On clique numbers of colored mixed graphs

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Abstract

An \((m, n)\)-colored mixed graph, or simply, an \((m, n)\)-graph is a graph having \(m\) different types of arcs and \(n\) different types of edges. A homomorphism of an \((m, n)\)-graph \(G\) to another \((m, n)\)-graph \(H\) is a vertex mapping that preserves adjacency; and the type and direction of the adjacency. An \((m, n)\)-relative clique of \(G\) is a vertex subset \(R\) whose images are always distinct under any homomorphism of \(G\) to any \(H\). The maximum cardinality of an \((m, n)\)-relative clique of a graph is called the \((m, n)\)-relative clique number of the graph. In this article, we explore the \((m, n)\)-relative clique numbers for three different families of graphs, namely, graphs having bounded maximum degree \(\Delta\), subcubic graphs, partial 2-trees and planar graphs and provide tight or close bounds in most cases.

Keywords: colored mixed graph, clique number, degree, subcubic, partial 2-trees, planar graphs.

1 Introduction

In 2000, Nešetřil and Raspaud \cite{10} generalized the concepts of graph homomorphism, coloring and chromatic number of a graph by introducing colored homomorphisms and \((m, n)\)-chromatic numbers of \((m, n)\)-graphs. Later, Bensmail, Duffy, and Sen \cite{1} introduced and studied two more parameters closely related to that of \((m, n)\)-chromatic number, namely, the \((m, n)\)-relative clique number and the \((m, n)\)-absolute clique number. While the latter \cite{1} had focused majorly on studying the \((m, n)\)-absolute clique number, the focus of this article is to study the \((m, n)\)-relative clique numbers for different families of graphs.

An \((m, n)\)-\textit{colored mixed graph}, or simply, an \((m, n)\)-graph \(G\) is a graph having \(m\) different types of arcs and \(n\) different types of edges. The set of vertices, arcs, edges, and the underlying undirected graph of \(G\) are denoted by \(V(G)\), \(A(G)\), \(E(G)\), and \textit{und}(\(G\)) respectively. In this article, we will restrict ourselves to \((m, n)\)-graphs whose underlying graph is a simple graph only.

Let us fix a convenient convention: whenever we speak of an \((m, n)\)-graph \(G\), we imagine that the arcs of \(G\) are labeled (colored) by one of the symbols \(1, 2, \ldots, m\), and that the edges of \(G\) are labeled by one of \(m+1, m+2, \ldots, m+n\). Equivalently, we shall think of \(\sigma\) to be a function from \(A(G) \rightarrow \{1, 2, \ldots, m\}\) and \(E(G) \rightarrow \{m+1, m+2, \ldots, m+n\}\) that denotes the labels on the arcs/edges. Thus, for an arc/edge \(uv\) with label \(k\), we have \(\sigma(uv) = k\). Furthermore, if \(uv\) is an arc, we shall set the convention of having \(\sigma(vu) = -\sigma(uv)\), and thus increasing the domain and range of \(\sigma\). However, as \(uv\) is an edge if and only if \(vu\) is an edge, we shall simply have \(\sigma(vu) = \sigma(uv)\).

A \textit{(colored) homomorphism} \cite{10} of an \((m, n)\)-graph \(G\) to an \((m, n)\)-graph \(H\) is a function \(f : V(G) \rightarrow V(H)\) such that for any \(uv \in A(G) \cup E(G)\), we have \(f(u)f(v) \in A(H) \cup E(H)\) and \(\sigma(uv) = \sigma(f(u)f(v))\). Moreover, whenever a homomorphism \(G\) to \(H\) is understood to exist, we simply denote it by \(G \rightarrow H\).

Then, the \((m, n)\)-\textit{chromatic number} of \(G\) is given by

\[\chi_{m,n}(G) = \min\{|V(H)| : G \rightarrow H\}.
\]
An \( (m, n) \)-relative clique \([1]\) \( R \) of \( G \) is a vertex subset, i.e. \( R \subseteq V(G) \), such that \( f(u) \neq f(v) \) for all distinct \( u, v \in R \) under any homomorphism \( f \) of \( G \) to any \((m, n)\)-graph \( H \). The \((m, n)\)-relative clique number of \( G \) is then given by

\[
\omega_r(m, n)(G) = \max\{|R| : R \text{ is an } (m, n)\text{-relative clique of } G\}.
\]

On the other hand, an \((m, n)\)-absolute clique \([1]\) \( A \) is an \((m, n)\)-graph such that \( \chi_{m,n}(A) = |V(A)| \); and the \((m, n)\)-absolute clique number of \( G \) is given by

\[
\omega_a(m, n)(G) = \max\{|V(A)| : A \text{ is subgraph of } G \text{ and is an } (m, n)\text{-absolute clique}\}.
\]

Given \( p \in \{\chi_{m,n}, \omega_r(m, n), \omega_a(m, n)\} \), for a family \( F \) of undirected simple graphs, the above mentioned parameters are defined as

\[
p(F) = \max\{p(G) : \text{und}(G) \in F\}.
\]

A direct observation \([1]\) from the definitions gives us a relation among the above three parameters, namely, for any \((m, n)\)-graph \( G \),

\[
\omega_a(m, n)(G) \leq \omega_r(m, n)(G) \leq \chi_{m,n}(G).
\]

To date, one of the major directions of research for studying colored homomorphisms of colored mixed graphs has been via establishing lower and upper bounds of \( \chi_{m,n}(F) \) for different well-known graph families \([10][8][4]\). Apart from that, for specific (small) values of \((m, n)\), the bounds of \( \omega_r(m, n)(F) \) and \( \omega_a(m, n)(F) \) have also been studied \([7][9][3]\). However, for general values of \((m, n)\), the bounds for \( \omega_r(m, n)(F) \) and \( \omega_a(m, n)(F) \) have not been studied extensively except for in \([1]\). This work is a sequel to \([1]\) where we majorly explore the parameter \( \omega_r(m, n)(F) \) for different interesting families of graphs.

In what follows, we present bounds of the two parameters namely the \((m, n)\)-absolute and the \((m, n)\)-relative clique numbers of some graph classes \( G \). In Section 2 some definitions and notation are given. Also we prove some basic lemmas which play important roles to prove the theorems in the subsequent sections. In Section 3 we prove upper bounds of the \((m, n)\)-absolute and the \((m, n)\)-relative clique numbers of graphs having bounded maximum degree. In Section 4 we study the same parameters for subcubic graphs and provide tight bounds for each \((m, n)\). In Section 5 we study the parameters for partial 2-trees and also partial 2-trees having girth at least \( g \) and provide tight bounds for all \( g \geq 3 \). In Section 6 we provide bounds of the same parameters for planar graphs having girth at least \( g \) for each value of \( g \geq 3 \). For \( g = 3, 4 \) we provide upper and lower bounds, but we do prove tight bounds for all other values of \( g \). In Section 7 we provide conclusive remarks along with two interesting conjectures.

## 2 Preliminaries

We follow West \([12]\) for all standard notation and terminology in graph theory, and as far as the non-standard notation are concerned, we introduce them here. If \( uv \) is an arc/edge or if \( vu \) is an arc having \( \sigma(uv) = \alpha \), then \( v \) is called an \( \alpha \)-neighbor of \( u \) and the set of all \( \alpha \)-neighbors of \( u \) is denoted by \( N^\alpha(u) \). Moreover, all \( \alpha \)-neighbors of \( u \) are said to agree on \( u \) by the adjacency type \( \alpha \).

A special 2-path connecting two vertices \( u \) and \( v \) is a 2-path \( uvw \) such that \( \sigma(uv) \neq \sigma(vw) \). In an \((m, n)\)-graph, a vertex \( u \) is said to see a vertex \( v \) if they are either adjacent, or are connected by a special 2-path. If \( u \) and \( v \) are connected by a special 2-path with \( w \) as the internal vertex, then it is said that \( u \) sees \( v \) through \( w \) or equivalently, \( v \) sees \( u \) through \( w \).

A particularly handy characterization of a relative clique is the following:

**Lemma 2.1.** \([1]\) Two distinct vertices of an \((m, n)\)-graph \( G \) are part of a relative clique if and only if they are either adjacent or connected by a special 2-path in \( G \).

