Multi-cluster problems: resonances, scattering and condensed states

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Abstract. This talk is mainly concerned with many-body resonances in nuclear physics. We extensively discuss the structure and reactions of multi-cluster systems using the complex scaling method (CSM). We expound three interesting problems in recent studies of multi-cluster systems. First, we discuss four- and five-body resonances in \(^7\)He, \(^7\)B systems, respectively. The observed states are well explained and many additional states are predicted. Second, the Coulomb breakup reactions of two-neutron halo nuclei and the \(\alpha-d\) scattering are investigated using a three-body model with CSM. Finally, we discuss the \(\alpha\)-condensate-like states and their symplectic excitation properties in three- and four-\(\alpha\) models for \(^{12}\)C and \(^{16}\)O, respectively.

1. Introduction

Cluster states in stable nuclei are usually located near the threshold of the relevant cluster breakup. Thus the physics of clustering has always necessitated the proper treatment of resonances and sometimes even the description of relevant reaction processes. This necessity is more enhanced for cluster states of neutron-rich nuclei, because even when the states of neutron-rich nuclei are located deeply below the cluster breakup thresholds, they are often near the neutron dissociation threshold. For the investigation of resonances over a wide region of the complex energy plane, the complex scaling method (CSM) [1, 2] has been used as a very powerful method in the description of many-body resonances.

A great advantage of CSM is that we can obtain resonance poles in the complex energy plane in the same way as a standard bound state owing to the \(L^2\) property of the resonance wave functions. In this way many-body systems can be treated as straightforward extensions of two-body calculations. We show the applications of CSM to the \(^4\)He+\(XN\) model [3, 4, 5, 6, 7], where \(N\) and \(X\) indicate a nucleon (neutron or proton) and their number \((X = 1, 2, 3, 4)\).

CSM transforms the scattering states on the positive energy axis into two separate groups: resonant states and residual continuum states on the rotated cuts. This is also an important advantage of CSM. We explain the extended completeness relation (ECR) [8], consisting of...
bound, resonant and rotated continuum states. For many-body systems, the rotated continuum states are further separated into sets of states lying on various rotated continua starting from different eigen-energies of subsystems. We apply CSM to the Green’s function, and extensively use the complex scaled Green’s function in calculations of scattering and reaction cross sections. In this paper, the Coulomb breakup cross section of two-neutron halo systems [9] and $\alpha + d$ scattering phase shifts [11] are discussed.

Furthermore, studies of the $\alpha$-condensate-like states of $^{12}\text{C}$ and $^{16}\text{O}$ as many-body resonances are also presented. For these many-$\alpha$ systems, the description of various excited states, by symplectic basis truncations, is shown.

2. Many-body Resonant States

2.1. Complex Scaling Method (CSM)

The wave function of a resonant state with a complex energy diverges exponentially in an asymptotic distance region as shown in Fig. 1. Then the norm and matrix elements cannot be defined in the usual manner. The difficulties of resonant states due to singularity at infinity have been overcome by regularization procedures, such as CSM [1, 2]. The complex scaling consists in the following transformation of the Jacobi coordinates and their conjugate momenta:

$$U(\theta) : \begin{align*} r_i &\rightarrow r_i e^{i\theta}, \\ p_i &\rightarrow p_i e^{-i\theta}, \end{align*} \quad (i = 1, \ldots, f),$$

where $\theta$ is a scaling angle common ($0 \leq \theta < \theta_{\text{max}}$) for all coordinates of the $f$ degrees of freedom. The maximum value $\theta_{\text{max}}$ is determined so as to keep the analyticity of the transformed function; for instance $\theta_{\text{max}} = \pi/4$ for a Gaussian potential.

