Learners that Leak Little Information

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Abstract

We study learning algorithms that are restricted to revealing little information about their input sample. Various manifestations of this notion have been recently studied. A central theme in these works, and in ours, is that such algorithms generalize. We study a category of learning algorithms, which we term \(d\)-bit information learners. These are algorithms whose output conveys at most \(d\) bits of information on their input.

We focus on the learning capacity of such algorithms: we prove generalization bounds with tight dependencies on the confidence and error parameters. We observe connections with well studied notions such as PAC-Bayes and differential privacy. For example, it is known that pure differentially private algorithms leak little information. We complement this fact with a separation between bounded information and pure differential privacy in the setting of proper learning, showing that differential privacy is strictly more restrictive.

We also demonstrate limitations by exhibiting simple concept classes for which every (possibly randomized) empirical risk minimizer must leak a lot of information. On the other hand, we show that in the distribution-dependent setting every VC class has empirical risk minimizers that do not leak a lot of information.

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1 Introduction

The amount of information an algorithm leaks on its input is a natural and important quantity to study. From a privacy perspective, we would like algorithms that convey as little information on their inputs as possible. This idea has been extensively studied in the context of differential privacy, as we survey below. From a statistical perspective, such an algorithm should not overfit its data, and from a learning perspective, it should generalize well.

In this paper, we consider this concept in the context of learning theory. However, it may also be interesting to study in other computational and algorithmic contexts.

We define a new class of learning algorithms we term \textit{d-bit information learners}. These are learning algorithms that abide by certain information leakage constraint; the mutual information between their input and output is at most \(d\). The definition naturally combines notions from information theory and from learning theory [19, 22]. Various related concepts have been recently studied (e.g. [11, 2, 25, 23]).

We start with a brief overview of this work (for definitions see Section 2).

\textbf{Low information yields generalization:} One of the seeds of our work is the idea that a learner that uses a small amount of information from its input should generalize well. We provide four different proofs of this statement (see Section 4). Each of the proofs highlights a different aspect of this basic property: one that translates low information to “independence”; one that is based on an “efficient use” of randomness in randomized algorithms; one that is based on stability; and one that uses the PAC-Bayes framework.

\textbf{Lower bounds for leakage:} We show that for some classes (like thresholds) every (possibly randomized) empirical risk minimizer (ERM) must leak a lot of information on its input. This means that even in very simple settings, learning may not always be possible if we restrict the information leakage of the algorithm.

\textbf{Upper bounds for leakage:} We provide a method for upper bounding the amount of information algorithms leak. We also define a generic learner \(A_H\) for a concept
class $\mathcal{H}$, and show that in a couple natural cases this algorithm conveys as little information as possible (up to some constant). This generic learner is proper and consistent (i.e. an ERM); it simply outputs a uniformly random hypothesis from the set of hypotheses that are consistent with the input sample.

**The distribution dependent setting:** Given the negative answers above, we turn to the setting of distribution dependent learning, where the distribution over the domain is known to the learner. We show that for any concept class with finite VC-dimension $d$ and for any distribution on the data domain, there exists a learning algorithm that, given $\tilde{O}(d)$ i.i.d. labeled examples as input, outputs with high probability an approximately correct function from the concept class, and the mutual information between the input sample and the output of the algorithm is $\tilde{O}(d)$.

**Connections to pure differential privacy:** Differential privacy, introduced by Dwork et al. [13], is a rigorous notion of privacy enabling a useful functionality that data holders may provide to their users. Pure differential privacy implies a bound on mutual information [21].

The following separation is a direct consequence of our work. For the class of point functions $\mathcal{P\mathcal{F}}$, it is known that any pure differentially private algorithm that properly learns this class must require number of examples that grows with the domain size [4]. On the other hand, we show that the generic ERM learner $A_{\mathcal{P\mathcal{F}}}$ leaks at most 2 bits of information and properly learns this class with optimal PAC-learning sample complexity.

**Sharp dependence on the confidence:** We prove that our derived upper bound on the generalization error has sharp dependence on the confidence parameter (Section 3.2). In particular, we show the existence of a learning problem and a $O(1)$-bit information learner such that the probability that the generalization error is below the trivial error is at most $1 - 1/m$, where $m$ is the size of the input sample.

**Amplifying confidence:** We show that the confidence of such learners can be amplified with only a small increase in the mutual information. For example, we

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1As opposed to a more relaxed notion known as approximate differential privacy (see Section 2 for a precise definition).
show that a learner that has small error with probability 2/3 can be transformed (in the standard way) into a learner that has a small error with probability at least $1 - \delta$ and the information cost of the new learner is at most a logarithmic factor (in $1/\delta$) larger.

1.1 Related work

**Mutual information for controlling bias in statistical analysis:** The connection between mutual information and statistical bias has been recently studied in [25] in the context of *adaptive* data analysis. In adaptive statistical analysis, the statistician/analyst conduct a sequence of analyses, where the choice and structure of each analysis depends adaptively on the outcomes of the previous analyses. Some of the results of [25] have been recently improved by Raginsky et al. [23].

**Differential privacy and generalization:** The role of differential privacy in controlling overfitting has been recently studied in several works (e.g. [12, 2, 24, 3]). The authors of [2] provide a treatment of differential privacy as a notion of distributional stability, and a tight characterization of the generalization guarantees of differential privacy.

**Max-information and approximate max-information:** Dwork et al. [11] introduced and studied the notions of max-information – a stronger notion than mutual information – and its relaxation, approximate max-information. They showed that these notions imply generalization and that pure differentially private algorithms exhibit low (approximate) max-information. Rogers et al. [24] showed that approximate differentially private algorithms also have low approximate max-information, and that the notion of approximate max-information captures the generalization properties (albeit with slightly worse parameters) of differentially private algorithms (pure or approximate).

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2Unlike max-information, the relaxed notion of approximate max-information is not directly related to mutual information; that is, boundedness of one does not necessarily imply the same for the other.
Connections to approximate differential privacy: De [10] has shown that the relaxed notion of approximate differential privacy does not necessarily imply bounded mutual information. In [21], it was also shown that if the dataset entries are independent, then approximate differential privacy implies a (weak) bound on the mutual information. Such bound has an explicit dependence of the domain size, which restricts its applicability in general settings. Unlike the case of pure differential privacy, an exact characterization of the relationship between mutual information and approximate differential privacy algorithms is not fully known even when the dataset distribution is i.i.d.

Bun et al. [8] showed that the sample complexity of properly learning thresholds under approximate differential privacy is $\Omega(\log^* (N))$, where $N$ is the domain size. Hence, their result asserts the impossibility of this task for infinite domains. In this work, we show a result of a similar flavor (albeit of a weaker implication) for the class of bounded information learners. Specifically, for the problem of proper PAC-learning of thresholds over a domain of size $N$, we show that the mutual information of any proper learning algorithm (deterministic or randomized) that outputs a threshold that is consistent with the input sample is $\Omega(\log \log N)$. This result implies that there are no consistent proper bounded information learners for thresholds over infinite domains.

This does not imply the non-existence of bounded information learners that are either non-consistent or non-proper. We leave this as an open question.

2 Preliminaries

Learning

We start with some basic terminology from statistical learning (for a textbook see [26]). Let $\mathcal{X}$ be a set called the domain, $\mathcal{Y} = \{0, 1\}$ be the label-set, and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ be the examples domain. A sample $S = ((x_1, y_1), \ldots, (x_m, y_m)) \in \mathcal{Z}^m$ is a sequence of examples. A function $h : \mathcal{X} \to \mathcal{Y}$ is called a hypothesis or a concept.

Let $\mathcal{D}$ be a distribution over $\mathcal{Z}$. The error of a hypothesis $h$ with respect to $\mathcal{D}$ is defined by $\text{err}(h; \mathcal{D}) = \mathbb{E}_{(x, y) \sim \mathcal{D}}[1[h(x) \neq y]]$. Let $S = ((x_1, y_1), \ldots, (x_m, y_m))$ be a sample. The empirical error of $h$ with respect to $S$ is defined by $\hat{\text{err}}(h; S) =$
\[ \frac{1}{m} \sum_{i=1}^{m} 1[h(x) \neq y]. \]

A hypothesis class \( \mathcal{H} \) is a set of hypotheses. A distribution \( \mathcal{D} \) is realizable by \( \mathcal{H} \) if there is \( h \in \mathcal{H} \) with \( \text{err}(h; \mathcal{D}) = 0 \). A sample \( S \) is realizable by \( \mathcal{H} \) if there is \( h \in \mathcal{H} \) with \( \hat{\text{err}}(h; S) = 0 \).

