Classification of magnons in Rotated Ferromagnetic Heisenberg model and their competing responses in transverse fields

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Competing orders is a general concept to describe various quantum phases and transitions in various materials. One efficient way to investigate competing orders is to first classify different class of excitations in a given quantum phase, then study their competing responses under various external probes. This strategy may not only lead to deep understanding of the quantum phase itself, but also its deep connections to various other quantum phases nearby. We implement this approach by studying the Rotated Ferromagnetic Heisenberg model (RFHM) in two different transverse fields $h_x$ and $h_z$ which can be intuitively visualized as studying spin-orbit couplings (SOC) effects in 2d Ising or anisotropic XY model in a transverse field. At a special SOC class, it was known that the RFHM at a zero fieldowns an exact ground state called Y-x state. It supports non only the commensurate C-C$_0$ and C-C$_x$ magnons, but also the in-commensurate C-IC magnons. These magnons are non-relativistic, not contained in the exact ground state, so need to be thermally excited. Their dramatic response under the longitudinal $h_z$ field was recently worked out by the authors. Here we find they respond very differently under the two transverse fields. Any $h_z$ ($h_x$) changes the collinear Y-x state to a canted co-planar YX-x (YZ-x) state which suffers quantum fluctuations. The C-C$_0$, C-C$_x$ and C-IC magnons sneak into the quantum ground state, become relativistic and play leading roles even at $T = 0$. We map out the boundaries among the C-C$_0$, C-C$_x$ and C-IC magnons, especially the detailed evolution of the C-IC magnons inside the canted phases. As $h_x$ ($h_z$) increases further, the C-C$_0$ magnons always win the competition and emerge as the seeds to drive a transition from the YX-x (YZ-x) to the X-FM ( Z-FM ) which is shown to be in the 3d Ising universality class. We evaluate their contributions to magnetization, specific heat, uniform and staggered longitudinal susceptibilities spin correlation functions which can be measured by various established experimental techniques. Some analogies with quantum fluctuations generated multiple vortices and multiple landscapes in quantum spin glass are mentioned. The implications to cold atom systems and materials with SOC are briefly discussed.

I. INTRODUCTION

A fundamental problem in any branch of physics is on the nature of the ground state. In strongly correlated electron systems, competing orders is a general concept to describe various quantum phases and transitions in various materials. In this work, we focus on competing excitations in a given quantum phase which could lead to a natural explanation of some competing orders. It is known a given quantum phase can still support different kinds of excitations with their own characteristics. These different classes of excitations are generated by quantum fluctuations inherent in the quantum ground state, so are intrinsic objects embedded in the ground state itself. Under various external probes, these excitations compete to emerge to drive the instability of the system into various other quantum phases through different universal class of quantum phase transitions. So classifying different classes of excitations of a given phase and investigating their behaviors under various external probes could lead to deep understandings not only on the nature of the ground state itself, but also its deep connections to various other quantum phases.

A quantum phase is characterized by its symmetry breaking and excitation spectrum [1,2]. For quantum spin or bosonic systems [3,4], gapless excitations indicate long-range correlations encoded in the quantum phase. External probes could open a positive gap to the excitation or induce a ”negative” gap which indicates a quantum phase transition to another phase. While gapped excitations [5,6] indicate short-ranged fluctuations encoded in the phase. The external probes such as magnetic fields, pressures, electric fields, etc may drive these gapped excitations near to a QCP, close their gaps and lead to their condensations into a new phase through a quantum phase transition. For fermionic systems [12–17], the quantum phase supports both fermionic excitations and collective bosonic excitations. The two sectors may compete to lead to various other quantum phases under various external probes. Due to the absence of any symmetry breaking, a topological phase ( such as quantum Hall state, spin liquids ) [8,9,11] is characterized by its topological orders and associated fractionalized excitations. The gap closings of these fractionalized excita-
tions could lead to another topological phase through a Topological phase transition. In this work, we only focus on quantum phases without topological orders and with only different kinds of bosonic excitations.

So far, the classifications of the quantum ground states in terms of symmetry breaking and possible resulting gapless Goldstone modes are well studied [1–5]. However, much less attention has been paid to classify gapped excitations in a quantum phase. In a previous work [7], the authors studied the Rotated Ferromagnetic Heisenberg model (RFHM) which is a new class of quantum spin model to describe quantum magnetisms in some cold atom systems or materials with strong spin-orbit coupling (SOC) [8–10]. For a specific SOC class, we identify a new spin-orbital entangled commensurate ground state: the Y-x state. It supports 3 kinds of magnons: commensurate magnons such as \( C - C_0, C - C_\pi \) and also a new gapped elementary excitation: in-commensurate magnon ( C-IC ) with its two gap minima continuously tuned by the SOC strength. They are gapped bosonic excitations taking non-relativistic dispersion with anisotropic effective mass \( m_x, m_y \). However, the Y-x ground state is an exact quantum ground state with no quantum fluctuations. So the \( C - C_0, C - C_\pi, C - IC \) magnons in the RFHM are extrinsic, not embedded in the ground state due to the absence of quantum fluctuations. They need to be excited by thermal fluctuations. Their parameters such as the minimum positions \( (0, \pm k_0^y) \), gap \( \Delta \), masses \( m_x, m_y \) can only be measured by various characteristics of the transverse structure factor at a finite \( T \): it is a Gaussian shape, peaked at \( (0, \pm k_0^y) \) with an exponentially suppressed amplitude \( e^{-\Delta/T} \), with a temperature dependent width \( \sigma_x = \sqrt{m_x(\beta)^T} \).

The existence of the C-IC above a commensurate phase is the most striking feature of the RFHM. They indicate the short-ranged in-commensurate order embedded in a long-range ordered commensurate ground state. They are the seeds driving possible transitions from commensurate to another commensurate phase with different spin-orbital structure or to an In-commensurate phase. An important question to ask is how to drag out these C-IC magnons, closing their gap and drive into new quantum phases through the condensation of these magnons? In a recent work [11], the authors showed that applying a uniform longitudinal Zeeman field \( h_y \) could do the job very well: the \( C - C_0, C - C_\pi, C - IC \) magnons compete to emerge under its effects to drive quantum phase transitions. It turns out that the C-IC always compete to emerge under its effects to drive quantum fluctuations generated, sneak into the YX-x state at two dual roles in dragging out the C-IC magnons, therefore drive into completely different phases and phase transitions than the longitudinal field \( h_y \) studied in [11]. Following [7, 11], in this work, we also focus along the solvable line \( (\alpha = \pi/2, \beta) \) of the RFHM in a transverse field Eqs[1]-[10] and Eqs[21]. Away from \( \alpha = \pi/2 \) will be briefly mentioned in the conclusion section and be presented in a separate publication. The two models can be considered as incorporating possible dramatic effects of SOC on well studied 2d Ising, anisotropic (or isotropic) quantum XY model in a transverse field. Note that the \( h_y \) field keeps the hidden \( U(1) \) symmetry \( \sum_i (-1)^x S_i^y \) of the Hamiltonian at the zero field, but \( h_x, h_z \) breaks it. This fact alone may lead to dramatic different competition among the magnons when they are subject to the longitudinal \( h_y \) or the two transverse fields \( h_x \) and \( h_z \). Indeed, in the longitudinal field \( h_y \), there is a Mirror transformation [11] relating \( (\beta, h_y) \) to \( (\pi/2 - \beta, h_y) \). So \( \beta = \pi/4 \) enjoys the Mirror symmetry. However, because \( h_x \) and \( h_z \) explicitly breaks the \( U(1) \) symmetry, so the mirror transformation does not work anymore in \( h_x \) and \( h_z \) case. However, in the \( h_x \) case, we will still able to find a generalized mirror transformation to characterize systematically the competitions among the magnons. Unfortunately, there is even no such generalized mirror transformation in the \( h_z \) case, so the competitions are mere intricate in the \( h_z \) field.

In the \( h_x \) field, any infinitesimal \( h_x \) will change the Y-x state into a canted YX-x state. In sharp contrast to the Y-x state which is an exact ground state free of quantum fluctuations. The YX-x state suffers quantum fluctuations. So at \( T = 0 \), these magnons are quantum fluctuations generated, sneak into the YX-x state and become important components embedded inside the quantum ground state. They stand for quantum fluctuations with intrinsic wavelength and frequency, so can be detected by spin structure factor even at \( T = 0 \). We also evaluate their contributions to magnetization, specific heat, uniform and staggered longitudinal susceptibilities at a finite temperature. Using the generalized mirror transformation, we map out the boundaries of the commensurate magnons C-C0, C-C\pi, and the in-commensurate magnons C-IC inside the YX-x canted phase. As \( h_x \) increases, the C-C0 magnons emerge and drives the quantum phase transition at a critical field \( h_{c_2}(\beta) \) from the YX-x phase to the X-FM phase. By identifying a suitable order parameter, performing spin wave expansion and symmetry analysis, we find it is in 3d Ising the universality class. Due to the enlarged \( U(1) \) symmetry, the transition at the Abelian \( \beta = \pi/2 \) point is driven by the simultaneous condensations of the C-C0 and C-C\pi magnons and is in the universality class of 3d XY model.

