Abstract. In vertex algebra theory, fusion rules are described as the dimension of the vector space of intertwining operators between three irreducible modules. We describe fusion rules in the category of weight modules for the Weyl vertex algebra. This way we confirm the conjecture on fusion rules based on the Verlinde algebra. We explicitly construct intertwining operators appearing in the formula for fusion rules. We present a result which relates irreducible weight modules for the Weyl vertex algebra to the irreducible modules for the affine Lie superalgebra $\hat{gl}(1|1)$.

1. Introduction

In the theory of vertex algebras and conformal field theory, determination of fusion rules is one of the most important problems. By a result by Y. Z. Huang [21] for a rational vertex algebra, fusion rules can be determined by using the Verlinde formula. However, although there are certain versions of Verlinde formula for a broad class of non-rational vertex algebras, so far there is no proof that fusion rules for such algebras can be determined by using the Verlinde formula. One important example is the singlet vertex algebra for $(1,p)$-models whose irreducible representations were classified in [2]. Verlinde formula for fusion rules was also presented by T. Creutzig and A. Milas in [12], but so far the proof was only given for the case $p = 2$ in [7]. We should also mention that the fusion rules and intertwining operators for some affine and superconformal vertex algebras were studied in [1], [4], [11] and [24].

In this paper we study the case of the Weyl vertex algebra, which we denote by $M$, also called the $\beta\gamma$ system in the physics literature. Its Verlinde type conjecture for fusion rules was presented by S. Wood and D. Ridout in [27]. Here, we present a short proof of Verlinde conjecture in this case. We prove the following fusion rules result:

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Theorem 1.1. Assume that $\lambda, \mu, \lambda + \mu \in \mathbb{C} \setminus \mathbb{Z}$. Then we have:

\begin{align}
(1.1) \quad & \rho_{\ell_1}(M) \times \rho_{\ell_2}(\widetilde{U}(\lambda)) = \rho_{\ell_1+\ell_2}(\widetilde{U}(\lambda)), \\
(1.2) \quad & \rho_{\ell_1}(\widetilde{U}(\lambda)) \times \rho_{\ell_2}(\widetilde{U}(\mu)) = \rho_{\ell_1+\ell_2}(U(\lambda + \mu)) + \rho_{\ell_1+\ell_2-1}(\widetilde{U}(\lambda + \mu)),
\end{align}

where $\widetilde{U}(\lambda)$ is an irreducible weight module, and $\rho_{\ell}, \ell \in \mathbb{Z}$, are the spectral flow automorphisms defined by (3.8).

The fusion rules (1.1) was proved in Proposition 3.1 and it is a direct consequence of the construction of H. Li [25]. The main contribution of our paper is vertex-algebraic proof of (1.2) which uses the theory of intertwining operators for vertex algebras and the fusion rules for the affine vertex superalgebra $V_1(gl(1|1))$.

We also prove a general irreducibility result which relates irreducible weight modules for the Weyl vertex algebra $M$ to irreducible weight modules for $V_1(gl(1|1))$ (see Theorem 5.1).

Theorem 1.2. Assume that $\mathcal{N}$ is an irreducible weight $M$–module. Then $\mathcal{N} \otimes F$ is a completely reducible $V_1(gl(1|1))$–module.

The construction of intertwining operators appearing in the fusion rules is based on two different embeddings of the Weyl vertex algebra $M$ into the lattice vertex algebra $\Pi(0)$. Then one $\Pi(0)$–intertwining operator gives two different $M$–intertwining operators. Therefore, both intertwining operators are realized as $\Pi(0)$–intertwining operators. Once we tensor the Weyl vertex algebra $M$ with the Clifford vertex algebra $F$, we can use the fusion rules for $V_1(gl(1|1))$ to calculate the fusion rules for $M$.

It is known that fusion rules can be determined by using fusion rules for the singlet vertex algebra (cf. [7], [12]). However, we believe that our methods, which use $V_1(gl(1|1))$, can be generalized to a wider class of vertex algebras. In our future work we plan to study the following related fusion rules problems:

- Connect fusion rules for higher rank Weyl vertex algebra with fusion rules for $V_1(gl(n|m))$.
- Extend fusion ring with weight modules having infinite-dimensional weight spaces (cf. Subsection 3.4) and possibly with irreducible Whittaker modules (cf. [8]).

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2. Fusion rules and intertwining operators

In this section we recall the definition of intertwining operators and fusion rules. More details can be found in [20], [19], [14], [10]. We also prove an important result on the action of certain automorphisms on intertwining operators. This result will enable us to produce new intertwining operators from the existing one.

Let $V$ be a conformal vertex algebra with the conformal vector $\omega$ and let $Y(\omega,z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$. We assume that the derivation in the vertex algebra $V$ is $D = L(-1)$. A $V$-module (cf. [26]) is a vector space $M$ endowed with a linear map $Y_M$ from $V$ to the space of $\text{End}(M)$-valued fields $a \mapsto Y_M(a,z) = \sum_{n \in \mathbb{Z}} a^M_{(n)} z^{-n-1}$ such that:

1. $Y_M(|0\rangle, z) = I_M$,
2. for $a, b \in V$,

$$z^{-1}_0 \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(a,z_1) Y_M(b,z_2) - z^{-1}_0 \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(b,z_2) Y_M(a,z_1)$$

$$= z^{-1}_2 \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(a,z_0) b, z_2).$$

Given three $V$-modules $M_1$, $M_2$, $M_3$, an intertwining operator of type $(M_3 \mid M_1 \mid M_2)$ (cf. [19], [20]) is a map $I : a \mapsto I(a,z) = \sum_{n \in \mathbb{Z}} a^I_{(n)} z^{-n-1}$ from $M_1$ to the space of $\text{Hom}(M_2, M_3)$-valued fields such that:

1. for $a \in V$, $b \in M_1$, $c \in M_2$, the following Jacobi identity holds:

$$z^{-1}_0 \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_{M_3}(a, z_1) I(b, z_2) c - z^{-1}_0 \delta \left( \frac{z_2 - z_1}{-z_0} \right) I(b, z_2) Y_{M_2}(a, z_1) c$$

$$= z^{-1}_2 \delta \left( \frac{z_1 - z_0}{z_2} \right) I(Y_{M_1}(a, z_0) b, z_2) c,$$

2. for every $a \in M_1$,

$$I(L(-1)a, z) = \frac{d}{dz} I(a, z).$$

We let $I(M_1 \mid M_2)$ denote the space of intertwining operators of type $(M_3 \mid M_1 \mid M_2)$, and set

$$N_{M_1,M_2}^{M_3} = \dim I(M_3 \mid M_1 \mid M_2).$$
When $N_{M_1, M_2}^{M_3}$ is finite, it is usually called a fusion coefficient.

Assume that in the category $K$ of $L(0)$-diagonalizable $V$-modules, irreducible modules $\{M_i \mid i \in I\}$, where $I$ is an index set, have the following properties

1. for every $i, j \in I$, $N_{M_i, M_j}^{M_k}$ is finite for any $k \in I$;
2. $N_{M_i, M_j}^{M_k} = 0$ for all but finitely many $k \in I$.

Then the algebra with basis $\{e_i \in I\}$ and product

$$e_i \cdot e_j = \sum_{k \in I} N_{M_i, M_j}^{M_k} e_k$$

is called the fusion algebra of $V, K$.

