HEAT CONTENT ASYMPTOTICS WITH TRANSMITTAL AND TRANSMISSION BOUNDARY CONDITIONS

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Abstract. We study the heat content asymptotics on a Riemannian manifold with smooth boundary defined by Dirichlet, Neumann, transmittal and transmission boundary conditions.
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1. Introduction

Let $M$ be a compact $m$ dimensional Riemannian manifold with smooth boundary $\partial M$. Let $D$ be an operator of Laplace type on a vector bundle $V$ over $M$. Let $B$ be a suitable local boundary condition and let $DB$ be the associated realization. Let $\phi \in C^\infty(V)$ describe the initial temperature distribution. The subsequent temperature distribution $u := e^{-tDB}\phi$ for $t \geq 0$ is described by the equations:

$$(\partial_t + D)u = 0, \ u(x; 0) = \phi, \text{ and } Bu = 0.$$

The specific heat $\rho$ is a section to the dual bundle $V^*$. Let $\beta(\phi, \rho, D, B)(t) := \int_M u \rho$ be the total heat energy content. As $t \downarrow 0$, there is a complete asymptotic expansion of the form

$$\beta(\phi, \rho, D, B)(t) \sim \sum_{n \geq 0} \beta_n(\phi, \rho, D, B)t^n/2;$$

the heat content coefficients $\beta_n(\phi, \rho, D, B)$ are locally computable.

If $DB$ is self-adjoint, then let $\{\phi_i, \lambda_i\}$ be a discrete spectral resolution. Let $\gamma_i(\phi) := \int_M \phi \phi_i$ be the associated Fourier coefficients. Then:

$$(1.2) \quad \beta(\phi, \rho, D, B)(t) = \sum_i e^{-t\lambda_i} \gamma_i(\phi) \gamma_i(\rho).$$

It is convenient to introduce a formalism to consider both Dirichlet and Robin boundary conditions at the same time. Suppose given a decomposition $\partial M = C_D \cup C_R$ of the boundary as the disjoint union of two closed (possibly empty) sets. Let $S$ be an auxiliary endomorphism of $V|_{C_R}$ and let $\phi_{\gamma m}$ be the covariant derivative of $\phi$ with respect to
the inward unit normal, where we use the natural connection which is induced on $V$ by $D$ - see Section 2 for details. We define

$$B_{DR} = B_D \oplus B_R$$

where $B_D \phi := \phi|_{C_D}$ and $B_R \phi := (\phi_m + S\phi)|_{C_R}$ are the pure Dirichlet and Robin operators respectively. In Section 2, we review previous results for the boundary conditions $B_{DR}$.

Transmittal and transfer boundary conditions will form the primary focus of this paper. Let $(M_\pm, g_\pm)$ be smooth compact $m$ dimensional Riemannian manifolds. We assume that $\Sigma = \partial M_+ = \partial M_-$ is a smooth $m-1$ dimensional manifold and that the induced metrics agree, i.e. $g_+|_\Sigma = g_-|_\Sigma$. Let $D_\pm$ be operators of Laplace type on vector bundles $V_\pm$ over $M_\pm$. Let $\nu_\pm$ be the inward unit normals of $\Sigma \subset M_\pm$; note that $\nu_+ = -\nu_-$. Let $\phi := (\phi_+, \phi_-)$ and $\rho := (\rho_+, \rho_-)$.

Suppose that $V_+|_\Sigma = V_-|_\Sigma$ and that there is given an auxiliary endomorphism $U$ of $V_\Sigma := V_+|_\Sigma$ serving as an impedance matching term. Let $\nabla^\pm$ be the natural connections defined by the operators $D_\pm$.

Let

$$B_1 \phi := \{ (\nabla_{\nu_+}^+ \phi_+)|_\Sigma - (\nabla_{\nu_-}^- \phi_-)|_\Sigma - U \phi_+|_\Sigma \}$$

(1.3)

Equivalently, $\phi$ satisfies the boundary conditions given in display (1.3) if and only if $\phi$ extends continuously across the interface $\Sigma$ and if the normal derivatives match, modulo the impedance matching term $U$. In Section 3, we determine the invariants $\beta_n$ for $n \leq 3$ for these boundary conditions, see Theorems 3.1 and 3.2 for details. The transmittal boundary operator $B_1 = B_1(U)$ is of relevance in the presence of distributional sources [7, 10, 13] as they have been considered, e.g., in the brane world scenario.

We shall also be studying boundary conditions which are defined by the boundary operator $B_2 = B_2(S)$:

$$B_2 \phi := \left\{ \begin{pmatrix} \nabla_{\nu_+}^+ + S_{++} & S_{+-} \\ S_{-+} & \nabla_{\nu_-}^- + S_{--} \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} \right\}|_\Sigma$$

(1.4)

where

$$S_{++}: V_+|_\Sigma \to V_+|_\Sigma, \quad S_{+-}: V_-|_\Sigma \to V_+|_\Sigma,$$

$$S_{-+}: V_+|_\Sigma \to V_-|_\Sigma, \quad S_{--}: V_-|_\Sigma \to V_-|_\Sigma.$$

