Quantum quench in the transverse field Ising chain: I. Time evolution of order parameter correlators

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Abstract. We consider the time evolution of order parameter correlation functions after a sudden quantum quench of the magnetic field in the transverse field Ising chain. Using two novel methods based on determinants and form factor sums respectively, we derive analytic expressions for the asymptotic behaviour of one- and two-point correlators. We discuss quenches within the ordered and disordered phases as well as quenches between the phases and to the quantum critical point. We give detailed accounts of both methods.

Keywords: correlation functions, form factors, integrable quantum field theory, exact results

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Quantum quench in the transverse field Ising chain I

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1. Introduction

The non-equilibrium dynamics of isolated quantum systems after a sudden ‘quench’ of a parameter characterizing their respective Hamiltonians is a subject currently under intensive theoretical and experimental investigation. Recent experiments on trapped ultra-cold-atomic gases [1]–[6] have established that these systems are sufficiently weakly coupled to their environments to allow the observation of essentially unitary non-equilibrium time evolution on very long timescales. This in turn provides the opportunity of investigating fundamental questions of many-body quantum mechanics, which in standard condensed matter systems are obscured by decoherence and dissipation. Two of the main questions raised in these works are (i) how fast correlations spread in quantum many-body systems and (ii) whether observables such as (multi-point) correlation functions generically relax to time-independent values and, if they do, what principles determine their stationary properties. The first issue was addressed in a seminal work by Lieb and Robinson [7], which established that in lattice many-body systems information has a finite speed of propagation and provided a bound on the maximal group velocity. In recent years an effective light-cone scenario has been proposed theoretically [8]–[12], was tested in many numerical computations [13]–[18] and was finally observed in cold-atom experiments [4].

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The relaxation towards stationary behaviour at first appears very surprising, because unitary time evolution maintains the system in a pure state at all times. The resolution of this apparent paradox is that in the thermodynamic limit, (finite) subsystems can and do display correlations characterized by a mixed state, namely the one obtained by tracing out the degrees of freedom outside the subsystem itself. In other words, the system acts as its own bath.

In groundbreaking (‘quantum Newton’s cradle’) experiments on the relaxation towards stationary states in systems of ultra-cold atoms, Kinoshita, Wenger and Weiss [6] demonstrated the importance of dimensionality and conservation laws for many-body quantum dynamics out of equilibrium. In essence, these experiments show that three-dimensional condensates relax quickly to a stationary state that is characterized by an effective temperature, a process known as ‘thermalization’, whereas the relaxation of quasi-one-dimensional systems is slow and towards an unusual non-thermal distribution. This difference has been attributed to the existence of additional (approximate) local conservation laws in the quasi-1D case, which are argued to constrain the dynamics in analogy with classical integrable systems. The findings of [6] sparked a tremendous theoretical effort aimed at clarifying the effects of quantum integrability on the non-equilibrium evolution in many-particle quantum systems [19–21], [9], [22–48]. Many of these studies are compatible with the widely held belief (see e.g. [19] for a comprehensive summary) that the reduced density matrix of any finite subsystem (which determines correlation functions of any local observables within the subsystem) of an infinite system can be described in terms of either an effective thermal (Gibbs) distribution or a so-called generalized Gibbs ensemble [20]. It has been conjectured that the latter arises for integrable models while the former is obtained for generic systems, and evidence supporting this view has been obtained in a number of examples [20–26], [29], [31–33]. On the other hand, several numerical studies [34, 36, 37, 44, 45] suggest that the full picture may well be more complex.

Moreover, open questions remain even with regards to the very existence of stationary states. For example, the order parameters of certain mean-field models have been shown to display persistent oscillations [49–54]. Non-decaying oscillations have also been observed numerically [36] in some non-integrable one-dimensional systems. This has given rise to the concept of ‘weak thermalization’, which refers to a situation where time-averaged quantities are thermal, while oscillations persist on short timescales. We note that questions related to thermalization and spreading of correlations have also been studied using holographic techniques [55–57].

Given that stationary behaviour is strictly speaking a property at infinite times in the thermodynamic limit (and these limits do not commute), it is important to have available analytic results that become exact in certain limits. To that end we consider here the non-equilibrium dynamics of the transverse field Ising chain

$$H(h) = -J \sum_{j=1}^{L} [\sigma_j^x \sigma_{j+1}^x + h \sigma_j^z],$$

(1)

where $\sigma_j^\alpha$ are the Pauli matrices at site $j$, $J > 0$ and we impose periodic boundary conditions $\sigma_{L+1}^\alpha = \sigma_1^\alpha$. The Hamiltonian (1) exhibits a $\mathbb{Z}_2$ symmetry of rotations around
the $z$-axis in spin space by $180^\circ$

$$\sigma_j^\alpha \to -\sigma_j^\alpha, \quad \alpha = x, y, \quad \sigma_j^z \to -\sigma_j^z.$$ (2)

The model (1) is a crucial paradigm of quantum critical behaviour and quantum phase transitions [58]. At zero temperature and in the thermodynamic limit it exhibits ferromagnetic ($h < 1$) and paramagnetic ($h > 1$) phases, separated by a quantum critical point at $h_c = 1$. For $h < 1$ and $L \to \infty$ there are two degenerate ground states related by the $\mathbb{Z}_2$ symmetry (2). Spontaneous symmetry breaking selects a unique ground state, in which spins align along the $x$-direction. On the other hand, for magnetic fields $h > 1$ the ground state is non-degenerate and as the magnetic field $h$ is increased spins align more and more along the $z$-direction. The order parameter for the quantum phase transition is the ground state expectation value $\langle \sigma_j^x \rangle$. In the following we shall refer to the ferromagnetic phase as the ordered phase, and to the paramagnetic one as the disordered phase. We note that the model (1) is (approximately) realized both in solids [59] and in systems of cold rubidium atoms confined in an optical lattice [60]. In the latter realization it is possible to investigate its non-equilibrium dynamics experimentally.

As shown in appendix A, the model can be diagonalized by a Jordan–Wigner transformation which maps the model to spinless fermions with local annihilation operators $c_j$, followed by a Fourier transform and a Bogoliubov transformation. In terms of the momentum space Bogoliubov fermions $\alpha_k$ the Hamiltonian reads

$$H(h) = \sum_k \varepsilon_h(k) \alpha_k^\dagger \alpha_k, \quad \varepsilon_h(k) = 2J\sqrt{1 + h^2 - 2h \cos(k)}.$$ (3)

For details and more precise definitions see appendix A.

1.1. Quench protocol and observables

In the following we focus on a global quantum quench of the magnetic field. We assume that the many-body system is prepared in the ground state $|\Psi_0\rangle$ of Hamiltonian $H(h_0)$. At time $t = 0$ the field $h_0$ is changed instantaneously to a different value $h$ and one then considers the unitary time evolution of the system characterized by the new Hamiltonian $H(h)$, i.e. the initial state $|\Psi_0\rangle$ evolves as

$$|\Psi_0(t)\rangle = e^{-itH(h)}|\Psi_0\rangle.$$ (4)

The above protocol corresponds to an experimental situation, in which a system parameter has been changed on a timescale that is small compared to any other timescale in the system. We note that this can be achieved in cold-atom experiments [1, 3, 4].

In this paper we study the one and two-point functions of the order parameter

$$\rho^x(t) = \frac{\langle \Psi_0(t)|\sigma_j^x|\Psi_0(t)\rangle}{\langle \Psi_0(t)\rangle \langle \Psi_0(t)\rangle},$$ (5)

$$\rho^{xx}(\ell, t) = \frac{\langle \Psi_0(t)|\sigma_j^x \sigma_j^x|\Psi_0(t)\rangle}{\langle \Psi_0(t)\rangle \langle \Psi_0(t)\rangle}, \quad \rho^{xx}(\ell, t) = \rho^{xx}(\ell, t) - \langle \rho^x(t) \rangle^2.$$ (6)

Although quantum quenches in the 1D Ising model have been the subject of many works [61]–[70], results on the time evolution of order parameter correlation functions were reported only very recently by us in a short communication [71]. In the following we...
give detailed derivations on the full asymptotic time and distance dependence of one- and two-point order parameter correlation functions in the thermodynamic limit for quenches within the two phases. These have been obtained by two novel, complementary methods, and we discuss both of them in detail. The first method is based on the determinant representation of correlation functions characteristic of free fermionic theories. The second is based on the form factor approach [72]–[76] and is applicable more generally to integrable quenches in interacting quantum field theories [77, 32]. We stress that this approach is qualitatively different from numerical approaches based on quantum integrability [78]–[80], [44, 46]. In particular, it allows us to obtain analytical results in the thermodynamic limit. New results not previously reported in [71] include expressions for the two-point function of the order parameter for quenches within the disordered phase and for quenches across the critical point.

1.2. The quench variables

As shown in appendix A both the initial and final Hamiltonians can be diagonalized by combined Jordan–Wigner and Bogoliubov transformations with Bogoliubov angles $\theta_0^k$ and $\theta_k$ respectively. The corresponding Bogoliubov fermions are related by a linear transformation (B.1) characterized by the difference $\Delta_k = \theta_k - \theta_0^k$. In order to parametrize the quench it is useful to introduce the quantity

$$
\cos \Delta_k = \frac{hh_0 - (h + h_0) \cos k + 1}{\sqrt{1 + h^2 - 2h \cos(k)} \sqrt{1 + h_0^2 - 2h_0 \cos(k)}}.
$$

We note that $\cos \Delta_k$ is invariant under the two transformations

$$(h_0, h) \rightarrow (h, h_0) \text{ and } (h_0, h) \rightarrow \left(\frac{1}{h_0}, \frac{1}{h}\right).$$

However, we stress that the quantum quench itself is not invariant under the maps (8). In the form factor approach a more natural quantity to consider is

$$K(p) = \tan[\Delta_p/2].$$

1.3. Timescales for two-point correlators

In this paper we are mainly concerned with equal time correlation functions of spins separated by a distance $\ell$, which we take to be much larger than the lattice spacing, i.e. $\ell \gg 1$. For fixed $\ell$ the time evolution is naturally divided into three regimes, which are determined by the propagation velocity of elementary excitations of the post-quench Hamiltonian $v(k) = d\varepsilon_h(k)/dk$. For a given final magnetic field $h$, the maximal propagation velocity is

$$v_{\text{max}} = \max_{k \in [-\pi, \pi]} |\varepsilon'_h(k)| = 2J \min[h, 1].$$

The three different regimes are:

- Short times $v_{\text{max}} t \ll \ell$.
- Intermediate times $v_{\text{max}} t \sim \ell$. This regime is of particular importance for both experiments and numerical computations. A convenient way of describing this regime is to consider evolution along a particular ‘ray’ $\kappa \ell = v_{\text{max}} t$ in space–time, see figure 1.
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Figure 1. Left panel: for intermediate times $v_{\text{max}} t \sim \ell$ the behaviour of $\rho^{\alpha \alpha}(\ell, t)$ is most conveniently determined by considering its asymptotic expansion around infinity (‘space–time scaling limit’) along the ray $v_{\text{max}} t = \kappa \ell$. This viewpoint is appropriate for any large, finite $t$ and $\ell$. Right panel: the asymptotic late-time regime is reached by considering time evolution at fixed $\ell$. To describe this regime one should consider an asymptotic expansion of $\rho^{\alpha \alpha}(\ell, t)$ around $t = \infty$ at fixed $\ell$.

In order to obtain a very accurate description of the dynamics at a particular point along this ray, one may then construct an asymptotic expansion in the single variable $\ell$ around the space–time scaling limit $v_{\text{max}} t, \ell \to \infty$, $\kappa$ fixed.

- Late times $v_{\text{max}} t \gg \ell$. This includes the limit $t \to \infty$ at fixed but large $\ell$. In this regime it is no longer convenient to consider evolution along a particular ray in space–time. In order to obtain accurate results for the late-time dynamics, one should construct an asymptotic expansion in $t$ around infinity, see figure 1.

Because of the horizon effect [9, 22] the short-time regime does not display interesting features. We therefore focus on intermediate and late times. The most convenient way of analysing the intermediate-time regime is via the space–time scaling limit. It is important to note that taking the limit $\kappa \to 0$ of results obtained in the space–time scaling limit is not expected to reproduce the short-time regime. Similarly, in general taking $\kappa \to \infty$ in the space–time scaling limit does not necessarily reproduce the late-time behaviour.

1.4. Organization of the paper

This is the first in a series of two papers of the dynamics in the transverse field Ising chain after a sudden quench of the magnetic field. The second paper, henceforth referred to as ‘paper II’, gives a detailed account of properties in the stationary state, i.e. at $t = \infty$. The present paper deals with the time evolution of observables and is organized in the following way. In section 2 we present a detailed summary of our main results. In section 3 we discuss the determinant approach for calculating two-point functions after quantum quenches in models with free fermionic spectra. Section 4 introduces the form factor approach to correlation functions in integrable models after quantum quenches and gives a detailed account of its application to one and two-point functions in the Ising chain. The scaling limit for quenches close to the critical point is constructed in sections 5
and 6 contains our conclusions. Our conventions for diagonalizing the Ising Hamiltonian with periodic boundary conditions are summarized in appendix A and the initial state is expressed in terms of eigenstates of the post-quench Hamiltonian in appendix B. Finally, appendices C and D deal with certain technical issues.

2. Summary and discussion of the results

In this section we present a comprehensive summary of our main results. Our analytic results are valid in the thermodynamic limit $L \to \infty$ and are obtained by two different methods.

(i) The first is based on representing correlation functions as determinants and then determining the asymptotic behaviour in the space–time scaling limit

$$t, \ell \to \infty, \frac{v_{\text{max}} t}{\ell} \text{ fixed.}$$

Results obtained by this method are exact.

(ii) The second method employs a Lehmann representation for correlation functions, which provides an expansion of the correlator in powers of the functions $K(k)$ (9). The Lehmann representation is recast as a low-density expansion and then the dominant terms at late times and large distances are summed to all orders in $K(k)$. The method exploits the existence of a small parameter, namely the average densities $n(k)$ of elementary excitations of the post-quench Hamiltonian $H(h)$ with momentum $k$ in the initial state $|\Psi_0(0)\rangle$ to

$$\frac{\langle \Psi_0(0) | n(k) | \Psi_0(0) \rangle}{\langle \Psi_0(0) | \Psi_0(0) \rangle} = \frac{K^2(k)}{1 + K^2(k)}.$$ 

A small quench is defined as being such that these densities are small for all $k$, i.e. $\max_k K^2(k) \ll 1$ (this does not necessarily imply that $h$ and $h_0$ have to be very close to each other). The form factor approach provides accurate results for small quenches at late times and large distances. In practice, the form factor method provides very good approximations to the exact result, except for quenches to or from the close vicinity of the critical point.

Analytic results obtained by the two methods are compared to a direct numerical evaluation of the determinant representations of correlation functions in the thermodynamic limit. Finite-size effects are concomitantly absent and for the purposes of the comparisons shown in the following figures the numerical results can therefore be considered to be exact.

2.1. One-point correlation function

For quenches starting in the disordered phase, i.e. $h_0 > 1$, the order parameter expectation value is zero for all times because the $\mathbb{Z}_2$ symmetry remains unbroken. For quenches that start and end in the ordered phase, i.e. $h_0, h < 1$, exact calculations based on the determinant approach show that the order parameter relaxes to zero exponentially at late
Figure 2. Expectation value of the order parameter after a quench within the ferromagnetic phase. Numerical data are compared with the asymptotic predictions from both determinants and form factors at order $O(K^2)$. The right panel shows the accuracy of the determinant result even very close to the critical point, while the form factor result ceases to be an accurate approximation because the density of excitations is no longer small. Numerical results are obtained by considering the cluster decomposition of the two-point function at distance $\ell = 180$. The left-hand panel shows that the short-time behaviour after the quench is sensitive to whether the transverse field is increased or decreased. If $1 > h > h_0$ then $\langle \sigma^x(t) \rangle$ decreases for short times, while it initially increases if $1 > h > h_0$.

\[ \langle \sigma^x_i(t) \rangle \simeq \left( C_{\text{FF}}^x \right)^{1/2} \exp \left[ t \int_0^\pi \frac{dk}{\pi} \varepsilon'_h(k) \ln |\cos \Delta_k| \right], \]  
(13)

where

\[ C_{\text{FF}}^x = \frac{1 - hh_0 + \sqrt{(1 - h^2)(1 - h_0^2)}}{2\sqrt{1 - hh_0}}. \]  
(14)

This result is obtained by applying the cluster decomposition principle to the two-point function (19) (see equation (69) in section 3 for the proof of equation (19)). The form factor approach gives (see section 4.2)

\[ \langle \sigma^x_i(t) \rangle \simeq (1 - h^2)^{1/8} \exp \left[ -t \int_0^\pi \frac{dk}{\pi} \varepsilon'_h(k) 2K^2(k) \right]. \]  
(15)

We note that

\[ \ln |\cos \Delta_k| = \ln \left[ \frac{1 - K^2(k)}{1 + K^2(k)} \right] = -2K^2(k) - \frac{2}{3}K^6(k) + \cdots, \]  
(16)

\[ \left( C_{\text{FF}}^x \right)^{1/2} \simeq (1 - h^2)^{1/8} + \frac{(h - h_0)^4}{64(1 - h^2)^{31/8}} + \cdots, \]

so that (15) is indeed the ‘low-density’ approximation to (13).

Figure 2 shows the comparison of the asymptotic result against exact numerical computation (obtained from cluster decomposition of the two-point function). It is evident...
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Figure 3. Expectation value of the order parameter after a quench from the ferromagnetic to the paramagnetic phase (inset) and its absolute value (main plot) in a logarithmic scale to show the exponential decay of the correlation. Numerical data from $h_0 = 1/2$ to $h = 5/3$ and 1.05 are compared with the asymptotic conjecture in equation (17), which is reported as a continuous line. The asymptotic formula is valid all the way up to the critical point. Data are obtained by considering the cluster decomposition of the two-point function at distance $\ell = 180$. That the asymptotic results become accurate already for small values of $t$. Equation (13) is asymptotically valid also for quenches to the critical point as is shown in the left panel of figure 2.

For a quench from the ferromagnetic $h_0 < 1$ to the paramagnetic $h > 1$ phase, we conjecture that the expectation value of the order parameter at late times $t$ is given by

$$\langle \sigma^x(t) \rangle = (C_{\text{FP}}^x)^{1/2} \left[ 1 + \cos(2\varepsilon_h(k_0)t + \alpha) + \cdots \right]^{1/2} \exp \left[ t \int_0^\pi \frac{dk}{\pi} \varepsilon_h(k) \ln |\cos \Delta_k| \right], \quad (17)$$

where $k_0$ is a solution of the equation $\cos \Delta k_0 = 0$, $\alpha(h, h_0)$ is an unknown constant, and

$$C_{\text{FP}}^x = \left[ \frac{h\sqrt{1-h_0^2}}{h+h_0} \right]^{1/2}. \quad (18)$$

The dots in equation (17) indicate subleading contributions. The conjecture (17) is compared to the numerically calculated one-point function (the one-point function is obtained by applying the cluster decomposition principle to the two-point function equation (32), see section 3.2.1) in figure 3. The agreement is clearly excellent. From a mathematical point of view the oscillating factor is a correction to the asymptotic behaviour, as is most clearly seen by considering $\log \langle |\sigma^x(t)| \rangle$. However, by virtue of its oscillatory nature, its presence obscures the leading behaviour and needs to be included in order to have a good description of the quench dynamics. In the limit $h \to 1$, $k_0$ goes to 0 and $\varepsilon_1(k_0) = 0$, signalling that the crossover between (13) and (17) is smooth. In particular, when approaching the critical point, the oscillation frequency decreases as shown in figure 3.

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2.2. Equal time two-point correlation function

2.2.1. Quench within the ferromagnetic phase. For a quench within the ordered phase, the determinant approach (detailed in section 3) leads to the following result for the two-point function in the space–time scaling limit \((t, \ell \to \infty)\) with \(v_{\text{max}}t/\ell\) fixed

\[
\rho_{\text{FF}}^{xx}(\ell, t) \simeq C_{\text{FF}}^x \exp \left[ \ell \int_0^{\pi} \frac{dk}{\pi} \ln |\cos \Delta_k| \theta_H(2\varepsilon'_h(k)t - \ell) \right] \times \exp \left[ 2t \int_0^{\pi} \frac{dk}{\pi} \varepsilon'_h(k) \ln |\cos \Delta_k| \theta_H(\ell - 2\varepsilon'_h(k)t) \right].
\]

Here \(\theta_H(x)\) is the Heaviside step function

\[
\theta_H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{else.} \end{cases}
\]

The constant \(C_{\text{FF}}^x\) is fixed by matching (19) to the corresponding result at infinite time \(t = \infty\), which is derived in paper II [81].

In the limit \(\ell \to \infty\) (19) gives the square of the result (13) for the one-point function. For times smaller than the Fermi time

\[
t_F = \frac{\ell}{2v_{\text{max}}},
\]

the first exponential factor in (19) equals 1. Thus, in the space–time scaling limit, connected correlations \emph{vanish identically} for times \(t < t_F\) and begin to form only after the Fermi time. This is a general feature of quantum quenches [9, 22] and has been recently observed in experiments on one-dimensional cold-atomic gases [4]. We stress that this by no means implies that the connected correlations are exactly zero for \(t < t_F\): in any model, both on the lattice and in the continuum there are exponentially suppressed terms (in \(\ell\)) which vanish in the scaling limit. The form factor approach gives the following result for large \(t\) and \(\ell\) (see section 4.3)

\[
\rho_{\text{FF}}^{xx}(\ell, t) \simeq (1 - h^2)^{1/4} \exp \left[ -2\ell \int_0^{\pi} \frac{dk}{\pi} K^2(k) \theta_H(2\varepsilon'_h(k)t - \ell) \right] \times \exp \left[ -4t \int_0^{\pi} \frac{dk}{\pi} \varepsilon'_h(k) K^2(k) \theta_H(\ell - 2\varepsilon'_h(k)t) \right].
\]

As expected, it gives the low-density approximation to the full result (19).

A comparison (for a typical quench from \(h_0 = 1/3\) to \(h = 2/3\)) between the asymptotic results (19) and (22) and numerical results for the correlation function at a finite but large distance (\(\ell = 30\)) is shown in figure 4. The numerical results are obtained by expressing the two-point correlator in the thermodynamic limit as the determinant of an \(\ell \times \ell\) matrix (see section 3) and then evaluating the determinant for different times. As we are concerned with equal time correlators only, we do not need to extract the two-point function from a cluster decomposition of the four-point function [61]. The agreement is clearly excellent. The ratio between the exact numerics and the analytic result (19) in the space–time scaling limit is shown in the inset of figure 4 for two values of \(\kappa = v_{\text{max}}t/\ell\). We see that the ratio approaches a constant for large \(\ell\). The corrections to this constant are seen to be oscillating. The right panel of figure 4 shows results for the connected two-point correlation function.
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Figure 4. Numerical data for a quench within the ferromagnetic phase from $h_0 = 1/3$ to $h = 2/3$. Left: the two-point function against the asymptotic prediction equation (19) for $\ell = 30$ (up to a multiplicative factor), showing excellent agreement in the scaling regime. Inset: ratio between the numerical data and asymptotic prediction (69). The leading correction is time independent, but subleading contributions oscillate. Right: the connected correlation function for the same parameters as on the left. For $t < t_F$, $\rho_{\ell, t}^{xx}(\ell, t)$ vanishes identically in the scaling regime.

In order to elucidate the relaxational behaviour of the two-point function, it is useful to follow [9, 22] and rewrite (19) in the form

$$\frac{\rho_{\ell, t}^{xx}(\ell, t)}{(\rho_{\ell, t}^{xx}(t))^2} \sim \exp \left[ \int_0^\pi \frac{dk}{\pi} \frac{2t}{\tau(k)} - \frac{\ell}{\xi(k)} \right] \theta_H \left( \frac{2t}{\tau(k)} - \frac{\ell}{\xi(k)} \right).$$

(23)

Here we have defined mode-dependent correlation lengths $\xi(k)$ and decay times $\tau(k)$ by

$$\xi^{-1}(k) = -\ln |\cos \Delta_k|, \quad \tau^{-1}(k) = -\varepsilon'_h(k) \ln |\cos \Delta_k|.$$  

(24)

We observe that these quantities are related by the velocity $v_k = \varepsilon'_h(k)$ of the momentum $k$ mode

$$v_k \tau(k) = \xi(k),$$

(25)

which allows us to rewrite the theta function in (23) as $\theta_H(2v_k t - \ell)$. The physical interpretation of (23) is then clear: a given mode contributes to the relaxational behaviour only if the distance $\ell$ lies within its forward ‘light cone’ (the factor of two multiplying the velocity is explained in [9, 22]). The form of the remaining factor then follows from the stationary behaviour: the time-dependent piece compensates the factor $(\rho_{\ell, t}^{xx})^{-2}$, while the time-independent part is fixed by the $t \to \infty$ value of the correlator. As we have already pointed out in our letter [71], this implies that the generalized Gibbs ensemble that characterizes the stationary state in fact determines the relaxational behaviour at late times as well.

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Figure 5. Numerical results for the connected two-point function for a quench from the ferromagnetic phase to a magnetic field very close to (left) and exactly at the critical point (right). The asymptotic behaviour agrees with the prediction (19) shown as continuous lines, but corrections are visible for smaller values of $\ell$.

Approach to infinite times within the space–time scaling regime. For the quench within the ferromagnetic phase, the infinite time limit at fixed $\ell$ gives the same result as the infinite time limit within the space–time scaling regime, i.e.

$$\lim_{t \to \infty} \frac{1}{\ell} \ln |\rho_{FF}^{xx}(\ell, t)| = \lim_{\kappa \to 0, \ell/t \to \text{fixed}} \frac{1}{\ell} \ln |\rho_{FF}^{xx}(\ell, t)|. \quad (26)$$

It is then useful to consider the approach to the stationary value within the space–time scaling regime result (19). In the limit $t \to \infty$, we have

$$\rho_{FF}^{xx}(\ell, t = \infty) \propto \exp \left( \ell \int_{0}^{\pi} \frac{dk}{k} \ln |\cos \Delta_k| \right). \quad (27)$$

The corrections to (27) for large, finite times arise from the modes with $\epsilon_k'(k) \approx 0$. For any $h \neq 1$, both modes with $k = 0$ and $\pi$ contribute to this correction and at the same order, since both $\ln |\cos \Delta_k|$ and the dispersion relation itself are quadratic at both points. A straightforward calculation gives for any $h, h_0 < 1$

$$\frac{\rho_{FF}^{xx}(\ell, t \gg \ell)}{\exp[\ell \int_{0}^{\pi} \frac{dk}{k} \ln |\cos \Delta_k|]} \propto 1 + \frac{(h - h_0)^2(1 - 2hh_0 + h_0^2)}{96\pi(1 - h_0^2)^2} \frac{\ell^4}{(v_{\max}t)^3} + \ldots. \quad (28)$$

2.2.2. Quenches from the ferromagnetic phase to the quantum critical point. If we adjust the constant $C_{FF}$ appropriately, equation (19) holds even for quenches to the quantum critical point. In figure 5 numerical results for the connected correlation function are compared with the asymptotic prediction for quenches very close to the critical point (left) and exactly to the critical point (right). In both cases, the asymptotic prediction is seen to become more accurate when $\ell$ is increased. However, it is clear from the figure that the asymptotic prediction works better for quenches exactly to the critical point than for quenches very close to it. This somewhat counterintuitive behaviour is readily understood as a property of subleading contributions to (19). We have already seen in the inset of figure 4 that there are subleading oscillating corrections. A more detailed analysis

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shows that they are power laws with an exponent that tends to zero upon approaching the critical point. This explains why the agreement of (19) with the numerical results is worse in figure 5 (left) than in figure 4. At the same time, as the exponent of the power-law correction tends to zero upon quenching ever closer to the critical point, the oscillation frequency approaches zero as well. For quenches exactly to the critical point the leading oscillating corrections are therefore absent: they have morphed into a renormalization of the constant amplitude multiplying (19). This is the reason why (19) works better for quenches exactly to the critical point than for quenches close to it.

For a quench to the critical point at \( h_c = 1 \), the result (19) should be compared with conformal field theory (CFT) predictions of Refs [9, 22], which give the following result valid in the scaling limit of the Ising model

\[
\lim_{h \to 1} \rho_{xx}(r, t) \propto \begin{cases} 
\exp\left[-\frac{\Delta_0}{2v}(r \theta(2vt - r) + 2vt \theta(r - 2vt))\right] & \text{for } vt < r/2 \\
\exp\left[-\frac{\Delta_0}{2v}(r \theta(2vt - r) + 2vt \theta(r - 2vt))\right] & \text{for } vt > r/2.
\end{cases}
\]

(29)

Here \( r \) is the physical distance, \( v \) the velocity characterizing the (strictly) linear dispersion relation \( \varepsilon_q = qv \) at the critical point, and \( \tau_0 \) the so-called extrapolation time. Equation (29) is valid for times and distances such that \((x/v, t) \gg \tau_0\). The scaling limit of the result (19) is constructed in section 5, and considering a quench to the critical point we obtain from (264)

\[
\lim_{h \to 1} \rho_{xx}(r, t) \propto \exp\left(-\frac{\Delta_0}{2v}(r \theta(2vt - r) + 2vt \theta(r - 2vt))\right),
\]

(30)

where \( \Delta_0 = J(1 - h_0) \). Comparing (30) to (29) we see that the two expressions agree if we take the extrapolation time to be

\[
\tau_0^{-1} = \frac{8\Delta_0}{\pi}.
\]

(31)

Outside the scaling limit the CFT expression (29) is not expected to provide a good approximation to the full result (19) because the nonlinearity of the dispersion relation becomes important.

