The Light-Cone Vacuum in 1+1 Dimensional Super-Yang-Mills Theory

F.Antonuccio\(^{(1)}\), O.Lunin\(^{(1)}\), S.Pinsky\(^{(1)}\), and S.Tsujimaru\(^{(2)}\)

\(^{(1)}\)Department of Physics,  
The Ohio State University,  
Columbus, OH 43210, USA

and

\(^{(2)}\)Max-Planck-Institut für Kernphysik,  
69029 Heidelberg, Germany

Abstract

The Discrete Light-Cone Quantization (DLCQ) of a supersymmetric \(SU(N)\) gauge theory in 1+1 dimensions is discussed, with particular emphasis given to the inclusion of all dynamical zero modes. Interestingly, the notorious ‘zero-mode problem’ is now tractable because of special supersymmetric cancellations. In particular, we show that anomalous zero-mode contributions to the currents are absent, in contrast to what is observed in the non-supersymmetric case. We find that the supersymmetric partner of the gauge zero mode is the diagonal component of the fermion zero mode. An analysis of the vacuum structure is provided and it is shown that the inclusion of zero modes is crucial for probing the phase properties of the vacua. In particular, we find that the ground state energy is zero and \(N\)-fold degenerate, and thus consistent with unbroken supersymmetry. We also show that the inclusion of zero modes for the light-cone supercharges leaves the supersymmetry algebra unchanged. Finally, we remark that the dependence of the light-cone Fock vacuum in terms of the gauge zero is unchanged in the presence of matter fields.
1 Introduction

A possibly surprising outcome of recent developments in string/M theory are the proposed connections between non-perturbative objects in string theory, and supersymmetric gauge theories in low dimensions [1, 2]. It is therefore of interest to study directly the non-perturbative properties of super-Yang-Mills theories in various dimensions.

Recently, a class of 1+1 [3, 4, 5, 6] and 2+1 [7] dimensional super Yang-Mills theories has been studied using a supersymmetric form of Discrete Light-Cone Quantization (SDLCQ). This formulation has the advantage of preserving supersymmetry after discretizing momenta, and admits a very natural and straightforward algorithm for extracting numerical bound state masses and wave functions [8, 9]. Although a technical necessity, the omission of zero-momentum modes in these numerical computations raises many doubts about the consistency of such a quantization scheme. Little is in fact known about the precise effects of dropping the zero-momentum mode at finite compactification radius, but it is generally believed that such effects disappear in the decompactified limit [3].

There are instances, however, when we would like to know the measurable effects of a finite spatial compactification [2]. In this work, we will deal with measurable effects that reflect the spatial compactification induced by DLCQ. This is accomplished by explicitly incorporating all the gauge zero mode degrees of freedom in the DLCQ formulation of a supersymmetric gauge theory. It turns out that this is tantamount to including a quantum mechanical degree of freedom corresponding to ‘quantized electric flux’ around the compact direction 1. The supersymmetric partner of this gauge degree of freedom is the diagonal component of the fermion zero mode. The implications of this on the vacuum structure of the theory is discussed.

The supersymmetric gauge theory we consider may be obtained by dimensionally reducing $\mathcal{N} = 1$ super-Yang-Mills from 2+1 to 1+1 dimensions [3]. The DLCQ formulation of this theory consists of an adjoint scalar field (represented as a $N \times N$ Hermitian matrix field), a corresponding adjoint fermion field, and several zero-mode (or quantum mechanical) degrees of freedom to be discussed later. To maintain supersymmetry one must impose periodic boundary conditions, so all the color degrees of freedom of the fermion and boson fields will have zero modes. In addition, periodic boundary conditions prevent us from adopting the light-cone gauge, $A^+ = 0$, so we choose instead the gauge $\partial_+ A^+ = 0$, which allows $A^+$ to have a zero mode. In addition, there are large gauge transformations, a Weyl transformation, and a color permutation symmetry which must be taken into account when constructing physical states of the theory. We briefly discuss this procedure.

In this work, we concentrate on the effect of including the quantum mechanical degree of freedom represented by the gauge zero mode. This zero mode corresponds to a quantized color electric flux that circulates around the compact direction $x^-$. The problems

---

1 There is also a fermion degree of freedom by virtue of supersymmetry.
2 i.e. arising from the the generators of SU(N).
associated with this zero mode have already been studied in two dimensional gauge theories involving adjoint scalars, and theories with adjoint fermions \cite{10, 11, 12, 13, 14}. The consequences of including these modes are quite drastic. These theories are known to possess anomalies in the diagonal components of the current, and therefore the charges in these theories are time dependent. This makes it difficult – if not impossible – to define a consistent theory. In contrast, owing to special supersymmetric cancellations between boson and fermion currents, no such anomalies arise in the supersymmetric theory studied here, and so a DLCQ formulation becomes sensible and tractable.

In general, one finds a contribution after normal ordering the Hamiltonian that is a function only of the gauge zero-mode. This term acts as a vacuum potential and leads to a non-zero vacuum energy. When the gauge theory without matter fields is solved, however, the only degree of freedom is the quantum mechanical gauge zero mode, in which the vacuum potential plays no role. The ground state energy is thus zero. However, this simple picture of the vacuum may be drastically altered if we consider the addition of matter. For the supersymmetric case studied here, we show that there is no vacuum potential, and that the ground state has zero energy even in the presence of matter fields.