If $S_{+-} = S_{-+} = 0$, then equation (1.4) decouples to define Robin boundary conditions. Note that we do not assume given an identification of $V_+|_\Sigma$ with $V_-|_\Sigma$; in particular, we can consider the situation when $\dim V_+ \neq \dim V_-$. In Section 4 we determine the heat content invariants $\beta_n$ for $n \leq 3$ for the heat transfer boundary conditions $B_2$, see Theorem 4.3.
The boundary conditions defined by equations (1.3) and (1.4) can be thought of as living on the singular manifold $M := M_+ \cup_\Sigma M_-$. Both boundary conditions are relevant to heat transfer problems between two media of different conductivities. Which boundary condition is to be applied depends on the details of the surface of separation $\Sigma$ between $M_+$ and $M_-$. Let $K_+$ and $K_-$ be the thermal conductivities of $M_+$ and $M_-$. The flux of heat is continuous over the interface $\Sigma$,
\[
(K_+ \nabla_{\nu_+}^+ \phi_+ + K_- \nabla_{\nu_-}^- \phi_-) \big|_{\Sigma} = 0.
\]
(1.5)
If the contact between the two media $M_+$ and $M_-$ is very intimate, in addition one assumes
\[
\phi_+ |_{\Sigma} = \phi_- |_{\Sigma},
\]
(1.6)
and boundary conditions of the type (1.3) are found. Otherwise, e.g., for surfaces pressed lightly together, in a linear approximation the flux of heat between $M_+$ and $M_-$ is proportional to their temperature difference. In this case, equation (1.3) has to be augmented by
\[
(K_+ \nabla_{\nu_+}^+ \phi_+) |_{\Sigma} = H(\phi_+ - \phi_-) |_{\Sigma}
\]
(1.7)
where $H$ is referred to as the surface conductivity. The boundary conditions (1.5) and (1.7) can be combined into the form of equation (1.4); see, for example, the discussion in [8].

2. Dirichlet and Robin boundary conditions

We begin by reviewing some of the basic invariance theory of operators of Laplace type. Let $(M, g)$ be a compact Riemannian manifold of dimension $m$. We suppose the boundary $\Sigma$ of $M$ is smooth. We adopt the Einstein convention and sum over repeated indices. Let
\[
D = -(g^{\mu\nu} \partial_\mu \partial_\nu + A^\mu \partial_\mu + B)
\]
be an operator of Laplace type on $C^\infty(V)$. The operator $D$ determines a natural connection $\nabla$ and a natural endomorphism $E$ such that we may express $D$ invariantly in the form:
\[
D = -\{Tr(\nabla^2) + E\};
\]
see [3] for details. Let $\Gamma$ be the Christoffel symbols of the metric. We may express the connection 1 form $\omega$ of $\nabla$ and the endomorphism $E$:
\[
\omega_\delta = \frac{1}{2} g_{\alpha\delta}(A^\nu + g^{\mu\sigma}\Gamma_{\mu\nu}^{\sigma}) \quad \text{and}
\]
(2.1)
\[
E = B - g^{\mu\nu}(\partial_\nu \omega_\mu + \omega_\mu \omega_\nu - \omega_\sigma \Gamma_{\nu\mu}^{\sigma}).
\]
Note that the connection defined by the dual operator $\tilde{D}$ on the dual bundle $V^*$ is the associated dual connection with connection 1 form given by $-\omega^*$; furthermore the associated endomorphism is $E^*$. 
We shall let Roman indices $a, b,$ etc. range from $1$ to $m-1$ and index a local coordinate frame for the tangent bundle of the boundary. Let $e_m$ be the inward unit normal and let indices $i, j,$ etc. range from $1$ to $m$ and index this augmented frame for $TM$. Let $L_{ab}$ be the second fundamental form, let $R_{ijkl}$ be the Riemann curvature tensor with the sign convention that $R_{1221} = +1$ for the unit sphere in $\mathbb{R}^3$. Let $\Omega$ be the curvature of the induced connection on $V$. Let $\{,\}$ and $\{,\}$ denote multiple covariant differentiation with respect to the Levi-Civita connection of the boundary and of the interior, respectively; these two connections differ by the second fundamental form.

The boundary operator $\partial$ is defined by the Dirichlet and Robin boundary condition $\partial_\partial = \partial_\partial^D$ and $\partial_\partial^R$ of locally computable invariants given by integrals over the interior and over the boundary. Let $\bar{D}$ and $\bar{B}$ be the dual operators on $C^\infty(V^*)$. The interior invariants are independent of the boundary condition and vanish if $n$ is odd. For $n \leq 3$, we have:

\[
\beta_0^C(\rho, \phi, D, \mathcal{B}) = \int_M \phi \cdot \rho, \quad \beta_1^C(\rho, \phi, D, \mathcal{B}) = 0, \\
\beta_2^C(\rho, \phi, D, \mathcal{B}) = -\int_M D\phi \cdot \rho, \quad \beta_3^C(\rho, \phi, D, \mathcal{B}) = 0.
\]

The heat content asymptotics $\beta_n$ defined by the Dirichlet and Robin boundary operator $\mathcal{B}_{DR}$ have been studied previously [1, 2, 3, 4, 11, 12, 14, 15, 16]. There are also results available in the singular setting, see for example [3, 4]. We summarize the results for $\beta_3$, $\beta_0$, $\beta_1$, $\beta_2$, and $\beta_3$:

**Theorem 2.1.**

1. $\beta_0(\phi, \rho, D, \mathcal{B}_{DR}) = \int_M \phi \rho$.
2. $\beta_1(\phi, \rho, D, \mathcal{B}_{DR}) = -\frac{2}{\sqrt{\pi}} \int_C \phi \rho$.
3. $\beta_2(\phi, \rho, D, \mathcal{B}_{DR}) = -\int_M D\phi \cdot \rho + \int_C \left\{ \frac{1}{2} L_{aa} \phi \rho - \phi \rho \right\}$
   
