FREE LOCALLY CONVEX SPACES AND THE ASCOLI PROPERTY

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Abstract. T. Banakh showed in [2] that if $X$ is a Dieudonné complete space, then the free locally convex space $L(X)$ on $X$ is an Ascoli space if and only if $X$ is a countable discrete space. We give an independent, short and clearer proof of Banakh’s result and remove the condition of being a Dieudonné complete space. Thus we have the following result: the free locally convex space $L(X)$ over a Tychonoff space $X$ is an Ascoli space if and only if $X$ is a countable discrete space.

1. Introduction

The study of topological properties of locally convex spaces is an active area of research attracting specialists both from topology and functional analysis. Especially important properties are those ones which generalize metrizability: the Fréchet–Urysohn property, sequentiality, the $k$-space property and countable tightness. These properties are intensively studied for function spaces, $(LM)$-spaces, strict $(LF)$-spaces and their strong duals, and Banach and Fréchet spaces in the weak topology, see for example [1, 6, 10, 13, 14, 16, 19] and reference therein.

For a Tychonoff space $X$, denote by $C_k(X)$ the space $C(X)$ of all real-valued continuous functions on $X$ endowed with the compact-open topology $\tau_k$. Being motivated by the classic Ascoli theorem we introduced and studied in [3] a new class of topological spaces, namely, the class of Ascoli spaces. A Tychonoff space $X$ is Ascoli if every compact subset $K$ of $C_k(X)$ is equicontinuous. By Ascoli’s theorem [4, Theorem 3.4.20], each $k$-space is Ascoli. So we have the following diagram

\[
\text{metric} \quad \xrightarrow{\text{Fréchet–Urysohn}} \quad \text{sequential} \quad \xrightarrow{k\text{-space}} \quad \text{Ascoli space},
\]

and none of these implications is reversible.

One of the most important classes of locally convex spaces is the class of free locally convex spaces. Recall (see [15]) that the free locally convex space $L(X)$ on a Tychonoff space $X$ is a pair consisting of a locally convex space $L(X)$ and a continuous map $i : X \to L(X)$ such that every continuous map $f$ from $X$ to a locally convex space $E$ gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is essentially unique. Topological properties of free locally convex spaces have been studied for example in [2, 5, 6, 20]. In particular, using pure topological methods we proved in [5] that $L(X)$ is a $k$-space if and only if $X$ is a countable discrete space (in this case $L(X)$ is even a sequential space). Note that (see [6]) a Tychonoff space $X$ is Ascoli if and only the canonical map $L(X) \to C_k(C_k(X))$ is an embedding of locally convex spaces.

The diagram and the aforementioned results motivate the following question posed in [10]: Is it true that $L(X)$ is an Ascoli space only if $X$ is a countable discrete space? A partial answer to this question was obtained by T. Banakh in [2, Theorem 10.11.9], where it is proved that if $X$ is a Dieudonné complete space, then $L(X)$ is an Ascoli space if and only if $X$ is a countable discrete space. We give an independent, short and clearer proof of Banakh’s result and remove the condition of being a Dieudonné complete space. Thus we obtain the following result complete answer to the above question:

2000 Mathematics Subject Classification. Primary 46A03, 54D50; Secondary 22A05, 54A25.
Key words and phrases. Free locally convex space, Ascoli property, barrelled space.
Theorem 1.1. For a Tychonoff space $X$, the space $L(X)$ is Ascoli if and only if $X$ is a countable discrete space.

We prove Theorem 1.1 in the next section using essentially functional-analytic methods.

