The background gauge renormalization of the first order formulation of the Yang-Mills theory is studied by means of the BRST identities. Despite the fact that certain improper diagrams which violate the BRST symmetry must be removed, the renormalizability may be indirectly deduced to all orders. This method involves rescalings and mixings of the fields, which lead to a renormalized effective action for the background field theory.

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I. INTRODUCTION

The background field method \[1-10\], is a formulation which allows to fix a gauge and evaluate the quantum corrections without breaking the background gauge symmetry. This is an efficient method for calculating the $\beta$-function and it has also been used in perturbative gravity \[11, 12\]. The main idea of this method is to write the gauge field $A_{\mu}^a$ which occurs in the Yang-Mills (YM) Lagrangian

$$\mathcal{L}_{YM} = \frac{-1}{4} \left( \partial_{\mu} A_{a}^{\nu} - \partial_{\nu} A_{a}^{\mu} + g f^{abc} A_{\mu}^{b} A_{\nu}^{c} \right)^2, \quad (1.1)$$

as $B_{\mu}^a + Q_{\mu}^a$, where $B_{\mu}^a$ is a background field and $Q_{\mu}^a$ is the quantum field. Then a gauge is chosen which suppresses the gauge invariance of the $Q_{\mu}^a$ field, but maintains the gauge invariance in terms of the $B_{\mu}^a$ field. The gauge-fixing term is made dependent upon $B_{\mu}^a$ as

$$\mathcal{L}_{GF} = \frac{-1}{2\xi} \left[ (\partial_{\mu} \delta_{a}^{ab} + g f^{abc} B_{\mu}^{c}) Q^{b\mu} \right]^2 \equiv \frac{-1}{2\xi} (D_{\mu}^{ab}(B) Q^{b\mu})^2, \quad (1.2)$$

where $\xi$ is a gauge fixing parameter and $D_{\mu}^{ab}(B)$ is the covariant derivative.

The above terms are invariant under the gauge transformations

$$\delta B_{\mu}^{a} = D_{\mu}^{ab}(B) \omega^{b}(x); \quad \delta Q_{\mu}^{a} = g f^{abc} Q_{\mu}^{b} \omega^{c}(x); \quad (1.3)$$

where $\omega^{c}(x)$ is an arbitrary infinitesimal parameter.

Thus, quantum calculations can be performed and explicit gauge invariance in the background field is maintained. Carrying out the path integral over $Q_{\mu}^{ab}$ yields the action $\Gamma[g,B]$. But this action is not the appropriate effective action for the background field because it leads to unwanted one-particle reducible graphs. In order to get the correct effective action $\Gamma[g,B]$, which generates the one-particle irreducible graphs, it is necessary to subtract from $\mathcal{L}_{YM}(B + Q)$ in \[1.1\], the terms which are linear in $Q$ \[3, 4, 7, 9\].

When the background method is used to two-loop order or higher, the sub-graphs are functionals of $B_{\mu}^a$ as well as of $Q_{\mu}^a$, leading to an action $\Gamma[g,B,Q]$ which has a background symmetry under \[1.3\]. This symmetry is not sufficient to fix the renormalization of $\Gamma[g,B,Q]$. In addition, one must also use the BRST symmetry \[13\] of this action. Although the omission of the linear terms in $Q$ preserves the background symmetry under \[1.3\], this operation breaks the BRST symmetry. Consequently, the effective action $\Gamma[g,B,Q]$ is no longer invariant under the BRST transformation. This poses a problem in applying BRST in the background method.
Calculations of quantum corrections in the standard second-order YM theory are generally involved, due to the presence in (1.1) of momentum-dependent three-point as well as four-point vertices. It is well known [14–20] that one may replace [1] by a simpler first order Lagrangian, provided one introduces in the theory another auxiliary field \( F^a_{\mu \nu} \). The corresponding first order Lagrangian may then be written as (see section 2)

\[
\mathcal{L}_{YM}^I(A, F) = \frac{1}{4} F^a_{\mu \nu} F^{\mu \nu \ a} - \frac{1}{2} f^a_{\mu \nu} f^a_{\mu \nu} (A).
\]

This simplifies the computations since the interaction term involves only a single cubic vertex \( \langle F A A \rangle \) which is momentum-independent. If we now substitute in (1.4), \( A^a_{\mu} \) by \( B^a_{\mu} + Q^a_{\mu} \) and proceed along the lines indicated above, one gets an action \( \Gamma[g, B, Q, F] \) which is BRST invariant and has a background symmetry under

\[
\delta B^a_{\mu} = D^a_{\nu} (B) \omega^b_{\mu}(x); \quad \delta Q^a_{\mu} = g f^{abc} B^b_{\mu} \omega^c(x); \quad \delta F^a_{\mu \nu} = g f^{abc} F^{b \mu \nu} \omega^c(x).
\]

