Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles

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Abstract

In this paper we study the characteristic or generating function of a certain discontinuous linear statistic of the Laguerre unitary ensembles and show that this is a particular fifth Painlevé transcendent in the variable \( t \), the position of the discontinuity. The proof of the ladder operators adapted to orthogonal polynomial with discontinuous weight announced sometime ago [13] is presented here, followed by the nonlinear difference equations satisfied by two auxiliary quantities and the derivation of the Painlevé equation.

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1. Introduction

In the theory of random matrix ensembles with unitary symmetry, the real eigenvalues \( \{x_j\}_{j=1}^n \) have the joint probability distribution

\[
P(x_1, \ldots, x_n) \, dx_1 \cdots dx_n = \frac{1}{n! D_n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{l=1}^n w_0(x_l) \, dx_l, \quad (1.1)
\]

where \( w_0(x) \) with \( x \in [\alpha, b] \) say, is strictly positive and satisfies a Lipshitz condition and has finite moments, that is, the existence of the integrals,

\[
\int_a^b x^j w_0(x) \, dx, \quad j \in \{0, 1, 2, \ldots\}.
\]

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Here $D_n$ is the normalization constant

$$D_n[u_0] = \frac{1}{n!} \int_{[a,b]^n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{l=1}^n w_0(x_l) \, dx_l,$$

so that

$$\int_{[a,b]^n} p(x_1, \ldots, x_n) \, dx_1 \cdots dx_n = 1.$$  \hfill (1.3)

We include the cases where $a$ may be $-\infty$ and/or $b$ may be $\infty$. For a comprehensive study of the theory of random matrices, see [20].

A linear statistics is a linear sum of a certain function $g$ of the random variable $x_j$:

$$\sum_{k=1}^n g(x_k).$$

The generating function of such a linear statistics is, by definition the average of $\exp(\lambda \sum_{k=1}^n g(x_k))$, with respect to the joint probability distribution (1.1), where $\lambda$ is a parameter, reads

$$\int_{[a,b]^n} \exp \left[ \lambda \sum_{k=1}^n g(x_k) \right] p(x_1, \ldots, x_n) \prod_{k=1}^n dx_k.$$

More generally, we can consider

$$\int_{[a,b]^n} \left[ \prod_{k=1}^n f(x_k) \right] p(x_1, \ldots, x_n) \prod_{k=1}^n dx_k.$$

If $f$ is a ‘smooth’ function then asymptotic formulas for large $n$ for the characteristic functions have been obtained for the Hermite case, where $w_0(x) = e^{-x^2}$, $x \in (-\infty, \infty)$ by Kac [16] and Akhiezer [1] and generalized by many authors. See [8] for a history of this problem. These results are continuous analogs of the classical Szegö limit theorem on Toeplitz determinants.

In the Laguerre case where $w_0(x) = x^\alpha e^{-x}$, $x \in [0, \infty)$ an analogous formula was found recently for smooth functions in [4]. However, results for the situations where $f$ has discontinuities are harder to come by. We mention here the original studies in [6] where $f$ has several discontinuities and which corresponds to the Hermite case. More general results can be found in [7, 9, 10, 21]. Also, results that correspond to $\alpha = \pm 1/2$ and large $n$ appear in [5].

In this paper we investigate the case where $n$ is finite, and $f$ is constant except for a jump at $t \in [0, \infty)$, and is of the form

$$f(x,t) = A + B \theta(x - t),$$

where $\theta(x)$ is 1 for $x > 0$ and 0 otherwise and $A \geq 0$ and $B > 0$. In the special case of linear statistics the function $g$ will take the form

$$g(x,t) := \theta(x - t) \ln \left( 1 + \frac{\beta}{2} \right) + \theta(t - x) \ln \left( 1 - \frac{\beta}{2} \right),$$

where $-1 < \frac{\beta}{2} < 1$. This corresponds to a function $f$ where

$$f(x,t) = \left( 1 - \frac{\beta}{2} \right)^{\lambda} + \left( 1 + \frac{\beta}{2} \right)^{\lambda} - \left( 1 - \frac{\beta}{2} \right)^{\lambda} \theta(x - t)$$

that is

$$A = \left( 1 - \frac{\beta}{2} \right)^{\lambda}.$$
and
\[ B = \left(1 + \beta^2\right)^{-\lambda} - \left(1 - \beta^2\right)^{-\lambda}. \]

We also point out that if \( A = 0 \) and \( B = 1 \) then we have the important case where we are computing the probability that all the eigenvalues are in the interval \([t, \infty)\). This case of course is what leads to the now well-known Tracy–Widom laws. More will be said about this later.

Our main tool will be to use the theory of orthogonal polynomials. Previously, in random matrix theory one made use of the orthogonal polynomials associated with the weight that defined the ensemble. Fundamental quantities were then described in terms of Fredholm determinants. While both the authors are very fond of determinants, in this work, we do not consider Fredholm determinants. Instead we consider the polynomials that are orthogonal to the perturbed weight, that is a regular or ‘nice’ weight multiplied by the discontinuous factor given in (1.7). In this manner we are able to use the results of the orthogonal polynomials to derive equations associated with the various statistics of interest.

The idea is that we write the multiple integral in (1.6) as a Hankel determinant. We then need to know information about the norms of the orthogonal polynomials. To understand this we need to know something about the recursion coefficients of the polynomials. This will lead us naturally to another pair of auxiliary quantities that depend on \( t \) and \( n \). In this paper they are called \( r_n \) and \( R_n \). Using these auxiliary quantities we are able to produce the second-order nonlinear differential equations satisfied by \( S_n = 1 - 1/R_n \) which turns out to be a particular fifth Painlevé transcendent, in addition to the Jimbo–Miwa–Okamoto \( \sigma \) form [15] satisfied by the logarithmic derivative of the Hankel determinant with respect to \( t \). We also wish to emphasize that the logarithmic derivative of the Hankel determinant can be computed very naturally in terms of our quantity \( r_n(t) \) and its derivative and the relationships between these quantities arise naturally using this approach.

