Bicartesian Coherence

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Abstract
Coherence is demonstrated for categories with binary products and sums, but without the terminal and the initial object, and without distribution. This coherence amounts to the existence of a faithful functor from a free category with binary products and sums to the category of relations on finite ordinals. This result is obtained with the help of proof-theoretic normalizing techniques. When the terminal object is present, coherence may still be proved if of binary sums we keep just their bifunctorial properties. It is found that with the simplest understanding of coherence this is the best one can hope for in bicartesian categories. The coherence for categories with binary products and sums provides an easy decision procedure for equality of arrows. It is also used to demonstrate that the categories in question are maximal, in the sense that in any such category that is not a preorder all the equations between arrows involving only binary products and sums are the same. This shows that the usual notion of equivalence of proofs in nondistributive conjunctive-disjunctive logic is optimally defined: further assumptions would make this notion collapse into triviality. (A proof of coherence for categories with binary products and sums simpler than that presented in this paper may be found in Section 9.4 of Proof-Theoretical Coherence, revised version of September 2007, http://www.mi.sanu.ac.yu/~kosta/coh.pdf.)

Mathematics Subject Classification (2000): 18A30, 18A15, 03G30, 03F05
Keywords: bicartesian categories, coherence, decidability of equality of arrows
1 Introduction

At the very beginning of his categorial proof-theoretical program, Lambek has formulated the idea that two proofs with the same premise and conclusion should be considered equivalent iff they have the same “generality” (see [10], p. 316, and [11], p. 89). The standard example of two proofs with a different generality is given by the first projection and the second projection from \( p \land p \) to \( p \); they generalize respectively to proofs from \( p \land q \) to \( p \) and from \( p \land q \) to \( q \). Lambek’s way of making the idea of generality precise ran into difficulties (see [12], p. 65). Szabo pursued the idea in his own way in [21] and [25].

We find that the most simple way to understand generality is to connect by a link propositional letters that must remain identical after generalizing, and not connect those that may differ. So for the first and the second projection proof from \( p \land p \) to \( p \) we would have the two diagrams

\[
\begin{array}{c}
  p \land p \\
  \downarrow \\
  p \\
\end{array} \quad \begin{array}{c}
  p \land p \\
  \downarrow \\
  p \\
\end{array}
\]

The two proofs would not be equivalent because these diagrams are different.

Such diagrams compose in an obvious way by composing links, and we have also obvious identity diagrams like, for example,

\[
\begin{array}{c}
  p \land ( q \lor p ) \\
  \mid \\
  p \land ( q \lor p )
\end{array}
\]

So diagrams, or a formal analogue of them, make a category. We call such categories graphical categories. In this paper, the graphical category will be the category of relations on finite ordinals.

Equivalence between proofs in intuitionistic logic is axiomatized independently of these diagrams in the typed lambda calculus and in various sorts of categories, like bicartesian closed categories. There, proofs are coded by typed lambda terms or by arrow terms, and two proofs are considered equivalent iff the coding terms are equal as lambda terms or as arrow terms in categories. This approach is rather standard nowadays, because lambda equality and equality of arrows in categories match pretty well the equivalence between proofs induced by normalization in natural deduction or cut elimination in sequent systems for intuitionistic logic.
The question then arises how this standard notion of equivalence relates to generality of proofs. This question can be exactly posed by asking whether for a freely generated category $C$ of some sort, like, for example, a freely generated bicartesian closed category, there is a faithful functor $G$ from $C$ to a graphical category $\mathcal{G}$, in which diagrams like the two diagrams above, or a formal analogue of these diagrams, would be arrows. In limit cases, the functor $G$ may be an isomorphism, but in general it is enough that it be faithful, which means that for two arrow terms $f, g : A \rightarrow B$ of $C$ we shall have

$$f = g \text{ in } C \quad \text{iff} \quad G(f) = G(g) \text{ in } \mathcal{G}.$$ 

From left to right, this equivalence follows from $G$’s being a functor, and from right to left it expresses the faithfulness of $G$.

Understood in this manner, Lambek’s generality idea merges with coherence questions in category theory, which were treated intensively when Lambek wrote the three papers cited above. Now, coherence in category theory has no doubt been understood in various ways. (We shall not try to survey here the literature on this question; for earlier works see [16].) Although Mac Lane’s paradoxical dictum “All diagrams commute” can be made precise in different ways, the paradigmatic results on coherence of [15] and [8] can be understood as faithfulness results, like the equivalence above, and this is how we understand coherence here. The graphical categories of these results, whose investigation starts with [7], are very well adapted to make precise Lambek’s idea of generality in logic.

The faithfulness equivalence above, which may also be understood as a coherence equivalence, is, from a logical point of view, a completeness equivalence. The freely generated category $C$ is syntax, i.e. a formal system, the graphical category $\mathcal{G}$ is a model, the functoriality of $G$, i.e. the implication from left to right, is soundness (here it is desirable that $G$ preserve also the particular structure of $C$, and not only identities and composition), and the implication from right to left is completeness proper. For this completeness result to be interesting, there should be a gain in passing from $C$ to $\mathcal{G}$. It is desirable that $\mathcal{G}$ be easy to handle, so that, for example, we may decide equality of arrows in $C$ by passing to $\mathcal{G}$, or that we may normalize arrow terms by referring to $\mathcal{G}$, without going through tedious syntactic reductions.

If we understand generality in $\mathcal{G}$ in the simplest fashion: “Connect by a link all propositional letters that must remain the same after generalizing”, the generality idea matches equality in categories that correspond to intuitionistic propositional logic only to a limited extent. (It fares better in linear logic, as the coherence result of [8] shows.) We have coherence, i.e. faithfulness, i.e. soundness and completeness, for categories with binary product, which covers the purely conjunctive fragment of logic, and for cartesian categories, where the terminal object is added, which covers the conjunctive fragment extended with the constant true proposition (see references in Section 4). By duality, this
covers also the purely disjunctive fragment, and this fragment extended with the constant absurd proposition.

We shall see in this paper that we have coherence also for categories with binary products and sums (i.e. coproducts), which covers the conjunctive-disjunctive fragment without distribution of conjunction over disjunction. We shall also see that we have coherence for cartesian categories, where the terminal object is present, extended with an operation on objects and arrows that keeps of sum (or of product) just its bifunctorial properties.

Coherence fails for bicartesian categories, where the terminal and the initial object have been added. If \( T \) is the terminal object, which corresponds to the constant true proposition, and \( \bot \) the initial object, which corresponds to the constant absurd proposition, both for the first and for the second projection proof from \( \bot \land \bot \) to \( \bot \) we have the empty diagram, and analogously for the first and the second injection proof from \( T \) to \( T \lor T \). However, neither the first and the second projection from \( \bot \land \bot \) to \( \bot \), nor the first and the second injection from \( T \) to \( T \lor T \), are equal in all bicartesian categories (see Section 4). While the first and the second projection from \( \bot \land \bot \) to \( \bot \) become equal in all bicartesian closed categories, the first and the second injection from \( T \) to \( T \lor T \) are equal in a bicartesian closed category iff the category has collapsed into a preorder, i.e. a category where all arrows with the same source and the same target are equal. (The equality between the first and the second projection from \( \bot \land \bot \) to \( \bot \) in bicartesian closed categories, which is a consequence of the existence of a right adjoint to the functor \( \bot \times \), implies that in bicartesian closed categories all arrows with the same source and the target \( \bot \) are equal; cf. Section 5 and [13], p. 67, Proposition 8.3. The equality between the first and the second injection from \( T \) to \( T \lor T \) yields preordering in bicartesian closed categories because in these categories all arrows are in one-to-one correspondence with arrows whose source is \( T \); cf. Section 5.) It is the completeness part of coherence that fails for bicartesian categories—the soundness part holds true.