CSM gives us a very simple way to construct resonant states by using the $L^2$ basis functions. Therefore, we can easily apply CSM to many-body systems. Diagonalizing the complex-scaled Hamiltonian $H(\theta) = U(\theta)HU(\theta)^{-1}$, we obtain resonant solutions with complex energies $E_R - i\Gamma/2$, which are stationary against variations of the $\theta$ value. An advantage of CSM is that matrix elements of a physical operator $\hat{O}$ can be calculated as

$$\langle O \rangle = \langle \tilde{\psi}_R^\theta | U(\theta)\hat{O} U(\theta)^{-1} | \psi_R^\theta \rangle,$$

where $\tilde{\psi}_R^\theta$ and $\psi_R^\theta$ are conjugate pairs in the bi-orthogonal basis. The matrix elements are generally complex, although the imaginary parts are small when the width of the resonant state is small. Then we can see a correspondence between the real part of the matrix element and the observed value. However, when the width is larger than the resonance energy, the matrix element does not correspond directly to an observable. We need to develop a new method to relate the complex energy resonant states to the physical states. In section 4, we show that

![Figure 1. Wave functions of a resonant state (a) before and (b) after complex scaling. Solid and broken lines are the real and imaginary parts of the wave function, respectively.](image)
the observed cross section can be calculated by complex resonance solutions and non-resonant continuum states in CSM.

2.2. Many-body decaying resonant states
As examples of many-body decaying resonances, we present our recent studies [3, 4, 5, 6, 7] on the neutron-rich He isotopes (A=5–8) and their proton-rich mirror nuclei. Those systems are described in the models of the α-core plus nucleons (4He+X N) as shown in Fig. 2. The Hamiltonian is given by

\[
H = \sum_{i=1}^{X+1} t_i - T_{cm} + \sum_{i=1}^{X} V_i^{\alpha N} + \sum_{i<j}^{X} V_{ij}^{NN} \\
= \sum_{i=1}^{X} \left[ \frac{p_i^2}{2\mu} + V_i^{\alpha N} \right] + \sum_{i<j}^{X} \left[ \frac{p_i \cdot p_j}{4M} + V_{ij}^{NN} \right],
\]

The second line contains the form used in the cluster orbital shell model (COSM) [12]. For the details of the interactions V_\alpha N_i and V_{ij} NN, see references [3, 4, 5, 6, 7]. The mirror nuclei are described by replacing valence neutrons with protons.

The results are presented in Fig. 3. The energy is measured from the threshold of separation into ^4He and nucleons. For A=5 (^5He, ^5Li), A=6 (^6He, ^6Be), A=7 (^7He, ^7B) and A=8 (^8He, ^8C) systems, two-, three-, four- and five-body decay resonances are described, respectively, under the correct boundary conditions. The observed bound and resonant states are well reproduced, and many theoretical predictions are made. Comparing spectra of the He isotopes and their mirror nuclei, we see good mirror symmetry. In ^7He, we found a broad 1/2− resonance at the low excitation energy of E_x = 1.05 MeV, while the experimental uncertainty is still large. Recently, there is a report [13] discussing the order of energy levels, which fully agrees with our result. The energy of the ^7B ground state is obtained to be E_r = 3.35 MeV, which agrees with the recently observed value, 3.38 MeV [14]. For A=8 nuclei, only 0^+ states have been calculated.
because a huge number of basis states are required by states with higher spins. The $0^+_1$ state of $^8$He is predicted to be a five-body resonance dominated by the $(p_{3/2})^2(p_{1/2})^2$ configuration, with an excitation energy of 6.3 MeV and a width of 3.2 MeV.

