Matrix Model Maps and Reconstruction of AdS SUGRA Interactions

Sera Cremonini
Michigan Center for Theoretical Physics, Randall Laboratory of Physics, The University of Michigan, Ann Arbor, MI 48109, USA
Robert de Mello Koch
Department of Physics, University of Witwatersrand, Wits, 2050, South Africa
Antal Jevicki
Department of Physics, Brown University, Providence, RI 02912, USA
(Dated: February 2, 2008)

Abstract

We consider the question of reconstructing (cubic) SUGRA interactions in AdS/CFT. The method we introduce is based on the matrix model maps (MMP) which were previously successfully employed at the linearized level. The strategy is to start with the map for 1/2 BPS configurations which is exactly known (to all orders) in the hamiltonian framework. We then use the extension of the matrix model map with the corresponding Ward identities to completely specify the interaction. A central point in this construction is the non-vanishing of off-shell interactions (even for highest-weight states).
I. INTRODUCTION

The question of reconstructing bulk supergravity (SUGRA) through the AdS/CFT correspondence is of considerable interest. Initially much insight into the correspondence was gained through the GKP-W holographic relation which states that correlators of Yang-Mills theory coincide with certain boundary to boundary amplitudes in supergravity. Indeed this was the scheme which provided some of the initial prescriptions for relating cubic supergravity interactions to gauge theory correlators. The holographic relation, however, has elements of an S-matrix relation, and one can ask what set of correlators contains all the information for reconstructing the theory in the bulk. Although some studies have been done along this direction, there are still some main questions left open. The issue/problem seems to be analogous to the question of reconstructing the off-shell theory from strictly on-shell data, a problem which is usually plagued by non-uniqueness. In addition there is the question of unitarity, namely the issue of securing a unitary and local evolution of the bulk theory. An alternative is to develop the construction directly in the hamiltonian framework, a method we consider in the present work. The basic building block of our construction will be the matrix model representation that was developed in the last few years beginning with the 1/2 BPS sector of the theory. This approach came for studies of 1/2 BPS correlators in gauge theory and the dual exact configurations in SUGRA. What emerged is a fermion droplet correspondence (see references therein). Its hamiltonian version given through collective field theory can serve as a starting point for reconstructing the full theory. Specifically, the strategy that we develop for the construction of the bulk interaction is then as follows: starting from the non-zero $c = 1$ collective field theory vertex we proceed with the action of raising operators to establish Ward identities that, as we argue, are capable of determining the full cubic vertex. The form of the raising and lowering operators can be deduced through the matrix model map (MMP) formulated in. The map of was given at the linearized level, and was shown to provide a mapping from eigenfunctions of matrix model
equations to those of AdS. As such our work represents an extension to the nonlinear level of the mapping introduced in [23]. The outline of this paper is as follows. In Section II we discuss the form of cubic interactions in supergravity as well as for the 1/2 BPS collective field theory. Here we also discuss and resolve issues that concern the comparison of the (vanishing) SUGRA vertex for the 1/2 BPS sector with the (nonvanishing) matrix model vertex. In Section III we review the linearized MMP of [23] in terms of canonical transformations on phase space. This version turns out to be useful for the nonlinear extension that we give in Section IV where we consider a simplified limit. Finally, in Section V we discuss the Ward identities and their ability to determine the vertex (from the initial highest-weight one). Throughout this paper we restrict our analysis to the AdS\(_2\) case, where the method can be presented in the simplest possible way.

II. CUBIC INTERACTIONS IN THE 1/2 BPS SECTOR

Our starting point is the direct hamiltonian level map that was accomplished in the (limited) 1/2 BPS sector of the theory. On the Yang-Mills side one has a (reduced) matrix model hamiltonian established in [16] and its collective field theory hamiltonian description. This is fully reproduced on the gravity side with the 1/2 BPS reduction of 10D SUGRA in [18]. In particular, the energy of the 10D geometries of the 1/2 BPS sector was shown [18] to be given by

\[
E = \int dx_1 \int dx_2 \left( x_1^2 + x_2^2 \right) u(x_1, x_2),
\]

(1)

where \(u(x_1, x_2)\) is a density function distinguishing between spacetime regions having different boundary conditions (“black” and “white” regions). The expression (1) is recognizable as the energy of fermions (corresponding to matrix eigenvalues) in a harmonic oscillator potential. In this language, \(u\) is responsible for differentiating between particles (fermion droplets) and holes. After identifying \(x_1 = x\) and \(x_2 = y\), and after performing the \(x_2\) integration over a “black” region (fermion droplet), the energy (1) can be shown...
to be equivalent to the collective field theory hamiltonian \[ H = \int dx \left( \frac{y^3}{3} - \frac{y'^3}{3} + x^2(y_+ - y_-) \right) \] of a one-matrix model described by
\[ H = \frac{1}{2} Tr(X^2 + P^2) \].

The Hermitian $N \times N$ matrix $X(t)$ depends only on time, and $P(t) = \dot{X}(t)$ denotes its conjugate momentum. The functions $y_+$ and $y_-$ describe the upper and lower profiles of the Fermi droplet. Furthermore, the matrix hamiltonian is related to (2) via $X = x$ and $P = y$. It is important to emphasize that the collective field formalism describes well the fully interacting theory of chiral primaries on $AdS_5 \times S^5$. To show this explicitly, we examine next the form of the cubic vertex as given by collective field theory. The dynamics of the resulting collective field theory can be directly induced from the much simpler dynamics of the one-dimensional matrix $X(t)$ (with eigenvalues $\lambda_i$), after a change to the density field obeying the following cubic collective hamiltonian:
\[ H_{\text{coll}} = \int dx \left( \frac{1}{2} \partial_x \Pi \phi \partial_x \Pi + \frac{\pi^2}{6} \phi^3 + \frac{1}{2} (x^2 - \mu) \phi \right) \].

The static ground state equation yields the background value $\phi_0$ for the field $\phi$. One can then introduce small fluctuations about the background, letting
\[ \phi(x, t) = \phi_0(x) + \frac{1}{\sqrt{\pi}} \partial_x \eta(x, t) \].

After expanding the hamiltonian one finds
\[ H = \int dx \left[ \pi \phi_0 \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_x \eta)^2 \right) + \frac{\pi^2}{6} (\partial_x \eta)^3 + \frac{\pi}{2} \Pi \partial_x \eta \Pi \right] \].

Note that the corresponding quadratic Lagrangian takes the form
\[ L_2 = \int dt \int dx \left[ \frac{\dot{\eta}^2}{\pi \phi_0} - \pi \phi_0 \eta_x^2 \right] \],

describing a massless particle in a gravitational background with metric
\[ g^0_{\mu\nu} = \left( \frac{1}{\pi \phi_0}, \pi \phi_0 \right) \].
The metric can be removed by an appropriate coordinate transformation. In terms of the “time of flight” coordinate \( \tau \), the hamiltonian then becomes

\[
H = \int d\tau \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_\tau \eta)^2 + \frac{1}{6\pi^2 \phi_0^2} ((\partial_\tau \eta)^3 + 3\Pi \partial_\tau \eta \Pi) \right].
\]

(9)

Furthermore, notice that this is the theory of a massless boson with a spatially dependent coupling, \( g_{\text{string}}(\tau) = \frac{1}{\pi \phi_0^2} \). Since we are interested in studying the (cubic) interaction terms, let’s concentrate on

\[
H^{(3)} = \int d\tau \frac{1}{6\pi^2 \phi_0^2} ((\partial_\tau \eta)^3 + 3\Pi \partial_\tau \eta \Pi).
\]

(10)

If we recall how the (standard) fields \( \alpha_\pm \) were introduced, \( \alpha_\pm(x,t) = \partial_x \Pi \pm \pi \phi(x,t) \), it is clear that they could have been expanded about the background in a similar way:

\[
\alpha_\pm = \pm \pi \phi_0 + \tilde{\alpha}_\pm.
\]

(11)

The cubic hamiltonian takes on a much simpler form in terms of \( \tilde{\alpha}_\pm \):

\[
H^{(3)} = \int d\tau \frac{1}{6\pi^2 \phi_0^2} (\tilde{\alpha}_+^3(\tau) - \tilde{\alpha}_-^3(\tau)).
\]

(12)

This can be simplified even further by combining \( \tilde{\alpha}_\pm \) into a single field \( \alpha(\tau) \) in the following way:

\[
\alpha(\tau) = \tilde{\alpha}_+(\tau) \quad \text{for} \quad \tau > 0,
\]

\[
= -\tilde{\alpha}_-(\tau) \quad \text{for} \quad \tau < 0,
\]

(13)

where we must now take \(-\pi < \tau < \pi\). Finally, expressing the cubic hamiltonian in terms of the new field \( \alpha \), we find

\[
H^{(3)} = \int_{-\pi}^{\pi} d\tau \frac{d\tau}{\phi_0^2} \alpha^3(\tau).
\]

(14)

Expanding \( \alpha \) into creation and annihilation operators gives

\[
\alpha(\tau) = \sum_n \sqrt{n} (e^{in\tau} a_n + e^{-in\tau} a_n^\dagger).
\]