We find that the supersymmetric partner of the gauge zero mode is given in terms of a diagonal component of the fermion zero mode, and therefore the inclusion of fermion zero modes are necessary for a consistent treatment of the theory. Since the fermion zero modes commute with the Hamiltonian they may be used to generate additional vacua. We find that there are in fact $N$ vacua consistent with the $SU(N)/Z_N$ reduced gauge invariance of the theory.

This paper is organized as follows. In Section 2, we briefly describe the DLCQ procedure of the 1+1 dimensional supersymmetric Yang-Mills theory in the modified light-cone gauge. In Section 3, the point splitting regularization designed to preserve symmetry under large gauge transformations is applied to the current operator. In Section 4, we discuss the vacuum structure of the theory by deriving the quantum mechanics of the zero modes. We conclude in Section 5 with a brief discussion.

## 2 Gauge Fixing in DLCQ

We consider the supersymmetric Yang-Mills theory in 1+1 dimensions \cite{15} which is described by the action

$$S = \int d^2x \, \text{tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \phi D^\mu \phi + i \bar{\Psi} \gamma^\mu D_\mu \Psi - 2ig\phi \bar{\Psi} \gamma_5 \Psi \right),$$

(1)

where $D_\mu = \partial_\mu + ig[A_\mu, \cdot]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. All fields are in the adjoint representation of the gauge group $SU(N)$. A convenient representation of the gamma matrices is $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$ and $\gamma^5 = \sigma_3$ where $\sigma^a$ are the Pauli matrices. In this representation the Majorana spinor is real. We use the matrix notation for $SU(N)$ so that $A^\mu_{ij}$ and $\Psi_{ij}$ are $N \times N$ traceless matrices.
We now introduce the light-cone coordinates \( x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1) \). The longitudinal coordinate \( x^- \) is compactified on a finite interval \( x^- \in [-L,L] \) and we impose periodic boundary conditions on all fields to ensure unbroken supersymmetry.

The light-cone gauge \( A^+ = 0 \) can not be used in a finite compactification radius, but the modified condition \( \partial_- A^+ = 0 \) is consistent with the light-like compactification. We can make a global rotation in color space so that the zero mode is diagonalized \( A^+_{ij}(x^+) = v_i(x^+)\delta_{ij} \) with \( \sum_i v_i = 0 \). The gauge zero modes correspond to a (quantized) color electric flux loops around the compactified space. The modified light-cone gauge is not a complete gauge fixing. We still have large gauge transformations preserving the gauge condition \( \partial_- A^+ = 0 \). There are two types of such transformations \( T_D \) and central conjugations \( T_C \). Their actions on the physical fields of the theory and complete gauge fixing will be discussed in the end of this section. Now we just mention that being discrete transformations, \( T_D \) and \( T_C \) don’t affect quantization procedure.

The quantization in the light–cone gauge with or without dynamical \( A^+ \) is widely explored in the literature, here we provide only the results which are useful for later purposes. The quantization proceeds in two steps. First, we must resolve the constraints to eliminate the redundant degrees of freedom. There are two constraints in the theory,

\[
-D_-^2 A^- = gJ^+, \\
\sqrt{2i}D_- \chi = g[\phi, \psi],
\]

where \( \Psi \equiv (\psi, \chi)^T \) and the current operator is

\[
J^+(x) = \frac{1}{i}[\phi(x), D^- \phi(x)] - \frac{1}{\sqrt{2}}\{\psi(x), \psi(x)\}. \tag{4}
\]

Different components of (2), (3) play different roles in the theory. First we look at diagonal zero modes of these equations. The diagonal zero mode of (3) gives us constraints on the physical fields:

\[
[\phi, \psi]_{ii}^0 = 0. \tag{5}
\]

There is no sum over \( i \) in above expression. As one can see this constraint leads to decoupling of \( \chi_{ii} \), this field plays the role of Lagrange multiplier for above condition. The same is true for \( A_{ii} \), the corresponding constraint is \( J_{ii} = 0 \). The reason we treated the diagonal zero modes of (2) and (3) separately is that for all other modes the \( D_- \) operator is invertible and instead of constraints on physical fields \( \psi \) and \( \phi \) one gets expressions for non-dynamical ones:

\[
A^- = -\frac{g}{D_-}J^+, \quad \chi = \frac{g}{i\sqrt{2}}D_-[\phi, \psi]. \tag{6}
\]

The next step is to derive the commutation relations for the physical degrees of freedom. As in the ordinary quantum mechanics, the zero mode \( v_i \) has a conjugate
momentum $p = 2L\partial_x v_i$ and the commutation relation is $[v_i, p_j] = i\delta_{ij}$. The off–diagonal components of the scalar field are complex valued operators with $\phi_{ij} = (\phi_{ji})^\dagger$. The canonical momentum conjugate to $\phi_{ij}$ is $\pi_{ij} = (D_\varphi)_{ji}$. They satisfy the canonical commutation relations

\[ [\phi_{ij}(x), \pi_{kl}(y)]_{x^+=y^+} = [\phi_{ij}(x), D_-\phi_{lk}(y)]_{x^+=y^+} = \frac{i}{2}(\delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl})\delta(x^- - y^-). \]  