4. $\beta_3(\phi, \rho, D, \mathcal{B}_{DR}) = -\frac{2}{\sqrt{\pi}} \int_C \left\{ -\frac{2}{3} D\phi \cdot \rho - \frac{2}{3} \phi \bar{D}\rho + \frac{1}{3} \phi a \rho a \right\}
   
   \left( -\frac{1}{3} E + \frac{1}{12} L_{aa} L_{bb} - \frac{1}{6} L_{ab} L_{ab} + \frac{1}{6} R_{amam} \right) \phi \rho
   
   + \frac{4}{3\sqrt{\pi}} \int_C \mathcal{B}_R \phi \cdot \bar{B}_R \rho.$

3. **The boundary operator $\mathcal{B}_j$**

We postpone for the moment the discussion of $\beta_3$. Using the chiral symmetry and the homogeneity of the invariants, we see:

**Theorem 3.1.** There exist universal constants so

1. $\beta_0(\phi, \rho, D, \mathcal{B}_j) = \int_{M_+} \phi_+ \rho_+ + \int_{M_-} \phi_- \rho_-$.
2. $\beta_1(\phi, \rho, D, \mathcal{B}_j) = \int \left\{ c_1 (\phi_+ \rho_+ + \phi_- \rho_-) + c_2 (\phi_+ \rho_- + \phi_- \rho_+) \right\}.
3. $\beta_2(\phi, \rho, D, \mathcal{B}_j) = -\int_{M_+} D\phi_+ \cdot \rho_+ - \int_{M_-} D\phi_- \cdot \rho_-
   
   + \int \left\{ c_3 (\phi_+ \rho_+ L_{aa}^+ + \phi_- \rho_- L_{aa}^-) + c_4 (\phi_+ \rho_+ L_{ab}^+ + \phi_- \rho_- L_{ab}^-) \right\}$.
Furthermore, (equation (17)) for details. We may therefore apply the relations of display (1.2) to see:

\[ \phi = U \]

that the operators \( D \) are the dual boundary conditions, then we have

that \( U \) is self-adjoint, see [10] (equation (17)) for details. We may therefore apply the relations of display (1.2) to see:

\[ \beta_n(\phi, \rho, D, B_1) = \beta_n(\rho, \phi, D, B_1). \]

More generally, if \( \tilde{D} \) is the formal adjoint of \( D \) on \( C^\infty(V^*) \) and if \( \tilde{B}_1 \) are the dual boundary conditions, then we have

(3.1) \[ \beta_n(\phi, \rho, D, B_1) = \beta_n(\rho, \phi, \tilde{D}, \tilde{B}_1). \]

The expression \( -\int_M D\phi \cdot \rho \) is not symmetric in \( \phi \) and \( \rho \). We use equation (3.1) and integrate by parts to see:

(3.2) \[ a_5 = a_6, \quad a_7 - a_9 = 1, \quad a_8 = a_{10}. \]

Doubling the manifold yields additional information. Let \( M_0 \) be a smooth Riemannian manifold with smooth boundary \( \Sigma \) and let \( D_0 \) be a self-adjoint operator of Laplace type over \( M_0 \). Let \( \{\tilde{\phi}_{D,i}, \lambda_{D,i}\} \) and \( \{\tilde{\phi}_{R,i}, \lambda_{R,i}\} \) be the discrete spectral resolutions for \( D_0 \) with Dirichlet (\( D \)) and Robin (\( R \)) boundary conditions over \( M_0 \). Let \( M^\pm := M_0 \) define the double. Extend the \( \tilde{\phi}_{D,i} \) to be odd and the \( \tilde{\phi}_{R,i} \) to be even:

\( \tilde{\phi}_{D,i}(x_\pm) = \pm \frac{1}{\sqrt{2}} \phi_{D,i}(x) \) and \( \tilde{\phi}_{R,i}(x_\pm) = \frac{1}{\sqrt{2}} \phi_{R,i}(x) \).

Set \( U = -2S \). It was shown in [10] that \( B_1 \phi_{D,i} = 0 \) and \( B_1 \phi_{R,i} = 0 \). Furthermore, \( \{\tilde{\phi}_{D,i}, \tilde{\phi}_{R,i}\} \) is a complete orthonormal basis for \( L^2(V) \) which defines the spectral resolution of \( D := (D_1^+, D_0^-) \). Decompose \( \phi = \phi_o + \phi_e \) and \( \rho = \rho_o + \rho_e \) as the sum of even and odd functions and let \( \tilde{\phi}_o, \tilde{\phi}_e, \tilde{\rho}_o, \) and \( \tilde{\rho}_e \) be the restrictions to \( M_0 = M_+ \). We then have

\[ \tilde{\phi}_o = \sum_i \gamma_{D,i} \tilde{\phi}_{D,i}, \quad \phi_o = \sqrt{2} \sum_i \gamma_{D,i} \tilde{\phi}_{D,i}, \]

\[ \tilde{\rho}_o = \sum_i \gamma_{D,i} \tilde{\rho}_{D,i}, \quad \rho_o = \sqrt{2} \sum_i \gamma_{D,i} \tilde{\rho}_{D,i}, \]

