Existence of minimizers and convergence of almost minimizers in ferromagnetic nanowires. An energy barrier for thick wires.

Davit Harutyunyan
Department of Mathematics of The University of Utah
May 2, 2014

Abstract
We study static 180 degree domain walls in infinite magnetic wires with bounded, $C^1$ and rotationally symmetric cross sections. We prove an existence of global minimizers for the energy of micromagnetics for any $C^1$ cross sections. We show, that under some asymmetry of the cross sections the whole sequence of almost minimizers converges to a minimizer of the limiting energy up to a rotation and a translation. An upper bound on the energy of micromagnetics is obtained in thick wires with rectangular cross sections as well, which is realized via a vortex wall.

Keywords: Nanowires; Magnetization reversal; Transverse wall; Vortex wall; Domain wall

Contents
1 Introduction 2
2 The main results 4
3 The oscillation preventing lemma 6
4 Existence of minimizers 10
5 Convergence of almost minimizers 15
5.1 Proof of Theorem 24 18
6 Upper and lower bounds for thick wires 20
1 Introduction

In the theory of micromagnetics to any domain $\Omega \in \mathbb{R}^3$ and a unit vector field (called magnetization) $m: \Omega \to S^2$ with $m = 0$ in $\mathbb{R}^3 \setminus \Omega$ the energy of micromagnetics is assigned:

$$E(m) = A_{ex} \int_{\Omega} |\nabla m|^2 + K_d \int_{\mathbb{R}^3} |\nabla u|^2 + Q \int_{\Omega} \varphi(m) - 2 \int_{\Omega} H_{ext} \cdot m,$$

where $A_{ex}, K_d, Q$ are material parameters, $H_{ext}$ is the externally applied magnetic field, $\varphi$ is the anisotropy energy density and $u$ is obtained from Maxwell’s equations of magnetostatics,

$$\begin{cases}
\text{curl} H_{ind} = 0 & \text{in } \mathbb{R}^3 \\
\text{div}(H_{ind} + m) = 0 & \text{in } \mathbb{R}^3,
\end{cases}$$

i.e., $u$ is a weak solution of

$$\triangle u = \text{div} m \quad \text{in } \mathbb{R}^3.$$

According to micromagnetics, stable magnetization patterns are described by the minimizers of the micromagnetic energy functional, see [5,6,12]. The study of magnetic wires and thin films has attracted significant attention in recent years, see [12,14,9,11,16,17,18,19,20,21,22] for wires and [3,5,6,7,8,13,14,15] for thin films. It has been suggested in [1] that magnetic nanowires can be effectively used as storage devices. When a homogenous external field is applied in the axial direction of a magnetic wire facing the homogenous magnetization direction (see Fig. 1), then at a critical strength of the field the reversal of the magnetization typically starts at one end of the wire creating a domain wall which starts moving along the wire. The domain wall separates the reversed and the not yet reversed parts of the wire (see Fig. 1). It is known that the magnetization pattern reversal time is closely related to the writing and reading speed of such a device, thus it is crucial to understand the magnetization reversal and switching processes. Several authors have numerically, experimentally and analytically observed two different magnetization modes in magnetic nanowires [9,11,10,16]. In [9] the magnetization reversal process has been studied numerically in cobalt nanowires by the Landau-Lifshitz-Gilbert equation. Two different domain wall types were observed. For thin cobalt wires with 10nm in diameter the transverse mode has been observed: the magnetization is constant on each cross section and is moving along the wire. For thick wires, with diameters bigger that 20nm the vortex wall has been observed: the magnetization is approximately tangential to the boundary and forms a vortex which propagates along the wire. In [11] the magnetization reversal process has been studied both numerically and experimentally. By considering a conical type wire so that the diameter of the cross section increases very slowly, the magnetization switching from the vortex wall to the transverse at a critical diameter has been observed, as the domain wall moves along the wire. The results in [9] and [11] were the same: in thin wires the transverse wall occurs, while in thick wires the vortex wall occurs.
Homogenius magnetization

180 degree domain wall

\[ H_{\text{ext}} \]

Figure 1.

In Figure 2 one can see the transverse and the vortex wall longitudinal and cross section pictures for wires with a rectangular cross section.

Figure 2.

The transverse wall The vortex wall

It has been observed that there is a distinctive crossover between two different modes, which occurs at a critical diameter of the wire and it was suggested that the magnetization switching process can be understood by analyzing the micromagnetics energy minimization problem for different diameters of the cross section. In [16], K. Kühn studied 180 degree static domain walls in magnetic wires with circular cross sections by an asymptotic analysis proving that indeed the transverse mode must occur in thin magnetic wires. It is also shown in [16] that for thick wires the vortex wall has the optimal energy scaling and that the minimal energy scales like \( R^2 \sqrt{\ln R} \). In [21] V.V.Slastikov and C.Sonnenberg studied a
similar problem for finite curved wires proving a $\Gamma$-convergence on energies as the diameter of the wire goes to zero. In [10], the author studied the same problem as K.Kühn in [16] and independently of [21] (see the submission and the publication dates of [10] and [21] respectively) extended some of the results proven in [16] for arbitrary wires with a rotational symmetry. In this paper we study the 180 degree static domain walls in magnetic wires with arbitrary bounded, $C^1$ and rotationally symmetric cross sections. We generalize the existence of minimizers result proven by K.Kühn for circular cross sections, to wires with arbitrary bounded and $C^1$ cross sections. For a class of domains we prove the convergence of almost minimizers which is a new and much deeper result and it does not follow from the $\Gamma$–convergence of the energies. It actually requires much deeper analysis of minimization problem of minimizing the energy of micromagnetics and its minimizers. We also construct a vortex wall that has an energy of order $d^2\sqrt{\ln d}$ for thick rectangular wires.

2 The main results

Assume $\Omega = \mathbb{R} \times \omega$, where $\omega \subset \mathbb{R}^2$ is a bounded $C^1$ domain. Consider the isotropic energy of micromagnetics without an external field like in [16,21,10],

$$E(m) = A_{ex} \int_\Omega |\nabla m|^2 + K_d \int_\mathbb{R} |\nabla u|^2.$$

By scaling of all coordinates one can achieve the situation where $A_{ex} = K_d$, so we will henceforth assume that $A_{ex} = K_d = 1$. Next we rescale the magnetization $m$ in the $y$ and $z$ coordinates such that the domain of the rescaled magnetization is fixed, i.e., if $d = \text{diam}(\omega)$, then set $\hat{m}(x, y, z) = m(x, dy, dz)$.

Denote

$$A(\Omega) = \{m : \Omega \to \mathbb{S}^2 : m \in H^1_{\text{loc}}(\Omega), E(m) < \infty\}.$$

We are interested in 180 degree domain walls, so set

$$\tilde{A}(\Omega) = \{m : \Omega \to \mathbb{S}^2 : m - \bar{e} \in H^1(\Omega)\},$$

where

$$\bar{e}(x, y, z) = \begin{cases} (-1, 0, 0) & \text{if } x < -1 \\ (x, 0, 0) & \text{if } -1 \leq x \leq 1 \\ (1, 0, 0) & \text{if } 1 < x \end{cases}.$$

The objective of this work will be studying the existence of minimizers in the minimization problem

$$\inf_{m \in \tilde{A}(\Omega)} E(m), \quad (2.1)$$

and the behavior of its almost minimizers, where the notion of "almost minimizers" will be defined later in Definition 2.3. The following existence theorem is a generalization of the corresponding theorem proven for circular cross sections in [16].

**Theorem 2.1 (Existence of minimizers).** For every bounded $C^1$ domain $\omega \in \mathbb{R}^2$ there exists a minimizer of $E$ is $\tilde{A}(\Omega)$. 

4
It has been shown for circular wires in [16] and later for any cross sections in [21], that as $d$ goes to zero, the rescaled energy functional $\frac{E(m)}{d^2}$, $\Gamma$-converges to a one dimensional energy $E_0(m^0)$ under the following notion of convergence of magnetization vectors:

**Definition 2.2.** The sequence $\{\hat{m}^n\} \subset A(\Omega)$ is said to converge to $m^0$ as $n$ goes to infinity if,

(i) $\nabla \hat{m}^n \rightharpoonup \nabla m^0$ weakly in $L^2(\Omega)$

(ii) $\hat{m}^n \to m^0$ strongly in $L^2_{\text{loc}}(\Omega)$.

The limit or reduced energy is given by

$$E_0(m) = \begin{cases} 
|\omega| \int_{\mathbb{R}} |\partial_x m|^2 \, dx + \int_{\mathbb{R}} m M_\omega m^T \, dx, & \text{if } m = m(x), \\
\infty, & \text{otherwise},
\end{cases} \quad (2.2)$$

Where $M_\omega$ is a symmetric matrix given by

$$M_\omega = -\frac{1}{2\pi} \int_{\partial \omega} \int_{\partial \omega} n(x) \otimes n(y) \ln |x - y| \, dx \, dy,$$

and $n = (0, n_2, n_3)$ is the outward unit normal to $\partial \omega$, see [21]. Since $M_\omega$ is symmetric it can be diagonalized by a rotation in the $OYZ$ plane. We choose the coordinate system such that $M_\omega$ is diagonal. Assume now $\omega$ is fixed and diam($\omega$) = 1. Actually, the $\Gamma$-convergence theorem implies the following two properties of the minimal energies and sequences of minimizers:

(i) \[
\lim_{d \to 0} \min_{m \in \tilde{A}(d \cdot \Omega)} \frac{E(m)}{d^2} = \min_{m \in A_0} E_0(m), \quad (2.3)
\]

where $A_0 = \{m: \mathbb{R} \to \mathbb{R}^3: |m| = 1, \; m(\pm \infty) = \pm 1\}$.

(ii) If $m^n$ is any sequence of minimizers with $m^n$ defined in $d \cdot \Omega$, then a subsequence of $\{\hat{m}^n\}$ converges to a minimizer of $E_0$ in the sense of Definition 2.2.

It turns out, that under some asymmetry condition on $\omega$ a stronger convergence holds, namely an $H^1$ convergence of the whole sequence of almost minimizers holds.

**Definition 2.3.** Let $\{d_n\}$ be a sequence of positive numbers such that $d_n \to 0$. A sequence of magnetizations $\{m^n\}$ defined in $d_n \cdot \Omega$ is called a sequence of almost minimizers if

$$\lim_{n \to \infty} \frac{E(m^n)}{d_n^2} = \min_{m \in A_0} E(m). \quad (2.4)$$

We are now ready to formulate the other result of the paper.
Theorem 2.4 (Convergence of almost minimizers). Let \( \{d_n\} \) be a sequence of positive numbers such that \( d_n \to 0 \). Assume that the domain \( \omega \) is so that \( M_\omega \) has three different eigenvalues. Then for any sequence of almost minimizers \( \{m^n\} \) defined in \( d_n \cdot \Omega \), there exist a sequence \( \{T_n\} \) of translations in the x direction and a sequence \( \{R_n\} \) of rotations in the OYZ plane, each of which is either the identity or the rotation by 180 degrees such that for \( \tilde{m}^n(x, y, z) = m^n(T_n(R_n(x, y, z))) \) for a minimizer \( m^\omega \) of \( E_0 \), there holds,

\[
\lim_{n \to \infty} \frac{1}{d_n} \|\tilde{m}^n - m^\omega\|_{H^1(\Omega_n)} = 0.
\]

We refer to Appendix for the definition of \( m^\omega \).

