FROM PINNED BILLIARD BALLS TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. We discuss the propagation of kinetic energy through billiard balls fixed in place along a one-dimensional segment. The number of billiard balls is assumed to be large but finite and we assume kinetic energy propagates following the usual collision laws of physics. Assuming an underlying stochastic mean-field for the expectation and the variance of the kinetic energy, we derive a coupled system of nonlinear partial differential equations assuming a stochastic energy re-distribution procedure. The system of PDEs has a number of interesting dynamical properties some of which are numerically simulated.

1. Introduction

This paper is concerned with the evolution of pseudo-velocities of “pinned billiard balls” introduced in [ABD23]. Pinned billiard balls do no move but they have pseudo-velocities which evolve according to the usual totally elastic collision laws for velocities of moving balls. We will take a step towards an “approximate” hydrodynamic limit model and the corresponding nonlinear partial differential equations (see Sections 4.6 and 4.7 for the explanation and discussion of the “approximate” nature of our project). In Section 2, we describe the pinned billiard balls model in detail, we present a conjecture stating its large scale behavior (modulated white noise hypothesis), we derive partial difference equations for the parameters of modulated white noise, and we indicate how partial difference equations lead to nonlinear partial differential equations. Section 3 is devoted to numerical results supporting the modulated white noise hypothesis. Section 4 contains the discussion of the basic properties of the PDEs informally derived in Remark 2.4 and ends with the discussion of some hydrodynamic limit results in the literature. Sections 5 and 6 contain proofs of the main rigorous mathematical results of this paper. The first of these sections is devoted to partial difference equations while the second one is devoted to partial differential equations.

2. Evolution of pinned billiard balls model parameters

We will present some computations inspired by a one-dimensional system of pinned billiard balls, a special case of a model introduced in [ABD23]. In a system of pinned billiard balls, the balls touch some other balls and have pseudo-velocities but they do
not move. The balls “collide,” i.e., their pseudo-velocities change according to the usual laws of totally elastic collisions.

In our case, the centers of the balls are arranged on a finite segment of the real line. Their centers are one unit apart and their radii are all equal to $1/2$, so there is a finite ordered set of balls, each touching its two neighbors (except for the two endpoints, where the balls have only one neighbor). See Fig. 1.

![Figure 1. Billard balls arranged along a one-dimensional line. The balls touch but are fixed for all time.](image)

The spacetime for the model is discrete, i.e., the velocities $v(x,t)$ are defined for $x = 1, 2, \ldots, n$ and $t = 0, 1, 2, \ldots$, where $x$ is the position (i.e., number) of the $x$-th ball. The evolution, i.e., pseudo-collisions of the balls and transformations of the velocities, is driven by an exogenous random process because the balls do not move and hence they cannot collide in the usual way.

First consider a simplified model in which pairs of adjacent balls are chosen randomly, i.e., in a uniform way, and form an i.i.d. sequence. Every time a pair of adjacent balls is chosen, the velocities become ordered, i.e., if the chosen balls have labels $x$ and $x + 1$ and the collision occurs at time $t$ then

$$v(x, t + 1) = \min(v(x, t), v(x + 1, t)),$$

(2.1)

$$v(x + 1, t + 1) = \max(v(x, t), v(x + 1, t)).$$

(2.2)

This agrees with the usual transformation rule for velocities of moving balls of equal masses undergoing totally elastic collisions. The evolution described above has been studied under the names of “random sorting networks” in [AHRV07], “oriented swap process” in [AHR09] and “TASEP speed process” in [AAV11]. It has been also called “colored TASEP.” For a related model featuring confined (but moving) balls, see [GG08a, GG08c, GG08b].

While the model described above is very natural and well motivated by physics, it is characterized by a property that is strictly limited to the one-dimensional collision systems—the set of all initial velocities is conserved. The velocities are only rearranged. In multidimensional pinned ball families energy packets will not be preserved. A typical collision will change two energy packets into two new energy packets of different sizes subject to obeying the conservation laws.

The model described below is a compromise between the one-dimensional and higher dimensional models. It is one-dimensional to make the analysis easier but it involves energy exchange to simulate multidimensional evolutions. In the following model the evolution of velocities in the one-dimensional family of pinned balls consists of a sequence of two-step transformations. In the first step we redistribute energy. In the
second step we reorder a pair of velocities. We start by generating an i.i.d. sequence 
\((x_t, t = 0, 1, 2, \ldots)\), with each \(x_t\) distributed uniformly in \(\{2, 3, \ldots, n - 1\}\).

**Step 2.1.** Suppose that velocities \(v(x, s)\) have been defined for \(s = 0, \ldots, t\) and all \(x = 1, 2, \ldots, n\). Consider the following equations for \(v_-(x_t - 1, t + 1), v_-(x_t, t + 1)\) and \(v_-(x_t + 1, t + 1)\), representing conservation of energy and momentum,

\[
\begin{align*}
(2.3) \quad v_-(x_t - 1, t + 1) + v_-(x_t, t + 1) + v_-(x_t + 1, t + 1) &= v(x_t - 1, t) + v(x_t, t) + v(x_t + 1, t), \\
(2.4) \quad v_-(x_t - 1, t + 1)^2 + v_-(x_t, t + 1)^2 + v_-(x_t + 1, t + 1)^2 &= v(x_t - 1, t)^2 + v(x_t, t)^2 + v(x_t + 1, t)^2.
\end{align*}
\]

Given \(v(x_t - 1, t), v(x_t, t)\) and \(v(x_t + 1, t)\), the set of solutions \((v_-(x_t - 1, t + 1), v_-(x_t, t + 1), v_-(x_t + 1, t + 1))\) forms a circle in three-dimensional space, since it is the intersection of a sphere with a two-dimensional plane. We use extra randomness, independent of everything else, to choose a point \((v_-(x_t - 1, t + 1), v_-(x_t, t + 1), v_-(x_t + 1, t + 1))\) uniformly on this circle. This completes the first step.

**Step 2.2.** In the second step, the above energy exchange is followed by reordering of a pair of velocities. Let \(\kappa_t\) be equal \(-1\) or \(1\), with equal probabilities, independent of everything else. If \(\kappa_t = -1\) then

\[
\begin{align*}
v(x_t - 1, t + 1) &= \min(v_-(x_t - 1, t + 1), v_-(x_t, t + 1)), \\
v(x_t, t + 1) &= \max(v_-(x_t - 1, t + 1), v_-(x_t, t + 1)), \\
v(x_t + 1, t + 1) &= v_-(x_t + 1, t + 1).
\end{align*}
\]

Otherwise,

\[
\begin{align*}
v(x_t - 1, t + 1) &= v_-(x_t - 1, t + 1), \\
v(x_t, t + 1) &= \min(v_-(x_t, t + 1), v_-(x_t + 1, t + 1)), \\
v(x_t + 1, t + 1) &= \max(v_-(x_t, t + 1), v_-(x_t + 1, t + 1)).
\end{align*}
\]

This completes the second step.

For all \(x \neq x_t - 1, x_t, x_t + 1\), we let \(v(x, t + 1) = v(x, t)\).

2.1. **Modulated white noise.** We will assume that the joint distribution of \(\{v(x, t), 1 \leq x \leq n, t \geq 0\}\) converges after appropriate rescaling to

\[
(2.5) \quad v(x, t) = \mu(x, t) + \sigma(x, t)W(x, t),
\]

when \(n\) goes to infinity. Here \(W(x, t)\) is spacetime white noise and \(\mu(x, t)\) and \(\sigma(x, t)\) are deterministic functions.

We will present numerical evidence for our assumption in Section 3. On the theoretical side, our assumption is questionable (see Sections 4.6), but we will defend it in Section 4.7 as a reasonable compromise between true hydrodynamic model and simplicity.
In our discrete model, “white noise” is a collection of i.i.d. standard normal random variables. The way we will formally work with this assumption is to note that
\[ \mathbb{E} v(x, t) = \mu(x, t) \]
and
\[ \mathbb{E} v(x, t)^2 = \mu(x, t)^2 + \sigma(x, t)^2 \]
from which it becomes possible to deduce both the values of \( \mu \) and \( \sigma \).

On the theoretical side, there is an immense literature on interacting particle systems and hydrodynamic limits. We list only one article and two books: [BDSG15, KL99, Spo91]. The model closest to ours seems to be that of “hot rods” introduced in [DF77, BDS83], but the balls are allowed to move in that model. For recent results on the hot rods model and a review of the related literature, see [FFGS23, FO23]. The linear version of our partial differential equations for \( \mu \) and \( \sigma \) is essentially the same as the equations (2.1) and (2.2) for the local density and the local current in [BDSG15]. We will discuss some results from [TV03] in Section 4.6.

2.2. Partial difference equations. Proving the hydrodynamic limit theorem for the pinned balls model is a major technical challenge. In this paper, we limit ourselves to a very modest step. We will derive formulas for the one-step evolution of parameters \( \mu \) and \( \sigma \) under the assumption that (2.5) holds.

We will use the following notation. For functions \( \tilde{\mu} : (0, 1) \times [0, \infty) \to \mathbb{R} \) and \( \tilde{\sigma} : (0, 1) \times [0, \infty) \to (0, \infty) \) we let for \( t \geq 0 \) and \( 1 \leq x \leq n \),
\[
\mu(x, t) = \tilde{\mu}\left(\frac{x-1}{n-1}, \frac{t}{n}\right) \quad \sigma(x, t) = \tilde{\sigma}\left(\frac{x-1}{n-1}, \frac{t}{n}\right),
\]
\[
\tilde{\mu}_{x,n}(x, t) = \frac{\partial}{\partial z} \tilde{\mu}(z, s) \bigg|_{z=(x-1)/(n-1), s=t/n} \quad \tilde{\sigma}_{x,n}(x, t) = \frac{\partial}{\partial z} \tilde{\sigma}(z, s) \bigg|_{z=(x-1)/(n-1), s=t/n}
\]
\[
\tilde{\mu}_{x,n}(x, t) = \frac{\partial}{\partial z^2} \tilde{\mu}(z, s) \bigg|_{z=(x-1)/(n-1), s=t/n} \quad \tilde{\sigma}_{x,n}(x, t) = \frac{\partial}{\partial z^2} \tilde{\sigma}(z, s) \bigg|_{z=(x-1)/(n-1), s=t/n}
\]
\[
\left(\tilde{\mu}(x, t)^2 + \tilde{\sigma}(x, t)^2\right)_{x,n} = \frac{\partial}{\partial z^2} \left(\tilde{\mu}(z, s)^2 + \tilde{\sigma}(z, s)^2\right) \bigg|_{z=(x-1)/(n-1), s=t/n}
\]
\[
\tilde{\mu}_{t,n}(x, t) = \frac{\partial}{\partial s} \tilde{\mu}(z, s) \bigg|_{z=(x-1)/(n-1), s=t/n}.
\]

**Theorem 2.3.** Suppose functions \( \tilde{\mu} : (0, 1) \times [0, \infty) \to \mathbb{R} \) and \( \tilde{\sigma} : (0, 1) \times [0, \infty) \to (0, \infty) \) are \( C^3 \) (with bounded third derivative). Assume that \( v(x, t) \) satisfies (2.5) at a fixed time \( t \geq 0 \) and for \( 1 \leq x \leq n \). Then for \( 2 \leq x \leq n - 1 \),
\[
(2.6) \quad (n - 2) \mathbb{E} (v(x, t + 1) - v(x, t)) = -\frac{1}{\sqrt{n-1}} \tilde{\sigma}_{x,n}(x, t) + \frac{2}{(n-1)^2} \tilde{\mu}_{x,n}(x, t) + O(n^{-3}),
\]
\[
(2.7) \quad (n - 2) \mathbb{E} (v(x, t + 1)^2 - v(x, t)^2)
\]
\[- \frac{2}{\sqrt{\pi}(n-1)} (\mu(x,t)\tilde{\sigma}_{x,n}(x,t) + \sigma(x,t)\tilde{\mu}_{x,n}(x,t)) \]
\[+ \frac{2}{(n-1)^2} (\tilde{\mu}(x,t)^2 + \tilde{\sigma}(x,t)^2)_{xx,n} + O(n^{-3}). \]

