1*} (f^*_1 L_{X_1} \otimes \omega_{\pi_X}) \simeq (R\pi_{1*} f^*_1 T_X)^\vee, \\
E_{X_1/X} := R\pi_{1*} (f^*_1 L_{X_1/X} \otimes \omega_{\pi_X}) \simeq E[2] \oplus O[1],
$$

where smoothness of $X_1$ is used in the second equality of the first line.

**Remark 1.4.** Since the stacks $\mathcal{X}$ and $X_1$ are smooth, obstruction theories $E_X$ and $E_{X_1}$ are perfect, i.e. of amplitude in $[-1, 0]$. On the other hand, obstruction theories $E_{X_0/X}$ and $E_{X_1/X}$ are of amplitude $[-2, -1]$, hence they are not perfect, and we do not know whether the obstruction theory $E_{X_0}$ is perfect, as $X_0$ is not assumed to be smooth.

**Definition 1.5.** The regularized obstruction theory for $M_{X_0}$ is defined as follows. We first take the cone

$$
F_{X_0} := \text{Cone} (O \to j^*_0 E_X),
$$

where the map is the composition of the inclusion $O \to E_{X_0/X}[-1] = E[1] \oplus O$ with the connecting morphism $E_{X_0/X}[-1] \to j^*_0 E_X$ from the exact triangle [1]. We then obtain a map of cones

$$
F_{X_0} \to \text{Cone} \left( L_{M_{X_0}/M_X}[-1] \to j^*_0 L_{M_X} \right).
$$

At last, from the exact triangle [1], we observe that the right-hand side above is quasi-isomorphic to $L_{M_{X_0}}$, giving us a morphism

$$
F_{X_0} \to L_{M_{X_0}}.
$$

**Remark 1.6.** In genus zero, the Hodge bundle is the zero vector bundle and the regularized obstruction theory $F_{X_0}$ is quasi-isomorphic to the Gromov–Witten obstruction theory $E_{X_0}$.

Definition 1.5 works as well for $M_{X_1}$. However, smoothness of $X_1$ yields the following equivalent definition.

**Definition 1.7.** The regularized obstruction theory for $M_{X_1}$ is defined as follows. We first take

$$
F_{X_1} := E_{X_1} \oplus E[1],
$$

and then use the map $E_{X_1} \to L_{M_{X_1}}$ and the composed morphism

$$
E \to (j^*_1 E_X)^{-1} \to (j^*_1 L_{M_X})^{-1} \to L_{M_{X_1}}^{-1}
$$

to get $F_{X_1} \to L_{M_{X_1}}$. Clearly, it is a perfect obstruction theory on $M_{X_1}$.

**Lemma 1.8.** The regularized obstruction theory $F_{X_0} \to L_{M_{X_0}}$ defines a perfect obstruction theory on $M_{X_0}$. 

Proof. Since the complex $E_X$ has amplitude in $[-1, 0]$, so does $j_0^*E_X$ and thus so does $F_{X_0}$.

Since the map $j_0 : \mathcal{M}_{X_0} \to \mathcal{M}_X$ is a closed immersion, then the cohomologies of the relative cotangent complex are

$$h^{-1}(L_{\mathcal{M}_{X_0}/\mathcal{M}_X}) = I/I^2 \quad \text{and} \quad h^0(L_{\mathcal{M}_{X_0}/\mathcal{M}_X}) = 0,$$

where $I$ is the coherent sheaf of ideals defining $j_0$. Since $E_X \to L_{\mathcal{M}_X}$ is an obstruction theory, then we have

$$h^{-1}(j_0^*E_X) \to h^{-1}(j_0^*L_{\mathcal{M}_X}) \quad \text{and} \quad h^0(j_0^*E_X) \simeq h^0(j_0^*L_{\mathcal{M}_X}).$$

Moreover, we have a surjection

$$\mathcal{O} \to I/I^2$$

between the (pullback of the) conormal sheaf of $0 \mapsto h^1$ and the conormal sheaf of the closed immersion $j_0$.

Furthermore, by unicity of the cone, we have the following commutative diagram

$$\begin{array}{ccc}
    h^{-1}(j_0^*E_X) \oplus \mathcal{O} & \xrightarrow{f} & h^0(j_0^*E_X) \\
    \downarrow & & \downarrow \simeq \\
    h^{-1}(j_0^*L_{\mathcal{M}_X}) \oplus I/I^2 & \xrightarrow{g} & h^0(j_0^*L_{\mathcal{M}_X})
\end{array}$$

where we introduce notations $f : \mathcal{O} \to h^0(j_0^*E_X)$ and $g : I/I^2 \to h^0(j_0^*L_{\mathcal{M}_X})$.

Let $U$ be an open subset of $\mathcal{M}_{X_0}$ and $x \in h^{-1}(j_0^*L_{\mathcal{M}_X})$ and $y \in I/I^2$ be two sections over $U$, such that $g(y) = 0$. Then, there exist $x' \in h^{-1}(j_0^*E_X)$ and $y' \in \mathcal{O}$ such that $x'$ is sent to $x$ and $y'$ is sent to $y$ by the second vertical map from the diagram. Then by the commutativity of the diagram, we have $f(y') = 0$ and thus $f(x' + y') = 0$, which proves surjectivity of $\text{ker}(f) \to \text{ker}(g)$.

