1. Introduction

Given a finite subgroup $G \subset \text{Gl}(V)$ of the linear group of a finite-dimensional complex vector space $V$, it is a well-studied problem to describe the structure of the symmetric algebra $B = S(V)$ as a representation of $G$, and also as a module over the ring of invariant differential operators $\mathcal{D} = \mathcal{D}_B^G \subset \mathcal{D}_B$ in the ring of differential operators on $B$, where we mention in passing that $\mathcal{D}$ is also the set of liftable differentiable operators with respect to the map $B^G \to B$ (see $[\text{Kno06}]$). In fact, the two perspectives are known to be equivalent; for a precise statement, see Proposition 2.2. The ring $\mathcal{D}$ inherits the natural grading of $B$, and we let $\mathcal{D}^0 \subset \mathcal{D}$ and $\mathcal{D}^- \subset \mathcal{D}$ be the invariant differential operators of degree 0 and strictly negative degree, respectively. Our first and main result is that there is for all such finite
groups a “lowest weight” description of the category of $\mathcal{D}$-submodules of $B$, where the ring $\mathcal{R} = \mathcal{D}^0/(\mathcal{D}^0 \cap (\mathcal{D}^\mathcal{D}^-))$ plays the role of “Cartan algebra”.

**Theorem 3.1.** The functor

$$N \mapsto N^{ann} = \text{Ann}_{\mathcal{D}^-}(N) = \{ n \in N \mid \mathcal{D}^- n = 0 \}$$

is an equivalence between the category of $\mathcal{D}$-submodules of $B$ and the category of $\mathcal{R}$-submodules of $B^{ann}$.

The $\mathcal{R}$-module $B^{ann}$ is also a subrepresentation of the space of $G$-harmonic polynomials, and is therefore finite-dimensional. As an example of what the theorem contains, we mention that for $G$ a generalized symmetric group, $\mathcal{R}$ is a quotient of the commutative algebra $R_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$. As a rather immediate consequence, isomorphism classes of simple $\mathcal{R}$-modules are 1-dimensional and classified by (ordered) partitions, hence by Theorem 3.1 the same classification applies to the simple $\mathcal{D}$-submodules of $B$ as well as to all representations of $G$. This is of course well-known but here it is a consequence of the explicit structure of $R_n$.

The $G$-representation $B^{ann}$, which contains a copy of all irreducibles (Prop. 2.2), has been studied under the name of the polynomial model [AA01, GO10], in particular when $G$ is a complex reflection group, with the aim to determine when each irreducible occurs with multiplicity 1; one then says that $B^{ann}$ is a Gelfand model. The $\mathcal{R}$-structure, however, seems not to have been exploited, in spite of the fact that the above theorem has the following nice immediate consequence, just using the fact that simple modules over commutative $\mathbb{C}$-algebras are one-dimensional:

**Theorem 3.14.** If $\mathcal{R}$ is commutative then $B^{ann}$ is a Gelfand model for $G$.

As already mentioned $\mathcal{R}$ is commutative for $G$ a generalized symmetric group $G(d, 1, n)$ (which includes all Weyl groups of type $A$ and $B$), as well as when $G$ is a dihedral group. Hence we have in particular a short and conceptual proof that $B^{ann}$ is a Gelfand model for these groups, a result due to [AB09] when $G = G(d, 1, n)$. Several authors have attempted to construct Gelfand models for $G(\rho, \epsilon, c, n)$ with $\epsilon > 1$, and it might be hoped that a study of $\mathcal{R}$ in this case would be similarly helpful.

One way of computing $\mathcal{D}^-$ and $\mathcal{R}$ is by utilizing the strong result by Levasseur and Stafford [LS95] that $\mathcal{D}$ is generated as an algebra by its two commutative subrings $B^G = S(V)^G$ and $S(V^*)^G$, where $V^* = \bigoplus_{i=1}^n \mathbb{C}\partial_i$ is the vector space of constant derivations and $V = \bigoplus_{i=1}^n \mathbb{C}x_i$. Ring generators $f_i(x)$ of $S(V)^G$ and $f_i(\partial) \in S(V^*)^G$ thus give generators of $\mathcal{D}$, but they also generate a Lie subalgebra $a \subset \mathcal{D}$, for which the PBW-theorem then is available. In the case of the generalized symmetric group a good choice is to let $f_i$ be power sums, which gives a basis of $a$ by elements of the form $\sum_{i=1}^n x_i^k \partial_i^l$, which we call power differential operators; these operators turn out to be amenable to effective computation. For our calculations with the dihedral group we use a different and more straightforward technique to get $\mathcal{D}^-$ and $\mathcal{R}$.

There is another context in which $B^{ann}$ occurs, though only implicitly, and without using differential operators, namely that of Macdonald-Spaltenstein-Lusztig induction of representations relative to an inclusion of finite groups $H \subset G$. In fact, this induction is best understood as an operation on $\mathcal{D}$-modules, described in Theorem 4.1 (which relies on Theorem 3.1), instead of $G$-modules; for the relation with the usual definition for groups, see Proposition 4.4. In our differential algebra context, MLS-restriction (instead of induction) will be

$$J^G_H : \text{Mod}_{\mathcal{D}_1}(B) \to \text{Mod}_{\mathcal{D}_2}(B), \quad N \mapsto J^G_H(N) = \mathcal{D}_2 \cdot \text{Ann}_{\mathcal{D}_1^-}(N),$$
where $\mathcal{D}_2 = \mathcal{D}^G_B \subset \mathcal{D}_1 = \mathcal{D}^H_B$. We exemplify the use of $J^G_H$ by constructing the simple components of the $\mathcal{D}_2$-module $B$ when $G$ is the generalized symmetric group.

Another application of Theorem 3.1 is to get an abstract branching rule (multiplicity 1) (Th. 5.2), exemplified with generalized symmetric groups, and a rather detailed decomposition of restricted modules for the symmetric group (Th. 5.6), providing a new proof of the classical branching rule using lowest weight arguments.

Our last application is to a new construction of Young bases for representations of the symmetric group, showing the close relation between the Jucys-Murphy elements $L_i = \sum_{j=1}^{i-1} (j \ i)$ (a sum of transpositions) in the group algebra of $S_n$ and the nabla operators $\nabla_i = x_i \partial_i$. Put $\mathcal{D}_n = \mathcal{D}^n_B$ and $B^n_{\text{ann}} = \text{Ann}_D^-(B)$, $1 \leq i \leq n$. Given a basis $\{v_j^{(i-1)}\}$ of $B^n_{i-1}$, by the branch rule the $R_i$-module $\text{Ann}_D^-(D_{i-1} v_j^{(i-1)})$ is multiplicity free, so a decomposition into simples gives a basis. Iterating this procedure one gets a canonical basis $\{v_T\}_{T \in S}$ of $B^n_{\text{ann}}$, indexed by the set of paths in a branching graph, and in turn are encoded by standard Young tableaux. Interestingly enough, it turns out that the canonical basis is the same as the Young basis (Th. 5.10). The weights of the commutative algebra generated by the $L_i$ that is used in [OV96] here has a natural and more immediate analogue in the multidegree of nabla operators. One can conclude from Theorem 5.6 and Theorem 5.10 that it is possible to build up the representation theory of the symmetric group from the action of nabla operators in the ring of differential operators.

In the final section we study the dihedral group $D_{2e}$ of order $2e$ acting on $C^2$. Noteworthy is the fact that $R$ for its cyclic subgroup $C_e$ is non-commutative (though still simple to describe), that moreover in this case the lowest weight space $N^{\text{ann}}$ of a certain simple module $N$ may have dimension strictly larger than 1, and that for this module MLS-restriction $J^G_H$ does not preserve simplicity, where $H = C_e$ and $G = D_{2e}$.

We conclude by the remark that though most of our examples are taken from reflection groups, they serve primarily as examples of the use of the setup. This setup, however, is valid quite generally, and we suspect it is worthwhile, e.g., to compute $R$ for other groups.

2. Preliminaries

2.1. Notation. We will throughout the paper assume that we have a finite subgroup $G$ of the general linear group $\text{GL}(V)$ of a complex finite dimensional vector space $V$, inducing a graded action on the graded polynomial algebra $B = S(V)$ (with $V$ in degree 1). The algebra of differential operators on $B$ is denoted by $\mathcal{D}_B$ (and sometimes $\mathcal{D}(V)$), which is the Weyl algebra in $n = \text{dim}_C V$ variables. The canonical map $V \otimes_C V^* \rightarrow \mathcal{C}$ can be extended to an isomorphism of left $S(V)$-module and right $S(V^*)$-module (not as rings)

\[(*) \quad S(V) \otimes_C S(V^*) \rightarrow \mathcal{D}_B, \quad p \otimes q \mapsto (b \mapsto p(x_1, \ldots, x_n)q(\partial_1, \ldots, \partial_n)(b)),\]

where $x_1, \ldots, x_n$ is a basis of $V$, $\partial_1, \ldots, \partial_n$ is dual basis of $V^*$, and $q(\partial_1, \ldots, \partial_n)(b)$ the usual action of a constant coefficient differential operator on a polynomial $b$. Note that as a Lie sub algebra of $\mathcal{D}_B$ (with the commutator as bracket) the homogeneous derivations can be identified with the general Lie algebra $V \otimes_C V^* = \text{gl}(V)$, and that this Lie subalgebra contains the canonical element $\nabla = x_1 \partial_1 + \cdots + x_n \partial_n$.

The adjoint action of $\nabla$ on $\mathcal{D}_B$ gives decomposition $\mathcal{D}_B = \oplus \mathcal{D}_B(n)$, where $\mathcal{D}_B(n) = \{P \in \mathcal{D}_B \mid [\nabla, P] = nP\}$; it gives $\mathcal{D}_B$ the same grading as the natural one that is induced by the identification ($*$), placing $V^*$ in degree $-1$ and $V$ in degree 1.
There is an induced action of $G$ on $D_B$ that can be described using (*), as coming from the canonical left action on $V^*$ and $V$. The algebra of invariant differential operators $D = D_B^G$ naturally acts on the invariant ring $A = B^G$, so there is a homomorphism $D \to D_A$.

2.2. Nabla operators. When $\dim V = 1$ the above construction gives us the Weyl algebra $\mathcal{D}(C) = C[x, \partial]$ in 1 variable. Its subspace $\mathcal{D}(C)^0 = \mathcal{D}(C)(0)$ of degree 0 has the basis $\{x^i\partial^j\}_{i,j \geq 0}$, where we in particular have the canonical element $\nabla = x\partial$.

The following easy result concerned with the isotypic component of the $D^G$ will prove useful.

Lemma 2.1. \hspace{1em} (1) $\mathcal{D}(C)^0 = C[\nabla]$. In particular there are polynomials $p_k \in C[t]$, $k = 0, 1, 2, \ldots$, such that $x^k\partial^k = p_k(\nabla)$.

(2) $[\nabla, x^i\partial^j] = (i - j)x^i\partial$.

(3) Assume that $[\nabla, v] = av$, where $v \in V$. Then, for any polynomial $p(t) \in C[t]$, $p(\nabla)v = vp(\nabla + a)$ and consequently also $p(\nabla - a)v = vp(\nabla)$.

2.3. Representations of groups and $D$-modules. The group algebra $\mathcal{D}[G]$ of $G$ with coefficients in $D$ consists of functions $\sum_{g \in G} P_g g : D \to D$, $g \mapsto P_g$, where the product is

$$\sum_{g_1 \in G} P_{g_1} g_1 \cdot \sum_{g_2 \in G} Q_{g_2} g_2 = \sum_{g \in G} \sum_{g_1, g_2 = g} (P_{g_1} Q_{g_2}) g.$$

Then $B$ is a $\mathcal{D}[G]$-module. Recall also that if $M$ is a semi-simple module over a ring $R$, and $N$ is an simple $R$-module, then the isotypic component $M_N$ of $M$ associated to $N$ is the sum $\sum N' \subset M$ of all $N' \subset M$ such that $N' \cong N$. Let $\hat{G}$ denote the set of isomorphism classes of irreducible complex $G$-representations.

Proposition 2.2. As a $\mathcal{D}[G]$-module, we have a decomposition into simple submodules

$$B = \bigoplus_{\chi \in \hat{G}} B_{\chi},$$

where each simple $B_{\chi}$ occurs with multiplicity one.

(1) This decomposition coincides with the decomposition of $B$ into isotypic components either as a representation of $G$ or as a $D$-module.

(2) If $B_{\chi}$ is the isotypic component of the irreducible $G$-representation $V_\chi$ and of the simple $D$-module $N_\chi$, respectively, then, as a $\mathcal{D}[G]$-module,

$$B_{\chi} \cong V_\chi \otimes_C N_\chi.$$

(Here the action on the right is given by $(g P)(v \otimes n) = gv \otimes Pn$, $g \in G$, $P \in \mathcal{D}$).

(3) In the situation in (2),

$$N_\chi \cong \text{Hom}_G(V_\chi, B),$$

as a $D$-module, and

$$V_\chi \cong \text{Hom}_C(N_\chi, B),$$

as a representation of $G$.

Proof. These results, though parts occur in [Mon80], may be found in [LS95, Lemma 3.3 and Thm. 3.4] and [Wal93, Prop.1.5 and Thm. 1.6].

If the isotypic component of an irreducible $G$-representation $V$ in $B$ coincides with the isotypic component of the $D$-module $N$, as in (2) above, we will write $N \sim_G V$. Note that as a direct corollary of (2), the isotypic component corresponding to a linear character $\phi : G \to \mathbb{C}^*$ is in itself a simple $D$-module. In this case the isotypic component is called the \textit{module of semi-invariants} associated to $\phi$. The
above results may also be viewed as consequences of the decomposition theorem of direct images in D-module theory; [Kïl].

Let $H$ be a subgroup of $G$, so that $D_B^G \subset D_B^H$. For a $D^H$-submodule $N$ of $B$, we let $\text{res}_{D_B^H}^G(N) = N$, where $N$ is considered as $D_B^G$-module by restriction to the subring. For an $H$-representation $V$ we let $\text{ind}_{D_B^H}^G V = C[G] \otimes_{C[H]} V$ be the induced representation of $G$.

**Proposition 2.3.** Assume that $V \sim_H N$ in the correspondence Proposition 2.2, (3) (with $G = H$). Then

\[ \text{res}_{D_B^H}^G(N) \sim_G \text{ind}_{D_B^H}^G V. \]

**Proof.** Put $\text{Loc}_G(V) = \text{Hom}_G(V, B)$ and $\Delta_H(N) = \text{Hom}_{D_B^H}(N, B)$, so that $\text{Loc}_G(V)$ is a $D_B^G$-module and $\Delta_H(N)$ is an $H$-representation. Then

\[ \text{Loc}_G \circ \text{ind}_{D_B^H}^G \circ \Delta_H(N) = \text{Hom}_G(C[G] \otimes_{C[H]} \text{Hom}_{D_B^H}(N, B), B) \]

\[ = \text{Hom}_H(\text{Hom}_{D_B^H}(N, B), \text{Hom}_G(C[G], B)) = \text{Hom}_H(\text{Hom}_{D_B^H}(N, B), B) = N, \]

where $N$ is only regarded as a $D_B^G$-module. \hfill $\square$

3. **Equivalence between $D$-modules and $D^\theta$-modules**

In (3.1) and (3.2) we present our main result, which is about studying $D$-modules $M$ by its lowest weight space $\text{Ann}_D(M)$, where the latter is a module over $\mathcal{R} = D^\theta/(D^\theta \cap (DD^-))$. In (3.3) we work out methods to compute $\mathcal{R}$ and $D^-$, which are also exemplified. Gelfand models are discussed in (3.4).

3.1. **Abstract equivalence.** We describe the equivalence first in a more general setting than we need, to facilitate the proof and to give a model that perhaps can be used elsewhere. If $M$ is an arbitrary module over a ring $R$, then $\text{Mod}_R(M)$ denotes the category with objects all $R$-submodules of $M$ and as morphisms all $R$-homomorphisms between these modules.

Assume that the element $\nabla \in D$ has an adjoint action on a $C$-algebra $D$, $P \mapsto [\nabla, P]$, which is semisimple, and that the semisimple decomposition gives a grading $D = \oplus D(n)$, where $P \in D(n)$ if $[\nabla, P] = nP$. We make the triangular decomposition

\[ D = D^- \oplus D^0 \oplus D^+, \]

where $D^- = \oplus_{n>0} D(n)$, $D^0 = D(0)$, and $D^+ = \oplus_{n>0} D(n)$. Define also the ring $\mathcal{R} = D^0/D^0 \cap (D D^-)$.

Define the functor

\[ \ell : \text{Mod}_D(M) \to \text{Mod}(\mathcal{R}), \]

\[ N \mapsto \ell(N) = \text{Hom}_D(D^-/D^- N, N) = \{ n \in N \mid D^- n = 0 \} \]

and the map

\[ \delta : \text{Mod}_\mathcal{R}(\ell(M)) \to \text{Mod}_D(M), \]

\[ V \mapsto \delta(V) = \text{Im}(D^-/D^- \otimes_\mathcal{R} V \to M) = D V. \]

Here $D/D^- D$ is a $(D, \mathcal{R})$-bimodule, so that one gets the adjoint pair of functors $(D/D^- \otimes_{\mathcal{R}}, \text{Hom}_D(D/D^- \otimes_{\mathcal{R}}))$, while $\delta$ in general does not give a functor on the category $\text{Mod}_\mathcal{R}(\ell(M))$. However, if $M$ is sufficiently nice we do get a functor.

In the main part of the paper we will use the more evocative and convenient notation $M^{ann} = \ell(M)$.
Theorem 3.1. Let $M$ be a semisimple $\mathcal{D}$-module which is semisimple over $\nabla$ and satisfying $\delta \circ \ell(M) = M$. Then $\ell : \text{Mod}_{\mathcal{D}}(M) \to \text{Mod}_{\ell(M)}(\ell(M))$ defines an isomorphism of categories, with inverse $\delta : \text{Mod}_{\ell(M)}(\ell(M)) \to \text{Mod}_{\mathcal{D}}(M)$.

Note that we actually have an isomorphism of categories in the theorem, not only an equivalence, and that this isomorphism preserves the subcategories with the same objects, but in which the morphisms are restricted to being inclusions of submodules.

With the support of a semi-simple $\mathbb{C}[\nabla]$-module is meant the set of non-zero eigenvalues of $\nabla$.

Lemma 3.2. Let $W$ be a simple $\mathcal{R}$-module, that is semi-simple as a $\mathbb{C}[\nabla]$-module, and which we also regard as a simple module over the ring $\mathcal{B} = \mathcal{D}^0 + \mathcal{D}^-$ by the projection $\mathcal{D}^0 \to \mathcal{R}$ and trivial action of $\mathcal{D}^-$. Then

1. The support of $W$ as a $\mathbb{C}[\nabla]$-module consists of one element.
2. $\mathcal{D} \otimes_{\mathcal{B}} V$ contains a unique maximal submodule.