**Remark 2.4.** We will outline how (2.6)-(2.7) lead to a system of PDEs.

We have not proved that (2.5) is the limiting distribution of the system when \( n \to \infty \) but we expect that

\[ \mathbb{E}(v(x,t+1) - v(x,t)) = \mu(x,t+1) - \mu(x,t) \]
\[ = \tilde{\mu}(x/n, t/n + 1/n) - \tilde{\mu}(x/n, t/n) \approx \tilde{\mu}_{t,n}(x,t). \]

We combine this with (2.6) to obtain

\[(n-2) \tilde{\mu}_{t,n}(x,t) \approx - \frac{1}{\sqrt{\pi}(n-1)} \tilde{\sigma}_{x,n}(x,t) + \frac{2}{(n-1)^2} \tilde{\mu}_{x,n}(x,t) + O(n^{-3}). \]

We rescale space and time, multiply both sides by \( (n-1)^2/2 \) and ignore the error terms so that

\[ \frac{(n-1)^2(n-2)}{2} \frac{\partial}{\partial t} \tilde{\mu}(x,t) = - \frac{n-1}{2\sqrt{\pi}} \frac{\partial}{\partial x} \tilde{\sigma}(x,t) + \frac{\partial^2}{\partial x^2} \tilde{\mu}(x,t). \]

We remove the factor \( \frac{(n-1)^2(n-2)}{2} \) from the left hand side by rescaling time and we obtain

\[ \frac{\partial}{\partial t} \tilde{\mu}(x,t) = - \frac{n-1}{2\sqrt{\pi}} \frac{\partial}{\partial x} \tilde{\sigma}(x,t) + \frac{\partial^2}{\partial x^2} \tilde{\mu}(x,t). \]

A completely analogous argument starting with (2.7) yields

\[ \frac{\partial}{\partial t} (\tilde{\mu}(x,t)^2 + \tilde{\sigma}(x,t)^2) \]
\[ = \frac{\partial^2}{\partial x^2} (\tilde{\mu}(x,t)^2 + \tilde{\sigma}(x,t)^2) - \frac{n-1}{\sqrt{\pi}} \left( \tilde{\mu}(x,t) \frac{\partial}{\partial x} \tilde{\sigma}(x,t) + \tilde{\sigma}(x,t) \frac{\partial}{\partial x} \tilde{\mu}(x,t) \right). \]

We set

\[ (2.8) \]
\[ \lambda = \frac{n-1}{2\sqrt{\pi}} \]

and change the notation from \( \tilde{\mu} \) and \( \tilde{\sigma} \) to \( \mu \) and \( \sigma \) to obtain the following form of these equations,

\[ (2.9) \]
\[ \mu_t = \Delta_x \mu - \lambda \sigma_x, \]
\[ (2.10) \]
\[ (\sigma^2 + \mu^2)_t = \Delta_x (\sigma^2 + \mu^2) - 2\lambda(\mu \sigma)_x. \]

It is easy to check that these equations are equivalent to the system (A) in Section 4.1, except that there are no boundary conditions here.

Boundary conditions must match the conservation laws (4.1)-(4.2) inherent in the billiards model. It is easy to check using (2.9)-(2.10) that if we assume that

\[ (2.11) \]
\[ \mu'(a) = \mu'(b) = 0, \]
\[ (2.12) \]
\[ \sigma(a) = \sigma(b) = 0, \]
then (4.1)-(4.2) are satisfied. We offer a heuristic justification for (2.11)-(2.12). Billiard ball velocities become ordered and stay ordered at the endpoints of the system because there are no constraints on one side preventing the increasing ordering of the velocities at the endpoints of the configuration. Hence, $\sigma$ instantaneously becomes and stays equal to zero at the endpoints. The derivative of $\mu$ is zero due to the averaging action of the Laplacian in the endpoint regions.

Note that the parameter $\lambda$ should be thought of as large. It represents the spatial size of the discrete system, so in simulations it typically takes a value larger than 1,000.

3. Numerical Examples

We will show the results of 100,000 simulations with 1,000 balls and compare them to numerical solutions of PDEs (2.9)-(2.10) obtained using a standard finite difference method with 3201 equispaced spatial grid points. First we show figures supporting the conjecture that the pinned balls system is represented by modulated white noise. Then we will present numerical evidence for the agreement between the evolution of parameters $\mu$ and $\sigma$ in the collision model and the PDEs.

We will present the results for only one representative set of initial conditions, namely,

$$\mu(x, 0) = \text{Erf} \left( 27(x - 1/2)^3 \right),$$

$$\sigma(x, 0) = \frac{[1 - \cos(2\pi x)]}{1000},$$

for $x \in [0, 1]$, where Erf denotes the error function. The functions were rescaled from the interval $[0, 1]$ to $[1, 1000]$ for the collision simulations.

We define $T$ as the time until the apparent total freeze, i.e., the time when the variance $\sigma^2$ is almost identically equal to 0 (compared to typical values in the main part of the evolution). We have approximately $T \approx 0.000120162$ for the timescale used in (2.9)-(2.10) with $\lambda = (n - 1)/(2\sqrt{\pi})$ (to match (2.8)) and $n = 1,000$ (the number of balls).

The distribution of the empirical white noise $W$ at time $0.37T$ estimated from the simulations matches the normal distribution quite well, according to Fig. 2. The values of the empirical white noise were calculated for a spatial position $x$ by subtracting the mean $\mu(x, 0.37T)$ and dividing by the standard deviation $\sigma(x, 0.37T)$, where the last two functions were evaluated as averages over all runs.

Correlations of the adjacent velocities should be equal to 0 assuming the white noise hypothesis. If we have a sample of size $n$ from the bivariate standard normal distribution then the density of the empirical correlation coefficient is

$$f(r) = \frac{(1 - r^2)^{(n-4)/2}}{B(1/2, (n-2)/2)},$$

where $B$ is the beta function. The standard deviation of this distribution is $1/\sqrt{n - 1}$. For $n = 1,000$, the standard deviation is about 0.032. We show in Fig. 3 that correlation values are not much larger than the theoretical value.

Fig. 4 shows that the joint distribution of the noise at adjacent sites is rotationally symmetric, as expected from white noise. The color is added to improve perception.
Figure 2. Empirical histogram (blue) of white noise values at the time $0.37T$. The standard normal density is drawn in red.

Figure 3. Correlation between white noise values at the distance $k$ at time $0.37T$, for $k = 1, \ldots, 50$, for a single run. The values of the empirical white noise were calculated for a spatial position $x$ by subtracting the mean $\mu(x, 0.37T)$ and dividing by the standard deviation $\sigma(x, 0.37T)$, where the last two functions were evaluated as averages over all runs.

Fig. 5 shows different stages of the evolution of $\mu$ and $\sigma$. The agreement between the moments $\mu$ and $\sigma$ estimated from simulations and the solutions to PDEs (2.9)-(2.10) is excellent. Fig. 5 supports our choice of the initial conditions—the resulting evolution of $\mu$ and $\sigma$ has interesting complexity.
Figure 4. Pairs of values of the noise \((W(300, 0.37T), W(301, 0.37T))\) for 100,000 runs of the simulation. The RGB scheme is \(((k/100,000)^5, 0, 1 - (k/100,000)^5)\) where \(k\) is the number of the simulation.
Figure 5. The moments $\mu$ and $\sigma$ at times $0.01T$, $0.2T$, $0.4T$, $0.6T$, $0.8T$ and $0.99T$ (top from left to right, then bottom left to right). The mean $\mu$ (dotted red) and $\sigma$ (dotted orange) were estimated by averaging values over 100,000 repetitions of the pinned balls model. The mean $\mu$ (solid blue) and $\sigma$ (solid green) were numerically computed using the equations (2.9)-(2.10). The curves were horizontally and vertically rescaled to show agreement.
4. A System of PDEs

4.1. The equations. We propose a suitable mean-field limit of a billiards model in terms of a coupled system of nonlinear partial differential equations as a way to describe its evolution. One of the two functions, \( \mu(x,t) \), measures the expected velocity of a billiard ball centered close to \( x \) at time \( t \) while \( \sigma(x,t) \) measures the standard deviation of velocities of balls close to \( x \) around time \( t \). This gives rise to an evolution of a coupled system of equations on an interval \([a, b]\). Throughout the paper the values of \( a \) and \( b \) will not be of further importance. The system is given by, for a parameter \( \lambda > 0 \),

\[
\begin{align*}
\mu_t &= \Delta \mu - \lambda \sigma_x, \\
\sigma_t &= \Delta \sigma - \lambda \mu_x + \frac{1}{\sigma}((\sigma_x)^2 + (\mu_x)^2), \\
\mu_x(a) &= \mu_x(b) = 0, \\
\sigma(a) &= \sigma(b) = 0
\end{align*}
\]

where \( \sigma \) is supposed to map \([a, b] \times [0, \infty)\) to \([0, \infty)\) since it will model standard deviation of a random process and thus needs to be nonnegative. The expression for \( \sigma_t \) contains a singularity \( 1/\sigma \) which will have to come into play since there are Dirichlet boundary conditions and \( \sigma(a) = 0 = \sigma(b) \), however, there is also a Laplacian term leading to smoothing which balances the blow-up. We see this as an indication that the phenomena of interest may perhaps be better studied in other coordinate systems and we will present two such coordinates changes in Sections 4.2 and 4.3. Formally, there are two conserved quantities

\[
\frac{d}{dt} \int_a^b \mu(x,t) dx = 0
\]

as well as

\[
\frac{d}{dt} \int_a^b \mu(x,t)^2 + \sigma(x,t)^2 dx = 0.
\]

These conditions correspond to two physical conservation laws: momentum and energy. Another formal observation is that for \( \lambda \gg 1 \), we can ignore the Laplacian term (which we expect to introduce additional dampening) and the nonlinear term and approximate

\[
\mu_t \sim -\lambda \sigma_x \quad \text{as well as} \quad \sigma_t \sim -\lambda \mu_x
\]

which would imply

\[
\mu_{tt} \sim \lambda^2 \mu_{xx} \quad \text{and} \quad \sigma_{tt} \sim \lambda^2 \sigma_{xx}
\]

suggesting that for large values of \( \lambda \) and smooth solutions (meaning a small Laplacian), the dynamics might be similar to that of the wave equation whenever the nonlinearity is small. However, we also note that \( \sigma \geq 0 \) which limits the extent to which the wave equation analogy can be applied.

4.2. Different coordinates I. The original equation has a \( 1/\sigma \) term and it is clear that \( \sigma \) will get close to 0 because it satisfies Dirichlet boundary conditions. This suggests
that it may be advantageous to work in a different coordinate system. We introduce
the function
\[ w(x, t) = \frac{1}{2} \sigma(x, t)^2. \]
We observe that the equation
\[ \sigma_t = \sigma_{xx} - \lambda \mu_x + \frac{1}{\sigma} ((\sigma_x)^2 + (\mu_x)^2) \]
can be rewritten as
\[ w_t = \sigma \sigma_t = \sigma \sigma_{xx} + (\sigma_x)^2 - \lambda \sigma \mu_x + (\mu_x)^2 \]
\[ = \Delta w - \lambda \sigma \mu_x + (\mu_x)^2. \]
This allows us to rewrite the system in these new coordinates as
\[
(B) = \begin{cases}
\mu_t - \Delta \mu = -\lambda (\sqrt{w})_x, \\
w_t - \Delta w = -\lambda \mu_x \sqrt{w} + (\mu_x)^2, \\
\mu_x(a) = \mu_x(b) = 0, \\
w(a) = w(b) = 0,
\end{cases}
\]
which will be a more convenient form. This representation also more clearly illustrates
why we would expect, at least for classical solutions, the system to preserve the non-
negativity of \( w \). If we start with non-negative initial conditions, then in local minima
where \( w \) starts getting close to 0 we have
\[ w_t - \Delta w \geq \frac{1}{2} (\mu_x)^2 \geq 0 \]
which propagates the non-negativity from the rest of the region and prevents the solution
from vanishing inside the domain.

