Index of continuous families of bounded linear operators in Banach spaces and application

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Abstract

In this paper, we define an analytical index for continuous families of Fredholm operators parameterized by a topological space $X$ into a Banach space $X$. We also consider the Weyl spectrum for continuous families of bounded linear operators and we study its continuity.

1 Index of continuous families of Fredholm operators

Let $X$ be an infinite dimensional Banach space, let $L(X)$ be the Banach algebra of bounded linear operators acting on $X$ and let $T \in L(X)$. We will denote by $N(T)$ the null space of $T$, by $\alpha(T)$ the nullity of $T$, by $R(T)$ the range of $T$ and by $\beta(T)$ its defect. If the range $R(T)$ of $T$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$), then $T$ is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator. A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. In the sequel $\Phi_+(X)$ (resp. $\Phi_-(X)$) will denote the set of upper (resp. lower) semi-Fredholm operators. If both of $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$.

In [1], we defined an analytical index for a continuous family of Fredholm operators parameterized by a topological space $X$ into a Hilbert space $H$, as a sequence of integers, extending naturally the usual definition of the index of a single Fredholm operator and we proved the homotopy invariance of this index. We proved also that if $X$ is a compact locally connected space, satisfying an homotopy condition, then the analytical index establishes an isomorphism between the homotopy equivalence classes of families of Fredholm operators $[X, Fred(H)]$ and the group $\mathbb{Z}^{n_c}$, where $n_c$ is the cardinal of the connected components of $X$, proving by this way a result similar to the theorem of Atiyah-Jänich [2, Theorem 3.40].

The motivation of [1], was the construction of an analytical index for continuous families of Fredholm operators parameterized by a topological space as a sequence

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of integers, extending naturally the usual index of a single Fredholm operator, which is an integer and avoiding the use of vector bundles.

Here, in this section, we extend the definition of the analytical index given in \[1\] to the case of a continuous family of Fredholm operators parameterized by a topological space \(X\) into a Banach space \(X\).

Define an equivalence relation \(R\) on the set \(f\text{dim}(X) \times f\text{cod}(X)\), where \(f\text{dim}(X)\) is the set of finite dimensional vector subspaces of \(X\) and \(f\text{cod}(X)\) is the set of finite codimension vector subspaces of \(X\) by:

\[
(E_1, F_1) R (E_1', F_1') \iff \text{dim}E_1 - \text{codim}F_1 = \text{dim}E_1' - \text{codim}F_1',
\]

where \(\text{dim}\) stands for dimension, while \(\text{codim}\) for codimension.

Since \(X\) is an infinite dimensional vector space, then the map:

\[
\Psi : f\text{dim}(X) \times f\text{cod}(X) / R \to \mathbb{Z},
\]

defined by \(\Psi([E_1, F_1]) = \text{dim}E_1 - \text{codim}F_1\), where \([E_1, F_1]\) is the equivalence class of the couple \((E_1, F_1)\), is a bijection. Moreover \(\Psi\) generate a commutative group structure on the set \([f\text{dim}(X) \times f\text{cod}(X)] / R\) and \(\psi\) is then a group isomorphism.

Consider now a family of Fredholm operators parameterized by a topological space \(X\), that is a continuous map \(T : X \to \text{Fred}(X)\), where \(\text{Fred}(X)\) is the set of Fredholm operators, endowed with the norm topology of \(L(X)\). We denote by \(T_x\) the image \(T(x)\) of an element \(x \in X\).

