SIMPLE LIE ALGEBRAS OF SMALL CHARACTERISTIC V.
THE NON-MELIKIAN CASE

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Abstract. Let $L$ be a finite-dimensional simple Lie algebra over an algebraically closed field $F$ of characteristic $p > 3$. We prove in this paper that if for every torus $T$ of maximal dimension in the $p$-envelope of $\text{ad} L$ in $\text{Der} L$ the centralizer of $T$ in $\text{ad} L$ acts triangulably on $L$, then $L$ is either classical or of Cartan type. As a consequence we obtain that any finite-dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 5$ is either classical or of Cartan type. This settles the last remaining case of the generalized Kostrikin-Shafarevich conjecture (the case where $p = 7$).

1. Introduction and preliminaries

Let $F$ be an algebraically closed field of characteristic $p > 3$. In this note we go through the relevant parts of the second author’s classification of finite-dimensional simple Lie algebras of characteristic $p > 7$ to check whether the results there need additions, modifications or supplementary proofs in order to apply in the present case where $p > 3$. It turned out that only few changes are necessary.

In what follows $L$ will always denote a finite-dimensional simple Lie algebra over $F$. We identify $L$ with the subalgebra $\text{ad} L$ of the derivation algebra $\text{Der} L$ and let $L_p$ be the $p$-envelope of $L$ in the restricted Lie algebra $\text{Der} L$. We denote by $T$ a torus of maximal dimension in $L_p$ and set

$$\tilde{H} := c_{L_p}(T) = \{x \in L_p \mid [t, x] = 0 \text{ for all } t \in T\}, \quad H := c_L(T) = \tilde{H} \cap L.$$  

A torus $T$ is called standard if $H^{(1)}$ consists of nilpotent derivations of $L$. We denote by $\Gamma(L, T)$ the set of roots of $L$ relative to $T$ (roots are nonzero linear functions $\gamma \in T^*$ such that $L_\gamma := \{x \in L \mid [t, x] = \gamma(t)x \text{ for all } t \in T\}$ is nonzero). We have root space decompositions

$$L = H \oplus \bigoplus_{\gamma \in \Gamma(L, T)} L_\gamma, \quad L_p = \tilde{H} \oplus \bigoplus_{\gamma \in \Gamma(L, T)} L_\gamma.$$

By [P-St 04, Corollary 3.7], only four types of roots can occur in simple Lie algebras of characteristic $p > 3$: solvable, classical, Witt, and Hamiltonian roots. In other words, for any $\gamma \in \Gamma(L, T)$ the semisimple quotient $L[\gamma] = L(\gamma) / \text{rad} L(\gamma)$ of the 1-section $L(\gamma) := H \oplus \bigoplus_{i \in F_p} L_{i\gamma}$ is either $(0)$ or $\mathfrak{sl}(2)$ or the Witt algebra or...
contains an isomorphic copy of the Hamiltonian algebra $H(2; 1)^{(2)}$ as an ideal of codimension $\leq 1$. The main result of [P-St 04] states that if $L_p$ contains a torus $T'$ of maximal dimension such that all roots in $\Gamma(L, T')$ are solvable or classical, then $L$ is either a classical Lie algebra or a filtered Lie algebra of type $S$ or $H$; see [P-St 04, Theorems C and D].

In this note we impose the following two assumptions on $L$:

- all tori of maximal dimension in $L_p$ are standard;
- the set of roots of any torus of maximal dimension in $L_p$ contains either a Witt root or a Hamiltonian root.

**Theorem 1.1.** If a finite dimensional simple Lie algebra $L$ over $F$ satisfies the above assumptions, then $L$ is isomorphic to a Lie algebra of Cartan type.

Combining Theorem 1.1 with [P-St 04, Theorems C and D] we derive:

**Theorem 1.2.** Let $L$ be a finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 3$ such that all tori of maximal dimension in $L_p$ are standard. Then $L$ is either classical or of Cartan type.

Due to [Wil 77, St 89a, P 94] the assumption on tori in Theorem 1.2 is fulfilled automatically when $p > 5$. In this case Theorem 1.2 can be rephrased as follows:

**Theorem 1.3.** Any finite dimensional simple Lie algebra over an algebraically closed field of characteristic $p > 5$ is either classical or of Cartan type.

Theorem 1.3 settles the last remaining case $p = 7$ of the Kostrikin–Shafarevich conjecture on the structure of finite-dimensional restricted simple Lie algebras over algebraically closed fields of characteristic $p > 5$; see [Ko-S 66]. In the early 80s, G.M. Melikian discovered a restricted simple Lie algebra of characteristic 5 which was neither classical nor of Cartan type, thereby showing that the restriction on $p$ in the Kostrikin–Shafarevich conjecture was necessary. In 1984, R.E. Block and R.L. Wilson succeeded to prove the Kostrikin–Shafarevich conjecture for algebraically closed fields of characteristic $p > 7$; see [B-W 88].

As far as the general classification problem for $p > 3$ is concerned, Theorem 1.2 leaves open the case where $p = 5$ and $L_p$ contains nonstandard tori of maximal dimension. This isolated case will be treated in [P-St 06], the last paper of the series.

2. **Two-sections of $L$**

In the next two sections our standing hypothesis is that $L$ is a finite dimensional simple Lie algebra such that all tori of maximal dimension in $L_p$ are standard. The second assumption of Sect. 1 will come into force in Sect. 4. We retain the notation introduced in [P-St 97], [P-St 99], [P-St 01], [P-St 04] with the following two exceptions: to match the notation of [St 04] we will denote the divided power algebra $A(m; \mathfrak{n})$ by $\mathcal{O}(m; \mathfrak{n})$ and the Melikian algebra $\mathfrak{g}(m, n)$ by $\mathcal{M}(m, n)$. 
Our first result extends [St 89b, Theorems 3.1, 1.7, 1.8] and [B-O-St, Corollary 1.9] which hold for \( p > 7 \).

**Theorem 2.1.** Let \( T \) be any torus of maximal dimension in \( L_p \).

(i) The subalgebra \( \tilde{H} = c_{L_p}(T) \) acts triangulably on \( L \).

(ii) For every \( \gamma \in \Gamma(L, T) \) the radical \( \text{rad} L(\gamma) \) is \( T \)-invariant and the factor algebra \( L[\gamma] = L(\gamma)/\text{rad} L(\gamma) \) is either zero or isomorphic to one of \( \mathfrak{sl}(2), W(1; 1), H(2; 1)^{(2)}, H(2; 1)^{(1)} \).

**Proof.** Since all tori of maximal dimension in \( L_p \) are assumed to be standard, the first statement is nothing but [P-St 04, Theorem 3.12], while the second statement is immediate from [P-St 04, Corollary 3.7]. \( \Box \)

Given a filtered Cartan type Lie algebra \( \mathfrak{g} \) (not necessarily simple) we denote by \( \mathfrak{g}(0) \) the standard maximal subalgebra of \( \mathfrak{g} \). When \( \mathfrak{g} = W(1; 1) \), we have \( \dim(\mathfrak{g}/\mathfrak{g}(0)) = 1 \), while when \( \mathfrak{g} \cong H(2; 1)^{(\epsilon)} \) with \( \epsilon = 1, 2 \), we have \( \dim(\mathfrak{g}/\mathfrak{g}(0)) = 2 \).

Theorem 2.1 shows that every 1-section \( L(\gamma) \) with \( \gamma \in \Gamma(L, T) \) contains a distinguished subalgebra \( Q(\gamma) \) such that \( Q(\gamma)/\text{rad} Q(\gamma) \) is either zero or isomorphic to \( \mathfrak{sl}(2) \). More precisely, the following holds:

(a) \( Q(\gamma) = L(\gamma) \) if \( L(\gamma) \) is solvable or \( L[\gamma] \cong \mathfrak{sl}(2) \);

(b) \( (Q(\gamma) + \text{rad} L(\gamma))/\text{rad} L(\gamma) = L[\gamma](0) \) if \( L[\gamma] \) is of Cartan type.

This is analogous to [St 89b, Proposition 1.9] and [B-O-St, Proposition 1.11].

Recall that a root \( \gamma \in \Gamma(L, T) \) is called proper, if \( Q(\gamma) \) is \( T \)-invariant, and improper otherwise; see [P-St 04]. This definition differs from that introduced by Block–Wilson. However, it agrees with the Block–Wilson definition when \( p > 7 \) and reflects better the desired properties of \( \gamma \) when \( p \in \{5, 7\} \) (the formal extension of the Block–Wilson definition to the case where \( p = 5 \) would imply that all Hamiltonian roots are improper, which is not what we want).

The following result is very important for the classification:

**Theorem 2.2.** Let \( L(\alpha, \beta) \) be any 2-section of \( L \) relative to a torus \( T \) of maximal dimension in \( L_p \), and \( I(\alpha, \beta) \) the maximal solvable ideal of \( T + L(\alpha, \beta) \). Let \( \psi \) denote the canonical homomorphism \( T + L(\alpha, \beta) \to (T + L(\alpha, \beta))/I(\alpha, \beta) \), and put \( K := \psi(L(\alpha, \beta)) \) and \( \overline{T} := \psi(T) \). Then one of the following holds:

1. \( L(\alpha, \beta) \) is solvable and \( K = \{0\} \);
2. \( S = \overline{T} + K \) where \( S \) is one of \( \mathfrak{sl}(2), W(1; 1), H(2; 1)^{(2)} \) or else \( S = H(2; 1)^{(2)} \) and \( S \subset \overline{T} + K \subset H(2; 1)^{(1)} \). Moreover, there exists a root \( \mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta \) such that \( K = \psi(L(\mu)) \);
3. \( S_1 \oplus S_2 \subset \overline{T} + K \subset (\text{Der } S_1)^{(1)} \oplus (\text{Der } S_2)^{(1)} \) where \( S_i \) is one of \( \mathfrak{sl}(2), W(1; 1), H(2; 1)^{(2)} \) for \( i = 1, 2 \);
4. \( K = FD \oplus H(2; 1)^{(2)} \) where either \( D \in \{0, D_H(x_1^{p-1}x_2^{p-1})\} \) or \( p = 5 \) and \( D = x_1^{p-1}\partial_2 \), and there is an automorphism \( \sigma \) of \( \text{Der } H(2; 1)^{(2)} \) such that \( \sigma(\overline{T}) = Fz_1\partial_1 \oplus Fz_2\partial_2 \) where \( z_i \in \{x_i, 1 + x_i\}, i = 1, 2 \).
\[ \text{(5)} \quad S \otimes \mathcal{O}(m; \underline{1}) \subset \overline{T} + K \subset \text{Der} (S \otimes \mathcal{O}(m; \underline{1})) \text{ where } S \text{ is one of } \mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)} \text{ and } m = 1, 2. \text{ Let } \pi_2 \text{ be the canonical projection from } \text{Der} (S \otimes \mathcal{O}(m; \underline{1})) \cong ((\text{Der} S) \otimes \mathcal{O}(m; \underline{1})) \rtimes \text{Id}_S \otimes W(m; \underline{1}) \text{ onto } W(m; \underline{1}). \text{ Then } \pi_2(K) \cong W(1; \underline{1}) \text{ if } m = 1 \text{ and } \pi_2(K) \cong H(2; \underline{1})^{(\epsilon)} \text{ if } m = 2, \text{ where } \epsilon = 1, 2; \]

\[ \text{(6)} \quad S \otimes \mathcal{O}(1; \underline{1}) \subset K \subset \tilde{\mathcal{S}} \otimes \mathcal{O}(1; \underline{1}), \text{ where } S \text{ is one of } \mathfrak{sl}(2), W(1; \underline{1}), H(2; \underline{1})^{(2)} \text{ and either } S = \tilde{\mathcal{S}} \text{ or } S = H(2; \underline{1})^{(2)} \text{ and } \tilde{\mathcal{S}} = H(2; \underline{1})^{(1)}. \text{ Moreover, } \overline{T} = F(h_0 \otimes 1) \oplus F(\text{Id}_\mathcal{S} \otimes (1 + x_1)\partial_1) \text{ for some toral element } h_0 \in S; \]

\[ \text{(7)} \quad S \subseteq \overline{T} + K \subseteq S_p \text{ where } S \text{ is one of the nonrestricted Cartan type Lie algebras } W(1; \underline{2}), H(2; \underline{1}; \Phi(\tau))^{(1)}, H(2; \underline{1}; \Delta) \text{ or else } \]

\[ H(2; (2, 1))^{(2)} \subset \overline{T} + K \subset H(2; (2, 1))_p; \]

\[ \text{(8)} \quad K \text{ is either a classical Lie algebra of type } A_2, B_2 \text{ or } G_2 \text{ or one of the restricted Cartan type Lie algebras } W(2; \underline{1}), S(3; \underline{1})^{(1)}, H(4; \underline{1})^{(1)}, K(3; \underline{1}). \]

\textbf{Proof.} 1) If } L(\alpha, \beta) \text{ is solvable, then we are in case (1). So assume from now that } L(\alpha, \beta) \text{ is nonsolvable.} 

Recall from [P-St 04] that } \text{rad}_T \L(\alpha, \beta) \text{ denotes the maximal } T\text{-invariant solvable ideal of } L(\alpha, \beta), \text{ and } L[\alpha, \beta] = L(\alpha, \beta)/\text{rad}_T L(\alpha, \beta). \text{ Since } \text{rad}_T L(\alpha, \beta) = I(\alpha, \beta) \cap L(\alpha, \beta), \text{ we have that } L[\alpha, \beta] \cong \psi(L(\alpha, \beta)) = K \hookrightarrow \overline{T} + K. \]

As in [P-St 04] we denote by } \tilde{\mathcal{S}} = \bigoplus_{i=1}^r \tilde{S}_i \text{ the sum of all minimal } T\text{-invariant ideals of } K = L[\alpha, \beta] \neq (0). \text{ Since the Lie algebra } \overline{T} + K \text{ is semisimple, it acts faithfully on } \tilde{\mathcal{S}}. \text{ We will identify } \overline{T} + K \text{ with a Lie subalgebra of } \text{Der } \tilde{\mathcal{S}}. \text{ As shown in [P-St 04, Sect. 4], we have that } r \in \{1, 2\}. \text{ Moreover, if } r = 2, \text{ then we are in case (3); see } [\text{P-St 04, Theorem 4.1}]. 

2) From now on assume that } \tilde{\mathcal{S}} = \tilde{\mathcal{S}}_1 \text{ is the unique minimal } \overline{T}\text{-invariant ideal of } K = L[\alpha, \beta]. \text{ Recall from [P-St 04] that } TR(\tilde{\mathcal{S}}) \leq TR(L[\alpha, \beta]) \leq 2. \]

Suppose } TR(\tilde{\mathcal{S}}) = 2. \text{ If the Lie algebra } \tilde{\mathcal{S}} \text{ is restrictable, then } [\text{P-St 04, Theorem 4.2}] \text{ shows that } \tilde{\mathcal{S}} = L[\alpha, \beta]. \text{ Therefore, if } \tilde{\mathcal{S}} \text{ is a classical Lie algebra or a restricted Lie algebra of Cartan type, then we are in case (8). As explained in [P-St 04, Sect. 4] there is a natural restricted homomorphism } \Psi_{\alpha, \beta}: T + L(\alpha, \beta)_p \rightarrow \text{Der } \tilde{\mathcal{S}} \text{ which maps } T + L(\alpha, \beta) \subset T + L(\alpha, \beta)_p \text{ onto } \overline{T} + K. \]

Suppose } \tilde{\mathcal{S}} \cong \mathcal{M}(1, 1). \text{ Then } \tilde{\mathcal{S}} = \text{Der } \tilde{\mathcal{S}}. \text{ Choose a two-dimensional nonstandard torus } \overline{T}' \text{ in } \tilde{\mathcal{S}}. \text{ There exists a torus } T' \text{ in the restricted Lie algebra } T + L(\alpha, \beta)_p \text{ such that ker } \alpha \cap \ker \beta \subset T' \text{ and } \Psi_{\alpha, \beta}(T') = \overline{T}'. \text{ By construction, } T' \text{ is then a nonstandard torus of maximal dimension in } L_p. \text{ Since no such tori can exist by our general assumption, we derive that } \tilde{\mathcal{S}} \not\cong \mathcal{M}(1, 1). \]
If $\widetilde{S}$ is a nonrestricted Cartan type Lie algebra, then [P-St 01, Theorem 1.1] yields that $\widetilde{S}$ is one of $W(1, 2), H(2; 1; \Phi(\tau))^{(1)}, H(2; 1; \Delta), H(2; (2, 1))^{(2)}. \text{ Applying [P-St 04, Theorem 4.2] shows that we are in case (7).} \]

3) It remains to consider the situation where $r = 1$ and $TR(\widetilde{S}) = 1$, which is ruled by [P-St 04, Theorem 4.4]. Due of Theorem 2.1, case (1) of that theorem is our case (2). Assuming case (2) of Theorem 4.4 in [P-St 04], part (b) of the proof in loc. cit. shows that there exists an automorphism $\sigma$ of the Lie algebra $\text{Der } H(2; 1)^{(2)}$ such that $\sigma(K) = H(2; 1)^{(2)} \oplus FD$ and $\sigma(T) = Fz_1\partial_1 \oplus F_{z_2}\partial_2$, where $D$ is as in case (4) of the present theorem and $z_i \in \{x_i, 1 + x_i\}, i = 1, 2.$

Now assume we are in case (3) of [P-St 04, Theorem 4.4]. Then $\widetilde{S} = S \otimes O(1; 1)$ where $S$ is one of $\mathfrak{sl}(2), W(1; 1), H(2; 1)^{(2)},$ and $\Psi_{\alpha, \beta}(T)$ is spanned by $h_0 \otimes 1$ and $\text{Id} \otimes (1 + x_1)\partial_1$ for some nonzero toral element $h_0 \in S$. Moreover, $K \subset (\text{Der } S) \otimes O(1; 1)$, To show that this is our case (6) we can assume that $S = H(2; 1)^{(2)}$ (in the other two cases $\text{Der } S = \text{ad } S$ and there is nothing to prove).

