The versal Deformation of an isolated toric Gorenstein Singularity

Klaus Altmann*
Dept. of Mathematics, M.I.T., Cambridge, MA 02139, U.S.A.
E-mail: altmann@math.mit.edu

Abstract

Given a lattice polytope \( Q \subseteq \mathbb{R}^n \), we define an affine scheme \( \bar{M} \) that reflects the possibilities of splitting \( Q \) into a Minkowski sum.

On the other hand, \( Q \) induces a toric Gorenstein singularity \( Y \), and we construct a flat family over \( \bar{M} \) with \( Y \) as special fiber. In case \( Y \) has an isolated singularity only, this family is versal.

Contents

1 Introduction 2
2 The Minkowski scheme of a lattice polytope 3
3 Proof of the statements of §2 7
4 The tautological cone over \( C(Q) \) 11
5 A flat family over \( \bar{M} \) 18
6 The Kodaira-Spencer map 22
7 The obstruction map 24
8 The components of the reduced versal family 34
9 Further examples 36

1Die Arbeit wurde mit einem Stipendium des DAAD unterstützt.
1 Introduction

(1.1) The whole deformation theory of an isolated singularity is encoded in its so-called versal deformation. For complete intersection singularities this is a family over a smooth base space - obtained by certain disturbances of the defining equations. As soon as we are leaving this class of singularities, the structure of the family or even the base space will be more complicated. It is well known that the base space might consist of several components or might be non-reduced. In (4) we will present a (three-dimensional) example of a singularity admitting a fat point as base space of its versal deformation.

(1.2) For two-dimensional cyclic quotient singularities (coinciding with the two-dimensional affine toric varieties), the computations of Arndt, Christophersen, Kollár/Shephard-Barron, Riemenschneider, and Stevens provide a description of the versal family - in particular, number and dimension of the components of the reduced base (they are smooth) are computed. Christophersen observed that the total spaces over these components are toric varieties again (cf. [Ch]). This suggests the conjecture that the entire deformation theory of affine toric varieties keeps inside this category. It should be a challenge to find the versal deformation, its base space, or the total spaces over the components by purely combinatorial methods.

(1.3) In the present paper we investigate the case of affine, toric Gorenstein singularities $Y$ given by some lattice polytope $Q$.

In §2 and §3 we start with describing an affine scheme $\bar{M}$ which seems to be interesting independently from the toric or deformation stuff. It describes the possibilities of splitting $Q$ into Minkowski summands. The underlying reduced space is an arrangement of planes corresponding to those Minkowski decompositions involving summands, that are lattice polytopes themselves.

In §4 we construct a flat family over $\bar{M}$ with the toric Gorenstein singularity $Y$ induced by $Q$ as special fiber. Computing the Kodaira-Spencer as well as the obstruction map shows that, in case that the singularity is isolated, the family is versal (nevertheless trivial for $\dim Q \geq 3$). In the general case, the Kodaira-Spencer map is an isomorphism onto the homogeneous part of $T^1_Y$ with the most interesting multidegree (cf. Theorem (5.2)), and the obstruction map is still injective (cf. Theorem (7.2)).

On the other hand, this family is embedded in a larger (non-flat) family that equals a morphism of affine toric varieties: The base space is given by the cone $C(Q)$ of
Minkowski summands of positive multiples of $Q$, and the total space comes from the
tautological cone over $C(Q)$ (cf. §4 and §5). In particular, for affine, toric, isolated
Gorenstein singularities, Christophersen’s observation (cf. (1.2)) keeps true (cf. §8).

Through the whole paper, an example accompanies the general theory. Further ex-
amples can be found in §9.

(1.4) Acknowledgements: I am very grateful to Duco van Straten and Theo de
Jong for constant encouragement and valuable hints.
This paper was written during a one-year-stay at MIT. I want to thank Richard
Stanley and all the other people who made it possible for me to work at this very
interesting and stimulating place.

2 The Minkowski scheme of a lattice polytope

(2.1) Let $Q \subseteq \mathbb{R}^n$ be a lattice polytope, i.e. the vertices are contained in
$\mathbb{Z}^n$. We will always assume that the edges do not contain any interior lattice points
(cf. (3.6)), hence, after choosing orientations they are given by primitive vectors
$d^1, \ldots, d^N \in \mathbb{Z}^n$.

Definition: For every 2-face $\varepsilon < Q$ we define its sign vector $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N) \in
\{0, \pm 1\}^N$ by

$$\varepsilon_i := \begin{cases} 
\pm 1 & \text{if } d^i \text{ is an edge of } \varepsilon \\
0 & \text{otherwise}
\end{cases}$$

such that the oriented edges $\varepsilon_i \cdot d^i$ fit to a cycle along the boundary of $\varepsilon$. This deter-
mines $\underline{\varepsilon}$ up to sign, and we choose one of both possibilities. In particular, $\sum_i \varepsilon_i d^i = 0$.

Example: Let us introduce the following example, which will be continued through
the paper:
For $Q$ we take the hexagon

$$Q_6 := \text{Conv}\{(0,0), (1,0), (2,1), (2,2), (1,2), (0,1)\} \subseteq \mathbb{R}^2.$$
Starting with \( d^1 := (0,0) \), the anticlockwise oriented edges are denoted by \( d^1, \ldots, d^6 \). As vectors they equal
\[
\begin{align*}
d^1 &= (1,0); & d^2 &= (1,1); & d^3 &= (0,1); \\
d^4 &= (-1,0); & d^5 &= (-1,-1); & d^6 &= (0,-1).
\end{align*}
\]

\( Q_6 \) is 2-dimensional, hence, it is its own unique 2-face \( \varepsilon = Q \). For \( Q_6 \) we take \( Q_6 = (1, \ldots, 1) \).

\[ (2.2) \quad \text{We define the vector space } V \subseteq \mathbb{R}^N \text{ by} \]
\[
V := V(Q) := \{(t_1, \ldots, t_N) \mid \sum_i t_i \varepsilon_i d^i = 0 \text{ for every 2-face } \varepsilon < Q\}.
\]

Then, \( C(Q) := V \cap \mathbb{R}^N_0 \) is a rational, polyhedral cone in \( V \), and its points correspond to the Minkowski summands of positive multiples of \( Q \): Given a point \( (t_1, \ldots, t_N) \in C(Q) \), the corresponding polytope \( Q_t \) is built by the edges \( t_i \cdot d^i \) instead of the plain \( d^i \) used in \( Q \) (cf. \( (2.1) \)).

For a Minkowski summand \( Q' \) of a positive multiple of \( Q \) we will denote its point in the cone by \( \varrho(Q') \in C(Q) \).

\[ (\text{Example: } \varrho(t \cdot Q) = (t, \ldots, t) \in C(Q) \subseteq V \subseteq \mathbb{R}^N.) \]

\[ (2.3) \quad \text{For each 2-face } \varepsilon < Q \text{ and for each integer } k \geq 1 \text{ we define the (vector valued) polynomial} \]
\[
g_{\varepsilon,k}(\underline{t}) := \sum_{i=1}^N t_i^k \varepsilon_i d^i.
\]

Using coordinates of \( \mathbb{R}^n \) the \( g_{\varepsilon,k}(\underline{t}) \) turn into regular polynomials - for each pair \( (\varepsilon, k) \) we will get two linearly independent ones.

We obtain an ideal
\[
\mathcal{J} := (g_{\varepsilon,k} \mid \varepsilon < Q, k \geq 1) \subseteq \mathcal{O}[t_1, \ldots, t_N],
\]
which defines an affine closed subscheme
\[
\mathcal{M} := \text{Spec } \mathcal{O}[\underline{t}] / \mathcal{J} \subseteq V_{\mathcal{O}} \subseteq \mathcal{O}^N.
\]

\[ \text{Example: For our hexagon } Q_6 \text{ introduced in } (2.1) \text{ we obtain} \]
\[
\mathcal{J} = (t_1^k + t_2^k - t_4^k - t_5^k, t_2^k + t_3^k - t_5^k - t_6^k \mid k \geq 1).
\]

Of course, finally many equations are sufficient to generate the ideal \( \mathcal{J} \) - but we can
even give an effective criterion to see which equations can be dropped:

Proposition: Let \( \varepsilon < Q \) be a 2-face. Then, \( \varepsilon \) is contained in a two-dimensional subspace of \( \mathbb{R}^n \), and this vector space comes with a natural lattice (the restriction of the big lattice \( \mathbb{Z}^n \)).

If \( \varepsilon \) is contained in two different strips defined by pairs of parallel lines of lattice-distance \( \leq k_0 \) each, then the equations \( g_{\varepsilon,k} \) \((k > k_0)\) are contained in the ideal generated by \( g_{\varepsilon,1}, \ldots, g_{\varepsilon,k_0} \).

Proof: cf. (3.3).

Corollary: If \( Q \) is contained in \( n \) linearly independent strips (defined by pairs of parallel hyperplanes) of lattice-thickness \( \leq k_0 \), then all polynomials \( g_{\varepsilon,k} \) with \( k > k_0 \) are superfluous.

Example: Obviously, \( Q_6 \) is contained in at least three strips of thickness 2. Hence, \( \mathcal{J} \) is generated in degree \( \leq 2 \):

\[
\mathcal{J} = \left( t_1 + t_2 - t_4 - t_5, \ t_2 + t_3 - t_5 - t_6, \ t_1^2 + t_2^2 - t_4^2 - t_5^2, \ t_2^2 + t_3^2 - t_5^2 - t_6^2 \right).
\]

\[ (2.4) \]
Denote by \( \ell \) the canonical projection

\[
\ell : \mathcal{A}^N \longrightarrow \mathcal{A}^N/\mathcal{A} \cdot (1, \ldots, 1) = \mathcal{A}^N/\mathcal{A} \cdot \varrho(Q).
\]

On the level of regular functions this corresponds to the inclusion \( \mathcal{A}[t_i - t_j | 1 \leq i, j \leq N] \subseteq \mathcal{A}[t_1, \ldots, t_N] \).

Theorem:

1. \( \mathcal{J} \) is generated by polynomials from \( \mathcal{A}[t_i - t_j] \), i.e. \( \mathcal{M} = \ell^{-1}(\mathcal{M}) \) for some affine closed subscheme \( \mathcal{M} \subseteq V_{\mathcal{A}/(1, \ldots, 1)} \mathcal{A}^N/\mathcal{A} \cdot \varrho(Q) \).

   \( \mathcal{M} \) is defined by the ideal \( \mathcal{J} \cap \mathcal{A}[t_i - t_j] \).

2. \( \mathcal{J} \subseteq \mathcal{A}[t_1, \ldots, t_N] \) is the smallest ideal (i.e. \( \mathcal{M} \) is the largest closed subscheme of \( \mathcal{A}^N \)) with

   (i) property (1) and

   (ii) containing the “toric” equations

\[
\prod_{i=1}^N t_i^{d_i} - \prod_{i=1}^N t_i^{c_i}, \quad \text{with}
\]
\[ d \in \mathbb{Z}^N \cap \text{span} \left\{ [\langle \varepsilon d^1, c \rangle, \ldots, \langle \varepsilon N d^N, c \rangle] | \varepsilon < Q \text{ 2-face, } c \in \mathbb{R}^n \} \].

(For an integer \( h \) we denote
\[ h^+ := \begin{cases} h & \text{if } h \geq 0 \\ 0 & \text{otherwise} \end{cases} ; \quad h^- := \begin{cases} 0 & \text{if } h \geq 0 \\ -h & \text{otherwise} \end{cases} \).}

Proof: cf. (3.4).

Example: Toric equations for \( Q_6 \) are for instance \( t_1 t_2 - t_4 t_5 \), \( t_2 t_3 - t_5 t_6 \), and \( t_1 t_6 - t_3 t_4 \).

(2.5) We want to describe the structure of the underlying reduced spaces of \( \mathcal{M} \) or \( \overline{\mathcal{M}} \).

First, we mention the following trivial observations concerning the cone \( C(Q) \):

(i) Minkowski summands \( Q' \) of \( Q \) (instead of a positive multiple of \( Q \)) are characterized by the property \( g(Q) - g(Q') \in \mathbb{R}_{\geq 0}^N \), i.e. all components of \( g(Q') \) have to be contained in the interval \([0, 1]\).

(ii) For a Minkowski summand \( Q' \) (of some positive multiple of \( Q \)) the property of being a lattice polytope is equivalent to the fact that \( g(Q') \in \mathbb{Z}^N \).

Now, let \( Q = R_0 + \ldots + R_m \) be a decomposition of \( Q \) into a Minkowski sum of \( m + 1 \) lattice polytopes. Then, the \( N \)-tuples \( g(R_0), \ldots, g(R_m) \) consist of numbers 0 and 1 only, and they sum up to \((1, \ldots, 1)\). In particular, the \((m + 1)\)-plane \( \mathcal{a}' \cdot g(R_0) + \ldots + \mathcal{a}' \cdot g(R_m) \subseteq \mathcal{a}'^N \) (or its \( m \)-dimensional image via \( \ell \)) is contained in \( \mathcal{M} \) (in \( \overline{\mathcal{M}} \), respectively).

Remark:

(1) Those \((m + 1)\)-plane (or its image via \( \ell \)) is given by the linear equations \( t_i - t_j = 0 \) (if \( d^i, d^j \) belong to a common summand \( R_v \)).

(2) Refinements of Minkowski decompositions (they form a partial ordered set) correspond to inclusions of the associated planes.

Theorem: \( \mathcal{M}_{\text{red}} \) and \( \overline{\mathcal{M}}_{\text{red}} \) equal the union of those flats corresponding to maximal Minkowski decompositions of \( Q \) into lattice summands.

Proof: cf. (3.4).

Example: \( \mathcal{M}(Q_6) \) and \( \overline{\mathcal{M}}(Q_6) \) are already reduced schemes - for non-reduced examples cf. §9. Let us study them directly:
The linear equations allow the following substitution:

\[
\begin{align*}
    t &= t_1 \\
    s_1 &= t_1 - t_3 \\
    s_2 &= t_4 - t_2 \\
    s_3 &= t_1 - t_4 \\
    t_6 &= t - s_1 - s_3.
\end{align*}
\]

The two quadratic equations transform into \( s_1 s_3 = s_2 s_3 = 0 \).

In particular, \( \mathcal{M} \) is the union of a line and a 2-plane - corresponding to the Minkowski decompositions

\[
\begin{align*}
    Q_6 &= \text{Conv}\{(0,0),(1,0),(1,1)\} + \text{Conv}\{(0,0),(0,1),(1,1)\} \quad \text{and} \\
    Q_6 &= \text{Conv}\{(0,0),(1,0)\} + \text{Conv}\{(0,0),(0,1)\} + \text{Conv}\{(0,0),(1,1)\}.
\end{align*}
\]

(2.6) \( \mathcal{M} \) (or \( M = \ell^{-1}(\mathcal{M}) \)) reflect the possibilities of Minkowski decompositions of \( Q \):

- The underlying reduced space encodes the decompositions of \( Q \) into lattice summands.
- Extremal decompositions into rational summands are hidden in the scheme structure of \( \mathcal{M} \).
- Its tangent space in 0 (the smallest affine space containing \( \mathcal{M} \)) equals \( V_a / \mathcal{I} \cdot \mathcal{I}(Q) \) - it is the vector space arising from the cone \( C(Q) \) of Minkowski summands by killing the summands homothetic to \( Q \).

Therefore, we will call \( \mathcal{M} \) the (affine) Minkowski scheme of \( Q \).

Remark: The ideals defining \( M \) and \( \mathcal{M} \) are homogeneous. Hence, there are projective versions of these schemes, too.

3 Proof of the statements of §2

(3.1) Using vectors \( c \in \mathbb{Z}^N \) (or selected \( c \in \mathbb{R}^N \)) we can evaluate the edges \( d_1, \ldots, d_N \) to get integers

\[
\begin{align*}
    d_1 := \langle \varepsilon_1 d_1, c \rangle, \ldots, d_N := \langle \varepsilon_N d_N, c \rangle
\end{align*}
\]

for every given 2-face \( \varepsilon < Q \). Doing so, the statements of §2 can be reduced to much simpler lemmas, which we will present here.

Then, all those lemmas are proved using the following recipe:
(i) Assume \( d_i = \pm 1 \) - then the lemmas reduce to well known facts concerning symmetric functions.

(ii) Move to the general case by specialization of variables.

