A regularity result for fixed points, with applications to linear response

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Received 15 May 2017, revised 7 December 2017
Accepted for publication 12 December 2017
Published 3 March 2018

Recommended by Dr Sebastien Gouezel

Abstract
In this paper, we show a series of abstract results on fixed point regularity with respect to a parameter. They are based on a Taylor development taking into account a loss of regularity phenomenon, typically occurring for composition operators acting on spaces of functions with finite regularity. We generalize this approach to higher order differentiability, through the notion of an n-graded family.

We then give applications to the fixed point of a nonlinear map, and to linear response in the context of (uniformly) expanding dynamics (theorem 3 and corollary 2), in the spirit of Gouëzel–Liverani.

Keywords: dynamical systems, fixed-point differentiability, linear response, expanding maps
Mathematics Subject Classification numbers: Primary: 37C30; 47H10; Secondary: 37F15

1. Introduction
The aim of this paper is to study the following regularity problem for a fixed point depending on a (multi-dimensional) parameter: Given three Banach spaces $B$, $X_0$, $X_1$, such that there exists a continuous, linear injection $j_0 : X_1 \hookrightarrow X_0$, we consider maps $F_i : U \times A_i \longrightarrow A_i$ ($i \in \{0, 1\}$), where $U \subset B$ open, $A_1 \subset X_1$ is closed and non-empty and $A_0 = j_0(A_1)$. We assume that for every $\phi_1 \in A_1$, every $u \in U$, $j_0 \circ F_1(u, \phi_1) = F_0(u, j_0(\phi_1))$ and the existence, for every $u \in U$, of a $\phi_1(u) \in A_1$, such that

$$F_1(u, \phi_1(u)) = \phi_1(u).$$

(1.1)

We consider situations where the fixed-point map $\phi_1 : u \in U \mapsto \phi_1(u) \in X_1$ has no particular regularity, yet when one takes the injection $\phi_0 = j_0 \circ \phi_1$, one gains some regularity.
When studying the regularity of fixed point map, the most natural tool at our disposal is the implicit function theorem, formulated in the Banach space setting. However, there are a number of cases where this approach fails, notably when the maps \( F_i \) are not continuously differentiable in the classical sense: this is the case when, for example, \( F \) is a composition operator.

One can give explicit examples, where some sort of regularity can be recovered, and much can be obtained through elementary methods: for instance, given \( \epsilon > 0 \), \( u \in [-\epsilon, \epsilon] \) and \( g \in C^1([-1, 1] \times [-\epsilon, \epsilon]) \) non identically zero, the operator \( F(u, \phi)(t) = \frac{1}{2}\phi(T(t)u) + g(t,u) \) initially defined on \([-\epsilon, \epsilon] \times C^1([-1, 1])\), is a contraction in its second variable \(^1\) when acting on \( C^\alpha([-1, 1]) \), for every \( \alpha \in [0, 1] \).

However, the fixed point map, \( u \in [-\epsilon, \epsilon] \mapsto \phi_0 \in C^\alpha([-1, 1]) \) is not continuous. Yet, the map \( u \in [-\epsilon, \epsilon] \mapsto \phi_0 \in C^\alpha([-1, 1]) \) is \( \alpha \)-Hölder (see appendix C).

Another kind of problem arises when one studies the differentiability of that fixed point map: if it is natural to define a partial differential \( \partial_u F(u, \phi) = \frac{1}{2}\phi'(\frac{u}{2}) + \partial_u g(\cdot, u) \) for every \( \phi \in C^1([-1, 1]) \), the corresponding operator \( \partial_u F(u, \cdot) \) does not define a partial differential with respect to \( u \) for \( F(u, \cdot) \) in the classical sense (as it is not a linear map from \( \mathbb{R} \) to \( C^1([-1, 1]) \)): a phenomenon one can consider as a loss of regularity.

Our main result, theorem 1, allows one to obtain differentiability results for the kind of fixed points problems outlined in this introduction. The full statement is given in section 2.1 and a proof in 2.2; it is based on a type of Taylor development (2.2), which can be interpreted as an analogue of the Gouëzel–Liverani spectral perturbation result [16, section 8.1]. The major improvement here is the possible application to fixed points of nonlinear maps.

We also discuss a generalization to higher order differentiability in section 2.3, by introducing the notion of graded family (definition 2).

In section 3 we propose an application of our result to a nonlinear situation, where the set of parameter lies in an infinite dimensional space: in short, we interpret the perturbation itself as a parameter, and study regularity of the solutions with respect to it. This example is somehow ‘minimal’, in the sense that it is the simplest non trivial, nonlinear example we could think of.

We then turn to an application to linear response for expanding dynamics, i.e. differentiability results and first-order variations for the absolutely continuous, invariant measure for a one parameter family of dynamics. This field has already been thoroughly studied, in various dynamical contexts: uniformly expanding maps on the circle [2] or on general Riemann manifold [3], intermittent maps of the interval [7], piecewise expanding maps of the interval [4–6].

In the hyperbolic case, one can cite Ruelle’s work on Axiom A systems [30, 32], see also the erratum by Jiang [20], Gouëzel–Liverani papers on Anosov and Axiom A systems [14, 16], and the 2004 paper of Dolgopyat on partially hyperbolic systems [11]. In a different vein, one can see the paper by Haider and Majda [18].

The ‘modern’ approach to linear response is based on the ‘weak spectral perturbation’ techniques developed in Gouëzel and Liverani papers (see Baladi’s monograph [3] and the original papers [14, 16], see also [15]). Our method allows to recover similar regularity results, as well as a linear response formula, and one can fruitfully compare our main results theorem 1–3, and corollary 2 to Gouëzel–Liverani’s paper [16, section 8.1, 8.3], and to Gouëzel’s paper [15, corollary 3.5, p 21] (see also theorems 2.36 and 2.38 in Baladi’s book [3]). Let us emphasize the differences and similarities:

- **Linear versus nonlinear**: if the ‘weak spectral perturbation’ theorem only applies to bounded, linear operators, our theorem 1 can also be applied to nonlinear maps (see

\[ \text{i.e. for every fixed } u \in [-\epsilon, \epsilon], \| F(u, \phi) - F(u, \psi) \|_{C^\alpha} \leq k_\alpha \| \phi - \psi \|_{C^\alpha} \text{ with } \sup_{u \in [-\epsilon, \epsilon]} k_\alpha < 1. \]
section 3). However, it is worth noting that when one does apply our theorem to (linear) transfer operators, the ‘Taylor development’ (2.2) becomes (8.3) in section 8.1 of [16] (i.e. the Taylor expansion assumption in Gouëzel–Liverani paper): this is made precise in the proof of lemma 2.

- **Parameter dimension**: our result is naturally formulated for a multi-dimensional (even infinite-dimensional) parameter, whereas Gouëzel–Liverani spectral theorem assume a one-dimensional parameter. Nonetheless, the latter can easily be extended to multi-dimensional parameter. It is not known whether it can be generalized to an infinite-dimensional parameter. We provide an application with an infinite-dimensional parameter in section 3.

- **Uniform Lasota–Yorke versus fixed point continuity**: the proper generalization of the uniform Lasota–Yorke inequalities (assumptions (8.1–8.2), [16]) in Gouëzel–Liverani result seems to be the continuity of the fixed point map. A notable difference in our approaches is that the spectral gap assumption is made on the largest Banach spaces, whereas our fixed point map existence and continuity assumption (i) in theorem 1) is on the smallest one. Otherwise, the scheme works in the same sense, i.e gain of one derivative when going to the next space.

- **Regularity results for the normalized eigenfunction**: [15, corollary 3.5, p 21] studies the regularity of the normalized eigenfunction \( \phi_{t} \) of a transfer operator \( (L_{t})_{t \in (-\delta, \delta)} \). It is shown that when the transfer operator acts on \( X_{0} \hookrightarrow X_{1} \) 2 Banach spaces with a Taylor expansion of the form (4.2), then \( \phi_{t} \) admits itself a Taylor expansion at \( t = 0 \) in \( X_{0} \):

\[
\| \phi_{t} - \phi_{0} - t \partial_{t} \phi_{0} \|_{X_{0}} = O(h^{2-\epsilon}) \quad \text{with} \quad \epsilon > 0 \quad \text{arbitrarily small, not depending on the spaces} \quad X_{0}, X_{1}.
\]

We obtain a very similar result in theorem 3, by applying theorem 1.

In order to keep the exposition to a reasonable length, we will not discuss applications of theorem 4 to higher-order response theory, nor to higher-order differentiation of the spectral data of the transfer operator. To the reader interested by this subject, we recommend [16, section 8.1] or [31].

A fair warning to our reader: throughout the text, constants are denoted by the letter C, whose numerical value changes from one occurrence to the next.

Recall that if \( \Omega \subset \mathbb{R}^{n} \) is an open subset, \( f \in C^{k}(\Omega) \), \( k \in \mathbb{N} \), \( \alpha \in (0, 1) \), \( r = k + \alpha > 0 \), we say that \( f \) is a \( r \)-map on \( \Omega \) if \( f \) is of class \( C^{k} \) on \( \Omega \) and its \( k \)th differential (seen as a \( k \)-multilinear map) is \( \alpha \)-Hölder. We endowed the space of \( r \) maps of \( \Omega \) with the norm

\[
\| f \|_{C^{r}} = \max(\| f \|_{C^{k}}, \sup_{x \neq y} \frac{\| D^{r}f(x) - D^{r}f(y) \|}{\| x - y \|^{\alpha}}).
\]

The author would like to thank the anonymous referees for their many suggestions and comments, which greatly helped improve both the presentation and mathematical content of the paper.

The author also acknowledges the support of the ESI in Vienna, where the redaction of this paper was started in May 2016.

Finally, the author would like to express his warmest thanks to Hans Henrik Rugh, for his constant support, his availability, and many fruitful conversations during the maturation of this work.

### 2. Differentiation and graded diagram

#### 2.1. Main results

This theorem can be thought of as a complement to the implicit function theorem. Besides the resemblance with [16, theorem 8.1] one can see an analogy with the Nash–Moser scheme [19], with the use of a (finite) scale of spaces.
**Definition 1 (Scale of Banach spaces).** Let \( n \geq 1 \). A family of Banach spaces \( X_0 \supset X_1 \supset \cdots \supset X_n \) is said to be a scale if the injective linear maps \( j_k : X_{k+1} \to X_k \) are bounded (i.e \( 0 \leq i \leq j \leq n \Leftrightarrow \|x_i\| \leq \|x_j\| \)).

We will denote a scale by \( X_0 \overset{j_0}{\to} X_1 \overset{j_1}{\to} \cdots \overset{j_{n-1}}{\to} X_n \), or simply by \( (X_n, \ldots, X_1, X_0) \).

Note that scales of spaces already appeared in [15, 16] and other previous works on spectral stability [8, 24].

**Theorem 1.** Let \( \mathcal{B}, X_0, X_1 \) be Banach spaces such that \( X_0 \overset{j_0}{\to} X_1 \).

Let \( A_1 \subset X_1 \) be closed and non-empty, and \( A_0 = j_0(A_1) \subset X_0 \).

Let \( u_0 \in \mathcal{B} \), and \( U \) a neighborhood of \( u_0 \) in \( \mathcal{B} \).

Consider continuous maps \( F_i : U \times A_i \to A_i \), where \( i \in \{0, 1\} \), with the following property:

\[
F_0(u, j_0(\phi_1)) = j_0 F_1(u, \phi_1) \tag{2.1}
\]

for all \( u \in U \), \( \phi_1 \in A_1 \).

Moreover, we will assume that:

(i) For every \( u \in U \), \( F_1(u, .) : A_1 \to A_1 \) admits a fixed point \( \phi_1(u) \in A_1 \).

