Abelian Hopfions of the $\mathbb{CP}^n$ model on $\mathbb{R}^{2n+1}$ and a fractionally powered topological lower bound

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Abstract

Regarding the Skyrme-Faddeev model on $\mathbb{R}^3$ as a $\mathbb{CP}^1$ sigma model, we propose $\mathbb{CP}^n$ sigma models on $\mathbb{R}^{2n+1}$ as generalisations which may support finite energy Hopfion solutions in these dimensions. The topological charge stabilising these field configurations is the Chern-Simons charge, namely the volume integral of the Chern-Simons density which has a local expression in terms of the composite connection and curvature of the $\mathbb{CP}^n$ field. It turns out that subject to the sigma model constraint, this density is a total divergence. We prove the existence of a topological lower bound on the energy, which, as in the Vakulenko-Kapitansky case in $\mathbb{R}^3$, is a fractional power of the topological charge, depending on $n$. The numerical construction of the simplest ring shaped un-knot Hopfion on $\mathbb{R}^5$ is also discussed.

1 Introduction

Hopfions are knotted, static, finite energy and topologically stable solutions to nonlinear differential equations. To date, the only known model supporting Hopfions is the Skyrme-Faddeev model [1], which is an $O(3)$ sigma model on $\mathbb{R}^3$. These Hopf soliton solutions are
constructed numerically [2, 3, 4] and the topological charge stabilising them is the volume integral of the Chern–Simons (CS) density [5]. This, namely being stabilised by the CS charge, is the distinguishing feature of the known [1] Hopfions on \( \mathbb{R}^3 \), as well as the higher dimensional ones to be introduced below.

In the definition of a Chern-Simons charge, the entities employed are the gauge field connection and curvature. But if these entities be the Yang-Mills connection and curvature, then the resulting Chern-Simons density is manifestly not a total divergence, and hence is not a candidate for a topological charge density. What confers the status of a topological charge density on the CS density is, that the latter be defined in terms of the composite gauge field connection and curvature of a suitable nonlinear sigma model described by a scalar field. In this case, it is possible to show that such a CS density is indeed total divergence \(^1\), which means that the volume integral of the CS density can be evaluated as a surface integral, rendering it a candidate for a topological charge density. This is done by subjecting the CS density to the variational principle taking the sigma model constraint into account. The resulting field equations turn out to be trivial, encoding no dynamics. We have described such topological charge densities as “essentially total derivative” below \(^2\). The role of the Chern–Simons (CS) density as the topological charge density, is the distinguishing feature of field theories that support Hopf solitons \(^3\).

Concerning the role of composite connections in the definition of the CS density, it is clear that these can be either Abelian or non-Abelian. As suggested by the title here, the present work is concerned exclusively with the case of Abelian composite connections.

A very important difference between the usual solitons and Hopfions is, that in the former the topological lower bound on the energy is established by the familiar Bogomol'nyi-type procedure employing the Chern-Pontryagin charge density (or its descendents). In the case of Hopfions on the other hand, this lower bound is established by the considerably more elaborate procedure of Vakulenko and Kapitansky \([7, 8]\), hinged instead on the Chern-Simons density, and employing a Sobolev space technique. The resulting topological lower bounds are expressed as fractional powers of the topological charge, namely the Chern-Simons charge. This type of lower bound is essential in forming knotted solitons.

Finally we note, in passing, that the CS densities used in the context of Hopfions are defined on the odd dimensional Euclidean spaces \( \mathbb{R}^{2n+1} \), and not on Minkowskian spacetimes in those dimensions. They encode only geometric and topological data and do not enter the Lagrangian in Minkowski space as in Ref [9]. As such, they are entirely different from the dynamical Chern-Simons terms appearing in field theories introduced in Ref [9].

The aim of this work is to propose a family of sigma models that we claim can support

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1. In the familiar case of the Skyrme-Faddeev model, the CS density is nonlocal and hence cannot be expressed in this way, as it stands. However, since the \( O(3) \) sigma model is equivalent to the \( \mathbb{C}P^1 \) model, this assertion remains true.

2. The Chern-Simons densities of sigma models supporting Hopfions share this property with the topological charge densities of the \( O(D+1) \) sigma models on \( \mathbb{R}^D \).

3. In contrast, the solitons of the Grassmannian and Yang-Mills models in all even dimensions \([6]\) are stabilised by Chern-Pontryagin (CP) charges, and the solitons of the Yang-Mills-Higgs models in all dimensions \([6]\) by the descendents of CP charges.
Hopfion solutions on $\mathbb{R}^{2n+1}$, the $n=1$ case being the well-known Skyrme-Faddeev Hopfion. We have referred to these as Abelian Hopfions since the composite connections of the sigma models in question, namely the $\mathbb{CP}^n$ models, are Abelian. The $\mathbb{CP}^n$ field systems are presented in section 2, where the general properties of the Chern-Simons densities are also discussed. In section 3 we propose models that can support finite energy Hopfion solutions, with special regard to section 4 following it, where the energy lower bounds for some of these models is established. In section 5 we deal with the detailed properties of the Chern-Simons charges, subjecting them to suitable symmetries that result in revealing the boundary values required of Hopfion field configurations. Some numerical results for the simplest Hopfions in $D=5$ dimensions are reported in section 6. Section 7 is devoted to a summary and a discussion of our results.

2 The $\mathbb{CP}^n$ fields on $\mathbb{R}^{2n+1}$

We start with the generic structure of models that can support Abelian Hopfion on $\mathbb{R}^{2n+1}$. These are described by complex $(n+1)$-tuplets

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \end{bmatrix} \equiv z_a ; \quad \bar{Z} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \vdots \\ \bar{z}_{n+1} \end{bmatrix} \equiv \bar{z}^a , \quad a = 1, 2, ..., n+1 ,$$

subject to the constraint

$$Z^\dagger Z \equiv \bar{z}^a z_a = 1 ,$$

taking their values in $U(n) \times U(1)$, parametrised by $2n$ real functions. In (1), $\bar{z}^a$ is the complex conjugate of $z_a$, transforming with an index that is contravariant to the covariant index of $z_a$, and $Z^\dagger$ in (2) is the transpose of $\bar{Z}$. This leads to the definition of the projection operator

$$P = (\mathbb{I} - Z Z^\dagger) \equiv (\delta^b_a - z_a \bar{z}^b) .$$

The most interesting feature of these models is that when the field $Z$ is subjected to a local $U(1)$ gauge transformation $g = e^{\pm i \Lambda(x)}$, under which the constraint (2) is invariant, then the quantity defined as

$$B_i = i Z^\dagger \partial_i Z , \quad i = 1, 2, ..., 2n+1 ,$$

transforms like an Abelian composite connection under $g(\Lambda)$,

$$B_i \rightarrow B_i \pm \partial_i \Lambda .$$

\footnote{Higher dimensional models supporting Hopfions were considered earlier \cite{10, 11}, but these are $O(2n+1)$ sigma models rather than the $\mathbb{CP}^n$ models proposed here. In contrast to the former, the CS density of the latter has a local expression.}
This leads to the definition of the covariant derivative of $Z$ and the \textit{Abelian curvature} of this connection,

\begin{align*}
D_i Z &= \partial_i Z + B_i Z, \quad (5) \\
G_{ij} &= \partial_i B_j - \partial_j B_i, \quad (6)
\end{align*}

with $D_i Z$ transforming covariantly under the action of $g$, and with $G_{ij}$ invariant.