**3.2** For the whole \( \S 3 \) we use the following notations:

Let \( d_1, \ldots, d_N \in \mathbb{Z} \) such that \( d_1, \ldots, d_M \geq 0, d_{M+1}, \ldots, d_N \leq 0 \), and \( \sum_{i=1}^{N} d_i = 0 \).

\[
\begin{align*}
g_k(t) &:= g_{d,k}(t) := \sum_{i=1}^{N} d_i t_i^k, \\
p(t) &:= p_{d}(t) := t_1^{d_1} \ldots t_M^{d_M} - t_{M+1}^{d_{M+1}} \ldots t_N^{d_N}.
\end{align*}
\]

Denote by \( \sigma_k \) and \( s_k \) the \( k \)-th elementary symmetric polynomial and the sum of the \( k \)-th powers of a given set of variables, respectively.

**Remark:** For \( 1 \leq i, j \leq M \) or \( M + 1 \leq i, j \leq N \), identifying the two variables \( t_i \) and \( t_j \) (i.e. switching from \( \mathcal{A}[t] \) to \( \mathcal{A}[t] / t_i - t_j \)) yields the following situation:

- \( t_i, t_j \) are replaced by a common new variable \( \tilde{t} \) (i.e. \( N \) is replaced by \( N - 1 \)),
- \( d_i, d_j \) are replaced by \( \tilde{d} := d_i + d_j \), but
- \( g_k(t), p(t) \) keep their shapes in the new set up.

In particular, the general situation can always be obtained via factorization from the special case \( d_1 = \ldots = d_M = 1; d_{M+1} = \ldots = d_N = -1 \) (and \( N = 2M \)). Renaming \( t_i = x_i, t_{M+i} = y_i (i \leq M) \) it looks like

\[
\begin{align*}
g_k(x, y) &= \left( \sum_{i=1}^{M} x_i^k \right) - \left( \sum_{i=1}^{M} y_i^k \right) = s_k(x) - s_k(y), \\
p(x, y) &= (x_1 \cdots x_M) - (y_1 \cdots y_M) = \sigma_M(x) - \sigma_M(y).
\end{align*}
\]

**3.3** **Lemma:** If \( k_0 := \sum_{i=1}^{M} d_i = - \sum_{i=M+1}^{N} d_i \), then the polynomials \( g_k \) (\( k > k_0 \)) are \( \mathcal{A}[t] \)-linear combinations of the \( g_1, \ldots, g_{k_0} \). (This implies Proposition (2.3).)

**Proof:** As previously discussed, we may regard the special case \( d_i = \pm 1 \). In particular, this implies \( k_0 = M \).

Now, for an arbitrary \( k (> M) \), the expression \( s_k(x) \) is a polynomial in either the \( \sigma_1(x), \ldots, \sigma_M(x) \) or the \( s_1(x), \ldots, s_M(x) \), say

\[
s_k(x) = P_k(s_1(x), \ldots, s_M(x)).
\]

Then,

\[
g_k(x, y) = s_k(x) - s_k(y) = P_k(s_1(x), \ldots, s_M(x)) - P_k(s_1(y), \ldots, s_M(y)),
\]

8
but for each monomial $s_1^{e_1} s_2^{e_2} \ldots s_M^{e_M}$ occurring in $P_k$, we have
\[
s_1(\underline{x})^{e_1} \cdot \ldots \cdot s_M(\underline{x})^{e_M} - s_1(\underline{y})^{e_1} \cdot \ldots \cdot s_M(\underline{y})^{e_M} = \\
= \sum_{v=1}^{M} \sum_{i=1}^{e_v} [s_v(\underline{x}) - s_v(\underline{y})] \cdot s_1(\underline{x})^{e_1} \cdot \ldots \cdot s_{v-1}(\underline{x})^{e_{v-1}} s_v(\underline{x})^{i-1} \cdot s_v(\underline{y})^{e_v} s_{v+1}(\underline{y})^{e_{v+1}} \ldots s_M(\underline{y})^{e_M} \\
= \sum_{v=1}^{M} g_v(\underline{x}, \underline{y}) \cdot \sum_{i=1}^{e_v} s_1(\underline{x})^{e_1} \cdot \ldots \cdot s_{v-1}(\underline{x})^{e_{v-1}} s_v(\underline{x})^{i-1} \cdot s_v(\underline{y})^{e_v} s_{v+1}(\underline{y})^{e_{v+1}} \ldots s_M(\underline{y})^{e_M},
\]
which proves the lemma. □

(3.4) Lemma:

(1) The ideal $\mathcal{J} := (g_k \mid k \geq 1) \subseteq \mathcal{O}[t_1, \ldots, t_N]$ is generated by polynomials in $t_i - t_1$ $(i = 2, \ldots, N)$ only.

(2) $\mathcal{J}$ is the smallest ideal generated by polynomials in $t_i - t_1$, which additionally contains $p$.

(This implies Theorem (3.3).)

Proof: (1) Replacing $t_i$ by $t_i - t_1$ as arguments in $g_k$ yields
\[
g_k(t_1 - t_1, \ldots, t_N - t_1) = \sum_{i=1}^{N} d_i (t_i - t_1)^k = \sum_{i=1}^{N} d_i \cdot \left( \sum_{v=0}^{k} (-1)^v t_i^v t_i^{k-v} \right) \\
= \sum_{v=0}^{k} (-1)^v t_i^v \cdot \left( \sum_{i=1}^{N} d_i t_i^{k-v} \right) = \sum_{v=0}^{k} (-1)^v t_i^v g_{k-v}(\underline{t}).
\]
In particular, $(g_k(\underline{t}) \mid k \geq 1)$ and $(g_k(\underline{t} - t_1) \mid k \geq 1)$ are the same ideals in $\mathcal{O}[\underline{t}]$.

(2) The polynomial rings $\mathcal{O}[\underline{t}]$ and $\mathcal{O}[t_1, \underline{t} - t_1]$ are equal, i.e. each polynomial $q(\underline{t})$ can uniquely be written as
\[
q(\underline{t}) = \sum_{v \geq 0} q_v(t_2 - t_1, \ldots, t_N - t_1) \cdot t_1^v.
\]
Moreover, if $J \subseteq \mathcal{O}[\underline{t}]$ is an ideal generated by polynomials in $\underline{t} - t_1$ only, then for each $q(\underline{t}) \in J$ the components $q_v$ are automatically contained in $J$, too.

Let us determine the components of the polynomial $p$ - we will start with our special case again:
\[
p(T + X, T + Y) = (T + X_1) \cdot \ldots \cdot (T + X_M) - (T + Y_1) \cdot \ldots \cdot (T + Y_M)
\]
has $\sigma_k(\underline{X}) - \sigma_k(\underline{Y})$ as coefficient of $T^{M-k}$ $(k = 1, \ldots, M)$. Now, there are a polynomial $P_k$ and a non-vanishing rational number $c_k$ (not depending on $M$) such that
\[
\sigma_k(\underline{X}) = P_k(s_1(\underline{X}), \ldots, s_{k-1}(\underline{X})) + c_k \cdot s_k(\underline{X}).
\]
As in the proof of the previous lemma we obtain
\[
\sigma_k(X) - \sigma_k(Y) = P_k(s_1(X), \ldots, s_{k-1}(X)) - P_k(s_1(Y), \ldots, s_{k-1}(Y)) + c_k \cdot s_k(X) - c_k \cdot s_k(Y)
\]
for some coefficients \(q_v\). Specialization - first by \(T \mapsto x_1, X_i \mapsto x_i - x_1, Y_i \mapsto y_i - x_1\), then followed by the usual one - shows that the ideal generated by the components \(p_v(t - t_1)\) of \(p\) equals \(J\).

\begin{proof}
The equations \(\sum_{i=1}^{N} d_i c^k_i = 0\) present 0 as a linear combination of the vectors \((c_i, c^2_i, c^3_i, \ldots)\). On the other hand, it is the Vandermonde that tells us that this linear combination has to be a trivial one, i.e. the sum of the coefficients \(d_i\) belonging to equal variables vanishes.
\end{proof}

\begin{enumerate}
\item[(3.5)] Lemma: Let \(c = (c_1, \ldots, c_N) \in \mathbb{A}^N\) be a point such that \(g_k(c) = 0\) for each \(k \geq 1\). Then, for every fixed \(c \in \mathbb{A}\), we have \(\sum_{c_i = c} d_i = 0\). (This implies Theorem (2.5).)
\end{enumerate}

\begin{proof}
The equations \(\sum_{i=1}^{N} d_i c^k_i = 0\) present 0 as a linear combination of the vectors \((c_i, c^2_i, c^3_i, \ldots)\). On the other hand, it is the Vandermonde that tells us that this linear combination has to be a trivial one, i.e. the sum of the coefficients \(d_i\) belonging to equal variables vanishes.
\end{proof}

\begin{enumerate}
\item[(3.6)] The polytope \(Q\) was assumed to have primitive edges only. Actually, we never needed this fact neither in the previous lemmata nor in their proofs. It is only important to translate these results into the language of Minkowski summands used in \(\mathcal{N}\).
\end{enumerate}

Droping this condition, similar constructions are possible. However, by declaring some or all lattice points contained in edges of \(Q\) to be additional, artificial vertices of \(Q\), several possibilities arise with equal rights. The two extremal cases (add either no or all possible generalized vertices) seem to be the most interesting ones.

Remark:

1. For a natural number \(g \in \mathbb{N}\), the polytopes \(Q\) (with some fixed set of possibly artificial vertices) and \(g \cdot Q\) (with the correponding set of vertices) induce the same Minkowski scheme \(\mathcal{M}\).

2. Let \(Q_1 \subseteq Q_2\) be the same polytopes with different sets of generalized vertices. Then, \(\mathcal{M}_1\) is a closed subscheme of \(\mathcal{M}_2\). It is defined by identifying the variables associated to those generalized edges of \(Q_2\), that are contained in the same generalized edge of \(Q_1\).

Conjecture: Let \(Q\) be a lattice polytope such that each extremal Minkowski summand of \(Q\) is a lattice polytope, too. Then, using all generalized vertices of \(Q\), the
affine schemes $\mathcal{M}$ and $\mathcal{M}$ are reduced.

In particular, if $Q$ is an arbitrary lattice polytope (with primitive edges), then $\mathcal{M}_Q$ would be embedded in some reduced $\mathcal{M}_{g,Q}$. The non-reduced structure of $\mathcal{M}_Q$ would arise as a germ of components visible in $\mathcal{M}_{g,Q}$ only.

4 The tautological cone over $C(Q)$

(4.1) In (2.2) we have introduced the cone $C(Q)$ of Minkowski summands of $\mathbb{R}_{\geq 0} \cdot Q$. For an element $(t_1,\ldots,t_N) \in C(Q)$ the corresponding summand $Q_t$ was built by the edges $t_i \cdot d^i$ ($i = 1,\ldots,N$). However, defining $Q_t$ as a particular polytope inside its translation class requires a closer look:

Assume that $0 \in \mathbb{R}^n$ coincides with some vertex of the lattice polytope $Q$. Then, each vertex $a$ of $Q$ can be reached from there by some walk along the edges of $Q$ - we obtain

$$a = \sum_{i=1}^N \lambda_i d^i \text{ for some } \lambda = (\lambda_1,\ldots,\lambda_N), \lambda_i \in \mathbb{Z}.$$  

Now, given an element $\underline{t} \in C(Q)$, we can define the corresponding vertex $a_{\underline{t}}$ (and finally the polytope $Q_{\underline{t}}$ as the convex hull of all of them) by

$$a_{\underline{t}} := \sum_{i=1}^N t_i \lambda_i d^i.$$  

(The linear equations defining $V = \text{span} C(Q)$ ensure that this definition does not depend on the particular path from 0 to $a$ through the 1-skeleton of $Q$.)

(4.2) Definition: The tautological cone $\tilde{C}(Q) \subseteq \mathbb{R}^n \times V \subseteq \mathbb{R}^{n+N}$ is defined as

$$\tilde{C}(Q) := \{(a, \underline{t}) | \underline{t} \in C(Q); a \in Q_{\underline{t}}\}.$$  

Remark: $\tilde{C}(Q)$ is (as $C(Q)$) a rational, polyhedral cone. It is generated by the pairs $(a_{\underline{t}}, \underline{t})$ with

- $a^i$ is a vertex of $Q$ and
- $\underline{t}^j$ is a fundamental generator of $C(Q)$.

(This follows from the simple rule $(a_{\underline{t}+\underline{t}'} + \underline{t} + \underline{t}') = (a_{\underline{t}}, \underline{t}) + (a_{\underline{t}'} + \underline{t}')$ for a vertex $a \in Q$ and $\underline{t}, \underline{t}' \in C(Q)$.)
Defining \( \sigma := \text{Cone}(Q) \subseteq \mathbb{R}^{n+1} \) by putting \( Q \) into the hyperplane \((t = 1)\), we obtain a fiber product diagram of rational polyhedral cones:

\[
\begin{array}{c}
[\sigma \subseteq \mathbb{R}^{n+1}] \\
\downarrow \text{pr}_{n+1} \\
\mathbb{R}_{\geq 0} \\
\end{array}
\begin{array}{c}
[\tilde{C}(Q) \subseteq \mathbb{R}^{n} \times V] \\
\downarrow \text{pr}_{V} \\
[C(Q) \subseteq V] \\
\end{array}
\]

(The vertical maps are projections onto the \((n+1)\)-th and the \(V\)-component, respectively. The inclusion \(i\) is given by \((t \cdot a; t) \mapsto (t \cdot a; t, \ldots, t)\).)

(4.3) The three cones \( \sigma = \text{Cone}(Q) \subseteq \mathbb{R}^{n+1} \), \( \tilde{C}(Q) \subseteq \mathbb{R}^{n} \times V \), and \( C(Q) \subseteq V \) define affine toric varieties called \( Y, X, \) and \( S \), respectively. The corresponding rings of regular functions are \( A(Y) = \mathcal{O}[\sigma^\vee \cap \mathbb{Z}^{n+1}] \), \( A(X) = \mathcal{O}[\tilde{C}(Q)^\vee \cap (\mathbb{Z}^{n} \times V^*_\mathbb{Z})] \), and \( A(S) = \mathcal{O}[C(Q)^\vee \cap V^*_\mathbb{Z}] \). These varieties come with the following maps:

(i) The diagram of (4.2) induces a fiber product diagram

\[
\begin{array}{c}
Y \xleftarrow{i} X \\
\downarrow \downarrow \pi \\
\mathcal{O} \xleftarrow{p} S \\
\end{array}
\]

Both horizontal maps are closed embeddings. (These claims will be checked in (4.5) and (4.8)(1).)

(ii) \( C(Q) = V \cap \mathbb{R}^{n}_{\geq 0} \) is contained in \( \mathbb{R}^{n}_{\geq 0} \), and the inclusion provides a morphism \( p : S \to \mathcal{O}^N \) defining functions \( t_1, \ldots, t_N \) on \( S \). The composition \( \mathcal{O} \hookrightarrow S \xrightarrow{p} \mathcal{O}^N \) sends \( t \) to \((t, \ldots, t)\).

Remark: \( Y \) is the affine toric Gorenstein singularity corresponding to the lattice polytope \( Q \). We will use the map \( \pi : X \to S \) to construct the versal deformation of \( Y \).

(4.4) To study the toric varieties \( Y, X, \) and \( S \) it is important to understand the dual cones of \( \sigma, \tilde{C}(Q), \) and \( C(Q) \), respectively. Let us start with the dual cone of \( \sigma \):

To each non-trivial \( c \in \mathbb{Z}^n \) we associate a vertex \( a(c) \) of \( Q \) and an integer \( \eta_0(c) \) meeting the properties

\[
\langle Q, -c \rangle \leq \eta_0(c) \quad \text{and} \quad \langle a(c), -c \rangle = \eta_0(c).
\]

For \( c = 0 \) we define \( a(0) := 0 \in \mathbb{R}^n \) and \( \eta_0(0) := 0 \in \mathbb{Z} \).
Remark:

(1) With respect to $Q$, $c \neq 0$ is the inner normal vector of the affine supporting hyperplane $\langle \bullet, -c \rangle = \eta_0(c)$ through $a(c)$. In particular, $\eta_0(c)$ is uniquely determined, while $a(c)$ is not.

(2) Since $0 \in Q$, the integers $\eta_0(c)$ are non-negative.