In the \( h_z \) field, by applying the hidden \( U(1) \) symmetry operator at the zero field case, we show that the \( h_z \) case can be mapped to RFHM in a staggered \( h_x \) field along the \( x \) direction. We also work out the corresponding phase diagram in this case. We show that any infinitesimal \( h_z \) will change the Y-x state into a canted YZ-x state,
then a phase transition into the Z-FM state at a critical field $h_c(x)(\beta)$ which is shown to be also in the universality class of 3d Ising model. We find the transition from the YZ-x canted phase to the Z-FM is always driven by the condensations of the C-C magnons. Unfortunately, the generalized mirror transformation used in the $h_x$ case does not work in the $h_z$ case anymore, this fact makes the landscapes of the C-IC magnons much more complicated in the $h_z$ case than the $h_x$ case. Even so, we are still able to map out the competing boundaries and detailed structures of the C-C$_0$, C-C$_\pi$ and C-IC magnons inside the YZ-x canted phase. Due to the enlarged U(1) symmetry, the transition at the two Abelian points $\beta = 0$ and $\beta = \pi/2$ point is driven by the condensation of $C - C_0$ and the simultaneous condensations of the C-C$_0$ and C-C$_\pi$ magnons respectively and is in the universality class of 3d XY model. In principle, all the thermodynamic quantities such as the magnetization, specific heat, uniform and staggered susceptibilities in the YZ-x canted phase to the Z-FM is always driven by the condensation of $C$-C magnons respectively and is in the universality class of the Hamiltonian.

In view of recent remarkable experimental realization of 2d Rashba or Dresselhaus SOC or any of their linear combinations in Fermi gas or spinor BEC \cite{13, 14}. The two models Eqn.1 and Eqn.29 can be realized in near future experiments. The results achieved in this work can be detected by various techniques such as specific heat \cite{23, 24}, In situ measurement \cite{27} and light or atom Bragg spectroscopy \cite{28, 24} respectively. They may also shed some lights to study magnetic orderings in some strongly correlated SOC materials \cite{6, 11} with $h_x, h_z$ play the roles of crystal fields.

II. TRANSVERSE FIELD $h_x$

The Rotated Ferromagnetic Heisenberg model (RFHM) \cite{7} in a transverse field along the $S_x$ direction is described by:

$$\mathcal{H} = -J \sum_i [S_i R_x(2\alpha) S_{i+x} + S_i R_y(2\beta) S_{i+y}] - H_x \sum_i S_i^x$$

where $J > 0$ is the ferromagnetic interaction and the sum is over a unit cell $i$ in a square lattice, the $R_x(2\alpha)$, $R_y(2\beta)$ are two SO(3) rotation matrices around $\hat{x}$, $\hat{y}$ spin axis by angle $2\alpha$, $2\beta$ putting on the two bonds $\hat{x}$, $\hat{y}$ respectively. Following the previous works \cite{7, 11}, we also focus along the solvable line ($\alpha = \pi/2, \beta$). At the zero field case $H_x = 0$, the ground state is the $Y - x$ state shown in the horizontal axis in Fig.1.

As shown in \cite{7}, at $H_x = 0$, the Hamiltonian has the Time reversal ($T$) symmetry, translational symmetry and 3 spin-orbital coupled $Z_2$ symmetries $P_x, P_y, P_z$, especially a spin-orbital coupled $U(1)$ symmetry. However, any $H_x$ will break all these symmetries except the $P_x$ symmetry: $S^x \rightarrow -S^x, S_y \rightarrow -S_y, S^z \rightarrow -S^z$ and the translational symmetry. It also keeps the combined $TP_y$ and $TP_z$ symmetries.

A. X-FM state and excitations in the strong field

To map out the phases of Eqn.1, it is instructive to start from the high field limit $H_x \gg J$. In this limit, the system is in X-FM phase with spin fully (classically) polarized to $S_x$ direction Fig.1. Obviously, the X-FM keeps all the symmetry of the Hamiltonian.

Under the global spin rotation $(S_x^+, S_y^+, S_z^+) \rightarrow (S_x^-, S_y^-, S_z^-)$, Hamiltonian becomes

$$\mathcal{H} = -J \sum_i [S_i R_z(\pi) S_{i+x} + S_i R_y(2\beta) S_{i+y}] - H_x \sum_i S_i^x$$

Introducing the Holstein-Primakoff (HP) bosons \cite{7, 11}, $S^+ = \sqrt{2S - a^\dagger a}$, $S^- = a^\dagger \sqrt{2S - a^\dagger a}$, $S^z = S - a^\dagger a$ to the linear spin wave (LSW) order at $1/S$, we map the Hamiltonian Eqn.2 to:

$$\mathcal{H} = E_0 + 2JS \sum_k [(h_x + \cos k_x + \cos^2 \beta (2 - \cos k_y)) a_k^\dagger a_k$$

$$+ \sin^2 \beta \cos k_y (a_k a_{-k} + a_k^\dagger a_{-k}^\dagger)/2]$$

where the classical ground state energy $E_0 = -2JS^2 \cos^2 \beta - H_x NS$ and we have introduced the dimensionless field $h_x = H_x/(2JS)$. Now the Hamiltonian can be diagonalized by a Bogoliubov transformation

$$\mathcal{H} = E_0 + 4JS \sum_k \omega_k a_k^\dagger a_k$$

where the ground-state energy at the order of $1/S$ is $E_0' = E_0 - 2JS \sum_k \omega_k$ and the energy spectrum is:

$$\omega_k = \sqrt{[h_x + \cos k_x + \cos^2 \beta (2 - \cos k_y)]^2 - \sin^4 \beta \cos^2 k_y}$$

where, for $0 < \beta < \pi/2$, one can identify that there is a unique minimum located at $k^0 = (k_x, k_y) = (\pi, 0)$ with the energy gap:

$$\Delta_x = \omega_{k=k^0} = \sqrt{h_x(h_x - 1 + \cos 2\beta)}$$

The gap vanishing condition leads to the critical field $h_c$:

$$h_{cx}(\beta) = 1 - \cos 2\beta = 2 \sin^2 \beta$$

which is shown in Fig.1. The gap vanishing at $k^0 = (k_x, k_y) = (\pi, 0)$ indicate a quantum phase transition into a spin-orbital correlated state with orbital order $(\pi, 0)$. It was known that at $h_x = 0$, the ground state $Y - x$ state also has the $(\pi, 0)$ orbital order. That indicates that there is only one phase transition and the state below $h_{cx}$ could be just the $YX - x$ state with a canted angle. As to be shown in the next subsection, we show that it is indeed
the YX-x cant state with the orbital order (π,0). So near the QPT, Δx ≈ (h_x − h_{cx})^{1/2}.

From Eqns. 5 we find the excitation spectrum around the minimum takes the relativistic form

\[ \omega_q = \sqrt{\Delta^2_\pi + v_x^2 q_x^2 + v_y^2 q_y^2}, \quad k = k^0 + q \]  

(8)

where

\[ v_x^2 = (2h_x - 1 + \cos 2\beta)/2 \]

\[ v_y^2 = [h_x + \cos 2\beta(h_x - 1 + \cos 2\beta)]/2 \]  

(9)

At h = h_{cx}, the critical velocities are \( v_x = v_y = \sin^2 \beta \). As long as \( \beta > 0 \), we obtain a non-zero critical velocity, which indicates a relativistic critical behavior with the dynamic exponent \( z = 1 \).

At the two Abelian points \( \beta = 0, \pi/2 \) the system has SU(2) symmetry in the rotated basis \( \tilde{S}_i = R(\hat{\beta}, \pi n_1) S_i \) and \( \tilde{S} \tilde{U}(2) \) with \( \tilde{S}_i R(\hat{\beta}, \pi n_2) \).

respectively. So Eqns. 11 can be mapped to a FM Heisenberg model in \( -h_x \sum_i \tilde{S}_i^x \) and \( -h_x \sum_i (-1)^x \tilde{S}_i^z \) (see Eqn. 10) respectively. So at \( \beta = 0 \), any \( h_x \) will pick up the \( X - FM \) phase as the exact ground state. At \( \beta = \pi/2 \), taking the result from 11, any \( h_x \) will lead to a spin-flop transition resulting into a U(1) symmetry breaking cant phase with one Goldstone mode φ. Then there is another transition to the X-FM at a finite \( h_x = 2 \). These results at the two Abelian points are consistent with the general result Eqn. 17 and shown in Fig. 11.

Similarly, at \( \beta = \pi/2, h_x = 0 \), transferring back to the original basis, the Hamiltonian Eqn. 11 has the SU(2) symmetry generated by \( \sum_i S_i^x, \sum_i (-1)^x S_i^y, \sum_i (-1)^{x+y} S_i^z \). At any \( h_x > 0 \), only \( \sum_i S_i^x \) remains as a conserved quantity. Obviously, the X-FM state keeps all symmetries of the Hamiltonian. Acting the conserved quantity \( e^{i\phi} \sum_i S_i^x \) on the minimum \( (\pi,0) \), changes nothing. So we conclude that at \( \beta = 0, h_x > 0 \), the system only have one minima located at \( (\pi,0) \) as shown in Fig. 11.

