Stochastic stability of Pollicott–Ruelle resonances

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Abstract
Pollicott–Ruelle resonances for chaotic flows are the characteristic frequencies of correlations. They are typically defined as eigenvalues of the generator of the flow acting on specially designed functional spaces. We show that these resonances can be computed as viscosity limits of eigenvalues of second order elliptic operators. These eigenvalues are the characteristic frequencies of correlations for a stochastically perturbed flow.

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(Some figures may appear in colour only in the online journal)

1. Introduction and statement of results

We consider an Anosov flow $\varphi_t = e^{tV}$ on a compact manifold $X$. For the Laplacian $\Delta_X \leq 0$ with respect to some metric on $X$, we define

$$P_\varepsilon = \frac{1}{i} V + i \varepsilon \Delta_X,$$  \hspace{1cm} (1.1)

For $\varepsilon = 0$ this operator is elliptic and hence has a discrete $L^2(X)$-spectrum $\{ \lambda_j(\varepsilon) \}_{j=0}^\infty$. However, for $\varepsilon = 0$ most of the $L^2$ spectrum is not discrete.

Following the seminal work of Ruelle [Ru] and Pollicott [Po], many authors investigated the discrete spectrum of $P_0$ acting on specially designed \textit{anisotropic Sobolev spaces} and the role of that spectrum in the expansion of correlations—see Blank–Keller–Liverani [BKL], Baladi–Tsujii [BaTs], Faure–Sjöstrand [FaSj], Faure–Tsujii [FaTs1, FaTs2], Gouëzel–Liverani.
We review a yet another approach based on [DyZw1] in section 3. These complex eigenvalues of \( P_0, \lambda = \infty \), are called Pollicott–Ruelle resonances. For perspectives on physical manifestations of these resonances see for instance Gaspard–Ramírez [GaRa] or Chekroun et al [CNKMG].

The purpose of this note is to show that Pollicott–Ruelle resonances can be defined as limits of \( \lambda_\varepsilon \) as \( \varepsilon \to +0 \). This can be considered a stochastic stability of resonances:

**Theorem 1.** Let \( P_\varepsilon \) be given by (1.1) and let \( \{ \lambda_\varepsilon \}_{\varepsilon=0}^\infty \) be the set of its \( L^2 \)-eigenvalues. If \( \{ \lambda_0 \}_{\varepsilon=0}^\infty \) is the set of the Pollicott–Ruelle resonances of the flow \( \varphi_\varepsilon \), then

\[
\lambda_\varepsilon \to \lambda_0, \quad \varepsilon \to 0+. 
\]

with convergence uniform for \( \lambda_0 \) in a compact set.

The nature of convergence is much more precise—see section 5. In particular the spectral projections depend smoothly on \( \varepsilon \in [0, \varepsilon_0] \) where \( \varepsilon_0 \) depends on the compact set. Also, when \( \lambda_0 \) is a simple resonance then for \( \varepsilon \) sufficiently small the map \( \varepsilon \mapsto \lambda_\varepsilon \) is smooth all the way up to \( \varepsilon = 0 \). As explained in the next paragraph \( \lambda_\varepsilon \to \lambda_0 \) when \( \varepsilon \to 0 + \). We also note the symmetry of \( \lambda_\varepsilon \)'s with respect to the imaginary axis (see figure 1). That follows from the fact that \( \overline{P_\varepsilon u} = -P_\varepsilon \overline{u} \) and thus

\[
(P_\varepsilon - \lambda)^{-1}u = -(P_\varepsilon + \overline{\lambda})^{-1}\overline{u}. \tag{1.2}
\]

The proof of theorem 1 relies on the fact that \( P_\varepsilon - \lambda \) is a Fredholm operator on the same anisotropic Sobolev spaces as \( P_0 - \lambda \), in a way which is controlled uniformly as \( \varepsilon \to 0 + \). This Fredholm property is established by the same methods as those used in [DyZw1] for the case of \( \varepsilon = 0 \). The key feature of the damping term \( i\varepsilon \Delta_x \) is that its imaginary part is nonpositive and thus the propagation of singularities theorem of Duistermaat–Hörmander (see (2.10)) still applies in the forward time direction. For \( \varepsilon < 0 \), the damping term is nonnegative and propagation of singularities applies in the negative time direction, which means that we have to consider the dual anisotropic Sobolev spaces \( H_{-\varepsilon G(0)} \) and the spectrum of \( P_0 \) on these spaces is given by \( \{ \overline{\lambda_0} \} \).

We remark that all the results of this paper are valid for the operators acting on sections of vector bundles arising in dynamical systems—see [DyZw1]. We consider the scalar case to make the notation, which is all that is affected, simpler.
Previously, stability of Pollicott–Ruelle resonances has been established for Anosov maps, $f : T^d \to T^d$, [BKL, Li2], following a very general argument of Keller–Liverani [KeLi]. In that case the Koopman operator $f^* : C^\infty(T^d) \to C^\infty(T^d)$ is replaced by a ‘noisy propagator’ $G_{\varepsilon}f^*$, where $G_{\varepsilon}u = g_{\varepsilon} * u$, $g_{\varepsilon} \to \delta_0$, $\varepsilon \to 0$. For general Anosov maps on compact manifolds a semi-classical proof was given by Faure–Roy–Sjöstrand [FRS, theorem 5]. Further refinements concerning dependence on $\varepsilon$ can be found in [GoLi1, section 8] and interesting applications were obtained by Gouëzel–Liverani [GoLi2] and Fannjiang–Nonnenmacher–Wołowski [FNW]. For a physics perspective on this see for instance Blum–Agam [BlAg] and Venegeroles [Ve].

For flows, Butterley–Liverani [BuLi1, BuLi2] showed that if a vector field depends smoothly on a parameter, then the spectrum of the transfer operator associated to the weight corresponding to the SRB measure is smooth in that parameter. Constantin–Kiselev–Ryzhik–Zlatoš [CRKR] established that solutions to the heat equation with a large transport term equidistribute after arbitrarily small times if and only if the flow corresponding to the transport term is mixing; this can be viewed as an analogue of our work for the $L^2$ spectrum on the real line instead of resonances.

A dynamical interpretation of $\lambda(\varepsilon)$’s can be formulated as follows. In terms of the operator $P_0$ the flow, $x(t) := \varphi_{\varepsilon}(x(0))$, is given by

$$ e^{-it\varepsilon\hat{f}(x)} = \hat{f}(x(t)), \quad \dot{x}(t) = -V_{\text{str}}, \quad x(0) = x. $$

For $\varepsilon > 0$ the evolution equation is replaced by the Langevin equation:

$$ e^{-it\varepsilon\hat{f}(x)} = \mathbb{E}[f\{x(t)\}], \quad \dot{x}(t) = -V_{\text{str}} + \sqrt{2\varepsilon} B(t), \quad x(0) = x, $$

where $B(t)$ is the Brownian motion corresponding to the metric $g$ on $X$ (presented here in an informal way; see [El]). This explains why considering $P_0$ corresponds to a stochastic perturbation of the deterministic flow. See also Kifer [Ki] for other perspective on random perturbations of dynamical systems.

We also remark that a result similar to theorem 1 is valid for scattering resonances: for $V \in L^\infty(\mathbb{R}^n; \mathbb{C})$ (and in greater generality) they appear as limits of eigenvalues of $-\Delta + V(x) - i\varepsilon|x|^2$ when $\varepsilon \to 0 +$, see [Zw2]. The proof is based on the method of complex scaling and is technically very different than the one presented here. The result however is exactly analogous with spacial infinity, $|x| \to \infty$, replacing the momentum infinity, $|\xi| \to \infty$.

Pretending that spectrum of $P_0$ is semisimple (algebraic multiplicities are equal to geometric multiplicities—see section 5 for the general statement), the relation to the eigenvalues $\lambda(\varepsilon)$ comes from considering long time behaviour: for any $f \in C^\infty(X)$, and $i > 0$,

$$ e^{-it\varepsilon\hat{f}(x)} = \sum_{\lambda_j > -A} e^{-it\varepsilon\lambda_j} u^*_j(x) \int_X v_j^*(y) f(y) d\nu_\varepsilon(y) + O(e^{-\delta|\varepsilon|}e^{-A}), $$

(1.3)

where $u^*_j, v_j^* \in C^\infty(X)$ are the eigenfunctions of $P_0$ and $P_0^*$ corresponding to $\lambda_j(\varepsilon)$. We note that there are no convergence problems as the number of $\lambda_j(\varepsilon)$ with imaginary parts above $-A$ is finite though the number will grow with $\varepsilon$. In fact, [JiZw] shows that the number of Pollicott–Ruelle resonances, $\lambda_j$, with $\Im \lambda_j > -A$ is always infinite if $A$ is sufficiently large.

The validity of a modification of (1.3) for $\varepsilon = 0$ is only known for contact Anosov flows (see section 6) and for $A > -\gamma_0/2$, where $\gamma_0$ is an averaged Lyapounov exponent (see (1.5)). That is due to Tsujii [Ts1, Ts2] who followed earlier advances by Dolgopyat [Do] and Liverani [Li1]. It is also a consequence of more general results obtained in [NoZw].