Define an equivalence relation \(c\) on the space \(X\) by setting that \(x \sim y\), if and only if \(x\) and \(y\) belongs to the same connected component of \(X\). Let \(X^c\) be the quotient space associated to this equivalence relation, let \(C(X, \text{Fred}(X))\) be the space of continuous maps from the topological space \(X\) into the topological space \(\text{Fred}(X)\) and let the map:

\[
q : C(X, \text{Fred}(X)) \to [f\text{dim}(X) \times f\text{cod}(X)] / R]^X^c,
\]

defined by \(q(T) = ([N(T_x), R(T_x)]_{x \in X^c}\) for all \(T \in C(X, \text{Fred}(X))\). Define also the map

\[
\Psi_X : [f\text{dim}(X) \times f\text{cod}(X)] / R]^X^c \to \mathbb{Z}^{n_c}
\]

by setting \(\Psi_X((Y_x)_{x \in X^c}) = (\Psi(Y_x))_{x \in X^c}\) for all \((Y_x)_{x \in X^c} \in [f\text{dim}(X) \times f\text{cod}(X)] / R]^X^c\).

Here \(n_c\) stands for the cardinal of the connected components of the topological space \(X\), assuming that the space \(X\) has at most a countable set of connected components.

**Definition 1.1.** The analytical index (or simply the index) of a family of Fredholm operators \(T : X \to \text{Fred}(X)\), parameterized by a topological space \(X\) is defined by \(\text{ind}(T) = \psi_X(q(T))\).

Explicitly, we have: \(\text{ind}(T) = \psi_T([N(T_x), R(T_x)]_{x \in X^c}) = (\text{ind}(T_x))_{x \in X^c}\).
Thus the index of a family of Fredholm operators \( T \) is a sequence of integers in \( \mathbb{Z}^n \), which may be a finite sequence or infinite sequence, depending on the cardinal of the connected components of \( X \).

**Theorem 1.2.** The index of a continuous family of Fredholm operators \( T \) parameterized by a topological space \( X \) is well defined as an element of \( \mathbb{Z}^n \). In particular if \( X \) is reduced to a single element, then the index of \( T \) is equal to the usual index of the Fredholm operator \( T \).

*Proof.* From the usual properties of the index \( [4, \text{Theorem 5.17}] \), we know that two Fredholm operators located in the same connected component of the set of Fredholm operators have the same index. Moreover, as \( T \) is continuous, the image of a connected component of the topological space \( X \), is included in a connected component of the set of Fredholm operators. This shows that the index of a family of Fredholm operators is well defined, and it is clear that if \( X \) is reduced to a single element, the index of \( T \) defined here is equal to the usual index of the single Fredholm operator \( T \).

**Definition 1.3.** A continuous family \( \mathcal{K} \) from \( X \) to \( L(X) \) is said to be compact if \( \mathcal{K}_x \) is compact for all \( x \in X \). The family \( \mathcal{K} \) is said to be of finite rank if \( \mathcal{K}_x \) is of finite rank for all \( x \in X \).

**Proposition 1.4.** i) Let \( T \in \mathcal{C}(X, \text{Fred}(X)) \) and let \( \mathcal{K} \) be a continuous compact family from \( X \) to \( L(X) \). Then \( T + \mathcal{K} \) is a Fredholm family and \( \text{ind}(T) = \text{ind}(T + \mathcal{K}) \).

ii) Let \( S, T \in \mathcal{C}(X, \text{Fred}(X)) \) be two Fredholm families, then the family \( ST \) defined by \( (ST)_x = S_x T_x \) is a Fredholm family and \( \text{ind}(ST) = \text{ind}(S) + \text{ind}(T) \).

*Proof.* This is clear from the usual properties of Fredholm operators.

**Theorem 1.5.** Assume that \( X \) is a compact topological space. Then the set \( \mathcal{K}C(X, L(X)) \) of continuous compact families from \( X \) to \( L(X) \) is a closed ideal in the Banach algebra \( \mathcal{C}(X, L(X)) \).

