REAL-VARIABLE CHARACTERIZATIONS
OF BERGMAN SPACES

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Abstract. In this paper, we give a survey of results obtained recently by the present authors on real-variable characterizations of Bergman spaces, which are closely related to maximal and area integral functions in terms of the Bergman metric. In particular, we give a new proof of those results concerning area integral characterizations through using the method of vector-valued Calderón-Zygmund operators to handle Bergman singular integral operators on the complex ball. The proofs involve some sharp estimates of the Bergman kernel function and Bergman metric.

1. Introduction

There is a mature and powerful real variable Hardy space theory which has distilled some of the essential oscillation and cancellation behavior of holomorphic functions and then found that behavior ubiquitous. A good introduction to that is [11]; a more recent and fuller account is in [1, 14, 16] and references therein. However, the real-variable theory of the Bergman space is less well developed, even in the case of the unit disc (cf. [12]).

Recently, in [7] the present authors established real-variable type maximal and area integral characterizations of Bergman spaces in the unit ball of \( C^n \). The characterizations are in terms of maximal functions and area functions on Bergman balls involving the radial derivative, the complex gradient, and the invariant gradient. Subsequently, in [8] we introduced a family of holomorphic spaces of tent type in the unit ball of \( C^n \) and showed that those spaces coincide with Bergman spaces. Moreover, the characterizations extend to cover Besov-Sobolev spaces. A special case of this is a characterization of \( H^p \) spaces involving only area functions on Bergman balls.

We remark that the first real-variable characterization of the Bergman spaces was presented by Coifman and Weiss in 1970’s. Recall that

\[
g(z, w) = \begin{cases} 
|z| - |w| + \left| 1 - \frac{1}{|z||w|} \langle z, w \rangle \right|, & \text{if } z, w \in \mathbb{B}_n \setminus \{0\}, \\
|z| + |w|, & \text{otherwise}
\end{cases}
\]

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is a pseudo-metric on \( \mathbb{B}_n \) and \((\mathbb{B}_n, \varrho, dv_\alpha)\) is a homogeneous space. By their theory of harmonic analysis on homogeneous spaces, Coifman and Weiss [11] can use \(\varrho\) to obtain a real-variable atomic decomposition for Bergman spaces. However, since the Bergman metric \(\beta\) underlies the complex geometric structure of the unit ball of \(\mathbb{C}^n\), one would prefer to real-variable characterizations of the Bergman spaces in terms of \(\beta\). Clearly, the results obtained in [7, 8] are such a characterization.

In this paper, we will give a detailed survey of results obtained in [7, 8]. Moreover, we will give a new proof of those results concerning area integral characterizations through using the method of vector-valued Calderón-Zygmund operator theory to handle Bergman singular integral operators on the complex ball. This paper is organized as follows. In Section 2, some notations and a number of auxiliary (and mostly elementary) facts about the Bergman kernel functions are presented. In Section 3, we will discuss real-variable type atomic decomposition of Bergman spaces. In particular, we will present the atomic decomposition of Bergman spaces with respect to Carleson tubes that was obtained in [7]. Section 4 is devoted to present maximal and area integral function characterizations of Bergman spaces. Finally, in Section 5, we will give a new proof of those results concerning the area integral characterizations obtained in [7, 8] using the argument of Calderón-Zygmund operator theory through introducing Bergman singular integral operators on the complex ball.

In what follows, \(C\) always denotes a constant depending (possibly) on \(n, q, p, \gamma\) or \(\alpha\) but not on \(f\), which may be different in different places. For two nonnegative (possibly infinite) quantities \(X\) and \(Y\), by \(X \lesssim Y\) we mean that there exists a constant \(C > 0\) such that \(X \leq CY\). We denote by \(X \approx Y\) when \(X \lesssim Y\) and \(Y \lesssim X\). Any notation and terminology not otherwise explained, are as used in [20] for spaces of holomorphic functions in the unit ball of \(\mathbb{C}^n\).

2. BERGMAN SPACES

Let \(\mathbb{C}\) denote the set of complex numbers. Throughout the paper we fix a positive integer \(n\), and let \(\mathbb{B}_n\) denote the open unit ball in \(\mathbb{C}^n\). The boundary of \(\mathbb{B}_n\) will be denoted by \(S_n\) and is called the unit sphere in \(\mathbb{C}^n\). Also, we denote by \(\overline{\mathbb{B}}_n\) the closed unit ball, i.e., \(\overline{\mathbb{B}}_n = \{z \in \mathbb{C}^n : |z| \leq 1\} = \mathbb{B}_n \cup S_n\). The automorphism group of \(\mathbb{B}^n\), denoted by \(\text{Aut}(\mathbb{B}^n)\), consists of all bi-holomorphic mappings of \(\mathbb{B}^n\). Traditionally, bi-holomorphic mappings are also called automorphisms.

For \(\alpha \in \mathbb{R}\), the weighted Lebesgue measure \(dv_\alpha\) on \(\mathbb{B}_n\) is defined by

\[
dv_\alpha(z) = c_\alpha(1 - |z|^2)\alpha dv(z)
\]

where \(c_\alpha = 1\) for \(\alpha \leq -1\) and \(c_\alpha = \Gamma(n + \alpha + 1)/[n!\Gamma(\alpha + 1)]\) if \(\alpha > -1\), which is a normalizing constant so that \(dv_\alpha\) is a probability measure on \(\mathbb{B}_n\).
In the case of $\alpha = -(n+1)$ we denote the resulting measure by
\[ d\tau(z) = \frac{dv(z)}{(1 - |z|^2)^{n+1}}, \]
and call it the invariant measure on $\mathbb{B}^n$, since $d\tau = d\tau \circ \varphi$ for any automorphism $\varphi$ of $\mathbb{B}^n$.

Recall that for $\alpha > -1$ and $p > 0$ the (weighted) Lebesgue space $L^p_\alpha(\mathbb{B}_n)$ (or, $L^p_\alpha$ in short) consists of measurable (complex) functions $f$ on $\mathbb{B}_n$ with
\[ \|f\|_{p, \alpha} = \left( \int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \right)^{\frac{1}{p}} < \infty. \]

The (weighted) Bergman space $A^p_\alpha$ is then defined as
\[ A^p_\alpha = H(\mathbb{B}_n) \cap L^p_\alpha, \]
where $H(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. When $\alpha = 0$ we simply write $A^p$ for $A^p_0$. These are the usual Bergman spaces. Note that for $1 \leq p < \infty$, $A^p_\alpha$ is a Banach space under the norm $\| \|_{p, \alpha}$.