A learning algorithm, or a learner is a (possibly randomized) algorithm \( \mathcal{A} \) that takes samples \( S \) as input and outputs hypotheses, denoted by \( \mathcal{A}(S) \). We say that \( \mathcal{A} \) learns \( \mathcal{H} \) if for every \( \epsilon, \delta > 0 \) there is a finite bound \( m = m(\epsilon, \delta) \) such that for every \( \mathcal{H} \)-realizable distribution \( \mathcal{D} \),

\[ \Pr_{S \sim \mathcal{D}^m} [\text{err}(\mathcal{A}(S); \mathcal{D}) \geq \epsilon] \leq \delta. \]

\( \epsilon \) is called the error parameter, and \( \delta \) the confidence parameter. \( \mathcal{A} \) is called proper if \( \mathcal{A}(S) \in \mathcal{H} \) for every realizable \( S \), and it is called consistent if \( \hat{\text{err}}(\mathcal{A}(S); S) = 0 \) for every realizable \( S \).

**Differential Privacy**

Differential privacy [13] is a standard notion for statistical data privacy. Despite the connotation perceived by the name, differential privacy is a distributional stability condition that is imposed on an algorithm performing analysis on a dataset. Algorithms satisfying this condition are known as differentially private algorithms. There is a vast literature on the properties of this class of algorithms and their design and structure (see, e.g. [14] for an in-depth treatment).

**Definition** (Differential privacy). Let \( \mathcal{X}, \mathcal{Z} \) be two sets, and let \( m \in \mathbb{N} \). Let \( \alpha > 0, \beta \in [0, 1) \). An algorithm \( \mathcal{A} : \mathcal{X}^m \rightarrow \mathcal{Z} \) is said to be \((\alpha, \beta)\)-differentially private if for all datasets \( S, S' \in \mathcal{X}^m \) that differ in exactly one entry, and all measurable subsets \( \mathcal{O} \subseteq \mathcal{Z} \), we have

\[ \mathbb{P}_A [\mathcal{A}(S) \in \mathcal{O}] \leq e^\alpha \mathbb{P}_A [\mathcal{A}(S') \in \mathcal{O}] + \beta \]

where the probability is taken over the random coins of \( \mathcal{A} \).

When \( \beta = 0 \), the condition is sometimes referred to as pure differential privacy (as opposed to approximate differential privacy when \( \beta > 0 \).)

\(^3\)In this paper we focus on learning in the realizable case.
The general form of differential privacy entails two parameters: \( \alpha \) which is typically a small constant and \( \beta \) which in most applications is of the form \( \beta = o(1/m) \).

Differential privacy has been shown to provide non-trivial generalization guarantees especially in the adaptive settings of statistical analyses (see, e.g., [12, 2, 11]). In the context of (agnostic) PAC-learning, there has been a long line of work (e.g. [18, 4, 5, 15, 8]) that studied differentially private learning and the characterization of the sample complexity of private learning in several settings. However, the picture of differentially private learning is very far from complete and there are still so many open questions. Vadhan [27] gives a good survey on the subject.

**Information Theoretic Measures**

Information theory studies the quantification and communication of information. In this work, we use the language of learning theory combined with information theory to define and study a new type of learning theoretic compression. Here are standard notions from information theory (for more background see the textbook [9]).

Let \( Y \) and \( Z \) be two discrete random variables. The entropy of \( Y \) measures the number of bits required to encode \( Y \) on average.

**Definition** (Entropy). The entropy of \( Y \) is defined as

\[
H(Y) = - \sum_y \Pr(Y = y) \log \Pr(Y = y),
\]

where \( \log = \log_2 \) and by convention \( 0 \log 0 = 0 \).

The mutual information between \( Y \) and \( Z \) is (roughly speaking) a measure for the amount of random bits \( Y \) and \( Z \) share on average. It is also a measure of their independence; for example \( I(Y; Z) = 0 \) iff \( Y \) and \( Z \) are independent.

**Definition** (Mutual information). The mutual information between \( Y \) and \( Z \) is defined to be

\[
I(Y; Z) = H(Y) + H(Z) - H(Y, Z).
\]

The Kullback-Leibler divergence between two measures \( \mu \) and \( \nu \) is a useful measure for the “distance” between them (it is not a metric and may be infinite).
Definition (KL-divergence). The KL-divergence between two measures \( \mu \) and \( \nu \) on \( \mathcal{X} \) is

\[
\text{KL} (\mu || \nu) = \sum_x \mu(x) \log \frac{\mu(x)}{\nu(x)}
\]

where \( 0 \log \frac{0}{0} = 0 \).

Mutual information can be written as the following KL-divergence:

\[
I (Y; Z) = \text{KL} (p_{Y,Z} || p_Y \cdot p_Z),
\]

where \( p_{Y,Z} \) is the joint distribution of the pair \((Y, Z)\), and \( p_Y \cdot p_Z \) is the product of the marginals \( p_Y \) and \( p_Z \).

PAC-Bayes

The PAC-Bayes framework considers distributions over hypotheses. It is convenient to think of a distribution \( P \) over hypotheses as a randomized hypothesis. We extend the notions of error to randomized hypotheses as follows

\[
\text{err}(Q; D) = \mathbb{E}_{h \sim Q} [\text{err}(h; D)],
\]

and

\[
\hat{\text{err}}(Q; S) = \mathbb{E}_{h \sim Q} [\hat{\text{err}}(h; S)].
\]

Fix some randomized hypothesis \( P \). The following theorem known as the PAC-Bayes bound gives a bound on the generalization error simultaneously for all randomized hypotheses \( Q \) in terms of their KL-divergence with \( P \).

**Theorem 2.1** ([20] [26]). Let \( D \) be a distribution over examples, and let \( P \) be a fixed but otherwise arbitrary distribution over hypotheses. Let \( S \) denote a set of \( m \) i.i.d. examples generated by \( D \). Then, the following event occurs with probability at least \( 1 - \delta \): for every distribution \( Q \) over hypotheses,

\[
\text{err}(Q; D) - \hat{\text{err}}(Q; S) \leq \sqrt{\text{KL}(Q||P) + \ln(m/\delta)} / m.
\]

Following the Bayesian reasoning approach, the distribution \( P \) can be thought of as the a priori perspective of the target concept, and after seeing its input sample \( S \) the learning process outputs the distribution \( Q \) (which may depend on \( S \)), which is
its a posteriori perspective of the target concept. The PAC-Bayes theorem bounds the generalization error of the algorithm in terms of the KL-divergence between the a priori and the a posteriori perspectives.

3 Information Learners

Here we define learners that leak little info on their inputs. We also discuss some of their basic properties and connections to other notion.

We start by setting some notation. Let \( \mathcal{A} \) be a (possibly randomized) learning algorithm. For every sample \( S \in (\mathcal{X} \times \{0, 1\})^m \), let \( W_\mathcal{A}(\cdot|S) \) denote the conditional distribution of the output of the algorithm \( \mathcal{A} \) given that its input is \( S \). When \( \mathcal{A} \) is deterministic, \( W_\mathcal{A}(\cdot|S) \) is a degenerate distribution. For a distribution \( \mathcal{D} \) over examples and \( m \in \mathbb{N} \), let \( \mathcal{P}_\mathcal{D} \) denote the marginal distribution of the output of \( \mathcal{A} \) when it takes an input sample of size \( m \) drawn i.i.d. from \( \mathcal{D} \), that is, \( \mathcal{P}_\mathcal{D}(h) = \mathbb{E}_{S \sim \mathcal{D}^m}[W_\mathcal{A}(h|S)] \) for every function \( h \).

**Definition 1** (Mutual information of an algorithm). We say that \( \mathcal{A} \) has mutual information of at most \( d \) bits (for sample size \( m \)) with respect to a distribution \( \mathcal{D} \),

\[
I(S; \mathcal{A}(S)) \leq d,
\]

where \( S \sim \mathcal{D}^m \).

**Definition 2** (\( d \)-bit information learner). A learning algorithm \( \mathcal{A} \) for \( \mathcal{H} \) is called a \( d \)-bit information learner if it has mutual information of at most \( d \) bits with respect to every realizable distribution (for every sample size).

We will later provide more variants of the definition above and discuss their connections to other notions (see Section 3.3).

3.1 Bounded Information Implies Generalization

The following theorem quantifies the generalization guarantees of \( d \)-bit information learners.

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4In this text we focus on Shannon’s mutual information, but other notions of divergence may be interesting to investigate as well.
Theorem 3.1. Let $\mathcal{A}$ be a learner that has mutual information of at most $d$ bits with a distribution $\mathcal{D}$, and let $S \sim \mathcal{D}^m$. Then, for every $\epsilon > 0$,

$$
\Pr_{\mathcal{A}, S} [ |\hat{\text{err}}(\mathcal{A}(S); S) - \text{err}(\mathcal{A}(S); \mathcal{D})| > \epsilon ] < \frac{d + 1}{2m\epsilon^2 - 1}
$$

where the probability is taken over the randomness in the sample $S$ and the randomness of $\mathcal{A}$.

Theorem 3.1 states a simple and basic property, and is proved in Section 4 (below we provide a proof-sketch for deterministic protocols). In fact, we provide four different proofs of the theorem, each highlight its generality.