2.2.3. Quenches from the ferromagnetic to the paramagnetic phase. We have not been able to carry out a full analytical calculation of the time evolution of the two-point function for a quench across the critical point. However, we conjecture\(^3\) that (see section 3.2.1)

\[
\rho_{FF}^r(\ell, t) \simeq C_{FF}^r \exp\left[\int_0^\pi \frac{dk}{\pi} 2\varepsilon'_h(k) t \ln |\cos \Delta_k|\right]
\]

\[
\times \begin{cases} 
\exp\left[\int_0^\pi \frac{dk}{\pi} (\ell - 2t\varepsilon'_h(k)) \ln |\cos \Delta_k| \theta_H(2\varepsilon'_h(k)t - \ell)\right] & t > t_F, \\
1 + \cos(2\varepsilon_h(k_0)t + \alpha) + \cdots & t < t_F.
\end{cases}
\]

(32)

where \( \alpha \) and \( k_0 \) are the same as in (17) (see also [81]) and the constant factor \( C_{FF}^r \) is given in (18). This prediction is compared with the numerically calculated correlation function in figure 6 (left) and the agreement is clearly very good. For \( t < t_F \) (32) is simply the square

\(^3\) Our conjecture is based on properties of the spectrum of the matrix \( \Gamma \) (50).
of the corresponding one-point function, which ensures that connected correlations vanish for \( t < t_F \) in the space–time scaling regime. This is in agreement with numerical results for the connected two-point function as shown in the right-hand panel of figure 6. As can be seen in figure 6, oscillations are present for \( t > t_F \) as well, but they display a rather fast decay in time towards the determinant result (19). As for the one-point function, the oscillating factor is a correction to the leading asymptotic behaviour in the space–time scaling limit, but it needs to be included to give a good description of the numerical data. Finally, we note that for \( h = 1 \) we have \( \varepsilon_1(k_0) = 0 \) and (32) reduces to (19).

2.2.4. Quench within the paramagnetic phase. For quenches within the paramagnetic phase the form factor approach gives the following result (see section 4.5)

\[
\rho_{PP}^{xx}(\ell, t) \simeq \rho_{PP}^{xx}(\ell, \infty) + (h^2 - 1)^{1/4} \sqrt{4J^2h} \int_{-\pi}^{\pi} \frac{dk}{\pi} \frac{K(k)}{\varepsilon_k} \sin(2t\varepsilon_k - k\ell)
\]

\[
\times \exp \left[ -2 \int_0^\pi \frac{dp}{\pi} K^2(p) (\ell + \theta_H(\ell - 2t\varepsilon'_p)[2t\varepsilon'_p - \ell]) \right] + \cdots. \tag{33}
\]

The regime of validity of (33) is sufficiently large values of \( \ell \) and \( t \) and ‘small’ quenches in the sense discussed at the beginning of section 2. We have not attempted to calculate the infinite time limit \( \rho_{PP}^{xx}(\ell, \infty) \) within the framework of the form factor approach, because its exact large-\( \ell \) asymptotics is known from the determinant approach to be [71, 81]

\[
\rho_{PP}^{xx}(\ell, \infty) \simeq C_{PP}^{xx}(\ell) e^{-\ell/\xi}, \tag{34}
\]

where \( C_{PP}^{xx}(\ell) \) is determined in paper II [81] and

\[
\xi^{-1} = \ln \left( \min[h_0, h_1] \right) - \ln \left[ \frac{h_1 h + h_0}{2hh_0} \right], \quad h_1 = \frac{1 + hh_0 + \sqrt{(h^2 - 1)(h_0^2 - 1)}}{h + h_0}. \tag{35}
\]
Quantum quench in the transverse field Ising chain I

Figure 7. Time evolution of order parameter correlators for quenches within the paramagnetic phase. Left: $\rho_{xx}(\ell=30,t)$ for a quench from $h_0 = \frac{3}{2}$ to $h = 2$. Right: $\rho_{xx}(\ell=30,t)$ for a quench from $h_0 = \infty$ to $h = 5$. The two-point function is seen to exhibit oscillatory power-law decay at late times. The form factor result (solid lines) is seen to give a very good description of the numerical data (points). The short-time regime is not shown as the correlators are exponentially small by virtue of the horizon effect.

As discussed in our previous letter [71], (34) is described by a general Gibbs ensemble. Based on the form factor result (33) one may speculate that the full answer may have the structure

$$
\rho_{xx}(\ell,t) \simeq \left[ C_{xx}(\ell) + (h^2 - 1)^{1/4}\sqrt{4J^2h} \int_{-\pi}^{\pi} \frac{dk}{\pi} \frac{K(k)}{\varepsilon_k} \sin(2t\varepsilon_k - k\ell) + \cdots \right] 
\times \exp \left[ -\int_{0}^{\pi} \frac{dp}{\pi} \ln \left[ \frac{1 + K^2(p)}{1 - K^2(p)} \right] (\ell + \theta_H(\ell - 2t\varepsilon'_p)[2t\varepsilon'_p - \ell]) \right] + \cdots .
$$

(36)

In figure 7 we compare the analytic result (33) to numerical results obtained for two different quenches within the paramagnetic phase. The agreement is seen to be excellent.

For strong quenches the form factor result is not expected to be quantitatively accurate. This can be seen in figure 8. In all cases, the two-point function is seen to display slowly decaying oscillatory behaviour on the timescales shown. At sufficiently large $t$ the decay is proportional to $t^{-3/2}$. This is in marked contrast to quenches within the ordered phase. The origin of this difference lies in the nature of the relaxational processes that drive the time evolution. The oscillatory behaviour seen in the paramagnetic phase arises from processes involving the annihilation of spin-flip excitations, while the smooth exponential behaviour seen in the ferromagnetic phase is related to the ballistic motion of domain wall excitations.

The structure of (33) implies the existence of a late-time crossover scale $t^*$, at which the second contribution becomes smaller than $\rho_{xx}(\ell,\infty)$. Using that $\rho_{xx}(\ell,\infty) \propto e^{-\ell/\xi}$ and that the second contribution decays as $(Jt)^{-3/2}$ at late times we may estimate $t^*$ as

$$
Jt^* \sim e^{2\ell/(3\xi)}.
$$

(37)
Figure 8. Time evolution of $\rho_{PP}(\ell = 60, t)$ for a quench from $h_0 = 5$ to $h = 5/4$. As the quench is no longer small, deviations from the form factor result become more pronounced.

For the cases shown in figure 7 this gives $Jt^* \sim 4114$ and $Jt^* \sim 10^{20}$ respectively ($Jt_F = 7.5$ in both cases). This means that in both cases the stationary behaviour characterised by the generalized Gibbs ensemble is revealed only at very late times.

Finally, we note that so far we have not been able to analyse the time evolution of order parameter correlators for quenches within the paramagnetic phase by means of the determinant approach.

2.2.5. Quenches from the paramagnetic to the ferromagnetic phase. Here we have not been able to obtain analytic results by either the determinant or the form factor approach. However, observations within the framework of the determinant approach (see section 3.2) suggest that for late times $t > t_F$ the leading asymptotic behaviour of the two-point function should be given by

$$
\rho_{PF}(\ell, t) \simeq C_{PF}(\ell) \exp \left[ \ell \int_0^\pi \frac{dk}{\pi} \ln |\cos \Delta_k| \theta_H(2\varepsilon'_h(k)t - \ell) \right] 
\times \exp \left[ 2t \int_0^\pi \frac{dk}{\pi} \varepsilon'_h(k) \ln |\cos \Delta_k| \theta_H(\ell - 2\varepsilon'_h(k)t) \right], \quad t > t_F.
$$

Here the factor $C_{PF}(\ell)$ is given by the $t = \infty$ results of [81]

$$
C_{PF}(\ell) = \left[ \frac{h_0 - h}{\sqrt{h_0^2 - 1}} \right]^{1/2} \cos \left( \ell \arctan \left[ \frac{\sqrt{(1-h^2)(h_0^2-1)}}{1 + h_0 h} \right] \right).
$$

We have tested this conjecture by comparing it to the numerical results and found it to hold. An example is shown in figure 9, where we plot the ratio of the numerically calculated correlation function and the analytic expression (19) for several values of the distance $\ell$. The ratio clearly approaches unity at late times. We note that the values of $\ell$ have been chosen in such a way that, for the particular quench considered (i.e. $h_0 = 3.12$ to $h = 0.5$), oscillations in $\ell$ are suppressed in the $t \rightarrow \infty$ limit. Our analytic methods do
Figure 9. Order parameter two-point function for a quench from the paramagnetic to the ferromagnetic phase (inset) and its ratio with the asymptotic prediction $\rho_{PF}^{xx}$ given in equation (38) (main plot). We consider a quench from $h_0 = (2 + 3\sqrt{2})/2$ to $h = 1/2$. For any $t > t_F$, the ratio approaches 1 with increasing $\ell$.

not currently provide an understanding of the quench dynamics for times shorter than the Fermi time, $t < t_F$.

2.3. Quenches to $h = 0$ or $\infty$

These quenches are special, because the post-quench Hamiltonian is classical in both cases. As a result the dynamics is anomalous. For example, for quenches to $h = 0$ correlation functions involving only $\sigma_j^x$ operators are time independent, because the final Hamiltonian $H(h)$ commutes with all $\sigma_j^x$. Correlation functions involving other operators do depend on time, but generally exhibit persistent oscillations. For instance, we have that

$$e^{iH(h=0)t}\sigma_j^x e^{-iH(h=0)t} = \cos^2(2Jt)\sigma_j^z + \frac{1}{2} \sin(4Jt)(\sigma_{j-1}^x + \sigma_{j+1}^x)\sigma_j^y - \sin^2(2Jt)\sigma_{j-1}^x\sigma_j^z\sigma_{j+1}^x,$$

and as a result $\langle \Psi(t)|\sigma_j^z|\Psi(t)\rangle$ does not approach a stationary value at late times.

3. Determinant approach: analytic derivation of the asymptotic two-point function for a quench within the ordered phase.

In this section we present the analytic derivation of equation (19) for the asymptotic behaviour in the space–time scaling regime ($t, \ell \gg 1$, but $t/\ell$ arbitrary) of the two-point correlation function $\rho_{xx}(\ell, t)$. Within this approach, it is convenient to replace the fermions $c_j$ in appendix A with the Majorana fermions

$$a_j^x = c_j + c_j^\dagger, \quad a_j^y = i(c_j^\dagger - c_j),$$

(41)
which satisfy the algebra \( \{a^x_l, a^x_n\} = 2 \delta_{ln}, \{a^y_l, a^y_n\} = 2 \delta_{ln}, \{a^x_l, a^y_n\} = 0 \). In terms of these Majorana fermions, the operator \( \sigma^x \) has the non-local representation

\[
\sigma^x = \prod_{j=1}^{\ell-1} (ia^y_j a^x_j) a^x_\ell. \tag{42}
\]

The two-point function of \( \sigma^x \) is then the expectation value of a string of Majorana fermions

\[
\rho^{xx}(\ell, t) = \left\langle \prod_{j=1}^{\ell} (-ia^y_j(t) a^x_{j+1}(t)) \right\rangle. \tag{43}
\]

The Ising Hamiltonian (1) can be written as (see appendix A)

\[
H = \frac{1}{2} \left[ 1 - \prod_{l=1}^{L} \sigma^z_l \right] H_o + \frac{1}{2} \left[ 1 + \prod_{l=1}^{L} \sigma^z_l \right] H_e, \tag{44}
\]

where \( H_o \) and \( H_e \) are quadratic Hamiltonians of the Jordan–Wigner fermions in the sectors with odd and even fermion number respectively (\( H_o/e \) both commute with \( \prod_{l=1}^{L} \sigma^z_l \)). For finite chains, the ground state of the Ising Hamiltonian is an eigenstate of \( H_e \). However, in the thermodynamic limit, the \( \mathbb{Z}_2 \) symmetry of \( H \) is spontaneously broken in the ordered phase \( h_0 < 1 \), where the ground state is a linear superposition of the ground states of \( H_e \) and \( H_o \)

\[
|\Psi(0)\rangle = \frac{|B(0)\rangle_{NS} + |B(0)\rangle_R}{2}. \tag{45}
\]

Thus, for a quench starting in the ordered phase, free-fermion techniques cannot be straightforwardly applied to the calculation of correlation functions involving generic operators. However, here we are interested only in the expectation value of even operators \( \hat{O}_e \) characterized by

\[
\left[ \prod_{j=1}^{L} \sigma^z_j \right] \hat{O}_e \left[ \prod_{l=1}^{L} \sigma^z_l \right] = \hat{O}_e. \tag{46}
\]

Hence we have

\[
\lim_{L \to \infty} \langle \Psi(t) | \hat{O}_e | \Psi(t) \rangle = \lim_{L \to \infty} \frac{\text{NS} \langle B(t) | \hat{O}_e | B(t) \rangle_{NS} + \text{R} \langle B(t) | \hat{O}_e | B(t) \rangle_{R}}{2} = \lim_{L \to \infty} \frac{\text{NS} \langle B(t) | \hat{O}_e | B(t) \rangle_{NS}}{2}, \tag{47}
\]

Crucially, the expectation value \( \text{NS} \langle B(t) | \hat{O}_e | B(t) \rangle_{NS} \) can be evaluated using Wick’s theorem. In contrast to even operators, expectation values of odd operators

\[
\left[ \prod_{j=1}^{L} \sigma^z_j \right] \hat{O}_o \left[ \prod_{l=1}^{L} \sigma^z_l \right] = -\hat{O}_o \tag{48}
\]

are significantly more difficult to determine [62]. The particular case of interest, \( \rho^{xx}(\ell, t) \), involves an even operator (cf equation (43)), and by straightforward application of Wick’s theorem one obtains a representation as the Pfaffian of a \( 2\ell \times 2\ell \) antisymmetric matrix [62]

\[
\rho^{xx}(\ell, t) = \text{pf}(\Gamma), \tag{49}
\]

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Thus, the eigenvector of a real antisymmetric matrix, its eigenvalues are complex conjugate pairs \( \pm \), and column indices, instead of the difference as in Toeplitz matrices. Indeed, as \( \bar{\Gamma} \) is a Hankel matrix (Hankel matrices have elements which depend only on the sum of row and column indices). It is customary to introduce the (block) symbol of the matrix \( \bar{\Gamma} \) via Fourier transform as follows:

\[
\bar{\Gamma} = \begin{bmatrix}
\Gamma_0 & \Gamma_{-1} & \cdots & \Gamma_{1-\ell} \\
\Gamma_{1} & \Gamma_0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\Gamma_{\ell-1} & \cdots & \cdots & \Gamma_0
\end{bmatrix}, \quad \Gamma_\ell = \begin{pmatrix}
-f_\ell & g_\ell \\
-g_{-\ell} & f_\ell
\end{pmatrix}.
\]

The elements of this matrix are the fermionic correlations (The identity in the second line of equation (52) is a consequence of reflection symmetry, which indeed implies that correlations should be invariant under the transformation

\[
a_i^x \to i \left( \prod_{j=1}^{\ell} \sigma_i^j \right) a_{i+1-\ell}^y, \quad a_i^y \to -i \left( \prod_{j=1}^{\ell} \sigma_i^j \right) a_{i+1-\ell}^x.
\]

For example, the Dzyaloshinskii–Moriya interaction breaks this symmetry.)

\[
g_n \equiv i \langle a_i^x a_{i+n-1}^y \rangle, \quad f_n + i \delta_n \equiv i \langle a_i^y a_{i+n}^x \rangle = i \langle a_{i+n}^y a_i^x \rangle \quad \forall \ell.
\]

The matrix \( \bar{\Gamma} \) is a block Toeplitz matrix because its constituent 2 \( \times \) 2 blocks depend only on the difference between row and column indices. It is customary to introduce the (block) symbol of the matrix \( \bar{\Gamma} \) via Fourier transform as follows:

\[
\Gamma_\ell = \begin{pmatrix}
-f_\ell & g_\ell \\
-g_{-\ell} & f_\ell
\end{pmatrix} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ik\bar{\Gamma}(k)}, \quad \text{with} \quad \bar{\Gamma}(k) = \begin{pmatrix}
-f(k) & g(k) \\
g(-k) & f(k)
\end{pmatrix},
\]

where the functions \( f(k) \) and \( g(k) \) are

\[
f(k) = i \sin \Delta_k \sin(2\varepsilon_k(h,k)t), \quad g(k) = -e^{i\theta_k-ik[\cos \Delta_k - i \sin \Delta_k \cos(2\varepsilon_k(h,k)t)]}.
\]

The spectrum of block Toeplitz matrix \( \bar{\Gamma} \) is the same as that of an \( \ell \times \ell \) Toeplitz + Hankel matrix (Hankel matrices have elements which depend only on the sum of row and column indices, instead of the difference as in Toeplitz matrices). Indeed, as \( \bar{\Gamma} \) is a real antisymmetric matrix, its eigenvalues are complex conjugate pairs \( \pm i\lambda_j \). Given an eigenvector \( \vec{V} \) of \( \bar{\Gamma} \) with eigenvalue \( i\lambda \), we can define the vector \( \vec{w} \) with \( \ell \) components \( w_i = V_{2i-1} \) (since \( \Gamma_{-1} = \sigma_y \Gamma_0 \sigma_y \), taking the even components would change the sign of \( \lambda \)). Thus, the \( \ell \) vectors \( \vec{w}_i \) are solutions of the eigenvalue problem

\[
(iT \pm H)\vec{w}_j = \mp i\lambda_j \vec{w}_j, \quad \begin{cases}
T_{ln} = f_{l-n} \\
H_{ln} = g_{l+n-L-1}.
\end{cases}
\]

In particular, this means that

\[
|\text{pf}(\bar{\Gamma})| = |\det[iT \pm H]|.
\]

The sign of the Pfaffian can be fixed by observing that, for any given \( \ell \), both the determinant of the Toeplitz + Hankel matrix and the Pfaffian of the block Toeplitz matrix are polynomials in \( f_l \) and \( g_l \) of the same degree. Thus, the sign is independent of the actual values of the matrix elements and can be determined by considering the simplest case \( T = 0 \) (which occurs at the initial time and at asymptotically late times after the quench), obtaining

\[
\rho^{xx}(\ell, t) = \text{pf}(\bar{\Gamma}) = (-1)^{\ell(\ell-1)/2} \det(H + iT) = (-1)^{\ell(\ell-1)/2} \det W,
\]

doi:10.1088/1742-5468/2012/07/P07016
where $W = H + iT$. The matrix $\tilde{\Gamma}$ is real and antisymmetric, so its eigenvalues are complex conjugate pairs of the form $\pm i\lambda_j$ with $j = 1 \cdots \ell$. By construction, $\lambda_j \in [-1, 1]$ (see e.g. [82]), and they are also the eigenvalues of the matrix $W = H + iT$. Thus we have

$$ (\rho^{xx})^2 = \det \tilde{\Gamma} = \det W^2 = \prod_{j=1}^{\ell} \lambda_j^2. \quad (58) $$

The evaluation of the correlation function $\rho^{xx}(\ell, t)$ for large $\ell$ is then equivalent to the asymptotic evaluation of the determinant of a block Toeplitz matrix or the sum of a Toeplitz and a Hankel matrix. Such matrices have been the subject of intense study by mathematicians and physicists for more than a century, and standard, rigorous techniques for calculating their determinants such as Szegő’s lemma and the Fisher–Hartwig conjecture are available; see, e.g., [83, 84]. However, these methods have been specifically designed for the evaluation of determinants of matrices whose elements do not depend explicitly on the matrix size. This is in contrast to our case, where we are interested in the scaling limit $\ell, t \to \infty$ with finite ratio $\ell/t$. Under these circumstances, each element of the matrix $\Gamma$ in the scaling limit depends on a parameter (namely $t$) which is proportional to the matrix dimension $2\ell$. This precludes the application of the aforementioned techniques for the asymptotic evaluation of these determinants. The two exceptions are the $t = 0$ case, where we recover the known equilibrium results, and the limit $t = \infty$. In order to deal with large values of $t$ we developed a completely novel approach, which follows the one we proposed in [66] for the entanglement entropy and is based on a multi-dimensional stationary phase approximation.

In the following we derive a fairly general result valid for any block $2 \times 2$ block Toeplitz matrix $\Gamma$ with a symbol $\hat{t}(k)$ that can be cast in the form

$$ \hat{t}(k) = n_x(k)\sigma_{z}^{(k)} + \vec{n}_{\perp}(k) \cdot \vec{\sigma}^{(k)}e^{2i\epsilon(k)\tau_{z}^{(k)}}, \quad \vec{n}_{\perp}(k) \cdot \dot{x} = 0. \quad (59) $$

Here the time $t$ is the only parameter proportional to the matrix size $2\ell$, $n_x, n_{\perp}$ are fixed but otherwise arbitrary and $\sigma^{(k)}$ denotes a local rotation of the Pauli matrices

$$ \sigma_{\alpha}^{(k)} \sim e^{i\hat{w}(k)\cdot \vec{\sigma}} \sigma_{\alpha} e^{-i\hat{w}(k)\cdot \vec{\sigma}}. \quad (60) $$

Our particular case of interest (53) corresponds to having $n_x^2 + |\vec{n}_{\perp}|^2 = 1$.

We first consider $\text{Tr}\Gamma^{2n}$ for positive integer $n$. We note that the traces of odd powers of $\Gamma$ vanish, because $\Gamma$ is a real antisymmetric matrix ($\Gamma$ is diagonalizable and for each non-zero eigenvalue there is another one with opposite sign). Our main result is that

$$ \lim_{\ell, t \to \infty \atop \ell/t \text{ fixed}} \frac{\text{Tr}[\Gamma^{2n}]}{2\ell} = \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \max\left(1 - 2|\epsilon'(k_0)|\frac{t}{\ell}, 0\right) n_x^2(k_0)^2 + |\vec{n}_{\perp}(k_0)|^2)^n \left(1 - 2|\epsilon'(k_0)|\frac{t}{\ell}, 1\right) n_x^2(k_0)^{2n}. \quad (61) $$

Given the result (61) it is possible to infer the asymptotic behaviour of more complicated quantities such as

$$ \text{Tr}\mathcal{F}(\Gamma^2) = \sum_n \mathcal{F}_n \text{Tr}[\Gamma^{2n}], \quad (62) $$

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where $\mathcal{F}(z)$ is an analytic function with power series expansion $\mathcal{F}(z) = \sum_n \mathcal{F}_n z^n$ around $z = 0$. Using equation (61) and interchanging the order of sum and integration, we have

$$
\lim_{t,\ell \to \infty} \frac{\text{Tr}[\mathcal{F}(\Gamma^2)]}{2\ell} = \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \max \left( 1 - 2|\varepsilon'(k_0)| \frac{t}{\ell}, 0 \right) \mathcal{F}(n_x(k_0)^2 + |\vec{n}_\perp(k_0)|^2)
+ \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \min \left( 2|\varepsilon'(k_0)| \frac{t}{\ell}, 1 \right) \mathcal{F}(n_x(k_0)^2). \quad (63)
$$

Clearly this approach is fully justified only as long as all eigenvalues of $\Gamma$ fall within the radius of convergence of the expansion of the function $\mathcal{F}(z)$ around $z = 0$. For example, in the case of the entanglement entropy, we have $\mathcal{F}(z) = -((1 + iz)/2) \ln((1 + iz)/2)$ [66] and since the eigenvalues of $\Gamma$ are of the form $\pm i\lambda_j$ with $\lambda_j \in [-1, 1]$ (63) holds (further generalizations to the entanglement of two blocks have also been considered [85]).

In the case at hand we have

$$
(\rho^{tx})^2 = \det \Gamma = \exp[\text{Tr} \ln \Gamma], \quad \text{i.e.} \quad \ln(\rho^{tx})^2 = \frac{1}{2} \text{Tr} \ln \Gamma^2. \quad (64)
$$

In order to use (62) and (63) we are therefore led to consider the function $\mathcal{F}(z) = \frac{1}{2} \ln z$. The latter has a branch point at $z = 0$ and the previous approach appears not to be applicable. In order to circumvent this problem we employ a power series expansion of the logarithm around $z = 1$

$$
\ln(\rho^{tx})^2 = \frac{1}{2} \text{Tr} \ln \Gamma^2 = \frac{1}{2} \text{Tr} \ln[1 + (\Gamma^2 - 1)] = \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \text{Tr}[(\Gamma^2 - 1)^m]. \quad (65)
$$

Application of (63) to the function $\mathcal{F}(z) = (z - 1)^m$ then results in

$$
\lim_{t,\ell \to \infty} \frac{\text{Tr}[(\Gamma^2 - 1)^m]}{2\ell} = \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \max \left( 1 - 2|\varepsilon'(k_0)| \frac{t}{\ell}, 0 \right) (n_x(k_0)^2 + |\vec{n}_\perp(k_0)|^2 - 1)^m
+ \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \min \left( 2|\varepsilon'(k_0)| \frac{t}{\ell}, 1 \right) (n_x(k_0)^2 - 1)^m. \quad (66)
$$

Finally, we can carry out the sum over $m$

$$
\lim_{t,\ell \to \infty} \frac{\text{Tr}[\ln \Gamma^2]}{2\ell} = \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \max \left( 1 - 2|\varepsilon'(k_0)| \frac{t}{\ell}, 0 \right) \ln(n_x(k_0)^2 + |\vec{n}_\perp(k_0)|^2)
+ \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \min \left( 2|\varepsilon'(k_0)| \frac{t}{\ell}, 1 \right) \ln(n_x(k_0)^2), \quad (67)
$$

which is exactly the same result as we would have obtained applying (63) directly to the function $\mathcal{F}(z) = \frac{1}{2} \ln z$. The reason for this lies in the simple algebraic dependence of (61) on $n$. The specialization of (67) to the longitudinal correlator $\rho^{tx}$ is now straightforward.

The symbol $\Gamma(k)$ of the block Toeplitz matrix $\Gamma$ can be cast in the form (59) with

$$
\hat{w}(k) = [\theta(k) - k] \hat{x}, \quad n_x^2(k) = \cos^2 \Delta_k, \quad |\vec{n}_\perp(k)|^2 = \sin^2 \Delta_k. \quad (68)
$$
Equation (67) then gives the asymptotic behaviour

$$\lim_{t, \ell \to \infty} \frac{1}{\ell} \ln |\rho^{xx}(\ell, t)| = \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \min\left(2|\epsilon'(k_0)|, 1\right) \ln |\cos \Delta k_0|.$$  \hspace{1cm} (69)

This is a rewriting of (19) and represents one of our main results.

In the reasoning leading up to (67) we have assumed that the eigenvalues of $\Gamma^2 - 1$ lie within the unit circle so that we can expand the logarithm in a power series. This is equivalent to the requirement that the eigenvalues $\lambda_j$ of $\Gamma$ are such that $0 < \lambda_j^2 \leq 1$. More precisely, if there exists an $x_0 > 0$ such that for all $\ell$ the eigenvalues $\lambda_j$ of $\Gamma$ fulfil

$$x_0 < \lambda_j^2 \leq 1,$$  \hspace{1cm} (70)

then the results (67) and (69) hold. In some cases of interest we find that, even though $\lambda_j$ are different from zero for any finite $\ell$, one or several eigenvalues approach zero in the limit $\ell \to \infty$. In these circumstances the steps leading up to (67) and (69) are no longer justified. However, even then (69) can provide useful information about the asymptotics of $\rho^{xx}(\ell, t)$. On general grounds\footnote{This form has been observed in various examples, see e.g. [22, 34, 66, 61, 77, 36, 56], even with oscillating factors.} we expect the latter to be of the form

$$\lim_{t, \ell \to \infty} \frac{1}{\ell} \ln |\rho^{xx}(\ell, t)| = -\ell \mu(t/\ell) - \alpha \ln(\ell) + c(t/\ell) + \cdots,$$  \hspace{1cm} (71)

and the question is under what circumstances $\mu(t/\ell)$ is given by (69). Under the assumption that the general structure (71) holds, this depends on the number $N_0$ of eigenvalues $\lambda_j$ that approach zero for $\ell \to \infty$, as well as on how quickly they tend to zero. If $N_0$ is finite and the corresponding eigenvalues approach zero sufficiently slowly with $\ell$, (69) will still be applicable. As an example, let us consider the case where only a single pair $\pm i \lambda_1$ of eigenvalues approaches zero for $\ell \to \infty$ in such a way that for large $\ell$ we have $|\lambda_1| \sim \ell^{-\beta}$. Since $\rho^{xx}(\ell, t) = \prod_j \lambda_j^{2j}$, the eigenvalue pair will contribute to the subleading logarithmic term in (71), but will leave the function $\mu(t/\ell)$ unchanged. On the other hand, if our pair of eigenvalues were to approach zero exponentially fast $|\lambda_1| \sim e^{-\ell \mu(t/\ell)}$, then it would contribute additively to the function $\mu(t/\ell)$ and (69) would cease to hold.

As far as our quench problem is concerned, we do not have a criterion that would establish a priori whether eigenvalues exponentially close to zero will be present. We find that for quenches within the ferromagnetic phase they are always absent. In the other cases, the picture emerging from numerical studies of the spectrum of $\bar{\Gamma}$ suggests that only for quenches starting in the disordered phase do eigenvalues approach zero exponentially fast, and then only a single pair $\pm i \lambda_0$ does so. Interestingly, we find that the contribution of all other eigenvalues is again captured by (67). At present we are not able to determine $\lambda_0$ analytically.

