ORIENTATION OF ALCOVES IN AFFINE WEAyL GROUPS

NATHAN CHAPELIER-LAGET

ABSTRACT. Let $W$ be an irreducible Weyl group and $W_a$ its affine Weyl group. In [2] the author introduced an affine variety $X_{W_a}$, called the Shi variety of $W_a$, whose integral points are in bijection with $W_a$. The set of irreducible components of $X_{W_a}$ provided results at the intersection of group theory, combinatorics and geometry. In this article we express the notion of orientation of alcoves in terms of the first group of cohomology of $W$ and in terms of the irreducible components of the Shi variety. We also provide modular equations in terms of Shi coefficients that describe efficiently the property of having the same orientation.

CONTENTS

1. Introduction 1
    1.1. General definitions 1
    1.2. The goals of this article 2

2. Background about the Shi variety 3

3. Cohomology of Weyl group 4
    3.1. Background about group cohomology 4
    3.2. Cohomology in degree 1 with coefficients in $A\Phi$ 4
    3.3. Concrete realization of $H^1(W,\mathbb{Z}\Phi)$ 7
        3.3.1. Type $A_n$ 8
        3.3.2. Type $B_n$ 9
        3.3.3. Type $C_n$ 9
        3.3.4. Type $D_n$ 10

4. Orientation of alcoves 11

References 14

1. Introduction

1.1. General definitions. Let $V$ be a Euclidean space with inner product $\langle-,-\rangle$. Let $\Phi$ be an irreducible crystallographic root system in $V$ with simple system $\Delta$. We set $\Delta = \{\alpha_1,\ldots,\alpha_n\}$ and $\Phi^+ = \{\beta_1,\ldots,\beta_m\}$ with $n = |\Delta|$ and $m = |\Phi^+|$. In this article, when we say “root system” it always means irreducible crystallographic root system.

Let $W$ be the Weyl group associated to $\mathbb{Z}\Phi$, that is the maximal (for inclusion) reflection subgroup of $O(V)$ admitting $\mathbb{Z}\Phi$ as a $W$-equivariant lattice. We identify $\mathbb{Z}\Phi$ and the group of its associated translations and we denote by $\tau_x$ the translation corresponding to $x \in \mathbb{Z}\Phi$.

Let $k \in \mathbb{Z}$, $\alpha \in \Phi$ and let $\alpha^\vee := \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ be the dual root of $\alpha$. We define the affine reflection $s_{\alpha,k} \in \text{Aff}(V)$ by $s_{\alpha,k}(x) = x - (\langle\alpha^\vee,x\rangle - k)\alpha$. The group generated by all the affine reflections $s_{\alpha,k}$ with $\alpha \in \Phi$ and $k \in \mathbb{Z}$ is called the affine Weyl group associated to $\Phi$. We denote it by

$$W_a = \langle s_{\alpha,k} \mid \alpha \in \Phi, k \in \mathbb{Z} \rangle.$$ 

It is also well known that $W_a \simeq \mathbb{Z}\Phi \rtimes W$. Therefore, any element $w \in W_a$ decomposes as $w = \tau_x \overline{w}$ where $x \in \mathbb{Z}\Phi$ and $\overline{w} \in W$. The element $\overline{w}$ is called the finite part of $w$.

Let $\alpha \in \Phi$, $k, m \in \mathbb{Z}$. We set the hyperplanes

$$H_{\alpha,k} = \{x \in V \mid s_{\alpha,k}(x) = x\} = \{x \in V \mid \langle x,\alpha^\vee\rangle = k\}$$

and the strips

$$H^p_{\alpha,k} = \{x \in V \mid k < \langle x,\alpha^\vee\rangle < k+p\}.$$
An alcove of $V$ is by definition a connected component of

$$V \setminus \bigcup_{\alpha \in \Phi^+, k \in \mathbb{Z}} H_{\alpha,k}.$$ 

We denote by $A_w$ the alcove defined by $A_w = \bigcap_{\alpha \in \Phi^+} H_{\alpha,0}^1$. It turns out that $W_a$ acts regularly on the set of alcoves. Therefore we have a bijective correspondence between the elements of $W_a$ and all the alcoves. This bijection is defined by $w \mapsto A_w$ where $A_w := wA_e$. We call $A_w$ the corresponding alcove associated to $w \in W_a$. Any alcove of $V$ can be written as an intersection of special strips, that is there exists a $\Phi^+$-tuple of integers $(k(w,\alpha))_{\alpha \in \Phi^+}$, that we call the Shi coefficients of $w$, such that

$$A_w = \bigcap_{\alpha \in \Phi^+} H_{\alpha,k(w,\alpha)}^1.$$ 

In [4] J.Y. Shi shows that the $\Phi^+$-tuple of integers $(k(w,\alpha))_{\alpha \in \Phi^+}$ subject to certain conditions characterizes entirely $w$. Based on this characterization, the author defined in [2] an affine variety $\tilde{X}_{W_a}$, called the Shi variety of $W_a$, whose integral points are in bijection with $W_a$.

We denote by $H^0(\tilde{X}_{W_a})$ the set of irreducible components of $\tilde{X}_{W_a}$. These components are affine subspace which are parameterized by a collection of integral vectors $\lambda \in \mathbb{N}^a$ that we call admitted vectors. The component associated to an admitted vector $\lambda$ is denoted by $X_{W_a}[\lambda]$ and if $\lambda \neq \gamma$ are both admitted then $X_{W_a}[\lambda] \cap X_{W_a}[\gamma] = \emptyset$ (we recall these notions in Section 2). Thus, each element $w \in W_a$ can be seen as an integral point of $\tilde{X}_{W_a}$, which will be denoted by $\iota(w) := (k(w,\alpha))_{\alpha \in \Phi^+}$, and lies in a specific component (see Theorem 2.3 for more details about the map $\iota$).

The action of the lattice $\mathbb{Z}\Phi$ on $W_a$ induces a natural action on each component ([2], Proposition 4.3). In particular it suggests that the only relevant structure on the components is the structure of $\mathbb{Z}\Phi$. This last observation, combined with the fact that $W_a \simeq \mathbb{Z}\Phi \times W$, gives the idea that it should be related to the group $H^1(W,Z\Phi)$, where the group structure is also controlled by $Z\Phi$.

1.2. The goals of this article. The first goal of this paper is to understand how the sections of the exact sequence (3) behave up to conjugacy when the space of coefficients $Z\Phi$ is replaced by a $A\Phi$ with $A$ a (commutative unitary) ring satisfying $Z \subset A \subset R$.

Another goal is to find the smaller $A$ (for inclusion) such that $H^1(W,A\Phi) = 0$. This is done in Section 3.2. We will say that there is an $A\Phi$-obstruction between two sections $s_1$ and $s_2$ if they don’t define the same class in $H^1(W,A\Phi)$.

Section 3.2 also expresses the $A\Phi$-conjugacy in terms of the Cartan matrix. We explain how the transpose of the Cartan matrix detects the obstructions between sections. In particular we give a general tool (Proposition 3.1) that allows us to give in Section 3.3 some modular equations describing the elements of $H^1(W,Z\Phi)$ in type $A,B,C,D$.

The main interest of this article, which is addressed in Section 4, consists of expressing the orientation of an alcove in terms of the Shi variety and $H^1(W,Z\Phi)$. We will see that the most interesting situation occurs in type $A$. To do so we show that to each element $w \in W_a$ one can associate a section $s_w$, which combined to the results of Section 3.2 yields Theorem 4.1. Then, using this theorem and the results of Section 3.3 we obtain the modular equations that characterize the orientations of the alcoves.

To motivate the reading we give a reformulation of Theorem 4.1 and Corollary 4.1 in type $A$ as follows:

Theorem 1.1. Let $w,w' \in W_a$. Then $A_w$ and $A_{w'}$ have the same orientation if and only if

1) The Shi vectors $\iota(w)$ and $\iota(w')$ are in the same irreducible component of $\tilde{X}_{W_a}$.
2) The sections $s_w$ and $s_{w'}$ define the same element in $H^1(W,Z\Phi)$.