Recall that the dual space of $A^1_\alpha$ is the Bloch space $B$ defined as follows (we refer to [20] for details). The Bloch space $B$ of $\mathbb{B}_n$ is defined to be the space of holomorphic functions $f$ in $\mathbb{B}_n$ such that
\[ \|f\|_B = \sup \{ |\overline{\nabla} f(z)| : z \in \mathbb{B}_n \} < \infty. \]

The dual space is semi-norm on $B$. $B$ becomes a Banach space with the following norm
\[ \|f\| = |f(0)| + \|f\|_B. \]

It is known that the Banach dual of $A^1_\alpha$ can be identified with $B$ (with equivalent norms) under the integral pairing
\[ \langle f, g \rangle_\alpha = \lim_{r \to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(z)}dv_\alpha(z), \quad f \in A^1_\alpha, \ g \in B. \]

(e.g., see Theorem 3.17 in [20].)

We define the so-called generalized Bergman spaces as follows (e.g., [19]). For $0 < p < \infty$ and $-\infty < \alpha < \infty$ we fix a nonnegative integer $k$ with $pk + \alpha > -1$ and define $A^p_\alpha$ as the space of all $f \in H(\mathbb{B}_n)$ such that
\[ (1 - |z|^2)^k R^k f \in L^p(\mathbb{B}_n, dv_\alpha). \]
One then easily observes that $A^p_\alpha$ is independent of the choice of $k$ and consistent with the traditional definition when $\alpha > -1$. Let $N$ be the smallest nonnegative integer such that $pN + \alpha > -1$ and define
\begin{equation}
(2.1) \quad \|f\|_{p, \alpha} = |f(0)| + \left( \int_{\mathbb{B}_n} (1 - |z|^2)^p |R^N f(z)|^p dv_\alpha(z) \right)^{\frac{1}{p}}, \quad f \in A^p_\alpha.
\end{equation}

Equipped with (2.1), $A^p_\alpha$ becomes a Banach space when $p \geq 1$ and a quasi-Banach space for $0 < p < 1$. Note that the family of the generalized Bergman spaces $A^p_\alpha$ covers most of the spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, which has been
extensively studied before in the literature under different names. For example, \( B^s_p = \mathcal{A}_p^s \) with \( \alpha = -(ps + 1) \), where \( B^s_p \) is the classical diagonal Besov space consisting of holomorphic functions \( f \) in \( \mathbb{B}_n \) such that \( (1 - |z|^2)^{k-s} R^k f \) belongs to \( L^p(\mathbb{B}_n, dv) \) with \( k \) being any positive integer greater than \( s \). It is clear that \( \mathcal{A}_p^s = B^s_p \) with \( s = -\frac{(\alpha + 1)}{p} \). Thus the generalized Bergman spaces \( \mathcal{A}_p^s \) are exactly the diagonal Besov spaces. On the other hand, if \( k \) is a positive integer, \( p \) is positive, and \( \beta \) is real, then there is the Sobolev space \( W^p_{k,\beta} \) consisting of holomorphic functions \( f \) in \( \mathbb{B}_n \) such that the partial derivatives of \( f \) of order up to \( k \) all belong to \( L^p(\mathbb{B}_n, dv) \) (cf. [2, 3, 6]). It is easy to see that these holomorphic Sobolev spaces are in the scale of the generalized Bergman spaces, i.e., \( W^p_{k,\beta} = \mathcal{A}_p^\alpha \) with \( \alpha = -\frac{pk - \beta + 1}{p} \) (e.g., see [19] for an overview). We refer to Arcozzi-Rochberg-Sawyer [4, 5], Tchoundja [17] and Volberg-Wick [18] for some recent results on such Besov spaces and more references.

Recall that \( D(z, \gamma) \) denotes the Bergman metric ball at \( z \)
\[
D(z, \gamma) = \{ w \in \mathbb{B}_n : \beta(z, w) < \gamma \}
\]
with \( \gamma > 0 \), where \( \beta \) is the Bergman metric on \( \mathbb{B}_n \). It is known that
\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n,
\]
whereafter \( \varphi_z \) is the bijective holomorphic mapping in \( \mathbb{B}_n \), which satisfies \( \varphi_z(0) = z, \varphi_z(z) = 0 \) and \( \varphi_z \circ \varphi_z = id \). If \( \mathbb{B}_n \) is equipped with the Bergman metric \( \beta \), then \( \mathbb{B}_n \) is a separable metric space. We shall call \( \mathbb{B}_n \) a separable metric space instead of \( (\mathbb{B}_n, \beta) \).

For reader’s convenience we collect some elementary facts on the Bergman metric and holomorphic functions in the unit ball of \( \mathbb{C}^n \).

**Lemma 2.1.** (cf. Lemma 1.24 in [20]) For any real \( \alpha \) and positive \( \gamma \) there exist constant \( C_\gamma \) such that
\[
C^{-1}_\gamma (1 - |z|^2)^{n+\alpha} \leq v_\alpha(D(z, \gamma)) \leq C_\gamma (1 - |z|^2)^{n+\alpha}
\]
for all \( z \in \mathbb{B}_n \).

**Lemma 2.2.** (cf. Lemma 2.20 in [20]) For each \( \gamma > 0 \),
\[
1 - |a|^2 \approx 1 - |z|^2 \approx |1 - \langle a, z \rangle|
\]
for all \( a \) in \( \mathbb{B}_n \) with \( z \in D(a, \gamma) \).

**Lemma 2.3.** (cf. Lemma 2.27 in [20]) For each \( \gamma > 0 \),
\[
|1 - \langle z, u \rangle| \approx |1 - \langle z, v \rangle|
\]
for all \( z \) in \( \mathbb{B}_n \) and \( u, v \) in \( \mathbb{B}_n \) with \( \beta(u, v) < \gamma \).
3. Atomic decomposition

We first recall the following “complex-variable” atomic decomposition for Bergman spaces due to Coifman and Rochberg [10] (see also [20], Theorem 2.30).

**Theorem 3.1.** Suppose \( p > 0, \alpha > -1, \) and \( b > n \max\{1, 1/p\} + (\alpha + 1)/p. \) Then there exists a sequence \( \{a_k\} \) in \( B_n \) such that \( A^p_\alpha \) consists exactly of functions of the form

\[
f(z) = \sum_{k=1}^\infty c_k \frac{(1 - |a_k|^2)^{(pb-n-1-\alpha)/p}}{(1 - \langle z, a_k \rangle)^b}, \quad z \in B_n,
\]

where \( \{c_k\} \) belongs to the sequence space \( \ell^p \) and the series converges in the norm topology of \( A^p_\alpha \). Moreover,

\[
\int_{B_n} |f(z)|^p dv_\alpha(z) \approx \inf \left\{ \sum_k |c_k|^p \right\},
\]

where the infimum runs over all the above decompositions.

