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IRREDUCIBLE COMPONENTS OF THE GLOBAL NILPOTENT CONE

TRISTAN BOZEC

ABSTRACT. This paper gives a combinatorial description of the set of irreducible components of the semistable locus of the global nilpotent cone, in genus $g \geq 2$.

Keywords: Moduli stacks of sheaves over curves of higher genus, Higgs sheaves, Lagrangian subvarieties, stability.

Mathematics Subject Classification (2010): 14H60

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INTRODUCTION

Given a smooth projective curve $X$ of genus $g$, the moduli stack of Higgs sheaves of rank $r$ and degree $d$ is known to be of dimension $2(g−1)r^2$. It can be viewed as the cotangent stack of the stack of coherent sheaves of class $(r, d)$ over $X$, and Laumon proved in [Lau88] that the substack $\Lambda_{r,d}$ of nilpotent Higgs pairs is Lagrangian (see also [Fal93, Gin01]). This substack, which is the $0$-fiber of the Hitchin map, is a global analog of the nilpotent cone and plays a critical role in the geometric Langlands program. The global nilpotent cone is highly singular, and one first interesting step toward its comprehension is the study of its set of irreducible components (see [BD, 2.10.3] for rather implicit results in this direction).

The stack of stable Higgs pairs is known to be smooth, and several results have been proved recently regarding the counting of the number of stable irreducible components of the global nilpotent cone. Its is known from [Hau05, Corollary 3.11] that the Poincaré polynomial of genus $g$ twisted character varieties, and hence of the diffeomorphic moduli spaces of stable Higgs bundles, is independent from the degree $d$, provided that it is coprime to the rank $r$. In [HRV08], Hausel and Rodriguez-Villegas establish several conjectures dealing with the $E$-polynomial (a specialization of the mixed Hodge polynomial) of these character varieties. In particular, they conjecture a combinatorial relation with the Kac polynomial $A_{g,r}$ of the quiver with one vertex and $g$ loops in dimension $r$, which counts absolutely indecomposable isoclasses of $g$-tuples of matrices over finite fields.

With a different perspective, Schiffmann establishes in [Sch16] that the number of absolutely indecomposable vector bundles of rank $r$ and degree $d$ over $\mathbb{F}_q$ (still over a curve $X$ of genus $g$) is given by an expression $A_{g,r,d}$, polynomial in the Weil numbers of $X$. These polynomials are therein proved to be related to the moduli space of stable Higgs bundles, and, for instance, the number of stable irreducible components of $\Lambda_{r,d}$ is given by $A_{g,r,d}(0)$.
In a recent work [Mel17], Mellit relates the formulas obtained in [HRV08, Sch16], and proves as a consequence that the polynomials $A_{g,r,d}$ and the $E$-polynomials of the moduli spaces of semistable Higgs pairs are both independent from $d$. A very particular consequence of this work is the equality $A_{g,r,d}(0) = A_{g,r}(1)$.

The aim of the present paper is to give a combinatorial description of the set of irreducible components of $\Lambda_{r,d}$, and explain which ones subsist in the subset of semistable components. It is motivated by the $W = P$ conjecture claimed by de Cataldo, Hausel and Migliorini [dCHM12]. In the light of the works above-mentioned, one can expect the polynomial $A_{g,r}$ to play a role in the understanding of the perverse filtration. Adding structure to the set of irreducible components could lead to an interpretation of each of the coefficients of $A_{g,r}$, rather than just their sum ($= A_{g,r}(1)$).

The first main result of this paper is Corollary 2.4, which states that the set of irreducible components of the global nilpotent cone is given by the very natural decomposition in twisted Jordan strata, which are smooth. It is based on a direct computation of the dimensions of these strata and previous works [Sch16, MS17]. Then we move on to the semistable locus and obtain Theorem 3.1 which gives purely combinatorial conditions on the twisted Jordan type to be semistable. The proof uses an analogous result from [BGPGH17] obtained in the context of moduli stacks of chains, and shows that semistability can be tested on the most ‘simple’ subsheaves - the ones built with iterated kernels and images. The proof is constructive and do not rely on the coprimality of $r$ and $d$, in particular we get in Corollary 3.3 that the attracting cells are irreducible in any case.

The Corollary 4.5 describes this set of semistable irreducible components in terms of integral polytopes, which sheds a new light on the quantity $A_{g,r}(1)$, whose behaviour is still very poorly understood.

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1. Recollection on coherent and Higgs sheaves

1.1. Coherent sheaves over a curve. Let $X$ be a smooth projective curve of genus $g$ over a field $k$. We will denote by $\text{Coh}$ the category of coherent sheaves over $X$, and by

$$[\mathcal{F}] = (\text{rank } \mathcal{F}, \text{deg } \mathcal{F}) \in \mathbb{H} = \{(r, d) \in \mathbb{N} \times \mathbb{Z} \mid d \geq 0 \text{ if } r = 0\}$$

the class of $\mathcal{F} \in \text{Coh}$. We will denote by $\text{Coh}_{\alpha} \subset \text{Coh}$ the subcategory of coherent sheaves of class $\alpha$. If $\alpha = (r, d)$, we write $r = \text{rank } \alpha$ and $d = \text{deg } \alpha$. For any $\alpha = (r, d) \in \mathbb{H}$ and $p \in \mathbb{Z}$, we set $\alpha(p) = (r, d + pr)$ so that if $\mathcal{F} \in \text{Coh}$ and $D$ is a divisor of degree $p$ over $X$ we have

$$[\mathcal{F}(D)] = [\mathcal{F} \otimes D] = [\mathcal{F}] (p).$$

We will use the usual slope defined on $\mathbb{H}$ by $\mu(r, d) = d/r \in \mathbb{Q} \cup \{\infty\}$ and we set $\mu(\mathcal{F}) = \mu([\mathcal{F}])$. We say that $\mathcal{F}$ is semistable if

$$\{0\} \subset \mathcal{G} \subset \mathcal{F} \Rightarrow \mu(\mathcal{G}) \leq \mu(\mathcal{F}),$$

stable if the right-hand side inequality is strict. Note that these notions coincide if $\text{deg } \mathcal{F}$ and rank $\mathcal{F}$ are coprime. We will use the following basic property.