(7)

On the other hand, the quantization of the diagonal component $\phi_{ii}$ needs care. As mentioned in [11], the zero mode of $\phi_{ii}$, the mode independent of $x^-$, is not an independent degree of freedom but obeys a certain constrained equation [10, 11, 13]. Except the zero mode, the commutation relation is canonical

\[ [\phi_{ii}(x), \partial_- \phi_{jj}(y)]_{x^+=y^+} = \frac{i}{2}(1 - \frac{1}{N})\delta_{ij}\left[\delta(x^- - y^-) - \frac{1}{2L}\right]. \]

(8)

The commutator of diagonal and non-diagonal elements of $\phi$ vanishes. The canonical anti-commutation relations for fermion fields are [3]

\[ \{\psi_{ij}(x), \psi_{kl}(y)\}_{x^+=y^+} = \frac{1}{\sqrt{2}}\delta(x^- - y^-)(\delta_{ii}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}). \]  

(9)

There are two differences between this expression and one from [3]. First one is technical: we consider commutators for $SU(N)$ group, this gives $1/N$ term. Second difference is that unlike [3] we include zero modes in the expansion of $\psi$, we also include such modes in non-diagonal elements of $\phi$.

Finally we return to the problem of complete gauge fixing. The actions of $T_D$ and $T_C$ on physical fields are given by [21, 13]:

\[ T_D: \quad v_i(x^+) \rightarrow v_i(x^+) + \frac{n_i\pi}{gL}, \quad n_i \in \mathbb{Z}, \quad \sum n_i = 0, \]

\[ \psi_{ij} \rightarrow \exp\left(\frac{\pi i(n_i - n_j)x^-}{L}\right)\psi_{ij}, \quad \phi_{ij} \rightarrow \exp\left(\frac{\pi i(n_i - n_j)x^-}{L}\right)\phi_{ij}; \]  

(10)

\[ T_C: \quad v_i(x^+) \rightarrow v_i(x^+) + \frac{\nu_i\pi}{gL}, \quad \nu_i = n(1/N - \delta_{iN}), \]

\[ \psi_{ij} \rightarrow \exp\left(\frac{\pi i(\nu_i - \nu_j)x^-}{L}\right)\psi_{ij}, \quad \phi_{ij} \rightarrow \exp\left(\frac{\pi i(\nu_i - \nu_j)x^-}{L}\right)\phi_{ij}. \]  

(11)

There are also permutations of the color basis $i \rightarrow P(i)$ which leave the theory invariant. These symmetries preserve the gauge condition $\partial_- A^+ = 0$, but two configurations related by $T_D$, $T_C$ or $P$ are equivalent. To fix the gauge completely one therefore considers $v_i$ only in the fundamental domain, other regions related with this domain by $T_D$, $T_C$ or $P$ give gauge “copies” of it [17]. The easiest thing to do is to describe the boundaries of fundamental domain imposed by displacements $T_D$: $-\frac{\pi}{2gL} < v_i < \frac{\pi}{2gL}$. The invariance under $T_C$ limits this region even more, but since we will not need the explicit form of fundamental domain, we do not discuss such limits for $SU(N)$. For the simplest case of
SU(2) the fundamental domain is given by 0 < v_1 = -v_2 < \frac{\pi}{2gL}$, the result for SU(3) can be found in [18]. The $P$ symmetries do not respect the fundamental domain, so they are not symmetries of gauge fixed theory. However there is one special transformation among $P$ which being accompanied with combination of $T_D$ and $T_C$ leaves fundamental domain invariant. Namely if $R$ is cyclic permutation of color indexes then there exists a combination $T$ of $T_D$ and $T_C$ such that $S = TR$ is the symmetry of gauge fixed theory.

The explicit form of $T$ depends on the rank of the group, for $SU(2)$ and $SU(3)$ it may be found in [18]. The operator $S$ satisfies the condition $S^N = 1$ and it was used in classifying the vacua [18, 14].

3 Current Operators

The resolution of the Gauss-law constraint (2) is a necessary step for obtaining the light-cone Hamiltonian. The expression for the current operator is, however, ill–defined unless an appropriate definition is specified, since the operator products are defined at the same point. We shall use the point–splitting regularization which respects the symmetry of the theory under the large gauge transformation.