The dual cone of $\sigma$ is defined as

$$\sigma^\vee := \{ r \in \mathbb{R}^{n+1} \mid \langle \sigma, r \rangle \geq 0 \}.$$  

By the definition of $\eta_0$, we have

$$\partial \sigma^\vee \cap \mathbb{Z}^{n+1} = \{ [c, \eta_0(c)] \mid c \in \mathbb{Z}^n \}.$$  

Moreover, if $c^1, \ldots, c^w \in \mathbb{Z}^n \setminus 0$ are those elements producing irreducible pairs $[c, \eta_0(c)]$ (i.e. not allowing any non-trivial lattice decomposition $[c, \eta_0(c)] = [c', \eta_0(c')] + [c'', \eta_0(c'')]$), then the elements

$$[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [0, 1]$$

form the minimal generator set for $\sigma^\vee \cap \mathbb{Z}^{n+1}$ as a semigroup. Among them are all pairs $[c, \eta_0(c)]$ corresponding to facets (i.e. top dimensional faces) of $Q$.

We obtain a closed embedding $Y \hookrightarrow \mathcal{A}^{w+1}$. The coordinate functions of $\mathcal{A}^{w+1}$ will be denoted by $z_1, \ldots, z_w, t$ corresponding to $[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [0, 1]$, respectively.

Example: We continue our example $Q_6$ from Section 4. Here, the facets of $Q_6$ equal its edges $d^1, \ldots, d^6$, and they are sufficient for producing all irreducible pairs $[c^1, \eta_0(c^1)], \ldots, [c^6, \eta_0(c^6)]$. We have

$$c^1 = [0, 1], \quad c^2 = [-1, 1], \quad c^3 = [-1, 0],$$
$$c^4 = [0, -1], \quad c^5 = [1, -1], \quad c^6 = [1, 0].$$

The corresponding vertices are (for instance)

$$a(c^6) = a(c^1) = (0, 0), \quad a(c^2) = a(c^3) = (2, 1), \quad a(c^4) = a(c^5) = (1, 2),$$

and we obtain

$$\eta_0(c^1) = 0, \quad \eta_0(c^2) = 1, \quad \eta_0(c^3) = 2, \quad \eta_0(c^4) = 2, \quad \eta_0(c^5) = 1, \quad \eta_0(c^6) = 0.$$
Thinking of $C(Q)$ as a cone in $\mathbb{R}^N$ instead of $V$ allows dualizing the equation $C(Q) = \mathbb{R}^N_{\geq 0} \cap V$ to get $C(Q)^\vee = \mathbb{R}^N_{\geq 0} + V^\perp$. Hence, for $C(Q)$ as a cone in $V$ we obtain

$$
C(Q)^\vee = \mathbb{R}^N_{\geq 0} + V^\perp / V^\perp = \text{Im} [\mathbb{R}^N_{\geq 0} \to V^*].
$$

As already happened with $\mathbb{R}^n$, we do not use different notations for $\mathbb{R}^N$ and its dual space. However, writing down vectors we try to use paranthesis and brackets for primal and dual ones, respectively.

The surjection $\mathbb{R}^N_{\geq 0} \to C(Q)^\vee$ induces a map $\mathbb{N}^N \to C(Q)^\vee \cap V^*_Z$, which does not need to be surjective at all. This leads to the following definition:

**Definition:** On $V^*_Z$ we introduce a partial ordering “$\succeq$” by

$$
\eta \succeq \eta' \iff \eta - \eta' \in \text{Im} [\mathbb{N}^N \to V^*_Z] \subseteq C(Q)^\vee \cap V^*_Z.
$$

On the geometric level, the non-saturated semigroup $\text{Im} [\mathbb{N}^N \to V^*_Z] \subseteq C(Q)^\vee \cap V^*_Z$ corresponds to the scheme theoretical image $\bar{S}$ of $p : S \to \mathcal{G}^N$, and $S \to \bar{S}$ is its normalization (cf. (5.2)).

The equations of $\bar{S} \subseteq \mathcal{G}^N$ are collected in the kernel of

$$
\mathcal{G}^N_{\{t_1, \ldots, t_N\}} = \mathcal{G}[\mathbb{N}^N] \xrightarrow{\varphi} \mathcal{G}[C(Q)^\vee \cap V^*_Z] \subseteq \mathcal{G}[V^*_Z],
$$

and it is easy to see that

$$
\text{Ker } \varphi = \left( \prod_{i=1}^N t_i^{d_i} - \prod_{i=1}^N t_i^{d'_i} \bigg| d \in \mathbb{Z}^N \cap V^\perp \right) \quad \text{with}
$$

$$
V^\perp = \text{span} \left\{ \left[ \langle \varepsilon_1 d^1, c \rangle, \ldots, \langle \varepsilon_N d^N, c \rangle \right] \bigg| \varepsilon < Q \text{ is a 2-face, } c \in \mathbb{R}^n \right\}.
$$

**Remark:** Using our new notations, we can reformulate Theorem (2.4) now: $\mathcal{M} \subseteq \mathcal{G}^N$ is the largest closed subscheme that is contained in $\bar{S}$ and, additionally, comes from $\mathcal{G}^N \setminus \mathcal{G} \cdot q(Q)$ via $\ell$.

On the other hand, dualizing the embedding $\mathbb{R}_{\geq 0} \hookrightarrow C(Q)$ yields

$$
C(Q)^\vee \cap V^*_Z \xrightarrow{\eta} \mathcal{G}[\mathbb{N}] \xrightarrow{\varphi} \mathcal{G}^N \implies \sum_i \eta_i
$$

at the level of semigroups. This map is surjective, even after restricting to the subset $\text{Im} [\mathbb{N}^N \to V^*_Z]$. All vectors $e_i$ corresponding to the functions $t_i$ map onto $1 \in \mathbb{N}$.

Geometrically this means that both maps $\mathcal{G} \to S$ and $\mathcal{G} \to \bar{S}$ are closed embeddings, and the corresponding ideals are $\left( x_\eta - x_{\eta'} \bigg| \eta, \eta' \in C(Q) \cap V^*_Z \text{ with } \sum_i \eta_i = \sum_i \eta'_i \right)$.
and \((t_i - t_j \mid 1 \leq i, j \leq N)\), respectively. In particular, we got a first contribution to
proof the claims made in (4.3)(i).

\[(4.6)\]

In the next two sections we take a closer look at the dualized cone \(\tilde{C}(Q)^\vee\).

**Definition:** For \(c \in \mathbb{Z}^n\) let \(\lambda^c = (\lambda_1^c, \ldots, \lambda_N^c) \in \mathbb{Z}^N\) describe some path from \(0 \in Q\) to \(a(c) \in Q\) through the 1-skeleton of \(Q\) (cf. (4.1)). Then,

\[\eta(c) := \left[ -\lambda_1^c(d^1, c), \ldots, -\lambda_N^c(d^N, c) \right] \in \mathbb{Z}^N\]

defines an element \(\eta(c) \in V^*_\mathbb{Z}\) not depending on the choice of the particular path \(\lambda^c\).

(Let \(\tilde{\lambda}^c\) be a different path from 0 to \(a(c)\) - it will differ from \(\lambda^c\) by some linear combination \(\sum_{\varepsilon < Q} g_{\varepsilon} \varepsilon\) \((g_{\varepsilon} \in \mathbb{Z}\) for 2-faces \(\varepsilon < Q)\) only. In particular,

\[\tilde{\lambda}_i^c(d^i, c) - \lambda_i^c(d^i, c) = \sum_{\varepsilon < Q} g_{\varepsilon} \varepsilon_i d^i, c\]

and we obtain \(\eta(c) - \eta(c) \in V^\perp\).)

**Lemma:**

(i) \(\eta(0) = 0 \in V^*_\mathbb{Z}\).

(ii) For all \(c \in \mathbb{Z}^n\) we have \(\eta(c) \succeq 0\) (in the sense of Definition (4.5)).

(iii) \(\eta\) is convex: \(\sum_v g_v \eta(c^v) \succeq \eta(\sum_v g_v c^v)\) for natural numbers \(g_v \in \mathbb{N}\).

(iv) \(\sum_{i=1}^N \eta_i(c) = \eta_0(c)\) for arbitrary \(c \in \mathbb{Z}^n\).

**Proof:**

(ii) \(a(c)\) is a vertex of \(Q\) providing minimal value of the linear function \(\langle \bullet, c \rangle\). In particular, we can choose a path \(\lambda^c\) from \(0 \in Q\) to \(a(c)\) such that this function decreases in each step, i.e. \(\lambda_i^c(d^i, c) \leq 0\) \((i = 1, \ldots, N)\).

(iii) We define the following paths through the 1-skeleton of \(Q\):

- \(\lambda := \text{path from } 0 \in Q \text{ to } a(\sum_v g_v c^v) \in Q\),
- \(\mu^v := \text{path from } a(\sum_v g_v c^v) \in Q \text{ to } a(c^v) \in Q\) such that \(\mu_i^v(d^i, c^v) \leq 0\) for each \(i = 1, \ldots, N\).

Then, \(\lambda^v := \lambda + \mu^v\) is a path from \(0 \in Q\) to \(a(c^v)\), and for \(i = 1, \ldots, N\) we obtain

\[
\sum_v g_v \eta_i(c^v) - \eta_i \left( \sum_v g_v c^v \right) = - \sum_v g_v (\lambda_i + \mu_i^v) \langle d^i, c^v \rangle + \lambda_i \left( d^i, \sum_v g_v c^v \right) \geq 0.
\]
(iv) By definition of $\Delta^c$ we have $\sum_{i=1}^{N} \lambda_i^c d_i^s = a(c)$. In particular,

$$\sum_{i=1}^{N} \eta_i(c) = - \sum_{i=1}^{N} \langle \lambda_i^c d_i^s, c \rangle = -\langle a(c), c \rangle = \eta_0(c).$$

\[\square\]

**Example:** In our hexagon $Q_6$ we choose the following paths from $(0,0)$ to the vertices $a(c^1), \ldots, a(c^6)$, respectively:

$$\Delta^6 = \Delta^1 := 0, \quad \Delta^2 = \Delta^3 := [1, 1, 0, 0, 0], \quad \Delta^4 = \Delta^5 := [1, 1, 1, 0, 0].$$

They provide

$$\eta(c^1) = [0, 0, 0, 0, 0, 0], \quad \eta(c^2) = [1, 0, 0, 0, 0, 0], \quad \eta(c^3) = [1, 1, 0, 0, 0, 0], \quad \eta(c^4) = [1, 1, 0, 0, 0, 0], \quad \eta(c^5) = [0, 0, 0, 0, 0, 0].$$

Since $[1, 0, -1, -1, 0, 1] = [d^1, [1, -1]], \ldots, [d^6, [1, -1]] \in V^\perp$, the vector $\eta(c^5)$ can be transformed into $[0, 0, 0, 0, 0, 1]$.

**Remark:** The definitions of $a(c), \eta_0(c)$, and $\eta(c)$ also make sense for general $c \in R^n$. Then, $\eta_0(c) \in R$ and $\eta(c) \in V^*$ do not need to be contained in the lattices anymore. The previous lemma will keep valid (even for $g_v \in R_{\geq 0}$ in (iii)), if the relation $"\geq 0"$ is replaced by the weaker version $"\in C(Q)^\vee\ast"$.

**(4.7) Proposition:**

1. $\tilde{C}(Q)^\vee = \{ [c, \eta] \in R^n \times V^* \mid \eta - \eta(c) \in C(Q)^\vee \}$

2. In particular, $[c, \eta(c)] \in \tilde{C}(Q)^\vee$, and moreover, it is the only preimage of $[c, \eta_0(c)] \in \sigma^\vee$ via the surjection $\tilde{i} : \tilde{C}(Q)^\vee \longrightarrow \sigma^\vee$.

3. $[c^1, \eta(c^1)], \ldots, [c^w, \eta(c^w)]$ and $C(Q)^\vee \cap V^*_Z$ (embedded as $[0, C(Q)^\vee]$) generate the semigroup $\tilde{C}(Q)^\vee \cap (Z^n \times V^*_Z)$. (For recalling the definition of the $c^1, \ldots, c^w$, cf. \[\square\]).

**Proof:** (1) Let $[c, \eta] \in R^n \times V^*$ be given; if some representative of $\eta$ in $R^N$ is needed, then it will be denoted by the same name. We have the following equivalences:

$$[c, \eta] \in \tilde{C}(Q)^\vee \iff \langle (Q_{\perp}, c), [c, \eta] \rangle \geq 0 \quad \text{for each } t \in C(Q)$$

$$\iff \langle Q_{\perp}, c \rangle + \langle t, \eta \rangle \geq 0 \quad \text{for each } t \in C(Q)$$

$$\iff \langle a(c), c \rangle + \langle t, \eta \rangle \geq 0 \quad \text{for each } t \in C(Q).$$
Using some path $\lambda^c$ we obtain:

$$[c, \eta] \in \tilde{C}(Q)^\vee \iff \sum_{i=1}^{N} t_i \langle d^i, c \rangle + \langle t, \eta \rangle \geq 0 \quad \text{for each } t \in C(Q)$$

$$\iff \sum_{i=1}^{N} t_i \cdot (\lambda^c_i \langle d^i, c \rangle + \eta_i) \geq 0 \quad \text{for each } t \in C(Q)$$

$$\iff [\lambda^c_1 \langle d^1, c \rangle + \eta_1, \ldots, \lambda^c_N \langle d^N, c \rangle + \eta_N] \in C(Q)^\vee.$$

(2) By part (1) we know that for a $[c, \eta] \in \tilde{C}(Q)^\vee$ it is possible to choose $R^N$-representatives for $\eta, \eta(c)$ such that $\eta_i \geq \eta_i(c)$ for $i = 1, \ldots, N$.

On the other hand, the two equalities $\sum_i \eta_i(c) = \eta_0(c)$ (cf. (iv) of the previous lemma) and $\sum_i \eta_i = \eta_0(c)$ (corresponding to the fact $[c, \eta] \mapsto [c, \eta_0(c)]$) imply $\eta = \eta(c)$ then.

(3) Let $[c, \eta] \in \tilde{C}(Q)^\vee$. Then, $[c, \eta_0(c)]$ is representable as a non-negative linear combination $[c, \eta_0(c)] = \sum_{v=1}^{w} p_v \langle c^v, \eta_0(c^v) \rangle$ ($p_v \in \mathbb{N}$ if $c \in \mathbb{Z}^n$). Since both elements $[c, \eta(c)]$ and $\sum_v p_v \langle c^v, \eta(c^v) \rangle$ are preimages of $[c, \eta_0(c)]$ via $i^v$, they must be equal by (2), and we obtain

$$[c, \eta] = [c, \eta(c)] + [0, \eta - \eta(c)] = \sum_v p_v \langle c^v, \eta(c^v) \rangle + [0, \eta - \eta(c)]. \quad \square$$

(4.8) Finally, we will take a short look at the geometrical situation reached at this point.

(1) The linear map

$$\tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V_\mathbb{Z}^*) \longrightarrow \sigma^\vee \cap \mathbb{Z}^{n+1}$$

$$[c, \eta] \mapsto [c, \sum_i \eta_i]$$

is surjective ($[c, \eta(c)] \mapsto [c, \eta_0(c)]; [0, e_i] \mapsto [0, 1]$). Since

$$x^{[c, \eta]} - x^{[c, \eta']} = x^{[c, \eta(c)]} \cdot (x^{[0, \eta - \eta(c)]} - x^{[0, \eta' - \eta(c)]}),$$

the kernel of the corresponding homomorphism between the semigroup algebras equals the ideal

$$\left( x^{[0, \eta]} - x^{[0, \eta']} \mid \sum_i \eta_i = \sum_i \eta'_i \right).$$

In particular, the map $Y \hookrightarrow X$ is a closed embedding. Moreover, comparing with the similar statement concerning $C(Q)^\vee$ and $\mathcal{N}$ at the end of (4.5), we obtain that the diagram of (4.3)(i) is a fiber product diagram, indeed.
(2) The elements $[c_1, \eta(c_1)], \ldots, [c_w, \eta(c_w)] \in \tilde{C}(Q)$ induce some regular functions $Z_1, \ldots, Z_w$ on $X$. They define a closed embedding $X \hookrightarrow \mathcal{O}^w \times S$ lifting the embedding $Y \hookrightarrow \mathcal{O}^{w+1}$ of (4.4). Moreover, for $i = 1, \ldots, N$, $Z_i$ is the only monomial function lifting $z_i$ from $Y$ to $X$.

