Non-Abelian Vortices, Super-Yang-Mills Theory and Spin(7)-Instantons

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Abstract

We consider a complex vector bundle $\mathcal{E}$ endowed with a connection $\mathcal{A}$ over the eight-dimensional manifold $\mathbb{R}^2 \times G/H$, where $G/H = SU(3)/U(1) \times U(1)$ is a homogeneous space provided with a never integrable almost complex structure and a family of SU(3)-structures. We establish an equivalence between $G$-invariant solutions $\mathcal{A}$ of the Spin(7)-instanton equations on $\mathbb{R}^2 \times G/H$ and general solutions of non-Abelian coupled vortex equations on $\mathbb{R}^2$. These vortices are BPS solitons in a $d = 4$ gauge theory obtained from $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in ten dimensions compactified on the coset space $G/H$ with an SU(3)-structure. The novelty of the obtained vortex equations lies in the fact that Higgs fields, defining morphisms of vector bundles over $\mathbb{R}^2$, are not holomorphic in the generic case. Finally, we introduce BPS vortex equations in $\mathcal{N} = 4$ super Yang-Mills theory and show that they have the same feature.
1 Introduction

Symmetries play an important role in physics and mathematics. In particular, it is often useful to study solutions of partial differential equations which are invariant under the action of some symmetry group. Invariant solutions to the original equations can be interpreted as ordinary solutions to a related set of equations on the orbit space of the group action obtained with the help of dimensional reduction.

In Yang-Mills theory in arbitrary dimensions, the $G$-equivariant dimensional reduction on Kähler manifolds of the form $M^{2n} \times G/H$, where $G/H$ is a Kähler homogeneous space for a compact Lie group $G$ with a closed subgroup $H$, induces a Yang-Mills-Higgs theory on $M^{2n}$ which is a quiver gauge theory [1]-[5]. Recall that a quiver is an oriented graph, i.e. a set of vertices $Q_0$ together with a set of arrows $Q_1$ between the vertices. A path in $Q = (Q_0, Q_1)$ is a sequence of arrows in $Q_1$ which compose. A relation $r$ of the quiver is a formal finite sum of paths and quivers with relations can naturally be associated with any Kähler coset space $G/H$ [4, 5]. In a $G$-equivariant dimensional reduction from $M^{2n} \times G/H$ to $M^{2n}$, one chooses a quiver $Q = (Q_0, Q_1)$ associated with $G/H$ and then to each vertex from $Q_0$ there corresponds a vector bundle over $M^{2n}$ and to each arrow from $Q_1$ a morphism (defined by a Higgs field) of two bundles from the set $Q_0$. The reduction of the associated first-order Hermitian-Yang-Mills equations yields non-Abelian quiver vortex equations [1]-[5]. The Seiberg-Witten monopole equations [6] for $n=2$ and the ordinary vortex equations [7] for $n=1$ are particular instances of quiver vortex equations. Recently, the formalism of $G$-equivariant reductions has been applied in a variety of contexts [8]-[12].

In this paper, we consider the manifold $X^8 = \mathbb{R}^2 \times SU(3)/U(1) \times U(1)$, and a complex vector bundle $\mathcal{E}$ over $X^8$. However, instead of the Hermitian-Yang-Mills equations on $\mathcal{E} \to X^8$ which can be considered along with the standard Kähler structures on $\mathbb{R}^2$ and $SU(3)/U(1) \times U(1)$ [10], we consider the Spin(7)-instanton equations [13, 14], the related SU(4)-instanton equations [13] and their dimensional reduction to $\mathbb{R}^2$. For that, on the coset space $SU(3)/U(1) \times U(1)$, we introduce a never integrable almost complex structure and a family of SU(3)-structures. These SU(3)-structures induce a Spin(7) $\supset$ SU(4) structure on the direct product manifold $X^8 = \mathbb{R}^2 \times SU(3)/U(1) \times U(1)$ with $\mathbb{R}^2 \cong \mathbb{C}$. Thus, our manifold $X^8$ is no longer Kähler and has a torsion. Using an SU(4)-structure on $X^8$, we rewrite the Spin(7)-instanton equations for a connection $A$ on $\mathcal{E}$ in the form of SU(4)-instanton equations.

The study of SU(3)-equivariant solutions to the SU(4)-instanton equations on the manifold $\mathbb{R}^2 \times SU(3)/U(1) \times U(1)$ yields, via a dimensional reduction, a new type of non-Abelian quiver vortex equations on $\mathbb{R}^2$. We write down these vortex equations explicitly for a simplest quiver associated with the coset space $SU(3)/U(1) \times U(1)$.\footnote{A generalization to arbitrary quiver is straightforward.} They are equations for connections $A^i$ on three vector bundles $E_i$ over $\mathbb{R}^2$, $i = 1, 2, 3$, linked by three homomorphisms $\phi_i$ (Higgs fields) which in general are not holomorphic. In fact, the Dolbeault operator $\bar{\partial}_A$ maps the Higgs fields $\phi_i$ to polynomials of Higgs fields corresponding to quiver relations $r_i$. This is a new potentiality of quiver vortex theory. Holomorphicity is restored when a connection $A$ on $\mathcal{E}$ satisfies not only the SU(4)-instanton equations but also the Hermitian-Yang-Mills equations that in general is not possible.\footnote{There are SU(4)-instanton connections which are not Hermitian-Yang-Mills.} Note that the considered quiver vortices are BPS solitons in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in ten dimensions reduced to $d = 2$. 

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\footnotesize

1 A generalization to arbitrary quiver is straightforward.

2 There are SU(4)-instanton connections which are not Hermitian-Yang-Mills.
Finally, we consider the bosonic sector of $\mathcal{N} = 4$ U$(k)$ super Yang-Mills (SYM) theory in Minkowski space $\mathbb{R}^{3,1}$ and introduce BPS vortex equations on $\mathbb{R}^2 \subset \mathbb{R}^{3,1}$. We show that these equations on a gauge potential $A$ on a U$(k)$-bundle over $\mathbb{R}^2$ and three complex matrix scalar fields $\phi_i$ are also related to the SU$(4)$-instanton equations in eight dimensions and have the same feature as the above quiver vortex equations, i.e. the Higgs fields $\phi_i$ are not holomorphic.