We also derive a discrete version of the \( \sigma \) form of a nonlinear second-order difference equation satisfied by the same logarithmic derivative. Our computations show that in fact the values of our generalized polynomials at the end points of the intervals are intimately related to the resolvent kernels found in the standard approach of Tracy and Widom. This is really not surprising, since we are all starting with the same multiple integral. Rather, our point is that computations can all be made by using only the very basic theory of orthogonal polynomials.

The Painlevé equation can be found in [27]. The second-order difference equation (4.29, theorem 8), as far as we know is a new equation.

In the following section the proof for a pair of ladder operators, and the associated supplementary conditions adapted to orthogonal polynomials with discontinuous weights which was announced sometime ago [13] will be provided. In section 3, a system of difference equations satisfied by two auxiliary quantities \( r_n \) and \( R_n \) (these will ultimately determine the recurrence coefficients for the orthogonal polynomials) are derived. In section 4 we derive a second-order nonlinear differential equation which turns out to be a particular fifth Painlevé transcendent. In the process we identify that the quantity
\[ S_n(t) := 1 - \frac{1}{R_n(t)} \]

to be such an equation. Furthermore we show that the logarithmic derivative of the generating function
\[ H_n(t) := t \frac{d}{dt} \ln G(n, t) = t \frac{d}{dt} \ln D_n(t) \]
satisfies both the continuous and discrete \( \sigma \) form of Painlevé V.
2. Ladder operators and supplementary conditions

According to the general theory of orthogonal polynomials of one variable, for a generic weight \( w \), the normalization constant (1.2) has the two more alternative representations

\[ D_n[w] := \det (\mu_{i+j})_{i,j=0}^{n-1} := \det \left( \int_a^b x^i w(x) \, dx \right)_{i,j=0}^{n-1} \]

(2.1)

\[ = \prod_{j=0}^{n-1} h_j, \]

(2.2)

where the determinant of the moment matrix \((\mu_{i+j})\) is the Hankel determinant. Here, \( [h_j]_{j=0}^n \) is the square of the \( L^2 \) norm of the sequence of (monic-)polynomials \( \{P_j(x)\}_{j=0}^n \) orthogonal with respect to \( w \) over \([a, b]\):

\[ \int_a^b P_i(x) P_j(x) w(x) \, dx = \delta_{i,j} h_j. \]

(2.3)

Therefore, with reference to (1.2) and (1.5) the quantity that we need to compute is

\[ G(t, n) = \frac{D_n[w]}{D_n[w_0]} = \frac{\prod_{j=0}^{n-1} h_j(t)}{\prod_{j=0}^{n-1} h_j}, \]

where \( w(x, t) := x^\alpha e^{-x(A + B \theta(x - t))} \), and \( h_k(t) \) is defined by

\[ \int_0^\infty \{P_k(x)\}^2 (A + B \theta(x - t)) x^\alpha e^{-x} \, dx = h_k(t). \]

(2.4)

We also denote

\[ D_n(t) := D_n[w(., t)]. \]

This leads to the generic problem of the characterization of polynomials orthogonal with respect to ‘smooth’ weights \( w_0(x) \) perturbed by a jump factor where the discontinuity is at \( t \). So if we write

\[ w_J(x, t) := A + B \theta(x - t), \quad A \geq 0, \quad A + B > 0 \]

(2.5)

then

\[ \int_a^b P_i(x) P_j(x) w_J(x, t) \, dx = \delta_{i,j} h_j(t). \]

(2.6)

It follows from the orthogonality relations that

\[ z P_n(z) = P_{n+1}(z) + a_n(t) P_n(z) + b_n(t) P_{n-1}(z). \]

(2.7)

This three-term recurrence relation, together with the ‘initial’ conditions,

\[ P_0(z) = 1, \quad b_0 P_{-1}(z) = 0, \]

generates the monic polynomials,

\[ P_n(z) = z^n + p_1(n, t) z^{n-1} + \cdots, \]

(2.8)

the first two of which are

\[ P_0(z) = 1, \quad P_1(z) = z - a_0(t) = z - \frac{\mu_1(t)}{\mu_0(t)}. \]

(2.9)
Note that due to the \( t \) dependence of the weight, the coefficients of the polynomials and the recurrence coefficients \( \alpha_n \) and \( \beta_n \) also depend on \( t \), the position of the jump. However, unless it is required we do not display the \( t \) dependence.

From (2.7) and (2.8), we find, for \( n \in \{0, 1, 2, \ldots\} \)

\[
\alpha_n = p_1(n, t) - p_1(n + 1, t),
\]

\[
\sum_{j=0}^{n-1} \alpha_j = -p_1(n, t), \tag{2.10}
\]

where \( p_1(0, t) := 0 \).

From (2.6) and (2.7) we have the well-known strictly positive expression

\[
\beta_n := \frac{h_n}{h_{n-1}}. \tag{2.11}
\]

Another consequence of the recurrence relation is the Christoffel–Darboux formula

\[
\sum_{k=0}^{n-1} \frac{P_k(x) P_k(y)}{h_k} = \frac{P_n(x) P_n(y) - P_n(y) P_n(x)}{h_{n-1}(x - y)}. \tag{C - D}
\]

The above basic information about orthogonal polynomials can be found in [26].

In this section, we give an account of a recursive algorithm for the determination of the \( \alpha_n, \beta_n \) for a given weight. This is based on a pair of ladder operators and the associated supplementary conditions to be denoted as \((S_1)\) and \((S_2)\). For a general ‘smooth’ weight the lowering and raising operators have been derived by many authors [3, 11, 12, 24]. We should like to note here Magnus’s contribution to this formalism [17–19]. Indeed, we have been motivated by the investigation of [19] where he obtained the large \( n \) behavior of the recurrence coefficients of a generalization of the Jacobi polynomials in which the standard Jacobi weight is perturbed by a ‘line’ analog to the Fisher–Hartwig singularity. We end the discussion about the ladder operators with the remark that the supplementary conditions for orthogonal polynomials on the unit circle was found in [2] and have been used to compute explicitly the Toeplitz determinants with Fisher–Hartwig symbols.

The following lemma gives a detailed proof of the ladder operators in the discontinuous case where the results were announced sometime ago [13].

**Lemma 1.** Let \( w_0(x), x \in [a, b] \) be a smooth weight function where the associated moments

\[
\int_a^b x^j w_0(x) \, dx, \quad j \in \{0, 1, 2, \ldots\} \tag{2.12}
\]

of all order exist.