The first proof is based on an information theoretic lemma, which roughly states that if the KL-divergence between two measures $\mu$ and $\nu$ is small then $\mu(E)$ is not much larger than $\nu(E)$ for every event $E$. The nature of this proof strongly resembles the proof of the PAC-Bayes bounds [26] and indeed the theorem can be derived from these standard bounds as well (see the fourth proof). The second proof is based on a method to efficiently “de-correlate” two random variables in terms of their mutual information; roughly speaking, this implies that an algorithm of low mutual information can only generate a small number of hypotheses and hence does not overfit. The third proof highlights an important connection between low mutual information and the stability of the learning algorithm. The fourth proof uses the PAC-Bayes framework.

We mention that the dependence on $\epsilon$ can be improved in the realizable case; if the algorithm always outputs a hypothesis with empirical error 0 then the bound on the right hand side can be replaced by $O\left(\frac{d + 1}{m}\right)$. As in similar cases, the reason for this difference stems from that estimating the bias of a coin up to additive error $\epsilon$ requires $m \approx \frac{1}{\epsilon^2}$ samples, but if the coin falls on heads w.p. $\epsilon$ the chance of seeing $m$ tails in a row is $(1 - \epsilon)^m \approx e^{-m\epsilon}$.

Sketch for Deterministic Algorithms

Here we sketch a proof of Theorem 3.1 for deterministic protocols. When $\mathcal{A}$ is deterministic, we have $I = I(S; \mathcal{A}(S)) = H(\mathcal{A}(S))$. Let $\mathcal{P}_D$ denote the distribution of $\mathcal{A}(S)$. Let $\mathcal{H}_0$ be the set of hypothesis $h$ so that $\mathcal{P}_D(h) \geq 2^{1/\delta}$. By Markov’s inequality, $\mathcal{P}_D(\mathcal{H}_0) \geq 1 - \delta$. In addition, the size of $\mathcal{H}_0$ is at most $2^{1/\delta}$. So Chernoff’s
inequality and the union bound imply that for every \( h \in \mathcal{H}_0 \) the empirical error is close to the true error for \( m \approx \frac{1}{\epsilon^2} \) (with probability at least \( 1 - \delta \)).

### 3.2 The Dependence on the Confidence

#### Sharpness of Bound

Standard bounds on the sample complexity of learning VC classes imply that to get \( \delta \leq \frac{1}{2} \) one must use at least \( m \geq \Omega(\frac{d}{\epsilon}) \) samples (see e.g. [26]). The dependence of \( \delta \) in the theorem is, however, less standard; in most cases it suffices for \( m \) to be logarithmic in \( \frac{1}{\delta} \) in order to get confidence at least \( 1 - \delta \), whereas here \( m \) should be proportional to \( \frac{1}{\delta} \) to get the same confidence level. This turns out to be sharp, as the following example shows.

**Proposition 3.2.** Let \( n \geq m \geq 4 \) be integers such that \( n \) is sufficiently large. Let \( \mathcal{X} = [n] \) and let \( \mathcal{D} \) be the uniform distribution on examples of the form \((x, 1): x \in [n]\). There is a deterministic learning algorithm \( A : (\mathcal{X} \times \{0, 1\})^m \to \{0, 1\}^\mathcal{X} \) with sample size \( m \) and mutual information \( O(1) \) so that

\[
\mathbb{P}\left[ |\hat{\text{err}}(A(S); S) - \text{err}(A(S); \mathcal{D})| \geq \frac{1}{2}\right] \geq \frac{1}{m}
\]

where \( S \) is generated i.i.d. from \( \mathcal{D} \) and \( f \).

The construction is based on the following claim.

**Claim 3.3.** For a sufficiently large \( n \), there are \( M = \Theta\left(\frac{2^m}{m}\right) \) subsets \( T_1, \ldots, T_M \) of \([n]\) each of size at most \( n/2 \) so that the \( \mathcal{D} \)-measure of \( T = \bigcup_i (T_i \times \{1\})^m \) is between \( 1/m \) and \( 4/m \).

**Proof.** Let \( T_1, \ldots, T_M \) be i.i.d. uniformly random subsets of \( \mathcal{X} \), each of size \( k = \lfloor n/2 \rfloor \). The size of each \( T_i^m \) is \( k^m \). For \( i \neq j \) we have

\[
\mathbb{E}\left[ |T_i^m \cap T_j^m| \right] = \mathbb{E}\left[ |T_i \cap T_j|^m \right] = \sum_{x_1, \ldots, x_m} \mathbb{E}\left[ 1_{x_1, \ldots, x_m \in T_i} \right] \mathbb{E}\left[ 1_{x_1, \ldots, x_m \in T_j} \right] \leq n^m \cdot \left( \frac{(n-m)}{n-k} \right)^2 + m^2 n^{m-1} \cdot 1,
\]

\(^5\text{Here and below } O(\cdot), \Omega(\cdot) \text{ and } \Theta(\cdot) \text{ mean up to some multiplicative universal constants.}\)
where in the last inequality, the first term corresponds to sequences \(x_1, \ldots, x_m\) where all elements are distinct, and the second term corresponds to sequences \(x_1, \ldots, x_m\) for which there are \(i \neq j\) such that \(x_i = x_j\). Now, since 
\[
\binom{n-m}{k} \frac{k(k-1) \cdots (k-m+1)}{n(n-1) \cdots (n-m+1)} \leq \left(\frac{k}{n}\right)^m,
\]
\[
\mathbb{E} \left[ |T_i^m \cap T_j^m| \right] \leq \frac{k^{2m}}{n^m} + m^2 n^{m-1}.
\]
Therefore, since 
\[
\sum |T_i^m| \geq |\cup_i T_i^m| \geq \sum |T_i^m| - \sum_{i \neq j} |T_i^m \cap T_j^m|,
\]
\[
Mk^m \geq \mathbb{E}[|T|] \geq Mk^m - \frac{M^2 k^{2m}}{2 n^m} - \frac{M^2}{2} m^2 n^{m-1} \geq \frac{M k^m}{2} - \frac{M^2}{2} m^2 n^{m-1},
\]
as long as \(M \leq n^m/k^m\). Hence, plugging \(M = \frac{4 m^m}{mk^m} = \Theta\left(\frac{2^m}{m}\right)\) yields that
\[
\frac{4}{m} \geq \mathbb{E}[|T|/n^m] \geq \frac{2}{m} - o(1),
\]
where the \(o(1)\) term approaches 0 as \(n\) approaches \(\infty\). So, for a sufficiently large \(n\) there is a choice of \(T_1, \ldots, T_M\) as claimed.

Proof of Proposition 3.3: The algorithm \(A\) is defined as follows. Let \(T_1, \ldots, T_M\) be the sets given in Claim 3.3. For each \(i \in [M]\), let \(h_i\) be the hypothesis that is 1 on \(T_i\) and 0 elsewhere. Given a sample \((x_1, 1), \ldots, (x_m, 1)\), the algorithm outputs \(h_i\), where \(i\) is the minimum index so that \(\{x_1, \ldots, x_m\} \subset T_i\); if no such index exists the algorithm outputs the all-ones function.

The empirical error of the algorithm is 0, and with probability \(p \in [1/m, 4/m]\) it outputs a hypothesis with true error at least 1/2. The amount of information it provides on its inputs can be bounded as follows: letting \(p_i\) be the probability that the algorithm outputs \(h_i\), we have
\[
I(S; A(S)) = H(A(S))
\]
\[
= (1 - p) \log \frac{1}{1-p} + p \sum_i \frac{p_i}{p} \log \frac{1}{p_i}
\]
\[
\leq (1 - p) \log \frac{1}{1-p} + p \log \frac{M}{p} \quad \text{(convexity)}
\]
\[
\leq O(1).
\]
We show that the same standard procedure used for boosting the confidence of learning algorithms can be used with information learners at a modest cost.

**Theorem 3.4 (Confidence amplification).** Let \(A\) be a \(d\)-bit information learner, and let \(D\) be a distribution on examples. Assume that given \(m_0\) examples from \(D\) to \(A\), then with confidence \(1/2\), it outputs a hypothesis with error at most \(\epsilon\). Then there is an algorithm \(B\) such that for every \(\delta > 0\),

- **given** \(m = m_0 \lceil \log(2/\delta) \rceil + \frac{2 \ln(4 \log(2/\delta)/\delta)}{4 \epsilon^2}\) examples from \(D\) to \(B\), with confidence at least \(1 - \delta\), it outputs a hypothesis with error at most \(\epsilon\)

- \(B\) is a \(\left(\log \log(2/\delta) + d \log(2/\delta)\right)\)-bit information learner.

**Proof.** We use the natural confidence amplification procedure. Set \(k = \lceil \log(2/\delta) \rceil\), and draw \(k\) subsamples \((S^{(1)}, \ldots, S^{(k)}) \triangleq S\), each of size \(m_0\), and another “validation” set \(T\) of size \(2 \ln(4 \log(2/\delta)/\delta)/\epsilon^2\). Run \(A\) independently on each of the \(k\) subsamples to output hypotheses \(\vec{h} = (h_1, \ldots, h_k)\). Next, we validate \(h_1, \ldots, h_k\) on \(T\) and output a hypothesis \(h^* \in \{h_1, \ldots, h_k\}\) with the minimal empirical error on \(T\).