**Proposition 1.1.** For any short exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

we have

$$[\mathcal{F}] = [\mathcal{E}] + [\mathcal{G}].$$


in \(\text{Coh}\), one of the following is true
\[
\mu(\mathcal{E}) < \mu(\mathcal{F}) < \mu(\mathcal{G})
\]
\[
\mu(\mathcal{E}) = \mu(\mathcal{F}) = \mu(\mathcal{G})
\]
\[
\mu(\mathcal{E}) > \mu(\mathcal{F}) > \mu(\mathcal{G}).
\]

The category \(\text{Coh}\) being hereditary, the Euler form is defined by
\[
\langle \mathcal{F}, \mathcal{G} \rangle = \dim \text{Hom}(\mathcal{F}, \mathcal{G}) - \dim \text{Ext}^1(\mathcal{F}, \mathcal{G}),
\]
and we will denote by \((-,-)\) its symmetrized version. It only depends on the class of the sheaves, and satisfies
\[
\langle (r, d), (r', d') \rangle = (1-g)rr' + rd' - r'd.
\]

1.2. **Higgs sheaves.** A Higgs sheaf is a pair \((\mathcal{F}, \theta)\), where \(\mathcal{F} \in \text{Coh}\) and \(\theta \in \text{Hom}(\mathcal{F}, \mathcal{F}(\Omega))\), \(\Omega\) being the canonical divisor of degree \(l = 2g - 2\). We will denote by \(M_\alpha\) the moduli stack of pairs \((\mathcal{F}, \theta)\) satisfying \([\mathcal{F}] = \alpha\), whose dimension is \(l(\text{rank } \alpha)^2\). A Higgs sheaf \((\mathcal{F}, \theta)\) is said to be semistable
\[
\begin{align*}
\{0\} & \subset \mathcal{G} \subset \mathcal{F} \\
\theta(\mathcal{G}) & \subset \mathcal{G}(\Omega)
\end{align*}
\]
if \(\mu(\mathcal{G}) \leq \mu(\mathcal{F})\), stable if the right-hand side inequality is strict. Semistability defines an open substack \(M^{sst}_\alpha \subset M_\alpha\).

1.3. **The global nilpotent cone.**

**Definition 1.2.** For any \((\mathcal{F}, \theta) \in M_\alpha\) and \(k \geq 1\), set
\[
\theta^k = \theta((k-1)\Omega) \circ \cdots \circ \theta(\Omega) \circ \theta : \mathcal{F} \to \mathcal{F}(k\Omega).
\]
A pair \((\mathcal{F}, \theta)\) is said to be nilpotent if \(\theta^k = 0\) for some \(k\), and we denote by
\[
\Lambda_\alpha = \{(\mathcal{F}, \theta) \in M_\alpha \mid (\mathcal{F}, \theta) \text{ nilpotent}\}
\]
the global nilpotent cone.

It is nothing but the zero fiber of the Hitchin map \(M_\alpha \to \bigoplus_{1 \leq i \leq r} H^0(X, \Omega^i)\), mapping \((\mathcal{F}, \theta)\) to the coefficients of the characteristic polynomial of \(\theta\). It is known, thanks to Laumon [Lau88], to be a Lagrangian substack of \(M_\alpha\), but its irreducible components are still not well understood (see [Ibid., Remarque 3.9 (iii)]). The aim of this article is to give a precise combinatorial description of these components, as well as the ones of the semistable locus \(M^{sst}_\alpha\).

2. **The twisted Jordan stratification**

2.1. **The setting.** In this section we will recall and make use of notations and results established in [MS17][Sch16]. Consider a nilpotent Higgs sheaf \((\mathcal{F}, \theta) \in \Lambda_{(r,d)}\), with \(r > 0\) but \(r\) and \(d\) not necessarily coprime. Set \(\mathcal{F}_k = \text{Im } \theta^k(-k\Omega)\) and denote by \(s\) the nilpotency index of \(\theta\). We have a chain of epimorphisms
\[
\mathcal{F}_0 \twoheadrightarrow \mathcal{F}_1(\Omega) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}_s(s\Omega) = \{0\}
\]
that allows us to define \(\mathcal{F}'_k = \ker(\mathcal{F}_k \to \mathcal{F}_{k+1}(\Omega))\). We also have a chain of inclusions
\[
\{0\} = \mathcal{F}_s \subset \mathcal{F}_{s-1} \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_0 = \mathcal{F}
\]
whose successive quotients are denoted by \(\mathcal{F}'_k = \mathcal{F}_k/\mathcal{F}_{k+1}\). These two chains induce the following ones
\[
\mathcal{F}'_0 \twoheadrightarrow \mathcal{F}'_1(\Omega) \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{F}'_s(s\Omega) = \{0\}
\]
\[
\{0\} = \mathcal{F}'_s \subset \mathcal{F}'_{s-1} \subset \cdots \subset \mathcal{F}'_1 \subset \mathcal{F}'_0,
\]
and we define
\[ \alpha_k = (r_k, d_k) = \ker \{ \mathcal{F}_{k-1}^\prime ((k-1)\Omega) \to \mathcal{F}_k^\prime (k\Omega) \}. \]

The family \( \alpha = (r, d) = (\alpha_k) \) is called the Jordan type of \((\mathcal{F}, \theta)\) and is denoted by \( J(\mathcal{F}, \theta) \). We will call \( s \) the length of \( \alpha \) or \( r \). One good way to understand its definition is to fill the triangular Young tableau \( T_s \) of size \( s \) in the following way (here with \( s=4 \)):

\[
\begin{array}{c c c c}
\alpha_4 & \alpha_3 & & \\
\alpha_4(-l) & \alpha_3 & & \\
\alpha_4(-2l) & \alpha_3(-l) & \alpha_2 & \\
\alpha_4(-3l) & \alpha_3(-2l) & \alpha_2(-l) & \alpha_1 \\
\end{array}
\]

and then notice that
\[ \mathcal{F}_k^\prime = \sum_{i>k} \alpha_i(-kl) \]
is the sum of the classes in the boxes of the \( k \)-th subdiagonal. Hence, \( \mathcal{F}_k = \sum_{i\geq k} \mathcal{F}_i^\prime \) corresponds to the region below this subdiagonal. Denote by \( R \) the sum of the classes of the boxes in a region \( R \). We have, for instance with \( s = 5 \),

\[
\mathcal{F}_1^\prime = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{bmatrix} \quad \mathcal{F}_2 = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{bmatrix} \quad \mathcal{F}_4 = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]

where the classes are summed over the blackened regions. In particular the sum over all boxes is \( \alpha = [\mathcal{F}] \) and we write \( \alpha \vdash \alpha \). This implies the following equality

\[
\sum_k kd_k = d + l \sum_k \frac{k(k-1)}{2} r_k.
\]

