ON A BROWNIAN EXCURSION LAW, I: CONVOLUTION REPRESENTATIONS

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Abstract: This paper studies Brownian motion subject to the occurrence of a minimal length excursion below a given excursion level. The law of this process is determined. The characterization is explicit and shows by a layer construction how the law is built up over time in terms of the laws of sums of a given set of independent random variables.

Key words: Brownian excursions, Brownian law subject to excursion conditions, Laplace transform and its inversion.

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1. Introduction

In this paper we identify a general Parisian-type excursion law and determine its structure in the Brownian case. The idea is that we do not just ask when a process passes a given level, but instead ask for how long it will thereafter stay on the same side of the level. These excursions may have any length and may occur at any time. Having settled on a minimal length for them, the question to ask is if a minimal-length excursion will occur during a given time span. This then yields a second source of randomness for the problem, and the object to be studied is its joint law with that of the given process; see Section 2 for how to make all this precise.

The principal difficulty is to obtain an explicit description of this joint law when the process to start with is given explicitly. The main contribution of the paper is to give such an explicit description when this process is Brownian motion; details are in Section 3.

Historically, a rigorous study of these questions seems to originate with [13]. Extensions and further applications are developed for example in [6] and the references there. Applications to insurance are emerging; see for example [5].

Regarding the structural understanding of the joint law, its Laplace transform with respect to time has been determined in the Brownian case; see [13, Appendix] with modifications in [11]. As made precise in Section 5, the Laplace transform permits one to separate the above two sources of randomness of the problem. This is effected in [13, Appendix] by recourse to the Brownian meander and the Azéma martingale; the key steps of the argument are recalled in Section 6. The result is quotients of higher transcendental functions as Laplace transform of the joint law, with the denominators corresponding to the law’s excursion source of stochasticity and the numerators to the law’s process source of stochasticity. This generalizes situations as encountered for example with Laplace transforms of first passage times of Brownian motion, which lead to laws in terms of theta functions.

Mathematically, the contribution of the paper is to provide analogous such representations to the present context. The basis for this is furnished by our finding in Appendix B that the above denominators satisfy a functional equation, and as its main technical contribution the paper turns this into a way for the analytic inversion of the corresponding quotients. With more details developed in Section 4, this then shows how the effect of the excursion source on the law is built up over time – by adding at any integer time $n+1$ a new layer of $n$ independent copies of a fixed explicit random variable. This picture is made rigorous in Section 9 yielding the explicit expressions for the joint law of Section 3.
To conclude, Section 10 indicates how our approach is complemented in the companion [12] to the present paper by constructive techniques for convolution representations.

2. Basic notions and facts: the general process excursion law

We address in the paper the occurrence of excursions of a minimal length duration of a given stochastic process below a given excursion level which take place within a given period of time, and ask to characterize the law of the new stochastic process thus obtained. We follow [11] to make this precise in two stages. A general framework is thus established in the present section which furnishes the basis for the Brownian case to be developed subsequently in the paper.

2.1 Basic setting: Let \( X \) be any real-valued stochastic process on the complete probability space \((\Omega, \mathcal{F}, Q)\) whose time set is the nonnegative reals \([0, \infty)\), so that \( X = (X_u)_{u \geq 0} \), and let it be adapted to the filtration \( \mathcal{F} = (\mathcal{F}_u)_{u \geq 0} \) on this space which satisfies the usual conditions.

As the primitives of the problem fix any time \( t \), specify any real \( a \), which is to take the rôle of the excursion level of the problem, and specify any real \( D > 0 \), which is to take the rôle of the minimal excursion duration.

Denote by \( T_{t,a} := T(X)_{t,a} \) the first passage time after time \( t \) to the level \( a \) of \( X \) given by \( T(X)_{t,a} = \inf\{s \geq t | X_s \in [a, \infty)\} \); for simplicity assume that \( T_{t,a} \) is a stopping time, and if \( t = 0 \) we drop reference to the time \( t \) from the notation.

2.2 Achievement time: We first recall how the excursion situation under consideration is encoded by the achievement time \( H_{a,t} := H(X)_{a,t} \), a random variable to be defined presently with the property that for any real \( T > t \) we have \( H_{t,a} \leq T \) iff there is in \([t, T]\) a subinterval of length at least \( D \) on which \( X \) only takes values below the level \( a \). Morally, \( H_{a,t} \) so is the upper bound of the first of these subintervals occurring after time \( t \). For its explicit construction, which is to follow, for any real \( u \geq 0 \) introduce \( \sigma_{a,u} = \sigma(X)_{a,u} \) to denote the smallest upper bound of subintervals as above of \([u, \infty)\), namely:

\[
\sigma_{a,u} = \inf\{s \in [u + D, \infty) | X_s \in (s-D, s) < a\} \quad u \in [0, \infty).
\]

We will need to distinguish the two cases when \( X_t \) is equal to or above the level \( a \), and when it is below this level.

If \( X_t \geq a \), the situation to be referred to as the first case, the question arises if \( X \) will become and remain smaller than \( a \) for all points in time of an interval of length at least \( D \) contained in \([t, \infty]\). We represent this first case by setting:

\[
H_{a,t} = \sigma_{a,t} \quad \text{if} \quad X_t \geq a.
\]

If \( X_t < a \), the situation to be referred to as the second case, a new idea is needed when \( X \) has been staying below \( a \) for some connected period of time already. This excursion needs to continue below \( a \) for a period of length \( \delta_t \in (0, D) \) only to reach the minimum duration \( D \) stipulated. Two subcases result which we formalize in terms of the location of \( t+\delta_t \) relative to the first passage time \( T_{t,a} = T(X)_{t,a} \). Firstly, if \( T_{t,a} > t+\delta_t \), the situation to be referred to as Subcase 1, then the process \( X \) continues to stay below \( a \) during the period of time from time \( t \) until time \( t+\delta_t \) a fortiori, whence the representation:

\[
H_{a,t} = t+\delta_t \quad \text{if} \quad X_t < a \quad \text{and} \quad T_{t,a} > t+\delta_t.
\]
Secondly, if $T_{t,a} \leq t + \delta_t$, the situation to be referred to as Subcase 2, the level $a$ is passed by $X$ earlier than time $t + \delta_t$, and the clock for reaching the minimum length $D$ is restarted at time $T_{t,a}$. This then puts us into the first case, albeit as of time $T_{t,a}$ instead of time $t$, whence the representation:

$$H_{a,t} = \sigma_{a,T_{t,a}} \quad \text{if } X_t < a \text{ and } T_{t,a} \leq t + \delta_t.$$ 

### 2.3 Excursion density:

The excursion law to be studied is then by definition the representing measure of the functional given by:

$$\phi \mapsto E[1_{\{H_{a,t} \leq T\}} \phi(X_T) \mid \mathcal{F}_t],$$

for any $Q$-measurable function $\phi$ on the reals $\mathbb{R}$; assuming absolute continuity with respect to Lebesgue measure in addition the concept to be studied is hence the excursion density $h_{a,t} := h(X)_{a,t}$ which by definition is the conditional density on $[t, \infty) \times \mathbb{R}$ characterized by:

$$E[1_{\{H_{a,t} \leq T\}} \phi(X_T) \mid \mathcal{F}_t] = \int_{\mathbb{R}} \phi(x) h_{a,t}(T, x) \, dx,$$

for any $Q$-measurable function $\phi$ on the reals $\mathbb{R}$; here notice $x \mapsto h_{a,t}(T, x) = 0$ for $T < \delta_t$ extending the definition of $\delta_t$ in Section 2.2 to the first case there by the convention $\delta_t := D$.