To simplify notation it is convenient to introduce the dimensionless variables $z_i = Lg v_i/\pi$ instead of quantum mechanical coordinates $v_i$ describing $A^+$. The mode–expanded fields at the light-cone time $x^+ = 0$ are

$$
\begin{align*}
\phi_{ij}(x) & = \frac{1}{\sqrt{4\pi}} \left( \sum_{n=0}^{\infty} a_{ij}(n) u_{ij}(n) e^{-i k_n x^-} + \sum_{n=1}^{\infty} a_{ji}^{\dagger}(n) u_{ij}(-n) e^{i k_n x^-} \right), & i \neq j, \\
\phi_{ii}(x) & = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left( a_{ii}(n) e^{-i k_n x^-} + a_{ii}^{\dagger}(n) e^{i k_n x^-} \right), \\
\psi_{ij}(x) & = \frac{1}{2\sqrt{2L}} \left( \sum_{n=0}^{\infty} b_{ij}(n) e^{-i k_n x^-} + \sum_{n=1}^{\infty} b_{ij}^{\dagger}(n) e^{i k_n x^-} \right),
\end{align*}
$$

where $k_n = n\pi/L$, $u_{ii}(n) = 1/\sqrt{|n - z_i + z_j|}$. The (anti)commutation relations for Fourier modes are found in [11, 12] and in our notation they take the form

$$
\begin{align*}
[a_{ij}(n), a_{kl}^{\dagger}(m)] &= \text{sgn}(n + z_j - z_i) \delta_{n,m} (\delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl}), \\
\{b_{ij}(n), b_{kl}^{\dagger}(m)\} &= \delta_{n,m} (\delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl})
\end{align*}
$$

The zero modes in above relations deserve special consideration. Although we formally wrote them as $a_{ij}(0)$ and $b_{ij}(0)$, these modes also act as creation operators because the conjugation of zero mode gives another zero mode:

$$
a_{ij}^{\dagger}(0) = a_{ji}(0), \quad b_{ij}^{\dagger}(0) = b_{ji}(0). \quad (14)$$

$u_{ij}(n)$ is well-defined in the fundamental domain. Similarly, $(D_-)^2$ in the Gauss-law constraint have no zero modes in this domain.
In particular the diagonal components of fermionic zero mode are real and we will use them later to describe the degeneracy of vacua. Now we concentrate our attention on non-diagonal zero modes. In the fundamental domain all $z_i$ are different, then one can always make them to satisfy the inequality $z_N < z_{N-1} < \ldots < z_1$ in this domain.

Such condition together with (13) leads to interpretation of $a_{ij}(0)$ as creation operator if $i < j$ and as annihilation operator otherwise. The situation for fermions is more ambiguous. One can consider $b_{ij}(0)$ as creation operator either when $i < j$ or when $i > j$, both assumptions are consistent with (13). Later we will explore each of these situations.

Let us now discuss the definition of singular operator products in the current (4). We define the current operator by point splitting:

$$ J^+ = \lim_{\epsilon \to 0} \left( J^+_{\phi}(x; \epsilon) + J^+_{\psi}(x; \epsilon) \right), $$

where the divided pieces are given by

$$ J^+_{\phi}(x; \epsilon) = \frac{1}{i} \left[ e^{-i \frac{\pi M}{2L} \phi(x^- - \epsilon) e^{i \frac{\pi M}{2L} D_- \phi(x^-)} \right], $$

$$ J^+_{\psi}(x; \epsilon) = -\frac{1}{\sqrt{2}} \left\{ e^{-i \frac{\pi M}{2L} \psi(x^- - \epsilon) e^{i \frac{\pi M}{2L} D_- \psi(x^-)} \right\}. $$

Here $M$ is diagonal matrix: $M = \text{diag}(z_1, \ldots, z_N)$. An advantage of this regularization is that the current transforms covariantly under the large gauge transformation.

To evaluate (16) and (17) we will generalize the approach used in [11, 12] to the SU(N) case. First let us calculate the vacuum average of bosonic current. Taking into account the interpretation of zero modes as creation–annihilation operators we obtain:

$$ \langle 0 | J^+_{ij \phi}(x; \epsilon) | 0 \rangle = \frac{1}{i} \langle 0 | e^{-i \frac{\pi}{2L} (z_i - z_k) \phi_{ik}(x^- - \epsilon) D_- \phi(x^-)} | 0 \rangle =$$

$$ = \frac{1}{4L} \sum_k \sum_{m>0} \left( e^{-i \frac{\pi}{2L} (z_i - z_k)} - e^{-i \frac{\pi}{2L} (z_k - z_j)} \right) e^{-im\epsilon} (\delta_{ij} - \frac{1}{N} \delta_{ik} \delta_{jk}) +$$

$$ + \frac{1}{4L} \sum_{k<j} e^{-i \frac{\pi}{2L} (z_i - z_k)} \delta_{ij} - \frac{1}{4L} \sum_{k>i} e^{-i \frac{\pi}{2L} (z_k - z_j)} \delta_{ij}. $$

Evaluating the sum and taking the limit one finds:

$$ \lim_{\epsilon \to 0} J^+_{ij \phi}(x; \epsilon) =: J^+_{ij \phi}(x) : = \frac{1}{4L} \left( z_i - (N + 1 - 2i) \right) \delta_{ij}, $$

where $: J^+_{ij \phi} :$ is the naive normal ordered currents. To be more precise, we have omitted the zero modes of the diagonal color sectors in which the notorious constrained zero mode [16] appears.
The result for fermionic current depends on our interpretation of zero modes as creation–annihilation operators and it is given by

$$\lim_{\epsilon \to 0} J^+_{ij}(x; \epsilon) =: J^+_{ij}(x) = -\frac{1}{4L} (z_i \mp (N + 1 - 2i)) \delta_{ij}. \quad (20)$$

