Isoperimetric upper bound for the first eigenvalue of discrete Steklov problems

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Abstract

We study upper bounds for the first non-zero eigenvalue of the Steklov problem defined on graphs with boundary. For graphs with boundary included in a Cayley graph associated to a group of polynomial growth, we give an upper bound for the first non-zero Steklov eigenvalue depending on the number of vertices of the graph and of its boundary. As a corollary, if the graph with boundary also satisfies a discrete isoperimetric inequality, we show that the first non-zero Steklov eigenvalue tends to zero as the number of vertices of the graph tends to infinity. This was recently shown by Han and Hua for the case of $\mathbb{Z}^n$. We obtain the result using metric properties of Cayley graphs associated to groups of polynomial growth.

1 Introduction

Let $M$ be a compact Riemannian manifold of dimension $n \geq 2$ with boundary $\partial M$. The Steklov problem on $M$ is

$$
\begin{cases}
\triangle u = 0 \quad \text{in } M \\
\frac{\partial u}{\partial n} = \sigma u \quad \text{on } \partial M
\end{cases}
$$

where $\triangle$ is the Laplace-Beltrami operator and $\frac{\partial u}{\partial n}$ is the outward normal derivative along the boundary $\partial M$. It is a well known result that if the boundary is sufficiently regular, the spectrum of the Steklov problem is discrete and its eigenvalues form a sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow \infty$.

An important question in studying the spectral geometry of the Steklov problem is to maximize its eigenvalues under a constraint on the volume of
the boundary or on the volume of the manifold. For simply-connected planar domains of fixed length of boundary, it has been shown by R. Weinstock that the disk maximizes $\sigma_1$. For bounded Lipschitz domains of fixed volume in $\mathbb{R}^n$, F. Brock proved that the ball maximizes $\sigma_1$. Several upper bounds have also been obtained for different families of manifolds where the volume or the volume of the boundary is fixed. In 2014, a survey of the literature on this question has been given in [7]. In 2017, it was shown in [1] that the Weinstock inequality holds in $\mathbb{R}^n$ in the class of convex sets.

In this article, we investigate isoperimetric upper bounds for $\sigma_1$ of the Steklov problem on graphs. The Steklov problem on graphs is a discrete analogue of the Steklov problem and has recently received attention in the literature. In [10] and [11], lower bounds for the first non-zero eigenvalue are given. A lower bound for higher eigenvalues is given in [3]. For subgraphs of integer lattices, an upper bound has been obtained by W. Han and B. Hua [9]. In [3], a relation between the eigenvalues of the Steklov problem on a manifold and the eigenvalues of a discrete problem is established. Hence, results in the discrete and in the Riemannian settings are closely related and the study of the discrete problem is a possible approach to understand the spectral geometry of the Steklov problem.

A graph with boundary is a pair $(\Gamma, B)$ where $\Gamma = (V, E)$ is a simple graph, that is, without loops or multiple edges, and $B \subset V$ is a subset of $V$, called the boundary, such that two vertices of $B$ are not joined by an edge. The Steklov problem on a graph with boundary $(\Gamma, B)$ is to find all $\sigma \in \mathbb{R}$ for which there exists a non-zero function $v$ on the vertices such that

$$
\begin{cases}
(\triangle v)(i) = 0 & \text{if } i \notin B \\
(\frac{\partial v}{\partial n})(i) = \sigma v(i) & \text{if } i \in B
\end{cases}
$$

where $\triangle$ is the discrete Laplacian, and $\frac{\partial v}{\partial n}$ is the discrete normal derivative (a definition of these objects can be found in Section 2). Its eigenvalues are $0 = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_{|B|-1}$.