4.3. Different coordinates II. There is yet another representation that leads to
slightly more complicated coupled boundary conditions but may be useful in other
ways that can be obtained by introducing a notion of energy as
\[ E(x, t) = \frac{1}{2} (\mu(x, t)^2 + \sigma(x, t)^2). \]
The equation for \( \mu \) can then be written as
\[ \mu_t - \Delta \mu = -\lambda (\sqrt{2E - \mu^2})_x. \]
A computation shows
\[
E_t = \mu \mu_t + \sigma \sigma_t \\
= \mu (\Delta \mu - \lambda \sigma_x) + \sigma \Delta \sigma - \lambda \sigma \mu_x + (\sigma_x)^2 + (\mu_x)^2 \\
= \mu \Delta \mu + (\mu_x)^2 + \sigma \Delta \sigma + (\sigma_x)^2 - \lambda (\mu \cdot \sigma)_x \\
= \Delta E - \lambda (\mu \sqrt{2E - \mu^2})_x.
\]
This leads to the coupled system
\[
(C) = \begin{cases}
\mu_t - \Delta \mu = -\lambda(\sqrt{2E - \mu^2})_x,
E_t - \Delta E = -\lambda(\sqrt{2E - \mu^2})_x,
\mu'(a) = \mu'(b) = 0,
E(a, t) = \frac{\mu(t, a)^2}{2},
E(b, t) = \frac{\mu(t, b)^2}{2}.
\end{cases}
\]

This system has the advantage of a simpler right-hand side (involving only a single derivative) at the cost of having coupled boundary conditions involving both \(E\) and \(\mu\).

4.4. Local well-posedness. We prove local well-posedness for a modified version of the PDEs in the second coordinate system. The system of equations is
\[
(B') = \begin{cases}
\mu_t - \Delta \mu = -\lambda D\sqrt{w},
w_t - \Delta w = -\lambda\mu_x\sqrt{w} + (\mu_x)^2,
\mu'(a) = \mu'(b) = 0,
w(a) = w(b) = 0,
\end{cases}
\]

where \(D\) is the Fourier multiplier corresponding to differentiation in \(x\). Whenever the original system of equations has a solution with \(w \geq 0\) and \(\sqrt{w} \in C^1\), the solution is also a solution of this system. To state our main theorem, we first define several function spaces. For \(f : [a, b] \times [0, T] \to \mathbb{R}\), let
\[
\|f\|_{L^p_{T,x}} = \left(\int_0^T \int_a^b |f(x, t)|^p\,dx\,dt\right)^{1/p}
\]
which we will use for \(p \in \{1, 2\}\). Analogously, we define the time-space Sobolev norm
\[
\|f\|_{H^{-1}_{x,T}} = \left(\int_0^T \int_a^b f(x, t)^2 + \left(\frac{\partial}{\partial x} f(x, t)\right)^2 \,dx\,dt\right)^{1/2}.
\]

We will also work with the associated dual space \(H_{T,T}^{-1}\).

Theorem 4.1 (Local well-posedness). For arbitrary initial data in \(H^1_{x,T} \times L^1_{x,T}\), there exists \(T > 0\) such that \((B')\) has a solution in \(H^1_{x,T} \times L^1_{x,T}\).

We expect this to be far from optimal and expect the system to have solutions that are much better behaved. This could be an interesting avenue for further research.

4.5. Comments on Dynamics. We observe that the system
\[
(B) = \begin{cases}
\mu_t - \Delta \mu = -\lambda(\sqrt{w})_x,
w_t - \Delta w = -\lambda\mu_x\sqrt{w} + \mu_x^2,
\mu_x(a) = \mu_x(b) = 0,
w(a) = w(b) = 0,
\end{cases}
\]
has a trivial solution: \(\mu = c > 0\) is constant and \(w = 0\). The purpose of this section is to quickly note some properties of the dynamics assuming the system has classical
solutions for all time.

Our first observation is that if the initial datum is not trivial in the sense that \( \mu \) is constant (and then \( w = 0 \)), then the dynamics stays nontrivial: the system will work towards achieving an equilibrium between \( \mu \) and the stochastic noise \( w \).

**Proposition 4.2.** Suppose that the initial data \( \mu(x,0) \) is not constant and \( w(x,0) = 0 \). Then \( \mu(x,t) \) does not converge to a constant function as \( t \to \infty \).

**Proof.** We use the conservation laws, which we call the momentum

\[
M = \int_{a}^{b} \mu(x,t) dx \quad \text{and} \quad E = \int_{a}^{b} \mu(x,t)^2 + 2w(x,t) dx.
\]

The Cauchy-Schwarz inequality implies that if \( w(x,0) = 0 \), then

\[
M^2 = \left( \int_{a}^{b} \mu(x,0) dx \right)^2 \leq (b-a) \int_{a}^{b} \mu(x,0)^2 dx = (b-a)E
\]

with equality if and only if \( \mu(x,0) \) is constant. If \( \mu(x,0) \) is therefore not constant, then, for some \( 0 < q < 1 \) we have the inequality \( M < q(b-a)E \) for all time. If \( \mu(x,t) \to c \) in \( H^1 \) as \( t \to \infty \), then \( w_t - \Delta w \) converges to 0 and \( w \) starts to behave like a heat equation which, with Dirichlet boundary conditions, forces \( w \to 0 \) which then violates \( M^2 < q(b-a)E \).

We quantify the notion that the equation reorders \( \mu \) in a weak sense: after some time, \( \mu \) will be at least as large on the right endpoint of the interval as on the left endpoint.

**Proposition 4.3.** Let \( c = -\inf_{a \leq y \leq b} \mu(y,0) \) and \( \tilde{\mu}(x,t) = \mu(x,t) + c \) for all \( x \) and \( t \). For any initial data and any \( \varepsilon > 0 \), there exists a time

\[
0 \leq t \leq \left( \int_{a}^{b} \tilde{\mu}(x,0) dx \right) \frac{b-a}{\varepsilon}
\]

such that

\[
\mu(b,t) \geq \mu(a,t) - \varepsilon.
\]

**Proof.** We consider the functional

\[
F(t) = \int_{a}^{b} x\tilde{\mu}(x,t) dx.
\]

We first observe that \( \tilde{M} := \int_{a}^{b} \tilde{\mu}(x,t) dx = \int_{a}^{b} \mu(x,t) dx + c(b-a) \) is constant in time. Since \( \tilde{\mu}(x,0) \) is non-negative,

\[
\tilde{M}a = \int_{a}^{b} a\tilde{\mu}(x,t) dx \leq \int_{a}^{b} x\tilde{\mu}(x,t) dx \leq \int_{a}^{b} b\tilde{\mu}(x,t) dx = \tilde{M}b
\]

which shows that \( F \) is bounded from above and from below for all time. It also implies that for all \( s, t > 0 \) we always have \( F(s) - F(t) \leq \tilde{M}(b-a) \). Moreover, we have

\[
F'(t) = \frac{d}{dt} \int_{a}^{b} x\tilde{\mu}(x,t) dx = \int_{a}^{b} x\tilde{\mu}_t(x,t) dx
\]
\[ = \int_a^b x(\Delta \mu - \lambda (\sqrt{w})_x) \, dx \]
\[ = - \int_a^b \mu_x(x, t) \, dt + \lambda \int_a^b \sqrt{w} \, dx \]
\[ = \mu(a, t) - \mu(b, t) + \lambda \int_a^b \sqrt{w} \, dx \]
\[ \geq \mu(a, t) - \mu(b, t). \]

Suppose now that \( \mu(b, t) \leq \mu(a, t) - \varepsilon \) for all \( 0 \leq t \leq T \). Then
\[
T \varepsilon \leq \int_0^T \mu(a, t) - \mu(b, t) \, dt \leq \int_0^T F'(t) \, dt \leq \tilde{M}(b - a)
\]
implying the desired upper bound on \( T \).

\( \square \)

**Remark 4.4.** The system of PDEs
\[
\begin{cases}
\mu_t = \Delta \mu - \lambda \sigma_x, \\
\sigma_t = \Delta \sigma - \lambda \mu_x + \frac{1}{\sigma}(\sigma_x^2 + \mu_x^2),
\end{cases}
\]
can be rescaled as
\[
\begin{cases}
\frac{1}{\lambda} \mu_t = \frac{1}{\lambda} \Delta \mu - \sigma_x, \\
\frac{1}{\lambda} \sigma_t = \frac{1}{\lambda} \Delta \sigma - \mu_x + \frac{1}{\lambda} \cdot \frac{1}{\sigma}(\sigma_x^2 + \mu_x^2).
\end{cases}
\]

We are mostly interested in large \( \lambda \) so the following limiting case is of interest,
\[
(4.9) \quad \begin{cases}
\frac{1}{\lambda} \mu_t = -\sigma_x, \\
\frac{1}{\lambda} \sigma_t = -\mu_x.
\end{cases}
\]

After rescaling time by \( \lambda \) we obtain (cf. \( (4.3)-(4.4) \)),
\[
(4.10) \quad \begin{cases}
\mu_t = -\sigma_x, \\
\sigma_t = -\mu_x.
\end{cases}
\]

The paper [BO23] contains a result on the existence of solutions to the last system with given terminal values.

### 4.6. Related hydrodynamic models.

We are grateful to Balint Tóth for the following remarks on related hydrodynamic limits (but we take responsibility for possible inaccuracies).

As shown in [TV03] there are some combinatorial conditions for the rates of local dynamics in interacting particle systems with several conservation laws, in order that the ergodic translation invariant measures be of product structure (see condition (C) in [TV03]). That paper is formulated in the context of discrete local observables but all arguments apply to continuous observables. In the present context this implies that in order to get the product Gaussians as Gibbs measures the rates of swaps \( (v(x, t), v(x + 1, t)) \rightarrow (v(x + 1, t), v(x, t)) \) must be chosen as \( (r(v(x, t)) - r(v(x + 1, t))) \mathbb{1}_{v(x, t) < v(x + 1, t)} \)
where \( r : \mathbb{R} \to \mathbb{R} \) is a non-decreasing and non-constant function (see section 2 of [TV03]). A natural choice could be \( r(u) = u \). This would lead to the PDE’s

\[
\begin{aligned}
\mu_t - \Delta \mu &= -\lambda(2E - \mu^2)_x, \\
E_t - \Delta E &= -\lambda(\mu 2E - \mu^2)_x,
\end{aligned}
\]

rather than our \((4.5)\):

\[
\begin{aligned}
\mu_t - \Delta \mu &= -\lambda(\sqrt{2} - \mu^2)_x, \\
E_t - \Delta E &= -\lambda(2E - \mu^2)_x.
\end{aligned}
\]

The Onsager relations discussed in [TV03] say that the thermodynamic Gibbs entropy (of the equilibrium measures of the microscopic system) expressed as a function of the conserved quantities is a Lax entropy for the PDE. This feature links the microscopic system (Gibbs entropy of the stationary/ergodic measures) with the macroscopic hydrodynamic PDE (Lax entropy). This feature fails to hold in our system and ansatz. It has been shown later in [GS11] that the Onsager relations are valid in wider generality than in [TV03]. Namely, there is no need for the product structure of the stationary measure. However, in the more general setting it is more difficult (if possible at all) to find explicit formulas for the Gibbs entropy.

4.7. Justification of our model. Although the results reviewed in Section 4.6 indicate that our model cannot lead to “modulated white noise” local equilibrium measure, we will try to justify our approach.