Proof. (1) is clear since $\mathcal{R}$ preserves any eigenspace of $\nabla$. Also $\nabla$ acts semi-simply on $\mathcal{D} \otimes_{\mathcal{B}} W$ as a derivation by $\nabla(Q \otimes v) = [\nabla, Q] \otimes v + Q \otimes \nabla v$. Since $W$ is simple, the support of any proper submodule of $\mathcal{D} \otimes_{\mathcal{B}} W$, regarded as $\mathbb{C}[\nabla]$-module, is disjoint from the support of $W$. The maximal proper submodule is then the sum of all proper submodules.

Proof of Theorem 3.1. All direct sums below are internal, and by an $\nabla$-isotypical component of $M$ associated to $\lambda$ we intend the subspace of $\mathcal{M}$ consisting of elements $m$ such that $\nabla \cdot m = \lambda m$. Let $B$ be a $\mathcal{R}$-submodule of $\ell(M)$ and $N$ a submodule of $M$.

(a) $\delta \circ \ell(N) = N$: If $N$ is a submodule of $M$, by semisimplicity there exists a module $N_1$ such that $M = N \oplus N_1$, so that

$$N \oplus N_1 = M = \delta(M) = \delta(\ell(N)) \oplus \delta(\ell(N_1))$$

and hence $N = \delta(\ell(N))$.

(b) $\ell \circ \delta(V) = V$: We note that it follows from the decomposition (T), that if $N \subset M$ is a simple $\mathcal{D}$-module, $\ell(N)$ contains only one isotypical component with respect to the action of $\nabla$. Assume first that $V$ contains only a single $\nabla$-isotypical component, and that $\delta(V) = \bigoplus_{i \in I} N_i$, where $N_i$ are simple $\mathcal{D}$-submodules of $M$. Hence

$$(\star) \quad \ell(\delta(V)) = \bigoplus_{i \in I} \ell(N_i).$$

Since $\delta(V) = V \oplus \mathcal{D}^+ V$, and $V \subset \ell(\delta(V))$, it is clear that $\ell(\delta(V)) = V \oplus V'$, where $V$ and $V'$ have different $\nabla$-isotypical components. Thus, there is a subset $I' \subset I$ such that $V = \bigoplus_{i \in I'} \ell(N_i)$. Then

$$\delta(V) = \bigoplus_{i \in I'} \delta(\ell(N_i)) = \bigoplus_{i \in I'} N_i = \bigoplus_{i \in I} N_i,$$

where the second equality follows from (a). Therefore $I = I'$ and so $\ell \circ \delta(V)$ on the right side of $(\star)$ equals $V$.

Assume then that $V = V_1 \oplus V_2$, where $V_1$ and $V_2$ have no common $\nabla$-isotypical component, and satisfy that $\ell \circ \delta(V_i) = V_i$, $i = 1, 2$. Then $\ell(\delta(V_1) \cap \delta(V_2)) \subset V_1 \cap V_2 = 0$, hence by (a) $\delta(V_1) \cap \delta(V_2) = 0$, so that

$$\delta(V_1 \oplus V_2) = \delta(V_1) \oplus \delta(V_2),$$

and, by assumption,

$$\ell(\delta(V)) = \ell(\delta(V_1)) \oplus \ell(\delta(V_2)) = V_1 \oplus V_2.$$
Since any $V$ may be decomposed as a $D^{\partial}$-module into isotypic components for $\nabla$, it follows by induction that $\ell \circ \delta(V) = V$.

(c) $\Hom_{R}(N_1, N_2) = \Hom_{R}(\ell(N_1), \ell(N_2))$: Since $\ell$ is additive and the category $\text{Mod}_D(M)$ is semisimple, it suffices to prove this when $N_1$ and $N_2$ are simple. If $V_1 \subset \ell(N_1)$, $V_1 \neq 0$, then $\ell(V_1) = N_1$; hence by (b) $V_1 = \ell(N_1)$; hence $\ell(N_1)$ is simple and for the same reason $\ell(N_2)$ is also simple. It is obvious that an isomorphism $\phi : N_1 \rightarrow N_2$ induces an isomorphism $\ell(N_1) \rightarrow \ell(N_2)$. Conversely, let $\psi : \ell(N_1) \rightarrow \ell(N_2)$ be a non-zero homomorphism, hence it is an isomorphism. There is a canonical inclusion homomorphism of $D^{\partial}$-modules $f : \ell(N_2) \rightarrow N_2$, so that we get a map of $D$-modules $\ell \circ \psi : \ell(N_1) \rightarrow N_2$. Hence we get a non-zero homomorphism of $D$-modules $h : D \otimes_{B} \ell(N_1) \rightarrow N_2$, which is surjective since $N_2$ is simple. We moreover have a surjective map $D \otimes_{B} \ell(N_1) \rightarrow N_1$. Since $\ell(N_1)$ is simple, by Lemma 3.2 $D \otimes_{B} \ell(N_1)$ has a unique maximal proper submodule. Therefore we get a unique isomorphism $N_1 \rightarrow N_2$ that extends $\psi$.

(d) $\Hom_{R}(V_1, V_2) = \Hom_{D}(\delta(V_1), \delta(V_2))$: Putting $N_1 = \delta(V_1)$ we have by (b) that $\ell(N_1) = V_i$, $i = 1, 2$. Hence by (c)

$\Hom_{R}(V_1, V_2) = \Hom_{R}(\ell(N_1), \ell(N_2)) = \Hom_{D}(\delta(V_1), \delta(V_2))$.

\[ \square \]

3.2. Equivalence for invariant rings. Theorem 3.1 applies immediately to algebras $D = D^{G}_{\partial}$ of invariant differential operators, where we use the notation of Section 2.1. The semi-simple adjoint action of the Euler operator $\nabla$ induces a $\mathbb{Z}$-grading, we have the decomposition (T), and by Proposition 2.2 $B$ is a semi-simple $D$-module. If $N \subset B$ is a simple $D$-submodule, the vector space $N_{a}$ of lowest degree homogeneous elements in $N$ will be annihilated by $D^{-}$, hence $N_{a} \subset B^{ann}$. Since this is true for any simple submodule of the semi-simple module $B$, we have $B = D \cdot B^{ann}$. Hence the conditions of Theorem 3.1 are obtained for $D$ and $M = B$. We can therefore immediately conclude most of the following basic result:

Corollary 3.3. Suppose that $G$ is a finite group acting on $B = S(V)$ and $D_{B}$. Put $D = D^{G}_{\partial}$ and $R = D^{\partial}/D^{\partial} \cap (DD^{-})$.

1. $B^{ann}$ is a finite-dimensional semisimple $R$-module.

2. There is an isomorphism of categories between the category of $D$-submodules of $B$ and the category of $R$-submodules of $B^{ann}$. The isomorphism is $N \mapsto N^{ann}$, where $N$ is a submodule of $B$, and its inverse is $V \mapsto DV$, where $V$ is an $R$-submodule of $B^{ann}$.

3. Simple $D$-submodules of $B$ correspond to simple $R$-submodules of $B^{ann}$.

4. Each simple $R$-submodule of $B^{ann}$ is concentrated in a single degree.

Proof. There remains to prove $\dim_{C} B^{ann} < \infty$. This follows from Proposition 2.2 and (A), together with the following facts: $\dim V_{\chi} < \infty$, $\text{Ann}_{D_{B}}(B_{\chi})$ is concentrated in one degree, and each homogeneous degree of $B$ is of finite dimension. \[ \square \]

Remark 3.4. The ring $S(V^{*})$ is isomorphic to the subring of constant differential operators in $D$ and $H = \text{Hom}_{S(V^{*})}(S(V^{*})^{\leq}, B)$ is the space of harmonic elements in $B$, so that $B^{ann} \subset H$. Since $H \cong B/m_{A}B$, this gives another argument for $\dim_{C} B^{ann} < \infty$. The space $H$ is isomorphic to the regular $G$-representation if and only if $G$ is a complex reflexion group [Ste64]. In this case the $A$-modules $N_{\chi}$ in Proposition 2.2 are free of rank $\dim_{C} V_{\chi}$.

To fix ideas we give a non-trivial example.

Example 3.5. Let the symmetric group $G = S_{3}$ act on $B = C[x_{1}, x_{2}, x_{3}]$ by permuting the variables; put $A = B^{G}$ and $D = D^{G}_{\partial}$. Let $\alpha_{ij} = x_{i} - x_{j}$, $i \neq j$.
be the equations of irreducible reflecting hyperplanes. Then $B^{\text{ann}} = C1 + C\alpha_{13} + C\alpha_{23} + C\alpha_{12} \alpha_{13} \alpha_{23}$ is a four-dimensional vector space, and the simple $\mathcal{R}$-modules are the one-dimensional vector spaces $N_0 = C$; $N_1 = C\alpha_{12} \alpha_{13} \alpha_{23}$ and $N_p = Cp$ where $p \in (\alpha_{13}, \alpha_{23})$. Corresponding representants for the three classes of simple $\mathcal{D}$-modules are $\mathcal{D} \cdot N_0 = A$, $\mathcal{D} \cdot N_1 = A\alpha_{12} \alpha_{13} \alpha_{23}$ and, selecting $p = \alpha_{12}$, $\mathcal{D} \cdot N_{1-2} = A(x_1 - x_2) \oplus A(x_1^2 - x_2^2)$. A complete description of the $\mathcal{R}$-module structure of $B^{\text{ann}}$ for the general symmetric group $S_n$ is given in Section 5.

### 3.3. Computation of $\mathcal{D}^-$, $\mathcal{D}^0$ and $\mathcal{R}$.

#### 3.3.1. General procedure using basic invariants.

Assume that $\{f_i\}$, $\{g_i\}$ are homogeneous generators of $S(V)^G$ and $S(V^*)^G$, respectively, where one observes that $\{g_i\} \subset \mathcal{D}^-$. Let $a$ be a graded Lie subalgebra of $\mathcal{D} = D(V)^G$ which contains the Lie algebra $\text{Lie} < f_i, g_j >$ that is generated by the set $\{f_i\} \cup \{g_i\}$.

Letting $U(a)$ be the enveloping algebra of $a$ we have a canonical homomorphism $j : U(a) \to \mathcal{D}$.

**Proposition 3.6.**

1. The homomorphism $j$ is surjective.
2. Let $\{r_k\}$ be a homogeneous basis of $a$ and $a^- = \sum \deg(r_k) < 0 C r_k$ be the subalgebra of elements of negative degree. Then $a^- \subset \mathcal{D}^- \subset Da^-$. 
3. Let $a^0$ be the subalgebra of degree $0$ in $a$. Then we have a surjective homomorphism $U(a^0) \to \mathcal{R}$.

In particular, if $a^0$ is commutative, then $\mathcal{R}$ is commutative.

It follows from (3) that $M^{\text{ann}} = \text{Ann}_{\mathcal{R}}(M)$ and from (4) that the $\mathcal{R}$-module structure of $M^{\text{ann}}$ come from its structure as $a^0$-module.

**Proof.** (1): This follows from the famous theorem of Levasseur and Stafford [LS95], stating that the sets $\{f_i\}$, $\{g_i\}$ together generate $\mathcal{D}$.

(2-3): Provide the homogeneous basis $\{r_k\}$ with a total ordering that is compatible with the degrees in the sense that $\deg(r_i) \leq \deg(r_j)$, when $i \geq j$. It follows from the Poincaré-Birkhoff-Witt theorem, using the homomorphism above, that any element $P \in \mathcal{D}$ can be expressed (non-uniquely) in the form

$$P = \sum \alpha_{i_1 \cdots \ldots i_r} r_{i_1} \cdots r_{i_r},$$

where $i_1 \geq \cdots \geq i_r$. Hence the factors in each term $r_{i_1} \cdots r_{i_r}$ have descending degree $\deg r_{i_1} \geq \cdots \geq \deg r_{i_r}$. If $P \in \mathcal{D}^-$ then the last factor in each term has $\deg(r_{i_r}) < 0$ and so $P \in Da^-$. This gives (2). Furthermore, when $P \in \mathcal{D}^0$ we can write

$$P = \sum \alpha_{i_1 \cdots \ldots i_r} r_{i_1} \cdots r_{i_r} \mod \mathcal{D}^0 \cap \mathcal{D}^-,$$

where $\deg(r_{i_1}) = \cdots = \deg(r_{i_r}) = 0$ and $\alpha_{i_1 \cdots \ldots i_r} \in \mathcal{C}$. This gives (3). \qed

#### 3.3.2. $\mathcal{R}$ for $G(m, e, n)$. As an example we will consider the irreducible imprimitive complex reflection groups $G = G(m, e, n)$, where $e$ and $m$ are positive integers such that $e \mid m$, and determine generators of $\mathcal{R}$ when $e = 1$. Let $V$ be a complex vector space of dimension $n$. Then

$$G = A(m, e, n) \rtimes S_n \subset \text{Gl}(V)$$

where $S_n$ is realized as permutation matrices and $A(m, e, n)$ as diagonal matrices whose entries belong to $\mu_m$, the group of $m$-roots of unity, such that their determinant belongs to $\mu_d \subset \mu_m$, where $d = m/e$, and in the semi-direct product $S_n$ acts on $A(m, e, n)$ by permutation. This means that $G$ can be realized as permutation matrices with entries in $\mu_m$, such that the product of the non-zero entries belongs to $\mu_d \subset \mu_m$; see [Bro10]. Here $G(1, 1, n) = S_n$ is of type $A_{n-1}$, $G(2, 1, n)$ is of type
Let \( B_n = C_n \); the dihedral group \( G(e,e,2) = I_2(e) \), where \( I_2(6) \) is of type \( G_2 \), is treated in more detail in (6.2); \( G(2,2,n) \) is of type \( D_n \), and \( G(de,e,1) = C_2 \) is a cyclic group.

Put \( \tilde{A} = A(m,1,n) \), \( A = A(m,e,n) \), and \( \tilde{G} = \tilde{A} \rtimes S_n = G(m,1,n) \) so that \( A \subset \tilde{A} \) and \( \tilde{G} \subset G \). Define the \( S_n \)-invariant elements \( \Theta = (\prod_{i=1}^n x_i^d)^l, \Psi = (\prod_{j=1}^l \partial_j)^d \), and \( h_i(x) = \sum_{j=1}^n x_j^d \).

**Lemma 3.7.**

\[
\begin{align*}
B^G &= C[f_1, \ldots, f_n] \\
B^G &= C[f_1, \ldots, f_{n-1}, \Theta]
\end{align*}
\]

where
\[
f_i = h_i(x_1^m, \ldots, x_n^m) = \sum_{j=1}^n x_j^m, \quad i = 1, \ldots, n
\]

\[
\begin{align*}
\mathcal{D}_n := \mathcal{D}_B^G &= C(f_i(x), f_i(\partial)) \\
\mathcal{D}_B^G &= C(f_i(x), f_i(\partial), \Psi, \Theta)
\end{align*}
\]

**Proof.** (1) This is of course well known, but let us at least sketch the argument. We have (as detailed below)
\[
B^{A(m,e,n) \rtimes S_n} = (B^{A(m,e,n)})^{S_n} = (C[x_1^m, \ldots, x_n^m, (x_1 \cdots x_n)^d])^{S_n}
\]

\[
= (C[(x_1 \cdots x_n)^d][x_1^m, \ldots, x_n^m])^{S_n} = C[(x_1 \cdots x_n)^d][f_1, \ldots, f_n]
\]

The second equality can be seen by first noting that if a polynomial is \( A(m,e,n) \)-invariant, then each of its monomial terms is invariant, and these are exactly given by powers of the monomials \( x_1^m, \ldots, x_n^m, (x_1 \cdots x_n)^d \). The \( n \) monomials \( x_i^m \) are algebraically independent, while the last is \( S_n \)-invariant and algebraically dependent on the other ones. Therefore the second equality on the second line follows from the well-known fact that \( \{ f_i \}_{i=1}^n \) is an algebraically independent set of generators of \( C[x_1^m, \ldots, x_n^m]^{S_n} \). When \( e > 1 \), then \( f_n \in C[f_1, \ldots, f_{n-1}, \Theta] \), while if \( e = 1 \) (so that \( m = d \) ), then \( \Theta \in C[f_1, \ldots, f_n] \), so that \( B^G = C[f_1, \ldots, f_n] \).

(2) This follows from the theorem of Levasseur and Stafford [LS95]. \qed

Lemma 3.7 means that we have good control of the invariants \( \{ f_i \} \) for the ring \( B^{G(m,e,n)} \) which are needed in Proposition 3.6. However, it is only in the case \( e = 1 \) that we obtain a really useful description of the Lie subalgebra \( \tilde{a} = \text{Lie} < f_i(x), f_i(\partial) > \) of \( \mathcal{D}_n \) (or more precisely of a Lie algebra containing \( \tilde{a} \)), using the basic invariants \( f_i(x) = h_i(x_1^m, \ldots, x_n^m) \) for \( B = S(V) \) and \( f_i(\partial) \) for its isomorphic ring \( S(V^*) \).

**Proposition 3.8.**

(1) The Lie algebra \( \tilde{a} \) is contained in a Lie algebra \( \tilde{a}' \) with the basis

\[
\left\{ \sum_{k \geq 0, l \geq 0} x_i^k \partial_i^l \right\}, \quad k \geq 0, l \geq 0, m \mid l - k
\]

If \( m = 1 \), then \( \tilde{a} = \tilde{a}' \).

(2) A basis of \( (\tilde{a}')^- \) is provided by the elements in (1) with \( 0 \leq k < l \). If \( z \in B^{ann} \), then its degree \( \deg(z) < nm \). In particular, if \( M \) is a \( \mathcal{D}_n \)-submodule of \( B \), then

\[
\text{Ann}_{\mathcal{D}_n}(M) = \{ z \in M \mid \sum_{i=1}^n x_i^k \partial_i^l \cdot z = 0, \text{ when } 0 \leq k < l \leq nm - 1, \ m \mid l - k \}.
\]
(3) We have
\[(a')^0 = C[\nabla_1, \ldots, \nabla_n]^{S_n},\]
and therefore \(a^0\) is commutative.

Let us agree to call \( \sum_{i=1}^{n} x_i^k \partial_i^l, m|(l - k)\), a power differential operator. Of particular interest is of course the case \(m = 1\), so that \(G = S_n\) and \(D_n\) is the ring of symmetric differential operators, and there are then no restrictions on \(l - k\).

**Lemma 3.9.** Let \(D(C)^{Lie} = C(x, \partial)^{Lie}\) be the Weyl algebra in 1 variable, considered as a Lie algebra. Let the cyclic group \(C_m\) act on \(x\) by a primitive \(m\)th root of unity, and thus inducing an action on \(D(C)^{Lie}\). Then the invariant algebra \((D(C)^{Lie})^{C_m} = C(x^m, \partial^m)^{Lie}\). If \(m = 1\), then this Lie algebra is generated by the set \(\{x^k, \partial^l\}_{0 \leq k \leq 3, 0 \leq l \leq 3}\).

**Remark 3.10.**

(1) Notice (in the proof) that if \(n > 2\) then \(a'\) is isomorphic to the Lie algebra \((D(C)^{C_m})^{Lie}\) in Lemma 3.9, and \((a')^0 \cong C[\nabla]^{Lie} \subset (D(C)^{Lie})^{C_m}\), where \(\nabla = x \partial_x\). The Lie subalgebra \((a')^-\) is not finitely generated.