The minus sign here corresponds to the case where \( b_{ij}(0) \) is a creation operator if \( i < j \) (i.e. the convention is the same as for the bosons) and plus corresponds to the opposite situation. As can be seen, \( J^+_{\phi} \) and \( J^+_{\psi} \) acquire extra \( z \) dependent terms, so called gauge corrections. Integrating these charges over \( x^- \), one finds that the charges are time dependent. Of course this is an unacceptable situation, and implies the need to impose special conditions to single out ‘physical states’ to form a sensible theory. The important simplification of the supersymmetric model is that these time dependent terms cancel, and the full current \( (20) \) becomes

$$J^+_{ij}(x) =: J^+_{ij}(x) : + : J^+_{ij}(x) : + C_i \delta_{ij}. \quad (21)$$

Depending on the convention for fermionic zero modes the \( z \) independent constants \( C_i \) either vanish or they are given by

$$C_i = -\frac{1}{2L} (N + 1 - 2i). \quad (22)$$

The regularized current is thus equivalent to the naive normal ordered current up to an irrelevant constant. Similarly, one can show that \( P^+ \) picks up gauge correction when the adjoint scalar or adjoint fermion are considered separately but in the supersymmetric theory it is nothing more than the expected normal ordered contribution of the matter fields.

In one sense these results are a consequence of the well known fact that the normal ordering constants in a supersymmetric theory cancel between fermion and boson contributions. The important point here is that these normal ordered constants are not actually constants, but rather quantum mechanical degrees of freedom. It is therefore not obvious that they should cancel. Of course, this property profoundly effects the dynamics of the theory.

## 4 Vacuum Energy

The wave function of the vacuum state for the supersymmetric Yang-Mills theory in 1+1 dimensions has already been discussed in the equal-time formulation \[20\]. An effective potential is computed in a weak coupling region as a function of the gauge zero mode by using the adiabatic approximation. Here we analyze the vacuum structure of the same theory in the context of the DLCQ formulation.

The presence of zero modes renders the light-cone vacuum quite nontrivial, but the advantage of the light-cone quantization becomes evident: the ground state is the Fock
vacuum for a fixed gauge zero mode and therefore our ground state may be written in the tensor product form
\[ |\Omega\rangle \equiv \Phi[z] \otimes |0\rangle, \]  
(23)
where we have taken the Schrödinger representation for the quantum mechanical degree of freedom \( z \) which is defined in the fundamental domain. In contrast, to find the ground state of the fermion and boson for a fixed value of the gauge zero mode turns out to be a highly nontrivial task in the equal-time formulation [20].

Our next task is to derive an effective Hamiltonian acting on \( \Phi[z] \). The light-cone Hamiltonian \( H \equiv P^- \) is obtained from energy momentum tensors, or through the canonical procedure:
\[
H = -\frac{g^2 L}{4\pi^2} \frac{1}{K(z)} \sum_i \frac{\partial}{\partial z_i} K(z) \frac{\partial}{\partial z_i} + 
\int_{-L}^{L} dx^- \text{tr} \left( -\frac{g^2}{2} J^+ \frac{1}{D^2} J^+ + \frac{ig^2}{2\sqrt{2}} [\phi, \psi] \frac{1}{D_-} [\phi, \psi] \right),
\]  
(24)
\[
K(z) = \prod_{i>j} \sin^2 \left( \frac{\pi(z_i - z_j)}{2} \right),
\]  
(25)
where the first term is the kinetic energy of the gauge zero mode, and in the second term the zero modes of \( D_- \) are understood to be removed. Note that the kinetic term of the gauge zero mode is not the standard form \(-d^2/dz^2\) but acquires a nontrivial Jacobian \( K \) which is nothing but the Haar measure of \( SU(N) \). The Jacobian originates from the unitary transformation of the variable from \( A^+ \) to \( v \), and can be derived by explicit evaluation of a functional determinant [21, 18]. In the present context it is found in [13]. Also we mention that Hamiltonian (24) seems to contain terms quadratic in diagonal zero modes \( \psi_{ii} \). However using constraint equations one can show that the total contribution of all such term vanishes. This also can be seen by using the fact that Hamiltonian is proportional to the square of supercharge (34).