To prove that $\text{coker}(f) \simeq \text{coker}(g)$, we apply the five lemma to the diagram

$$\begin{array}{ccc}
    \mathcal{O} & \xrightarrow{f} & h^0(j_0^*E_X) \xrightarrow{\text{ker}(f)} \text{coker}(f) \xrightarrow{0} \text{coker}(f) \xrightarrow{0} \\
    \downarrow \simeq & & \downarrow \simeq \\
    I/I^2 & \xrightarrow{g} & h^0(j_0^*L_{\mathcal{M}_X}) \xrightarrow{\text{ker}(g)} \text{coker}(g) \xrightarrow{0} \text{coker}(g) \xrightarrow{0}
\end{array}$$

As a consequence, we have proved that the morphism

$$F_{X_0} \to \text{Cone} \left( L_{\mathcal{M}_{X_0}/\mathcal{M}_X}[-1] \to j_0^*L_{\mathcal{M}_X} \right) \simeq L_{\mathcal{M}_{X_0}}$$

is an obstruction theory. \hfill \Box

**Definition 1.9.** We call regularized virtual cycle of $\mathcal{M}_{X_0}$ (resp. of $\mathcal{M}_X$) the virtual fundamental cycle $[\mathcal{M}_{X_0}, F_{X_0}] \in A_*(\mathcal{M}_{X_0})$ (resp. $[\mathcal{M}_X, F_X] \in A_*(\mathcal{M}_X)$) obtained by Behrend–Fantechi from the perfect obstruction theory $F_{X_0}$ (resp. $F_X$).

We also call Gromov–Witten virtual cycle of $\mathcal{M}_X$, the virtual fundamental cycle $[\mathcal{M}_X, E_X] \in A_*(\mathcal{M}_X)$ obtained by Behrend–Fantechi from the perfect obstruction theory $E_X$. 

Lemma 1.10. In the smooth case, the regularized virtual cycle equals the Hodge–
Gromov–Witten virtual cycle up to a sign. Precisely, for the DM stack $X_1$, we have
the relation
\[ [M_{X_1}, F_{X_1}] = (-1)^g \lambda_g \cdot [M_{X_1}, E_{X_1}] \in \mathbb{A}_*(\mathcal{M}_{X_1}), \]
where $\lambda_g := c_{\text{top}}(E)$ is the top Chern class of the Hodge bundle and $g$ is the genus
of curves involved in a given connected component of the moduli space.

Proof. The virtual fundamental class $[M_{X_1}, E_{X_1}]$ is the intersection of the intrinsic
normal cone $C_{0 M_{X_1}}$ of $M_{X_1}$ with the zero section of $h^1/h^0(E^\vee_{X_1})$, and similarly for
$[M_{X_1}, F_{X_1}]$. Since $F_{X_1} := E_{X_1} \oplus \mathbb{E}[1]$, we get
\[ h^1/h^0(F^\vee_{X_1}) \simeq h^1/h^0(E^\vee_{X_1}) \times \text{Spec}(\text{Sym} \mathbb{E}). \]

Therefore, we have
\[ [M_{X_1}, F_{X_1}] = 0^1_{h^1/h^0(F^\vee_{X_1})}[\mathcal{C}_{M_{X_1}}] \]
\[ = 0^1_{\text{Spec}(\text{Sym} \mathbb{E})} h^1/h^0(E^\vee_{X_1})[\mathcal{C}_{M_{X_1}}] \]
\[ = 0^1_{\text{Spec}(\text{Sym} \mathbb{E})} [M_{X_1}, E_{X_1}] \]
\[ = \lambda_g \cap [M_{X_1}, E_{X_1}]. \]

\[ \Box \]

1.2. Regular Specialization Theorem. First, we compare regularized virtual
cycles of $M_{X_0}$ and of $M_{X_1}$.

1.2.1. Pull-backs from the regular family.

Proposition 1.11. The regularized virtual cycle associated to a fiber of a regular
$\mathbb{A}^1$-family does not depend on the fiber. Precisely, we have equalities
\[ j^!_{0} [M_{X_1}, F_{X_1}] = [M_{X_0}, F_{X_0}] \in \mathbb{A}_*(\mathcal{M}_{X_0}), \]
\[ j^!_{1} [M_{X_1}, E_{X_1}] = [M_{X_1}, E_{X_1}] \in \mathbb{A}_*(\mathcal{M}_{X_1}). \]

Proof. Since the varieties 0 and $\mathbb{A}^1$ are smooth, by [3, Proposition 5.10], it is enough
to find a compatibility datum relative to $0 \to \mathbb{A}^1$ for $E_X$ and $F_{X_0}$, see [3, Definition
5.8], that is a triple $(\phi, \psi, \chi)$ of derived morphisms giving rise to a morphism of
exact triangles

\[ j^*_0 E_X \xrightarrow{\phi} F_{X_0} \xrightarrow{\psi} q^*_0 L_{0/\mathbb{A}^1} \xrightarrow{\chi} j^*_0 E_X[1] \]
\[ j^*_0 L_{M_{X_0}} \xrightarrow{} L_{M_{X_0}} \xrightarrow{} L_{M_{X_0}/M_X} \xrightarrow{} j^*_0 L_{M_{X}}[1] \]

The existence of the compatibility datum follows from the exact triangles of cones
\[ \mathcal{O} \to j^*_0 E_X \to \text{Cone}(\mathcal{O} \to j^*_0 E_X) \to \mathcal{O}[1] \]
\[ L_{M_{X_0}/M_X}[-1] \to j^*_0 L_{M_X} \to \text{Cone} \left( L_{M_{X_0}/M_X}[-1] \to j^*_0 L_{M_X} \right) \to L_{M_{X_0}/M_X} \]
and from the quasi-isomorphism $q^*_0 L_{0/\mathbb{A}^1} \simeq \mathcal{O}[1]$. The same holds for $X_1$. \[ \Box \]

Lemma 1.12. The morphism $q: \mathcal{M}_X \to \mathbb{A}^1$ is proper.
Proof. Since every morphism from a nodal curve to the affine line is a contraction to a point, then we have an isomorphism between the moduli space \( \mathcal{M}_X \) of stable maps to \( X \) and the moduli space \( \mathcal{M}_{g,n}(\mathcal{X}/\mathbb{A}^1) \) of relative stable maps to the \( \mathbb{A}^1 \) family \( p: \mathcal{X} \to \mathbb{A}^1 \). Therefore, by [1, Section 8.3], the morphism \( q: \mathcal{M}_X \to \mathbb{A}^1 \) is proper. □