The modification in (1.3) is needed since the corresponding $u_j$’s are now new distributions and the expansion provides fine aspects of the decay of correlations. Let $d\mu(\chi)$ be the volume form obtained from the contact form on $X$, $\mu(X) = 1$. For $f, g \in C^\infty(X)$ and any $\delta > 0$,
\[
\int_X [e^{-i\theta f}] \cdot (g(x)) d\mu(x) = \int_X f(x) d\mu(x) \int_X g(x) d\mu(x) + \sum_{-\frac{1}{2} \gamma \lambda < 0} \epsilon^{i\lambda_0 v_0} C_{ij}(e^{-\frac{i}{2} \gamma_0 - \delta}),
\]
(1.4)

where \(\gamma_0\) is the minimal asymptotic growth rate of the unstable Jacobian, that is the largest constant such that for each \(\delta > 0\)

\[
|\det(\varphi_{t}, \delta, 0)| \leq C e^{-\gamma_0 - \delta t}, \quad t \geq 0; \quad \varphi_{t} = e^{-i\phi_0} : C^\infty(X) \rightarrow C^\infty(X),
\]
(1.5)

with \(E_u(x) \subset T_x X\) the unstable subspace of the flow at \(x\)—see section 2. Now \(u_j\) and \(v_j\) are distributional eigenfunctions of \(P_0\) and \(P_0^*\) \(WF(u_j) \subset E_u^*\) and \(WF(v_j) \subset E_v^*\). Here again we make the simplifying assumption that the spectrum is semisimple; that is always the case for geodesic flows in constant negative curvature as shown by Dyatlov–Faure–Guillarmou [DFG, theorem 3].

Hence it is natural to ask the question if the gap \(\gamma_0/2\) is uniform with respect to \(\epsilon\), that is, if the expansion (1.3) with \(\Lambda > -\frac{1}{2}(\gamma_0 - \epsilon)\) uniformly approaches the expansion (1.4). That is indeed a consequence of the next theorem:

**Theorem 2.** Suppose that \(X\) is an odd dimensional compact manifold and that \(V \in C^\infty(X; \mathbb{T}X)\) generates a contact Anosov flow. There exists a constant \(s_0\) such that for any \(\delta > 0\) there exist \(N_0, R > 0\) such that for all \(\epsilon > 0\),

\[
(P_\epsilon - \lambda)^{-1} = O(\lambda^{s_0}) : H^s(X) \rightarrow H^{-s}(X),
\]
(1.6)

for \(\gamma_0\) defined in (1.5) and \(\lambda \in [R, \infty) \rightarrow \left[0, \frac{1}{2}(\gamma_0 - \delta)\right]\).

The same estimate is true for \(\lambda \in (-\infty, -R] \rightarrow \left[0, \frac{1}{2}(\gamma_0 - \delta)\right]\) by recalling (1.2). Since on the compact set \([-R, R] \rightarrow \left[0, \frac{1}{2}(\gamma_0 - \delta)\right]\), \(\lambda_\epsilon(\epsilon)\) converge uniformly to \(\lambda_1\)’s, we see that for \(\epsilon\) small enough the number of eigenvalues of \(P_\epsilon\) in that set is independent of \(\epsilon\). We should remark that the estimate (1.6) can be made more precise by using microlocally weighted reviews mentioned in section 4—see (6.10).

The proof of theorem 2 combines the approach of Faure–Sjöstrand [FaSj] and [DyZw1] with the work on resonance gaps for general differential operators [NoZw]. As in that paper we also use the resolvent gluing method of Datchev–Vasy [DaVa].

For a class of maps on \(\mathbb{T}^2\) a similar result has been obtained by Nakano–Wittsten [NaWi].

**Negative examples.** It is important to point out that the existence of a discrete limit set for the eigenvalues of the operator \(P_\epsilon\) is very special to chaotic flows and for mixed flows could only hold under some special domain restrictions. The simplest ‘counterexample’ is given by considering \(X = S^1 \times S^1\) with \(V = \partial_x + \alpha \partial_y, \quad x, y \in S^1 := \mathbb{R}/2\pi \mathbb{Z}\). When \(\alpha\) is irrational then accumulation points of the spectrum of \(P_\epsilon\) as \(\epsilon \rightarrow 0 + 0\) form the lower half plane. When \(\alpha = plq\) with \(p\) and \(q, r\) coprime the limit set is equal to \(\mathbb{Z}lq - i[0, \infty)\).

A more interesting example is given by the geodesic flow on the torus, \(\mathbb{T}^2 = S^1 \times S^1\) with the flat metric. That is a contact flow on the unit cotangent bundle \(S^*\mathbb{T}^2 = \mathbb{T}^2_{\nu, \alpha} \times S^1_{\nu, \alpha}\) generated by and it is generated by

\[
V = \cos \theta \partial_{\nu} + \sin \theta \partial_{\nu},
\]

Defining \(P_\epsilon\) using the flat Laplacian \(n\) on \(T^3\), and by expanding in Fourier modes in \(x\) we see that
\[ \text{Spec}(P_\epsilon) = \bigcup_{n\in\mathbb{Z}^2} \text{Spec}(P_\epsilon(n)), \quad P_\epsilon(n) := n_1 \cos \theta + n_2 \sin \theta - i\epsilon(n_1^2 + n_2^2 + D_\theta^2) \]

\[ D_\theta = \frac{1}{i} \partial_\theta. \]

We rewrite the operator \( P_\epsilon(n) \) as follows:

\[ P_\epsilon(n) = -i\epsilon D_\theta^2 + |n|\cos(\theta - \delta_n) - i|n|^2 \epsilon, \quad \delta_n = \tan^{-1}(n_1/n_2). \]

For \( n = 0 \) the spectrum is simply \(-i\epsilon m^2, m \in \mathbb{Z}\) and it accumulates on the negative imaginary axis. For \( n \neq 0 \) the asymptotic behaviour of the spectrum is determined by the asymptotic behaviour of the spectrum of the semiclassical operator

\[ Q(h) := (hD_\theta)^2 + i \cos \theta, \quad h^2 := \epsilon |n|. \]

That has been determined by Galtsev–Shafarevitch \([GaSh]\) who showed that as \( h \to 0 \) the spectrum concentrates on a rotated 'Y' shape with the vertices at \( \pm i \) and the junction at a special value \( E^* \approx 0.85 \).

This shows that the accumulation points in the case of the generator of the geodesic flow on the two torus regularized using the flat Laplacian are given by

\[ -i[0, \infty) \cup \bigcup_{n \in \mathbb{Z}^2 \setminus \{0,0\}} \{ z : |\Re z| \leq n, \quad \Im z = -E^*|n| + E^*|\Re z| \}, \]

and part of this set is shown in figure 2.

2. Preliminaries

We review some definitions and basic facts mostly to fix notation and to provide references. The needed results from microlocal/semiclassical analysis are presented in detail in \([DyZw1, \text{section 2.3, appendix C}]\) and we will rely on the presentation given there.

**Notation.** We use the following notation: \( f = O_\ell(g)_H \) means that \( \|f\|_H \leq C_{\ell g} \) where the norm (or any seminorm) is in the space \( H \), and the constant \( C_{\ell g} \) depends on \( \ell \). When either \( \ell \) or \( H \) are absent then the constant is universal or the estimate is scalar, respectively. When \( G = O_\ell(g)_{H_0} \to H_2 \) then the operator \( G : H_1 \to H_2 \) has its norm bounded by \( C_{\ell g} \).

2.1. Dynamical systems

In this paper \( X \) is a compact manifold and \( \varphi : X \to X \) a \( C^\infty \) flow, \( \varphi = \exp tV \), \( V \in C^\infty(X; TX) \). The flow is Anosov if the tangent space to \( X \) has a continuous decomposition \( TX = E_\theta(x) \oplus E_u(x) \oplus E_s(x) \) which is invariant, \( d\varphi_t(x)E_\theta(x) = E_\theta(\varphi_t(x)), E_\theta(x) = \mathbb{R}V(x) \), and for some \( C \) and \( \theta > 0 \) fixed

\[ |d\varphi_t(x)v|_{\varphi_t(x)} \leq Ce^{-\theta |t|} |v|_x, \quad v \in E_\theta(x), \quad t < 0, \]

\[ |d\varphi_t(x)v|_{\varphi_t(x)} \leq Ce^{-\theta |t|} |v|_x, \quad v \in E_\theta(x), \quad t > 0. \]  

(2.1)

where \( |*| \) is given by a smooth Riemannian metric on \( X \).

Following Faure–Joststrand \([FaSj]\) we exploit the analogy between dynamical systems and quantum scattering, with the fiber (\( \xi \)) infinity playing the role of \( x \)-infinity in scattering theory. The pull-back map can be written analogously to the Schrödinger propagator

\[ \varphi_{-t} = e^{-it\theta_0}, \quad P_0 := \frac{1}{i} V. \]
The symbol of $P_0$ and its Hamiltonian flow are

$$p(x, \xi) = \xi(V_x), \quad e^{H_p}(x, \xi) = (\varphi(t)_x, (T_0d\varphi(t)_x))^{-1}\xi.$$  

Here $H_p$ denotes the Hamilton vector field of $p$: $\omega(\bullet, H_p) = dp$, where $\omega = d(\xi dx)$ is the symplectic form on $T^*X$.

In the study of $P_0$ we need the dual decomposition of the cotangent space:

$$T^*_X = E^{00}_x(x) \oplus E^{01}_x(x) \oplus E^{10}_x(x),$$  

where $E^{00}_x(x), E^{01}_x(x), E^{10}_x(x)$ are symplectic annihilators of $E_0(x) \oplus E_a(x), E_0(x) \oplus E_b(x),$ and $E_0(x) \oplus E_b(x)$. Hence they are dual to to $E_0(x), E_a(x), E_b(x)$.