As shown in section 4, these symmetries are sufficient to ensure the renormalizability of \( \Gamma[g, B, Q, F] \). But the subtraction of the terms linear in \( Q \), which leads to the correct effective action \( \bar{\Gamma}[g, B, Q, F] \), breaks the BRST symmetry. Nevertheless, as discussed in secs. 3 and 4, the renormalizability of \( \Gamma[g, B, Q, F] \) can be used in an indirect way to renormalize to all orders the effective action \( \Gamma[g, B, Q, F] \), which generates the one-particle irreducible graphs. To this end, we employ a similar approach to that used in [9] for the second-order YM theory. In Sec. 5, we give a short discussion of the results and point out a possible application of this method to the quantum gauge theory of gravity. In Appendix A, a functional equation for the one-particle irreducible generating functional is derived. In Appendices B and C we explicitly show, by computing the \( \langle F B J \rangle \) and \( \langle F B Q \rangle \) of Green functions in the background formalism is

\[
Z[B] = N \int \mathcal{D}Q \mathcal{D}c \mathcal{D}\bar{c} \exp i \int d^4x \left\{ \mathcal{L}_{YM}(B + Q) - \frac{1}{2\xi} [D^a_\mu (B) Q^a_\mu]^2 - [\bar{c} D^a_\mu (B)] \cdot [D^a_\mu (B + Q) c] \right\}, \tag{2.1}
\]

where \( c \) and \( \bar{c} \) are ghost fields and we have suppressed the colour indices by using the notation \( B_\mu \cdot Q_\nu \equiv B^a_\mu Q^a_\nu \) and \( (B_\mu \wedge Q_\nu)^a \equiv f^{abc} B^b_\mu Q^c_\nu \). To convert (2.1) to the first order form, we introduce in the normalization constant \( N \) the factor

\[
\int \mathcal{D}F \exp \frac{i}{4} \int d^4x F_{\mu \nu} \cdot F^{\mu \nu} = \int \mathcal{D}F \exp \frac{i}{4} \int d^4x \left[ F_{\mu \nu} - f_{\mu \nu}(B + Q) \right]^2 \tag{2.2}
\]

where we made a shift in the integration variable. Substituting this factor in (2.1), leads to the cancellation of \( \mathcal{L}_{YM}(B + Q) \), with the result

\[
Z[B] = N \int \mathcal{D}Q \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}F \exp i \int d^4x \left\{ \frac{1}{4} F_{\mu \nu} \cdot F^{\mu \nu} - \frac{1}{2} F^{\mu \nu} \cdot f_{\mu \nu}(B + Q) \right\}
\]

\[
- \frac{1}{2\xi} [D^a_\mu (B) Q^a_\mu]^2 - [\bar{c} D^a_\mu (B)] \cdot [D^a_\mu (B + Q) c] \} . \tag{2.3}
\]

One may now introduce an interacting current via \( J_\mu \cdot Q^\mu \), thus defining a generating functional for connected Green functions \( W[B, J] = -i \log Z[B, J] \) and hence, by a Legendre transform an action \( \bar{\Gamma}[g, B] \). But this action is not the appropriate one for the background method, since it leads to some 1P-reducible graphs. This shortcoming may be avoided by subtracting from \( \mathcal{L}(B + Q) \) in (2.1), the terms linear in \( Q \), which yields the generating functional

\[
\bar{Z}[B] = N \int \mathcal{D}Q \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}F \exp i \int d^4x \left\{ \mathcal{L}_{YM}(B + Q) - Q_\nu \cdot D_\mu (B) f^{\mu \nu}(B) - \mathcal{L}_{YM}(B) \right\}
\]

\[
- \frac{1}{2\xi} [D^a_\mu (B) Q^a_\mu]^2 - [\bar{c} D^a_\mu (B)] \cdot [D^a_\mu (B + Q) c] \} , \tag{2.4}
\]

where we have subtracted also the \( \mathcal{L}_{YM}(B) \) term, which is not relevant for our purposes. To convert (2.4) into a first order form, it is convenient to introduce in \( N \) the factor (compare with (2.2))

\[
\int \mathcal{D}F \exp \frac{i}{4} \int d^4x \left[ F_{\mu \nu} - [\partial_\mu Q_\nu - \partial_\nu Q_\mu + g (Q_\mu \wedge Q_\nu + Q_\mu \wedge B_\nu + B_\mu \wedge Q_\nu)] \right]^2 \tag{2.5}
\]
which leads to several cancellations in (2.4), with the result
\[ Z[B] = N \int DQ Dc D\bar{c} D\bar{F} \exp i \int d^4x \left\{ \frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - \frac{1}{2} \bar{f}_{\mu\nu} \cdot [f_{\mu\nu}(Q) + g(B_\mu \wedge Q_\nu + Q_\mu \wedge B_\nu)] \\
- \frac{1}{2} g f_{\mu\nu}(B) \cdot (Q_\mu \wedge Q_\nu) - \frac{1}{2\xi} [D_\mu(B)Q_\mu]_+^2 - [\bar{c}D_\mu(B) \cdot [D^{\mu}(B + Q)c)] \right\}. \] (2.6)