Item 1 in the above theorem follows directly from standard analysis (Chernoff plus union bound). To prove item 2, first we note that \((S, h_1, \ldots, h_k)\) is independent of \(T\). So,

\[
I(h^*; S, T) \leq I\left(h^*, \vec{h}; S, T\right)
= I(\vec{h}; S, T) + I(h^*; S, T|\vec{h})
= I(\vec{h}; S) + \left[I(h^*; T|\vec{h}) + I(h^*; S|T, \vec{h})\right]
\leq \sum_{i=1}^{k} I(h_i; S^{(i)}) + H(h^*|h_1, \ldots, h_k) + 0
\leq \log(k) + dk.
\]
3.3 Related Notions

Occam’s Razor

The above extends the classical Occam’s razor generalization bound [6], which we describe next: assume a fixed encoding of hypotheses in $\mathcal{H}$ by bit strings. The complexity of a hypothesis is the bit-length of its encoding. A learning algorithm for $\mathcal{H}$ is called an Occam-algorithm with parameters $c, \alpha$ if for every realizable sample of size $m$ it produces a consistent hypothesis of complexity at most $n^c m^\alpha$

**Theorem 3.5** ([6]). Let $A$ be an Occam-algorithm with parameters parameters $c \geq 1$ and $0 \leq \alpha < 1$. Let $\mathcal{D}$ be a realizable distribution, let $f \in \mathcal{H}$ be such that $\text{err}(f; \mathcal{D}) = 0$, and let $n$ denote the complexity of $f$. Then,

$$\Pr_{S \sim \mathcal{D}^m}[\text{err}(A(S); \mathcal{D}) \geq \epsilon] \leq \delta,$$

as long as $m$ is at least some $O \left( \frac{\log(1/\delta)}{\epsilon} + (n^c/\epsilon)^{1/(1-\alpha)} \right)$.

To relate Occam’s razor to our theorem, observe that an Occam-algorithm is in particular a $O(n^c m^\alpha)$-bit information learner (since its output hypothesis is encoded by $O(n^c m^\alpha)$ bits), which implies (by Theorem 3.1) that the probability of it outputting a function with true error more than $\epsilon$ is at most $O \left( \frac{n^c m^\alpha}{\epsilon^2} \right)$. The bound can be improved by standard confidence-boosting techniques (see Section 3.2 below).

**Sample Compression Schemes**

Why do $d$-bit information learners generalize? Why does the empirical loss of such a learner is concentrated near its true loss? In a nutshell, this is because the dependence of the output hypothesis on the input sample is bounded by at most $d$ bits of mutual information.

This resembles the notion of sample compression schemes [19]. Sample compression schemes correspond to learning algorithms whose output hypothesis is determined by a small subsample of the input. For example, support vector machines output a separating hyperplane that is determined by the support vectors.

Both sample compression schemes and information learners quantify (in different ways) the property of limited dependence between the output hypothesis and the in-
put sample. It is therefore natural to ask how do these two notions relate to each other.

It turns out that not every sample compression scheme of constant size also leaks a constant number of bits. Indeed, in Section 5 it is shown that there is no ERM for thresholds that is an \( O(1) \)-bits information learner. On the other hand, there is an ERM for it that is based on a sample compression of size \( O(1) \).

**PAC-Bayes**

It is natural to analyze learners that leak little information within the PAC-Bayes framework. Technically, this follows from expressing the mutual information between random variables \( X, Y \) in terms of the KL-divergence between \( X|Y \) and \( X \): let \( A \) be a \( d \)-bit information learner for \( H \), and let \( D \) be a realizable distribution. Therefore,

\[
d \geq I(S; A(S)) = \mathbb{E}_{S \sim D^m} \left[ \text{KL}(W_A(\cdot|S) || P_D) \right]
\]

(recall that \( P_D \) denotes the marginal distribution of \( A(S) \) for \( S \sim D^m \)). Therefore, by Markov’s inequality, with probability \( 1 - \delta \) it holds that

\[
\text{KL}(W_A(\cdot|S) || P_D) \leq d/\delta.
\]

Now we can apply the standard PAC-Bayes generalization bound (Theorem 2.1) and deduce that with probability at least \( 1 - \delta \) over the choice of the sample, the expected generalization error is

\[
O \left( \sqrt{\frac{d}{\delta} \ln \left( \frac{m}{\delta} \right)} / m \right),
\]

where the expectation is taken over \( h \sim W_A(\cdot|S) \). Indeed, this follows by choosing the prior \( P \) in the PAC-Bayes bound to be the marginal distribution \( P_D \) of the output of \( A \), and choosing the posterior \( Q \) to be the conditional distribution \( W_A(\cdot|S) \). Hence, from the observation above, it follows that, with probability at least \( 1 - \delta \) over the choice of the sample, a \( d \)-bit learner has expected generalization error \( \tilde{O}(\sqrt{d/m}) \).

**Pure Differential Privacy**

Pure differential privacy is a stronger notion than bounded mutual information: *Any pure \( \alpha \)-differentially private algorithm (see Section 2) with input sample of size \( m \)
has mutual information $O(\alpha m)$. This statement follows directly from [21]. It also follows from the implication of pure differential privacy on max-information studied in [11]. Typically, $\alpha$ is a small constant in most applications of differential privacy, and hence, in general, the above bound on mutual information depends on $m$.

4 Bounded Information Implies Generalization

In this section we present four proofs of Theorem 3.1 (some of the arguments are just sketched).

Proof I: Mutual Information and Independence

The first proof of Theorem 3.1 we present uses the following lemma, which allows to control a distribution $\mu$ by a distribution $\nu$ as long as it is close to it in KL-divergence.

Lemma 4.1. Let $\mu$ and $\nu$ be probability distributions on a finite set $X$ and let $E \subset X$. Then,

$$
\mu(E) \leq \frac{\text{KL}(\mu||\nu) + 1}{\log(1/\nu(E))}.
$$

The lemma enables us to compare between events of small probability: if $\nu(E)$ is small then $\mu(E)$ is also small, as long as $\text{KL}(\mu||\nu)$ is not very large.

The bound given by the lemma above is tight, as the following example shows. Let $X = [2n]$ and let $E = [n]$. For each $x \in X$, let

$$
\mu(x) = \frac{1}{2n}
$$

and let

$$
\nu(x) = \begin{cases} 
1/n^2 & x \in E, \\
(n-1)/n^2 & x \notin E.
\end{cases}
$$

Thus, $\mu(E) = \frac{1}{2}$ and $\nu(E) = \frac{1}{n}$. But on the other hand

$$
\text{KL}(\mu||\nu) = \frac{1}{2} \log \left( \frac{n}{2} \right) + \frac{1}{2} \log \left( \frac{1}{2} \right) + o(1),
$$

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so

\[
\frac{1}{2} = \mu(E) \geq \lim_{n \to \infty} \frac{\text{KL}(\mu||\nu) + 1}{\log(1/\nu(E))} = \frac{1}{2}.
\]

Similar examples can be given when \(\text{KL}(\mu||\nu)\) is constant.

We now change the setting to allow it to apply more naturally to the setting of information learners.

**Lemma 4.2.** Given a distribution \(\mu\) on the space \(X \times Y\) and an event \(E\) that satisfies \(\mu_X(E_y) < \alpha\) for all \(y \in Y\) then \(\mu(E) \leq \frac{I(X;Y)+1}{\log(1/\alpha)}\), where \(\mu_X\) is the marginal distribution of \(X\) and \(E_y = \{x : (x,y) \in E\}\) is a fiber of \(E\) over \(y\).

The lemma enables us to bound the probability of an event \(E\), if we have a bound on the probability of its fibers over \(y\) measured with the marginal distribution of \(X\).

This lemma can be thought of as a generalization of the extremal case where \(X\) and \(Y\) are independent (i.e. \(I(X,Y) = 0\)). In this case, the lemma corresponds to the following geometric statement in the plane: if the width of every \(x\)-parallel fiber of a shape is at most \(\alpha\) and its height is bounded by 1 then its area is also at most \(\alpha\). The bound given by the above lemma is \(\frac{1}{\log(1/\alpha)}\), which is weaker, but the lemma applies more generally when \(I(X,Y) > 0\). In fact, the bound is tight when the two variables are highly dependent; e.g. \(X = Y\) and \(X \sim U([n])\). In this case, the probability of the diagonal \(E\) is 1, while \(I(X;Y) = \log(n)\) and \(\alpha = 1/n\). So indeed \(1 = \mu(E) \approx \frac{\log(n)+1}{\log(n)}\).

We now use the lemmas to prove the theorem.