The Abelian Chern-Simons (CS) density on $\mathbb{R}^{2n+1}$ is then readily defined in terms of the quantities (5) and (6),

\[
\Omega_{CS} \simeq \varepsilon_{i_1 i_2 \ldots i_{2n+1}} B_{i_{2n+1}} G_{i_1 i_2} G_{i_3 i_4} \ldots G_{i_{2n-1} i_{2n}}.
\]

When it is subjected to arbitrary variations of $Z^\dagger$ (or $Z$), with adequate account taken of the constraint (2), this yields the variational equation

\[
\varepsilon_{i_1 i_2 \ldots i_{2n+1}} D_{i_{2n+1}} Z G_{i_1 i_2} G_{i_3 i_4} \ldots G_{i_{2n-1} i_{2n}} = 0,
\]

which on the face of it is a nontrivial equation of motion for $Z$, meaning that the CS density $\Omega_{CS}$ is not a total divergence. This however is not true. It can be verified by direct calculation, with the constraint (2) taken account of, that $\Omega_{CS}$ is indeed a total divergence and that (8) is actually trivial and encodes no dynamical information.

\section{The $\mathbb{CP}^n$ “Skyrme-Faddeev” models on $\mathbb{R}^{2n+1}$}

We refer to the models supporting Hopfions as “Skyrme-Faddeev” models, to distinguish them from the “Skyrme” models that support usual Skyrmions. Both are sigma models, for example the Skyrme-Faddeev model and the Skyrme model on $\mathbb{R}^3$ are both $O(3)$ sigma models. These however differ from each other in that the highest order kinetic term in the former is the quartic Skyrme term, while that in the latter is the sextic Skyrme term. These are respectively the squares of a 2-form and a 3-form field constructed from the partial derivatives of the scalar sigma model field. The highest order form that can be constructed is limited by the number of independent real functions parametrising the sigma model field.

In the case of the $\mathbb{CP}^n$ fields here, this number is $2n$ so that the highest order kinetic term in the model supporting a Hopfion on $\mathbb{R}^{2n+1}$ is the square of a 2n-, or $(D-1)$-form field. This contrasts with the situation of, for example, the (usual) Skyrmions of the $O(D + 1)$ sigma models on $\mathbb{R}^D$. In that case the number of independent functions is $D$, so that the highest order kinetic term is the square of a $D$-form field.

Irrespective of what symmetries the models are subjected to, their energy density functionals must be consistent with Derrick’s scaling requirement. This is so that the energy of the Hopfion solutions be finite. In this respect, the situation is the same as that for the usual Skyrme models, except for the fact that the order of the highest order kinetic term for the Skyrme-Faddeev models is of lower order as explained in the previous paragraph.

\footnote{It should be remarked here that in the case of the $O(D + 1)$ sigma models on $\mathbb{R}^D$, the situation is much simpler, with the analogue of Eqn. (8) displaying a vanishing left hand side.}
The knotted solitons supported by these systems are topologically stable and their energies are bounded from below by \( n \)-dependent fractional powers of the Chern-Simons charge.

### 3.1 The \( \mathbb{C}P^1 \) “Skyrme-Faddeev” models on \( \mathbb{R}^3 \)

The most general model supporting finite energy solutions, consistent with the Derrick scaling requirement is

\[
E_3 = \kappa_0^0 V + \frac{1}{2} \kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4} \kappa_2^4 G_{ij}^2 ,
\]

with \( D_i Z \) and \( G_{ij} \) given by (5) and (6). The constants \( \kappa_0, \kappa_1, \) and \( \kappa_2 \) each have the dimension of length, and \( V \) is some pion mass type potential, which can most naturally be chosen to be

\[
V = 1 + Z^\dagger \sigma_3 Z .
\]

It is well known that a \( \mathbb{C}P^1 \) sigma model is equivalent to the corresponding \( O(3) \) sigma model. In the present case, the \( \mathbb{C}P^1 \) system (9) with \( \kappa_0 = 0 \), is equivalent to the \( O(3) \) Skyrme-Faddeev model. The extension of the latter with the potential term (10) has been studied recently in [12], where Hopfion solutions are constructed.

The virial identity resulting from the scaling requirement that must be satisfied is

\[
3 \| V \| + \| D_i Z \|^2 - 2 \| G_{ij} \|^2 = 0 ,
\]

where the dimensional constants and the detailed normalisations have been suppressed, and where each of the quantities \( \| . \|^2 \) is the positive definite integral of the corresponding density in (2).

The lower bound on the energy of (9) has been established long ago in [7]. The presence of the potential term (10) does not influence this lower bound.

### 3.2 The \( \mathbb{C}P^2 \) “Skyrme-Faddeev” models on \( \mathbb{R}^5 \)

The most general model supporting finite energy solutions in this dimension, consistent with the Derrick scaling requirement is

\[
E_5 = \kappa_0^0 V + \frac{1}{2} \kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4} \kappa_2^4 G_{ij}^2 + \frac{1}{8} \kappa_3^8 (G_{[ij} D_{k]} Z)^\dagger (G_{[ij} D_{k]} Z) + \frac{1}{16} \kappa_4^8 G_{ijkl}^2 ,
\]

with the 4–form (composite) curvature \( G_{ijkl} \) being the totally antisymmetrised two-fold product of the 2–form (composite) curvature \( G_{ij} \), and the square brackets on the indices

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6The \( O(3) \) field \( \phi^a, a = 1, 2, 3 \), subject to the constraint \( |\phi^a|^2 = 1 \) is given by

\[
\phi^a = Z^\dagger \sigma^a Z
\]

in terms of the Pauli matrices \( \sigma^a \) and the field \( Z \) in (11) for \( n = 1 \).
implying cyclic symmetry. The constants $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ and $\kappa_4$ each have the dimension of length, and $V$ is some pion mass type potential. According to the scaling requirement for finite energy, it is necessary to retain at least one of the constants $(\kappa_1, \kappa_2)$ and at least one of the constants $(\kappa_3, \kappa_4)$, with the option of setting the rest equal to zero.

The virial identity resulting from the scaling requirement that must be satisfied is

$$5\|V\| + 3\|D_i Z\|^2 + \|G_{ij}\|^2 - \|(G_{[ij} D_{k]} Z)\|^2 - 3\|G_{ijkl}\|^2 = 0,$$

where the dimensional constants and the detailed normalisations have been suppressed, and where each of the quantities $\|\cdot\|^2$ is the positive definite integral of the corresponding density in $(12)$.

### 3.3 The $\mathbb{C}P^3$ “Skyrme-Faddeev” models on $\mathbb{R}^7$

The most general model supporting finite energy solutions here, consistent with the Derrick scaling requirement is

$$\mathcal{E}_7 = \kappa_0^3 V + \frac{1}{2} \kappa_1^2 D_i Z^\dagger D_i Z + \frac{1}{4} \kappa_2^4 G_{ij}^2 + \frac{1}{8} \kappa_3^6 (G_{[ij} D_{k]} Z)^\dagger (G_{[ij} D_{k]} Z) + \frac{1}{16} \kappa_4^8 G_{ijkl}^2 + \frac{1}{32} \kappa_5^{10} (G_{[ijkl} D_{m]} Z)^\dagger (G_{[ijkl} D_{m]} Z) + \frac{1}{32} \kappa_6^{12} G_{ijklm}^2 \tag{14}$$

with the $6$-form (composite) curvature $G_{ijklm}$ being the totally antisymmetrised three-fold product of the $2$-form (composite) curvature $G_{ij}$, and the square brackets on the indices again implying total antisymmetrisation. The constants $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5$ and $\kappa_6$ each have the dimension of length, and $V$ is some pion mass type potential. According to the scaling requirement for finite energy, it is necessary to retain at least one of the constants $(\kappa_1, \kappa_2, \kappa_3)$ and at least one of the constants $(\kappa_4, \kappa_5, \kappa_6)$, with the option of setting the rest equal to zero.

The virial identity resulting from the scaling requirement that must be satisfied is

$$7\|V\| + 5\|D_i Z\|^2 + 3\|G_{ij}\|^2 + \|(G_{[ij} D_{k]} Z)\|^2 - \|(G_{ijkl} D_{m]} Z)\|^2 - 3\|G_{ijklm}\|^2 = 0,$$

where the dimensional constants and the detailed normalisations have been suppressed, and where each of the quantities $\|\cdot\|^2$ is the positive definite integral of the corresponding density in $(14)$.