2. Proof of Theorem 1.1

We start from some necessary definitions and notations. Set $\mathbb{N} := \{1, 2, \ldots \}$. The closure of a subset $A$ of a Tychonoff space $X$ is denoted by $\overline{A}$ or $\text{cl}(A)$. The support of a function $f \in C(X)$ is denoted by $\text{supp}(f)$. Recall that the sets

$$[K; \varepsilon] := \{ f \in C(X) : |f(x)| < \varepsilon \forall x \in K \},$$

where $K$ is a compact subset of $X$ and $\varepsilon > 0$, form a base at zero of the compact-open topology $\tau_e$ of $C_k(X)$. If $E$ is a locally convex space, the polar of a subset $A$ of $E$ is denoted by $A^\circ := \{ \chi \in E' : |\chi(x)| \leq 1 \forall x \in A \}$.

For every Tychonoff space $X$, the set $X$ forms a Hamel basis for $L(X)$ and the map $i$ is a topological embedding, see [18, 20], so we shall identify $x \in X$ with its image $i(x) \in L(X)$.

Recall that a Tychonoff space $X$ is Dieudonné complete if the universal uniformity $U_X$ on $X$ is complete. For numerous characterizations of Dieudonné complete spaces see Section 8.5.13 of [4]. The Dieudonné completion $\hat{X}$ of $X$ is the completion of the uniform space $(X, U_X)$.

Denote by $M_c(X)$ the space of all real regular Borel measures on $X$ with compact support. It is well-known that the dual space of $C_k(X)$ is $M_c(X)$, see [13, Proposition 7.6.4]. For every $x \in X$, we denote by $\delta_x \in M_c(X)$ the evaluation map (Dirac measure), i.e. $\delta_x(f) := f(x)$ for every $f \in C(X)$. The total variation norm of a measure $\mu \in M_c(X)$ is denoted by $\|\mu\|$. Denote by $\tau_\varepsilon$ the polar topology on $M_c(X)$ defined by the family of all equicontinuous pointwise bounded subsets of $C(X)$. We shall use the following deep result of Uspenski [20].

Theorem 2.1 (20). Let $X$ be a Tychonoff space and let $\mu X$ be the Dieudonné completion of $X$. Then the completion $\hat{L}(X)$ of $L(X)$ is topologically isomorphic to $(M_c(\mu X), \tau_\varepsilon)$.

In what follows we shall also identify elements $x \in X$ with the corresponding Dirac measure $\delta_x \in M_c(X)$. We need the following corollary of Theorem 2.1 noticed in [7].

Corollary 2.2 (7). Let $X$ be a Dieudonné complete space. Then $(M_c(X), \tau_\varepsilon)' = C_k(X)$.

We shall use the following fact, see Lemmas 5.10.2 and 5.10.3 and Theorem 5.10.4 of [17] (this fact is also proved in the “if” part of the Ascoli theorem [4, Theorem 3.4.20]).

Proposition 2.3. Let $X$ be a Tychonoff space and $A$ be an equicontinuous pointwise bounded subset of $C(X)$. Then the $\tau_\varepsilon$-closure of $A$ is $\tau_k$-compact and equicontinuous.

Recall that a locally convex space $E$ is called semi-Montel if every bounded subset of $E$ is relatively compact.

Proposition 2.4. Let $X$ be a Dieudonné complete space and let $K$ be a $\tau_e$-closed subset of $M_c(X)$. Then the following assertions are equivalent:

(i) $K$ is $\tau_e$-compact;

(ii) $K$ is $\tau_e$-bounded;

(iii) there is a compact subset $C$ of $X$ and $\varepsilon > 0$ such that $K \subseteq [C; \varepsilon]^{\circ}$.

In particular, the space $(M_c(X), \tau_\varepsilon)$ is a semi-Montel space.

