Chiral gravity in two dimensions

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ABSTRACT: It is shown that conformal matter with $c_L \neq c_R$ can be consistently coupled to two-dimensional 'frame' gravity. The theory is quantized, following David, and Distler and Kawai, using the derivation of their ansatz due to Mavromatos and Miramontes, and D’Hoker and Kurzepa. New super-selection rules are found by requiring SL(2,C) invariance of correlation functions on the plane. There is no analogue of the $c = 1$ barrier found in non-chiral non-critical strings. A non-critical heterotic string is constructed—it has 744 states in its spectrum, transforming in the adjoint representation of $(E_8)^3$. Correlation functions are calculated in this example.
1. Introduction

Conformal matter in two dimensions couples to quantum gravity via the conformal anomaly. The coupling is characterized by the central charge of the conformal matter. In Euclidean space, conformal matter can be constructed with different values of the central charge for holomorphic and anti-holomorphic fields. It is natural to ask how such theories, with $c_L \neq c_R$, interact with quantum gravity. Since these theories have a gravitational anomaly, in addition to the conformal anomaly, there must be degrees of freedom other than the conformal factor that become dynamical. This is related to an additional reason for interest in chiral gravity—it is widely believed that topological excitations ‘disorder’ the world-sheet geometry when induced gravity is coupled to matter with $c > 1$. The question then is what governs the dynamics of such degrees of freedom. Chiral gravity, of course, is naturally a theory with additional geometric degrees of freedom on the world-sheet, and its study may lead to insights into possible continuum descriptions of physics above $c = 1$. Indeed, we shall find that chiral gravity does involve disordered world-sheet geometry, in a precise manner.

Recall that local gravitational anomalies can be exchanged for local Lorentz anomalies. A theory with a gravitational anomaly has no generally covariant interpretation, but a theory with a Lorentz anomaly is a theory in which the gauge invariance associated with local frame rotations is absent. Thus, just as the conformal anomaly provides dynamics for the scale factor of the frame, the Lorentz anomaly provides dynamics for local Lorentz rotations. In two dimensions, this is particularly natural, for one has

$$e^\pm \to \exp (\rho \pm i\chi) e^\pm,$$

under combined scale ($\rho$) and Lorentz ($\chi$) transformations. It is the action that governs $\rho$ and $\chi$ that we shall study in this paper. We shall follow the David–Distler–Kawai (DDK) approach[1]—this has the virtue of being applicable to arbitrary topology, and the practical advantage of allowing the use of conformal field theory techniques. The geometric meaning of induced chiral gravity becomes somewhat more transparent as well.

This paper is organized as follows: in sect. 2, we set some conventions and normalizations, and remind the reader of the calculation of the DDK change-of-measure due to Mavromatos and Miramontes, and D’Hoker and Kurzepa[2,3]. We show that the analogous factor in the case of chiral gravity is the square of that in non-chiral gravity. In sect. 3 we use this to work out the conformal field theory representation of chiral gravity. In the next section we consider some global issues—it is shown that the conformal field theory action requires careful definition in order to preserve SL(2,C) invariance on the plane. In particular, the fields associated with the classical conformal or Lorentz modes are multivalued, with super-selection rules governing the deviation from being single-valued. In sect. 5 we discuss operators which are constructed purely from the fields in the gravity sector, and examine some of the critical exponents. In sect. 6 we give an example, the $j$-string, and compute the partition function and correlation functions in this example. Sect. 7 contains some concluding remarks.

We note that the coupling of chiral matter to quantum gravity was studied by Oz, Pawelczyk and Yankielowicz[4], who worked in the light-cone gauge introduced by Polyakov[5].
2. Review and conventions

We will work with the Euclidean theory in this paper. The frame is specified locally by two 1-forms, \( \{e^\pm\} \), with the metric

\[
g = \frac{1}{2}[e^+ \otimes e^- + e^- \otimes e^+].
\]

The spin connection is obtained from

\[
de^\pm \mp i\omega e^\pm = 0. \tag{2}
\]

The * operator is defined by

\[
*e^\pm = \mp ie^\pm, \quad *(e^+ e^-) = -2i, \quad *1 = \frac{i}{2}e^+ e^-.
\tag{3}
\]

The local area element is \( \frac{i}{2}e^+ e^- \).

Under Lorentz and conformal transformations, (1), \( \omega \) changes as

\[
\omega \rightarrow * \omega + d\rho + d\chi. \tag{4}
\]

The curvature is \( \mathcal{R} \equiv d\omega \), so \( \delta \mathcal{R} = -d * d\rho = * \Delta \rho \), where \( \Delta \equiv -d * d * - * d * d \), is the positive definite Laplacian. \( \mathcal{R} \) is invariant under Lorentz transformations, and is related to the scalar curvature \( R \) by \( \frac{i}{2}e^+ e^- R = 2\mathcal{R} \).

Peculiar to two dimensions is the fact that \( *\omega \) is a 1-form, and from (4) \( U \equiv d*\omega \) behaves much as \( \mathcal{R} \) does, with the roles of \( \rho \) and \( \chi \) reversed. Explicitly, \( \delta U = d * d\chi = - * \Delta \chi \). Therefore \( U \) is locally invariant under conformal transformations. For future reference define a scalar \( U \) via \( \frac{i}{2}e^+ e^- U \equiv 2U \). In terms of the spin connection, one has \( U = 2\nabla^\mu \omega_\mu \). The same arguments that imply that \( \int \mathcal{R} \) is a topological invariant imply that \( \int U \) is also an invariant. It is easy to see that \( \int U = 0 \), since one can always choose a frame with a divergenceless spin connection (see sect. 4). However, when \( \int \phi U \) occurs in an action, and \( \phi \) is not necessarily single-valued, surprising super-selection rules arise. The ‘connection’ \( *\omega \) corresponds to a reduction of the structure group of the frame bundle to multiplication by positive real numbers, just as \( \omega \) corresponds to a reduction to SO(2).

Integration over the matter degrees of freedom, using a diffeomorphism-invariant regularization, produces an effective action for the zweibein (see, \textit{e.g.}, ref. 6),

\[
S_0 = \frac{1}{24\pi} \left[ \int * \mathcal{R} \frac{1}{\Delta}(c R + ic \mathcal{U}) \right] \equiv \frac{c_+}{3} S_L - \frac{c_-}{3} S_U, \tag{5}
\]

where \( c_\pm = (c_R \pm c_L)/2 \). In background gauge, \( e^\pm = \exp(\rho \pm i\chi)\hat{e}^\pm \), the dependence of \( S_0 \) on the Weyl and Lorentz factors is purely local

\[
K_L(\rho) = S_L(\hat{e}, \rho, \chi) - S_L(\hat{e}, 0, 0) = \int \frac{d^2z}{2\pi} \left\{ \partial_\rho \bar{\partial}_\rho + \frac{1}{4} \sqrt{g} \mathcal{R} \rho \right\},
\]

\[
K_U(\rho, \chi) = S_U(\hat{e}, \rho, \chi) - S_U(\hat{e}, 0, 0) = \int \frac{d^2z}{2\pi} \left\{ -\partial_\rho \bar{\partial}_\chi - \frac{1}{8} \sqrt{g} \left( \mathcal{R} \chi - \mathcal{U} \rho \right) \right\}. \tag{6}
\]
An important (diffeomorphism-invariant) local counterterm can be added to the action, namely
\[ S_{\text{loc}} = \frac{1}{8\pi} \int \omega * \omega. \]