### 3.4 The general case: $\mathbb{C}P^n$ models on $\mathbb{R}^{2n+1}$

It is convenient here to introduce a dedicated notation, whereby the the $p$-form fields and curvatures are denoted by

$$G(0) = V, \ G(1) = D_i Z, \ G(2) = G_{ij}, \ G(3) = G(2)^2 G(1), \ldots, \ G(2n - 1), \ G(2n) \tag{16}$$

such that the energy density functional can be written as

$$\mathcal{E}_{2n+1} = G(0) + |G(1)|^2 + G(2)^2 + |G(3)|^2 + \cdots + |G(2n - 1)|^2 + G(2n)^2. \tag{17}$$
The virial identity in this case is
\[(2n + 1)\|G(0)\|^2 + (2n - 1)\|G(1)\|^2 + (2n - 3)\|G(2)\|^2 + \cdots + \|G(2n - 3)\|^2 - \|G(2n - 4)\|^2 - \cdots - (2n - 3)\|G(2n - 1)\|^2 - (2n - 1)\|G(2n)\|^2 = 0. \quad (18)\]

Anticipating our considerations in the next section, where a topological bound on the energy is established, we point out that in all but the \(n = 1\) case the most general models proposed feature the two terms constructed from the composite curvature 2-form \(G(2)\) and the \(2n\)-form composite curvature \(G(2n)\). The lower bound established is for the systems consisting of these two kinetic terms, and remains valid when the other positive definite terms are added.

### 4 A topological lower bound

In \(2n + 1\) dimensions, use \(G_{i_1 i_2 \cdots i_{2n}}\) to denote the \(2n\)-form curvature which is formed from taking the totally antisymmetric \(n\)-fold product of the 2-form curvature \(G_{ij}\). Then
\[
\Omega_{CS} = \epsilon^{i_1 j_1 \cdots j_{2n}} B_i G_{j_1 j_2 \cdots j_{2n}},
\]
so that
\[
Q = \int \Omega_{CS}
\]
is the topological charge, where the integral is understood to be evaluated over the domain space \(\mathbb{R}^{2n+1}\). The construction of \(G_{i_1 i_2 \cdots i_{2n}}\) leads naturally to the point-wise bound
\[
|G_{i_1 i_2 \cdots i_{2n}}| \leq C |G_{ij}|^n,
\]
where and in the sequel we use \(C\) to denote a generic positive constant.

Consider the energy functional
\[
E = \int (G_{ij}^2 + G_{i_1 i_2 \cdots i_{2n}}^2)
\]
We shall establish the following fractional-exponent topological lower bound
\[
E \geq C |Q|^{\frac{2n+1}{2n+2}}, \quad n \in \mathbb{N}.
\]
To prove this lower bound, we first apply the Cauchy–Schwarz inequality to get
\[
|Q| \leq C \left( \int |B_i|^p \right)^{\frac{1}{p}} \left( \int |G_{j_1 j_2 \cdots j_{2n}}|^q \right)^{\frac{1}{q}},
\]
where \(p, q > 1\) satisfies
\[
\frac{1}{p} + \frac{1}{q} = 1.
\]
Next, we recall the Sobolev inequality over $\mathbb{R}^m$:

$$\|f\|_p \leq C \|\nabla f\|_2,$$  \hspace{1cm} (26)

where we use $\| \cdot \|_p$ to denote the usual $L^p$-norm for a function defined over the space $\mathbb{R}^m$ and $p$ satisfies the relation

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{m} = \frac{m - 2}{2m}. \hspace{1cm} (27)$$

Setting $m = 2n + 1$ in (27), we see that $p$ is given by

$$p = \frac{2(2n + 1)}{2n - 1}. \hspace{1cm} (28)$$

Inserting (28) into (26), we have

$$\|f\|_{\frac{2(2n+1)}{2n-1}} \leq C \|\nabla f\|_2. \hspace{1cm} (29)$$

Moreover, restricting to the Coulomb gauge, $\partial_i B_i = 0$, there holds

$$\int G^2_{ij} = \int (\partial_i B_j - \partial_j B_i)^2 = \int |\nabla B_i|^2. \hspace{1cm} (30)$$

In view of (29) and (31), we obtain

$$\left( \int |B_i|^{\frac{2(2n+1)}{2n-1}} \right)^{\frac{2n-1}{2(2n+1)}} \leq C \left( \int G^2_{ij} \right)^{\frac{1}{2}}. \hspace{1cm} (31)$$

On the other hand, in view of (25) and (28) we see that

$$q = \frac{2(2n + 1)}{2n + 3}, \hspace{1cm} (32)$$

and in view of (24) and (31) we conclude that the bound

$$|Q| \leq C \left( \int G^2_{ij} \right)^{\frac{1}{2}} \left( \int |G_{j_1j_2\ldots j_{2n}}|^{\frac{2(2n+1)}{2n+3}} \right)^{\frac{2n+3}{2(2n+1)}}$$

is valid.

To proceed further, we seek constants $\alpha, \beta > 0$ and $s, t > 1$ such that

$$\begin{align*}
\alpha + \beta &= \frac{2(2n+1)}{2n+3}, \\
\frac{1}{s} + \frac{1}{t} &= 1, \\
\alpha s &= \frac{2}{n}, \\
\beta t &= 2. 
\end{align*} \hspace{1cm} (34)$$
If the system (34) has a solution, then we can use the Cauchy–Schwarz inequality to reduce the second integral on the right-hand side of (33) into

\[
\int |G_{j_1j_2\cdots j_{2n}}|^{\frac{2(2n+1)}{2n+3}} = \int |G_{j_1j_2\cdots j_{2n}}|^{\alpha+\beta} \\
\leq \left( \int |G_{j_1j_2\cdots j_{2n}}|^{\alpha^s} \right)^{\frac{1}{s}} \left( \int |G_{j_1j_2\cdots j_{2n}}|^{\beta^t} \right)^{\frac{1}{t}} \\
= \left( \int |G_{j_1j_2\cdots j_{2n}}|^{\frac{2}{n}} \right)^{\frac{1}{s}} \left( \int |G_{j_1j_2\cdots j_{2n}}|^{2} \right)^{\frac{1}{t}}.
\] (35)

Fortunately, it is not hard to see that when \( n \neq 1 \) the system (34) may be uniquely solved to yield

\[
\begin{align*}
\alpha &= \frac{4}{(n-1)(2n+3)}, \\
\beta &= \frac{2(2n^2-n-3)}{(n-1)(2n+3)}, \\
s &= \frac{(n-1)(2n+3)}{2n^2-n-3}, \\
t &= \frac{(n-1)(2n+3)}{2n^2-n-3}.
\end{align*}
\] (36)

Substituting (36) into (35) and applying (21), we arrive at

\[
\int |G_{j_1j_2\cdots j_{2n}}|^{\frac{2(2n+1)}{2n+3}} \leq \left( \int |G_{ij}|^{2n} \right)^{\frac{1}{2(2n-1)(2n+3)}} \left( \int |G_{j_1j_2\cdots j_{2n}}|^{\frac{2n^2-n-3}{2(2n-1)(2n+1)}} \right)^{\frac{2n^2-n-3}{2(2n-1)(2n+1)}}.
\] (37)

Now inserting (37) into (33) and applying (22), we get

\[
|Q| \leq C \left( \int G_{ij}^2 \right)^{\frac{1}{2n-1}(n-1)(2n+1)} \left( \int |G_{j_1j_2\cdots j_{2n}}|^{2n^2-n-3} \right)^{\frac{1}{2(2n-1)(2n+1)}} \\
\leq CE^{\frac{1}{2n-1}(n-1)(2n+1)} \cdot \frac{2n^2-n-3}{2n^2-n-3} \\
= CE^{\frac{2(n+1)}{2n+1}}, \quad n \neq 1.
\] (38)

In other words, the energy-topological charge inequality (23) is proved when \( n \neq 1 \). However, when \( n = 1 \), the classical Vakulenko–Kapitansky inequality holds. Thus we see that (23) is valid for all \( n \in \mathbb{N} \).