A special class of Anosov flows is given by contact Anosov flows. In that case $X$ is a contact manifold, that is a manifold equipped with a contact 1-form $\alpha$: that means that if the dimension of $X$ is $2k - 1$ then $(d\alpha)^{k} \wedge \alpha$ is non-degenerate. A contact flow is the flow generated by the Reeb vector field $V$:

$$\alpha(V) = 1, \quad d\alpha(V, \bullet) = 0.$$  

For an example of a non-Anosov contact flow see figure 3. An important class of examples of Anosov contact flows is obtained from negatively curved Riemannian manifolds $(M, g)$:

$$X = S^{k}M := \{(z, \zeta) \in T^*M; |\zeta| = 1\}, \quad \alpha = \zeta dz|_{S^{k}M}.$$  

2.2. Wave front set of distributions and operators

Semiclassical quantization on a compact manifold [DyZw1, appendix C], [Zw1, chapter 14] is central to our analysis.
Let $X$ be a compact manifold and $h \in (0, 1)$ a parameter (the asymptotic parameter in the semiclassical analysis). A family of $h$-dependent distributions $u \in \mathcal{D}'(X)$ is called $h$-tempered if for some $N$, $-\kappa < u_h \in H^N$. A phase space description of singularities of $u$ is given by the wave front set:

\[ \text{WF}_h(u) = T^*X, \]

where $T^*X$ is the fiber-radially compactified cotangent bundle, a manifold with interior $T^*X$ and boundary,

\[ \partial T^*X = S^*X = (T^*X \setminus 0)/\mathbb{R}^+, \quad \kappa : T^*X \setminus 0 \longrightarrow S^*X = \partial T^*X. \]

(2.4)

In addition to singularities, $\text{WF}_h$ measures oscillations on the $h$-scale. We also refer to it as the microsupport of $u$ or as having $u$ microlocalized to some region containing $\text{WF}_h(u)$—see section C.2 [DyZw1] for the definitions.

For families of $(h$-tempered) operators we define the wave front set $\text{WF}_h(B)$ using the Schwartz kernel of $B, K_B$:

\[ \text{WF}_h(B) = \{(x, \xi, y, -\eta) : (x, y, \xi, \eta) \in \text{WF}_h(K_B)\}. \]

This convention guarantees that $\text{WF}_h(I) = \Delta_{T^*X}$ is the diagonal, $\{(x, \xi, x, \xi)\}$, in $T^*X \times T^*X$.

2.3. Pseudodifferential operators

We only use the standard class of semiclassical pseudodifferential operators, $\Psi^m_h(X)$ with the symbol map $m$, for which

\[ \text{Figure 3.} \text{ An illustration of chaotic and stochastic trajectories: we consider the Nosé–Hoover oscillator [PHV] which is possibly the simplest chaotic system: } W = x_2 \partial_{x_1} + (-x_1 + x_3 \partial_{x_3}) \partial_{x_2} + (1 - x_3^2) \partial_{x_3}, \text{ } x \in \mathbb{R}^3. \text{ The vector field } V = e^{i \omega t} \partial_{x_3} \text{ is the Reeb vector field for the contact form } \alpha = e^{-i \omega t} (2x_2 dx_1 + dx_3). \text{ On the left the Poincaré section } \{x_3 = 0\} \text{ showing the chaotic sea and islands of quasi-periodicity (each colour corresponds to a numerical iteration of a single point). On the right a chaotic trajectory and the stochastic trajectory, } \varepsilon = 0.01, \text{ with the same initial data. We stress that the results of our paper do not apply to mixed systems and this example is meant as an illustration of chaotic and stochastic trajectories. However, as in [CNKMG], Pollicott–Ruelle resonances are expected to be relevant for mixed systems as well.} \]
0 \longrightarrow h\Psi^m_{h^{-1}}(X) \hookrightarrow \Psi^m_h(X) \xrightarrow{\partial_h} S^m(X)/hS^{m-1}(X) \longrightarrow 0,

is a short exact sequence of algebra homomorphisms and

\[ S^m(X) := \{ a \in C^\infty(T^*X) : \partial_x^j \partial_{\xi}^k a(x, \xi) = O_h((\xi^{m-|\beta|})) \} \]

(where we were informal about coordinates on \( X \)).

One of our uses of the pseudodifferential calculus is that for \( \chi \in C^\infty_0(\mathbb{R}) \), the operator \( \chi - \Delta_h \), defined via spectral theory on \( L^2 \), is pseudodifferential in the class \( \Psi^{-hN}_0 \) for each \( N \), and \( \sigma_0(\chi - \Delta_h) = \chi((\xi^2)_{\mathbb{C}}) \)– see [Zw1, theorem 14.9]. Moreover, we implicitly use in the analysis of the operator \( \tilde{P}(\lambda) \) in section 4 that the \( S^0 \)-seminorms of the full symbol of \( \chi - \Delta_h \) are controlled by the \( S^0(\mathbb{R}) \)-seminorms of \( \chi \). To see that, we use the proof of [Zw1, theorem 14.9] to write the full symbol of \( \chi - \Delta_h \) in the form (see [Dy, propositions 2.2 and 2.4] for details)

\[
\sum_{j,k} h^j \sum_{j,k} \chi^{(j\xi_k^2)} a_{j,k}(x, \xi), \quad a_{j,k} \in S^{2j-2}(T^*X).
\]

If we control \( \sup_{\lambda \in \mathbb{R}} \chi^{(j\xi_k^2)}(\lambda) \) for all \( k \geq 0 \), then we control \( \chi^{(j\xi_k^2)} \) in \( S^{-2k} \) and thus we control (2.5) in \( S^0 \).

The semiclassical Sobolev spaces on \( X \) are defined as

\[
H^k_s(X) = (I - h^2\Delta_h)^{-k/2}L^2(X) \subset \mathcal{D}'(X),
\]

for a choice of a Laplacian \( \Delta_h \leq 0 \) on \( X \) and with the inner product inherited from \( L^2 \).

For \( A \in \Psi^m_h(X) \) the elliptic set \( \text{ell}_h(A) \subset T^*X \) is defined as the set of \( (x, \xi) \in T^*X \) such that \( \langle (\xi^2)^{-1}(A)(x', \xi') \rangle \geq c > 0 \) for \( h \) small enough and all \( (x', \xi') \in T^*X \) in a neighbourhood of \( (x, \xi) \). We recall [DyZw1, proposition 2.4]:

**Proposition 2.1.** Suppose that \( P \in \Psi^0(X) \) and that \( u(h) \in \mathcal{D}'(X; \mathcal{E}) \) be \( h \)-tempered. Then

\[
WF_h(u) \cap \text{ell}_h(P) \subset WF_h(Pu).
\]

If \( A \in \Psi^m_h(X) \) and \( WF_h(A) \subset \text{ell}_h(P) \), then for each \( m \),

\[
\|Au\|_{H^m_h(X)} \leq C\|Pu\|_{H^m_{h^{-1}}(X)} + O(h^\infty).
\]

2.4. Propagation estimates

The crucial components of the proofs of theorems 1 and 2 are propagation results presented in [DyZw1, section 2.3] and proved in [DyZw1, section C.3].

We start by recalling a modification of the result of Duistermaat–Hörmander:

**Proposition 2.2.** Assume that \( \tilde{P} \in \Psi^0_h(X) \) and the semiclassical principal symbol, \( \sigma_0(\tilde{P}) \in S^0_h \), has a representative \( \tilde{P} - iq \), where for some \( \delta > 0 \),

\[
\tilde{P} = p + O(h^\delta)(\xi^{2i})_{T^*X}, \quad p(x, \xi) = tp(x, \xi) \in \mathbb{R}, \quad \|\xi\|_{\delta} \geq 1, \quad t \geq 1, \quad q \geq 0.
\]

Let \( e^{it\xi} \) be the Hamiltonian flow of \( p \) on \( T^*X \) and \( u(h) \in \mathcal{D}'(X; \mathcal{E}) \) be an \( h \)-tempered family of distributions. Then (see figure 4):
1. Assume that $A, B, C \in \Psi^{(0)}(X)$ and for each $(x, \xi) \in \WF_{h}(A)$, there exists $T > 0$ with $e^{-iH_{t}}(x, \xi) \in \ell l_{h}(C)$ and $e^{iH_{t}}(x, \xi) \in \ell l_{h}(B)$ for $t \in [-T, 0]$. Then for each $m$,

$$
\| Au \|_{\ell l^{m}(X; E)} \leq K \| Cu \|_{\ell l^{m}(X; E)} + K h^{-1} \| B Pu \|_{\ell l^{m}(X; E)} + O(h^{\infty}).
$$

(2.10)

2. If $\gamma(t)$ is a flow line of $H_{p}$, then for each $T > 0$,

$$
\gamma(-T) \not\in \WF_{h}(u), \quad \gamma([-T, 0]) \cap \WF_{h}(Pu) = \emptyset \implies \gamma(0) \not\in \WF_{h}(u).
$$

(2.11)

**Proof.** We explain the modifications needed in the proof of [DyZw1, proposition 2.5] where $p = p$. We again construct the escape function $f$ using the homogeneous part of the symbol given by $p$. The difference $p - \tilde{p}$ produces an additional $O(h^{\infty})$ term in the operator $T_{t}$ of [DyZw1], which is uniform in the parameter $\epsilon$ of [DyZw1] (note that in [DyZw1] the letter $\epsilon$ has a different meaning than in the current paper). The $H^{m-1/4}$ norm should be replaced by the $H^{m-1}$ norm on the right-hand side of [DyZw1, (C.12)], which leads to the same modification on the right-hand side of [DyZw1, (C.5)]; the rest of the proof is carried out the same way as in [DyZw1].