(i) If we repeat Step 2.1 multiple times, say \( n^\alpha \) times, for every single Step 2.2, this will basically affect only \( \lambda \), and it will change its value from order \( n \) to \( n^{1-\alpha} \). Assuming \( \alpha \in (0, 1) \), we can expect that the local equilibrium measure will be close to Gaussian because Step 2.1 is locally a mixing process on the sphere.

The postulate that white noise governs the evolution of \( v(x,t) \) is motivated by the postulate of equidistribution of energy. Due to conservation of energy, \( \sum_{x \in N} v(x,t)^2 \) is more or less constant over small time intervals in a small neighborhood \( N \). Hence, for a fixed \( t \), one expects the vector \( \{v(x,t), x \in N\} \) to be approximately uniformly distributed over the sphere and, therefore, to be approximately i.i.d. normal.

(ii) Step 2.2 may introduce correlation between values of \( v \) at various locations for a fixed time. Specifically, we may expect negative correlation between adjacent sites because Step 2.2 orders the values in the increasing manner. However, Step 2.2 is associated with the energy redistribution in Step 2.1 on three adjacent sites. Hence, one can expect that the correlation between the first two and the last two sites in the triplet would annihilate the correlation effects in the difference equation calculations. One can expect much lower correlation effects from the sites at least two units away.

Simulations discussed in Section 3 (see especially Fig. 3) suggest that correlations are small.

(iii) Although our assumption that the local equilibrium measure is modulated white noise is questionable in view of the results reviewed in Section 4.6, the excellent agreement between simulations of \( \mu \) and \( \sigma \) and numerical solutions of our PDEs shown in
Fig. 5 justifies our model as a very good, if not perfect, approximation of the true hydrodynamic limit.

(iv) Starting with (4.11) and proceeding as in Remark 4.4, in particular, “sending \( \lambda \to \infty \)” as in (4.9)-(4.10), we would obtain a system of Burgers-like equations

\[
\begin{align*}
\mu_t &= -\sigma_x \sigma, \\
\sigma_t &= -\mu_x \sigma.
\end{align*}
\]

rather than the transport equations (4.10). The equations (4.12) develop shocks (see [Smo94, Ser99] and, therefore, present a major technical challenge for the hydrodynamic limit theory. On the other hand, the transport equations (4.10) combined with the freezing condition, i.e., \( \mu_t = \mu_x = \sigma_t = \sigma_x = 0 \) whenever \( \sigma = 0 \), are tractable and display interesting behavior—this has been shown in [BS24]. Hence, the present approach seems to have intrinsic pure mathematical value independent of the physical applications.

(v) The paper [FT04] is about a very closely related system and the corresponding hydrodynamic limit, going even beyond the shocks. That model can be represented in our terms as follows: particles have three types of velocities, \(-1, 0, +1\). Otherwise the dynamics is very similar to ours: momentum and kinetic energy are conserved, particles move according to their velocities, with only “monotone” swaps allowed. But the rate of near-neighbor swaps depends on the values of the velocities, unlike in our system. The system of PDEs obtained in the hydrodynamic limit is the so-called Leroux-system. These PDEs develop shocks, unlike our limiting PDEs.

5. Proof of partial difference equations

**Proof of Theorem 2.3** Fix \( t \geq 0 \) and \( 3 \leq x \leq n - 2 \). For a fixed \( 2 \leq y \leq n - 1 \), \( \mathbb{P}(x_t = y) = 1/(n-2) \). In view of (2.5), \( \mathbb{E} v(y, t) = \mu(y, t) \) for all \( y \). Since \( x_t \) could be \( x - 1, x \) and \( x + 1 \) with equal probabilities, symmetry and Step 2.1 (see especially (2.3)) show that

\[
\begin{align*}
\mathbb{E} v_-(x, t + 1) &= \frac{n - 5}{n - 2} \mathbb{E} v(x, t) \\
+ \mathbb{P}(x_t = x - 1) \frac{1}{3} (\mathbb{E} v(x - 2, t) + \mathbb{E} v(x - 1, t) + \mathbb{E} v(x, t)) \\
+ \mathbb{P}(x_t = x) \frac{1}{3} (\mathbb{E} v(x - 1, t) + \mathbb{E} v(x, t) + \mathbb{E} v(x + 1, t)) \\
+ \mathbb{P}(x_t = x + 1) \frac{1}{3} (\mathbb{E} v(x, t) + \mathbb{E} v(x + 1, t) + \mathbb{E} v(x + 2, t)) \\
= \frac{n - 5}{n - 2} \mu(x, t) \\
+ \frac{1}{n - 2} \cdot \frac{1}{3} (\mu(x - 2, t) + \mu(x - 1, t) + \mu(x, t)) \\
+ \frac{1}{n - 2} \cdot \frac{1}{3} (\mu(x - 1, t) + \mu(x, t) + \mu(x + 1, t)) \\
+ \frac{1}{n - 2} \cdot \frac{1}{3} (\mu(x, t) + \mu(x + 1, t) + \mu(x + 2, t))
\end{align*}
\]
\[
\begin{align*}
&= \mu(x, t) - \frac{3}{n - 2} \mu(x, t) \\
&\quad + \frac{1}{n - 2} \cdot \frac{1}{3} (\mu(x - 2, t) + 2\mu(x - 1, t) + 3\mu(x, t) + 2\mu(x + 1, t) + \mu(x + 2, t)) \\
&= \mu(x, t) \\
&\quad + \frac{1}{n - 2} \cdot \frac{1}{3} (\mu(x - 2, t) + 2\mu(x - 1, t) - 6\mu(x, t) + 2\mu(x + 1, t) + \mu(x + 2, t)),
\end{align*}
\]
so
\[
(n - 2) \mathbb{E}(v_-(x, t + 1) - v(x, t)) = (n - 2) \mathbb{E}(v_-(x, t + 1) - \mu(x, t))
\]
(5.1)
\[
= \frac{1}{3} (\mu(x - 2, t) + 2\mu(x - 1, t) - 6\mu(x, t) + 2\mu(x + 1, t) + \mu(x + 2, t)).
\]

Our model, encapsulated in (2.5), implies that \( \mathbb{E} v(y, t)^2 = \mu(y, t)^2 + \sigma(y, t)^2 \) for all \( y \). We use symmetry, Step 2.1 and (2.4) to obtain the following formula analogous to (5.1),
\[
(5.2) \quad (n - 2) \mathbb{E}(v_-(x, t + 1)^2 - v(x, t)^2) = \frac{1}{3} \left( \sigma(x - 2, t)^2 + \mu(x - 2, t)^2 + 2(\sigma(x - 1, t)^2 + \mu(x - 1, t)^2) \\
- 6(\sigma(x, t)^2 + \mu(x, t)^2) + 2(\sigma(x + 1, t)^2 + \mu(x + 1, t)^2) \\
+ \sigma(x + 2, t)^2 + \mu(x + 1, t + 1)^2 \right).
\]

Let
\[
\hat{\mu}(x, t) = (n - 2) \mathbb{E}(v(x, t + 1) - v_-(x, t + 1)),
\]
(5.3)
\[
\mathcal{E}(x, t) = (n - 2) \mathbb{E}(v(x, t + 1)^2 - v_-(x, t + 1)^2).
\]
(5.4)

Then, after adding \( \hat{\mu} \) and \( \mathcal{E} \) to both sides of (5.1) and (5.2), respectively, we obtain
\[
(n - 2) \mathbb{E}(v(x, t + 1) - v(x, t))
\]
(5.5)
\[
= \frac{1}{3} (\mu(x - 2, t) + 2\mu(x - 1, t) - 6\mu(x, t) + 2\mu(x + 1, t) + \mu(x + 2, t)) + \hat{\mu}(x, t),
\]
\[
= \frac{2}{(n - 1)^2} \tilde{\mu}_{xx,n}(x, t) + O(n^{-4}) + \hat{\mu}(x, t),
\]
(5.6)
\[
(n - 2) \mathbb{E}(v(x, t + 1)^2 - v(x, t)^2)
\]
(5.6)
\[
= \frac{1}{3} \left( \sigma(x - 2, t)^2 + \mu(x - 2, t)^2 + 2(\sigma(x - 1, t)^2 + \mu(x - 1, t)^2) \\
- 6(\sigma(x, t)^2 + \mu(x, t)^2) + 2(\sigma(x + 1, t)^2 + \mu(x + 1, t)^2) \\
+ \sigma(x + 2, t)^2 + \mu(x, t + 1)^2 \right) + \mathcal{E}(x, t)
\]
\[
= \frac{2}{(n - 1)^2} (\tilde{\mu}(x, t)^2 + \tilde{\sigma}(x, t)^2)_{xx,n} + O(n^{-4}) + \mathcal{E}(x, t).
\]
Next, we observe that by Step 2.2
\begin{align*}
\tilde{\mu}(x, t) &= (n - 2) \mathbb{E}((v_-(x - 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x - 1, \kappa_t = 1) \\
&\quad \times \mathbb{P}(x_t = x - 1, \kappa_t = 1) \\
+ (n - 2) \mathbb{E}((v_-(x - 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x, \kappa_t = -1) \\
&\quad \times \mathbb{P}(x_t = x, \kappa_t = -1) \\
+ (n - 2) \mathbb{E}((v_-(x + 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x+1,t+1) > v_-(x,t+1)} \mid x_t = x, \kappa_t = 1) \\
&\quad \times \mathbb{P}(x_t = x + 1, \kappa_t = 1) \\
+ (n - 2) \mathbb{E}((v_-(x + 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x+1,t+1) > v_-(x,t+1)} \mid x_t = x + 1, \kappa_t = -1) \\
&\quad \times \mathbb{P}(x_t = x + 1, \kappa_t = -1) \\
= &\frac{1}{2} \mathbb{E}((v_-(x - 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x - 1, \kappa_t = 1) \\
+ \frac{1}{2} \mathbb{E}((v_-(x - 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x, \kappa_t = -1) \\
+ \frac{1}{2} \mathbb{E}((v_-(x + 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x+1,t+1) > v_-(x,t+1)} \mid x_t = x, \kappa_t = 1) \\
+ \frac{1}{2} \mathbb{E}((v_-(x + 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x+1,t+1) > v_-(x,t+1)} \mid x_t = x + 1, \kappa_t = -1).
\end{align*}

If \( x_t = x \) then the sequence \((v_-(x - 1, t + 1), v_-(x, t + 1), v_-(x+1, t+1))\) is exchangeable—this follows from the definition given in Step 2.1. Hence, the two middle terms in the above formula cancel each other and we obtain
\begin{equation}
\tilde{\mu}(x, t) = \frac{1}{2} \mathbb{E}((v_-(x - 1, t + 1) - v_-(x, t + 1)) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x - 1, \kappa_t = 1).
\end{equation}

A similar calculation yields
\begin{equation}
\mathcal{E}(x, t) = \frac{1}{2} \mathbb{E}((v_-(x - 1, t + 1)^2 - v_-(x, t + 1)^2) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x - 1, \kappa_t = 1)
\end{equation}

\begin{equation}
+ \frac{1}{2} \mathbb{E}((v_-(x + 1, t + 1)^2 - v_-(x, t + 1)^2) \mathbb{1}_{v_-(x+1,t+1) > v_-(x,t+1)} \mid x_t = x + 1, \kappa_t = -1).
\end{equation}

Define \( a(x, t) \) and \( r(x, t) \geq 0 \) by
\begin{equation}
a(x, t) = \frac{1}{3} (v(x - 1, t) + v(x, t) + v(x + 1, t)),
\end{equation}

\begin{equation}
r(x, t) = \mathbb{E}((v_-(x - 1, t + 1)^2 - v_-(x, t + 1)^2) \mathbb{1}_{v_-(x-1,t+1) > v_-(x,t+1)} \mid x_t = x - 1, \kappa_t = 1)
\end{equation}

\begin{equation}
+ \mathbb{E}((v_-(x + 1, t + 1)^2 - v_-(x, t + 1)^2) \mathbb{1}_{v_-(x+1,t+1) > v_-(x,t+1)} \mid x_t = x + 1, \kappa_t = -1).
\end{equation}
Thus, (5.7) and (5.12) imply that
\begin{equation}
(5.13)
\end{equation}
\begin{align}
(5.11) & \quad = \frac{4}{9} \left( (v(x - 1, t) - a(x, t))^2 + (v(x, t) - a(x, t))^2 + (v(x + 1, t) - a(x, t))^2 \right) \\
& \quad - v(x - 1, t)v(x) - v(x - 1, t)v(x + 1, t) - v(x, t)v(x + 1, t).
\end{align}

If \( x_t = x \) then the vector \((v_{-}(x - 1, t + 1), v_{-}(x, t + 1), v_{-}(x + 1, t + 1))\) is distributed uniformly on the circle in \( \mathbb{R}^3 \) given by the parametric formula,
\begin{equation}
(\begin{aligned}
(a(x, t) + r(x, t) \sin \theta, a(x, t) + r(x, t) \sin(\theta + 2\pi/3), a(x, t) + r(x, t) \sin(\theta + 4\pi/3)) \end{aligned})
\end{equation}
for \( \theta \in [0, 2\pi) \).

