Abstract

The complexity in large-scale optimization can lie in both handling the objective function and handling the constraint set. In this respect, stochastic Frank-Wolfe algorithms occupy a unique position as they alleviate both computational burdens, by querying only approximate first-order information from the objective and by maintaining feasibility of the iterates without using projections. In this paper, we improve the quality of their first-order information by blending in adaptive gradients. Starting from the design of adaptive gradient algorithms, we propose to solve the occurring constrained optimization subproblems very incompletely via a fixed and small number of iterations of the Frank-Wolfe algorithm (often times only 2 iterations), in order to preserve the low per-iteration complexity. We derive convergence rates and demonstrate the computational advantage of our method over the state-of-the-art stochastic Frank-Wolfe algorithms on both convex and nonconvex objectives.

1 Introduction

Consider the constrained finite-sum optimization problem

\[
\min_{x \in \mathcal{C}} \left\{ f(x) := \frac{1}{m} \sum_{i=1}^{m} f_i(x) \right\},
\]

where \( \mathcal{C} \subset \mathbb{R}^n \) is a compact convex set and \( f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R} \) are smooth convex functions. We assume that \( m \) is so large that exact function and gradient evaluations of the objective \( f \) are significantly less efficient than their stochastic evaluations, and that projections onto \( \mathcal{C} \) are significantly more expensive than linear optimizations over \( \mathcal{C} \). Such situations are encountered in, e.g., large-scale regression and classification problems over \( \ell_p \)-balls, nuclear norm-balls, or structured polytopes. In this setting, stochastic Frank-Wolfe algorithms are the method of choice.

The Frank-Wolfe algorithm (FW) (Frank and Wolfe, 1956), a.k.a. conditional gradient algorithm (Levitin and Polyak, 1966), is a simple projection-free first-order algorithm for constrained minimization. At each iteration, it calls a linear minimization oracle \( v_t \leftarrow \text{arg min}_{v \in \mathcal{C}} \langle \nabla f(x_t), v \rangle \) given by the current gradient \( \nabla f(x_t) \) and moves in the direction of \( v_t \) by convex combination, updating the new iterate \( x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t) \) where \( \gamma_t \in [0, 1] \). Hence, \( x_{t+1} \in \mathcal{C} \) by convexity of \( \mathcal{C} \) and there is no need to ensure feasibility via projections back onto \( \mathcal{C} \). In short, FW avoids the projection step of gradient descent by moving in the direction of a
point $v_t$ minimizing over $C$ the linear approximation of $f$ at $x_t$. The Stochastic Frank-Wolfe algorithm (SFW) simply replaces the gradient $\nabla f(x_t)$ in the input to the linear minimization oracle with a stochastic estimator $\tilde{\nabla} f(x_t) \leftarrow (1/b_t) \sum_{i=1}^{i b_t} \nabla f_i(x_t)$, where $i_1, \ldots, i_{b_t}$ are sampled i.i.d. uniformly at random from $[1, m]$. When the batch-sizes scale as $b_t = \Theta(t^2)$, SFW converges with rate $O(1/t)$ (Hazan and Luo, 2016).

Many variants have been proposed to improve the practical efficiency of SFW, also converging at a rate of $O(1/t)$. From a theoretical standpoint, a popular measure of efficiency in large-scale optimization is the number of gradient evaluations required to achieve $\varepsilon$-convergence. To this end, the Stochastic Variance-Reduced Frank-Wolfe algorithm (SVRF) (Hazan and Luo, 2016) integrates variance reduction (Johnson and Zhang, 2013; Zhang et al., 2013) in the estimation of the stochastic gradients to improve the batch-size rate to $b_t = \Theta(t)$. The STOchastic variance-Reduced Conditional gradient sliding algorithm (STORC) (Hazan and Luo, 2016) builds on the Conditional Gradient Sliding algorithm (Lan and Zhou, 2016) and further reduces the total number of gradient evaluations by half an order of magnitude, although STORC is not as competitive as SVRF in practice (Hazan and Luo, 2016, Section 5). This may be because SVRF obtains more progress per gradient evaluation or because the analysis of STORC is more precise. When the objective is additively separable in the data samples, Négiar et al. (2020) present a Constant batch-size Stochastic Frank-Wolfe algorithm (CSFW) where the batch-sizes do not need to grow over time, i.e., $b_t = \Theta(1)$; in practice, they set $b_t \leftarrow \lfloor m/100 \rfloor$.

Hence, the number of gradient evaluations required to achieve convergence, although appealing for its theoretical insight, may not necessarily reflect the relative performances of different algorithms in practice. In this paper, we take a different route and leverage recent advances in optimization to improve the performance of SFW (and variants) by focusing on a better use of first-order information. To the best of our knowledge, it has not yet been explored how to take advantage of adaptive gradients (Duchi et al., 2011; McMahan and Streeter, 2010), which have been very successful in modern large-scale learning (see, e.g., Dean et al. (2012)), in projection-free optimization. Adaptive gradient algorithms consist in setting entry-wise step-sizes based on first-order information from past iterates. An interpretation of the success of these methods is that they provide a feature-specific learning rate, which is particularly useful when informative features from the dataset are present in the form of rare events. From an optimization standpoint, these adaptive step-sizes fit better to the loss landscape and alleviate the struggle with ill-conditioning, without requiring access to second-order information.

**Contributions.** Inspired by the method of adaptive gradients, we present a new template for improving the performance of stochastic Frank-Wolfe algorithms; in particular, it applies to all the aforementioned variants. In essence, we propose to solve the constrained optimization subproblems occurring in the adaptive gradient algorithms very incompletely via a fixed and small number of $K$ iterations of the Frank-Wolfe algorithm. Hence, contrary to the classical literature on inexact projections, we choose not to worry about the accuracy of the solutions thus obtained, and aim at designing a particularly efficient method. We establish convergence guarantees and demonstrate the computational advantage of our method over the state-of-the-art stochastic Frank-Wolfe algorithms on both convex and nonconvex objectives.

**Outline.** We start with background materials on stochastic Frank-Wolfe algorithms and adaptive gradient algorithms in Section 2. In Section 3, we motivate our approach and present our method through a generic template. We further propose specific implementations and analyze their respective convergence properties. We end that section with practical recommendations and we report computational experiments in Section 4. We conclude the paper with some final remarks in Section 5. All proofs can be found in Appendix D. Appendix B contains an analysis of the sensitivity to $K$. 

2
2 Preliminaries

2.1 Notation and definitions

We work in the Euclidean space \((\mathbb{R}^n, \langle \cdot, \cdot \rangle)\) equipped with the standard inner product. We consider Problem (1) where \(C \subseteq \mathbb{R}^n\) is a compact convex set and \(f_1, \ldots, f_m : \mathbb{R}^n \to \mathbb{R}\) are smooth functions. Unless declared otherwise, we will also assume that they are convex. The objective function is \(f := (1/m) \sum_{i=1}^m f_i\).

Let \(L\) be the gradient algorithm (Levitin and Polyak, 1966). The update by convex combination in Line 4 ensures that \(\langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2\) is a first-order condition key to the convergence analyses: let \(v \in \mathbb{R}^n\) be chosen in Line 3 so that the sequence of iterates converges, as the iterates move in the directions \(v_t - x_t\) satisfying a sufficient first-order condition key to the convergence analyses: let \(x^* \in \text{arg min}_C f\), then

\[
\langle \nabla f(x_t), v_t - x_t \rangle = \langle \nabla f(x_t), v_t - x_t \rangle + \langle \nabla f(x_t) - \nabla f(x_t), v_t - x_t \rangle \\
\leq \langle \nabla f(x_t), x^* - x_t \rangle + \langle \nabla f(x_t) - \nabla f(x_t), x^* - v_t \rangle \leq \langle \nabla f(x_t) - \nabla f(x_t), x^* - v_t \rangle \\
= \langle \nabla f(x_t), x^* - x_t \rangle + \langle \nabla f(x_t) - \nabla f(x_t), x^* - v_t \rangle \\
\leq -(f(x_t) - f(x^*)) + \langle \nabla f(x_t) - \nabla f(x_t), x^* - v_t \rangle,
\]

by convexity of \(f\). The second term can be controlled by properties of the gradient estimator \(\nabla f(x_t)\), often by using first the Cauchy-Schwarz or Hölder’s inequality.

### Template 1: Stochastic Frank-Wolfe

**Input**: Start point \(x_0 \in C\), step-sizes \(\gamma_t \in [0, 1]\).

1. for \(t = 0\) to \(T - 1\) do
2. Update the gradient estimator \(\nabla f(x_t)\)
3. \(v_t \leftarrow \text{arg min}_{v \in C} \langle \nabla f(x_t), v \rangle\)
4. \(x_{t+1} \leftarrow x_t + \gamma_t(v_t - x_t)\)
5. end for

In order to fit FW to the large-scale finite-sum setting of Problem (1), many stochastic Frank-Wolfe algorithms have been developed. Most of them follow Template 1 and differ only in how they update the
Table 1: Gradient estimator updates in stochastic Frank-Wolfe algorithms. The indices $i_1, \ldots, i_b$ are sampled i.i.d. uniformly at random from $[1, m]$.