5 Imposition of symmetries on \( \mathbb{C}P^n \) models on \( \mathbb{R}^{2n+1} \)

The imposition of symmetries on a system obeying partial differential equations (PDE) serves the purpose of lowering the order of the PDE’s. This is useful in the analytic proof of existence of some solutions, and is often necessary in the numerical construction of solutions. Indeed in the last section, we present some numerical results for a set of second order PDE’s with dependence of two variables only, which result from a suitable imposition of symmetry.
There is however another reason for the imposition of symmetries, relating to the evaluation of the topological charge. We know that the \( \mathbb{CP}^n \) field \( Z \) on \( \mathbb{R}^{2n+1} \) is parametrised by \( 2n \) real functions, and since the Chern-Simons density is (effectively) a total divergence, then the number of effective space coordinates for the field configuration describing a topologically stable Hopfion must also be \( 2n \). The effective space coordinates must reflect the symmetry of the Hopfion in question. In general, this symmetry can be quite complicated and the resulting evaluation of the integral giving the topological charge may not be transparent. It is our aim here to introduce a symmetry imposition criterion that renders the statement of the boundary values of the Hopfion field configuration consistent with integer topological charge, transparent.

The topological charge, which is the volume integral of the CS density, is evaluated as a surface integral. Our criterion in the imposition of symmetries is that in its final form, after the imposition of symmetry, the CS density be expressed as a totally antisymmetrised product of the residual space derivatives of the residual array of functions parametrising the system, \( i.e. \) that it be expressed as a total divergence explicitly.

As \( per \) this criterion, the minimal imposition of symmetry is axial symmetry in any one of the 2-planes in \( \mathbb{R}^{2n+1} \). This results in decreasing the number of residual space coordinates to \( 2n \), that being the number of independent real functions parametrising the \( \mathbb{CP}^n \) field \( Z \) (and hence \( B_i \) and the CS density), namely \( 2n \). We refer to this as the minimal imposition of symmetry.

The maximal imposition of symmetry, consistent with our criterion is that of the imposition of azimuthal symmetry in each of the \( n \), 2-planes in \( \mathbb{R}^{2n+1} \), \( i.e. \) \( n \)-fold azimuthal symmetry.

It is clear that in between the minimal, namely mono-azimuthal or axial symmetry, and the maximal \( n \)-fold azimuthal symmetry, one can impose any number \( N (\leq n) \) of azimuthal symmetries. In the following, we shall ignore these and concentrate only the maximal and the minimal cases.

### 5.1 Imposition of (minimal) mono-azimuthal symmetry

As explained, the function \( Z \) must be subjected to at least one azimuthal symmetry, and this is the case considered here. (We ignore intermediate cases where more than one but less than \( n \) azimuthal symmetries can be imposed.)

We will apply axial symmetry to only one 2-plane in \( \mathbb{R}^{2n+1} \), say to the \( x_\alpha = (x_{2n}, x_{2n+1}) \) plane. We denote the radial coordinate in this plane by \( \zeta = \sqrt{|x_\alpha|^2} \), the azimuthal angle by \( \phi \) and the vortex number by \( n \). The coordinate on \( \mathbb{R}^{2n+1} \) is denoted by \( x_i \), with \( i = 1, 2, \ldots, (2n + 1) \). Denoting the coordinates in the plane where axial symmetry is imposed by \( x_\alpha \), we split the index \( i \) as

\[
  x_i = (x_\alpha, x_\alpha), \quad \alpha = 2n, 2n + 1, \quad a = 1, 2, \ldots, (2n - 1).
\]
The mono-azimuthally symmetric Ansatz for \( Z \) on \( \mathbb{R}^{2n+1} \) is
\[
Z = \begin{pmatrix}
  a_1 + ib_1 \\
  a_2 + ib_2 \\
  a_3 + ib_3 \\
  \vdots \\
  a_n + ib_n \\
  c e^{i n \varphi}
\end{pmatrix}, \tag{39}
\]
which subject to the constraint (2) satisfies
\[
(a_1^2 + a_2^2 + \cdots + a_n^2) + (b_1^2 + b_2^2 + \cdots + b_n^2) + c^2 = 1, \tag{40}
\]
resulting in the maximal number of \( 2n \) independent real functions parametrizing \( Z \), each of these depending on the \( 2n \) variables \( (x_a, \zeta) \), \( a = 1, 2, \ldots, 2n - 1 \).

Substitution of (39) in the Abelian CS density (7) yields a result of the form
\[
\Omega_{CS} \simeq \frac{n}{\zeta} c \cdot \det \begin{vmatrix}
  a_1 & b_1 & a_2 & b_2 & \cdots & a_n & b_n & c \\
  a_{1,1} & b_{1,1} & a_{2,1} & b_{2,1} & \cdots & a_{n,1} & b_{n,1} & c_{1} \\
  a_{1,2} & b_{1,2} & a_{2,2} & b_{2,2} & \cdots & a_{n,2} & b_{n,2} & c_{2} \\
  a_{1,3} & b_{1,3} & a_{2,3} & b_{2,3} & \cdots & a_{n,3} & b_{n,3} & c_{3} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  a_{1,2n-1} & b_{1,2n-1} & a_{2,2n-1} & b_{2,2n-1} & \cdots & a_{n,2n-1} & b_{n,2n-1} & c_{2n-1} \\
  a_{1,\zeta} & b_{1,\zeta} & a_{2,\zeta} & b_{2,\zeta} & \cdots & a_{n,\zeta} & b_{n,\zeta} & c_{\zeta}
\end{vmatrix}, \tag{41}
\]
where \( a_{1,\zeta} = \partial_\zeta a_1 \), \( a_{1,a} = \partial_x a_1 \), with \( a = 1, 2, \ldots, 2n - 1 \), etc.

The determinant (41) is not explicitly a total divergence but if subjected to the variational principle taking account of the constraint (40), yields no nontrivial equation of motion, \textit{i.e.} it is “essentially total divergence”.

Of course, if (39) is reparametrised in a convenient polar parametrisation satisfying (40), then the reduced CS density will become “explicitly total divergence”. It is worthwhile giving some examples of such polar parametrisations, to help illustrate the boundary values that should be satisfied for the (topological) charge integral to take integer values. To this end, consider the two examples with \( n = 2, D = 5 \) and \( n = 3, D = 7 \). There
\[
Z = \begin{pmatrix}
  a_1 + ib_1 \\
  a_2 + ib_2 \\
  a_3 + ib_3 \\
  c e^{i n \varphi}
\end{pmatrix} = \begin{pmatrix}
  \sin \frac{1}{2} f \sin g e^{i \alpha} \\
  \sin \frac{1}{2} f \cos g e^{i \beta} \\
  \cos \frac{1}{2} f e^{i n \varphi}
\end{pmatrix} \tag{42}
\]
and
\[
Z = \begin{pmatrix}
  a_1 + ib_1 \\
  a_2 + ib_2 \\
  a_3 + ib_3 \\
  c e^{i n \varphi}
\end{pmatrix} = \begin{pmatrix}
  \sin \frac{1}{2} f \sin g_1 \cos g_2 e^{i \alpha} \\
  \sin \frac{1}{2} f \sin g_1 \sin g_2 e^{i \beta} \\
  \sin \frac{1}{2} f \cos g_1 e^{i \gamma} \\
  \cos \frac{1}{2} f e^{i n \varphi}
\end{pmatrix}, \tag{43}
\]
\footnote{We omit the \( n = 1, D = 3 \) case since this is identical with the \( n = 1, D = 3 \) case considered in section 5.2.1 next.}
respectively, which automatically satisfy the constraints (40). Each of the functions in (42) and (43) depends on the $2n$ variables $0 \leq \zeta \leq \infty$ and $-\infty \leq x_a \leq \infty$, $a = 1, 2, \ldots, 2n - 1$, for $n = 2$ and $n = 4$ respectively.