*Proof.* Recall that \( \mathcal{C}(X, L(X)) \) is a unital algebra with the usual properties of addition, scalar multiplication and multiplication defined by:

\[
(\lambda S + T)_x = \lambda S_x + T_x, \quad (ST)_x = S_x T_x, \quad \forall (S, T) \in \mathcal{C}(X, L(X))^2, \forall x \in X, \forall \lambda \in \mathbb{C}.
\]

The unit element of \( \mathcal{C}(X, L(X)) \) is the constant function \( I \) defined by \( I_x = I \) the identity of \( X \), for all \( x \in X \). Moreover as \( X \) is compact, then if we set \( \|T\| = \sup_{x \in X} \|T_x\|, \forall T \in \mathcal{C}(X, L(X)) \), then \( \mathcal{C}(X, L(X)) \) equipped with this norm is a Banach algebra. Similarly \( \mathcal{C}(X, L(X)/K(X)) \) equipped with the norm \( \|T\| = \sup_{x \in X} \|PT_x\| \) is a unital Banach algebra, where \( P : L(X) \to L(X)/K(X) \) is the usual projection from \( L(X) \) onto the Calkin algebra \( L(X)/K(X) \).

It is clear that \( \mathcal{K}C(X, L(X)) \) is an ideal of \( \mathcal{C}(X, L(X)) \). Assume now that \( (T_n)_n \) is a sequence in \( \mathcal{K}C(X, L(X)) \) converging in \( \mathcal{C}(X, L(X)) \) to \( T \). Then \( ((T_n)_n)_n \) converges to \( T_x \), as each \( (T_n)_x \) is compact, then \( T \in \mathcal{K}C(X, L(X)) \).
Remark 1.6. In the same way as in the case of the Calkin algebra, Theorem \[1.5\] generates the new Banach algebra \( C(\mathbb{X}, L(X))/KC(\mathbb{X}, L(X)) \). Moreover, there is a natural injection \( \overline{\Pi} : C(\mathbb{X}, L(X))/KC(\mathbb{X}, L(X)) \to C(\mathbb{X}, L(X)/K(X)) \) defined by \( \overline{\Pi}(\mathcal{T}) = P\mathcal{T} \), where \( \mathcal{T} \) is the equivalence class of the element \( \mathcal{T} \) of \( C(\mathbb{X}, L(X)) \) in \( C(\mathbb{X}, L(X))/KC(\mathbb{X}, L(X)) \) and \( P : L(X) \to L(X)/K(X) \) is the natural projection.

Open questions:

1. Given an element \( S \in C(\mathbb{X}, L(X)/K(X)) \), does there exist a continuous family \( \mathcal{T} \in C(\mathbb{X}, L(X)) \) such that \( \overline{\Pi}(\mathcal{T}) = S \)?

2. Let \( H \) be a Hilbert space and \( K \) be a continuous compact family in \( C(\mathbb{X}, L(H)) \). Does there exist a sequence \( (K_n)_n \) of continuous finite rank families in \( (C(\mathbb{X}, L(H)) \) such that \( \lim_{n \to \infty} K_n = K \)?

Theorem 1.7. Assume that \( \mathbb{X} \) is a compact topological space and let \( \mathcal{T} \in C(\mathbb{X}, L(X)) \). Then \( \mathcal{T} \) is a Fredholm family if and only if \( P\mathcal{T} \) is invertible in the Banach algebra \( C(\mathbb{X}, L(X)/K(X)) \).

Proof. Assume that \( \mathcal{T} \) is a Fredholm family, then for all \( x \in X \), \( \mathcal{T}_x \) is a Fredholm operator. Thus \( P\mathcal{T}_x \) is invertible in \( L(X)/K(X) \). Let \( (P\mathcal{T}_x)^{-1} \) be its inverse, then the family \( (P\mathcal{T})^{-1} \) defined by \( (P\mathcal{T})^{-1}(x) = (P\mathcal{T}_x)^{-1} \) is a continuous family, because the inversion is a continuous map in the Banach algebra \( L(X)/K(X) \), and \( (P\mathcal{T})^{-1} \) is the inverse of \( P\mathcal{T} \) in the Banach algebra \( C(\mathbb{X}, L(X)/K(X)) \).