(2) When \(n = 2\) and \(m = 1\) then the Lie algebra \(\bar{a} = C \cdot 1 + C \nabla + \sum_{i=1}^{2} (Cf_i(x) + C \partial f_i(\partial))\) is finite-dimensional and \(a^0 = C \cdot 1 + C \nabla\).

(3) One may ask when the Lie algebras \(\bar{a}\) are isomorphic for different choices of basic sets of invariants. For instance, when \(n = 2\) and \(m = 1\) using the basic invariants \(e_1 = x_1 + x_2\), \(e_2 = x_1 x_2\) we get \(a^{(1)} = C \cdot 1 + C \nabla + \sum_{i=1}^{2} C e_i(x) + C e_i(\partial)\), so that \(a^{(1)} \not\cong a\) but still \(a^{(1)} \cong \bar{a}\).

**Proof.** The equality \(D(C)^{C_m} = C(x^m, \partial^m)\) follows from the theorem of Levassow and Stafford [LS95] (already when \(m = 3\) it is a nontrivial fact that \(x \partial \in C(x^m, \partial^m)\)), so that \((D(C)^{Lie})^{C_m} = C(x^m, \partial^m)^{Lie}\). It is elementary to see, however, that the set \(\{x^k \partial^l\}_{m|(l-k)}\) is a basis of \((D(C)^{Lie})^{C_m}\). Assume now that \(m = 1\) and let \(\mathfrak{g}\) be the Lie algebra that is generated by the elements \(\{x^r, \partial^s\}_{0 \leq r \leq 3, 0 \leq s \leq 3}\). Since \([\partial^2, x^3] = 6x^2 \partial + 6x\) and \([\partial^3, x^2] = 6x \partial^2 + 6x\) it follows that \(E_x = x^2 \partial, E_\partial = x \partial^2 \in \mathfrak{g}\). Now \([x^{k_1} \partial^{l_1}, x^{k_2} \partial^{l_2}] = (l_1 k_2 - l_2 k_1)x^{k_1 + k_2 - 1} \partial^{l_1 + l_2 - 1} + (l.o.)\) where "1.o." signifies a linear combination of terms \(x^r \partial^s\) where \(r < k_1 + k_2 - 1, s < l_1 + l_2 - 1\). In particular, 
\[E_x, x^k \partial^l = (k - 2l)x^{k+1} \partial^l + (l.o.), \quad [E_\partial, x^k \partial^l] = (2k - l)x^k \partial^{l+1} + (l.o.).\]
A straightforward induction in \(k\) and \(l\) now shows that \(\mathfrak{g} = D(C)^{Lie}\).

**Proof of Proposition 3.8.** (1): This is essentially a 1-variable assertion, based first on the fact that
\[\[x_i^{k} \partial_i^j, x_i^{k} \partial_i^{j+1}\] = \sum_{j=1}^{r} c_j x_i^{k} \partial_i^{j} \partial_i^{j+1},\]
where \(r = \max(\min(l, k_1), \min(k, l_1))\) (unless the Lie bracket is 0), where of course the coefficients \(c_j\) do not depend on \(i\), and secondly that variables do not mix in Lie brackets since \(x_i^{k} \partial_i^j, x_j^{k} \partial_j^j = 0\) when \(j \neq i\). If \(m|(l - k)\) and \(m|(l_1 - k_1)\), then \(m | (k + k_1 - j - (l + l_1 - j))\). This implies that the bracket of two power differential operators is a linear combination of power differential operators, so that the vector space \(\bar{a}'\) that is spanned by such differential operators forms a Lie algebra, and clearly \(\bar{a} \subset \bar{a}'\). Assume that \(m = 1\). By Lemma 3.9 the set \(\{x_i^{k}, \partial_i^j\}_{0 \leq k, 0 \leq l \leq 3}\) generates \(D(C)^{Lie}\), which implies that the corresponding set of power differential operators \(\{x_i^{k}, \partial_i^j\}_{0 \leq k, 0 \leq l \leq 3}\) generates \(\bar{a}\), i.e. \(\bar{a}' = \bar{a}\).
(2): It is evident by definition that \((\bar{a}')^-\) has the given basis of power differential operators \(p_{k,l} = \sum_{i=1}^{n} x_i^k \partial_i^l, \quad l > k, \; m \mid (l - k).\) Since \((\bar{a}')^- \subset \mathcal{D}^-_n\), it is also evident that if \(z \in \text{Ann}_{\mathcal{D}^-_n}(M)\), then \(p_{k,l} \cdot z = 0\), when \(0 \leq k < l \leq nm, \; m \mid (l - k).\) Assume now the converse, that \(p_{k,l} \cdot z = 0\) for such \(l\) and \(k\), so that in particular \(p_{0,m} \cdot z = 0\) for \(l = 0, \ldots, n\). We have
\[
Q(\partial^{m}) = \prod_{i=1}^{n} (\partial_i m - \partial_i^m) = \partial^{nm} + \sum_{i=1}^{n} (-1)^i e_i(\partial_1 m, \ldots, \partial_n m)\partial^{(n-i)m},
\]
where the elementary symmetric polynomials
\[
e_i = e_i(\partial_1^m, \ldots, \partial_n^m) \in \mathcal{C}[\partial_1^m, \ldots, \partial_n^m]_{\mathcal{S}_n} = \mathcal{C}[p_{0,m}, \ldots, p_{0,m}]_+,
\]
so that \(e_i \cdot z = 0\). Since \(Q(\partial^{m}) = 0\), it follows that \(\partial_i^{nm} \cdot z = 0\) and hence \(\deg_i(z) < nm, \; i = 1, \ldots, n\). Therefore \(p_{k,l} \cdot z = 0\) also when \(l \geq nm\). Since \(\mathcal{D}^-_n \subset \mathcal{D}^-_n(\bar{a}')^-,\) by Proposition 3.6 (2), it follows that \(\mathcal{D}^-_n \cdot z = 0\).

(3): Clearly \(f_j(\nabla_1, \ldots, \nabla_n)\) is a power sum of degree 0 so it belongs to \((\bar{a}')^0\); hence \(C[\nabla_1, \ldots, \nabla_n]_{\mathcal{S}_n} \subset (\bar{a}')^0\). Conversely, since \(x_i^k \cdot \partial_i = p_k(\nabla_i)\) (see Lemma 2.1) it is also clear that \((\bar{a}')^0 \subset C[\nabla_1, \ldots, \nabla_n]_{\mathcal{S}_n}. \square\)

In general \(\mathfrak{a}\) is an extension of \(\bar{a}\) by the elements \(\Theta\) and \(\Psi\), similarly to Lemma 3.7, and these elements will mix the variables, making it considerably more difficult to describe bases of \(\mathfrak{a}, \; \mathfrak{a}^-\) and \(\mathfrak{a}^0\). The cases \(n = 2\) and \(e \geq 1\) are studied in Section 6.

3.3.3. Using \(\mathfrak{gl}(V)\) to determine generators of \(\mathcal{R}\). There is another way to think of (3) in Lemma 3.8. We have inclusions
\[
\mathfrak{h} \subset \mathfrak{gl}(V) \subset \mathcal{D}^0(V)
\]
where \(\mathfrak{h}\) is a Cartan algebra in the general Lie algebra \(\mathfrak{gl}(V)\), and we have a surjective map \(l : U(\mathfrak{gl}(V)) \rightarrow \mathcal{D}^0(V)\), where \(U(\mathfrak{gl}(V))\) is the enveloping algebra of \(\mathfrak{gl}(V)\). Since \(G\) is finite the induced (and same noted) map
\[
l : U(\mathfrak{gl}(V))^G \rightarrow \mathcal{D}^0\]
is again surjective. The maximal subgroup of \(\text{Gl}(V)\) that preserves the Cartan algebra \(\mathfrak{h}\) is of the form \(T \times S_n\), where \(S_n\) is the symmetric group and the torus \(T\), is the maximal subgroup that leaves \(\mathfrak{h}\) invariant, where moreover \(T\) acts trivially on \(\mathfrak{h}\). Thus, if \(G \subset T \times S_n\) and \(\bar{G}\) is the image of \(G\) in \(S_n\), then \(\bar{G}\) preserves \(\mathfrak{h}\), and we have the commutative subring
\[
l(S(\mathfrak{h})^G) \subset \mathcal{D}^0.
\]
This is in particular true when \(G = G(m, e, n)\), where \(A(m, e, n) \subset T\), so that we get the subring \(l(S(\mathfrak{h})^{S_n})\). Lemma 3.8 therefore implies

Proposition 3.11. If \(G = G(m, 1, n)\), then
\[
l(S(\mathfrak{h})^{S_n}) \mod \mathcal{D}^0 \cap (\mathcal{D} \cdot \mathcal{D}^-) = \mathcal{R}.
\]

We may also use \(l\) and invariant theory of commutative rings, to find algebra generators of \(\mathcal{D}^0\). (This will be the method used in Section 6, for the cyclic and dihedral groups.)

The natural order filtration \(\{\mathcal{D}_n(V)\}\) of \(\mathcal{D}(V)\) is \(G\)-invariant, \(G \cdot \mathcal{D}_n(V) \subset \mathcal{D}_n(V)\) and therefore induces a filtration \(\mathcal{D}_n^G = \mathcal{D}_n(V)^G \cap \mathcal{D}^0\) of the subring \(\mathcal{D}^0 \subset \mathcal{D} \subset \mathcal{D}(V)\). Similarly, the enveloping algebra \(U((\mathfrak{gl}(V))\) is also provided with a natural filtration \(\{U_n(\mathfrak{gl}(V))\}\) such that \(G \cdot U_n(\mathfrak{gl}(V)) \subset U_n(\mathfrak{gl}(V))\), and thus induces a filtration \(\{U_n(\mathfrak{gl}(V))^G\}\) of \(U(\mathfrak{gl}(V))^G\). Put
\[
\mathfrak{g}^*(\mathcal{D}^0) = \bigoplus_{n \geq 0} \frac{\mathcal{D}_n^G}{\mathcal{D}_n^{G+1}}
\]
(where $\text{gr}^0(D^0) = S(V)^G$). By the Poincaré-Birkhoff-Witt theorem
\[
\bigoplus_{n \geq 1} \frac{U_{n+1}(\mathfrak{gl}(V))}{U_n(\mathfrak{gl}(V))} = S^*(\mathfrak{gl}(V)),
\]
and therefore
\[
\bigoplus_{n \geq 1} \frac{U_{n+1}(\mathfrak{gl}(V))^G}{U_n(\mathfrak{gl}(V))^G} = S^*(\mathfrak{gl}(V))^G,
\]
since taking $G$-invariants is an exact functor. By the same reason, we have that
\[
I(U_n(\mathfrak{gl}(V))^G) = D_n^0,
\]
so there is a surjective homomorphism
\[
I^\rho : S^*(\mathfrak{gl}(V))^G \to \text{gr}^*(D^0).
\]
Since a homogeneous set of generators of $\mathfrak{gl}^*(D^0)$ can be lifted to generators of $D^0$, we conclude that to get generators of $D^0$ it suffices to compute generators of $S^*(\mathfrak{gl}(V))^G = S^*(V \otimes C V^*)^G$, where $G$ acts diagonally on $V \otimes C V^*$. We summarize this in a lemma:

**Lemma 3.12.** Let $\{a_i\}_{i \in I}$ be a set of homogeneous elements in $S(V \otimes C V^*)^G$ that generates $S(V \otimes C V^*)^G$, and let $\{a_i\}_{i \in I}$ be a subset of $U(\mathfrak{gl}(V))$ such that $a_i$ represents $\hat{a}_i$. Then $\{\hat{a}_i\}_{i \in I}$ is a generating subset of $D^0$.

**Remark 3.13.** The map $I^\rho$ is in fact a homomorphism of Poisson algebras. Using the Poisson product one can sometimes, for example when $G$ is a Weyl group with no factors of type $E_7$, prove that generators of the subrings $S^*(V)^G$ and $S^*(V^*)^G$ together generate the Poisson algebra $S^*(V \otimes V^*)^G$, which then can be lifted to generators of $D(V)^G$; see [Wal93, LS95]. For $G = A(e,e,1) \subset \text{Gl}(C^3)$ one does not get generators of the whole Poisson algebra in this way, but even so lifts of the generators of $S^*(V)^G$ and $S^*(V^*)^G$ do give generators of $D(V)^G$; see [LS95, p.371].

3.4. *Gelfand models and $\mathcal{R}$.** The space $B^{ann}$ has been considered by other authors, under the name of the polynomial model, in the context of finding *Gelfand models* of a finite group $G$ (see e.g. [AB09, AB14, GO10]). Such a model is defined to be a $G$-representation that is a direct sum of a representative of each isotypical class in $\hat{G}$. Now, by Proposition 2.2, we have
\[
B^{ann} \cong \bigoplus_{\chi \in \hat{G}} V_\chi \otimes N^{ann}_\chi,
\]
so $B^{ann}$ is a Gelfand model if and only if, for all $\chi \in \hat{G}$, $N^{ann}_\chi$ is a 1-dimensional complex vector space. We note the relation to fake degree [GJ11, 5.3.3], which is the Poincaré polynomial $P_\chi(t) = \sum \dim C N_\chi(i)t^i$ of the graded vector space $\hat{N}_\chi := N_\chi/m_4 N_\chi = \oplus_1 N_\chi(i)$. In terms of the fake degree, $B^{ann}$ is a Gelfand model if and only if, the least non-zero coefficient of all $P_\chi(t)$ is 1. The fake degree has been calculated for Coxeter groups; this is used in [GO10] to give a uniform proof of the fact that $B^{ann}$ is a Gelfand model when $G$ is a finite Coxeter group not of type $D_2n$, $n \geq 2$, $E_7$ or $E_8$.

We do know by the correspondence above in Corollary 3.3 that $N^{ann}_\chi$ is a simple $\mathcal{R}$-module. Since the only simple modules over a commutative algebra over $C$ are 1-dimensional, we have the following immediate result, that gives a simple proof of a main result in [AB09].

**Theorem 3.14.** If $\mathcal{R}$ is commutative, then $B^{ann}$ is a Gelfand model of $G$. In particular this is true for $G = G(m,1,n)$. 

Remark 3.15. The quotient $R \to \hat{R} = D^0/(\text{Ann}_D(B^{ann})) = \text{End}_D(B^{ann})$, implying that $B^{ann}$ is a Gelfand model if and only if $\hat{R}$ is commutative. One may ask whether the same connection between commutativity and the fact that $B^{ann}$ is a Gelfand model holds for $R$, as for $\hat{R}$. When $G = G(e,e,2)$, a dihedral group, we prove in Proposition 6.5 that $R$ is commutative and hence $B^{ann}$ is a Gelfand model, but on the other hand we will see in Proposition 6.3 that $R$ is not commutative for the action of a cyclic group on $C^2$, and that $B^{ann}$ is then not a Gelfand model.

4. Macdonald-Lusztig-Spaltenstein restriction for $D$-modules

In (4.1) we present the MLS-restriction functor, which generalises and clarifies a construction in group theory. This is applied to the case of generalized symmetric groups in (4.3), after a discussion in (4.2) of the polynomials $B$ considered as a module over the ring $R_n = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$. 

4.1. MLS-restriction. Suppose we have an inclusion of graded algebras $D_2 \subset D_1$ of the type in Section 3.1, and a $D_1$-module $M$ which is semisimple both as $D_1$- and $D_2$-module, and moreover that $\text{Ann}_{D_1}(M)$ generates $M$ over $D_2$. Denote the category of $D_i$-submodules of $M$ by $\text{Mod}_{D_i}(M)$, $i = 1, 2$, and similarly the equivalent categories of $D_1^0$-submodules of $M^{ann}_i := \text{Ann}_{D_i}(M)$ by $\text{Mod}_{D_1^0}(M^{ann}_i)$. Since $D_2 \subset D_1$ we have

$$M^{ann}_1 \subset M^{ann}_2$$

and since also $D_2^0 \subset D_1^0$, there is a restriction functor

$$\text{Mod}_{D_1^0}(M^{ann}_1) \to \text{Mod}_{D_2^0}(M^{ann}_2)$$

that takes $V \subset M^{ann}_1$ to $V \subset M^{ann}_2$. This functor corresponds by the category equivalence in Corollary 3.3 to the functor

$$J_+ : \text{Mod}_{D_1}(M) \to \text{Mod}_{D_2}(M), \quad N \mapsto D_2 \cdot N^{ann}.$$

Theorem 4.1. Let $N$ be a $D_1$-submodule of $M$. 

1. If the restriction of $N^{ann}_i$ to a $D_2^0$-module is simple, then $J_+(N)$ is a simple $D_2$-module. This holds for example if $N^{ann}_i$ is a 1-dimensional complex vector space.

2. If $N^{ann}_i$ is regarded as a $D_2^0$-module by restriction, then

$$N^{ann}_1 = (J_+(N))^{ann}_2.$$

Notice that the assumption in (1) implies, by Theorem 3.1, that $N$ is simple.

Proof. (1): Since $N^{ann}_1$ is a simple $D_2^0$-module, it follows by Theorem 3.1 that $D_2 \cdot N^{ann}_1$ is a simple $D_2$-module.

(2): By definition, $J_+$ is just restriction from $D_1^0$ to $D_2^0$ on submodules of $M^{ann}$, so that using the category equivalence of Theorem 3.1 we get $(J_+(N))^{ann}_2 = (D_2 \cdot N^{ann})^{ann}_2 = N^{ann}_1$. 

We apply the above construction to the ring $B = S(V)$, where $V$ is a representation of a finite group $G$, and $H \subset G$ is a subgroup, so that $B$ is both a $G$- and $H$-representation. Set $D_B = D(V)$ and

$$D_2 = D_B^0 \subset D_1 = D_B^H.$$

Letting $M = B$, by Corollary 3.3 we are in the above situation, and the functor $J_+$ is given by:

Definition 4.2. Define the functor

$$J^G_H : \text{Mod}_{D_1}(B) \to \text{Mod}_{D_2}(B), \quad N \mapsto J^G_H(N) = D_2 \cdot \text{Ann}_{D_1}(N).$$

We call $J^G_H$ the differential MLS-restriction.
We record the behaviour of the differential MLS-restriction in a chain of subgroups.

**Lemma 4.3.** Let \( G_2 \subset G_1 \subset G \) be an inclusion of finite groups and \( V \) be a representation of \( G \). Then we have
\[
J_{G_2}^G = J_{G_1}^G \circ J_{G_2}^{G_1}.
\]

**Proof.** Letting \( D = D_B^G \subset D_2 \subset D_B^{G_1} \subset D_1 = D_B^{G_2} \), we have
\[
J_{G_1}^G \circ J_{G_2}^{G_1}(N) = \mathcal{D} \operatorname{Ann}_{D_1}(D_1 \operatorname{Ann}_{D_2}(N)) = \mathcal{D} \operatorname{Ann}_{D_2}(N) = J_{G_2}^G(N),
\]
where the second equality follows from Corollary 3.3 since the \( D_1 \)-module \( \operatorname{Ann}_{D_2}(N) \) is by restriction \( D_1 \subset D_2 \).

