ASYMPTOTIC BEHAVIOR FOR STOCHASTIC PLATE EQUATIONS WITH MEMORY AND ADDITIVE NOISE ON UNBOUNDED DOMAINS

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Abstract. In this paper we study asymptotic behavior of a class of stochastic plate equations with memory and additive noise. First we introduce a continuous cocycle for the equation and establish the pullback asymptotic compactness of solutions. Second we consider the existence and upper semicontinuity of random attractors for the equation.

1. Introduction. Consider the following stochastic plate equations with memory and additive noise defined in the entire space $\mathbb{R}^n$:

$$u_{tt} + h(u_t) + \Delta^2 u + \int_0^\infty \mu(s)\Delta^2(u(t) - u(t-s))\,ds + \lambda u + f(x,u) = g(x,t) + \alpha \phi(x) \frac{dW}{dt} \quad x \in \mathbb{R}^n, \quad t > \tau$$

with the initial value conditions

$$u(x,\tau) = u_0(x), \quad u_t(x,\tau) = u_1(x), \quad x \in \mathbb{R}^n, \quad t \leq \tau,$$

where $\tau \in \mathbb{R}, u = u(x,t)$ is a real-valued function in $\mathbb{R}^n \times [\tau, +\infty), \lambda > 0$ and $\alpha$ are constants, the memory kernel $\mu$, nonlinear functions $f$ and $h$ satisfy certain conditions, $g(x,\cdot)$ and $\phi$ are given functions in $L^2_{loc}(\mathbb{R}; H^1(\mathbb{R}^n))$ and $H^2(\mathbb{R}^n) \cap H^3(\mathbb{R}^n)$, respectively, $W(t)$ is a two-sided real-valued Wiener process on a probability space.

Plate equations have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theories of solid mechanics. The study of the long-time dynamics of plate equations has become an outstanding area in the field of the infinite-dimensional dynamical system. While the attractors is regarded as a proper notation to describe the long-time dynamics of solutions. To the best of our knowledge, there have been many works on the investigation of the attractors for the plate equations over the last few years. For instance, if the random term is vanished, $\mu = 0$ and $g(x,t) = g(x)$, then (1.1)-(1.2) change into a deterministic autonomous plate equation. The existence and uniqueness of the

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global attractor of the corresponding dynamical system was studied in [27, 29, 34, 35, 26, 1, 9, 10, 11, 7, 13]; besides, the uniform attractor of the dynamical system generated by the non-autonomous plate equation was established in [28].

For the stochastic plate equations, if \( \mu = 0 \) and the forcing term \( g(x, t) = g(x) \), then the existence of a random attractor of (1.1)-(1.2) on bounded domain have been proved in [15, 16, 12, 14]; if \( \mu \neq 0 \), the existence of random attractors for plate equations with memory and additive white noise on bounded domain were considered in [19, 20]. Recently, on the unbounded domain, the authors investigated the asymptotic behavior for stochastic plate equation with additive noise and multiplicative noise (see [33, 32, 30, 31] for details). To the best of our knowledge, it is not considered by any predecessors for the stochastic plate equation with additive noise and memory on unbounded domain.

Motivated by above literatures, the goal of the present paper is to study the existence and upper semicontinuity of random attractors for non-autonomous stochastic equation (1.1)-(1.2). By applying the abstract results in [22], we will prove the stochastic plate equation (1.1)-(1.2) has tempered random attractors in \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu, 2} \) (the definition of \( \mathfrak{R}_{\mu, 2} \) see Section 3).

In general, the existence of global random attractor depends on some kind compactness (see, e.g., [2, 3, 4, 6, 24]). To prove the existence of random attractors for (1.1)-(1.2) in \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu, 2} \), we must establish the pullback asymptotical compactness of solutions. Involving to our problem (1.1)-(1.2), there are two essential difficulties in verifying the compactness. One difficulty is that Sobolev embeddings are not compact on unbounded domain, we cannot get the desired asymptotic compactness directly from the regularity of solutions. We here overcome the difficulty by using the uniform estimates on the tails of solutions outside a bounded ball in \( \mathbb{R}^n \) and the splitting technique, see [24, 25] for details. Another one difficulty is caused by the memory kernel, because there is no applicable compact embedding property in the “history” space. In this case, we solve it with the help of a useful result in [17]. For our purpose, we introduce a new variable and an extend Hilbert space.

The framework of this paper is as follows. In the next Section, we recall a sufficient and necessary criterion for existence of pullback attractors for cocycle or non-autonomous random dynamical systems. In Section 3, we define a continuous cocycle for Eq.(1.1)-(1.2) in \( H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \times \mathfrak{R}_{\mu, 2} \), then we derive all necessary uniform estimates of solutions in Section 4. In Section 5, we prove the existence and uniqueness of tempered random attractor for the non-autonomous stochastic strongly damped plate equation. Finally, in Section 6, we prove the upper semicontinuity of random attractors as \( \alpha \) to zero.

Throughout the paper, the letters \( c \) and \( c_i \) (\( i = 1, 2, \ldots \)) are generic positive constants which may change their values from line to line or even in the same line.

2. Preliminaries. In this section, we recall some basic concepts related to random attractors for stochastic dynamical systems.

Let \( X \) be a separable Banach space and \( (\Omega, \mathcal{F}, \mathcal{P}) \) be the standard probability space, where \( \Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \} \), \( \mathcal{F} \) is the Borel \( \sigma \)-algebra induced by the compact open topology of \( \Omega \), and \( \mathcal{P} \) is the Wiener measure on \( (\Omega, \mathcal{F}) \). There is a classical group \( \{ \theta_t \}_{t \in \mathbb{R}} \) acting on \( (\Omega, \mathcal{F}, \mathcal{P}) \) which is defined by
\[
\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \text{for all } \omega \in \Omega, \; t \in \mathbb{R}.
\] (2.1)
We often say that \( (\Omega, \mathcal{F}, \mathcal{P}, \{ \theta_t \}_{t \in \mathbb{R}}) \) is a parametric dynamical system.
The following four definitions and one proposition are from [22].

**Definition 2.1.** A mapping \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X \) is called a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}}) \) if for all \( t \in \mathbb{R} \), \( \omega \in \Omega \) and \( t, s \in \mathbb{R}^+ \), the following conditions (1)-(4) are satisfied:

1. \( \Phi(t, \tau, \omega, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X \) is \((\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X)), \mathcal{B}(X))\)-measurable;
2. \( \Phi(0, \tau, \omega, \cdot) \) is the identity on \( X \);
3. \( \Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot) \);
4. \( \Phi(t, \tau, \omega, \cdot) : X \to X \) is continuous.

Hereafter, we assume \( \Phi \) is a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}}) \), and \( \mathcal{D} \) is the collection of all tempered families of nonempty bounded subsets of \( X \) parameterized by \( t \in \mathbb{R} \) and \( \omega \in \Omega \):

\[
\mathcal{D} = \{ D = \{ D(\tau, \omega) \subseteq X : D(\tau, \omega) \neq \emptyset, \tau \in \mathbb{R}, \omega \in \Omega \} \}.
\]

\( \mathcal{D} \) is said to be tempered if there exists \( x_0 \in X \) such that for every \( c > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the following holds:

\[
\lim_{t \to -\infty} e^{ct} d(D(\tau + t, \theta_t \omega), x_0) = 0. \tag{2.2}
\]

Given \( D \in \mathcal{D} \), the family \( \Omega(D) = \{ \Omega(D, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is called the \( \Omega \)-limit set of \( D \) where

\[
\Omega(D, \tau, \omega) = \bigcap_{r \geq 0, t \geq \tau} \bigcup_{s \geq 0} \Phi(t, \tau - t, \theta_{s} \omega, D(\tau - t, \theta_{s} \omega)). \tag{2.3}
\]

The cocycle \( \Phi \) is said to be \( \mathcal{D} \)-pullback asymptotically compact in \( X \) if for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), the sequence

\[
\{ \Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, x_n) \}_{n=1}^{\infty}
\]

has a convergent subsequence in \( X \) \((2.4)\)

whenever \( t_n \to \infty \), and \( x_n \in D(\tau - t_n, \theta_{-t_n} \omega) \) with \( \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \).

**Definition 2.2.** A family \( K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) is called a \( \mathcal{D} \)-pullback absorbing set for \( \Phi \) if for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \) and for every \( D \in \mathcal{D} \), there exists \( T = T(D, \tau, \omega) > 0 \) such that

\[
\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega) \text{ for all } t \geq T. \tag{2.5}
\]

If, in addition, \( K(\tau, \omega) \) is closed in \( X \) and is measurable in \( \omega \) with respect to \( \mathcal{F} \), then \( K \) is called a closed measurable \( \mathcal{D} \)-pullback absorbing set for \( \Phi \).

**Definition 2.3.** A family \( \mathcal{A} = \{ A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) is called a \( \mathcal{D} \)-pullback attractor for \( \Phi \) if the following conditions (1)-(3) are fulfilled: for all \( t \in \mathbb{R}^+ \), \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

1. \( A(\tau, \omega) \) is compact in \( X \) and is measurable in \( \omega \) with respect to \( \mathcal{F} \).
2. \( \mathcal{A} \) is invariant, that is,
3. For every \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \),

\[
\lim_{t \to \infty} d_H(\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0, \tag{2.7}
\]

where \( d_H \) is the Hausdorff semi-distance given by \( d_H(F, G) = \sup_{u \in F} \inf_{v \in G} ||u - v||_X \), for any \( F, G \subset X \).
As in the deterministic case, random complete solutions can be used to characterize the structure of a $D$-pullback attractor. The definition of such solutions are given below.

**Definition 2.4.** A mapping $\Psi : \mathbb{R} \times \mathbb{R} \times \Omega \to X$ is called a random complete solution of $\Phi$ if for every $\tau \in \mathbb{R}^+$, $s, \tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\Phi(t, \tau + s, \theta_\omega, \Psi(s, \tau, \omega)) = \Psi(t + s, \tau, \omega). \quad (2.8)$$

If, in addition, there exists a tempered family $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ such that $\Psi(t, \tau, \omega)$ belongs to $D(\tau + t, \theta_\omega)$ for every $t \in \mathbb{R}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then $\Psi$ is called a tempered random complete solution of $\Phi$.

**Proposition 2.1.** Suppose $\Phi$ is $D$-pullback asymptotically compact in $X$ and has a closed measurable $D$-pullback absorbing set $K$ in $D$. Then $\Phi$ has a unique $D$-pullback attractor $A$ in $\mathcal{D}$ which is given by, for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$A(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{D \in \mathcal{D}} \Omega(D, \tau, \omega) \quad (2.9)$$

$$= \{\Psi(0, \tau, \omega) : \Psi \text{ is a tempered random complete solution of } \Phi\}. \quad (2.10)$$

### 3. Cocycles for stochastic plate equation.

In this section, we outline some basic settings about (1.1)-(1.2) and show that it generates a continuous cocycle in $E = H^2 \times L^2 \times \mathfrak{R}_{\mu,2}$.

Let $-\Delta$ denote the Laplace operator in $\mathbb{R}^n$, $A = \Delta^2$ and $D(A) = H^4(\mathbb{R}^n)$. We can define the powers $A^r$ of $A$ for $r \in \mathbb{R}$. The space $V_r = D(A^r)$ is a Hilbert space with the following inner product and norm

$$(u, v)_r = (A^r u, A^r v), \quad \|u\|_r = \|A^r u\|, \quad \forall u, v \in V_r.$$ 

In particular, $V_0 = L^2(\mathbb{R}^n)$, $V_1 = H^1(\mathbb{R}^n)$, $V_2 = H^2(\mathbb{R}^n)$.

For brevity, the notation $(\cdot, \cdot)$ for $L^2$-inner product will also be used for the notation of duality pairing between dual spaces.

