In search of the true vacuum: natural ordering, $\gamma$ condensate and the last renormalization

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Abstract: With the idea of canceling the leading divergence in vacuum energy of $\varphi^4$ field theory a parameter is introduced that interpolates between free Hamiltonian with or without normal ordering. This leads to a condensate ground state having an arbitrary number of particle-particle pairs. In addition to the usual states, the condensate supports the states of negative energy and negative norm. An explicit expression for the condensate state is derived and perturbation theory with this state investigated. The propagator is modified off the mass shell while unchanged on the mass shell. Lowest order correction to the vacuum energy is calculated and conditions for cancelation of the leading divergence investigated. One possible solution is that all radiative corrections in this formulation vanish. The other possible solution implies a phase transition above the coupling of $(2\pi)^2$ and the condensate non-analytical in the coupling constant. Possible implications are discussed.

Keywords: Renormalization, regularization and renormalons, Nonperturbative Effects.

Dedicated to... if you want.
1. Introduction and Motivation

Let us start with a simple pedagogical example. Consider the motion of a particle in a potential well shown in a figure 1. Assume that the potential is not precisely known to a physicist and he/she uses parabolic, or quadratic approximation.

The physicist expands around the origin believing it to be a minimum. It is not and they are ample signs of it. Among others, the correction to the energy calculated in the said approximation is infinite. Indeed, the divergent correction to the energy is standard sign of expanding around the wrong "minimum”. That is so everywhere in physics, everywhere that is, except in Quantum Field Theory (abbreviated QFT). There one accepts the infinite energy and consequently speaks of the cosmological constant problem [1, 2, 3, 4]. The problem is, that an infinite, or very large, vacuum energy is incompatible with the general relativity where energy is the source of gravitational effects. The gravitational effects of a large vacuum energy have not been seen and something is wrong either with our estimates of vacuum energy or with our idea that energy is the source of gravity. The myriad proposals to deal with the problem are given in the numerous references of [2, 3, 4]. Here this problem is related to the structure of vacuum of QFT.

\[ V(x) \]

\[ x \]

Figure 1: Potential for a particle motion full line, quadratic approximation to the potential dashed line

\(^1\)This potential for a quantum mechanical particle is zero space and one time dimensional analog of the Higgs potential.
The idea here is that, like in quantum mechanics, one should solve for the ground state. Of course, one uses the interaction picture, but that has the problem with the Haag’s theorem [5]. Also the success of the spontaneous symmetry breaking idea [6, 7] seems to imply that vacuum is some kind of condensate. The condensate states are generally not analytical in the coupling constant (the prime example being BCS ground state [8]) so one can not obtain those states perturbatively from the non-interacting ground state.

In the next section the discussion of the need for the last renormalization is given. In section three an explicit proposal for a particular condensate ground state is given in the case of $\phi^4$ theory. In section four the perturbation theory with the above ground state is discussed, while in section five nontrivial vacuum solutions are presented. The last section is reserved for the conclusion and outlook. An appendix is devoted to the condensate state defined in terms of creation operators and the usual ground state.

2. The Last Renormalization

In QFT one deals with the infinite energy of free oscillators by the procedure called normal ordering, which forces vanishing of the ground state energy to zeroth order in the coupling constant. That one does by changing the order of operators in Hamiltonian and Lagrangian [9]. However, in calculating the propagator, the inverse of the quadratic part of the Lagrangian, one uses different ordering procedure since using the normal ordering would lead to a vanishing propagator. These procedures, different orderings for quadratic part of Lagrangian depending on what is calculated, are standard in QFT. One pays the price in having the perturbatively calculated correction to the vacuum energy infinite to any order in the coupling constant. Note that the degree of the divergence is $VA^4$, where $V$ is normalization volume using periodic boundary conditions while $\Lambda$ is the large momentum cut-off. This degree is the same in the divergence of vacuum energy removed by the normal ordering procedure and in the corrections to vacuum energy in perturbative calculation. The divergence of vacuum energy is indeed the most divergent quantity of all ultraviolet divergent quantities in QFT. The hope is that if one can make somehow this finite, all less divergent quantities (various renormalization constants) which diagrammatically are sub-diagrams of vacuum to vacuum diagrams, could be made finite too.

The relationship between the vacuum energy and ultraviolet divergences in QFT was discussed by Allcock [11], and made explicit by the supersymmetric theories which have vanishing vacuum energy and milder ultraviolet divergences [12, 13]. For example, the Wess-Zumino model [14] is finite save for the wave function renormalization [15], while e.g. the $N = 4$ supesymmetric Young-Mills is completely finite [16]. For review of divergences in supersymmetric theories see, e.g. [17].

Here I consider the simple self interacting scalar $\phi^4$ field theory. To make the theory finite it is not sufficient that the vacuum energy be made finite. This is clear from the supersymmetric theories which have the vanishing vacuum energy, many of which still have divergences. Therefore it is not enough just to render the connected vacuum to vacuum graphs, figure 2, finite. One also needs the functional derivative of this with

\[ \text{Here I am generally following the notation of this textbook.} \]
Figure 2: Connected vacuum to vacuum diagrams in $\phi^4$ field theory giving perturbative corrections to vacuum energy density

respect to propagator to be finite. That means if one has a functional of propagators in $k$ space, $F(\Delta(k_j))$ given by integrals over various $k$'s one varies each propagator by $\eta \delta(k_j - p)$ and subtracts unvaried expression and divides by $\eta$ and finally performs $\eta \to 0$ limit. One derivative applied to vacuum to vacuum diagram cuts a line of the graph giving the propagator correction, while two derivatives cut two lines giving scattering correction figure 3, etc. That is, functional derivatives with respect to the propagator cut lines and

one needs in addition to vacuum energy being finite, its functional derivatives with respect to propagator to be finite. In $\varphi^4$ theory one needs first four derivatives of the vacuum energy with respect to the propagator to be finite to ensure the finiteness of the theory. One can presumably write analogous derivative relations for other field theories.