| Algorithm   | Update $\hat{\nabla} f(x_t)$ in Line 2 | Additional information |
|-------------|----------------------------------------|------------------------|
| SFW         | $\frac{1}{b_t} \sum_{i=i_1}^{i_b} \nabla f_i(x_t)$ | $\varnothing$          |
| SVRF        | $\nabla f(\tilde{x}_t) + \frac{1}{b_t} \sum_{i=i_1}^{i_b} (\nabla f_i(x_t) - \nabla f_i(\tilde{x}_t))$ | $\tilde{x}_t$ is the last snapshot iterate |
| SPIDER-FW   | $\nabla f(\tilde{x}_t) + \frac{1}{b_t} \sum_{i=i_1}^{i_b} (\nabla f_i(x_t) - \nabla f_i(x_{t-1}))$ | $\tilde{x}_t$ is the last snapshot iterate |
| ORGFW       | $\frac{1}{b_t} \sum_{i=i_1}^{i_b} \nabla f_i(x_t) + (1 - \rho_t) \left( \hat{\nabla} f(x_{t-1}) - \frac{1}{b_t} \sum_{i=i_1}^{i_b} \nabla f_i(x_{t-1}) \right)$ | $\rho_t$ is the momentum parameter |
| CSFW        | $\tilde{\nabla} f(x_{t-1}) + \sum_{i=i_1}^{i_b} \left( \frac{1}{m} f'_i(\langle a_i, x_t \rangle) - [\alpha_{t-1}]_i \right) a_i$ | Assumes separability of $f$ as and $[\alpha_t]_i \leftarrow \begin{cases} \left( \frac{1}{m} \right) f'_i(\langle a_i, x_t \rangle) & \text{if } i \in \{i_1, \ldots, i_b\} \\ [\alpha_{t-1}]_i & \text{else} \end{cases}$ |

gradient estimator $\hat{\nabla} f(x_t)$ (Line 2). In Table 1, we report the strategies adopted in the Stochastic Frank-Wolfe algorithm (SFW) (Hazan and Luo, 2016), the Stochastic Variance-Reduced Frank-Wolfe algorithm (SVRF) (Hazan and Luo, 2016), the Stochastic Path-Integrated Differential Estimator Frank-Wolfe algorithm (SPIDER-FW) (Yurtsever et al., 2019; Shen et al., 2019), the Online stochastic Recursive Gradient-based Frank-Wolfe algorithm (ORGFW) (Xie et al., 2020), and the Constant batch-size Stochastic Frank-Wolfe algorithm (CSFW) (Négiar et al., 2020). SFW is the natural extension of FW to the large-scale setting of Problem (1), SVRF and SPIDER-FW integrate variance reduction based on the works of Johnson and Zhang (2013) and Fang et al. (2018) respectively, ORGFW uses a form of momentum inspired by Cutkosky and Orabona (2019), and CSFW takes advantage of the additive separability of the objective function in the data samples, when applicable, following the design of Schmidt et al. (2017).

2.3 The Adaptive Gradient algorithm

The Adaptive Gradient algorithm (AdaGrad) (Duchi et al., 2011) (see also McMahan and Streeter (2010)) is presented in Algorithm 2. The new iterate $x_{t+1}$ is computed in Line 4 by solving a constrained convex quadratic minimization subproblem.

Algorithm 2 Adaptive Gradient (AdaGrad)

**Input:** Start point $x_0 \in \mathcal{C}$, offset $\delta > 0$, learning rate $\eta > 0$.

1. $t = 0$ to $T - 1$ do
2. Update the gradient estimator $\hat{\nabla} f(x_t)$
3. $H_t \leftarrow \text{diag} \left( \delta I + \sqrt{\sum_{s=0}^{t} \hat{\nabla} f(x_s)^2} \right)$
4. $x_{t+1} \leftarrow \arg\min_{x \in \mathcal{C}} \eta \langle \hat{\nabla} f(x_t), x \rangle + \frac{1}{2} \| x - x_t \|^2_{H_t}$
5. end for
All operations in Line 3 are entry-wise in $\mathbb{R}^n$. The matrix $H_t \in \mathbb{R}^{n \times n}$ is diagonal and satisfies for all $i, j \in [1, n]$,

$$
[H_t]_{i,j} = \begin{cases} 
\delta + \sqrt{\sum_{s=0}^{t} [\nabla f(x_s)]_i^2} & \text{if } i = j, \\
0 & \text{if } i \neq j.
\end{cases}
$$

(3)

The default value for the offset hyperparameter is $\delta \leftarrow 10^{-8}$.

Note that by first-order optimality condition (Polyak, 1987, Section 7.1.2, Theorem 3), the subproblem in Line 4 is equivalent to a projection in the metric $\| \cdot \|_{H_t}$:

$$
x_{t+1} \leftarrow \text{arg min}_{x \in C} \| x - (x_t - \eta H_t^{-1} \nabla f(x_t)) \|_{H_t}.
$$

(4)

Ignoring the constraint set $C$ for ease of exposition, we obtain

$$
x_{t+1} \leftarrow x_t - \eta H_t^{-1} \nabla f(x_t),
$$

i.e., for every feature $i \in [1, n]$,

$$
[x_{t+1}]_i \leftarrow [x_t]_i - \frac{\eta [\nabla f(x_t)]_i}{\delta + \sqrt{\sum_{s=0}^{t} [\nabla f(x_s)]_i^2}}.
$$

(5)

Thus, the offset $\delta$ prevents from dividing by zero, and we can see that the step-size automatically scales with the geometry of the problem. In particular, infrequent features receive large step-sizes whenever they appear, allowing the algorithm to notice these rare but potentially very informative features.

The family of adaptive gradient algorithms originated with AdaGrad and expanded with RMSProp (Tieleman and Hinton, 2012), AdaDelta (Zeiler, 2012), Adam (Kingma and Ba, 2015), AMSGrad (Reddi et al., 2018), and, e.g., AdaBound (Luo et al., 2019; Keskar and Socher, 2017), with each new variant addressing some flaws in the previous ones: vanishing step-sizes, incomplete theory, generalization performance (Wilson et al., 2017), etc. For example, RMSProp uses an exponential moving average instead of a sum in Line 3 in order to avoid vanishing step-sizes, since the sum in the denominator of (5) can grow too fast for features with dense gradients.

3 Frank-Wolfe with adaptive gradients

3.1 Our approach

When minimizing over a constraint set, each iteration of AdaGrad can be relatively expensive as it needs to solve the subproblem

$$
\min_{x \in C} \eta \langle \nabla f(x_t), x \rangle + \frac{1}{2} \| x - x_t \|_{H_t}^2,
$$

(6)

given in Line 4. By (4), this is equivalent to a non-Euclidean projection of the unconstrained step $x_t - \eta H_t^{-1} \nabla f(x_t)$. Thus, we could reduce the complexity of AdaGrad by avoiding this projection and moving in the direction of $\text{arg min}_{v \in C} \langle G_t, v \rangle$, where $-G_t = -\eta H_t^{-1} \nabla f(x_t)$ denotes the unconstrained descent direction of AdaGrad, as was done in FW for gradient descent with $-G_t = -\nabla f(x_t)$. However, by doing so we may lose the precious properties of the descent directions of AdaGrad, as the directions returned by $\text{arg min}_{v \in C} \langle G_t, v \rangle$ can be significantly different from $-G_t$ (Combettes and Pokutta, 2020); see Polyak (1987, Figure 32) for an early illustration of the phenomenon.
Thus, instead of avoiding the subproblem (6), we can consider solving it incompletely via a projection-free algorithm. Following Lan and Zhou (2016), at each iteration we could use FW to solve (6) until some specified accuracy $\phi_t$ is reached, which we check via the duality gap $\max_{v \in C} (y^T f(x_t) + H_t(x - x_t), v)$ (Jaggi, 2013) (Fact C.1). The solution to this procedure would then constitute the new iterate $x_{t+1}$. The subproblem (6) is easy to address since the objective is a simple convex quadratic function, so we can evaluate its exact gradient cheaply and derive the optimal step-size in any descent direction.

However, in order to provide nice theoretical analyses, the sequence of accuracies $\phi_t \in [0, T - 1]$ would need to decay to zero relatively fast, which means that we are back to solving the subproblems completely. This is very time-consuming and overkill in practice. Therefore, instead we propose to perform a fixed number of $K$ iterations on the subproblems, where $K$ is chosen small, e.g., $K \sim 5$. Hence, we choose to leverage just a small amount of information from the adaptive metric $H_t$, and claim that this will be enough in practice.

### 3.2 The algorithm

We now present our method via a generic template in Template 3. We allow the matrix $H_t$ to be relatively general, the only requirements being that it is diagonal and that its entries are clipped to some hand-designed values $[\lambda_t^-, \lambda_t^+]$, as done in Luo et al. (2019). Hence, we can apply the AdaGrad update (3), but we can also apply any other variant. Lines 4-10 apply $K$ iterations of FW on

$$
\min_{x \in C} \left\{ Q_t(x) := f(x_t) + \langle \nabla f(x_t), x - x_t \rangle + \frac{1}{2\eta_t} \| x - x_t \|_H^2 \right\}.
$$

This is exactly the subproblem (6) with a time-varying learning rate $\eta_t > 0$. We denote by $y_k^{(t)}$ for $k \in [0, K]$ the iterates on the subproblem (7), starting from $y_0^{(t)} \leftarrow x_t$ (Line 4) and ending at $x_{t+1} \leftarrow y_K^{(t)}$ (Line 11). The step-size $\gamma_k^{(t)}$ in Line 8 is optimal in the sense that $\gamma_k^{(t)} = \arg \min_{\gamma \in (0, \eta_t]} Q_t(y_k^{(t)} + \gamma (v_k^{(t)} - y_k^{(t)}))$ (Lemma D.1), where the upper bound $\gamma_t$ ensures convergence of the sequence $(x_t)_{t \in [0, T]}$.

**Template 3** Frank-Wolfe with adaptive gradients

**Input:** Start point $x_0 \in C$, bounds $0 < \lambda_t^- \leq \lambda_{t+1}^- \leq \lambda_{t+1}^+ \leq \lambda_t^+$, number of inner iterations $K \in \mathbb{N} \setminus \{0\}$, learning rates $\eta_t > 0$, step-size bounds $\gamma_t \in [0, 1]$.

1. **for** $t = 0$ to $T - 1$ **do**
2.  Update the gradient estimator $\nabla f(x_t)$
3.  Update the diagonal matrix $H_t$ and clip its entries to $[\lambda_t^-, \lambda_t^+]$
4.  $y_0^{(t)} \leftarrow x_t$
5.  **for** $k = 0$ to $K - 1$ **do**
6.   $\nabla Q_t(y_k^{(t)}) \leftarrow \nabla f(x_t) + \frac{1}{\eta_t} H_t(y_k^{(t)} - x_t)$
7.   $v_k^{(t)} \leftarrow \arg \min_{v \in C} \langle \nabla Q_t(y_k^{(t)}), v \rangle$
8.   $\gamma_k^{(t)} \leftarrow \min \left\{ \eta_t \frac{\langle \nabla Q_t(y_k^{(t)}), y_k^{(t)} - v_k^{(t)} \rangle}{\| y_k^{(t)} - v_k^{(t)} \|_H^2}, \gamma_t \right\}$
9.   $y_{k+1}^{(t)} \leftarrow y_k^{(t)} + \gamma_k^{(t)} (v_k^{(t)} - y_k^{(t)})$
10. **end for**
11. $x_{t+1} \leftarrow y_K^{(t)}$
12. **end for**

In Sections 3.3-3.5, we propose specific implementations of Template 3, where gradients are estimated as done in SFW, SVRF, and CSFW (Table 1). The matrices $H_t$, $t \in [0, T - 1]$, can still be very general. The derived algorithms are named AdaSFW, AdaSVRF, and AdaCSFW respectively, and we analyze their convergence rates. We follow the assumptions from Section 2.1; in particular, unless declared otherwise, $f_1, \ldots, f_m$ are smooth convex functions.
3.3 SFW with adaptive gradients

We present AdaSFW in Algorithm 4. It simply estimates the gradient by averaging over a minibatch.