Proof. (i)$\Rightarrow$(ii) is clear. Let us prove that (ii)$\Rightarrow$(iii). Since $X$ being a Dieudonné complete space is a $\mu$-space, $C_k(X)$ is barrelled by the Nachbin–Shirota theorem. This fact and Corollary 2.2 imply that $K$ is equicontinuous. So there is a compact subset $C$ of $X$ and $\varepsilon > 0$ such that $K \subseteq [C; \varepsilon]^{\circ}$. To prove (iii)$\Rightarrow$(i) we note first that $[C; \varepsilon]^{\circ}$ is equicontinuous and weak-$*$ compact by the Alaoglu theorem.
Therefore \([C; \varepsilon]^{\circ}\) is compact in the precompact-open topology \(\tau_{pc}\) on \(M_c(X)\) by Proposition 3.9.8 of [12]. It immediately follows from Proposition 2.3 that \(\tau_{e} \leq \tau_{k} \leq \tau_{pc}\). Hence \([C; \varepsilon]^{\circ}\) is \(\tau_{e}\)-compact. Thus \(K\) being closed is also \(\tau_{e}\)-compact.

We shall use also the following proposition to show that a space is not Ascoli.

**Proposition 2.5** ([10]). Assume that a Tychonoff space \(X\) admits a family \(U = \{U_i : i \in I\}\) of open subsets of \(X\), a subset \(A = \{a_i : i \in I\} \subseteq X\) and a point \(z \in X\) such that

(i) \(a_i \in U_i\) for every \(i \in I\);

(ii) \(|\{i \in I : C \cap U_i \neq \emptyset\}| < \infty\) for each compact subset \(C\) of \(X\);

(iii) \(z\) is a cluster point of \(A\).

Then \(X\) is not an Ascoli space.

In Corollary 1.14 of [6] we show that the weak-* dual space of an infinite-dimensional metrizable barrelled space is not an Ascoli space. The next proposition complements this result and the construction given in its proof will play an essential role also in the proof of the main result.

**Proposition 2.6.** Let \(E\) be an infinite-dimensional barrelled space and let \(T\) be a locally convex vector topology on \(E'\) such that \((E', T)' = E\). Assume that \(H\) is a linear subspace of \((E', T)\) such that \(H\) is a normal space and

(i) there is a sequence \(\{\chi_n : n \in \mathbb{N}\}\) in \(H\) converging to a vector \(\chi_0 \in E' \setminus H\);

(ii) there is a sequence \(\{\eta_n : n \in \mathbb{N}\}\) in \(H\) and a vector \(\eta_0 \in H\) such that

\[\eta_0 \in \overline{\{\eta_n : n \in \mathbb{N}\}} \setminus \{\eta_n : n \in \mathbb{N}\};\]

(iii) there is a bounded sequence \(\{x_n : n \in \mathbb{N}\}\) in \(E\) such that

\[|\chi_n(x_n)| \leq 1, \ |\eta_n(x_n)| = 1\] and \(\eta_0(x_n) = 0\) for every \(n \in \mathbb{N}\).

Then \(H\) is not an Ascoli space.

**Proof.** By (i), \(\chi_n \to \chi_0 \in E' \setminus H\). Hence, for every \(m \in \mathbb{N}\), the sequence \(S_m = \{\frac{1}{m+1}\chi_n : n \in \mathbb{N}\}\) is closed and discrete in \(H\). As \(H\) is normal, \(S_m\) is \(C\)-embedded in \(H\), i.e. every real function on \(S_m\) can be extended to a continuous function on the whole \(H\). Take a continuous function \(G_m : H \to \mathbb{R}\) such that \(G_m(\frac{1}{m+1}\chi_n) = n\) for every \(n \in \mathbb{N}\). For every \(n, m \in \mathbb{N}\), set

\[\tilde{V}_{n,m} := G_m^{-1}\left((n - 0.1, n + 0.1)\right) - \frac{1}{m + 1}\chi_n.\]

Then, for every \(m \in \mathbb{N}\), the family

\[V_m := \left\{\frac{1}{m + 1}\chi_n + \tilde{V}_{n,m} : n \in \mathbb{N}\right\}\]

of open subsets of \(H\) is discrete in \(H\) (i.e., every \(h \in H\) has a neighborhood \(U_h\) which intersects with at most one element of the family \(V_m\)).