3. Extended completeness relation and complex scaled Green’s function

The result of the complex scaling [Eq. (1)] in the momentum and energy planes is illustrated in Fig. 4. A semicircle in the upper half momentum plane is inclined clockwise by an angle $\theta$, and then some resonance poles in the fourth quadrant are enclosed within the inclined semicircle. Integrating the Green’s function along the inclined semicircle [15], we obtain an extended completeness relation (ECR) [8]

$$1 = \sum_B^n |\psi^\theta_B \rangle \langle \tilde{\psi}^\theta_B| + \sum_R^n |\psi^\theta_R \rangle \langle \tilde{\psi}^\theta_R| + \int_{L_\theta} dk |\psi^\theta_{k^\theta} \rangle \langle \tilde{\psi}^\theta_{k^\theta}|,$$

(4)

where $\psi^\theta_B$, $\psi^\theta_R$ and $\psi^\theta_{k^\theta}$ are bound, resonant and continuum solutions of the complex-scaled Schrödinger equation, respectively. The momentum $\tilde{k}$ in the bra-states is $\tilde{k}^B,R = -k^*_{B,R}$ for bound (B) and resonant (R) states, and $\tilde{k}^\theta = k^\theta_{k^\theta}$ for continuum states. The continuum states $\psi^\theta_{k^\theta}$ are solutions of the complex-scaled Schrödinger equation for momenta $k^\theta$ along the branch cut $L_\theta$ rotated from the original position of the real momentum axis, and describe the broad resonance contributions from poles not included within the semicircle.

The above discussion is for a single channel case, but we can postulate a similar ECR for many-body systems as well:

$$1 = \sum_B^n |\psi^\theta_B \rangle \langle \tilde{\psi}^\theta_B| + \sum_R^n |\psi^\theta_R \rangle \langle \tilde{\psi}^\theta_R| + \sum_{cb} \int_{L_\theta(cb)} dk_{cb} |\psi^\theta_{k^\theta_{cb}} \rangle \langle \tilde{\psi}^\theta_{k^\theta_{cb}}| + \sum_{cr} \int_{L_\theta(cr)} dk_{cr} |\psi^\theta_{k^\theta_{cr}} \rangle \langle \tilde{\psi}^\theta_{k^\theta_{cr}}|,$$

(5)

where the suffices $cb$ and $cr$ represent different channels consisting of bound subsystems and resonant subsystems, respectively.

We take a Green’s function satisfying the outgoing boundary condition:

$$G^{(+)}(E; \vec{r}, \vec{r}') = \langle \vec{r}' | \frac{1}{E - H + i\epsilon} | \vec{r} \rangle.$$

Its complex-scaled version is given by

$$G^\theta(E; \vec{r}, \vec{r}') = \langle \vec{r}' | \frac{1}{E - H(\theta)} | \vec{r} \rangle.$$

(7)

Figure 4. The rotated Cauchy integral contours in the momentum and energy planes in CSM.
Here we used a superscript ‘θ’ instead of ‘(+’) and dropped iǫ from the operator on the right-hand side, because $H(θ)$ automatically satisfies the outgoing boundary condition due to CSM without any additional prescription.

4. Nuclear reactions to many-body final states
Here we discuss nuclear reactions leading to many-body final states, which satisfy outgoing boundary conditions. Such a many-body final state can be easily described by the complex scaling method. As examples we show two successful results for the Coulomb breakup reaction of a two-neutron halo nucleus and the $α$+deuteron scattering with the deuteron breakup and rearrangement processes taken into account.

4.1. Coulomb breakup reactions of two-neutron halo nuclei
The Coulomb breakup experiment is a unique method to investigate the exotic properties of neutron halo nuclei, which are weakly bound systems. Many studies have been done on the two-neutron halo nuclei $^6$He and $^{11}$Li, but a deep understanding of the relation between the exotic structure and characteristic reaction mechanism has not been obtained yet. Since the final states of the two-neutron halo breakup are three-body continuum states, it is very appropriate to apply CSM to the Coulomb breakup reactions.