Theorem 2.5 (Bounds for thick wires). Let \( \Omega = \mathbb{R} \times [-d, d] \times [-l, l] \) and \( c \geq 1 \). Then there exists \( d_1 > 0 \) such that if \( l \geq d > d_1 \) and \( l \leq cd \), then

\[
C_1d^2\sqrt{\ln d} \leq \min_{m \in A(\Omega)} E(m) \leq C_2d^2 \sqrt{\ln d},
\]

where \( C_1 \) and \( C_2 \) depend only on \( c \).

3 The oscillation preventing lemma

In this section we prove a lemma, that will be crucial in proving both existence and convergence of almost minimizers results. The lemma bounds the oscillations of a magnetization \( m \) and the total measure of the set where \( m \) develops oscillations by the energy of \( m \). Uzing the idea of Kohn and Slastikov in \[14\] of the dimension reduction in thin domains, define

\[
\tilde{m}(x) = \int_\omega m(x, y, z) \, dy \, dz.
\]

Using the definition of \( M_\omega^1 \) it is straightforward to show that \( M_\omega^1 \) is positive definite, where \( M_\omega^1 \) is the lower right \( 2 \times 2 \) block of \( M_\omega \). Denote for convenience

\[
M_\omega^1 = \begin{bmatrix} \alpha_2 & 0 \\ 0 & \alpha_3 \end{bmatrix}.
\]

It has been explicitly shown in \[10\] Corollary 3.7.5] and implicitly in \[21\] Proof of Lemma 4.1], that the inequality below holds uniformly in \( m \): \( (d \cdot \Omega) \to \mathbb{S}^2 \):

\[
\frac{E(m)}{d^2} \geq \int_\Omega |\nabla m|^2 + \alpha_2 \int_\mathbb{R} |\tilde{m}_2|^2 + \alpha_3 \int_\mathbb{R} |\tilde{m}_3|^2 + O(1),
\]

as \( d \) goes to zero.

Lemma 3.1. Assume \( m^d \in A(d \cdot \Omega) \). Then there exists \( d_0 > 0 \) such that,

\[
\int_\mathbb{R} (|\tilde{m}_2|^2 + |\tilde{m}_3|^2) \leq \frac{2E(m)}{d^2 \min(\alpha_2, \alpha_3)}, \quad \text{if} \quad d \leq d_0
\]

\[
\int_\mathbb{R} (|\tilde{m}_2|^2 + |\tilde{m}_3|^2) \leq \frac{2 \max \left( \frac{d}{d_0}, \left( \frac{d}{d_0} \right)^3 \right) E(m^d)}{dd_0 \min(\alpha_2, \alpha_3)}, \quad \text{if} \quad d > d_0
\]
Proof. Due to inequality (3.1) there exists $d_0 > 0$ such that for $d \leq d_0$ we have
\[
\frac{2E(m)}{d^2} \geq \alpha_2 \int_{\mathbb{R}} |\bar{m}_2|^2 + \alpha_3 \int_{\mathbb{R}} |\bar{m}_3|^2,
\]
and inequality (3.2) follows. Assume now $d > d_0$. It is straightforward that if $m^d(\cdot) \in A(d \cdot \Omega)$ then $m^d(x, y, z) = m^d(tx, ty, tz) \in A(\frac{d}{t} \cdot \Omega)$ with $E(m^d_t) = t E_{ex}(m^d) + t^3 E_{mag}(m^d)$, where $E_{ex}(m) = \int_{\Omega} |\nabla m|^2$ is the exchange energy and $E_{mag}(m) = \int_{\mathbb{R}^3} |\nabla u|^2$ is the magnetostatic energy, thus we get on one hand,
\[
E(m^d) \leq \max(t, t^3) E(m^d).
\]
But on the other hand we have
\[
\int_{\mathbb{R}} (|\bar{m}_2^d|^2 + |\bar{m}_3^d|^2) = \frac{1}{t} \int_{\mathbb{R}} (|\bar{m}_{t2}|^2 + |\bar{m}_{t3}|^2),
\]
thus we obtain choosing $t = \frac{d}{d_0}$ and taking into account (3.2) and (3.4),
\[
\int_{\mathbb{R}} (|\bar{m}_2^d|^2 + |\bar{m}_3^d|^2) \leq \frac{2d_0 E(m^d)}{dd_0 \min(\alpha_2, \alpha_3)} \leq \frac{2 \max \left( \frac{d}{d_0}, \left( \frac{d}{d_0} \right)^3 \right) E(m^d)}{dd_0 \min(\alpha_2, \alpha_3)}
\]
which completes the proof. \qed

Next we prove a simple estimate between $m$ and $\bar{m}$ that will be useful in the proof of the oscillation preventing lemma.

Lemma 3.2. For any $m \in A(\Omega)$ there holds
\[
\int_{\omega} (|m|^2 - |\bar{m}|^2) = \int_{\omega} |m - \bar{m}|^2 \leq C_p d^2 \int_{\omega} |\nabla_{yz} m| \quad \text{for all} \quad x \in \mathbb{R},
\]
where $C_p$ is the Poincaré constant of $\omega$.

Proof. We have for any $x \in \mathbb{R}$
\[
\int_{\omega} (m - \bar{m}) = \int_{\omega} m - |\omega| \cdot \bar{m}(x) = 0,
\]
thus by the Poincaré inequality we get
\[
\int_{\omega} |m|^2 = \int_{\omega} |\bar{m}|^2 + \int_{\omega} |m - \bar{m}|^2 + 2 \bar{m}(x) \int_{\omega} (m - \bar{m}) = \int_{\omega} |\bar{m}|^2 + \int_{\omega} (m - \bar{m})^2 \leq \int_{\omega} |\bar{m}|^2 + C_p d^2 \int_{\omega} |\nabla_{yz} m|,
\]
the proof is complete now. \qed
Lemma 3.3 (Oscillation preventing lemma). Let \( m \in A(\Omega) \) and let \( \alpha, \beta, \rho \in \mathbb{R} \) such that \(-1 < \alpha < \beta < 1\) and \( 0 < \rho < 1\). Assume \( \mathcal{R} \) is a family of disjoint intervals \((a, b)\) satisfying the conditions

\[
\{\bar{m}_1(a), \bar{m}_1(b)\} = \{\alpha, \beta\} \quad \text{and} \quad |\bar{m}_1(x)| \leq \rho, \quad x \in (a, b).
\]

Then,

(i) \[
\text{card}(\mathcal{R}) \leq M \quad \text{and} \quad \sum_{(a, b) \in \mathcal{R}} (b - a) \leq M, \quad (3.5)
\]

where \( M \) is a constant depending on \( \alpha, \beta, \rho, \omega \) and \( E(m) \).

(ii) The component \( \bar{m}_1 \), satisfies \( \lim_{x \to \pm \infty} |\bar{m}_1(x)| = 1 \).

Proof. Let us first prove the second inequality in (3.5). The function \( \bar{m} \) is a one variable weakly differentiable function therefore it is locally absolutely continuous in \( \mathbb{R} \). For any \((a, b) \in \mathcal{R}\), we have by Lemma 3.2 and by the assumption of the lemma,

\[
|\omega|(b - a) = \int_{(a, b) \times \omega} |m|^2 \\
\leq \int_{(a, b) \times \omega} |\bar{m}|^2 + C_p d^2 \int_{(a, b) \times \omega} |\nabla \bar{m}|^2 \\
\leq \rho^2 |\omega|(b - a) + \int_{(a, b) \times \omega} (\bar{m}_2^2 + \bar{m}_3^2) + C_p d^2 \int_{(a, b) \times \omega} |\nabla m|^2.
\]

Summing up the inequalities for all \((a, b) \in \mathcal{R}\) we get,

\[
|\omega| \sum_{(a, b) \in \mathcal{R}} (b - a) \leq \rho^2 |\omega| \sum_{(a, b) \in \mathcal{R}} (b - a) + \int_{\Sigma} (\bar{m}_2^2 + \bar{m}_3^2) + C_p d^2 \int_{\Sigma} |\nabla m|^2 \\
\leq \rho^2 |\omega| \sum_{(a, b) \in \mathcal{R}} (b - a) + \int_{\Omega} (\bar{m}_2^2 + \bar{m}_3^2) + C_p d^2 \int_{\Omega} |\nabla m|^2,
\]

where \( \Sigma = \bigcup_{(a, b) \in \mathcal{R}} (a, b) \times \omega \). By virtue of Lemma 3.1 we have

\[
\int_{\Omega} (\bar{m}_2^2 + \bar{m}_3^2) \leq C_1,
\]