It is rather typical for special objects to make trouble for coherence results. Such was the case for the unit object in symmetric monoidal closed categories (see [8]), and such is the case in bicartesian categories with the terminal and the initial object. (Special objects may also cause trouble for normalizing terms; see, for example, [13], p. 88.) So, for some purposes, it may be unwise to subsume the terminal and the initial object under a generalized concept of finite product and sum, where they are the nullary cases. This may obscure matters.

Even the soundness part of coherence fails for distributive categories with binary products and sums, and for distributive bicartesian categories (for these categories see [14], pp. 222-223 and Session 26, and [1]). If we have a distributivity isomorphism from \( p \land (q \lor r) \) to \( (p \land q) \lor (p \land r) \), then the composition of diagrams
yields the diagram on the left-hand side below, which doesn’t amount to the identity diagram on the right-hand side:

![Diagram](image)

In distributive bicartesian categories we also have that $p \land \bot$ is isomorphic to $\bot$, but the composition of two empty diagrams for this isomorphism is not equal to the identity diagram from $p \land \bot$ to $p \land \bot$.

Without disjunction, but with implication, the situation is not better. Both the soundness part and the completeness part of coherence fail for cartesian closed categories. For soundness, we have the counterexample

![Counterexample](image)

while for completeness, counterexamples may be constructed along the lines of [22]. (Both soundness and completeness would fail if we had only implication, but the associated categories, which correspond to the lambda calculus without product types, are not usually considered.)

So it seems that with the results of this paper we have reached the limits of coherence for the simple approach to generality in logic. This doesn’t mean that another approach, with a more subtly built graphical category $\mathcal{C}$, couldn’t vindicate Lambek’s idea in wider fragments of logic.

The main coherence result we are going to establish here is useful for a particular purpose. Recently, Cockett and Seely set forth in [2] a syntactical decision procedure obtained via cut elimination for equality of arrow terms in freely generated categories with finite products and sums, without distribution,
which amount to bicartesian categories\textsuperscript{1}. (That there is such a procedure is claimed in [23], p. 69, but we have found that difficult to check.) We don’t cover the whole ground of [2], since coherence fails iff we add the empty product and the empty sum, i.e. the terminal and the initial object. However, as far as it goes, for equations between terms involving only nonempty finite products and sums, which covers the major part of the categories in question, our coherence yields a very simple graphical decision procedure, in which we find a clear advantage over syntactical procedures, even when they are entirely explicit.

We use our coherence result for categories with binary products and sums to demonstrate in the last section of this paper that these categories are maximal, in the sense that if such a category satisfies every instance of any equation not satisfied in freely generated categories of this sort, then this category is a preorder. Analogous results hold for cartesian and cartesian closed categories (see [3], [20] and [4]). Such results are interesting for logic, because they show that our choice of equations is optimal. These equations are wanted, because they are induced by normalization of proofs, and no equation is missing, because any further equation would lead to collapse: all arrows with the same source and the same target would be equal. Our maximality result in the last section shows optimal the choice of equations assumed for conjunctive-disjunctive logic in the absence of distribution.

The literature on bicartesian categories, without distribution, and without closure, i.e. exponentiation, does not seem very rich, though, of course, the notions of product and sum (coproduct) are explained in every textbook of category theory. An early reference we know about, but which does not cover matters we are treating, is [6]. Of course, nondistributive lattices have been extensively studied, but though this topic is related to bicartesian categories, categorial studies are on a different level. In the literature on nonclassical logics, one encounters very much studied logics where conjunction and disjunction make a nondistributive lattice structure—such is, for instance, linear logic—, but the purely conjunctive-disjunctive fragment is not usually separated and considered categorially. (A sequent system for nondistributive conjunction and disjunction is considered towards the end of [19] from the point of view of algebraic logic, but categories are not mentioned there.)

\section{Free Bicartesian Categories}

The propositional language $\mathcal{P}$ is generated from a set of \textit{propositional letters} $\mathcal{L}$ with the nullary connectives, i.e. propositional constants, I and O, and the binary connectives $\times$ and $\dagger$. The fragments $\mathcal{P}_{\times,\dagger}, \mathcal{P}_{\times,\dagger}$ etc. of $\mathcal{P}$ are obtained by keeping only those formulae of $\mathcal{P}$ that contain the connectives in the index. For the propositional letters of $\mathcal{P}$, i.e. for the members of $\mathcal{L}$, we use the schematic

\textsuperscript{1}We are grateful to Robert Seely and Robin Cockett for having sent us the draft of [2], which prompted the writing of this paper.
letters \(p, q, \ldots, p_1, \ldots\), and for the formulae of \(\mathcal{P}\), or of its fragments, we use the schematic letters \(A, B, \ldots, A_1, \ldots\).

Next we define inductively the *terms* that will stand for the arrows of the free bicartesian category \(\mathcal{C}\) generated by \(\mathcal{L}\). Every term has a *type*, which is a pair \((A, B)\) of formulae of \(\mathcal{P}\). That a term \(f\) is of type \((A, B)\) is written \(f : A \to B\).

The *atomic* terms of \(\mathcal{C}\) are for every \(A\) of \(\mathcal{P}\)

\[
1_A : A \to A,
\]

\[
k_A : A \to I,
\]

\[
l_A : O \to A.
\]

The terms \(1_A\) are called *identities*. The other terms of \(\mathcal{C}\) are generated with the following operations on terms, which we present by rules so that from the terms in the premises we obtain the terms in the conclusion:

\[
\begin{align*}
\frac{f : A \to B \quad g : B \to C}{g \circ f : A \to C} & \quad \frac{f : C \to A}{L^1_A f : C \to A + B} \\
\frac{f : B \to C}{K^1_B f : A \times B \to C} & \quad \frac{f : C \to B}{L^2_A f : C \to A + B} \\
\frac{f : C \to A \quad g : C \to B}{(f, g) : C \to A \times B} & \quad \frac{f : A \to C \quad g : B \to C}{[f, g] : A + B \to C}
\end{align*}
\]

We use \(f, g, \ldots, f_1, \ldots\) as schematic letters for terms of \(\mathcal{C}\).

The category \(\mathcal{C}\) has as objects the formulae of \(\mathcal{P}\) and as arrows equivalence classes of terms so that the following equations are satisfied for \(i \in \{1, 2\}\):

\[
\begin{align*}
\text{(cat 1)} & \quad 1_B \circ f = f \circ 1_A = f, \\
\text{(cat 2)} & \quad h \circ (g \circ f) = (h \circ g) \circ f,
\end{align*}
\]

\[
\begin{align*}
\text{(K1)} & \quad g \circ K^1_A f = K^1_A (g \circ f), & \quad \text{(L1)} & \quad L^1_A g \circ f = L^1_A (g \circ f), \\
\text{(K2)} & \quad K^2_A g \circ (f_1, f_2) = g \circ f_1, & \quad \text{(L2)} & \quad [g_1, g_2] \circ L^1_A f = g_1 \circ f, \\
\text{(K3)} & \quad (g_1, g_2) \circ f = (g_1 \circ f, g_2 \circ f), & \quad \text{(L3)} & \quad g \circ [f_1, f_2] = [g \circ f_1, g \circ f_2], \\
\text{(K4)} & \quad (K^1_B 1_A, K^2_A 1_B) = 1_{A \times B}, & \quad \text{(L4)} & \quad [L^1_B 1_A, L^2_A 1_B] = 1_{A + B}, \\
\text{(k)} & \quad \text{for } f : A \to 1, \ f = k_A, & \quad \text{(l)} & \quad \text{for } f : O \to A, \ f = l_A.
\end{align*}
\]