Let $K$ be a category of $V$-modules. Let $M_1, M_2$ be irreducible $V$-modules in $K$. Given an irreducible $V$-module $M_3$ in $K$, we will say that the fusion rule

$$(2.3) \quad M_1 \times M_2 = M_3$$

holds in $K$ if $N_{M_1, M_2}^{M_3} = 1$ and $N_{M_1, M_2}^{M_j} = 0$ for any other irreducible $V$-module $R$ in $K$ which is not isomorphic to $M_3$.

We say that an irreducible $V$-module $M_1$ is a simple current in $K$ if $M_1$ is in $K$ and, for every irreducible $V$-module $M_2$ in $K$, there is an irreducible $V$-module $M_3$ in $K$, such that the fusion rule $(2.3)$ holds in $K$ (see [14]).

Recall that for any automorphism $g$ of $V$, and any $V$-module $(M, Y_M(\cdot, z))$, we have a new $V$-module $M \circ g = M^g$, such that $M^g \cong M$ as a vector space and the vertex operator $Y_M^g$ is given by $Y_M^g(v, z) := Y_M(gv, z)$, for $v \in V$. Namely, the only axiom we have to check is the Jacobi identity, and we have:

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M^g(a, z_1) Y_M^g(b, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M^g(b, z_2) Y_M^g(a, z_1)$$

$$= z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(ga, z_1) Y_M(gb, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(gb, z_2) Y_M(ga, z_1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(\cdot, z_0) gb, z_2)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M^g(Y^g(\cdot, z_0) gb, z_2).$$

Therefore, $M^g$ is a $V$-module. The following proposition shows that automorphism $g$ also produces a new intertwining operator.

**Proposition 2.1.** Let $g$ be an automorphism of the vertex algebra $V$ satisfying the condition

$$(2.4) \quad \omega - g(\omega) \in \text{Im}(D).$$

Let $M_1, M_2, M_3$ be $V$-modules and $I(\cdot, z)$ an intertwining operator of type $(M_3^{M_3}_{M_1, M_2})$. Then we have an intertwining operator $I^g$ of type $(M_3^{M_3}_{M_1, M_2})$, such
that $I^g(b, z_1) = I(b, z_1)$, for all $b \in M_1$. Moreover,

$$N^M_{M_1, M_2} = N^{M_1^g, M_2^g}_{M_1, M_2}.$$ 

**Proof.** We have:

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y^g_{M_3}(a, z_1) I^g(b, z_2) c - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) I^g(b, z_2) Y^g_{M_2}(a, z_1) c$$

$$= z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_{M_3}(ga, z_1) I(b, z_2) c - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) I(b, z_2) Y_{M_2}(ga, z_1) c$$

$$= z_0^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) I(Y_{M_3}(ga, z_0)b, z_2) c$$

$$= z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_2} \right) I(Y^g_{M_1}(a, z_0)b, z_2) c.$$

Set

$$Y^g(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)^g z^{-n-1}.$$ 

Since $g(\omega) = \omega + Dv$ for certain $v \in V$, we have that

$$g(\omega)_0 = \omega_0 + (Dv)_0 = \omega_0 = L(-1).$$

This implies that $L(-1)^g = L(-1)$. Hence for $a \in M_1$ we have

$$I^g(L(-1)^g a, z) = I^g(L(-1)a, z) = I(L(-1)a, z) = \frac{d}{dz} I(a, z) = \frac{d}{dz} I^g(a, z).$$

Therefore, $I^g$ has the $L(-1)$-derivation property and $I^g$ is an intertwining operator of type $(M_1^g, M_2^g).$

\[\square\]

**Remark 2.1.** If $V$ is a vertex operator algebra and $g$ an automorphism of $V$, then $g(\omega) = \omega$ and the condition (2.4) is automatically satisfied. In our applications, $g$ will only be a vertex algebra automorphism such that $g(\omega) \neq \omega$, yet the condition (2.4) will be satisfied.

3. The Weyl vertex algebra

3.1. The Weyl vertex algebra. The Weyl algebra $\hat{A}$ is an associative algebra with generators

$$a(n), a^*(n) \quad (n \in \mathbb{Z})$$

and relations

$$(3.5) [a(n), a^*(m)] = \delta_{n+m, 0}, \quad [a(n), a(m)] = [a^*(m), a^*(n)] = 0 \quad (n, m \in \mathbb{Z}).$$

Let $M$ denote the simple Weyl module generated by the cyclic vector $1$ such that

$$a(n)1 = a^*(n+1)1 = 0 \quad (n \geq 0).$$
As a vector space,
\[ M \cong \mathbb{C}[a(-n), a^*(-m) \mid n > 0, m \geq 0]. \]

There is a unique vertex algebra \((M, Y, 1)\) (cf. [17], [23], [16]) where the vertex operator map is
\[ Y : M \to \text{End}(M)[[z, z^{-1}]] \]
such that
\[
Y(a(-1)1, z) = a(z), \quad Y(a^*(0)1, z) = a^*(z), \quad a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n)z^{-n}.
\]

In particular we have:
\[
Y(a(-1)a^*(0)1, z) = a(z)^+a^*(z) + a^*(z)a(z)^-,
\]
where
\[
a(z)^+ = \sum_{n \leq -1} a(n)z^{n-1}, \quad a(z)^- = \sum_{n \geq 0} a(n)z^{-n-1}.
\]

Let \(\beta := a(-1)a^*(0)1\). Set \(\beta(z) = Y(\beta, z) = \sum_{n \in \mathbb{Z}} \beta(n)z^{-n-2}\). Then \(\beta\) is a Heisenberg vector in \(M\) of level \(-1\). This means that the components of the field \(\beta(z)\) satisfy the commutation relations
\[
[\beta(n), \beta(m)] = -n\delta_{n+m,0} \quad (n, m \in \mathbb{Z}).
\]
Also, we have the following formula
\[
[\beta(n), a(m)] = -a(n + m), \quad [\beta(n), a^*(m)] = a^*(n + m).
\]

The vertex algebra \(M\) admits a family of Virasoro vectors
\[
\omega_\mu = (1 - \mu)a(-1)a^*(-1)1 - \mu a(-2)a^*(0)1 \quad (\mu \in \mathbb{C})
\]
of central charge \(c_\mu = 2(6\mu(\mu - 1) + 1)\). Let
\[
L^\mu(z) = Y(\omega, z) = \sum_{n \in \mathbb{Z}} L^\mu(n)z^{-n-2}.
\]
This means that the components of the field \(L(z)\) satisfy the relations
\[
[L^\mu(n), L^\mu(m)] = (n - m)L^\mu(n + m) + \frac{n^3 - n}{12} \delta_{n+m,0}c_\mu.
\]
For \(\mu = 0\), we write \(\omega = \omega_0\), \(L^\mu(n) = L(n)\). Then \(c_\mu = 2\). Clearly
\[
\omega_\mu = \omega - \mu \beta(-2)1.
\]
Since \((\beta(-2)1)_0 = (D\beta)_0\), we have that
\[
L^\mu(-1) = L(-1), \quad \text{for every } \mu \in \mathbb{C}.
\]
For \(n, m \in \mathbb{Z}\) we have
\[
[L(n), a(m)] = -ma(n + m), \quad [L(n), a^*(m)] = -(m + n)a^*(n + m).
\]
In particular,
\[ [L(0), a(m)] = -ma(m), [L(0), a^*(m)] = -ma^*(m). \]