B. YX-x canted state below \( h_{cx} \).

1. Classical YX-x canted phase at \( h < h_{cx} \).

When \( \beta = \pi/2 \) in the \( \tilde{S} \tilde{U}(2) \) basis \( \tilde{S}_i = R_x(i\pi R_y(i\pi) S_i \), the Hamiltonian Eqn. 11 takes the form:

\[ \mathcal{H} = -J \sum_{ij} S_i \cdot S_j - H_x \sum_i (-1)^x \tilde{S}_i \]  

(10)

When \( 0 < H_x < H_{xc} \), the classical state takes the form:

\[ \hat{S}_i = S \left( (-1)^x \cos \theta, \sin \theta \cos \phi, \sin \theta \sin \phi \right) \]  

(11)

where \( \phi \) is the Goldstone mode.

Reverting back to the original basis, we obtain the classical state in the original basis:

\[ S_i = S \left( \cos \theta, (-1)^x \sin \theta \cos \phi, (-1)^{x+y} \sin \theta \sin \phi \right) \]  

(12)

Note that although we obtained Eq. 11 and Eq. 12 at \( \beta = \pi/2 \), the same ansatz hold for \( 0 < \beta < \pi/2 \) whose classical ground energy is:

\[ E_c = -2NJS^2 (1 + h_x \cos \theta - \sin^2 \beta \cos^2 \theta - \cos^2 \beta \sin^2 \theta \sin^2 \phi) \]  

(13)

It is easy to see that any deviation from \( \beta = \pi/2 \) explicitly breaks the \( U(1) \) symmetry at \( \beta = \pi/2 \), so picks...
up $\phi = 0$ and leads to the YX-x canted state:

$$ S_i = S(\cos \theta, (1)^i \sin \theta, 0) $$

which indeed has the $(\pi, 0)$ order as indicated from the magnon condensations from the X-FM studied in the subsection A.

Substituting $\phi = 0$ in Eq.13 leads to the classical ground-state energy

$$ E_c = -2NJS^2(1 + h_x \cos \theta - \sin^2 \beta \cos^2 \theta) $$

whose minimization leads to the canted angle:

$$ \cos \theta = \frac{h_x}{2 \sin^2 \beta} < 1, \quad \text{when } h_x < h_{cx} $$

which always has a solution as long as $h_x < h_{cx}$.

Only when $h = h_{cx}, \theta = 0$, it becomes the X-FM phase. The fact that we achieved the same critical field $h_{cx}$ from the X-FM state Eq.7 above it and the YX-x state Eq.16 below it indicate that there is only one phase transition with the critical field $h = h_{cx}$ shown in Fig.1. Note that from above $h > h_{cx}$, we achieved it by the LSW at the order of $1/S$. while, from below $h > h_{cx}$, we achieved it just by the classical ground-state energy minimization at $S = \infty$.

In sharp contrast, in the $h_y$ case [11], there are two critical fields $h_{c1} < h_{c2}$, there is an intermediate IC-SKX phase between the two critical fields.

2. Spin wave analysis in the YX-x Canted state

Again performing the global spin rotation $(S_x^+ \xi, S_y^+ \xi, S_z^+ \xi) \rightarrow (S_x^0, S_y^0, S_z^0)$, then applying the spin rotation $R_x(\theta)$ for the $A$-sublattice and $R_x(-\theta)$ for the $B$-sublattice lead to:

$$ \mathcal{H} = -J \sum_{i \in A} [S_i R_x(\pi) S_{i+x} + S_i R_x(\theta) R_y(2\beta) R_x(-\theta) S_{i+y}] $$

$$ + J \sum_{i \in B} [S_i R_x(\pi) S_{i+x} + S_i R_x(-\theta) R_y(2\beta) R_x(\theta) S_{i+y}] $$

$$ - H_x \sum_{i \in A} [\sin \theta S_i^y \cos \theta S_i^z] $$

$$ - H_x \sum_{i \in B} [-\sin \theta S_i^y \cos \theta S_i^z] $$

Introducing the Holstein-Primakoff (HP) bosons $S^+ = \sqrt{2S-a^\dagger a}$, $S^- = a^\dagger \sqrt{2S-a}$, $S^z = S - a^\dagger a$ for sublattice $A$ and $S^+ = \sqrt{2S-b^\dagger b}$, $S^- = b^\dagger \sqrt{2S-b}$, $S^z = S - b^\dagger b$ for sublattice $B$, we map the Hamiltonian Eq.17 at the order 1/$S$ to

$$ \mathcal{H} = E_c + 2JS \sum_k [(A_k + B_k) a_k^\dagger a_k + (A_k - B_k) b_k^\dagger b_k] $$

$$ + C_k (a_k^\dagger b_k + b_k^\dagger a_k) + D_k (a_k a_{k-a} + b_k b_{k-b} + h.c.) $$

where $E_c$ is the classical ground state energy Eq.15 and

$$ A_k = 2 - (\cos^2 \beta - \sin^2 \beta \sin^2 \theta) \cos k_y $$

$$ B_k = \sin 2\beta \sin \theta \sin k_y $$

$$ C_k = \cos k_x $$

$$ D_k = \sin^2 \beta \cos^2 \theta \cos k_y $$

The Hamiltonian Eq.18 can be diagonalized by a Bogoliubov transformation

$$ \mathcal{H} = E'_c + 4JS \sum_k (\omega_k^+ a_k^\dagger a_k + \omega_k^- b_k^\dagger b_k) $$

where $E'_c = E_c - 2JS \sum_k (\omega_k^+ + \omega_k^-)$ is the ground state energy up to the order of $1/S$ and the energy spectra are:

$$ \omega_k^\pm = \sqrt{A_k^2 + B_k^2 + C_k^2 - D_k^2 \pm 2 \sqrt{A_k^2 (B_k^2 + C_k^2) - B_k^2 D_k^2}} $$

from which one can determine the minimum positions.

We found there are three regimes inside the YX-x Canted state: C-C0 regime, C-IC regime, and C-C\pi regime which, at $h_x = 0$, reduce to the three regimes identified in [7]. Among the three regimes, only C-C0 regime sits just below the transition line $h_{cx}$, so the transition from the YX-x state to the X-FM is driven by the condensations of the C-C0 magnons only.

Because the transition from YX-x Canted state to X-FM state is driven by the condensation of the C-C0 magnons. We find that just below the phase boundary, the C-C0 magnons take also the relativistic form around $k_0 = (0, 0)$:

$$ \omega_q = \sqrt{\Delta_0^2 + v_x^2 q_x^2 + v_y^2 q_y^2}, \quad k = q + k_0 $$

where

$$ \Delta_0 = (1 - \cos 2\beta)(1 - \cos 2\beta - \frac{h_x^2}{2 \sin^2 \beta}), $$

$$ v_x^2 = 2 \sin^2 \beta - (\frac{h_x}{2 \sin \beta})^2, $$

$$ v_y^2 = \frac{2 \sin^2 \beta - (\frac{h_x}{2 \sin \beta})^2}{\cos 2\beta - \sin^2 \beta \cos^2 \beta + \frac{h_x^2}{4 \sin^2 \beta} + \frac{h_x^2 \cos^2 \beta}{\sin^2 \beta}} $$

$$ + \frac{h_x}{2 \sin \beta} \left[1 + \sin^2 \beta (\frac{1 - \frac{h_x^2}{4 \sin^2 \beta}}{2 - \cos 2\beta - (\frac{h_x}{2 \sin \beta})^2})\right] $$

At $h = h_{cx}$, the critical velocities are $v_{x,c} = v_{y,c} = \sin^2 \beta$ which are the same as those achieved from X-FM from above the $h_{cx}$ in Eq.9. Near $h_{cx}$, $\Delta \sim (h_{cx} - h)^{1/2}$. Now we can check the consistence of the orbital orders on both sides of $h_{cx}$. The YX-x state has the orbital order $(\pi, 0)$, the $C - C_0$ has the orbital order $(0, 0) = (\pi, 0)$ in the RBZ. So its condensation on the top of YX-x could lead to the two orbital orders either $(\pi, 0) + (0, 0) = (\pi, 0)$ or $(\pi, 0) + (\pi, 0) = (0, 0)$ in the EBZ. The (0, 0) order is nothing but that of the X-FM in Fig.1.

The competition between C-C0 and C-C\pi gives the boundary between C-C0 and C-C\pi where they become degenerate:

$$ h_{0\pi} = 2 \sin \beta \sqrt{- \cos 2\beta} < h_{cx} $$

(24)
where $\beta^* \sim 0.330482\pi < \beta < \pi/2$.

The competition between C-C$_0$ and C-IC is given by the condition: \( \frac{\partial^2 \omega}{\partial k_y^2} \big|_{k=(0,0)} = 0 \). That between C-C$_\pi$ and C-IC is given by the condition: \( \frac{\partial^2 \omega}{\partial k_y^2} \big|_{k=(0,\pi)} = 0 \). We find that the three boundaries (dashed lines) in Fig.1 meet at the same point \((\beta^* = 0.330482\pi, h_y^* = 1.19921)\). The fine structure near this point is shown in Fig.2.