(15)
Rewriting (14) in terms of creation and annihilation operators we find several terms, but we would like to restrict our attention to the one containing \( a_1 a_2 a_3^\dagger \):

\[
H^{(3)} = \sqrt{n_1 n_2 n_3} \int_{-\pi}^{\pi} \frac{d\tau}{\sin^2 \tau} e^{i(n_1 + n_2 - n_3)\tau} a_1 a_2 a_3^\dagger + \ldots
= -\sqrt{n_1 n_2 n_3} \int \left( \frac{d}{d\tau} \cot \tau \right) e^{i(n_1 + n_2 - n_3)\tau} a_1 a_2 a_3^\dagger + \ldots
= i(n_1 + n_2 - n_3) \sqrt{n_1 n_2 n_3} \int d\tau \cot \tau e^{i(n_1 + n_2 - n_3)\tau} a_1 a_2 a_3^\dagger + \ldots
\]

where we have implicitly used the fact that the boundary term cancels. Introducing \( z = e^{i\tau} \) and letting \( n \equiv n_1 + n_2 - n_3 \), the integral above becomes

\[
I = \int_{-\pi}^{\pi} d\tau \cot \tau e^{in\tau} = \int dz \frac{z^{n-1}}{(z-1)(z+1)} (z^2 + 1),
\]

which has simple poles at \( z_k = 0, \pm 1 \). Evaluating the integral we find that the only non-zero contribution from the residues occurs for \( n > 0 \) and even, and is given by \( \sum_k \text{Res}(f, z_k) = 2 \), yielding

\[
H^{(3)} = -4\pi \sqrt{n_1 n_2 n_3} (n_1 + n_2 - n_3) a_1 a_2 a_3^\dagger + \ldots.
\]

We should note that the vertex vanishes when \( (n_1 + n_2 - n_3) = 0 \), which is the on-shell energy conservation condition. We mention here the relevance of an Euclidean picture which was established recently in \[24\]. It corresponds to the inverted harmonic oscillator model of the \( c=1 \) string theory \[25\]. The analogue cubic hamiltonian interaction was shown capable of reconstructing the non-critical string amplitudes at both tree and loop level. The relevance of this S-matrix to AdS/CFT (and comparison with 1/2 BPS correlators) was shown in \[24\]. For completeness in the rest of this section we discuss the comparison of (17) with the SUGRA vertex obtained by studying three-point functions of chiral primaries on \( AdS_5 \times S^5 \). Next, we outline the main steps of such a comparison, and leave a detailed discussion to Section II A below. The typical 3-point (cubic) SUGRA interaction on backgrounds of the form \( AdS_n \times S^m \) is given by the overlap of bulk wavefunctions,

\[
H_3 = (\Delta_3 - \Delta_1 - \Delta_2) \int_{AdS} d^{n-1}x \sqrt{g_{AdS}} g_{AdS}^{ij} f_i f_j \bar{f}_k \int_{S} d^{m}y \sqrt{g_{S}} Y^l f^l \bar{Y}^k, \quad (18)
\]
with \( f_I(x) \) and \( Y^I(y) \) denoting eigenfunctions on \( AdS_n \) and \( S^m \) respectively. The total wavefunctions \( \psi(x,y) = \sum_I f_I(x)Y^I(y) \) obey the linearized equation
\[
(\square_{AdS} + \square_S)\psi = 0.
\]

Understanding the cubic interaction then relies on understanding bulk properties of \( AdS \). From the GKP-W map one has the “holographic” formula
\[
H_3 \sim (\Delta_3 - \Delta_1 - \Delta_2) C(I_1, I_2, I_3),
\]
where \( C(I_1, I_2, I_3) \) are coefficients in the 3-point correlator. For the highest-weight states one has that their energy \( \Delta \) is given by the angular momenta \( \Delta = j \). One also has
\[
C(j_1, j_2, j_3) \propto \delta_{j_1+j_2,j_3},
\]
which is the R-charge conservation condition. We find that the \( \delta \)-function forces the (highest-weight) vertex to vanish, \( V^{h.w.} = 0 \). We emphasize that this implies that the holographic vertex is equal to 0 both on and off-shell. On the other hand, the collective vertex is seen to be non-vanishing off-shell and can therefore serve as the starting point for a raising-lowering procedure that one can apply to highest-weight states.

A. Chiral Primary Interactions in \( AdS_5 \times S^5 \)

Let’s now examine the full interacting theory of chiral primaries, with the ultimate goal of showing agreement with the collective field calculation. We consider the case of \( AdS_5 \times S^5 \), which has been studied in \cite{5}. The equation of motion for the chiral primary field \( s \), of mass \( m^2 = j(j - 4) \), was found to be of the form
\[
(\nabla_\mu \nabla^- - m_J^2)s^I = \sum_{J,K} \left( D_{IJK}s^J s^K + E_{IJK} \nabla_\mu s^J \nabla^\mu s^K + F_{IJK} \nabla(\mu \nabla^\nu)s^J \nabla_{(\mu} \nabla_{\nu)}s^K \right),
\]

1 We note that in Appendix A we will show in more detail the origin of the energy-conserving \( \delta \)-function in the 3-point function.
where \( \mu \) denotes AdS\(_5\) coordinates, and the sphere dependence has already been integrated out. For the explicit form of the coefficients \( D, E \) and \( F \) we refer the reader to [5]. The derivative terms can be removed by the following field redefinition

\[
s^J = s'^I + \sum_{J,K} \left( J_{IJK} s'^J s'^K + L_{IJK} \nabla^\mu s'^J \nabla_\mu s'^K \right),
\]

(21)

where \( L_{IJK} = \frac{1}{2} F_{IJK} \) and \( J_{IJK} = \frac{1}{2} E_{IJK} + \frac{1}{4} F_{IJK} (m_1^2 - m_j^2 - m_3^2 + 8) \). The field redefinition dramatically simplifies the equation of motion, which becomes

\[
(\nabla_\mu \nabla^\mu - m_1^2) s^I = \sum_{J,K} \lambda_{IJK} s^J s^K,
\]

(22)

where \( \lambda_{IJK} = D_{IJK} - (m_j^2 + m_3^2 - m_1^2) J_{IJK} - \frac{2}{3} L_{IJK} m_j^2 m_3^2 \). Finally, after plugging in the coefficients \( D, E \) and \( F \), the action for the chiral primary \( s \) becomes

\[
S = \int d^5x \sqrt{-g} \left[ -\nabla s^I \nabla s^I - m_1^2 |s|^2 - \frac{1}{2} \lambda_{IJK} (s^I s^J s^K + c.c) \right],
\]

(23)

where \( m^2 = j(j - 4) \), and the coupling constant [5] is

\[
\lambda_{123} = (j_3 - j_1 - j_2) 2\kappa \frac{\sqrt{j_1 j_2 j_3 (j_3^2 - 1)(j_3 + 2)(j_3 - 2)}}{\sqrt{(j_1^2 - 1)(j_2^2 - 1)(j_1 + 2)(j_2 + 2)}} f_{123},
\]

\[
f_{123} = \frac{1}{\sqrt{2\pi}^3} \frac{\sqrt{(j_1 + 1)(j_1 + 2)(j_2 + 1)(j_2 + 2)}}{\sqrt{(j_3 + 1)(j_3 + 2)}}.
\]

(24)

The coefficient \( f_{123} \) comes from the overlap integral over spherical harmonics on \( S^5 \). In global coordinates, the highest-weight state on \( AdS_5 \times S^5 \) takes the form

\[
s = \frac{\sqrt{\Delta} (\Delta - 1)}{\pi (\cosh \mu)^\Delta}.
\]

(25)

The matrix element of the cubic hamiltonian for the action (23) is then given by

\[
<3|H_3|12> = \frac{1}{2^{3/2}\pi} \frac{(\Delta_1 - 1)(\Delta_2 - 1)(\Delta_3 - 1)}{(\Delta_3 - 1)(\Delta_3 - 2)} G_{123} \delta(j_1 + j_2 - j_3)
\]

\[
= (\Delta_3 - \Delta_1 - \Delta_2) \frac{\sqrt{\Delta_1 \Delta_2 \Delta_3}}{N} \delta(j_1 + j_2 - j_3).
\]

(26)

Since \( \Delta = j \), this agrees with the collective field theory vertex [17]

\[
H^{(3)} \sim -4\pi \sqrt{n_1 n_2 n_3} (n_1 + n_2 - n_3) a_1 a_2 a_3^\dagger,
\]

(27)
apart from the absence of the $\delta$-function coming from conservation of angular momentum. Thus, we have shown that the collective field theory vertex is contained in the gravity description. We will discuss at a later stage the origin of two different pictures (for 1/2 BPS states) related to the appearance of a delta function term in the vertex.