**Algorithm 4 AdaSFW**

**Input:** Start point $x_0 \in \mathcal{C}$, batch-sizes $b_t \in \mathbb{N}\backslash \{0\}$.

1: for $t = 0$ to $T - 1$ do
2: $i_1, \ldots, i_{b_t} \overset{i.i.d.}{\sim} \mathcal{U}\left([1, m]\right)$
3: $\nabla f(x_t) \leftarrow \frac{1}{b_t} \sum_{i=t}^{i_{t+1}} \nabla f_i(x_t)$
4: Execute Lines 3-11 of Template 3
5: end for

**Theorem 3.1.** Consider AdaSFW (Algorithm 4) with $b_t \leftarrow \left(G(t+2)/(LD)\right)^2$, $\eta_t \leftarrow \lambda_t^- / L$, and $\gamma_t \leftarrow 2/(t+2)$, and let $\kappa := \lambda_0^+/\lambda_0^-$. Then for all $t \in [1, T]$,

$$E[f(x_t)] - \min_c f \leq \frac{2LD^2(K + 1 + k)}{t + 1}.$$

**Remark 3.2.** Theorem 3.1 simply states that we need to scale the batch-sizes as $b_t = \Theta(t^2)$. We do not need to search for the values of $G$, $L$, or $D$ in practice. The same holds for SFW (Hazan and Luo, 2016).

We propose in Theorem 3.3 a convergence analysis of AdaSFW on nonconvex objectives. We measure convergence via the duality gap $g: x \in \mathcal{C} \mapsto \max_{v \in \mathcal{C}} \langle \nabla f(x), x - v \rangle$ (Jaggi, 2013) as done in, e.g., Lacoste-Julien (2016); Reddi et al. (2016). The duality gap satisfies $g(x) \geq 0$, $g(x) = 0$ if and only if $x$ is a stationary point, and, when $f$ is convex, $g(x) \geq f(x) - \min_c f$ (Fact C.1).

**Theorem 3.3.** Suppose that $f_1, \ldots, f_m$ are not necessarily convex. Consider AdaSFW (Algorithm 4) with $b_t \leftarrow \left(G/(LD)\right)^2(t + 1)$, $\eta_t \leftarrow \lambda_t^- / L$, and $\gamma_t \leftarrow 1/(t + 1)^{1/2 + \nu}$ where $\nu \in [0, 1/2]$ and let $\kappa := \lambda_0^+/\lambda_0^-$. For all $t \in [0, T]$, let $X_t$ be sampled uniformly at random from $\{x_0, \ldots, x_t\}$. Then for all $t \in [0, T - 1]$,

$$E[g(X_t)] \leq \frac{(f(x_0) - \min_c f) + LD^2(K + 1 + k/2)S}{(t + 1)^{1/2 - \nu}},$$

where $S := \sum_{s=0}^{+\infty} 1/(s + 1)^{1 + \nu} \in \mathbb{R}^+$. Alternatively, if the time horizon $T$ is fixed, then with $b_t \leftarrow (G/(LD))^2T$ and $\gamma_t \leftarrow 1/\sqrt{T}$,

$$E[g(X_{T-1})] \leq \frac{(f(x_0) - \min_c f) + LD^2(K + 1 + k/2)}{\sqrt{T}}.$$

**Remark 3.4.** In the first setting of Theorem 3.3, if $\nu \leftarrow 0.05$ for example, then $E[g(X_t)] = \mathcal{O}(1/t^{0.45})$ and $S \approx 20.6$.

3.4 SVRF with adaptive gradients

We present AdaSVRF in Algorithm 5. At every iteration $t = s_k$, $k \in \mathbb{N}$, it computes the exact gradient of the iterate, saves it into memory, then builds the gradient estimator $\nabla f(x_t)$ in the following iterations $t \in [s_k + 1, s_{k+1} - 1]$ from this snapshot. Compared to AdaSFW, the variance $E[\|\nabla f(x_t) - \nabla f(x_i)\|^2]$ of the estimator is effectively reduced. The snapshot iterate for $x_t$ is denoted by $\tilde{x}_t$.

**Theorem 3.5.** Consider AdaSVRF (Algorithm 5) with $s_k \leftarrow 2^k - 1$, $b_t \leftarrow 24(K + 1 + \kappa)(t + 2)$ where $\kappa := \lambda_0^+ / \lambda_0^-$, $\eta_t \leftarrow \lambda_t^- / L$, and $\gamma_t \leftarrow 2/(t + 2)$. Then for all $t \in [1, T]$,

$$E[f(x_t)] - \min_c f \leq \frac{2LD^2(K + 1 + \kappa)}{t + 2}.$$
Algorithm 5 AdaSVRF

Input: Start point $x_0 \in C$, snapshot times $s_k < s_{k+1}$ with $s_0 = 0$, batch-sizes $b_t \in \mathbb{N}\setminus\{0\}$.

1: for $t = 0$ to $T - 1$ do
2: if $t \in \{s_k \mid k \in \mathbb{N}\}$ then
3: \quad $\tilde{x}_t \leftarrow x_t$
4: \quad $\nabla f(x_t) \leftarrow \nabla f(\tilde{x}_t)$
5: else
6: \quad $\tilde{x}_t \leftarrow \tilde{x}_{t-1}$
7: \quad $i_1, \ldots, i_{b_t} \overset{i.i.d.}{\sim} U([1, m])$
8: \quad $\nabla f(x_t) \leftarrow \nabla f(\tilde{x}_t) + \frac{1}{b_t} \sum_{i=i_1}^{i_{b_t}} (\nabla f_i(x_t) - \nabla f_i(\tilde{x}_t))$
9: end if
10: Execute Lines 3-11 of Template 3
11: end for

Remark 3.6. Consider the setting of Theorem 3.5 with the more general strategy $s_k \leftarrow 2^k + 2k_0 - 2k_0$ and $b_t \leftarrow 8(2^{k_0+1} + 1)(K + 1 + \kappa)(t + 2)$ where $k_0 \in \mathbb{N}$. Then the same result holds. Choosing $k_0 > 0$ is useful in practice to avoid computing exact gradients too many times in the early iterations.

3.5 CSFW with adaptive gradients

Here we assume the objective function to be additively separable in the data samples. The problem is

$$\min_{x \in C} \left\{ f(x) := \frac{1}{m} \sum_{i=1}^{m} f_i((a_i, x)) \right\},$$

where $f_1, \ldots, f_m : \mathbb{R} \to \mathbb{R}$ are smooth convex functions and $a_1, \ldots, a_m \in \mathbb{R}^n$ are the data samples. Thus, $\nabla f(x) = (1/m) \sum_{i=1}^{m} f_i^\prime((a_i, x))a_i$. We present AdaCSFW in Algorithm 6. It estimates the gradient with the quantity $\tilde{\nabla} f(x_t) = \sum_{i=i_1}^{i_{b_t}} [\alpha_t]_i a_i$ by iteratively updating entries of the vector $\alpha_t \in \mathbb{R}^m$.

Algorithm 6 AdaCSFW

Input: Start point $x_0 \in C$, batch-size $b \in \mathbb{N}\setminus\{0\}$.

1: $\alpha_{-1} \leftarrow 0 \in \mathbb{R}^m$
2: $\tilde{\nabla} f(x_{-1}) \leftarrow 0 \in \mathbb{R}^n$
3: for $t = 0$ to $T - 1$ do
4: \quad $i_1, \ldots, i_{b_t} \overset{i.i.d.}{\sim} U([1, m])$
5: \quad for $i = 1$ to $m$ do
6: \quad \quad if $i \in \{i_1, \ldots, i_{b_t}\}$ then
7: \quad \quad \quad $[\alpha_t]_i \leftarrow \frac{1}{m} f_i^\prime((a_i, x_t))$
8: \quad \quad else
9: \quad \quad \quad $[\alpha_t]_i \leftarrow [\alpha_{t-1}]_i$
10: \quad \quad end if
11: \quad end for
12: $\tilde{\nabla} f(x_t) \leftarrow \tilde{\nabla} f(x_{t-1}) + \sum_{i=i_1}^{i_{b_t}} ([\alpha_t]_i - [\alpha_{t-1}]_i) a_i$
13: Execute Lines 3-11 of Template 3
14: end for
Theorem 3.7. Consider AdaCSFW (Algorithm 6) with $\eta_t \leftarrow m\lambda_t^+/(L\|A\|^2_2)$ and $\gamma_t \leftarrow 2/(t+2)$, and let $\kappa := \lambda_0^+ / \lambda_0^-$. Then for all $t \in [1, T]$, \[
abla f(x_t) - \min_{c} f(c) \leq \frac{2L}{t+1} \left( 4K(K+1)D_k^1 D_k^\infty \left( \frac{1}{b} - \frac{1}{m} \right) + \|A\|^2_2 D_k^2 \right) + \frac{2(K+1)D_k^\infty (m/b)^2}{t(t+1)} \left( \|f'(x_0) - a_0\|_1 + \frac{16KLD_k^1}{b} \right).
\]

3.6 Practical recommendations

We end this section with some practical recommendations. Following Remark 3.6, for convex objectives (Section 4.1) we set $k_0 \leftarrow 4$ in SVRF and AdaSVRF; for nonconvex objectives (Section 4.2), we took snapshots once per epoch. In all variants of Template 3, we used the AdaGrad strategy (3) for the adaptive metric $H_t$ and did not clip its entries (i.e., $\lambda_i^+ \leftarrow \delta$ and $\lambda_i^- \leftarrow \lambda_i^+ \leftarrow \delta$ arbitrarily large). The offset was set to the default value $\delta \leftarrow 10^{-8}$. We picked a constant value for the learning rate $\eta_t$, tuned in the range $\{10^{i/2} | i \in \mathbb{Z}\}$ by starting from $\{10^{i/2} | i \in \{-2, 0, 2\}\}$ and then narrowing the search space to $\{10^{i/2} | i \in \{t_{\text{best} - 1}, t_{\text{best}}, t_{\text{best} + 1}\}\}$, repeating if necessary. We did not bound the step-sizes $\gamma_k^t$, i.e., we set $\gamma_t \leftarrow 1$. Either way, we noticed that the bounds $\gamma_t$ obtained from the theoretical analyses were not active in our experiments. Lastly, we found $K \sim 5$ to be a good default value in general, as it provides both low complexity and very high performance. A sensitivity analysis is reported in Appendix B.