Conversely if \( P\mathcal{T} \) is invertible in the Banach algebra \( C(\mathbb{X}, L(X)/K(X)) \), then there exists \( S \in C(\mathbb{X}, L(X)/K(X)) \) such that \( (P\mathcal{T})S = S(P\mathcal{T}) = \overline{\mathcal{T}} \), where \( \overline{\mathcal{T}} \) is defined by \( \overline{\mathcal{T}}_x = PI \), for all \( x \in \mathbb{X} \). Thus \( (P\mathcal{T}_x)S_x = S_x(P\mathcal{T}_x) = PI \). Thus \( P\mathcal{T}_x \) is invertible in the Calkin algebra \( L(X)/K(X) \), \( \mathcal{T}_x \) is a Fredholm operator and \( \mathcal{T} \in C(\mathbb{X}, Fred(X)) \).

Definition 1.8. Let \( S, \mathcal{T} \) be in \( \in C(\mathbb{X}, Fred(X)) \). We will say that \( S \) and \( \mathcal{T} \) are Fredholm homotopic, if there exists a map \( \Phi : [0, 1] \times X \to L(X) \) such that \( \Phi(0, x) = S_x, \Phi(1, x) = \mathcal{T}_x \) and \( \Phi(t, x) \) is a Fredholm operator, for all \( (t, x) \in [0, 1] \times X \).

Theorem 1.9. Let \( S, \mathcal{T} \) be two Fredholm homotopic elements of \( C(\mathbb{X}, Fred(X)) \). Then \( ind(\mathcal{T}) = ind(S) \).

Proof. Since \( S \) and \( \mathcal{T} \) are Fredholm homotopic, there exists a continuous map \( h : \mathbb{X} \times [0, 1] \to Fred(X) \) such that \( h(x, 0) = S_x(x) \) and \( h(x, 1) = T(x) \) for all \( x \in \mathbb{X} \). For a fixed \( x \in \mathbb{X} \), the map \( h_x : [0, 1] \to Fred(X) \), defined by \( h_x(t) = h(x, t) \) is a continuous path in \( Fred(X) \) linking \( S_x \) to \( \mathcal{T}_x \). Since \( [0, 1] \) is connected and since the usual index is constant on connected sets of \( Fred(X) \), then \( ind(S_x) = ind(T_x) \). So \( q(S) = q(T) \) and then \( ind(S) = ind(T) \).
Theorem 1.10. Let $X$ be a compact topological space. Then the index is a continuous locally constant function from $C(X, Fred(X))$ into the group $\mathbb{Z}^{nc}$.

Proof. Let $T \in C(X, Fred(X))$, then $\forall x \in X, \exists \epsilon_x > 0$, such that $B(T_x, \epsilon_x) \subset Fred(X)$, because $Fred(X)$ is open in $L(X)$. Then the index is constant on $B(T_x, \epsilon_x)$, because $B(T_x, \epsilon_x)$ is compact. We have $X \subset \bigcup_{x \in X} T^{-1}(B(T_x, \epsilon_x))$. Since $X$ is compact, there exists $x_1, ... x_n$ in $X$ such that $X \subset \bigcup_{i=1}^n T^{-1}(B(T_{x_i}, \epsilon_{x_i}))$. Let $\epsilon = min\{\frac{\epsilon_i}{2}, 1 \leq i \leq n\}$ the minimum of the $\frac{\epsilon_i}{2}, 1 \leq i \leq n$, and let $S \in C(X, Fred(X))$ such that $||T - S|| < \frac{\epsilon}{2}$. If $x \in X$, then $||T_x - S_x|| < \frac{\epsilon}{2}$ and there exists $i, 1 \leq i \leq n$, such that $x \in T^{-1}(B(T_{x_i}, \epsilon_{x_i}))$. Then $||S_x - T_{x_i}|| \leq ||S_x - T_x|| + ||T_x - T_{x_i}|| < \epsilon/2 + \epsilon_{x_i}/2 \leq \epsilon_{x_i}$. So $ind(S_x) = ind(T_{x_i}) = ind(T_x)$. Hence the index is a locally constant function, in particular it is a continuous function.

