Convergence of the Environment Seen from Geodesics in Exponential Last-Passage Percolation

James B. Martin ∗ Allan Sly † Lingfu Zhang ‡

Abstract

A well-known question in the planar first-passage percolation model concerns the convergence of the empirical distribution along geodesics. We demonstrate this convergence for an explicit model, directed last-passage percolation on $\mathbb{Z}^2$ with i.i.d. exponential weights, and provide explicit formulae for the limiting distributions, which depend on the asymptotic direction. For example, for geodesics in the direction of the diagonal, the limiting weight distribution has density $(1/4 + x/2 + x^2/8)e^{-x}$, and so is a mixture of Gamma(1, 1), Gamma(2, 1) and Gamma(3, 1) distributions with weights $1/4$, 1/2, and 1/4 respectively. More generally, we study the local environment as seen from vertices along the geodesics (including information about the shape of the path and about the weights on and off the path in a local neighborhood). We consider finite geodesics from $(0, 0)$ to $n\rho$ for some vector $\rho$ in the first quadrant, in the limit as $n \to \infty$, as well as the semi-infinite geodesic in direction $\rho$. We show almost sure convergence of the empirical distributions along the geodesic, as well as convergence of the distribution around a typical point, and we give an explicit description of the limiting distribution.

We make extensive use of a correspondence with TASEP as seen from a single second-class particle for which we prove new results concerning ergodicity and convergence to equilibrium. Our analysis relies on geometric arguments involving estimates for the last-passage time, available from the integrable probability literature.

Contents

1 Introduction ........................................ 2
   1.1 Model definition and main results ....................... 3
   1.2 A roadmap of our arguments .......................... 5
   1.3 Some previous arguments ............................ 6
   1.4 Further applications and questions ................... 6

2 Preliminaries ....................................... 7
   2.1 Semi-infinite geodesics and the Busemann function .... 7
   2.2 Competition interface and a hole-particle pair in TASEP ... 8
   2.3 Second-class particle and the stationary distribution .. 9
   2.4 Estimates for last-passage percolation ................. 11

3 The limiting distribution ............................... 14

∗Department of Statistics, University of Oxford, e-mail: martin@stats.ox.ac.uk
†Department of Mathematics, Princeton University, e-mail: asly@princeton.edu
‡Department of Mathematics, Princeton University, e-mail: lingfuz@math.princeton.edu
In this article we study exactly solvable models of planar directed last-passage percolation (LPP), an instance of the more general Kardar-Parisi-Zhang (KPZ) universality class, which dates back to the seminal work of [KPZ86]. The KPZ universality class has been a major topic of interest both in statistical physics and in probability theory in recent decades. In [KPZ86], the authors predicted universal scaling behaviour for a large number of planar random growth processes, including first-passage percolation and corner growth processes; in particular, it is predicted that these models have length fluctuation exponent $1/3$ and transversal fluctuation exponent $2/3$. Since then, rigorous progress has been made only in a handful of cases. The first breakthrough was made by Baik, Deift and Johansson [BDJ99] when they established $n^{1/3}$ fluctuations of the length of the longest increasing path from $(0,0)$ to $(n,n)$ in a homogeneous Poissonian field on $\mathbb{R}^2$, and also established...
the GUE Tracy-Widom scaling limit. Then Johansson proved a transversal fluctuation exponent of \(2/3\) for that model, and also \(n^{1/3}\) fluctuations and a Tracy-Widom scaling limit for directed last-passage percolation on \(\mathbb{Z}^2\) with i.i.d. geometric or exponential weights \[\text{Joh00a, Joh00b}.\] For these models such results could be obtained due to their exactly solvability, using exact distributional formulae from algebraic combinatorics, random matrix theory, or queueing theory in some cases. Since then there have been tremendous developments in achieving a detailed understanding of these exactly solvable models, with notable progress concerning scaling limits (e.g. the recent works of \[\text{MQR17, DOV18}\]). For surveys in this direction, see e.g. \[\text{Cor12, QR14, Zyg18}\].

In another related direction, there has been great interest in studying planar first passage percolation with general weights. Such models are also conjectured to be in the KPZ universality class, but much less is known due to the lack of exact formulae. The geometry of the set of geodesics has been an important tool in the study of these models; see e.g. \[\text{New95, ADH17}\]. When trying to understand the behavior of large finite or infinite geodesics, a well-known open question is whether the empirical distributions along geodesics converge; see e.g. \[\text{Hot15}\] where it is proposed by Hoffman during a 2015 American Institute of Mathematics workshop. Recently, Bates gave an affirmative answer to this question for various abstract dense families of weight distributions \[\text{Bat20}\]. The proof uses a variational formula, and does not rely on any exactly solvable structure.

In this paper we study the limiting local behaviour for LPP in the exactly solvable case. We focus on LPP on \(\mathbb{Z}^2\) with i.i.d. exponential weights. Rather than the weights along geodesics, we consider the more general ‘empirical environment’ around vertices, along a finite or semi-infinite geodesic, and we show that it converges to a deterministic measure. By the environment we mean the weights of nearby points, and the path of the geodesic through them. In particular, this positively answers the question of Hoffman for a first explicit model. Our approach is different from \[\text{Bat20}\] and relies on information provided by the exactly solvable structure. In addition to proving convergence results, we also give an explicit description of the limiting distribution, which depends on the direction of the geodesic. Using this description one can compute any limiting local statistics of the geodesic, and we give some first examples in this paper.

A particular exactly solvable input that we use is the connection between LPP and the totally asymmetric exclusion process (TASEP), dating back to \[\text{Ros81}\]. We use the correspondence between a semi-infinite geodesic and the trajectory of a second-class particle, as developed in a series of works \[\text{FP05, FMP09, Pim16}\]. Then in order to understand the local environment around the geodesic, we study the stationary distribution of TASEP as seen from a second-class particle. Models involving second-class particles have proven powerful in understanding the evolution of the TASEP \[\text{FKS91, Fer92, DJLS93, BCS06, BS10, Sch21}\], and stationary distributions for multi-type systems have been widely studied \[\text{DJLS93, Spe94, FPK94, Ang06, FM07, EFM09, AAMP11}\]. See also \[\text{Fer18}\] for a recent survey of related ideas.

### 1.1 Model definition and main results

We study the exponential weight planar directed last-passage percolation (LPP) model, which is defined as follows. To each vertex \(v \in \mathbb{Z}^2\) we associate an independent weight \(\xi(v)\) with \(\text{Exp}(1)\) distribution. For two vertices \(u, v \in \mathbb{Z}^2\), we say \(u \leq v\) if \(u\) is coordinate-wise less than or equal to \(v\). For such \(u, v\) and any up-right path \(\gamma\) from \(u\) to \(v\), we define the \emph{passage time} of the path to be

\[
T(\gamma) := \sum_{w \in \gamma} \xi(w).
\]

Then almost surely there is a unique up-right path from \(u\) to \(v\) that has the largest passage time. We call this path the \emph{geodesic} \(\Gamma_{u,v}\), and call \(T_{u,v} := T(\Gamma_{u,v})\) the \emph{passage time from \(u\) to \(v\)}. For each
In this section, we define the empirical distribution of the environment along a directed path \( \gamma \), and we define the empirical distribution along a semi-infinite geodesic as

\[
\mu_{\gamma} := \frac{1}{|\Gamma_{u,v}|} \sum_{w \in \Gamma_{u,v}} \delta_{(\xi(w), \Gamma_{u,v} - w)}.
\]

Similarly, we define the empirical distribution along a semi-infinite geodesic as

\[
\mu_{v,r} := \frac{1}{2r + 1} \sum_{i=1}^{2r+1} \delta_{(\xi(\Gamma_{v}[i]), \Gamma_{v} - \Gamma_{v}[i])},
\]

for any \( v \in \mathbb{Z}^2 \), \( \rho \in (0,1) \), and \( r \in \mathbb{Z}_{>0} \). We will show that these empirical distributions converge. For each \( \rho \), there is a limiting distribution \( \nu^\rho \) on \( \mathbb{R}^{\mathbb{Z}^2} \times \{0,1\}^{\mathbb{Z}^2} \), which is explicit and will be defined in Section 3.

For any \( n \in \mathbb{Z} \) we denote \( n^\rho := \left( \left[ \frac{2(1-\rho)^2n}{\rho^2+(1-\rho)^2} \right], \left[ \frac{2\rho^2n}{\rho^2+(1-\rho)^2} \right] \right) \). We also let \( n = n^{1/2} = (n,n) \), and in particular we have \( 0 = (0,0) \). For the next four theorems we fix any \( \rho \in (0,1) \).

**Theorem 1.1.** In probability \( \mu_{0,n^\rho} \rightarrow \nu^\rho \) as \( n \rightarrow \infty \).

**Theorem 1.2.** Almost surely \( \mu_{0,r}^{\rho} \rightarrow \nu^\rho \) as \( r \rightarrow \infty \).

We also have convergence of distributions.

**Theorem 1.3.** The law of \( (\xi(\Gamma_{0}[i]), \Gamma_{0} - \Gamma_{0}[i]) \) converges to \( \nu^\rho \) as \( i \rightarrow \infty \).

**Theorem 1.4.** For each \( 0 < \alpha < 2 \), the law of \( (\xi(\Gamma_{0,n^\rho}[|\alpha n|]), \Gamma_{0,n^\rho} - \Gamma_{0,n^\rho}[|\alpha n|]) \) converges to \( \nu^\rho \) as \( n \rightarrow \infty \).

For the limiting distribution of \( \nu^\rho \) to be defined in Section 3, the construction is explicit, and here we give the formula for the distribution function at the origin.

**Proposition 1.5.** For \( (\xi, \gamma) \sim \nu^\rho \), we have \( \mathbb{P}[\xi(0) > h] = \left( 1 + \frac{\rho(1-\rho)h}{(1-\rho)^2+h} \right) (1 + \rho(1-\rho)h)e^{-h} \).

The distribution of \( \xi(0) \) given in Proposition 1.5 is a mixture of Gamma(1,1), Gamma(2,1) and Gamma(3,1) distributions. In the case \( \rho = 1/2 \), for example, the weights of this mixture are 1/4, 1/2, and 1/4 respectively, and the distribution of \( \xi(0) \) can be interpreted as that of \( 2 \min(E_1 + E_2, E_3 + B_E) \) with \( B \sim \text{Bernoulli}(1/2) \) and \( (E_i)_{1 \leq i \leq 4} \) i.i.d. ~ \text{Exp}(1) independently of \( B \). Related but slightly less simple representations can be given for general \( \rho \). See the discussion at the end of Section 3.

From the distribution \( \nu^\rho \) and the convergence results, we can obtain information about the geodesics. The following result is such an example.

**Proposition 1.6.** Denote by \( N_{n,\rho} \) the number of ‘corners’ along \( \Gamma_{0,n^\rho} \); that is, the number of \( v \in \mathbb{Z}^2 \) such that \( \{ v - (1,0), v, v + (0,1) \} \subset \Gamma_{0,n^\rho} \), or \( \{ v - (0,1), v, v + (1,0) \} \subset \Gamma_{0,n^\rho} \). Then we have \( \frac{N_{n,\rho}}{2n} \rightarrow \frac{2\rho^2(1-\rho)^2(1+2\rho-2\rho^2)}{(1-\rho)^2+\rho^2} \) in probability, as \( n \rightarrow \infty \).
For example, for $\rho = 1/2$, the proportion of steps of the geodesic which are ‘corners’ converges to $3/8$.

In our proofs of the above results we will use the connection between LPP and the TASEP, which can be described as a Markov process $(\eta_t)_{t \in \mathbb{R}}$ on $\{0, 1\}^\mathbb{Z}$, where $\eta_t(x) = 1$ means that there is a particle at site $x$ at time $t$, whereas $\eta_t(x) = 0$ means that there is a hole at site $x$ at time $t$. If there is a particle at site $x$ and a hole at site $x+1$, they switch at rate 1. We shall consider TASEP with a single ‘second-class particle’, which can switch with a hole to the right of it, or with a (normal) particle to the left of it. We prove a corresponding result for TASEP with a single second-class particle as well, which may be of independent interest.

**Theorem 1.7.** $\lim_{t \to \infty} \Phi_t^\rho = \Psi^\rho$.

Here $\Phi_t^\rho$ and $\Psi^\rho$ are measures on $\{0, 1\}^\mathbb{Z}$ to be defined in Section 2.3 and we describe them here. Consider TASEP with a single second-class particle, where initially there is a second-class particle at the origin, and any other site has a (normal) particle with probability $\rho$ independently. Then $\Phi_t^\rho$ is the law of such TASEP at time $t$, as seen from the second-class particle. The measure $\Psi^\rho$ is the stationary distribution of a TASEP as seen from a second-class particle with density $\rho$. In proving this theorem, we will also show that this stationary process is ergodic in time (Proposition 5.3).

### 1.2 A roadmap of our arguments

There are two main ingredients in our proofs of the above results: geometry of the geodesics in exponential LPP, and the TASEP as seen from a second-class particle.

For each $\rho \in (0, 1)$ there is a (density $\rho$) stationary distribution for TASEP, where for each site there is a particle with probability $\rho$ and a hole with probability $1 - \rho$ independently (i.e. i.i.d. Bernoulli). Such i.i.d. Bernoulli TASEP corresponds to a growing interface in $\mathbb{Z}^2$, which (when rotated by $\pi/4$) is a random walk at any time. Dividing the interface into two competing clusters, this gives a competition interface which corresponds to a semi-infinite geodesic; see e.g. [FMP09, Pim16]. On the other hand, in TASEP such a competition interface corresponds to a second-class particle. Thus, the environment seen from a semi-infinite geodesic corresponds to TASEP as seen from a single second-class particle.

We will first construct the limiting distribution $\nu^\rho$ in Section 3 using the stationary measure of TASEP as seen from a single second-class particle, as described in [FFK94]; we then prove Proposition 1.5 and 1.6 assuming the convergence results.

For the remaining results we take the following approach. In Section 4 we prove Theorem 1.7 by first proving a weaker version involving convergence in an averaged sense (Proposition 4.1), and then upgrading using the connection with LPP and geometric arguments. In Section 5 we prove a weaker version of Theorem 1.2 involving convergence in probability (Theorem 5.1). This is done by using Theorem 1.7 and ergodicity of the stationary process, which we prove as Proposition 5.3. From then on we work completely in the LPP setting. In Section 6 we prove Theorem 1.1 using Theorem 5.1 by covering a finite geodesic with an infinite one.

The next few sections rely on a generalization of Theorem 1.1, which is Proposition 7.1 and which says that for a family of geodesics whose endpoints vary range along two anti-diagonal segments, their empirical distributions converge jointly (in probability). The proof is via taking a finite (i.e. whose size does not grow) dense subset of these geodesics, and showing that each geodesic in the family can be mostly covered by them. Using this result, in Section 8 we prove Theorem 1.3 and 1.4 by showing that the distributions at neighboring points are close to each other. In Section 9, by covering long or semi-infinite geodesics by short ones, we prove that the empirical distribution concentrates exponentially fast, and thus upgrade Theorem 5.1 to Theorem 1.2.
1.3 Some previous arguments

In the development of this work there were some alternative arguments that was previously taken – we briefly discuss the arguments here.

Compared to the current arguments, the major difference of the other approach is that, working just from the LPP perspective, one could prove convergence of the empirical distribution for finite geodesics, i.e. Theorem 1.1 without identifying the limit. The idea is to study the passage times $T_{0,u}$ and $T_{v,n}$, where $u,v$ vary in the lines $x + y = an$ and $x + y = (a + \varepsilon)n$ for some $a \in (0,2)$ and small $\varepsilon > 0$. These two point-to-line profiles are independent, each converges (after rescaling) to the Airy$_2$ process [BF08, BP08], which is locally like a Brownian motion. Then it can be shown that in small neighborhoods of the intersections of the geodesic with these two lines, the point-to-line profiles (after rescaling) are similar to the maximum of the sum of two independent Brownian motions. (In recent work [DSV20], such behaviour is observed for geodesics in the directed landscape, the limiting object of the exponential LPP model.) One can also show that the geodesic between these lines is stable with respect to small perturbations of the line-to-point profiles. From this we could deduce that, for different vertices along the geodesic, the local statistics are similar, and that the correlation decays fast when the distance between the vertices is of order $n$. Thus we prove concentration of the empirical distribution via decay of the variance. To show that it converges, and to prove convergence for semi-infinite geodesics, we would need to cover long geodesics with short ones, using arguments similar to those in Section 6–9. Finally, from there to get an explicit description of the limiting distribution, we still need to use the stationary distribution of TASEP with a single second-class particle.

The main reason we take the current path is for simplicity of arguments. For instance, it is highly technical to establish asymptotic independence of environments around different (far away) vertices along the geodesic, and also the stability of geodesics (with respect to the point-to-line profiles). The convergence to $R - B, R + B$ would involve computations using the KPZ fixed point formulae from [MQR17]. These arguments are now replaced by proving convergence of TASEP as seen from a second-class particle, although the proof of the TASEP convergence still relies on the LPP setting and geometric arguments (see Section 4). It is also more natural to start by showing convergence for semi-infinite geodesics, and deduce the result for finite ones (as opposed to the other way round), given the input from TASEP as seen from a second-class particle. The more geometric arguments later in Section 6–9 are more inherited and adapted from the previous approach.

1.4 Further applications and questions

With the limiting measure $\nu^\rho$ one can get any local information along the geodesics. Before closing the introduction we discuss some questions, which can potentially be answered using our explicit descriptions of $\nu^\rho$, either as direct applications or require some further analysis.

The first question is communicated to us by Alan Hammond. Given that a vertex on the geodesic has a large weight, how would the local environment behave? For a vertex with a large weight, it would force the geodesic to go through it. Thus we expect that conditional on this, weights of nearby vertices are distributed like i.i.d. exponential random variables. From the TASEP aspect, a large weight corresponds to a long waiting time between two jumps of the second class particle, and this is mostly due a ‘jam’ in TASEP, i.e., a consecutive sequence of particles to the right of the second class particle, and a sequence of holes to the left. This resembles a ‘reversed’ step initial condition.

A related question is about vertices near but off the geodesic. Using our formulation of $\nu^\rho$ it can
be verified that, for those next to the geodesic, their weights are strictly stochastically dominated by \( \text{Exp}(1) \). It is then interesting to see if the distribution converges to \( \text{Exp}(1) \) as the distance increases to infinity.

The next question is about a slightly different setting, that of LPP with i.i.d. geometric weights. The main difference is that in that case the weights are discrete, and the geodesics are not necessarily unique. However, one could still consider the ‘rightmost’ geodesics. Geometric LPP corresponds to discrete-time TASEP, and one can similarly construct a stationary measure for such a TASEP as seen from a single second-class particle. For a correspondence with the rightmost geodesic, one takes a second-class particle which is prioritized to jump to the right rather than to the left. One can similarly construct a limiting distribution, and thus get local information about the environment along the rightmost geodesics. One question that would be interesting to study is the proportion of ‘unique geodesic vertices’. For fixed endpoints (or for one fixed endpoint and a fixed direction), take the intersection of all the geodesics, and call the vertices in that intersection ‘unique geodesic vertices’. Do these unique geodesic vertices asymptotically make up a positive proportion of the vertices? Furthermore, does the proportion converge in probability, and can we compute the limit explicitly? We think such questions are related to the convergence of the environment around the rightmost geodesic, because we expect that a vertex on the path is unlikely to be ‘locally unique’ without being a unique geodesic vertex in the sense mentioned above. Anomalous ‘locally but not globally unique’ vertices should make up a vanishing proportion of the path in the limit.

Another direction concerns the scaling limit of the measure \( \nu^\rho \). In a recent work by Dauvergne, Sarkar, and Virág, the authors zoom in around the geodesic in the directed landscape, constructing a local environment [DSV20, Theorem 1.1]. It is reasonable to expect that when zooming out, the measure \( \nu^\rho \) would converge to the local environment constructed there. Also, once this is established, we would like to see if our explicit description of \( \nu^\rho \) could be used to get some explicit information about the local environment and geodesics in the directed landscape (e.g. [DSV20, Problem 4]).

2 Preliminaries

We start by setting up some basic notation to be used for the rest of the text. For any \( x, y \in \mathbb{R} \) we denote \( x \vee y = \max\{x, y\} \), and \( x \wedge y = \min\{x, y\} \), and \( [x, y] \) is the set \( \{x, y\} \cap \mathbb{Z} \). For each \( u = (a, b) \in \mathbb{Z}^2 \), denote \( d(u) = a + b \) and \( ad(u) = a - b \). For \( n \in \mathbb{Z} \) we denote \( \mathbb{I}_n = \{u \in \mathbb{Z}^2 : d(u) = 2n\} \).

2.1 Semi-infinite geodesics and the Busemann function

For each \( \rho \in (0, 1) \), and any \( u, v \in \mathbb{Z}^2 \), we denote \( B^\rho(u, v) := T_{u,c} - T_{v,c} \), where \( c \in \mathbb{Z}^2 \) is the coalescing point of \( \Gamma^\rho_u \) and \( \Gamma^\rho_v \), i.e. \( c \) is the vertex in \( \Gamma^\rho_u \cap \Gamma^\rho_v \) with the smallest \( d(c) \). Such \( B^\rho \) is called the Busemann function in direction \( \rho \). We also write \( G^\rho(u) = B^\rho(0, u) \). This Busemann function satisfies the following properties.

1. For each \( u, v, w \in \mathbb{Z}^2 \), \( B^\rho(u, v) + B^\rho(v, w) = B^\rho(u, w) \). In particular, \( B^\rho(u, v) = G^\rho(v) - G^\rho(u) \).
2. For each \( (a, b) \in \mathbb{Z}^2 \), \( G^\rho(a, b) = G^\rho(a + 1, b) \wedge G^\rho(a, b + 1) - \xi(a, b) \).
3. For any down-right path \( U = \{u_k\}_{k \in \mathbb{Z}} \), the random variables \( B^\rho(u_k, u_{k-1}) \) are independent. The law of \( B^\rho(u_k, u_{k-1}) \) is \( \text{Exp}(\rho) \) if \( u_k = u_{k-1} - (0, 1) \), and is \( -\text{Exp}(1-\rho) \) if \( u_k = u_{k-1} + (1, 0) \).
4. Denote \( \xi^\vee(a, b) = G^\rho(a, b) - G^\rho(a - 1, b) \vee G^\rho(a, b - 1) \) for \( (a, b) \in \mathbb{Z}^2 \), then \( \{\xi^\vee(u)\}_{u \in \mathbb{Z}^2} \) are i.i.d. \( \text{Exp}(1) \).

The first two properties are by definition. For the third property, a proof can be found in \[\text{Sep20}\]. The last property comes from [FMP09, Lemma 4.2] (see also [BCS06]).
2.2 Competition interface and a hole-particle pair in TASEP

We next discuss the competition interface starting from the origin (see e.g. [FP05, FMP09]).

We consider an up-right growing interface: for each \( t \geq 0 \), we say that a vertex \( u \in \mathbb{Z}^2 \) is occupied by time \( t \) if \( G^\rho(u) \leq t \), and we denote by \( I_t \) the set of vertices occupied by time \( t \). Then the waiting times in this growing interface are given by \( \{ \xi^\rho(u) \}_{u \in \mathbb{Z}^2} \), which are i.i.d. \( \text{Exp}(1) \); thus \((I_t)_{t \in \mathbb{R}}\) is a Markov process, such that given \( I_t \), each vertex \( u \not\in I_t \) with \( u - (0,1), u - (1,0) \in I_t \) becomes occupied with rate 1 independently.

We next define two clusters \( C_1 \) and \( C_2 \): let \( \{(a,0) : a \in \mathbb{N}\} \subset C_1 \), and \( \{(0,b) : b \in \mathbb{N}\} \subset C_2 \). For any \((a,b) \in \mathbb{Z}^2 \) with \( a,b > 0 \), let its ‘parent’ be either \((a-1,b)\) or \((a,b-1)\), whichever is occupied later; then \((a,b)\) is in the same cluster as its parent. We can then define the boundary of this competition clusters, \( Z = \{ Z[i] \}_{i \in \mathbb{N}} \), as follows: first let \( Z[1] = (1/2,1/2) \), and then let \( Z[i+1] = Z[i] + (1,0) \) if \( G^\rho(Z[i] + (1/2,-1/2)) \leq G^\rho(Z[i] + (-1/2,1/2)) \), and \( Z[i+1] = Z[i] + (0,1) \) otherwise. From this recursive construction we have that \( Z = \Gamma^\rho_0 + (1/2,1/2) \), for the latter one also satisfies the same recursive relation. In words, the competition interface defined from \( \{ \xi^\rho(u) \}_{u \in \mathbb{Z}^2} \) is equivalent to the semi-infinite geodesic defined from \( \{ \xi(u) \}_{u \in \mathbb{Z}^2} \). We also define the process \( \{ p_t \}_{t \geq 0} \), such that \( p_t \) is the last vertex in \( \Gamma^\rho_0 \) with \( G^\rho(p_t) \leq t \) (see Figure 2).

We now describe the connection between LPP and TASEP (denoted as a Markov process on \( \{0,1\}^Z \)). We take the following initial conditions: let \( \eta_0(0) = 0 \) and \( \eta_0(1) = 1 \), and for any other \( x \) let \( \eta_0(x) = 1 \) with probability \( \rho \) and \( \eta_0(x) = 0 \) with probability \( 1 - \rho \) independently. We label the holes by \( Z \) from left to right, with the one at site 0 labeled 0; and label the particles by \( Z \) from right to left, with the one at site 1 labeled 0. For any \( a,b \in \mathbb{Z} \) such that at time 0 the particle labeled \( b \) is to the left of the hole labeled \( a \), we denote \( L(a,b) \) as the time when the particle interchanges with the hole. Then we have \( L(a,b) = G^\rho(a,b) \) in distribution jointly, see e.g. [FMP09, Section 4.2]. We could couple this TASEP with LPP so that this equality holds almost surely.

As in [FP05], we keep track of a ‘hole-particle pair’ in TASEP, which is a hole with a particle next to it in the right. At \( t = 0 \) it is the hole at site 0 and particle at site 1. Whenever the particle is interchanged with a hole to the right, we move this pair to the right; and whenever the hole is interchanged with a particle to the left, we move this pair to the left. Recall the process \( \{ p_t \}_{t \geq 0} \),
then we have the follow lemma from [FP05], which says that the trajectory of this ‘hole-particle pair’ can be mapped to the competition interface.

**Lemma 2.1.** Under the above coupling, for the hole-particle pair at time $t$, let $b_t$ be the label of the particle and $a_t$ be the label of the hole. Then $p_t = (a_t, b_t)$.

Note that $a_t$ is also the number of times the pair moved to the right, and $b_t$ is the number of times that pair moved to the left. Then at time $t$ the hole-particle pair is at site $a_t - b_t$ and $a_t - b_t + 1$. We denote $\eta'_t$ as $\eta_t$ centered by the hole-particle pair. Specifically, we denote $\eta'_t(x) = \eta_t(x + a_t - b_t)$, where $a_t - b_t$ is the location of the hole in the pair, at time $t$. This process $\eta'_t$ is also a Markov process, and we denote $\Phi^\rho_t$ as the measure of $\eta'_t$ (see Figure 3).

![Figure 3](image-url)  
Figure 3: An illustration of the evolution of $\eta_t$ (and the centered version $\eta'_t$): the numbers above the particles/holes are the labels, which increase from left to right for holes, and decrease from left to right for particles. The yellow boxes indicate the tracked hole-particle pairs.

### 2.3 Second-class particle and the stationary distribution

The ‘hole-particle pair’ in Section 2.2 also plays the role of a second-class particle in the TASEP, as shown by [FP05]. Such a TASEP with second-class particles is a Markov process on $\{1, *, 0\}^\mathbb{Z}$, where the symbols 1, *, and 0 represent particles, second-class particles, and holes respectively. Any second-class particle can exchange places with a hole to its right, or with a (normal) particle to its left. We will see that the process $(\eta'_t)_{t \geq 0}$ described above corresponds to a TASEP as seen from an isolated second-class particle.

Stationary measures for TASEP as seen from a second-class particle are constructed in [FFK94]; we now give a construction which yields one of the measures they describe (specifically, one with equal asymptotic density in both directions). Let $Y_1(x), x \geq 1$ and $Y_2(x), x \geq 1$ be independent collections of i.i.d. Bernoulli($\rho$) random variables. Let $R_1(x) = \sum_{y=1}^{x} Y_1(y)$ and $R_2(x) = \sum_{y=1}^{x} Y_2(y)$.

Then we can define a symmetric random walk $W$ by

$$W(x) = R_2(x) - R_1(x) \quad \text{(2.1)}$$
for \( x \geq 0 \). We define also 
\[
M(x) = \sup_{0 \leq y \leq x} W(y),
\]  
(2.2)
and \( \mathcal{E} = \{ x \geq 1 : M(x) > M(x - 1) \} \), and let \( E(x) = \# \{ y \in [1, x] : y \in \mathcal{E} \} \).

Then we can see \( M(x) - W(x) \) as a symmetric random walk with steps in \( \{-1, 0, 1\} \) and reflected to stay non-negative. The points of \( \mathcal{E} \), i.e. the points of increase of \( E \), are those times where reflection occurs. We have that \( x \in \mathcal{E} \) iff \( M(x - 1) = W(x - 1), Y_2(x) = 1 \), and \( Y_1(x) = 0 \). By well-known properties of symmetric random walk, we can obtain that as \( x \to \infty \), \( \mathbb{P}(x \in \mathcal{E}) \) decays like \( x^{-1/2} \), while \( |\mathcal{E} \cap [0, x]|/x^{1/2} \) converges in distribution to a random variable supported on \( (0, \infty) \).