We also have
\[
[\ker \theta^k] - [\ker \theta^{k-1}] = [\mathcal{F}_0] - [\mathcal{F}_k (k\Omega)] - \{ [\mathcal{F}_0] - [\mathcal{F}_{k-1} ((k-1)\Omega)] \}
= [\mathcal{F}_{k-1} ((k-1)\Omega)] - [\mathcal{F}_k (k\Omega)]
= \sum_{j>\geq k-1} \alpha_j((k-1-l)l) - \sum_{j>\geq k} \alpha_j((k-l)l)
= - \sum_{j>\geq k-1} \alpha_j(l(j-k+1)(j-k)/2)
+ \sum_{j>\geq k} \alpha_j(l(j-k)(j-k-1)/2)
= \sum_{j>\geq k} \alpha_j((k-j)l)
\]

which corresponds to the \( k \)-th (from the bottom) horizontal strip. Graphically, we have for instance

\[
[\ker \theta] = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{bmatrix} \quad [\ker \theta^3] = \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\]
We call canonical the subsheaves of $\mathcal{F}$ obtained by intersections and sums of the $\mathcal{F}_k$ and the $\ker \theta^k$. The corresponding regions in the Young tableau are the ones saturated in the west, south and south-east directions (note that there is a mistake in the corresponding statement in [Sch16, 3.1]). We denote by $\mathcal{R}$ this set of regions, which we will also call canonical. The slope of the sheaf $\mathcal{F}_R$ corresponding to a region $R \in \mathcal{R}$ is given by

$$
\mu(\alpha)(R) = \frac{d_R}{r_R} = \sum_{\square \in R} \deg \square \sum_{\square \in R} \rank \square
$$

with respect to the filling (2.1) by $\alpha = (r, d) = J(\mathcal{F}, \theta)$ (we will call $\mu(\alpha)(R)$ the $\alpha$-slope of $R$). For instance

$$
\mu(\ker \theta^{2} \cap \mathcal{F}_{2} + \ker \theta \cap \mathcal{F}_{1}) = \mu(\alpha) = \begin{pmatrix}
2 & 3 & 4 & 5
\end{pmatrix} =
\frac{2d_5 + 2d_4 + d_3 + d_2 - 7lr_5 - 5lr_4 - 2lr_3 - lr_2}{2r_5 + 2r_4 + r_3 + r_2}.
$$

The definition of the Jordan type yields a stratification $\Lambda_\alpha = \bigsqcup \alpha \vdash \alpha$ where $\Lambda_\alpha = J^{-1}(\alpha)$.

### 2.2. Irreducible components.

In this section, we will study the map of stacks $\pi_\alpha : \Lambda_\alpha \to \prod_k \Coh_{\alpha_k}$

$$(\mathcal{F}, \theta) \mapsto \big( \ker \{ \mathcal{F}_{k-1}^j ((k-1)\Omega) \to \mathcal{F}_k^j (k\Omega) \} \big)_k$$

for any Jordan type $\alpha$. Denote by $F_\alpha$ the stack of chains of epimorphisms $\mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_s = \{0\}$ satisfying $\alpha_k = |\ker \{ \mathcal{H}_{k-1} \to \mathcal{H}_k \}|$ and write $\pi_\alpha = \rho_\alpha \circ \chi_\alpha$ where

$\chi_\alpha : \Lambda_\alpha \to F_\alpha$

$$(\mathcal{F}, \theta) \mapsto (\mathcal{F}_0^j \to \mathcal{F}_1^j (\Omega) \to \cdots \to \mathcal{F}_s^j (s\Omega))$$

and

$\rho_\alpha : F_\alpha \to \prod_k \Coh_{\alpha_k}$

$$(\mathcal{H}_0 \to \mathcal{H}_1 \to \cdots \to \mathcal{H}_s) \mapsto \big( \ker \{ \mathcal{H}_{k-1} \to \mathcal{H}_k \} \big)_k.$$ 

The following equalities are obtained in [MS17, Proposition 5.2] and [Sch16, 3.1] respectively.

**Proposition 2.3.** The maps $\chi_\alpha$ and $\rho_\alpha$ are iterations of vector bundle stacks and their respective relative dimensions are

$$
d_{\chi_\alpha} = -\sum_k \langle \mathcal{F}_k^j, \mathcal{F}_k^{j+1} \rangle
$$

$$
d_{\rho_\alpha} = -\sum_{i<j} \langle \alpha_j, \alpha_i \rangle.
$$

**Corollary 2.4.** The set of irreducible components of $\Lambda_\alpha$ is

$$(\text{Irr } \Lambda_\alpha) = \{ \Lambda_\alpha | \alpha \vdash \alpha \}.$$
Proof. From Proposition 2.3 we know that the $\Lambda_\alpha$ are irreducible, thus we just have to prove that they have the same dimension. From Proposition 2.3 and the fact that $\dim \text{Coh}_\alpha = -\langle \alpha, \alpha \rangle$, we get

$$\dim \Lambda_\alpha = -\sum_{i > k} \langle \alpha_i (-kl), \mathcal{F}_{k+1} \rangle - \sum_{i \leq j} \langle \alpha, \alpha_i \rangle,$$

Now, since $[\ker \theta^k] = \sum_{j \leq k-1} \mathcal{F}_j (j \Omega)$, we have

$$[\mathcal{F}_k^+] = ([\ker \theta^{k+1}] - [\ker \theta^k]) (-kl) = \sum_{j > k} \alpha_j (1 - j)l),$$

thus

$$\dim \Lambda_\alpha = -\sum_{i > j \geq k} \langle \alpha_i (1 - j)l \rangle - \sum_{i \leq j} \langle \alpha, \alpha_i \rangle$$

$$= -\sum_{i > j \geq k} \langle \alpha_i + l(k + 1 - j)r_i r_j \rangle - \sum_{i \leq j} \langle \alpha, \alpha_i \rangle$$

$$= -\sum_{i \geq j} \left( j \langle \alpha_i, \alpha_j \rangle - l \frac{j(j - 1)}{2} r_i r_j \right)$$

$$- \sum_{i < j} \left( i \langle \alpha_i, \alpha_j \rangle + \frac{i(i + 1 - 2j)}{2} r_i r_j \right)$$

$$= -\sum_{i < j} \langle \alpha_i, \alpha_j \rangle - \sum_i \left( i \langle \alpha_i, \alpha_i \rangle - \frac{i(i - 1)}{2} r_i^2 \right)$$

$$+ l \sum_{i < j} (i - 1) r_i r_j$$

$$= (g - 1)r^2$$

as expected. □

3. The Semistable Locus

We know that the irreducible components of $\Lambda_\alpha$ form a subset of $\text{Irr} \Lambda_\alpha$ which can be now identified, thanks to Corollary 2.4, with the set of all Jordan types of size $\alpha$. We say that a Jordan type is semistable if it appears in $\text{Irr} \Lambda_\alpha$. The aim of this section is to prove the following.