Projecting the light-cone Hamiltonian onto the Fock vacuum sector we obtain the quantum mechanical Hamiltonian
\[
H_0 = -\frac{g^2 L}{4\pi^2} \frac{1}{K(z)} \sum_i \frac{\partial}{\partial z_i} K(z) \frac{\partial}{\partial z_i} + V_{JJ} + V_{\phi\psi},
\]  
(26)
where the reduced potentials are defined by
\[
V_{JJ} \equiv -\frac{g^2}{2} \int_{-L}^{L} dx^- \langle \text{tr} J^+ \frac{1}{D^2} J^+ \rangle,
\]  
(27)
\[
V_{\phi\psi} \equiv \frac{ig^2}{2\sqrt{2}} \int_{-L}^{L} dx^- \langle \text{tr} [\phi, \psi] \frac{1}{D_-} [\phi, \psi] \rangle,
\]  
(28)
respectively. As stated in the previous section, the gauge invariantly regularized current turns out to be precisely the normal ordered current in the absence of the zero modes. It
is now straightforward to evaluate $V_{JJ}$ and $V_{\phi\psi}$ in terms of modes. One finds that they cancel among themselves as expected from the supersymmetry:

$$V_{JJ} = -V_{\phi\psi} = \frac{g^2 L}{16\pi^2} \left[ \sum_{n,m=1}^{\infty} \sum_{i,j,k} \left( \frac{1}{n - z_i + z_k} \frac{1}{m + z_j - z_k} \right) - \sum_{n,m=1}^{\infty} \frac{N_{mn}}{mn} + \sum_{n=1}^{\infty} \sum_{i,j} \frac{1}{(n - z_i + z_k)(z_j - z_k)} \sum_{k>j} \frac{1}{(n + z_j - z_k)(z_k - z_i)} \right] + \sum_{i,j} \sum_{i>k>j} \frac{1}{(z_k - z_i)(z_j - z_k)}. \tag{29}$$

This cancellation was found as the result of formal manipulations with divergent series like ones in the right hand side of the last formula. Such transformations are not well defined mathematically and as the result they may lead to the finite "anomalous" contribution. The famous chiral anomaly initially was found as the result of careful analysis of transformations analogous to ones we just performed \[22\]. However if one considers derivatives of $V_{JJ}$ or $V_{\phi\psi}$ with respect to any $z_i$ then all the sums become convergent, the order of summations becomes interchangeable and as the result the derivatives of $V_{JJ} + V_{\phi\psi}$ vanish. Thus if there is any anomaly in the expression above it is given by $z$-independent constant. Such constant in the Hamiltonian would correspond to the shift of energy levels and usually it is ignored. However in supersymmetric case there is a natural choice for such constant: in order for vacuum to be supersymmetric it should be zero. Below we assume that SUSY is not broken, then we expect that (29) is true.

Thus we arrive at

$$H_0 = -\frac{g^2 L}{4\pi^2} \frac{1}{K(z)} \sum_i \frac{\partial}{\partial z_i} K(z) \frac{\partial}{\partial z_i}. \tag{30}$$

The relevant solutions of this equation should be finite in the fundamental domain, this requirement leads to discrete spectrum due to the fact that Jacobian vanishes on the boundary of this domain. However the operator $H_0$ is elliptic, and therefore it can’t have negative eigenvalues. If the eigenvalue problem

$$H_0 \Phi(z) = E \Phi(z) \tag{31}$$

has a solution for $E = 0$, this solution corresponds to the ground state of the theory. It is easy to see that such solution exists and it is given by $\Phi(z) = \text{const}$. We have thus found that the ground state has a vanishing vacuum energy, suggesting that the supersymmetry is not broken spontaneously.

\[\text{\footnotemark[4]}\text{some authors prefer to rewrite this to include the measure in the definition of the wave function and then in SU(2) for example the ground state wave function is a sin}\]
5 Supersymmetry and Degenerate Vacua.

As we saw in the previous section supersymmetry leads to the cancellation of the anomaly terms in current operator. However these terms played an important role in the description of $Z_N$ degeneracy of vacua \[14\], so we should find another explanation of this fact here. It appears that fermionic zero modes give a natural framework for such treatment.

First we will generalize the supersymmetry transformation given in \[3\] to the present case, i.e. we include $A^+$ and the zero modes of fermions. The naive SUSY transformations spoil the gauge fixing condition, so we combine them with compensating gauge transformation following \[3\]. In three dimensional notation (spinors have two components and indices go from 0 to 2) the result reads:

$$\delta A_\mu = \frac{i}{2}\bar{\varepsilon}\gamma_\mu\Psi - \frac{1}{2}D_\mu\bar{\varepsilon}\gamma_\mu, \quad (32)$$

$$\delta\Psi = \frac{1}{4}F_{\mu\nu}\gamma^{\mu\nu}\varepsilon - \frac{g}{2}\bar{\varepsilon}\gamma_- \frac{1}{D_-}\Psi, (33)$$

The difference between above expression and those in \[3\] is that we include the zero modes. Namely we defined $\Psi$ as the complete field with all the zero modes included and $\bar{\Psi}$ as fermion without diagonal zero modes. The introducing of $\bar{\Psi}$ is necessary, because diagonal zero modes form the kernel of operator $D_-$, so $\frac{1}{D_-}$ is not defined on this subspace.