\[ \phi_e = \sum_i \gamma_{R,i} \tilde{\phi}_{R,i}, \quad \phi_e = \sqrt{2} \sum_i \gamma_{R,i} \tilde{\phi}_{R,i}, \]

\[ \tilde{\rho}_e = \sum_i \gamma_{R,i} \tilde{\rho}_{R,i}, \quad \rho_e = \sqrt{2} \sum_i \gamma_{R,i} \tilde{\rho}_{R,i}. \]
Consequently by equation (1.2),
\[ \beta(\phi, \rho, D, B_1)(t) = 2\beta(\tilde{\phi}_o, \tilde{\rho}_o, D_0, B_D)(t) + 2\beta(\tilde{\phi}_e, \tilde{\rho}_e, D_0, B_R)(t) \]
(3.3) \[ \beta_n(\phi, \rho, D, B_1) = 2\beta_n(\tilde{\phi}_o, \tilde{\rho}_o, D_0, B_D) + 2\beta_n(\tilde{\phi}_e, \tilde{\rho}_e, D_0, B_R). \]

This relation continues to hold even if \( D_0 \) is not self-adjoint. Thus
\[
\begin{align*}
2a_1 + 2a_2 &= 0, \\
2a_3 + 2a_4 + 2a_5 + 2a_6 &= 0, \\
2a_7 + 2a_8 &= 2, \\
2a_9 + 2a_{10} &= 0, \\
-4a_{11} - 4a_{12} &= 2,
\end{align*}
\]
(3.4)

Take arbitrary metrics on \( M_{\pm} \) and let \( D_\pm \) be the scalar Laplacian. Take \( \phi = 1 \) and \( U = 0 \). Then \( D\phi = 0 \) and \( B_1\phi = 0 \) so \( e^{-tD_B}, \phi = \phi \).

Thus \( \beta_n(1, \rho, D, B_1) = 0 \) for \( n \geq 1 \). Take \( \rho_+ = 0 \). The terms \( \rho_+, \rho_+L^+_{aa}, \rho_+\nu_+ \) can then be specified arbitrarily. This yields:
\[
\begin{align*}
a_1 + a_2 &= 0, \\
a_3 + a_6 &= 0, \\
a_4 + a_5 &= 0, \\
a_9 + a_{10} &= 0.
\end{align*}
\]
(3.5)

This allows for the determination of the multipliers \( a_1, \ldots, a_{12} \). However, in order to provide further checks and because it will be useful later, we give one final property. Let \( N_{\pm} := [0, 1] \) be the interval. Let \( M_{\pm} := [0, 1] \times S^1 \) be the cylinder with the metrics
\[
ds^2 = dr^2 + e^{2f_+(r)} d\theta^2
\]
where the real functions \( f_\pm \) vanish on \( \partial\{[0, 1]\} \). Let \( f_{\pm,r} := \partial_r f_\pm \). Let
\[
\begin{align*}
D_{\pm,N} := & - (\partial_r^2 + f_{\pm,r} \partial_r) \text{ on } N_{\pm} \text{ and} \\
D_{\pm,M} := & - (\partial_r^2 + f_{\pm,r} \partial_r + e^{-2f_\pm} \partial_\theta^2) \text{ on } M_{\pm}.
\end{align*}
\]

Then \( D_{\pm,M} \) is the scalar Laplacian on \( M_{\pm} \). The second fundamental form vanishes on \( N_{\pm} \) while \( L^\pm = -f_{\pm,r} \) is the second fundamental form on \( M_{\pm} \). The connection forms defined by these two operators differ:
\[
\begin{align*}
\omega^N_r &= \frac{1}{2} f_r \text{ on } V, & \omega^N_r &= -\frac{1}{2} f_r \text{ on } V^*, \\
\omega^M_r &= 0 \text{ on } V, & \omega^M_r &= 0 \text{ on } V^*.
\end{align*}
\]

To compensate for this difference, we let
\[
U^N = \frac{1}{2} (f_{+,r} + f_{-,r}) \text{ on } \partial N \text{ and } U^M = 0 \text{ on } \partial M.
\]

The volume forms also differ:
\[
dvol^N = dr \text{ on } N_{\pm} \text{ and } dvol^M = e^{f_\pm} dr d\theta \text{ on } M_{\pm}.
\]

We let \( \phi_\pm \) and \( \rho_\pm \) be constants. We then have:
\[
e^{-tD^M_{B_1}} \phi = e^{-tD^N_{B_1}} \phi \text{ so}
\]
(3.6) \[ \beta_n(\phi, \rho, D^M, B_{1}^M) = 2\pi \beta_n(\phi, e^f \rho, D^N, B_{1}^N). \]
We take $f_+ = f$ and $f_- = 0$. On the cylinder, the only invariant that plays a role in the computation of $\beta_2(\phi, \rho, D, B_1)$ is $L_{aa}^+ = -f_r$. On the interval, the only invariants that play a role are $U = \frac{1}{2} f_r$, the connection 1 form $\omega_r = \frac{1}{2} f_r$ on $V$, the dual connection one form $-\frac{1}{2} f_r$ on $V^*$, and the endomorphism $E = -\frac{1}{2} f_r^2 - \frac{1}{2} f_{rr}$; the interior invariants vanish as
\[
D_M(\phi) = 0, \quad D_N(\phi) = 0, \quad \bar{D}_M(\rho) = 0, \quad \bar{D}_N(\rho \ell \rho) = 0.
\]
Note that on $\partial N$, $(\rho \ell \rho)_{\nu \nu} = \frac{1}{2} f_r \rho_+$. Thus equation (3.6) implies:
\[
f_r(-a_3 \phi_+ \rho_+ - a_4 \phi_- \rho_+ - a_5 \phi_+ \rho_- - a_6 \phi_- \rho_-)
= \frac{1}{2} f_r (a_7 \phi_+ \rho_+ + a_8 \phi_+ \rho_- + a_9 \phi_- \rho_+ + a_{10} \phi_- \rho_- + a_{11} (\phi_+ \rho_+ + \phi_- \rho_-) + a_{12} (\phi_+ \rho_- + \phi_- \rho_+))
\]
and consequently
\[
-2a_3 = a_7 + a_9 + a_{11}, \quad -2a_4 = a_{11}, \quad -2a_5 = a_8 + a_{12}, \quad -2a_6 = a_{10} + a_{12}.
\]