### 2.4 Normalizations:

We develop a normalized form of the excursion law. The normalization is based on an emulation of the restarting at a fixed stopping of Markov processes. The construction starts from postulating the existence of a process $X^*$ which satisfies

$$X^*_{u \uparrow} = X_{t+u} - X_t, \quad u \in [0, \infty),$$

as well as $X^*(0) = 0$, and which is independent of $\mathcal{F}_t$; examples for processes $X$ for which these normalized processes exist thus include Brownian motion, Lévy processes which creep across levels, or continuous Markov processes which are homogeneous in their time and their space variable. The construction of Section 2.2 applied to $X^*$ then yields for any real $b$ the normalized achievement time $H^*_b := H(X^*)_{b,0}$ which satisfies

$$H_{a,t} = t + H^*_{a-X_t},$$

as well as

$$E[1_{\{H_{a,t} \leq u+t\}} \phi(X_{u+t}) \mid \mathcal{F}_t] = E[1_{\{H^*_{a-X_t} \leq u\}} \phi(X_t + X^*_u)],$$

for any measurable map $\phi$ on $\mathbb{R}$, treating the time-$t$ value $X_t$ of $X$ as a real number. Granting absolute continuity with respect to Lebesgue measure in addition as in Section 2.3 above, densities $h^*_b$ on $[0, \infty) \times \mathbb{R}$ are hence defined by:

$$E[1_{\{H^*_b \leq u\}} \phi^*(X^*_u)] = \int_{\mathbb{R}} \phi^*(y) h^*_b(u, y) \, dy,$$

for any $Q$-measurable function $\phi^*$ on $\mathbb{R}$. By construction these normalized excursion densities $h^*_b$ satisfy $h^*_b([0, \delta_0) \times \mathbb{R}) = \{0\}$ with $\delta_0$ the minimal excursion duration remaining as in Section 2.3 above, and are related to the excursion densities $h_{a,t}$ of Section 2.3 by:

$$h_{a,t}(u+t, y+X_t) = h^*_{a-X_t}(u, y),$$

for any real $u \geq 0$ and $y$. Granting existence, the study of the former law is hence reduced to the latter one. Translating the first and second case of Section 2.2 this explicitly asks to characterize $h^*_b$, for any real $b$, in the following Cases I and II respectively:

(I) We have $b \leq 0$ and ask if there will be a length $D$ time interval $I_D$ on which $X^*|_{I_D} < b$. 

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II. We have $b > 0$ and ask if $X^*|_{[0,\delta_0)} < b$ where the positive real $\delta_0 < D$ is the Section 2.2 minimal excursion length remaining.

Here Case II further decomposes into two subcases according to the relative position of $\delta_0$ to $T_b^*$, the first passage time of $X^*$ to the level $b$, as follows. If moreover $T_b^* > \delta_0$, we are in subcase (II-1), otherwise we are in subcase (II-2).

The paper initiates a study of the structure of $h_b^*$ along these lines when $X$ is Brownian motion; our main results here are to be described in the next Section 3 as a first step.

3. Statement and discussion of main results: convolution representations of the excursion law in the Brownian case

In this section we formulate and discuss our main results about the normalized excursion law $h_b^*$ of Section 2.4 in the Brownian case. We delineate in Sections 3.2 and 3.3 how, with hindsight, it emerges from a core relationship pinned down in Section 4 by taking into account the additional interrelations entailed by the Case I and Case II situations. Our discussion is in terms of convolutions of laws, with representing functions listed in Section 3.1 to follow.

3.1 Set up and functions: We modify the Section 2.1 set up as follows. First we let the probability space $(\Omega, \mathcal{F}, Q)$ there be equipped with the standard filtration $\mathcal{F} = (\mathcal{F}_u | u \in [0, \infty))$ of the Brownian motion $W$ on it. Then let $X = W$ and, consequently, let $X^* = W^*$ in Section 2.4 with $W^*$ now being the standard Brownian motion independent of time-$t$ information $\mathcal{F}_t$ that is obtained by time-$t$ restarting of $W$.

It is convolution representations for $h_b^*$ which thus result. They are in terms of three main classes of functions. Referring to Appendix A for more detail about them, these are as follows. Firstly, the functions $\nu_n$ on $(0, \infty)$ given for any integer $n \geq 1$ by the $n$-fold convolutions on $(0, \infty)$:

$$\nu_n = \nu^{*(n)}$$

where $\nu(u) = (2/\sqrt{\pi})\sqrt{u}/(2u+1)$, $u \in [0, \infty)$, and by $\nu_0 = \nu^{(0)}$ being set equal to the Dirac delta function at 0; for $n \geq 1$ note $\nu_n = \text{law}(nV)$ with $nV$ the $n$-fold sum of independent copies of a random variable $V$ with $\nu = \text{law}(V)$. Secondly, for any real $a$ and $c$, the functions $\rho_{a,c}$ on $(0, \infty)$ given by:

$$\rho_{a,c} = \left\{ \begin{array}{ll}
\sqrt{D}\chi_{|a+c|} \ast \nu & c \leq 0, \\
\sqrt{D}\chi_{|a|} \ast g_c & c > 0;
\end{array} \right.$$

they are convolutions on $(0, \infty)$ in terms of two additional functions on $(0, \infty)$ that depend on the complex parameter $\alpha$ with $|\text{arg}(\alpha)| \leq \pi/4$ and for any real $u \geq 0$ are given by:

$$\chi_\alpha(u) = (1/\sqrt{\pi u}) \exp\left(-(-\alpha/2)^2/u\right)$$

$$g_\alpha(u) = \exp\left(-\frac{\alpha^2}{2(1+2u)}\right)\left\{h(u)\exp\left(-\frac{\alpha^2}{4u(1+2u)}\right) + \frac{\alpha}{1+2u}\text{Erfc}\left(\frac{\alpha}{\sqrt{4u(1+2u)}}\right)\right\}.$$ 

Thirdly, for any excursion level $b$ the function $\beta = \beta_b$ on the real line $R$ given by:

$$\beta(y) = (b-y)/\sqrt{D}.$$ 

3.2 Case I results: We first assume the situation of Case I to hold as it is described in Section 2.4; we are thus asking about the occurrence of an excursion of duration at
least $D$ below the level $b \leq 0$ at some point in the future. The structure of the normalized excursion law $h^*_b$ here is governed by the sign of $\beta$ on the ‘state space’ by way of the functions $\rho_{a,c}$. Our precise result is as follows.

**Theorem 3.1:** In the above Case I setting we have for any reals $u > 0$ and $y$ the convolution sum representation:

$$h^*_b(u, y) = \frac{1}{2D} \sum_{1 \leq n < \frac{b}{D}} \frac{(-1)^{n-1}}{(2\pi)^{n/2}} \left( \rho_{b^*, \beta(y)} \ast \nu^*(n-1) \right) \left( \frac{1}{2} \left( \frac{u}{D} - n \right) \right),$$

where the sum is over all integers $n \geq 1$ satisfying $n < u/D$, and where $b^* = b/\sqrt{D}$.

With a proof in Section 9.1 our results make explicit how the law is built up over time by the addition of a new layer at integer points in time. Anticipating the discussion in Sections 5 and 6 this is a principal feature. It gives expression to the effect of the second source of stochasticity $X_2 = W^*(H^*_b)$, on the other hand, is encoded by the structure of the layers to be added. Here it gives rise to a fixed density $\rho = \rho_{b^*, \beta(y)}$ which, at each time $n$, is convolved with $\nu_{n-1}$, the ($n-1$)-fold convolution of the density $\nu$ originating with the above first source of stochasticity. The structure of the function $\rho$ depends on the state variable $y$, and gives expression to the position of $y$ relative to the excursion level $b$ by way of the function $\beta$.