A category \(\mathcal{C}'\) isomorphic to \(\mathcal{C}\) is obtained with the same objects, and terms defined inductively as follows. The atomic terms are for every \(A\) and every \(B\) of \(\mathcal{P}\)

\[
1_A : A \to A,
\]

\[
k_A : A \to I,
\]

\[
l_A : O \to A,
\]

\[
k^1_{A, B} : A \times B \to A,
\]

\[
l^1_{A, B} : A \to A + B,
\]

\[
k^2_{A, B} : A \times B \to B,
\]

\[
l^2_{A, B} : B \to A + B,
\]

\[
w_A : A \to A \times A,
\]

\[
m_A : A + A \to A,
\]

\[
7
\]
and we have the following operations on terms:

\[
\begin{array}{c}
f : A \to B \quad g : B \to C \\
g \circ f : A \to C \\
f \times g : A \times C \to B \times D \\
f + g : A + C \to B + D
\end{array}
\]

On these terms we impose the equations (cat 1), (cat 2), (k), (l) and

\[
\begin{align*}
(\times 1) \ & \ 1_A \times 1_B = 1_{A \times B}, \\
(\times 2) \ & \ (g_1 \circ g_2) \times (f_1 \circ f_2) = (g_1 \times f_1) \circ (g_2 \times f_2), \\
(k^1) \ & \ k_{B_1,B_2}^1 \circ (f_1 \times f_2) = f_1 \circ k_{A_1,A_2}^1, \\
(w) \ & \ w_B \circ f = (f \times f) \circ w_A, \\
(kw1) \ & \ k_{A,A}^1 \circ w_A = 1_A, \\
(kw2) \ & \ (k_{A,B}^1 \times k_{A,B}^2) \circ w_{A \times B} = 1_{A \times B}, \\
(+1) \ & \ 1_A + 1_B = 1_{A+B}, \\
(+2) \ & \ (g_1 \circ g_2) + (f_1 \circ f_2) = (g_1 + f_1) \circ (g_2 + f_2), \\
(l^1) \ & \ (f_1 + f_2) \circ l_{A_1,A_2}^1 = l_{B_1,B_2}^1 \circ f_i, \\
(l^2) \ & \ f \circ m_A = m_B \circ (f + f), \\
(lm1) \ & \ m_A \circ l_{A,A}^1 = 1_A, \\
(lm2) \ & \ m_{A+B} \circ (l_{A,B}^1 + l_{A,B}^2) = 1_{A+B}.
\end{align*}
\]

The isomorphism of $C$ and $C'$ is shown with the definitions

\[
\begin{align*}
k_{A,B}^1 &= \text{def. } K_{B}^1 1_A, \quad l_{A,B}^1 = \text{def. } L_{B}^1 1_A, \\
k_{A,B}^2 &= \text{def. } K_{A}^2 1_B, \quad l_{A,B}^2 = \text{def. } L_{A}^2 1_B, \\
w_A &= \text{def. } \langle 1_A, 1_A \rangle, \quad m_A = \text{def. } [1_A, 1_A], \\
f \times g &= \text{def. } \langle K_{B}^1 f, K_{A}^2 g \rangle, \quad f + g = \text{def. } [L_{B}^1 f, L_{A}^2 g].
\end{align*}
\]

The free $\times, +$-categories $C_{x,+}$ and $C'_{x,+}$ generated by $\mathcal{L}$ have as objects formulae of $P_{x,+}$. In that case, the terms $k_A$ and $l_A$ are missing, and the associated equations (k) and (l) are omitted. The remaining equations are as in $C$ and $C'$. The categories $C_{x,+}$ and $C'_{x,+}$ are also isomorphic. We obtain similarly the free cartesian category $C_{x,1}$ generated by $\mathcal{L}$, whose objects are from $P_{x,1}$, and the isomorphic category $C_{x,1}^\prime$. In that case, we omit all the terms, operations and equations tied to $+$ and $O$. For the free category with binary product $C_x$ generated by $\mathcal{L}$, and the isomorphic category $C_x'$, we omit $I$, $k_A$ and $(k)$ from $C_{x,1}$ and $C'_{x,1}$ respectively.
The categories \( C_{x,1}^+ \) and \( C_x^+ \) are obtained by extending \( C_{x,1} \) and \( C_x \) respectively with the connective \( + \), the operation on terms \( + \), and the equations \((+1)\) and \((+2)\). We obtain \( C_{x,1}^+ \) and \( C_x^+ \), which are isomorphic to \( C_{x,1}^+ \) and \( C_x^+ \) respectively, by extending in the same manner \( C_{x,1}^+ \) and \( C_x^+ \) respectively.

The categories \( C_{+,O} \) and \( C_+ \) are isomorphic to \( C_{+,O}^{op} \) and \( C_+^{op} \) respectively. We obtain the categories \( C_{+,O}^+ \) and \( C_+^+ \), which are isomorphic to \( C_{+,O}^+ \) and \( C_+^+ \) respectively, by extending \( C_{+,O} \) and \( C_+ \) with the connective \( \times \), the operation on terms \( \times \), and the equations \((\times1)\) and \((\times2)\). We have also isomorphic primed versions of \( C_{+,O}^+ \) and \( C_+^+ \).

Up to a certain point in our exposition (noted in Section 4), we shall distinguish \( C \) from \( C' \), and analogously for the other categories derived from \( C \), which we have now introduced. We do that until the statements of our auxiliary results are tied to the nonprimed or to the primed version of the category in question. Once the necessity for this distinction ceases, we refer to both of these isomorphic categories by the nonprimed name.

### 3 Cut Elimination

We prove the following theorem for \( C \).

**Cut Elimination.** *Every term is equal to a composition-free term.*

**Proof.** Take a subterm \( g \circ f \) of a term such that both \( f \) and \( g \) are composition-free. We call such a term a topmost cut. We show that \( g \circ f \) is either equal to a composition-free term, or it is equal to a term all of whose compositions occur in topmost cuts of strictly smaller length than the length of \( g \circ f \). The possibility of eliminating compositions in topmost cuts, and hence every composition, follows by induction on the length of topmost cuts.

The cases where \( f \) or \( g \) is \( 1_A \), or \( f \) is \( l_A \), or \( g \) is \( k_A \), are taken care of by \((cat 1)\), \((l)\) and \((k)\). The cases where \( f \) is \( K_A^i f' \) or \( g \) is \( L_A^i g' \) are taken care of by \((K1)\) and \((L1)\). And the cases where \( f \) is \([f_1, f_2] \) or \( g \) is \( \langle g_1, g_2 \rangle \) are taken care of by \((L3)\) and \((K3)\).

The following cases remain. If \( f \) is \( k_A \), then \( g \) is of a form covered by cases we dealt with above.

If \( f \) is \( \langle f_1, f_2 \rangle \), then \( g \) is either of a form covered by cases above, or \( g \) is \( K_A^i g' \), in which case we apply \((K2)\).

If \( f \) is \( L_A^i f' \), then \( g \) is either of a form covered by cases above, or \( g \) is \( [g_1, g_2] \), in which case we apply \((L2)\). This covers all possible cases.

In this proof we have used all the equations assumed for \( C \) except \((cat 2)\), \((K4)\) and \((L4)\).

A portion of this proof suffices to demonstrate Cut Elimination for \( C_{x,+,1} \), \( C_{x,1} \) and \( C_x \). By duality, we also have Cut Elimination for \( C_{+,O} \) and \( C_+ \). To
demonstrate Cut Elimination for \( C_{\times,1}^+ \) we have to consider the following additional cases.