Corollary 1.1. Assume that $W = W(A_n)$. Let $w,w' \in W_a$. Then $A_w$ and $A_{w'}$ have the same orientation if and only if one has in $Z/(n+1)Z$ the following equality (where $\overline{k}(w,\alpha) := \overline{k}(w,\alpha)$)

$$\sum_{j=1}^n j\overline{k}(w,\alpha_j) = \sum_{j=1}^n j\overline{k}(w',\alpha_j).$$
2. Background about the Shi variety

We recall in this section some necessary material. If $\mathcal{I}$ is an ideal of a polynomial ring $k[X_1, \ldots, X_r]$ we denote $V(\mathcal{I}) = \{ x \in k^r \mid P(x) = 0 \ \forall \ P \in \mathcal{I} \}$. All the following definitions were introduced in [2]. We denote $\mathbb{Z}[X_\Delta] := \mathbb{Z}[X_{\alpha_1}, \ldots, X_{\alpha_n}]$ and $\mathbb{Z}[X_{\Phi^+}] := \mathbb{Z}[X_{\beta_1}, \ldots, X_{\beta_m}]$. For $w \in W_a$ and $Q \in \mathbb{Z}[X_\Delta]$ we denote

$$Q(w) := Q(k(w, \alpha_1), \ldots, k(w, \alpha_n)).$$

For short, when we need to involve the simple Shi coefficients $k(w, \alpha_i)$, we will often write $Q(w) = Q((k(w, \alpha_i)), \alpha_i \in \Delta)$ or just $Q((k(w, \alpha_i)))$ if there is no confusion.

The following theorem is Shi’s characterization of the elements $w \in W_a$ by their $\Phi^+$-tuples of integers.

**Theorem 2.1** ([4], Theorem 5.2). Let $A = \bigcap_{\alpha \in \Phi^+} H_{\alpha,k}^1$ with $k_\alpha \in \mathbb{Z}$. Then $A$ is an alcove, if and only if, for all $\alpha, \beta \in \Phi^+$ satisfying $\alpha + \beta \in \Phi^+$, we have the following inequality

$$||\alpha||^2 k_\alpha + ||\beta||^2 k_\beta + 1 \leq ||\alpha + \beta||^2 (k_{\alpha+\beta} + 1) \leq ||\alpha||^2 k_\alpha + ||\beta||^2 k_\beta + ||\alpha||^2 + ||\beta||^2 + ||\alpha + \beta||^2 - 1.$$

**Remark 2.1.** In type $A$ Theorem 2.1 has an easy reformulation: Let $A = \bigcap_{\alpha \in \Phi^+} H_{\alpha,k}^1$ with $k_\alpha \in \mathbb{Z}$. Then $A$ is an alcove, if and only if, for all $\alpha, \beta \in \Phi^+$ satisfying $\alpha + \beta \in \Phi^+$, we have the following inequality

$$k_\alpha + k_\beta \leq k_{\alpha+\beta} \leq k_\alpha + k_\beta + 1.$$

The following theorem decomposes the Shi coefficients as polynomial equations.

**Theorem 2.2** ([2], Theorem 4.1, Lemma 4.1). Let $w \in W_a$. Then for all $\theta \in \Phi^+$ there exists a linear polynomial $P_\theta \in \mathbb{Z}[X_\Delta]$ with positive coefficients and $\lambda_\theta(w) \in [0, h(\theta^\vee) - 1]$ such that

$$(2)\quad k(w, \theta) = P_\theta(w) + \lambda_\theta(w).$$

Moreover, the polynomial $P_\theta$ satisfies

$$\theta^\vee = P_\theta(\alpha_1^\vee, \ldots, \alpha_n^\vee).$$

**Definition 2.1.** Let $\theta \in \Phi^+$. Write $I_\theta := [0, h(\theta^\vee) - 1]$. Notice that if $\theta$ is a simple root then $I_\theta = \{0\}$. For any root $\theta \in \Delta$ we set $P_\theta = X_\theta$ and $\lambda_\theta = 0$. We denote by $P_\theta[I_\theta]$ the polynomial $P_\theta + \lambda_\theta - X_\theta \in \mathbb{Z}[X_{\Phi^+}]$. We define the ideal $J_{W_a}$ of $\mathbb{R}[X_{\Phi^+}]$ as $J_{W_a} := \sum_{\alpha \in \Phi^+} (\prod_{\lambda_\alpha \in I_\alpha} P_\alpha[\lambda_\alpha])$. We define the affine variety $X_{W_a}$ to be:

$$X_{W_a} := V(J_{W_a}).$$

**Definition 2.2.** We say that $v = (v_\alpha)_{\alpha \in \Phi^+} \in \mathbb{N}^m$ is an admissible vector (or just admissible) if it satisfies the boundary conditions, that is if for all $\alpha \in \Phi^+$ one has $v_\alpha \in I_\alpha$. For instance, all the $\lambda := (\lambda_\alpha)_{\alpha \in \Phi^+}$ coming from the polynomials $P_\alpha[\lambda_\alpha]$ give rise to admissible vectors. Furthermore, each admissible vector arises this way. We will write $\lambda$ instead of $(\lambda_\alpha)_{\alpha \in \Phi^+}$.

**Definition 2.3.** Let $\lambda$ be an admissible vector. We denote

$$J_{W_a}[\lambda] := \sum_{\alpha \in \Phi^+} (P_\alpha[\lambda_\alpha]) = (P_\alpha[\lambda_\alpha], \alpha \in \Phi^+),$$

$$X_{W_a}[\lambda] := V(J_{W_a}[\lambda]).$$

**Definition 2.4.** We will denote $S[W_a]$ as the system of all the inequalities coming from Theorem 2.1. Let $\lambda$ be an admissible vector. We say that $\lambda$ is admitted if it satisfies the system $S[W_a]$. Let $w \in W_a$. For an admitted vector $\lambda$, we write $\lambda(w) = \lambda$ if and only if $\iota(w) \in X_{W_a}[\lambda]$.

If $Y \subset \mathbb{R}^m$ we denote by $Y(\mathbb{Z})$ the set of integral points of $Y$. Finally, we have the following result which gives the paramaterization of the elements of $H^0(\hat{X}_{W_a})$ in terms of the admitted vectors.

**Theorem 2.3** ([2], Theorem 4.3). The map $\iota : W_a \rightarrow X_{W_a}(\mathbb{Z})$ defined by $w \mapsto (k(w, \alpha))_{\alpha \in \Phi^+}$ induces by corestriction a bijective map from $W_a$ to the integral points of a subvariety of $X_{W_a}$, denoted $\hat{X}_{W_a}$, which we call the Shi variety of $W_a$. This subvariety is nothing but $\hat{X}_{W_a} = \bigcup_{\lambda \text{ admitted}} X_{W_a}[\lambda]$. 


3. Cohomology of Weyl group

In this section we show how the irreducible components of the Shi variety together with the first group of cohomology of \( W \) can be used to express the orientation of the alcoves.

Let \( S = \{ s_1, \ldots, s_n \} \) be the set of Coxeter generators of \( W \) associated to \( \Delta \) and \( m_{ij} := \text{ord}(s_is_j) \). The Cartan matrix \( C_\Phi = (h_{ij})_{i,j} \) associated to \( \Phi \) is the matrix of \( GL_n(\mathbb{R}) \) defined by \( h_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle \). It is well known that \( C_\Phi \) is invertible since it is a change of basis matrix.

The decomposition \( W_a \cong \mathbb{Z}\Phi \rtimes W \) induces the following short exact sequence
\[
\begin{align*}
1 \longrightarrow & \mathbb{Z}\Phi \longrightarrow \mathbb{Z}\Phi \rtimes W \longrightarrow W \longrightarrow 1,
\end{align*}
\]
where \( i \) and \( \pi \) are the morphisms defined by \( i(x) = \tau_x \) and \( \pi(w) = \varpi \).

