STOCHASTIC DYNAMICS RELATED TO PLANCHEREL MEASURE ON PARTITIONS

ALEXEI BORODIN AND GRIGORI OLSHANSKI

Abstract. Consider the standard Poisson process in the first quadrant of the Euclidean plane, and for any point \((u, v)\) of this quadrant take the Young diagram obtained by applying the Robinson–Schensted correspondence to the intersection of the Poisson point configuration with the rectangle with vertices \((0, 0)\), \((u, 0)\), \((u, v)\), \((0, v)\). It is known that the distribution of the random Young diagram thus obtained is the poissonized Plancherel measure with parameter \(uv\).

We show that for \((u, v)\) moving along any southeast–directed curve \(C\) in the quadrant, these Young diagrams form a Markov process \(\Lambda_C\) with continuous time.

We also describe \(\Lambda_C\) in terms of jump rates.

Our main result is the computation of the dynamical correlation functions of such Markov processes and their bulk and edge scaling limits.

Introduction

For any \(n = 1, 2, \ldots\), consider a measure on the set of all partitions of \(n\) which assigns to a partition \(\lambda\) the square of the dimension of the corresponding irreducible representation of the symmetric group \(S(n)\), divided by \(|S(n)| = n!\). The classical Burnside formula implies that this is a probability measure (the sum of all weights equals one). It is usually referred to as the \(n\)th Plancherel measure.

As was independently shown by Logan–Shepp [LS] and Vershik–Kerov [VK1], [VK3], for large \(n\) the random partitions distributed according to the \(n\)th Plancherel measure have a typical (limit) shape. More detailed information about local behavior of the random partitions in different regions of the limit shape was later obtained in Baik–Deift–Johansson [BDJ1], [BDJ2], Okounkov [Ok], Borodin–Okounkov–Olshanski [BOO], Johansson [Jo1] for the “edge” of the limit shape, and in Borodin–Okounkov–Olshanski [BOO] for the “bulk” of the limit shape.

One key observation that allowed to perform such a detailed analysis was that the mixture of the Plancherel measures with different \(n\)’s by a Poisson distribution, the so–called poissonized Plancherel measure, has a nice algebraic structure: it defines a determinantal point process\(^2\) [BOO], [Jo1].

In this work we construct stationary Markov processes on the set of all partitions which have the poissonized Plancherel measures as their invariant distributions. We prove that for any finite number of time moments, the corresponding joint

\footnote{A different proof was later found by Kerov, see [IO].}

\footnote{This means that its correlation functions can be written as minors of a suitable matrix called the correlation kernel.}
distribution of the same number of random partitions defines a determinantal point process, and we compute its correlation kernel.

As in the “static” case, there are two limit transitions, at the edge and in the bulk of the limit shape. The corresponding limit of the correlation kernel at the edge turns out to be the well–known extended Airy kernel, while the limiting kernel in the bulk appears to be a new one.

As a matter of fact, we obtain these results for more general, nonstationary Markov processes on partitions, which change the value of the poissonization parameter of the poissonized Plancherel measure with time. This allows us to interpret the results in terms of the Poisson process in a quadrant and its projection by the Robinson–Schensted correspondence.

Our results also extend to more general measures on partitions, the so–called z–measures [BO1]. Markov processes related to these measures are studied in detail in our paper [BO2]. The Plancherel measures may be viewed as appropriate limits of the z–measures, and we use this connection extensively in our proofs: the results of §3 are obtained from the similar results of [BO2] by a degeneration.

Our work was largely inspired by previous papers due to Okounkov–Reshetikhin [OR], Präähofer–Spohn [PS], and Johansson [Jo2]. Thus, some of the results below may already be known to experts. In particular, the results of section 3 can be obtained using the formalism of Schur processes [OR] (this is not true for the z–measures, however), and for certain special cases of our Markov processes the determinantal structure of the correlation functions and the edge scaling limit were obtained by Präähofer–Spohn [PS] in their work on polynuclear growth processes.

Acknowledgements. This research was partially conducted during the period the first author (A. B.) served as a Clay Mathematics Institute Research Fellow. He was also partially supported by the NSF grant DMS-0402047. The second author (G. O.) was supported by the CRDF grant RM1-2543-MO-03.

1. Construction of Markov processes

As in Macdonald [Ma] we identify partitions and Young diagrams. By $\mathbb{Y}_n$ we denote the set of partitions of a natural number $n$, or equivalently, the set of Young diagrams with $n$ boxes. By $\mathbb{Y}$ we denote the set of all Young diagrams, that is, the disjoint union of the finite sets $\mathbb{Y}_n$, where $n = 0, 1, 2, \ldots$ (by convention, $\mathbb{Y}_0$ consists of a single element, the empty diagram $\emptyset$). Given $\lambda \in \mathbb{Y}$, let $|\lambda|$ denote the number of boxes of $\lambda$ (so that $\lambda \in \mathbb{Y}_{|\lambda|}$), and let $\ell(\lambda)$ be the number of nonzero rows in $\lambda$ (the length of the partition).

For two Young diagrams $\lambda$ and $\mu$ we write $\mu \nearrow \lambda$ (equivalently, $\lambda \searrow \mu$) if $\mu \subset \lambda$ and $|\mu| = |\lambda| - 1$, or, in other words, $\mu$ is obtained from $\lambda$ by removing one box.