Now we define a configuration \( \sigma \) on the non-negative half-line, which copies \( Y_1 \) except at points of \( \mathcal{E} \). We set \( \sigma(0) = * \) and, for \( x \geq 1 \),
\[
\sigma(x) = \begin{cases} 
1 & \text{if } Y_1(x) = 1 \\
0 & \text{if } Y_1(x) = 0 \text{ and } x \notin \mathcal{E} \\
* & \text{if } Y_1(x) = 0 \text{ and } x \in \mathcal{E}.
\end{cases}
\]

(There is a natural interpretation involving the departure process of a discrete-time \( M/M/1 \) queue – see [FM07].) We wish to extend to give a configuration \( \sigma(x), x \in \mathbb{Z} \) on the whole line. We can do this in two equivalent ways:

1. Note that \( \sigma(x), x \geq 0 \) is a renewal process with renewals at points \( x \) where \( \sigma(x) = * \), i.e. where \( x \in \mathcal{E} \). Between successive renewal points, we see an i.i.d. sequence of finite strings in \( \bigcup_{n \geq 0} \{0, 1\}^n \). We can extend to a renewal process on the whole line by extending this sequence of i.i.d. strings, separated by stars, leftward from the origin also.

2. Alternatively, we can exploit the symmetry of the TASEP under exchanging holes/particles and left/right. Write \( \pi_\rho \) for the distribution defined above on \( \sigma(x), x \geq 0 \). Now generate another configuration \( \tilde{\sigma}(x), x \geq 0 \) from \( \pi_{1-\rho} \), independently of \( \sigma \), and for \( x \geq 1 \) set
\[
\tilde{\sigma}(-x) = \begin{cases} 
1 & \text{if } \tilde{\sigma}(x) = 0 \\
0 & \text{if } \tilde{\sigma}(x) = 1 \\
* & \text{if } \tilde{\sigma}(x) = *.
\end{cases}
\]

(The equivalence of these two definitions follows from the random walk construction above. If we look at the configuration between 0 and the first * to the right of 0, we obtain a finite string of holes and particles whose distribution is invariant under exchanging both left/right and hole/particle; this invariance comes from the invariance under reflection of the random walk path beginning and ending at level 0 which is used to construct the configuration.)

We also extend the definition of \( \mathcal{E} \) to the whole line, by saying \( x \in \mathcal{E} \) whenever \( \sigma(x) = * \).

Now we have a distribution of \( \sigma(x), x \in \mathbb{Z} \). Note immediately that \( \sigma(x), x \geq 1 \) is independent of \( \sigma(x), x \leq -1 \). This distribution is stationary for the TASEP with second-class particles, as seen from one of the second-class particles.

**Proposition 2.2 ([FFK94 Theorem 1]).** Let \( (\sigma_t)_{t \geq 0} \) be TASEP with second-class particles, started from \( \sigma_0 = \sigma \). Suppose that at time \( t \), the second-class particle starting from the origin is at site \( l_t \), and let \( \sigma^0_t \) be \( \sigma_t \) shifted by \( l_t \), i.e. \( \sigma^0_t(x) = \sigma_t(x + l_t) \). Then for each \( t \geq 0 \), \( \sigma^0_t \) has the same distribution as \( \sigma \).

Now we have a distribution of \( \sigma(x), x \in \mathbb{Z} \). Note immediately that \( \sigma(x), x \geq 1 \) is independent of \( \sigma(x), x \leq -1 \).
Given a configuration $\sigma$, there are two related projections of it which involve setting all the * symbols except for the one at the origin to be either 1s or 0s.

1. The simpler one consists of setting all * symbols on positive sites to be 0, and all * symbols on negative sites to be 1. This gives a configuration distributed according to product measure, in which the non-zero sites are i.i.d. Bernoulli($\rho$).

2. Alternatively, we can follow the opposite rule of setting all * symbols on positive sites to be 1 and all * symbols on negative sites to be 0. Specifically, define a configuration $\vec{\eta}$ by $\vec{\eta}(0) = *$ and for $x \neq 0$,

$$\vec{\eta}(x) = \begin{cases} 0 & \text{if } \sigma(x) = 0, \text{ or if } \sigma(x) = * \text{ and } x < 0 \\ 1 & \text{if } \sigma(x) = 1, \text{ or if } \sigma(x) = * \text{ and } x > 0. \end{cases}$$

This gives a configuration which, compared to product measure of Bernoulli($\rho$), has a bias towards particles on positive sites and towards holes on negative sites. This bias decays as one gets further away from the origin.

Theorem 2 of [FFK94] says that the distribution of $\vec{\eta}(x)$ is stationary for the TASEP as seen from an isolated second-class particle. The bias above reflects the tendency created by the dynamics of the process for the second-class particle to get stuck behind particles and to get stuck in front of holes.

The combination of the two projections above gives a coupling between the configuration $\vec{\eta}^*$ and a configuration with product measure in which the discrepancies are precisely the non-zero members of $E$. The fact that the intersection of $E$ with $[0, x]$ grows on the order of $\sqrt{x}$ can be used to show that product measure and the stationary distribution of the TASEP as seen from an isolated second-class particle are mutually singular.

For later calculation, it will be useful to look at the position of the first hole to the right of the origin in $\vec{\eta}^*$ (and similarly the first particle to the left).

Let $X_+ = \min\{x \geq 1 : \vec{\eta}^*(x) = 0\}$, which is also $\min\{x \geq 1 : \sigma(x) = 0\}$. From the random walk construction of $\sigma(x)$, $x > 0$, one gets that

$$X_+ = \min\{x \geq 1 : Y_1(x) = 0\}, \text{ and for some } y \in [1, x], Y_2(y) = 0\}.$$ 

That is, to find $X_+$ we look for the first 0 in the process $Y_2$, and then we look for the first 0 in the process $Y_1$ from then on. Since all the variables $Y_1(1)$ and $Y_2(x)$ are i.i.d. Bernoulli($\rho$), this gives that $X_+ + 1$ is the sum of two Geometric($1 - \rho$) random variables, and so

$$P(X_+ = k) = k(1 - \rho)^2 \rho^{k-1} \quad (2.3)$$

for $k \geq 1$. Similarly if $X_-$ is the location of the first particle to the left of the origin, then

$$P(X_- = -k) = k\rho^2(1 - \rho)^{k-1}. \quad (2.4)$$

Finally we go back to the hole-particle representation of the second-class particle. Take $\vec{\eta} : \mathbb{Z} \to \{0, 1\}$, such that $\vec{\eta}(0) = 0$, $\vec{\eta}(1) = 1$; for $x > 1$ let $\vec{\eta}(x) = \vec{\eta}(x - 1)$, and for $x < 0$ let $\vec{\eta}(x) = \vec{\eta}^*(x)$. Denote by $\Psi^\rho$ as the measure of $\vec{\eta}$. Then $\Psi^\rho$ is the stationary measure of the Markov process of the TASEP as seen from a hole-particle pair, having the same transition probabilities as $(\eta^\rho_t)_{t \geq 0}$ (defined at the end of Section 2.2). One of our results (Theorem 1.7) asserts that $\Psi^\rho$ is the limit of $\Phi^\rho_t$ as $t \to \infty$.

### 2.4 Estimates for last-passage percolation

In this subsection we set up notation for the LPP model and give some useful estimates, which are mostly from the literature.
For \( a, b \in \mathbb{Z} \) and \( \rho \in (0, 1) \), we denote
\[
\langle a, b \rangle_\rho = \left( \frac{2(1 - \rho)^2a}{\rho^2 + (1 - \rho)^2}, b, \frac{2p^2a}{\rho^2 + (1 - \rho)^2} - b \right).
\]
Then \( \langle n, 0 \rangle_\rho = n^\rho \). We also write \( \langle a, b \rangle = \langle a, b \rangle_{1/2} = (a + b, a - b) \).

We start with estimates on passage times. We have that \( T_0,(m,n) \) has the same law as the largest eigenvalue of \( X^*X \) where \( X \) is an \((m + 1) \times (n + 1)\) matrix of i.i.d. standard complex Gaussian entries (see \[Joh00a\] Proposition 1.4). Hence we get the following one-point estimates from \[LR10\] Theorem 2.

**Theorem 2.3.** There exist constants \( C, c > 0 \), such that for any \( m \geq n \geq 1 \) and \( x > 0 \), we have
\[
\mathbb{P}[T_0,(m,n) - (\sqrt{m} + \sqrt{n})^2 \geq xn^{1/2}n^{-1/6}] \leq Ce^{-cx}.
\]
In addition, for each \( \psi > 1 \), there exist \( C', c' > 0 \) depending on \( \psi \) such that if \( \frac{m}{n} < \psi \), we have
\[
\mathbb{P}[T_0,(m,n) - (\sqrt{m} + \sqrt{n})^2 \geq xn^{1/3}] \leq C'e^{-c'\min\{x^{3/2},xn^{1/3}\}},
\]
and as a consequence
\[
|\mathbb{E}T_0,(m,n) - (\sqrt{m} + \sqrt{n})^2| \leq C'n^{1/3}.
\]
We also have the following parallelogram estimate.

**Proposition 2.4 ([BGZ21] Theorem 4.2).** Let \( U \) be the parallelogram whose one pair of sides have length \( 2n^{2/3} \) and are aligned with \( \mathbb{L}_0 \) and \( \mathbb{L}_n \) respectively, and let their midpoints being \((m, -m)\) and \( n \). Let \( U_1, U_2 \) be the part of \( U \) below \( \mathbb{L}_{n/3} \) and above \( \mathbb{L}_{2n/3} \) respectively. For each \( \psi < 1 \), there exist constants \( C, c > 0 \) depending only on \( \psi \), such that when \( |m| < \psi n \),
\[
\mathbb{P}\left[ \sup_{u \in U_1, v \in U_2} |T_{u,v} - \mathbb{E}T_{u,v}| \geq xn^{1/3} \right] \leq Ce^{-cx}.
\]
Such a result is first proved as \[BSS14\] Proposition 10.1, 10.5, in the setting of Poissonian DLPP. In the setting of exponential DLPP a proof is given in \[BGZ21\] Appendix C, following the ideas in \[BSS14\].

We will also need the following estimate on the coalescing probability of two geodesics, for finite and semi-infinite geodesics respectively.

**Proposition 2.5 ([Zha20]).** For each \( \psi \in (0, 1) \), there exists \( C > 0 \), such that
\[
\mathbb{P}[\Gamma_{0,(n,b_1)} \cap \mathbb{L}_{n-m} = \Gamma_{0,(n,b_2)} \cap \mathbb{L}_{n-m}] > 1 - Cm^{-2/3}|b_1 - b_2|
\]
for any \( n, m \in \mathbb{N}, b_1, b_2 \in \mathbb{Z} \), such that \( m < n/3, |b_1|, |b_2| < \psi n \).

**Proposition 2.6 ([BSS19] Theorem 2).** For any \( \rho \in (0, 1) \), there is a constant \( C > 0 \), such that for any \( r \in \mathbb{N} \), and \( k > 1 \), we have \( \mathbb{P}[\Gamma^0_0 \cap \mathbb{L}_{[r^{1/2}k]} \neq \Gamma^0_{0,r} \cap \mathbb{L}_{[r^{1/2}k]}] < Ck^{-2/3} \).

We next give some estimates on transversal fluctuation of geodesics.

**Lemma 2.7.** For each \( \psi \in (0, 1) \), there exist constants \( C, c \) such that the following is true. For any \( r < n \in \mathbb{N} \) large enough, and \( |b| < \psi n \), let \( \langle r, b' \rangle \) be the vertex in \( \mathbb{L}_r \) that is closest to the straight line connecting \( 0 \) and \( \langle n, b \rangle \), and let \( \langle r, b'' \rangle \) be the intersection of \( \mathbb{L}_r \) with \( \Gamma_{0,(n,b)} \). Then we have
\[
\mathbb{P}[|b'' - b'| > xr^{2/3}] < Ce^{-cxr} \quad \text{for any } x > 0.
\]

This is a slightly generalized version of \[Zha20\] Proposition 2.3], and the proof follows in exactly the same way. See also \[BSS19\] Theorem 3]. We omit the proof here. We have a similar estimate for semi-infinite geodesics as well.
Lemma 2.8. For any \( \rho \in (0,1) \), there exist \( C, c > 0 \) such that the following is true. Let \( b_r \in \mathbb{Z} \) such that \( \Gamma_0^b[2r+1] = \langle r, b_r \rangle, \rho \), then \( \mathbb{P}[[b_r] > x^{2/3}] < C e^{-c x^3} \) for any \( x > 0 \).

Proof. For simplicity of notation we denote \( T_{u,v}^* = T_{u,v} - \xi(v) \) for any vertices \( u \leq v \). The event \( |b_r| > x^{2/3} \) implies that there exists \( b \in \mathbb{Z} \) such that \( |b| > x^{2/3} \), and \( T_{0,(r,b),\rho}^* + B^\rho((r,b),\rho) > T_{0,r^*}^* \). For this we denote \( L_j := \{(r,b),\rho : |b| - 2j r^{2/3} | \leq r^{2/3}\} \) for \( j \in \mathbb{Z}, |j| \leq r^{1/3} \), and we have

\[
\mathbb{P}\left[ \max_{u \in L_j} T_{0,u}^* + B^\rho(u,\rho) > T_{0,r^*}^* \right]
\]

where \( c_0 > 0 \) is a small enough constant. We claim that each term in the right hand side is bounded by \( e^{-c_j^3} \), for some constant \( c > 0 \); then by summing over \( j \in \mathbb{Z} \) for \( 2|j| > x - 1 \) we get the conclusion. For the first term, when \( 2|j| < 0.9r^{1/3} \) we apply Proposition 2.3 when \( 0.9r^{1/3} \leq 2|j| \leq r^{1/3} \), we use that \( T_{0,u}^* \leq T_{0,-\lfloor 0.1r \rfloor}^* \) and then apply Proposition 2.3 to \( \max_{u \in L_j} T_{0,-\lfloor 0.1r \rfloor,u}^* \).

For the second term we apply Theorem 2.3. For the last term, we use 2.7 to conclude that \( \mathbb{E}[T_{0,u}^*] - \mathbb{E}[T_{0,r^*}^*] \leq c_1 r^{1/3} - b(\rho^{-1} - (1 - \rho)^{-1}) \) for some constant \( c_1 > 0 \); and use that \( b \mapsto B^\rho((r,b),\rho) - b(\rho^{-1} - (1 - \rho)^{-1}) \) is a (two-sided) centered random walk.

In addition to the above one-point bounds, we also quote the following uniform bound on transversal fluctuation.

Lemma 2.9 ([BGZ21] Proposition C.9). For each \( \psi \in (0,1) \) there exist constants \( C, c \) such that the following is true. For \( x > 0, n \in \mathbb{N} \), and \( |b| < \psi n \), consider the rectangle whose one pair of opposite edges are given by the segment connecting \( 0, [x n^{2/3}] \) and \( 0, [x n^{2/3}] \), and the segment connecting \( \langle n, b - [x n^{2/3}] \rangle \) and \( \langle n, b + [x n^{2/3}] \rangle \). Then the probability that the geodesic \( \Gamma_{0,(n,b)} \) exits this rectangle is at most \( C e^{-c x^3} \).

Combining these transversal fluctuation estimates we get the following, for finite and semi-infinite geodesics respectively.

Corollary 2.10. For each \( \psi \in (0,1) \), there exist constants \( C, c \) such that the following is true. Take any \( r < n \in \mathbb{N} \) large enough, \( x > 0 \), and \( |b| < \psi n \). Let \( \langle r, b' \rangle \) be the vertex in \( L_r \) that is closest to the straight line connecting \( 0 \) and \( \langle n, b \rangle \). Consider the parallelogram whose one pair of opposite edges are given by the segment connecting \( 0, [x r^{2/3}] \) and \( 0, [x r^{2/3}] \), and the segment connecting \( \langle r, r' - [x r^{2/3}] \rangle \) and \( \langle r, r' + [x r^{2/3}] \rangle \). Then with probability \( 1 - C e^{-c x^3} \), the geodesic \( \Gamma_{0,(r,b')} \) below \( \mathbb{L}_r \) is contained in that rectangle.

Corollary 2.11. For each \( \psi \in (0,1) \), there exist constants \( C, c \) such that the following is true. Take any \( r \in \mathbb{N} \) large enough and any \( x > 0 \). Consider the parallelogram whose one pair of opposite edges are given by the segment connecting \( 0, [-x r^{2/3}] \) and \( 0, [x r^{2/3}] \), and the segment connecting \( \langle r, r' - [x r^{2/3}] \rangle \) and \( \langle r, r' + [x r^{2/3}] \rangle \). Then with probability \( 1 - C e^{-c x^3} \), the part of the geodesic \( \Gamma_{0} \) below \( \mathbb{L}_r \) is contained in that rectangle.

Finally, we have the following estimate on the passage time along a semi-infinite geodesic.

Lemma 2.12. For each \( \psi \in (0,1) \), there exist constants \( C, c \) such that the following is true. Take any \( l > 0 \). Let \( u_l \) be the first vertex in \( \Gamma_{0}^{b} \) above the line \( \{(1 - \rho) y + (1 - \rho)^2 l, -\rho y + \rho^2 l : y \in \mathbb{R}\} \). Then \( \mathbb{P}[[T_{0,u_l} - l] > x^{2 l^{1/3}}] < C e^{-c x^3} \) for any \( x > 0 \).
Proof. Let $C, c > 0$ denote large and small constants depending only on $\rho$, and below the values can change from line to line. Let $U$ be the parallelogram whose four vertices are $\langle cl, -Cxl^2/3 \rangle_\rho$, $\langle cl, Cxl^2/3 \rangle_\rho$, $\langle cl, -Cxl^2/3 \rangle_\rho$, $\langle cl, Cxl^2/3 \rangle_\rho$ (round to the nearest lattice points). By Corollary 2.11 with probability at least $1 - C e^{-cx}$ we have $u_* \in U$. Note that for any $v \in U$ that is within distance 1 to the line $\{(1 - \rho)y + (1 - \rho)^2 l, -py + \rho^2 l\} : y \in \mathbb{R}$, we have $|E T_{0,v} - l| < C x l^2/3$. Then by covering $0$ and all such $U$ with $C x$ parallelograms, each of size in the order of $l \times l^2/3$, and applying Proposition 2.4, we get the conclusion. \hfill \square

Combining the above two results we get the following.

Corollary 2.13. For each $\rho \in (0, 1)$, there exist constants $C, c$ such that the following is true. Take any $l > 0$, and let $u_*$ be the last vertex in $\Gamma^*_0$ with $T_{0,u_*} < 1$. Then with probability greater than $1 - C e^{-cx}$, $u_*$ is between the line $\{((1 - \rho)y + (1 - \rho)^2 l, -py + \rho^2 l) : y \in \mathbb{R}\}$ and $\{((1 - \rho)y + (1 - \rho)^2 l, -py + \rho^2 l) : y \in \mathbb{R}\}$, thus below $L(t)$ when $x < l^{1/3}$; and $\Gamma^*_{0,u_*}$ is contained in the rectangle whose one pair of opposite edges are given by the segment connecting $\langle 0, [-x l^2/3] \rangle_{\rho}$ and $\langle 0, [x l^2/3] \rangle_{\rho}$, and the segment connecting $\langle l, [-x l^2/3] \rangle_{\rho}$ and $\langle l, [x l^2/3] \rangle_{\rho}$.

3 The limiting distribution

We can now define the measures $\nu^\rho$ from $\Psi^\rho$. Let $(\tilde{\eta}_t)_{t \in \mathbb{R}}$ be the stationary Markov process of TASEP centered by a hole-particle pair, i.e., the law of $\tilde{\eta}_t$ for each $t \in \mathbb{R}$ is $\Psi^\rho$. Note that $(\tilde{\eta}_t)_{t \in \mathbb{R}}$ and $(\eta^\rho_{t})_{t \geq 0}$ (from Section 2.2) have the same transition probability but different initial conditions. The idea is to construct the competition interface from $(\tilde{\eta}_t)_{t \in \mathbb{R}}$, in a way as described in Section 2.2 then take the environment as seen around the origin. This would be the environment along the geodesic $\Gamma^*_0$, as seen at a uniform time; i.e. from $p_t$ of a uniform $t$, where $p_t$ is defined as the last vertex in $\Gamma^*_0$ with $G^\rho(p_t) \leq t$. To get the environment as seen from a uniform vertex $\nu^\rho$ we would do a reweighting.

We use $\tilde{\Psi}^\rho$ to denote the measure of $(\tilde{\eta}_t)_{t \in \mathbb{R}}$. As above, we label the particles from right to left, and the holes from left to right, such that at time 0 the particle at site 1 and the hole at site 0 are both labeled 0. Let $\overline{L}(a,b)$ be the time when the particle labeled $b$ is switched with the hole labeled $a$, and let $\overline{Z}(a,b) = \overline{L}(a+1,b) \land \overline{L}(a,b+1) - \overline{L}(a,b)$. We define $\overline{\eta} \subset \mathbb{Z}^2$ as the collection of all $(a,b)$, such that there is a time when the particle labeled $b$ is at site 1 and the hole labeled $a$ is at site 0. We let $\tilde{\nu}^\rho$ be the measure given by the law of $(\{\overline{Z}(a,b)\}_{(a,b) \in \mathbb{Z}^2}, \overline{\eta})$ under $\tilde{\Psi}^\rho$.

We then let $\Psi^\rho$ be the measure $\tilde{\Psi}^\rho$ conditional on $\overline{L}(0) = 0$, i.e. let $d\Psi^\rho = \lim_{\epsilon \to 0^+} \frac{1}{\tilde{\Psi}^\rho_{\overline{L}(0) = -\epsilon}} d\tilde{\Psi}^\rho$. As $(\tilde{\eta}_t)_{t \in \mathbb{R}}$ under $\Psi^\rho$ is a Markov process, the limit could be computed as $\tilde{\Psi}^\rho$ conditional on that there is a jump of the hole-particle pair at time 0; i.e. first reweight $\tilde{\Psi}^\rho$ by $1 [\overline{\eta}_{0-}(2) = 0] + 1 [\overline{\eta}_{0-}(-1) = 1]$, the events where a jump is allowed, then let the jump happen at time 0. More precisely, we can define the limit as follows. We have

$$
\Psi^\rho = \frac{\mathbb{P}_{\tilde{\Psi}^\rho} [\overline{\eta}_{0-}(2) = 0] \Psi^\rho_{\overline{\eta}_{0-}(2) = 0} + \mathbb{P}_{\tilde{\Psi}^\rho} [\overline{\eta}_{0-}(-1) = 1] \Psi^\rho_{\overline{\eta}_{0-}(-1) = 1} \mathbb{P}_{\tilde{\Psi}^\rho} [\overline{\eta}_{0-}(2) = 0] + \mathbb{P}_{\tilde{\Psi}^\rho} [\overline{\eta}_{0-}(-1) = 1]}{(1 - \rho)^2 \Psi^\rho_{\overline{\eta}_{0-}(2) = 0} + \rho^2 \Psi^\rho_{\overline{\eta}_{0-}(-1) = 1}} = \frac{(1 - \rho)^2 \Psi^\rho_{\overline{\eta}_{0-}(2) = 0} + \rho^2 \Psi^\rho_{\overline{\eta}_{0-}(-1) = 1}}{(1 - \rho)^2 + \rho^2},
$$

where $\Psi^\rho_{\overline{\eta}_{0-}(2) = 0}$ (resp. $\Psi^\rho_{\overline{\eta}_{0-}(-1) = 1}$) is $\tilde{\Psi}^\rho$ conditional on that a jump of the hole-particle pair to the left (resp. to the right) happens at time 0. More precisely, we define these measures as follows. Let $(\overline{\eta}_t)_{t \in \mathbb{R}} \sim \Psi^\rho_{\overline{\eta}_{0-}(2) = 0}$, then the process $(\overline{\eta}_t)_{t < 0}$ has distribution given by

$$
\frac{1 [\overline{\eta}_{0-}(2) = 0]}{\mathbb{P}_{\tilde{\Psi}^\rho} [\overline{\eta}_{0-}(2) = 0]},
$$

14
and given \( \eta_0 \), we let \( \eta \) be that \( \eta_0(-1) = \eta_0(0) = 0, \eta_0(1) = 1, \) and \( \eta_0(x) = \eta_0(x + 1) \) for any \( x \notin \{-1, 0, 1\} \); and \( (\eta_t)_{t \geq 0} \) is the Markov process of TASEP as seen from a hole-particle pair. Similarly, for \( (\eta_t)_{t \in \mathbb{R}} \sim \Psi^\rho,2 \), then the process \( (\eta_t)_{t < 0} \) has distribution given by

\[
\frac{1[\eta_0(-1) = 1]}{P_{\Psi^\rho}[\eta_0(-1) = 1]};
\]

and given \( \eta_0 \), we have \( \eta_0(0) = 0, \eta_0(1) = \eta_0(2) = 1, \) and \( \eta_0(x) = \eta_0(x - 1) \) for any \( x \notin \{0, 1, 2\} \); and \( (\eta_t)_{t \geq 0} \) is the Markov process of TASEP as seen from a hole-particle pair.

From this construction, the laws of \( \eta_0 \) under \( \Psi^\rho,1 \) and \( \Psi^\rho,2 \) can also be described as follows. Let \( \Psi^\rho \) be the law of \( \{\eta_0(x + 1)\}_{x \in \mathbb{N}} \) and \( \Psi^\rho \) be the law of \( \{\eta_0(x - 1)\}_{x \in \mathbb{N}} \), for \( \eta_0 \sim \Psi^\rho \). Under \( \Psi^\rho,1 \), there is \( \eta_0(-1) = \eta_0(0) = 0, \eta_0(1) = 1, \) and \( \{\eta_0(x + 1)\}_{x \in \mathbb{N}} \sim \Psi^\rho \) and \( \{\eta_0(x - 1)\}_{x \in \mathbb{N}} \sim \Psi^\rho \), and are independent. Under \( \Psi^\rho,2 \), there is \( \eta_0(0) = 0, \eta_0(1) = \eta_0(2) = 1, \) and \( \{\eta_0(x + 2)\}_{x \in \mathbb{N}} \sim \Psi^\rho \) are independent.

We let \( \nu^\rho \) be the measure given by the law of \( \{\xi(a, b)\}_{(a, b) \in \mathbb{Z}^2} \) under \( \Psi^\rho \). By Lemma \( 3.2 \) below we can see that \( \xi(0) \) has exponential tail under \( \nu^\rho \), so \( E_{\nu^\rho}[\xi(0)] < \infty \). Then we show that \( \nu^\rho \) is \( \nu^\rho \) reweighted by \( \xi(0) \).

**Lemma 3.1.** We have \( d\nu^\rho = \frac{\xi(0)d\nu^\rho}{E_{\nu^\rho}[\xi(0)]} \) when the hole-particle pair jumps; this is a stationary point process in \( \mathbb{R} \). Then \( \nu^\rho \) corresponds to the environment as seen from the hole-particle at a typical jump-time. On the other hand, \( \nu^\rho \) corresponds to the environment as seen from the hole-particle at its position during the interval containing time 0. Because of the ‘inspection effect’, this is not the position at a typical jump – rather it is biased by the length of the interval in the point process containing time 0, which is \( \xi(0) \).

**Proof of Lemma 3.2.** For each \( s > 0 \), we let \( \Psi^\rho_s \) be the measure of \( \Psi^\rho \) conditional on \( \{L(u)\}_{u \in \mathbb{T}} \), i.e., let \( d\Psi^\rho_s = \lim_{\epsilon \to 0} \epsilon^{-1}[s-\epsilon < L(0) < s] d\Psi^\rho \). Since \( \Psi^\rho_s \) is stationary, we have \( 1[s-\epsilon < L(0) < s]d\Psi^\rho = \left(1[L(0) > -\epsilon]\mathbb{E}_\nu[\xi(0)] > -s\right) d\Psi^\rho \circ \mathcal{F}_s \), where \( \mathcal{F}_s \) is the translation operator: for any process \( P = (P_u)_{u \in \mathbb{R}} \), we denote \( P \) as the process \( (P_{u+w})_{w \in \mathbb{R}} \). This implies that

\[
P_{\Psi^\rho}[L(0) = s]d\Psi^\rho_s = P_{\Psi^\rho}[L(0) = 0](1[\xi(0) > -s]d\Psi^\rho) \circ \mathcal{F}_s,
\]

where \( P_{\Psi^\rho}[L(0) = s] = \lim_{\epsilon \to 0} \epsilon^{-1}P_{\Psi^\rho}[s-\epsilon < L(0) < s] \) and \( P_{\Psi^\rho}[L(0) = 0] = \lim_{\epsilon \to 0} \epsilon^{-1}P_{\Psi^\rho}[L(0) > -\epsilon] \) are the probability densities. By integrating over \( s > 0 \), the left hand side is \( d\Psi^\rho \); under which the law of \( \{\xi(a, b)\}_{(a, b) \in \mathbb{Z}^2} \) is \( \nu^\rho \). For the right hand side, we note that the laws of \( \{\xi(a, b)\}_{(a, b) \in \mathbb{Z}^2} \) are the same under \( 1[\xi(0) > -s]d\Psi^\rho \circ \mathcal{F}_s \) or \( \mathcal{F}_s \xi(0) > -s \) and \( d\Psi^\rho \). Therefore, by integrating over \( s < 0 \) and considering the law of \( \{\xi(a, b)\}_{(a, b) \in \mathbb{Z}^2} \), we get \( \mathbb{E}_{\nu^\rho}[\xi(0)] = 0 \xi(0)d\nu^\rho \). Thus we conclude that \( d\nu^\rho = \mathbb{E}_{\nu^\rho}[\xi(0)] \xi(0)d\nu^\rho \). Since \( \nu^\rho \) and \( \nu^\rho \) are probability measures, by integrating both sides we get that \( \mathbb{E}_{\nu^\rho}[\xi(0)] = 1 \), and the conclusion follows.

We now use the above construction of \( \nu^\rho \) to compute local statistics of the geodesics (assuming the main results of this paper). We start with the following computations on the next jump-times.

**Lemma 3.2.** For any \( h \geq 0 \) we have

\[
P_{\Psi^\rho,1}[L(1, 0) > h] = (1 + (1 - \rho)ph)e^{-(1 - \rho)h},
\]

\[
P_{\Psi^\rho,1}[L(0, 1) > h] = (1 + \rho h)e^{-\rho h},
\]

\[
P_{\Psi^\rho,2}[L(1, 0) > h] = (1 + (1 - \rho)h)e^{-(1 - \rho)h},
\]

\[
P_{\Psi^\rho,2}[L(0, 1) > h] = (1 + (1 - \rho)ph)e^{-\rho h}.
\]
Proof. Let $D_+ = \min\{x \geq 1 : \mathcal I_0(x + 1) = 0\}$, the number of particles between the origin and the leftmost hole at a positive site. Similarly let $D_- = \min\{x \geq 1 : \mathcal I_0(-x) = 1\}$, the number of holes to the right of the rightmost particle at a negative site, up to and including the origin.