The whole spirit of renormalization is to equate a renormalized quantity with an experimentally measured value. Since a finite vacuum energy density have been measured [18, 19, 20, 21], one should apply the renormalization procedure to the vacuum energy too. One needs to see how one could implement what could be called the last renormalization:

*renormalization of the vacuum energy in such a way that all renormalization constants become finite.* A proposal of what could be done, follows in the next section.

3. $\gamma$ Vacuum

Consider the standard $\phi^4$ field theory in four space-time dimensions

$$\mathcal{L} = \frac{1}{2} \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \varphi}{\partial x^\mu} - \frac{1}{2} m^2 \varphi^2 + \frac{1}{2} g \varphi^4.$$  \hspace{1cm} (3.1)

Define a condensate ground state by the relation:

$$\left( a_k^\dagger a_k + \frac{\gamma}{2} \right) \left| v >,\gamma \right. = 0.$$  \hspace{1cm} (3.2)

Here, e.g. $a_k^\dagger$ is normalized creation operator (here I am not following the notation of Bjorken and Drell instead I am following [22], with the exception of the covariant normalization of states), with normalization $\sqrt{V} a_k^\dagger = b_k^\dagger$ where $[b_k, b_{k'}^\dagger] = \delta_{k,k'} V$ with box
normalization used and V normalization volume. In the continuum limit, the limit \( V \to \infty \) is performed. In this \( \gamma \) is a parameter that vanishes for normal ordering, while it is one to the lowest order for natural ordering (meaning without any additional ordering, see e.g. [10]). The other values of \( \gamma \) interpolate between this cases. Since the ultraviolet divergences of free oscillators and perturbative corrections are of the same type \( (V \Lambda^4) \), one can in principle adjust \( \gamma \) so the energy of free oscillators cancels the leading ultraviolet divergence of the perturbative correction to vacuum energy to any order, all order corrections to the vacuum energy having the same \( V \Lambda^4 \) dependance. Starting with the natural ordering and \( \gamma = 1 \) and changing \( \gamma \) while keeping the natural ordering one gets free energy proportional to \( 1 - \gamma \). With correction one expects

\[
\gamma = 1 + \text{const } g + \text{higher order corrections.} \tag{3.3}
\]

Here \( g \) is the coupling constant and the value of \( \gamma \) is adjusted to cancel the leading \( \Lambda^4 \) divergence of the vacuum energy. One hopes that canceling of next to leading singularities leads to a relationships between mass and the coupling constant of the theory, possibly involving renormalized (and experimentally measured) vacuum energy. It would be shown later that the radiative correction to the energy also depends on \( 1 - \gamma \). The idea is here that like in quantum mechanics, starting with an appropriate initial state (which can be non-analytical in the coupling constant) one calculates the corrections to the ground state order by order in the coupling constant. Note that to the lowest order the two ground states with different \( \gamma \)'s differ by an infinite energy and as a result there is a superselection rule between words built upon those states [5, 23], the origin of the Haag’s theorem. I shall discuss an explicit realization of the \( \gamma \) condensate shortly.

Examine now some properties of the \( \gamma \) vacuum. It is clear that the definition of \( \gamma \) vacuum (3.2) introduces the states of negative norm in the theory. Assuming the \( \gamma \) vacuum to be normalized one obtains from (3.2)

\[
\gamma < v \left( a_k^+ a_k + \frac{\gamma}{2} \right) |v>_{\gamma} = 0 \quad \text{from which } \quad \|a_k|v>_{\gamma}\|^2 = -\frac{\gamma}{2}. \tag{3.4}
\]

As will become clear from an explicit expression for the \( \gamma \) vacuum one could have non vanishing states of an arbitrary number of \( a_k \) operators acting on the state, odd number of \( a_k \)'s giving negative norm, as above. Note that negative norm states have been both introduced [24] and criticized [25] relatively early in the history of field theory, for a review see [26].

As we will see shortly, the role of the negative norm states is to provide high momentum subtraction in the theory, so perhaps their unusual properties should not be a surprise. We have one well established example of states doing subtraction: the role of the ghost states is to act like a negative degrees of freedom [27, 28], designed to subtract the overcounting of the gauge degrees of freedom. Those states do not satisfy the second Pauli postulate [29], so that the fields do not commute for a space like intervals, and that is certainly unusual. One can easily see that the negative norm states are also states of negative energy, and thus the states violate the first Pauli postulate. An energy-parity symmetry with the negative energy ghost sector have been discussed by Kaplan and Sundrum [30] in relation
To cosmological constant problem. Here one does not have energy-parity symmetry: the negative energy states have negative norm for odd number of particles, and it would be shown that the magnitude of the norm of positive and negative energy states is different for $\gamma \neq 1$.

With $\gamma$ vacuum one defines the number operator of particles with momentum $k$ by

$$N_k = a_k^\dagger a_k + \frac{\gamma}{2}$$

while the total number operator is

$$N = \sum_k \left( a_k^\dagger a_k + \frac{\gamma}{2} \right).$$

It is assumed that the operators $N$ and $N_k^\gamma$ are well defined acting on states built by repeated application of $a_k^\dagger$ or $a_k$ on $|v\rangle_\gamma$. This one can use to show that $a_k^\dagger|v\rangle_\gamma$ represents one particle state by

$$N a_k^\dagger|v\rangle_\gamma = a_k^\dagger N|v\rangle_\gamma + [N, a_k^\dagger] |v\rangle_\gamma = a_k^\dagger|v\rangle_\gamma.$$ 

Thus we can call $a_k^\dagger|v\rangle_\gamma$ one particle state. Analogously, one justifies calling the $a_k|v\rangle_\gamma$ minus one particle state.