4 Computational experiments

4.1 Convex objectives

We compare our method to SFW, SVRF (Hazan and Luo, 2016), SPIDER-FW (Yurtsever et al., 2019; Shen et al., 2019), ORGFW (Xie et al., 2020), and CSFW (Négiar et al., 2020) on three standard convex optimization problems. We apply Template 3 to the best performing variant, demonstrating its flexibility and consistent performance. We evaluate the algorithms in both function value $f(x_t)$ and duality gap $\max_{v \in C} \langle \nabla f(x_t), x_t - v \rangle$. When $\min_C f$ is unknown, the duality gap serves as a measure of convergence and as a stopping criterion (Fact C.1). For the batch-sizes, we follow the recommendations given by the theoretical analyses of the respective algorithms. In SFW and SVRF, by Remark 3.2 we set $b_t \sim t^2/\sqrt{m}$ and $b_t \sim t$ respectively, making sure $b_t$ does not grow too fast and stays small compared to the full batch-size $m$. We have $b_t \sim \max\{2^k | t + 1 \geq 2^k, k \in \mathbb{N}\}$ in SPIDER-FW, and $b_t \sim |m/100| \sim |m/100|$ in ORGFW and CSFW, following Négiar et al. (2020) for algorithms where the batch-sizes do not need to grow over time.

4.1.1 Support vector classification

We start with a support vector classification experiment from Duchi (2018, Equation (4.3.11)). Since our work only deals with smooth objective functions, we smoothen the hinge loss by taking its square, as done in, e.g., Zhang and Oles (2001). The problem is

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \max\{0, 1 - y_i \langle a_i, x \rangle\}^2$$

s.t. $\|x\|_2 \leq \tau$,

where the data is generated as follows. For every $(i, j) \in [1, m] \times [1, n]$, let $a_{i,j} = 0$ with probability $1 - 1/j$, else $a_{i,j} = \pm 1$ equiprobably. Thus, the data matrix $A \in \mathbb{R}^{m \times n}$ has significant variability in the frequency of the features. Then let $u \sim U((-1, 1)^n)$, and $y_i = \text{sign}(\langle a_i, u \rangle)$ with probability 0.95 else $y_i = -\text{sign}(\langle a_i, u \rangle)$. We set $m = 20000$, $n = 1000$, $\tau = 1$, and $K = 2$ and $\eta \leftarrow 10^{-3/2}$ in AdaCSFW. The results are presented in Figure 1.

---

1For Frank-Wolfe on nonsmooth objectives, see Argyriou et al. (2014).
4.1.2 Linear regression

We consider a linear regression experiment on the YearPredictionMSD dataset (Bertin-Mahieux et al., 2011), available at https://archive.ics.uci.edu/ml/datasets/YearPredictionMSD. The goal is to predict the release years $y_1, \ldots, y_m$ of songs from their audio features $a_1, \ldots, a_m \in \mathbb{R}^n$. We include a sparsity-inducing constraint via the $\ell_1$-norm:

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} (y_i - \langle a_i, x \rangle)^2$$ \hspace{2cm} \text{s.t. } \|x\|_1 \leq 100.$$

We have $m = 463715$, $n = 90$, and we set $K \leftarrow 2$ and $\eta \leftarrow 10^{1/2}$ in AdaSVRF. The results are presented in Figure 2.

4.1.3 Logistic regression

We consider a text categorization experiment on the RCV1 dataset (Lewis et al., 2004). We use the pre-processed version for binary classification from the LIBSVM library (Chang and Lin, 2011), available at https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/binary.html#rcv1.binary, and adopt a logistic regression model with a sparsity-inducing constraint via the $\ell_1$-norm:

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \ln(1 + \exp(-y_i \langle a_i, x \rangle))$$ \hspace{2cm} \text{s.t. } \|x\|_1 \leq 100,$$

where $y_i \in \{-1, +1\}$. We have $m = 20242$, $n = 47236$, and we set $K \leftarrow 5$ and $\eta \leftarrow 10^2$ in AdaCSFW. The results are presented in Figure 3.
4.2 Nonconvex objectives

We compare our method to SFW, SVRF, ORGFW, and SPIDER-FW on the training of three neural networks. CSFW is not applicable here. Analyses of these algorithms in the nonconvex setting are provided in Reddi et al. (2016); Yurtsever et al. (2019); Xie et al. (2020). Since variance reduction can be ineffective in the training of deep neural networks (Defazio and Bottou, 2019), we run AdaSFW only. In addition, we propose a variant with momentum inspired by Adam (Kingma and Ba, 2015) and AMSGrad (Reddi et al., 2018), named AdamSFW; see Appendix A for details. In line with the practice of deep learning, we use constant batch-sizes in all algorithms. Experiments were logged with Weights & Biases (Biewald, 2020). The results are averaged over 5 runs and we plot shaded areas representing ±1 standard deviation.
Except on the MNIST dataset, our method is the only one to improve the performance of SFW. The state-of-the-art methods perform worse, although we used transform locking as recommended by Defazio and Bottou (2019) to improve the performance of the variance-reduced methods.

4.2.1 MNIST dataset

We train a convolutional neural network (CNN) on the MNIST dataset (Le Cun et al., 1998), available at http://yann.lecun.com/exdb/mnist/, for 10 epochs. The CNN has two $3 \times 3$ convolutional layers with 32 and 64 channels respectively, one $2 \times 2$ max-pooling layer, one fully-connected hidden layer of 128 units, and ReLU activations. Each layer is constrained into an $\ell_1$-ball with $\ell_2$-diameter equal to 60 times the expected $\ell_2$-norm of the Glorot uniform initialized values. We set $K \leftarrow 50$ and $\eta \leftarrow 10^{-3/2}$ in AdaSFW, and $K \leftarrow 50$, $\beta_m \leftarrow 0.9$, $\beta_s \leftarrow 0.999$, and $\eta \leftarrow 10^{-2}$ in AdamSFW. The results are presented in Figure 4.

![Figure 4: Convolutional neural network on the MNIST dataset.](image)

4.2.2 IMDB dataset

We train a neural network with one fully-connected hidden layer of 64 units and ReLU activations on the IMDB dataset (Maas et al., 2011) for 20 epochs. We use the 8 185 subword representation from TensorFlow, available at https://www.tensorflow.org/datasets/catalog/imdb_reviews#imdb_reviews_subwords8k. This is a natural language processing (NLP) task and the data is highly sparse. Each layer is constrained into an $\ell_\infty$-ball with $\ell_2$-diameter equal to 6 times the expected $\ell_2$-norm of the Glorot uniform initialized values. We set $K \leftarrow 2$ and $\eta \leftarrow 10^{-5/2}$ in AdaSFW, and $K \leftarrow 5$, $\beta_m \leftarrow 0.9$, $\beta_s \leftarrow 0.99$, and $\eta \leftarrow 10^{-3}$ in AdamSFW. The results are presented in Figure 5.

Contrary to the MNIST experiment, we can see here that SVRF and AdamSFW overfit quite strongly. AdamSFW in particular converges very fast on the training set and hits its maximum test accuracy very early on, which can be favorable if we consider using early stopping. AdaSFW, despite optimizing slowly over the training set, yields the best test accuracy, converging both very fast and to a higher value than any other algorithm.
4.2.3 CIFAR-10 dataset

We train a CNN on the CIFAR-10 dataset (Krizhevsky, 2009), available at https://www.cs.toronto.edu/~kriz/cifar.html, for 100 epochs. It has three $3 \times 3$ convolutional layers with 32, 64, and 64 channels respectively, two $2 \times 2$ max-pooling layers, one fully-connected hidden layer of 64 units, and ReLU activations. Each layer is constrained into an $\ell_{\infty}$-ball with $\ell_2$-diameter equal to 200 times the expected $\ell_2$-norm of the Glorot uniform initialized values. We set $K \leftarrow 10$ and $\eta \leftarrow 10^{-3/2}$ in AdaSFW, and $K \leftarrow 5$, $\beta_m \leftarrow 0.9$, $\beta_s \leftarrow 0.999$, and $\eta \leftarrow 10^{-7/2}$ in AdamSFW. The results are presented in Figure 6. AdaSFW and AdamSFW strongly outperform the other methods.

Figure 5: Neural network with one fully-connected hidden layer on the IMDB dataset.

Figure 6: Convolutional neural network on the CIFAR-10 dataset.
5 Final remarks

We have proposed a new method for large-scale constrained optimization by augmenting stochastic Frank-Wolfe algorithms through adaptive gradients, provided theoretical guarantees, and demonstrated its computational advantage over the state-of-the-art Frank-Wolfe algorithms in a wide range of experiments. We believe that constraint sets have often times been overlooked in machine learning and in the training of neural networks for they may require expensive projections to ensure feasibility. Through this work, by improving the performance of projection-free algorithms on several tasks, we hope to promote the use of constraint sets and to foster research in this direction. One question that remains open is whether the number $K$ of inner steps yields a form of regularization for the adaptive gradient algorithms, a phenomenon that we observed experimentally.

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A AdamSFW: AdaSFW with momentum

In Algorithm 7, inspired by Adam (Kingma and Ba, 2015) and AMSGrad (Reddi et al., 2018), we propose a variant of AdaSFW (Algorithm 4) with momentum which we used in our neural network training experiments (Section 4.2). The batch-size $b \in \mathbb{N}\backslash\{0\}$ and the learning rate $\eta > 0$ could be chosen as time-varying. Following Reddi et al. (2018), we require $\beta_m < \sqrt{\beta_s}$, with default values $\beta_m \leftarrow 0.9$ and $\beta_s \leftarrow 0.99$ or $\beta_s \leftarrow 0.999$. All operations in Line 8 are entry-wise in $\mathbb{R}^n$. Notice the presence of the momentum term $m_t$ instead of $\tilde{\nabla} f(x_t)$ in Line 11: the subproblem addressed by AdamSFW is

$$\min_{x \in \mathcal{C}} \left\{ Q_t(x) := f(x_t) + \langle m_t, x - x_t \rangle + \frac{1}{2\eta} \|x - x_t\|_{H_t}^2 \right\}.$$ 

Algorithm 7 AdamSFW

**Input:** Start point $x_0 \in \mathcal{C}$, batch-size $b \in \mathbb{N}\backslash\{0\}$, momentum parameters $\beta_m, \beta_s \in [0, 1]$, offset $\delta > 0$, number of inner iterations $K \in \mathbb{N}\backslash\{0\}$, learning rate $\eta > 0$.