### 3.1. Proof of the formula for the trace of integer powers of $\Gamma$

The first step of our calculation is to derive an appropriate integral representation for $\text{Tr} \Gamma^{2n}$ by trading matrix multiplications for additional integrations. The basic idea is
easily explained for a product of two \( \Gamma \) matrices. Each block element is given by
\[
\Gamma_{lm} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i(l-n)k} \hat{t}(k),
\]
where the \( 2 \times 2 \) block \( \hat{t}(k) \) is of the form (59). Hence the block matrix elements of \( \Gamma^2 \) can be written as
\[
(\Gamma^2)_{lm} = \sum_{m=1}^{\ell} \Gamma_{lm} \Gamma_{mn} = \int_{[-\pi,\pi]^2} \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} e^{i(k_1-nk_2)} \hat{t}(k_1) \cdot \hat{t}(k_2) \sum_{m=1}^{\ell} e^{im(k_1-k_2)}.
\]
The sum over \( m \) can now be replaced by an integral using
\[
e^{-i(\ell+1)k/2} \sum_{m=1}^{\ell} e^{imk} = \frac{\ell}{2} \int_{-1}^{1} d\xi \frac{k}{2\sin(k/2)} e^{i\ell \xi k/2}.
\]
The generalization to \( \Gamma^{2n} \) is straightforward, and replacing the trace in an analogous way we obtain the following integral representation:
\[
\text{Tr}[\Gamma^{2n}] = \left( \frac{\ell}{2} \right)^{2n} \int_{[-\pi,\pi]^n} \frac{d^{2n}k}{(2\pi)^{2n}} \int_{[-1,1]^{2n}} d^{2n}\xi C(\{k\}) e^{i\sum_{j=0}^{2n-1} \xi_j(k_{j+1}-k_j)/2} F(\{k\}).
\]
Here the functions appearing under the integrals are
\[
C(\{k\}) = \prod_{j=0}^{2n-1} \frac{k_j - k_{j-1}}{2\sin[(k_j - k_{j-1})/2]},
\]
\[
F(\{k\}) = \text{Tr} \left[ \prod_{i=0}^{2n-1} n_x(k_i) \sigma_x^{(k)} + \vec{n}_\perp(k_i) \cdot \sigma^{(k)} e^{2i\epsilon_i \sigma_z^{(k)}} \right],
\]
the trace is over the remaining \( 2 \times 2 \) blocks and \( \epsilon_i = \epsilon(k_i) \). Performing the change of variables
\[
\zeta_0 = \xi_1, \quad \zeta_i = \xi_{i+1} - \xi_i \quad i \in [1, n-1].
\]
Equation (75) can be rewritten in the form
\[
\text{Tr}[\Gamma^{2n}] = \left( \frac{\ell}{2} \right)^{2n} \int_{[-\pi,\pi]^n} \frac{d^{2n}k}{(2\pi)^{2n}} \int_{R_\zeta} \frac{d^{2n}\xi C(\{k\})e^{-i\sum_{j=1}^{2n-1} \xi_j(k_j-k_0)/2} F(\{k\})}.\]
Here the domain of integration \( R_\zeta \) is determined by the conditions
\[
R_\zeta : -1 \leq \sum_{j=0}^{k-1} \zeta_j \leq 1 \quad \forall k \in [1, 2n].
\]
As the integrand in equation (79) is independent of \( \zeta_0 \), we can carry out the \( \zeta_0 \) integration, which gives
\[
\text{Tr}[\Gamma^{2n}] = \left( \frac{\ell}{2} \right)^{2n} \int_{[-\pi,\pi]^n} \frac{d^{2n}k}{(2\pi)^{2n}} \int d^{2n-1}\zeta \mu(\{\zeta\}) C(\{k\}) e^{-i\sum_{j=1}^{2n-1} \zeta_j(k_j-k_0)/2} F(\{k\}).
\]
Here the function $\mu(\{\zeta\})$ is the measure of the domain of $\zeta_0$ under the constraints (80)

$$\mu(\{\zeta\}) = \max \left[ 0, \min_{j \in \{0,2n-1\}} \left[ 1 - \sum_{k=1}^{j} \zeta_k \right] + \min_{j \in \{0,2n-1\}} \left[ 1 + \sum_{k=1}^{j} \zeta_k \right] \right]. \quad (82)$$

It can be shown that $\mu(\{\zeta\})$ is symmetric with respect to an arbitrary permutation of the variables $\zeta$. Since we are interested in the behaviour for $\ell \gg 1$ and the phase in the integral is proportional to the large parameter $\ell$, the asymptotic behaviour can be obtained using a multi-dimensional stationary phase approximation. The main idea behind this method is that the leading contribution to the integral arises from the neighbourhoods of the points in which the phase is stationary. As the symbol is independent of the integration variables $\zeta$, the stationarity of these variables implies

$$k_j \approx k_0, \quad j = 1, \ldots, 2n - 1. \quad (83)$$

We may replace any $k_j$ with $k_0$ everywhere except in rapidly oscillating terms such as the $e^{2i\epsilon t}$ factors in the symbol. We call this the localization rule. This rule allows us in particular to drop the factor $C(\{\zeta\})$, cf (76), as it is equal to one at the stationary points $C(\{\zeta\}) \approx 1$

$$\text{Tr}[T^{2n}] \approx \left( \frac{\ell}{2} \right)^{2n} \int_{[-\pi,\pi]^{2n}} \frac{d^{2n}k}{(2\pi)^{2n}} \int d^{2n-1} \zeta \mu(\{\zeta\}) e^{-i \sum_{j=1}^{2n-1} \zeta_j (k_j - k_0)/2} F(\{\zeta\}). \quad (84)$$

The localization rule furthermore allows us to substitute the integration variables in $n_x$ and $n_\perp$ with $k_0$, and to remove the local rotation in $f(\{\zeta\})$, i.e. under the integral we may replace

$$F(\{\zeta\}) \rightarrow G(\{\zeta\}) \equiv \text{Tr} \prod_{i=0}^{2n-1} \left[ n_x(k_0) \sigma_x + n_\perp(k_0) \cdot \vec{\sigma} e^{2i\epsilon t \sigma_x} \right]. \quad (85)$$

As is shown in appendix C, the product under the trace can be rewritten as

$$\sum_{p=0}^{2n} n_x^{2n-p} |n_\perp|^p (i \sigma_z)^p \sum_{1 \leq j_1 < j_2 < \ldots < j_p \leq 2n} (-1)^{\sum_{m=1}^{2n} j_m} e^{2i \sigma_x \sum_{m=1}^{p} (-1)^{p-m} \epsilon_{m-1}}. \quad (86)$$

Each term in the sums over $j_1 \ldots j_p$ gives the same contribution to the integral (84) as can be shown by changing integration variables

$$k'_1 = k_{j_1}, k'_2 = k_{j_2}, \ldots, k'_p = k_p, k'_l = k_l \quad \text{for all} \quad l \notin \{j_1, \ldots, j_p\},
\zeta'_1 = \zeta_{j_1}, \zeta'_2 = \zeta_{j_2}, \ldots, \zeta'_p = \zeta_p, \zeta'_l = \zeta_l \quad \text{for all} \quad l \notin \{j_1, \ldots, j_p\}, \quad (87)$$

and then invoking the invariance of $\mu(\{\zeta\})$ in equation (82) under any permutation of its variables. When evaluating the trace in (86) we may therefore replace $j_m \rightarrow m$. We call this the contraction rule. Noting that only the terms with even $p$ have a non-vanishing trace (because $\text{Tr} \sigma_x e^{3i \sigma_x} = 0$) and then applying the contraction rule allows us to replace

$$G(\{\zeta\}) \rightarrow \sum_{p=0}^{n} \binom{n}{p} n_x^{2n-2p} n_\perp^{2p} 2 \cos \left( 2t \sum_{m=1}^{2p} (-1)^{m} \epsilon_{m-1} \right). \quad (88)$$

Here the binomial takes into account the number of terms giving identical contributions to the integral and we have used that $\text{Tr} e^{3i \sigma_x} = 2 \cos a$. Inserting this expression into (84)

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we arrive at
\[
\text{Tr}[\Gamma^{2n}] = \ell \left(\frac{\ell}{2}\right)^{2n-1} \sum_{p=0}^{n} \left(\begin{array}{c} n \\ p \end{array}\right) \int_{[-\pi,\pi]^{2n}} \frac{d^{2n} k}{(2\pi)^{2n}} \int d^{2n-1} \zeta \mu(\{\zeta\}) \\
\times n_{x}^{2n-2k_{e}^{2}} n_{\perp}^{2k_{e}} e^{-i\pi \sum_{j=1}^{2n-1} \zeta_{j}(k_{j}-k_{0})/2 + 2i\pi \sum_{j=0}^{2n-1} (-1)^{j} \varepsilon_{j}},
\] (89)
where we replaced the cosine with a complex exponential using the symmetry of the integral under the change of variables \( \vec{k} \to -\vec{k} \) and \( \vec{\zeta} \to -\vec{\zeta} \).

We are now in a position to employ a stationary phase approximation in order to extract an exact asymptotic result in the limit of large \( \ell \). As the phase of the integral is stationary along a one-dimensional smooth curve, application of the stationary phase method is not a simple matter. For a two-dimensional integral whose phase is stationary on a one-dimensional variety, the solution can be found in [86]. Higher-dimensional integrals with a curve of stationary points are only partially treated in [87], where it is demonstrated that it is possible to isolate the integration in \( k_{0} \) and perform a standard multi-dimensional stationary phase approximation for the remaining integrals. To apply this idea to our case, we rewrite equation (89) as
\[
\text{Tr}[\Gamma^{2n}] = \ell \left(\frac{\ell}{2}\right)^{2n-1} \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array}\right) \int_{-\pi}^{\pi} \frac{dk_{0}}{2\pi} n_{x}(k_{0})^{2n-2} n_{\perp}(k_{0})^{2l} \Lambda_{n,l}(k_{0}),
\] (90)
where the functions \( \Lambda_{n,l} \) are given by
\[
\Lambda_{n,l}(k_{0}) = \int_{[-\pi,\pi]^{2n-1}} \frac{d^{2n-1} k}{(2\pi)^{2n-1}} \int d^{2n-1} \zeta \mu(\{\zeta\}) e^{-i\pi \sum_{j=1}^{2n-1} \zeta_{j}(k_{j}-k_{0})/2 + 2i\pi \sum_{j=0}^{2n-1} (-1)^{j} \varepsilon_{j}}. 
\] (91)
We now use the standard multi-dimensional phase approximation to evaluate \( \Lambda_{n,l}(k_{0}) \).

The general result for the large-\( \ell \) asymptotic behaviour of a multi-dimensional integral over rapidly oscillating functions, whose phase is stationary at an isolated point \( \vec{x}_{0} \) detached from the boundary, is [86]
\[
\int_{D} d^{N} x p(\vec{x}) e^{iQ(\vec{x})} = \left(\frac{2\pi}{\ell}\right)^{N/2} p(\vec{x}_{0}) |\text{det} A|^{-1/2} \exp \left[ i\pi q(\vec{x}_{0}) + \frac{i\pi\sigma_{A}}{4} \right],
\] (92)
where \( A_{ij} = \partial_{x_{j}}\partial_{x_{i}} q|_{\vec{x}_{0}} \) is the Hessian matrix of \( f(\vec{x}) \) evaluated in \( \vec{x}_{0} \), and \( \sigma_{A} \) is the signature of the matrix \( A \), i.e. the difference between the number of the positive and negative eigenvalues. The stationarity conditions for the phases of the integrands in \( \Lambda_{n,l}(k_{0}) \) are
\[
\vec{k}_{j} = k_{0}, \quad j = 1, \ldots, 2n - 1,
\vec{\zeta}_{j} = \frac{4}{\ell} (-1)^{j} \varepsilon_{j}, \quad j = 1, \ldots, 2l - 1,
\vec{\zeta}_{j} = 0, \quad j = 2l, \ldots, 2n - 1.
\] (93)
Using an ordering of integration variables where the \( \zeta \)s are placed before the \( k \)s, the Hessian is of the form
\[
A = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & M \end{pmatrix}.
\] (94)

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Hence the eigenvalues $a^{(i)}_{\pm}$ of $A$ are related to the eigenvalues $\mu_i$ of the matrix $M$ by $a^{(i)}_{\pm} = (\mu_i \pm \sqrt{\mu_i^2 + 4})/4$. The signature of $A$ is $\sigma_A = 0$ and its determinant is $\det A = -4^{1-2n}$. At the stationary point the phase in the integral vanishes, because there is an even number of $\varepsilon$s with alternating signs. Finally, the value of $\mu(\{\zeta\})$ at the stationary point is found to be

$$
\mu(\{\tilde{\zeta}\}) = \begin{cases} 2 \max[0, l - 2|\varepsilon'(k_0)|t] & l \neq 0, \\ 2 & l = 0. \end{cases} \quad (95)
$$

Putting everything together, the stationary phase approximation gives the following result for the functions $\Lambda_{n,l}(k_0)$

$$
\Lambda_{n,l}(k_0) = \left(\frac{2}{\ell}\right)^{2n-1} \mu(\{\tilde{\zeta}\}). \quad (96)
$$

Inserting (96) into (90) then gives

$$
\text{Tr}[\Gamma^{2n}] = \ell \sum_{l=0}^{\ell} \left(\frac{n}{l}\right) \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} n_x(k_0)^{2n-2l} n_{\perp}(k_0)^{2l} \mu(\{\tilde{\zeta}\}), \quad (97)
$$

and substituting the value (95) for $\mu(\{\tilde{\zeta}\})$ we arrive at

$$
\text{Tr}[\Gamma^{2n}] = 2 \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} \max(\ell - 2|\varepsilon'(k_0)|t, 0)(n_x(k_0)^2 + |\vec{n}_{\perp}(k_0)|^2)n^{n} 
+ 2 \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} (\ell - \max(0, \ell - 2|\varepsilon'(k_0)|t)) n_x(k_0)^{2n}. \quad (98)
$$

This is equivalent to (61). A less general version of this result was previously presented in [66].

### 3.2. On the applicability of stationary phase method

Equation (69) is our main result obtained with the determinant approach. It is based on a multi-dimensional stationary phase approximation for $\text{Tr}[\tilde{\Gamma}^{2n}]$. The result (98) is then used in combination with a series expansion in order to obtain the asymptotic behaviour of $\det \tilde{\Gamma}$, which in turn gives the square of the longitudinal correlation function $\rho^{xx}$. As we have already discussed, this series expansion is possible only if there is no eigenvalue of the matrix $\Gamma$ that approaches zero in the large $\ell$ limit. While this restriction appears to be quite simple, we do not have an analytic method that allows us to predict for what kind of quenches zero eigenvalues exist in the $\ell \to \infty$ limit. To address this question we therefore have carried out numerical studies of the spectrum of $W$ (which is related to the spectrum of $\Gamma$ by (58)) for different quenches. We have to distinguish between four cases: (1) FM $\rightarrow$ FM; (2) PM $\rightarrow$ PM; (3) PM $\rightarrow$ FM and (4) FM $\rightarrow$ PM, where FM and PM denote the ferromagnetic and paramagnetic phases respectively. In figure 10 we report the (absolute values of) smallest eigenvalues of the matrix $W$ for particular examples of the four cases. Extensive numerical studies suggest that the spectra are similar for all quenches of a given type, i.e. (1)–(4). The qualitative features emerging from figure 10 can be summarized as the following.

1. FM $\rightarrow$ FM: all eigenvalues are well separated from zero.
Figure 10. Typical absolute values of the low lying eigenvalues of the matrix $W$ in the four kinds of quench. In (a) and (b) $\ell = 30$, while in (c) and (d) $\ell = 60$.

(a) For a quench within the ordered phase, all eigenvalues are well separated from zero. Before the Fermi time (up to finite distance corrections to $t_F$) the eigenvalues are almost degenerate in pairs. (b) For a quench within the disordered phase, a single eigenvalue is very close to the real axis (exponentially close for $t < t_F$ and as a power law for $t > t_F$). (c) Quench from the disordered to the ordered phase. For $t < t_F$ a single eigenvalue is exponentially close to the real axis. For $t > t_F$ the distance of the smaller eigenvalue from the real axis scales like a power law. (d) Quench from the ordered to the disordered phase. All eigenvalues are doubly degenerate. All the smallest eigenvalues approach zero as a power in $\ell$ for any time. For $t < t_F$ the smallest eigenvalue periodically crosses zero.

(2) $\text{PM} \rightarrow \text{PM}$: a single eigenvalue of $W$ is close to zero. For $t < t_F$ it approaches zero exponentially fast in $\ell$, while for $t > t_F$ it tends to zero only as a power law.

(3) $\text{PM} \rightarrow \text{FM}$: for both $t < t_F$ and $t > t_F$ several eigenvalues scale to zero like power laws in $\ell$. For $t < t_F$ the smallest eigenvalue approaches zero exponentially fast in $\ell$.

(4) $\text{FM} \rightarrow \text{PM}$: several eigenvalues tend to zero in a power-law fashion in $\ell$ for both $t > t_F$ and $t < t_F$. For $t < t_F$ the smallest eigenvalue crosses zero periodically.

The numerical analysis suggests that (69) can be applied for all quenches of type (1) for large $t$ and all corrections to (69) are small. In case (4) our result (69) still gives the dominant behaviour, but there are additional important oscillating power-law corrections.
to $\rho^{xx}(\ell, t)$ that we have not been able to determine analytically. For quenches originating in the paramagnetic phase, (69) gives the dominant contribution only if $t > t_F$, but there are important power-law corrections to $\rho^{xx}(\ell, t)$ beyond the accuracy of our analysis.

These properties of the spectrum suggest that, for quenches starting from the disordered phase, the product of all eigenvalues except the only one which is close to zero (i.e. $\rho^{xx}/\lambda_0$) should be given by equation (69). Figure 11 shows that this is indeed the case for quenches to both disordered and ordered phases for any $t$. Thus, if we were able to obtain a prediction for the smallest eigenvalue, we would completely characterize these quenches as well.

Finally, we note that the gross features of the spectrum of $W$ (and hence $\bar{\Gamma}$) appear to be related to the winding number (around zero) of the (block) symbol (53). We observe that the winding number of $g(k)$ (which is only a part of $\bar{\Gamma}(k)$) depends on the quench parameters in the following way:

1) if $|h|, |h_0| < 1$, the winding number is zero;
2) if $|h_0|, |h| > 1$, the winding number is equal to unity;
3) if $|h_0| < 1$ and $|h| > 1$, the winding number oscillates with $t$ between zero and unity.

In particular it remains zero for any $t > t_F$.
4) If $|h_0| > 1$ and $|h| < 1$, the winding number oscillates with $t$ between $-1$ and $1$.

We see that the winding number vanishes when there is no eigenvalue exponentially close to zero, but the reverse does not hold. For (block) Toeplitz matrices with $\ell$ independent elements the generalization of the Szego lemma to symbols with non-vanishing winding number is complicated and not known in general (see e.g. [83, 84]).

3.2.1. A closer look at quenches from the ordered to the disordered phase. For quenches from the ordered to the disordered phase, the last panel of figure 10 shows that all
eigenvalues of $W$ move coherently in time. In particular, the eigenvalues closest to zero oscillate with the same frequency about zero until the Fermi time $t_F$. After $t_F$ they do not cross zero anymore and the smallest one remains separated from zero as a power law in $\ell$. These qualitative features of the spectrum would suggest that for $t < t_F$, $\rho_{xx}(t, \ell)$ should oscillate in time with the same frequency as the smallest eigenvalues. While it is not easy to put this on solid ground, we have been able to identify it as the frequency with which the winding number of $q(k)$ changes. We indeed find that the frequency is determined by the mode $k_0$ such that $\cos \Delta_{k_0} = 0$. Thus for $t < t_F$ we heuristically predict

$$\langle \sigma_1^x(t) \sigma_{\ell+1}^x(t) \rangle \sim C_\ell^x [1 + \cos(2\epsilon_{k_0} t + \alpha)] \exp \left[ \int_{-\pi}^{\pi} \frac{dk_0}{2\pi} 2|\epsilon'_k| t \ln |\cos \Delta_k| \right] t < t_F,$$

where $\alpha$ is an undetermined phase shift. The comparison of this prediction with the direct computation of the determinant has already been shown in figure 6. For times larger than $t_F$, equation (69) matches well the evolution of the two-point function, up to oscillatory decaying corrections (see figure 6), which is consistent with the presence of one eigenvalue exponentially close to zero.

4. Form factor approach

While the procedure set out in section 3 can be generalized to any model with a free-fermion representation [88], it cannot be applied to quenches in interacting integrable models. In order to overcome these limitations we have generalized the form factor approach to correlation functions in integrable quantum field theories [72]–[76] to quantum quenches. A characteristic feature of integrable models is the existence of a basis of scattering states of elementary excitations, which are simultaneous exact eigenstates of the Hamiltonian and the momentum operator. These states can be characterized by $n$ momenta $k_1, \ldots, k_n$ and in general also other quantum numbers $a_1, \ldots, a_n$

$$H|k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n} = \left[ \sum_{j=1}^n \varepsilon_{a_j}(k_j) \right] |k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n},$$

$$P|k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n} = \left[ \sum_{j=1}^n k_j \right] |k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n}.$$  

Ground state correlators of local operators $\mathcal{O}$ can be expressed in a Lehmann representation

$$\langle 0| \mathcal{O}(t, x) \mathcal{O}^\dagger(0, 0) |0 \rangle = \sum_{n=0}^\infty \sum_{k_1, \ldots, k_n}_{a_1, \ldots, a_n} \langle 0| \mathcal{O}(0, 0) |k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n}^\dagger |k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n} |0 \rangle^2 e^{-i \sum_{j=1}^n \varepsilon_{a_j}(k_j) t - k_j x},$$

where $|0\rangle$ denotes the ground state and in our notations corresponds to particle number zero. Using the known expressions for the form factors $\langle 0| \mathcal{O}(0, 0) |k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n}^\dagger |k_1, \ldots, k_n\rangle_{a_1, \ldots, a_n}$ the spectral sum can be calculated to very high accuracy (for large $t$ and $x$) by taking into account only terms involving a small number $n$ of particles. Recently, the form factor approach has been generalized to low temperature correlation functions [89, 90] (see
also \[91, 92\] for related work)

\[
\frac{\text{Tr} \left[ e^{-\beta H} \mathcal{O}(t,x) \mathcal{O}^\dagger(0,0) \right]}{\text{Tr} \left[ e^{-\beta H} \right]}.
\]

(102)

The numerator in (102) can now be expanded in a Lehmann representation as follows:

\[
\sum_{n,m=0}^\infty \sum_{a_1, \ldots, a_n \ b_1, \ldots, b_m} \left| b_{m,b_1} \langle p_m, \ldots, p_1 | \mathcal{O}(0,0) | k_1, \ldots, k_n \rangle_{a_1, \ldots, a_n} \right|^2 \times e^{-\beta \sum_{l=1}^m \varepsilon_{b_l}(p_l)} e^{-\sum_{j=1}^n \varepsilon_{a_j}(k_j)t-k_jx} e^{i \sum_{l=1}^m \varepsilon_{b_l}(p_l)t-p_lx}.
\]

(103)

Using exact results for the form factors \[b_{m,b_1} \langle p_m, \ldots, p_1 | \mathcal{O}(0,0) | k_1, \ldots, k_n \rangle_{a_1, \ldots, a_n} [72, 93]\], one needs to sum an infinite number of terms in the Lehmann expansion in order to obtain the correlation function at late times and large distances. Such a resummation is possible at low temperatures \[T \ll \Delta\], where \[\Delta\] is the spectral gap, because the density of excitations in the state of thermal equilibrium constitutes a natural small parameter in this case [89].

The situation after a quantum quench bears many similarities to the finite temperature case. When dealing with a ‘small’ quench in a gapped theory there exists a regime in which the density of excitations (of the post-quench Hamiltonian) in the initial state is small. It is then natural to use this small parameter in order to carry out a low-density expansion for the observables of interest. Let us consider a quantum quench \[H_0 \rightarrow H\], where \(\mathcal{H}\) describes an integrable scattering theory: we prepare the system in the ground state \(|\Psi_0 \rangle\) of \(H_0\) and then consider time evolution by \(H\). We are interested in observables such as

\[
\frac{\langle \Psi_0(t) | \mathcal{O}(x) \mathcal{O}^\dagger(0) | \Psi_0(t) \rangle}{\langle \Psi_0 | \Psi_0 \rangle}.
\]

(104)

Any translationally invariant initial state can be expressed in terms of the eigenstates of \(H\) as

\[
|\Psi_0(t) \rangle = \sum_{n=0}^\infty \sum_{k_1, \ldots, k_n \ a_1, \ldots, a_n} e^{-\sum_{j=1}^n \varepsilon_{a_j}(k_j)t} F_n(\{k_j, a_j\}) |k_1, \ldots, k_n \rangle_{a_1, \ldots, a_n},
\]

\[
F_n(\{k_j, a_j\}) = \sum_{a_{n-1}, a_1} \langle k_n, \ldots, k_1 | \Psi(0) \rangle.
\]

(105)

A particular class of initial states is given by

\[
|\Psi_0(0) \rangle = \sum_{n=0}^\infty \frac{i^n}{n!} \sum_{p_1, \ldots, p_n} \left[ \prod_{j=1}^n K^{a_j b_j}(p_j) \right] | -p_1, p_1, \ldots, -p_n, p_n \rangle_{a_1, b_1, \ldots, a_n, b_n},
\]

(106)

where summation over the indices \(a_j, b_k\) is implied. As is shown in appendix B, quantum quenches of the transverse field in the TFIM automatically lead to initial states of the form (106). For general integrable quantum field theories, (106) are known to describe boundary states [94] and correspond to situations where the initial state can be viewed as a boundary condition (in Euclidean space) that is compatible with quantum integrability. Given an
initial state of the form (106), the basic idea is to employ a Lehmann representation in terms of the exact eigenstates of the post-quench Hamiltonian $H$. For a two-point function we have

$$\langle \Psi_0(t) | \mathcal{O}(x) \mathcal{O}^\dagger(0) | \Psi_0(t) \rangle = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n!m!} \prod_{j=1}^{n} K^{a_j b_j} (p_j) \prod_{l=1}^{m} K^{c_l d_l} (k_l) \right)^{*} \times \sum_{s=0}^{\infty} \sum_{q_1, \ldots, q_s} \frac{1}{s!} c_{m, d_m, \ldots, c_1, d_1} \left\langle k_m, -k_m, \ldots, k_1 - k_1 | \mathcal{O}(t, x) | q_1, \ldots, q_s \right\rangle f_1, \ldots, f_s \times f_s, \ldots, f_1 \left\langle q_s, \ldots, q_1 | \mathcal{O}^\dagger(t, 0) | -p_1, p_1, \ldots, -p_n, p_n \right\rangle a_1 b_1, \ldots, a_n b_n. \quad (107)$$

The form factors entering this expression are known, for both the TFIM and a variety of integrable quantum field theories [72, 93]. The normalization itself has the following Lehmann representation:

$$\langle \Psi_0(t) | \Psi_0(t) \rangle = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \sum_{p_1, \ldots, p_n} \prod_{j=1}^{n} K^{a_j b_j} (p_j) \left[ K^{c_j d_j} (k_j) \right]^* \times c_{m, d_m, \ldots, c_1, d_1} \left\langle k_n, -k_n, \ldots, k_1 - k_1 \right| -p_1, p_1, \ldots, -p_n, p_n \left\rangle a_1 b_1, \ldots, a_n b_n. \quad (108)$$

In order to extract the large-time and distance asymptotics of (107) it is necessary to sum an infinite number of terms. In analogy to the finite-temperature case we proceed as follows.

- Consider all contributions at a given order in the functions $K^{ab}(q)$.
- At each order we encounter two types of divergence: (i) infinite-volume divergences that arise because in the thermodynamic limit the scattering states are normalized to delta functions—these singular contributions are ultimately compensated by corresponding divergences in the normalization (108) and are dealt with by an appropriate subtraction procedure, cf [89] for the finite temperature case; (ii) ‘infrared’ divergences, i.e. contributions that diverge as $t \to \infty$ or $x \to \infty$.
- We isolate the terms with the strongest infrared divergences at each order in $K^{ab}(q)$ and then sum up all these contributions; cf. [95] for a similar calculation in the finite-temperature case.

