Photon surfaces with equipotential time-slices

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Abstract
Photon surfaces are timelike, totally umbilic hypersurfaces of Lorentzian spacetimes. In the first part of this paper, we locally characterize all possible photon surfaces in a class of static, spherically symmetric spacetimes that includes Schwarzschild, Reissner–Nordström, Schwarzschild-anti de Sitter, etc., in \( n + 1 \) dimensions. In the second part, we prove that any static, vacuum, “asymptotically isotropic” \( n + 1 \)-dimensional spacetime that possesses what we call an “equipotential” and “outward directed” photon surface is isometric to the Schwarzschild spacetime of the same (necessarily positive) mass, using a uniqueness result by the first named author.

1 Introduction

One of the cornerstone results in the theory of black holes (in 3+1 dimensions) is the static black hole uniqueness theorem, first due to Israel \([14]\) for a single horizon and later due to Bunting and Masood-ul-Alam \([2]\) for multiple horizons, which establishes the uniqueness of the Schwarzschild spacetime among all static, asymptotically flat, black hole solutions to the vacuum Einstein equations. See Heusler’s book \([13]\) and Robinson’s review article \([18]\) for a complete list of references on further contributions.

A well-known, and intriguing, feature of (positive mass) Schwarzschild spacetime is the existence of a photon sphere, namely, the timelike cylinder \( P \) over the \( \{ r = (nm)^{\frac{1}{n-2}}, t = 0 \} \ n - 1 \)-sphere. \( P \) has the property of being null totally geodesic in the sense that any null geodesic tangent to \( P \) remains in \( P \), i.e., \( P \) traps all light rays tangent to it.

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In [4], the first named author introduced, and made a study of the notion of a photon sphere for general static spacetimes (see also [9] in the spherically symmetric case). Based on this study, in [5], she adapted Israel’s argument (which requires the static lapse function to have nonzero gradient) to obtain a photon sphere uniqueness result, thereby establishing the uniqueness of the Schwarzschild spacetime among all static, asymptotically flat solutions to the vacuum Einstein equations which admit a single photon sphere. Subsequent to that work, by adapting the argument of Bunting and Masood-ul-Alam [2], the authors [7] were able to improve this result by, in particular, avoiding the gradient condition and allowing a priori multiple photon spheres. For further results on photon spheres, in particular uniqueness results in the electrovacuum case and the case of other matter fields, see for example [6,12,15,19,21–26].

In this paper we will be concerned with the notion of photon surfaces in spacetimes (see [9, 17] for slightly more general versions of this notion.) A photon surface in an $n + 1$-dimensional spacetime $(\mathbb{S}^{n+1}, g)$ is a timelike hypersurface $P^n$ which is null totally geodesic, as described above. As was shown in [9,17], a timelike hypersurface $P^n$ is a photon surface if and only if it is totally umbilic. By definition, a photon sphere in a static spacetime is a photon surface $P^n$ along which the static lapse function $N$ is constant; see Section 2 for details. While the Schwarzschild spacetime (in dimension $n + 1$, $n \geq 3$, and with $r > (2m)^{\frac{1}{n-2}}$, $m > 0$) admits a single photon sphere, it admits infinitely many photon surfaces of various types as briefly described in Section 3.

In Section 3, we derive the relevant ODE’s describing spherically symmetric photon surfaces for a class of static, spherically symmetric spacetimes, which includes Schwarzschild, Reissner–Nordström, and Schwarzschild-AdS. In the generic case, a photon surface in this setting is given by a formula for the derivative of the radius-time-profile curve $r = r(t)$; see Theorem 3.5. This formula is used in [8] to give a detailed qualitative description of all spherically symmetric photon surfaces in (subextremal) Reissner–Nordström spacetime, which also includes (positive mass) Schwarzschild spacetime by setting the charge $q = 0$. In addition, in Section 3 we present a result which shows, for generic static, isotropic spacetimes which includes positive mass Schwarzschild and sub-extremal Reissner–Nordström, that, apart from some (partial) timelike hyperplanes, all photon surfaces are isotropic, see Theorem 3.8 and Corollary 3.9. As a consequence, one obtains a complete characterization of all photon surfaces in Reissner–Nordström and Schwarzschild.

Finally, in Section 4 we obtain a new rigidity result pertaining to photon surfaces, rather than just to photon spheres. We prove that any static, vacuum, asymptotically isotropic spacetime possessing a (possibly disconnected) “outward directed” photon surface inner boundary with the property that the static lapse function $N$ is constant on each component of each time-slice $\Sigma^{n-1}(t) := P^n \cap \{t = \text{const.}\}$ must necessarily be a Schwarzschild spacetime of positive mass, with the photon surface being one of the spherically symmetric photon surfaces in Schwarzschild classified in Section 3. We call such photon surfaces equipotential. This generalizes static vacuum photon sphere uniqueness to certain photon surfaces and to higher dimensions.
The proof makes use of a new higher dimensional uniqueness result for the
Schwarzschild spacetime due to the first named author [3]; see Section 4 for a
statement. This result generalizes in various directions the higher dimensional
Schwarzschild uniqueness result of Gibbons et al. [11]. In particular, it does not a
priori require the spacetime to be vacuum or static.

2 Preliminaries

The static, spherically symmetric $(n+1)$-dimensional Schwarzschild spacetime
of mass $m \in \mathbb{R}$, with $n \geq 3$, is given by $(\Sigma^{n+1} := \mathbb{R} \times (\mathbb{R}^n \setminus B_{r_m}(0)), \bar{g})$, where the Lorentzian
metric $\bar{g}$ is given by

$$
\bar{g} = -N^2 dt^2 + N^{-2} dr^2 + r^2 \Omega, \quad N = \left(1 - \frac{2m}{r^{n-2}}\right)^{1/2}, \quad (2.1)
$$

with $\Omega$ denoting the standard metric on $\mathbb{S}^{n-1}$, and $r_m := (2m)^{\frac{1}{n-2}}$ for $m > 0$ and
$r_m := 0$ for $m \leq 0$, see also Tangherlini [20]. For $m > 0$, the timelike, cylindrical
hypersurface $\mathcal{P}^n := \{r = (nm)^{\frac{1}{n-2}}\}$ is called the photon sphere of the Schwarzschild
spacetime because any null geodesic (or “photon”) $\gamma : \mathbb{R} \to \Sigma^{n+1}$ that is tangent to
$\mathcal{P}^n$ for some parameter $\tau_0 \in \mathbb{R}$ is necessarily tangent to it for all parameters $\tau \in \mathbb{R}$. In
particular, the Schwarzschild photon sphere is a timelike hypersurface ruled by null
godesics spiraling around the central black hole of mass $m > 0$ “at a fixed distance”.

The Schwarzschild photon sphere can be seen as a special case of what is called a
“photon surface” [9,17] in a general spacetime (or smooth Lorentzian manifold):

Definition 2.1 (Photon surface). A timelike embedded hypersurface $P^n \hookrightarrow \Sigma^{n+1}$ in a
spacetime $(\Sigma^{n+1}, g)$ is called a photon surface if any null geodesic initially tangent to
$P^n$ remains tangent to $P^n$ as long as it exists or in other words if $P^n$ is null totally
godesic.

The one-sheeted hyperboloids in the Minkowski spacetime (Schwarzschild spacetime
with $m = 0$) are also examples of photon surfaces, see Section 3. It will be useful
to know that, by an algebraic observation, being a null totally geodesic timelike
hypersurface is equivalent to being an umbilic timelike hypersurface:

Proposition 2.2 ([9, Theorem II.1], [17, Proposition 1]). Let $(\Sigma^{n+1}, g)$ be a spacetime and
$P^n \hookrightarrow \Sigma^{n+1}$ an embedded timelike hypersurface. Then $P^n$ is a photon surface if
and only if it is totally umbilic, that is, if and only if its second fundamental form is
pure trace.

As stated above, the Schwarzschild spacetime is “static”, by the following definition.
Definition 2.3 (Static spacetime). A spacetime \((\mathcal{L}^{n+1}, g)\) is called (standard) static if it is a warped product of the form
\[
\mathcal{L}^{n+1} = \mathbb{R} \times M^n, \quad g = -N^2 dt^2 + g, \tag{2.2}
\]
where \((M^n, g)\) is a smooth Riemannian manifold and \(N: M^n \to \mathbb{R}^+\) is a smooth function called the (static) lapse function of the spacetime.

Remark 2.4 (Static spacetime cont., (canonical) time-slices). We will slightly abuse standard terminology and also call a spacetime static if it is a subset (with boundary) of a warped product static spacetime \((\mathbb{R} \times M^n, g = -N^2 dt^2 + g)\), \(\mathcal{L}^{n+1} \subseteq \mathbb{R} \times M^n\), to allow for inner boundary \(\partial \mathcal{L}\) not arising as a warped product. We will denote the (canonical) time-slices \(\{t = \text{const.}\}\) of a static spacetime \((\mathcal{L}^{n+1}, g)\), \(\mathcal{L}^{n+1} \subseteq \mathbb{R} \times M^n\) by \(M^n(t)\) and continue to denote the induced metric and (restricted) lapse function on \(M^n(t)\) by \(g, N\), respectively.

In the context of static spacetimes, we will use the following definition of “photon spheres”, extending that of \([4, 5, 9]\). Consistently, the Schwarzschild photon sphere clearly is a photon surface in the Schwarzschild spacetime in this sense.

Definition 2.5 (Photon sphere). Let \((\mathcal{L}^{n+1}, g)\) be a static spacetime, \(P^n \hookrightarrow \mathcal{L}^{n+1}\) a photon surface. Then \(P^n\) is called a photon sphere if the lapse function \(N\) of the spacetime is constant along each connected component of \(P^n\).

For our discussions in Sections \([3, 4]\), we will make use of the following definitions.

Definition 2.6 (Equipotential photon surface). Let \((\mathcal{L}^{n+1}, g)\) be a static spacetime, \(P^n \hookrightarrow \mathcal{L}^{n+1}\) a photon surface. Then \(P^n\) is called equipotential if the lapse function \(N\) of the spacetime is constant along each connected component of each time-slice \(\Sigma^{n-1} := P^n \cap M^n(t)\) of the photon surface.

Definition 2.7 (Outward directed photon surface). Let \((\mathcal{L}^{n+1}, g)\) be a static spacetime, \(P^n \hookrightarrow \mathcal{L}^{n+1}\) a photon surface arising as the inner boundary of \(\mathcal{L}^{n+1}\), \(P^n = \partial \mathcal{L}\), and let \(\eta\) be the “outward” unit normal to \(P^n\) (i.e. the normal pointing into \(\mathcal{L}^{n+1}\)). Then \(P^n\) is called outward directed if the \(\eta\)-derivative of the lapse function \(N\) of the spacetime is positive, \(\eta(N) > 0\), along \(P^n\).