One can use $N$ operator to show vanishing of $\gamma < v|a_k^\dagger|v\rangle_\gamma$ matrix element viz.

$$\gamma < v|a_k^\dagger|v\rangle_\gamma = \gamma < v|N a_k^\dagger|v\rangle_\gamma = \gamma < v|N a_k^\dagger|v\rangle_\gamma = 0$$

which vanishes through action of the operator $N$ on the left $\gamma < v$. Analogously one shows vanishing of average value of any odd number of $a_k^\dagger$ and $a_k's$. By the similar method (using $N_k^\gamma$) one ensures the vanishing of $\gamma < v|a_k^\dagger a_k'|v\rangle_\gamma$ for $k \neq k'$.

One can use the the standard commutation relation between a's, namely $[a_{k'}, a_k^\dagger] = \delta_{k,k'}$, to calculate the norm of a multiparticle state $(a_k^\dagger)^n|v\rangle_\gamma$. The result is

$$\| (a_k^\dagger)^n |v\rangle_\gamma \|^2 = (n - \frac{\gamma}{2})(n - 1 - \frac{\gamma}{2}) \cdots (1 - \frac{\gamma}{2}) = \prod_{j=1}^{n} (j - \frac{\gamma}{2}).$$

Note that this expression tends to the usual $n!$ as $\gamma$ tends to zero. Also note that this norm is decreasing with increasing $\gamma$, vanishing for $\gamma = 2$. One can similarly calculate the norms of the $(a_k)^n|v\rangle_\gamma$ states, and the result is

$$\|(a_k)^n |v\rangle_\gamma \|^2 = (-1)^n (\frac{\gamma}{2} + n - 1)(\frac{\gamma}{2} + n - 2) \cdots \frac{\gamma}{2} = \prod_{j=0}^{n-1} (\frac{\gamma}{2} + j).$$

Again this vanishes as usual as $\gamma$ tends to zero. The magnitude of this function is an increasing function of $\gamma$. Note also that for $\gamma = 1$ the norms of $|-n_k\rangle_\gamma \equiv (a_k)^n|v\rangle_\gamma$ state and $|n_k\rangle_\gamma \equiv (a_k^\dagger)^n|v\rangle_\gamma$ state are equal for even number of particles, while for an odd number of particles they are equal in magnitude and different in sign.
The matrix elements are very similar to usual. For any combination of operators \( a_k \) and \( a_{k'} \) where \( N \left( a_k a_{k'} \ldots a_{k_1} a_{k_1'} \ldots \right) |v> > \gamma = q \left( a_k a_{k'} \ldots a_{k_1} a_{k_1'} \ldots \right) |v> > \gamma \) and \( q \neq 0 \) one can show that the scalar product with \( \gamma < v \) vanishes, simply by

\[
\gamma < v \left( a_k a_{k'} \ldots a_{k_1} a_{k_1'} \ldots \right) |v> > \gamma = \frac{1}{q} \left( \gamma < v |N a_k a_{k'} \ldots a_{k_1} a_{k_1'} \ldots |v> > \gamma \right) = 0. \tag{3.11}
\]

In particular that implies that the matrix element of odd number of any kind of \( a \)’s vanishes as does \( \gamma < v |a_k^\dagger a_k|v> > \gamma \) for \( k \neq k' \). The later matrix element is non-vanishing only for \( k = k' \). Thus some simple matrix elements are

\[
\gamma < v |a_k^\dagger a_k|v> > \gamma = -\frac{\gamma}{2} \delta_{k,k'} \quad \gamma < v |a_k^\dagger a_{k'}|v> > \gamma = \left( 1 - \frac{\gamma}{2} \right) \delta_{k,k'}. \tag{3.12}
\]

For matrix elements of more operators, one obtains some terms in addition to the standard ones, namely

\[
\gamma < v |a_k a_{k_2} a_{k_3} a_{k_4}^\dagger a_{k_5} a_{k_6}^\dagger |v> > \gamma = \left( \delta_{k_1,k_3} \delta_{k_2,k_4} + \delta_{k_1,k_4} \delta_{k_2,k_3} \right) \left( 1 - \frac{\gamma}{2} \right)^2 + \delta_{k_1,k_2} \delta_{k_2,k_3} \delta_{k_3,k_4} \frac{\gamma}{2} \left( 1 - \frac{\gamma}{2} \right). \tag{3.13}
\]

The matrix elements for different order of operators can be easily obtained by commutation. Note that the last term, that is different then standard, becomes of the measure zero in the continuum limit. Also it vanishes in the limit \( \gamma \) tends to zero (or two). All matrix element of more \( a \)’s have analogous terms, always proportional to \( \frac{\gamma}{2} \left( 1 - \frac{\gamma}{2} \right) \). Let us just quote the result for six operators

\[
\gamma < v |a_k a_{k_2} a_{k_3} a_{k_4} a_{k_5}^\dagger a_{k_6} |v> > \gamma = \left( 1 - \frac{\gamma}{2} \right)^3 \left( \delta_{1,4} \delta_{2,5} \delta_{3,6} + \text{perm} 1,2,3 \right) + \frac{\gamma}{2} \left( 1 - \frac{\gamma}{2} \right)^2 \times \left( \delta_{1,4} \delta_{2,3,5,6} + \text{combinations} \right) + (3.14)
\]

\[
+ \gamma \left( 1 - \frac{\gamma}{2} \right) \left( \gamma - 1 \right) \delta_{1,2,3,4,5,6}.
\]

Here, for example \( \delta_{2,3,5,6} \) stands for \( \delta_{k_2,k_3} \delta_{k_5,k_6} \) and analogous notation for other indices. Note that all nonstandard terms are of measure zero in the continuum limit and also vanish as \( \gamma \) tends to zero. Again one can obtain the matrix elements for other orders of operators simply by commutation.