### 4.1. Finite-volume form factors for the TFIM

As is clear from the above discussion, the basic building blocks are the form factors. For the case of the transverse field Ising chain these naturally depend on the precise choice of basis of free-fermion scattering states $| k_1, \ldots, k_m \rangle_{R, NS}$. For a particular such choice, the finite-volume form factors of the spin operators $\sigma^x_l$ have been determined in [96]–[98]. The non-vanishing matrix elements are

$$\langle q_1, \ldots, q_{2n} | \sigma^x_l | p_1, \ldots, p_m \rangle_R = e^{-i(l/12) (q_j - \sum_{j=1}^{m} p_n)} \times i^{n+m/2} (A^2 J^2 \xi^2) (m-2n)^2/4 \sqrt{\xi \xi T \prod_{j=1}^{2n} \left( \frac{e^{\eta_j}}{L^2 q_j} \right)^{1/2} \prod_{l=1}^{m} \left( \frac{e^{-\eta_l}}{L^2 p_l} \right)^{1/2}}$$

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\[
\times \prod_{j<l=1}^{2n} \left[ \frac{\sin((q_j - q_{j'})/2)}{\xi_{q_jq_{j'}}} \right] \prod_{l<k=1}^{m} \left[ \frac{\sin((p_l - p_{l'})/2)}{\xi_{p lp_{l'}}} \right]
\times \prod_{j=1}^{2n} \prod_{l=1}^{m} \left[ \frac{\xi_{q_j p_l}}{\sin((q_j - p_l)/2)} \right]
\]

(109)

where \( m \) is even (odd) for \( h < 1 \) (\( h > 1 \)) and

\[
\xi_k = 2J \sqrt{1 + h^2 - 2h \cos k}, \quad \xi_{k,k'} = \frac{\xi_k + \xi_{k'}}{2}, \quad \xi = |1 - h^2|^{1/4},
\]

\[
\xi_T = \prod_{q \in \text{NS}, p \in \text{R}} (\xi_{q,p})^{1/2} \prod_{q,q' \in \text{NS}} (\xi_{q,q'})^{-1/4} \prod_{p,p' \in \text{R}} (\xi_{p,p'})^{-1/4}, \quad e^{\eta k} = \prod_{q \in \text{NS}} \xi_{k,q} \prod_{p \in \text{R}} \xi_{k,p}.
\]

(110)

For large \( L \) we have with exponential accuracy (in \( L \))

\[
\xi_T \approx 1 \quad e^{\eta k} \approx 1.
\]

(111)

As we are interested in the thermodynamic limit we shall set these terms equal to unity in what follows (their deviations from unity give rise to subleading contributions). We stress that as a consequence of the different quantization conditions of \( R \) and NS momenta there are no singularities in equation (109) as long as \( L \) is (large but) finite. The free fermionic basis used in [98] differs from the one discussed in appendix A. However, the initial state can still be represented in terms of eigenstates of the post-quench Hamiltonian \( H(h) \) using (B.8) and (B.10), provided that \( K(k) \) is chosen appropriately.

4.2. Quench within the ferromagnetic phase: time evolution of the order parameter

We now consider a quench within the ordered phase: we prepare the system in the ground state of the Hamiltonian \( H(h_0) \) and at time \( t = 0 \) suddenly change the magnetic field from \( h_0 \) to \( h \), where \( h_0, h < 1 \). We are interested in the case where the \( \mathbb{Z}_2 \) symmetry of \( H(h_0) \) is broken spontaneously in the ground state \( |\Psi_0\rangle \). This is possible only in the thermodynamic limit. On the other hand, we would like to keep the length \( L \) of the system very large but finite in our calculations for technical reasons. In order to be able to work in a large, finite volume, we therefore take our initial state to be of the form

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2}} \left( |0; h_0\rangle_{\text{NS}} + |0; h_0\rangle_{\text{R}} \right),
\]

(112)

where \( |0; h_0\rangle_{\text{NS}} \) and \( |0; h_0\rangle_{\text{R}} \) are the ground states of \( H(h_0) \) in the sectors with even/odd numbers of fermions respectively, see appendix A.4. For finite system size \( L \), the state (112) is a particular linear combination of the ground state and first excited state of the Hamiltonian \( H(h_0) \). On the other hand, in the thermodynamic limit \( L \to \infty \) (112) becomes the symmetry broken ground state of \( H_0 \). As is shown in appendix B the time-evolved initial state

\[
|\Psi_0(t)\rangle = e^{-iH(h)t} |\Psi_0\rangle
\]

(113)

can be expressed in terms of eigenstates of \( H(h) \) as

\[
|\Psi_0(t)\rangle = \frac{1}{\sqrt{2}} \left[ \frac{|B(t)\rangle_{\text{NS}}}{\sqrt{\text{NS}(B|B\rangle_{\text{NS}})} + \frac{|B(t)\rangle_{\text{R}}}{\sqrt{\text{R}(B|B\rangle_{\text{R}})}} \right],
\]

(114)

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where

\[ |B(t)\rangle_a = e^{-iE_0t} \exp \left[ i \sum_{0<p\in a} K(p) e^{-2i\epsilon_p t} b^\dagger_p b_p \right] |0; h\rangle_a, \quad a = R, NS. \tag{115} \]

For the choice of basis underlying (109) the function \( K(k) \) is given by

\[ K(k) = \frac{\sin(k)(h - h_0)}{\epsilon_h(k)\epsilon_h(k)(2J)^{-2} + 1 + hh_0 - (h + h_0) \cos(k)}. \tag{116} \]

We note that this agrees with what would be obtained using the choice of fermions presented in (appendix A). The expectation value of \( \sigma^x_\ell \) is then given by

\[ \langle \Psi_0(t)|\sigma^x_\ell|\Psi_0(t)\rangle = Re R\langle B(t)|\sigma^x_\ell|B(t)\rangle_{NS} + NS\langle B(t)|\sigma^x_\ell|B(t)\rangle_R \tag{117} \]

We note that the diagonal contributions vanish

\[ a\langle B(t)|\sigma^x_\ell|B(t)\rangle_a = 0, \quad a = R, NS, \tag{118} \]

because

\[ e^{-i\pi N} \sigma^x_\ell e^{i\pi N} = -\sigma^x_\ell, \tag{119} \]

where \( \hat{N} \) is the fermion number operator (A.6).

The normalization is readily evaluated using the explicit representation (115)

\[ NS\langle B|B\rangle_{NS} = \exp \left[ \sum_{0<q\in NS} \log(1 + K^2(q)) \right], \]
\[ R\langle B|B\rangle_R = \exp \left[ \sum_{0<q\in R} \log(1 + K^2(p)) \right], \tag{120} \]

so that, for large \( L \), the norms \( R_{NS}\langle B|B\rangle_{NS} \) are approximately equal (we note that \( K(0) = K(\pi) = 0 \))

\[ R\langle B|B\rangle_R \approx NS\langle B|B\rangle_{NS} \approx \exp \left( L \int_0^\pi \frac{dk}{2\pi} \log(1 + K^2(k)) \right). \tag{121} \]

Importantly, we have

\[ \frac{R\langle B|B\rangle_R}{NS\langle B|B\rangle_{NS}} = 1 + \mathcal{O}(e^{-\alpha L}), \tag{122} \]

where \( \alpha \) is a positive constant. Equation (122) allows us to simplify the expression (117) for the one-point function to

\[ \langle \Psi_0(t)|\sigma^x_\ell|\Psi_0(t)\rangle = Re \frac{R\langle B(t)|\sigma^x_\ell|B(t)\rangle_{NS}}{R\langle B|B\rangle_R} + \mathcal{O}(L^{-2}). \tag{123} \]

As we are interested in the thermodynamic limit, this representation is most convenient and we shall use it in the following. The normalization can be expanded in powers of

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\[ K \text{ as} \]
\[
\mathcal{R}(B|B)_{R} = 1 + \sum_{0<k \in R} K^{2}(k) + \frac{1}{2} \left[ \sum_{0<k \in R} K^{2}(k) \right]^{2} - \frac{1}{2} \sum_{0<k \in R} K^{4}(k) + \cdots \\
\equiv 1 + \sum_{n=1}^{\infty} \Upsilon_{2n}, \tag{124}
\]

where \( \Upsilon_{2n} \) collects all the terms in which \( 2n \) functions \( K(k) \) are multiplied together. The following representation of \( \Upsilon \) where \( \Upsilon \) appears:

\[
\Upsilon_{2n} = \frac{1}{n!} \sum_{0<k_{1}, \ldots, k_{n} \in R} \prod_{j=1}^{n} K^{2}(k_{j}). \tag{125}
\]

Here \( \sum' \) indicates that the sum is only over terms with \( k_{l} \neq k_{m} \) for \( l \neq m = 1, \ldots, n \). Equation (124) is a formal series expansion, as each term grows as \( L^{n} \), i.e. diverges in the thermodynamic limit. The divergences in the normalization (124) are mirrored by infinite-volume divergences in the numerator of (117). Using (115) to expand \( \langle B(t)|\sigma_{t}|B(t)\rangle_{R} \) in powers of the function \( K(k) \) gives

\[
\text{Re} \left[ \langle B(t)|\sigma_{t}|B(t)\rangle_{R} \right] = 2m_{R}^{*} \left[ 1 + R_{1}(t) + [R_{2}(t) + \Upsilon_{2}] + [R_{3}(t) + \Upsilon_{4} + R_{1}(t)] \right. \\
\left. + [R_{4}(t) + \Upsilon_{2} R_{2}(t) + \Upsilon_{4}] + \cdots \right], \tag{126}
\]

where \( m_{R}^{*} \equiv \langle \sigma_{t} \rangle_{0}/2 \) is the magnetization at the initial time and the \( R_{n}(t) \) are finite in the thermodynamic limit. In terms of the ‘connected’ contributions \( R_{n}(t) \) the order parameter expectation value (123) is expressed as

\[
\langle \Psi_{0}(t)|\sigma_{t}|\Psi_{0}(t) \rangle = 2m_{R}^{*} \left[ 1 + \sum_{n=1}^{\infty} R_{n}(t) \right]. \tag{127}
\]

Equation (127) constitutes a linked cluster expansion, where the contribution \( R_{n}(t) \) is of order \( \mathcal{O}(K^{n}) \). As is shown in appendix B (see equation (B.12)) physically the formal expansion in powers of \( K(q) \) corresponds to a low-density expansion, where the small parameter is the density of excitations of the post-quench Hamiltonian in the initial state.

The next step is to determine the functions \( R_{n}(t) \). Expanding the boundary states in \( \langle B(t)|\sigma_{t}|B(t)\rangle_{R} \) we obtain

\[
\langle B(t)|\sigma_{t}|B(t)\rangle_{R} = \sum_{l,n=0}^{\infty} \sum_{q_{1}, \ldots, q_{n} \in \mathcal{R}} \prod_{j=1}^{n} K(q_{j}) \prod_{i=1}^{l} K(p_{i}) e^{2i t (\sum_{i=1}^{n} \epsilon_{q_{i}} - \sum_{j=1}^{l} \epsilon_{p_{j}})} \\
\times \langle B(t)|q_{1}, \ldots, q_{n}|\sigma_{t}|p_{1}, \ldots, p_{l} \rangle_{R} \\
\equiv \sum_{l,n=0}^{\infty} C_{(n,l)}. \tag{128}
\]

In the following we consider the first few terms in the expansion (128). We shall find that most terms exhibit long-time (infrared) divergences, the strongest of which occur in ‘diagonal’ contributions \( n = l \). We shall determine and then sum these leading singularities to all orders in the expansion (128).
4.2.1. Order $O(K^0)$ contribution ($n = l = 0$). The zero-particle contribution in (128) is the ground state (of the post-quench Hamiltonian $H(h)$) expectation value

$$C_{(0|0)} = 2m^*_0 = \sqrt{\xi} = (1 - h^2)^{1/8}. \quad (129)$$

4.2.2. Order $O(K)$ contributions. At first order in $K$ there are two contributions

$$C_{(1|0)} = i \sum_{0 < q \in NS} K(q)e^{2i\varepsilon_q}NS(-q, q|\sigma_\ell^x|0)_R = 4J^2h\sqrt{\xi}e^{1\over L} \sum_{0 < q \in NS} {K(q)\sin q \over \varepsilon_q^2}e^{2i\varepsilon_q},$$

$$C_{(0|1)} = -i \sum_{0 < p \in R} K(p)e^{-2i\varepsilon_p}NS(0|\sigma_\ell^x|p, -p)_R = 4J^2h\sqrt{\xi}e^{1\over L} \sum_{0 < p \in R} {K(p)\sin p \over \varepsilon_p^2}e^{-2i\varepsilon_p}. \quad (130)$$

In the limit $L \to \infty$ we can turn the momentum sums in (130) into integrals, which gives

$$I(t) = {1 \over \sqrt{\xi}} \lim_{L \to \infty} [C_{(1|0)} + C_{(0|1)}] = 8J^2h \int_0^\pi {dk \over 2\pi} {K(k)\sin k \over \varepsilon_k^2} \cos(2t\varepsilon_k). \quad (131)$$

For late times we can evaluate this integral by a stationary phase approximation

$$I(t) = A_0 {\cos(2\varepsilon_0 t + 3\pi/4) \over t^{3/2}} - A_\pi {\cos(2\varepsilon_\pi t - 3\pi/4) \over t^{3/2}} + o(t^{-3/2}), \quad (132)$$

where

$$A_k = {2hJ^2K'(k) \over \sqrt{\pi\varepsilon_k^2|\varepsilon_k'|^{3/2}}}, \quad k = 0, \pi. \quad (133)$$

Comparison of (128) and (126) then allows us to identify the function $R_1(t)$ in the thermodynamic limit

$$R_1(t) = I(t). \quad (134)$$

4.2.3. Order $O(K^2)$ contributions. To order $O(K^2)$ there are two types of contribution. $C_{(2|0)}$ and $C_{(0|2)}$ are finite in the thermodynamic limit and well behaved at late times. They do not play a significant role in the following and we therefore refrain from presenting explicit expressions. The most important contribution to order $O(K^2)$ is given by

$$C_{(1|1)} = \sum_{0 < q \in NS \atop 0 < p \in R} K(q)K(p)e^{-2i(\varepsilon_p - \varepsilon_q)}NS(-q, q|\sigma_\ell^x|p, -p)_R. \quad (135)$$

The relevant form factor is, cf. equation (109),

$$NS(-q, q|\sigma_\ell^x|p, -p)_R \simeq {4\sqrt{\xi} \over L^2} {\varepsilon_q^4 \over \varepsilon_p^2 \varepsilon_p^2} \sin q \sin p \varepsilon_q \varepsilon_p, \quad (136)$$

where we have defined

$$c_{kk'} \equiv {1 \over \cos k - \cos k'}. \quad (137)$$

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For large $t$ and $L$ the momentum sum in the NS sector can be evaluated using lemma 1 (D.1) and retaining only the leading terms in $t$ and $L$. This results in

$$C_{(1)1} = \sqrt{\xi} \sum_{0<p\in\mathbb{R}} K^2(p) \left[ 1 - \frac{4t\varepsilon'_p}{L} \right] + \cdots. \quad (138)$$

Crucially, this contribution exhibits an infinite-volume divergence (for $L \to \infty$) as well as an infrared divergence (for $t \to \infty$). To $O(1)$ (in both $L$ and $t$) (138) can be expressed as

$$C_{(1)1} = \sqrt{\xi} \Upsilon_2 - \sqrt{\xi} \int_0^\pi \frac{dk}{\pi} K^2(k) 2t\varepsilon'_k + \cdots, \quad (139)$$

where $\Upsilon_2$ is defined in (124). By comparing (139)–(126) we can determine the function $R_2(t)$ for large $t$

$$R_2(t) = -\int_0^\pi \frac{dk}{\pi} K^2(k) 2t\varepsilon'_k + \cdots, \quad (140)$$

where the dots denote terms that are subleading in $t$ at late times. We note that the leading term (140) arises from the region $p \approx q$ in (135). This observation will be important in what follows.

4.2.4. Order $O(K^3)$ contributions. To this order there are several contributions. $C_{(3)0}$ and $C_{(0)3}$ are completely regular and do not play an important role in the following. The evaluation of the other contributions follows closely the calculation for $C_{(1)1}$ and results in

$$C_{(1)2} + \text{h.c.} = \text{Re} \sum_{0<k_{1,2}\in\text{NS}} \sum_{0<p\in\mathbb{R}} iK(k_1)K(k_2)K(p)e^{2it[\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_p]}$$

$$\times_{\text{NS}} (-k_1, k_1, -k_2, k_2|\sigma^x_m|p, -p)_{\text{R}}$$

$$= \text{Re}(C_{(0)1}) \left[ \Upsilon_2 - \frac{1}{L} \sum_{0<p\in\mathbb{R}} 4t\varepsilon'_p K^2(p) \right] + \cdots, \quad (141)$$

$$C_{(2)1} + \text{h.c.} = \text{Re} \sum_{0<k_{1,2}\in\mathbb{R}} \sum_{0<p\in\text{NS}} iK(k_1)K(k_2)K(p)e^{2it[\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_p]}$$

$$\times_{\text{R}} (-k_1, k_1, -k_2, k_2|\sigma^x_m|p, -p)_{\text{NS}}$$

$$= \text{Re}(C_{(1)0}) \left[ \Upsilon_2 - \frac{1}{L} \sum_{0<p\in\text{NS}} 4t\varepsilon'_p K^2(p) \right] + \cdots.$$

Comparison with (126) then gives

$$R_3(t) = -I(t) \int_0^\pi \frac{dp}{\pi} 2t\varepsilon'_p K^2(p) + \cdots, \quad (142)$$

where the dots indicate contributions that are subleading in $t$. We note that (142) arises from the regions $k_{1,2} \approx p$ in (141).

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4.2.5. Order $O(K^4)$ contribution $C_{(2|2)}$. The most important contribution at order $O(K^4)$ is $C_{(2|2)}$ and we now discuss its evaluation in some detail. From (128) we have

$$
C_{(2|2)} = \frac{1}{2\pi^2} \sum_{0 < q, q' \in \text{NS}} K(q)K(q')K(p)K(p')e^{-2i\theta(\varepsilon_p + \varepsilon_{p'} - \varepsilon_q - \varepsilon_{q'})} 
\times \text{NS}(-q, q, -q', q'|\sigma^T_p)_{p, -p, p', -p'}_{R},
$$

where the form factor is given by

$$
\text{NS}(-q, q, -q', q'|\sigma^T_p)_{p, -p, p', -p'}_{R} = \frac{16\sqrt{\xi}}{L^4} \frac{\epsilon^4_{q,q'}\epsilon^4_{q,p'}\epsilon^4_{q',p'}}{\varepsilon^2_q - \varepsilon^2_{q'} - \varepsilon^2_p - \varepsilon^2_{p'}} \sin q \sin q' \sin p \sin p' \frac{c^2_{q,p}c^2_{q,p'}c^2_{q,q'}}{c^2_{q,p'}}.
$$

Each factor $c_{qq'}$ is associated with a singularity in the thermodynamic limit and it is useful to isolate these poles as functions of the NS-sector momenta, e.g.

$$
\begin{align*}
C_{p,q}C_{p',q'} &= C_{p,p'}(c_{p,q} - c_{p,q}), \\
\frac{c^2_{p,q}c^2_{p',q'}}{c^2_{q,p'}} &= \frac{c^2_{p,q}c^2_{p',q'} - 2c_{p,q}c_{p,q}c_{p',q'} - 2c_{p,q}c_{p,q}c_{p',q'}}{c^2_{q,p'}}.
\end{align*}
$$

Using the symmetry of (143) under exchange of $p$ and $p'$ the contribution of the last term in (145) can be reexpressed by substituting

$$
\frac{c^2_{q,p}c^2_{q,p'}c^2_{q',q'}}{c^2_{q,p'}} \rightarrow 2c^2_{q,p}c^2_{q',p'} - 4c^2_{pp'}c_{pp'} + 4c^2_{pp'c_{pp'}}.
$$

The sums over the NS momenta can then be carried out using lemma 1 (D.1) and lemma 5 (D.10). We begin with the contribution due to the first term in (146), which we denote by $C^{|2|2}_{(2)}$. Here the superscript indicates the pole structure, namely two double poles. The NS-sector momentum sums in (143) are performed by using (D.1) and retaining only the leading terms in $L$ and $t$, which gives

$$
C^{|2|2}_{(2)} = \frac{\sqrt{\xi}}{2} \sum_{0 < p, p' \in \mathbb{R}} K^2(p) \left[ 1 - \frac{4t\varepsilon_p'}{L} \right] K^2(p') \left[ 1 - \frac{4t\varepsilon_{p'}}{L} \right] + \cdots
$$

$$
= \frac{\sqrt{\xi}}{2} [R_2(t) + Y_2] - \frac{\sqrt{\xi}}{2} \sum_{0 < p \in \mathbb{R}} K^4(p) \left[ 1 - \frac{8t\varepsilon_p}{L} \right] + \cdots.
$$

Working out the contributions arising from the other terms in (146) is more involved. Carrying out the sums over $q$ and $q'$ using (D.10) and retaining only the leading contributions in $L$ and $t$ gives

$$
-4c_{pp'}c_{pq}c_{q'} \rightarrow 4\frac{\sqrt{\xi}}{L^2} \sum_{0 < p, p' \in \mathbb{R}} K^3(p)K(p')e^{-2i\theta(\varepsilon_{p'p} - \varepsilon_p)} \frac{\epsilon^4_{p,p}}{\varepsilon^2_{p} - \varepsilon^2_{p'}} \sin p \sin p' c_{pp'}^2,
$$

$$
4c_{pp'}c_{pq}c_{q'} \rightarrow -4\frac{\sqrt{\xi}}{L^2} \sum_{0 < p, p' \in \mathbb{R}} K^2(p)K^2(p') \sin p \sin p' c_{pp'}.
$$

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The leading contribution at late times and large $L$ can then be extracted by using lemma 2a (D.2)

$$C_{(2|2)}^{[1,1]} = -\frac{4\sqrt{\xi}}{L} \sum_{0 < \epsilon \in \mathbb{R}} K^4(p) t^\epsilon + \cdots.$$  \hspace{1cm} (149)

Here the superscript $[1,1]$ indicates that we are considering the contribution from two single poles. Putting everything together we arrive at the following result for $C_{(2|2)}$:

$$C_{(2|2)} = C_{(2|2)}^{[2,2]} + C_{(2|2)}^{[1,1]} = \sqrt{\xi} \left( \frac{1}{2} (R_2(t))^2 + \Upsilon_2 R_2(t) + \Upsilon_4 \right) + \cdots.$$  \hspace{1cm} (150)

This allows us to identify the leading contribution to the term $R_4(t)$ in (126) as

$$R_4(t) = \frac{1}{2} (R_2(t))^2 + \cdots.$$  \hspace{1cm} (151)

4.2.6. Exponentiation of the contributions $C_{(n|n)}$. Our analysis of the first few orders in powers of $K$ reveals the general structure of the expansion (128): at each order (except the very lowest ones) there are late-time divergences that become stronger at higher orders. Moreover, the leading singularities are found in the ‘diagonal’ contributions $C_{(n|n)}$ and we shall now isolate these singular terms. At order $\mathcal{O}(K^{2n})$ we have

$$C_{(n|n)} = \frac{1}{n!^2} \sum_{q_1, \ldots, q_n \in \text{NS}} K(q_j) K(p_j) e^{2it \sum_{j=1}^n (\epsilon_{q_j} - \epsilon_{p_j})} \times \text{NS}(-q_1, q_1, \ldots, -q_n, q_n | \sigma_j^T | p_1, -p_1 \ldots, p_n, -p_n)_R,$$

where the form factor in the limit of large $L$ is given by

$$\text{NS}(-q_1, q_1, \ldots, -q_n, q_n | \sigma_j^T | p_1, -p_1 \ldots, p_n, -p_n)_R = \frac{4^n \sqrt{\xi}}{L^{2n}} \prod_{j=1}^n \left( \frac{\sin(q_j) \sin(p_j)}{q_j} \right) \prod_{j < j'}^n \frac{1}{\epsilon_{q_j} \epsilon_{q_{j'}} \epsilon_{p_j} \epsilon_{p_{j'}}}.$$  \hspace{1cm} (152)

The most singular terms at late times arise from the regions in momentum space

$$p_j \approx q_{P(j)}.$$  \hspace{1cm} (153)

where $P$ is a permutation of $1, \ldots, n$. There are $n!$ such contributions with only double poles and they are all equal. They are determined by replacing

$$\frac{\prod_{j=1}^n \prod_{i=1}^{n} c_{q_i p_i}^2}{\prod_{j < j'}^n c_{q_j q_{j'}} c_{q_j p_{j'}} c_{q_{j'} p_{j'}}} \rightarrow n! \prod_{i=1}^{n} c_{q_i p_i}^2 + \cdots.$$  \hspace{1cm} (154)

and then carrying out the NS-sector momentum sums using (D.1). This gives

$$C_{(n|n)}^{[2, \ldots, 2]} = \frac{1}{n!} \frac{4^n \sqrt{\xi}}{L^n} \sum_{0 < p_1, \ldots, p_n \in \mathbb{R}} \prod_{i=1}^{n} K(p_i)^2 \left[ \frac{L}{4} - t^\epsilon_{p_i} \right] + \cdots.$$  \hspace{1cm} (155)
This is in agreement with (138) and (147). Using (125) we see that the t-independent part of (156) equals Υ_{2n}, so that
\[
\sum_{n=0}^{\infty} C_{(n|n)}^{(2,\ldots,2)} \bigg|_{t=0} = \sqrt{\xi} \exp \left[ \sum_{0 < p \in \mathbb{R}} \log \left( 1 + K^2(p) \right) \right] + \cdots. \tag{157}
\]
For \( t > 0 \) we may invert the steps used to express (120) in terms of (125), which gives
\[
\sum_{n=0}^{\infty} C_{(n|n)}^{(2,\ldots,2)} = \sqrt{\xi} \exp \left[ \sum_{0 < p \in \mathbb{R}} \log \left( 1 + K^2(p) \left[ 1 - \frac{4t\epsilon_p^\prime}{L} \right] \right) \right] + \cdots
\]
\[
= \sqrt{\xi} R \langle B | B \rangle_R \exp \left[ -t \int_0^{\pi} dk \frac{K^2(k)}{\pi 1 + K^2(k)} [2\epsilon'(k)] \right] + \cdots. \tag{158}
\]
The contribution of all these terms to the one-point function of the order parameter is thus
\[
\text{Re} \sum_{n=0}^{\infty} \frac{C_{(n|n)}^{(2,\ldots,2)}}{\langle B | B \rangle_R} = \sqrt{\xi} \exp \left[ -t \int_0^{\pi} dk \frac{K^2(k)}{\pi 1 + K^2(k)} [2\epsilon'(k)] \right] + \cdots. \tag{159}
\]

4.2.7. Exponentiation of the contributions \( C_{(n\pm1|n)} \). The leading terms (at late times and large \( L \)) in \( C_{(n\pm1|n)} \) can be summed to all orders in a similar way to our treatment of \( C_{(n|n)} \). The contributions \( C_{(n+1|n)} \) are given by
\[
\text{Re} C_{(n+1|n)} = \frac{1}{(n+1)!n!} \text{Im} \sum_{0 < q_1, \ldots, q_n, q \in \mathbb{NS}} \sum_{0 < p_1, \ldots, p_n \in \mathbb{R}} \text{K}(q) \left[ \prod_{j=1}^{n} \text{K}(q_j) \text{K}(p_j) \right] e^{2it\epsilon_q + 2it\sum_{i=1}^{n} (\epsilon_{q_i} - \epsilon_{p_i})}
\]
\[
\times \text{NS}(-q_1, q_1, \ldots, -q_n, q_n, -q, q\sigma_\uparrow^2 | p_1, -p_1, \ldots, p_n, -p_n)_R, \tag{160}
\]
where the form factors in the limit of large \( L \) are
\[
\text{NS}(-q_1, q_1, \ldots, -q_n, q_n, -q, q\sigma_\uparrow^2 | p_1, -p_1, \ldots, p_n, -p_n)_R
\]
\[
= -4ihf^2 \frac{4^n \sqrt{\xi}}{L^{2n+1}} \prod_{j,l=1}^{n} \epsilon_{q_j,p_l} \prod_{k=1}^{n} \epsilon_{-q_k}\epsilon_{-2}^{2m}
\]
\[
\times \prod_{j < j'=1}^{n} \epsilon_{q_j,q_{j'}}^{-4} \epsilon_{p_j,p_{j'}}^{-4} \prod_{i=1}^{n} \sin(q_i) \sin(p_i) \prod_{j \neq j'=1}^{n} \epsilon_{q_j,q_{j'}}^{2} \epsilon_{p_j,p_{j'}}^{2} \tag{161}
\]
Following the same reasoning that led to (155), we focus on the contributions arising from the regions
\[
p_j \approx \tilde{q}_j, \quad j = 1, \ldots, n, \tag{162}
\]
where \((\tilde{q}_1, \ldots, \tilde{q}_{n+1})\) is an arbitrary permutation of \((q_1, \ldots, q_n, q)\). All of these contributions are the same and hence the leading behaviour at late times can be extracted by the
substitution
\[
\prod_{j=1}^{n} \prod_{j' < 1}^{n} \frac{c_{q,p_{j}}^2}{c_{q,p_{j'}}^2} \prod_{j=1}^{n} \frac{c_{q,p_{j}}^2}{c_{q}^2} \to (n+1)! \prod_{i=1}^{n} c_{q,p_{i}}^2 + \ldots .
\] (163)

The NS-sector momentum sums can be carried out using (D.1), which gives
\[
\text{Re} C_{(n+1|n)}^{[2, \ldots, 2]} = - \frac{1}{n!} \frac{4^{n+1} h J^2 \sqrt{\xi}}{L^{n+1}} \sum_{0 < q < \xi, 0 < p_1 \ldots, p_n < R} K(q) \cos(2t \xi_q) \frac{\sin q}{\xi_q} \prod_{i=1}^{n} K(p_i)^2 \left[ \frac{L}{4} - t \xi_{p_i} \right] + \ldots
\]
\[
= - 4hJ^2 C_{(n|n)}^{[2, \ldots, 2]} \int_{0}^{\pi} \frac{dk}{2\pi} K(k) \cos(2t \xi_k) \frac{\sin k}{\xi_k^2} + \ldots .
\] (164)

Summing over \( n \) then results in
\[
\text{Re} \sum_{n=0}^{\infty} C_{(n+1|n)}^{[2, \ldots, 2]} = \sqrt{\xi} I(t) \exp \left[ - \frac{t}{\pi} \int_{0}^{\pi} \frac{dk}{1 + K^2(k)} [2 \xi'(k)] \right] + \ldots ,
\] (165)

where \( I(t) \) is given by (132). The analysis for the contributions \( C_{(n|n+1)} \) is completely analogous and finally leads to the following result:
\[
\text{Re} \sum_{n=0}^{\infty} \frac{C_{(n+1|n)} + C_{(n|n+1)}}{(B|B)^{R}} = \sqrt{\xi} I(t) \exp \left[ - \frac{t}{\pi} \int_{0}^{\pi} \frac{dk}{1 + K^2(k)} [2 \xi'(k)] \right] + \ldots ,
\] (166)

We see that by virtue of the extra factor \( I(t) \) this is always subleading compared to the ‘diagonal’ contribution (159).