If $K \not\subset H(2; 1)^{(2)} \otimes O(1; 1)$, then there is a root $\mu \in \Gamma(L, T) \cap (\mathbb{F}_p\alpha + \mathbb{F}_p\beta)$ such that $K(\mu) \not\subset H(2; 1)^{(2)} \otimes O(1; 1)$. Restricting the composite

$\bar{\psi} : L(\alpha, \beta) \rightarrow K \rightarrow (\text{Der } S) \otimes O(1; 1) \rightarrow \text{Der } S$

to the 1-section $L(\mu)$ and using the above description of $\Psi_{\alpha, \beta}(T)$ it is easy to observe that $K(\mu) \cong \bar{\psi}(L(\mu))$ is sandwiched between $S$ and $\text{Der } S$. Consequently, $\mu$ is a Hamiltonian root and $K(\mu) \cong L[\mu]$. Theorem 2.1 now yields $K(\mu) \cong H(2; 1)^{(6)}$ where $\epsilon \in \{1, 2\}$. If $\epsilon = 2$, then $K(\mu) = K(\mu)^{(\infty)} \subset \widetilde{S}$ contrary to our choice of $\mu$. Hence $\epsilon = 1$. Since $K(\mu) \not\subset H(2; 1)^{(2)} \otimes O(1; 1)$, it must be that $\bar{\psi}(L(\mu)) = H(2; 1)^{(1)}$ by Theorem 2.1. This proves that $K \subset H(2; 1)^{(1)} \otimes O(1; 1)$, as claimed.

In case (4) of [P-St 04, Theorem 4.4] there exist $\mu, \nu \in \Gamma(L, T)$ such that $L[\mu, \nu] \cong \mathcal{M}(1, 1)$. But then, as before, $L_p$ contains a nonstandard torus of maximum dimension, violating our general assumption. Case (5) of [P-St 04, Theorem 4.4] is included into case (5) of the present theorem.

Finally, consider case (6) of [P-St 04, Theorem 4.4]. Let $h_0$ be a nonzero toral element of $S = H(2; 1)^{(2)}$ and choose a torus $T'$ in $T + L(\alpha, \beta)_p$ with dim $T' = \text{dim } T$ and $h_0 \otimes 1 \in \Psi_{\alpha, \beta}(T')$. Note that ker $\alpha \cap \ker \beta \subset T'$ and $T'$ is standard by our general assumption. Choose $\mu \in \Gamma(K, \Psi_{\alpha, \beta}(T'))$ with $\mu(h_0 \otimes 1) = 0$ and regard it as a linear function on $T'$ vanishing on ker $\alpha \cap \ker \beta$. Then

$\varepsilon_S(h_0) \otimes O(2; 1) \subset K(\mu) \subset (\varepsilon_{\text{Der } S}(h_0) \otimes O(2; 1)) \times \text{Id} \otimes \pi_2(K).$

Let $\varrho$ be the canonical projection $(\varepsilon_{\text{Der } S}(h_0) \otimes O(2; 1)) \times \text{Id} \otimes \pi_2(K) \rightarrow \pi_2(K).$ Note that $\varrho$ is $T'$-equivariant and maps the 1-section $K(\mu)$ onto $\pi_2(K)$. By [P-St 04, Theorem 4.4(6)], $\pi_2(K)$ is sandwiched between $H(2; 1)^{(2)}$ and $H(2; 1)$. This implies that $\varrho(K(\mu))$ is semisimple. As $\varrho(K(\mu))$ is a homomorphic image of the 1-section $L(\mu)$, it must be that $\varrho(K(\mu)) \cong L[\mu]$. Theorem 2.1 now yields $\pi_2(K) = \varrho(K(\mu)) \cong H(2; 1)^{(\epsilon)}$ where $\epsilon \in \{1, 2\}$, completing the proof. \(\square\)
We have now established (for \( p > 3 \)) a refined version of [St 92, Theorem II.2] which is the corrected version of [St 89b, Theorem 6.3] (or, equivalently, of [B-O-St, Theorem 1.15]). In what follows we will need a description of \( \overline{T} \) in the respective cases.

**Proposition 2.3.** With the assumptions and notations of Theorem 2.2, \( \overline{T} \) has the following properties in the respective cases:

1. \( \overline{T} = (0) \);
2. \( \overline{T} \subset S \) and \( \dim \overline{T} = 1 \);
3. \( \overline{T} = Fh_1 \oplus Fh_2 \) where \( h_i \in S_i, \ i = 1, 2 \);
4. \( \overline{T} \) is conjugate to \( Fz_1 \partial_1 \oplus Fz_2 \partial_2 \) where \( z_i \in \{ x_1, 1 + x_1 \}, \ i = 1, 2 \);
5. \( \overline{T} = F(h \otimes 1) \oplus F(d \otimes 1 + \text{Id} \otimes t) \) where \( h \in S \) and \( t \in \pi_2(K) \) are nonzero toral elements, \( d \in \text{Der} S \) is toral, and \( [d, h] = 0 \);
6. \( \overline{T} = F(h \otimes 1) \oplus F\text{Id} \otimes (1 + x_1) \partial_1 \) where \( h \in S \setminus (0) \); and \( h^{[p]} = h \);
7. \( \overline{T} \subset S_p \) and, moreover, \( \overline{T} \cap S = (0) \) when \( S \cong H(2; 1; \Phi(\tau))^{(1)} \) and \( \dim \overline{T} \cap S = 1 \) otherwise;
8. \( \overline{T} \subset K \).

**Proof.** (a) If \( K = (0) \), then \( T + L(\alpha, \beta) \) is solvable, hence coincides with \( I(\alpha, \beta) \).

Therefore, \( \overline{T} = (0) \).

(b) From now on suppose \( K \neq (0) \). Then \( (0) \neq \overline{T} + K = (T + L(\alpha, \beta))/I(\alpha, \beta) \) is semisimple, hence acts faithfully on its socle \( \tilde{S} \). Note that

\[ \tilde{S} \subset (\overline{T} + K)^{(1)} \subset K \subset \text{Der} \tilde{S}. \]

We regard \( \overline{T} + K \) as a Lie subalgebra of \( \text{Der} \tilde{S} \). The restricted homomorphism \( \Psi_{\alpha, \beta} : T + L(\alpha, \beta)_p \rightarrow \text{Der} \tilde{S} \) introduced in [P-St 04, Sect. 4] then maps \( T \) onto \( \overline{T} \). It also maps \( L(\alpha, \beta)_p \subset L_p \) onto the \( p \)-envelope of \( K \) in \( \text{Der} \tilde{S} \).

The latter \( p \)-envelope will be denoted by \( K_p \). It contains the \( p \)-envelope \( \tilde{S}_p \) of \( S \) in \( \text{Der} \tilde{S} \).

Since \( \tilde{S} \subset \overline{T} + \tilde{S}_p \subset \text{Der} \tilde{S}, \) the restricted Lie algebra \( \overline{T} + \tilde{S}_p \), is centerless. In conjunction with [St 04, Theorem 1.2.6(3)] this shows that if \( \overline{T} \) is a torus of maximal dimension in \( \overline{T} + K_p \), then \( \overline{T} \cap \tilde{S}_p \) is a torus of maximal dimension in \( \tilde{S}_p \).

In view of [St 04, Theorem 1.2.8(3) and Theorem 1.3.11(3)] we also have that

\[ 1 \leq TR(\overline{T} + K_p) \leq TR(T + L(\alpha, \beta)_p) \leq TR(L_p(\alpha, \beta)) \leq 2. \]

(c) Let \( \overline{T} + K \) be as in case (2) of Theorem 2.2. Then \( \overline{T} + K = S \) where \( S = \tilde{S} = \tilde{S}_p \) is one of \( \mathfrak{sl}(2), W(1; 1), H(2; 1)^{(1)} \) or else \( \overline{T} + K \) is sandwiched between \( S = H(2; 1)^{(2)} \) and \( H(2; 1)^{(1)} \). In any event, \( \overline{T} + K \) is semisimple, restricted, and has absolute toral rank 1. It follows that \( \dim \overline{T} = 1 \) and \( \overline{T} \) is a torus of maximal dimension in \( \overline{T} + K = \overline{T} + K_p \). But then \( S \cap \overline{T} \) is a torus of maximal dimension in \( S = \tilde{S}_p \), by our concluding remark in part (b). Hence \( \overline{T} \subset S \), by dimension reasons.
(d) In case (4) of Theorem 2.2 we have \( \dim \mathcal{T} = 2 \). In cases (3) and (5)–(8), we have that either \( TR(\tilde{S}) = 2 \) or the Lie algebra \( \tilde{S} \) is not simple. From this it follows that in the remaining cases of Theorem 2.2 the torus \( \mathcal{T} = \Psi_{\alpha,\beta}(T) \) acts on \( \tilde{S} \) as a two-dimensional torus of derivations. Indeed, otherwise there would exist a root \( \gamma \in \Gamma(L, T) \) such that \( \mathcal{T} + K = \mathcal{T} + K(\gamma) \) and this would imply that \( \tilde{S} = K(\gamma)^{(\infty)} \) is simple; see Theorem 2.1. By our concluding remark in part (b), \( \mathcal{F} \cap \tilde{S} \) is a torus of maximal dimension in \( \tilde{S}_p \).

(e) Suppose we are in case (3) of Theorem 2.2. Then \( \tilde{S} = S_1 \oplus S_2, TR(S) = 2 \), and \( \tilde{S} = \tilde{S}_p \). Since \( \dim(\mathcal{T} \cap \tilde{S}_p) = TR(\tilde{S}) \) due to our concluding remark in part (b), it must be that \( T \subset S_1 \oplus S_2 \).

(f) If \( \mathcal{T} + K \) is as in case (4), then \( \mathcal{T} \) already has the required form in view of Theorem 2.2.

(g) Suppose we are in case (5) of Theorem 2.2. Because \( S \otimes O(m; 1)_1 \) is a [\( p \)]-nilpotent ideal of \( \tilde{S} = S \otimes O(m; 1) \), it follows that
\[
\tilde{S}_p = S \otimes O(m; 1) + S_p \otimes F = \tilde{S}
\]
and \( TR(\tilde{S}) = TR(S) = 1 \). The discussion in part (b) now yields \( \mathcal{T} \cap \tilde{S}_p = F\tilde{h} \) for some nonzero toral element \( \tilde{h} \in \tilde{S} \). According to [P-St 99, Theorem 2.6], there is an automorphism \( \varphi \) of \( \text{Der} \, \tilde{S} \) such that \( \varphi(\mathcal{T}) \subset (\text{Der} \, S) \otimes F \rtimes \text{Id} \otimes W(m; 1) \). Since \( \varphi \) preserves the socle of \( \text{Der} \, \tilde{S} \), it must be that
\[
\varphi(\tilde{h}) \in (S \otimes O(m; 1)) \cap ((\text{Der} \, S) \otimes F) = S \otimes F.
\]
In other words, \( \varphi(\tilde{h}) = h \otimes 1 \) for some nonzero toral element \( h \in S \).

Now let \( \tilde{t} \) be any (nonzero) toral element such that \( \mathcal{T} = F\tilde{h} \oplus F\tilde{t} \). Write \( \varphi(\tilde{t}) = d \otimes 1 + \text{Id} \otimes t \) with \( d \in \text{Der} \, S \) and \( t \in W(m; 1) \). It is straightforward to see that \( d \) and \( t \) are both toral, and \( [d, h] = 0 \).

It remains to show that \( t \) is a nonzero element of \( \pi_2(K) \). If \( t = 0 \), then \( \mathcal{T} \) lies in \( (\text{Der} \, S) \otimes O(m; 1) \), that is \( \pi_2(\mathcal{T}) = (0) \). Since \( \mathcal{T} \) is a torus of maximal dimension in \( \mathcal{T} + K_p \), we apply [St 04, Theorem 1.2.8(4)] to get \( TR(\pi_2(K)) = 0 \). But Theorem 2.2 says that in the present case \( \pi_2(K) \) is a semisimple restricted Lie algebra of absolute toral rank one. Hence \( F\tilde{t} \) is a torus of maximal dimension in \( \pi_2(\mathcal{T} + K_p) = F\tilde{t} + \pi_2(K) \). As \( \pi_2(K) \) is a restricted ideal of \( \pi_2(\mathcal{T} + K_p) \), we conclude that \( t \in \pi_2(K) \); see [St 04, Theorem 1.2.6(3)].

(h) If \( \mathcal{T} + K \) is as in case (6), then \( \mathcal{T} \) already has the required form in view of Theorem 2.2.

(i) Suppose \( \mathcal{T} + K \) is as in case (7). If \( S \) is one of \( W(1; 2), H(2; 1; \Phi(\tau))^{(1)}, H(2; 1; \Delta) \), then Theorem 2.2 says that \( T \subset S_p \). Suppose \( S = W(1; 2) \). As \( W(1; 2) \) has codimension 1 in \( W(1; 2)_p \), it must be that \( \mathcal{T} \cap W(1; 2) \neq (0) \). Since all weight spaces of the \( \mathcal{T} \)-module \( W(1; 2) \) are one-dimensional, by [St 92, Theorem V.2.2(2)], we have that \( \dim(\mathcal{T} \cap S) = 1 \) in this case. If \( S = H(2; 1; \Phi(\tau))^{(1)}, \)
it is clear that the restricted Lie algebra

\[ H = H(2; 1; \Delta) \]

such that \( x \) implies \( T \cap S \neq (0) \). Suppose \( T \) has at least \( p^2 - 2 \) weights on \( S \). Since \( \dim S = p^2 \), we then have

\[ S = T \oplus \bigoplus_{\gamma \in \Gamma(S,T)} S_\gamma, \quad |\Gamma(S,T)| = p^2 - 2, \quad \dim S_\gamma = 1 \quad (\forall \gamma \in \Gamma(S,T)). \]

But then the subset \( T \cup \bigcup_{\gamma \in \Gamma(S,T)} S_\gamma^{[p]} \) spans a diagonalizable Lie subalgebra of \( S_p \) containing \( T \). Since \( T \) is a torus of maximal dimension in \( S_p \), we get \( \bigcup_{\gamma \in \Gamma(S,T)} S_\gamma^{[p]} \subset T \). However, this would imply that \( S \) is a restrictable Lie algebra, which is false. We conclude that \( \dim(T \cap S) = 1 \) in the present case.

Now consider the case where \( S = H(2; (2, 1))^{(2)} \). It is well-known (and easily seen) that \( S_p = F\partial^p \oplus S \) and \( H(2; (2, 1))_p = F\partial^p \oplus H(2; (2, 1)) \). But then it is clear that the restricted Lie algebra \( H(2; (2, 1))_p/S_p \) is \( [p] \)-nilpotent. As \( T \subset H(2; (2, 1))_p \) by Theorem 2.2, we now derive that \( T \subset S_p \). Since \( S \) has codimension 1 in \( S_p \), we have \( T \cap S \neq (0) \), while part (f) of [B-W 88, (10.1.1)] yields \( T \not\subset S \). Therefore, \( \dim(T \cap S) = 1 \).

(j) Finally, suppose \( T + K \) is as in case (8) of Theorem 2.2. Then \( K = S \) is a restricted simple Lie algebra of absolute toral rank 2. Hence \( \dim(T \cap S) = 2 \) by our discussion at the end of part (b). In other words, \( T \subset K \).

3. Optimal Tori

Given a torus \( T \) of maximal dimension in \( L_p \) we denote by \( \Gamma_p(L, T) \) the subset of all proper roots in \( \Gamma(L, T) \). Note that if \( \gamma \in \Gamma(L, T) \) is proper, then \( \Gamma(L, T) \cap \mathbb{F}_p\gamma \subset \Gamma_p(L, T) \). We say that \( T \) is an optimal torus if the number

\[ r(T) := |\Gamma(L, T) \setminus \Gamma_p(L, T)| \]

is minimal possible; see [P-St 01, p. 242].

From now on we fix an \( \mathbb{F}_p \)-linear map \( \xi : F \to F \) such that \( \xi^p - \xi = \text{Id}_F \) (such a map is unique up to an \( \mathbb{F}_p \)-valued additive function on \( F \)). The process of toral switching (based on the ideas of [Win 69], [Wil 83], [P 86]) has been described in detail in [P-St 99, Sect. 2]. Given \( x \in L_\alpha \), where \( \alpha \in \Gamma(L, T) \), we denote by \( T_x \) the linear span of all \( t_x := t - \alpha(t)(x + x^p + \cdots + x^{p^{m-1}}) \) with \( t \in T \) (here \( m = m(x) \) is the smallest nonnegative integer with the property that \( x^{p^m} \in T \) ). By [P 86], \( T_x \) is a torus of maximal dimension in \( L_p \) and \( \Gamma(L, T_x) = \{ \gamma_x | \gamma \in \Gamma(L, T) \} \), where

\[ \gamma_x(t_x) = \gamma(t) - (\xi \circ \gamma)(x^{p^m})\gamma(t) \quad (\forall t \in T). \]

We say that \( T_x \) is obtained from \( T \) by the elementary switching corresponding to \( x \). By [P 86], there exists an invertible linear operator \( E_x = E_{x,\xi} \in \text{GL}(L_p) \) such that \( E_{\xi}(T_x) = E_x(T_x) \) and \( L_{\gamma_x} = E_x(L_{\gamma}) \) for all \( \gamma \in \Gamma(L, T) \). Moreover,
$E_x$ is a lopynomial in $\text{ad } x$. The operator $E_x$ is often referred to as a generalized Winter exponential.

It is immediate from the explicit form of generalized Winter exponetials that $L(\alpha_x) = L(\alpha)$ and $L(\alpha_x, \beta_x) = L(\alpha, \beta)$ for all roots $\beta \in \Gamma(L, T)$; see [P-St 99, pp. 218–222] for more detail. Recall that the torus $T_x$ is standard by our general assumption. Solvable and classical roots are proper by definition. If $\alpha$ is Witt or Hamiltonian, then one can always find an element $x \in \bigcup_{i \neq 0} L_{\alpha i}$ such that the root $\alpha_x \in \Gamma(L, T_x)$ is proper.

Our main goal in this section is to show that if $T$ is an optimal torus, then all roots in $\Gamma(L, T)$ are proper, and describe the action of optimal tori on 2-sections.