1. $m_{t-1}, s_{t-1}, \bar{s}_{t-1} \leftarrow 0, 0, 0$
2. for $t = 0$ to $T - 1$ do
3. $i_1, \ldots, i_b \sim \mathcal{U}(\{1, m\})$
4. $\tilde{\nabla} f(x_t) \leftarrow \frac{1}{b} \sum_{i=1}^{n} \nabla f_{i}(x_t)$
5. $m_t \leftarrow \beta_m m_{t-1} + (1 - \beta_m) \tilde{\nabla} f(x_t)$
6. $s_t \leftarrow \beta_s s_{t-1} + (1 - \beta_s) \tilde{\nabla} f(x_t)^2$
7. $\bar{s}_t \leftarrow \max\{s_{t-1}, s_t\}$
8. $H_t \leftarrow \text{diag}(\delta \bar{s} + \sqrt{s_t})$
9. $y_{0}^{(t)} \leftarrow x_t$
10. for $k = 0$ to $K - 1$ do
11. $\nabla Q_t(y_{k}^{(t)}) \leftarrow m_t + \frac{1}{H_t} (y_{k}^{(t)} - x_t)$
12. $v_{k}^{(t)} \leftarrow \arg \min_{v \in \mathcal{C}} \langle \nabla Q_t(y_{k}^{(t)}), v \rangle$
13. $\gamma_{k}^{(t)} \leftarrow \min \left\{ \eta_t \frac{\langle \nabla Q_t(y_{k}^{(t)}), y_{k}^{(t)} - v_{k}^{(t)} \rangle}{\|y_{k}^{(t)} - v_{k}^{(t)}\|_{H_t}^2}, 1 \right\}$
14. $y_{k+1} \leftarrow y_{k}^{(t)} + \gamma_{k}^{(t)} (v_{k}^{(t)} - y_{k}^{(t)})$
15. end for
16. $x_{t+1} \leftarrow y_{K}^{(t)}$
17. end for

B Sensitivity to the number $K$ of inner iterations

We report in Figures 7-12 the sensitivity to $K$ in the respective computational experiments (Section 4). In most cases, we can see that for large values of $K$, the method becomes less efficient in CPU time. This validates our approach, detailed in Section 3.1, since $K \gg 1$ represents solving the subproblems (almost) completely, as done in AdaGrad.
Figure 7: Sensitivity of AdaCSFW to $K$ on the support vector classification experiment (Section 4.1.1).

Figure 8: Sensitivity of AdaSVRF to $K$ on the linear regression experiment (Section 4.1.2).
Figure 9: Sensitivity of AdaCSFW to $K$ on the logistic regression experiment (Section 4.1.3).

Figure 10: Sensitivity of AdaSFW (left) and AdamSFW (right) to $K$ on the MNIST dataset experiment (Section 4.2.1).
Figure 11: Sensitivity of AdaSFW (left) and AdamSFW (right) to $K$ on the IMDB dataset experiment (Section 4.2.2).

Figure 12: Sensitivity of AdaSFW (left) and AdamSFW (right) to $K$ on the CIFAR-10 dataset experiment (Section 4.2.3).
C  The Frank-Wolfe duality gap

We report in Fact C.1 some well-known properties of the Frank-Wolfe duality gap (Jaggi, 2013).

**Fact C.1.** Let $C \subset \mathbb{R}^n$ be a compact convex set, $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and $x \in C$. The Frank-Wolfe duality gap of $f$ at $x$ over $C$ is $g(x) := \max_{v \in C} \langle \nabla f(x), x - v \rangle$ and satisfies

(i) $g(x) \geq 0$,

(ii) $g(x) = 0 \iff x$ is a stationary point,

(iii) $f(x) - \min_C f \leq g(x)$ if $f$ is convex.

**Proof.** (i) Let $w \in \arg\min_{v \in C} \langle \nabla f(x), x - v \rangle$. We have

$$g(x) = \max_{v \in C} \langle \nabla f(x), x - v \rangle = \langle \nabla f(x), x - w \rangle = \langle \nabla f(x), x \rangle - \langle \nabla f(x), w \rangle \geq 0,$$

by definition of $w$.

(ii) We have

$$0 = g(x) = \max_{v \in C} \langle \nabla f(x), x - v \rangle \geq \langle \nabla f(x), x - y \rangle,$$

for all $y \in C$. Therefore, there exists no descent direction for $f$ at $x$ over $C$. The converse is trivial.

(iii) Let $x^* \in \arg\min_C f$. By convexity of $f$,

$$f(x) - \min_C f = f(x) - f(x^*) \leq \langle \nabla f(x), x - x^* \rangle \leq \max_{v \in C} \langle \nabla f(x), x - v \rangle = g(x),$$

since $x^* \in C$. 

D  Proofs

D.1  The algorithm

**Lemma D.1.** Consider Template 3 and let $t \in [0, T - 1]$. For all $k \in [0, K - 1]$,

$$Q_t(y_{k+1}^{(t)}) = \min_{\gamma \in [0, \gamma_t]} Q_t(y_k^{(t)} + \gamma (v_k^{(t)} - y_k^{(t)})),$$

In particular, $Q_t(y_{k+1}^{(t)}) \leq Q_t(y_k^{(t)})$ and $Q_t(y_1^{(t)}) \leq Q_t(y_0^{(t)} + \gamma_t (v_0^{(t)} - y_0^{(t)}))$. 

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Proof. Let \( k \in [0, K - 1] \) and \( \varphi_k^{(t)} : \gamma \in \mathbb{R} \mapsto Q_t(y_k^{(t)} + \gamma(v_k^{(t)} - y_k^{(t)})) \). Then \( \varphi_k^{(t)} \) is a convex quadratic and is minimized at

\[
\gamma^* := \eta_t \frac{\langle \nabla Q_t(y_k^{(t)}), y_k^{(t)} - v_k^{(t)} \rangle}{\|y_k^{(t)} - v_k^{(t)}\|^2_H}.
\]

Since \( v_k^{(t)} \in \arg \min_{v \in C} \langle \nabla Q_t(y_k^{(t)}), v \rangle \) and \( y_k^{(t)} \in C \), we have \( \langle \nabla Q_t(y_k^{(t)}), y_k^{(t)} - v_k^{(t)} \rangle \geq 0 \) so \( \gamma^* \geq 0 \). Thus, \( \varphi_k^{(t)} \) is a decreasing function over \([0, \gamma^*]\). Since \( y_k^{(t)} = \min\{\gamma^*, \gamma_t\} \), we obtain

\[
Q_t(y_{k+1}^{(t)}) = \min_{\gamma \in [0, \gamma_t]} Q_t(y_k^{(t)} + \gamma(v_k^{(t)} - y_k^{(t)})).
\]

Lemma D.2. Consider Template 3. For all \( t \in [0, T - 1] \),

\[
\|x_{t+1} - x_t\|_2 \leq KD\gamma_t.
\]

Proof. Let \( t \in [0, T - 1] \). We have

\[
x_{t+1} - x_t = y_{K}^{(t)} - y_0^{(t)}
\]

and, by a straightforward induction on \( k \in [0, K] \),

\[
y_{K}^{(t)} - y_0^{(t)} = \sum_{k=0}^{K-1} \left( \prod_{\ell=k+1}^{K-1} (1 - \gamma^{(t)}_{\ell}) \right) \gamma^{(t)}_k (v_{k}^{(t)} - x_t).
\]

Since for all \( k \in [0, K - 1] \),

\[
0 \leq \gamma^{(t)}_k \leq \gamma_t \leq 1,
\]

we obtain

\[
\|x_{t+1} - x_t\|_2 \leq \sum_{k=0}^{K-1} \left( \prod_{\ell=k+1}^{K-1} (1 - \gamma^{(t)}_{\ell}) \right) \gamma^{(t)}_k \|v_{k}^{(t)} - x_t\|_2
\]

\[
\leq \sum_{k=0}^{K-1} 1 \cdot \gamma_t \cdot D
\]

\[
= K\gamma_t D.
\]

D.2 SFW with adaptive gradients

Lemma D.3 is adapted from Hazan and Luo (2016, Appendix B).

Lemma D.3. Consider AdaSFW (Algorithm 4). For all \( t \in [0, T - 1] \),

\[
\mathbb{E}[\|\nabla f(x_t) - \nabla f(x_t)\|_2] \leq \frac{G}{\sqrt{b_t}}.
\]
Theorem D.4 (Theorem 3.1). Consider AdaSFW (Algorithm 4) with \( b_t \leftarrow (G(t + 2)/(LD))^2 \), \( \eta_t \leftarrow \lambda_t^+ / L \), and \( \gamma_t \leftarrow 2/(t + 2) \), and let \( \kappa := \lambda_t^+ / \lambda_0 \). Then for all \( t \in [1, T] \),

\[
E[f(x_t)] - \min_f f \leq \frac{2LD^2(K + 1 + \kappa)}{t + 1}.
\]

Proof. Let \( t \in [0, T - 1] \). By (2),

\[
\frac{L}{2} \| \cdot \|^2 \leq \frac{L}{2\lambda_t} \| \cdot \|^2_{H_t} = \frac{1}{2\eta_t} \| \cdot \|^2_{H_t}
\]

and

\[
\frac{1}{2\eta_t} \| \cdot \|^2_{H_t} \leq \frac{\lambda_t^+}{2\lambda_t} \| \cdot \|^2 \leq \frac{L\lambda_t^+}{2} \| \cdot \|^2 \leq \frac{L}{2} \| \cdot \|^2.
\]