Let us now give an explicit expression for \(|v> > \gamma\) in terms of the standard vacuum \(|0>\),

\[
|v> > \gamma = \prod_{k_1} \left( \int_0^\infty e^{-a_{k_1}^\dagger a_{k_1} t_{k_1}} \frac{dt_{k_1}}{t_{k_1}^{\frac{3}{2}}} \right) |0> . \tag{3.15}
\]

Since this is a tensor product of different \( k \)'s one can have various \( C(k) \) inserted in the product, I am using one as the simplest. Note that the limits for \( \gamma \) are

\[
0 < \gamma < 2. \tag{3.16}
\]

Parameter \( \gamma \) starting from one to the lowest order should not change through radiative corrections by one or more. The derivations of the above equations are given in the appendix.
4. Perturbation theory with $\gamma$ Vacuum

The first step in developing perturbation theory is to calculate the propagator. Defining

$$i\Delta_\gamma(x) \equiv \langle v|T(\varphi(x)\varphi(0)|v \rangle = i \left(1 - \gamma^2\right) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ikx}$$

$$= i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ikx} - \frac{\gamma}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ikx} = (4.1)$$

Here I am using the notation of Bjorken and Drell, $T$ stands for time ordered product, $\Delta_F(x)$ for Feynman propagator, while $\Delta_1(x)$ is even solution for the homogenous Klein-Gordon equation. The above equation is derived by expanding the field $\varphi$ in terms of creation and annihilation operators and using the matrix elements (3.12). We see here that the change of the usual Feynman propagator does not effect an external (on mass shell) line. The change of the propagator comes from two sources: first is the change in the normalization of states, multiplying the usual propagator by $\left(1 - \gamma^2\right)$, second is the effect of negative norm states, propagating with the acausal propagator characterized with $-i\epsilon$ prescription in going around poles, the opposite sign to the usual Feynman prescription. This makes the difference in continuation to the Euclidean $k_0$, resulting in different signs which leads to some cancelations. In going to Euclidean $k_0$ one starts with expressions of the form

$$\int d^3k \oint f(k_0, \vec{k}) dk_0 = 0$$

where $\oint$ is integral over closed path in a complex $k_0$, contour being such to avoid singularities. For contours see the enclosed figure. The

**Figure 4:** The contours for Feynman and for acausal propagator

contours for Feynman propagator and for acausal propagator for negative norm states are different and where for Feynman propagator one gets

$$\int d^3k \int_{-\infty}^{\infty} f(k_0, \vec{k}) dk_0 = i \int d^3k \int_{-\infty}^{\infty} f(iy, \vec{k}) dy$$

(4.2)

for the acausal propagator one gets
\[ \int d^3 k \int_{-\infty}^{\infty} f \left( k_0, \vec{k} \right) dk_0 = -i \int d^3 k \int_{-\infty}^{\infty} f \left( -iy, \vec{k} \right) dy. \quad (4.3) \]

Here \( y \) plays the role of Euclidean \( k_0 \).

Consider now the second order correction to the scattering amplitude, see figure 5.

**Figure 5:** correction to scattering amplitude

In this one can follow the textbook derivation [31] and use dimensional regularization. Consider the integral

\[ F_{\text{reg}} \left( p^2 \right) = \int \frac{d^n k}{(2\pi)^n} \Delta_\gamma (k) \Delta_\gamma (k + p). \quad (4.4) \]

The integrals one and four we can deal with by standard methods [31], only the rotation to Euclidean momenta go by different contours. Lets add the integrals two and three

\[ F_1 = \left( 1 - \frac{\gamma}{2} \right)^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon (k + p)^2 - m^2 + i\epsilon} \]

\[ F_2 = \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon (k + p)^2 - m^2 - i\epsilon} \]

\[ F_3 = \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 - i\epsilon (k + p)^2 - m^2 + i\epsilon} \]

\[ F_4 = \left( \frac{\gamma}{2} \right)^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 - i\epsilon (k + p)^2 - m^2 - i\epsilon}. \]

The integrals one and four we can deal with by standard methods [31], only the rotation to Euclidean momenta go by different contours. Lets add the integrals two and three

\[ F_2 + F_3 = \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon (k + p)^2 - m^2 - i\epsilon} + \]

\[ + \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 - i\epsilon (k + p)^2 - m^2 + i\epsilon} = \]

\[ \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon (k + p)^2 - m^2 - i\epsilon} + \]

\[ + \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 - i\epsilon (k + p)^2 - m^2 + i\epsilon} + \]

\[ + \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 - i\epsilon (k + p)^2 - m^2 + i\epsilon} + \left( 1 - \frac{\gamma}{2} \right) \left( \frac{\gamma}{2} \right) \Delta^2 I_1, \]

where

\[ I_1 = \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 + \epsilon^2 [(k + p)^2 - m^2]^2 - \epsilon^2}. \quad (4.7) \]

For nonzero \( p \) the last term of 4.6 vanishes as \( \epsilon \) when \( \epsilon \) tends to zero while the other two terms can be added to \( F_1 \) and \( F_4 \) respectively. One can calculate the explicit expressions following Brown, [31], the result for the amplitude is

\[ F_{\text{reg}} \left( p^2 \right) = i (1 - \gamma) \Gamma (2 - n/2) (\frac{\mu}{2\pi})^{n-4} \int_0^1 \left( \frac{m^2 +\alpha (1-\alpha)p^2}{4\pi\mu^2} \right)^{n/2-2} d\alpha, \quad (4.8) \]
where \( \mu \) is an arbitrary mass scale. The key of the matter is that the result is proportional to \( i(1 - \gamma) \). That means that starting with \( \gamma = 1 \) in lowest approximation, the radiative correction vanishes and the loop corrections are zero. Moreover, since other more involved corrections are obtained by integrating this expression, those vanish too.