As usual, a spacetime \((\mathcal{L}^{n+1}, g)\) is said to be vacuum or to satisfy the Einstein vacuum equation if
\[
\mathcal{Ric} = 0 \tag{2.3}
\]
on \(\mathcal{L}^{n+1}\), where \(\mathcal{Ric}\) denotes the Ricci curvature tensor of \((\mathcal{L}^{n+1}, g)\). For a static spacetime, the Einstein vacuum equation \((2.3)\) is equivalent to the static vacuum equations
\[
N \, \text{Ric} = \nabla^2 N \tag{2.4}
\]
\[
R = 0 \tag{2.5}
\]
on $M^n$, where $\text{Ric}$, $R$, and $\nabla^2$ denote the Ricci and scalar curvature, and the covariant Hessian of $(M^n, g)$, respectively. Combining the trace of (2.4) with (2.5), one obtains the covariant Laplace equation on $M^n$,

$$\triangle N = 0.$$  \tag{2.6}

It is clear that, provided (2.4) holds, (2.5) and (2.6) can be interchanged without losing information. Of course, the Schwarzschild spacetime $(\mathbb{R} \times M^n, g)$ is vacuum and thus (2.4) and (2.6) hold for the Schwarzschild spatial metric $g = N^{-2} dt^2 + r^2 \Omega$ and lapse $N$ on its canonical time-slice $M^n = \mathbb{R}^n \setminus \overline{B_r}(0)$.

Curvature quantities of a spacetime $(\mathcal{L}^{n+1}, g)$ such as the Riemann curvature endomorphism $R^m$, the Ricci curvature tensor $\text{Ric}$, and the scalar curvature $R$ will be denoted in gothic print. The Lorentzian metric induced on a timelike embedded hypersurface $P^n \hookrightarrow \mathcal{L}^{n+1}$ will be denoted by $\sigma$, the (outward, see Definition 2.7) unit normal by $\eta$, and the corresponding second fundamental form and mean curvature by $h$ and $H = \text{tr}_\sigma h$, respectively. With this notation, Proposition 2.2 can be restated to state that a photon surface is characterized by

$$h = \frac{\bar{H}}{n} \sigma.$$  \tag{2.7}

To set sign conventions: $h(X, Y) = \sigma(p^X \nabla X \eta, Y)$ for vectors $X, Y$ tangent to $P$.

If the spacetime $(\mathcal{L}^{n+1}, g)$ is static, its time-slices $M^n(t)$ have vanishing second fundamental form $K = 0$ by the warped product structure, or, in other words, the time-slices are totally geodesic. The time-slices of a photon surface $P^n \hookrightarrow \mathcal{L}^{n+1}$ will be denoted by $\Sigma^{n-1}(t) := P^n \cap M^n(t)$, with induced metric $\sigma_\sigma = \sigma(t)$, second fundamental form $h = h(t)$, and mean curvature $H = H(t) = \text{tr}_{\sigma(t)} h(t)$ with respect to the outward pointing unit normal $\nu = \nu(t)$. As an intersection of a totally geodesic time-slice and a totally umbilic photon surfaces, $\Sigma^{n-1}(t)$ is necessarily totally umbilic, and we have

$$h(t) = \frac{H(t)}{n-1} \sigma(t).$$  \tag{2.8}

Our choice of sign of the mean curvature is such that the mean curvature of $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ is positive with respect to the outward unit normal in Euclidean space.

The following proposition will be useful to characterize photon surfaces in vacuum spacetimes.

**Proposition 2.8 (\cite{5} Proposition 3.3).** Let $n \geq 2$ and let $(\mathcal{L}^{n+1}, g)$ be a smooth semi-Riemannian manifold possessing a totally umbilic embedded hypersurface $P^n \hookrightarrow \mathcal{L}^{n+1}$. If the semi-Riemannian manifold $(\mathcal{L}^{n+1}, g)$ is Einstein, or in other words if $\text{Ric} = \Lambda g$ for some constant $\Lambda \in \mathbb{R}$, then each connected component of $P^n$ has constant mean curvature $\bar{H}$ and constant scalar curvature $R = \frac{n-1}{n} \bar{H}^2$,

$$R_p = (n + 1 - 2\tau)\Lambda + \tau \frac{n-1}{n} \bar{H}^2,$$  \tag{2.9}

where $\tau := g(\eta, \eta)$ denotes the causal character of the unit normal $\eta$ to $P^n$.  

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In particular, connected components of photon surfaces in vacuum spacetimes (Λ = 0) have constant mean curvature and constant scalar curvature, related via

\[ R_p = \frac{n-1}{n} \delta^2. \tag{2.10} \]

We will now proceed to define and discuss the assumption of asymptotic flatness and asymptotic isotropy of static spacetimes.

**Definition 2.9 (Asymptotic flatness).** A smooth Riemannian manifold \((M^n, g)\) with \(n \geq 3\) is called asymptotically flat if the manifold \(M^n\) is diffeomorphic to the union of a (possibly empty) compact set and an open end \(E^n\) which is diffeomorphic to \(\mathbb{R}^n \setminus B\), \(\Phi = (x^i): E^n \to \mathbb{R}^n \setminus B\), where \(B\) is some centered open ball in \(\mathbb{R}^n\), and

\[ (\Phi^*g)_{ij} - \delta_{ij} = O_k(r^{1-\frac{n}{2}-\varepsilon}) \tag{2.11} \]

\[ \Phi^*R = O_0(r^{-n-\varepsilon}) \tag{2.12} \]

for \(i, j = 1, \ldots, n\) on \(\mathbb{R}^n \setminus B\) as \(r := \sqrt{(x^1)^2 + \cdots + (x^n)^2} \to \infty\) for some \(k \in \mathbb{Z}\), \(k \geq 2\) and \(\varepsilon > 0\). Here, \(\delta\) denotes the flat Euclidean metric, and \(\delta_{ij}\) its components in the Cartesian coordinates \((x^i)\).

A static spacetime \((\Sigma^{n+1} = \mathbb{R} \times M^n, g = -N^2 dt^2 + g)\) is called asymptotically flat if its Riemannian base \((M^n, g)\) is asymptotically flat as a Riemannian manifold and, in addition, its lapse function satisfies

\[ N - 1 = O_{k+1}(r^{1-\frac{n}{2}-\varepsilon}) \tag{2.13} \]

on \(\mathbb{R}^n \setminus B\) as \(r \to \infty\), with respect to the same coordinate chart \(\Phi\) and numbers \(k \in \mathbb{Z}\), \(k \geq 2\), \(\varepsilon > 0\). We will abuse language and call \(\Sigma^{n+1} \subseteq \mathbb{R} \times M^n\) asymptotically flat, as long as \(\Sigma^{n+1}\) has timelike inner boundary \(\partial \Sigma\).

One can expect a well-known result by Kennefick and Ó Murchadha [16] to generalize to higher dimensions, which would assert that static vacuum asymptotically flat spacetimes are automatically “asymptotically isotropic” in suitable asymptotic coordinates. Here, we will resort to assuming asymptotic isotropy, leaving the higher dimensional generalization of this result to be dealt with elsewhere.

**Definition 2.10 (Asymptotic isotropy [3]).** A smooth Riemannian manifold \((M^n, g)\) of dimension \(n \geq 3\) is called asymptotically isotropic (of mass \(m\)) if the manifold \(M^n\) is diffeomorphic to the union of a (possibly empty) compact set and an open end \(E^n\) which is diffeomorphic to \(\mathbb{R}^n \setminus B\), \(\Psi = (y^i): E^n \to \mathbb{R}^n \setminus B\), where \(B\) is some centered open ball in \(\mathbb{R}^n\), and if there exists a constant \(m \in \mathbb{R}\) such that

\[ (\Psi^*g)_{ij} - (\bar{g}_m)_{ij} = O_2(s^{1-n}), \tag{2.14} \]
for \( i, j = 1, \ldots, n \) on \( \mathbb{R}^n \setminus \overline{B} \) as \( s := \sqrt{(y^1)^2 + \cdots + (y^n)^2} \to \infty \), where

\[
\tilde{g}_m := \varphi_m^{\frac{4}{n-2}}(s) \delta, \\
\varphi_m(s) := 1 + \frac{m}{2s^{n-2}}
\]

(2.15) (2.16)
denotes the spatial Schwarzschild metric in isotropic coordinates.

A static spacetime \((\mathcal{L}^{n+1} = \mathbb{R} \times M^n, g = -N^2 dt^2 + g)\) is called asymptotically isotropic (of mass \( m \)) if its Riemannian base \((M^n, g)\) is asymptotically isotropic of mass \( m \in \mathbb{R} \) as a Riemannian manifold and, in addition, its lapse function \( N \) satisfies

\[
N - \tilde{N}_m = O_2(s^{1-n})
\]

(2.17)
on \( \mathbb{R}^n \setminus \overline{B} \) as \( s \to \infty \), with respect to the same coordinate chart \( \Psi \) and mass \( m \). Here, \( \tilde{N}_m \) denotes the Schwarzschild lapse function in isotropic coordinates, given by

\[
\tilde{N}_m(s) := \frac{1 - \frac{m}{2s^{n-2}}}{1 + \frac{m}{2s^{n-2}}}
\]

(2.18)

As before, we will abuse language and call \( \mathcal{L}^{n+1} \subseteq \mathbb{R} \times M^n \) asymptotically isotropic as long as it has timelike inner boundary.

Here, we have rewritten the Schwarzschild spacetime, spatial metric, and lapse function in isotropic coordinates via the radial coordinate transformation

\[
r =: s \varphi_m^{\frac{2}{n-2}}(s).
\]

(2.19)

For \( m > 0 \), this transformation bijectively maps \( r \in (r_m, \infty) \mapsto s \in (s_m, \infty) \), with \( s_m := (\frac{m}{2})^{\frac{1}{n-2}} \). For \( m = 0 \), this transformation is the identity on \( \mathbb{R}^+ \), while for \( m < 0 \), it only provides a coordinate transformation for \( r \) suitably large, namely corresponding to \( s > \left(\frac{|m|}{2}\right)^{\frac{1}{n-2}} \).

Remark 2.11. A simple computation shows that the parameter \( m \) in Definition 2.10 equals the ADM-mass of the Riemannian manifold \((M^n, g)\) defined in \([1]\).