The dependence on the Weyl and Lorentz factors of this term is
\[ K_{\text{loc}}(\rho, \chi) = S_{\text{loc}}(\hat{e}, \rho, \chi) - S_{\text{loc}}(\hat{e}, 0, 0) = \int \frac{d^2 z}{2\pi} \left\{ \partial \rho \partial \rho + \partial \chi \partial \chi - \frac{1}{4} \sqrt{\hat{g}} (\hat{U} \chi - \hat{R} \rho) \right\}. \]

It is important to keep in mind that the effective action, (5), is derived from the anomalous conservation equation by integration. It follows therefore that a constant of integration can be added to the action, if needed. Also, this action is not globally defined on an arbitrary surface. Using (6,7), we shall produce a local conformal field theory expression that agrees with eq. (5) in background gauge, and then observe that the conformal field theory is well-defined on an arbitrary surface about suitable choices of background zweibein. As explained in sect. 4, additional requirements arise in the definition of the functional integral from the requirement of conformal invariance, that are not immediately apparent from a consideration of the action alone.

We now briefly describe the calculation of ref.’s 2,3. On a two-dimensional surface, \( \Sigma \), the non-critical non-chiral induced gravity partition function is
\[ Z \equiv \int \frac{Dg}{\text{vol.}(\text{Diff}_\Sigma)} \exp \left( \frac{c_m}{3} S_L[g] \right) \]

where \( c_m \) is the central charge of the non-chiral matter theory. Choosing conformal gauge, \( g_{\mu\nu} = e^{2\rho} \hat{g}_{\mu\nu}(m) \), and fixing diffeomorphisms à la Faddeev-Popov, this becomes
\[ Z = \int dm D\rho \exp \left( \frac{c_m - 26}{3} K_L(\rho) \right), \]

where \( dm \) stands for an integration over the moduli space of \( \Sigma \). The problem is now to understand the measure \( D\rho \), which is supposed to be the Riemannian measure induced by

\[ (\delta \rho, \delta \rho) = \int d^2 x e^{2\rho} \sqrt{\hat{g}} (\delta \rho)^2. \]

In ref.’s 2,3, this problem was tackled by comparing the problem to finite-dimensional vector space inner products and measures related by linear transformations. It follows that, if \( D_0 \rho \) is the translation-invariant measure induced by

\[ (\delta \rho, \delta \rho)_0 = \int d^2 x \sqrt{\hat{g}} (\delta \rho)^2, \]

one has \( D\rho = D_0 \rho \text{Det} \left( e^{\rho(z_1)+\rho(z_2)} \delta(z_1, z_2) \right) \). The functional determinant is singular, but a careful heat kernel regularization\[2\] using Ward identities for Weyl invariance\[3\] leads to

\[ \text{Det} = \exp \left( \frac{1}{3} K_L(\rho) \right). \]
In our case, we wish to perform an integration over all zweibein on a two-dimensional surface, Σ. The functional measure is defined in terms of the reparametrization invariant distance on the space of frames given by

\[ (\delta e, \delta e) = \int d^2 x |e| \frac{1}{2} (\delta_{ab} g^{\mu\nu} + \lambda_1 e_a^{\mu} e_b^{\nu} + \lambda_2 e_a^{\nu} e_b^{\mu}) \delta e_a^\mu \delta e_b^\nu. \]

Fixing diffeomorphisms by choosing a fiducial background frame, \( e^\pm = \exp(\rho \pm i\chi) \hat{e}^\pm \), leaves

\[ (\delta e, \delta e) = \int d^2 x |\hat{e}| e^{2\rho} \left[ (1 + 2\lambda_1 + \lambda_2) (\delta \rho)^2 + (1 - \lambda_2) (\delta \chi)^2 \right]. \]

This result is positive definite for \( \lambda_2 < 1 \) and \( 2\lambda_1 + \lambda_2 > -1 \). In the following, we will set \( \lambda_1 = 0 = \lambda_2 \) since nothing will depend on these constants. Thus variations of the conformal and Lorentz factors can be considered independently with the separate line elements

\[ (\delta \rho, \delta \rho) = \int d^2 x |\hat{e}| e^{2\rho} (\delta \rho)^2 \]

\[ (\delta \chi, \delta \chi) = \int d^2 x |\hat{e}| e^{2\rho} (\delta \chi)^2 \]

Changing (9) to the translation invariant measure for \( \rho \) is precisely the problem discussed above, and so contributes a factor of (8). The measure (10) on \( \chi \) is already translation invariant, and as usual, one can eliminate the \( \rho \) dependence by shifting the world-sheet metric by a conformal factor. This transformation yields the same factor (8). Hence it follows that the total determinant in this case is just the square of (8). It should not come as a surprise that the Lorentz factor does not appear in the determinant, since it does not affect the line element.

Therefore the partition function for chiral induced gravity is

\[ Z = \int \frac{D e}{\text{vol.}(\text{Diff}_\Sigma)} \exp \left( \frac{c_+}{3} S_L - i\frac{c_-}{3} S_U - \frac{\xi}{3} S_{\text{loc}} \right) \]

\[ = \int d\rho d\chi D_0 \rho D_0 \chi \exp \left( \frac{c_+ - 24}{3} K_L - i\frac{c_-}{3} K_U - \frac{\xi}{3} K_{\text{loc}} \right), \]

where \( d\rho \) stands for a possible integration over additional Lorentz moduli, discussed in sect. 4. Note especially that the measure in (11) is not divided by the volume of the group of Lorentz transformations—such a division would be incorrect when \( c_L \neq c_R \), since the theory is not invariant under Lorentz transformations in this case. This is exactly analogous to the difference between the non-critical and critical string measures. Also note that for the purposes of the present discussion, we have assumed that a local counterterm has been introduced to produce a vanishing cosmological constant on the world-sheet.

3. Conformal field theory

We are now in a position to use conformal field theory. Define \( \phi^1 \equiv \rho \) and \( \phi^2 \equiv \chi \). The functional integral in (11) takes the form \( \int d\rho d\chi e^{-S_{\text{cft}}} \), with

\[ S_{\text{cft}} = \int \frac{d^2 z}{2\pi} \left[ \bar{\partial} \phi^i \partial \phi^j G_{ij} - \frac{1}{4} \sqrt{g} (\hat{\mathcal{R}} Q_j - i\hat{U} P_j) \phi^j \right], \]
where
\[ 3G_{ij} \equiv \begin{pmatrix} 24 + \xi - c_+ & -ic_-/2 \\ -ic_-/2 & \xi \end{pmatrix}, \quad 3Q_i \equiv \begin{pmatrix} -24 - \xi + c_+ \\ ic_-/2 \end{pmatrix}, \quad 3P_i \equiv \begin{pmatrix} c_-/2 \\ i\xi \end{pmatrix}. \] (13)

The stress tensors associated with this action are
\[ T = -\frac{1}{2} [G_{ij} \partial \phi^i \partial \phi^j + (Q_j + P_j) \partial^2 \phi^j], \]
\[ \bar{T} = -\frac{1}{2} [G_{ij} \bar{\partial} \phi^i \bar{\partial} \phi^j + (Q_j - P_j) \bar{\partial}^2 \phi^j]. \] (14)

The fields in this theory have a non-diagonal propagator
\[ \langle \phi^i(z, \bar{z}) \phi^j(w, \bar{w}) \rangle = -G^{ij} \ln|z - w|^2 \] (15)
where \( G^{ij} \) is the inverse of \( G_{ij} \). The central charge computed from (14) is
\[ c_{e,R} = 2 + 3G_{ij}(Q_i + P_i)(Q_j + P_j) = 26 - c_R, \] (16)
and similarly for \( R \to L \). Therefore in accord with expectations à la DDK[1], the total central charge vanishes in both the holomorphic and anti-holomorphic sectors. The parameter \( \xi \) does not appear in this expression for the central charge of the combined conformal and Lorentz induced action. However, \( \xi \) will affect physical exponents. Bastianelli considered a conformal field theory similar to (12) for chiral bosons[7].