4.2.8. Form factor result for the one-point function \( \langle \Psi_0(t)|\sigma_{\bar{\ell}}^{x}|\Psi_0(t) \rangle \). Combining the results in the previous subsections we conclude that
\[
\langle \Psi_0(t)|\sigma_{\bar{\ell}}^{x}|\Psi_0(t) \rangle \sim \sqrt{\xi} \left[ 1 + I(t) + O(t^{-1}) \right] e^{-t/\tau},
\] (167)

where the decay time \( \tau \) is given by
\[
\tau^{-1} = \int_{0}^{\pi} \frac{dk}{\pi} K^2(k) [2 \xi'(k)] + O(K^6).
\] (168)

Here we have used (151) to infer the absence of \( O(K^4) \) corrections to the decay time. The corrections in the prefactor are expected to be \( O(t^{-1}) \) as there are such (subleading) contributions in the various terms \( C_{(n|j)} \) we have calculated above. We note that these terms are larger than \( I(t) \) at late times as \( I(t) \sim t^{-3/2} \).

The result (168) is in agreement with the one obtained by applying the cluster decomposition principle on the asymptotic result for the two-point function obtained in the determinant approach (cf equation (69)), which gives
\[
\tau^{-1} = \int_{0}^{\pi} \frac{dk}{\pi} \log \left| \frac{1 + K^2(k)}{1 - K^2(k)} \right| \xi'(k).
\] (169)

Indeed, expanding (169) in the limit where \( K^2(k) \ll 1 \) gives precisely (168). We note that (168) is a fairly good approximation for a wide range of magnetic fields \( h_0 \) and \( h \). For example, for a quench from \( h_0 = 0.1 \) to \( h = 0.8 \) the relative error is less than 1%. This shows the effectiveness of the form factor approach.
4.3. Two-point function in the ordered phase

We now turn to the time evolution of the two-point function for a quench within the ordered phase. The quantity we want to evaluate is

\[ \rho^{xx}(t, \ell) = \langle \Psi_0(t) | \sigma_{m+\ell}^x \sigma_m^x | \Psi_0(t) \rangle \]

\[ = \frac{R(\langle B(t) | \sigma_{m+\ell}^x \sigma_m^x | B(t) \rangle_R)}{2R(\langle B | B \rangle_R)} + \frac{\text{NS} \langle B(t) | \sigma_{m+\ell}^x \sigma_m^x | B(t) \rangle_{\text{NS}}}{2\text{NS} \langle B | B \rangle_{\text{NS}}}. \quad (170) \]

In the large-\( L \) limit both terms contribute equally, so that

\[ \lim_{L \to \infty} \rho^{xx}(t, \ell) = \lim_{L \to \infty} \frac{R(\langle B(t) | \sigma_{m+\ell}^x \sigma_m^x | B(t) \rangle_R)}{R(\langle B | B \rangle_R)}. \quad (171) \]

We note that as a consequence of reflection symmetry the sign of \( \ell \) does not matter, i.e. \( \rho^{xx}(t, \ell) = \rho^{xx}(t, -\ell) \). We shall therefore assume from now on that \( \ell \geq 0 \).

The Lehmann representation for the numerator on the right-hand side of equation (171) is

\[ \text{R}(\langle B(t) | \sigma_{m+\ell}^x \sigma_m^x | B(t) \rangle_R) = \sum_{l,n=0}^{\infty} \frac{(n-l)!}{n!!} \sum_{0<k_1,\ldots,k_n \in \text{R}} \sum_{q_1,\ldots,q_r \in \text{NS}} \frac{1}{r!} \left[ \prod_{j=1}^{n} K(k_j) \right] \left[ \prod_{i=1}^{l} K(p_i) \right] \]

\[ \times e^{-2i|\sum_{j=1}^{n} \epsilon_{k_j} - \sum_{j=1}^{n} \epsilon_{k_j}|^2} \left[ \langle -k_1, k_1, \ldots, -k_n, k_n | \sigma_{m+\ell}^x q_1, \ldots, q_r \rangle_{\text{NS}} \right] \]

\[ \times \text{NS}(q_1, \ldots, q_r | p_1, \ldots, p_l, -p_l) \]

\[ \equiv \sum_{l,n,r=0}^{\infty} D_{(2n,|r|=2l)}. \quad (173) \]

As in the case of the one-point function we focus on the terms with the strongest possible singularities in the form factors for a given order in the formal expansion in powers of \( K(p) \). These are obtained by taking \( 2n = 2l = r \).

4.3.1. Contributions at \( \mathcal{O}(K^0) \). These are equal to the ground state two-point function

\[ \text{R}(\langle 0 | \sigma_{m+\ell}^x \sigma_m^x | 0 \rangle_R). \quad (174) \]

Using a Lehmann representation to evaluate (174) gives rise to contributions

\[ D_{(0,2s=0)} = \sum_{0<q_1,\ldots,q_{2s} \in \text{NS}} \frac{1}{(2s)! R(\langle 0 | \sigma_{m+\ell}^x q_1, \ldots, q_{2s} \rangle_{\text{NS}} \langle q_{2s}, \ldots, q_1 | \sigma_m^x | 0 \rangle_R). \quad (175) \]

For \( s = 0 \) we have

\[ D_{(0,0,0)} = (2m_0^2)^2 = \xi = (1 - h^2)^{1/4}, \]

while for \( s = 1 \) we obtain

\[ D_{(0,2)} = \frac{2h^2 J_4^2 \xi}{\pi^2} \int_{-\pi}^{\pi} dq_1 dq_2 \frac{\sin^2((q_1 - q_2)/2)}{\varepsilon_{q_1 q_2}^2 \varepsilon_{q_1 q_2}} e^{i(q_1 + q_2)} + \mathcal{O}(L^{-1}) \propto h^{-2\ell}. \quad (177) \]

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We see that at large $j$ $D_{(02|0)}$ is exponentially small compared to $D_{(00|0)}$. The contributions $D_{(02s|0)}$ with $s > 1$ are suppressed by additional exponential factors. We conclude that for large distances $\ell$ we only need to retain $D_{(00|0)}$.

4.3.2. Contributions at $O(K^2)$. At order $O(K^2)$ the largest contribution at large distances and late times arises from

$$
D_{(2|2|2)} = \sum_{0<k,p\in\mathbb{R}} \frac{1}{2} \sum_{q_1,q_2\in\mathbb{NS}} K(k)K(p)e^{-2i[\varepsilon_p-\varepsilon_k]}e^{i(q_1+q_2)}R\langle-k,k|\sigma^x_m|q_1,q_2\rangle_{\mathbb{NS}}
$$

$$
\times \langle\mathbb{NS}|q_2,q_1|\sigma^x_m|p,-p\rangle_R = D_{(2|2|2)}^{(1)} + D_{(2|2|2)}^{(2)},
$$

(178)

where $D_{(2|2|2)}^{(1,2)}$ denote the contributions arising from terms with $k \neq p$ and $k = p$ respectively. Using (D.4) to carry out the summations over $q_{1,2}$ we arrive at

$$
D_{(2|2|2)}^{(1)} = \frac{2\xi}{L^2} \sum_{0<k\neq p\in\mathbb{R}} e^{-2i[\varepsilon_p-\varepsilon_k]}(\varepsilon_k + \varepsilon_p)^2 \varepsilon_k^2 \varepsilon_p^2 (\cos k - \cos p)^2 \left[\sin k \sin p e^{-2k}\right]
$$

$$
\times \varepsilon_k \varepsilon_p \left[\sin^2\left(\frac{k-p}{2}\right) \cos(\ell(k + p)) - \sin^2\left(\frac{k+p}{2}\right) \cos(\ell(k - p))\right] + \cdots
$$

(179)

We note that in this expression there is no singularity if we consider the limit $k - p \to 0$. Nevertheless for large $\ell$ and $t$ the leading contribution to the double sum arises from the vicinity of $k = p$. In order to isolate this contribution we turn the sum over $k$ into an integral using the Euler–Maclaurin sum formula and then deform the integration contour into the upper half-plane. As $\varepsilon_p > 0$ for all $p > 0$ the contribution of the first term in square brackets will be negligible for large $t$. The same holds true for the second contribution as long as $2\varepsilon_p t \pm \ell > 0$. On the other hand, if $2\varepsilon_p t \pm \ell < 0$ we need to deform the contour into the lower half-plane (in the variable $k$). In doing so we acquire a contribution from the double pole at $p = k$. The residue is dominated by the factors involving $t$ and $\ell$ as both of these are assumed to be large, so we end up with

$$
D_{(2|2|2)}^{(1)} = -4\xi \sum_{0<p\in\mathbb{R}} K^2(p)\theta_H(\ell - 2\varepsilon_p t) \left[2\varepsilon_p t - \ell\right] + \cdots,
$$

(180)

where $\theta_H(x)$ denotes the Heaviside step function. The second contribution $D_{(2|2|2)}^{(2)}$ is given by

$$
D_{(2|2|2)}^{(2)} = \frac{8\xi}{L^4} \sum_{0<k\in\mathbb{R}} \sin^2(k) K^2(k) \sum_{q_1,q_2\in\mathbb{NS}} e^{i(q_1+q_2)} \sin^2((q_1 - q_2)/2) \varepsilon_{q_1}^2 \varepsilon_{q_2}^2 c_{q_1,k}^2 c_{q_2,k}^2.
$$

(181)

Using (D.6) to carry out the summations over $q_{1,2}$ we arrive at

$$
D_{(2|2|2)}^{(2)} = \xi \sum_{0<k\in\mathbb{R}} K^2(k) \left[1 - \frac{4\ell}{L}\right] + \cdots
$$

(182)

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where we again have only retained the leading terms at large $L$ and $\ell$. Adding the two contributions (180) and (182) we obtain

$$D_{(2|2)}(t, \ell) = \xi \left[ \Upsilon_2 - \frac{4}{L} \sum_{k \in \mathbb{R}} K^2(k) \left( \theta_H(\ell - 2t\epsilon'_k) [2t\epsilon'_k - \ell] + \ell \right) \right] + \cdots$$

$$= \xi \left[ \Upsilon_2 - \frac{4}{L} \sum_{k > 0} K^2(k) \left( 2t\epsilon'_k \theta_H(\ell - 2t\epsilon'(k)) + \ell \theta_H(2t\epsilon'_k - \ell) \right) \right] + \cdots. \quad (183)$$

This contains a part $\Upsilon_2$ that diverges in the thermodynamic limit, but which will again be compensated by the denominator in (171).

### 4.4. Exponentiation of $D_{(2n|2n)}$}

The dominant contribution at order $O(K^{2n})$ is again the ‘diagonal’ one

$$D_{(2n|2n)} = \frac{\xi}{(n!)^2(2n)! L^{4n}} \sum_{k_1,\ldots,k_n \in \mathbb{R}} \prod_{r=1}^{n} \left[ \sin(k_r)K(k_r)e^{2it\epsilon_k} \sin(p_r)K(p_r)e^{-2it\epsilon_p} \right]$$

$$\times \prod_{l < l'}^{n} \frac{1}{4\epsilon_{k_l k_{l'}}^2} \frac{1}{4\epsilon_{p_{l'} p_l}^2} \sum_{q_1,\ldots,q_{2n} \in \text{NS}} \left[ \prod_{r=1}^{2n} e^{t\epsilon_{q_{r}}} \right]$$

$$\times \prod_{l < l'}^{2n} \sin^2((q_l - q_{l'})/2) \prod_{r=1}^{2n} 4\epsilon_{k_r q_{r}}^2 e^{2it\epsilon_{k_r q_{r}}} e_{p_r q_{r}} e_{c_{k_r} q_{r}} e_{c_{p_r} q_{r}}. \quad (184)$$

In order to extract the large time and distance behaviour of (184) we need to focus on the regions where the form factors exhibit the strongest singularities. For the case $n = 1$ discussed in section 4.3.2 these regions were

- $q_1 \approx \pm k, \quad q_2 \approx \mp p$ and $p \approx k$, 
- $q_2 \approx \pm k, \quad q_1 \approx \mp p$ and $p \approx k. \quad (185)$

For general $n$ we need to focus on the regions

$$\left( \begin{array} {c} k_{Q_1} \\ \vdots \\ k_{Q_n} \end{array} \right) \approx \left( \begin{array} {c} p_1 \\ \vdots \\ p_n \end{array} \right), \quad \left( \begin{array} {c} q_{S_1} \\ \vdots \\ q_{S_n} \end{array} \right) \approx \left( \begin{array} {c} \sigma_1 k_{Q_1} \\ -\sigma_1 p_1 \\ \vdots \\ \sigma_n k_{Q_n} \\ -\sigma_n p_n \end{array} \right). \quad (186)$$

where $\sigma_j = \pm$ and $(Q_1,\ldots,Q_n), (S_1,\ldots,S_{2n})$ are permutations of $(1,2,\ldots,n)$ and $(1,2,\ldots,2n)$ respectively. By symmetry all these regions contribute equally, which gives rise to a combinatorial factor of $n!(2n)!$. We therefore focus on the single region where $(Q_1,\ldots,Q_n) = (1,\ldots,n)$ and $(S_1,\ldots,S_{2n}) = (1,\ldots,2n)$. We start by expressing (184)
in the form

\[
D_{(2n|2n|2n)} = \frac{1}{L^{3n}} \sum'_{k_1, \ldots, k_n \in R} e^{2it\sum_{i=1}^{n}(\varepsilon_{k_i}-\varepsilon_{p_i})} \sum_{q_1, \ldots, q_{2n} \in NS} f(k_1, \ldots, k_n; p_1, \ldots, p_n|q_1, \ldots, q_{2n}) \\
\times \frac{n}{\sum} e^{i(q_{2r-1}+q_{2r})\ell} c_{k_r,q_{2r-1}} c_{k_r,q_{2r}} c_{p_r,q_{2r-1}} c_{p_r,q_{2r}},
\]

(187)

and then carrying out the sums over the \(q\)s using (D.5) and (D.9). When doing this we need to distinguish the cases \(k_r \neq p_r\) and \(k_r = p_r\) as they give rise to different kinds of singularity in the \(q\) summations. For \(k_r = p_r\) the summation over both \(q_{2r-1}\) and \(q_{2r}\) gives a leading contribution

\[
\frac{1}{L^3} \sum_{q_{2r-1},q_{2r} \in NS} e^{i(q_{2r-1}+q_{2r})\ell} c_{p_r,q_{2r-1}} c_{p_r,q_{2r}} f(k_1, \ldots, k_n; p_1, \ldots, p_n|q_1, \ldots, q_{2n}) \bigg|_{p_r=k_r} \\
= \frac{L}{16} \left(1 - \frac{2\ell}{L}\right)^2 \left(\hat{f}(p_r,-p_r) + \hat{f}(-p_r,p_r)\right) \left(\frac{1}{\sin^4(p_r)}\right)_{k_r=p_r} + \cdots,
\]

(188)

where we have defined

\[
\hat{f}(p, q) = f(k_1, \ldots, k_n; p_1, \ldots, p_n|q_1, \ldots, q_{2r-2}, p, q, q_{2r+1}, \ldots, q_{2n}).
\]

(189)

The extra factor \(1/L\) in (188) is present as we are considering a single term in the sum over \(k_r\). On the other hand, for \(p_r \neq k_r\) we obtain

\[
\frac{1}{L^3} \sum_{0<k_r \in R} e^{2it(\varepsilon_{k_r}-\varepsilon_{p_r})} \sum_{q_{2r-1},q_{2r} \in NS} e^{i(q_{2r-1}+q_{2r})\ell} c_{k_r,q_{2r-1}} c_{k_r,q_{2r}} c_{p_r,q_{2r-1}} c_{p_r,q_{2r}} \\
\times f(k_1, \ldots, k_n; p_1, \ldots, p_n|q_1, \ldots, q_{2n}) \\
= \frac{1}{L} \sum_{0<k_r \in R} e^{2it(\varepsilon_{k_r}-\varepsilon_{p_r})} \left\{ -e^{-i(\varepsilon_{p_r}-\varepsilon_{k_r})} \hat{f}(p_r,-k_r) + e^{i(\varepsilon_{p_r}-\varepsilon_{k_r})} \hat{f}(p_r, k_r) \right\} + \frac{\hat{f}(p_r,-p_r) + \hat{f}(-p_r,p_r)}{4\sin^4(p_r)} + p_r \leftrightarrow k_r \right\} + \cdots
\]

(190)

Similarly to our treatment of the \(n = 1\) case in section 4.3.2, we can now carry out the summation over \(k_r\) with the result

\[-\theta_H(\ell - 2t\varepsilon'_{p_r})[2t\varepsilon'_{p_r} - \ell] \left[\frac{\hat{f}(p_r,-p_r) + \hat{f}(-p_r,p_r)}{4\sin^4(p_r)}\right]_{k_r=p_r} + \cdots
\]

(191)

Combining the two contributions (188) and (191) we find that the leading terms at large \(\ell\) and \(t\) are

\[
\left[\frac{L}{4} - \theta_H(\ell - 2t\varepsilon'_{p_r})[2t\varepsilon'_{p_r} - \ell]\right] \left[\frac{\hat{f}(p_r,-p_r) + \hat{f}(-p_r,p_r)}{4\sin^4(p_r)}\right]_{k_r=p_r} + \cdots.
\]

(192)
Carrying out all $q_s$ and $k_l$ summations in the same way gives

\[ D_{(2n|2n)} = \frac{\xi}{n! L^n} \sum_{p_1, \ldots, p_n \in \mathbb{R}} \prod_{r=1}^{n} \left[ L - 4\ell - 4\theta_H(\ell - 2t\varepsilon'_{p_r})[2t\varepsilon'_{p_r} - \ell] \right] \frac{1}{16 \sin^4(p_r)} \times \sum_{\sigma_1, \ldots, \sigma_r = \pm} f(p_1, \ldots, p_n; p_1, \ldots, p_n|\sigma_1 p_1, -\sigma_1 p_1, \ldots, \sigma_n p_n, -\sigma_n p_n) + \cdots \]

As for the one-point function the sum over $n$ can be taken by inverting the steps used to express (120) in terms of (125), which gives

\[ \sum_{n=0}^{\infty} D_{(2n|2n)} = \xi \exp \left( \sum_{0 < p \in \mathbb{R}} \ln \frac{1 + K^2(p)}{1 - \frac{4\ell}{L} - \theta_H(\ell - 2t\varepsilon'_{p})[2t\varepsilon'_{p} - \ell]} \right) + \cdots \]

\[ \simeq \xi_{\text{NS}} \langle B|B\rangle_{\text{NS}} \exp \left[ -\frac{4}{L} \sum_{0 < p \in \mathbb{R}} K^2(p) \left( \ell + \theta_H(\ell - 2t\varepsilon'_{p})[2t\varepsilon'_{p} - \ell] \right) \right] + \cdots \]  

(194)

Here we have retained only the $O(K^2)$ term in the exponent as the higher orders are beyond the accuracy of our calculation. This then gives the desired result for the two-point function (171) in the ordered phase

\[ \lim_{L \to \infty} \rho^{xx}(t, \ell) \simeq (1 - h^2)^{1/4} \exp \left[ -2 \int_{0}^{\pi} \frac{dp}{\pi} K^2(p) \left( \ell + \theta_H(\ell - 2t\varepsilon'_{p})[2t\varepsilon'_{p} - \ell] \right) \right] + \cdots \]  

(195)

Equation (195), including the prefactor, is expected to be accurate at late times and large distances as long as we do not quench too close to the critical point.

### 4.5. Two-point function for quenches in the disordered phase $h_0, h > 1$

As a consequence of the $\mathbb{Z}_2$ symmetry the one-point function of $\sigma^x_i$ is identically zero for quenches within the disordered phase, i.e. $h_0, h > 1$. We therefore turn to the two-point function. As shown in appendix A the ground state in the paramagnetic phase is the NS vacuum $|0; h_0\rangle_{\text{NS}}$. Hence the two-point function after the quench is equal to

\[ \rho^{xx}(t, \ell) = \frac{\langle B(t)|\sigma^x_{n+1} \sigma^x_n|B(t)\rangle_{\text{NS}}}{\langle B|B\rangle_{\text{NS}}} \]  

(196)

In the basis underlying the expression (109) for the form factors, the boundary states are again given by (115), but now the function $K(k)$ is given by

\[ K(k) = -\frac{\sin(k)(h - h_0)}{\varepsilon_{h_0}(k)\varepsilon_{h}(k)(2J)^{-2} + 1 + hh_0 - (h + h_0)\cos(k)}. \]

(197)

The Lehmann representation of the numerator is again given by (173) if we replace $R \to \text{NS}$. The difference from the ordered phase is that now only form factors with odd numbers $r$ of particles in the intermediate states are non-vanishing.
4.5.1. Order $\mathcal{O}(K^0)$ contributions. The $\mathcal{O}(K^0)$ contribution obtained upon expanding the boundary states in powers of $K$ is

$$NS\langle 0|\sigma^x_{m+\ell}(t)\sigma^x_m(t)|0\rangle_{NS} = NS\langle 0|\sigma^x_{m+\ell}\sigma^x_m|0\rangle_{NS}.$$  (198)

This is equal to the static zero temperature correlator in equilibrium. Inserting a resolution of the identity we conclude that the large-$\ell$ behaviour in the $L \to \infty$ limit is determined by one-particle intermediate states

$$D_{(0|0)}(t,\ell) = \xi \sqrt{4J^2h} \sum_{q \in \mathbb{R}} \frac{e^{iq\ell}}{\varepsilon_q} = 2JA \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{e^{iq\ell}}{\varepsilon_q} + \mathcal{O}(L^{-1})$$  (199)

where we have defined

$$A = \xi \sqrt{h}.$$  (200)

The integral can be carried out approximately by bending the contour around the branch cut of the energy in the upper half-plane, which gives

$$D_{(0|0)}(t,\ell) \approx \xi \sqrt{\frac{h}{\pi(h^2-1)}} \frac{h^{-|q|}}{\sqrt{\ell}} \equiv D(\ell).$$  (201)

We conclude that the leading contribution in $K(q)$ is exponentially small, which means that we need to evaluate all subleading terms with the same exponential accuracy as well. The contributions due to 3, 5, $\ldots$ particle intermediate states are small for large $\ell$ when compared to (201)

$$D_{(0|2s+1)} \propto h^{-(2s+1)\ell},$$  (202)

and can hence be ignored for our purposes.

4.5.2. $\mathcal{O}(K)$ contributions. The leading contributions to order $K$ are obtained by taking single-particle intermediate states into account only. This gives

$$D_{(2|1)} + D_{(0|2)} = \frac{4JA}{L} \sum_k \frac{K(k)}{\varepsilon_k} \sin(2t\varepsilon_k - k\ell) + \cdots.$$  (203)

The corrections to the rhs of (203) can be evaluated using contour techniques but are negligible at large distances. As a function of $t$ for fixed $n$ $D_{(2|1)} + D_{(0|2)}$ displays oscillatory behaviour on top of a slow $t^{-3/2}$ power-law decay in time. This can be seen by turning the sum in (203) into an integral and evaluating the latter by means of a saddle-point approximation, which gives

$$D_{(2|1)} + D_{(0|2)} \approx \frac{2JA}{\sqrt{\pi}} \sum_{a=\pm} K(k_a) \text{Im} \left[ \frac{e^{2it\varepsilon_{k_a} - ik_a\ell}}{\sqrt{-i\varepsilon''_{k_a}}} t \right],$$  (204)

where

$$k_{\pm} = \text{arccos} \left( \frac{x^2 \pm \sqrt{x^4 - x^2(1 + h^2) + h^2}}{h} \right), \quad x = \frac{\ell}{4Jt}.$$  (205)
In the limit $\ell/t \to 0$ this can be simplified further,

$$D_{(2|1)0} + D_{(0|1)2} \simeq -\frac{\ell}{(2Jt)^{3/2}} \left[ \frac{h+1}{h-1} \right]^{1/4} \frac{h-h_0}{4h(h_0-1)\sqrt{\pi}}$$

$$\times \sin \left( 4Jt(h-1) + \frac{\pi}{4} - \frac{h-1}{4hJ} \ell^2 \right)$$

$$- \frac{\ell}{(2Jt)^{3/2}} \left[ \frac{h-1}{h+1} \right]^{1/4} \frac{h-h_0}{4h(h_0+1)\sqrt{\pi}} \sin \left( 4Jt(h+1) - \frac{\pi}{4} + \frac{h+1}{4hJ} \ell^2 \right).$$

(206)

4.5.3. $O(K^2)$ contributions. The next contributions we need to consider are second order in $K(k)$ and arise from

$$\sum_{0<k,p\in\text{NS}} K(k)K(p)_{\text{NS}} \langle -k,k|\sigma_m^x(t)\sigma_m^x(t)|p,-p \rangle_{\text{NS}}.$$

(207)

Inserting a resolution of the identity between the spin operators we see that for large $\ell$ and $t$ the dominant contributions are generated by one-particle and three-particle intermediate states. The former is given by

$$D_{(2|1)2} = \sum_{k,p>0} \sum_{q} K(k)K(p)_{\text{NS}} \langle -k,k|\sigma_m^x(t)\sigma_m^x(t)|p,-p \rangle_{\text{NS}}.$$

(208)

As in our analysis of the two-point function for quenches within the ordered phase we again have to consider the two cases $k \neq p$ and $k = p$ separately. Denoting the corresponding contributions by $D_{(2|1)2}^{(1)}$ and $D_{(2|1)2}^{(2)}$ respectively, we find in the limit of large $\ell$, $t$ and $L \to \infty$

$$D_{(2|1)2}^{(2)} = \frac{2JA}{L^2} \sum_{k\in\text{NS}} K^2(k) \frac{e^{k|\ell|}}{\varepsilon_k} + \ldots,$$

(209)

$$D_{(2|1)2}^{(1)} = \frac{8JA}{L^2} \sum_{0<k\neq p\in\text{NS}} \frac{K(k)K(p)}{\varepsilon_k^2 \varepsilon_p^2} \frac{\sin(k)\sin(p)}{\varepsilon_k^2 - \varepsilon_p^2} \left[ \frac{\varepsilon_k\sin(k\ell)}{\sin(k)} - \frac{\varepsilon_p\sin(p\ell)}{\sin(p)} \right] e^{2it[\varepsilon_k - \varepsilon_p]}$$

$$+ \ldots.$$ 

(210)

In the thermodynamic limit (210) can be written as a double integral, but we have not succeeded in simplifying it in a useful way. The contribution $D_{(2|1)2}^{(2)}$ is time independent and exponentially small in $\ell$. In contrast to this, $D_{(2|1)2}^{(1)}$ displays power-law decay in $t$ for fixed $\ell$

$$D_{(2|1)2}^{(1)} \propto t^{-3}.$$

(211)

We now turn to the $O(K^2)$ contribution involving three-particle intermediate states

$$D_{(2|3)2} = \frac{1}{4} \sum_{0<k,p\in\text{NS}} \sum_{q_1,q_2,q_3\in\text{R}} K(k)K(p)_{\text{NS}} \langle -k,k|\sigma_m^x(t)|q_1,q_2,q_3 \rangle_{\text{R}}$$

$$\times \langle q_3,q_2,q_1|\sigma_m^x(t)|p,-p \rangle_{\text{NS}}.$$

(212)

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We denote the contributions from \( k \neq p \) and \( k = p \) by \( D_{(2|3|2)}^{(1)} \) and \( D_{(2|3|2)}^{(2)} \), respectively. After some lengthy calculations we find

\[
D_{(2|3|2)}^{(2)} = D_{(0|1|0)} + \sum_{0<k<\mathrm{NS}} K^2(k) \left[ 1 - \frac{4\ell}{L} \right] + \frac{4JA}{L} \sum_{0<k<\mathrm{NS}} \frac{K^2(k)}{\varepsilon_k} \cos(\ell k) - 4(h^2 - 1)D(\ell) \sum_{0<k<\mathrm{NS}} \frac{4J^2K^2(k)}{\varepsilon_k^2} + \ldots.
\]

(213)

This is time independent and contains a piece that diverges with the volume as expected.