The terminology is motivated by the fact that \( J_H^G \) is closely related to Macdonald-Lusztig-Spaltenstein induction for group representations, which is defined in the following manner [Mac72, LS79]; see also [GJ11, 5.2], and in particular for the construction of the representations of \( S_n \) and the generalized symmetric group, see [GJ11, 5.4] and [ATY97]. Suppose that \( W \) is a representation of \( H \) with \( W \)-isotypic component \( B_W \) in \( B \). Let \( d_W \) be the least integer such that the homogeneous component \( B_{W}^{d_W} \) of degree \( d_W \) is nonzero, and assume that \( B_{W}^{d_W} \) is isomorphic to \( W \). Then the MLS-induced representation is defined as the isomorphism class of the representation
\[
j_H^G(W) = C[G]B_W^{d_W} \subset B,
\]
and it is not difficult to prove that it is an irreducible \( G \)-representation (see [loc. cit.] or Proposition 4.4 below). In this way one gets a partially defined map \( j_H^G : H \rightarrow G \).

Note that it makes sense to extend the definition by dropping the condition \( B_W^{d_W} \cong W \), but then \( C[G]B_W^{d_W} \) need not be irreducible. However, using the \( D_1 \)-module structure one may still keep track of the decomposition of \( j_H^G(W) \), as described in the following proposition. We use the notation in Definition 4.2.

**Proposition 4.4.** Let \( W \) be an irreducible representation over \( H \) and \( N \) be a \( D_1 \)-module such that \( N \sim_H W \) (Prop. 2.2). Assume that the restriction of \( N^{ann} = \operatorname{Ann}_{D_1}(N) \) to a \( D_2 \)-module is simple, and put \( r = \dim C N^{ann} \). Then we have:

1. \( j_H^G(W) = (W_1)^r \), where \( W_1 \) is an irreducible \( G \)-representation,
2. \( W_1 \sim_G j_H^G(N) \), so if \( r = 1 \), then \( j_H^G(W) \sim_G j_H^G(N) \).

Notice that the condition on \( N^{ann} \) is trivially satisfied when \( r = 1 \).

**Proof.** That \( N \sim_H W \) means that the \( W \)-isotypic component \( B_W \) of \( B \) is isomorphic to \( W \otimes_C N \) as a \( D_1[H] \)-module, for some (simple) \( D_1 \)-submodule \( N \subset B \) (Prop. 2.2). Since \( N^{ann} = N^{d_W} \) is a simple \( D_2 \)-module, Theorem 4.1 implies that the \( D_2 \)-module \( N_1 = j_H^G(N) = D_2N^{d_W} \subset B \) is simple; hence by Proposition 2.2
\[
C[G]N_1 \cong W_1 \otimes_C N_1
\]
for some irreducible \( G \)-representation \( W_1 \). In particular \( j_H^G(W) = W_1 \otimes_C N^{d_W} \), since \( N^{d_W} = N^{ann} = N^{ann} = N^{d_W} \). This implies (1) and (2).

One should note the slight conceptual difference: MLS-restriction of \( D \)-modules as defined above takes submodules of \( B \) to submodules of \( B \), but MLS-induction of \( G \)-representations takes (certain) isomorphism classes of irreducible representations to isomorphism classes of irreducible representations. Our definition is partly motivated by the fact that we are interested in the actual generators of the irreducible \( D \)-submodules, not only the isomorphism classes. In the rest of the section we will exemplify this for the generalized symmetric group.
4.2. Simple $R$-modules and partitions. Before we can describe MLS-restriction for the generalized symmetric group, we need to understand the simple submodules of $B$ over the algebra $R_n = \mathbb{C}[h]^S_n$ that appeared in Proposition 3.11, mapping surjectively to $\mathcal{R}$.

A multi-index is a function $\alpha : [n] = \{1, \ldots, n\} \rightarrow \mathbb{N}$, and this determines the monomial $x^\alpha = \prod x_i^{\alpha(i)}$. Since $\nabla_i(x^\alpha) = \alpha(i) x^\alpha$ it follows that the algebra $\mathbb{C}[h]$ acts multiplicity-free on $B$, where $M_\alpha = \mathbb{C} x^\alpha$ is the unique simple submodule of $B$ in its isomorphism class. Therefore $M_\alpha$ is also an $R_n$-module, necessarily simple, and $B$ is a semi-simple $R$-module. Using partitions, it is easy to describe when $M_\alpha \cong M_{\beta}$ as $R_n$-modules.

The function $\alpha$ has fibres $P_\alpha(i) = \{ j : \alpha(j) = i \}$, that induce a partition of the set $[n] = \cup_{i \geq 0} P_\alpha(i)$. Note that some of the sets may be empty and that the order of the sets in the partition is significant; we will call a sequence $\alpha$ an integer

\[ \lambda = \{ \lambda_1, \ldots, \lambda_r \} \] 

for the generalized symmetric group, we need to understand the simple submodules $\lambda^\alpha$. Moreover, we used the description of $\mathcal{R}$ in Proposition 3.11, mapping surjectively to $\mathcal{R}$. The first assertion is already motivated. The mapping $M_\alpha$ is the unique simple submodule of $B$ in its isomorphism class. Therefore $M_\alpha$ is also an $R_n$-module, necessarily simple, and $B$ is a semi-simple $R$-module. Using partitions, it is easy to describe when $M_\alpha \cong M_{\beta}$ as $R_n$-modules.

The function $\alpha$ has fibres $P_\alpha(i) = \{ j : \alpha(j) = i \}$, that induce a partition of the set $[n] = \cup_{i \geq 0} P_\alpha(i)$. Note that some of the sets may be empty and that the order of the sets in the partition is significant; we will call a sequence $\alpha$ an integer

\[ \lambda = \{ \lambda_1, \ldots, \lambda_r \} \] 

such that $\cup P_i = [n]$ an ordered partition. Note also that $\alpha$ is determined by $P_\alpha = (P_\alpha(i))_{i \geq 0}$. Similarly, a sequence $(\lambda_1, \ldots, \lambda_r)^\times$, is called an ordered partition of an integer $n$, denoted by $\lambda = (\lambda_i)_{i \geq 1} = (\lambda_1, \ldots, \lambda_r)$ $\vdash^n$ $n$, if the integers $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_r = n$. Given an ordered partition $P = (P_i)_{i = 1}^r$ of the set $[n]$, the partition $\lambda^P = (\lambda^P_1, \ldots, \lambda^P_r)$ $\vdash^n n$ is defined by $\lambda^P_i = |P_i|$. In particular, we put $\lambda^\alpha = \lambda^{P_\alpha}$, and say that $\lambda^\alpha$ is the ordered partition of the integer $n$, associated to $P_\alpha$.

The unordered partition $\tilde{P} = \{ P_i \}$ of $[n]$ corresponding to an ordered partition $P = (P_i)_{i \geq 1}$ is the set of subsets such that $P_i \neq \emptyset$. Similarly, the unordered partition $\tilde{\lambda} = \{ \lambda_i \}$ of an ordered partition $\lambda \vdash^n n$ is the set of nonzero elements in the sequence $\lambda$.

The action of the symmetric group on the set $[n]$ induces an action on the set of multiindices $\alpha$ by $\sigma \cdot \alpha = \alpha \circ \sigma^{-1}$, so that $P_{\sigma \cdot \alpha}(i) = \sigma(P_\alpha(i))$. Clearly then $P_\alpha$ and $P_{\beta}$ belong to the same orbit under the symmetric group if and only if $|P_\alpha(i)| = |P_{\beta}(i)|$ for $i = 0, 1, \ldots$, that is, exactly when the induced partitions of numbers $\lambda^\alpha = \lambda^\beta$.

Proposition 4.5. (1) For any simple $R_n$-module $M \subset B$ there is a multiindex $\alpha$ such that $M \cong M_\alpha$. There is an isomorphism $M_\alpha \cong M_{\beta}$ if and only if there exists $\sigma \in S_n$ such that $\sigma \cdot \alpha = \beta$ $\iff$ $\lambda^\alpha = \lambda^\beta$.

(2) The decomposition of the $R_n$-module $B$ into isotypical components is

\[ B = \bigoplus_{\lambda \vdash n} B_\lambda \]

where for each ordered partition $\lambda$ we have an isotypical component $B_\lambda$ of the form

\[ B_\lambda = \bigoplus_{\alpha \in \Omega_\lambda} \mathbb{C} x^\alpha, \]

and $\Omega_\lambda = \{ \alpha : [n] \rightarrow \mathbb{N} \mid \lambda^\alpha = \lambda \}$.

Proof. (1): The first assertion is already motivated. The mapping $x^\alpha$ to $x^{\sigma \cdot \alpha}$ defines an isomorphism $M_\alpha \rightarrow M_{\sigma \cdot \alpha}$ of $R_n$-modules. Conversely, if $M_\alpha \cong M_{\beta}$, then, using the description of $\mathcal{R}$ in Proposition 3.11, $p(\alpha(1), \ldots, \alpha(n)) = p(\beta(1), \ldots, \beta(n))$.

\footnote{This should be regarded as an infinite sequence where $P_i = \emptyset$ when $i > r$, for some integer $r$.}

\footnote{Again this is regarded as an infinite sequence such that $\lambda_i = 0$ when $i > r$.}

\footnote{If one thinks of the ordered partition of an integer $n$ as a sequence of columns with $|P_\alpha(0)|, |P_\alpha(1)|$ boxes, the relation between the concepts of ordered partitions $P$ and integers $\lambda \vdash^n n$ is similar to the one between Young tableaux and Young diagrams [Sag01, 2.1].}
for all symmetric polynomials \( p(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]^{S_n} \). This implies that \( \beta(i) = \alpha(\sigma(i)) \) for some \( \sigma \in S_n \).

(2) Immediate from (1).

The set \( \Omega_\lambda \) is the set of \( S_n \)-orbits of multi-indices \( \alpha : [n] \to \mathbb{N} \) for a given \( \lambda = \lambda^\alpha \). Select an ordered partition \( P^\lambda = (P_i^\lambda)_{i=1}^{\ell} \) of \([n]\) such that \( |P_i^\lambda| = \lambda_i \). Then \( \Omega_\lambda \cong \mathbb{S}_n \setminus G_P \), where \( G_P = \prod S(P_i^\lambda) \) and \( S(P_i^\lambda) \) is the symmetric group of the set \( P_i^\lambda \).

### 4.3. Decomposition of \( B \) for some complex reflection groups.

There is a striking use of \( R \) for the construction of simple \( \mathcal{D}^G_B \)-modules for generalized symmetric groups \( G = G(d, 1, n) = A(d, 1, n) \times S_n = A \times S_n \). The main point of our proof is that the description of the simple modules of \( R \) in Proposition 4.5 makes it easy to determine when, for simple \( \mathcal{D}^G_B \) modules \( L_1 \neq L_2 \), we have \( J_{G_P}(L_1) \neq J_{G_P}(L_2) \), where \( H = A \times G_P \) and \( G_P \) is a Young subgroup of \( S_n \). For the equivalent group representation case (using Proposition 2.2), the results about the symmetric group \( d = 1 \) go back to Specht [Spe35], see also [Pee75], and for the generalized symmetric group \( d > 1 \) see [ATY97].

To a multi-index \( \alpha : [n] \to \mathbb{N} \) we have associated an ordered partition \( P_\alpha \) of \([n]\) (4.2). Say that an unordered partition \( P = \{P_j\}, [n] = \bigcup P_j \) is an \( \alpha \)-partition if \( \bigcup P_j = P_\alpha \). Let \( \lambda^P = n \) be the (unordered) integer partition that is determined by \( P \); so that \( \lambda^P \) can be visualized by a sequence of at most \( n \) Young diagrams, each one of cardinality \( |P_\alpha(i)| \).

Let \( S(\Omega) \) be the symmetric group of a subset \( \Omega \) in \([n]\). Given a multi-index \( \alpha : [n] \to \mathbb{N} \) we put \( G^\alpha = A \times \prod S(P_\alpha(i)) \) and given an \( \alpha \)-partition \( P \) put \( G_\beta^P = A \times \prod S(P_j(i)) \subset G^\alpha \).

**Proposition 4.6.**

1. The simple \( \mathcal{D}^A_B \)-submodules of \( B \) are of the form \( N_\alpha = B^A x^\alpha \), where \( \alpha : [n] \to \{d-1\} \). If \( N_\beta = B^A x^\beta \), \( \beta : [n] \to \{d-1\} \), is another such module, then \( N_\beta \cong N_\alpha \iff \alpha = \beta \).

2. Let \( P \) be an \( \alpha \)-partition and define the polynomial \( s^\alpha_P = x^\alpha \), where \( s^\alpha_P \) is the Jacobian of the map \( B^{G^P} \to B^A \). Then

\[
N^\alpha_P = \mathcal{D}^G_B s^\alpha_P = B^{G^P} s^\alpha_P
\]

is a simple \( \mathcal{D}^{G^P}_B \)-module.

3. The module \( \mathcal{M}^\beta_P = J_{G_P}(N^\alpha_P) = \mathcal{D}^G_B s^\alpha_P \) is a simple \( \mathcal{D}^G_B \)-submodule of \( B \).

4. Let \( \beta : [n] \to \{d-1\} \) be another multi-index and \( Q \) be a \( \beta \)-partition. Then

\[
\mathcal{M}^\beta_P \cong \mathcal{M}^\beta_Q \iff \lambda^P = \lambda^Q \text{ and } \beta \in S_n \cdot \alpha.
\]

5. Let \( M \) be a simple \( \mathcal{D}^G_B \)-submodule of \( B \). Then there exists a multi-index \( \alpha : [n] \to \{d-1\} \) and an \( \alpha \)-partition \( P \) such that \( M \cong \mathcal{M}^\alpha_P \).

**Remark 4.7.**

1. The group \( \prod S(P_\alpha(i)) \) is the inertial group of the \( \mathcal{D}^A_B \)-module \( N_\alpha \) with respect to the homomorphism \( B^{G^P} \to B^A \) (see [Käl]) and subgroups of the form \( \prod S(P_j(i)) \) are its parabolic subgroups, i.e. subgroups that preserves some closed point in Spec \( B^A \).

2. Let \( G_P = \prod S(P_j) \subset S_n \) be the Young group of a partition \( P = \{P_j\} \) of \([n]\). The Jacobian of the invariant map \( B^{G^P} \to B \) is independent (up to a multiplicative constant) of the choice of homogeneous coordinates in the polynomial ring \( B^{G^P} \); in the calculation below we will use that it may be taken as \( s_P = \prod_{i,j} (x_k - x_l) \). Similarly, for a 2-step partition \( P = \{P_{ij}\} \), we may take the van der Monde determinants

\[
s_P = \prod_{ij} \prod_{k < l \in P_{ij}} (x_k^l - x_l^k).
\]
A key step in the proof of Proposition 4.6 is the following relation between an integer partition \( \lambda^P \vdash n \) that comes from a partition \( P \) of the set \([n]\) and the partition \( \lambda^P \vdash n \), where \( \alpha \) is a multiindex that occurs in an expansion of the Specht polynomial \( s_P \) of \( P \). In our context, the result identifies which isomorphism class of \( \mathcal{R} \)-modules the Specht polynomial corresponds to.

Let \( \lambda^P \vdash n \) denote the conjugate of a partition \( \lambda \vdash n \), i.e. the partition whose \( i \)th part \( \lambda_i^P \) is the number of \( m \) with \( \lambda_m \geq i \). The conjugate of a partition \( P = \{ P_i \} \) is \( P^c = \{ Q_i \} \) where \( Q_i = \{ j \in [n] \mid j \in P_s \} \) belongs to at least \( i \) different \( P_s \), so that \( \lambda_P^P = \lambda_P^c \).

**Lemma 4.8.** Let \( x^\alpha \) be a non-zero term in an expansion of \( s_P \). Then \( (\lambda^\alpha)^c = \lambda^P \). More precisely, in the notation of Proposition 4.5, we have \( s_P \in B_\lambda \), where the ordered partition \( \lambda = ([Q_1], \ldots, [Q_s]) \vdash n \) and \( Q \) is the conjugate of \( P \).

Proof. If \( P = \{ P_j \}_{j=1}^r \) we write \( \lambda^P = \{ n_1, \ldots, n_r \} \vdash n \), where \( n_j = |P_j| \). Expanding the Specht polynomial of one subset \( P_j \)

\[
\prod_{1 \leq k < i < n_j} (x_k - x_i) = \sum c_{\alpha_i} x^{\alpha_i}, \quad \alpha_i : [n_j] \to \mathbb{N},
\]

then if \( c_{\alpha_i} \neq 0 \), it follows that the (unordered) set \( \{ \alpha_i(1), \alpha_i(2), \ldots, \alpha_i(n_j) \} = \{ n_j - 1, n_j - 2, \ldots, 1, 0 \} \); hence \( \lambda^{\alpha_i} = \{ 1, 1, \ldots, 1 \} \vdash n_j \) (so that \( \lambda^{\alpha_i} = \{ n_j \}^c \)). Since

\[
s_P = \prod_{j=1}^r s_{P_j} = \prod_{j=1}^r (\sum c_{\alpha_i} x^{\alpha_i}) = \sum a_\alpha x^\alpha
\]

it follows that if \( a_\alpha \neq 0 \), then \( \lambda^\alpha = |\{ \alpha(i) = i \}| \) is the number of subsets \( P_j \) with \( |P_j| > i \). This implies that \( \lambda^\alpha = (\lambda^P)^c \). \( \Box \)

**Proof of Proposition 4.6.** (1): We have

\[
B^A = C[\{ x_1^a, \ldots, x_n^d \}] \subset D^A_B = C[\{ x_1^a, \ldots, x_n^d, \partial_{x_1}^d, \ldots, \partial_{x_n}^d \}],
\]

so that \( C[\{ \partial_{x_1}^d, \ldots, \partial_{x_n}^d \}] \subset (D^A_B)^- \subset \sum_{i=1}^{n} D^A_B \partial_{x_i}^d \) and hence \( \text{Ann}(D^A_B) \cdot (B) = \{ x^\alpha \mid \alpha : [n] \to [d - 1] \} \). Notice that \( C[\{ \partial_{x_1}^d, \ldots, \partial_{x_n}^d \}] \subset D^A_B \). If \( Cx^\alpha \cong Cx^\beta \) as \( C[\{ \partial_{x_1}^d, \ldots, \partial_{x_n}^d \}] \)-modules, then \( \alpha = \beta \). Therefore \( B = \oplus D^A_B x^\alpha \), where the sum runs over multi-indices \( \alpha : [n] \to [d - 1] \), and \( N_\alpha \cong N_{\beta} \) implies \( \alpha = \beta \). It is straightforward to see that \( B^A \) is a simple \( D^A_B \)-module, implying that each \( N_\alpha \) is also simple.

Another way to see that \( N_\alpha \) is a simple \( D^A_B \)-module is to appeal to \( A \)-semiinvariants, which is in a sense more easy to see. This is what we will have to do in (2) below.

(2): The element \( s_{P \times x^\alpha} \) defines a \( G_p^r \)-semiinvariant \( \chi : G_p^r \to C^A \), and \( \chi \) generates the \( D^A_{G_p^r} \)-module of all semi-invariants associated to \( \chi \). It then follows from Proposition 2.2 that \( N_\alpha^A = D^A_{G_p^r} s_P x^\alpha = B^A_{G_p^r} s_P x^\alpha \) and that this is a simple \( D^A_{G_p^r} \)-module.