If \( f \) is \( k_A \) or \( \langle f_1, f_2 \rangle \), then \( g \) cannot be of the form \( g_1 + g_2 \). If \( f \) is \( f_1 + f_2 \), and \( g \) is not of a form already covered by cases in the proof above, then \( g \) is of the form \( g_1 + g_2 \), in which case we apply \((+2)\). This covers all possible cases. A portion of this proof suffices to demonstrate Cut Elimination for \( C_{\times,1}^+ \). By duality, we also obtain Cut Elimination for \( C_{+,O}^\times \) and \( C_{+}^\times \).

A composition-free term of \( C_{\times,1}^+ \) is reduced to normal form with the following reductions:

\[
\begin{array}{ll}
\text{redexes} & \text{contracta} \\
1_{A \times B} & \langle K_1^A 1_A, K_2^A 1_B \rangle \\
1_{A + B} & 1_A + 1_B \\
1_i & k_i \\
K_i^A \langle f, g \rangle & \langle K_i^A f, K_i^A g \rangle \\
K_1^B k_A & k_{A \times B} \\
K_2^B k_B & k_{A \times B}
\end{array}
\]

These reductions are strongly normalizing. To show that, let \( n_1 \) be the number of connectives \( \times, + \) and \( \text{I} \) in the indices of identities, and let \( n_2 \) be the number of pairs of brackets \( \langle, \rangle \) and \( k \) terms within the scope of an operation \( K_C^i \) (not necessarily the immediate scope). Let the degree of a term be \( (n_1, n_2) \), and let these degrees be lexicographically ordered. Then every reduction decreases the degree.

The reduction from \( 1_1 \) to \( k_1 \) and the last two reductions enable us to reduce every term different from \( k_A \) of type \( A \rightarrow 1 \) to \( k_A \). We disregard these three reductions to reduce to normal form composition-free terms of \( C_{\times,1}^+ \).

All the reductions above are covered by equations of \( C_{\times,1}^+ \). For the fourth reduction we have the following derivation in \( C_{\times,1}^+ \):

\[
K_i^A \langle f, g \rangle = \langle f, g \rangle \circ K_i^A 1_B, \quad \text{by (cat 1) and (K1)}
\]

\[
= \langle K_i^A f, K_i^A g \rangle, \quad \text{by (K3), (K1) and (cat 1)}.
\]

4 Coherence

We shall now define a graphical category \( \mathcal{G} \) into which \( \mathcal{C} \) can be mapped. The objects of \( \mathcal{G} \) are finite ordinals. An arrow \( f : n \rightarrow m \) of \( \mathcal{G} \) will be a binary relation from \( n \) to \( m \), i.e. a subset of \( n \times m \) with domain \( n \) and codomain \( m \). The identity \( 1_n : n \rightarrow n \) of \( \mathcal{G} \) is the identity relation on \( n \), and composition of arrows is composition of relations.

For an object \( A \) of \( \mathcal{C} \), let \( |A| \) be the number of occurrences of propositional letters in \( A \). For example, \(|(p \times (q + p)) + (1 \times p)| \) is 4.
We now define a functor $G$ from $\mathcal{C}'$ to $\mathcal{G}$ such that $G(A) = |A|$. It is clear that $G(A \times B) = G(A + B) = |A| + |B|$. We define $G$ on arrows inductively:

$$G(\mathbf{1}_A) = \{(x, x) : x \in |A|\} = \mathbf{1}_{|A|},$$
$$G(k^1_{A,B}) = \{(x, x) : x \in |A|\},$$
$$G(k^2_{A,B}) = \{(x + |A|, x) : x \in |B|\},$$
$$G(w_A) = \{(x, x) : x \in |A|\} \cup \{(x, x + |A|) : x \in |A|\},$$
$$G(k_A) = \emptyset,$$
$$G(l^1_{A,B}) = \{(x, x) : x \in |A|\},$$
$$G(l^2_{A,B}) = \{(x, x + |A|) : x \in |B|\},$$
$$G(m_A) = \{(x, x) : x \in |A|\} \cup \{(x + |A|, x) : x \in |A|\},$$
$$G(l_A) = \emptyset,$$
$$G(g \circ f) = G(g) \circ G(f),$$

and for $f : A \to B$ and $g : C \to D$,

$$G(f \times g) = G(f + g) = G(f) \cup \{(x + |A|, y + |B|) : (x, y) \in G(g)\}.$$

Though $G(\mathbf{1}_A)$, $G(k^1_{A,B})$ and $G(l^1_{A,B})$ are the same as sets of ordered pairs, in general they have different domains and codomains, the first being a subset of $|A| \times |A|$, the second a subset of $(|A| + |B|) \times |A|$, and the third a subset of $|A| \times (|A| + |B|)$. We have an analogous situation in some other cases.

It is easy to draw $G(f)$ diagrammatically. For example, for $G(m_{p+q} \circ (l^1_{p,q} + l^2_{p,q}))$ we have

$$\begin{array}{ccc}
\text{(p + q)} & \text{(p + q)} & \text{l^1_{p,q} + l^2_{p,q}} \\
p + q & p + q & m_{p+q} \\
\text{which is equal to} & j & \text{1}_{p+q}
\end{array}$$

It is also easy to check that $G$ is a functor from $\mathcal{C}'$ to $\mathcal{G}$. We show by induction on the length of derivation that if $f = g$ in $\mathcal{C}'$, then $G(f) = G(g)$ in $\mathcal{G}$. (Of course, $G$ preserves identities and composition.) Since, $\mathcal{C}'$ and $\mathcal{C}$ are isomorphic we also have a functor from $\mathcal{C}$ to $\mathcal{G}$.

For the bicartesian structure of $\mathcal{G}$ we have that the operations $\times$ and $+$ on objects are both addition of ordinals, the operations $\times$ and $+$ on arrows coincide and are defined by the clauses for $G(f \times g)$ and $G(f + g)$, and the terminal and
the initial object also coincide: they are both the ordinal zero. The category \( \mathcal{G} \) has zero arrows, namely, the arrows

\[
\begin{array}{cccc|cc}
   n & \overset{0}{\rightarrow} & 0 & \overset{0}{\rightarrow} & 0 & \overset{0}{\rightarrow} & m \\
G(A) & G(I) & G(O) & G(B)
\end{array}
\]

which composed with any other arrow give another zero arrow. It is easy to see that the bicartesian category \( \mathcal{G} \) is a linear category in the sense of [14] (see p. 279). The functor \( G \) from \( \mathcal{C} \) to \( \mathcal{G} \) is not just a functor, but a bicartesian functor; namely, a functor that preserves the bicartesian structure of \( \mathcal{C} \).

We also have functors defined analogously to \( G \), which we call \( G \) too, from \( \mathcal{C}_{x,+} \), \( \mathcal{C}_{x,1}^+ \) and \( \mathcal{C}_x^+ \) to \( \mathcal{G} \). These functors, which are defined officially for the primed versions of these categories, are obtained from the definition of \( G \) above by just rejecting clauses that are no longer applicable. For these last three functors we shall show that they are faithful. By duality, we also have faithful functors \( G \) from \( \mathcal{C}_{+,0}^+ \) and \( \mathcal{C}_{+}^+ \) to \( \mathcal{G} \).