The distribution of $D_+$ under $\Psi_{\rho,1}$ is that of $X_+$ given by (2.3), while the distribution of $D_+$ under $\Psi_{\rho,2}$ is that of $X_++1$ (which is the distribution of the sum of two independent Geometric($1-\rho$) random variables).

Similarly the distribution of $D_-$ under $\Psi_{\rho,1}$ is that of $X_-$ at (2.4), while the distribution of $D_-$ under $\Psi_{\rho,1}$ is that of $X_-+1$.

In order for the particle which is at site 1 at time 0 to jump, the hole starting at site $D_+ + 1$ must exchange places with each of the $D_+\ $ particles starting in $[1, D_+ - 1]$. So given $D_+$, the distribution of $\mathcal I(1, 0)$ is the sum of $D_+$ independent Exp(1) random variables; that is, a Gamma($D_+, 1$) distribution. A random variable $V$ with Gamma($k$) distribution has $\mathbb E[e^{-sv}] = (1+s)^{-k}$, and from this we obtain, for any $s > -1 + \rho$,

$$\mathbb E_{\Psi_{\rho,1}}[e^{-s(\mathcal I(1,0))}] = \sum_{k=1}^{\infty} k(1-\rho)^2 \rho^{k-1}(1+s)^{-k} = \frac{(1+s)(1-\rho)^2}{(1+s-\rho)^2},$$

which can be shown to match the expression for $\mathbb P_{\Psi_{\rho,1}}[\mathcal I(1,0) > h]$ given in the statement.

Similarly, in order for the hole which is at site 0 at time 0 to jump, the particle starting at site $-D_-$ must exchange places with each of the $D_-$ holes starting in $[-D_--1, 0]$. One obtains

$$\mathbb E_{\Psi_{\rho,1}}[e^{-s(\mathcal I(0,1))}] = \sum_{k=1}^{\infty} k \rho^2 (1-\rho)^k (1+s)^{-k+1} = \frac{\rho^2}{(\rho+s)^2},$$

which matches the desired expression for $\mathbb P_{\Psi_{\rho,1}}[\mathcal I(0,1) > h]$.

Analogous calculations give the probabilities under $\Psi_{\rho,2}$.

Now we compute the law of $\xi(0)$ for $\xi \sim \nu^\rho$.

Proof of Proposition 1.5. It suffices to compute the law of $\mathcal I(1,0) \land \mathcal I(0,1)$, under the measure $\Psi^\rho = \frac{(1-\rho)^2 \Psi_{\rho,1} + \rho^2 \Psi_{\rho,2}}{(1-\rho)^2 + \rho^2}$. Note that under either $\Psi_{\rho,1}$ or $\Psi_{\rho,2}$, the random variables $\mathcal I(1,0)$ and $\mathcal I(0,1)$ are independent. Thus by Lemma 3.2 we get that

$$\mathbb P_{\Psi_{\rho,1}}[\mathcal I(1,0) \land \mathcal I(0,1) > h] = (1 + \rho h)(1 + \rho(1-\rho)h) e^{-h},$$

and

$$\mathbb P_{\Psi_{\rho,2}}[\mathcal I(1,0) \land \mathcal I(0,1) > h] = (1 + (1-\rho)h)(1 + (1-\rho)h) e^{-h}.$$ 

Thus the conclusion follows.

Assuming Theorem 1.1 we can also compute the proportion of ‘corners’ in geodesics.

Proof of Proposition 1.6. Assuming Theorem 1.1 we have

$$\frac{N_{n,\rho}}{2n} \to \mathbb P_{\nu^\rho}[\{(0,0),(0,1),(-1,0)\} \subset \mathcal I] + \mathbb P_{\nu^\rho}[\{(0,0),(0,-1),(1,0)\} \subset \mathcal I],$$

as $n \to \infty$. From the construction of $\nu^\rho$, this equals

$$\frac{(1-\rho)^2 \mathbb P_{\Psi_{\rho,1}}[\mathcal I(1,0) > \mathcal I(0,1)] + \rho^2 \mathbb P_{\Psi_{\rho,2}}[\mathcal I(1,0) < \mathcal I(0,1)]}{(1-\rho)^2 + \rho^2}.$$

Using that $\mathcal I(1,0) - \mathcal I(0,0)$ and $\mathcal I(0,1) - \mathcal I(0,0)$ are independent under either $\Psi_{\rho,1}$ or $\Psi_{\rho,2}$, by
Lemma 3.2 we have
\[ Pr_{\Psi_{\rho}}[\tau(1,0) > \tau(0,1)] = \rho^2(1 + 2\rho - 2\rho^2), \]
\[ Pr_{\Psi_{\rho}}[\tau(1,0) < \tau(0,1)] = (1 - \rho)^2(1 + 2\rho - 2\rho^2). \]
Thus the conclusion follows.

We finish this section by giving an outline of alternative derivation of the formulas in Proposition 1.5 and Proposition 1.6, which also leads to representations of the type mentioned after the statement of Proposition 1.5.

Note that under \( \Psi_{\rho,2} \), \( D_+ \) takes values in \{2, 3, \ldots\} and has the distribution of the sum of two independent Geometric random variables with parameter \( 1 - \rho \). Given \( D_+ \), the random variable \( \tau(1,0) \) is the sum of \( D_+ \) independent \( \text{Exp}(1) \) random variables. From this, \( \tau(1,0) \) has the same distribution as the sum of two \( \text{Exp}(1 - \rho) \) random variables, or equivalently of \( \frac{1}{1-\rho}(E_1 + E_2) \) for \( E_1, E_2 \) i.i.d. \( \sim \text{Exp}(1) \).

Meanwhile under \( \Psi_{\rho,2} \), \( D_- \) takes values in \{1, 2, \ldots\} and has the distribution of 1 less than the sum of two independent Geometric(\( \rho \)) random variables. Note that if \( X \sim \text{Geom}(\rho) \), then \( X - 1 \overset{d}{=} BX \) where \( B \sim \text{Ber}(\rho) \) independently of \( X \). We obtain that \( \tau(0,1) \) has the distribution of \( \frac{1}{\rho}(E_3 + BE_4) \), for \( B \sim \text{Ber}(\rho) \) and \( E_3, E_4 \) i.i.d. \( \sim \text{Exp}(1) \) independently of \( B \).

Note \( \tau(0,1) \) and \( \tau(1,0) \) are independent under \( \Psi_{\rho,2} \). So we can combine the previous two paragraphs to get that the distribution of \( \tau(0) = \min(\tau(0,1), \tau(1,0)) \) under \( \Psi_{\rho,2} \) is that of
\[
\min \left\{ \frac{1}{1-\rho}(E_1 + E_2), \frac{1}{\rho}(E_3 + BE_4) \right\}
\]
for \( B \sim \text{Ber}(\rho) \) and \( (E_i)_{1 \leq i \leq 4} \) i.i.d. \( \sim \text{Exp}(1) \) independently of \( B \).

We continue in the particular case \( \rho = 1/2 \). Then the distribution of \( \tau(0) \) is the same under \( \Psi_{\rho,1} \) as under \( \Psi_{\rho,2} \), and so its distribution under \( \Psi_{\rho} \) is again the same, that of \( 2 \min \{ (E_1 + E_2), (E_3 + BE_4) \} \) for \( B \sim \text{Ber}(1/2) \) and \( (E_i)_{1 \leq i \leq 4} \) i.i.d. \( \sim \text{Exp}(1) \) independently of \( B \).

By elementary arguments involving the memoryless property of exponentials, this distribution can be seen to be a \((1/4, 1/2, 1/4)\) mixture of \( \text{Gamma}(1,1) \), \( \text{Gamma}(2,1) \) and \( \text{Gamma}(3,1) \) distributions.

A similar but slightly more involved argument can be made for the case of general \( \rho \), to give that the distribution of \( \tau(0) \) is again a mixture of \( \text{Gamma}(1,1) \), \( \text{Gamma}(2,1) \) and \( \text{Gamma}(3,1) \) distributions, now with weights
\[
\left( \frac{\rho^4 + (1-\rho)^4}{\rho^2 + (1-\rho)^2}, \frac{2\rho(1-\rho)}{\rho^2 + (1-\rho)^2}, \frac{2\rho^2(1-\rho)^2}{\rho^2 + (1-\rho)^2} \right).
\]
As a function of \( \rho \in [0,1] \), this distribution is stochastically increasing on \([0, 1/2]\), and symmetric around \( 1/2 \).

4 Convergence of TASEP as seen from a single second-class particle

As in the previous section, we take \((\eta_t)_{t \geq 0}\) as TASEP with initial condition \( \eta_0(0) = 0, \eta_0(1) = 1 \), and \( \eta_0(x) \) being i.i.d. Bernoulli \( \rho \) for any \( x \in \mathbb{Z} \setminus \{0, 1\} \). Let \((\eta_t^\rho)_{t \geq 0}\) be \( \eta_t \) as seen from the hole-particle pair, and let \( \Phi_t^\rho \) be the measure of \( \eta_t^\rho \). Also let \( \Psi^\rho \) be the stationary measure of TASEP as seen from a single hole-particle pair. The goal of this section is to show the convergence of \( \Phi_t^\rho \) to \( \Psi^\rho \) as \( t \to \infty \), i.e., to prove Theorem 1.7.
4.1 TASEP convergence in the average sense

\[ η_t^L \sim Φ^ρ_t \]

\[ η_t^∗ \]

\[ \sigma_t^{b,0} \overset{d}{=} \sigma \]

\[ \bar{\eta}^∗ \]

\[ \bar{\eta} \sim Ψ^ρ \]

Figure 4: A coupling between \( Ψ^ρ \) and \( Φ^ρ_t \) via \( \sigma_t^{b,0} \). The red numbers are labels of second-class particles. Here \( \bar{\eta} \) and \( η_t^L \) are the same on \([-9, 9]\).

In this subsection we prove the convergence of \( Φ^ρ_t \) to \( Ψ^ρ \) in the averaged sense.

**Proposition 4.1.** We have \( T^{-1} \int_0^T Φ^ρ_t dt \to Ψ^ρ \) in distribution, as \( T \to ∞ \).

Recall the stationary measure for TASEP as seen from a second-class particle \( σ \), constructed in Section 2.3. Also recall that we have the following two projections of \( σ \): first, if we set all \( * \) symbols on positive sites to be 0, and all \( * \) symbols on negative sites to be 1, we get i.i.d. Bernoulli(\( ρ \)) on all non-zero sites; second, if we set all \( * \) symbols on positive sites to be 1, and all \( * \) symbols on negative sites to be 0, we get a distribution which is stationary for the TASEP as seen from an isolated second-class particle.

Now recall TASEP (\( σ_t \)) with (infinitely many) second-class particles, and starting from \( σ_0 = σ \). At time 0, we label all the second-class particles with \( \mathbb{Z} \) from right to left, such that the one at the origin is labeled 0. We consider two ways where the labels evolve.

- **Rule (a):** for all second-class particles, the labels never change.
- **Rule (b):** two second-class particles labeled \( i > j \), if they are at sites \( x \) and \( x + 1 \), then with rate 1 they exchange their labels.

We note that when forgetting the labels, the dynamic is unchanged. For each \( i \in \mathbb{Z} \) and \( t \geq 0 \), we denote \( l^{a,i}_t \) as the site of the second-class particle labeled by \( i \) at time \( t \), under Rule (a). Then for each \( i \in \mathbb{Z} \) we have \( l^{a,i}_t > l^{a,i+1}_t \), and there is no other second-class particle between sites \( l^{a,i}_t \) and \( l^{a,i+1}_t \). We also denote \( l^{b,i}_t \) as the site of the second-class particle labeled by \( i \) at time \( t \), under Rule (b).

Define \( σ_t^{a,i}, σ_t^{b,i} : \mathbb{Z} \to \{0, 1, *\} \) as \( σ_t^{a,i}(x) = σ_t(x + l_t^{a,i}) \) and \( σ_t^{b,i}(x) = σ_t(x + l_t^{b,i}) \), which is \( σ_t \) as seen from the second-class particle labeled by \( i \), under each rule. As \( σ \) is a renewal process, and that \( σ \) is stationary, we have that \( σ_t^{a,i} \) has the same distribution as \( σ \). We next show that the same is true for \( σ_t^{b,i} \).

**Lemma 4.2.** For each \( i \in \mathbb{Z} \) and \( t \geq 0 \), \( σ_t^{b,i} \) has the same distribution as \( σ \).
Proof. Take any measurable set \( B \subset \{0,1,*\}^\mathbb{Z} \), it suffices to show that \( \mathbb{P}[\sigma_t^{b,i} \in B] = \mathbb{P}[\sigma \in B] \).

We fix \( t \geq 0 \). Since for each \( i \in \mathbb{Z}, \sigma_t^{a,i} \) has the same distribution as \( \sigma \), we have
\[
\lim_{N \to \infty} \frac{1}{2N+1} \mathbb{E}[[\{ -N \leq i \leq N : \sigma_t^{a,i} \in B \}]] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \mathbb{P}[\sigma_t^{a,i} \in B] = \mathbb{P}[\sigma \in B].
\]
As each second-class particle jumps with rate at most 1, for any \( \epsilon > 0 \) we can find \( M > 0 \), such that \( \mathbb{P}[|l_t^{b,i} - l_t^{0,i}| > M] < \epsilon \) for any \( i \in \mathbb{Z} \). For each \( i \) with \( |i| < N - M \), if \( |l_t^{b,i} - l_t^{0,i}| \leq M \), we must have \( l_t^{b,i} \in \{ l_t^{a,j} : -N \leq j \leq N \} \), since this set contains all locations of second-class particles in \( [l_t^{a,i} - M, l_t^{a,i} + M] \). We then have that
\[
\mathbb{E}[[\{ l_t^{b,i} : -N \leq i \leq N \} \setminus \{ l_t^{a,i} : -N \leq i \leq N \}]] < 2M + \epsilon(2N + 1).
\]
Thus since \( \epsilon \) is arbitrarily taken, we have
\[
\lim_{N \to \infty} \frac{1}{2N+1} \sum_{i=-N}^{N} \mathbb{P}[\sigma_t^{b,i} \in B] = \lim_{N \to \infty} \frac{1}{2N+1} \mathbb{E}[[\{ -N \leq i \leq N : \sigma_t^{b,i} \in B \}]] = \mathbb{P}[\sigma \in B].
\]
Now that \( \sigma \) is a renewal process, \( \sigma_t^{b,i} \), thus \( \sigma_t^{b,i} \), has the same distribution for all \( i \). From the above equation the conclusion follows.

We define \( \eta_t^{*,i} : \mathbb{Z} \to \{0,1,*\} \) from \( \sigma_t^{b,i} \), by identifying all second-class particles whose labels are \( < i \) with holes, and all second-class particles whose labels are \( > i \) with particles. Formally, we let \( \eta_t^{*,i}(0) = * \), and \( \eta_t^{*,i}(x) = 1 \) for any \( x \) such that \( \sigma_t(x + l_t^{b,i}) = 1 \), or \( x = l_t^{b,j} - l_t^{0,i} \) for some \( j > i \); and \( \eta_t^{*,i}(x) = 0 \) such that \( \sigma_t(x + l_t^{b,i}) = 0 \), or \( x = l_t^{b,j} - l_t^{0,i} \) for some \( j < i \).

From such construction, \( \eta_t^{*,0} \geq 0 \) is TASEP starting from i.i.d. Bernoulli(\( \rho \)) on all non-zero sites, as seen from the only second-class particle. Thus we immediately get the following.

Lemma 4.3. Take \( \eta_t^{i} : \mathbb{Z} \to \{0,1\} \) such that \( \eta_t^{i}(0) = 0 \), \( \eta_t^{i}(1) = 1 \), and \( \eta_t^{i}(x) = \eta_t^{*,0}(x) \) for \( x < 0 \), \( \eta_t^{i}(x) = \eta_t^{*,0}(x-1) \) for \( x > 1 \). Then the law of \( \eta_t^{i} \) is given by \( \Phi^\rho_t \).

Now we finish the proof of Proposition 4.4 via a coupling between \( \eta_t^{i} \) and \( \eta_t \sim \Psi^\rho_t \).

Proof of Proposition 4.4. It suffices to show that for any finitely supported set \( B \subset \{0,1\}^\mathbb{Z} \), we have \( T^{-1} \int_0^T \Phi^\rho_t(B)dt \to \Psi^\rho(B) \). Below we fix \( B \) and assume that it is supported on \( [-L,L+1] \).

Recall the construction of \( \Psi^\rho \) from \( \sigma \). By Lemma 4.2, from \( \sigma_t^{b,0} \), we get \( \Psi^\rho \) (the stationary for the TASEP as seen from an isolated second-class particle, from Section 2.3) by identifying all * with 1 in \( \mathbb{Z}_+ \) and all * with 0 in \( \mathbb{Z}_- \). Next, by replacing the * at the origin by a 0 \(-1\) pair we get \( \Psi^\rho \), by distribution \( \Psi^\rho \). Thus by Lemma 4.3 and comparing the procedures of getting \( \Psi^\rho \) and \( \Phi^\rho_t \) from \( \sigma_t^{b,0} \), we have
\[
|\Phi^\rho_t(B) - \Psi^\rho(B)| \leq \mathbb{P}[\{ l_t^{b,i} - l_t^{0,i} : i > 0 \} \cap [-L,0] = \{ l_t^{b,i} - l_t^{0,i} : i < 0 \} \cap [0,L] = 0].
\]
Then we have
\[
\int_0^T |\Phi^\rho_t(B) - \Psi^\rho(B)|dt \leq \sum_{i \in \mathbb{N}} \int_0^T \mathbb{P}[l_t^{b,i} - l_t^{0,i} \in [-L,0] \cup \{ \infty \} | l_t^{b,i} - l_t^{0,i} \in [0,L] \cup \{ \infty \}]dt.
\]
For each \( i \in \mathbb{N} \) we recursively define a sequence of stopping times: let \( T_{i+1} = \inf\{ t \geq 0 : l_t^{b,i} - l_t^{0,i} \in [-L,0] \cup \{ \infty \} \}; \) and given \( T_{i+1} < \infty \), let \( T_{i+1} = \inf\{ t \geq T_{i+1} + 1 : l_t^{b,i} - l_t^{0,i} \in [-L,0] \cup \{ \infty \} \}. \) Note that there exists \( \delta > 0 \) depending only on \( L \), such that \( \mathbb{P}[l_t^{b,i} - l_t^{0,i} \in [-L,0] | l_t^{b,i} - l_t^{0,i} \in [0,L] \cup \{ \infty \}] > \delta \); and if \( l_t^{b,i} - l_t^{0,i} \) for some \( t_0 \geq 0 \), then \( l_t^{b,i} - l_t^{0,i} \) for any \( t > t_0 \). Then we have \( \mathbb{P}[T_{i+1} \geq T | T_{i+1} < T] > \delta \),
and
\[
\int_0^T \mathbb{P}[\ell_t^i - \ell_t^0 \in [-L, 0]] dt \leq \sum_{n=1}^{\infty} \mathbb{P}[T_{i,n} < T] \leq \sum_{n=1}^{\infty} (1 - \delta)^{n-1} \mathbb{P}[T_{i,1} < T] = \delta^{-1} \mathbb{P}[T_{i,1} < T].
\]

Next we bound \(\sum_{i \in \mathbb{N}} \mathbb{P}[T_{i,1} < T]\). Take any \(\varepsilon > 0\). From the renewal construction of \(\sigma\), we have that \(\ell_t^0 - \ell_t^0\) is the sum of \(i\) i.i.d. random variables, each with infinite expectation. Thus we have
\[
\lim_{T \to \infty} \mathbb{P}[\ell_t^0 - \ell_t^0 < cT] = 0.
\]

Since each label moves with rate at most 1, given \(\{\ell_t^0\}_{i \in \mathbb{Z}}\) satisfying \(\ell_t^0 - \ell_t^0 < 3T\), for each \(j \in \mathbb{Z}_{\geq 0}\) the probability that \(T_{[cT]} + j, 1 < T\) is bounded by the probability that the sum of \([3T] + j - L\) independent \(\text{Exp}(1/2)\) random variables is less than \(T\). Summing up such probabilities for all \(j\) we get
\[
\lim_{T \to \infty} \sum_{j > cT} \mathbb{P}[T_{i,1} < 0] = 0.
\]

From these we get
\[
\limsup_{T \to \infty} \sum_{i \in \mathbb{N}} \int_0^T \mathbb{P}[\ell_t^i - \ell_t^0 \in [-L, 0]] dt - \delta^{-1} \varepsilon T \leq 0.
\]

Similarly we have
\[
\limsup_{T \to \infty} \sum_{i \in \mathbb{Z}_{-}} \int_0^T \mathbb{P}[\ell_t^i - \ell_t^0 \in [0, L]] dt - \delta^{-1} \varepsilon T \leq 0.
\]

Adding them up we get
\[
\limsup_{T \to \infty} T^{-1} \int_0^T |\Phi_t^\rho(B) - \Psi^\rho(B)| dt \leq \delta^{-1} \varepsilon.
\]

Since \(\varepsilon > 0\) is arbitrarily taken, the conclusion follows. \(\square\)

### 4.2 Convergence to the stationary distribution

In the next two subsections we upgrade Proposition 4.4 to Theorem 1.7. The general idea is to show that \(\Phi_t\) and \(\Phi_{t+s}\) are close when \(s\) is much smaller than \(t\).

**Proposition 4.4.** For any \(N \in \mathbb{N}\) and \(\rho \in (0, 1)\), there is a constant \(C > 0\) such that the following is true. Take any \(s, t > C\) with \(t < s^{1.01}\), and any function \(f : \{0, 1\}^{[-N,N]} \to [0, 1]\), regarded as a function on \(\{0, 1\}^Z\), we have \(|\Phi_t^\rho(f) - \Phi_{t+s}^\rho(f)| < C(s/t)^{1/30}\).

Using this we can deduce Theorem 1.7.

**Proof of Theorem 1.7.** Take any \(N \in \mathbb{N}\) and \(f : \{0, 1\}^{[-N,N]} \to [0, 1]\), regarded as a function on \(\{0, 1\}^Z\), it suffices to show that
\[
\lim_{t \to \infty} \Phi_t^\rho(f) = \Psi^\rho(f). \quad (4.1)
\]

Take any \(\delta > 0\). By Proposition 1.1 we have that \((\delta t)^{-1} \int_0^{\delta t} \Phi_{t+s}^\rho(f) ds \to \Psi^\rho(f)\) as \(t \to \infty\). By Proposition 4.4 we have for any \(t > C\),
\[
|\Phi_t^\rho(f) - (\delta t)^{-1} \int_0^{\delta t} \Phi_{t+s}^\rho(f) ds| < C\delta^{1/30} + (C \vee t^{1/0.01})(\delta t)^{-1},
\]
where \(C\) depends only on \(\rho\) and \(N\). Thus \(\limsup_{t \to \infty} |\Phi_t^\rho(f) - \Psi^\rho(f)| \leq C\delta^{1/30}\), and by sending \(\delta \to 0\) we get (4.1). \(\square\)
The idea of proving Proposition 4.4 is to couple two copies of $(\eta^t_0)_{t \geq 0}$. For this we recall the set up in Section 2.2.

Let $(\eta^-_0)_{t \geq 0}$ and $(\eta^+_0)_{t \geq 0}$ be two copies of TASEP, with $\eta^-_0(0) = \eta^+_0(0) = 0$ and $\eta^-_0(1) = \eta^+_0(1) = 1$; and all $\eta^-_0(x), \eta^+_0(x)$ for $x \in \mathbb{Z} \setminus \{0, 1\}$ are i.i.d. Bernoulli $\rho$. In both copies, we label the holes by $\mathbb{Z}$ from left to right, with the hole at site $0$ at time $0$ labeled $0$; and label the particles by $\mathbb{Z}$ from right to left, with the particle at site $1$ at time $0$ labeled $0$. Keeping track of the hole-particle pair, we denote $p^-_i = (a^-_i, b^-_i)$ and $p^+_i = (a^+_i, b^+_i)$ as the label of the tracked particle and hole at time $t$ in $(\eta^-_t)_{t \geq 0}$ and $(\eta^+_t)_{t \geq 0}$ respectively, and set $\eta^-_t(x) = \eta^-_t(x + a^-_t - b^-_t)$, $\eta^+_t(x) = \eta^+_t(x + a^+_t - b^+_t)$ for any $t \geq 0, x \in \mathbb{Z}$. Then $\eta^-_t, \eta^+_t$ have distribution $\Phi^0_t$.

For any $a, b \in \mathbb{Z}$, if in $\eta^-_0$ (resp. $\eta^+_0$) the particle with label $b$ is to the left of the hole with label $a$, we denote $L^-(a, b)$ (resp. $L^+(a, b)$) as the time when they switch; otherwise we set $L^-(a, b) = 0$ (resp. $L^+(a, b) = 0$). For each $t \geq 0$ denote

$$
I_t^- := \{ u \in \mathbb{Z}^2 : L^+(u) \leq t \},
$$

$$
I_t^+ := \{ u \in \mathbb{Z}^2 : L^-(u) \leq t \},
$$

$$
\partial I_t^- := \{ u \in I_t^+ : L^+(u + (1, 0)) \vee L^+(u + (0, 1)) > t \},
$$

$$
\partial I_t^+ := \{ u \in I_t^- : L^-(u + (1, 0)) \vee L^-(u + (0, 1)) > t \}.
$$

For the rest of this subsection we fix $s > 0$. Take any $r \in \mathbb{N}$. For any coupling between $\eta^-_0$ and $\eta^+_s$, we denote $A$ as the event where

$$
\eta^-_0(x) = \eta^+_s(x), \forall x \in \mathbb{Z}, |x| > r; \quad \sum_{x = -r}^{r} \eta^-_0(x) = \sum_{x = -r}^{r} \eta^+_s(x).
$$

Under $A$, we can find a (unique) $p^* = (a^*, b^*) \in \mathbb{Z}^2$, such that $I_0^- \cap \{ u \in \mathbb{Z}^2 : |ad(u)| > r \} = (I_s^- - p^*) \cap \{ u \in \mathbb{Z}^2 : |ad(u)| > r \}$; in particular, the sets $I_0^-$ and $I_s^- - p^*$ differ by a finite number of vertices.

**Lemma 4.5.** There is a coupling of $\eta^-_0$ and $\eta^+_s$ such that $\mathbb{P}[A] > 1 - C(rs^{-2/3})^{-1/10}$ when $Cs^{2/3} < r < s^{2/3 + 0.01}$ and $s > C$, where $C > 0$ is a constant depending only on $\rho$.

We leave the construction of this coupling to the next subsection, and we now couple $(\eta^-_0)_{t \geq 0}$ and $(\eta^+_s)_{t \geq 0}$ assuming such coupling between $\eta^-_0$ and $\eta^+_s$.

The construction of the coupling for these two copies of TASEP is via the connection with LPP. For any $u \notin I^-_0$ we denote $\xi^-_\updownarrow(u) = L^- - L^-(u - (1, 0)) \vee L^-(u - (0, 1))$; and for any $u \notin I^+_0$ we denote $\xi^+\updownarrow(u) = L^+(u) - L^+(u - (1, 0)) \vee L^+(u - (0, 1))$. In addition, for $u \notin I^+_s$ we denote $\xi^+\updownarrow,\downarrow(u) = L^+(u) - L^+(u - (1, 0)) \vee L^+(u - (0, 1)) \vee s$. Note that for any $u \notin I^+_s$, we have $\xi^+\updownarrow,\downarrow(u) = \xi^+\updownarrow,\downarrow(u)$ unless both $u - (1, 0), u - (0, 1) \in I^+_s$. Given $\eta^-_0$ and $\eta^+_s$ under the coupling from Lemma 4.5, if $A$ does not hold, we just couple $(\eta^-_0)_{t \geq 0}$ and $(\eta^+_s)_{t \geq 0}$ arbitrarily. If $A$ holds, then for any $u \in \mathbb{Z}^2$ such that $u \notin I_0^-$ and $u + p^* \notin I^+_s$, we let $\xi^-\updownarrow(u) = \xi^+\updownarrow,\downarrow(u + p^*)$. Such coupling exists because conditioned on $I^-_0$, the random variables $\xi^-\updownarrow(u)$ are i.i.d. Exp(1) for all $u \notin I^-_0$; and conditioned on $I^+_s$, the random variables $\xi^+\updownarrow,\downarrow(u)$ are i.i.d. Exp(1) for all $u \notin I^+_s$. We shall bound the total variation distance between $\Phi^0_t$ and $\Phi^0_{t+s}$ using this coupling.

As discussed in Section 2.2, $L^-$ (resp. $L^+$) has the same law as a Busemann function in LPP restricted to $\mathbb{Z}^2 \setminus I^-_0$ (resp. $\mathbb{Z}^2 \setminus I^+_0$). From this we can construct the semi-infinite geodesics. For any $u \in (\mathbb{Z}^2 \setminus I^-_0) \cup \partial I^-_0$, we define the semi-infinite geodesic $\Gamma^\uparrow_u$ recursively, by letting $\Gamma^\uparrow_u[1] = u$, and for each $i \in \mathbb{N}$ letting $\Gamma^\uparrow_u[i + 1] = \text{argmin}_{v \in \{ \Gamma^\uparrow_u[i] + (1, 0), \Gamma^\uparrow_u[i] + (0, 1) \}} L^- (v)$; and similarly we define $\Gamma^\downarrow_u$ for any $u \in (\mathbb{Z}^2 \setminus I^+_0) \cup \partial I^+_0$.

