1. Introduction

Lusztig defined in [Lus91] Lagrangian subvarieties of the cotangent stack to the moduli stack of representations of a quiver associated to any Kac-Moody algebra. The proof of the Lagrangian character of these varieties was obtained via the study of some natural stratifications of each irreducible component, and then proceeding by induction. This particular structure on the set of irreducible components made it possible for Kashiwara and Saito in [KS97] to relate this variety to the usual quantum group associated to Kac-Moody algebras, via the notion of crystals. This later led Lusztig in [Lus00] to define a semicanonical basis of this quantum group, indexed by the irreducible components of these Lagrangian varieties.

There are more and more evidences of the relevance of the study of quivers with loops. A particular class of such quivers are the comet-shaped quivers, which have recently been used by Hausel, Letellier and Rodriguez Villegas in their study of the topology of character varieties, where the number of loops at the central vertex is the genus of the considered curve (see [HRV08] and [HLRV13]). We can also see quivers with loops appearing in a work of Nakajima relating quiver varieties with branching (see [Nak09]), as in the work of Okounkov and Maulik about quantum cohomology (see [MO12]).

Kang, Kashiwara and Schiffmann generalized these varieties in the framework of generalized Kac-Moody algebras in [KKS09], using quivers with loops. In this case, one has to impose a somewhat unnatural restriction on the regularity of the maps associated to the loops.

In this article we define a generalization of such Lagrangian varieties in the case of arbitrary quivers, possibly carrying loops. As opposed to the Lagrangian varieties constructed by Lusztig, which consisted in nilpotent representations, we have to consider here slightly more general representations. That this is necessary is already clear from the Jordan quiver case. Note that our Lagrangian variety is strictly larger than the one considered in [KKS09] and has many more irreducible
components. Our proof of the Lagrangian character is also based on induction, but with non trivial first steps, consisting in the study of quivers with one vertex but possible loops. From our proof emerges a new combinatorial structure on the set of irreducible components, which is more general than the usual crystals, in that there are now more operators associated to a vertex with loops, see [5.3].

We finally consider, following [Lus00], a convolution algebra of constructible functions on our varieties, and construct a family of constructible functions naturally attached to the irreducible components. In a forthcoming work we will relate this convolution algebra to some explicit "Kac-Moody type" algebra.

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2. Quiver Varieties

Let $Q$ be a quiver, defined by a set of vertices $I$ and a set of oriented edges $\Omega = \{ h : s(h) \to t(h) \}$. We denote by $\bar{h} : t(h) \to s(h)$ the opposite arrow of $h \in \Omega$, and $\bar{Q}$ the quiver $(I, H = \Omega \cup \bar{\Omega})$, where $\bar{\Omega} = \{ \bar{h} | h \in \Omega \}$: each arrow is replaced by a pair of arrows, one in each direction, and we set $\epsilon(h) = 1$ if $h \in \Omega$, $\epsilon(h) = -1$ if $h \in \bar{\Omega}$. We denote by $H_{\text{loop}}$ the set of loops (edges such that $s(h) = t(h)$) and $I_{\text{loop}} = s(H_{\text{loop}})$. We work over the field of complex numbers $\mathbb{C}$.

For any pair of $I$-graded $\mathbb{C}$-vector spaces $V = (V_i)_{i \in I}$ and $V' = (V'_i)_{i \in I}$, we set:

$$\bar{E}(V, V') = \bigoplus_{h \in H} \text{Hom}(V_{s(h)}, V'_{t(h)}).$$

For any dimension vector $\alpha = (\alpha_i)_{i \in I}$, we fix an $I$-graded $\mathbb{C}$-vector space $V_\alpha$ of dimension $\alpha$, and put $\bar{E}_\alpha = \bar{E}(V_\alpha, V_\alpha)$. The space $\bar{E}_\alpha = \bar{E}(V_\alpha, V_\alpha)$ is endowed with a symplectic form:

$$\omega_\alpha(x, x') = \sum_{h \in H} \text{Tr}(\epsilon(h)x_hx'_h)$$

which is preserved by the natural action of $G_\alpha = \prod_{i \in I} GL_{\alpha_i}(\mathbb{C})$ on $\bar{E}_\alpha$. The associated moment map $\mu_\alpha : \bar{E}_\alpha \to g_\alpha = \oplus_{i \in I} \text{End}(V_\alpha)_i$ is given by:

$$\mu_\alpha(x) = \sum_{h \in H} \epsilon(h)x_hx_h.$$

Here we have identified $g_\alpha^*$ with $g_\alpha$ via the trace pairing.

**Definition 2.1.** An element $x \in \bar{E}_\alpha$ is said to be seminilpotent if there exists an $I$-graded flag $W = (W_0 = \{0\} \subset \ldots \subset W_r = V_\alpha)$ of $V_\alpha$ such that:

$$x_h(W_\bullet) \subseteq W_{\bullet-1} \text{ if } h \in \Omega, \quad x_h(W_\bullet) \subseteq W_\bullet \text{ if } h \in \bar{\Omega}.$$ 

We put $\Lambda(\alpha) = \{ x \in \mu_\alpha^{-1}(0) \mid x \text{ seminilpotent} \}$.