Let \( w_0(a) = w_0(b) = 0 \), and \( v_0(x) := -\ln w_0(x) \).

The lowering and raising operators for polynomials orthogonal with respect to

\[
w(x) := w_0(x) w_J(x, t)
\]

are

\[
P_n(z) = -B_n(z) P_n(z) + \beta_n A_n(z) P_{n-1}(z), \tag{2.13}
\]

\[
P_n^\prime(z) = [B_n(z) + v_0^\prime(z)] P_{n-1}(z) - A_{n-1}(z) P_n(z), \tag{2.14}
\]

where

\[
A_n(z) := \frac{R_n(t)}{z - t} + \frac{1}{h_n} \int_a^b \frac{v_0^\prime(z) - v_0^\prime(y)}{z - y} P_n^2(y) w(y) \, dy \tag{2.15}
\]
\begin{align}
\mathcal{B}_n(z) &:= \frac{r_n(t)}{z - t} + \frac{1}{h_{n-1}} \int_a^b \frac{v'_0(z) - v'_0(y)}{z - y} P_n(y) P_{n-1}(y) w(y) \, dy \\
\mathcal{R}_n(t) &:= B \frac{w_0(t)}{h_n(t)} (P_n(t, t))^2 \\
\mathcal{R}_n(t) &:= B \frac{w_0(t)}{h_{n-1}(t)} P_n(t, t) P_{n-1}(t, t),
\end{align}

where

\[ P_n(t, t) := P_n(z, t) \big|_{z = t}. \]

Here, \( w_0(x) \) is well defined since \( w_0(x) \) is supposed to be strictly positive for \( x \in [a, b] \).

**Proof.** We start from

\[ P'_n(z) = \sum_{k=0}^{n-1} C_{nk} P_k(z), \]

where \( C_{nk} \) is determined from the orthogonality relations

\[ C_{nk} = \frac{1}{h_k} \int_a^b P'_n(y) P_k(y) w(y) \, dy. \]

Therefore,

\[ P'_n(z) = \sum_{k=0}^{n-1} \frac{P_k(z)}{h_k} \int_a^b P'_n(y) P_k(y) w(y) \, dy \]

\[ = -\sum_{k=0}^{n-1} \int_a^b \frac{P_k(z)}{h_k} P_n(y)[P'_n(y) w(y) + P_k(y)[B \delta(y - t) w_0(y) + w'_0(y) w_J(y, t)]] \, dy \]

\[ = -\int_a^b P_n(y) \sum_{k=0}^{n-1} \frac{P_k(z) P_k(y)}{h_k} \left[ B w_0(y) \delta(y - t) + \frac{w'_0(y)}{w_0(y)} w(y) \right] \, dy \]

\[ = -\int_a^b P_n(y) \sum_{k=0}^{n-1} \frac{P_k(z) P_k(y)}{h_k} \left[ B w_0(y) \delta(y - t) + [v'_0(z) - v'_0(y)] w(y) \right] \, dy \]

\[ = -\int_a^b P_n(y) \frac{P_n(z) P_{n-1}(y) - P_n(y) P_{n-1}(z)}{h_{n-1}(z - y)} \left[ B \delta(y - t) w_0(y) \right. \]

\[ + [v'_0(z) - v'_0(y)] w(y) \] \, dy, \]

where we have used integration by parts, (C-D), the definition of \( v_0 \), (2.11) and that

\[ \int_a^b P_n(y) P_k(y) w(y) \, dy = 0, \quad k = 0, 1, 2, \ldots, n - 1, \]

to arrive at the above. A little simplification produces (2.15), and (2.16) follows from the straightforward application of the recurrence relations. \( \square \)

**Remark 1.** If \( w_0(a) \neq 0, w_0(b) \neq 0 \), the terms

\[ w(y) \left. \frac{1}{h_n(t)} (P_n(y, t))^2 \right|_{y=a}^b \quad \text{and} \quad w(y) \left. \frac{P_n(y, t) P_{n-1}(y, t)}{h_{n-1}(t)} \right|_{y=a}^b \]

are to be added into the definition of \( \mathcal{A}_n(z) \) and \( \mathcal{B}_n(z) \), respectively.
Remark 2. If there are several jumps at \( t_1, \ldots, t_N \) then the first term of (2.15) and (2.16) should be replaced by

\[
\sum_{j=1}^{N} \frac{R_{n,j}(t_j; t)}{z-t_j} = \sum_{j=1}^{N} \frac{r_{n,j}(t_j; t)}{z-t_j},
\]

where

\[
R_{n,j}(t_j; t) := B_{j} \frac{w_0(t_j)}{B_n(t)} \left(P_n(t_j; t)\right)^2
\]

and

\[
r_{n,j}(t_j; t) := B_{j} \frac{w_0(t_j)}{B_n(t)} P_n(t_j; t) P_{n-1}(t_j; t)
\]

\( t := (t_1, \ldots, t_N). \)

As in the case of the smooth weight the ‘coefficients’ \( A_n(z) \) and \( B_n(z) \) that appear in the ladder operators satisfy two identities valid for all \( z \in \mathbb{C} \cup \{\infty\} \), which we gather in the following lemma.

Lemma 2. The functions \( A_n(z) \) and \( B_n(z) \) satisfy the following identities which hold for all \( z \):

\[
B_{n+1}(z) + B_n(z) = (z-a_n) A_n(z) - v_0(z) \quad (S_1)
\]

\[
1 + (z-a_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z). \quad (S_2)
\]

Proof. By a direct computation using the definition of \( A_n(z) \) and \( B_n(z) \). \( \square \)

It turns out that a suitable combination of (S1) and (S2) produces an identity involving \( \sum_{j=0}^{n-1} A_j(z) \), from which further insight into the recurrence coefficients may be gained.