Theorem 1.11. Let $X$ be a compact topological space. Then the set $C(X, Fred(X))$ is an open subset of the Banach algebra $C(X, L(X))$ endowed with the uniform norm $||T|| = sup_{x \in X} ||T_x||$.

Proof. Let $T \in C(X, Fred(X))$, then $\forall x \in X, \exists \epsilon_x > 0$, such that $B(T_x, \epsilon_x) \subset Fred(X)$. We have $X \subset \bigcup_{x \in X} T^{-1}(B(T_x, \epsilon_x))$. Since $X$ is compact, there exists $x_1, ... x_n$ in $X$ such that $X \subset \bigcup_{i=1}^n T^{-1}(B(T_{x_i}, \epsilon_{x_i}))$. Let $\epsilon = min\{\frac{\epsilon_i}{2}, 1 \leq i \leq n\}$ the minimum of the $\frac{\epsilon_i}{2}, 1 \leq i \leq n$. Let $S \in C(X, L(X))$ such that $||T - S|| < \frac{\epsilon}{2}$. If $x \in X$, then $||T_x - S_x|| < \frac{\epsilon}{2}$ and there exists $i, 1 \leq i \leq n$, such that $x \in T^{-1}(B(T_{x_i}, \epsilon_{x_i}))$. Then $||S_x - T_{x_i}|| \leq ||S_x - T_x|| + ||T_x - T_{x_i}|| < \epsilon/2 + \epsilon_{x_i}/2 \leq \epsilon_{x_i}$ and $S_x$ is a Fredholm operator. Therefore $S \in C(X, Fred(X))$ and $C(X, Fred(X))$ is open in $C(X, L(X))$.

Alternatively, we can see that $C(X, Fred(X)) = \Pi^{-1}( (C(X, L(X))/K(X))^{inv})$, where $(C(X, L(X))/K(X))^{inv}$ is the open group of invertible elements of the unital Banach algebra $C(X, L(X)/K(X))$ and $\Pi : C(X, L(X)) \rightarrow C(X, L(X)/K(X))$ is the map defined by $\Pi(T) = PT$, for all $T \in C(X, L(X)/K(X))$.

Corollary 1.12. Let $X$ be a compact topological space and $p \in \mathbb{Z}^{nc}$. Then the set $C_p(X, Fred(X))$ of the continuous Fredholm families of index $p$ is an open subset of the Banach algebra $C(X, L(X))$.

Proof. As the set $\{p\}$ is open in $\mathbb{Z}^{nc}$ and the index is a continuous function, from Theorem [1.10] it follows that $C_p(X, Fred(X))$ is an open subset of the Banach algebra $C(X, L(X))$.

Theorem 1.13. Let $T \in C(X, L(X))$ and $f$ an analytic function in a neighborhood of the spectrum $\sigma(T)$ of $T$ which is non-constant on any connected component of $\sigma(T)$. Then:

1. $f(\sigma(T)) = \sigma(\phi(T))$, in particular $f(T)$ is a Fredholm family if and only if $f(\lambda) \neq 0$, for all $\lambda \in \sigma(T)$. 


2. If \( f(T) \) is a Fredholm family, then \( \text{ind}(f(T)) = (\sum_{n \in \mathbb{N}} n\alpha_{n,x})_{\overline{T} \in \mathcal{X}} \), where for all \( x \in \mathbb{X}, \alpha_{n,x} \) is the number of zeros of \( f \) on the set \( \{ \lambda \in \sigma(T_x) \mid T_x - \lambda I \in Fred(X) \text{ and } \text{ind}(T_x - \lambda I) = n \} \), counted according to their multiplicity.