for some \( C_1 \) depending on \( \omega \) and \( E(m) \). Therefore we obtain

\[
\sum_{(a, b) \in \mathcal{R}} (b - a) \leq \frac{C_1 + C_p d^2 E(m)}{|\omega|(1 - \rho^2)}. \quad (3.6)
\]
Next we have for any point \((y, z) \in \omega\) and any interval \((a, b) \in \mathbb{R}\),
\[
\int_a^b |\partial_x m_1(x, y, z)|^2 \, dx \geq \frac{1}{b-a} \left( \int_a^b |\partial_x m_1(x, y, z)| \, dx \right)^2,
\]
Thus integrating over \(\omega\) we get
\[
\int_{(a, b) \times \omega} |\partial_x m_1|^2 \, d\xi \geq \frac{1}{b-a} \int_{\omega} \left( \int_a^b |\partial_x m_1(x, y, z)| \, dx \right)^2 \, dy \, dz
\]
\[
\geq \frac{1}{b-a} \int_{\omega} |m_1(a, y, z) - m_1(b, y, z)|^2 \, dy \, dz
\]
\[
\geq \frac{1}{|\omega|(b-a)} \left( \int_{\omega} (m_1(a, y, z) - m_1(b, y, z)) \, dy \, dz \right)^2
\]
\[
= \frac{|\omega|}{b-a} (\alpha - \beta)^2,
\]
thus
\[
\int_{(a, b) \times \omega} |\partial_x m_1|^2 \, d\xi \geq \frac{|\omega|}{b-a} (\alpha - \beta)^2.
\]
Summing up the last inequalities for all \((a, b) \in \mathbb{R}\) we arrive at
\[
\sum_{(a, b) \in \mathbb{R}} \frac{1}{b-a} \leq \frac{1}{|\omega|} (\alpha - \beta)^2 \int_{\Sigma} |\partial_x m_1|^2 \, d\xi
\]
\[
\leq \frac{1}{|\omega|} (\alpha - \beta)^2 \int_{\Sigma} |\nabla m|^2 \, d\xi
\]
\[
\leq \frac{E(m)}{|\omega|(\alpha - \beta)^2},
\]
thus
\[
\sum_{(a, b) \in \mathbb{R}} \frac{1}{b-a} \leq \frac{E(m)}{|\omega|(\alpha - \beta)^2}. \tag{3.7}
\]
Combining now (3.6) and (3.7) we obtain,
\[
\sum_{(a, b) \in \mathbb{R}} \left( \frac{1}{b-a} + b-a \right) \leq \frac{1}{|\omega|} \left( \frac{E(m)}{(\alpha - \beta)^2} + \frac{C_1 + C_p d^2 E(m)}{1 - \rho^2} \right) := M(\alpha, \beta, \rho, \omega, E(m)). \tag{3.8}
\]
The last inequality and the inequality \(\frac{1}{b-a} + b-a \geq 2\) yield \(M(\alpha, \beta, \rho, \omega, E(m)) \geq 2\text{card}(\mathbb{R})\), which finishes the proof of the first part. It is clear that
\[
|m_1(x)| = \frac{1}{|\omega|} \left| \int_{\omega} m_1(x, y, z) \, dy \, dz \right| \leq \frac{1}{|\omega|} \int_{\omega} |m_1(x, y, z)| \, dy \, dz \leq 1
\]
thus
\[
0 \leq 1 - \bar{m}^2_1(x) \leq 1, \quad x \in \mathbb{R}.
\]
By virtue of Lemma 3.1 and Lemma 3.2 we have,

$$\int_\Omega (1 - \bar{m}_1^2) \, d\xi \leq \int_\Omega (\bar{m}_2^2 + \bar{m}_3^2) \, d\xi + C_p d^2 E(m) < \infty,$$

thus

$$\int_\mathbb{R} (1 - \bar{m}_x^2) \, dx < \infty. \quad (3.9)$$

The integrand is continuous and positive thus for any $0 < \delta < 1$ and $N > 0$ there exists $x_\delta > N$ such that $|\bar{m}_1(x_\delta)| > 1 - \frac{\delta}{2}$. Therefore there exists an increasing sequence $\{x_n\}$ such that $x_n \to \infty$ and $|\bar{m}_1(x_n)| > 1 - \frac{\delta}{2}$. Thus for infinitely many indices $n$ one has one of the following: $\bar{m}_1(x_n) > 1 - \frac{\delta}{2}$ or $\bar{m}_1(x_n) < -1 + \frac{\delta}{2}$. Assume that for a subsequence (not relabeled) there holds $\bar{m}_1(x_n) > 1 - \frac{\delta}{2}$. Let us then show, that $\bar{m}_1(x) > 1 - \delta$ for all $x > N_\delta$ and some $N_\delta$. Assume in the contrary that for an increasing sequence $(\bar{x}_n)_{n \in \mathbb{N}}$ with $\bar{x}_n \to \infty$ one has $\bar{m}_1(\bar{x}_n) \leq 1 - \delta$. We construct an infinite family of disjoint intervals $(a_n, b_n)$ such that the value of $\bar{m}_1$ at one end of $(a_n, b_n)$ is less or equal than $1 - \delta$ and at the other end is bigger than $1 - \frac{\delta}{2}$ for all $n \in \mathbb{N}$. We start with taking the smallest $n$ such that $\bar{x}_n > x_1$ and denote it by $\bar{n}_1$ and set $a_1 = x_1, b_1 = \bar{x}_1$. In the second step we take the smallest $n$ such that $x_n > b_1$ and denote it by $n_2$ and then we take the smallest $n$ such that $\bar{x}_n > x_{n_2}$ and denote it by $\bar{n}_2$ and set $a_2 = x_{n_2}$ and $b_2 = \bar{x}_{n_2}$. This process will never stop, thus the intervals $(a_n, b_n)$ are constructed such that $\bar{m}_x(a_n) > 1 - \frac{\delta}{2}$ and $\bar{m}_x(b_n) < 1 - \delta$. Since $\bar{m}_x$ is continuous in $\mathbb{R}$ the new sequence of disjoint intervals $(\bar{a}_n, \bar{b}_n)$ where $\bar{a}_n = \sup \{ x \in (a_n, b_n) \mid \bar{m}_x(x) \geq 1 - \frac{\delta}{2} \}$ and $\bar{b}_n = \inf \{ x \in (a_n, b_n) \mid \bar{m}_x(x) \leq 1 - \delta \}$ have the properties $\bar{m}_1(\bar{a}_n) = 1 - \frac{\delta}{2}$, $\bar{m}_1(\bar{b}_n) = 1 - \delta$ and $|\bar{m}_x(x)| \leq 1 - \frac{\delta}{2}$ for all $x \in [\bar{a}_n, \bar{b}_n]$ which contradicts (3.5). The same can be done for $-\infty$.

\[ \square \]

**Remark 3.4.** If $m \in \tilde{A}(\Omega)$ then $\lim_{x \to \pm \infty} \bar{m}_1(x) = \pm 1$.

**Proof.** By Lemma 3.3 we have $\lim_{x \to \pm \infty} |\bar{m}_1(x)| = 1$. Since $\bar{m}_1(x)$ is continuous and $\bar{m} - \bar{e} \in H^1(\Omega)$, then the proof follows. \[ \square \]

**Remark 3.5.** If $|m| = 1$ and $E_0(m) < \infty$ then $\lim_{x \to \pm \infty} |m_1(x)| = 1$.

**Proof.** The proof is analogues to the proof of property (ii) in Lemma 3.3. \[ \square \]

## 4 Existence of minimizers

We start by proving a simple compactness lemma that will be crucial in the proof of the existence theorem.

**Lemma 4.1.** Assume that the sequence of magnetizations $\{m^n\}$ defined in the same domain $\Omega$ satisfies and $E(m^n) \leq C$ for some constant $C$. Then there exists a magnetization $m^0 : \Omega \to S^2$ such that for a subsequence of $\{m^n\}$ (not relabeled) the following statements hold:

- (i) $\nabla m^n \rightharpoonup \nabla m^0$ weakly in $L^2(\Omega)$
(ii) \( m^n \to m^0 \) strongly in \( L^2_{\text{loc}}(\Omega) \)

(iii) \( E(m^0) \leq \lim \inf E(m^n) \).

Proof. Let \( u_n \) be a weak solution of \( \Delta u = \text{div} m^n \). From \( \int_{\Omega} |\nabla m^n|^2 + \int_{\mathbb{R}^3} |\nabla u^n|^2 \leq C \) we get by a standard compactness argument that, \( \nabla m^n \rightharpoonup \nabla m^0 \) in \( L^2(\Omega) \), \( \nabla u_n \rightharpoonup g \) in \( L^2(\mathbb{R}^3) \) and \( m^n \to m^0 \) in \( L^2_{\text{loc}}(\Omega) \), for the same subsequence (not relabeled) of \( \{\nabla m^n\} \) and \( \{\nabla u_n\} \) and some \( f \in L^2(\Omega) \) and \( g \in L^2(\mathbb{R}^3) \). We extend \( m^0 \) outside \( \Omega \) as zero. The identities

\[
\int_{\Omega} m^n \cdot \nabla \varphi = \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \varphi \quad \text{for all } n \in \mathbb{N} \text{ and } \varphi \in C^\infty_0(\mathbb{R}^3),
\]

will then yield

\[
\int_{\Omega} m^0 \cdot \nabla \varphi = \int_{\mathbb{R}^3} g \cdot \nabla \varphi \quad \text{for all } \varphi \in C^\infty_0(\mathbb{R}^3).
\]

Since \( g \in L^2(\mathbb{R}^3) \) then the Helmholtz projection of \( g \) onto the subspace of gradient fields in \( L^2(\mathbb{R}^3) \) will have the form \( \nabla u_0 \), will satisfy \( \|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leq \|g\|_{L^2(\mathbb{R}^3)} \) and will be a weak solution of \( \Delta u = \text{div} \, m^0 \) which is equivalent to

\[
\int_{\mathbb{R}^3} g \cdot \nabla \varphi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \varphi \quad \text{for all } \varphi \in C^\infty_0(\mathbb{R}^3),
\]

thus we get

\[
\int_{\mathbb{R}^3} m^0 \cdot \nabla \varphi = \int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \varphi \quad \text{for all } \varphi \in C^\infty_0(\mathbb{R}^3)
\]

which means that \( u_0 \) is a weak solution of

\[
\Delta u = \text{div} m^0.
\]

Therefore from the weak convergence \( \nabla m^n \rightharpoonup \nabla m^0 \) and \( \nabla u_n \rightharpoonup g \) we obtain,

\[
\|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leq \|g\|_{L^2(\mathbb{R}^3)} \leq \lim \inf_{n \to \infty} \|\nabla u_n\|_{L^2(\mathbb{R}^3)}
\]

\[
\|\nabla m^0\|_{L^2(\mathbb{R}^3)} \leq \lim \inf_{n \to \infty} \|\nabla m^n\|_{L^2(\mathbb{R}^3)}
\]

which yields \( E(m^0) \leq \lim \inf_{n \to \infty} E(m^n) \).