The breakup cross section of the Coulomb response is assumed to be determined by $E1$ transitions and is described as

$$\frac{d\sigma}{dE} = \frac{16\pi^3}{9\hbar c} N_{E1}(E_\gamma) \frac{dB(E1)}{dE},$$

where $N_{E1}(E_\gamma)$ is the virtual photon number and

$$\frac{dB(E1)}{dE} = \frac{1}{2J_{gr} + 1} \sum_\nu \langle \tilde{\psi}_{gr} | \hat{E1} | \psi_\nu \rangle \langle \tilde{\psi}_\nu | \hat{E1} | \psi_i \rangle \delta(E - E_\nu)$$

$$= -\frac{1}{\pi} \text{Im} \left[ \frac{1}{2J_{gr} + 1} \sum_\nu \langle \tilde{\psi}_{gr}^\theta | \hat{E1}(\theta) | \psi_\nu^\theta \rangle \langle \tilde{\psi}_\nu^\theta | \hat{E1}(\theta) | \psi_i^\theta \rangle \right].$$

The last equation is calculated with CSM, where $\psi_{\nu}^\theta$ are solutions belonging to the eigenenergy $E_{\nu}^\theta$ and satisfy the outgoing boundary condition. It should be noted that the result is independent of the parameter $\theta$.

Figure 5. The $1^-$ energy eigenvalue distribution (left-hand panel) in the complex-scaled $^4$He+$n+n$ model and the Coulomb breakup reaction cross section (right-hand panel) $E1$ transition strength distributions. For details, see text.
In Fig. 5, we show the result of the calculated Coulomb breakup cross section for the two-neutron halo nucleus \(^6\)He using a \(^4\)He + \(n + n\) model [9]. In the left-hand panel, the eigenenergy distribution of \(1^-\) states obtained with \(\theta = 30^\circ\) is shown, where squares and triangles indicate the two-body continuum states of \(^5\)He(3\(^2^-\)) + \(n\) and \(^5\)He(1\(^2^-\)) + \(n\), respectively. Circles indicate the three-body continuum states of \(^4\)He + \(n + n\). In the right-hand panel, we show \(E1\) transition strength distributions, where dashed, dotted and dash-dotted lines are contributions from the \(^5\)He(3\(^2^-\)) + \(n\), \(^5\)He(1\(^2^-\)) + \(n\) and \(^4\)He + \(n + n\) continuum states, respectively. The thick solid line indicates the total strength distribution. The position of the arrow stands for the two-body threshold energy of the \(^5\)He(3\(^2^-\)) + \(n\) channel. We can see that the contribution from the continuum states of \(^5\)He(3\(^2^-\)) + \(n\) is dominant in the cross section, and the total strength well reproduces the observed Coulomb dissociation cross sections [10] of \(^6\)He as shown in Fig. 6.

Similar calculations for \(^{11}\)Li are presented by Kikuchi at this conference.

4.2. Beyond the continuum discretized coupled channels

The method of continuum discretized coupled channels (CDCC) has been employed in studies of nuclear reactions in which the projectile or the target nucleus is broken up [18]. Although the CDCC method is successful in two-body breakup problems, it is difficult to apply it to three-body breakup reactions of two-neutron halo nuclei because three-body scattering states are needed explicitly in order to construct the smoothing factors for the discretized cross sections. This problem has been solved recently by applying CSM to the calculation of the smoothing factors [19].

Another interesting problem in the CDCC method is its extension to various reactions including a rearrangement process. This problem has also been discussed by using CSM recently, and applied to the \(\alpha + d\) scattering [11]. Using ECR and the complex-scaled Green’s function, the \(\alpha + d\) scattering state is expressed as the solution of a Lippmann-Schwinger equation

\[
\langle \Psi(-) \rangle = \langle \Phi_0 \rangle + \sum_{\nu} \langle \Phi_0 \rangle \hat{V} U^{-1}(\theta) |\psi^{\nu}_0 \rangle \frac{1}{E - E^{\nu}_0} \langle \tilde{\psi}^{\nu}_0 | U(\theta),
\]

where \(\Phi_0\) is a solution of the asymptotic Schrödinger equation \(H_0 \Phi_0 = E \Phi_0\). The total Hamiltonian is given by \(H = H_0 + \hat{V}\), where \(\hat{V}\) is an interaction. When we employ an \(\alpha + p + n\) model with the asymptotic Hamiltonian belonging to the \(\alpha + d\) channel, the second term of Eq. (10) describes the deuteron breakup with intermediate rearrangements of \(p\) or \(n\).