We also define the ‘downward semi-infinite geodesics’. For any $u \in \mathbb{Z}^2 \setminus I^-_0$, we define $\Gamma^\downarrow_u$ by
letting $\Gamma^{-\vee}_u[1] = u$, and for each $i \in \mathbb{N}$ letting $\Gamma^{-\vee}_u[i+1] = \operatorname{argmax}_{v \in \{\Gamma^{-\vee}_u[i]-(0,1),\Gamma^{-\vee}_u[i]-(0,1)\}} L^-(v)$, until $\Gamma^{-\vee}_u[i] - (1,0), \Gamma^{-\vee}_u[i] - (0,1) \notin I_0^-$. Equivalently and by induction, we could also define such downward semi-infinite geodesics as following. For any $u,v \notin I_0^-$ with $u \leq v$, we let $T_{u,v}^{+\vee}$ and $\Gamma_{u,v}^{+\vee}$ as the maximum passage time and geodesic from $u$ to $v$, using the weights $\xi^{+\vee}$. Then for $v \notin I_0^-$ we have $L^-(v) = \max_{u \in I_0^+, u \leq v} T_{u,v}^{+\vee}$, and $\Gamma^{+\vee}_v$ equals $\Gamma_{u,v}^{+\vee}$ for $u$ where the maximum is achieved. Similarly we define $\Gamma_{u,v}^{+\vee}$ for any $u \notin I_0^-$, and $T_{u,v}^{+\vee}$ and $\Gamma_{u,v}^{+\vee}$ for any $u,v \notin I_0^+$ with $u \leq v$, using the weights $\xi^{+\vee}$. In addition, we also define $T_{u,v}^{+\vee,s}$ and $\Gamma_{u,v}^{+\vee,s}$ for $u \notin I_0^+$ with $u \leq v$, using the weights $\xi^{+\vee,s}$. For any $v \notin I_0^+$ we have $L^+(v) - s = \max_{u \in I_s^+, u \leq v} T_{u,v}^{+\vee,s}$, and $\Gamma^{+\vee,s}_v \setminus I_s^+$ is $\Gamma_{u,v}^{+\vee,s}$ for $u$ where the maximum is achieved.

A quick observation is that almost surely there is a ‘non-crossing’ property for semi-infinite geodesics and downward semi-infinite geodesics: for any $\Gamma_u^-$ and $\Gamma_v^{-\vee}$, we cannot find $w \in \mathbb{Z}^2$ with $w, w - (1,0) \in \Gamma_u^-$ and $w, w + (0,1) \in \Gamma_v^{-\vee}$ simultaneously. This is because, from the construction of $\Gamma_u^-$ and $\Gamma_v^{-\vee}$, the event $w, w - (1,0) \in \Gamma_u^-$ implies that $L^-(w) \leq L^-(w + (-1,1))$, while $w, w + (0,1) \in \Gamma_v^{-\vee}$ implies that $L^-(w) \geq L^-(w + (-1,1))$. Similarly we cannot find $w \in \mathbb{Z}^2$ such that $w, w - (0,1) \in \Gamma_u^-$ and $w, w + (1,0) \in \Gamma_v^{-\vee}$ simultaneously. The same is true for the semi-infinite geodesics and downward semi-infinite geodesics defined using $L^+$. Take $m \in \mathbb{N}$, $m > r$, we now define events depending on the parameters $m$ and $r$. Let $\mathcal{B}_-$ be the event where

$$\exists u_{-1}, u_{-2} \in \partial I_0^-, \ ad(u_{-1}) < -r, ad(u_{-2}) > r, \Gamma_{u_{-1},u_{-2}} \cap \mathbb{L}_m = \Gamma_{u_{-1},u_{-2}} \cap \mathbb{L}_m;$$

and let $\mathcal{B}_+ \subset \mathcal{A}$ be the event where

$$\exists u_{+1}, u_{+2} \in \partial I_s^+, \ ad(u_{+1}) < ad(p^*) - r, ad(u_{+2}) > ad(p^*) + r, \Gamma_{u_{+1},u_{+2}} \cap \mathbb{L}_m = \Gamma_{u_{+1},u_{+2}} \cap \mathbb{L}_m.$$ 

See Figure 5 for the events.

**Lemma 4.6.** Under $\mathcal{B}_- \cap \mathcal{B}_+$, almost surely we have that $\{L^-(u) = L^+(u + p^*)\} \text{ for any } u \in \mathbb{Z}^2$ with $d(u) > 2m$, and $p_t^- = p_{t+s}^+ - p^*$ for any $t > 0$ with $d(p_t^-) > 2m$.

**Proof.** Since $r < m$, under $\mathcal{A}$ we have $\{u \in \mathbb{Z}^2 : d(u) > 2m\} \cap I_0^- = \{u \in \mathbb{Z}^2 : d(u) > 2m\} \cap (I_s^+ - p^*)$. Define $U := \mathbb{Z}^2 \setminus (I_0^- \cup (I_s^+ - p^*))$. By the above ‘non-crossing’ property, under $\mathcal{A} \cap \mathcal{B}_-$ we have $\Gamma_u^{-\vee} \subset U$ for all $u \in U$ with $d(u) > 2m$; and under $\mathcal{A} \cap \mathcal{B}_+$ we have $\Gamma_u^{+\vee} - p^* \subset U$ for any $u \in U + p^*$ with $d(u) > 2m$.

Now take any $u \in U$. We claim that if $\Gamma_u^{-\vee} \subset U$ and $\Gamma_u^{+\vee} - p^* \subset U$, we must have that $\Gamma_u^{-\vee} = \Gamma_u^{+\vee} \setminus I_s^+ - p^*$ and $L^-(u) = L^+(u + p^*) - s$. Indeed, from the coupling we have we have $\xi^{-\vee}(v) = \xi^{+\vee,s}(v + p^*)$ for any $v \in U$, so both $\Gamma_u^{-\vee}$ and $\Gamma_u^{+\vee} \setminus I_s^+ - p^*$ are the path in $U$ ending at $u$ with the maximum total weight. Then almost surely these paths must be the same with the same weights. From the claim the first conclusion follows.

For any $t > 0$ we have that $p_t^-$ is the last vertex in $\Gamma_t^0 \cap I_t^-$ (Lemma 2.1). Similarly we have that $p_{t+s}^+$ is the last vertex in $\Gamma_t^+ \cap I_t^+$. It then suffices to show that

$$\Gamma_0^0 \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\} + p^* = \Gamma_0^+ \cap \{u \in \mathbb{Z}^2 : d(u) > 2m + d(p^*)\}. \quad (4.2)$$

Indeed, by the non-crossing property, $\Gamma_0^0 \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\}$ is determined by $\Gamma_u^{-\vee}$ for all $u$ with $d(u) > 2m$ and $L^-(u) > 0$; more precisely, $\Gamma_0^- \cap \{u \in \mathbb{Z}^2 : d(u) > 2m\} + (1/2,1/2)$ is the boundary of $u \in \mathbb{Z}^2$ with $d(u) > 2m + 1$, with lower end of $\Gamma_u^{-\vee}$ to the upper-left or lower-right of 0. For the same reason, under $\mathcal{B}_+$, $\Gamma_0^+ \cap \{u \in \mathbb{Z}^2 : d(u) > 2m + d(p^*)\} + (1/2,1/2)$ is the boundary of $u \in \mathbb{Z}^2$ with $d(u) > 2m + d(p^*) + 1$, with lower end of $\Gamma_u^{+\vee} \setminus I_s^+ \cap (2m + d(p^*) + 1)$, upper-left or lower-right of $p^*$. Thus by the above claim we get (4.2).
For the proof of Proposition 4.4 we set up some additional notation (which will be used in the next subsection as well). For each \( x \in \mathbb{R} \) we denote
\[
\mathbb{V}_x := \{ ((1 - \rho)^2 y + x, \rho^2 y - x) : y \in \mathbb{R} \},
\]
and
\[
\mathbb{H}_x := \{ ((1 - \rho)y + (1 - \rho)\rho^2 x, -\rho y + \rho^2 x) : y \in \mathbb{R} \}.
\]
For any set \( A \subset \mathbb{R} \) we denote \( \mathbb{V}_A := \bigcup_{x \in A} \mathbb{V}_x \) and \( \mathbb{H}_A := \bigcup_{x \in A} \mathbb{H}_x \). Besides, since \( L^+ \) (resp. \( L^- \)) has the same law as a Busemann function in LPP, we can also define \( \Gamma^+_u \) (resp. \( \Gamma^-_u \)) for \( u \notin I_0^+ \cup \partial I_0^+ \) (resp. \( u \notin I_0^- \cup \partial I_0^- \)) as the semi-infinite geodesics in the coupled LPP.

**Proof of Proposition 4.4.** In this proof we let \( C, c > 0 \) be large and small constants which depend only on \( \rho \) and \( N \), and the values can change from line to line.

Take \( m = [t/10] \) and \( r = [s^{1/3} t^{1/3}] \). We show that \( \eta_t' \) equals \( \eta_{t+s}' \) in \([-N, N]\) with probability \( > 1 - C(s/t)^{1/30} \), when \( t, s > C \). We could assume that \( t/s \) is large enough (depending on \( \rho \)), since otherwise we would have \( 1 - C(s/t)^{1/30} < 0 \).

We first lower bound \( \mathbb{P}[\mathcal{B}_-] \) and \( \mathbb{P}[\mathcal{B}_+] \). Without loss of generality we assume that \( \rho \leq 1/2 \). Denote \( v_1 = (-[4r\rho^{-2}], 0) \) and \( v_2 = (0, -[4r\rho^{-2}]) \), and we take \( u_{-1} \) to be the last vertex in \( \Gamma^-_{v_1} \cap I_0^- \), and \( u_{-2} \) to be the last vertex in \( \Gamma^-_{v_2} \cap I_0^- \). Then \( u_{-1} \in \{(a, b) \in \mathbb{Z}^2 : a < 0, b \geq 0 \} \), and \( u_{-2} \in \{(a, b) \in \mathbb{Z}^2 : a \geq 0, b < 0 \} \). Then by Corollary 2.11 we have \( \mathbb{P}[ad(u_{-1}) < -r], \mathbb{P}[ad(u_{-2}) > r] > 1 - Ce^{-cr^{1/3}} \). By Proposition 2.6 we have \( \mathbb{P}[\Gamma^-_{v_1} \cap L_m = \Gamma^-_{v_2} \cap L_m] > 1 - Crm^{-2/3} \). Thus we conclude that \( \mathbb{P}[\mathcal{B}_+] > 1 - Crm^{-2/3} \).

For \( \mathbb{P}[\mathcal{B}_+] \), we take \( u_{+1} \) to be the last vertex in \( \Gamma^-_{v_1} \cap I^+_s \), and \( u_{+2} \) to be the last vertex in \( \Gamma^-_{v_2} \cap I^+_s \). Since \( |ad(p^*) - ad(p_s^*)| \leq r \) under \( A \), the events \( ad(u_{+1}) < ad(p^*) - r \) and \( ad(u_{+2}) > ad(p^*) + r \) are implied by \( ad(u_{+1}) < ad(p_s^*) - 2r \) and \( ad(u_{+2}) > ad(p_s^*) + 2r \), respectively.
Let \( A = \mathbb{V}(-r, r) \cap \mathbb{H}(s-r^2/4, s+r^2/4) \). By Corollary 2.11 we have \( \mathbb{P}[p^+_s \in A] > 1 - Ce^{-crs^{-2/3}} \). When \( p^+_s \in A \), we must have \( ad(u_{+1}) < ad(p^+_s) - 2r \) and \( ad(u_{+2}) > ad(p^+_s) + 2r \), unless \( u_{+1} \in A' \) or \( u_{+2} \in A' \), where

\[
A' = A + \{ u \in \mathbb{R}^2 : |d(u)|, |ad(u)| \leq 2r \}.
\]

By Corollary 2.11 we have \( \mathbb{P}[u_{+1} \in A', u_{+2} \in A'] > 1 - Ce^{-crs^{-2/3}} \). Thus we conclude that \( \mathbb{P}[ad(u_{+1}) < ad(p^+_s) - r], \mathbb{P}[ad(u_{+2}) > ad(p^+_s) + r] > 1 - Ce^{-crs^{-2/3}} \). By Proposition 2.6 we have \( \mathbb{P}[\Gamma^+_v \cap \mathbb{L}_m = \Gamma^+_w \cap \mathbb{L}_m] > 1 - Crm^{-2/3} \). Thus we conclude that \( \mathbb{P}[A \setminus B_+] < Crm^{-2/3} \).

By Corollary 2.13 we have \( \mathbb{P}[d(p^-_s) > 2m + N] > 1 - Ce^{-ct} \). By Lemma 4.5 and the above estimates, we have that \( \eta_{-s}^+ \) equals \( \eta_{s}^+ \) in \([-N, N]\) with probability \( > 1 - Crm^{-2/3} - C(rs^{-2/3})^{-1/10} \). Thus the conclusion follows.

### 4.3 The initial step coupling

This subsection is devoted to proving Lemma 1.5.

We define \( (\tau_i)_{i \in \mathbb{R}} \) as the process of a stationary TASEP with density \( \rho \). Then for any \( t \in \mathbb{R} \), we have \( \tau_i(x) \) being Bernoulli \( \rho \) for each \( x \in \mathbb{Z} \) independently. Our strategy is to construct a coupling between \( \eta_{t}^+ \) and \( \tau_i \), where (with high probability) \( \eta_{t}^+ \) and \( \tau_i \) are identical outside \([-r, r]\), and have the same number of particles in \([-r, r]\). It would be straightforward to couple \( \tau_i \) and \( \eta_{t}^+ \) since both are Bernoulli \( \rho \) on \( \mathbb{Z} \setminus \{0, 1\} \).

We denote \( \alpha = (rs^{-2/3})^{1/5}, \) and \( r_i = \alpha s^{2/3}, \) for \( i = 1, 2, 3, 4 \). Below we assume that \( \alpha \) and \( s \) are large enough depending on \( \rho \), and also \( \alpha < r^{0.01} \).

Recall (from the previous subsection) the definitions of \( L^+, \xi^{+, \vee}, I^+_t \) for \( t \geq 0 \), the geodesics \( \Gamma^+_u \) for \( u \in I^+_0 \cup \partial I^+_0 \) and \( \Gamma^+_{u, \vee} \) for \( u \notin I^+_0 \); and also \( T^+_{u, v} \) and \( \Gamma^+_{u, \vee} \) for any \( u, v \notin I^+_0 \) with \( u \leq v \), using the weights \( \xi^{+, \vee} \).

We will use the following extensions of these constructions. First, as in the previous subsection, we define \( \Gamma^+_u \) for \( u \notin I^+_0 \cup \partial I^+_0 \), using that \( L^+ \) has the same distribution as a Busemann function. Second, we extend \( \xi^{+, \vee} \) to \( I^+_0 \), so that \( \xi^{+, \vee}(u) \) is i.i.d. \( \text{Exp}(1) \) for each \( u \in \mathbb{Z}^2 \), and is independent of \( I^+_0 \). Using the weights \( \xi^{+, \vee} \) on \( \mathbb{Z}^2 \) we can define \( T^+_{u, v} \) and \( \Gamma^+_{u, \vee} \) for any \( u \leq v \). Note that these two extensions are not related and we would not use the joint law of them; so the joint law is taken arbitrarily.

A straightforward way of coupling \( (\eta_{t}^+)_{t \geq 0} \) and \( (\tau_i)_{i \geq 0} \) would be first to couple \( \eta_{t}^+ \) with \( \tau_0 \), and then to let them evolve using the same set of exponential waiting times. However, this would not give the desired construction, since \( (\eta_{t}^+)_{t \geq 0} \) is centered at the hole-particle pair. Thus we need to first shift \( \eta_{t}^+ \) by \( ad(p^+_s) \) and then couple with \( \tau_0 \). However, the vertex \( p^+_s \) depends on the evolution, although mostly rely on the evolution around the hole-particle pair. Thus we take the following approach: we construct the coupling by first sample the evolution of \( (\eta_{t}^+)_{t \geq 0} \) around the pair, then get a good approximation of \( p^+_s \). Using that we could shift \( \tau_0 \), and couple the rest waiting times together.

Recall that the vertex \( p^+_s \) is also the last vertex in \( \Gamma^+_0 \cap I^+_s \). Denote \( P = \mathbb{V}((-r, r), (1, 0)) \). Using \( I^+_0 \cap P \) and \( \{ \xi^+(u) \}_{u \in P \setminus I^+_0} \) we defined a ‘restricted version’ of \( p^+_s \) as following. First we define \( L^P \), by letting \( L^P(u) = 0 \) for \( u \in I^+_0 \cup (\mathbb{Z}^2 \setminus P) \), and setting \( L^P(u) = L^P(u - (1, 0)) \vee L^P(u - (0, 1)) + \xi^{+, \vee}(u) \) recursively for each \( u \in P \setminus I^+_0 \). We then define \( \Gamma^P_0 \), by letting \( \Gamma^P_0[1] = 0 \), and

\[
\Gamma^P_0[i + 1] = \arg\min_{v \in \{ \Gamma^P_0[i] \cup \{(1, 0)\}, \Gamma^P_0[i] \cup \{(0, 1)\}\} \cap P} L^P(v)
\]

for each \( i \in \mathbb{N} \). We let \( p^P \) be the last vertex in \( \{ u \in \Gamma^P_0 : L^P(u) \leq s \} \). Denote \( M = ad(p^P) \). Then \( M \) is determined by \( I^+_0 \cap P \) and \( \{ \xi^{+, \vee}(u) \}_{u \in P \setminus I^+_0} \).
Lemma 4.7. \( \mathbb{P}[p^P = p^+_s] > 1 - Ce^{-c\alpha} \) for some constants \( c, C > 0 \) depending only on \( \rho \).

Proof. Consider the event \( \mathcal{D}_1 \) where
\[
\exists u_1, u_2 \in \partial I^+_0, \quad \Gamma^+_u \cap I^+_s \subset V_{(-r_1, -r_1/2)}, \quad \Gamma^+_u \subset V_{(r_1/2, r_1)}.
\]
Assume that \( \mathcal{D}_1 \) holds, let \( S \) be the set consisting of vertices in \( I^+_s \setminus I^+_0 \) between \( \Gamma^+_{u_1} \) and \( \Gamma^+_{u_2} \). We note that for each \( u \in S \), there is
\[
\max_{v \in \{u-(1, 0), u-(0, 1)\}} L^+(v) = \max_{v \in \{u-(1, 0), u-(0, 1)\} \cap S} L^+(v).
\]
Thus by induction we have \( L^+(u) = L^P(u) \) for each \( u \in S \). Let \( \mathcal{D}_2 \) be the event where \( \Gamma^+_0 \cap I^+_s \subset V_{(-r_1/2, r_1/2)} \). Then under \( \mathcal{D}_1 \cap \mathcal{D}_2 \) we have \( \Gamma^+_0 \cap I^+_s = \{ u \in \Gamma^+_0 : L^P(u) \leq s \} \), and in particular \( p^P = p^+_s \).

Denote
\[
u_1 = \left( -\frac{3((1 - \rho)^2 + \rho^2)}{4 \rho^2} r_1, 0 \right), \quad \nu_2 = \left( 0, -\frac{3((1 - \rho)^2 + \rho^2)}{4(1 - \rho^2)} r_1 \right),
\]
and take \( u_1 \) as any vertex in \( \Gamma^+_{u_1} \cap \partial I^+_0 \) and \( u_2 \) as any vertex in \( \Gamma^+_{u_2} \cap \partial I^+_0 \). We let \( \mathcal{D}_3 \) be the event where \( d(p^+_s) < 2s \), let \( \mathcal{D}_4 \) be the event where \( \Gamma^+_{u_1} \cap \{ u : d(u) < 4s \} \subset V_{(-r_1, -r_1/2)} \) and \( \Gamma^+_{u_2} \cap \{ u : d(u) < 4s \} \subset V_{(r_1/2, r_1)} \). Since \( u_1, u_2 \) cannot be greater than \( p^+_s \) in each coordinate, under \( \mathcal{D}_2 \cap \mathcal{D}_3 \cap \mathcal{D}_4 \) we have \( d(u_1), d(u_2) < 4s \) and \( \mathcal{D}_1 \) holds.

Thus we conclude that \( \mathbb{P}[p^P = p^+_s] \geq \mathbb{P}[\mathcal{D}_1 \cap \mathcal{D}_2] \geq \mathbb{P}[\mathcal{D}_2 \cap \mathcal{D}_3 \cap \mathcal{D}_4] \). By Corollary 2.13 we have \( \mathbb{P}[\mathcal{D}_2 \cap \mathcal{D}_3] > 1 - Ce^{-c r_1 s^{-2}/3} \); by Corollary 2.11 we have \( \mathbb{P}[\mathcal{D}_4] > 1 - Ce^{-c r_1 s^{-2}/3} \). Thus the conclusion follows.

Now we couple \( (\tau_t)_{t \geq 0} \) with \( (\eta^{+, \prime}_t)_{t \geq 0} \) using the following steps.

1. First take \( \eta^{+, \prime}_0(x) \) being i.i.d. Bernoulli \( \rho \), for each \( x \in [-r_2, r_2] \setminus \{0, 1\} \). Then by our choice of \( r_1, r_2 \), the set \( I^+_0 \cap \partial \rho \) is determined. We next sample \( \xi^{+, \prime}(u) \) for \( u \in P \), and then the number \( M = ad(p^P) \) is determined.

2. Let \( \tau_0(x - M) \) be i.i.d. Bernoulli \( \rho \) for each \( x \in [-r_2, r_2] \).

3. For each \( x = [-r_2], [-r_2] - 1, \ldots \), we take \( \eta^{+, \prime}_0(x) \) and \( \tau_0(x - M) \) being Bernoulli \( \rho \) independently, until for some \( x_* \in \mathbb{Z} \) there is \( \sum_{x = x_*}^0 \eta^{+, \prime}_0(x) - \tau_0(x - M) = 0 \). Then for each \( x < x_* \), we take \( \eta^{+, \prime}_0(x) = \tau_0(x - M) \) being Bernoulli \( \rho \) independently.

4. For each \( x = [r_2], [r_2] + 1, \ldots \), we take \( \eta^{+, \prime}_0(x) \) and \( \tau_0(x - M) \) being Bernoulli \( \rho \) independently, until for some \( x_* \in \mathbb{Z} \) there is \( \sum_{x = x_*}^x \eta^{+, \prime}_0(x) - \tau_0(x - M) = 0 \). Then for each \( x > x_* \) we take \( \eta^{+, \prime}_0(x) = \tau_0(x - M) \) being Bernoulli \( \rho \) independently.

5. Take \( \xi^{+, \prime}(u) \) to be i.i.d. Exp(1) for \( u \notin P \).

Up to now we have constructed a joint law of \( \tau_0 \) and \( (\eta^{+, \prime}_t)_{t \geq 0} \), each with the desired marginal distribution. Also \( \tau_0, M, \) and \( \{\xi^{+, \prime}(u)\} \) are mutually independent.

In \( \tau_0 \), we label the holes by \( \mathbb{Z} \) from left to right, and the particles by \( \mathbb{Z} \) from right to left, such that for \( |x| \) large enough, the particle (or hole) at site \( x - M \) has the same label as the particle (or hole) at site \( x \) in \( \eta^{+, \prime}_0 \). Let \( L^+(a, b) \) be the time when particle with label \( b \) switches with label \( a \), if in \( \tau_0 \) this particle is to the left of this label, and let \( L^-(a, b) = 0 \). For each \( t \geq 0 \) denote \( I^+_t := \{ u \in \mathbb{Z}^2 : L^+(u) \leq t \} \), and \( \partial I^+_t := \{ u \in I^+_t : L^+(u + (1, 0)) \) or \( L^+(u + (0, 1)) > t \}. \)

25
From the above construction, we have $I_0^+$ and $I_0^-$ differ by at most finitely many vertices. For any $u \in (\mathbb{Z}^2 \setminus I_0^+) \cup \partial I_0^+$, we also define the ‘semi-infinite geodesic’ $\Gamma_u^+$ recursively, by letting $\Gamma_u^+[1] = u$, and for each $i \in \mathbb{N}$ letting $\Gamma_u^+[i + 1] = \text{argmin}_{v \in (\Gamma_u^+[i] + (1, 0), \Gamma_u^+[i] + (0, 1))} L^+(v)$. Note that unlike $L^+$, there is no coupling between $L^+$ with a Busemann function in LPP, thus these $\Gamma_u^+$ are not actual geodesics.

6. We couple the evolution of $\tau$ with $(\eta^+_t)_{t \geq 0}$, by letting $L^+(u) - L^+(u - (1, 0)) \lor L^+(u - (0, 1)) = \xi^+(\eta^+_t(u))$ for any $u \in \mathbb{Z}^2 \setminus P$ with $L^+(u) > 0$; and $L^+(u) - L^+(u - (1, 0)) \lor L^+(u - (0, 1))$ for $u \in P$ be i.i.d. Exp(1), independent of $\xi^{+, +}$.

Now we have a coupling between $(\eta^+_t)_{t \geq 0}$ and $(\tau_t)_{t \geq 0}$.

We denote $\mathcal{E}_1$ as the event where for any $x < -r$, $\tau_s(x) = \eta^+_s(x)$, and the particle or hole at site $x$ has the same label for $\tau$ and $\eta^+$; denote $\mathcal{E}_2$ as the event where the same is true for any $x > r$. We shall lower bound $\mathbb{P}[\mathcal{E}_1]$ and $\mathbb{P}[\mathcal{E}_2]$.

For this we consider the following events. We denote $\mathcal{E}_3$ as the event where there exists a vertex $u_+ \in \partial I_0^+$, such that $ad(u_+) < x_*$ and $\Gamma_{u+}^+ \cap I_s^+ \subset \mathbb{V}_{(-r_4, -r_1)}$, and $a'_+ > (1 - \rho)^2 s - r_4$ for $u'_+ = (a'_+, b'_+)$ being the last vertex in $\Gamma_{u+}^+ \cap I_s^+$. We also denote $\mathcal{E}_4$ to be the event such that for each $u = (a, b) \in \partial I_s^+$ with $ad(u) < M - r + 1$, we have $a < (1 - \rho)^2 s - r_4$, and $u \in \mathbb{V}_{(-\infty, -r_4)}$. Analogly we denote $\mathcal{E}_3^+$ as the event where there exists a vertex $u_+ \in \partial I_0^+$ with $ad(u_+) < x_*$ and $\Gamma_{u+}^+ \cap I_s^+ \subset \mathbb{V}_{(-r_4, -r_1)}$, and $a'_+ > (1 - \rho)^2 s - r_4$ for $u'_+ = (a'_+, b'_+)$ being the last vertex in $\Gamma_{u+}^+ \cap I_s^+$. We denote $\mathcal{E}_4^+$ as the event where for each $u = (a, b) \in \partial I_s^+$ with $ad(u) < M - r + 1$, we have $a < (1 - \rho)^2 s - r_4$, and $u \in \mathbb{V}_{(-\infty, -r_4)}$. See Figure 6 for an illustration of these events.

**Lemma 4.8.** $\{p^P = p^+_s\} \cap \mathcal{E}_3 \cap \mathcal{E}_4 \cap \mathcal{E}_3^+ \cap \mathcal{E}_4^+ \subset \mathcal{E}_1$.

**Proof.** Under $\mathcal{E}_3$, let $S_+$ be the set consisting of vertices $u = (a, b) \in \mathbb{Z}^2 \setminus I_0^+$ and above $\Gamma_{u+}^+$, with $a < (1 - \rho)^2 s - r_4$. We then have

$$\max_{v \in \{u - (1, 0), u - (0, 1)\}} L^+(v) = \max_{v \in \{u - (1, 0), u - (0, 1)\} \cap S_+} L^+(v), \quad \forall u \in S_+,$$
and thus by induction we have $L^+(u) \leq L^r(u)$ for each $u \in S_+$. 

Under $E^r_3$, let $S_\tau$ be the set consisting of vertices $u = (a, b) \in \mathbb{Z}^2 \setminus I^r_0$ and above $\Gamma^r_{u\tau}$ with $a < (1 - \rho)^2 s - r_4$. Then similarly, by induction we have $L^+(u) \geq L^r(u)$ for each $u \in S_\tau$.

Suppose that $E_3 \cap E_4 \cap E^r_3 \cap E^r_4$ holds and $p^P = p^+_s$. Take any $x \in \mathbb{Z}$ with $x < -r$, and assume that $\eta^+_s(x) = 1$. Then there is $v \in \partial I^+_s$ such that $ad(v) = ad(p^+_s) + x - 1$ and $v + (1, 0) \not\in I^+_s$. By $p^P = p^+_s$ we have $M = ad(p^+_s)$, so $ad(v) = M + x - 1$. The second coordinate of $v$ is the label of the particle at site $x$ (in $\eta^+_s$). From $E_3 \cap E_4$ we have $v, v + (1, 0) \in S_+,$ thus

$$L^r(v + (1, 0)) \geq L^+(v + (1, 0)) > s.$$ (4.3)

If $\tau_s(x) = 0$, then there is $v^r \in \partial I^+_s$ such that $ad(v^r) = M + x$ and $v^r + (0, 1) \not\in I^+_s$. Then $ad(v + (1, 0)) = ad(v^r)$. By (4.3) and $L^+(v^r) \leq s$ we have $d(v) > d(v^r)$. From $E^r_3 \cap E^r_4$ we have $v^r, v^r + (0, 1) \in S_\tau$, then $L^+(v^r + (0, 1)) > s \geq L^+(v)$, implying that $d(v) < d(v^r)$ since $ad(v^r + (0, 1)) = ad(v)$. This is a contradiction.

Thus we must have $\tau_s(x) = 1$, and there is $v^r \in \partial I^+_s$ such that $ad(v^r) = M + x - 1$ and $v^r + (0, 1) \not\in I^+_s$. The second coordinate of $v^r$ is the label of the particle at site $x$ (in $\tau_s$). From $E^r_3 \cap E^r_4$ we have $v^r, v^r + (1, 0) \in S_\tau$, so

$$L^r(v^r + (1, 0)) \geq L^+(v^r + (1, 0)) > s \geq L^+(v).$$ (4.4)

Since $ad(v) = ad(v^r)$, from (4.4) we have $d(v^r) \geq d(v)$; and from (4.3) and $L^r(v^r) \leq s$ we have $d(v^r) \leq d(v)$. Then we must have $d(v^r) = d(v)$ and $v^r = v$, so the label of the particle at site $x$ is the same for $\eta^+_s$ and $\tau_s$.

By similar arguments, when $\eta^+_s(x) = 0$ we must have $\tau_s(x) = 0$, and the label of the hole at site $x$ is the same for $\eta^+_s$ and $\tau_s$. Thus we have that the event $E_1$ holds.