**Lemma 2.2.** The variety $\Lambda(\alpha)$ is isotropic.

**Proof.** We proceed as in [KKS09] 2.1, using the following general fact:
Proposition 2.3. Let $X$ be a smooth algebraic variety, $Y$ a projective variety and $Z$ a smooth closed algebraic subvariety of $X \times Y$. Consider the Lagrangian subvariety $\Lambda = T^*_Z(X \times Y)$ of $T^*(X \times Y)$. Then the image of the projection $q : \Lambda \cap (T^*X \times T^*_Y) \to T^*X$ is isotropic.

We apply this result to $X = \bigoplus_{h \in \Omega} \text{End}(V_{\alpha(h)}, V_{\alpha(\bar{h})})$, $Y$ the 1-graded flag variety of $V_\alpha$ and:

$$Z = \{(x, W) \in X \times Y \mid x(W_\bullet) \subseteq W_{\bullet-1}\}.$$ 

In this case, we get:

$$T^*X = \tilde{E}_\alpha$$

$$T^*Y = \{(W, \xi) \in Y \times g_\alpha \mid \xi(W_\bullet) \subseteq W_{\bullet-1}\}$$

$$\Lambda = \left\{ (x, W, \xi) \left| \begin{array}{l}
\xi = \sum_{h \in H} \epsilon(h)x_hx_h \\
\forall h \in \Omega, x_h(W_\bullet) \subseteq W_{\bullet-1} \text{ and } x_\bar{h}(W_\bullet) \subseteq W_\bullet
\end{array} \right. \right\}$$

$$\text{Im } q = \left\{ x \in \tilde{E}_\alpha \left| \begin{array}{l}
\mu_\alpha(x) = 0 \text{ and there exists } W \in Y \text{ such that} \\
\forall h \in \Omega, x_h(W_\bullet) \subseteq W_{\bullet-1} \text{ and } x_\bar{h}(W_\bullet) \subseteq W_\bullet
\end{array} \right. \right\}$$

hence $\Lambda(\alpha) \subseteq \text{Im } q$, which proves the lemma. \qed

3. The Case of the Jordan Quiver

This case is very well known. For $\alpha \in \mathbb{N}$, we have:

$$\Lambda(\alpha) = \{(x, y) \in (\text{End } \mathbb{C}^\alpha)^2 \mid x \text{ nilpotent and } [x, y] = 0\} = \bigcup_\lambda T^*_{\mathcal{O}_\lambda}(\text{End } \mathbb{C}^\alpha),$$

where $\mathcal{O}_\lambda$ is the nilpotent orbit associated to the partition $\lambda$ of $\alpha$. Therefore $\Lambda(\alpha)$ is a Lagrangian subvariety of $(\text{End } \mathbb{C}^\alpha)^2$, and its irreducible components are the closures of the conormal bundles to the nilpotent orbits.

4. The Case of the Quiver with One Vertex and $g \geq 2$ Loops

For $\alpha \in \mathbb{N}$, $\Lambda(\alpha)$ is the subvariety of $(\text{End } \mathbb{C}^\alpha)^{2g}$ with elements $(x_i, y_i)_{1 \leq i \leq g}$ such that:

$\triangleright$ there exists a flag $W$ of $\mathbb{C}^\alpha$ such that $x_i(W_\bullet) \subseteq W_{\bullet-1}$ and $y_i(W_\bullet) \subseteq W_\bullet$;

$\triangleright$ $\sum_{1 \leq i \leq g} [x_i, y_i] = 0$.

We will often forget the index $1 \leq i \leq g$ in the rest of this section, which is dedicated to the proof of the following theorem:

Theorem 4.1. The subvariety $\Lambda(\alpha) \subseteq (\text{End } \mathbb{C}^\alpha)^{2g}$ is Lagrangian. Its irreducible components are parametrized by the compositions $w = (0 = w_0 < w_1 < \ldots < w_r = \alpha)$ of $\alpha$.