Following Dafermos [5], we introduce a Hilbert “history” space $\mathfrak{R}_{\mu,2} = L^2_{\mu}(\mathbb{R}^+, V_2)$ with the inner product

$$(\eta_1, \eta_2)_{\mu,2} = \int_0^\infty \mu(s)(\Delta \eta_1(s), \Delta \eta_2(s))ds, \quad \forall \eta_1, \eta_2 \in \mathfrak{R}_{\mu,2},$$

and new variables

$$\eta(x, t, s) = u(x, t) - u(x, t - s), \quad (x, s) \in \mathbb{R}^n \times \mathbb{R}^+, \quad t \geq 0.$$ 

By differentiation we have

$$\eta_t(x, t, s) = -\eta_s(x, t, s) + u_t(x, t), \quad (x, s) \in \mathbb{R}^n \times \mathbb{R}^+, \quad t \geq 0.$$ 

Denote $E = H^2 \times L^2 \times \mathfrak{R}_{\mu,2}$, with the Sobolev norm

$$\|y\|_{H^2 \times L^2 \times \mathfrak{R}_{\mu,2}} = (\|v\|^2 + \|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2})^{\frac{1}{2}}, \quad \text{for } y = (u, v, \eta)^\top \in E. \quad (3.1)$$

Let $\xi = u_t + \varepsilon u$, where $\varepsilon$ is a small positive constant whose value will be determined later, then (1.1)-(1.2) can be rewritten as the equivalent system

$$\begin{cases}
\frac{du}{dt} + \varepsilon u = \xi, \\
\frac{d\xi}{dt} - \varepsilon \xi + (\lambda + \varepsilon^2 + A)u + h(\xi - \varepsilon u) + \int_0^\infty \mu(s)A\eta(s)ds + f(x, u) = g(x, t) + \alpha \phi(x) \frac{dW}{dt}, \\
\eta_t + \eta_s = u_t,
\end{cases} \quad (3.2)$$
with the initial value conditions
\[ u(x, \tau) = u_0(x), \quad \xi(x, \tau) = \xi_0(x), \quad \eta(x, \tau, s) = \eta_0(x, s) = u(x, \tau) - u(x, \tau - s), \quad (3.3) \]
where \( \xi_0(x) = u_1(x) + \varepsilon u_0(x), \ x \in \mathbb{R}^n, \ s \in \mathbb{R}^+. \)

**Assumption I.** Assume that the memory kernel function \( \mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+), \)
nonlinear functions \( h \in C^1(\mathbb{R}) \) and \( f \in C^1(\mathbb{R}) \) satisfy the following conditions:

1. For all \( s \in \mathbb{R}^+ \) and some \( \delta > 0, \)
   \[ \mu(s) \geq 0, \quad \mu'(s) + \delta \mu \leq 0, \quad (3.4) \]
   Note that (3.4) implies \( m_0 \equiv \| \mu \|_{L^1(\mathbb{R}^+)} = \int_0^\infty \mu(s)ds > 0. \)

2. Let \( F(x, u) = \int_0^u f(x, s)ds \) for \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}, \) there exist positive constants \( c_i (i = 1, 2, 3, 4), \)
   such that
   \[ |f(x, u)| \leq c_1 |u|^p + \phi_1(x), \ \phi_1 \in L^2(\mathbb{R}^n), \quad (3.5) \]
   \[ f(x, u)u - c_2 F(x, u) \geq \phi_2(x), \ \phi_2 \in L^1(\mathbb{R}^n), \quad (3.6) \]
   \[ F(x, u) \geq c_3 |u|^{p+1} - \phi_3(x), \ \phi_3 \in L^1(\mathbb{R}^n), \quad (3.7) \]
   \[ |\partial f/\partial u(x, u)| \leq \beta, \quad |\partial f/\partial x(x, u)| \leq \phi_4(x), \ \phi_4 \in L^2(\mathbb{R}^n), \quad (3.8) \]
   where \( \beta > 0, \ 1 \leq p \leq \frac{n+4}{n-4}. \) Note that (3.5) and (3.6) imply
   \[ F(x, u) \leq c(|u|^2 + |u|^{p+1} + \phi_1^2 + \phi_2). \quad (3.9) \]

3. There exist two constants \( \beta_1, \beta_2 \) such that
   \[ h(0) = 0, \quad 0 < \beta_1 \leq h'(v) \leq \beta_2 < \infty. \quad (3.10) \]

By (3.4), the space \( \mathcal{R}_{\mu, r} = L^2(\mathbb{R}^+, V_r)(r \in \mathbb{R}) \) is a Hilbert space of \( V_r \)-valued functions on \( \mathbb{R}^+ \) with the inner product and norm
\[
(\eta_1, \eta_2)_{\mu, r} = \int_0^\infty \mu(s)(A^\top \eta_1(s), A^\top \eta_2(s))ds, \\
\|\eta\|_{\mu, r}^2 = \int_0^\infty \mu(s)(A^\top \eta(s), A^\top \eta(s))ds, \\
\end{align*}
\]for all \( \eta, \eta_1, \eta_2 \in V_r, \)

and on \( \mathcal{R}_{\mu, r}, \) the linear operator \( -\partial_s \) has domain
\( D(-\partial_s) = \{ \eta \in H^1_\mu(\mathbb{R}^+, V_r) : \eta(0) = 0 \} \)
where \( H^1_\mu(\mathbb{R}^+, V_r) = \{ \eta : \eta(s), \partial_s \eta \in L^2_\mu(\mathbb{R}^+, V_r) \}. \)

For our purpose, it is convenient to convert the problem (1.1)-(1.2) (or (3.2)-(3.3)) into a deterministic system with a random parameter, and then show that it generates a cocycle over \( \mathbb{R} \) and \( (\Omega, F, \mathcal{P}, \{ \theta_t \}_{t \in \mathbb{R}}). \)

We identify \( \omega(t) \) with \( W(t), \) i.e., \( \omega(t) = W(t) = W(t, x), \ t \in \mathbb{R}. \) Set \( v(t) = \xi(t) - \alpha \phi \omega(t), \) we obtain the equivalent system of (3.2)-(3.3),
\[
\begin{align*}
\frac{dv}{dr} + \varepsilon u &= v + \alpha \phi \omega(t), \\
\frac{du}{dr} - \varepsilon v + (\lambda + \varepsilon^2 + A)u + \int_0^\infty \mu(s)A\eta(s)ds + f(x, u) &= g(x, t) \\
\eta_t + \eta_s &= u_t, \quad \eta_t + \eta_s = u_t, \quad (3.11) \end{align*}
\]with the initial value conditions
\[ u(x, \tau) = u_0(x), \quad v(x, \tau) = v_0(x), \quad \eta(x, \tau, s) = \eta_0(x, s) = u(x, \tau) - u(x, \tau - s), \quad (3.12) \]
where \( v_0(x) = \xi_0(x) - \alpha \phi \omega(t), \ x \in \mathbb{R}^n. \)
The well-posedness of the deterministic problem (3.11)-(3.12) in $H^2 \times L^2 \times \mathbb{R}_{t,2}$ can be established by standard methods as in [18, 21], more precisely, if Assumption I and Assumption II below are fulfilled, then we can prove the following Lemma.

**Lemma 3.1.** Put $\varphi^{(\alpha)}(t + \tau, \tau, \theta_{-\tau} \omega, \varphi^{(\alpha)}_0) = (u(t + \tau, \tau, \theta_{-\tau} \omega, u_0), v(t + \tau, \tau, \theta_{-\tau} \omega, v_0), \eta(t + \tau, \theta_{-\tau} \omega, \eta_0, s))^\top$, where $\varphi^{(\alpha)}_0 = (u_0, v_0, \eta_0)^\top$, and let conditions (3.4)-(3.8) and (3.10) hold. Then for every $\omega \in \Omega$, $\tau \in \mathbb{R}$ and $\varphi^{(\alpha)}_0 \in E(\mathbb{R}^n)$, problem (3.11)-(3.12) has a unique $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable solution $\varphi^{(\alpha)}(\cdot, \tau, \omega, \varphi^{(\alpha)}_0) \in C([\tau, \infty), E(\mathbb{R}^n))$ with $\varphi^{(\alpha)}(\tau, \tau, \omega, \varphi^{(\alpha)}_0) = \varphi^{(\alpha)}_0$, $\varphi^{(\alpha)}(t, \tau, \omega, \varphi^{(\alpha)}_0) \in E(\mathbb{R}^n)$ being continuous in $\varphi^{(\alpha)}_0$ with respect to the usual norm of $E(\mathbb{R}^n)$ for each $t > \tau$. Moreover, for every $(t, \tau, \omega, \varphi^{(\alpha)}_0) \in \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$, the mapping

$$
\Phi(\alpha)(t, \tau, \omega, \varphi^{(\alpha)}_0) = \varphi^{(\alpha)}(t + \tau, \tau, \theta_{-\tau} \omega, \varphi^{(\alpha)}_0) \quad (3.13)
$$

generates a continuous cocycle from $\mathbb{R}^+ \times \mathbb{R} \times \Omega \times E(\mathbb{R}^n)$ to $E(\mathbb{R}^n)$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

Introducing the homeomorphism $P(\theta_\omega)(u, v, \eta)^\top = (u, v + z(\theta_\omega), \eta)^\top, (u, v, \eta)^\top \in E(\mathbb{R}^n)$ with an inverse homeomorphism $P^{-1}(\theta_\omega)(u, v, \eta)^\top = (u, v - z(\theta_\omega), \eta)^\top$. Then, the transformation

$$
\Phi(\alpha)(t, \tau, \omega, (u_0, \xi_0, \eta_0)) = P(\theta_\omega)\Phi(\alpha)(t, \tau, \omega, (u_0, v_0, \eta_0))P^{-1}(\theta_\omega) \quad (3.14)
$$

generates a continuous cocycle with (3.2)-(3.3) over over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathcal{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

Note that these two continuous cocycles are equivalent. By (3.14), it is easy to check that $\Phi(\alpha)$ has a random attractor provided $\Phi(\alpha)$ possesses a random attractor. Then, we only need to consider the continuous cocycle $\Phi(\alpha)$.

Next we make another assumption:

**Assumption II.** We assume that $\sigma$, $\varepsilon$, $\delta$ and $g(x, t)$ satisfy the following conditions:

$$
\sigma = \min\{\varepsilon, \frac{\delta}{2}, \frac{\varepsilon \sigma_2}{2}\}, \quad (3.15)
$$

$$
\lambda + \varepsilon^2 - \beta_2 \varepsilon > 0 \quad \text{and} \quad \beta_1 > 4\varepsilon + \frac{3\beta^2}{\varepsilon(\lambda + \varepsilon^2 - \beta_2 \varepsilon)}. \quad (3.16)
$$

Moreover,

$$
\int_{-\infty}^0 e^{\sigma s}\|g(\cdot, \tau + s)\|^2 ds < \infty, \quad \forall \tau \in \mathbb{R}, \quad (3.17)
$$

and

$$
\lim_{k \to \infty} \int_{-\infty}^0 e^{\sigma s} \int_{|x| \geq k} |g(x, \tau + s)|^2 dx ds = 0, \quad \forall \tau \in \mathbb{R}, \quad (3.18)
$$

where $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

Given a bounded nonempty subset $B$ of $E$, we write $\|B\| = \sup_{\phi \in B} \|\phi\|_E$. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets of $E$ such that for every $\tau \in \mathbb{R}, \omega \in \Omega$,

$$
\lim_{s \to -\infty} e^{\sigma s} \|D(\tau + s, \theta_\omega)\|_{E}^2 = 0. \quad (3.19)
$$

Let $\mathcal{D}$ be the collection of all such families, that is,

$$
\mathcal{D} = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies } (3.19)\}. \quad (3.20)
$$
4. Uniform estimates of solutions. In this section, we conduct uniform estimates on the weak solutions of the stochastic plate equations (3.2)-(3.3) defined on $\mathbb{R}^n$, through the converted random equation (3.11)-(3.12), for the purposes of showing the existence of a pullback absorbing sets and the pullback asymptotic compactness of the random dynamical system.

We define a new norm $\|\cdot\|_E$ by

$$\|Y\|_E = (\|v\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)\|u\|^2 + \|\Delta u\|^2 + \|\eta\|_{\mu,2})^{\frac{1}{2}}, \quad \text{for } Y = (u, v, \eta) \in E.$$  \hfill (4.1)

It is easy to check that $\|\cdot\|_E$ is equivalent to the usual norm $\|\cdot\|_{H^2 \times L^2 \times \mathcal{D}_{\mu,2}}$ in (3.1).