**Lemma 4.9.** $\mathbb{P}[E_3], \mathbb{P}[E^r_3] > 1 - C\alpha^{-1/2}$, and $\mathbb{P}[E_4], \mathbb{P}[E^r_4] > 1 - Ce^{-\alpha}$, for constants $c, C > 0$ depending only on $\rho$.

**Proof.** We shall write the proof for the estimates of $\mathbb{P}[E_3]$ and $\mathbb{P}[E_4]$, and the approach we take here applies to the estimates for $\mathbb{P}[E^r_3]$ and $\mathbb{P}[E^r_4]$, essentially verbatim. We will use $c, C > 0$ to denote...
small and large enough constants depending only on $\rho$, and their values can change from line to line.

We consider the following events (see Figure 7).

$E_5$: $x_*>-r_3$.

$E_6$: $V_{(jr_4,jr_4)} \cap \partial I_0^+ \subset \mathbb{H}_{(j\alpha r_4^{1/2},j\alpha r_4^{1/2})}$ for each $j \in \mathbb{N}$.

$E_7$: $V_{(-6r_3,-r_3)} \cap \partial I_s^+ \subset \mathbb{H}_{(-(s-2\alpha r_4^{1/2},s+2\alpha r_4^{1/2})}$.

$E_8$: Let $u_1 = 2s((1-\rho)^2,\rho^2) + (-5(1-\rho)r_3,5\rho r_3)$ and $u_2 = 2s((1-\rho)^2,\rho^2) + (-2(1-\rho)r_3,2\rho r_3)$ (round to the nearest lattice point). Then $\Gamma_{u_1}^{\alpha,\nu} \subset V_{(-6r_3,-3r_3)}$, and $\Gamma_{u_2}^{\Sigma,\nu} \subset V_{(-3r_3,-r_3)}$.

We have $E_5 \cap E_6 \cap E_7 \cap E_8 \subset \mathcal{E}_3$. Indeed, we take any $u_+ \in \partial I_0^+ \cap V_{(-4r_3,-3r_3)}$, and let $u_+ = (a_+^j,b_+^j)$ be the last vertex in $\Gamma_{u_+}^{\alpha,\nu} \cap I_s^+$. By $E_6$, and note that $r_3 > C\alpha r_4^{1/2}$ by our choice of the parameters, we have $ad(u_+) < -r_3$. So under $E_5 \cap E_6$ we have $ad(u_+) < x_*$. Since $\Gamma_{u_+}^{\alpha,\nu}$ does not cross $\Gamma_{u_+}^{\alpha,\nu}$ or $\Gamma_{u_+}^{\Sigma,\nu}$, by $E_7 \cap E_8$ we have that $\Gamma_{u_+}^{\alpha,\nu} \cap I_s^+ \subset V_{(-6r_3,-r_3)}$ and $u_+ \in \mathbb{H}_{(s-2\alpha r_4^{1/2},s+2\alpha r_4^{1/2})} \cap V_{(-6r_3,-r_3)}$.

Thus we get $a_+^j > (1-\rho)^2s-r_4$. It remains to estimate the probabilities of these events and take a union bound.

The number $-x_*$ is just the time of a symmetric random walk hitting 0 after $r_2$. Thus $P{E_5} \geq 1 - C\alpha r_2^{1/2}r_3^{-1/2} = 1 - C\alpha^{-1/2}$.

The event $E_6$ is again an estimate on the hitting probability of a random walk. For each $j \in \mathbb{N}$ we have $P{V_{(jr_4,jr_4)} \cap \mathbb{H}_{(j\alpha r_4^{1/2},j\alpha r_4^{1/2})}} \geq 1 - Ce^{-cj\alpha}$; so when $\alpha > C$ we have $P{E_6} \geq 1 - Ce^{-c\alpha}$.

We let $S_1 = V_{(-r_4,r_4)} \cap \mathbb{H}_{(\alpha r_4^{1/2},\infty)}$ and $S_1 = V_{(-r_4,r_4)} \cap \mathbb{H}_{(\alpha r_4^{1/2},\infty)}$, and for each $j \geq 2$ we let

$$S_j = V_{(-jr_4,(j-1)r_4]} \cap \mathbb{H}_{(j\alpha r_4^{1/2},\infty)}$$

$$S_j = V_{(-jr_4,(j-1)r_4]} \cap \mathbb{H}_{(j\alpha r_4^{1/2},\infty)}$$

Let $S_* = \cup_j \mathbb{N}S_j$ and $S_* = \cup_j \mathbb{N}S_j$. For $E_7$, we consider the following events:

$E_7^1$: $T_{u,v}^{\alpha,\nu} < s$ for any $v \in V_{(-6r_3,-r_3)} \cap \mathbb{H}_{(-\infty,s-2\alpha r_4^{1/2}}$ and $u \in S_*$, $u,v \in \mathbb{Z}^2$.

$E_7^2$: For any $v \in V_{(-6r_3,-r_3)} \cap \mathbb{H}_{(s+2\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$, there exists $u \in V_{(-6r_3,-r_3)} \cap \mathbb{H}_{(\alpha r_4^{1/2},\infty)} \cap \mathbb{Z}^2$ such that $T_{u,v}^{\alpha,\nu} > s$.

Then under $E_7^1 \cap E_7^2 \cap E_6$, we have that $V_{(-6r_3,-r_3)} \cap \mathbb{H}_{(-\infty,s-2\alpha r_4^{1/2}} \cap \mathbb{Z}^2 \subset I_s^+$, and $V_{(-6r_3,-r_3)} \cap \mathbb{H}_{(s+2\alpha r_4^{1/2},\infty)} \cap I_s^+ = \emptyset$, thus $E_7$ holds.

To lower bound $P{E_7^1}$, we just need to consider $T_{u,v}^{\alpha,\nu}$, for $v \in \mathbb{Z}^2$ within distance 1 from $V_{(-6r_3,-r_3)} \cap \mathbb{H}_{s-2\alpha r_4^{1/2}}$, and $u \in \mathbb{Z}^2$ within distance 1 from $V_{(-jr_4,(j-1)r_4]} \cap \mathbb{H}_{-jr_4^{1/2}}$, for each $j \in \mathbb{N}$. By (2.7), we have $E_{T_{u,v}^{\alpha,\nu}} < s - c\alpha r_4^{1/2} - c(j-1)r_4^{1/2}s^{-1}$. Then if the slope of the line connecting $u,v$ is between $\frac{\rho^2}{10(1-\rho^2)}$ and $\frac{(1-\rho)^2}{10\rho^2}$, we apply Proposition (2.4) by covering these pairs with parallelogram of size $Cs \times Cs^{2/3}$, and for all other such pairs we apply Theorem (2.5). We conclude that $P{E_7^1} > 1 - C(r_3 s^{-2/3})(r_3 s^{-2/3})e^{-c\alpha}$ when $\alpha > C$.

For $P{E_7^2}$, we need to consider $T_{u,v}^{\alpha,\nu}$, for $v \in \mathbb{Z}^2$ within distance 1 from $V_{(-6r_3,-r_3)} \cap \mathbb{H}_{s+2\alpha r_4^{1/2}}$, and $u = v - (s + \alpha r_4^{1/2}((1-\rho)^2,\rho^2)$ (round up to the nearest lattice point). By (2.7) we have
\( \mathbb{E} T_{u,v}^{+,V} > s + c \alpha r_4^{1/2} \). Again we apply Proposition 2.4 by covering these \( u, v \) with parallelogram of size \( C_s \times C_s^{2/3} \), and we conclude that \( \mathbb{P}[E''_u] > 1 - C'r_3 s^{-2/3} e^{-c\alpha} \) when \( \alpha > C \).

For \( \mathcal{E}_8 \), we denote \( u_3 \) as the lower end point of \( \Gamma_{u_3}^{+,V} \). Consider the event \( \mathcal{E}_8^* \), where for any \( u \not\in S_u \cup \mathbb{V}(-5r_3-2r_3,-5r_3-r_3) \), there is \( T_{u,u_1}^{+,V} < 2s - 4\alpha r_4^{1/2} \); and \( T_{u_3,u_1}^{+,V} > 2s - 4\alpha r_4^{1/2} \), where \( u_1 = u_3 - (2s - 2\alpha r_4^{1/2})((1 - \rho)_2, \rho_2)^2 \). By Lemma 2.9 and Corollary 2.10, we have \( \mathcal{E}_8^* \cap \mathcal{E}_8^0 \) we must have \( u_3 \in \mathbb{V}(-5r_3-2r_3,-5r_3+r_3+2) \). We next lower bound \( \mathbb{P}[\mathcal{E}_8^0] \). By (2.7) we have \( \mathbb{E} T_{u_3,u_1}^{+,V} > 2s - 3\alpha r_4^{1/2} \). For \( T_{u,u_1}^{+,V} \) with \( u \not\in S_u \cup \mathbb{V}(-5r_3-2r_3,-5r_3+r_3) \), again we just need to consider \( u \in \mathbb{Z}^2 \) within distance 1 from \( \mathbb{V}(-jr_4,(j-1)r_4 \cup (j-1)r_4, jr_4) \cap \mathbb{H}_{-jr_4^{1/2}} \), for each \( j \in \mathbb{N} \). By (2.7), when \( j \geq 2 \) we have
\[
\mathbb{E} T_{u,u_1}^{+,V} < 2s - (c(j-1))^2 r_4^{s-1} < 2s - 5\alpha r_4^{1/2} - c(j-1)^2 r_4^{s-1}/2;
\]
and when \( j = 1 \) we have
\[
\mathbb{E} T_{u,u_1}^{+,V} < 2s - cr_4^{s-1} < 2s - 5\alpha r_4^{1/2}
\]
for \( u \not\in \mathbb{V}(-5r_3-2r_3,-5r_3+r_3) \). Here we used that \( r_4^{s-1} < 2s - 5\alpha r_4^{1/2} \), due to our choice of the parameters. Then (similar to lower bounding \( \mathbb{P}[\mathcal{E}_8^*] \) and \( \mathbb{P}[E''_u] \)) we can lower bound \( \mathbb{P}[\mathcal{E}_8^0] \) by covering these \( u \) and \( u_1 \) with parallelograms of size \( C_s \times C_s^{2/3} \), and applying Proposition 2.4 and Theorem 2.3. We conclude that \( \mathbb{P}[\mathcal{E}_8^0] > 1 - C'r_3 s^{-2/3} e^{-c\alpha} \) when \( \alpha > C \).

Now we take \( u_4, u_5 \) as the intersection of \( \mathbb{H}_{-cr_4^{1/2}} \) with \( \mathbb{V}(-5r_3-2r_3) \) and \( \mathbb{V}(-5r_3+2r_3) \), respectively (round to the nearest lattice point). By Lemma 2.4 and Corollary 2.10 we have
\[
\mathbb{P}[\Gamma_{u_4, u_1}^{+,V} \cap \mathbb{H}_{-cr_4^{1/2}}] < \mathbb{P}[\mathbb{V}(-5r_3-2r_3,-5r_3-2r_3)] > 1 - Ce^{-cr_4^{1/2} - 3/4^{1/3}},
\]
and
\[
\mathbb{P}[\Gamma_{u_5, u_1}^{+,V} \cap \mathbb{H}_{-cr_4^{1/2}}] < \mathbb{P}[\mathbb{V}(-5r_3+2r_3,-5r_3+2r_3)] > 1 - Ce^{-cr_4^{1/2} - 3/4^{1/3}},
\]
and
\[
\mathbb{P}[\Gamma_{u_4, u_1}^{+,V} \subset \mathbb{V}(-6r_3,-4r_3]) \setminus \mathbb{P}[\Gamma_{u_5, u_1}^{+,V} \subset \mathbb{V}(-6r_3,-4r_3)] > 1 - Ce^{-cr_4^{1/2} - 3/4^{1/3}}.
\]
Note that when the above four events happen, and \( u_3 \in \mathbb{V}(-5r_3-2r_3,-5r_3+2r_3) \), we have that \( \Gamma_{u_4, u_1}^{+,V} = \Gamma_{u_5, u_1}^{+,V} \) is between \( \Gamma_{u_4, u_1}^{+,V} \) and \( \Gamma_{u_5, u_1}^{+,V} \) by ordering of geodesics, and \( \Gamma_{u_4, u_1}^{+,V} \subset \mathbb{V}(-5r_3,-4r_3) \). We can argue similarly for \( \Gamma_{u_5, u_1}^{+,V} \). Then with \( \mathbb{P}[\mathcal{E}_8^0] > 1 - Ce^{-c\alpha} \) we conclude that \( \mathbb{P}[\mathcal{E}_8^0] < (C e^{-c\alpha} + e^{-cr_4^{2/3} - 3/4^{1/3}} + e^{-cr_3^{2/3} + 3/4^{1/3}}) < Ce^{-c\alpha} \).

Taking together the lower bounds for \( \mathbb{P}[\mathcal{E}_8^0], \mathbb{P}[\mathcal{E}_6], \mathbb{P}[\mathcal{E}_7], \mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_8] \), we conclude that \( \mathbb{P}[\mathcal{E}_8] > 1 - C\alpha^{1/2} \).

We next bound \( \mathbb{P}[\mathcal{E}_4] \). For this we define two more events.

**\( \mathcal{E}_9 \):** \( |M - (1 - 2\rho)s| < r_2 \).

**\( \mathcal{E}_{10} \):** For any \( u = (a, b) \) with \( ad(u) < (1 - 2\rho)s + r_2 - r + 1 \) and \( a \geq (1 - \rho)^2 s - r_4 \), we have \( u \not\in I_s^+ \).

Obviously we have \( \mathcal{E}_9 \cap \mathcal{E}_{10} \subset \mathcal{E}_4 \).

By Corollary 2.13 we have \( \mathbb{P}[\mathcal{E}_9] > 1 - C'e^{-cr_3^{2s-1/3}} \). For \( \mathcal{E}_{10} \), we take \( u_* = (a_*, b_*) \) where \( a_* = [(1 - \rho)^2 s - r_4] \) and \( b_* = a_* - [(1 - 2\rho)s + r_2 - r] \), and \( \mathcal{E}_{10} \) is equivalent to \( L^+(u_*) > s \). Denote \( u_*' = (r_4, [r_4]) \) are \( S^* \), so under \( \mathcal{E}_6 \) we have \( u_*' \not\in I_0^+ \). Thus under \( \mathcal{E}_6 \setminus \mathcal{E}_{10} \) we have \( T_{u_*', u_*}^{+,V} \leq s \); then \( \mathbb{P}[\mathcal{E}_6 \setminus \mathcal{E}_{10}] \leq \mathbb{P}[T_{u_*', u_*}^{+,V} \leq s] < Ce^{-cr_3^{2s-1/3}} \), where the last inequality is by Theorem 2.3. Thus we conclude that \( \mathbb{P}[\mathcal{E}_4] > 1 - Ce^{-c\alpha} \).

**Proof of Lemma 4.3.** Let \( C > 0 \) be a constant depending only on \( \rho \). By Lemma 4.7 and 4.8 and 4.9 we have \( \mathbb{P}[\mathcal{E}_4] > 1 - C\alpha^{1/2} \), and by symmetry we also have \( \mathbb{P}[\mathcal{E}_2] > 1 - C\alpha^{1/2} \). Under \( \mathcal{E}_1 \cap \mathcal{E}_2 \) we have \( \tau_s(x) = \eta_s^+(x) \) for any \( |x| > r \), and \( \sum_{x = -r}^{r} \tau_s(x) = \sum_{x = -r}^{r} \eta_s^+(x) \).
We couple $\tau_s$ with $\eta_0^\prime$ as following. For each $x = 1, 2, \ldots$, we take $\eta_0^\prime(x)$ and $\tau_s(x)$ being Bernoulli $\rho$ independently, until for some $x^* \in \mathbb{Z}$ there is $\sum_{x=1}^{x^*} \eta_0^\prime(x) - \tau_s(x) = 0$. Then for each $x > x^*$ we take $\eta_0^\prime(x) = \tau_s(x)$ being Bernoulli $\rho$ independently. For each $x = 0, -1, \ldots$, we take $\eta_0^\prime(x)$ and $\tau_s(x)$ being Bernoulli $\rho$ independently, until for some $x_* \in \mathbb{Z}$ there is $\sum_{x=x_*}^{0} \eta_0^\prime(x) - \tau_s(x) = 0$. Then for each $x < x_*$ we take $\eta_0^\prime(x) = \tau_s(x)$ being Bernoulli $\rho$ independently.

Under this coupling, when $x_* > -r$ and $x^* < r$ we have $\tau_s(x) = \eta_0^\prime(x)$ for any $|x| > r$, and $\sum_{x=-r}^{r} \tau_s(x) = \sum_{x=-r}^{r} \eta_0^\prime(x)$. Also, since partial sums of $\eta_0^\prime(x) - \tau_s(x)$ are symmetric random walks, we have $P[\tau_* > -r], P[\tau^* \geq r] < Cr^{-1/2}$. Thus we have

$$P[A^*] < P[\mathcal{E}_1^*] + P[\mathcal{E}_2^*] + P[\mathcal{E}_3^*] = 0$$

and the conclusion follows.

5 Convergence along semi-infinite geodesics: ergodicity of the stationary distribution

In this section we prove convergence in law along semi-infinite geodesics, which is a weaker version of Theorem 1.2.

Theorem 5.1. For each $\rho \in (0, 1)$, we have in probability $\mu^\rho_{0;r} \to \nu^\rho$, as $r \to \infty$.

As before, we take $(\eta_t)_{t \geq 0}$ as the TASEP with density $\rho$, conditioned on $\eta_0(0) = 0, \eta_0(1) = 1$, and let $(\eta^\prime_t)_{t \geq 0}$ be $\eta_t$ as seen from the hole-particle pair (corresponding to a second-class particle). Let $\eta = (\eta_t)_{t \in \mathbb{R}}$ be the stationary process of TASEP as seen from a hole-particle pair, and $\Psi^\rho$ be its measure. Also let $\Psi^\rho$ be the measure of $\eta^\prime_0$.

For any process $P = (P_w)_{w \in \mathbb{R}}$ and $t \in \mathbb{R}$, we denote $\mathcal{R}P$ as the process $(P_{t+w})_{w \in \mathbb{R}}$. By Lemma 2.1 and 3.1, we can deduce Theorem 5.1 from the following result. To make it well-defined, we extend $(\eta^\prime_t)_{t \geq 0}$ to $\eta^\prime = (\eta^\prime_t)_{t \in \mathbb{R}}$, such that $\eta^\prime_t = \eta^\prime_0$ for each $t < 0$.

Proposition 5.2. For any function $f$ on $\{0, 1\}^{\mathbb{Z} \times \mathbb{R}}$ that is bounded and measurable, we have

$$T^{-1} \int_0^T f(\mathcal{R}\eta^\prime)dt \to \tilde{\Psi}^\rho(f)$$

in probability.

By Birkhoff’s Theorem, this proposition follows from Theorem 1.7 and the following ergodicity result.

Proposition 5.3. The process $\eta = (\eta_t)_{t \in \mathbb{R}}$ is ergodic in time.

Assuming this we prove Proposition 5.2 now.

Proof of Proposition 5.2. Without loss of generality we assume that $0 \leq f \leq 1$, and for some $s > 0$, it is $\mathcal{F}_s$-measurable, where $\mathcal{F}_s$ is the $\sigma$-algebra generated by $A \times \{0, 1\}^{\mathbb{Z} \times \{0, 1\} \cup \{s, \infty\}}$ for all measurable $A \subset \{0, 1\}^{\mathbb{Z} \times \{(-\infty, s] \cup [s, \infty)\}}$. Take any $\delta > 0$, then by Birkhoff’s Theorem and Proposition 5.3, we can find $r$ large enough such that $P \left[ \left| r^{-1} \int_0^r f(\mathcal{R}\eta^\prime)dt - \tilde{\Psi}^\rho(f) \right| > \delta \right] < \delta$. For each $t > 0$, denote $\chi_t = 1 \left[ \left| r^{-1} \int_{t+s}^{t+s+r} f(\mathcal{R}\eta^\prime)dw - \tilde{\Psi}^\rho(f) \right| > \delta \right]$. Then by Theorem 1.7, we have

$$\lim_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} E[\chi_i] = P \left[ \left| r^{-1} \int_0^r f(\mathcal{R}\eta^\prime)dt - \tilde{\Psi}^\rho(f) \right| > \delta \right] < \delta.$$
This implies that for any $N$ large enough, we have $\mathbb{P}[\sum_{i=0}^{N-1} X_i > \sqrt{\delta}N] < \sqrt{\delta}$, thus
\[
\mathbb{P} \left[ \left( N\rho \right)^{-1} \int_0^{N\rho + s} f(\mathcal{R}_t') dt - \bar{\Psi}^\rho(f) \right] > \sqrt{\delta} + \delta \right] < \delta,
\]
which implies our conclusion since $\delta > 0$ is arbitrary.

It now remains to prove Proposition 5.3. Our key step is the following coupling of two copies of $\Psi^\rho$.

**Lemma 5.4.** For any $L \in \mathbb{N}$ and $\epsilon > 0$, there exist an integer $M > L$, and a coupling between $\Psi^\rho$ and itself, such that the following is true. Let $\bar{\Psi}^{(1)}$ and $\bar{\Psi}^{(2)}$ be sampled from this coupling, then

1. Restricted to $[-L, L]$, $\bar{\Psi}^{(1)}$ and $\bar{\Psi}^{(2)}$ are independent.

2. With probability $1 - \epsilon$, $\bar{\Psi}^{(1)}$ and $\bar{\Psi}^{(2)}$ have the same number of particles in $[-M, 0]$ and in $[0, M]$, and $\bar{\Psi}^{(1)}$ and $\bar{\Psi}^{(2)}$ are identical on $\mathbb{Z} \setminus [-M, M]$.

Recall that we defined the configuration $\bar{\Psi}'(x)$ for $x > 0$ in terms of two independent collections of i.i.d. Bernoulli($\rho$) random variables $Y_1(x), x \geq 1$ and $Y_2(x), x \geq 1$.

For $x > 0$, define
\[
\bar{Y}_1(x) = \bar{\Psi}'(x) = \begin{cases} 1, & Y_1(x) = 1 \text{ or } x \in \mathcal{E} \\ 0, & Y_1(x) = 0 \text{ and } x \notin \mathcal{E} \end{cases}
\]
\[
\bar{Y}_2(x) = \begin{cases} 0, & Y_2(x) = 0 \text{ or } x \in \mathcal{E} \\ 1, & Y_2(x) = 1 \text{ and } x \notin \mathcal{E} \end{cases}.
\]

We also define
\[
\bar{R}_1(x) = \sum_{y=1}^{x} \bar{Y}_1(y) = R_1(x) + E(x)
\]
\[
\bar{R}_2(x) = \sum_{y=1}^{x} \bar{Y}_2(y) = R_2(x) - E(x).
\]

We have that $\bar{R}_1(x) - \bar{R}_2(x) = 2M(x) - W(x)$, where $W$ and $M$ are defined in (2.1) and (2.2). In particular note that $\bar{R}_1(x) \geq \bar{R}_2(x)$ for all $x$.

Note that $\bar{R}_1(x)$ is the number of particles of $\bar{\Psi}'(x)$ in the interval $[1, x]$. The process $\bar{R}_1$ is certainly not Markov. However, we will exploit the fact that the process $(\bar{R}_1(x), \bar{R}_2(x)), x \geq 0$ is a Markov chain.

Consider the transition function $T : \mathbb{Z}_\leq^2 \times \mathbb{Z}_\geq^2 \to [0, 1]$ defined by
\[
T((a, b), (a + 1, b + 1)) = \rho^2,
\]
\[
T((a, b), (a + 1, b)) = \rho(1 - \rho) \frac{a - b + 2}{a - b + 1},
\]
\[
T((a, b), (a, b + 1)) = \rho(1 - \rho) \frac{a - b}{a - b + 1},
\]
\[
T((a, b), (a, b)) = (1 - \rho)^2.
\]

**Lemma 5.5.** The process $(\bar{R}_1, \bar{R}_2)$ is a Markov chain in $\mathbb{Z}_\geq^2$ with transition probability $T$.  

31
Proof. For any \( x \geq 0 \), we show that

\[
\mathbb{P}\{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, E(x) = h\}
= \rho^{r_1(x) + r_2(x)} (1 - \rho)^{2x - r_1(x) - r_2(x)} \quad (5.1)
\]

for any integers \( \{r_1(y)\}, \{r_2(y)\} \) and \( h \) such that

1. \( r_1(0) = r_2(0) = 0 \),
2. \( r_1(y) - r_1(y - 1), r_2(y) - r_2(y - 1) \in \{0, 1\} \), and \( r_1(y) \geq r_2(y) \) for any \( 1 \leq y \leq x \),
3. \( 0 \leq h \leq r_1(x) - r_2(x) \).

We prove this by induction on \( x \). The base case is trivial, and now we assume that it is true for \( x \), and consider \( x + 1 \).

Note that we have \( h + 1 \in \mathcal{E} \) if the following three conditions all hold: (i) \( M(x) = W(x) \) (i.e. \( \overline{R}_1(x) = \overline{R}_2(x) = E(x) \)); (ii) \( Y_1(x + 1) = 0 \); (iii) \( Y_2(x + 1) = 1 \). In that case we have \( \overline{R}_1(x + 1) = \overline{R}_1(x) + 1, \overline{R}_2(x + 1) = \overline{R}_2(x) \), and \( E(x + 1) = E(x) + 1 \). In any other case we have \( \overline{R}_1(x + 1) = \overline{R}_1(x) + Y_1(x + 1), \overline{R}_2(x + 1) = \overline{R}_2(x) + Y_2(x + 1), \) and \( E(x + 1) = E(x) \).

Denote \( y_1(x + 1) = r_1(x + 1) - r_1(x) \) and \( y_2(x + 1) = r_2(x + 1) - r_2(x) \). From the above transition we have that when \( h \leq r_1(x) - r_2(x) \),

\[
\mathbb{P}\{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, E(x + 1) = h\}
= \mathbb{P}\{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, E(x) = h\}
\times \mathbb{P}[Y_1(x + 1) = y_1(x + 1), Y_2(x + 1) = y_2(x + 1)],
\]

where the second probability on the right-hand side equals \( \rho^{y_1(x + 1) + y_2(x + 1)} (1 - \rho)^{2y_1(x + 1) - y_2(x + 1)} \).

When \( h > r_1(x) - r_2(x) \), we must have that \( h = r_1(x) - r_2(x) + 1 \) and \( y_1(x + 1) = 1, y_2(x + 1) = 0 \), and that

\[
\mathbb{P}\{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, E(x + 1) = h\}
= \mathbb{P}\{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, E(x) = h - 1\}
\times \mathbb{P}[Y_1(x + 1) = 0, Y_2(x + 1) = 1],
\]

where the second probability on the right-hand side equals \( \rho(1 - \rho) \), which also equals \( \rho^{y_1(x + 1) + y_2(x + 1)} (1 - \rho)^{2y_1(x + 1) - y_2(x + 1)} \). Thus by the induction hypothesis (5.1) for \( x \), we get (5.1) for \( x + 1 \).

Finally, by summing over all \( h \), we conclude that

\[
\mathbb{P}\{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, E(x) = h\}
= (r_1(x) - r_2(x) + 1)\rho^{r_1(x) + r_2(x)} (1 - \rho)^{2x - r_1(x) - r_2(x)}.
\]

Using this we conclude that

\[
\mathbb{P}[\overline{R}_1(x + 1) = r_1(x + 1), \overline{R}_2(x + 1) = r_2(x + 1) | \{\overline{R}_1(y)\} = \{r_1(y)\}, \overline{R}_2(y)\} = \{r_2(y)\}, \}
= \frac{r_1(x + 1) - r_2(x + 1) + 1}{r_1(x) - r_2(x) + 1} \rho^{y_1(x) + y_2(x)} (1 - \rho)^{2y_1(x) - y_2(x)},
\]

which implies the conclusion.

We have the following mixing property of this Markov chain.

**Lemma 5.6.** For any \( u, v \in \mathbb{Z}_2 \), we have \( \lim_{n \to \infty} \| T^n(u, \cdot) - T^n(v, \cdot) \|_1 = 0 \).
Proof. Let \((\overline{R}_1^{(1)}, \overline{R}_2^{(1)})\) and \((\overline{R}_1^{(2)}, \overline{R}_2^{(2)})\) be two copies of the Markov chain with transition probabilities \(\mathbf{T}\), starting from \(u\) and \(v\) respectively. For \(i = 1, 2\), denote \(A^{(i)} = \overline{R}_1^{(i)} + \overline{R}_2^{(i)} - x\) and \(B^{(i)} = \overline{R}_1^{(i)} - \overline{R}_2^{(i)}/2\). Then for any \(x \in \mathbb{Z}_{\geq 0}\) we have
\[
\mathbb{P}[A^{(i)}(x) + 1 = A^{(i)}(x) + 1, B^{(i)}(x) + 1 = B^{(i)}(x) | A^{(i)}(x), B^{(i)}(x)] = \rho^2,
\]
\[
\mathbb{P}[A^{(i)}(x) + 1 = A^{(i)}(x) - 1, B^{(i)}(x) + 1 = B^{(i)}(x) | A^{(i)}(x), B^{(i)}(x)] = (1 - \rho)^2,
\]
\[
\mathbb{P}[A^{(i)}(x) + 1 = A^{(i)}(x), B^{(i)}(x) + 1 = B^{(i)}(x) + 1 | A^{(i)}(x), B^{(i)}(x)] = \rho(1 - \rho) \frac{B^{(i)}(x) + 2}{B^{(i)}(x) + 1},
\]
\[
\mathbb{P}[A^{(i)}(x) + 1 = A^{(i)}(x), B^{(i)}(x) + 1 = B^{(i)}(x) - 1 | A^{(i)}(x), B^{(i)}(x)] = \rho(1 - \rho) \frac{B^{(i)}(x)}{B^{(i)}(x) + 1}.
\]

Notice that at each step, for each \(i = 1, 2\), exactly one of \(A^{(i)}\) and \(B^{(i)}\) changes.