Proof of Theorem 2.1. We adopt the direct method of proving an existence of a minimizer. The idea is starting with any minimizing sequence, we construct another minimizing sequence that has a limit in \( \tilde{A}(\Omega) \) in the sense of Lemma 4.1. Let \( \{m^n\} \) be a minimizing sequence, i.e.,

\[
\lim_{n \to \infty} E(m^n) = \inf_{m \in \tilde{A}(\Omega)} E(m).
\]

First of all note that minimization problem (2.1) is invariant under translations in the \( x \) direction, that is if \( m \in \tilde{A}(\Omega) \) then obviously \( m_{c}(x, y, z) = m(x - c, y, z) \in \tilde{A}(\Omega) \) and
$E(m_n) = E(m)$. We have that $|E(m^n)| \leq M$ for some $M$ and for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ consider the sets $A_n$, $B_n$ and $C_n$ defined as follows:

$$A_n = \left\{ x \in \mathbb{R} : -1 \leq \bar{m}_n^1(x) < -\frac{1}{2} \right\}$$

$$B_n = \left\{ x \in \mathbb{R} : -\frac{1}{2} \leq \bar{m}_n^1(x) \leq \frac{1}{2} \right\}$$

$$C_n = \left\{ x \in \mathbb{R} : \frac{1}{2} < \bar{m}_n^1(x) \leq 1 \right\}$$

Since $\bar{m}_n^1$ is continuous in $\mathbb{R}$ then for all $n \in \mathbb{N}$, $A_n$, $B_n$ and $C_n$ are a finite or countable union of disjoint intervals. We distinguish two types of intervals in $B_n$. A composite interval $(a, b) \subset B_n$ is said to be of the first type if $|\bar{m}_n^1(a) - \bar{m}_n^1(b)| = 1$, and of the second type otherwise. By Lemma 3.3 the sum of the lengths of all intervals, as well as the number of the first type intervals in $B_n$ is bounded by a number $s$ depending only on $M$ and $\omega$, i.e., a constant not depending on $n$. Consider two cases:

**CASE1. There are no second type intervals in $B_n$ for all $n \in \mathbb{N}$.**

Let us paint all the points of $A_n$, $B_n$ and $C_n$ with respectively black, yellow and red color for all $n \in \mathbb{N}$. We call the increasing sequence $\{n_k\} \subset \mathbb{N}$ "good" if for every $k \in \mathbb{N}$ there exist two intervals $(a^k_1, a^k_2) \subset A_{n_k}$ and $(c^k_1, c^k_2) \subset C_{n_k}$ such that

$$a^k_2 - a^k_1 \to +\infty, \quad c^k_2 - c^k_1 \to +\infty, \quad 0 < c^k_1 - a^k_2 \leq C$$

for a constant $C$ not depending on $k$. The endpoints $a^k_1$ and $c^k_2$ can also take values $-\infty$ and $+\infty$ respectively. If $\{n_k\}$ is "good", the subsequence $\{m^{n_k}\}$ will also be called "good". We show, that any minimizing sequence $\{m^n\} \subset \bar{A}(\Omega)$ can be translated in the $x$ coordinate such that the new sequence contains a "good" subsequence. For every fixed $n$ there are some black, yellow and red intervals between $(-\infty, a_n)$ and $(c_n, +\infty)$. Note that there is obviously at least one yellow interval between any two black and any two red ones, thus the number of both black and yellow intervals is at most $s + 1$, hence the number of all intervals in the $n$-th family is bounded by the same number $S = 3s + 2$ for all $n$. Let us number both the red and the black intervals in any family of intervals. Let us prove the proposition below, which is a reformulation of our problem:

**Proposition.** Assume a sequence of natural numbers $l_n$ and a sequence of families of $l_n$ disjoint intervals on the real line paired with black and red color are given for all $n \in \mathbb{N}$. Assume $l_n \leq l$ and the sum of the lengths of $l_n - 1$ gaps between the intervals in the $n$-th family is bounded by the same number $M$ for all $n$. Assume furthermore that for any $n$, the far left placed interval is black and the far right placed interval is red and their lengths tend to $\infty$ as $n$ goes to infinity. Then there exists a subsequence $\{n_k\}$ and two associated intervals $(a^k_1, a^k_2)$ and $(c^k_1, c^k_2)$ in the $n_k$-th family such that $(a^k_1, a^k_2)$ is black, $(c^k_1, c^k_2)$ is red, and
\[ a^k_2 - a^k_1 \to +\infty, \quad c^k_2 - c^k_1 \to +\infty \quad 0 < c^k_1 - a^k_2 \leq M_1 \] (4.1)

for a constant \( M_1 \) and all \( k \in \mathbb{N} \).

**Proof of proposition.** The case \( l = 2 \) is evident. Assume that the proposition is true for \( l \leq N \) and let us prove it for \( l = N + 1 \). Since \( l \geq 3 \), in every family there are at least two black intervals of the same color. Assume that for infinitely many indices \( n \) there are at least two black intervals in the \( n \)-th family. Consider the far right placed black intervals for all such families. There are two possible cases:

**Case 1.** For a subsequence their lengths tend to \(+\infty\).

In this case we can omit all the intervals placed on their left side which leads to a situation with less intervals in every family (in such a subsequence) fulfilling the requirements of the proposition, so by induction the existence of a "good" subsequence is proven.

**Case 2.** Their lengths are bounded by the same constant.

In this case we can remove this intervals and this will lead us to a situation with less intervals in all families fulfilling the requirements of the statements so by the induction the existence of a "good" subsequence is proven.

Let us get now back to our situation. If we remove all the yellow intervals from the real line for all \( n \in \mathbb{N} \) then the families of the black and the red intervals fulfill the requirements of the proposition, thus the existence of a "good" sequence is proven. Take the two intervals \([a^k_1, a^k_2]\) and \([c^k_1, c^k_2]\) for all \( k \in \mathbb{N} \) and denote the the "good" subsequence of magnetizations again by \( \{m^k\} \) which will also be a minimizing sequence. Let us translate \( m^k \) by a factor of \( a^k_2 \) and denote

\[ m^k_{\text{good}}(x, y, z) = m^k(x + a^k_2, y, z). \]

Then \( \{m^k_{\text{good}}\} \) is a minimizing sequence and furthermore denoting \( a^k_3 = a^k_2 - a^k_1 \), \( c^k_3 = c^k_1 - a^k_2 \) and \( c^k_4 = c^k_2 - a^k_2 \), we obtain,

\[ m^k_{\text{good}}(x) \leq -\frac{1}{2} \quad \text{for} \quad x \in [-a^k_3, 0] \quad \text{and} \quad m^k_{\text{good}}(x) \geq \frac{1}{2} \quad \text{for} \quad x \in [c^k_3, c^k_4], \] (4.2)
\[ a^k_3 \to \infty, \quad c^k_4 - c^k_3 \to \infty, \quad 0 < c^k_3 \leq M_1. \] (4.3)

Owing to Lemma 4.1 one can extract a subsequence from \( \{m^k_{\text{good}}\} \) (not relabeled) with a limit \( m^0 \in A(\Omega) \). Let us now prove that conditions (4.2) and (4.3) imply that \( m^0 \in \bar{A}(\Omega) \). We have for any fixed \( R > 0 \),
\[
\int_{-R}^{R} |\bar{m}^0_1 - \bar{m}^k_{good1}| \, dx = \frac{1}{|\omega|} \int_{-R}^{R} \int_{\omega} (m^0_1 - m^k_{good1}) \, dy \, dz \, dx
\]
\[
\leq \frac{1}{|\omega|} \int_{-R}^{R} \int_{\omega} |m^0_1 - m^k_{good1}| \, dy \, dz \, dx
\]
\[
\leq \frac{1}{|\omega|} \left( 2R|\omega| \cdot \int_{[-R,R] \times \omega} |m^0_1 - m^k_{good1}|^2 \, d\xi \right)^{\frac{1}{2}}
\]
\[
= \sqrt{\frac{2R}{|\omega|} \cdot \|m^0_1 - m^k_{good1}\|_{L^2([-R,R] \times \omega)} \to 0}
\]
as \(k \to \infty\) because of the strong convergence \(m^k_{good} \to m^0\) in \(L^2_{loc}(\Omega)\). Therefore a subsequence of \(\{\bar{m}^k_{good1}(x)\}\) converges pointwise to \(\bar{m}^1_1(x)\) almost everywhere in \([-R, R]\). Giving \(R\) all natural values and applying a diagonal argument we establish that a subsequence of \(\{\bar{m}^k_{good1}(x)\}\) converges pointwise to \(\bar{m}^0_1(x)\) almost everywhere in \(\mathbb{R}\), therefore
\[
\bar{m}^0_1(x) \leq -\frac{1}{2} \text{ a.e. in } (-\infty, 0) \text{ and } \bar{m}^0_1(x) \geq \frac{1}{2} \text{ a.e. in } [M_1, +\infty) \quad (4.4)
\]
Let us now show that conditions \(E(m^0) < \infty\) and \((4.4)\) imply \(m^0 \in \bar{A}(\Omega)\). We have by the triangle inequality
\[
\|\nabla (m^0 - \bar{e})\|_{L^2(\Omega)}^2 \leq 2\|\nabla m^0\|_{L^2(\Omega)}^2 + 2\|\nabla \bar{e}\|_{L^2(\Omega)}^2 \leq 2E(m^0) + 4|\omega| < \infty,
\]
thus it remains to prove that \(m^0 - \bar{e} \in L^2(\Omega)\). We have again by the triangle inequality and by Lemma 3.2
\[
\|m^0 - \bar{e}\|_{L^2(\Omega)}^2 \leq 2\|\bar{m}^0 - \bar{e}\|_{L^2(\Omega)}^2 + 2\|m^0 - \bar{m}^0\|_{L^2(\Omega)}^2
\]
\[
\leq 2Cp^2\|\nabla m^0\|_{L^2(\Omega)}^2 + 2\|\bar{m}^0 - \bar{e}\|_{L^2(\Omega)}^2
\]
\[
\leq 2Cp^2E(m^0) + 2\|\bar{m}^0 - \bar{e}\|_{L^2(\Omega)}^2,
\]
thus it remains to prove that \(\bar{m}^0 - \bar{e} \in L^2(\Omega)\). One can assume without loss of generality that \(M_1 \geq 1\) in \((4.4)\). We calculate,
\[
\int_{\Omega} |\bar{m}^0 - \bar{e}|^2 = \int_{[-1,M_1] \times \omega} |\bar{m}^0 - \bar{e}|^2 + \int_{[-\infty,-1] \times \omega} |\bar{m}^0 - \bar{e}|^2 + \int_{[M_1,\infty] \times \omega} |\bar{m}^0 - \bar{e}|^2 = I_1 + I_2 + I_3.
\]
The estimation of \(I_1, I_2\) and \(I_3\) is straightforward:
\[
I_1 \leq 4(1 + M_1)|\omega|.
\]
Due to condition (4.4) and Lemma 3.2 we have,

\[ I_2 = \int_{[-\infty, -1] \times \omega} (1 + |\bar{m}|^2 + 2|\tilde{m}|) \]

\[ = 2 \int_{[-\infty, -1] \times \omega} (1 + \bar{m}) + \int_{[-\infty, -1] \times \omega} (|m|^2 - |\bar{m}|^2) \]

\[ \leq 2 \int_{[-\infty, -1] \times \omega} (1 + \bar{m})(1 - \bar{m}) + C \rho^2 \int_{[-\infty, -1] \times \omega} |\nabla m|^2 \]

\[ = 2 \int_{[-\infty, -1] \times \omega} (|m|^2 - |\bar{m}|^2) + 2 \int_{[-\infty, -1] \times \omega} (|\bar{m}|^2 + |\tilde{m}|^2) + C \rho^2 \int_{[-\infty, -1] \times \omega} |\nabla m|^2 \]

\[ \leq 3C \rho^2 \int_{[-\infty, -1] \times \omega} |\nabla m|^2 + 2 \int_{[-\infty, -1] \times \omega} (|\bar{m}|^2 + |\tilde{m}|^2). \]

Analogue analysis for \( I_3 \) gives

\[ I_3 \leq 3C \rho^2 \int_{[M_1, \infty] \times \omega} |\nabla m|^2 + 2 \int_{[M_1, \infty] \times \omega} (|\bar{m}|^2 + |\tilde{m}|^2). \]

Therefore combining the estimates for \( I_1, I_2 \) and \( I_3 \) and taking into account Lemma 3.1 we discover \( I_1 + I_2 + I_3 < \infty \) as wished. CASE1 is now established.