**Lemma 3.1.** Assume that \( W = \bigoplus_{\ell \in \mathbb{Z}_{\geq 0}} W(\ell) \) is a \( \mathbb{Z}_{\geq 0} \)-graded \( M \)-module with respect to \( L(0) \). Then
\[ L(0) \equiv 0 \text{ on } W(0). \]

**Proof.** Since \( W \) is \( \mathbb{Z}_{\geq 0} \)-graded with top component \( W(0) \), the operators \( a(n), a^*(n) \) must act trivially on \( W(0) \) for all \( n \in \mathbb{Z}_{>0} \). Since \( L(z) =: \partial a^*(z)a(z) := \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} \), we have
\[ L(0) = \sum_{n=1}^{\infty} n(a^*(-n)a(n) - a(-n)a^*(n)) \equiv 0 \text{ (on } W(0)). \]

The Lemma holds. \( \Box \)

**Remark 3.1.** One can show that Zhu’s algebra \( A(M) \cong A_1 \) and that \([\omega] = 0 \) in \( A(M) \). This can give a second proof of Lemma 3.1.

**Definition 3.1.** A module \( W \) for the Weyl algebra \( \hat{A} \) is called restricted if the following condition holds:
- For every \( w \in W \), there is \( N \in \mathbb{Z}_{\geq 0} \) such that
  \[ a(n)w = a^*(n)w = 0, \quad \text{for } n \geq N. \]

3.2. **Automorphisms of the Weyl algebra.** Denote by \( \text{Aut}(\hat{A}) \) the group of automorphisms of the Weyl algebra \( \hat{A} \). For any \( f \in \text{Aut}(\hat{A}) \), and \( \hat{A} \)-module \( N \), one can construct \( \hat{A} \)-module \( f(N) \) as follows:
\[ f(N) := N \quad \text{as vector space, and action is } x.v = f(x)v \quad (v \in N). \]

For \( f, g \in \text{Aut}(\hat{A}) \), we have
\[ (f \circ g)(N) = g(f(N)). \]

For every \( s \in \mathbb{Z} \) the Weyl algebra \( \hat{A} \) admits the following automorphism
\[ \rho_s(a(n)) = a(n+s), \quad \rho_s(a^*(n)) = a^*(n-s). \]

Then \( \rho_s \) is an automorphism of \( \hat{A} \) which can be lifted to an automorphism of the vertex algebra \( M \). Automorphism \( \rho_s \) is called spectral flow automorphism.

Assume that \( U \) is any restricted module for \( \hat{A} \). Then \( \rho_s(U) \) is also a restricted module for \( \hat{A} \) and \( \rho_s(U) \) is a module for the vertex algebra \( M \).

Let \( \mathcal{K} \), be the category of weight \( M \)-modules such that the operators \( \beta(n), n \geq 1 \), act locally nilpotent on each module \( N \) in \( \mathcal{K} \). Applying the automorphism \( \rho_s \) to the vertex algebra \( M \), we get \( M \)-module \( \rho_s(M) \), which
is a simple current in the category $\mathcal{K}$. The proof is essentially given by H. Li in [25, Theorem 2.15] in a slightly different setting.

**Proposition 3.1.** [25] Assume that $N$ is an irreducible weight $M$–module in the category $\mathcal{K}$. Then the following fusion rules hold:

$$\rho_{s_1}(M) \times \rho_{s_2}(N) = \rho_{s_1+s_2}(N) \quad (s_1, s_2 \in \mathbb{Z}).$$

**Proof.** First we notice that if $N$ is an irreducible $M$–module in $\mathcal{K}$, we have the following fusion rules:

$$M \times N = N.$$  \hfill (3.10)

Using [25], one can prove that $\rho_a(M)$ is constructed from $M$ as:

$$(\rho_a(M), Y_a(\cdot, z)) := (M, Y(\Delta(-s\beta, z) \cdot, z)),$$  \hfill (3.11)

where

$$\Delta(v, z) := z^{v_0} \exp \left( \sum_{n=1}^{\infty} \frac{v_n}{-n}(z)^{-n} \right).$$

Assume that $N_i$, $i = 1, 2, 3$ are irreducible modules in $\mathcal{K}$. By [25, Proposition 2.4] from an intertwining operator $I(\cdot, z)$ of type $(N_3 N_1 N_2)$, one can construct intertwining operator $I_{s_2}(\cdot, z)$ of type $(N_1 \rho_{s_2}(N_2))$, where

$$I_{s_2}(v, z) := I(\Delta(-s_2\beta, z)v, z) \quad (v \in N_1).$$

Now, the fusion rules (3.9) follows easily from the above construction using (3.10). \hfill \square

Consider the following automorphism of the Weyl vertex algebra

$$g : \quad M \rightarrow M$$

$$a \mapsto -a^*, \quad a^* \mapsto a$$

Assume that $U$ is any $M$–module. Then $U^g = U \circ g$ is generated by the following fields

$$a_g(z) = -a^*(z), \quad a_g^*(z) = a(z).$$

As an $\hat{A}$–module, $U^g$ is obtained from $U$ by applying the following automorphism $g$:

$$a(n) \mapsto -a^*(n + 1), \quad a^*(n) \mapsto a(n - 1).$$

This implies that

$$g = \rho_{-1} \circ \sigma = \sigma \circ \rho_1$$

where $\sigma$ is the automorphism of $\hat{A}$ determined by

$$a(n) \mapsto -a^*(n), \quad a^*(n) \mapsto a(n).$$
The automorphism \( g \) is then a vertex algebra automorphism of order 4. Denote by \( \sigma_0 \) the restriction of \( \sigma \) on \( A_1 \). Using (3.11)-(3.13) we get the following result:

**Lemma 3.2.** Assume that \( W \) is an irreducible \( M \)-module. Then

\[
W^g \cong \rho_1(\sigma(W)).
\]

**Proof.** We have that as an \( \hat{A} \)-module:

\[
W^g = (\rho_{-1} \circ \sigma)(W).
\]

Since \( \rho_{-1} \circ \sigma = \sigma \circ \rho_1 \), by applying (3.7) we get:

\[
W^g = (\sigma \circ \rho_1)(W) = \rho_1(\sigma(W)).
\]

The proof follows. \( \square \)

### 3.3. Weight modules for the Weyl vertex algebra.

**Definition 3.2.** A module \( W \) for the Weyl vertex algebra \( M \) is called **weight** if the operators \( \beta(0) \) and \( L(0) \) act semisimply on \( W \).

Clearly, vertex algebra \( M \) is a weight module. We will now construct a family of weight modules.

- Recall that the first Weyl algebra \( A_1 \) is generated by \( x, \partial_x \) with the commutation relation

\[
[\partial_x, x] = 1.
\]

- For every \( \lambda \in \mathbb{C} \),

\[
U(\lambda) := x^{\lambda} \mathbb{C}[x, x^{-1}]
\]

has the structure of an \( A_1 \)-module.

- \( U(\lambda) \) is irreducible if and only if \( \lambda \in \mathbb{C} \setminus \mathbb{Z} \).