Furthermore, the map \( u \in U \mapsto \phi_1(u) \in X_1 \) is continuous.

(ii) Let \( \phi_0(u) = j_0(\phi_1(u)) \).

For some \( (u_0, \phi_0(u_0)) \in U \times j_0(A_1) \), there exists \( P_0 = P_{u_0, \phi_0} \in L(\mathcal{B}, X_0) \).

\[
Q_0 = Q_{u_0, \phi_0} \in L(j_0(X_1), X_0)
\]

such that

\[
F_0(u_0 + h, \phi_0 + z_0) - F_0(u_0, \phi_0) = P_0 h + Q_0 z_0 + (\|h\|_{\mathcal{B}} + \|z_0\|_{X_0}) \epsilon(h, z_1)
= 0 \tag{2.2}
\]

where \( h \in \mathcal{B} \) satisfies \( u_0 + h \in U' \), \( z_1 \in A_k \) \( z_0 = j_0(z_1) \in A_0 \), and \( \epsilon(h, z_1) \xrightarrow{(h,z_1) \to (0,0)} 0 \).

(iii) \( \text{Id} - Q_0 \in L(j_0(X_1), X_0) \) can be extended to a bounded, invertible operator of \( X_0 \) into itself.

Then the following holds:

(i)’ Let \( \phi_0(u) = j_0(\phi_1(u)) \). The map \( u \in U \mapsto \phi_0(u) \in X_0 \) is differentiable at \( u = u_0 \).

(ii)’ Its differential satisfies

\[
D_u \phi(u_0) = (\text{Id} - Q_0)^{-1} P_0.
\tag{2.3}
\]

**Remark 1.** If one were to take \( \epsilon(h, z_1) \) in (2.2) depending only upon \( h \), one could recover a condition similar to [16, section 8.1, 8.3] (see lemma 2).

It can seem artificial to include a statement about continuity of the map \( u \in U \mapsto \phi_1(u) \in X_1 \) without further explanation. One of the cases where such an assumption can be rigorously justified is when one of the iterates of \( F_1 : U \times A_1 \to A_1 \), say \( F^n_1 \) is a contraction w.r.t its second variable, a classical result in fixed point theory:

**Proposition 1.** Let \( \mathcal{B}, X \) be Banach spaces, \( U \subset \mathcal{B} \) an open set and \( A \subset X \) closed, non-empty. Let \( F : U \times A \to A \) be a continuous map, such that there exists \( n \in \mathbb{N} \) for which \( F^n \) is a contraction w.r.t respect to its second variable.

\( i.e. \) there exists a bounded, linear operator \( D_u \phi_0(u_0) : \mathcal{B} \to X_0 \) such that \( \|\phi_0(u_0 + h) - \phi_0(u_0) - D_u \phi_0(u_0) h\|_{X_0} \to 0 \) for all \( h \in \mathcal{B} \) such that \( u_0 + h \in U \).
Then for every \( u \in U \), \( F(u, \cdot) \) admits a unique fixed point \( \phi_u \in A \), and furthermore the map \( u \in U \mapsto \phi_u \in X \) is continuous.

**Proof of proposition 1.** We can apply the Banach contraction principle to \( F^n : U \times A \to A \), and thus obtain the existence of a fixed point \( \phi(u) \in A \) for every \( u \in U \). We also have:

\[
\|\phi(u) - \phi(u_0)\|_X = \|F^n(u, \phi(u)) - F^n(u_0, \phi(u_0))\|_X \tag{2.4}
\]

\[
= \|F^n(u, \phi(u)) - F^n(u_0, \phi(u)) + F^n(u_0, \phi(u)) - F^n(u_0, \phi(u_0))\|_X \tag{2.5}
\]

\[
\leq C\|\phi(u) - \phi(u_0)\|_X + \|F^n(u, \phi(u)) - F^n(u_0, \phi(u))\|_X \tag{2.6}
\]

with \( C < 1 \), so that:

\[
\|\phi(u) - \phi(u_0)\|_X \leq \frac{1}{1-C}\|F^n(u, \phi(u)) - F^n(u_0, \phi(u))\|_X. \tag{2.7}
\]

We can now conclude with the continuity of \( F : U \times A \to A \).

Remark that if we were to demand a stronger condition on the regularity of \( F \) with respect to \( u \), say Hölder-continuity or Lipschitz continuity, the fixed point map \( u \in U \mapsto \phi(u) \in X \) would mirror that condition.

In section 4, we illustrate the abstract theorem 1 by applying it to a positive, linear transfer operator \( L_u \), associated with a family \( \{T_u\}_{u \in U} \) of \( C^{1+\alpha} \) expanding maps on a Riemann manifold \( X \), acting on \( C^{1+\alpha}(X) \), and who admits an isolated, simple eigenvalue \( \lambda_u \) of maximal modulus. It requires to work with the nonlinear map \( F : U \times C^{1+\alpha}(X) \), defined for \( u \in U \) a neighborhood of \( u_0 \in B \) and \( \phi \notin \ker L_u \ell_{u_0} \), by

\[
F(u, \phi) = \frac{L_u \phi}{\int L_u \phi d\ell_{u_0}} \tag{2.8}
\]

where \( \ell_u \) (resp. \( \phi_u \)) is the left (resp. right) eigenvector of \( L_u \), chosen so that \( \int L_u \phi_u d\ell_{u_0} = \lambda_u \). For \( u \in U \), we chose \( \phi_u \) so that \( \langle \ell_u, \phi_u \rangle = 1 \) (this will prove useful in section 4.2).

This (nonlinear) renormalization originates from cone contraction theory, and has been used e.g in [34, 35]. Satisfying assumption (iii) in theorem 1 is the main reason why one is lead to introduce (2.8): indeed, working with the naive guess \( \lambda_u^{-1} L_u \) (for which \( \phi_u \) is an obvious fixed point) cannot give a bounded and invertible second partial differential, by definition of an eigenvalue...

It is also worth noting that the normalized maps \( F \) satisfy condition (i) in theorem 1 thanks to proposition 1. More precisely, we are able to establish the following:

**Theorem 2.** For every \( 0 \leq \beta < \alpha, u \in U \), one has

- \( F(u, \cdot) \) acts continuously (and even analytically) on \( C^{1+\alpha}(X)^* := \{ f \in C^{1+\alpha}(X), \ f \geq 0 \text{ and } f \neq 0 \} \).
- Consider \( F(u, \cdot) : C^{1+\alpha}(X)^* \mapsto C^{1+\beta}(X)^* \). Then \( u \in U \mapsto F(u, \cdot) \) is Hölder continuous, with exponent \( \gamma := \alpha - \beta \).
- \( F(u, \cdot) \) admits a unique fixed point \( \phi(u) \in C^{1+\alpha}(X)^* \), and \( u \in U \mapsto \phi(u) \in C^{1+\beta}(X) \) is \( \gamma \)-Hölder.

We establish this result in section 4.2. It also establishes the first assumption of theorem 1, and is therefore instrumental in proving the following:
**Theorem 3.** Let \( 0 \leq \beta < \alpha < 1 \), \( u_0 \in \mathcal{B} \), \( \mathcal{U} \) a neighborhood of \( u_0 \), \( (T_u)_{u \in \mathcal{U}} \) be a family of \( C^{1+\alpha} \), expanding maps of a Riemann manifold \( X \). For each \( u \in \mathcal{U} \), let \( \mathcal{L}_u \) be a weighted transfer operator on \( C^{1+\alpha}(X) \), associated with \( T_u \) defined by (4.1).

Let \( \lambda_u > 0 \) be its dominating eigenvalue, \( \phi(u) \in C^{1+\alpha}(X) \), \( \ell_u \in (C^{1+\alpha}(X))^* \) be the associated eigenvectors of \( \mathcal{L}_u \) and \( \mathcal{L}_u^* \) respectively. We denote by \( \Pi_u \) the associated spectral projector, and let \( R_u = \mathcal{L}_u - \lambda_u \Pi_u \) (see appendix A).

Then the following holds true:

- The map \( u \in \mathcal{U} \mapsto \phi(u) \in C^\beta(X) \) is differentiable.
- We have the following linear response formula for the derivative with respect to \( u \) at \( u = u_0 \):

\[
D_u \phi(u_0) = \frac{1}{\lambda_{u_0}} (\text{Id} - \lambda_{u_0}^{-1} R_{u_0})^{-1} (\text{Id} - \Pi_{u_0}) \partial_u \mathcal{L}_u \big|_{u = u_0}. 
\]

(2.9)

We establish this result in section 4.3, by applying theorem 1 to \( F \) acting on the scale \((C^{1+\beta}(X), C^\beta(X))\) for any \( 0 < \beta < \alpha \). We show that \( F \) satisfies to a Taylor expansion of the form (2.2), with (see formulas (4.25) and (4.22))

\[
P_0 = \frac{1}{\lambda_{u_0}} (\text{Id} - \Pi_{u_0}) \partial_u \mathcal{L}_u \big|_{u = u_0} 
\]

\[
Q_0 = \frac{1}{\lambda_{u_0}} \mathcal{L}_{u_0} - \Pi_{u_0}.
\]

**2.2. Taking the first derivative: a proof of theorem 1**

Thanks to assumption (ii), we can estimate the difference \( z_0(h) = \phi_0(u_0 + h) - \phi_0(u_0) \) for \( h \in \mathcal{B}, u_0 + h \in \mathcal{U} \).

\[
\phi_0(u_0 + h) - \phi_0(u_0) = F_0(u_0 + h, \phi_0(u_0) + h) - F_0(u_0, \phi_0) = F_0(u_0 + h, \phi_0(u_0) + z_0) - F_0(u_0, \phi_0) = P_0 h + Q_0 z_0(h) + (\|h\|_\mathcal{B} + \|z_0(h)\|_{X_0}) \epsilon(h, z_1)
\]

thus, by (iii):

\[
z_0(h) = (\text{Id} - Q_0)^{-1} P_0 h + (\text{Id} - Q_0)^{-1} (\|h\|_\mathcal{B} + \|z_0(h)\|_{X_0}) \epsilon(h, z_1). 
\]

(2.10)

Now, remark that:

- By continuity of \( u \in \mathcal{U} \rightarrow \phi_0(u) \in \mathcal{X}_1 \) (which is assumption (i)), we have \( \lim_{h \to 0} z_1(h) = 0 \) in \( X_1 \), so that \( \epsilon(h, z_1(h)) = \epsilon(h) \to 0 \) in \( X_0 \) as \( h \to 0 \) in \( \mathcal{B} \).

- \( (\text{Id} - Q_0)^{-1} \epsilon(h, z_1) \|h\|_\mathcal{B} = o(h) \) in \( X_0 \) as \( h \to 0 \) in \( \mathcal{B} \).