We define

$$\xi_\mu = \begin{pmatrix} r \sin \theta \cos \varphi_1 \\ r \sin \theta \sin \varphi_1 \\ r \cos \theta \cos \varphi_2 \\ r \cos \theta \sin \varphi_2 \end{pmatrix}$$

(44)

with $0 \leq \varphi_1 \leq 2\pi$, $0 \leq \varphi_2 \leq 2\pi$ and $0 \leq \theta \leq \frac{\pi}{2}$ for (42), and,

$$\xi_\mu = \begin{pmatrix} r \sin \theta_1 \cos \theta_2 \cos \varphi_1 \\ r \sin \theta_1 \cos \theta_2 \sin \varphi_1 \\ r \sin \theta_1 \sin \theta_2 \cos \varphi_2 \\ r \sin \theta_1 \sin \theta_2 \sin \varphi_2 \\ r \cos \theta_1 \cos \varphi_3 \\ r \cos \theta_1 \sin \varphi_3 \end{pmatrix}$$

(45)

with $0 \leq \varphi_1 \leq 2\pi$, $0 \leq \varphi_2 \leq 2\pi$, $0 \leq \varphi_3 \leq 2\pi$, $0 \leq \theta_1 \leq \frac{\pi}{2}$ and $0 \leq \theta_2 \leq \frac{\pi}{2}$ for (43).

The boundary values required of the functions in $(f, g, \alpha, \beta)$ in (42) and the functions $(f, g_1, g_2, \alpha, \beta, \gamma)$ in (43), such that the topological charge be integer (with suitable normalisation) can be stated very naturally. As in the corresponding maximal symmetry cases considered in sections 5.2.1, 5.2.2, and 5.2.3, above, the function $f$ is assigned the asymptotic value

$$\lim_{r \to \infty} f = 0.$$ 

The rest of the functions in (42)-(43) are then assigned, respectively

$$\lim_{r \to \infty} g = \theta, \quad \lim_{r \to \infty} \alpha = m_1 \varphi_1, \quad \lim_{r \to \infty} \beta = m_2 \varphi_2.$$  

(46)

$$\lim_{r \to \infty} g_1 = \theta_1, \quad \lim_{r \to \infty} g_2 = \theta_2, \quad \lim_{r \to \infty} \alpha = m_1 \varphi_1, \quad \lim_{r \to \infty} \beta = m_2 \varphi_2, \quad \lim_{r \to \infty} \gamma = m_3 \varphi_3.$$  

(47)

the case of arbitrary dimension following by induction.

The Chern-Simons topological charges in the generic case will be of the form

$$Q \simeq n m_1 m_2 \ldots m_n \pi^{n+1}.$$  

5.2 Imposition of (maximal) $n$-fold azimuthal symmetry

In these cases the residual systems are parametrised by $(n + 1)$-dimensional spatial coordinates. Then, the application of Gauss’ Theorem on the volume integral of the CS density, expressed as a total divergence, results in $n$ dimensional angular integrals yielding the CS charges. This is carried out explicitly for $n = 1, 2, 3$, and the generic case is deduced by induction.
5.2.1 Chern-Simons charge on $\mathbb{R}^3$ subject to axial symmetry

This is the usual Skyrme-Faddeev example, which we present here by way of illustrating the extrapolations in the subsequent cases in higher dimensions.

Clearly, in this case the maximal and minimal levels of symmetry impositions coincide since there is only one 2-plane in $\mathbb{R}^3$. Subject to this symmetry, the most general Ansatz is

$$Z = \left[ \begin{array}{c} a + ib \\ c e^{i\varphi} \end{array} \right] \equiv \left[ \begin{array}{c} \sin \frac{f}{2} e^{i\alpha} \\ \cos \frac{f}{2} e^{i\varphi} \end{array} \right] \quad (48)$$

where the functions $a$, $b$, $c$, $f$ and $\alpha$ all depend on both $\rho = \sqrt{|x_\alpha|^2}$ and $z \equiv x_3$, $\alpha = 1, 2$.

The Abelian Chern–Simons density (7) on $\mathbb{R}^3$ is

$$\Omega_{\text{CS}}^{(3)} = \varepsilon_{mij} B_m G_{ij} \quad (49)$$

Substituting the azimuthally symmetric Ansatz (48) into (49) yields the simple expression

$$\Omega_{\text{CS}}^{(3)} = - 4 \frac{n}{\rho} c \cdot \text{det} \left| \begin{array}{ccc} a & b & c \\ a,\rho & b,\rho & c,\rho \\ a,z & b,z & c,z \end{array} \right| \quad (50)$$

It is clear that if any one of the functions $a$, $b$, and $c$ vanishes, $\Omega_{\text{CS}}^{(3)}$ vanishes.

Using the trigonometric parametrisation in the Ansatz (48), $\Omega_{\text{CS}}^{(3)}$ reduces to

$$\Omega_{\text{CS}}^{(3)} = \frac{4}{3} \frac{n}{\rho} (F_{\rho \alpha} \alpha_z + \text{antisymm}.(\rho, z)) \quad (51)$$

where $F$ is the function

$$F(\rho, z) = \cos^3 \frac{f}{2} \quad (52)$$

The topological charge can then be expressed as,

$$Q = \frac{4}{3} 2\pi \int \Omega_{\text{CS}}^{(3)} \rho d\rho dz = \frac{4}{3} 2\pi n \int F_{[\rho \alpha, z]} \rho d\rho dz \quad (53)$$

In analogy with the coordinates $\xi_\mu$ used in (44)-(45), we denote the coordinates in the half plane $(\rho, z) = \xi_\mu$, $\mu = 1, 2$, i.e.,

$$\xi_\mu = \left( \begin{array}{c} r \sin \psi \\ r \cos \psi \end{array} \right) \quad (54)$$

with $0 \leq \psi \leq \pi$, the volume integral (53) can be rewritten as follows

$$Q = \frac{4}{3} 2\pi n \int \varepsilon_{\mu\nu} \partial_\nu F \partial_\nu \alpha d^2 \xi \quad = \quad \frac{4}{3} 2\pi n \int (F \hat{x}_i \varepsilon_{\mu\nu} \partial_\nu \alpha) \bigg|_{r \to \infty} \hat{\xi}_\mu dS \quad = \quad \frac{4}{3} 2\pi \int_{\psi=0}^{\psi=\pi} F \partial_\psi \alpha |_{r \to \infty} d\psi \quad (55)$$

13
where $\psi$ is the polar angle in the $(\rho, z)$ half plane.

Requiring the field configurations in question have the asymptotic values

$$
\lim_{r \to \infty} f(r, \theta) = 0, \quad \lim_{r \to \infty} \alpha(r, \theta) = m\pi,
$$

the integral (55) results in

$$
Q = \frac{8}{3} n m \pi^2.
$$

### 5.2.2 Chern-Simons charge on $\mathbb{R}^5$ subject to bi-azimuthal symmetry

The Ansatz we use for the field (1) on $\mathbb{R}^5$ is

$$
Z = \begin{bmatrix}
a + ib \\
c_1 e^{in_1 \varphi} \\
c_2 e^{in_2 \chi}
\end{bmatrix} \equiv \begin{bmatrix}
\sin \frac{1}{2} f e^{i\alpha} \\
\cos \frac{1}{2} f \sin g e^{in_1 \varphi} \\
\cos \frac{1}{2} f \cos g e^{in_2 \chi}
\end{bmatrix},
$$

(57)

the functions $a, b, c_1, c_2$ depending on the variables $\rho = \sqrt{|x_\alpha|^2}$, $\sigma = \sqrt{|x_A|^2}$ with $\alpha = 1, 2$, $A = 3, 4$ and $z \equiv x_5$. $\varphi$ and $\chi$ are the azimuthal angles in the $(x_1, x_2)$ and $(x_3, x_4)$ planes respectively, $(n_1, n_2)$ being the winding (vortex) numbers of these planes respectively.