We have obtained the following commutative diagram:

$$
\begin{array}{ccc}
Y & \hookrightarrow & \mathcal{O}^w \times \mathcal{O} \\
\downarrow \otimes & & \downarrow \Delta \\
X & \hookrightarrow & \mathcal{O}^w \times S \\
\downarrow & \overset{p}{\rightarrow} & \downarrow \\
S & \overset{p}{\rightarrow} & \mathcal{O}^N \\
\end{array}
$$

\[ \rightarrow \mathcal{O}^{N-1}. \]

## 5 A flat family over $\tilde{M}$

(5.1) **Theorem:** Denote by $\tilde{X}$ and $\tilde{S}$ the scheme theoretical images of $X$ and $S$ in $\mathcal{O}^w \times \mathcal{O}^N$ and $\mathcal{O}^N$, respectively. Then,

1. $X \rightarrow \tilde{X}$ and $S \rightarrow \tilde{S}$ are the normalization maps.
2. $\pi : X \rightarrow S$ induces a map $\tilde{\pi} : \tilde{X} \rightarrow \tilde{S}$, and $\pi$ can be recovered from $\tilde{\pi}$ via base change $S \rightarrow \tilde{S}$.
3. Restricting to $M \subseteq \tilde{S}$ and composing with $\ell$ turns $\tilde{\pi}$ into a family

$$
\tilde{X} \times_S M \overset{\tilde{\pi}}{\longrightarrow} M \overset{\ell}{\longrightarrow} \tilde{M}.
$$

It is flat in $0 \in \tilde{M} \subseteq \mathcal{O}^{N-1}$, and the special fiber equals $Y$.

The proof of this theorem will fill §5.

(5.2) The ring of regular functions $A(\tilde{S})$ is given as the image of the map $\mathcal{O}[t_1, \ldots, t_N] \rightarrow A(S)$. Since $\mathbb{Z}^N \twoheadrightarrow V_\mathbb{Z}^+$ is surjective, the rings $A(\tilde{S}) \subseteq A(S) \subseteq \mathcal{O}[V_\mathbb{Z}^+]$ have the same field of fractions.

On the other hand, while $t$-monomials with negative exponents are involved in $A(S)$, the surjectivity of $\mathbb{R}^+_0 \twoheadrightarrow C(Q)^c$ tells us that sufficiently high powers of those monomials always come from $A(\tilde{S})$. In particular, $A(S)$ is normal over $A(\tilde{S})$.

$A(\tilde{X})$ is given as the image $A(\tilde{X}) = \text{Im}(\mathcal{O}[Z_1, \ldots, Z_w, t_1, \ldots, t_N] \rightarrow A(X))$. Since $A(X)$ is generated by $Z_1, \ldots, Z_w$ over its subring $A(S)$ (cf. Proposition (4.4)(3)), the same arguments as for $S$ and $\tilde{S}$ apply. Hence, Part (1) of the previous theorem
is proved.

(5.3) Recalling that \( z_1, \ldots, z_w, t \in A(Y) \) stand for the monomials with exponents \([c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], \) \([0,1] \in \mathcal{C}(Q)^\vee \cap V_{x,X}^* \), respectively, we obtain the following equations defining \( Y \subseteq \mathcal{A}^{w+1} \):

\[
     f_{(a,b,a,\beta)}(\mathbf{z}, t) := t^\alpha \prod_{v=1}^{w} z_v^{a_v} - t^\beta \prod_{w=1}^{w} z_v^{b_v}
\]

with \( a, b \in \mathbb{N}^w : \sum_v a_v c^v = \sum_v b_v c^v \) and

\[
     \alpha, \beta \in \mathbb{N} : \sum_v a_v \eta_0(c^v) + \alpha = \sum_v b_v \eta_0(c^v) + \beta.
\]

Example: The singularity \( Y_6 \) induced by the hexagon \( Q_6 \) equals the cone over the Del Pezzo surface of degree 6 (obtained by blowing up three points of \((P^2, O(3))\)). As a closed subset of \( \mathcal{A}^7 \), it is given by the following 9 equations:

\[
     f_{(e_1,e_2+e_2,1,0)} = z_1 - z_6 z_2, \quad f_{(e_2,e_2+e_3,1,0)} = z_2 - z_1 z_3, \quad f_{(e_3,e_2+e_4,1,0)} = z_3 - z_2 z_4,
\]

\[
     f_{(e_4,e_3+e_5,1,0)} = z_4 - z_3 z_5, \quad f_{(e_5,e_4+e_6,1,0)} = z_5 - z_4 z_6, \quad f_{(e_6,e_5+e_6,1,0)} = z_6 - z_5 z_1, \quad f_{(e_7,e_6+e_2,1,0)} = t_2 - z_1 z_4, \quad f_{(e_8,e_2+e_5,1,0)} = t_2 - z_2 z_5, \quad f_{(e_9,e_6+e_2,1,0)} = t_2 - z_3 z_6.
\]

(5.4) Defining \( c := \sum_v a_v c^v = \sum_v b_v c^v \) we can lift the equations of \( Y \) to the following elements of \( \mathcal{A}[Z_1, \ldots, Z_w, t_1, \ldots, t_N] \to A(\hat{S})[Z_1, \ldots, Z_w] \):

\[
     F_{(a,b,a,\beta)}(\mathbf{z}, \mathbf{t}) := f_{(a,b,a,\beta)}(\mathbf{z}, t_1) - Z^{[c, \eta(c)]} \cdot \left( t^{\alpha_1 + \sum_v a_v \eta(c^v)} - t^{\beta_1 + \sum_v b_v \eta(c^v)} \right) \cdot t^{-\eta(c)}.
\]

Remark:

1. The symbol \( Z^{[c, \eta(c)]} \) means \( \prod_{v=1}^{w} Z_v^{p_v} \) with natural numbers \( p_v \in \mathbb{N} \) such that \([c, \eta(c)] = \sum_v p_v [c^v, \eta(c^v)]\) or equivalently \([c, \eta_0(c)] = \sum_v p_v [c^v, \eta_0(c^v)]\). This condition does not determine the coefficients \( p_v \) uniquely - choose one of the possibilities.

2. By part (iii) of Lemma (5.3), we have \( \sum_v a_v \eta(c^v) \), \( \sum_v b_v \eta(c^v) \geq \eta(c) \). In particular, representatives of the \( \eta \)'s can be chosen such that all \( t \)-exponents occurring in monomials of \( F \) are non-negative, i.e. \( F \in A(\hat{S})[Z_1, \ldots, Z_w] \).

3. Mapping \( F \) to \( A(X) = \oplus_{[c, \eta]} \mathcal{A}[x^{[c, \eta]}] \) runs through all elements of \( \tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V_{x,X}^*) \); \( Z_v \mapsto x^{[c_v, \eta(c_v)]} \), \( t_i \mapsto x^{[0,\eta_i]} \) yields

\[
     F_{(a,b,a,\beta)} = \left( x^{\alpha_1} \prod_{v=1}^{w} Z_v^{a_v} - Z^{[c, \eta(c)]} \prod_{v=1}^{w} Z_v^{a_v} \eta(c^v) - \eta(0) \right) - \left( x^{\beta_1} \prod_{v=1}^{w} Z_v^{b_v} - Z^{[c, \eta(c)]} \prod_{v=1}^{w} Z_v^{b_v} \eta(c^v) - \eta(0) \right)
\]

\[
     \mapsto \left( x^{\alpha_1} - \prod_{v=1}^{w} Z_v^{a_v} \right) - x^{[c, \eta(c)]+\alpha[0,\eta]} \prod_{v=1}^{w} Z_v^{a_v} \eta(c^v) - \eta(0) \right) - \left( x^{\beta_1} - \prod_{v=1}^{w} Z_v^{b_v} \right) - x^{[c, \eta(c)]+\beta[0,\eta]} \prod_{v=1}^{w} Z_v^{b_v} \eta(c^v) - \eta(0) \right) = 0 - 0 = 0,
\]

19
i.e. the polynomials $F_{(a,b,a,\beta)}$ are equations for $\bar{X} \subseteq \mathcal{A}^w \times \bar{S}$.

**Example:** In the hexagon example, we obtain the following liftings:

\[
\begin{align*}
F_{(e_1,e_6+e_2,1,0)} &= (Z_1 t_1 - Z_6 Z_2) - Z_1 (t_1 - t_1) = Z_1 t_1 - Z_6 Z_2, \\
F_{(e_2,e_1+e_3,1,0)} &= (Z_2 t_1 - Z_1 Z_3) - Z_2 (t_1^2 - t_1 t_2) t_1^{-1} = Z_2 t_2 - Z_1 Z_3, \\
F_{(e_3,e_2+e_4,1,0)} &= (Z_3 t_1 - Z_2 Z_4) - Z_3 (t_2^2 - t_1 t_2) t_1^{-1} t_1^{-1} = Z_3 t_3 - Z_2 Z_4, \\
F_{(e_4,e_3+e_5,1,0)} &= (Z_4 t_1 - Z_3 Z_5) - Z_4 (t_2 t_3 - t_2 t_3) t_2^{-1} t_2^{-1} = Z_4 t_4 - Z_3 Z_5, \\
F_{(e_5,e_4+e_6,1,0)} &= (Z_5 t_1 - Z_4 Z_6) - Z_5 (t_1 t_6 - t_2 t_3) t_6^{-1} = Z_5 t_5 - Z_4 Z_6, \\
F_{(e_6,e_5+e_1,1,0)} &= (Z_6 t_1 - Z_5 Z_1) - Z_6 (t_1 - t_6) = Z_6 t_6 - Z_5 Z_1, \\
F_{(e_0,e_1+e_4,2,0)} &= (t_1^2 - Z_1 Z_4) - (t_1^2 - t_2 t_3) = t_2 t_3 - Z_1 Z_4 = t_5 t_6 - Z_1 Z_4, \\
F_{(e_0,e_2+e_5,2,0)} &= (t_1^2 - Z_2 Z_5) - (t_1^2 - t_3 t_4) = t_3 t_4 - Z_2 Z_5, \\
F_{(e_0,e_3+e_6,2,0)} &= (t_1^2 - Z_3 Z_6) - (t_1^2 - t_1 t_2) = t_1 t_2 - Z_3 Z_6.
\end{align*}
\]

(5.5) To obtain a complete list of equations defining $\bar{X} \subseteq \mathcal{A}^w \times \bar{S}$, we have to regard the kernel of the homomorphism $A(\bar{S})[Z_1,\ldots,Z_w] \rightarrow A(\bar{X}) \subseteq A(X)$. It is generated by the binomials

\[t^a Z_1^a \cdots Z_w^a - t^b Z_1^b \cdots Z_w^b\] such that

\[
\sum_v a_v [c^v,\eta(c^v)] + [0,\mu] = \sum_v b_v [c^v,\eta(c^v)] + [0,\mu],
\]

i.e. \(c := \sum_v a_v c^v = \sum_v b_v c^v\)

\[
\sum_v a_v \eta(c^v) + \eta = \sum_v b_v \eta(c^v) + \mu.
\]

However,

\[
t^a Z^a - t^b Z^b = t^a \cdot (\Pi_v Z_v^a - Z_v^a) - t^a \cdot \left(\sum_v a_v \eta(c^v) - \eta(c)\right)
\]

\[
= t^a \cdot F_{(a,p,0,\alpha)} - t^b \cdot F_{(b,p,0,\beta)}
\]

with \(p \in \mathbb{N}^w\) such that \(\sum_v p_v [c^v,\eta(c^v)] = [c,\eta(c)]\), \(\alpha = \sum_v a_v \eta_0(c^v) - \eta_0(c)\), and \(\beta = \sum_v b_v \eta_0(c^v) - \eta_0(c)\).

In particular, \(\text{Ker}(A(\bar{S})[Z] \rightarrow A(X))\) is generated by the polynomials $F_{(a,b,a,\beta)}$ introduced in (3.3).

**Remark:**

1. The inaccuracy caused by writing $Z_v^{[c,\eta(c)]}$ for some undetermined $Z_1^{p_1} \cdots Z_w^{p_w}$ (with $\sum_v p_v [c^v,\eta(c^v)] = [c,\eta(c)]$) does not matter: Choosing other coefficients $q_v$ with the same property yields

\[
Z_1^{q_1} \cdots Z_w^{q_w} - Z_1^{q_1} \cdots Z_w^{q_w} = F_{(p,q,0,0)}(Z,t) = f_{(p,q,0,0)}(Z,t).
\]
There are three types of relations between the functions $\bar{f}(t, \bar{\eta})$:

1. $\bar{f}(r, b, \alpha, \beta) + \bar{f}(r, b, \alpha, \gamma) = \bar{f}(a, b, \alpha)$
   with $\sum_v a_v c^v = \sum_v r_v c^v = \sum b_v c^v$

2. $\sum_v a_v \eta_0(c^v) + \alpha = \sum_v r_v \eta_0(c^v) + \gamma = \sum b_v \eta_0(c^v) + \beta$.
   For this relation, the same equation between the $F$'s is true.

3. $t \cdot \bar{f}(a, b, \alpha, \beta) = \bar{f}(a+b+r, \alpha, \beta+1)$
   lifts to $t \cdot F(a, b, \alpha, \beta) = F(a+b+r, \alpha, \beta+1)$.

4. $z^r \cdot \bar{f}(a, b, \alpha, \beta) = \bar{f}(a+r, b+r, \alpha, \beta)$.
   With $c := \sum_v a_v c^v = \sum b_v c^v$, $\bar{c} := c + \sum r_v c^v$ we obtain

$$Z^r \cdot F(a, b, \alpha, \beta) - F(a+r, b+r, \alpha, \beta) =$$

$$= Z^{\bar{c}, \bar{\eta}(\bar{c})} \cdot \left( t^{a+1} + \sum_v a_v \eta(c^v) + \sum_v r_v \eta(c^v) - t^{\beta+1} + \sum_v b_v \eta(c^v) + \sum_v r_v \eta(c^v) \right) - \sum \eta(c^v) - t^{\bar{c}, \eta(\bar{c})} \cdot \left( t^{a+1} + \sum v a_v \eta(c^v) - t^{\beta+1} + \sum v b_v \eta(c^v) \right) \cdot t^{-\eta(c)}$$

$$= \left( t^{a+1} + \sum_v a_v \eta(c^v) - \eta(c) - t^{\beta+1} + \sum v b_v \eta(c^v) - \eta(c) \right) \cdot \left( u^{c} + \sum v r_v \eta(c^v) - \eta(\bar{c}) \right) \cdot Z^{\bar{c}, \eta(\bar{c})} - Z^{c, \eta(c)} Z^r \right).$$

Now, the inequalities

$$\sum_v a_v \eta(c^v), \sum v b_v \eta(c^v) \geq \eta(c) \quad \text{and} \quad \eta(c) + \sum v r_v \eta(c^v) - \eta(\bar{c}) \geq 0$$

imply that

- the first factor is contained in the ideal defining $0 \in \mathcal{M}$, and
- the second factor is an equation of $\bar{X} \subseteq \mathcal{C}^w \times \bar{S}$ (called $F_{(q,p+r, \xi,0)}$ in (5.4)).
In particular, we have found a lift for the third relation, too.

The proof of Theorem (5.1) is complete.

(5.7) **Example:** For $Y_6$, the previously constructed family is contained in $\mathcal{A}^6 \times \mathcal{A}^6 \xrightarrow{\text{pr}_i} \mathcal{A}^6 / \mathcal{A} \cdot (1, \ldots, 1)$. Its base space is defined by the 4 equations mentioned at the end of (2.3), and for the total space, the 9 equations of (5.4) have to be added.

### 6 The Kodaira-Spencer map

(6.1) Denote by $E \subseteq \sigma^\vee \cap \mathbb{Z}^{n+1}$ the minimal generating set

$$E := \{[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [0, 1]\}$$

mentioned in (3.3). To each vertex $a^j \in Q$ (or equally named fundamental generator $a^j := (a^j, 1) \in \sigma$) and each element $R \in \mathbb{Z}^{n+1}$ we associate the subset

$$E^R_j := E^R_{a^j} := \{r \in E \mid \langle a^j, r \rangle < \langle a^j, R \rangle\}.$$

**Theorem:** (cf. [Al 1]) The vector space $T^1_Y$ of infinitesimal deformations of $Y$ is $\mathbb{Z}^{n+1}$-graded, and in degree $-R$ it equals

$$T^1_Y(-R) = \left( L_\sigma \left( \bigcup_j E^R_j \right) / \sum_j L_\sigma(E^R_j) \right)^*$$

($L(\ldots)$ denotes the vector space of linear relations).