The contribution \( D_{(2|3|2)}^{(1)} \) is given by

\[
D_{(2|3|2)}^{(1)} = \frac{2JA}{L^5} \sum_{0<k\neq p<\mathrm{NS}} K(k)K(p) \frac{\sin(k)\sin(p)}{\varepsilon_k^2\varepsilon_p^2} \cos(2\ell[k - \varepsilon_p])
\]

\[
\times \frac{1}{6} \sum_{q_1,q_2,q_3 \in \mathbb{R}} \left[ \prod_{j<l} \sin^2((q_j - q_l)/2) \right] \left[ \prod_{m=1}^3 \frac{\varepsilon_{k,q_m}^2\varepsilon_{p,q_m}^2\varepsilon_{q_m}^2}{\varepsilon_{q_m}^2} \right] e^{i2\ell[k - \varepsilon_p]}
\]

(214)

Carrying out the sums over \( q_j \) using the same techniques as for the other contributions we eventually arrive at

\[
D_{(2|3|2)}^{(1)} = \frac{8JA}{L^2} \sum_{0<k\neq p<\mathrm{NS}} K(k)K(p) \frac{\varepsilon_k^2\varepsilon_p^2}{\varepsilon^2(k)} \cos(k) - \cos(p) \left[ \frac{\varepsilon_k \sin(k\ell)}{\sin(k)} - \frac{\varepsilon_p \sin(p\ell)}{\sin(p)} \right] e^{i2\ell[k - \varepsilon_p]}
\]

+ \ldots.

(215)

The large time and distance behaviour is thus the same as for \( D_{(2|1|2)} \).

In summary, the combined \( \mathcal{O}(K^2) \) contributions can be divided into two categories.

(i) Time-independent contributions

\[
4JA \int_{-\pi}^{\pi} \frac{dk}{2\pi} \frac{K^2(k)}{\varepsilon_k} e^{ik\ell} + D_{(0|1|0)} \left[ \gamma^2 - 4\ell \int_0^\pi \frac{dk}{2\pi} K^2(k) \right]
\]

\[-4(h^2 - 1)D(\ell) \int_0^\pi \frac{dk}{2\pi} \frac{4J^2K^2(k)}{\varepsilon^2(k)}.
\]

(216)

(ii) Time-dependent oscillatory contributions

\[
4JA \int_0^\pi \frac{dk}{2\pi} \frac{K(k)K(p)}{\varepsilon^2(k)} \varepsilon^2(p) \frac{\sin(k)\sin(p)}{\cos(k) - \cos(p)} \left[ \frac{\varepsilon_k \sin(k\ell)}{\sin(k)} - \frac{\varepsilon_p \sin(p\ell)}{\sin(p)} \right] e^{i2\ell[k - \varepsilon_p]}
\]

(217)

4.5.4. \( \mathcal{O}(K^3) \) contributions. In order to infer the structure of higher-order oscillatory contributions we determine the following \( \mathcal{O}(K^3) \) term:

\[
D_{(4|3|2)} + D_{(2|3|4)} = \frac{JA}{3L^6} \Re \sum_{0<k_1,2,3<\mathrm{NS} \atop q_1,2,3<\mathbb{R}} K(k_1)K(k_2)K(p) \frac{\sin k_1 \sin k_2 \sin p}{\varepsilon^2(k_1)\varepsilon^2(k_2)\varepsilon^2(p)} g_{k_1,k_2}^2 g_{k_1,-k_2}
\]

\[
\times \frac{g_{q_1,q_2,q_3}}{\varepsilon_{q_1}\varepsilon_{q_2}\varepsilon_{q_3}} e^{i2\ell[k_1 + \varepsilon_{k_2} - \varepsilon_p]} \prod_{j=1}^3 2\varepsilon_{p,q_j}^2 c_{p,q_j}^2 e^{i\ell\varepsilon_j} \prod_{l=1}^2 2\varepsilon_{k_1,q_l}^2 c_{k_1,q_l}.
\]

(218)
where we have defined
\[ g_{p,k} = \frac{2 \sin((p-k)/2)}{\varepsilon_p + \varepsilon_k}. \] (219)

(i) Contributions for \( p \neq k_{1,2} \). We first consider only the contributions with \( p \neq k_{1,2} \) and denote them by \( D^{(1)}_{(4/3|2)} + D^{(1)}_{(2/3|4)} \). In order to carry out the sums over \( q_j \) it is useful to rewrite the product of pole factors \( \prod_{i=1}^{3} \prod_{j=1}^{3} c_{k_i q_i} c_{k_j q_j} \) using the identity
\[ c_{k_1 q} c_{k_2 q} c_{p q} = c_{k_2 k_1} \left\{ c_{p k_1} \left[ c_{k_1 q} - c_{p q} \right] - c_{p k_2} \left[ c_{k_2 q} - c_{p q} \right] \right\}. \] (220)
The fully decomposed expression reads
\[ \prod_{j=1}^{3} 8 c_{k_1 q_j} c_{k_2 q_j} c_{p q_j} = \left[ 8 c_{k_2 k_1} \right]^3 \left\{ -c_{p k_1}^3 \left[ \frac{1}{2} \tilde{C}_{k_1,k_1,p} - \tilde{C}_{k_1,k_1,1} - \frac{1}{2} \tilde{C}_{p,p,k_1} + C_{p,p,p} \right] \right. \\
+ c_{p k_1}^2 c_{p k_2} \left[ \tilde{C}_{k_1,k_2,p} - \tilde{C}_{k_2,k_1,p} + 3 C_{p,p,p} \right] \\
- \frac{1}{2} \tilde{C}_{p,p,k_2} + \frac{1}{2} \tilde{C}_{k_1,p,k_1} - \frac{1}{2} \tilde{C}_{k_2,k_1,k_2} - \left\{ k_1 \leftrightarrow k_2 \right\}, \] (221)
where we have defined
\[ c_{k_1,k_2,k_3} = \prod_{j=1}^{3} c_{k_j,q_j}, \quad \tilde{C}_{k_1,k_2,k_3} = \sum_{p \in S_3} C_{k_1,k_2,k_3} \] (222)
We then can carry out the sums over \( q_j \) using lemma 4 (D.6) and retaining only the pole contributions.

(A) The combined contributions to (218) arising from \( c_{p k_1}^3 \tilde{C}_{k_1,k_1,p}, c_{p k_2} \tilde{C}_{k_2,k_2,p}, c_{p k_1} c_{p k_2} \tilde{C}_{k_1,k_2,k_3} \) and \( c_{p k_1} c_{p k_2} \tilde{C}_{k_2,p,k_2} \) the decomposition (221) are of the form
\[ D^{(1,1)}_{(4/3|2)} + D^{(1,1)}_{(2/3|4)} = \frac{16 J A}{L^3} \sum_{0<k_{1,2} \neq p \in NS} K(k_1) K(k_2) K(p) e^{2i t [\varepsilon(k_1) + \varepsilon(k_2) - \varepsilon_p]} \times \frac{\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_p^2}{\varepsilon_{k_1} \varepsilon_{k_2} c_{k_1 k_2}} \left[ c_{p k_1} - c_{p k_2} \right] \sin(\ell p) + \text{h.c.} + \cdots. \] (223)

(B) The contributions due to \( c_{p k_1}^3 \tilde{C}_{p,p,k_1} \) and \( c_{p k_2} \tilde{C}_{p,p,k_2} \) in (221) are
\[ D^{(1,2)}_{(4/3|2)} + D^{(1,2)}_{(2/3|4)} = \frac{8 J A}{L^3} \sum_{0<k_{1,2} \neq p \in NS} K(k_1) K(k_2) K(p) e^{2i t [\varepsilon(k_1) + \varepsilon(k_2) - \varepsilon_p]} \frac{\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_p^2}{\varepsilon_{k_1} \varepsilon_{k_2}} c_{k_1 k_2} \times \sin p \left[ \sin(\ell k_1) \sin k_2 \varepsilon_{k_2} c_{k_1 k_2} - \left\{ k_1 \leftrightarrow k_2 \right\} \right] + \text{h.c.} + \cdots. \] (224)

(C) The contributions of the \( c_{p k_1} c_{p k_2} \tilde{C}_{p,p,k_1} \) and \( c_{p k_1} c_{p k_2} \tilde{C}_{p,p,k_2} \) terms are
\[ D^{(1,3)}_{(4/3|2)} + D^{(1,3)}_{(2/3|4)} = \frac{16 J A}{L^3} \sum_{0<k_{1,2} \neq p \in NS} K(k_1) K(k_2) K(p) e^{2i t [\varepsilon(k_1) + \varepsilon(k_2) - \varepsilon_p]} \frac{\varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_p^2}{\varepsilon_{k_1} \varepsilon_{k_2}} c_{k_1 k_2} \times c_{k_1 k_2} \sin p \left[ \sin(\ell k_1) \sin k_2 \varepsilon_{k_2} c_{k_1 k_2} - \left\{ k_1 \leftrightarrow k_2 \right\} \right] + \text{h.c.} + \cdots. \] (225)
(D) The contributions of the $c_{pk_1}^2 c_{pk_2} \check{C}_{p,p,k_2}$ and $c_{pk_1}^2 c_{pk_2} \check{C}_{p,p,k_1}$ terms are

$$
D_{(4|3|2)}^{(1,4)} + D_{(2|3|4)}^{(1,4)} = -\frac{8J A}{L^3} \sum_{0 < k_1, 2 \neq p \notin NS} \frac{K(k_1)K(k_2)K(p)}{\varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{p}} e^{2i(\varepsilon_{k_1} - \varepsilon_{p})} \varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_{p}^2 \\
\times c_{k_2,k_1} \sin p \left[ \sin(\ell k_1) \sin k_2 \varepsilon_{k_2}^2 c_{p,k_2}^2 - \{k_1 \leftrightarrow k_2\} \right] + \text{h.c.} + \cdots .
$$

(E) The contributions of the $c_{pk_1}^2 c_{pk_2} \check{C}_{k_1,k_1,k_2}^1$ and $c_{pk_1}^2 c_{pk_2} \check{C}_{k_2,k_2,k_1}^1$ terms are

$$
D_{(4|3|2)}^{(1,5)} + D_{(2|3|4)}^{(1,5)} = \frac{8J A}{L^3} \sum_{0 < k_1, 2 \neq p \notin NS} \frac{K(k_1)K(k_2)K(p)}{\varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{p}} e^{2i(\varepsilon_{k_1} - \varepsilon_{p})} \varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_{p}^2 \\
\times \frac{c_{pk_1} c_{pk_2}}{c_{k_1 k_2}} \sin p \left[ \sin(\ell k_2) \sin k_1 \varepsilon_{k_1}^2 c_{p,k_1} - \{k_1 \leftrightarrow k_2\} \right] + \text{h.c.} + \cdots .
$$

(F) Finally, the contributions of the $c_{pk_1}^2 c_{pk_2} \check{C}_{k_1,k_1,k_2,p}^1$ and $c_{pk_1}^2 c_{pk_2} \check{C}_{k_2,k_2,k_1,p}^1$ terms are

$$
D_{(4|3|2)}^{(1,6)} + D_{(2|3|4)}^{(1,6)} = -\frac{32J A}{L^3} \sum_{k_1, 2 \neq p \notin NS} \frac{K(k_1)K(k_2)K(p)}{\varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{p}} \cos(2t(\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{p})) \\
\times \varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_{p}^2 \sin(\ell k_1 + k_2 + p),
$$

where here the momentum sums are over the entire Brillouin zone.

The contributions from all other terms are subleading. We observe that the leading terms at large $\ell$ and $t$ in (224)–(227) combine to give

$$
\sum_{a=2}^{5} D_{(4|3|2)}^{(1,a)} + D_{(2|3|4)}^{(1,a)} \sim -\frac{64J A}{L^3} \sum_{0 < k_1, 2 \neq p \notin NS} \frac{K(k_1)K(k_2)K(p)}{\varepsilon_{k_1} \varepsilon_{k_2} \varepsilon_{p}} \cos(2t(\varepsilon_{k_1} + \varepsilon_{k_2} - \varepsilon_{p})) \\
\times \varepsilon_{k_1}^2 \varepsilon_{k_2}^2 \varepsilon_{p}^2 \sin k_2 \sin p c_{pk_2} [c_{pk_2} - c_{pk_1}] \sin(\ell k_1) + \cdots .
$$

Both (229) and (228) can be simplified further, because for large $t$ and $\ell$ the dominant contributions arise from the ‘double pole’ factors and can be extracted using lemmas 2a and 2b of appendix D. This leaves us with

$$
D_{(4|3|2)}^{(1,6)} + D_{(2|3|4)}^{(1,6)} = -\frac{16J A}{L^2} \sum_{0 < p \notin NS} \sum_{k \notin NS} K(k) \sin(2t\varepsilon_k - \ell \varepsilon) K^2(p) \\
\times \left[ \frac{L}{6} - \left( (2t\varepsilon_p - \ell) \theta_H (2t\varepsilon_p - \ell) \right) \right] + \cdots ,
$$

$$
\sum_{a=2}^{5} D_{(4|3|2)}^{(1,a)} + D_{(2|3|4)}^{(1,a)} = \frac{16J A}{L^2} \sum_{0 < p \notin NS} \sum_{k \notin NS} K(k) \sin(2t\varepsilon_k - \ell \varepsilon) K^2(p) \left[ \frac{L}{6} - 2t\varepsilon_p^\prime \right] + \cdots .
$$
Combining (230) and (231) we arrive at our final result for the leading asymptotics of the $p \neq k_{1,2}$ contributions to (218)

$$D^{(1)}_{(4|3|2)} + D^{(1)}_{(2|3|4)} = \left[ \frac{4J_A}{L} \sum_{k \in \text{NS}} \frac{K(k)}{\varepsilon_k} \sin(2t\varepsilon_k - \ell k) \right]$$

$$\times \left[ - \frac{4}{L} \sum_{0 < p \in \text{NS}} K^2(p) [2t\varepsilon'_p - \ell] \theta_H(\ell - 2t\varepsilon'_p) \right] + \cdots$$

$$= \left[ D_{(2|1|0)} + D_{(0|1|2)} \right] \left[ - \frac{4}{L} \sum_{0 < p \in \text{NS}} K^2(p) [2t\varepsilon'_p - \ell] \theta_H(\ell - 2t\varepsilon'_p) \right] + \cdots \quad (232)$$

(ii) Contributions with $p = k_1$ or $k_2$. Using the symmetry under $k_1 \leftrightarrow k_2$ we can express these contributions in the form

$$D^{(2)}_{(2|3|4)} + D^{(2)}_{(4|3|2)} = - \frac{1}{6} \sum_{0 < k_{1,2} \in \text{NS}} \sum_{q_1, q_2} K^2(k_1) K(k_2) e^{2t\varepsilon(k_2)}$$

$$\times \sum_{\text{NS}} \langle -k_1, k_1, -k_2, k_2 | \sigma^x_{m+1} | q_1, q_2, q_3 \rangle R \langle q_3, q_2, q_1 | \sigma^x_{m} | k_1, -k_1 \rangle_{\text{NS}} + \text{h.c.} \quad (233)$$

In order to carry out the $q_j$ sums we rewrite the pole factors as

$$\prod_{j=1}^{3} 8c_{k_1 q_j} c_{k_2 q_j} = (8c_{k_2 k_1})^3 \left\{ C^2(k_1, k_1, k_1) - c_{k_2 k_1} b(k_1, k_2 | q_1) c_{k_1 q_2} c_{k_1 q_3} - c_{k_2 k_1} c_{k_1 q_1} b(k_1, k_2 | q_2) c_{k_1 q_3} - c_{k_2 k_1} c_{k_1 q_1} c_{k_2 q_2} b(k_1, k_2 | q_3) + c_{k_1 k_2} c_{k_1 q_1} b(k_1, k_2 | q_1) b(k_1, k_2 | q_3) + c_{k_1 k_2} c_{k_1 q_2} b(k_1, k_2 | q_1) b(k_1, k_2 | q_2) + c_{k_1 k_2} c_{k_1 q_3} b(k_1, k_2 | q_1) b(k_1, k_2 | q_3) \right\} \quad (234)$$

where we have defined

$$b(k_1, k_2 | q) = c_{k_1 q} - c_{k_2 q}. \quad (235)$$

The various terms in (234) contribute in qualitatively different ways, depending on their structure when viewed as functions of (the complex variables) $q_1$, $q_2$ and $q_3$.

1. Terms with only simple poles give rise to contributions of order $\mathcal{O}(L^{-1})$ and can be ignored.

2. Terms with only double poles arise from the first contribution on the rhs of (234). Using (D.6) to carry out the sums over $q_j$ we obtain after some calculations

$$D^{(2,1)}_{(4|3|2)} + D^{(2,1)}_{(2|3|4)} = \frac{1}{L} \left[ 1 - \frac{4\ell}{L} \right] D(\ell) \sum_{0 < k_{1,2} \in \text{NS}} K^2(k_1) K(k_2) \cos(2t\varepsilon_k) c_{k_2 k_1}$$

$$+ \frac{4J_A}{L} \sum_{k \in \text{NS}} \frac{K^3(k)}{\varepsilon_k} \sin(2t\varepsilon_k - \ell k) + \cdots \quad (236)$$

which can be simplified further if required.
(3) Terms with only one double pole arise from the fifth, sixth and seventh contributions on the rhs of (234). Using (D.6) and (D.4) to carry out the sums over \( q_j \) we obtain after some calculations

\[
D_{(4|3|2)}^{(2,2)} + D_{(2|3|4)}^{(2,2)} = \frac{16 JA}{L^2} \sum_{0<k_1 \neq k_2 \in \text{NS}} K^2(k_1)K(k_2) \cos(2t\varepsilon_{k_2}) \sin(k_2) \cos(k_1\ell) \\
\times c_{k_1 k_2} \frac{\varepsilon_{k_1}^2}{\varepsilon_{k_2}} + \cdots.
\]  

(237)

Here we can carry out the \( k_1 \) sum using lemma 3 (D.4), which gives

\[
D_{(4|3|2)}^{(2,2)} + D_{(2|3|4)}^{(2,2)} = \frac{4 JA}{L} \sum_{k \in \text{NS}} \frac{K^3(k)}{\varepsilon_k} \sin(2t\varepsilon_k - \ell k) + \cdots.
\]  

(238)

(4) Terms with two double poles arise from the second, third and fourth contributions on the rhs of (234). Using (D.6) and (D.4) to carry out the sums over \( q_j \), we obtain after some calculations

\[
D_{(4|3|2)}^{(2,3)} + D_{(2|3|4)}^{(2,3)} = \left[ \frac{4 JA}{L} \sum_{k \in \text{NS}} \frac{K(k_2)}{\varepsilon_{k_2}} \sin(2t\varepsilon_{k_2} - \ell k_2) \right] \\
\times \sum_{0<k_1 \in \text{NS}} K^2(k_1) \left[ 1 - \frac{4\ell}{L} \right] + \cdots.
\]  

(239)

This is in fact the leading contribution.

Combining (232) and (239) we arrive at the following result for our \( O(K^3) \) contributions:

\[
D_{(4|3|2)} + D_{(2|3|4)} = [D_{(2|1|0)} + D_{(0|1|2)}] \frac{4}{L} \sum_{0<p \in \text{NS}} K^2(p) \\
\times \left[ \frac{L}{4} - \ell - [2t\varepsilon'(p) - \ell]\theta_H(\ell - 2t\varepsilon'(p)) \right] \\
+ \frac{8 JA}{L} \sum_{k \in \text{NS}} \frac{K^3(k)}{\varepsilon_k} \sin(2t\varepsilon_k - \ell k) + \cdots.
\]  

(240)

We see that as expected there is a contribution that diverges in the thermodynamic limit. However, in addition there are terms that become very large for \( \ell \gg 1 \).

4.5.5. Resummation of leading \( D_{(2n|2n+1|2n)} + D_{(2n|2n+1|2n+2)} \) contributions. The above calculation of the \( O(K^3) \) contribution shows that the dominant contributions arise from the regions

\[
k_{Q_1} \approx p, \quad \begin{pmatrix} q_{s_1} \\ q_{s_2} \\ q_{s_3} \end{pmatrix} \approx \begin{pmatrix} \sigma_1 k_{Q_1} \\ -\sigma_1 p \\ \sigma_2 k_{Q_2} \end{pmatrix}, \quad \sigma_j = \pm,
\]  

(241)

where \((Q_1, Q_2)\) and \((S_1, S_2, S_3)\) are permutations of \((1, 2)\) and \((1, 2, 3)\) respectively. Motivated by this observation we therefore consider the analogous regions for the
\( \mathcal{O}(K^{2n+1}) \) contributions \( D_{(2n+2|2n+1|2n)} \) and \( D_{(2n|2n+1|2n+2)} \). They are

\[
\left( \begin{array}{c} k_{Q_1} \\
\vdots \\
k_{Q_n} \end{array} \right) \sim \left( \begin{array}{c} p_1 \\
\vdots \\
p_n \end{array} \right), \quad \left( \begin{array}{c} q_{S_1} \\
\vdots \\
q_{S_{2n-1}} \\
q_{S_{2n}} \end{array} \right) \sim \left( \begin{array}{c} \sigma_1 k_{Q_1} \\
-\sigma_1 p_1 \\
\vdots \\
\sigma_n k_{Q_n} \\
-\sigma_n p_n \\
\sigma_{n+1} k_{Q_{n+1}} \end{array} \right),
\]

(242)

where \( \sigma_j = \pm \) and \( (Q_1, \ldots, Q_{n+1}), (S_1, \ldots, S_{2n+1}) \) are permutations of \( (1, 2, \ldots, n+1) \) and \( (1, 2, \ldots, 2n+1) \) respectively. Our goal is to determine the contribution of the regions (242) to

\[
D_{(2n+2|2n+1|2n)} + \text{h.c.} = C \Re \sum_{k_1, \ldots, k_{n+1} \in \mathbb{NS}} \left[ \prod_{r=1}^{n+1} \frac{\sin(k_r) K(k_r) e^{2i\varepsilon k_r}}{\varepsilon^2_{k_r}} \right] \\
\times \left[ \prod_{s=1}^{n} \frac{\sin(p_s) K(p_s) e^{-2i\varepsilon p_s}}{\varepsilon^2_{p_s}} \right] \\
\times \prod_{l<l'}^2 c_{k_l k_{l'}} c_{k_{l'} k_l} \prod_{m<m'}^2 c_{p_m p_{m'}} c_{p_{m'} p_m} \sum_{q_1, \ldots, q_{2n} \in \mathbb{NS}} \left[ \prod_{r=1}^{2n+1} \frac{e^{i\varepsilon q_r}}{\varepsilon_{q_r}} \right] \prod_{l<l'}^2 g_{q_l q_{l'}}^2 \\
\times \prod_{s=1}^{2n+1} 2\varepsilon_{k_{n+1} q_s} c_{k_{n+1} q_s},
\]

(243)

where the constant \( C \) is

\[
C = \frac{4JA}{L^{n+2}(n+1)!n!(2n+1)!}.
\]

(244)

As all regions (242) contribute equally, we focus on the case \( (Q_1, \ldots, Q_{n+1}) = (1, \ldots, n+1), (S_1, \ldots, S_{2n+1}) = (1, \ldots, 2n+1) \) and multiply the result by a combinatorial factor \( (n+1)!2n+1)! \). We first carry out the summations over \( q_1, \ldots, q_{2n} \) and \( k_1, \ldots, k_n \) by following the analogous calculation for the ordered phase, see section 4.4. Finally, we carry out the sum over \( q_{2n+1} \) using (D.4). This results in

\[
D_{(2n+2|2n+1|2n)} + \text{h.c.} = -\frac{2A}{n!} \sum_{p_1, \ldots, p_n \in \mathbb{NS}} \left[ \prod_{r=1}^{n} K^2(p_r) \left[ 1 - \frac{4\ell}{L} - \frac{4}{L} \theta_H(\ell - 2t\varepsilon_{p_r}) \right] [2t\varepsilon_{p_r} - \ell] \right] \\
\times \frac{4J}{L} \sum_{0<k_{n+1} \in \mathbb{NS}} \frac{K(k_{n+1})}{\varepsilon_{k_{n+1}}} \sin(k_{n+1}\ell) \cos(2t\varepsilon_{k_{n+1}}) + \cdots \\
= \frac{1}{n!} \sum_{p_1, \ldots, p_n \in \mathbb{NS}} \left[ \prod_{r=1}^{n} K^2(p_r) \left[ 1 - \frac{4\ell}{L} - \frac{4}{L} \theta_H(\ell - 2t\varepsilon_{p_r}) \right] [2t\varepsilon_{p_r} - \ell] \right] \\
\times \left[ D_{(2|1|0)} + D_{(0|1|2)} \right] + \cdots.
\]

(245)
The sum over \( n \) can be again taken by inverting the steps used to express (120) in terms of (125), which gives

\[
\sum_{n=0}^{\infty} D_{(2n+2)(2n+1)2n} + \text{h.c.} = [D_{(2)10} + D_{(0)12}]
\]

\[
\times \exp \left( \sum_{0<p \in \text{NS}} \ln \left[ 1 + K^2(p) \left( 1 - \frac{4}{L} \left( \ell + \theta_H \left( \ell - 2t\epsilon_p' \right) \right) \right) \right] \right) + \cdots
\]

\[
\approx [D_{(2)10} + D_{(0)12}]_{\text{NS}} (B|B)_{\text{NS}}
\]

\[
\times \exp \left[ -\frac{4}{L} \sum_{0<p \in \text{NS}} K^2(p) \left( \ell + \theta_H \left( \ell - 2t\epsilon_p' \right) \right) \right].
\]

(246)

In the last step we have only retained the \( O(K^2) \) term in the exponent in order not to exceed the accuracy of our calculation.

4.5.6. Full answer for the two-point function in the disordered phase. The above results suggest that the two-point function consists of two parts

\[
\lim_{L \to \infty} \rho^{xx}(t, \ell) = F_1(\ell) + F_2(t, \ell).
\]

(247)

The two kinds of contribution are

- an oscillating time-dependent contribution arising from the terms \( D_{(2n+2)(2n+1)2n} + \text{h.c.} \)

\[
F_2(t, \ell) \approx 2J_A \int_{-\pi}^{\pi} \frac{dk}{\pi} K(k) \sin(2t\epsilon_k - k\ell)
\]

\[
\times \exp \left[ -2 \int_0^{\pi} \frac{dp}{\pi} K^2(p) \left( \ell + \theta_H \left( \ell - 2t\epsilon_p' \right) \right) \right]
\]

(248)

- an exponentially small, time-independent contribution arising from the ‘diagonal’ terms

\[
F_1(\ell) = \frac{1}{\text{NS}(B|B)_{\text{NS}}} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{0<k_1, \ldots, k_n \in \text{NS}} \left[ \prod_{j=1}^{n} K^2(k_j) \right]
\]

\[
\times \text{NS}(-k_n, k_n, \ldots, -k_1, k_1|\sigma_{m+\ell}^x |k_1, -k_1, \ldots, k_n, -k_n)_{\text{NS}}.
\]

(249)

The \( O(K^0) \) and \( O(K^2) \) have been evaluated in sections 4.5.1 and 4.5.3 respectively and are given by

\[
F_1(\ell) \approx 2J_A \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{iqt} \left[ 1 + 2K^2(q) - 4\ell \int_{0}^{\pi} \frac{dk}{2\pi} K^2(k) \right]
\]

\[
- 4(h^2 - 1) \int_{0}^{\pi} \frac{dk}{2\pi} \frac{4J^2 K^2(k)}{\epsilon^2(k)} D(\ell).
\]

(250)

It is shown in appendix E that the ‘pair ensemble’ average (249) is equal to the average in the generalized Gibbs ensemble, which was previously calculated in [71]. Hence we

\[
doi:10.1088/1742-5468/2012/07/P07016
\]
conclude that
\[ F_1(\ell) \simeq C_{PP}(\ell) \exp \left( -\frac{\xi}{\ell} \right), \]
\[ \xi^{-1} = \ln \left( \min[h_0, h_1] \right) - \ln \left[ \frac{1 + hh_0 + \sqrt{(h^2 - 1)(h_0^2 - 1)}}{2hh_0} \right], \]  
(251)
where \( h_1 = (1 + hh_0 + \sqrt{(h^2 - 1)(h_0^2 - 1)})/(h + h_0) \) and the large distance behaviour of \( C_{PP}(\ell) \) is determined in paper II.

5. Scaling limit of the Ising model

So far we have focused on the quench dynamics in the transverse field Ising lattice model. In the vicinity of the quantum critical point at \( h = 1 \) a quantum field theory description applies; see, e.g., [99]. As quantum quenches in integrable fields theories are of great current interest, we now present explicit expressions for the quench dynamics in the field theory limit.