CASE2. There are some second type intervals in \( B_n \) for some \( n \).

Removing all the second type yellow intervals from the real line we can regard the rest as a real line without gaps simply by shifting all the intervals to the left hand side such that after that operation no overlap occurs and there is no gap left. Precisely, we shift each interval to the left by a factor equal to the sum of the lengths of the gaps between that interval and \(-\infty\). During that operation we unify the black and red intervals with the neighboring intervals of the same color but we regard the possible neighboring first type yellow intervals as separate. We get a situation like in CASE1 and therefore we can prove the existence of a "good" subsequence. It is easy to show that since that sum of the lengths of the second type yellow intervals in each family is bounded by the same constant then the in Lemma 4.1 described limit of the obtained "good" subsequence will belong to \( \tilde{A}(\Omega) \) and hence will be an energy minimizer in \( \tilde{A}(\Omega) \). The proof is complete now.

5 Convergence of almost minimizers

Throughout this section we will consider a sequence of domain-magnetization-energy triples \((\Omega_n, m^n, E(m^n))_{n \in \mathbb{N}}\) such that \( \Omega_n = \mathbb{R} \times (d_n \cdot \omega) \), \( m^n \in \tilde{A}(\Omega_n) \), \( d_n \rightarrow 0 \) and \( \lim_{n \rightarrow \infty} \frac{E(m^n)}{d_n^2} = \min_{m \in A} E_0(m) \), i.e., \( \{m^n\} \) is a sequence of almost minimizers. Assume furthermore that \( \omega \) has 180 degree rotational symmetry and the matrix \( M_\omega \) has three different eigenvalues, i.e., \( \alpha_2 \neq \alpha_3 \), hence one can assume without loss of generality, that \( \alpha_2 < \alpha_3 \). Note that due to (2.4) we have

\[ E(m^n) \leq C d_n^2 \text{ for all } n. \quad (5.1) \]
Let $\{\hat{m}^n\}$ converge to some $m^0(x) \in \tilde{A}(\Omega)$ in the sense of Definition 2.2 then

(i) $\lim_{n \to \infty} \|\nabla \hat{m}^n\|_{L^2(\Omega)} = \|\nabla m^0\|_{L^2(\Omega)}$,

(ii) $\lim_{n \to \infty} \|\hat{m}^n_2\|_{L^2(\Omega)} = \|m^0_2\|_{L^2(\Omega)}$, \quad $\lim_{n \to \infty} \|\hat{m}^n_3\|_{L^2(\Omega)} = \|m^0_3\|_{L^2(\Omega)}$.

Proof. The inequality $\liminf_{n \to \infty} \|\nabla \hat{m}^n\|_{L^2(\Omega)} \geq \|\nabla m^0\|_{L^2(\Omega)}$ is trivial, while the inequality $\liminf_{n \to \infty} \|\hat{m}^n_2\|_{L^2(\Omega)} \geq \|m^0_2\|_{L^2(\Omega)}$ follows from the convergence $m^0_2 \to m^0_2$ in $L^2_{loc}(\Omega)$. We have furthermore by Lemma 5.2 and by (5.1) that,

$$\|\hat{m}^n_2 - \hat{m}^n_2\|_{L^2(\Omega)} = \frac{1}{d_n^2} \|\nabla m^n - \nabla m^n\|_{L^2(\Omega)}$$

$$\leq C_p \|\nabla m^n\|_{L^2(\Omega)}^2$$

$$\leq C_p C d_n^2,$$

thus

$$\|\hat{m}^n_2 - \hat{m}^n_2\|_{L^2(\Omega)} \to 0.$$ (5.2)

Therefore we get $\liminf_{n \to \infty} \|\hat{m}^n_2\|_{L^2(\Omega)} \geq \|m^0_2\|_{L^2(\Omega)}$ and a similar inequality for $m^0_3$ is also fulfilled. It remains to only show the opposite inequalities with $\limsup$. It is clear that $\|\nabla \hat{m}^n\|_{L^2(\Omega)} \leq \|\nabla \hat{m}^n\|_{L^2(\Omega)}$, thus it suffices to prove that $\limsup_{n \to \infty} \|\nabla \hat{m}^n\|_{L^2(\Omega)} \leq \|\nabla m^0\|_{L^2(\Omega)}$. Assume now in contradiction that one of the three inequalities with $\limsup$, we intend to prove, fails. Therefore we have owing to (3.1), that for some $\delta > 0$ there holds,

$$\limsup_{n \to \infty} \frac{E(m^n)}{d_n^2} \geq \max \left( \limsup_{n \to \infty} \|\nabla \hat{m}^n\|_{L^2(\Omega)}^2 + \liminf_{n \to \infty} \alpha_2 \|\hat{m}^n_2\|_{L^2(\Omega)}^2 + \liminf_{n \to \infty} \alpha_3 \|\hat{m}^n_3\|_{L^2(\Omega)}^2, \right.$$

$$\liminf_{n \to \infty} \|\nabla \hat{m}^n\|_{L^2(\Omega)}^2 + \limsup_{n \to \infty} \alpha_2 \|\hat{m}^n_2\|_{L^2(\Omega)}^2 + \limsup_{n \to \infty} \alpha_3 \|\hat{m}^n_3\|_{L^2(\Omega)}^2, \left. \right.$$}

$$\liminf_{n \to \infty} \|\nabla \hat{m}^n\|_{L^2(\Omega)}^2 + \liminf_{n \to \infty} \alpha_2 \|\hat{m}^n_2\|_{L^2(\Omega)}^2 + \limsup_{n \to \infty} \alpha_3 \|\hat{m}^n_3\|_{L^2(\Omega)}^2)$$

$$\geq E_0(m^0) + \delta$$

$$\geq \min_{m \in A_0} E_0(m) + \delta,$$

which contradicts (2.4). The lemma is proved now.

Corollary 5.2. Let $\{m^n\}$ and $m^0$ be as in Lemma 5.1. Then

(i) $\lim_{n \to \infty} \|\hat{m}^n_2\|_{L^2(\Omega)} = \|m^0_2\|_{L^2(\Omega)}$, \quad $\lim_{n \to \infty} \|\hat{m}^n_3\|_{L^2(\Omega)} = \|m^0_3\|_{L^2(\Omega)}$.

Proof. It follows from Lemma 5.1 and equality (5.2).

Lemma 5.3. Let $\{m^n\}$ and $m^0$ be as in Lemma 5.1. Then

(i) $\lim_{n \to \infty} \|\nabla \hat{m}^n - \nabla m^0\|_{L^2(\Omega)} = 0$

(ii) $\lim_{n \to \infty} \|\hat{m}^n_2 - m^0_2\|_{L^2(\Omega)} = 0$, \quad $\lim_{n \to \infty} \|\hat{m}^n_3 - m^0_3\|_{L^2(\Omega)} = 0$.  

16
Proof. The inequality \( \liminf_{n \to \infty} \| \nabla \dot{m}^n \|_{L^2(\Omega)} \geq \| \nabla m^0 \|_{L^2(\Omega)} \) ia a consequence of the weak convergence \( \nabla \dot{m}^n \to \nabla m^0 \). The opposite inequality \( \limsup_{n \to \infty} \| \nabla \dot{m}^n \|_{L^2(\Omega)} \leq \| \nabla m^0 \|_{L^2(\Omega)} \) has been proven in the proof of Lemma 5.1. Therefore \( \limsup_{n \to \infty} \| \nabla \dot{m}^n \|_{L^2(\Omega)} = \| \nabla m^0 \|_{L^2(\Omega)} \) which combined with the weak convergence \( \dot{m}^n \to m^0 \) gives (i). Fix now \( l > 0 \). We have by virtue of Corollary [5.2]

\[
\limsup_{n \to \infty} \int_{\Omega} |\dot{m}^n_2 - m^0_2|^2 \leq \limsup_{n \to \infty} \int_{[-l,l] \times \omega} |\dot{m}^n_2 - m^0_2|^2 + \limsup_{n \to \infty} \int_{\Omega \setminus ([[-l,l] \times \omega)} |\dot{m}^n_2 - m^0_2|^2 \\
\leq 2 \limsup_{n \to \infty} \int_{\Omega \setminus ([[-l,l] \times \omega)} (|\dot{m}^n_2|^2 + |m^0_2|^2) \\
\leq 2 \limsup_{n \to \infty} \int_{\Omega} (|\dot{m}^n_2|^2 + |m^0_2|^2) - 2 \liminf_{n \to \infty} \int_{[-l,l] \times \omega} (|\dot{m}^n_2|^2 + |m^0_2|^2) \\
= 4|\omega| \int_{\mathbb{R} \setminus [-l,l]} |m^0_1(x)|^2 \, dx.
\]

From the arbitrariness of \( l \) we get the validity of the first equality in (ii). The proof of the second equality in (ii) is straightforward.

\[ \square \]

Lemma 5.4. Let \( \{ m^n \} \) and \( m^0 \) be as in Lemma 5.1. Assume in addition that for some \( N \in \mathbb{N} \) and \( l > 0 \) we have for all \( n \geq N \)

\[
\dot{m}^n_1(x) \leq 0, \quad x \in (-\infty, -l] \quad \text{and} \quad \dot{m}^n_1(x) \geq 0, \quad x \in [l, +\infty).
\]

Then

\[
\lim_{n \to \infty} \| \dot{m}^n - m^0 \|_{H^1(\Omega)} = 0.
\]

Proof. By Lemma 5.3 it suffices to show that \( \lim_{n \to \infty} \| \dot{m}^n_1 - m^0_1 \|_{L^2(\Omega)} = 0 \). Since \( m^0(x) \in \tilde{A}(\Omega) \) then due to Remark 3.4 there exists \( l_1 > 0 \) such that

\[
m^0_1(x) \leq -\frac{1}{2}, \quad x \in (-\infty, l_1] \quad \text{and} \quad m^0_1(x) \geq \frac{1}{2}, \quad x \in [l_1, +\infty).
\]

For any fixed \( l_2 > \max(l, l_1) \) we have,

\[
\int_{\Omega} |\dot{m}^n_1 - m^0_1|^2 = \int_{[-l_2, l_2] \times \omega} |\dot{m}^n_1 - m^0_1|^2 + \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} |\dot{m}^n_1 - m^0_1|^2.
\]

The first summand converges to zero and we have furthermore that \( \| \dot{m}^n_1 - \dot{m}^n_1 \|_{L^2(\Omega)} \to 0 \), thus it suffices to show that

\[
\lim_{n \to \infty} \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} |\dot{m}^n_1 - m^0_1|^2 = 0.
\]