3.1. Background about group cohomology. Let \( A \) be a \( G \)-module. We define \( A^G \) to be the submodule of \( G \)-invariants, that is, the subgroup of \( A \) defined by \( A^G := \{ a \in A \mid ga = a \text{ for all } g \in G \} \). The functor \( F : A \rightarrow A^G \) from the category of \( G \)-modules to the category of abelian groups is covariant and left-exact. Therefore, we can define its right derived functors \( R^iF \) and by definition the \( i \)-th cohomology group of \( G \) with coefficients in \( A \) is \( H^n(G,A) := R^nF(A) \).

Another description of group cohomology in degree 1 is given by 1-cocycles and 1-coboundaries. A \( n \)-cochain is a map \( f : G \rightarrow A \) and the set of \( n \)-cochains is denoted by \( C^n(G,A) \). For \( n = 0 \) we have \( C^0(G,A) = A \) and we denote by \( f_a \) \((a \in A)\) the elements of \( C^0(G,A) \). The set \( C^n(G,A) \), endowed with the addition induced from \( A \), is a group.

The boundary map \( \partial_1 : C^0(G,A) \rightarrow C^1(G,A) \) is defined by \( \partial_0f_a(x) = xa - a \) and the boundary map \( \partial_1 : C^1(G,A) \rightarrow C^2(G,A) \) is defined by \( \partial_1f(x,y) = xf(y) - f(xy) + f(x) \). The 1-cocycles are the elements \( f \in C^1(G,A) \) satisfying \( \partial_1f = 0 \), and the 1-coboundaries are the elements \( f \in C^1(G,A) \) such that \( f = \partial_0f_a \) for some \( a \in A \). The set of 1-cocycles is denoted by \( Z^1(G,A) \) and the set of 1-coboundaries by \( B^1(G,A) \). These two sets are subgroups of \( C^1(G,A) \) and \( B^1(G,A) \subset Z^1(G,A) \).

Therefore, we can define \( H^1(G,A) \) by
\[
H^1(G,A) = Z^1(G,A)/B^1(G,A).\]

Let \( B \) be a \( G \)-module. Let us consider the following exact sequence:
\[
\begin{align*}
1 \longrightarrow & B \xrightarrow{i} B \rtimes G \xrightarrow{\pi} G \longrightarrow 1.
\end{align*}
\]
A section of \( \pi \) is a morphism \( s : G \rightarrow E \) such that \( \pi \circ s = id_G \). We denote by \( \text{Sec}_B(\pi) \) the set of sections of \( \pi \). If \( B = \mathbb{Z}\Phi \) we will just write \( \text{Sec}(\pi) \).

Two sections \( s \) and \( s' \) are called \( B \)-conjugated if there exists \( b \in B \) such that for all \( g \in G \) we have \( s'(g) = i(b)s(g)i(b)^{-1} \). We denote this by \( s \sim_B s' \), or just \( s \sim s' \) if there is no confusion.

Let \( \varphi : G \rightarrow \text{Aut}(B) \) be a morphism of groups. \( B \) inherits a structure of \( G \)-module where the action is given by \( g.b := \varphi(g)(b) \). The following result is also well known and will be of interest for us (see [3], Theorem 4.4 for a good reference).

**Theorem 3.1.** The set of sections up to \( B \)-conjugacy of the exact sequence
\[
1 \longrightarrow B \xrightarrow{i} B \rtimes G \xrightarrow{\pi} G \longrightarrow 1
\]
is in bijection with \( H^1(G,B) \).

3.2. Cohomology in degree 1 with coefficients in \( A\Phi \). This section is dedicated to the study of the cohomology in degree 1 of a Weyl group \( W \) with coefficients in \( A\Phi \).

Let us set once for all \( A \) to be a commutative unitary ring such that \( \mathbb{Z} \subset A \subset \mathbb{R} \). Throughout this section we will use the following short exact sequence
\[
\begin{align*}
1 \longrightarrow & A\Phi \xrightarrow{i} A\Phi \rtimes W \xrightarrow{\pi} W \longrightarrow 1.
\end{align*}
\]
We can now state the main results of this section:

**Proposition 3.1.** Let \( s_1 \) and \( s_2 \) be two different sections of \( (4) \) defined by \( s_1(s_i) = \tau_x, s_i \) and \( s_2(s_i) = \tau_y, s_i \) with \( x_i, y_i \in A\Phi \) for \( i = 1, \ldots, n \). Then we have equivalence between:

i) \( s_1 \sim s_2 \).

ii) There exists \( z \in A\Phi \) such that \( \forall \ i = 1, \ldots, n, \ y_i - x_i = \langle z, \alpha_i^\vee \rangle \).
Theorem 3.2. The following points are equivalent:

i) $H^1(W, A\Phi) = 0$.

ii) $C_\Phi \in GL_n(A)$.

Theorem 3.3. Let $f_\Phi$ be the index of connection of $\Phi$ and $\mathbb{Z}_{f_\Phi} := \mathbb{Z}[\frac{1}{f_\Phi}]$. Then the following points are equivalent:

i) $H^1(W, A\Phi) = 0$ with $A$ minimal.

ii) $A = \mathbb{Z}_{f_\Phi}$.

Example 3.1. Let us first illustrate this with an example. Let us take $W = W(B_2)$. The Cartan matrix of $B_2$ and its inverse are

$$C_{B_2} = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad ^tC_{B_2}^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix}.$$ 

Let $s_1$ and $s_2$ be two sections of $W(B_2)$. We know via Lemma 3.1 that we can identify $s_1$ with a point $(x_1, x_2) \in Z\alpha_1 \times Z\alpha_2$ and $s_2$ with a point $(y_1, y_2) \in Z\alpha_1 \times Z\alpha_2$. Denote $x_i = a_i \alpha_i$ and $y_i = b_i \alpha_i$. Moreover we know that

$$s_1 \sim s_2 \iff \exists z \in Z\alpha_1 \times Z\alpha_2 \text{ such that } \begin{cases} x_1 - y_1 = (z, \alpha_1') \alpha_1 \\ x_2 - y_2 = (z, \alpha_2') \alpha_2. \end{cases}$$

It follows that

$$s_1 \sim s_2 \iff \exists z \in Z\alpha_1 \times Z\alpha_2 \text{ such that } ^tC_{B_2} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}.$$ 

Thus, we must have $z_1$ and $z_2$ both integral, and such that

$$\begin{cases} 2z_1 = 2(a_1 - b_1) + 2(a_2 - b_2) \\ 2z_2 = 2(a_1 - b_1) + a_2 - b_2. \end{cases}$$

These arithmetical conditions show how $H^1(W(B_2), Z\Phi)$ behaves, and we see in this particular situation that we do not have $H^1(W(B_2), Z\Phi) = 0$. Indeed, by taking for example $a_1 = b_1 = 1$, $a_2 = 3$ and $b_2 = 2$ we have $2z_2 = 1$, which is impossible since $z_2$ is integral. We thus have an obstruction between $s_1$ and $s_2$, which implies that $H^1(W(B_2), Z\Phi) \neq 0$.

However, if we consider the equations in the ring $\mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}]$ instead of $\mathbb{Z}$, the previous obstruction no longer exists and it follows that $H^1(W(B_2), \mathbb{Z}_2\Phi) = 0$. Furthermore, the ring $\mathbb{Z}_2$ is the smallest one that satisfies the last equality.

Lemma 3.1. Let $s$ be a section of the exact sequence $(\xi)$ defined by $s(s_i) = \tau_{x_i} s_i$ for $s_i \in S$ and $x_i \in A\Phi$. Then the map $\xi$ from $\text{Sec}_{A\Phi}(\pi)$ to $A\alpha_1 \times \cdots \times A\alpha_n$ defined by $\xi(s) = (x_1, \ldots, x_n)$ is a bijection.