C. Evolution of the C–IC magnons inside the C–IC regime in Fig.1

A generalized Mirror symmetry about the contour \( k_y^0 = \pm \pi/2 \).

![Figure 2](image_url)

FIG. 2. (Color online) The evolution of the C-IC magnons in Fig.1 zoomed \((\times 10^3)\) around \((\beta_0 = 0.330458\pi, h_x = 1.19899)\) and \((\beta^* = 0.330482\pi, h_y^* = 1.19921)\). There is a generalized mirror symmetry around \( k_y^0 = \pm \pi/2 \). The minimum at \((0, \pm k_y^0)\) and its mirror image at \((0, \pm (\pi - k_y^0))\) symmetrically located on the two sides of \( k_y^0 = \pm \pi/2 \) must end in the regime \( \beta_0 < \beta < \beta^* \). The three segments of the contour line \( k_y^0 = \pm \pi/2 \) are explained in the text.

In the longitudinal field \( h_y \) which keeps the spin-orbital coupled \( U(1) \) symmetry \([11]\), there is a Mirror transformation relating \((\beta, h_y)\) to \((\pi/2 - \beta, h_y)\). So \( \beta = \pi/4 \) enjoys the Mirror symmetry. However, because \( h_x \) and \( h_z \) explicitly breaks the \( U(1) \) symmetry, so the mirror transformation does not work anymore in \( h_x \) and \( h_z \) case. Even so, it would be important to first understand the constant contour minimum at \( k_y^0 = \pm \pi/2 \). In the \( h_x \) case, it seems there is a “generalized” Mirror transformation relating the minimum at \((0, \pm k_y^0)\) to its associated mirror image at \((0, \pm (\pi - k_y^0))\) as shown in Fig.2 while the \( k_y^0 = \pm \pi/2 \) is the self-dual line which starts at \((\beta = \pi/4, h_x = 0)\). Unfortunately, in contrast to the \( h_y \) case, it is difficult to find the exact form of such a “generalized” Mirror transformation in terms of \((\beta, h_x)\). Its form in terms of the contour \( k_y^0 \) would be enough to analyze the structure of the C-IC regime in Fig.2 at least to the order of \( 1/S \). However, as to be shown in the next section, there is no “generalized” Mirror symmetry in the \( h_z \) case.

As shown in the Appendix B, the minimum contour \( k_y^0 = \pm \pi/2 \) can be determined by the equation

\[
\hbar_{\pi/2}(\beta) = 2\sin \beta \sqrt{1 - \cos 2\beta}
\]

where \( 0.25\pi < \beta < \beta_0 \approx 0.330458\pi \).

If comparing Eqn.25 with the C-C$_0$/C-C$_\pi$ boundary, we will find out they have the same form but different domains. In fact, one can extend Eq.(25) to the whole domain \( 0.25\pi < \beta < 0.5\pi \) where we have two special \( \beta: \beta_0 \approx 0.330458\pi, \beta_* \approx 0.330482\pi \). For \( 0.25\pi < \beta < \beta_0 \), Eqn.25 describe the minimum contour \( k_y^0 = \pi/2 \) shown in Fig.6 for \( \beta_* < \beta < 0.5\pi \), it describes the C-C$_0$/C-C$_\pi$ boundary. What happens when \( \beta_0 < \beta < \beta_* \) is shown in Fig.7 and summarized below.

As shown in the Fig.2 the constant contour line at \( k_y^0 = \pi/2 \) can be divided into 3 segments: (1) \( \pi/4 < \beta < \beta_0 \approx 0.330458\pi \), \( k_y^0 = \pm \pi/2 \) is indeed a minimum as shown in Fig.6 (2) \( \beta_0 < \beta < \beta^* \approx 0.330482\pi \), \( k_y^0 = \pm \pi/2 \) becomes a local maximum, \( k_y^0 = 0, \pi \) are also local maximum. There are 4 minima \((0, \pm k_y^0)\) and \((0, \pm (\pi - k_y^0))\) symmetrically located on the two sides of \( k_y^0 = \pm \pi/2 \) as shown in Fig.7. At \( \beta_* \), the second derivatives of the spectrum at \( k_y^0 = \pm \pi/2 \) vanish. (3) \( \beta^* < \beta < \pi/2 \), C-C$_0$ and C-C$_\pi$ become two degenerate minima, with \( k_y^0 = \pm \pi/2 \) being still the maximum as shown in Fig.2.

At \( \beta^* \), the second derivatives of the spectrum at \( k_y^0 = 0, \pi \) vanish.

So all the two mirror related minima \((0, \pm k_y^0)\) and \((0, \pm (\pi - k_y^0))\) must end in the regime \( \beta_0 < \beta < \beta^* \) shown in Fig.2.

D. The transition from the YX-x canted state to the X-FM at \( T = 0 \) and finite \( T \).

1. The Zero temperature transitions:

The transition from the YX-x canted state to the X-FM is characterized by the order parameter is \( M_y(T = 0) = \langle S^y \rangle \). As said at the beginning of Sec.II, the Hamiltonian Eqn.11 has \( P_x \) symmetry: \( S^x \rightarrow S^x, k_y \rightarrow -k_y, S^y \rightarrow -S^y, S^\tau \rightarrow -S^\tau \) and the translational symmetry. The X-FM respects both symmetry, so \( M_y(T = 0) = 0 \), but the YX-x states breaks both, but keeps its combination \( P_x \times (x \rightarrow x + 1) \) as shown in the appendix A, so \( M_y(T = 0) \neq 0 \). Due to the spin-orbital locking, destroying the \( M_y(T = 0) = \langle S^y \rangle \) order will also restore the translational symmetry along \( x \) direction. As shown in Eqn.8 and 22, there are relativistic canted \( C - C_0 \) magnons on both sides indicating the dynamic exponent \( z = 1 \). So we conclude that the transition is in the 3d Ising universality class. The RG flow is controlled by a fixed point on the phase boundary shown in Fig.3.

At \( h_y = 0 \), the \( Y - x \) collinear state is the exact eigenstate \([7]\), so \( M_y(T = 0) = S \). The ground state itself con-
tains no information on the $C - C_0$, $C - C_\pi$ and $C - IC$ magnons. As shown in Sec.II-B, any $h_x \neq 0$ transfers the Y-x state into the $XY - x$ canted state and also introduces quantum fluctuations. The canted angle of the classical $XY$-x state is given in Eqn.16. The ground state itself contains information on the $C - C_0$, $C - C_\pi$ and $C - IC$ magnons. They all compete and move to the phase transition boundary.

From the classical $XY$-x state Eqn.14 with the canted angle Eqn.16 and Eqn.18, we find that they reduce the order parameter below its classical value:

$$M_y(T = 0) = M_c[1 - \frac{1}{2N} \sum_k \left( \frac{1}{\omega_k^2} + \frac{1}{\omega_k^2} \right)]$$

(26)

where $M_c = S\sqrt{1 - (h/h_{cx})^2}$ is the classical order parameter.

When approaching the phase boundary $h_{cx} = 2\sin^2 \beta$, the quantum fluctuations get stronger and stronger, finally, the $C - C_0$ wins the competition, the order parameter vanishes as $M_y(T = 0) \sim (h_{cx} - h)^{\beta_3}$ with the 3d Ising exponent $\beta_{3d} \sim 0.31$. Eqn.26 leads to $M_y(T = 0) \sim \Delta \sim (h_{cx} - h)^{1/2}$ with $\beta_{MF} = 1/2$. Near the critical line $h_{cx}$, the $C - C_0$ magnon gap $\Delta$ on both sides owns the critical scaling $\Delta \sim |h - h_{cx}|^{1/2}$ which gives the mean field exponent $\nu_{MF}$. Note that $\nu = 0.64$ for the 3d Ising model. In principle, one may achieve these exact exponents from the SWE to infinite orders in the quantum fluctuations get stronger and stronger, a gap $\Delta_{ic} \sim |h - h_{cx}|^{1/2}$.

$\omega_k = \sqrt{\Delta_{ic}^2 + v_x^2 q_x^2 + v_y^2 q_y^2}$, $k = (0, \pm k_{y0})$

(27)

whose detailed behaviors along the $k_{y0} = \pm \pi/2$ are shown in Fig.4 and Fig.7. They dominate the contributions to the magnetization and the specific heat when $T \ll \Delta_{ic}$:

$$C_m(T) \sim \frac{\Delta_{ic}^3}{2\pi v_x v_y T} e^{-\Delta_{ic}/T}$$

$$M_y(T) \sim M_y(T = 0) - \frac{T^2}{2\pi v_x v_y} e^{-\Delta_{ic}/T}$$

(28)

where $M_y(T = 0)$ is the zero temperature staggered magnetization given in Eq.26.

Following the procedures in [7], one can also evaluate the uniform and staggered susceptibilities along the $y$ direction, and various dynamic spin correlation functions. Especially, we expect that the C-IC magnons will lead to two split peaks located at $(0, \pm k_{y0})$ in the transverse spin structure factors $S^z(k)$. All these physical quantities can be measured by specific heat [23, 24], In situ measurement [27] and light or atom Bragg spectroscopy [28, 29] respectively.