That an analogously defined functor \( G \) exists from \( \mathcal{C}_{x,1} \) to \( \mathcal{G} \), and is faithful, has been announced in [9] (p. 129) and proved in [17] (Theorem 2.2), [26] (Theorem 8.2.3, p. 207), [18] and [3]. The functor \( G \) from \( \mathcal{C}_{x,1} \) maps \( \mathcal{C}_{x,1} \) into the subcategory of \( \mathcal{G} \) whose arrows are relations converse to functions; in other words, \( G \) maps \( \mathcal{C}_{+,0} \) into the subcategory of \( \mathcal{G} \) whose arrows are functions.

It is clear that the functor \( G \) from \( \mathcal{C} \) to \( \mathcal{G} \) is not full, since there are no arrows in \( \mathcal{C} \) from \( I \) to \( O \). This functor is also not faithful. The counterexamples that show that are

\[
G(k_1^{0,0}) = G(k_2^{0,0}) = \emptyset,
\]

\[
G(l_1^{1,1}) = G(l_2^{1,1}) = \emptyset,
\]

whereas \( k_1^{0,0} = k_2^{0,0} \) and \( l_1^{1,1} = l_2^{1,1} \) don’t hold in \( \mathcal{C} \). That these equations don’t hold in \( \mathcal{C} \) is demonstrated by the bicartesian category \( \text{Set} \) of sets with functions, with cartesian product \( \times \), disjoint union \( + \), singleton \( I \) and empty set \( O \). In \( \text{Set} \) we don’t have \( l_1^{1,1} = l_2^{1,1} \), and in the bicartesian category \( \text{Set}^{op} \) we don’t have \( k_1^{0,0} = k_2^{0,0} \). The same counterexamples show that the functors \( G \) from \( \mathcal{C}_{x,+} \) to \( \mathcal{G} \), and from \( \mathcal{C}_{x,+,0} \) to \( \mathcal{G} \), are not faithful.

To prove the faithfulness of \( G \) from \( \mathcal{C}_{x,1}^+ \) we need the following lemma.

**Lemma 4.1.** If \( f, g : A \to B \) are composition-free terms of \( \mathcal{C}_{x,1}^+ \) in normal form and \( G(f) = G(g) \) in \( \mathcal{G} \), then \( f \) and \( g \) are the same term.

**Proof.** We proceed by induction on the length of \( f \). If \( f \) is \( 1_p \), then \( g \) must be \( 1_p \), and if \( f \) is \( k_A \), then \( g \) must be \( k_A \). If \( f \) is \( K_{A_1}^{i_1} \ldots K_{A_n}^{i_n} f' \) for \( n \geq 1 \), and \( f' \) is either \( 1_p \) or \( f'_1 + f'_2 \), then with the help of \( G(f) = G(g) \) we conclude that \( g \) too must be of the form \( K_{A_1}^{i_1} \ldots K_{A_n}^{i_n} g' \) for \( g' \) either \( 1_p \) or \( g'_1 + g'_2 \). In the latter case, we apply the induction hypothesis. We apply the induction hypothesis also when \( f \) is \( \langle f_1, f_2 \rangle \) or \( f_1 + f_2 \). \( \square \)
As a corollary we obtain the following proposition.

**UNIQUENESS OF COMPOSITION-FREE NORMAL FORM FOR \( C_{\times,1}^+ \).** If \( f = g \) in \( C_{\times,1}^+ \) for \( f \) and \( g \) in composition-free normal form, then \( f \) and \( g \) are the same term.

**PROOF.** From \( f = g \) it follows that \( f \) and \( g \) are of the same type and that \( G(f) = G(g) \) in \( \mathcal{G} \). Then we apply Lemma 4.1. \( \square \)

Note that we have established this uniqueness without appealing to the Church-Rosser property for our reductions. Another corollary of Lemma 4.1 is that \((\text{cat} \ 2)\) can be derived from the remaining equations of \( C_{\times,1}^+ \). To derive \( h \circ (g \circ f) = (h \circ g) \circ f \), we reduce both sides to cut-free normal form, which is done without using \((\text{cat} \ 2)\). An analogous lemma, which implies Uniqueness of Composition-Free Normal Form and derivability of \((\text{cat} \ 2)\), can also be established for \( C_{\times}^+, C_{\times,1} \) and \( C_{\times} \).

We can now establish the following coherence proposition.

**FAITHFULNESS OF \( G \) FROM \( C_{\times,1}^+ \).** If \( f, g : A \to B \) are terms of \( C_{\times,1}^+ \) and \( G(f) = G(g) \) in \( \mathcal{G} \), then \( f = g \) in \( C_{\times,1}^+ \).

**PROOF.** Suppose \( f, g : A \to B \) are terms of \( C_{\times,1}^+ \), and \( f' \) and \( g' \) are the composition-free normal forms of \( f \) and \( g \) respectively. Then from \( G(f) = G(g) \), \( G(f) = G(f') \) and \( G(g) = G(g') \) we obtain \( G(f') = G(g') \), and therefore, by Lemma 4.1, it follows that \( f' \) and \( g' \) are the same term. Hence \( f = g \) in \( C_{\times,1}^+ \). \( \square \)

A portion of this proof suffices to demonstrate that the functor \( G \) from \( C_{\times}^+ \) to \( \mathcal{G} \) is also faithful, and we can also demonstrate in the same manner the faithfulness of the functor \( G \) from \( C_{\times,1} \) to \( \mathcal{G} \), or from \( C_{\times} \) to \( \mathcal{G} \), but this is already known, as we noted above. By duality, we also have the faithfulness of \( G \) from \( C_{\times,0}, C_{+,1}, C_{+,0} \) and \( C_0 \). It remains to demonstrate that the functor \( G \) from \( C_{\times,1}^+ \) to \( \mathcal{G} \) is also faithful.

For a term of \( C' \) of the form \( f_n \circ \ldots \circ f_1 \), for some \( n \geq 1 \), where \( f_i \) is composition-free we shall say that it is factorized. By using \((\times 2)\), \((+1)\) and \((\text{cat} \ 1)\) it is easy to show that every term of \( C' \) is equal to a factorized term of \( C' \). A subterm \( f_1 \) in a factorized term \( f_n \circ \ldots \circ f_1 \) is called a factor.

A term of \( C' \) where all the atomic terms are identities will be called a complex identity. According to \((\times 1)\), \((+1)\) and \((\text{cat} \ 1)\), every complex identity is equal to an identity. A factor which is a complex identity will be called an identity factor. It is clear that if \( n > 1 \), we can omit in a factorized term every identity factor, and obtain a factorized term equal to the original one.

A term of \( C_{\times,1}^+ \) is called a K-term if it is a term of \( C_{\times}^+ \) and it is not a
complex identity. A term of $C_{x,+}'$ is called an $L$-term iff it is a term of $C_{x,+}'$ and it is not a complex identity. Remember that the terms of $C_{x,+}'$ have the atomic terms $1_A, k^i_{A,B}$ and $w_A$, and the operations on terms $\circ, \times$ and $+$; the terms of $C_{x,+}'$ have the atomic terms $1_A, l^i_{A,B}$ and $m_A$, and the same operations on terms.

A term of $C_{x,+}'$ is said to be in $K-L$ normal form iff it is of the form $g \circ f : A \to B$ for $f$ a $K$-term or $1_A$ and $g$ an $L$-term or $1_B$. Note that $K-L$ normal forms are not unique, since $(m_A \times m_A) \circ w_{A+A}$ and $m_{A \times A} \circ (w_A + w_A)$, which are both equal to $w_A \circ m_A$, are both in $K-L$ normal form.

We can prove the following proposition.