Let $\dim \lambda$, the dimension of $\lambda$, be the number of all standard tableaux of shape $\lambda$. Equivalently, $\dim \lambda$ is the dimension of the irreducible representation of the symmetric group $S(|\lambda|)$ labelled by $\lambda$. A convenient explicit formula for $\dim \lambda$ is

$$\dim \lambda = \frac{n!}{\prod_{i=1}^{N} (\lambda_i + N - i)!} \prod_{1 \leq i < j \leq N} (\lambda_i - i - \lambda_j + j), \quad \lambda \in \mathbb{Y}_n,$$

where $N$ is an arbitrary integer $\geq \ell(\lambda)$ (the above expression is stable in $N$).
For $\lambda \in Y_n, \mu \in Y_{n-1}$ set
\[
p^\downarrow(n, \lambda; n-1, \mu) = \begin{cases} 
\frac{\dim \mu}{\dim \lambda}, & \mu \nless \lambda, \\
0, & \text{otherwise},
\end{cases}
\]
and for $\lambda \in Y_n, \nu \in Y_{n+1}$ set
\[
p^\uparrow(n, \lambda; n+1, \nu) = \begin{cases} 
\frac{\dim \nu}{\dim \lambda(n+1)}, & \lambda \nless \nu, \\
0, & \text{otherwise}.
\end{cases}
\]

Then we have (see Vershik–Kerov [VK2])
\[
\sum_{\mu \in Y_{n-1}} p^\downarrow(n, \lambda; n-1, \mu) = 1, \\
\sum_{\nu \in Y_{n+1}} p^\uparrow(n, \lambda; n+1, \nu) = 1.
\]

The $n$th Plancherel measure is a probability measure on the finite set $Y_n$ which is defined by
\[
M^{(n)}(\lambda) = \frac{\left(\dim \lambda\right)^2}{n!}, \quad \lambda \in Y_n,
\]
see [VK2]. The Plancherel measures with various indices $n$ are related to each other by means of the “down probabilities” $p^\downarrow(n, \lambda; n-1, \mu)$ and the “up probabilities” $p^\uparrow(n, \lambda; n+1, \nu)$, as follows (see [VK2])
\[
M^{(n-1)}(\mu) = \sum_{\lambda \in Y_n} M^{(n)}(\lambda) p^\downarrow(n, \lambda; n-1, \mu),
\]
\[
M^{(n+1)}(\nu) = \sum_{\lambda \in Y_n} M^{(n)}(\lambda) p^\uparrow(n, \lambda; n+1, \nu).
\]

Note also the following relation:
\[
M^{(n)}(\lambda) p^\uparrow(n, \lambda; n+1, \nu) = M^{(n+1)}(\nu) p^\downarrow(n+1, \nu; n, \lambda).
\]

Consider the Poisson distribution on the set $Z_+ = \{0, 1, 2, \ldots\}$, with parameter $\theta > 0$:
\[
\text{Poisson}_\theta(n) = e^{-\theta} \frac{\theta^n}{n!}, \quad n \in Z_+.
\]

Mixing all measures $M^{(n)}$ together by means of the Poisson distribution (1.4) we obtain a probability measure on the set $Y$. We denote it by $M_\theta$ and call it the poissonized Plancherel measure with parameter $\theta$:
\[
M_\theta(\lambda) = e^{-\theta} \theta^n \left(\frac{\dim \lambda}{n!}\right)^2, \quad n = |\lambda|.
\]

We are going to define a stationary Markov process $\Lambda_\theta = \Lambda_\theta(t)$ with discrete state space $Y$ and continuous time $t \in \mathbb{R}$, and such that $M_\theta$ is an invariant measure of $\Lambda_\theta$. Moreover, $\Lambda_\theta$ is reversible with respect to $M_\theta$.

The trajectories of $\Lambda_\theta$ are step functions (in other words, piece–wise constant functions) $\Lambda(t)$ of variable $t \in \mathbb{R}$, with values in $Y$. We say that a trajectory $\Lambda(t)$ makes a jump at a moment $t$ if the left limit $\Lambda(t^-) = \lim_{t' \uparrow t} \Lambda(t')$ differs from the right limit $\Lambda(t^+) = \lim_{t' \downarrow t} \Lambda(t')$. We reserve the notation $\Lambda_\theta(t)$ to denote the random trajectory.
Definition 1.1 (jump rates of $\Lambda_\theta$). For any $t \in \mathbb{R}$ and any $\lambda \in \mathbb{Y}_n$ we have by definition: conditional on $\Lambda_\theta(t^-) = \lambda$, the probability that $\Lambda_\theta(\cdot)$ makes a jump to a diagram $\mu \in \mathbb{Y}_{n-1}$ in the time interval $[t, t+dt]$ is equal to $R^\downarrow(n, \lambda; n-1, \mu)dt + o(dt)$, where
\begin{equation}
R^\downarrow(n, \lambda; n-1, \mu) = n p^\downarrow(n, \lambda; n-1, \mu). \tag{1.6}
\end{equation}
Likewise, the (conditional) probability of jumping to a diagram $\nu \in \mathbb{Y}_{n+1}$ in the time interval $[t, t+dt]$ is equal to $R^\uparrow(n, \lambda; n+1, \nu)dt + o(dt)$, where
\begin{equation}
R^\uparrow(n, \lambda; n+1, \nu) = \theta p^\uparrow(n, \lambda; n+1, \nu). \tag{1.7}
\end{equation}
Finally, any other jumps in $[t, t+dt]$ are excluded (with probability $1 - o(dt)$). For obvious reasons we refer to (1.6) and (1.7) as to the jump rates.