Proceeding as in the previous case and making a Legendre transform, leads to the correct effective action \( \bar{\Gamma}[g, B] \) in the background field. Let us now compare the Lagrangians which appear in the exponentials of Eqs. (2.3) and (2.6). We have, respectively
\[ \mathcal{L}'(g, B, Q, F) = \frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - \frac{1}{2} \bar{f}_{\mu\nu} \cdot f_{\mu\nu}(B + Q) - \frac{1}{2\xi} [D_\mu(B)Q_\mu]_+^2 - [\bar{c}D_\mu(B) \cdot [D^{\mu}(B + Q)c] \] (2.7)
and
\[ \mathcal{L}^\prime(g, B, Q, F) = \mathcal{L}'(g, B, Q, F) + \frac{1}{2}(F_{\mu\nu} - gQ_\mu \wedge Q_\nu) \cdot f^{\mu\nu}(B). \] (2.8)

The difference between these Lagrangians involves terms which are invariant under the background transformation \( \Delta B \). The second contribution on the right side of (2.8) subtracts from \( \mathcal{L}' \) a term which leads to 1P-reducible graphs. On the other hand, the last term in (2.8), which is induced by the linear term in \( Q_\mu \), subtracted in (2.4), is necessary to obtain correct physical results. For example, using the Feynman rules derived in Appendix B, we have evaluated in Appendix C the divergent part of the background field self-energy. In a space-time of dimension \( d = 4 - 2\epsilon \), we get to one-loop order
\[ \Pi^{ab}_{\mu\nu}(k) \bigg|_{UV} = -\frac{11}{3} i\delta^{ab} \frac{Ng^2}{16\pi^2\epsilon} \left( k_\mu k_\nu - k^2 \eta_{\mu\nu} \right) \] (2.9)
This transverse form is independent of the gauge-fixing parameter and leads to the expected result for the \( \beta \)-function
\[ \beta = \frac{11}{3} \frac{Ng^3}{16\pi^2}. \] (2.10)

We note here that the unsubtracted Lagrangian (2.7) would lead instead to a transverse, but gauge dependent self-energy for the background field (see Appendix C).

III. THE ACTIONS \( \Gamma \) AND \( \bar{\Gamma} \)

We remark that \( \mathcal{L}'(g, B, Q, F) \) in (2.7) with the gauge-fixing term left out, is also invariant under the BRST transformations
\[ \Delta B = 0; \ \Delta Q_\mu = D_\mu(B + Q)c; \ \Delta c = -\frac{1}{2} gc \wedge ct; \ \Delta \bar{c} = 0; \ \Delta F_{\mu\nu} = gF_{\mu\nu} \wedge ct, \] (3.1)
where \( \tau \) is an infinitesimal anti-commuting constant. Let us now add to \( \mathcal{L}'(g, B, Q, F) \) in (2.7) the Zinn-Justin source terms \( U, V, W \), which are useful for setting up the BRST equations [21] and omit the gauge-fixing term (1.2). This leads to the zeroth order action
\[ \Gamma^{(0)}(g, B, Q, c, \bar{c}, F; U, V, W) = \int d^4x \mathcal{L} \] (3.2)
where \( c, \bar{c}, U, V, W \) transform under the background transformations (1.5) in the same way as \( Q \), and
\[ \mathcal{L} = \frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} - \frac{1}{2} F_{\mu\nu} \cdot f_{\mu\nu}(B + Q) + gW_{\mu\nu} \cdot (F_{\mu\nu} \wedge c) + [U_\mu + D_\mu(B)c] \cdot [D^{\mu}(B + Q)c] - \frac{1}{2} gV \cdot (c \wedge c) \] (3.3)
It may be verified that \( \mathcal{L} \) is invariant under the BRST transformations (3.1), provided that the sources remain unchanged, so that \( \Gamma^{(0)} \) obeys the BRST equations (where \( Q \) actually stands for the mean value of the quantum field)
\[ \int d^4x \left[ \frac{\delta \Gamma^{(0)}}{\delta F_{\mu\nu}} \cdot \frac{\delta \Gamma^{(0)}}{\delta W_{\mu\nu}} + \frac{\delta \Gamma^{(0)}}{\delta Q_\mu} \cdot \frac{\delta \Gamma^{(0)}}{\delta U^{\mu}} + \frac{\delta \Gamma^{(0)}}{\delta c} \cdot \frac{\delta \Gamma^{(0)}}{\delta V} \right] = 0 \] (3.4)
\[
\frac{\delta \Gamma^{(0)}}{\delta \bar{c}} - D_\mu(B) \frac{\delta \Gamma^{(0)}}{\delta U_\mu} = 0. \tag{3.5}
\]

Equation (3.5) is a consequence of the fact that the Lagrangian (3.3) depends on \(U_\mu\) only through the combination \(U_\mu + D_\mu(B)\bar{c}\). Moreover, equation (3.4) may be understood by rewriting it, with the help of (3.1), in the alternative form

\[
\int d^4x \left[ \frac{\delta \Gamma^{(0)}}{\delta F_{\mu\nu}} \cdot \Delta F_{\mu\nu} + \frac{\delta \Gamma^{(0)}}{\delta Q_\mu} \cdot \Delta Q_\mu + \frac{\delta \Gamma^{(0)}}{\delta \bar{c}} \cdot \Delta \bar{c} = 0 \right] \tag{3.6}
\]

which reflects the invariance of \(\Gamma^{(0)}\) under the BRST transformations (3.1).

