A photon can encode several bits of information based on an alphabet of its time of arrival, energy, and polarization. Heisenberg’s uncertainty principle places a limit on measuring pairs of physical properties of a particle, limiting the maximal information efficiency to < 59 bits per photon in practice, and < 171 bits per photon at Planck energy, at a data rate of one photon per second.

1. INTRODUCTION

Photons are precious in interstellar communications due to the large distances involved (Hippke 2017a). Therefore, it is important to understand photon information efficiency (PIE): how many bits can be encoded into each photon? Using an alphabet based on the photon’s dimensions (time of arrival, energy, and polarization), encoding schemes can be built. An introduction on how to calculate the number of bits per photon as a function of the alphabet size, noise, and losses was given in Hippke (2017c). Now, we examine the ultimate limits based on Heisenberg’s uncertainty principle, atomic surface smoothness limits, and Planck energy.

2. UNCERTAINTY PRINCIPLE

Heisenberg’s uncertainty principle states that “the more precisely the position is determined, the less precisely the momentum is known in this instant, and vice versa” (Heisenberg 1927), so that ∆E ∆t ≥ ℏ/2 where ∆E is the standard deviation of the particle energy, ∆t is the time it takes the expectation value to change by one standard deviation, and ℏ is the reduced Planck constant. A photon pulse with a temporal width ∆t can therefore not be monochromatic, but has a spectrum. Both are related through a Fourier transform, and it can be shown that (Griffiths 2004; Rullière 2005)

$$\Delta t_{\text{min}} \geq K \frac{\lambda_0^2}{\Delta \lambda c}$$

(1)

where λ₀ is the central wavelength, ∆λ is the width of the spectrum (FWHM), c is the speed of light and K ≈ 0.441 for a Gaussian pulse shape. For example, an optical laser pulse (λ₀ = 500 nm) with a 10% bandwidth (∆λ = 50 nm) has a minimum width of ∆t_{min} ≈ 7.3 × 10^{-15} s, or ≈ 7 fs.

3. MINIMUM WAVELENGTH

Decreasing λ₀ allows for shorter ∆t. A practical limit comes from the fact that polished surfaces of diffraction-limited telescope mirrors or lenses require a surface smoothness smaller than the wavelength (Rayleigh 1879; Danjon & Couder 1935), precisely < λ/4 peak-to-valley as well as < λ/14 root-mean-square surface accuracy (Strehl 1894; Smith 2008). The physical limit of materials is set by the atomic radius ≈ 0.1 nm (Bohr 1913). Near the limit, X-ray focusing limits arise from scattering at the electrons of the atomic shell (Kirkpatrick & Baez 1948; Suzuki 2004) with limits of ≈ 0.03 nm (Yu et al. 1999). This results in a theoretical physical limit for usable optics of λ ≈ 0.5 nm (Hippke & Forgan 2017). The limit can be surpassed by beam-forming with electromagnetic fields, e.g. using a free electron laser, however such methods are not energetically competitive (Hippke 2017b). In any case, the highest possible is the Planck energy, E_P = √ℏc^5/G with λ₀ ≈ 10^{-34} m. Higher energy particles would directly collapse into a black hole.

4. MAXIMUM BANDWIDTH

Increasing bandwidth ∆λ allows for shorter ∆t (although PIE limits will be dominated by λ₀). Interstellar dispersion and scattering place upper limits on the usable wavelength, and therefore on the bandwidth.

Earth’s ionosphere is opaque for frequencies v < 10 MHz (λ > 30 m), and the ionized interstellar medium in the galaxy absorbs radio signals v < 2 MHz (λ > 150 m) (Condon & Romań 2016). In addition, interstellar dispersion broadens pulses λ > μm (Taylor & Cordes 1993; Shostak 2011) to

$$\Delta t_{\text{disp}} = 4 \times 10^{-15} \text{DM} \lambda^2 c^5 \text{ s}$$

(2)

where a typical value for the dispersion measure is DM = 1 over 100 pc in the solar neighborhood, and DM = 100 over kpc towards the galactic center (Yao et al. 2017). This limits pulse widths to ∆t_{disp} > 10^{-8} s at λ = m over pc distances to the nearest stars which is stricter than the uncertainty limit, ∆t_{min} ≈ 10^{-10} s. Dedispersion is imperfect, because the true interference is unknown and changes with time (Alder 2012). Photon counting at radio frequencies is difficult near the the quantum limit due to the low energy per photon.
5. INFORMATION EFFICIENCY