(3): Put \( D_1 = D^A_{G_p^r} \) and \( D_2 = D^A_B \). Then \( C s_{\alpha}^A \) forms a 1-dimensional \( D^A_B \)-module, so the assertion follows from Theorem 4.1, (1).

(4): If \( M^A_Q \cong M^A_{P} \), then

\[
C s_{Q x^\beta} \cong C s_{P x^\alpha}
\]

as \( R_\alpha \)-modules. Since \( s_P, s_Q \in B^A \) it follows from (1) and Proposition 4.5 (1) that there exists \( \sigma \in S_n \) such that \( \beta \equiv \sigma(\alpha \mod d) \), and since \( \alpha, \beta : [n] \to [d - 1] \) this implies that \( \beta = \sigma \alpha \). Therefore we have isomorphisms, where the second one comes from the action of \( \sigma \),

\[
C s_{Q x^\beta} \cong C s_{P x^\alpha} \cong C s_{\sigma P x^\beta},
\]

and hence \( C s_{Q} \cong C s_{\sigma P} \cong C s_P \); hence by Lemma 4.8, \( \lambda^Q = \lambda^P \).

Conversely, if \( \lambda^Q = \lambda^P \) and \( \beta \in S_n \cdot \alpha \), then there exists \( \sigma \in S_n \) such that \( \beta = \sigma \alpha \) and \( Q = \sigma P \), and hence the \( D^A_{G_p^r} \)-homomorphism \( \sigma : B \to B \) induces an isomorphism \( M^A_Q \cong M^A_P \).
(5): First note that if $\lambda^\alpha \neq \lambda^\beta \vdash n$, then $\beta \not\in S_n \cdot \alpha$, so that the corresponding modules are non-isomorphic. It follows that the set of nonisomorphic simple $D_B^G$-submodules that arise in (4) is parametrised by a sequence of at most $d$ Young diagrams. This agrees with the parametrisation of the set of conjugacy classes $\Cl(G(d, 1, n))$ [Osi54]. □

Consider now $G = G(m, e, n) = A \times S_n$ with $e > 1$, and put $D = D_B^G = (D_B^A)^{S_n}$. A possible strategy to construct the simple $D$-submodules of $B$ is as follows. Let $\{N_i\}_{i=1}^r$ be a set of representatives of the simple $D_B^A$-submodules of $B$ (this is done for $A = A(e,c,2)$ in Section 6.1). Let $G_i \subset S_n$ be the inertial group of $N_i$, $i = 1, \ldots, r$, and put $G_i = A \rtimes G_i$. Let $\{M_{ij}\}$ be a set of representatives of the simple $D_{B_i}^G$-submodules of $B$. To find such modules $M_{ij}$ it is natural to consider parabolic subgroups $G_{ij}^i \subset G_i$, let $s_{P,i}$ be the Jacobian of the invariant map $B^{G_{ij}^i} \to B$, and expect that $M_{ij}$ is contained in the composition series of $J_G^{G_{ij}^i}(D_B^{G_{ij}^i} s_{P,i})$.

Then we can construct the $D$-module

$$M_{ij} = J_G^{G_i}(N_i) \otimes_B J_G^{G_{ij}^i}(M_{ij}).$$

Since $B$ is semisimple over $D$ and $B$ is free over $B^G$, $G$ being a reflection group, it follows that the $D$-submodules $J_G^{G_i}(M_{ij})$ and $J_G^{G_i}(N_i)$ also are free over $B^G$. An interesting problem would be to understand the decomposition of $M_{ij}$ into simples. When $e = 1$, so that $B^A$ is polynomial ring, we are in the situation of Proposition 4.6 where these modules are already studied, albeit expressed differently.

5. The branch rule for $S_n$

We start with a fairly general condition in (5.1) that ensures that the restriction of a simple $D_1$-module to a module over a subring $D_2 \subset D_1$ is multiplicity free. In (5.2) we give a proof of the classical branching rule for the symmetric group $S_n$, expressed in terms of $D$-modules and based on a lowest weight argument, where $D = D_B^{S_n}$. In (5.3) and (5.4) we discuss the branching graph of the $D$-module $B$ and provide $B^{an}$ with canonical bases, which turn out to coincide with Young bases.

5.1. The generalized symmetric groups. The generalized symmetric group $G_n = G(m, 1, n) = A(m, 1, n) \rtimes S_n$ acts on $\overline{B} = \mathbb{C}[x_1, \ldots, x_n]$ by permuting the coordinates and by multiplying by $m$th roots of unities. It contains the subgroup $G_{n-1} \subset G_n$ of elements that fix the variable $x_n$. Let $D_n = D_B^{G_n}$ be the ring of invariant differential operators, so that $D_n \subset D_{n-1}$. Letting $B_{n-1} = \mathbb{C}[x_1, \ldots, x_{n-1}]$, we note that

$$D_{n-1} = \bar{D}_{n-1}[x_n, \partial_n],$$

where $\bar{D}_{n-1} = D_{B_{n-1}}^{G_{n-1}}$.

The branch rule for representations of the generalized symmetric groups, describing induction from $G_{n-1}$ to $G_n$-representations, is the second statement in Proposition 5.1 below. By Proposition 2.2 it is equivalent to the first statement on restriction from $D_{n-1}$ to $D_n$-modules, which we will see is a consequence of the determination of $\mathcal{R}$ for the generalized symmetric group in Proposition 3.11.\footnote{The correspondence between operations like induction and restrictions for representations and direct and inverse images of $D$-modules is described in more detail in [Kål].}

**Proposition 5.1.** Let $N$ be a simple $D_{n-1}$-submodule of $B$ and $V$ be the corresponding irreducible $G_n$-representation, so that $V \sim_{G_n} N$ in Proposition 2.2.
(1) The restriction of \( N \) to a \( D_n \)-module \( \text{res}(N) \) by the inclusion \( D_n \subset D_{n-1} \) splits into a direct sum

\[ \text{res}(N) = \bigoplus M_i \]

of pairwise non-isomorphic simple submodules.

(2) The induced representation \( \text{ind}_{G_{n-1}}^{G_n}(V) \) splits into a sum

\[ \text{ind}_{G_{n-1}}^{G_n}(V) = \bigoplus W_i, \]

of pairwise non-isomorphic irreducible representations, where \( W_i \sim_{G_n} M_i \).

We put \( D_B^+(k) = \{ P \in D_B \mid [\nabla_n, P] = kP \} \) and \( D_B^- = \bigoplus_{k<0} D_B^+(k) \). Let \( \tilde{a}_n = \sum_{k \geq 0, l \geq -k} C_{p_{k,l}^{(n)}} \) be the Lie algebra in Proposition 3.8, where we have the power differential operator \( p_{k,l}^{(n_i)} = \sum_{i=1}^{n_i} x_i^k \partial_i \). We notice here the following:

(i) \( x_n, \partial_n \in D_{n-1} \),
(ii) \( \tilde{a}_{n-1} \subset D_{n-1} \subset D_{n-1} + \tilde{a}_{n-1} + D_B \partial_n \),
(iii) \( \tilde{a}_{n-1} \subset D_{n-1} + D_B^- \),

where (ii) is a consequence of Proposition 3.6, and (iii) follows from the relation \( p_{k,l}^{(n_i)} = p_{k,l-1}^{(n_i)} + x_n^l \partial_n \).

Proposition 5.1 is a consequence of (i-iii) and the fact that the algebra \( R = D_n^0/(D_n^0 \cap (D_n D_n)) \) is commutative. In the theorem below, which encodes this argument, we consider general graded subrings \( D = \bigoplus_{k \in \mathbb{Z}} D(k) = D^+ \oplus D^0 \oplus D^- \subset D_B \) as in Section 3.1.

**Theorem 5.2.** Let \( D_2 \subset D_1 \) be an inclusion of graded subrings of \( D_B \) as above, where \( x_n, \partial_n \in D_1 \), and put \( D_1 = (D_1)^{\partial_n} = \{ P \in D_1 \mid [\partial_n, P] = 0 \} \). Assume that there exists a graded Lie subalgebra \( a_1 \) of \( D_1 \) such that:

1. \( a_1^- \subset D_1^- \subset D_1 a_1^- + D_B \partial_n \),
2. \( a_1^- \subset D_2^- + D_B^- \).

Let \( N \) be a simple \( D_1 \)-submodule of \( B \) such that \( \dim_{\mathbb{C}} \text{Ann}_{D_1^-}(N) = 1 \) and assume that for any simple \( D_2 \)-submodule \( M \subset N \) we have \( \dim_{\mathbb{C}} \text{Ann}_{D_2^-}(M) = 1 \); these two conditions are satisfied if \( R_1 \) and \( R_2 \) are commutative (\( R_k = D_k^0/(D_k^0 \cap (D_k D_k^-)) \)). Then it follows that the restriction \( \text{res}(N) \) to a \( D_2 \)-module splits into a direct sum

\[ \text{res}(N) = \bigoplus M_i \]

of pairwise non-isomorphic simple submodules.

**Remark 5.3.**

1. It would be interesting to find applications of Theorem 5.2 in other situations. Let \( H \subset G \subset \text{Gl}(V) \) be an inclusion of finite groups. Put \( D_2 = D_B^H \subset D_1 = D_B^H \), \( R_H = D_1^H/(D_1^H \cap (D_1 D_1^H)) \), \( R_G = D_2^H/D_1^H \cap (D_2 D_1^H) \). Assume that \( H \) fixes the variable \( x_n \) and that \( a_1 \) is a graded Lie subalgebra of \( D_1 \) such that (1) and (2) in Theorem 5.2 are satisfied. If now \( R_H \) and \( R_G \) are commutative, it follows as in the proof below that any irreducible representation of \( G \) restricts to a multiplicity free representation of \( H \).

2. Note that \( R_1 = D_1^H/D_1^0 \cap (D_1 D_1^-) \), since \( \partial_n \in D_1^- \).

**Proof of Proposition 5.1.** We know that the algebras \( R_i \) are commutative by Propositions 3.8 and 3.6, and the above remark. Putting \( a_1 = a_{n-1} \), (i) implies the first and (ii-iii), where the ring \( D_{n-1}^- \) is a subring of \( D_1 \), implies the conditions (1-2) in Theorem 5.2, hence we get (1). The corresponding assertion (2) for representations follows since \( \text{ind}_{G_{n-1}}^{G_n}(V) \sim_{G_n} \text{res}(N) \) (Prop. 2.3). □
Proof of Theorem 5.2. If $R$ is commutative and $M$ is a simple $D_r$-module, then Theorem 3.1 implies that $\dim \text{Ann}_{D_r}(M) = 1$. So assume $\text{Ann}_{D_r}(N) = Cy$ for some homogeneous elements $y \in N$. Similarly for any simple $D_2$-submodule $M \subset \text{res}(N)$ we have $\text{Ann}_{D_2}(M) = Cz$, for a homogeneous polynomial $z \in M$, and we then put $\deg(M) = \deg(z)$. Since $V \in D_2$ it follows that if $M_1$ is another simple submodule of $N$ and $\deg(M) \neq \deg(M_1)$, then $M \neq M_1$. Conversely, we will prove that if $\deg(M) = \deg(M_1)$, then $M = M_1$. Expand $z = y_0 + y_1 x_n + \cdots + y_\alpha x_n^\alpha$, where $\partial_n(y_\alpha) = 0$, $y_\alpha \in N_1$ since $x_n, \partial_n \in D_1$, and $y_\alpha \neq 0$. Let $r_1 \in a_1^\alpha$, so that by (2), $r_1 = r_2 + r^{(n)}$, where $r_2 \in D_2$, $r^{(n)} \in D_B^{x_\alpha}$. Since $r_2(z) = 0$, we have (as detailed below)

$$r_1(z) = r_1(y_\alpha)x_n^\alpha + (\text{I.o. in } x_n) = r^{(n)}(y_0 + y_1 x_n + \cdots + y_\alpha x_n^\alpha).$$

The first equality follows since $r_1 \in D_1$, so that $r_1 = \sum c_{\gamma,b}(x')\gamma(\partial')^b\partial_n^b$, where $x' = (x_1, \ldots, x_{n-1})$, $\partial' = (\partial_1, \ldots, \partial_{n-1})$, $|\beta| + b > |\gamma|$, and $\partial_n(y_\alpha) = 0$. Therefore $r_1(y_\alpha) = 0$. Hence by (1), $y_\alpha \in \text{Ann}_{D_1}(N)$, and therefore $Cy_\alpha = \text{Ann}_{D_1}(N) = Cy$.

We have also $\text{Ann}_{D_2}(M_1) = Cz'$ for some homogeneous polynomial $z'$, and it suffices now, since $M$ and $M_1$ are simple, to prove that $Cz = Cz'$ when $\deg(z') = \deg z$. We expand $z' = y'_\alpha x_n^\alpha + (\text{I.o. in } x_n)$ in the same way as $z$ above, and by the same argument as before we have $Cy'_\alpha = \text{Ann}_{D_1}(N) = Cy$. Therefore $\deg(y'_\alpha) = \deg(y_\alpha)$, and as $\deg(z) = \deg(z')$, it follows also that $a = a'$. Multiplying $z'$ by a complex number so that $y'_\alpha = y_\alpha$, it suffices now to see that $z = z'$. Assume on the contrary that

$$z - z' = y'_\alpha x_n^\alpha + (\text{I.o. in } x_n) \neq 0.$$ 

Since $z$ and $z'$ are homogeneous of equal degree it follows that $\deg(z - z') = \deg(z)$. Clearly $b < a$, and again we have $y'_\alpha \in \text{Ann}_{D_1}(N)$ so that $\deg(y'_\alpha) = \deg(y_\alpha)$, implying that $\deg(z - z') < \deg(z)$, which is a contradiction. Therefore $z = z'$.

5.2. The symmetric group. The symmetric group $S_n$ is a subgroup of $G_n$, so we have actions of $S_n$ and its subgroup $S_{n-1}$ on both $B$ and $D_B$, and we now put instead $D_n = D_B^{S_n} \subset D_{n-1} = D_B^{S_{n-1}}$. We want to describe the decomposition (1) in Proposition 5.1 in more detail when $m = 1$, which, by Proposition 2.2, also implies the very well-known branching rule for the symmetric group.

Remark 5.4. In [Pee75, JK81, Th. 2.4.3] the proof of the branching rule for representations of the symmetric group requires the non-trivial fact that the standard Specht polynomials of shape $\lambda \vdash n$ form a basis of a simple $S_n$-module $V_\lambda$. The proof below is instead based on Corollary 3.3 and Proposition 4.6, where the latter shows that any simple $R_n$-submodule of $B^{ann}$ is isomorphic to $ks\rho$ for some Specht polynomial $s\rho$. The fact that the standard Specht polynomials, indexed by the standard Young tableaux, form a basis is then an immediate consequence of the branching rule, as described in Section 5.3.

We assume now that every partition $\lambda \vdash n$ is ordered, defining a function $\lambda : \{1, 2, \ldots\} \rightarrow \mathbb{N}$ such that $\lambda(i) \geq \lambda(i + 1)$; this is the same as associating a Young diagram to $\lambda$.

We already know that $B^{ann}$ is a Gelfand module, but we can be more precise.

Corollary 5.5. The $S_n$-representation $B^{ann}$ is multiplicity free and is canonically decomposed

$$B^{ann} = \bigoplus_{\lambda \vdash n} V_\lambda.$$
where the representations $V_\lambda$ are irreducible and of the form

\[ V_\lambda = k[S_n]s_P = \{ p \in B \mid \sum_{i=1}^n x_i^k \partial_i^l p = 0, 0 \leq k \leq n - 1, \text{ and } p = \sum_{\lambda_\alpha = \lambda^\alpha} c_{\alpha}x^\alpha \}, \]

where $s_P$ is the Specht polynomial of a partition $P$ of $[n]$ such that $\lambda P = \lambda$ (see Remark 4.7).

**Proof.** By Proposition 4.6 any simple $\mathcal{D}_n$-submodule of $B$ is isomorphic to a module of the form $\mathcal{D}_n s_P$ and $\mathcal{D}_n s_p \cong \mathcal{D}_n s_Q$ for another partition $Q$ of $[n]$ if and only if $\lambda = \lambda^Q$. If $x^\alpha$ is a monomial term in $s_P$ then $\lambda^\alpha = \lambda^Q$. By Proposition 2.2 it follows that $B^{ann} = \oplus_{\lambda^\alpha = \lambda^Q} k[S_n]s_P$. Putting $V_\lambda = k[S_n]s_P$, where $\lambda = \lambda_P$, it follows that any monomial term $x^\alpha$ in any polynomial that belongs to $V_\lambda$ satisfies $\lambda^\alpha = \lambda\alpha$. The description of $B^{ann}$ follows from Proposition 3.8, (2). $\square$

One says that a box in a Young diagram is **addable** if one gets another Young diagram by adding a box.

**Theorem 5.6.** Let $N_\lambda = \mathcal{D}_{n-1}V_\lambda$ be a simple $\mathcal{D}_{n-1}$-module corresponding to the partition $\lambda \vdash n - 1$ in Proposition 4.6, where $kv_\lambda = \text{Ann}_{\mathcal{D}_{n-1}}(N_\lambda)$, and let $\text{res}(N_\lambda)$ denote its restriction to $\mathcal{D}_n$-module.

1. Then

\[ \text{res}(N_\lambda) = \bigoplus_{\mu^\alpha = \lambda^\alpha} N_\mu \]

is multiplicity-free and the simple direct composants $N_\mu$ correspond to all partitions $\mu \vdash n$ that can be formed by adding a box to $\lambda$.

2. Assume that $N_\mu$ corresponds to adding a box to the $r$th row of the Young diagram of $\lambda$. Then the decomposition in (1) is determined by submodules $N_\mu \subseteq N_\lambda$ where $N_\mu$ is generated by a homogeneous lowest weight polynomial of the form $v_\mu = x^n_{\alpha}v_\lambda + v'_\mu \in \text{Ann}_{\mathcal{D}_n}(N_\lambda)$, where $a = \lambda(r)$. We have $\deg(v_\mu) = \sum i\lambda(i) + a - l$, where $l$ is the number of $i$ such that $\lambda(i) \neq 0$.

That a box is addable means more precisely the following. Given an ordered partition $\lambda \vdash n - 1$ and an integer $r$ one gets the function $\mu : \{1, 2, \ldots \} \rightarrow \mathbb{N}$ by $\mu(r) = \lambda(r) + 1$, $\mu(i) = \lambda(i)$, $i \neq r$, then the index $r$ is addable if it again is non-increasing. We note that an index $r$ is addable to $\lambda \vdash n - 1$ if and only if the index $a = \lambda(r)$ is addable to the conjugate partition $\lambda^c \vdash n - 1$; we then also write $a \in \lambda$.