Lemma 3. \( A_n(z) \), \( B_n(z) \) and \( \sum_{j=0}^{n-1} A_j(z) \) satisfy the identity

\[
[ B_n(z) ]^2 + v_0(z) B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z). \quad (S'_2)
\]

Proof. Multiply (S2) by \( A_n(z) \) and replace \( (z-a_n) A_n(z) \) in the resulting equation by \( B_{n+1}(z) + B_n(z) + v_0(z) \). See (S1). This leads to

\[
[B_{n+1}(z)]^2 - [B_n(z)]^2 + v_0(z) [B_{n+1}(z) - B_n(z)] + A_n(z)
\]

\[
= \beta_{n+1} A_{n+1}(z) A_n(z) - \beta_n A_n(z) A_{n-1}(z).
\]

Taking a telescopic sum of the above equation from 0 to \( n-1 \) with the ‘initial’ conditions, \( B_0(z) = 0 \) and \( \beta_0 A_{n-1}(z) = 0 \), we have (S'2). \( \square \)

Let \( y = P_n(z) \) we find, by eliminating \( P_{n-1}(z) \) from the raising and lowering operators, the second-order differential equation.

Lemma 4.

\[
y''(z) - \left( v_0(z) + \frac{A_n'(z)}{A_n(z)} \right) y'(z) + \left( B_n'(z) - B_n(z) \frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z) \right) y(z) = 0. \quad (2.19)
\]

Proof. By a straightforward computation using (2.15), (2.16) and (S'2). \( \square \)

Recalling (2.17) and (2.18) we note that if \( v_0'(z) \) is rational in \( z \) then the difference kernel, \( [v_0'(z) - v_0(y)]/(z-y) \), is rational in \( z \) and \( y \). Consequently, (S1) and (S'2) may be put to good use to obtain a system of difference equations satisfied by the auxiliary quantities \( R_n \) and \( r_n \) and the recurrence coefficients \( \alpha_n \) and \( \beta_n \). This will be clear in the following section.
3. Recurrence coefficients and difference equations

For the problem at hand, 
\[ w_0(x) = x^\alpha e^{-x}, \quad x \in [0, \infty), \quad v_0(x) := -\ln w_0(x) = -\alpha \ln x + x \]
and for \( \alpha > 0, \ w_0(0) = 0. \) Note that \( w(\infty) = 0. \) An easy computation gives 
\[ \frac{v'(z) - v'(y)}{z - y} = \frac{\alpha}{z} \] 
Using these and integration by parts we have the following lemma.

Lemma 5.

\begin{align*}
A_n(z) &= \frac{R_n(t)}{z - t} + \frac{1 - R_n(t)}{z}, \quad (3.1) \\
B_n(z) &= \frac{r_n(t)}{z - t} - \frac{n + r_n(t)}{z}, \quad (3.2)
\end{align*}

where 
\begin{align*}
R_n(t) &:= Bw_0(t) \frac{(P_n(t, t))^2}{h_n(t)}, \quad (3.3) \\
r_n(t) &:= Bw_0(t) \frac{P_n(t, t)P_{n-1}(t, t)}{h_{n-1}(t)}. \quad (3.4)
\end{align*}

Proof. Through integration by parts, we find 
\[ \frac{\alpha}{\alpha} \int_0^\infty y^{\alpha - 1} e^{-y} w_J(y; t) \{ P_n(y, t) \}^2 \, dy = h_n(t) - Bw_0(t) \{ P_n(t, t) \}^2 \] 
(3.5) 
and we have used the fact that 
\[ \frac{\partial}{\partial x} P_n(x, t) = n P_{n-1}(x, t) + \text{lower degree} \]
to arrive at (3.6). From (3.5) and (3.6) and the definitions of \( A_n(z) \) and \( B_n(z), \) (3.1)–(3.4) follow.

Substituting (3.1) and (3.2) into \( (S_1) \), we find by equating the residues 
\[ r_{n+1} + r_n = R_n(t - \alpha_n) \quad (3.7) \]
\[ -(r_{n+1} + r_n) = 2n + 1 + \alpha - \alpha_n (1 - R_n). \quad (3.8) \]

Lemma 6.

\begin{align*}
\alpha_n &= 2n + 1 + \alpha + t R_n, \quad (3.9) \\
r_{n+1} + r_n &= R_n(t - \alpha_n). \quad (3.10)
\end{align*}
Proof. Equations (3.7)+(3.8) implies (3.9) and we restate (3.7) as (3.10).

Substituting (3.1) and (3.2) into (S′2), we find, after some elementary but messy computations,

\[
[B_n(z)]^2 + v'_0(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \frac{r_n^2}{(z-t)^2} + \frac{(n+r_n)(\alpha + n + r_n)}{z^2} + \sum_{j=0}^{n-1} R_j + r_n \left[1 - \frac{2 \alpha}{t} - \frac{2(n+r_n)}{t} \right] + \frac{1}{z} \left[n - \sum_{j=0}^{n-1} R_j + (n + r_n) \left(\frac{2r_n}{t} - 1\right) + \alpha t \right] \tag{3.11}
\]

and

\[
\beta_n A_n(z) A_{n-1}(z) = \frac{\beta_n R_n R_{n-1}}{(z-t)^2} + \frac{\beta_n (1 - R_n)(1 - R_{n-1})}{z^2} + \frac{1}{t} \left(\frac{1}{z - t} - \frac{1}{z}\right) \beta_n [(1 - R_n) R_{n-1} + (1 - R_{n-1}) R_n] \tag{3.12}
\]

Now (S′2) implies the following lemma.

**Lemma 7.** For a fixed \(t\), the quantities \(r_n, R_n, \beta_n\) satisfy the equations

\[
r_n^2 = \beta_n R_n R_{n-1} \tag{3.13}
\]

\[
(n + r_n)(n + \alpha + r_n) = \beta_n (1 - R_n)(1 - R_{n-1}) \tag{3.14}
\]

\[
\sum_{j=0}^{n-1} R_j + r_n \left[1 - \frac{\alpha}{t} - \frac{2(n + r_n)}{t} \right] = \frac{\beta_n}{t} [(1 - R_n) R_{n-1} + (1 - R_{n-1}) R_n] \tag{3.15}
\]

**Proof.** Equations (3.13)–(3.15) are obtained by equating residues of (S′2). \( \square \)

In the following lemma an expression is found for \(\beta_n\) in terms of \(r_n\) and \(R_n\).