### 3.3 Case II results:

We next address the structure of the normalized excursion law $h^*_b$ in the situation of Case II of Section 2.4; recall that this setting is characteristic for the problem by asking about the excursion of Brownian motion $W^*$ to continue below the level $b > 0$ for a period of length at least $\delta := \delta_0 > 0$ smaller than $D$. We here obtain $h^*_b$ as a four term sum:

$$h^*_b(u, y) = \sum_{j=1}^2 \sum_{k=1}^2 h^*_{b,j,k},$$

where the single summands are functions of the time variable $u > 0$ and the state variable $y$ as follows. Pertaining to the situations where the first source of stochasticity $X_1 = H^*_b$ (see the discussion of Section 3.2) is in Case II-1,

$$h^*_{b,1,1}(u, y) = \frac{Q^*(T_b > \delta)Q^*(T_b \leq \delta)}{D} \frac{1_{(\delta, \infty)}(u)}{\sqrt{2\pi(u-\delta)}} \int_{\delta}^{\infty} x \exp \left( - \frac{x^2}{2D} - \frac{(b-x-y)^2}{2(u-\delta)} \right) dx;$$

$$h^*_{b,1,2}(u, y) = Q^*(T_b > \delta)1_{(\delta, \infty)}(u)\varphi_{b,u}(y);$$

here $\varphi_{b,u}$ is a translate of the transition density from 0 to $y < b$ of Brownian motion killed at the first hitting time of $b$ and living on $(-\infty, b)$ as follows:

$$\varphi_{b,u}(y) = \frac{1}{\sqrt{2\pi u}} \exp \left( - \frac{y^2}{2u} \right) - \frac{1}{\sqrt{2\pi u}} \exp \left( - \frac{(y-2b)^2}{2u} \right),$$

where $Q^*(T_b > \delta) = (1-\text{Erfc})(b/\sqrt{2\delta})$ and $Q^*(T_b \leq \delta) = \text{Erfc}(b/\sqrt{2\delta})$, and where the integral factor of $h^*_{b,1,1}$ is expressible in terms of Erfc and its derivative as well; see the representation for the function $h^*_{b,3}$ in [11, p.4].

The remaining two summands have been known at the level of their Laplace transforms. By inverting these transforms in Sections 9.2 and 9.3 we are now able to describe in the next two results how their structure is built up over time. This once more proceeds by expressing them as sums of convolutions in terms of Section 3.1 functions. Our first result pertains to the summand $h^*_{b,2,1}$ as follows.
Theorem 3.2: In the above Case II situation, with excursion level $b > 0$, we have for any reals $u > 0$ and $y$ the convolution sum representation:

$$h_{b,2,1}(u,y) = \frac{1}{2D} \sum_n \frac{(-1)^{n-1}}{(2\pi)^{n/2}} \int_0^{\delta \wedge (u-Dn)} \mu_b(dw) \left( \rho_b(\cdot, y) \ast \nu^{*(n-1)} \ast \left( \frac{u-Dn-w}{2D} \right) \right),$$

where the sum is over all integers $n \geq 1$ satisfying $n < u/D$.

With a proof in Section 9.2 this theorem refers to a situation where the first stochasticity source $X_1 = H_b^*$ is now considered on the set of events where is expected to be restarted. This is given expression to by integration with respect to $\mu_b$, the law of the first passage time of Brownian motion to the level $b$, as expressed by the results

$$\mu_b(dw) = \psi_{b\sqrt{2}}(w) dw;$$

see Section A.1. Conditional on that influence, the excursion law is built up at positive integer time points as follows. At each time $n$ add the new term obtained by convolving $\nu_{n-1}$, the $(n-1)$-fold convolution of the density $\nu$, with a fixed density $\rho_b(\cdot, y)$. Here the function $\rho_b$ on $(0, \infty) \times \mathbb{R}$ is explicitly given by the following integral:

$$\rho_b(\tau, y) = 2 \int_{\mathbb{R}} \varphi_{b,\delta}(x) N_{0,D\tau}(|x-y|/\sqrt{2}) \, dx,$$

where $N_{0,v}(\xi) = (1/\sqrt{2\pi v}) \int_{(-\infty,\xi]} \exp(-x^2/(2v)) \, dx$ denotes the normal distribution with mean equal to 0 and variance equal to $v$. The functions $\rho_b$ originate with the restriction of the second source of stochasticity $X_2 = W^*(H_b^*)$ to the set of all events where a restarting of $H_b^*$ will not happen.

The situation for the final fourth summand $h_{b,2,2}^*$ differs from the above set up in regard to the stochasticity source $X_2 = W^*(H_b^*)$: here it is considered on the set of all events where a restarting of its argument $H_b^*$ is to take place. Conditional on the restarting of $X_1$ as above we therefore are in the situation of Section 3.2, and hence seek to characterize the occurrence of future length $D$ excursions below the level 0. This situation is encoded by the Section 3.1 functions $\rho_{0,\beta(y)}$ and our precise result is as follows.

Theorem 3.3: In the above Case II situation, with excursion level $b > 0$, we have for any reals $u > 0$ and $y$ the convolution sum representation:

$$h_{b,2,2}(u,y) = \frac{Q^*(T_b \leq \delta)}{2D} \sum_n \frac{(-1)^{n-1}}{(2\pi)^{n/2}} \int_0^{\delta \wedge (u-Dn)} \mu_b(dw) \left( \rho_{0,\beta(y)} \ast \nu^{*(n-1)} \ast \left( \frac{u-Dn-w}{2D} \right) \right),$$

where the sum is over all integers $n \geq 1$ satisfying $n < u/D$.

This result is proved Section 9.3, as the final step of our argument which starts in Section 4.

4. The key result for Laplace inversion

At the heart of the structure of the normalized excursion law $h_b^*$ as expressed by the results of Section 3 is the reconstruction of the law of $H_b^*$, the level-0 normalized achievement time. This reconstruction is to be established in this section, and asserts $H_b^*$ morally to originate as (the weak limit of) an infinite sum of independent copies of a fixed random variable, to
be denoted by $V$, in which only finitely many summands contribute when looked at in a pointwise sense. Here $V$ is pinned down in terms of its distribution by:

$$\text{law}(V) = \nu \quad \text{where} \quad \nu(u) = (2/\sqrt{\pi}) \sqrt{u} / (2u+1), \quad u \in [0, \infty);$$

our construction, however, necessitates two auxiliary random variable summands as follows.

**Theorem 4.1:** For any real $u > 0$, we have the equality of laws of pairwise independent random variables:

$$\text{law} \left( A_R + H^*_R \right)(u) = \sum_{1 \leq n < 2} \frac{(-1)^{n-1}}{(2\pi)^{n/2}} \text{law} \left( B_R + (n-1)V \right) \left( u - \frac{1}{2} n \right),$$

where $A_R$ and $B_R$ are random variables such that on some complex half-plane $\{ \text{Re} z > z_0 \}$ with $z_0 \geq 0$ we have:

$$E[\exp(-zA_R)] = R(z) \quad \text{and} \quad E[\exp(-zB_R)] = R(z) / \sqrt{z},$$

for an analytic function $R$ which for some $a > \frac{1}{2}$ satisfies $R(z) = O(|z|^{-a})$ as $|z| \to \infty$. The starting point of this result is provided by joining the equality

$$E[\exp(-zH^*_R)] = 1 / \Psi(\sqrt{2Dz}), \quad \text{Re} (z) > 0,$$

anticipated from Section 6.4 in terms of the function $\Psi(w) = \int_{(0, \infty)} x \exp(-x^2/2+wx) \, dx$ of Appendix B, with the symmetry property of $H^*_0$ expressed by the functional equation

$$\Psi(w) = \Psi(-w) + \sqrt{2\pi} w \exp\left(\frac{1}{2}w^2\right),$$

for any complex $w$, furnished by the key identity of Appendix B. This enables a representation of the expectation by resolution as a geometric series:

$$\frac{R(z)}{\Psi(\sqrt{z})} = R(z) \frac{f(z)}{1-p(z)} = \frac{R(z)}{\sqrt{2\pi z}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}(n+1)z\right) N_1(z)^n,$$

where $f(z) = (1/\sqrt{2\pi z})e^{-z/2}$ and $p(z) = -f(z)\Psi(-\sqrt{z})$, where $N_1(z) = \Psi(-\sqrt{z}) / \sqrt{z}$ from Appendix A, and granting that for some $a_0 > 0$ the first two functions are smaller than 1 in absolute value when $\text{Re} (z) \geq a_0$. Granting too that Laplace inversion of this sum can then be effected term by term, the assertion of Theorem 4.1 follows noting $\mathcal{L}^{-1}(N_1^n) = \nu_n$ from Proposition A.1. With the analytical facts implied by the Appendix B leading term expansion of $\Psi$, omitting further detail the proof of Theorem 4.1 is complete.