Let \( F_{x,t} \) denote the \( \sigma \)-field generated by \( v(x - 1, t), v(x, t) \) and \( v(x + 1, t) \). We have
\begin{equation}
(5.12)
\end{equation}
\begin{align}
& \quad \mathbb{E}(v_{-}(x - 1, t + 1) - v_{-}(x, t + 1)) | x_t = x - 1, \kappa_t = 1, F_{x-1,t} = 1 \nonumber \\
& \quad = -\frac{\sqrt{3}}{\pi} r(x - 1, t).
\end{align}

The following formula is analogous,
\begin{equation}
(5.13)
\end{equation}
\begin{align}
\mathbb{E}(v_{-}(x + 1, t + 1) - v_{-}(x, t + 1)) | x_t = x + 1, \kappa_t = -1, F_{x+1,t} = 1 = \frac{\sqrt{3}}{\pi} r(x + 1, t).
\end{align}

Thus, (5.7) and (5.12) imply that
\begin{equation}
(5.13)
\end{equation}
\begin{align}
\hat{\mu}(x, t) = \frac{\sqrt{3}}{2\pi} \mathbb{E}(r(x - 1, t) - r(x + 1, t)).
\end{align}

Similarly, we see that
\begin{equation}
(5.14)
\end{equation}
\begin{align}
& \quad \mathbb{E}(v_{-}(x - 1, t + 1)^2 - v_{-}(x, t + 1)^2) | x_t = x - 1, \kappa_t = 1, F_{x-1,t} = 1 \\
& \quad = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left( (a(x - 1, t) + r(x - 1, t) \sin(\theta + 2\pi/3))^2 \\
& \quad - (a(x - 1, t) + r(x - 1, t) \sin(\theta + 4\pi/3))^2 \right) d\theta \\
& \quad = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 2a(x - 1, t)r(x - 1, t)(\sin(\theta + 2\pi/3) - \sin(\theta + 4\pi/3)) d\theta
\end{align}
It follows from (5.11) that
\[ = \alpha \] and the joint density of three independent normal random variables with parameters 

Additionally, an analogous calculation to the previous one yields

Hence we see that (5.8) and (5.14) together imply that

Recall that the density of a normal random variable with mean \( \alpha \) and variance \( \beta^2 \) is

and the joint density of three independent normal random variables with parameters \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \) is

It follows from (5.11) that

where

Let

According to (5.13),

\[
\hat{\mu}(x, t) = \frac{\sqrt{3}}{2\pi} \mathbb{E}(r(x - 1, t) - r(x + 1, t)) \\
= \frac{\sqrt{3}}{2\pi} \int_{\mathbb{R}^3} \frac{2}{3} \sqrt{u^2 + y^2 + z^2 - uy - yz - uz} f_{\alpha_-, \beta_-}(u, y, z) \, du \, dy \, dz
\]
\[-\frac{\sqrt{3}}{2\pi} \int_{\mathbb{R}^3} \frac{2}{3} \sqrt{u^2 + y^2 + z^2 - uy - yz - uz} f_{\alpha_+, \beta_+}(u, y, z) \, du \, dy \, dz.\]

According to (5.15),

\[(5.24)\]
\[E(x, t) = \frac{\sqrt{3}}{\pi} \mathbb{E}(a(x - 1, t - 1)r(x - 1, t - 1) - a(x + 1, t - 1)r(x + 1, t - 1)) \]
\[= \frac{\sqrt{3}}{\pi} \int_{\mathbb{R}^3} \frac{1}{3}(u + y + z) \frac{2}{3} \sqrt{u^2 + y^2 + z^2 - uy - yz - uz} f_{\alpha_-, \beta_-}(u, y, z) \, du \, dy \, dz \]
\[- \frac{\sqrt{3}}{\pi} \int_{\mathbb{R}^3} \frac{1}{3}(u + y + z) \frac{2}{3} \sqrt{u^2 + y^2 + z^2 - uy - yz - uz} f_{\alpha_+, \beta_+}(u, y, z) \, du \, dy \, dz.\]

In view of (5.23) and (5.24), we need to compute (or at least estimate) integrals of the form

\[(5.25)\]
\[I_1(\alpha, \beta) := \int_{\mathbb{R}^3} \sqrt{\hat{u}^2 + \hat{y}^2 + \hat{z}^2 - \hat{u}\hat{y} - \hat{y}\hat{z} - \hat{u}\hat{z}} f_{\alpha, \beta}(\hat{u}, \hat{y}, \hat{z}) \, d\hat{u} \, d\hat{y} \, d\hat{z},\]

\[(5.26)\]
\[I_2(\alpha, \beta) := \int_{\mathbb{R}^3} (\hat{u} + \hat{y} + \hat{z}) \sqrt{\hat{u}^2 + \hat{y}^2 + \hat{z}^2 - \hat{u}\hat{y} - \hat{y}\hat{z} - \hat{u}\hat{z}} f_{\alpha, \beta}(\hat{u}, \hat{y}, \hat{z}) \, d\hat{u} \, d\hat{y} \, d\hat{z}.\]

Given \(\alpha_1, \alpha_2, \alpha_3\) and \(\varepsilon > 0\), let \(\alpha = (\alpha_1 + \alpha_2 + \alpha_3)/3\). It is elementary to check that there exist unique \(\delta_1\) and \(\gamma_1\) such that

\[(5.27)\]
\[\alpha = (\alpha_1, \alpha_2, \alpha_3) = (\alpha - \varepsilon \delta_1 + \varepsilon^2 \gamma_1, \alpha - 2\varepsilon \gamma_1, \alpha + \varepsilon \delta_1 + \varepsilon^2 \gamma_1).\]

The following formula is analogous,

\[(5.28)\]
\[\beta = (\beta_1, \beta_2, \beta_3) = (\beta - \varepsilon \delta_2 + \varepsilon^2 \gamma_2, \beta - 2\varepsilon \gamma_2, \beta + \varepsilon \delta_2 + \varepsilon^2 \gamma_2),\]

with \(\beta\) denoting the average of \(\beta\). We will assume that \(\beta_k > 0\) for \(k = 1, 2, 3\).

According to Lemma 5.1

\[(5.29)\]
\[I_1(\alpha, \beta) = \frac{\sqrt{3\pi \beta}}{2} \left[ 1 + \frac{\varepsilon^2}{2\beta^2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) \right] + O(\varepsilon^3),\]

\[(5.30)\]
\[I_2(\alpha, \beta) = 3\alpha \frac{\sqrt{3\pi \beta^2}}{2} \left[ 1 + \frac{\varepsilon^2}{2\beta^2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) \right] + 2\varepsilon^2 \sqrt{3\pi \delta_1 \delta_2} + O(\varepsilon^3).\]

Recall (5.19)–(5.22) and assume that

\[(5.31)\]
\[\alpha_- = (\mu(x - 2, t), \mu(x - 1, t), \mu(x, t))\]
\[= (\mu(x, t) - 2\varepsilon \delta_1, \mu(x, t) - \varepsilon \delta_1 - 3\varepsilon^2 \gamma_1, \mu(x, t)),\]

\[(5.32)\]
\[\beta_- = (\sigma(x - 2, t), \sigma(x - 1, t), \sigma(x, t))\]
\[= (\sigma(x, t) - 2\varepsilon \delta_2, \sigma(x, t) - \varepsilon \delta_2 - 3\varepsilon^2 \gamma_2, \sigma(x, t)),\]

\[(5.33)\]
\[\alpha_+ = (\mu(x, t), \mu(x + 1, t), \mu(x + 2, t))\]
\[= (\mu(x, t), \mu(x, t) + \varepsilon \delta_3 - 3\varepsilon^2 \gamma_3, \mu(x, t) + 2\varepsilon \delta_3),\]

\[(5.34)\]
\[\beta_+ = (\sigma(x, t), \sigma(x + 1, t), \sigma(x + 2, t))\]
\[= (\sigma(x, t), \sigma(x, t) + \varepsilon \delta_4 - 3\varepsilon^2 \gamma_4, \sigma(x, t) + 2\varepsilon \delta_4).\]
Then \((5.29)\) implies that
\[
I_1(\alpha_-, \beta_-)
\]
\[
= \frac{(\sigma(x, t) - \epsilon \delta_2 - \epsilon^2 \gamma_2) \sqrt{3\pi}}{2} + \frac{\sqrt{3\pi}}{2} \frac{\epsilon^2}{2(\sigma(x, t) - \epsilon \delta_2 - \epsilon^2 \gamma_2)} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) + O(\epsilon^3),
\]
\[
I_1(\alpha_+, \beta_+)
\]
\[
= \frac{(\sigma(x, t) + \epsilon \delta_4 - \epsilon^2 \gamma_4) \sqrt{3\pi}}{2} + \frac{\sqrt{3\pi}}{2} \frac{\epsilon^2}{2(\sigma(x, t) + \epsilon \delta_4 - \epsilon^2 \gamma_4)} \left( \delta_3^2 + \frac{\delta_4^2}{2} \right) + O(\epsilon^3).
\]
Recall \((5.23)\) and \((5.25)\) to see that
\[
\hat{\mu}(x, t) = \frac{1}{\sqrt{3\pi}} \int_{\mathbb{R}^3} \sqrt{u^2 + y^2 + z^2 - uy - yz - uz} f_{\alpha_-, \beta_-}(u, y, z) du dy dz
\]
\[
- \frac{1}{\sqrt{3\pi}} \int_{\mathbb{R}^3} \sqrt{u^2 + y^2 + z^2 - uy - yz - uz} f_{\alpha_+, \beta_+}(u, y, z) du dy dz
\]
\[
= \frac{1}{\sqrt{3\pi}} (I_1(\alpha_-, \beta_-) - I_1(\alpha_+ , \beta_+))
\]
\[
= \frac{\sigma(x, t) - \epsilon \delta_2 - \epsilon^2 \gamma_2}{2\sqrt{\pi}} + \frac{\epsilon^2}{2\sqrt{\pi}} \frac{\epsilon^2}{2(\sigma(x, t) - \epsilon \delta_2 - \epsilon^2 \gamma_2)} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) + O(\epsilon^3)
\]
\[
- \frac{\sigma(x, t) + \epsilon \delta_4 - \epsilon^2 \gamma_4}{2\sqrt{\pi}} - \frac{\epsilon^2}{2\sqrt{\pi}} \frac{\epsilon^2}{2(\sigma(x, t) + \epsilon \delta_4 - \epsilon^2 \gamma_4)} \left( \delta_3^2 + \frac{\delta_4^2}{2} \right) + O(\epsilon^3)
\]
\[(5.35) \quad = - \frac{1}{2\sqrt{\pi}} \epsilon (\delta_2 + \delta_4) + \epsilon^2 \frac{1}{2\sqrt{\pi}} (-\gamma_2 + \gamma_4) + O(\epsilon^3).
\]
The terms involving \(\delta_1^2\) and \(\delta_3^2\) canceled each other in the above calculation. More precisely, they combined into an \(O(\epsilon^3)\) quantity because, essentially, \(\delta_1\) and \(\delta_3\) are derivatives at the nearby locations; see \((5.36)-(5.39)\) below for more details. A similar remark applies to the cancellation of terms involving \(\delta_2^2\) and \(\delta_4^2\).