(6.2) There is a special degree $R^* = [0, 1] \in \mathbb{Z}^{n+1}$ corresponding to the affine hyperplane containing $Q$. The associated subsets of $E$ equal

$$E^R_{a^j} = E \cap (a^j)_{\perp} = \{[c^v, \eta_0(c^v)] \mid \langle a^j, -c^v \rangle = \eta_0(c^v)\}.$$

In (3.3), for each $c \in \mathbb{Z}^n$, we have defined the linear form $\eta(c) \in V^*_\mathbb{Z}$. Restricted to the cone $C(Q)$, it maps $\underline{t}$ to $\text{Max}(Q_{\underline{t}} - c) = \langle a(c)_{\underline{t}}, -c \rangle$. This induces the following bilinear map:

$$\Phi : V_{\mathbb{Z}}(1, \ldots, 1) \times L_{\mathbb{Z}}(E \cap \partial \sigma^\vee) \rightarrow \mathbb{Z}$$

$$\underline{t} \quad , \quad q \quad \mapsto \quad \sum_{v,i} t_i q_v \eta_i(c^v).$$
we obtain \( \sum_v q_v \eta_i(c^v) = \sum_v q_v \eta_0(c^v) = 0 \) since \( q \in L_{\mathbb{Z}}(E \cap \partial \sigma^\vee) \).

Moreover, if \( q \) comes from one of the submodules \( L_{\mathbb{Z}}(E_{j}^{R^*}) \subseteq L_{\mathbb{Z}}(E \cap \partial \sigma^\vee) \), we obtain

\[
\Phi(q, t) = \sum_v q_v \cdot \text{Max}(Q_v - c^v) = \sum_v q_v \cdot \langle a^v_i, -c^v \rangle = \langle a^v_i, -\sum_v q_v c^v \rangle = 0.
\]

**Theorem:** The Kodaira-Spencer map of the family \( \tilde{X} \times_S \mathcal{M} \to \tilde{\mathcal{M}} \) equals the map

\[
T_0 \tilde{\mathcal{M}} = V\mathcal{E}/\langle 1, \ldots, 1 \rangle \to \left( L_{\mathcal{E}}(E \cap \partial \sigma^\vee)/\sum_j L_{\mathcal{E}}(E_{j}^{R^*}) \right)^* = T_Y^1(-R^*)
\]

induced by the previous pairing. Moreover, this map is an isomorphism.

**Proof:** Using the same symbol \( \mathcal{J} \) for the ideal \( \mathcal{J} \subseteq \mathcal{A}[t_1, \ldots, t_N] \) and the intersection \( \mathcal{J} \cap \mathcal{A}[t_i - t_j | 1 \leq i, j \leq N] \) (cf. (2.4)), our family corresponds to the flat \( \mathcal{A}[t_i - t_j] / \mathcal{J} \)-module \( \mathcal{A}[\mathbb{Z}, \mathcal{L}] / (\mathcal{J}, F_\bullet(\mathbb{Z}, \mathcal{L})) \).

Now, we fix a non-trivial tangent vector \( \mathcal{L}^0 \in V\mathcal{E} \). Via \( t_i \mapsto t + t^0_i \varepsilon \) it induces the infinitesimal family given by the flat \( \mathcal{A}[\varepsilon] / \varepsilon^2 \)-module

\[
A_{\mathcal{E}}^0 := \mathcal{A}[\varepsilon, t, \varepsilon] / (\varepsilon^2, F_\bullet(\mathbb{Z}, t + \mathcal{L}^0)).
\]

To obtain the associated \( A(Y) \)-linear map \( I / I^2 \to A(Y) \) (\( I := (f_\bullet(\mathbb{Z}, t)) \) denotes the ideal of \( Y \) in \( \mathcal{A}^{w+1} \)), we have to compute the images of \( f_\bullet(\mathbb{Z}, t) \) in \( \varepsilon A(Y) \subseteq A_{\mathcal{E}}^0 \) and divide them by \( \varepsilon \):

Using the notations of (2.4) and (2.6), in \( A_{\mathcal{E}}^0 \) it holds

\[
0 = F_{(a, b, \alpha, \beta)}(\mathbb{Z}, t + t^0_i \varepsilon) = f_{(a, b, \alpha, \beta)}(\mathbb{Z}, t + t^0_i \varepsilon) - \varepsilon [\varepsilon^a_1 (\varepsilon^b_1) \cdot t^0 \varepsilon^{a+1} + \varepsilon [\varepsilon^a_1 (\varepsilon^b_1) \cdot \varepsilon^0 (c) - \varepsilon (c) t + t^0_i \varepsilon] t^0].
\]

The relation \( \varepsilon^2 = 0 \) yields

\[
f_{(a, b, \alpha, \beta)}(\mathbb{Z}, t + t^0_i \varepsilon) = f_{(a, b, \alpha, \beta)}(\mathbb{Z}, t) + \varepsilon \cdot (\alpha t^{a-1} t^0_i \varepsilon^a - \beta t^{b-1} t^0_i \varepsilon^b),
\]

and similarly we can expand the other terms. Eventually, we obtain

\[
f_{(a, b, \alpha, \beta)}(\mathbb{Z}, t) = -\varepsilon t^0_i (\alpha t^{a-1} \varepsilon^a - \beta t^{b-1} \varepsilon^b + \varepsilon [\varepsilon^a_1 (\varepsilon^b_1) t^0 + \varepsilon [\varepsilon^a_1 (\varepsilon^b_1) \cdot \varepsilon^0 (c) - \varepsilon (c) t + t^0_i \varepsilon] t^0] t^0_i (\varepsilon^a_1 (\varepsilon^b_1) \cdot \varepsilon^0 (c) - \varepsilon (c) t + t^0_i \varepsilon) + \sum_i t^0_i (\varepsilon^0 (a_v - b_v) \eta_i (c^v) )]
\]

\[= \varepsilon \cdot x \sum a_v [c^v, \eta_i (c^v)] + \alpha - 1, \quad \sum_i t^0_i \left( \sum_v (a_v - b_v) \eta_i (c^v) \right). \]
(In $\varepsilon A(Y)$ we were able to replace the variables $t$ and $z_i$ by $x^{[0, 1]}$ and $x^{[e^\alpha, \eta_0(e^\alpha)]}$, respectively.)

On the other hand, we use Theorem (3.4) of [Al 3]: Fixing $R^* \in \mathbb{Z}^{n+1}$, it is the element of $L_{\mathcal{E}}(E \cap \partial \mathcal{E})^*$ given by $q \mapsto \sum_{i,v} t^0_i q_v \eta_i(c^v)$ that corresponds to the infinitesimal deformation of $T^1_Y(-R^*)$ defined by the map

$$J / I^2 \longrightarrow A(Y)$$

$$t^\alpha z^a - t^\beta z^b \mapsto \left( \sum_{i,v} t^0_i (a_v - b_v) \eta_i(c^v) \right).$$

\[ \square \]

(6.3) To discuss the meaning of the homogeneous part $T^1_Y(-R^*)$ inside the whole vector space $T^1_Y$, we have to look at the results of [Al 2]:

If $\dim T^1_Y < \infty$ (for instance, if $Y$ has an isolated singularity), then

1. $T^1_Y = T^1_Y(-R^*)$, but
2. $T^1_Y = 0$ for $\dim Y \geq 4$.

In particular, the interesting cases arise from 2-dimensional lattice polygons $Q$ with primitive edges only. The corresponding 3-dimensional toric varieties $Y$ have an isolated singularity, and the Kodaira-Spencer map $T_0 \mathcal{M} \rightarrow T^1_Y$ is an isomorphism.

If $T^1_Y$ has infinite dimension, then this comes from the existence of infinitely many non-trivial homogeneous pieces $T^1_Y(-R)$. Whenever $\langle a^j, R \rangle \leq 1$ holds for all vertices $a^j \in Q$, we have

$$T^1_Y(-R) = V_{\mathcal{E}}(\text{conv}\{a^j \mid \langle a^j, R \rangle = 1\}),$$

i.e. $T^1_Y(-R)$ equals the vector space of Minkowski summands of some face of $Q$. ($T^1_Y(-R) = 0$ for all other $R \in \mathbb{Z}^{n+1}$.)

In particular, $T^1_Y(-R^*)$ is a typical, but nevertheless extremal and perhaps the most interesting part of $T^1_Y$.

7 The obstruction map

(7.1) Dealing with obstructions in the deformation theory of $Y$ involves the $A(Y)$-module $T^2_Y$. Usually, it is defined in the following way:

Let $m := \{ ([a, \alpha], [b, \beta]) \in \mathbb{N}^{w+1} \times \mathbb{N}^{w+1} \mid \sum_v a_v c^v = \sum_v b_v c^v; \sum_v a_v \eta_0(c^v) + \alpha = \sum_v b_v \eta_0(c^v) + \beta \}$

24
denote the set parametrizing the equations \( f_{(a,b,\alpha,\beta)} \) generating the ideal \( I \subseteq \mathcal{A}[\underline{z},t] \) of \( Y \). Then,
\[
\mathcal{R} := \ker (\mathcal{A}[\underline{z},t]^m \to I)
\]
is the module of linear relations between these equations; it contains the submodule \( \mathcal{R}_0 \) of the so-called Koszul relations.

**Definition:** \( T^2_Y := \text{Hom} (\mathcal{R}/\mathcal{R}_0, A(Y)) / \text{Hom} (\mathcal{A}[\underline{z},t]^m, A(Y)) \).

Now, we have a similar theorem for \( T^2_Y \) as we had in (6.1) for \( T^1_Y \); in particular, we use the notations introduced there.

**Theorem:** (cf. [Al 3]) The vector space \( T^2_Y \) is \( \mathbb{Z}^{n+1} \)-graded, and in degree \( -R \) it equals
\[
T^2_Y(-R) = \left( \frac{\ker \left( \bigoplus_j L\sigma(E^R_j) \to L\sigma(E) \right)}{\text{Im} \left( \bigoplus_{(a',a)<Q} L\sigma(E^R_{i} \cap E^R_{j}) \to \bigoplus_i L\sigma(E^R_{i}) \right)} \right)^*.
\]

(7.2)

In this section we build up the so-called obstruction map. It detects all possible infinitesimal extensions of our family over \( \overline{M} \) to a flat family over some larger base space. We follow the explanation given in §4 of \([JS]\).

As before,
\[
\mathcal{J} = (g_{e,k}(t - t_1) | e < Q, k \geq 1) = (g_{d,k}(t - t_1) | d \in V^\perp \cap \mathbb{Z}^N, k \geq 1) \subseteq \mathcal{A}[t_i - t_j]
\]
denotes the homogeneous ideal of the base space \( \overline{M} \). Let
\[
\tilde{\mathcal{J}} := (t_i - t_j)_{i,j} \cdot \mathcal{J} + \mathcal{J}_1 \cdot \mathcal{A}[t_i - t_j] \subseteq \mathcal{A}[t_i - t_j | 1 \leq i, j \leq N].
\]

Then, \( W := \tilde{\mathcal{J}} / \tilde{\mathcal{J}} \) is a finitely dimensional, \( \mathbb{Z} \)-graded vector space \((W = \bigoplus_{k \geq 2} W_k, \) and \( W_k \) is generated by the polynomials \( g_{d,k}(t - t_1) \)). It comes as the kernel in the exact sequence
\[
0 \to W \to \mathcal{A}[t_i - t_j] / \tilde{\mathcal{J}} \to \mathcal{A}[t_i - t_j] / \mathcal{J} \to 0.
\]

Identifying \( t \) with \( t_1 \) and \( \underline{z} \) with \( \underline{Z} \), the tensor product with \( \mathcal{A}[\underline{z},t] \) (over \( \mathcal{A} \)) yields the important, exact sequence
\[
0 \to W \otimes_{\mathcal{A}} \mathcal{A}[\underline{z},t] \to \mathcal{A}[\underline{z},t] / \tilde{\mathcal{J}} \cdot \mathcal{A}[\underline{z},t] \to \mathcal{A}[\underline{z},t] / \mathcal{J} \cdot \mathcal{A}[\underline{z},t] \to 0.
\]

Now, let \( s \) be any relation with coefficients in \( \mathcal{A}[\underline{z},t] \) between the equations \( f_{(a,b,\alpha,\beta)} \), i.e.
\[
\sum s_{(a,b,\alpha,\beta)} f_{(a,b,\alpha,\beta)} = 0 \quad \text{in} \quad \mathcal{A}[\underline{z},t].
\]
By flatness of our family (cf. (5.6)), the components of \( s \) can be lifted to \( O[Z, t] \) obtaining an \( \tilde{s} \) such that

\[ \lambda(s) := \sum \tilde{s}_{(a,b,\alpha,\beta)} F_{(a,b,\alpha,\beta)} \rightarrow 0 \quad \text{in} \quad O[Z, t] \big/ J \cdot O[Z, t]. \]

In particular, each relation \( s \in R \) induces some element \( \lambda(s) \in W \otimes_{A} O[Z, t] \), which is well defined after the additional projection to \( W \otimes_{A} A(Y) \). This procedure describes a certain element \( \lambda \in T^{2}_{Y} \otimes_{A} W = \text{Hom}(W^{*}, T^{2}_{Y}) \) called the obstruction map.

**Theorem:** The obstruction map \( \lambda : W^{*} \rightarrow T^{2}_{Y} \) is injective.

**Corollary:** If \( \text{dim} T^{1}_{Y} < \infty \), our family equals the versal deformation of \( Y \). In general, we could say that it is “versal in degree \(-R^{*}\)”. 

**Proof:** In (6.2) we have proved that the Kodaira-Spencer map is an isomorphism (at least onto the homogeneous piece \( T^{1}_{Y}(-R^{*}) \)). By a criterion also described in [JS], this fact combined with injectivity of the obstruction map implies versality. \( \square \)

The remaining part of \( \S 7 \) contains the proof of the previous theorem.

(7.3) We have to improve the notations of \( \S 4 \) and \( \S 5 \). Since \( \tilde{M} \subseteq \tilde{S} \subseteq O^{N} \), we were able to use the toric equations (cf. (4.4)) during computations modulo \( J \). In particular, the exponents \( \eta \in \mathbb{Z}^{N} \) of \( t \) needed be known modulo \( V^{\perp} \) only; it was enough to define \( \eta(c) \) as elements of \( V_{\mathbb{Z}}^{*} \).

However, to compute the obstruction map, we have to deal with the smaller ideal \( \tilde{J} \subseteq J \). Let us start with refining the definitions of (4.6):

(i) For each vertex \( a \in Q \) we choose the following paths through the 1-skeleton of \( Q \):

- \( \lambda(a) := \) path from \( 0 \in Q \) to \( a \in Q \).
- \( \mu^{v}(a) := \) path from \( a \in Q \) to \( a(c^{v}) \in Q \) such that \( \mu^{v}(a) \langle d_{i}, c^{v} \rangle \leq 0 \) for each \( i = 1, \ldots, N \).
- \( \lambda^{v}(a) := \lambda(a) + \mu^{v}(a) \) is then a path from \( 0 \in Q \) to \( a(c^{v}) \), which depends on \( a \).

(ii) For each \( c \in \mathbb{Z}^{n} \) we use the vertex \( a(c) \) to define

\[ \eta^{v}(c) := \left[ -\lambda_{1}(a(c)) \langle d^{1}, c \rangle, \ldots, -\lambda_{N}(a(c)) \langle d^{N}, c \rangle \right] \in \mathbb{Z}^{N} \]

and

\[ \eta^{v}(c^{v}) := \left[ -\lambda_{1}^{v}(a(c)) \langle d^{1}, c^{v} \rangle, \ldots, -\lambda_{N}^{v}(a(c)) \langle d^{N}, c^{v} \rangle \right] \in \mathbb{Z}^{N}. \]
(iii) For each $c \in \mathbb{Z}^n$ we fix a representation $c = \sum_v p_v^c e^v$ ($p_v^c \in \mathbb{N}$) such that
\[\eta_0(c) = \sum_v p_v^c \eta_0(e^v).\] (That means, $c$ is represented only by those generators $e^v$ that define faces of $Q$ containing the face defined by $c$ itself.)

**Remark:** Let $a \in \mathbb{N}^w$. Denoting $c := \sum_v a_v e^v$ we obtain $\sum_v a_v f^c(e^v) - \eta^c(c) \in \mathbb{N}^N$ by arguments as in Lemma $(\text{??})$.
Moreover, for the special representation $c = \sum_v p_v^c e^v$, the equation $\sum_v p_v^c \eta^c(e^v) = \eta^c(c)$ is true.

Now, we improve the definition of the polynomials $F_\bullet(Z,t)$ given in $(\text{??})$. Let $a, b \in \mathbb{N}^w, \alpha, \beta \in \mathbb{N}$ such that
\begin{align*}
c := \sum_v a_v e^v = \sum_v b_v e^v \quad \text{and} \quad \sum_v a_v \eta_0(e^v) + \alpha = \sum_v b_v \eta_0(e^v) + \beta.
\end{align*}
Then,
\begin{align*}
F_{(a,b,\alpha,\beta)}(Z,t) := f_{(a,b,\alpha,\beta)}(Z,t_1) - Z^c \cdot \left( t^{\alpha_1} + \sum_v a_v \eta_0(e^v) - \eta(c) - t^{\beta_1} + \sum_v b_v \eta_0(e^v) - \eta(c) \right).
\end{align*}

(7.4) We have to discuss the same types of relations as we did in $(\text{??})$. Since there is only one single element $c \in \mathbb{Z}^n$ involved in the relations (i) and (iii), computing modulo $\overline{J}$ instead of $J$ makes no difference in these cases - we always obtain $\lambda(s) = 0$.