One has to have in mind that we are using neither normal ordering nor Wick’s theorem here and one has to consider corrections to propagator and vacuum energy with the lines starting and finishing at the same vertex, see figure.

Calculating directly (strictly speaking one does not use time ordering in these lines, but the results with, or without, time ordering are the same) one obtains the propagator correction

\[
i(1 - \gamma) \int \frac{d^\eta k_E}{(2\pi)^\eta} \frac{1}{k_E^2 + m^2}. \tag{4.9}
\]

Here \( k_E \) is Euclidean momentum and this vanishes for \( \gamma = 1 \). It is perhaps simpler to show this in x space, instead in k space. One has

\[
i\Delta\gamma(x) = i\Delta F(x) - \frac{\gamma}{2}\Delta_1(x) = \frac{1}{2}(1 - \gamma)\Delta_1(x) + i\epsilon(t)2 \Delta(x).
\]

When \( \gamma = 1 \) the first part is zero, while the function \( \Delta(x) \) is odd and vanishes at \( x = 0 \). So the lines starting and finishing at the same point do not contribute.

One here obtains an unusual result: The radiative corrections for the \( \phi^4 \) field theory vanish with this formulation. This theory is in some sense trivial. The arguments for triviality of the \( \phi^4 \) theory are well known [32, 33, 34, 35, 36, 37, 38], [39, 40, 41, 42, 43, 44, 45, 46], both analytically and through computer simulation, however the arguments here are perturbative and thus different. In contrast the standard perturbative treatment [31] certainly gives nontrivial correction to the scattering amplitude. Here one obtains perturbatively the vanishing of the radiative corrections; the \( \gamma \) vacuum brings agreement between perturbative calculation and other methods.

5. Nontrivial \( \gamma \)

Now let us investigate the vacuum energy for \( \gamma \neq 1 \). In that case one has, to lowest order, vacuum energy proportional to \( 1 - \gamma \). To that order the vacuum energy is given by

\[
E_0 = \frac{1}{2}(1 - \gamma) \int \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2} = \\
\left[ \frac{1}{2}(1 - \gamma) \int_0^\Lambda k^2 \sqrt{k^2 + m^2} dk \right] = \\
\left[ \frac{1}{2}(1 - \gamma)\Lambda^4 \left( \sqrt{1 + a^2(2 + a^2)} + a^2 \ln \frac{a}{1 + \sqrt{1 + a^2}} \right) \right]. \tag{5.1}
\]

Here \( a \equiv \frac{m}{\Lambda} \). All logarithmic divergences are represented by \( \ln a \). Note that here we are using three dimensional high momentum cut-off \( \Lambda \), four dimensional is not convenient since we are on the mass hyperboloid. With three dimensional high momentum cut-off, one

\[\text{using the notation and formulae from Bjorken and Drell field theory, p 388.}\]
expects some Lorentz invariance violation close to the cut-off momentum. That violation was not seen \[47, 48, 49\], which means that cut-off is high, in the parlance of this model, \(a\) is small.

One can now calculate the correction to the vacuum energy to order \(g\), using the vacuum to vacuum diagram from the figure 6, just the average value (using \(|\nu| > \gamma\)) of the interaction term. After some calculation one obtains

\[
E_1 = \frac{3g(1-\gamma)^2}{2^3\pi^4} I^2 \quad \text{where} \quad I = \int_0^\Lambda \frac{k^2}{2\sqrt{k^2 + m^2}} \, dk.
\]  

(5.2)

That integral can be calculated from the previous one \(E_0\) integral 5.1 by differentiation with respect to \(m^2\). Calculating this one obtains

\[
E_1 = \frac{3g(1-\gamma)^2}{(2\pi)^4} \frac{\Lambda^4}{8} f_2(a)
\]

(5.3)

where

\[
f_2(a) \equiv \frac{1}{1+a^2} \left[1 + 3a^2/4 - \frac{a^4}{4(1+\sqrt{1+a^2})} + \frac{a^2}{4} \sqrt{1+a^2} + a^2 \sqrt{1+a^2} \ln \frac{a}{1+\sqrt{1+a^2}} \right]^2.
\]

(5.4)

Note that \(f_2(a)\) is nonnegative, and \(f_2(0) = 1\). One then obtains

\[
E_0 + E_1 = \frac{\Lambda^4}{8} \left[\frac{1-\gamma}{(2\pi)^2} f_1(a) + \frac{3g(1-\gamma)^2}{(2\pi)^4} f_2(a)\right].
\]

(5.5)

Here \(f_2(a)\) is defined above, while \(f_1(a)\) is given by

\[
f_1(a) \equiv \left[\sqrt{1+a^2(2+a^2)} + a^4 \ln \frac{a}{1+\sqrt{1+a^2}} \right].
\]

(5.6)

Note that \(f_1(0) = 2\). Obviously, \(\gamma = 1\) does make the \(\Lambda^4\) term vanish, however this is not the only possibility. The other possibility (to the first order in \(g\)) is given by

\[
f_1(a) + \frac{3g(1-\gamma)}{(2\pi)^2} f_2(a) = 0.
\]

(5.7)

Solving for \(\gamma\) one obtains

\[
\gamma = 1 + \frac{(2\pi)^2}{3g} \frac{f_1(a)}{f_2(a)}.
\]