2 SU(3)-structures on the homogeneous space SU(3)/U(1)×U(1)

SU(3)-structure on 6-manifolds. Let $G$ be a closed subgroup of the orthogonal group SO$(d)$. A $G$-structure on an oriented Riemannian manifold $(X^d, g)$ of dimension $d$ is a reduction of the structure group SO$(d)$ of the tangent bundle of $X^d$ to the subgroup $G$. In particular, for a six-dimensional manifold $(X^6, g)$, an SU$(3)$-structure on $X^6$ is determined by a pair $(\omega, \Omega)$, where $\omega$ is a non-degenerate two-form (an almost symplectic structure) and $\Omega$ is a decomposable complex three-form such that

$$\omega \wedge \Omega = 0 \quad \text{and} \quad \Omega \wedge \bar{\Omega} = -\frac{4i}{3} \omega \wedge \omega \wedge \omega .$$

Indeed, the above complex three-form

$$\Omega = \Theta^1 \wedge \Theta^2 \wedge \Theta^3$$

determines an almost complex structure $J$ on $X^6$ such that

$$J \Theta^i = i \Theta^i \quad \text{for} \quad i = 1, 2, 3 ,$$

i.e. forms $\Theta^i$ span the space of forms of type $(1,0)$. Hence, $c_1(X^6) = 0$ and the (3,0)-form $\Omega$ is a global section of the canonical bundle of $X^6$. For more details see e.g. [15, 16] and references therein.

The form $\omega$ is of type $(1,1)$ by virtue of (2.1) and $g = \omega J$ is an almost Hermitian metric. We choose

$$g = \Theta^1 \Theta^1 + \Theta^2 \Theta^2 + \Theta^3 \Theta^3 \quad \text{and} \quad \omega = \frac{i}{2} \left( \Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2 + \Theta^3 \wedge \Theta^3 \right) .$$

We assume that $X^6$ is not a Calabi-Yau manifold and an almost complex structure $J$ is not integrable. Examples of such manifolds are nearly Kähler [15, 16] and nearly Calabi-Yau [17] ones.

Twistor space $\mathcal{T}(\mathbb{CP}^2)$. We will consider a non-integrable almost complex structure $J$ and SU$(3)$-structures on the flag manifold SU$(3)/U(1)\times U(1)$ which is the twistor space $\mathcal{T}(\mathbb{CP}^2)$ of the projective plane $\mathbb{CP}^2$. It is fibred over $\mathbb{CP}^2$ with the canonical projection

$$\pi : \mathcal{T}(\mathbb{CP}^2) \to \mathbb{CP}^2$$

and the Riemann sphere $\mathbb{CP}^1$ as a typical fibre. We will endow this twistor space with a one-parameter family of metric $\{g_\sigma\}_{\sigma \in (0, \infty)}$ such that this space is a nearly Kähler manifold for a special choice of the parameter $\sigma$. Although the geometry of the coset space $\mathcal{T}(\mathbb{CP}^2) = SU(3)/U(1)\times U(1)$ is well-known, we describe it briefly by using local coordinates for fixing our notation and further applications.

Coset representatives. Consider the principal bundle

$$SU(3) \to \mathcal{T}(\mathbb{CP}^2)$$

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with the structure group \( \text{U}(1) \times \text{U}(1) \). Let \( y^\alpha, \alpha = 1, 2 \), be local complex coordinates on \( \mathbb{C}P^2 \) and \( \zeta \) a local complex coordinate on the typical fibre \( \mathbb{C}P^1 \) of (2.5). Then a local section of the fibration (2.6) (a representative element for the coset) is given by the \( 3 \times 3 \) matrix
\[
\hat{V} = \gamma^{-1} \begin{pmatrix} 1 & -T^\dagger & 0 \\ T & W & 0 \\ 0 & 0 & h \end{pmatrix} \in \text{SU}(3),
\]
where
\[
T := \begin{pmatrix} \tilde{y}_2^\dagger \\ y_1^\dagger \end{pmatrix}, \quad W := \gamma \cdot 1_2 - \frac{1}{\gamma + 1} TT^\dagger \quad \text{and} \quad \gamma = (1 + T^\dagger T)^{\frac{1}{2}} = (1 + y^\alpha \tilde{y}^\alpha)^{\frac{1}{2}}
\]
and therefore
\[
W^\dagger = W, \quad WT = T \quad \text{and} \quad W^2 = \gamma^2 \cdot 1_2 - TT^\dagger,
\]
(2.8) obey
\[
\text{Flat connection on } \mathcal{T}(\mathbb{C}P^2). \text{ Consider now a trivial complex vector bundle } \mathcal{T}(\mathbb{C}P^2) \times \mathbb{C}^4 \to \mathcal{T}(\mathbb{C}P^2) \text{ endowed with a flat connection}
\]
\[
\hat{A} = \hat{V}^{-1} d\hat{V} =: \begin{pmatrix} 2b & -\frac{1}{\gamma} \hat{\theta}^\dagger \\ \frac{1}{\sqrt{\gamma}} \hat{B} & \end{pmatrix},
\]
(2.11) where
\[
\hat{\theta} = h^3 \theta = \frac{1}{\gamma + 1} \begin{pmatrix} \Theta_1 \Theta_2^2 \\ \Omega^2 \Theta_1 \end{pmatrix}, \quad \hat{\theta}^\dagger = \theta^\dagger h = (\Theta^2 \Theta^1),
\]
(2.12) and
\[
\hat{B} = h^3 B h + h^3 d h =: \begin{pmatrix} \hat{a} + & -\frac{1}{2\gamma} \Theta^3 \\ \frac{1}{2\gamma} \Theta^2 & -\hat{a} + \end{pmatrix} - b \cdot 1_2.
\]
(2.13) with
\[
\hat{a} + = \frac{1}{1 + \zeta} \left\{ (1 - \zeta) a_+ + \zeta b_+ - \zeta \bar{b}_+ + \frac{1}{2} (\bar{\zeta} d \zeta - \zeta d \bar{\zeta}) \right\},
\]
(2.14) \[
\Theta^3 = \frac{2\Lambda}{1 + \zeta} \left( d \zeta + b_+ + 2 \zeta a_+ + \zeta^2 b_+ \right).
\]
(2.15) Here \( \theta^1, \theta^2, b, a_+ \) and \( b_+ \) are defined by formulae [18]
\[
b = \frac{1}{4\gamma^2} (T^\dagger dT - dT^\dagger T),
\]
(2.16) \[
\theta = \frac{2\Lambda}{\gamma} W d T = \begin{pmatrix} \theta^2 \\ \theta^1 \end{pmatrix} = \frac{2\Lambda}{\gamma} \begin{pmatrix} d \tilde{y}^2 \\ d y^1 \end{pmatrix} - \frac{2\Lambda}{\gamma^2 (\gamma + 1)} \begin{pmatrix} \tilde{y}^2 \\ y^1 \end{pmatrix} (y^1 d y^1 + y^2 d \tilde{y}^2),
\]
(2.17) \[
\begin{pmatrix} a_+ & -\bar{b}_+ \\ b_+ & -a_+ \end{pmatrix} := B + b \cdot 1_2 = \frac{1}{\gamma^2} (W d W - T d T^\dagger - \frac{1}{2} dT^\dagger T - \frac{1}{2} T^\dagger d T) + b \cdot 1_2.
\]
(2.18)
Note that $\theta^1$ and $\theta^2$ are local orthonormal basis of (1,0)-forms on $\mathbb{C}P^2$. The real parameters $\Lambda$ and $R$, inserted in the definition of $\Theta^1, \Theta^2$ and $\Theta^3$, respectively, characterize ‘sizes’ of $\mathbb{C}P^2$ and $\mathbb{C}P^1 \hookrightarrow \mathcal{T}(\mathbb{C}P^2) = SU(3)/U(1) \times U(1)$ from (2.5).