The knowledge of the jump rates makes it possible (in our concrete case) to define uniquely a transition function
\[ P_{\Lambda_\theta}(t, \lambda; s, \kappa) = \text{Prob}\{\Lambda_\theta(s) = \kappa \mid \Lambda_\theta(t) = \lambda\}, \quad s > t, \quad \lambda, \kappa \in \mathbb{Y}, \]
which depends only on $s - t$. The poissonized Plancherel measure is compatible with the transition function,
\[ \sum_{\lambda \in \mathbb{Y}} M_\theta(\lambda) P_{\Lambda_\theta}(t, \lambda; s, \kappa) = M_\theta(\kappa), \]
which allows us to define the Markov process in question. Moreover we can define the process not only on a half-line $[t_0, +\infty) \subset \mathbb{R}$ but on the whole real line, that is, we can construct a probability measure on the set $\{\Lambda(t)\}$ of $\mathbb{Y}$-valued step functions $\Lambda(t)$ defined for all $t \in \mathbb{R}$.

Since the transition function is translation invariant in time, the process is stationary (that is, the above measure on the set $\{\Lambda(t)\}$ is invariant under shifts of time, $t \to t + \text{const}$). Finally, the process turns out to be reversible (that is, the measure on $\{\Lambda(t)\}$ is also invariant under the time reversion $t \to -t$).

Remark 1.2. The above definition of the Markov process $\Lambda_\theta$ can be rephrased as follows. Introduce an auxiliary Markov process $N_\theta$: a birth–death process on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, which can be defined by the “down” and “up” jump rates
\[ R^\downarrow(n; n-1) = n, \quad R^\uparrow(n; n+1) \equiv \theta. \tag{1.8} \]

The key property of $N_\theta$ is that it has the Poisson distribution (1.4) as the invariant measure. The birth–death process $N_\theta$ governs the jumps of $\Lambda_\theta$: each moment $N_\theta(\cdot)$ jumps down, say, from $n$ to $n-1$, the trajectory $\Lambda_\theta(\cdot)$ makes a jump from $\mathbb{Y}_n$ to $\mathbb{Y}_{n-1}$ (and the target Young diagram $\mu \in \mathbb{Y}_{n-1}$ is chosen according to the “down” probabilities $p^\downarrow(n, \lambda; n-1, \mu)$, where $\lambda$ stands for the preceding state). Likewise, when $N_\theta(\cdot)$ jumps up, say, from $n$ to $n+1$, the trajectory $\Lambda_\theta(\cdot)$

\[ \text{Let us note that the relevant step functions have only finitely many jumps on finite time intervals $[t, s] \subset \mathbb{R}$. Markov processes with such a property (finitely many jumps in finite time) are sometimes called regular.} \]

\[ \text{Recall that this distribution is precisely the “mixing” measure used in the definition of the poissonized Plancherel measure } M_\theta. \]
makes a jump from $\lambda$ to a random diagram $\nu \in \mathbb{Y}_{n+1}$ chosen according to the “up” probabilities $p^\uparrow(n, \lambda; n+1, \nu)$. The fact that $M_\theta$ is the invariant measure is deduced from the fact that the Poisson distribution is the invariant measure of $N_\theta$ and from relations (1.2). The reversibility property of $\Lambda_\theta$ is deduced from the reversibility of $N_\theta$ and relation (1.3).

The above construction of the Markov process can be generalized. The idea is to use more general birth–death processes, with time–dependent jump rates.

Let $\mathbb{R}^2_{>0}$ denote the open first quadrant of the Euclidean plane $\mathbb{R}^2$ with coordinates $u > 0, v > 0$. Consider a parameterized curve $C = (u(t), v(t))$ in $\mathbb{R}^2_{>0}$ subject to the following conditions: the functions $u(t) > 0, v(t) > 0$ are continuous and piece–wise continuously differentiable; the curve is directed southeast, that is, $\dot{u}(t) \geq 0, \dot{v}(t) \leq 0$, and $\dot{u}(t)$ and $\dot{v}(t)$ do not vanish simultaneously (here the dot means derivative with respect to $t$, and $t$ is interpreted as time). Such curves $C$ will be called admissible.

We modify formulas (1.8) as follows

$$R^\downarrow(n; n-1; t) = -n \frac{\dot{v}(t)}{v(t)}, \quad R^\uparrow(n; n+1; t) = \dot{u}(t)v(t). \quad (1.9)$$

Note that if $C$ is the hyperbola $uv = \theta$ parameterized by $t = \ln u$ then (1.9) reduces to (1.8). In the general case we set

$$\theta(t) = u(t)v(t). \quad (1.10)$$

There exists a birth–death process determined by the jump rates (1.9); we denote it by $N_C$. Let

$$P_{N_C}(t, n; s, m) = \text{Prob}\{N_C(s) = m \mid N_C(t) = n\} \quad (1.11)$$

be the transition function of $N_C$ (here $s > t$ and $n, m \in \mathbb{Z}_+$). The process $N_C$ is no longer stationary in time, and instead of a single Poisson distribution with fixed parameter $\theta$ (see (1.4)) we deal with the whole family of such distributions indexed by the varying parameter $\theta(t)$ given by (1.10). These distributions are compatible with the transition function (1.11):

$$\sum_{n \in \mathbb{Z}_+} e^{-\theta(t)} \frac{(\theta(t))^n}{n!} P_{N_C}(t, n; s, m) = e^{-\theta(s)} \frac{(\theta(s))^m}{m!}, \quad s > t, \quad m \in \mathbb{Z}_+.$$

Therefore, we can make the assumption that

$$\text{Prob}\{N_C(t) = n\} = e^{-\theta(t)} \frac{(\theta(t))^n}{n!}$$

for any $t$.