For the convenience of reference, we are going to name a particular notion that will be used time and again in our proofs. Let \( F \) be a graph family. A critical \((m, n)\)-relative clique \( H \) of \( F \) is an \((m, n)\)-graph \( H \) satisfying the following properties:

(i) \( \text{und}(H) \in F \),

(ii) \( \omega_r(m, n)(H) = \omega_a(m, n)(H) \).

(iii) \( \chi_{m,n}(H) = |V(H)| \).

(iv) There exists an \((m, n)\)-relative clique of \( F \) that is strictly larger than \( H \).
(ii) \(\omega_r(m,n)(H) = \omega_r(m,n)(F)\),

(iii) \(\omega_r(m,n)(H^*) < \omega_r(m,n)(F)\) if \(|V(H^*)|, |E(\text{und}(H^*))| < (|V(H)|, |E(\text{und}(H))|)\) in the dictionary ordering, where \(H^* \in F\).

Whenever we consider a critical \((m,n)\)-relative clique \(H\) of graph family \(F\), we shall assume that \(H\) contains a relative clique \(R\) of cardinality \(\omega_r(m,n)(H)\). To that end, the vertices of \(R\) are called good vertices and those of \(S = V(H) \setminus R\) are called the helper vertices. Therefore, whenever we consider a critical relative clique \(H\) for a graph family \(F\), the symbols \(R\) and \(S\) and the terms good and helper vertices should make sense. Due to the minimality of \(H\), we can directly observe the following:

**Lemma 2.2.** Any critical \((m,n)\)-relative clique \(H\) of a subgraph closed family \(F\) is connected and \(S\) is an independent set.

**Proof.** If \(H\) is not connected, then one of its components, say, \(H_1\) must have \(\omega_r(m,n)(H) = \omega_r(m,n)(H_1)\). This contradicts the criticality of \(H\), and thus \(H\) is connected.

If there is an arc or an edge \(e\) between two vertices of \(S\), then note that \(e\) does not contribute to a pair of vertices of \(R\) seeing one another. Thus, we will have \(\omega_r(m,n)(H) = \omega_r(m,n)(H - \{e\})\), contradicting the criticality of \(H\). Hence, \(S\) is an independent set. \(\square\)

Let \(G \in F\) be any graph having a vertex \(u\) of degree at most \(k\). Let \(G^*\) be the graph obtained by adding edges between every non-adjacent pair of neighbors of \(u\) and then deleting the vertex \(u\). If the family \(F\) is such that, for any \(G \in F\), we also have \(G^* \in F\), then we call \(F\) a \(k\)-closed graph family.

**Lemma 2.3.** Let \(H\) be a critical \((m,n)\)-relative clique of a \(k\)-closed family \(F\). Then any vertex in \(H\) having degree \(k\) or less is a good vertex.

**Proof.** If there is a helper \(h\) of degree \(k\) or less in \(H\), then we can delete \(h\) and make its non-adjacent neighbors adjacent to each other by adding some extra arcs/edges to obtain \(H^* \in F\). Note that, \(R\) is still an \((m,n)\)-relative clique in \(H^*\). This contradicts the fact that \(H\) is a critical \((m,n)\)-relative clique. \(\square\)

### 3 Graphs with bounded maximum degree

Let \(G_\Delta\) denote the family of graphs having maximum degree \(\Delta\). It is known that [2] the \((m,n)\)-chromatic number of \(G_\Delta\) is lower bounded by \((2m + n)^\frac{1}{2}\Delta^2\) and for connected graphs, is also upper bounded by \(2(\Delta - 1)^2m + (2m + n)\Delta - 2 + 2\). This means that the \((m,n)\)-chromatic number of \(G_\Delta\) is exponential in \(\Delta\).

However, while studying the values of the \((m,n)\)-absolute and the \((m,n)\)-relative clique numbers of \(G_\Delta\), we find that their values are of a drastically smaller order of \(\Omega(\Delta^2)\).

More interestingly, we find that the values of the parameters are also connected to the famous degree-diameter problem. So, if \(v(2, \Delta)\) denotes the maximum number of vertices of a graph of diameter 2 and maximum degree \(\Delta\), the connection is the following.

**Theorem 3.1.** For all \((m,n) \neq (0,1)\), we have

(i) \(\nu(2, \Delta) = \omega_a(m,n)(G_\Delta) \leq \omega_r(m,n)(G_\Delta) \leq \Delta^2 + 1\) for all \(\Delta < 2m + n\).

(ii) \(\omega_a(m,n)(G_\Delta) \leq \omega_r(m,n)(G_\Delta) \leq \Delta^2 + 1\) for \(\Delta = 2m + n\).

(iii) \(\omega_a(m,n)(G_\Delta) \leq \omega_r(m,n)(G_\Delta) \leq \lfloor \frac{2m + n - 1}{2} \Delta^2 \rfloor + \Delta + 1\) for all \(\Delta > 2m + n\).

The proof of the above result will be contained across several lemmas in this section.

**Lemma 3.2.** For all \((m,n) \neq (0,1)\) and all \(\Delta \geq 0\), we have \(\omega_r(m,n)(G_\Delta) \leq \Delta^2 + 1\).

**Proof.** Let \(G \in G_\Delta\) with an \((m,n)\)-relative clique \(R\). It is enough to show that \(|R| \leq \Delta^2 + 1\).

Take any \(u \in R\). As \(\Delta\) is the maximum degree of \(G\), \(|N(u)| \leq \Delta\). Moreover, each vertex of \(R \setminus N[u]\) must be adjacent to a vertex of \(N(u)\) in order to see \(u\). Observe that, any vertex of \(N(u)\) can have at most \((\Delta - 1)\) neighbors other than \(u\). Therefore,

\[|R| \leq 1 + |N(u)| + |\cup_{v \in N(u)} (N(v) \setminus \{u\})| \leq (\Delta + 1) + \Delta(\Delta - 1) = \Delta^2 + 1.\]

That completes the proof. \(\square\)
Given an \((m,n)\)-graph \(G\), let \(G^2\) be the simple graph with \(V(G^2) = V(G)\) and \(E(G^2) = \{uv : u \text{ sees } v \text{ in } G\}\). Moreover, for any graph \(G\) and any \(X \subseteq V(G)\), we write \(G[X]\) to denote the subgraph of \(G\) induced by \(X\). Then, we have,

**Lemma 3.3.** If \(G\) is an \((m,n)\)-colored mixed graph with maximum degree \(\Delta\), then \(G^2\) is \((\lceil (2m+n-1)\Delta^2 \rceil + \Delta)\)-degenerate for all \((m,n) \neq (0,1)\).

**Proof.** For convenience, let us assume that \(p = 2m + n\). Thus, it is enough to prove that, for any \(X \subseteq V(G) = V(G^2)\), the subgraph \(G^2[X]\) has at least one vertex of degree less than or equal to \((\lceil (p-1)\Delta^2 \rceil + \Delta)\). We prove this using discharging.

Assume that the initial charge \(ch(x)\) of a vertex \(x\) of \(G^2\) is

\[
ch(x) = \begin{cases} 
\frac{\Delta^2}{p} & \text{if } x \in X, \\
0 & \text{if } x \notin X.
\end{cases}
\]

Thus, the total charge of the graph is \(\sum_{x \in V(G^2)} ch(x) = \frac{\Delta^2|X|}{p}\).

Next we are going to present the discharging rule.

\((R1)\): Every vertex of \(X\) donates a charge of \(\frac{\Delta^2}{p}\) to each of its neighbors in \(G^2\).

Let \(ch^*(x)\) be the updated charge of the vertices of \(G^2\) after applying \((R1)\). Observe that a vertex \(x\) with \(k\) neighbors in \(X\) has a charge \(ch^*(x) \geq \frac{\Delta^2}{p}\) as it has received \(\frac{\Delta^2}{p}\) charge from each of its \(k\) neighbors belonging to \(X\) by \((R1)\).