Define the bilinear form $\langle \cdot, \cdot \rangle : B \otimes C B \rightarrow C, \langle f, g \rangle = (f(\partial)g)|_{x=0}$, so that the monomials form an orthogonal basis with respect to $\langle \cdot, \cdot \rangle$, and define the antiisomorphism $t : \mathcal{D}_B \rightarrow \mathcal{D}_B$ by $x_i^t = \partial_i$, $\partial_i^t = -x_i$, and $(PQ)^t = Q^tP^t$. Then we have:

1. $\langle Pf, g \rangle = \langle f, P^t g \rangle$, $f, g \in B$, $P \in \mathcal{D}_B$.
2. $(\mathcal{D}^n)^t = \mathcal{D}^n$, $(\mathcal{D}_n^t)^t = \mathcal{D}^n$, and $\mathcal{D}_n^t = \mathcal{D}_n$.
3. If $f$ and $g$ are homogeneous of different degrees, then $\langle f, g \rangle = 0$. More precisely, if $f_i = \sum_{\alpha \in \Gamma_i} c_\alpha x^\alpha \in B_i$, $\mathcal{D}_n^t \subseteq B_i$, where the set of multiindices $\Gamma_1 \cap \Gamma_2 = \emptyset$, then $\langle f_1, f_2 \rangle = 0$.
4. $\langle f, f \rangle \neq 0$, $f \in B$.

Notice that this follows from (ii) that if $v \in B^{ann}$ and $w \in B$ is such that $\langle w, v \rangle = 0$, then $\langle \mathcal{D}_n^t w, v \rangle = 0$.

**Proof.** a) **Multiplicity 1:** (This is implied by Proposition 5.1 but we give a separate proof) If $N \subset N_\lambda$, where $N$ is a simple $\mathcal{D}_n$-submodule, by Corollary 3.3 and Proposition 3.8 $\text{Ann}_{\mathcal{D}_{n-1}}(N)$ is a simple module over the commutative ring
We assert that \[ v_n = v_{n-1}x_n^a + w \quad (w \text{ is of lower order in } x_n) \]

where

\[
v_n \in \text{Ann}_{\mathcal{D}^-}(N_\lambda) = \{ \mu \in N_\lambda \mid \sum_{i=1}^n x_i^k \partial_i^l(\mu) = 0, 0 \leq k < l \leq n-1 \}. \]

(Prop. 3.8). Therefore \( \sum_{i=1}^{n-1} x_i^k \partial_i^l(v_{n-1}) = 0 \), when \( 0 \leq k < l \leq n-2 \), so that \( v_{n-1} \in \text{Ann}_{\mathcal{D}^-}(N_\lambda) = kv_\lambda \). We can therefore assume, after multiplying \( v_n \)
by a constant, that \( v_{n-1} = \lambda \). Assume that \( N' \subset N_\lambda \) and \( N' \cong N \). Again \( \text{Ann}_{\mathcal{D}^-}(N') = kv'_n \).

Then clearly \( \deg v_n = \deg v'_n \), and after multiplying by a constant we get the expansion

\[ v'_n = x_n^a v_\lambda + w'. \]

We assert that \( v_n = v'_n \). Assuming the contrary,

\[
0 \neq v_n - v'_n = w - w' = ex_n^a v_\lambda + w' \quad \text{deg}(v_n - v'_n) = a + \deg sQ
\]

But \( b < a \), which results in the contradiction \( \deg(v_n - v'_n) < a + \deg sQ \).

b) Existence of submodules: We have a natural exhaustive filtration by \( \mathcal{D}_n \)-submodules

\[ N_0 = \mathcal{D}_n v_\lambda \subset \cdots \subset N_j = \sum_{i \leq j} \mathcal{D}_n x_n^i v_\lambda \subset \cdots \subset N_\lambda \]

Clearly \( N_0 \) is a simple \( \mathcal{D}_n \)-module, so assume that \( a > 0 \). We assert:

\[
a \in \lambda \iff \text{there exists } v_a = x_n^a v_\lambda + w_a \in \text{Ann}_{\mathcal{D}^-}(N_a) \text{ and } kv_a \text{ is a simple } R_n \text{-module.}
\]

\[ \iff \text{There exists a partition } P \text{ of } [n-1] \text{ such that } \lambda^P = \lambda \text{ and } kv_\lambda = ksp \text{ as } R_{n-1}-\text{module. Then if } x^\alpha \text{ is a monomial term in } v_\lambda \text{ it follows that } x^\alpha = \lambda^c \text{ for } n-1, \]

where \( \lambda^c \) is the conjugate of \( \lambda \) (see Lemma 4.8). If there exists a vector \( v_a \) as stated, so that \( x^i = x_n^a x^\alpha \) is a monomial term in \( v_a \), then \( \lambda^c = n \). This implies that \( a = (\lambda^c)^i(i) = \lambda(i) \) for some index \( i \). (We can also say that \( a \) is admissible to \( \alpha \) if \( a \in \lambda = (\lambda^c)^c \).

\[ \Rightarrow \text{We can assume that } v_\lambda = sQ \text{ for some partition } Q = \{ Q_i \} \text{ of } [n-1] \text{ such that } \lambda Q = \lambda. \]

If \( a = \lambda = \dim Q \), then \( \lambda Q = \lambda \). If \( a = \lambda = \dim Q_i \), then \( \lambda Q = \lambda \).

Notice that \( sP \) is not a semi-invariant of the Young subgroup of \( Q \), so that \( sP \notin N_\lambda \).

Since \( \{ x^\alpha \} \) is an orthogonal basis for \( B \) we have \( \langle sP, x_s n^a sQ \rangle = 0 \).

Moreover, \( \deg sP = a + \deg sQ > \deg(x_n^a sQ) = i + \deg sQ, \) when \( i < a \), so that

\[ \langle sP, x_n^a sQ \rangle = \langle \mathcal{D}_n sP, x_n^a sQ \rangle = \langle (\mathcal{D}_n^+ + \mathcal{D}_n^0) sP, x_n^a sQ \rangle = 0, \]

i.e. \( sP \perp N_{a-1} \). Therefore \( x_n^a sQ \notin N_{a-1} \), hence there exists elements of the form \( v_b \) in \( \text{Ann}_{\mathcal{D}_n}(N_a) \neq 0 \) where \( b \geq a \). Now if \( v \in \text{Ann}_{\mathcal{D}_n}(N_a) \) is homogeneous, then \( \deg Pu \geq \deg v \) when \( P \in \mathcal{D}_n \); since moreover \( N_a = \text{Ann}_{\mathcal{D}_n}(N_a) \) it follows that there exists an element in \( \text{Ann}_{\mathcal{D}_n}(N_a) \) of the form \( v_a \) such that \( kv_a \) is \( R_n \)-simple.

It follows from a) and b) that

\[
N_\lambda = \bigoplus_{a \in \lambda} \mathcal{D}_n v_a.
\]
It remains to see that $\deg v_\lambda = \sum_i i\lambda(i) - l$, which we prove by induction in $n - 1$. The vector $v_\lambda$ arises from some $v_{\lambda'} \in \text{Ann}_{D_{n-2}}(B)$ where $\lambda' \vdash n - 2$, in the form $v_\lambda = x_n^\lambda v_{\lambda'} + \cdots$. By induction
\[
\deg v_{\lambda'} = \sum i\lambda'(i) - l', \quad l' = \text{number of } i \text{ such that } \lambda'(i) \neq 0.
\]
This implies that
\[
\deg v_\lambda = a' + \deg(v_{\lambda'}) = a' + \sum i\lambda(i) - l' = \sum \lambda(i) - l.
\]
since $a' = \lambda'(j)$ for some $j$, so that $\lambda(j) = \lambda'(j) + 1$, $\lambda(i) = \lambda'(i), i \neq j$ (treat the cases $a' = 0$ and $a' \neq 0$ separately). \[\square\]

5.3. The branching graph. Let us start with the general situation of inclusions of graded rings of the type in Section 3.1
\[D_n \subset D_{n-1} \subset \cdots \subset D_2 \subset D_1,\]
so that in particular $\nabla \in D_i$ for all $i$. Let $M$ be a simple $D_1$-module such that its restriction to $D_i$-module is semisimple and $D_i \text{Ann}_{D_{n-1}}(M) = M$, $i = 1, \ldots, n$ (see Theorem 3.1 and Section 3.2). Let $C_i$ be the set of isomorphism classes of simple $D_i$-submodules of $M$. The branching graph $B(M)$ of $M$ (or oriented Bratelli diagram) is defined as follows. Its set of vertices is $\bigcup_{n \geq 1} C_i$ and there are $\dim_M \text{Hom}_{D_i}(N_\lambda, \text{res}^{D_i}_{D_{i+1}} N_\mu)$ directed edges from the vertex $\mu \in C_{i+1}$ to the vertex $\lambda \in C_i$ (where $N_\lambda, N_\mu$ are representative modules for $\lambda$ and $\mu$), and there are no other edges. Let us agree to say that a vertex $\lambda$ in $B(M)$ has the level $i$ if $\lambda \in C_i$, and write $|\lambda| = i$. Write $T > T'$ when $T$ and $T'$ are directed paths in $B(M)$ with a common first vertex (which normally is the root $C_1$ of $B(M)$; this is a singleton set) and the last vertex of $T'$ is joined by an edge with the last vertex of $T$. It is a fundamental problem to give a combinatorial description of the oriented rooted tree $B(M)$, given parametrisations of the sets $C_i$.

We will give such a description of $B = B(B)$ when $D_i = D^n_B$, so that by Theorem 5.2 there is at most one edge between two vertices. The Young graph $\mathcal{Y}$ is the oriented graph whose set of vertices is $\bigcup_{n \geq 1} \mathcal{P}(i)$, where $\mathcal{P}(i)$ is the set of partitions $\lambda \vdash i$. There is an edge from the vertex $\lambda' \in \mathcal{P}(i - 1)$ to $\lambda \in \mathcal{P}(i)$ if the Young diagram of $\lambda$ is obtained from $\lambda'$ by adding 1 addable box, and there are no other edges in $\mathcal{Y}$. It is easy to see that the paths in $\mathcal{Y}$ are in correspondence with standard Young tableau.

**Proposition 5.7.** $B$ is isomorphic to $\mathcal{Y}$.

**Proof.** Since $C_i$ is parametrised by the set of partitions of the integer $i$ (Prop. 4.6) it is clear that the cardinality of the set of vertices in $B$ agrees with that of $\mathcal{Y}$. That the edges agree follow from Theorem 5.6. \[\square\]

In fact, the branching graph $B$ is isomorphic to the branching graph $C$ of the sequence of group inclusions $S_1 \subset S_2 \subset \cdots \subset S_{i-1} \subset S_i \subset \cdots \subset S_n$, described in detail in [OV96] (see also [Kle05]). The vertex set of $C$ is $\bigcup_{i \geq 1} S_i$ and there is an edge from the vertex $\mu \in S_{i+1}$ to the vertex $\lambda \in S_i$ if $\dim_C \text{Hom}_{S_i}(\text{res}_{S_{i+1}}^{S_i} V^\lambda, V^\mu) = 1$ ($\leq 1$ by Proposition 5.1), where $\text{res}_{S_{i+1}}^{S_i} V^\lambda$ is the restriction of a representative $V^\lambda$ of $\lambda$ to a representation of $S_{i+1}$, and there are no other edges in $C$. It follows from Proposition 2.3 and Frobenius reciprocity that the oriented graphs $B$ and $C$ are isomorphic.

**Remark 5.8.** In [OV96, Th. 6.7] it is proven that $C$ is isomorphic to $\mathcal{Y}$, which together with Proposition 2.3 also implies Proposition 5.7 (and vice versa, Proposition 5.7 implies $\mathcal{Y} \cong C$). Notice that in our setup, where first the branch rule in
Theorem 5.6 is proven rather explicitly, one immediately gets that \( C \) is isomorphic to \( Y \). The proof of [loc cit] is based on an identification of paths in \( C \) with paths in \( Y \), using the notion of weights of a GZ-algebra as a bridge, so that the complete branching rule in Theorem 5.6 (transcribed to groups) is achieved at the same time as [loc cit] is established.

Below we will make no distinction between a path in the branching graph and a path in the Young graph, so that paths in \( B \) of length \( n \) are identified with standard tableaux of size \( n \).

5.4. The canonical basis of \( B^{ann} \). Let as above \( D_i = D_i^{\urcorner} = D(V_i^{\urcorner}) \subset C D(V_i) \), where \( V_i = \sum_j c_{ij} x_j \) and \( V_i' = \sum_{j=1}^n c_{ij} x_j \). We have \( R_i = D_i^{\urcorner}/(D_i^{\urcorner} \cap D_i) \cong C[\nabla_1, \ldots, \nabla_i]/S_i \), so that \( B^{ann} = \text{Ann}_D(B) \) is a finite-dimensional semisimple \( R_i \)-module. Let \( i > 1 \) be an integer and \( C_i-1 = \{ v_{i-1}^j \} \) be a basis of \( B^{ann} \) which is compatible with its isotypic decomposition as \( R_{i-1} \)-module. Now the vector space \( \text{Ann}_D(B) = \text{Ann}_D(D_i^{\urcorner}) \) is a multiplicity free \( R_i \)-module (Props. 5.1 and 3.1), so we can select a basis \( C_i = \{ v_{i-1}^j \} \) such that the \( R_i \)-modules \( C_i \) are simple and mutually non-isomorphic for different \( k \); hence this basis is unique up to scalars. Then \( C_i = \bigcup_j C_j^i \) is a canonical basis of \( B^{ann} \) given the basis of \( D_i^{\urcorner} \). Since \( \dim_C B^{ann} = 1 \) it follows by iteration that we get a basis \( C = \{ v_T \} \) of \( B^{ann} \) which is unique up to scalars, and where the basis elements \( v_T \) are indexed by paths \( T \) of length \( n \) in the branching graph \( B \). By Proposition 5.7 the set of paths \( T \) from the root of the graph \( B \) with the same endpoint \( \lambda \in \mathcal{P}(n) \) can be parametrised by the set of standard tableaux of shape \( s(T) = \lambda \). We have a decomposition

\[
B^{ann} = \bigoplus_{\lambda \in \mathcal{P}(n)} V_\lambda, \quad V_\lambda = \bigoplus_{s(T) = \lambda} C_{v_T},
\]

where \( V_\lambda \) are the \( R_\lambda \)-isotypical components of \( B^{ann} \), and they form also the irreducible representations of \( S_n \).

Example 5.9. Let \( n = 3 \). The decomposition into isotypical components in terms of a canonical (Young) basis is of the form

\[
\begin{align*}
B_2^{ann} &= V_{\{1,1\}} \oplus V_{\{2\}} = C \oplus C(x_1 - x_2), \\
B_3^{ann} &= V_{\{1,1,1\}} \oplus V_{\{2,1\}} \oplus V_{\{1,1,1\}}, \quad V_{\{1,1,1\}} = C, \\
V_{\{2,1\}} &= C(x_1 + x_2 - 2x_3) \oplus C(x_1 - x_2), \quad V_{\{3\}} = C(x_1 - x_2)(x_1 - x_3)(x_2 - x_3).
\end{align*}
\]

Young bases of an irreducible \( S_n \)-representation \( V_\lambda \) were first introduced in [You77]; see also [JK81, §3.2]. The construction of a basis of \( \mathcal{P}(n) \) is a representation of a group by using nested sequences of subgroups, so that the restriction in each step is multiplicity free, was developed for the groups \( SO(n) \) and \( U(n) \) in [GC50b, GC50a], and one refers often therefore to Gelfand-Zetlin bases. In [Mum83] and [Juc74], independently (see also [OV96]), Young bases were constructed using the algebra \( A(n) \) that is generated by the so-called Jucys-Murphy elements \( \{ L_i \}_{i=2}^n \subset C[S_n] \), where \( L_i = \sum_{j=1}^{n-1} j i \) (\( j i \) is a transposition). Such bases \( \{ v_T \}_{T \in \mathcal{S}_n} \subset V_\lambda \) are indexed by the set \( \mathcal{S}_\lambda \) of standard tableaux of shape \( \lambda \vdash n \), and are characterised by the fact that \( V_\lambda = \oplus C_{v_T} \) where each \( C_{v_T} \) forms a simple \( A(n) \)-module, and \( C_{v_T} \neq C_{v_T'} \) when \( T_1 \neq T_2 \); they are uniquely determined by this condition up to scalars. Since \( B^{ann} = \oplus V_\lambda \) is multiplicity free and hence canonically decomposed into irreducible \( S_n \)-representations (Th. 3.14) we also get a unique (up to scalars) basis of \( B^{ann} \) by taking the union of Young bases of the irreducibles \( V_\lambda \).

Let \( \{ v_T \}_{T \in \mathcal{S}} \) be the canonical basis of \( B^{ann} \), where \( \mathcal{S} \) is the set of standard tableaux of size \( n \) (or set of paths of length \( n \) in the branching graph).
Theorem 5.10. A canonical basis of $B^{an}$ is the same as a Young basis of the $S_n$-representation $B^{an}$. In particular, the canonical basis vectors are common eigenvectors of the Jucys-Murphy elements in $C[S_n]$.

Proof. It suffices, by the above description of Young bases to see that $A(n)\nu_T = C\nu_T$, which we prove by induction over $n$. It is evidently true when $n = 1$, so assume that $A(n-1)\nu_{T'} = C\nu_{T'}$ when $T'$ is a path of length $n - 1$. We have to prove that if $C\nu_T$ is a simple $R_n$-submodule of $\text{Ann}_{D_n}(D_{n-1}\nu_{T'})$, then $A(n)\nu_T = C\nu_T$.

Since $[A(n-1), D_{n-1}] = 0$ and $[A(n-1), D_n] = 0$ it follows by induction that $A(n-1)\nu_T \subset \text{Ann}_{D_n}(D_{n-1}\nu_{T'})$. Therefore $C\nu_T$ and $C\nu_{T'}$, for $i \leq n-1$, are $R_n$-submodules of $\text{Ann}_{D_n}(D_{n-1}\nu_{T'})$ of equal support, since $[L_i, R_n] = 0$. By the branch rule it follows that the $R_n$-module $\text{Ann}_{D_n}(D_{n-1}\nu_{T'})$ is multiplicity free, which implies that $L_{i}\nu_T \subset C\nu_T$; hence $A(n-1)\nu_T = C\nu_T$. We can write $L_n = Z_n + Z_{n-1}$ where $Z_n$ and $Z_{n-1}$ belong to the center of $C[S_n]$ and $C[S_{n-1}]$, respectively. Since $A(n-1)$ is a maximal commutative subalgebra of $C[S_{n-1}]$ [DG89], it follows that $Z_{n-1} \subset A(n-1)$, so that $Z_{n-1}\nu_T \subset C\nu_T$. The element $Z_n$ acts by a scalar on the simple $D_n[S_n]$-module $B_i$ in Proposition 2.2. Since the vector $\nu_T$ generates a simple $D_n$-module it is contained in precisely one isotypical component $B_i$. Hence $Z_n\nu_T \subset C\nu_T$. This completes the proof that $A(n)\nu_T = C\nu_T$. \Box

Consider now an expansion

$$v_T = cx^{\alpha} + \text{l.o.},$$

where we use the reverse lexicographic ordering of the multiindices, so that $\alpha > \beta$ if for some integer $1 \leq m \leq n$ we have $\alpha(i) \geq \beta(i)$ for $m \leq i \leq n$ and $\alpha(m) > \beta(m)$. Thus “l.o.” signifies a sum of monomial terms $x^\beta$ such that $\alpha \succ \beta$.