First we show that the cocycle $\Phi_{\tau}$ has a pullback $\mathcal{D}$-absorbing set in $\mathcal{D}$.

**Lemma 4.1.** Under Assumptions I and II, for every $\tau \in \mathbb{R}, \omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$ the solution of problem (3.11)-(3.12) satisfies

$$\|\varphi^{(\alpha)}(\tau, \tau - t, \theta_{-\tau} \omega, \varphi_0^{(\alpha)})\|_E^2 \leq R_1(\alpha, \tau, \omega),$$

and $R_1(\alpha, \tau, \omega)$ is given by

$$R_1(\alpha, \tau, \omega) = M + M \int_{-\infty}^{\tau} e^{\sigma(s-\tau)} \|g(x, s)\|^2 ds + \lambda^2 \int_{-\infty}^{0} e^{\sigma s} (1 + \|\omega(s)\|^2 + \|\omega(s)\|^{p+1}) ds,$$

where $M$ is a positive constant depending on $\lambda, \sigma, \beta_1, \beta_2, \varepsilon, m_0$, but independent of $\tau, \omega, D, \alpha$.

**Proof.** Taking the inner product of the second equation of (3.11) with $v$ in $L^2(\mathbb{R}^n)$, we find that

$$\frac{1}{2} \frac{d}{dt} \|v\|^2 \leq \varepsilon \|v\|^2 + (\lambda + \varepsilon^2)(u, v) + (Av, v) + \int_{0}^{\infty} \mu(s) (A\eta(s), v) ds + (f(x, u), v)$$

$$= (g(x, t), v) - (h(v + \alpha \phi(t) - \varepsilon u), v) + \varepsilon \alpha(\phi, v) \omega(t).$$

By the first equation of (3.11), we have

$$v = u_t - \alpha \phi(t) + \varepsilon u.$$ \hfill (4.4)

By (3.10) and Lagrange's mean value theorem, we have

$$- (h(v + \alpha \phi(t) - \varepsilon u), v)$$

$$= - (h(v + \alpha \phi(t) - \varepsilon u) - h(0), v)$$

$$= - (h'(\overline{\theta}) (v + \alpha \phi(t) - \varepsilon u), v)$$

$$\leq - \beta_1 \|v\|^2 - (h'(\overline{\theta}) (\alpha \phi(t) - \varepsilon u), v)$$

$$\leq - \beta_1 \|v\|^2 + \beta_2 \alpha |\omega(t)||\phi||v|| + h'(\theta) \varepsilon(u, v)$$

$$\leq - \beta_1 \|v\|^2 + \frac{\beta_1 - \varepsilon}{6} \|v\|^2 + \alpha^2 |\omega(t)|^2 ||\phi||^2 + h'(\theta) \varepsilon(u, v),$$

where $\theta$ is between 0 and $v + \alpha \phi(t) - \varepsilon u$.
By (3.10) and (4.4), we get
\[ h'(\vartheta)\varepsilon(u, v) \]
\[ = h'(\vartheta)\varepsilon(u, u_t - \alpha \phi \omega(t) + \varepsilon u) \]
\[ \leq \beta_2 \varepsilon \cdot \frac{1}{2} \frac{d}{dt} \|u\|^2 + \beta_2 \varepsilon^2 \|u\|^2 + \beta_2 \varepsilon \|\omega(t)\| \|\phi\| \|\varepsilon\| u \]
\[ \leq \beta_2 \varepsilon \cdot \frac{1}{2} \frac{d}{dt} \|u\|^2 + \beta_2 \varepsilon^2 \|u\|^2 + \frac{1}{4} \varepsilon (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|u\|^2 + c \alpha^2 \|\omega(t)\|^2 \|\phi\|^2. \]

Substituting (4.4) into the third, fourth and fifth terms on the left-hand side of (4.3), we find that
\[ (\lambda + \varepsilon^2)(u, v) \]
\[ = (\lambda + \varepsilon^2)(u, u_t - \alpha \phi \omega(t) + \varepsilon u) \]
\[ \geq \frac{1}{2} (\lambda + \varepsilon^2) \frac{d}{dt} \|u\|^2 + \varepsilon (\lambda + \varepsilon^2) \|u\|^2 - (\lambda + \varepsilon^2) |\alpha| |\omega(t)| \|\phi\| \|\varepsilon\| u \]
\[ \geq \frac{1}{2} (\lambda + \varepsilon^2) \frac{d}{dt} \|u\|^2 + \varepsilon (\lambda + \varepsilon^2) \|u\|^2 - \frac{1}{4} \varepsilon (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|u\|^2 - c \alpha^2 \|\omega(t)\|^2 \|\phi\|^2, \]
\[ (Au, v) = (\Delta u, \Delta v) = (\Delta u, u_t - \alpha \phi \omega(t) \Delta \phi + \varepsilon \Delta u) \]
\[ \geq \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \varepsilon \|\Delta u\|^2 - |\alpha| |\omega(t)| \|\Delta \phi\| \|\Delta u\| \]
\[ \geq \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \frac{3 \varepsilon}{4} \|\Delta u\|^2 - \frac{\alpha^2}{\varepsilon} \|\omega(t)\|^2 \|\Delta \phi\|^2; \]
\[ \int_0^\infty \mu(s)(A\eta(s), v)ds \]
\[ = \int_0^\infty \mu(s)(\Delta^2 \eta(s), v)ds \]
\[ = \int_0^\infty \mu(s)(\Delta \eta(s), \Delta(u_t - \alpha \phi \omega(t) + \varepsilon u))ds \]
\[ = \int_0^\infty \mu(s)(\Delta \eta(s), \Delta u_t)ds - \alpha \omega(t) \int_0^\infty \mu(s)(\Delta \eta(s), \Delta \phi)ds + \varepsilon \int_0^\infty \mu(s)(\Delta \eta(s), \Delta u)ds. \]

Using the third equation of (3.11), then integrating by parts with respect to s, we get
\[ \int_0^\infty \mu(s)(\Delta \eta(s), \Delta u_t)ds \geq \frac{1}{2} \frac{d}{dt} \|\eta\|_{\mu, 2}^2 + \frac{\delta}{2} \|\eta\|_{\mu, 2}^2. \]

Using Young’s inequality, we have
\[ - \alpha \omega(t) \int_0^\infty \mu(s)(\Delta \eta(s), \Delta \phi)ds \geq - \frac{\delta}{8} \|\eta\|_{\mu, 2}^2 - \frac{2 m_0 \alpha^2}{\delta} |\omega(t)|^2 \|\Delta \phi\|^2 \]
and
\[ \varepsilon \int_0^\infty \mu(s)(\Delta \eta(s), \Delta u)ds \geq - \frac{\delta}{8} \|\eta\|_{\mu, 2}^2 - \frac{2 m_0 \varepsilon^2}{\delta} \|\Delta u\|^2. \]

Combining with (4.10)-(4.12) and (4.9), we get
\[ \int_0^\infty \mu(s)(A\eta(s), v)ds \geq \frac{1}{2} \frac{d}{dt} \|\eta\|_{\mu, 2}^2 + \frac{\delta}{4} \|\eta\|_{\mu, 2}^2 - \frac{2 m_0 \alpha^2}{\delta} |\omega(t)|^2 \|\Delta \phi\|^2 - \frac{2 m_0 \varepsilon^2}{\delta} \|\Delta u\|^2. \]
Using the Cauchy-Schwarz inequality and Young’s inequality, we have
\[ \varepsilon \alpha(\phi(t), v) \leq \varepsilon |\alpha| |\phi(t)||v||v| \leq c\alpha^2 |\phi|^2 |\phi(t)|^2 + \frac{\beta_1 - \varepsilon}{6} |v|^2, \]  
(4.14)
and
\[ (g, v) \leq \|g\|\|v\| \leq \frac{3}{2(\beta_1 - \varepsilon)} \|g\|^2 + \frac{\beta_1 - \varepsilon}{6} |v|^2. \]  
(4.15)

Let \( \tilde{F}(x, u) = \int_{\mathbb{R}^n} F(x, u)dx. \) Then for the last term on the left-hand side of (4.3) we have
\[ (f(x, u), v) = (f(x, u), u_\varepsilon - \alpha \phi(t) + \varepsilon u) = \frac{d}{dt} \tilde{F}(x, u) + \varepsilon (f(x, u), u) - \alpha (f(x, u), \phi(t)). \]  
(4.16)

By condition (3.6) we get
\[ (f(x, u), u) \geq c_2 \tilde{F}(x, u) + \int_{\mathbb{R}^n} \phi_2(x)dx. \]  
(4.17)
Following from condition (3.5) and (3.7), we obtain
\[
\alpha(f(x, u), \phi(t)) \leq |\alpha| \int_{\mathbb{R}^n} (c_1 |u|^p + \phi_1(x))|\phi(t)|dx
\]
\[ \leq |\alpha| |\phi_1(x)||\phi(t)| + c_1 |\alpha|(\int_{\mathbb{R}^n} |u|^{p+1}dx)^{\frac{1}{p+1}} |\phi(t)|
\]
\[ \leq |\alpha| |\phi_1(x)||\phi(t)| + c_1 |\alpha|(\int_{\mathbb{R}^n} (F(x, u) + \phi_3(x))dx)^{\frac{1}{p+1}} |\phi(t)|
\]
\[ \leq \frac{1}{2} |\phi_1(x)|^2 + \frac{\alpha^2}{2} |\phi|^2 |\phi(t)|^2 + \varepsilon c_2 \tilde{F}(x, u) + \frac{\varepsilon c_2}{2} \int_{\mathbb{R}^n} \phi_3(x)dx + c\alpha^2 |\phi|_{H^2}^p |\phi(t)|^{p+1}. \]  
(4.18)