Remark 2.12. One can analogously define asymptotically flat and isotropic Riemannian manifolds and static spacetimes with multiple ends \( E_l^n \) (and associated masses \( m_l \) in the latter case).

With these definitions at hand, let us point out that photon spheres are always outward directed in static, vacuum, asymptotically isotropic spacetimes, a fact which is a straightforward generalization to higher dimensions of \([7\) Lemma 2.6 and Equation (2.13)]:

Lemma 2.13. Let \( P^n \hookrightarrow \mathcal{L}^{n+1} \) be a photon sphere in a static vacuum asymptotically flat spacetime \((\mathcal{L}^{n+1}, g)\). Then \( P^n \) is outward directed.
3 Photon surfaces in a class of static, spherically symmetric spacetimes

In this section, we will give a local characterization of photon surfaces in a certain class $S$ of static, spherically symmetric spacetimes $(\mathbb{R} \times M^n, g) \in S$, which includes the $n+1$-dimensional Schwarzschild spacetime. We will first locally characterize the spherically symmetric photon surfaces in $(\mathbb{R} \times M^n, g) \in S$ in Theorem 3.5 and then show in Theorem 3.8 and in particular in Corollary 3.9 that there are essentially no other photon surfaces in spacetimes $(\mathbb{R} \times M^n, g) \in S$. As mentioned in the Introduction, these results have been used in [8] to give a detailed description of all photon surfaces in Reissner–Nordström and Schwarzschild.

The class $S$ is defined as follows: Let $(\mathbb{R} \times M^n, g) \in S$ be a smooth Lorentzian spacetime such that

$$M^n = \mathcal{I} \times S^{n-1} \ni (r, \xi)$$

(3.1)

for an open interval $\mathcal{I} \subseteq (0, \infty)$, finite or infinite, and so that there exists a smooth, positive function $f: \mathcal{I} \to \mathbb{R}$ for which we can express the spacetime metric $g$ as

$$g = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\Omega$$

(3.2)

in the global coordinates $t \in \mathbb{R}$, $(r, \xi) \in \mathcal{I} \times S^{n-1}$, where $\Omega$ denotes the canonical metric on $S^{n-1}$ of area $\omega_{n-1}$. A Lorentzian spacetime $(\mathbb{R} \times M^n, g) \in S$ is clearly spherically symmetric and moreover naturally (standard) static via the hypersurface orthogonal, timelike Killing vector field $\partial_t$.

**Remark 3.1.** Note that we do not assume that spacetimes $(\mathbb{R} \times M^n, g) \in S$ satisfy any kind of Einstein equations or have any special type of asymptotic behavior towards the boundary of the radial interval $\mathcal{I}$, such as being asymptotically flat or asymptotically hyperbolic as $r \nearrow \sup \mathcal{I}$, or such as forming a regular minimal surface as $r \searrow \inf \mathcal{I}$.

**Remark 3.2.** As $\partial_t$ is a Killing vector field, the time-translation of any photon surface in a spacetime $(\mathbb{R} \times M^n, g) \in S$ will also be a photon surface in $(\mathbb{R} \times M^n, g)$. As all spacetimes $(\mathbb{R} \times M^n, g) \in S$ are also time-reflection symmetric (i.e. $t \to -t$ is an isometry), the time-reflection of any photon surface in $(\mathbb{R} \times M^n, g)$ will also be a photon surface in $(\mathbb{R} \times M^n, g)$.

While the form of the metric (3.2) is certainly non-generic even among static, spherically symmetric spacetimes, the class $S$ contains many important examples of spacetimes, such as the Minkowski and Schwarzschild spacetime, the Reissner–Nordström spacetime, the Schwarzschild-anti de Sitter spacetime, etc., (in $n+1$ dimensions), each for a specific choice of $f$.

Before we proceed with characterizing photon surfaces in spacetimes in this class $S$, let us first make the following natural definition.
Definition 3.3. Let $(\mathbb{R} \times M^n, g) \in S$. A connected, timelike hypersurface $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$ will be called spherically symmetric if, for each $t_0 \in \mathbb{R}$ for which the intersection $M^n(t_0) := P^n \cap \{t = t_0\} \neq \emptyset$, there exists a radius $r_0 \in \mathcal{I}$ (where $M^n = \mathcal{I} \times S^{n-1}$) such that

$$M^n(t_0) = \{t_0\} \times \{r_0\} \times S^{n-1} \subset \{t_0\} \times M^n.$$  \hspace{1cm} (3.3)

A future timelike curve $\gamma : I \rightarrow P^n$, parametrized by arclength on some open interval $I \subset \mathbb{R}$, is called a radial profile of $P^n$ if $\gamma' \in \text{span}\{\partial_t, \partial_r\} \subset T_{\gamma'}(\mathbb{R} \times M^n)$ on $I$ and if the orbit of $\gamma$ under the rotation generates $P^n$.

With this definition at hand, we will now prove the following lemma which will be used in the proof of Theorem 3.5.

Lemma 3.4. Let $(\mathbb{R} \times M^n, g) \in S$ and let $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$ be a spherically symmetric timelike hypersurface. Assume $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$ has radial profile $\gamma : I \rightarrow P^n$, which may be written as, $\gamma(s) = (t(s), r(s), \xi_*) \in \mathbb{R} \times \mathcal{I} \times S^{n-1}$ for some fixed $\xi_* \in S^{n-1}$.

If $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$ is a photon surface, i.e. is totally umbilic with umbilicity factor $\lambda$, then the following first order ODEs holds on $I$

$$i = \frac{\lambda r}{f(r)} ,$$

$$(\dot{r})^2 = \lambda^2 r^2 - f(r),$$

where $\lambda$ is constant (and where $\dot{\cdot} = \frac{d}{ds}$). Conversely, provided $\dot{r} \neq 0$, if the ODE’s hold, with $\lambda$ constant, then $P$ is a photon surface with umbilicity factor $\lambda$.

Proof. To simplify notation we write $P$ for $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$ and $f$ for $f(r)$. As in Section 2, let $p$ and $\mathfrak{h}$ denote the induced metric and second fundamental form of $P$, respectively.

Set $e_0 = \dot{\gamma}$, and extend it to all of $P$ by making it invariant under the rotational symmetries. Thus, $e_0$ is the future directed unit tangent vector field to $P$ orthogonal to each time slice $\{t(s) = \text{const.}\}$, $s \in I$. In terms of coordinates we have

$$e_0 = i \frac{\partial}{\partial t} + \dot{r} \frac{\partial}{\partial r} .$$

Let $\eta$ be the outward pointing unit normal field to $P$. From (3.2) and (3.6) we obtain

$$\eta = \frac{\dot{r}}{f} \partial_t + if \partial_r .$$

Claim: $P$ is a photon surface, with umbilicity factor $\lambda = \frac{\xi_*}{\theta}$, if and only if $e_0$ satisfies,

$$\mathfrak{h} \nabla_{e_0} e_0 = \lambda \eta .$$

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Proof of the claim. Extend $e_0$ to an orthonormal basis $\{e_0, e_1, \ldots, e_n\}$ in a neighborhood of an arbitrary point in $P$. Thus, each $e_I$, $I = 1, \ldots, n$, where defined, is tangent to the time slices. A simple computation then gives,

$$p\nabla_{e_I} \eta = \frac{\dot{r}}{f} \nabla_{e_I} \partial_t + if \nabla_{e_I} \partial_r = \frac{i f}{r} e_I, \quad (3.9)$$

from which it follows that

$$h(e_I, e_J) = \lambda p(p\nabla_{e_I} \eta, e_J) = \lambda \delta_{IJ}, \quad I, J = 1, \ldots, n, \quad (3.10)$$

where $\delta_{IJ}$ is the Kronecker delta, and

$$\lambda = \frac{f}{r} \dot{t}. \quad (3.11)$$

Similarly,

$$h(e_0, e_I) = h(e_I, e_0) = p(p\nabla_{e_0} \eta, e_0) = 0. \quad (3.12)$$

Hence,

$$[h(e_I, e_J)]_{I, J=0,\ldots,n} \quad (3.13)$$

is a diagonal matrix with $h(e_I, e_I) = \lambda$ for $I = 1, \ldots, n$. It remains to consider $h(e_0, e_0)$.

The profile curve $\gamma$, and its rotational translates, are ‘longitudes’ in the ‘surface of revolution’ $P$. As such, each is a unit speed geodesic in $P$, from which it follows that,

$$p\nabla_{e_0} e_0 = \ell \eta \quad (3.14)$$

for some scalar $\ell$. This implies that,

$$h(e_0, e_0) = p(p\nabla_{e_0} \eta, e_0) = -p(p\nabla_{e_0} e_0, \eta) = -\ell. \quad (3.15)$$

From this and (3.13) we conclude that $P$ is a photon surface if and only if $\ell = \lambda = \frac{\ell}{r} \dot{t}$, which establishes the claim.

Using the coordinate expressions for $e_0$, $\eta$ and $\lambda$, a straight forward computation shows that (3.8), with $\lambda = \frac{\ell}{r} \dot{t}$, is equivalent to the following system of second order ODE’s in the coordinate functions $t = t(s)$ and $r = r(s)$,

$$\dddot{t} + \frac{f'}{f} \dot{t}^2 = \frac{\dot{r}}{r} \dot{t}, \quad (3.16)$$

$$\dddot{r} + \frac{ff'}{2} \dot{t}^2 - \frac{f'}{2f} r^2 = \frac{(\dot{f})^2}{r}. \quad (3.17)$$

Now assume $P$ is a photon sphere with umbilicity factor $\lambda$, so that, in particular, (3.16) and (3.11) hold. Treating (3.16) as a first order linear equation in $\dot{t}$, we have

$$\dddot{t} + \left(\frac{f'}{f} \dot{r} - \frac{\dot{r}}{r} \dot{t}\right) \dot{t} = 0$$
which, multiplying through by the integrating factor \( f \), gives 
\[ \frac{d}{ds} \left( f \frac{\dot{r}}{r} \right) = 0 \]
so that (3.4) holds, with \( \lambda = \frac{\dot{r}}{r} > 0 \) a constant on \( P \). The assumption that \( \gamma \) is parameterized with respect to arc length, gives
\[ f(\dot{r})^2 - \frac{1}{f}(\dot{r})^2 = 1. \] (3.18)
Together with (3.4), we see that (3.5) also holds.