**SU(3)-structures on $\mathcal{T}(\mathbb{C}P^2)$**. From flatness of the connection (2.11) we obtain

\[ \begin{align*}
\mathbf{d}b &= -\frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^\dagger - \Theta^2 \wedge \Theta^\dagger) , \\
\mathbf{d}a_+ &= -\frac{1}{8\Lambda^2} (\Theta^1 \wedge \Theta^\dagger + \Theta^2 \wedge \Theta^\dagger - \frac{2}{\sigma} \Theta^3 \wedge \Theta^\dagger) 
\end{align*} \]  

(2.19)

with

\[ \sigma := \frac{R^2}{\Lambda^2} , \]  

(2.20)

and the structure equations

\[ \begin{align*}
\mathbf{d} \left( \begin{array}{c} 
\Theta^1 \\
\Theta^2 \\
\Theta^3 
\end{array} \right) &= \left( \begin{array}{ccc}
\hat{a}_+ + 3b & 0 & 0 \\
0 & \hat{a}_+ - 3b & 0 \\
0 & 0 & -2\hat{a}_+ 
\end{array} \right) \wedge \left( \begin{array}{c} 
\Theta^1 \\
\Theta^2 \\
\Theta^3 
\end{array} \right) + \frac{1}{2R} \left( \begin{array}{c} 
\Theta^2 \wedge \Theta^3 \\
\Theta^3 \wedge \Theta^1 \\
\sigma \Theta^1 \wedge \Theta^2 
\end{array} \right) , 
\end{align*} \]  

(2.21)

where the first term defines $u(1) \oplus u(1)$ torsional connection and the last term defines the Nijenhuis tensor (torsion) with components $N_{jk}^i$ and their complex conjugate. Namely, we have

\[ N_{23}^1 = N_{31}^2 = \frac{1}{2R} \]  

and \[ N_{12}^3 = \frac{R}{2\Lambda^2} = \frac{\sigma}{2R} . \]  

(2.22)

We see that the parameters $\Lambda$ and $R$ enter in all formulae but of real importance is only their ratio (2.20). For instance, consider the forms $\Omega$ and $\omega$ defined by (2.2) and (2.4). From (2.21) it follows that

\[ \begin{align*}
\mathbf{d}\omega &= \frac{1}{2R} (2+\sigma) \Im \Omega , \\
\mathbf{d}\Omega &= \frac{1}{2R} (\Theta^2 \wedge \Theta^3 \wedge \Theta^3 \wedge \Theta^3 \wedge \Theta^1 \wedge \Theta^1 \wedge \Theta^1 \wedge \Theta^1 + \sigma \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^3 ) ,
\end{align*} \]  

(2.23)

and we have a one-parameter family $(\omega, \Omega)$ of SU(3)-structures on SU(3)/U(1)×U(1) such that for $\sigma = 1$ this manifold is nearly Kähler (see e.g. [15, 16, 17] and references therein).

### 3 SU(4)-instanton equations in eight dimensions

**Instanton equations in $d > 4$**. The concept of Yang-Mills instantons in four dimensions can be generalized by considering first-order equations for gauge potentials in spaces of dimension greater than four [19]-[24], [13, 14], [25]-[29]. Most of these equations naturally appear in superstring theory and M-theory as the conditions for the survival of at least one unbroken supersymmetry in low-energy effective field theory in $d \leq 4$ dimensions. Generically BPS-type first-order gauge equations in higher dimensions can only be defined on manifolds with a $G$-structure. For instance, on Kähler, Calabi-Yau and hyper-Kähler manifolds one can define the Hermitian-Yang-Mills equations [21, 22]. In $d = 7$ one should consider manifolds with a $G_2$-structure [13, 25, 27, 28]. In $d = 8$ one should consider a reduction of the holonomy group to Spin(7), SU(4), Sp(2)Sp(1) or Sp(2) subgroups of SO(8) [13, 14, 23], [25]-[28] and so on. Some solutions of various type of the above-mentioned

\[ ^3 \text{Overall scaling parameter is not essential, } R \text{ can be fixed to some number.} \]
first-order gauge equations were found e.g. in [30, 31]. Stud ying the geometry of moduli spaces of Yang-Mills instantons in $d > 4$ is considered as an important task [13, 26, 27, 28].