We say that a graph with boundary $(\Gamma' = (V', E'), B)$ is included in a graph $\Gamma = (V, E)$ if $V' \subset V$ and $E' \subset E$. A particular case of graphs with boundary included in a given graph $\Gamma = (V, E)$ are what we call the subgraphs of $\Gamma$. They are defined from a subset $\Omega$ of $V$ in the following way: given a graph $\Gamma = (V, E)$ and a subset $\Omega \subset V$, we define $\delta \Omega := \{i \in \Omega^c : i \sim j \text{ for some } j \in \Omega\}$, $\Omega := \Omega \cup \delta \Omega$, and $E(\Omega, \bar{\Omega}) := \{\{i, j\} \in E : i \in \Omega, j \in \bar{\Omega}\}$; then $(\Gamma' = (\Omega, E(\Omega, \bar{\Omega})), \delta \Omega)$ is a graph with boundary that we call subgraph...
of $\Gamma = (V, E)$. The distinction between subgraphs of a given graph $\Gamma = (V, E)$ and graphs with boundary included in $\Gamma = (V, E)$, without being necessarily a subgraph, is illustrated in Section 2. This distinction is important in this article because some of the given results require us to consider subgraphs of a given graph.

We recall that we are interested in upper bounds for $\sigma_1$. A first remark is that without any constraint on $(\Gamma, B)$, it is easy to find examples (see, e.g., Example 1 in [11]) showing that it is not possible to bound from above $\sigma_1$ uniformly in terms of the inverse of the number of vertices of the graph. In [9], W. Han and B. Hua brought forward the idea of considering subgraphs of $\mathbb{Z}^n$. In this setting, they obtain an upper bound for $\sigma_1$, which proves that for a sequence $\{\Omega_n\}_{n \in \mathbb{N}}$ of finite subsets in $\mathbb{Z}^n$ satisfying $|\Omega_n| \to \infty$, we have that $\sigma_1$ of $\Omega_n$ tends to 0 as $n$ tends to infinity. To prove this, Han and Hua use a very interesting method to reduce to the Euclidean case. In the present article, we study wider families of graphs and show that the result can be generalized with a more direct proof using qualitative metric geometry methods.

The idea is to consider graphs with boundary included in a Cayley graph of a group with polynomial growth (we recall the notions of geometric group theory that we use in Section 2.2). Our main result is the following (Theorem 1, Section 3.1). Given a Cayley graph $\Gamma = (V, E)$ of a group with polynomial growth of order $D$, there exists a constant $C(\Gamma) > 0$ such that for any graph with boundary $(\Gamma' = (V', E'), B)$ included in $\Gamma$,

$$\sigma_1(\Gamma', B) \leq C(\Gamma) \frac{|V'|^{\frac{D-2}{D}}}{|B|}.$$  

Because the subgraphs of a Cayley graph of a group with polynomial growth satisfy a discrete isoperimetric inequality, we can deduce the following two corollaries for subgraphs of $\Gamma = (V, E)$ given by a subset $\Omega \subset V$ (Corollaries 2 and 3, Section 3.2): if $D \geq 2$, there exists $C(\Gamma) > 0$ such that for any subset $\Omega$ of $V$

$$\sigma_1(\Omega) \leq C(\Gamma) \frac{1}{|\delta \Omega|^{\frac{1}{D-1}}};$$

there exists $C(\Gamma) > 0$ such that for any subset $\Omega$ of $V$

$$\sigma_1(\Omega) \leq C(\Gamma) \frac{1}{|\Omega|^\frac{1}{D}}.$$
A direct consequence is that for a sequence of subgraphs such that the number of vertices tends to infinity, $\sigma_1$ tends to zero.

In $\mathbb{Z}^n$ this last result corresponds to Corollary 1.4 in [9] but we do not give explicit constants as were given in [9]. The proof of our main result essentially uses the control of the growth function of the Cayley graph. The method was inspired by the methods used in [2]. In the contrast to the proof of the result of Han and Hua, the proof of our result is direct because it does not use known results for domains in Euclidean space. A straightforward example of a Cayley graph of a group of polynomial growth that is different from $\mathbb{Z}^n$ is a Cayley graph associated to the discrete Heisenberg group of dimension 3, which has polynomial growth of order 4. Many other examples exist (see Example 3) where the result holds.