By (4.17)-(4.18), we get
\[ \varepsilon (f(x, u), u) - (f(x, u), \phi(t)) \geq \frac{\varepsilon c_2}{2} \tilde{F}(x, u) + \varepsilon \int_{\mathbb{R}^n} \phi_2(x)dx - \frac{1}{2} |\phi_1(x)|^2 - \frac{\alpha^2}{2} |\phi|^2 |\phi(t)|^2
\]
\[ - \frac{\varepsilon c_2}{2} \int_{\mathbb{R}^n} \phi_3(x)dx - c\alpha^2 |\phi|_{H^2}^p |\phi(t)|^{p+1}. \]  
(4.19)
Substitute (4.5)-(4.8), (4.13)-(4.16) and (4.19) into (4.3) to obtain
\[ \frac{1}{2} \frac{d}{dt} (|v|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u|^2 + |\Delta u|^2 + |\eta|^2_{\mu,2} + 2\tilde{F}(x, u))
\]
\[ + \frac{\varepsilon}{2} |v|^2 + \frac{\varepsilon}{2} (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u|^2 + \varepsilon(\frac{3}{4} - \frac{2m_0 \epsilon}{\delta}) |\Delta u|^2 + \frac{\delta}{4} |\eta|_{\mu,2}^2 + \frac{\varepsilon c_2}{2} \tilde{F}(x, u)
\]
\[ \leq \frac{2\varepsilon - \beta_1}{2} |v|^2 + c\alpha^2 (1 + |\phi(t)|^2 + |\phi(t)|^{p+1} + c|g|^2. \]  
(4.20)
Choosing \( \varepsilon \) small enough such that \( \frac{3}{4} - \frac{2m_0 \epsilon}{\delta} \geq \frac{1}{2} \), then by (3.15) we obtain
\[ \frac{d}{dt} (|v|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u|^2 + |\Delta u|^2 + |\eta|^2_{\mu,2} + 2\tilde{F}(x, u))
\]
\[ + \sigma (|v|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u|^2 + |\Delta u|^2 + |\eta|^2_{\mu,2} + 2\tilde{F}(x, u)) \]
\[ \leq c|g|^2 + c\alpha^2 (1 + |\phi(t)|^2 + |\phi(t)|^{p+1}). \]  
(4.21)
Multiplying (4.21) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we have
\[
e^{\sigma t} (\|v(\tau, \tau - t, \omega, v_0)\|^2 + (1 + \varepsilon^2 - \beta_2 \varepsilon) \|u(\tau, \tau - t, \omega, u_0)\|^2 \\
+ \|\Delta u(\tau, \tau - t, \omega, u_0)\|^2 + \|\eta(\tau, \tau - t, \omega, \eta_0, s)\|_{\mu,2}^2 + 2\tilde{F}(x, (\tau, \tau - t, \omega, u_0))) \\
\leq e^{\sigma(t-t)} (\|v_0\|^2 + (1 + \varepsilon^2 - \beta_2 \varepsilon) \|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\tilde{F}(x, u_0)) \\
+ c \int_{\tau-t}^{\tau} e^{\sigma s} \|g(x,s)\|^2 ds + 2\alpha^2 \int_{\tau-t}^{\tau} e^{\sigma s}(1 + |\omega(s)|^2 + |\omega(s)|^{p+1}) ds.
\]
Replacing $\omega$ by $\theta_{-\tau} \omega$ in the above we obtain, for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$, and $\omega \in \Omega,$
\[
\|v(\tau, \tau - t, \theta_{-\tau} \omega, v_0)\|^2 + (1 + \varepsilon^2 - \beta_2 \varepsilon) \|u(\tau, \tau - t, \theta_{-\tau} \omega, u_0)\|^2 \\
+ \|\Delta u(\tau, \tau - t, \omega, u_0)\|^2 + \|\eta(\tau, \tau - t, \theta_{-\tau} \omega, \eta_0, s)\|_{\mu,2}^2 + 2\tilde{F}(x, u(\tau, \tau - t, \theta_{-\tau} \omega, u_0)) \\
\leq e^{\sigma(t-t)} (\|v_0\|^2 + (1 + \varepsilon^2 - \beta_2 \varepsilon) \|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\tilde{F}(x, u_0)) \\
+ c \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)} \|g(x,s)\|^2 ds + 2\alpha^2 \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}(1 + |\theta_{-\tau} \omega(s)|^2 + |\theta_{-\tau} \omega(s)|^{p+1}) ds.
\]
Again, by (3.10), we get
\[
\tilde{F}(x,u_0) \leq c(1 + \|u_0\|^2 + \|u_0\|^{p+1}).
\]
Therefore, for the first term on the right-hand side of (4.22), we have
\[
e^{-\sigma t} (\|v_0\|^2 + (1 + \varepsilon^2 - \beta_2 \varepsilon) \|u_0\|^2 + \|\Delta u_0\|^2 + \|\eta_0\|_{\mu,2}^2 + 2\tilde{F}(x, u_0)) \\
\leq c e^{-\sigma t}(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^{p+1}}^2 + \|\eta_0\|_{\mu,2}^2).
\]
Since that $(u_0, v_0, \eta_0)^T \in D(\tau - t, \theta_{-\tau} \omega)$ and $D \in D$, then we find
\[
\lim_{t \to +\infty} e^{-\sigma t}(\|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^{p+1}}^2 + \|\eta_0\|_{\mu,2}^2) = 0.
\]
Therefore, there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$,
\[
e^{-\sigma t}(1 + \|v_0\|^2 + \|u_0\|_{H^2}^2 + \|u_0\|_{H^{p+1}}^2 + \|\eta_0\|_{\mu,2}^2) \leq 1. 
\]
For the last term on the right-hand side of (4.17), we find
\[
2\alpha^2 \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}(1 + |\theta_{-\tau} \omega(s)|^2 + |\theta_{-\tau} \omega(s)|^{p+1}) ds \\
\leq 2\alpha^2 \int_{-\infty}^{0} e^{\sigma s}(1 + |\omega(s)|^2 + |\omega(s)|^{p+1}) ds \\
\leq \frac{2\alpha^2}{\sigma} + 2\alpha^2 \int_{-\infty}^{0} e^{\sigma s}(|\omega(s)|^2 + |\omega(s)|^{p+1}) ds.
\]
It is worth mentioning that $\omega(s)$ has at most linear growth at $|s| \to \infty$, which combines (3.19), we can have
\[
2\alpha^2 \int_{\tau-t}^{\tau} e^{\sigma(s-\tau)}(1 + |\theta_{-\tau} \omega(s)|^2 + |\theta_{-\tau} \omega(s)|^{p+1}) ds \to \frac{2\alpha^2}{\sigma}, \quad (t \to \infty).
\]
In order to complete the proof, we still need to estimate the fourth term on the left-hand side of (4.22). Thanks to (3.7), we obtain that, for all $t \geq 0$,

$$-2F(x, u(t, \tau - t, \theta_{-\tau} \omega, u_0)) \leq 2 \int_{\mathbb{R}^n} \phi_0 dx.$$  \hspace{1cm} (4.26)

Then it follows from (4.23)-(4.26), we get (4.2).

The following lemma will be used to show the uniform estimates of solutions as well as to establish pullback asymptotic compactness.

**Lemma 4.2.** Under Assumptions I and II, for every $\tau \in \mathbb{R}, \omega \in \Omega, D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D},$ there exists $T = T(\tau, \omega, D) > 0$ such that for all $t \geq T$ the solution of problem (3.11)-(3.12) satisfies

$$\|A^{1/2} \varphi^{(\alpha)}(\tau, t, \theta_{-\tau} \omega, \varphi_0^{(\alpha)})\|_{L^2}^2 \leq R_2(\alpha, \tau, \omega),$$

and $R_2(\alpha, \tau, \omega)$ is given by

$$R_2(\alpha, \tau, \omega) = e^{-\sigma t}(\|A^{1/2} u_0\|^2 + \|A^{1/2} \nu_0\|^2 + \|A^{1/2} \eta_0\|^2) + c \int_{-\infty}^{\tau} e^{\sigma(s-t)} \|g(x,s)\|^2_1 ds + \alpha^2 \int_{-\infty}^{0} e^{\sigma s}(1 + |\omega(s)|^2) ds.$$  \hspace{1cm} (4.27)

**Proof.** Taking the inner product of the second equation of (3.11) with $A^{1/2} v$ in $L^2(\mathbb{R}^n)$, we find that

$$\frac{1}{2} \frac{d}{dt} \|A^{1/2} v\|^2 - \varepsilon \|A^{1/2} v\|^2 + (\lambda + \varepsilon^2)(u, A^{1/2} v) + (Au, A^{1/2} v)$$

$$= (g(x, t), A^{1/2} v) - (h(v + \alpha \omega(t) - \varepsilon u), A^{1/2} v) + \varepsilon \alpha(\phi, A^{1/2} v) \omega(t).$$

Similar to the proof of Lemma 4.1, we have the following estimates:

$$-(h(v + \alpha \omega(t) - \varepsilon u), A^{1/2} v)$$

$$\leq -\beta_1 \|A^{1/2} v\|^2 + \frac{\beta_1 - \varepsilon}{6} \|A^{1/2} v\|^2 + \alpha^2 \|\omega(t)\|^2 \|A^{1/2} \phi\|^2 + h'(\vartheta)\varepsilon(u, A^{1/2} v),$$  \hspace{1cm} (4.29)

$$h'(\vartheta)\varepsilon(u, A^{1/2} v)$$

$$\leq \beta_2 \varepsilon \cdot \frac{1}{2} \frac{d}{dt} \|A^{1/2} u\|^2 + \beta_2 \varepsilon^2 \|A^{1/2} u\|^2 + \frac{1}{6} \varepsilon (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|A^{1/2} u\|^2 + \varepsilon \alpha^2 \|\omega(t)\|^2 \|A^{1/2} \phi\|^2,$$  \hspace{1cm} (4.30)

$$\|A^{1/2} u\|$$

$$\geq \frac{\lambda + \varepsilon^2}{2} \frac{d}{dt} \|A^{1/2} u\|^2 + \delta (\lambda + \varepsilon^2) \|A^{1/2} u\|^2 - \frac{1}{6} \varepsilon (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|A^{1/2} u\|^2 - \varepsilon \alpha^2 \|\omega(t)\|^2 \|A^{1/2} \phi\|^2,$$  \hspace{1cm} (4.31)

$$\|A^{1/2} v\|$$

$$\geq \frac{\lambda + \varepsilon^2}{2} \frac{d}{dt} \|A^{1/2} v\|^2 + \frac{3}{4} \|A^{1/2} u\|^2 - \frac{\alpha^2}{\varepsilon} \|\omega(t)\|^2 \|A^{1/2} \phi\|^2,$$  \hspace{1cm} (4.32)

$$\alpha \vartheta(u, A^{1/2} v)$$

$$\leq \frac{1}{2} \frac{d}{dt} \|A^{1/2} \eta\|^2_{\mu, 2} + \frac{\delta}{4} \|A^{1/2} \eta\|^2_{\mu, 2} - \frac{2m \alpha^2}{\delta} \|\omega(t)\|^2 \|A^{1/2} \phi\|^2 - \frac{2m \varepsilon \alpha^2}{\delta} \|A^{1/2} u\|^2,$$  \hspace{1cm} (4.33)

$$\varepsilon \alpha(\phi \omega(t), A^{1/2} v) \leq \alpha^2 \|A^{1/2} \phi\|^2 |\omega(t)|^2 + \frac{\beta_1 - \varepsilon}{6} \|A^{1/2} v\|^2,$$  \hspace{1cm} (4.34)
Choosing \( \varepsilon \) small enough such that \( \frac{3}{4} - \frac{2m_0}{G} \geq \frac{1}{2} \), then by (3.15) we obtain
\[
\frac{d}{dt}(\|A^\frac{1}{2}v\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)\|A^\frac{1}{2}u\|^2 + \|A^\frac{1}{2}\eta\|^2_{\mu,2})
+ \sigma(\|A^\frac{1}{2}v\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)\|A^\frac{1}{2}u\|^2 + \|A^\frac{1}{2}\eta\|^2_{\mu,2})
\leq c\|g\|^2 + c|\omega(t)|^2.
\]
Lemma 4.3. Under Assumptions I and II, for every $\zeta > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D$, there exists $T = T(\tau, \omega, D, \zeta) > 0$, $K = K(\tau, \omega, \zeta) \geq 1$ such that for all $t \geq T$, $k \geq K$, the solution of problem (3.11)-(3.12) satisfies

$$\|\varphi^{(n)}(\tau, \tau - t, \theta_{-\tau} \omega, \varphi^{(n)}_0)\|_{L^2(\mathbb{R}^n \setminus \mathbb{B}_k)}^2 \leq \zeta,$$

where for $k \geq 1$, $\mathbb{B}_k = \{x \in \mathbb{R}^n : |x| \leq k\}$ and $\mathbb{R}^n \setminus \mathbb{B}_k$ is the complement of $\mathbb{B}_k$.

Proof. Choose a smooth function $\rho$, such that $0 \leq \rho \leq 1$ for $s \in \mathbb{R}$, and

$$\rho(s) = \begin{cases} 0, & \text{if } 0 \leq |s| \leq 1, \\ 1, & \text{if } |s| \geq 2, \end{cases}$$

(4.41)

and there exist constants $\mu_1$, $\mu_2$, $\mu_3$, $\mu_4$ such that $|\rho'(s)| \leq \mu_1$, $|\rho''(s)| \leq \mu_2$, $|\rho'''(s)| \leq \mu_3$, $|\rho''''(s)| \leq \mu_4$ for $s \in \mathbb{R}$. Taking the inner product of the second equation of (3.11) with $\rho(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^n)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx - \varepsilon \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + (\lambda + \varepsilon^2) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})uv dx$$

$$+ \int_{\mathbb{R}^n} (Au)v \rho(\frac{|x|^2}{k^2})dx + \int_{\mathbb{R}^n} -\mu(s)A\eta(s)\rho(\frac{|x|^2}{k^2})v ds + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x, u) dx$$

$$\varepsilon \alpha \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\phi \omega(t) dx - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(h(v + \alpha \phi \omega(t) - \varepsilon u) v dx$$

$$+ \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})g(x, t) dx.$$ 

(4.42)

Similar to (4.5), we have

$$- \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})(h(v + \alpha \phi \omega(t) - \varepsilon u) v dx$$

$$\leq - \beta_1 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + h'(\sigma)\varepsilon \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v dx + \beta_2 |\alpha| \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\phi||\omega(t)||v| dx.$$ 

(4.43)

Taking (4.43) into (4.42), we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx - (\varepsilon - \beta_1) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + \int_{\mathbb{R}^n} (Au)v \rho(\frac{|x|^2}{k^2})dx$$

$$+ (\lambda + \varepsilon^2 - h'(\sigma))\varepsilon \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v dx$$

$$+ \int_{\mathbb{R}^n} \mu(s)A\eta(s)\rho(\frac{|x|^2}{k^2})v ds + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(x, u) dx$$

Thus the proof is completed. 