For every \(n, m \in \mathbb{N}\), set

\[W_{n,m} := \{h \in H : |h(x_n)| < 1\}\] and \(V_{n,m} := \tilde{V}_{n,m} \cap W_{n,m}\),

and put

\[a_{n,m} := \frac{1}{m + 1}\chi_n + m(\eta_n - \eta_0)\] and \(U_{n,m} := a_{n,m} + V_{n,m}\).

Finally we set \(A := \{a_{n,m} : n, m \in \mathbb{N}\}\) and \(U := \{U_{n,m} : n, m \in \mathbb{N}\}\). We show that the families \(A\), \(U\) and the point \(z = 0 \in H\) satisfy (i)-(iii) of Proposition 2.5.

**Claim 1.** \(0 \in \mathcal{A} \setminus A\).
Indeed, (2.1) implies
\[
|a_{n,m}(x_n)| = \left| \frac{1}{m+1} \chi_n(x_n) + m \cdot \eta_n(x_n) \right| \geq m - \frac{1}{m+1} > 0,
\]
and hence $0 \notin A$. To show that $0 \in \overline{A}$ take arbitrary a neighborhood $W$ of $0$ in $(E', T)$ and choose an absolutely convex neighborhood $U$ of $0$ in $(E', T)$ such that $3U \subseteq W$. By (i), choose an $N \in \mathbb{N}$ such that $\chi_n \in \chi_0 + U$ for every $n \geq N$, and choose an $m_0 \in \mathbb{N}$ such that $\frac{1}{m_0+1} \chi_0 \in U$. Now, by (ii), for the chosen $N$ and $m_0$, take an $n_0 > N$ such that $m_0(\eta_{n_0} - \eta_0) \in U$. Noting that $\frac{1}{m_0+1} U \subseteq U$ we obtain
\[
a_{n_0,m_0} = \frac{1}{m_0+1}(\chi_{n_0} - \chi_0) + \frac{1}{m_0+1} \chi_0 + m_0(\eta_{n_0} - \eta_0) \in U + U + U \subseteq W.
\]
Thus $0 \in \overline{A}$ and the claim is proved.

Claim 2. For every compact subset $K$ of $H$, the set $\{(n,m) \in \mathbb{N} \times \mathbb{N} : K \cap U_{n,m} \neq \emptyset \}$ is finite.

Indeed, let $K$ be a compact subset of $H$. Then $K$ is compact and hence bounded in $(E', T)$. Since $(E', T)' = E$ we obtain that $K$ is weak-\* bounded in $E'$. As $E$ is barrelled, there is an open neighborhood $O$ of zero in $E$ such that $K \subseteq O^\circ$.

Set $Z := \{x_n : n \in \mathbb{N}\}$. Since $Z$ is bounded we can choose $C > 0$ such that $Z \subseteq C \cdot O$. Then
\[
O^\circ \subseteq C \cdot Z^\circ = \{\chi \in E' : |\chi(x_n)| \leq C \ \forall n \in \mathbb{N}\}.
\]
Now, if $\chi = a_{n,m} + h \in U_{n,m}$ with $h \in W_{n,m}$, (2.1) and (2.2) imply
\[
|\chi(x_n)| \geq |a_{n,m}(x_n)| - |h(x_n)| \geq m|\eta_n(x_n)| - \frac{1}{m+1}|\chi_n(x_n)| - |h(x_n)| > m - 2.
\]
Therefore, by (2.3), if $m > C + 2$ then $K \cap U_{n,m} \subseteq O^\circ \cap U_{n,m} = \emptyset$. For a fixed natural number $m \leq C + 2$, $K \cap U_{n,m}$ is nonempty only for a finite number of $n$ because the family $V_n$ is discrete. The claim is proved.

Finally, Claims 1 and 2 imply that (i)-(iii) of Proposition 3.6 are satisfied, and thus $H$ is not an Ascoli space.