We solve the relations of displays (3.2), (3.4), (3.5), and (3.7) to complete the determination of $\beta_3$, $\beta_1$, and $\beta_2$ in this setting by determining the unknown coefficients to complete the proof of Theorem 3.1. 

Let $\omega_a := \nabla^+_a - \nabla^-_a$ on $V$ and $\bar{\omega}_a = -\omega_a^*$ on $V^*$; this is a chiral tensor that changes sign if we interchange the roles of $\pm$ or of $V$ and $V^*$. We determine $\beta_3$ in this setting:

**Theorem 3.2.**

1. There exist universal constants so $\beta_3(\phi, \rho, D, B_1)$
\[
= \frac{1}{\sqrt{\pi}} \int \Sigma \{ a_{20} (D_+ \phi_+ \cdot \rho_+ + \phi_+ \cdot \bar{D}_+ \rho_+ + D_- \phi_- \cdot \rho_- + \phi_- \cdot \bar{D}_- \rho_-)
+ a_{21} (D_+ \phi_+ \cdot \rho_+ + \phi_+ \cdot \bar{D}_+ \rho_+ + D_- \phi_- \cdot \rho_- + \phi_- \cdot \bar{D}_- \rho_-)
+ a_{22} (\omega_a \nabla^+_a \phi_+ \cdot \rho_+ - \omega_a \nabla^+_a \phi_- \cdot \rho_- - \omega_a \phi_+ \cdot \bar{\nabla}_a^+ \rho_+ + \omega_a \phi_- \cdot \bar{\nabla}_a^+ \rho_-)
+ a_{23} (\omega_a \nabla^-_a \phi_+ \cdot \rho_+ - \omega_a \nabla^-_a \phi_- \cdot \rho_- + \omega_a \phi_+ \cdot \bar{\nabla}_a^- \rho_+ - \omega_a \phi_- \cdot \bar{\nabla}_a^- \rho_-)
+ a_{24} (\nabla^+_a \phi_+ \cdot \nabla^+_a \rho_+ + \nabla^+_a \phi_- \cdot \nabla^+_a \rho_-)
+ a_{25} (\nabla^-_a \phi_+ \cdot \nabla^-_a \rho_+ - \nabla^-_a \phi_- \cdot \nabla^-_a \rho_-)
+ a_{26} (\nabla^+_a \phi_+ \cdot \nabla^+_a \rho_+ + \nabla^-_a \phi_- \cdot \nabla^-_a \rho_-)
+ a_{27} (\nabla^-_a \phi_+ \cdot \nabla^-_a \rho_+ - \nabla^-_a \phi_- \cdot \nabla^-_a \rho_-)
+ a_{28} U (\partial_{\nu_+} (\phi_+ \rho_+) + \partial_{\nu_-} (\phi_- \rho_-))
+ a_{29} U (\nabla^-_a \phi_- \cdot \rho_+ + \phi_- \cdot \nabla^-_a \rho_+ + \nabla^+_a \phi_+ \cdot \rho_- + \phi_+ \cdot \nabla^+_a \rho_-)
+ a_{30} (L_{aa}^+ \partial_{\nu_+} (\phi_+ \rho_+) + L_{aa}^- \partial_{\nu_-} (\phi_- \rho_-))
+ a_{31} (L_{aa}^+ \partial_{\nu_+} (\phi_+ \rho_+) + L_{aa}^- \partial_{\nu_-} (\phi_- \rho_-))
+ a_{32} (L_{aa}^+ (\nabla^+_a \phi_+ \cdot \rho_+ + \phi_+ \nabla^+_a \rho_+) + L_{aa}^- (\nabla^-_a \phi_+ \cdot \rho_+ + \phi_+ \nabla^-_a \rho_-))
+ a_{33} (L_{aa}^+ (\nabla^+_a \phi_+ \cdot \rho_+ + \phi_+ \nabla^+_a \rho_+) + L_{aa}^- (\nabla^-_a \phi_+ \cdot \rho_+ + \phi_+ \nabla^-_a \rho_-))
+ a_{34} \omega_d \bar{\omega}_d (\phi_+ \rho_+ + \phi_- \rho_-) + a_{35} \omega_d \bar{\omega}_d (\phi_+ \rho_- + \phi_- \rho_+)
\]
}
We use the relations of equation (3.1) to simplify the format at the outset and derive (1). We shall use the functorial properties involved to determine the unknown coefficients and prove (2).