By using an analogous method to that employed in the usual first order formulation of the YM theory [18] one can show that the action \(\Gamma\) satisfies to all orders the BRST equation

\[
\frac{\delta \Gamma}{\delta \bar{c}} - D_\mu(B) \frac{\delta \Gamma}{\delta U_\mu} = 0. \tag{3.7b}
\]

But as we have explained, \(\Gamma\) is not the correct action for the background method. In order to get the appropriate action \(\bar{\Gamma}\), one must instead start from the Lagrangian (2.8), where the last two terms are not BRST invariant, and use a similar procedure to that which led to the action (3.2). We then find that \(\bar{\Gamma}^{(0)}\) may be obtained from \(\Gamma^{(0)}\) by the operation \(\Omega^{(0)}\)

\[
\bar{\Gamma}^{(0)} \equiv \Omega^{(0)}(g, Q, F)\Gamma^{(0)} = \Gamma^{(0)} - \int d^4x \left\{ (F_{\mu\nu} - g Q_\mu \wedge Q_\nu) \cdot \left[ \frac{\delta \Gamma^{(0)}}{\delta F_{\mu\nu}} \right]_{F=Q=0} \right\}. \tag{3.8}
\]

We note that this operator preserves the background gauge invariance under (1.5), but breaks the BRST symmetry under (3.1). (This may be seen by noticing that the term in the square bracket is proportional to \(\mu + \bar{c}\).) It follows that \(\bar{\Gamma}^{(0)}\) does not satisfy the BRST equations. The generalization of the above relation, to higher orders, will be examined in the next section.

**IV. RENORMALIZATION**

As we pointed out, we study first the renormalization of \(\Gamma\), which requires both the background invariance as well as the BRST symmetry. Since the background field \(B\) appears explicitly in the BRST Eq. (3.7b), we need to fix its renormalization. To this end we remark that in consequence of the background symmetry under (1.5), the renormalized action which is got by functionally integrating over \(Q, c, \bar{c}, F\) and setting the sources to zero, must have the form

\[
\Gamma_R[g, B] = -\frac{1}{4} \int d^4x \left( \partial_\mu B_\nu - \partial_\nu B_\mu + g B_\mu \wedge B_\nu \right)^2. \tag{4.1}
\]

This may be obtained from the bare action \(\Gamma^{(0)}(g^{(0)}, B^{(0)})\), by the re-scalings

\[
g^{(0)} = Z_g g; \quad B^{(0)}_\mu = Z_B^{1/2} B_\mu = Z_g^{-1} B_\mu. \tag{4.2}
\]

Thus the background invariance ties these two renormalizations by the relation \(Z_B^{1/2} Z_g = 1\), which is an important virtue of the background field method.

One must also re-scale in \(\Gamma^{(0)}(g^{(0)}, B^{(0)}, Q^{(0)}, c^{(0)}, \bar{c}^{(0)}, F^{(0)}, U^{(0)}, V^{(0)}, W^{(0)})\) the fields

\[
Q^{(0)}_\mu = Z_Q^{1/2} Q^{(0)}_\mu; \quad c^{(0)} = Z^{1/2} c^{(0)}; \quad \bar{c}^{(0)} = \bar{Z}^{1/2} \bar{c}. \tag{4.3}
\]

As shown in [18], the renormalization of the first order formulation of the YM theory requires a re-scaling as well as a mixing of the \(F_{\mu\nu}\) field

\[
F_{\mu\nu}^{(0)} = Z_F^{1/2} F_{\mu\nu} + Z_{FQ} f_{\mu\nu}(Q) \tag{4.4}
\]
where \( f_{\mu\nu} \) is defined in (1.1) and both \( Z_{FQ}^{1/2} - 1; Z_{FQ} \) are of order \( \hbar \). Similarly, in the renormalization process, the bare sources \( U^{(0)}, V^{(0)} \) and \( W^{(0)} \) will also undergo appropriate re-scalings and mixings which relate these to the renormalized sources \( U, V \) and \( W \). All such transformations preserve the BRST invariance. Using this gauge symmetry together with the Lorentz invariance, one can show [22, 23] that the renormalized action \( \Gamma_R \) must be similar to \( \Gamma^{(0)} \) in Eq. (3.2), but it must include all the allowed re-scalings and mixings. Thus, it should have the form

\[
\Gamma_R(g, B, Q, c, \bar{c}, F; U, V, W) = \Gamma^{(0)}(g^{(0)}, B^{(0)}, Q^{(0)}, c^{(0)}, \bar{c}^{(0)}, F^{(0)}; U^{(0)}, V^{(0)}, W^{(0)}),
\]

(4.5)

where the bare quantities can be expressed in terms of the renormalized ones as indicated above.