In the next part of the proof, it will be convenient to use the notation \(\epsilon = 1/(n - 1)\).

We relate \(\delta_k\)’s and \(\gamma_k\)’s to derivatives of \(\hat{\mu}\) and \(\hat{\sigma}\) as follows.
\[(5.36) \quad 2\epsilon \delta_1 = \mu(x, t) - (\mu(x, t) - 2\epsilon \delta_1) = \mu(x, t) - \mu(x - 2, t)
\]
\[
= \frac{2}{n - 1} \hat{\mu}_{x, n}(x, t) - \frac{1}{2} \cdot \frac{4}{(n - 1)^2} \hat{\mu}_{xx, n}(x, t) + O(n^{-3}).
\]
Similarly,
\[(5.37) \quad 2\epsilon \delta_3 = (\mu(x, t) + 2\epsilon \delta_3) - \mu(x, t) = \mu(x + 2, t) - \mu(x, t)
\]
\[
= \frac{2}{n - 1} \hat{\mu}_{x, n}(x, t) + \frac{1}{2} \cdot \frac{4}{(n - 1)^2} \hat{\mu}_{xx, n}(x, t) + O(n^{-3}),
\]
\[(5.38) \quad \epsilon^2 \gamma_1 = \frac{1}{2(n - 1)^2} \hat{\mu}_{xx, n}(x, t) + O(n^{-3}),
\]
\[(5.39) \quad \epsilon^2 \gamma_3 = \frac{1}{2(n - 1)^2} \hat{\mu}_{xx, n}(x, t) + O(n^{-3}).
\]
Analogous formulas hold for $\delta_k$ and $\gamma_k$ for $k = 2, 4$. This and (5.35) imply that

$$\hat{\mu}(x, t) = -\frac{1}{2\sqrt{\pi}} \varepsilon(\delta_2 + \delta_4) + \varepsilon^2 \frac{1}{2\sqrt{\pi}} (-\gamma_2 + \gamma_4) + O(\varepsilon^3)$$

(5.40)

$$= -\frac{1}{\sqrt{\pi}(n-1)} \tilde{\sigma}_{x_n}(x, t) + O(n^{-3}).$$

Next we use (5.30) to obtain

$$I_2(\alpha_-, \beta_-) = \frac{(\mu(x, t) - \varepsilon \delta_1 - \varepsilon^2 \gamma_1)(\sigma(x, t) - \varepsilon \delta_2 - \varepsilon^2 \gamma_2)3\sqrt{3\pi}}{2} + \frac{3\sqrt{3\pi} \varepsilon^2 (\mu(x, t) - \varepsilon \delta_1 - \varepsilon^2 \gamma_1)}{2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) + \frac{2\varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2 + O(\varepsilon^3)}{3\pi},$$

$$I_2(\alpha_+, \beta_+) = \frac{(\mu(x, t) + \varepsilon \delta_2 - \varepsilon^2 \gamma_2)(\sigma(x, t) + \varepsilon \delta_4 - \varepsilon^2 \gamma_4)3\sqrt{3\pi}}{2} + \frac{3\sqrt{3\pi} \varepsilon^2 (\mu(x, t) + \varepsilon \delta_2 - \varepsilon^2 \gamma_2)}{2} \left( \delta_3^2 + \frac{\delta_4^2}{2} \right) + \frac{2\varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2 + O(\varepsilon^3)}{3\pi},$$

Recall (5.24) and (5.26) to see that,

$$\mathcal{E}(x, t) = \frac{\sqrt{3}}{\pi} \int_{\mathbb{R}^3} \frac{1}{3} (u + y + z) \frac{2}{3} \sqrt{u^2 + y^2 + z^2 - w y - y z - u z f_{\alpha_-, \beta_-}(u, y, z) d u d y d z}$$

$$- \frac{\sqrt{3}}{\pi} \int_{\mathbb{R}^3} \frac{1}{3} (u + y + z) \frac{2}{3} \sqrt{u^2 + y^2 + z^2 - w y - y z - u z f_{\alpha_+ \beta_+}(u, y, z) d u d y d z}$$

$$= \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} I_2(\alpha_-, \beta_-) - \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} I_2(\alpha_+, \beta_+)$$

$$= \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} \cdot \frac{3\sqrt{3\pi} (\mu(x, t) - \varepsilon \delta_1 - \varepsilon^2 \gamma_1)(\sigma(x, t) - \varepsilon \delta_2 - \varepsilon^2 \gamma_2)}{2} + \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} \cdot \varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2$$

$$+ \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} \cdot \frac{3\sqrt{3\pi} \varepsilon^2 (\mu(x, t) - \varepsilon \delta_1 - \varepsilon^2 \gamma_1)}{2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) + \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} \cdot \varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2$$

$$- \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} \cdot \frac{3\sqrt{3\pi} (\mu(x, t) + \varepsilon \delta_2 - \varepsilon^2 \gamma_2)(\sigma(x, t) + \varepsilon \delta_4 - \varepsilon^2 \gamma_4)}{2} - \frac{2\sqrt{3}}{3\pi} \cdot \frac{1}{3} \cdot \varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2$$

$$+ O(\varepsilon^3)$$

$$= \frac{1}{\sqrt{\pi}} (\mu(x, t) - \varepsilon \delta_1 - \varepsilon^2 \gamma_1)(\sigma(x, t) - \varepsilon \delta_2 - \varepsilon^2 \gamma_2)$$

$$+ \frac{1}{\sqrt{\pi}} \frac{\varepsilon^2 (\mu(x, t) - \varepsilon \delta_1 - \varepsilon^2 \gamma_1)}{2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right)$$

$$- \frac{1}{\sqrt{\pi}} (\mu(x, t) + \varepsilon \delta_3 - \varepsilon^2 \gamma_3)(\sigma(x, t) + \varepsilon \delta_4 - \varepsilon^2 \gamma_4).$$
Proof. We make the change of variables 
\[ u = (\hat{u} - \alpha)/\beta, \quad y = (\hat{y} - \alpha)/\beta, \quad z = (\hat{z} - \alpha)/\beta. \]
Then,
\[ (\hat{u} - \alpha^2)/\beta^2 = u^2 + 2\epsilon u \delta_1 + \delta_2 u + \frac{\epsilon^2}{\beta^2} [3\delta_2^2 - 2\beta\gamma_2]u^2 - (2\beta\gamma_1 - 4\delta_1\delta_2)u + \delta_1^2 \]
\[ + (1 + u^2)O(\epsilon^3), \]
so
\[ \frac{1}{\sqrt{\pi}} \varepsilon^2(\delta_1 \alpha_2 - \delta_3 \delta_4 - \gamma_1 \alpha(x, t) - \gamma_2 \alpha(x, t) + \gamma_3 \alpha(x, t) + \gamma_4 \alpha(x, t)) = O(\epsilon^3), \]
by combining this estimate with (5.6) and similarly combining (5.40) with (5.5), we obtain (2.6)-(2.7).

By combining this estimate with (5.6) and similarly combining (5.40) with (5.5), we obtain (2.6)-(2.7).

5.0.1. Estimates for \( I_1 \) and \( I_2 \). Recall integrals defined in (5.25)-(5.26):
\[ I_1(\alpha, \beta) = \int_{\mathbb{R}^3} \sqrt{\hat{u}^2 + \hat{y}^2 + \hat{z}^2 - \hat{u}\hat{y} - \hat{y}\hat{z} - \hat{u}\hat{z}} f_{\alpha, \beta}(\hat{u}, \hat{y}, \hat{z}) d\hat{u} d\hat{y} d\hat{z}, \]
\[ I_2(\alpha, \beta) = \int_{\mathbb{R}^3} (\hat{u} + \hat{y} + \hat{z}) \sqrt{\hat{u}^2 + \hat{y}^2 + \hat{z}^2 - \hat{u}\hat{y} - \hat{y}\hat{z} - \hat{u}\hat{z}} f_{\alpha, \beta}(\hat{u}, \hat{y}, \hat{z}) d\hat{u} d\hat{y} d\hat{z}. \]
We will provide estimates for the two integrals needed in the proof of Theorem 2.3.
Recall the following notation from (5.27)-(5.28). Given \( \alpha_1, \alpha_2, \alpha_3 \) and \( \epsilon > 0 \), let
\[ \alpha = (\alpha_1 + \alpha_2 + \alpha_3)/3, \]
\[ \beta = (\beta_1, \beta_2, \beta_3) = (\beta - \epsilon\delta_2 + \epsilon^2\gamma_2, \beta - 2\epsilon^2\gamma_2, \beta + \epsilon\delta_2 + \epsilon^2\gamma_2), \]
with \( \beta \) denoting the average of \( \beta \). Assume that \( \beta_k > 0 \) for \( k = 1, 2, 3 \).

**Lemma 5.1.** We have
\[ I_1(\alpha, \beta) = \frac{3\sqrt{\pi} \beta^2}{2} \left[ 1 + \frac{\epsilon^2}{2\beta^2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) \right] + O(\epsilon^3), \]
\[ I_2(\alpha, \beta) = 3\alpha \frac{\sqrt{3\pi} \beta^2}{2} \left[ 1 + \frac{\epsilon^2}{2\beta^2} \left( \delta_1^2 + \frac{\delta_2^2}{2} \right) \right] + \epsilon^2 \sqrt{3\pi} \delta_1 \delta_2 + O(\epsilon^3). \]

**Proof.** We make the change of variables 
\[ u = (\hat{u} - \alpha)/\beta, \quad y = (\hat{y} - \alpha)/\beta, \quad z = (\hat{z} - \alpha)/\beta. \]
Then,
\[ (\hat{u} - \alpha^2)/\beta^2 = u^2 + 2\epsilon u \delta_1 + \delta_2 u + \frac{\epsilon^2}{\beta^2} [3\delta_2^2 - 2\beta\gamma_2]u^2 - (2\beta\gamma_1 - 4\delta_1\delta_2)u + \delta_1^2 \]
\[ + (1 + u^2)O(\epsilon^3), \]
\[ (\hat{y} - \alpha_2)^2 / \beta_2^2 = y^2 + \frac{4 \varepsilon^2}{\beta} [\gamma_2 y^2 + \gamma_1 y] + (1 + y^2) O(\varepsilon^4), \]
\[ (\hat{z} - \alpha_3)^2 / \beta_3^2 = z^2 - 2 \varepsilon \frac{\delta_1 + \delta_2 z}{\beta} + \frac{\varepsilon^2}{\beta^2} \left[ (3 \delta_2^2 - 2 \beta \gamma_2) z^2 - (2 \beta \gamma_1 - 4 \delta_1 \delta_2) z + \delta_1^2 \right] \]
\[ + (1 + z^2) O(\varepsilon^3). \]

We make another change of variables,
\[ s = (z - u) / \sqrt{2}, \quad b = (y - (u + z) / 2) \sqrt{2 / 3}, \quad \text{and} \quad w = (u + y + z) / \sqrt{3}, \]
and we note that \( s, b \) and \( w \) form an orthonormal coordinate system if \( u, y \) and \( z \) do.

Hence the Jacobian of the transformation \( (u, y, z) \to (s, b, w) \) is equal to 1.