Because inside the $XY$-x phase in Fig.1 the RG flows to the fixed point $(\beta = \pi/4, h = 0)$, so the finite temperature transition from the $XY - x$ canted phase to the X-FM is in the same universality class as that at zero field case discussed in [3]. Its nature remains to be determined in a separate publication [24]. Of course, at the Abelian $\beta = \pi/2$ point, it is in the 2d $XY$ universality class.


III. TRANSVERSE FIELD $h_z$

The RFHM in a transverse field along $S_z$ direction is described by

$$\mathcal{H} = -\sum_i [S_i R_x(\phi) S_{i+\pm} + S_i R_y(2\beta) S_{i\mp}] - H_z \sum_i S_i^z$$  \hspace{1cm} (29)$$

By applying the hidden $U(1)$ symmetry operator $\hat{T}$ at the zero field case, we show that the $h_z$ case can be mapped to the RFHM in a staggered $h_z$ field along the $x$ direction. However, as expected the staggered $h_z$ could make dramatic difference than the uniform case discussed in the last section.

Similar to the analysis below Eqn.1, one can see the Hamiltonian Eqn.29 has the translational symmetry and the $P_z$ symmetry: $k_x \rightarrow -k_x, S^x \rightarrow -S^x, k_y \rightarrow -k_y, S^y \rightarrow -S^y, S^z \rightarrow S^z$ which is also equivalent to a joint $\pi$ rotation of both the spin and the orbital around the $\hat{z}$ axis. It also keeps $TP_x$ and $TP_y$ symmetry.

A. Z-FM state and excitations in the strong field

In a strong transverse field $H_z \gg J$, the system is in Z-FM phase with spin classically fully polarized to $S_z$ direction with quantum fluctuations shown in Fig.4. Introducing the HP bosons $S^+ = \sqrt{2S} - a^\dagger a$, $S^- = a^\dagger \sqrt{2S} - a^\dagger a$, $S^z = S - a^\dagger a$, we map Hamiltonian Eq.29 to the order of $1/S$:

$$\mathcal{H} = E_0 + 2JS \sum_k [(h_z - 2\sin^2\beta - \cos^2\beta \cos k_y) a_k^\dagger a_k + \sin^2\beta \cos k_y - \cos k_x)](a_k a_{-k} + a_{-k}^\dagger a_k^\dagger)/2$$  \hspace{1cm} (30)$$

where the classical ground state energy $E_0 = 2JS^2 \sin^2\beta - H_z NS$ and the dimensionless field $h_z = H_z/(2JS)$. Now the Hamiltonian can be diagonalized by a Bogoliubov transformation

$$\mathcal{H} = E'_0 + 4JS \sum_k \omega_k a_k^\dagger a_k$$  \hspace{1cm} (31)$$

where the ground-state energy at the order of $1/S$ is $E'_0 = E_0 - 2JS \sum_k \omega_k$ and the spin-wave dispersion takes the form

$$\omega_k = \sqrt{(h_z - 2\sin^2\beta - \cos^2\beta \cos k_y)^2 - (\sin^2\beta \cos k_y - \cos k_x)^2}$$  \hspace{1cm} (32)$$

where, for $0 < \beta < \pi/2$, one can identify there is a unique minimum located at $k_0 = (k_x, k_y) = (\pi, 0)$ with the gap:

$$\Delta_x = \omega_{k_0} = \sqrt{h_z(h_z - 3 + \cos 2\beta)}$$  \hspace{1cm} (33)$$

and the critical field strength is given by the gap vanishing condition:

$$h_{cz}(\beta) = 3 - \cos 2\beta = 2 + 2 \sin^2\beta$$  \hspace{1cm} (34)$$

which is shown in Fig.4.

The Excitation around the minimum takes the relativistic form

$$\omega_q = \sqrt{\Delta_x^2 + v_x^2 q_x^2 + v_y^2 q_y^2}, \quad k = k^0 + q$$  \hspace{1cm} (35)$$

where

$$v_x^2 = 1 + \sin^2\beta, \quad v_y^2 = (h - 1) \cos^2\beta + 2 \sin^4\beta$$  \hspace{1cm} (36)$$

and the critical velocities are $v_{x,c}^2 = v_{y,c}^2 = 1 + \sin^2\beta$. In contrast the HP case, here, the $v_{x,c}$ and $v_{y,c}$ do not vanish even at $\beta = 0$. The gap vanishing at $k^0 = (k_x, k_y) = (\pi, 0)$ indicate a quantum phase transition into a spin-orbital correlated state with orbital order $(\pi, 0)$. It was known that at $h_z = 0$, the ground state $Y - x$ state also has the $(\pi, 0)$ orbital order. That indicates that there is only one phase transition and the state below $h_{cz}$ could be just the $YZ - x$ state with a canted angle. As to be shown in the next subsection, we show that it is indeed
the YZ – x state with the orbital order (π, 0). Near the QPT, \( \Delta_\pi \sim (h_z - h_{cz})^{1/2} \).

At the two Abelian points \( \beta = 0, \pi/2 \) the system has SU(2) symmetry in the rotated basis \( \tilde{S}_i = R(\tilde{x}, \pi a) S_i \) and \( \tilde{S}_\theta = R(\tilde{\theta}, \pi b) R(\tilde{y}, \pi a) S_i \) respectively. So Eqn.29 can be mapped to a FM Heisenberg model in \(-h_z \sum_i (-1)^i \tilde{S}_i^z \) (see Eqn.37) and \(-h_x \sum_i (-1)^{i_x} \tilde{S}_i^x \) (see Eqn.44) respectively. So at \( \beta = 0, \pi/2 \), taking the result from \( \beta = \pi/2 \), any \( h_z \) will lead to a spin-flop transition resulting into a U(1) symmetry breaking canted phase with one Goldstone mode \( \phi \). Then there is another transition to the Z-FM at a finite \( h_z = 2 \) respectively. These results at the two Abelian points are consistent with the general result Eqn.44 and shown in Fig.3.

For \( (\beta = 0, h_z = 0) \), transferring back from the \( \tilde{S}_i \) basis to the original basis, the Hamiltonian Eq.29 has the SU(2) symmetry which is generated by \( \sum_i S_i^z \), \( \sum_i (-1)^i S_i^y \), and \( \sum_i (-1)^i S_i^x \). When \( h_z > 0 \), only \( \sum_i (-1)^i S_i^y \) remains as a conserved quantity. Obviously, the Z-FM state keeps all symmetry from the Hamiltonian. By acting the conserved quantity \( e^{i\phi} \sum_i (-1)^i S_i^y \) on the minima \((0, 0)\) will generate another minimum at \((0, 0)\). So we conclude that at \( (\beta = 0, h_z > 0) \), the system has two minima located at \((0, 0)\) and \((\pi, 0)\) as shown in Fig.4.

For \( (\beta = \pi/2, h_z = 0) \), transferring back from the \( \tilde{S}_i \) basis to the original basis, the Hamiltonian Eq.29 has the SU(2) symmetry which is generated by \( \sum_i (-1)^i S_i^y \), \( \sum_i (-1)^i S_i^x \), and \( \sum_i (-1)^i S_i^x \). When \( h_z > 0 \), only \( \sum_i (-1)^i S_i^x \) remain as a conserved quantity. Acting the conserved quantity \( e^{i\phi} \sum_i (-1)^i S_i^x \) on the minima \((0, \pi)\), generates another minimum at \((0, \pi)\). So we conclude that at \( (\beta = \pi/2, h_z > 0) \), the system has two minima located at \((\pi, 0)\) and \((0, \pi)\) as shown in Fig.4.

B. The Co-planar YZ-x Canted state below \( h_{cz} \):

1. Classical YZ-x canted phase at \( h < h_{cz} \)

(a) Approaching from the left Abelian point \( \beta = 0 \).

At \( \beta = 0 \), in the \( \tilde{S}_i \) basis \( \tilde{S}_i = R_\phi(i_x, \pi) S_i \), the Hamiltonian Eq.29 takes the form:

\[
\mathcal{H} = -J \sum_{(ij)} \tilde{S}_{ij} \cdot \tilde{S}_{ij} - H_z \sum_i (-1)^i \tilde{S}_i^z \quad (37)
\]

When \( 0 < H_z < H_{cz} \) the classical state in the \( \tilde{S}_i \) basis is:

\[
\tilde{S}_i = S(\sin \theta \cos \phi, \sin \theta \sin \phi, (-1)^i \cos \theta) \quad (38)
\]

Reverting back to original basis leads to the classical state in original basis:

\[
S_i = S(\sin \theta \cos \phi, \sin \theta \sin \phi, (-1)^i \cos \theta) \quad (39)
\]

Although we obtained Eq.38 and Eq.39 at \( \beta = 0 \), the same ansatz hold for \( 0 < \beta < \pi/2 \) whose classical ground energy is:

\[
E_c = -2NJS^2[1 - (1 + \sin^2 \beta) \cos^2 \theta + h_z \cos \theta - \sin^2 \beta \sin^2 \theta \cos^2 \phi] \quad (40)
\]

It is easy to see that any \( \beta > 0 \) explicitly breaks the U(1) symmetry at \( \beta = 0 \), so picks up \( \phi = \pi/2 \) and leads to the classical YZ-x canted state:

\[
S_i = S(0, (-1)^i \sin \theta, \cos \theta) \quad (41)
\]

with the corresponding classical ground state energy

\[
E_c = -2NJS^2[1 - (1 + \sin^2 \beta) \cos^2 \theta + h_z \cos \theta] \quad (42)
\]

Minimization of Eq.42 leads to the canted angle:

\[
\cos \theta = \frac{h_z}{2(1 + \sin^2 \beta)} < 1, \quad \text{when} \quad h_z < h_{cz} \quad (43)
\]

which always has a solution as long as \( h_z < h_{cz} \).