This propagation result is applied away from the radial sinks and sources given by $\kappa(E_{+}^{*})$ and $\kappa(E_{+}^{*})$ where $\kappa$ is the projection in (2.4) and $E_{+}^{*}$ are from (2.2). Near $\kappa(E_{+}^{*})$ we use radial estimates obtained in the context of scattering theory by Melrose [Me, propositions 9,10] (see also Vasy [Va, propositions 2.3, 2.4]). These less standard estimates guarantee regularity of $u$ near sources/sinks, provided that $u$ lies in a sufficiently high/low Sobolev space.

Let $p$ satisfy the assumptions in (2.9). Assume that $L \subset T^{*}X \setminus 0$ is a closed conic set invariant under the flow $e^{tH_{p}}$. It is called a radial source if there exists an open conic neighbourhood $U$ of $L$ with the following properties valid for some constant $\theta > 0$:

$$
d(\kappa(e^{-tH_{p}}(U)), \kappa(L)) \to 0 \quad \text{as } t \to +\infty;
$$

$$
(x, \xi) \in U \implies |e^{-tH_{p}}(x, \xi)| \geq C^{-1}e^{\theta t},
$$

(2.12)

for any norm on the fibers.

A radial sink is defined analogously, reversing the direction of the flow.

We now have a propagation estimate near radial sources. It shows that $P u$ controls $u$ there for sufficiently regular solutions:

**Proposition 2.3.** Let $P \in \Psi^{(0)}(X)$ and assume that $\sigma_{0}(P)$ has a representative of the form $p - iq$ and $p$ and $Q$ satisfy (2.9). Assume that $L \subset T^{*}X \setminus 0$ is a radial source for the flow of $H_{p}$. Then there exists $m_{0} > 0$ such that (see figure 5(a))

1. For each $B \in \Psi^{(0)}(X)$ elliptic on $\kappa(L) \subset S^{*}X = \partial T^{*}X$, there exists $A \in \Psi^{(0)}(X)$ elliptic on $\kappa(L)$ such that if $u(\cdot) \in \mathcal{D}^{*}(X; E)$ is $h$-tempered, then for each $m \geq m_{0}$

$$
Au \in H^{m}_{h} \implies \| Au \|_{\ell l^{m}_{h}} \leq K h^{-1} \| B Pu \|_{\ell l^{m}_{h}} + O(h^{\infty}).
$$

(2.13)
2. If $u(h) \in D'(X; E)$ is $h$-tempered and $B \in \Psi^0_h(X)$ is elliptic on $\kappa(L)$, then

$$Bu \in H^m_\infty, \WF_h(Pu) \cap \kappa(L) = \emptyset \implies \WF_h(u) \cap \kappa(L) = \emptyset .$$

(2.14)

The second result shows that for sufficiently low regularity we have a propagation result at radial sinks analogous to (2.10).

**Proposition 2.4.** Assume that $P \in \Psi^1_h(X)$ is as in proposition 2.3 and $L \subset T^*X \setminus 0$ is a radial sink. Then there exists $m_0 > 0$ such that for each $B \in \Psi^0_h(X)$ elliptic on $\kappa(L)$, there exists $A \in \Psi^0_h(X)$ elliptic on $\kappa(L)$ and $C \in \Psi^0_h(X)$ with $\WF_h(C) \subset eL_0(B) \setminus \kappa(L)$, such that if $u(h) \in D'(X)$ is $h$-tempered, then for each $m \leq -m_0$ (see figure 5(b))

$$\|Au\|_{H^m_\infty} \leq K\|Cu\|_{H^m_\infty} + K h^{-1}\|BPu\|_{H^m_\infty} + C(h^\infty).$$

(2.15)

The proofs of propositions 2.3 and 2.4 can be found in [DyZw1, section C.3].

### 3. Definition of Pollicott–Ruelle resonances

The resonances for Anosov flows are defined as spectra of the generator of the flow acting on suitably modified spaces—see Baladi–Tsujii [BaTs], Faure–Sjöstrand [FaSj], Gouëzel–Liverani [GoLi1, Li2], and references given there.

Here we follow [DyZw1, sections 3.1 and 3.2] where the spaces are defined using microlocal weights with simple properties:

$$H_{Gibh}(X) := \exp(-sG(x, hD))L^2(X), \quad G \in \Psi^0_h(X),$$

$$\sigma_h(G) = (1 - \psi_0(x, \xi))m_{G}(x, \xi) \log|\xi|, \quad (3.1)$$

where $\psi_0 \in C^\infty_c(T^*X, [0, 1])$ is 1 near $\{\xi = 0\}$, $m_G(x, \xi) \in C^\infty(T^*X \setminus 0, [-1, 1])$ is homogeneous of degree 0 and satisfies

$$m_{G}(x, \xi) = \begin{cases} 1 & \text{near $E^\theta_h$}, \\ -1 & \text{near $E^3_{-h}$} \end{cases} \quad (3.2)$$

The existence of such $m_{G}$ is shown in [DyZw1, lemma C.1]. For convenience we choose $|\xi|^2$ to be the same metric as in the definition of the Laplacian $-\Delta_x$. We can also assume that for some $\chi_0 \in C^\infty_c(\mathbb{R}), \chi_0 \equiv 1$ near 0,

$$G(x, hD) = (1 - \chi_0(-h^2\Delta_x))G(x, hD). \quad (3.3)$$
(Simply multiply $G(x,hD)$ by $(1 - \widetilde{\chi}_0(-h^2\Delta_x))$ for $\widetilde{\chi}_0 \in C_c^\infty$ such that if $|\xi| \in \text{supp} \chi_0$ then $\psi_0(x,\xi) = 1$ and then choose $\chi_0$ so that $\text{supp} \chi_0 \subset \chi_0^{-1} \{1\}$.

We note that as a set $H_{s,G(h)}$ is independent of $h$ and that for some $N$ and $C$,

$$h^N \|u\|_{H^s_G(h)} / C \leq \|u\|_{H^s_G(h)} \leq C h^{-N} \|u\|_{H^s_G(h)} .$$

We also need a version of weighted Sobolev spaces associated to $H_{s,G(h)}$:

$$H^s_{G(h)} := \exp(-G_{\tau}(x,hD))L^2(X), \quad G_{\tau,\tau} \in \Psi^{0,0}_h(X),$$

$$\sigma_0(G_{\tau,\tau}) = (1 - \psi_0(x,\xi))(\text{sm}_G(x,\xi) + r) \log|\xi|^r .$$

We can also assume that (3.3) holds for $G_{\tau,r}$ as well.

The spaces with $\tau \neq r$ will be used to control applications of differential operators:

$$\Psi^{\mu}_h(X) \ni A : H^s_{G(h)}(X) \rightarrow H^{s-m}_{G(h)}(X).$$

Since (see [DyZw1, (3.9)])

$$H^{s}_h(sG_{\tau}) = s \log|\xi|^r H_{\tau}m_G + \mathcal{O}(1)G_h^s$$

we can use the estimates reviewed in section 2.4 as in the proof of [DyZw1, proposition 3.4]. That shows that for any $r \in \mathbb{R}, \lambda \in D(0,R), s > s_0 = s_0(R, r)$ and $0 < h < h_0$,

$$(hP_0 - iQ - h\lambda)^{-1} = \mathcal{O}(1/h) : H^s_{G(h)} \rightarrow H^{s-m}_{G(h)}$$

Here $Q$ is a complex absorbing operator

$$Q = \chi(-h^2\Delta_x), \quad \chi \in C_c^\infty((-2, 2); [0, 1]), \quad \chi(t) = 1, \quad t \in [-1, 1].$$

It is introduced to damp the trapped set which, on $p^{-1}(0)$, is equal to the zero section. Writing $P_0 - \lambda = h^{-1}(I + iQ(hP_0 - iQ - h\lambda)^{-1})(hP_0 - iQ - h\lambda)$, and noting that

$$Q(hP_0 - iQ - \lambda h)^{-1} : H^s_{G(h)} \rightarrow C^\infty(X),$$

is compact as an operator $H^s_{G(h)} \rightarrow H^s_{G(h)}$, analytic Fredholm theory (see for instance [Zw1, theorem D.4]) shows that $(P_0 - \lambda)^{-1}$ is a meromorphic family:

**Proposition 3.1.** For $\lambda \in D(0,R)$ and $s > s_0 = s_0(R)$,

$$(P_0 - \lambda)^{-1} : H^s_{G(h)} \rightarrow H^s_{G(h)},$$

is a meromorphic family of operators with poles of finite rank. These poles are independent of $s$ and are called Pollicott–Ruelle resonances.

The mapping property (3.9) also shows that the operator there is of trace class. Combined with Gohberg–Sigal theory (see for instance [DyZw2, (C.4.6)]) this gives the following characterization of Pollicott–Ruelle resonances:

**Proposition 3.2.** Let $R > 0$ and assume that $s > s_0(R)$. For $0 < h < h_0(R, s)$ define

$$D_R(\lambda) := \det_{H^s_{G(h)}}(I + iQ(hP_0 - iQ - \lambda h)^{-1}), \quad \lambda \in D(0,R).$$

Then Pollicott–Ruelle resonances in $D(0,R)$ are given, with multiplicities, by the zeros of $D_R$. 