We consider also the support of the simple $R_n$-module $C\nu_T$. The support of any $R_n$-submodule of $B$ is a subset of $Z_0^c \subset C^n = \text{Spec } R_n$, where $C^n$ is identified with $Hom_C(CT_1 + \cdots + CT_n, C)$ and $t_i = \sum_{j=1}^n \nabla_j$. We embed $Z_0^c \subset Z^n$ by $(\gamma_1, \ldots, \gamma_n) \mapsto (\gamma_1, \ldots, \gamma_n, 0, \ldots, 0)$. If $T$ is a path of length $n$ we let $\{\gamma_T\} \subset Z^n$ be the support of the $R_n$-module $C\nu_T$.

Let $F_{n+1}(x_1, \ldots, x_n)$ be the polynomial such that

$$h_{n+1}(x_1, \ldots, x_n) = F_{n+1}(h_1(x), \ldots, h_n(x)),$$

where the power sums $h_i$ are defined in (3.3.1).

For a directed path $T$ of length $n$ in $B$ we let $\lambda_T \vdash n$ be the integer partition that corresponds to the endpoint of $T$ at the level $n$.

Proposition 5.11. Assume that $T > T'$ so that $T'$ is a path of length $n - 1$. Given $\alpha_T$ and $\gamma_T$, the set of possible $\alpha_T$ and $\gamma_T$ is

1. $\alpha_T = \alpha_{T'} + (0, \ldots, a)$
2. $\gamma_T(i) = \begin{cases} \gamma_{T'}(i) + a, & 1 \leq i < n, \\ F_n(\gamma_{T'}(1), \ldots, \gamma_{T'}(n-1)) + a^n, & i = n, \end{cases}$

where $a \in \lambda_T \vdash n - 1$, or $a = 0$.

Proof. Let $N = D_{n-1}\nu_{T'}$ and $M = D_n\nu_T$, where $\nu_T \in \text{Ann}_{D_{n-1}}(N)$ and $\nu_T \in \text{Ann}_{D_n}(N)$. As in the proof of Theorem 5.2 we have $\text{deg } M = \text{deg } N + a$, where $v_T = cx_a^\nu_T + \text{l.o.}$, and any simple submodule of $N$ is determined by $\text{deg } M$, and hence by the integer $a$. By Theorem 5.6 the possible $a$ are determined by the branch rule, which means that $a \in \lambda_T$, or $a = 0$.

(2): Put $t_i^{(n)} = \sum_{j=1}^n \nabla_j$, so that $t_i^{(n)} = t_i^{(n-1)} + \nabla_i$. Since $v_T = cx_a^\nu_T + \text{l.o.}$ is an eigenvector of $t_i^{(n)}$ and $v_{T'}$ is an eigenvector of $t_i^{(n)}$, it follows that when $i < n$,
then
\[ t_i^{(n)}v_T = (\gamma_T(i) + a^i)v_T. \]
and
\[ t_n^{(n)}v_T = (F_n(t_i^{n-1}, \ldots, t_n^{n-1}) + \nabla_n)v_T = (F_n(\gamma_T(1), \ldots, \gamma_T(n-1)) + a^n)v_T. \]

We can compare with the spectrum \( t_T : [n] \to \mathbb{N} \) of the Jucys-Murphy elements
\( L_i, L_i v_T = t_T(i) v_T \), given by \( t_T(i) = s - r \), where \( r \) and \( s \) are the row and column
number of the box containing \( i \) [Mur81] (the number \( s - r \) is called the “class”,
“content”, or “residue” of the box of \( i \)). We thus have:
\[ \alpha_T(i) = r - 1 \quad \text{and} \quad t_T(i) = s - r. \]

By Proposition 5.11 one can read off \( \alpha_T \) and \( \gamma_T \) (and \( t_T \)) from the standard tableau
\( T \). The value \( \alpha_T(i) \) equals the number of boxes above the box of \( i \) in \( T \), where
the box of \( i \) is inserted in \( T' \) at an addable position. It is therefore clear that the map
\[ \mathcal{B} \to (\alpha : [n] \to \mathbb{N}) : T \to \alpha_T \]
is injective. One can similarly recover \( T \) from \( \gamma_T \) by considering the successive
removing of the boxes of \( n, n - 1, \ldots, 2 \) from \( T \). We can conclude that paths in the
branching graph \( \mathcal{B} \) are determined by the multi-index \( \alpha_T \) (i.e. eigen-values of the
\( x_i \partial_i \)) or the eigen-values of the elements \( t_i \). One can compare to the
description of the branching graph \( C \) in [OV96] in terms of the spectrum of the Jucys-Murphy
elements \( L_i \).

The relation \( v_T = cz^n v_T + \text{l.o.} \) also makes it straightforward to recover the
canonical basis at one level from a higher level basis.

**Proposition 5.12.** Embed \( S_k \) in \( S_n \) such that \( S_k \) fixes the variables \( x_{k+1}, \ldots, x_n \).
Select for each path \( T' \) of length \( k \) a path \( T \) of length \( n \) that starts with \( T' \), and let
\( \{v_T\} \) be the corresponding subset of a canonical basis of \( C[x_1, \ldots, x_n]^\text{ann} \).
Expand
\[ v_T = x_n^{a_n} x_{n-1}^{a_{n-1}} \cdots x_{k+1}^{a_{k+1}} v_{T'} + \text{l.o.}, \]
where \( \text{l.o.} \) signifies terms of lower order than \( (a_{k+1}, \ldots, a_n) \) in the reverse lexicographic ordering of the set of multiindices \([n-k] \to \mathbb{N}\). Then \( \{v_T\} \) is a canonical basis of \( C[x_1, \ldots, x_n]^\text{ann} \). Moreover, the exponent \( a_i \) equals the number of boxes in
\( T \) above the box of the integer \( i \).

We have already motivated the last assertion in Proposition 5.12. The proof of the
remaining part follows from the following lemma.

**Lemma 5.13.** Let \( p \) be a homogeneous polynomial in \( B \) and \( 1 \leq i < n \) an integer.
Make the expansion
\[ p = \sum p_{\beta_i} x^{\beta_i}, \]
where \( p_{\beta_i} \in C[x_1, \ldots, x_i], \beta_i : [n - i] \to \mathbb{N}, \) and \( x^{\beta_i} = \prod_{j=1}^{n-i} x_i^{\beta_i(j)}. \)

(1) If \( R_n \cdot p = C p \), then \( R_i \cdot p_{\beta_i} = C p_{\beta_i} \).

(2) Assume that \( \beta_n : [n - i] \to \mathbb{N} \) is maximal in the reverse lexicographic order
of multiindices such that \( p_{\beta} \neq 0 \) in the above expansion. If \( p \in \text{Ann}_{D_n}(B) \),
then \( p_{\beta_n} \in \text{Ann}_{D_n}(B) \).

**Proof.** By iteration it suffices to prove (1) and (2) when \( i = n - 1 \).

(1): We have an index \( \beta_{n-1} : [1] \to \mathbb{N} \) determined by a single integer \( \beta_{n-1}(1) = j \).
Consider the expansions
\[ p = \sum c_{a} x^{a} = \sum_{j=1}^{a} p_{\beta_{n-1}} x_{n}^{j} = \sum_{j=1}^{a} (\sum_{\gamma_j} d_{\gamma_j} x^{\gamma_j}) x_{n}^{j}. \]
Since \( R_n \cdot p_n = C p_n \) the multi-indices \( \alpha \) with \( c_\alpha \neq 0 \) are of the form \( \sigma \cdot \beta \) for some fixed \( \beta \) with \( \sigma \) running over some elements in \( S_n \). This implies that the multi-indices \( \gamma : [n-1] \to \mathbb{N} \) are of the form \( \sigma \cdot \beta \) for some fixed \( \beta \) and \( \sigma \) runs over elements in \( S_{n-1} \). This implies that \( R_{n-1} p_{\beta n-1} = C p_{\beta n-1} \).

(2): Since \( \hat{p}_{k,l}^{(n)} = \hat{p}_{k,l}^{(n-1)} + x_k \partial_l^\alpha \in D_n^{(n)} \) we have
\[
0 = \hat{p}_{k,l}^{(n)}(p) = \hat{p}_{k,l}^{(n-1)}(\beta) x_{\beta^k} + p_{\beta} x_{\beta^k} \partial_l^{\alpha}(x_{\beta^k}) + (\text{l.o. in } x') = \hat{p}_{k,l}^{(n-1)}(\beta) x_{\beta^k} + (\text{l.o. in } x'),
\]
implying that \( \hat{p}_{k,l}^{(n-1)}(\beta) = 0 \). By Lemma 3.8
\[
D_{n-1} = \sum_{0 \leq k < l < n} D_{n-1} \hat{p}_{k,l}^{(n-1)} ,
\]
implying that \( D_{n-1} \cdot p_{\beta} = 0 \).

**Proposition 5.14.** Let \( S \) be the set of standard tableaux of size \( n \) and \( \{v_T\}_{T \in S} \) be a canonical (Young) basis of \( B^{ann} \). This induces a decomposition of \( B \) as \( D_n \)-module
\[
B = \bigoplus_{T \in S} N_T ,
\]
where the \( D_n \)-module \( N_T = D_n v_T \) is simple. The \( A \)-module \( N_T \) is free of rank equal to the number of standard tableaux of shape \( s(T) = \lambda \).

**Proof.** By Proposition 2.2 \( B = \oplus B_\lambda \) where the \( B_\lambda \) are isotypical components of \( B \) and a subset \( B_\lambda \subset \{v_T\}_{T \in S} \) gives a basis of \( B^{ann}_\lambda = \mathcal{V}_\lambda \). Since \( D_n v_T \) is a simple \( D_n \)-module (Th. 3.1) it follows that \( B_\lambda = \oplus v_T \in B D_n v_T \). Since \( B \) is free over \( A \), it follows by semisimplicity that \( D_n v_T \) is also free. It is well-known that as \( A[G] \)-module \( B \cong A[G] \), the regular representation, which, by Proposition 2.2, implies that \( \text{rank } N_T = \dim_{\mathbb{C}} \mathcal{V}_\lambda \), where \( \mathcal{V}_\lambda \) has a basis \( \{v_T\} \) indexed by all paths \( T \) that end at \( \lambda \); and the number of such paths equals the number of standard tableaux of shape \( \lambda \).

By Lemma 3.6, (2), and Lemma 3.8, (2), (with \( m = 1 \)) we have \( \mathfrak{a}^- \subset D_n^- = \sum_{0 \leq l < k < n} D_n \mathfrak{a}^- \), and \( t, v_T = \gamma_T(t_i) v_T \). Hence
\[
\sum_{0 \leq k < l < n} D_n (\sum_{i=1}^n x_i \partial_i^\alpha) + \sum_{i=1}^n D_n (t_i - \gamma_T(t_i)) \subset \text{Ann}_{D_n}(v_T).
\]
This is however not an equality, and it would be interesting to find a complete description of \( \text{Ann}_{D_n}(v_T) \). Since \( N_T \) is a free module over \( A \) of finite rank it follows that for fixed \( k > l \) the set \( \{\hat{p}_{k,l}^{(n)}\}_{0 \leq r} \) is not linearly independent over \( A \) for sufficiently high \( r \). Let for \( 0 < l < k < n \)
\[
P_{k,l} = p_{k,l}^{(n)} + a_1 p_{k+1,l}^{(n)} + \cdots + a_o \in D_n^+ \cap \text{Ann}_{D_n}(v_T),
\]
where \( a_i \in A \), and \( r \) is of minimal degree \( r \geq 1 \). One may ask if
\[
\text{Ann}_{D_n}(v_T) = \sum_{0 \leq k < l < n} D_n (\sum_{i=1}^n x_i \partial_i^\alpha) + \sum_{i=1}^n D_n (t_i - \gamma_T(t_i)) + \sum_{0 < l < k < n} D_n P_{k,l},
\]
and also what is the precise form of the \( P_{k,l} \)?

**Remark 5.15.** Murphy [Mur81] expressed idempotents \( E_T \) in terms of the Jucys-Murphy elements \( L_i \), so that if \( \{s_T\} \) is the standard Specht basis and one puts \( v_T = E_T s_T \), then \( \{v_T\} \) is a Young basis, where the two bases of \( B^{ann} \) are transformed into one another by a unimodular triangular matrix. Still, it would be desirable to find a concrete decomposition of the \( K \)-module \( \text{Ann}_{D_n}(D_{n-1} v_{T_{i-1}}) \), perhaps by refining the proof of Theorem 5.6, and thus getting a direct construction of \( \{v_T\}_{T \in S} \) from \( \{v_{T_{i-1}}\}_{T_{i-1} \in S_{i-1}} \), where \( S_i \) is the set of standard tableaux of size \( i \). The complexity
of finding a basis of \( \text{Ann}_{D^*}(D_{i-1}v_{T_{i-1}}) \) using a direct linear algebra approach is determined by the degrees \( \deg v_{T_{i-1}} = |\alpha_{T_{i-1}}| \) and \( \deg v_T = |\alpha_T| \); see the example below.

Let us take Example 5.9 one step further.

**Example 5.16.** Consider the tableau \( T = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix} \) by adding the element 4 to the subset \( \{3\} \subset \{3\} \). We have \( \alpha_T = (0,1,0) \) and \( \gamma_T = (1,1,1) \). By Lemma 5.11 \( \alpha_T = (0,1,0,1) \) and, since \( F_4(y_1,y_2,y_3) = \frac{1}{6}y_1^3 - \frac{1}{2}y_1^2y_2 + \frac{3}{4}y_1y_2^2 + \frac{4}{3}y_2^3 \), so that \( F_4(1,1,1) = 1 \),

\[ \gamma_T = (1,1,1,0) + (1,1,1,0) + (0,0,0,1 + F_4(1,1,1)) = (2,2,2,2) \]

We also can see that \( \ell_T = (0, -1, 1, 0) \). The canonical basis \( v_T = 2x_3 - (x_1 + x_2) \) (we ignore scalars in \( v_T \) and \( v_T \)). Put \( p_{i,l}^{(m)} = \sum_{i=1}^{m} x_i^l \). A moments reflections shows that the following Ansatz suffices

\[ v_T = x_4 v_T + v_T(0) = (x_4 + c_1p_{1,0}^{(3)} + c_2p_{2,1}^{(3)})v_T, \]

where \( v_T(0) \) is independent of \( x_4 \). The conditions are

\[ \begin{align*}
  p_{0,1}^{(4)} v_T &= [p_{0,1}^{(4)} + c_1p_{1,0}^{(3)} + c_2p_{2,1}^{(3)}]v_T = (1 + 3c_1 + 2c_2) v_T = 0 \\
  p_{1,2}^{(4)} v_T &= [p_{1,2}^{(4)} + c_1p_{1,0}^{(3)} + c_2p_{2,1}^{(3)}]v_T = (2x_4c_1 + c_2p_{1,1}^{(3)} + c_2(3p_{2,2}^{(3)} + 2p_{1,1}^{(3)}))v_T \\
  &= 0,
\end{align*} \]

which give

\[ v_T = (x_4 + \frac{1}{3}(p_{1,0}^{(3)} - p_{2,1}^{(3)}))v_T = 2x_4 x_3 - x_4 x_2 - x_4 x_1 - x_3 x_2 - x_3 x_1 + 2x_2 x_1, \]

where the expansion is in reverse lexicographic order. To exemplify Proposition 5.12, we have

\[ v_T = x_4 v_T + v_T(0) = x_4 (x_3 v_T + v_T(0)) + v_T(0) = x_4 x_3 v_T + w, \]

where \( T > T' > T'' \), so that for \( k = 3 \), \( v_{T'} = 2x_3 - (x_1 + x_2) \), and for \( k = 2 \), \( v_{T''} = 2 \) (again, we do not keep track of scalars).

6. **Cyclic and Dihedral groups**

We will in this section study invariants of the dihedral group \( G = G(e, e, 2) = A(e, e, 2) \rtimes S_2 \), already discussed in (3.3.2), and its normal cyclic subgroup \( C_e = A(e, e, 2) = (1, \rho, \rho^2, \ldots, \rho^{e-1}) \), where the action of \( G \) is.

\[ \begin{align*}
  \rho(x_1) &= x_1, \\
  \rho(x_2) &= \epsilon^{-1} x_2, \quad \text{and} \\
  \sigma(x_1) &= x_2
\end{align*} \]

on a basis \( \{x_1, x_2\} \) of \( V = \mathbb{C}^2 \), where \( \epsilon = \exp(2\pi i/e) \). We will in particular describe \( R \) and its module \( B^{\text{ann}} \) for these two groups, as well as describe the MLS-restriction functor induced by \( C_e \subset G \). The results on invariants and semi-invariants are of course classical and surely date back to Gordan, but we include the computations for completeness. The invariant rings are

\[ A_1 = B^{C_e} = \mathbb{C}[x_1^2, x_2^2, x_1 x_2] \quad \text{and} \quad A_2 = B^G = \mathbb{C}[x_1^2 + x_2^2, x_1 x_2] \]

(see the proof of Lemma 3.7). Notice that the cyclic group \( C_e \) is not generated by complex reflexions of \( V \), so that \( A_1 \) is not a polynomial ring, in contrast to \( A_2 \). The ring \( A_1 \) corresponds to one of the two infinite series of Kleinian surface singularities; it may be of interest to also study the other infinite series of \( B^G \), where \( G \) is the binary dihedral extension of \( G \), see [Dol].
The invariant differential operators are denoted \( \mathcal{D}_1 = \mathcal{D}_H^{C_e} \) and \( \mathcal{D}_2 = \mathcal{D}_H^{U} \), so that \( \mathcal{D}_2 \subset \mathcal{D}_1 \).

6.1. **The cyclic group** \( C_e \). Since \( C_e \) is abelian, each \( C_e \)-isotypic component of \( B \) is by Proposition 2.2 a simple \( \mathcal{D}_1 \)-module. Hence in order to find simple \( \mathcal{D}_1 \)-modules it suffices to find the semi-invariants of \( C_e \). Let

\[
\chi_i : C_e \to \mathbb{C}^*, \quad \chi_i(e) = e^i, \quad 0 \leq i < e,
\]

be the linear characters of \( C_e \).

**Lemma 6.1.** The isotypic component associated to \( \chi_i \in \hat{C}_e \) is

\[
N_i = A_1x_1^i + A_1x_2^{e-i}.
\]

In particular,

\[
B = \bigoplus_{0 \leq i \leq e-1} N_i,
\]

is the decomposition of \( B \) into simple \( \mathcal{D}_1 \)-modules.

Notice that \( N_i \) is not free over \( A_1 \) even though \( A_1 \) and \( N_i \) are simple \( \mathcal{D}_1 \)-modules.