Notations 4.2. For $(x_i, y_i) \in \Lambda(\alpha)$, we define $W^0(x_i, y_i) = \mathbb{C}^\alpha$, then by induction $W^{k+1}(x_i, y_i)$ the smallest subspace of $\mathbb{C}^\alpha$ containing $\sum x_i(W^k(x_i, y_i))$ and stable by $(x_i, y_i)$. By seminilpotency, we can define $r$ to be the first power such that $W^r(x_i, y_i) = \{0\}$. Although $r$ depends on $(x_i, y_i)$ we don’t write it explicitly. We also set $W_k(x_i, y_i) = W^{r-k}(x_i, y_i)$. 
Let:
\(w(x_i, y_i) = (0 = w_0(x_i, y_i) < w_1(x_i, y_i) < \ldots < w_r(x_i, y_i) = \alpha)\)
denotes the tuple of dimensions associated to the flag \(W_{\alpha}(x_i, y_i)\). For every composition \(w = (0 = w_0 < w_1 < \ldots < w_r = \alpha)\) of \(\alpha\), we define a locally closed subvariety:
\[
\Lambda(w) = \{ (x_i, y_i) \in \Lambda(\alpha) \mid \dim W_{\alpha}(x_i, y_i) = w \} \subseteq \Lambda(\alpha).
\]
Then, if \(\delta = (\delta_1, \ldots, \delta_{r-1}) \in \mathbb{N}^{r-1}\), let \(\Lambda(w)_\delta \subseteq \Lambda(w)\) be the locally closed subvariety defined by:
\[
\left( \dim \left( \bigcap_{1 \leq i \leq g} \ker \{ X \mapsto y_i^{(k+1)} X - X y_i^{(k)} \} \right) \right)_{1 \leq k \leq r-1} = \delta,
\]
where:
\[
y_i^{(k)} \in \text{End} \left( \frac{W_k(x_i, y_i)}{W_{k-1}(x_i, y_i)} \right)
\]
is induced by \(y_i\) and:
\[
X \in \text{Hom} \left( \frac{W_k(x_i, y_i)}{W_{k-1}(x_i, y_i)}, \frac{W_{k+1}(x_i, y_i)}{W_k(x_i, y_i)} \right).
\]
Set \(l = w_r - w_{r-1}\), then:
\[
\hat{\Lambda}(w)_\delta = \left\{ (x_i, y_i, X, \beta, \gamma) \mid (x_i, y_i) \in \Lambda(w)_\delta \right. \\
W_{r-1}(x_i, y_i) \oplus \mathfrak{X} = \mathbb{C}^n \\
\beta : W_{r-1}(x_i, y_i) \cong \mathbb{C}^{w_{r-1}} \text{ and } \gamma : \mathfrak{X} \cong \mathbb{C}^l \}
\]
and:
\[
\pi_{w, \delta} \bigg| \hat{\Lambda}(w)_\delta \to \Lambda(w^-)_{\delta^-} \times (\text{End } \mathbb{C}^l)^g \\
(x_i, y_i, X, \beta, \gamma) \mapsto (\beta_* (x_i, y_i)|_{W_{r-1}}, \gamma_*(y_i) X) \bigg)
\]
where \(w^- = (w_0 < w_1 < \ldots < w_{r-1})\) and \(\delta^- = (\delta_1, \ldots, \delta_{r-2})\). Let finally \((\Lambda(w^-)_{\delta^-} \times (\text{End } \mathbb{C}^l)^g)_{w, \delta}\) denotes the image of \(\pi_{w, \delta}\).

**Proposition 4.3.** The morphism \(\pi_{w, \delta}\) is smooth over its image, with connected fibers of dimension \(\alpha^2 + (2g - 1)(\alpha - l) + \delta_{r-1}\) whenever \(\Lambda(w)_\delta \neq \emptyset\).

**Proof.** Let \((x_i, y_i, z_i) \in (\Lambda(w^-)_{\delta^-} \times (\text{End } \mathbb{C}^l)^g)_{w, \delta}\). Let \(\mathfrak{M}\) and \(\mathfrak{X}\) be two supplementary subspaces of \(\mathbb{C}^n\) such that \(\dim \mathfrak{X} = l\), together with two isomorphisms:
\[
\beta : \mathfrak{M} \cong \mathbb{C}^{w_{r-1}} \text{ and } \gamma : \mathfrak{X} \cong \mathbb{C}^l.
\]
We identify \(x_i, y_i\) and \(z_i\) with \(\beta^* (x_i, y_i)\) and \(\gamma^* z_i\), and define an element \((X_i, Y_i)\) in the fiber of \((x_i, y_i, z_i)\) by setting:
\[
(X_i, Y_i)_{\mathfrak{M}} = (x_i, y_i) \\
(X_i, Y_i)_{\mathfrak{X}} = (0, z_i) \\
(X_i, Y_i)_{\mathfrak{M}}^{(\mathfrak{M})} = (u_i, v_i) \in \text{Hom}(\mathfrak{X}, \mathfrak{M})^{2g}.
\]
Then:
\[
\mu_\alpha (X_i, Y_i) = 0 \iff \phi(u_i, v_i) = \sum_{i=1}^{g} (x_i v_i + u_i z_i - y_i u_i) = 0,
\]
and, for $X \in \text{Hom}(\mathfrak{M}, \mathfrak{X})$:

\[
\forall(u_i, v_i), \quad \text{Tr}(X \phi(u_i, v_i)) = 0 \iff \begin{cases} \forall i, \forall u_i, \quad \text{Tr}(X(u_i z_i - y_i u_i)) = 0 \\
\forall i, \forall v_i, \quad \text{Tr}(X x_i v_i) = 0 \\
\forall i, \forall u_i, \quad \text{Tr}((z_i X - X y_i) u_i) = 0 \\
\forall i, \forall v_i, \quad \text{Tr}(X x_i v_i) = 0 \\
\forall i, \quad z_i X = X y_i \\
\forall i, \quad X x_i = 0 \\
\end{cases}
\]

where $X^{(r-1)}$ denotes the map $\mathfrak{M}/\mathfrak{W}_{r-2}(x_i, y_i) \to \mathfrak{X}$ induced by $X$. Since $(x_i, y_i, z_i)$ is in the image of $\pi_{w,\delta}$, then the image of $\phi$ is of codimension $\delta_{r-1}$, and thus its kernel is of dimension $(2g - 1)l(\alpha - l) + \delta_{r-1}$.