Now we can construct a Markov process $\Lambda_C$ with state space $\mathbb{Y}$ precisely as in Remark 1.2. It should be added that we assume that conditional on $N_C(t) = n$, the distribution of $\Lambda_C(t)$ coincides with the $n$th Plancherel measure. This implies that

$$\text{Prob}\{\Lambda_C(t) = \lambda\} = M_{\theta(t)}(\lambda)$$

for any $\lambda \in \mathbb{Y}$.

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5. Any birth–death process is reversible.
Remark 1.3. One can show that the Markov process obtained from $\Lambda_C$ by the reversion of time $t \to -t$ has a similar form, $\Lambda_{\tilde{C}}$, where the curve $\tilde{C}$ is the image of $C$ under the transposition of coordinate axes. That is, $\tilde{C}$ is given by

\[ \tilde{u}(t) = v(-t), \quad \tilde{v}(t) = u(-t). \]

Remark 1.4. A reparametrization of the curve $C$ (which leaves it admissible) leads simply to a reparametrization of time in the corresponding Markov process $\Lambda_C$. There is a distinguished parametrization of $C$ which is unique within an additive constant:

\[ t = \frac{1}{2} (\ln u - \ln v) + \text{const}, \quad (u, v) \in C. \tag{1.12} \]

We call $t$ the interior time along the curve. If $C$ is a hyperbola $uv = \theta$, then the interior time coincides with the natural time in the stationary Markov process $\Lambda_\theta$.

Remark 1.5. There are two particular cases of nonstationary Markov processes $\Lambda_C$ which can be called the descending and the ascending ones. By definition, they are obtained when we take as $C$ a vertical or horizontal line, respectively. These processes correspond to pure death or pure birth processes, respectively. More generally, one considers broken lines $C$ with alternating vertical and horizontal segments. Note that any admissible curve can be approximated by such broken lines, which suggests the idea that a general Markov process $\Lambda_C$ can be approximated, in an appropriate sense, by processes with alternating descending and ascending fragments.

2. Interpretation of Markov processes via Poisson process in quadrant

In this section we describe a nice interpretation of the Markov processes $\Lambda_C$. As in §1, we are dealing with the quadrant $\mathbb{R}^2_{>0} \subset \mathbb{R}^2$ with coordinates $u > 0, v > 0$.

Definition 2.1. Let $\pi$ be an $n$–point configuration in $\mathbb{R}^2_{>0}$ such that no two points lie on the same vertical or horizontal line. We assign to $\pi$ a permutation $\sigma$ in the symmetric group $S(n)$ and then a Young diagram $\lambda \in \mathbb{Y}_n$, as follows. Let $u_1 < \cdots < u_n$ and $v_1 < \cdots < v_n$ be the $u$– and $v$–coordinates of the points in $\pi$. By definition, the permutation $\sigma$ determines a matching of these coordinates. That is, the points in $\pi$ are of the form $(u_i, v_{\sigma(i)})$, where $i = 1, \ldots, n$. Next, to obtain $\lambda$ we apply the Robinson–Schensted algorithm (RS for short). Recall (see, e.g., Sagan’s book [Sa, §3.3 and §3.8]) that RS establishes an explicit bijection between permutations $\sigma \in S(n)$ and pairs $(P, Q)$ of standard Young tableaux of the same shape $\lambda \in \mathbb{Y}_n$, and we just take this diagram $\lambda$.

Consider the Poisson process $\Pi$ in the quadrant $\mathbb{R}^2_{>0}$ with constant density 1. We also denote by $\Pi$ the random point configuration in $\mathbb{R}^2_{>0}$ produced by the process. We can assume that no two points in $\Pi$ lie on the same vertical or horizontal line, because this condition holds for almost all configurations $\Pi$.

Definition 2.2. To any point $(u, v) \in \mathbb{R}^2_{>0}$ we assign a random permutation $\sigma_{\Pi}(u, v)$ and a random Young diagram $\lambda_{\Pi}(u, v)$, both depending on a realization $\Pi$ of the Poisson process, as follows. Let $\square(u, v)$ denote the rectangle with vertices
Let $(u, v), (u, 0), (0, v), (0, 0)$, and let $\Pi(u, v) = \Pi \cap \square(u, v)$ be the random point configuration in this rectangle. Then we set $\pi = \Pi(u, v), n = |\pi|$, and apply Definition 2.1.