**K-L Normalization.** Every term of $C_{x,+}'$ is equal in $C_{x,+}'$ to a term of $C_{x,+}'$ in $K-L$ normal form.

**Proof.** Suppose $f : B \to C$ is a composition-free $K$-term and $g : A \to B$ is a composition-free $L$-term. We show by induction on the length of $f \circ g$ that

\[(*) \quad f \circ g = g' \circ f' \text{ or } f \circ g = f' \text{ or } f \circ g = g'\]

for $f'$ a composition-free $K$-term and $g'$ a composition-free $L$-term.

We shall not consider below cases where $g$ is $m_B$, which are easily taken care of by $(m)$. The following cases remain.

If $f$ is $k^i_{C,E}$ and $g$ is $g_1 \times g_2$, then we use $(k^i)$. If $f$ is $w_B$, then we use $(w)$. If $f$ is $f_1 \times f_2$ and $g$ is $g_1 \times g_2$, then we use $(\times 2)$, the induction hypothesis, and perhaps (cat 1).

Finally, if $f$ is $f_1 + f_2$, then we have the following cases. If $g$ is $l^{i_1}_{B_1,B_2}$, then we use $(l^i)$. If $g$ is $g_1 + g_2$, then we use $(+2)$, the induction hypothesis, and perhaps (cat 1). This proves $(*)$.

Every term of $C_{x,+}'$ is equal to an identity or to a factorized term $f_n \circ \ldots \circ f_1$ without identity factors. Every factor $f_i$ of $f_n \circ \ldots \circ f_1$ is either a $K$-term or an $L$-term or, by (cat 1), $(\times 2)$ and $(+2)$, it is equal to $f''_i \circ f'_i$ where $f'_i$ is a $K$-term and $f''_i$ is an $L$-term. For example, $(k^1_{A,B} \times l^1_{C,D}) + w_E$ is equal to

\[((1_A \times l^{1}_{C,D}) + 1_{E \times E}) \circ ((k^1_{A,B} \times 1_C) + w_E).\]

Then it is clear that by applying $(*)$ repeatedly, and by applying perhaps (cat 1) at the end, we obtain a term in $K-L$ normal form. ∎

Note that to reduce a term of $C_{x,+}'$ to $K-L$ normal form we have used in this proof all the equations of $C_{x,+}'$ except $(kw_1)$, $(kw_2)$, $(lm_1)$ and $(lm_2)$.

A term of $C'$ is called a $K$-term iff $l_A$, $l^i_{A,B}$ and $m_A$ don’t occur in it and it is not a complex identity. A term of $C'$ is called an $L$-term iff $k_A$, $k^i_{A,B}$ and $w_A$ don’t occur in it and it is not a complex identity. The definition of $K-L$ normal form is as above. Then we can prove $K-L$ Normalization for $C'$ too. It is enough to consider in the induction that establishes $(*)$ in the proof above the
additional cases where \( f \) is \( k_B \) or \( g \) is \( l_B \), which are easily taken care of by (k) and (l).

From now on we shall make no distinction any more between the categories \( \mathcal{C} \) and \( \mathcal{C}' \). These categories are isomorphic, and both will be called \( \mathcal{C} \). When we refer, for example, to the term \( k^1_{A,B} \) of \( \mathcal{C} \), we refer to the term defined as in Section 2, and analogously in other cases. We proceed in the same way in making no distinction between other categories derived from \( \mathcal{C} \) and their isomorphic primed versions. We can then prove the following lemma.

**Lemma 4.2.** Let \( f : A_1 \times A_2 \to B \) be a term of \( \mathcal{C}_x^+ \). If for every \((x, y) \in G(f)\) we have \( x \in |A_1| \), then \( f \) is equal in \( \mathcal{C}_x^+ \) to a term of the form \( f' \circ k^1_{A_1,A_2} \), and if for every \((x, y) \in G(f)\) we have \( x - |A_1| \in |A_2| \), then \( f \) is equal in \( \mathcal{C}_x^+ \) to a term of the form \( f' \circ k^2_{A_1,A_2} \).

**Proof.** We proceed by induction on the length of \( B \). If \( B \) is a propositional letter or \( B_1 + B_2 \), then by Cut Elimination \( f \) must be equal to a term of the form \( f' \circ k^1_{A_1,A_2} \). The condition on \( G(f) \) dictates whether \( i \) here is 1 or 2.

If \( B = B_1 \times B_2 \), and for every \((x, y) \in G(f)\) we have \( x \in |A_1| \), then for \( k^1_{B_1,B_2} \circ f : A_1 \times A_2 \to B_i \), for every \((x, z) \in G(k^1_{B_1,B_2} \circ f)\) we have \( x \in |A_1| \).

So, by the induction hypothesis,

\[ k^1_{B_1,B_2} \circ f = f_1 \circ k^1_{A_1,A_2}. \]

Hence

\[ f = \langle k^1_{B_1,B_2} \circ f, k^2_{B_1,B_2} \circ f \rangle = \langle f_1, f_2 \rangle \circ k^1_{A_1,A_2}. \]

We reason analogously if for every \((x, y) \in G(f)\) we have \( x - |A_1| \in |A_2| \). \( \square \)

We can prove analogously the following dual lemma.

**Lemma 4.3.** Let \( f : A \to B_1 + B_2 \) be a term of \( \mathcal{C}_x^+ \). If for every \((x, y) \in G(f)\) we have \( y \in |B_1| \), then \( g \) is equal in \( \mathcal{C}_x^+ \) to a term of the form \( l^1_{B_1,B_2} \circ g' \), and if for every \((x, y) \in G(f)\) we have \( y - |B_1| \in |B_2| \), then \( g \) is equal in \( \mathcal{C}_x^+ \) to a term of the form \( l^2_{B_1,B_2} \circ g' \).

We shall next prove the following coherence proposition.

**Faithfulness of \( G \) from \( \mathcal{C}_x^+ \).** If \( f, g : A \to B \) are terms of \( \mathcal{C}_x^+ \) and \( G(f) = G(g) \) in \( G \), then \( f = g \) in \( \mathcal{C}_x^+ \).

**Proof.** We proceed by induction on the length of \( A \) and \( B \). In the basis of this induction, when both \( A \) and \( B \) are propositional letters, we conclude by Cut Elimination that \( f \) and \( g \) exist iff \( A \) and \( B \) are the same propositional letter \( p \), and \( f = g = 1_p \) in \( \mathcal{C}_x^+ \). (We could conclude the same thing by
interpreting $C_{x,+}$ in conjunctive-disjunctive logic.) Note that we didn’t need here the assumption $G(f) = G(g)$.

If $A$ is $A_1 + A_2$, then $f \circ l_{A_1,A_2}^1$ and $g \circ l_{A_1,A_2}^1$ are of type $A_1 \rightarrow B$, while $f \circ l_{A_1,A_2}^2$ and $g \circ l_{A_1,A_2}^2$ are of type $A_2 \rightarrow B$. We also have

$$G(f \circ l_{A_1,A_2}^1) = G(f) \circ G(l_{A_1,A_2}^1) = G(g) \circ G(l_{A_1,A_2}^1) = G(g \circ l_{A_1,A_2}^1),$$

whence, by the induction hypothesis,

$$f \circ l_{A_1,A_2}^1 = g \circ l_{A_1,A_2}^1$$
in $C_{x,+}$. Then we infer that

$$[f \circ l_{A_1,A_2}^1, f \circ l_{A_1,A_2}^2] = [g \circ l_{A_1,A_2}^1, g \circ l_{A_1,A_2}^2],$$
from which it follows that $f = g$ in $C_{x,+}$. We proceed analogously if $B$ is $B_1 \times B_2$.