**Lemma 3.1.** Let $K = L[\alpha, \beta]$ and $\mathcal{T}$ be as in Theorem 2.2, and suppose that we are not in case (5) of that theorem. Let $u \in L_\alpha$, and assume that $\alpha \notin \Gamma_p(L, T)$ and $\alpha_u \in \Gamma_p(L, T_u)$. Then $|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|$.

**Proof.** We denote by $\bar{u}$ the image of $u$ in $K$.

1. Since $L(\gamma) \cap \Gamma(\alpha, \beta) \subset Q(\gamma)$ for all $\gamma \in \Gamma(L, T)$, it suffices to show that under the above assumptions we have $|\Gamma_p(K, \mathcal{T}_u)| > |\Gamma_p(K, \mathcal{T})|$. If $K = (0)$, then the root $\alpha$ is solvable, hence proper. So this case is impossible.

2. Let $K$ be as in case (2) of Theorem 2.2. Then $K = \psi(L(\mu))$ for some $\mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$. Since all roots in $(\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus \mathbb{F}_p\mu$ are solvable, hence proper, it must be that $K = \psi(L(\alpha)) = \psi(L(\alpha_u))$, implying $|\Gamma(K, \mathcal{T}_u)| = |\Gamma_p(K, \mathcal{T}_u)|$.

Since $\alpha$ is improper, the result follows.

3. Let $K$ be as in case (3) of Theorem 2.2. Then $\mathcal{T} = (\mathcal{T} \cap S_1) \oplus (\mathcal{T} \cap S_2)$ by Proposition 2.3. Hence there exist $\mu, \mu' \in \Gamma(L, T)$ such that $K = K(\mu) + K(\mu')$ and $[K_{i\mu}, K_{j\mu'}] = 0$ for all $i, j \in \mathbb{F}_p^*$. All roots in $(\mathbb{F}_p\alpha + \mathbb{F}_p\beta) \setminus (\mathbb{F}_p\mu \cup \mathbb{F}_p\mu')$ are solvable. Hence $\alpha \in \mathbb{F}_p\mu \cup \mathbb{F}_p\mu'$. No generality will be lost by assuming that $\alpha = \mu'$. Since $K_{i\mu} = E_{\bar{u}}(K_{i\mu})$ for all $i \in \mathbb{F}_p$, the preceding remark yields

$$K(\mu_u) = E_{\bar{u}}(K(\mu)) = e_{K(\mu)}(\mathcal{T}_u) \oplus \bigoplus_{i \in \mathbb{F}_p} K_{i\mu}$$

(as $E_{\bar{u}}$ is a polynomial in ad $\bar{u}$, we have that $E_{\bar{u}}(y) = y$ for all $y \in \bigcup_{i \neq 0} K_{i\mu}$). It follows that $\mu$ is proper if and only if $\mu_u$ is. Since $\alpha = \mu'$ is improper and $\mu_u$ is proper, the result follows.

4. Let $K$ be as in case (4) of Theorem 2.2. Then $K = FD \oplus H(2; 1)^{(2)}$ and there is an automorphism $\sigma$ of the Lie algebra $\text{Der } H(2; 1)^{(2)}$ such that $\sigma(\mathcal{T})$ is one of

$$T_0 = Fx_1 \partial_1 \oplus Fx_2 \partial_2, \quad T_1 = F(1 + x_1) \partial_1 \oplus Fx_2 \partial_2, \quad T_2 = F(1 + x_1) \partial_1 \oplus F(1 + x_2) \partial_2.$$ 

Replacing $K$ by its isomorphic copy $\sigma(K) \subset H(2; 1)$ we may assume that $\sigma = \text{Id}$. Theorem III.4 in [St 92] describes the 1-sections of $\text{Der } H(2; 1)^{(2)}$ relative to $\sigma(\mathcal{T})$, hence the 1-sections of $K$ relative to $\mathcal{T}$ (the deliberations in [St 92, Sect. III] only
require that \( p > 2 \). It is immediate from the description in [St 92, Sect. III] that all improper roots in \( \Gamma(K, \overline{T}) \) are Witt (no root in \( \Gamma(K, \overline{T}) \) is Hamiltonian by dimension reasons).

If \( \sigma(\overline{T}) = T_0 \), then [St 92, Theorem III.5] shows that all roots in \( \Gamma(K, \overline{T}) \) are proper. As \( \sigma \not\in \Gamma_p(L, T) \), this case is impossible.

Suppose \( \overline{T} = T_1 \). Let \( \mu \) be any \( \overline{T} \)-root of \( K \) such that \( \mu(x_2 \partial_2) \not\in \{0, 1\} \). Then in the notation of [St 92, Proposition III.3] we have \( b \not\in \{0, 1\} \). Hence \( b \geq 2 \), in which case that proposition yields \( H(2; 1)_\mu \subset H(2; 1)_{1(0)} \). Thus if \( \mu(x_2 \partial_2) \neq 0 \), then there is a unique \( i_0 \in \mathbb{F}_p^* \) with \( H(2; 1)_{i_0\mu} \not\subset H(2; 1)_{(0)} \). Moreover, loc. cit. shows that \( H(2; 1)_{i_0\mu} \cap H(2; 1)_{(0)} \) has codimension 1 in \( H(2; 1)_{i_0\mu} \). From this it is easy to deduce that \( \overline{T} \) normalizes a solvable subalgebra of codimension \( \leq 1 \) in \( K(\mu) \). As a consequence, \( \mu \) is a proper root in \( \Gamma(K, \overline{T}) \).

Applying these deliberations to our improper root \( \alpha \) we obtain \( \alpha(x_2 \partial_2) = 0 \). Then \( H(2; 1)(\alpha) \) is spanned by \( \{k(1 + x_1)^{k-1}x_2 \partial_2 - (1 + x_1)^k\partial_1 \mid k \in \mathbb{F}_p \} \).

Therefore, \( K(\alpha) \) is isomorphic to the Witt algebra \( W(1; 1) \). Since \( \alpha_u \) is a proper root in \( \Gamma(L, T_u) \), the torus \( T_u = T_u \) must normalize \( H(2; 1)_{(0)} \). But then \( T_u \) is conjugate to \( T_0 \) under the automorphism group of \( \text{Der} H(2; 1)^{(2)} \). Our remarks earlier in the proof now show that all roots in \( \Gamma(K, \overline{T_u}) \) are proper.

Suppose \( \overline{T} = T_2 \). Let \( \mu \) be any root in \( \Gamma(K, \overline{T}) \) and assume by symmetry that \( \mu((1 + x_1)\partial_1) \neq 0 \). Set

\[
a := \frac{\mu((1 + x_2)\partial_2)}{\mu((1 + x_1)\partial_1)}, \quad b := \frac{\mu((1 + x_1)\partial_1) - \mu((1 + x_2)\partial_2)}{\mu((1 + x_1)\partial_1)}.
\]

Clearly, \( a, b \in \mathbb{F}_p \). If \( b = 0 \), then it is easy to see that \( H(2; 1)(\mu) \) is solvable. If \( b \neq 0 \), then the 1-section \( H(2; 1)(\mu) \) is spanned by all \( k(1 + x_1)^{k-1}(1 + x_2)^{ka+b}\partial_2 - (ka+b)(1 + x_1)^k(1 + x_2)^{ka+b-1}\partial_1 \) with \( k \in \mathbb{F}_p \). It follows from this description that \( H(2; 1)(\mu) \cong W(1; 1) \) and \( \overline{T} \) does not normalize the unique subalgebra of codimension 1 in \( H(2; 1)(\mu) \). Since \( H(2; 1)(\mu) \subset H(2; 1)^{(2)} \), this implies that in the present case any root in \( \Gamma(K, \overline{T}) \) is either solvable or improper. Moreover, the inequality \( |\Gamma_p(K, \overline{T})| \leq p - 1 \) holds.

Since \( \alpha_u \) is a proper Witt root in \( \Gamma(K, \overline{T_u}) \), the above discussion shows that the torus \( T_u \) is not conjugate to \( T_2 \). Hence \( T_u \) is conjugate to either \( T_0 \) or \( T_1 \). But then \( |\Gamma_p(K, \overline{T_u})| > p - 1 \) by our remarks earlier in the proof. The result follows.

(5) Case (5) of Theorem 2.2 cannot occur by our general assumption.

(6) Let \( K \) be as in case (6) of Theorem 2.2. Since \( \tilde{S} \) is a restricted ideal of Der \( \tilde{S} \), we have that \( t_u - t \in \tilde{S} \) for all \( t \in \overline{T} \). Since \( L(\alpha, \beta) \) is a 2-section for \( T_u \), the pair \((K, \overline{T_u})\) appears in Proposition 2.3(6). Hence we may assume that

\[
T_u = F(h \otimes 1) \oplus F\text{Id} \otimes (1 + x_1)\partial_1.
\]
For $k \in \mathbb{F}_p$ put $\hat{S}_k := \{x \in \hat{S} \mid [h, x] = kx\}$ and $S_k := \hat{S}_k \cap S$. Given $\mu \in \Gamma(K, \mathcal{T}_u)$ set $a = a(\mu) := \mu(h \otimes 1)$ and $b = b(\mu) := \mu(\text{Id} \otimes (1 + x_1)\partial_1)$. Then $K_\mu = K \cap (\hat{S}_a \otimes (1 + x_1)^b)$.

If $a = a(\mu) = 0$, then $K(\mu) \subset \mathfrak{c}_{\hat{S}}(h) \otimes \mathcal{O}(1; \underline{1})$. As $Fh$ is a maximal torus of $S$ and $\hat{S}/S$ is solvable, $K(\mu)$ is solvable too. Then $\mu$ is proper. If $a = a(\mu) \neq 0$, then $K(\mu) = \bigoplus_{i=0}^{p-1} K \cap (\hat{S}_a \otimes (1 + x_1)^i)$. The evaluation map $\text{ev} : \hat{S} \otimes \mathcal{O}(1; \underline{1}) \rightarrow \hat{S}$ taking $x \otimes f \in \hat{S} \otimes \mathcal{O}(1; \underline{1})$ to $f(0) \cdot x \in \hat{S}$ is a Lie algebra homomorphism. It is injective on $K(\mu)$ and the image $\text{ev}(K(\mu))$ is sandwiched between $S$ and $\hat{S}$. From this it is immediate that when $a = a(\mu) \neq 0$, the root $\mu$ is proper in $\Gamma(K, \mathcal{T}_u)$ if and only if either $S = \mathfrak{sl}(2)$ or $S$ is one of $W(1; \underline{1}), H(2; \underline{1})^{(2)}$ and $h$ normalizes the standard maximal subalgebra of $S$.

Since $\alpha$ is improper and $u \in L_\alpha$, the 1-section $K(\alpha) = K(\alpha_u)$ is neither solvable nor classical. Hence $a(\alpha_u) \neq 0$ and $S \neq \mathfrak{sl}(2)$. The above discussion now shows that $h$ normalizes the standard maximal subalgebra of $S$. This, in turn, yields that all roots in $\Gamma(K, \mathcal{T}_u)$ are proper. Consequently, $|\Gamma(K, \mathcal{T}_u)| > |\Gamma(K, \mathcal{T})|$.

(7) Let $K$ be as in case (7) of Theorem 2.2. If $S = W(1; 2)$, then [St 92, Sect. V] shows that all roots in $\Gamma(K, \mathcal{T}_u)$ are proper (it is proved in loc. cit. that for any two-dimensional torus $t$ in $S_p$ either $\Gamma(S_p, t) = \Gamma_p(S_p, t)$ or $\Gamma_p(S_p, t) = \emptyset$). Thus the result holds in this case.

If $S = H(2; (2, 1))^{(2)}$, then for any two-dimensional torus $t$ in $S_p$ all roots in $\Gamma(S_p, t)$ are solvable, hence proper; see [St 92, Theorem VII.3]. Since $\alpha$ is improper, this case cannot occur. If $S = H(2; 1; \Delta)$, then $Fx_1\partial_1 + Fx_2\partial_2$ is a toral Cartan subalgebra of $S_p$. Applying [P-St 99, Corollary 2.10] shows that $\mathcal{T}$ is a toral Cartan subalgebra of $S_p$ as well. Now [Wil 83, Proposition 4.9] gives the result. If $S = H(2; (2, 1))^{(2)}$, then [B-W 88, Lemma 10.1.1] applies and gives the result.

(8) Finally, suppose $K$ is as in case (8) of Theorem 2.2. If $K$ is classical, then all roots in $\Gamma(K, \mathcal{T})$ are classical, hence proper. Thus this case cannot occur. If $K = W(2; \underline{1})$, then $Fx_1\partial_1 + Fx_2\partial_2$ is a toral Cartan subalgebra of $K$. But then $\mathcal{T}$ is a toral Cartan subalgebra of $K$ too; see [P-St 99, Corollary 2.10]. As before, [Wil 83, Proposition 4.9] gives the result. If $K$ is one of $S(3; \underline{1})^{(1)}$, $H(4; \underline{1})^{(1)}$, $K(3; \underline{1})^{(1)}$, then the deliberations of [B-W 88, (5.8)] apply and yield the result. 

\begin{lemma}
Let $K = L[\alpha, \beta]$ and $\mathcal{T} = F(h \otimes 1) \oplus F(d \otimes 1 + \text{Id} \otimes t)$ be as in case (5) of Theorem 2.2 and Proposition 2.3. If $S$ is of Cartan type, let $S_{(0)}$ denote the standard maximal subalgebra of $S$. If $S = \mathfrak{sl}(2)$, put $S_{(0)} = S$. Let $u \in L_\alpha$ and $\mu \in \Gamma(K, \mathcal{T})$. Then the following hold:

1. If $\mu(h \otimes 1) = 0$, then $\mu$ is proper if and only if $t \in W(m; \underline{1})_{(0)}$.

2. Suppose $\mu(h \otimes 1) \neq 0$. If $t \notin W(m; \underline{1})_{(0)}$, then $\mu$ is proper if and only if $h \in S_{(0)}$. If $t \in W(m; \underline{1})_{(0)}$, then $\mu$ is proper if and only if the torus
\(\overline{T}\) normalizes the maximal compositionally classical subalgebra of \(S(\mu) \cong \overline{S}(\mu)/(S \otimes \mathcal{O}(m; 1))_{(1)}(\mu)\).

(3) If \(h \in S_{(0)}\) and \(t \in W(m; 1)_{(0)}\), then all roots in \(\Gamma(K, \overline{T})\) are proper.

(4) Assume that \(\alpha \notin \Gamma_p(L, T)\) and \(\alpha_u \in \Gamma_p(L, T_u)\). Then \(|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|\).

**Proof.** In proving this lemma, it will be convenient to work with a slightly more general two-dimensional torus

\[ \mathcal{R} := F(h \otimes 1) \oplus F(D + \text{Id} \otimes t), \]

where \(D \in (\text{Der} S) \otimes \mathcal{O}(m; 1)\), \(h\) and \(t\) are toral elements of \(S\) and \(W(m; 1)\), respectively, and \([D, h \otimes 1] = 0\). Recall that \(\overline{S} = S \otimes \mathcal{O}(m; 1)\) and \(m \in \{1, 2\}\).

(1) Let \(\mu \in \Gamma(K, \mathcal{R})\) be such that \(\mu(h \otimes 1) = 0\). Then \(\overline{S}(\mu) \subseteq \mathcal{O}(h) \otimes \mathcal{O}(m; 1)\).

Since \(Fh\) is a maximal torus of \(S\), the subalgebra \(\mathcal{O}(h)\) is nilpotent. Consequently, \(\overline{S}(\mu)\) is a nilpotent ideal of \(K(\mu)\). But then the preimage of \(\overline{S}(\mu)\) under the canonical homomorphism \(L(\alpha, \beta) \rightarrow K\) lies in the radical of \(L(\mu)\). Since the subalgebra \(\pi_2(K)^{(1)}\) of \(W(m; 1)\) is isomorphic to one of \(W(1, 1; 2)\), \(H(2, 1)^{(2)}\), it follows that \(\mu \in \Gamma(L, T)\) is proper if and only if \(Ft\) normalizes the standard maximal subalgebra of \(\pi_2(K)^{(1)}\).

If \(m = 1\), then \(\pi_2(K) = W(1, 1; 2)\). In this case \(\mu\) is proper if and only if \(t \in W(m; 1)_{(0)}\). Suppose \(m = 2\). Then the subalgebra \(\pi_2(K)^{(1)} \cap W(2; 1)_{(0)}\) has codimension \(\leq 2\) in \(\pi_2(K)^{(1)}\) and is either solvable or isomorphic to \(sl(2)\) modulo its radical (because \(sl(2)\) is the semisimple quotient of \(W(2, 1)_{(0)}\)). This shows that \(\pi_2(K)^{(1)} \cap W(2; 1)_{(0)}\) is the standard maximal subalgebra of \(\pi_2(K)^{(1)} \cong H(2, 1)^{(2)}\). Then again \(\mu \in \Gamma(L, T)\) is proper if and only if \(t \in W(m; 1)_{(0)}\).

(2) Now let \(\mu \in \Gamma(K, \mathcal{R})\) be such that \(\mu(h \otimes 1) \neq 0\). Then \(K_{ip} \subseteq \overline{S}\) for all \(i \in \mathbb{F}_p^*,\) whence \(K(\mu) = \mathcal{O}_K(\mathcal{R}) + S(\mu)\). As \(\mathcal{O}_K(\mathcal{R})\) is nilpotent, \(K(\mu)^{(\infty)} = S(\mu)^{(\infty)}\).

Combining this with Theorem 2.1(ii) we now derive that

\[ L[\gamma]^{(\infty)} \cong K(\gamma)^{(\infty)}/(K(\gamma)^{(\infty)} \cap \text{rad } K(\gamma)) \cong S[\mu]^{(\infty)}\]

This shows that \(\mu \in \Gamma_p(K, \mathcal{R})\) if and only if \(\mu \in \Gamma_p(\overline{S}, \mathcal{R})\). Recall that the evaluation map \(ev: \overline{S} = S \otimes \mathcal{O}(m; 1) \rightarrow S, \ x \otimes f \mapsto f(0) \cdot x\), is a Lie algebra homomorphism whose kernel \(S \otimes \mathcal{O}(m; 1)_{(1)}\) is a nilpotent ideal of \(\overline{S}\).