Let $s = \{0, 1, 1/2, \ldots, 1/n, \ldots\}$ be a convergent sequence with the usual topology induced from $\mathbb{R}$. Then the space $L(s)$ is incomplete by Theorem 2.1. This essential fact was used firstly by the author in [5] Proposition 3.1 to show that $L(s)$ is not a $k$-space. The incompleteness of $L(s)$ was used by T. Banakh [2] Lemma 10.11.6 to show that $L(s)$ is not an Ascoli space. In the proof of Step A of the main result below we use an analogous idea, see also the main construction in the proof of Proposition 3.6 of [8].

**Proof of Theorem 2.1.** If $X$ is countable and discrete, then $L(X)$ is sequential and hence an Ascoli space.

Conversely, assume that $L(X)$ is an Ascoli space. Lemma 2.7 of [9] states that if $H$ is a dense subgroup of a topological group $G$ and has the Ascoli property, then also $G$ is an Ascoli space. Since $L(X)$ is a dense subspace of $L(\mu X)$ by Theorem 2.3, the last fact implies that $L(\mu X)$ is also an Ascoli space. So it is sufficient to prove that $\mu X$ is a countable discrete space. Thus in what follows we shall assume that $X$ is a Dieudonné complete space.

**Step A. The space $X$ does not contain infinite compact subsets.**

Suppose for a contradiction that $X$ contains an infinite compact subset $K$. Let $S = \{\eta_n : n \in \mathbb{N}\}$ be a one-to-one sequence in $K$ and let $\eta_0$ be a cluster point of $S$. We assume that $\eta_0 \notin S$. Fix a sequence $\{\lambda_k : k \in \mathbb{N}\}$ of positive numbers such that $\sum_{k=1}^{\infty} \lambda_k = 1$. For every $n \in \mathbb{N}$, set
\[
\chi_n := \sum_{k=1}^{n} \lambda_k \eta_k \in L(X).
\]
Claim A.1. $\chi_n$ converges to $\chi_0 := \sum_{k=1}^{\infty} \lambda_k \eta_k$ in $(M_c(X), \tau_e) = \overline{L(X)}$.

Indeed, let $Q$ be a pointwise bounded and equicontinuous subset of $C(X)$. Then $Q|_K := \{f|_K : f \in Q\}$ is a pointwise bounded and equicontinuous subset of $C(K)$. Proposition 2.4 implies that the pointwise closure of $Q|_K$ is a compact subset of the Banach space $C(K)$. Thus $Q|_K$ is uniformly bounded, i.e., there is a $B > 0$ such that

$$|f(x)| \leq B, \quad \forall x \in K, \quad \forall f \in Q.$$ 

Choose an $n_0 \in \mathbb{N}$ such that $B \cdot \sum_{k>n_0} \lambda_k < 1$. Then

$$|(\chi_0 - \chi_n)(f)| \leq \sum_{k>n} \lambda_k |f(\eta_k)| < \sum_{k>n} \lambda_k B < 1, \quad \forall f \in Q,$$

and hence $\chi_n \in \chi_0 + Q^o$ for every $n > n_0$. Thus $\chi_n \to \chi_0$ in $(M_c(X), \tau_e)$ and the claim is proved.

Claim A.2. Basic construction. By Claim A.1 and the definition of the sequence $S$, the sequences $\{\chi_n : n \in \mathbb{N}\}$, $\{\eta_n : n \in \mathbb{N}\}$ and the points $\chi_0$ and $\eta_0$ satisfy (i)-(ii) of Proposition 2.6 for $E = C_k(X)$ and $\mathcal{T} = \tau_e$ (see Corollary 2.2). Note also that the space $C_k(X)$ is barrelled by the Nachbin–Shirota theorem. Since $X$ is Tychonoff, for every $n \in \mathbb{N}$, one can find a continuous function $f_n : X \to [0,1]$ such that $\eta_n(f_n) = f_n(\eta_n) = 1$ and $\eta_0(f_n) = f_n(\eta_0) = 0$. Then, for every $n \in \mathbb{N}$, we have

$$(2.4) \quad |\chi_n(f_n)| \leq \sum_{k=1}^{n} \lambda_k \cdot \eta_k(f_n) \leq \sum_{k=1}^{n} \lambda_k \cdot f_n(\eta_k) \leq \sum_{k=1}^{n} \lambda_k < 1.$$ 