Proof. Let us first show that $\xi$ is well defined, that is $x_i \in A\alpha_i$ for all $i$. We know that $W$ is a group defined by generators and relations. The relations are given by $s_i^2 = e$ and some braid relations. The braid relations are either of type $st = ts$ or $sts = tsts$ or $tstst = tsts$ for all generators $s, t \in S$. Since $s$ is a morphism, it must preserve all of these relations. Therefore, the relation $s_i^2 = e$ implies that $\tau_{x_i} s_i \tau_{x_i} s_i = e$, that is $\tau_{x_i + s_i(x_i)} s_i^2 = e$ and then $x_i + s_i(x_i) = 0$. Consequently we must have $s_i(x_i) = -x_i$, which means that $x_i \in H_{\alpha_i}^1 = R\alpha_i$. It follows that $(x_1, \ldots, x_n) \in \prod_{i=1}^n A\alpha_i$. Thus, for all $i = 1, \ldots, n$ there exists $k_i \in A$ such that $x_i = k_i \alpha_i$. We claim that any choice of $(k_1, \ldots, k_n) \in A^n$ gives rise to a section. From this claim it is clear that the map $\xi$ is a bijection.

Proof of the claim. It is almost direct to show that $\text{ord}((s_{\alpha_i k_i} s_{\alpha_j k_j})^d) = \text{ord}((s_{\alpha_i} s_{\alpha_j})^d)$ for all $d \in \mathbb{N}$. In particular for $d = 2, 3, 4, 6$, since $s_{\alpha_i k_i} = \tau_{k_i \alpha_i} s_{\alpha_i} = \tau_{x_i} s_i$, we see that the braid relations are preserved under the section $s$. It follows that there is no constraint on the choices of the $k_i$. This ends the proof.
Proof of Proposition 3.1. Since $s_1$ and $s_2$ are different there exists at least one index $j$ such that $x_j \neq y_j$. By definition, $s_1$ is $A\Phi$-conjugated to $s_2$ if and only if there exists $z \in A\Phi$ such that for all $w \in W$ one has $s_2(w) = \tau_z s_1(w)\tau_{-z}$. It turns out that $\tau_z s_1(w)\tau_{-z} = \tau_{(id-w)(z)} s_1(w)$, indeed since $s_1(w) \in A\Phi \times W$ there exists $u \in A\Phi$ such that $s_1(w) = \tau_u w$. Therefore we have

$$\tau_z s_1(w)\tau_{-z} = \tau_z \tau_u w \tau_{-z} = \tau_z \tau_u \tau_{-w(z)} w = \tau_z \tau_{-w(z)} \tau_u w = \tau_{(id-w)(z)} s_1(w).$$

Hence, it follows

$$s_1 \sim s_2 \iff \exists z \in A\Phi \text{ such that } \forall w \in W, \ s_2(w) = \tau_{(id-w)(z)} s_1(w) \iff \exists z \in A\Phi \text{ such that } \forall s_i \in S, \ s_2(s_i) = \tau_{(id-s_i)(z)} s_1(s_i) \iff \exists z \in A\Phi \text{ such that } \forall s_i \in S, \ y_i - x_i = (id - s_i)(z).$$

It turns out that we also have the opposite direction. Indeed, let us write $w = t_1 t_2 \ldots t_p$ with $t_i \in S$ such that it is a reduced expression. Via the following equalities we have

$$s_1(w) = s_1(t_1) s_1(t_2) \ldots s_1(t_p) = \tau_{x_1 t_1} \tau_{x_2 t_2} \ldots \tau_{x_p t_p} = \tau_{x_1 t_1 + t_1 t_2 + \ldots + t_{p-1} t_p} t_1 t_2 \ldots t_p,$$

and

$$s_2(w) = s_2(t_1) s_2(t_2) \ldots s_2(t_p) = \tau_{y_1 t_1} \tau_{y_2 t_2} \ldots \tau_{y_p t_p} = \tau_{y_1 t_1 + y_1 y_2 + \ldots + y_1 y_{p-1} y_p} t_1 t_2 \ldots t_p.$$

It follows that

$$s_2(w) = \tau_{(id-w)(z)} s_1(w) \iff \tau_{y_1 t_1 + y_1 y_2 + \ldots + y_1 y_{p-1} y_p} t_1 t_2 \ldots t_p = \tau_{(id-w)(z)} \tau_{x_1 t_1 + t_1 t_2 + \ldots + t_{p-1} t_p} t_1 t_2 \ldots t_p \iff y_1 + y_1 y_2 + \ldots + y_1 y_{p-1} y_p = z - w(z) + x_1 t_1 + t_1 t_2 + \ldots + t_{p-1} t_p \iff z - w(z) = (y_1 - x_1) + t_1 (y_2 - x_2) + \ldots + t_{p-1} (y_p - x_p).$$

However, with the assumption $y_i - x_i = (id - t_i)(z)$ for all $i = 1, \ldots, p$, it follows that

$$z - w(z) = z - t_1(z) + t_1(z) + t_1 t_2(z) + t_1 t_2 t_3(z) - \ldots - t_1 t_2 \ldots t_{p-1}(z) + t_1 t_2 \ldots t_{p-1}(z) - w(z) = z - t_1(z) + t_1(z) - t_1 t_2(z) + \ldots + t_1 t_2 \ldots t_{p-1}(z) - t_1 t_2 \ldots t_{p-1}(z) - w(z) = y_1 - x_1 + t_1 (y_2 - x_2) + \ldots + t_1 t_2 \ldots t_{p-1} (y_p - x_p).$$

Therefore we have the equivalence

$$s_1 \sim s_2 \iff \exists z \in \mathbb{Z}\Phi \text{ such that } \forall s_i \in S, \ y_i - x_i = (id - s_i)(z).$$

We conclude the proof with the following equation

$$(id - s_i)(z) = z - s_i(z) = z - (z, \alpha_i \gamma) \alpha_i = (z, \alpha_i \gamma) \alpha_i.$$\hspace{1cm} \Box$$

Proof of Theorem 3.2. Let us first show the direction ii) implies i). By Proposition 3.1 we know that two sections $s_1$ and $s_1$ of (4) are $A\Phi$-conjugated if and only if there exists $z \in A\Phi$ such that $\forall i = 1, \ldots, n, \ y_i - x_i = (z, \alpha_i \gamma)$ (where the $x_i$ and $y_i$ are the same as in Proposition 3.1). We see then how the Cartan matrix is involved:

$$s_1 \sim s_2 \iff \exists z \in A\Phi \text{ such that } \begin{cases} x_1 - y_1 = (z, \alpha_1 \gamma) \alpha_1 \\ \vdots \\ x_n - y_n = (z, \alpha_n \gamma) \alpha_n \\ a_1 - b_1 = (z, \alpha_1 \gamma) \\ \vdots \\ a_n - b_n = (z, \alpha_n \gamma). \end{cases}$$

$$\iff \exists z \in A\Phi \text{ such that }$$
Writing \( z = z_1 \alpha_1 + \cdots + z_n \alpha_n \) it follows that

\[
(5) \quad s_1 \sim s_2 \iff \exists z \in A\Phi \text{ such that } t C_{\Phi} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix}.
\]

With the assumption: \( C_{\Phi} \in GL_n(A) \), we also have \( t C_{\Phi}^{-1} \in GL_n(A) \) and then there is no constraint for the choices of \( z_1, \ldots, z_n \) in \( A \). Thus, there is no obstruction between \( s_1 \) and \( s_2 \), which implies that \( s_1 = s_2 \) in \( H^1(W, A\Phi) \). Hence, there is an only one element in \( H^1(W, A\Phi) \), which implies the first direction.

Let us show now the other direction. Assume that \( C_{\Phi} \notin GL_n(A) \). Since \( C_{\Phi} \in GL_n(\mathbb{R}) \) with its coefficients in \( \mathbb{Z} \), the previous assumption implies in particular that \( t C_{\Phi}^{-1} \notin GL_n(A) \), and then there exists a coefficient \( q_{ij} \) of \( t C_{\Phi}^{-1} \) such that \( q_{ij} \notin A \). Let \( s_1 \) be the section with corresponding point \( x := (0, \ldots, 1, \ldots, 0) \in A\alpha_1 \times \cdots \times A\alpha_n \) where 1 is in position \( i \), and \( s_2 \) be the trivial section. It turns out that \( s_1 \sim s_2 \), if and only if, there exists \( z = z_1 \alpha_1 + \cdots + z_n \alpha_n \in A\Phi \) such that \( t C_{\Phi} z = x - 0 = x \).