Theorem 3.1. Assume that $g \geq 2$. A Jordan type $\alpha \vdash (r,d)$ is semistable if and only if for every nontrivial canonical subregion $R \in \mathcal{R}$ we have

$$\mu^\alpha (R) \leq \frac{d}{r}. \quad (3.2)$$

Note that this condition is strictly numerical and obviously necessary. This result is optimal in the way that it says that it is sufficient to test (generic) semistability on the most trivial $\theta$-stable subsheaves. To prove it, we will use the results of [BGPGH17], which deals with the moduli stack of chains $\mathcal{E}_s \to \ldots \to \mathcal{E}_1$. In this article is obtained an analogous result in the way that it gives necessary and sufficient conditions on the numerical invariants $(n_s, p_s) = [\mathcal{E}_s]$ of chains for these to be generically semistable (we will call semistable
types of chains such invariants). The conditions obtained therein are somewhat unnatural, in the way that they do not correspond to proper subchains - see [Ibid., Remark 2.11]. However, in this section (c.f. Proposition 3.11), we will build a injection between Jordan types satisfying (3.2) and semistable types of chains, which are related to the semistable components of the nilpotent cone in the following way. We consider stability of chains with respect to the $\alpha_{\text{Higgs}}$-slope

$$
\mu_{\alpha_{\text{Higgs}}} (\mathcal{E}_s) = \frac{\sum_{1 \leq i \leq s} (\deg(\mathcal{E}_i) + \alpha_{\text{Higgs}} \text{ rank}(\mathcal{E}_i))}{\sum_{1 \leq i \leq s} \text{ rank}(\mathcal{E}_i)}
$$

where $\alpha_{\text{Higgs}} = ((i - 1)(2g - 2))_{1 \leq i \leq s}$. Then the direct sum $\bigoplus_k \mathcal{E}_k((k - 1)\Omega)$ of a semistable chain yields a Higgs pair which is a semistable fixed point under the action $t.(\mathcal{F}, \theta) = (\mathcal{F}, t\theta)$ of $G^*$ on $M_{r,d}$, where $\sum_k [\mathcal{E}_k((k - 1)\Omega)] = (r,d)$. Denote by $C_{n^*, p^*}$ the substack of fixed points associated to chains of type $(n^*, p^*)$, and by $C_{n^*, p^*}^{-}$ the corresponding attracting variety

$$
C_{n^*, p^*}^{-} = \{ ((\mathcal{F}, \theta) | \lim_{t \rightarrow 0} t.(\mathcal{F}, \theta) \in C_{n^*, p^*} \}.
$$

It is known (see e.g. [BGPGH17] 6) that the closures of these attracting varieties, for $(n^*, p^*)$ of semistable type, are unions of irreducible components of $\Lambda^{\text{sst}}_{r,d}$. Hence, building the injection announced will imply Theorem 3.1 as all inequalities in the following chain will have to be equalities:

$$
\# \text{ Irr } \Lambda_{r,d}^{\text{sst}} \leq \# \{ \alpha \triangleright (r, d) | (3.2) \} \\
\leq \# \{ \text{semistable types } (n^*, p^*) | \sum_k (n_k, p_k)((k - 1)l) = (r, d) \} \\
\leq \# \text{ Irr } \Lambda_{r,d}^{\text{sst}}.
$$

The last equality can be stated as

**Corollary 3.3.** The closures of the semistable attracting cells $C_{n^*, p^*}^{-}$ are irreducibly regardless of the coprimality of $r$ and $d$.

This was only known in the coprime case.

Fix for now a type $\alpha$ of length $s$ satisfying (3.2), and $(\mathcal{F}, \theta) \in \Lambda_{s}$. We introduce a couple of notions before rephrasing the main result of [BGPGH17] in our context (c.f. Proposition 3.7).

**Definition 3.4.** We call 1-flags the flags $R_s = (\emptyset = R_0 \subset R_1 \subset \cdots \subset R_t = T_s)$ of subregions of $T_s$ such that:

(i) $t = s$;
(ii) $R_k \in \mathcal{R}$;
(iii) if the number of boxes in a given column of $T_s$ is increased by 1 from $R_k$ to $R_{k+1}$, the same must be true for every column on its left.

We call strips the (noncanonical) subregions $S_k = R_k \setminus R_{k-1}$, and 1-chains the chains $\mathcal{E}_s = (E_s \rightarrow \ldots \rightarrow E_1)$ associated to a given 1-flag $R_s$, where $E_{k} = T_{R_k} / T_{R_{k-1}}(-k(1)\Omega)$, and the morphisms are induced by $\theta$.

**Remark 3.5.**

- A 1-chain $\mathcal{E}_s$ depends on the data $(R_s, \mathcal{F}, \theta)$, but its type $(n^*, p^*) = [\mathcal{E}_s]$ only depends on $(R_s, \alpha)$.
- The 1- notation comes from the fact that $ht(S_k) = 1$ (the height being the number of boxes in the higher column), which is why $\theta$ induces morphisms $\mathcal{E}_{k+1} \rightarrow \mathcal{E}_k$.
- Each strip has a box on the left border of $T_s$.

**Example 3.6.**
The flag of regions \((s = 3)\)

\[
\begin{array}{c}
\framebox{1} & \framebox{2} & \framebox{3} \\
\framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9}
\end{array}
\]

is not a 1-flag, as condition (iii) is not satisfied by \(R_1 \subset R_2\).

The flag of regions \((s = 3)\)

\[
\begin{array}{c}
\framebox{1} & \framebox{2} & \framebox{3} \\
\framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9}
\end{array}
\]

is not a 1-flag, as condition (ii) is not satisfied by \(R_2\) (region not saturated in the south-east direction).

Consider a 1-flag \(R_*\) and the associated 1-chain \(E_*\) (again, the pair \((F, \theta)\) is fixed for now). We will denote by \(|E_k|\) the number of boxes in \(S_k\) and set \((n_k, p_k) = [E_k]\).