Let us regard the relation $s := \left[ z^r \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha,\beta)} = 0 \right] \quad (r \in \mathbb{N}^w)$. We will use the following notations:

- $c := \sum_v a_v e^v = \sum_v b_v e^v; \quad p := p^c; \quad \eta := \eta^c$;
- $\tilde{c} := \sum_v (a_v + r_v) e^v = \sum_v (b_v + r_v) e^v = \sum_v (p_v + r_v) e^v; \quad q := p^\tilde{c}; \quad \tilde{\eta} := \eta^\tilde{c}$;
- $\xi := \sum_i \left( \sum_v (p_v + r_v) \tilde{\eta}_i(e^v) \right) - \tilde{\eta}(\tilde{c}) = \sum_v (p_v + r_v) \eta_0(e^v) - \eta_0(c)$.

Using the same lifting of $s$ to $\tilde{s}$ as in $(\text{??})$ yields
\begin{align*}
\lambda(s) &= Z^r \cdot F_{(a,b,\alpha,\beta)} - F_{(a+r,b+r,\alpha,\beta)} - \\
&\quad - \left( t^{\alpha_1} + \sum_v a_v \eta_0(e^v) - \eta(c) - t^{\beta_1} + \sum_v b_v \eta_0(e^v) - \eta(c) \right) \cdot F_{(q,p+r,\xi,0)}
\end{align*}

\begin{align*}
&= -Z^{p+r} \cdot \left( t^{\alpha_1} + \sum_v (a_v - p_v) \eta(e^v) - t^{\beta_1} + \sum_v (b_v - p_v) \eta(e^v) \right) + \\
&\quad + Z^q \cdot \left( t^{\alpha_1} + \sum_v (a_v + r_v - q_v) \tilde{\eta}(e^v) - t^{\beta_1} + \sum_v (b_v + r_v - q_v) \tilde{\eta}(e^v) \right) - \\
&\quad - \left( t^{\alpha_1} + \sum_v (a_v - p_v) \eta(e^v) - t^{\beta_1} + \sum_v (b_v - p_v) \eta(e^v) \right) \cdot \left( Z^q \cdot \sum_v (p_v + r_v - q_v) \tilde{\eta}(e^v) - Z^{p+r} \right)
\end{align*}

\begin{align*}
&= Z^q \cdot \left( t^{\alpha_1} + \sum_v (a_v + r_v - q_v) \tilde{\eta}(e^v) - t^{\alpha_1} + \sum_v (p_v + r_v - q_v) \tilde{\eta}(e^v) + \sum_v (a_v - p_v) \eta(e^v) \right) - \\
&\quad - Z^q \cdot \left( t^{\beta_1} + \sum_v (b_v + r_v - q_v) \tilde{\eta}(e^v) - t^{\beta_1} + \sum_v (p_v + r_v - q_v) \tilde{\eta}(e^v) + \sum_v (b_v - p_v) \eta(e^v) \right).
\end{align*
As in (3.4)(iii), we can see that $\lambda(s)$ vanishes modulo $J$ (or even in $A(\bar{S})$) - just identify $\eta$ and $\tilde{\eta}$.

(7.5) In (3.2) we already mentioned the isomorphism

$$W \otimes \mathcal{O}[\bar{S},t] \sim \mathcal{O}[\bar{Z},t] / J \cdot \mathcal{O}[\bar{Z},t]$$

obtained by identifying $t$ with $t_1$ and $\bar{S}$ with $\bar{Z}$. Now, with $\lambda(s)$, we have obtained an element of the right hand side, which has to be interpreted as an element of $W \otimes \mathcal{O}[\bar{S},t]$.

**Lemma:** Let $A, B \in \mathbb{N}_N$ such that $d := A - B \in V^\perp$ (i.e. $t^A - t^B \in J \cdot \mathcal{O}[\bar{Z},t]$). Then, via the previously mentioned isomorphism, $t^A - t^B$ corresponds to the element

$$\sum_{k \geq 1} c_k \cdot g_{d^+}(t - t_1) \cdot t^{k_0 - k} \in W \otimes \mathcal{O}[\bar{S},t]$$

($k_0 := \sum_i A_i; c_k$ are the constants occurred in (3.4)). In particular, the coefficients from $W_k$ vanish for $k > k_0$.

**Proof:** First, we remark that it is allowed to assume that $A = d^+, B = d^-$, i.e. $t^A - t^B = p_d(t)$ (cf. (3.2)). (Otherwise we could write this binomial as $t^A - t^B = t^C \cdot (t^{d^+} - t^{d^-})$ ($C \in \mathbb{N}_N$),

and since

$$t^C = (t_1 + [t - t_1])^C \equiv t_1^{\sum_i C_i} \pmod{t_i - t_j}$$

we would obtain

$$t^A - t^B \equiv t_1^{\sum_i C_i} \cdot (t^{d^+} - t^{d^-}) \pmod{\bar{J}}.$$}

In (3.4) we have seen that

$$p_d(t) = \sum_{k=1}^{k_0} t_1^{k_0 - k} \cdot \left( \sum_{v=1}^{k-1} q_{v,k} (t - t_1) \cdot g_{d^+}(t - t_1) + c_k \cdot g_{d^+}(t - t_1) \right)$$

(with $k_0 := \sum_i d_i^+$. Since $q_{v,k}(t - t_1) \in (t_i - t_j) \cdot \mathcal{O}[t_i - t_j]$, this implies

$$p_d(t) \equiv \sum_{k=1}^{k_0} t_1^{k_0 - k} \cdot c_k \cdot g_{d^+}(t - t_1) \pmod{\bar{J}}.$$}

On the other hand, for $k > k_0$, Lemma (3.3) tells us that $g_{d^+}(t - t_1)$ is a $\mathcal{O}[t_i - t_j]$-linear combination of the elements $g_{d^+}(t - t_1), \ldots, g_{d_{k_0}}(t - t_1)$. Then, the degree $k$
On the other hand, for each $g$ given in (7.6). For the first one we obtain

$$
\sum_{k \geq 1} c_k \cdot g_{d_k(t - t_1)} \cdot z^q \cdot t^{k_0 - k}
$$

with $d_k := \sum_v (a_v - b_v) \cdot \left( \bar{\eta}(c^v) - \eta(c^v) \right)$, $k_0 := \alpha + \sum_v (a_v + r_v) \eta_0(c^v) - \eta_0(\bar{c})$.

Remark:\qquad \square

Corollary: Transferred to $W \otimes _A \mathcal{O}[\mathbb{Z}, t]$, the element $\lambda(s)$ equals

$$
\sum_{k \geq 1} c_k \cdot g_{d_k(t - t_1)} \cdot z^q \cdot t^{k_0 - k}
$$

with $d_k := \sum_v (a_v - b_v) \cdot \left( \bar{\eta}(c^v) - \eta(c^v) \right)$, $k_0 := \alpha + \sum_v (a_v + r_v) \eta_0(c^v) - \eta_0(\bar{c})$.

The coefficients vanish for $k > k_0$.

Proof: We apply the previous lemma to both summands of the $\lambda(s)$-formula of (7.4). For the first one we obtain

$$
d^a = [\alpha e_1 + \sum_v (a_v + r_v - q_v) \bar{\eta}(c^v)] - [\alpha e_1 + \sum_v (p_v + r_v - q_v) \bar{\eta}(c^v) + \sum_v (a_v - p_v) \eta(c^v)]
$$

$$
k_0 = \sum_i \left( \alpha e_1 + \sum_v (a_v + r_v - q_v) \bar{\eta}(c^v) \right)
$$

$$
= \alpha + \sum_v (a_v + r_v - q_v) \eta_0(c^v) = \alpha + \sum_v (a_v + r_v) \eta_0(c^v) - \eta_0(\bar{c}).
$$

$k_0$ has the same value for both the $a$- and $b$-summand, and

$$
d = d^a - d^b = \sum_v (a_v - p_v) \cdot \left( \bar{\eta}(c^v) - \eta(c^v) \right) - \sum_v (b_v - p_v) \cdot \left( \bar{\eta}(c^v) - \eta(c^v) \right)
$$

$$
= \sum_v (a_v - b_v) \cdot \left( \bar{\eta}(c^v) - \eta(c^v) \right). \quad \square
$$

(7.6) Now, we try to approach the obstruction map $\lambda$ from the opposite direction. Using the description of $T^2_Y$ given in (7.4) we construct an element of $T^2_Y \otimes _A W$, that afterwards turns out to equal $\lambda$.

For a path $g \in \mathbb{Z}^N$ along the edges of $Q$, we will denote by

$$
d(g, c) := [\langle g_1 d^1, c \rangle, \ldots, \langle g_N d^N, c \rangle] \in \mathbb{Z}^N
$$

the vector showing the behaviour of $c \in \mathbb{Z}^n$ passing each particular edge. If, moreover, $g$ comes from a closed path, $d(g, c)$ is also contained in $V^\perp$.

On the other hand, for each $k \geq 1$, we can use the $d_i$’s from $V^\perp$ to get elements $g_{d_k(t - t_1)} \in W_k$ generating this vector space. Composing both procedures we obtain, for each closed path $g \in \mathbb{Z}^N$, a map

$$
g^{(k)}(g, \bullet) : \mathbb{R}^n \quad c \quad \mapsto \quad V^\perp \quad \mapsto \quad W_k
$$

Remark: \qquad 29
(1) Taking the sum over all 2-faces we get a surjective map
\[ \sum_{\varepsilon < Q} g^{(k)}(\varepsilon, \bullet) : \oplus_{\varepsilon < Q} \mathbb{Q}^n \rightarrow W_k. \]

(2) Let \( c \in \mathbb{Z}^n \) (having integer coordinates is very important here). If \( g^1, g^2 \in \mathbb{Z}^N \) are two paths each connecting vertices \( a, b \in Q \) such that

\[ \bullet |\langle a, c \rangle - \langle b, c \rangle| \leq k - 1 \quad \text{and} \]
\[ \bullet c \text{ is monotonic along both paths (i.e. } \langle g^1_i d^i, c \rangle; \langle g^2_i d^i, c \rangle \geq 0 \text{ for } i = 1, \ldots, N), \]

then \( g^1 - g^2 \in \mathbb{Z}^N \) will be a closed path yielding \( g^{(k)}(g^1 - g^2, c) = 0 \) in \( W_k \).

**Proof:** The reason for (1) is the fact that the elements \( d(\varepsilon, c) \) \( (\varepsilon < Q \) 2-face; \( c \in \mathbb{Z}^n) \) generate \( V^\perp \) as a vector space.

For the proof of (2), we look at \( d := d(g^1 - g^2, c) \). Since \( d_i = \langle g^1_i d^i, c \rangle - \langle g^2_i d^i, c \rangle \) is the difference of two non-negative integers, we obtain \( d_i^+ \leq \langle g^1_i d^i, c \rangle \).

Hence,
\[ \sum_i d_i^+ \leq \sum_i \langle g^1_i d^i, c \rangle = \langle b, c \rangle - \langle a, c \rangle \leq k - 1, \]

and as in (1,5) we obtain \( g_{d,k}(t - t_1) \in \tilde{J} \) by Lemma (1,5). \( \square \)

Using the notations introduced in (1,5) we obtain for \( R := kR^*, k \geq 2 \)
\[ E^{kR^*}_j = \{ [c^v, \eta_0(c^v)] | \langle a^j, c^v \rangle + \eta_0(c^v) \leq k - 1 \} \cup \{ R^* \} \subseteq \sigma^v \cap \mathbb{Z}^{n+1}. \]

Then, we can define the following linear maps :
\[ \psi^{(k)}_j : L(E^{kR^*_j}) \rightarrow W_k, \]
\[ q \rightarrow \sum_v q_v \cdot g^{(k)}(\Lambda(a^j) + \mu^v(a^j) - \Lambda(a(c^v)), c^v). \]

(The \( q \)-coordinate corresponding to \( R^* \in E^{kR^*}_j \) is not used in the definition of \( \psi^{(k)}_j \).)

**Lemma:** Let \( \langle a^i, a^j \rangle < Q \) be an edge of the polyhedron \( Q \). Then, on \( L(E^{kR^*_i} \cap E^{kR^*_j}) = L(E^{kR^*_i}) \cap L(E^{kR^*_j}) \), the maps \( \psi^{(k)}_i \) and \( \psi^{(k)}_j \) coincide.

In particular (cf. Theorem (2,4)), the \( \psi^{(k)}_j \)'s induce a linear map \( \psi^{(k)} : T^*_Y(-kR^*) \rightarrow W_k \).

**Proof:** Let \( q \in L(E^{kR^*_i} \cap E^{kR^*_j}) \). Moreover, we denote by \( g^{ij} \in \mathbb{Z}^N \) the path consisting of the single edge running from \( a^i \) to \( a^j \).

Then,
\[ \psi^{(k)}_i(q) - \psi^{(k)}_j(q) = \sum_v q_v \cdot g^{(k)}(\Lambda(a^i) + \mu^v(a^i) - \Lambda(a(c^v)) + \mu^v(a^j), c^v) \]
\[ = g^{(k)}(\Lambda(a^i) - \Lambda(a^j), \sum_v q_v c^v) + \sum_v q_v \cdot g^{(k)}(\mu^v(a^i) - \mu^v(a^j) - g^{ij}, c^v), \]

30
and both summands vanish for several reasons. The first one is killed simply by the equality \( \sum_v q_v c^v = 0 \). For the second one we can use (2) of the previous remark: If \( q_v \neq 0 \), then the assumption about \( q \) implies the inequalities

\[
0 \leq \langle a^i, c^v \rangle - \langle a(c^v), c^v \rangle ; \quad \langle a^i, c^v \rangle - \langle a(c^v), c^v \rangle \leq k - 1 .
\]

Hence, assuming w.l.o.g. \( \langle a^i, c^v \rangle \geq \langle a^j, c^v \rangle \), we can take \( g^1 := -\mu^v(a^i) - \varrho^j \) and \( g^2 := -\mu^v(a^i) \) to see that \( g^{(k)} (\mu^v (a^i) - \mu^v (a^i)) = 0 \).

### (7.7) Proposition

\( \sum_{k \geq 1} c_k \psi^{(k)} \) equals \( \lambda^* \), the adjoint of the obstruction map.

**Proof:** In Theorem (3.5) of [1] we gave a dictionary between the two \( T^2 \)-formulas mentioned in [1]. Using this result we can find an element of \( \text{Hom}(\mathcal{R}/\mathcal{R}, W_k \otimes A(Y)) \) representing \( \psi^{(k)} \in T^2_k \otimes W_k \) - it sends relations of type (i) (cf. [1]) to 0 and deals with relations of type (ii) and (iii) in the following way:

\[
[\xi^r t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \mapsto \psi^{(k)}(a - b) \cdot x \sum_v (a_v + r_v) [c^v, \eta_0 (c^v)] + (\alpha + \gamma - k) R^*,
\]

if

\[
\langle (Q, 1), \sum_v (a_v + r_v) [c^v, \eta_0 (c^v)] + (\alpha + \gamma - k) R^* \rangle \geq 0 ,
\]

and \( j \) is such that

\[
\langle (a^j, 1), \sum_v a_v [c^v, \eta_0 (c^v)] + (\alpha - k) R^* \rangle < 0 ;
\]

otherwise the relation is sent to 0 (in particular, if there is not any \( j \) meeting the desired property).

On \( Q \), the linear forms \( c := \sum_v a_v c^v \) and \( \tilde{c} = \sum_v (a_v + r_v) c^v \) admit their minimal values at the vertices \( a(c) \) and \( a(\tilde{c}) \), respectively. Hence, we can transform the previous formula into

\[
[\xi^r t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \mapsto \psi^{(k)}(a - b) \cdot x \sum_v (a_v + r_v) [c^v, \eta_0 (c^v)] + (\alpha + \gamma - k) R^*,
\]

if

\[
\sum_v (a_v + r_v) \eta_0 (c^v) - \eta_0 (\tilde{c}) + (\alpha + \gamma - k) =
\]

\[
= (\langle (a(\tilde{c}), 1), \sum_v (a_v + r_v) [c^v, \eta_0 (c^v)] + (\alpha + \gamma - k) R^* \rangle \geq 0 ,
\]

\[
\sum_v a_v \eta_0 (c^v) - \eta_0 (c) + (\alpha - k) =
\]

\[
= (\langle (a(c), 1), \sum_v a_v [c^v, \eta_0 (c^v)] + (\alpha - k) R^* \rangle < 0
\]

(and mapping to 0 otherwise).