By Theorem 3.1 we conclude that for any \( \alpha > -1, \) \( A^p_\alpha \) as a Banach space is isomorphic to \( \ell^p \) for every \( 1 \leq p < \infty. \)

Now we turn to the real-variable atomic decomposition of Bergman spaces. To this end, we need some more notations as follows.

For any \( \zeta \in S_n \) and \( r > 0, \) the set \( Q_r(\zeta) = \{z \in B_n : d(z, \zeta) < r\} \) is called a Carleson tube with respect to the nonisotropic metric \( d. \) We usually write \( Q = Q_r(\zeta) \) in short.

As usual, we define the atoms with respect to the Carleson tube as follows: for \( 1 < q < \infty, \) \( a \in L^q(B_n, dv_\alpha) \) is said to be a \((1, q)_\alpha\)-atom if there is a Carleson tube \( Q \) such that

1. \( a \) is supported in \( Q; \)
2. \( \|a\|_{L^q(B_n, dv_\alpha)} \leq v_\alpha(Q)^{1/q-1}; \)
3. \( \int_{B_n} a(z) dv_\alpha(z) = 0. \)

The constant function 1 is also considered to be a \((1, q)_\alpha\)-atom.

Note that for any \((1, q)_\alpha\)-atom \( a, \)

\[
\|a\|_{1, \alpha} = \int_Q |a| dv_\alpha \leq v_\alpha(Q)^{1-1/q}\|a\|_{q, \alpha} \leq 1.
\]

Then, we define \( A^{1,q}_\alpha \) as the space of all \( f \in A^1_\alpha \) which admits a decomposition

\[
f = \sum_i \lambda_i P_\alpha a_i \quad \text{and} \quad \sum_i |\lambda_i| \leq C_q \|f\|_{1, \alpha},
\]
where for each $i$, $a_i$ is an $(1, q)_\alpha$-atom and $\lambda_i \in \mathbb{C}$ so that $\sum_i |\lambda_i| < \infty$. We equip this space with the norm
\[
\|f\|_{A^{1,q}_\alpha} = \inf \left\{ \sum_i |\lambda_i| : f = \sum_i \lambda_i P_\alpha a_i \right\}
\]
where the infimum is taken over all decompositions of $f$ described above.

It is easy to see that $A^{1,q}_\alpha$ is a Banach space.

**Theorem 3.2.** Let $1 < q < \infty$ and $\alpha > -1$. For every $f \in A^{1}_\alpha$ there exist a sequence $\{a_i\}$ of $(1, q)_\alpha$-atoms and a sequence $\{\lambda_i\}$ of complex numbers such that
\[
f = \sum_i \lambda_i P_\alpha a_i \quad \text{and} \quad \sum_i |\lambda_i| \leq C_q \|f\|_{1,\alpha}.
\]
Moreover,
\[
\|f\|_{1,\alpha} \approx \inf \sum_i |\lambda_i|
\]
where the infimum is taken over all decompositions of $f$ described above and “$\approx$” depends only on $\alpha$ and $q$.

Theorem 3.2 is proved in [7] via duality.

**Remark 3.1.** One would like to expect that when $0 < p < 1$, $A^p_\alpha$ also admits an atomic decomposition in terms of atoms with respect to Carleson tubes. However, the proof of Theorem 3.2 via duality cannot be extended to the case $0 < p < 1$. At the time of this writing, this problem is entirely open.

As mentioned in Introduction, by their theory of harmonic analysis on homogeneous spaces, Coifman and Weiss [11] have obtained a real-variable atomic decomposition in terms of $q$ for Bergman spaces in the case $0 < p \leq 1$.

4. Real-variable characterizations

4.1. Maximal functions. As is well known, maximal functions play a crucial role in the real-variable theory of Hardy spaces (cf. [16]). In [7], the authors established a maximal-function characterization for the Bergman spaces. To this end, we define for each $\gamma > 0$ and $f \in \mathcal{H}(\mathbb{B}_n)$:
\[
(M_\gamma f)(z) = \sup_{w \in D(z, \gamma)} |f(w)|, \quad \forall z \in \mathbb{B}_n.
\]
The following result is proved in [7].

**Theorem 4.1.** Suppose $\gamma > 0$ and $\alpha > -1$. Let $0 < p < \infty$. Then for any $f \in \mathcal{H}(\mathbb{B}_n)$, $f \in A^p_\alpha$ if and only if $M_\gamma f \in L^p(\mathbb{B}_n, d\nu_\alpha)$. Moreover,
\[
\|f\|_{p,\alpha} \approx \|M_\gamma f\|_{p,\alpha},
\]
where “$\approx$” depends only on $\gamma, \alpha, p$, and $n$. 
The norm appearing on the right-hand side of (4.2) can be viewed as an analogue of the so-called nontangential maximal function in Hardy spaces. The proof of Theorem 4.1 is fairly elementary, using some basic facts and estimates on the Bergman balls.

**Corollary 4.1.** Suppose $\gamma > 0$ and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and $k$ be a nonnegative integer such that $pk + \alpha > -1$. Then for any $f \in \mathcal{H}(\mathbb{B}_n)$, $f \in \mathcal{A}_p^{\alpha}$ if and only if $M_\gamma(\mathcal{R}^k f) \in L^p(\mathbb{B}_n, dv_\alpha)$, where

$$M_\gamma(\mathcal{R}^k f)(z) = \sup_{w \in D(z,\gamma)} |(1 - |w|^2)^k \mathcal{R}^k f(w)|,$$  

$z \in \mathbb{B}_n$.

Moreover,

$$\|f\|_{p,\alpha} \approx |f(0)| + \|M_\gamma(\mathcal{R}^k f)\|_{p,\alpha},$$

where "\approx" depends only on $\gamma, \alpha, p, k,$ and $n$.

To prove Corollary 4.1, one merely notices that $f \in \mathcal{A}_p^{\alpha}$ if and only if $\mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_\alpha+pk)$ and applies Theorem 4.1 to $\mathcal{R}^k f$ with the help of Lemma 2.2.