(a) Suppose \(t \notin W(m; 1)_{(0)}\). Then [P-St 99, Theorem 2.6] shows that there exists \(\sigma \in \text{Aut } \overline{S}\) such that \(\sigma(\mathcal{R}) = F(h' \otimes 1) \oplus \text{Id } \otimes (1 + x_1)\partial_1\) for some nonzero toral \(h' \in S\). Since \(\mathcal{R} \cap \overline{S} = F(h \otimes 1)\), we can assume (after rescaling \(h'\) if necessary) that \(\sigma(h) = h'\). Set \(\mu' := \mu \circ \sigma^{-1}\), an element in \(\Gamma(\overline{S}, \sigma(\mathcal{R}))\). Since \(\overline{S}(\mu') = \sigma(\overline{S}(\mu))\), we have that \(\mu \in \Gamma_p(\overline{S}, \mathcal{R})\) if and only if \(\mu' \in \Gamma_p(\overline{S}, \sigma(\mathcal{R}))\).

Put \(a := \mu'(h' \otimes 1)\) and \(b := \mu'(\text{Id } \otimes (1 + x_1)\partial_1)\). Note that \(a \in \mathbb{F}_p^*\) by our present assumption on \(\mu\). Let \(u \in S\) be such that \([h', u] = ru\). Then \(r \in \mathbb{F}_p^*\)
(for $h'$ is toral) and $u \otimes (1 + x_1)^{rb/a} \in \tilde{S}(\mu')$. From this it is immediate that the evaluation map takes $\tilde{S}(\mu')$ onto $S$. Since $\tilde{S}(\mu') \cap \ker \text{ev} \subset \text{rad} \tilde{S}(\mu')$, the maximal compositionally classical subalgebra of $\tilde{S}(\mu')$ is mapped onto that of $S$. It follows that $\mu' \in \Gamma_p(\tilde{S}, \sigma(\mathcal{R}))$ if and only if $h' \in S_{(0)}$. Since the subalgebra $S_{(0)}$ is invariant under all automorphisms of $S$, we obtain that in the present case $\mu \in \Gamma_p(L, T)$ if and only if $h \in S_{(0)}$.

(b) Now suppose $t \in W(m; \mathbb{1})_{(0)}$. Then $\mathcal{R}$ preserves the ideal $\ker \text{ev}$. As a consequence, the evaluation map induces a natural Lie algebra homomorphism $\Phi: \mathcal{R} + \tilde{S} \rightarrow \text{Der} S$. Since $\tilde{S}(\mu) \cap \ker \Phi \subset \text{rad} \tilde{S}(\mu)$, it is straightforward that $\mu \in \Gamma_p(\tilde{S}, \mathcal{R})$ if and only if $\Phi(\mathcal{R})$ normalizes the maximal compositionally classical subalgebra of $\Phi(\tilde{S}(\mu)) \subset S$. Since $\Phi(\tilde{S}(\mu)) \cong \tilde{S}(\mu)/(S \otimes \mathcal{O}(m; \mathbb{1})_{(1)})(\mu)$, part (2) follows.

(3) Next assume that $h \in S_{(0)}$ and $t \in W(m; \mathbb{1})_{(0)}$, and suppose that $\mu$ is an improper root in $\Gamma(K, \mathcal{R})$. Part (1) shows that $\mu(h \otimes 1) \neq 0$, while part (2b) says that $\Phi(\mathcal{R})$ does not normalize the maximal compositionally classical subalgebra of $S(\mu) \cong \tilde{S}(\mu)/(S \otimes \mathcal{O}(m; \mathbb{1})_{(1)})(\mu)$. In particular, $\tilde{S}_\mu \not\subset S \otimes \mathcal{O}(m; \mathbb{1})_{(1)}$ for some $j \in \mathbb{P}_a$. Note that $h \in \Phi(\mathcal{R})$. If $\Phi(\mathcal{R}) = Fh$, we have that $S = S(\mu)$. But then $h \in S_{(0)}$ normalizes the maximal compositionally classical subalgebra of $S(\mu)$, contrary to our choice of $\mu$.

Therefore, $\dim \Phi(\mathcal{R}) = 2$. This implies that $S = H(2; \mathbb{1})_{(2)}$. Since $\Phi(\mathcal{R}) \cap S = Fh \subset S_{(0)}$, it follows from [St 92, Proposition III.1.5] that $\Phi(\mathcal{R})$ is conjugate under $\text{Aut} S$ to the torus $F x_1 \partial_1 \oplus F x_2 \partial_2$. But then all roots in $\Gamma(S, \Phi(\mathcal{R}))$ are proper by [St 92, Theorem III.5], contrary to part (2b) and our choice of $\mu$. This contradiction proves that all roots in $\Gamma(K, \mathcal{R})$ are proper.

(4) We now apply the above results to $\overline{T}$. Assume that $\alpha \not\in \Gamma_p(L(\alpha, \beta), T)$ and $\alpha_u \in \Gamma_p(L(\alpha, \beta), T_u)$, where $u \in L_u$. Let $\bar{u}$ denote the image of $u$ in $K = L(\alpha, \beta)$. Regard $\alpha$ as a $\overline{T}$-root of $K$. Our assumption on $\alpha$ and $u$ implies that $\bar{u} \neq 0$ and $\mathbb{F}_p^* \alpha \subset \Gamma(K, \overline{T})$.

(a) Suppose $\alpha$ vanishes on $\overline{T} \cap \tilde{S}$. As $\alpha$ is improper, part (1) yields $t \not\in W(m; \mathbb{1})_{(0)}$. According to [P-St 99, Theorem 2.6], we can assume without loss of generality that $\overline{T} = F(h' \otimes 1) \oplus F \text{Id} \otimes (1 + x_1) \partial_1$. If $h' \not\in S_{(0)}$, then parts (1) and (2) yield $\Gamma_p(K, \overline{T}_u) = \emptyset$. Then $|\Gamma_p(K, \overline{T}_u)| \geq p - 1 > |\Gamma_p(K, \overline{T})|$, forcing $|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|$.

So from now we assume that $h' \in S_{(0)}$. Since $\alpha$ vanishes on $\overline{T} \cap \tilde{S}$, we have $\alpha(h' \otimes 1) = 0$. Note that the torus $\overline{T}_u$ is spanned by the elements $(h' \otimes 1)\bar{u}$ and $(\text{Id} \otimes (1 + x_1)\partial_1)\bar{u}$. As $\alpha(h' \otimes 1) = 0$, our discussion at the beginning of this section shows that $(h' \otimes 1)\bar{u} = h' \otimes 1$ and $\alpha_u(h' \otimes 1) = 0$. Besides, $(\text{Id} \otimes (1 + x_1)\partial_1)\bar{u} = D + \text{Id} \otimes t'$ for some $D \in (\text{Der} S) \otimes \mathcal{O}(m; \mathbb{1})$ and $t' \in W(m; \mathbb{1})$. Since $\alpha_u$ is proper and vanishes on $\overline{T}_u \cap \tilde{S} = F(h' \otimes 1)$, part (1) shows that
t' \in W(m; 1)_{(0)}$. But then part (3) implies that all roots in $\Gamma(K, \overline{T}_u)$ are proper, yielding $|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|$.

(b) Suppose $\alpha(h \otimes 1) \neq 0$. Then $\bar{u} \in [h \otimes 1, K_\alpha] \subset \tilde{S}_\alpha$. If $t \notin W(m; 1)_{(0)}$, then, as before, it can be assumed that $\overline{T} = F(h' \otimes 1) \oplus F\text{Id} \otimes (1 + x_1)\partial_1$ and $\alpha(h' \otimes 1) \neq 0$; see [P-St 99, Theorem 2.6]. Since $\alpha$ is improper, part (2) shows that $h \notin S_{(0)}$. But then all roots in $\Gamma(K, \overline{T})$ are improper, by (1) and (2), implying $|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|$.

So assume $t \in W(m; 1)_{(0)}$. Then all roots in $\Gamma(K, \overline{T})$ vanishing on $h' \otimes 1$ are proper by (1). Let $\Phi: \overline{T} + \tilde{S} \to \text{Der } S$ be the Lie algebra homomorphism from part (2b), so that $\Phi(\tilde{S}) = (S \otimes \mathcal{O}(m; 1))/\bigl(S \otimes \mathcal{O}(m; 1)\bigr) \cong S$, $\Phi(h' \otimes 1) = h'$, and $\Phi(d \otimes 1 + \text{Id} \otimes t) = d$. As $\ker \Phi$ is solvable, a root $\nu \in \Gamma(K, \overline{T})$ with $\nu(h' \otimes 1) \neq 0$ is proper if and only if $\Phi(\overline{T})$ normalizes the maximal compositionally classical subalgebra of the 1-section $S(\nu) \cong \tilde{S}(\nu)/\left(S \otimes \mathcal{O}(m; 1)\right)(\nu)$. Since $\alpha$ is improper, the above discussion shows that $\tilde{S}(\alpha) \not\subset S \otimes \mathcal{O}(m; 1)_{(1)}$.

Since $\bar{u} \in \tilde{S}_\alpha$ and $S$ is reductive, we have that $(h' \otimes 1)_{\bar{u}} \in \tilde{S}$ and $(\text{Id} \otimes t)_{\bar{u}} = D' + \text{Id} \otimes t$ for some $D' \in (\text{Der } S) \otimes \mathcal{O}(m; 1)$. Set $h'' := \Phi((h' \otimes 1)_{\bar{u}})$, the image of $(h' \otimes 1)_{\bar{u}}$ in $(S \otimes \mathcal{O}(m; 1))/\bigl(S \otimes \mathcal{O}(m; 1)\bigr)$. It is immediate from the proof of [P-St 99, Lemma 2.5] that $\overline{T}_u$ is conjugate under $\text{Aut } \tilde{S}$ to the torus $F(h'' \otimes 1) \oplus F(D'' + \text{Id} \otimes t)$ for some $D'' \in (\text{Der } S) \otimes \mathcal{O}(m; 1)$.

Suppose $\Phi(\overline{T}) = Fh'$. Then $S = S(\alpha) = S(\alpha_{\bar{u}})$. Since $\alpha_{\bar{u}}$ is a proper root, it must be that $h'' \in S_{(0)}$. Combining the preceding remark with (3) we now obtain that all roots in $\Gamma(K, \overline{T}_u)$ are proper. Then $|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|$.

Now suppose $\dim \Phi(\overline{T}) = 2$. Then $S = H(2; 1)_{(2)}$ and $\Phi(\overline{T})$ is conjugate under $\text{Aut } S$ to one of the tori $T_i$, $i = 0, 1, 2$, from part (4) of the proof of Lemma 3.1. By our earlier remarks, $\alpha$ is an improper root in $\Gamma(S, \Phi(\overline{T}))$. Therefore, $\Phi(\overline{T})$ is not conjugate to $T_0$ and $\alpha \in \Gamma(S, \Phi(\overline{T}))$ is Witt; see [St 92, Sect. III] for more detail.

If $\Phi(\overline{T})$ is conjugate to $T_1$, then part (4) of the proof of Lemma 3.1 shows that $\overline{T}_u$ normalizes $S_{(0)}$. Since the latter is invariant under all automorphisms of $S$, it follows that $h'' \in S_{(0)}$. But then all roots in $\Gamma(K, \overline{T}_u)$ are proper by (3). If $\Phi(\overline{T})$ is conjugate to $T_2$, then part (4) of the proof of Lemma 3.1 implies that there is a $\kappa \in \Gamma(K, \overline{T})$ such that $\Gamma_p(K, \overline{T}) \subset \mathbb{F}_p^\kappa$. Since $\alpha_u$ is a proper Witt root in $\Gamma(S, \Phi(\overline{T}_u))$, part (4) of the proof of Lemma 3.1 also shows that $|\Gamma_p(S, \Phi(\overline{T}_u))| > p - 1$. But then $|\Gamma_p(K, \overline{T})| \leq p - 1 < |\Gamma_p(S, \Phi(\overline{T}_u))| \leq |\Gamma_p(K, \overline{T}_u)|$, proving that $|\Gamma_p(L(\alpha, \beta), T_u)| > |\Gamma_p(L(\alpha, \beta), T)|$ in all cases.

\begin{proof}

The proof of [B-W 88, Proposition 10.4.1] applies without changes, since for that proof one only needs the conclusions of Lemmas 3.1 and 3.2.
\end{proof}

\textbf{Theorem 3.3.} If $T$ is an optimal torus in $L_p$, then all roots in $\Gamma(L, T)$ are proper.

\begin{proof}

The proof of [B-W 88, Proposition 10.4.1] applies without changes, since for that proof one only needs the conclusions of Lemmas 3.1 and 3.2.
\end{proof}
Corollary 3.4. Let $T$ be an optimal torus in $L_p$. With the notations of Theorem 2.2 we have the following description of $\mathcal{T}$ in the respective cases of that theorem:

1. $\mathcal{T} = (0)$.

2. $\mathcal{T} \subset S$ is conjugate under an automorphism of $S$ to $Fx_1\partial_1$ if $S = W(1;\underline{1})$ and to $F(x_1\partial_1 - x_2\partial_2)$ if $S = H(2;\underline{1})^{(2)}$.

3. $\mathcal{T} = Fh_1 \oplus Fh_2$ where $h_i \in S_i$. Moreover, for $i = 1, 2$, the torus $Fh_i$ is conjugate under $\text{Aut} S_i$ to $Fx_1\partial_1$ if $S_i = W(1;\underline{1})$ and to $F(x_1\partial_1 - x_2\partial_2)$ if $S_i = H(2;\underline{1})^{(2)}$.

4. $\mathcal{T}$ is conjugate under an automorphism of $S$ to the torus $Fx_1\partial_1 \oplus Fx_2\partial_2$.

5. Let $\{e_0, h_0, f_0\}$ be a standard basis of $\mathfrak{sl}(2)$. For $s = 1, 2$, let $y_1, \ldots, y_s$ be the generating set of $\mathfrak{O}(s;\underline{1})$ contained in $\mathfrak{O}(s;\underline{1})^{(2)}$, and let $D_1, \ldots, D_s \in W(s;\underline{1})$ be such that $D_i(y_j) = \delta_{ij}$ for all $1 \leq i, j \leq s$. Then $\mathcal{T}$ is conjugate under $\text{Aut} \tilde{S}$ to one of the following tori:

   - $\text{span}\{h_0 \otimes 1, \text{Id} \otimes x_1\partial_1\}$ if $m = 1$ and $S = \mathfrak{sl}(2)$,
   - $\text{span}\{y_1D_1 \otimes 1, \text{Id} \otimes x_1\partial_1\}$ if $m = 1$ and $S = W(1;\underline{1})$,
   - $\text{span}\{(y_1D_1 - y_2D_2) \otimes 1, r(y_1D_1 + y_2D_2) \otimes 1 + \text{Id} \otimes x_1\partial_1\}$ with $r \in \mathbb{F}_p$, if $m = 1$ and $S = H(2;\underline{1})^{(2)}$.

   - $\text{span}\{h_0 \otimes 1, \text{Id} \otimes (x_1\partial_1 - x_2\partial_2)\}$ if $m = 2$ and $S = \mathfrak{sl}(2)$,
   - $\text{span}\{y_1D_1 \otimes 1, \text{Id} \otimes (x_1\partial_1 - x_2\partial_2)\}$ if $m = 2$ and $S = W(1;\underline{1})$,
   - $\text{span}\{(y_1D_1 - y_2D_2) \otimes 1, r(y_1D_1 + y_2D_2) \otimes 1 + \text{Id} \otimes (x_1\partial_1 - x_2\partial_2)\}$, $r \in \mathbb{F}_p$, if $m = 2$ and $S = H(2;\underline{1})^{(2)}$.

6. $\mathcal{T} = F(\text{Id} \otimes 1 + x_1)\partial_1$ where $h \in S \setminus (0)$ and $h^{[p]} = h$. If $S$ is of Cartan type, then $h \in S_{(0)}$.

7. $\mathcal{T} \subset S_p$ and $\dim \mathcal{T} \cap S = q$ where $q = 0$ if $S = H(2;\underline{1};\Phi(\tau))^{(1)}$ and $q = 1$ otherwise;

8. $\mathcal{T} \subset S_{(0)}$, where $S_{(0)} = S$ if $S$ is classical and $S_{(0)}$ is the standard maximal subalgebra of $S$ if $S$ is of Cartan type.

Proof. (1) is clear.

(2) According to Proposition 2.3, one has $\mathcal{T} \subset S$ and $\dim \mathcal{T} = 1$. Therefore, $S \cong L[\gamma]^{(\infty)}$ for some $\gamma \in \Gamma(L, T)$. Suppose $S$ is of Cartan type. By Theorem 3.3, $\gamma$ is a proper $T$-root. It follows that $\mathcal{T} \subset S_{(0)}$. The statement now follows from Demushkin’s theorem; see [St 04, (7.5)]. Part (3) is analogous to (2). Part (4) follows from [St 92, Theorem III.5(1)].

(5) By Proposition 2.3, $\mathcal{T}$ is conjugate under $\text{Aut} \tilde{S}$ to $F(h \otimes 1 + F(d \otimes 1 + \text{Id} \otimes t)$, where $h \in S$, $t \in \pi_2(K) \subset W(m;\underline{1})$ are nonzero toral elements, $d$ is a toral element of $\text{Der} S$, and $[h, d] = 0$. Since all roots in $\Gamma(K, \mathcal{T})$ are proper, by
Theorem 3.3, Lemma 3.2 implies that \( t \in W(m; 1)_{(0)} \) and \( h \in S_{(0)} \). Put \( \mathcal{R} := Fh + Fd \), a torus in \( \text{Der} S \).

If \( \dim \mathcal{R} = 1 \), then \( d \) is a scalar multiple of \( h \); so we can assume that \( d = 0 \). In this case there exist \( \phi \in \text{Aut} \mathcal{O}(m; 1) \) satisfying \( \phi \circ H(2; 1)_{(2)} \circ \phi^{-1} = H(2; 1)_{(2)} \) if \( m = 2 \) and \( \sigma \in \text{Aut} S \) such that \( \sigma(h) \) and \( \phi \circ t \circ \phi^{-1} \) are nonzero multiples of \( h_0 \) and \( x_1 \partial_1, y_1 D_1 \) and \( x_1 \partial_1, y_1 D_1 - y_2 D_2 \) and \( x_1 \partial_1, h_0 \) and \( x_1 \partial_1 - x_2 \partial_2, y_1 D_1 \) and \( x_1 \partial_1 - x_2 \partial_2, y_1 D_1 - y_2 D_2 \) and \( x_1 \partial_1 - x_2 \partial_2 \) in the six respective cases (when \( S = H(2; 1)_{(2)} \) and \( m = 2 \), one should also keep in mind Demushkin’s theorem mentioned above). Clearly, the desired normalization can be achieved with the help of \( \sigma \otimes \phi \in \text{Aut} \tilde{S} \). Note that \( r = 0 \) when \( \dim \mathcal{R} = 1 \).