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4. Microlocal bounds on the modified operator

Let \( P \) be given by (1.1) and let \( Q \) be the complex absorbing operator (3.8). The goal of this section is to prove that for \( 0 < h < h_0 \) and \( \varepsilon < h/C \) the operator \( hP - iQ - h\lambda \) is invertible on the same weighted spaces on which \( hP_0 - iQ - h\lambda \) is invertible. Note that for \( \varepsilon > 0 \), \( hP - iQ - h\lambda \) is a Fredholm operator \( H_{G(h)}^2 \rightarrow H_{G(h)}^0 \) of index 0 by the standard elliptic theory applied to the conjugation of this operator by \( e^{G(x,D)} \) (see (4.6) below and [HöIII, theorem 19.2.1]).

We first prove an elliptic estimate, which does not involve the parameter \( h \):

**Lemma 4.1.** Suppose that \( \chi \in C_c^\infty ((-2, 2), [0, 1]) \) satisfies \( \chi_1 = 1 \) on \([-1, 1]\), and put \( \chi_0(t) := \chi_1(3t) \). Then for \( \lambda \in \mathbb{D} \) and \( \varepsilon \)

\[
\left\| (1 - \chi_0(-\varepsilon^2 \Delta_x)u) \right\|_{L^2} \leq C\varepsilon \left\| (1 - \chi_0(-\varepsilon^2 \Delta_x))(P_\lambda - \lambda u) \right\|_{L^2} + O(\varepsilon^2) \left\| u \right\|_{L^2}.
\]

(4.1)

**Proof.** In (3.3) we can assume that \( \supp \chi \cap \varepsilon = \emptyset \); changing \( \psi_0 \) corresponds to changing \( \psi_1 \) in the definition of \( H_{G(1)}^1 \) that produces an equivalent norm (see [Zw1, theorem 8.8]).

The weight of the space \( H_{G(1)}^1 \) is not smooth at the zero section when one considers the \( \varepsilon \)-quantization. To counteract this problem, we introduce a new, \( \varepsilon \)-dependent, norm on \( H_{G(1)}^1 \) using a modified weight:

\[
\sigma_\varepsilon(G(x, \varepsilon D)) := (1 - \chi_0(|\xi|/\varepsilon)) \log(|\xi|/\varepsilon) m_G(x, \xi) \text{ mod } \varepsilon \log(1/\varepsilon) S^{-1} + \theta^* X.
\]

(4.3)

(We used here the homogeneity of \( m_G : m_G(x, \xi/\varepsilon) = m_G(x, \xi) \).)

We now claim that for \( j = 1, 2 \)

\[
(e^{G(x,D)} - e^{G(x,D)}(1 - \chi_j(-\varepsilon^2 \Delta_x))) = O(\varepsilon^2) D(x) \rightarrow C^\infty(X).
\]

(4.4)

This can be rewritten as the following identity for \( t = s \):

\[
(e^{G(x,D)} - e^{G(x,D)}(1 - \chi_j(-\varepsilon^2 \Delta_x))) = O(\varepsilon^2) D(x) \rightarrow C^\infty(X).
\]

Differentiating the left-hand side in \( t \), we obtain

\[
e^{G(x,D)} C(t,s), \quad C(t,s) = \chi_0(-\varepsilon^2 \Delta_x) G(x, D) e^{(s-t)G(x,D)}(1 - \chi_j(-\varepsilon^2 \Delta_x)).
\]

We now consider \( C(t,s) \) as an operator in \( \Psi^{-1} \). Since \( \supp \chi_0 \cap \supp (1 - \chi_j) = \emptyset \), we see that the all the terms in the symbolic composition formula for the four factors in \( C(t,s) \) vanish. The remainder (estimated, for instance, as in [Zw1, (9.3.7)]) is of size \( \varepsilon^N \) for any any \( N \). Hence \( C(t,s) \in e^{\infty} \Psi^{-\infty}(X) \) and consequently

\[
e^{G(x,D)} C(t,s) \in e^{\infty} \Psi^{-\infty}(X),
\]

uniformly for bounded \( t, s \). Integration then gives (4.4).
By (4.4), we may replace the $H_{sG}^1$ norms in (4.1) by the $\| \cdot \|_{*, \varepsilon}$ norms. We now consider our operator in the $\varepsilon$-pseudodifferential calculus:

$$
\varepsilon(P - \lambda) \in \Psi_{s \varepsilon}^2, \quad p_s(x, \xi) := \sigma_\varepsilon(\varepsilon P) = -i |\xi|^2 \bar{\xi} + \xi(V).
$$

This operator is elliptic in the class $\Psi_{s \varepsilon}^2$ for $\xi \neq 0$. By the choice of $\chi_j$'s, we see that both $\varepsilon(P - \lambda) \in \Psi_{s \varepsilon}^2$ and $(1 - \chi(\varepsilon^2 \Delta g)) \in \Psi_{s \varepsilon}^2$ are elliptic on $\text{WF}(1 - \chi(\varepsilon^2 \Delta g))$. Hence the estimate (4.1) holds for $s = 0$—see proposition 2.1 above.

To prove (4.1) for the $\| \cdot \|_{*, \varepsilon}$-norms, we consider conjugated operators:

$$
P_{s, \varepsilon} := e^{iG(x, \varepsilon, D)} P e^{-iG(x, \varepsilon, D)}, \quad A_{s, \varepsilon}(x, hD) := e^{iG(x, \varepsilon, D)(1 - \chi(\varepsilon^2 \Delta g))} e^{-iG(x, \varepsilon, D)},
$$

and need to prove that

$$
\sigma_\varepsilon(\varepsilon P_{s, \varepsilon}) = -i |\xi|^2 \bar{\xi} + \xi(V) + \varepsilon s H_{s \varepsilon} \sigma_\varepsilon(G) + i \varepsilon H_{t \varepsilon} \rho_{s \varepsilon}(G) \mod \varepsilon S^1(T^*X).
$$

Recalling (4.3) we see that

$$
|H_{t \varepsilon} \sigma_\varepsilon(G) + H_{t \varepsilon} \rho_{s \varepsilon}(G)| \leq C \log(1/\varepsilon)(\xi)^{1 +}.
$$

Hence, $\varepsilon P_{s, \varepsilon} - \varepsilon \lambda$ is elliptic in $\Psi_{s \varepsilon}^2$ on the set $|\xi| > \delta$ for any $\delta > 0$. Composition of pseudodifferential operators in $\Psi_{s \varepsilon}^2$ shows that

$$
\text{WF}(A_{s, \varepsilon}) \subset \{|\xi| > 1\} \subset \text{ell}_{s \varepsilon}(\varepsilon P_{s, \varepsilon} - \varepsilon \lambda), \quad \text{WF}(I - A_{s, \varepsilon}) \cap \text{WF}(A_{s, \varepsilon}) = \emptyset.
$$

We can apply proposition 2.1 again to obtain (4.5) and hence (4.1). We turn to the question of invertibility of $\varepsilon h P - iQ - \lambda h$ and suppose that

$$
(hP - iQ - \lambda h)u = f.
$$

For $\varepsilon < h/C$ we have $(1 - \chi(\varepsilon^2 \Delta g))Q = 0$. Hence in view of (3.4) and (4.1),

$$
|| \chi(\varepsilon^2 \Delta g) ||_{H_{\varepsilon \delta \eta 0}}^2 + || \chi(\varepsilon^2 \Delta g) ||_{H_{\varepsilon \delta \eta 0}}^2 \leq C \varepsilon^N || f ||_{H_{\varepsilon \delta \eta 0}} + O(h^{-N} \varepsilon^\infty) || u ||_{H_{\varepsilon \delta \eta 0}}, \quad (4.7)
$$

for $\lambda \in D(0, R)$, $\varepsilon < h/C$, and some $N$ depending on $s$. Put

$$
P_{1}(\lambda) := \frac{h}{i} \bar{V} + i \varepsilon h \Delta \chi(\varepsilon^2 \Delta g) - iQ + \lambda h, \quad (4.8)
$$
Then
\[ \hat{P}_\epsilon(\lambda) u = -i\hbar \Delta_x (1 - \chi_\epsilon(-\epsilon^2 \Delta_x)) u + f =: F. \] (4.9)

From (4.7) we see immediately that
\[ \|F\|_{u_{2225}} \lesssim \hbar^{-N} \|f\|_{u_{2225}} + O(h^{-N} \epsilon^\infty) \|u\|_{u_{2225}}, \] (4.10)

where \( N \) depends on \( s \).

The operator \( \hat{P}_\epsilon(\lambda) \) on the left-hand side of (4.9) is an \( h \)-pseudodifferential operator in \( \Psi^1_h \) and
\[ \sigma_\epsilon(\hat{P}_\epsilon(\lambda)) = (V_\epsilon) - i|\xi\chi_\epsilon(\frac{\xi^2}{h^2} + \xi) \frac{\xi}{h} - i\chi(|\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\xi\x
\[ \|A_2u\|_{H^\infty} = \|A_2u\|_{H^\infty} + O(h^{\gamma})\|u\|_{H^\infty}, \quad \|B_2f\|_{H^\infty} = \|B_2f\|_{H^\infty} + O(h^{\gamma})\|u\|_{H^\infty}, \]

and similarly for \(C_2\), so that again the estimate (4.13) is valid with \(H^\infty\) replaced by \(H^\infty\).