Moreover, if we denote by $u_i^{(r-1)}$ the map $\mathfrak{X} \to \mathfrak{M}/\mathfrak{W}_{r-2}(x_i, y_i)$ induced by $u_i$, $W^i(X_i, Y_i) = \mathfrak{M}$ if and only if the space spanned by the action of $(y_i^{(r-1)})_i$ on $\sum_i \text{Im} u_i^{(r-1)}$ is $\mathfrak{M}/\mathfrak{W}_{r-2}(x_i, y_i)$. This condition defines an open subset of ker $\phi$.

We end the proof noticing that the set of elements $(\mathfrak{M}, \mathfrak{X}, \beta, \gamma)$ is isomorphic to $GL_0(\mathbb{C})$.

\[\square\]

Proposition 4.4. The variety $\Lambda(w)_0$ is not empty.

Proof. Fix $W$ of dimension $w$ and define $x_1$ such that

\[
x_1(W_\bullet) \subseteq W_{\bullet-1}
\]

We define inductively an element $y_1$ stabilizing $W$ such that:

- the action of $y_1^{(k)}$ on $\text{Im} \left( x_1|_{W_k/W_{k-1}} \right)$ spans $W_k/W_{k-1}$;
- Spec $y_1^{(k)} \cap \text{Spec} y_1^{(k-1)} = \emptyset$.

We finally set $x_2 = -x_1$, $y_2 = y_1$ and $x_i = y_i = 0$ for $i > 2$. This yields an element $(x_1, y_1)$ in $\Lambda(w)_0$. \[\square\]

Corollary 4.5. For any $w = (0 = w_0 < w_1 < \ldots < w_r = \alpha)$, $\Lambda(w)_0$ is irreducible of dimension $g\alpha^2$.

Proof. We argue by induction on $r$. If $w = (0 = w_0 < w_1 = \alpha)$, we have $\Lambda(w)_0 = \Lambda(w) = (\text{End } \mathbb{C}^\alpha)^g$ which is irreducible of dimension $g\alpha^2$. For the induction step, 4.3 and 4.4 ensure us that $\Lambda(w)_0$ is irreducible of dimension:

\[
\alpha^2 + (2g - 1)l(\alpha - l) + \text{dim}(\Lambda(w^-)_0 \times (\text{End } \mathbb{C}^l)^g)_{w,0}
\]

\[
= \alpha^2 + (2g - 1)l(\alpha - l) + g(\alpha - l)^2 + gl^2
\]

since $(\Lambda(w^-)_0 \times (\text{End } \mathbb{C}^l)^g)_{w,0}$ is a non-empty subvariety of $\Lambda(w^-)_0 \times (\text{End } \mathbb{C}^l)^g$, irreducible of dimension $g(\alpha - l)^2 + gl^2$ by our induction hypothesis. Moreover, $\Lambda(w)_0 \to \Lambda(w)_0$.
being a principal bundle with fibers of dimension \( \alpha^2 - l(\alpha - l) \), we get that \( \Lambda(w)_0 \) is irreducible of dimension
\[
\alpha^2 + (2g - 1)l(\alpha - l) + g(\alpha - l)^2 + gl^2 - \alpha^2 + l(\alpha - l) = ga^2.
\]

\[
\square
\]

**Lemma 4.6.** Let \( V \) and \( W \) be two vector spaces, and \( k \geq 0 \). For any \((u, v) \in \text{End} V \times \text{End} W\), we set:
\[
\mathcal{C}(u, v) = \{x \in \text{Hom}(V, W) \mid xu = vx\}
\]
\[
(\text{End} V \times \text{End} W)_k = \{(u, v) \in \text{End} V \times \text{End} W \mid \dim \mathcal{C}(u, v) = k\}.
\]
Then we have
\[
\text{codim}(\text{End} V \times \text{End} W)_k \geq k.
\]

**Proof.** The restrictions of an endomorphism \( a \) to a generalized eigenspace associated to an eigenvalue \( \eta \) will be denoted \( a_\eta = \eta \text{id} + \tilde{a}_\eta \). As usual, the nilpotent orbit associated to a partition \( \xi \) will be denoted \( O\xi \). We have:
\[
\text{codim}(\text{End} V \times \text{End} W)_k
\]
\[
= \text{codim}\{(u, v) \mid \sum_{\alpha, \beta} \dim \mathcal{C}(u_\alpha, v_\beta) = k\}
\]
\[
= \text{codim}\{(u, v) \mid \sum_{\alpha \in \text{Spec} u, \beta \in \text{Spec} v} \dim \mathcal{C}(u_\alpha, v_\beta) = k\}
\]
\[
= \text{codim}\{(u, v) \mid \sum_{\alpha} \dim \mathcal{C}(\tilde{u}_\alpha , \tilde{v}_\alpha) = k\}
\]
\[
= \text{codim}\left\{(u, v) \mid (\tilde{u}_\alpha , \tilde{v}_\alpha) \in O\lambda_\alpha \times O\mu_\alpha \sum_{\alpha} \sum_{j}(\lambda'_\alpha)_j(\mu'_\alpha)_j = k\right\}
\]

