Research Article

Pick’s Theorem in Two-Dimensional Subspace of $\mathbb{R}^3$

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In the Euclidean space $\mathbb{R}^3$, denote the set of all points with integer coordinate by $\mathbb{Z}^3$. For any two-dimensional simple lattice polygon $P$, we establish the following analogy version of Pick’s Theorem,

$$k(I(P) + \frac{1}{2}B(P) - 1),$$

where $B(P)$ is the number of lattice points on the boundary of $P$ in $\mathbb{Z}^3$, $I(P)$ is the number of lattice points in the interior of $P$ in $\mathbb{Z}^3$, and $k$ is a constant only related to the two-dimensional subspace including $P$.

1. Introduction

In the Euclidean plane $\mathbb{R}^2$, a lattice point is one whose coordinates are both integers. A lattice polygon is a polygon with all vertices on integer coordinates. The area $A(P)$ of a simple lattice polygon $P$ can be given by celebrated Pick’s theorem [1]

$$A(P) = I(P) + \frac{1}{2}B(P) - 1,$$  \hspace{1cm} (1)

where $B(P)$ is the number of lattice points on the boundary of $P$ and $I(P)$ is the number of lattice points in the interior of $P$.

Pick’s formula can be used to compute the area of a lattice polygon conveniently.

For example, in Figure 1, $I(P) = 60$, $B(P) = 15$. Then, the area of the polygon is $A(P) = 60 + 15 - 1 = 74$.

There are many papers concerning Pick’s theorem and its generalizations [2–5], which mostly be discussed in two dimensions.

Unfortunately, Pick theorem is failed in three dimensions. In 1957, John Reeve found a class of tetrahedra, named as Reeve tetrahedra later, whose vertices are

$$(0,0,0)^T, (1,0,0)^T, (0,1,0)^T, (1,1,r)^T,$$  \hspace{1cm} (2)

where $r$ is a positive integer.

All Reeve tetrahedra contain the same number of lattice points, but their volumes are different.

In this note, we discussed Pick’s theorem in two-dimensional subspace of $\mathbb{R}^3$. For any $(a, b, c)^T \in \mathbb{Z}^3$ with $(a, b, c) = 1$, that is, the greatest common factor of $a, b, c$ is one, denote by $K$, $ax + by + cz = 0$, the two-dimensional subspace of $\mathbb{R}^3$. Then we established the following theorem.

Theorem 1. If $P$ is simple lattice polygon in the $K$, then the area of $P$ is

$$k \left( I(P) + \frac{1}{2}B(P) - 1 \right),$$  \hspace{1cm} (3)

where $B(P)$ is the number of lattice points on the boundary of $P$ in $\mathbb{Z}^3$, $I(P)$ is the number of lattice points in the interior of $P$ in $\mathbb{Z}^3$, and $k$ is the constant $(a^3 + ab^2)\sqrt{a^2 + b^2 + c^2}$.

Remark 2. Although the simple lattice polygon $P$ is in the two-dimensional subspace $K$, the lattice points in $P$ belong to $\mathbb{Z}^3$.

Let $(a, b, c)^T = (1, 0, 0)^T$ in the Theorem; then we can get Pick’s theorem in some coordinate plane of $\mathbb{R}^3$.

Corollary 3. If $P$ is simple lattice polygon in the $K$, whose normal vector is $(1, 0, 0)^T$, then the area of $P$ is

$$I(P) + \frac{1}{2}B(P) - 1.$$  \hspace{1cm} (4)
2. Proof of Main Result

For any \((a, b, c)^T \in \mathbb{Z}^3\) with \((a, b, c) = 1\), there is a two-dimensional subspace of \(\mathbb{R}^3\)

\[ ax + by + cz = 0, \tag{5} \]

whose normal vector is just \((a, b, c)^T\). We denote this two-dimensional subspace by \(K\).

By the theory of linear equations system, \((-b, a, 0)^T\) and \((-c, 0, a)^T\) are two linearly independent solutions of (5). We denote \((-b, a, 0)^T\) by \(\alpha\) and \((-c, 0, a)^T\) by \(\beta\). Obviously, \(\alpha\) and \(\beta\) are also the basis of \(K\).

**Lemma 4.** For any \((a, b, c)^T \in \mathbb{Z}^3\) with \((a, b, c) = 1\), there exists the lattice basis with the minimal area in the two-dimensional subspace \(K\).