We now have to consider the case of \(A_3 \in \Psi^0(h)\) microlocalized away from \(\kappa(E^+_{\omega}) \cup \kappa(E^-_{\omega})\). For that we need to see that the conjugated operator satisfies the assumptions of the Duistermaat–Hörmander propagation theorem (proposition 2.2). As in (4.6) we have

\[ \tilde{P}_c(\lambda) := e^{-G(h)}\tilde{P}(\lambda)e^{-G(h)} = \tilde{P}(\lambda) - \text{i}h^2 \frac{1}{\hbar} [G(h), \tilde{P}(\lambda)] + O(h^{\gamma})\Psi_{h^{-1}}, \]

where now, as the operators \(\tilde{P}(\lambda)\) and \(G(h)\) are uniformly bounded in \(\Psi^0(h)\) and \(\Psi^0(h^{-1})\), respectively, the error is in \(\Psi_{h^{-1}}\). Hence we have

\[ \sigma_h(P_c(\lambda)) = p_{c,(x, \xi) - iq_{c,(x, \xi)}(h\Psi^0(h))} \]

where, with \(p(x, \xi) := \xi(V_x)\), away from \(\xi = 0\) we can take

\[ p_{c,(x, \xi)} = p(x, \xi) - \text{i}h^2 \log[A_\lambda \kappa H_{\text{osc}} \left( |\kappa_x| \chi_{\lambda} \left( \frac{\xi^2}{\hbar^2} \right) \left| \frac{\xi}{\hbar} \right| \right)], \]

\[ q_{c,(x, \xi)} = \chi(|\xi \frac{\xi}{\hbar}| + |\kappa_x| \chi_{\lambda} \left( \frac{\xi^2}{\hbar^2} \right) \left| \frac{\xi}{\hbar} - \text{i}h^2 \log[A_\lambda \kappa H_{\text{osc}}(x, \xi)] \right) \geq 0. \]
We note that \( \tilde{p} := p_{\delta, \epsilon} = p + \mathcal{O}(h)_{\delta, \epsilon} \) satisfies the assumptions of proposition 2.2 with \( \delta = 1 \) and \( d_{\delta, \epsilon}, \epsilon \leq 0 \). Hence the propagation estimate (2.10) applies.

As in the proof of [DyZw1, proposition 3.4], combining (4.12), (4.13), proposition 2.2, and the elliptic estimate (proposition 2.1) we obtain uniformly in \( \epsilon \),
\[
\|u\|_{L_{\delta, \epsilon}} \leq C h^{-1} \|\tilde{p}(\lambda)u\|_{H_{\delta, \epsilon}^s} + \mathcal{O}(h^\infty) \|u\|_{H_{\delta, \epsilon}^s}, \quad 0 < h < h_0(R), \quad s > s_0(R), \quad \lambda \in D(0, R), \quad 0 < \epsilon \leq h,
\]
for any \( N \) and that implies (4.11), finishing the proof. □

We now fix \( h < h_0 \) and apply lemma 4.2 to (4.9). That and (4.10) give
\[
\|u\|_{L_{\delta, \epsilon}} \leq C h^{-N} \|f\|_{L_{\delta, \epsilon}} + \mathcal{O}(h^{-N} e^\infty) \|u\|_{L_{\delta, \epsilon}}
\]
and the \( \mathcal{O}(h^{-N} e^\infty) \) can be absorbed into the left-hand side for \( \epsilon/h \) small enough.

We summarize the result of this section in

**Proposition 4.3.** Let \( P_0 \) be given by (1.1) and \( Q \) by (3.8). Suppose that \( \lambda \in D(0, R) \) and that
\[
0 \leq \epsilon \leq h C_0.\text{ Then there exist } h_0 = h_0(R), s_0 = s_0(R), \text{ (independent of } \epsilon) \text{ such that for } 0 < h < h_0 \text{ and } s > s_0(R)
\]
\[
h P_0 - iQ - h \lambda : H_{\delta, \epsilon}^2(h) \to H_{\delta, \epsilon}^2(h),
\]
is invertible and for some constants \( C \) and \( N \) independent of \( \epsilon \),
\[
\|(h P_0 - iQ - h \lambda)^{-1}\|_{L_{\delta, \epsilon}} \leq C h^{-N}.
\]

**Remark.** Same statement is true if we replace the spaces \( H_{\delta, \epsilon}^2(h) \) with \( H_{\delta, \epsilon}^r(h) \) for some fixed \( r \). Indeed, this amounts to replacing \( sm_G \) by \( sm_G + r \) in the weight \( G \). The proof of lemma 4.1 remains unchanged. As for lemma 4.2, its proof uses the inequality \( Hm_{\delta, \epsilon} \leq 0 \) (which is still true), as well as the fact that \( H_{\delta, \epsilon}^r(h) \) is equivalent to \( H_{\delta, \epsilon}^s(h) \) microlocally near \( E^s_* \) and to \( -H_{\delta, \epsilon}^s(h) \) microlocally near \( E^s_* \). The space \( H_{\delta, \epsilon}^r(h) \) is equivalent to \( H_{\delta, \epsilon}^s(h) \) near \( E^s_* \) and to \( -H_{\delta, \epsilon}^s(h) \) near \( E^s_* \); for \( s \) large enough depending on \( r \) and \( R \), lemma 4.2 still holds.

5. Stochastic approximation of Pollicott–Ruelle resonances

In this section we prove theorem 1. Using proposition 4.3 we see that for \( \lambda \in D(0, R) \), we have the following expression for the meromorphic continuation of the resolvent of \( P_0 \):
\[
(P_0 - \lambda)^{-1} = h(h P_0 - iQ - h \lambda)^{-1}(I + K(\lambda, \epsilon))^{-1} : H_{\delta, \epsilon} \to H_{\delta, \epsilon},
\]
where
\[
K(\lambda, \epsilon) := iQ(h P_0 - iQ - h \lambda)^{-1} : H_{\delta, \epsilon} \to H_{\delta, \epsilon},
\]
is of trace class and depends holomorphically on \( \lambda \)—see (3.9). Here \( 0 < h < h_0, 0 \leq \epsilon \leq \epsilon_0 := h C_0 \) and \( s > s_0 \) with \( h_0 \) and \( s_0 \) depending on \( R \). We fix \( h \) and drop it in the notation for \( H_{\delta, \epsilon} \).

As in proposition 3.2 we see that the spectrum of \( P_0 \) in \( D(0, R) \) is given (with multiplicities) by the zeros of the following Fredholm determinant:
\[
D_\delta(\lambda, \epsilon) := \det_{H_{\delta, \epsilon}}(I + K(\lambda, \epsilon)).
\]
Note that, since \( Q \) is compactly microlocalized, \( K(\lambda, \epsilon) \) acts \( H_{\delta, \epsilon} \to H_{\delta, \epsilon}^N \) for all \( N \). It follows that \( D_\delta(\lambda, \epsilon) \) is equal to the \( H^N \) determinant of \( I + K(\lambda, \epsilon) \) for each \( N \geq s \).
To analyze the determinant $D_R(\lambda, \varepsilon)$, we apply the following two lemmas. We use the notation $f \in C^k([a, b])$ to mean that $f$ and its derivative $f'$ are continuous in $[a, b]$; here $f'(a), f'(b)$ are the left and right derivatives of $f$ at those points. By induction we then define $C^k([a, b])$ and $C^\infty([a, b])$.

**Lemma 5.1.** Let $R$ and $h$ be fixed so that (5.1) is valid. Then for every $k$ there exists $s_1 = s_1(k, R)$ such that for $s \geq s_1$

$$K(\lambda, \varepsilon) \in C^k([0, \varepsilon_0], \text{Hol}(D(0, R), \mathcal{L}(H', H'))),$$

(5.4)

where $H' = H'(X)$ are Sobolev spaces and $\mathcal{L}$ denotes the space of trace class operators.

**Proof.** We first show that the identity

$$\partial_\varepsilon (hP_x - iQ - h\lambda)^{-1} = -i h (hP_x - iQ - h\lambda)^{-1} \Delta_x (hP_x - iQ - h\lambda)^{-1}$$

(5.5)

is true for $\varepsilon \in [0, \varepsilon_0]$ in the space $\text{Hol}(D(0, R), \mathcal{B}(H'_{sg}, H'_{sg}^{-4}))$, for each $r$ and for $s$ large enough depending on $R$ and $r$. Here $\mathcal{B}$ stands for the class of bounded operators with operator norm. Indeed, for each $\varepsilon, \varepsilon' \in [0, \varepsilon_0]$, 

$$\frac{(hP_x - iQ - h\lambda)^{-1} - (hP_x - iQ - h\lambda)^{-1} \Delta_x (hP_x - iQ - h\lambda)^{-1}}{\varepsilon - \varepsilon'}$$

(5.6)

where the right-hand side of the equation is uniformly bounded in $\varepsilon, \varepsilon'$ as an operator $H'_{sg} \to H'^{-2}_{sg}$. Here we used

$$(hP_x - iQ - h\lambda)^{-1} \in \mathcal{B}(H'_{sg}, H'_{sg}^{'-4}), \quad (hP_x - iQ - h\lambda)^{-1} \in \mathcal{B}(H'_{sg}^{'-2}, H'^{-2}_{sg}),$$

(see proposition 4.3 and the remark following it) and the fact that $\Delta_x$ is bounded $H'_{sg} \to H'^{-2}_{sg}$. Now, (5.6) implies that $(hP_x - iQ - h\lambda)^{-1}$ is Lipschitz (and thus continuous) as an operator $H'_{sg} \to H'^{-2}_{sg}$. Passing to the limit $\varepsilon' \to \varepsilon$ in (5.6), we obtain (5.5) in the class $\mathcal{B}(H'_{sg}, H'^{-4}_{sg})$. Holomorphy in $\lambda$ follows automatically from the holomorphy of each of the operators involved.