The Abelian Chern–Simons density (7) on $\mathbb{R}^5$ is

$$
\Omega_{\text{CS}}^{(5)} = \varepsilon_{ijkl} B_m G_{ij} G_{kl}.
$$

(58)

Substituting the bi-azimuthally symmetric Ansatz (57) into (58) yields the simple expression

$$
\Omega_{\text{CS}}^{(5)} = 32 \frac{n_1 n_2}{\rho \sigma} c_1 c_2 \cdot \det \begin{vmatrix}
a & b & c_1 & c_2 \\
a_{\rho} & b_{\rho} & c_{1,\rho} & c_{2,\rho} \\
a_{\sigma} & b_{\sigma} & c_{1,\sigma} & c_{2,\sigma} \\
a_z & b_z & c_{1,z} & c_{2,z}
\end{vmatrix}.
$$

(59)

Substituting the trigonometric parametrisation in (57), namely the parametrisation in which the sigma model constraint is already imposed, (59) simplifies to

$$
\Omega_{\text{CS}}^{(5)} = -\frac{n_1 n_2}{\rho \sigma} \left[ \partial_\rho F \partial_\sigma G \partial_z \alpha + \text{cycl. symm.}(\rho, \sigma, z) \right].
$$

(60)

with the definitions

$$
F(\rho, z) = \cos f + \frac{1}{2} \cos 2f, \quad G = \cos 2g.
$$

(61)

In analogy with the coordinates $\xi_\mu$ used in (44)-(45) and (54), we denote the coordinates in the quarter sphere $(\rho, \sigma, z) = \xi_\mu$, $\mu = 1, 2, 3$,

$$
\xi_\mu = \begin{pmatrix}
r \sin \psi \sin \theta \\
r \sin \psi \cos \theta \\
r \cos \psi
\end{pmatrix}
$$

(62)
with $0 \leq \psi \leq \pi$ and $0 \leq \theta \leq \frac{\pi}{2}$, the volume integral of (60), namely the charge, can be expressed as

$$Q = -(2\pi)^2 n_1 n_2 \int \varepsilon_{\mu\nu\lambda} \partial_\mu G \partial_\nu G \partial_\lambda \alpha \, d^3 \xi$$

$$= -(2\pi)^2 n_1 n_2 \int \varepsilon_{\mu\nu\lambda} (F \partial_\nu G \partial_\lambda \alpha) \, \left. \frac{\hat{\xi}_\mu dS}{r \to \infty} \right|$$

in an obvious notation where $dS = r^2 \sin \psi \, d\psi d\theta$, and where we have applied Gauss’ Theorem.

The result is

$$Q = 4\pi^2 n_1 n_2 \int_{\psi=0}^{\pi} \int_{\theta=0}^{\frac{\pi}{2}} F (\partial_\nu G \partial_\theta \alpha - \partial_\nu \alpha \partial_\theta G) \, \left. \frac{\hat{\xi}_\mu dS}{r \to \infty} \right| d\psi d\theta. \quad (63)$$

Requiring the field configurations in question have the asymptotic values

$$\lim_{r \to \infty} f = 0 \quad , \quad \lim_{r \to \infty} g = \theta \quad , \quad \lim_{r \to \infty} \alpha = m\pi, \quad (65)$$

(63) yields the following charge

$$\Omega_{CS}^{(5)} = -12 n_1 n_2 m \pi^3. \quad (66)$$

5.2.3 Chern-Simons charge on $\mathbb{R}^7$ subject to tri-azimuthal symmetry

The Anstaz we use is for the field (1) on $\mathbb{R}^7$ is

$$Z = \begin{bmatrix} a + ib \\ c_1 e^{i n_1 \varphi} \\ c_2 e^{i n_2 \chi} \\ c_3 e^{i n_3 \xi} \end{bmatrix} \equiv \begin{bmatrix} \sin \frac{1}{2} f \, e^{i\alpha} \\ \cos \frac{1}{2} f \, \sin g \cos h \, e^{i n_1 \varphi} \\ \cos \frac{1}{2} f \, \sin g \sin h \, e^{i n_2 \chi} \\ \cos \frac{1}{2} f \, \cos g \, e^{i n_3 \xi} \end{bmatrix} \quad (67)$$

the functions $a, b, c_1, c_2, c_3$ depending on the variables $\rho = \sqrt{|x_\alpha|^2}, \sigma = \sqrt{|x_A|^2}, \tau = \sqrt{|x_a|^2}$ with $\alpha = 1, 2, A = 3, 4, a = 5, 6$ and $z \equiv x_7$. $\varphi, \chi$ and $\xi$ are the azimuthal angles in the $(x_1, x_2), (x_3, x_4)$ and $(x_5, x_6)$ planes respectively, $(n_1, n_2, n_3)$ being the winding (vortex) numbers of each plane respectively.

The Chern–Simons density (7) on $\mathbb{R}^7$, is

$$\Omega_{CS}^{(7)} = \varepsilon_{pijklmn} B_p G_{ij} G_{kl} G_{mn} \quad (68)$$

Substituting the tri-azimuthally symmetric Ansatz (67) into (68) yields the simple expression

$$\Omega_{CS}^{(7)} = 96 \frac{n_1 n_2 n_3}{\rho \sigma \tau} c_1 c_2 c_3 \cdot \det \begin{vmatrix} a & b & c_1 & c_2 & c_3 \\ a_\rho & b_\rho & c_1,\rho & c_2,\rho & c_3,\rho \\ a_\sigma & b_\sigma & c_1,\sigma & c_2,\sigma & c_3,\sigma \\ a_\tau & b_\tau & c_1,\tau & c_2,\tau & c_3,\tau \\ a_z & b_z & c_1,z & c_2,z & c_3,z \end{vmatrix}. \quad (69)$$
Substituting the trigonometric parametrisation in (67) (i.e. the parametrisation in which the sigma model constraint is already imposed) reduces the charge, namely the volume integral of (69) to the simple expression
\[ \Omega_{\text{CS}} = -4 \frac{n_1 n_2 n_3}{\rho \sigma \tau} \left[ \partial_\rho F \partial_\sigma G \partial_\tau H \partial_\alpha + \text{tot. antisymm.}(\rho, \sigma, \tau, z) \right], \]
where
\[ F(\rho, z) = \cos^6 \frac{f}{2}, \quad G = \frac{1}{4} (1 - \cos 2g)^2, \quad H = \cos 2h. \]

In analogy with the coordinates \( \xi_\mu \) used in (44)-(45) and (54)-(62), we denote the coordinates in the sextant of the hypersphere \( (\rho, \sigma, \tau, z) = \xi_\mu, i = 1, 2, 3, 4, \)
\[ \xi_\mu = \begin{pmatrix} r \sin \psi \sin \theta_1 \sin \theta_2 \\ r \sin \psi \sin \theta_1 \cos \theta_2 \\ r \sin \psi \cos \theta_1 \\ r \cos \psi \end{pmatrix}, \]
with \( 0 \leq \psi \leq \pi, \ 0 \leq \theta_1 \leq \frac{\pi}{2} \) and \( 0 \leq \theta_2 \leq \frac{\pi}{2} \), the volume integral of (70), namely the charge, can be expressed as
\[ Q = -4 (2\pi)^3 n_1 n_2 n_3 \int \varepsilon_{\mu \nu \tau \lambda} \partial_\mu F \partial_\nu G \partial_\tau H \partial_\lambda \alpha d^4 \xi \]
\[ = -4 (2\pi)^3 n_1 n_2 n_3 \int \varepsilon_{\mu \nu \tau \lambda} \left( F \partial_\nu G \partial_\tau H \partial_\lambda \alpha \right) \bigg|_{r \to \infty} dS \]
(73)
in an obvious notation where \( dS = r^3 \sin^2 \psi \sin \theta_1 d\psi d\theta_1 d\theta_2 \), and where we have applied Gauss' Theorem. The result is
\[ Q = 8 \pi^3 n_1 n_2 n_3 \int_{\psi = 0}^{\pi} \int_{\theta_1 = 0}^{\frac{\pi}{2}} \int_{\theta_2 = 0}^{\frac{\pi}{2}} F \left( \partial_\nu G \partial_\theta_1 H \partial_\theta_2 \alpha + \text{cycl.}(\psi, \theta_1, \theta_2) \right) \bigg|_{r \to \infty} d\psi d\theta_1 d\theta_2. \]
(74)
Finally, requiring the boundary values
\[ \lim_{r \to \infty} g = \theta_1, \quad \lim_{r \to \infty} h = \theta_2, \quad \lim_{r \to \infty} \alpha = m \pi, \]
(75)
(73) yields the following charge
\[ \Omega_{\text{CS}}^{(5)} = 64 n_1 n_2 n_3 m \pi^4. \]
(76)