2 Preliminaries

2.1 The Steklov problem on graphs

Let $\Gamma = (V, E)$ be a graph. The boundary of a subset $\Omega \subset V$ is the set $\delta \Omega := \{i \in \Omega^c : i \sim j \text{ for some } j \in \Omega\}$ where $i \sim j$ signifies that $\{i, j\} \in E$. This set is sometimes called vertex boundary in the literature (there is also a notion of edge boundary that we will not use here). Given two subsets $\Omega_1$, $\Omega_2 \subset V$, the set of edges between these two subsets is $E(\Omega_1, \Omega_2) := \{e = \{i, j\} \in E : i \in \Omega_1, j \in \Omega_2\}$. The degree of a vertex $i$ is denoted $d(i)$. In this article, we will always consider simple graphs, that is without loops or multiple edges. The distance between two vertices $i$ and $j$ is the number of edges in the shortest path joining $i$ and $j$. It is denoted $d(i, j)$.

**Definition 1.** A graph with boundary is a pair $(\Gamma, B)$, where $\Gamma = (V, E)$ is a simple graph and $B \subset V$ is a subset of $V$ such that $E(B, B) = \emptyset$. We call $B$ the boundary of the graph and $B^c$ the interior.

**Remark 1.** A subset $\Omega \subset V$ of the set of vertices of a graph $\Gamma = (V, E)$ defines a graph with boundary. Indeed, if we denote $\bar{\Omega} := \Omega \cup \delta \Omega$ and consider the graph $\Gamma' := (\bar{\Omega}, E(\Omega, \bar{\Omega}))$, then $(\Gamma', \delta \Omega)$ satisfies the definition.

**Definition 2.** Given a graph $\Gamma = (V, E)$, a graph with boundary induced by a subset $\Omega \subset V$ is called a subgraph of $\Gamma$. 
Given a graph $\Gamma = (V, E)$, we say that a graph with boundary $(\Gamma' = (V', E'), B)$ is included in $\Gamma$ if $V' \subset V$ and $E' \subset E$. A particular case of graphs with boundary included in $\Gamma$ are the subgraphs of $\Gamma$, that is, those induced by a subset $\Omega \subset V$. Only in this case do we call them subgraphs of $\Gamma = (V, E)$. In our main result, we consider graphs with boundary included in a given graph; in Section 3.2, we give corollaries for subgraphs. Figure 1 illustrates the difference between the two objects (the bigger vertices are boundary vertices).

![Figure 1: Subgraph of $\mathbb{Z}^2$ and graph with boundary included in $\mathbb{Z}^2$.](image)

Let $(\Gamma, B)$ be a graph with boundary. The space of all real functions defined on the vertices $V$, denoted by $\mathbb{R}^V$, is the Euclidean space of dimension $|V|$. Similarly, the space of real functions defined on the vertices of the boundary, denoted $\mathbb{R}^B$, is the Euclidean space of dimension $|B|$. We denote by $1_B$ the matrix of the orthogonal projection onto $\mathbb{R}^B \subset \mathbb{R}^V$.

The Laplacian $\Delta$ of a function $v \in \mathbb{R}^V$ is defined by

$$\left(\Delta v\right)(i) = \sum_{j \sim i} (v(i) - v(j)).$$

A function $v \in \mathbb{R}^V$ is called harmonic if

$$\left(\Delta v\right)(i) = \sum_{j \sim i} (v(i) - v(j)) = 0 \quad \forall i \notin B.$$

The normal derivative operator $\partial v / \partial n : \mathbb{R}^V \to \mathbb{R}^B$ is defined by

$$\left(\frac{\partial v}{\partial n}\right)(i) = \sum_{j \in B, j \sim i} (v(i) - v(j)) \quad i \in B.$$

Since there are never edges between two boundary vertices (see Definition 1), we remark that $\left(\frac{\partial v}{\partial n}\right)(i) = \left(\Delta v\right)(i)$. 