- For \( h \) small enough in \( \mathcal{B} \)-norm,

\[
\| (\text{Id} - Q_0)^{-1} \epsilon(h) \|_{X_0} \leq \frac{1}{2}.
\]

(2.11)

Thus, taking the \( X_0 \)-norm in (2.10) and choosing \( h \) small enough in \( \mathcal{B} \)-norm, we obtain:
\[ \|z_0(h)\|_{X_0} \leq \| (I - Q_0)^{-1} P_0 h \|_{X_0} + \| (I - Q_0)^{-1} \varepsilon(h, z_1) \|_{X_0} \| h \|_B + \frac{1}{2} \|z_0(h)\|_{X_0} \]

\[ \frac{1}{2} \|z_0(h)\|_{X_0} \leq \| (I - Q_0)^{-1} P_0 h \|_{X_0} + \| (I - Q_0)^{-1} \varepsilon(h, z_1) \|_{X_0} \| h \|_B \] (2.12)

and thus:

\[ z_0(h) = O(h). \] (2.13)

Following (2.13), the second term of the sum in the right hand term of (2.10) becomes:

\[ (I - Q_0)^{-1} (\| h \|_B + O(h)) \varepsilon(h) = o(h). \] (2.14)

Finally, in the \( X_0 \)-topology,

\[ z_0(h) = (I - Q_0)^{-1} P_0 h + o(h) \] (2.15)

and thus \( u \in U \rightarrow \phi_0(u) \in X_0 \) is differentiable at \( u = u_0 \) and

\[ D_u \phi_0(u_0) = (I - Q_0)^{-1} P_0. \] (2.16)

### 2.3. Higher differentiability and graded diagram

In order to differentiate the fixed point map we have to consider an argument coming from a smaller, more ‘regular’ space. More precisely, we showed that if there is, for every \( u \in U \) a \( \phi_1(u) \in X_1 \) such that \( F(u, \phi_1(u)) = \phi_1(u) \), then \( u \mapsto \phi_0(u) = j_0(\phi_1(u)) \in j_0(X_1) \subset X_0 \) is differentiable.

We aim to iterate this approach to differentiate further the fixed point map with respect to the parameter. In order to do so, we define a notion of an \( n \)-graded family as such:

**Definition 2 (Graded family).** Let \( n \geq 1 \) be an integer, and consider a Banach space \( B \), a scale \( X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n \), \( U \subset \mathcal{B} \) an open subset, \( A_n \subset X_n \) a closed, non-empty subset.

For \( 0 \leq k \leq l < n \), we denote by \( j_{k,l} \) the bounded linear map \( j_k \circ j_{k+1} \circ \ldots \circ j_l : X_{l+1} \rightarrow X_k \), and by \( j_k \circ \ldots \circ j_0 : X_n \rightarrow X_k \).

Define, for \( i \in \{0, \ldots, n - 1 \} \), \( A_i = j_{i,n-1}(A_n) \), and continuous maps \( F_i : U \times A_i \rightarrow A_i \), \( i \in \{0, \ldots, n\} \) such that:

(i) For every \( u \in U \), \( \phi_1 \in A_1 \), \( j_i(F_{i+1}(u, \phi_{i+1})) = F_i(u, j_k(\phi_{i,k})) \).

(ii) There exists \( (u, \phi_n) \in U \times X_n \) such that for every \( h \in B \) such that \( u + h \in U \), every \( z_n \in X_n \) such that \( \phi_n + z_n \in A_n \), for every \( 1 \leq k \leq n \), \( F_{n-k}(u + h, \phi_{n-k} + z_{n-k}) - F_{n-k}(u, \phi_{n-k}) = \sum_{\ell=1}^k \sum_{i+j=k-\ell+1} Q^{(i,j)}(u, \phi_{n-\ell+1})[h, z_{n-\ell}] + R_n(h, z_n) \)

where for every pair \((i, j)\) so that \( i + j = k \),

- \( Q^{(i,j)}(u, \phi_k) \in L(B^i \times X^j_{k-1}, X_{n-k}) \) is a \( \ell \)-linear map.
- \( R_n \in C^0(B \times X_n, X_{n-k}) \) is such that \( \| R_n(h, z_n) \|_{X_{n-k}} = o(\|h\|_B; \|z_n\|^{\frac{1}{2}}_{X_{n-k}}) \).
We call a family of maps \((F_i)_{i \in \{0, \ldots, n\}}\) acting on \(B, X_0 \xrightarrow{h} X_1 \xrightarrow{h} \cdots \xrightarrow{h} X_n\) and satisfying (i)–(ii), an \(n\)-graded family.

**Lemma 1.** Let \((F_i)_{i \in \{0, \ldots, n\}}\) be an \(n\)-graded family.

Then for every \(1 \leq k \leq n\), whenever \(\phi_n \in C^k(U, A_1), \ldots, \phi_{n-k+1} = j_{n-k+1}(\phi_n) \in C^{k-1}(U, A_{n-k+1})\), the map \(u \in U \mapsto j_{n-k} \circ F_n(u, \phi_n(u)) \in X_{n-k}\) is \(k-1\) times differentiable, and has the following property: there exists \(R^{(k-1)}(u) \in L^{k-1}(B, X_{n-k})\) such that for every \(u \in U\)

\[
D_{\alpha}^{k-1} [j_{n-k} \circ F_n](u, \phi_n(u)) = R^{(k-1)}(u) + Q^{(0,1)}_{\alpha, \phi_{n-k+1}} D_{\alpha}^{k-1} \phi_{n-k}(u) \tag{2.18}
\]

where

1. \(u \in U \mapsto R^{(k-1)}(u) \in L^{k-1}(B, X_k)\) is differentiable.
2. \(u \in U \mapsto Q^{(0,1)}(u, \phi_{n-k+1}(u)) \in L(X_{n-k})\) is differentiable.

**Proof of Lemma 1.** From (2.17), one can write:

\[
j_{n-k} \circ [F_n(u + h, \phi_n(u + h)) - F_n(u, \phi_n(u))] = F_n(u + h, \phi_{n-k}(u + h)) - F_n(u, \phi_{n-k}(u))
\]

\[
= \sum_{\ell=1}^k \sum_{j+\ell+j \in \ell+1} Q^{(j)}(u, \phi_{n-\ell-1}(h, \phi_{n-\ell}(u + h) - \phi_{n-\ell}(u)) + R_u(h, \phi_{n-\ell}(u + h) - \phi_{n-\ell}(u)).
\tag{2.19}
\]

From our assumptions, for every \(\ell \in \{1, \ldots, k-1\}, \phi_{n-\ell}\) is \(\ell\) times differentiable on \(U\), so that one can write for every \(u \in U\) and \(h \in B\) such that \(u + h \in U\),

\[
\phi_{n-\ell}(u + h) - \phi_{n-\ell}(u) = D_u \phi_{n-\ell}(u) h + \cdots + D_u \phi_{n-\ell}(u) h + o(||h||^\ell) \tag{2.20}
\]

the term in \(o(||h||^\ell)\) being understood in \(X_{n-\ell}\).

The same Taylor development (at order \(k - 1\)) holds for \(\phi_{n-k} = j_{n-k}(\phi_{n-k+1})\). Injecting (2.20) in (2.19) establishes first that \(j_{n-k} \circ F_n(u, \phi_n)\) is \((k-1)\) times differentiable.

Secondly, from the variety of terms it yields, we only choose the terms that are \(k-1\) linear in \(h\): this gives us the \(k\)-1 differential with respect to \(u\), written as (2.18), along with the following explicit expression for \(R^{(k-1)}\):

\[
R^{(k-1)}(u) = \sum_{\ell=2}^{k-2} \sum_{i=0}^\infty \sum_{j+\ell+j \in \ell+1} Q^{(j)}(u, \phi_{n-\ell-1}(u)) \left[ h, D_u \phi_{n-\ell-1}(u), \ldots, D_u \phi_{n-\ell-1}(u) \right]
\]

\[
+ \sum_{j+\ell+j \in \ell+1} Q^{(j)}(u, \phi_{n-\ell}(u)) \left[ h, D_u \phi_{n-2} \right]. \tag{2.21}
\]

From there, it easy to check differentiability with respect to \(u\), as the previous expression only involves terms of indices \(n-\ell\) with at most \(\ell-1\) derivatives.

**Theorem 4.** Let \((F_i)_{i \in \{0, \ldots, n\}}\) be a \(n\)-graded family. Let \(u \in U\). We make the following assumptions:
For every \( u \in U \), \( F_n(u, \cdot) \) admits a fixed point \( \phi_n(u) \). Furthermore, we assume that the map \( u \in U \mapsto \phi_n(u) \in X_n \) is continuous.

For every \( 0 \leq k \leq n - 1 \), \( Id - Q_{\phi_n(u)}^{(k)} \) is an invertible, bounded operator of \( X_k \).

Then for every \( 1 \leq k \leq n \) the fixed point map \( u \in U \mapsto \phi_{n-k}(u) \in X_{n-k} \) is \( k \) times differentiable, and one has the following formula for its differential:

\[
D_k^\phi \phi_{n-k}(u) = (Id - Q_{\phi_n(u)}^{(k)}(u, \phi_{n-k+1}(u)))^{-1} R_k(u).
\] (2.22)

Furthermore, when \( u \in U \mapsto (Q_{\phi_n(u)}^{(1)}(u, \phi_{n-k+1}(u)), R_k(u)) \) is continuous, then so is \( u \in U \mapsto D_k^\phi \phi_{n-k}(u) \), i.e. the fixed point map \( \phi_{n-k} \) is \( C^k \).

**Proof of theorem 4.** The continuity statement is obvious. We present a proof by finite and descending induction.

- For \( k = 1 \), the differentiability of \( u \in U \mapsto \phi_{n-1}(u) \) at \( u = u_0 \) is simply theorem 1.
- For \( k = 2 \).
  - For every \( h \in B, u_0 + h \in U \), one has, thanks to the case \( k = n - 1 \) and assumption (2.18):
    \[
    D_n(j_{n-2,n-1} \circ F_n)(u, \phi_n(u)).h = D_n F_{n-2}(u, \phi_{n-2}(u)).h
    \] (2.23)
    \[
    = Q_{\phi_n(u)}^{(1)}(u, \phi_{n-1}(u)).h + Q_{\phi_n(u)}^{(0,1)}(u, \phi_{n-1}(u)) D_n \phi_{n-2}(u).h
    \] (2.24)
  - Note that \( \phi_{n-2}(u) \) is, for every \( u \in U \), a fixed point of \( F_{n-2}(u, \cdot) \), so that
    \[
    D_n F_{n-2}(u), \phi_{n-2}(u).h = D_n \phi_{n-2}(u).h.
    \] (2.25)
  - This last equality yields,
    \[
    D_n \phi_{n-2}(u).h = Q_{\phi_n(u)}^{(1)}(u, \phi_{n-1}(u)).h + Q_{\phi_n(u)}^{(0,1)}(u, \phi_{n-1}(u)) D_n \phi_{n-2}(u).h
    \] (2.26)
    \[
    (Id - Q_{\phi_n(u)}^{(0,1)})(u, \phi_{n-1}(u)) D_n \phi_{n-2}(u).h = Q_{\phi_n(u)}^{(1,0)}(u, \phi_{n-1}(u)).h
    \] (2.27)
    \[
    D_n \phi_{n-2}(u_0).h = (Id - Q_{\phi_n(u)}^{(0,1)}(u, \phi_{n-1}(u)))^{-1} Q_{\phi_n(u)}^{(1,0)}(u, \phi_{n-1}(u)).h
    \] (2.28)
  - By (2.18) in definition 1,
    \[
    \begin{cases}
    u \in U \mapsto Q_{\phi_n(u)}^{(1,0)}(u, \phi_{n-1}(u)) \\
    u \in U \mapsto Q_{\phi_n(u)}^{(0,1)}(u, \phi_{n-1}(u))
    \end{cases}
    \]
    are differentiable at \( u = u_0 \), between the Banach spaces \( B \) and \( L(B, X_{n-2}) \) (respectively \( L(X_{n-2}) \)).
  - By the previous equality, \( u \in U \mapsto D_n \phi_{n-2}(u) \) is differentiable at \( u = u_0 \), i.e \( u \in U \mapsto \phi_{n-2}(u) \in X_{n-2} \) is a twice differentiable map at \( u = u_0 \).
  - Let \( 3 \leq k \leq n \).
    - Assume the property:
      \[
      u \in U \mapsto \phi_{n-k+1}(u) = j_{n-k+1}(\phi_n(u)) \in X_{n-k+1} \text{ is a } k - 1 \text{ times differentiable map.}
      \]
    - Then, by lemma 1, (2.18) one can write, for the \( k \)-1 differential of \( u \mapsto j_{n-k} \circ F_n(u, \phi_n(u)) \)
\[ D_u^{k-1} \phi_{n-k}(u) = D_u^{k-1} \varphi_u \circ F_u(u, \phi_u(u)) = R^{(k-1)}(u) + Q^{(0,1)}(u, \phi_{n-k+1}(u))D_u^{k-1} \phi_{n-k}(u). \]

Thus, we obtain by the invertibility assumption,

\[ D_u^{k-1} \phi_{n-k}(u) = (\text{Id} - Q^{(0,1)}(u, \phi_{n-k+1}(u)))^{-1} R^{(k-1)}(u). \]

By virtue of lemma 1, one obtains the differentiability of \( u \in \mathcal{U} \mapsto D_u^{k-1} \phi_{n-k}(u) \), and therefore, that the map \( u \in \mathcal{U} \mapsto \phi_{n-k}(u) \in X_{n-k} \) is \( k \) times differentiable, with the announced formula.