Suppose now $A$ is $A_1 \times A_2$ or a propositional letter, and $B$ is $B_1 + B_2$ or a propositional letter, but $A$ and $B$ are not both propositional letters. Then, by Cut Elimination, $f$ is equal either in $C_{x,+}$ to a term of the form $f' \circ k_{A_1,A_2}^1$, or to a term of the form $l_{B_1,B_2} \circ f'$. Suppose $f = f' \circ k_{A_1,A_2}^1$. Then for every $(x, y) \in G(f)$ we have $x \in |A_1|$. (We reason analogously when $f = f' \circ k_{A_1,A_2}^2$.)

By K-L normalization, $g = g_2 \circ g_1$ in $C_{x,+}$ for $g_1 : A_1 \times A_2 \rightarrow C$ a term of $C_{x,+}^+$ and $g_2$ a term of $C_{x,+}^\times$. Since $g_2 : C \rightarrow B$ is a term of $C_{x,+}^\times$, and hence $K^\times$ (that is, $k^\times$) does not occur in it, for every $z \in |C|$ we have a $y \in |B|$ such that $(z, y) \in G(g_2)$. If for some $(x, z) \in G(g_1)$ we had $x \not\in |A_1|$, then for some $(x, y) \in G(g_2 \circ g_1)$ we would have $x \not\in |A_1|$, but this is impossible since $G(g_2 \circ g_1) = G(g_2) = G(f)$. So for every $(x, z) \in G(g_1)$ we have $x \in |A_1|$. Then, by Lemma 4.2, $g_1 = g'_1 \circ k_{A_1,A_2}^1$ in $C_{x,+}^\times$, and hence in $C_{x,+}$ too. Therefore, $g = g_2 \circ g'_1 \circ k_{A_1,A_2}^1$ in $C_{x,+}$.

Because of the particular form of $G(k_{A_1,A_2}^1)$, we can infer from $G(f) = G(g)$ that $G(f') = G(g_2 \circ g'_1)$, but since $f'$ and $g_2 \circ g'_1$ are of type $A_1 \rightarrow B$, by the induction hypothesis we have $f' = g_2 \circ g'_1$ in $C_{x,+}$, and hence $f = g$. When $f = l_{B_1,B_2} \circ f'$, we reason analogously and apply Lemma 4.3. □

With the help of the Faithfulness of $G$ from $C_{x,-}$ it is easy to establish, for example, that in $C_{x,+}$

$$\langle\langle f_1, f_2, g_1, g_2 \rangle\rangle = \langle\langle f_1, g_1, f_2, g_2 \rangle\rangle,$$
for which we have the diagrams
or that in $C_{\times,+}$

$$((k^1_{A,B} + k^1_{C,D}) \times (k^2_{A,B} + k^2_{C,D})) \circ w_{(A \times B) + (C \times D)} = m_{(A+C) \times (B+D)} \circ ((l^1_{A,C} \times l^1_{B,D}) + (l^2_{A,C} \times l^2_{B,D})),$$

for which we have the diagrams

Each line in such a diagram stands for a family of parallel lines, one for each propositional letter in the schemata $A$, $B$, $C$ and $D$. 

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In general, to verify whether for \( f, g : A \rightarrow B \) in the language of \( \mathcal{C}_{x,+} \) we have \( f = g \) in \( \mathcal{C}_{x,+} \) it is enough to draw \( G(f) \) and \( G(g) \), and check whether they are equal, which is clearly a finite task. So we have here an easy decision procedure for the equations of \( \mathcal{C}_{x,+} \).

5 Maximality

We shall now show that categories with binary products and sums, which we call \( x,+ \)-categories, are maximal in the sense that if any equation \( f = g \) in the language of the free \( x,+ \)-category \( \mathcal{C}_{x,+} \) doesn’t hold in \( \mathcal{C}_{x,+} \) holds in a \( x,+ \)-category \( \mathcal{B} \), then \( \mathcal{B} \) is a preorder, i.e. a category where all arrows with the same source and the same target are equal. That \( f = g \) holds in \( \mathcal{B} \) means that it holds universally with respect to objects. Namely, propositional letters in the indices of \( f \) and \( g \) are assumed to be variables, and \( f = g \) holds iff it holds for every assignment of objects to these variables. (This sort of universal holding is quite natural in logic, and elsewhere in mathematics: it is usually taken for granted when one says that a formula with variables “holds”. In the lambda calculus this universal holding is sometimes called “typical ambiguity”.) An analogous maximality is proved for \( \mathcal{C}_{x,1} \) and \( \mathcal{C}_{x} \) (and, by duality, for \( \mathcal{C}_{+,0} \) and \( \mathcal{C}_{+} \)) in [3], and for cartesian closed categories in [20] and [4].

Suppose \( A \) and \( B \) are formulae of \( \mathcal{P}_{x,+} \) in which only \( p \) occurs as a propositional letter. If for \( f, g : A \rightarrow B \), we have \( G(f) \neq G(g) \), then for some \( x \in |A| \) and some \( y \in |B| \) we have \( (x, y) \in G(f) \) and \( (x, y) \notin G(g) \), or vice versa. Suppose \( (x, y) \in G(f) \) and \( (x, y) \notin G(g) \). For every subformula \( C \) of \( A \) and every formula \( D \) let \( A^D_C \) be the formula obtained from \( A \) by replacing the particular occurrence of the subformula \( C \) by \( D \). It is easy to see that for every subformula \( A_1 + A_2 \) of \( A \) we have an arrow \( h(l^i_{A_1,A_2}) \) of \( \mathcal{C}_{x,+} \) built with \( l^i_{A_1,A_2} \), identity arrows and the operations on arrows \( \times \) and \( + \), such that \( f \circ h(l^i_{A_1,A_2}) \) and \( g \circ h(l^i_{A_1,A_2}) \) are of type \( A^{A_1 + A_2} \rightarrow B \).

We say that \( x \in \omega \) belongs to a subformula \( C \) of \( A \) iff the \( x \)-th occurrence of propositional letters in \( A \), counting from the left, is in \( C \). If \( x \) happens to belong to \( A_1 \), we take care above to choose \( h(l^i_{A_1,A_2}) \), and if it belongs to \( A_2 \), we choose \( h(l^j_{A_1,A_2}) \). If \( x \) belongs to neither, we choose \( h(l^k_{A_1,A_2}) \) arbitrarily. By repeated compositions of \( f \) and \( g \) with such \( h(l^i_{A_1,A_2}) \) arrows, for every \( + \) in \( A \), we obtain two arrows \( f', g' : p \times \ldots \times p \rightarrow B \) of \( \mathcal{C}_{x,+} \) such that parentheses are somehow associated in \( p \times \ldots \times p \), and for some \( (z, y) \in G(f') \) we have \( (z, y) \notin G(g') \). The formula \( p \times \ldots \times p \) may be only \( p \). We may further compose \( f' \) and \( g' \) with natural isomorphisms of the types \( (C_1 \times C_2) \times C_3 \rightarrow C_1 \times (C_2 \times C_3) \), \( C_1 \times (C_2 \times C_3) \rightarrow (C_1 \times C_2) \times C_3 \) and \( C_1 \times C_2 \rightarrow C_2 \times C_1 \), which are definable in \( \mathcal{C}_x \), and hence also in \( \mathcal{C}_{x,+} \), in order to obtain two arrows \( f'', g'' : p \times A' \rightarrow B \) or \( f'', g'' : p \rightarrow B \) such that \( A' \) is of the form \( p \times \ldots \times p \) with parentheses somehow associated, and \( (0, y) \in G(f'') \) but \( (0, y) \notin G(g'') \). The functor \( G \) maps the
natural associativity and commutativity isomorphisms into bijections.