**Ξ-anti-self-duality.** BPS-type instanton equations in more than four dimensions can be introduced as follows. Let $(X^d, g)$ be an oriented Riemannian manifold of dimension $d$ and $Ξ$ a differential form of degree $d - 4$ on $X^d$. Consider a complex vector bundle $E$ over $X^d$ endowed with a connection $A$. The Ξ-anti-self-dual gauge equations are defined [27] as the first-order equations,

$$ *F = -Ξ \wedge F , $$

on a connection $A$ with the curvature $F = dA + A \wedge A$. Here $*$ is the Hodge star operator.

Differentiating (3.1), we obtain

$$ d*F + A \wedge *F - *F \wedge A + *H \wedge F = 0 , $$

where the 3-form $H$ is defined by the formula

$$ H := *dΞ . $$

Equations (3.2) differ from the standard Yang-Mills equations by the last term with a 3-form $H$ which can be identified with a totally antisymmetric torsion. This torsion term naturally appears in string theory (a lot of references can be found in [18]).

**Spin(7)-instantons.** Let us consider a Riemannian manifold $(X^8, g)$ of dimension 8, and let $\tilde{Ξ}$ be a 4-form which defines an almost Spin(7)-structure on $X^8$. A Spin(7)-instanton is defined as a connection $\tilde{A}$ on a complex vector bundle $\tilde{E}$ over $X^8$ such that its curvature $\tilde{F}$ satisfies the equations

$$ *\tilde{F} = -\tilde{Ξ} \wedge \tilde{F} , $$

with an almost Spin(7)-structure $\tilde{Ξ}$ on $X^8$. For more details see e.g. [13, 14, 28, 33, 34].

**SU(4)-instantons.** Suppose that an almost Spin(7)-manifold $(X^8, g)$ allows an almost complex structure $J$ and an SU(4)-structure, i.e. $c_1(X^8) = 0$. Then on $X^8$ there exists a non-degenerate (4,0)-form $\tilde{Ω}$ and an (1,1)-form $\tilde{ω}$ such that

$$ \tilde{Ω} = Θ^1 \wedge Θ^2 \wedge Θ^3 \wedge Θ^4 \quad \text{and} \quad \tilde{ω} = \frac{1}{2} (Θ^1 \wedge Θ^2 + Θ^3 \wedge Θ^4 + Θ^2 \wedge Θ^3 + Θ^4 \wedge Θ^1) , $$

with

$$ J Θ^A = i Θ^A \quad \text{for} \quad A = 1, ..., 4 , $$

i.e. $Θ^A$'s are (1,0)-forms with respect to $J$. For such a case the 4-form $\tilde{Ξ}$ - an almost Spin(7)-structure - can be written as

$$ \tilde{Ξ} = \frac{1}{2} \tilde{ω} \wedge \tilde{ω} - \text{Re} \tilde{Ω} . $$

The inclusion SU(4) $\subset$ Spin(7) allows us to reduce the Spin(7)-instanton equations (3.4) with $\tilde{Ξ}$ given in (3.7) to SU(4)-instanton equations [27]. Let $\tilde{E}$ be a zero-degree bundle, i.e. $c_1(\tilde{E}) = 0$

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4In four dimensions, (3.1) is reduced to $*F = -F$.

5One can omit the word ‘almost’ if $\tilde{Ξ}$ is closed [14, 27, 28, 32]. However, the case $dΞ \neq 0$ is also of interest and with an air of importance [33, 28, 34].

6For an integrable almost complex structure $J$ we obtain a Calabi-Yau 4-fold.
and therefore $\text{tr} \tilde{F} = 0$. Then there exists a Hermitian rank-$k$ vector bundle $\mathcal{E}$ with a connection $\mathcal{A}$ such that $\tilde{F} = \mathcal{F} - \frac{1}{k} (\text{tr} \mathcal{F}) \cdot 1_k$, where $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is the curvature of $\mathcal{A}$. Note that $\frac{1}{2k} \text{tr} \mathcal{F}$ represents the first Chern class $c_1(\mathcal{E})$ in $H^2(X^8, \mathbb{R})$. Recall that on $X^8$ we are given a $(4,0)$-form $\tilde{\Omega}$ and its complex conjugate $\bar{\tilde{\Omega}}$ induces an anti-linear involution $\ast_{\tilde{\Omega}} : \Lambda^{0,2}(X^8) \to \Lambda^{0,2}(X^8)$ so that one can introduce a self-dual part

$$\mathcal{F}^{0,2}_+ := \frac{1}{2} (\mathcal{F}^{0,2} + \ast_{\tilde{\Omega}} \mathcal{F}^{0,2})$$

(3.8)

of $\mathcal{F}^{0,2}$. Assume now that $\text{tr} \mathcal{F}$ is harmonic and $\text{tr} \mathcal{F}^{0,2}_+ = 0$ [27]. Then Spin(7)-instanton equations (3.4) for $\tilde{\mathcal{A}} = \mathcal{A} - \frac{1}{k} (\text{tr} \mathcal{A}) \cdot 1_k$ are equivalent to the equations

$$i \tilde{\omega} \lrcorner \mathcal{F} = \tilde{\lambda} \cdot 1_k \quad \text{and} \quad \mathcal{F}^{0,2}_+ = 0 \ ,$$

(3.9)

where $\tilde{\omega} \lrcorner$ denotes a contraction of $\mathcal{F}$ with a bivector dual to $\tilde{\omega}$ [15] and $\tilde{\lambda} \in \mathbb{R}$ is related with $c_1(\mathcal{E})$.\footnote{For $c_1(\mathcal{E}) = 0$ one has $\tilde{\lambda} = 0$. In this case one can identify $\mathcal{E}$ and $\tilde{\mathcal{E}}$.} Equations (3.9) are called SU(4)-instanton equations [13, 27]. In the basis of $(1,0)$-forms $\Theta^A$ and $(0,1)$-forms $\bar{\Theta}^\bar{A}$ they can be written as

$$2\delta^{AB} \mathcal{F}_{AA} = \tilde{\lambda} \cdot 1_k \quad \text{and} \quad \mathcal{F}_{AB} + \frac{1}{2} \varepsilon_{ABCD} \mathcal{F}_{CD} = 0 \ ,$$

(3.10)

where $\varepsilon_{ABCD}$ is the totally skew-symmetric tensor.