3. A nonlinear application

In this section we give an application of theorem 1 to the study of a fixed point of a nonlinear map. Note also that the parameters lie in an infinite dimensional space.

Consider the interval \( I = [-1,1] \), and let \( C^{1,1}(I) \) be the set of \( C^1 \) map on \( I \) with Lipschitz derivative, endowed with the norm \( \| f \|_{1,1} = \max(\| f \|_{C^1}, \sup_{x \in I} \frac{|f(x) - f(y)|}{|x-y|}) \), which makes it a Banach space. Define the map \( F : C^{1,1}(I) \times C^{1,1}(I) \to C^{1,1}(I) \) by

\[ F(u, \phi) = \frac{1}{2} \phi \circ \phi + u. \]  

(3.1)

We will show the following:

**Theorem 5.** Let \( I, C^{1,1}(I) \), and \( F : C^{1,1}(I) \times C^{1,1}(I) \to C^{1,1}(I) \) be as above. One has:

(i) Let \( \mathcal{U} = B_{C^{1,1}}(0, r') \) be an open ball in \( C^{1,1}(I) \). There is \( r, r' \in (0, 1) \), such that for every \( u \in \mathcal{U}, B_{C^{1,1}}(0, r) \), \( F(u, \cdot) \) is a contraction of \( B_{C^{1,1}}(0, r) \) in the \( C^1 \) topology: therefore it admits a fixed point \( \varphi_u \in B_{C^{1,1}}(0, r) \), and furthermore the map \( u \in \mathcal{U} \mapsto \varphi_u \in C^1(I) \) is continuous.

(ii) \( F \) acting on the scale \( (C^1(I), C^0(I)) \) satisfies a development of the form (2.2). Therefore the map \( u \in \mathcal{U} \mapsto \varphi_u \in C^1(I) \) is differentiable.

**Proof of theorem 5.**

(i) It is a straightforward computation: for every \( u \in \mathcal{U} \), one has

\[ \| F(u, \phi) \|_{\infty} \leq \frac{\| \phi \|_{\infty}}{2} + \| u \|_{\infty} \]

\[ \| D_1 F(u, \phi) \|_{\infty} \leq \frac{\| \phi' \|_{C^0}^2}{2} + \| u' \|_{\infty} \]

\[ \| D_1 F(u, \phi) \|_{L_1} \leq \frac{\| \phi' \|_{C^0} \| \phi' \|_{L_1} (1 + \| \phi' \|_{\infty}) + \| u' \|_{L_1}. \]

Therefore we should choose \( r, r' \) such that \( \frac{r}{2} + r' \leq r, \frac{r}{2} + r' \leq r \) and \( \frac{r}{2} (1 + r) + r' \leq r \). This conditions, which admits obvious solutions, insure us that \( F(u, \cdot) \) preserves \( B_{C^{1,1}}(0, r) \). From now on, we fix \( r, r' \) so that those conditions are satisfied.
We now show that \( \|F(u, \phi) - F(u, \psi)\|_{C^1} \leq k\|\phi - \psi\|_{C^1} \), when \( \phi, \psi \in B_{2,1}(0, r) \). It is noteworthy that here, \( k \) is independent of \( u \). One has:

\[
\|F(u, \phi) - F(u, \psi)\|_{C^1} \leq \frac{1}{2}(1 + \|\phi'\|_\infty)\|\phi - \psi\|_{C^1}
\]

\[
\|D_tF(u, \phi) - D_tF(u, \psi)\|_{C^1} \leq \frac{1}{2}(\|\phi'\|_\infty + |\phi'|_{Lip}\|\phi'\|_\infty + \|\phi'\|_\infty)\|\phi - \psi\|_{C^1}
\]

so that one need to impose the following conditions on \( r ; \frac{1 + r}{2} < 1 \), \( 2 + r^2 < 1 \).

Not only do these conditions clearly have solutions, they are also compatible with the conditions imposed on \( r \) in (i). From now on, we assume that \( r, r' \) satisfy both sets of conditions.

Thus, for every \( u \in B_{2,1}(0, r') \), \( F(u, \cdot) : B_{2,1}(0, r) \to B_{2,1}(0, r) \) is a contraction in the \( C^1 \) topology. Hence it admits a fixed point \( \varphi_u \in B_{2,1}(0, r) \), and the map \( u \in U \mapsto \varphi_u \in C^1(I) \) is continuous (and even Lipschitz) by proposition 1.

- One can write, for \( u, h \in C^{1,1}(I) \) such that \( u, u + h \in U \) and \( \phi, z \in C^1(I) \),

\[
F(u + h, \phi + z) - F(u, \phi) = h + \frac{1}{2}[\phi' \circ \phi, z + z \circ \phi] + (z' \circ \phi), z + \|z\|_\infty \epsilon_0(z)
\]

where \( \|\epsilon_0(z)\|_\infty \to 0 \) as \( \|z\|_\infty \to 0 \). From there it is clear that with:

\[
P_{u, \phi}h = h
\]

\[
Q_{u, \phi}z = \frac{1}{2}[\phi' \circ \phi, z + z \circ \phi]
\]

\[
\epsilon(h, z_1) = (z' \circ \phi), z + \|z\|_\infty \epsilon_0(z) = (z' \circ \phi), z_0 + \|z_0\|_\infty \epsilon_0(z_0).
\]

\( F \) satisfies a development of the form (2.2).

To conclude, we need to establish the invertibility (and boundedness of the inverse) of \( Q_{u, \phi} \in C^0(I) \).

It is easy to see that for every \( \phi \in B_{C^{1,1}}(0, r) \), \( \|Q_{u, \phi}z\|_\infty \leq \frac{1}{2}(1 + r)\|z\|_\infty \), so that \( \|Q_{u, \phi}\|_{C^0} < 1 \) whenever \( r < 1 \) (which is insured by the sets of conditions in (i), (ii)). Therefore its Neumann series converges in \( C^0(I) \), and \( Id - Q_{u, \phi} \) has a bounded inverse in \( C^0(I) \) for every \( \phi \in B_{C^{1,1}}(0, r) \).

4. Application to linear response for expanding maps

As a second application of our main result theorem 1, we study the linear response problem, in the context of smooth uniformly expanding maps.

More precisely, our strategy is the following:

- We first show regularity results (Hölder and Lipschitz continuity, differentiability in the sense of (2.2)) for the transfer operator \( L_u \) acting on Hölder spaces, with respect to \( u \); see lemma 2.
- We then establish theorem 2 by a direct argument (see section 4.2).
- We finally prove theorem 3 by applying theorem 1 to the map \( F \) defined by (2.8), acting on the scale \( (C^{1+\beta}(X), C^1(X)) \) (see section 4.3).
4.1. Perturbations of the transfer operator

Let \( d \geq 1, \; \epsilon > 0, \; \mathcal{U} = (-\epsilon, \epsilon)^d, \; 0 < \alpha < 1 \) and \((T_\alpha)_{\alpha} \subset \mathcal{U} \in C^{1+\alpha} \) be a \( C^{1+\alpha} \) family of \( C^{1+\alpha} \) expanding maps. For example, \( T_\alpha \) can be a \( C^{1+\alpha} \) perturbation of an original expanding map \( T_0 \); by (iii) in proposition \( 3, \; T_\alpha \) is also expanding for \( u \in \mathcal{U} \) small enough.

Let \( g : \mathcal{U} \times X \to \mathbb{R} \) be a \( C^{1+\alpha} \) map. For every \( u \in \mathcal{U} \), define the associated transfer operators (e.g. on \( L^\infty(X) \)) by

\[
L_u \phi(x) = \sum_{\gamma, T_\gamma = x} g(u, y) \phi(y). \tag{4.1}
\]

Recall that the spectral features of interest appears when the transfer operator acts on Hölder spaces (see appendix A). In the next proposition, we study the regularity of \( L_u \) with respect to the parameter \( u \).

**Lemma 2 (Regularity of the perturbed transfer operator).** Let \( 0 \leq \beta < \alpha < 1 \), and \( \gamma := \alpha - \beta \). Let \( X, \mathcal{U} \) and \( g, T_\alpha, L_u \) be as above.

- \( u \in \mathcal{U} \mapsto L_u \in L(C^{1+\alpha}(X), C^{1+\beta}(X)) \) is \( \gamma \)-Hölder.
- In particular, it is a continuous map.
- For every \( h \in \mathcal{B} \) such that \( u_0 + h \in \mathcal{U} \), every \( 0 \leq \beta \leq \alpha \), we can define a bounded operator \( \partial_u L(u_0, \cdot) : C^{1+\beta}(X) \to C^\beta(X) \), such that for every \( \phi \in C^{1+\beta}(X) \),

\[
\mathcal{L}(u_0 + h, \phi) - \mathcal{L}(u_0, \phi) - \partial_u \mathcal{L}(u_0, \phi) h = \|h\|_{C^\beta}(h), \tag{4.2}
\]

with \( \epsilon(h) \to 0 \) in \( C^\beta(X) \).

Furthermore, \( \mathcal{L} \) satisfies (2.2) in theorem 1, with the scale \( (C^{1+\beta}(X), C^\beta(X)) \).

**Proof.** By a standard argument (see [17, 28]), one can construct a family of open sets covering \( X \), small enough to be identified with open sets in \( \mathbb{R}^{\dim(X)} \), and such that on each of this open sets, the transfer operator is a (finite) sum of operators of the form \( \mathcal{W}_v \phi := (g_v \phi) \circ \psi_v \), with \( \phi \in C^{1+\alpha}(W), \; \psi_v \in C^{1+\alpha}(U \times V, W) \) is a contraction in its second variable (and a local inverse branch of \( T_\alpha \)), \( g \in C^{1+\alpha}(U \times W) \) with compact support, and \( V \), \( W \) open sets in \( \mathbb{R}^{\dim(X)} \). We will apply the results of appendix B to the operators \( \mathcal{W}_v \).

For the first item, one needs to estimate, for \( \phi \in C^{1+\alpha}(W), \; \|\mathcal{W}_v - \mathcal{W}_\gamma\|_{C^{1+\gamma}} = \max(\|W_v - W_\gamma\|_{C^{1+\gamma}}, \|D_v(W_v - W_\gamma)\|_{C^{1+\gamma}}) \).

Assume first that the weight \( g \) is independent of the parameter. Then by lemma 1, (6.3),

\[
\|W_v - W_\gamma\|_{C^{1+\gamma}} \leq C \|\phi\|_{C^{1+\alpha}} \|u - v\|^\gamma \tag{4.3}
\]

with \( C = C(\alpha, \beta, \|g\|_{C^\gamma}, \|\psi_v\|_{C^\gamma}, \|\psi_v\|_{C^{1+\alpha}, L^\alpha, L_\alpha, L^\alpha_\gamma}) \).