Finally, we must relate the renormalized action \( \Gamma_R \) for the background field to \( \Gamma_R \). To this end, we note that all the above transformations preserve as well the background gauge symmetry. Thus, one may define a renormalized operator by

\[
\Omega_R(g, Q, F) = \Omega^{(0)}(g^{(0)}, Q^{(0)}, F^{(0)}) = \Omega^{(0)}[Z_g g; Z_Q^{1/2} Q^\mu; Z_F^{1/2} F_{\mu\nu} + Z_{FQ} f_{\mu\nu}(Q)]
\]

(4.6)

which reduces to lowest order to \( \Omega^{(0)}(g, B, F) \) in Eq. (3.8) and maintains to all orders the background gauge invariance. Hence, to higher orders, the appropriate generalization of Eq. (3.8) may be written in the form

\[
\bar{\Gamma}_R = \Omega_R(g, Q, F) \Gamma_R
\]

(4.7)

which allows to deduce the renormalized effective action \( \bar{\Gamma}_R \) by the application of the operation \( \Omega_R \) to \( \Gamma_R \).

V. CONCLUSION

Background field quantization has some appealing features, especially when considering the renormalization of gauge theories. The relation between the coupling constant and the background field renormalization (4.2), has been exploited in explicit calculations in the Standard Model [24, 25]. In a four dimensional space-time, this relation leads to an observable quantity. Thus, in this case there is no reason why the divergent part of the background field self-energy should be gauge-independent. In higher dimensions, the YM theory is non-renormalizable and then it is no longer possible to directly relate \( \langle BB \rangle \) to an observable quantity. However, one may define a renormalized operator by

\[
\Omega_R(g, Q, F) = \Omega^{(0)}(g^{(0)}, Q^{(0)}, F^{(0)}) = \Omega^{(0)}[Z_g g; Z_Q^{1/2} Q^\mu; Z_F^{1/2} F_{\mu\nu} + Z_{FQ} f_{\mu\nu}(Q)]
\]

which involves only a finite number of interacting cubic vertices [16, 17] and allows one to introduce a graviton propagator that is both traceless and transverse [26, 27].

Appendix A: Generating function for 1PI Green’s Functions

The necessity of subtracting terms linear in the quantum field \( Q^a_{\mu} \) from the Lagrangian \( \mathcal{L}_{YM}(B+Q) \) when computing \( \tilde{Z} \) in Eq. (2.4) can be clarified by considering directly the path integral for the 1PI generating functional \( \hat{\Gamma}[Q, B] \) [28]. Rather than working with the generating functional appearing in Eq. (2.4) that follows from background field quantization, we consider the 1PI generating functional that arises with conventional quantization. This generating functional \( \hat{\Gamma}[f] \), which depends on the average of the quantum field \( \phi \), is related to \( \Gamma[Q, B] \) arising in the background field method by [3, 4]

\[
\hat{\Gamma}[f] = \Gamma[Q = 0, B = f]
\]

(A1)
If we consider the field $\phi$ with a Lagrangian $L(\phi)$, then, in the Euclidean space

$$Z[J] = \int D\phi \exp \left\{ -\frac{1}{\hbar} \int dx [L(\phi) + J\phi] \right\}$$

$$\equiv \exp \left\{ -\frac{1}{\hbar} W[J] \right\}$$

leads to a generating functional for 1PI diagrams

$$\Gamma[f] = W[J] - \int dx f(x) J(x),$$

where

$$f(x) = \frac{\delta W[J]}{\delta J(x)},$$

$$J(x) = -\frac{\delta \Gamma[f]}{\delta f(x)}.$$  \hspace{1cm} (A4b)

Together, Eqs. (A2) and (A3) lead to

$$\exp \left\{ -\frac{1}{\hbar} \Gamma[f] \right\} = \int D\phi \exp \left\{ -\frac{1}{\hbar} \int dx [L(\phi) + J(\phi - f)] \right\}$$

which, upon making the shift $\phi \rightarrow \phi + f$, becomes

$$\int D\phi \exp \left\{ -\frac{1}{\hbar} \int dx [L(f + \phi) + J\phi] \right\}.$$  \hspace{1cm} (A6)

In Eq. (A6), $J(x)$ is no longer independent as it is in Eq. (A2); it is a function of $f(x)$ on account of Eq. (A4b).