It is evident that the sequence $\{f_n : n \in \mathbb{N}\}$ is also bounded in $C_k(X)$. Therefore (2.1) and the condition (iii) of Proposition 2.6 are satisfied as well.

For every $n, m \in \mathbb{N}$, set

$$a_{n,m} := \frac{1}{m+1} \chi_n + m(\eta_n - \eta_0),$$

and put $A := \{a_{n,m} : n, m \in \mathbb{N}\}$. By Claim 1 of the proof of Proposition 2.6 (which uses only conditions (i)-(iii) of that proposition) we obtain

$$0 \in \overline{A} \setminus A.$$ 

Now let $p : X \to \beta X$ be a natural embedding of $X$ into the Stone–Cech compactification $\beta X$ of $X$. Then $p$ extends uniquely to a continuous linear injective operator from $L(X)$ to $L(\beta X)$ which is also denoted by $p$. For every $n \in \mathbb{N}$, denote by $F_n : \beta X \to [0,1]$ the unique extension of $f_n$ onto $\beta X$. By (2.4) we have

$$|p(\chi_n)(F_n)| = \left| \sum_{k=1}^{n} \lambda_k \cdot F_n(p(\eta_k)) \right| = \left| \sum_{k=1}^{n} \lambda_k \cdot f_n(\eta_k) \right| < 1,$$

and

$$|p(\eta_n)(F_n)| = |F_n(p(\eta_n))| = |\eta_n(f_n)| = 1,$$

and

$$p(\eta_0)(F_n) = F_n(p(\eta_0)) = \eta_0(f_n) = 0 \quad \text{for every } n \in \mathbb{N}.$$ 

These three estimates, Claim A.1 and (2.5) imply that the sequences $\{p(\chi_n) : n \in \mathbb{N}\}$, $\{p(\eta_n) : n \in \mathbb{N}\}$ and $\{F_n : n \in \mathbb{N}\}$ satisfy (i), (ii) and (iii) of Proposition 2.6 respectively, with respect to the points $p(\chi_0)$ and $p(\eta_0)$. Recall that $L(\beta X)$ is a dense proper subspace of $(M_c(\beta X), \tau_e)$ by Theorem 2.1 and $(M_c(\beta X), \tau_e)' = C_k(\beta X)$ by Corollary 2.2. Further, since $\beta X$ is compact, $C_k(\beta X)$ is a Banach space and the space $L(\beta X)$ is Lindelöf by Proposition 5.2 of [15]. Therefore we can use the construction in the proof of Proposition 2.6 to find a family $U^\beta = \{U^\beta_{n,m} : n, m \in \mathbb{N}\}$ of open sets in $L(\beta X)$ such that (1) $p(a_{n,m}) \in U^\beta_{n,m}$ for every $n, m \in \mathbb{N}$, and (2) the families $p(A)$, $U^\beta$ and the point $0 = p(0) \in L(\beta X)$ satisfy conditions (i)-(iii) of Proposition 2.5. But then (see also (2.5))
the families $A, U = \{ p^{-1}(U_{n,m}^\beta) : n, m \in \mathbb{N} \}$ and $z = 0 \in L(X)$ satisfy (i)-(iii) of Proposition 2.5 as well. Thus $L(X)$ is not an Ascoli space, a contradiction.

Below we assume that $X$ does not contain infinite compact subsets and hence $L(X) = (M_c(X), \tau_e)$ by Theorem 2.1.