(5.8)

Note that this is non-analytical in \(g\) at \(g = 0\), having dependence \(\frac{1}{g}\). Since this \(\gamma \neq 1\), one gets nonzero scattering amplitude correction equation (4.8) different from the trivial case \(\gamma = 1\). Also we have the limits for the value of \(\gamma\), \(0 < \gamma < 2\), putting that in the above expression for \(\gamma\) one obtains

\[
\frac{g}{2} > \frac{(2\pi)^2}{3} \frac{f_1(a)}{2f_2(a)} \sim \frac{(2\pi)^2}{3} \left(1 - 2a^2 \ln(a/2) \right).
\]

(5.9)

The last behavior is for small \(a\). That means that there is a possibility of phase transition in this model at \(g/2\) greater then \(\frac{(2\pi)^2}{3}\), although the energy is not lower then \(\gamma = 1\) case.
The theory should be checked in computer simulations in this range of the coupling for any sign of phase transition.

We proceeded here assuming the the coefficient \( \Lambda^4 \) of the vacuum energy vanishes. However, since the expression for vacuum energy equation (5.5) is proportional to \( \Lambda^4 \), in that approximation, the whole vacuum energy vanishes. Now let's check what happens if we presume finite vacuum energy density, as evidenced by observation [18, 19, 20, 21]. Using the expression for the vacuum energy to first order in the coupling (5.5), one obtains

\[
E_0 + E_1 = \frac{\Lambda^4}{8} \left( 1 - \frac{3g(1 - \gamma)}{(2\pi)^2} f_2(a) \right) = E_m. \tag{5.10}
\]

Here one can take \( E_m \) a measured vacuum energy density (of the order of \((10^{-3}eV)^4\)) or some other value in a model calculation. The \( f_1(a) \) and \( f_2(a) \) are defined previously, equations (5.6), (5.4). By this equation we are changing problem from the one of cancelation, to the one of fine tuning the constants so equation (5.10) is satisfied. Defining \( X \equiv \frac{1 - \gamma}{(2\pi)^2} \) one obtains a quadratic equation for \( X \)

\[
X f_1(a) + 3gX^2 f_2(a) = \epsilon \quad \epsilon \equiv \frac{8E_m}{\Lambda^4}. \tag{5.11}
\]

Two solutions for \( X \) give two solutions for \( \gamma \). One solution bit smaller then one (for positive \( \epsilon \)) a bit greater then one for negative \( \epsilon \), and analytical function of \( g \). The other greater then one, non-analytical function of \( g \). Those are given by

\[
\gamma_{(1,2)} = 1 + \frac{(2\pi)^2 f_1(a)}{6gf_2(a)} \times \left( 1 \mp \sqrt{1 + 12g\epsilon \frac{f_2(a)}{(f_1(a))^2}} \right). \tag{5.12}
\]

Here \( \epsilon \) is very small and expanding the above solution in series one obtains

\[
\gamma_1 = 1 - \frac{(2\pi)^2 \epsilon}{f_1(a)} + 0(\epsilon^2) \quad \gamma_1 = 1 + \frac{(2\pi)^2 f_1(a)}{3gf_2(a)} + 0(\epsilon). \tag{5.13}
\]

Using the limits on \( \gamma \), and expanding the square root for small \( \epsilon \) one obtains the limits

\[
| \epsilon/f_1(a) | < \frac{1}{(2\pi)^2} \tag{5.14}
\]

for the first sign and the same limit as previously, equation (5.9), for the second sign. Therefore by fine tuning the constants, one can obtain the finite energy density within this model. Given the smallness of \( \epsilon \) the first sign is very close to \( \gamma = 1 \) case with no radiative corrections. Again the theory has the other sector with nontrivial radiative corrections and specific range of the coupling. Both \( \gamma < 1 \) (for positive \( \epsilon \)) function analytical in \( g \), and \( \gamma > 1 \), function non-analytical in \( g \) are possible.

6. Conclusion and Outlook

It is shown here that handling of the worst \( (\Lambda^4) \) singularity in the vacuum energy is possible with the condensate defined by (3.2), starting from natural instead of normal ordering.
Triviality of the $\varphi^4$ theory is explained perturbatively for $\gamma = 1$. However for $\gamma \neq 1$ there is a possibility of a non-trivial state non analytical in the coupling constant equation (5.8) and one solution of equation (5.12), and those can not be obtained by the standard perturbation theory.

The moral of this story is the existence of negative energy sector of QFT that has a subtractive role. Out of four integrals needed to bring perturbative calculation in agreement with non-perturbative methods (equation (4.5)) the standard method recognizes only one; that leads to disagreement between the methods which was obviated in this paper. The corollary to this moral is that no symmetry is needed to ameliorate the ultraviolet divergences of QFT, what is needed is the proper solution for the ground state.

There are, however, some unanswered questions. First, what about next ($\Lambda^2$) singularity that appears in the propagator, or mass, correction for $\gamma \neq 1$? If we keep the standard Feynman diagram calculation there is nothing to cancel the divergent term given in equation (4.9) and one would have to resort to the standard mass correction counter term. However, the propagator correction is the functional derivative with respect to propagator of the vacuum energy (that holds for all orders of diagrams) and one can take that as the method of defining the propagator corrections. In that case one also has to take the functional derivative of the zero point energy (5.1) with respect to the propagator, $\delta E_0/\delta \Delta(p)$. The first term is just function of $p$ while the second term is obtained by taking the derivative of $E_0$ with respect to $m^2$ and then the functional derivative of $m^2$ with respect to $\Delta(p)$. To first order in $g$ the derivative of $m^2$ with respect to $\Delta(p)$ is just the coupling constant and one has the exactly the structure of the term needed to cancel the propagator infinity. Analogously by taking second derivative of $E_0$ with respect to $\Delta(p)$ one obtains the logarithmically divergent term needed to regularize the correction to scattering. Calculating in this manner may lead to cancelations.