The construction of Definition 2.2 is well known. It was widely used in the literature since Hammersley’s paper [Ha]. Note that for a fixed point $(u, v)$, the number $n = |\Pi(u, v)|$ has Poisson distribution (1.4) with parameter $\theta = uv$, and that $\lambda(\Pi(u, v))$ is distributed according to the poissonized Plancherel measure $M_\theta$. Now we let $(u, v)$ vary.

**Theorem 2.3.** Let $C$ be an admissible curve in the sense of §1 and let a point $(u, v) = (u(t), v(t))$ move along $C$. Let, as above, $\Pi$ be the random Poisson point configuration in $\mathbb{R}_{>0}^2$, and consider the $\mathbb{Y}$-valued stochastic process $\tilde{\Lambda}_C$ with random trajectories $\lambda(\Pi(u(t), v(t)))$, where the random Young diagram $\lambda(\Pi(u(t), v(t)))$ is afforded by Definition 2.2.

The process $\tilde{\Lambda}_C$ is a Markov process, equivalent to the Markov process $\Lambda_C$ of §1.

By Theorem 2.3, each Markov process $\Lambda_C$ can be interpreted as a certain projection of the Poisson process in the quadrant. Actually, a more precise result holds. Assume that the curve $C$ satisfies the conditions

$$\lim_{t \to -\infty} u(t) = 0, \quad \lim_{t \to +\infty} v(t) = 0,$$

and let $D \subset \mathbb{R}_{>0}^2$ denote the subgraph of $C$, that is, the part of the quadrant which is below and on the left of $C$. For instance, in the case of the stationary process $\Lambda_\theta$, one has $D = \{(u, v) \in \mathbb{R}_{>0}^2 | uv < \theta\}$.

**Theorem 2.4.** In the situation of Theorem 2.3, assume additionally that condition (2.1) is satisfied.

Then the construction of Theorem 2.3 provides a measure space isomorphism between the realizations of the Poisson process in the domain $D$ and the trajectories of $\tilde{\Lambda}_C$.

Clearly, any trajectory $\{\lambda(\Pi(u(t), v(t)))\}_{t \in \mathbb{R}}$ of $\tilde{\Lambda}_C$ depends only on the restriction $\Pi|_D$ of the corresponding Poisson configuration $\Pi$ to the domain $D$. It turns out that, conversely, $\Pi|_D$ can be reconstructed from $\{\lambda(\Pi(u(t), v(t)))\}_{t \in \mathbb{R}}$. This implies the theorem.

**Remark 2.5.** Let, as in Theorem 2.3, a point $(u, v) = (u(t), v(t))$ move along an admissible curve $C$ and replace $\lambda(\Pi(u, v))$ by $\sigma(\Pi(u, v))$, see Definition 2.2. Then we obtain a random process taking values in permutations $\sigma \in S(n)$ with varying $n$. One can show that this is again a Markov process. Clearly, it “covers” the Markov process $\Lambda_C$ of Theorem 2.3 (the latter is a projection of the former). On the other hand, Theorem 2.4 implies a somewhat paradoxical claim that the projection $\sigma \mapsto \lambda$ given by the algorithm RS defines a measure space isomorphism between the trajectories of both processes.

**Remark 2.6.** In case the curve $C$ is a straight line $u + v = \text{const}$, the corresponding Markov process $\Lambda_C = \tilde{\Lambda}_C$ was earlier described in very different terms by Prähofer–Spohn [PS], see also Remarks 3.4 and 4.5.
Remark 2.7. The assumption that the curve $C$ goes in southeast direction is crucial for the Markov property in Theorem 2.3. This can be demonstrated on the following simple example. Consider three points in the quadrant: $a = (1,1)$, $b = (2,1)$, and $c = (2,2)$. Then, conditional on $\lambda_{11}(b)$ is the one–box diagram, the random diagrams $\lambda_{11}(a)$ and $\lambda_{11}(c)$ are not independent. This shows that on the broken line going northeast, from $a$ to $b$ to $c$, the Markov property does not hold.

3. Dynamical correlation functions

Consider the lattice of (proper) half–integers $Z' = \mathbb{Z} + \frac{1}{2} = \{\ldots, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$. We can write $Z' = Z'_\lor \cup Z'_\land$, where $Z'_\land$ consists of all negative half–integers and $Z'_\lor$ consists of all positive half–integers. For any $\lambda \in \mathbb{Y}$ we set

$$L(\lambda) = \{\lambda_i - i + \frac{1}{2} \mid i = 1, 2, \ldots\} \subset Z'.$$

For instance, $L(\emptyset) = Z'_\land$. The correspondence $\lambda \mapsto L(\lambda)$ is a bijection between the Young diagrams $\lambda$ and those (infinite) subsets $L \subset Z'$ for which the symmetric difference $\Delta \lambda L'$ is a finite set with equally many points in $Z'_\lor$ and $Z'_\land$.

We regard $L(\lambda)$ as a point configuration on the lattice $Z'$. Assume we are given a probability measure $M$ on $\mathbb{Y}$. Then we can speak about the random diagram $\lambda$ and hence about the random point configuration $L(\lambda)$. The $n$th correlation function of $M$ is defined as follows

$$\rho_n(x_1, \ldots, x_n) = \text{Prob}\{x_1, \ldots, x_n \in L(\lambda)\},$$

where $n = 1, 2, \ldots$ and $x_1, \ldots, x_n$ are pairwise distinct points of $Z'$. In other words, the correlation functions tell us what is the probability that the random point configuration $L(\lambda)$ contains a given finite set of points. The collection of all correlation functions determines the initial probability measure $M$ uniquely.