Therefore, it follows that

\[
\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = t C_{\Phi}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} q_{11} \\ \vdots \\ q_{in} \end{pmatrix}.
\]

Since \( z_i \in A \) and \( q_{ij} \notin A \), the sections \( s_1 \) and \( s_2 \) define two different elements in \( H^1(W, A\Phi) \). This is impossible since \( H^1(W, A\Phi) = 0 \). Therefore, we must have \( C_{\Phi} \in GL_n(A) \). □

\textbf{Proof of Theorem 3.3.} It is well known that the determinant of \( C_{\Phi} \) is \( f_{\Phi} \) (see [1] Ch. VI, § 1, exercise 7). Consequently \( C_{\Phi}^{-1} = \frac{1}{\det(C_{\Phi})} D = \frac{1}{f_{\Phi}} D \) where \( D \in M_n(\mathbb{Z}) \). Therefore, because of Theorem 3.2, if \( A = \mathbb{Z}_{f_{\Phi}} \) we have \( C_{\Phi} \in GL_n(A) \) and it follows that \( H^1(W, \mathbb{Z}_{f_{\Phi}} \Phi) = 0 \).

Let us show now the direction i) implies ii). Since \( H^1(W, A\Phi) = 0 \), Theorem 3.3 tells us that \( C_{\Phi} \in GL_n(A) \), as well as its inverse. Therefore, all the coefficients of \( C_{\Phi}^{-1} \) are in \( A \). However, all the coefficients of \( C_{\Phi}^{-1} \) are of the form \( \frac{d}{f_{\Phi}} \) with \( d \in \mathbb{Z} \). We claim then that \( \frac{1}{f_{\Phi}} \notin A \). Indeed, a short observation (see [5] together with its appendix for the exceptional cases) of the inverses of the Cartan matrices in type \( A, B, C, E_6, E_7, E_8 \), \( F_4 \) and \( G_2 \) shows that there are always a coefficient of the form \( \frac{d}{f_{\Phi}} \) and another one of the form \( \frac{d+1}{f_{\Phi}} \) in the same Cartan matrix. From this, the claim becomes obvious. In type \( D_n \) the previous statement fails but we have that \( \frac{2(n-2)}{f_{\Phi}} \) and \( \frac{2}{f_{\Phi}} \in A \). This implies in particular that \( \frac{2}{f_{\Phi}} \in A \). Moreover in type \( D_n \), one has \( f_{\Phi} = 4 \). Hence \( 1/2 \in A \) and then \( 1/4 \) as well.

Therefore, in each situation \( \mathbb{Z}[\frac{1}{f_{\Phi}}] \subset A \). Since \( A \) is supposed to be minimal and since \( H^1(W, \mathbb{Z}_{f_{\Phi}} \Phi) = 0 \), one has \( A = \mathbb{Z}_{f_{\Phi}} \). □

\textbf{3.3. Concrete realization of} \( H^1(W, \mathbb{Z}\Phi) \). Thanks to Lemma 3.1, setting \( A = \mathbb{Z} \), we know that the sections of \( \pi \) are in bijective correspondence with the elements of \( \mathbb{Z}\alpha_1 \times \cdots \times \mathbb{Z}\alpha_n \). We also know that the elements of \( H^1(W, \mathbb{Z}\Phi) \) are in bijection with the sections of \( \pi \) up to \( \mathbb{Z}\Phi \)-conjugacy. Furthermore, the condition of being \( \mathbb{Z}\Phi \)-conjugated, when considered as a condition on pairs of elements in \( \mathbb{Z}\alpha_1 \times \cdots \times \mathbb{Z}\alpha_n \), corresponds to the solvability of a system of linear equations defined by the transpose of the Cartan matrix (this is embodied through Proposition 3.1).

In this section we investigate in type \( A, B, C, D \) how this equivalence relation behaves and we give the equations that will allow us to express the orientation of an alcove in terms of its Shi vector.

First of all, notice that when the index of connection is 1, the Cartan matrix is invertible in \( \mathbb{Z} \), which implies via Theorem 3.3 that \( H^1(W, \mathbb{Z}\Phi) = 0 \). Hence, if \( \Phi \) is of type \( G_2, F_4 \) or \( E_8 \), the cohomology in degree 1 is trivial.

Let \( s_1, s_2 \in \text{Sec}(\pi) \). Let \( x_1, \ldots, x_n \in \mathbb{Z}\alpha_1 \times \cdots \times \mathbb{Z}\alpha_n \) be the corresponding point of \( s_1 \) and \( (y_1, \ldots, y_n) \) be the corresponding point of \( s_2 \) (both via Lemma 3.1). Denote \( x_i = a_i \alpha_i \) and \( y_i = b_i \alpha_i \). We denote by \( d_i(s_1, s_2) \), or just \( d_i \) if there is no confusion, the number \( d_i := a_i - b_i \). We also define

\[
d := d_1 \alpha_1 + \cdots + d_n \alpha_n.
\]
3.3.1. Type $A_n$. In this section $W = W(A_n)$ and $\Phi = A_n$. We denote by $C := C_{A_n}$ the Cartan matrix of $A_n$. The coefficients of $C^{-1}$ are known (see [5]) and are given by

$$(C^{-1})_{ij} = \frac{(n + 1)\min(i, j) - ij}{n + 1}.$$ 

In particular we see that the inverse of the Cartan matrix in type $A$ is symmetric, hence $tC^{-1} = C^{-1}$.

**Proposition 3.2.** The sections $s_1$ and $s_2$ define two different elements in $H^1(W, \mathbb{Z}\Phi)$ if and only if $\overline{d_1} + 2\overline{d_2} + \cdots + n\overline{d_n} \neq 0$ in $\mathbb{Z}/(n + 1)\mathbb{Z}$.

**Proof.** By (5) we know that $s_1 \sim s_2$ if and only if there exists $z \in \mathbb{Z}e_1 \times \cdots \times \mathbb{Z}e_n$ such that $z = tC^{-1}d$, that is if and only if $z = C^{-1}d$. Denote $z = z_1e_1 + \cdots + z_ne_n$. We thus obtain the following system (S)

$$
\begin{align*}
(n + 1)z_1 &= nd_1 + (n - 1)d_2 + \cdots + d_n \\
(n + 1)z_2 &= (n - 1)d_1 + (2n - 2)d_2 + \cdots + 2d_n \\
&\vdots \\
(n + 1)z_k &= [(n + 1)\min(k, 1) - k]d_1 + [(n + 1)\min(k, 2) - 2k]d_2 + \cdots + [(n + 1)\min(k, n) - nk]d_n \\
&\vdots \\
(n + 1)z_n &= d_1 + 2d_2 + \cdots + nd_n.
\end{align*}
$$