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Definition 3. The Steklov problem on a graph with boundary is the eigenvalue problem
\[ \Delta v = \sigma 1_B v \]
where \( v \not\equiv 0 \) and \( \sigma \) is a spectral parameter.

As shown in [11], the solutions of this problem coincide with the eigenvalues of the discrete Dirichlet-to-Neumann operator defined in [10]. They form a sequence \( 0 = \sigma_0 \leq \sigma_1 \leq \ldots \leq \sigma_{b-1} \), where \( b = |B| \). The associated Rayleigh quotient is, for \( v \in \mathbb{R}^V \),
\[
R(v) := \frac{\langle v, \Delta v \rangle}{\langle 1_B v, 1_B v \rangle} = \frac{\sum_{i \sim j} (v(i) - v(j))^2}{\sum_{i \in B} v(i)^2}.
\]

We have the following variational characterization of the eigenvalues:
\[
\sigma_j = \min_{E} \max_{v \in E, v \not\equiv 0} \left\{ \frac{\sum_{i \sim j} (v(i) - v(j))^2}{\sum_{i \in B} v(i)^2} \right\},
\]
where \( E \) is the set of all linear subspaces of \( \mathbb{R}^V \) of dimension \( j + 1 \).

Remark 2. It is easy to see that the multiplicity of the eigenvalue 0 corresponds to the number of connected components of the graph with boundary (see Proposition 2.1 in [9]).

2.2 Groups with polynomial growth and Cayley graphs

In this article, we work in the setting of Cayley graphs of groups with polynomial growth. We recall here the definitions and the geometric group theory notions that we will use. For further details on this topic, one can see e.g. [5].

Let \( G \) be a finitely generated infinite discrete group and \( S = \{g_1, ..., g_k\} \) a generating set of \( G \). For \( n \in \mathbb{N}^* \), we denote the ball of radius \( n \) \( B(n) := \{ x \in G : x = g_{i_1}^{\epsilon_1} \ldots g_{i_n}^{\epsilon_n}, i_1, ..., i_n \in \{1, ..., k\}, \epsilon_j = \pm 1 \} \). The growth function of \( G \) is \( V(n) := |B(n)| \). If there exist \( D \in \mathbb{N}^* \) and \( C > 0 \) such that
\[
C^{-1} n^D \leq V(n) \leq C n^D,
\]
we say that the growth rate is polynomial of order \( D \). Since the growth rate does not depend on the choice of generating set, we can speak of the growth type of a group.
Let \( G \) be a group and \( S \) a generating set that does not contain the identity element of the group and is symmetric, that is, satisfies \( S = S^{-1} \). The Cayley graph \( \Gamma = \Gamma(G, S) \) associated to \((G, S)\) is the graph with vertices \( V = G \) and edges \( E = \{ \{x, y\} : x, y \in V \) and \( \exists s \in S \) such that \( y = xs \}. \) Since \( S \) is symmetric and does not contain the identity element, the graph is simple, and since \( S \) is a generating set of \( G \), the graph is connected.

Let \( \Gamma = \Gamma(G, S) \) be a Cayley graph of a group with polynomial growth of order \( D \). Then, a ball \( B(y, n) = \{x \in V : d(x, y) \leq n\} \) in \( \Gamma \) has \( V(n) \) vertices. Therefore, we say that the graph has polynomial growth of order \( D \) to signify that \( C^{-1}n^D \leq |B(y, n)| \leq Cn^D \).

We now give two properties of Cayley graphs with polynomial growth that we will need to prove our results.

**Lemma 1.** Let \( \Gamma = (V, E) \) be a graph with polynomial growth of order \( D \). Let \( a, b \in \mathbb{R}^*_+ \) and \( B(x, aR) \) be a ball in \( \Gamma \) of radius \( aR \). Then \( \exists N \in \mathbb{N}^* \) such that \( B(x, aR) \) is the union of \( N \) balls of radius \( bR \) and this number does not depend on \( R \). More precisely, we can take \( N = \lceil C^2(\frac{2a+b}{b})^D \rceil \) where \( C \) is a constant satisfying \( C^{-1}n^D \leq V(n) \leq Cn^D \).