Adding the coboundary \( h \in \text{Hom}(\mathcal{D} [\xi, t]^m, W_k \otimes A(Y)) \)

\[
h_{(a,\alpha),(b,\beta)} := \begin{cases} 
\psi^{(k)}(a - b) \cdot x \sum_v a_v [c^v, \eta_0 (c^v)] + (\alpha - k) R^* & \text{for } \sum_v a_v \eta_0 (c^v) - \eta_0 (c) + \alpha \geq k , \\
0 & \text{otherwise}
\end{cases}
\]
does not change the class in $T_Y^2(-kR^*)$ (still representing $\psi^{(k)}$), but improves the representative from $\text{Hom}(\mathcal{R}/\mathcal{R}_0, W_k \otimes \mathcal{A}(Y))$. It still maps type-(i)-relations to 0, and moreover

$$[z^T \cdot t^r \cdot f_{(a,b,a,\beta)} - f_{(a+r,b+r,a+\gamma,b+\gamma)} = 0] \mapsto$$

$$\mapsto \begin{cases} \left( \psi^{(k)}_{a(c)}(a-b) - \psi^{(k)}_{a(c)}(a-b) \right) \cdot x^{\sum_v (a_v + r_v) [c^v, \eta_0(c^v)] + (a+\gamma-k)R^*} & \text{for } k_0 + \gamma \geq k \\ 0 & \text{otherwise} \end{cases}$$

(with $k_0 = \alpha + \sum_v (a_v + r_v) \eta_0(c^v) - \eta_0(\tilde{c})$).

By definition of $\psi_j^{(k)}$ and $g^{(k)}$ we obtain

$$\psi^{(k)}_{a(c)}(a-b) - \psi^{(k)}_{a(c)}(a-b) =$$

$$= \sum_v (a_v - b_v) \cdot g^{(k)} \left( \Delta(a(c)) + \sum_v (a_v - b_v) \right)$$

$$= \sum_v (a_v - b_v) \cdot g^{(k)} \left( \Delta'(a(c)) - \sum_v (a_v - b_v) \right)$$

$$= \sum_v (a_v - b_v) \cdot \left( g^{(k)}(\tilde{c}) - \sum_v (a_v - b_v) \right) \sum_v (a_v + r_v) [c^v, \eta_0(c^v)] + (a+\gamma-k)R^* \in \mathcal{A}(Y). \quad \Box$$

It remains to show that the summands $\psi^{(k)}$ of $\lambda^*$ are indeed surjective maps from $T_Y^2(-kR^*)^*$ to $W_k$. We will do so by composing them with auxiliary surjective maps $p^k : \oplus_{\varepsilon < Q} \mathcal{F}^{m+1} \longrightarrow T_Y^2(-kR^*)^*$ yielding $\psi^{(k)} \circ p^k = \sum_{\varepsilon < Q} g^{(k)}(\mathcal{F} \bullet)$. Then, the result follows from the first part of Remark (7.6).

In §6 of [Al 3] we used a short exact sequence of complexes called

$$0 \longrightarrow L_{\mathcal{A}(E^R)} \longrightarrow (\mathcal{A}^{E^R}) \longrightarrow \text{span}_{\mathcal{A}(E^R)} \rightarrow 0$$

to obtain from Theorem (7.4) an isomorphism

$$T_Y^2(-R) \cong \left( \frac{\text{Im} \left[ \oplus_{\varepsilon < Q} \mathcal{F}^{m+1} \rightarrow \oplus_{(a',a') < Q} \mathcal{F}^{m+1} \right]}{\text{Im} \left[ \oplus_{\varepsilon < Q} \text{span}_{\mathcal{A}(E^R)} \left( \cap_{\varepsilon < E_j^R} \right) \rightarrow \oplus_{(a',a') < Q} \mathcal{F}^{m+1} \right]} \right)^*.$$
map $p^k$ just mentioned. Taking a closer look at the construction of [Al 3] §6, we can give an explicit description of $p^k$; eventually we will be able to compute $\psi^{(k)} \circ p^k$.

Let us fix some 2-face $\varepsilon < Q$. Assume that $d^1, \ldots, d^M$ are its counterclockwise orientated edges, i.e. the sign vector $\varepsilon$ looks like $\varepsilon_i = 1$ for $i = 1, \ldots, M$ and $\varepsilon_j = 0$ otherwise. Moreover, we denote the vertices of $\varepsilon < Q$ by $a^1, \ldots, a^M$ such that $d^i$ runs from $a^i$ to $a^{i+1}$ ($M + 1 := 1$).

Starting with a $[c, \eta_0] \in \mathcal{C}^{n+1}$ (and, as just mentioned, only the $c \in \mathcal{C}^n$ is essential) we have to proceed as follows:

(i) For $i = 1, \ldots, M$ we represent $[c, \eta_0]$ as a linear combination of elements of $E_i^{kR^*} \cap E_{i+1}^{kR^*}$. (This corresponds to the lifting from span$(E^R)$ to $(\mathcal{C}^{E^R})_\bullet$)$
\[ [c, \eta_0] = \sum_v q_{iv} [c^v, \eta_0(c^v)] + q_i [0, 1], \]

and $q_{iv} \neq 0$ implies $[c^v, \eta_0(c^v)] \in E_i^{kR^*} \cap E_{i+1}^{kR^*}$, i.e.
\[ \langle a^i, c^v \rangle + \eta_0(c^v) \leq k - 1; \quad \langle a^{i+1}, c^v \rangle + \eta_0(c^v) \leq k - 1. \]

(ii) We map the result to $\bigoplus_{i=1}^M \mathcal{C}^{E_i^{kR^*}}$ by taking successive differences (corresponding to the application of the differential in the complex $(\mathcal{C}^{E^R})_\bullet$). The result is automatically contained in Ker $(\bigoplus_i L(E_i^{kR^*}) \to L(E))$, and its $i$-th summand is the linear relation
\[ \sum_v (q_{i,v} - q_{i-1,v}) \cdot [c^v, \eta_0(c^v)] + (q_i - q_{i-1}) \cdot [0, 1] = 0. \]

(iii) Finally, we apply $\psi^{(k)}$ to obtain
\[
\psi^{(k)}(p^k(c)) = \sum_{i=1}^M \sum_v (q_{i,v} - q_{i-1,v}) \cdot g^{(k)}(\Delta(a^i) - \Delta(a(c^v)) + \mu^v(a^i), c^v) \\
= \sum_{i,v} g^{(k)}(\Delta(a^i) - \Delta(a(c^v)) + \mu^v(a^i), q_{i,v} c^v) - \\
- \sum_{i,v} g^{(k)}(\Delta(a^{i+1}) - \Delta(a(c^v)) + \mu^v(a^{i+1}), q_{i,v} c^v). \\
= \sum_{i,v} g^{(k)}(\lambda(a^i) - \lambda(a^{i+1}) + \mu^v(a^i) - \mu^v(a^{i+1}), q_{i,v} c^v). 
\]

Similar to the proof of Lemma (3) we introduce the path $q^i$ consisting of the single edge $d^i$ only. Then, if $q_{iv} \neq 0$ and w.l.o.g. $\langle a^i, c^v \rangle \geq \langle a^{i+1}, c^v \rangle$, the pair of paths $\mu^v(a^i)$ and $\mu^v(a^{i+1}) + q^i$ meets the assumption of Remark (3)(2) (cf. (i)). Hence,
we can proceed as follows:

\[
\psi^{(k)}(p^k(c)) = \sum_{i,v} g^{(k)}(\Delta(a^i) - \Delta(a^{i+1}) + \varrho_i, q_{iv} c^v) + \\
+ \sum_{i,v} g^{(k)}(\mu_v(a^i) - \mu_v(a^{i+1}) - \varrho_i, q_{iv} c^v)
\]

\[
= \sum_{i=1}^M g^{(k)}(\Delta(a^i) - \Delta(a^{i+1}) + \varrho_i, \sum_v q_{iv} c^v)
\]

\[
= g^{(k)}(\sum_{i=1}^M \varrho_i, c)
\]

\[
= g^{(k)}(\varrho, c).
\]

Hence, Theorem (7.2) is proven.

8 The components of the reduced versal family

(8.1) The components of the reduced base space \( \tilde{\mathcal{M}}_{\text{red}} \) correspond to maximal decompositions of \( Q \) into a Minkowski sum \( Q = R_0 + \ldots + R_m \) with lattice polytopes \( R_k \subseteq \mathbb{R}^m \) as summands. Intersections of components are obtained by the finest Minkowski decompositions of \( Q \), that are coarser than all the involved maximal ones.

Theorem: Fix such a Minkowski decomposition. Then, the corresponding component (or intersection of components) \( \tilde{\mathcal{M}}_0 \) is isomorphic to \( \mathcal{A}^{m+1}/\mathcal{A} \cdot (1, \ldots, 1) \), and the restriction \( X_0 \to \mathcal{A}^m \) of the versal family can be described as follows:

(i) Define the cone

\[
\tilde{\sigma} := \text{Cone} \left( \bigcup_{k=0}^m (R_k \times \{e^k\}) \right) \subseteq \mathbb{R}^{n+m+1},
\]

it contains \( \sigma = \text{Cone} (Q \times \{1\}) \subseteq \mathbb{R}^{n+1} \) via the diagonal embedding \( \mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+m+1} ((a,1) \mapsto (a;1,\ldots,1)) \). The inclusion \( \sigma \subseteq \tilde{\sigma} \) induces a closed embedding of the affine toric varieties defined by these cones - this gives \( Y \hookrightarrow X_0 \).

(ii) The projection \( \mathbb{R}^{n+m+1} \longrightarrow \mathbb{R}^{n+1} \) provides \( m+1 \) regular functions on \( X_0 \), i.e. we obtain a map \( X_0 \to \mathcal{A}^{m+1} \). Composing this map with \( \ell : \mathcal{A}^{m+1} \longrightarrow \mathcal{A}^{m+1}/\mathcal{A} \cdot (1, \ldots, 1) \) yields the family.

We will use the (8.2) and (8.3) to prove the theorem.

(8.2) We already know (cf. (2.5)) that both the space

\[
\mathcal{M}_0 = \mathcal{A}^{m+1}/\mathcal{A} \cdot (1, \ldots, 1) \subseteq \mathcal{A}^N/\mathcal{A} \cdot (1, \ldots, 1)
\]

34
and its pullback $\mathcal{M}_0 \subseteq \mathcal{G}^N$ are given by the equations $t_i - t_j = 0$ (if $d^i, d^j$ belong to a common summand $R_k$ of $Q$).

There is a chain of inclusions $\mathcal{M}_0 \subseteq \mathcal{M} \subseteq \bar{\mathcal{S}} \subseteq \mathcal{G}^N$, and the map $\mathcal{M}_0 \hookrightarrow \bar{\mathcal{S}}$ factorizes through an embedding $\mathcal{M}_0 \hookrightarrow S$. It is given by the surjection of $\mathcal{G}$-algebras

$$\mathcal{G}[C(Q)^\vee \cap V^*_Z] \xrightarrow{\psi} \mathcal{G}[T_0, \ldots, T_m] \xrightarrow{\bar{\mathcal{M}} \mapsto T_k \text{ with } d^i \in R_k}$$

coming from the linear map

$$\mathbb{R}^{m+1} \xrightarrow{c^k} \mathbb{V} \xrightarrow{\sum_{d \in R_k} e^j} \mathbb{R}^N.$$

The matrix of $\mathbb{R}^{m+1} \xrightarrow{e^k} \mathbb{R}^N$ equals the incidence matrix between edges and Minkowski summands of $Q$, and the space $\mathcal{M}_0 = \mathcal{G}^{m+1}$ corresponds to the cone $\mathbb{R}^{m+1} = \mathbb{R}^{m+1} \cap C(Q)$.

(8.3) The family $X_0 \rightarrow \tilde{\mathcal{M}}_0$ arises from $\bar{X} \times_S \mathcal{M} \rightarrow \tilde{\mathcal{M}}$ via base change $\mathcal{M}_0 \mapsto \tilde{\mathcal{M}}$. We obtain

$$X_0 = (\bar{X} \times_S \mathcal{M}) \times_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}}_0 = \bar{X} \times_S (\mathcal{M} \times_{\tilde{\mathcal{M}}} \tilde{\mathcal{M}}_0) = \bar{X} \times_S \mathcal{M}_0$$

$$= \bar{X} \times_S (S \times_S \mathcal{M}_0) = (\bar{X} \times_S S) \times_S \mathcal{M}_0 = \bar{X} \times_S \mathcal{M}_0.$$

Hence, with $\tilde{\sigma}$ as defined is the theorem, it remains to show that

$$\mathcal{G}[\tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V^*_Z)] \xrightarrow{\psi} \mathcal{G}[\tilde{\sigma}^\vee \cap \mathbb{Z}^{n+m+1}]$$

$$\mathcal{G}[C(Q)^\vee \cap V^*_Z] \xrightarrow{\psi} \mathcal{G}[N^{m+1}] = \mathcal{G}[T_0, \ldots, T_m]$$

is a tensor product diagram:

(i) $\tilde{\sigma}$ is the preimage of $\mathbb{R}^{m+1} \subseteq C(Q)$ via the projection $\tilde{C}(Q) \rightarrow C(Q)$. In particular, $\tilde{\sigma} \subseteq \tilde{C}(Q)$ causes a surjective map $\psi_{\mathbb{R}} : \tilde{C}(Q)^\vee \rightarrow \tilde{\sigma}^\vee$.

(ii) To show surjectivity at the level of lattices (i.e. surjectivity of $\psi$) we start with an element $[c, \eta] \in \tilde{C}(Q)^\vee$ and suppose its image $\psi_{\mathbb{R}}([c, \eta])$ to be contained in $\tilde{\sigma}^\vee \cap \mathbb{Z}^{n+m+1}$.

In particular, $c \in \mathbb{Z}^n$, and we obtain $[c, \eta(c)] \in \tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V^*_Z)$ implying that

$$[0, \eta - \eta(c)] = [c, \eta] - [c, \eta(c)] \in [0, C(Q)^\vee] \subseteq \tilde{C}(Q)^\vee$$

maps to an element of $N^{m+1} \subseteq \tilde{\sigma}^\vee \cap \mathbb{Z}^{n+m+1}$.

On the other hand, surjectivity of $C(Q)^\vee \cap V^*_Z \rightarrow N^{m+1}$ causes that this element can be reached by some $[0, \bar{\eta}] \in [0, C(Q)^\vee \cap V^*_Z]$, too.

Hence, $[c, \eta(c) + \bar{\eta}] \in \tilde{C}(Q)^\vee \cap (\mathbb{Z}^n \times V^*_Z)$ is a lattice-preimage of $\psi_{\mathbb{R}}([c, \eta])$. 

35
(iii) The same methods applies for showing that \( \text{Ker} \psi \) is generated by the same elements as \( \text{Ker} (\mathcal{F}[C(Q) \vee \cap V_Z^*] \rightarrow \mathcal{F}[N^{m+1}]) \): 

If \([c^1, \eta^1], [c^2, \eta^2] \in \tilde{C}(Q) \vee (\mathbb{Z}^n \times V_Z^*) \) have the same image in \( \mathcal{F}[\tilde{\sigma} \vee \cap \mathbb{Z}^n+m+1] \) (i.e. \( x^{[c_1, \eta_1]} - x^{[c_2, \eta_2]} \in \text{Ker} \psi \)), then \( c^1 = c^2 \), and the elements 

\[
\mu_1 := \eta^1 - \eta(c^1), \quad \mu_2 := \eta^2 - \eta(c^2) \in C(Q) \vee V_Z^*
\]

have the same image in \( \mathcal{F}[N^{m+1}] \).