Now consider the case where \( \dim \mathcal{R} = 2 \). Since all \( T \)-roots of \( L \) are proper, so are all \( \mathcal{R} \)-roots of \( S = H(2; 1) \); see Lemma 3.2(2). If \( m = 1 \) (resp., \( m = 2 \)), then \( t \) is a nonzero toral element in \( W(1; 1)_{(0)} \) (resp., in \( H(2; 1)_{(2)}_{(0)} \)). As before, there exists \( \phi \in \text{Aut} \mathcal{O}(m; 1) \) satisfying \( \phi \circ H(2; 1)_{(2)} \circ \phi^{-1} = H(2; 1)_{(2)} \) if \( m = 2 \) such that \( \phi \circ t \circ \phi^{-1} = x_1 \partial_1 \) (resp., \( \phi \circ t \circ \phi^{-1} = x_1 \partial_1 - x_2 \partial_2 \)) when \( m = 1 \) (resp., \( m = 2 \)). According to [St 92, Theorem III.5(1)], there exists \( \sigma \in \text{Aut} S \) such that

\[
\sigma(\mathcal{R}) = F(y_1 D_1) \oplus F(y_2 D_2), \quad \sigma(h) \in \mathbb{F}_p^{*}(y_1 D_1 - y_2 D_2).
\]

Subtracting a multiple of \( h \) from \( d \) if necessary we may assume that \( \sigma(d) = r(y_1 D_1 + y_2 D_2) \) for some \( r \in F^{*} \). Since \( \overline{T} \) is two-dimensional and contains \( (d \otimes 1 + \text{Id} \otimes t)^p = d^{[p]} \otimes 1 + \text{Id} \otimes t \), it can only be that \( r^p = r \), that is \( r \in \mathbb{F}_p^{*} \). As before, the desired normalization can now be achieved with the help of \( \sigma \otimes \phi \in \text{Aut} \tilde{S} \).

Part (6) is analogous to (2). Part (7) has already been proved; see Proposition 2.3.

(8) Assume that \( S \) is a restricted Lie algebra of Cartan type. As all \( \overline{T} \)-roots of \( S \) are proper by Theorem 3.3, the statement follows from the discussion in [St 92, Sect. IX].

We now have generalized the main results of [St 89b] to the case where \( p > 3 \). In particular, we have classified all \( T \)-semisimple quotients of 2-sections that occur in \( L \); see Theorem 2.2. In fact, our list is more precise than the list in [St 92, Theorem II.2] which is the revised version of [St 89b, Theorem 6.3]. For \( p > 3 \), all roots with respect to optimal tori in \( L_p \) are proper (Theorem 3.3). We recall that our definition of “properness” differs from that introduced by Block and Wilson.

4. The subalgebra \( Q(L, T) \)

Our next goal is to show that all deliberations of [B-O-St] are valid for \( p > 3 \). Theorem 1.15 of [B-O-St] can now replaced by the stronger Theorem 2.2 of the present work, and Theorem 1.16 of [B-O-St] can be substituted by the stronger Corollary 3.4.
Lemmas 2.1–2.4 of [B-O-St] generalize easily to the case where \( p > 3 \): the proofs of Lemmas 2.1 and 2.2 require no changes, while Lemmas 2.3 and 2.4 are even easier to prove now, as we acquired more information on \( T \).

Let us look at the proof of [B-O-St, Lemma 2.5]. Suppose \( S \sim W(1; 2) \). Since all roots are proper, \( T \) cannot be as in [St 92, Theorems V.2 and V.3]. So \( T \) is as in [St 92, Theorem V.4], hence the result follows. Suppose \( S \sim H(2; 1; \Phi(\tau))^{(1)} \). Then [St 92, Theorem VII.3] yields that all 1-sections of \( S \) are solvable. This is the claim. Suppose \( S \sim H(2; (2, 1))^{(2)} \). Then Corollary VI.3 and Theorem VI.4 of [St 92] imply the statement of [B-O-St, Lemma 2.5] in this case. Note that, apart from straightforward computations, only [B-W 88, (10.1.1)] is used in [St 92, Sect. VI], and that holds for \( p > 3 \). When \( S \sim H(2; 1; \Delta) \), the proof of Lemma 2.5 in [B-O-St] relies only on elementary observations and [B-W 88, (11.1.3)], which holds for \( p > 3 \).

The proof of Lemma 2.6 in [B-O-St] requires no changes (for a more elaborate computation see [St 92, Sect. IX]). Corollary 2.7 of [B-O-St] holds for \( p > 3 \) too.

Thus all results of [B-O-St, Sect. 2] hold for \( p > 3 \). It is now a matter of routine to check that all results of [B-O-St, Sect. 3] remain valid for \( p > 3 \) as well. As a consequence, we obtain the following:

**Theorem 4.1.** Suppose all tori of maximal dimension in \( L_p \) are standard and let \( T \) be an optimal torus in \( L_p \). If \( \alpha \in \Gamma(L, T) \) is classical or solvable, define \( Q_\alpha := L_\alpha \). If \( \alpha \in \Gamma(L, T) \) is Witt or Hamilton, denote by \( L[\alpha]_{(0)} \) the standard maximal subalgebra of \( L[\alpha] \) and define

\[
Q_\alpha := \psi_\alpha^{-1}(L[\alpha]_{(0)}) \cap L_\alpha,
\]

where \( \psi_\alpha : L(\alpha) \rightarrow L[\alpha] \) stands for the canonical homomorphism. Then

\[
Q = Q(L, T) := H \oplus \bigoplus_{\gamma \in \Gamma(L, T)} Q_\gamma
\]

is a \( T \)-invariant subalgebra of \( L \).

Note that \( L = Q \) if and only if all roots in \( \Gamma(L, T) \) are solvable or classical. As another consequence, we obtain:

**Theorem 4.2.** Let \( T \) be an optimal torus in \( L_p \). Given \( \alpha, \beta \in \Gamma(L, T) \) set \( Q(\alpha, \beta) := Q \cap L(\alpha, \beta) \) and let \( J(\alpha, \beta) \) denote the maximal solvable ideal of \( T + Q(\alpha, \beta) \). Let \( g_{\alpha, \beta} : T + Q(\alpha, \beta) \rightarrow (T + Q(\alpha, \beta))/J(\alpha, \beta) \) be the canonical homomorphism, and set \( M := g_{\alpha, \beta}(Q(\alpha, \beta)) \). Then one of the following hold:

(A) \( M = (0) \);
(B) \( M \cong \mathfrak{sl}(2) \);
(C) \( M \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \);
(D) \( M \cong \mathfrak{sl}(2) \otimes O(1; 1) \);
(E) \( M \cong H(2; 1; \Phi(\tau))^{(1)} \);

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(F) \( M \) is classical of type \( A_2, B_2 \) or \( G_2 \).

The main result of [P-St 04] tells us that if \( L = Q \), then \( L \) is either classical or of Cartan type. So from now on we will assume that \( Q(L,T) \neq L \) for any optimal torus \( T \subset L_p \).

Our next goal is to show that \( T + Q \) is a maximal subalgebra of \( T + L \subset L_p \). To do this we will need a result on roots in \( \Gamma(L,T) \) sticking out of the subalgebra \( Q \). The union of all such roots is denoted by \( \Phi_- \) in [St 93]. The set \( \Phi_- \cap \Gamma(L(\alpha,\beta),T) \) is described in [St 93, Theorem 1.9] for all types of 2-sections \( K = L[\alpha,\beta] \) that occur in Theorem 2.2. It should be stressed here that the proof of Theorem 1.9 in [St 93] only relies on [B-W 88, (10.1.1)], [B-O-St, Lemma 2.5] and [St 92, III, IV.4, IV.5, V.4, VII, IX]. Therefore, it holds for \( p > 3 \).

**Theorem 4.3.** \( T + Q \) is a maximal subalgebra of \( T + L \).

**Proof.** If \( T + Q = T + L \), then all roots in \( \Gamma(L,T) \) are solvable or classical. Since this case has been excluded, \( T + Q \) is a proper subalgebra of \( T + L \). Let \( G \) be a subalgebra of \( T + L \) such that \( T + Q \subset G \). Note that \( G \) is \( T \)-invariant. There exists a nonsolvable, nonclassical root \( \alpha \in \Gamma(L,T) \) such that \( G \cap L(\alpha) \neq Q(\alpha) \). As \( Q(\alpha) \) is a maximal subalgebra of \( L(\alpha) \), it must be that \( L(\alpha) \subset G \).

(a) Pick \( \beta \in \Gamma(L,T) \setminus \mathbb{P}_p^\alpha \) and consider the 2-section \( L(\alpha,\beta) \) and its \( T \)-semisimple quotient \( K = L[\alpha,\beta] \). We let \( \psi : L(\alpha,\beta) \to K \) denote the canonical homomorphism, and put \( \overline{G} := \psi(G \cap L(\alpha,\beta)) \).

We will now go through the eight cases of Theorem 2.2. If \( K \) is as in case (6) or as in case (7) with \( S \) being one of \( W(1;2) \) or \( H(2; (2,1))^{(2)} \), then \( \Gamma(K,\overline{T}) \) contains a root \( \delta \) which vanishes on \( \psi(H) \). Indeed, in case (6) this is easily deduced from Corollary 3.4(6), so assume that we are in case (7). Combining [St 93, Theorem 1.9] with Theorem 2.2(7) one observes that \( \psi(H) \) normalizes the standard maximal subalgebra \( S_{(0)} = \psi(Q(\alpha,\beta)) \cap K^{(1)} \). Since the subalgebras \( W(1;2)_{(0)} \) and \( H(2; (2,1))_{(0)} \) are restricted, Theorem 2.2(7) yields that \( TR(\psi(H),K) = 1.0 \). Hence \( T \cap \psi(H)_p \) is spanned by a single toral element. On the other hand, it follows from [St 92] that \( |\Gamma(K,\overline{T})| = p^2 - 1 \) in the present case. This shows that there is a root \( \delta \in \Gamma(K,\overline{T}) \) which vanishes on \( \psi(H) \).

The 1-section \( L(\delta) \) is nilpotent, hence \( K(\delta) \subset \psi(Q(\alpha,\beta)) \subset \overline{G} \). By [B-W 88, (10.1.1), (11.1.1)] and [St 92, Theorem IV.5(3)], the root space \( K_\delta \) contains an element \( x \) with \( \alpha(x^{[p]}) \neq 0 \). The adjoint endomorphism \( \text{ad} x \) acts invertibly on each subspace \( \bigoplus_{i \in \mathbb{P}_p} K_{s\alpha + \iota \delta} \) with \( s \in \mathbb{F}_p^* \). As \( L(\alpha) \cup L(\delta) \subset \overline{G} \), this yields \( K \subset \overline{G} \).

(b) Now suppose that \( K \) has type different from the ones considered in part (a). As \( \alpha \) is Witt or Hamiltonian, it must be that \( \Gamma(K/\overline{G},\overline{T}) \subset \Phi_- \setminus \mathbb{P}_p^\alpha \). The description of \( \Phi_- \cap \Gamma(L(\alpha,\beta),T) \) in [St 93, Theorem 1.9] now entails that \( \Gamma(K/\overline{G},\overline{T}) \) has one of the following types:

\[ \emptyset, \ \{\mu\}, \ \{\pm \mu\}, \ \{\mu,\nu\}, \ \{\mu,\pm \nu\}, \]
where \( \mu \) and \( \nu \) are \( \mathbb{F}_p \)-independent roots in \( \Gamma(L(\alpha \beta), T) \setminus \mathbb{F}_p \alpha \). Consequently, the \( \alpha \)-string through each of the displayed roots contains no more than two roots from \( \Gamma(K/G, T) \). As \( p \geq 5 \), there exists \( j_0 \in \mathbb{F}_p^* \) such that \([K_{j_0 \alpha}, K] \subset \overline{G}\). On the other hand, the derived subalgebra of \( K(\alpha)/\text{rad} K(\alpha) \) is isomorphic to either \( W(1; \underline{1}) \) or \( H(2; \underline{1}^{(2)}) \); see Theorem 2.1(ii). From this it is immediate that \( K(\alpha) = \ker \tau + \text{rad} K(\alpha) \), where \( \tau \) denotes the representation of \( K(\alpha) \) in \( K/G \) induced by the adjoint action of \( L \).

We now interpret this information globally. Set \( N := L(\alpha)^{(\infty)} \), a subalgebra of \( G \). Let \( \beta \) be any \( T \)-root of \( L \) with \( \beta \not\in \mathbb{F}_p \alpha \). If \( K = L[\alpha, \beta] \) is as in part (a) of this proof, then \( K = \overline{G} \), forcing \( L(\beta) \subset G \). Otherwise, \([N, L(\beta)] \subset G \) by the discussion above. As a result, \([N, T + L] \subset G \). Applying [B-O-St, Lemma 4.1] (which holds in all characteristics), we now deduce that \( N \) acts nilpotently on \( L \). But then \( N \) acts nilpotently on itself contrary to the fact that \( N = N^{(1)} \neq (0) \). This contradiction shows that \( G = T + L \) and hence that \( T + Q \) is a maximal subalgebra of \( T + L \).

**Corollary 4.4.** If \( Q = Q(L, T) \) is solvable, then \( L \cong W(1; \underline{n}) \) for some \( \underline{n} \) and \( Q \subset L_{(0)} \cong W(1; \underline{n})_{(0)} \).

**Proof.** By Theorem 4.3, \( Q \) is a maximal \( T \)-invariant subalgebra of \( L \). As \( Q \) is solvable, [St 04, Corollary 9.2.13] says that \( L \) is one of \( \mathfrak{s}(2), W(1; \underline{n}), H(2; \underline{n}, \Phi)^{(2)} \) (up to isomorphism). Moreover, the proof of Corollary 9.2.13 in [St 04] shows that either \( L = \mathfrak{s}(2) \) or \( Q \) is a Cartan subalgebra of \( L \) or \( L = W(1; \underline{n}) \) and \( Q \subset W(1; \underline{n})_{(0)} \). Since \( Q \neq L \), the first possibility cannot occur. If \( Q \) is a Cartan subalgebra of \( L \), then every 1-section of \( Q \) relative to \( T \) is nilpotent. But then it immediate from the definition of \( Q \) and Theorem 2.1(ii) that every 1-section of \( L \) relative to \( T \) is solvable. However, in this case \( Q = L \) which is impossible as \( L \) is simple. Thus, \( L = W(1; \underline{n}) \) and \( Q \subset W(1; \underline{n})_{(0)} \) as stated. \( \square \)

5. The associated graded algebra

In this section we will go through [St 93] in order to extend the results there to our present situation. All references to [St 93] will underlined; for example, Lemma 2.3 will indicate that we refer to Lemma 2.3 of [St 93]. We will adopt the notation of [St 93]; in particular, \( \Phi = \Phi(L, T) \) will denote the set \( \Gamma(L, T) \cup \{0\} \).

Lemma 1.1 and Corollary 1.2 are valid in all characteristics; see [St 04, §1.2]. Theorem 1.3 holds for \( p > 3 \); see Theorem 2.1(ii). Theorem 1.4 is our Theorem 2.2, and Theorem 1.5 is covered by the stronger Corollary 3.4 of the present paper. Theorem 1.6 is easily deduced from our Theorem 4.3 and Corollary 4.4, while Theorem 1.7 is our Theorem 4.2. Lemma 1.8 is often referred to as Schue’s lemma; it holds all characteristics. Theorem 1.9 holds for \( p > 3 \); see our discussion before the proof of Theorem 4.3. Corollary 1.10 follows from Theorem 1.9, hence remains valid in our present setting. The proofs of Theorem 1.11 and Theorem 1.12 only need Theorems 1.4 and 1.7. Therefore, these results remain
true for \( p > 3 \). Lemma 1.13 does not require any restrictions on \( p \); see [St 04, Proposition 1.3.7 and Corollary 1.3.8]. Lemma 1.14 (which is a reformulation of [B-W 88, (3.1.2)]) does not need any restrictions on \( p \) either. Summarizing, all results of [St 93, Sect. 1] are valid for \( p > 3 \).

The proofs of Lemmas 2.1 and 2.2 work for \( p > 3 \). Note that Lemma 2.2(4) holds if \( \mu \in \Delta \). Indeed, if \( \mu \) vanishes on \( H \), then [P-St 04, Theorem 3.1] implies that \( L(\mu) \) acts triangulably on \( L \). However, one has to make some changes on pp. 15, 16 of [St 93].

Lemma 2.3 (new parts).

(3) If \( TR(H, L(\alpha, \beta, \kappa)) \geq 2 \), then \( \dim L(\alpha, \beta, \kappa)/Q(\alpha, \beta, \kappa) \leq 4p \).

(4) Assume that \( \kappa \in \Phi_{[-1]} \). If \( \sum_{i,j \in \mathbb{F}_p} \dim L_{\kappa+i\alpha+j\beta}/Q_{\kappa+i\alpha+j\beta} \geq 5p \), then

(a) \( \alpha, \beta \in \Delta \);

(b) \( \kappa(Q_{\alpha+j\beta}) \neq 0 \) for all \( i, j \in \mathbb{F}_p \);

(c) \( L_{\kappa+i\alpha+j\beta} \nsubseteq Q \) for all \( i, j \in \mathbb{F}_p \);

(d) \( \sum_{i,j \in \mathbb{F}_p} \dim L_{\kappa+i\alpha+j\beta}/Q_{\kappa+i\alpha+j\beta} = p^2 \).