Conversely, now assume that (3.4), (3.5) hold, with \( \lambda = \frac{\dot{r}}{r} = \text{const.} \), and, in addition, that \( \dot{r} \neq 0 \). Differentiating (3.4) with respect to \( s \), and then using (3.11) easily implies (3.16). Differentiating (3.5) with respect to \( s \), then using (3.11) and dividing out by \( \dot{r} \) gives,
\[ \ddot{r} + \frac{f'}{2} = \frac{(f\dot{r})^2}{r}. \] (3.19)
Together with (3.18) (which follows from (3.4) and (3.5)), this implies (3.17). We have shown that (3.16) and (3.17) hold, from which it follows that (3.8) holds with \( \lambda = \frac{\dot{r}}{r} \). Invoking the claim then completes the proof of Lemma 3.4.

From Lemma 3.4 we obtain the following.

**Theorem 3.5.** Let \( (\mathbb{R} \times M^n, g) \in \mathcal{S} \) and let \( P^n \hookrightarrow (\mathbb{R} \times M^n, g) \) be a spherically symmetric timelike hypersurface. Assume that \( P^n \hookrightarrow (\mathbb{R} \times M^n, g) \) is a photon surface, with umbilicity factor \( \lambda \), i.e.
\[ \mathfrak{h} = \lambda p \]
where \( p \) and \( \mathfrak{h} \) are the induced metric and second fundamental form of \( P^n \hookrightarrow (\mathbb{R} \times M^n, g) \), respectively. Let \( \gamma : I \to P^n \) be a radial profile for \( P^n \) and write \( \gamma(s) = (t(s), r(s), \xi_s) \in \mathbb{R} \times I \times S^{n-1} \) for some \( \xi_s \in S^{n-1} \).

Then \( \lambda \) is a positive constant and either \( r \equiv r_* \) along \( \gamma \) for some \( r_* \in I \) at which the photon sphere condition
\[ f'(r_*)r_* = 2f(r_*) \] (3.20)
holds, \( \lambda = \frac{\sqrt{f(r_*)}}{r_*} \), and \( (P^n, p) = (\mathbb{R} \times S^{n-1}, -f(r_*), dt^2 + r_*^2 \Omega) \) is a cylinder and thus a photon sphere, or \( r = r(t) \) can globally be written as a smooth, non-constant function of \( t \) in the range of \( \gamma \) and \( r = r(t) \) satisfies the photon surface ODE
\[ \left( \frac{dr}{dt} \right)^2 = \frac{f(r)(\lambda^2 r^2 - f(r))}{\lambda^2 r^2}. \] (3.21)

Conversely, whenever the photon sphere condition \( f'(r_*)r_* = 2f(r_*) \) holds for some \( r_* \in I \), then the cylinder \( (P^n, p) = (\mathbb{R} \times S^{n-1}, -f(r_*), dt^2 + r_*^2 \Omega) \) is a photon sphere in \( (\mathbb{R} \times M^n, g) \) with umbilicity factor \( \lambda = \frac{\sqrt{f(r_*)}}{r_*} \). Also, any smooth, non-constant solution \( r = r(t) \) of (3.21) for some constant \( \lambda > 0 \) gives rise to a photon surface in \( (\mathbb{R} \times M^n, g) \) with umbilicity factor \( \lambda \).
Proof. From Lemma 3.4, we know that $\lambda$ is a positive constant. Moreover, we know that $t = t(s)$ and $r = r(s)$ satisfy equations (3.4) and (3.5).

In the case when $r \equiv r_*$ for some constant $r_*$, these equations immediately imply,

$$
\dot{t} = \frac{1}{\sqrt{f(r_*)}}, \quad \lambda = \frac{\sqrt{f(r_*')}}{r_*}.
$$

(3.22)

Further, (3.17) implies

$$
f'(r_*) r_* = 2 f(r_*)
$$

(3.23)

In the general case, Equations (3.4) and (3.5) clearly imply (3.21). The converse statements are easily obtained from (3.21) and the unit speed condition (3.18).

Remark 3.6. In view of Remark 3.2, note that in the ‘either’ case, the photon sphere is time-translation and time-reflection invariant in itself. In the ‘or’ case, note that the photon surface ODE (3.21) is time-translation and time-reflection invariant and will thus allow for time-translated and time-reflected solutions corresponding to the same $\lambda > 0$.

Example 3.7. Choosing $(\mathbb{R} \times M^n, g) = (\mathbb{R}^{1,n}, \mathfrak{m})$, where $\mathfrak{m}$ is the Minkowski metric and $f: (0, \infty) \to \mathbb{R}: r \mapsto 1$, the photon sphere condition cannot be satisfied for any $r_* \in (0, \infty)$ so that every spherically symmetric photon surface in the Minkowski spacetime must satisfy the ODE (3.21) which reduces to

$$
\left(\frac{dr}{dt}\right)^2 = \frac{\lambda^2 r^2 - 1}{\lambda^2 r^2} \quad \iff \quad r(t) = \sqrt{\lambda^{-2} + (t - t_0)^2} \text{ for some } t_0 \in \mathbb{R}
$$

and describes the rotational one-sheeted hyperboloids of radii $\lambda^{-1}$ for any $0 < \lambda < \infty$. This is of course consistent with the well-known fact that the only timelike, totally umbilic hypersurfaces in the Minkowski spacetime are, apart from (parts of) timelike hyperplanes, precisely (parts of) these hyperboloids and their spatial translates, the formula for which explicitly displays the time-translation and time-reflection invariance of the photon surface characterization problem.

Note that the photon sphere condition is satisfied precisely at the well-known photon sphere radius $r_* = (nm)^{\frac{1}{n-2}}$ in the $n + 1$-dimensional Schwarzschild spacetime where $f(r) = 1 - \frac{2m}{r}$ for $m > 0$ and $r > r_H = (2m)^{\frac{1}{n-2}}$ and there is no photon sphere radius for $m \leq 0$ and $r > 0$. While there are no other photon spheres, there are many non-cylindrical photon surfaces in the Schwarzschild spacetime. The analysis in [8], based on Theorem 3.5, shows that, up to time translation and time reflection (cf. Remarks 3.2 and 3.6), there are basically five one scaling parameter classes of non-cylindrical spherically symmetric photon surfaces in the Schwarzschild spacetime (as well as in the Reissner–Nordström spacetime); the profile curves for representatives from each class are depicted in Figure 1. In each case, they approach asymptotically the event horizon $r = r_H$ and/or the photon sphere $r = r_*$ and/or become asymptotically null at infinity.

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Figure 1: Profile curves for various types of spherically symmetric photon spheres in Schwarzschild spacetime. These are grouped according to a certain scaling parameter related to the umbilicity factor $\lambda$; see [8] for details.

Using quite different methods, in [10], the same types of photon surfaces are found in a $2+1$-dimensional spacetime obtained by dropping an angle coordinate from $3+1$-dimensional Schwarzschild.

The question naturally arises: What about non-spherically symmetric photon surfaces? This is addressed in the following theorem, see in particular Corollary 3.9.

**Theorem 3.8.** Let $n \geq 3$, $I \subseteq \mathbb{R}^+$ an open interval, $D^n := \{ y \in \mathbb{R}^n | ||y|| = s \in I \}$, and let $\bar{N}, \varphi: I \to \mathbb{R}^+$ be smooth, positive functions. Consider the static, isotropic spacetime

$$\left( \mathbb{R} \times D^n, \bar{g} = -\bar{N}^2 dt^2 + \varphi^{\frac{4}{n-2}} \delta \right) \quad (3.24)$$

of lapse $\bar{N} = \bar{N}(s)$ and conformal factor $\varphi = \varphi(s)$. We write $\bar{g} := \varphi^{\frac{4}{n-2}} \delta$. A timelike hypersurface $P^n$ in $(\mathbb{R} \times D^n, \bar{g})$ is called isotropic if $P^n \cap \{ t = \text{const.} \} = S^{n-1}_s(0) \subset D^n$ for some radius $s(t) \in I$ for every $t$ for which $P^n \cap \{ t = \text{const.} \} \neq \emptyset$. A (partial) centered vertical hyperplane in $(\mathbb{R} \times D^n, \bar{g})$ is the restriction of a timelike hyperplane in the Minkowski spacetime containing the $t$-axis to $\mathbb{R} \times D^n$, i.e. a set of the form

$$\{(t, y) \in \mathbb{R} \times D^n | y \cdot u = 0 \} \quad (3.25)$$

for some fixed Euclidean unit vector $u \in \mathbb{R}^n$, where $\cdot$ denotes the Euclidean inner product. Centered vertical hyperplanes are totally geodesic in $(\mathbb{R} \times D^n, \bar{g})$.

Assume furthermore that the functions $\bar{N}$ and $\varphi$ satisfy

$$\frac{\bar{N}'(s)}{\bar{N}(s)} \neq \frac{2\varphi'(s)}{(n-2)\varphi(s)} \quad (3.26)$$

for all $s \in I$. Then any photon surface in $(\mathbb{R} \times D^n, \bar{g})$ is either (part of) an isotropic photon surface or (part of) a centered vertical hyperplane.
Corollary 3.9. Let \( n \geq 3, \ m > 0, \) and consider the \( n+1 \)-dimensional Schwarzschild spacetime of mass \( m \). Then any connected photon surface is either (part of) a centered vertical hyperplane as described above or (part of) a spherically symmetric photon surface as described in Theorem 3.5.

Proof. (of Corollary 3.9) Recall the isotropic form of the Schwarzschild spacetime (2.15), (2.16), and (2.18), with \( I = (s_m, \infty) \), and note that (3.26) corresponds to \( s^{n-2} \neq \frac{m}{2(n-1)} \) (which can be quickly seen when exploiting \( \tilde{N} = \frac{2-n}{r^2} \)). This, however, is automatic as \( (\frac{m}{2(n-1)})^{\frac{2}{n-2}} < s_m = (\frac{m}{2})^{\frac{n-1}{n-2}} < s \).

Remark 3.10. Condition (3.26) can be interpreted geometrically as follows: If \( \tilde{N}'(s) = \frac{2\phi'(s)}{(n-2)\phi(s)} \) for \( s \in J, \ J \subseteq I \) an open interval, then there exists a positive constant \( A > 0 \) such that \( \phi^{-\frac{2}{n-2}}(s) = A\tilde{N}(s) \) for all \( s \in J \), where we used that \( \tilde{N} \) and \( \phi \) are positive. This shows that \( g = -\tilde{N}^2dt^2 + \phi^{-\frac{4}{n-2}}\delta = \phi^{-\frac{4}{n-2}}(-A^2dt^2 + \delta) \) or in other words the static, isotropic spacetime \( (\mathbb{R} \times (J \times S^{n-1}), g) \) is globally conformally flat and hence possesses additional photon surfaces corresponding to the totally geodesic timelike hyperplanes that do not contain the \( t \)-axis and to the spatially translated totally umbilic rotational one-sheeted hyperboloids of the (time-rescaled) Minkowski spacetime, see Example 3.7.