**Step B. The space $X$ is discrete.**

Suppose for a contradiction that $X$ is not discrete. Proposition 2.4 of [7] states that there exist an infinite cardinal $\kappa$, a point $z \in X$, a family $\{ g_i \}_{i \in \kappa}$ of continuous functions from $X$ to $[0,2]$ and a family $\{ U_i \}_{i \in \kappa}$ of open subsets of $X$ such that

$(\alpha)$ $\text{supp}(g_i) \subseteq U_i$ for every $i \in \kappa$;

$(\beta)$ $U_i \cap U_j = \emptyset$ for all distinct $i, j \in \kappa$;

$(\gamma)$ $z \notin U_i$ for every $i \in \kappa$ and $z \in \text{cl}(\bigcup_{i \in \kappa} \{ x \in X : g_i(x) \geq 1 \})$.

Set $E := (M_c(X), \tau_e)$. By Corollary 2.2 we have $E' = (M_c(X), \tau_e)' = C(X)$. So we can consider the family $\{ g_i \}_{i \in \kappa}$ as a subset of the dual space $C(X)$ of $E$, in particular $\{ g_i \}_{i \in \kappa} \subseteq C_k(E)$. Denote by $0$ the zero function on $X$.

**Claim B.1. The set $K := \{ g_i \}_{i \in \kappa} \cup \{ 0 \}$ is a compact subset of $C_k(E)$.**

Indeed, fix arbitrarily a compact subset $Z$ of $E$ and $\delta > 0$. We have to show that all but finitely many of functions $g_i$s belong to the neighborhood $[Z; \delta]$ of $0$ in $C_k(E)$. By Proposition 2.4 we can assume that $Z = [C; \varepsilon]^\circ$, where $C$ is a compact subset of $X$ and $\varepsilon > 0$. Therefore

$$Z = [C; \varepsilon]^\circ = \{ \mu \in M_c(X) : |\mu(f)| \leq 1 \forall f \in [C; \varepsilon] \} = \{ \mu \in M_c(X) : \text{supp}(\mu) \subseteq C \text{ and } ||\mu|| \leq 1/\varepsilon \}.$$ 

Since, by Step A, the compact set $C$ is finite, (a) and (b) imply that $g_i|_C = 0$ and hence $g_i \in [Z; \delta]$ for all but finitely many indices $i \in \kappa$. Thus $K$ is compact.

**Claim B.2. The compact set $K$ is not equicontinuous.**

Indeed, fix arbitrarily a $\tau_e$-neighborhood $W = [K; \varepsilon] = \{ \mu \in M_c(X) : |\mu(f)| < \varepsilon \forall f \in K \}$ of zero in $M_c(X)$, where $K$ is a pointwise bounded and equicontinuous subset of $C(X)$ and $\varepsilon > 0$. Choose a neighborhood $V$ of the point $z$ in $X$ such that

$$|f(x) - f(z)| < \varepsilon/2, \quad \forall x \in V, \forall f \in K.$$ 

Then $|(\delta_x - \delta_z)(f)| < \varepsilon/2$ for every $x \in V$ and each $f \in K$. Therefore $\delta_x - \delta_z \in W$ for every $x \in V$. By (γ), there are $i \in \kappa$ and $x_i \in U_i$ such that $x_i \in V$ and $g_i(x_i) \geq 1$. Hence (note that $g_i(z) = 0$ for every $i \in \kappa$ since $z \notin U_i$)

$$|(\delta_{x_i} - \delta_z)(g_i)| = g_i(x_i) - g_i(z) = g_i(x_i) \geq 1.$$ 

Thus $K$ is not equicontinuous.

Now Claims B.1 and B.2 imply that the space $E$ is not Ascoli. This contradiction shows that $X$ must be discrete.

**Step C. By Step B, the space $X$ is discrete. Therefore $X$ is countable by Theorem 3.2 of [6] which states that $L(X)$ is not an Ascoli space for every uncountable discrete space $X$. □**

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