**Proof of Theorem 3.1.** Let \(\mu\) be the distribution on pairs \((S,h)\) where \(S\) is chosen i.i.d. from \(\mathcal{D}\) and \(h\) is the output of the algorithm given \(S\). Let \(E\) be the event of error; that is,

\[
E = \left\{ (S,h) : \left| \text{err}(h;\mathcal{D}) - \widehat{\text{err}}(h;S) \right| > \epsilon \right\}.
\]

Using Chernoff’s inequality, for each \(h\),

\[
\mu_S(E_h) \leq 2 \cdot \exp\left(-2m\epsilon^2\right),
\]

where \(E_h\) is the fiber of \(E\) over function \(h\). Lemma 4.2 implies

\[
\mu(E) \leq \frac{I(S;A(S)) + 1}{2m\epsilon^2 - 1},
\]

\(\square\)
We now prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1

\[
\text{KL}(\mu || \nu) = -\mu(E) \sum_{x \in E} \frac{\mu(x)}{\mu(E)} \cdot \log \left( \frac{\nu(x)}{\mu(x)} \right) - \mu(E^c) \sum_{x \in E^c} \frac{\mu(x)}{\mu(E^c)} \cdot \log \left( \frac{\nu(x)}{\mu(x)} \right)
\]

\[
\geq -\mu(E) \cdot \log \left( \sum_{x \in E} \frac{\mu(x)}{\mu(E)} \nu(x) \right) - \mu(E^c) \cdot \log \left( \sum_{x \in E^c} \frac{\mu(x)}{\mu(E^c)} \nu(x) \right)
\]

\[
= -\mu(E) \cdot \log \left( \frac{\nu(E)}{\mu(E)} \right) - \mu(E^c) \cdot \log \left( \frac{\nu(E^c)}{\mu(E^c)} \right)
\]

\[
\geq -\mu(E) \cdot \log (\nu(E)) - \mu(E^c) \cdot \log (\nu(E^c)) - 1
\]

where (a) follows by convexity, and (b) holds since the binary entropy is at most one.

Proof of Lemma 4.2

By Lemma 4.1, for each \(y\),

\[
\mu(X|Y=y) = \mu(E_y) \leq \frac{\text{KL}(\mu_{X|Y=y}||\mu_X) + 1}{\log \left( \frac{1}{\mu_X(E_y)} \right)} \leq \frac{\text{KL}(\mu_{X|Y=y}||\mu_X) + 1}{\log (1/\alpha)}.
\]

Taking expectation over \(y\) yields

\[
\mu(E) \leq \frac{I(X;Y) + 1}{\log (1/\alpha)}.
\]

Proof II: De-correlating

The second proof we present allows to “de-correlate” two random variables in terms of the mutual information. The lemma follows from [16, 7].

Lemma 4.3. Let \(\mu\) be a probability distribution over \(\mathcal{X} \times \mathcal{Y}\), where \(\mathcal{X}\) and \(\mathcal{Y}\) are finite sets. Let \((X,Y)\) be chosen according to \(\mu\). Then there exists a random variable \(Z\) such that:

1. \(Z\) is independent of \(X\).
2. $Y$ is a deterministic function of $(X, Z)$.

3. $H(Y|Z) \leq I(X, Y) + \log(I(X, Y) + 1) + O(1)$.

The lemma can be interpreted as follows. Think of $X$ as sampled from some unknown process, and of $Y$ as the output of a randomized algorithm on input $X$. Think of $Z$ as the random coins of the algorithm. The lemma says that there is a way to sample $Z$, before seeing $X$, in a way the preserves functionality (namely, so that $(X, Y)$ are correctly distributed) and so that for an average $Z$, the randomized algorithm outputs only a small number of $Y$’s.

Theorem 3.1 now follows similarly to the proof sketch for deterministic protocol in Section 3.1, since conditioned on $Z$, the algorithm uses only a small number of outputs (on average).

**Proof III: Stability**

The third proof of Theorem 3.1 we provide is based on stability. The parameters obtained from this proofs are slightly different. For a $d$-bit information learner $\mathcal{A}$, we prove that

$$\mathbb{E}_{\mathcal{A},S} [\text{err}(\mathcal{A}(S); \mathcal{D}) - \hat{\text{err}} (\mathcal{A}(S); S)] < \sqrt{\frac{d}{m}}$$

(1)

where $S$ is the input sample of $m$ examples drawn i.i.d. from $\mathcal{D}$.

We start by setting some notation. For random variables $X, Y$, we use the notation $d_{\text{TV}}(X, Y)$ to denote the total variation (i.e., the statistical distance) between the distributions of $X$ and $Y$. For clarity, we also twist the notation we used earlier a little bit, and use $\text{KL}(X||Y)$ to denote the KL-divergence between the distributions of $X$ and $Y$. Also, in the following, we use the notation $(X, Y)$ to denote the joint distribution of $X$ and $Y$, and $X \times Y$ to denote the product distribution resulting from the marginal distributions of $X$ and $Y$.

Finally, as typical in stability arguments, for any sample $S$ and any example $z \in \mathcal{X} \times \{0, 1\}$, we will use the notation $S^{(i,z)}$ to denote the set resulting from replacing the $i$-th example in $S$ by $z$.

The proof relies on the following two lemmas.
Lemma 4.4. If \( I(A(S); S) \leq d \), then

\[
\frac{1}{m} \sum_{i=1}^{m} \sqrt{I(A(S); S_i)} \leq \sqrt{\frac{d}{m}}
\]

where \( S_i \) denotes the \( i \)-th example in \( S \).

Lemma 4.5. For any \( i \in [m] \), we have

\[
\sqrt{I(A(S); S_i)} \geq \mathbb{E}_z [d_{TV}(A(S^{(i,z)}), A(S))]
\]

where \( z = (x, y) \sim D \) independently from \( S \).

Proof of Lemma 4.4. By the independence of the samples \( S_1, \ldots, S_m \) and the fact that conditioning reduces entropy,

\[
I(A(S); S) \geq \sum_{i=1}^{m} I(A(S); S_i)
\]

By the Cauchy-Schwartz inequality,

\[
\sum_{i=1}^{m} \sqrt{I(A(S); S_i)} \leq \sqrt{m \sum_{i=1}^{m} I(A(S); S_i))}.
\]

Proof of Lemma 4.5.

\[
\sqrt{I(A(S); S_i)} = \sqrt{KL((A(S), S_i) \parallel A(S) \times S_i)}
\]

\[
\geq d_{TV}((A(S), S_i), A(S) \times S_i) \quad \text{(Pinsker’s inequality)}
\]

\[
= d_{TV}((A(S^{(i,z)}), z), A(S) \times z)
\]

\[
= \mathbb{E}_z [d_{TV}(A(S^{(i,z)}), A(S))].
\]

where the third step follows from the fact that \((A(S), S_i)\) and \((A(S^{(i,z)}), z)\) are identically distributed, and the fact that \( S_i \) and \( z \) are identically distributed.

We are now ready to prove \( \Pi \). Recall that for any example \( S_i = (x_i, y_i) \), we have

\[
\hat{\text{err}}(A(S); S_i) \triangleq 1(A(S)(x_i) \neq y_i),
\]

where \( A(S)(x_i) \) denotes the label of the output.
hypothesis $\mathcal{A}(S)$ on $x_i$. Let $z$ denote a fresh example $(x, y) \sim \mathcal{D}$ independent of $S$. Let $\mathcal{U}$ denote the uniform distribution over $[m]$.

The two lemmas above imply that

$$E_{i \sim \mathcal{U}, z} [d_{TV} (\mathcal{A}(S^{(i,z)}), \mathcal{A}(S))] \leq \sqrt{\frac{d}{m}}.$$  

It follows that for any $\bar{z} \in \mathcal{X} \times \{0, 1\}$, we must have

$$E_{i \sim \mathcal{U}, z} [d_{TV} (\hat{\text{err}} (\mathcal{A}(S^{(i,z)}); \bar{z}), \hat{\text{err}} (\mathcal{A}(S); \bar{z}))] \leq \sqrt{\frac{d}{m}},$$

which is equivalent to

$$E_{i \sim \mathcal{U}, z} \left[ E_{S, A} \left[ \hat{\text{err}} (\mathcal{A}(S^{(i,z)}); S) - \hat{\text{err}} (\mathcal{A}(S); S_i) \right] \right] \leq \sqrt{\frac{d}{m}},$$

which implies

$$E_{i \sim \mathcal{U}, z} \left[ E_{S, A} \left[ \hat{\text{err}} (\mathcal{A}(S); S_i) \right] \right] \leq \sqrt{\frac{d}{m}}.$$ 

Finally, we use the fact that $\hat{\text{err}} (\mathcal{A}(S^{(i,z)}); S_i)$ and $\hat{\text{err}} (\mathcal{A}(S); z)$ are identically distributed to get

$$E_{i \sim \mathcal{U}} \left[ E_{S, z, A} \left[ \hat{\text{err}} (\mathcal{A}(S); z) \right] \right] \leq \sqrt{\frac{d}{m}},$$

which leads directly to (1).