By smoothness of \( f \) and (8),

\[
f(x_{t+1}) \leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2} \| x_{t+1} - x_t \|^2
\]

\[
\leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{1}{2\eta_t} \| x_{t+1} - x_t \|^2_{H_t}
\]

\[
= f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{1}{2\eta_t} \| x_{t+1} - x_t \|^2_{H_t} + \langle \nabla f(x_t) - \nabla f(x_t), x_{t+1} - x_t \rangle
\]

\[
= Q_t(x_{t+1}) + \langle \nabla f(x_t) - \nabla f(x_t), x_{t+1} - x_t \rangle
\]

\[
= Q_t(y^{(t)}) + \langle \nabla f(x_t) - \nabla f(x_t), x_{t+1} - x_t \rangle
\]

\[
\leq Q_t(y_0^{(t)} + \gamma_t(v_t - y_0^{(t)})) + \| \nabla f(x_t) - \nabla f(x_t) \|_2 \| x_{t+1} - x_t \|_2,
\]

by Lemma D.1 and the Cauchy-Schwarz inequality. Recall that \( y_0^{(t)} = x_t \) and let \( v_t := v_0^{(t)} \). Then,

\[
f(x_{t+1}) \leq Q_t(x_t + \gamma_t(v_t - x_t)) + \| \nabla f(x_t) - \nabla f(x_t) \|_2 \| x_{t+1} - x_t \|_2
\]

\[
= f(x_t) + \gamma_t \langle \nabla f(x_t), v_t - x_t \rangle + \frac{\gamma_t^2}{2\eta_t} \| v_t - x_t \|^2_{H_t} + \| \nabla f(x_t) - \nabla f(x_t) \|_2 \| x_{t+1} - x_t \|_2.
\]

Let \( x^\ast \in \text{arg} \min_{x \in c} f \). Since \( \nabla Q_t(y_0^{(t)}) = \nabla f(x_t) \), we have \( v_t \in \text{arg} \min_{v \in c} \langle \nabla f(x_t), v \rangle \) so

\[
\langle \nabla f(x_t), v_t - x_t \rangle \leq \langle \nabla f(x_t), x^\ast - x_t \rangle
\]

\[
\leq \langle \nabla f(x_t), x^\ast - x_t \rangle + \langle \nabla f(x_t) - \nabla f(x_t), x^\ast - x_t \rangle
\]

\[
\leq f(x^\ast) - f(x_t) + \| \nabla f(x_t) - \nabla f(x_t) \|_2 \| x^\ast - x_t \|_2,
\]

by convexity of \( f \) and the Cauchy-Schwarz inequality. Let \( \varepsilon_t := f(x_t) - \min_f f \) for all \( t \in [0, T] \). Combining (10) and (11), subtracting both sides by \( \min_{x \in c} f \), and taking the expectation, we obtain

\[
E[\varepsilon_{t+1}] \leq (1 - \gamma_t)E[\varepsilon_t] + \gamma_t E[\| \nabla f(x_t) - \nabla f(x_t) \|_2 \| x^\ast - x_t \|_2] + \frac{\gamma_t^2}{2\eta_t} E[\| v_t - x_t \|^2_{H_t}]
\]

\[
+ E[\| \nabla f(x_t) - \nabla f(x_t) \|_2 \| x_{t+1} - x_t \|_2]
\]

\[
\leq (1 - \gamma_t)E[\varepsilon_t] + \gamma_t E[\| \nabla f(x_t) - \nabla f(x_t) \|_2] + \frac{\gamma_t^2}{2} 2LD^2 + E[\| \nabla f(x_t) - \nabla f(x_t) \|_2] KD\gamma_t,
\]

where we used (9) and Lemma D.2. By Lemma D.3, and with \( b_t = (G(t + 2)/(LD))^2 \) and \( \gamma_t = 2/(t + 2) \),

\[
E[\varepsilon_{t+1}] \leq (1 - \gamma_t)E[\varepsilon_t] + \gamma_t \frac{G}{\sqrt{\eta_t}} D + \frac{\gamma_t^2}{2} 2LD^2 + \frac{G}{\sqrt{\eta_t}} KD\gamma_t
\]

\[
= \frac{t}{t + 2} E[\varepsilon_t] + \frac{2}{(t + 2)^2} LD^2 + \frac{2}{(t + 2)^2} LD^2 + \frac{2}{(t + 2)^2} LD^2 + \frac{LD}{t + 2} 2KD
\]

\[
= \frac{t}{t + 2} E[\varepsilon_t] + \frac{C}{(t + 2)^2},
\]
where \( C := 2LD^2(K + 1 + \kappa) \). Thus,

\[
(t + 1)(t + 2)E[\varepsilon_{t+1}] \leq (t + 1)E[\varepsilon_t] + \frac{C(t + 1)}{t + 2},
\]

so, by telescoping,

\[
t(t + 1)E[\varepsilon_t] \leq 0 \cdot 1 \cdot E[\varepsilon_0] + \sum_{s=0}^{t-1} \frac{C(s + 1)}{s + 2} \leq Ct
\]

for all \( t \in [1, T] \). Therefore,

\[
E[\varepsilon_t] \leq \frac{C}{t + 1}
\]

for all \( t \in [1, T] \).

**Theorem D.5** (**Theorem 3.3**). Suppose that \( f_1, \ldots, f_m \) are not necessarily convex. Consider AdaSFW (**Algorithm 4**) with \( b_t \leftarrow (G/(LD))^2(t + 1), \eta_t \leftarrow \lambda_t^{-1}/L \), and \( \gamma_t \leftarrow 1/(t + 1)^{1/2 + \nu} \) where \( \nu \in [0, 1/2] \), and let \( \kappa := \lambda_0^2/\lambda_0^2 \). For all \( t \in [0, T] \), let \( X_t \) be sampled uniformly at random from \( \{x_0, \ldots, x_t\} \). Then for all \( t \in [0, T-1] \),

\[
E[g(X_t)] \leq \frac{(f(x_0) - \min_C f) + LD^2(K + 1 + \kappa/2)S}{(t + 1)^{1/2 - \nu}}
\]

where \( S := \sum_{s=0}^{t-1} 1/(s + 1)^{1+\nu} \in \mathbb{R}_+ \). Alternatively, if the time horizon \( T \) is fixed, then with \( b_t \leftarrow (G/(LD))^2T \) and \( \gamma_t \leftarrow 1/\sqrt{T} \),

\[
E[g(X_{T-1})] \leq \frac{(f(x_0) - \min_C f) + LD^2(K + 1 + \kappa/2)}{\sqrt{T}}.
\]

**Proof.** For all \( t \in [0, T] \), let \( E_{X_t} \) denote the conditional expectation with respect to the realization of \( X_t \) given \( \{x_0, \ldots, x_t\} \). Recall that \( E_t \) denotes the expectation with respect to all the randomness in the system. Let \( t \in [0, T-1] \). By (10),

\[
f(x_{t+1}) \leq f(x_t) + \gamma_t(\nabla f(x_t), v_t - x_t) + \frac{\gamma_t^2}{2\eta_t} \|v_t - x_t\|^2_{\mathcal{H}_t} + \|\nabla f(x_t) - \nabla f(x_t)\|_{\mathcal{H}_t} \|x_{t+1} - x_t\|.
\]

Let \( w_t \in \text{arg min}_{v \in C}(\nabla f(x_t), v) \) and note that \( g(x_t) = \langle \nabla f(x_t), x_t - w_t \rangle \). Then, since \( v_t \in \text{arg min}_{v \in C}(\nabla f(x_t), v) \),

\[
f(x_{t+1}) \leq f(x_t) + \gamma_t(\nabla f(x_t), w_t - x_t) + \frac{\gamma_t^2}{2\eta_t} \|v_t - x_t\|^2_{\mathcal{H}_t} + \|\nabla f(x_t) - \nabla f(x_t)\|_{\mathcal{H}_t} \|x_{t+1} - x_t\| \leq f(x_t) - \gamma_t g(x_t) + \gamma_t \|\nabla f(x_t) - \nabla f(x_t)\|_{\mathcal{H}_t} \|x_{t+1} - x_t\|_2
\]

where we used the Cauchy-Schwarz inequality, (9), and Lemma D.2 in the last inequality. By Lemma D.3, we obtain

\[
E[f(x_{t+1})] \leq E[f(x_t)] - \gamma_t E[g(x_t)] + \frac{G}{\sqrt{b_t}} D + \frac{\gamma_t^2}{2\eta_t} D^2 + \frac{G}{\sqrt{b_t}} KD\gamma_t
\]

for all \( t \in [0, T-1] \). Therefore,

\[
E[\varepsilon_t] \leq \frac{C}{t + 1}
\]

for all \( t \in [1, T] \).
so, by telescoping,

\[
\sum_{s=0}^{t} \gamma_s E[g(x_s)] \leq E[f(x_0)] - E[f(x_{t+1})] + \sum_{s=0}^{t} \frac{LD^2(K + 1 + \kappa/2)}{(s + 1)^{1+\nu}} \\
\leq \left( f(x_0) - \min_{\mathcal{C}} f \right) + LD^2(K + 1 + \kappa/2)S
\]

and

\[
\sum_{s=0}^{t} \gamma_s E[g(x_s)] \geq \gamma_t \sum_{s=0}^{t} E[g(x_s)] \\
= \gamma_t (t + 1) E\left[ \sum_{s=0}^{t} \frac{1}{t + 1} g(x_s) \right] \\
= (t + 1)^{1/2-\nu} E[E_t[g(X_t)]] \\
= (t + 1)^{1/2-\nu} E[g(X_t)],
\]

by the law of total expectation. Therefore,

\[
E[g(X_t)] \leq \left( f(x_0) - \min_{\mathcal{C}} f \right) + LD^2(K + 1 + \kappa/2)S
\]

At (13), alternatively,

\[
E[f(x_{t+1})] \leq E[f(x_{t})] - \frac{1}{\sqrt{T}} E[g(x_t)] + \frac{LD^2}{T} + \frac{1}{T} L^2 D^2 + \frac{KLD^2}{T}
\]

so, by telescoping,

\[
\frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} E[g(x_t)] \leq E[f(x_0)] - E[f(x_T)] + LD^2(K + 1 + \kappa/2) \\
\leq \left( f(x_0) - \min_{\mathcal{C}} f \right) + LD^2(K + 1 + \kappa/2)
\]

and

\[
\frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} E[g(x_t)] = \frac{T}{\sqrt{T}} E\left[ \sum_{t=0}^{T-1} \frac{1}{T} g(x_t) \right] \\
= \sqrt{T} E[E_{T-1}[g(X_{T-1})]] \\
= \sqrt{T} E[g(X_{T-1})],
\]

by the law of total expectation. Therefore,

\[
E[g(X_{T-1})] \leq \left( f(x_0) - \min_{\mathcal{C}} f \right) + LD^2(K + 1 + \kappa/2) \frac{1}{\sqrt{T}}.
\]