In particular we are interested in supersymmetry transformations for $A^+$ and fermionic zero modes. Performing a mode expansion one can check that diagonal elements of matrix $[\frac{1}{D_-}\psi, \bar{\psi}]$ vanish, then from (32) we get:

$$\delta A_{ii}^+ = \frac{i}{\sqrt{2}}\bar{\varepsilon}_{++}^T\psi_{ii},$$

$$\delta\psi_{ii} = -2\partial_+ A_{ii}^+\bar{\varepsilon}_{++}. (33)$$

This expression is written in two component notation and the decomposition of spinor $\varepsilon$: $\varepsilon = (\varepsilon_{++}, \varepsilon_{--})^T$ is used. Note that since $\bar{\varepsilon}Q = \sqrt{2}(\varepsilon_{++}Q + \varepsilon_{--}Q^+)$ the fields involved in transformations (33) don’t contribute to $Q^+$, this is consistent to the fact that being $x^-$ independent they don’t contribute to $P^+$. The equations (33) look like supersymmetry transformation for the quantum mechanical system built from free bosons and free fermions. In fact as one can see the supercharge $Q^-$ is the sum of supercharge for the quantum mechanical system and from the QFT without diagonal zero modes:

$$Q^- = -2g\int dx^- tr(J^+\frac{1}{D_-}\psi) + 4Ltr(\partial_+ A^+\psi). (34)$$

Calculating $(Q^-)^2$ and writing the momentum conjugate to $A^+$ as differential operator \[7\] we reproduce Hamiltonian \[24\]. Note that $\psi$ there has all the zero modes in it. The

\textsuperscript{5}using Schrodinger coordinate representation for quantum mechanical degree of freedom - note that the QFT term has non-trivial dependence on the quantum mechanical coordinate.
square of another supercharge

\[ Q^+ = 2 \int dx^- \text{tr}(\psi D_\phi) \]  

(35)
gives \( P^+ \) while the anti-commutator of \( Q^- \) with \( Q^+ \) is proportional to the constraint (33) and thus vanishes.

One can check that although \([0\psi_{ii}, H]\) does not vanish, this commutator annihilates Fock vacuum \(|0\rangle\), then it also annihilates \(|\Omega\rangle\). In section 2 we mentioned that \(0\chi_{ii}\) decouples from the theory, and therefore it commutes with Hamiltonian. Thus acting on the vacuum state \(|\Omega\rangle\) by diagonal elements of either \(0\psi\) or \(0\chi\) we get states annihilated by \(P^-\) and \(P^+\) (the latter statement is obvious since zero modes commute with momentum). Not all such states however may be considered as vacua. Although we fixed the gauge in section 2, the theory still has residual symmetry \(P\), corresponding to permutations of the color basis. Physical states are constructed from operator acting on the physical vacuum \(|\Omega\rangle\) and both the operators and the physical vacuum must be invariant under \(P\). Such objects can always be written as combinations of traces. The candidates for the vacuum state may have any combination of \(0\psi\) and \(0\chi\) inside the trace, here and below we consider only diagonal components of zero modes. Since \(0\chi\) is not dynamical we have the usual c–number relation

\[ \{0\chi_{ii}, 0\chi_{jj}\} = 0 \]  

(36)
instead of canonical anti-commutator, so \(0\chi 0\chi = 0\). From the relations (13) one finds:

\[ 0\psi 0\psi = \frac{1}{4L\sqrt{2}}(1 - \frac{1}{N}), \]  

(37)
also we have \(0\chi 0\psi = -0\psi 0\chi\). Using all these relations and the \(SU(N)\) conditions \(\text{tr}(0\psi) = 0\) and \(\text{tr}(0\chi) = 0\) we find that the only nontrivial trace involving only zero modes is \(\text{tr}(0\psi 0\chi)\). Then the family of vacua is given by:

\[ \left(\text{tr}(0\psi 0\chi)\right)^n|\Omega\rangle, \quad 0 \leq n \leq N - 1. \]  

(38)
The region for \(n\) is determined taking into account the fact that \(0\chi\) is anti-commuting field with \(N - 1\) independent components. Thus we explained the \(Z_N\) degeneracy of vacua first mentioned in [23].

6 Discussion

The theory we consider here is an \(\mathcal{N} = 1\) super-Yang-Mills theory with one adjoint fermion and one adjoint scalar with periodic boundary conditions in \(x^-\). The boundary
conditions reduce the gauge group to $SU(N)/Z_N$ and give rise to diagonal gauge zero modes. We find that supersymmetry requires diagonal fermion zero modes which are the supersymmetric partner of the gauge zero mode. These zero modes behave as quantum mechanical degrees of freedom. When we include these zero modes in the supercharge we find that the super-algebra is unchanged.

In general, when one normal orders the operators of the theory one finds contributions that depend only on this quantum mechanical degree of freedom. These terms are anomalies and profoundly effect the structure of the theory. In theories with only fermions or only bosons, these anomalies yield time dependent charges and a non-zero vacuum energy. In the supersymmetric theory presented here, these anomalies are seen to cancel and the operators are all well behaved. In particular, the charges are time independent and the ground state is the same as the ground state for the theory without matter. The energy of the ground state is zero leaving the supersymmetry unbroken. We show that the fermion zero modes can be used to construct an $N$-fold set of degenerate vacua. This is expected for a theory with the reduced gauge symmetry $SU(N)/Z_N$.

It is expected that there will be constrained zero modes which we do not consider here. They are not dynamical degrees of freedom but can introduce new interactions which could lead to supersymmetry breaking in the same way that they are known to spontaneously break the $Z_2$ symmetry in the simple $\phi^{4}_{1+1}$ theory.