**Lemma 8.** In terms of \(r_n\) and \(R_n\), the off-diagonal recurrence coefficient \((\beta_n)\) reads

\[
\beta_n = \frac{1}{1 - R_n} \left[ r_n(2n + \alpha) + n(n + \alpha) + \frac{r_n^2}{R_n} \right] \tag{3.16}
\]

**Proof.** We eliminate \(\beta_n R_n R_{n-1}\) from (3.13) and (3.14) to find

\[
r_n(2n + \alpha) + n(n + \alpha) = \beta_n (1 - R_n - R_{n-1}) \tag{3.17}
\]

\[
= \beta_n (1 - R_n) - \frac{r_n^2}{R_n}. \tag{3.18}
\]

In the last step we have used (3.13) to replace \(\beta_n R_n - 1\) by \(r_n^2/R_n\). \( \square \)

We note that \(B > 0\) can always be satisfied for the proper range of \(\lambda\).

Equation (3.9) states that \(c_n\) is linear in \(R_n\) up to a linear form in \(n\), together with (3.10) and (3.16), when combined with say, (3.13) provide us with a pair of nonlinear difference equations satisfied by \(r_n\) and \(R_n\). We state this in the following theorem.
Theorem 1. The quantities $r_n$ and $R_n$ satisfy the difference equations

$$
\begin{align*}
r_{n+1} + r_n &= R_n(t - 2n - \alpha - 1 - tR_n) \\
r_n^2 \left( \frac{1}{R_n R_{n-1}} - \frac{1}{R_n} - \frac{1}{R_{n-1}} \right) &= r_n(2n + \alpha) + n(n + \alpha)
\end{align*}
$$

(3.19) (3.20)

with the ‘initial’ conditions

$$
\begin{align*}
r_0(t) &= 0 \\
R_0(t) &= \frac{Bt^\alpha e^{-t}}{h_0(t)} \\
h_0(t) &= \left( 1 - \frac{\beta}{2} \right)^\lambda \Gamma(1 + \alpha) + \left[ \left( 1 + \frac{\beta}{2} \right)^\lambda - \frac{\beta}{2} \right] \int^{\infty}_t x^\alpha e^{-x} \, dx.
\end{align*}
$$

(3.21) (3.22) (3.23)

Proof. This is simply a restatement of (3.10) and (3.13) with (3.9) and (3.16).

We shall see that $(S'_2)$ automatically performs finite sums in ‘local’ form, of the quantities $R_n$ and $\alpha_n$. This will be seen later to be relevant in the evaluation of the derivative of \( \ln D_n(t) \) with respect to $t$ and the derivation of the Painlevé transcendent.

Theorem 2.

$$
\begin{align*}
t \sum_{j=0}^{n-1} R_j &= -tr_n - n(n + \alpha) + \beta_n \\
\sum_{j=0}^{n-1} \alpha_j &= -p_1(n) = \beta_n - tr_n.
\end{align*}
$$

(3.24) (3.25)

Proof. From (3.15), we have

$$
\begin{align*}
t \sum_{j=0}^{n-1} R_j &= r_n [2(n + r_n) + \alpha - t] + \beta_n [R_n + R_{n-1} - 2R_n R_{n-1}] \\
&= r_n [2(n + r_n) + \alpha - t] + \beta_n [R_n + R_{n-1}] - 2r_n \\
&= r_n [2(n + r_n) + \alpha - t] + \beta_n - r_n(2n + \alpha) - n(n + \alpha) - 2r_n \\
&= -tr_n - n(n + \alpha) + \beta_n.
\end{align*}
$$

The second equality of (3.26) follows from (3.13) and the third equality follows from (3.17). Equation (3.25) follows from (3.9) and the second equality of (2.10).

4. $P_V(0, -\frac{\alpha^2}{2}, 2n + 1 + \alpha, -\frac{1}{2})$

In this section we shall discover which of the auxiliary quantities defined as the residues of the rational functions $A_n(z)$ and $B_n(z)$ is a Painlevé transcendent.

This will be obtained from a pair of Toda equations which shows that the Hankel determinant is the $\tau$-function and these when suitably combined with the difference equations produce our $P_V$.
Taking the derivative of $h_n(t)$ with respect to $t$, we find
\[
\frac{d}{dt} \ln h_n(t) = -Bw_0(t) \frac{(P_n(t, t))^2}{h_n(t)} = -R_n(t),
\]
and consequently we have the following theorem.

**Theorem 3.**
\[
-t \frac{d}{dt} \ln D_n(t) = -t \sum_{j=0}^{n-1} \frac{d}{dt} \ln h_j(t)
\]
\[
= t \sum_{j=0}^{n-1} R_j = -p_1(n, t) - n(n + \alpha).
\]

**Proof.** The proof is obvious. \(\square\)

The following lemma gives the derivative of $p_1(n, t)$ with respect to $t$.

**Lemma 9.**
\[
\frac{d}{dt} p_1(n, t) = r_n(t).
\]

**Proof.** Note the $t$ dependence of $p_1(n, t)$. Taking a derivative of
\[
0 = \int_0^\infty P_n(x) P_{n-1}(x) w_j(x, t) w_0(x) \, dx,
\]
with respect to $t$, produces
\[
0 = -Bw_0(t) P_n(t, t) P_{n-1}(t, t) + \int_0^\infty \left[ \frac{d}{dt} p_1(n, t)x^{n-1} + \ldots \right] P_{n-1}(x) w_j(x, t) w_0(x) \, dx
\]
\[
= -Bw_0(t) P_n(t, t) P_{n-1}(t, t) + h_{n-1} \frac{d}{dt} D_1(n, t)
\]
and (4.3) follows. \(\square\)

We expect $D_n(t)$ to satisfy the Toda molecule equation [25] and this should indicate the emergence of a Painlevé transcendent. The question that we will address is ‘Which quantity is satisfied by this particular Painlevé transcendent?’