For \( n \geq N \) we have

\[
\int_{\Omega \setminus ([-l_2, l_2] \times \omega)} |\dot{m}^n_1 - m^0_1|^2 \leq \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} (|\dot{m}^n_1|^2 - |m^0_1|^2) \\
\leq \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} (|\dot{m}^n_1|^2 - |m^0_1|^2) + \int_{\Omega \setminus ([-l_2, l_2] \times \omega)} (|\dot{m}^n_1|^2 - |m^0_1|^2).
\]

17
The first summand converges to zero, for the second summand we have by Lemma 5.2

$$\limsup_{n \to \infty} \int_{\Omega \setminus [-l_2,l_2] \times \omega} \left| \dot{m}_1^n - m_1^0 \right|^2 \leq \limsup_{n \to \infty} \int_{\Omega \setminus [-l_2,l_2] \times \omega} \left( \left| \dot{m}_2^n \right|^2 + \left| \dot{m}_3^n \right|^2 + \left| m_2^0 \right|^2 + \left| m_3^0 \right|^2 \right)$$

$$\leq 2 \int_{\Omega \setminus [-l_2,l_2] \times \omega} \left( \left| m_2^0 \right|^2 + \left| m_3^0 \right|^2 \right),$$

which converges to zero as $l_2$ goes to infinity.

\[\square\]

**Lemma 5.5.** Let $0 < \epsilon < 1$ and let the sequence of intervals $([b_n^1, b_n^2])_{n \in \mathbb{N}}$ be such that

$$\dot{m}_1^n(b_n^1) = -1 + \epsilon, \quad \dot{m}_1^n(b_n^2) = 1 - \epsilon.$$  

Then for sufficiently big $n$ there holds

$$\dot{m}_1^n(x) < -1 + 2\epsilon, \quad x \in (-\infty, b_n^1], \quad \dot{m}_1^n(x) > 1 - 2\epsilon, \quad x \in [b_n^2, +\infty),$$

$$-1 + \frac{\epsilon}{2} < m_1^n(x) < 1 - \frac{\epsilon}{2}, \quad x \in [b_n^1, b_n^2].$$

**Proof.** Assume in contradiction that for a subsequence $\{n_k\}$ (not relabeled) there is a point $b_n^3 \in (-\infty, b_n^1)$ such that $\dot{m}_1^n(b_n^3) \geq -1 + 2\epsilon$. Since $\dot{m}_1^n(-\infty) = -1$ and $\dot{m}_1^n$ is continuous we can without loss of generality assume that $\dot{m}_1^n(b_n^3) = -1 + 2\epsilon$. Utilizing Lemma A.3 for the intervals $(-\infty, b_n^3], [b_n^3, b_n^1], [b_n^1, +\infty)$ and (3.1) we discover,

$$\frac{E(m^n)}{d_n^2} \geq \int_{\Omega} \left| \nabla \dot{m}_1^n \right|^2 + \alpha_2 \int_{\mathbb{R}} \left| m_2^n \right|^2 + \alpha_3 \int_{\mathbb{R}} \left| m_3^n \right|^2 + o(1)$$

$$\geq 2\sqrt{\alpha_2 |\omega| \left( 2\epsilon + |\epsilon| + |2 - 2\epsilon| \right)} + o(1)$$

$$= (4 + 2\epsilon) \min_{m \in A_0} E_0(m) + o(1),$$

which contradicts the almost minimizing property of $\{m^n\}$. Similarly we get the bounds near $\infty$ and in $[b_n^1, b_n^2]$.

\[\square\]

### 5.1 Proof of Theorem 2.4

**Proof.** The proof splits into some steps:

**Step1.** Let us prove that if a sequence of magnetizations converges to some $m^0 \in \hat{A}(\Omega)$ in the sense of Definition 2.2 and satisfies conditions (2.4) and $\dot{m}_2^n(x_0) \geq 0$ for some $x_0 \in \mathbb{R}$ and for big $n$ then $m_2^0(x_0) \geq 0$.

We have due to (2.4), that

$$\int_{\Omega_n} \left| \partial_x \dot{m}_1^n \right|^2 \leq \int_{\Omega_n} \left| \partial_x m_1^n \right|^2 \leq C d_n^2,$$
Thus
\[ \int_{\mathbb{R}} |\partial_x \tilde{m}^n(x)|^2 \, dx \leq \frac{C}{|\omega|} \]
which yields that the sequence \( \{\tilde{m}^n\} \) is equicontinuous in \( \mathbb{R} \), and therefore by the Arzela-Ascoli theorem \( \{\tilde{m}^n(x)\} \) has a subsequence with a uniform limit in the interval \([x_0-1, x_0+1]\). It is trivial that the limit is \( m^0 \), and thus \( \tilde{m}^n_2(x_0) \geq 0 \) yields \( m^0_2(x_0) \geq 0 \). Evidently, the same sing preserving property holds for the first and the third components of \( \tilde{m}^n \) and also for the opposite sign. This means in particular that if \( \tilde{m}^n_1(x_0) = 0 \) for big \( n \) then \( m^0_1(x_0) = 0 \).

**Step 2.** In the second step we construct the sequences \( \{T_n\} \) and \( \{R_n\} \). Note first, that the change of variables mentioned in the theorem translates the domain \( \Omega \) to itself and preserves the energy, thus the minimization problem \((2.1)\) is invariant under that kind of transformations. Let us now evaluate the constant in estimate (3.8). The constant \( C_1 \) in (3.8) comes from Lemma 3.1 and is given by
\[
C_1 = \frac{2d_n^2 E(m^n)}{\alpha_2 d_n^2} \leq \frac{2Cd_n^2}{\alpha_2},
\]
for big \( n \). Thus we get
\[
M(\alpha, \beta, \rho, \omega_n, E(m^n)) = \frac{1}{|\omega_n|} \left( \frac{E(m^n)}{(\alpha - \beta)^2} + \frac{C_1 + C_\rho d_n^2 E(m^n)}{1 - \rho^2} \right) \\
\leq \frac{1}{|\omega|} \left( \frac{C}{(\alpha - \beta)^2} + \frac{2Cd_n^2 + C_\rho C_\alpha d_n^2}{1 - \rho^2} \right) \\
\leq M_1
\]
uniformly in \( n \). Next we choose the intervals \([b^1_n, b^2_n]\) to be as in Lemma 5.5 with \( \epsilon = \frac{1}{3} \), which is possible due to the continuity of \( \tilde{m}^n_1 \) and the fact that \( \tilde{m}^n_1(\pm \infty) = \pm 1 \). Owing to Lemma 5.5 we get
\[
\tilde{m}^n_1(x) < -\frac{1}{3}, \quad x \in (-\infty, b^1_n], \quad \tilde{m}^n_1(x) > \frac{1}{3}, \quad x \in [b^2_n, +\infty), \quad (5.3)
\]
\[
-\frac{5}{6} < m^1_n(x) < \frac{5}{6}, \quad x \in [b^1_n, b^2_n]. \quad (5.4)
\]
Therefore, we obtain by the uniform estimate on \( M(\alpha, \beta, \rho, \omega_n, E(m^n)) \) and by the estimate (3.8) of Lemma 3.3 that for sufficiently big \( n \) there holds,
\[
b^2_n - b^1_n \leq M_1. \quad (5.5)
\]
Let now \( x_n \in [b^1_n, b^2_n] \) be such that \( \tilde{m}^n_1(x_n) = 0 \). For any \( n \in \mathbb{N} \) we choose \( T_n \) to be the translation by \( x_n \) and the rotation \( R_n \) to be the identity if \( \tilde{m}^n_2(x_n) \geq 0 \) and the rotation by 180 degree otherwise. In the last step we prove that the whole sequence \( \{\tilde{m}^n\} \) converges to \( m^\omega \) in \( H^1(\Omega) \).

**Step 3.** For convenience of notation we will omit the "tilde" in \( \tilde{m}^n \). We are now ready to prove that \( \|\tilde{m}^n - m^\omega\|_{H^1(\Omega)} \to 0 \) as \( n \to \infty \). Assume in contradiction that for a subsequence
(not relabeled) $\|\hat{m}^n - m^\omega\|_{H^1(\Omega)} \geq \delta > 0$ for some $\delta$. Like in the proof of Lemma 4.1 we can show that a subsequence of $\{\hat{m}^n\}$ converges to some $m^0$ in the sense of Definition 2.2. By the $\Gamma$-convergence theorem we then have $E_0(m^0) \leq \liminf_{n \to \infty} \frac{E(m^n)}{d_n^2}$ thus

$$E_0(m^0) = \min_{m \in A_0} E_0(m).$$

(5.6)

Next we have by the sign-preserving property of Step1 and by bounds (5.3)–(5.5), that

$$\hat{m}^0_1(x) \leq -\frac{1}{3}, \quad x \in (-\infty, -M_1], \quad \hat{m}^0_1(x) \geq \frac{1}{3}, \quad x \in [M_1, +\infty).$$

(5.7)

Invoking now Remark 3.5 and the properties (5.6) and (5.7) we discover $m^0(\pm\infty) = \pm 1$, which yields

$$m^0 \in A_0,$$

(5.8)

i.e., $m_0$ is a minimizer of the minimization problem (A.3). Again, by the sign-preserving property we have $m^0_1(0) = 0$ and $m^0_2(0) \geq 0$, thus by the analysis on the minimization problem (A.3) in Appendix, we establish that actually $m^0$ and $m^\omega$ coincide. Note, finally, that the requirements of Lemma 5.4 are satisfied, thus we get

$$\lim_{n \to \infty} \|\hat{m}^n - m^\omega\|_{H^1(\Omega)} = \lim_{n \to \infty} \|\hat{m}^n - m^0\|_{H^1(\Omega)} = 0,$$

which is a contradiction. The theorem is proven now.

We mention that it is easy to see that any rectangle that is not a square and any ellipse that is not a circle satisfies the condition $0 < \alpha_2 < \alpha_3$. This condition shows that the cross section $\omega$ does not have many rotational symmetries in some sense. For instance, if $\omega$ has a 90 degree rotational symmetry, then one can show that $\alpha_2 = \alpha_3$. It is also worth mentioning that one can prove a modified version of Theorem 2.4 in the case when $\omega$ is a disc or a canonical polygon with even number of vertices, namely due to the symmetry it is not true that any of the rotations $R_n$ is either the identity the rotation by 180 degree, but one can prove their existence. In conclusion we state that Theorem 2.4 shows that in thin wires energy minimizers with a 180 degree domain wall are transverse (Neél) walls that have the shape of $m^\omega$.