Lemma 1.13. If we have a torus action on the family \( \mathcal{X} \), then the pullback map on Chow rings

\[
i_1^*: A^*(\mathcal{X}) \to A^*(\mathcal{X}_1)
\]

is surjective. Moreover, we have commutative diagrams

\[
\begin{array}{ccc}
\mathcal{M}_{X_1} & \xrightarrow{j_1} & \mathcal{M}_X \\
\circ & \downarrow & \circ \\
X_1 & \xrightarrow{i_1} & \mathcal{X}
\end{array}
\quad \begin{array}{ccc}
\mathcal{M}_{X_1} & \xrightarrow{j_1} & \mathcal{M}_X \\
\circ & \downarrow & \circ \\
\mathcal{M}_{g,n} & \xrightarrow{r_{X_1}} & \mathcal{M}_{g,n}
\end{array}
\]

where the maps \( \text{ev}_X \) and \( \text{ev}_{X_1} \) are the evaluation maps and the maps \( r_X \) and \( r_{X_1} \) remember only the coarse curve and stabilize it.

Proof. Let \( Z \subset X_1 \) be a closed substack and let \( T \) be the torus. Using the \( T \)-action on the total space \( \mathcal{X} \), we define \( Z^{\text{ext}} \) as the closure of

\[
\{ \lambda \cdot z \in \mathcal{X} | z \in Z \text{ and } \lambda \in T \} \subset \mathcal{X}
\]

with reduced stack structure. Then we get \( i_1^*[Z^{\text{ext}}] = [Z] \). Commutativities of the diagrams are obvious. □

Remark 1.14. The surjectivity part in Lemma 1.13 may fail when we replace \( X_1 \) by \( X_0 \), but the part on commutative diagrams still holds.

Remark 1.15. In all moduli spaces above, we consider curves with arbitrary genus, degree, number of markings, and isotropy type at markings. Hence, these moduli spaces are heavily disconnected. We write subscripts to indicate restrictions to a (bunch of connected) component of the moduli space. For instance, the moduli space of stable maps to \( \mathcal{X} \) from genus-\( g \) n-marked curves is \( (\mathcal{M}_X)_{g,n} \).

Proposition 1.11 works as well when adding the subscript \( (g,n) \). Furthermore, we can also add isotropies, since we have closed immersions of inertia stacks

\[
IX_0 \subset IX \quad \text{and} \quad IX_1 \subset IX \quad \text{with} \quad IX = \bigsqcup_{\rho} X_{\rho}.
\]

It does not compare isotropies for \( X_0 \) and for \( X_1 \). Nevertheless, in the case when an isotropy \( \rho \) of \( \mathcal{X} \) is contained in \( X_1 \) but not in \( X_0 \) (or in \( X_0 \) but not in \( X_1 \)), then the moduli space \( (\mathcal{M}_{X_0})_{\rho} \) is empty and its regularized virtual cycle \( [\mathcal{M}_{X_0}, F_{X_0}]_{\rho} \) is zero. Proposition 1.11 is still valid.

Remark 1.16. It is not straightforward to compare curve classes for \( \mathcal{X} \), \( X_0 \), and \( X_1 \). We would need to fix a curve class \( \beta \) in \( \mathcal{X} \) and to consider a sum of all curve classes in \( X_0 \) (resp. in \( X_1 \)) whose pushforward in \( \mathcal{X} \) is \( \beta \). Unfortunately, since \( \mathcal{X} \) is an \( \mathbb{A}^1 \)-family and since \( X_0 \) and \( X_1 \) are fibers, every curve class in \( X_0 \) or \( X_1 \) pushes-forward to the zero class in \( \mathcal{X} \). To solve this issue, we introduce an ambient space which contains every fiber of \( \mathcal{X} \).
1.2.2. Ambient space. From now on, we assume we have a smooth proper DM stack $\mathcal{P}$ with an embedding of $\mathbb{A}^1$-families $\mathcal{X} \hookrightarrow \mathcal{P} \times \mathbb{A}^1$, i.e. every fiber of $\mathcal{X}$ lies in $\mathcal{P}$. In particular, we have push-forward maps

$$H_2(X_t) \rightarrow H_2(\mathcal{P}), \text{ for every } t \in \mathbb{A}^1.$$  

We also have maps $\mathcal{M}_{X_t} \rightarrow \mathcal{M}_\mathcal{P}$, that we can decompose in terms of curve classes. Precisely, for every $\beta \in H_2(\mathcal{P})$, we have

$$\bigcup_{\beta' \in H_2(X_t) \text{ with } \beta' = \beta \in H_2(\mathcal{P})} \mathcal{M}_{X_t}(\beta') \rightarrow \mathcal{M}_\mathcal{P}(\beta).$$

As a consequence, Proposition 1.11 becomes the following.