For the poissonized Plancherel measure $M_\theta$ defined in (1.5) the correlation functions were found independently by Borodin–Okounkov–Olshanski [BOO] and Johansson [Jo1]:

Theorem 3.1. (i) The correlation functions of the measure $M_\theta$ have determinantal form

$$\rho_n(x_1, \ldots, x_n) = \det_{1 \leq i, j \leq n} [K(x_i, x_j)],$$

where $n = 1, 2, \ldots$ and

$$K(x, y) = \sqrt{\theta} \frac{J_{\frac{1}{2}}(2\sqrt{\theta}) J_{y+\frac{1}{2}}(2\sqrt{\theta}) - J_{\frac{1}{2}}(2\sqrt{\theta}) J_{x+\frac{1}{2}}(2\sqrt{\theta})}{x - y}. \quad (3.1)$$

Here $J_{\cdot}(\cdot)$ denotes the Bessel function which is regarded as a function of its index $z$. When $x = y$, the indeterminacy in formula (3.1) is resolved by l’Hôpital’s rule.

(ii) The kernel can also be written in the form

$$K(x, y) = \sum_{a \in Z'_\lor} J_{x+a}(2\sqrt{\theta}) J_{y+a}(2\sqrt{\theta}). \quad (3.2)$$

The function $K(x, y)$ on $Z' \times Z'$ is called the discrete Bessel kernel.

We are now going to state an analog of Theorem 3.1 for the Markov processes $\Lambda_C$. The correspondence $\lambda \mapsto L(\lambda)$ allows us to regard each trajectory $\Lambda(t)$ in $\mathbb{Y}$ as a varying in time point configuration $L(\Lambda(t))$. In this picture, a jump consists in shifting one of the points of the configuration by ±1.
Theorem 3.3. Let $C = (u(t), v(t))$ be an admissible curve in $\mathbb{R}^2_0$ with its canonical parametrization (1.12) given by interior time $t$, let $\theta(t) = u(t)v(t)$ as in (1.10), and let $\Lambda_C$ be the corresponding Markov process.

(i) The dynamical correlation functions of $\Lambda_C$ have determinantal form

$$\rho_n(t_1, x_1; \ldots; t_n, x_n) = \det_{1 \leq i, j \leq n} [K_C(t_i, x_i; t_j, x_j)],$$

where $n = 1, 2, \ldots$ and $K_C(s; x; t; y)$ is a kernel on $(\mathbb{R} \times \mathbb{Z}') \times (\mathbb{R} \times \mathbb{Z}')$ which can be written as a double contour integral

$$K_C(s; x; t; y) = \frac{e^{s-t}}{(2\pi i)^2} \int_{\omega_1} \int_{\omega_2} e^{\sqrt{\theta(s)}(\omega_1 - \omega_1^{-1}) + \sqrt{\theta(t)}(\omega_2 - \omega_2^{-1})} e^{s-t \omega_1 \omega_2 - 1} \omega_1^{-\frac{1}{2}} \omega_2^{-\frac{1}{2}} d\omega_1 d\omega_2$$

(3.3)

where $\{\omega_1\}$ and $\{\omega_2\}$ are any two contours which go around $\theta$ in positive direction and satisfy the following condition:

- if $s \geq t$, so that $e^{s-t} \geq 1$, then the contour $\{\omega_1\}$ must contain the contour $\{e^{-s} \omega_2^{-1}\}$;
- if $s < t$, so that $e^{s-t} < 1$, then, on the contrary, the contour $\{\omega_1\}$ must be contained in the contour $\{e^{-s} \omega_2^{-1}\}$.

(ii) The kernel can also be written in the form

$$K_C(s; x; t; y) = \pm \sum_{a \in \mathbb{Z}'} e^{-a|s-t|} J_{x+a}(2\sqrt{\theta(s)}) J_{y+a}(2\sqrt{\theta(t)})$$

(3.4)

where the plus sign is chosen if $s \geq t$ whereas the minus sign is chosen if $s < t$.

One can verify that if $s = t$ then (3.3) can be reduced to (3.1), and it is immediate that (3.4) turns into (3.2).

Note the asymmetry between the conditions $s \geq t$ and $s < t$. One can show that

$$\lim_{s \to t} K_C(s; x; t; y) = K_C(t; x; t; y) = \lim_{s \to t} K_C(s; x; t; y) + \delta_{xy}$$

This agrees with the fact that

$$\sum_{a \in \mathbb{Z}'} J_{x+a}(2\sqrt{\theta}) J_{y+a}(2\sqrt{\theta}) = \delta_{xy}.$$

Remark 3.4. For the curves $u + v = \text{const}$, formula (3.4) was earlier proved by Prähöfer and Spohn, [PS, (3.52)].
4. Scaling limits

The study of the asymptotic behavior of the random Young diagrams distributed according to the \( n \)th Plancherel (1.1) as \( n \to \infty \), or according to the poissonized Plancherel measure (1.5) as \( \theta \to \infty \), is an important and nontrivial problem which has many different aspects. We refer the reader to [BOO], [IO] for a general discussion and relevant references, and restrict ourselves here to considering two types of the asymptotics: in the middle (bulk) and at the edge of the Young diagrams distributed according to \( M \) and Markov processes introduced in §1.