- Note that \( a(0), a^*(0) \) generate a subalgebra of the Weyl algebra, which is isomorphic to the first Weyl algebra \( A_1 \). Therefore \( U(\lambda) \) can be treated as an \( A_1 \)-module by letting \( a(0) = \partial_x, a^*(0) = x \).

- By applying the automorphism \( \sigma_0 \) on \( U(\lambda) \) we get

\[
\sigma_0(U(\lambda)) \cong U(-\lambda).
\]

Indeed, let \( z_{-\mu-1} = \frac{x^{\mu}}{\Gamma(\mu+1)} \), where \( \mu \in \mathbb{C} \setminus \mathbb{Z} \). Then

\[
\sigma_0(a(0)).z_{-\mu} = -\frac{x^{\mu}}{\Gamma(\mu)} = -\mu \frac{x^{\mu}}{\Gamma(\mu + 1)} = -\mu z_{-\mu-1}.
\]

\[
\sigma_0(a^*(0)).z_{-\mu} = (\mu - 1)\frac{x^{\mu-2}}{\Gamma(\mu)} = \frac{x^{\mu-2}}{\Gamma(\mu - 1)} = z_{-\mu+1}.
\]
• Define the following subalgebras of $\hat{A}$:
  \[ \hat{A}_{\geq 0} = \mathbb{C}[a(n), a^*(m) \mid n, m \in \mathbb{Z}_{\geq 0}], \]
  \[ \hat{A}_{< 0} = \mathbb{C}[a(-n), a^*(-n) \mid n \in \mathbb{Z}_{\geq 1}]. \]

• The $A_1$–module structure on $U(\lambda)$ can be extended to a structure of $\hat{A}_{\geq 0}$–module by defining
  \[ a(n)|_{U(\lambda)} = a^*(n)|_{U(\lambda)} = 0 \quad (n \geq 1). \]

• Then we have the induced module for the Weyl algebra:
  \[ \widehat{U}(\lambda) = \hat{A} \otimes \hat{A}_{\geq 0} \ U(\lambda) \]
  which is isomorphic to
  \[ \mathbb{C}[a(-n), a^*(-n) \mid n \geq 1] \otimes U(\lambda) \]
  as a vector space.

**Proposition 3.2.** For every $\lambda \in \mathbb{C} \backslash \mathbb{Z}$, $\widehat{U}(\lambda)$ is an irreducible weight module for the Weyl vertex algebra $M$.

**Proof.** The proof follows from Lemma 3.1 and the fact that $\widehat{U}(\lambda)$ is a $\mathbb{Z}_{\geq 0}$–graded $M$–module whose top component is an irreducible module for $A_1$. \qed

Applying Lemma 3.2 we get:

**Corollary 3.1.** For every $\lambda \in \mathbb{C} \backslash \mathbb{Z}$ and $s \in \mathbb{Z}$ we have
  \[ \widehat{U}(\lambda)^g \cong \rho_1(\widehat{U}(\lambda)), \quad \left(\rho_{-s+1}(\widehat{U}(\lambda))\right)^g \cong \rho_s(\widehat{U}(\lambda)). \]

3.4. **More general weight modules.** A classification of irreducible weight modules for the Weyl algebra $\hat{A}$ is given in [18]. Let us describe here a family of weight modules having infinite–dimensional weight spaces.

Take $\lambda, \mu \in \mathbb{C} \backslash \mathbb{Z}$. Let
  \[ U(\lambda, \mu) = x_1^\lambda x_2^\mu \mathbb{C}[x_1, x_2, x_1^{-1}, x_2^{-1}]. \]

Then $U(\lambda, \mu)$ is an irreducible module for the second Weyl algebra $A_2$ generated by $\partial_1, \partial_2, x_1, x_2$. Note that $A_2$ can be realized as a subalgebra of $\hat{A}$ generated by $\partial_2 = a(1), \partial_1 = a(0), x_2 = a^*(-1), x_1 = a^*(0)$. Then we have the irreducible $\hat{A}$–module $\widehat{U}(\lambda, \mu)$ as follows. Let $B$ be the subalgebra of $\hat{A}$ generated by $a(i), a^*(j), i \geq 0, j \geq -1$. Consider $U(\lambda, \mu)$ as a $B$–module such that $a(n), a^*(m)$ act trivially for $n \geq 2, m \geq 1$. Then by $[18],$

  \[ \widehat{U}(\lambda, \mu) = \hat{A} \otimes_B U(\lambda, \mu) \]

is an irreducible $\hat{A}$–module. As a vector space:

  \[ \widehat{U}(\lambda, \mu) \cong \mathbb{C}[a(-n-1), a^*(-m-2) \mid n, m \in \mathbb{Z}_{\geq 0}] \otimes U(\lambda, \mu) \]
  \[ \cong a^*(0)^j a^*(-1)^m \mathbb{C}[a(-n-1), a^*(-m) \mid n, m \in \mathbb{Z}_{\geq 0}]. \]
ON FUSION RULES FOR THE WEYL VERTEX ALGEBRA

Since \(\widetilde{U}(\lambda, \mu)\) is a restricted \(\hat{A}\)-module we get:

**Proposition 3.3.** \(\widetilde{U}(\lambda, \mu)\) is an irreducible weight module for the Weyl vertex algebra \(M\).

One can see that the weight spaces of the module \(\widetilde{U}(\lambda, \mu)\) are all infinite-dimensional with respect to \((\beta(0), L(0))\). In particular, vectors

\[a(-1)^m a^*(0)\lambda + 2m a^*(-1)\mu - m, \quad m \in \mathbb{Z}_{\geq 0},\]

are linearly independent and they belong to the same weight space.

**Remark 3.2.** Note that modules \(\widetilde{U}(\lambda, \mu)\) are not in the category \(\mathcal{K}\), and therefore Proposition 3.1 can not be applied in this case.

4. The vertex algebra \(\Pi(0)\) and the construction of intertwining operators

In this section we present a bosonic realization of the weight modules for the Weyl vertex algebra. We also construct intertwining operators using this bosonic realization.

4.1. The vertex algebra \(\Pi(0)\) and its modules. Let \(L\) be the lattice

\[L = \mathbb{Z}\alpha + \mathbb{Z}\beta, \quad \langle \alpha, \alpha \rangle = -\langle \beta, \beta \rangle = 1, \quad \langle \alpha, \beta \rangle = 0,\]

and \(V_L = M_{\alpha, \beta}(1) \otimes \mathbb{C}[L]\) the associated lattice vertex superalgebra, where \(M_{\alpha, \beta}(1)\) is the Heisenberg vertex algebra generated by fields \(\alpha(z)\) and \(\beta(z)\) and \(\mathbb{C}[L]\) is the group algebra of \(L\). We have its vertex subalgebra

\[\Pi(0) = M_{\alpha, \beta}(1) \otimes \mathbb{C}[\mathbb{Z}(\alpha + \beta)] \subset V_L.\]

There is an injective vertex algebra homomorphism \(f : M \to \Pi(0)\) such that

\[f(a) = e^{\alpha + \beta}, \quad f(a^*) = -\alpha(-1)e^{-\alpha - \beta}.\]

We identify \(a, a^*\) with their image in \(\Pi(0)\). We have (cf. [16])

\[M \cong \text{Ker}_{\Pi(0)} e_0^\alpha.\]

The Virasoro vector \(\omega\) is mapped to

\[\omega = a(-1)a^*(-1)1 = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \beta(-2))1.\]

Note also that

\[g(\omega) = -a(-2)a^* = \omega_\mu = \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 - \beta(-2))1. \quad (\mu = 1).\]

Since

\[(4.14) \quad \Pi(0) = \mathbb{C}[\mathbb{Z}(\alpha + \beta)] \otimes M_{\alpha, \beta}(1)\]
is an irreducible $\Pi(0)$–module.