Now if \( g \) also depends on \( u \in \mathcal{U} \), computing \( \|W_v - W_\gamma\|_{C^{1+\gamma}} \) with \( \phi \in C^{1+\alpha} \) would yield an additional term of the form \( \|g(u, \cdot) - g(v, \cdot)\|_{C^{1+\gamma}} \), whose \( C^{1+\beta} \) norm would be bounded by \( C\|\phi\|_{C^{1+\alpha}} \|u - v\|^\gamma \), with \( C \) a constant.

Thus, \( u \in \mathcal{U} \mapsto L_u \in L(C^{1+\alpha}(X), C^{1+\beta}(X)) \) is (locally) \( \gamma \)-Hölder.

Let \( \phi \in C^{1+\alpha}(W) \). The \( C^1 \) regularity of the inverse branches (w.r.t to \( u \)) allows one to consider the (partial) differential of \( W \) with respect to \( u \). Again, assume for the time being that \( g \) does not depends on \( u \). Define \( \chi_u : X \to L(B, TX) \) such that \( D_u \psi_u = -\chi_u \circ \psi_u \), one gets:

\[
\partial_u W(u, \phi) = [Dg(\psi_u) \circ D_u \psi_u]. \phi \circ \psi_u + g \circ \psi_u [D\phi(\psi_u) \circ D_u \psi_u]. \tag{4.4}
\]

The previous formula defines a bounded operator \( \partial_u W \in L(B, L(C^{1+\alpha}(W), C^\alpha(W))) \), by virtue of lemma B.2.
One can easily extend the former to \( L_u \), and define a ‘partial differential’ \( \partial_u \), taking value in \( \mathcal{L}(B, L(C^{1+\alpha}(X), C^0(X))) \). To what extend is it a ‘true’ partial differential? To answer that question one has to estimate \( \| \mathcal{L}(u_0 + \h, \phi) - \mathcal{L}(u_0, \phi) - \partial_u \mathcal{L}(u_0, \phi).h \|_{C^0} \), for \( \phi \in C^{1+\alpha}(X) \).

Let \( x \in X \). One has

\[
[\mathcal{W}_{u_0+k}: \phi - \mathcal{W}_{u_0}: \phi - \partial_u \mathcal{W}(u_0, \phi).h](x) = (I) + (II) + (III)
\]

where

\[
(I) = \phi(\psi(u_0, x))[g(\psi(u_0, x + h)) - g(\psi(u_0, x))] = g(\psi(u_0, x)) \circ \chi_{u_0}(x).h
\]

\[
(II) = g(\psi(u_0, x))[\phi(\psi(u_0, x + h)) - \phi(\psi(u_0, x))] = D \phi(\psi(u_0, x)) \circ \chi_{u_0}(x).h
\]

\[
(III) = [\phi(\psi(u_0 + h, x)) - \phi(\psi(u_0, x))][g(\psi(u_0, x + h)) - g(\psi(u_0, x))].
\]

By lemma B.3, (B.8), and lemma B.2, (B.5) (I), (II) and (III) can be bounded as follows:

\[
\| (I) \|_{C^0} \leq C \| \phi \|_{C^0} \| h \|^{1+\gamma} \| g \|_{C^{1+\alpha}}
\]

\[
\| (II) \|_{C^0} \leq C \| g \|_{C^0} \| h \|^{1+\gamma} \| \phi \|_{C^{1+\alpha}}
\]

\[
\| (III) \|_{C^0} \leq C \| h \|^{2} \| \phi \|_{C^{1+\alpha}} \| g \|_{C^{1+\alpha}}.
\]

From the latter 3, it is straightforward that

\[
\mathcal{L}(u_0 + \h, \phi) - \mathcal{L}(u_0, \phi) - \partial_u \mathcal{L}(u_0, \phi).h = \| \h \|_{\mathcal{E}(h, \| g \|_{C^{1+\alpha}}, \| \phi \|_{C^{1+\alpha}})}
\]

where \( \mathcal{E}(h, \| g \|_{C^{1+\alpha}}, \| \phi \|_{C^{1+\alpha}}) = \mathcal{O}(\| h \|_{\mathcal{E}}) \).

Let us now show that \( \mathcal{L} \) satisfies the Taylor expansion (2.2) in the assumptions of theorem 1.

We start by recalling the following Taylor estimate, found in [10]4:

Letting \( E, F, G \) be Banach spaces, \( U \subset E, V \subset F \) be open sets, \( 0 \leq \beta < \alpha < 1, \) and \( (f, h) \in C^{1+\beta}(U, V) \) (g, \( k ) \in C^{1+\alpha}(V, G) \), one has

\[
(g + k) \circ (f + h) = g \circ f + k \circ f + [dg \circ f].h + R_{g/k}(h,k)
\]

where there exists some \( 0 < \rho < 1 \) such that the remainder term \( R_{g/k}(h,k) \) satisfies

\[
\| R_{g/k}(h,k) \|_{C^0} \leq C(\| h \|_{C^{1+\rho}} + \| h \|_{C^{1+\alpha}} \| k \|_{C^{1+\alpha}}).
\]

This, together with the definition of \( \partial_u \mathcal{W}_{u_0} \), yields for \( (\phi, z) \in C^{1+\alpha}(W) \)

\[
\mathcal{W}_{u_0+k}(\phi + z) - \mathcal{W}_{u_0}(\phi) - \partial_u \mathcal{W}(u_0, \phi).h - \mathcal{W}_{u_0}(z)
\]

\[
= D(g) \circ \psi_{u_0}(\psi_{u_0+h} - \psi_{u_0} - h) + R_1(\psi_{u_0+h} - \psi_{u_0}, g, z)
\]

where \( R_1 = R_{g, \psi_{u_0}} \) from (4.7). We start by bounding the first term. One has

\[
\psi_{u_0+h} - \psi_{u_0} - h = \int_0^1 [\partial_u \psi(u_0 + th) - \partial_u \psi(u_0)].hdt
\]

which leads us to estimate a term of the form \( \| df(\psi(u_0)). \int_0^1 [\partial_u \psi(u_0 + th) - \partial_u \psi(u_0)].hdt \|_{C^0} \).

Following the trick used in the proof of lemma B.3, we get

3 From the previous bounds, one can even conclude that the map \( u \in U \mapsto \mathcal{L}(u, \phi) \in C^0(X) \) is \( C^{1+\gamma} \) for \( \phi \in C^{1+\alpha}(X) \), which is precisely the conclusion drawn from the Taylor development at first order in Gouëzel–Liverani’s paper [16, section 8.1, 8.3].
4 We specifically refer to estimate (6.7) after theorem 6.10.
\[ \| df(\psi(u_0)) \|_\infty \leq \frac{\| df \|_{C^\alpha} \| \psi(u_0) \|_{C^\alpha}^2 + C_2 \| df \|_{C^\alpha}}{1 + \gamma}. \]

(4.10)

Now for $R$, we write, following estimate (4.7):
\[ \| R \|_{C^\alpha} \leq M \| h \|^{1+\rho} + \| h \| \cdot (C_1 \| z \|_{C^{1+\alpha}} + C_0 \| z \|_{C^\alpha}) \]
(4.11)

with $C_1$, $C_2$ depending on $\alpha$, $\| g \|_{C^n}$, $\| g \|_{C^{1+\alpha}}$.

Therefore, we obtained the following bound for (4.8):
\[ M \| h \|^{1+\rho} + M' \| h \|^{1+\gamma} + C_1 \| h \| \cdot \| z \|_{C^{1+\alpha}} + C_2 \| h \| \cdot \| z \|_{C^\alpha} = \| h \| + \| z \|_{C^\alpha} \epsilon(h, z_{1+\alpha}) \]
(4.12)

where $z_{1+\alpha}$ is $z$ in $C^{1+\alpha}$ topology and $\epsilon(h, z_{1+\alpha}) \to 0$ in $C^\beta(X)$.

In the case of a weight $g$ depending on the parameter $u$, the partial derivative $\partial_u \mathcal{V}$ is given by
\[ \partial_u \mathcal{V}(u, \phi) = ([D_u(g)(u)] \phi) \circ \psi(u) + D_x(g \phi) \circ \psi(u).D_u \psi(u). \]
(4.13)

Thus, the Taylor expansion at $(u_0, \phi)$ now has an additional term
\[ ([g(u_0 + h) - g(u_0)] - D_u(g)(u_0)h) \circ \psi(u_0) \]

This term can be bounded (in $C^\beta$-norm), with upper bound of the form $C \| g \|_{C^{1+\alpha}} \| h \|^{1+\gamma}$, where $C = (\| \psi(u_0) \|_{C^{1+\alpha}}, \| \phi \|_{C^{1+\alpha}})$ is a constant, as outlined in lemma B.3.

It follows that the transfer operator defined in (4.1) also has a Taylor expansion of the form (2.2).

**Remark 2.** The previous regularity results are given for $\mathcal{L}_u$ acting on the scale $(C^{1+\beta}(X), C^{\phi}(X))$, $0 < \beta < \alpha \leq 1$. Following the method outlined in [10], and using theorem 4, one can show (by induction) that $\mathcal{L}_u$ acting on the scale $C^{k+\beta}(X), C^{k-j+\beta}(X)$ has a Taylor development of the form (2.17) at order $j$, with $0 \leq j < k$ integers.

4.2. Hölder continuity of the spectral data: proof of theorem 2

This section is devoted to establish theorem 2, by a direct argument. Note that this type of result is already known for a one-dimensional parameter, with previous works on spectral stability [8, 24], or in the context of piecewise expanding maps of the interval [23].

Let $0 \leq \beta < \alpha < 1$, and $(T_u)_{u \in U}$ be a family of $C^{1+\alpha}$ expanding maps, on a Riemann manifold $X$. Let $g : X \to \mathbb{R}$ be a positive $C^{1+\alpha}$ function.

It follows from Ruelle theorem [28] that the transfer operator $(\mathcal{L}_u)_{u \in U}$ admits a spectral gap in $C^{1+\alpha}(X)$. Let $\lambda_u$ be the dominating eigenvalue of $\mathcal{L}_u$, $\phi_u \in C^{1+\alpha}(X)$ (resp $\ell_u \in (C^{1+\alpha}(X))^*$) be the right (resp left) eigenvector of $\mathcal{L}_u$ associated with $\lambda_u$, chosen so that $\langle \ell_u, \phi_u \rangle = 1$. Let $F : U \times C^{1+\alpha}(X)$, defined for $u \in U$ and $\phi \notin \ker \mathcal{L}_u$, by equation (2.8).

Note that $F$ trivially inherits every regularity property of $(u, \phi) \in U \times C^{1+\alpha}_+(X)$ $\to \mathcal{L}_u \phi$, so in particular it is $\gamma$-Hölder in $u \in U$ when considered as an operator from $C^{1+\alpha}_+(X)$ to $C^{1+\beta}_+(X)$. Hence the first point.

5 Note that we only need the positivity of the weight to insure the simplicity of the maximal eigenvalue.
The second item follows from the former remark and the fact that \( \ell_u \) admits a bounded extension to \( C^{1+\beta}(X) \), for every \( 0 \leq \beta < \alpha \) (see [28]).