The scaling limit of the transverse field Ising chain is \((a_0 \text{ is the lattice spacing}) [99] \]
\[ J \to \infty, \quad h \to 1, \quad a_0 \to 0, \]  
(252)
while keeping fixed both the gap \( \Delta \) and the velocity \( v \)
\[ 2J|1 - h| = \Delta, \quad 2Ja_0 = v. \]  
(253)
In this limit the dispersion and Bogoliubov angle become
\[ \varepsilon(q) = \sqrt{\Delta^2 + v^2q^2}, \]  
(254)
\[ \theta_q(h) \to \text{sgn}(h - 1) \arctan \left( \frac{vq}{\Delta} \right). \]  
(255)
In our quench problem both the initial and the final magnetic field are scaled to the critical point, i.e. we need to take
\[ h_0 \to 1, \quad J|1 - h_0| = \Delta_0 = \text{fixed}. \]  
(256)
In this limit the \( K \)-matrix turns into
\[ K(q) = \tan \left[ \frac{\arctan (vq/\Delta) - \arctan(vq/\Delta_0)}{2} \right]. \]  
(257)
Here the physical momentum is defined as
\[ q = \frac{k}{a_0}, \quad -\infty < q < \infty. \]  
(258)
The Hamiltonian describing the scaling limit is expressed in terms of Majorana fermions as
\[ H = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \left[ \frac{iv}{2} (\bar{\psi} \partial_x \psi - \psi \partial_x \bar{\psi}) - i\Delta \psi \bar{\psi} \right]. \]  
(259)
The order parameter in the scaling limit must be defined as
\[ \sigma(x) \propto |1 - h^2|^{-1/8} \sigma_n^x. \]  
(260)

\textit{doi:10.1088/1742-5468/2012/07/P07016}
where \( x = na_0 \). It is customary to choose the normalization of the field \( \sigma(x) \) such that

\[
\lim_{x \to 0} \langle 0 | \sigma(x) \sigma(0) | 0 \rangle = \frac{1}{|x|^{1/4}},
\]

which implies that

\[
\sigma_j^x \to 2^{1/24} e^{1/8} \mathcal{A}^{-3/2} a_0^{1/8} \sigma(x),
\]

where

\[
\mathcal{A} = 1.282 \, 427 \, 129 \, 100 \, 62 \cdots.
\]

### 5.1. Ordered phase

The result for the two-point function after a quench within the ordered phase in the scaling limit is

\[
\frac{\langle \psi_0(t) | \sigma(x) \sigma(0) | \psi_0(t) \rangle}{\langle \psi_0 | \psi_0 \rangle} \propto \exp \left( \int_0^\infty \frac{dq}{\pi} \ln \left[ \frac{1 - K^2(q)}{1 + K^2(q)} \right] \left[ x \theta(2 \varepsilon'(q)t - x) + 2 \varepsilon'(q) t \theta(x - 2 \varepsilon'(q)t) \right] \right),
\]

where \( x = na_0 \). This expression is well defined because at large \( q \) we have

\[
K(q) \sim \frac{\Delta - \Delta_0}{2vq}.
\]

In the stationary state we have

\[
\lim_{t \to \infty} \frac{\langle \psi_0(t) | \sigma(x) \sigma(0) | \psi_0(t) \rangle}{\langle \psi_0 | \psi_0 \rangle} \propto \exp \left( -\frac{x}{\zeta} \right),
\]

where

\[
\zeta^{-1} = \frac{\Delta + \Delta_0}{2v} - \frac{\sqrt{\Delta \Delta_0}}{v}.
\]

### 5.2. Disordered phase

For quenches within the disordered phase we can take the scaling limits of the results (248) and (249), which give

\[
\frac{\langle \psi_0(t) | \sigma(x) \sigma(0) | \psi_0(t) \rangle}{\langle \psi_0 | \psi_0 \rangle} \propto \mathcal{F}_1(x) + \mathcal{F}_2(t, x),
\]

where the time-dependent part dominates except at very late times and is given by

\[
F_2(t, x) \approx 2v \int_{-\infty}^{\infty} \frac{dq}{2\pi} K(q) \sin(2t \varepsilon(q) - qx) \times \exp \left( \int_0^\infty \frac{dq}{\pi} \ln \left[ \frac{1 - K^2(q)}{1 + K^2(q)} \right] \left[ x \theta(2 \varepsilon'(q)t - x) + 2 \varepsilon'(q) t \theta(x - 2 \varepsilon'(q)t) \right] \right).
\]
The stationary state component is
\[ \mathcal{F}_1(x) \sim \exp \left( -\frac{x}{\zeta} \right), \tag{270} \]
where now
\[ \zeta^{-1} = \frac{\Delta + \Delta_0}{2v} - \frac{\sqrt{\Delta\Delta_0}}{v} + \frac{\min(\Delta_0, \sqrt{\Delta\Delta_0})}{v}. \tag{271} \]

6. Conclusions

In this work we have derived analytic expressions for the time evolution of one- and two-point functions in the transverse field Ising chain after a sudden quench of the magnetic field. To do so we have developed two novel methods based on determinants and form factor sums respectively. The former is applicable to quenches in models with free fermionic spectrum and our analysis generalizes straightforwardly e.g. to the spin-1/2 XY chain in a magnetic field [88]. Results obtained by this method are exact for asymptotically large times and distances in what we call the space–time scaling limit \((t, \ell \to \infty, \text{keeping their ratio fixed})\). The form factor approach is applicable more generally to integrable quenches [32] in integrable quantum field theories [100] such as the sine–Gordon model. It is furthermore straightforwardly extended to the study of non-equal time correlation functions [101]. The form factor method provides approximate results that become exact in the limit of small quenches, defined by the requirement that the density of excitations (of the post-quench Hamiltonian) in the initial state is low. We observe that the difference of the form factor and exact results for quenches within either the ferromagnetic or paramagnetic phase are generally very small, except for quenches originating or terminating in the close vicinity of the quantum critical point.

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Appendix A. Diagonalization of the transverse field Ising model

In this appendix we summarize the diagonalization of the TFIM [102] with periodic boundary conditions
\[ H(h) = -J \sum_{j=1}^{L} [\sigma_j^x \sigma_{j+1}^x + h \sigma_j^z], \tag{A.1} \]
where \(\sigma_j^\alpha\) are the Pauli matrices at site \(j\) and
\[ \sigma_{L+1}^\alpha = \sigma_1^\alpha, \quad \alpha = x, y, z. \tag{A.2} \]
The dimensionless constant $h$ describes the coupling with an external magnetic field $Jh$. The quantum Ising chain is mapped to a model of spinless fermions by means of a Jordan–Wigner transformation. Defining $\sigma_j^\pm = (\sigma_j^x \pm i\sigma_j^y)/2$, we construct spinless fermion creation and annihilation operators by

$$c_j^\dagger = \prod_{j=1}^{j-1} \sigma_j^z \sigma_j^\dagger, \quad \{c_j, c_j^\dagger\} = \delta_{jj}. \quad (A.3)$$

The inverse transformation is

$$\sigma_j^z = 1 - 2c_j^\dagger c_j, \quad \sigma_j^x = \prod_{l=1}^{j-1} (1 - 2c_l^\dagger c_l)(c_j + c_j^\dagger). \quad (A.4)$$

The Hamiltonian can be expressed in terms of the fermions as

$$H(h) = -J \sum_{j=1}^{L} [c_j^\dagger - c_j][c_{j+1} + c_{j+1}^\dagger] - Jh \sum_{j=1}^{L} c_j c_j^\dagger - c_j^\dagger c_j$$

$$- J e^{i\pi \hat{N}} (c_L^\dagger - c_L)(c_1 + c_1^\dagger), \quad (A.5)$$

where

$$\hat{N} = \sum_{j=1}^{L} c_j^\dagger c_j. \quad (A.6)$$

As $[H, e^{i\pi \hat{N}}] = 0$ we may diagonalize the two operators simultaneously. The Hamiltonian is block diagonal, $H = H_e \oplus H_o$, where $H_e/o$ act on the subspaces of the Fock space with an even/odd number of fermions respectively.

### A.1. Even fermion number

In the sector with an even number of fermions we have $e^{i\pi \hat{N}} = 1$ and the Hamiltonian can be written in the form

$$H_e(h) = -J \sum_{j=1}^{L} [c_j^\dagger - c_j][c_{j+1} + c_{j+1}^\dagger] - Jh \sum_{j=1}^{L} c_j c_j^\dagger - c_j^\dagger c_j, \quad (A.7)$$

where we have imposed antiperiodic boundary conditions on the fermions

$$c_{L+1} = -c_1. \quad (A.8)$$

The Hamiltonian $H_e$ is diagonalized by going to Fourier space

$$c(k_n) = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} c_j e^{iknj}, \quad (A.9)$$

where $k_n$ are quantized according to (A.8)

$$k_n = \frac{2\pi(n + 1/2)}{L}, \quad n = -\frac{L}{2}, \ldots, \frac{L}{2} - 1. \quad (A.10)$$

The antiperiodic sector is commonly referred to as the Neveu–Schwarz (NS) sector. Following this nomenclature we introduce the notation $k \in \text{NS}$ to describe the set (A.10).
Introducing Bogoliubov fermions by
\begin{align}
    c(k_n) &= \cos(\theta_{k_n}/2)\alpha_{k_n} + i\sin(\theta_{k_n}/2)\alpha^\dagger_{-k_n}, \\
    c^\dagger(-k_n) &= i\sin(\theta_{k_n}/2)\alpha_{k_n} + \cos(\theta_{k_n}/2)\alpha^\dagger_{-k_n},
\end{align}
where the Bogoliubov angle fulfills
\begin{equation}
    e^{i\theta_k} = \frac{h - e^{ik}}{\sqrt{1 + h^2 - 2h\cos k}},
\end{equation}
the Hamiltonian becomes diagonal
\begin{equation}
    H_e(h) = \sum_{n=-L/2}^{L/2-1} \varepsilon(k_n) \left[ \alpha^\dagger_{k_n}\alpha_{k_n} - \frac{1}{2} \right].
\end{equation}
Here the dispersion relation is
\begin{equation}
    \varepsilon_k = 2J\sqrt{1 + h^2 - 2h\cos k}.
\end{equation}
A basis for the Fock space in the sector with even fermion number is then given by
\begin{equation}
    |k_1, \ldots, k_{2m}; h\rangle_{NS} = \prod_{j=1}^{2m} \alpha^\dagger_{k_j}|0; h\rangle_{NS}, \quad k_j \in \text{NS},
\end{equation}
where the fermion vacuum $|0; h\rangle_{NS}$ is the state annihilated by all $\alpha_{k_j} (j = -L/2, \ldots, L/2 - 1)$.

**A.2. Odd fermion number**

In the sector with an odd number of fermions we have $e^{i\pi N} = -1$. The Hamiltonian can again be written in the form
\begin{equation}
    H_o(h) = -J \sum_{j=1}^{L} [c^\dagger_j - c_j][c_{j+1} + c^\dagger_{j+1}] - Jh \sum_{j=1}^{L} c_j c^\dagger_{j+1} - c^\dagger_{j} c_j,
\end{equation}
but now we have to impose periodic boundary conditions on the fermions
\begin{equation}
    c_{L+1} = c_1.
\end{equation}
In Fourier space we therefore now have
\begin{equation}
    c(p_n) = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} c_j e^{ip_n j},
\end{equation}
where $p_n$ are quantized according to (A.17)
\begin{equation}
    p_n = \frac{2\pi n}{L}, \quad n = -\frac{L}{2}, \ldots, \frac{L}{2} - 1.
\end{equation}
The periodic sector is known as the Ramond sector and we shall denote the set (A.19) by \( p_n \in \mathbb{R} \). Defining Bogoliubov fermions \( \alpha_{p_n} \) for \( p_n \neq 0 \) by

\[
\begin{align*}
    c(p_n) &= \cos(\theta_{p_n}/2)\alpha_{p_n} + i \sin(\theta_{p_n}/2)\alpha_{-p_n}^\dagger, \\
    c^\dagger(-p_n) &= i \sin(\theta_{p_n}/2)\alpha_{p_n} + \cos(\theta_{p_n}/2)\alpha_{-p_n}^\dagger,
\end{align*}
\]

(A.20)

we can express the Hamiltonian as

\[
H_0(h) = \sum_{n = -L/2}^{L/2-1} \varepsilon(p_n)[\alpha_{p_n}^\dagger \alpha_{p_n} - \frac{1}{2}] - 2J(1 - h)[\alpha_0^\dagger \alpha_0 - \frac{1}{2}].
\]

(A.21)

A basis of the subspace of the Fock space with odd fermion numbers is then given by

\[
|p_1, \ldots, p_{2m+1}; h\rangle = \prod_{j=1}^{2m+1} \alpha_{p_j}^\dagger |0; h\rangle_R, \quad p_j \in \mathbb{R},
\]

(A.22)

where the fermion vacuum \( |0\rangle_R \) is the state annihilated by all \( \alpha_{p_j} \) (\( j = -L/2, \ldots, L/2-1 \)).

A.3. Paramagnetic phase \( h > 1 \)

Here the ground state is

\[
|0\rangle_{NS}.
\]

(A.23)

A complete set of states is then given by

\[
|p_1, \ldots, p_{2m+1}; h\rangle_R = \prod_{k_j \in \mathbb{R}}^{2m} \alpha_{k_j}^\dagger |0; h\rangle_R,
\]

\[
|k_1, \ldots, k_{2m}; h\rangle_{NS} = \prod_{p_j \in \mathbb{NS}} \alpha_{k_j}^\dagger |0; h\rangle_{NS}.
\]

(A.24)

The Hamiltonians can be written as

\[
\begin{align*}
    H_v(h) &= \sum_{k_n \in \mathbb{NS}} \varepsilon(k_n) \alpha_{k_n}^\dagger \alpha_{k_n} + E_0^{NS}(h), \\
    H_o(h) &= \sum_{p_n \in \mathbb{R}} \varepsilon(p_n) \alpha_{p_n}^\dagger \alpha_{p_n} + E_0^R(h),
\end{align*}
\]

(A.25)

where \( E_0^a(h) = \sum_{q \in a} \varepsilon(q) \), \( a = \mathbb{R}, \mathbb{NS} \).

A.4. Ferromagnetic phase \( h < 1 \)

As the zero momentum mode has negative energy it is useful to perform a particle–hole transformation

\[
\alpha_0 \rightarrow \alpha_0^\dagger.
\]

(A.26)
Redefining the Ramond vacuum as the state that is annihilated by all $\alpha_{p_n}$ after the particle–hole transformation, we can construct a complete set of states as

$$|k_1, \ldots, k_{2m}; h\rangle_R = \prod_{k_j \in R} \alpha^\dagger_{k_j} |0; h\rangle_R,$$

$$|p_1, \ldots, p_{2m}; h\rangle_{NS} = \prod_{p_j \in NS} \alpha^\dagger_{p_j} |0; h\rangle_{NS}. \quad (A.27)$$

The Hamiltonians are then again given by (A.25), but now

$$E_{0}^{NS}(h) = \sum_{k \in NS} \varepsilon_k, \quad E_{0}^{R}(h) = \sum_{p \in R} \varepsilon_p - 2J(1 - h). \quad (A.28)$$

For large $L$ we have $E_{0}^{NS}(h) - E_{0}^{R}(h) = O(L^{-1})$, so that there are two low energy states

$$|0; h\rangle_R, \quad |0; h\rangle_{NS}. \quad (A.29)$$

As long as $L$ is finite the ground state is $|0; h\rangle_{NS}$. On the other hand, in the thermodynamic limit the states (A.29) become degenerate and by spontaneous symmetry breaking one of the two combinations

$$\frac{1}{\sqrt{2}} [|0; h\rangle_R \pm |0; h\rangle_{NS}] \quad (A.30)$$

is selected as the ground state.

**Appendix B. Initial state**

As described in appendix A the Hamiltonian $H(h_0)$ can be diagonalized by a Bogoliubov transformation. Let us denote the corresponding Bogoliubov fermions by $\tilde{\alpha}_k$, the Bogoliubov angle by $\theta_k^0$ and the NS vacuum by $|0; h_0\rangle_{NS}$. Similarly, the Hamiltonian $H(h)$ is diagonalized by the Bogoliubov transformation (A.11) and its lowest energy state in the even fermion sector is $|0; h\rangle_{NS}$. As both sets of Bogoliubov fermions are given in terms of the same spinless fermions $c_j$ and $c_j^\dagger$ ($j = 1, \ldots, L$), they can be expressed in terms of one another by

$$\tilde{\alpha}_{k_n} = \cos \left( \frac{\theta_{k_n} - \theta_{k_n}^0}{2} \right) \alpha_{k_n} + i \sin \left( \frac{\theta_{k_n} - \theta_{k_n}^0}{2} \right) \alpha^\dagger_{-k_n}. \quad (B.1)$$

As both sets of fermions can be used to construct a basis of the even Fock space we can express $|0; h_0\rangle_{NS}$ in the form

$$|0; h_0\rangle_{NS} = \sum_{n=0}^{\infty} \sum_{k^{(1)}, \ldots, k^{(n)} \in NS} f_{k^{(1)}, \ldots, k^{(n)}} \prod_{j=1}^{n} \alpha^\dagger_{k^{(j)}} |0; h\rangle_{NS}. \quad (B.2)$$

Using the expression (B.1) in the condition

$$\tilde{\alpha}_{k_n} |0; h_0\rangle_{NS} = 0, \quad (B.3)$$

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allows determination of the coefficients $f_{k^{(1)},\ldots,k^{(n)}}$. A simple calculation gives

$$|0; h_0\rangle_{\text{NS}} = \frac{1}{\mathcal{N}_{\text{NS}}} \exp \left[ i \sum_{p \in \text{NS}} K(p) \alpha_p^\dagger \alpha_p^\dagger \right] |0; h\rangle_{\text{NS}},$$

(B.4)

where $\mathcal{N}_{\text{NS}}$ is a normalization constant and the function $K(k)$ is given by

$$K(k) = \tan \left( \frac{\theta_k - \theta_0^k}{2} \right).$$

(B.5)

The equations of motion for $\alpha_k$ imply that

$$\alpha_k(t) = e^{it\epsilon_k} \alpha_k(0),$$

(B.6)

so that [61]

$$e^{-itH_{e(h)}} |0; h_0\rangle_{\text{NS}} = \frac{|B(t)\rangle_{\text{NS}}}{\sqrt{\text{NS} \langle B|B \rangle_{\text{NS}}}},$$

(B.7)

where

$$|B(t)\rangle_{\text{NS}} = e^{-itE_0^{\text{NS}}} \exp \left[ i \sum_{0 < p \in \text{NS}} e^{-2it\epsilon_p} K(p) \alpha_p^\dagger \alpha_p^\dagger \right] |0; h\rangle_{\text{NS}}.$$  

(B.8)

Similarly, one can show that

$$e^{-itH_{o(h)}} |0; h_0\rangle_{\text{R}} = \frac{|B(t)\rangle_{\text{R}}}{\sqrt{\text{R} \langle B|B \rangle_{\text{R}}}},$$

(B.9)

where

$$|B(t)\rangle_{\text{R}} = e^{-itE_0^{\text{R}}} \exp \left[ i \sum_{0 < p \in \text{R}} e^{-2it\epsilon_p} K(p) \alpha_p^\dagger \alpha_p^\dagger \right] |0; h\rangle_{\text{R}}.$$  

(B.10)

The states (B.8) and (B.10) are of the same form as boundary states in integrable scattering theories [94, 9].

A physical interpretation of the function $K(k)$ (B.5) is obtained as follows. The density of post-quench Bogoliubov fermions $\alpha_k^\dagger \alpha_k$ in the initial state is given by

$$a\langle 0; h_0| \alpha_k^\dagger \alpha_k |0; h_0\rangle_a = \frac{1}{\mathcal{N}_a^{\text{NS}}} \langle 0; h_0 \prod_{k \in a} (1 - iK(k)\alpha_k^\dagger \alpha_k) \alpha_k^\dagger \alpha_k \left( 1 + iK(k)\alpha_k^\dagger \alpha_k^\dagger \right) |0; h_0\rangle_a$$

$$= \frac{K^2(k)}{1 + K^2(k)}, \quad a = \text{NS, R}.$$  

(B.11)

This implies that in the case where $K(k)$ is uniformly small in $k$ we have

$$\langle \Psi_0| \alpha_k^\dagger \alpha_k |\Psi_0\rangle = K^2(k) + \mathcal{O}(K^4),$$

(B.12)
where $|\Psi_0\rangle$ is the initial state of our quantum quench. Physically, the small parameter characterizing the expansion in powers of $K(k)$ is therefore the density of excitations of the post-quench Hamiltonian $H(h)$ induced by the quantum quench.

**Appendix C. Proof of the product formula**

We need to evaluate the product of the $2 \times 2$ matrices

$$
\Pi_{2n}(\{a_i\}) \equiv \prod_{i=1}^{2n} M(a_i), \quad \text{with} \quad M(a) = A\sigma_x + B\sigma_y e^{ia\sigma_z}.
$$

(C.1)

Compared to equation (85) in the main text we have $A = n_x(k_0)$, $B = |\bar{n}_\perp(k_0)|$, $a_i = 2\epsilon_{i-1} t$ and we choose (without loss of generality) $\bar{n}_\perp = \bar{y}$. Using the algebra of the Pauli matrices, it is straightforward to calculate

$$
\Pi_2(a_1, a_2) = M(a_1) \cdot M(a_2) = A^2 I + B^2 e^{i(a_2-a_1)\sigma_x} + iAB\sigma_z(e^{ia_2\sigma_x} - e^{ia_1\sigma_x}),
$$

(C.2)

where $I$ is the $2 \times 2$ identity matrix. A slightly longer exercise is required to calculate $\Pi_4$ and to obtain

$$
\Pi_4 = A^4 I + iA^3 B\sigma_z \sum_{j=1}^{4} (-1)^j e^{ia_1\sigma_x} + A^2 B^2 \sum_{1 \leq j_1 < j_2 \leq 4} (-1)^{j_1+j_2+1} e^{i(a_{j_2}-a_{j_1})\sigma_x} \\
+ iAB^2\sigma_z \sum_{1 \leq j_1 < j_2 < j_3 \leq 4} (-1)^{j_1+j_2+j_3+1} e^{i(a_{j_3} - a_{j_2} + a_{j_1})\sigma_x} + B^4 e^{i(a_4-a_3+a_2-a_1)\sigma_x}.
$$

(C.3)

From these two first examples, it should be clear that the general structure of $\Pi_{2n}$ is

$$
\Pi_{2n} = \sum_{p=0}^{2n} A^{2n-p} B^p (i\sigma_z)^p \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq 2n} (-1)^{\sum_{k=1}^{p} j_k} \exp \left( i \sum_{k=1}^{p} (-1)^{p-k} a_{j_k} \sigma_x \right) \cdot e^{i2\epsilon x_k f(k)}.
$$

(C.4)

Having this conjecture, it is straightforward (but requires some algebra) to prove it by induction, showing that it is compatible with the recurrence relation

$$
\Pi_{2n+2}(a_1, \ldots, a_{2n+2}) = \Pi_{2n}(a_1, \ldots, a_{2n}) \cdot \Pi_2(a_{2n+1}, a_{2n+2}).
$$

(C.5)

**Appendix D. Useful relations**

**Lemma 1.** For any function $f(z)$ that is $2\pi$ periodic and analytic in a strip around the real axis, we have for $0 < k \in \mathbb{R}$

$$
\frac{1}{L} \sum_{q_n \in \mathbb{R}_{>0}} \frac{f(q_n) e^{2i\epsilon(q_n)}}{(cos k - cos q_n)^2} = \left[ \frac{L}{4} - t\epsilon'(k) \right] \frac{e^{2i\epsilon x_k f(k)}}{\sin^2 k} + ie^{2i\epsilon x_k f'(k)} \sum_{k=1}^{p} (-1)^{p-k} a_{j_k} \sigma_x \\
- \oint_{\gamma} \frac{f(z) e^{2i\epsilon(z)t}}{2\pi (cos z - cos k)^2 (1 + e^{iLz})},
$$

where the integration is along a closed contour encircling the interval $[0, \pi]$. 

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Lemma 2a. For any function $f(z)$ that is $2\pi$ periodic and analytic in a strip around the real axis, we have for $0 < k \in \mathbb{R}$

\[
\frac{1}{L} \sum_{q_n \in R > 0} \frac{f(q_n)e^{2i\epsilon(q_n)}}{(\cos k - \cos q_n)^2} = \left[ \frac{L}{12} \left( 1 - \frac{6i \cot k}{L} \right) - t\epsilon'(k) \right] \frac{e^{2i\epsilon k} f(k)}{\sin^2 k}
\]

\[
+ i \frac{e^{2i\epsilon k} f'(k)}{2\sin^2 k} + \oint \frac{dz}{2\pi} \frac{f(z)e^{2i\epsilon(z)l}}{2(\cos z - \cos k)^2 (1 - e^{iLz})}.
\]  

The analogous equation for momenta in the NS sector is the following.

Lemma 2b. For any function $f(z)$ that is $2\pi$ periodic and analytic in a strip around the real axis, we have for $0 < k \in \text{NS}$

\[
\frac{1}{L} \sum_{q_n \in \text{NS} > 0} \frac{f(q_n)e^{2i\epsilon(q_n)}}{(\cos k - \cos q_n)^2} = \left[ \frac{L}{12} \left( 1 + \frac{6i \cot k}{L} \right) - t\epsilon'(k) \right] \frac{e^{2i\epsilon k} f(k)}{\sin^2 k}
\]

\[
- i \frac{e^{2i\epsilon k} f'(k)}{2\sin^2 k} + \oint \frac{dz}{2\pi} \frac{f(z)e^{2i\epsilon(z)l}}{2(\cos z - \cos k)^2 (1 + e^{iLz})}.
\]  

Lemma 3. For large $j \gg 1$ and any function $f(z) = f(z + 2\pi)$ that is analytic in a strip around the real axis, we have

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{q_n \in \text{NS}} \frac{f(q_n)e^{ijq_n}}{\cos k - \cos q_n} = \frac{e^{-ikj} f(-k) - e^{ikj} f(k)}{2i \sin k} + \mathcal{O}(e^{-\gamma j}),
\]  

where $\gamma$ is a positive constant.

Proof. Using contour integration, we have

\[
\frac{1}{L} \sum_{q_n} \frac{f(q_n)e^{ijq_n}}{\cos k - \cos q_n} = \oint \frac{dz}{2\pi} \frac{f(z)e^{ijz}}{\cos z - \cos k}(1 + e^{iLz}) + \frac{e^{-ikj} f(-k) - e^{ikj} f(k)}{2i \sin k},
\]  

where we have used the quantization conditions (A.10) and (A.19) and where the integration is along an anticlockwise contour encircling the interval $[-\pi, \pi]$ in such a manner that no singularities of $f(z)$ lie within it. The contribution of the part of the contour below the real axis tends to zero in the limit $L \to \infty$ because $|e^{ij(L\pm j)}| \gg 1$. As $j > 0$, the part of the contour above the real axis can be deformed as shown in figure D.1. The contributions of the pieces parallel to the imaginary axis cancel due to the periodicity of $f(z)$, while the remaining part has the exponentially small bound given in (D.4).
Lemma 4. For any function $f(z)$ that is $2\pi$ periodic and analytic in a strip around the real axis, we have

$$
\frac{1}{L} \sum_{q_n \in \text{NS}} \frac{f(q_n)e^{ijq_n}}{(\cos k - \cos q_n)^2} = -\frac{e^{-ikj}f'(-k) + e^{ikj}f(k)}{2i\sin^2 k}
+ \left[ \frac{L}{4} - \frac{j}{2} \right] \frac{e^{-ikj}f(-k) + e^{ikj}f(k)}{\sin^2 k} + \mathcal{O}(e^{-\gamma j}).
$$

(D.6)

Proof. Using contour integration, we have

$$
\frac{1}{L} \sum_{q_n \in \text{NS}} \frac{f(q_n)e^{ijq_n}}{(\cos k - \cos q_n)^2} = \frac{e^{-ikj}f'(-k) + e^{ikj}f(k)}{2i\sin^2 (k)} \left[ \frac{L}{4} - \frac{j}{2} \right]
+ i \frac{e^{-ikj}f'(-k) + e^{ikj}f(k)}{2\sin^2 (k)} - \oint_{\Gamma} \frac{f(z)e^{ijz}}{2\pi} \frac{dz}{\cos z - \cos k (1 + e^{iLz})},
$$

(D.7)

where the integration is along a closed contour encircling the interval $[-\pi, \pi]$. As for $\text{Im}(z) < 0$ we have

$$
\lim_{L \to \infty} e^{i(j-L)z} = 0,
$$

(D.8)

only the upper part of the contour contributes in the $L \to \infty$ limit. Using that $f(z)$ is analytic in some strip around the real axis, the part of the contour above the real axis can be deformed as shown in figure D.1. The contributions of the pieces parallel to the imaginary axis cancel due to the periodicity of $f(z)$, while the remaining part is exponentially small in $j$. In the limit of large $L$ and $j$, the first term in (D.7) can be neglected, so that

$$
\frac{1}{L} \sum_{q_n \in \text{NS}} \frac{f(q_n)e^{ijq_n}}{(\cos k - \cos q_n)^2} \approx \frac{e^{-ikj}f(-k) + e^{ikj}f(k)}{2i\sin^2 (k)} \left[ \frac{L}{4} - \frac{j}{2} \right].
$$

(D.9)

Lemma 5. For any function $f(z)$ that is $2\pi$ periodic and analytic in a strip around the real axis, we have

$$
\lim_{L \to \infty} \frac{1}{L} \sum_{0 < q \in \text{NS}} \frac{f(q)e^{2\pi \sigma q \gamma t}}{\cos k - \cos q} = \frac{i\sigma f(k)e^{2i\pi \sigma k \gamma t}}{2i\sin k} + \oint_{\Gamma} \frac{f(z)e^{2i\pi \sigma z \gamma t}}{2\pi \cos k - \cos z},
$$

(D.10)

where $\sigma = \pm$ and $0 < k \in \mathbb{R}$.