Of course, getting the finiteness of a field theory in theories with unusual states is nothing new (see e.g. [52, 53, 54]), what is new is the motivation for negative norm states, coming from the specific condensate vacuum. As opposed to other papers the hermiticity of the Hamiltonian and standard commutation relations are kept here (which means the standard microscopic causality [55]), the price paid is violating both the positivity of energy (the first Pauli postulate) and positivity of norm. Note however that with Lorentz invariant normalization the added term $2E$ makes the norm positive.

Second, the unitarity of the theory with these states is not discussed, for an earlier example of discussion of unitarity of theories with a non-positive norm see [56]. In my opinion, those "renormalization ghost" states violating the first Pauli postulate are necessary for the complete set of states, and take full part in resolution of unity. Those states are propagating forward in time with negative energy and it is not clear to me are there any other effects of those states besides the high momentum subtraction. Given that those are negative energy states perhaps the states are tachyonic although in the usual formulation [57] tachyonic field has different commutation rules. Physics of negative energy states, as well as physics of vacuum structure, still has to be worked out.

4one expects $(p^2 - m^2)^2$ term but making Hamiltonian in theories with higher derivatives is involved [50, 51] and I will not calculate it here
As outlook lets discuss (speculative) possibilities for further research. The key here is to introduce the fermions: for fermions the sign of the vacuum energy is opposite to that of bosons and that helps regularization of the vacuum energy. Note that for fermions the negative energy states describe antiparticles, and what one can glimpse from expressions for norms for the positive energy states (3.9), \( \gamma \) decreasing with \( \gamma \), and negative energy states (3.10) increasing with \( \gamma \) is the possible reason for the observed particle-antiparticle asymmetry: the situation is not symmetrical since vacuum energy density is different from zero, and \( \gamma \neq 1 \). What one has is akin to energy parity symmetry \([30]\) (with negative energy sector having negative norm and subtractive effect) softly broken by the nonzero, albeit small, vacuum energy density.

Presumably the higher order corrections makes an equation of the higher order then quadratic, which leaves the room for a richer vacuum structure, several solutions for \( \gamma \) describing different vacua of the same energy\(^{6}\) and possibly the different generations of particles. The same field with non-trivial vacuum structure may, perhaps, describe generations.

If these speculations are valid, the study of the structure of vacuum is worthwhile; that should not be a surprise since for Euclidean field theory what one studies is like a statistical sum in statistical physics, a quantity of prime importance. What may be hiding in the vacuum structure is the information about particle spectrum. All this has to be extended to systems of interacting bosons and fermions, and appropriate symmetry breaking condensate searched for while protecting (to a good approximation) the Lorentz invariance. The finiteness of the vacuum energy density should be a requirement given the observed small vacuum energy density. In that one would be helped by the opposite signs of vacuum energies of bosons and fermions. So in addition to (and perhaps concurrently with) renormalizability, the existence of the appropriate vacuum state(s) may be the condition for a realistic field theory. If one is able to make the theory finite in the process, the question of Weisskopf \([58]\) on the soundness of renormalization procedure will be answered affirmatively. The measure of success of such a program would be the decreasing of the bewildering number of constants describing the standard model of particle physics \([59]\).

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\(^{5}\) having in mind that fermions have opposite sign of vacuum energy

\(^{6}\) this is important since it avoids the superselection rule between the states built over these states
A. Standard Vacuum and $\gamma$ Vacuum

Here the relation between the standard field theory vacuum and the condensate $\gamma$ vacuum is discussed. Act with $a_k^\dagger a_k$ on the expression for the $\gamma$ condensate state, (3.15)

\[
a_k^\dagger a_k|x> \equiv a_k^\dagger a_k \prod_{k_1} \left( \int_{0}^{\infty} e^{-\langle a_k a_k^\dagger \rangle_k t_{k_1}^\beta dt_{k_1}} \right) |0> =
\]

\[
a_k^\dagger a_k \left( \int_{0}^{\infty} e^{-\langle a_k a_k^\dagger \rangle_k t_{k}^\beta dt} \right) |0_k> \prod_{k_1 \neq k} \left( \int_{0}^{\infty} e^{-\langle a_k a_k^\dagger \rangle_k t_{k_1}^\beta dt_{k_1}} \right) |0> . \quad (A.1)
\]

Here $|x>$ is written instead of $|v>\gamma$ until the equality of the two is proven, and ditto $\beta \rightarrow 1$ instead of $\frac{\gamma}{4} \rightarrow 1$. Since we are dealing with the tensor product of expressions for the different k’s, one can separate

\[
a_k^\dagger a_k \left( \int_{0}^{\infty} e^{-\langle a_k a_k^\dagger \rangle_k t_{k}^\beta dt} \right) |0_k> =
\]

\[
a_k^\dagger \int_{0}^{\infty} \left[ a_k, e^{-\langle a_k a_k^\dagger \rangle_k t} \right] t^\beta dt |0> =
\]

\[
2 \int_{0}^{\infty} -a_k^\dagger a_k e^{-\langle a_k a_k^\dagger \rangle_k t} t^\beta dt |0_k> , \quad (A.2)
\]

using partial integration with

\[
dv = -a_k^\dagger a_k e^{-\langle a_k a_k^\dagger \rangle_k t} dt \quad \text{and} \quad t^\beta = u \quad \text{one obtains for the above expression}
\]

\[
-2\beta \int_{0}^{\infty} e^{-\langle a_k a_k^\dagger \rangle_k t} t^\beta dt |0_k> +2t^\beta e^{-\langle a_k a_k^\dagger \rangle_k |0_k>} |0> \]

\[
|0> . \quad (A.3)
\]