III. MATRIX MODEL MAPS

Our goal is to extend the hamiltonian formulation from the highest-weight states of the bubbling 1/2 BPS configuration. In global coordinates $AdS_5 \times S^5$ can be written as

\[ ds^2 = (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_3^2) + (\sin^2 \theta d\theta^2 + d\phi^2 + \cos^2 \theta d\tilde{\Omega}_3^2), \]

and, as we will show in more detail later, the chiral primary fields fluctuations read

\[ \delta g \sim \left( \frac{\cos \theta}{\cosh \rho} e^{i\phi} \right)^j, \]

a highest-weight state of the isometry algebra. The collective droplet vertex represents an off-shell interaction of such fluctuations. The basic strategy that we will employ is then to use the resulting non-vanishing three-point interaction as a starting point for reconstructing the full 3-point vertex, i.e. involving more general states. The first ingredient in this program is the reconstruction of linearized wavefunctions:

\[ \psi_{jnm}(t, \rho, \theta, \phi) \sim L_n^m J_{-m}^j \left( \frac{\cos \theta}{\cosh \rho} e^{i\phi} \right). \]

This was done in [23]. From the interactions of chiral primaries, we will develop an analogous raising-lowering Ward identity which relates $V_{j_1n_1,j_2n_2,j_3n_3}$, the vertex for two-matrix states, to $V_{j_1j_2j_3}$, the one-matrix vertex. Toward this end it is important to describe the inclusion of the 1/2 BPS “bubbling” configurations of $AdS_5 \times S^5$ in the two matrix (coordinate) picture. In the 2D coordinate space $(X_1, X_2)$, where the hamiltonian is given by $H = \frac{1}{2}(X_1^2 + X_2^2 + P_1^2 + P_2^2)$ and the angular momentum by $J = X_1 P_2 - X_2 P_1$, one can introduce complex coordinates

\[ Z = \frac{X_1 + iX_2}{\sqrt{2}}, \quad \bar{Z} = \frac{X_1 - iX_2}{\sqrt{2}}, \]
with corresponding conjugate momenta
\[ \Pi = \frac{P_1 + iP_2}{\sqrt{2}}, \quad \bar{\Pi} = \frac{P_1 - iP_2}{\sqrt{2}}. \] (32)

Switching to creation and annihilation operators,
\[ Z = \frac{1}{\sqrt{2}}(A^\dagger + B), \quad \bar{Z} = \frac{1}{\sqrt{2}}(A + B^\dagger), \] (33)
\[ \Pi = -i\frac{1}{\sqrt{2}}(A^\dagger - B), \quad \bar{\Pi} = i\frac{1}{\sqrt{2}}(A - B^\dagger), \] (34)

the Hamiltonian and angular momentum generators become
\[ H = Tr(A^\dagger A + B^\dagger B), \]
\[ J = Tr(A^\dagger A - B^\dagger B). \] (35)

So 1/2 BPS states having \( H = J \) are described by a truncation to the sector where \( B = 0 \), and only \( A \) oscillators remain. This condition can be translated into having a single matrix \( X = (A + A^\dagger) \), with conjugate momentum \( P = i(A - A^\dagger) \). In the phase space (matrix model) one has the corresponding canonical transformation
\[ X = \frac{X_1 + P_2}{\sqrt{2}}, \quad Y = \frac{X_1 - P_2}{\sqrt{2}}, \]
\[ P_X = \frac{P_1 - X_2}{\sqrt{2}}, \quad P_Y = \frac{P_1 + X_2}{\sqrt{2}}, \] (36)

with the fact that
\[ J = X_1 P_2 - X_2 P_1 = \frac{1}{2}(X^2 + P_X^2) - \frac{1}{2}(Y^2 + P_Y^2) = \tilde{J}. \] (37)

In the matrix model picture (which from now on we denote by a \textit{tilde}), the R-charge transformation is not a coordinate transformation, but rather a canonical transformation (dynamical symmetry). This gives an explanation of the origin of the non-conservation in the 3-vertex of the cubic collective field theory that we have noted earlier: in the matrix model formulation we have two representations that are related by a canonical
transformation. Next, we describe the matrix model map associated with LLM (one-matrix model), followed by the construction of [23] which extends it to two matrices. The LLM map is given by one central formula

\[ Z(x_1, x_2, y) = \frac{y^2}{\pi} \int_D dx'_1 dx'_2 \frac{u(x'_1, x'_2, 0)}{[(\vec{x} - \vec{x'})^2 + y^2]^2}, \]  

(38)

where the integral is defined over a domain \( D \) and \( u(x_1, x_2, 0) = \pm \frac{1}{2} \). It is a non-linear map since the dynamical degree of freedom on the right hand side is not \( u(x_1, x_2, 0) \), but the boundary of the domain. Linearization leads to the following (linear) relationship (for a detailed derivation see [23]):

\[ \delta g = \frac{1}{2\pi} \int_0^{2\pi} d\tau \frac{1 - 4a^2 - a^4 + 4a^3 \cos(\tau - \phi)}{[1 + a^2 - 2a \cos(\tau - \phi)]^2} \delta \alpha(\tau), \]

(39)

where \( a = \frac{\cos \theta}{\cosh \rho} \). On the right-hand side of the equation we have the small fluctuation \( \delta \alpha(\tau) \) of the one-matrix collective field described by

\[ H_2 = \frac{1}{2} \int dx \pi \phi_0(x) \left( \Pi^2 + (\partial_x \eta)^2 \right) = \int d\tau (\delta \alpha(\tau))^2, \]

(40)

with

\[ \phi_0(x) = \frac{1}{\pi} \sqrt{\mu - x^2}, \]

(41)

and \( d\tau = \frac{dx}{\pi \phi_0(x)} \). On the left-hand side of (39) \( \delta g \) denotes the fluctuation of a SUGRA chiral primary. In the notation of [23], \( t \) and \( \rho \) denote AdS coordinates, while \( \theta \) and \( \phi \) are sphere angles. For \( \delta \alpha(\tau) \sim e^{ij\tau} \) one gets

\[ \delta g \sim \left( \frac{\cos \theta}{\cosh \rho} e^{i\phi} \right)^j, \]

(42)

the correct chiral primary fields fluctuations. More precisely, denoting the Kernel by \( K_{LLM}(\rho, \theta, \phi; \tau) \), one finds (see [23] for more details):

\[ \delta g(t, \rho, \theta, \phi) = \frac{e^{ijt}}{2\pi} \int_0^{2\pi} d\tau K_{LLM}(\rho, \theta, \phi; \tau) \delta \alpha(\tau). \]

(43)

This is a one-dimensional map from the space \( \tau = \int \frac{dx}{\pi \phi_0(x)} \) of a matrix model to the subspace of \( AdS_5 \times S^5 \) given by \( \frac{\cos \theta}{\cosh \rho} e^{i\phi} \). The extension of the linearized LLM map to
the two-matrix case was given in \cite{23} and starts with the matrix observable
\[ \psi(x, n) = Tr\left( (\delta(x - (A + A^\dagger))B^n)_{SYM} \right). \] (44)

This then leads to an eigenvalue problem
\[ \tilde{K}\psi = \omega \psi, \] (45)

with solution
\[ \tilde{\psi}_{jn}(\tau, \sigma) = \sin((j + 2n)\tau) e^{i\sigma}, \quad \omega_{jn} = j + 2n. \] (46)

Through a kernel constructed in \cite{23}, this maps into a non-trivial eigenfunction on AdS space:
\[ \psi_{jn}(t, \rho, \phi, \theta) = e^{i\omega_{jn}t} \cos^j \theta \frac{1}{4\pi^2} \int_0^{2\pi} d\tau \int_0^{2\pi} d\sigma K_2(\rho, \phi; \tau, \sigma) \tilde{\psi}_{jn}. \] (47)

Notice that the map is 2 ↔ 2, mapping the two coordinates \( \tau, \sigma \) of the matrix model into the spacetime coordinates \( \rho, \phi \). Furthermore, we have the following two remarks about the kernel \( K_2 \). First, when applied to the states with \( n = 0 \), it reduces to the kernel associated with the LLM map. Second, the map is essentially a reduction to action-angle variables associated with the non-trivial AdS Laplacian.