We have
\[ \Pi(0) e^{r\beta + (n+\lambda)(\alpha + \beta)} = \frac{1 - r}{2} (r + 2(n + \lambda)) e^{r\beta + (n+\lambda)(\alpha + \beta)}, \]
and for $\mu = 1$
\[ \Pi(0)^{\mu} e^{r\beta + (n+\lambda)(\alpha + \beta)} = \frac{1}{2} r (r + 2(n + \lambda)) e^{r\beta + (n+\lambda)(\alpha + \beta)}. \]

**Proposition 4.1.** Assume that $\ell \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$. Then as $M$–modules:

1. $\Pi(\ell)(\lambda) \cong \rho_{-\ell+1}(\check{U}(\lambda))$,
2. $\Pi(\ell)(\lambda)^g \cong \rho(\ell,\lambda)(\check{U}(\lambda))$.

**Proof.** Assume first that $r = 1$. Then $\Pi((\lambda))$ is a $\mathbb{Z}_{\geq 0}$–graded $M$–module whose lowest component is
\[ \Pi((\lambda))(0) \cong \mathbb{C}[\beta + (\mathbb{Z} - \lambda)(\alpha + \beta)] \cong U(\lambda). \]
Now Lemma 3.1 and Corollary 3.1 imply that $\Pi((\lambda))$ is an irreducible $M$–module isomorphic to $\check{U}(\lambda)$. Modules $\Pi_{\ell}(\lambda)$ can be obtained from $\Pi_{\ell}(\lambda)$ by applying the spectral flow automorphism $\rho_{-\ell+1} := e^{(\ell-1)\beta}$. Using Corollary 3.1 we get
\[
\Pi_{\ell}(\lambda)^g = \left( \rho_{-\ell+1}(\check{U}(\lambda)) \right)^g = \rho_{\ell - 1}(\check{U}(\lambda)) = \rho_{\ell}(\check{U}(\lambda)).
\]
The proof follows. \qed

**4.2. Construction of intertwining operators.**

**Proposition 4.2.** For every $\ell_1, \ell_2 \in \mathbb{Z}$ and $\lambda, \mu \in \mathbb{C}$ there exist non-trivial intertwining operators of types
\[
(\rho_{\ell_1+\ell_2}(U(\mu + \lambda))) = \begin{pmatrix}
\rho(\ell_1 + \ell_2)(U(\lambda + \mu)) & \rho(\ell_2)(U(\mu)) \\
\rho(\ell_1)(U(\lambda)) & \rho(\ell_2)(U(\mu))
\end{pmatrix},
\]
in the category of weight $M$–modules.

**Proof.** By using explicit bosonic realization, as in [15], one can construct a unique nontrivial intertwining operator $I(\cdot, z)$ of type
\[
\begin{pmatrix}
\Pi_{s_1+s_2}(\lambda_1 + \lambda_2) \\
\Pi_{s_1}(\lambda_1) & \Pi_{s_2}(\lambda_2)
\end{pmatrix}
\]
in the category of $\Pi(0)$–modules such that
\[ e^{s_1\beta + \lambda_1(\alpha + \beta)} e^{s_2\beta + \lambda_2(\alpha + \beta)} = e^{s_1 + s_2\beta + (\lambda_1 + \lambda_2)(\alpha + \beta)} \quad (\nu \in \mathbb{C}). \]
By restriction, this gives a non-trivial intertwining operator in the category of weight $M$–modules. Taking the embedding $f : M \to \Pi(0)$ and applying Corollary 4.1 we conclude the operator (4.16) gives the intertwining operator of type

\[
(\rho_{-s_1-s_2+1}(U(-\lambda_1 - \lambda_2)), \rho_{-s_1+1}(U(-\lambda_1)), \rho_{-s_2+1}(U(-\lambda_2))),
\]

which for $\ell_1 = -s_1 + 1$, $\ell_2 = -s_2 + 1$, $\lambda = -\lambda_1$, $\mu = -\lambda_2$ gives the first intertwining operator. By using action of the automorphism $g$ and Corollary 4.1 we get the following intertwining operator

\[
(\rho_{s_1+s_2}(U(\lambda_1 + \lambda_2)), \rho_{s_1}(U(\lambda_1)), \rho_{s_2}(U(\lambda_2))),
\]

which for $\ell_1 = s_1$, $\ell_2 = s_2$, $\lambda = \lambda_1$, $\mu = \lambda_2$ gives the second intertwining operator.

**Remark 4.1.** Intertwining operators in Proposition 4.2 are realized on irreducible $\Pi(0)$–modules. This result can be read as

\[
\Pi_{\ell_1}(\lambda) \times \Pi_{\ell_2}(\mu) \supseteq \Pi_{\ell_1+\ell_2-1}(\lambda + \mu) + \Pi_{\ell_1+\ell_2}(\lambda + \mu).
\]

In the category of $M$–modules, we have non-trivial intertwining operators

\[(4.17) \begin{pmatrix} \Pi_{\ell_1+\ell_2-1}(\lambda + \mu) \\ \Pi_{\ell_1}(\lambda) \end{pmatrix} \frac{1}{\Pi_{\ell_2}(\mu)},
\]

which are not $\Pi(0)$–intertwining operators.

**Remark 4.2.** Note that $(M,Y,1)$ is a conformal vertex algebra with the conformal vector

\[
\omega = a(-1)a^*(-1)1.
\]

Note that the intertwining operators (4.15) satisfy the $L(-1)$–derivative property. Intertwining operators (4.16) satisfy this property by using the lattice realization as before, and intertwining operators (4.17) satisfy it by Proposition 2.1, using the facts that $g(\omega) = \omega_1$ and $L^\mu(-1) = L(-1)$, for $\mu = 1$. Moreover, using relation (3.6) we see that the $L^\mu(-1)$–derivation property holds for every $\mu \in \mathbb{C}$, for all intertwining operators constructed above.

5. The vertex algebra $V_1(gl(1|1))$ and its modules

5.1. **On the vertex algebra** $V_1(gl(1|1))$. We now recall some results on the representation theory of $gl(1|1)$ and $gl(1|1)$. The terminology follows [13] Section 5].
Let \( g = gl(1|1) \) be the complex Lie superalgebra generated by two even elements \( E, N \) and two odd elements \( \Psi^\pm \) with the following (super)commutation relations:
\[
[\Psi^+, \Psi^-] = E, \ [E, \Psi^\pm] = [E, N] = 0, \ [N, \Psi^\pm] = \pm \Psi^\pm.
\]
Other (super)commutators are trivial. Let \((\cdot, \cdot)\) be the invariant supersymmetric invariant bilinear form such that
\[
(\Psi^+, \Psi^-) = (\Psi^- , \Psi^+) = 1, \ (N, E) = (E, N) = 1.
\]
All other products are zero.