If we now expand

$$L(f + \phi) = L(f) + L'(f)\phi + \frac{1}{2!} L''(f)\phi^2 + \frac{1}{3!} L'''(f)\phi^3 + \frac{1}{4!} L^{IV}(f)\phi^4$$

then Eq. (A6) becomes

$$\exp \left\{ -\frac{1}{\hbar} \Gamma[f] \right\} = \exp \left\{ -\frac{1}{\hbar} \int dx L(x) \right\} \exp \left\{ -\frac{1}{\hbar} \int dx \left[ \frac{1}{3!} L'''(f) \left( -\hbar \frac{\delta}{\delta j(x)} \right)^3 + \frac{1}{4!} L^{IV}(f) \left( -\hbar \frac{\delta}{\delta j(x)} \right)^4 \right] \right\}$$

$$\int D\phi \exp \left\{ -\frac{1}{\hbar} \int dx \left[ L''(f)\phi^2 + j(x)\phi(x) \right] \right\},$$

where by Eqs. (A4) and (A7)

$$j(x) = L'(f(x)) - \frac{\delta \Gamma[f]}{\delta f(x)}.$$  \hspace{1cm} (A9)

If we now make the loop expansion of $\Gamma[f]$ so that

$$\Gamma[f] = \Gamma_0[f] + \hbar \Gamma_1[f] + \hbar^2 \Gamma_2[f] + \ldots$$

then upon matching powers of $\hbar$ in Eq. (A8) we obtain

$$\Gamma_0[f] = \int dx L(f),$$  \hspace{1cm} (A11)

$$\Gamma_1[f] = -\frac{1}{2} \log \text{det } L''(f),$$  \hspace{1cm} (A12)
\[ \Gamma_2[f] = \frac{1}{2} \int dxdy \frac{\delta \Gamma_1}{\delta f(x)} \Delta(x - y) \frac{\delta \Gamma_1}{\delta f(y)} + \frac{1}{2} \int dxdy \frac{\delta^3 \mathcal{L}}{\delta f^3(x)} \Delta(0) \Delta(x - y) \frac{\delta \Gamma_1}{\delta f(y)} - \frac{1}{8} \int dxdy \frac{\delta^3 \mathcal{L}}{\delta f^3(x)} \frac{\delta^3 \mathcal{L}}{\delta f^3(y)} (\Delta(0))^2 \Delta(x - y) - \frac{1}{3!2!} \int dxdy \frac{\delta^3 \mathcal{L}}{\delta f^3(x)} \frac{\delta^3 \mathcal{L}}{\delta f^3(y)} (\Delta(x - y))^3 + \frac{1}{8} \int dx \frac{\delta^4 \mathcal{L}}{\delta f^4(x)} (\Delta(0))^2, \] (A13)

where

\[ \Delta(x - y) = (\mathcal{L}'(f))^{-1} \] (A14)

and by Eq. (A12)

\[ \frac{\delta \Gamma_1}{\delta f(x)} = -\frac{1}{2} \left( \Gamma \frac{1}{\mathcal{L}'(f)} \right) \frac{\delta^3 \mathcal{L}}{\delta f^3(x)}. \] (A15)

Upon using (A15), we see that \( \Gamma_2[f] \) reduces to the last two terms on the right side of Eq. (A13) which can be represented graphically by

\[ \text{+} \]

which are the two 1PI graphs in background field theory quantization.

In Refs [29–31] it is shown that the \( S \)-matrix is independent of the vertices generated by dependence of gauge fixing on the background field, both with covariant gauge fixing of Eq. (1.2) and with other non-covariant gauges [30, 31].

### Appendix B

Here we present the Feynman rules which arise from argument of the exponential \( iS \) in (2.6). The bilinear terms in the quantum fields \( F \) and \( Q \) can be expressed in matrix form as follows (here we follow [16])

\[ \frac{1}{2} \left( Q_\mu, F^\sigma_{\lambda\rho} \right) \left( -\frac{1}{2} \left( \frac{\partial \eta^{\mu\nu}}{\partial x^\rho} - \partial^\nu \eta^{\lambda\rho} \right) \frac{1}{4} \left( \partial^\rho \eta^{\sigma\kappa} - \partial^\kappa \eta^{\rho\sigma} \right) \right) \left( F_{\sigma}^{\alpha} \right). \] (B1)

The inverse of the matrix appearing in Eq. (B1) is

\[ \Delta(\partial) = \left( \begin{array}{c} \frac{1}{2} \left( \frac{\partial \eta^{\mu\nu} - (1 - \xi) \partial^\mu \partial^\nu}{\partial x^\rho} \right) - \frac{1}{2} \left( \partial^\rho \eta^{\sigma\kappa} - \partial^\kappa \eta^{\rho\sigma} \right) \\ \frac{1}{2} \left( \partial^\rho \eta^{\sigma\kappa} - \partial^\kappa \eta^{\rho\sigma} \right) \end{array} \right). \] (B2)

where

\[ L^{\lambda\sigma,\rho\kappa}(\partial) = \frac{1}{2} \left( \partial^\lambda \partial^\rho \eta^{\sigma\kappa} + \partial^\sigma \partial^\kappa \eta^{\lambda\rho} - \partial^\lambda \partial^\kappa \eta^{\sigma\rho} - \partial^\sigma \partial^\rho \eta^{\lambda\kappa} \right). \] (B3b)