**Remark 4.2:** The proof of Theorem 4.1 can be seen as being effected by transfer to the framework of the Itô theory of excursions of Brownian motion. Denoting the objects of this theory by $e$, this perspective then in particular affords an interpretation of the function $p$ at the heart of our inversion in terms of volumes of the Itô measure $n$ as follows:

$$p(\sqrt{2D\lambda}) = (\sqrt{2}/2) \frac{\mathcal{L}(\varphi_D)(\lambda)}{n(\{ e \geq 0 \text{ containing a } \lambda\text{-marked point} \} ), \quad \lambda > 0,$$

setting $\varphi_D(t) = 1_{\{ t > D \}} n(\{ e \geq 0 \text{ of lifetime}(e) > t \})$, for any real $t > 0$.

5. **Laplace transforms of the excursion law:** principal results

In this section we identify a principal structure of the excursion law enforced when working with processes which are restartable twice in the sense of Section 2.4. The thrust is to seek
giving expression to the effects of the two sources of stochasticity featured in Section 3 by
admitting the use of transform methods in time direction, and to identify a framework of
sufficient conditions for rendering this rigorous; this framework is then to be verified in
particular when working with Brownian motion in the further development of the paper.

We therefore concentrate now on the Section 2.4 normalized excursion laws \( h_b^* \) associated
with Markov processes \( X \) restartable-at-a-stopping-time twice (with the processes that
result denoted by \( X^* \) and \( X^{**} = (X^*)^* \) respectively), and, for simplicity, assume absolute
continuity with respect to Lebesgue measure of all random variables under consideration
to hold as well. For the study of the existence of \( h_b^* \) we adopt the Laplace transform with
respect to time, \( \mathcal{L} \); see Appendix A for pertinent notation and concepts.

5.1 Basic Laplace transform: Our starting point is furnished by the representation
of \( h_b^* \) in terms of the transition densities of \( X \) as follows:

\[
h_b^*(u, y) = E \left[ 1_{\{H_b^* < u\}} p_X \left( u - H_b^*, X_{H_b^*}^*, y \right) \right],
\]

for any reals \( u > 0 \) and \( y \), where \( p_X(\tau, \xi, dx) \) denotes the density of \( X \) conditional on time-
\( \tau \) information subject to \( \xi = X_t \). To obtain this representation write in Section 2.3 the
expectation defining \( h_b^* \) as an iterated one by taking it conditional on time-\( H_b^* \) information
first, and then express these conditional expectations by time-\( H_b^* \) restarting of \( X^* \) in terms
of the transition densities \( p_X \).

Two sources of stochasticity are thus identified to determine the structure of \( h_b^* \), namely
\( H_b^* \) and \( X^*(H_b^*) \). They can be separated on taking Laplace transforms with respect to
time once their independence is granted, and the following result is seen to hold then.

**Proposition 5.1:** Let the Markov process \( X \) be twice restartable and assume \( H_b^* \) and
\( X^*(H_b^*) \) to be independent. Then we have for any real \( y \) the Laplace transform identity:

\[
\mathcal{L} \left( h_b^*(\cdot, y) \right)(z) = E \left[ \exp (-z H_b^*) \right] E \left[ \mathcal{L} \left( p_X(\cdot, X_{H_b^*}^*, y) \right)(z) \right],
\]

for any complex \( z \) with \( \text{Re} (z) \geq z_0 \), in the sense of measurable functions with both sides
either \( \infty \) or finite.

As a next step we distinguish the effects of the Section 2.3 Cases I and II on \( H_b^* \). In (II-1)
observe \( H_b^* = \delta_0 \). In (I) and (II-2) a reduction occurs to a time-0 and excursion-level-0
case I situation by way of the decomposition of \( H_b^* \) into independent random variables:

\[
H_b^* = T_b^* + H_{0,0}^*
\]

which gives expression to the very construction of \( H_b^* \) in these cases as follows: the con-
struction refers to the process \( X^{**} \) obtained by restarting of \( X^* \) at time \( T_b^* \), the
first passage time of \( X^* \) to the level \( b \), and starting from \( T_b^* \) it proceeds by measuring \( H_{0,0}^{**} \), the
achievement time associated with \( X^{**} \) by the Section 2.2 construction in the case I situation
where \( a = 0 \) and \( t = 0 \) there. Observe that \( T_b^* \) is equal in law to \( T_b \), the first passage time
of \( X \) to the level \( b \), and \( H_{0,0}^{**} \) is equal in law to \( H_0^* \), the Section 2.4 normalized achievement
time for \( X^* \) with \( b = 0 \) there. Effects thereof are explained in Sections 5.2 and 5.3.

5.2 Case I specializations: First consider the Case I situation as left in Section 5.1.
Using the independence of the summands in the decomposition of \( H_b^* \) derived there, the
Laplace transform of Proposition 5.1 is here checked to become a product of three Laplace
transforms as follows.
Proposition 5.2: Let the Markov process $X$ be twice restartable. Assume a Case I situation in which the random variables $H_b^*$ and $X^*(H_b^*)$ are independent and the Laplace transforms $\mathcal{L}(h_b^*(\cdot, y))$ for any real $y$ are well-defined on the half-plane $\{\text{Re}(z) > z_0\}$ within the right-hand complex half-plane. Then, for any complex $z$ with $\text{Re}(z) > z_0$,

$$\mathcal{L}(h_b^*(\cdot, y))(z) = E[e^{-zH_b^*}] E[e^{-zT_b^*}] E\left[\mathcal{L}(p_X(\cdot, b + X^*_b, y))(z)\right].$$

5.3 Case II specializations: Concentrate on the Case II situation as left in Section 5.1. Referring to Section 2.3 we hence ask about the excursion of $X^*$ to continue below the level $b > 0$ for a period of length at least $\delta := \delta_0$ which is positive and smaller than $D$: $\delta \in (0, D)$. The principal structure of the Laplace transform of the normalized excursion law $h_b^*$ here mirrors an additional characteristic feature: the relative position of $\delta$ to $T_b^*$, the first passage time of $X^*$ to the level $b$. This affords a decomposition of the sample space into the disjoint subsets $A_1 = \{T_b^* \geq \delta\}$ and $A_2 = \{T_b^* < \delta\}$, corresponding to the Subcases II-1 and II-2 of Section 2.4 respectively. In Proposition 5.1 factors of the Laplace transform thus decompose into two term sums. A four term decomposition of the Laplace transform results whose summands are Laplace transforms as well. We hence revert to the Brownian setting of Section 3, and as a second step in the proof of the explicit the law of $H_b^*$ and $X^*(X_b^*)$. We look at this in the Brownian case.

Proposition 5.3: Let the Markov process $X$ be twice restartable. Assume a Case II situation in which the random variables $H_b^*$ and $X^*(H_b^*)$ are independent and the Laplace transforms $\mathcal{L}(h_b^*(\cdot, y))$ for any real $y$ are well-defined on the half-plane $\{\text{Re}(z) > z_0\}$ within the right-hand complex half-plane. Then the functions $h_b^*$ are well-defined and afford the four-term decomposition:

$$h_b^* = \sum_{j=1}^2 \sum_{k=1}^2 h_{b,j,k}^*.$$

Explicitly we here have:

$$h_{b,1,1}^*(u, y) = Q(T_b^* > \delta) \mathbf{1}_{\{u > \delta\}} E\left[\mathbf{1}_{\{M(X^*) < b\}} p_X(u - \delta, X^*_\cdot, y)\right],$$

$$h_{b,1,2}^*(u, y) = Q(T_b^* > \delta) \mathbf{1}_{\{u > \delta\}} E\left[\mathbf{1}_{\{T^*_b \leq \delta\}} p_X(u - \delta, b + X^*_H^*, y)\right],$$

for any real $u > 0$ and $y$, using the running maximum $M(X^*) = \max\{X^*_\tau | \tau \in [0, \delta]\}$. The remaining two functions are for any real $y$ characterized by the three-factor Laplace transform identities on $\{\text{Re}(z) > z_0\}$ as follows:

$$\mathcal{L}(h_{b,2,1}^*(\cdot, y))(z) = E[e^{-zH_b^*}] E\left[\mathbf{1}_{\{T_b^* \leq \delta\}} e^{-zT_b^*}\right] E\left[\mathbf{1}_{\{M(X^*) < b\}} \mathcal{L}(p_X(\cdot, X^*_\cdot, y))(z)\right],$$

$$\mathcal{L}(h_{b,2,2}^*(\cdot, y))(z) = E[e^{-zH_b^*}] E\left[\mathbf{1}_{\{T_b^* \leq \delta\}} e^{-zT_b^*}\right] E\left[\mathbf{1}_{\{T_b^* \leq \delta\}} \mathcal{L}(p_X(\cdot, b + X^*_H^*, y))(z)\right].$$

This result is essentially extracted from [13, Appendix Section 8.3] and ultimately makes explicit the effects of the Section 5.1 restarting at $T_b^*$ decomposition of $H_b^*$ once more. Seen in conjunction with Proposition 5.2, rendering $H_b^*$ explicit thus reduces to making explicit the law of $H_b^*$ and $X^*(X_b^*)$. We look at this in the Brownian case.