Let \( \hat{g} = gl(1|1) = g \otimes \mathbb{C}[t, t^{-1}] + CK \) be the associated affine Lie superalgebra with the commutation relations
\[
[x(n), y(m)] = [x, y](n + m) + n\delta_{n+m,0}K,
\]
where \( K \) is central and for \( x \in g \) we set \( x(n) = x \otimes t^n \). Let \( V_k(g) \) be the associated simple affine vertex algebra of level \( k \).

Let \( V_{r,s} \) be the Verma module for the Lie superalgebra \( g \) generated by the vector \( v_{r,s} \) such that \( Nv_{r,s} = rv_{r,s} \), \( Ev_{r,s} = sv_{r,s} \). This module is a 2–dimensional module and it is irreducible iff \( s \neq 0 \). If \( s = 0 \), \( V_{r,s} \) has a 1–dimensional irreducible quotient, which we denote by \( A_r \).

We will need the following tensor product decompositions:
\[
\begin{align*}
A_{r_1} \otimes A_{r_2} &= A_{r_1 + r_2}, & A_{r_1} \otimes V_{r_2,s_2} &= V_{r_1+r_2, s_2}, \\
V_{r_1,s_1} \otimes V_{r_2,s_2} &= V_{r_1+r_2, s_1+s_2} \oplus V_{r_1+r_2-1, s_1+s_2} \ (s_1 + s_2 \neq 0), \\
V_{r_1,s_1} \otimes V_{r_2,-s_1} &= \mathcal{P}_{r_1+r_2},
\end{align*}
\]
where \( \mathcal{P}_r \) is the 4–dimensional indecomposable module which appears in the following extension
\[
0 \to V_{r,0} \to \mathcal{P}_r \to V_{r-1,0} \to 0.
\]

Let \( \hat{V}_{r,s} \) denote the Verma module of level 1 induced from the irreducible \( gl(1|1) \)–module \( V_{r,s} \). If \( s \notin \mathbb{Z} \), then \( \hat{V}_{r,s} \) is an irreducible \( V_1(gl(1|1)) \)–module. If \( s \in \mathbb{Z} \), \( \hat{V}_{r,s} \) is reducible and its structure is described in [13, Section 5.3].

By using the tensor product decomposition (5.19) we get the following result on fusion rules for \( V_1(g) \)–modules:

**Proposition 5.1.** Let \( r_1, r_2, s_1, s_2 \in \mathbb{C}, \ s_1, s_2, s_1 + s_2 \notin \mathbb{Z} \). Assume that there is a non-trivial intertwining operator
\[
\left( \begin{array}{c}
\hat{V}_{r_3,s_3} \\
\hat{V}_{r_1,s_1} \\
\hat{V}_{r_2,s_2}
\end{array} \right)
\]
in the category of \( V_1(g) \)–modules. Then \( s_3 = s_1 + s_2 \) and \( r_3 = r_1 + r_2 \), or \( r_3 = r_1 + r_2 - 1 \).
Recall that the Clifford algebra is generated by $\psi (r), \psi^* (s)$, where $r, s \in \mathbb{Z} + \frac{1}{2}$, with relations

\begin{align}
[\psi (r), \psi^* (s)] &= \delta_{r+s,0}, \\
[\psi (r), \psi (s)] &= [\psi^* (r), \psi^* (s)] = 0, \quad \text{for all } r, s.
\end{align}

(5.21)

Note that the commutators (5.21) are actually anticommutators because both $\psi (r)$ and $\psi^* (s)$ are odd for every $r$ and $s$.

The Clifford vertex algebra $F$ is generated by the fields

$$
\psi (z) = \sum_{n \in \mathbb{Z}} \psi (n + \frac{1}{2}) z^{-n-1},
$$

$$
\psi^* (z) = \sum_{n \in \mathbb{Z}} \psi^* (n + \frac{1}{2}) z^{-n-1}.
$$

As a vector space,

$$
F \cong \bigwedge \left( \left\{ \psi (r), \psi^* (s) \mid r, s < 0 \right\} \right)
$$

Let $V_{\mathbb{Z} \gamma}$ be the lattice vertex algebra associated to the lattice $\mathbb{Z} \gamma \cong \mathbb{Z}$, $\langle \gamma, \gamma \rangle = 1$. By using the boson–fermion correspondence, we have that $F \cong V_{\mathbb{Z} \gamma}$, and we can identify the generators of the Clifford vertex algebra as follows (cf. [22]):

$$
\psi := e^\gamma, \psi^* = e^{-\gamma}
$$

Now we define the following vertex superalgebra:

$$
SI(0) = \Pi(0) \otimes F \subset V_L,
$$

and its irreducible modules

$$
SI_r (\lambda) = \Pi_r (\lambda) \otimes F = \Pi(0).e^{r \beta + \lambda (\alpha + \beta)}.
$$

Let $U = M \otimes F$. Using [22, Section 5.8] we define the vectors

$$
\Psi^+ := e^{\alpha + \beta + \gamma} = a (-1) \psi, \quad \Psi^- := -\alpha (-1) e^{-\alpha - \beta - \gamma} = a^* (0) \psi^*,
$$

$$
E := \gamma + \beta, \quad N := \frac{1}{2} (\gamma - \beta).
$$

Then the components of the fields

$$
X(z) = Y(X, z) = \sum_{n \in \mathbb{Z}} X(n) z^{-n-1}, \quad X \in \{ \Psi^+, \Psi^-, E, N \}
$$

satisfy the commutation relations for the affine Lie algebra $\hat{\mathfrak{g}} = gl(1|1)$, so that $M \otimes F$ is a $\hat{\mathfrak{g}}$–module of level 1. (See also [5] for a realization of $gl(1|1)$ at the critical level).
The Sugawara conformal vector is
\[
\omega_{c=0} = \frac{1}{2}(N(-1)E(-1) + E(-1)N(-1) - \Psi^+(1)\Psi^-(1) + \Psi^-(1)\Psi^+(1) + E(-1)^2)1
\]
(5.22)
\[
= \frac{1}{2}(\beta(-1) + \gamma(-1))(\gamma(-1) - \beta(-1)) + \alpha(-1)(\alpha(-1) + \beta(-1) + \gamma(-1))
- \frac{1}{2}((\alpha(-1) + \beta(-1) + \gamma(-1))^2 + (\alpha(-2) + \beta(-2) + \gamma(-2)))
+ \frac{1}{2}(\beta(-1) + \gamma(-1))^2 + \frac{1}{2}(\beta(-2) + \gamma(-2))
= \frac{1}{2}(\alpha(-1)^2 - \alpha(-2) - \beta(-1)^2 + \gamma(-1)^2)
= \omega_{c=-1} + \frac{1}{2}\gamma(-1)^2 \quad (\omega_{c=-1} = \omega_{1/2})
\]

5.2. **Construction of irreducible \( V_1(\mathfrak{g}) \)-modules from irreducible \( M \)-modules.** Let \( V_1(\mathfrak{g}) \) be the simple affine vertex algebra of level 1 associated to \( \mathfrak{g} \).