**Hermitian-Yang-Mills equations.** Note that $\mathcal{F}^{0,2}_+ \neq 0$ for SU(4)-instantons according to (3.9) and (3.10). However, on an almost complex manifold $X^8$ one can also introduce the Hermitian-Yang-Mills equations [21, 22, 35],

$$i \tilde{\omega} \lrcorner \mathcal{F} = \tilde{\lambda} \cdot 1_k \quad \text{and} \quad \mathcal{F}^{0,2} = 0 \ ,$$

(3.11)

which impose restrictions on a connection $\mathcal{A}$ on a complex vector bundle $\mathcal{E} \to X^8$ stronger than equations (3.9). Any solution $\mathcal{A}$ of (3.11) solves (3.9) but the converse is not true. According to R.Bryant [35], any connection $\mathcal{A}$ on $\mathcal{E}$ which satisfies (3.11) defines a pseudo-holomorphic structure $\bar{\partial}_A$ on $\mathcal{E}$. Solutions $\mathcal{A}$ of (3.11) are called the Hermitian-Yang-Mills connections. In the case of integrable almost complex structure $\mathcal{J}$ the Hermitian-Yang-Mills connections $\mathcal{A}$ define (poly)stable holomorphic bundles $\mathcal{E}$ [21, 22].

**Manifold $G/H \times \mathbb{R}^2$.** Let us consider the direct product

$$G/H \times \mathbb{R}^2 \ ,$$

(3.12)

where $G/H$ is the homogeneous space SU(3)/U(1)$\times$U(1) described in section 2 or any other coset space with an SU(3)-structure, e.g. Sp(2)/Sp(1)$\times$U(1), $S^6$ or $S^3 \times S^3$. So, on $G/H$ we have an SU(3)-structure $(\omega, \Omega)$, a Hermitian metric $g$ and never integrable almost complex structure $\mathcal{J}$. On the manifold (3.12) we introduce an almost complex structure $\tilde{\mathcal{J}} = (\mathcal{J}, j)$, where $j$ is the canonical (integrable) almost complex structure on the space $\mathbb{R}^2$ with coordinates $x^7, x^8$. Namely, $j$ is defined so that

$$\Theta^4 := dz^4 = dx^7 + i dx^8$$

(3.13)

is a $(1,0)$-form on $\mathbb{R}^2 \cong \mathbb{C}$.\footnote{For $c_1(\mathcal{E}) = 0$ one has $\tilde{\lambda} = 0$. In this case one can identify $\mathcal{E}$ and $\tilde{\mathcal{E}}$.}
On $G/H \times \mathbb{R}^2$, we introduce forms
\[
\tilde{\Omega} = \Omega \wedge \Theta^4 = \Theta^1 \wedge \Theta^2 \wedge \Theta^3 \wedge \Theta^4, \quad (3.14)
\]
\[
\tilde{\omega} = \omega + \frac{i}{2} \Theta^4 \wedge \Theta^4 = \frac{i}{2} (\Theta^1 \wedge \Theta^1 + \Theta^2 \wedge \Theta^2 + \Theta^3 \wedge \Theta^3 + \Theta^4 \wedge \Theta^4), \quad (3.15)
\]
and the metric
\[
\tilde{g} = g + \Theta^4 \Theta^4 = \Theta^1 \Theta^1 + \Theta^2 \Theta^2 + \Theta^3 \Theta^3 + \Theta^4 \Theta^4. \quad (3.16)
\]
Thus, $G/H \times \mathbb{R}^2$ is an 8-dimensional Riemannian manifold with an SU(4)-structure.\(^8\) Hence, on the manifold (3.12) one can introduce the 4-form (3.7) and the Spin(7)-instanton equations (3.4). These equations can also be rewritten in the form (3.9) of SU(4)-instanton equations.

4 Vortex equations associated with SU(3)/U(1)×U(1)

In this section we will consider Yang-Mills theory with SU(3)-equivariant gauge fields on the manifold $X^8 = G/H \times \mathbb{R}^2$ with $G=SU(3)$ and $H=U(1)\times U(1)$. The group $G$ acts trivially on $\mathbb{R}^2 \cong \mathbb{C}$ and in the standard way by isometries on $G/H$.

**Invariant connection.** Let $\mathcal{E} \to X^8$ be an SU(3)-equivariant complex vector bundle of rank $k$ over $X^8$ and $\mathcal{A}$ a $u(k)$-valued local form of SU(3)-equivariant\(^9\) connection on $\mathcal{E}$. Such a connection $\mathcal{A}$ is given by naturally extending the flat connection $\hat{\mathcal{A}}$ on the bundle $\hat{\mathcal{E}} \to G/H$ from (2.11) [10]. In the simplest case of a quiver bundle $E^{0,1}$ associated with the fundamental representation $\mathbb{C}^3$ of SU(3) we get

\[
\mathcal{A} = \begin{pmatrix}
A^1 \otimes 1 + 1_{k_1} \otimes 2b & -\frac{i}{2\pi} \phi_1^\dagger \otimes \Theta^2 & -\frac{i}{2\pi} \phi_1 \otimes \Theta^1 \\
\frac{1}{2\pi} \phi_2 \otimes \Theta^2 & A^2 \otimes 1 + 1_{k_2} \otimes (\hat{a}_+ - b) & -\frac{i}{2\pi} \phi_3 \otimes \Theta^3 \\
\frac{1}{2\pi} \phi_1^\dagger \otimes \Theta^1 & -\frac{1}{2\pi} \phi_3 \otimes \Theta^3 & A^3 \otimes 1 - 1_{k_3} \otimes (\hat{a}_+ + b)
\end{pmatrix}, \quad (4.1)
\]