Thanks to Definition 3.4, each 1-flag can be seen as the permutation \(\sigma\) on \(s\) elements given by

\[
(\sigma(1), \ldots, \sigma(s)) = (|E_1|, \ldots, |E_s|).
\]

It will be convenient to represent 1-flags with horizontal strips, for instance write

\[
R_* = \begin{array}{c c c c c c}
\framebox{1} & \framebox{2} & \framebox{3} & \framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9} & \framebox{10} & \framebox{11} & \framebox{12} \\
\framebox{13} & \framebox{14} & \framebox{15} & \framebox{16} & \framebox{17} & \framebox{18}
\end{array}
\]

for the 1-flag associated to \(\sigma = (3, 2, 5, 4, 1, 6)\).

Take two integers \(1 \leq k < j \leq s\). Note that \(n_j < n_k\) implies \(|E_j| < |E_k|\). When \(n_j < \min\{n_k, \ldots, n_{j-1}\}\), the \(\alpha\)-Higgs-slope of the chain (see [BGPGH17, Definition 2.10])

\[
E_s \rightarrow \ldots \rightarrow E_j = \cdots = E_j \rightarrow E_{k-1} \rightarrow \ldots \rightarrow E_1
\]

is the \(\alpha\)-slope of the (noncanonical!) subregion obtained by only considering the \#\(S_j\) leftmost boxes in each \(S_t, k \leq t \leq j\). We denote by \(\tilde{R}_{jk}\) the complementary of this region. For instance if \(s = 6\) and \(\sigma = (3, 2, 5, 4, 1, 6)\), and if we represent the strips \(S_t\) horizontally, we get

\[
R_* = \begin{array}{c c c c c c}
\framebox{1} & \framebox{2} & \framebox{3} & \framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9} & \framebox{10} & \framebox{11} & \framebox{12} \\
\framebox{13} & \framebox{14} & \framebox{15} & \framebox{16} & \framebox{17} & \framebox{18}
\end{array}
\]

\[
\Rightarrow R_3^4 = \begin{array}{c c c c c c}
\framebox{1} & \framebox{2} & \framebox{3} & \framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9} & \framebox{10} & \framebox{11} & \framebox{12} \\
\framebox{13} & \framebox{14} & \framebox{15} & \framebox{16} & \framebox{17} & \framebox{18}
\end{array}, \quad R_2^6 = \begin{array}{c c c c c c}
\framebox{1} & \framebox{2} & \framebox{3} & \framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9} & \framebox{10} & \framebox{11} & \framebox{12} \\
\framebox{13} & \framebox{14} & \framebox{15} & \framebox{16} & \framebox{17} & \framebox{18}
\end{array}
\]

where the black boxes are the ones contained in the region. Note that in order to take the \(\alpha\)-slope, one has to project the boxes in the south direction, and then proceed to the previously mentioned filling (2.1) of \(T_s\). For instance in our example \(\mu^\alpha(R_3^4) = \mu(\alpha_2(-l))\).

Similarly, when \(n_k < \min\{n_{k+1}, \ldots, n_j\}\), the \(\alpha\)-Higgs-slope of the chain

\[
E_s \rightarrow \ldots \rightarrow E_{j+1} \rightarrow E_k = \cdots = E_k \rightarrow \ldots \rightarrow E_1
\]

is the \(\alpha\)-slope of the subregion obtained by only considering the \#\(S_k\) leftmost boxes in each \(S_t, k \leq t \leq j\). We denote by \(\tilde{R}_{k}\) the complementary of this region, and with the same example \(\sigma = (3, 2, 5, 4, 1, 6)\), we have

\[
\tilde{R}_2^6 = \begin{array}{c c c c c c}
\framebox{1} & \framebox{2} & \framebox{3} & \framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9} & \framebox{10} & \framebox{11} & \framebox{12} \\
\framebox{13} & \framebox{14} & \framebox{15} & \framebox{16} & \framebox{17} & \framebox{18}
\end{array}, \quad \tilde{R}_5^6 = \begin{array}{c c c c c c}
\framebox{1} & \framebox{2} & \framebox{3} & \framebox{4} & \framebox{5} & \framebox{6} \\
\framebox{7} & \framebox{8} & \framebox{9} & \framebox{10} & \framebox{11} & \framebox{12} \\
\framebox{13} & \framebox{14} & \framebox{15} & \framebox{16} & \framebox{17} & \framebox{18}
\end{array},
\]

where \(\mu^\alpha(\tilde{R}_5^6) = \sum_{1 \leq t \leq 5} \alpha_t\). The article [BGPGH17] gives necessary and sufficient conditions (C0,C1,C2,C3) for a type \((n_*, p_*)\) to be semistable, meaning that generically,
chains of this type are semistable with respect to the $\alpha^{\text{Higgs}}$-slope. In our context, it yields the following.

**Proposition 3.7.** A 1-chain $E_\bullet$ is of semistable type if it satisfies

$$n_{i-1} = n_i \Rightarrow p_i \leq p_{i-1} \quad (C_0)$$

and if for any $1 \leq k < j \leq s$, the associated 1-flag $R_\bullet$ satisfies

$$\mu^\alpha(R_k) \leq \mu \quad (C_k)$$

$$n_j < \min\{n_k, \ldots, n_{j-1}\} \Rightarrow \mu^\alpha(R'_k) > \mu \quad (C'_k)$$

$$n_k < \min\{n_{k+1}, \ldots, n_j\} \Rightarrow \mu^\alpha(R''_k) \leq \mu \quad (C''_k)$$

where $\mu = d/r = \mu(\mathcal{F})$.

**Remark 3.8.**

- Note first that (3.2) \Rightarrow (C_k) for any 1-flag $R_\bullet$.
- Also, if $\sigma$ is the permutation associated to a 1-chain $E_\bullet$ of type satisfying $n_{i-1} = n_i$ but $p_{i-1} < p_i$, one can always multiply on the left by the transposition $(i-1, i)$ in order to satisfy the condition $(C0)$ - without impacting any of the other conditions.

We are going to construct a 1-chain satisfying this set of conditions, under the assumption (3.2). We define recursively a particular 1-flag. Assuming that $R_k$ is constructed, set $R_k \in \mathcal{R}$ such that $\text{ht}(R_k \setminus R_k) = 1$. We define $R_k+1 \in \mathcal{R}$ as the minimal region such that $R_k \subseteq R_{k+1} \subseteq R_k$ and $\mu^\alpha(R_k \setminus R_{k+1}) \leq \mu$ (with the convention $\mu(\emptyset) = -\infty$). This construction ensures that $R_\bullet$ satisfies $(C'_k)$ for any (admissible) $k, j$. Indeed, if $r < t$ satisfies $n_t < n_r$, we have

$$\mu^\alpha(\mathcal{S}_r \setminus \{\text{the } \#S_t \text{ leftmost boxes}\}) > \mu$$

by definition (every item in Definition 3.4 is crucial!). This is still true after the eventual application of transpositions ensuring $(C0)$. For instance, if $\sigma = (3, 2, 5, 4, 1, 6)$, $r = 2$, $t = 5$, we necessarily have

$$\mu^\alpha(\text{the } \#S_2 \text{ leftmost boxes}) > \mu$$

otherwise $S_2$ would have been reduced to one box.