The non-diagonal propagator for these fields (15) makes calculations somewhat cumbersome. It is more convenient to produce a basis where the ‘metric’ \( G_{ij} \) is diagonal, by taking linear combinations of \( \rho \) and \( \chi \). (Using linear combinations produces a trivial Jacobian in the functional measure.) Such a diagonal basis is by no means unique, and we briefly illustrate two choices below. We refer to these two cases as the \( \rho \) theory and the \( \chi \) theory, since the bases of fields are chosen to be \((\rho, \theta \equiv \chi - ic_- \rho/2\xi)\) and \((\zeta \equiv \rho - ic_- \chi/2(24 + \xi - c_+), \chi)\), respectively.

i. The \( \rho \) theory
In this case, we diagonalize the kinetic terms by defining \( \theta \equiv \chi - ic_- \rho/2\xi \).† Now
\[ S_{\text{eft}} = \int \frac{d^2z}{2\pi} \left[ \frac{D}{3\xi} (\partial \rho \partial \bar{\rho} + \frac{1}{4} \sqrt{g} \hat{R} \rho) + \frac{\xi}{3} \{ \partial \theta \bar{\partial} \theta - \frac{1}{4} \sqrt{g} (\hat{U} + \frac{ic_-}{2\xi} \hat{R}) \bar{\theta} \} \right], \] (17)
where \( D \equiv \det 3G \). Explicitly, one has
\[ \frac{D}{\xi} = 24 - c_+ + \xi + \frac{c_-^2}{4\xi}. \] (18)
If we assign \((\phi^1, \phi^2) = (\rho, \theta)\), (12,14,15,16) are unchanged upon replacing (13) with
\[ 3G^*_{ij} \equiv \begin{pmatrix} D/\xi & 0 \\ 0 & \xi \end{pmatrix}, \quad 3Q_i \equiv \begin{pmatrix} -D/\xi \\ ic_-/2 \end{pmatrix}, \quad 3P_i \equiv \begin{pmatrix} 0 \\ i\xi \end{pmatrix}. \] (19)

† Here, we assume that \( \xi \neq 0 \).
We compute the weights of various exponentials
\[
e^{\alpha \rho(z)} : \Delta_\alpha = -\frac{3}{2} \frac{\alpha \xi}{D} \left( \alpha - \frac{D}{3 \xi} \right) \quad e^{i k \theta(z)} : \Delta_k = \frac{3}{2} \frac{k}{\xi} \left( k + \frac{c_0}{6} + \frac{\xi}{3} \right)
\]
\[
e^{\tilde{\alpha} \rho(z)} : \Delta_{\tilde{\alpha}} = -\frac{3}{2} \frac{\tilde{\alpha} \xi}{D} \left( \tilde{\alpha} - \frac{D}{3 \xi} \right) \quad e^{i \tilde{k} \theta(z)} : \Delta_{\tilde{k}} = \frac{3}{2} \frac{\tilde{k}}{\xi} \left( \tilde{k} + \frac{c_0}{6} - \frac{\xi}{3} \right)
\]

Above, we have explicitly introduced separate momenta for the holomorphic and anti-holomorphic exponentials because the discussion in the next section suggests that we should be considering operators which introduce cuts.† Even if \( \tilde{k} = k \), \( \Delta_{\tilde{k}} - \Delta_k = k \). This spin for an operator \( \exp[i k (\theta_R + \theta_L)] \) results from the coupling of \( \theta \) to \( \hat{U} \).

Given a matter operator of weight \((\Delta_L, \Delta_R)\), one may expect that it acquires exponential dressings by \( \rho \) and \( \theta \) to make it a \((1,1)\) operator, as in non-chiral gravity[1]. For a given \( k \), \( \alpha \) is determined to be
\[
\alpha = \frac{1}{2} \left[ \frac{D}{3 \xi} \pm \sqrt{\frac{D^2}{9 \xi^2} - \frac{8 D}{3 \xi} (1 - \Delta_R - \Delta_k)} \right], \quad (21)
\]
or alternatively for fixed \( \alpha \), \( k \) is
\[
k = -\frac{1}{6} \left[ \frac{c_0}{2} + \xi \pm \sqrt{\left( \frac{c_0}{2} + \xi \right)^2 + 24 \xi (1 - \Delta_R - \Delta_\alpha)} \right]. \quad (22)
\]

Therefore one finds a one parameter family of dressings in the holomorphic sector. Similar formulæ relate \( \tilde{\alpha} \) and \( \tilde{k} \) in the anti-holomorphic sector. While two branches have been indicated in eq.'s (21,22) for possible gravitational dressings, the negative sign yields the correct results in the limit \( \xi \to \infty \). We will show that the latter corresponds to the classical limit of chiral gravity in sect. 7. The exponential dressings we have considered here are the simplest possible dressings. Because of the spin inherent in the \( \theta \) exponentials, we expect that it is possible to find other primary fields constructed from \( \rho \) and \( \theta \), which can dress matter operators to produce weight \((1,1)\) operators (see sect. 5).

ii. The \( \chi \) theory

An alternate diagonalization comes from the choice \((\phi^1, \phi^2) = (\zeta \equiv \rho - ic_0 \chi/2X, \chi)\) where \( X \equiv 24 + \xi - c_0 \). In this case, one replaces (13) with
\[
3G_{ij} \equiv \begin{pmatrix} X & 0 \\ 0 & D/X \end{pmatrix}, \quad 3Q_i \equiv \begin{pmatrix} -X \\ 0 \end{pmatrix}, \quad 3P_i \equiv \begin{pmatrix} c_0/2 \\ iD/X \end{pmatrix}, \quad (23)
\]
in (12,14,15,16). \( \chi \) couples only to \( \hat{U} \), and the coupling of \( \zeta \) to \( \hat{U} \) vanishes as \( c_0 \to 0 \). Also, in contrast to the \( \rho \) theory, the limit \( \xi \to 0 \) is nonsingular in this case.

† In sect. 4, we will find \( \tilde{\alpha} = \alpha \), so that the formulæ presented here are slightly more general than needed.
As in the previous case, one can calculate gravitational dressings for matter fields. Here, we will only note that in general the weight of an operator is given by
\[
\left[ e^{k_i \phi^i} \right]_R = -\frac{1}{2} \left\{ G^{ij} k_i (k_j + Q_j + P_j) \right\},
\]
while replacing \( P_j \) by \( -P_j \) above yields the weight of an exponential of anti-holomorphic fields.