Next we conduct uniform estimates on the tail parts of the solutions for large space variables when time is sufficiently large in order to prove the pullback asymptotic compactness of the cocycle associated with Eqs. (3.11)-(3.12) on the unbounded domain $\mathbb{R}^n$. 

$$\leq ce^{-\omega t} (|A^1_1v_0|^2 + |A^2_2u_0|^2 + |A^3_3\eta_0|^2_{\mu, 2})$$

$$+ c \int_{\mathbb{R}^n} e^{\alpha(s+\tau)}|g(x, s)|^2 ds + c\alpha^2 \int_{-\infty}^0 e^{\alpha s}(1 + |\omega(s)|^2) ds.$$ 

(4.39)
\[
\leq (\varepsilon + \beta_2) |\alpha| \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\omega(t)||v|dx + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) g(x, t)vdx \\
\leq \frac{\beta_1 - \varepsilon}{3} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v|^2 dx + c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\phi||\omega(t)||v|dx + \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) g(x, t)vdx.
\]

(4.44)

For the fourth term on the left-hand side of (4.44), we have

\[
(\lambda + \varepsilon^2 - h'(\vartheta))\varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|dx
\]

\[
= (\lambda + \varepsilon^2 - h'(\vartheta))\varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( \frac{1}{2} \frac{du}{dt} + \varepsilon u - \alpha \omega(t) u \right) dx
\]

\[
= (\lambda + \varepsilon^2 - h'(\vartheta))\varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( \frac{1}{2} \frac{du}{dt} + \varepsilon u^2 - \alpha \omega(t) u \right) dx
\]

\[
\geq (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \left( \frac{1}{2} \frac{du}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|^2 dx + \varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|^2 dx \right)
\]

\[
- (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |\alpha| \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\phi| |\omega(t)||u|dx
\]

\[
\geq (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \left( \frac{1}{2} \frac{du}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|^2 dx + \varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|^2 dx \right)
\]

\[
- \frac{\varepsilon}{2} (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|^2 dx - c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\phi|^2 |\omega(t)||v|^2 dx.
\]

(4.45)

For the third term on the left-hand side of (4.44), we have

\[
\int_{\mathbb{R}^n} (Au) \rho \left( \frac{|x|^2}{k^2} \right) vdx
\]

\[
= \int_{\mathbb{R}^n} (Au) \rho \left( \frac{|x|^2}{k^2} \right) \left( \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) dx
\]

\[
= \int_{\mathbb{R}^n} (\Delta^2 u) \rho \left( \frac{|x|^2}{k^2} \right) \left( \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) dx
\]

\[
= \int_{\mathbb{R}^n} (\Delta u) \Delta \left( \rho \left( \frac{|x|^2}{k^2} \right) \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) dx
\]

\[
= \int_{\mathbb{R}^n} (\Delta u) \left( \frac{2}{k^2} \rho \left( \frac{|x|^2}{k^2} \right) \nabla \left( \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) + \rho \left( \frac{|x|^2}{k^2} \right) \Delta \left( \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) \nabla \right) dx
\]

\[
+ 2 \cdot \frac{2}{k^2} \rho \left( \frac{|x|^2}{k^2} \right) \nabla \left( \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) + \rho \left( \frac{|x|^2}{k^2} \right) \Delta \left( \frac{du}{dt} + \varepsilon u - \alpha \omega(t) \right) dx
\]
+ \varepsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \Delta u^2 \, dx - |\alpha| \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} |\Delta u| |\Delta \phi| |\omega(t)| \, dx \\
\geq - \frac{\mu_1}{k^2} + 4\mu_2 \left( \|\Delta u\|^2 + \|v\|^2 \right) \frac{2\sqrt{2}\mu_1}{k} \left( \|\Delta u\|^2 + \|v\|^2 \right) + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} |\Delta u|^2 \, dx \\
+ \varepsilon \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} |\Delta u|^2 \, dx - \frac{\varepsilon}{4} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} |\Delta u|^2 \, dx - c \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} |\Delta \phi|^2 |\omega(t)| \, dx. \quad (4.46)

For the fifth term on the left-hand side of (4.44), we have

\[
\int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \, v \, ds \, dx \\
= \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \left( \frac{du}{dt} + \varepsilon u - \alpha \phi(t) \right) \, ds \, dx \\
= \int_{\mathbb{R}^n} \left( \int_0^\infty \mu(s)A(s) \Delta \left( \rho \frac{|x|^2}{k^2} \left( \frac{du}{dt} + \varepsilon u - \alpha \phi(t) \right) \right) \, ds \right) \, dx \\
= \int_{\mathbb{R}^n} \left( \int_0^\infty \mu(s)A(s) \Delta \left( \rho \frac{|x|^2}{k^2} \left( \frac{du}{dt} + \varepsilon u - \alpha \phi(t) \right) \right) \, ds \right) \, dx \\
+ 2 \cdot \frac{2|\varepsilon|}{k^2} \rho \frac{|x|^2}{k^2} \left( \frac{du}{dt} + \varepsilon u - \alpha \phi(t) \right) \Delta \left( \frac{du}{dt} - \varepsilon u - \alpha \phi(t) \right) \, ds \, dx \\
\geq - \int_{k < x < \sqrt{2}k} \frac{2\mu_1}{k^2} + 4\mu_2 \left( \frac{2|\varepsilon|}{k^2} \rho \frac{|x|^2}{k^2} \right) \int_0^\infty \mu(s)A(s) \Delta \left( \rho \frac{|x|^2}{k^2} \right) \, ds \, dx \\
- \int_{k < x < \sqrt{2}k} 4\mu_2 \rho \frac{|x|^2}{k^2} \mu(s)A(s) \nabla v \, ds \, dx + \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx \\
+ \varepsilon \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx - |\alpha| \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta \phi \, ds \, dx \\
\geq - \frac{2\mu_1 + 8\mu_2}{k^2} \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx - \frac{4\sqrt{2}\mu_1}{k} \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \nabla v \, ds \, dx \\
+ \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx + \varepsilon \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx \\
- |\alpha| \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta \phi \, ds \, dx. \quad (4.47)

Using Young’s inequality, we get

\[
- \frac{2\mu_1 + 8\mu_2}{k^2} \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx \geq - \frac{\mu_1}{k^2} + 4\mu_2 \left( \|\eta\|_{L^2}^2 + m_0 \|\eta\|^2 \right), \quad (4.48)
\]

and

\[
- \frac{4\sqrt{2}\mu_1}{k} \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \nabla v \, ds \, dx \geq - \frac{2\sqrt{2}\mu_1}{k} \left( \|\eta\|_{L^2}^2 + m_0 \|\eta\|^2 \right). \quad (4.49)
\]

Integrating by parts with respect to s and using (3.4), we obtain

\[
\int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx \\
= \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta (\eta_0 + \eta_s) \, ds \, dx \\
\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \|\eta(t)\|_{L^2}^2 \, dx + \frac{\delta}{2} \int_{\mathbb{R}^n} \rho \frac{|x|^2}{k^2} \|\eta(t)\|_{L^2}^2 \, dx. \quad (50.50)
\]

\[
\varepsilon \int_{\mathbb{R}^n} \int_0^\infty \mu(s)A(s) \rho \frac{|x|^2}{k^2} \Delta u \, ds \, dx
\]
By (4.44)-(4.56), we have

\[
\begin{align*}
&\geq -\frac{\delta}{8} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\eta(s)|^2_{\mu,2} dx - \frac{2m_0\varepsilon^2}{\delta} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\Delta u|^2 dx. \\
&\quad - |\alpha| \int_{\mathbb{R}^n} \int_0^\infty \mu(s)|\Delta \eta(s)| \rho \left( \frac{|x|^2}{k^2} \right) |\Delta \phi| \omega(t) ds dx \\
&\quad \geq -\frac{\delta}{8} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\eta(s)|^2_{\mu,2} dx - \frac{2m_0\alpha^2}{\delta} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\Delta \phi|^2 |\omega(t)|^2 dx. \\
&\quad (4.51)
\end{align*}
\]

For the sixth term on the left-hand side of (4.44), we have

\[
\begin{align*}
&\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) f(x, u) v dx \\
&= \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) f(x, u) \left( \frac{du}{dt} + \varepsilon u - \alpha \phi \omega(t) \right) dx \\
&= \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) F(x, u) u dx + \varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) f(x, u) u dx - \alpha \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) f(x, u) \phi \omega(t) dx. \\
&\quad (4.53)
\end{align*}
\]

Similar to (4.17) and (4.18) in Lemma 4.1, we have

\[
\begin{align*}
&\alpha \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) f(x, u) \phi \omega(t) dx \\
&\quad \leq \frac{1}{2} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\phi_1|^2 dx + \frac{\alpha^2}{2} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\phi|^2 |\omega(t)|^2 dx \\
&\quad + \frac{\varepsilon c_2}{2} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |F(x, u) + \phi_3| dx + c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\phi|^{p+1} |\omega(t)|^{p+1} dx. \\
&\quad (4.55)
\end{align*}
\]

\[
\begin{align*}
&\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) g(x, t) v dx \leq \frac{3}{2(\beta_1 - \varepsilon)} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |g(x, t)|^2 dx + c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v|^2 dx. \\
&\quad (4.56)
\end{align*}
\]

By (4.44)-(4.56), we have

\[
\begin{align*}
&\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v|^2 dx + \left( \lambda + \varepsilon^2 - \beta_2 \varepsilon \right) |u|^2 + |\Delta u|^2 + |\eta(s)|^2_{\mu,2} + 2F(x, u) dx \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v|^2 dx + \frac{\varepsilon}{2} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |u|^2 dx \\
&\quad + \varepsilon \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\Delta u|^2 dx + \frac{\delta}{4} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |\eta(s)|^2_{\mu,2} dx \\
&\quad + \frac{\varepsilon c_2}{2} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) F(x, u) dx \\
&\quad \leq \frac{\mu_1 + 4\mu_2}{k^2} (|\Delta u|^2 + |v|^2) dx + \frac{2\sqrt{2}\mu_1}{k} (|\Delta u|^2 + |\nabla v|^2) dx \\
&\quad + \left( \frac{\mu_1 + 4\mu_2}{k^2} + \frac{2\sqrt{2}\mu_1}{k} \right) (|\eta(s)|^2_{\mu,2} + m_0 |u|^2) \\
&\quad + c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) (|\phi_1|^2 + |\phi_2| + |\phi_3| + |g|^2 + |\omega(t)|^{p+1} |\phi|^{p+1}) dx \\
&\quad + c |\omega(t)|^2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) (|\phi|^2 + |\Delta \phi|^2) dx. \\
&\quad (4.57)
\end{align*}
\]
Since that $\phi \in H^2(\mathbb{R}^n)$, $\phi_1 \in L^2(\mathbb{R}^n)$, $\phi_2 \in L^1(\mathbb{R}^n)$, $\phi_3 \in L^1(\mathbb{R}^n)$, we obtain that there exists $K_1 = K_1(\tau, \zeta) \geq 1$ such that for all $k \geq K_1$,

\[
\begin{align*}
&c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |\phi_1|^2 + |\phi_2| + |\phi_3| + |\omega(t)|^{p+1} |\phi|^{p+1} \right) dx \\
&\quad + c|\omega(t)|^2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) (|\phi|^2 + |\Delta \phi|^2) dx \\
&= c \int_{|x| \geq k} \rho \left( \frac{|x|^2}{k^2} \right) \left( |\phi_1|^2 + |\phi_2| + |\phi_3| + |\omega(t)|^{p+1} |\phi|^{p+1} \right) dx \\
&\quad + c|\omega(t)|^2 \int_{|x| \geq k} \rho \left( \frac{|x|^2}{k^2} \right) (|\phi|^2 + |\Delta \phi|^2) dx \\
&\leq c \int_{|x| \geq k} \left( |\phi_1|^2 + |\phi_2| + |\phi_3| + |\omega(t)|^{p+1} |\phi|^{p+1} \right) dx + c|\omega(t)|^2 \int_{|x| \geq k} \left( |\phi|^2 + |\Delta \phi|^2 \right) dx \\
&= c\zeta^2 (1 + |\omega(t)|^2 + |\omega(t)|^{p+1}) \tag{4.58}
\end{align*}
\]