We apply Theorem [2.7] and equation (3.3) to see:

\[
\begin{align*}
2a_{20} + 2a_{21} &= 0, \\
2a_{24} + 2a_{25} &= 16, \\
2a_{26} + 2a_{27} &= 0, \\
-4a_{28} - 4a_{29} &= 16, \\
a_{30} + a_{31} + a_{32} + a_{33} &= 0, \\
a_{36} + a_{37} + a_{38} + a_{39} + 2a_{40} &= 0, \\
a_{41} + a_{42} + a_{43} + a_{44} + 2a_{45} &= 0, \\
2a_{46} + 2a_{47} + 4a_{48} &= 0, \\
8a_{49} + 8a_{50} &= 16, \\
2a_{51} + 2a_{52} + 4a_{53} &= 0, \\
2a_{54} + 2a_{55} + 4a_{56} &= 0, \\
2a_{57} + 2a_{58} + 4a_{59} &= 0,
\end{align*}
\]

\[
\begin{align*}
2a_{20} - 2a_{21} &= 16, \\
2a_{24} - 2a_{25} &= 0, \\
2a_{26} - 2a_{27} &= -8, \\
2a_{28} - 2a_{29} &= 0, \\
2a_{30} + 2a_{31} - 4a_{32} - 4a_{33} &= 0, \\
a_{36} + a_{37} + a_{38} - 2a_{39} - 2a_{40} &= -1, \\
a_{41} + a_{42} + a_{43} - 2a_{44} - 2a_{45} &= 2, \\
2a_{46} + 2a_{47} - 2a_{48} &= 0, \\
2a_{49} - 2a_{50} &= 0, \\
2a_{51} + 2a_{52} - 4a_{53} &= 8, \\
2a_{54} + 2a_{55} - 4a_{56} &= 0, \\
2a_{57} + 2a_{58} - 4a_{59} &= 4.
\end{align*}
\]
Take arbitrary metrics on $M_\pm$ and let $D_\pm$ be the scalar Laplacian. Take $\phi = 1$ and $U = 0$. Then $\phi$ satisfies transmittal boundary conditions. Thus $\beta_n(1, \rho, D, B_1) = 0$ for $n \geq 1$. Take $\rho_- = 0$. This yields:

\begin{align*}
a_{30} + a_{32} &= 0, \quad a_{31} + a_{33} = 0, \\
a_{36} + a_{40} &= 0, \quad a_{38} + a_{40} = 0, \quad a_{37} + a_{39} = 0, \\
a_{41} + a_{45} &= 0, \quad a_{43} + a_{45} = 0, \quad a_{42} + a_{44} = 0, \\
a_{51} + a_{53} &= 0, \quad a_{52} + a_{53} = 0, \quad a_{54} + a_{56} = 0, \\
a_{55} + a_{56} &= 0, \quad a_{57} + a_{59} = 0, \quad a_{58} + a_{59} = 0.
\end{align*}

Let $D_\pm(\varepsilon) = D_\pm - \varepsilon$. Then $D_\pm(\varepsilon) = D_\pm - \varepsilon$ and $E_\pm(\varepsilon) = E_\pm + \varepsilon$. As $e^{-tD_1(\varepsilon)} = e^{-t\varepsilon}e^{-tD_1}$, $\beta(\phi, \rho, D(\varepsilon), B_1)(t) = e^{t\varepsilon}\beta(\phi, \rho, D, B_1)(t)$ and hence $\partial_\varepsilon\beta_n|_{\varepsilon=0} = \beta_{n-2}$. Thus studying the coefficients of the terms $\{\phi_+\phi_+, \phi_+\phi_-\}$ leads to the relations:

\begin{align*}
\frac{1}{6\sqrt{\pi}}\{-2a_{20} + a_{51} + a_{52}\} &= a_1 = -\frac{1}{\sqrt{\pi}}, \\
\frac{1}{6\sqrt{\pi}}\{-2a_{21} + 2a_{53}\} &= a_2 = \frac{1}{\sqrt{\pi}}.
\end{align*}

We use separation of variables to generate additional relationships among the coefficients. First, we study flat metrics. Let $(r, \theta)$ be the usual parameters on $M_\pm := [0, 1] \times S^1$. Let

$$D_\pm := -(\partial_r^2 + \partial_\theta^2 + 2\varepsilon_{\pm} \partial_\theta)$$

where $\varepsilon_{\pm} = \varepsilon_{\pm}(\theta)$. Let $N_\pm := [0, 1]$ and let $D_\pm := -\partial_r^2$. Let $\phi_{\pm}$ and $\rho_{\pm}$ be constant. Let $U_0$ be constant. Separation of variables and an application of equation (2.33) and Theorem 2.1 then yields:

$$e^{-tD_1(\varepsilon)}\phi = e^{-tD_\pm(\varepsilon)}\phi$$

$$\beta_3(\phi, \rho, D, B_1)_M = 2\pi\beta_3(\phi, \rho, D, B_1)_N$$

$$= 4\pi\beta_3(\phi, \rho, D, B_R)_N + 4\pi\beta_3(\phi, \rho, D, B_D)_N + 0.$$
We can also get information from the divergence terms. We now let \( \phi = (1, 1) \) and \( \rho = (\rho_+ (\theta), 0) \). We work modulo \( O(\varepsilon^2) \) and use the fact that \( a_{20} + a_{21} = 0 \) to show:

\[
0 = \int_{\Sigma} \left\{ \left( \partial_\theta \rho_+ \right) - a_{22}(\varepsilon_+ - \varepsilon_-) - a_{23}(\varepsilon_+ - \varepsilon_-) + a_{26}\varepsilon_+ + a_{27}\varepsilon_- \right\} + \rho_+ \left\{ -a_{51} \partial_\theta \varepsilon_+ - a_{52} \partial_\theta \varepsilon_- - a_{53} \partial_\theta (\varepsilon_+ + \varepsilon_-) \right\}
\]