D.3 SVRF with adaptive gradients

Lemma D.6 is a slight modification of Hazan and Luo (2016, Lemma 1).
**Lemma D.6.** Consider AdaSVRF (Algorithm 5). For all $t \in [0, T-1]$,

$$
E[\|\hat{\nabla} f(x_t) - \nabla f(x_t)\|^2] \leq \frac{4L}{b_t} \left( E[f(x_t) - \min_c f] + E \left[ f(\tilde{x}_t) - \min_c f \right] \right).
$$

**Proof.** Let $t \in [0, T-1]$, $E_t$ denote the conditional expectation with respect to the realization of $i_1, \ldots, i_{b_t}$ given all the randomness in the past (hence, $\tilde{x}_t$ and $x_t$ are given), and $x^* \in \arg\min_C f$. For all $i \in \{i_1, \ldots, i_{b_t}\}$,

$$
E_t[\|\nabla f(x_t) - (\nabla f(\tilde{x}_t) + \nabla f_i(x_t) - \nabla f_i(\tilde{x}_t))\|^2] \\
\leq 2E_t[\|\nabla f(x_t) - \nabla f_i(x_t)\|^2] + E_t[\|\nabla f_i(\tilde{x}_t) - \nabla f_i(x_t)\|^2] \\
\leq 2E_t[\|\nabla f_i(x_t) - f_i(x^*) \|^2] + E_t[\|\nabla f_i(x_t) - f_i(x^*) \|^2] \\
\leq 4L(E_t[f_i(x_t) - f_i(x^*) - \langle \nabla f_i(x^*) , x_t - x^* \rangle] + E_t[f_i(x_t) - f_i(x^*) - \langle \nabla f_i(x^*) , \tilde{x}_t - x^* \rangle] \\
= 4L(f(x_t) - f(x^*) - \langle \nabla f(x^*) , x_t - x^* \rangle + f(\tilde{x}_t) - f(x^*) - \langle \nabla f(x^*) , \tilde{x}_t - x^* \rangle),
$$

where we used $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$. Since $E_t[\nabla f_i(x)] = \nabla f(x)$ for all $i \in \{i_1, \ldots, i_{b_t}\}$ and $x \in \{x^*, \tilde{x}_t, x_t\}$, then the first term above is the variance of $\nabla f_i(x_t) - \nabla f_i(x^*)$ and the second term above is the variance of $\nabla f_i(\tilde{x}_t) - \nabla f_i(x^*)$, both with respect to $E_t$. The variance of a random variable being upper bounded by its second moment, we obtain

$$
E_t[\|\nabla f(x_t) - (\nabla f(\tilde{x}_t) + \nabla f_i(x_t) - \nabla f_i(\tilde{x}_t))\|^2] \leq 4L(f(x_t) - f(x^*) + f(\tilde{x}_t) - f(x^*)).
$$

By the law of total expectation,

$$
E[\|\nabla f(x_t) - \hat{\nabla} f(x_t)\|^2] = E \left[ \frac{1}{b_t} \sum_{i=1}^{b_t} \|\nabla f(x_t) - (\nabla f(\tilde{x}_t) + \nabla f_i(x_t) - \nabla f_i(\tilde{x}_t))\|^2 \right] \\
\leq \frac{1}{b_t^2} \sum_{i=1}^{b_t} E[\|\nabla f(x_t) - (\nabla f(\tilde{x}_t) + \nabla f_i(x_t) - \nabla f_i(\tilde{x}_t))\|^2] \\
\leq \frac{4L}{b_t} (E[f(x_t) - f(x^*)] + E[f(\tilde{x}_t) - f(x^*)]).
$$

\[ \square \]

**Theorem D.7** ((Theorem 3.5)). Consider AdaSVRF (Algorithm 5) with $s_k \leftarrow 2^{-k} - 1$, $b_t \leftarrow 24(K+1+\kappa)(t+2)$ where $\kappa := \lambda h_0 / \lambda_0$, $\eta_t \leftarrow \lambda_t / L$, and $\gamma_t \leftarrow 2/(t+2)$. Then for all $t \in [1, T]$,

$$
E[|f(x_t)| - \min_c f \leq \frac{2LD^2(K+1+\kappa)}{t+2}.
$$

**Proof.** We proceed by strong induction. Let $\epsilon_t := f(x_t) - \min_c f$ for all $t \in [0, T]$. By (12),

$$
E[\epsilon_{t+1}] \leq (1 - \gamma_t)E[\epsilon_t] + \gamma_t E[\|\hat{\nabla} f(x_t) - \nabla f(x_t)\|_2]D + \frac{\gamma_t^2}{2} LD^2 \leq E[\|\nabla f(x_t) - \hat{\nabla} f(x_t)\|_2]KD \gamma_t,
$$

for all $t \in [0, T-1]$. If $t = 0$ then, since $s_0 = 0$, we have $\hat{\nabla} f(x_0) = \nabla f(x_0)$ so

$$
E[\epsilon_1] \leq (1 - \gamma_0)E[\epsilon_0] + \frac{\gamma_0^2}{2} LK \leq \frac{LD^2 \kappa}{2},
$$

(14)
because $\gamma_0 = 1$, so the base case holds. Suppose (14) holds for all $t' \in [1, t]$ for some $t \in [1, T - 1]$. There exist $k, \ell \in \mathbb{N}$ such that $t = s_k + \ell$ and $\ell \leq s_{k+1} - s_k - 1$. That is, $s_k$ is the last snapshot time and $\tilde{x}_t = x_{s_k}$.

Note that this implies $\ell \leq 2^k - 1 = s_k$ so $t + 2 = s_k + \ell + 2 \leq 2s_k + 2 \leq 2(s_k + 2)$. By Lemma D.6,

$$E[\|\tilde{\nabla} f(x_t) - \nabla f(x_t)\|_2^2] \leq \frac{4L}{b_t} \left( E[\|f(x_t) - \min f\|] + E[\|\tilde{x}_t - \min f\|] \right)$$

$$\leq \frac{4L}{b_t} \left( \frac{2LD^2(K + 1 + \kappa)}{t + 2} + \frac{2LD^2(K + 1 + \kappa)}{s_k + 2} \right)$$

$$\leq \frac{4L}{b_t} \left( \frac{2LD^2(K + 1 + \kappa)}{t + 2} + \frac{4LD^2(K + 1 + \kappa)}{t + 2} \right)$$

$$= \frac{24L^2D^2(K + 1 + \kappa)}{b_t(t + 2)}$$

$$= \left( \frac{LD}{t + 2} \right)^2,$$

since $b_t = 24(K + 1 + \kappa)(t + 2)$. By Jensen’s inequality,

$$E[\|\tilde{\nabla} f(x_t) - \nabla f(x_t)\|_2^2] \leq \sqrt{E[\|\tilde{\nabla} f(x_t) - \nabla f(x_t)\|_2^2]}$$

$$\leq \frac{LD}{t + 2}.$$

Thus, with $\gamma_t = 2/(t + 2)$,

$$E[\varepsilon_{t+1}] \leq \frac{t}{t + 2} \frac{2LD^2(K + 1 + \kappa)}{t + 2} + \frac{2}{t + 2} \frac{LD}{t + 2} D + \frac{2}{(t + 2)^2} LK^2D^2 + \frac{LD}{t + 2} KD \frac{2}{t + 2}$$

$$= \frac{2LD^2(K + 1 + \kappa)t + 2LD^2 + 2LD^2K + 2KLD^2}{(t + 2)^2}$$

$$= \frac{2LD^2(K + 1 + \kappa)(t + 1)}{(t + 2)^2}$$

$$\leq \frac{2LD^2(K + 1 + \kappa)}{t + 3}.$$

\[\square\]

### D.4 CSFW with adaptive gradients

Let

$$A := \begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \end{pmatrix} \in \mathbb{R}^{m \times n} \quad \text{and} \quad f'(x) := \frac{1}{m} \sum_{i=1}^m f'_i(a_i, x)e_i$$

for all $x \in \mathbb{R}^n$, where $(e_1, \ldots, e_m)$ denotes the canonical basis of $\mathbb{R}^m$. Thus,

$$\nabla f(x) = \frac{1}{m} \sum_{i=1}^m f'_i(a_i, x)a_i = A^T f'(x)$$

(15)

for all $x \in \mathbb{R}^n$.

Lemma D.8 is adapted from Négier et al. (2020, Lemmata 2 and 3) and uses Lemma D.2.
Lemma D.8. Consider AdaCSFW (Algorithm 6). For all \(t \in [1, T]\),

\[
E_t[\|f'(x_t) - \alpha_t\|_1] \leq \left(1 - \frac{b}{m}\right) \left(\|f'(x_{t-1}) - \alpha_{t-1}\|_1 + \frac{KLD_A}{m} \gamma_{t-1}\right).
\]

where \(E_t\) denotes the conditional expectation with respect to the realization of \(i_1, \ldots, i_b\) given all the randomness in the past. Thus, for all \(t \in [0, T]\),

\[
E[\|f'(x_t) - \alpha_t\|_1] \leq \left(1 - \frac{b}{m}\right)^t \|f'(x_0) - \alpha_0\|_1 + \frac{2KLD_A}{m} \left(\left(1 - \frac{b}{m}\right)^{t/2} \ln \left(\frac{t}{2} + 1\right) + \frac{2(m/b - 1)}{t + 2}\right).
\]

Proof. Let \(t \in [1, T]\) and \(i_1, \ldots, i_b\) be the indices sampled at iteration \(t\). For all \(i \in [1, m]\), we have

\[
[a_t]_i = \begin{cases} 
(1/m)f'_i((a_i, x_t)) & \text{if } i \in \{i_1, \ldots, i_b\}, \\
[a_{t-1}]_i & \text{if } i \notin \{i_1, \ldots, i_b\}.
\end{cases}
\]