We now couple these processes inductively in the following way. Let \(N \in \mathbb{N}\). Suppose that we are given \(A^{(1)}(x), A^{(2)}(x)\) and \(B^{(1)}(x), B^{(2)}(x)\), we choose \(A^{(1)}(x+1), A^{(2)}(x+1)\) and \(B^{(1)}(x+1), B^{(2)}(x+1)\) in the following way. First, we let \(B^{(1)}(x+1) = B^{(1)}(x)\) if and only if \(B^{(2)}(x+1) = B^{(2)}(x)\). In the case where \(B^{(1)}(x+1) = B^{(1)}(x)\), if \(A^{(1)}(x) \neq A^{(2)}(x)\) we take \(A^{(1)}(x+1) = A^{(1)}(x+1)\) and \(B^{(1)}(x+1) = B^{(2)}(x+1)\) independently; and if \(A^{(1)}(x) = A^{(2)}(x)\) we always take \(A^{(1)}(x+1) = A^{(2)}(x+1)\). In the case where \(B^{(1)}(x+1) \neq B^{(1)}(x)\), if \(B^{(1)}(x) = B^{(2)}(x)\) we take \(B^{(1)}(x+1) = B^{(2)}(x+1)\); if \(B^{(1)}(x) \neq B^{(2)}(x)\) and \(\max_{0 \leq y \leq x} B^{(1)}(y) < N\), we couple \(B^{(1)}(x+1)\) and \(B^{(2)}(x+1)\) such that almost surely \(|B^{(1)}(x+1) - B^{(2)}(x+1)| \leq |B^{(1)}(x) - B^{(1)}(x+1)|\); if \(B^{(1)}(x) \neq B^{(2)}(x)\) and \(\max_{0 \leq y \leq x} B^{(1)}(y) \geq N\), we take \(B^{(1)}(x+1)\) and \(B^{(2)}(x+1)\) independently.

Under this coupling, we see that \(A^{(1)}(x) = A^{(2)}(x)\) for all large enough \(x\), since \(A^{(1)}\) and \(A^{(2)}\) are independent random walks until they are equal.

Take any \(\epsilon > 0\). We claim that when \(N\) is large enough depending on \(u, v, \epsilon\), with probability at least \(1 - \epsilon\) we have \(B^{(1)}(x) = B^{(2)}(x)\) for some large enough \(x\), thus for all large \(x\). Let \(x_0 = \min\{x \in \mathbb{Z}_{\geq 0} : B^{(1)}(x) = N\}\). We have \(x_0 < \infty\) almost surely, since \(B^{(1)}\) dominates a simple random walk. From the coupling we must have \(|B^{(1)}(x_0) - B^{(2)}(x_0)| \leq |B^{(1)}(0) - B^{(2)}(0)| \leq \|u - v\|\).

For \(i = 1, 2\), let \(V^{(i)} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}\) be a random walk satisfying \(V^{(i)}(0) = B^{(i)}(x_0)\), and
\[
\mathbb{P}[V^{(i)}(x + 1) = V^{(i)}(x) | V^{(i)}(x)] = \rho^2 + (1 - \rho)^2,
\]
\[
\mathbb{P}[V^{(i)}(x + 1) = V^{(i)}(x) + 1 | V^{(i)}(x)] = \rho(1 - \rho),
\]
\[
\mathbb{P}[V^{(i)}(x + 1) = V^{(i)}(x) - 1 | V^{(i)}(x)] = \rho(1 - \rho).
\]

Also we let \(V^{(1)}\) and \(V^{(2)}\) be independent, until \(V^{(1)}(x_1) = V^{(2)}(x_1)\) for some \(x_1 > 0\), and let \(V^{(1)}(x) = V^{(2)}(x)\) for all \(x > x_1\). For some \(N_1\) large enough (depending on \(u, v, \epsilon\)) we have \(\mathbb{P}[x_1 < N_1] > 1 - \epsilon/2\). We next take \(N\) large enough (depending on \(N_1, \epsilon\)) such that with probability \(1 - \epsilon/2\) we can couple \(V^{(1)}, V^{(2)}\) with \(B^{(1)}, B^{(2)}\), so that \(V^{(1)}(x) = B^{(1)}(x_0 + x)\) and \(V^{(2)}(x) = B^{(2)}(x_0 + x)\) for \(0 \leq x \leq N_1\). Thus the claim follows. Since \(\epsilon\) is arbitrarily taken, we have that the conclusion follows.

For the process \((\overline{R}_1, \overline{R}_2)\) starting from \((0, 0)\), denote \(\mathcal{F}\) as its measure. From the above lemma we could construct a coupling between \(\mathcal{F}\) and itself, as follows.

**Lemma 5.7.** For any \(L \in \mathbb{N}\) and \(\epsilon > 0\), there exist an integer \(M > L\), and a coupling between \(\mathcal{F}\) and itself, such that the following is true. Let \((\overline{R}_1^{(1)}, \overline{R}_2^{(1)})\) and \((\overline{R}_1^{(2)}, \overline{R}_2^{(2)})\) be sampled from this coupling, then

1. Restricted to \([0, L]\), \((\overline{R}_1^{(1)}, \overline{R}_2^{(1)})\) and \((\overline{R}_1^{(2)}, \overline{R}_2^{(2)})\) are independent.
2. $\mathbb{P}[\overline{R}_1(1)(M) = \overline{R}_2(1)(M), \overline{R}_2(1)(M) = \overline{R}_2(2)(M)] > 1 - \epsilon$.

Proof. We construct the coupling by first allowing $(\overline{R}_1(1), \overline{R}_2(1))$ and $(\overline{R}_1(2), \overline{R}_2(2))$ to evolve independently for the first $L$ steps. Then conditioned on $(\overline{R}_1(1)(L), \overline{R}_2(1)(L))$ and $(\overline{R}_1(2)(L), \overline{R}_2(2)(L))$, we couple $(\overline{R}_1(1)(M), \overline{R}_2(1)(M))$ and $(\overline{R}_1(2)(M), \overline{R}_2(2)(M))$ to maximize the probability that they coincide. The conclusion follows from Lemma 5.6 by taking $M$ large enough, since there are only finitely many possible values of $(\overline{R}_1(1)(L), \overline{R}_2(1)(L))$ and $(\overline{R}_1(2)(L), \overline{R}_2(2)(L))$. □

Proof of Lemma 5.2. From the coupling in Lemma 5.1, we get the desired coupling. The configurations of particles to the right and to the left of the origin are independent of each other under $\Psi^\rho$, and the configuration of particles to the right of the origin is a function of $\overline{R}_1$; we can apply an analogous construction for the configuration to the left. □

Proof of Proposition 5.3. We assume the contrary, i.e., $\overline{\eta}$ is not ergodic. Then there is a measurable set $B \subset \{0,1\}^\mathbb{Z}$ invariant under the Markov process, with $0 < \Psi^\rho(B) < 1$. Let $\overline{\eta} \sim \Psi^\rho$. Take $L \in \mathbb{N}$ and we consider the random variable $\chi_L(\overline{\eta}) = \mathbb{P}[\overline{\eta} \in B \mid \{\overline{\eta}(x)\}_{x \in [-L,L]}]$. Note that this a martingale in $L$, and almost surely converges to $\mathbb{E}[\overline{\eta} \in B]$. Thus for any $\epsilon > 0$, we can take $L$ large enough, such that $\mathbb{P}[|\chi_L(\overline{\eta}) - \mathbb{E}[\overline{\eta} \in B]| > \epsilon] < \epsilon$.

For the above $L$ and $\epsilon$, we take $M$ and the coupling as given by Lemma 5.3. Suppose that $\overline{\eta}^{(1)}, \overline{\eta}^{(2)}$ are sampled from this coupling. By the first property of the coupling, and that $\chi_L$ only depends on the configuration in $[-L,L]$, we have

$$\mathbb{P}[\chi_L(\overline{\eta}^{(1)}) > 1 - \epsilon, \chi_L(\overline{\eta}^{(2)}) < \epsilon] > \Psi^\rho(B)(1 - \Psi^\rho(B)) - 2\epsilon.$$ 

Since $\Psi^\rho(\{\eta : \chi_L(\eta) < \epsilon\} \cap B) < \epsilon$ and $\Psi^\rho(B \setminus \{\eta : \chi_L(\eta) > 1 - \epsilon\}) < \epsilon$, we have

$$\mathbb{P}[\overline{\eta}^{(1)} \in B, \overline{\eta}^{(2)} \not\in B] > \Psi^\rho(B)(1 - \Psi^\rho(B)) - 4\epsilon.$$

Using the second property of the coupling, and by taking $\epsilon$ small enough, we have that with probability $\Psi^\rho(B)(1 - \Psi^\rho(B)) - 5\epsilon > 0$, $\overline{\eta}^{(1)} \in B$ and $\overline{\eta}^{(2)} \not\in B$, and $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ are identical on $\mathbb{Z} \setminus [-M,M]$.

We next couple two TASEP starting from $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ at time 0, such that interchanges happen between neighboring sites with the same Poisson clock. With positive probability the following happens: from time 0 to time 1, no exchange happens between sites $-M - 1$, $-M$, and between sites $M$, $M + 1$; and exchanges happen between sites $x, x + 1$, sequentially for $x = -M, \ldots, M - 1$, and repeat this for $2M$ times. Conditioned on $\overline{\eta}^{(1)} \in B$ and $\overline{\eta}^{(2)} \not\in B$, and $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ are identical on $\mathbb{Z} \setminus [-M,M]$, and the above event on the exchanging clocks, at time 1 the two processes starting from $\overline{\eta}^{(1)}$ and $\overline{\eta}^{(2)}$ would be identical. However, as $B$ and $\{0,1\}^\mathbb{Z} \setminus B$ are assumed to be invariant under the Markov process (of TASEP as seen from a hole-particle pair), we get a subset of $\{0,1\}^\mathbb{Z}$ with positive $\Psi^\rho$ measure, and is contained (up to a measure zero set) in both $B$ and $\{0,1\}^\mathbb{Z} \setminus B$. This is a contradiction. □

6 From semi-infinite geodesics to point-to-point geodesics

In this section we fix $\rho \in (0,1)$. From the convergence along semi-infinite geodesics (Theorem 5.1), we deduce the following result, which is an extension of Theorem 1.1.

**Theorem 6.1.** Let $\{b_n\}_{n \in \mathbb{N}}$ be a sequence of integers such that $\limsup_{n \to \infty} n^{-2/3}|b_n| < \infty$. Then in probability we have $\mu_{0,(n,b_n),\rho} \to \nu^\rho$ as $n \to \infty$. 

34
The idea is to construct semi-infinite geodesics which largely cover the finite ones.

Recall $G^\rho$, the Busemann function in the direction of $\rho$. We consider the following event: for any $b \in \mathbb{Z}$ with $h^{-1}n^{2/3} < |b| < hn^{2/3}$, there is $G^\rho((0,b)) + b(\rho^{-1} - (1 - \rho)^{-1}) > h^{-1}n^{1/3}$; and for $b \in \mathbb{Z}$ with $|b| \geq hn^{2/3}$, there is $G^\rho((0,b)) + b(\rho^{-1} - (1 - \rho)^{-1}) > -bn^{-1/3}$. Denote this event by $\mathcal{E}_{h,n}$, we show that its probability is lower bounded uniformly in $n$.

**Lemma 6.2.** For any $h > 0$, there is $\delta > 0$ such that $\mathbb{P}(\mathcal{E}_{h,n}) > \delta$ for all $n$ large enough.

**Proof.** Denote $F(b) = -G^\rho((0,b)) - b(\rho^{-1} - (1 - \rho)^{-1})$, then $F$ is a (two-sided) random walk, where each step is centered with exponential tail. By independence of all the steps, we have

$$\mathbb{P}[\mathcal{E}_{h,n}] \geq \mathbb{P}\left[ \max_{h^{-1}n^{2/3} < |b| < hn^{2/3}} F(b) < -hn^{1/3} \right]$$

$$\times \mathbb{P}\left[ \max_{b \geq hn^{2/3}} F(b) - F(hn^{2/3}) - bn^{-1/3} < hn^{1/3} \right]$$

$$\times \mathbb{P}\left[ \max_{b \leq -hn^{2/3}} F(b) - F(-hn^{2/3}) + bn^{-1/3} < hn^{1/3} \right].$$

As the process converges to Brownian motion (weakly in the uniform topology) in compact sets, the first factor in the right hand side is lower bounded by a constant. We next lower bound the factor in the second line, and the third line could be lower bounded in a similar way. It is at least

$$\mathbb{P}\left[ \max_{b \in \mathbb{N}} F(b) - bn^{-1/3} < hn^{1/3} \right] \geq \mathbb{P}\left[ \max_{0 \leq b \leq I n^{2/3}} F(b) - bn^{-1/3} < hn^{1/3} \right]$$

$$- \sum_{i=I}^{\infty} \mathbb{P}\left[ \max_{in^{2/3} < b \leq (i+1)n^{2/3}} F(b) \geq (i + h)n^{1/3} \right].$$

where $I$ is a large constant. As $n \to \infty$, the first term in the right hand side converges to the probability that a Brownian motion is bounded below a (sloped) line in $[0,I]$, and such probability is lower bounded uniformly in $I$. For the summation in the second line, the term for each $i$ is upper bounded by

$$\mathbb{P}\left[ F([in^{2/3}]) \geq (i + h)n^{1/3}/2 \right] + \mathbb{P}\left[ \max_{0 \leq b \leq [n^{2/3}]} F(b) \geq (i + h)n^{1/3}/2 \right]$$

$$\leq \mathbb{P}\left[ F([in^{2/3}]) \geq (i + h)n^{1/3}/2 \right] + 2\mathbb{P}\left[ F([n^{2/3}]) \geq (i + h)n^{1/3}/2 \right],$$

where the inequality is by reflection principle. By a Bernstein type estimate for sum of independent variables with exponential tails, this could be bounded by $Ce^{-ci}$ for some $C, c > 0$, independent of $n$. Thus by taking $I$ large enough the conclusion follows. \hfill \qed

**Proof of Theorem 6.1.** Take any $s \in \mathbb{N}$ and $f : \mathbb{R}[-s,s]^2 \to \mathbb{R}$ with $0 \leq f \leq 1$, regarded as a function on $\mathbb{R}^2$, we shall prove that $\mu_{(n,b_n)}(f) \to \nu^\rho(f)$ in probability.

In this proof we use $C, c$ to denote large or small universal positive constants, whose value may change from line to line. We then have that $|b_n| < Cn^{2/3}$ for any $n \in \mathbb{N}$. For simplicity of notation we denote $T^*_{u,v} = T_{u,v} - \xi(v)$ for any vertices $u \leq v$.

We denote $A_{h,n}$ as the following event: for any $b \in \mathbb{Z}$, we have

- $T^*_{0,(n,b)} + b(\rho^{-1} - (1 - \rho)^{-1}) > ET_{0,(n,0)} - hn^{1/3}/2$, if $|b - b_n| \leq h^{-1}n^{2/3}$;

- $T^*_{0,(n,b)} + b(\rho^{-1} - (1 - \rho)^{-1}) < ET_{0,(n,0)} + hn^{1/3}/2$, if $h^{-1}n^{2/3} < |b - b_n| < hn^{2/3}$;

- $T^*_{0,(n,b)} + b(\rho^{-1} - (1 - \rho)^{-1}) < ET_{0,(n,0)} + hn^{1/3}/2 - bn^{-1/3}$, if $|b - b_n| \geq hn^{2/3}$.
By splitting \( \mathbb{L}_n \) into segments of length \( n^{2/3} \), and using Proposition 2.3 and Theorem 2.38 we have \( \mathbb{P}[^{A_h,n}_{a,b}] > 1 - e^{-cn^2} \), for \( n \) and \( h \) large enough.

We also denote \( \mathcal{B}_{h,n} \) as the following event:

\[
\Gamma_{0,A_{h,n} \cap \mathbb{L}_n(1 \pm h^{-1})} \cap \mathbb{L}_n \cap \mathbb{L}_n(1 \pm h^{-1}) = \Gamma_{0,\mathbb{L}_n} \cap \mathbb{L}_n(1 \pm h^{-1}).
\]

By Proposition 2.5 we have \( \mathbb{P}[\mathcal{B}_{h,n}] > 1 - Ch^{-1/3} \), for \( h < cn^{2/3} \) and \( h, n \) large enough.

Denote \( \langle n, b_n \rangle_{\rho} \) as the intersection of \( \Gamma_{0,\mathbb{L}_n} \) with \( \mathbb{L}_n \). By Theorem 5.1 we have \( \mu_{\langle n, b_n \rangle_{\rho}} = \nu_{\rho} \) in probability; i.e. we can find a sequence \( \{\epsilon_n\}_{n \in \mathbb{Z}_{\geq 0}} \), such that \( \epsilon_n \to 0 \), and

\[
\mathbb{P}[^{\mu_{\langle n, b_n \rangle_{\rho}}}_{\nu_{\rho}}(f) - \nu_{\rho}(f) < \epsilon_n] \to 1,
\]

We denote \( \mathcal{E}_{h,n}^\prime \) as \( \mathcal{E}_{h,n} \) translated by \( \langle n, b_n \rangle_{\rho} \). Let \( \{h_n\}_{n \in \mathbb{Z}_{\geq 0}} \) be a sequence with \( h_n \to \infty \). By Lemma 6.2 we can choose this sequence to make \( \mathbb{P}[\mathcal{E}_{h,n}^\prime] = \mathbb{P}[\mathcal{E}_{h,n}] \) decay to zero slowly enough, such that

\[
\mathbb{P}[^{\mu_{\langle n, b_n \rangle_{\rho}}}_{\nu_{\rho}}(f) - \nu_{\rho}(f) < \epsilon_n | \mathcal{E}_{h,n}^\prime] \to 1.
\]

Since \( \mathbb{P}[A_{h,n}] > 1 - e^{-cn^2} \), \( \mathbb{P}[\mathcal{B}_{h,n}] > 1 - Ch^{-1/3} \), and \( A_{h,n}, \mathcal{B}_{h,n} \) are independent of \( \mathcal{E}_{h,n}^\prime \), we have

\[
\mathbb{P}[A_{h,n}, \mathcal{B}_{h,n}, \mu_{\langle n, b_n \rangle_{\rho}}(f) - \nu_{\rho}(f) < \epsilon_n | \mathcal{E}_{h,n}^\prime] \to 1.
\]

However, under \( A_{h,n} \cap \mathcal{B}_{h,n} \cap \mathcal{E}_{h,n}^\prime \), there must be \( |b_n - b_n'| < h_n^{-1}n^{2/3} \), and \( \Gamma_{0,\langle n, b_n \rangle_{\rho}} \cap \mathbb{L}_n \cap \mathbb{L}_n(1 \pm h^{-1}) = \Gamma_{0,\langle n, b_n \rangle_{\rho}} \cap \mathbb{L}_n \cap \mathbb{L}_n(1 \pm h^{-1}) \), by monotonicity of geodesics. Thus \( \mu_{\langle n, b_n \rangle_{\rho}}(f) - \nu_{\rho}(f) < \epsilon_n \) implies that

\[
|\mu_{\langle n, b_n \rangle_{\rho}}(f) - \nu_{\rho}(f)| < \epsilon_n + h_n^{-1}.
\]

So we have

\[
\mathbb{P}[^{\mu_{\langle n, b_n \rangle_{\rho}}}_{\nu_{\rho}}(f) - \nu_{\rho}(f) < \epsilon_n | h_n^{-1} | \mathcal{E}_{h,n}^\prime] \to 1.
\]

Note that for \( v \in \Gamma_{0,\langle n, b_n \rangle_{\rho}} \) with \( d(v) < n - 4s \), \( f(v) \) is independent of \( \mathcal{E}_{h,n}^\prime \). Thus we conclude that \( \mathbb{P}[\mu_{\langle n, b_n \rangle_{\rho}}(f) - \nu_{\rho}(f) < \epsilon_n + h_n^{-1} + 2s/n] \to 1 \). This implies that \( \mu_{\langle n, b_n \rangle_{\rho}}(f) \to \nu_{\rho}(f) \) in probability.

\[\square\]

7 Parallelogram uniform covering

The goal of this section is to prove the following upgraded version of Theorem 6.1. It will be the key input for the next two sections.

**Proposition 7.1.** For any \( \rho \in (0, 1) \), \( h > 0 \), \( s \in \mathbb{N} \), and any bounded function \( f : \mathbb{R}^{[-s,s]^2} \to \mathbb{R} \), regarded as a function on \( \mathbb{R}^{2} \), we have

\[
\max_{a,b \in \mathbb{Z}, |a|, |b| < nhn^{2/3}} \mu_{\langle n, b \rangle_{\rho}}(f), \min_{a,b \in \mathbb{Z}, |a|, |b| < nhn^{2/3}} \mu_{\langle n, b \rangle_{\rho}}(f) \to \nu_{\rho}(f),
\]

in probability.

The idea of the proof of this proposition is to take a finite family of geodesics, and show that for any geodesic from some \( \langle 0, a \rangle_{\rho} \) to \( \langle n, b \rangle_{\rho} \), with \( |a|, |b| < nhn^{2/3} \), it could be covered by one of the geodesics in this family. For simplicity of notation, below we write the proof for \( \rho = 1/2 \), while the more general case follows essentially verbatim.

We collect some ingredients in the proof. The first one concerns continuity of the function \( (a, b) \mapsto T_{\langle 0, a \rangle_{\rho}, \langle n, b \rangle_{\rho}} \).
denotes the intersection of \( L_{u,v} \). For \( d \) large enough, and \( \ad \) small enough parameter \( c \) when \( \gamma \), there exists a directed path \( \Gamma \) intersections of \( \Gamma_{u,v} \) with width \( t\ell^{2/3} \) below \( L_{\ell} \) or above \( L_{n-\ell} \).

\[ \Gamma_{u,v} \]

Figure 8: The complement of the event \( T_{l,t}^0(n,b) \): the geodesic \( \Gamma_{0,(n,b)} \) is restricted within the green boxes with width \( t\ell^{2/3} \), below \( L_{\ell} \) or above \( L_{n-\ell} \).

**Lemma 7.2.** There exist constants \( C, c \) such that the following is true. For \( h > 0, \theta < 1 < t \), we have

\[
P \left[ \max_{|a|,|a'|,|b|,|b'|<Chn^{2/3}} \left| T_{(0,a),(n,b)} - T_{(0,a'),(n,b')} \right| > t\theta^{1/2} - 0.01 n^{1/3} + C \theta n^{1/3} \right] < Ce^{-ct}
\]

when \( n \) is large enough (depending on \( h, \theta, t \)).

We next bound the transversal fluctuation of geodesics. Recall the anti-diagonal distance \( ad(u) = a - b \), for any vertex \( u = (a, b) \). For vertices \( u < v \), and \( 0 \leq l \leq d(v) - d(u), t > 1 \), let \( T_{l,t}^u \) be the following event: there exists \( m \in \mathbb{Z} \) with \( d(u) \leq 2m \leq d(u) + 2l \), such that if \( w \) denotes the intersection of \( L_m \) with \( \Gamma_{u,v} \), then \( |ad(w) - ad(u)| \geq 2t\ell^{2/3} \); or there exists \( m \in \mathbb{Z} \) with \( d(v) - 2l \leq 2m \leq d(v) \), and \( |ad(w) - ad(v)| \geq 2t\ell^{2/3} \) where \( w \) denotes the intersection of \( L_m \) with \( \Gamma_{u,v} \) (see Figure 3).

**Lemma 7.3.** For \( h > 0 \), there exist constants \( C, c \) such that the following is true. For any \( 0 \leq l \leq n \) large enough, and \( |b| < hn^{2/3}, t > 1 \), we have \( P[T_{l,t}^0(n,b)] < Ce^{-ct} \).

The proof of the lemma is by applying Corollary 2.10 twice, and we omit the proof here.

Our next lemma wishes to establish that, for a geodesic and a path with a good weight, it is unlikely for them being disjoint and staying together for a while.

For any vertices \( u < v \), and \( M, l \in \mathbb{N}, m \in \mathbb{Z} \) with \( d(u) \leq 2m < 2m + 2Ml \leq d(v) \), and a small enough parameter \( c_0 > 0 \), we denote \( D_{M,l,m} \) as the following event (see Figure 9): (1) for the intersections of \( \Gamma_{u,v} \) with \( L_m \) and \( L_{m+Ml} \), denoted as \( u', v' \), we have \( |ad(u') - ad(v')| < 2M^{5/6}l^{2/3} \); (2) there exists a directed path \( \gamma \) from \( L_m \) to \( L_{m+Ml} \), such that

- \( \gamma \) is disjoint from \( \Gamma_{u,v} \).
- The weight of \( \gamma \) (i.e. \( T(\gamma) \)) is at least \( 4Ml - c_0 Ml^{1/3} \).

37
For each $i = 0, 1, \ldots, M$, \[ \| \Gamma_{u,v} \cap \mathbb{L}_{m+i} - \gamma \cap \mathbb{L}_{m+i} \|_1 < 2c_0 l^{2/3}. \]

**Lemma 7.4.** For $h > 0$, there exist constants $C, c$ such that the following is true. If $l > C$, $c_0 < c$, $|b| < hn^{2/3}$, and $0 \leq m < m + M l \leq n$, we have $P[D_{u,v}^{0,\langle n, b \rangle}] < Ce^{-cM}$.

The last ingredient we need is to bound the probability of multiple peaks in the sum of two independent point-to-line profiles.

We denote $T_{u,v}^* = T_{u,v} - \xi(v)$ for any vertices $u \leq v$; i.e. removing the weight of the last vertex in the geodesic $\Gamma_{u,v}$. For any vertices $u < v$, and $m \in \mathbb{Z}$ with $d(u) \leq 2m \leq d(v)$, and $\eta, t > 0$, we denote $\mathcal{M}_{u,v}^{0,\langle n, b \rangle}$ as the following event: there exist $-g \leq b_1 < b_2 < b_3 < b_4 < b_5 < b_6 \leq g$, with $b_2 - b_1, b_3 - b_2, b_4 - b_3, b_5 - b_4, b_6 - b_5 \geq \eta$, such that $T_{u,v} = T_{u,\langle m, b_i \rangle} + T_{\langle m, b_i \rangle, v}$, and $T_{u,v} < T_{u,\langle m, b_i \rangle} + T_{\langle m, b_i \rangle, v} + t\eta^{1/2}, \forall i = 2, 3, 4, 5, 6$.

**Lemma 7.5.** For $h > 0$, there exist constants $C, c$ such that the following is true. For any $\theta > 0$, $0 < t < 1$, $h^{-1} < \alpha < 1 - h^{-1}$, $|\beta| < h$, we have $P[\mathcal{M}_{\theta n^{2/3}, t, \langle \alpha n \rangle, 2\eta n^{2/3}}^0] < Ct^{5-0.01}$, for $n$ large enough depending on $h, \theta, t, \alpha, \beta$.

We assume these lemmas and prove Proposition 7.1. We shall take a large finite family of geodesics, and assume there is a geodesic that is not covered by them. Our strategy is to show that this has one of the following two implications: either there are two geodesics stay close for a while; or there are several (almost) geodesics, whose end points are close, but their intersections with some line $\mathbb{L}_m$ are far from each other, implying ‘multiple peaks’. Each of these two scenarios is unlikely to happen, using the lemmas above.

We set up the events to be used in the proof of Proposition 7.1. From now on we take parameters $\delta_0 < \delta_1 < \delta_2 < \delta_3 < \delta_4 < \delta_5 < \delta$, whose values are to be determined. We fix the parameter $h$, and let $n$ be large enough depending on all these parameters. Without lose of generality we assume that (by
our choice of parameters and \( n \) \( \delta_0 n, \delta_0 n^{2/3}, h n^{2/3}, h^{-1} n, \delta^{-1} \), and each \( \delta_i^{-1} \delta_{i+1} \) for \( i \in \{0, 1, 2, 3, 4\} \) are integers.

We take two families of vertices \( \mathcal{P}_1 := \{ i \delta_0 n, j \delta_0 n^{2/3} : i, j \in \mathbb{Z}, 0 \leq i \leq \delta_0^{-1}/3, |j| < 4 \delta_0^{-1}\} \); and \( \mathcal{P}_2 := n - \mathcal{P}_1 \). We shall show that with high probability, each geodesic \( \Gamma_{(0,a),(n,b)} \) for some \( |a|, |b| < h n^{2/3} \) is largely covered by a geodesic between vertices in \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \).

Below we use \( C, c > 0 \) to denote large and small constants, which can only depend on \( h \), and the values may change from line to line to line. Consider the following events.

- Let \( \mathcal{T} \) be the union of \( \mathcal{T}_{L,\delta^{-1}} \), for all \( u \in \mathcal{P}_1, v \in \mathcal{P}_2 \), and \( l \in \delta_0 n \mathbb{Z}, 0 \leq l < d(v) - d(u) \). By Lemma \ref{lem:3} we have \( \mathbb{P}[\mathcal{T}] < C \delta_0^{-5} e^{-c \delta^{-1}} \).

- Let \( \mathcal{T}' = \mathcal{T}^{(0,h n^{2/3}),\langle n,h n^{2/3}\rangle} \cup \mathcal{T}^{(0,-h n^{2/3}),\langle n,-h n^{2/3}\rangle} \cup \mathcal{T}^{(0,3 h n^{2/3}),\langle n,3 h n^{2/3}\rangle} \cup \mathcal{T}^{(0,-3 h n^{2/3}),\langle n,-3 h n^{2/3}\rangle} \). Then by Lemma \ref{lem:3} we can make \( \mathbb{P}[\mathcal{T}'] \) arbitrarily small by taking \( h \) large.

- Let \( \mathcal{F} \) be the event where
  \[ |T^{(i \delta_0 n, a),(j \delta_0 n, b)} - T^{(i \delta_0 n, a'),(j \delta_0 n, b')}| > \delta_0^{1/2 - 0.02} n^{1/3} \]

for some integers \( 0 \leq i < j \leq \delta_0^{-1} \), and \( |a|, |a'|, |b|, |b'| \leq 2 h n^{2/3} \) with \( |a - a'|, |b - b'| \leq \delta_0 n^{2/3} \). By Lemma \ref{lem:2} we have \( \mathbb{P}[\mathcal{F}] < C \delta_0^{-2} e^{-c \delta^{-0.01}} \).

- Let \( \mathcal{D} \) be the union of \( \mathcal{D}^{(0,\delta^{-7} l, m)} \) for all \( u \in \mathcal{P}_1, v \in \mathcal{P}_2, l \in \{ \delta_0 n : i = 1, 2, 3, 4, 5 \}, m \in \delta_0 n \mathbb{Z} \), such that \( d(u) < 2 m < 2 m + 2 \delta^{-7} l \leq d(v) \). By Lemma \ref{lem:4} we have \( \mathbb{P}[\mathcal{D}] < C \delta_0^{-5} e^{-c \delta^{-7}} \).