**Proof.**

1. As we have \( \sigma_F(T) = \bigcup_{x \in \mathbb{X}} \sigma_F(T_x) \), and \( f \) is an analytic function in a neighborhood of the spectrum \( \sigma(T) = \bigcup_{x \in \mathbb{X}} \sigma(T_x) \), then \( f \) is analytic in a neighborhood \( \sigma(T_x) \). From [3, Theorem 1], we obtain \( f(\sigma_F(T_x)) = \sigma_F(f(T_x)) \). Thus:

\[
\sigma_f(\mathcal{F}) = f(\bigcup_{x \in \mathbb{X}} \sigma_F(T_x)) = \bigcup_{x \in \mathbb{X}} f(\sigma_F(T_x)) = \bigcup_{x \in \mathbb{X}} \sigma_F(f(T_x)) = \sigma_F(f(\mathcal{F})).
\]

2. As \( f \) is non-constant on any connected component of \( \sigma(T) \), then it has a finite number of zeros in \( \sigma(T) \), say \( \lambda_1, \ldots, \lambda_k \), with multiplicities \( n_1, n_2, \ldots, n_k \). This imply in particular that the series \( \sum_{n \in \mathbb{N}} n\alpha_{n,x} \) are finite sums, for all \( x \in \mathbb{X} \).

Moreover, there exists a non-vanishing analytic function \( g \) on \( \sigma(T) \), such that \( f(\lambda) = g(\lambda) \prod_{i=1}^{n} (\lambda - \lambda_i)^{n_i} \). Thus \( g(T) \) is invertible, \( \text{ind}(g(T)) = 0 \) and

\[
\text{ind}(f(T)) = \text{ind}(\prod_{i=1}^{n} (T - \lambda_i I)^{n_i}) = \sum_{i=1}^{n} n_i \text{ind}(T - \lambda_i I) = \left( \sum_{i=1}^{n} n_i \text{ind}(T_x - \lambda_i I) \right)_{\overline{T} \in \mathcal{X}} = \left( \sum_{i=1}^{n} \text{ind}(g(T_x)(T_x - \lambda_i I)^{n_i}) \right)_{\overline{T} \in \mathcal{X}} = \left( \text{ind}(f(T_x)) \right)_{\overline{T} \in \mathcal{X}} ,
\]

because \( g(T_x) \) is invertible, for all \( x \in \mathbb{X} \). Applying [3, Theorem 1, c] to \( f(T_x) \), we obtain

\[
\text{ind}(f(T_x)) = \sum_{n \in \mathbb{N}} n\alpha_{n,x}.
\]

Observe that by the Fredholmness of \( f(T) \), the integer \( \sum_{n \in \mathbb{N}} n\alpha_{n,x} \) is constant on \( \overline{T} \), for all \( x \in \mathbb{X} \). Thus we get the desired result.

## 2 On the continuity of the Weyl spectrum

The definition of an index for continuous families of Fredholm operators as a sequence of integers, enable us to define the Weyl spectrum \( \sigma_W(T) \) for a continuous family of bounded linear operators \( T \). In [3], upper and lower semi-continuity properties of the map \( T \to \sigma_W(T) \), were studied in the case of single bounded linear operators. Here, we study similar questions for continuous families of bounded linear operators. For the definitions of upper semi-continuity, lower semi-continuity, continuity, lower limit (\( \lim \inf \)), upper limit (\( \lim \sup \)) and limit (\( \lim \)) of sets, we refer the reader to [3].