By (3.17), we have

\[
\begin{align*}
&c \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) g^2(x, t) dx \\
&\quad \leq c \int_{|x| \geq k} g^2(x, t) dx. \tag{4.59}
\end{align*}
\]

Choosing $\varepsilon$ small enough such that \( \frac{2}{3} - \frac{2mu_2}{s} \geq \frac{1}{2} \), then by (3.16), (4.58) and (4.59), we have that for all $k \geq K_1$,

\[
\begin{align*}
&\frac{d}{dt} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) (|v|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u|^2 + |\Delta u|^2 + |\eta(s)|_{\mu,2}^2 + 2F(x, u, \omega, \eta)) dx \\
&\quad + \sigma \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) (|v|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u|^2 + |\Delta u|^2 + |\eta(s)|_{\mu,2}^2 + 2F(x, u)) dx \\
&\leq 2\mu_1 + 8\mu_2 \left( (|\Delta u|^2 + |v|^2) dx + \frac{4\sqrt{2}\mu_1}{k} (|\Delta u|^2 + \|v\|^2) dx \\
&\quad + \frac{2\mu_1 + 8\mu_2}{k^2} + \frac{4\sqrt{2}\mu_1}{k} (|\eta|^2_{\mu,2} + m_0 |v|^2) \\
&\quad + c\zeta (1 + |\omega(t)|^2 + |\omega(t)|^{p+1}) + c \int_{|x| \geq k} g^2(x, t) dx. \tag{4.60}
\end{align*}
\]

Multiplying (4.60) by $e^{\sigma t}$ and then integrating over $(\tau - t, \tau)$, we find

\[
\begin{align*}
&\int_{\tau}^{\tau+t} \rho \left( \frac{|x|^2}{k^2} \right) (|v(\tau, \tau - t, \omega, \eta_0)|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u(\tau, \tau - t, \omega, u_0)|^2 \\
&\quad + |\Delta u(\tau, \tau - t, \omega, u_0)|^2 + |\eta(\tau, \tau - t, \omega, \eta_0)|^2_{\mu,2} + 2F(x(\tau, \tau - t, \omega, u_0))) dx \\
&\leq e^{-\sigma t} \int_{\tau}^{\tau+t} \rho \left( \frac{|x|^2}{k^2} \right) \left( |u_0|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon)|u_0|^2 + |\Delta u_0|^2 + |\eta_0|_{\mu,2}^2 + 2F(x, u_0) \right) dx \\
&\quad + \frac{2\mu_1 + 8\mu_2}{k^2} \int_{\tau-t}^{\tau} e^{\sigma(s-t)} \left( |\Delta u(s, \tau - t, \omega, u_0)|^2 + |v(s, \tau - t, \omega, v_0)|^2 \right) ds \\
&\quad + \frac{4\sqrt{2}\mu_1}{k} \int_{\tau-t}^{\tau} e^{\sigma(s-t)} \left( |\Delta u(s, \tau - t, \omega, u_0)|^2 + |\eta(s, \tau - t, \omega, \eta_0)|^2_{\mu,2} \right) ds \\
&\quad + \left( \frac{2\mu_1 + 8\mu_2}{k^2} + \frac{4\sqrt{2}\mu_1}{k} \right) \int_{\tau-t}^{\tau} e^{\sigma(s-t)} \left( |\eta(s, \tau - t, \omega, \eta_0)|^2_{\mu,2} \right) ds
\end{align*}
\]
Following from (4.62)-(4.63), Lemma 4.1 and Lemma 4.2 that there exists 
\[ \leq (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |u(\tau, t, \theta_{\tau}, \omega)|^2 + |\Delta u(\tau, t, \theta_{\tau}, \omega, u_0)|^2 \]
+ |\Delta u(\tau, t, \theta_{\tau}, \omega, u_0)|^2 + |\eta(\tau, t, \theta_{\tau}, \eta_0, s)|^2_{\mu,2} \]
+ 2 F(x, u(\tau, t, \theta_{\tau}, \omega, u_0)) dx \] 

Replacing \( \omega \) by \( \theta_{\tau}\omega \), it then follows from (4.61) that 
\[
\int_{\mathbb{R}^n} \rho\frac{|x|^2}{k^2} \left( |v(\tau, \tau - t, \theta_{\tau}, \omega, v_0)|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |u(\tau, t, \theta_{\tau}, \omega)|^2 \right) dx 
\]
+ \[ |\Delta u(\tau, t, \theta_{\tau}, \omega, u_0)|^2 + |\eta(\tau, t, \theta_{\tau}, \eta_0, s)|^2_{\mu,2} \]
+ 2 F(x, u(\tau, t, \theta_{\tau}, \omega, u_0)) dx \]

\[
\leq c_\varepsilon + e^{-\sigma t} \int_{\mathbb{R}^n} \rho\frac{|x|^2}{k^2} \left( |v(\tau, \tau - t, \theta_{\tau}, \omega, v_0)|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |u(\tau, t, \theta_{\tau}, \omega)|^2 \right) dx 
\]
+ \[ |\Delta u(\tau, t, \theta_{\tau}, \omega, u_0)|^2 + |\eta(\tau, t, \theta_{\tau}, \eta_0, s)|^2_{\mu,2} \]
+ 2 F(x, u(\tau, t, \theta_{\tau}, \omega, u_0)) dx 
\]

By (3.18), we see that there exists \( K_2 = K_2(\tau, \zeta) \geq K_1 \) such that for all \( k \geq K_2 \), 
\[
\int_{\mathbb{R}^n} \rho\frac{|x|^2}{k^2} \left( |v(\tau, \tau - t, \theta_{\tau}, \omega, v_0)|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |u(\tau, t, \theta_{\tau}, \omega)|^2 \right) dx 
\]
+ \[ |\Delta u(\tau, t, \theta_{\tau}, \omega, u_0)|^2 + |\eta(\tau, t, \theta_{\tau}, \eta_0, s)|^2_{\mu,2} \]
+ 2 F(x, u(\tau, t, \theta_{\tau}, \omega, u_0)) dx \]

Following from (4.62)-(4.63), Lemma 4.1 and Lemma 4.2 that there exists \( T_1 = T_1(\tau, \omega, D, \zeta) > 0 \) such that for all \( t \geq T_1, k \geq K_2 \), 
\[
\int_{\mathbb{R}^n} \rho\frac{|x|^2}{k^2} \left( |v(\tau, \tau - t, \theta_{\tau}, \omega, v_0)|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |u(\tau, t, \theta_{\tau}, \omega)|^2 \right) dx 
\]
+ \[ |\Delta u(\tau, t, \theta_{\tau}, \omega, u_0)|^2 + |\eta(\tau, t, \theta_{\tau}, \eta_0, s)|^2_{\mu,2} \]
+ 2 F(x, u(\tau, t, \theta_{\tau}, \omega, u_0)) dx \]
\[
\leq c(1 + \int_{-\infty}^{0} e^{\sigma s}(|\omega(s)|^2 + |\omega(s)|^{p+1}) ds + \int_{-\infty}^{T} \int_{|x| \geq k} e^{\sigma(s-\tau)} g^2(x,s) dx ds),
\]
where \((u_0, v_0, \eta_0)^T \in D(\tau - t, \theta - \omega)\).

Note that (3.7) holds, then we find
\[
-2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) F(x,u) dx \leq 2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \phi_3 dx \leq 2 \int_{|x| \geq k} \rho \left( \frac{|x|^2}{k^2} \right) \phi_3 dx,
\]
which along with \(\phi_3 \in L^1(\mathbb{R}^n)\), we obtain that there exists \(K_3 = K_3(\tau, \zeta) \geq K_2\) such that for all \(k \geq K_3\),
\[
-2 \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) F(x,u) dx \leq \zeta. \tag{4.65}
\]

Then from (4.64)-(4.65), we get that there exists \(T_2 = T_2(\tau, \omega, D, \zeta) > T_1\) such that for all \(t \geq T_2\) and \(k \geq K_3\),
\[
\left| \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) \left( |v(\tau,\tau-t,\theta-\omega,v_0)|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) |u(\tau,\tau-t,\theta-\omega,u_0)|^2 \right. \right.
\]
\[
+ |\Delta u(\tau,\tau-t,\theta-\omega,u_0)|^2 + |\eta(\tau,\tau-t,\theta-\omega,\eta_0,\eta)|^2_{\mu,2} \right| \leq c(1 + \int_{-\infty}^{0} e^{\sigma s}(|\omega(s)|^2 + |\omega(s)|^{p+1}) ds + \int_{-\infty}^{T} \int_{|x| \geq k} e^{\sigma(s-\tau)} g^2(x,s) dx ds), \tag{4.66}
\]
which completes the proof. \(\Box\)

In order to obtain the precompactness of the solutions for (3.11)-(3.12) in bounded domain \(B_{2k}\) later, we decompose \(\varphi^\alpha = (u,v,\eta)^T\) of (3.11)-(3.12) into \(\varphi^{(\alpha)} = \varphi^{(\alpha)}_L + \varphi^{(\alpha)}_N\), where \(\varphi^{(\alpha)}_L = (u_L,\xi_L,\eta_L)^T\) and \(\varphi^{(\alpha)}_N = (u_N,\xi_N,\eta_N)^T\) solve, respectively,
\[
\begin{align*}
\frac{du_L}{dt} + \varepsilon u_L &= \xi_L, \\
\frac{d\xi_L}{dt} - \varepsilon \xi_L + \lambda u_L + f_0^\infty \mu(s) A\xi_L(s) ds &= 0, \\
\eta_L + \eta_L, s &= u_L, t, \\
u(x,\tau) &= u_0(x), \quad v(x,\tau) = v_0(x), \\
\eta(x,\tau, s) &= \eta_0(x, s) = u(x, \tau) - u(x, \tau - s)
\end{align*}
\tag{4.67}
\]
and
\[
\begin{align*}
\frac{du_N}{dt} + \varepsilon u_N &= v_N + \alpha f_0^\infty \mu(s) A\eta_N(s) ds + f(x,u), \\
\frac{dv_N}{dt} - \varepsilon v_N + \h(u_{N,t}) + (\lambda + \varepsilon^2 + A) u_N + f_0^\infty \mu(s) A\eta_N(s) ds &= g(x,t) + \varepsilon \alpha \varphi(t), \\
\eta_{N,t} + \eta_{N,s} &= u_{N,t}.
\end{align*}
\tag{4.68}
\]

For the solutions of equations (4.67) and (4.68), by Lemma 4.1 and Lemma 4.2, we can easily get the following estimates and regularity results, respectively.

**Lemma 4.4.** Assume that (3.4) and (3.10) hold. Then for any \((u_L,\xi_L,\eta_L)^T\) of the solution of (4.67) satisfies
\[
\|\varphi^{(\alpha)}_L(\tau,\tau-t,\theta-\omega,\varphi^{(\alpha)}_L)|_{E(\mathbb{B}_{2k})}^2 \to 0, \quad \text{when} \quad t \to \infty. \tag{4.69}
\]
5. Random attractors. In this section, we prove existence and uniqueness of D-pullback attractors for the stochastic system (3.11)-(3.12). First we also need the following results to prove the asymptotic compactness about memory term as well as the existence of random attractors.