In Fig. 7, we show results of phase shifts for the \(\alpha + d\) elastic scattering with \(J^\pi = 3^+\), 2\(^+\) and 1\(^+\). The solid lines are the calculations including all terms in Eq. (10). They reproduce the observed phase shifts well. The dashed lines show the results with only the deuteron breakup included. The interaction is obviously not attractive enough, but the overall trends are very similar to the solid lines. Furthermore, we elucidate the phase shifts including the elastic channel alone described by the first term in Eq. (10). They are presented by the dotted lines, which belonging to different spin values (seen at the bottom) almost coincide, and show no attractive behaviour.
From the results in Fig. 7, it is found that the deuteron breakup has a significant effect on the production of the resonances of $^{6}$Li in the $\alpha + d$ scattering, and the coupling to the rearrangement channels of $^{5}$He+$p$ and $^{5}$Li+$n$ has a sizable effect on the determination of the resonance positions.

Figure 7. Effects of deuteron breakup and rearrangement on the phase-shift results.

5. Alpha-condensate-like states in $^{12}$C and $^{16}$O
The $\alpha$-condensate-like states in $^{12}$C and $^{16}$O are typical examples of multi-cluster resonances in highly excited energy regions. Kurokawa [20, 21, 22] calculated many three-$\alpha$ resonances in the 3$\alpha$ orthogonality condition model (OCM). The three-$\alpha$ system is a Borromean system, which has no bound states in any of its subsystems. In $^{12}$C, except for the ground $0^+$ and an excited $2^+$ states, all excited states are resonant states above the three-$\alpha$ threshold. The structures of the observed levels of $^{12}$C have been thoroughly investigated and many resonant states are predicted as shown in Fig. 8[22]. The most interesting prediction is the $0^+_3$ state [21, 22] above the Hoyle ($0^+_2$) state, which has recently been observed by Itoh et al [24, 25]. It has turned out that this broad resonant state is like an excited Hoyle state.

The structure of the Hoyle state has been discussed in connection with $\alpha$-condensate-like states [27, 17]. This $0^+_2$ state has been shown to have a large monopole strength of transition from the ground state. This property is understood from the viewpoint of the cluster symplectic excitation. Yoshida has recently investigated the symplectic structure and monopole strength in $^{12}$C [28]. His recent multi-alpha cluster model calculations for $^{12}$C and $^{16}$O are presented in his talk at this conference.

6. Summary
In this talk, we discussed three problems of multi-cluster systems investigated recently. Since cluster states are usually located near the relevant breakup threshold, proper treatment of resonances and the description of the relevant reaction processes are required. To facilitate the discussion, we briefly explained the powerful method of complex scaling. As for the first problem, we discussed many-body resonances in He isotopes and their mirror nuclei using the $^4$He+$XN$ model. The completeness relation for the bound and scattering states were extended so as to separate the scattering states into resonant states and rotated continuum states by applying the complex scaling. The second problem to be dealt with was the three-body Coulomb breakup cross section and the extension of the continuum discretized coupled channels method. This was treated by using the extended completeness relation and the complex-scaled Green’s
Figure 8. Energy levels for the low-lying states of $^{12}\text{C}$ in comparison with the experiment and other $3\alpha$ theoretical calculations of the $3\alpha$ GCM [23]. Experimental values of the $2^+_2$ and $0^+_3$ states are given in [24, 25], and other experimental data are taken from [26].

function. The third group of problems discussed was that of the $\alpha$-condensate-like states and the symplectic excitations in the three- and four-$\alpha$ models for $^{12}\text{C}$ and $^{16}\text{O}$, respectively.

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