Finally, we comment on a duality between these two diagonalized theories. The following substitutions in (23) take the \( \chi \) theory to the \( \rho \) theory:
\[
\chi \rightarrow -i \rho, \quad \zeta \rightarrow i \theta, \quad X \leftrightarrow -\xi, \quad \hat{U} \leftrightarrow i \hat{R}.
\]
Further considering the weights of exponential operators in the two theories, one finds
\[
\left[ e^{\alpha \rho + i k \theta} \right]_R = \left[ e^{k \zeta + i \alpha \chi} \right]_R \quad \text{and} \quad \left[ e^{\tilde{\alpha} \rho + i \tilde{k} \theta} \right]_L = \left[ e^{-\tilde{k} \zeta - i \tilde{\alpha} \chi} \right]_L,
\]
with the above substitution \( X \leftrightarrow -\xi \). Thus the \( \chi \) and \( \rho \) theories are isomorphic for appropriate values of \( X \) and \( \xi \).

4. Global issues

Thus far we have evaluated local quantities in the conformal field theory, and have therefore had no need to consider the definition of the terms in the action on a non-trivial manifold. \textit{A priori} it is not clear that expressions such as \( \int \omega \ast \omega \) can be defined in an invariant manner over the entire surface considered. We will now demonstrate that the conformal field theory action is well-defined on any orientable surface.

The problem is as follows: On a higher genus surface, the action, (12), is evaluated by a sum of integrals on a set of overlapping coordinate patches. In general on an overlap between adjacent patches, the background zweibeins \( \hat{e}^\pm \) are related by both coordinate and Lorentz transformations. Now in order to be well-defined, the action should be unchanged if the boundaries between the integrals are shifted. This is clearly the case for the kinetic terms and the couplings to \( \hat{R} \), which are Lorentz invariant and covariant under coordinate changes. Recall though that under a Lorentz transformation, \( \delta \hat{U} = 2 \ast \delta \ast \delta \hat{\chi} \). Thus moving boundaries would appear to change the contributions of the couplings to \( \hat{U} \). A simple resolution of this problem is easily found since we are only interested in complex coordinates for conformal field theory. Gauss showed that locally there exist coordinates such that the metric is conformally flat,
\[
\hat{g} = \frac{1}{2} e^{2 \hat{\rho}} (dz \otimes d\bar{z} + d\bar{z} \otimes dz).
\]
This implies that locally there exist zweibein such that \( \hat{U} = 0 \), since we can choose
\[
e^+ \equiv e^{\hat{\rho}} dz \quad \text{and} \quad e^- \equiv e^{\hat{\rho}} d\bar{z}.
\tag{24}
\]
In general, one can allow for additional holomorphic Lorentz transformations between patches. In either case, \( \hat{U} \) is invariant in the overlaps leading to a well-defined action. Henceforth, we make the choice of a Gaussian background zweibein (24) with \( \hat{U} = 0 \).
It remains to determine the moduli associated with the integration over ‘all zweibein’. Since phases cancel when zweibein are tensored to produce the metric, there are global phases that must be integrated over, above and beyond the moduli associated with the integration over conformal equivalence classes of metrics. In the classical geometry, one can realize these phases by shifting the spin connection, \( \omega \rightarrow \omega + \sum_{i=1}^{2g} \lambda_i \beta^i \), where \( \beta^i \) is a basis for the harmonic differentials on the genus \( g \) surface. Since \( \beta^i \) are closed and divergenceless, both \( \mathcal{R} \) and \( \mathcal{U} \) are unaffected by this shift. Given any closed contour around a particular nontrivial cycle though, one acquires an additional phase: \( \int_a \omega \rightarrow \int_a \omega + \lambda_a \). These global phases are most conveniently incorporated into the Lorentz field \( \chi \). To be precise, one lets

\[
d\chi = d\tilde{\chi} + \sum_{i=1}^{2g} \lambda_i \beta^i .
\]

where \( \tilde{\chi} \) is a (single-valued) function on the surface. In contrast since the \( \beta^i \) are not exact forms, \( \chi \) must now be multivalued on the surface (or alternatively, \( \chi \) contains discontinuities). The measure for the Lorentz moduli, \( dn = \prod d\lambda_i \), would then be included as a part of the functional measure \( D\chi \). These global phases are associated with the nontrivial cycles on the higher genus surfaces. Additional nontrivial cycles occur in correlation functions surrounding the operator insertions. The above discussion then suggests that we allow \( \chi \) to have discontinuities around such cycles. Such cuts would be produced by dressing the matter operators with exponentials of the form: \( \exp[i\rho k \chi + i\tilde{k} \chi_L] \). Below, we will see that these expectations motivated by the classical geometry must be slightly modified in the quantum theory.

Usually momentum (non)conservation super-selection rules for correlation functions are derived in the path integral framework. One would consider integrating the constant mode of the fields \( \phi^i \). These modes vanish in the kinetic terms, and in the \( \hat{U} \) interactions as well since \( \int \hat{U} = 0 \). The coupling to the background curvature gives a contribution proportional to the Euler constant of the surface. Thus in a correlation function with a number of exponential insertions \( \exp[k^{(a)}_i \phi^i] \), integrating the constant mode (over an appropriate contour) yields a delta function requiring \( \sum_a k^{(a)}_i = -Q_i(1-g) \). A slightly more general calculation allowing for multivalued fields produced by operator insertions \( \exp[k^{(a)}_i \phi^i_R + \tilde{k}^{(a)}_i \phi^i_L] \), would appear to yield \( \sum_a(k^{(a)}_i + \tilde{k}^{(a)}_i) = -2Q_i(1-g) \) and \( \sum_a(k^{(a)}_i - \tilde{k}^{(a)}_i) = 0 \). In fact, we show below that these naive results are incorrect.

The correct momentum conservation rules can be deduced by a consideration of SL(2,C) invariance on the plane. Consider a correlation function of exponential operators. For the purposes of an explicit example, we assume that the matter theory is a set of chiral bosons \( X^i \) with no background charges. The holomorphic part of the operators takes the form, \( \exp[k^{(a)}_i \phi^i + i\gamma^{(a)}_i X^i] \), where demanding that they have weight 1 requires

\[
2 = \gamma^{(a)}_i \gamma^{(a)}_i - G^{ij} k^{(a)}_i (k^{(a)}_j + Q_j + P_j) .
\]

Correlation functions may be written as \( \mathcal{A} \equiv \int \mathcal{A}_R \mathcal{A}_L \), with the holomorphic part of the correlation function measure taking the form

\[
\mathcal{A}_R = \prod_a \mathcal{D} z^{(a)} \prod_{a<b} (z^{(a)} - z^{(b)})^\Delta_{ab} , \quad \text{where } \Delta_{ab} = \gamma^{(a)}_i \gamma^{(b)}_i - G^{ij} k^{(a)}_i k^{(b)}_j .
\]
Now $\mathcal{A}_R$ should be invariant* under an $\text{SL}(2,\mathbb{C})$ transformation: $z^{(a)} \mapsto (az^{(a)} + b)/(cz^{(a)} + d)$ where $ad - bc = 1$. This transformation takes $dz^{(a)} \mapsto dz^{(a)}/(cz^{(a)} + d)^2$, $(z^{(a)} - z^{(b)}) \mapsto (z^{(a)} - z^{(b)})/[(cz^{(a)} + d)(cz^{(b)} + d)]$, and hence

$$\mathcal{A}_R \rightarrow \mathcal{A}_R \prod_a (cz^{(a)} + d)^{-2} \prod_e((cz^{(e)} + d)(cz^{(f)} + d))^{-\Delta_{ef}}.$$ 