**Proof.** Let \( \{y_i\}_{i=1}^m \) be a maximal subset of vertices in \( B(x, aR) \) such that \( d(y_i, y_j) \geq bR \) for \( i \neq j \). Then \( \bigcup_{i=1}^m B(y_i, bR) \supset B(x, aR) \) and, by the triangle inequality, \( B(y_i, \frac{bR}{2}) \cap B(y_j, \frac{bR}{2}) = \emptyset \). This implies

\[
\sum_{i=1}^m |B(y_i, \frac{bR}{2})| \leq |B(x, (a + \frac{b}{2})R)|. \tag{2}
\]

Since the graph has polynomial growth of order \( D \), we know that there exists \( C \) such that \( C^{-1}n^D \leq |B(z, n)| \leq Cn^D \forall z \in V \). We approximate the volume of the balls in equation (2) using the latter inequality and we obtain that \( m \leq C^2(\frac{2a+b}{b})^D \).

The second property is a discrete isoperimetric inequality.

**Proposition 1.** Let \( \Gamma = (V, E) \) be a graph with polynomial growth of order \( D \). There exists \( C \) such that for any subset \( \Omega \subset V \), \( \partial \Omega \) its boundary, and \( \overline{\Omega} := \Omega \cup \partial \Omega \), we have that

\[
\frac{|\overline{\Omega}|^{\left(D-1\right)}}{|\partial \Omega|} \leq C. \tag{3}
\]
For the proof of this proposition, we refer to [4]. In fact, the result that we give corresponds to the first particular case of Theorem 1 of [4], but formulated in the setting of Cayley graphs.

3 Isoperimetric upper bound for \( \sigma_1 \) in Cayley graphs with polynomial growth

For the Steklov problem on graphs with boundary without any additional geometric constraint, \( \sigma_1 \) does not admit a uniform upper bound. But if we assume that the degree of the graph, \( d \), is bounded, it is easy to see that \( \sigma_1 \leq d \). In the following, we will work in a setting that allows us to deduce upper bounds for \( \sigma_1 \) in terms of the inverse of the number of vertices of the graph or in terms of the inverse of the number of vertices of the boundary.

3.1 Main result

We give an upper bound for \( \sigma_1 \) which holds for every graph with boundary included in a given Cayley graph associated to a group with polynomial growth. In section 3.2, we will give corollaries for subgraphs.

**Theorem 1.** Let \( \Gamma = (V, E) \) be a Cayley graph with polynomial growth of order \( D \). There exists \( C(\Gamma) > 0 \) such that for any graph with boundary \((\Gamma' = (V', E'), B)\) included in \( \Gamma \), we have

\[
\sigma_1(\Gamma', B) \leq C(\Gamma) \frac{|V'|}{|B|}. 
\]

The proof consists of finding two regions of the subgraph with a sufficient number of vertices of the boundary, then building test functions, evaluating their Rayleigh quotient, and using the variational characterization in order to obtain an upper bound for \( \sigma_1 \).

**Proof.** By Lemma [\( ] \) there exists \( c_1 \) such that a ball of radius \( 3R \) in \( \Gamma \) is the union of \( c_1 \) balls of radius \( \frac{1}{3}R \). From now on, we will assume \( |B| > c_1 + 1 \). If \( |B| \leq c_1 + 1 \), the result is trivially true because \( \sigma_1 \) is bounded from above by the degree of the graph. We define
\[ \alpha := \frac{|B|}{c_1 + 1}. \]

Let \( x \in V \). We set
\[ r_x := \min \{ r \in \mathbb{N} : |B(x, r) \cap B| \geq \alpha \} \]
and
\[ R := \min_{x \in V} r_x. \]