where $A^1$, $A^2$ and $A^3$ are $u(k_1)$-, $u(k_2)$- and $u(k_3)$-valued gauge potentials on complex vector bundles $E_1$, $E_2$ and $E_3$ over $\mathbb{R}^2$ with ranks $k_1$, $k_2$ and $k_3$, respectively, such that $k_1 + k_2 + k_3 = k = \text{rank} \mathcal{E}$. The bi-fundamental scalar fields $\phi_1 \in \text{Hom}(E_3,E_1)$, $\phi_2 \in \text{Hom}(E_1,E_2)$ and $\phi_3 \in \text{Hom}(E_2,E_3)$ can be identified with sections (Higgs fields) of the bundles $E_1 \otimes E_3^\vee$, $E_2 \otimes E_1^\vee$ and $E_3 \otimes E_2^\vee$, respectively. Note that fields $A^i$ and $\phi_i$ depend only on coordinates $x^7, x^8$ of $\mathbb{R}^2$ and $\phi_i^\dagger$ is a Hermitian conjugate of $\phi_i$, $i = 1, 2, 3$. Ansatz for the invariant connection $\mathcal{A}$ associated with arbitrary representation $C^{p,q}$ of SU(3) can be written down by using the formulae from [10]. However, in this short paper we intentionally will consider only the simplest case for exemplifying non-typical properties of new quiver vortex equations.

**$C^{0,1}$-quiver bundle.** The fundamental representation $\mathbb{C}^3$ ($\sim C^{0,1}$) of SU(3) decomposes as $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ after the restriction to $U(1)\times U(1)$. To this decomposition there corresponds a $U(1)\times U(1)$-equivariant bundle

\[
E^{0,1} = E_1 \otimes \mathbb{C} \oplus E_2 \otimes \mathbb{C} \oplus E_3 \otimes \mathbb{C} \quad (4.2)
\]

---

\(^8\)In fact, the torsional connection on $G/H \times \mathbb{R}^2$ has holonomy contained in SU(3) due to trivial holonomy along the subspace $\mathbb{R}^2$.

\(^9\)This means a generalized SU(3)-invariance, i.e. invariance under SU(3)-isometries up to gauge transformations [36, 37, 38, 4]. For transition functions one considers a compensating change of a trivialization.
over $\mathbb{R}^2$ along with appropriate bundle morphisms $\phi_i$ between $E_i$ given by the quiver diagram

$$
\begin{array}{ccc}
E_2 & \rightarrow & E_3 \\
\downarrow & & \downarrow \\
E_1 & \rightarrow & E_3
\end{array}
$$

We also introduce the homomorphisms

$$r_1 := \phi_1 - \phi_2 \phi_3^\dagger, \quad r_2 := \phi_2 - \phi_3 \phi_1^\dagger \quad \text{and} \quad r_3 := \phi_3 - \phi_2 \phi_1^\dagger$$

(4.4)
corresponding to the quiver relations.

In (4.2), factors $\mathbb{C}$ denote the trivial $U(1) \times U(1)$-equivariant complex line bundles over $\mathbb{R}^2$ arising from the decomposition $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Later we will see that in general the diagram (4.3) is not commutative due to non-integrability of the Dolbeault operator on vector bundles over the coset space $SU(3)/U(1)$ with non-integrable almost complex structure. This will result in the proportionality of $\partial_A \phi_i$ to$^{10}$ the polynomials $r_i$ from (4.4) corresponding to the quiver relations.

**Invariant field strength tensor.** For the curvature$^{11}$ $\mathcal{F} = dA + A \wedge A = (\mathcal{F}^{ij})$ of the invariant connection $A$ given by (4.1) we obtain

$$
\begin{align*}
\mathcal{F}^{11} & = F^{1} - \frac{1}{4\pi^2} (1_{k_1} - \phi_1 \phi_1^\dagger) \Theta^1 \wedge \Theta^1 + \frac{1}{4\pi^2} (1_{k_1} - \phi_2 \phi_2^\dagger) \Theta^2 \wedge \Theta^2, \\
\mathcal{F}^{22} & = F^{2} - \frac{1}{4\pi^2} (1_{k_2} - \phi_2 \phi_2^\dagger) \Theta^2 \wedge \Theta^2 + \frac{1}{4\pi^2} (1_{k_2} - \phi_3 \phi_3^\dagger) \Theta^3 \wedge \Theta^3, \\
\mathcal{F}^{33} & = F^{3} + \frac{1}{4\pi^2} (1_{k_3} - \phi_3 \phi_3^\dagger) \Theta^3 \wedge \Theta^3 - \frac{1}{4\pi^2} (1_{k_3} - \phi_2 \phi_2^\dagger) \Theta^3 \wedge \Theta^3, \\
\mathcal{F}^{13} & = \frac{1}{2\pi} (d\phi_1 + A^1_1 \phi_1 - \phi_1 A^3_1) \wedge \Theta^1 - \frac{1}{2\pi} (d\phi_1 + A^1_2 \phi_2 - \phi_2 A^3_2) \wedge \Theta^3 - \frac{1}{2\pi} (d\phi_1 + A^1_3 \phi_3 - \phi_3 A^3_3) \wedge \Theta^3, \\
\mathcal{F}^{21} & = \frac{1}{2\pi} (d\phi_2 + A^2_2 \phi_2 - \phi_2 A^1_2) \wedge \Theta^2 + \frac{1}{2\pi} (d\phi_2 + A^2_3 \phi_3 - \phi_3 A^1_3) \wedge \Theta^1, \\
\mathcal{F}^{32} & = \frac{1}{2\pi} (d\phi_3 + A^3_3 \phi_3 - \phi_3 A^2_3) \wedge \Theta^3 + \frac{1}{2\pi} (d\phi_3 + A^3_2 \phi_2 - \phi_2 A^2_2) \wedge \Theta^2,
\end{align*}
$$

(4.5)-(4.10)

plus their Hermitian conjugates $\mathcal{F}^{ij} = -(\mathcal{F}^{ij})^\dagger$ for $i \neq j$, $i, j, ... = 1, 2, 3$. In deriving (4.5)-(4.10) we used various relations following from the flatness of the connection (2.11). In (4.5)-(4.10) $F^i = dA^i + A^i \wedge A^i$ is the curvature of a connection $A^i$ on the complex vector bundle $E_i$.