5.2.4 Chern-Simons charge on \( \mathbb{R}^{2n+1} \) subject to \( n \)-fold azimuthal symmetry

It is clear now that by induction, the appropriate imposition of symmetry is the application of azimuthal symmetry in each of the \( n \) planes in \( \mathbb{R}^{2n+1} \). The resulting reduced subsystem will now be an \( n + 1 \) dimensional system of PDE's, parametrised by \( n + 2 \) functions
\[ a, b, c_1, c_2, \ldots, c_n, \]
only \( n + 1 \) of which are independent, subject to the sigma model constraint
\[
a^2 + b^2 + c_1^2 + c_2^2 + \cdots + c_n^2 = 1. \tag{77}
\]
The reduced Chern-Simons density will then take the form
\[
\Omega_{\text{CS}}^{(2n+1)} \simeq \frac{n_1 n_2 \ldots n_n}{\rho_1 \rho_2 \ldots \rho_n} c_1 c_2 \ldots c_n \cdot \det \begin{vmatrix} a & b & c_1 & \cdots & c_n \\ a_{\rho_1} & b_{\rho_1} & c_{1,\rho_1} & \cdots & c_{n,\rho_1} \\ a_{\rho_2} & b_{\rho_2} & c_{1,\rho_2} & \cdots & c_{n,\rho_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{\rho_n} & b_{\rho_n} & c_{1,\rho_n} & \cdots & c_{n,\rho_n} \\ a_{z} & b_{z} & c_{1,z} & \cdots & c_{n,z} \end{vmatrix}, \tag{78}
\]
where of course \( z = x_{n+1} \).

The quantity \((78)\) is not as it stands explicitly a total divergence, but imposing the sigma model constraint \((77)\) appropriately, it turns out to be “essentially total divergence”. Alternatively, reparametrising it in terms of functions that satisfy the constraint, e.g. the polar parametrisations used above, it does become a total divergence explicitly.

The topological charge in this generic case is of the form
\[
Q \simeq n_1 n_2 \ldots n_m \pi^{n+1}, \tag{79}
\]
where \( n_1, n_2, \ldots, n_n \) are the vorticities in each of the \( n, 2 \)-planes.

6 **Hopfions in \( D = 5 \): Numerical results for a restricted Ansatz**

The existence of a topological lower bound in itself does not guarantee that the model possesses nontrivial solutions. However, in the absence of exact solutions in closed form, the only recourse is to construct the configurations which minimise the energy density functional, in this case \((17)\), **numerically**.

For \( n = 1 \), i.e. the \( \mathbb{CP}^1 \) “Skyrme-Faddeev” models on \( \mathbb{R}^3 \), this problem has been approached by various authors. Restricting to configurations within the Ansatz \((48)\), we mention here only the early work \([13],[2],[14]\), where evidence has been presented for the existence of smooth minimum energy configurations in the static axially symmetric sector. For the solutions with the lowest topological charge, the energy density is maximal at the origin and the energy density isosurfaces are squashed spheres, while in other cases the axially symmetric solutions have toroidal structure (see e.g. \([4]\) for more details together with an extensive list of relevant references).

On general grounds, we expect that the general model \((17)\) would present solutions with rather similar properties also in higher dimension \( n > 1 \). However, the numerical construction of such solutions is much harder in this case. In what follows, we report some numerical results which can be viewed as an indication for the existence of finite energy.
Hopfions of a $\mathbb{CP}^2$ “Skyrme-Faddeev” model on $\mathbb{R}^5$. The study of such configuration has been performed for a simplified version of the general bi-azimuthally symmetric Ansatz (57),

$$Z = \begin{bmatrix} a(\rho, \sigma, z) + ib(\rho, \sigma, z) \\ c_1(\rho, \sigma, z) e^{im_1\varphi} \\ c_2(\rho, \sigma, z) e^{im_2\chi} \end{bmatrix}$$

with $\rho$ and $\sigma$ the radial variables in the $(x_1, x_2)$ and $(x_3, x_4)$ planes respectively, while $z \equiv x_5$. In terms of these coordinates, the $D = 5$ line element reads

$$ds^2 = d\rho^2 + \rho^2 d\varphi^2 + d\sigma^2 + \sigma^2 d\chi^2 + dz^2.$$  

(81)

The next step here is to consider the coordinate transformation

$$\rho = r \sin \psi \sin \theta, \quad \sigma = r \sin \psi \cos \theta, \quad z = r \cos \psi$$

(with $0 \leq r < \infty$, $0 \leq \psi \leq \pi$, $0 \leq \theta \leq \pi/2$), such that (81) becomes

$$ds^2 = dr^2 + r^2 (d\psi^2 + \sin^2 \psi d\Omega_3^2), \quad \text{with} \quad d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\varphi^2 + \cos^2 \theta d\chi^2.$$  

(83)

Then the functions $a, b, c_1, c_2$ in (80) would depend on $r, \psi, \theta$. Remarkably it turns out that

$$a = a(r, \psi), \quad b = b(r, \psi), \quad c_1 = c(r, \psi) \sin \theta, \quad c_2 = c(r, \psi) \cos \theta$$

is a consistent truncation of the general model, provided that

$$n_1 = n_2 = 1.$$  

(85)

This restrictive Ansatz greatly reduces the complexity of the system and makes the numerical construction of at least the lowest topological charge solutions, possible.

In this approach, the problem reduced to solving a set of three partial differential equations with dependence on only two coordinates, for the functions $a, b, c$ subject to the constraint

$$a^2 + b^2 + c^2 = 1.$$  

(86)

As usual, these equations result by varying (12) w.r.t. the functions $a, b, c$, the constraint (86) being imposed by using the Lagrange multiplier method (see e.g. [4]). The boundary conditions satisfied by the functions $a, b$ and $c$ are

$$a \big|_{r=0} = -1, \quad b \big|_{r=0} = 0, \quad c \big|_{r=0} = 0, \quad a \big|_{r=\infty} = 1, \quad b \big|_{r=\infty} = 0, \quad c \big|_{r=\infty} = 0,$$

$$\partial_\psi a \big|_{\psi=0,\pi} = 0, \quad \partial_\psi b \big|_{\psi=0,\pi} = 0, \quad c \big|_{\psi=0,\pi} = 0.$$  

(87)

This is similar to the factorization of the $\theta$–dependence on the $S^3$ sphere employed in the scalar field Ansätze in [13, 16].
Figure 1: The energy density $E_5$ is plotted for the $D = 5$ Hopfion with $n_1 = n_2 = m = 1$, as a function of the coordinates $z = r \cos \psi$, $R = r \sin \psi$.