6. Review: Laplace transforms in the Brownian case

The general structure of the Laplace transforms of $h_b^*$ has been identified in Section 5. We now revert to the Brownian setting of Section 3, and as a second step in the proof of the
Section 3 results recall how to make the Section 5.2 and 5.3 Laplace transforms of $h_b^*$ explicit when $X = W^*$ is Brownian motion there. The argument follows [13, Appendix]; it is based on the structure theory of the Brownian meander and the Azéma martingale as developed in [1] and [2], with another exposition in [14, Section 12.3].

6.1 Brownian motion at normalized achievement time: As a first step in obtaining the Laplace transforms this section reports how the assumptions of the Section 4 results are satisfied in the Brownian case as follows.

**Fact 6.1:** For any real $b$, the random variables $H_b^*$ and $W^*(H_b^*)$ are independent, and for the law of the absolute value of the latter we have on $\mathbb{R}$:

$$|W_{H_b^*}^*(dx) = 1_{(0,\infty)}(x) x \exp \left(-\frac{1}{2}x^2\right) dx.$$ 

Simplifying notation $B = W^*$ we recall how this is based on the Brownian meander which for any real $T > 0$ is the process $m_T$ given by

$$m_T(u) = \left|W^*(g_T + u(t-g_T))\right|/\sqrt{T-g_T}, \quad u \in [0,1];$$

Here $g_T = \sup \{s \mid s \leq T \text{ and } B(s) = 0\}$ is the last time before $T$ where $B$ is 0. The idea is to study $m_T$ with $T = H_b^*$. Since $\mathcal{F}_{g_T} \subseteq \mathcal{F}_T$ by [10, XII (3.2) Lemma, p. 464], it is sufficient to show that $m_T$ is independent of $\mathcal{F}_{g_T}$ and $\text{sgn}(B_T)$, and has the law indicated independently of $u$. Since $g_T = Tg_1$ by Brownian scaling, a further reduction occurs to the case $T = 1$ where these independence properties are taken care of by the following result.

**Fact 6.2:** With respect to the measure $Q^* = (|B_1|/\text{E}[|B_1|])Q$, the process $m_1$ is a dimension 3 Bessel process $BES(3)$ which is independent of $\mathcal{F}_{g_1}$.

In [2, §Théorème, p. 293] or [14, Section 12.3.2] this is proved as a consequence of a generalized Girsanov argument; the idea is to thus construct a Brownian motion $\beta$ such that $m_1$ satisfies $m_1(u) = \beta(u) + \int_0^u (1/m_1(s)) ds$ as it characterizes Bessel processes of dimension 3 started at 0 at time 0.

As indicated in [2, p. 294] or [14, Step 3, p. 45], the independence properties of Fact 6.2 entail a relation going back at least to [7] as follows.

**Fact 6.3:** We have $\text{law}(m_1)(u,x) = \sqrt{\pi/2} x^{-1} \text{law}(BES(3))(1,x)$ for any $u$ in [0,1] and any real $x > 0$, where $\text{law}(BES(3))(1,x) = 2^{-3/2} \Gamma(3/2)^{-1} x^2 \exp(-x^2/2)$.

6.2 Case I Laplace transforms: Keeping the concepts and the notation of Section 6.1, we look at the consequences of the results there for the Laplace transforms of the normalized excursion law $h_b^*$ in Case I, and indicate how a reduction occurs to the Laplace transform of the Section 2.4 normalized achievement time $H_0^*$. Indeed, as a consequence of Fact 6.1 the independence assumptions for Proposition 5.2 are satisfied. The law of Brownian motion at time $H_0^*$ as it enters into the Laplace transform representation there is explicitly known by Fact 6.1 as well. Summarizing the computations of [11, Sections 8 and 9] following [13, Appendix Sections 8.3.1 and 8.3.2], we thus have for any complex $z$ with $\text{Re}(z) > 0$ the representation:

$$\mathcal{L}(h_b^*(\cdot,y))(z) = E^*[\exp(-zH_0^*)] R_I(y, \sqrt{2Dz}),$$

in the sense of measurable functions, where the factor
\[ R_1(y, \sqrt{2Dz}) \overset{\text{def}}{=} E^*[\exp(-zT^*_b)] E^*[\mathcal{L}(\chi_{|X_2-y|})(2z)] \]
\[ = \exp \left( b^* \sqrt{2Dz} \right) (\sqrt{D} G_{\beta(y)})(2Dz), \]
with \( b^* = b/\sqrt{D} \leq 0 \), is well-defined and finite on \( \{ \text{Re}(z) > 0 \} \). Here recall the functions \( \chi_a \) from Section A.1, \( G_a \) from Section A.3, and \( \beta \) from Section 3.1.

### 6.3 Case II Laplace transforms:
Keeping the concepts and the notation of Section 6.1, we study the Case II situation. Here a reasoning analogous to that of Section 6.2 yields the Laplace transform representation of \( h^*_b \) of Proposition 5.3 in the sense of measurable functions. A reduction occurs of this law to that of determining two Laplace transforms. The computations of [11, Section 10] following [13, Appendix Section 8.3.3] yield for the first of these, the Laplace transform of the density \( h^*_{b,2,1} \), the representation:

\[ \mathcal{L}(h^*_{b,2,1}(\cdot, y))(z) = \int_0^\delta \exp(-zw) E^*[\exp(-zH^*_0)] R_{2,1}(y, \sqrt{2Dz}) \mu_b(dw), \]
for any complex \( z \) with \( \text{Re}(z) > 0 \), in the sense of measurable functions. Here the factor

\[ R_{2,1}(y, \sqrt{2Dz}) \overset{\text{def}}{=} E^*[1\{T^*_b > \delta\} \mathcal{L}(\chi_{|X_2-y|})(2z)] \]

\[ = \sqrt{D} \int_R \mathcal{L}(\chi_{|x-y|/\sqrt{D}})(2Dz) \varphi_{b,\delta}(x)dx \]
is well-defined and finite on \( \{ \text{Re}(z) > 0 \} \), and the remaining concepts are as follows: from Section A.1 we have the density \( \mu_b \) of \( T^*_b \), the first passage time of Brownian motion \( W^* \) to the level \( b \), as well as the functions \( \chi_a \), while the function \( \varphi_{b,\delta} \) is from Section 3.3.

For the second Laplace transform in question, the one of the density \( h^*_{b,2,1} \), we have for any complex \( z \) with \( \text{Re}(z) > 0 \) the representation:

\[ \mathcal{L}(h^*_{b,2,2}(\cdot, y))(z) = \int_0^\delta \exp(-zw) E^*[\exp(-zH^*_0)] R_{2,2}(y, \sqrt{2Dz}) \mu_b(dw), \]
in the sense of measurable functions, where the factor

\[ R_{2,2}(y, \sqrt{2Dz}) \overset{\text{def}}{=} E^*[1\{T^*_b \leq \delta\} \mathcal{L}(\chi_{|X_2-y|})(2z)] \]

\[ = \sqrt{D} Q^* (T^*_b \leq \delta) G_{\beta(y)}(2Dz) \]
is well-defined and finite on \( \{ \text{Re}(z) > 0 \} \). Here again recall the Section A.3 function \( G_a \).