Then, we have that \( \forall x \in V, |B(x, R - 1) \cap B| < \alpha \) and there exists \( x_0 \) such that \( |B(x_0, R) \cap B| \geq \alpha \). We remark that \( R \geq 1 \). Since \( B(x, R - 1) \geq B(x, \frac{1}{2}) \) we have that \( B(x_0, 3R) \) is the union of \( c_1 \) balls of radius \( R - 1 \). This implies
\[ |B(x_0, 3R) \cap B| < c_1 \alpha \]
and consequently
\[ |B(x_0, 3R)^c \cap B| = |B| - |B(x_0, 3R) \cap B| \]
\[ > |B| - c_1 \alpha \]
\[ = |B| - c_1 \frac{|B|}{c_1 + 1} \]
\[ = \frac{|B|}{c_1 + 1} = \alpha. \]

Hence, we have found two regions, \( B(x_0, R) \) and \( B(x_0, 3R)^c \), such that
\[ |B(x_0, R) \cap B| \geq \alpha \]
and
\[ |B(x_0, 3R)^c \cap B| > \alpha. \]

We define two test functions, one with support \( B(x_0, 2R) \), and the other with support \( B(x_0, 2R)^c \).
\[ f_1(y) = \begin{cases} 
1 & \text{if } y \in B(x_0, R) \\
1 - \frac{k}{R} & \text{if } k := d(y, B(x_0, R)) \leq R \\
0 & \text{otherwise}, 
\end{cases} \]

\[ f_2(y) = \begin{cases} 
1 & \text{if } y \in B(x_0, 3R)^c \\
1 - \frac{k}{R} & \text{if } k := d(y, B(x_0, 3R)^c) \leq R \\
0 & \text{otherwise}. 
\end{cases} \]

We consider the linear subspace \( W \) of \( \mathbb{R}^V \) generated by \( f_1 \) and \( f_2 \). The variational characterization of equation (1) gives

\[ \sigma_1 \leq \max_{v \in W} R(v). \]

Since \( f_1 \) and \( f_2 \) have disjoint support, it implies

\[ \sigma_1 \leq \max\{R(f_1), R(f_2)\}. \]

\( R(f_1) \) can be evaluated in the following way. The denominator is

\[ \sum_{i \in B} f_1(i)^2 = |B(x_0, R) \cap B| \geq \alpha = \frac{|B|}{c_1 + 1}. \]

The only edges contributing to the sum in the numerator \( \sum_{i \sim j} (f_1(i) - f_1(j))^2 \) are the ones in \( B(x_0, 2R) \setminus B(x_0, R) \). In this annulus, for two adjacent vertices, we have that \( (f_1(i) - f_1(j))^2 \leq \frac{1}{R^2} \). Moreover, the number of edges in this annulus is smaller than or equal to the number of edges in \( B(x_0, 3R) \). Hence we have

\[ \sum_{i \sim j} (f_1(i) - f_1(j))^2 \leq \sum_{i \sim j, i,j \in B(x_0, 3R)} \frac{1}{R^2}. \]