Appendix E. ‘Pair ensemble’ averages

The ‘diagonal’ terms in the Lehmann representation based on the eigenstates of the post-quench Hamiltonian $H(h)$ give rise to a time-independent contribution to the expectation value $\langle \Psi_0(0)|\mathcal{O}|\Psi_0(0) \rangle$, which we call the ‘pair ensemble’. It is defined as the average

$$
\langle \mathcal{O} \rangle_{\text{PE}} \equiv 
\frac{1}{\text{NS}(B|B)_{\text{NS}}}
\sum_{n=0}^{\infty} \frac{1}{n!} \sum'_{0 < k_1, \ldots, k_n \in \text{NS}} \left[ \prod_{j=1}^{n} K^2(k_j) \right]
\times \text{NS}(-k_n, k_n, \ldots, -k_1, k_1; h|\mathcal{O}|k_1, -k_1, \ldots, k_n, -k_n; h)_{\text{NS}}.
$$

(E.1)
Averages of the form (E.1) can be represented using a density matrix as \( \langle O \rangle_{PE} = \text{tr}(\rho_{PE} O) \), where
\[
\rho_{PE} = (1 - n_0)(1 - n_n) \prod_{0 < k \in \mathbb{NS}} \left( \frac{\mathbb{I}_k - n_k \mathbb{I}_{-k} - \mathbb{I}_k n_{-k}}{1 + K^2(k)} + n_k n_{-k} \right). \tag{E.2}
\]
Here \( n_q = \alpha_q^\dagger \alpha_q \) are the Bogoliubov fermion number operators in momentum space and \( \mathbb{I}_k \) is the identity in the subspace with momentum \( k \). The only non-vanishing expectation values of the pair ensemble are
\[
\langle n_{q_1} n_{q_2} \cdots n_{q_m} \rangle_{PE} = \prod_{|q_j| \in S} \frac{K^2(|q_j|)}{1 + K^2(|q_j|)}, \tag{E.3}
\]
where \( S \) is the set of all momenta \( q_j \) with mutually distinct magnitudes, i.e.
\[
0 < |q_r| \neq |q_s| < \pi \quad \forall q_r, q_s \in S. \tag{E.4}
\]
We note that \( \langle n_k n_{-k} O \rangle = \langle n_k O \rangle \), since the ensemble describes pairs of particles with opposite momenta.

As shown in section 3, lattice spin operators can be expressed as products of the Majorana fermions \( a_j^x, a_j^y \), which are related to the Bogoliubov fermions diagonalizing the post-quench Hamiltonian \( H(h) \) by
\[
a_j^x = \frac{1}{\sqrt{L}} \sum_k e^{-ikj} e^{i\theta_k/2} [\alpha_k + \alpha_k^\dagger],
a_j^y = \frac{1}{\sqrt{L}} \sum_k e^{-ikj} e^{-i\theta_k/2} [\alpha_k^\dagger - \alpha_k]. \tag{E.5}
\]
As the only non-zero expectation values in the pair ensemble are of the form (E.3), we can express a general average in the form
\[
\langle \prod_{r=1}^{2n} a_j^b \rangle_{PE} = \frac{1}{L^n} \sum_{k_1, \ldots, k_n} \sum_{s=0}^n \chi_b^{(s)}(k_1, \ldots, k_n|\vec{j}) \langle \prod_{u=1}^s n_{k_u} \rangle_{PE}, \tag{E.6}
\]
where \( \chi_b^{(s)}(k_1, \ldots, k_n|\vec{j}) \) are well behaved functions of the momenta.

The generalized Gibbs ensemble for the Ising model is defined as
\[
\langle O \rangle_{GGE} = \frac{1}{Z_{GGE}} \text{tr}[e^{-\sum_q \lambda_q n_q} O], \tag{E.7}
\]
where \( Z_{GGE} = \text{tr}[e^{-\sum_q \lambda_q n_q}] \) and
\[
\lambda_q = -\log(K^2(q)). \tag{E.8}
\]
By taking the trace over a basis of eigenstates of \( H(h) \), we conclude that the only non-zero averages are
\[
\langle n_{q_1} n_{q_2} \cdots n_{q_m} \rangle_{GGE} = \prod_{j=1}^n \frac{K^2(q_j)}{1 + K^2(q_j)}. \tag{E.9}
\]
The corresponding density matrix is
\[ \rho_{\text{GGE}} = \prod_{k \in \text{NS}} \frac{1}{1 + K^2(k)} \sum_{k_1, \ldots, k_n} |k_1 \rangle \langle k_1| \cdots |k_n \rangle \langle k_n| \]

(E.10)

We observe that \( \langle n_{k_1} \cdots n_{k_m} \rangle_{\text{GGE}} = \langle n_{k_1} \rangle_{\text{GGE}} \langle n_{k_2} \cdots n_{k_m} \rangle_{\text{GGE}} \) for any set of distinct momenta \( k_1, k_2, \ldots, k_m \), and unlike the case in the pair ensemble Wick’s theorem applies in the generalized Gibbs ensemble. General averages can be expressed as
\[ \langle 2^{2n} \rangle_{\text{GGE}} = \frac{1}{L^n} \sum_{k_1, \ldots, k_n} \sum_{s=0}^{n} \chi^{(s)}_{k_1, \ldots, k_n} \langle \Pi_1 \cdots \rangle_{\text{GGE}}, \]

(E.11)

where the functions \( \chi^{(s)}_{k_1, \ldots, k_n} \) are the same as in (E.6). As the averages \( \langle \Pi_1 \cdots \rangle_{\text{GGE}} \) and \( \langle \Pi_1 \cdots \rangle_{\text{PE}} \) are the same unless at least two of the \( |k_u| \)s coincide, and such contributions are suppressed by factors of \( 1/L \), we conclude that
\[ \lim_{L \to \infty} \langle 2^{2n} \rangle_{\text{PE}} = \langle 2^{2n} \rangle_{\text{GGE}}. \]

(E.12)

The above arguments show that averages of operators that are local in space are the same in both ensembles. However, non-local operators have in general different averages, e.g. \( \langle n_k n_{-k} \rangle_{\text{GGE}} = \langle n_k n_{-k} \rangle_{\text{PE}} \).

References

[1] Greiner M, Mandel O, Hänisch T W and Bloch I, Collapse and revival of the matter wave field of a Bose–Einstein condensate, 2002 Nature 419 51
[2] Hofferberth S, Lesanovsky I, Fischer B, Schumm T and Schmiedmayer J, Non-equilibrium coherence dynamics in one-dimensional Bose gases, 2007 Nature 449 324
[3] Trotzky S, Chen Y-A, Flesch A, Mc Culloch I P, Schollwöck U, Eisert J and Bloch I, Probing the relaxation towards equilibrium in an isolated strongly correlated 1D Bose gas, 2012 Nature Phys. 8 325 arXiv:1101.2659
[4] Cheneau M, Barmettler P, Poletti D, Endres M, Schauss P, Fukuhara T, Gross C, Bloch I, Kollath C and Kuhr S, Light-cone-like spreading of correlations in a quantum many-body system, 2012 Nature 481 484
[5] Gring M, Kuhntert M, Langen T, Kitagawa T, Rauer B, Schreitl M, Mazets I, Smith D A, Demler E and Schmiedmayer J, Relaxation dynamics and pre-thermalization in an isolated quantum system, 2011 arXiv:1112.0013
[6] Kinosita T, Wenger T and Weiss D S, A quantum Newton’s cradle, 2006 Nature 440 900
[7] Lieb E H and Robinson D W, The finite group velocity of quantum spin systems, 1972 Commun. Math. Phys. 28 251
[8] Calabrese P and Cardy J, Evolution of entanglement entropy in one-dimensional systems, 2005 J. Stat. Mech. P04010
[9] Calabrese P and Cardy J, Time-dependence of correlation functions following a quantum quench, 2006 Phys. Rev. Lett. 96 136801
[10] Calabrese P and Cardy J, Entanglement and correlation functions following a local quench: a conformal field theory approach, 2007 J. Stat. Mech. P10004
[11] Sotiriadis S and Cardy J, Quantum quench in interacting field theory: a self-consistent approximation, 2010 Phys. Rev. B 81 134305
[12] Stéphan J-M and Dubail J, Local quantum quenches in critical one-dimensional systems: entanglement, the Loschmidt echo, and light-cone effects, 2011 J. Stat. Mech. P08019
[13] Manmana S R, Wessel S, Noack R M and Muramatsu A, Time evolution of correlations in strongly interacting fermions after a quantum quench, 2009 Phys. Rev. B 79 155104
[14] Lauchli A and Kollath C, Spreading of correlations and entanglement after a quench in the one-dimensional Bose–Hubbard model, 2008 J. Stat. Mech. P05018

doi:10.1088/1742-5468/2012/07/P07016
Effect of suddenly turning on the interactions in the Luttinger model

Cazalilla M A, Quantum quenches in extended systems

Calabrese P and Cardy J, Thermalization and its mechanism for generic isolated quantum systems, 2011 Rev. Mod. Phys. 83 863

Rigol M, Dunjko V and Olshanii M, Relaxation in a completely integrable many-body quantum system: an ab initio study of the dynamics of the highly excited states of lattice hard-core bosons, 2007 Phys. Rev. Lett. 98 50405

Rigol M, Dunjko V and Olshanii M, Thermalization and its mechanism for generic isolated quantum systems, 2008 Nature 452 854

Calabrese P and Cardy J, Quantum quenches in extended systems, 2007 J. Stat. Mech. P06008

Cazalilla M A, Effect of suddenly turning on the interactions in the Luttinger model, 2006 Phys. Rev. Lett. 97 156403

Iucci A and Cazalilla M A, Quantum quench dynamics of the sine-Gordon model in some solvable limits, 2010 New J. Phys. 12 055019

Iucci A and Cazalilla M A, Quantum quench dynamics of the Luttinger model, 2009 Phys. Rev. A 80 063619

Manmana S R, Wessel S, Noack R M and Muramatsu A, Strongly correlated fermions after a quantum quench, 2007 Phys. Rev. Lett. 98 210405

Cramer M, Dawson C M, Eisert J and Osborne T J, Quenching, relaxation, and a central limit theorem for quantum lattice systems, 2008 Phys. Rev. Lett. 100 030602

Cramer M and Eisert J, A quantum central limit theorem for non-equilibrium systems: exact local relaxation of correlated states, 2010 New J. Phys. 12 055020

Barthel T and Schollwöck U, Dephasing and the steady state in quantum many-particle systems, 2008 Phys. Rev. Lett. 100 100601

Cramer M, Flesch A, McCulloch I A, Schollwoeck U and Eisert J, Exploring local quantum many-body relaxation by atoms in optical superlattices, 2008 Phys. Rev. Lett. 101 063001

Flesch A, Cramer M, McCulloch I P, Schollwoeck U and Eisert J, Probing local relaxation of cold atoms in optical superlattices, 2008 Phys. Rev. A 78 033608

Kollar M and Eckstein M, Relaxation of a one-dimensional Mott insulator after an interaction quench, 2008 Phys. Rev. A 78 013626

Sotiriadis S, Calabrese P and Cardy J, Quantum quench from a thermal initial state, 2009 Europhys. Lett. 87 20002

Roux G, Quenches in quantum many-body systems: one-dimensional Bose–Hubbard model reexamined, 2009 Phys. Rev. A 79 021608

Roux G, Finite size effects in global quantum quenches: examples from free bosons in an harmonic trap and the one-dimensional Bose–Hubbard model, 2010 Phys. Rev. A 81 053604

Sotiriadis S, Fioretto D and Mussardo G, Zamolodchikov-Faddeev algebra and quantum quenches in integrable field theories, 2012 J. Stat. Mech. P02017

Fioretto D and Mussardo G, Quantum quenches in integrable field theories, 2010 New J. Phys. 12 055015

Brandino G P, De Luca A, Konik R M and Mussardo G, Quench dynamics in randomly generated extended quantum models, 2012 Phys. Rev. B 85 214435

Kollath C, Laeuchli A and Altman E, Quench dynamics and non equilibrium phase diagram of the Bose–Hubbard model, 2007 Phys. Rev. Lett. 98 180601

Biroli G, Kollath C and Laeuchli A, Effect of rare fluctuations on the thermalization of isolated quantum systems, 2010 Phys. Rev. Lett. 105 250401

Banuls M C, Cirac J I and Hastings M B, Strong and weak thermalization of infinite non-integrable quantum systems, 2011 Phys. Rev. Lett. 106 050405

Gogolin C, Mueller M P and Eisert J, Absence of thermalization in nonintegrable systems, 2011 Phys. Rev. Lett. 106 040401

Rigol M and Fitzpatrick M, Initial-state dependence of the quench dynamics in integrable quantum systems, 2011 Phys. Rev. A 84 033640

Caneva T, Canovi E, Rossini D, Santoro G E and Silva A, Applicability of the generalized Gibbs ensemble after a quench in the quantum Ising chain, 2011 J. Stat. Mech. P07015

doi:10.1088/1742-5468/2012/07/P07016
[40] Cazalilla M A, Iucci A and Chung M-C, Thermalization and quantum correlations in exactly solvable models, 2012 Phys. Rev. E 85 011133
[41] Rigol M and Srednicki M, Alternatives to eigenstate thermalization, 2012 Phys. Rev. Lett. 108 110601
[42] Zhang J M, Cui F C and Hu J, Dynamical predictive power of the generalized Gibbs ensemble revealed in a second quench, 2012 Phys. Rev. E 85 041138
[43] Santos L F, Polkovnikov A and Rigol M, Entropy of isolated quantum systems after a quench, 2011 Phys. Rev. Lett. 107 040601
[44] Mossel J and Caux J-S, Exact time evolution of space- and time-dependent correlation functions after an interaction quench in the 1D Bose gas, 2012 arXiv:1201.1885
[45] Grisins P and Mazets I, Thermalization in a one-dimensional integrable system, 2011 Phys. Rev. A 84 053635
[46] Caux J-S and Konik R M, Numerical renormalization based on integrable theories: quantum quenches and their corresponding generalized Gibbs ensembles, 2012 arXiv:1203.0901
[47] Mossel J and Caux J-S, Generalized TBA and generalized Gibbs, 2012 J. Phys. A: Math. Theor. 45 255001
[48] Canovi E, Rossini D, Fazio R, Santoro G E and Silva A, Quantum quenches, thermalization and many-body localization, 2011 Phys. Rev. B 83 094431
[49] Barankov R A and Levitov L S, Synchronization in the BCS pairing dynamics as a critical phenomenon, 2006 Phys. Rev. Lett. 96 230403
Yuzbashyan E A and Dzero M, Dynamical vanishing of the order parameter in a fermionic condensate, 2006 Phys. Rev. Lett. 96 230404
Yuzbashyan E A, Tsypliyatov O and Altshuler B L, Relaxation and persistent oscillations of the order parameter in fermionic condensate, 2006 Phys. Rev. Lett. 96 097005
[50] Barankov R A and Levitov L S, Synchronization in the BCS pairing dynamics as a critical phenomenon, 2006 Phys. Rev. Lett. 96 230403
Yuzbashyan E A and Dzero M, Dynamical vanishing of the order parameter in a fermionic condensate, 2006 Phys. Rev. Lett. 96 230404
Yuzbashyan E A, Tsypliyatov O and Altshuler B L, Relaxation and persistent oscillations of the order parameter in fermionic condensate, 2006 Phys. Rev. Lett. 96 097005
[51] Gambassi A and Calabrese P, Quantum quenches as classical critical films, 2011 Europhys. Lett. 95 66007
[52] Balasubramanian V, Bernardini A, de Boer J, Copland N B, Craps B, Keski-Vakkuri E, Muller B, Schafer A, Shigemori M and Staessens W, Thermalization of strongly coupled field theories, 2011 Phys. Rev. Lett. 106 191601
Balasubramanian V, Bernardini A, de Boer J, Copland N B, Craps B, Keski-Vakkuri E, Muller B, Schafer A, Shigemori M and Staessens W, Thermalization of strongly coupled field theories, 2011 Phys. Rev. D 84 026010
[53] Eckstein M, Kollar M and Werner P, 2009 Phys. Rev. Lett. 103 056403
Schiò M and Fabrizio M, Time-dependent mean field theory for quench dynamics in correlated electron systems, 2010 Phys. Rev. Lett. 105 076401
Schiò M and Fabrizio M, Quantum quenches in the hubbard model: time dependent mean field theory and the role of quantum fluctuations, 2011 Phys. Rev. B 83 165105
[54] Gambassi A and Calabrese P, Quantum quenches as classical critical films, 2011 Europhys. Lett. 95 66007
[55] Balasubramanian V, Bernardini A, de Boer J, Copland N B, Craps B, Keski-Vakkuri E, Muller B, Schafer A, Shigemori M and Staessens W, Thermalization of strongly coupled field theories, 2011 Phys. Rev. Lett. 106 191601
Balasubramanian V, Bernardini A, de Boer J, Copland N B, Craps B, Keski-Vakkuri E, Muller B, Schafer A, Shigemori M and Staessens W, Holographic thermalization, 2011 Phys. Rev. D 84 026010
[56] Abajo-Arrastia J, Aparicio J and Lopez E, Holographic evolution of entanglement entropy, 2010 J. High Energy Phys. JHEP11(2010)149
Aparicio J and Lopez E, Evolution of two-point functions from holography, 2011 arXiv:1109.3571
[57] Das S R, Nishioka T and Takayanagi T, Probe branes, time-dependent couplings and thermalization in AdS/CFT, 2010 J. High Energy Phys. JHEP07(2010)071
Allais A and Tonni E, Holographic evolution of the mutual information, 2012 J. High Energy Phys. JHEP01(2012)102
Balasubramanian V, Bernardini A, Copland N, Craps B and Galli F, Thermalization of mutual and tripartite information in strongly coupled two dimensional conformal field theories, 2011 Phys. Rev. D 84 105017
Keranen V, Keski-Vakkuri E and Thorlacius L, Thermalization and entanglement following a nonrelativistic holographic quench, 2012 Phys. Rev. D 85 026005
[58] Sachdev S, 2001 Quantum Phase Transitions (Cambridge: Cambridge University Press)
Coldea R, Tennant D A, Wheeler E M, Wawrzynska E, Prabhakaran D, Telling M, Habicht K, Smeibidl P and Kiefer K, Quantum criticality in an Ising chain: experimental evidence for emergent E8 symmetry, 2010 Science 327 177
[59] Simon J, Bakr W S, Ma R, Tai M E, Preiss P M and Greiner M, Quantum simulation of antiferromagnetic spin chains in an optical lattice, 2011 Nature 472 307
Rossini D, Silva A, Mussardo G and Santoro G, Effective thermal dynamics following a quench in a spin chain, 2009 Phys. Rev. Lett. 102 127204

doi:10.1088/1742-5468/2012/07/P07016
Quantum quench in the transverse field Ising chain I

Rossini D, Suzuki S, Mussardo G, Santoro G E and Silva A, Long time dynamics following a quench in an integrable quantum spin chain: local versus non-local operators and effective thermal behaviour, 2010 Phys. Rev. B 82 144302

Barouch E, McCoy B and Dresden M, Statistical mechanics of the XY model. I, 1970 Phys. Rev. A 2 1075
Barouch E and McCoy B, Statistical mechanics of the XY model. II, 1971 Phys. Rev. A 3 786
Barouch E and McCoy B, Statistical mechanics of the XY model. III, 1971 Phys. Rev. A 3 2137

Igloi F and Rieger H, Long-range correlations in the nonequilibrium quantum relaxation of a spin chain, 2000 Phys. Rev. Lett. 85 3233

Igloi F and Rieger H, Quantum relaxation after a quench in systems with boundaries, 2011 Phys. Rev. Lett. 106 035701

Sengupta K, Powell S and Sachdev S, Quench dynamics across quantum critical points, 2004 Phys. Rev. A 69 053616

Fagotti M and Calabrese P, Evolution of entanglement entropy following a quantum quench: analytic results for the XY chain in a transverse magnetic field, 2008 Phys. Rev. A 78 010306

Silva A, The statistics of the work done on a quantum critical system by quenching a control parameter, 2008 Phys. Rev. Lett. 101 120603

Gambassi A and Silva A, Statistics of the work in quantum quenches, universality and the critical casimir effect, 2011 arXiv:1106.2671

Campos Venuti L and Zanardi P, Unitary equilibrations: probability distribution of the Loschmidt echo, 2010 Phys. Rev. A 81 022113

Campos Venuti L, Jacobson N T, Santra S and Zanardi P, Exact infinite-time statistics of the loschmidt echo for a quantum quench, 2011 Phys. Rev. Lett. 107 010403

Foini L, Cugliandolo L F and Gambassi A, Fluctuation-dissipation relations and critical quenches in the transverse field Ising chain, 2011 Phys. Rev. B 84 212404

Rieger H and Igloi F, Semiclassical theory for quantum quenches in finite transverse Ising chains, 2011 Phys. Rev. B 84 165117

Calabrese P, Essler F H L and Fagotti M, Quantum quench in the transverse field Ising chain, 2011 Phys. Rev. Lett. 106 227203

Smirnov F A, 1992 Form Factors in Completely Integrable Models of Quantum Field Theory (Singapore: World Scientific)

Cardy J L and Mussardo G, Form factors of descendent operators in perturbed conformal field theories, 1990 Nucl. Phys. B 340 387

Yurov V P and Zamolodchikov Al B, Correlation functions of integrable 2-D* models of relativistic field theory. Ising model, 1991 Int. J. Mod. Phys. A 6 3419

Lukyanov S, Free field representation for massive integrable models, 1995 Commun. Math. Phys. 167 183

Babujian H, Fring A, Karowski M and Zapletal A, Exact form factors in integrable quantum field theories: the sine-Gordon model, 1999 Nucl. Phys. B 538 535

Delfino G, Integrable field theory and critical phenomena. The Ising model in a magnetic field, 2004 J. Phys. A: Math. Gen. 37 R15

Mussardo G, 2009 Statistical Field Theory, An Introduction to Exactly Solved Models in Statistical Physics (Oxford: Oxford University Press)

Essler F H L and Konik R M, Applications of massive integrable quantum field theories to problems in condensed matter physics, 2005 Jan Kogan Memorial Collection From Fields to Strings: Circumnavigating Theoretical Physics ed M Shifman, A Vainshtein and J Wheater (Singapore: World Scientific) [ arXiv:cond-mat/0412421 ]

Barmettler P, Punk M, Gritsev V, Demler E and Altman E, Relaxation of antiferromagnetic order in spin-1/2 chains following a quantum quench, 2009 Phys. Rev. Lett. 102 130603

Barmettler P, Punk M, Gritsev V, Demler E and Altman E, Quantum quenches in the anisotropic spin-1/2 Heisenberg chain: different approaches to many-body dynamics far from equilibrium, 2010 New J. Phys. 12 055017

Faribault A, Calabrese P and Caux J-S, Quantum quenches from integrability: the fermionic pairing model, 2009 J. Stat. Mech. P03018
Faribault A, Calabrese P and Caux J-S, Bethe Ansatz approach to quench dynamics in the Richardson model, 2009 J. Math. Phys. 50 095212

Gritsev V, Rostunov T and Demler E, Exact methods in analysis of nonequilibrium dynamics of integrable models: application to the study of correlation functions in nonequilibrium 1D Bose gas, 2010 J. Stat. Mech. P05012

Ng R and Sorensen E S, Exact real-time dynamics of quantum spin systems using the positive-P representation, 2011 J. Phys. A: Math. Theor. 44 065305

doi:10.1088/1742-5468/2012/07/P07016
[81] Calabrese P, Essler F H L and Fagotti M, Quantum quench in the transverse field Ising chain II: stationary state properties, 2012 arXiv:1205.2211, J. Stat. Mech. at press.
[82] Vidal G, Latorre J I, Rico E and Kitaev A, Entanglement in quantum critical phenomena, 2003 Phys. Rev. Lett. 90 227902
Latorre J I, Rico E and Vidal G, Ground state entanglement in quantum spin chains, 2004 Quant. Inf. Comp. 4 048.
[83] Fisher M E and Hawting R E, Toeplitz determinants: some applications, theorems, and conjectures, 1968 Adv. Chem. Phys. 15 333
Au-Yang H and McCoy B, Theory of layered Ising models. II. Spin correlation functions parallel to the layering, 1974 Phys. Rev. B 10 3885
Basor E L and Tracy C A, The Fisher-Hartwig conjecture and generalizations, 1991 Physica A 177 167
Forrester P J and Frankel N E, Applications and generalizations of Fisher–Hartwig asymptotics, 2004 J. Math. Phys. 45 2003
Karlovich A Yu, Asymptotics of block Toeplitz determinants generated by factorable matrix functions with equal partial indices, 2007 Math. Nachr. 280 1118
Deift P, Its A and Krasovsky I, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities, 2011 Ann. Math. 174 1243
[84] Boettcher A and Widom H, Szegő via Jacobi Toeplitz determinants: some applications, theorems, and conjectures, 2006 Linear Algebra Appl. 419 656
[85] Fagotti M and Calabrese P, Entanglement entropy of two disjoint blocks in XY chains, 2010 J. Stat. Mech. P04016
[86] Wong R, 2001 Asymptotic Approximations of Integrals (Philadelphia: SIAM)
[87] Fagotti M, unpublished
[88] Vidal G, Latorre J I, Rico E and Kitaev A, Ground state entanglement in quantum spin chains, 2004 Quant. Inf. Comp. 4 048.
[89] Forrester P J and Frankel N E, Applications and generalizations of Fisher–Hartwig asymptotics, 2004 J. Math. Phys. 45 2003
Karlovich A Yu, Asymptotics of block Toeplitz determinants generated by factorable matrix functions with equal partial indices, 2007 Math. Nachr. 280 1118
Deift P, Its A and Krasovsky I, Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities, 2011 Ann. Math. 174 1243
[90] Kormos M and Fagotti M, Finite-temperature lineshapes in gapped quantum spin chains, 2008 Phys. Rev. B 78 100403
Essler F H L and Konik R M, Finite-temperature dynamical correlations in massive integrable quantum field theories, 2009 J. Stat. Mech. P09018
[91] Doyon B, Finite-temperature form factors in the free Majorana theory, 2005 J. Stat. Mech. P11006
Doyon B, Finite-temperature form factors: a review, 2007 Sigma 3 11
[92] Leclair A and Mussardo G, 1999 Nucl. Phys. B 482 579
Leclair A and Mussardo G, 1999 Nucl. Phys. B 552 624
Konik R M, 2003 Phys. Rev. B 68 104435
Reyes S A and Tsvelik A, 2006 Phys. Rev. B 73 220405(R)
[93] Ghoshal S and Zamolodchikov A B, Boundary S-matrix and boundary state in two-dimensional integrable field theory, 1994 Int. J. Mod. Phys. A 9 3841
Ghoshal S and Zamolodchikov A B, Boundary S-matrix and boundary state in two-dimensional integrable field theory, 1994 Int. J. Mod. Phys. A 9 E4353
[94] Leclair A and Mussardo G, 1999 Nucl. Phys. B 482 579
Leclair A and Mussardo G, 1999 Nucl. Phys. B 552 624
Konik R M, 2003 Phys. Rev. B 68 104435
Reyes S A and Tsvelik A, 2006 Phys. Rev. B 73 220405(R)
[95] James A J A, Essler F H L and Konik R M, Finite temperature dynamical structure factor of alternating Heisenberg chains, 2008 Phys. Rev. B 78 094411
Goetze W D, Karahasanovic U and Essler F H L, Low-temperature dynamical structure factor of the two-leg spin-1/2 Heisenberg ladder, 2010 Phys. Rev. B 82 104417
[96] Ghoshal S and Zamolodchikov A B, Boundary S-matrix and boundary state in two-dimensional integrable field theory, 1994 Int. J. Mod. Phys. A 9 3841
Ghoshal S and Zamolodchikov A B, Boundary S-matrix and boundary state in two-dimensional integrable field theory, 1994 Int. J. Mod. Phys. A 9 E4353
[97] Altschuler B L, Konik R M and Tsvelik A M, Low temperature correlation functions in integrable models: derivation of the large distance and time asymptotics from the form factor expansion, 2006 Nucl. Phys. B 739 311
[98] Ghoshal S and Zamolodchikov A B, Boundary S-matrix and boundary state in two-dimensional integrable field theory, 1994 Int. J. Mod. Phys. A 9 3841
Ghoshal S and Zamolodchikov A B, Boundary S-matrix and boundary state in two-dimensional integrable field theory, 1994 Int. J. Mod. Phys. A 9 E4353
[99] Altschuler B L, Konik R M and Tsvelik A M, Low temperature correlation functions in integrable models: derivation of the large distance and time asymptotics from the form factor expansion, 2006 Nucl. Phys. B 739 311
[100] Bugrij A, Correlation function of the two-dimensional Ising model on the finite lattice. I, 2001 Theor. Math. Phys. 127 528
Bugrij A and Lisovvy O, Spin matrix elements in 2D Ising model on the finite lattice, 2003 Phys. Lett. A 319 390
[101] von Gehlen G, Iorgov N, Pakuliak S, Shadura V and Tykhyy Y, Form-factors in the Baxter–Bazhanov–Stroganov model II: Ising model on the finite lattice, 2008 J. Phys. A: Math. Theor. 41 095003

doi:10.1088/1742-5468/2012/07/P07016
Quantum quench in the transverse field Ising chain I

[98] Iorgov N, Shadura V and Tykhyy Yu, Spin operator matrix elements in the quantum Ising chain: fermion approach, 2011 J. Stat. Mech. P02028

[99] see e.g. Itzykson C and Drouffe J-M, 1989 Statistical Field Theory (Cambridge: Cambridge University Press)

[100] Schuricht D and Essler F H L, 2012 J. Stat. Mech. P04017

[101] Essler F H L, Evangelisti S and Fagotti M, in preparation

[102] Lieb E, Schultz T and Mattis D, Two soluble models of an antiferromagnetic chain, 1961 Ann. Phys., NY 16 407

Pfeuty P, The one-dimensional Ising model with a transverse field, 1970 Ann. Phys., NY 57 79