(b) Approaching from the right Abelian point \( \beta = \pi/2 \).

In fact, one can reach the same results in Eqn.41 and 42 from the right Abelian point. In the \( \tilde{S}_i \) basis \( \tilde{S}_i = R_\phi(i_x, \pi) R_\theta(i_y, \pi) S_i \), the Hamiltonian in Eq.29 at \( \beta = \pi/2 \) takes the form:

\[
\mathcal{H} = -J \sum_{(ij)} \tilde{S}_{ij} \cdot \tilde{S}_{ij} - H_z \sum_i (-1)^{i_x+i_y} \tilde{S}_i^z \quad (44)
\]

When \( 0 < H_z < H_{cz} \), the classical ground state is:

\[
\tilde{S}_i = S(\sin \theta \cos \phi, \sin \theta \sin \phi, (-1)^i \cos \theta) \quad (45)
\]

Reverting back to the original basis leading to the classical ground state in the original basis

\[
S_i = S(\sin \theta \cos \phi, \sin \theta \sin \phi, (-1)^i \sin \theta \sin \phi, \cos \theta) \quad (46)
\]

with the classical ground state energy

\[
E_c = -2NJS^2[1 - (1 + \sin^2 \beta) \cos^2 \theta + h_z \cos \theta - \cos^2 \beta \sin^2 \theta \cos^2 \phi] \quad (47)
\]

Obviously, any \( \beta < \pi/2 \) picks up \( \phi = \pi/2 \). Then Eqn.46 and Eqn.47 reduce to Eqn.41 and Eqn.42 respectively.

2. Spin wave analysis in the YZ-x Canted state

Starting from the classical YZ-x state Eqn.41 and using similar procedures to obtain Eqn.21 we obtain the spin-wave dispersion:
\[
\omega_k^\pm = \sqrt{A_k^2 + B_k^2 + C_k'^2 - C_k''^2 - D_k^2 \pm 2 \sqrt{(A_k^2 - D_k^2)B_k^2 + (A_kC_k' - C_k''D_k)^2}}
\]

where the expressions of \(A_k, B_k, D_k\) are listed in Eqn. 10 and

\[
\begin{align*}
C_k' &= \sin^2 \theta \cos k_x \\
C_k'' &= \cos^2 \theta \cos k_x
\end{align*}
\]

(49)

where one can see \(C_k' + C_k'' = C_k = \cos k_x\) listed in Eqn. 10.

Of course, the \(k\) in Eqn. 10 in the \(H_z\) field is different from that in Eqn. 10 in the \(H_x\) field.

From Eqn. 18, one can determine the minimum positions inside the YZ-x state. The general structure of Fig. 4 is similar to the \(h_x\) case Fig. 1. However, due to the lack of generalized mirror symmetry as in the \(h_x\) case, the detailed landscape of the C-IC regime in Fig. 4 is much more complicated than that in the \(h_x\) case. In this subsection, we only outline the general structure. In the next subsection and appendix C, we describe details of the shape of the C-IC regime in Fig. 4.

In Fig. 4, we still found there are three regimes inside the YZ-x Canted state: C-C0 regime, C-IC regime, and C-Cp regime which, at \(h_x = 0\), reduce to the three regimes identified in 7. Among the three magnons, only C-C0 wins the game and drives the transition, so the transition from the YZ-x state to the Z-FM is driven by the condensations of the C-C0 magnons only. The C-IC magnons still loses to the C-C0 in the competition.

Now we can check the consistence of the orbital orders on both sides of \(h_{yz}\). The YZ-x state has the orbital order \((\pi, 0)\), the \(C - C_0\) has the orbital order \((0, 0) = (\pi, 0)\) in the RBZ. So its condensation on the top of YZ-x could lead to the two orbital orders either \((\pi, 0) + (0, 0) = (\pi, 0)\) or \((\pi, 0) + (\pi, 0) = (0, 0)\) in the EBZ. The \((0, 0)\) order is nothing but that of the Z-FM in Fig. 4.

C. Fine structure of the C-IC magnons inside the C-IC regime in Fig. 4

As shown in appendix C, the line \(h_3'\) in Fig. 4 is determined by setting the first derivative of dispersion vanishing at \((0, k_0^y = \pi/2)\). The line \(h_2\) and \(h_3\) are determined by the condition that \(C - C_0\) and \(C - C_\pi\) become degenerate. There is one crossing point \((\beta_0, h_0)\) between \(h_3'\) and \(h_2\) in Fig. 4.

In the \(h_x\) case discussed in Sec. II, both conditions are the same, so lead to just one single line with the 3 different segments in Fig. 2 presented in Sec. II-C. However, in the \(h_x\) case, there are two different conditions which leads to three different lines \(h_3', h_2, h_3\), which make the detailed shape of the C-IC regime more complicated than that in \(h_x\) case.

Along the \(h_3'\), the minimum at \(k_0^y = \pi/2\), stays as the (local) minima until \(\beta_{\text{flat}} \sim 0.33\pi\) where the second derivative of the dispersion at \((0, k_0^y = \pi/2)\) vanishes, then it becomes a maximum after \(\beta > \beta_{\text{flat}}\). (In fact, before getting to \(\beta_{\text{flat}} \sim 0.33\pi\), there is another point (let’s call it \(\beta_{\pi}\) in Fig. 4) where the \((0, k_0^y = \pi/2)\) is just a local minimum, while the \(C - C_\pi\) becomes the global minimum). Then \(C - C_\pi\) becomes the minimum, while \(C - C_0\) becomes the maximum, then until \(C - C_\pi\) and \(C - C_0\) becomes degenerate at \(\beta_0 \sim 0.33729\pi\) in Fig. 4. After \(\beta > \beta_0\), it moves into the \(C - C_0\) regime where \(C - C_0\) becomes the minimum. \(h_3'\) rises above \(h_2\) line. So \(\beta_0 \sim 0.33729\pi\) is determined by setting \(h_2 = h_3'\) as shown in Fig. 4.

So in practice, the \(h_3'\) can be split into two segments \(\pi/4 < \beta < \beta_\pi\), \((0, k_0^y = \pi/2)\) is the minimum position.
\beta_x < \beta < \beta_0$, $C - C_\pi$ becomes the minimum position. (So $\beta_{\text{lat}}$ is really not that important anymore). Then when $\beta_0 < \beta < \pi/2$, $h'_2$ rises above $h_2$, moves into the $C - C_0$ regime. Then we have to use the $h_2$ line to delineate the $C - C_0$ and $C - C_\pi$ boundary.

So the $C - IC$ boundary along $(0, h'_0 = \pi/2)$ happens at $(\beta_x, h_x)$ where it enters into $C - C_\pi$. In principle, one can determine the whole $C - IC$ boundary in the whole $YZ - x$ phase. Indeed, we determine the $C - IC$ boundary along the line $h_2$ and $h_3$ in Fig.6. Connecting all the special points along the three lines $h_2, h_3, h'_3$ in Fig.5 and Fig.6 in the appendix C. and also $\beta_1, \beta_2$ at $h_x = 0$ lead to Fig.6 and also the evolution around $(\beta_0, h_0)$ in Fig.11.

D. The transition from the $YZ - x$ canted state to the Z-FM at $T = 0$ and finite $T$.

1. The $T = 0$ transitions

The transition from the $YZ - x$ canted state to the Z-FM at $T = 0$ is still characterized by the order parameter $M_y(T = 0) = \langle S^y \rangle$. As said at the beginning of Sec.III, the Hamiltonian Eqn.1 has the translational symmetry and the $P_z$ symmetry: $k_x \rightarrow -k_x, S^x \rightarrow -S^x, k_y \rightarrow -k_y, S^y \rightarrow -S^y, S^z \rightarrow S^z$. The Z-FM respects both symmetry, so $M_y(T = 0) = 0$, but the YZ-x states breaks both, so still keeps the combination $P_z \times (x \rightarrow x + 1)$, so $M_y(T = 0) \neq 0$. Due to the spin-orbital locking, destroying the $M_y(T = 0) = \langle S^y \rangle$ order will also restore the translational symmetry along $x$ direction. Similar to the $h_x$ case, there are relativistic gapped $C - C_0$ magnons on both sides indicating the dynamic exponent $z = 1$. So we conclude that the transition is also in the 3d Ising universality class. The LSWE only leads to the mean field exponent $\beta_{\text{MF}} = 1/2, \nu_{\text{MF}} = 1/2$.