The ultimate noiseless quantum information efficiency is (Holevo 1973; Giovannetti et al. 2004)

\[
\text{PIE} = g(\eta M) \text{ (bits per photon)}
\]

where \( M \) is the number of photons per mode, \( \eta \) is the receiver efficiency and

\[
g(x) = (1 + x) \log_2(1 + x) - x \log_2 x
\]

so that \( g(x) \) is a function of \( \eta M \). Without noise and for a perfect receiver (\( \eta = 1 \)) we can approximate

\[
\text{PIE} \approx \log_2(M^{-1})
\]

valid to within 5 % for \( M^{-1} > 10 \). Heisenberg’s uncertainty principle is an integral part of the Holevo bound, as the number of modes is finite for any finite amount of energy, time and space.

When we decrease \( \lambda_0 \) towards its minimum, the shortest \( \Delta t_{\text{min}} \) occurs for \( \Delta \lambda = \lambda_0 \), i.e. at 100 % bandwidth. When approximating \( K = 0.5 \) in Equation 1 we get

\[
\Delta t_{\text{min}} \geq 0.5 \frac{\lambda_0^2}{\lambda_0 c} = \frac{\lambda_0}{2c}.
\]

Each photon in a communication shall be transmitted (and received) within a finite amount of time, \( t_{\text{dur}} \). Within this time, the number of modes for this photon is then

\[
M = \frac{\lambda_0}{2c t_{\text{dur}}}.
\]

We can now insert Equation 7 into Equation 5 and get

\[
\text{PIE} \lesssim \log_2 \left( \frac{2c t_{\text{dur}}}{\lambda_0} \right) \text{ (bits per photon)}
\]

For \( \lambda_0 = 1 \text{ nm and } t_{\text{dur}} = 1 \text{ s we find } C_{\text{ult}} \approx 59 \text{ bits per photon, and 171 bits per photon at Planck energy. Including two alternative states in polarizations would double } M^{-1} \text{ and thus increase PIE by } \log_2(2) = 1 \text{ bit per photon.}

6. TRADE-OFF BETWEEN INFORMATION EFFICIENCY AND DATA RATE

Maximizing PIE comes at the cost of low dimensional information efficiency (DIE), measured in bits per mode, and thus data rate (in physical units: bits/sec/Hz). At the ultimate Holevo limit, PIE relates to DIE as (Dolinar et al. 2011) \( \text{PIE} = \text{DIE} \times M^{-1} \). Figure 1 (left) shows the trade-off between PIE and DIE.

When one photon leverages \( 10^{18} \) time slots within one second, then only one photon can be received in this channel at the given PIE. If instead \( 10^{18} \) colors are used, the arrival of the photon can only be measured to within this one second. Thus, data rates are slow, but can be traded for higher DIE and thus higher data rates (but lower PIE). Conversely, one could further increase PIE by sending even fewer photons, e.g. one per year, which gives an additional \( 3 \times 10^7 \) one-second slots, and thus increases PIE by \( \approx 23 \) bits per photon.

7. BIT ERRORS

The quantum nature of physical systems imposes an impossibility to discriminate perfectly between states. The lowest error probability allowed by quantum mechanics is the Helstrom (1976) bound

\[
P_{\text{Helstrom}} = \frac{1}{2} \left( 1 - \sqrt{1 - e^{-4 \text{PIE}^{-1}}} \right).
\]

The performance of practical receiver implementations (Wittmann et al. 2008; Gallion & Mendieta 2011; Cariolaro 2015) converge towards the bound in different
regimes (Kennedy 1973; Dolinar 1973). The resulting bit error rate as a function of PIE is shown in Figure 1 (right), and can be treated with software forward error correction at the expense of data rate (Moon 2005; Huang 2009). The extra overhead from using optimal correction codes is small (few percent).

8. DISCUSSION AND CONCLUSION

In a practical interstellar communication, the distance between transmitter and receiver will be large causing significant diffractive beam widening, so that only a small fraction (typically $\ll 10^{-10}$) of the transmitted photons are collected by the receiver, ideally exactly one per pulse. As the source is unresolved in realistic interstellar communications, we neglected spatial encoding.