Let \( \phi_u \in C^{1+\alpha}(X)^* \) be an eigenvector for \( \lambda_u \), the dominating eigenvalue of \( \mathcal{L}_u \). Then one has

\[
F(u, \phi_u) = \frac{\lambda_u \phi_u}{\lambda_u \langle \ell_u, \phi_u \rangle} = \frac{\phi_u}{\langle \ell_u, \phi_u \rangle}.
\]

(4.14)

For every \( u \in \mathcal{U} \), fix a \( \phi_u \in \ker(\lambda_u - \mathcal{L}_u) \) such that \( \langle \ell_u, \phi_u \rangle = 1 \). Such a \( \phi_u \) is unique in \( \ker(\lambda_u - \mathcal{L}_u) \) and verifies

\[
F(u, \phi_u) = \phi_u.
\]

(4.15)

so that \( F(u, .) \) has a unique fixed point \( \phi_u \) in \( C^{1+\alpha}(X)^* \) for every \( u \in \mathcal{U} \).

Remark that for every \( k \in \mathbb{N}^* \), for every \( u \in \mathcal{U} \), every \( \phi \notin \ker((\mathcal{L}_u)^k \ell_u) \),

\[
F^k(u, \phi) = \frac{\mathcal{L}^k_u(\phi)}{\langle \ell_u, \mathcal{L}^k_u(\phi) \rangle}
\]

(4.16)

by an immediate induction

Now note that, for every \( k \in \mathbb{N}^* \), \( u \in \mathcal{U} \),

\[
\phi(u) - \phi(u_0) = F^k(u, \phi(u)) - F^k(u_0, \phi(u)) + F^k(u_0, \phi(u)) - F^k(u_0, \phi(u_0))
\]

and that

\[
F^k(u_0, \phi(u)) - F^k(u_0, \phi(u_0)) = \frac{\mathcal{L}^k_u(\phi(u))}{\langle \ell_u, \mathcal{L}^k_u(\phi(u)) \rangle} - \phi(u_0) = \lambda_u^{-k} R^k_u(\phi(u) - \phi(u_0)).
\]

(4.17)

(4.18)

Recall that there is a \( 0 < \sigma < 1 \) such that \( \|\lambda_u^{-k} R^k_u\|_{C^{1+\beta}(X)} \leq C\sigma^k \) (see appendix A), so that for \( k \) large enough, one has

\[
\|F^k(u_0, \phi(u)) - F^k(u_0, \phi(u_0))\|_{C^{1+\beta}} \leq \frac{1}{2} \|\phi(u) - \phi(u_0)\|_{C^{1+\beta}}.
\]

(4.19)

From there, (4.17) yields

\[
\|\phi(u) - \phi(u_0)\|_{C^{1+\beta}} \leq C_{k,u} \|u - u_0\| + \frac{1}{2} \|\phi(u) - \phi(u_0)\|_{C^{1+\beta}}
\]

\[
\|\phi(u) - \phi(u_0)\|_{C^{1+\beta}} \leq 2C_{k,u} \|u - u_0\|\gamma
\]

where \( C_{k,u} = \|F^k(u, \phi(u))\|_{C^{1+\beta}}. \) Thus, \( u \in \mathcal{U} \mapsto \phi(u) \in C^{1+\beta}(X) \) is \( \gamma \)-Hölder.

4.3. Differentiability of the spectral data: proof of theorem 3

Let \( 0 \leq \beta < \alpha < 1 \). This section is devoted to establish theorem 3 by applying theorem 1 to the map \( F \) from (2.8) acting on the scale \( (C^{1+\beta}(X), C^\beta(X)) \).

The first hypothesis, i.e existence, for every \( u \in \mathcal{U} \), of a fixed point \( \phi_u \) for the map \( F(u, .) : C^{1+\alpha}(X)^* \to C^{1+\alpha}(X)^* \) from (2.8) and continuity of the map \( u \in \mathcal{U} \mapsto \phi_u \in C^{1+\beta}(X) \), has already been addressed in theorem 2.

We now turn to assumption (ii). We showed the perturbed Taylor development for \( \mathcal{L} \) acting on \( (C^{1+\beta}(X), C^\beta(X)) \) in lemma 2: it immediately follows that \( F \) acting on the scale \( (C^{1+\beta}(X), C^\beta(X)) \) satisfies the perturbed Taylor development (2.2).
We now check assumption (iii). We start by remarking for every \( z \in C^{1+\beta}(X) \),
\[
Q_{u,\phi}z = \frac{1}{\langle \ell_m, \mathcal{L}(u, \phi) \rangle} \left[ \mathcal{L}(u, z) \langle \ell_m, \ell_m \rangle - \mathcal{L}(u, \phi) \langle \ell_m, \ell_m \rangle \right].
\]
(4.20)
Thus, for \( \phi = \phi_u \), we obtain
\[
Q_{u,\phi_u}z = \frac{1}{\lambda_u} (\mathcal{L}(u, z) - \langle \ell_m, \mathcal{L}(u, z) \rangle \phi_u)
\]
(4.21)
and for \( u = u_0 \):
\[
Q_{u_0,\phi_{u_0}}z = \frac{1}{\lambda_{u_0}} \mathcal{L}(u_0) - \Pi_{u_0} = \frac{1}{\lambda_{u_0}} R_{u_0}
\]
(4.22)
where \( \Pi_{u_0}z = \langle \ell_m, z \rangle \phi_{u_0} \), \( z \in C^{1+\beta}(X) \) is the spectral projector on the (one-dimensional) eigenspace associated to \( \lambda_{u_0} \). It is also noteworthy that the previous expression is independent of \( \phi_{u_0} \).

From (4.22), one sees that there is a \( N \geq 1 \) such that \( \| Q_{u_0}^N \|_{C^0} \leq C \sigma^N \), for some \( C > 0 \) and \( \sigma \in (0,1) \) (see appendix A, (A.2)): therefore its Neumann series converges towards \((Id - Q_{u_0})^{-1}\).

This proves (iii) in the assumptions of theorem 1.

We can therefore conclude that
\[
\text{If } \phi_u \in C^{1+\beta}(X), u \in \mathcal{U} \mapsto \phi_u \in C^\beta(X) \text{ is differentiable,}
\]
and that its differential satisfies
\[
D_{\phi} \phi(u_0) = (Id - Q_{u_0,\phi_{u_0}})^{-1} P_{u_0,\phi_{u_0}}.
\]
(4.23)
Furthermore,
\[
P_{u_0,\phi_{u_0}} = \frac{\partial_\phi \mathcal{L}(u, \phi)}{\langle \ell_m, \mathcal{L}(u, \phi) \rangle} - \frac{\langle \ell_m, \partial_\phi \mathcal{L}(u, \phi) \rangle}{\langle \ell_m, \mathcal{L}(u, \phi) \rangle^2} \mathcal{L}(u, \phi)
\]
(4.24)
which simplifies, for \( (u, \phi) = (u_0, \phi_{u_0}) \), to
\[
P_{u_0,\phi_{u_0}} = \frac{1}{\lambda_{u_0}} \left( \partial_\phi \mathcal{L}(u_0, \phi_{u_0}) - \langle \ell_m, \partial_\phi \mathcal{L}(u_0, \phi_{u_0}) \rangle \phi_{u_0} \right)
\]
(4.25)
\[
= \frac{1}{\lambda_{u_0}} (Id - \Pi_{u_0}) \circ \partial_\phi \mathcal{L}(u_0, \phi_{u_0}).
\]
(4.26)
This, together with (4.22), proves formula (2.9).

**Corollary 1 (Same setting as theorem 3).** The real valued map \( u \in \mathcal{U} \mapsto \lambda_u \) is differentiable.

**Proof.** Let \( u_0 \in B \), and \( \mathcal{U} \subset B \) be a neighborhood of \( u_0 \). Given the normalization chosen for \( \ell_m \) and \( \phi_u \) (see section 2.1, (2.8)) for every \( u \in \mathcal{U} \) one has
\[
\lambda_u = \langle \ell_m, \mathcal{L}(u, \phi_u) \rangle.
\]
(4.27)
Thus, injecting (2.2) and using the Hölder continuity (resp differentiability) of \( u \in \mathcal{U} \mapsto \phi_u \in C^{1+\beta}(X) \) (resp \( C^\beta(X) \)), one gets the desired conclusion. \( \square \)
Corollary 2 (Same setting as theorem 3). Let \( m_u \) be defined on \( C^0(X) \) by \( m_u(f) = \langle \ell_u, f \phi_u \rangle \). Then it is a Radon measure, and for every \( f \in C^0(X) \), the map \( u \in \mathcal{U} \mapsto m_u(f) \) is \( C^1 \).

Proof. By a standard positivity argument (see [1]) we extend continuously \( \ell_u \) to \( C^0(X) \). It naturally follows that \( m_u \) is a Radon measure. \( \square \)

For \( s \in D(0, 1) \subset \mathbb{C}, \ u \in \mathcal{U} \) and \( A \in C^{1+\alpha}(X) \), we introduce the parameter
\[
\mathbf{u} = (s, u) \in D(0, 1) \times \mathcal{U} \subset \mathbb{C} \times \beta \text{ and the weighted transfer operator (with weight } e^\ell, \ g : X \to \mathbb{R}) \ L_u \text{ defined on } C^{1+\alpha}(X) \text{ by }
\]
\[
L_u \phi = L_{s,u} = L_u(e^{s\phi}). \tag{4.28}
\]

Note that \( L_{s,u} \) is an analytical perturbation of \( L_u \) (at a fixed \( u \in \mathcal{U} \)). Hence, \( L_{s,u} \) also has a spectral gap for \( s \in D(0, r) \), with \( r = r(u) \) small enough (see [21]), and we will write \( \lambda_{s,u}, \ \phi_{s,u} \) for its simple, maximal eigenvalue and the associated eigenvector (which is not necessarily a positive function, nor even a real valued one).

It follows from Ruelle theorem [28] that \( \lambda_{s,u} = e^{P(s,u)} \) with \( P(s,u) \) the topological pressure associated with the dynamic \( T_u \) and the weight \( e^{4\beta s} \).

We now state a version of a well-known formula (see [33]), connecting topological pressure and the expectation of the observable \( A \) under the Gibbs measure \( m_u \), suited to our needs.

Proposition 2. Let \( u \in \mathcal{U} \). The map \( s \in D(0, r_u) \mapsto P(s,u) \) is analytical and one has
\[
\partial_s P(0, u) = m_u(A). \tag{4.29}
\]

Proof. Fix \( u \in \mathcal{U} \). For \( s \in D(0, r) \), with \( r = r(u) \) small enough, one can write \( L_{s,u} \phi_{s,u} = e^{P(s,u)} \phi_{s,u} \). The first statement follows from analytic perturbation theory, see [21], as well as analyticity of \( s \mapsto \ell_{s,u} \), with \( \ell_{s,u} \) the eigenform for \( \lambda_{s,u} \).

Furthermore, from the normalization \( \langle \ell_{s,u}, \phi_{s,u} \rangle = 1 \), one gets \( \langle \ell_{s,u}, L_{s,u} \phi_{s,u} \rangle = e^{P(s,u)} \) and by differentiating this last equality with respect to \( s \), one has
\[
\partial_s P(s,u)e^{P(s,u)} = \left( \langle \partial_s \ell(s,u), \phi_{s,u} \rangle + \langle \ell_{s,u}, \partial_s \phi_{s,u} \rangle \right) e^{P(s,u)} + \langle \ell_{s,u}, \partial_s L_{s,u} \phi_{s,u} \rangle. \tag{4.30}
\]

From \( \langle \ell_{s,u}, \phi_{s,u} \rangle = 1 \), one gets \( (I) = 0 \).