At the two Abelian points $\beta = 0$ (or $\beta = \pi/2$), starting from $h > h_{cz}$, as shown in (11), due to the enlarged $U(1)$ symmetry, the transition is driven by the simultaneous condensations of the two degenerate minima at $(0, 0)$ and $(\pi, 0)$ (or $(0, \pi)$ and $(\pi, 0)$) shown in Fig.4 and is in the universality class of 3d XY model. From below $h < h_{cz}$, at $\beta = 0$, it is just the condensation of $C - C_0$ magnons, at $\beta = \pi/2$, it is a simultaneous condensations of $C - C_0$ and $C - C_\pi$ magnons, so the transition is also in the 3d XY universality class. After considering the above differences, the $T = 0$ RG flow diagram is similar to Fig.4.

2. The finite temperature behaviors and transitions

Because inside the YZ-x phase in Fig.4 the RG flows to the fixed point $(\beta = \pi/4, h_x = 0)$, so the finite temperature transition at $T_x$ from the $YZ - x$ canted phase to the Z-FM is in the same universality class as that at zero field case discussed in [2]. Its nature remains to be determined. Of course, at the two Abelian points $\beta = 0, \pi/2$, it is in the 2d XY universality class. At the $T = 0$ phase boundary in Fig.4 $T_x = 0$. All the physical quantities at $T \ll T_x$ can be similarly evaluated as in $h_x$ case.

IV. DISCUSSIONS AND CONCLUSIONS

The C-IC magnons in the zero field RFHM stand for short-range In-commensurate orders embedded in a long-ranged ordered commensurate phase [3]. In order to transfer the short-ranged In-commensurate orders into long-range ordered ones, one need to drag out these C-IC and then drive them into condensations. However, as shown in (11) and this work, these C-IC response quite differently to the $h_y$ and $h_x, h_z$ field. In the $h_y$ case, at a small $h_y < h_{c1}$, the $Y - x$ state stay as the exact ground state, so $C - C_0, C - IC$ and C-IC remain extrinsic, detached form the exact ground state and need to be thermally excited. As $h_y \rightarrow h_{c1}^{-}$, the C-IC always emerge as the driving seeds to lead to various IC-SKX phase through a line of fixed points at $h_y = h_{c1}^{-}$. However, in both $h_x$ and $h_z$ case, the C-IC always lose to C-IC which is the driving seeds to lead to X-FM and Z-FM respectively. In fact, one can also group $h_y$ and $h_x$ as an in-plane field, while $h_z$ as the perpendicular field. In the in-plane case, there is Mirror symmetry or a generalized mirror symmetry respectively to characterize the competition among the magnons. While in the perpendicular field, there is no such mirror symmetry.

In the $(\beta, h_y)$ phase diagram, the IC-SKX phase is surrounded by 4 other phases: the two commensurate co-planar canted phases at the left and right in the SOC parameters, two collinear phases in the low and high field. The two canted phases and the IC-SKX phases break the $U(1)$ symmetry spontaneously, so support a gapless excitation. The transition from the canted phase to the Y-FM in the high field is in the 3d XY transition class, controlled by the RG fixed point at the two Abelian points. This is in sharp contrast to the YX-x (YZ-x ) canted phase to the high field X-FM (Z-FM in Fig.4 (Fig.4)) which is in the 3d Ising transition class, controlled by the RG fixed point in the middle of the phase boundary (see Fig.6 for the $h_x$ case) instead of at the two Abelian points. As stressed in (11) and appendix A, in the $h_y$ case, the IC-SKX is due to the condensations of non-relativistic C-IC at a single minima $(0, k'_0)$ from $h < h_{c1}$, so the transition has the dynamic exponent $z = 2$. On the experimental side, the IC-SKX phase match rather naturally and precisely the incommensurate, counter-rotating (in A/B sublattice), non-coplanar magnetic orders detected on iridates $\alpha, \beta, \gamma$-Li$_2$IrO$_3$ [6]. Both $h_x$ and $h_z$ explicitly breaks the spin-orbital coupled $U(1)$ symmetry of the RFHM at a zero field. The YX-x or YZ-x phase supports only gapped magnons. Unfortunately, as shown in Fig.4 and Fig.4 the relativistic C-IC with at least two minima at $(0, \pm k'_0)$ always lose to $C - C_0$, so can not emerge to drive any phase transitions. There is only one transition which is driven by the condensation of $C - C_0$ and is in the 3d Ising universality class. Of course, the Finite temperature transitions in $h_y$ and $h_x, h_z$ cases are also quite different.

It is easy to see why the transition from YX-x to X-FM in Fig.4 and YZ-x to Z-FM in [4] have to go through
C – \( C_0 \) instead of \( C – C_\tau \). This is because YX-x or YZ-x have the orbital order \((\pi,0)\), the \( C – C_0 \) has the orbital order \((0,0) = (\pi,0) \) in the RBZ. So its condensation on the top of YX-x or YZ-x order could lead to two orbital orders either \((\pi,0) + (0,0) = (\pi,0)\) or \((\pi,0) + (\pi,0) = (0,0)\) in the EBZ. The \((0,0)\) order is nothing but that of the X-FM in Fig 1 or Z-FM in 3. However, the \( C – C_\tau \) has the orbital order \((0,\pi) = (\pi,\pi)\) in the RBZ. So its condensation on the top of YX-x or YZ-x order could lead to two orbital orders either \((\pi,0) + (0,\pi) = (\pi,\pi)\) or \((\pi,0) + (\pi,\pi) = (0,\pi)\) in the EBZ, none of the two contains the \((0,0)\) order.

It is instructive to compare the C-IC magnons with quantum fluctuations generated vortices in \( p/q \) filling Boson Hubbard models 24 25, those in high \( T_c \) superconductors 30 31 and exciton superfluids in Bilayer or trilayer quantum Hall systems 12 13. The vortices are gapped topological excitations inside a superfluid, there are at least \( q \) degenerate minima in their dispersions which transform to each other under the projective representation of the Magnetic space group (MSG). So the gap closing (or condensations) of the \( q \) minima lead to various kinds of lattice symmetry breaking insulating states. So these quantum fluctuations generated vortices are short-range translational symmetry breaking insulating orders embedded inside the translational invariant superfluid states. Even inside the superfluid state, they are the crucial ingredients of the superfluid ground state and are generated by the intrinsic quantum fluctuations. Their condensations tuned by interactions spark quantum phase transitions into various neighboring insulating states breaking various translational symmetries of lattices. Of course, vortices are bosons and satisfy boson statistics. Here, these \( C – C_0 \) and \( C – IC \) gapped magnons inside the YX-x or YZ-x state play similar roles as the vortices inside a translational invariant superfluid state. They are the crucial ingredients of the YX-x or YZ-x state and are generated by the intrinsic quantum fluctuations. Their condensations tuned by various Zeeman fields spark quantum phase transitions into various neighboring spin-orbital correlated commensurate or incommensurate phases. The salient feature of the C-IC magnons is that they may condense at any in-commensurate wavevector leading to incommensurate spin-orbital correlated magnetic phases. This indeed what happens in the \( h_z \) Zeeman field studied in 11. However, in the \( h_x, h_z \) fields studied in this paper, they are eliminated before their possible condensations.

In a recent preprint 17, we studied Rotated Antiferromagnetic Heisenberg model (RAFHm) which is the fermionic analog of the RFHM 14. We found that the \( C – C_0, C – IC \) magnons in the RAFHM are also intrinsic ones generated by quantum fluctuations, take relativistic dispersion and already embedded in the ground state. Their parameters such as the minimum positions \( \{0, \pm k_y^0\} \), gap, velocities \( v_x, v_y \) can be precisely measured by the peak positions, the width and Lorentizan shape of the transverse structure factor at \( T = 0 \) respectively.

In this sense, the relativistic \( C – C_0, C – IC \) magnons in the Y-Y state in the RAFHM at zero field resemble those in the YX-x and YZ-x canted state studied in this paper.

The multiple local (meta-stable) or global minima structure of the C-IC magnons shown in Fig 4,5,6 indicate some short-ranged quantum fluctuations with multiple length scales. These complex structure is intrinsic and embedded in the quantum ground state, which may resemble the complex multiple local minimum landscapes in quantum spin glass 32 33. However, the former is SOC induced, the latter is due to quenched disorders. So the SOC may induce some similar complex phenomena as the disorders.

In this work, we only focus along \((\alpha = \pi/2, \beta)\). Obviously, it is important to study how these magnons response when \( \alpha \neq \pi/2 \). We expect that turning on \( \alpha – \pi/2 \) will lead to new competitions different from all the three Zeeman field cases.

In this work, we only focus on quantum phases with only bosonic excitations and without topological orders. As said in the introduction, in fermionic systems 12 17, the quantum phase supports both fermionic excitation and collective bosonic excitations. The two sectors may compete to lead to various other quantum phases under various external probes. In a recent preprint 30, we studied the attractive Hubbard model with Rashba or Dresselhaus spin-orbit coupling in a 2d square lattice subject to a perpendicular \( h_z \) field which is the weak coupling and negative interaction cousin of Eqn 29. We find it is the touching (or gap closing) of fermionic quasi-particle excitations which signify a topological transition from a topological SF to a trivial one or to a band insulator. Obviously, a fermionic quasi-particle can not condense, but they could change the topological winding numbers, therefore spark topological transitions. It remains much more challenging to study topological transitions driven by condensations of fractionized particles satisfying Abelian or non-Abelian statistics 3,4. Unfortunately, in contrast to bosonic or fermionic excitations, one may not be able to treat these fractionized particles as independent particles due to their long-range entanglements mediated by Abelian or non-Abelian Chern-Simons interactions 3,5,30,37,38.