IV. NON-LINEAR ANALYSIS

We now come to the main consideration of this work and address the question of a non-linear extension. In this section we will also address the issue concerning the presence of delta function constraint in the 1/2 BPS interaction vertex. To simplify the discussion we start by considering what we refer to as the non-relativistic model, which will allow us to present the main steps of our argument in explicit terms. Recall that in Section III we distinguished between the matrix model picture (i.e. the tilde representation with matrices \( X, Y \) and conjugate momenta \( P_X, P_Y \)) and the coordinate space \( (X_1, X_2) \). In the non-relativistic approximation one directly replace the matrices with the corresponding coordinates, a procedure that is simple to implement based on density fields. In Section III
we described (at the matrix level) the canonical transformation relating the two pictures in question, with
\[ \tilde{H} = \frac{1}{2} (x^2 + y^2 + p_x^2 + p_y^2) , \]
(48)
and similarly for \( H \). The linear map (in the non-relativistic approximation) which relates the two representations reads
\[ \tilde{\psi}(x, y) = \int dx_1 \int dx_2 K(\tilde{x}, \vec{x}) \psi(x_1, x_2) , \]
(49)
where the kernel is given by
\[ K(x, y; x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-ix_2 \frac{x-y}{\sqrt{2}}} \delta(x_1 - \frac{x+y}{\sqrt{2}}) . \]
(50)
It corresponds to a canonical transformation such that
\[ J = x_1 p_2 - x_2 p_1 = \frac{1}{2} (p_x^2 + x^2) - \frac{1}{2} (p_y^2 + y^2) = \tilde{J} , \]
(51)
related to the change to the matrix model picture discussed in Section III. It maps (matrix model) eigenfunctions
\[ \tilde{\psi}_{jn}(x, y) = e^{-(x^2+y^2)} H_{j+n}(x) H_n(y) \]
(52)
into (space-time) eigenfunctions
\[ \psi_{jn}(r, \phi) = \frac{e^{ij\phi}}{\sqrt{2\pi}} L^j_n(r) , \]
(53)
where in the space-time picture \( x_1 + ix_2 = re^{i\phi} \). The vertices in the two pictures are denoted by \( V \) and \( \tilde{V} \) and are given by the overlap integral of three eigenfunctions:
\[ V_{j_1n_1,j_2n_2,j_3n_3} = (\Delta_1 - \Delta_2 - \Delta_3) \int \frac{d^2 \vec{x}}{\phi_0(x)} \psi_{j_1n_1}(r, \phi) \psi_{j_2n_2}(r, \phi) \psi_{j_3n_3}(r, \phi) , \]
(54)
with \( \psi_{jn} \) given in (53), and similarly for \( \tilde{V} \). The 3-point overlap \( V_{j_1n_1,j_2n_2,j_3n_3} \) will then be roughly of the form
\[ V_3 \sim \int d\phi \ e^{i(-j_1+j_2+j_3)\phi} \int dr \ L_{n_1}^{j_1} L_{n_2}^{j_2} L_{n_3}^{j_3} \]
\[ \sim \delta(-j_1+j_2+j_3) \int dr \ L_{n_1}^{j_1} L_{n_2}^{j_2} L_{n_3}^{j_3} \equiv \delta(-j_1+j_2+j_3) V , \]
(55)
and (still) yield a conserving delta function. Let’s briefly sketch what happens in the case of the \textit{tilde} representation, with eigenfunctions now given by (52). The overlap integral takes the form

\[ \tilde{V}_3 \sim \int dx e^{-3x^2} H_{j_1 + n_1}(x) H_{j_2 + n_2}(x) H_{j_3 + n_3}(x) \int dy e^{-3y^2} H_{n_1}(y) H_{n_2}(y) H_{n_3}(y). \] (56)

As one can verify, written in this basis the vertex no longer has a conserving \( \delta \)-function. Thus, as we commented earlier, the vertex \( V \) has R-charge conservation

\[ V_{j_1n_1j_2n_2j_3n_3} = \delta_{j_1,j_2+j_3} V, \] (57)

while \( \tilde{V} \) does not. This is explained by the different action of the R-charge operator \( \hat{J} \) in the two pictures. While in the present case one can easily show that for 2-point overlaps

\[ \int dx_1 dx_2 \psi_{jn} \psi_{j'n'} = \int dx dy \tilde{\psi}_{jn} \tilde{\psi}_{j'n'}, \] (58)

one cannot do that for the 3-point function overlap. In fact one can show explicitly that

\[ V_{j_1n_1j_2n_2j_3n_3} \neq \tilde{V}_{j_1n_1j_2n_2j_3n_3}. \] (59)

The basic theorem that we will establish in what follows is that the two hamiltonians

\[ H = \omega_{jn} A_{jn}^\dagger A_{jn} + (V_{j_1n_1j_2n_2j_3n_3} A_{j_1n_1}^\dagger A_{j_2n_2} A_{j_3n_3} + \text{h.c.}) \] (60)

and

\[ \tilde{H} = \sum_{jn} \omega_{jn} \tilde{A}_{jn}^\dagger \tilde{A}_{jn} + \]

\[ + \sum_{\{j',s,n's\}} \left( \tilde{V}_{j_1n_1j_2n_2j_3n_3}^{(1)} \tilde{A}_{j_1n_1}^\dagger \tilde{A}_{j_2n_2} \tilde{A}_{j_3n_3} + \tilde{V}_{j_1n_1j_2n_2j_3n_3}^{(2)} \tilde{A}_{j_1n_1}^\dagger \tilde{A}_{j_2n_2}^\dagger \tilde{A}_{j_3n_3} + \text{h.c.} \right) \] (61)

are in fact equivalent, with a non-linear canonical transformation relating them. To demonstrate this statement, namely the fact that (60) and (61) match, we would like to perform the following field redefinition:

\[ \tilde{A}_N = A_N + c_{NMP} A_M A_P + d_{NMP} A_M^\dagger A_P + e_{NMP} A_M^\dagger A_P^\dagger. \] (62)
We have simplified the notation by using the index $N$ to denote all quantum numbers $(j, n)$. The hamiltonian in the tilde representation with this more compact notation takes the form

$$\tilde{H} = \tilde{H}_2 + \tilde{H}_3 = \sum_N \omega_N \tilde{A}_N \tilde{A}_N + \sum_{(N,M,P)} \left( \tilde{V}^{(1)}_{NMP} \tilde{A}_N \tilde{A}_M \tilde{A}_P + \tilde{V}^{(2)}_{NMP} \tilde{A}_N \tilde{A}_M \tilde{A}_P + h.c. \right). \quad (63)$$

Under the field redefinition (62), the quadratic part $\tilde{H}_2$ yields additional cubic terms, and the total hamiltonian becomes

$$\tilde{H} = \sum_N \omega_N \tilde{A}_N \tilde{A}_N + \sum_{N,M,P} \left[ (\tilde{V}^{(1)}_{NMP} + \omega_N c_{NMP} + \omega_P \bar{d}_{PMN}) \tilde{A}_N \tilde{A}_M \tilde{A}_P + \right.$$

$$+ (\tilde{V}^{(2)}_{NMP} + \omega_N e_{NMP}) \tilde{A}_N \tilde{A}_M \tilde{A}_P + h.c. \right] \quad (64)$$

If we want this to match (60), we need the following conditions on the coefficients of the field redefinition:

$$\omega_N e_{NMP} = -\tilde{V}^{(2)}_{NMP}, \quad \tilde{V}^{(1)}_{NMP} + \omega_N c_{NMP} + \omega_P \bar{d}_{PMN} = V_{NMP}. \quad (65)$$

Furthermore, we can obtain additional constraints on $c_{NMP}, d_{NMP}$ and $e_{NMP}$ by imposing appropriate commutation relations:

$$[\tilde{A}_N, \tilde{A}_{N'}] = 0, \quad (66)$$

$$[\tilde{A}_N, \tilde{A}_{N'}^\dagger] = \delta_{N,N'}. \quad (67)$$

Requiring (66) yields

$$d_{N'NM} = d_{NN'M}, \quad e_{N'MN} - e_{NMN'} + e_{N'NM} - e_{NN'M} = 0. \quad (68)$$

The remaining commutation relation (67) gives

$$d_{NMN'} + \bar{c}_{N'MN} + \bar{c}_{N'NM} = 0,$$

$$\bar{d}_{N'MN} + c_{N'MN'} + c_{NN'M} = 0. \quad (69)$$
This entirely fixes the non-linear redefinition (62), showing that one can in fact connect the two hamiltonians.

To summarize, we have described how the matrix level canonical transformation induces changes at the nonlinear level. One has two related pictures, one in which the R-symmetry is implemented as a coordinate symmetry (with the corresponding delta function) and another where the R-symmetry is dynamical, given by a canonical transformation. We have shown the equivalence of these two pictures through a nonlinear field transformation. Related field transformations have been identified previously at the Lagrangian level in [8].

V. WARD IDENTITIES AND VERTEX RECONSTRUCTION

Our main goal is to establish that, starting from the vertex of highest-weight states, it is possible to build the vertex for more general states that are reachable by (in this case) $SL(2)$ raising/lowering procedure. Specifically, we will develop an identity that will allow us to generate such non-trivial vertices, by making use of the available Ward identities. We will again start from the simplified (non-relativistic) model discussed in Section [IV]. This will then be followed by a discussion on the form of Ward identities in the AdS case.