Proof. (3) If \( L_{i\kappa+j\alpha+k\beta} \subset Q \) for all nonzero \( (i, j, k) \in \mathbb{F}_p^3 \), then we are done. Thus, replacing \( \kappa \) by a suitable root in \( \mathbb{F}_p^*\kappa + \mathbb{F}_p\alpha + \mathbb{F}_p\beta \), we may assume that \( L_\kappa \nsubseteq Q \). Since \( TR(H, L(\alpha, \beta, \kappa)) \geq 2 \), we may assume, renaming \( \alpha \) if necessary, that \( \alpha \) and \( \kappa \) are are linearly independent on \( H \). We take \( \beta \in \Delta \) if \( TR(H, L(\alpha, \beta, \kappa)) \leq 2 \), and we take \( \beta \) to be independent of \( \alpha \) and \( \kappa \) as functions on \( H \) if \( TR(H, L(\alpha, \beta, \kappa)) \geq 3 \).

Given \( \ell \in \mathbb{F}_p \) put \( \rho_\ell := \alpha + \ell \beta \). Then in both cases \( \kappa \) and \( \rho_\ell \) are \( \mathbb{F}_p \)-independent as functions on \( H \). Since \( L(\alpha, \beta, \kappa) = \sum_{\ell \in \mathbb{F}_p} L(\kappa, \rho_\ell) + L(\kappa, \beta) \), we have that

\[
\dim L(\alpha, \beta, \kappa)/Q(\alpha, \beta, \kappa) \leq \sum_{\ell \in \mathbb{F}_p} \left( \dim L(\kappa, \rho_\ell)/Q(\kappa, \rho_\ell) - \dim L(\kappa)/Q(\kappa) \right) + \dim L(\kappa, \beta)/Q(\kappa, \beta).
\]

As \( L_\kappa \nsubseteq Q \), the root \( \kappa \) is either Witt or Hamiltonian. It follows that \( \kappa \) vanishes on \( H \cap \text{rad}_T L(\kappa, \rho_\ell) \subset H \cap \text{rad}_L(\kappa) \) for all \( \ell \in \mathbb{F}_p \). If \( \rho_\ell \) does not vanish on \( H \cap \text{rad}_T L(\kappa, \rho_\ell) \), then \( L_{i\kappa+j_\rho+K_\ell} \subset \text{rad}_T L(\kappa, \rho_\ell) \) for all \( i \in \mathbb{F}_p \) and \( j \in \mathbb{F}_p^* \). Then \( L[\kappa, \rho_\ell] = L[\kappa, \rho_\ell](\kappa) \) is of type (2); see Theorem 2.2. If both \( \kappa \) and \( \rho_\ell \) vanish on \( H \cap \text{rad}_T L(\kappa, \rho_\ell) \), then they are linearly independent as elements in \( \Gamma(L[\kappa, \rho_\ell], T) \), forcing \( TR(\psi(H), L[\kappa, \gamma_\ell]) = 2 \). In this situation Theorem 2.2 shows that \( L[\kappa, \rho_\ell] \) cannot be be of type \( (1) \), \( (2) \) or \( (6) \). If \( L[\kappa, \rho_\ell] \) is of type \( (7) \) and \( S \) is one of \( W(1; 2, H(2; 2, 1; 1; 1, \Delta)) \), then one of the roots in \( \Gamma(L[\kappa, \rho_\ell], T) \) vanishes on \( \psi(H) \); see [St 92, (V.4), (V5), (VI.4), (VII.4)]. Since in the present situation \( \kappa \) and \( \rho_\ell \) are linearly independent on \( \psi(H) \), this is not the case.

Thus, each 2-section \( L[\kappa, \rho_\ell] \) must be of type \( (2), (3), (4), (5), (8) \) or of type \( (7) \) with \( S = H(2; 1, \Delta) \). Theorem 1.9 now implies the following: If \( \kappa \) is Witt,
then only $\kappa$ and at most three more roots stick out of $Q(\kappa, \rho)$. Also, at most $p$ roots stick out of $Q(\kappa, \beta)$. Then $\dim L(\alpha, \beta, \kappa)/Q(\alpha, \beta, \kappa) \leq 3p + p = 4p$. If $\kappa$ is Hamiltonian, then only $\pm \kappa$ and at most two more roots stick out of $Q(\kappa, \rho)$, whereas the number of roots sticking out $Q(\kappa, \beta)$ is at most $2p$. In this case we have $\dim L(\alpha, \beta, \kappa)/Q(\alpha, \beta, \kappa) \leq 2p + 2p = 4p$. The claim follows.

(4) Let $n$ be the number of 2-sections $L(\kappa, \mu)$ with $\mu \in \mathbb{F}_p\alpha + \mathbb{F}_p\beta$ such that $L[\kappa, \mu]$ is either of type (6) or of type (7) with $S$ being $W(1; 2)$ or $H(2; (2, 1))^{(2)}$. Our present assumption on $\kappa$ is slightly weaker than that in the original Lemma 2.3(4). Arguing as in the original proof, we obtain that $5p \leq (3 + n)p + 3(1 - n)$. Then $(n - 2)p \geq 3(n - 1)$, forcing $n \notin \{0, 1, 2\}$. Hence $n \geq 3$, and we can proceed as in the original proof; see [St 93, p. 16].

Lemma 2.4 is an immediate consequence [St 91b, Proposition 2.2] which, in turn, relies on [St 92, Theorem IV.3(1)], some standard considerations, and a version of our Corollary 3.4 (proved in [St 92]). Since [St 92, Theorem IV.3] is true in all characteristics, Lemma 2.4 holds for $p > 3$.

**Lemma 2.5** (new proof). Suppose $\beta \in \Phi$ and $\alpha \in \Phi \setminus \Delta$, and put $J := J(Q, T)$. Then $[J_\alpha, Q_\beta]$ acts triangulably on $L$.

**Proof.** First suppose that $\alpha = i\mu$ and $\beta = j\mu$ for some $\mu \in \Phi$ and $i, j \in \mathbb{F}_p$. Then $\mathbb{F}_p\mu \cap \Delta = \emptyset$. If $i = -j$, then Lemma 2.2(4) yields the assertion. If $i \neq -j$, the assertion follows from Lemma 2.2(3). So from now on we may assume that $\alpha$ and $\beta$ are $\mathbb{F}_p$-independent.

Suppose the assertion is not true. Then there is $\kappa \in \Phi_-$ with $L_\kappa \not\subset Q$ and $\kappa([J_\alpha, Q_\beta]) \neq 0$. We have that $L_{\kappa + \alpha + \beta} = ([J_\alpha, Q_\beta], L_\kappa) \not\subset Q$. Lemma 2.3(1) now shows that $TR(H, L(\alpha, \beta, \kappa)) \geq 2$. The new version of Lemma 2.3(3) then yields

$$\sum_{i,j \in \mathbb{F}_p^*} \dim L_{\kappa + i\alpha + j\beta}/Q_{\kappa + i\alpha + j\beta} \leq 4p,$$

implying that the adjoint $(T + Q(\alpha, \beta))$-module $\bigoplus_{i,j \in \mathbb{F}_p} L_{\kappa + i\alpha + j\beta}$ has a composition factor of dimension $< p^2$. We call it $V$ and denote by $\rho$ the corresponding representation of $T + Q(\alpha, \beta)$ in $\mathfrak{gl}(V)$.

In view of Lemma 2.4, $J \cap Q(\alpha, \beta)$ is a solvable ideal of $T + Q(\alpha, \beta)$. If $(J \cap Q(\alpha, \beta))^{(1)} \subset \ker \rho$, we set $I := J \cap Q(\alpha, \beta)^{(1)} + \ker \rho$. Otherwise, choose $k \geq 1$ maximal subject to the condition $(J \cap Q(\alpha, \beta))^{(k)} \not\subset \ker \rho$, and set $I := (J \cap Q(\alpha, \beta))^{(k)} + \ker \rho$. Since $[J_\alpha, Q_\beta] \not\subset \ker \rho$, it follows that in either case $I$ is an ideal of $Q(\alpha, \beta)$ satisfying $I^{(1)} \subset \ker \rho$ and $I \not\subset \ker \rho$.

It is easy to see that $I$ acts nilpotently on $Q(\alpha, \beta)/\ker \rho$. As $Q_\beta \not\subset \ker \rho$ and $Q_{\alpha + \beta} \not\subset \ker \rho$, we have that $\alpha(I_\gamma) = \beta(I_\gamma) = 0$ for all $\gamma \in \Phi$. By general representation theory, there is a linear function $\chi$ on $I$ such that $\rho(u) - \chi(u)1_{\mathfrak{gl}(V)}$ is a nilpotent operator for all $u \in I$. It is immediate from the preceding remark that $\chi(u) = \kappa(u)$ for all $u \in \bigcup_{\gamma \in \Phi} I_\gamma$. 

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Parts (a) and (b) of the original proof of Lemma 2.5 do not use any restriction on $p$. So let us take a look at part (c): If there exists a nonzero $\lambda \in \Phi$ with $I_\lambda \not\subseteq \ker \rho$, then part (b) of the original proof shows that $I_\lambda + \ker \rho$ is an ideal of $T + Q(\alpha, \beta)$ acting nonnilpotently on $V$. The above discussion then yields that $\kappa(I_\lambda) \neq 0$. Since $[I, Q(\alpha, \beta)] \subset \ker \rho$ by part (b), it follows that

$$Q(\alpha, \beta) \subset \{u \in T + Q(\alpha, \beta) \mid \chi([I, u]) = 0\} \subseteq T + Q(\alpha, \beta).$$

By general representation theory, the $(T + Q(\alpha, \beta))$-module $V$ is then induced from its $Q(\alpha, \beta)$-submodule $V_0$ of dimension $\leq p^{-1} \dim V \leq 4$.

Let $x_\beta \in Q_\beta$ be such that $\kappa([J_\alpha, x_\beta]) \neq 0$. Note that $J \cap Q(\alpha, \beta) + Fx_\beta$ is a solvable subalgebra of $Q(\alpha, \beta)$. Since $\dim V_0 < p$, it must act trianugally on $V_0$. Combining this with the Engel–Jacobson theorem one now observes easily that $(J \cap Q(\alpha, \beta) + Fx_\beta)^{(1)} \subset J$ acts nilpotently on $V$. As a consequence, $[J_\alpha, x_\beta]$ acts nilpotently on $V$, implying $\kappa([J_\alpha, x_\beta]) = 0$. This contradiction shows that $I \subset H + \ker \rho$. The rest of the original proof works for $p > 3$.

The original proofs of Theorem 2.6, Corollary 2.7 and Corollary 2.8 go through for $p > 3$ (in fact, the proof of Corollary 2.8 can be streamlined by making use of Corollary 2.7(3)). The original proof of Theorem 2.9 only requires one very minor adjustment in the middle of p. 21 in [St 93]:

“If $\dim W \geq 5p$, then assertion (4a) of the new Lemma 2.3 yields $\alpha \in \Delta$, a contradiction. Thus $\dim W < 5p \leq p^2$.”

Thus, all results in [St 93, Sect. 2] essentially remain true for $p = 5, 7$. Next, inspection shows that apart from standard results valid in all characteristics, the arguments in [St 93, Sect. 3] rely only on results of [St 92] and [B-O-St] valid for $p > 3$, on Theorems 1.9 and 2.6, and on Lemma 2.3(5) which we did not change. Hence all arguments and results in [St 93, Sect. 3] remain valid for $p > 3$.

In order to show that the proofs in [St 93, Sect. 4] generalize to the case where $p > 3$ we first recall that all results of [St 91a] are valid for $p > 3$; see [P-St 04, Sect. 5]. It should also be noted that [B-O-St, (4.7)] holds for $p > 3$; see Corollary 4.4. Since [St 90, (2.4)] and [St 91b, (4.5)] can now be substituted by [P-St 04, Theorem D], inspection shows that all results used in [St 93, Sect. 4] are valid for $p > 3$. Thus, what remains to be revised in [St 93, Sect. 4] is the proof of Claim 4 in Theorem 4.6 (this proof uses the inequality $p - 1 > 5$ which is no longer available in our situation).

**New proof.** Let $\overline{H}$ denote the image of $H$ in $G_{[0]}$. As the $T$-root spaces of $G_{(-1)}$ are 1-dimensional, by part (3) of the proof, the subalgebra $\overline{H}$ is spanned by $\overline{h}_i$, $i = 1, 2$, and $\dim \overline{H} = 2$. By parts (2) and (3) of the proof, we have that $\overline{h}_2 \in G_{[0]}^{(1)}$ and $\text{tr}(\text{ad}_{G_{(-1)}} \overline{h}_1) \neq 0$. It follows that $\overline{H} \cap G_{[0]}^{(1)} \subset F\overline{h}_2$. Let $b$ denote the invariant symmetric bilinear form on $G_{[0]}$ given by

$$b(x, y) := \text{tr}(\text{ad}_{G_{(-1)}} x \circ \text{ad}_{G_{(-1)}} y) \quad (\forall x, y \in G_{[0]}).$$
The radical $G_{(0)}^+ := \{ x \in G_{(0)} \mid b(x, G_{(0)}) = 0 \}$ of $b$ is an ideal of $G_{(0)}$. Suppose $G_{(0)}^+ \cap G_{(0)}^{(1)} \cap \overline{H} = (0)$. Then $\tilde{h}_2 \in G_{(0)}^+$, hence $b(\tilde{h}_2, \tilde{h}_2) = 0$. Recall from part (3) of the proof that

$$G_{(-1)} = G_{(-1), \tilde{b}} + G_{(-1), -\tilde{b}} + \sum_{-1 \leq i \leq 1} G_{(-1), -\tilde{a} + i\tilde{b}}$$

and $\tilde{a}(h_2) = 0$, $\tilde{b}(h_2) = -1$. Since $p \geq 5$ and $\dim G_{(-1), \gamma} = 1$ for all weights $\gamma$ of $G_{(-1)}$, this can only happen if $G_{(-1)} = G_{(-1), \tilde{a}}$. But then in the notation of [St 93, Sect. 4] we have $G = G_{(-1)} = Fz_0 \oplus G'_{(0)}$. Recall that $G'_{(0)}$ is the normalizer of $Fu_{p-2} \oplus J$, where $J = \sum_{i \geq 1} G_{[i]}$, and we are assuming that $J$ is a maximal ideal of $G$. From this it is immediate that the subalgebra $G'_{(0)} / J$ of the simple Lie algebra $G / J$ acts triangulably on $G / J$. However, $G'_{(0)} / J$ contains a copy of $\mathfrak{sl}(2)$ spanned by the images of $D(x^2), D(xy)$ and $D(y^2)$ in $G'_{(0)}/J$.

Therefore, $G_{(0)}^+ \cap G_{(0)}^{(1)} \cap \overline{H} = (0)$, implying $G_{(0)}^+ \cap G_{(0)}^{(1)} = \bigoplus_{\gamma \neq 0} (G_{(0)}^+ \cap G_{(0)}^{(1)})_{\gamma}$. Since all elements in $\bigcup_{\gamma \neq 0} G_{(0), \gamma}$ act nilpotently on $G_{(-1)}$, the Engel–Jacobson theorem yields that $G_{(0)}^+ \cap G_{(0)}^{(1)}$ acts nilpotently on $G_{(-1)}$. Since $G_{(-1)}$ is an irreducible $G_{(0)}$-module, $G_{(0)}^+ \cap G_{(0)}^{(1)} = (0)$ necessarily holds. But then $b$ is nondegenerate on $G_{(0)}^{(1)}$, forcing $G_{(0)} = G_{(0)}^{(1)} \oplus C$ where $C$ is the orthogonal complement to $G_{(0)}^{(1)}$ in $G_{(0)}$. As $[G_{(0)}^{(1)}, C] \subset C$ and $C^{(1)} \subset C_{(0)}^{(1)}$, it must be that $C$ is a central ideal of $G_{(0)}$. On the other hand, the center of $G_{(0)}$ has dimension $\leq 1$ by Schur’s lemma, and $\tilde{h}_1 \notin G_{(0)}^{(1)}$ by our remarks earlier in the proof. Hence $C$ coincides with the center of $G_{(0)}$, and Claim 4 follows. \hfill \Box

We have already mentioned that Proposition 2.2 of [St 91b] remains true for $p > 3$. As a consequence, Lemma 2.4(1) of [St 91b] is valid for $p > 3$. Then one can see by inspection that Lemma 5.1 remains true for $p > 3$ as well.

Part (a) of the original proof of Lemma 5.2 has to be modified, however: in the course of the proof one has to show that a certain torus $R$ is optimal, but the argument used in [St 93] does not extend to the case where $p = 5, 7$. The argument below will justify that $R$ is optimal for $p > 3$.

We begin as in [St 93, p. 46] and establish the existence of a root $\kappa$ with

$$\kappa \notin \Delta, \ \kappa(L_{\alpha}) \neq 0, \ \alpha([L_\kappa, L_{\beta - \kappa}] \neq 0.$$

Put $K := \bigcap_{n \geq 0} L(\alpha, \beta, \kappa)^{(n)}$ and let $I$ be a maximal ideal of $K$. Put $G := K/I$ and let $G_p$ be the $p$-envelope of the simple Lie algebra $K$ in $\text{Der} K$. Let $\varphi : K \to G$ denote the canonical homomorphism.

New part of the proof. (a) Suppose $\text{TR}(G) = 2$ and put $N := \varphi(K(\beta))$. Then one shows as in the original proof that $N$ is a triangulable Cartan subalgebra of $G$ with $\text{TR}(N, G) = \text{TR}(G) = 2$. Let $N_p$ be the $p$-envelope of $N$ in $G_p$, and let $R$ denote the unique maximal torus of $G_p$ contained in $N_p$. Note that $\dim R = 2$. 