### 6.4 The achievement time Laplace transform:
This section establishes the Laplace transform of the normalized achievement time \( H^*_0 \) by recalling ideas of [13, Appendix]. The principal finding is that the thus encoded excursion theoretic aspect of the problem leads to higher transcendental function denominators in the Laplace transform as follows:

\[ E[\exp(-zH^*_0)] = 1/\Psi(\sqrt{2Dz}), \quad \text{Re}(z) > 0, \]
with \( \Psi \) the Section A.2 function. The significance of this result at this stage is that it provides the missing factor in the Laplace transforms of Sections 6.2 and 6.3 and also shows all Laplace transforms to be finite on \( \{ \text{Re}(z) > 0 \} \).

The equivalent form of this result to be proved is the following key relation:

\[ E\left[ \exp\left(-\frac{1}{2}z^2H^*_0\right) \right] \Psi(z\sqrt{D}) = \Psi(0) = 1, \]
for any complex \( z \) with \( \text{Re}(z^2) > 0 \). Taking up the discussion of Section 6.1, the key relation is based on the Azéma martingale \( \mu \). This process is defined in terms of the
Brownian meander of Section 6.1 by way of the equality:

\[ W_u^* = m_u(1) \mu_u, \quad u \in [0, \infty); \]

it furnishes a martingale with respect to \( \mathbf{F}^+ \), the progressive enlargement of the Brownian filtration \( \mathbf{F} \) by the sign of \( W^* \) whose time-\( u \) step is given by \( \mathbf{F}^+(u) = \mathcal{F}_g_u \lor \sigma(\text{sgn} W_u^*) \), for any \( u \geq 0 \). Taking stochastic exponentials of this defining equality using the independence results of Fact 6.1 at time 1 there,

\[ E \left[ \exp \left( zW_t^* - \frac{1}{2} z^2 t \right) \mid \mathcal{F}^+(g_t) \right] = \exp \left( -\frac{1}{2} z^2 t \right) \Psi(z \mu_t), \]

for any real \( z > 0 \). For fixed such \( z \), the right-hand side of the equality at time \( t = H_0^* \) is a martingale by appealing to forms of the optional stopping theorem. The expectation of this martingale stopped at \( H_0^* \) is equal to its time-0 expectation which is equal to \( \Psi(0) \), and hence equals 1. Use the independence of \( H_0 \) and \( W^*(H_0^*) \) to obtain the key relation.

7. Identification of the Cases I and II inversion problems

We obtain the results of Section 3 in three steps from those of Sections 5 and 6 proceeding by reduction to the situation addressed by Theorem 4.1. Summarizing Sections 5 and 6, this section provides the first of these steps and identifies the functional relations to be considered. Since these transcribe the effects of two sources of stochasticity in two principal situations it is two representations which thus result as follows.

**Lemma 7.1:** For any reals \( u > 0 \) and \( y \), we have in Case I:

\[ h_b^*(u, y) = \frac{1}{2D} \mathcal{L}^{-1} \left( \frac{R_1(y, \sqrt{z})}{\Psi(\sqrt{z})} \right) \left( \frac{u}{2D} \right). \]

**Lemma 7.2:** For any reals \( u > 0 \) and \( y \) and with \( k \) in \( \{1, 2\} \), we have in Case II:

\[ h_{b,2,k}^*(u, y) = \frac{1}{2D} \int_0^\delta \mu_b(dw) \mathcal{L}^{-1} \left( \exp \left( -\frac{w}{2D} \right) \frac{R_{2,k}(y, \sqrt{z})}{\Psi(\sqrt{z})} \right) \left( \frac{u-w}{2D} \right), \]

\[ = \frac{1}{2D} \int_0^{\min\{\delta, u\}} \mu_b(dw) \mathcal{L}^{-1} \left( \frac{R_{2,k}(y, \sqrt{z})}{\Psi(\sqrt{z})} \right) \left( \frac{u-w}{2D} \right). \]

Here Lemma 7.1 summarizes the findings of Section 6.2 and Lemma 7.2 those of Section 6.3, after a change of variables in both cases.

8. Identification of the auxiliary random variables

On comparison with Theorem 4.1 the results of Section 7 suggest immediate candidates for the functions \( R \) governing the auxiliary random variables \( A_R \) and \( B_R \) to be introduced there. As a second step of the inversion procedure for the Section 3 results this section therefore identifies these functions and verifies their pertinent properties. These functions, it might be worth recalling from Section 6, transcribe the effects of two sources of stochasticity and their interrelations. In what follows let \( y \) denote an arbitrary real.

**Lemma 8.1:** The function \( R_1(y, \sqrt{z}) \) on the right-hand complex half-plane \( \{ \text{Re}(z) > 0 \} \) is obtained as a Laplace transform and \( R_1(y, \sqrt{z}) = O(z^{-3/2}) \) as \( z \) tends to \( \infty \) there.

**Lemma 8.2:** We have \( \mathcal{L}^{-1}(R_1(y, \sqrt{z})) = \sqrt{D} \rho_{b^*,\beta(y)}. \)

Proving these two results together we have to look at the functions \( R(z) = R_1(y, \sqrt{z}) = \sqrt{D} \exp (b^* z) G_{\beta(y)}(z) \) where \( b^* = b/\sqrt{D} \leq 0 \), and we address their inversion right away by
replicating the functions $\rho_{a,c}$ of Section 3.1 with $a = b^*$ and $c = \beta(y)$. First let $\beta(y) \leq 0$. Setting $\alpha(y) = -(b^* + \beta(y))$, then appeal to Remark A.3 to obtain:

$$\mathcal{L}^{-1} \left( \frac{R(z)}{\sqrt{z}} \right) = \sqrt{D} \mathcal{L}^{-1} \left( \frac{\exp(-\alpha(y)\sqrt{z})}{\sqrt{z}} N_1(z) \right) = \sqrt{D} \chi_{\alpha(y)} * \nu$$

using Section A.1 for inverting the first factor and Proposition A.1 for inverting the second one. In the case $\beta(y) > 0$, on the other hand,

$$\mathcal{L}^{-1} \left( \frac{R(z)}{\sqrt{z}} \right) = \sqrt{D} \mathcal{L}^{-1} \left( \frac{\exp(-|b^*|\sqrt{z})}{\sqrt{z}} G_{\beta(y)}(z) \right) = \sqrt{D} \chi_{|b^*|} * g_{\beta(y)}$$

now using Proposition A.2 for inverting the second factor. The proof of Lemma 8.2 is complete. From this discussion the asymptotic behaviour of $R$ near $\infty$ required is immediate if $b^*$ or $\beta(y)$ are not 0. Otherwise, this behaviour is inherited from the one of $N_1$ by way of Appendix B, and the proof of Lemma 8.1 is complete as well.

**Lemma 8.3:** The function $R_{2,1}(y, \sqrt{z})$ on $\{\text{Re}(z) > 0\}$ is obtained as a Laplace transform and $R_{2,1}(y, \sqrt{z}) = O(z^{-3/2})$ as $z$ tends to $\infty$ there.

**Lemma 8.4:** We have $\mathcal{L}^{-1}(R_{2,1}(y, \sqrt{z})) = \sqrt{D} \rho_{0,\beta(y)}$.

These are special cases of Lemma 8.1 and 8.2 respectively; from the definitions in Sections 3.3 and 6.3 recall how the functions $R_{2,1}$ arise from functions $R_1$ with $b^* = 0$.

**Lemma 8.5:** The function $R_{2,2}(y, \sqrt{z})$ on $\{\text{Re}(z) > 0\}$ is obtained as a Laplace transform and $R_{2,2}(y, \sqrt{z}) = O(z^{-1})$ as $z$ tends to $\infty$ there.

**Lemma 8.6:** We have $\mathcal{L}^{-1}(R_{2,2}(y, \sqrt{z}))(u) = \rho_b(u, y)$ for any real $u > 0$.

Setting $R(z) = R_{2,2}(y, \sqrt{z})$ recall $R(z) = \sqrt{D} \int_{\mathbb{R}} \mathcal{L}(\chi_{|x-y|}/\sqrt{D})(z)\varphi_{b,\delta}(x) \, dx$ from Section 6.3 in establishing these two results.