We have the following gradation:
\[
\mathcal{U} = \bigoplus \mathcal{U}^\ell, \quad E(0)|_{\mathcal{U}^\ell} = \ell \text{ Id}.
\]
We will present an alternative proof of the following result:

**Proposition 5.2.** [22] We have:
\[
V_1(\mathfrak{g}) \cong \mathcal{U}^0 = \text{Ker}_{M \otimes F}E(0).
\]

**Proof.** Let \( \tilde{V}_1(\mathfrak{g}) \) be the vertex subalgebra of \( \mathcal{U}^0 \) generated by \( \mathfrak{g} \). Assume that \( \tilde{V}_1(\mathfrak{g}) \neq \mathcal{U}^0 \). Then there is a subsingular vector \( v_{r,s} \notin \mathbb{C}1 \) for \( \tilde{\mathfrak{g}} \) of weight \((r,s)\) such that for \( n > 0 \):
\[
\Psi^+(0)v_{r,s} \in \tilde{V}_1(\mathfrak{g}), \quad X(n)v_{r,s} \in \tilde{V}_1(\mathfrak{g}), \quad X \in \{E,N,\Psi^\pm\}
\]
\[
E(0)v_{r,s} = sv_{r,s}, \quad N(0)v_{r,s} = rv_{r,s}.
\]
In other words, \( v_{r,s} \) is a singular vector in the quotient \( \tilde{\mathcal{U}}^0 = \tilde{\mathcal{U}}^0/\tilde{V}_1(\mathfrak{g}) \). Since \( E(0) \) acts trivially on \( \mathcal{U}^0 \), we conclude that \( s = 0 \). Recalling the expression for the Virasoro conformal vector \( (5.22) \), we get that in \( \tilde{\mathcal{U}}^0 \):
\[
L_{c=0}(0)v_{r,0} = (\omega_{c=0})_1v_{r,0} = \frac{1}{2}(2N(0)E(0) - E(0) + E(0)^2)v_{r,0} = 0.
\]
This implies that \( v_{r,0} \) has the conformal weight 0 and hence must be proportional to \( 1 \). A contradiction. Therefore, \( \mathcal{U}^0 = \tilde{V}_1(\mathfrak{g}) \). Since \( \mathcal{U}^0 \) is a simple vertex algebra, we have that \( \tilde{V}_1(\mathfrak{g}) = V_1(\mathfrak{g}) \).

We can extend this irreducibility result to a wide class of weight modules. The proof is similar to the one given in [3, Theorem 6.2].

**Theorem 5.1.** Assume that \( \mathcal{N} \) is an irreducible weight module for the Weyl vertex algebra \( M \), such that \( \beta(0) \) acts semisimply on \( \mathcal{N} \):
\[
\mathcal{N} = \bigoplus_{s \in \mathbb{Z}+\Delta} \mathcal{N}^s, \quad \beta(0)|\mathcal{N}^s \equiv s \text{Id} \quad (\Delta \in \mathbb{C}).
\]
Then $\mathcal{N} \otimes F$ is a completely reducible $V_1(\mathfrak{g})$–module:

$$\mathcal{N} \otimes F = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(N), \quad \mathcal{L}_s(N) = \{v \in \mathcal{N} \otimes F \mid E(0) v = (s + \Delta) v\},$$

and each $\mathcal{L}_s(N)$ is irreducible $V_1(\mathfrak{g})$–module.

**Proof.** Clearly $\mathcal{L}_s(N)$ is a $\mathcal{U}^0(= V_1(\mathfrak{g}))$–module. It suffices to prove that each vector $w \in \mathcal{L}_s(N)$ is cyclic. Since $\mathcal{N} \otimes F$ is a simple $\mathcal{U}$–module, we have that $\mathcal{U} \cdot w = \mathcal{N} \otimes F$. On the other hand, $\mathcal{N} \otimes F$ is $\mathbb{Z}$–graded $\mathcal{U}$–module so that

$$\mathcal{U}^r \cdot \mathcal{L}_s(N) \subset \mathcal{L}_{r+s}(N), \quad (r, s \in \mathbb{Z}).$$

This implies that $\mathcal{U}^r \cdot w \subset \mathcal{L}_{r+s}(N)$ for each $r \in \mathbb{Z}$. Therefore $\mathcal{U}^0 \cdot w = \mathcal{L}_r(N)$. The proof follows.

As a consequence we get a family of irreducible $V_1(\mathfrak{g})$–modules:

**Corollary 5.1.** Assume that $\lambda, \mu \in \mathbb{C} \setminus \mathbb{Z}$. Then for each $s \in \mathbb{Z}$ we have:

1. $\mathcal{L}_s(\mathcal{U}(\lambda))$ is an irreducible $V_1(\mathfrak{g})$–module,
2. $\mathcal{L}_s(\mathcal{U}(\lambda, \mu))$ is an irreducible $V_1(\mathfrak{g})$–module.

We will prove in the next section that $\mathcal{L}_s(\mathcal{U}(\lambda))$ are irreducible highest weight modules. But one can see that $\mathcal{L}_s(\mathcal{U}(\lambda, \mu))$ have infinite-dimensional weight spaces. A detailed analysis of the structure of these modules will appear in our forthcoming papers (cf. [9]).

6. **THE CALCULATION OF FUSION RULES**

In this section we will finish the calculation of fusion rules for the Weyl vertex algebra $M$.

We will first identify certain irreducible highest weight $\hat{\mathfrak{h}}$–modules.

**Lemma 6.1.** Assume that $r, n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, $\lambda + n \notin \mathbb{Z}$. Then $e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)}$ is a singular vector in $\Pi_r(\lambda)$ and

$$U(\hat{\mathfrak{h}})e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} \cong \tilde{V}_{r + \frac{1}{2}(\lambda + n), -\lambda - n},$$

$$L(0)e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = \frac{1}{2}(1 - 2r)(n + \lambda)e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)}.$$ 

**Proof.** By using standard calculation in lattice vertex algebras we get for $m \geq 0$

$$\Psi^+(m)e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = e^{\alpha + \beta + \gamma} e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = 0,$$

$$\Psi^-(m + 1)e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = (-(\alpha(-1)e^{-\alpha - \beta - \gamma})_m^{-1} e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = 0,$$

$$E(m)e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = -(\lambda + n)\delta_{m,0} e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)},$$

$$N(m)e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)} = \frac{1}{2}(2r + \lambda + n)\delta_{m,0} e^{r(\beta + \gamma) + (\lambda + n)(\alpha + \beta)}.$$
Therefore \( e^{r(\beta+\gamma)+\lambda(n+\beta)(\alpha+\beta)} \) is a highest weight vector for \( \hat{\mathfrak{g}} \) with highest weight \( (r + \frac{1}{2}(\lambda + n), -\lambda - n) \) with respect to \((N(0), E(0))\). This implies that \( U(\hat{\mathfrak{g}}) \) is isomorphic to a certain quotient of the Verma module \( \mathcal{V}_{r + \frac{1}{2}(\lambda + n), -\lambda - n} \). But since, \( \lambda + n \notin \mathbb{Z} \), \( \mathcal{V}_{r + \frac{1}{2}(\lambda + n), -\lambda - n} \) is irreducible and therefore \((6.23)\) holds. Relation \((6.24)\) follows by applying the expression \( r \in \mathbb{Z} \). Applying Theorem 5.1 we see that each \( \Pi_r(\lambda) \) is irreducible \( M \otimes F^{-}\text{module} \):

\[
\Pi_r(\lambda) \cong \bigoplus_{s \in \mathbb{Z}} U(\hat{\mathfrak{gl}}(1|1)).
\]

\[
(6.25)
\]