The following identities are easy to verify,
\[ (\hat{u} - \alpha_1)^2 / \beta_1^2 + (\hat{y} - \alpha_2)^2 / \beta_2^2 + (\hat{z} - \alpha_3)^2 / \beta_3^2 = \]
\[ = s^2 + b^2 + w^2 - 2 \varepsilon \left( \frac{\delta_1 \sqrt{2s}}{\beta} + \sqrt{\frac{8 \delta_2 s (w - b / \sqrt{2})}{\beta}} \right) \]
\[ + \frac{\varepsilon^2}{\beta^2} \left[ 2 \delta_1^2 + \frac{8 \sqrt{3}}{\delta_1 \delta_2} \left( w - \frac{b}{\sqrt{2}} \right) \right] \]
\[ + \sqrt{24 \gamma_1 \beta b + (3 \delta_2^2 - 2 \beta \gamma_2)(s^2 + b^2 + w^2) + (2 \beta \gamma_2 - \delta_2^2)(\sqrt{2b} + w)^2} \]
\[ + (1 + s^2 + b^2 + w^2) O(\varepsilon^3). \]

For ease of exposition, we define
\[ \omega^2 := s^2 + b^2 + w^2, \]
\[ p_0 := w - \frac{b}{\sqrt{2}}, \]
\[ p_1 := \frac{\delta_1 \sqrt{2}}{\beta} + \sqrt{\frac{8 \delta_2 p_0}{\beta}}, \]
and
\[ p_2 := \frac{1}{\beta_2^2} \left[ 2 \delta_1^2 + \frac{8 \sqrt{3}}{\delta_1 \delta_2} \delta_2 p_0 + \sqrt{24 \gamma_1 \beta b + (3 \delta_2^2 - 2 \beta \gamma_2) \omega^2 + (2 \beta \gamma_2 - \delta_2^2)(\sqrt{2b} + w)^2} \right]. \]

Then (5.48) takes the form
\[ \frac{(\hat{u} - \alpha_1)^2}{\beta_1^2} + \frac{(\hat{y} - \alpha_2)^2}{\beta_2^2} + \frac{(\hat{z} - \alpha_3)^2}{\beta_3^2} = \omega^2 + 2 \varepsilon s p_1 - \varepsilon^2 p_2 - \varepsilon^3 R, \]
where the remainder satisfies $\varepsilon^3 R = \varepsilon^3 R(\varepsilon) = (1 + s^2 + b^2 + w^2)O(\varepsilon^3)$. We combine this formula with (5.25), (5.47) and the formula for the normal density to obtain

$$I_1(\alpha, \beta) = \sqrt{\frac{3}{2} \frac{\beta^4}{(2\pi)^{3/2} \beta_1 \beta_2 \beta_3}} \times \int_{\mathbb{R}^3} \sqrt{b^2 + s^2} \exp \left( -\omega^2/2 + \varepsilon s p_1 - (\varepsilon^2/2)p_2 - (\varepsilon^3/2)R \right) \ ds \ db \ dw. \tag{5.49}$$

We apply Lemma 5.2 to obtain

$$I_1(\alpha, \beta) = \sqrt{\frac{3}{2} \frac{\beta^4}{(2\pi)^{3/2} \beta_1 \beta_2 \beta_3}} \times \int_{\mathbb{R}^3} (1 + \varepsilon sp_1 - (\varepsilon^2/2)(p_2 - p_1^2 s^2) + \tilde{R}) \sqrt{b^2 + s^2} \exp \left( -\omega^2/2 \right) \ ds \ db \ dw, \tag{5.50}$$

where $\tilde{R} = O(\varepsilon^3)$.

The term in the integrand in (5.50) which is linear in $\varepsilon$ is an odd function of $s$. By symmetry it integrates to zero. Similar reasoning applies to the mixed terms in the quadratic term. Specifically, we can eliminate terms in $p_2$ and $p_1^2$ that contain $b, w$ or $bw$. Thus,

$$I_1(\alpha, \beta) = \sqrt{\frac{3}{2} \frac{\beta^4}{(2\pi)^{3/2} \beta_1 \beta_2 \beta_3}} \times \int_{\mathbb{R}^3} (1 - \frac{\varepsilon^2}{2}(\hat{p}_2 - \hat{p}_1 s^2) + \tilde{R}) \sqrt{b^2 + s^2} \exp \left( -\omega^2/2 \right) \ ds \ db \ dw, \tag{5.51}$$

where

$$\hat{p}_1 := 2\frac{\delta_1^2}{\beta^2} + 8\delta_2 \frac{w^2 + b^2/2}{3\beta^2}$$

and

$$\hat{p}_2 := \frac{1}{\beta^2} \left[ 2\delta_1^2 + (3\delta_2^2 - 2\beta \gamma_2)w^2 - (\delta_2^2 - 2\beta \gamma_2)(2b^2 + w^2) \right].$$

Now, we consider the integral

$$P(q_1, q_2, q_3) := \int_{\mathbb{R}^3} \sqrt{b^2 + s^2} \exp \left( -\frac{s^2 + b^2 + w^2}{2} - q_1 s - q_2 b - q_3 w \right) \ ds \ db \ dw. \tag{5.52}$$

Performing the integral over $w$, we see that

$$\frac{P(q_1, q_2, q_3)}{\sqrt{2\pi}} = \exp(q_3^2/2) \int_{\mathbb{R}^2} \sqrt{b^2 + s^2} \exp \left( -\frac{s^2 + b^2}{2} - q_1 s - q_2 b \right) \ ds \ db. \tag{5.53}$$

Going into polar coordinates yields

$$\frac{P(q_1, q_2, q_3)}{\sqrt{2\pi}} = \exp(q_3^2/2) \int_0^{2\pi} \int_0^\infty \rho^2 \exp \left( -\rho^2/2 - \rho(q_1 \cos(\theta) + q_2 \sin(\theta)) \right) \ d\theta \ d\rho. \tag{5.54}$$

We note that for any fixed $(q_1, q_2)$ we can find $\Delta \theta$ so that

$$q_1 \cos(\theta) + q_2 \sin(\theta) = q_{1,2} \cos(\theta'),$$
where $\theta' = \theta + \Delta \theta$ and $q_{1,2} := \sqrt{q_1^2 + q_2^2}$. In these new variables,
\[
\frac{P(q_1, q_2, q_3)}{\sqrt{2\pi}} = \exp(q_3^2/2) \int_0^\infty \int_0^{2\pi} \rho^2 \exp\left(-\rho^2/2 - \rho q_{1,2} \cos(\theta')\right) \, d\theta' \, d\rho.
\]
Performing the inner integral, we find
\[
\frac{P(q_1, q_2, q_3)}{(2\pi)^{3/2}} = \exp(q_3^2/2) \int_0^\infty \rho^2 e^{-\rho^2/2} I_0(\rho q_{1,2}) \, d\rho,
\]
where $I_0(\rho q_{1,2})$ is the modified Bessel function of the first kind.

Next, we use identity 6.618.4 from Gradshteyn and Ryzhik [GR07]:
\[
\int_0^\infty e^{-\lambda x^2} I_{\nu}(\eta x) \, dx = \frac{\sqrt{\pi}}{2\sqrt{\lambda}} e^{\eta^2/(8\lambda)} I_{\nu^2/(8\lambda)}(\eta^2).
\]
After differentiating with respect to $\lambda$ and then setting $\lambda = 1/2$ and $\nu = 0$, we obtain
\[
\int_0^\infty x^2 e^{-x^2/2} I_0(\eta x) \, dx = \frac{\sqrt{2\pi}}{4} e^{\eta^2/4} \left[\eta^2 (I_0(\eta^2/4) + I_1(\eta^2/4)) + 2I_0(\eta^2/4)\right].
\]
Putting this together with our previous expression for $P$, we arrive at
\begin{equation}
(5.53) \quad \frac{P(q_1, q_2, q_3)}{(2\pi)^2} = \frac{1}{4} e^{q_3^2/2} e^{q_2^2/4} \left[2 q_{1,2}^2 \left(I_0 \left(\frac{q_{1,2}^2}{4}\right) + I_1 \left(\frac{q_{1,2}^2}{4}\right)\right) + 2I_0 \left(\frac{q_{1,2}^2}{4}\right)\right].
\end{equation}

It follows from (5.52) that
\[
-\partial_{q_1} P(q_1, q_2, q_3) = \int_{\mathbb{R}^3} s \sqrt{b^2 + s^2} \exp \left(-\frac{s^2 + b^2 + w^2}{2} - q_1 s - q_2 b - q_3 w\right) \, ds \, db \, dw.
\]
Similar formulas hold if we differentiate with respect to $q_2$ or $q_3$, or we take higher derivatives with respect to these variables. When we evaluate the last expression at $(q_1, q_2, q_3) = (0, 0, 0)$, we obtain
\[
\int_{\mathbb{R}^3} s \sqrt{b^2 + s^2} \exp \left(-\frac{s^2 + b^2 + w^2}{2}\right) \, ds \, db \, dw.
\]
These remarks imply that for any polynomial $p(s, b, w)$,
\begin{equation}
(5.54) \quad I_p := \int_{\mathbb{R}^3} p(s, b, w) \sqrt{s^2 + b^2} \, ds \, db \, dw = p(-\partial_{q_1}, -\partial_{q_2}, -\partial_{q_3}) P(q_1, q_2, q_3)\big|_{(q_1, q_2, q_3) = (0, 0, 0)}.
\end{equation}
We now combine this formula and (5.53), and use standard recurrence formulas for derivatives of the modified Bessel functions to obtain
\begin{align}
(5.55) \quad & \int_{\mathbb{R}^3} \sqrt{s^2 + b^2} \exp \left(-\frac{s^2}{2} - \frac{b^2}{2} - \frac{w^2}{2}\right) \, ds \, db \, dw = \frac{(2\pi)^2}{2}, \\
(5.56) \quad & \int_{\mathbb{R}^3} w^2 \sqrt{s^2 + b^2} \exp \left(-\frac{s^2}{2} - \frac{b^2}{2} - \frac{w^2}{2}\right) \, ds \, db \, dw = \frac{(2\pi)^2}{2}, \\
(5.57) \quad & \int_{\mathbb{R}^3} s^2 \sqrt{s^2 + b^2} \exp \left(-\frac{s^2}{2} - \frac{b^2}{2} - \frac{w^2}{2}\right) \, ds \, db \, dw = \frac{3(2\pi)^2}{4}.
\end{align}
We now apply (5.55)-(5.60) to (5.51) to obtain
\[ I_1(\alpha, \beta) = \sqrt{\frac{3}{2}} \left( \frac{\beta^4}{(2\pi)^2} + \frac{2\pi}{\beta} \right) \left[ 1 + \frac{\varepsilon^2}{2\beta^2} \left( \delta_1^2 + \delta_2^2 \right) \right] + O(\varepsilon^3). \]

Expanding the prefactor \( \beta^4/(\beta_1\beta_2\beta_3) \) into a series in \( \varepsilon \), we see that
\[ I_1(\alpha, \beta) = \sqrt{3\pi\beta} \left[ 1 + \frac{\varepsilon^2}{2\beta^2} \left( \delta_1^2 + \delta_2^2 \right) \right] + O(\varepsilon^3), \]
which completes the proof of (5.45).