A. Non-relativistic Model

Recall that the hamiltonian of the non-relativistic model is given by
\[ H = \frac{x^2 + y^2}{2} + \frac{p_x^2 + p_y^2}{2}, \]  
(70)
or, in terms of creation and annihilation operators,
\[ H = a^\dagger a + b^\dagger b. \]  
(71)
Let’s introduce complex variables
\[ z = \frac{a^\dagger + b}{\sqrt{2}}, \quad \bar{z} = \frac{a + b^\dagger}{\sqrt{2}}, \]  
(72)
and corresponding conjugate momenta
\[ \Pi = -i \partial_z = i \frac{a^\dagger - b}{\sqrt{2}}, \quad \bar{\Pi} = -i \partial_{\bar{z}} = -i \frac{a - b^\dagger}{\sqrt{2}}. \] (73)

These expressions can be combined to obtain
\[ a = \frac{1}{\sqrt{2}} (\bar{z} + \partial_z), \quad b = \frac{1}{\sqrt{2}} (z + \partial_{\bar{z}}). \] (74)

The wavefunctions are then given by
\[ |J, n> \equiv (a^\dagger)^J (b^\dagger)^n |0>, \] (75)

and the generators \( l_+, l_-, l_0 \) by
\[ l_+ = \frac{1}{2} (\partial_z \partial_{\bar{z}} + z \bar{z} - 1) - \frac{z}{2} \partial_z - \frac{\bar{z}}{2} \partial_{\bar{z}}, \] \( l_- = \frac{1}{2} (\partial_z \partial_{\bar{z}} + z \bar{z} + 1) + \frac{z}{2} \partial_z + \frac{\bar{z}}{2} \partial_{\bar{z}}, \] \( l_0 = -\partial_z \partial_{\bar{z}} + z \bar{z} - 1. \) (76, 77, 78)

As we mentioned earlier, our goal is to use these generators to derive an identity for the cubic vertex, which is given by
\[ \int d^2x \frac{1}{\sqrt{\phi_0(x)}} \eta \eta l_0 \eta, \] (79)

with \( \bar{x} = (x, y) \). If we plug the standard mode expansion
\[ \eta = \sqrt{2} \sum_{J=1}^\infty \sum_{n=0}^\infty (\bar{c}_{J,n} \psi_{J,n} + c_{J,n} \bar{\psi}_{J,n}) \] (80)

into the vertex we find
\[ \sum_{J=1}^\infty \sum_{n=0}^\infty \left[ \bar{c}_{J,n_1} c_{J_2,n_2} c_{J_3,n_3} \int dx \int dy \frac{1}{\sqrt{\phi_0(x)}} \psi_{J_1,n_1} \psi_{J_2,n_2} \bar{\psi}_{J_3,n_3} \cdots \right] P, \] (81)

where we denote by \( P \) the prefactor coming from the action of \( l_0 \) on the wavefunctions.

Next, we would like to use the fact that
\[ \psi_{J,n} = \frac{l_+}{\sqrt{(J+n)n}} \psi_{J,n-1}, \] (82)
and focus on
\[ V_1 = \int d^2 x \frac{1}{\sqrt{\phi_0(x)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} \psi_{J_3,n_3}. \]  

(83)

Using (76), the vertex term above becomes
\[ V_1 = \int d^2 x \frac{1}{2\sqrt{\phi_0(x)} \sqrt{n_3(J_3 + n_3)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} (\partial_z \partial_{\bar{z}} + z \bar{z} - 1 - z \partial_z - \bar{z} \partial_{\bar{z}}) \psi_{J_3,n_3-1}. \]  

(84)

Let’s treat each term in \( V_1 \) separately. We start from
\[ T_1 \equiv \int d^2 x \frac{1}{2\sqrt{\phi_0} \sqrt{n_3(J_3 + n_3)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} \partial_z \partial_{\bar{z}} \psi_{J_3,n_3-1} \]
\[ = \int d^2 x \frac{1}{2\sqrt{\phi_0} \sqrt{n_3(J_3 + n_3)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} [-l_0 + z \bar{z} - 1] \psi_{J_3,n_3-1} \]
\[ = \int d^2 x \frac{1}{2\sqrt{\phi_0} \sqrt{n_3(J_3 + n_3)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} [-J_3 - 2n_3 + z \bar{z} - 1] \psi_{J_3,n_3-1}. \]  

(85)

We then look at the term
\[ T_2 \equiv - \int d^2 x \frac{1}{2\sqrt{\phi_0} \sqrt{n_3(J_3 + n_3)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} z \partial_z \psi_{J_3,n_3-1} \]
\[ = \int d^2 x \frac{1}{2\sqrt{n_3} \sqrt{(J_3 + n_3)}} \psi_{J_3,n_3-1} \bar{\psi}_{J_2,n_2} \left[ \frac{1}{2\sqrt{\phi_0}} + \frac{z}{\sqrt{\phi_0}} \partial_z + \frac{z}{2} \partial_z \phi_0^{-1/2} \right] \bar{\psi}_{J_1,n_1} + \]
\[ + \int d^2 x \frac{1}{2\sqrt{n_3} \sqrt{(J_3 + n_3)}} \psi_{J_3,n_3-1} \bar{\psi}_{J_2,n_2} \left[ \frac{1}{2\sqrt{\phi_0}} + \frac{z}{\sqrt{\phi_0}} \partial_z + \frac{z}{2} \partial_z \phi_0^{-1/2} \right] \psi_{J_1,n_1}. \]  

(86)

Similarly,
\[ T_3 \equiv - \int d^2 x \frac{1}{2\sqrt{\phi_0} \sqrt{n_3(J_3 + n_3)}} \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} \bar{z} \partial_{\bar{z}} \psi_{J_3,n_3-1} \]
\[ = \int d^2 x \frac{1}{2\sqrt{n_3} \sqrt{(J_3 + n_3)}} \psi_{J_3,n_3-1} \bar{\psi}_{J_2,n_2} \left[ \frac{1}{2\sqrt{\phi_0}} + \frac{z}{\sqrt{\phi_0}} \partial_z + \frac{z}{2} \partial_z \phi_0^{-1/2} \right] \bar{\psi}_{J_1,n_1} + \]
\[ + \int d^2 x \frac{1}{2\sqrt{n_3} \sqrt{(J_3 + n_3)}} \psi_{J_3,n_3-1} \bar{\psi}_{J_2,n_2} \left[ \frac{1}{2\sqrt{\phi_0}} + \frac{z}{\sqrt{\phi_0}} \partial_z + \frac{z}{2} \partial_z \phi_0^{-1/2} \right] \psi_{J_1,n_1}. \]  

(87)
We now collect all terms and, using the definitions of $l_0$ and $l_-$, find

$$V_1 = \int d^2 x \frac{\psi_{J_3,n_3-1}}{\sqrt{\phi_0} \sqrt{n_3(1 + n_3)}} \times$$

$$\left[ \psi_{J_3,n_3-1} [l_+ + \frac{l_0}{2} - z\bar{z} + \frac{1}{2} + \frac{z \sqrt{\phi_0} \partial_z \phi_0^{-1/2}}{4} + \frac{\bar{z} \sqrt{\phi_0} \partial_{\bar{z}} \phi_0^{-1/2}}{4}] \bar{\psi}_{J_1,n_1} + \\
+ \bar{\psi}_{J_1,n_1} [l_+ + \frac{l_0}{2} - z\bar{z} + \frac{1}{2} + \frac{z \sqrt{\phi_0} \partial_z \phi_0^{-1/2}}{4} + \frac{\bar{z} \sqrt{\phi_0} \partial_{\bar{z}} \phi_0^{-1/2}}{4}] \bar{\psi}_{J_2,n_2} + \\
+ \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} [z\bar{z} - 1 - \frac{2n_3 + J_3^2}{2}] \right]. \quad (88)$$

We can rewrite the vertex above in the following way:

$$\int d^2 x \frac{\bar{\psi}_{J_1,n_1} \psi_{J_2,n_2}}{\sqrt{\phi_0} \sqrt{n_3(1 + n_3)}} (l_+ + \frac{l_0}{2}) \bar{\psi}_{J_3,n_3-1} =$$

$$= \int d^2 x \frac{\bar{\psi}_{J_1,n_1} \psi_{J_2,n_2}}{\sqrt{\phi_0} \sqrt{n_3(1 + n_3)}} (l_+ + \frac{l_0}{2}) \bar{\psi}_{J_3,n_3-1} + \int d^2 x \frac{\bar{\psi}_{J_3,n_3-1} \psi_{J_1,n_1}}{\sqrt{\phi_0} \sqrt{n_3(1 + n_3)}} (l_+ + \frac{l_0}{2}) \bar{\psi}_{J_2,n_2} +$$

$$+ \int d^2 x \frac{\bar{\psi}_{J_1,n_1} \psi_{J_2,n_2}}{\sqrt{\phi_0} \sqrt{n_3(1 + n_3)}} [z\bar{z} + \frac{z \sqrt{\phi_0} \partial_z \phi_0^{-1/2}}{2} + \frac{\bar{z} \sqrt{\phi_0} \partial_{\bar{z}} \phi_0^{-1/2}}{2}] \bar{\psi}_{J_3,n_3-1} \quad (89)$$

The last line vanishes, since $\phi_0 = e^{-2z\bar{z}}$. Thus, we find the following identity:

$$\int d^2 x \frac{1}{\sqrt{\phi_0}} \left[ l_+^{(3)} - l_+^{(1)} - l_+^{(2)} + \frac{l_0^{(3)} - l_0^{(1)} - l_0^{(2)}}{2} \right] \bar{\psi}_{J_1,n_1} \psi_{J_2,n_2} \bar{\psi}_{J_3,n_3-1} = 0. \quad (90)$$

Notice that this identity can be used to relate the vertex for single-matrix states (highest-weight states with $n_i = 0$) to vertices of multi-matrix states. In this sense, it provides a generating mechanism for constructing non-trivial interactions starting from the (simpler) 1/2 BPS sector of the theory.