We have restricted our considerations here to a particular truncation of the energy density functional (12), with $\kappa_0 = \kappa_1 = \kappa_3 = 0$, i.e. a model consisting of the terms $G_{ij}^2$ and $G_{ijkl}^2$ only. This is the simplest model consistent with the Derrick scaling requirement for finite energy.

Subject to this restrictive symmetry, these terms simplify to the expressions

$$G_{ij}^2 = 8 \left( \frac{1}{r^2} (a_r b_r - b_r a_r)^2 + \frac{c^2}{r^2 \sin^2 \psi} (c^2 + \frac{1}{r^2} c_r)^2 + \frac{c^4}{r^4 \sin^4 \psi} \right), \quad (88)$$

and

$$G_{ijkl}^2 = \frac{48 c^4}{r^4 \sin^4 \psi} \left( G_{ij}^2 - \frac{8 c^4}{r^4 \sin^4 \psi} \right), \quad (89)$$

with the action (which coincides with the total mass-energy):

$$E = 2\pi^2 \int dr d\psi r^4 \sin^3 \psi \left( \frac{1}{4} \kappa_2^4 G_{ij}^2 + \frac{1}{16} \kappa_4^8 G_{ijkl}^2 \right). \quad (90)$$

The equations satisfied by the functions $a, b, c$ are very complicated and are not presented here.

Our numerical approach is similar to that employed in [4] to construct various solutions, in particular axially symmetric Hopfions in $D = 3$, by directly solving the field equations.
The numerical calculations were performed with the software package CADSOL, based on
the Newton-Raphson method [17]. In this case, the field equations are first discretized on
a nonequidistant grid and the resulting system is solved iteratively until convergence is
achieved. In this scheme, a new radial variable \( x = r/(1 + r) \) is introduced which maps
the semi-infinite region \([0, \infty)\) to the closed region \([0, 1]\).

Our numerical results strongly suggest the existence of a solution of the model satisfying
the boundary conditions (87). A distinguishing property of this configuration is that the
function \( a^2 + b^2 \) vanishes on a circle of finite radius \( \kappa_4 = 1/24 \) in the equatorial 'plane' \( \psi = \pi/2 \), such
that \( m = 1 \) in the asymptotic conditions (65). The profile of the energy density of the
solution is plotted in Figure 1. One can see that the typical energy density isosurfaces are
squashed spheres (this is similar to the \( D = 3 \) solutions with the lowest topological
charges [13]). However, a 'toroidal' shape is obtained when considering instead isosurfaces
close to the maximal value of the energy density (e.g. \( E_5 = 7000 \)).

Here we have to emphasize that given the complexity of, even the simplified system,
the accuracy for the configurations supplied by the solver is not really satisfactory. For
any grid choice, we could not reduce the typical numerical error (as supplied by the solver
or based on the virial identity (13)) below 10%. However, based on the experience with
related problems, we judge this to be a numerical problem mainly due to the extreme
nonlinearity of the model and to the lack of good starting profiles. As such, we interpret
our results only as an indication for the existence of the lowest charge solution (i.e. with
\( n_1 = n_2 = m = 1 \)) of the \( D = 5 \) model (12). We do, however, expect the qualitative
features mentioned above to be confirmed by a more accurate numerical treatment of the
problem.

7 Summary and discussion

We have proposed a family of sigma models on \( \mathbb{R}^{2n+1} \) that may support knotted solitons
solutions, and have established an energy lower bound for these. The topological charge
stabilising these solitons is the Chern-Simons charge, namely the volume integral of the
Abelian Chern-Simons density, in the given dimension. Like in the case of the Faddeev-
Skyrme Hopfion, this lower bound is expressed as a fractional power of the topological
charge.

The sigma models in question are the most general \( \mathbb{CP}^n \) models on \( \mathbb{R}^{2n+1} \) displaying a
potential terms and all possible (kinetic) “Skyrme terms”. The latter are the squares of
\( N \)-forms constructed from the derivatives of the \( \mathbb{CP}^n \) fields, the highest of these being the
\( n \)-form. (This contrasts with the case of the usual Skyrmions on \( \mathbb{R}^D \), where the highest

\[ \text{Note that any point there represents in fact a round three-sphere.} \]

\[ \text{The data plotted in Figure 1 has been found for } \kappa_2^2 = 0.04, \kappa_4^4 = 1/24, \text{ the total mass-energy of this } \]

\[ \text{configuration being } E = 20.45. \text{ Although the model has no free parameters, a choice of } \kappa_2, \kappa_4 \text{ around } \]

\[ \text{these values leads to a faster convergence of our numerical algorithm. However, we have verified that, } \]

\[ \text{within the numerical errors, the same results are found for other choices of these parameters in the same } \]

range.
order “Skyrme term” is the square of a 2D-form, this being of 2 orders higher than that of the Hopfion.)

The criterion used in the choice of the sigma models is that the Chern-Simons density be expressed as a total divergence. This is so that the volume integral of this density, which is the topological charge, be evaluated easily as a surface integral, resulting in the transparent calculation of the topological charge. The Chern-Simons densities in question are constructed from the composite Abelian connections (and their curvatures) of the $\mathbb{CP}^n$ fields on $\mathbb{R}^{2n+1}$, and because of this we have chosen to refer to the Hopfions pertaining to these models, as “Abelian Hopfions”. These densities are manifestly total divergence when the sigma model constraint is imposed. Otherwise, the are “essentially total divergence” in the sense that they yield no equations of motion when subjected to the variational principle, with the constraint imposed. We note that this is precisely the situation with the Faddeev-Skyrme Hopfion on $\mathbb{R}^3$. The latter is a $O(3)$ sigma model, which is is classically the same as the $\mathbb{CP}^1$ model via the well known equivalence of these two models.

The $\mathbb{CP}^n$ systems considered are subjected to spatial symmetries. We have invoked the criterion that after imposition of symmetry, the residual Chern-Simons density is expressed as a totally antisymmetrised product of the partial derivatives of the residual functions parametrising the system. This enables the transparent imposition of suitable boundary values of the Hopfion field configurations, consistent with integer topological charge. It turns out that the minimal symmetry necessary is that of one azimuthal (axial) symmetry, applied in any one 2-plane in $\mathbb{R}^{2n+1}$. We have also considered what we have called the case of maximal imposition of symmetry, involving the application of azimuthal symmetry in each of the $n$, 2-planes in $\mathbb{R}^{2n+1}$. We have implemented both these types of symmetry and have stated the boundary conditions that must be satisfied for the Chern-Simons charge to be integer. It is clearly possible to impose several intermediate levels of symmetry in each case, the number of these increasing with $n$. These we have eschewed in the present work.

We have supplied an analogue of the classic analysis of Vakulenko and Kapitansky for the Faddeev-Skyrme Hopfion on $\mathbb{R}^3$, to establish an energy lower bound on the Hopfions of the $\mathbb{CP}^n$ models proposed here. It is worth pointing out that the fractional-exponent topological lower bound of the energy derived here is valid in all odd dimensions which is in sharp contrast with that derived in all the Hopf dimensions, $D = 4n - 1$, aimed at directly extending the Faddeev model in [10, 11]. The reason for such a big distinction is that our topological invariant here is the pure Chern-Simons invariant while that employed in [10, 11] is the Hopf invariant which can only be defined in $4n - 1$ dimensions. In this sense, our work here complements that in [10, 11], in that it covers all odd dimensions.

As a final remark, we note that the cornerstone of our considerations is the definition of the topological charge, namely the volume integral of the Chern-Simons density. The latter quantity is defined in terms of the composite connection (and curvature) of a nonlinear sigma model. In the present work, we restricted our attention to Abelian such connections and curvatures, and it turned out that the $\mathbb{CP}^n$ sigma models on $\mathbb{R}^{2n+1}$ were precisely suited for this purpose. It is obvious that the same considerations can be applied to Chern-Simons densities defined in terms of non-Abelian connections. In that case, the
relevant sigma models are Grassmannian like systems. We shall report on such Hopfions elsewhere.

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