Therefore, if there exists a $z$ satisfying this system, since all the $z_i$ are integral, it must also satisfy for all $k = 1, \ldots, n$ the following condition

$$
\sum_{j=1}^{n} [(n + 1)\min(k, j) - kj]d_j \in (n + 1)\mathbb{Z},
$$

which is equivalent to the following equation in $\mathbb{Z}/(n + 1)\mathbb{Z}$

$$
\sum_{j=1}^{n} [(n + 1)\min(k, j) - kj]d_j = 0.
$$

We claim now that if $d_1 + 2d_2 + \cdots + nd_n \in (n + 1)\mathbb{Z}$ then we have $\sum_{j=1}^{n} [(n + 1)\min(k, j) - kj]d_j \in (n + 1)\mathbb{Z}$ for all others indexes $k$. Since $d_1 + 2d_2 + \cdots + nd_n \in (n + 1)\mathbb{Z}$ there exists $r \in \mathbb{Z}$ such that $d_1 + 2d_2 + \cdots + nd_n = (n + 1)r$, that is $d_1 = (n + 1)r - 2d_2 - \cdots - nd_n$. Therefore

$$
\sum_{j=1}^{n} [(n + 1)\min(k, j) - kj]d_j = (n + 1 - k)(n + 1)r - 2d_2 - \cdots - nd_n + \sum_{j=2}^{n} [(n + 1)\min(k, j) - kj]d_j
$$

$$
= r(n + 1 - k)(n + 1) + \sum_{j=2}^{n} [(n + 1)\min(k, j) - kj - j(n + 1 - k)]d_j
$$

$$
= r(n + 1 - k)(n + 1) + \sum_{j=2}^{n} [(n + 1)(\min(k, j) - j)]d_j
$$

$$
= (n + 1)\left[r(n + 1 - k) + \sum_{j=2}^{n} [\min(k, j) - j]d_j\right].
$$

To summarize, if there exists $z \in \mathbb{Z}\Phi$ such that (S) is satisfied then we must have

$$
\sum_{j=1}^{n} [(n + 1)\min(k, j) - kj]d_j = 0
$$

for all $k = 1, \ldots, n$, which is equivalent to the equation $\overline{d_1} + 2\overline{d_2} + \cdots + n\overline{d_n} = 0$. Therefore, if $\overline{d_1} + 2\overline{d_2} + \cdots + n\overline{d_n} \neq 0$ there doesn’t exist $z \in \mathbb{Z}\Phi$ such that $z = tC^{-1}d$, whence $s_1$ and $s_2$ define two different classes in $H^1(W, \mathbb{Z}\Phi)$. 

Conversely, if \( s_1 \) and \( s_2 \) define two different elements in \( H^1(W, \mathbb{Z}\Phi) \) we don’t have \( z \in \mathbb{Z}\Phi \) satisfying \((S)\). However, if \( \overline{d}_1 + 2\overline{d}_2 + \cdots + nd\overline{a}_n = 0 \) it follows that \( \frac{1}{n+1}\sum_{j=1}^{n} [(n+1)\min(k,j) - k\overline{d}_j]\) for all \( k = 1, \ldots, n \). Thus, by setting \( z_k := \frac{1}{n+1}\sum_{j=1}^{n} [(n+1)\min(k,j) - k\overline{d}_j]\) for all \( k = 1, \ldots, n \) we have built an element in \( \mathbb{Z}\Phi \) such that \( z = t^{-1}C^{-1}d \), i.e. \( s_1 \) and \( s_2 \) are equivalent, which is impossible according to our assumption. \( \square \)

**Remark 3.1.** Assume that \( n + 1 \) is a prime number. Then for each pair of sections \( s_1 \) and \( s_2 \) that satisfy \( d_i(s_1, s_2) = d_j(s_1, s_2) \) for all \( i, j \), we have \( s_1 = s_2 \).

Indeed let \( s_1 \) and \( s_2 \) be two sections satisfying \( d_i(s_1, s_2) = d_j(s_1, s_2) \) for all \( i, j \). Since \( n + 1 \) is a prime number, the polynomial \( X^2 + 2X + \cdots + nX \) admits all the elements of \( \mathbb{Z}/(n+1)\mathbb{Z} \) as solutions. In particular, the equality \( \overline{d}_1 + 2\overline{d}_2 + \cdots + nd\overline{a}_n = 0 \) is true. Therefore, because of Proposition 3.2 these two sections define the same element in \( H^1(W(A_n), \mathbb{Z}\Phi) \).

Note that if \( n + 1 \) is not prime, the above result does not necessarily hold. Indeed, assume that \( n = 3 \) and let us take \( s_1 \) that we identify with \((3, 5)\) and \( s_2 \) with \((6, 7, 8)\). We have here \( d_i(s_1, s_2) = 3 \) for \( i = 1, 2, 3 \). Moreover \( \overline{d}_1 + 2\overline{d}_2 + 3\overline{d}_3 = 3 \neq 0 \). Hence \( s_1 \neq s_2 \).

**3.3.2. Type \( B_n \).** In this section \( W = W(B_n) \) and \( \Phi = B_n \). We denote by \( C := C_{B_n} \) the Cartan matrix of \( B_n \). The coefficients of \( C^{-1} \) are known (see \([5]\)) and are given by

\[
(C^{-1})_{ij} = \frac{\min(i, j)}{1 - \min(0, n - i - 1)} = \begin{cases} 
\min(i, j) & \text{if } i < n \\
\frac{i}{2} & \text{if } i = n
\end{cases} \quad 1 \leq i, j \leq n.
\]

**Proposition 3.3.** The sections \( s_1 \) and \( s_2 \) define two different elements in \( H^1(W, \mathbb{Z}\Phi) \) if and only if \( \overline{d}_n = 1 \) in \( \mathbb{Z}/2\mathbb{Z} \).

**Proof.** The proof is almost the same as the \( A_n \) case above. The corresponding system \((S)\) is in this situation as follows

\[
\begin{align*}
2z_1 &= 2d_1 + 2d_2 + \cdots + 2d_{n-1} + d_n \\
& \vdots \\
2z_k &= 2\min(k, 1)d_1 + 2\min(k, 2)d_2 + \cdots + 2\min(k, n-1)d_{n-1} + kd_n \\
& \vdots \\
2z_n &= 2d_1 + 2d_2 + \cdots + 2(n-1)d_{n-1} + nd_n.
\end{align*}
\]

Therefore, \( s_1 \sim s_2 \) if and only if \( kd_n \in 2\mathbb{Z} \) for all \( k = 1, \ldots, n \), that is if and only if \( \overline{d}_n = 1 \) in \( \mathbb{Z}/2\mathbb{Z} \). This ends the proof. \( \square \)

**3.3.3. Type \( C_n \).** In this section \( W = W(C_n) \) and \( \Phi = C_n \). We denote by \( C := C_{C_n} \) the Cartan matrix of \( C_n \). The coefficients of \( C^{-1} \) are known (see \([5]\)) and are given by

\[
(C^{-1})_{ij} = \frac{\min(i, j)}{1 - \min(0, n - j - 1)} = \begin{cases} 
\min(i, j) & \text{if } j < n \\
\frac{i}{2} & \text{if } j = n
\end{cases} \quad 1 \leq i, j \leq n.
\]

**Proposition 3.4.** Write \( I_n := \{ k \in [1, n] \mid k \text{ is odd} \} \). The sections \( s_1 \) and \( s_2 \) define two different elements in \( H^1(W, \mathbb{Z}\Phi) \) if and only if \( \sum_{k \in I_n} \overline{d}_k = 1 \) in \( \mathbb{Z}/2\mathbb{Z} \).

**Proof.** The proof is almost the same as the \( A_n \) case above. The corresponding system \((S)\) is in this situation as follows

\[
\begin{align*}
2z_1 &= 2d_1 + 2d_2 + \cdots + 2d_n \\
& \vdots \\
2z_k &= 2\min(k, 1)d_1 + 2\min(k, 2)d_2 + \cdots + 2\min(k, n)d_n \\
& \vdots \\
2z_n &= d_1 + 2d_2 + \cdots + nd_n.
\end{align*}
\]
It is obvious that $2\min(k,1)d_1 + \cdots + 2\min(k,n)d_n \in 2\mathbb{Z}$ for all $k = 1, \ldots, n - 1$. Therefore, the only equation that matters in this system is the last one: $2z_n = d_1 + 2d_2 + \cdots + nd_n \in 2\mathbb{Z}$. This equation transferred in $\mathbb{Z}/2\mathbb{Z}$ becomes $\sum_{k \in I_n} k\overline{d_k} = 0$ and then $\sum_{k \in I_n} \overline{d_k} = 0$. The result follows.