Unfortunately, $R_\bullet$ might not satisfy $(C''_k)$, and we are going to make some mutations in order to make the (finite) number of unfulfilled conditions decrease. Consider $k$ such that $(C''_k)$ is unfulfilled for some $j$, and maximizing $|E_k|$. Then pick $j$ minimal such that $(C''_k)$ is unfulfilled. Denote by $E_\bullet$ and $\sigma$ the chain and permutation associated to $R_\bullet$, and set

$$\rho^\sigma_k(E_\bullet) = (|E_1|, \ldots, |\mathcal{E}_{k-1}|, |\mathcal{E}_k|, \ldots, |\mathcal{E}_s|)$$

where the symbol $\sim$ means that we remove the underneath entry (this is just the multiplication on the right by the cycle $(k, \ldots, j)$). Denote by $\rho^\sigma_k(E_\bullet)$ and $\rho^\sigma_k(R_\bullet)$ the corresponding chain and flag. If $\sigma = (3, 2, 5, 4, 1, 6)$, $k = 2, j = 4$, we have

$$\rho^\sigma_2R_\bullet = \begin{array}{c}
\hline
\hline
\hline
\hline
\hline
\end{array}$$

We prove the sturdiness of this process through the rather technical following lemma.

**Lemma 3.10.** In the setting described above, $\rho^\sigma_kR_\bullet$ still satisfies all conditions $(C''_k)$. 
Proof. Set \( m_k = \text{rank}(\rho_k^J \mathcal{C}_k) \) and consider \( k' < j' \) such that \( m_{j'} < \min\{m_k, \ldots, m_{j'-1}\} \).

Case 1. If \( j' \leq k - 1 \) or \( k' \geq j + 1 \), we have \((\rho^j_k R)^{j'}_{k'} = R^{j'}_{k'}\) hence the condition \((C^j_{k'})\) is satisfied because it is for \( R_* \).

Case 2. If \( k' < j' \leq j - 1 \), we have \((\rho^j_k R)^{j'}_{k'} = R^{j'+1}_{k'+1}\) hence the condition \((C^j_{k'})\) is satisfied because \((C^j_{k'+1})\) is for \( R_* \).

For instance if \( \sigma = (3, 2, 5, 4, 1, 6) \), \( k = 2 \), \( j = 4 \), and \( k' = 2 \), \( j' = 3 \):

\[
(\rho^2_3 R)^{3}_{3} = \quad \text{and } R^3_3 = \\
\]

have same \( \alpha \)-slope = \( \mu(\alpha_2(-l)) \).

Case 3. If \( k' < j = j' \leq j - 1 \), we have \( \mu^\alpha((\rho^j_k R)^{j'}_{k'}) > \mu \) thanks to the property (3.9) with \( t = k + 1 \) and \( r = k', \ldots, k - 1 \).

Case 4. If \( k' < k < j' \leq j - 1 \), we have \((\rho^j_k R)^{j'}_{k'} = (\rho^j_k R)^{j'}_{k'} \cup Z\) where the first piece has its \( \alpha \)-slope > \( \mu \) thanks to Case 2, and the second piece \( Z \) too for the same reasons than in Case 3.

Case 5. If \( k' \leq k - 1 \) and \( j' \geq j + 1 \), since then \( [k, j] \subset [k', j'] \), we have again \((\rho^j_k R)^{j'}_{k'} = R^{j'}_{k'}\).

For instance if \( \sigma = (3, 2, 5, 4, 1, 6) \), \( k = 2 \), \( j = 4 \), and \( k' = 1 \), \( j' = 5 \):

\[
(\rho^2_3 R)^{5}_3 = \quad \text{and } R^5_1 = \\
\]

have same \( \alpha \)-slope.

The two following Cases 6 & 7 are the most problematic \( a \ priori \): we assume \( j = j' \). We are going to use the actual definition of \( j \): the smallest integer \( k' \) such that \((\mathcal{C}^j_{k'})\) is unfulfilled.

Case 6. If \( j' = j \) and \( k' \geq k \). We have

\[
R^{k'}_k \cup (\rho^j_k R)^{j'}_{k'} = R^j_k, \\
\]

where

\[
\mu^\alpha(R^{k'}_k) \leq \mu < \mu^\alpha(R^j_k) \\
\]

by minimality of \( j \), hence \( \mu^\alpha((\rho^j_k R)^{j'}_{k'}) > \mu \), as expected.

For instance if \( \sigma = (3, 2, 5, 4, 1, 6) \), \( k = 2 \), \( j = 4 \), and \( k' = 3 \), \( j' = 4 \):

\[
(\rho^2_3 R)^{4}_3 = \quad (R^4_2 = \quad (R^3_2 = \\
\]

has a \( \alpha \)-slope > \( \mu \).

Case 7. If \( j' = j \) and \( k' \leq k - 1 \), we have

\[
(\rho^j_k R)^{j'}_{k'} = R^{k'}_k \cup R^j_k \\
\]

where \( \mu^\alpha(R^j_k) > \mu \) since, by definition of \( k, j \), \((\mathcal{C}^j_{k})\) is unfulfilled, and \( \mu^\alpha(R^{k'}_k) > \mu \) (since \((\mathcal{C}^j_{k'})\) is satisfied by \( R_* \)), hence we are also done.
For instance if $\sigma = (3, 2, 5, 4, 1, 6)$, $k = 2$, $j = 4$, and $k' = 1$, $j' = 4$:

\[
(r_2^4 R_1^4)^j = (\mathcal{F}_1^4 = \{ \cdots \}) \cup (R_2^4 = \{ \cdots \})
\]

has an $\alpha$-slope $> \mu$.