On the other hand, reconsidering the proof of Theorem 3.8 one finds that the assumption (3.26) is not (fully) necessary (it is only needed for the reasoning after (3.35) and (3.51) for the planar and the spherical case, respectively). Hence, Theorem (3.8) gives a full characterization of photon surfaces in nowhere locally conformally flat static, isotropic spacetimes.

Remark 3.11. A static, isotropic spacetime \( (\mathbb{R} \times D^n, -\tilde{N}^2dt^2 + \phi^{-\frac{4}{n-2}}\delta) \) can be globally rewritten as a spacetime of class \( S \) if and only if \( \tilde{N}^2(s) = (1 + \frac{2\phi'(s)}{(n-2)\phi(s)})^2 > 0 \) for all \( s \in I \). In this case, the photon sphere and photon surface conditions on the isotropic radius profile \( s = S_* \) and \( s = S(t) \), Equations (3.57) and (3.60), reduce to the much simpler photon sphere and photon surface conditions for the area radius profile \( r = r_* \) and \( r = r(t) \), Equations (3.20) and (3.21), respectively.

The main reason for switching into the isotropic picture lies in the spatial conformal flatness allowing to easily describe centered vertical hyperplanes and to exclude photon surfaces that are not centered vertical hyperplanes nor isotropic.

Proof of Theorem 3.8. Let \( P^n \) be a connected photon surface in a static, isotropic spacetime \( (\mathbb{R} \times D^n, g = -\tilde{N}^2dt^2 + \phi^{-\frac{4}{n-2}}\delta) \). As before, set \( M^n(t) := \{ t = \text{const.} \} \). Let \( T := \{ t \in \mathbb{R} | P^n \cap M^n(t) \neq \emptyset \} \) and note that \( T \) is an open, possibly infinite, interval. Set \( \Sigma^{n-1}(t) := P^n \cap M^n(t) \) for \( t \in T \). As timelike and spacelike submanifolds are always transversal, \( \Sigma^{n-1}(t) \) is a smooth surface. Furthermore, \( \Sigma^{n-1}(t) \) is umbilic in \( M^n(t) \) by time-symmetry of \( M^n(t) \), or in other words because the second fundamental form of \( M^n(t) \) in a static spacetime vanishes. As \( (M^n(t), \tilde{g}) \) is conformally...
flat, exploiting the conformal invariance of umbilicity, the only umbilic hypersurfaces in \((M^n(t), \tilde{g})\) are the conformal images of pieces of Euclidean round spheres and pieces of Euclidean hyperplanes. Slightly abusing notation and denoting points in the spacetime by their isotropic coordinates, by continuity and by connectedness of \(P^n\), \((\Sigma^{n-1}(t))_{t \in T}\) is thus either a family of pieces of spheres

\[\Sigma^{n-1}(t) \subseteq \{ y \in \mathbb{R}^n \mid ||y - c(t)|| = S(t) \}\]  

(3.27)

with centers \(c(t) \in \mathbb{R}^n\) and radii \(S(t) > 0\) for all \(t \in T\), or a family of pieces of hyperplanes

\[\Sigma^{n-1}(t) \subseteq \{ y \in \mathbb{R}^n \mid y \cdot u(t) = a(t) \}\]  

(3.28)

for some \(\delta\)-unit normal vectors \(u(t) \in \mathbb{R}^n\) and altitudes \(a(t) \in \mathbb{R}\) for all \(t \in T\), where \(|\cdot|\) and \(\cdot\) denote the Euclidean norm and inner product, respectively. The (outward, where appropriate) unit normal \(\eta\) to \(P^n\) can be written as \(\eta = \alpha \nu + \beta \partial_t\), with \(\alpha > 0\), recalling that \(\nu\) denotes the (outward, where appropriate) unit normal to \(\Sigma^{n-1}(t)\) in \(M^n(t)\). By time-symmetry of \(M^n(t)\), the second fundamental form \(\mathcal{h}\) of \(P^n\) in the spacetime restricted to \(\Sigma^{n-1}(t)\) can be expressed in terms of the second fundamental form \(h\) of \(\Sigma^{n-1}(t)\) in \(M^n(t)\) via \(\mathcal{h}|_{T\Sigma^{n-1}(t) \times T\Sigma^{n-1}(t)} = \alpha h\). By umbilicity, \(\mathcal{h} = \lambda p\), \(p\) denoting the induced metric on \(P^n\), this implies

\[h = \frac{\lambda}{\alpha} \sigma,\]  

(3.29)

where \(\sigma\) is the induced metric on \(\Sigma^{n-1}(t)\). We will treat the planar and the spherical cases separately. We will denote \(t\)-derivatives by \(\cdot^t\) and \(s\)-derivatives by \(\cdot\).

**Planar case:** Let \(\Sigma^{n-1}(t)\) be as in (3.28) for all \(t \in T\). We will show that \(a(t) = 0\) and \(\dot{u}(t) = 0\) for all \(t \in T\), and moreover that \(\lambda = 0\) along \(P^n\). This then implies that \(P^n\) is contained in a centered vertical hyperplane with unit normal \(\eta = \nu = \varphi^{-\frac{2}{n-2}}(s) u^i \partial_{y^i}\) and moreover that all centered vertical hyperplanes are totally geodesic as every centered vertical hyperplane can be written in this form.

For each \(t \in T\), extend \(u(t)\) to a \(\delta\)-orthonormal basis \(\{e_1(t) = u(t), e_2(t), \ldots e_n(t)\}\) of \(\mathbb{R}^n\) in such a way that \(e_1(t)\) is smooth in \(t\) for all \(I = 2, \ldots, n\). Then clearly \(X_I(t, y) := e_I^t(t) \partial_{y^k}\) is tangent to \(P^n\) for all \(I = 2, \ldots, n\) and \(\{X_I(t, \cdot)\}_I^n\) is an orthogonal frame for \(\Sigma^{n-1}(t)\) with respect to \(\tilde{g}\) by conformal flatness. To find the missing (spacetime-)orthogonal tangent vector to \(P^n\), consider a curve \(\mu(t) = (t, y(t))\) in \(P^n\) with tangent vector \(\dot{\mu}(t) = \partial_t + \dot{y}(t) \partial_{y^i}\). Let capital latin indices run from 2, \ldots, \(n\). Now decompose \(\dot{y}(t) = \rho(t) u(t) + \xi^i(t) e_i(t) \in \mathbb{R}^n\). By (3.28), we find \(\rho(t) = \dot{y}(t) \cdot u(t) = \dot{a}(t) - y(t) \cdot \dot{u}(t)\). Hence, a future pointing tangent vector to \(P^n\) orthogonal in the spacetime to all \(X_I\) is given by

\[X_I(t, y) := \partial_t + (\dot{a}(t) - y \cdot \dot{u}(t)) \dot{u}(t) \partial_{y^i},\]  

(3.30)
Hence, (3.35) simplifies to using (3.31) and (3.30), we get

\[ \eta(t, y) = \frac{\varphi^{\frac{n-2}{2}}(s) (\dot{a}(t) - y \cdot \dot{u}(t)) \partial_t + \frac{\tilde{N}(s)}{\varphi^{\frac{n-2}{2}}(s)} u^i(t) \partial_y^i}{\sqrt{\tilde{N}^2(s) - \varphi^{\frac{n-2}{2}}(s) (\dot{a}(t) - y \cdot \dot{u}(t))^2}}. \]  

(3.31)

In other words, using that \( \nu(t, y) = \varphi^{-\frac{n-2}{2}}(s) u^i(t) \partial_y^i \), we have

\[ \alpha(t, y) = \frac{\tilde{N}(s)}{\sqrt{\tilde{N}^2(s) - \varphi^{\frac{n-2}{2}}(s) (\dot{a}(t) - y \cdot \dot{u}(t))^2}}, \]  

(3.32)

\[ \beta(t, y) = \frac{\varphi^{\frac{n-2}{2}}(s) (\dot{a}(t) - y \cdot \dot{u}(t))}{\tilde{N}(s) \sqrt{\tilde{N}^2(s) - \varphi^{\frac{n-2}{2}}(s) (\dot{a}(t) - y \cdot \dot{u}(t))^2}}, \]  

(3.33)

We are now in a position to compute the second fundamental forms explicitly and take advantage of the umbilicity of \( P^n \). Using \( u(t) \cdot \dot{e}_J(t) = -\dot{u}(t) \cdot e_J(t) \) for all \( t \in T \), the condition \( h(X_1, X_J) = 0 \) gives

\[ -\dot{u}(t) \cdot e_J(t) + \frac{1}{s} \left\{ \frac{2\varphi'(s)}{(n-2)\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)} \right\} (\dot{a}(t) - y \cdot \dot{u}(t)) y \cdot e_J(t) = 0 \]  

(3.34)

for \( J = 2, \ldots, n \). As \( \{u(t), e_J(t)\}_{J=2}^n \) is a \( \delta \)-orthonormal frame, this is equivalent to

\[ \frac{1}{s} \left\{ \frac{2\varphi'(s)}{(n-2)\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)} \right\} (\dot{a}(t) - y \cdot \dot{u}(t)) (y - a(t) u(t)) = \ddot{u}(t). \]  

(3.35)

As \( \Sigma^{n-1}(t) \) has dimension \( n - 1 \), (3.35) tells us that \( \ddot{u}(t) = 0 \) for all \( t \in T \) by linear dependence considerations (or otherwise if the term in braces \{ \ldots \} vanishes). Hence, (3.35) simplifies to

\[ \left\{ \frac{2\varphi'(s)}{(n-2)\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)} \right\} \dot{a}(t) (y - a(t) u) = 0 \]  

(3.36)

so that, for a given \( t \in T \), again using that \( \Sigma^{n-1}(t) \) has dimension \( n - 1 \) and linear dependence considerations, we find \( \dot{a}(t) = 0 \) if the term in braces \{ \ldots \} does not vanish along \( P^n \), i.e. when assuming (3.26).