In particular, \( x^{\mu_1} - x^{\mu_2} \in \text{Ker} (\mathcal{F}[C(Q) \vee \cap V_Z^*] \rightarrow \mathcal{F}[N^{m+1}]) \), and

\[
x^{[c_1, \eta_1]} - x^{[c_2, \eta_2]} = x^{[c_1, \eta(c^1)]} \cdot \left( x^{\mu_1} - x^{\mu_2} \right).
\]

(8.4) Example: At the end of (2.5) we presented two decompositions of \( Q_6 \) into a Minkowski sum of lattice summands. Let us describe now the restrictions of the versal family to the associated components of \( \tilde{\mathcal{M}} \):

(i) Putting the two triangles \( R_0, R_1 \) into two parallel planes contained in \( IR^3 \) yields an octahedron as the convex hull of the whole configuration. Then, \( \tilde{\sigma} \) is the (4-dimensional) cone over this octahedron

\[
\tilde{\sigma} = \langle (0,0;1,0), (1,0;1,0), (1,1;1,0), (0,0;0,1), (0,1;0,1), (1,1;0,1) \rangle.
\]

(ii) Looking at the second decomposition, we have to put three line segments \( R_0, R_1, R_2 \) on three parallel 2-planes in general position inside the affine space \( IR^4 \). Taking the convex hull of this configuration yields a 4-dimensional polytope that is dual to \((\text{triangle}) \times (\text{triangle})\).

Again, \( \tilde{\sigma} \) is the (5-dimensional) cone over this polytope

\[
\tilde{\sigma} = \langle (0,0;1,0,0), (1,0;1,0,0), (0,0;0,1,0), (0,1;0,1,0), (0,0;0,0,1), (1,1;0,0,1) \rangle.
\]

The total spaces over the components arise as the toric varieties defined by \( \tilde{\sigma} \). In our example, they equal the cones over \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) and \( \mathbb{P}^2 \times \mathbb{P}^2 \), respectively.

9 Further examples

(9.1) Three examples of toric Gorenstein singularities arise as cones over the Del Pezzo surfaces obtained by blowing up \((\mathbb{P}^2, \mathcal{O}(3))\) in one, two, or three points, respectively. They correspond to the following polygons:
Let us discuss these three examples:

(iv) The edges equal

\[ d^1 = (1, 0), \ d^2 = (1, 2), \ d^3 = (-2, -1), \ d^4 = (0, -1), \]

and they imply the following equations of the versal base space as closed subscheme of \( \mathcal{A}^4/\mathcal{A} \cdot (1,1,1,1) \):

\[
t_1 + t_2 = 2t_3, \quad t_3 + t_4 = 2t_2, \quad t_1^2 + t_2^2 = 2t_3^2, \quad t_2^2 + t_4^2 = 2t_3^2.
\]

Using the two linear equations, only two coordinates \( t := t_1, \varepsilon := t_1 - t_3 \) are sufficient. (We get the \( t_i \)'s back by \( t_1 = t, \quad t_2 = t - 2\varepsilon, \quad t_3 = t - \varepsilon, \quad t_4 = t - 3\varepsilon \).

Then, the two quadratic equations transform into \( 2\varepsilon^2 = 0 \), i.e. the versal base space is a fat point.

On the other hand, \( Q_4 \) does not allow any splitting into a Minkowski sum involving lattice summands only. This reflects the triviality of the underlying reduced space. (Cf. (9.2).)

(v) The polygon \( Q_5 \) allows the decomposition into the sum of a triangle and a line segment. In particular, the reduced base space of the versal deformation of \( Y_5 \) has to be a line.

We compute the true base space: \( d^1 = (1, 1), \ d^2 = (-1, 1), \ d^3 = (-1, 0), \ d^4 = (0, -1), \ d^5 = (1, -1) \) yield the equations

\[
t_1 - t_3 = t_2 - t_5 = t_4 - t_1 \quad \text{and} \quad t_1^2 - t_3^2 = t_2^2 - t_5^2 = t_4^2 - t_1^2.
\]

With \( t := t_1, \ s_1 := t_1 - t_3, \ s_2 := t_1 - t_2 \) and \( t_1 = t, \ t_2 = t - s_2, \ t_3 = t - s_1, \ t_4 = t + s_1, \ t_5 = t - s_1 - s_2 \), they turn into

\[ s_1^2 = 2s_1s_2 = 0. \]

(vi) This example was spread in the paper.

(9.2) We will use the polygon \( Q_4 := \text{Conv} \{(0,0), (1,0), (2,2), (0,1)\} \) of (iv) for a more detailed demonstration how the theory works. In particular, we will describe the versal family of \( Y_4 \) over \( \text{Spec} \mathcal{A}^4/\mathcal{A} \cdot (1,1,1,1) \).
(1) The \((t, \varepsilon)\)-coordinates of \(V\) correspond to the linear map
\[
\begin{pmatrix}
1 & 0 \\
1 & -2 \\
1 & -1 \\
1 & -3
\end{pmatrix} : \mathbb{R}^2 \to V \hookrightarrow \mathbb{R}^4.
\]
We obtain
\[
C(Q_4) = \{(a, b) \in \mathbb{R}^2 \mid a \geq 0, a - 2b \geq 0, a - b \geq 0, a - 3b \geq 0\}
\]
\[
= \{(a, b) \in \mathbb{R}^2 \mid a \geq 0, a - 3b \geq 0\}
\]
\[
= \langle [1, 0], [1, -3] \rangle = \langle (0, -1), (3, 1) \rangle \subseteq \mathbb{R}^2,
\]
and the map \(N^4 \to C(Q_4) \cap V_{\mathbb{Z}}\) sends \(e_1, e_2, e_3, e_4\) to \([1, 0], [1, -2], [1, -1], [1, -3]\), respectively. In particular, this map is surjective, i.e. \(S_4 = \bar{S}_4\) and \(X_4 = \bar{X}_4\).

(2) To compute the tautological cone \(\tilde{C}(Q_4)\), we need the Minkowski summands associated to the two fundamental generators of \(C(Q_4)\):
\[
(Q_4)_{(0, -1)} = \text{Conv}\{(0, 0), (2, 4), (0, 3)\}, \quad (Q_4)_{(3, 1)} = \text{Conv}\{(0, 0), (3, 0), (4, 2)\}.
\]
Hence,
\[
\tilde{C}(Q_4) = \langle (0, 0; 0, -1); (2, 4; 0, -1); (0, 3; 0, -1); (0, 0; 3, 1);
(3, 0; 3, 1); (4, 2; 3, 1) \rangle.
\]

(3) Now, we have all information to obtain the versal family of \(Y_4 = \text{Spec} \mathcal{A}[\text{Cone}(Q_4) \cap \mathbb{Z}^3]\):
- Restrict the family \(\text{Spec} \mathcal{A}[\tilde{C}(Q_4) \cap \mathbb{Z}^4] \to \text{Spec} \mathcal{A}[C(Q_4) \cap \mathbb{Z}^3] \subseteq \mathcal{A}^4\) to the subspace \(\mathcal{A}^2 \simeq V_{\mathcal{A}} \subseteq \mathcal{A}^4\), i.e. use the \((t, \varepsilon)\)-coordinates instead of \((t_1, t_2, t_3, t_4)\).
- Compose the result with the projection \(\mathcal{A}^2 \twoheadrightarrow \mathcal{A}^1 \ ((t, \varepsilon) \mapsto \varepsilon)\). That means, we do no longer regard \(t\) as a coordinate of the base space.
- Finally, we restrict our family to the closed subscheme defined by the equation \(\varepsilon^2 = 0\).

(4) To obtain equations, we could either take a closer look to the family constructed so far, or we can proceed more directly as described in (3.3), (3.4), and (3.5):
- Computing the minimal generator set of the semigroup \(\text{Cone}(Q_4) \cap \mathbb{Z}^3\), we get the elements \([c^v; \eta_0(c^v)]\):
  \[
  [c^1; \eta_0^1] = [0, 1; 0], \quad [c^2; \eta_0^2] = [-1, 1; 1], \quad [c^3; \eta_0^3] = [-2, 1; 2],
  [c^4; \eta_0^4] = [-1, 0; 2], \quad [c^5; \eta_0^5] = [0, -1; 2], \quad [c^6; \eta_0^6] = [1, -2; 2],
  [c^7; \eta_0^7] = [1, -1; 1], \quad [c^8; \eta_0^8] = [1, 0; 0].
  \]
Finally, we restrict the family to the versal base space by switching to \( q \in \mathcal{A}^9 \).

(The sum of the three components of the vectors are always 1. In the figure we have drawn the first two coordinates.)

\(-\ Y_4 \subseteq \mathcal{A}^9\) is defined by the following 20 equations:

\[
\begin{align*}
    t^2 - z_4 z_8, & \quad \eta = [1, 0, 0, 0], \\
    t^2 - z_1 z_5, & \quad \eta = [1, 0, 0, 0], \\
    t^2 - z_2 z_7, & \quad \eta = [2, 0, 0, 0], \\
    z_2 t - z_3 z_8, & \quad \eta = [0, 0, 0, 0], \\
    z_2 t - z_1 z_4, & \quad \eta = [0, 0, 0, 0], \\
    z_3 t - z_2 z_4, & \quad \eta = [0, 0, 0, 0], \\
    z_4 t - z_2 z_5, & \quad \eta = [0, 0, 0, 0], \\
    z_5 t - z_4 z_7, & \quad \eta = [0, 0, 0, 0], \\
    z_5 t - z_2 z_6, & \quad \eta = [0, 0, 0, 0], \\
    z_6 t - z_5 z_7, & \quad \eta = [0, 0, 0, 0], \\
    z_7 t - z_5 z_8, & \quad \eta = [0, 0, 0, 0], \\
    z_8 t - z_1 z_7, & \quad \eta = [0, 0, 0, 0], \\
    z_1 z - z_2^2, & \quad \eta = [0, 0, 0, 0], \\
    z_2 z - z_3^2, & \quad \eta = [0, 0, 0, 0], \\
    z_3 z - z_4^2, & \quad \eta = [0, 0, 0, 0].
\end{align*}
\]

\(-\) Choosing paths from \((0, 0) \in Q_4\) to the other vertices, we obtain the list

\[
\begin{align*}
    \eta^1 & = [0, 0, 0, 0], \\
    \eta^2 & = [1, 0, 0, 0], \\
    \eta^3 & = [2, 0, 0, 0], \\
    \eta^4 & = [1, 1, 0, 0] = [0, 0, 2, 0], \\
    \eta^5 & = [0, 2, 0, 0] = [0, 0, 1, 1], \\
    \eta^6 & = [0, 0, 0, 2], \\
    \eta^7 & = [0, 0, 0, 1], \\
    \eta^8 & = [0, 0, 0, 0].
\end{align*}
\]

\(-\) Now, we can lift our 20 equations to the ring \( \mathcal{A}[Z_1, \ldots, Z_8, t_1, \ldots, t_4] \):

\[
\begin{align*}
    t_1 t_2 - Z_4 Z_8, & \quad t_2 - Z_4 Z_5, \\
    Z_2 t_1 - Z_3 Z_8, & \quad Z_2 t_2 - Z_1 Z_4, \\
    Z_4 t_1 - Z_2 Z_5, & \quad Z_4 t_2 - Z_3 Z_7, \\
    Z_5 t_3 - Z_4 Z_7, & \quad Z_5 t_4 - Z_2 Z_6, \\
    Z_7 t_3 - Z_5 Z_8, & \quad Z_7 t_4 - Z_1 Z_6, \\
    Z_3 Z_5 - Z_4^2, & \quad Z_4 Z_6 - Z_5^2, \\
    Z_6 Z_8 - Z_7^2, & \quad Z_5 Z_6 - Z_4 Z_5.
\end{align*}
\]

\(-\) Finally, we restrict the family to the versal base space by switching to the \((t, \varepsilon)\)-coordinates and obeying the equation \(\varepsilon^2 = 0\). Moreover, \(t\) is no longer a coordinate of the base space:

\[
\begin{align*}
    t(t - 2\varepsilon) - z_4 z_8, & \quad t(t - 4\varepsilon) - z_1 z_5, \\
    z_1 t - z_2 z_8, & \quad z_2 t - z_3 z_8, \\
    z_3 (t - 2\varepsilon) - z_2 z_4, & \quad z_4 (t - \varepsilon) - z_3 z_7, \\
    z_5 (t - \varepsilon) - z_4 z_7, & \quad z_5 (t - 2\varepsilon) - z_2 z_6, \\
    z_7 (t - \varepsilon) - z_5 z_8, & \quad z_7 (t - 3\varepsilon) - z_1 z_6, \\
    z_1 z_3 - z_2^2, & \quad z_3 z_5 - z_4^2, \\
    z_6 z_8 - z_7^2, & \quad z_3 z_6 - z_4 z_5.
\end{align*}
\]
(9.3) At last we want to present an example involving more than only quadratic equations for the versal base space. Let \( Q_8 \) be the “regular” lattice 8-gon, it is contained in two strips of lattice thickness 3.

\[ Q_8 \]

Polygon \( Q_8 \)

\( Q_8 \) admits three maximal Minkowski decompositions into a sum of lattice summands:

\[
(i) \quad Q_8 = \dot{\phantom{i}} \ddot{\phantom{i}} \ddot{\phantom{i}} + \ddot{\phantom{i}} \dddot{\phantom{i}} + \dddot{\phantom{i}} \dddot{\phantom{i}} + \underrightarrow{\underline{\phantom{i}}} \quad (ii) \quad Q_8 = \dot{\phantom{i}} \dddot{\phantom{i}} \dddot{\phantom{i}} + \dddot{\phantom{i}} \dddot{\phantom{i}} + \dddot{\phantom{i}} \dddot{\phantom{i}} + \dddot{\phantom{i}} \dddot{\phantom{i}} \\
(iii) \quad Q_8 = \dot{\phantom{i}} \dddot{\phantom{i}} \dddot{\phantom{i}} + \dddot{\phantom{i}} \dddot{\phantom{i}} + \dddot{\phantom{i}} \dddot{\phantom{i}} \]
The decompositions (i), (ii) and (i), (iii) are refinements of the coarser decompositions

\[ Q_8 = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} + \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \quad \text{and} \quad Q_8 = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} + \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array},
\]
respectively.

These facts translate directly into the geometry of the reduced base space of the versal deformation of \( Q_8 \):

- It is embedded in some affine space \( \mathcal{A}^5 \) and equals the union of a 3-plane with two 2-planes (through \( 0 \in \mathcal{A}^5 \)).
- The two 2-planes each have a common line with the 3-dimensional component. However, they intersect each other in \( 0 \in \mathcal{A}^5 \) only.

On the other hand, we can write down the equations of the true versal base space (as a closed subscheme of \( \mathcal{A}^8/\mathcal{A} \cdot (1,\ldots,1) \)):

\[ t_1^k + t_2^k + t_3^k = t_4^k + t_5^k + t_6^k, \quad t_2^k + t_3^k + t_4^k = t_6^k + t_7^k + t_8^k \quad (k = 1, 2, 3). \]

References

[Al 1] Altmann, K.: Computation of the vector space \( T^1 \) for affine toric varieties. J. Pure Appl. Algebra (to appear).

[Al 2] Altmann, K.: Toric \( \mathcal{A} \)-Gorenstein Singularities. Preprint 9/1993 Humboldt-Universität Berlin; e-print alg-geom/9403003.

[Al 3] Altmann, K.: Obstructions in the deformation theory of toric singularities. Preprint M.I.T., April 1994; preprint 6? “Europäisches Singularitäten-Projekt”; e-print alg-geom/9405008.

[Ar] Arndt, J.: Verselle Deformationen zyklischer Quotientensingularitäten. Dissertation, Universität Hamburg, 1988.

[Ch] Christophersen, J.A.: On the Components and Discriminant of the Versal Base Space of Cyclic Quotient Singularities. In: Singularity Theory and its Applications, Warwick 1989, Part I: Geometric Aspects of Singularities, pp. 81-92, Springer-Verlag Berlin Heidelberg, 1991 (LNM 1462).
[JS] Jong, T. de, Straten, D. van: On the deformation theory of rational surface singularities with reduced fundamental cycle.
J. Algebraic Geometry 3, 117-172 (1994).

[Ke] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings I.
Lecture Notes in Mathematics 339, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[KS] Kollár, J., Shepherd-Barron, N.I.: Threefolds and deformations of surface singularities.
Invent. math. 91, 299-338 (1988).

[Od] Oda, T.: Convex bodies and algebraic geometry.
Ergebnisse der Mathematik und ihrer Grenzgebiete (3/15), Springer-Verlag, 1988.

[Ri] Riemenschneider, O.: Deformationen von Quotientensingularitäten (nach zyklischen Gruppen).
Math. Ann. 209 (1974), 211-248.

[St] Stevens, J.: On the versal deformation of cyclic quotient singularities.
In: Singularity Theory and its Applications, Warwick 1989, Part I: Geometric Aspects of Singularities, pp. 302-319, Springer-Verlag Berlin Heidelberg, 1991 (LNM 1462).