### B. Interactions in $AdS$

We now move on to the case of real interest, interactions in $AdS \times S$. For simplicity we consider $AdS_2 \times S^2$. The generators are given by

$$l_+ = i [\cos \rho \partial_\rho + i \sin \rho \partial_\lambda], \quad l_0 = i \partial_t \quad (91)$$
and the eigenfunctions (denoted by $\phi^\lambda_n$ to distinguish them from those of the non-relativistic model) by

$$
\phi^\lambda_n(t, \rho) = c(\lambda) \sqrt{\frac{n!}{\Gamma(n + 2\lambda)}} e^{-i(n+\lambda)(t+\frac{\pi}{2})}(\cos \rho)^\lambda C_n^\lambda(\sin \rho),
$$

with

$$
c(\lambda) = \frac{\Gamma(\lambda) 2^{\lambda-1}}{\sqrt{\pi}}.
$$

Starting from $\int d\rho \left[ l_1^{(1)} \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3} \right]$ at $t = 0$ and integrating by parts with respect to $\rho$ we find

$$
i \int d\rho \sin \rho \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3} + i \int d\rho \sin \rho \left[ l_0^{(2)} + l_0^{(3)} - l_0^{(1)} \right] \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3}
$$

$$
+ \int d\rho \left[ l_-^{(2)} + l_-^{(3)} - l_+^{(1)} \right] \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3} = 0
$$

We can eliminate the $\sin \rho$ terms from the recursion relation by using

$$
\sin \rho = \frac{l_- - l_+}{2il_0}
$$

on the "1" leg to obtain:

$$
\int d\rho \left[ \frac{l_-^{(1)} - l_+^{(1)}}{2(n_1 + \lambda_1)} \right] \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3} - \int d\rho \left[ l_-^{(2)} + l_-^{(3)} - l_+^{(1)} \right] \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3}
$$

$$
+ \int d\rho \left[ \frac{l_-^{(1)} - l_+^{(1)}}{2(n_1 + \lambda_1)} \right] [n_1 + \lambda_1 + n_2 + \lambda_2 + n_3 + \lambda_3] \tilde{\phi}^{\lambda_1}_{n_1} \phi^{\lambda_2}_{n_2} \phi^{\lambda_3}_{n_3} = 0.
$$

1. Use of the Ward Identity

We can use this recursion relation to evaluate the overlap integral of a product of any three eigenfunctions given the overlap of highest-weight eigenfunctions. Inserting $n_1 = n_1$, $n_2 = n_3 = 0$ into the Ward identity and using ($C_0^\lambda = 1$, $C_1^\lambda(x) = 2\lambda x$)

$$
l_- \phi^\lambda_m = e^{-it} \sqrt{m(m-1+2\lambda)} \phi^\lambda_{m-1}, \quad l_- \tilde{\phi}^\lambda_n = -e^{-it} \sqrt{(n+1)(n+2\lambda)} \tilde{\phi}^\lambda_{n+1},
$$

$$
l_+ \phi^\lambda_n = -e^{+it} \sqrt{n(n-1+2\lambda)} \phi^\lambda_{n-1},
$$

(97)
we obtain
\[
\int d\rho \phi_{n_{1}+1}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} = \frac{1 - n_{1} - \lambda_{1} + \lambda_{2} + \lambda_{3}}{1 + n_{1} + \lambda_{1} + \lambda_{2} + \lambda_{3}} \sqrt{n_{1}(n_{1} - 1 + 2\lambda_{1})} \times 
\times e^{2it} \int d\rho \phi_{n_{1} - 1}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{0}^{\lambda_{3}}.
\]

(98)

This relation allows us to determine \( \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{0}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} \) for any \( n_{1} \), once we know its value for \( n_{1} = 0, 1 \). To obtain the value when \( n_{1} = 1 \), insert \( n_{1} = n_{2} = n_{3} = 0 \) into the Ward identity. The resulting identity implies that \( \int d\rho \phi_{1}^{\lambda_{1}} \phi_{0}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} = 0 \). Next, set \( n_{1} = n_{1}, n_{2} = n_{2} \) and \( n_{3} = 0 \). In this case, we find

\[
\alpha_{1} \int d\rho \phi_{n_{1}+1}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} + \alpha_{2} \int d\rho \phi_{n_{1}-1}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} + \alpha_{3} \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{n_{2}-1}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} = 0,
\]

(99)

where

\[
\alpha_{1} = -e^{-it} \sqrt{(n_{1} + 1)(n_{1} + 2\lambda_{1})} \frac{1 + n_{1} + \lambda_{1} + n_{2} + \lambda_{2} + \lambda_{3}}{2(n_{1} + \lambda_{1})},
\]

(100)

\[
\alpha_{2} = e^{it} \sqrt{n_{1}(n_{1} - 1 + 2\lambda_{1})} \frac{1 - n_{1} - \lambda_{1} + n_{2} + \lambda_{2} + \lambda_{3}}{2(n_{1} + \lambda_{1})},
\]

(101)

\[
\alpha_{3} = -e^{-it} \sqrt{n_{2}(n_{2} - 1 + 2\lambda_{2})}.
\]

(102)

If we set \( n_{2} = 1 \) we have

\[
\alpha_{1} \int d\rho \phi_{n_{1}+1}^{\lambda_{1}} \phi_{1}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} + \alpha_{2} \int d\rho \phi_{n_{1}-1}^{\lambda_{1}} \phi_{1}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} + \alpha_{3} \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{0}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} = 0,
\]

(103)

which (starting from \( n_{1} = 0 \)) determines \( \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{0}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} \) for all \( n_{1} \). Next, set \( n_{2} = 2 \) to obtain

\[
\alpha_{1} \int d\rho \phi_{n_{1}+1}^{\lambda_{1}} \phi_{2}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} + \alpha_{2} \int d\rho \phi_{n_{1}-1}^{\lambda_{1}} \phi_{2}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} + \alpha_{3} \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{1}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} = 0,
\]

(104)

which (starting from \( n_{1} = 0 \)) fixes \( \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{2}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} \) for all \( n_{1} \). Continuing in this way, it is clear that we can determine \( \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{0}^{\lambda_{3}} \), for all \( n_{1}, n_{2} \). Finally, set \( n_{1} = n_{1}, n_{2} = n_{2} \) and \( n_{3} = n_{3} \). In this case, we find

\[
\alpha_{1} \int d\rho \phi_{n_{1}+1}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{n_{3}}^{\lambda_{3}} + \alpha_{2} \int d\rho \phi_{n_{1}-1}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{n_{3}}^{\lambda_{3}} + \alpha_{3} \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{n_{2}-1}^{\lambda_{2}} \phi_{n_{3}}^{\lambda_{3}} + \alpha_{4} \int d\rho \phi_{n_{1}}^{\lambda_{1}} \phi_{n_{2}}^{\lambda_{2}} \phi_{n_{3} - 1}^{\lambda_{3}} = 0,
\]

(105)
where
\[
\alpha_1 = -e^{-it} \sqrt{(n_1 + 1)(n_1 + 2\lambda_1)} \frac{1 + n_1 + \lambda_1 + n_2 + \lambda_2 + n_3 + \lambda_3}{2(n_1 + \lambda_1)}, \tag{106}
\]
\[
\alpha_2 = e^{it} \sqrt{n_1(n_1 - 1 + 2\lambda_1)} \frac{1 - n_1 - \lambda_1 + n_2 + \lambda_2 + n_3 + \lambda_3}{2(n_1 + \lambda_1)}, \tag{107}
\]
\[
\alpha_3 = -e^{-it} \sqrt{n_2(n_2 - 1 + 2\lambda_2)}, \tag{108}
\]
\[
\alpha_4 = -e^{-it} \sqrt{n_3(n_3 - 1 + 2\lambda_3)}. \tag{109}
\]

If we take \(n_2 = 0\) and \(n_3 = 1\) we can determine \(\int d\rho \bar{\phi}^\lambda n_1 \phi^\lambda_0 \phi^\lambda_1\) for all \(n_1\). Setting \(n_2 = 1\) and \(n_3 = 1\), we can find \(\int d\rho \bar{\phi}^\lambda n_1 \phi^\lambda_0 \phi^\lambda_1\) for all \(n_1\). Next, set \(n_2 = 2\) and \(n_3 = 1\) to get \(\int d\rho \bar{\phi}^\lambda n_1 \phi^\lambda_0 \phi^\lambda_2\) for all \(n_1\). Inching one step at a time we can determine the full vertex.