Therefore invariance of $\mathcal{A}_R$ requires for each $a$

$$2 = -\sum_{b \neq a} \Delta_{ab} = -\gamma^{(a)}_i \sum_{b \neq a} \gamma^{(b)}_i + G^{ij} k^{(a)}_i \sum_{b \neq a} k^{(b)}_j .$$

(26)

Momentum conservation for the matter fields requires $\sum_b \gamma^{(b)}_i = 0$, while similarly in the gravity theory one expects to find $\sum_b k^{(b)}_i = -\Lambda_i$ for some constants $\Lambda_i$. Using these conservation equations, (26) becomes

$$2 = \gamma^{(a)}_i \gamma^{(a)}_i - G^{ij} k^{(a)}_i (k^{(a)}_j + \Lambda_j) ,$$

which is precisely the weight 1 condition (25) when $\Lambda_i = Q_i + P_i$. Hence we have deduced the superselection rule

$$\sum_a k^{(a)}_i = -Q_i - P_i$$

(27)

in the holomorphic sector. The same calculation for anti-holomorphic exponentials, $\exp[\tilde{\gamma}^{(a)}_i \phi^i + i\tilde{\gamma}^{(a)}_i X^i]$, yields

$$\sum_a \tilde{k}^{(a)}_i = -Q_i + P_i$$

(28)

These results are in contradiction with those produced by a naive path integral approach since here we have $\sum_a (k^{(a)}_i - \tilde{k}^{(a)}_i) = -2P_i$, which is nonvanishing in general. A more careful treatment of the $\hat{U}$ coupling in the presence of multivalued fields may reproduce (27,28) from a path integral calculation.

Now if one does not allow any cuts to occur in the gravity sector fields ($i.e., k^{(a)}_j = \tilde{k}^{(a)}_j$), all correlation functions in the plane will vanish since (27) and (28) cannot then be satisfied simultaneously. One might only insist $\rho$ have no cuts, but allow $\chi$ to be multivalued following the considerations of the classical geometry given above. All planar correlation functions vanish in this case as well, as can be seen from the discussion of the original basis of fields in sect. 3 where $\rho$ couples to $\hat{U}$ with $P_1 = c_\rho/6$.

These problems arise because of the $\hat{U}$ interactions in (12). Thus we are lead to the natural suggestion that one should allow for discontinuities in the linear combination of fields that couples to $\hat{U}$, namely $\theta = \chi - ic_\rho/2\xi$. We will also insist $\rho$ be single-valued, as in the classical geometry. Thus the $\rho$ theory in sect. 3 provides the natural basis in which to study correlation functions. The exponential dressings determined from (20,21,22) should

* In general, of course, $\mathcal{A}_{R(L)}$ need only be invariant up to complex conjugate phases.
have $\tilde{\alpha} = \alpha$, but this still leaves one free parameter, namely the phase or cut introduced in $\theta$. Explicitly, the super-selection rules are

\[ \sum_a \alpha^{(a)} = \frac{D}{3\xi} (1-g), \]
\[ \sum_a k^{(a)} = \left[ -\frac{c_-}{6} - \frac{\xi}{3} \right] (1-g), \]
\[ \sum_a \tilde{k}^{(a)} = \left[ -\frac{c_-}{6} + \frac{\xi}{3} \right] (1-g). \]

(29)

These rules are extended to arbitrary genus surfaces, as would result by standard factorization arguments. Such an extension is easily verified to be correct for $g = 1$. Finally note that the original discussion of the Lorentz moduli arising on such higher genus surfaces should be modified by replacing the field $\chi$ by $\theta$.

On a Euclidean world-sheet, it is natural to identify $\chi$ with $\chi + 2\pi$ (i.e., a constant $2\pi$ rotation leaves the zweibein invariant). Implementing this identification results in various quantization conditions. For instance, $c_- \in 6\mathbb{Z}$, so that $e^{-S_{\text{eff}}}$ only acquires a phase of $\exp(2\pi i n)$ when $\chi$ is shifted by $2\pi$, due to the curvature interaction. We will ignore any such identification though, because it would be incorrect if one regards this theory as the analytic continuation of a Minkowskian world-sheet theory, where the Lorentz group is non-compact.

Global Lorentz and diffeomorphism anomalies are distinct. Global diffeomorphism invariance translates into the modular invariance of the partition function defining the theory, while global Lorentz transformations are not an issue for the non-compact form of the Lorentz group. In sect. 6 we shall see that rather intriguing theories are selected by this criterion of modular invariance.

5. More operators

The classical theory had zweibein and a metric, so one might be interested in operators in the quantum theory with similar transformation properties. For instance, the zweibein transform as fields of weight $(1,0)$ or $(0,1)$. From (20), we see that a $(1,0)$ operator of the form, $\exp[\alpha \rho + i k \theta_R + i \tilde{k} \theta_L]$, has

\[ k = -\frac{1}{6} \left[ \frac{c_-}{2} + \xi \pm \sqrt{\left( \frac{c_-}{2} + \xi \right)^2 - 24\xi \Delta_\alpha} \right] \]
\[ \tilde{k} = -\frac{1}{6} \left[ \frac{c_-}{2} - \xi \pm \sqrt{\left( \frac{c_-}{2} - \xi \right)^2 - 24\xi (\Delta_\alpha - 1)} \right]. \]

Such operators will prove useful as dressing fields in the next section. Due to the spin inherent to $\theta$ exponentials a solution introducing no discontinuities is possible with $k = \tilde{k} = -1$. Similarly a weight $(1,1)$ operator is produced if in the expression for $k$ above, $\Delta_\alpha$ is replaced by $(\Delta_\alpha - 1)$. These operators correspond to introducing a puncture in the surface, which is
also the origin of a cut in $\theta$. In this case, producing a local operator (i.e., no cuts) requires that $k = \bar{k} = 0$, and
\[ \alpha = \frac{1}{6} \left[ \frac{D}{\xi} \pm \sqrt{\frac{D}{\xi} \left( \frac{D}{\xi} - 24 \right)} \right]. \quad (30) \]
Agreement with the classical limit, $\xi \to \infty$, requires choosing the negative branch. This local operator provides a cosmological constant operator, which could be added as a new perturbative interaction to (12). It may be of interest to investigate the deformed theory à la Goulian and Li[8].

This cosmological constant operator is the quantum analog of an operator that exists in the classical geometry. One might ask if there are other classical operators, which may have quantum analogs in this way. The answer is an affirmative, since we need only insist that these operators be diffeomorphism invariant, but they need be neither Weyl nor Lorentz invariant. With only one derivative, one might consider: $d\bar{e}^\pm, d\pm e^\pm, \omega e^\pm$, and $\omega \pm e^\pm$. In the classical geometry using (2,3), one shows that only two of these operators are distinct: $de^+ = \partial(e^{\rho+i\chi})d\bar{z}dz$ and $de^- = \partial(e^{\rho-i\chi})d\bar{z}dz$. Unfortunately, these are total derivatives locally, and hence will not provide interesting interactions. One can confirm that in the quantum gravity theory there are two primary (1,1) operators in the form of an exponential and a single holomorphic or anti-holomorphic derivative, that these operators are indeed total derivatives (except for special values of $\xi$), and that they do in fact coincide with the above expressions in the classical limit, $\xi \to \infty$.