**Proof.** Since \( x_1x_2, x_1^i, x_2^j \in A_1 \), any monomial may be written as

\[
a = bx_2^j, \quad \delta = 1, 2, \quad 0 \leq j \leq e - 1, \quad b \in A_1.
\]

Such a monomial belongs to the \( \chi_i \)-isotypic component if and only if \( j = i \), when \( \delta = 1 \), or \( e - j = i \), when \( \delta = 2 \). This implies that \( N_i \) is the \( \chi_i \)-isotypical component of \( B \). \( \square \)

To determine \( N_i^{ann} \), notice that if \( i \neq e/2 \) then the lowest non-zero degree component of \( N_i \) is spanned by either \( x_1^i \), when \( 0 \leq i < e/2 \), or \( x_2^{e-i} \), when \( 0 \leq e - i < e/2 \). These components are hence one-dimensional and will therefore coincide with \( N_i^{ann} \) by Corollary 3.3 (3). This gives most of the following result:

**Proposition 6.2.**

\[
N_i^{ann} = \begin{cases} 
C_x^i & 0 \leq i \leq (e - 1)/2, \\
C_x^i + C_{x_2}^{e-i} & i = e_1, \text{ when } e = 2e_1 \text{ is an even number,} \\
C_{x_2}^{e-i} & (e - 1)/2 < i \leq e - 1.
\end{cases}
\]

**Proof.** It remains to treat \( i = e_1 \) when \( e = 2e_1 \) is even. Then the lowest non-zero component is \( W = C_{x_1}^{e_1} + C_{x_2}^{e_1} \), and to see that \( W = N_i^{ann} \) it suffices to see that \( W \) is a simple \( \mathcal{R}_1 \)-module. This follows since \( \mathcal{D}_1^0 \) contains the elements

\[
\nabla_1, \nabla_2, (x_1 \partial_2)^{e_1}, (x_2 \partial_1)^{e_1},
\]

and these elements span the four-dimensional space \( \text{End}_\mathbb{C}(W) \). \( \square \)

Next we will describe \( \mathcal{R}_1 = \mathcal{D}_1^0 / \mathcal{D}_1^0 \cap (\mathcal{D}_1 \cdot \mathcal{D}_1) \). By the just shown existence of a simple \( \mathcal{D}_1 \)-module \( N \) such that \( N^{ann} \) is not of dimension 1, we know by Theorem 3.14 that, for even \( e \), \( \mathcal{R}_1 \) is non-commutative. However, in the case of \( e \) odd, \( \mathcal{R}_1 \) will be seen to be commutative.

**Proposition 6.3.** Let \( C_e \) be the cyclic group acting on \( \mathbb{C}^2 \) as above and consider the canonical composed homomorphism

\[
\Pi : \mathbb{C}[\nabla_1, \nabla_2] \to \mathcal{D}_1^0 \to \mathcal{R}_1 = \frac{\mathcal{D}_1^0 \mathcal{D}_1^0}{\mathcal{D}_1^0 \cap (\mathcal{D}_1 \cdot \mathcal{D}_1)}.
\]

Then we have:
(1) If $e$ is odd, then $\mathcal{D}_1^0$ is generated as an algebra by
\[
\pi, \pi_2, (x_1 \partial_2)^e, (x_2 \partial_1)^e,
\]
where $\pi$ is surjective, and hence $\mathcal{R}_1$ is a commutative ring.

(2) If $e = 2e_1$ is even, $\mathcal{D}_1^0$ is generated as an algebra by the elements
\[
\pi_1, \pi_2, t_1 = (x_1 \partial_2)^{e_1}, t_2 = (x_2 \partial_1)^{e_1}.
\]

Let $\mathcal{E} = \text{Im}(\Pi)$ be the algebra that is formed as the image of $C[\pi_1, \pi_2]$.
Then $\mathcal{R}_1$ is generated by $1, t_1$ and $t_2$ as left (or right) module over $\mathcal{E}$.

Proof. We will use the method in (3.12), and therefore start by computing $S(V \otimes_C V)^{C_e}$. Here $V \otimes_C V^* = V_0 \oplus V_1$ where $V_0 = C[\pi_1, \pi_2] = C[\pi_1 \otimes \pi_2]$, $V_1 = C[b_1 + Cb_1]$, $b_1 = x_1 \partial_2$, and $b_2 = x_2 \partial_1$; $C_e$ acts trivially on $V_0$ and $\rho(b_1) = e^2 b_1, \rho(b_2) = e^{-2} b_2$.

If $e$ is odd $e^2$ is just another primitive $e$th root of unity, and
\[
S(V \otimes_C V)^{C_e} = (S(V_0) \otimes C S(V_1))^{C_e} = S(V_0) \otimes C S(V_1)^{C_e} = C[\pi_1, \pi_2] \otimes C [b_1 b_2, b_1^2, b_2^2].
\]

If $e = 2e_1$ is even, then $e^2$ is an $e_1$th primitive root of unity and
\[
S(V \otimes_C V)^{C_e} = C[\pi_1, \pi_2] \otimes C [b_1 b_2, b_1^{e_1}, b_2^{e_1}].
\]

The first assertion in (1) follows since for the map $l$ in (3.3.3) we have $l(b_i b_2) = l(\pi_1 \pi_2) = 1$, and the second follows since the elements $(x_1 \partial_2)^e = x_1^e \partial_2^e$ and $(x_2 \partial_1)^e = x_2^e \partial_1^e$ belong to $\mathcal{D}_1^0 \cap \mathcal{D}_1^0$.

Now consider (2), so $e = 2e_1$, and $t_1 = l(b_i)$. The elements $t_1^2$ and $t_2^2$ belong to $\mathcal{D}_1 \mathcal{D}_1^0$, while $t_1 t_2 = (x_1 \partial_2)^{e_1} = x_1^{e_1} \partial_2^{e_1}$, and $t_2 t_1 = (x_2 \partial_1)^{e_1} = x_2^{e_1} \partial_1^{e_1}$ (see Lemma 2.1) all belong to $\mathcal{E}$. This implies that $\mathcal{R}_1$ is generated as a left $\mathcal{E}$-module by $1, t_1$ and $t_2$. \hfill \Box

Remark 6.4. Since $\pi_1, \pi_2 = x_1 x_2 \partial_1 \partial_2, p_e(\pi_1) = x_1^e \partial_1^e$ and $p_e(\pi_2)$ (Lem. 2.1) belong to $\mathcal{D}_1^0 \cap \mathcal{D}_1^0$, the kernel of $\Pi$ contains $\pi_1, \pi_2$ and $p_e(\pi_1), p_e(\pi_2)$, implying that $\mathcal{R}_1$ is artinian. However, we do not know if $\mathcal{E}$ is generated by these three elements.

6.2. The dihedral group. Now we will consider the dihedral group $G = C_e \rtimes S_2$ and the invariant map $A_2 \to B$. In the decomposition of $B$ as $D_2$-module the following modules will occur:
\[
M_0 = A_2, \quad M_e = D_2(x_1^e - x_2^e), \quad M_{i} = D_2(x^i_1 - x^i_2), \quad M_i = D_2(x^i_1 + x^i_2),
\]
\[
M_{i} = D_2(x_1^i - x_2^i), \quad M_{i} = D_2(x_1^i + x_2^i)
\]

where the latter two modules are defined only when $e = 2e_1$ is an even number, and we notice that $M_e$ is generated by the Jacobian $J = x_1^e - x_2^e$ of the invariant map.

Proposition 6.5. Let $A_2 = B^G = C[x_1^2 + x_2^2, x_1 x_2]$. 

(1) $B$ has the basis $1, x_1^i, x_2^i, x_1^i - x_2^i, 1 \leq i \leq e - 1$, over $A_2$.

(2) The $D_2$-module $B$ has the following decomposition into simple components
\[
B = A_2 \oplus M_e \oplus (\bigoplus_{1 \leq i < e/2} (M_i + M_i^*)) \oplus M_{i} \oplus M_{i}^*
\]

where the two terms that contain $e_1$ only occur when $e = 2e_1$ is an even number.

(3) We have an isomorphism $M_i \cong M_i^*$, $1 \leq i < e/2$, where the modules have rank 2 over $A_2$, and these are the only isomorphisms between the modules in (2). In the case of odd $e$, the only $D_0^2$-submodules of $B$ rank 1 as $A_2$-modules are the non-isomorphic $M_0$ and $M_e$, while if $e = 2e_1$ is an even number, then also $M_i$ and $M_i^*$ have rank 1, and are non-isomorphic.
(4) In the decomposition of the $D^0_2$-module $B^{ann}$ the simple components are

$$M^{ann} = \begin{cases}
  Cx^i, & M = M^i_1, \ 0 < i < e/2, \\
  Cx^j, & M = M^j_1, \ 1 < i < e/2, \\
  C(x^i_1 - x^j_2), & M = M^i_1, \\
  C(x^i_1 + x^j_2), & M = M^i_1, (e = 2e_1 \text{ is even}), \\
  C(x^{e_1}_1 - x^{e_2}_2), & M = M^i_1, (e = 2e_1 \text{ is even}).
\end{cases}$$

Proof. Lemma 3.7 gives the equality $B^G = C[x^i_1 + x^j_2, x_1, x_2]$, which implies (1).

We first prove

$$(*) \quad B^{ann} = \text{Ann}_{D^+_2}(B) = C \cdot 1 + \sum_{i=1}^{e/2} (Cx^i_1 + Cx^i_2) + C(x^i_1 - x^i_2).$$

We have $B^{ann} \subset \text{Ann}_{D(V^*)^0_2}(B)$, where the space of harmonic polynomials (see Remark 3.4)

$$\text{Ann}_{D(V^*)^0_2}(B) = \{ p \in B \mid \partial_i \partial_j p = (\partial^i_1 + \partial^i_2)p = 0 \} = C \cdot 1 + \sum_{i=1}^{e-1} (Cx^i_1 + Cx^i_2) + C(x^i_1 - x^i_2).$$

To see this, note that the first condition $\partial_i \partial_j p = 0$ implies that a harmonic polynomial $p$ contains no mixed terms, and that $c_1 x^i_1 + c_2 x^j_2$ is killed by $(\partial^i_1 + \partial^i_2)$ only if $c_2 = -c_1$. Let $p = a_0 + \sum_{i=1}^{e-1} (a_i x^i_1 + b_i x^i_2) + (c x^i_1 - x^i_2) \in \text{Ann}_{D(V^*)^0_2}(B)$. Assume that $j < e/2$, so $c - j > j$. Then $r(x^j_1) = x^j_1 = 0$, so $p(x^j_1) = 0$. Therefore, the left side of $(*)$ is a subset of the right side.

Conversely, Proposition 6.2 implies that the right side of $(*)$ is contained in $\text{Ann}_{D^+_2}(B) \subset \text{Ann}_{D^+_2}(B)$, since $D^+_2 \subset D^-_1$.

Since $D^0_2 \subset D^0_1$ and by Proposition 6.2 $D^0_1$ preserves the 1-dimensional spaces $Cx^i_1, Cx^i_2$ when $0 < i < e/2$, and also the space $C(x^i_1 - x^i_2)$, they define simple $D^0_2$-modules. When $e = 2e_1$ is an even number, then $D^0_1$ and $S_2$ acts on the space $Cx^i_1 + Cx^i_2$, which can be split according to the $S_2$-action into two simple $D^0_2$-modules $C(x^i_1 + x^j_2)$, $C(x^i_1 - x^j_2)$. This proves the decomposition of $B^{ann}$ in (4).

By Corollary (3.3) each of the above simple $D^0_2$-modules generates a simple $D^+_2$-submodule of $B$, which implies (2).

It remains to see (3), and for this it suffices, by Corollary (3.3), to see that the assertions hold for the terms in (3). The element $\sigma \in G$ induces an isomorphism $D^0_2 x^i_1 \cong D^0_2 x^j_2$. Assume first that $e$ is an odd number. Then the eigenspace decomposition of $B^{ann}$ with respect to the element $\nabla = \nabla_1 + \nabla_2 \in D^0_2$, all have multiplicity 1, hence they are non-isomorphic as $D^0_2$-modules as well. When $e = 2e_1$ is an even number, then the $\nabla$-eigenspace $C(x^i_1 + x^j_2) \oplus C(x^i_1 - x^j_2)$ has multiplicity 2, with different eigenvalue $e_1$ from the other ones. Now since the element $s = (x_1 \partial^i_2)^{e_1} + (x_2 \partial^i_2)^{e_1}$ belongs to $D^0_2$ and

$$s(x^i_1 + x^j_2) = e_1(x^{e_1}_1 + x^{e_2}_2), \quad s(x^i_1 - x^j_2) = -e_1(x^{e_1}_1 - x^{e_2}_2),$$

we conclude that (3) is also true.
it follows that the two $D_2^0$-modules $C(x_1^1 \pm x_2^1)$ are non-isomorphic.

**Proposition 6.6.** If $e$ is an odd integer, then the canonical composed homomorphism

$$\Pi : C[\nabla] \to D_2^0 \to R_2 = \frac{D_2^0}{D_2^0 \cap (D_2 \cdot D_2^0)}$$

is surjective. If $e = 2e_1$ is an even integer, then $R_2$ is generated as a module over $E = \Pi(C[\nabla])$ by the elements 1 and $(x_1 \partial_2)^{e_1} + (x_2 \partial_1)^{e_2} \notin E$. In either case $R_2$ is commutative.

**Proof.** Similarly to the proof of Proposition 6.3 we will use the method in Lemma 3.12 to compute $D_2^0$. Here the $G$-representation $V \otimes_\mathbb{C} V^* = V_0 \oplus V_1$, where $V_0 = \mathbb{C}V_1 + \mathbb{C}V_2$ and $V_1 = \mathbb{C}b_1 + \mathbb{C}b_2$, $b_1 = x_1 \partial_2$, and $b_2 = x_2 \partial_1$. The action of $G$ on $V_0$ is through the map $G \to S_2$, and and $S(V_0) = S(V_0)^{\mathbb{C}} \otimes S(V_0)^{\mathbb{C}}$ where the character $\chi : G \to \mathbb{C}^*$ is given by $\chi(\rho) = 1$ and $\chi(\sigma) = -1$. The action of $G = C_\ast \times S_2$ on $V_1$ is similar to that on $V$, namely $\rho(b_1) = e^2$ and $\rho(b_2) = e^{-2}, \sigma(b_1) = b_2, \sigma(b_2) = b_1$.

The invariants are therefore

$$(*)$$

$S(V \otimes_\mathbb{C} V^*)^G = (S(V_0) \otimes_\mathbb{C} S(V_1))^G = ((S(V_0)^G \otimes_\mathbb{C} S(V_1)^G)) \oplus (S(V_0) \otimes_\mathbb{C} S(V_1))_\chi$

$$= S(V_0)^G \otimes_\mathbb{C} S(V_1)^G \oplus (S(V_0)^G \otimes_\mathbb{C} S(V_1)^G) \cdot ((\nabla_1 - \nabla_2) \otimes (b_1^2 - b_2^2)), $$

where the last step follows since the semi-invariants are given by

$$S(V_0)\chi = S(V_0)^G(\nabla_1 - \nabla_2), \quad \text{and} \quad S(V_1)\chi = S(V_1)^G(b_1^2 - b_2^2).$$

From this we can get generators of $S(V \otimes_\mathbb{C} V^*)^G$, which give rise to the following generators of $D_2^0$ (as described in Proposition 6.3)

$$1, \nabla, \nabla_1 \nabla_2, b_1^e + b_2^e, b_1 b_2 + b_2 b_1$$

and $(\nabla_1 - \nabla_2)(b_1^e - b_2^e)$,

where $e' = e_1 = e/2$ if $e$ is an even number, and otherwise $e' = e$. Not all these generators are required to generate $R_2$. First we have $\nabla_1 \nabla_2 \in D_2^0 \cap (D_2\bar{D}_2)$, secondly $b_1 b_2 + b_2 b_1 = 2\nabla_1 \nabla_2 + \nabla \equiv \nabla$, where the congruence is modulo $D_2^0 \cap (D_2 \bar{D}_2)$, and third,

$$(\nabla_1 - \nabla_2)(b_1^e - b_2^e) = e'(b_1^e + b_2^e) + (x_1^e \partial_2^e - 1 + x_2^e \partial_1^{e-1})\partial_1 \partial_2$$

$$- x_1 x_2(x_2^e \partial_1^{e-1} + x_1^e \partial_2^{e+1}) \equiv e'(b_1^e + b_2^e).$$

Finally, $b_1^e + b_2^e$ maps to $E$, since

$$2(b_1^e + b_2^e) = (x_1^1 + x_2^1)(\partial_1^e + \partial_2^e) - (x_1^e \partial_1^e + x_2^e \partial_2^e) \equiv p(\nabla_1, \nabla_2)$$

where $p(t_1, t_2)$ is a symmetric polynomial so that $p(\nabla_1, \nabla_2) \equiv q(\nabla)$ for some polynomial $q(t)$. It follows that $E = R_2$ when $e' = e$ is odd. If $e' = e_1 = e/2$ and $e$ is even, the residue class $s$ of $s = b_1^e + b_2^e \in D_2^0$ does not belong to $E$. This follows since $\nabla$ acts as a constant on the 2-dimensional space $C(x_1^1 + x_2^1) + C(x_1^1 - x_2^1) \subset D^{an}$, while $s$ splits this space into two eigenspaces with distinct eigenvalues; see the end of the proof of Proposition 6.5. Since $[\nabla, D_2^0] = 0$ it follows that $sE \subset E\bar{s}$, so it remains to prove that $s^2 \in E$. This follows since

$$s^2 = b_1^e + b_2^e + x_1^e \partial_2^e x_2^e \partial_1^e + x_2^e \partial_1^e x_1^e \partial_2^e$$

where we already have seen that the projection of $b_1^e + b_2^e$ in $R_2$ belongs to $E$ and the projection of the remaining part of the right side also belongs to $E$ since $x_1^e \partial_2^e x_2^e \partial_1^e + x_2^e \partial_1^e x_1^e \partial_2^e \in C[\nabla_1, \nabla_2]^{an}$, and also recalling that $\nabla_1 \nabla_2 \in D_2^0 \cap (D_2 \cap D_2^0)$. \qed
Remark 6.7. For \( e = 1 \) the dihedral group is just \( S_2 \), so in that case \( R \) is actually generated by just \( \nabla \) (compare Proposition 3.11). For \( e = 3 \), the dihedral group is again a symmetric group \( S_3 \), but now acting on \( C^2 \), as compared to the 3-dimensional permutation representation studied in Proposition 3.11.

6.3. MLS-restriction. We can now see precisely how the MLS-restriction behaves between the cyclic group \( C_e \) and the dihedral group \( D_{2e} = A(e, e, 2) \times S_2 \), by considering how the simple modules over \( D^0_e \) in Proposition 6.2 behave on restriction to the subring \( D^1_e \):

**Corollary 6.8.** Put \( J = J_{D^0_e}^e \).

1. \( J(N_i) = M^1_{e'} \), for \( 0 \leq i < e/2 \) and \( J(N_i) = M^2_{e-i} \), for \( e/2 < i < e \).
2. \( J(N_{e-i}) = M^1_{e'} \oplus M^2_{e-i} \) (\( e = 2e_1 \) is even).
3. The simple \( D_2 \)-submodules of \( B \) are either isomorphic to the image under MLS-restriction of a simple \( D_1 \)-module, or are modules of semi-invariants belonging to a linear character of \( D_{2e} \). The latter case occurs for \( M^1_{e'} \), \( M^2_{e-i} \) (\( e = 2e_1 \) is even) and \( M_e \) (all \( e \)).

**Proof.** This follows from Propositions 6.2 and 6.5. \( \square \)

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