Let \(\pi(x)\) be the number of special 2-paths going through a degree \(k\) vertex \(x\) and linking two vertices in \(X\). Let \(\beta, \gamma \in \{-m, -(m-1), \ldots, -1, 1, 2, \ldots, m+n\}\) be distinct. Then, the number of special 2-paths through \(x\) due to its \(\beta\)-neighbors and \(\gamma\)-neighbors is equal to \(|N^\beta(x)| \cdot |N^\gamma(x)|\). Therefore, the total number of special 2-paths through \(x\) is equal to

\[
\pi(x) \leq \sum |N^\beta(x)| \cdot |N^\gamma(x)|,
\]

where \(\beta < \gamma\) are two of the \(p\) symbols from \(\{-m, -(m-1), \ldots, -1, 1, 2, \ldots, m+n\}\).

Observe that, by maximizing the sum, we can conclude that

\[
\pi(x) \leq \left\lceil \frac{(p)^2 k^2}{2} \right\rceil = \left\lceil \frac{(p-1)k^2}{2p} \right\rceil.
\]

Since \(k \leq \Delta\) and \(ch^*(x) \geq \frac{\Delta^2}{p}\), we have

\[
\pi(x) \leq \frac{(p-1)k^2}{2p} \leq \frac{(p-1)k^2}{2p} \leq \frac{(p-1)ch^*(x)}{2}.
\]

Note that, the above equation provides an upper bound on the number of edges in \(G^2[X]\) that are there due to special 2-paths through \(x\). On the other hand, by the Handshaking lemma, \(G[X]\) has at most \(\frac{\Delta}{2}|X|\) edges, those which \(G^2[X]\) inherits from \(G[X]\). Furthermore, as the total charge of the graph is constant, we have \(\sum_{x \in V(G^2)} ch(x) = \sum_{x \in V(G^2)} ch^*(x)\). Hence, the total number of edges in \(G^2[X]\) will be at most

\[
\frac{\Delta}{2}|X| + \frac{p-1}{2} \sum_{x \in V(G^2)} ch^*(x) = \left(\frac{p-1}{p} \cdot \Delta^2 + \Delta\right) \frac{|X|}{2}.
\]

Thus, there exists at least one vertex in \(G^2[X]\) having degree at most

\[
\left(\frac{p-1}{p} \cdot \Delta^2 + \Delta\right).
\]

This completes the proof. \(\square\)
The above result is a generalization of a result due to Gonçalves, Raspaud and Shalu. Next we are going to show that, indeed, we have \( \nu(2, \Delta) = \omega_{a(m,n)}(G_{\Delta}) \) for all \( \Delta < 2m+n \).

**Lemma 3.4.** For all \( \Delta < 2m+n \), we have \( \nu(2, \Delta) = \omega_{a(m,n)}(G_{\Delta}) \).

**Proof.** Notice that there must exist an \( (m,n) \)-absolute clique \( H \in G_{\Delta} \) satisfying \( |V(H)| = \omega_{a(m,n)}(G_{\Delta}) \). Therefore, \( \text{und}(H) \) is a graph with diameter at most 2 and maximum degree at most \( \Delta \). Hence, \( \omega_{a(m,n)}(G_{\Delta}) = |V(H)| \leq \nu(2, \Delta) \).

On the other hand, there exists an undirected graph \( U \) on \( \nu(2, \Delta) \) vertices having diameter 2 and maximum degree \( \Delta \). We will construct an \( (m,n) \)-absolute clique \( U^* \) whose underlying graph is \( U \).

Observe that, \( U \) is a \( (\Delta+1) \)-edge colorable due to Vizing’s Theorem [12]. Consider a proper edge coloring of \( U \) using \( 1,2,\ldots, \Delta+1 \). Notice that, as \( 2m+n > \Delta \), it is possible to find non-negative integers \( m_1 \leq m \) and \( n_1 \leq m + n - m_1 \) satisfying \( 2m_1 + n_1 = \Delta + 1 \).

Now, for each \( i \leq m_1 \), consider the subgraph of \( U \) induced by the edge colors \( (2i-1) \) and \( 2i \). For convenience, call it \( U_i \). Notice that, each vertex of \( U_i \) has degree at most 2. This means that \( U_i \) is a disjoint union of paths and cycles. Now, replace each of these paths/cycles by a directed path/cycle having arcs of label \( i \) to (partially) construct \( U^* \). Moreover, for \( m_1 < j \leq m_1 + n_1 \), consider the subgraph \( U_j \) induced by the edge color \( 2m_1 + j \). The graph \( U_j \) is a matching. We replace each edge of \( U_j \) by an arc/edge of label \( j \) (orientation of arcs can be chosen randomly) to construct \( U^* \).

Observe that, the \( U^* \), so obtained, is an \( (m,n) \)-graph. Moreover, every vertex \( x \) of \( U^* \) has \( \sigma(xy) \neq \sigma(xz) \) for distinct \( y, z \in N(x) \). Therefore, \( U^* \) is indeed an \( (m,n) \)-absolute clique having \( \text{und}(U^*) = U \). \( \Box \)

Now, we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** The result follows directly from Lemmas 3.2, 3.3 and 3.4. \( \Box \)

Das, Prabhu and Sen [8] had shown that \( \omega_{a(1,0)}(G_{\Delta}) = \Omega(\Delta^2) \). They proved the lower bound using examples. The same examples will imply a quadratic (in \( \Delta \)) lower bound for \( \omega_{a(m,n)}(G_{\Delta}) \) for all \( (m,n) \) where \( m \geq 1 \). Therefore, together with Theorem 3.1 we can conclude that \( \omega_{a(m,n)}(G_{\Delta}) = \Theta(\Delta^2) \) and \( \omega_{r(m,n)}(G_{\Delta}) = \Theta(\Delta^2) \) whenever \( m \geq 1 \).

## 4 Subcubic graphs

In this section, we study the same parameters for subcubic graphs and we manage to provide tight bounds in each case. That is, this section is dedicated to a special case \( \Delta = 3 \) of the previous section.

**Theorem 4.1.** For all \( (m,n) \neq (0,1) \) we have

(i) \( \omega_{a(1,0)}(G_3) = \omega_{r(1,0)}(G_3) = 7 \).

(ii) \( \omega_{a(0,n)}(G_3) = \omega_{r(0,n)}(G_3) = 8 \) for \( n = 2,3 \).

(iii) \( \omega_{a(m,n)}(G_3) = \omega_{r(m,n)}(G_3) = 10 \) for \( (m,n) = (1,1) \) and for \( 2m+n \geq 4 \).

We prove Theorem 4.1 in a series of lemmas.

**Lemma 4.2.** For \( (m,n) = (1,1) \) and for \( 2m+n \geq 4 \), we have \( \nu(2,3) = \omega_{a(m,n)}(G_3) = \omega_{r(m,n)}(G_3) = 10 \).

**Proof.** We have \( \omega_{r(m,n)}(G_3) \leq 10 \) by Lemma 5.2 for any \( (m,n) \neq (0,1) \). Also, as \( \nu(2,3) = 10 \), uniquely realized by the Petersen Graph, we have \( \omega_{a(m,n)}(G_3) = \omega_{r(m,n)}(G_3) = 10 \) for \( 2m+n \geq 4 \) due to Theorem 3.1(i).

In the case that \( (m,n) = (1,1) \), the proof rests in showing that it is possible to convert the Petersen graph into an \( (1,1) \)-absolute clique (see Fig 1(a)). \( \Box \)

As \( \omega_{a(1,0)}(G_3) = \omega_{r(1,0)}(G_3) = 7 \) is proved by Das, Prabhu and Sen [8], we are only left with the cases when \( (m,n) = (0,2) \) and \( (0,3) \).

**Lemma 4.3.** We have \( \omega_{a(0,2)}(G_3) \geq 8 \) and \( \omega_{r(0,3)}(G_3) \leq 8 \).
Proof. The first bound is implied by the $(0, 2)$-absolute clique on 8 vertices depicted in Fig. 1(b).

For the second bound, let $H$ be a critical $(0, 3)$-relative clique of $G$. By Lemma 3.2 we have that $|R| \leq 10$.

Let $u$ be a good vertex of $H$. Note that, it can have at most three good neighbors and six good second neighbors. Thus, to have $|R| = 10$, all nine of its neighbors and second neighbors of $u$ must be good. In particular, this implies that any good vertex is non-adjacent to a vertex of $S$ and hence, $S = \emptyset$. Therefore, $H$ must be a $(0, 3)$-absolute clique.