**Definition 2.1.** Let \( T \in \mathcal{C}(\mathbb{X}, L(\mathbb{X})) \). Then \( T \) is called a Weyl family if it is a Fredholm family of index 0.
The Weyl spectrum $\sigma_W(\mathcal{T})$ of $\mathcal{T}$ is defined by $\sigma_W(\mathcal{T}) = \{ \lambda \in \mathbb{C} : \mathcal{T} - \lambda I$ is not a Weyl family $\}.$

It’s easily seen that $\sigma_W(\mathcal{T}) = \bigcup_{x \in \mathcal{X}} \sigma_W(\mathcal{T}_x),$ see [1] Theorem 3.5 for more details.

**Theorem 2.2.** The map $\mathcal{T} \rightarrow \sigma_W(\mathcal{T}),$ is an upper semi-continuous function from $\mathcal{C}(\mathcal{X}, L(X))$ into closed subsets of $\mathbb{C}.$

**Proof.** Since the index is a continuous function, then for all $\mathcal{T} \in \mathcal{C}(\mathcal{X}, L(X))$, $\sigma_W(\mathcal{T})$ is closed.

Now, let $\mathcal{T} \in \mathcal{C}(\mathcal{X}, L(X))$ and let $\mathcal{T}_n$ in $\mathcal{C}(\mathcal{X}, L(X))$ such that $\lim_{n \rightarrow \infty} \mathcal{T}_n = \mathcal{T}.$ If $\lambda \notin \sigma_W(\mathcal{T}),$ then $\mathcal{T} - \lambda I$ is a Weyl family. From Corollary [1.12] there exists $\eta > 0,$ such that if $S \in \mathcal{C}(\mathcal{X}, L(X))$, satisfy $\|\mathcal{T} - \lambda I - S\| < \eta,$ then $S$ is a Weyl family. Since $\lim_{n \rightarrow \infty} \mathcal{T}_n = \mathcal{T},$ there exists an integer $N$ such that for all integer $n \geq N,$ we have $\|\mathcal{T}_n - I - (\mathcal{T} - \lambda I)\| < \frac{\eta}{2}.$ If $|\lambda - \mu| < \frac{\eta}{2},$ then $\|\mathcal{T} - \lambda I - (\mathcal{T}_n - \mu I)\| < \eta.$ Hence $\mathcal{T}_n - \mu I$ is a Weyl family. This imply that $\lambda \notin \limsup_{n \rightarrow \infty} \sigma_W(\mathcal{T}_n)$ and so $\limsup_{n \rightarrow \infty} \sigma_W(\mathcal{T}_n) \subset \sigma_W(\mathcal{T}).$

This proves that the map $\mathcal{T} \rightarrow \sigma_W(\mathcal{T})$ is upper semi-continuous.

**Theorem 2.3.** Let $\mathcal{T}_n, \mathcal{T}$ in $\mathcal{C}(\mathcal{X}, L(X))$ such that $\lim_{n \rightarrow \infty} \mathcal{T}_n = \mathcal{T}.$ If $\lim \sigma(\Pi(\mathcal{T}_n)) = \sigma(\Pi(\mathcal{T})), \text{ then } \lim_{n \rightarrow \infty} \sigma_W(\mathcal{T}_n) = \sigma_W(\mathcal{T}).$

**Proof.** Recall that $\Pi : \mathcal{C}(\mathcal{X}, L(X)) \rightarrow \mathcal{C}(\mathcal{X}, L(X)/K(X))$ is the map defined by $\Pi(\mathcal{T}) = P\mathcal{T},$ for all $\mathcal{T} \in \mathcal{C}(\mathcal{X}, L(X)/K(X)),$ where $P : L(X) \rightarrow L(X)/K(X)$ is the natural projection.