Up to replace \( A \) by \( A \circ T \), \( \partial_s L_{s,u} \phi_{s,u} = AL_{s,u} \phi_{s,u} = e^{P(s,u)}A \phi_{s,u} \), so that we get
\[
(II) = e^{P(s,u)} \langle \ell_{s,u}, A \phi_{s,u} \rangle. \text{ Finally, one has, at } s = 0 \text{ equation } (4.29).
\]

Fix a \( u_0 \in \mathcal{U} \); thus \( \lambda_{0,u_0} = \lambda_{u_0} > 0 \).

One easily has, for all \( y \in X \),
\[
L_{s,u} \phi(y) = \sum_{x \in T^{-1}_s y} e^{sA(x) + s} \phi(x).
\]

From theorem 2, it holds that there is a neighborhood \( D(0, r) \times B(u_0, \delta) \) such that \( (s,u) \in D(0, r) \times B(u_0, \delta) \) implies \( |\lambda_{s,u} - \lambda_{u_0}| \leq \frac{\lambda_{u_0}}{r} \).

In particular, \( r \) is independent of \( u \) and \( \lambda_{s,u} \) is a positive real number. Hence \( P(s,u) \) is correctly defined, and continuous with respect to \( u \in B(u_0, \delta) \), for \( s \in D(0, r) \).
From theorem 3, it holds that there is a neighborhood $D(0, r') \times B(u_0, \delta')$ on which $(s, u) \mapsto P(s, u)$ is $C^1$. In particular, $\partial_u P(s, u)$ exists and is continuous with respect to $u \in B(u_0, \delta')$ for $s \in D(0, r')$. Once again, $r'$ is a priori independent of $u$.

From analytical perturbation theory, it holds that $s \in D(0, r''') \mapsto P(s, u)$ is analytical for $u \in B(u_0, \delta'')$, where $r''' = \min(r, r')$ and $\delta'' = \min(\delta, \delta')$. Therefore, one can write, following Cauchy formula and (4.29)

$$m_u(A) = \int_{C(0, r''')} \frac{P(s, u)}{s^2} \, ds$$

where $C(0, r''')$ is the circle of radius $r'''$ centered at 0.

By Lebesgue’s theorem, $u \in B(u_0, \delta'') \mapsto m_u(A)$ is a $C^1$ map. Up to a change in constants, this can be done for every $u_0 \in U$, thus concluding this proof.

### Appendix A. Spectrum of expanding maps on H"older spaces

Recall that a $C^1$, expanding dynamic on a Riemann manifold $X$ is a map $T : X \to X$ such that there exists a $\lambda > 1$, and for every $x \in X$, every $v \in T_xX$, $\|DT(x).v\| \geq \lambda\|v\|$, where $TX$ is endowed with a norm field $(\|\cdot\|_x)_{x \in X}$.

We recall a few useful properties of expanding maps in this setting:

**Proposition 3.** Let $(X, g), T$ be as above. Then:

(i) $T$ is a local diffeomorphism at every $x \in X$.

(ii) For every $y \in X$, $T^{-1}\{y\}$ is a finite set.

(iii) The set of $C^1$ expanding maps is open in the $C^1$-topology. Moreover, it is structurally stable.

The study of expanding maps started with the pioneering paper of Shub [36]. One can find proof of the proposition claims in Shub’s paper, or in the monograph [22]. The study of their ergodic properties was started by [25] where it is shown that every $C^2$ expanding map of a compact manifold has an invariant measure.

Defining the (weighted) transfer operator associated to $(T, g)$ by

$$L \phi(x) = \sum_{y, T^{-1}y = x} g(y) \phi(y)$$

where $g : X \to \mathbb{R}$, $C^1$ map, acting on the space $C^0(X)$, one can link statistical properties of the dynamic to spectral properties of $L$ acting on an appropriate Banach space [1, 3, 27]. As a result, the spectral picture of transfer operators for expanding maps has been thoroughly investigated, in the works of David Ruelle [28, 29], Carlangelo Liverani [26, 27], the 2000 monograph by Viviane Baladi [1], or in a 2003 paper by Gundlach and Latushkin [17].

For example, SRB measures (which are physically relevant invariant measures, see [38]) and linear response formulas (first-order variation of the SRB measure w.r.t a real parameter) can be computed from spectral data of the transfer operator [3, 20, 27, 30, 32], decay of correlations can be linked to convergence of $L^n$ towards its spectral projectors [1, 26].

The proper spectral setting is encapsulated in the notion *spectral gap*: the operator $L$ acting on the Banach space $B$ has a spectral gap if:
• There exists a simple, isolated eigenvalue $\lambda$ of maximal modulus, i.e. $|\lambda| = \rho(L|B)$, called the dominating eigenvalue.

• The rest of the spectrum is contained in a disk centered at 0 and of radius strictly smaller than $\rho(L|B)$.

In this case, one has the following decomposition:

$$L\phi = \lambda \Pi(\phi) + R(\phi).$$  \hspace{1cm} (A.2)

In addition, the bounded operator $R$ has the following property: There exists $0 < \sigma < 1$, $C > 0$ such that $\|\lambda^{-n}R^n\|_B \leq C\sigma^n$.

Although $L$ does not have nice spectral properties on $C_0(X)$ [28], a classical theorem of Ruelle [28, 29] shows that, assuming a little more regularity for the dynamic, the transfer operator admits a spectral gap on the Banach spaces $(C^r(X))_{r>0}$.

The proof relies on fine estimates on the (essential) spectral radius, established through Lasota–Yorke inequalities. Those estimates were refined by Gundlach and Latushkin, in the paper [17], where they give an exact formula for the essential spectral radius of the transfer operator acting on $C^r(X)$ for $r \in \mathbb{R}_+$. A spectral gap can be obtained through other techniques, notably ‘cone contraction’ based on abstract results of Birkhoff [9]: this approach was first applied in [13] and successfully extended by Liverani [26]. Clear and complete account of those works can be found in the monographs by Viana [37] or by Baladi [1]. Let us also mention the approach of Fan and Jiang [12].

**Appendix B. Estimates for composition operators on Hölder spaces**

It is a well-established fact that $(C^{k+\alpha}(\Omega), \|\cdot\|_{C^{k+\alpha}})$ is a Banach space.

For $\Omega$ an open set in $\mathbb{R}^n$, and $0 \leq \beta \leq \alpha < 1$, one has the compact embedding:

$$C^{k+\alpha}(\Omega) \subset C^{k+\beta}(\Omega).$$

The proof of this compact embedding relies on the Arzelà–Ascoli theorem and the following interpolation inequality:

**Theorem B.1.** Let $E, F$ be Banach spaces, $U \subset E$ an open subset. Let $0 \leq \alpha < \beta < \gamma < 1$ and $k \in \mathbb{N}$.

Denote by $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$. Then for every $f \in C^{k+\gamma}(U, F)$, one has

$$\|f\|_{C^{k+\beta}} \leq M_{\alpha}\|f\|_{C^{k+\alpha}}^{\mu}\|f\|_{C^{k+\gamma}}^{1-\mu},$$  \hspace{1cm} (B.1)

We refer to [10] for a proof.

The main object of this section is to address the regularity problem for composition operators: $g \mapsto [f \mapsto f \circ g]$ in Hölder spaces. An important inspiration for the results presented here is a paper by de la Llave and Obaya, [10], particularly the following result:

**Theorem B.2 ([10], proposition 6.2, (iii)).** Let $E, F, G$ be Banach spaces, and $U \subset E$, $V \subset F$ open subsets. Let $k \geq 1$, $0 \leq \gamma < 1$ and $t = k + \gamma$. Let $s > t$ and $r \geq t$, and let $U \subset C^s(U, F)$. Then for every $g_1 \in U$, there exists $\delta, \rho, M > 0$, such that for every $f \in C^r(V, G)$, every $g_2 \in C^t(U, F)$ which verifies $\|g_1 - g_2\|_C \leq \delta$, one has $g_2 \in U$, and

$$\|f \circ g_1 - f \circ g_2\|_C \leq M\|f\|_C\|g_1 - g_2\|_C^\rho$$  \hspace{1cm} (B.2)

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The estimates we establish in the following (lemmas 1–B.3) are parameter variants of this theorem. They are used to prove lemma 2, which in turn is key for using theorem 1 to prove theorem 3.

In the first lemma, $g \mapsto [f \mapsto f \circ g]$ is Hölder continuous from $C^{1+\alpha}$ to $C^{1+\beta}$ with exponent $\gamma := \alpha - \beta$

**Lemma B.1.** Let $B, E, F, G$ be Banach spaces, $U \subset B$, $V \subset E$, $W \subset F$ be open domains. Let $0 < \beta < \alpha < 1$, $\psi \in C^0(U \times V, W)$ such that for every $u \in U$, $\psi_u = \psi(u, \cdot) \in C^{1+\alpha}(V, W)$, and every $x \in V$, $u \mapsto \psi(u, x)$ is Lipschitz continuous, $u \mapsto D\psi_u(x)$ is $\alpha$-Hölder. Let $f \in C^{1+\alpha}(W, G)$.

Denote by $\left\{ \begin{array}{l} L_0 = \sup_{u \in U} \| \psi_u \|_{\text{Lip}} \\ L'_0 = \sup_{x \in V} \| \psi(x) \|_{\text{Lip}} \\ L'_\alpha = \sup_{x \in V} \| D\psi_u(x) \|_{C^\alpha} \end{array} \right.$ 

Let $u, v \in U$. Then $f \circ \psi_u, f \circ \psi_v$ are $C^{1+\beta}$ maps, and we have

$$\| f \circ \psi_u - f \circ \psi_v \|_{C^{1+\beta}} \leq C \| f \|_{C^{1+\alpha}} \| u - v \|^{\gamma}$$

(B.3)

with $C = C(\alpha, \| f \|_{C^1}, \| \psi_u \|_{C^1}, \| \psi_v \|_{C^{1+\alpha}}, L_0, L'_0, L'_\alpha)$.

**Proof.** We want to estimate $\| f \circ \psi_u - f \circ \psi_v \|_{C^{1+\beta}} = \max(\| f \circ \psi_u - f \circ \psi_v \|_{C^1}, \| D_\gamma f \circ \psi_u - D_\gamma f \circ \psi_v \|_{C^\alpha})$.

For $x \in V$, one has:

$$\| Df(\psi_u(x)) \circ D\psi_u(x) - Df(\psi_v(x)) \circ D\psi_v(x) \| \\
\leq \| Df(\psi_u(x)) - Df(\psi_v(x)) \| \cdot \| D\psi_u(x) \|_{C^\alpha} + \| Df(\psi_v(x)) \| \cdot \| D\psi_v(x) - D\psi_u(x) \| \\
\leq (\| f \|_{C^{1+\alpha}} \| \psi_u \|_{C^1}(L'_0)^{\alpha} + \| f \|_{C^1} L'_\alpha) \| u - v \|^{\alpha}.$$

For the Hölder norm $\| D_\gamma f \circ \psi_u - D_\gamma f \circ \psi_v \|_{C^\alpha}$, we have the following:

Let $x, x' \in V$, such that $\| x - x' \| \leq \| u - v \|$. Then

$$\| Df(\psi_u(x)) \circ D\psi_u(x) - Df(\psi_v(x')) \circ D\psi_v(x') \| \\
\leq \| Df(\psi_u(x)) - Df(\psi_v(x')) \| \cdot \| D\psi_u(x) \|_{C^\alpha} + \| Df(\psi_v(x')) \| \cdot \| D\psi_u(x) - D\psi_v(x') \| \\
\leq \| f \|_{C^{1+\alpha}} \| \psi_u(x) - \psi_v(x') \|^{\alpha} + \| f \|_{C^1} L'_\alpha \| x - x' \|^{\alpha} \\
\leq (\| f \|_{C^{1+\alpha}} L'_0 + \| f \|_{C^1} L'_\alpha) \| x - x' \|^{\alpha} \| u - v \|^{\alpha-\beta}.$$

Similarly in the case $\| x - x' \| \geq \| u - v \|$, one has:

$$\| Df(\psi_u(x)) \circ D\psi_u(x) - Df(\psi_v(x)) \circ D\psi_v(x) \| \\
\leq (\| f \|_{C^1} L'_\alpha + \| f \|_{C^{1+\alpha}} \| \psi_u \|_{C^1}(L'_0)^{\alpha}) \| u - v \|^{\gamma} \| x - x' \|^{\beta}.$$

Thus,

$$| D_\gamma f \circ \psi_u - D_\gamma f \circ \psi_v |_{C^\alpha} \leq 2(\| f \|_{C^{1+\alpha}} \max(L'_0, (L'_0)^{\alpha}) + \| f \|_{C^1} \max(L'_\alpha, (L'_0)^{\alpha})) \| u - v \|^{\gamma}$$

(B.4)

and (6.3) readily follows.
Note that the previous lemma yields Hölder continuity for \( g \mapsto [f \mapsto f \circ g] \) from \( C^{1+\alpha} \) to \( C^{1+\beta} \), for \( g \in C^{1+\alpha}(V) \). One could easily follow the method outlined for the proof of theorem B.2 to establish our previous result from \( C^{k+\alpha}(\Omega) \) to \( C^{k+\beta}(\Omega) \), for every \( k \geq 1 \) and every \( 0 \leq \beta < \alpha < 1 \).