**Lemma 5.1.** ([17]) Let $X_0$, $X$, $X_1$ be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, the first injection being compact. Let $Y \subset L^2_\mu(\mathbb{R}^+, X)$ satisfy the following hypotheses:

(i) $Y$ is bounded in $L^2_\mu(\mathbb{R}^+, X_0) \cap H^1_\mu(\mathbb{R}^+, X_1)$;

(ii) $\sup_{\eta \in Y} \|\eta(s)\|^2_X \leq K_0$, $\forall \ s \in \mathbb{R}^+$ for some $K_0 > 0$.

Then $Y$ is relatively compact in $L^2_\mu(\mathbb{R}^+, X)$.

Note that for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $t_m \geq 0$,

$$
\eta_N(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega),s) = \begin{cases}
\left\{ u_N(\tau,\tau-t_m,\theta_{-\tau}\omega,u_N(\theta_{-t_m}\omega)) \right. & \text{for } s \leq t, \\
-u_N(\tau-s,\tau-t_m,\theta_{-\tau+s}\omega,u_N(\theta_{-t_m+s}\omega)) & \text{for } s \geq t,
\end{cases}
$$

(5.1)

and

$$
\eta_{N,s}(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega)) = \begin{cases}
u_N,s(\tau-s,\tau-t_m,\theta_{-\tau+s}\omega,u_N(\theta_{-t_m+s}\omega)) & \text{for } s \leq t, \\
0 & \text{for } s \geq t.
\end{cases}
$$

(5.2)

Then from Lemma 4.5 and (5.1)-(5.2), it follows that

$$
\max\{\|\eta_N(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega),s)\|^2_{\mu,1},\|\eta_N(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega),s)\|_{\mu,2}\}
$$

$$
\leq 2R_2(\alpha, \tau, \omega), \forall \ s \geq 0,
$$

(5.3)

this implies that $\{\eta_N(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega),s)\}_{m=1}^{\infty}$ is bounded in $L^2_\mu(\mathbb{R}^+, V_3) \cap H^1_\mu(\mathbb{R}^+, V_1)$. For brevity, we denote $\hat{B}(\tau, \omega) = \{\eta_N(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega),s)\}_{m=1}^{\infty}$. Again, by Lemma 4.1, Lemma 4.5 and (5.1), we have

$$
\sup_{\eta \in \hat{B}(\tau, \omega), s \geq 0} \|\triangle \eta(s)\|^2 = \sup_{t_m \geq 0} \sup_{\varnothing(\theta_{-t_m}\omega) \in D(\tau-t_m, \theta_{-\tau}\omega)} \|\triangle \eta_N(\tau,\tau-t_m,\theta_{-\tau}\omega,\eta_N(\theta_{-t_m}\omega),s)\|^2
$$

$$
\leq 2R_1(\alpha, \tau, \omega).
$$

(5.4)

Thus, by (3.4) and (5.4), it follows that for any $\eta \in \hat{B}(\tau, \omega)$,

$$
\|\eta(s)\|^2_{\mu,2} = \int_0^{+\infty} \mu(s)\|\triangle \eta(s)\|^2ds \leq 2R_2(\alpha, \tau, \omega) \int_0^{+\infty} e^{-\delta s}ds \leq \frac{2R_1(\alpha, \tau, \omega)}{\delta},
$$

(5.5)
which shows that \( \{ \eta_N(\tau, \tau - t_m, \theta_{-\tau}, \eta_{N,0}(\theta_{-t_m}, \omega), s) \}_{m=1}^{\infty} \subset L^2(\mathbb{R}^+, H^2(\mathbb{B}_{2k_1})) \) is a bounded subset. By Lemma 4.4 and 5.1, we know that the sequence \( \{ \eta(\tau, \tau - t_m, \theta_{-\tau}, \eta_0(\theta_{-t_m}, \omega), s) \}_{m=1}^{\infty} \) is compact in \( L^2(\mathbb{R}^+, H^2(\mathbb{B}_{2k_1})) \).

Then we apply the lemmas shown in Section 4 to prove the asymptotic compactness of solutions of (3.11)-(3.12) in \( E \).

**Lemma 5.2.** Under Assumptions I and II, for every \( \tau \in \mathbb{R}, \omega \in \Omega \), the sequence of solutions of (3.11)-(3.12), \( \{ \varphi^{(\alpha)}(\tau, \tau - t_m, \theta_{-\tau}, \varphi^{(\alpha)}_0) \}_{m=1}^{\infty} \) has a convergent subsequence in \( E \) whenever \( t_m \to \infty \) and \( \varphi^{(\alpha)}_0 \in D(\tau - t_m, \theta_{-t_m}) \) with \( D \in \mathcal{D} \).

**Proof.** By Lemma 4.3, for every \( \zeta > 0 \), there exist \( k_0 = k_0(\tau, \omega, \zeta) \geq 1 \) and \( m_2 = m_2(\tau, \omega, D, \zeta) \geq m_1 \) such for all \( m \geq m_1 \),

\[
|| \varphi^{(\alpha)}(\tau, \tau - t_m, \theta_{-\tau}, \varphi^{(\alpha)}_0) ||_{E(\mathbb{R}^n \cup \mathbb{B}_{k_0})} \leq \zeta, (5.6)
\]

By Lemma 4.5, there exist \( k_1 = k_1(\tau, \omega) \geq k_0 \), such that

\[
\begin{align*}
&\| A_1^{1/2} u_N(\tau, \tau - t_m, \theta_{-\tau}, u_N, 0(\theta_{-t_m}, \omega)) \|_{H^2(\mathbb{B}_{2k_1})}^2 \\
+ &\| A_1^{1/2} \nu_N(\tau, \tau - t_m, \theta_{-\tau}, \nu_N, 0(\theta_{-t_m}, \omega)) \|_{L^2(\mathbb{B}_{2k_1})}^2 \\
\leq & R_2(\alpha, \tau, \omega),
\end{align*}
\]

which along with the compact embedding \( H^3(\mathbb{B}_{2k_1}) \times H^1(\mathbb{B}_{2k_1}) \hookrightarrow H^2(\mathbb{B}_{2k_1}) \times L^2(\mathbb{B}_{2k_1}) \), we know the sequences \( \{ (u_N(\tau, \tau - t_m, \theta_{-\tau}, u_N, 0(\theta_{-t_m}, \omega)), \nu_N(\tau, \tau - t_m, \theta_{-\tau}, \nu_N, 0(\theta_{-t_m}, \omega))) \}_{m=1}^{\infty} \) is precompact in \( H^2(\mathbb{B}_{2k_1}) \times L^2(\mathbb{B}_{2k_1}) \). By Lemma 4.4, we deduce that the sequences \( \{ (u(\tau, \tau - t_m, \theta_{-\tau}, u_0(\theta_{-t_m}, \omega)), \nu(\tau, \tau - t_m, \theta_{-\tau}, \nu_0(\theta_{-t_m}, \omega))) \}_{m=1}^{\infty} \) is precompact in \( H^2(\mathbb{B}_{2k_1}) \times L^2(\mathbb{B}_{2k_1}) \) and the sequences \( \{ \eta(\tau, \tau - t_m, \theta_{-\tau}, \eta_0(\theta_{-t_m}, \omega), s) \}_{m=1}^{\infty} \) is precompact in \( L^2_{\mu}(\mathbb{R}^+, H^2(\mathbb{B}_{2k_1})) \).

Thus, \( \{ \varphi^{(\alpha)}(\tau, \tau - t_m, \theta_{-\tau}, \varphi^{(\alpha)}_0) \}_{m=1}^{\infty} \) is precompact in \( E(\mathbb{B}_{2k_1}) \), this together with (5.6) implies \( \{ \varphi^{(\alpha)}(\tau, \tau - t_m, \theta_{-\tau}, \varphi^{(\alpha)}_0) \}_{m=1}^{\infty} \) has a convergent subsequence in \( E(\mathbb{R}^n) \).

**Theorem 5.3.** Under Assumptions I and II, the random dynamical system \( \Phi_\alpha \) generated by the stochastic plate equation (3.11)-(3.12) has a unique pullback \( \mathcal{D} \)-attractor \( \mathcal{A}_\alpha = \{ \mathcal{A}_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \) in the space \( E \).

**Proof.** Note that the cocycle \( \Phi_\alpha \) is pullback \( \mathcal{D} \)-asymptotically compact in \( E \) by Lemma 5.2. On the other hand, the cocycle \( \Phi_\alpha \) has a pullback \( \mathcal{D} \)-absorbing set by Lemma 4.1. Then the existence and uniqueness of a pullback \( \mathcal{D} \)-attractor of \( \Phi_\alpha \) follow from Proposition 2.1 immediately.

6. Upper semicontinuity of pullback attractors. First, we present a criteria concerning upper semicontinuity of non-autonomous random attractors with respect to a parameter in [23].

**Theorem 6.1.** Let \( (X, || \cdot ||_X) \) be a separable Banach space, \( \Phi_\alpha \) be a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathcal{P}, \{ \theta_t \}_{t \in \mathbb{R}}) \). Suppose that (i) \( \Phi_\alpha \) has a closed measurable random absorbing set \( K_0 = \{ K_0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}(X) \) and a unique random attractor \( \mathcal{A}_\alpha = \{ \mathcal{A}_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}(X) \).

(ii) There exists a map \( \zeta : \mathbb{R} \to \mathbb{R} \) such that for each \( \tau \in \mathbb{R}, \omega \in \Omega, K_0(\tau) = \{ u \in X : ||u||_X \leq \zeta(\tau) \} \) and

\[
\limsup_{\alpha \to 0} ||K_0(\tau, \omega)||_X = \limsup_{\alpha \to 0} \limsup_{x \in K_0(\tau, \omega)} ||x||_X \leq \zeta(\tau). \quad (6.1)
\]


(iii) There exists \( \alpha_0 > 0 \), such that for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\bigcup_{|\alpha| \leq \alpha_0} A_\alpha(\tau, \omega) \text{ is precompact in } X.
\]
(iv) For \( t > 0, \tau \in \mathbb{R}, \omega \in \Omega, \alpha_n \to 0 \) when \( n \to \infty \), and \( x_n, x_0 \in X \) with \( x_n \to x_0 \) when \( n \to \infty \), it holds
\[
\lim_{n \to \infty} \Phi_{\alpha_n}(t, \tau, \omega)x_n = \Phi_0(t, \tau)x_0.
\]

Then for \( \tau \in \mathbb{R}, \omega \in \Omega \),
\[
d_H(A_\alpha(\tau, \omega), A_0(\tau)) = \sup_{u \in A_\alpha(\tau, \omega)} \inf_{v \in A_0(\tau)} \|u - v\| \to 0, \quad \text{as } \alpha \to 0. \quad (6.3)
\]

Next, we will use Theorem 6.1 to consider an upper semicontinuity of random attractors \( A_\alpha(\tau, \omega) \) when \( \alpha \to 0 \). To indicate the dependence of solutions on \( \alpha \), we will respectively write the solutions of problem (3.10)-(3.12) as \( u^{(\alpha)}, v^{(\alpha)} \) and \( \eta^{(\alpha)} \), that is, \((u^{(\alpha)}, v^{(\alpha)}, \eta^{(\alpha)})\) satisfies
\[
\begin{aligned}
\frac{du^{(\alpha)}}{dt} + \varepsilon u^{(\alpha)} &= u^{(\alpha)} + \alpha \phi_\omega(t), \\
\frac{d\eta^{(\alpha)}}{dt} - \varepsilon \eta^{(\alpha)} &= (\lambda + \varepsilon^2 + A)u^{(\alpha)} + \int_0^\infty \mu(s)A\eta^{(\alpha)}(s)ds + f(x, u^{(\alpha)}) = g(x, t) \\
-\varepsilon \eta^{(\alpha)} &= h(v^{(\alpha)} + \alpha \phi_\omega(t) - \varepsilon u^{(\alpha)}) + \varepsilon \alpha \phi_\omega(t), \\
\eta^{(\alpha)}(x, \tau, s) &= \eta^{(\alpha)}(x, s) = u^{(\alpha)}(x, \tau) - u^{(\alpha)}(x, \tau - s).
\end{aligned}
\]

When \( \alpha = 0 \), the random problem (6.4) reduces to a deterministic one:
\[
\begin{aligned}
\frac{du^{(0)}}{dt} + \varepsilon u^{(0)} &= v^{(0)}, \\
\frac{d\eta^{(0)}}{dt} - \varepsilon \eta^{(0)} &= (\lambda + \varepsilon^2 + A)u^{(0)} + \int_0^\infty \mu(s)A\eta^{(0)}(s)ds + f(x, u^{(0)}) = g(x, t) \\
-\varepsilon \eta^{(0)} &= h(v^{(0)} - \varepsilon u^{(0)}), \\
\eta^{(0)}(x, \tau, s) &= \eta^{(0)}(x, s) = u^{(0)}(x, \tau) - u^{(0)}(x, \tau - s).
\end{aligned}
\]

Accordingly, by Theorem 5.3 the deterministic and non-autonomous system \( \Phi_0 \) generated by (6.5) is readily verified to admit a unique \( \mathcal{D}_0(E(\mathbb{R}^n)) \)-pullback attractor \( A_0(\tau) \).