**Proof IV: PAC-Bayes**

The fourth proof is straightforward via the connection between information learners and the PAC-Bayes framework. Using the same notation as in the section above (Proof III):

**Theorem 4.6.** Assuming $m \geq 5 \frac{(d+1)}{\epsilon^2} \ln \left( \frac{d+1}{\epsilon} \right)$, for every $\epsilon > 0$,

$$P_{S} \left[ E_{A} [\text{err}(\mathcal{A}(S)); \mathcal{D}) - \hat{\text{err}} (\mathcal{A}(S); S)] > \epsilon \right] < \frac{d + 1}{m \epsilon^2},$$

where the probability is taken over the randomness in the sample $S$ and the expectation inside the probability is taken over the random coins of $\mathcal{A}$.
The proof of Theorem 4.6 follows from the discussion in Section 3.3 on the connection to PAC-Bayes. We just rephrased the statement of the bound that follows from PAC-Bayes in Theorem 2.1 so that it has a similar form to the statement of Theorem 3.1.

5 Thresholds Leak Information

In this section we show that any proper consistent learner for the class of thresholds can not leak little information with respect to all realizable distributions $D$. Namely, we find for every such algorithm $A$ a realizable distribution $D$ so that $I(S; A(S))$ is large.

Let $X = [2^n]$ and $T \subseteq \{0, 1\}^X$ be the set of all thresholds; that is $T = \{f_k\}_{k \in [2^n]}$ where

$$f_k(x) = \begin{cases} 0 & x < k, \\ 1 & x \geq k. \end{cases}$$

**Theorem 5.1.** For any consistent and proper learning algorithm $A$ for $T$ with sample size $m$ there exists a realizable distribution $D = D(A)$ so that

$$I(S; A(S)) = \Omega \left( \frac{\log n}{m^2} \right) = \Omega \left( \frac{\log \log |X|}{m^2} \right),$$

where $S \sim D^m$.

The high-level approach is to identify in $A$ a rich enough structure and use it to define the distribution $D$. Part of the difficulty in implementing this approach is that we need to argue on a general algorithm, with no specific structure. A different aspect of the difficulty in defining $D$ stems from that we can not adaptively construct $D$, we must choose it and then the algorithm gets to see many samples from it.

5.1 Warm Up

We first prove Theorem 5.1 for the special case of deterministic learning algorithms. Let $A$ be a consistent deterministic learning algorithm for $T$. Define the $2^n \times 2^n$ upper triangular matrix $M$ as follows: for all $i < j$,

$$M_{ij} = k$$
where $f_k$ is the output of $\mathcal{A}$ on a sample of the form

$$S_{ij} = \left( (1,0), \ldots, (1,0), (i,0), (j,1) \right)$$

and $M_{ij} = 0$ for all $i \geq j$.

The matrix $M$ summarizes the behavior of $\mathcal{A}$ on some of its inputs. Our goal is to identify a sub-structure in $M$, and then use it to define the distribution $\mathcal{D}$.

We start with the following lemma.

**Lemma 5.2.** Let $Q \in \text{Mat}_{2^n \times 2^n}(\mathbb{N})$ be a symmetric matrix that has the property that for all $i, j$:

$$\min\{i,j\} \leq Q_{ij} \leq \max\{i,j\} \quad \text{(i)}$$

Then $Q$ contains a row with at least $n + 1$ different values (and hence also a column with $n + 1$ different values).

**Proof.** The proof is by induction on $n$. In the base case $n = 1$ we have

$$Q = \begin{bmatrix} 1 & x \\ x & 2 \end{bmatrix}$$

and the lemma indeed holds.

For the induction step, let

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \in \text{Mat}_{2^{n+1} \times 2^{n+1}}(\mathbb{N})$$

where $Q_1, Q_2, Q_3, Q_4 \in \text{Mat}_{2^n \times 2^n}(\mathbb{N})$.

All the values in $Q_1$ are in the interval $[1, 2^n]$ and $Q_1$ is also a symmetric matrix and satisfies property (i). So $Q_1$ contains some row $r$ with at least $n + 1$ distinct values in the interval $[1, 2^n]$.

Similarly, all the values in $Q_4$ are in the interval $[2^n + 1, 2^{n+1}]$, and we can write $Q_4$ as

$$Q_4 = Q_4' + 2^n \cdot J$$

where $J$ is the all-1 matrix and $Q_4' = Q_4 - (2^n \cdot J)$ is a symmetric matrix satisfying property (i). From the induction hypothesis it follows that $Q_4$ contains a column $k$ with at least $n + 1$ different values in the interval $[2^n + 1, 2^{n+1}]$. 

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Now consider the value $Q_{rk}$. If $(Q_{2})_{rk} \in [1, 2^n]$ then the column in $Q$ corresponding to $k$ contains $n + 2$ different values. Otherwise, the row corresponding to $r$ contains $n + 2$ different values.

Next, consider the matrix

$$Q = M + M^t + \text{diag}(1, 2, 3, \ldots, 2^n).$$

It is a symmetric matrix satisfying property (i), and so it contains a row $r$ with at least $n + 1$ distinct values. If row $r$ contains at least $\frac{n}{2}$ distinct values above the diagonal then row $r$ in $M$ also contains $\frac{n}{2}$ distinct values above the diagonal. Otherwise, row $r$ contains at least $\frac{n}{2}$ distinct values below the diagonal, and then column $r$ in $M$ contains $\frac{n}{2}$ distinct values above the diagonal.

The proof proceeds by separately considering each of the two cases (row or column):

**Case 1:** $M$ contains a row $r$ with $\frac{n}{2}$ distinct values. Let $k_1, \ldots, k_{n/2}$ be columns such that all the values $M_{rk_i}$ are distinct. Let $D$ be the distribution that gives the point 1 mass of $1 - \frac{1}{m-2}$ and evenly distributes the remaining mass is on the values $r, k_1, \ldots, k_{n/2}$. The function labeling the examples is chosen to be $f_{r+1} \in \mathcal{T}$.

Let $E$ be the indicator random variable of the event that the sample $S$ is of the form

$$S = \left( (1, 0), \ldots, (1, 0), (r, 0), (k_i, 1) \right)$$

for some $k_i$. The probability of $E = 1$ is at least

$$\left( 1 - \frac{1}{m-2} \right)^{m-2} \cdot \left( \frac{1}{2(m-2)} \right)^2 = \Omega \left( \frac{1}{m^2} \right).$$

From the definition of $M$, when $E = 1$ the algorithm outputs $h = f_{M_{rk_i}}$ where $k_i$ is uniformly distributed uniformly over $\frac{n}{2}$ values, and so $H(\mathcal{A}(S)|E = 1) = \log(n/2)$. This yields

$$I(S; \mathcal{A}(S)) = I(S, E; \mathcal{A}(S))$$

$$\geq I(S; \mathcal{A}(S)|E)$$

$$= H(\mathcal{A}(S)|E)$$

$$\geq \Pr[E = 1]H(\mathcal{A}(S)|E = 1)$$

$$\geq \Omega \left( \frac{\log n}{m^2} \right).$$
Case 2: $M$ contains a column $k$ with $\frac{n}{2}$ distinct values. Let $r_1, \ldots, r_{n/2}$ be rows such that all the values $M_{r_i,k}$ are distinct. Now, consider the event that

$$S = \left( (1,0), \ldots, (1,0), (r_i,0), (k,1) \right)$$

for some $r_i$. The rest of the argument is the same as for the previous case.

## 5.2 Framework for Lower Bounding Mutual Information

Here we describe a simple framework that allows to lower bound the mutual information between two random variables.

### Standard Lemmas

**Lemma 5.3.** For any two distribution $p$ and $q$, the contribution of the terms with $p(x) < q(x)$ to the divergence is at least $-1$:

$$- \sum_{x:p(x)<q(x)} p(x) \log \frac{p(x)}{q(x)} > -1.$$

**Proof.** Let $E$ denote the subset of $x$’s for which $p(x) < q(x)$. Then we have

$$\sum_{x \in E} p(x) \log \frac{p(x)}{q(x)}$$

$$\geq -p(E) \cdot \sum_{x \in E} p(x|E) \log \frac{q(x)}{p(x)}$$

$$\geq -p(E) \cdot \log \sum_{x \in E} p(x|E) \frac{q(x)}{p(x)}$$

$$= -p(E) \cdot \log \frac{q(E)}{p(E)}$$

$$\geq p(E) \cdot \log \frac{p(E)}{e}.$$

For $0 \leq z \leq 1$, $z \log z$ is maximized when its derivative is 0: $\log e + \log x = 0$. So the maximum is attained at $z = 1/e$, proving that $p(E) \log p(E) \geq -\log e > -1$.  