One could ask what to expect for the composition operator from \( C^{1+\alpha} \) to \( C^{\alpha} \). We present the following estimate, a natural extension of the previous result:

**Lemma B.2.** Let \( B,E,F,G \) be Banach spaces, \( U \subset B \), \( V \subset E \), \( W \subset F \), \( W \subset G \) be open subsets.

Let \( 0 \leq \alpha < 1 \) and \( \psi \in C^{1+\alpha}(U \times V, W) \), \( f \in C^{1+\alpha}(W,G) \).

Then for every \( u_0 \in U \), and every \( h \in B \) such that \( u_0 + h \in U \), the maps \( f \circ \psi(u_0 + h) \) are \( \alpha \)-Hölder and one has the estimate:

\[
\| f \circ \psi(u_0 + h) - f \circ \psi(u_0) \|_{C^\alpha} \leq C \| f \|_{C^{1+\alpha}} \| h \|_B
\]

with \( C = C(\alpha, \| \psi \|_{C^1}, \| \psi \|_{C^{1+\alpha}}) \).

**Proof.** It is a straightforward consequence of the mean value theorem. Taking the \( C^\alpha \)-norm, one has for every \( x \in V \):

\[
\| f \circ \psi(u_0 + h) - f \circ \psi(u_0) \|_{C^\alpha} \leq \| h \| \int_0^1 \| Df(\psi(u_0 + th)) \|_{Lip} \cdot D\psi(u_0 + th) \|_{C^\alpha} \, dt.
\]

It is enough to establish the Lipschitz continuity that we wanted. Yet it is convenient to get a more precise estimate of \( \| Df(\psi(u)) \|_{Lip} \cdot D\psi(u_0 + th) \|_{C^\alpha} \), for \( u \in U \).

Letting \( x, x' \in W \), and taking the operator norm, one gets

\[
\| Df(\psi(u,x)) \circ D\psi(u,x) - Df(\psi(u,x')) \circ D\psi(u,x') \|
\leq \| Df(\psi(u,x)) - Df(\psi(u,x')) \| \cdot \| D\psi(u,x) \| + \| Df(\psi(u,x')) \| \cdot \| D\psi(u,x) - D\psi(u,xv) \|
\leq \| f \|_{C^{1+\alpha}} \| D\psi(u,x) \|_{C^\alpha} \| \psi_u \|_{C^1} + \| f \|_{C^1} \| D\psi \|_{C^\alpha} \| x - x' \|_\alpha
\]

so that

\[
\| Df(\psi(u)) \circ D\psi(u) \|_{C^\alpha} \leq \| f \|_{C^{1+\alpha}} \| D\psi \|_{C^\alpha} \| \psi_u \|_{C^1} + \| f \|_{C^1} \| D\psi \|_{C^\alpha}.
\]

It is desirable to complete the previous lemmas with a differentiability result. In that spirit, we show the following:

**Lemma B.3.** Let \( B,E,F,G \) be Banach spaces, \( U \subset B \), \( V \subset E \), \( W \subset F \), \( W \subset G \) be open subsets.

Let \( 0 \leq \beta < \alpha < 1 \) and \( \psi \in C^{1+\alpha}(U \times V, W) \), \( f \in C^{1+\alpha}(W,G) \).

Denote by

\[
\left\{ \begin{array}{l}
L_0 = \sup_{u \in U} \| \psi(u,.) \|_{Lip} \quad L_0' = \sup_{x \in \Omega} \| \psi(.,x) \|_{Lip} \\
L_{1,\alpha} = \sup_{u \in U} \| D\psi(u,.) \|_{C^\alpha} \quad L_{1,\alpha}' = \sup_{x \in \Omega} \| D\psi(.,x) \|_{C^\alpha}
\end{array} \right.
\]

Let \( u_0 \in U \), and \( h \in \mathbb{R}^d \) such that \( u_0 + h \in U \). Then \( f \circ \psi(u_0) \), \( f \circ \psi(u_0 + h) \), \( D\psi(u_0 + h) \) are \( C^3 \) maps, and we have
\[ \| f \circ \psi(u_0 + h) - f \circ \psi(u_0) - D_u(f \circ \psi)(u_0)h \|_{C^\alpha} \leq C \| f \|_{C^{\alpha+\gamma}} \| h \|^{1+\gamma} \]

with \( C = C(u_0, \alpha, \| f \|_{C^1}, L_0, L'_0, L_{1,\alpha}, L'_{1,\alpha}). \)

**Proof.** Using the mean value theorem and taking the norm, one can write:

\[ \| f \circ \psi(u_0 + h) - f \circ \psi(u_0) - D_u(f \circ \psi)(u_0)h \|_{C^\alpha} \leq \| h \| \int_0^1 \| Df(\psi(u_0 + th)) \circ D_u \psi(u_0 + th) - Df(\psi(u_0)) \circ D_u \psi(u_0) \|_{C^\alpha} \| h \| \, dt. \]

To estimate \( \| Df(\psi(u_0 + th)) \circ D_u \psi(u_0 + th) - Df(\psi(u_0)) \circ D_u \psi(u_0) \|_{C^\alpha}, \) we apply the same method we used to establish (6.3).

Letting \( x, x' \in V, u, v \in U, \) such that \( \| u - v \| \leq \| x - x' \| \) one obtains:

\[ \| Df(\psi(u_0, x)) \circ D_u \psi(u_0, x) - Df(\psi(u, x)) \circ D_u \psi(u, x) \|_{C^\alpha} \leq \left( \| f \|_{C^{\alpha+\gamma}} \| D_u \psi \|_{L^0} + \| f \|_{C^1} \| L_{1,\alpha} \| \right) \| u - v \|^{\gamma}. \]

Similarly, in the case \( \| x - x' \| < \| u - v \| \)

\[ \| Df(\psi(u_0, x)) \circ D_u \psi(u_0, x) - Df(\psi(u_0, x')) \circ D_u \psi(u_0, x') \|_{C^\alpha} \leq \left( \| f \|_{C^{\alpha+\gamma}} \| D_u \psi \|_{L^0} + \| f \|_{C^1} \| L_{1,\alpha} \| \right) \| u - v \|^{\gamma}. \]

Finally, one has

\[ \| Df(\psi(u)) \circ D_u \psi(u) - Df(\psi(v)) \circ D_u \psi(v) \|_{C^\alpha} \leq 2\| f \|_{C^{\alpha+\gamma}} \| D_u \psi \|_{L^0} \max(L_0, L'_0) + \| f \|_{C^1} \max(L_{1,\alpha}, L'_{1,\alpha}). \]

Injecting this last estimate in (B.9), one gets the following:

\[ \| f \circ \psi(u_0 + h) - f \circ \psi(u_0) - D_u(f \circ \psi)(u_0)h \|_{C^\alpha} \leq \| h \| \int_0^1 \| f \|_{C^{\alpha+\gamma}} \| h \|^{\gamma} \, dt \]

\[ = C \| h \|^{1+\gamma} \| f \|_{C^{\alpha+\gamma}} \frac{1}{\Gamma(1+\gamma)}. \]

which gives the promised result with \( C' = \frac{C}{\Gamma(1+\gamma)}. \)

**Appendix C. An elementary example**

Let \( I = [-1, 1], \) and consider the Banach space \( C^0(I). \) Let \( 0 < \epsilon < 1, \) and define the family of maps \( (F_u)_{u \in [-\epsilon, \epsilon]} \) by:

\[ F_u(\phi)(t) = \frac{1}{2} \phi\left(\frac{t+u}{2}\right) + g(t, u) \]

with \( g : I \times [0, \epsilon] \to \mathbb{R} \) a non-zero \( C^1 \) map, such that \( g(t, \cdot) \in C^0(0, \frac{1}{2}). \) Being a contraction of \( C^0(I), \) \( F_u \) admits a fixed point, say \( \phi_u, \) by the Banach contraction principle. But what about the regularity of \( u \in I \mapsto \phi_u \in C^0(I)? \) On the \( C^0 \) space, \( u \mapsto F_u \) is not even continuous.
Nevertheless, if we consider the same operator $F(u,.) : C^\alpha(I) \to C^\alpha(I)$, with $\alpha \in (0, 1)$, and the immersion $I : C^\alpha(I) \to C^\alpha(I)$, we see that:

$$
|F(u, \phi) - F(u, \psi))(t) - (F(u, \phi) - F(u, \psi))(t')| \leq \frac{1}{2^{1+\alpha}} \| \phi - \psi\|_{C^\alpha}|t - t'|^\alpha.
$$

It follows that,

$$
\|F(u, \phi) - F(u, \psi)\|_{C^\alpha} \leq \frac{1}{2^{1+\alpha}} \| \phi - \psi\|_{C^\alpha},
$$

and $F(u,.) : C^\alpha(I) \to C^\alpha(I)$ is a contraction; by the Banach contraction principle, this map has a fixed point in $C^\alpha(I)$ for all $u \in I$, say $\phi(u)$. Note also that

$$
|F(u, \phi))(t) - F(u', \phi))(t)| \leq \left( \frac{1}{2^{1+\alpha}} \| \phi\|_{C^\alpha} + \|g\|_{C^\alpha}\right) |u - u'|^\alpha
$$

so $u \in I \mapsto F(u,.) \in C^\alpha(I)$ is a $\alpha$-Hölder map. Finally,

$$
\| \phi(u) - \phi(u')\|_{C^\alpha} = \|F(u, \phi(u)) - F(u', \phi(u'))\|_{C^\alpha}
= \|F(u, \phi(u)) - F(u', \phi(u)) + F(u', \phi(u)) - F(u', \phi(u'))\|_{C^\alpha}
\leq \left( \frac{1}{2^{1+\alpha}} \| \phi(u)\|_{C^\alpha} + \|g\|_{C^\alpha}\right) |u - u'|^\alpha + \frac{1}{2} \| \phi(u) - \phi(u')\|_{C^\alpha}.
$$

Hence

$$
\| \phi(u) - \phi(u')\|_{C^\alpha} \leq \left( \frac{1}{2^{1+\alpha}} \| \phi(u)\|_{C^\alpha} + 2\|g\|_{C^\alpha}\right) |u - u'|^\alpha
$$

and $u \mapsto \phi(u) \in I(C^\alpha)$ is locally $\alpha$-Hölder.

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