**Bulk.** We start with recalling the following statement.

**Theorem 4.1 ([BOO, Section 3]).** The correlation functions of the measure \( M \) have the following limit as \( \theta \to \infty \): Fix \( c \in (-2, 2) \) and let \( x_0(\theta) \in \mathbb{Z}' \) be such that

\[
x_0(\theta) = c \cdot \sqrt{\theta} + o(\sqrt{\theta}), \quad \theta \to \infty.
\]

Further, for any \( n = 1, 2, \ldots \) and arbitrary \( x_1, \ldots, x_n \in \mathbb{Z} \), set

\[
x_i(\theta) = x_0(\theta) + x_i, \quad i = 1, \ldots, n.
\]

Then

\[
\lim_{\theta \to \infty} \rho_n(x_1(\theta), \ldots, x_n(\theta)) = \det_{1 \leq i, j \leq n} [S_c(x_i - x_j)],
\]

where

\[
S_c(r) = \frac{\sin(\arccos(c/2) \cdot r)}{\pi r}.
\]

The function \( S_c(x - y) \) on \( \mathbb{Z} \times \mathbb{Z} \) is called the discrete sine kernel.

In Theorem 4.1, the pairwise distances \( x_i(\theta) - x_j(\theta) \) do not vary as \( \theta \to +\infty \); if instead this one supposes that for some \( i \) and \( j \), the distance between two points \( x_i(\theta) \) and \( x_j(\theta) \) from the bulk\(^6\) tends to infinity together with \( \theta \), then the events of finding particles (=elements of \( L(\lambda) \)) at these locations become asymptotically independent.

Our goal is to extend Theorem 4.1 to the Markov processes \( \Lambda_C \).

Let us return for a moment to the interpretation of \( \Lambda_C \) via the Poisson process in the first quadrant, see §2. Theorem 4.1 deals with Young diagrams \( \lambda_{\Pi}(u, v) \) sitting at the points \((u, v)\) such that \( uv = \theta \to \infty \). Now let us assume that such a point \((u, v)\) sits on an admissible curve.

It turns out that the bulk of \( \lambda_{\Pi}(u, v) \) will have nontrivial correlations with the bulks of the Young diagrams corresponding to other points of the curve if the distance to these points in the \((u, v)\)-plane remains finite as \( \theta \to \infty \). If we assume that we are far enough from the boundary of the quadrant \((u/v\) is bounded away from zero and infinity) then the interior time change between such points has to be of order \( 1/\sqrt{\theta} \).

**Theorem 4.2.** Let \( C_0 = (u_0(t), v_0(t)) \) be a family of admissible curves in \( \mathbb{R}_+^2 \) with their canonical parametrizations given by the interior time (1.12). Here \( \theta > 0 \) is a parameter, and we assume that there exists a constant \( T \in \mathbb{R} \) such that \( u_0(T)v_0(T) = \theta \). For instance, we may take as the curves \( C_0 \) the hyperbolas \( uv = \theta \).

\(^6\)meaning that \( x_i(\theta)/\sqrt{\theta} \) and \( x_j(\theta)/\sqrt{\theta} \) are bounded away from the edges \(-2\) and \(2\).
The dynamical correlation functions of $\Lambda_{C_\theta}$ have the following limit as $\theta \to +\infty$.

Fix an arbitrary $c \in (-2, 2)$ and let $x_0(\theta) \in \mathbb{Z}'$ be such that
$$x_0(\theta) = c \cdot \sqrt{\theta} + o(\sqrt{\theta}), \quad \theta \to \infty.$$ 

Let $n = 1, 2, \ldots$, and let $\tau_1, \ldots, \tau_n \in \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{Z}$ be arbitrary. Further, assume that
$$t_i(\theta) = T + \tau_i/\sqrt{\theta} + o(1/\sqrt{\theta}).$$

Then
$$\lim_{\theta \to \infty} \rho_n(t_1(\theta), x_0(\theta) + x_1; \ldots; t_n(\theta), x_0(\theta) + x_n) = \det_{1 \leq i, j \leq n} [S_c(\tau_i - \tau_j; x_i - x_j)],$$
where
$$S_c(h; r) = \frac{1}{2\pi i} \int_{\{\omega\}} e^{-h(\omega^2 - c)} \frac{d\omega}{\omega^{r+1}}, \quad (4.2)$$
and $\{\omega\}$ is a contour in $\mathbb{C}$ going from the point $e^{-i\phi}$ to the point $e^{i\phi}$, $\phi = \arccos(c/2)$, in such a way that it passes to the right of the origin if $h \geq 0$, and to the left of the origin if $h < 0$.

Note that the limit correlations do not depend on the choice of the curves $C_\theta$.

It is readily verified that when $h = 0$, (4.2) coincides with (4.1).

Another, somewhat similar extension of the discrete sine kernel called incomplete beta kernel was obtained by Okounkov–Reshetikhin [OR, Section 3].

**Edge.** We now concentrate our attention on the asymptotics of the correlation functions at the edge of the Young diagrams, where $x_i(\theta) \sim \pm 2\sqrt{\theta}$. Due to the symmetry of our measures with respect to transposition of Young diagrams, which also swaps the edges $2\sqrt{\theta}$ and $-2\sqrt{\theta}$, it suffices to consider one of the edges.