Iterating (5.5), we see that for each $r$, each $k > 0$, and for $s$ large enough depending on $R$, $r$ and $k$,

$$(hP_x - iQ - h\lambda)^{-1} \in C^k([0, \varepsilon_0], \text{Hol}(D(0, R), \mathcal{B}(H'_{sg}, H'^{-4k}_{sg}))).$$

To obtain (5.4) we recall the definition (5.2) of $K(\lambda, \varepsilon)$, take $r = 0$, note that $H'$ embeds into $H_{sg}$ and that the operator $Q$ is compactly microlocalized and thus of trace class $H'^{-4k}_{sg} \to H'^{r}$. \hfill \square

**Lemma 5.2.** Suppose that $(X_j)_{j=0}^{\infty}$ is a nested family of Hilbert spaces, $X_{j+1} \subset X_j$. Let

$$K(\varepsilon) : X_j \to \bigcap_{\varepsilon' = 0}^{\infty} X_{\varepsilon'},$$

(5.7)

be a family of operators such that $K \in C^k([0, \varepsilon_0], \mathcal{L}(X_0, X_0))$. Then

$$F(\varepsilon) := \det_{X_0}(I + K(\varepsilon)) \in C^\infty([0, \varepsilon_0]).$$
Proof. Because of (5.7) we see that $\det_x(I + K(\varepsilon))$ is independent of $j$ and hence we only need to prove that $\det_x(I + K(\varepsilon)) \in \mathcal{C}([0, \varepsilon])$, for any $j$. For $j = 1$ we note that $\partial_\varepsilon F(\varepsilon) = F(\varepsilon) \text{tr}_x((I + K(\varepsilon))^{-1} \partial_\varepsilon K(\varepsilon))$. The operators $\varepsilon \mapsto F(\varepsilon)(I + K(\varepsilon))^{-1}$ form a continuous family of uniformly bounded operators (see for instance [DyZw2, (B.7.4)]). Hence, $|\partial_\varepsilon F(\varepsilon)| \leq C \|\partial_\varepsilon K(\varepsilon)\|_{\mathcal{C}(X, X_1)}$. Higher order derivatives are handled similarly and smoothness of $F$ follows.

Applying this lemma with $X_0 = H^{s,j}(R^n)$ where $s_1$ comes from lemma 5.1, we see that $\varepsilon \mapsto D_\varepsilon(\lambda, \varepsilon)$ is a smooth function of $\varepsilon \in [0, \varepsilon_0]$ with values in $\text{Hol}(D(0, R^n))$. Rouché’s theorem implies that the zeros are continuous in $\varepsilon$ up to $0$, proving theorem 1. If $\mu_0$ is a simple zero of $D_\varepsilon(\lambda, 0)$ then for $0 \leq \varepsilon < \varepsilon_0$, $D_\varepsilon(\lambda, \varepsilon)$ has a unique zero, $\mu(\varepsilon)$, close to $\mu_0$. Smoothness of $D_\varepsilon$ in $\varepsilon$ shows that

$$\mu(\varepsilon) \in C^\infty([0, \varepsilon_0]).$$

When the zeros are not simple (in particular, when the eigenvalues of $P_0$ are not semisimple) the situation is potentially quite complicated. However we have smoothness of spectral projectors:

**Proposition 5.3.** Suppose that $\mu_0 \in D(0, R - 1)$ is an eigenvalue of $P_0 : H_{sp}(X) \to H_{sp}(X)$, $s \geq s_0(R)$, and that multiplicity of $\mu_0$ is $m$:

$$m = \text{tr} \Pi_0, \quad \Pi_0 = \frac{1}{2\pi i} \oint_\gamma (\lambda - P_0)^{-1} d\lambda,$$

where $\gamma : [0, 2\pi) \ni t \mapsto \mu_0 + \delta e^{it}$, and $\delta$ is small enough.

Then there exists $\varepsilon_0$ and $\delta$ such that for $0 < \varepsilon \leq \varepsilon_0$, $P_\varepsilon$ has exactly $m$ eigenvalues in $D(\mu_\varepsilon, \delta)$:

$$\text{tr} \Pi_\varepsilon = m, \quad \Pi_\varepsilon := \frac{1}{2\pi i} \oint_\gamma (\lambda - P_\varepsilon)^{-1} d\lambda, \quad \Pi_\varepsilon^2 = \Pi_\varepsilon,$$  \hspace{1cm} (5.8)

and $\Pi_\varepsilon \in C^\infty([0, \varepsilon_0], \mathcal{L}(C^\infty(X), D(X)))$. More precisely, the projections $\Pi_\varepsilon$ have rank $m$ and for each $j$ there exists $s_j$ such that

$$\Pi_\varepsilon \in C([0, \varepsilon], \mathcal{L}(H_{sp}, H_{sp})) \subset C^j([0, \varepsilon], \mathcal{L}(H^{s_j}, H^{s_j})).$$  \hspace{1cm} (5.9)

**Proof.** From the analysis of the determinants, we already know that there exist $\varepsilon_0$, $\delta$ such that for $0 \leq \varepsilon \leq \varepsilon_0$, $\lambda \mapsto D_\varepsilon(\lambda, \varepsilon)$ has no zeros on $[\lambda - \mu_0] = \delta$ and has exactly $m$ zeros in $D(\mu_\varepsilon, \delta)$. Hence the spectral projectors are well defined by the formula in (5.8) and their rank is equal to $m$. To consider regularity, we choose $h$ sufficiently small (depending on $R$) and write

$$(P_\varepsilon - \lambda)^{-1} = (P_\varepsilon - \mu_0)^{-1}Q(\lambda - P_\varepsilon)^{-1}Q(\mu_0)^{-1}(P_\varepsilon - \mu_0)^{-1}Q(\lambda - P_\varepsilon)^{-1}Q(\mu_0)^{-1}(P_\varepsilon - \lambda)^{-1}.$$  \hspace{1cm}

Since the first term is holomorphic in $\lambda \in D(0, R)$ we have

$$\Pi_\varepsilon := -\frac{1}{2\pi h} \oint_\gamma (\lambda - P_\varepsilon)^{-1}Q(\lambda - P_\varepsilon)^{-1}d\lambda.$$

Also

$$(\lambda - P_\varepsilon)^{-1} = O_{R, s_0}(1) : H_{sp}' \to H_{sp}', \quad s \geq s_0(R, r), \quad \lambda \in \partial D(\mu_0, \delta).$$

Hence the same argument as in the proof of lemma 5.1 shows $j$-fold differentiability of $\Pi_\varepsilon$ as bounded operators $H_{sp} \to H_{sp}$.  \hspace{1cm} $\Box$
6. Stochastic stability in the case of contact Anosov flows

We now turn to the proof of theorem 2. The first result concerns values of $\varepsilon$ larger than $h^2$. Here we do not need to make the contact assumption on the flow.

**Lemma 6.1.** Let $P_{\varepsilon}$ be given by (1.1). There exist $K_0 > 0$ and $h_0 > 0$ such that for any $\gamma > 1$, $h$ and $\varepsilon$ satisfying

$$0 < K_0 \gamma h^2 < \varepsilon, \quad 0 < h < h_0,$$

we have

$$(hP_{\varepsilon} - z)^{-1} = C\left(\frac{1}{\sqrt{\varepsilon}}\right) : L^2(X) \to L^2(X), \quad z \in \left[\frac{1}{2}, \frac{3}{2}\right] - i[0, \gamma h].$$

(6.1)

In particular $hP_{\varepsilon}$ does not have any spectrum with $|z - 1| < \frac{1}{2}$ and $\Im z > -\gamma h$.

**Remark.** The lemma shows that for any fixed $\varepsilon$ the number of eigenvalues of $P_{\varepsilon}$ in $\lambda > -C$ is finite. In fact the rescaling from $z$ to $\lambda$ shows that there are only finitely many eigenvalues of $P_{\varepsilon}$ in $[\Im \lambda > -\varepsilon|\Im \lambda|^2/C_0]$, for some fixed $C_0$. This leads to an easy justification of the expansion (1.3). We also see that a gap $\Im z > -\gamma h$ for any $\gamma$ is valid for $\varepsilon > C(\gamma)h^2$. Hence in what follows we will assume that $\varepsilon = \mathcal{O}(h^2)$.