**Proposition 1.17.** For every genus $g$, number of markings $n$, curve class $\beta \in H_2(\mathcal{P})$, and isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $\mathcal{X}$, we have

$$j_0^!\left[\mathcal{M}_{X, E}, X|_{g,n,\beta, \underline{\rho}}\right] = \sum_{\beta_0 \in H_2(X_0) \text{ with } \beta_0 = \beta \in H_2(\mathcal{P})} [\mathcal{M}_{X_0, E}|_{g,n,\beta_0, \underline{\rho}}] \in A_*(\mathcal{M}_{X_0}),$$

and

$$j_1^!\left[\mathcal{M}_{X, E}, X|_{g,n,\beta, \underline{\rho}}\right] = \sum_{\beta_1 \in H_2(X_1) \text{ with } \beta_1 = \beta \in H_2(\mathcal{P})} [\mathcal{M}_{X_1, E}|_{g,n,\beta_1, \underline{\rho}}] \in A_*(\mathcal{M}_{X_1}).$$

1.2.3. Correlators. In this subsection, we fix a genus $g$, a number of markings $n$, isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $\mathcal{X}$, and curve classes $\beta \in H_2(\mathcal{P})$, $\beta_0 \in H_2(X_0)$, and $\beta_1 \in H_2(X_1)$ satisfying

$$\beta_0 = \beta \in H_2(\mathcal{P}) \text{ and } \beta_1 = \beta \in H_2(\mathcal{P}).$$

We also fix $a_1, \ldots, a_n \in \mathbb{N}$ and $u_1, \ldots, u_n \in A^*(\mathcal{I})$ such that

$$u_i \in A^*(\mathcal{X}_{\mathcal{P}}) \subset A^*(\mathcal{I})$$

where $\mathcal{X}_{\mathcal{P}}$ is the component of the inertia stack of $\mathcal{X}$ with isotropy $\rho_i$. Furthermore, we denote by $\psi_i$ the usual psi-class on the moduli space of stable curves, i.e. the first Chern class of the cotangent line of the curve at the $i$-th marking, and by $\lambda_g$ the top Chern class of the (pull-back of the) Hodge bundle.

**Definition 1.18.** A Gromov–Witten correlator of $X_1$ is

$$\langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1} := q_{1*}\left( [\mathcal{M}_{X_1, E}|_{g,n,\beta_1, \underline{\rho}}] \cdot \prod_{i=1}^n j_1^* \text{ev}_X^*(u_i) \cdot r_{X_1}(\psi_i^{a_i}) \right) \in \mathbb{Q},$$

A Hodge–Gromov–Witten correlator of $X_1$ is

$$\langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1, \lambda_g} := q_{1*}\left( \lambda_g \cdot [\mathcal{M}_{X_1, E}|_{g,n,\beta_1, \underline{\rho}}] \cdot \prod_{i=1}^n j_1^* \text{ev}_X^*(u_i) \cdot r_{X_1}(\psi_i^{a_i}) \right) \in \mathbb{Q}. $$

A relative Gromov–Witten correlator of $p: \mathcal{X} \rightarrow \mathbb{A}^1$ is

$$\langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_0}^{X, \text{rel}} := q_*\left( [\mathcal{M}_{X, E}|_{g,n,\beta_0, \underline{\rho}}] \cdot \prod_{i=1}^n \text{ev}_X^*(u_i) \cdot r_{X}(\psi_i^{a_i}) \right) \in A^0(\mathbb{A}^1) \simeq \mathbb{Q}.$$  

A regularized Gromov–Witten correlator of $X_0$ is

$$\langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_0}^{X_0, \text{reg}} := q_{0*}\left( [\mathcal{M}_{X_0, E}|_{g,n,\beta_0, \underline{\rho}}] \cdot \prod_{i=1}^n j_0^* \text{ev}_{X_0}^*(u_i) \cdot r_{X_0}(\psi_i^{a_i}) \right) \in \mathbb{Q}.$$
A regularized Gromov–Witten correlator of $X_1$ is
\[ \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1,\text{reg}} := q_1^* \left( [M_{X_1}, F_{X_1}]_{g,n,\beta_1} \cdot \prod_{i=1}^n \int_0^1 \beta_i^* \text{ev}_i^* (u_i) \cdot r_{X_1} (\psi_i^{a_i}) \right) \in \mathbb{Q}. \]

**Definition 1.19.** We call ambient theory the special case where we take isotropies $\rho$ in $\mathcal{P}$ and insertions $u_i \in A^*(P_\rho) \subset A^*(I\mathcal{X})$, where pull-back is taken under the map $\mathcal{X} \rightarrow \mathcal{P} \times \mathbb{A}^1 \rightarrow \mathcal{P}$.

From Lemma 1.10, we see that
\[ \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1,\text{reg}} = (-1)^g \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1,\lambda_0}, \]
and from Proposition 1.11, we obtain
\[ \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1,\text{reg}} = \sum_{\beta_0 \in H_2(X_0) \text{ with } \beta_0 = \beta H_2(X)} \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_0}^{X_0,\text{reg}}. \]

We sum up with the following statement.

**Theorem 1.20** (Regular Specialization Theorem). Let $\mathcal{X}$ be a regular $\mathbb{A}^1$-family whose fibers are embedded in a smooth proper DM stack $\mathcal{P}$. For every genus $g$, number of markings $n$, isotropies $\rho = (\rho_1, \ldots, \rho_n)$ in $X_1$, curve class $\beta \in H_2(\mathcal{P})$, integers $a_1, \ldots, a_n \in \mathbb{N}$, and insertions $u_1, \ldots, u_n \in A^*(I\mathcal{X})$ with $u_i \in A^*(X_{1,\rho_i})$, there exist liftings $v_1, \ldots, v_n \in A^*(I\mathcal{Y})$ with $v_i = u_i$ and we have
\[ \sum_{\beta_1 \in H_2(X_1) \text{ with } \beta_1 = \beta H_2(\mathcal{P})} \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1,\lambda_0} = (-1)^g \sum_{\beta_0 \in H_2(X_0) \text{ with } \beta_0 = \beta H_2(\mathcal{P})} \langle \tau_{a_1}(v_1) \cdots \tau_{a_n}(v_n) \rangle_{g,n,\beta_0}^{X_0,\text{reg}}. \]

**Remark 1.21.** It may be difficult to work with general insertions $u_1, \ldots, u_n$ and to find liftings $v_1, \ldots, v_n$, but it is easy to work with the ambient theory, as $u_i$ and $v_i$ are pulled-back from $A^*(I\mathcal{P})$.