**Proof of Lemma 8.5** With the Section A.1 Laplace transforms $\mathcal{L}(\chi_a)$ defined on the right-hand half-plane, the function $R$ is defined there as well. To determine its asymptotics there, express $R$ by a direct computation in terms of the function Erfc as follows:

$$\sqrt{z} \, R(z) = \sqrt{2\delta D} \left( \frac{\sqrt{\pi}}{2} \right) \sum_{\varepsilon \in \{\pm 1\}} \exp \left( \delta z + \varepsilon y \sqrt{2z} \right) \text{Erfc} \left( \frac{1}{\sqrt{2\delta}} \left( \delta \sqrt{2z} + \varepsilon y \right) \right).$$

Using the leading term of the asymptotic expansion of Erfc on the right-hand half-plane (see [9, Section 2.2]) the right-hand side of this expression is checked to behave like a scalar multiple of $1/\sqrt{z}$ as $z$ tends to $\infty$ there. Hence $R(z) = O(z^{-1})$, as desired.

To exhibit $R$ as a Laplace transform, and to thus complete the proof, it is tempting to argue that its Laplace inverse is obtained by inversion under the sign of the defining integral. This turns out to be correct proceeding in two steps as follows. First delete an $\varepsilon$-neighbourhood of $y$ from the domain of integration of the defining integral, and effect Laplace inversion by inversion under the integral sign. Identify the integrand thus obtained as a continuous function bounded by an integrable function independent of $y$, and let $\varepsilon$ shrink to 0.

**Proof of Lemma 8.6** The last two step argument in proving Lemma 8.5 extends to the situation of Lemma 8.6. Reminding the Section A.1 function $\varphi_\alpha$, we here obtain

$$\mathcal{L}^{-1} \left( \frac{R(z)}{\sqrt{z}} \right)(\tau) = \sqrt{D} \int_{\mathbb{R}} \varphi_{b,\delta}(x) \text{Erfc} \left( \frac{1}{2} \frac{|x-y|}{\sqrt{D\tau}} \right) dx = \rho_b(\tau, y),$$

where the last equality results noting $\text{Erfc} \left( z/\sqrt{2D\tau} \right) = 2N_{0,D\tau}(z)$. The proof of Lemma 8.6 is complete as well.
9. Proof of the excursion density results

Summarizing the development up to now, this section establishes the three convolution sum representation results, Theorems 3.1, 3.2 and 3.3, for the normalized excursion law and its two distinguished summands respectively. Referring to Section 7, each of these functions is the inverse of an explicitly given Laplace transform, and the principal idea is to effect these inversions analytically by applying Theorem 4.1. Apart from collecting terms the task at this point therefore reduces to two things. First, to establish for the numerators $R$ of the Laplace transforms of Section 7 the asymptotic behaviour required in Theorem 4.1. Second, to identify the Laplace inverses of these numerators on division by the complex square root. This we address in turn in the three sections to follow.

9.1 Proof of Theorem 3.1: To establish the Theorem 3.1 description of the normalized excursion density $h^*_b$ in Case I, we start from the Laplace inversion problem of Lemma 7.1. The idea is to apply Theorem 4.1 with the functions $R(z) = R_1(y, \sqrt{z}) = \sqrt{D} \exp(b^*z)G_{\beta(y)}(z)$ on $\{\text{Re}(z) > 0\}$. With the assumptions for this result satisfied by Lemma 8.1 indeed,

$$h^*_b(u, y) = \frac{1}{2D} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2\pi)^{n/2}} \mathcal{L}^{-1} \left( \frac{R(z)}{\sqrt{z}} \right) * \nu_{n-1} \left( \frac{u}{2D} - \frac{n}{2} \right),$$

where the summation is over the finitely many positive integers $n$ satisfying $n < u/D$. The Laplace inverses here are taken care of by Lemma 8.2 to be equal to $\rho_{b^*, \beta(y)}$, as desired.

A technical problem occurs on substitution of these results in Lemma 7.2. This is because one has to integrate there with respect to $\mu_b(dw)$, and the integration variable enters into the respective number of the summands. Noting that these numbers are smaller than $u/D$, however, the representation of Theorem 3.2 is checked to follow, and the proof of this result is complete.

9.2 Proof of Theorem 3.2: To establish the Theorem 3.2 description of the summand $h^*_{b,2,1}$ of the excursion law in Case II, we start from the inversion problem of Lemma 7.2 with $k = 1$ there. Recall that this asks to integrate a Laplace inverse with respect to a density, and therefore we first look at the inversion problem alone. The idea is to apply for this inversion Theorem 4.1 with the function $R(z) = R_{2,1}(y, \sqrt{z})$. With the assumptions of this result satisfied by Lemma 8.5 indeed,

$$\mathcal{L}^{-1} \left( \frac{R(z)}{\Psi(\sqrt{z})} \right)(\tau) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2\pi)^{n/2}} \mathcal{L}^{-1} \left( \frac{R(z)}{\sqrt{z}} \right) * \nu_{n-1} \left( \tau - \frac{n}{2} \right),$$

for any $\tau > 0$, where only summands to indices $n < 2\tau$ are not 0. The Laplace inverses here are then taken care of by Lemma 8.6 to be equal to $\rho_{0, \beta(y)}$, as desired.

9.3 Proof of Theorem 3.3: Establishing the Theorem 3.3 description of the summand $h^*_{b,2,2}$ of the excursion law in Case II essentially reduces to the problem considered in Section 9.1. In fact, writing out the second Laplace inversion problem of Lemma 7.2 with $k = 2$ there gives the representation:

$$h^*_{b,2,2}(u, y) = \frac{1}{2\sqrt{D}} \int_{0}^{mu} \mathcal{L}^{-1} \left( \frac{G_{\beta(y)}(z)}{\Psi(\sqrt{z})} \right) \left( \frac{u-w}{2D} \right) \mu_b(dw),$$

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where \( m_u = \min\{\delta, u\} \). The Laplace inverses here are those of Sections 4.1 and 9.1 when \( b^* = 0 \) there, as formalized in Lemma 8.3 and 8.4. On inspection, substitution of the latter result’s Laplace inverses yields the desired representation on harmonizing the number of summands as in Section 9.2. The proof of Theorem 3.3 is complete.

10. Vista

The paper has introduced a Parisian-style excursion law and determined its structure in the Brownian case, thus taking up and extending a development initiated in [13]. The characterization of our Brownian excursion law in terms of sums of independent random variables, however, does not lend itself readily to actual work with the law as it would be desirable for addressing the declared motivation from finance of [13], for example.

![Figure 10.1. Comparison of Deltas of Down-and-In call [-] and Parisian Down-and-In call [o]](image)

Two methods for explicit handling of laws of sums of independent random variables are thus developed in the companion [12] to the present paper. Anticipating here some of the results there, for the Parisian barrier options proposed in [13] the methods are found to furnish effective and stable ways not just for valuation of the but also for hedging; this in particular so in situations where standard barrier options build up large Deltas ‘near their barriers’; a typical example is featured in Figure 10.1 where Parisian barrier options permit a reduction of Deltas by some factor 5. We have thus come full circle, with the results of the present paper instrumental for this.

Appendix A. Laplace transform pairs

This appendix collects pertinent Laplace transform pairs. Here the Laplace transform is the linear operator \( \mathcal{L} \) on the continuous functions of exponential type on \((0, \infty)\), the positive reals, defined as follows: it associates with any such function \( f \) the function \( \mathcal{L}(f) \) given by:

\[
\mathcal{L}(f)(z) = \int_0^\infty \exp(-zu)f(u)\,du,
\]

for any complex \( z \) in a half-plane contained sufficiently deep within the right-hand complex half-plane \( \{z|\text{Re} (z) > 0\} \). The maps \( \mathcal{L}(f) \) are analytic on such half-planes, and the operator \( \mathcal{L} \) is an injection with inverse \( \mathcal{L}^{-1} \), the inverse Laplace transform; see [3] or [4] for more detail. We moreover work with the principal branch of the complex logarithm on \( \mathbb{C} \setminus (-\infty,0] \), the complex plane \( \mathbb{C} \) cut along the non-positive reals \((-\infty,0] \).