Let us now compute the umbilicity factor \( \lambda \), exploiting that \( u \) and \( a \) are constant. Note that \( e_I \) is also constant, and (3.32) and (3.33) reduce to \( \alpha = 1 \) and \( \beta = 0 \), and, using (3.31) and (3.30), we get \( \eta = \nu \) and \( X_1 = \partial_t \). As \( e_I \) is independent of \( y \), we find

\[ h(X_I, X_J) = \frac{2\varphi'(s)}{(n-2)s\varphi^{\frac{n-2}{2}}(s)} a \delta_{IJ} \]  

(3.37)

\[ \text{for } I, J = 1, \ldots, n. \]
so that by (3.29), the photon surface umbilicity factor $\lambda$ satisfies
\[
\lambda(t, y) = \lambda(y) = \frac{2\varphi'(s)}{(n-2)s\varphi^{\frac{n-2}{2}}(s)} a
\]
and is in particular independent of $t$. From $h(X_1, X_1) = \lambda p(X_1, X_1)$, we find
\[
\lambda(y) = \frac{\tilde{N}'(s)}{s\tilde{N}(s)\varphi^{\frac{n-2}{2}}(s)} a.
\]
Thus, (3.38) and (3.39) combine to
\[
\lambda(y) = \frac{2\varphi'(s)}{(n-2)s\varphi^{\frac{n-2}{2}}(s)} a = \frac{\tilde{N}'(s)}{s\tilde{N}(s)\varphi^{\frac{n-2}{2}}(s)} a
\]
which implies $a = 0$ and indeed $\lambda(y) = \lambda = 0$ is also independent of $y$, when assuming that (3.26) holds along $P^n$. This shows that centered vertical hyperplanes are totally geodesic and that any photon surface $P^n$ as in (3.28) along which (3.26) holds is (part of) a centered vertical hyperplane.

**Spherical case:** Let $\Sigma^{n-1}(t)$ be as in (3.27) for all $t \in T$. We will show that $c(t) = 0$ for all $t \in T$. This then implies that $P^n$ is contained in an isotropic photon surface as desired, namely in a photon sphere with isotropic radius $s = S_*$ satisfying (3.57) or with isotropic radius profile $s = S(t)$ as in (3.60). We will use the abbreviation
\[
u(t,y) := \frac{y - c(t)}{S(t)}
\]
(3.41) to reduce notational complexity.

A straightforward computation shows that the outward unit normal $\nu$ to $\Sigma^{n-1}(t)$ in $M^n(t)$ is given by
\[
\nu = \varphi^{-\frac{2}{n-2}}(s) u'(t, y) \partial_{y^i}.
\]
(3.42)
Now choose a smooth $\delta$-orthonormal system of vectors $e_I(t, y)$ along $P^n$ such that $e_I(t, y) \cdot u(t, y) = 0$ for all $(t, y) \in P^n$ and set $X_I(t, y) := e^I(t, y) \partial_{y^k}$ for all $(t, y) \in P^n$ and all $I = 2, \ldots, n$ so that $\{X_I(t, \cdot)\}_{I=2}^n$ is an orthogonal frame for $\Sigma^{n-1}(t)$ with respect to $\tilde{g}$ by conformal flatness. To find the missing (spacetime-)orthogonal tangent vector to $P^n$, consider a curve $\mu(t) = (t, y(t))$ in $P^n$ with tangent vector $\dot{\mu}(t) = \partial_t + \dot{y}^i(t) \partial_{y^i}$. Let capital latin indices again run from 2, \ldots, $n$. Now decompose
\[
\dot{y}(t) = \rho(t)u(t, y(t)) + \xi^I(t)e_I(t, y(t)) \in \mathbb{R}^n.
\]
(3.43)
By (3.27) and the fact that $\frac{d}{dt} |u(t, y(t))|^2_\delta = 0$ for all $t \in T$, we find
\[
\rho(t) = \dot{y}(t) \cdot u(t, y(t)) = \dot{c}(t) \cdot u(t, y(t)) + \dot{S}(t).
\]
(3.44)
Hence, a future pointing tangent vector to \( P^n \) orthogonal in the spacetime to all \( X_i \) is given by

\[
X_1(t, y) := \partial_t + \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right) u^i(t, y) \partial_{y^i}, \tag{3.45}
\]

so that we have constructed a smooth orthogonal tangent frame \( \{X_i\}_{i=1}^n \) for \( P^n \). Hence we can compute the outward (spacetime) unit normal to \( P^n \) to be

\[
\eta(t, y) = \frac{\varphi_{\pi/2}^{\pi/2}(s) \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right) \partial_t + \frac{\tilde{N}(s)}{\varphi_{\pi/2}^{\pi/2}(s)} \cdot u^i(t, y) \partial_{y^i}}{\sqrt{\tilde{N}^2(s) - \varphi_{\pi/2}^{\pi/2}(s) \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right)^2}}. \tag{3.46}
\]

In other words, using that \( \nu(t, y) = \varphi_{\pi/2}^{\pi/2}(s) u^i(t, y) \partial_{y^i} \), we have

\[
\alpha(t, y) = \frac{\tilde{N}(s)}{\sqrt{\tilde{N}^2(s) - \varphi_{\pi/2}^{\pi/2}(s) \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right)^2}}, \tag{3.47}
\]

\[
\beta(t, y) = \frac{\varphi_{\pi/2}^{\pi/2}(s) \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right)}{\tilde{N}(s)\sqrt{\tilde{N}^2(s) - \varphi_{\pi/2}^{\pi/2}(s) \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right)^2}}. \tag{3.48}
\]

Let us first collect the following explicit formulas

\[
e^i_j(t, y) \left( e_{j, y^i}(t, y) \right) \cdot u(t, y) = -\frac{e^i_j(t, y) e_{j}(t, y) \cdot y_{y^i}}{S(t)} = -\frac{\delta_{ij}}{S(t)}, \tag{3.49}
\]

\[
u^i(t, y) \left( e_{j, y^i}(t, y) \right) \cdot u(t, y) = -\frac{\nu^i(t, y) e_{j}(t, y) \cdot y_{y^i}}{S(t)} = 0 \tag{3.50}
\]

that will simplify our computations later.

Now, let us compute the second fundamental forms explicitly. Using (3.50) and partial integration, we find that the umbilicity condition \( b(X_1, X_j) = 0 \) gives

\[
0 = u(t, y) \cdot \dot{e}_j(t, y) + \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right) u_i(t, y) u^k(t, y)
\]

\[
\times \left[ e^i_{j, y^i}(t, y) + \frac{2\varphi'(s)}{(n-2)s\varphi(s)} \left( y_k e^i_j(t, y) + y \cdot e_j(t, y) \delta^i_k - y^i e_j(t, y) \right) \right]
\]

\[
- \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right) \frac{\tilde{N}'(s)}{s\tilde{N}(s)} y \cdot e_j(t, y)
\]

\[
= \frac{\dot{c}(t)}{S(t)} \cdot e_j(t, y) + \left( \dot{c}(t) \cdot u(t, y) + \dot{S}(t) \right) \frac{1}{s} \left\{ \frac{2\varphi'(s)}{(n-2)s\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)} \right\} y \cdot e_j(t, y)
\]

\[18\]
for all $J = 2, \ldots, n$. As $\{u(t, y), e_J(t, y)\}_{J=2}^n$ is a $\delta$-orthonormal frame, this turns out to be equivalent to

$$-\frac{\dot{c}(t)}{S(t)} = \left(\dot{c}(t) \cdot u(t, y) + \dot{S}(t)\right) \frac{1}{s} \left\{\frac{2\varphi'(s)}{(n-2)\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)}\right\} y$$

$$-\left(\left(\dot{c}(t) \cdot u(t, y) + \dot{S}(t)\right) \frac{1}{s} \left\{\frac{2\varphi'(s)}{(n-2)\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)}\right\} y \cdot u(t, y) + \dot{c}(t) \cdot u(t, y)\right) u(t, y).$$

(3.51)

As $\Sigma^{n-1}(t)$ has dimension $n - 1$ for all $t \in T$, (3.51) tells us by linear dependence considerations that $\dot{c}(t) = 0$ for all $t \in T$. Consequentially, (3.51) simplifies to

$$0 = \dot{S}(t) \left\{\frac{2\varphi'(s)}{(n-2)\varphi(s)} - \frac{\tilde{N}'(s)}{\tilde{N}(s)}\right\} (y - (y \cdot u(t, y)) u(t, y)).$$

(3.52)

Assuming (3.26), the term in braces $\{\ldots\}$ does not vanish and (3.52) implies that, for a fixed $t \in T$, $\dot{S}(t) = 0$ or $y = (y \cdot u(t, y)) u(t, y)$ for all $y \in \Sigma^{n-1}(t)$. But $y = (y \cdot u(t, y)) u(t, y)$ for all $y \in \Sigma^{n-1}(t)$ is equivalent to $c = 0$ and $S(t) = s$ for all $y \in \Sigma^{n-1}(t)$, again by linear dependence considerations and as $\Sigma^{n-1}(t)$ has dimension $n - 1$. In other words, assuming (3.26), we now know that $c = 0$ unless $S$ is constant along $P^n$ in which case a constant center $c \neq 0$ is potentially possible.

Let us now continue with our computation of the conformal factor $\lambda$, using the simplification $\dot{c}(t) = 0$ for all $t \in T$. We first treat the case $\dot{S}(t) = 0$ for all $t \in T$: We know $S(t) = S = s$. Moreover, (3.47) and (3.48) give $\alpha = 1$, $\beta = 0$ and by (3.46) and (3.45) $n = v$ and $X_1 = \partial_t$. Moreover, $u(t, y) = u(y)$ and $e_J(t, y)$ and hence $X_J$ are independent of $t$. By (3.49) and $y \cdot u(y) = c \cdot u(y) + S$ via (3.41), we find

$$h(X_1, X_J) = \frac{\varphi^{\frac{n-2}{2}}(S)}{S} \left(1 + \frac{2\varphi'(S)}{(n-2)\varphi(S)} (c \cdot u(y) + S)\right) \delta_{IJ}.$$

(3.53)

so that, by (3.29), the photon surface umbilicity factor $\lambda$ satisfies

$$\lambda(t, y) = \frac{1}{S\varphi^{\frac{n-2}{2}}(S)} \left(1 + \frac{2\varphi'(S)}{(n-2)\varphi(S)} (c \cdot u(y) + S)\right)$$

(3.54)

and thus in particular $\lambda(t, y) = \lambda(y)$ independent of $t$. Similarly, from $h(X_1, X_1) = \lambda p(X_1, X_1)$, we find

$$\lambda(y) = \frac{\tilde{N}'(S)}{S\tilde{N}(S)\varphi^{\frac{n-2}{2}}(S)} (c \cdot u(y) + S)$$

(3.55)
and hence
\[
1 + \left\{ \frac{2\varphi'(S)}{(n-2)\varphi(S)} - \frac{\tilde{N}'(S)}{\tilde{N}(S)} \right\} (c \cdot u(y) + S) = 0 \tag{3.56}
\]
for all \( y \in \Sigma^{n-1}(t) \) and all \( t \in T \). As \( \Sigma^{n-1}(t) \) has dimension \( n - 1 \) and \( S \) is constant, we conclude that \( c \cdot u(y) \) is constant and hence by (3.41) that \( c \cdot y \) must be constant along \( P^n \). Using again that \( \Sigma^{n-1}(t) \) has dimension \( n - 1 \), this leads to \( c = 0 \) as desired. Hence, the isotropic radii \( s = S_* \) for which this photon sphere can occur are the solutions of the implicit photon sphere equation
\[
1 + \left\{ \frac{2\varphi'(S_*)}{(n-2)\varphi(S_*)} - \frac{\tilde{N}'(S_*)}{\tilde{N}(S_*)} \right\} S_* = 0 \tag{3.57}
\]
provided such solutions exist.