**Corollary 1.22** (Regular Specialization Theorem in genus zero). Under the same assumptions as before, we have
\[ \sum_{\beta_1 \in H_2(X_1) \text{ with } \beta_1 = \beta H_2(\mathcal{P})} \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_1}^{X_1} = \sum_{\beta_0 \in H_2(X_0) \text{ with } \beta_0 = \beta H_2(\mathcal{P})} \langle \tau_{a_1}(v_1) \cdots \tau_{a_n}(v_n) \rangle_{g,n,\beta_0}^{X_0,\text{reg}}. \]

1.2.4. **Torus action.** In this subsection, we assume we have a torus action on the $\mathbb{A}^1$-family $\mathcal{X}$ leaving the fiber $X_0$ invariant. Denote by $T$ the torus. Then, we get a $T$-action on the moduli spaces $M_{\mathcal{X}}$ and $M_{X_0}$, and the perfect obstruction theories $E_{\mathcal{X}}$ on $M_{\mathcal{X}}$ and $F_{X_0}$ on $M_{X_0}$ are also $T$-equivariant.

**Notation 1.23.** For a DM stack $\mathcal{Y}$ with a $T$-action, we denote by $i: \mathcal{Y}_T \rightarrow \mathcal{Y}$ the fixed locus. For a $T$-equivariant perfect obstruction theory $E_{\mathcal{Y}}$ on $\mathcal{Y}$, we denote by $N_{\mathcal{Y}_T}^{\mathcal{Y}_{\text{vir}}}$ the moving part of its restriction to the fixed locus $\mathcal{Y}_T$, and by $E_{\mathcal{T}}$ the fixed part, which is a perfect obstruction theory on the fixed locus $\mathcal{Y}_T$. 
Proposition 1.24 (Localization formula, [21 Equation (8)]). Let $\mathcal{Y}$ be a $\text{DM}$ stack with a $T$-action and a $T$-equivariant perfect obstruction theory $E \rightarrow L_\mathcal{Y}$. Let $A^*_T(\mathcal{Y})$ denote the $T$-equivariant Chow ring of $\mathcal{Y}$ and $t$ denote the $T$-equivariant parameter. Introduce the ring

$$A^*_T(\mathcal{Y})_t := A^*_T(\mathcal{Y}) \otimes \mathbb{Q}[t] \mathbb{Q}[t, t^{-1}]$$

obtained by inverting the parameter $t$. Then the virtual localization formula is

$$[\mathcal{Y}, E] = t_* \left( \frac{[\mathcal{Y}_T, E_T]}{e(N_{\text{vir}}^T)} \right),$$

where $e$ denotes the equivariant Euler class.

Remark 1.25. In our situation, the fixed locus $X_T$ lies in the central fiber $X_0$ and we have $X_T = X_{0T}$. Moreover, it is a smooth $\text{DM}$ stack and we denote it by $X_T$.

Theorem 1.26 (Equivariant Regular Specialization Theorem). Let $X$ be a $T$-equivariant regular $\mathbb{A}^1$-family whose fibers are embedded in a smooth proper $\text{DM}$ stack $\mathcal{P}$ and where the torus action leaves the central fiber invariant. For every genus $g$, number of markings $n$, isotropies $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ in $X_1$, curve class $\beta \in H_2(\mathcal{P})$, integers $a_1, \ldots, a_n \in \mathbb{N}$, and insertions $u_1, \ldots, u_n \in A^*(IX_1)$ with $u_i \in A^*(X_{1,\rho_i})$, there exist $T$-equivariant liftings $v_1, \ldots, v_n \in A^*_T(I\mathcal{X})$ with $t^*_i(v_i) = u_i$ and we have

$$\sum_{\beta_i \in H_2(X_1) \text{ with } \beta_i - \beta \in H_2(\mathcal{P})} \langle \tau_{a_1}(u_1) \cdots \tau_{a_n}(u_n) \rangle_{g,n,\beta_i}^{X_0,\lambda_0} = \lim_{t \rightarrow 0} \sum_{\beta_0 \in H_2(X_0) \text{ with } \beta_0 - \beta \in H_2(\mathcal{P})} (-1)^g \times$$

$$\int_{[\mathcal{M}_{\mathcal{X}_T, E_T}]_{g,n,\beta_0}^{\mathcal{X}_T}} \prod_{i=1}^n \frac{\text{ev}_T^*(v_i) \cdot r_T^*(\psi_T^*)}{e(N_{\text{vir}}^T)},$$

where $\text{ev}_T = \text{ev}_X \circ j_0 \circ t_{\mathcal{X}_T}$ and $r_T = r_{X_0} \circ t_{\mathcal{X}_T}$, and $[\mathcal{M}_{\mathcal{X}_T, E_T}]$ is the Gromov–Witten virtual fundamental cycle of the moduli space of stable maps to the smooth $\text{DM}$ stack $X_T$.

2. Smooth hypersurfaces in weighted projective spaces

2.1. All-genus localization formula. Denote by $\mathbb{P}(\underline{w}) = \mathbb{P}(w_1, \ldots, w_N)$ the weighted projective space given by weights $w_1, \ldots, w_N \in \mathbb{N}^*$. It comes naturally with the action of a torus $T = (\mathbb{C}^*)^N$. Denote the equivariant parameters by $\underline{t} = (t_1, \ldots, t_N)$. For any integer $d \in \mathbb{Z}$ and any character $\chi \in \text{Hom}(T, \mathbb{C})$, there is a $T$-equivariant line bundle $\mathcal{O}_\chi(d)$.