---

<sup>6</sup>We include the proof for completeness.
Lemma 5.4 (Data processing). Let $X, Y, Z$ be random variables such that $X - Y - Z$ form a Markov chain; that is, $X$ and $Z$ are independent conditioned on $Y$. Then

$$I(X; Y) \geq I(X; Z)$$

Proof. The chain rule for mutual information yields that

$$I(X; Y) = I(X; Z) + I(X; Y|Z) - I(X; Z|Y) \quad \text{(chain rule)}$$

$$= I(X; Z) + I(X; Y|Z) \quad \text{(X - Y - Z)}$$

$$\geq I(X; Z). \quad \text{(information is non-negative)}$$

The Framework

The following lemma is the key tool in proving a lower bound on the mutual information.

Lemma 5.5. Let $n \in \mathbb{N}$, and let $p_1, \ldots, p_n$ be probability distributions over the set $[n]$ such that for all $i \in [n],

$$p_i(i) \geq \frac{1}{2} \quad \text{(iv)}$$

Let $U$ be a random variable distributed uniformly over $[n]$. Let $T$ be a random variable over $[n]$ that results from sampling an index $i$ according to $U$ and then sampling an element of $[n]$ according to $p_i$. Then

$$I(U; T) = \Omega(\log n).$$

Proof.

$$I(U; T) = \sum_{i=1}^n \sum_{t=1}^n p_U(i)p_i(t) \log \frac{p_i(t)}{p_T(t)}$$

$$= \sum_i p_U(i)p_i(i) \log \frac{p_i(i)}{p_T(i)} + \sum_i \sum_{t \neq i} p_U(i)p_i(t) \log \frac{p_i(t)}{p_T(t)}.$$
Consider the first sum (the “diagonal”):

$$
\sum_i p_U(i)p_i(i) \log \frac{p_i(i)}{p_T(i)} = \frac{1}{n} \sum_i p_i(i) \log \frac{p_i(i)}{p_T(i)} \\
\geq \frac{1}{n} \sum_i \frac{1}{2} \log \frac{1}{2} \\
\geq \frac{1}{n} \cdot \frac{n}{2} \log \frac{n}{2} \quad \text{(log-sum inequality)} \\
= \frac{1}{2} \log \frac{n}{2}.
$$

Finally, Lemma 5.3 implies that the second sum (the “off-diagonal”) is at least $-1$. 

We need to generalize the previous lemma as follows.

**Lemma 5.6.** Let $p_1, \ldots, p_n$ be probability distributions over $\mathcal{X}$. Let $S_1, \ldots, S_n \subset \mathcal{X}$ be pairwise disjoint events such that for all $i \in [n],$

$$
p_i(S_i) \geq \frac{1}{2} \quad \text{(v)}
$$

Let $U$ be a random variable distributed uniformly over $[n]$. Let $W$ be a random variable taking value in $\mathcal{X}$ that results from sampling an index $i$ according to $U$ and then sampling an element of $\mathcal{X}$ according to $p_i$. Then,

$$
I(U; W) = \Omega(\log n).
$$

**Proof.** Let $S_0 = \mathcal{X} \setminus (S_1 \cup \cdots \cup S_n)$, and let $T$ be the random variable taking values in $\mathbb{N}$ defined by $T(W) = i$ iff $W \in A_i$. Hence, $T$ satisfies the conditions of Lemma 5.5. Furthermore, we have that the random variables $U, W, T$ form the following Markov chain: $U \rightarrow W \rightarrow T$. We thus conclude that

$$
I(U; W) \geq I(U; T) = \Omega(\log n)
$$

where the inequality is according to the data processing inequality (Lemma 5.4) and the equality follows from Lemma 5.5.
5.3 Proof for General Case

We start with the analog of Lemma 5.2 from the warm up. Let $\Delta([2^n])$ be the set of all probability distribution over $[2^n]$. Let $Q \in \text{Mat}_{2^n \times 2^n}(\Delta([2^n]))$, i.e., $Q$ is a $2^n \times 2^n$ matrix where each cell contains a probability distribution.

**Lemma 5.7.** Assume that $Q$ is symmetric and that it has the property that for all $i, j$,
\[ \text{supp}(Q_{ij}) \subseteq [\min\{i, j\}, \max\{i, j\}]\]  
(ii)
Then, $Q$ contains a row with $n + 1$ distributions $p_1, \ldots, p_{n+1}$ such that there exist pairwise disjoint sets $S_1, \ldots, S_{n+1} \subset [2^n]$ so that for all $i \in [n + 1]$,
\[ p_i(S_i) \geq \frac{1}{2} \]  
(iii)
(and hence it also contains such a column).

**Proof.** Again, the proof by induction on $n$. The base case is easily verified. For the step, let
\[ Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix} \]
where $Q_1, Q_2, Q_3, Q_4 \in \text{Mat}_{2^n \times 2^n}(\Delta([2^n]))$.

The matrix $Q_1$ is symmetric, it satisfies property (ii), and that the supports of all the entries in $Q_1$ are contained in $[2^n]$. So by induction $Q_1$ contains a row $r$ with probability functions $p_1, \ldots, p_{n+1}$ and pairwise disjoint sets $A_1, \ldots, A_{n+1} \subset [2^n]$ that satisfy property (iii).

Similarly, one sees that $Q_4$ satisfies property (iii) as well. Namely, $Q_4$ contains a column $k$ with probabilities $q_1, \ldots, q_{n+1}$ and pairwise disjoint sets $B_1, \ldots, B_{n+1} \subset [2^n + 1, 2^{n+1}]$ with $q_i(B_i) \geq \frac{1}{2}$ for all $i$.

We now consider the probability distribution $Q_{rk}$. If $Q_{rk}([2^n]) \geq \frac{1}{2}$ then we define $q_{n+2} = Q_{rk}$ and $B_{n+2} = [2^n]$. Thus, column $k$ of $Q$ satisfies property (iii) with probabilities $q_1, \ldots, q_{n+2}$ and sets $B_1, \ldots, B_{n+2}$. Otherwise, we choose $p_{n+2} = Q_{rk}$ and $A_{n+2} = [2^n + 1, 2^{n+1}]$. In this case, the row of $Q$ corresponding to $r$ satisfies property (iii) with probabilities $p_1, \ldots, p_{n+2}$ and sets $A_1, \ldots, A_{n+2}$.

In the previous section we proved that property (iii) yields a lower bound on mutual information. We are prepared to prove the desired lower bound for probabilistic algorithms.
Proof of Theorem 5.1. Let \( A \) be a consistent and proper learning algorithm for \( T \). Let \( M \in \text{Mat}_{2^n \times 2^n}(\Delta([2^n])) \) be the matrix whose \((i, j)\) entry for \( i \neq j \) is the probability distribution \( A \) induces on \( T \) with input

\[
\left( (1, 0), \ldots, (1, 0), (\min \{i, j\}, 0), (\max \{i, j\}, 1) \right)_{m-2}
\]

and whose \((i, i)\) entry for all \( i \) is the degenerate distribution that assigns probability 1 to \( i \). The matrix \( M \) is symmetric, and because \( A \) is consistent and proper it follows that \( M \) satisfies property (ii). By Lemma 5.7 therefore \( M \) contains a row \( r \) with probabilities \( p_1, \ldots, p_{n+1} \) for which there are pairwise disjoint sets \( S_1, \ldots, S_{n+1} \subseteq [2^n] \) such that \( p_i(S_i) \geq \frac{1}{2} \) for all \( i \).

We assume without loss of generality that the probabilities \( p_1, \ldots, p_{n/2} \) are located on row \( r \) above the diagonal in cells \((r, k_1), \ldots, (r, k_{n/2})\); the symmetric case can be handled similarly.

We now construct the probability \( D \) over \( X \). The probability of 1 is \( 1 - \frac{1}{m-2} \) and rest of the mass is uniformly distributed on \( r, k_1, \ldots, k_{n/2} \). Consider the indicator random variable \( E \) of the event that \( S \) is of the form

\[
S = \left( (1, 0), \ldots, (1, 0), (r, 0), (k_i, 1) \right)_{m-2}
\]

for some \( k_i \). For every \( S \) in this event, denote by \( i_S \) the index so that \( k_{i_S} \) is in the last example of \( S \). Again, \( \Pr[E = 1] \geq \Omega(1/m^2) \). Finally, as in the warm up,

\[
I(S; A(S)) \geq I(S; A(S)|E) \\
\geq \Pr[E = 1] \cdot I(S; A(S)|E = 1) \\
\geq \Pr[E = 1] \cdot I(i_S; A(S)|E = 1) \\
\overset{(*)}{=} \Omega \left( \frac{\log n}{m^2} \right),
\]

where (*) is justified as follows: Given that \( E = 1 \), we know that \( i_S \) is uniformly distributed on \([n/2]\). Furthermore, \( A(S) \) is the result of sampling a hypothesis according to the distribution \( p_{i_S} \). The lower bound hence follows from Lemma 5.6.

\footnote{We assume for simplicity that 1 is not \( r \) or one of the \( k_i \)’s.}
6 A Generic Information Learner

Here we give a simple and generic information learner that provides sharp upper bounds for some basic concept classes. For example, we get an ERM that is a $O(\log \log N)$-bit information learner for the class of thresholds over $[N]$; this bound is indeed tight given the lower bound in Section 5.