We set \( U_M = U_0 \) constant. As the connection forms defined by \( D^N \) and \( D^M \) are different, we set \( U_N = U_0 + \frac{1}{2} \sum_i (f_{i,+} r + f_{i,-} r) \). Let \( \phi \pm \) and \( \rho \pm \) be constant. The argument used to establish equation (3.6) then generalizes immediately to yield:

\[
(3.8) \quad \beta_3(\phi, \rho, D_M, B_{1,M}) = (2\pi)^2 \beta_3(\phi, e^{\sum_i f_i \rho}, D_N, B_{1,N}).
\]

We compute on \( \partial M \):

\[
\begin{align*}
\Gamma_{mab}^\pm &= \frac{1}{2} \partial_m g_{ab}^\pm = e^{2f_{a,\pm}} f_{a,\pm, r} \delta_{ab}, \\
\Gamma_{amb}^\pm &= -\frac{1}{2} \partial_m g_{ab}^\pm = -e^{2f_{a,\pm}} f_{a,\pm, r} \delta_{ab}, \\
\Gamma_{amb}^\pm &= -\Gamma_{amb}^\pm = e^{2f_{a,\pm}} f_{a,\pm, r} \delta_{ab}, \\
\Gamma_{amb}^b &= g_{ab} \Gamma_{amb}^\pm = f_{a,\pm, r} \delta_{ab}, \\
L_{ab}^\pm &= (\nabla_a \partial_b, \partial_m) = \Gamma_{oab}^\pm = -e^{2f_{a,\pm}} f_{a,\pm, r} \delta_{ab}, \\
R_{\pm jk}^j &= (\nabla_a \nabla_m \nabla^\pm, \partial_m, \partial_{a j}) = -\Gamma_{\pm k}^\pm \partial_{b, \partial_m} - (\nabla_{\pm m}^\pm f_{a,\pm, r} \partial_{a, \partial_m}) = \Delta_{\pm m}^\pm f_{a,\pm, r} \partial_{a, \partial_m} = f_{a,\pm, rr} - f_{a,\pm, r}^2.
\end{align*}
\]

(We do not compute \( R_{ijj}^\pm \) as we have shown \( a_{54} = a_{55} = a_{56} = 0 \) so these terms do not appear). Let \( f_\pm = \sum_i f_{i,\pm} \). We compute on \( \partial N \):

\[
\begin{align*}
\omega_{\pm}^\pm &= \frac{1}{2} f_{\pm, r}, \\
U_N &= U_0 + \frac{1}{2} (f_{+, r} + f_{-, r}), \\
\nabla_{\pm}^\pm \rho_{\pm} &= \frac{1}{2} f_{\pm, \pm} \phi_{\pm}, \\
D_{\pm}^N \phi_{\pm} &= 0,
\end{align*}
\]

We use equation (3.8) to derive the following equations from the coefficients of the indicated monomials:
We now let $\rho = \rho(r)$. Equation (3.8) continues to hold and yields:

$$-a_{31} = \frac{1}{2} a_{28} \quad (f_{1,+} \phi_+ \rho_+)$$
$$-a_{33} = \frac{1}{2} a_{25} + \frac{1}{2} a_{29} \quad (f_{1,+} \phi_+ \partial_\nu_+ \rho+)$$

We combine the relations given above to complete the determination of $\beta_3$ by determining the constants $a_{20} - a_{59}$ and complete the proof of Theorem 3.2. $\square$

4. THE BOUNDARY CONDITION $B_2$

Again, we begin our discussion by studying $\beta_0$, $\beta_1$, and $\beta_2$. Note that terms such as $\phi_+ \rho_-$ cannot occur since we do not assume an identification of $V_+ | \Sigma$ with $V_- | \Sigma$.

Lemma 4.1. There exist universal constants so

1. $\beta_0(\phi, \rho, D, B_1) = \int_{M_+} \phi_+ \rho_+ + \int_{M_-} \phi_- \rho_-.$
2. $\beta_1(\phi, \rho, D, B_2) = \int_{\Sigma} b_1(\phi_+ \rho_+ + \phi_- \rho_-).$
3. $\beta_2(\phi, \rho, D, B_2) = -\int_{M_+} D \phi_+ \cdot \rho_+ - \int_{M_-} D \phi_- \cdot \rho_- + \int_{\Sigma} \{b_2(\phi_+ \rho_+ L^a+ + \phi_- \rho_- L^a-) + b_3(\phi_+ \rho_+ L_{ab} - \phi_- \rho_- L_{ab}) + b_4(\phi_+ \rho_+ + \phi_- \rho_-) + b_5(\phi_+ \rho_+ + \phi_- \rho_-) + b_6(S_+ \phi_- + S_- \phi_+ \cdot \rho_+ + S_+ \phi_+ \cdot \rho_-)\}.$
Taking $S_{++} = 0$ and $S_{--} = 0$ forces the boundary conditions given in equation (1.4) to decouple and defines Robin boundary conditions on $M_+$ and on $M_-$ separately. We use Theorem 2.1 to see:

$$b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \quad b_4 = 1, \quad b_5 = 0, \quad b_6 = 1.$$  