As the Petersen graph is the only cubic graph on 10 vertices having diameter at most 2, and as any two non-adjacent vertices of the Petersen graph are connected by a unique 2-path, we can conclude that $H$ is a $(0, 3)$-absolute clique if and only if there exists a proper 3-edge-coloring of $H$. However, we know that the chromatic index of the Petersen graph is 4 and so, it is not possible to have a 3-edge-coloring of $H$.

Therefore, $\omega_r(0, 3)(G) \leq 9$. Moreover, due to the handshaking lemma, there does not exist any cubic graph on 9 vertices. Hence, $\omega_r(0, 3)(G) \leq 8$. \hfill $\square$

Finally, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. The proof of (ii) and (iii) follow directly from Lemmas 4.3 and 4.2 respectively. \hfill $\square$

5 Partial 2-trees

For the family of outerplanar graphs, Bensmail, Duffy and Sen [11] provided a tight bound of $3(2m + n) + 1$ for its $(m, n)$-relative clique number. Here we consider a superfamily of it, namely, the family of partial 2-trees, or equivalently, the family of $K_4$-minor-free graphs and provide tight bounds in this case too. Furthermore, we also provide tight bounds for some sparse subfamilies of partial 2-trees. As, usually, many bounds and values of such parameters are the same for the families of outerplanar graphs and partial 2-trees, the difference in our case comes as a surprise.

Let $T^2_g$ denote the family of partial 2-trees having girth at least $g$, where girth of a graph is the length of its smallest cycle. All bounds are tight in this section.

Theorem 5.1. For all $(m, n) \neq (0, 1)$, we have

(i) $\omega_a(m, n)(T^2_g) = \omega_r(m, n)(T^2_g) = (2m + n)^2 + (2m + n) + 1$.

(ii) $\omega_a(m, n)(T^2_4) = \omega_r(m, n)(T^2_4) = (2m + n)^2 + 2$.

(iii) $\omega_a(m, n)(T^2_5) = \omega_r(m, n)(T^2_5) = \max(2m + n + 1, 5)$ for $(m, n) \neq (0, 2)$.

(iv) $\omega_a(0, 2)(T^2_5) = 3$ and $\omega_r(0, 2)(T^2_5) = 4$.

(v) $\omega_a(m, n)(T^2_g) = \omega_r(m, n)(T^2_g) = (2m + n) + 1$ for all $g \geq 6$. 

\hfill $6$
We will handle the higher girth cases first. To begin with, let us prove a result for trees.

**Proposition 5.2.** For the family $\mathcal{T}$ of trees, we have
\[
\omega_a(m, n)(\mathcal{T}) = \omega_r(m, n)(\mathcal{T}) = (2m + n) + 1
\]
for all $(m, n) \neq (0, 1)$.

**Proof.** Let $H$ be a critical $(m, n)$-relative clique of $\mathcal{T}$. We know that helpers cannot have degree 1 or less due to Lemma 2.3. Thus, there exists a good vertex $u$ of degree one, with a neighbor $v$ (say) in $H$. Every good second neighbor $w$ of $u$ must have $\sigma(uw) \neq \sigma(vw)$. Thus, two good neighbors of $v$ cannot have the same type of adjacency with $v$. Moreover, there cannot be any arc/edge or special 2-path, other than the one through $v$, between the neighbors of $v$, as $H$ is a tree. Therefore, $v$ can have at most $(2m + n)$ good neighbors.

On the other hand, the star graph with $2m + n$ leaves, each having a distinct type of adjacency with the central vertex gives the lower bound. Hence we are done.

Next, let us handle the case of graphs having girth at least 7. To do so, let us define $F_g$ as the family of all graphs having girth $g$.

**Proposition 5.3.** For $g \geq 7$, we have
\[
\omega_a(m, n)(F_{g-1}) = \omega_r(m, n)(F_g) = (2m + n) + 1
\]
for all $(m, n) \neq (0, 1)$.

**Proof.** Let $H$ be a critical $(m, n)$-relative clique of $F_g$ having girth $g \geq 7$. Suppose $H$ contains a cycle $C = v_1v_2 \cdots v_kv_1$ of length $k$, where $k \geq 7$. By Lemma 2.2, as two helpers can never be adjacent, $C$ must have a good vertex, say, $v_1$ without loss of any generality.

Note that, if we add an arc/edge or a special 2-path between $v_1$ and either $v_4$ or $v_5$, then a cycle of length 6 or less is created. Thus, both $v_4$ and $v_5$ must be helpers. However, as $v_4$ and $v_5$ are adjacent, this is a contradiction to Lemma 2.2. Thus, $H$ does not contain a cycle, which implies that $H$ is a tree.

If however, $H \in F_{g-1}$ is an $(m, n)$-absolute clique and $C = v_1v_2 \cdots v_lv_1$, where $l \geq 6$, is a cycle, we can see that it is not possible for $v_1$ to see $v_4$ without violating the girth restrictions. Thus, $H$ is a tree in this case too.

Hence, the proof follows from Proposition 5.2.

Now, let us examine the situation for $(m, n)$-relative clique number of girth 6 graphs.

**Lemma 5.4.** Let $H$ be a critical $(m, n)$-relative clique of $F_6$ having a 6-cycle $C = v_1v_2 \cdots v_6v_1$, where $v_1$ is a good vertex. Then, $v_3$ and $v_5$ are also good vertices while $v_2, v_4$ and $v_6$ are helpers. Moreover, any good vertex $v$, other than $v_1, v_3$ and $v_5$, must be connected to each of the latter by internally disjoint special 2-paths.

**Proof.** Notice that two antipodal vertices of $C$ cannot be good vertices, as an arc/edge or special 2-path connecting them will contradict the girth restrictions. Thus, $v_4$ is a helper as it is antipodal to $v_1$. This implies that $v_3$ and $v_5$ are good due to Lemma 2.2. By the same argument then, $v_2$ and $v_6$ are helpers, as they are antipodals to $v_3$ and $v_5$ respectively.

As for the last part of the Lemma, observe that the only way for a vertex $v$ that is not part of the cycle $C$ to be able to see $v_1, v_3$ and $v_5$ are through distinct helpers due to the girth restrictions.

**Lemma 5.5.** We have
\[
\omega_r(m, n)(\mathcal{T}_6^2) = (2m + n) + 1
\]
for all $(m, n) \neq (0, 1)$.

**Proof.** Let $H$ be a critical $(m, n)$-relative clique of $\mathcal{T}_6^2$. If $H$ contains a 6-cycle and $\omega_r(m, n)(H) \geq 4$, then it will force a $K_4$-minor in $H$ by Lemma 5.4. As partial 2-trees are $K_4$-minor-free, this is a contradiction. Therefore, if $H$ contains a 6-cycle, we must have $\omega_r(m, n)(H) \leq 3 \leq (2m + n) + 1$ for all $(m, n) \neq (0, 1)$.

On the other hand, if $H$ does not contain any 6-cycle, then $H \in \mathcal{T}_g^2$, where $g \geq 7$, and we are done due to Proposition 5.3.
This, along with Proposition 5.2 proves Theorem 5.1(v). Now, we are going to handle the case where the girth is at least 5. However, during the proof, we shall show something useful for planar graphs also.

**Lemma 5.6.** Let $H$ be a critical $(m, n)$-relative clique of $F_5$ having a 5-cycle $C = v_1v_2 \cdots v_5v_1$ where $v_1, v_3$ and $v_4$ are good vertices. Then any other good vertex $v$, not contained in $C$, must be either adjacent or connected by internally disjoint special 2-paths with $v_1, v_3$ and $v_4$. Moreover, the above-mentioned paths connecting $v$ to $v_3$ and $v_4$ must be special 2-paths only.

**Proof.** Follows from the girth restrictions directly. □

Now, we are ready to handle the girth 5 case.

**Lemma 5.7.** We have

$$\omega_{a(m,n)}(T_5^2) = \omega_{r(m,n)}(T_5^2) = \max(2m + n + 1, 5)$$

for all $(m, n) \neq (0, 1), (0, 2)$.