Finally, we remark that the properties of Matrix String Theory – which is defined as $1+1$ $N=8$ super-Yang-Mills theory on a circle – depend crucially on the measurable effects produced by the space-like compactification. These effects are intimately tied with the dynamics of non-perturbative objects in Type IIA string theory known as D0 branes. It would be interesting to consider the DLCQ formulation of the same Yang-Mills theory, and to establish – if possible – any connection with the Matrix String proposal. The simplicity of the light-cone Fock vacuum, owing to special supersymmetry cancellations, might present a tractable approach to non-perturbative string theory.

Acknowledgments
S.T. wishes to thank S.Tanimura for helpful discussions. S.T. would like to thank Ohio State University for hospitality, where this work was begun.

References

[1] T.Banks, W.Fischler, S.Shenker and L.Susskind, Phys.Rev. D55, (1997), 5112-5128.

[2] L.Motl, “Proposals on Non-Perturbative Superstring Interactions,” hep-th/9701023; T.Banks and N.Seiberg, Nucl.Phys. B497, 41 (1997); R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl.Phys.B500, (1997), 43-61.

6If enough supersymmetry is present, one expects certain interactions not to be renormalized. This would be tantamount to a cancellation of such constrained zero-mode degrees of freedom.
[3] Y.Matsumura, N.Sakai and T.Sakai, *Phys. Rev.* D52 (1995) 2446.

[4] F.Antonuccio, O.Lunin, S.Pinsky, *Phys.Lett.* B429 (1998) 327-335; F.Antonuccio, O.Lunin, S.Pinsky, *Phys.Rev.* D58 (1998) 085009.

[5] F.Antonuccio, O.Lunin, H.C.Pauli, S.Pinsky, and S.Tsujimaru, “The DLCQ Spectrum of N=(8,8) Super Yang-Mills”, [hep-th/9806133](http://arxiv.org/abs/hep-th/9806133) (to appear in *Phys.Rev.* D); F.Antonuccio, H.C.Pauli, S.Pinsky and S.Tsujimaru, “DLCQ Bound States of N=(2,2) Super Yang-Mills at Finite and Large N”, [hep-th/9808120](http://arxiv.org/abs/hep-th/9808120) (to appear in *Phys.Rev.* D).

[6] F.Antonuccio, O.Lunin, S.Pinsky, “On Exact Supersymmetry in DLCQ”, [hep-th/9809163](http://arxiv.org/abs/hep-th/9809163) (to appear in *Phys.Lett.* B).

[7] F. Antonuccio, O. Lunin, S. Pinsky, “Super Yang-Mills at Weak, Intermediate and Strong Coupling”, (submitted to Phys. Rev. D); [hep-th/9811083](http://arxiv.org/abs/hep-th/9811083).

[8] H.C.Pauli and S.J.Brodsky, *Phys.Rev.* D32 (1985) 1993, 2001.

[9] S.J.Brodsky, H.C.Pauli, and S.S.Pinsky, *Phys. Rept.* 301 (1998) 299-486.

[10] A.C.Kalloniatis, H.C.Pauli and S.S.Pinsky, *Phys. Rev*. D50 (1994) 6633-6639. H.C.Pauli, A.C.Kalloniatis and S.S.Pinsky, *Phys. Rev.* D52 (1995) 1176-1189.

[11] S.Pinsky and A.Kalloniatis, *Phys.Lett* B 365 (1996) 225-232.

[12] S.Pinsky and D.Robertson, *Phys.Lett* B 379 (1996) 169-178.

[13] A.C.Kalloniatis, *Phys.Rev* D54 (1996) 2876-2888. A.S.Mueller, A.C.Kalloniatis and H.C.Pauli, *Phys.Lett* B435 (1998) 189-198.

[14] G.McCartor, D.G.Robertson and S.Pinsky *Phys.Rev* D56 (1997) 1035-1049.

[15] S.Ferrara, *Lett. Nuovo. Cimen* 13 (1965) 629.

[16] T.Maskawa and K.Yamawaki, *Prog.Theor.Phys.* 56 (1976) 270.

[17] V.N.Gribov, *Nucl. Phys.* B139 (1978) 1.

[18] F.Lenz, M.Shifman and M.Thies, *Phys. Rev.* D51 (1995) 7060-7082.

[19] C. Bender, S. Pinsky and B. van de Sande, *Phys. Rev.* D48 (1993) 816.

[20] H.Oda, N.Sakai and T.Sakai, *Phys. Rev.* D55 (1997) 1079.

[21] F.Lenz, H.W.L.Naus and M.Theis, *Ann. Phys. (N.Y.)* 233 (1994) 317-373.

[22] S.Adler, *Phys. Rev.* 177 (1969) 2426.
[23] E. Witten, *Nuovo Cim.* A51 (1979) 325.

[24] M. Burkardt, F. Antonuccio, and S. Tsujimaru, “Decoupling of Zero-Modes and Covariance in the Light-Front Formulation of Supersymmetric Theories”, (to appear in Phys.Rev.D); hep-th/9807035.