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Suppose \( L(\alpha, \beta) \) is not solvable. Then one shows as in the original proof that

\[
L[\alpha, \beta]^{(1)} \cong H(2; 1; \Phi(\tau))^{(1)} \cong (L(\alpha, \beta)/\text{rad } L(\alpha, \beta))^{(1)}.
\]

Our choice of \( \kappa \) then implies that \( TR(K) = 3 \). Hence \( I \) is a \( T \)-invariant ideal of \( K \). If \( \beta(I_\alpha) \neq 0 \) or \( \alpha(I_\beta) \neq 0 \), then \( K_{i\alpha+j\beta} \subset I \) for all nonzero \((i, j) \in \mathbb{F}_p^2 \) (one should keep in mind here that \( \beta(K_\alpha) \neq 0 \) and \( \alpha(K_\beta) \neq 0 \)). This yields \( TR(I) \geq 2 \) forcing \( TR(G) \leq 1 \), a contradiction. Thus, it must be that \( \beta(I_\alpha) = \alpha(I_\beta) = 0 \). Therefore,

\[
\beta(G_\alpha) \neq 0, \quad \alpha(G_\beta) \neq 0, \quad I \cap K(\alpha, \beta) \subset H + \text{rad } K(\alpha, \beta).
\]

Since \( \alpha, \beta \in \Delta \) and \( TR(G) = 2 \), this entails \( G = G(\alpha, \beta) \). Since \( \kappa \not\in \Delta \), we now deduce that \( K_\kappa \subset I \). But then \( \alpha([I_\kappa, L_{\beta-\kappa}]) = \alpha([K_\kappa, L_{\beta-\kappa}]) = \alpha([L_\kappa, L_{\beta-\kappa}]) \neq 0 \), showing that \( G_{i\alpha+j\beta} \subset I \) for all \( i \in \mathbb{F}_p^* \) and \( j \in \mathbb{F}_p \). As this contradicts our assumption that \( TR(G) = 2 \), we deduce that \( L(\alpha, \beta) \) is solvable.

We now intend to show that all \( R \)-roots of \( G \) are proper. The 1-sections of \( G \) relative to \( R \) are related to the 2-sections of \( K \) relative to \( T \) as follows: Let \( \mu \) be any \( T \)-root of \( K \). Since \( I \) is \( K(\beta) \)-stable and \( \dim R = 2 \), the map \( \varphi \) takes the subspace \( K_\mu := \bigoplus_{i \in \mathbb{F}_p} K_{\mu+i\beta} \) onto a root space relative to \( R \). Conversely, every \( R \)-root space of \( G \) is of the form \( \varphi(K_\mu) \) for some \( \mu \in \langle \alpha, \beta, \kappa \rangle \). The nonzero roots \( \tilde{\mu} \in \Phi(G, R) \) correspond to those \( \mu \in \Phi(K, T) \) which are \( \mathbb{F}_p \)-independent of \( \beta \).

Let \( \tilde{\mu} \) be a nonzero root in \( \Phi(G, R) \). As \( G \) is a simple Lie algebra and \( R \) is a torus of maximal dimension in \( G_\mu \), Theorem 2.1 shows that the derived subalgebra \( U := G[\tilde{\mu}]^{(1)} \) of \( G[\tilde{\mu}] = G(\tilde{\mu})/\text{rad } G(\tilde{\mu}) \) is either \((0) \) of one of \( \mathfrak{sl}(2) \), \( W(1; 1) \), \( H(2; 1) \). Furthermore, \( U \) has codimension \( \leq 1 \) in \( G[\tilde{\mu}] \). If \( U \) is either \((0) \) or \( \mathfrak{sl}(2) \), then \( \tilde{\mu} \) is solvable or classical, hence proper. So from now on we may assume that \( U \) is either \( W(1; 1) \) or \( H(2; 1) \).

By analyzing the list of semisimple quotients in Theorem 2.2 one finds out that \( L[\mu, \beta] \) can only be of type \((2), (3), (4), (6), \) or \((7) \). Indeed, in case \((1) \) the Lie algebra \( L(\mu, \beta) \) is solvable, hence \( U = (0) \). In case \((5) \) no 1-section in \( L[\mu, \beta] \) is nilpotent, for otherwise one of the roots in \( \Gamma(L[\mu, \beta], T) \) would vanish on \( h \otimes 1 \in \psi(H) \) (the notation of Proposition 2.3(6)). But then the centralizer of \( h \otimes 1 \) in \( L[\mu, \beta] \) would be nilpotent, contrary to the description in Theorem 2.2(6). In case \((8) \) the inclusion \( T \subset \psi(H) \) holds as \( L[\mu, \beta] \) is simple and restricted. Since \( U \) is of Cartan type, the Lie algebra \( L[\mu, \beta] \) cannot be classical. As a consequence, in both cases \((5) \) and \((8) \) the set \( \Phi(L[\mu, \beta], T) \) contains a nonzero multiple of \( \beta \). This, however, contradicts our assumption that \( \beta \) vanishes on \( H \).

It is immediate from the definition of \( K \) that \( K(\mu, \beta) \) is an ideal of \( L(\mu, \beta) \) containing \( \bigcap_{n \geq 0} L(\mu, \beta)^{(n)} \). Let \( \pi: K(\mu, \beta) \to L[\mu, \beta] \) denote the restriction to \( K(\mu, \beta) \subset L(\mu, \beta) \) of the canonical homomorphism \( \psi: L(\mu, \beta) \to L[\mu, \beta] \), and \( \varphi_{\mu, \beta}: K(\mu, \beta) \to G \) the restriction to \( K(\mu, \beta) \subset K \) of the epimorphism \( \varphi \). As explained above, \( \varphi_{\mu, \beta} \) takes \( K(\mu, \beta) \) onto the 1-section \( G(\tilde{\mu}) \) with respect to \( R \).
Composing \( \varphi_{\mu, \beta} \) with the canonical homomorphism \( G(\mu) \to G[\mu] \) we obtain a surjective Lie algebra map \( \nu: K(\mu, \beta) \to G[\mu] \).

Let \( \tilde{S} \) denote the socle of \( L[\mu, \beta] \) and \( Q[\mu, \beta] \) the maximal compositionally classical subalgebra of \( L[\mu, \beta] \). Put

\[
\nu := \dim L[\mu, \beta]/Q[\mu, \beta].
\]

Consider cases (2), (4) and (7). In each of these cases \( \tilde{S} \) is a simple Lie algebra and the quotient \( L[\mu, \beta]/\tilde{S} \) is nilpotent. Then \( \pi^{-1}(\tilde{S}) \) is simple modulo its radical and \( \pi(K(\mu, \beta)) \supset \tilde{S} \), by our earlier remarks. Since \( G[\mu] \) is semisimple with simple socle \( U = G[\mu]^{(1)} \), it must be that \( (\nu \circ \pi^{-1})(\tilde{S}) \) is either \((0)\) or \(U\). As \( L[\mu, \beta]/\tilde{S} \) is nilpotent, the first possibility cannot occur. Therefore,

\[
(\nu \circ \pi^{-1})(\tilde{S}) = \tilde{S} \cong \tilde{S}, \quad \ker \nu \subset \ker \pi.
\]

Consequently, \( \ker \nu = \text{rad } K(\mu, \beta) \). By our earlier remarks, \( \tilde{S} \) is either \( W(1; \mathbf{1}) \) or \( H(2; \mathbf{1})^{(2)} \). We denote by \( \tilde{S}_{(0)} \) the standard maximal subalgebra of \( \tilde{S} \).

As \( TR(U) = 1 \), case (7) is impossible. It is easily seen that in case (2) we have \( e = 1 \) if \( \tilde{S} \) is Witt and \( e = 2 \) if \( \tilde{S} \) is Hamiltonian. Lemma 2.2 of [B-O-St] holds for \( p > 3 \) and shows that in case (4) we have \( e = 2 \) and \( \tilde{S} = H(2; \mathbf{1})^{(2)} \).

Furthermore, \( Q[\mu, \beta] \cap \tilde{S} = \tilde{S}_{(0)} \) in both cases. Let \( M := (\nu \circ \pi^{-1})(Q[\mu, \beta]) \), a subalgebra of \( G[\mu] \). Since \( \ker \nu \) is solvable, \( M \) has the following properties:

1. \( M/\text{rad } M \) is either \((0)\) or \( \mathfrak{sl}(2) \);
2. \( \dim G[\mu]/M \leq e \);
3. \( M \cap U \subset U_{(0)} \).

Then \( M \) coincides the maximal compositionally classical subalgebra of \( G[\mu] \). Moreover, since \( \beta \) is solvable, it must be that \( K(\beta) \subset \pi^{-1}(Q[\mu, \beta]) \). This implies that \( R \) normalizes \( M \). Consequently, \( \tilde{\mu} \) is a proper \( R \)-root.

Now suppose we are in case (6). Then \( \tilde{S} = S \otimes \mathcal{O}(1; \mathbf{1}) \), where \( S \) is one of \( \mathfrak{sl}(2) \), \( W(1; \mathbf{1}) \), \( H(2; \mathbf{1})^{(2)} \), and \( L[\mu, \beta]/\tilde{S} \) is nilpotent. As \( \tilde{S} \) is perfect and \( \tilde{S}/\text{rad } \tilde{S} \) is simple, this gives \( (\nu \circ \pi^{-1})(\tilde{S}) \) is \( \tilde{S} \) and \( \ker \nu \subset \ker \pi \). Hence \( \ker \nu = \text{rad } K(\mu, \beta) \).

We now proceed as before. Let \( \mathfrak{n} \) denote the centralizer of \( S \otimes \mathcal{O}(1; \mathbf{1})_{(p-1)} \) in \( L[\mu, \beta] \). Since \( S \otimes \mathcal{O}(1; \mathbf{1}) \subset L[\mu, \beta] \subset \tilde{S} \otimes \mathcal{O}(1; \mathbf{1}) \) by Theorem 2.2(6), it is straightforward that \( \mathfrak{n} = \text{rad } L[\mu, \beta] \). Then \( \pi^{-1}(\mathfrak{n}) \subset \ker \nu \) by the preceding remark. Using [B-O-St, Lemma 2.4] (which holds for \( p > 3 \)) one observes that when \( S \) is Witt (resp., Hamiltonian), the subalgebra \( Q[\mu, \beta] + \mathfrak{n} \) has codimension 1 (resp., 2) in \( L[\mu, \beta] \). As \( \pi^{-1}(\mathfrak{n}) \subset \ker \nu \), it follows that \( (\nu \circ \pi^{-1})(Q[\mu, \beta]) \) is a maximal compositionally classical subalgebra of \( G[\mu] \). It contains \( \nu(K(\beta)) \) because \( \beta \) is solvable. Therefore, \( \tilde{\mu} \) is a proper \( R \)-root.

Finally, suppose we are in case (3). Then \( L[\mu, \beta]/\tilde{S} \) is nilpotent and \( \tilde{S} = S_i \oplus S_2 \), where \( S_i \in \{ \mathfrak{sl}(2), W(1; \mathbf{1}), H(2; \mathbf{1})^{(2)} \} \) for \( i = 1, 2 \). Since \( L[\mu, \beta] \subset \text{Der } \tilde{S} \), the Lie algebra \( \pi(K(\mu, \beta)) \supset \tilde{S} \) is semisimple. Hence \( \ker \pi = \text{rad } K(\mu, \beta) \). By
Proposition 2.3(3), we have $T \subset \psi(H)$, which implies that no nonzero root in $\Phi(L[\mu, \beta], T)$ vanishes on $\psi(H)$. It follows that $K_{i \beta} \subset \ker \pi$ for all $i \in \mathbb{F}_p^*$. As $\nu(K(\mu, \beta))$ is semisimple, we have $\ker \pi = \text{rad } K(\mu, \beta) \subset \ker \nu$. As $TR(G[\tilde{\mu}]) = 1$, either $(\nu \circ \pi^{-1})(S_1) = 0$ or $(\nu \circ \pi^{-1})(S_2) = 0$. No generality will be lost by assuming that the latter case occurs. Then $(\nu \circ \pi^{-1})(S_1) = U \cong S_1$.

Denote by $m$ the centralizer of $S_1$ in $L[\mu, \beta]$. This is an ideal of $L[\mu, \beta]$ containing $S_2$. Since $m/S_2$ is solvable, the preceding remark implies that $\pi^{-1}(m) \subset \ker \nu$. When $S_2$ is Witt (resp., Hamiltonian), the subalgebra $Q[\mu, \beta] + m$ has codimension 1 (resp., 2) in $L[\mu, \beta]$. As in the previous case we now obtain that $(\nu \circ \pi^{-1})(Q[\mu, \beta])$ is a maximal compositionally classical subalgebra of $G[\tilde{\mu}]$. It contains $\nu(K(\beta))$ because $\beta$ is solvable. Thus, all $R$-roots of $G$ are proper. Now proceed as in the original proof of Lemma 5.2. □

The rest of Section 5 and most of Section 6 are essentially self-contained: they rely only on earlier results in [St 93] and all arguments hold for $p > 3$. However, in what follows we will need a slightly different version of Theorem 6.7.

According to Theorem 3.5, the maximal subalgebra $T + Q$ of $T + L$ gives rise to a long standard filtration in $T + L$, and the corresponding graded Lie algebra $G := \text{gr}(T + L)$ has a unique minimal ideal $A(L, T)$. Furthermore, the Lie algebra $A(L, T) = \bigoplus_{i \in \mathbb{Z}} A[i]$ is graded and there exist a nonnegative integer $m$ and a simple graded Lie algebra $S(L, T) = \bigoplus_{i \in \mathbb{Z}} S[i]$ such that

$$A(L, T) = S(L, T) \otimes O(m; 1), \quad A[i] = S[i] \otimes O(m; 1) \quad (\forall i \in \mathbb{Z})$$

as graded Lie algebras. Our next result describes the graded component $S[0]$ of $S(L, T)$.

**Theorem 5.1.** The Lie algebra $S[0]$ is one of the following:

(a) 1-dimensional;

(b) classical simple;

(c) $\mathfrak{sl}(kp)$, $\mathfrak{gl}(kp)$ or $\mathfrak{pgl}(kp)$ for some $k \geq 1$;

(d) $S[0] \oplus C$ where $C = C(S[0])$ is 1-dimensional and $S[0]$ is either classical simple or $\mathfrak{pgl}(kp)$ for some $k \geq 1$.

**Proof.** (1) We know from Proposition 3.10 and Proposition 6.5(4) that $H[0] = \mathfrak{c}_{S[0]}(T)$ is an abelian Cartan subalgebra of $S[0]$. By Proposition 6.1(3), the semisimple quotients of the 2-sections of $S[0]$ relative to $H[0]$ are of types (0), $A_1$, $A_1 \times A_1$, $A_2$, $C_2$ or $G_2$. In view of Lemma 6.2(2) and Proposition 6.5(1) we have that $\text{rad } S[0](\alpha) \subset H[0]$ for every root $\alpha \in \Phi(S[0], H[0])$. Consequently, $S[0](\alpha)$ is nonsolvable if $\alpha \neq 0$. By Proposition 3.10 and Proposition 6.5(2), the torus $T$ acts on $S[0]$ and $H[0]$ distinguishes the weight spaces of $S[0]$ relative to $T$. Since those are 1-dimensional, by properties of $Q(L, T)$, we derive that every $S[0]$-submodule of $S[0]$ is $T$-stable. The construction of $S(L, T)$ now yields that $S[0]$ is an irreducible $S[0]$-module. But then $\text{rad } S[0] \subset H[0]$ acts on $S[0]$ by scalar operators. As a result, $\text{rad } S[0] = C(S[0])$. 

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Suppose $\alpha, \beta \in \Phi(S_0, H_0)$ are linearly independent. As 
\[
\text{rad } S_0(\bar{\gamma}) = \{h \in H_0 | \bar{\gamma}(h) = 0\} \quad (\forall \bar{\gamma} \in \Phi(S_0, H_0) \setminus \{0\})
\]
and $\ker \bar{\alpha} \cap \ker \bar{\beta}$ has codimension 2 in $H_0$, it must be that $S_0[\bar{\alpha}, \bar{\beta}] \not\cong \mathfrak{sl}(2)$. But then $S_0[\bar{\alpha}, \bar{\beta}]$ has type $A_1 \times A_1$, $C_2$ or $G_2$. This implies that $\bar{\alpha}$ and $\bar{\beta}$ are linearly independent as functions on $H_0 \cap S_0^{(1)}$. As $\dim S_0 \gamma = 1$ for any nonzero $\gamma \in \Phi(S_0, H_0)$, it follows that every ideal of $S_0^{(1)}$ is $H_0$-stable.

Since $S_0 = S_0^{(1)} + H_0$ and $H_0$ is abelian, the derived subalgebra $S_0^{(1)}$ is perfect. The preceding remark implies that $\text{rad } S_0^{(1)} = C(S_0) \cap S_0^{(1)}$. Put $g := S_0^{(1)}/\text{rad } S_0^{(1)}$ and let $h$ denote the image of $H_0 \cap S_0^{(1)}$ in $g$. Obviously, $g$ is perfect and semisimple. The above discussion shows that $h$ is an abelian Cartan subalgebra of $g$ and the pair $(g, h)$ satisfies the Mills–Seligman axioms. Since $p > 3$, the main result of [M-Se 57] enables us to conclude that $g$ is a direct sum of classical simple Lie algebras.

(2) Theorem 3.3(4) yields $Q_{(p-2)} \neq (0)$. Furthermore, the proof of Theorem 3.3 shows that $Q_{(p-1)} \neq (0)$ provided that $L \neq Q_{(-1)}$. It also shows that if $L = Q_{(-1)}$, then there exist root vectors $x \in L \setminus Q$ and $u \in Q_{(1)}$ with $[x, y], Q_{(p-2)} \neq (0)$. Since $G_{(-1)} \subset A(L, T)$, it follows that $A_{(p-2)} \neq (0)$. As a consequence, $S_3 \neq (0)$.

Now Lemmas 12.4.2–12.4.4 of [B-W 88] apply and yield that $g$ is simple or zero. At this point we can refer to [B-W 88, Corollary 12.4.7] to complete the proof. (All our references to [B-W 88, Sect. 12] work for $p > 3$; we are not interested in the $p$-structure of $S_0$ which is also discussed in [B-W 88].)

Now we are ready to determine the Lie algebra $S(L, T)$.

**Theorem 6.7** (new). Let $L$ be a finite-dimensional simple Lie algebra over $F$ and suppose that all tori of maximal dimension in $L_\mu$ are standard. Let $T$ be an optimal torus in $L_\mu$ and assume that $Q(L, T) \neq L$. Let $A(L, T)$ be the minimal ideal of the graded Lie algebra $g = \text{gr}(T + L)$, and $S(L, T)$ the simple graded Lie algebra such that $A(L, T) \cong S(L, T) \otimes \mathfrak{O}(m; 1)$. Then $S(L, T)$ is a restricted simple Lie algebra of Cartan type.