One might consider classical operators of the general form
\[ [\nabla_z; n][\nabla_{\bar{z}}; m] \exp(\alpha \rho + ik \chi) \, dzd\bar{z} \quad (31) \]
where $[\nabla_z; n]$ (resp. $[\nabla_{\bar{z}}; m]$) is a sum of arbitrary terms each containing a total of $n$ (resp. $m$ anti-) holomorphic covariant derivatives acting on $\rho$ and/or $\chi$. Following ref. 9, one constructs operators which are invariant under conformal reparametrizations: In a local complex coordinate patch, the nonvanishing components of the zweibein and inverse-metric are $e^+ z = e^{\rho+i\chi}$, $e^- \bar{z} = e^{\rho-i\chi}$ and $g^{z\bar{z}} = 2e^{-2\rho}$. Since $[\nabla_z; n]$ acts as a tensor of weight $(0,n)$ under analytic reparametrizations, one can form an invariant scalar, $(e^- \bar{z} g^{z\bar{z}})^n [\nabla_z; n] \sim e^{-n\rho-in\chi} [\nabla_z; n]$. Similarly, $(e^+ z g^{z\bar{z}})^m [\nabla_{\bar{z}}; m] \sim e^{-m\rho+im\chi} [\nabla_{\bar{z}}; m]$ and $e^{+}e^{-} \sim e^{2\rho}dzd\bar{z}$ are invariant. Combining these formulæ, one constructs an invariant operator in (31) by setting
\[ k = m - n \quad \text{and} \quad \alpha = 2 - m - n. \quad (32) \]
As a simple example, $n = 2$, $m = 0$ yields $k = -2$, $\alpha = 0$, and $[\nabla_z; 2]$ is a sum of five possible terms, but two linear combinations yield total derivatives. Hence there are three new nontrivial interactions, which might have analogs in the quantum theory. Working with the basis, $(\phi^1, \phi^2) = (\rho, \theta)$, in the conformal field theory, one finds a single (1,1) operator of the form
\[ \left[ \eta_{ij} \partial_\phi^i \partial_\phi^j + \psi_i \partial_\phi^2 \phi^i \right] \exp \left( q_i \phi^i \right) \]
where $q_2 = ik = -2i$ and
\[ q_1 = \alpha = \frac{1}{2} \left[ \frac{D}{3\xi} - \sqrt{\frac{D^2}{9\xi^2} + \frac{8D}{3\xi} (\Delta_k + 1)} \right]. \]

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Here only the solution for $\alpha$, with the correct classical limit as $\xi \to \infty$, is displayed. Writing the ‘polarization’ tensor as $\eta = \hat{\eta} + \frac{1}{2}(\eta \otimes \psi + \psi \otimes \eta)$, one also requires that

$$
\hat{\eta}_{ij} G^{jk}(Q_k + P_k + q_k) = \frac{1}{6} G^{jk} \hat{\eta}_{jk} q_i \quad \text{and} \quad G^{ij} \hat{\eta}_{ij} + 3 C^{ij} \psi_i (Q_j + P_j + q_j) = 0.
$$

Generically, the solution of these equations is unique up to the homogeneous solution $\psi_0$ of the latter equation (i.e., set $\hat{\eta} = 0$), which corresponds to a total derivative operator. The spin exponent $k = -2$ is not renormalized in the quantum operator. This property holds in general for such operators. Thus we have found a quantum analog for one of the three classical expressions. Presumably more quantum interactions could be constructed if one considers a general ansatz which allows for mixing with the background connection and curvature[10], or alternatively with the ghost current[11].

Apparently these new operators are examples of discrete states in chiral gravity. The determination of the complete spectrum is beyond the scope of the present paper. It should be noted that the spectrum appears to depend on $\xi$ in a complicated manner. Further the above discussion focussed on local operators, which might appear as interactions in the action. When one allows nontrivial phases in $\theta$, a continuum of new operators is produced for each one of these interactions. It may be of interest to investigate the analog of the Witten ground ring[12] in these theories.

We conclude this section by briefly addressing the question of the critical ‘dimension’ for chiral gravity. One might begin by demanding the reality of the string susceptibility as in [1]. Define the area operator, $\int d^2 x \sqrt{\hat{g}} e^{\alpha \rho}$, with $\alpha$ given by (30). This definition is chosen to involve only a local dressing operator, and to make no explicit reference to the Lorentz field, which should be irrelevant to defining the area, in analogy to the classical geometry. Then one considers the fixed area partition function

$$
Z(A) = \left\langle \delta \left( \int d^2 x \sqrt{\hat{g}} e^{\alpha \rho} - A \right) \right\rangle = 0.
$$

Here a vanishing result occurs, except for genus one surfaces, because the super-selection rules for $\theta$ in (29) are not satisfied. (For $g = 1$, one finds $Z(A) \propto A^{\Gamma - 3}$ with $\Gamma = 2$, exactly as in nonchiral gravity[1].) To properly fix the two $\theta$ zero mode integrals, one can consider inserting punctures with dressings which absorb the appropriate $\theta$-momenta. If one introduces a single puncture, there is a unique dressing which yields a nonzero result for $g = 0, 1$. $P_0 = \int d^2 x \sqrt{g} \exp[\beta \rho + i k \theta_R + i \tilde{k} \theta_L]$ with $\beta = \alpha$ precisely as in the area operator, and $k = (g - 1)(c_- / 6 + \xi / 3)$ and $\tilde{k} = (g - 1)(c_- / 6 - \xi / 3)$. Now one has

$$
Z'(A) = \left\langle P_0 \delta \left( \int d^2 x \sqrt{\hat{g}} e^{\alpha \rho} - A \right) \right\rangle \propto A^{\Gamma - 2}
$$

where

$$
\Gamma = 2 + \frac{g - 1}{12} \left[ \frac{D}{\xi} + \sqrt{\frac{D}{\xi} \left( \frac{D}{\xi} - 24 \right)} \right],
$$

for $g = 0, 1$. This is precisely analogous to the result for nonchiral gravity[1] with the replacement $(25 - c_m) \to D/\xi$. If we make the assumption that $D/\xi$ is positive so that the
\(\rho\)-action (17) has a positive coefficient, then from (18) requiring that \(\Gamma\) be real imposes the restriction
\[
\frac{D}{\xi} - 24 = \xi + \frac{c^2}{4\xi} - c_+ > 0. \tag{34}
\]
Now for any fixed values of \(c_\pm\), there will exist values of \(\xi\) for which this inequality is satisfied. Therefore chiral gravity has no barriers for the allowed values of central charges in the matter sector.

Of course, this conclusion relies on results for only \(g = 0, 1\), since \(Z'(A)\) always vanishes for \(g > 1\). Introducing two punctures yields nontrivial results for higher genera as well. In this case, one finds that \(Z''(A) \propto A^{\Gamma-1}\) where \(\Gamma = \Gamma + (\alpha_+ + \alpha_- - 2\alpha)/\alpha\). Here, \(\Gamma\) is given by (33), while \(\alpha_\pm = \frac{1}{6} \left[ D/\xi \pm \sqrt{DY_\pm/\xi} \right]\) with
\[
Y_\pm = \left[ 2g^2 - 1 \pm 2g\sqrt{g^2 - 1} \right] \xi + \left[ 2g^2 - 1 \mp 2g\sqrt{g^2 - 1} \right] \frac{c_+}{4\xi} - c_+
\]
for \(g \geq 1\). Thus one finds that the genus dependence of the string susceptibility is far more complicated than the simple linear dependence found in nonchiral gravity[1]. Further requiring real exponents by imposing \(Y_\pm > 0\), only produces constraints which are less restrictive than (34). Finally note that we have ignored ghost zero modes in this entire discussion, but that a more rigorous account can be given completely equivalent to that appearing in ref. 1.