Precomposing the torus action with a stable map, we obtain a torus action on the moduli space $\mathcal{M}(\mathbb{P}(\underline{w}))$ of stable maps to the weighted projective space. Its perfect obstruction theory is $T$-equivariant and there is an equivariant virtual fundamental cycle

$$[\mathcal{M}(\mathbb{P}(\underline{w}))]_{\text{vir}}^{\mathcal{L}} \in A^*_T(\mathcal{M}(\mathbb{P}(\underline{w})))$$

in the equivariant Chow ring of the moduli space, whose non-equivariant limit $\underline{L} \rightarrow 0$ gives back the virtual fundamental cycle. We refer to [12] for a detailed construction of the equivariant Chow ring. Moreover, the derived object $R\pi_*f^*\mathcal{O}_\chi(d)$, where $\pi$ is the projection map from the universal curve and $f$ is the universal stable map, is also $T$-equivariant and its equivariant Euler class lies in

$$e_\mathcal{L}(R\pi_*f^*\mathcal{O}_\chi(d)) \in A^*_T(\mathcal{M}(\mathbb{P}(\underline{w})))_{\mathcal{L}}.$$
that is the equivariant Chow ring in which we invert \( t_1, \ldots, t_N \).

Unfortunately, the following expression

\[
e_t(R\pi_* f^* \mathcal{O}(d) \cup [\mathcal{M}(\mathbb{P}(w))]^{vir,t} \in A^*_t(\mathcal{M}(\mathbb{P}(w))))
\]

does not admit a non-equivariant limit \( t \to 0 \), unless the convexity condition holds and thus \( R\pi_* f^* \mathcal{O}(d) = \pi_* f^* \mathcal{O}(d) \) is a vector bundle.

**Remark 2.1.** Convexity holds in genus zero under the Gorenstein condition: \( w_j | d \) for all \( j \). In that case, the non-equivariant limit \( t \to 0 \) gives back the virtual cycle \([\mathcal{M}(X)]^{vir}\) of the moduli space of stable maps to a smooth degree-\( d \) hypersurface \( X \subset \mathbb{P}(w) \).

**Remark 2.2.** In the next theorem, we use the Hodge bundle \( E \) (pulled-back to) on the moduli space of stable maps to \( \mathbb{P}(w) \). We will choose a specific torus action on \( E \), that we express in terms of the equivariant parameters \( t_1, \ldots, t_N \).

**Remark 2.3.** For the line bundle \( \mathcal{O}(d) \), we take the trivial character \( \chi \) on \( \mathcal{O}(1) \) and then take its \( d \)-th power. It means that \( \mathcal{O}(d) \) has weight \( -dt_j w_j \) in the affine chart \( x_j = 1 \). We refer to Remark 2.7 below for another description of the action used on the line bundle \( \mathcal{O}(d) \).

**Convention 2.4.** In the next theorem, we will reduce the torus action to a \( \mathbb{C}^* \)-action by expressing variables \( t_1, \ldots, t_N \) in terms of a single \( \mathbb{C}^* \)-equivariant parameter \( t \). We denote by

\[
[M(\mathbb{P}(w))]^{vir,t} \in A^C_*(\mathcal{M}(\mathbb{P}(w)))
\]

the corresponding \( \mathbb{C}^* \)-equivariant virtual fundamental cycle.

As an application of our Regular Specialization Theorem, we prove the following.

**Theorem 2.5 (Hodge–Gromov–Witten theory of hypersurfaces).** We fix \( g, n \in \mathbb{N} \) such that \( 2g - 2 + n > 0 \), \( \beta \in \mathbb{N} \), and isotropies \( \underline{\rho} = (\rho_1, \ldots, \rho_n) \) in \( \mathbb{P}(w) \). Let \( X \subset \mathbb{P}(w) \) be a smooth hypersurface of degree \( d \). We assume

1. there exist positive integers \( a_1, \ldots, a_N \) such that
   \[
   a_j w_j + w_{j+1} = d \quad \text{for} \quad j < N \quad \text{and} \quad a_N w_N = d.
   \]

   Take the following specialization of the torus action to a \( \mathbb{C}^* \)-action
   \[
   t_{j+1} = (-a_1) \cdots (-a_j)t,
   \]

   for all \( 1 \leq i \leq N \), where \( t_{N+1} \) refers to the action on the Hodge bundle \( E \). Then we have the following non-equivariant limit
   \[
   e_!(R\pi_* f^* \mathcal{O}(d)) \cup [\mathcal{M}_{g,\underline{\rho}}(\mathbb{P}(w), \beta)]^{vir,t} \cup c_{t_{N+1}}(E^\vee) \xrightarrow{t \to 0} e(E^\vee) \cup [\mathcal{M}_{g,\underline{\rho}}(X, \beta)]^{vir}
   \]

   in the Chow ring of the moduli space of stable maps to the weighted projective space. Here, the equivariant Euler class \( e_!(R\pi_* f^* \mathcal{O}(d)) \) is defined after localization\(^3\), see Remark 2.7.

**Remark 2.6.** The arithmetic condition \(^4\) on the weights and the degree can be rephrased as “there exists a degree-\( d \) hypersurface in \( \mathbb{P}(w) \) defined by a chain polynomial”.

\(^3\)It is similar to the definition of the formal quintic, see \[31\]

\(^4\)
Remark 2.7. By the arithmetic condition \( [1] \), there is a \( \mathbb{C}^* \)-invariant (singular) hypersurface of degree \( d \)

\[
X_0 = \{ x_1^{a_1} x_2 + \cdots + x_N^{a_N-1} x_N = 0 \} \subset \mathbb{P}(w),
\]
and the line bundle \( \mathcal{O}(d) \) in Theorem 2.3 is its normal line bundle. Therefore, it comes with a \( \mathbb{C}^* \)-action. To be more precise, look at the weights on fibers over the fixed locus, which consists of all coordinate points in \( \mathbb{P}(w) \). At the point \((0, \ldots, x_j = 1, \ldots, 0) \in \mathbb{P}(w)\), the \( \mathbb{C}^* \)-action has weight \(-\frac{d w_j}{w_j}\), as was announced in Remark 2.3. As a consequence, the meaning of the formula in Theorem 2.3 is to first write the equivariant virtual cycle \( [\mathcal{M}_{g, \lambda}(\mathbb{P}(w), \beta)]^{\text{vir}, t} \) as a sum over dual graphs corresponding to fixed loci in the moduli space, see [14][21][33], and then multiply each term of the sum by the equivariant Euler class of \( R\pi_* f^* \mathcal{O}(d) \) with appropriate weight, and further by the Hodge class.