4.2. **Area integral functions.** In order to state the real-variable area integral characterizations of the Bergman spaces, we require some more notation. For any $f \in \mathcal{H}(\mathbb{B}_n)$ and $z = (z_1,\ldots,z_n) \in \mathbb{B}_n$ we define

$$\mathcal{R} f(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k}$$

and call it the radial derivative of $f$ at $z$. The complex and invariant gradients of $f$ at $z$ are respectively defined as

$$\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1},\ldots,\frac{\partial f(z)}{\partial z_n}\right)$$

and

$$\tilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0).$$

Now, for fixed $\gamma > 0$ and $1 < q < \infty$, we define for each $f \in \mathcal{H}(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$:

1. The radial area function

$$A_\gamma^{(q)}(\mathcal{R} f)(z) = \left(\int_{D(z,\gamma)} |(1 - |w|^2)^q \mathcal{R} f(w)|^q d\tau(w)\right)^{\frac{1}{q}}.$$  

2. The complex gradient area function

$$A_\gamma^{(q)}(\nabla f)(z) = \left(\int_{D(z,\gamma)} |(1 - |w|^2)^q \nabla f(w)|^q d\tau(w)\right)^{\frac{1}{q}}.$$  

3. The invariant gradient area function

$$A_\gamma^{(q)}(\tilde{\nabla} f)(z) = \left(\int_{D(z,\gamma)} |\tilde{\nabla} f(w)|^q d\tau(w)\right)^{\frac{1}{q}}.$$  

The following theorem is proved in [7].
Theorem 4.2. Suppose \( \gamma > 0, 1 < q < \infty, \) and \( \alpha > -1. \) Let \( 0 < p < \infty. \) Then, for any \( f \in \mathcal{H}(\mathbb{B}_n) \) the following conditions are equivalent:

(a) \( f \in \mathcal{A}_p^q. \)
(b) \( A_{\gamma}^{(q)}(\mathcal{R}f) \) is in \( L^p(\mathbb{B}_n, dv_\alpha). \)
(c) \( A_{\gamma}^{(q)}(\nabla f) \) is in \( L^p(\mathbb{B}_n, dv_\alpha). \)
(d) \( A_{\gamma}^{(q)}(\tilde{\nabla} f) \) is in \( L^p(\mathbb{B}_n, dv_\alpha). \)

Moreover, the quantities

\[
|f(0)| + \|A_{\gamma}^{(q)}(\mathcal{R}f)\|_{p,\alpha}, \quad |f(0)| + \|A_{\gamma}^{(q)}(\nabla f)\|_{p,\alpha}, \quad |f(0)| + \|A_{\gamma}^{(q)}(\tilde{\nabla} f)\|_{p,\alpha},
\]

are all comparable to \( \|f\|_{p,\alpha}, \) where the comparable constants depend only on \( \gamma, q, \alpha, p, \) and \( n. \)

In particular, taking the equivalence of (a) and (b), one obtains

\[
\|f\|_{p,\alpha} \approx |f(0)| + \|A_{\gamma}^{(q)}(\mathcal{R}f)\|_{p,\alpha},
\]

which looks tantalizingly simple. However, the authors know no simple proof of this fact even in the case of the usual Bergman space on the unit disc.

Corollary 4.2. Suppose \( \gamma > 0, 1 < q < \infty, \) and \( \alpha \in \mathbb{R}. \) Let \( 0 < p < \infty \) and \( k \) be a nonnegative integer such that \( pk + \alpha > -1. \) Then for any \( f \in \mathcal{H}(\mathbb{B}_n), \)

\[
A_{\gamma}^{(q)}(\mathcal{R}^k f)(z) = \left( \int_{D(z,\gamma)} (1 - |w|^2)^k \mathcal{R}^k f(w) |^q d\tau(w) \right)^{\frac{1}{q}}.
\]

Moreover,

\[
\|f\|_{p,\alpha} \approx |f(0)| + \|A_{\gamma}^{(q)}(\mathcal{R}^{k+1} f)\|_{p,\alpha},
\]

where "\( \approx \)" depends only on \( \gamma, q, \alpha, p, k, \) and \( n. \)

To prove Corollary 4.2, one merely notices that \( f \in \mathcal{A}_p^q \) if and only if \( \mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_{\alpha + pk}) \) and applies Theorem 4.2 to \( \mathcal{R}^k f \) with the help of Lemma 2.2.

4.3. Tent spaces. The basic functional used below is the one mapping functions in \( \mathbb{B}_n \) to functions in \( \mathbb{B}_n, \) given by

\[
A_{\gamma}^{(q)}(f)(z) = \left( \int_{D(z,\gamma)} |f(w)|^q d\tau(w) \right)^{\frac{1}{q}}
\]

if \( 1 < q < \infty, \) and

\[
A_{\gamma}^{(\infty)}(f)(z) = \sup_{w \in D(z,\gamma)} |f(w)|, \quad \text{when } q = \infty.
\]

Then, the "holomorphic space of tent type" \( \mathcal{T}_{q,\alpha}^p \) in \( \mathbb{B}_n \) is defined as the holomorphic functions \( f \) in \( \mathbb{B}_n \) so that \( A_{\gamma}^{(q)}(f) \in L^p_0, \) when \( 0 < p \leq \infty \) and \( \alpha > -1, \gamma > 0, 1 < q \leq \infty. \) The corresponding classes are then equipped
with a norm (or, quasi-norm) \( \| f \|_{T_{p,\alpha}^q} = \| A^{(q)}_\gamma(f) \|_{p,\alpha} \). This motivation aries from the tent spaces in \( \mathbb{R}^n \), which were introduced and developed by Coifman, Meyer and Stein in [9].

The case of \( q = \infty \) and \( 0 < p < \infty \) was studied in Section 4.1 (see [7] for details). Actually, the resulting tent type spaces \( T^\infty_{q,\alpha} \) is Bergman spaces \( \mathcal{A}_\alpha^p \). It is clear that \( T^\infty_{q,\alpha} \) with \( 1 < q < \infty \) is imbedded in Bloch space. On the other hand, \( T^p_{q,\alpha} \) are Banach spaces when \( p \geq 1 \).

It is well known that the Hardy-Littlewood maximal function operator has played important role in harmonic analysis. To cater our estimates, we use two variants of the non-central Hardy-Littlewood maximal function operator acting on the weighted Lebesgue spaces \( L^p_n(B_n) \), namely,

\[
(4.9) \quad M^{(q)}_\gamma(f)(z) = \sup_{z \in D(w,\gamma)} \left( \frac{1}{v_\alpha(D(w,\gamma))} \int_{D(w,\gamma)} |f|^q dv_\alpha \right)^{\frac{1}{q}}
\]

for \( 0 < q < \infty \). We simply write \( M_\gamma(f)(z) := M^{(1)}_\gamma(f)(z) \).

The following result is proved in [8].