**Theorem 4.** The Hankel determinant $D_n(t)$ satisfy the following differential-difference or the Toda molecule equation [25]:
\[
t^2 \frac{d^2}{dt^2} \ln D_n(t) = -n(n + \alpha) + \frac{D_{n+1}(t)D_{n-1}(t)}{D_n^2(t)}.
\]

**Proof.** Taking a derivative of (4.2) with respect to $t$ and (4.3) imply
\[
\frac{d}{dt} \left( t \frac{d}{dt} \ln D_n(t) \right) = r_n.
\]
Now substitute $r_n$ given above into (3.24) to find
\[
t \sum_{j=0}^{n-1} R_j = -t \frac{d}{dt} \left[ t \frac{d}{dt} \ln D_n(t) \right] - n(n + \alpha) + \beta_n.
\]
\[
= -t \frac{d}{dt} \ln D_n(t),
\]
where the last equality comes from (4.2). Equation (4.4) follows if we recall
\[ \beta_n = \frac{h_n}{h_{n-1}} = \frac{D_{n+1}D_{n-1}}{D_n^2}, \]
since \( D_n = h_0 \cdots h_{n-1}. \)
\( \square \)

We now state a pair of somewhat non-standard Toda equations.

**Lemma 10.** The recurrence coefficients \( \alpha_n \) and \( \beta_n \) satisfy for \( n \in \{1, 2, \ldots\} \) the differential-difference equations
\[
\begin{align*}
\beta_n'(t) &= (R_{n-1} - R_n)\beta_n \\
\alpha_n'(t) &= r_n - r_{n-1},
\end{align*}
\]
with \( r_0(t) \) and \( R_0(t) \) given by (3.22) and (3.23), respectively.

**Proof.** These equations are an immediate consequence of (4.1), (2.11), (4.3) and the first equality (2.10).
\( \square \)

To discover the \( P_V \) of our problem. We first state two preliminary lemmas describing the \( t \) evolution of \( r_n \) and \( R_n \).

**Lemma 11.** For a fixed \( n \), \( R_n(t) \) satisfy the Riccati equation
\[
tr_n' = 2r_n + (2n + \alpha - t + tR_n)R_n.
\]

**Proof.** We begin with \((T_2)\) and replace \( r_{n+1} \) by \( R_n(t - \alpha_n) - r_n \). See (3.7). This leaves
\[
\alpha_n' = 2r_n - (t - \alpha_n)R_n.
\]
After eliminating \( \alpha_n \) in favor of \( R_n \) with (3.9) we have (4.5).
\( \square \)

**Lemma 12.** For a fixed \( n \), \( r_n(t) \) satisfy the Riccati equation
\[
tr_n' = \frac{1 - 2R_n}{R_n(1 - R_n)} (r_n)^2 - (2n + \alpha) \frac{R_n r_n}{1 - R_n} - n(n + \alpha) \frac{R_n}{1 - R_n}.
\]

**Proof.** By equating (3.24) to the last equality of (4.2), we find
\[
p_1(n, t) = tr_n - \beta_n.
\]
Taking a derivative of the above equation with respect to \( t \) and noting (4.3) we see that
\[
tr_n = \beta_n' = [R_{n-1} - R_n]\beta_n = \frac{r_n^2 R_n}{R_n} = \beta_n R_n,
\]
and use have been made of \((T_2)\) and (3.13) to obtain the last two equalities. Equation (4.6) follows if we express \( \beta_n \) in terms of \( r_n \) and \( R_n \) using (3.16).
\( \square \)

The following theorem shows that \( R_n \) is up to a linear fractional transformation of a particular \( P_V \).

**Theorem 5.** The quantity
\[
S_n(t) := 1 - \frac{1}{R_n(t)}
\]
satisfies
\[
S_n'' = \frac{3S_n - 1}{2S_n(1 - S_n)} (S_n')^2 - \frac{S_n'}{t} - \frac{\alpha^2 (S_n - 1)^2}{2t^2 S_n} + (2n + 1 + \alpha) \frac{S_n}{t} - \frac{1}{2} \frac{S_n(S_n + 1)}{S_n - 1},
\]
which is \( P_V(0, -\alpha^2/2, 2n + 1 + \alpha, -1/2). \)
In terms of the recurrence coefficient $\alpha_n(t)$, we have
\[ S_n(t) = \frac{\alpha_n(t) - (2n + \alpha + 1) - t}{\alpha_n(t) - (2n + \alpha + 1)}. \] (4.9)

**Proof.** Eliminate $r_n(t)$ from (4.5) and (4.6) and with $R_n = 1/(1 - S_n)$ gives (4.8). We have followed the convention of [14]. □

**Remark 3.** Note that for $n = 0$, (4.8) is satisfied by
\[ S_0(t) = 1 - \frac{1}{R_0(t)}, \]
where $R_0(t)$ is given by (3.22) and (3.23) and ultimately in terms of an incomplete gamma function—a special case of the Kummer function of the second kind. Furthermore, since $r_0(t) = 0$, it can be verified that $R_0(t)$ also satisfy (4.5) at $n = 0$.

We may express the logarithmic derivative of $D_n(t)$ with respect to $t$ in the so-called Jimbo–Miwa–Okamoto $\sigma$ form. This is described in the following theorem.

**Theorem 6.** Let
\[ H_n(t) := t \frac{d}{dt} \ln D_n(t), \] (4.10)
then
\[ (t H''_n)^2 = 4(H'_n)^2[H_n - n(n + \alpha) + (2n + \alpha - t)H'_n + H_n] + [H'_n]^2. \] (4.11)

**Proof.** First we express $r_n(t)$ and $\beta_n(t)$ in terms of $H_n$ and its derivatives. From (3.24) and (4.2), we have
\[ -H_n = -tr_n + \beta_n - n(n + \alpha) \]
\[ = -p_1(n, t) - n(n + \alpha). \] (4.12)

Taking a derivative of (4.2) with respect to $t$ and recalling (4.3), we have
\[ r_n = H'_n, \] (4.13)
and with the first equality of (4.12) and (4.13), we find
\[ \beta_n = tH'_n - H_n + n(n + \alpha). \] (4.14)

Now a derivative of (4.14) with respect to $t$ and $(T_1)$ gives
\[ (tH''_n) - H'_n = tH''_n \]
\[ = \beta''_n = (R_{n-1} - R_n) \beta_n \]
\[ = r^2_n R_n - \beta_n R_n. \] (4.15)

Here we have made use of (3.13) to arrive at the last equality. Therefore, we have a quadratic equation in $R_n$:
\[ r^2_n - \beta_n R_n = tH''_n. \] (4.16)

There is another quadratic equation in $R_n$ which is a restatement of (3.16):
\[ r^2_n + \beta_n R_n = \beta_n - (2n + \alpha)r_n - n(n + \alpha). \] (4.17)
Now we solve for $R_n$ and $1/R_n$ from (4.16) and (4.17) and find
\[
\frac{2R_n^2}{R_n} = \beta_n - (2n + \alpha)r_n - n(n + \alpha) + tH_n''
\]
\[
2\beta_n R_n = \beta_n - (2n + \alpha)r_n - n(n + \alpha) - tH_n''.
\]
Equation (4.10) follows from the product of the above two equations,
\[
4\beta_n r_n^2 = [\beta_n - (2n + \alpha)r_n - n(n + \alpha)]^2 - (tH_n'')^2,
\]
and (4.13) and (4.14).