Let us now treat the other case \( c = 0 \): We find \( X_1 = \partial_t + \dot{S}(t)u(t, y)\partial_y \) by (3.45) and indeed \( e_J(t, y) = e_J(y) \) is independent of \( t \) as \( u(t, y) = \frac{y}{S(t)} \). Moreover, \( s = S(t) \) holds for all \( (t, y) \in P^n \). Thus using (3.49), we can compute
\[
h(X_1, X_J) = \frac{\varphi^{\frac{n^2}{2}}(S(t))}{S(t)} \left( 1 + \frac{2\varphi'(S(t))}{(n-2)\varphi(S(t))} S(t) \right) \delta_{IJ} \tag{3.58}
\]
so that, by (3.29), the photon surface umbilicity factor \( \lambda \) satisfies
\[
\lambda(t, y) = \frac{\tilde{N}(S(t)) \left( 1 + \frac{2\varphi'(S(t))}{(n-2)\varphi(S(t))} S(t) \right)}{S(t)\varphi^{\frac{n^2}{2}}(S(t))\sqrt{\tilde{N}^2(S(t)) - \varphi^{\frac{4}{n^2}}(S(t)) \dot{S}^2(t)}} \tag{3.59}
\]
from which we see that \( \lambda(t, y) = \lambda(t) \) only depends on \( t \). From the remaining umbilicity condition \( h(X_1, X_1) = \lambda p(X_1, X_1) \), we obtain
\[
\lambda(t) = \frac{\tilde{N}(S(t))\varphi^{\frac{n^2}{2}}(S(t))}{\sqrt{\tilde{N}^2(S(t)) - \varphi^{\frac{4}{n^2}}(S(t)) \dot{S}^2(t)}} \left( \tilde{N}'(S(t))\tilde{N}(S(t)) + \dot{S}(t) + \left\{ \frac{2\varphi'(S(t))}{(n-2)\varphi(S(t))} - \frac{2\tilde{N}'(S(t))}{\tilde{N}(S(t))} \right\} \dot{S}^2(t) \right)
\]
and can conclude that the implicit equation
\[
\left( 1 + \frac{2\varphi'(S(t))}{(n-2)\varphi(S(t))} S(t) \right) \left( \tilde{N}^2(S(t)) - \varphi^{\frac{4}{n^2}}(S(t)) \dot{S}^2(t) \right)
= S(t)\tilde{N}'(S(t))\tilde{N}(S(t))
+ S(t)\varphi^{\frac{1}{n^2}}(S(t)) \left( \dot{S}(t) + \left\{ \frac{2\varphi'(S(t))}{(n-2)\varphi(S(t))} - \frac{2\tilde{N}'(S(t))}{\tilde{N}(S(t))} \right\} \dot{S}^2(t) \right) \tag{3.60}
\]
holds for the isotropic radius profile \( s = S(t) \). \( \square \)
4 A rigidity result for photon surfaces with equipotential time-slices

As discussed in the previous section, the Schwarzschild spacetime of mass \( m > 0 \) in \( n + 1 \) dimensions possesses not only the well-known photon sphere at \( r = (nm)^{\frac{1}{n-2}} \) but also many other photon surfaces. Except for the planar ones, all of these Schwarzschild photon surfaces are spherically symmetric and thus in particular equipotential as defined in Section 2. In this section, we will prove the following theorem which can be considered complementary to Corollary 3.9 in the context of static, vacuum, asymptotically flat spacetimes.

**Theorem 4.1.** Let \((\mathcal{L}^{n+1}, g)\) be a static, vacuum, asymptotically isotropic spacetime of mass \( m \). Assume that \((\mathcal{L}^{n+1}, g)\) is geodesically complete up to its inner boundary \( \partial \mathcal{L} \), which is assumed to be a (possibly disconnected) photon surface, \( \partial \mathcal{L} =: P^n \). Assume in addition that \( P^n \) is equipotential, outward directed, and has compact time-slices \( \Sigma^{n-1}(t) = P^n \cap M^n(t) \). Then \((\mathcal{L}^{n+1}, g)\) is isometric to a suitable piece of the Schwarzschild spacetime of mass \( m \), and in fact \( m > 0 \). In particular, \( P^n \) is connected, and is (necessarily) a spherically symmetric photon surface in Schwarzschild spacetime.

The proof relies on the following theorem by the first named author.

**Theorem 4.2 ([3]).** Assume \( n \geq 3 \) and let \( M^n \) be a smooth, connected, \( n \)-dimensional manifold with non-empty, possibly disconnected, smooth, compact inner boundary \( \partial M = \bigcup_{i=1}^{I} \Sigma_{i}^{n-1} \). Let \( g \) be a smooth Riemannian metric on \( M^n \). Assume that \((M^n, g)\) has non-negative scalar curvature

\[ R \geq 0, \]

and that it is geodesically complete up to its inner boundary \( \partial M \). Assume in addition that \((M^n, g)\) is asymptotically isotropic with one end of mass \( m \in \mathbb{R} \). Assume that the inner boundary \( \partial M \) is umbilic in \((M^n, g)\), and that each component \( \Sigma_i^{n-1} \) has constant mean curvature \( H_i \) with respect to the outward pointing unit normal \( \nu_i \). Assume furthermore that there exists a function \( u: M^n \to \mathbb{R} \) with \( u > 0 \) away from \( \partial M \) which is smooth and harmonic on \((M^n, g)\),

\[ \Delta u = 0. \]

We ask that \( u \) is asymptotically isotropic of the same mass \( m \), and such that \( u|_{\Sigma_i^{n-1}} \equiv: u_i \) and the normal derivative of \( u \) across \( \Sigma_i^{n-1} \), \( \nu_i(u)|_{\Sigma_i^{n-1}} \equiv: \nu(u)_i \), are constant on each \( \Sigma_i^{n-1} \). Finally, we assume that for each \( i = 1, \ldots, I \), we are either in the semi-static horizon case

\[ H_i = 0, \quad u_i = 0, \quad \nu(u)_i \neq 0, \quad (4.1) \]
or in the true CMC case $H_i > 0$, $u_i > 0$, and there exists $c_i > \frac{n-2}{n-1}$ such that

$$R_{\sigma_i} = c_i H_i^2, \quad (4.2)$$

$$2\nu(u)_i = \left( c_i - \frac{n-2}{n-1} \right) H_i u_i, \quad (4.3)$$

where $R_{\sigma_i}$ denotes the scalar curvature of $\Sigma_{i}^{n-1}$ with respect to its induced metric $\sigma_i$.

Then $m > 0$ and $(M^n, g)$ is isometric to a suitable piece $(\tilde{M}_m \setminus B_S(0), \tilde{g}_m)$ of the (isotropic) Schwarzschild manifold of mass $m$ with $S \geq s_m$. Moreover, $u$ coincides with the restriction of $\tilde{u}_m$ (up to the isometry) and the isometry is smooth.

**Remark 4.3 (Generalization).** Our proof of Theorem 4.1 makes use of the static vacuum Einstein equations, (2.4), (2.5) and (2.6). In fact, as we will see in the proof below, it is sufficient to ask that the vacuum Einstein equations hold in a neighborhood of $P^n$; outside this neighborhood it suffices that $\triangle N = 0$ and that the dominant energy condition $R \geq 0$ holds.

**Remark 4.4 (Multiple ends).** As Theorem 4.2 generalizes to multiple ends (see [3]), Theorem 4.1 also readily applies in the case of multiple ends satisfying the decay conditions (2.14) and (2.17) with potentially different masses $m_i$ in each end $E_i$. Note that, in each end $E_i$, it is necessary that both $g_{ij}$ and $N$ have the same mass $m_i$ in their expansions.

**Remark 4.5 (Discussion of $\eta(N) > 0$).** The assumption that $P^n$ is outward directed, $\eta(N) > 0$ (hence $dN \neq 0$) along $P^n$, can be removed if instead, one assumes that $m > 0$ a priori and that $P^n$ is connected. Using the Laplace equation $\triangle N = 0$ and the divergence theorem as well as the asymptotics (2.14) and (2.17), one computes

$$\frac{1}{\omega_{n-1}} \int_{\Sigma^{n-1}} \nu(N) \, dA = m > 0,$$

where $\Sigma^{n-1} := P^n \cap \{ t = \text{const} \}$ and $\omega_{n-1}$ is the volume of $(S^{n-1}, \Omega)$, see [4, Definition 4.2.1] for the $n = 3$ case; the argument is identical in higher dimensions. From this and connectedness of $\Sigma^{n-1}$, we can deduce that $\nu(N) > 0$ and thus in particular $\eta(N) > 0$ at least in an open subset of $\Sigma^{n-1}$. However, we will see in the proof of Theorem 4.1 that this necessarily implies that $\nu(N) \equiv \text{const} > 0$ in this open neighborhood (noting that all computations performed there are purely local). As $\Sigma^{n-1}$ is connected, we obtain $\nu(N) \equiv \text{const} > 0$ and thus in particular $\eta(N) > 0$ everywhere on $\Sigma^{n-1}$, see Equation (4.6) below.