Remark 2.8. Theorem 2.3 yields an explicit formula for Hodge–Gromov–Witten invariants of \( X \) as a sum over dual graphs. However, it is not as straightforward as the formula for the projective space \( \mathbb{P}^N \) given in [21]. Indeed, although the \( \mathbb{C}^* \)-fixed locus in \( X_0 \) consists of all \( N \) coordinate points, the 1-dimensional fixed orbits are not isolated. The reason is we are using a \( \mathbb{C}^* \)-action on \( \mathbb{P}(w) \) instead of the full torus action. In particular, it is not clear whether we recover the result of Theorem 2.5 when we replace the \( \mathbb{C}^* \)-equivariant virtual cycle by the \( T \)-equivariant one and then express variables \( t_1, \ldots, t_N \) in terms of \( t \). Nevertheless, we can use the method described in [14] Section 2] to compute the sum over dual graphs.

Proof. As Gromov–Witten theory is invariant under smooth deformations, we can take the degree-\( d \) hypersurface \( X \) to be the zero locus of the chain polynomial

\[
P = x_1^{a_1} x_2 + \cdots + x_N^{a_N-1} x_N + x_N^{a_N}.
\]

Let us define the regular \( \mathbb{A}^1 \)-family

\[
\mathcal{X} = \{ x_1^{a_1} x_2 + \cdots + x_N^{a_N-1} x_N + x_N^{a_N} s = 0 \} \subset \mathbb{P}(w_1, \ldots, w_N) \times \mathbb{A}^1.
\]

It is equipped with a \( \mathbb{C}^* \)-action with weight \( p_j \) on \( x_j \) and \( p_{N+1} \) on \( s \) satisfying \( p_1 = 1 \) and \( p_{j+1} = (-a_1) \cdots (-a_j) \) for \( 1 \leq j \leq N \).

The fiber \( X_1 \) at \( s = 1 \) equals the smooth hypersurface \( X \) and the fiber \( X_0 \) at \( s = 0 \) has exactly one singular point \( \text{Sing}(X_0) = (0, \ldots, 0, 1) \in \mathbb{P}(w_1, \ldots, w_N) \). The \( \mathbb{C}^* \)-fixed locus is the same as for \( \mathbb{P}(w) \), i.e. it is given by all \( N \) coordinate points. Then, the virtual normal bundle in the localization formula differ for \( \mathcal{M}(\mathbb{P}(w)) \) and for \( \mathcal{M}(X_0) \), exactly by the terms \( R\pi_* f^* \mathcal{O}(d) \) and \( -E \). Indeed, the first comes from the normal bundle of the hypersurface \( X_0 \) in the ambient space \( \mathbb{P}(w) \) and the second comes from the trivial normal bundle of \( X_0 \) inside \( \mathcal{X} \). Note that by the cone construction in the definition of the regularized perfect obstruction theory, we indeed only get the contribution of \(-E\) and not of \( O - E \). \( \square \)

2.2. Genus-zero Gromov–Witten theory. Since the Hodge class \( \lambda_g \) equals 1 in genus 0, we obtain the following interesting corollary.

Corollary 2.9 (Genus-zero Gromov–Witten theory of hypersurfaces). Under assumptions and notations of Theorem 2.3 but with \( g = 0 \), we have the following non-equivariant limit

\[
e^\mathbb{L}(R\pi_* f^* \mathcal{O}(d)) \cup [\mathcal{M}_{0, \lambda}(\mathbb{P}(w), \beta)]^{\text{vir}, t} \longrightarrow_{t \to 0} [\mathcal{M}_{0, \lambda}(X, \beta)]^{\text{vir}}
\]
Remark 2.10. In genus zero, we can partially relax the arithmetic condition (4) by allowing the hypersurface to be defined by a (Thom–Sebastiani) sum of chain polynomials, i.e. with disjoint sets of variables,

\[ P = x_1^{a_1} x_2 + \cdots + x_{N-1}^{a_{N-1}} x_N + x_N^{a_N} + y_1^{b_1} y_2 + \cdots + y_{M-1}^{b_{M-1}} y_M + y_M^{b_M} + \cdots \]

Let us call the associated arithmetic condition (4'). One then gets the product of the virtual cycle with as many Hodge classes as the number of chain polynomials. In genus zero, we still recover the virtual cycle. But it is of no use in positive genus as we have \( \lambda_2^g = 0 \in A_*(\overline{M}_{g,n}) \).

Remark 2.11. Remark 2.10 is a particular case of the following: replacing the regular \( \mathbb{A}^1 \)-family in our Regular Specialization Theorem by a regular \( \mathbb{A}^m \)-family, we then need to replace the Hodge bundle \( E \) by a direct sum \( E \oplus m \) with \( m \) copies. Therefore, we get the virtual cycle cap with \( \lambda_m^g \), which equals 1 in genus 0 and 0 in positive genus.

Definition 2.12. A DM stack \( Y \) is called regularizable if there is an embedding \( Y \hookrightarrow X \) as a fiber in a regular \( \mathbb{A}^m \)-family \( X \) for some integer \( m \). Genus-zero Gromov–Witten theory of \( Y \) is then defined using regularized virtual cycle and is independent of the choice of embedding \( X \), see Remark 1.6. A substack \( Y \subset P \) of a smooth DM stack \( P \) is called regularizable inside \( P \) if we can choose the family \( X \) above as a subfamily of the trivial family \( P \times \mathbb{A}^m \).