**Theorem 4.3.** Suppose \( \gamma > 0, 0 < q < \infty, \) and \( \alpha > -1 \). Let \( 0 < p < \infty \). Then for any \( f \in \mathcal{H}(B_n) \), the following conditions are equivalent:

1. \( f \in \mathcal{A}_\alpha^p \).
2. \( A^{(q)}_\gamma(f) \) is in \( L^p(B_n, dv_\alpha) \).
3. \( M^{(q)}_\gamma(f) \) is in \( L^p(B_n, dv_\alpha) \).

Moreover,

\[
\| f \|_{\mathcal{A}_\alpha^p} \approx \| f \|_{T^p_{q,\alpha}} \approx \| M^{(q)}_\gamma(f) \|_{p,\alpha},
\]

where the comparable constants depend only on \( \gamma, q, \alpha, p, \) and \( n \).

Note that the Bergman metric \( \beta \) is non-doubling on \( B^n \) and so \( (B_n, \beta, dv_\alpha) \) is a non-homogeneous space. The proof of the above theorem does involve some techniques of non-homogeneous harmonic analysis developed in [15].

**Corollary 4.3.** Suppose \( \gamma > 0, 1 < q < \infty, \) and \( \alpha \in \mathbb{R} \). Let \( 0 < p < \infty \) and \( k \) be a nonnegative integer such that \( pk + \alpha > -1 \). Then for any \( f \in \mathcal{H}(B_n) \), \( f \in \mathcal{A}_\alpha^p \) if and only if \( A^{(q)}_\gamma(\mathcal{R}^k f) \) is in \( L^p(B_n, dv_\alpha) \) if and only if \( M^{(q)}_\gamma(\mathcal{R}^k f) \) is in \( L^p(B_n, dv_\alpha) \), where

\[
(4.10) \quad A^{(q)}_\gamma(\mathcal{R}^k f)(z) = \left( \int_{D(z,\gamma)} \left( 1 - |w|^2 \right)^k \mathcal{R}^k f(w) d\tau(w) \right)^{\frac{1}{q}}
\]

and

\[
(4.11) \quad M^{(q)}_\gamma(\mathcal{R}^k f)(z) = \sup_{z \in D(w,\gamma)} \left( \int_{D(w,\gamma)} \left| 1 - |u|^2 \right|^k \mathcal{R}^k f(u) \left( \frac{dv_\alpha(u)}{v_\alpha(D(w,\gamma))} \right) \right)^{\frac{1}{q}}
\]
Moreover,
\begin{equation}
\|f\|_{p,\alpha} \approx |f(0)| + \|A^{(q)}(\mathcal{R}^k f)\|_{p,\alpha} \approx |f(0)| + \|M^{(q)}_\gamma(\mathcal{R}^k f)\|_{p,\alpha},
\end{equation}
where \(\approx\) depends only on \(\gamma, q, \alpha, p, k,\) and \(n.\)

To prove Corollary 4.3, one merely notices that \(f \in A^\alpha_p\) if and only if \(\mathcal{R}^k f \in L^p(B_n, d\nu_{\alpha+pki})\) and applies Theorem 4.3 to \(\mathcal{R}^k f\) with the help of Lemma 2.2. When \(\alpha > -1,\) we can take \(k = 1\) and then recover Theorem 4.2.

As mentioned in Section 2, the family of the generalized Bergman spaces \(A^\alpha_p\) covers most of the spaces of holomorphic functions in the unit ball of \(\mathbb{C}^n,\) such as the classical diagonal Besov space \(B^s_p\) and the Sobolev space \(W^{p,k}_\beta.\)

In particular, \(H^s_p = A^\alpha_{p-1}\) with \(\alpha = -2s - 1,\) where \(H^s_p\) is the Hardy-Sobolev space defined as the set
\[
\left\{ f \in H(B_n) : \|f\|_{H^s_p}^p = \sup_{0 < r < 1} \int_{S_n} |(I + \mathcal{R})^sf(r\zeta)|^p d\sigma(\zeta) < \infty \right\}.
\]
Here,
\[
(I + \mathcal{R})^sf = \sum_{k=0}^{\infty} (1+k)^sf_k
\]
if \(f = \sum_{k=0}^{\infty} f_k\) is the homogeneous expansion of \(f.\) There are several real-variable characterizations of the Hardy-Sobolev spaces obtained by Ahern and Bruna [1] (see also [3]). These characterizations are in terms of maximal and area functions on the admissible approach region
\[
D_\alpha(\eta) = \left\{ z \in B_n : |1 - (z, \eta)| < \frac{\alpha}{2}(1 - |z|^2) \right\}, \quad \eta \in S_n, \quad \alpha > 1.
\]
Evidently, Corollary 4.3 present new real-variable descriptions of the Hardy-Sobolev spaces in terms of the Bergman metric. A special case of this is a characterization of the usual Hardy space \(H^p = A^p_{p-1}\) itself.

5. Bergman integral operators

5.1. Vector-valued kernels and Calderón-Zygmund operators on homogeneous spaces. Recall that a quasimetric on a set \(X\) is a map \(\rho\) from \(X \times X\) to \([0, \infty)\) such that

1. \(\rho(x, y) = 0\) if and only if \(x = y;\)
2. \(\rho(x, y) = \rho(y, x);\)
3. there exists a positive constant \(C \geq 1\) such that
\[
\rho(x, y) \leq C[\rho(x, z) + \rho(z, y)], \quad \forall x, y, z \in X,
\]
(the quasi-triangular inequality).

For any \(x \in X\) and \(r > 0,\) the set \(B(x, r) = \{ y \in X : \rho(x, y) < r \}\) is called a \(\rho\)-ball of center \(x\) and radius \(r.\)

A space of homogeneous type is a topological space \(X\) endowed with a quasimetric \(\rho\) and a Borel measure \(\mu\) such that
(a) for each $x \in X$, the balls $B(x, r)$ form a basis of open neighborhoods of $x$ and, also, $\mu(B(x, r)) > 0$ whenever $r > 0$;
(b) (doubling property) there exists a constant $C > 0$ such that for each $x \in X$ and $r > 0$, one has
\[
\mu(B(x, 2r)) \leq C \mu(B(x, r)).
\]

$(X, \rho, \mu)$ is called a space of homogeneous type or simply a homogeneous space. We will usually abusively call $X$ a homogeneous space instead of $(X, \rho, \mu)$. We refer to [11, 16] for details on harmonic analysis on homogeneous spaces.