**Proof.** The assertion (1) follows from the fact that \( \Pi_r(\lambda) \) is an irreducible \( M^{-}\text{module} \) (cf. Proposition 5.1). Note next that the operator \( E(0) = \beta(0) + \gamma(0) \) acts semi–simply on \( M \otimes F \):

\[
M \otimes F = \bigoplus_{s \in \mathbb{Z}} (M \otimes F)^{(s)}, \quad (M \otimes F)^{(s)} = \{v \in M \otimes F | E(0)v = -sv\}.
\]

In particular, \( (M \otimes F)^{(0)} \cong V_1(\mathfrak{g}) \) (cf. [22] and Proposition 5.2). But \( E(0) \) also defines the following \( \mathbb{Z}^{-}\text{gradation on } \Pi_r(\lambda) \):

\[
\Pi_r(\lambda) = \bigoplus_{s \in \mathbb{Z}} \Pi_r(\lambda)^{(s)}, \quad \Pi_r(\lambda)^{(s)} = \{v \in \Pi_r(\lambda) | E(0)v = (-s - \lambda)v\}.
\]

Applying Theorem 5.1 we see that each \( \Pi_r(\lambda)^{(s)} \) is an irreducible \( (M \otimes F)^{(0)} \cong V_1(\mathfrak{g})^{-}\text{module} \). Using Lemma 6.1 we see that it is an irreducible highest weight \( \hat{\mathfrak{g}} \)-module with highest weight vector \( e^{r(\beta+\gamma)+\lambda(n+\beta)(\alpha+\beta)} \). The proof follows. \( \square \)

**Theorem 6.2.** Assume that \( \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \mathbb{C} \setminus \mathbb{Z}, r_1, r_2, r_3 \in \mathbb{Z} \). Assume that there is a non-trivial intertwining operator of type

\[
\begin{pmatrix}
\Pi_{r_3}(\lambda_3) \\
\Pi_{r_1}(\lambda_1) & \Pi_{r_2}(\lambda_2)
\end{pmatrix}
\]

in the category of \( M \otimes F^{-}\text{modules} \). Then \( \lambda_3 = \lambda_1 + \lambda_2 \) and \( r_3 = r_1 + r_2 \), or \( r_3 = r_1 + r_2 - 1 \).
Proof. Assume that $I$ is an non-trivial intertwining operator of type

$$
\begin{pmatrix}
S\Pi_r^3(\lambda_3) \\
S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)
\end{pmatrix},
$$

Since $S\Pi_r(\lambda)$ are simple $M \otimes F$–modules, we have that for every $s_1, s_2 \in \mathbb{Z}$:

$I(e^{r_1(\beta + \gamma) + (\lambda_1 + s_1)(\alpha + \beta)}, z)e^{r_2(\beta + \gamma) + (\lambda_2 + s_2)(\alpha + \beta)} \neq 0$.

Here we use the well-known result which states that for every non-trivial intertwining operator $I$ between three irreducible modules we have that $I(v, z)w \neq 0$ (cf. [15, Proposition 11.9]). Note that $e^{r_i(\beta + \gamma) + \lambda_i(\alpha + \beta)}$ is a singular vector for $\hat{g}$ which generates $V_1(g)$–module $\hat{V}_{r_i + \frac{1}{2}\lambda_i, -\lambda_i}$, $i = 1, 2$.

The restriction of $I(\cdot, z)$ on

$$
\hat{V}_{r_1 + \frac{1}{2}\lambda_1, -\lambda_1} \otimes \hat{V}_{r_2 + \frac{1}{2}\lambda_2, -\lambda_2}
$$

gives a non-trivial intertwining operator

$$
\begin{pmatrix}
S\Pi_r^3(\lambda_3) \\
S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)
\end{pmatrix}
$$

in the category of $V_1(g)$–modules. Proposition 5.1 implies that then

$\hat{V}_{r_1 + r_2 + \frac{1}{2}(\lambda_1 + \lambda_2), -\lambda_1 - \lambda_2}$ or $\hat{V}_{r_1 + r_2 + \frac{1}{2}(\lambda_1 + \lambda_2) - 1, -\lambda_1 - \lambda_2}$

has to appear in the decomposition of $S\Pi_r^3(\lambda_3)$ as a $V_1(g)$–module. Using decomposition (6.25) we get that there is $s \in \mathbb{Z}$ such that

$\lambda + s = \lambda_1 + \lambda_2$, $r_3 = r_1 + r_2$,

and of (6.26) is

$r_1 + r_2 + \frac{1}{2}(\lambda_1 + \lambda_2) = r_3 + \frac{1}{2}(\lambda + s)$,

and of (6.27) is

$r_1 + r_2 - 1 + \frac{1}{2}(\lambda_1 + \lambda_2) = r_3 + \frac{1}{2}(\lambda + s)$.

Solution of (6.26) is

$\lambda + s = \lambda_1 + \lambda_2$, $r_3 = r_1 + r_2$,

and of (6.27) is

$\lambda + s = \lambda_1 + \lambda_2$, $r_3 = r_1 + r_2 - 1$.

Since $S\Pi_r(\lambda) \cong S\Pi_r(\lambda + s)$ for $s \in \mathbb{Z}$, we can take $s = 0$. Thus, $\lambda_3 = \lambda_1 + \lambda_2$ and $r_3 = r_1 + r_2$ or $r_3 = r_1 + r_2 - 1$. The claim holds.

By using the following natural isomorphism of the spaces of intertwining operators (cf. [5, Section 2]):

$I_{M \otimes F}(\begin{pmatrix} S\Pi_r^3(\lambda_3) \\
S\Pi_{r_1}(\lambda_1) \quad S\Pi_{r_2}(\lambda_2)\end{pmatrix}) \cong I_{M}(\begin{pmatrix} \Pi_{r_3}(\lambda_3) \\
\Pi_{r_1}(\lambda_1) \quad \Pi_{r_2}(\lambda_2)\end{pmatrix})$,

Theorem 6.2 implies the fusion rules result in the category of modules for the Weyl vertex algebra $M$ (see also [27, Corollary 6.7], for a derivation of the same fusion rules using Verlinde formula).
Corollary 6.1. Assume that \( \lambda_1, \lambda_2, \lambda_1 + \lambda_2 \in \mathbb{C} \setminus \mathbb{Z} \), \( r_1, r_2, r_3 \in \mathbb{Z} \). There exists a non-trivial intertwining operator of type

\[
\left( \begin{array}{c}
\Pi_{r_3}(\lambda_3) \\
\Pi_{r_1}(\lambda_1) \\
\Pi_{r_2}(\lambda_2)
\end{array} \right)
\]

in the category of \( M \)-modules if and only if \( \lambda_3 = \lambda_1 + \lambda_2 \) and \( r_3 = r_1 + r_2 \) or \( r_3 = r_1 + r_2 - 1 \).

The fusion rules in the category of weight \( M \)-modules are given by

\[
\Pi_{r_1}(\lambda_1) \times \Pi_{r_2}(\lambda_2) = \Pi_{r_1+r_2}(\lambda_1 + \lambda_2) + \Pi_{r_1+r_2-1}(\lambda_1 + \lambda_2).
\]

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