In practice, the maximum information efficiency is limited by non-zero thermal noise. For $\lambda = \mu m$ and 100% bandwidth, $\Delta t_{\text{min}} \approx 10^{-15}$ s which encodes $C_{\text{ult}} \approx 49$ bits per photon. Current laboratory demonstrations achieve 19.3 bits per photon, or $\approx 40\%$ of the maximum, limited by dark noise in a $^3$He-cooled (350 mK) superconducting detector (section 3.1 in Farr et al. 2013).

Acknowledgments MH is thankful to Vittorio Giovannetti and David G. Messerschmitt for useful discussions.

REFERENCES

Alder, B. 2012, Radio Astronomy, Methods in computational physics (Elsevier Science)

Bohr, N. 1913, Philosophical Magazine, 26, 476

Cariolaro, G. 2015, Quantum Communications, Signals and Communication Technology (Springer International Publishing)

Condon, J., & Ransom, S. 2016, Essential Radio Astronomy, Princeton Series in Modern Observational Astronomy (Princeton University Press)

Danjon, A., & Couder, A. 1935, Lunettes et telescopces - Theorie, conditions d'emploi, description, reglage (Edite par Librairie Scientifique et Technique)

Dolinar, S., Birnbaum, K. M., Erkmen, B. I., & Moision, B. 2011, ArXiv e-prints, arXiv:1104.2643 [quant-ph]

Dolinar, S. J. 1973, An optimum receiver for the binary coherent state quantum channel, Tech. rep., MIT Research Laboratory of Electronics, Quarterly Progress Report 111

Farr, W. H., Choi, J. M., & Moision, B. 2013, in Proc. SPIE, Vol. 8610, Free-Space Laser Communication and Atmospheric Propagation XXV, 861006

Gallion, P., & Mendieta, F. J. 2011, in Proc. SPIE, Vol. 8065, SPIE Eco-Photonics 2011: Sustainable Design, Manufacturing, and Engineering Workforce Education for a Green Future, 80650F

Giovannetti, V., Guha, S., Lloyd, S., et al. 2004, Physical Review Letters, 92, 027902

Griffiths, D. J. 2004, Introduction to Quantum Mechanics (Prentice Hall International)

Heisenberg, W. 1927, Zeitschrift für Physik, 43, 172

Helstrom, C. W. 1976, Quantum Detection and Estimation Theory, Mathematics in Science and Engineering (Elsevier Science)

Hippke, M. 2017a, ArXiv e-prints, arXiv:1706.03795 [astro-ph.IM]

—. 2017b, ArXiv e-prints, arXiv:1711.07962 [astro-ph.IM]

—. 2017c, ArXiv e-prints, arXiv:1712.05682 [astro-ph.IM]

Hippke, M., & Forgan, D. H. 2017, ArXiv e-prints, arXiv:1711.05761 [astro-ph.IM]

Holevo, A. S. 1973, Problemy Peredachi Informatsii, 9, 3

Huang, J. 2009, Software Error Detection through Testing and Analysis, Wiley InterScience (Wiley)

Kennedy, R. S. 1973, A Near-Optimum Receiver for the Binary Coherent State Quantum Channel, Tech. rep., MIT Research Laboratory of Electronics, Quarterly Progress Report 111

Kirkpatrick, P., & Baez, A. V. 1948, Journal of the Optical Society of America (1917-1983), 38, 766

Moon, T. 2005, Error Correction Coding: Mathematical Methods and Algorithms (Wiley)

Rayleigh, L. 1879, Philosophical Magazine Series 5, 8, 261

Rullière, C., ed. 2005, Femtosecond Laser Pulses (Springer New York)

Shostak, S. 2011, Acta Astronautica, 68, 366

Smith, W. J. 2008, Modern Optical Engineering: The Design of Optical Systems, Fourth Edition (The McGraw-Hill Companies)

Strehl, K. 1894, Theorie des Fernrohrs. Auf Grund der Beugung des Lichts. 1 Teil. (Nabu Press)

Suzuki, Y. 2004, Japanese Journal of Applied Physics, 43, 7311

Taylor, J. H., & Cordes, J. M. 1993, ApJ, 411, 674

Wittmann, C., Takeoka, M., Cassemino, K. N., et al. 2008, Physical Review Letters, 101

Yao, J. M., Manchester, R. N., & Wang, N. 2017, ApJ, 835, 29

Yu, J., Namba, Y., Ngoi, B. K., & Zhong, Z. 1999, in Proc. 14th ASPE Annual Meeting, Monterey, 368