**Proof.** We fix the volume form on $X$ induced by the metric $g$, so that the operator $\Delta_g$ is symmetric on $L^2(X)$. Take $u \in H^0(X)$ and denote $\mathcal{A} := -\varepsilon fh P_z u u$; then

$$\mathcal{A} = \mathcal{A}(V u, u) = \frac{h}{i}(V u, u)_{L^2} - i h \varepsilon \|\nabla u\|_{L^2}^2 - z\|u\|_{L^2}^2.$$ 

Taking the real part, we get

$$\Re \mathcal{A} = h f(\Im z) - \Re \varepsilon \|\nabla u\|_{L^2}^2.$$ 

Since $\Im z \geq \frac{1}{2}$ and $V$ is a vector field, we find for some constant $C$ independent of $h, z, \varepsilon$,

$$\|u\|_{L^2}^2 \leq C f\|f\|_{L^2} \|u\|_{L^2}^2 + C h \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2,$$

which implies

$$\|u\|_{L^2}^2 \leq C f\|f\|_{L^2} + C h \|\nabla u\|_{L^2}^2.$$ 

(6.2)

Now, taking the imaginary part, we get for $F := \frac{1}{2} \text{div} V \in C^\infty(X)$,

$$\Im \mathcal{A} = h f(\Im z) - \Re \varepsilon \|\nabla u\|_{L^2}^2 -(\Im z)\|u\|_{L^2}^2.$$ 

Since $\Im z \geq -\gamma h$ and $F$ is a bounded function, we get

$$h \varepsilon \|\nabla u\|_{L^2}^2 \leq C f\|f\|_{L^2} \|u\|_{L^2}^2 + (C + \gamma) h \|u\|_{L^2}^2,$$

which implies

$$\|\nabla u\|_{L^2} \leq Ch^{-1}e^{-1/2} f\|f\|_{L^2} + (C + \sqrt{\gamma})e^{-1/2} \|u\|_{L^2}.$$ 

(6.3)
Combining (6.2) and (6.3), we get
\[ \|u\|_{L^2} \leq C e^{-1/2} \|f\|_{L^2} + Ch\left(C + \sqrt{\gamma}\right)\varepsilon^{-1/2}\|u\|_{L^2}. \]

For \( K_0 \) large enough and \( \varepsilon > K_0^2 N^2 \), we have \( Ch\left(C + \sqrt{\gamma}\right)\varepsilon^{-1/2} < \frac{1}{2} \), implying (6.1).

To prove theorem 2 we follow [NoZw]. We first prove a result in which damping is introduced near the fiber infinity in \( T^*X \). For that we introduce a complex absorbing operator
\[ \hat{P}_t := -\Delta h + W(\xi) h \],
where \( f \in C^\infty(\mathbb{R}) \) satisfies the following conditions:
\[ f \geq 0, \quad |f(t)| \leq C_k f^{1-\alpha}, \quad f(t) \equiv 0 \text{ for } t \leq C_0, \quad f(t) \equiv 1 \text{ for } t \geq 2 C_0 \]
for some \( \alpha < \frac{1}{2} \) and some large constant \( C_0 \). The technical condition on \( f(t) \) is useful for comparing the propagators of \( \hat{P}_t \) and \( P_0 \)—see [NoZw, appendix].

With \( P_0 \) given by (1.1) we now consider
\[ \hat{P}_t := h P_0 - i W(\xi), \]
Unlike in sections 4 and 5 we will now work near a fixed rescaled energy level \( z = h \lambda \sim 1 \) rather than near the zero energy.

The next result is an almost immediate application of [NoZw, theorem 2]:

**Lemma 6.2.** Suppose that the flow \( \varphi_t : X \to X \) is a contact Anosov flow (see (2.3)), \( \hat{P}_t \) is given by (6.6) and that \( \varepsilon = O(h^2) \). Let \( \gamma_0 \) be the averaged Lyapounov exponent defined in (1.5). Then for any \( \delta > 0 \) and \( s \) there exist \( h_0, c_0, C_{1} \), such that for \( 0 < h < h_0 \),
\[ \| (\hat{P}_t - z)^{-1} \|_{L^2 \to L^2} \leq C h^{-1 + \alpha_0} \log(1/h), \]
for
\[ z \in \left[ \frac{1}{2}, \frac{3}{2} \right] - i h [0, \gamma_0 / 2 - \delta]. \]

**Remark.** The bound (6.7) is more precise than the bound (1.6) which corresponds to \( O(h^{-N}) \). It is obtained by interpolation between the bound \( 1/3z \) for \( z > 0 \) and the polynomial bound \( O(h^{-N}) \)—see [Bu, lemma 4.7], [TaZw, lemma 2]. Using the fact that \( H_{st} \) are complex interpolation spaces [Ca] the estimate (1.6) can be refined to a form similar to (6.7).

**Proof.** Put \( W = -(\varepsilon/h) h^2 \Delta_k + W_0 \) (where \( W_0 \) appearing in the definition of \( \hat{P}_t \) is given by (6.4)). Note that \( \hat{P}_t = h P_0 - i W \). We have \( W \in \Psi_0(X) \), \( W \geq 0 \), and since \( \varepsilon = O(h^2) \),
\[ w := c_0(W) = c_0(W_0) = f(|\xi|^2). \]
Hence \( P := h P_0 \) and \( W \) satisfy the assumptions [NoZw, (1.9), (1.10)]. The only difference is that \( P \in \Psi_0(X) \), so that in the notation of [NoZw], \( k = 2 \) and \( m = 1 \). Replacing \( k \) with \( m \) in the ellipticity condition [NoZw, (1.9)] does not change the proofs in [NoZw] (in particular it does not affect [NoZw, proposition A.3]): all the arguments are microlocal near the (compact) trapped set
\[ \hat{K} := \{(x, \xi) : \rho_0(x, \xi) - 1 < 1/2, \quad \exp(i H_{st})(x, \xi) \to \infty, t \to \pm \infty\}, \]
\[ p(x, \xi) = \xi(V_x). \]

Since \( \varphi_t \) is a contact Anosov flow, the trapped set is normally hyperbolic in the sense of [NoZw, (1.14)–(1.17)]—see [NoZw, section 9]. Hence we can apply [NoZw, theorem 2] and obtain the bound (6.7). □

We are now ready for

Proof of theorem 2. We first note that (1.6) follows by rescaling \( \lambda = z h \) from a semiclassical estimate between the weighted spaces (we recall that \( \| \| \leq \| \| \))

\[ (hP_z - z)^{-1} = O(h^{-N}) : H_{\sigma(G)1}(X) \to H_{\sigma(G)1}(X), \] (6.10)

for the same range of \( z \)'s as in (6.8). By lemma 6.1, we can assume that \( \varepsilon = O(h^{2}). \)

To prove (6.10) we follow the strategy as in [NoZw, section 9] combined with the estimates of section 4. For that we choose \( Q \) in (3.8) and \( W_0 \) in (6.4), so that for the weight \( G \) in (3.1) and the trapped set \( K \) defined in (6.9) we have

\[ \text{WF}_d(G) \cap \text{WF}_d(G) = \hat{K} \cap \text{WF}_d(I - Q) = \text{WF}_d(G) \cap \text{WF}_d(W_0) = \emptyset. \]

Since \( \hat{K} \) is compact that is possible by modifying the conditions on \( \chi \) in (3.8) and by increasing \( C_0 \) in (6.5).

To stay close to the notation of [NoZw, section 9] we now put \( \varepsilon = +\Delta - \infty \varphi \). To apply the gluing argument of Datchev–Vasy [DaVa] as in [NoZw, section 9] we check that the conclusions of [NoZw, lemma 9.19] are valid. First,

\[ (P_\infty - z)^{-1} = O(h^{-N}) : H_{\sigma(G)1}(X) \to H_{\sigma(G)1}(X), \quad \Im z > -\gamma(1/2), \quad |\Re z - 1| < 1/2, \] (6.11)

is proved similarly as (4.16). Indeed, lemma 4.1 holds for \( \lambda = O(e^{-1/2}) \), and this condition is true since \( \lambda = O(h^{-1}) \) and \( \varepsilon = O(h^{2}) \). The proof of lemma 4.2 goes through as in the case \( \lambda = O(1) \). The proof of (4.16) works as before, using again that \( \varepsilon = O(h^{2}). \)

Next, we need the propagation statement,

\[ u = (P_\infty - z)^{-1}f, \quad \text{WF}_d(f) \cap \partial T^*X = \emptyset \implies \text{WF}_d(u) \setminus \left( \text{WF}_d(f) \cup \partial T^*X \right) \subseteq \exp([0, \infty)H_p)(\text{WF}_d(f) \cap \partial^{-1}(\partial T^{\infty})), \]

where \( p(x, \xi) = \xi(V_x). \) This statement follows from the propagation theorems reviewed in the proof of lemma 4.2; note that \( (hP_\infty - z)^{-1}u = O(h^{\infty})e^z \) since \( \varepsilon = O(h^{2}) \) and \( \text{WF}_d(f) \) does not intersect the fiber infinity.

We can now follow the gluing argument of the resolvent estimates on \( (P_\infty - z)^{-1} \) and \( (P_\infty - z)^{-1} \) (given in (6.7)) as in [NoZw, section 9] to obtain (6.10). In the notation of [NoZw, section 9] the parametrix for \( (hP_z - z)^{-1} \) is given by

\[ F(z) := A_1(P_\infty - z)^{-1}A_0 + B_1(P_\infty - z)^{-1}B_0, \]

where \( A_i, I - B_j \in \Psi^{\text{comp}}(X) \) are suitably chosen, with \( \text{WF}_d(A_i) \cap \text{WF}(G) = \emptyset. \) Away from the microsupport of \( G \), the spaces \( H_{\sigma(G)}(h) \) are microlocally equivalent to \( L^2 \). Hence the \( L^2 \) estimates on \( (P_\infty - z)^{-1} \) imply the \( H_{\sigma(G)}(h) \) estimates on \( F(z) \). The gluing argument of [DaVa] as recalled in [NoZw, section 8] concludes the proof of (6.10). □
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