**Proof.** The area of parallelogram generated by \(\alpha\) and \(\beta\) is

\[
\frac{1}{\sqrt{a^2+b^2+c^2}} \begin{vmatrix} a & -b & -c \\ b & a & 0 \\ c & 0 & a \end{vmatrix} = \frac{a^2+ab^2+ac^2}{\sqrt{a^2+b^2+c^2}} = a\sqrt{\frac{a^2+b^2+c^2}{a^4+b^2+c^2}}.
\]

Denote \((a, b, c)^T\) by \(n\). For any lattice basis in \(K\), \(k_1\alpha + k_2\beta\) and \(l_1\alpha + l_2\beta\), where \(k_i, l_i \in \mathbb{Z}\) \((i = 1, 2)\) and \(|\begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix}| = 0\). The area of parallelogram generated by \(k_1\alpha + k_2\beta\) and \(l_1\alpha + l_2\beta\) is

\[
\frac{1}{\sqrt{a^2+b^2+c^2}} \begin{vmatrix} k_1 & k_2 \beta \ \vdots \ l_1 & l_2 \beta \\ n & (n, \alpha, \beta) \end{vmatrix} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{vmatrix} 1 & 0 & 0 & k_1 & l_1 \\ 0 & k_1 & l_1 \end{vmatrix},
\]

where \(|\begin{vmatrix} k_1 & k_2 \beta \ \vdots \ l_1 & l_2 \beta \\ n & (n, \alpha, \beta) \end{vmatrix}|\) denote the determinant of \(n, k_1\alpha + k_2\beta\), and \(l_1\alpha + l_2\beta\).

Thus the lattice basis \(k_1\alpha + k_2\beta\) and \(l_1\alpha + l_2\beta\) have the minimal area if and only if \(|\begin{vmatrix} k_1 & l_1 \\ k_2 & l_2 \end{vmatrix}| = 1\).

Let \(k_1 = 1, k_2 = 0, l_1 = 0, l_2 = 1,\) and \(\alpha, \beta\) are the lattice basis with the minimal area in the two-dimensional subspace \(K\).

**Lemma 5.** For any \((a, b, c)^T \in \mathbb{Z}^3\) with \((a, b, c) = 1\), there exists the orthogonal lattice basis in the two-dimensional subspace \(K\).

**Proof.** By Lemma 4, \(\alpha, \beta\) are the lattice basis with the minimal area in the two-dimensional subspace \(K\). By Schmidt orthogonalization, let

\[
y_1 = \alpha = (-b, a, 0)^T,
\]

\[
y_2 = \beta - \frac{(\beta, y_1)}{(y_1, y_1)} = (-c, 0, a)^T - \frac{bc}{a^2+b^2} y_1
\]

\[= (-c, 0, a)^T - \left(\frac{-b^2c}{a^2+b^2}, \frac{abc}{a^2+b^2}, \frac{b^2c}{a^2+b^2}\right)^T \tag{8}
\]

where \((\beta, y_1)\) denote the usual inner product of \(\beta, y_1\) in \(\mathbb{R}^3\). Thus

\[
y_1 = y_1 = \alpha = (-b, a, 0)^T,
\]

\[
y_2 = (a^2+b^2) y_2 = (-a^2c, -abc, a^2+ab^2)^T
\]

are the orthogonal lattice basis in the two-dimensional subspace \(K\).

**Proof of Theorem.** By Lemma 5, \(\eta_1, \eta_2\) are the orthogonal lattice basis in the two-dimensional subspace \(K\).

The area of parallelogram generated by \(\eta_1, \eta_2\) is

\[
\frac{1}{\sqrt{a^2+b^2+c^2}} \begin{vmatrix} a & -b & -a^2c \\ b & a & -abc \\ c & 0 & a^3+ab^2 \end{vmatrix} = \frac{a^2+a^3b^2+ab^2c^2+a^3c^2+a^3b^2+ab^4}{\sqrt{a^2+b^2+c^2}}
\]

\[= a \left(\frac{a^4+2a^2b^2+b^4+(a^2+b^2)c^2}{\sqrt{a^2+b^2+c^2}}\right)
\]

\[= a \left(\frac{a^2+b^2}{\sqrt{a^2+b^2+c^2}}\right) \left(\frac{a^2+b^2}{\sqrt{a^2+b^2+c^2}}\right)
\]

\[= \frac{a^3+ab^2}{\sqrt{a^2+b^2+c^2}},
\]

which just is the constant \(k\) in the theorem.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.
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