Case 8. We are left with $j' \geq j + 1$ and $k \leq k' \leq j$. Set

\[
Z_p = S_p \setminus \{ \text{the } \#S_j' \text{ leftmost boxes} \}
\]

\[
Z = Z_k \cup \bigcup_{k'+1 \leq p \leq j} Z_p
\]

\[
Z^+ = Z_k(j - k) \cup \bigcup_{k'+1 \leq p \leq j} Z_p(-1)
\]

where $Z_p(h)$ the region obtained after translating $h$ times in the north direction the (possibly empty) subregion

\[
\{ \#S_k \text{ leftmost boxes} \} \setminus \{ \text{the } \#S_j' \text{ leftmost boxes} \} \subseteq Z_p.
\]

From (2.1), we see that

\[
\mu^\alpha(Z^+) = \mu^\alpha(Z) + \lambda((j - k) - (j - k'))l
\]

\[
= \mu^\alpha(Z) + \lambda(k' - k)l
\]

\[
\geq \mu^\alpha(Z)
\]

where $\lambda \in [0, 1]$.

We have $n_{j'} = m_{j'} < m_j = n_k$ and $k < j'$, hence from (3.9), we know that $\mu^\alpha(Z_k) > \mu$. For the same reason, we also have $\mu^\alpha(Z_p) > \mu$ for $k' + 1 \leq p \leq j$, hence $\mu^\alpha(Z) > \mu$. Now

\[
(r_2^4 R_1^4)^j = Z^+ \cup (r_2^4 R_1^4)^j_{j'+1}
\]

with $\mu^\alpha((r_2^4 R_1^4)^j_{j'+1}) > \mu$ from Case 1, and $\mu^\alpha(Z^+) \geq \mu^\alpha(Z) > \mu$, and we are done.

For instance if $\sigma = (3, 2, 5, 4, 1, 6)$, $k = 2$, $j = 4$, and $k' = 3$, $j' = 5$:

\[
\mu^\alpha((r_2^3 R_1^5)^5) = \mu^\alpha(Z^+) = \mu^\alpha(\{ \cdots \}) \geq \mu^\alpha(Z) = \mu^\alpha(\{ \cdots \}) > \mu.
\]

As explained at the beginning of the section, the following completes the proof of Theorem 3.11

**Proposition 3.11.** There is an injective map $\kappa$ from the set of Jordan types $\alpha \vdash (r, d)$ satisfying (3.2) to the set of semistable types of chains $\mathcal{E}_\bullet$ such that $\sum_{k} [\mathcal{E}_k((k-1)\Omega)] = (r, d)$.

**Proof.** We keep the same setting: we start with $(\mathcal{F}, \theta) \in \Lambda_{\alpha}$, build $R_\bullet$ satisfying every $(C_{k'}^j)$ and apply $\rho_k^j$ if $(\mathcal{C}_{k'}^j)$ is unfulfilled, for $k$ maximizing $|\mathcal{E}_k|$, and $j$ minimal with respect to this $k$. We iterate the process in order to reduce the number of unfulfilled $(C_{k'}^j)$. It might be necessary to apply $\rho_k^j$ to $\rho_k^j \mathcal{E}_\bullet$ for some $l$, which would affect the same maximal strip of size $|\mathcal{E}_k|$. Since this strip can not go north forever, there is a finite composition of permutations $\rho_k = \cdots \rho_j \rho_k' \rho_k^j$ that only affects the strip of size $|\mathcal{E}_k|$ such that, from Lemma 3.10, $\rho_k \mathcal{E}_\bullet$ satisfies every $(C_{k'}^j)$ and every $(\mathcal{C}_{k'}^j)$.
We then proceed by descending induction on $|E_k|$. Assume that a 1-flag $R_\bullet$ is defined, satisfying all $|C_j^k|$, and all $|C_h^k|$ for $h \in H$ such that $k \not\in H \ni h$ implies $|E_k| < |E_h|$. Consider $k$ maximizing these $|E_k|$, $k \not\in H$, such that $|C_j^k|$ is unfulfilled for some $j$. Pick a minimal such $j$ and apply $\rho_{j,k}^\prime$. The only arguments in the proof of Lemma 3.10 that may eventually no longer work are the ones invoking (3.9), i.e. in Cases 3 & 8. But

$$\mu^\alpha(S_r \setminus \{\text{the } \# S_r \text{ leftmost boxes}\}) > \mu$$

is true as long as the relative position of the strips $S_r$ and $S_t$ is the same as in the initial 1-flag $R_\bullet$. Proceeding by descending induction ensures that this is true in all involved Cases (which are not Cases 6 & 7!). It is also important to notice that a mutation can not make a strip pass north of a shorter one. Consider for instance the case $s = 3$, and assume that after applying potential mutations to the strips of size 3 and 2 (in this order!), one gets the following 1-flag

$$R_\bullet' = \begin{array}{c}
\rule{1cm}{0.5mm}
\rule{1cm}{0.5mm}
\end{array}$$

and that the condition $(C_2^3)$ is unfulfilled, i.e.

$$\mu^\alpha \left( \begin{array}{c}
\rule{1cm}{0.5mm}
\rule{1cm}{0.5mm}
\end{array} \right) = \mu(\alpha_2) > \mu$$

so that we need to consider

$$\rho_{2,3}^\prime R_\bullet' = \begin{array}{c}
\rule{1cm}{0.5mm}
\rule{1cm}{0.5mm}
\end{array}$$

Now consider condition $(C_1^3)$, which fits in Case 3 of the proof of Lemma 3.10: we need to check that

$$\mu^\alpha \left( \begin{array}{c}
\rule{1cm}{0.5mm}
\rule{1cm}{0.5mm}
\end{array} \right) > \mu$$

which is ensured by (3.9) if initially the strip of size 3 was beneath the strip of size 2. This is true from the shape of $R_\bullet'$: the mutation affecting the strip of size 2 can not have made it pass north of the strips of size 1 and 3 (can only pass strips of larger size).

Hence, after a finite number of steps, all conditions $(C_1^k)$ and $(C_h^k)$ will be satisfied, and Proposition 3.7 ensures that the 1-chain thus produced is of semistable type. Let $(n_\bullet, p_\bullet)$ be this type, the map $\kappa : \alpha \mapsto (n_\bullet, p_\bullet)$ hence built is injective because any type of 1-chain clearly characterizes $\alpha$. □

One could wonder how geometric this bijection is. We state the following conjecture.

**Conjecture 3.12.** For any semistable Jordan type $\alpha$, we have $\overline{\Lambda_\alpha} = \overline{C_{\kappa(\alpha)}}$.