2. Check of the Ward Identity

To check the action of the generators we checked:
\[
l_- \phi_0^\lambda = i [\cos \rho \partial_\rho + i \sin \rho \partial_\theta] \left( c(\lambda) \sqrt{\frac{1}{\Gamma(2\lambda)}} \right) e^{-i\lambda(t + \frac{\pi}{2})} (\cos \rho)^\lambda = 0, \tag{110}
\]
\[
l_- \bar{\phi}_1^\lambda = i [\cos \rho \partial_\rho + i \sin \rho \partial_\theta] \left( c(\lambda) \sqrt{\frac{1}{\Gamma(1 + 2\lambda)}} \right) e^{i(1 + \lambda)(t + \frac{\pi}{2})} (\cos \rho)^\lambda \lambda \sin \rho \right) = -\sqrt{2(1 + 2\lambda)} \bar{\phi}_2^\lambda e^{-it}, \tag{111}
\]
\[
l_+ \bar{\phi}_1^\lambda = i [\cos \rho \partial_\rho - i \sin \rho \partial_\theta] \left( c(\lambda) \sqrt{\frac{1}{\Gamma(1 + 2\lambda)}} \right) e^{i(1 + \lambda)(t + \frac{\pi}{2})} \lambda \sin \rho \right) = -\sqrt{2\lambda} \bar{\phi}_0^\lambda e^{it}. \tag{112}
\]

As a partial check of the results of the previous section, we will evaluate (98) for \(n_1 = 1\) and explicitly verify that it is correct. After setting \(n_1 = 1\) we have
\[
\int d\rho \bar{\phi}_2^\lambda \phi_0^\lambda \phi^\lambda_0 = \frac{\lambda_2 + \lambda_3 - \lambda_1}{2 + \lambda_1 + \lambda_2 + \lambda_3} \sqrt{2\lambda_1} \frac{\sqrt{2\lambda_1} e^{2it}}{\sqrt{2(1 + 2\lambda_1)}} \int d\rho \bar{\phi}_0^\lambda \phi_0^\lambda \phi^\lambda_0. \tag{113}
\]
Now,
\[
\int d\rho \, \bar{\phi}_2^\lambda \phi_0^\lambda \phi_0^0 = c(\lambda_1)c(\lambda_2)c(\lambda_3) \sqrt{\frac{2!}{\Gamma(2 + 2\lambda_1)\Gamma(2\lambda_2)\Gamma(2\lambda_3)}} \, e^{-i(\lambda_2 - \lambda_3 - \lambda_1)(t + \frac{1}{2}\pi)} \\
\times e^{2i(t + \frac{1}{2}\pi)} \int d\rho \, (\cos \rho)^{\lambda_1 + \lambda_2 + \lambda_3} C_2^{\lambda_1}(\sin \rho).
\] (114)

Using
\[
\sqrt{\frac{2!}{\Gamma(2 + 2\lambda_1)\Gamma(2\lambda_2)\Gamma(2\lambda_3)}} = \sqrt{\frac{2!}{(1 + 2\lambda_1)2\lambda_1}} \sqrt{\frac{1}{\Gamma(2\lambda_1)\Gamma(2\lambda_2)\Gamma(2\lambda_3)}},
\]
\[
e^{2i(t + \frac{1}{2}\pi)} = -e^{2it},
\]
\[
\int d\rho \, (\cos \rho)^{\lambda_1 + \lambda_2 + \lambda_3} C_2^{\lambda_1}(\sin \rho) = \frac{\lambda_1(\lambda_1 - \lambda_2 - \lambda_3)}{2 + \lambda_1 + \lambda_2 + \lambda_3} \int d\rho \, (\cos \rho)^{\lambda_1 + \lambda_2 + \lambda_3},
\] (115)
it is trivial to verify the identity.

In conclusion, in this section we have demonstrated the existence of (SL(2)) Ward identities. We have shown that these identities contain the necessary information to specify the cubic interaction vertex for general states from the knowledge of the vertex for the highest-weight states. Since we have shown that the one-matrix collective field theory correctly describes the latter case, we therefore have a scheme of reconstruction of the full vertex. This discussion was presented in the simplest AdS$_2$ framework; it is clear, however, that this procedure is valid in general. Nevertheless it will be important to develop the details in the higher dimensional case. In particular, there should be significant information on interactions in the 1/4 BPS sector where progress has recently been accomplished at the SUGRA level [20, 26].

VI. ACKNOWLEDGMENTS

One of us (AJ) would like to thank Tamiaki Yoneya for many discussions on the present topic and the hospitality of the Theory Group at The University of Tokyo, Komaba during part of this work. We also thank Aristomenis Donos for discussions. The work of SC is supported in part by the Michigan Society of Fellows. The work of RdMK is supported by
APPENDIX A: THE VERTEX IN THE TWO REPRESENTATIONS

In this appendix we show in some detail the origin of the energy-conserving $\delta$-function in the 3-point function in the original, non-tilde representation. Recall that a typical cubic term in the Hamiltonian takes the form

$$H_3^{(1)} = \frac{\omega_3(\omega_3 - \omega_1 - \omega_2)}{\sqrt{\omega_1\omega_2\omega_3}} A_1 A_2 A_3^\dagger \int d^D X \sqrt{-g} g^{tt} \psi_1 \psi_2 \bar{\psi}_3, \quad (A1)$$

where the integral is over the spatial coordinates only, and the $AdS_{d+1} \times S^{d+1}$ wavefunction is given by $\psi(x, y) = \sum_I f_I(x) Y^I(y)$. Here $x$ and $y$ denote $AdS_{d+1}$ and $S^{d+1}$ coordinates respectively. Furthermore, in general we have $f(x) = f(t, \rho, \Omega_{d-1}) = e^{-i\omega t} \Psi(\rho, \Omega_{d-1})$. Incorporating the (trivial) time dependence $e^{-i\omega t}$ into the creation/annihilation operators of (A1), we see that the 3-vertex

$$V_3 = \int d^D X \sqrt{-g} g^{tt} \psi_1 \psi_2 \bar{\psi}_3$$

(A2)

can be written as

$$V_3 = \int d^d x \sqrt{-g_{AdS}} g^{tt} \Psi_1 \Psi_2 \bar{\Psi}_3 \int d^{d+1} y \sqrt{g_S} Y_1 Y_2 \bar{Y}_3 \equiv F_{123} G_{123}, \quad (A3)$$

where we defined

$$F_{123} \equiv \int d^d x \sqrt{-g_{AdS}} g^{tt} \Psi_1 \Psi_2 \bar{\Psi}_3, \quad G_{123} \equiv \int d^{d+1} y \sqrt{g_S} Y_1 Y_2 \bar{Y}_3. \quad (A4)$$

For simplicity, we now restrict ourselves to $AdS_2 \times S^2$, with (global coordinates) metric

$$ds^2 = -\sec \rho^2 dt^2 + \sec \rho^2 d\rho^2 + \sin \theta^2 d\phi^2 + d\theta^2. \quad (A5)$$

On the sphere one has

$$\Box_{S^2} Y^m_j = -j(j+1) Y^m_j, \quad (A6)$$

24
where \( Y^m_j(\phi, \theta) = \tilde{N}^m_j e^{im\phi} P^m_j(\cos \theta), \) \( \bar{m} = -j, -j + 1, \ldots, j \) and \( \tilde{N}^m_j \) is the proper normalization. The AdS wavefunctions which satisfy the wave equation

\[
\Box_{\text{AdS}_2} f = \cos \rho^2 (-\partial_t^2 + \partial_{\rho}^2) f = m^2 f
\]  

are given by

\[
f_{\omega, \lambda}(t, \rho) = N^\lambda_{\omega, \lambda} e^{-i\omega t} (\cos \rho)^\lambda C^j_{\omega, \lambda}(\sin \rho), \quad \omega = \lambda + n, \quad n = 0, 1, 2, \ldots .
\]  

(A8)

Here \( C^j_{\omega, \lambda}(\sin \rho) \) are Gegenbauer polynomials, \( N^\lambda_{\omega, \lambda} \) is a normalization factor and \( \lambda \) is related to the mass of the field (also note \( 0 \leq \rho < \frac{\pi}{2} \)). For chiral primaries the mass turns out to be \( m^2 = j(j - 1) \), and the highest weight state is given by \( \lambda = j \) and \( n = 0 \):

\[
\psi_{\text{h.w.}}(t, \rho, \phi, \theta) = N^j_0 \tilde{N}^j_j e^{-ijt} (\cos \rho)^j e^{ij\phi} P^j_j(\cos \theta).
\]  

(A9)

In order for the spherical harmonics and the Gegenbauer polynomials to be \( \delta \)-function normalized we must take

\[
\tilde{N}^m_j = \sqrt{\frac{(2j + 1)(j - \bar{m})!}{4\pi(j + \bar{m})!}}, \quad N^\lambda_n = \frac{\Gamma(\lambda)2^{\lambda - 1/2}}{\sqrt{\pi}} \sqrt{\frac{n!(n + \lambda)}{\Gamma(n + 2\lambda)}}.
\]  

(A10)