Let $E$ be a Banach space. Let $L^p(\mu, E)$ be the usual Bochner-Lebesgue space for $1 \leq p \leq \infty$, and let $L^{1,\infty}(\mu, E)$ be defined by
\[
L^{1,\infty}(\mu, E) := \{ f : X \mapsto E | f \text{ is strongly measurable such that } \|f\|_{L^{1,\infty}(\mu, E)} < \infty \},
\]
where $\|f\|_{L^{1,\infty}(\mu, E)} := \sup_{t>0} t \mu(\{x \in X : \|f(x)\|_E > t\})$. Note that $\|f\|_{L^{1,\infty}}$ is not actually a norm in the sense that it does not satisfy the triangle inequality. However, we still have
\[
\|cf\|_{L^{1,\infty}(\mu, E)} = |c| \|f\|_{L^{1,\infty}(\mu, E)} \text{ and } \|f + g\|_{L^{1,\infty}(\mu, E)} \leq 2(\|f\|_{L^{1,\infty}(\mu, E)} + \|g\|_{L^{1,\infty}(\mu, E)})
\]
for every $c \in \mathbb{C}$ and $f, g \in L^{1,\infty}(\mu, E)$.

If $E = \mathbb{C}$ we simply write $L^p(\mu, E) = L^p(\mu)$ and $L^{1,\infty}(\mu, E) = L^{1,\infty}(\mu)$.

Fix $m > 0$ (not necessarily an integer). Define $\triangle = \{(x, x) : x \in X\}$. A vector-valued $m$-dimensional Calderón-Zygmund kernel with respect to $\rho$ is a continuous mapping $K : X \times X \setminus \triangle \mapsto E$ for which we have
(a) there exists a constant $C_1 > 0$ such that
\[
\|K(x, y)\|_E \leq \frac{C_1}{\rho(x, y)}, \quad \forall x, y \in X \times X \setminus \triangle;
\]
(b) there exist constants $0 < \varepsilon \leq 1$ and $C_2, C_3 > 0$ such that
\[
\|K(x, y) - K(x', y)\|_E + \|K(y, x) - K(y, x')\|_E \leq C_2 \frac{\rho(x, x')^{\varepsilon}}{\rho(x, y)^{m+\varepsilon}}
\]
whenever $x, x', y \in X$ and $\rho(x, x') \leq C_3 \rho(x, y)$.

Given a vector-valued $m$-dimensional Calderón-Zygmund kernel $K$, we can define (at least formally) a Calderón-Zygmund singular integral operator associated with this kernel by
\[
Tf(x) = \int_X K(x, y)f(y)d\mu(x).
\]

**Proposition 5.1.** Let $E$ be a Banach space. If a Calderón-Zygmund singular integral operator $T$ is bounded from $L^q(\mu)$ into $L^q(\mu, E)$ for some fixed $1 \leq q < \infty$, then $T$ can be extended to an operator on $L^p(\mu)$ for every $1 \leq p < \infty$ such that
(a) $T$ is $L^p$-bounded for every $1 < p < \infty$, i.e., $\|Tf\|_{L^p(\mu, E)} \leq C_p \|f\|_{L^p(\mu)}$;
(b) $T$ is of weak type $(1,1)$, i.e., $\|Tf\|_{L^{1,\infty}(\mu, E)} \leq C_p \|f\|_{L^{1}(\mu)}$ for all $f \in L^1(\mu)$;
(c) $T$ is bounded from $L^\infty(\mu)$ into $\text{BMO}(X, \rho, \mu; E)$;
(d) $T$ is bounded from $H^1(X, \rho, \mu)$ into $L^1(\mu)$.

$H^1(X, \rho, \mu)$ and $\text{BMO}(X, \rho, \mu; E)$ can be defined in a natural way, see [11] for the details. This result must be known for experts in the field of vector-valued harmonic analysis, and the proof can be obtained by merely modifying the proof of Theorem V.3.4 in [13].

5.2. Bergman singular integral operators. We now turn our attention to the special case of the unit ball $B_n$. Recall that

$$
\varrho(z, w) = \begin{cases} 
|z| - |w| + \frac{1 - |\langle z, w \rangle|}{|z||w|}, & \text{if } z, w \in \mathbb{B}_n \setminus \{0\}, \\
|z| + |w|, & \text{otherwise.}
\end{cases}
$$

It is known that $\varrho$ is a pseudo-metric on $\mathbb{B}_n$ and $(\mathbb{B}_n, \varrho, v_\alpha)$ is a homogeneous space for $\alpha > -1$ (e.g., Lemma 2.10 in [17]).

Let $E$ be a Banach space. Suppose $\alpha > -1$. We are interested in vector-valued Bergman type integral operators on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. More precisely, we are interested in Bergman type integral operators whose kernels with values in $E$ satisfy the following estimates

$$
(5.1) \quad \|K(z, w)\|_E \leq \frac{C}{\varrho(z, w)^{n+1+\alpha}}, \quad \forall (z, w) \in \mathbb{B}_n \times \mathbb{B}_n \setminus \{(\zeta, \zeta) : \zeta \in \mathbb{B}_n\},
$$

and

$$
(5.2) \quad \|K(z, w) - K(z, \zeta)\|_E + \|K(w, z) - K(\zeta, z)\|_E \leq \frac{C \varrho(w, \zeta)^\beta}{\varrho(z, \zeta)^{n+1+\alpha+\beta}},
$$

for $z, w, \zeta \in \mathbb{B}_n$ so that $\varrho(z, \zeta) \geq \delta \varrho(w, \zeta)$, with some (fixed) $\alpha > -1, \delta > 0$, and $0 < \beta \leq 1$. That is, $K$ is $(n + 1 + \alpha)$-dimensional Calderón-Zygmund kernel $K$ with values in $E$ on the homogeneous space $(\mathbb{B}_n, \varrho, v_\alpha)$.

Once the kernel has been defined, then a $\alpha$-time Bergman singular integral operator $T$ is defined as a Calderón-Zygmund singular integral operator with a vector-valued kernel $K$ by

$$
T f(z) = \int_{\mathbb{B}_n} K(z, w) f(w) dv_\alpha(w), \quad z, w \in \mathbb{B}_n.
$$

If $T$ is bounded from $L^p_\alpha$ into $L^p(\mathbb{B}_n, v_\alpha; E)$ for any $1 < p < \infty$, we call it a $\alpha$-time Bergman integral operator (BIO). We denote by $\text{BIO}_\alpha(E)$ all such operators. If $E = \mathbb{C}$ we write $\text{BIO}_\alpha(\mathbb{C}) = \text{BIO}_\alpha$.