From (B2) we obtain the following expressions for the momentum space propagators of the quantum fields

\[ Q^a_{\mu} \bigotimes^{\otimes k} Q^b_{\nu} = \frac{-i\delta^{ab}}{k^2 + i0} \left[ \eta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 + i0} \right], \] (B4)

\[ F^a_{\lambda\sigma} \bigotimes^{\otimes k} F^b_{\rho\kappa} = i\delta^{ab} \left[ \eta_{\lambda\rho} \eta_{\sigma\kappa} - \eta_{\lambda\kappa} \eta_{\sigma\rho} - \frac{1}{k^2 + i0} (k_\lambda k_\rho \eta_{\sigma\kappa} + k_\kappa k_\rho \eta_{\lambda\rho} - k_\lambda k_\kappa \eta_{\rho\sigma} - k_\sigma k_\rho \eta_{\lambda\kappa}) \right]. \] (B5)
and

\[ Q^a \rightarrow k \quad F^{ab}_{\mu \nu} = -\frac{\delta^{ab}}{k^2 + i0} (k_\rho \eta_{\rho \mu} - k_\kappa \eta_{\kappa \mu}). \tag{B6} \]

Similarly, the quadratic term for the ghost fields yields

\[ e^a - k \to -e^b = \frac{i\delta^{ab}}{k^2 + i0}. \tag{B7} \]

From the interaction terms in (2.6) we obtain

\[ B^b_{\mu \nu} f^{ab}_{\rho \kappa} = -\frac{i g}{2} f^{abc} (\eta_{\mu \lambda} \eta_{\nu \sigma} - \eta_{\mu \sigma} \eta_{\nu \lambda}), \tag{B8} \]

\[ B^b_{\rho \sigma} f^{ab}_{\mu \nu} = -\frac{i g}{2} f^{abc} (\eta_{\mu \lambda} \eta_{\nu \sigma} - \eta_{\mu \sigma} \eta_{\nu \lambda}), \tag{B9} \]

\[ B^b_{\nu \rho} f^{ab}_{\mu \lambda} = -\frac{i g}{2} f^{abc} (\eta_{\mu \lambda} \eta_{\nu \sigma} - \eta_{\mu \sigma} \eta_{\nu \lambda}), \tag{B10} \]

\[ B^b_{\lambda \rho} f^{ab}_{\mu \nu} = -\frac{g}{\xi} f^{abc} (\eta_{\mu \lambda} \eta_{\nu \sigma} - \eta_{\mu \sigma} \eta_{\nu \lambda}), \tag{B11} \]

\[ B^b_{\rho \sigma} f^{ab}_{\mu \lambda} = \frac{-ig^2}{\xi} (f^{ace \lambda \eta \mu \rho \sigma} + f^{ade \mu \rho \sigma \eta \lambda \mu}) \tag{B12} \]

\[ B^b_{\mu \nu} f^{ab}_{\rho \kappa} = g f^{abc} (p + q)_\mu. \tag{B13} \]
where we are using the momentum space representation $f_{\mu\nu}^{a} = i\delta^{ab}(k\mu\eta_{\nu}\beta - k\nu\eta_{\mu}\beta)B^{b\beta}(k)$.

Appendix C

The one-loop contributions to the two-point function $\langle BB \rangle$ are given by the Feynman diagrams of fig. 1 (we are not including tadpole diagrams which arises from the vertices $\text{[B12]}$ and $\text{[B16]}$ in the appendix B). After the loop momentum integration, the result can only depend (by covariance) on the two tensors $\eta_{\mu\nu}$ and $k\mu k\nu$. A convenient tensor basis is

$$T_{\mu\nu}^{1} = k\mu k\nu - k^{2}\eta_{\mu\nu} \quad \text{and} \quad T_{\mu\nu}^{2} = k\mu k\nu$$

so that each diagram in figure 1 can be written as $\Pi^{I \mu\nu}_{\mu\nu}(k) = Ng^{2}\delta_{\mu\nu}\Pi^{I}_{\mu\nu}(k)$ (we are using $f^{a mn}f^{bnm} = N\delta^{ab}$), where

$$\Pi^{I}_{\mu\nu}(k) = \sum_{i=1}^{2} T_{\mu\nu}^{i}(k)C_{I}^{i}(k); \quad I = a, b, c \ldots h.$$  

The coefficients $C_{I}^{j}$ can be obtained solving the following system of two algebraic equations

$$\sum_{i=1}^{2} T_{\mu\nu}^{i}(k)T^{j\mu\nu}(k)C_{I}^{i}(k) = \Pi^{I}_{\mu\nu}(k)T^{j\mu\nu}(k) \equiv J^{I j}(k); \quad j = 1, 2.$$  

Using the Feynman rules for $\Pi^{I}_{\mu\nu}(k)$ the integrals on the right hand side have the following form

$$J^{I j}(k) = \int \frac{d^{d}p}{(2\pi)^{d}}s^{I j}(p, q, k).$$

where $q = p + k$; $p$ is the loop momentum, $k$ is the external momentum and $s^{I j}(p, q, k)$ are scalar functions. Using the relations

$$p \cdot k = (q^{2} - p^{2} - k^{2})/2,$$
$$q \cdot k = (q^{2} + k^{2} - p^{2})/2,$$
$$p \cdot q = (p^{2} + q^{2} - k^{2})/2.$$  