Thus,
\[
\text{codim}(\text{End} V \times \text{End} W)_k \geq k
\]
\[
\iff \sum_{\alpha} (\text{codim} O\lambda_\alpha + \text{codim} O\mu_\alpha - 1) \geq \sum_{\alpha} \sum_{j}(\lambda'_\alpha)_j(\mu'_\alpha)_j
\]
\[
\iff \sum_{\alpha} (\sum_{j}(\lambda'_\alpha)_j + \sum_{j}(\mu'_\alpha)_j - 1) \geq \sum_{\alpha} \sum_{j}(\lambda'_\alpha)_j(\mu'_\alpha)_j
\]
which is clear. \( \square \)

**Proposition 4.7.** If \( \delta \neq 0 \), we have \( \dim \Lambda(w)_\delta < ga^2 \).

**Proof.** It’s enough to show that if \( \delta > 0 \), we have:
\[
\dim(\Lambda(w^-)_\delta \times (\text{End} C^d)^q)_{w, \delta} + \delta \rightarrow 1 < \dim(\Lambda(w^-)_0 \times (\text{End} C^d)^q).
\]
This is a consequence of the previous lemma (recall that \( g \geq 2 \)). Indeed, if we set :
\[
((\text{End} V)^q \times (\text{End} W)^q)_k = \{(u_i, v_i) \mid \dim \cap_i \mathcal{C}(u_i, v_i) = k\},
\]
we have:
\[
((\text{End} V)^q \times (\text{End} W)^q)_k \subseteq \prod_{i=1}^q (\text{End} V \times \text{End} W)_{k_i}
\]
for some \( k_i \geq k \), and thus:
\[
\text{codim}(\text{End} V)^q \times (\text{End} W)^q)_k
\]
\[
\geq \sum_i \text{codim}(\text{End} V \times \text{End} W)_{k_i} \geq \sum_i k_i \geq gk > k.
\]
The following proposition concludes the proof of theorem 4.1.

**Proposition 4.8.** Every irreducible component of \( \Lambda(w) \) is of dimension larger than \( g\alpha^2 \).

**Proof.** We first prove the result for the following variety:

\[
\tilde{\Lambda}(w) = \{(x_i, y_i) | x_i(W_\bullet) \subseteq W_{\bullet-1} \text{ and } y_i(W_\bullet) \subseteq W_\bullet \}
\]

where \( Y_w \) denotes the variety of flags of \( \mathbb{C}^n \) of dimension \( w \). We use the following notations, analogous to 2.2:

\[
X = \{(x_i)_{1 \leq i \leq g} \in (\text{End}\mathbb{C}^\alpha)^g \}
\]

\[
Z = \{(x_i)_{1 \leq i \leq g}, W | x_i(W_\bullet) \subseteq W_{\bullet-1} \} \subseteq X \times Y_w.
\]

We get:

\[
T^*X = \{(x_i, y_i)_{1 \leq i \leq g} \in (\text{End}\mathbb{C}^\alpha)^{2g} \}
\]

\[
T^*Y_w = \{(W, K) \in Y_w \times \text{End}\mathbb{C}^\alpha | K(W_\bullet) \subseteq W_{\bullet-1} \}
\]

and:

\[
T^*_Z(X \times Y_w) = \left\{ (x_i, y_i), F, K \bigg| \begin{array}{l}
\sum_{1 \leq i \leq g} x_i, y_i = K \\
x_i(W_\bullet) \subseteq W_{\bullet-1} \text{ and } y_i(W_\bullet) \subseteq W_\bullet
\end{array} \right\}
\]

which is a pure Lagrangian subvariety of \( T^*(X \times Y_w) \), of dimension \( g\alpha^2 + \dim Y_w \). Since \( T^*Y_w \) is irreducible of dimension \( 2 \dim Y_w \), the irreducible components of the fibers of \( T^*_Z(X \times Y_w) \to T^*Y_w \) are of dimension larger than \( g\alpha^2 - \dim Y_w \). We denote by \( \tilde{\Lambda}_W \) the fiber above \( (W, 0) \), and by \( P \) the stabilizer of \( W \) in \( G_\alpha \). Since \( G_\alpha \) and \( P \) are irreducible, we get that the components of:

\[
\tilde{\Lambda}(w) = G_\alpha \times P \tilde{\Lambda}_W
\]

are of dimension larger than \( \dim Y_w + (g\alpha^2 - \dim Y_w) = g\alpha^2 \).