### 3.3.4. Type $D_n$. In this section $W = W(D_n)$ and $\Phi = D_n$. We denote by $C := C_{D_n}$ the Cartan matrix of $D_n$. It turns out that $C$ is symmetric, this is why we just give the coefficients of $C^{-1}$ with entries $1 \leq i \leq j \leq n$

$$(C^{-1})_{ij} = \begin{cases} i & \text{if } 1 \leq i \leq j \leq n - 2 \\
\frac{i}{2} & \text{if } i < n - 1, j = n - 1 \text{ or } n \\
\frac{i - n}{2} & \text{if } i = n - 1, j = n \\
\frac{i}{2} & \text{if } i = j = n - 1 \text{ or } n. \end{cases}$$

A better visualization of this matrix is given by:

$$C^{-1} = \frac{1}{4} \begin{pmatrix} 4 & 4 & 4 & \cdots & 4 & 2 & 2 \\
4 & 8 & 8 & \cdots & 8 & 4 & 4 \\
4 & 8 & 12 & \cdots & 12 & 6 & 6 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
4 & 8 & 12 & \cdots & 4(n-2) & 2(n-2) & 2(n-2) \\
2 & 4 & 6 & \cdots & 2(n-2) & n & 2(n-2) \\
2 & 4 & 6 & \cdots & 2(n-2) & 2(n-2) & n \end{pmatrix}.$$ 

### Proposition 3.5. Let $I_{n-2} := \{k \in [1,n-2] \mid k \text{ is odd}\}$. The sections $s_1$ and $s_2$ define two different elements in $H^1(W,\mathbb{Z}\Phi)$ if and only if we have the following points

i) $\overline{d_{n-1}} \neq \overline{d_n}$ in $\mathbb{Z}/2\mathbb{Z}$.

ii) $\sum_{k \in I_{n-2}} 2\overline{d_k} + nd_{n-1} + (n-2)d_n \neq 0$ in $\mathbb{Z}/4\mathbb{Z}$.

iii) $\sum_{k \in I_{n-2}} 2\overline{d_k} + (n-2)d_{n-1} + nd_n \neq 0$ in $\mathbb{Z}/4\mathbb{Z}$.

**Proof.** The first line of the corresponding system $(S)$ is given by:

$$4z_1 = 4d_1 + 4d_2 + \cdots + 4d_{n-2} + 2d_{n-1} + 2d_n.$$

Thus, we must have $4d_1 + 4d_2 + \cdots + 4d_{n-2} + 2d_{n-1} + 2d_n \in 4\mathbb{Z}$, that is $2d_{n-1} + 2d_n \in 4\mathbb{Z}$, which is equivalent to $\overline{d_{n-1}} = \overline{d_n}$ in $\mathbb{Z}/2\mathbb{Z}$. We see then that if $\overline{d_{n-1}} = \overline{d_n}$, the $(n-2)$ first equations of $(S)$ can be satisfied.

The penultimate equation of $(S)$ is

$$4z_{n-1} = \sum_{k=1}^{n-2} 2kd_k + nd_{n-1} + (n-2)d_n.$$ 

Therefore, this equality compels us to have

$$\sum_{k=1}^{n-2} 2kd_k + nd_{n-1} + (n-2)d_n \in 4\mathbb{Z},$$

that is $\sum_{k=1}^{n-2} 2\overline{d_k} + \overline{d_{n-1}} + (n-2)\overline{d_n} = 0$ in $\mathbb{Z}/4\mathbb{Z}$. If $k$ is even $2k\overline{d_k} = 0$, and if $k$ is odd $2k\overline{d_k} = 2\overline{d_k}$. Hence we have

$$\sum_{k \in I_{n-2}} 2\overline{d_k} + \overline{d_{n-1}} + (n-2)\overline{d_n} = 0.$$ 

The reasoning is exactly the same for the last equation. This concludes the proof. \qed
Theorem 4.1. \(A_{w'} = \langle w, \alpha \rangle\) is a notion that describes in the same time what are the walls of \(A_{w'}\), that is the hyperplanes supporting the faces of \(A_{w'}\), and what are the generators \(s \in S_a\) such that crossing a \(s\)-face of \(A_{w'}\) means crossing a specific supporting hyperplane of \(A_{w'}\). We illustrate this idea in Figure 1.

![Figure 1](image)

**Figure 1.** In this example: \(W(\mathcal{B}_2)\), all the possible shapes for an alcove are given by the 4 alcoves corresponding to the elements \(e, s_0, s_0s_1, s_0s_1s_2\). The alcove \(A_x\) doesn’t have the same orientation as the 3 other alcoves since its walls are different from those of \(A_{w'}, A_y\) and \(A_z\). Moreover, the alcoves \(A_{w'}\) and \(A_z\) have the same orientation, whereas \(A_y\) doesn’t.

The formal definition of the orientation is the following: two alcoves \(A_{w'}\) and \(A_{w''}\) have the same orientation if and only if \(w' = \tau_x w\) for some \(x \in Z\Phi\). Since the irreducible components of \(\tilde{X}_W\) are stable under the action of \(Z\Phi\) we can restrict the definition of having the same orientation to the components, namely \(A_{w'}\) and \(A_{w''}\) can have the same orientation only if \(\iota(w)\) and \(\iota(w')\) belong to the same component.

Let \(w \in W_a\). We define \(s_w\) to be the section given by \(s_w(s_i) = \tau_{k(w, \alpha_i)}\alpha_i s_i\) for all \(\alpha_i \in \Delta\).

**Theorem 4.1.** Let \(w, w' \in W_a\) such that \(\iota(w)\) and \(\iota(w')\) are in the same component. Then \(A_w\) and \(A_{w'}\) have the same orientation if and only if \(s_w\) and \(s_{w'}\) define the same element in \(H^1(W, Z\Phi)\).

**Proof.** By [4, Theorem 3.3], we know that \(k(\tau_x u, \alpha) = k(\alpha) + \langle x, \alpha \rangle\) for any \(\alpha \in \Phi^+\) and any \(u \in W\). This formula extends easily to any \(u \in W_a\). Indeed for \(u = \tau_y\) we have

\[
k(\tau_x u, \alpha) = k(\tau_x \tau_y, \alpha) = k(\alpha) + \langle x + y, \alpha \rangle = k(\alpha) + \langle x, \alpha \rangle + \langle y, \alpha \rangle = k(\alpha) + \langle x, \alpha \rangle.
\]

By definition \(A_w\) and \(A_{w'}\) have the same orientation if and only if there exists \(x \in Z\Phi\) such that \(k(w', \alpha) = k(\tau_x w, \alpha)\) for any \(\alpha \in \Phi^+\), that is if and only if \(k(w', \alpha) = k(w, \alpha) + \langle x, \alpha \rangle\) for any \(\alpha \in \Phi^+\). In particular \(k(w', \alpha) = k(w, \alpha) = \langle x, \alpha \rangle\) for any \(\alpha \in \Delta\).

It follows that if \(A_w\) and \(A_{w'}\) have the same orientation then by Proposition 3.1 we have \(s_w \sim s_{w'}\), that is \(s_w = s_{w'}\) in \(H^1(W, Z\Phi)\).

Conversely, if \(s_w \sim s_{w'}\) then by Proposition 3.1 there exists \(x \in Z\Phi\) such that \(k(w', \alpha) - k(w, \alpha) = \langle x, \alpha \rangle\) for any \(\alpha \in \Delta\). By Theorem 2.2 we know that for any \(\theta \in \Phi^+\) we have \(k(w, \theta) = P_\theta(w) + \lambda_\theta(w)\) and \(k(w', \theta) = P_\theta(w') + \lambda_\theta(w')\). Since \(\iota(w)\) and \(\iota(w')\) are in the same component it follows that
\[ \lambda_\theta(w) = \lambda_\theta(w') \] for any \( \theta \in \Phi^+ \). In particular we have
\[
k(w', \theta) - k(w, \theta) = P_\theta(w') - P_\theta(w) = P_\theta(\{k(w, \alpha)\}, \alpha \in \Delta) - P_\theta(\{k(w', \alpha)\}, \alpha \in \Delta)
= P_\theta(\{k(w', \alpha) - k(w, \alpha)\}, \alpha \in \Delta) \quad \text{(by linearity of } P_\theta) 
= P_\theta(\{\langle x, \alpha^\vee \rangle\}, \alpha \in \Delta) 
= \langle x, P_\theta(\{\alpha^\vee\}, \alpha \in \Delta) \rangle \quad \text{(by linearity of } P_\theta) 
= \langle x, \theta^\vee \rangle. \quad \text{(by Theorem 2.2)}
\]

Therefore \( k(w', \theta) = k(w, \theta) + \langle x, \theta^\vee \rangle \) for any \( \theta \in \Phi^+ \), that is \( k(w', \theta) = k(\tau_x w, \theta) \) for any \( \theta \in \Phi^+ \). Thus \( w' = \tau_x w \) with \( x \in \mathbb{Z}\Phi \), which means that \( A_{w'} \) and \( A_w \) have the same orientation. \( \square \)

In the following corollary, whose proof stems directly from Theorem 4.1 and Section 3.3, the simple root \( \alpha_i \) is in each case the \( i \)th simple root with the same conventions as in Section 3.3. We also use the notation \( \overline{K}(w, \alpha) := k(w, \alpha) \) for any \( \alpha \in \Phi^+ \) and any \( w \in W_a \).