Because the graph has polynomial growth of order \( D \), there exists \( c_2 > 0 \) such that \( |B(x_0, 3R)| \leq c_2(3R)^D \). We recall that the graph is the Cayley graph defined by a group \( G \) and a generating set \( S \) of \( G \). The degree of the graph is \( |S| = |B(y, 1)| \leq c_2 \). By the handshaking lemma, \( E(B(x_0, 3R), B(x_0, 3R)) \leq \frac{1}{2}|B(x_0, 3R)||S| \leq \frac{1}{2}c_2^3(3R)^D := c_3R^D \). Consequently, for \( D = 1 \) or \( D = 2 \), we have
If $D \geq 3$, we note that we have the following equality
\[
\sum_{i \sim j, i,j \in B(x_0,3R)} \frac{1}{R^2} = \left( \sum_{i \sim j, i,j \in B(x_0,3R)} \frac{1}{R^{D}} \right)^2 \left( \sum_{i \sim j, i,j \in B(x_0,3R)} 1 \right)^{\frac{D-2}{2}}
\]
The left factor is bounded by a constant:
\[
\left( \sum_{i \sim j, i,j \in B(x_0,3R)} \frac{1}{R^{D}} \right)^2 \leq c_4^2.
\]
For the right factor, we have
\[
\left( \sum_{i \sim j, i,j \in B(x_0,3R)} 1 \right)^{\frac{D-2}{2}} \leq \left( \frac{c_2^2}{2} |V'| \right)^{\frac{D-2}{2}},
\]
and we obtain
\[
\sum_{i \sim j, i,j \in B(x_0,3R)} \frac{1}{R^2} \leq c_4 \left( \frac{c_2}{2} |V'| \right)^{\frac{D-2}{2}} =: c_5 |V'|^{\frac{D-2}{2}}.
\]
Hence, the numerator of the Rayleigh quotient satisfies
\[
\sum_{i \sim j} (f_1(i) - f_1(j))^2 \leq \max\{c_3, c_5\} |V'|^{\frac{D-2}{2}} =: c_6 |V'|^{\frac{D-2}{2}}.
\]
The Rayleigh quotient of $f_1$ becomes
\[
R(f_1) = \frac{\sum_{i \sim j} (f_1(i) - f_1(j))^2}{\sum_{i \in B} f_1(i)^2} \leq \frac{(c_1 + 1)c_6 |V'|^{\frac{D-2}{2}}}{|B|} =: c_7 |V'|^{\frac{D-2}{2}} / |B|.
\]
By the definition of the test functions, the same upper bound can be obtained for $f_2$. We conclude that
\[
\sigma_1 \leq \max\{R(f_1), R(f_2)\} \leq c_7 |V'|^{\frac{D-2}{2}} / |B|.
\]
Remark 3. The proof is qualitative rather than quantitative since the goal here is not to find an optimal constant (the constant depends on the generating set of the group).

Example 1. An example of a group with polynomial growth of order $D$ is $\mathbb{Z}^D$.

Example 2. The Heisenberg group over $\mathbb{Z}$,

$$\text{Heis}(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\} ,$$

is an example of a group with polynomial growth of order 4, which is not quasi-isometric $\mathbb{Z}^4$. Hence, for the Steklov problem on a graph with boundary $(\Gamma' = (V', E'), B)$ included in a Cayley graph associated to the Heisenberg group, $\sigma_1$ is bounded from above by $C(\text{Heis}(\mathbb{Z}))\frac{|V'|}{|B|}$.

Example 3. An important theorem due to M. Gromov characterizes finitely generated groups of polynomial growth. It says that that a group is of polynomial growth if and only if it has a nilpotent subgroup of finite index. Lattices in nilpotent Lie groups, which are finitely generated and themselves nilpotent are other examples where the theorem holds (for the existence of such lattices, see e.g. [12] and [6]).

Corollary 1. Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order 2 and let $(\Gamma', B)$ be a graph with boundary included in $\Gamma$. Then, there exists $C(\Gamma) > 0$ such that

$$\sigma_1(\Gamma', B) \leq C(\Gamma) \frac{1}{|B|} .$$

Proof. We remark that the corollary corresponds to Theorem 1 when $D = 2$.

This shows that in this particular case, for a sequence $\{(\Gamma'_n, B_n)\}_{n \in \mathbb{N}}$ of graphs with boundary satisfying $|B_n| \to \infty$, we have that $\sigma_1$ tends to 0 as $n$ tends to infinity.
3.2 Application to subgraphs

Graphs with boundary that are subgraphs in a Cayley graph with polynomial growth satisfy the discrete isoperimetric inequality given in Proposition 1. Using this result, we give two corollaries of Theorem 1. Since subgraphs are determined by a subset $\Omega$ of the set of vertices of the graph, we can speak of $\sigma_1$ of $\Omega$.