We extend this result to \( \Lambda(w) \), noticing that:

\[
\begin{align*}
\Lambda(w) & \hookrightarrow \tilde{\Lambda}(w) \\
(x_i, y_i) & \mapsto (x_i, y_i, W_\bullet(x_i, y_i))
\end{align*}
\]

defines an open embedding. \( \square \)

5. **The General Case**

For every \( \alpha, \beta \in \mathbb{N}^I \) and \( j \in I \), we put:

\[
\langle \alpha, \beta \rangle = \sum_{h \in \Omega} \alpha_{s(h)} \beta_{t(h)}
\]

\[
\langle \alpha, \beta \rangle = \sum_{i \in I} \alpha_i \beta_i
\]

\[
e_j = (\delta_{i,j})_{i \in I}
\]
**Definition 5.1.** For every subset $A \subseteq I$, and every $x \in \Lambda(\alpha)$, we denote by $\mathcal{I}_A(x)$ the subspace of $V_\alpha$ spanned by the action of $x$ on $\oplus_{i \notin A} V_i$. Then, for $\underline{l} = (l_i)_{i \in A}$, we set:

$$\Lambda(\alpha)_{\underline{l}} = \{ x \in \Lambda(\alpha) \mid \text{codim} \mathcal{I}_A(x) = \underline{l} \}.$$  

In the case where $A$ is a singleton $\{i\}$, $\underline{l}$ is of the form $(\delta_{i,j})_{j \in I}$ and we write $\Lambda(\alpha)_{i,l}$ instead of $\Lambda(\alpha)_{\{i\},\underline{l}}$.

**Remark 5.2.** By the definition of seminilpotency, we have:

$$\Lambda(\alpha) = \bigcup_{i,l \geq 1} \Lambda(\alpha)_{i,l}.$$  

Indeed, if $x \in \Lambda(\alpha)$, there exists an $I$-graded flag $(W_0 \subset \ldots \subset W_r = \mathbb{C}^n)$ such that $(x,W)$ satisfies $\exists \tilde{l}$. Therefore there exists $i \in I$ and $l > 0$ such that $W_r / W_{r-1} \cong V_{l_i}$, and thus $x \in \Lambda(\alpha)_{i,l}$.

**Proposition 5.3.** There exists a variety $\tilde{\Lambda}(\alpha)_{\underline{l}}$ and a diagram:

$$\begin{array}{ccc}
\tilde{\Lambda}(\alpha)_{\underline{l}} & \xrightarrow{q_{\underline{l}}} & \Lambda(\alpha - \underline{l})_{\underline{\underline{l}}} \\
\Lambda(\alpha)_{\underline{l}} & \xrightarrow{p_{\underline{l}}} & \Lambda(\alpha - \underline{l})_{\underline{\underline{l}}} \times \Lambda(\underline{l})
\end{array}$$

such that $p_{\underline{l}}$ and $q_{\underline{l}}$ are smooth with connected fibers, inducing a bijection:

$$\text{Irr} \Lambda(\alpha)_{\underline{l}} \xrightarrow{\sim} \text{Irr} \Lambda(\alpha - \underline{l})_{\underline{\underline{l}}} \times \text{Irr} \Lambda(\underline{l}).$$

**Proof.** In this proof we will denote by $I(V,V')$ the set of $I$-graded isomorphisms between two $I$-graded spaces $V$ and $V'$ of same $I$-graded dimension. We set:

$$\tilde{\Lambda}(\alpha)_{\underline{l}} = \left\{ (x, \mathcal{X}, \beta, \gamma) \mid \begin{array}{l}
\mathcal{X} \text{ I-graded and } \mathcal{I}_A(x) \oplus \mathcal{X} = V_\alpha \\
\beta \in I(\mathcal{I}_A(x), V_{\alpha-l}) \text{ and } \gamma \in I(\mathcal{X}, V_l)
\end{array} \right\}$$

and:

$$p_{\underline{l}} \left| \tilde{\Lambda}(\alpha)_{\underline{l}} \rightarrow \Lambda(\alpha - \underline{l})_{\underline{\underline{l}}} \times \Lambda(\underline{l}) \\
(x, \mathcal{X}, \beta, \gamma) \mapsto (\beta_{\star}(x_{\mathcal{I}_A(x)}), \gamma_{\star}(x_{\mathcal{X}})) \right.$$

We study the fibers of $p_{\underline{l}}$: take $y \in \Lambda(\alpha - \underline{l})_{\underline{\underline{l}}}$ and $z \in \Lambda(\underline{l})$ and consider $\mathcal{I}$ and $\mathcal{X}$ two supplementary $I$-graded subspaces of $V_\alpha$ such that $\dim \mathcal{X} = \underline{l}$, together with two isomorphisms:

$$\beta \in I(\mathcal{I}, V_{\alpha-\underline{l}}) \text{ and } \gamma \in I(\mathcal{X}, V_l).$$

We identify $\gamma$ and $z$ with $\beta_{\star}y$ and $\gamma_{\star}z$, and we define a preimage of $x$ by setting $x_{I}^{\gamma} = y$, $x_{\mathcal{X}}^{\gamma} = z$ and $x_{\mathcal{I}_A(x)}^{\beta} = \eta \in E(\mathcal{X}, \mathcal{I})$. In order to get $\mu_{\alpha}(x) = 0$, $\eta$ must satisfy the following relation for every $i \in I$:

$$\phi_i(\eta) = \sum_{h \in H : s(h) = i} \epsilon(h)(y_{h} \eta_{h} + \eta_{h} z_{h}) = 0.$$
We need to show that \( \phi = \bigoplus_{i \in I} \phi_i \) is surjective to conclude. Consider \( \xi \in \bigoplus_{i \in I} \text{Hom}(I_i, \mathcal{X}_i) \) such that for every \( \eta : \sum_{i \in I} \text{Tr}(\phi_i(\eta_i) \xi_i) = 0. \)