6 Upper and lower bounds for thick wires

Throughout this section we assume that $l, d \geq 1$ and are comparable to each other. For convenience we will assume that $d = l$. We prove Theorem 2.5 by constructing a vortex wall that has an energy of order $d^2 \sqrt{\ln d}$ following the idea of DeSimone, Konh, Müller and Otto in [5], namely it is the idea of divergence free fields, that are tangential to the boundary, which are preferable for the magnetostatic energy. Assume $L > 0$ and denote by $\Omega_L$ the domain $[-L, L] \times [-d, d] \times [-d, d]$. Take the rectangular parallelepiped $\Omega_L$ and cut off from it the two cones with the vertex at $(0, 0, 0)$ and the bases $-L \times [-d, d] \times [-d, d]$ and $L \times [-d, d] \times [-d, d]$ respectively and denote the obtained domain by $R_L$. The main
diagonals of $\Omega_L$ divide $R_L$ into four parts. Taking into account the orientation in the plane $OYZ$ denote that parts by $R_{L}^{up}$, $R_{L}^{right}$, $R_{L}^{bottom}$ and $R_{L}^{left}$ respectively. We first construct a magnetization $\tilde{m}$ that has infinite exchange energy but a magnetostatic energy easy to bound. Consider the following vector field:

$$\tilde{m} = \begin{cases} 
\left( \sin \frac{\pi dx}{2L_z}, \cos \frac{\pi dx}{2L_z}, 0 \right) & \text{in } R_{L}^{up} \\
\left( \sin \frac{\pi dy}{2L_y}, 0, -\cos \frac{\pi dy}{2L_y} \right) & \text{in } R_{L}^{right} \\
\left( -\sin \frac{\pi dx}{2L_z}, -\cos \frac{\pi dx}{2L_z}, 0 \right) & \text{in } R_{L}^{bottom} \\
\left( -\sin \frac{\pi dy}{2L_y}, 0, \cos \frac{\pi dy}{2L_y} \right) & \text{in } R_{L}^{left} 
\end{cases}$$

Note that the vector field $(0, \tilde{m}_2, \tilde{m}_3)$ is divergence free (see cross section Figure 2.1).

A cross section for $\tilde{m}$

Figure 2.1

Therefore we have

$$\text{div}\tilde{m} = \frac{\partial\tilde{m}_1}{\partial x} \geq 0 \quad \text{in } \Omega.$$ 

We have furthermore that $u$ is the weak solution of $\triangle u = \text{div}\tilde{m}$, thus

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi = -\int_{\Omega} \varphi \cdot \text{div}\tilde{m} + \int_{\partial\Omega} \varphi \tilde{m} \cdot \nu,$$

where $\nu$ is the outward unit normal to $\partial\Omega$. Note that $\tilde{m}$ is tangential to the boundary of $\Omega$ thus substituting $\varphi = u$ in the above equality, we find the magnetostatic energy of $\tilde{m}$,

$$\int_{\mathbb{R}^3} |\nabla u|^2 = -\int_{\Omega} u \cdot \text{div}\tilde{m} = \int_{\Omega} \int_{\Omega} \Gamma(\xi - \xi_1) \frac{\partial\tilde{m}_1(\xi)}{\partial x} \frac{\partial\tilde{m}_1(\xi_1)}{\partial x} \, d\xi \, d\xi_1,$$

where $\Gamma(\xi) = \frac{1}{4\pi|\xi|}$ is the Green map in $\mathbb{R}^3$. Note that the integrand is zero in the complement of $R_L$, so we first estimate it if the first integration is done over $R_L^{up}$. We have in $R_L^{up}$,

$$0 \leq \frac{\partial\tilde{m}_1(\xi)}{\partial x} = \frac{\pi d}{2L_z} \cos \frac{\pi dx}{2L_z} \leq \frac{\pi d}{2L_z},$$
thus
\[
\int_{\mathbb{R}^d} \Gamma(\xi - \xi_1) \frac{\partial \tilde{m}_1(\xi)}{\partial x} \, d\xi \leq \int_0^d \frac{\pi d}{2Lz} \, dz \int_{-\frac{Lz}{d}}^{\frac{Lz}{d}} \int_{-z}^{z} \Gamma(\xi - \xi_1) \, dy \, dx.
\]

Due to Lemma A.2 we have,
\[
\int_{-\frac{Lz}{d}}^{\frac{Lz}{d}} \int_{-z}^{z} \Gamma(\xi - \xi_1) \, dy \, dx \leq \frac{10z}{4\pi} \left(1 + \ln \frac{L}{d}\right)
\]

and
\[
\int_{\mathbb{R}^d} \Gamma(\xi - \xi_1) \frac{\partial \tilde{m}_1(\xi)}{\partial x} \, d\xi \leq \frac{5d^2}{4L} \left(1 + \ln \frac{L}{d}\right).
\]

The integrals over the other parts of \(R_L\) have the same upper bound, thus we obtain
\[
E_{\text{mag}}(\tilde{m}) \leq \frac{20d^4}{L} \left(1 + \ln \frac{L}{d}\right). \tag{6.1}
\]

The reason for \(\tilde{m}\) having an infinite exchange energy is that it has singularities on the part of the boundary of \(R_L\) that belongs to \(\Omega_L\). We ignore for a moment this boundary charges and compute \(E_{\text{ex}}(\tilde{m})\) taking into account only the volume charges. We have formally by direct calculation that,
\[
E_{\text{ex}}^{\text{formal}}(\tilde{m}) = 4\int_0^d \frac{\pi d^2}{4L^2z^2} \int_{-\frac{Lz}{d}}^{\frac{Lz}{d}} \int_{-z}^{z} \left(1 + \frac{x^2}{z^2}\right) \, dy \, dx \, dz \leq \]
\[
\leq 4\int_0^d \frac{\pi d^2}{4L^2z^2} \int_{-\frac{Lz}{d}}^{\frac{Lz}{d}} \int_{-z}^{z} \left(1 + \frac{L^2}{d^2}\right) \, dy \, dx \, dz \]
\[
= 4\pi^2 \left(\frac{d^2}{L} + L\right),
\]

thus
\[
E_{\text{ex}}^{\text{formal}}(\tilde{m}) \leq 4\pi^2 \left(\frac{d^2}{L} + L\right). \tag{6.2}
\]

In the next step we build a magnetization \(m\) with a finite exchange energy by slightly modifying \(\tilde{m}\) near the singularity points. It works as follows: First take the planes \(\{z = \frac{d}{d-1}y\}\) and \(\{z = -\frac{d-1}{d}y\}\). To get a continuous \(m\) from \(\tilde{m}\) we change \(\tilde{m}\) in the following two regions: The first one is the intersection of \(\Omega_L\) with the region between the planes \(\{z = \frac{d}{d-1}y\}\) and \(\{z = y\}\) and the second one is the intersection of \(\Omega_L\) with the region between the planes \(\{z = -\frac{(d-1)}{d}y\}\) and \(\{z = -y\}\). For more transparency see Figures 2.2 and 2.3.
A longitudinal section \( \{ z = c > 0 \} \)

**Figure 2.2**

A cross section for \( m \).

**Figure 2.3**

Denote the upper part of the first new narrow region (where \( z \geq 0 \)) by \( \Omega_{L,1}^{up} \) and the bottom part by \( \Omega_{L,1}^{bottom} \). We make the same notation also for the second narrow region. Finally, define the magnetization \( m \) in \( \Omega_{L,1}^{up} \) by

\[
m(x, y, z) = (\sin \frac{\pi dx}{2Lz}, \cos \frac{\pi dx}{2Lz} \sin \frac{\pi d(z - y)}{2z}, -\cos \frac{\pi dx}{2Lz} \cos \frac{\pi d(z - y)}{2z}).
\]

The definition of \( m \) in the other three regions is analogues. Note that the vector field \( m \) has now a single singularity at the origin. Owing to Lemma [A.1] we have by direct calculation

\[
|E_{mag}(m) - E_{mag}(\tilde{m})| \leq \|m - \tilde{m}\|_{L^2(\Omega_L)}^2 + 2\|m - \tilde{m}\|_{L^2(\Omega_L)} \sqrt{E_{mag}(\tilde{m})}
\]

23
\[
\leq 16dL + 16\sqrt{5d^2\sqrt{d\ln L}}. 
\] (6.3)

Using the inequalities \(|y| \leq z\) and \(|x| \leq \frac{L}{d}z\) in \(\Omega_{L,1}^{\text{up}}\), it is not difficult to estimate,

\[
|\partial_y m_2|^2 + |\partial_z m_2|^2 + |\partial_y m_3|^2 + |\partial_z m_3|^2 \leq \frac{\pi^2}{4z^2}(2d^2 + 1) \quad \text{in} \quad \Omega_{L,1}^{\text{up}}.
\]

We can calculate now,

\[
\int_{\Omega_{L,1}^{\text{up}}} \frac{1}{z^2} \, d\xi = 2 \int_0^L \int_\frac{d}{L}^d \int_\frac{z}{L}^z \frac{1}{z^2} \, dy \, dz = \frac{1}{d} \int_0^L (\ln L - \ln x) \, dx = \frac{L}{d}.
\]

Therefore we obtain

\[
|E_{\text{formal}}^\text{ex}(\tilde{m}) - E_{\text{ex}}(m)| \leq 4 \int_{\Omega_{L,1}^{\text{up}}} (|\partial_y m_2|^2 + |\partial_z m_2|^2 + |\partial_y m_3|^2 + |\partial_z m_3|^2) \, d\xi \leq 2\pi^2dL + \frac{\pi^2L}{d}. \quad (6.4)
\]

Finally combining estimates (6.1)-(6.4) and choosing \(L = d^{\frac{3}{2}}\sqrt{\ln d}\) we arrive at

\[
E(m) \leq 150d^{\frac{3}{2}}\sqrt{\ln d}.
\]

It has been shown in [16] that there exists a number \(R_0 > 0\) such that if \(R \geq R_0\) then the minimal energy is bigger than a constant times \(R^2\sqrt{\ln R}\), where the cross section of the domain \(\Omega\) is a disc with radius \(R\). It is easily seen that the proof in there works also for a rectangular cross section, thus the proof of Theorem 2.5 finishes.

A Appendix

In appendix we recall some well-know facts, in particular the study of the minimization problem \(\min_{m \in A_0} E_0(m)\). We start with a simple lemma.

Lemma A.1. For any magnetizations \(m_1, m_2 \in A(\Omega)\) there holds

\[
|E_{\text{mag}}(m_1) - E_{\text{mag}}(m_2)| \leq \|m_1 - m_2\|_{L^2(\Omega)}^2 + 2\|m_1 - m_2\|_{L^2(\Omega)} \sqrt{E_{\text{mag}}(m_1)}
\]

Proof. The proof is elementary and can be found in [16].