**Proof.** Observe that $\nu(2, 2) = 5$ and it is realized by the 5-cycle. So, taking $U$ as a 5-cycle in the proof of Lemma 5.4 we can transform the 5-cycle into an $(m, n)$-absolute clique for $2m + n \geq 3$. Moreover, for $(m, n) = (1, 0)$, the directed 5-cycle is a $(1, 0)$-absolute clique. Thus, $5 \leq \omega_{a(m,n)}(T_5^2) \leq \omega_{r(m,n)}(T_5^2)$ for $(m, n) \neq (0, 1), (0, 2)$.

Moreover, if $H$ contains a 5-cycle, then the former must also contain at least 3 good vertices due to Lemma 5.2. Thus, if $\omega_{r(m,n)}(H) \geq 3$, then $H$ will have a $K_4$-minor by Lemma 5.6. This is a contradiction, as $H$ is a partial 2-tree and is, hence, $K_4$-minor-free. Therefore, we are done in the case when $H$ contains a 5-cycle.

However, if $H$ does not contain a 5-cycle, then we are done by Lemma 5.6, Proposition 5.2 and Proposition 5.3. □

This proves Theorem 5.1(iii) and leaves us with the corner cases of figuring out the values of $\omega_{a(0,2)}(T_5^2)$ and $\omega_{r(0,2)}(T_5^2)$.

**Lemma 5.8.** We have

$$\omega_{a(0,2)}(T_5^2) = 3 \quad \text{and} \quad \omega_{r(0,2)}(T_5^2) = 4.$$

**Proof.** Let $H \in T_5^2$ be a $(0, 2)$-absolute clique. As 5-cycle is not an underlying graph of a $(0, 2)$-absolute clique, by Lemma 5.4, Proposition 5.2 and Proposition 5.3, $\omega_{a(0,2)}(T_5^2) = 3$.

Next, let $C = v_1v_2v_3v_4v_1$ be a 5-cycle and let $\sigma(v_1v_2) = \sigma(v_3v_4) = \sigma(v_5v_1) = 1$ and $\sigma(v_2v_3) = \sigma(v_4v_5) = 2$.

Then, $(v_1, v_2, v_3, v_4)$ is a $(0, 2)$-relative clique of $C$. Since $C \in T_5^2$, $\omega_{r(m,n)}(T_5^2) \geq 4$. Now, let $H$ be a critical $(0, 2)$-relative clique of $T_5^2$ such that it has a 5-cycle $C$. Then, $H$ has at least 3 good vertices by Lemma 2.2 and at most 4 good vertices by the girth restrictions. So, $|R| \geq 5$ implies that there exists a vertex $v \in R$ which is not part of $C$. This, in turn, implies a $K_4$-minor in $H$ by Lemma 5.6. A contradiction to $H$ being a partial 2-tree. Thus, $\omega_{r(0,2)}(T_5^2) = |R| \leq 4$ when $H$ has a 5-cycle. If however, $H$ does not have a 5-cycle, then, again, using Lemma 5.6 and Proposition 5.3, $\omega_{r(0,2)}(T_5^2) = |R| \leq 3 < 4$, a contradiction to $\omega_{r(m,n)}(T_5^2) \geq 4$; and so, the case does not arise.

This proves Theorem 5.1(iv).

As we have finished dealing with the higher girth cases, let us move to the lower girth cases now. We start with proving the lower bounds.

**Lemma 5.9.** There exists an $(m, n)$-absolute clique on $(2m + n)^2 + (2m + n) + 1$ vertices whose underlying graph is a partial 2-tree.

**Proof.** Take an $(m, n)$-graph $X$ whose underlying graph is $K_{1,2m+n}$ with the degree $(2m + n)$ vertex being $u$. Also, assign adjacencies in such a way that $\sigma(uv) \neq \sigma(vu)$ for all $v_1, v_2 \in N(u)$. This is an $(m, n)$-absolute clique on $(2m + n) + 1$ vertices.

Now, take $(2m + n)$ copies of $X$ and name them $X_{-m}, X_{-(m-1)}, \cdots X_{-1}, X_1, X_2, \cdots, X_{m+n}$. Furthermore, take a new vertex $x$ and make it adjacent to all vertices of $X_i$ in such a way that $N^i(x) = X_i$ for all $i \in \{-m, -(m-1), \cdots, -1, 1, 2, \cdots, m+n\}$. Note that this new graph so obtained is an $(m, n)$-absolute clique on $(2m + n)^2 + (2m + n) + 1$ vertices whose underlying graph is a partial 2-tree. □
Next is the turn to prove the lower bound for triangle-free partial 2-trees.

**Lemma 5.10.** There exists an \((m, n)\)-absolute clique on \((2m + n)^2 + 2\) vertices whose underlying graph is a triangle-free partial 2-tree.

**Proof.** Take an \((m, n)\)-graph \(X\) whose underlying graph is \(K_{2, (2m + n)^2}\) with the degree \((2m + n)^2\) vertices being \(u_1\) and \(u_2\). Now, assign adjacencies in such a way that

\[
(\sigma(u_1v_1), \sigma(u_2v_1)) \neq (\sigma(u_1v_2), \sigma(u_2v_2))
\]

for all \(v_1, v_2 \in N(u_1) \cap N(u_2)\). This is an \((m, n)\)-absolute clique on \((2m + n)^2 + 2\) vertices whose underlying graph is a triangle-free partial 2-tree. \(\Box\)

With both the lower bounds proved, we will now engage in proving the upper bounds. However, before that, we will describe some common groundwork for both the cases.

Let \(H_g\) be a critical \((m, n)\)-relative clique of \(T_m^n\), where \(g \in \{3, 4\}\). Therefore, by Lemma 2.8, \(H_g\) contains a degree two good vertex \(u\) with neighbors \(u_1\) and \(u_2\) (say). Moreover, every good vertex is adjacent to \(u_1\) or \(u_2\) by Lemma 2.1. Let \(Y\) be the set of good vertices adjacent to both \(u_1\) and \(u_2\), \(Y_1\) be the set of good vertices adjacent to \(u_1\) but not to \(u_2\), and \(Y_2\) be the set of good vertices adjacent to \(u_2\) but not to \(u_1\). Note that, as \(u \in Y\), we have \(|Y| \geq 1\). Also, without loss of generality, assume that \(|Y_1| \geq |Y_2|\). Based on these notations and nomenclatures, we present some lemmas that will lead us to the coveted proofs for the upper bounds.

**Lemma 5.11.** It is possible to have at most \((2m + n + 1)\) good neighbors in \(N^\alpha(v)\) for some \(v \in V(H_g)\), where \(\alpha \in \{-m, -(m - 1), \ldots, -1, 1, 2, \ldots, m + n\}\). Moreover, if \(N^\alpha(v)\) has exactly \((2m + n + 1)\) good neighbors, then the latter induce a star.

**Proof.** Suppose that a particular planar embedding of \(H_g\) is given and \(|N^\alpha(v) \cap R| \geq 2m + n + 2\). Let the good \(\alpha\)-neighbors \(v_1, v_2, \ldots, v_t\) \((t \geq 2m + n + 2)\) of \(v\) be arranged in a clockwise manner around \(v\) in the said embedding. Thus, in particular, \(v_1\) sees \(v_3\) either by some special 2-path through some \(w\) (which may be one of the \(v_i\)'s) or directly by adjacency. Hence, \(v_2\) is forced to see all \(v_j\)’s \((j \geq 4)\) either through \(v_1\) or \(v_3\) or \(w\). In any case, all \(v_i\)’s are adjacent to some vertex \(w\) (which may be one of the \(v_i\)’s) other than \(v\). Therefore, \(N^\alpha(v) \cap N(w)\) contains at least \((2m + n + 1)\) good vertices. This implies that at least two good vertices are inside \(N^\alpha(v) \cap N^\beta(w)\) where \(\beta \in \{-m, -(m - 1), \ldots, -1, 1, 2, \ldots, m + n\}\). Then, these two vertices see each other either by being adjacent or by some 2-path between them. Notice that this creates a \(K_4\)-minor in \(H_g\). Thus, \(N^\alpha(v)\) can have at most \((2m + n + 1)\) good neighbors. For the moreover part, notice that, if \(w\) as above is not one of the \(v_i\)’s and if \(N^\alpha(v)\) contains only \((2m + n + 1)\) neighbors, then also \(N^\alpha(v) \cap N(w)\) is forced to contain at least \((2m + n + 1)\) good vertices, eventually, resulting in a \(K_4\)-minor in \(H_g\). \(\Box\)

A direct corollary follows:

**Corollary 5.12.** It is possible to have at most \(2m + n + 1\) good neighbors in \(N^\alpha(v)\) for some \(v \in V(H_4)\) where \(\alpha \in \{-m, -(m - 1), \ldots, -1, 1, 2, \ldots, m + n\}\).