By working dually on every $x$ in $B$ using $h(k_{B_1,B_2})$, and by composing perhaps further with natural associativity and commutativity isomorphisms of $+$, we obtain two arrows $f''$ and $g''$ of $C_{x,+}$ of type $p \times A' \rightarrow p + B'$ for $A'$ of the form $p \times \ldots \times p$ and $B'$ of the form $p + \ldots + p$, or of type $p \times A' \rightarrow p$, or of type $p \rightarrow p + B'$, such that $(0,0) \notin G(f'')$ and $(0,0) \notin G(g'')$. (We cannot obtain that $f''$ and $g''$ are of type $p \rightarrow p$, since otherwise, by Cut Elimination, $g$ would not exist.)

With the help of $w_p$ we can define in $C_{x,+}$ the arrow $h^x : p \rightarrow p \times \ldots \times p$ such that for every $x \in [p \times \ldots \times p]$ we have $(0,x) \in G(h^x)$. We define analogously with the help of $m_p$ the arrow $h^+ : p + \ldots + p \rightarrow p$ such that for every $x \in [p + \ldots + p]$ we have $(x,0) \in G(h^+)$. (The arrows $h^x$ and $h^+$ may be $1_p : p \rightarrow p$.)

If $f''$ and $g''$ are of type $p \times A' \rightarrow p + B'$, let $f^1, g^1 : p \times p \rightarrow p + p$ be defined by

\[
\begin{align*}
  f^1 &= \text{def.} (1_p + h^+) \circ f'' \circ (1_p \times h^x), \\
g^1 &= \text{def.} (1_p + h^+) \circ g'' \circ (1_p \times h^x).
\end{align*}
\]

By Cut Elimination we have that $G(f^1)$ and $G(g^1)$ are singletons. If $(1,0)$ or $(1,1)$ belong to $G(g^1)$, then for $f^*, g^* : p \times p \rightarrow p$ defined as $m_p \circ f^1$ and $m_p \circ g^1$, respectively, we have $(0,0) \in G(f^*)$ and $(0,0) \notin G(g^*)$. If $(0,1)$ or $(1,1)$ belong to $G(g^1)$, then for $f^*, g^* : p \rightarrow p + p$ defined as $f^1 \circ w_p$ and $g^1 \circ w_p$, respectively, we have $(0,0) \in G(f^*)$ and $(0,0) \notin G(g^*)$.

If $f''$ and $g''$ are of type $p \times A' \rightarrow p$, then for $f^*, g^* : p \times p \rightarrow p$ defined as $f'' \circ (1_p \times h^x)$ and $g'' \circ (1_p \times h^x)$, respectively, we have $(0,0) \in G(f^*)$ and $(0,0) \notin G(g^*)$.

If $f''$ and $g''$ are of type $p \rightarrow p + B'$, then for $f^*, g^* : p \rightarrow p + p$ defined as $(1_p + h^+) \circ f''$ and $(1_p + h^+) \circ g''$, respectively, we have $(0,0) \in G(f^*)$ and $(0,0) \notin G(g^*)$. In all that, we have by Cut Elimination that $G(f^*)$ and $G(g^*)$ are singletons.

In cases where $f^*$ and $g^*$ are of type $p \times p \rightarrow p$, by Cut Elimination, by the conditions on $G(f^*)$ and $G(g^*)$, and by the functoriality of $G$, we obtain that $f^* = k^1_{p,p}$ and $g^* = k^2_{p,p}$. So from $f = g$ we can derive $k^1_{p,p} = k^2_{p,p}$. In cases where $f^*$ and $g^*$ are of type $p \rightarrow p + p$, by Cut Elimination, by the conditions on $G(f^*)$ and $G(g^*)$, and by the functoriality of $G$, we obtain that $f^* = l^1_{p,p}$ and $g^* = l^2_{p,p}$. So from $f = g$ we can derive $l^1_{p,p} = l^2_{p,p}$. In any case, from $f = g$ we can derive $k^1_{p,p} = k^2_{p,p}$ or $l^1_{p,p} = l^2_{p,p}$. If either of these two equations holds in a $\times,+\text{-category } B$, then $B$ is a preorder. For $h_1, h_2 : C \rightarrow D$ in $B$ we have

\[
\begin{align*}
k^1_{D,D} \circ (h_1, h_2) &= k^2_{D,D} \circ (h_1, h_2), \\
[h_1, h_2] \circ l^1_{C,C} &= [h_1, h_2] \circ l^2_{C,C},
\end{align*}
\]

from which $h_1 = h_2$ follows. (We have said that the holding of $k^1_{p,p} = k^2_{p,p}$ or $l^1_{p,p} = l^2_{p,p}$ is understood universally with respect to objects of $B$, so that we may replace $p$ by any object of $B$.)

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It remains to remark that if for any $f, g : A \rightarrow B$ of $\mathcal{C}_{x,+}$ we have that $f = g$ doesn’t hold in $\mathcal{C}_{x,+}$, then by the Faithfulness of $G$ from $\mathcal{C}_{x,+}$ we have $G(f) \neq G(g)$. If we take the substitution instances $f'$ and $g'$ of $f$ and $g$ obtained by replacing uniformly every propositional letter in $A$ and $B$ by $p$, then we obtain again that $G(f') \neq G(g')$. If $f = g$ holds in a $x,+$-category $\mathcal{B}$, then $f' = g'$ holds too, and hence $\mathcal{B}$ is a preorder, as we have shown above.

This maximality result means that all equations in the language of $\mathcal{C}_{x,+}$ that don’t hold in $\mathcal{C}_{x,+}$ can be derived from each other with the help of the equations of $\mathcal{C}_{x,+}$. This result is analogous to the syntactic completeness of the classical propositional calculus discovered by Bernays and Hilbert (see [27], p. 341), which is called Post Completeness. The Faithfulness of $G$ from $\mathcal{C}_{x,+}$ amounts to a semantical completeness result (soundness is provided by $G$ being a functor).

As a consequence of the maximality of $x,+,$-categories we obtain that $\mathcal{C}_{x,+}$ is a subcategory of $\mathcal{C}$. By Cut Elimination we may conclude that it is a full subcategory of $\mathcal{C}$. This means that the equations between terms of $\mathcal{C}$ make a conservative extension of the equations of $\mathcal{C}_{x,+}$; namely, if an equation in the language of $\mathcal{C}_{x,+}$ holds in $\mathcal{C}$, then it holds already in $\mathcal{C}_{x,+}$. This applies also to any other category that is not a preorder whose equations extend those of $\mathcal{C}_{x,+}$: for example, the free distributive bicartesian category generated by $\mathcal{L}$, or the free bicartesian closed category generated by $\mathcal{L}$.

So we may use the decision procedure provided by our Faithfulness of $G$ from $\mathcal{C}_{x,+}$ in order to check equations of arrows in these extensions of $\mathcal{C}_{x,+}$, provided the terms of these arrows are terms of $\mathcal{C}_{x,+}$.

Bicartesian categories are not maximal in the sense in which $x,+,$-categories are. Set is a bicartesian category that is not a preorder in which $k_{1}^{1} \circ O_{1} = k_{2}^{2} \circ O_{1}$ holds, though this equation does not hold in every bicartesian category. (Another example of such an equation is $l_{0}^{0} \circ k_{0}^{0} = 1_{0} \circ A$.) In a companion to this paper [5] we shall show that it is enough to add $k_{1}^{1} = k_{2}^{2}$ to $\mathcal{C}_{x,+}$ to obtain coherence. We shall show that we have only a restricted form of coherence when we add to $\mathcal{C}$ this equation and $l_{1}^{1} = l_{1}^{2}$. However, this last equation does not hold in Set.

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