**Corollary 4.1.** Let \( w, w' \in W_a \) such that \( \iota(w) \) and \( \iota(w') \) are in the same component. Then \( A_w \) and \( A_{w'} \) have the orientation if and only if

**Type A:** \( \sum_{j=1}^{n} j \overline{K}(w, \alpha_j) = \sum_{j=1}^{n} j \overline{K}(w', \alpha_i) \) in \( \mathbb{Z}/(n+1)\mathbb{Z} \).

**Type B:** \( \overline{K}(w, \alpha_n) = \overline{K}(w', \alpha_n) \) in \( \mathbb{Z}/2\mathbb{Z} \).

**Type C:** \( \sum_{j \in I_n} \overline{K}(w, \alpha_j) = \sum_{j \in I_n} \overline{K}(w', \alpha_i) \) in \( \mathbb{Z}/2\mathbb{Z} \).

**Type D:** One of the following situations appear

1) \( \overline{K}(w, \alpha_n) = \overline{K}(w', \alpha_n) \) in \( \mathbb{Z}/2\mathbb{Z} \).

2) \( \sum_{i \in I_{n-2}} 2\overline{K}(w, \alpha_i) + n\overline{K}(w, \alpha_{n-1}) + (n - 2)\overline{K}(w, \alpha_{n-1}) = \sum_{i \in I_{n-2}} 2\overline{K}(w', \alpha_i) + n\overline{K}(w', \alpha_{n-1}) + (n - 2)\overline{K}(w', \alpha_{n-1}) \) in \( \mathbb{Z}/4\mathbb{Z} \).

3) \( \sum_{i \in I_{n-2}} 2\overline{K}(w, \alpha_i) + (n - 2)\overline{K}(w, \alpha_{n-1}) + n\overline{K}(w, \alpha_{n-1}) = \sum_{i \in I_{n-2}} 2\overline{K}(w', \alpha_i) + (n - 2)\overline{K}(w', \alpha_{n-1}) + n\overline{K}(w', \alpha_{n-1}) \) in \( \mathbb{Z}/4\mathbb{Z} \).

**Example 4.1** (Type A). Set \( V = \mathbb{R}^{n+1} \) with canonical basis \( \{e_1, \ldots, e_{n+1}\} \). A way to describe the roots of \( A_n \) is by \( \Phi = \{ \pm(e_i - e_j) \mid 1 \leq i < j \leq n + 1 \} \) with simple system
\[
\Delta = \{ \alpha_i := e_i - e_{i+1} \mid 1 \leq i \leq n \},
\]
and positive roots
\[
\Phi^+ = \{ e_i - e_j \mid 1 \leq i < j \leq n + 1 \}.
\]

A convenient way to write a Shi vector \( v = (v_{ij})_{1 \leq i < j \leq n+1} \) is by putting its coordinates in a pyramidal shape. For example:

\[
v_{16} \quad v_{15} \quad v_{26}
\]
\[
v_{14} \quad v_{25} \quad v_{36}
\]
\[
v_{13} \quad v_{24} \quad v_{35} \quad v_{46}
\]
\[
v_{12} \quad v_{23} \quad v_{34} \quad v_{45} \quad v_{56}
\]

**Figure 2.** Presentation of the coordinates of a Shi vector for \( n = 5 \) where \( v_{ij} \) is the coordinate over the position \( e_i - e_j \in \Phi^+ \). With this presentation the coefficients over the simple roots are on the first line of the pyramid.

By (1) we have \( v_{ij} = v_{ik} + v_{kj} + \delta_{ikj}(v) \) where \( \delta_{ikj} \in \{0, 1\} \). Another way to see the relation “being in the same component of the Shi variety” is via the coefficients \( \delta_{ikj} \). Indeed, two Shi vectors \( v, v' \) are in the same component if and only if \( \delta_{ikj}(v) = \delta_{ikj}(v') \) for all \( 1 \leq i < k < j \leq n + 1 \).
Since these two Shi vectors are in the same component, we can now ask whether they have the same orientation or not. By Corollary 4.1 we conclude that the corresponding alcoves don’t have the same orientation since in \( \mathbb{Z}/6\mathbb{Z} \) one has
\[
1 + 2 \cdot -2 + 3 \cdot 0 + 4 \cdot 3 + 5 \cdot 7 = 2,
\]
and
\[
0 + 2 \cdot -1 + 3 \cdot 2 + 4 \cdot 4 + 5 \cdot -3 = 5.
\]

**Example 4.2 (Type B).** Let \( \{e_1, e_2\} \) be the canonical basis of \( \mathbb{R}^2 \). In this case the positive root system is given by \( \Phi^+ = \{e_1 - e_2, e_2, e_1 + e_2\} \) with simple system \( \Delta = \{\alpha_1 := e_1 - e_2, \alpha_2 := e_2\} \).

By Corollary 4.1, two alcoves \( A_w, A_w' \) of \( W(\tilde{B}_2) \) belonging to the same component have the same orientation if and only if \( k(w, e_2) = k(w, e_2) \) in \( \mathbb{Z}/2\mathbb{Z} \), which can be checked on Figure 5 with the conventions of Figure 4.

\[
k(w, e_1 + e_2) \quad k(w, e_1 - e_2)
\]
\[
k(w, e_1) = k(w, e_1) \quad k(w, e_2)
\]
\[
k(w, e_1 - e_2) \quad k(w, e_2)
\]

**Figure 4.** On the left hand side is the presentation of the Shi vectors of \( W(\tilde{B}_2) \) ordered by height as in Figure 2, while on the right hand side is the way we put the Shi coefficients inside each alcove in Figure 5.

**Figure 5.** The two orbits in \( \tilde{X}_{W(\tilde{B}_2)}[s_1] \) with \( \iota(s_1) = (-1, 0, 0, 0) \) starting from \( k(s_1, e_2) \) and going anticlockwise.
Acknowledgements. We thank Matthew Dyer and Hugh Thomas for providing helpful comments. This work was partially supported by NSERC grants, by the LACIM and by the CNRS.

References

[1] N. Bourbaki. Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968. 7

[2] Nathan Chapelier-Laget. Shi variety corresponding to an affine Weyl group. arXiv preprint: 2010.04310, 2020. 1, 2, 3

[3] Jean-Pierre Serre. Groupes finis. Cours à l’école normale supérieure de jeunes filles, 1978/1979. 4

[4] Jian Yi Shi. Alcoves corresponding to an affine Weyl group. J. London Math. Soc. (2), 35(1):42–55, 1987. 2, 3, 11

[5] Wei Yangjiang and Yi Ming Zou. Inverses of Cartan matrices of Lie algebras and Lie superalgebras. Linear Algebra and its Applications, 521:283–298, 2017. 7, 8, 9

(Nathan Chapelier-Laget) Institut Denis Poisson, Université de Tours et Orléans, CNRS, Tours, France
Email address: nathan.chapelier@gmail.com