Lemma A.2. For any \(0 < s \leq r\) denote \(R(s,r) = [-s,s] \times [-r,r]\). Then for all points \((y_1, z_1) \in \mathbb{R}^2\) there holds

\[
I = \int_{R(s,r)} \frac{dy \, dz}{\sqrt{(y - y_1)^2 + (z - z_1)^2}} < 10s \left(1 + \ln \frac{r}{s}\right).
\]

24
Proof. It is clear that if we replace the point \((y_1, z_1)\) by its closest point to \(R(s, r)\), then the integral may only increase. Thus one can without loss of generality assume that \((y_1, z_1) \in R(s, r)\). We have that

\[
I \leq \frac{\int_{R(2s,2r)} dy \, dz}{\sqrt{y^2 + z^2}} = \int_{R(2s,2s)} \frac{dy \, dz}{\sqrt{y^2 + z^2}} + \int_{R(2s,2r) \setminus R(2s,2s)} \frac{dy \, dz}{\sqrt{y^2 + z^2}}
\]

\[
\leq \frac{1}{4} \int_{D_4(0)} \frac{dy \, dz}{\sqrt{y^2 + z^2}} + 8s \int_{2s} \frac{dy}{y}
\]

\[
= 2\sqrt{2\pi} s + 8s \ln \frac{r}{s}
\]

\[
< 10s \left(1 + \ln \frac{r}{s}\right).
\]

\[\square\]

Lemma A.3. Assume that \(\omega \subset \mathbb{R}^2\) is a bounded Lipschitz domain. Then for any interval \((a, b) \subset \mathbb{R}\), positive \(\alpha\) and a unit vector field \(f \in H^1((a, b) \times \omega, \mathbb{R}^3)\) there holds:

\[
\int_{(a,b) \times \omega} |\partial_x f|^2 + \alpha^2 \int_{(a,b) \times \omega} (|f_2|^2 + |f_3|^2) \geq 2\alpha|\omega||f_1(a) - f_1(b)|.
\]

(The endpoints \(a\) and \(b\) can take values \(-\infty\) and \(\infty\) respectively).

Proof. Fix a point \((y, z) \in \omega\) and consider the vector field \(f\) on the segment with endpoints \((a, y, z)\) and \((b, y, z)\). Being an \(H^1\) vector field, \(f\) must be absolutely continuous on that segment as a function of one variable, thus denoting

\[
f_1(x, y, z) = \sin \varphi(x), f_2(x, y, z) = \cos \varphi(x) \cos \theta(x), f_3(x, y, z) = \cos \varphi(x) \sin \theta(x)
\]

we obtain that \(\varphi\) and \(\theta\) are differentiable a.e. in \([a, b]\). Therefore we can calculate,

\[
\int_{(a,b) \times (y,z)} |\partial_x f(\xi)|^2 \, dx + \alpha^2 \int_{(a,b) \times (y,z)} (|f_2(\xi)|^2 + |f_3(\xi)|^2) \, dx
\]

\[
= \int_a^b (\varphi'^2(x) + \theta'^2(x) \cos^2 \varphi(x)) \, dx + \alpha^2 \int_a^b \cos^2 \varphi(x) \, dx
\]

\[
\geq 2\alpha \left| \int_a^b \varphi(x) \cos \varphi(x) \, dx \right|
\]

\[
= 2\alpha|f_1(a, y, z) - f_1(b, y, z)|.
\]

An integration of the obtained inequality over \(\omega\) completes the proof. \[\square\]

Recall now that analogues to the proof of the above lemma one can determine the minima of the energy functional

\[
E_\alpha(m) = \int_{\mathbb{R}} |\partial_x m(x)|^2 \, dx + \alpha \int_{\mathbb{R}} (|m_y(x)|^2 + |m_z(x)|^2) \, dx
\]

25
for any \( \alpha > 0 \) in the admissible set
\[
A_0 = \{ m : \mathbb{R} \to \mathbb{R}^3 : |m| = 1, m(\pm \infty) = \pm 1 \}.
\]
The minimizer is unique up to a translation in the \( x \) coordinate and a rotation in the \( OYZ \) plane and is given by
\[
m^{\alpha, \beta} = \left( \frac{e^{2\sqrt{\alpha}x} \cdot \beta - 1}{e^{2\sqrt{\alpha}x} \cdot \beta + 1}, \frac{2\sqrt{\beta}e^{\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x} \cdot \beta + 1} \cos \theta, \frac{2\sqrt{\beta}e^{\sqrt{\alpha}x}}{e^{2\sqrt{\alpha}x} \cdot \beta + 1} \sin \theta \right). \tag{A.1}
\]
Set \( m^\alpha := m^{\alpha, 1} \) to have \( m^\alpha(x(0)) = 0 \). The minimal value of \( E_\alpha \) in \( A_0 \) will be \( 4\sqrt{\alpha} \). Let us now find the minimal value and the minima of the reduced problem for any \( \omega \) under the condition
\[
0 < \alpha_2 < \alpha_3, \tag{A.2}
\]
i.e., consider the minimization problem
\[
\min_{m \in A_0} E_0(m). \tag{A.3}
\]
Observe that,
\[
E_0(m) = |\omega| \int_{\mathbb{R}} |\partial_x m(x)|^2 \, dx + \alpha_2 \int_{\mathbb{R}} m_2^2(x) + \alpha_3 \int_{\mathbb{R}} m_3^2(x)
\geq |\omega| \int_{\mathbb{R}} |\partial_x m(x)|^2 \, dx + \alpha_2 \int_{\mathbb{R}} m_2^2(x) + \alpha_2 \int_{\mathbb{R}} m_3^2(x)
\geq 4\sqrt{|\alpha_2 \omega|},
\]
and the minimum is realized by
\[
m^\omega = \left( \frac{e^{2\sqrt{\alpha_2 \omega}x} - 1}{e^{2\sqrt{\alpha_2 \omega}x} + 1}, \frac{2e^{\sqrt{\alpha_2 \omega}x}}{e^{2\sqrt{\alpha_2 \omega}x} + 1}, 0 \right), \tag{A.4}
\]
where \( \alpha_\omega = \frac{\alpha_2}{|\omega|} \). All other minimizers of \( E_0 \) are obtained via translations and 180 degree rotations of \( m^\omega \).

**Acknowledgement**

The present results are part of the author’s PhD thesis. The author is very grateful to his thesis supervisor Dr. Prof. S. Müller for suggesting the topic and for many fruitful discussions. He is also thankful to the Max-Planck Institute for Mathematics in the Sciences in Leipzig, Germany, and HCM in Mathematic in Bonn, Germany for supporting his graduate studies.

**References**

[1] D. Atkinson, G. Xiong, C. Faulkner, D. Allwood, D. Petit, and R. Cowburn. Magnetic domain wall logic. *Science*, 309:1688-1692, 2005.
[2] G. Beach, C. Nistor, C Knustom, M. Tsoi, and J. Erskine. Dynamics of field-driven domain-wall propagation in ferromagnetic nanowires. Nat. Mater., 4:741-744, 2005.

[3] A. Capella, C., Melcher and F. Otto. Wave type dynamics in ferromagnetic thin film and the motion of Neél walls. Nonlinearity, 20:2519 - 2537, 2007.

[4] G. Carbou and S. Labbé. Stability for static walls in ferromagnetic nanowires. Discrete Contin. Dyn. Syst. Ser. B, 6(2): 273-290(electronic), 2006

[5] Antonio Desimone, Robert V. Kohn, Stefan Müller, and Felix Otto. A reduced theory for thin-film micromagnetics. Comm. Pure Appl. Math., 55(11):1408-1460, 2002.

[6] Antonio DeSimone, Robert V. Kohn, Stefan Müller, Felix Otto, and Rudolf Schäfer. Two-dimensional modelling of soft ferromagnetic films. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci., 457(2016):2983-2991, 2001.

[7] C. J. García-Cervera and Weinan E. Effective dynamics for ferromagnetic thin films. J. Appl. Phys., 90:370-374, 2001.

[8] C. J. García-Cervera . Néel walls in low anisotropy symmetric double layers. SIAM J. Appl. Math., 65:1726–1747, 2005

[9] H. Forster, T. Schrefl, D. Suess, W. Scholz, V. Tsiantos, R. Dittrich, and J. Fidler. Domain wall motion in nanowires using moving grids. J. Appl. Phys., 91:6914-6919, 2002.

[10] D. Harutyunyan. On the Γ-convergence of the energies and the convergence of almost minimizers in infinite magnetic cylinders, Dissertation, Universitäts und Landesbibliothek Bonn, Submitted in June 2011, published in 2012, http://hss.ulb.uni-bonn.de/2012/2886/2886.htm

[11] R. Hertel and J. Kirschner. Magnetization reversal dynamics in nickel nanowires. Physica B, 343:206-210, 2004.

[12] A Hubert and R. Schäfer. Magnetic Domains. The analysis of Magnetic Microstructures. ISBN 3-540-64108-4, Springer-Verlag Berlin-Heidelberg New York, 1998

[13] R. V. Kohn and V.V. Slastikov. Effective dynamics for ferromagnetic thin films: a rigorous justification. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461(2053):143-154, 2005.

[14] R. V. Kohn and V. Slastikov. Another thin-film limit in micromagnetics. Arch. Ration. Mech. Anal., 178:227-245, 2005.

[15] M. Kurzke. Boundary vortices in thin magnetic films. Calc. Var. Partial Differ. Equ., 26:1–28, 2006.

[16] K. Kühn. Reversal modes in magnetic nanowires. Ph.D. Thesis, Max-Planck-Institut für Mathematik, Germany, 2007.
[17] Y. Nakatani, and A. Thaiville. Domain-wall dynamics in nanowires and nanostripes. *In Spin dynamics in confined magnetic structures. 3rd Volume 101 of Topics in applied physics, pages 161-205. Springer Verlag, Berlin, 2006*

[18] K. Nielsch, R.B. Wehrspohn, J. Barthel, J. Kirschner, U. Gösele, S.F. Fischer, and H. Kronmüller. Hexagonally ordered 100nm period nickel nanowire arrays. *Appl. Phys. Lett.*, 79(9):1360-1362, 2001.

[19] L. Piraux, J.M. George, J.F. Despres, C. Leroy, E.Ferain, R. Legras, K. Ounadjela, and A. Fert. Giant magnetoresistance in magnetic multilayered nanowires. *Appl. Phys. Lett.*, 65(19):2484-2486, 1994.

[20] D. Sanchez. Behavior of Landau-Lifshitz equation in a ferromagnetic wire. *Math. Models Methods Appl. Sci.* 32:167–205, 2009.

[21] V.V. Slastikov and C. Sonnenberg. Reduced models for ferromagnetic nanowires. *IMA J. Appl. Math.* 77,N2.220-235, 2012.

[22] R. Wieser, U. Nowak, and K. D. Usadel. Domain wall mobility in nanowires: Trensverse versus vortex walls. *Phys. Rev. B*, 69:0604401, 2004.