That means if

\[
2t^\beta e^{-a_k a_k^\dagger |0_k>} |0_k> \big|_{0}^{\infty} = 0 \quad (A.4)
\]

then

\[
a_k^\dagger a_k |x> = -2\beta |x> \quad (A.5)
\]

or $\gamma = 4\beta$ , $|x> = |v> \gamma$. Obviously for $\beta > 0$ the expression (A.4) vanishes at $t = 0$. To check what happens at $t \rightarrow \infty$ we need to check

\[
\| e^{-a_k a_k^\dagger t} |0_k> \|^2 < 0 |e^{-a_k a_k^\dagger t} e^{-a_k a_k^\dagger t} |0> \]

\[
as \quad t \rightarrow \infty . \quad (A.6)
\]

Here we use the closed algebra of the dynamical group of harmonic oscillator [60] given by the elements

\[
A = \frac{a_k^2}{2\sqrt{2}} \quad B = \frac{a_k^2}{2\sqrt{2}} \quad C = \frac{a_k a_k^\dagger + a_k^\dagger a_k}{4}. \quad (A.7)
\]

The commutation relations are

\[
[A, B] = C \quad [A, C] = A \quad [B, C] = -B. \quad (A.8)
\]
Defining

\[ x \equiv 2\sqrt{2}t \]

we write

\[ \Omega(x) \equiv e^{-Ax}e^{-Bx} = e^{-B\beta(x)}e^{-C\delta(x)}e^{-A\alpha(x)}. \]  

(A.9)

For \( t \to \infty \) we are interested in behavior of \( \delta(x) \), as \( x \to \infty \). Differentiating (A.9) with respect to \( x \), one obtains using repetitive commutation

\[
- \left[ (1 + x^2/2) A + B - x C \right] \Omega(x) = \]

(A.10)

\[
= - \left[ \alpha' e^\delta A + \left( \beta' + \delta' \beta + \alpha' e^\delta \frac{\beta^2}{2} \right) B + \left( \delta' + \alpha' e^\delta \beta \right) C \right] \Omega(x) \tag{A.11}
\]

Comparing the left and right sides of the above formula one obtains a system of differential equations for \( \alpha, \beta \), and \( \delta \). The initial conditions are \( \alpha(0) = 0 \), \( \beta(0) = 0 \), \( \delta(0) = 0 \). The equations could be written as

\[
1 + x^2/2 = \alpha' e^\delta
\]

(A.12)

\[
1 = \beta' + \delta' \beta + (1 + x^2/2)\frac{\beta^2}{2}
\]

\[
-x = \delta' + (1 + x^2/2)\beta.
\]

One can show the asymptotic behavior for large \( x \) (and large \( t \)) is

\[
\beta \sim -\frac{2}{x} - \frac{4}{x^3} + 0(x^{-5})
\]

and

\[
\delta' \sim \frac{4}{x} + 0(x^{-3})
\]

leading to

\[
\delta \sim 4\ln(const\ x) + 0(x^{-2}\ln x)\]

(A.13)

That enables us to conclude

\[
< 0|e^{-B\beta} e^{-C\delta} e^{-A\alpha}|0 > = < 0|e^{-C\delta}|0 > = \frac{1}{e^{\delta/4}} \sim \frac{1}{x} \quad \text{for} \quad x \to \infty.
\]

(A.14)

That could be used for

\[
\| t^\beta e^{\alpha^2 t^2} \|^2 \sim \frac{t^{2\beta}}{t}
\]

(A.15)

as \( t \to \infty \).

Thus we need \( 0 < \beta < 1/2 \) which gives

\[
0 < \gamma < 2
\]

(A.16)

Finally let us calculate the norm of \(|v >\gamma \) state. Define

\[
\Omega(x, y) = e^{-Ax}e^{-By} = e^{-\beta(x,y)}e^{-C\delta(x,y)}e^{-A\alpha(x,y)},
\]

(A.17)
where A, B, C are the previously defined operators. Differentiating with respect to x and y and using methods analogous to already shown one can show that $\delta = 2ln(1 - xy/2)$. That implies
\[
< 0|e^{-Ax}e^{-Bx}|0 > = < 0|e^{-C\delta}|0 > = < 0|e^{-\delta/4}|0 > = < 0|0 > \frac{1}{\sqrt{1 - xy/2}}.
\] (A.18)

Therefore
\[
\gamma < v|v > = \prod_k \int_0^\infty \int_0^\infty (t\tau)^{\gamma/4-1} \frac{1}{\sqrt{1 - 4t\tau}} dt d\tau < 0|0 >_k.
\] (A.19)

The integral
\[
\int_0^\infty \int_0^\infty (t\tau)^{\gamma/4-1} \frac{1}{\sqrt{1 - 4t\tau}} dt d\tau
\] (A.20)
do not converge since by change of the variables
\[
\rho = t\tau, \quad d\tau = d\rho/t
\]
one obtains
\[
\int_0^\infty \frac{dt}{t} \int_0^\infty \frac{1}{\sqrt{1 - 4\rho}} (\rho)^{\gamma/4-1} d\rho.
\] (A.21)

Here the second integral converges for $0 < \gamma < 2$ while the first integral is divergent. The question is, is the norm of $|v >_\gamma$ finite and the norm of $|0 >_k$ vanishing or the norm of $|v >_\gamma$ divergent and norm of $|0 >_k$ finite? The point of view of this paper is that matrix elements built over $|v >_\gamma$ exist, while the standard vacuum $|0>$ having $\gamma = 0$ is suspect.

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