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Appendix A: Energy spectrum symmetry analysis of \( YX-x \) state in \( h_x \) case and \( YZ-x \) state in \( h_z \) case.

As said at the beginning of Sec.II, the RFHM in the \( h_x \) transverse field Eqn 1 enjoys the \( P_x \) symmetry: \( S^x \rightarrow S^x, k_y \rightarrow –k_y, S^y \rightarrow –S^y, S^z \rightarrow –S^z \) and the translational symmetry. The \( YX-x \) state breaks both
the $P_z$ symmetry and the translational symmetry by one lattice site ($x \rightarrow x+1$), but keeps the combination of the two $P_x \times (x \rightarrow x+1)$. So the excitation spectrum must have the $k_y \rightarrow -k_y$ symmetry. This is indeed respected by the LSW spectrum shown in Fig.2, 3, and 9.

Very similarly, as said at the beginning of Sec.III, the RFHM in the $h_z$ transverse field Eqn.25 enjoys the translational symmetry and the $P_z$ symmetry: $k_x \rightarrow -k_x, S^x \rightarrow -S^x, k_y \rightarrow -k_y, S^y \rightarrow -S^y, S^z \rightarrow S^z$ which is also equivalent to a joint $\pi$ rotation of both the spin and the orbital around the $\hat{z}$ axis. The $YZ - x$ state breaks both the $P_z$ symmetry and the translational symmetry by one lattice site ($x \rightarrow x+1$), but keeps the combination of the two $P_z \times (x \rightarrow x+1)$. So the excitation spectrum must have the $k_y \rightarrow -k_y$ symmetry also. This is indeed respected by the LSW spectrum shown in Fig.8 and 9.

The zero field RFHM studied in [7] has the translational symmetry and the $T$, $P_x$, $P_y$ and $P_z$ symmetry. The $Y$-$x$ state breaks all these symmetries except the $P_y$; however, it still keeps $P_x \times (x \rightarrow x+1)$ symmetry, so the excitation spectrum must have the $k_y \rightarrow -k_y$ symmetry also, as indeed respected by the LSW spectrum shown in [7]. However, the energy spectrum in the longitudinal $h_y$ field studied in [10] has no such $k_y \rightarrow -k_y$ symmetry anymore. The RFHM in the longitudinal $h_y$ field enjoys the translational symmetry and the $P_y$ symmetry: $S^y \rightarrow S^y, k_x \rightarrow -k_x, S^x \rightarrow -S^x, S^z \rightarrow S^z$. The $Y-x$ state keeps $P_y$ symmetry, but breaks the translational symmetry by one lattice site ($x \rightarrow x+1$). So the excitation spectrum may not have the $k_y \rightarrow -k_y$ symmetry. Indeed, the $h_y$ field will just pick one the two degenerate minima $\pm k_y^0$ and condense it at $h = h_{c1}$ as shown in Fig. 1 in [11].

Appendix B: The evolution of $C-IC$ in $h_z$ case

As motivated in Sec.II-C, we like to investigate possible ”generalized” mirror symmetry around $y = \pi/2$. So we apply a shift $\tilde{k} = (0, \pi/2) + q$ to the dispersion Eqn.21 and get

$$A_q = 2 + (\cos^2 \beta - \sin^2 \beta \sin^2 \theta) \sin q_y$$
$$B_q = \sin 2\beta \sin \theta \cos q_y$$
$$C_q = \cos q_x$$
$$D_q = -\sin^2 \beta \cos^2 \theta \sin q_y$$

(B1)

It is easy to see that the only term which is not mirror symmetric with respect to $q_y = 0$ is contained in $A_q$. ($D_q$ has no problem because it is squared in Eqn.21) Making the spectrum mirror symmetric with respect to $q_y = 0$ dictates:

$$\cos^2 \beta - \sin^2 \beta \sin^2 \theta = 0$$

(B2)

Plugging in the Eqn.10 leads to Eqn.25.

(Eqn.25) is obtained demanding that the energy spectrum is symmetric with respect to $k_y^0 = \pi/2$, so it guarantees it must be an extreme (either minimum or maximum) at $k_y^0 = \pi/2$ and also the degeneracy condition $\omega_{k}(0,0) = \omega_{k}(\pi, \pi)$. This explains why Eq.25 also contains the C-C$_0$/C-C$_{C}$ boundary Eqn.24.

Appendix C: The evolution of $C-IC$ in $h_z$ case

Following the procedures in the $h_z$ case, we will first determine the boundary between $C-C_0$ and $C-C_{C}$ by
setting $\omega^2_k(0,0) = \omega^2_k(0,\pi)$. Using Eqn 48, we find it has 4 positive roots $h_1, h_2, h_3, h_4$ and 4 negative roots. After comparing with numerical results, we find only the two roots $h_2$ and $h_3$ are physical:

$$h_2 = \sqrt{2(3-\cos 2\beta)(1-\cos 2\beta)}$$

$$h_3 = (3-\cos 2\beta)\sqrt{-\frac{2\cos 2\beta}{1+\cos 2\beta}}$$

Setting $h_2 = h_3$ leads to $\beta = \beta_0 = 0.295296\pi$ as shown in Fig.5. When $0.25\pi < \beta < \beta_0$, $h = h_3$, when $\beta_0 < \beta < \pi/2$, $h = h_2$.

Next we determine the constant contour at $k_y = \pi/2$, thus we need solve

$$0 = \frac{\partial \omega^2_k}{\partial k_y} \bigg|_{k=(0,\pi/2)} \implies c_8 h^8 + c_6 h^6 + c_4 h^4 + c_2 h^2 + c_0 = 0$$

where the coefficients $c_8, c_6, c_4, c_2, c_0$ are functions of $\beta$. This equation also has 4 positive roots $h'_1, h'_2, h'_3, h'_4$ and 4 negative roots. We find only $h'_3$ is a physical solution.

Its analytic expression is complicated, so we only show its numerical solution in the Fig.5. Setting $h_2 = h'_3$ leads to $\beta = \beta_0 = 0.333729\pi$. The three lines $h_2, h_3, h'_3$ and their crossings are drawn in Fig.5.

Since we set $0 = \frac{\partial \omega^2_k}{\partial k_y} \bigg|_{k=(0,\pi/2)}$, the dispersion around $k_y = \pm \pi/2$ changes as shown in Fig.8.

We can summarize the evolution along $h'_3$ line in the following: Along $h'_3$, when $0.25\pi < \beta < \beta_{flat}$, $(0, \pi/2)$ is a local minimum; when $\beta_{flat} < \beta < \pi/2$, $(0, \pi/2)$ is a local maximum.

Along $h'_3$, when $0.25\pi < \beta < \beta_{12}$, $(0, 0)$ is a local maximum; when $\beta_{12} < \beta < \pi/2$, $(0, 0)$ is a local minimum.

Along $h'_3$, when $0.25\pi < \beta < \beta_{11}$, $(0, \pi)$ is a local maximum; when $\beta_{11} < \beta < \pi/2$, $(0, \pi)$ is a local minimum.

The relation between these $\beta$ is $\beta_{12} < \beta_{flat} < \beta_{11}$.

If $0.25\pi < \beta < \beta_{12}$, $(0, \pi/2)$ is a global minimum;

If $\beta_{12} < \beta < \beta_{flat}$, we need compare $(0, \pi/2)$ with $(0, \pi)$;

If $\beta_{11} < \beta < \beta_{f}$, $(0, \pi/2)$ is a global minimum;

If $\beta_{flat} < \beta < \beta_{11}$, $(0, \pi)$ is a global minimum;

If $\beta < \beta_{f}$, $(0, \pi)$ is a global minimum;

As summarized in Sec.III-C, if $0.25\pi < \beta < \beta_{f}$, $(0, \pi/2)$ is a global minimum; if $\beta_{f} < \beta < \beta_{flat}$, $(0, \pi)$ is a global minimum; if $\beta_{flat} < \beta < \pi/2$, $(0, 0)$ is a global minimum. The final result is shown in Fig.9.

The minimum structure along $h_2$ and $h_3$ are shown in Fig.8 and B respectively.
Combining all the special points along the three lines $h_2, h_3, h'_3$ in Fig.5 and Fig.9 and also $\beta_1, \beta_2$ at $h_x = 0$ lead to Fig.4 and the evolution around $(\beta_0, h_0)$ in Fig.10.

![Diagram](image_url)

FIG. 10. (Color online) Fine structure of C-C$_0$, C-IC, C-C$_\pi$ boundaries around $(\beta_0, h_0)$. It is reached by connecting those special points along the three lines $h_2$ (solid and dashed brown), $h_3$ (solid and dashed blue), $h'_3$ (solid red) in Fig.5 and Fig.9. The thick solid black line is the phase boundary $h_{cz}$.

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