For highest-weight states on \( \text{AdS}_2 \times S^2 \) the overlap integrals are (defining \( j \equiv j_1 + j_2 + j_3 \)):

\[
F_{j_1, j_2, j_3} = \int_0^\pi d\rho \sqrt{-g_{\text{AdS}_2}} g^{tt} \Psi_{j_1} \Psi_{j_2} \Psi_{j_3}
\]

\[
= N^{j_1}_0 N^{j_2}_0 N^{j_3}_0 \int d\rho (\cos \rho)^j C^{j_1}_0 C^{j_2}_0 C^{j_3}_0
\]

\[
= N^{j_1}_0 N^{j_2}_0 N^{j_3}_0 \left( \frac{\sqrt{\pi} \Gamma(\frac{1}{2} + \frac{j}{2})}{2 \Gamma(1 + \frac{j}{2})} \right),
\]  

(A11)

\[
G_{j_1, j_2, j_3} = \int d^2 \Psi \sqrt{g_{S^2}} Y_{j_1}^{j_1} Y_{j_2}^{j_2} Y_{j_3}^{j_3}
\]

\[
= \delta(j_1 + j_2 - j_3) \tilde{N}^{j_1}_{j_1} \tilde{N}^{j_2}_{j_2} \tilde{N}^{j_3}_{j_3} \frac{1}{(2\pi)^{3/2}} \int_0^\pi d\theta \sin \theta P^{j_1}_{j_1} P^{j_2}_{j_2} P^{j_3}_{j_3}
\]

\[
= \delta(j_1 + j_2 - j_3) \frac{(-1)^j \sqrt{\pi}}{(2\pi)^{3/2}} \prod_{i=1}^3 \frac{\Gamma(1 + \frac{j_i}{2}) \Gamma(\frac{3}{2} + \frac{j_i}{2})}{\Gamma(\frac{3}{2} + \frac{j_i}{2})},
\]  

(A12)
where we used $C_j^0 = 1$ and $P_j^j(x) = (-1)^j(2j - 1)!!(1 - x^2)^{j/2}$. Note the appearance of the $\delta(j_1 + j_2 - j_3)$ term in $G_{j_1 j_2 j_3}$, coming from the $\int d\phi e^{ij_1 \phi_1 + \phi_2 - \phi_3}$ integral. After some Gamma function cancellations, we are left with the following 3-vertex:

$$F_{j_1 j_2 j_3} G_{j_1 j_2 j_3} = \delta(j_1 + j_2 - j_3) \frac{(-1)^j \pi}{(2\pi)^{3/2}} \prod_{i=1}^3 ((2j_i - 1)!! N_0^{j_i} \bar{N}^{j_i}_{j_i}) \frac{1}{j + 1}, \quad (A13)$$

where we used $\Gamma(n/2) = \sqrt{2\pi} (n - 2)!! 2^{-n/2}$. Finally, using

$$N_0^{j_i} = \frac{\Gamma(j_i) 2^{j_i}}{\sqrt{2\pi} \Gamma(2j_i)}, \quad \bar{N}^{j_i}_{j_i} = \sqrt{\frac{2j_i + 1}{2(2j_i)!}}, \quad \text{and} \quad (2j_i - 1)!! = \frac{(2j_i)!}{2^{j_i} (j_i)!}, \quad (A14)$$

we find

$$(2j_i - 1)!! N_0^{j_i} \bar{N}^{j_i}_{j_i} = \sqrt{\frac{2j_i + 1}{2\pi}}. \quad (A15)$$

The final expression for the 3-point function overlap is

$$V_3 \equiv F_{j_1 j_2 j_3} G_{j_1 j_2 j_3} = \delta(j_1 + j_2 - j_3) \left[ \frac{(-1)^j}{(8\pi^2)(j + 1)} \sqrt{(2j_1 + 1)(2j_2 + 1)(2j_3 + 1)} \right]. \quad (A16)$$

Thus, (for the simple case of $AdS_2 \times S^2$) we have explicitly shown the origin of the delta-function term, and presented the final expression for the 3-point overlap. This calculation can be repeated for the more general wavefunctions given in (53). The 3-point overlap will then be roughly of the form

$$V_3 \sim \int d\phi e^{ij_1 \phi_1 + j_2 \phi_2 - \phi_3} \int dr L_{n_1}^{j_1} L_{n_2}^{j_2} L_{n_3}^{j_3} \sim \delta(j_1 + j_2 - j_3) \int dr L_{n_1}^{j_1} L_{n_2}^{j_2} L_{n_3}^{j_3}, \quad (A17)$$

and still yield a conserving delta function. On the other hand, in the case of the tilde representation the wavefunctions are

$$\tilde{\psi}(x, y) = e^{-(x^2 + y^2)} H_{j+n}(x) H_n(y), \quad (A18)$$

and the overlap integral takes the form

$$V_3 \sim \int dx e^{-3x^2} H_{j_1+n_1}(x) H_{j_2+n_2}(x) H_{j_3+n_3}(x) \int dy e^{-3y^2} H_{n_1}(y) H_{n_2}(y) H_{n_3}(y). \quad (A19)$$

26
As one can verify, written in this basis the vertex no longer has a conserving \( \delta \)-function.

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].

[2] H. Nastase, “Introduction to AdS-CFT,” [arXiv:0712.0689] [hep-th].

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[4] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[5] S. Lee, S. Minwalla, M. Rangamani and N. Seiberg, “Three-point functions of chiral operators in \( D = 4, N = 4 \) SYM at large \( N \),” Adv. Theor. Math. Phys. 2, 697 (1998) [arXiv:hep-th/9806074].

[6] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, “Comments on 4-point functions in the CFT/AdS correspondence,” Phys. Lett. B 452, 61 (1999) [arXiv:hep-th/9808006].

[7] N. R. Constable, D. Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnikov and W. Skiba, JHEP 0207, 017 (2002) [arXiv:hep-th/0205089].

[8] G. Arutyunov and S. Frolov, ” Some cubic couplings in type IIB supergravity on \( \text{AdS}(5) \times S^{**5} \) and three point functions in \( \text{SYM}(4) \) at large \( N \),” Phys. Rev. D 61, 064009 (2000) [arXiv:hep-th/9907085].

[9] A. Jevicki, M. Mihailescu and S. Ramgoolam, “Gravity from CFT on \( S^{**N(X)} \): Symmetries and interactions” [arXiv:hep-th/9907144].

[10] D. E. Berenstein, J. M. Maldacena and H. S. Nastase, “Strings in flat space and pp waves from \( N = 4 \) super Yang Mills,” JHEP 0204, 013 (2002) [arXiv:hep-th/0202021].

[11] S. Cremonini and A. Donos, Phys. Rev. D 70, 044016 (2004) [arXiv:hep-th/0402052].

[12] M. Asano, Y. Sekino and T. Yoneya, “PP-wave holography for Dp-brane backgrounds,”
[13] S. Dobashi, H. Shimada and T. Yoneya, “Holographic reformulation of string theory on AdS(5) x S**5 background in the PP-wave limit,” Nucl. Phys. B 665, 94 (2003) [arXiv:hep-th/0209251].

[14] S. Lee and R. Russo, “Holographic cubic vertex in the pp-wave,” Nucl. Phys. B 705, 296 (2005) [arXiv:hep-th/0409261].

[15] K. Skenderis and M. Taylor, “Anatomy of bubbling solutions,” JHEP 0709, 019 (2007) [arXiv:0706.0216 [hep-th]].

[16] S. Corley, A. Jevicki and S. Ramgoolam, “Exact correlators of giant gravitons from dual N = 4 SYM theory,” Adv. Theor. Math. Phys. 5, 809 (2002) [arXiv:hep-th/0111222].

[17] D. Berenstein, “A toy model for the AdS/CFT correspondence,” JHEP 0407, 018 (2004) [arXiv:hep-th/0403110].

[18] H. Lin, O. Lunin and J. M. Maldacena, “Bubbling AdS space and 1/2 BPS geometries,” JHEP 0410, 025 (2004) [arXiv:hep-th/0409174].

[19] L. Maoz and V. S. Rychkov, “Geometry quantization from supergravity: The Case of 'Bubbling AdS,'” JHEP 0508, 096 (2005) [arXiv:hep-th/0508059].

[20] B. Chen et al., “Bubbling AdS and droplet descriptions of BPS geometries in IIB supergravity,” JHEP 0710, 003 (2007) [arXiv:0704.2233 [hep-th]].

[21] A. Jevicki and B. Sakita, “The quantum collective field method and its application to the planar limit,” Nucl. Phys. B 165, 511 (1980).

[22] R. de Mello Koch, A. Jevicki and J. P. Rodrigues, “Collective string field theory of matrix models in the BMN limit,” Int. J. Mod. Phys. A 19, 1747 (2004) [arXiv:hep-th/0209155].

[23] A. Donos, A. Jevicki and J. P. Rodrigues, “Matrix model maps in AdS/CFT,” Phys. Rev. D 72, 125009 (2005) [arXiv:hep-th/0507124].

[24] A. Jevicki and T. Yoneya, “1/2-BPS Correlators as c = 1 S-matrix,” JHEP 0703, 001 (2007) [arXiv:hep-th/0612262].

[25] A. Jevicki, “Development in 2-d string theory” [arXiv:hep-th/9309115].
[26] A. Donos, “A Description of 1/4 BPS configurations in minimal type IIB SUGRA,” arXiv:hep-th/0606199.