The examples that we keep in mind are the Bergman projection operator $P_\alpha$ from $L^2_\alpha$ onto $A^2_\alpha$, which can be expressed as

$$
P_\alpha f(z) = \int_{\mathbb{B}_n} K_\alpha(z, w) f(w) dv_\alpha(w), \quad \forall f \in L^1(\mathbb{B}_n, dv_\alpha),
$$

where

$$
(5.4) \quad K_\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n \text{ with } \alpha > -1.
$$
Indeed, we have

**Proposition 5.2.** (Proposition 2.13 in [17])

(i) there exists a constant $C_1 > 0$ such that

$$|K_\alpha(z, w)| \leq \frac{C_1}{\varrho(z, w)^{n+1+\alpha}}, \quad \forall z, w \in \mathbb{B}_n.$$  

(ii) There are two constants $C_2, C_3 > 0$ such that for all $z, w, \zeta \in \mathbb{B}_n$ satisfying

$$\varrho(z, \zeta) > C_2 \varrho(w, \zeta)$$

one has

$$|K_\alpha(z, w) - K_\alpha(z, \zeta)| \leq C_3 \frac{\varrho(w, \zeta)^{\frac{1}{2}}}{\varrho(z, \zeta)^{n+1+\alpha+\frac{1}{2}}}.$$  

It is well known that $P_\alpha$ extends to a bounded operator on $L^p_{\alpha}$ for $1 < p < \infty$ (e.g., Theorem 2.11 in [20]). Thus, we have $P_\alpha \in \text{BIO}_\alpha$. This fact will be also concluded from the following result, which is clearly a special case of Proposition 5.1.

**Theorem 5.1.** Let $E$ be a Banach space and $\alpha > -1$. Suppose $T$ is a Calderón-Zygmund singular integral operator associated with a kernel satisfying (5.1) and (5.2). If $T$ is bounded on $L^q(v_\alpha)$ for some fixed $1 < q < \infty$, then $T$ is bounded from $L^p(E)$ into $L^p(E)$ for every $1 < p < \infty$, and is of weak type $(1, 1)$.

5.3. Area functions as vector-valued Bergman integral operators.

Given $\gamma > 0$ and $1 < q < \infty$. Let $E = L^q(\mathbb{B}_n, \chi_{D(0,\gamma)} d\tau)$. We consider the operator

$$(5.5) \quad [T_{\text{tent}} f(z)](w) = \int_{\mathbb{B}_n} \frac{f(u) d\nu_\alpha(u)}{(1 - \langle \varphi_z(w), u \rangle)^{n+1+\alpha}}, \quad \forall z, w \in \mathbb{B}_n,$$

with the kernel

$$K_{\text{tent}}(z, u)(w) = \frac{1}{(1 - \langle \varphi_z(w), u \rangle)^{n+1+\alpha}}.$$  

By the reproduce kernel formula (e.g., Theorem 2.2 in [20]) we have

$$[T_{\text{tent}} f(z)](w) = f(\varphi_z(w)), \quad \forall f \in \mathcal{H}(\mathbb{B}_n),$$

and hence

$$\|T_{\text{tent}} f(z)\|_E = A_{\gamma}^{(q)}(f)(z),$$

for any $f \in \mathcal{H}(\mathbb{B}_n)$.

**Theorem 5.2.** Let $\gamma > 0, \alpha > -1, 1 < q < \infty$, and $1 < p < \infty$. Then $T_{\text{tent}} \in \text{BIO}_\alpha(E)$. Consequently,

$$\|A_{\gamma}^{(q)}(f)\|_{L^p_{\alpha}} \lesssim \|f\|_{L^p_{\alpha}}, \quad \forall f \in \mathcal{A}_\alpha^{(p)}(\mathbb{B}_n).$$
Proof. Let $f \in L^q_\alpha(B_n)$. Then
\[
\|T_{\text{tent}} f\|_{L^q(v_\alpha, E)}^q = \int_{B_n} \int_{B_n} \left| \int_{B_n} f(u)dv_\alpha(u) \right|^q (1 - \langle \varphi_{\tau}(w), u \rangle)^{n+1+\alpha} \chi_{D(0,\gamma)}(w)d\tau(w)dv_\alpha(z)
\]
\[
= \int_{B_n} \int_{B_n} |P_\alpha f(w)|^q \chi_{D(z,\gamma)}(w)d\tau(w)dv_\alpha(z)
\]
\[
\approx \|P_\alpha f\|_{L^q_\alpha}^q \lesssim \|f\|_{L^q_\alpha}^q
\]
by the $L^q$-boundedness of $P_\alpha$ for $1 < q < \infty$. This concludes that $T_{\text{tent}}$ is bounded from $L^q_\alpha$ into $L^q(v_\alpha, E)$.

By Theorem 5.1, it remains to show that $K_{\text{tent}}$ satisfies the conditions (5.1) and (5.2). It is easy to check the condition (5.1). Indeed, by Lemmas 2.1 and 2.3 and Proposition 5.2 (i) we have
\[
\|K_{\text{tent}}(z, u)\|_E = \left( \int_{B_n} \frac{1}{|1 - \langle \varphi_{\tau}(w), u \rangle|^{2(n+1+\alpha)}\chi_{D(0,\gamma)}(w)d\tau(w)} \right)^{\frac{1}{2}}
\]
\[
= \left( \int_{D(z,\gamma)} \frac{1}{|1 - \langle w, u \rangle|^{2(n+1+\alpha)}d\tau(w)} \right)^{\frac{1}{2}}
\]
\[
\lesssim \frac{1}{|1 - \langle z, u \rangle|^{n+1+\alpha}}
\]
\[
\leq \frac{C}{\varrho(z, u)^{n+1+\alpha}}.
\]
This concludes that $K_{\text{tent}}$ satisfies (5.1).

To check the condition (5.2), we need the following variant of Proposition 5.2 (ii).

Lemma 5.1. There exist two constants $C_1, C_2 > 0$ such that for all $z, u, \zeta \in B_n$ satisfying
\[
\varrho(z, \zeta) > C_1 \varrho(u, \zeta)
\]
one has
\[
|K_\alpha(w, u) - K_\alpha(w, \zeta)| \leq C_2 \frac{\varrho(u, \zeta)^{\frac{1}{2}}}{\varrho(z, \zeta)^{n+1+\alpha+\frac{1}{2}}},
\]
for all $w \in D(z, \gamma)$.