**Proof.** Having \(2m + n + 1\) good neighbors in \(N^\alpha(v)\) will result in triangles in \(H_4\) due to the moreover part of Lemma 5.11. \(\Box\)

After the above two structural results, we are now ready to handle the particular situation when \(Y_2 = 0\).

**Lemma 5.13.** If \(|Y_2| = 0\), then \(H_3\) has at most \((2m + n)^2 + (2m + n) + 1\) good vertices.

**Proof.** Let \(H'_3\) be the graph obtained by adding an edge between \(u_1\) and \(u_2\) (if already not present) in \(H_3\). As \(u_1\) and \(u_2\) are both adjacent to the vertex \(u\) of degree two, observe that, \(H'_3\) is also \(K_4\)-minor-free like \(H_3\). Now, since \(|Y_2| = 0\), therefore \(u_1\) is adjacent to all good vertices in \(H'_3\). Thus, by Lemma 5.11, \(H_3\) can have at most \((2m + n)^2 + (2m + n) + 1\) good vertices. \(\Box\)

**Lemma 5.14.** If \(|Y_2| = 0\), then \(H_4\) has at most \((2m + n)^2 + 2\) good vertices.
Proof. If \( |Y_2| = 0 \), then \( u_1 \) is adjacent to all good vertices, except maybe \( u_2 \). Thus, by Corollary 5.12, \( N(u_1) \) contains at most \((2m+n)^2\) good vertices.

Now, to handle the cases when \( |C_2| \geq 1 \), we need to establish a few more structural properties of \( H_g \).

**Lemma 5.15.** If \( |N(v_1) \cap N(v_2)| \geq 3 \) for some \( v_1, v_2 \in V(H_g) \), then it is possible to have at most one good vertex in \( N^\alpha(v_1) \cap N^\beta(v_2) \), where \( \alpha, \beta \in \{-m, -(m-1), \ldots, -1, 1, 2, \ldots, m+n\} \).

**Proof.** Suppose \( |N(v_1) \cap N(v_2)| \geq 3 \) and \( N^\alpha(v_1) \cap N^\beta(v_2) \) contains at least two good vertices. Then, these good vertices must see each other either by adjacency or by some special 2-path through a \( w \notin \{v_1, v_2\} \), thus forcing a \( K_4 \)-minor in \( H_g \).

**Lemma 5.16.** Let \( |Y_2| \geq 1 \). In \( H_g \), if \( u_i \) has a good \( \alpha \)-neighbor in \( Y \), then \( u_i \) cannot have a good \( \alpha \)-neighbor in \( Y_i \) for all \( i \in \{1, 2\} \).

**Proof.** We shall prove this for \( i = 1 \), as the proof for \( i = 2 \) is similar.

Let \( u' \) be a good \( \alpha \)-neighbor of \( u_1 \) in \( Y \) and let \( v \) be a good \( \alpha \)-neighbor of \( u_1 \) in \( Y_1 \). As the only way for \( u \) to see \( v \) is by a special 2-path through \( u_1 \), it is not possible to have \( u = u' \). Thus, \( u' \) sees \( v \) either by adjacency or by some special 2-path through a \( w \notin \{u, u_1, u_2\} \).

Moreover, as \( |Y_2| \geq 1 \), there is a good vertex \( v' \) in \( C_2 \). Notice that, it is not possible to have \( v' = w \), as that forces a \( K_4 \)-minor in \( H_g \). However, as \( u' \) must see \( v \), the options for that to happen are either via an arc/edge between them or via a special 2-path between them through some \( u' \notin \{u, u_1, u_2\} \). Now, both the options produce \( K_4 \)-minors in \( H_g \), expect in the case that \( w' = u' \) in \( H_3 \); and so, we turn our attention to it.

So, let \( g = 3 \) and \( u' = u' \). Notice that, in this case, each vertex of \( Y_1 \) must see each vertex of \( Y_2 \) through \( u' \) to avoid creating a \( K_4 \)-minor. Moreover, since \( H_3 \) is \( K_4 \)-minor-free, this makes \( Y, Y_1 \) and \( Y_2 \) independent, and disallows any adjacency or special 2-paths between vertices of

(i) \( Y \setminus \{u'\} \) and \( Y_i \) and
(ii) \( Y_1 \) and \( Y_2 \).

Thus, the only way for the vertices of \( (Y \setminus \{u'\}) \cup Y_1 \cup Y_2 \) to see each other is by a special 2-path through exactly one of \( u' \), \( u_1 \) and \( u_2 \). Thus, the adjacency types of the arcs/edges between \( u' \) and vertices in \( Y_1 \) are different from those of the arcs/edges between \( u' \) and vertices in \( Y_2 \). Similarly, the adjacency types of the arcs/edges between \( u_i \) and vertices in \( (Y \setminus \{u'\}) \) are different from those of the arcs/edges between \( u_i \) and vertices in \( Y_1 \).

So, assume that \( u_i \) has \( p_i \) different types of adjacencies with the vertices in \( (Y \setminus \{u'\}) \); and suppose that \( u' \) has \( q \) different types of adjacencies with the vertices in \( Y_1 \). Therefore, taking \( p = 2m+n \) for convenience, we must have

\[
|R| \leq p_1p_2 + (p - p_1)q + (p - p_2)(p - q) + 3 \leq p^2 + p_1p_2 - p_1q - p_2(p - q) + 3.
\]

Now, without loss of generality, assuming that \( p_1 \geq p_2 \), we have

\[
|R| \leq p^2 + p_1p_2 - p_2p + 3 = p^2 + p_2(p_1 - p) + 3 \leq p^2 + 2 < p^2 + p + 1,
\]

as \( 1 \leq p_2 \leq p_1 < p \) since \( u \in Y \). This is a contradiction, as \( |R| \geq p^2 + p + 1 \) due to Lemma 5.9.

These two structural results now enable us to handle the case \( |Y_2| = 1 \).

**Lemma 5.17.** If \( |Y_2| = 1 \), then \( H_3 \) has at most \((2m+n)^2 + (2m+n)+1\) good vertices.

**Proof.** Let \( v \) be the good vertex in \( Y_2 \). This implies that \( u_2 \) can have at most \((2m+n-1)\) different types of adjacencies with the good vertices in \( Y \) due to Lemma 5.16. Moreover, let \( u_1 \) have \( p_1 \leq 2m+n \) different types of adjacencies with the good vertices in \( Y \). Thus, \( u_1 \) can have at most \((2m+n-p_1)\) different types of adjacencies with the good vertices in \( Y_1 \). Furthermore, notice that, as \( u \in Y \), we have \( p_1 \geq 1 \)

Thus, there can be at most \( p_1(2m+n-1) \) good vertices in \( Y \) due to Lemma 5.15 and at most \((2m+n-p_1)(2m+n+1) \) good vertices in \( Y_1 \) due to Lemma 5.11. The only other possible good vertices
in $H_3$ are $u_1, u_2$, and $v$. Thus, taking $p = 2m + n$ for convenience, the total number of good vertices in $H_3$ is
\[ |R| \leq p_1(p - 1) + (p - p_1)(p + 1) + 3 = p^2 + p - 2p_1 + 3 \leq p^2 + p + 1. \]
The last inequality uses the fact that $p_1 \geq 1$, as $u \in Y$. \hfill \square

**Lemma 5.18.** If $|C_4| = 1$, then $H_4$ has at most $(2m + n)^2 + 2$ good vertices.

**Proof.** Let $v$ be the good vertex in $Y_2$. This implies that $u_2$ can have at most $(2m + n - 1)$ different types of adjacencies with the good vertices in $Y$ due to Lemma 5.10. Moreover, let $u_1$ has $p_1 \leq 2m + n$ different types of adjacencies with the good vertices in $Y$. Thus, $u_1$ can have at most $(2m + n - p_1)$ different types of adjacencies with the good vertices in $Y_1$. Furthermore, notice that, as $u \in Y$, we have $p_1 \geq 1$.