6. The \(j\)-string

We now construct a simple example of a heterotic non-critical string. Consider the holomorphic conformal field theory associated with the \(E_8 \times E_8 \times E_8\) root lattice. One may view this as described by 24 chiral bosons, or as 48 chiral fermions with appropriate sums over spin structures. This choice has \(c_\pm = 12\), which simplifies calculations of correlation functions. Many choices of 24-dimensional chiral lattices are possible (see, e.g., ref. 13), but we choose \((E_8)^3\) for concreteness, and since it gives the amusing result that the partition function is the \(j\) invariant[14] on the torus.

To calculate the partition function, at genus 1, one must account for several contributions[10]. First, the modular invariant measure is \(d^2\tau/(\text{Im}\tau)^2\). The contribution coming from the Faddeev-Popov Jacobian, for fixing diffeomorphisms, is: \(\text{Im}\tau|\eta(q)|^4\), where \(q \equiv \exp(2\pi i\tau)\), and \(\eta(q)\) is the Dedekind \(\eta\)-function[15]. The integration over the matter fields yields
\[
j(\tau) \equiv \left( \frac{\Theta_{E_8}(q)}{\eta(q)^8} \right)^3 = \left( \frac{\Theta_2(q)^2 + \Theta_3(q)^2 + \Theta_4(q)^2}{\eta(q)^8} \right)^3
\]
where \(\Theta_i(q) = \Theta_i(z = 0|\tau)\) are Jacobi theta functions[15]. The \(\rho\) contribution is the same as that for a free scalar field: \((\text{Im}\tau)^{-1/2}|\eta(q)|^{-2}\).

The \(\theta\) contribution is more interesting because of the Lorentz moduli. Covering the torus with a fixed coordinate patch, \(0 \leq \sigma^1, \sigma^2 \leq 1\), the world-sheet metric becomes \(ds^2 = \)\(\dagger\ We ignore the unconstrained integrals over the \(\rho\) and \(\theta\) zero modes in the present discussion.
\(|d\sigma^1 + \tau d\sigma^2|^2 \). We introduce arbitrary phases in \(\theta\) around the \(\sigma^i\) cycles by setting

\[
\theta(\sigma^1, \sigma^2) = \tilde{\theta}(\sigma^1, \sigma^2) + 2\pi \lambda_i \sigma^i.
\]

Here \(\tilde{\theta}\) is a function on the torus, and the second term incorporates the multivalued contribution of the Lorentz moduli. The contribution to the partition function is then found to be

\[
Z_{\theta} = \frac{1}{(\text{Im}\tau)^{1/2} |\eta(q)|^2} \int_{-\infty}^{\infty} d\lambda_1 d\lambda_2 \exp \left( -\frac{\pi \xi |\lambda_2 - \lambda_1\tau|^2}{6 \text{Im}\tau}\right)
\]

where we have included the phase integral, which one may note is invariant under modular transformations. The integral takes a particularly simple form upon shifting \(\hat{\lambda}_2 = \lambda_2 - \lambda_1 \text{Re}\tau\)

\[
Z_{\theta} = \frac{1}{(\text{Im}\tau)^{1/2} |\eta(q)|^2} \int_{-\infty}^{\infty} d\lambda_1 d\hat{\lambda}_2 \exp \left( -\frac{\pi \xi}{6} \left\{ \frac{\hat{\lambda}_2^2}{\text{Im}\tau} + \text{Im}\tau\lambda_1^2 \right\} \right)
\]

Therefore after the phase integral is performed, the final \(\theta\) contribution is also simply that of a free scalar field (up to an arbitrary normalization).

Combining all of these contributions yields

\[
Z_{\text{torus}} = \int_{\mathcal{F}} \frac{d^2 \tau}{(\text{Im}\tau)^2} J(\tau)
\]

where \(\mathcal{F}\) is a fundamental domain for the action of the modular group on the Poincaré upper-half-plane. To understand the spectrum of the theory, we investigate the region of small \(q\), equivalently large \(\text{Im}\tau\), where

\[
j(\tau) \sim \frac{1}{q} + 744 + \ldots
\]

The integration over \(\text{Re}\tau\) projects out any term in eq. (37) with a non-zero power of \(q\), including the tachyon. The only term left is the constant 744, indicating that there are 744 states in this string theory, 720 corresponding to winding number states in the 24–dimensional lattice, and 24 coming from the maximal torus of the group, the so-called oscillator states. If one counted the states before performing the phase integral in (35), each of the 744 states corresponds to a continuum with arbitrary phases around the \(\sigma^1\) cycle.

We now consider some sample correlation functions in this theory. From the partition function, it appears that the physical states correspond to (0,1) operators in the matter sector. Thus we construct exponential (1,0) dressings, as discussed in sect. 5. First though in the present theory with \(c_{\pm} = 12\) in the \((\rho, \hat{\theta})\) basis, \(G_{ij} = \text{diag}(x + 2/x, x^2)\) where \(x = \sqrt{\xi/3}\). Therefore it is extremely convenient to rescale the fields to a new basis (\(\hat{\rho} = (x + 2/x)\rho, \hat{\theta} = x\theta\)) for which

\[
G_{ij} = \delta_{ij}, \quad Q_i \equiv \left( -\frac{x + 2/x}{2i/x} \right), \quad P_i \equiv \left( 0 \frac{ix}{ix} \right).
\]
Now there are four possible (0,1) operators of the form \( \exp[\alpha \hat{\rho} + i k \hat{\theta}_R + i \tilde{k} \hat{\theta}_L] \), but we restrict our attention to the two with good semiclassical limits. For fixed \( k \), they correspond to

i. \( \tilde{k} = k + x, \alpha = -k \)

ii. \( \tilde{k} = -2/x - k, \alpha = -k \).

The first introduces a fixed phase in the \( \theta \) field with \( k - \tilde{k} = -x \), while the phase of the second dressing varies continuously with \( k \), \( k - \tilde{k} = 2/x + 2k \). The momentum super-selection rules for correlation functions are: \( \sum_{\alpha} a^{(a)} = x + 2/x \), \( \sum_{k} k^{(a)} = -x - 2/x \) and \( \sum_{\tilde{k}} k^{(a)} = x - 2/x \).

One can begin by considering three-point amplitudes of some given set of matter operators. For a fixed set, there would be eight possible amplitudes corresponding to all of the different combinations of dressings. Some of these amplitudes vanish since they combine dressings which are incompatible with the momentum conservation rules. The amplitudes which survive are those which involve either two (i) and one (ii) dressings, or one (i) and two (ii). In those cases where the amplitude is non-vanishing, one finds that as expected the results are SL(2, \( \mathbb{C} \)) invariant, but also independent of how the various dressings are combined with the matter operators. There are two classes of nonvanishing amplitudes: those with three winding number states, \( \exp(i \gamma^{(a)} \cdot X_R) \), which yield simply \( \delta(\gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)}) \), and those with two winding number states and one oscillator state, \( i \beta \cdot \partial X_R \), which yield \( \beta \cdot \gamma^{(1)} \delta(\gamma^{(1)} + \gamma^{(2)}) \).