Then we have for every edge \( h \) such that \( s(h) = i, t(h) = j \) and for every \( \eta_h : \text{Tr}(\eta_h \xi_i y_i h) - \text{Tr}(\eta_h z_i h \xi_j) = 0. \)

But the member of the left is equal to:
\[
\text{Tr}(\eta_h \xi_i y_i h) - \text{Tr}(\eta_h z_i h \xi_j) = \text{Tr}(\eta_h (\xi_i y_i h - z_i h \xi_j)),
\]
hence \( \xi_i y_i h = z_i h \xi_j \) and therefore \( \ker \xi \) is stable by \( y \). Moreover, \( \mathcal{X}_i = \{ 0 \} \) if \( i \notin A \) so \( \ker \xi_i = I_i \) if \( i \notin A \). As \( \text{codim} J_A(y) = 0 \), we get \( \xi = 0 \), which finishes the proof.

We can now state the following theorem, which answers a question asked in \([L_1]\):

**Theorem 5.4.** The subvariety \( \Lambda(\alpha) \) of \( \bar{E}_\alpha \) is Lagrangian.

**Proof.** Since this subvariety is isotropic by \([2.2]\) we just have to show that the irreducible components of \( \Lambda(\alpha) \) are of dimension \( \langle \alpha, \alpha \rangle \). We proceed by induction on \( \alpha \), the first step corresponding to the one vertex quiver case which has already been treated: we have seen that \( \Lambda(le_i) \) is of dimension \( (le_i, le_i) \).

Next, consider \( C \in \text{Irr} \Lambda(\alpha) \) for some \( \alpha \). By \([5.2]\), there exists \( i \in I \) and \( l \geq 1 \) such that \( C \cap \Lambda(\alpha)_{i,l} \) is dense in \( C \). Let \( \hat{C} = (C_1, C_2) \) the couple of irreducible components corresponding to \( C \) via the bijection obtained in \([5.3]\) in the case \( A = \{ i \} \) and \( \ell = le_i \):
\[
\text{Irr} \Lambda(\alpha)_{i,l} \sim \text{Irr} \Lambda(\alpha - le_i)_{i,0} \times \text{Irr} \Lambda(le_i).
\]
We also know by the proof of \([5.3]\) that the fibers of \( p_{A,l} \) are of dimension:
\[
\langle \alpha, \alpha \rangle + \langle \alpha - \ell, \ell \rangle - \langle \ell, \alpha - \ell \rangle.
\]
Since \( q_{A,l} \) is a principal bundle with fibers of dimension \( \langle \alpha, \alpha \rangle - \langle \ell, \alpha - \ell \rangle \), we get:
\[
\dim C = \dim \hat{C} + \langle \alpha - \ell, \ell \rangle - \langle \ell, \alpha - \ell \rangle = \dim \hat{C} + (\alpha - le_i, le_i) + (le_i, \alpha - le_i).
\]
But \( \Lambda(\alpha - le_i)_{i,0} \) is open in \( \Lambda(\alpha - le_i) \), so we can use our induction hypothesis and the first step to write:
\[
\dim \hat{C} = (\alpha - le_i, \alpha - le_i) + t^2(e_i, e_i)
\]
and thus obtain:
\[
\dim C = (\alpha - le_i, \alpha - le_i) + t^2(e_i, e_i) + (\alpha - le_i, le_i) + (le_i, \alpha - le_i) = (\alpha, \alpha).
\]
6. CONSTRUCTIBLE FUNCTIONS

Following [Lus00], we denote by $\mathcal{M}(\alpha)$ the $\mathbb{Q}$-vector space of constructible functions $\Lambda(\alpha) \to \mathbb{Q}$, which are constant on any $G_\alpha$-orbit. Put $\mathcal{M} = \bigoplus_{\alpha \geq 0} \mathcal{M}(\alpha)$, which is a graded algebra once equipped with the product $\ast$ defined in [Lus00] 2.1.

For $Z \in \mathrm{Irr} \Lambda(\alpha)$ and $f \in \mathcal{M}(\alpha)$, we put $\rho_Z(f) = c$ if $Z \cap f^{-1}(c)$ is an open dense subset of $Z$.

If $i \in I^\text{loop}$ and $(l)$ denotes the trivial composition or partition of $l$, we denote by $1_{i,l}$ the characteristic function of the associated irreducible component $Z_i(l) \subseteq \mathrm{Irr} \Lambda(\omega_i)$ (the component of elements $x$ such that $x_h = 0$ for any loop $h \in \Omega$). If $i \notin I^\text{loop}$, we just denote by $1_i$ the function mapping to 1 the only point in $\Lambda(\omega_i)$.

We have $1_{i,l} \in \mathcal{M}(\omega_i)$ for $i \in I^\text{loop}$ and $1_i \in \mathcal{M}(\omega_i)$ for $i \notin I^\text{loop}$. We denote by $\mathcal{M}_0 \subseteq \mathcal{M}$ the subalgebra generated by these functions.