### 4. POLYTOPAL DESCRIPTION

We first reformulate Theorem 3.1 in order to use a slightly different set of inequalities to characterize the semistability. For any region $R \in \mathcal{R}$ we denote by $R_k$ the height of its $k$-th column, counted from the right. For instance

$$R = \begin{array}{c}
\rule{1cm}{0.5mm}
\rule{1cm}{0.5mm}
\end{array} \Rightarrow (R_5, R_4, R_3, R_2, R_1) = (2, 2, 1, 1, 0).$$

We set $\mathcal{R}_p = \{R \in \mathcal{R} \mid R_1 = \ldots = R_{p-1} = 0 \neq R_p\}$ and $\mathcal{R}_{>1} = \cup_{p>1} \mathcal{R}_p$. Note that from the definition of $\mathcal{R}$, there is a bijective map $\mathcal{R} \to \mathcal{R}$ sending $R = (R_k)$ to
\[ \bar{R} = (k - R_k). \] For instance
\[ \bar{R} = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array} \quad \text{and we see that this map swaps } \mathcal{R}_1 \text{ and } \mathcal{R}_{>1}. \] For any Jordan type \((r, d) \rightarrow (r, d)\) and any region \(R \in \mathcal{R}\), we finally define
\[ p_R(r) = \sum_{\square \in R} \text{rank } \square = \sum_k R_k r_k. \]

**Proposition 4.1.** A Jordan type \(\alpha = (r, d) \vdash (r, d)\) is semistable if and only if for any region \(R \in \mathcal{R}_{>1}\) we have
\[ p_R(r) d + l \sum_k \frac{R_k(R_k - 1)}{2} r_k \leq \sum_k R_k(R_k + k - 1) r_k. \]

**Remark 4.3.** Note that the bounds do not depend on \(d\) and that \(d_1\) never appears in the central term. Hence, because of (2.2), it is only subject to
\[ d_1 = d - \sum_{k \geq 2} kd_k + l \sum_k \frac{k(k - 1)}{2} r_k. \]

**Proof.** First note that
\[ d_R = \sum_{\square \in R} \text{deg } \square = \sum_k R_k d_k - l \sum_k \frac{R_k(R_k + k - 1)}{2} r_k \]
so that (3.2) is equivalent to
\[ \sum_k R_k d_k \leq p_R(r) d + l \sum_k \frac{R_k(R_k + k - 1)}{2} r_k. \]
Also,
\[ d_R = d - d_R - l \sum_k R_k(R_k - 1) r_k \]
so that (3.2) with respect to \(\bar{R}\) is equivalent to
\[ d - d_R - l \sum_k R_k(R_k - 1) r_k \leq p_R(r) d - (1 - p_R(r)) d \]
\[ \Leftrightarrow p_R(r) d + l \sum_k \frac{R_k(R_k - 1)}{2} r_k \leq \sum_k R_k d_k. \]
This concludes the proof since \(\mathcal{R} = \mathcal{R}_1 \sqcup \mathcal{R}_{>1}\). □

Assume that \(r > 0\). Then if \((\mathcal{F}, \theta) \in A_{\alpha}^{\ast}\) has type \(\alpha\) of length \(s\), we necessarily have \(r_s > 0\) since \(r_s = \text{rank } \mathcal{F}_{s-1} \subset \mathcal{F}\).

**Corollary 4.5.** Fix a partition \(r \vdash r\) of length \(s\). The set of degree vectors \(d = (d_1, \ldots, d_s)\) such that \((r, d) \vdash (r, d)\) is semistable is the intersection \(\mathcal{P}_{r,d}^Z\) of the integral lattice \(Z^s\) with a convex \((s - 1)\)-polytope \(\mathcal{P}_{r,d}\).

**Proof.** The polytope \(\mathcal{P}_{r,d}\) is defined by the set of linear inequalities (4.2) together with the hypothetical extra conditions \(d_k \geq 0\) every time we have \(r_k = 0\). It is \((s - 1)\)-dimensional because of (4.4). Also because of this equation, note that a facet may be given by the equation
\[ \sum_{k \geq 2} kd_k = d + l \sum_k \frac{k(k - 1)}{2} r_k \]
if \( r_1 = 0 \).

We get the following correspondence between polytopes associated to different degrees.

**Proposition 4.7.** Consider \( d, d' \in \mathbb{Z} \) and fix a partition \( \tau \vdash \rho \) of length \( s \). The translation \( \tau = (d - d')\rho/\rho' \) of \( \mathbb{R}^s \) induces a bijection \( \mathcal{P}_{\tau,d} \sim \mathcal{P}_{\tau,d'} \).

**Proof.** First rewrite (4.2) in the following way

\[
(4.8) \quad b^+_R(r) = \frac{1}{2} \sum_k R_k (R_k + k - 1) r_k,
\]

where \( b^+_R(r) = l \sum_k R_k r_k, \quad b^-_R(r) = l \sum_k R_k (R_k - 1) r_k, \quad \tau \cdot R = \sum_k R_k r_k, \quad d \cdot R = \sum_k R_k d_k. \)

Note that since we consider \( R \in \mathbb{R}_{>1} \) we can replace \( d \) by \( d^* = (0, d_2, \ldots, d_s) \) everywhere. Finally, also note that if \( r_1 = 0 \) the equation (4.6) fits in this study since it can be written

\[
(1) \quad d^* - \frac{r}{r^*} \cdot T_s = l \sum_k k(k-1) r_k,
\]

where \( T_s = (k) \) corresponds to the full tableau.

It is not clear how this bijection restricts to integral points, but we know, as explained in the introduction, that thanks to [Mel17] we have the following.

**Theorem 4.9.** The quantity \( \sum_{\tau \vdash \rho} \# \mathcal{P}_{\tau,d} \) does not depend on degrees \( d \) coprime to \( \tau \), and is equal to \( A_{\rho,\tau}(1) \).

The Proposition 4.7 together with the rather explicit description (4.8) however shed an interesting light on Mellit’s independence result, given our direct geometric approach of the global nilpotent cone, and its combinatorial flavor.

In fact, Rodríguez Villegas defined in [RV11] a refinement \( A_{\rho,\tau} \) of \( A_{\rho,\tau} \) for each partition \( \tau \) of \( \rho \), satisfying \( A_{\rho,\tau}(q) = \sum_{\tau \vdash \rho} A_{\rho,\tau}(q) \), and establishes closed formulas for the quantities \( A_{\rho,\tau}(1) \). Based on computations for small values of \( l(\tau) \) and the case \( \tau = (1^n) \) established by Reineke [Rei12, §7], the following is expected.

**Conjecture 4.10.** We have \( \# \mathcal{P}_{\tau,d} = A_{\rho,\tau}(1) \).

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