**Quiver vortex equations.** Let us substitute the field strength matrix elements (4.5)-(4.10) and their Hermitian conjugates into the SU(4)-instanton equations (3.10). After this substitution we get non-Abelian coupled vortex equations

$$
\begin{align*}
\partial_x \phi_1 + A^1_{1} \phi_1 - \phi_1 A^2_1 & = \frac{1}{2\pi} (\phi_1 - \phi_2 \phi_3^\dagger), \\
\partial_x \phi_2 + A^2_{2} \phi_2 - \phi_2 A^3_2 & = \frac{1}{2\pi} (\phi_2 - \phi_3 \phi_1^\dagger), \\
\partial_x \phi_3 + A^3_{3} \phi_3 - \phi_3 A^2_3 & = \frac{2\pi}{2\pi} (\phi_3 - \phi_1 \phi_2^\dagger), \\
F_{22}^{1} & = \frac{1}{2\pi} (\lambda \cdot 1_{k_1} - \sigma \phi_1 \phi_1^\dagger + \sigma \phi_2 \phi_2^\dagger), \\
F_{22}^{2} & = \frac{1}{2\pi} ((\lambda+\sigma-1)1_{k_2} - \sigma \phi_2 \phi_2^\dagger + \phi_3 \phi_3^\dagger), \\
F_{22}^{3} & = \frac{1}{2\pi} ((\lambda-\sigma+1)1_{k_3} - \phi_3 \phi_3^\dagger + \sigma \phi_1 \phi_1^\dagger),
\end{align*}
$$

(4.11)-(4.16)

$^{10}$Here, $\partial_A$ is the Dolbeault operator on the vector bundle $E^{0,1}$ over $\mathbb{R}^2$.

$^{11}$By $i, j$ we are numbering $k_i \times k_j$ blocks in $\mathcal{F}$. 

8
where $\lambda := 2\tilde{\lambda}R^2$ and $z := z^A = x^7 + i x^8$, $\bar{z} := \bar{z}^\dot{A}$. Recall that $\sigma = R^2/\Lambda^2$. Note that one can consider these equations not only on $\mathbb{R}^2$ but also on $\mathbb{R} \times S^1$ and on a complex torus $T^2 = S^1 \times S^1$.

Let us denote by $\tilde{\partial}_A$ the Dolbeault operator acting on $\phi_i$ in (4.11)-(4.13). From these equations we see that the Higgs fields $\phi_i$, defining homomorphisms of the bundles $E_i \to \mathbb{R}^2$, are not holomorphic. In fact, $\tilde{\partial}_A \phi_i$ is proportional to the quiver relation $r_i$ defined in (4.4),

$$\tilde{\partial}_A \phi_i = \kappa r_i ,$$

where $\kappa = d\bar{z}/2R$ for $i = 1, 2$ and $\kappa = \sigma d\bar{z}/2R$ for $i = 3$. This reflects the fact that the almost complex structure $\tilde{J}$ on $\mathbb{R}^2 \times \text{SU}(3)/\text{U}(1) \times \text{U}(1)$ is not integrable, the bundle $\tilde{E}$ is not (pseudo)holomorphic and therefore $\tilde{F}^{0,2}$ must not vanish. Of course, on the SU(3)-equivariant connection $A$ on $E$ one can impose the Hermitian-Yang-Mills equations (3.11) which are stronger than (3.9). Then one shall get the standard quiver vortex equations

$$\tilde{\partial}_A \phi_i = 0 \quad \text{and} \quad r_i = 0$$

(4.18) together with (4.14)-(4.16). However, there are solutions to (4.14)-(4.17) which are not reduced to solutions of (4.14)-(4.16) and (4.18).

**Particular cases.** Let us put $\sigma = 1$ when the SU(3)-structure on SU(3)/U(1)×U(1) becomes nearly Kähler. Then the quiver vortex equations (4.11)-(4.16) become more ‘symmetric’:

$$D_{\bar{z}} \phi_i = \frac{1}{2\sqrt{2}} r_i \quad \text{with} \quad d\bar{z} D_{\bar{z}} \phi_i := \tilde{\partial}_A \phi_i ,$$

$$F^1_{\bar{z}z} = \frac{1}{4R^2} (\lambda \cdot 1_{k_1} - \phi_1 \phi_1^\dagger + \phi_2 \phi_2^\dagger) ,$$

$$F^2_{\bar{z}\bar{z}} = \frac{1}{4R^2} (\lambda \cdot 1_{k_2} - \phi_2 \phi_2^\dagger + \phi_3 \phi_3^\dagger) ,$$

$$F^3_{zz} = \frac{1}{4R^2} (\lambda \cdot 1_{k_3} - \phi_3 \phi_3^\dagger + \phi_1 \phi_1^\dagger) .$$

Choosing in these equations $k_1 = k_2 = k_3 := m$ and $\phi_1 = \phi_2 = \phi_3 =: \phi$, we obtain the vortex equations

$$\tilde{\partial}_{\bar{z}} \phi + [A_{\bar{z}}, \phi] = \frac{1}{2\sqrt{2}} (\phi - \phi^\dagger \phi^\dagger) \quad \text{and} \quad F_{\bar{z}z} = \frac{1}{4R^2} (\lambda \cdot 1_m - [\phi, \phi^\dagger]) ,$$

(4.20)

where $F = dA + A \wedge A$ and $A := A_1 = A_2 = A_3$. Here, the equality of $A$’s follows from the equations for $F^i$, $i = 1, 2, 3$.

Finally, for $\lambda = 0$ and $[\phi, \phi^\dagger] = 0$, we obtain $F = 0 = A$ and (4.20) is reduced to the equation

$$\tilde{\partial}_{\bar{z}} \phi = \frac{1}{2\sqrt{2}} (\phi - \phi^\dagger \phi^\dagger)$$

(4.21)

on a matrix-valued scalar field. This equation obviously has nontrivial solutions that proves the existence of nonholomorphic solutions to the quiver vortex equations (4.11)-(4.16).