(C5a)  
(C5b)  
(C5c)
the scalars $s^{ij}(p,q,k)$ can be reduced to combinations of powers of $p^2$ and $q^2$. As a result, the integrals $J^{ij}(k)$ can be expressed in terms of combinations of the following well known integrals

$$I_{lm} = \int \frac{1}{(2\pi)^d \left(p^2\right)^l \left(q^2\right)^m} \frac{i (k^2)^{d/2-l-m}}{4\pi^{d/2}} \frac{\Gamma(l + m - d/2) \Gamma(d/2-l) \Gamma(d/2 - m)}{\Gamma(l) \Gamma(m) \Gamma(d-l-m)}, \quad (C6)$$

where powers $l$ and $m$ greater than one may only arise from the terms proportional to $1 - \xi$ in the gluon propagator (see Eq. \(B4\)). The only non-vanishing (ie non tadpole) integrals are

$$I_{11} = \frac{i (k^2)^{d/2-2} \Gamma \left(2 - \frac{d}{2}\right) \Gamma \left(\frac{d}{4} - 1\right)^2}{2^{d-4} \pi^{d/2}} \frac{\Gamma(d-2)}{\Gamma(d-2)} \frac{(2 - d)(d-2)}{k^2} I^{11}, \quad (C7a)$$

$$I_{12} = I_{21} = \frac{(3-d)}{k^2} I^{11}, \quad (C7b)$$

$$I_{22} = \frac{(3-d)(6-d)}{k^4} I^{11}. \quad (C7c)$$

Implementing the above described procedure as a straightforward computer algebra code, we readily obtain the following exact results for $C_1^a$ and $C_2^a$

$$C_1^a = \left[ \frac{1}{4} (d-2) \xi + \frac{(2-d)(d-2)}{4(d-1)} \right] I^{11}; \quad C_2^a = \frac{d-2}{4} I^{11}. \quad (C8a)$$
\[ C^b_1 = \frac{1}{4} I^{11}; \quad C^b_2 = \frac{1-d}{4} I^{11} \]
(C8b)
\[ C^c_1 = \left[ \frac{1}{8} (d-4) \xi^2 - \frac{\xi}{2} - \frac{d}{8} \right] I^{11}; \quad C^c_2 = 0 \]
(C8c)
\[ C^d_1 = \left[ \frac{1}{2} (d-3) \xi + \frac{1-d}{2} \right] I^{11}; \quad C^d_2 = 0 \]
(C8d)
\[ C^e_1 = \frac{1}{1-d} I^{11}; \quad C^e_2 = 0 \]
(C8e)
\[ C^f_1 = \frac{1}{4 (d-1)} I^{11}; \quad C^f_2 = -\frac{1}{4} I^{11} \]
(C8f)
\[ C^g_1 = \left[ \frac{1}{2 (d-1)} - 1 \right] I^{11}; \quad C^g_2 = \frac{1}{2} I^{11} \]
(C8g)
\[ C^h_1 = -\frac{1}{2} (\xi + 1) I^{11}; \quad C^h_2 = 0 \]
(C8h)

Adding all the diagrams, we obtain the following transverse result for the one-loop contribution to \( \langle BB \rangle \)
\[ \Pi^ab_{\mu\nu} = N g^2 \delta^ab \left[ \frac{d-4}{8} \xi^2 + \frac{3 (d-4)}{4} \xi + \frac{1}{2 - 2d} - \frac{7d}{8} \right] I^{11} (k_\mu k_\nu - k^2 \eta_{\mu\nu}) . \]
(C9)

Finally, using \( d = 4 - 2\epsilon \) we obtain the following contribution for the UV pole (\( I^{11} \approx 1/(16\pi^2\epsilon) \))
\[ \Pi^ab_{\mu\nu}(k)_{UV} = -\frac{11}{3} i \delta^ab \frac{N g^2}{16\pi^2\epsilon} (k_\mu k_\nu - k^2 \eta_{\mu\nu}) . \]
(C10)

It is also interesting the note that graphs (c), (d) and (h), Fig. 1, which contains the linear part of \( f^a_{\mu\nu} \), give the following UV gauge dependent contribution
\[ -\left( \frac{\xi}{2} + \frac{5}{2} \right) i \delta^ab \frac{N g^2}{16\pi^2\epsilon} (k_\mu k_\nu - k^2 \eta_{\mu\nu}) . \]
(C11)

This result shows that the terms containing \( f^a_{\mu\nu}(B) \) in Eq. [2.4] are indeed necessary in order to have a consistent gauge independent result, such as Eq. (C10). Since \( f^a_{\mu\nu} \) has been induced by the subtraction from \( \mathcal{L}_{YM}(B + Q) \), in [1.1], of the terms which are linear in \( Q \), this calculation of \( \langle BB \rangle \) provides an explicit example of the consistency of the subtraction prescription.

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