Thus, there can be at most $p_1(2m + n - 1)$ good vertices in $Y$ due to Lemma 5.10 and there can be at most $(2m + n - p_1)(2m + n)$ good vertices in $Y_1$ due to Lemma 5.11. The only other possible good vertices in $H_4$ are $u_1, u_2$, and $v$. Thus, taking $p = 2m + n$ for convenience, the total number of good vertices in $H_4$ is
\[ |R| \leq p_1(p - 1) + (p - p_1)p + 3 = p^2 - p_1 + 3 \leq p^2 + 2. \]
The last inequality uses the fact that $p_1 \geq 1$, as $u \in Y$. \hfill \square

We are now left with one last structural result which will allow us to handle the case when $|Y_2| \geq 2$.

**Lemma 5.19.** If $|Y_2| \geq 2$, then in $H_5$, all good vertices of $Y_1$ and $Y_2$ are adjacent to a vertex $w$. Moreover, if $w$ has a good $\alpha$-neighbor in $Y_1$, then $w$ cannot have a good $\alpha$-neighbor in $Y_2$.

**Proof.** Let us fix a planar embedding of $H_5$ and assume that in that embedding $u_1, u_2$ are arranged in a clockwise order around $u$. Furthermore, let $u_{11}, u_{12}, \ldots, u_{1t}$ denote the good neighbors of $u_1$ in $Y_1$, arranged in an anti-clockwise order around $u$ and let $u_{21}, u_{22}, \ldots, u_{2s}$ denote the good neighbors of $u_2$ in $Y_2$ arranged in a clockwise order around $u_2$. Notice that, $u_{11}$ must see $u_{22}$ either by adjacency or by a special 2-path through a vertex $w$ (which maybe one of the $u_{ij}s$). This will force all $u_{ij}s$ to be adjacent to $w$ in order to see each other and also maintain the planarity of $H_5$.

For the moreover part, observe that, any adjacency or a special 2-path between a $u_{1i}$ and a $u_{2j}$ through some vertex other than $w$ produces a $K_4$-minor in $H_5$. \hfill \square

Finally, we are ready to prove the upper bound for the case when $|Y_2| \geq 2$.

**Lemma 5.20.** If $|Y_2| \geq 2$, then $H_5$ has at most $(2m + n)^2 + 2$ good vertices.

**Proof.** According to Lemma 5.19, there exists a $w$ which is adjacent to all good vertices in $Y_1 \cup Y_2$. Moreover, suppose that $w$ has $q_1 \leq 2m + n$ different types of adjacencies with the good vertices in $Y_1$. Then, $w$ can have at most $(2m + n - q_1)$ different types of adjacencies with the good vertices in $Y_2$ due to the moreover part of Lemma 5.19.

Further, suppose that $u_i$ has $p_i \leq 2m + n$ different types of adjacencies with the good vertices in $Y_i$ where $i \in \{1, 2\}$. This implies that $u_i$ has at most $(2m + n - p_i)$ different types of adjacencies with the good vertices in $Y_i$ due to Lemma 5.16. Now, let us calculate the possible number of good vertices in $H_5$ using Lemma 5.15 and along with the observation that the only possible good vertices not contained in $Y \cup Y_1 \cup Y_2$ are $u_1$ and $u_2$. Therefore, taking $p = 2m + n$ for convenience, we have
\[ |R| \leq p_1p_2 + (p - p_1)q_1 + (p - p_2)(p - q_1) + 2 \leq p^2 + p_1p_2 - p_1q_1 - p_2(p - q_1) + 2. \]
Let us, with out loss of any generality, assume that $p_1 \geq p_2$ in the above inequality. Thus, we get
\[ |R| \leq p^2 + p_1p_2 - p_2q_1 - p_2(p - q_1) + 2 = p^2 + p_2(p_1 - p) + 2 \leq p^2 + 2. \]
The last inequality uses the fact that $1 \leq p_2 \leq p_1 < p$, as $u \in Y$. \hfill \square

Finally, we are ready to prove Theorem 5.1

**Proof of Theorem 5.1.** The proof of (i) directly follows from Lemmas 5.4, 5.13, 5.17 and 5.20. The proof of (ii) directly follows from Lemmas 5.10, 5.13, 5.18 and 5.20. The proof of (iii) directly follows from Lemma 5.7. The proof of (iv) directly follows from Lemma 5.8 and the proof of (v) directly follows from Proposition 5.2 and Lemma 5.5. \hfill \square
6 Planar graphs

Let $\mathcal{P}_g$ denote the family of planar graphs having girth at least $g$. We provide lower and upper bounds for the $(m,n)$-relative clique numbers of $\mathcal{P}_g$ for each value of $g \geq 3$.

**Theorem 6.1.** For all $(m,n) \neq (0,1)$ we have

(i) $3(2m+n)^2 + (2m+n) + 1 \leq \omega_{(m,n)}(\mathcal{P}_3) \leq \omega_{r(m,n)}(\mathcal{P}_3) \leq 42(2m+n)^2 - 11$.

(ii) $(2m+n)^2 + 2 = \omega_{d(m,n)}(\mathcal{P}_4) \leq \omega_{r(m,n)}(\mathcal{P}_4) \leq 14(2m+n)^2 + 1$.

(iii) $\max(2m+n+1,5) = \omega_{a(m,n)}(\mathcal{P}_5) \leq \omega_{r(m,n)}(\mathcal{P}_5) = \max(2m+n+1,6)$.

(iv) $2m+n+1 = \omega_{o(m,n)}(\mathcal{P}_6) \leq \omega_{r(m,n)}(\mathcal{P}_6) = \max(2m+n+1,4)$.

(v) $\omega_{a(m,n)}(\mathcal{P}_g) = \omega_{r(m,n)}(\mathcal{P}_g) = (2m+n) + 1$ for $g \geq 7$.

Let us start with proving some structural properties first.

**Lemma 6.2.** Let $H$ be a critical $(m,n)$-relative clique of $\mathcal{P}_g$ for $g = 3$ or $4$. Then the number of independent good vertices of $H$ agreeing on another vertex $v$ can be at most $2(2m+n)$, where $2m+n \geq 3$.

**Proof.** Let $v$ be a vertex of $H$ and let $v_1, v_2, \ldots, v_t$ be independent good vertices agreeing on $v$ by the adjacency type $\alpha$. Assume a fixed planar embedding of $H$ and suppose, without loss of generality, that in the said embedding $v_1, v_2, \ldots, v_t$ are arranged around $v$ in a clockwise manner. Also, note that $t \geq 7$ as $2m+n \geq 3$.

Now $v_1$ must see $v_3$ by a special $2$-path through some $w$ which is not equal to either $v$ or any of the $v_i$'s. Notice that, every $v_i$, for $i \geq 4$, must see $v_2$ through $w$, as $H$ is planar. Therefore, given this structure, every $v_i$ must see $v_{i+k}$ by a special $2$-path through $w$ for all $k \geq 2$ (here, the $+$ operation on the indices is taken modulo $t$). Therefore, $N^\beta(w)$ can contain at most two vertices from $v_1, v_2, \ldots, v_t$ for any $\beta \in \{-m, -(m-1), \ldots, -1, 1, 2, \ldots, (m+n)\}$ and hence, the result. 

As a corollary to the above lemma, we have the following:

**Lemma 6.3.** Let $H$ be a critical $(m,n)$-relative clique of $\mathcal{P}_g$ for $g = 3$ or $4$. Then the number of independent good neighbors of a vertex $v$ of $H$ is at most $2(2m+n)^2$.

**Proof.** Follows from Lemma 6.2 and the pigeonhole principle.

Now, we are ready to prove the upper bounds of Theorem 6.1(i) and (ii).