**Corollary 2.** Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order $D \geq 2$. There exists $C(\Gamma) > 0$ such that for any subset $\Omega$ of the set of vertices $V$ we have

$$\sigma_1(\Omega) \leq C(\Gamma) \frac{1}{|\delta \Omega|^{\frac{D}{D-1}}}.$$ 

**Proof.** By the isoperimetric inequality in Proposition 1 there exists $c_1 > 0$ such that $|\bar{\Omega}|^{\frac{D-1}{D}} \leq c_1$, where $\bar{\Omega} = \delta \Omega \cup \Omega$. We raise the latter inequality to the power of $\frac{D-2}{D-1}$ and obtain $|\bar{\Omega}|^{\frac{D-2}{D-1}} \leq (c_1|\delta \Omega|)^{\frac{D-2}{D-1}} =: c_2|\delta \Omega|^{\frac{D-2}{D-1}}$. By Theorem 1 there exists $c_3$ such that $\sigma_1 \leq c_3\frac{|\bar{\Omega}|^{\frac{D-2}{D-1}}}{|\delta \Omega|^{\frac{D-2}{D-1}}}$. Consequently,

$$\sigma_1 \leq c_3 \frac{|\bar{\Omega}|^{\frac{D-2}{D-1}}}{|\delta \Omega|^{\frac{D-2}{D-1}}} \leq c_3 c_2 \frac{|\delta \Omega|^{\frac{D-2}{D-1}}}{|\delta \Omega|^{\frac{D-2}{D-1}}} = c_3 c_2 \frac{1}{|\delta \Omega|^{\frac{D}{D-1}}} =: c_4 \frac{1}{|\delta \Omega|^{\frac{D}{D-1}}}.$$ 

\[\square\]

**Remark 4.** For $D = 1$, we remark that by Theorem 1 we have that $\sigma_1(\Omega) \leq C(\Gamma) \frac{1}{|\Omega|^2}$.

**Corollary 3.** Let $\Gamma = (V, E)$ be a Cayley graph with polynomial growth of order $D$. There exists $C(\Gamma) > 0$ such that for any subset $\Omega$ of the set of vertices $V$ we have

$$\sigma_1(\Omega) \leq C(\Gamma) \frac{1}{|\bar{\Omega}|^{\frac{D}{D-1}}}.$$ 

where $\bar{\Omega} = \delta \Omega \cup \Omega$.

**Proof.** By the isoperimetric inequality in Proposition 1 there exists $c_1 > 0$ such that $|\bar{\Omega}|^{\frac{D-1}{D}} \leq c_1$. By Theorem 1 there exists $c_2$ such that $\sigma_1 \leq c_2\frac{|\bar{\Omega}|^{\frac{D-1}{D}}}{|\delta \Omega|^{\frac{D-2}{D-1}}}$. Hence, we have
\[
\sigma_1 \leq c_2 \frac{|\Omega|^{\frac{D-2}{|\delta\Omega|}}}{|\delta\Omega|} = c_2 \frac{|\Omega|^{\frac{D-1}{|\delta\Omega|}}|\Omega|^{\frac{1}{|\delta\Omega|}}}{|\delta\Omega|} \leq c_2 c_1 |\bar{\Omega}|^{\frac{1}{|\delta\Omega|}} =: c_3 \frac{1}{|\Omega|^{\frac{1}{D}}}.
\]

\[\square\]

**Remark 5.** Since \(\bar{\Omega} = \delta\Omega \cup \Omega\), we also have \(\sigma_1 \leq C(\Gamma) \frac{1}{|\Omega|^{\frac{1}{D}}}\) and \(\sigma_1 \leq C(\Gamma) \frac{1}{|\delta\Omega|^{\frac{1}{D}}}\) but this last bound is weaker than Corollary 2.

**Remark 6.** In a Cayley graph with polynomial growth of order \(D\), for a sequence \(\{\Omega_n\}_{n \in \mathbb{N}}\) of finite subsets satisfying \(|\Omega_n| \to \infty\), we have that \(\sigma_1(\Omega_n)\) tends to 0 as \(n\) tends to infinity.

**Remark 7.** For subgraphs of \(\mathbb{Z}^n\), the result of Corollary 3 was recently obtained by Han and Hua (see Corollary 1.4 in [2]), who also give an explicit constant.

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