- Let \( \mathcal{H} \) denote the event where there exists some \( m \in \delta_0 n \mathbb{Z}, 0 \leq m \leq n \), and \( l \in \{ \delta_0 n : i = 1, 2, 3, 4, 5 \}, |a|, |b| < 4 h n^{2/3} \), \( |a - b| < \delta_0^{-0.2} n^{2/3} \), such that
  \[ T^{(m,a),(m + \delta^{-7} l, b)} < 4 \delta^{-7} l - c_0 \delta^{-0.2} n^{1/3} \]

where \( c_0 \) is the same as in the event \( \mathcal{D} \). By applying Lemma \ref{lem:4} via splitting the lines \( \mathbb{L}_m \) and \( \mathbb{L}_{m + \delta^{-7} l} \) into segments of length \( \delta_0 n^{2/3} \), we have \( \mathbb{P}[\mathcal{H}] < C \delta_0^{-3} e^{-c \delta^{-8}} \).

- Let \( \mathcal{M} \) be the union of \( \mathcal{M}^{(0,\delta^{-1} h n^{2/3})} \) for all \( u \in \mathcal{P}_1 \cap \mathbb{L}_0, v \in \mathcal{P}_2 \cap \mathbb{L}_m \), and \( \alpha \in \delta_1 \mathbb{Z} \) with \( h^{-1} < \alpha < 1 - h^{-1} \), and \( c_0 \) is the same as in the event \( \mathcal{D} \). By Lemma \ref{lem:5} we have \( \mathbb{P}[\mathcal{M}] < C \delta_0^{-3} e^{-c \delta^{-8}} \).

**Proof of Proposition \ref{prop:7.1}**. As stated above we write the proof for \( \rho = 1/2 \) for simplicity of notation. We also assume that \( h \in \mathbb{N} \), since the result for a larger \( h \) implies it for a smaller \( h \).

We denote \( \mathcal{E} := \mathcal{T}^c \cap \mathcal{T'}^c \cap \mathcal{F}^c \cap \mathcal{D}^c \cap \mathcal{H}^c \cap \mathcal{M}^c \).

**Claim.** Under \( \mathcal{E} \) the following holds: for any \( |a|, |b| < h n^{2/3} \), there exist \( u \in \mathcal{P}_1 \) and \( v \in \mathcal{P}_2 \), with \( d(u) < 4 h^{-1} n \) and \( d(v) > (1 - 4 h^{-1}) n \), such that

\[ \mathbb{L}_{2 h^{-1} n} \cap \Gamma_{u,v} = \mathbb{L}_{2 h^{-1} n} \cap \Gamma_{(0,a),(n,b)}, \mathbb{L}_{(1-2 h^{-1}) n} \cap \Gamma_{u,v} = \mathbb{L}_{(1-2 h^{-1}) n} \cap \Gamma_{(0,a),(n,b)}. \]

We first deduce the conclusion assuming this claim. For any fixed set of parameters, by Theorem \ref{thm:1} and sending \( n \to \infty \), we have in probability

\[ \max_{u \in \mathcal{P}_1, v \in \mathcal{P}_2} |\mu_{u,v}(f) - \nu^{1/2}(f)| \to 0. \]

Thus since \( f \) is a function on \( \mathbb{R}^{[-s,a]} \), for \( n \) such that \( \delta_0 n, \delta_0 n^{2/3}, h n^{2/3}, h^{-1} n \in \mathbb{Z} \), as \( n \to \infty \) we have

\[ \mathbb{P}[\mathcal{E}, \max_{a,b \in \mathbb{Z}, |a|, |b| < h n^{2/3}} |\mu_{(0,a),(n,b)}(f) - \nu^{1/2}(f)| > 10 h^{-1} ||f||_{\infty}] \to 0. \quad (7.1) \]
For all $n \in \mathbb{N}$, denote $n' \leq n$ be the largest number such that $\delta_0 n', \delta_0 n'^{2/3}, h n'^{2/3}, h^{-1} n' \in \mathbb{Z}$, and $E'$ as $E$ for $n'$ instead of $n$. Then as $n \to \infty$ we have $(n-n')/n \to 1$, and

$$\mathbb{P}\left[ E', \left( \mathcal{T}_{n-n', h n'^{2/3}} \cup \mathcal{T}_{n-n', h n'^{2/3}} \right)^c \right],$$

$$\max_{a, n, b, c < h n'^{2/3}/2} |\mu_{(a, b)}(f) - \nu^{1/2}(f)| > 11 h^{-1} \| f \|_\infty \to 0. \quad (7.2)$$

This is because when $n$ is large enough, using the second event in the first line and monotonicity of geodesics, for each $|a|, |b| < h n'^{2/3}/2$, the geodesic $\Gamma_{(a, b)}$ must intersect $\mathbb{L}_n'$ at $\langle n', b' \rangle$ for some $b'$ satisfying $|b'| < h n'^{2/3}$. Then using (7.2) we get (7.3).

We next choose the parameters. We take $L \in \mathbb{N}$, and let $\delta = L - 1$, and $\delta_i = L - 100^{i-1}$ for $i = 0, 1, 2, 3, 4, 5$. From the above discussions, by first taking $h$ large enough, then taking $L$ large (depending on $h$), we could make $\mathbb{P}[E^c]$ arbitrarily small. Using this with Lemma 7.3 from (7.2), we get for any $\epsilon > 0$ we could take $h$ large enough, such that

$$\limsup_{n \to \infty} \max_{a, b, c < h n'^{2/3}/2} |\mu_{(a, b)}(f) - \nu^{1/2}(f)| > 11 h^{-1} \| f \|_\infty < \epsilon,$$

which implies the conclusion.

Now we prove the above claim. Assume that $E$ holds, and fix $a, b$ such that $|a|, |b| < h n'^{2/3}$. Then by $\mathcal{T}^c$, and monotonicity of geodesics, the geodesic $\Gamma_{(a, b)}$ is contained in the rectangle whose four vertices are $\langle 0, 2 h n'^{2/3} \rangle$, $\langle 0, -2 h n'^{2/3} \rangle$, $\langle n, 2 h n'^{2/3} \rangle$, $\langle n, -2 h n'^{2/3} \rangle$.

Let $b^+$ be the smallest number with $b^+ \in \delta_0 n'^{2/3} \mathbb{Z}$ and $b^+ \geq b$. We first show that we can find $u^* \in \mathbb{P}_1$ with $d(u^*) < 4 h^{-1} n$, such that there exists $u \in \Gamma_{(a, b)}$ with $d(u) = d(u^*)$ and $d(u) \leq d(u^*)$ and $a u^+_{\langle n, b^+ \rangle} \cap \Gamma_{(a, b)}$ intersects $\Gamma_{(a, b)}$ before $\mathbb{L}_{2 h^{-1} n}$.

For this we argue by contradiction, and assume that no such $u^*$ exists. We take $a_5$ as the smallest number such that $a_5 \in \delta_5 \mathbb{Z}$ and $a_5 > h^{-1}$, and take $u_5 \in \mathbb{P}_1 \cap \mathbb{L}_{a_5 n}$, being the first one on or to the right of $\Gamma_{(a, b)}$. In other words, we take $u_5 = (a_5 n, a_5^+)$, where $a_5$ is the smallest number in $\delta_0 n'^{2/3} \mathbb{Z}$ such that $2 a_5 \geq \delta_0 \Gamma_{(a, b)} \cap \mathbb{L}_{a_5 n}$. Then we have that $|a_5| \leq 2 h n'^{2/3}$. Consider the path $\Gamma_{u_5, (n, b^+)}$. Again by $\mathcal{T}^c$ and monotonicity of geodesics, this path is restricted within the rectangle whose four vertices are $\langle 0, 4 h n'^{2/3} \rangle$, $\langle 0, -4 h n'^{2/3} \rangle$, $\langle n, 4 h n'^{2/3} \rangle$, $\langle n, -4 h n'^{2/3} \rangle$. Suppose it intersects $\mathbb{L}_{(a_5 + \delta_5 n_5)}$ at vertex $u_5 = (a_5 n, a_5^+)$. Then $a_5^+ \leq 4 h n'^{2/3}$. By $\mathcal{T}^c$ we have that

$$|a_5 - a_5^+| < \delta^{-1} (\delta^{-1} \delta_5 n_5)^{2/3} \leq \delta^{-7} \delta_5 n_5 \frac{2 c_0 (\delta_5 n_5)^{2/3}}{2 c_0 (\delta_5 n_5)^{2/3}} < \delta^{-6} \delta_5 n_5^{2/3}.$$

Then we have $\mathcal{T}_{u_5, w_5} \geq 4 \delta^{-5} \delta_5 n_5 - c_0 \delta^{-6} (\delta_5 n_5)^{1/3}$ by $\mathcal{H}^c$. Note that for the path $\Gamma_{u_5, (n, b^+)}$, by the assumption above, it is disjoint from $\Gamma_{(a, b)}$ before $\mathbb{L}_{2 h^{-1} n}$. As $a_5 + \delta^{-7} \delta_5 < 2 h^{-1}$, we have that $\Gamma_{u_5, w_5}$ is disjoint from $\Gamma_{(a, b)}$. By $\mathcal{D}^c$, the event $\mathcal{D}_{\delta^{-7} \delta_5 n_5} \cap \mathbb{L}_{a_5 n}$ does not happen, so there must exist $0 \leq j_5 \leq \delta^{-5}$, such that for the intersections of $\mathbb{L}_{(a_5 + j_5 \delta_5 n_5)}$ with $\Gamma_{(a, b)}$ and $\Gamma_{u_5, w_5}$, their $\| \cdot \|_1$ distance is at least $2 c_0 (\delta_5 n_5)^{2/3}$.

We next take $a_4 = a_5 + j_5 \delta_5$, and take $u_4 \in \mathbb{P}_1 \cap \mathbb{L}_{a_4 n}$, being the first one on or to the right of $\Gamma_{(a, b)}$. Using the same arguments we could find $0 \leq j_4 \leq \delta^{-7}$, such that for the intersections of $\mathbb{L}_{(a_4 + j_4 \delta_5 n_5)}$ with $\Gamma_{(a, b)}$ and $\Gamma_{u_4, (n, b^+)}$, their $\| \cdot \|_1$ distance is at least $2 c_0 (\delta_4 n_4)^{2/3}$. Similarly we can find $0 \leq j_2, j_3 \leq \delta^{-7}$, and $a_3 = a_4 + j_2 \delta_4$, $a_2 = a_3 + j_3 \delta_3$, $a_1 = a_2 + j_2 \delta_2$, and vertices $u_3, u_2, u_1$. For the intersections of $\mathbb{L}_{(a_1 + j_3 \delta_3 n_5)}$ with $\Gamma_{(a, b)}$ and $\Gamma_{u_1, (n, b^+)}$, their $\| \cdot \|_1$ distance is at least $2 c_0 (\delta_1 n_1)^{2/3}$, for $i = 1, 2, 3$ (see Figure 10).

Take $a_0 = a_1 + j_1 \delta_1$. Suppose that the intersections of $\mathbb{L}_{a_0 n}$ with $\Gamma_{(a, b)}$ and $\Gamma_{u_i, (n, b^+)}$ are $\langle a_0 n, b_0 \rangle$ and $\langle a_0 n, b_i \rangle$, for $i = 1, 2, 3, 4, 5$. By $\mathcal{T}^c$, and considering $\Gamma_{(a, b)}$ above $\mathbb{L}_{a_0 n}$ and $\Gamma_{u_i, (n, b^+)}$, we have $b_i - b_0 \leq \delta_0 n_0^{2/3} + 2 \delta^{-1} (a_0 - a_i)^{2/3} n_0^{2/3}$; and considering $\Gamma_{(a, b)}$ and $\Gamma_{u_i, (n, b^+)}$
above \( \mathbb{L}_{\alpha_1-1} \), we have \( b_i - b_0 \geq c_0(\delta_1 n)^{2/3} - 2\delta^{-1}(\alpha_0 - \alpha_i) n^{2/3} \). By the choice of our parameters we have \( \alpha_0 - \alpha_i \leq \delta^{-7} \sum_{i=1}^i \delta_i' < 2\delta^{-7}\delta_i \). Then we have
\[
c_0(\delta_i n)^{2/3} - 2\delta^{-1}(2\delta^{-7}\delta_i-1)n^{2/3} \leq b_i - b_0 \leq \delta_0 n^{2/3} + 2\delta^{-1}(2\delta^{-7}\delta_i-1)n^{2/3},
\]
and we get that \( b_i - b_{i-1} \geq c_0(\delta_i n)^{2/3} \), for each \( i = 1, 2, 3, 4, 5 \). As \( |b_0| \leq 2hn^{2/3} \) by \( T^\infty \), we have that \(-2hn^{2/3} \leq b_0 < b_5 \leq 4hn^{2/3}\).

Let \( a^-, b^- \) be the largest numbers with \( a^-, b^- \in \delta_0 n^{2/3}\mathbb{Z} \) and \( a^- \leq a, b^- \leq b \). By using \( F^c \) repeatedly, we have
\[
T^\infty (0, a^-), (\alpha_0, b_0) + T^\infty (\alpha_0, b_0), (n, b^-)
\geq T^\infty (0, a), (\alpha_0, b_0) + T^\infty (\alpha_0, b_0), (n, b) - 2\delta_0^{1/2-0.02} n^{1/3}
= T^\infty (0, a), (n, b) - 2\delta_0^{1/2-0.02} n^{1/3}
\geq T^\infty (0, a^-), (n, b^-) - 3\delta_0^{1/2-0.02} n^{1/3};
\]

Figure 10: An illustration of the geodesics \( \Gamma_{u_i, (n, b^+)} \) for \( i = 5, 4, 3 \). Their intersections with \( \mathbb{L}_{\alpha_0n} \) are separated by \( c_0(\delta_1 n)^{2/3} \).
and for each $i = 1, 2, 3, 4, 5$, let $u'_i$ be the intersection of $L_{\alpha \alpha n}$ with $\Gamma_{(0, a), (n, b)}$, we have

$$T^{*}_{(0, a^-), (\alpha \alpha n, b_i)} + T^{*}_{(\alpha \alpha n, b_i), (n, b^-)}$$

$$\geq T^{*}_{(0, a^-), (\alpha \alpha n, b_i)} + T^{*}_{(\alpha \alpha n, b_i), (n, b^+)} - 2(\delta_0^{1/2} - 0.02 n^{1/3})$$

$$= T^{*}_{(0, a^-), u'_i} + T^{*}_{u'_i, (\alpha \alpha n, b_i)} + T^{*}_{(\alpha \alpha n, b_i), (n, b^+)} - 2(\delta_0^{1/2} - 0.02 n^{1/3})$$

$$\geq T^{*}_{(0, a^-), u'_i} + T^{*}_{u'_i, (\alpha \alpha n, b_i)} + T^{*}_{(\alpha \alpha n, b_i), (n, b^+)} - 3(\delta_0^{1/2} - 0.02 n^{1/3})$$

$$= T^{*}_{(0, a^-), u'_i} + T^{*}_{u'_i, (n, b^+)} - 3(\delta_0^{1/2} - 0.02 n^{1/3})$$

$$\geq T^{*}_{(0, a^-), u'_i} + T^{*}_{u'_i, (n, b^+)} - 4(\delta_0^{1/2} - 0.02 n^{1/3})$$

$$= T^{*}_{(0, a^-), (n, b^+)} - 4(\delta_0^{1/2} - 0.02 n^{1/3})$$

$$\geq T^{*}_{(0, a^-), (n, b^-)} - 5(\delta_0^{1/2} - 0.02 n^{1/3}).$$

Note that if $(\alpha \alpha n, b^-)$ is the intersection of $\Gamma_{(0, a^-), (n, b^-)}$ with $L_{\alpha \alpha n}$, then $b^- \leq b_0$. Also note that $\alpha_0 \geq \alpha_5 > h^{-1}$, and $\alpha_0 \leq \alpha_5 + 2\delta^{-7}\delta_5 \leq h^{-1} + \delta_5 + 2\delta^{-7}\delta_5 < 2h^{-1}$. Thus the above inequalities contradict with $\mathcal{M}^c$. Thus we conclude that there exists $u^* \in \mathcal{P}_1$ with $d(u^*) < 4h^{-1}n$, such that there is $u \in \Gamma_{(0, a), (n, b)}$ with $d(u) = d(u^*)$ and $d(u) \leq d(u^*) \leq d(u) + 2h_0n^{2/3}$, and $U_{u, u^*}$ intersects $\Gamma_{(0, a), (n, b)}$ before $L_{2h^{-1}n}$.

Using the same arguments, there exists $v^* \in \mathcal{P}_1$ with $d(v^*) > (2 - 4h^{-1})n$, such that there is $v \in \Gamma_{(0, a), (n, b)}$ with $d(v) = d(v^*)$ and $d(v) \leq d(v^*) \leq d(v) + 2h_0n^{2/3}$, and $U_{v, v^*}$ intersects $\Gamma_{(0, a), (n, b)}$ after $L_{(1-2h^{-1})n}$. If $U_{u, v^*}$ also intersects $\Gamma_{(0, a), (n, b)}$ before $L_{2h^{-1}n}$, we must have $\Gamma_{u, v^*}$ coincides with $\Gamma_{(0, a), (n, b)}$ between $L_{2h^{-1}n}$ and $L_{(1-2h^{-1})n}$, so the claim follows; otherwise, $\Gamma_{u, v^*}$ is disjoint from $\Gamma_{(0, a), (n, b)}$ before $L_{2h^{-1}n}$, we must have that $\Gamma_{u, v^*}$ intersects $\Gamma_{(0, a), (n, b)}$ after $L_{(1-2h^{-1})n}$, since otherwise $\Gamma_{u, v^*}$ and $\Gamma_{u^*, (n, b^+)}$ would intersect twice, so we conclude that $\Gamma_{u^*, (n, b^+)}$ coincides with $\Gamma_{(0, a), (n, b)}$ between $L_{2h^{-1}n}$ and $L_{(1-2h^{-1})n}$, and the claim also follows.

For the next a few subsections we prove the lemmas used in the proof of Proposition $7.1$

### 7.1 Continuity of passage times and multiple peaks

In this subsection we prove Lemma $7.2$ and $7.3$. For both of them we use the convergence of the point-to-line profile to the Airy$_2$ process, which is a stationary ergodic process minus a parabola. Such convergence in the sense of finite dimensional distributions is from [BF08, BP08]. Using the so-called slow decorrelation phenomenon, and proving equicontinuity of the point-to-line profile, it also follows that the weak convergence holds in the topology of uniform convergence on compact sets [BGZ21, FO18]. More precisely, let $A_2$ denote the stationary Airy$_2$ process on $\mathbb{R}$, and let us define the stochastic process $L : \mathbb{R} \to \mathbb{R}$ by

$$L(x) := A_2(x) - x^2.$$

We quote the following result.

**Theorem 7.6** ([BGZ21, Theorem 3.8]). As $n \to \infty$, we have

$$2^{-4/3}n^{-1/3} \left( T_{0, (n, (2n)^{2/3})} - 4n \right) \Rightarrow L(x)$$

weakly in the topology of uniform convergence on compact sets.

We shall also use the following (quantitative) comparison between the Airy$_2$ process, and a Brownian motion.
For $K \in \mathbb{R}, d > 0$, let $\mathcal{B}^{[K,K+d]}$ denote the law of a Brownian motion with diffusivity 2 taking value 0 at $K$ and restricted to $[K, K + d]$. Let $\mathcal{L}^{[K,K+d]}$ denote the random function on $[K, K + d]$ defined by

$$\mathcal{L}^{[K,K+d]}(x) := \mathcal{L}(x) - \mathcal{L}(K), \ \forall x \in [K, K + d].$$

Let $C([K, K + d], \mathbb{R})$ denote the set of all real valued continuous functions defined on $[K, K + d]$ which vanish at $K$. The following result can be obtained from [CHH].

**Theorem 7.7 (CHH Theorem 1.1).** There exists an universal constant $G > 0$ such that the following holds. For any fixed $M > 0$, there exists $a_0 = a(M)$ such that for all intervals $[K, K + d] \subset [-M, M]$ and for all measurable $A \subset C([K, K + d], \mathbb{R})$ with $0 < \mathcal{B}^{[K,K+d]}(A) = a \leq a_0$,

$$\mathbb{P}\left(\mathcal{L}^{[K,K+d]}(A) \leq a \exp \left\{ GM \left( \log a^{-1} \right)^{5/6} \right\} \right).$$

Now we prove Lemma 7.2. We start with the following estimate on deviations when moving one end point.

**Lemma 7.8.** There are constants $C, c$ such that the following holds. For any $h \in \mathbb{R}$, $\theta < 1 < t$, we have

$$\mathbb{P}\left[ \max_{hn^{2/3} < b, b' < h(n+1)n^{2/3}} |T_0, (n, b) - T_0, (n, b')| > t\theta^{1/2} - 0.011h^{1/3} + C(|h| + 1)\theta h^{n^{1/3}} \right] < C e^{-ct}$$

for $n$ large enough (depending on $h, \theta, t$).

**Proof.** By Theorem 7.6 it suffices to bound

$$\mathbb{P}\left[ \max_{2^{-2/3}h < x, x' < 2^{-3/2}h + 1, |x - x'| < 2^{-3/2}h} |\mathcal{L}(x) - \mathcal{L}(x')| > 2^{-4/3}t\theta^{1/2} - 0.01 + 2^{-4/3}C(|h| + 1)\theta \right].$$

When $C > 2$ we have $|x^2 - x'^2| < 2^{-4/3}C(|h| + 1)\theta$ for all $x, x'$ we take the max over. Then by stationarity of $\mathcal{A}_2$, we can bound this probability by

$$\mathbb{P}\left[ \max_{0 < x, x' < 2^{-2/3}h, |x - x'| < 2^{-3/2}h} |\mathcal{L}(x) - \mathcal{L}(x')| > 2^{-4/3}t\theta^{1/2} - 0.01 \right].$$

Note that the event now only relies on $\mathcal{L}^{[0, 2^{-2/3}]}$. Using modulus of continuity for Brownian motions, and Theorem 7.7 we can bound this by $C e^{-ct}$ as desired. –

**Proof of Lemma 7.2.** First, note that we have the following inequality for passage times:

$$T_{\langle 0, a \rangle}, (n, b) - T_{\langle 0, a' \rangle}, (n, b') \geq T_{\langle 0, a' \rangle}, (n, b) - T_{\langle 0, a' \rangle}, (n, b')$$

for any $a \leq a', b \leq b'$. Indeed, if we take the geodesics $\Gamma_{\langle 0, a \rangle}, (n, b)$ and $\Gamma_{\langle 0, a' \rangle}, (n, b)$, then they must intersect. By switching the paths after their first intersection, we get two directed paths, between $\langle 0, a \rangle, (n, b)$ and $\langle 0, a' \rangle, (n, b)$, and their total weight remains unchanged and equals $T_{\langle 0, a' \rangle}, (n, b) + T_{\langle 0, a \rangle}, (n, b')$. Thus we get the above inequality from the definition of last passage times.

From this, for any $|a|, |a'|, |b|, |b'| < hn^{2/3}$ we have

$$|T_{\langle 0, a \rangle}, (n, b) - T_{\langle 0, a' \rangle}, (n, b')| \leq |T_{\langle 0, (hn^{2/3}) \rangle}, (n, b) - T_{\langle 0, (hn^{2/3}) \rangle}, (n, b')| + |T_{\langle 0, (hn^{2/3}) \rangle}, (n, b) - T_{\langle 0, (hn^{2/3}) \rangle}, (n, b')| + + |T_{\langle 0, a \rangle}, (n, -[hn^{2/3}]) - T_{\langle 0, a' \rangle}, (n, -[hn^{2/3}])| + |T_{\langle 0, a \rangle}, (n, [hn^{2/3}]) - T_{\langle 0, a' \rangle}, (n, [hn^{2/3}])|$$

By symmetry, now it suffices to bound

$$\mathbb{P}\left[ \max_{|b|, |b'| < hn^{2/3}, |b - b'| < \theta n^{2/3}} |T_{\langle 0, (hn^{2/3}) \rangle}, (n, b) - T_{\langle 0, (hn^{2/3}) \rangle}, (n, b')| > \frac{1}{4} (t\theta^{1/2} - 0.011h^{1/3} + C\theta h^{n^{1/3}}) \right].$$
For this we split \( \{0,b\} : |b| < hn^{2/3} \) into segments of length \( n^{2/3} \), and apply Lemma 7.8 to each of them to get the desired bound.

We next prove Lemma 7.5. Again, using Theorem 7.6 we reduce the problem to Airy_2 processes, and then by applying Theorem 7.7 we could just prove the result for Brownian motions.

**Proof of Lemma 7.3.** Denote \( \mathcal{L}_{\alpha,\beta} : \mathbb{R} \to \mathbb{R} \) as the process given by
\[
\mathcal{L}_{\alpha,\beta}(x) := \alpha^{1/3}\mathcal{L}(\alpha^{-2/3}x) + (1 - \alpha)^{1/3}\mathcal{L}'((1 - \alpha)^{-2/3}(x - 2^{-2/3}\beta)),
\]
where \( \mathcal{L}' \) is an independent copy of \( \mathcal{L} \). By Theorem 7.6 it suffices to bound the probability of the following event: there exist \(-2^{1/3}h < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < 2^{1/3}h\), with \( x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5 > 2^{-2/3}\theta \), such that \( x_1 = \text{argmax}_{[-2^{1/3}h,2^{1/3}h]} \mathcal{L}_{\alpha,\beta} \), and
\[
\mathcal{L}_{\alpha,\beta}(x_i) < \mathcal{L}_{\alpha,\beta}(x_{i-1}) + 2^{-4/3}t\theta^{1/2}, \quad \forall i = 2, 3, 4, 5, 6.
\]
One could replace \( \mathcal{L}_{\alpha,\beta} \) by a (two-sided) Brownian motion with diffusivity \( 2\alpha^2 + (2(1 - \alpha))^2 \), and study the probability of the same event. By Lemma 7.9 below this probability is bounded by \( C^5 \) for \( C \) depending on \( h \). Finally, we apply Theorem 7.7 and get the desired bound.

We finally bound the events on Brownian motion.

**Lemma 7.9.** There exists a constant \( C > 0 \), such that for any \( t, \theta > 0 \), the following event holds with probability at most \( C\theta^5 \). For \( W : [-2,2] \to \mathbb{R} \) being a two sided Brownian motion, there are \(-1 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < 1\), with \( x_2 - x_1, x_3 - x_2, x_4 - x_3, x_5 - x_4, x_6 - x_5 > \theta \), such that \( x_1 = \text{argmax}_{[-2,2]} W \), and
\[
W(x_1) < W(x_i) + t\theta^{1/2}, \quad \forall i = 2, 3, 4, 5, 6.
\]

**Proof.** Fix \( T_i \in [-1,1] \), and let \( \mathcal{E} \) be the event where \( W(T_i) = \text{max}_{[-2,2]} W \). For \( i = 2, 3, 4, 5, 6 \), let \( T_i = \text{min}\{x \geq T_{i-1} + \theta : W(x) \geq W(x_1) - t\theta^{1/2}\} \). It suffices to show that \( \mathbb{P}[T_6 \leq 1 | \mathcal{E}] < C\theta^5 \) for some constant \( C > 0 \). For \( i = 2, 3, 4, 5, 6 \), conditioned on \( \mathcal{E} \) and the event \( T_{i-1} \leq 1 \), and given the values of \( T_{i-1} \) and \( W(T_{i-1}) - W(T_i) \), the process \( x \mapsto W(T_{i-1} + x) - W(T_i) \) on \([0,2-T_{i-1}]\) has the same law of \( W' \), which is a Brownian motion on \([0,2-T_{i-1}]\) starting from \( W'(0) = W(T_{i-1}) - W(T_i) \) and conditioned below zero (for \( i = 2 \) this degenerates to a Brownian meander). Using reflection principle we have that \( \mathbb{P}[\text{max}_{[0,2-T_{i-1}]} W' \geq -t\theta^{1/2}] < C' t \) for some \( C' > 0 \). So we have that \( \mathbb{P}[T_i \leq 1 | \mathcal{E}, T_{i-1} \leq 1] < C't \). Since \( T_i \leq 1 \) is implied by \( T_{i-1} \leq 1 \) for each \( i \), we have \( \mathbb{P}[T_6 \leq 1 | \mathcal{E}] < (C't)^5 \), which implies the conclusion.

### 7.2 Disjoint paths

In this subsection we prove Lemma 7.4. The idea is to show that for a path restricted to be close to another (deterministic) path for a while, its weight is unlikely to be small (compared to that of a geodesic with the same end points). We then use the FKG inequality to move from a deterministic path to a geodesic.

**Lemma 7.10.** For sufficiently small \( c_0 > 0 \), there is \( c_1 > 0 \), such that for large enough \( l \) and any \( r \in \mathbb{Z} \), we have
\[
\mathbb{E} \left[ \max_{0 \leq a,b \leq c_0 t^{2/3}} T_{(0,a),(l,r+b)} \right] < 4l - c_1 t^{1/3}.
\]
Proof. Take \( u = (-[c_0^{3/2} l], 0) \) and \( v = (l + [c_0^{3/2} l], r') \), where \( r' \) is the number in \([c_0^{2/3}] \mathbb{Z} \) with \( r \leq r' \leq r + c_0^{2/3} \). Note that

\[
E \left[ \max_{0 \leq a, b \leq c_0^{2/3}} T_{(0,a), (l+r,b)} \right] \leq E T_{u,v} - E \left[ \min_{0 \leq a \leq c_0^{2/3}} T_{u,(-1,a)} \right] - E \left[ \min_{0 \leq b \leq c_0^{2/3}} T_{(l+1,r+b),v} \right].
\]

By Proposition 2.4 we have

\[
E \left[ \min_{0 \leq a \leq c_0^{2/3}} T_{u,(-1,a)} \right], E \left[ \min_{0 \leq b \leq c_0^{2/3}} T_{(l+1,r+b),v} \right] \geq 4c_0^{3/2} l - Cc_0^{1/2} l^{1/3}
\]

for some constant \( C > 0 \). We also claim that for \( l \) sufficiently large,

\[
E T_{u,v} \leq 4(l + 2c_0^{3/2} l) - C' l^{1/3}
\]

for some \( C' > 0 \). When \( l^{-2/3} |r| > C'' \) for some \( C'' \) depending on \( C' \), (7.3) is by (2.7). When \( l^{-2/3} |r| \leq C'' \), for each \( l \) there are at most \( 3C''/c_0 \) possible numbers \( r' \) can take. For each of them, by Theorem 7.3 the corresponding \( T_{u,v} \) after rescaling converges (as \( l \to \infty \)) to one point of the Airy process, whose law is given by the GUE Tracy-Widom distribution. Thus (7.3) follows since the GUE Tracy-Widom distribution has negative expectation. By choosing \( c_0 \) sufficiently small we complete the proof.