Thus,

\[
E_t[\|f'(x_t) - \alpha_t\|_1] = \sum_{i=1}^m E_t\left[ \left| \frac{1}{m} f'_i((a_i, x_t)) - [a_t]_i \right| \right]
= \sum_{i=1}^m \left(1 - \frac{b}{m}\right) \left| \frac{1}{m} f'_i((a_i, x_t)) - [a_{t-1}]_i \right|
= \left(1 - \frac{b}{m}\right) \|f'(x_t) - \alpha_{t-1}\|_1. \tag{16}
\]

Then, by the triangular inequality and \(L\)-smoothness of \(f_1, \ldots, f_m\),

\[
\|f'(x_t) - \alpha_{t-1}\|_1 \leq \|f'(x_t) - f'(x_{t-1})\|_1 + \|f'(x_{t-1}) - \alpha_{t-1}\|_1
\]

\[
= \sum_{i=1}^m \frac{1}{m} |f'_i((a_i, x_t)) - f'_i((a_i, x_{t-1}))| + \|f'(x_{t-1}) - \alpha_{t-1}\|_1
\]

\[
\leq \frac{L}{m} \sum_{i=1}^m |a_i, x_t - x_{t-1}| + \|f'(x_{t-1}) - \alpha_{t-1}\|_1
\]

\[
= \frac{L}{m} \|A(x_t - x_{t-1})\|_1 + \|f'(x_{t-1}) - \alpha_{t-1}\|_1. \tag{17}
\]

Now, similarly to the proof of Lemma D.2,

\[
\|A(x_t - x_{t-1})\|_1 \leq \left\| \sum_{k=0}^{K-1} \left( \prod_{t=k+1}^{K-1} (1 - \gamma_{t-1}) \right) \gamma_{k(t-1)} A(v_k^{(t-1)} - x_{t-1}) \right\|_1
\]

\[
\leq \sum_{k=0}^{K-1} 1 \cdot \gamma_{t-1} \cdot \|A(v_k^{(t-1)} - x_{t-1})\|_1
\]

\[
\leq K \gamma_{t-1} D_A^A.
\]

Together with (16) and (17), we obtain

\[
E_t[\|f'(x_t) - \alpha_t\|_1] \leq \left(1 - \frac{b}{m}\right) \left(\|f'(x_{t-1}) - \alpha_{t-1}\|_1 + \frac{KLD_A}{m} \gamma_{t-1}\right).
\]

The second result follows as in Négiar et al. (2020, Lemma 3). \(\square\)
Theorem D.9 (Theorem 3.7). Consider AdaCSFW (Algorithm 6) with \( \eta_t \leftarrow m \lambda_t^- / (L \| A \|_2^2) \) and \( \gamma_t \leftarrow 2 / (t + 2) \), and let \( \kappa := \lambda_0^+ / \lambda_0^- \). Then for all \( t \in [1, T] \),
\[
\mathbb{E}[f(x_t)] - \min_c f \leq \frac{2L}{t + 1} \left( 4K(K + 1)D_A^4D_{\infty}^4 \left( \frac{1}{b} - \frac{1}{m} \right) + \frac{\kappa \| A \|_2^2 D_2^2}{m} \right) \\
+ \frac{2(K + 1)D_A^4(m/b)^2}{t(t + 1)} \left( \| f'(x_0) - \alpha_0 \|_1 + \frac{16KLD_A^4}{b} \right).
\]

Proof. By \((L/m)\)-smoothness of \( \varphi: \xi \in \mathbb{R}^m \mapsto (1/m) \sum_{i=1}^m \varphi_i(\xi_i) \) (Négia et al., 2020, Proposition 2), we have
\[
\varphi(Ax_{t+1}) \leq \varphi(Ax_t) + \langle \nabla \varphi(x_t), A(x_{t+1} - x_t) \rangle + \frac{L}{2m} \| A(x_{t+1} - x_t) \|_2^2,
\]
i.e.,
\[
f(x_{t+1}) \leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2m} \| A(x_{t+1} - x_t) \|_2^2
\]
\[
\leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2m} \| A \|_2^2 \| x_{t+1} - x_t \|_2
\]
\[
\leq f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{L}{2m} \| A \|_2^2 \| x_{t+1} - x_t \|_H^2
\]
\[
= f(x_t) + \langle \nabla f(x_t), x_{t+1} - x_t \rangle + \frac{1}{2\eta_t} \| x_{t+1} - x_t \|_H^2
\]
\[
= f(x_t) + \langle \tilde{\nabla} f(x_t), x_{t+1} - x_t \rangle + \frac{1}{2\eta_t} \| x_{t+1} - x_t \|_H^2 + \langle \nabla f(x_t) - \tilde{\nabla} f(x_t), x_{t+1} - x_t \rangle
\]
\[
= Q_t(x_{t+1}) + \langle \nabla f(x_t) - \tilde{\nabla} f(x_t), x_{t+1} - x_t \rangle
\]
by definition of \( \| A \|_2 \) and \((2)\). Thus, with \( v_t := v_0^{(t)} \) and since \( x_{t+1} = y_{K}^{(t)}, x_t = y_0^{(t)} \), by Lemma D.1 we have
\[
f(x_{t+1}) \leq Q_t(x_t + \gamma_t(v_t - x_t)) + \langle \nabla f(x_t) - \tilde{\nabla} f(x_t), x_{t+1} - x_t \rangle
\]
\[
= f(x_t) + \gamma_t \langle \tilde{\nabla} f(x_t), v_t - x_t \rangle + \gamma_t^2 \| v_t - x_t \|_H^2 + \langle f'(x_t) - \alpha_t, A(x_{t+1} - x_t) \rangle
\]
by \((15)\). By Hölder’s inequality,
\[
f(x_{t+1}) \leq f(x_t) + \gamma_t \langle \tilde{\nabla} f(x_t), x^* - x_t \rangle + \gamma_t^2 \lambda_t^+ M \| v_t - x_t \|_2 + \| f'(x_t) - \alpha_t \|_1 \| A(x_{t+1} - x_t) \|_\infty
\]
\[
= f(x_t) + \gamma_t \langle \nabla f(x_t), x^* - x_t \rangle + \gamma_t \langle \tilde{\nabla} f(x_t) - \nabla f(x_t), x^* - x_t \rangle
\]
\[
+ \gamma_t^2 \lambda_t^+ M \| A \|_2^2 \| v_t - x_t \|_2 + \| f'(x_t) - \alpha_t \|_1 \| A(x_{t+1} - x_t) \|_\infty
\]
so, by Lemma D.8,
\[
\mathbb{E}[\xi_{t+1}] \leq (1 - \gamma_t)\mathbb{E}[\xi_t] + \gamma_t \mathbb{E}[\| f'(x_t) - \alpha_t \|_1 D_A^4] + \gamma_t^2 \frac{KL \| A \|_2^2 D_2^2}{2m} + \mathbb{E}[\| f'(x_t) - \alpha_t \|_1] K D_A^4 \gamma_t
\]
\[
= (1 - \gamma_t)\mathbb{E}[\xi_t] + \gamma_t (K + 1) D_\infty A^4 \mathbb{E}[\| f'(x_t) - \alpha_t \|_1] + \gamma_t^2 \frac{KL \| A \|_2^2 D_2^2}{2m}
\]
\[
\leq \frac{t}{t + 2} \mathbb{E}[\xi_t] + \frac{2(K + 1)D_\infty A^4}{t + 2} \left( 1 - \frac{b}{m} \right)^t \| f'(x_0) - \alpha_0 \|_1 +
\]
\[
+ \frac{4K(K + 1)LD_A^4 A^4 D_\infty}{m(t + 2)} \left( 1 - \frac{b}{m} \right)^{t/2} \ln \left( \frac{t}{2} + 1 \right) + \frac{2(m/b - 1)}{t + 1} + \frac{2KL \| A \|_2^2 D_2^2}{m(t + 2)^2}.
\]
Thus, multiplying both sides by \((t + 1)(t + 2)\),

\[
(t + 1)(t + 2)E[\epsilon_{t+1}] \leq t(t + 1)E[\epsilon_t] + 2(K + 1)D_\infty^A(t + 1) \left(1 - \frac{b}{m}\right)^t \|f'(x_0) - \alpha_0\|_1 \\
+ \frac{4K(K + 1)LD_1^AD_\infty^A(t + 1)}{m} \left(1 - \frac{b}{m}\right)^{t/2} \ln \left(\frac{t}{2} + 1\right) \\
+ \frac{8K(K + 1)LD_1^AD_\infty^A(m/b - 1)}{m} + 2\kappa L\|A\|_2^2D_2^2 \frac{t + 1}{t + 2}
\]

Telescoping, we obtain for all \(t \in [1, T]\),

\[
t(t + 1)E[\epsilon_t] \leq 0 \cdot 1 \cdot E[\epsilon_0] + 2(K + 1)D_\infty^A\|f'(x_0) - \alpha_0\|_1 \sum_{s=0}^{t-1} (s + 1) \left(1 - \frac{b}{m}\right)^s \\
+ \frac{4K(K + 1)LD_1^AD_\infty^A}{m} \sum_{s=0}^{t-1} (s + 1) \left(1 - \frac{b}{m}\right)^{s/2} \ln \left(\frac{s}{2} + 1\right) \\
+ \frac{2L}{m} \left(4K(K + 1)D_1^AD_\infty^A \left(\frac{m}{b} - 1\right) + \kappa \|A\|_2^2D_2^2\right) \sum_{s=0}^{t-1} \frac{s + 1}{s + 2} \\
\leq 2(K + 1)D_\infty^A\|f'(x_0) - \alpha_0\|_1 \cdot \left(\frac{m}{b}\right)^2 \\
+ \frac{4K(K + 1)LD_1^AD_\infty^A}{m} \cdot 8 \left(\frac{m}{b}\right)^3 \\
+ \frac{2L}{m} \left(4K(K + 1)D_1^AD_\infty^A \left(\frac{m}{b} - 1\right) + \kappa \|A\|_2^2D_2^2\right) t.
\]

Therefore,

\[
E[\epsilon_t] \leq \frac{2(K + 1)D_\infty^A\|f'(x_0) - \alpha_0\|_1(m/b)^2}{t(t + 1)} \\
+ \frac{32K(K + 1)LD_1^AD_\infty^A(m/b)^2}{bt(t + 1)} \\
+ \frac{2L}{m} \frac{4K(K + 1)D_1^AD_\infty^A(m/b - 1) + \kappa \|A\|_2^2D_2^2}{t + 1}.
\]