Theorem 6.2. Assume Assumptions I and II hold. Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\lim_{\alpha \to 0} \text{dist}_{E(\mathbb{R}^n)}(A_\alpha(\tau, \omega), A_0(\tau)) = 0.
\]

Proof. (i) By Lemma 4.1 and Theorem 5.3, \( \Phi_\alpha \) has a closed measurable random absorbing set \( E_\alpha = \{ E_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}(E(\mathbb{R}^n)) \), where \( E_\alpha(\tau, \omega) = \{ \varphi(\alpha) \in E(\mathbb{R}^n) : \| \varphi(\alpha) \|_{E(\mathbb{R}^n)}^2 \leq R_1(\alpha, \tau, \omega) \} \), and a unique random attractor \( A_\alpha = \{ A_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}(E(\mathbb{R}^n)) \), for each \( \tau \in \mathbb{R}, \omega \in \Omega \), \( A_\alpha(\tau, \omega) \subseteq E_\alpha(\tau, \omega) \).
(ii) Given $|\alpha| \leq 1$, by (4.2), we have
\[ R_1(\alpha, \tau, \omega) \leq R_1(1, \tau, \omega) < \infty, \]
and
\[ \limsup_{\alpha \to 0} R_1(\alpha, \tau, \omega) \leq R_1(1, \tau, \omega). \]
So, for every $\tau \in \mathbb{R}, \omega \in \Omega$,
\[ \limsup_{\alpha \to 0} \|E_\alpha(\tau, \omega)\| = \limsup_{\alpha \to 0} \sup_{x \in E_\alpha(\tau, \omega)} \|x\|_{E(\mathbb{R}^n)} \leq R_1^2(\tau, \omega). \]  
(6.6)
Let $E_1(\tau, \omega) = \{ \varphi(\alpha) \in E(\mathbb{R}^n) : \|\varphi(\alpha)\|^2_{E(\mathbb{R}^n)} \leq R_1(1, \tau, \omega) \}$, then
\[ \bigcup_{|\alpha| \leq 1} A_\alpha(\tau, \omega) \subseteq \bigcup_{|\alpha| \leq 1} E_\alpha(\tau, \omega) \subseteq E_1(\tau, \omega). \]  
(6.7)
(iii) Given $|\alpha| \leq 1$. Let us prove the precompactness of $\bigcup_{|\alpha| \leq 1} A_\alpha(\tau, \omega)$ for every $\tau \in \mathbb{R}, \omega \in \Omega$. For one thing, by (6.7), Lemma 4.3 and the invariance of $A_\alpha(\tau, \omega)$, for every $\zeta > 0, \alpha > 0, \tau \in \mathbb{R}, \omega \in \Omega$, there exist $T = T(\tau, \omega, E_1, \alpha, \zeta) > 0, K = K(\tau, \omega, \alpha, \zeta) \geq 1$, such that for all $t \geq T, k \geq K$, the solution $\varphi(\alpha)$ of (6.4) satisfies
\[ \sup_{\varphi(\alpha) \in \bigcup_{|\alpha| \leq 1} A_\alpha(\tau, \omega)} \|\varphi(\alpha)(\tau, t - t, \theta)\|_{E(\mathbb{R}^n \setminus B_k)} \leq \zeta. \]
For another, by (6.7) we find that the set $\bigcup_{|\alpha| \leq 1} A_\alpha(\tau, \omega)$ is precompact in $E(B_k)$ and hence $\bigcup_{|\alpha| \leq 1} A_\alpha(\tau, \omega)$ is precompact in $E(\mathbb{R}^n)$.
(iv) Let $\varphi^{(0)} = (u^{(0)}, v^{(0)}, \eta^{(0)})$ be a solution of (6.5) with initial data $\varphi^{(0)}_0 = (u^{(0)}_0, v^{(0)}_0, \eta^{(0)}_0)$, and $U = u^{(e)} - u^{(0)}, V = v^{(e)} - v^{(0)}, W = \eta^{(e)} - \eta^{(0)}$. We now prepare to establish convergence of solutions of the stochastic system (3.11)-(3.12) when the intensity of noise $\alpha \to 0$. It follows from (6.4) and (6.5) that
\[
\begin{cases}
\frac{dU}{dt} + \varepsilon V = V + \alpha \phi(t), \\
\frac{dV}{dt} - \varepsilon U + (\lambda + \varepsilon^2 + A)U + \int_0^\infty \mu(s)(AW(s) + f(x, u^{(s)}) - f(x, u^{(0)})) \\
= h(v^{(0)} - \varepsilon u^{(0)}) - h(v^{(s)} + \alpha \phi(t) - \varepsilon u^{(s)}) + \varepsilon \alpha \phi(t), \\
W_t + W_s = U_t, \\
U(x, t) = U_0(x), \quad V(x, t) = V_0(x), \\
W(x, t, s) = W_0(x, s).
\end{cases}
\]  
(6.8)
Taking the inner product of the second equation of (6.8) with $V$ in $L^2(\mathbb{R}^n)$, we find that
\[
\frac{1}{2} \frac{d}{dt} \|V\|^2 - \varepsilon \|V\|^2 + (\lambda + \varepsilon^2)(U, V) + (AU, V) + \int_0^\infty \mu(s)(AW(s), V)ds \\
+ \left( f(x, u^{(s)}) - f(x, u^{(0)}), V \right) \\
= - \left( h(v^{(s)} + \alpha \phi(t) - \varepsilon u^{(s)}) - h(v^{(0)} - \varepsilon u^{(0)}), V \right) + \varepsilon \alpha(\phi, V)\omega(t).
\]  
(6.9)
Similar to Lemma 4.1, we now estimate the terms in (6.9) as follows:

\[- \left( h(v^{(\alpha)}) + \alpha\phi\omega(t) - \varepsilon u^{(\alpha)}) - h(u^{(0)} - \varepsilon u^{(0)}), V \right) \]

\[ \leq - \beta_1 \|V\|^2 + \frac{\beta_1 - \varepsilon}{4} \|V\|^2 + \frac{\beta_2 \varepsilon^2}{2(\beta_1 - \varepsilon)} \|\omega(t)\|^2 \|\phi\|^2 + h'(\varepsilon)\varepsilon(U, V), \]  

(6.10)

\[ h'(\varepsilon)\varepsilon(U, V) \leq \frac{\beta_2 \varepsilon}{2} \frac{d}{dt} \|U\|^2 + \frac{\varepsilon \lambda + \varepsilon^3 + 3\beta_2 \varepsilon^2}{4} \|U\|^2 + c\alpha^2 \|\omega(t)\|^2 \|\phi\|^2, \]  

(6.11)

\[ (\lambda + \varepsilon^2)(U, V) \geq \frac{\lambda + \varepsilon^2}{2} \frac{d}{dt} \|U\|^2 + \frac{3\varepsilon \lambda + 3\varepsilon^3 + \beta_2 \varepsilon^2}{4} \|U\|^2 - c\alpha^2 \|\omega(t)\|^2 \|\phi\|^2, \]  

(6.12)

\[ (AU, V) \geq \frac{1}{2} \frac{d}{dt} \|AU\|^2 + \frac{3\varepsilon}{4} \|\Delta U\|^2 - \frac{\alpha_1}{\varepsilon} \|\omega(t)\|^2 \|\Delta \phi\|^2, \]  

(6.13)

\[ \varepsilon \alpha(\phi, V)\omega(t) \leq \varepsilon |\alpha| |\omega(t)| \|\phi\| \|V\| \leq c\alpha^2 \|\phi\|^2 |\omega(t)|^2 + \frac{\beta_1 - \varepsilon}{4} \|V\|^2, \]  

(6.14)

\[ \int_0^\infty \mu(s) (AW(s), V) ds \]

\[ \geq \frac{1}{2} \frac{d}{dt} \|W\|^2_{\mu,2} + \frac{\delta}{4} \|W\|^2_{\mu,2} - \frac{2m_\omega \alpha^2}{\delta} \|\omega(t)\|^2 \|\Delta \phi\|^2 - \frac{2m_\omega \varepsilon^2}{\delta} \|\Delta U\|^2, \]  

(6.15)

\[ (f(x, u^{(\alpha)}) - f(x, u^{(0)}), V) \leq \beta \|U\| \|V\| \leq c \left( \|V\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|U\|^2 \right). \]  

(6.16)

It follows from (6.9)-(6.18) that

\[ \frac{d}{dt} \left( \|V\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|U\|^2 + \|\Delta U\|^2 + \|W\|^2_{\mu,2} \right) \]

\[ \leq c \left( \|V\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|U\|^2 + \|\Delta U\|^2 + \|W\|^2_{\mu,2} \right) + c\alpha^2 \left( 1 + |\omega(t)|^2 \right), \]  

(6.17)

where \( c \) is a positive constant independent of \( \alpha \).

Applying Gronwall’s inequality to (6.17) from \( t \) to \( T \), we have

\[ \|V(t, \tau, \omega, V_0)\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|U(t, \tau, \omega, U_0)\|^2 + \|\Delta U(t, \tau, \omega, U_0)\|^2 \]

\[ + \|W(t, \tau, \omega, W_0, s)\|^2_{\mu,2} \]

\[ \leq c e^{(t - \tau)} \left( \|V_0\|^2 + (\lambda + \varepsilon^2 - \beta_2 \varepsilon) \|U_0\|^2 + \|\Delta U_0\|^2 + \|W_0\|^2_{\mu,2} \right) \]

\[ + c\alpha^2 \int_\tau^t e^{\varepsilon (t - s)} (1 + |\omega(s)|^2) ds. \]

Then, for all \( t \in [\tau, \tau + T] \) \( (T > 0) \), we get

\[ \|u^{(\alpha)}(t, \tau, \omega, \eta_0, V_0) - u^{(0)}(t, \tau, \omega, \eta_0, v_0)\|^2_{\mu_r(\omega, \tau)} + \|v^{(\alpha)}(t, \tau, \omega, \eta_0) - v^{(0)}(t, \tau, \omega, \eta_0)\|^2 \]

\[ + \|\phi^{(\alpha)}(t, \tau, \omega, \eta_0, s) - \phi^{(0)}(t, \tau, \omega, \eta_0, s)\|^2_{\mu_r(\omega, \tau)} \]

\[ \leq c e^{\varepsilon (t - \tau)} \left( \|u^{(0)}(t, \tau, \omega, \eta_0)\|^2_{\mu_r(\omega, \tau)} + \|v^{(0)}(t, \tau, \omega, \eta_0)\|^2 + \|\phi^{(0)}(t, \tau, \omega, \eta_0)\|^2 \right) \]

\[ + c\alpha^2 \int_\tau^t e^{\varepsilon (t - s)} (1 + |\omega(s)|^2) ds, \]  

(6.18)

which along with (i), (ii), (iii) and Theorem 6.1 completes the proof.
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