Incidentally, $R_n$ has two alternative representations:
\[
R_n = \frac{tH_n'' + (2n + \alpha - t)H_n' + H_n}{2\left[H_n - n(n + \alpha) - tH_n''\right]},
\]
(4.18)
\[
\frac{1}{R_n} = \frac{tH_n'' - (2n + \alpha - t)H_n' - H_n}{2(H_n')^2}.
\]
(4.19)

The ‘discrete’ structure inherited from the recurrence relations (2.7) induces a discrete analog of the $\sigma$ form, namely, a nonlinear second-order difference equation in $n$ satisfied by $H_n$ for a fixed $t$; we believe such a discrete form is new and may have been missed in previous similar studies perhaps because the recurrence relations were not sufficiently exploited. We note here that our derivation of (4.11) bypasses a third-order equation and without having to identify a first integral which reduces the order by 1.

We also note that equation (4.11) was first discovered by Tracy and Widom in [27] (which in our problem corresponds to $A = 0$ and $B = 1$) and just as was done in their paper for the Hermite case one can also rescale to obtain the Painlevé III equation corresponding to the Bessel kernel or ‘hard edge scaling’. We change variables $t \rightarrow s/4n$, $H_n \rightarrow \sigma$ use (4.11) and keep only the highest order terms to obtain
\[
(s\sigma'')^2 = 4\sigma(\sigma')^2 - 4s(\sigma')^3 - s(\sigma')^2 + \sigma\sigma' + \alpha^2(\sigma')^2.
\]

Finally, we point out that the above analysis shows that the resolvent kernel used in the Tracy–Widom approach can be directly related to the orthogonal polynomials defined on $(t, \infty)$. In fact, if we denote $\tilde{R}(t, t)$ as the resolvent kernel defined in [27] then
\[
t\tilde{R}(t, t) = H_n(t) = -tr_n - n(n + \alpha) + \beta_n.
\]

Thus,
\[
t\tilde{R}(t, t) = -t\frac{Bw_0 P_n(t, t) P_{n-1}(t, t)}{h_{n-1}(t)} - n(n + \alpha) + \frac{h_n(t)}{h_{n-1}(t)}.
\]
The term $\beta_n$ can also be written using (3.16). In addition, we have that
\[
\frac{d}{dt}(t\tilde{R}(t, t)) = r_n = \frac{Bw_0 P_n(t, t) P_{n-1}(t, t)}{h_{n-1}(t)}.
\]
In other words, we have found an identity for the resolvent kernel in terms of the values at the end points of the normalized orthogonal polynomials.

**Theorem 7.** The auxiliary quantities $R_n$ and $r_n$ are expressed in terms of $H_n$ and $H_{n+1}$ as follows:
\[
tR_n = H_n - H_{n+1}
\]
(4.20)
\[ tr_n = \frac{[H_n - n(n + \alpha)](t + H_{n+1} - H_{n-1}) + tn(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha}. \]  
(4.21)

The discrete analog of the \( \sigma \) form satisfied by \( H_n \) results from the substitution of (4.20) and (4.21) into
\[ (tr_n)^2 = [n(n + \alpha) + tr_n - H_n] (t R_n)^2 + t R_n (H_{n+1} + H_{n-1} - 2H_n). \]  
(4.22)

**Proof.** Taking a first-order difference on the second equality of (4.2) together with (2.10) and (3.9) implies (4.20). \( \square \)

We rewrite (3.24) and give
\[ \beta_n = n(n + \alpha) + tr_n - H_n. \]  
(4.23)

We will now find another equation expressing \( \beta_n \) in terms of \( r_n, R_n, H_n, H_{n \pm 1} \). Taking a first-order difference on (4.20) gives
\[ t(R_n - R_{n-1}) = 2H_n - H_{n+1} - H_{n-1}. \]
Now multiply the above equation by \( R_n \) and make use of (3.13) we find
\[ t R_n^2 - \frac{tr_n^2}{\beta_n} = (2H_n - H_{n+1} - H_{n-1}) R_n, \]
and therefore
\[ \frac{1}{\beta_n} = \frac{t R_n^2 -(2H_n - H_{n+1} - H_{n-1}) R_n}{tr_n^2}. \]  
(4.24)

Therefore, the product of (4.23) and (4.24) implies (4.21), which leaves us the job of finding a further expression of \( r_n \) in terms of \( H_n \) and \( H_{n \pm 1} \). For this purpose, we rewrite (3.17) as
\[ \beta_n (1 - R_n - R_{n-1}) = (2n + \alpha) r_n + n(n + \alpha). \]
Now substitute \( \beta_n \) given in (4.23) into the above resulting a linear equation in \( r_n \):
\[ r_n [(t - t R_n - t R_{n-1}) - 2n - \alpha] = [H_n - n(n + \alpha)](1 - R_n - R_{n-1}) + n(n + \alpha). \]
With \( t R_n \) as in (4.20), we have (4.21).

We summarize our results in the following theorem.