**Vortices in $\mathcal{N} = 4$ super-Yang-Mills theory.** Consider $\mathcal{N} = 1$ SYM theory with the gauge group U($k$) on ten-dimensional flat space with the metric

$$g_{10} = -dt^2 + ds^2 + dz d\bar{z} + \Theta^1 \Theta^1 + \Theta^2 \Theta^2 + \Theta^3 \Theta^3 ,$$

(4.22)

where $t, s, z$ and $\bar{z}$ are coordinates on Minkowski space $\mathbb{R}^{3,1}$ and $\{\Theta^i, \Theta^\jmath\}$ is the orthonormal complex frame on flat complex ‘internal’ space $\mathbb{R}^6 \cong \mathbb{C}^3$ (or complex torus $T^6$). Considering trivial
dimensional reduction (invariance under translations) of $\mathcal{N} = 1$ SYM theory from $d = 10$ to $d = 4$, we obtain the standard $\mathcal{N} = 4$ SYM theory in Minkowski space. Let us now reduce the bosonic sector of this theory to the space $\mathbb{R}^2 \cong \mathbb{C}$ with coordinates $z, \bar{z}$ and assume that $\mathcal{A}_i = A_s = 0$. Then after turning on a Fayet-Iliopoulos parameter $\lambda := \frac{1}{2} \bar{\lambda}$ in the Lagrangian, we see that the first-order BPS equations are dimensional reduction of the SU(4)-instanton equations (3.10) to $d = 2$. Namely, we get the vortex equations

$$F_{z\bar{z}} = \lambda \cdot 1_k - [\phi_1, \phi_i^\dagger] - [\phi_2, \phi_i^\dagger] - [\phi_3, \phi_i^\dagger], \quad (4.23)$$

$$\bar{\partial}_z \phi_i + [A_z, \phi_i] = \frac{1}{2} \varepsilon_{ijk} [\phi_j^\dagger, \phi_k^\dagger], \quad (4.24)$$

where $F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] \in u(k)$ and $\phi_i$ are $k \times k$ complex matrices.\footnote{Of course, one can assume that $\text{tr} F = 0$ and consider equations (4.23), (4.24) with $\lambda = 0$.} Note that $\Phi_i := \text{Re} \phi_i$ and $\Phi_{i+3} := \text{Im} \phi_i$ take values in $u(k)$ and belong to the fundamental representation of SO(6)$\cong$SU(4). For transition to the SU(4) notation $\{\Phi_i, \Phi_{i+3}\} \rightarrow \Phi_{AB} = -\Phi_{BA}$ one should simply convert $\Phi_i, \Phi_{i+3}$ with the chiral part $\sigma_{AB}^i, \sigma_{AB}^{i+3}$ of $\gamma$-matrices in $d = 6$: $\Phi_{AB} := \Phi_i \sigma_{AB}^i + \Phi_{i+3} \sigma_{AB}^{i+3}$.

Non-Abelian BPS vortices can and do occur in $\mathcal{N} = 2$ supersymmetric gauge theories (see e.g. [39] and references therein). Contrary to the $\mathcal{N} \leq 2$ case, vortices in $\mathcal{N} = 4$ SYM theory are described by non-holomorphic Higgs fields subject to the equations (4.23), (4.24). These non-Abelian vortex equations allow further (algebraic) reductions. For instance, if we put $\lambda = 0$ and assume $[\phi_1, \phi_i^\dagger] = 0$, then from (4.23) it follows that $F = 0 = A$ and (4.24) are reduced to the equations

$$\bar{\partial}_z \phi_i = \frac{1}{2} \varepsilon_{ijk} [\phi_j^\dagger, \phi_k^\dagger], \quad (4.25)$$

which generalize the celebrated Nahm equations and coincide with them for anti-hermitian $\phi_i$.

5 Conclusions

In this paper we have considered the Spin(7)-instanton equations on the almost complex manifold $\mathbb{R}^2 \times \text{SU}(3)/U(1) \times U(1)$ with a family of SU(4)-structures parametrized by $\sigma \in (0, +\infty)$. On such a manifold Spin(7)-instanton equations are reduced to the SU(4)-instanton equations. We considered an SU(3)-equivariant connection $\mathcal{A}$ on a complex vector bundle $\mathcal{E}$ over $\mathbb{R}^2 \times \text{SU}(3)/U(1) \times U(1)$ corresponding to a simplest quiver associated with the coset space $\text{SU}(3)/U(1) \times U(1)$. It has been shown that for symmetric connections $\mathcal{A}$ the SU(4)-instanton equations on $\mathbb{R}^2 \times \text{SU}(3)/U(1) \times U(1)$ are reduced to quiver vortex equations on $\mathbb{R}^2$ with non-holomorphic coupled Higgs fields $\phi_i, \quad i = 1, 2, 3$. It was shown that ‘deviation’ from the holomorphicity is given by the quiver relations $r_i$, i.e. $\bar{\partial}_A \phi_i = \sigma r_i$. From these equations it follows that the quiver diagram is not commutative until $\bar{\partial}_A \phi_i = 0$ and $r_i = 0$ separately. This happens when the connection $\mathcal{A}$ on $\mathcal{E}$ satisfies the more restrictive Hermitian-Yang-Mills equations. In general, the same is true for any quiver with relations associated with the space $\text{SU}(3)/U(1) \times U(1)$. It was shown that the obtained quiver vortex equations can further be reduced to a matrix kink-type equations (cf. [31, 40]).

We have introduced BPS vortex equations in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and shown that they are also related with the SU(4)-instanton equations on flat eight-dimensional space. It would be interesting to consider the above correspondence between non-Abelian vortices on $\mathbb{R}^2$ and symmetric instantons on $\mathbb{R}^2 \times X^6$ for other 6-dimensional spaces with an SU(3)-structure, e.g. $\text{Sp}(2)/Sp(1) \times U(1), \quad S^6$ and others, as well as to construct exact solutions to the obtained vortex equations.
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