Examples. Smooth DM stacks are regularizable via a trivial family. The hypersurface \( X_0 \) from Remark 2.7 is singular but it is regularizable inside \( P(\mathbb{w}) \). Every hypersurface in a projective space is regularizable inside the projective space. The quartic orbifold curve

\[ \{x^4 y + y^3 z = 0\} \subset \mathbb{P}(1,1,2) \]

is not regularizable inside \( \mathbb{P}(1,1,2) \).

In the following proposition, we interpret Condition (4') as the existence of a regularizable hypersurface admitting a \( \mathbb{C}^* \)-action whose fixed points are isolated.

Proposition 2.13. Let \( X \subset P(\mathbb{w}) \) be a regularizable hypersurface inside weighted projective space, such that there is a \( \mathbb{C}^* \)-action on \( P(\mathbb{w}) \) leaving \( X \) invariant and whose fixed points are isolated. Then \( X \) is singular and Condition (4') from Remark 2.10 is fulfilled. Conversely, whenever Condition (4') is fulfilled, there exists such a hypersurface in \( P(\mathbb{w}) \).

Proof. For every hypersurface \( X = \{P = 0\} \subset P(\mathbb{w}) \), denoting by \( \mathcal{P} \) the set of monomials of \( P \), we see easily that

\[
\begin{align*}
(1,0,\ldots,0) \notin X & \iff \exists m \in \mathbb{N}^*, \ x_1^m \in \mathcal{P}, \\
(1,0,\ldots,0) \in X - \text{Sing}(X) & \iff \exists m \in \mathbb{N}^*, j \neq 1, \ x_1^m x_j \in \mathcal{P}.
\end{align*}
\]

Moreover, if \( (1,0,\ldots,0) \in \text{Sing}(X) \), then we have

\[ X \text{ is regularizable inside } P(\mathbb{w}) \implies w_1 | d. \]

Therefore, whenever \( X \) is regularizable inside \( P(\mathbb{w}) \), we have, for every \( 1 \leq j \leq N \), either \( w_j | d \) or a monomial \( x_j^{d_j} x_k \in \mathcal{P} \), with possibly \( k = j \).
Furthermore, assume $X$ is invariant under a $C^*$-action on $\mathbb{P}(w)$ with weights $p_1, \ldots, p_N$. If we have $(1, 0, \ldots, 0) \notin X$, then we get $w_1 p_j = w_j p_1$ for all variable $x_j$ involved in the polynomial $P$, and fixed points are not isolated (unless $P$ is a Fermat monomial). Thus, if the $C^*$-action has only isolated fixed points, there are no Fermat monomials in $\mathcal{P}$ (unless $P$ is a Fermat monomial but then every weight $w_j$ divides the degree $d$, so Condition (4') is fulfilled).

As a consequence, if $X$ is as in the statement, then it contains all coordinate points. Introduce the set $T \subset \{1, \ldots, N\}^2$ defined by

$$(i, j) \in T \iff \exists a_i \in \mathbb{N}^*, \; x_i^{a_i} x_j \in \mathcal{P}.$$ 

By the discussion above, $T$ does not intersect the diagonal and for every $i$ such that $w_i$ does not divide $d$, there is at least one $j$ such that $(i, j) \in T$. We view $T$ as a directed graph and we check easily that if we have a loop in $T$, i.e. $j_1, \ldots, j_m$ such that $(j_1, j_2), \ldots, (j_m, j_1)$ are in $T$, then the $C^*$-action is trivial on $\mathbb{P}(w_{j_1}, \ldots, w_{j_m}) \subset \mathbb{P}(w)$. Moreover, if we have two edges colliding, i.e. $i, j, k$ such that $(i, j)$ and $(k, j)$ are in $T$, then $w_j p_k = w_k p_j$ and the $C^*$-action is trivial on $\mathbb{P}(w_j, w_k) \subset \mathbb{P}(w)$. Therefore, there is a directed subgraph in $T$ consisting of a disjoint union of directed lines. Each line corresponds to a chain polynomial without its last Fermat monomial, so that Condition (4') is fulfilled and $X$ is singular.

Conversely, if Condition (4') is fulfilled, then we take a (Thom–Sebastiani) sum of chain polynomials $\tilde{P}$. Up to renaming variables, we can assume Fermat monomials of $\tilde{P}$ are $y_1^{b_1}, \ldots, y_m^{b_m}$. Then we define

$$\tilde{P} = \tilde{P} + (s_1 - 1)y_1^{b_1} + \cdots + (s_m - 1)y_m^{b_m} \quad \text{and} \quad P = \tilde{P}_{|s_1 = \cdots = s_m = 0}.$$ 

Then the singular hypersurface $X = \{ P = 0 \} \subset \mathbb{P}(w)$ is invariant under a $C^*$-action on $\mathbb{P}(w)$ whose fixed points are isolated and the family $\mathcal{X} = \{ \tilde{P} = 0 \} \subset \mathbb{P}(w) \times \mathbb{A}^m$ is regular.

**Proposition 2.14.** Let $\mathcal{Y}$ be a regularizable DM stack and let $\mathcal{X}$ be a regular affine family containing $\mathcal{Y}$ as a fiber. Then every fiber is a regularizable DM stack and the genus-zero Gromov–Witten theory is independent of the fiber. □

**Remark 2.15.** A special feature of genus-zero Gromov–Witten theory is that we do not need a globally-defined torus action on the affine family to apply the Equivariant Regular Specialization Theorem \cite{1, 20} a torus action on the central fiber is enough.

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