**Theorem 8.** Let \( D_n(t) \) be the Hankel determinant associated with the Laguerre weight perturbed by a jump factor, and
\[ H_n(t) := \frac{d}{dt} \ln D_n(t). \]
Then the recurrence coefficients are
\[ \alpha_n(t) = (2n + \alpha + 1) = \frac{t^2 H_n'' + [(2n + \alpha)t - t^2] H'_n + t H_n}{2[H_n - n(n + \alpha) - t H'_n]}, \]  
(4.25)
\[ \beta_n(t) = n(n + \alpha) = t H'_n - H_n, \]  
(4.26)

where \( 2n + 1 + \alpha \) and \( n(n + \alpha) \) are the ‘unperturbed’ recurrence coefficients and \( H_n \) satisfies a nonlinear differential equation in the Jimbo–Miwa–Okamoto \( \sigma \) form
\[ (t H_n'')^2 = 4(H_n')^2[H_n - n(n + \alpha) - t H'_n] + [(2n + \alpha - t) H'_n + H_n]^2. \]
For the same $H_n$, the recurrence coefficients are
\begin{align}
\alpha_n(t) - (2n + \alpha + 1) &= H_n - H_{n+1}, \quad (4.27) \\
\beta_n(t) - n(n + \alpha) &= \frac{H_n(2n + \alpha) - n(n + \alpha)(H_{n+1} - H_{n-1})}{t + H_{n+1} - H_{n-1} - 2n - \alpha}, \quad (4.28)
\end{align}
where $H_n$ satisfies the discrete $\sigma$ form of a nonlinear difference equation
\begin{align}
\left\{ \frac{[H_n - n(n + \alpha)](t + H_{n+1} - H_{n-1}) + tn(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha} \right\}^2 \\
= \left\{ \frac{(2n + \alpha)[H_n - n(n + \alpha)] + tn(n + \alpha)}{t + H_{n+1} - H_{n-1} - 2n - \alpha} \right\}(H_n - H_{n+1})(H_{n-1} - H_n), \quad (4.29)
\end{align}
Note that since $\alpha_n(t)$ and $\beta_n(t)$ have two alternative representations, $H_n(t)$ satisfies two more differential-difference equations, (4.25) = (4.27) and (4.26) = (4.28).

We end this paper with a discussion on the relationship between our PV and the difference equations (3.19) and (3.20). We would like to thank the second referee for supplying us the background material part of which is reproduced here.

The fifth Painlevé equation $P_V(a, b, c, d = -1/2)$
\begin{align}
y'' &= \left( \frac{1}{2y} + \frac{1}{y - 1} \right)(y')^2 - \frac{y'}{t} + \frac{(y - 1)^2}{t^2} \left( ay + \frac{b}{y} \right) + c \frac{y}{t} + d \frac{y(y + 1)}{y - 1}, \quad = \frac{d}{dt}
\end{align}
is equivalent to the Hamiltonian system $\mathcal{H}_V$:
\begin{align}
q' &= \frac{\partial H}{\partial p}, \quad tp' = -\frac{\partial H}{\partial q},
\end{align}
with the time-dependent Hamiltonian $H = H(p, q, t)$:
\begin{align}
tH &= p(p + t)q(q - 1) + \alpha_2qt - \alpha_3pq - \alpha_1p(q - 1),
\end{align}
where
\begin{align}
a &= \alpha_0^2 \frac{1}{2}, \quad b = -\alpha_3^2 \frac{1}{2}, \quad c = \alpha_0 - \alpha_2, \quad d = -\frac{1}{2}, \\
\alpha_0 &:= 1 - \alpha_1 - \alpha_2 - \alpha_3
\end{align}
and
\begin{align}
y &= 1 - \frac{1}{q}.
\end{align}
The Hamiltonian structure was studied in [22] and the $\tau$-function is defined such that
\begin{align}
\frac{d}{dt} \ln \tau = H.
\end{align}
The extended affine Weyl group $W(A_3^{(1)}) = \langle s_0, s_1, s_2, s_3, \pi \rangle$ of the Weyl group type $A_3^{(1)}$ acts as bi-rational symmetries on $P_V$ and induces Backlund transformations on the solutions of $P_V$. Here the $s_i$’s and $\pi$ are the generators. See [22] for the study of Weyl group actions on $P_V$.

For example, the action of $s_0$
\begin{align}
\alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle = \{ -\alpha_0, \alpha_1 + \alpha_0, \alpha_2, \alpha_3 + \alpha_0 \}, \\
s_0(q) &= q + \frac{\alpha_0}{p + t}, \\
s_0(p) &= p.
\end{align}
leaves $P_V$ or the Hamiltonian system $H_V$ invariant. We refer the readers to [22, 28] for information on Weyl group actions and ((4.3), [28]) which lists the bi-rational transformations.

To proceed further, consider a parallel transformation $l : \mathbf{a} = (a_0, a_1, a_2, a_3) \mapsto \mathbf{a} \oplus (1, 0, -1, 0)$.

From a direct computation, we may verify that the variables $q$ and $r := pq(q - 1)$ satisfy the following system of difference equations:

$$l(r) + r = q(a_2 - a_0 + t - tq) - a_1 \quad (4.30)$$

$$\left( \frac{1}{q} - 1 \right) \left( \frac{1}{l^{-1}(q)} - 1 \right) = \frac{(r - a_2)(r - a_2 - a_3)}{r(r + a_1)} \quad (4.31)$$

these seems to the $d - P_{\text{III}}$ of (205, [23]) in disguise.

In terms of $H$ our auxiliary parameter $r$ reads

$$r = \frac{d}{dt}(tH). \quad (4.32)$$

In our problem we have $P_V(0, -\alpha^2/2, 2n + 1 + \alpha, -1/2)$, which implies

$$(a_0, a_1, a_2, a_3) = (1 + n, 0, -n - \alpha, \alpha).$$

If

$$l^n(q) = q_n =: R_n$$

$$l^n(r) =: r_n,$$

for $n \in \{0, 1, 2, \ldots \}$, then a direct computation shows that

$$r_{n+1} + r_n = R_n(-\alpha - 2n - 1 + t - tR_n), \quad (4.33)$$

$$\left( \frac{1}{R_n} - 1 \right) \left( \frac{1}{R_{n-1}} - 1 \right) = \frac{(r_n + n + \alpha)(r_n + n)}{r_n^2} \quad (4.34)$$

are equivalent to (3.19) and (3.20), respectively.

We should like to mention here that (3.19) and (3.20) and other equations are derived entirely from orthogonality and the immediate consequence—the recurrence relations.

In view of (3.32) we see that the logarithmic derivative of the generating function $G(n, t) = D_n(t)$ is the $\tau$-function of our $P_V$. We end this paper with the final remark: equation (4.4) is essentially the same as the Toda equation among a $\tau$-sequence discovered by Okamoto [22].

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