We use an argument similar to that used to establish display (3.5) to determine $b_7$. Let $D_\pm$ be the scalar Laplacians on manifolds $M_\pm$. Let $\phi_+ = \phi_- = 1$, let $S_{++} = S_{--} = 1$, and let $S_{+-} = S_{-+} = -1$. Then $B_2\phi = 0$ and $D\phi = 0$ so $\beta_n = 0$ for $n \geq 0$. Thus:

$$b_6 - b_7 = 0.$$  

In view of the remarks noted above, we see that:

**Lemma 4.2.** There exist universal constants so

$$\beta_3(\phi, \rho, D, B_2) = \frac{4}{3\sqrt{\pi}} \int_B B_2 \phi \cdot \tilde{B}_2 \rho$$

$$+ b_{10}(S_{+-}S_-\phi_+ \cdot \rho_+ + S_{-+}S_+\phi_- \cdot \rho_-)$$

$$+ b_{11}(S_{-+}S_-\phi_+ \cdot \rho_- + S_{+-}S_+\phi_- \cdot \rho_+)$$

$$+ b_{12}(S_{+-}S_-\phi_+ \cdot \rho_- + S_{-+}S_+\phi_- \cdot \rho_+)$$

$$+ b_{13}(S_{-+}S_-\phi_+ \cdot \rho_- + S_{+-}S_+\phi_- \cdot \rho_+ + S_{+-}\phi_- \cdot \rho_+ + S_{-+}\phi_- \cdot \rho_+)$$

$$+ b_{14}(S_{-+}S_-\phi_+ \cdot \rho_- + S_{+-}S_+\phi_- \cdot \rho_+ + S_{+-}\phi_- \cdot \rho_+ + S_{-+}\phi_- \cdot \rho_+)$$

$$+ b_{15}(L_+ \phi_+ \cdot \rho_- + L_- \phi_- \cdot \rho_+ + L_+ \phi_- \cdot \rho_+)$$

$$+ b_{16}(L_- \phi_+ \cdot \rho_- + L_+ \phi_- \cdot \rho_+ + L_+ \phi_- \cdot \rho_+).$$

Let $D_\pm$ be the scalar Laplacians on manifolds $M_\pm$. Let $\phi_+ = 1$, $\phi_- = 1$, let $S_{++} = a$, $S_{+-} = -a$, $S_{--} = b$, and $S_{-+} = -b$. Then $\phi$ satisfies transmittal boundary conditions and is harmonic so $\beta_n = 0$ for $n \geq 0$. Consequently taking $\rho_- = 0$ yields the equations:

$$b_{10} - b_{12} = 0, \quad b_{11} = b_{14} = b_{15} = b_{16} = 0.$$  

We work with $m = 1$ and $D_\pm = -\partial_x^2$. Suppose that $\phi_\pm = a_\pm x + b_\pm$ is such that $\phi$ satisfies $B_2\phi = 0$, i.e.

$$\varepsilon a_+ + S_{++}b_+ + S_{+-}b_- = 0$$

$$\varepsilon a_- + S_{--}b_- + S_{-+}b_+ = 0$$

where $\varepsilon(0) = +1$ and $\varepsilon(1) = -1$. We choose $S_*$ and $b_*$ arbitrarily and use equation (1.4) to determine $a_*$. Let $\rho_- = 0$. Since $\beta_n = 0$ for $n > 0$,

$$b_{10}S_{+-}S_-b_+ + b_{12}S_{-+}S_-b_- + b_{13}\varepsilon S_{+-}a_- = 0,$$

$$(b_{10} - b_{13})S_{++}S_-b_+ + (b_{12} - b_{13})S_{-+}S_-b_- = 0$$

so

$$b_{10} = b_{13}, \quad \text{and} \quad b_{12} = b_{13}.$$
We double the manifold to complete our determination. Suppose given an operator \( D_0 \) of Laplace type on a manifold \( M_0 \) with boundary \( \Sigma \). Let an initial condition \( \phi_0 \) be given and let \( u_0 \) solve equation (1.1) with the boundary operator \( B_R \). Let \( M_\pm := M_0 \) and \( D_\pm := D_0 \). Then \( u_\pm := u_0 \) and \( \phi_\pm := \phi_0 \) solves equation (1.1) with the boundary operator \( B_2 \) defined by equation (1.4) with \( S_{++} = S_{--} = 0 \) and \( S_{+-} = S_{-+} = S \). Thus

\[
\beta_n(\phi_0, \rho_+, \rho_-, D_0, B_R) = \beta_n(\phi, \rho, D, B_2).
\]

We may now conclude \( b_{10} = b_{13} = 0 \). This proves

**Theorem 4.3.**

1. \( \beta_0(\phi, \rho, D, B_2) = \int_M \phi \rho \).
2. \( \beta_1(\phi, \rho, D, B_2) = 0. \)
3. \( \beta_2(\phi, \rho, D, B_2) = -\int_M D\phi \cdot \rho + \int_\Sigma B_2 \phi \cdot \rho. \)
4. \( \beta_3(\phi, \rho, D, B_2) = \frac{4}{3\sqrt{\pi}} \int_\Sigma B_2 \phi \cdot \bar{B}_2 \rho. \)

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