Since the map $\mathcal{T} \rightarrow \sigma_W(\mathcal{T})$ is an upper semi-continuous function, it is enough to prove that $\sigma_W(\mathcal{T}) \subset \liminf_{n \rightarrow \infty} \sigma_W(\mathcal{T}_n).$ Suppose that $\lambda \notin \liminf_{n \rightarrow \infty} \sigma_W(\mathcal{T}_n),$ so that there is a neighborhood $V$ of $\lambda$ that does not intersect infinitely many $\sigma_W(\mathcal{T}_n).$ Since $\sigma(\Pi(\mathcal{T}_n)) \subset \sigma_W(\mathcal{T}_n),$ then $V$ does not intersect infinitely many $\sigma(\Pi(\mathcal{T}_n)).$ Hence $\lambda \notin \lim_{n \rightarrow \infty} \sigma(\Pi(\mathcal{T}_n)) = \sigma(\Pi(\mathcal{T}))$ and then $\mathcal{T} - \lambda I$ is a Fredholm family. As the index is a continuous function, we have $\text{ind}(\mathcal{T} - \lambda I) = 0$ and $\lambda \notin \sigma_W(\mathcal{T}).$

**Theorem 2.4.** Let $\mathcal{T}_n, \mathcal{T}$ in $\mathcal{C}(\mathcal{X}, L(X))$ such that $\lim_{n \rightarrow \infty} \mathcal{T}_n = \mathcal{T}.$ Then $\lim_{n \rightarrow \infty} \sigma_W(\mathcal{T}_n) = \sigma_W(\mathcal{T}),$ in each of the following cases:

1. $\mathcal{T}_n \mathcal{T} = \mathcal{T} \mathcal{T}_n,$ for each $n.$
2. $\sigma(\mathcal{T})$ is totally disconnected.
3. $X$ is Hilbert space and $\forall n, \mathcal{T}_{n,x}$ and $\mathcal{T}_x$ are normal operators for all $x \in \mathcal{X}.$

**Proof.** As $\sigma_W(\mathcal{T}) = \bigcup_{x \in \mathcal{X}} \sigma_W(\mathcal{T}_x)$ and $\sigma((\Pi(\mathcal{T}))) = \bigcup_{x \in \mathcal{X}} \sigma((\Pi(\mathcal{T}_x))),$ one of the following conditions holds:
1. \( T_{n,x} T_x = T_x T_{n,x} \), for each \( n \) and for all \( x \in X \),

2. \( \sigma(T_x) \) is totally disconnected, for all \( x \in X \).

3. \( X \) is Hilbert space and \( \forall n \) and for all \( x \in X \), \( T_{n,x} \) and \( T_x \) are normal operators.

From [6, Corollary, p. 209], in each of these cases, we have: \( \lim_{n \to \infty} \sigma(\Pi(T_{n,x})) = \sigma(\Pi(T_x)) \), for all \( x \in X \). From [6, Theorem 2], we have \( \lim_{n \to \infty} \sigma_w(T_{n,x}) = \sigma_w(T_x) \) and from [5, Formula 3a, p.336], we have \( \bigcup_{x \in X} \liminf_{n \to \infty} \sigma_w(T_{n,x}) \subseteq \liminf_{n \to \infty} (\bigcup_{x \in X} \sigma_w(T_{n,x})) \).

As we have \( \liminf_{n \to \infty} \sigma_w(T_{n,x}) = \lim_{n \to \infty} \sigma_w(T_{n,x}) = \sigma_w(T_x), \forall x \in X \), then \( \bigcup_{x \in X} \sigma_w(T_{n,x}) \subseteq \liminf_{n \to \infty} (\bigcup_{x \in X} \sigma_w(T_{n,x})). \) Since \( \bigcup_{x \in X} \sigma_w(T_x) = \sigma_w(T) \) and \( \bigcup_{x \in X} \sigma_w(T_{n,x}) = \sigma_w(T_n) \), then \( \sigma_w(T) \subseteq \liminf_{n \to \infty} \sigma_w(T_n) \). As we have already from Theorem 2.2 \( \limsup_{n \to \infty} \sigma_w(T_n) \subseteq \sigma_w(T) \), then \( \lim_{n \to \infty} \sigma_w(T_n) = \sigma_w(T) \).

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