**Lemma 6.4.** For $(m,n)$ such that $2m+n \geq 3$, we have $\omega_{r(m,n)}(\mathcal{P}_3) \leq 42(2m+n)^2 - 11$.

**Proof.** Towards a contradiction, let $\omega_{r(m,n)}(\mathcal{P}_3) > 42(2m+n)^2 - 11$ and let $H$ be a critical $(m,n)$-relative clique of $\mathcal{P}_3$. We now claim that it is always possible to find a good vertex in $H$ whose degree in $\text{und}(H)$ is at most 7. The proof of the claim goes as follows:

As planar graphs are 5-degenerate, there exists a vertex in $H$ of degree at most 5. If that vertex is a good vertex, then we are done. Otherwise, every good vertex of $H$ has a minimum degree of 6. Moreover, as planar graphs are 3-closed, every helper vertex of $H$ must have a minimum degree of 4 due to Lemma 2.3. Now, Jendrol’ and Voss [6] had shown that any planar graph with minimum degree at least four must have an edge $uv$ such that $\text{deg}(u) + \text{deg}(v) \leq 11$. So, if every good vertex of $H$ is of degree at least 6, then at least one of $u$ and $v$ must be a helper. So, without loss of generality, let $u$ be a helper vertex. This implies that $v$ must be a good vertex by Lemma 2.2. Moreover, as $u$ is of degree at least 4, it implies that $v$ has a degree of at most 7 in $\text{und}(H)$; and that proves the claim.

Notice that the good vertices in the second neighborhood of $v$ induce an outerplanar subgraph of $\text{und}(H)$ and is, hence, 3-colorable. As there are at least $42(2m+n)^2 - 10$ good vertices in $H$ and at most 7 good vertices can be neighbors of $v$, there must be at least $14(2m+n)^2 - 6$ independent good vertices in the second neighborhood of $v$ by the pigeonhole principle. Then, again by the pigeonhole principle, at least one neighbor of $v$ must have a minimum of $2(2m+n)^2 + 1$ independent good neighbors (including $v$), which is a contradiction to Lemma 6.2.
A similar proof works for establishing the upper bound for the relative clique number for triangle-free planar graphs as well.

**Lemma 6.5.** For \((m, n)\) such that \(2m + n \geq 3\), we have \(\omega_{r(m,n)}(P_4) \leq 14(2m + n)^2 + 1\).

**Proof.** Let \(H\) be a critical \((m, n)\)-relative clique of \(P_4\). Now, delete each helper in \(H\) of degree at most three and make its neighbors adjacent to each other (if not adjacent already). This will result in a planar graph which is not necessarily triangle-free. However, again using Jendrol’ and Voss [5], one can find a good vertex \(v\) in the new graph having degree at most 7. Observe that, the same vertex \(v\) has at most degree 7 in \(H\) as well.

As \(H\) is triangle-free, any neighborhood of a vertex is independent in \(H\). Thus, each neighbor of \(v\) can have at most \(2(2m + n)^2\) good neighbors including \(v\); and hence, the result. \(\square\)

Next, let us find the exact value for a triangle-free planar \((m,n)\)-absolute clique.

**Lemma 6.6.** For \((m, n)\) such that \(2m + n \geq 3\), we have \(\omega_{a(m,n)}(P_4) \leq (2m + n)^2 + 2\).

**Proof.** Take a multigraph having three vertices \(x, y, z\) with \(a\) edges between \(x\) and \(y\), \(b\) edges between \(y\) and \(z\), and one edge between \(x\) and \(z\). Now subdivide all its edges, except \(xz\), exactly once. The simple graph so obtained is denoted by \(C_{a,b}\).

As per Plesnık [11], the only triangle-free planar graphs having diameter two are: \(K_{1,1}, K_{2,4}\) and \(C_{a,b}\). As a triangle-free planar \((m,n)\)-absolute clique is one among these three types of graphs, observe that its maximum order is realized by \(K_{2, (2m+n)^2}\). \(\square\)

After this, let us consider the higher girth cases and start with planar graphs with girth at least 6.

**Lemma 6.7.** For \(2m + n \geq 3\), we have \(\omega_{r(m,n)}(P_6) = 2m + n + 1\).

**Proof.** For \(2m + n \geq 3\), \(\omega_{r(m,n)}(P_6) \geq (2m + n) + 1\) due to Lemma 5.3 as partial 2-trees are planar graphs in particular.

Let \(H\) be a critical \((m,n)\)-relative clique of \(P_6\). If \(H\) contains a 6-cycle, then \(\omega_{r(m,n)}(H) \geq 5\) will force a \(K_5\)-minor in \(H\) by Lemma 5.3. As planar graphs cannot have a \(K_5\)-minor, \(\omega_{r(m,n)}(H) \leq 4\), if \(H\) contains a 6-cycle.

On the other hand, if \(H\) does not contain any 6-cycle, then \(H \in F_7\) and we are done due to Proposition 5.3 and the fact that \((2m + n) + 1 \geq 4\). \(\square\)

Next, we consider the family of planar graphs with girth at least 5.

**Lemma 6.8.** For \(2m + n \geq 3\), we have \(\omega_{r(m,n)}(P_5) = \max(2m + n + 1, 6)\).

**Proof.** For \(2m + n \geq 3\), \(\omega_{r(m,n)}(P_5) \geq (2m + n) + 1\) due to Lemma 5.3 as partial 2-trees are planar graphs in particular. On the other hand, \(\omega_{r(m,n)}(P_5) \geq 6\) due to the following construction: take an \((m,n)\)-absolute clique whose underlying graph is the 5-cycle (existence follows from Lemma 5.4) and connect it to a sixth vertex using internally disjoint special 2-paths.

Let \(H\) be a critical \((m,n)\)-relative clique of \(P_5\). If \(H\) contains a 5-cycle, then \(\omega_{r(m,n)}(H) \geq 7\) will force a \(K_5\)-minor in \(H\) by Lemma 5.0. As planar graphs cannot have a \(K_5\)-minor, \(\omega_{r(m,n)}(H) \leq 6\), if \(H\) contains a 5-cycle.

On the other hand, if \(H\) does not contain any 5-cycle, then we are done due to Lemma 6.7. \(\square\)

Finally, we are ready to prove Theorem 6.1.

**Proof of Theorem 6.1.** The proof of (i) directly follows from Lemma 6.4. The proof of (ii) directly follows from Lemma 6.5. The proof of (iii) directly follows from Lemma 6.8. The proof of (iv) directly follows from Lemma 6.7 and the proof of (v) directly follows from Proposition 5.8. \(\square\)
7 Conclusions

This work may be regarded as the first systematic study of the \((m, n)\)-relative clique number. We explored the \((m, n)\)-relative clique number of subcubic graphs, graphs with bounded degree, partial 2-trees and planar graphs of girth at least \(g\), where \(g \geq 3\). In case of subcubic graphs and partial 2-trees having girth \(g\), where \(g \geq 3\), we have provided the exact bounds for all cases. In case of planar graphs having girth \(g\), we were unable to provide tight bounds for the cases \(g = 3, 4\). However, based on our experience of finding the bounds in those cases we would like to conjecture the following tight bounds.

**Conjecture 7.1.** For the family \(\mathcal{P}_3\) of planar graphs,

\[
\omega_{a(m,n)}(\mathcal{P}_3) = \omega_{r(m,n)}(\mathcal{P}_3) = 3(2m+n)^2 + (2m+n) + 1.
\]

Notice that our conjecture strengthens the conjecture by Bensmail, Duffy and Sen [1] which claimed only \(\omega_{a(m,n)}(\mathcal{P}_3) = 3(2m+n)^2 + (2m+n) + 1\). We make a similar conjecture for triangle-free planar graphs also. Notice that, as in this case we have already found out the exact value of \(\omega_{a(m,n)}(\mathcal{P}_4)\), the conjecture only concerns the value of \(\omega_{r(m,n)}(\mathcal{P}_4)\).

**Conjecture 7.2.** For the family \(\mathcal{P}_4\) of triangle-free planar graphs,

\[
\omega_{r(m,n)}(\mathcal{P}_4) = 2(2m+n)^2 + 2.
\]

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