The proof can be obtained by slightly modifying the proof of Proposition 2.13 (2) in [17] with the help of Lemmas 2.2 and 2.3. We omit the details.
Now we turn out to proceed our proof. Suppose \( z, u, \zeta \in \mathbb{B}_n \). Note that
\[
\|K_{\text{tent}}(z,u) - K_{\text{tent}}(z,\zeta)\|_E
= \left( \int_{\mathbb{B}_n} \left| \frac{1}{(1 - \langle \varphi_z(w), u \rangle)^{n+1+\alpha} - \frac{1}{(1 - \langle \varphi_z(w), \zeta \rangle)^{n+1+\alpha}} \right|^2 \chi_D(0,\gamma)(w) d\tau(w) \right)^{\frac{1}{2}}
= \left( \int_{\chi_D(0,\gamma)} \left| \frac{1}{(1 - \langle w, u \rangle)^{n+1+\alpha} - \frac{1}{(1 - \langle w, \zeta \rangle)^{n+1+\alpha}} \right|^2 d\tau(w) \right)^{\frac{1}{2}}
= \left( \int_{\chi_D(0,\gamma)} |K_{\alpha}(w,u) - K_{\alpha}(w,\zeta)|^2 d\tau(w) \right)^{\frac{1}{2}}.
\]
Then, by Lemma 5.1 there exist two constants \( C_1, C_2 > 0 \) such that for all \( z, u, \zeta \in \mathbb{B}_n \),
\[
\|K_{\text{tent}}(z,u) - K_{\text{tent}}(z,\zeta)\|_E \leq C_2 \frac{\varrho(u,\zeta)^{\frac{1}{2}}}{\varrho(z,\zeta)^{n+1+\alpha+\frac{1}{2}}}
\]
whenever \( \varrho(z,\zeta) > C_1 \varrho(u,\zeta) \).

On the other hand, since
\[
\|K_{\text{tent}}(u,z) - K_{\text{tent}}(\zeta,z)\|_E
= \left( \int_{\mathbb{B}_n} \left| \frac{1}{(1 - \langle \varphi_u(w), z \rangle)^{n+1+\alpha} - \frac{1}{(1 - \langle \varphi_u(w), \zeta \rangle)^{n+1+\alpha}} \right|^2 \chi_D(0,\gamma)(w) d\tau(w) \right)^{\frac{1}{2}}
= \left( \int_{\chi_D(0,\gamma)} |K_{\alpha}(\varphi_u(w),z) - K_{\alpha}(\varphi_u(w),\zeta)|^2 d\tau(w) \right)^{\frac{1}{2}}
\]
and
\[
\varrho(\varphi_u(w), \varphi_\zeta(w)) \lesssim |1 - \langle \varphi_u(w), \varphi_\zeta(w) \rangle| \approx |1 - \langle u, \zeta \rangle|, \quad \forall w \in D(0,\gamma),
\]
by Lemma 2.3 and the inequality
\[
\varrho(z,w) \lesssim |1 - \langle z, w \rangle|
\]
(e.g., Eq.(6) in [17]), then by slightly modifying the proof of Proposition 2.13 (2) in [17] we can prove that
\[
\|K_{\text{tent}}(u,z) - K_{\text{tent}}(\zeta,z)\|_E \lesssim \frac{\varrho(u,\zeta)^{\frac{1}{2}}}{\varrho(z,\zeta)^{n+1+\alpha+\frac{1}{2}}}.
\]
The details are left to readers. This completes the proof. \( \square \)

Evidently, we can define:

(i) The radial area integral operator
\[
T_{\text{radial}}f(z)(w) = \int_{\mathbb{B}_n} K_{\text{radial}}(z,u)(w)f(u)dv_\alpha(u), \quad \forall z, w \in \mathbb{B}_n,
\]
with the Bergman kernel
\[
K_{\text{radial}}(z,u)(w) = (n + 1 + \alpha) \frac{(1 - |\varphi_z(w)|^2)(\varphi_z(w),u)}{(1 - \langle \varphi_z(w), u \rangle)^{n+2+\alpha}}, \quad \forall z, u, w \in \mathbb{B}_n.
\]
It is easy to check that
\[[T_{\text{radial}}f(z)](w) = (1 - |\varphi_z(w)|^2)Rf(\varphi_z(w))\]
for any \(f \in \mathcal{H}(\mathbb{B}_n)\).

(ii) The complex gradient area integral operator
\[(5.7) \quad [T_{\text{grad}}f(z)](w) = \int_{\mathbb{B}_n} K_{\text{grad}}(z,u)(w)f(u)dv_\alpha(u), \quad \forall z, w \in \mathbb{B}_n,\]
with the Bergman kernel
\[K_{\text{grad}}(z,u)(w) = \frac{(n + 1 + \alpha)(1 - |\varphi_z(w)|^2)\bar{u}}{(1 - \langle \varphi_z(w), u \rangle)^{n+2+\alpha}}, \quad \forall z, u, w \in \mathbb{B}_n.\]

It is easy to check that
\[[T_{\text{grad}}f(z)](w) = (1 - |\varphi_z(w)|^2)\nabla f(\varphi_z(w))\]
for any \(f \in \mathcal{H}(\mathbb{B}_n)\).

(iii) The invariant gradient area integral operator
\[(5.8) \quad [T_{\text{invgrad}}f(z)](w) = \int_{\mathbb{B}_n} K_{\text{invgrad}}(z,u)(w)f(u)dv_\alpha(u), \quad \forall z, w \in \mathbb{B}_n,\]
with the Bergman kernel
\[K_{\text{invgrad}}(z,u)(w) = (n + 1 + \alpha)\frac{(1 - |\varphi_z(\omega)|^2)^{n+1+\alpha}\varphi_z(\omega)(u)}{|1 - \langle \varphi_z(\omega), u \rangle|^{2(n+1+\alpha)}}, \quad \forall z, u, w \in \mathbb{B}_n.\]

It is easy to check that
\[[T_{\text{invgrad}}f(z)](w) = \tilde{\nabla} f(\varphi_z(w))\]
for any \(f \in \mathcal{H}(\mathbb{B}_n)\).

Similarly, we have

**Theorem 5.3.** Let \(\gamma > 0, 1 < q < \infty,\) and \(\alpha > -1.\) Then \(T_{\text{radial}}, T_{\text{grad}},\) and \(T_{\text{invgrad}}\) are all in \(BIO_\alpha(E)\). Consequently, \(A_{\gamma}^{(q)}(\mathcal{R} f), A_{\gamma}^{(q)}(\nabla f),\) and \(A_{\gamma}^{(q)}(\tilde{\nabla} f)\) are all bounded on \(A_p^\alpha\) for every \(1 < p < \infty.\)

The proof is the same as that of Theorem 5.2 and the details are omitted.

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