Proof. We will show that the conditions (a)-(d) of the Recognition Theorem apply to the graded Lie algebra $S(L, T)$; see [B-G-P, Theorem 0.1] and [St 04, Theorem 5.6.1].

(a) Theorem 5.1 shows that the graded component $S_0$ of $S(L, T)$ satisfies condition (a) of the Recognition Theorem.

(b) In part (1) of the proof of Theorem 5.1 it was explained that $S_{[-1]}$ is an irreducible $S_0$-module. This means that condition (b) of the Recognition Theorem holds for $S(L, T)$.

(c) It follows from the definition of a standard filtration that the Lie algebra $\bigoplus_{i<0} G_i$ is generated by its subspace $G_{[-1]}$. Consequently, the Lie algebra
\( \bigoplus_{i<0} S_{[i]} \) is generated by \( S_{[-1]} \). If \( [x, S_{[-1]}] = (0) \) for some nonzero \( x \in \bigoplus_{i\geq 0} S_{[i]} \), then the subspace \( \bigoplus_{i\geq 0} S_{[i]} \) contains a nonzero ideal of \( S(L,T) \). Since this contradicts the simplicity of \( S(L,T) \) we derive that condition (c) of the Recognition Theorem holds for \( S(L,T) \).

(d) Now suppose that \( [x, S_{[1]}] = (0) \) for some nonzero \( x \in S_{[-j]} \) with \( j \geq 0 \). Since the subspace

\[
Y_{[-j]} := \{ y \in S_{[-j]} \mid [y, S_{[1]}] = (0) \}
\]

is \( S_{[0]} \)-stable, we may assume that \( x \in S_{[-j],\alpha} \) is a root vector for \( T \).

Suppose \( j = 0 \). Take any \( h \in Y_{[0]} \cap H_{[0]} \) and any \( \gamma \in \Phi_{[-1]}(S,T) \). It follows from the definition of \( Q(L,T) \) and \textbf{Theorem 3.10} that there exist \( u \in S_{[-1],\gamma} \) and \( v \in S_{[1],-\gamma} \) with \([u,v] \neq 0\). As mentioned in the proof of Theorem 5.1, the abelian Lie algebra \( H_{[0]} \) acts on the weight spaces of \( S_{[-1]} \) relative to \( T \). Since all these weight spaces are 1-dimensional and \([h,v] = 0\), we then have

\[
\gamma(h)[u,v] = [[h,u],v] = -[[u,[h,v]]].
\]

It follows that \( h \) annihilates \( S_{[-1]} \). But then \( h = 0 \), forcing \( Y_{[0]} \cap H_{[0]} = (0) \).

Combining this with \textbf{Proposition 6.1} and the Engel–Jacobson theorem, we now deduce that the ideal \( Y_{[0]} \) of \( S_{[0]} \) acts nilpotently on \( S_{[-1]} \). The irreducibility of \( S_{[-1]} \) yields \( Y_{[0]} = (0) \).

Suppose \( j = 1 \). Since \( Y_{[-1]} \) is an \( S_{[0]} \)-submodule of \( S_{[-1]} \) and \([S_{[-1]}, S_{[1]}] \neq (0)\) by part (c) of this proof, the irreducibility of \( S_{[-1]} \) now yields \( Y_{[-1]} = (0) \).

Suppose \( j = 2 \). Then the 1-section \( L(\alpha) \) fits into a 2-section \( L(\alpha, \beta) \) whose semisimple quotient is isomorphic to \( K(3;1) \); see \textbf{Proposition 3.4}. Since \( S_{[-2],\alpha} = FX \), it follows from \textbf{Lemma 3.1(1)} that \([x,G_{[1]}] \neq (0)\). Because \( G_{[1]} = S_{[1]} \otimes O(m;1) \) and \( x \) identifies with an element in \( S_{[-2]} \otimes F \subset S_{[-2]} \otimes O(m;1) \), we now obtain that \([x,S_{[1]}] \neq (0)\), a contradiction. Hence \( Y_{[-2]} = (0) \). As \( Y_{[-j]} = (0) \) for \( j > 2 \), by \textbf{Proposition 3.4}, we have proved that all conditions of the Recognition Theorem are satisfied for \( S(L,T) \).

(e) Applying the Recognition Theorem we obtain that \( S(L,T) \) is either classical or of Cartan type or a Melikian Lie algebra. As explained in part (2) of the proof of Theorem 5.1, we have that \( S_{[3]} \neq (0) \). So \( S(L,T) \) has an unbalanced grading, hence cannot be classical. The natural grading of any Melikian algebra has depth 3 and height \( > 3 \). As \( S_{[-3]} = (0) \), it follows that \( S(L,T) \) is not of Melikian type.

We conclude that the graded Lie algebra \( S(L,T) \) is isomorphic to a Cartan type Lie algebra \( X(r;\underline{s})^{(2)} \) regarded with its natural grading (here \( X \in \{W,S,H,K\} \) and \( \underline{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r \)).

To show that \( S(L,T) \) is restricted we take any root vector \( x \in S_{[i],\alpha} \) with 

\( i \in \{-1, -2\} \)

and let \( \beta \) be any \( T \)-root of \( S = S(L,T) \). The semisimple quotients of the 2-sections of \( S \) relative to \( T \) are described in \textbf{Theorem 3.11}. Furthermore, the proof of \textbf{Theorem 3.11} in conjunction with Theorem 5.1 shows that
2-sections of type (7) do not occur. Since \( \alpha \in \Phi \), it also follows from the proof of Theorem 3.11 that
\[
(\text{ad } x)^p(S(\alpha, \beta)) \subset \text{rad}_T S(\alpha, \beta) \subset \bigoplus_{i \geq 0} S[i](\alpha, \beta).
\]
But then \((\text{ad } x)^p\) maps \(S(L, T)\) into \(\bigoplus_{i \geq 0} S[i]\). Since \(S(L, T) \cong X(r; \mathfrak{s})^{(2)}\) as graded Lie algebras, this forces \(s = 1\). As a consequence, \(S(L, T)\) is restricted; see [St 04, Corollary 7.2.3] for example. \(\square\)

6. Classification results

Similar to [St 93, Sect. 7] the determination of the Lie algebra \(S(L, T)\) allows one to classify a large family of finite-dimensional simple Lie algebras. Note that all results and arguments used in [St 93, Sect. 7] are valid for \(p > 3\).

**Theorem 6.1.** (cf. [St 93, Theorem 7.3]). Let \(L\) be a finite-dimensional simple Lie algebra over \(F\) such that all tori of maximal dimension in \(L_p\) are standard. Let \(T\) be an optimal torus in \(L_p\) and suppose that \(Q(L, T) \neq L\) and \(L[\alpha, \beta]^{(1)} \neq H(2; 1; \Phi(\tau))^{(1)}\) for any two roots \(\alpha, \beta \in \Gamma(L, T)\). Then \(L\) is isomorphic to a Cartan type Lie algebra and \(Q(L, T)\) is contained in the standard maximal subalgebra of \(L\).

**Proof.** One argues as in the original proof of Theorem 7.3 in [St 93] to construct a maximal subalgebra \(L(0)\) containing \(Q(L, T)\) and to show that the pair \((L, L(0))\) satisfies all conditions of the Recognition Theorem for filtered Lie algebras; see [St 04, Theorem 5.6.2]. The argument in [St 93, pp. 57, 58] shows that \(L = L_{(-2)}\). This implies that \(L\) is not isomorphic to a Melikian algebra. Since \(S(L, T)\) is of Cartan type, it follows from the construction of \(L(0)\) that \(L_{(p-2)} \neq (0)\). But then \(L\) cannot be classical. By the Recognition Theorem, \(L\) must be isomorphic as a filtered Lie algebra to a Cartan type Lie algebra regarded with its standard filtration. This completes the proof. \(\square\)

We continue assuming that all tori of maximal dimension in \(L\) are standard and \(Q(L, T) \neq L\). In view of Theorem 6.1 we can also assume now that for any optimal torus \(T\) in \(L_p\) there are \(\alpha, \beta \in \Gamma(L, T)\) such that \(L[\alpha, \beta]^{(1)} = H(2; 1; \Phi(\tau))^{(1)}\). For \(p > 7\), the simple Lie algebras with these properties are classified in [St 94]. We will go through the arguments in [St 94] to verify whether they are still valid for \(p = 5, 7\). All our references to [St 94] will be underlined.

We have already shown in [P-St 04] that the results of [St 91a] hold for \(p = 5, 7\). Note that [St 91a] is the main prerequisite to [St 94]. Inspection shows that all results and arguments used in [St 94] are valid for \(p > 5\). In fact, only one minor issue in [St 94, Sect. 2] requires our attention; it arises when \(p = 5\).

**Proposition 2.4** (new parts).

(3a) If \(\alpha \in \Phi_{[-1]}\), \(\beta \in \Phi_{[0]}\) and \(\alpha + \beta \in \Phi_{[0]}\), then \(\alpha\) is Hamiltonian, \(p = 5\), and \(\beta = 2\alpha\).
(3d) \([Q_\mu, L_\lambda] \cap Q \subset Q_{(1)}\) for all \(\lambda \in \Phi_{[-1]}\) and \(\mu \in \Delta\).

**Proof.** (3a) The assertion follows from [St 04, Lemma 5.5.1].
(3d) Recall that \(\Phi_{[-1]}(\text{gr}(T + L), T) = \Phi_{[-1]}(S, T) + \Delta\); see Proposition 2.4(2). Since \(\mu + \lambda \in \Phi_{[-1]} + \Delta = \Phi_{[-1]}\) and \(\Phi_{[-1]} \cap \Phi_0 = \emptyset\) by Proposition 2.4(3b), the statement follows. \(\square\)

As a result of the above changes we have to modify slightly the statement and the original proof of Proposition 2.5: the subspace \(V_0\) from Proposition 2.5 has to be selected in a more sophisticated fashion.

**Proposition 2.5** (new proof).

1) There exists a \(T\)-invariant subspace \(V_{-1} \subset L\) such that

\[
L = V_{-1} + Q, \quad V_{-1} \cap Q = (0).
\]

2) There exists a \(T\)-invariant subspace \(V_0 \subset Q\) such that

\[
Q_{(1)} \cap V_0 = (0) \quad \text{and} \quad (V_0 + Q_{(1)})/Q_{(1)} = A(L, T)_{[0]}.
\]

3) For \(i = -1, 0\), let \(R_i \subset V_i\) denote the preimage of \(S_{[i]} \otimes \mathfrak{O}(m; L)_{(1)}\) under the linear isomorphism \(V_i \rightarrow (V_i + Q_{(i+1)})/Q_{(i+1)} = A(L, T)_{[i]}\). Then the following statements hold:

a) \(V_{-1} \subset \sum_{\mu \in \Phi_{[-1]}} L_\mu\);

b) \(\sum_{\mu \not\in \Delta} Q_\mu \subset V_0 + Q_{(1)}\);

c) \(Q = V_0 + Q(\Delta) + Q_{(1)}\) and \(V_0 \subset \sum_{\mu \in \Phi_0} Q_\mu + Q_{(1)}\);

d) \([V_{-1}, V_{-1}] \subset Q\);

e) \([T + V_0 + Q(\Delta), V_{-1}] \subset V_{-1} + Q_{(1)}\);

f) \([Q_{(1)}, V_{-1}] \subset V_0 + Q_{(1)}\);

g) \([T + Q, V_0] \subset V_0 + Q_{(1)}\);

h) \([V_0, R_{-1}] \subset R_{-1} + Q_{(1)} \subset [V_0, R_{-1}] + Q_{(1)}\);

i) \([V_0, R_0] \subset R_0 + Q_{(1)} \subset [V_0, R_0] + Q_{(1)}\);

j) \(V_0 + Q_{(1)}\) is an ideal of \(Q\);

k) \([R_0, V_{-1}] \subset R_{-1} + Q_{(1)} \subset [R_0, V_{-1}] + Q_{(1)}\).

**Proof.** The original proof goes through for \(p > 5\). So we assume from now that \(p = 5\). We choose \(V_{-1}\) as in the original proof. Then assertions 1), 3a) and 3d) hold. Let \(\Phi'_0\) denote the set of all nonzero \(T\)-roots of \(A(L, T)_{[0]}\). For every \(\mu \in \Phi'_0\) choose a nonzero \(u_\mu \in Q_\mu\) such that \(A(L, T)_{[0], \mu} = F u_\mu\), where \(u_\mu\) stands for the coset of \(u_\mu\). If \(\mu\) is not Hamiltonian, set \(v_\mu := u_\mu\). If \(\mu\) is Hamiltonian, then \(\pm 3\mu\) are \(T\)-weights of \(L/Q\), so that \(V_{-1, \pm 3\mu} \neq (0)\). Pick \(u_{\pm 3\mu} \in V_{-1, \pm 3\mu} \setminus \{0\}\). As \(p = 5\), we have \([u_{3\mu}, u_\mu] \in L_{-\mu}\). Then \([u_{3\mu}, u_\mu] = ru_{-\mu} + q_{-\mu}\) for some \(r \in F\) and \(q_{-\mu} \in Q_{(1),-\mu}\). Since \(\text{rad} \ L(\mu) \subset Q_{(1)}\) and the image of \(L(\mu) \cap Q_{(1)}\) in
\(L(\mu) \subset H(2;1)(1)\) contains \(H(2;1)(2)\), it is easy to see that there is \(w_\mu \in Q_{(1),\mu}\) such that
\[
[u_{3\mu}, w_\mu] \equiv ru_{-\mu} \pmod{Q_{(1)}}.
\]
Put \(v_\mu := u_\mu - w_\mu\). Then \([u_{3\mu}, v_\mu] \in Q_{(1)}\). Now set
\[
V_0 := (H_0 \otimes F) \oplus \bigoplus_{\mu \in \Phi^\prime} Fv_\mu.
\]
By construction, we have \(Q_{(1)} \cap V_0 = (0)\) and \((V_0 + Q_{(1)})/Q_{(1)} = A(L,T)_0\). As \(Q_{(1)} \cap V_0 = (0)\), this yields assertion 2). In view of the new Proposition 2.4(3a)
our choice of \(V_0\) ensures that
\[
[V_0, V_{-1}] \subset V_{-1} + Q_{(1)}.
\]
To prove assertions 3b), 3c), 3f), 3g) and 3j) one can argue as in the original proof. Assertion 3e) follows from the new Proposition 2.4(3d) and the displayed inclusion. Assertions 3h), 3i) and 3k) follows from the displayed inclusion and 3e) by the same argument as in the original proof. □

**Lemma 2.6** (new). Let \(u \in L_\alpha\), \(f \in (V_0 + Q(\Delta))_\beta\) and \(v \in L_{\alpha+\beta}\) be such that \(u \not\in Q\) and \([f, u] - v \in Q\). Then \([f, u] - v \in Q_{(1)}\).

*Proof.* Write \(u = u_\alpha + u'\) with \(u_\alpha \in V_{-1}\) and \(u' \in Q_{\alpha}\). Then \([f, u_\alpha] \in V_{-1} + Q_{(1)}\) thanks to Proposition 2.5(3e), while Proposition 2.4(3b) yields \(u' \in Q_{(1)}\). As \(Q_{(1)}\) is an ideal of \(Q\) and \(V_0 + Q(\Delta) \subset Q\), we have \([f, u'] \in Q_{(1)}\). So it remains to show that \([f, u_\alpha] - v \in Q_{(1)}\).

If \(v \not\in Q\), then the coset of \(v\) spans \(L_{\alpha+\beta}/L_{\alpha+\beta} \cap Q_{(1)}\), again by Proposition 2.4(3b). Hence the assertion holds in this case. Now suppose \(v \in Q\). If \(v \in Q_{(1)}\), we are done; so suppose for a contradiction that \(v \not\in Q_{(1)}\). Then the new Proposition 2.4(3d) yields that \(\alpha\) is Hamiltonian, \(p = 5\), and \(\beta = 2\alpha\). On the other hand, it is immediate from Corollary 2.3(3) that \(2\Phi_{[-1]} \cap \Delta = \emptyset\). This forces \(f \in V_0\). But then
\[
[f, u_\alpha] \in [V_{-1}, V_0] \cap Q \subset Q_{(1)},
\]
by our choice of \(V_0\). This completes the proof. □

With these substitutions, all arguments in [St 94] work. As a result, we obtain

**Theorem 6.4** (new). Let \(L\) be a finite-dimensional simple Lie algebra over \(F\) such that all tori of maximal dimension in \(L_p\) are standard. Let \(T\) be an optimal torus in \(L_p\) and suppose that \(Q(L,T) \neq L\) and \(L[\alpha, \beta]^{(1)} \cong H(2;1;\Phi(\tau))^{(1)}\) for some \(\alpha, \beta \in \Gamma(L,T)\). Then \(L\) is isomorphic to a Lie algebra of Cartan type.

We summarize the results of this paper and of [P-St 04] as follows:

**Theorem 6.2.** Let \(L\) be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic \(p > 3\). If all tori of maximal dimension in the the semisimple \(p\)-envelope \(L_p\) of \(L\) are standard, then \(L\) is isomorphic to a Lie algebra of classical or Cartan type.
Proof. If $Q(L, T) = L$, then all roots in $\Gamma(L, T)$ are either solvable or classical. So the assertion follows from [P-St 04, Theorems C and D] in this case. If $Q(L, T) \neq L$, the assertion follows from Theorem 6.1 and the new Theorem 6.4. □

Remark 6.1. The argument in the last two paragraphs of [P-St 04, p. 792] can be streamlined as follows: When $\alpha$ is solvable with $\alpha(H) \neq 0$, Proposition 3.8 of [P-St 04] yields that $[L_{\alpha}, L_{-\alpha}]$ consists of $p$-nilpotent elements of $L_p$. But then $[L_{\alpha}, L_{-\alpha}] \subset \text{nil } H \subset \text{nil } \tilde{H}$, contrary to our choice of $\alpha$. Therefore, $[L_{\gamma}, L_{-\gamma}] \subset \text{nil } \tilde{H}$ whenever $\gamma(\tilde{H}^{(1)}) \neq 0$.

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