The four-point amplitudes produce more interesting results, as we illustrate here. Consider the following correlation function

\[
\mathcal{A} = \int d^2 z \langle \bar{c} \tilde{c} V_1 e^{i \gamma^{(1)} \cdot X_R(z^{(1)})} c \bar{c} V_1 e^{i \gamma^{(2)} \cdot X_R(z^{(2)})} \rangle 
\]

\[
\times \langle \bar{c} \tilde{c} V_1 e^{i \gamma^{(3)} \cdot X_R(z^{(3)})} c \bar{c} V_1 e^{i \gamma^{(4)} \cdot X_R(z^{(4)})} \rangle ,
\]

where \( \bar{c} \tilde{c} \) are ghost dressings for the fixed operators, \( V_i(i) \) are gravitational dressings of type i (ii) given above, and \( \exp[i \gamma^{(a)} \cdot X_R] \) are winding state operators in the matter sector with \( \gamma^{(a)} \cdot \gamma^{(a)} = 2 \). Momentum conservation requires \( \sum \gamma^{(a)} = 0 \) in the matter sector, while the super-selection rules, for the gravity sector given above, restrict the momenta to

\( z^{(1)}: k = q, \tilde{k} = q + x, \alpha = -q \)

\( z^{(2)}: k = p, \tilde{k} = p + x, \alpha = -p \)

\( z^{(3)}: k = x/2 - 1/x, \tilde{k} = -1/x - x/2, \alpha = -x/2 + 1/x \)

\( z^{(4)}: k = -3x/2 - 1/x - p - q, \tilde{k} = -x/2 - 1/x - p - q, \alpha = 3x/2 + 1/x + p + q \).

Again, one can explicitly verify that the result is SL(2, \( \mathbb{C} \)) invariant, and fixing \( z^{(a)} = (\infty, 1, z, 0) \) yields

\[
\mathcal{A} = \int d^2 z \frac{(1 - z)^{\gamma^{(2)} \cdot \gamma^{(3)}} z^{\gamma^{(4)} \cdot \gamma^{(3)}} (1 - \bar{z})^{1 - x^2 - p x}}{z^{x^2 + x(p+q)}} .
\]

On the holomorphic side of this integral, one always has integral exponents since \( \gamma^{(a)} \cdot \gamma^{(b)} = -2, -1, 0, 1, 2 \), while on the anti-holomorphic side, arbitrary exponents arise as \( p \) and \( q \) are

\[\dagger\] These results use Kronecker \( \delta \)-functions, since the winding number vectors are discrete.
varied. Thus in general the amplitude contains cuts as are characteristic of correlators of the gravity dressings. As $z$ approaches 1, one finds that for $(\gamma^{(2)} \cdot \gamma^{(3)} , p) = (-1, -x/2)$ or $(-2, 1/x - x/2)$, $A$ factorizes with the appropriate three-point amplitudes on $|1-z|^{-1}$ or $|1-z|^{-2}$ singularities, respectively. Such singularities are characteristic of poles in the Virasoro-Shapiro amplitudes[16]. Similar results arise as $z$ approaches 0 or $\infty$, as well.

7. Concluding remarks

We have examined two-dimensional quantum gravity coupled to chiral matter, and found that for fixed values of $c_\pm$, there is in fact a family of theories labelled by the free parameter, $\xi$. One might think of this result in analogy to the cosmological constant, which arises as a free parameter in nonchiral gravity. The effects of $\xi$ are far more intricate: It does not appear in the central charge of the combined conformal and Lorentz induced action, eq. (16), nor in the partition function. However, $\xi$ does affect critical exponents, see e.g., eq. (33), the spectrum of discrete states, and the positions of poles in amplitudes. Actually, $\xi$ should be counted as only one of undetermined couplings, which arise in association with the new interactions discussed in sect. 5.

We should note that in the light-cone analysis of chiral gravity by ref. 4, it is concluded that $\xi$ is quantized to be $\xi = ic_-/2$. Of course, we have found no evidence of such a quantization condition in our analysis. This discrepancy may arise because the stress tensor for the Lorentz field appears to be misidentified in ref. 4.

One might think that $\xi$ should be fixed in analogy to the analysis of the chiral gauge anomaly in two dimensions[17], in which no undetermined parameters arise. While the formal expressions for the chiral gauge and gravitational anomalies seem similar, it is important to keep in mind that the spin connection drops out of the action for fermions in 1+1 dimensions, so the determinant of interest possesses a dependence on the spin connection only via quantum effects. In the case of the chiral gauge anomaly, the determinant depends explicitly on one component of the connection—it therefore may be reasonable to demand that the effective action also only involve the appropriate component, thus fixing the analogue of $\xi$.

It is interesting to further investigate the effects of $\xi$ by calculating the central charges of $\rho$ and $\theta$, separately,

$$c_\rho = 25 - c_+ + \xi + \frac{c^2}{4\xi}$$

$$c_{\theta,R} = 1 - c_- - \xi - \frac{c^2}{4\xi}$$

$$c_{\theta,L} = 1 + c_- - \xi - \frac{c^2}{4\xi}$$

With $\xi \to \infty$, $c_\rho \to \infty$ as expected for the semiclassical limit of the gravity theory. In this limit, the propagators for both $\rho$ and $\theta$ vanish as can be seen from (15,19). As a result, exponential operators are not renormalized from the classical expressions derived in accord with the analysis for (31) and (32). For example, the area operator becomes simply $\int d^2x \sqrt{\hat{g}} e^{2\rho}$. As well in this limit, $\theta \to \chi$, so that the discontinuities all occur in the classical Lorentz phase. Thus one completely recovers the classical world-sheet geometry in the limit $\xi \to \infty$. 

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From (38), it would appear that $\xi \to 0^+$ produces a limit completely analogous to $\xi \to \infty$. In fact, this limit must be taken with care in the $(\rho, \theta)$ basis, as can be seen from the fact that the kinetic term for $\theta$ vanishes in (17). Of course, the theory is perfectly well-defined at $\xi = 0$, and no problems arise for the original $(\rho, \chi)$ basis, or the alternate diagonal basis of $(\zeta, \chi)$. While one still has $\langle \rho(z, \bar{z}) \rho(w, \bar{w}) \rangle = 0$, so that the area operator is unrenormalized in this limit, exponentials involving both $\rho$ and $\chi$ do not take their classical form. Further with this limit, it is $\rho$ which becomes multivalued. Hence while $\xi \to 0^+$ may allow for a semiclassical treatment of $\rho$, the interpretation of the geometry is unclear in this limit.

A physically consistent analysis required some restrictions on $\xi$ in the form of the inequality (34), but no barriers appeared for the matter theories. Note as well that one can easily produce theories with space-time tachyons, which do not yield unphysical critical exponents. Thus at this level, the properties of space-time tachyons and physical consistency on the world-sheet, appear to be divorced in the present theory, in contrast to nonchiral gravity[18]. Clearly, the appearance of the Lorentz field has drastic effects on the quantum theory of the world-sheet geometry. The most pressing question would appear to be to understand the complete space of physical states[19].

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