For the next lemma, as above we denote \( T_{u,v} = T_{u,v} - \xi(v) \) for any vertices \( u \leq v \in \mathbb{Z}^2 \).

**Lemma 7.11.** For \( l, M \in \mathbb{N} \), and any \( r_0, \ldots, r_M \in \mathbb{Z} \) such that \( |r_0 - r_M| < M^{5/6}/l^{2/3} \), we have

\[
P \left[ \max_{0 \leq a_0, \ldots, a_M \leq c_0^{2/3}} \sum_{i=0}^{M-1} T^*_{(l,r_i+a_i), ((i+1), l,r_{i+1}+a_{i+1})} \geq 4Ml - cM^{1/3} \right] < Ce^{-cM}
\]

for some absolute constants \( c, C > 0 \), when \( l \) is large enough.

**Proof.** In this proof we let \( C, c > 0 \) be large and small enough constants, and their values can change from line to line.

We denote \( S_i = \max_{0 \leq a_i, a_{i+1} \leq c_0^{2/3}} T^*_{(l,r_i+a_i), ((i+1), l,r_{i+1}+a_{i+1})} \) for each \( 0 \leq i \leq M - 1 \). By Lemma 7.10 we have \( E[S_i] \leq 4l - c_1 l^{1/3} \), where \( c_0, c_1 > 0 \) are some constants.

Next we apply Proposition 2.4. When \( |r_i - r_{i+1}| \leq 0.9l \) we could directly apply it; and when \( |r_i - r_{i+1}| > 0.9l \), the slope condition may not be satisfied, thus we use the fact that \( T^*_{(l,r_i+a_i), ((i+1), l,r_{i+1}+a_{i+1})} < T^*_{(l,r_i+a_i), ((i+1), l+0.1l,r_{i+1}+a_{i+1})} \), and upper bound the later using Proposition 2.4. We conclude that \( P[S_i] > 4l + xl^{1/3} < Ce^{-cx} \) for any \( x > 0 \).

Note that \( S_i \) for each \( i \) are independent. Thus by a Bernstein type bound on sum of independent variables with exponential tails, we have

\[
P \left[ \max_{0 \leq a_0, \ldots, a_M \leq c_0^{2/3}} \sum_{i=0}^{M-1} T^*_{(l,r_i+a_i), ((i+1), l,r_{i+1}+a_{i+1})} \geq 4Ml - \frac{c_1}{2} M^{1/3} \right] < Ce^{-cM}.
\]

Then the conclusion follows.

**Proof of Lemma 7.2.** Take any directed path \( \Gamma \) from \( u \) to \( v \), such that for the intersections of \( \Gamma_{u,v} \) with \( \mathbb{L}_m \) and \( \mathbb{L}_{m+M} \), denoted as \( u', v' \), we have \( |ad(u') - ad(v')| < 2M^{5/6}/l^{2/3} \). Denote \( D_{\Gamma} \) as the following event: there exists a directed path \( \gamma \) from \( \mathbb{L}_m \) to \( \mathbb{L}_{m+M} \), such that

- \( \gamma \) is disjoint from \( \Gamma \).
- The weight of \( \gamma \) (i.e. \( T(\gamma) \)) is at least \( 4Ml - c_0 M^{1/3} \).
- For each \( i = 0, 1, \ldots, M \), \( \| \Gamma \cap \mathbb{L}_{m+il} - \gamma \cap \mathbb{L}_{m+il} \|_1 < 2c_0 l^{2/3} \).

45
Now we consider the event $\Gamma_{u,v} = \Gamma$. Under this event we have $D_{M,l,m}^{0,(n,b)} = D_{r}$. Also, $\Gamma_{u,v} = \Gamma$ is a negative event of the field on $\mathbb{Z}^2 \setminus \Gamma$, while $D_{r}$ is determined by the field on $\mathbb{Z}^2 \setminus \Gamma$, and is a positive event of the field on $\mathbb{Z}^2 \setminus \Gamma$. By the FKG inequality we have $$\mathbb{P}[D_{M,l,m}^{0,(n,b)} | \Gamma_{u,v} = \Gamma] = \mathbb{P}[D_{r} | \Gamma_{u,v} = \Gamma] \leq \mathbb{P}[D_{r}].$$ By Lemma 7.11, when $c_0 < c$ we have $\mathbb{P}[D_{r}] \leq C e^{-cM}$, for some constants $C, c > 0$. By averaging over all $\Gamma$ we get the conclusion. \qed

8 Convergence of one point distribution

In this section we prove Theorem 1.3 and 1.4. The general idea is to show that the law of the value of a specific vertex in the geodesic is close to that of nearby vertices along the geodesic; and this is achieved by a coalescing argument. Then we use Proposition 7.1 to argue that certain time average is close to the stationary one.

We start by deducing the following estimate on coalescence of geodesics, which is the main step towards proving Theorem 1.3. It directly follows from Proposition 2.6 and Lemma 2.8.

**Lemma 8.1.** For each $\rho \in (0,1)$, there is a constant $C > 0$, such that for any $r \in \mathbb{N}$, and $k > 2$, we have $\mathbb{P}[\Gamma_{\rho}^{0} \cap L_{[r]} \neq \gamma \cap L_{[r]}] < C \log(k)k^{-2/3}$, and $\mathbb{P}[\Gamma_{\rho}^{0} \cap L_{[r]} \neq \gamma \cap L_{[r]}] < C \log(k)k^{-2/3}$, where $\gamma = \Gamma_{\rho}^{0}_{(|r/2|,0)}$ or $\gamma = \Gamma_{\rho}^{0}_{(|r/2|,0)+1,0}$.

**Proof.** Denote the intersections of $\Gamma_{\rho}^{0}$ and $\gamma$ with $L_{r}$ as $\langle r,b_{r}\rangle_{\rho}$ and $\langle r,b'_{r}\rangle_{\rho}$. By Lemma 2.8 and Proposition 2.6 there is a constant $C_{0} > 0$ such that

$$\mathbb{P}[|b_{r}|,|b'_{r}| \leq C_{0} \log(k)r^{2/3}] < 1 - C_{0}k^{-1},$$

and

$$\mathbb{P}[\Gamma_{r,\rho}[C_{0} \log(k)r^{2/3}]_{1,\rho} \cap L_{[r]} \neq \Gamma_{r,\rho}[C_{0} \log(k)r^{2/3}+1,\rho] \cap L_{[r]}] < C_{0}^2 \log(k)(k-1)^{-2/3}.$$ 

Thus the conclusion follows by monotonicity of geodesics. \qed

**Proof of Theorem 1.3.** Take any $s \in \mathbb{N}$ and any measurable function $f : \mathbb{R}^{[-s,s]} \to [0,1]$, regarded as a function on $\mathbb{R}^{2s}$, we shall show that $\lim_{i \to \infty} \mathbb{E} f(\xi(\Gamma_{0}^{0}[i])) = \nu^{\rho}(f)$.

For $i, r \in \mathbb{N}$ and $k > 2$ with $i > 2r$, we consider $\Gamma_{0}^{0}$ with $\gamma = \Gamma_{\rho}^{0}_{(|r/2|,0)}$ (when $r$ is even) or $\gamma = \Gamma_{\rho}^{0}_{(|r/2|,0)+1,0}$ (when $r$ is odd). By Lemma 8.3 we have $\xi(\Gamma_{0}^{0}[i]) = \xi(\gamma[i-r])$, with probability at least $1 - C \log(k)^{-2/3}$. Since $\xi(\gamma[i-r])$ has the same law as $\xi(\Gamma_{0}^{0}[i-r])$, we must have that

$$\mathbb{E} f(\xi(\Gamma_{0}^{0}[i])) - \mathbb{E} f(\xi(\Gamma_{0}^{0}[i-r])) \leq C \log(k)^{-2/3}.$$ 

By averaging over $r$ for $0 \leq r \leq i/4k$, we have

$$\mathbb{E} f(\xi(\Gamma_{0}^{0}[i])) - \mathbb{E} \mu_{\Gamma_{0}^{0}[i],\Gamma_{0}^{0}[i]}(f) \leq C \log(k)^{-2/3}.$$ 

By Lemma 2.8 and Proposition 7.1 for any fixed $k > 0$, we have $\mu_{\Gamma_{0}^{0}[i],\Gamma_{0}^{0}[i]}(f) \to \nu^{\rho}(f)$ in probability as $i \to \infty$. Thus we have that $\limsup_{i \to \infty} |\mathbb{E} f(\xi(\Gamma_{0}^{0}[i])) - \nu^{\rho}(f)| \leq C \log(k)^{-2/3}$. Since $k$ can be arbitrarily large, the conclusion holds. \qed

The proof of Theorem 1.4 is similar, while for finite geodesics we use Lemma 2.7 and Proposition 2.5 instead of Lemma 2.8 and Proposition 2.6.

**Lemma 8.2.** For each $\rho \in (0,1)$, there is a constant $C > 0$, such that for any $r, n \in \mathbb{N}$ and $k > 2$, with $n \geq 2r$, we have $\mathbb{P}[\Gamma_{0,n}^{0} \cap L_{[r]} \neq \gamma \cap L_{[r]}] < C \log(k)k^{-2/3}$, and $\mathbb{P}[\Gamma_{0,n}^{0} \cap L_{[r]} \neq \gamma \cap L_{[r]}] < C \log(k)k^{-2/3}$, where $\gamma = \Gamma_{\rho}^{0}_{(|r/2|,0)+1,0}$ or $\gamma = \Gamma_{\rho}^{0}_{(|r/2|,0)+1,0}$. 

46
where the last inequality is by Proposition 2.5. Then the conclusion follows by monotonicity of geodesics (see Figure 11).

Proof. Since $n \geq 2rk$, we just show $\mathbb{P}[\Gamma_{0,n^\rho} \cap \mathbb{L}_{\{rk\}} \neq \mathbb{L}_{\{rk\}}] < Ck^{-2/3}$, and by symmetry the other inequality would follow.

Denote the intersections of $\Gamma_{0,n^\rho}$ and $\gamma$ with $\mathbb{L}_r$ as $\langle r, b_-, \rangle_{\rho}$ and $\langle r, b'_-, \rangle_{\rho}$, respectively; and the intersections of $\Gamma_{0,n^\rho}$ and $\gamma$ with $\mathbb{L}_{n-r}$ as $\langle n-r, b_+, \rangle_{\rho}$ and $\langle n-r, b'_+, \rangle_{\rho}$, respectively. There is a constant $C_0 > 0$, such that

$$\mathbb{P}[|b_-|, |b'_-| \leq C_0 \log(k)r^{2/3}], \mathbb{P}[|b_+|, |b'_+| \leq C_0 \log(k)r^{2/3}] > 1 - C_0k^{-1}$$

by Lemma 2.7 and

$$\mathbb{P}[\Gamma_{\langle r, -\lfloor C_0 \log(k)r^{2/3} \rfloor + 1 \rangle_{\rho}, (n-r, -\lfloor C_0 \log(k)r^{2/3} \rfloor + 1 \rangle_{\rho}} \cap \mathbb{L}_{\{rk\}} \neq \Gamma_{\langle r, -\lfloor C_0 \log(k)r^{2/3} \rfloor + 1 \rangle_{\rho}, (n-r, -\lfloor C_0 \log(k)r^{2/3} \rfloor + 1 \rangle_{\rho}} \cap \mathbb{L}_{\{rk\}}] < C_0^2 \log(k)(k-1)^{-2/3},$$

where the last inequality is by Proposition 2.5. Then the conclusion follows by monotonicity of geodesics (see Figure 11).

Proof of Theorem 1.4. Take any $s \in \mathbb{N}$ and any measurable function $f : \mathbb{R}^{[-s,s]} \rightarrow [0,1]$, regarded as a function on $\mathbb{R}^{2s}$, now we shall show that $\lim_{n \rightarrow \infty} \mathbb{E}f(\xi \{\Gamma_{0,n^\rho}([\alpha n])\}) = \nu^\rho(f)$.

Without loss of generality we assume that $\alpha \leq 1$. For $n,r \in \mathbb{N}$ and $k > 2$ with $\alpha n > 2rk$, we consider $\gamma = \Gamma_{\langle [r/2], 0 \rangle_{\rho}, n^\rho + \langle [r/2], 0 \rangle_{\rho}}$ (when $r$ is even) or $\gamma = \Gamma_{\langle [r/2], 0 \rangle_{\rho} + (1,0), n^\rho + \langle [r/2], 0 \rangle_{\rho} + (1,0)}$ (when $r$ is odd). By Lemma 8.2 we have

$$\mathbb{P}[\xi \{\Gamma_{0,n^\rho}([\alpha n])\} = \xi \{\gamma([\alpha n] - r)\}] \geq 1 - C \log(k)k^{-2/3}.$$
Since $\xi\{\gamma([\alpha n] - r)\}$ has the same law as $\xi\{\Gamma_{0,n^\rho}([\alpha n] - r)\}$, we must have that

$$|\mathbb{E} f(\xi\{\Gamma_{0,n^\rho}([\alpha n])\}) - \mathbb{E} f(\xi\{\Gamma_{0,n^\rho}([\alpha n] - r)\})| \leq C \log(k)k^{-2/3}.$$  

By averaging over $r$ for $0 \leq r \leq \alpha n/4k$, we have

$$|\mathbb{E} f(\xi\{\Gamma_{0,n^\rho}([\alpha n])\}) - \mathbb{E} \mu_{\Gamma_{0,n^\rho}([\alpha n] - [\alpha n/4k]),\Gamma_{0,n^\rho}([\alpha n])}(f)| \leq C \log(k)k^{-2/3}.$$  

By Lemma 2.7 and Proposition 7.1 for fixed $k$ we have $\mu_{\Gamma_{0,n^\rho}([\alpha n] - [\alpha n/4k]),\Gamma_{0,n^\rho}([\alpha n])}(f) \to \nu^\rho(f)$ in probability as $n \to \infty$. Thus we have that $\limsup_{n \to \infty} |\mathbb{E} f(\xi\{\Gamma_{0,n^\rho}([\alpha n])\}) - \nu^\rho(f)| \leq C \log(k)k^{-2/3}$. Then the conclusion follows since $k$ is arbitrarily taken.

\section{Exponential concentration via counting argument}

Using a covering argument, we can prove the following exponential concentration of the empirical distribution, for both finite or semi-infinite geodesics.

\textbf{Proposition 9.1.} For each $\rho \in (0,1)$ $s \in \mathbb{Z}_{\geq 0}$, and any bounded function $f : \mathbb{R}^{[-s,s]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, and any $\epsilon > 0$, we have

$$\mathbb{P}[|\mu_{0,r}^\rho(f) - \nu^\rho(f)| > \epsilon] < Ce^{-cr},$$

for $r$ large enough, and $C,c > 0$ depending on $\rho,s,f,\epsilon$.

\textbf{Proposition 9.2.} Let $\{b_n\}_{n \in \mathbb{N}}$ be a sequence of integers such that $\lim_{n \to \infty} n^{-2/3}|b_n| < \infty$. Then for any $s \in \mathbb{Z}_{\geq 0}$, any bounded function $f : \mathbb{R}^{[-s,s]^2} \to \mathbb{R}$, regarded as a function on $\mathbb{R}^{\mathbb{Z}^2}$, and any $\epsilon > 0$, we have

$$\mathbb{P}[|\mu_{0,(n,b_n)\rho}(f) - \nu^\rho(f)| > \epsilon] < Ce^{-cn},$$

for $n$ large enough, and $C,c > 0$ depending on $\rho,s,f,\epsilon$.

From Proposition 9.1 we can deduce Theorem 1.2

\textbf{Proof of Theorem 7.2} By Proposition 9.1 for any bounded $f : \mathbb{R}^{[-s,s]^2} \to \mathbb{R}$ and $\epsilon > 0$, we have that $\sum_{\alpha \in \mathbb{N}} \mathbb{P}[|\mu_{0,r}^\rho(f) - \nu^\rho(f)| > \epsilon] \leq \infty$; so almost surely, there exists some (random) $r_0$ such that $|\mu_{0,r}^\rho(f) - \nu^\rho(f)| \leq \epsilon$ for any $r > r_0$. Thus we have that $\mu_{0,r}^\rho(f) \to \nu^\rho(f)$ almost surely. The conclusion follows by taking all $s \in \mathbb{N}$, and $f$ over all characteristic functions $1_{[\prod_{i \in [-s,s]^2}(-\infty, x_i)]}$, for $\{x_i\}_{i \in [-s,s]^2} \in \mathcal{Q}^{[-s,s]^2}$.

To prove these exponential concentration bounds (Proposition 9.1 and 9.2), we cover the geodesics with short finite ones, and use Proposition 7.1.

For the rest of this section we fix $\rho \in (0,1)$. We take $m \in \mathbb{N}$, and assume that $m^{2/3} \in \mathbb{Z}$. For each $i,j \in \mathbb{Z}$ we denote $L_{i,j}$ as the segment joining $(im, (2j + 1)m^{2/3})_\rho$ and $(im, (2j + 1)m^{2/3})_\rho$. For each sequence $j_0, j_1, \ldots, j_k$, we let $P_{j_0,\ldots,j_k}$ be the collection of paths from $L_{j_0,j_k}$ to $L_{k,j_k}$, intersecting each $L_{i,j_i}$, $0 \leq i \leq k$. For any $k \in \mathbb{N}$ and $D > 0$, we also denote $P_{k,D}$ as the union of all $P_{j_0,j_1,\ldots,j_k}$ such that $j_0 = 0$ and $\sum_{i=1}^k (j_i - j_{i-1})^2 > Dk$.

\textbf{Lemma 9.3.} There exists $c_0 > 0$, such that when $m,k,D$ are large enough,

$$\mathbb{P}\left[ \exists \gamma \in P_{k,D}, T(\gamma) > \frac{2km}{(1 - \rho)^2 + \rho^2} - (b_+ - b_-)(\rho^{-1} - (1 - \rho)^{-1}) - c_0 Dkm^{1/3} \right] < e^{-ck},$$

where $b_-, b_+ \in \mathbb{Z}$ such that $(0,b_-), (km,b_+)_\rho$ are the intersections of $\gamma$ with $\mathbb{L}_0, \mathbb{L}_{km}$, respectively.
For each $i \in \mathbb{Z}$, $x > 0$,

$$
\mathbb{E}\left[ \max_{(0,b) \in L_{0,0}, \langle m,b' \rangle \in L_{1,j}} T_{0,b}(0,b)_\rho \langle m,b' \rangle_\rho + (b' - b)(\rho^{-1} - (1 - \rho)^{-1}) \right] < \frac{2m}{(1 - \rho)^2 + \rho^2} + (C_1 - c_1 j^2)m^{1/3},
$$

$$
P\left[ \max_{(0,b) \in L_{0,0}, \langle m,b' \rangle \in L_{1,j}} T_{0,b}(0,b)_\rho \langle m,b' \rangle_\rho + (b' - b)(\rho^{-1} - (1 - \rho)^{-1}) \right] > \frac{2m}{(1 - \rho)^2 + \rho^2} + (x - c_1 j^2)m^{1/3} \right] < C_1 e^{-c_1 x}.
$$

Proof. First, by Theorem 2.4 and 2.7, and fundamental computations, there exist $C_1, c_1 > 0$ such that for $m$ large enough and any $j \in \mathbb{Z}$, $x > 0$,

$$
\max_{\gamma \in P_{j_0,j_1,...,j_k}} T(\gamma) \leq \sum_{i=1}^{k-1} \max_{v \in L_{u,v}} T_{u,v}^* + \max_{v \in L_{k,j,k}} T_{u,v}
$$

Here $T_{u,v} = T_{u,v} - \xi(v)$ for any $u \leq v \in \mathbb{Z}^2$. Then by a Bernstein type estimate for independent random variables with exponential tails, we have

$$
P\left[ \max_{\gamma \in P_{j_0,j_1,...,j_k}} T(\gamma) + (b_+ - b_-)(\rho^{-1} - (1 - \rho)^{-1}) \right] > \frac{2m k}{(1 - \rho)^2 + \rho^2} - \frac{c_1}{2} D k m^{1/3}
$$

for any $D > 1$ and $j_0 = 0$, $\sum_{i=1}^{k} (j_i - j_{i-1})^2 > D k$, where $C_2, c_2 > 0$ are constants, $\langle 0, b_+ \rangle, \langle km, b_- \rangle$ are the intersections of $\gamma$ with $L_{0,0}, L_{km}$. Summing over all such sequences $j_0, j_1, \ldots, j_k$, the right hand side is bounded by

$$
C_2 e^{-c_2 D k} \left( \sum_{j \in \mathbb{Z}} e^{-c_2 j^2} \right)^k.
$$

By taking $D$ large so that $e^{c_2 D/2} > \sum_{j \in \mathbb{Z}} e^{-c_2 j^2}$, we get the conclusion. \hfill \square

Proof of Proposition 9.1. For any $u \leq v \in \mathbb{Z}^2$, denote

$$
\mu_{u,v}^* := \frac{1}{|\Gamma_{u,v}|^2 - 1} \sum_{w \in \Gamma_{u,v}, w \neq v} \delta \xi(w);
$$

i.e. it is the empirical distribution along $\Gamma_{u,v}$, excluding the last vertex $v$. Without loss of generality we also assume that $0 \leq f \leq 1$.

Take $D$ large enough, and then $m$ large, as required by Lemma 9.3, and also $m$ is large enough such that

$$
P\left[ \max_{|a||b| < \epsilon^{-2} m^{2/3}} |\mu_{0,0}^* \rho_{\langle m,b \rangle}(f) - \nu^\rho(f)| > \epsilon^2 \right] < \epsilon,
$$

by Proposition 7.1. Here $\epsilon$ is a small number depending on $D, \epsilon$ and to be determined. Take a sequence $j_0, \ldots, j_k$, such that $j_0 \equiv 0$ and $\sum_{i=1}^{k} (j_i - j_{i-1})^2 \leq D k$. We let $I' \subset \{1, \ldots, k\}$ be the collection of indices such that $|j_i - j_{i-1}| < \epsilon^{-2}/2 - 1$ for each $i \in I'$. Then $|I'| > (1 - \epsilon/2) k$, when $\epsilon$ is small enough (depending on $D$). Next we let $I \subset I'$ such that for each $i \in I$,

$$
\max_{u \in L_{i-1,j_{i-1}}, v \in L_{i,j_i}} |\mu_{u,v}^*(f) - \nu^\rho(f)| \leq \epsilon^2.
$$

For each $i \in I'$ we have $i \in I$ with probability at least $1 - \epsilon$, and this is independent for each $i$. Then by Chernoff bound and taking $\epsilon$ small enough (depending on $D, \epsilon$), we can make $P[|I'|-|I| > \epsilon^2 k] < (D + 1)^{-2k}$. Let $\gamma$ be the path consisting of the first $2 km + 1$ vertices of $\Gamma^\rho_0$; i.e. $\gamma$ is the part
of $\Gamma_0^\rho$ on and between $\mathbb{L}_0$ and $\mathbb{L}_{km}$. Given that $\gamma \in P_{j_0,\ldots,j_k}$, and $|I'| - |I| \leq \epsilon^2 k$, for any $r \in \mathbb{N}$ with $km \leq r < (k + 1)m$ we must have that $|\mu_0^\rho(f) - \nu^\rho(f)| \leq \epsilon/2 + \epsilon^2 + 1/(k + 1)$. So when $k > \epsilon^{-2}$ we have

$$P[\gamma \in P_{j_0,\ldots,j_k}, |\mu_0^\rho(f) - \nu^\rho(f)| > \epsilon] < (D + 1)^{-2k}.$$ 

Thus by summing over all sequences $j_0, \ldots, j_k$ with $j_0 = 0$, $\sum_{i=1}^k (j_i - j_{i-1})^2 \leq Dk$, we have

$$P[\gamma \notin P_{k,D}, |\mu_0^\rho(f) - \nu^\rho(f)| > \epsilon] < \left(\frac{Dk + k - 1}{k - 1}\right)(D + 1)^{-2k} < e^{-ck}$$

for some $c > 0$ depending on $D$.

Now it remains to bound $P[\gamma \in P_{k,D}]$. By Lemma 9.3 we have

$$P[\gamma \in P_{k,D}] < e^{-c_0 k} + P\left[T(\gamma) \leq \frac{2km}{(1 - \rho)^2 + \rho^2} - b_+(\rho^{-1} - (1 - \rho)^{-1}) - c_0 Dkm^{1/3}\right],$$

(9.1)

where $(km, b_+)_\rho$ is the intersection of $\Gamma_0^\rho$ with $\mathbb{L}_{km}$. When the event in the right hand side of (9.1) happens, we must have that (at least) one of the following happens:

- $|b_+| > km^{2/3},$
- $\max_{|b| \leq km^{2/3}} B^{\rho}(\langle km, b \rangle_\rho, \langle km, 0 \rangle_\rho) - b(\rho^{-1} - (1 - \rho)^{-1}) \geq c_0 Dkm^{1/3}/3,$
- $T^*_{0,(km,0)_\rho} \leq \frac{2km}{(1 - \rho)^2 + \rho^2} - c_0 Dkm^{1/3}/2,$

where $T^*_{0,(km,0)_\rho} = T_{0,(km,0)_\rho} - \xi((km, 0)_\rho)$. To see this, we assume the contrary, i.e. none of the above three events happen. If we let the first intersection of $\Gamma_0^\rho$ with $\Gamma^\rho_{0,(km,0)_\rho}$ be $w$, then we must have

$$T(\gamma) > T_{0,w} - T_{(km,b_+)_\rho,w} \geq T^*_{0,(km,0)_\rho} + T_{(km,0)_\rho,w} - T_{(km,b_+)_\rho,w} > \frac{2km}{(1 - \rho)^2 + \rho^2} - b_+(\rho^{-1} - (1 - \rho)^{-1}) - 5c_0 Dkm^{1/3}/6,$$

which contradicts with the event in the right hand side of (9.1). Finally, we claim that we can bound the probability of each of the three events by $e^{-c'k}$, for $c' > 0$ and $k$ large enough, depending on $m, D$. For the first event the bound is by Lemma 2.8. For the second event, note that $b \mapsto B^{\rho}(\langle km, b \rangle_\rho, \langle km, 0 \rangle_\rho) - b(\rho^{-1} - (1 - \rho)^{-1})$ is a (two-sided) centered random walk; for the third event, use Theorem 2.3. Thus the conclusion follows.

Proof of Proposition 9.2. The first half of this proof follows the same way as the proof of Proposition 9.1, and we conclude that the following is true when $D$ is large enough and then $m$ is large enough. Suppose that $km \leq n < (k + 1)m$ for some $k \in \mathbb{N}$, $k > \epsilon^{-2}$. Let $\gamma$ be the path from $\mathbb{L}_0$ to $\mathbb{L}_{km}$, consisting of the first $2km + 1$ vertices of $\Gamma_{0,(n,b_n)_\rho}$. Then we have

$$P[\gamma \notin P_{k,D}, |\mu_{0,(n,b_n)_\rho}(f) - \nu^\rho(f)| > \epsilon] < \left(\frac{Dk + k - 1}{k - 1}\right)(D + 1)^{-2k} < e^{-ck}$$

(9.2)

for some $c > 0$ depending on $D$. It remains to bound $P[\gamma \in P_{k,D}]$. By Lemma 9.3 we have

$$P[\gamma \in P_{k,D}] < e^{-c_0 k} + P\left[T(\gamma) \leq \frac{2km}{(1 - \rho)^2 + \rho^2} - b_+(\rho^{-1} - (1 - \rho)^{-1}) - c_0 Dkm^{1/3}\right],$$

where $(km, b_+)_\rho$ is the intersection of $\Gamma_{0,(n,b_n)_\rho}$ with $\mathbb{L}_{km}$. When the event in the right hand side of (9.2) happens, we must have that (at least) one of the following happens:

- $\max_{b \in \mathbb{L}_{km}} T_{(km,b)_\rho,(n,b_n)_\rho} - (b - b_n)(\rho^{-1} - (1 - \rho)^{-1}) \geq c_0 Dkm^{1/3}/3,$
\[ T^*_0, \langle km, b_n \rangle_\rho \leq \frac{2km}{(1-\rho)^2 + \rho^2} - b_n(\rho^{-1} - (1-\rho)^{-1}) - c_0 D km^{1/3}/2, \]

where \( T^*_0, r_\rho = T_0, r_\rho - \xi(r^\rho) \). To see this, we assume the contrary, i.e. none of the above events happen. Then we must have

\[
T(\gamma) > T_{0, (n,b_n)} - T_{(km, b_+)\rho, \langle n,b_n \rangle_\rho} \geq T^*_0, \langle km, b_n \rangle_\rho - T_{(km, b_+)\rho, \langle n,b_n \rangle_\rho} \\
> \frac{2km}{(1-\rho)^2 + \rho^2} - b_+(\rho^{-1} - (1-\rho)^{-1}) - 5c_0 D km^{1/3}/6,
\]

which contradicts with the event in the right hand side of (9.2). Finally, we claim that we can bound the probability of each of the two events by \( e^{-c'k} \), for \( c' > 0 \) and \( k \) large enough, depending on \( m, D \). For the first event, note that \( n - km < m \), then \( |b - b_n| < 2m \), and the bound can be obtained by taking a union bound over all directed paths from \( \mathbb{L}_{km} \) to \( \langle n, b_n \rangle_\rho \). For the second event, apply Theorem 2.3. Thus the conclusion follows.

\[ \square \]

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