A.1 The following three standard Laplace transforms on \( \{z|\text{Re} (z) > 0\} \) from [4, Beispiel 8, p. 50f] originate with the heat equation:

\[
\mathcal{L}(\psi_\alpha)(z) = \exp(-\alpha\sqrt{z}) \quad \text{where} \quad \psi_\alpha(u) = \frac{\alpha}{2\sqrt{\pi u^3}} \exp\left(-\frac{\alpha^2}{4u}\right),
\]
for any real \( u > 0 \). Here \( \text{Erfc}(\xi) = (2/\sqrt{\pi}) \int_{|\xi|}^{\infty} \exp(-x^2) \, dx \), for any complex \( \xi \), is the complementary error function, and \( \alpha \) is any complex with \( |\arg(\alpha)| \leq \pi/4 \) such that \( \text{Re}(\alpha) > 0 \) for \( \psi_\alpha \). Hence the law \( \mu_\alpha \) of the first passage time of Brownian motion to the level \( b \) is given on the real line \( \mathbb{R} \) by: \( \mu_\alpha(dw) = \psi_\alpha \sqrt{b} \, \delta_\alpha(w) \, dw \).

### A.2

As the first of two sets of functions to be considered define the functions \( N_n \) on \( \mathbb{C} \setminus (-\infty, 0] \) for any integer \( n \geq 0 \) by:

\[
N_n = N_1^n \quad \text{where} \quad N_1(z) = \Psi(-\sqrt{z})/\sqrt{z},
\]

for any \( z \) in \( \mathbb{C} \setminus (-\infty, 0] \). Following \([16]\), the function \( \Psi \) here is the generalization of the normal distribution given by the integral:

\[
\Psi(w) = \int_{0}^{\infty} x \exp\left(-\frac{1}{2}x^2 + wx\right) dx, \quad w \in \mathbb{C}.
\]

The second set is furnished by the functions \( \nu_n = \nu^{(n)}_\nu \) on \([0, \infty)\) already used in Section 3. Let \( \nu_0 = \nu^{(0)}_\nu \) be the Dirac delta function at 0 and define \( \nu_n \) for any integer \( n \geq 1 \) as an \( n \)-fold convolution on \([0, \infty)\) by:

\[
\nu_n = \nu^{(n)}_\nu \quad \text{where} \quad \nu(u) = (2/\sqrt{\pi}) \, \sqrt{u}/(2u+1), \quad u \in [0, \infty).
\]

### Proposition A.1:

We have \( \mathcal{L}(\nu_n) = N_n \) on \( \{\text{Re}(z) > 0\} \), for any integer \( n \geq 0 \).

### A.3

This section concentrates on generalizations of the Section A.2 function \( \Psi \). For any real \( \alpha \) these are the functions \( F_\alpha \) and \( G_\alpha \) on \( \mathbb{C} \setminus (-\infty, 0] \) given by:

\[
F_\alpha(z) = \int_{0}^{\infty} x \exp\left(-\frac{1}{2}x^2 - |\alpha-x|\sqrt{z}\right) dx,
\]

and

\[
G_\alpha(z) = F_\alpha(z)/\sqrt{z}.
\]

In terms of the Section A.2 function \( h \) and the Section A.1 complementary error function \( \text{Erfc} \) moreover define the function \( g_\alpha \) on \([0, \infty)\) by:

\[
g_\alpha(u) = \exp\left(-\frac{\alpha^2/2}{1+2u}\right) \left\{ h(u) \exp\left(-\frac{\alpha^2}{4u(1+2u)}\right) + \frac{\alpha}{1+2u} \text{Erfc}\left(\frac{\alpha}{\sqrt{4u(1+2u)}}\right)\right\},
\]

for any \( u \geq 0 \). Then, in particular all \( G_\alpha \) are Laplace transforms as follows.

### Proposition A.2:

We have \( \mathcal{L}(g_\alpha) = G_\alpha \) on \( \{\text{Re}(z) > 0\} \).

### Remark A.3:

If \( \alpha \leq 0 \) we have \( F_\alpha(z) = \exp(\alpha \sqrt{z}) \Psi(-\sqrt{z}) \).

### Appendix B. Further properties of the function \( \Psi \)

This appendix develops pertinent properties of the function \( \Psi \) which from Section A.2 for any complex number \( w \) is given by the integral: \( \Psi(w) = \int_{0}^{\infty} x \exp\left(-\frac{1}{2}x^2 + wx\right) dx \).

Developing the linear exponential factor of the integrand of \( \Psi \) in its series and integrating the resulting series term by term, yields the following series expansion:

\[
\Psi(w) = \sum_{n=0}^{\infty} a_n \, w^n \quad \text{where} \quad a_n = (2^n/2/n!) \, \Gamma\left(\frac{1}{2}(n+2)\right).
\]
This series is absolutely convergent for any complex number $w$, and its convergence is uniform on compact sets. As a first application it yields the key identity:

$$\Psi(w) = \Psi(-w) + \sqrt{2\pi} w \exp\left(\frac{1}{2} w^2\right)$$

which connects the values of $\Psi$ on the right-hand half-plane with those on the left-hand half-plane and has been noted in an equivalent form in establishing [1, Proposition 1 point 2), p. 94]; the identity can also be obtained by partial integration of the defining integrals for $\Psi(w)$ and $\Psi(-w)$ respectively by way of the partial integration identity:

$$\Psi(-\sqrt{2}w) = 1 - \sqrt{\pi} w \exp(w^2) \text{Erfc}(w).$$

This identity is the basis of the leading term expansion for $\Psi$ on the left hand half plane:

$$\Psi(-w) = 1/w^2 + R_2(w) \quad \text{where} \quad |R_2(w)| \leq 6/|w|^4,$$

for any complex $w$ with $\text{Re}(w) > 0$, which is a special case of a general uniform asymptotic expansion of $\Psi$ on the left-hand half-plane.

References:

[1] J. Azéma, M. Yor: Étude d’une martingale remarquable, Sém. Proba. XXIII, LNMS 1372, 88–130, Heidelberg: Springer 1989.
[2] J. Azéma, M. Yor: Sur les zéros des martingales continues, Sém. Proba. XXVI, LNMS 1526, 248–306, Heidelberg: Springer 1992.
[3] R. Beals: Advanced mathematical analysis, New York: Springer, 1973.
[4] G. Doetsch: Handbuch des res so have the Laplace transform representation there. r Laplace Transformation I, Basel: Birkhäuser, 1971.
[5] R. Loeffen, I. Czarna, Z. Palmowski: Parisian ruin probabilities for spectrally negative Lévy processes, arXiv: 1102.4055v1 (2011).
[6] L. Gauthier: Excursion height- and length-related stopping times, and applications to finance, Adv. Appl. Prob. 34 (2002), 846–868.
[7] J.P. Imhof: Density factorization for Brownian motion and the three dimensional Bessel processes and applications, J. Appl. Prob. 21 (1984), 500–510.
[8] I. Karatzas, S. Shreve: Brownian motion and stochastic calculus 2nd ed., New York: Springer, 1991.
[9] N.N. Lebedev: Special functions and their applications, New York: Dover, 1972.
[10] D. Revuz, M.Yor: Continuous martingales and Brownian motion 2nd ed., Heidelberg: Springer, 1994.
[11] M. Schröder: Brownian excursions and Parisian barrier options: a note, J. Appl. Prob. 40 (2003), 855–864.
[12] M. Schröder: On a Brownian excursion law, II: Three methods for handling convolutions of probability laws.
[13] M. Yor, M. Jeanblanc–Picqué, M. Chesney : Brownian excursions and Parisian barrier options, Adv. Appl. Prob. 29 (1997), 165–184.
[14] M. Yor: Some aspects of Brownian motion, Part II, Basel et al.: Birkhäuser, 1997.

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