**Theorem 4.3** ([BOO, Section 4], [Jo1]). The correlation functions of $M_\theta$ have the following scaling limit as $\theta \to \infty$: For any $n = 1, 2, \ldots$ and any $x_1, \ldots, x_n \in \mathbb{R}$, let $x_1(\theta), \ldots, x_n(\theta) \in \mathbb{Z}'$ be such that
$$x_i(\theta) = 2\sqrt{\theta} + x_i' \theta^{1/2} + o(\theta^{1/2}), \quad i = 1, \ldots, n.$$ 

Then
$$\lim_{\theta \to \infty} (\theta^{1/2})^n \rho_n(x_1(\theta), \ldots, x_n(\theta)) = \det_{1 \leq i, j \leq n} [\mathcal{A}(x_i, x_j)],$$
where
$$\mathcal{A}(x, y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x - y},$$
and $Ai(x)$ is the Airy function.

Note that the factor $(\theta^{1/2})^n$ comes from the scaling of the space variable $x$.

The function $\mathcal{A}(x, y)$ on $\mathbb{R}^2$ is called the Airy kernel. Another useful formula for the Airy kernel is
$$\mathcal{A}(x, y) = \int_0^\infty Ai(x + a)Ai(y + a)da.$$ 

Once again, let us return to the interpretation of $\Lambda_C$ through the Poisson process. It turns out that the edges of the Young diagrams sitting on a curve $(u(t), v(t))$ with $uv = \theta \to \infty$ will have nontrivial correlations if the distance between the points in the $(u, v)$–plane grows as $\theta^{1/2}$. This means that if we are away from the coordinate axes then the interior time change is of order $\theta^{-1/2}$. 

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Theorem 4.4. Let $C_0 = (u_0(t), v_0(t))$ be a family of admissible curves in $\mathbb{R}_{>0}^2$ with their canonical parametrizations given by the interior time (1.12). Here $\theta > 0$ is a parameter, and we assume that there exists a constant $T \in \mathbb{R}$ such that $u_0(T)v_0(T) = \theta$. For instance, we may take as the curves $C_0$ the hyperbolas $uv = \theta$.

The dynamical correlation functions of $\Lambda_{C_0}$ have the following limit as $\theta \to +\infty$. For arbitrary $\{\tau_i\}_{i=1}^n \subset \mathbb{R}$, choose $t_i(\theta)$ such that

$$t_i(\theta) = T + \tau_i \theta^{-\frac{1}{6}} + o\left(\theta^{-\frac{1}{6}}\right), \quad i = 1, \ldots, n.$$ 

Further, for arbitrary $x_1, \ldots, x_n \in \mathbb{R}$, choose $x_1(\theta), \ldots, x_n(\theta) \in \mathbb{Z}'$ such that

$$x_i(\theta) = 2\sqrt{u_0(t_i(\theta))v_0(t_i(\theta))} + x_i \theta^\frac{1}{6} + o\left(\theta^\frac{1}{6}\right), \quad i = 1, \ldots, n.$$ 

Then

$$\lim_{\theta \to \infty} \left(\theta^\frac{1}{6}\right)^n \rho_n(t_1(\theta), x_1(\theta); \ldots; t_n(\theta), x_n(\theta)) = \det_{1 \leq i,j \leq n} [A(\tau_i - \tau_j; x_i, x_j)],$$

where

$$A(\tau; x, y) = \begin{cases} \int_0^{\infty} e^{-\tau a} Ai(x + a) Ai(y + a) da, & \tau \geq 0, \\ -\int_0^{\infty} e^{-|\tau| a} Ai(x - a) Ai(y - a) e^{-\tau a} da, & \tau < 0. \end{cases}$$

The kernel $A(\tau; x, y)$ is called the extended Airy kernel. It is stationary in time.

Remark 4.5. A special case of Theorem 4.4 with $C_0$ being the lines $u + v = \text{const}$ was proved in Prähofer–Spohn [PS], also Johansson [Jo2].

The next statement uses the notion of the Airy process, introduced in [PS], see also Johansson [Jo2].

Corollary 4.6. Let $\{(u_\theta(t), v_\theta(t) \mid t \in (T - \epsilon, T + \epsilon)\}$ be a family of admissible curves in $\mathbb{R}_{>0}^2$ with their canonical parametrizations, and $u_\theta(T)v_\theta(T) = \theta$. Denote by $l(t, \theta)$ the length of the first row of the random Young diagram $\lambda_1(u_\theta(t), v_\theta(t))$ and set $t(\tau) = T + \tau \theta^{-\frac{1}{6}}$. Then as $\theta \to \infty$, the random variable

$$L(\tau) = \frac{l(t(\tau), \theta) - 2\sqrt{u_\theta(t(\tau))v_\theta(t(\tau))}}{\theta^\frac{1}{6}}$$

converges, as a function of $\tau$, to the Airy process.

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A. BORODIN: Mathematics 253-37, Caltech, Pasadena, CA 91125, U.S.A.,

E-mail address: borodin@caltech.edu

G. OLSHANSKI: Dobrushin Mathematics Laboratory, Institute for Information Transmission Problems, Bolshoy Karetny 19, 127994 Moscow GSP-4, RUSSIA.

E-mail address: olsh@online.ru