**Proof.** We write $M^n(t)$ for the time slice $\{ t \} \times M$ (cf. Remark 2.4), and consider each connected component $P^n_i$, $i = 1, \ldots I$ of $P^n$ separately. For the component of $P^n_i$ under consideration, let $\Sigma^{n-1}_i(t) := P^n_i \cap M^n(t)$. We will drop the explicit reference to $i$ in what follows and only start reusing it toward the end of the proof, where we bring in global arguments.
Let $\nu$ denote the outward unit normal to $\Sigma^{n-1}(t) \hookrightarrow (M^n(t), g)$, pointing to the asymptotically isotropic end. Let $\eta$ denote the outward unit normal of $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$. As $N := u(t)$ on $\Sigma^{n-1}(t)$ and because we assumed $\eta(N) > 0$, and hence $\nu(N) > 0$ (see (4.6) below) on $P^n$, we have

$$\nu = \frac{\text{grad } N}{|\text{grad } N|} = \frac{\text{grad } N}{|dN|} = \frac{\text{grad } N}{\nu(N)}.$$ (4.4)

Now let $\mu(s) = (s, x(s))$ be a curve in $P^n$, i.e. $N \circ \mu(s) = u(s)$. This implies by chain rule that $dN(\dot{\mu}) = u$. If $\dot{\mu}(t) \perp \Sigma^{n-1}(t)$, the tangent vector of $\mu$ can be computed explicitly as

$$Z := \dot{\mu} = \partial_t + \dot{x} = \partial_t + \frac{\dot{u}}{\nu(N)}\nu \in \Gamma(TP^n).$$ (4.5)

Expressed in words, $Z$ is the vector field going “straight up” along $P^n$.

The explicit formula (4.5) for $Z$ allows us to explicitly compute the spacetime unit normal $\eta$ to $P^n$, too: It has to be perpendicular to $Z$ and its projection onto $M^n$ has to be proportional to $\nu$. From this, we find

$$\eta = \frac{\nu + \frac{\dot{u}}{u^2 \nu(N)^2} \partial_t}{\sqrt{1 - \frac{\dot{u}^2}{u^2 \nu(N)^2}}}. \quad (4.6)$$

From umbilicity of $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$, it follows that the corresponding second fundamental form $\mathfrak{h}$ of $P^n \hookrightarrow (\mathbb{R} \times M^n, g)$ satisfies

$$\mathfrak{h} = \frac{1}{n} \mathfrak{S} p,$$ (4.7)

where $p$ is the induced metric on $P^n$ and $\mathfrak{S} := \text{tr}_p \mathfrak{h}$. From Proposition 3.3 in [5], we know that $\mathfrak{S} \equiv \text{const}$. Equation (4.7) implies in particular, that, for any tangent vector fields $X, Y \in \Gamma(T\Sigma^{n-1}(t))$, we have

$$\mathfrak{h}(X, Y) = \frac{1}{n} \mathfrak{S} \sigma(X, Y),$$ (4.8)

where $\sigma$ denotes the induced metric on $\Sigma^{n-1}(t)$. Now extend $X, Y$ arbitrarily smoothly.
along $P^n$ such that they remain tangent to $\Sigma^n_{t-1}$. We compute
\[ h(X, Y) = -\mathcal{g}(g\nabla_X Y, \eta) \]
\[ = -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}} \mathcal{g}\left(g\nabla_X Y, \nu + \frac{\dot{u}}{u^2\nu(N)^2}\partial_t\right) \]
\[ = -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}} \left\{ \mathcal{g}(g\nabla_X Y, \nu) + \frac{\dot{u}}{u^2\nu(N)^2} \mathcal{g}(g\nabla_X \partial_t) \right\} \]
\[ = -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}} \left\{ \mathcal{g}(g\nabla_X Y, \nu) - \frac{\dot{u}}{u^2\nu(N)^2} K(X, Y) \right\} \]
\[ = -\frac{1}{\sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}}} h(X, Y), \quad (4.9) \]
where $K = 0$ denotes the second fundamental form of $M^n(t) \hookrightarrow (\mathbb{R} \times M^n, \mathcal{g})$ and $h$ denotes the second fundamental form of $\Sigma^{n-1}_{t-1} \hookrightarrow (\{t\} \times M^n, \mathcal{g})$. In particular, $\Sigma^{n-1}_{t-1} \hookrightarrow (\{t\} \times M^n, \mathcal{g})$ is umbilic and its mean curvature $H$ inside $M^n$ can be computed as
\[ H = \frac{n - 1}{n} \bar{h} \sqrt{1 - \frac{\dot{u}^2}{u^2\nu(N)^2}} \quad (4.10) \]
when combining (4.8) with (4.9). Of course then $h = \frac{1}{n-1} H\sigma$. We will now proceed to show that $H$ and $\nu(N)$ are constant for each fixed $t$. Consider first the contracted Codazzi equation for $\Sigma^{n-1}_{t-1} \hookrightarrow (\{t\} \times M^n, \mathcal{g})$. It gives us
\[ \text{Ric}_\mathcal{g}(X, \nu) = \frac{n - 2}{n - 1} X(H). \quad (4.11) \]
On the other hand, using the static equation gives us
\[ X(\nu(N)) \equiv X(\nu(N)) - g\nabla_X \nu(N) \]
\[ = g\nabla^2 N(X, \nu) \]
\[ = N g\text{Ric}(X, \nu) \]
\[ = \frac{n - 2}{n - 1} u X(H). \quad (4.12) \]
Furthermore, (4.10) allows us to compute

\[ X(H) = \frac{(n - 1) \mathfrak{F}}{n \sqrt{1 - \frac{\dot{u}^2}{u^2 \nu(N)^2}}} \frac{\dot{u}^2}{u^2 \nu(N)^3} X(\nu(N)) \]

(4.13)

Assume \( X(H) \neq 0 \) in some open subset \( U \subset \Sigma^{n-1}(t) \). Then in \( U \), we have

\[ nu \nu(N)^3 \sqrt{1 - \frac{\dot{u}^2}{u^2 \nu(N)^2}} = (n - 2) \frac{\dot{u}^2}{u^2 \nu(N)^2} \]

\[ \iff \quad \nu(N)^6 \left( 1 - \frac{\dot{u}^2}{u^2 \nu(N)^2} \right) = \frac{(n - 2)^2 \dot{u}^2 \mathfrak{F}^2}{n^2 u^2} \]

\[ \Rightarrow \quad \nu(N)^6 - \frac{\dot{u}^2}{u^2} \nu(N)^4 - \frac{(n - 2)^2 \dot{u}^4}{n^2 u^2} = 0. \]

This is a polynomial equation for \( \nu(N) \) with coefficients that only depend on \( t \). As a consequence, \( \nu(N) \) has to be constant in \( U \). However, from (4.12), we know that then also \( H \) has to be constant in \( U \), a contradiction to \( X(H) \neq 0 \) in \( U \). Thus, \( H \) and by (4.12) also \( \nu(N) \) are constants along \( \Sigma^{n-1}(t) \) and only depend on \( t \). From now on, we will drop the explicit reference to \( t \) and also go back to using \( N \) instead of \( u(t) \) as the remaining part of the proof applies to each \( t \) separately.

Thus, each \( \Sigma^{n-1} = \Sigma^{n-1}(t) \) is an umbilic, CMC, equipotential surface in \( M^n \) with \( \nu(N) \) constant, too. From the usual decomposition of the Laplacian on functions and the static vacuum equation (2.6), we find

\[ 0 = \Delta N = \Delta_{\sigma} N + N \text{Ric}(\nu, \nu) + H \nu(N) = N \text{Ric}(\nu, \nu) + H \nu(N) \]

so that \( \text{Ric}(\nu, \nu) = -\frac{H \nu(N)}{N} \) must also be constant along \( \Sigma^{n-1} \). Plugging this into the contracted Gauß equation and using \( R = 0 \), we obtain

\[ -2 \text{Ric}(\nu, \nu) = R_{\sigma} - \frac{n - 2}{n - 1} H^2 \]

\[ \iff \quad R_{\sigma} = \frac{2H \nu(N)}{N} + \frac{n - 2}{n - 1} H^2. \quad (4.14) \]

This shows that \( (\Sigma^{n-1}, \sigma) \) also has constant scalar curvature. Now define the constant \( c > \frac{n - 2}{n - 1} \) by

\[ c := \frac{n - 2}{n - 1} + \frac{2\nu(N)}{NH}. \quad (4.15) \]
Together with (4.14), this definition of $c$ ensures

$$R_{\sigma} = cH^2,$$

(4.16)

$$2\nu(N) = \left(c - \frac{n - 2}{n - 1}\right)HN.$$

(4.17)

Let us summarize: Fix $t$ and keep dropping the explicit reference to it. Then each component of $(\Sigma^{n-1}, \sigma) \hookrightarrow (M^n, g)$ is umbilic, CMC, equipotential, has constant scalar curvature and constant $\nu(N)$, and all these constants together satisfy Equations (4.16) and (4.17) with constant $c$ given by (4.15) which is potentially different for each component of $\Sigma^{n-1}$. Recall the assumption that $(M^n, g)$ is geodesically complete up to its inner boundary so in particular $(M^n \setminus K, g)$ is geodesically complete up to $\Sigma^{n-1}$, where $K$ is the compact set such that $\Sigma^{n-1} = \partial (M^n \setminus K)$. Moreover, $(M^n \setminus K, g)$ satisfies the static vacuum equations. Altogether, these facts ensure that Theorem 4.2 applies. Thus $(M^n \setminus K, g)$ is isometric to a spherically symmetric piece of the spatial Schwarzschild manifold of mass $m$ given by the asymptotics (2.14), (2.17), and $N$ corresponds to the Schwarzschild lapse function of the same mass $m$ under this isometry. The area radius $r$ of the inner boundary $\Sigma^{n-1}$ in the spatial Schwarzschild manifold is determined by

$$R_{\sigma} = \frac{(n-1)(n-2)}{r^2}.$$  

Thus, recalling the dependence on $t$, the manifold $(\{t\} \times (M^n \setminus K(t)), g)$ outside the photon surface time-slice $\Sigma^{n-1}(t)$ is isometric to a piece of the spatial Schwarzschild manifold of mass $m$ with inner boundary area radius $r(t)$. In particular, $\Sigma^{n-1}(t)$ is a connected sphere and we find $m > 0$ (as per Theorem 1.1 in [3]). Recombining the time-slices $\Sigma^{n-1}(t)$ to the photon surface $P^n$, this shows that the part of the spacetime $(\mathbb{R} \times M^n, g)$ lying outside the photon surface $P^n$ is isometric to a piece of the Schwarzschild spacetime of mass $m$, and $m > 0$ necessarily. Moreover, $P^n$ is connected and its isometric image in the Schwarzschild spacetime is spherically symmetric with radius profile $r(t)$. Moreover, $P^n$ is connected and its isometric image in the Schwarzschild spacetime is spherically symmetric with radius profile $r(t)$.

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1 We point out in connection with Remarks 4.4 and 4.3 that the assumptions of Theorem 4.2 keep being met if we start with several ends and the static vacuum equations only holding near $\Sigma^{n-1}$, with $\Delta N = 0$ everywhere in $M^n$ as all the above computations and arguments were purely local near $P^n$. 

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