**Definition (Generic information learner).** The generic algorithm $A_H$ for a hypothesis class $H$ acts as follows. Given a realizable sample $S$, the algorithm first finds the set of all hypotheses in $H$ that are consistent with $S$, and then it simply outputs a uniformly random hypothesis from that set.

This algorithm is well defined for finite $H$ and it is proper and consistent. Specifically, if $H$ has finite VC-dimension then $A_H$ PAC-learns the class with the standard sample complexity of VC classes (when there are no information constraints).

6.1 A Method for Upper Bounding Information

The following simple lemma provides a useful method for upper bounding mutual information. Let $H \subseteq \{0,1\}^X$ and $m \in \mathbb{N}$. Let $A : (X \times \{0,1\})^m \rightarrow H$ be an algorithm that takes a sample $S$ of $m$ examples and outputs a hypothesis $h = A(S) \in H$. Recall that $W_A(\cdot | S)$ denotes the distribution of $A(S)$ when the input sample is $S$, and that $P_D(\cdot) = \mathbb{E}_{S \sim D^m}[W_A(\cdot | S)]$ denotes the marginal distribution of the output of $A$ with respect to input distribution $D$.

**Lemma 6.1.** For every distribution $Q$ over $H$, we have

$$I(S; A(S)) = \mathbb{E}_{S \sim D^m} [KL(W_A(\cdot | S) \mid P_D)] \leq \max_{S \in \text{Supp}(D^m)} \text{KL}(W_A(\cdot | S) \mid Q).$$

**Proof.** Observe that for any distribution $Q$ over $H$,

$$I(S; A(S)) = \mathbb{E}_{S \sim D^m} [KL(W_A(\cdot | S) \mid Q)] - \text{KL}(P_D \mid Q)$$

$$\leq \mathbb{E}_{S \sim D^m} [KL(W_A(\cdot | S) \mid Q)]$$

$$\leq \max_{S \in \text{Supp}(D^m)} \text{KL}(W_A(\cdot | S) \mid Q)$$

(if the KL-divergence is infinite, the upper bound trivially follows).
6.2 Examples

We demonstrate the behavior of the generic algorithm in a couple of simple cases.

Thresholds

Consider the class of thresholds $\mathcal{T}$ over $\mathcal{X} = [N]$ and the generic algorithm $A_T$.

**Theorem 6.2.** For every sample size, the generic algorithm $A = A_T$ is a $(\log \log(N) + O(1))$-bit information learner.

**Proof.** Suppose the labelling function is $f_k(x) = 1_{x \geq k}$. Consider the following distribution $Q$ over $\mathcal{H}$:

$$Q(t) = \frac{c}{(1 + |k - t|) \log(N)},$$

where $c \geq 1/2$ is the normalizing constant. One can verify this lower bound on $c$ by noting that

$$\sum_{t \in [N]} \frac{1}{1 + |k - t|} \leq \int_{t=1}^{N+1} \frac{1}{1 + |k - t|} dt \leq 2 \log(N).$$

Now, by plugging this choice of $Q$ into Lemma 6.1 and noting that for any realizable sample $S$, the distribution $W_A(\cdot|S)$ is uniform over $f_t$ for $t \in \{x_1, x_1 + 1, \ldots, x_2\}$ for some $x_1 \leq k \leq x_2$ in $\mathcal{X}$, we can reach the desired bound:

$$\text{KL}(W_A(\cdot|S) || Q) = \log \log(N) + \sum_{x_1 \leq t \leq x_2} \frac{1}{x_2 - x_1 + 1} \log \frac{1 + |k - t|}{c(x_2 - x_1 + 1)} \leq \log \log(N) + 1,$$

since $\frac{1 + |k - t|}{c(x_2 - x_1 + 1)} \leq 2$.

Given our results in Section 5 we note that the above bound is indeed tight.
**Point Functions**

We can learn the class of point functions $\mathcal{P}\mathcal{F}$ on $\mathcal{X} = [N]$ with at most 2 bits of information. The learning algorithm is again the generic one.

**Theorem 6.3.** For every realizable distribution $\mathcal{D}$, and for every sample size $m \leq N/2$, the generic algorithm $\mathcal{A} = \mathcal{A}_{\mathcal{P}\mathcal{F}}$ has at most 2 bits of mutual information with respect to $\mathcal{D}$.

**Proof.** Suppose, without loss of generality, that the target concept is $f_1(x) = 1_{\{x=1\}}$. Pick $Q$ in the bound of Lemma 6.1 as follows:

$$Q(x) = \begin{cases} 
\frac{1}{2} & x = 1, \\
\frac{1}{2(N-1)} & x \neq 1.
\end{cases}$$

Let $S$ be a sample. If 1 appears in $S$ then

$$\text{KL}(W_{\mathcal{A}}(\cdot|S) \mid Q) = 1 \cdot \log \frac{1}{1/2} = 1.$$ 

If 1 does not appear in $S$ then $W_{\mathcal{A}}(\cdot|S)$ is uniform on a subset of the form $\{1_{x=i} : i \in \mathcal{X}'\}$ for some $\mathcal{X}' \subset \mathcal{X}$ that contains 1 of size $k \geq N - m \geq N/2$, so

$$\text{KL}(W_{\mathcal{A}}(\cdot|S) \mid Q) \leq \frac{1}{k} \log \frac{2}{k} + \frac{k-1}{k} \log \frac{2(N-1)}{k} \leq 2.$$ 

$\square$

The above result together with known properties of VC classes and [4, Corollary 1] imply a separation between the family of proper information learners and the family of pure differentially private proper learners.

**Corollary 6.4.** There is a proper 2-bit information learner for the class of point functions over $[N]$ with sample size $O\left(\frac{\log(1/\delta)}{\epsilon}\right)$. On the other hand, for any $\alpha > 0$, any $\alpha$-differentially private algorithm that properly learns point functions over $[N]$ requires $\Omega\left(\frac{\log N + \log(1/\delta)}{\alpha}\right)$ examples.
The Distribution-Dependent Setting

As was shown in Section 5, there are classes of VC dimension 1 (thresholds on a domain of size \( N \)) for which every proper consistent learner must leak at least \( \Omega(\log \log N) \) bits of information on some realizable distributions.

Here, we consider the distribution-dependent setting; that is, we assume the learner knows the marginal distribution \( D_X \) on inputs (but it does not know the target concept). We show that in this setting every VC class can be learned with relatively low information.

**Theorem 7.1.** Given the size \( m \) of the input sample, a distribution \( D_X \) over \( X \) and \( H \subset \{0,1\}^X \) with VC-dimension \( d \), there exists a consistent, proper, and deterministic learner with \( O(d \log(m+1)) \)-bits of information (for \( H \)-realizable samples).

Before proving the theorem, we discuss a somewhat surprising phenomenon. In a nutshell, the theorem says that for every distribution there is a deterministic algorithm with small entropy. It is tempting to “conclude” using von Neumann’s minimax theorem that this implies that there is a randomized algorithm that for every distribution has small information. This “conclusion” however is false, as the threshold example shows.

**Proof.** Let \( \epsilon_k = (1/(m+1))^k \) for \( k > 0 \). For each \( k \), pick an \( \epsilon_k \)-net \( N_k \) for \( H \) with respect to \( D \) of minimum size; that is, for every \( h \in H \) there is \( f \in N_k \) so that 
\[
\Pr_{x \sim D_X}(h(x) \neq f(x)) \leq \epsilon_k.
\]
A result of Haussler [17] states that the size of \( N_k \) is at most 
\[
(4e^2/\epsilon_k)^d \leq (4e^2m)^kd.
\]

The algorithm works as follows: given an input sample \( S \), the algorithm checks if \( N_1 \) contains a consistent hypothesis. If there is, it outputs it. Otherwise, it checks in \( N_2 \), and so forth. As we explain below, the probability that the algorithm stops is one (even when \( X \) is infinite). Denote by \( K \) the value of \( k \) in which the algorithm stops.

Bound the entropy of the output as follows:
\[
H(A(S)) \leq H(A(S)|K) + H(K)
\]
\[
\leq \sum_k \Pr[K = k] \cdot kd \log(4e^2(m+1)) - H(K).
\]

The labelling function \( f \in H \) is at distance of at most \( \epsilon_k \) from \( N_k \), so
\[
\Pr[K \leq k] \geq (1 - \epsilon_k)^m \geq 1 - \frac{1}{(m+1)^{k-1}}.
\]
which implies \( \Pr[K = k + 1] \leq \frac{1}{(m+1)^k} \leq \frac{1}{2^k} \) (this in particular implies that the algorithm terminates with probability one). Hence,

\[
\sum_k \Pr[K = k] \cdot kd \log(4e^2(m + 1)) \leq O(d \log(m + 1))
\]

and \( H(K) \leq O(1) \).

\[\square\]

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