**Lemma 6.1.** Suppose $Q$ has one vertex $\circ$ and $g \geq 1$ loop(s). For every $Z \in \mathrm{Irr} \Lambda(\alpha)$ there exists $f \in \mathcal{M}_0(\alpha)$ such that $\rho_Z(f) = 1$ and $\rho_{Z'}(f) = 0$ for $Z' \neq Z$.

**Proof.** We denote by $Z_w$ the irreducible component associated to the partition (resp. composition) $w$ of $\alpha$ $g = 1$ (resp. $g \geq 2$). By convention, if $g = 1$, $Z_w$ will denote the component associated to the orbit $O_w$ defined by:

$$x \in O_w \iff \dim \ker x^l = \sum_{1 \leq k \leq l} w_k,$$

where we see now compositions as (non ordered) tuples of $\mathbb{N}_{>0}$. If $g \geq 2$, we remark that by trace duality, we can assume that $Z_w$ is the closure of $\Lambda_w$ defined by:

$$(x_i, y_i)_{1 \leq i \leq g} \in \Lambda_w \iff \dim K_i = \sum_{1 \leq k \leq i} w_k$$

where we define by induction $K_0 = \{0\}$, then $K_{j+1}$ as the biggest subspace of $\cap_i x_i^{-1}(K_i)$ stable by $(x_i, y_i)$. From now on, $w = (w_1, \ldots, w_r)$ will denote indistinctly a partition or a composition depending on the value of $g$. We define an order by:

$$w \preceq w'$$

if and only if for any $i \geq 1$ we have $\sum_{1 \leq k \leq i} w_k \leq \sum_{1 \leq k \leq i} w'_k$.

Therefore, setting $\bar{1}_w = 1_{w_r} \ast \cdots \ast 1_{w_1}$, where $1_l = 1_{\omega_i,l}$, we get:

$$x \in Z_w, \bar{1}_w(x) \neq 0 \Rightarrow w' \preceq w.$$

For $w = (\alpha)$ we have $\bar{1}_w = 1_{\alpha}$ which is the characteristic function of $Z_w$, and we put $1_w = \bar{1}_w$ in this case. Then, by induction:

$$1_w = \bar{1}_w - \sum_{w' \prec w} \rho_{Z_{w'}}(\bar{1}_w)1_{w'}$$

has the expected property. \hfill \Box

**Notations 6.2.**

- From now on, if $w$ corresponds to an irreducible component of $\Lambda(|w|\omega_i)$, we will note $1_{i,w}$ the function corresponding to $1_w$ in the previous proof.
- For $Z \in \mathrm{Irr} \Lambda(\alpha)_{i,l}$, we set $\epsilon_i(Z) = l$. 


Proposition 6.3. For every $Z \in \text{Irr} \Lambda(\alpha)$, there exists $f \in \mathcal{M}_o(\alpha)$ such that $ho_Z(f) = 1$ and $ho_{Z'}(f) = 0$ if $Z' \neq Z$.

Proof. We proceed as in [Lus00, lemma 2.4], by induction $\alpha$. The first step consists in 6.1. Then, consider $Z \in \text{Irr} \Lambda(\alpha)$. There exists $i \in I$ and $l > 0$ such that $Z \cap \Lambda(\alpha)_{i,l}$ is dense in $Z$.

We know proceed by descending induction on $l$. There’s nothing to say if $l > \alpha_i$. Otherwise, let $(Z', Z_w) \in \text{Irr} \Lambda(\alpha - le_i)_{i,0} \times \text{Irr} \Lambda(le_i)$ be the pair of components corresponding to $Z$, then, by induction hypothesis on $\alpha$, there exists $g \in \mathcal{M}_o(\alpha - le_i)$ such that $\rho_{Z'}(g) = 1$ and $\rho_Y(g) = 0$ if $Z' \neq Y \in \text{Irr} \Lambda(\alpha - le_i)$.

Then we set $\tilde{f} = 1_{i,w} \ast g \in \mathcal{M}_o(\alpha)$, and get:

- $\rho_Z(\tilde{f}) = 1$,
- $\rho_Z(\tilde{f}) = 0$ if $Z' \in \text{Irr} \Lambda(\alpha) \setminus Z$ satisfies $|\epsilon_i(Z')| = l$,
- $\tilde{f}(x) = 0$ if $x \in \Lambda(\alpha)_{i,l}$ so that $\rho_{Z'}(\tilde{f}) = 0$ if $\epsilon_i(Z') < l$.

If $\epsilon_i(Z') > l$, we use the induction hypothesis on $l$: there exists $f_{Z'} \in \mathcal{M}_o(\alpha)$ such that $\rho_{Z'}(f_{Z'}) = 1$ and $\rho_{Z''}(f_{Z'}) = 0$ if $Z'' \in \text{Irr} \Lambda(\alpha) \setminus Z'$. We end the proof by setting:

$$f = \tilde{f} - \sum_{Z': \epsilon_i(Z') > l} \rho_{Z'}(\tilde{f}) f_{Z'}.$$

\[\square\]

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Faculté des sciences d’Orsay, Bât 425, Université de Paris-Sud
F-91405 Orsay Cedex, France,
e-mail: tristan.bozec@math.u-psud.fr