Note that the integral in (5.42) has the extra factor of \( \hat{u} + \hat{y} + \hat{z} \) compared to (5.41). Since \( \hat{u} + \hat{y} + \hat{z} = 3\alpha + \sqrt{3}\beta w \), we use (5.50) to see that
\[ I_2(\alpha, \beta) = \sqrt{3} \left( \frac{\beta^4}{(2\pi)^{3/2}\beta_1\beta_2\beta_3} \right) \int_{\mathbb{R}^3} \left( 3\alpha + \sqrt{3}\beta w \right) \frac{1}{\sqrt{b^2 + s^2}} \left( 1 + \varepsilon \exp \left( \frac{\varepsilon^2}{2} \right) (p_2 - p_1^2 s^2) + \tilde{R}(\varepsilon^3) \right) \exp \left( -\omega^2/2 \right) ds \, db \, dw, \]
\[ = 3\alpha I_1 + \frac{3}{\sqrt{2}} \left( \frac{\beta^5}{(2\pi)^{3/2}\beta_1\beta_2\beta_3} \right) \int_{\mathbb{R}^3} w \sqrt{b^2 + s^2} \left( 1 + \varepsilon \exp \left( \frac{\varepsilon^2}{2} \right) (p_2 - p_1^2 s^2) + \tilde{R}(\varepsilon^3) \right) \exp \left( -\omega^2/2 \right) ds \, db \, dw. \]

Let \( \tilde{I}_2 \) denote the second term in the previous expression. Looking at the parity of each term in the integrand, we see that
\[ \tilde{I}_2(\alpha, \beta) = \sqrt{3} \frac{4\varepsilon^2 \delta_1 \delta_2 \beta^3}{2 (2\pi)^{3/2}\beta_1\beta_2\beta_3} \int_{\mathbb{R}^3} w^2 \sqrt{b^2 + s^2} (1 - s^2) \exp \left( -\omega^2/2 \right) ds \, db \, dw + O(\varepsilon^3). \]

Using (5.56) and (5.60), it follows that
\[ \tilde{I}_2 = -\frac{(2\pi)^2}{2} \sqrt{3} \frac{4\varepsilon^2 \delta_1 \delta_2 \beta^3}{2 (2\pi)^{3/2}\beta_1\beta_2\beta_3} \left( 1 - \frac{3}{2} \right) + O(\varepsilon^3) = \varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2 + O(\varepsilon^3). \]

Combining this with (5.61) and (5.62), we obtain
\[ I_2(\alpha, \beta) = 3\alpha \frac{\sqrt{3\pi\beta^2}}{2} \left[ 1 + \frac{\varepsilon^2}{2\beta^2} \left( \delta_1^2 + \delta_2^2 \right) \right] + \varepsilon^2 \sqrt{3\pi} \delta_1 \delta_2 + O(\varepsilon^3). \]

This completes the proof. \( \square \)
5.0.2. Series expansion. This subsection presents a technical result—a series expansion with an explicit error estimate needed in the proof of Lemma 5.1.

**Lemma 5.2.** Define $\gamma$ by

\[
\gamma(\varepsilon) = \gamma(\varepsilon, u, y, z) = \left(\frac{u + \tilde{\delta}_1 \varepsilon - \tilde{\gamma}_1 \varepsilon^2}{1 - \tilde{\delta}_2 \varepsilon + \tilde{\gamma}_2 \varepsilon^2}\right)^2 + \left(\frac{y + 2\tilde{\gamma}_1 \varepsilon^2}{1 + 2\tilde{\gamma}_2 \varepsilon^2}\right)^2 + \left(\frac{z - \tilde{\delta}_1 \varepsilon - \tilde{\gamma}_1 \varepsilon^2}{1 + \tilde{\delta}_2 \varepsilon + \tilde{\gamma}_2 \varepsilon^2}\right)^2.
\]

and $M$ by

\[
M = \max\{1, |\tilde{\delta}_1|, 2|\tilde{\gamma}_1|, |\tilde{\delta}_2|, 2|\tilde{\gamma}_2|\}.
\]

Further suppose that $\varepsilon$ is chosen so that $M \varepsilon \leq 1/4$. Then

\[
\exp\left(-\frac{\gamma(\varepsilon)}{2}\right) = 1 + sp_1 \varepsilon - (\varepsilon^2/2)(p_2 - p_1 s^2) \exp\left(-\rho^2/2\right) + \tilde{R},
\]

where

\[
|\tilde{R}| \leq \frac{(500 \varepsilon M)^3}{6} (\rho^2 + 1)^3 e^{-\rho^2/9},
\]

with $\rho^2 = u^2 + y^2 + z^2$, and where

\[
p_0 := w - \frac{b}{\sqrt{2}},
\]

\[
p_1 := \tilde{\delta}_1 \sqrt{2} + \sqrt{\frac{8}{3}} \tilde{\delta}_2 p_0,
\]

\[
p_2 := \left[2\tilde{\delta}_1^2 + \frac{8}{\sqrt{3}} \tilde{\delta}_1 \tilde{\delta}_2 p_0 + \sqrt{24\tilde{\gamma}_1 b + (3\tilde{\delta}_2 - 2\tilde{\gamma}_2)\rho^2 + (2\tilde{\gamma}_2 - \tilde{\delta}_2\sqrt{2b + w})^2}\right]
\]

and

\[
s = (z - u)/\sqrt{2}, \quad b = (y - (u + z)/2)\sqrt{2/3}, \quad \text{and} \quad w = (u + y + z)/\sqrt{3}.
\]

**Proof.** The proof consists of a sequence of estimates for derivatives of elementary functions. Their proofs, elementary but tedious, are omitted.

Let

\[
f(\varepsilon) = (1 + a_2 \varepsilon + b_2 \varepsilon^2)^{-2}.
\]

If $\varepsilon \leq 1/(4M_1)$ where

\[
M_1 = \max\{1, |a_2|, |b_2|\}
\]

then

\[
|f(\varepsilon)| \leq 4,
\]

\[
|f'(\varepsilon)| \leq 4(12)M_1,
\]

\[
|f''(\varepsilon)| \leq 4(12)^2 M_1^2,
\]

\[
|f'''(\varepsilon)| \leq 4(12)^3 M_1^3.
\]

Let

\[
g(\varepsilon) = (a_0 - a_1 \varepsilon - b_1 \varepsilon^2)^2.
\]

If $\varepsilon \leq 1/(4M_2)$ where

\[
M_2 = \max\{1, |a_1|, |b_1|\}
\]

then

\[
|g(\varepsilon)| \leq 4(a_0^2 + 1),
\]
\[ |g'(\varepsilon)| \leq 4(12)M_2(a_0^2 + 1), \]
\[ |g''(\varepsilon)| \leq 4(12)^2M_3^2(a_0^2 + 1), \]
\[ |g'''(\varepsilon)| \leq 4(12)^3M_3^3(a_0^2 + 1). \]

Let \( f(\varepsilon) \) and \( g(\varepsilon) \) be as in (5.65)-(5.66). If \( \varepsilon \leq 1/(4M_3) \) where
\[ M_3 = \max\{1, |a_1|, |b_1|, |a_2|, |b_2|\} \]
then
\[ |(fg)'(\varepsilon)| \leq 400(a_0^2 + 1)M_3, \]
\[ |(fg)''(\varepsilon)| \leq 1000 (a_0^2 + 1)M_3^2, \]
\[ |(fg)'''(\varepsilon)| \leq 250000 (a_0^2 + 1)M_3^3. \]

Let \( \gamma \) be as in (5.63). If \( \varepsilon \leq 1/(4M) \) where
\[ M = \max\{1, |\tilde{\delta}_1|, 2|\tilde{\gamma}_1|, |\tilde{\delta}_2|, 2|\tilde{\gamma}_2|\} \]
then
\[ |\gamma'(\varepsilon)| \leq 400(u^2 + y^2 + z^2 + 3)M, \]
\[ |\gamma''(\varepsilon)| \leq 1000 (u^2 + y^2 + z^2 + 3)M^2, \]
\[ |\gamma'''(\varepsilon)| \leq 250000 (u^2 + y^2 + z^2 + 3)M^3. \]

If \( \varepsilon \leq 1/(4M) \) then
\[ \left| \frac{d^3}{d\varepsilon^3} e^{-\gamma(\varepsilon,u,y,z)/2} \right| \leq \frac{500^3}{2} M^3(\rho^2 + 1)^3 e^{-\gamma(\varepsilon,u,y,z)/2}. \]

It can be shown that
\[ \gamma \geq \frac{2}{9} [u^2 + y^2 + z^2 - 3] \]
so we obtain a new estimate for the derivative,
\[ \left| \frac{d^3}{d\varepsilon^3} e^{-\gamma(\varepsilon,u,y,z)/2} \right| \leq 500^3 M^3(\rho^2 + 1)^3 e^{-(u^2+y^2+z^2)/9}. \]

The proposition follows from this estimate, noting that the right-hand side of (5.64) is (apart from the remainder) the quadratic Taylor approximation to the left-hand side about \( \varepsilon = 0 \). \( \square \)

6. Proof of Local Wellposedness

This section contains the proof of Theorem 4.1.
6.1. **Preliminary considerations.** Recall that we are concerned with the system

\begin{align}
(B') = \begin{cases}
\mu_t - \Delta \mu = -\lambda D \sqrt{|w|}, \\
w_t - \Delta w = -\lambda \mu_x \sqrt{|w|} + (\mu_x)^2, \\
\mu_x(a, t) = \mu_x(b, t) = 0, \\
w(a, t) = w(b, t) = 0.
\end{cases}
\end{align}

Our proof of the existence of a solution for any initial condition will be based on a fixed-point argument. Recall definitions of function spaces given in (4.7)-(4.8). Observe that when \( w \in L^1 \), then by the definition of \( L^p \) spaces

\[ \int_a^b \sqrt{|w|}^2 \, dx = \int_a^b |w| \, dx = \|w\|_{L^1} \]

and so \( \sqrt{|w|} \in L^2 \). The action of applying the Fourier multiplier \( D \) thus leads to a shift in the Sobolev space: since \( \sqrt{|w|} \in L^2 \) we have for any test function \( \phi \in H^1 \) from the self-adjointness of Fourier multipliers that

\[ \int_a^b \left( D \sqrt{|w|} \right) \phi \, dx = \int_a^b \sqrt{|w|} (D\phi) \, dx \leq \|\sqrt{|w|}\|_{L^2} \|D\phi\|_{L^2} \leq \|\sqrt{|w|}\|_{L^2} \|\phi\|_{H^1} \]

and therefore \( D \sqrt{|w|} \in H^{-1} \). This shows that the right-hand side of (6.1) is in \( H^{-1} \). Given some \( \mu \in H^1 \), we note that \( \mu_x \in L^2 \) as well as \( \sqrt{|w|} \in L^2 \) which implies \( \mu_x \sqrt{|w|} \in L^1 \) and \( \mu_x^2 \in L^1 \). Hence the right-hand side of (6.2) is in \( L^1 \).

6.2. **Linear Estimates.** We start by recalling some standard estimates for the heat equation: these are all classical (see, for example, [Hun14]).

**Lemma 6.1.** For any \( f \in H^{-1}_{T,x} \) and \( g \in H^1 \) with \( g(a) = g(b) = 0 \), there exists a unique weak solution \( u \in C([0, T]; L^2(a, b)) \cap H^1_{x,T} \) of the equation

\[ \frac{\partial u}{\partial t} - \Delta u = f, \]

\[ u(a, t) = u(b, t) = 0, \]

\[ u(x, 0) = g(x), \]

satisfying

\[ \frac{1}{2} \|u(T)\|_{L^2}^2 + \|u_x\|_{L^2_{x,T}}^2 \leq \|f\|_{L^2_{x,T}} \|u\|_{L^2_{x,T}} + \frac{1}{2} \|u(0)\|_{L^2}^2. \]

**Proof.** Multiplying both sides with \( u \) we obtain

\[ (\partial u / \partial t) - (\Delta u)u = fu. \]

Integrating on both sides and using that \( u \) vanishes on the boundary we have, for each \( t > 0 \), that

\[ \frac{1}{2} \frac{d}{dt} \int_a^b u^2 \, dx + \int_a^b (u_x)^2 \, dx = \int_a^b fu \, dx. \]
Note that the integral on the right hand side makes sense for almost all $t$ since $f \in H^{-1}_{T,x}$ and $u \in H^1_{x,T}$. Integrating in time over the interval $0 \leq t \leq T$, we obtain
\begin{equation}
(6.4) \quad \frac{1}{2} \int_a^b u(x,T)^2 \, dx + \int_0^T \int_a^b (u_x)^2 \, dx \, dt = \int_0^T \int_a^b f u \, dx \, dt + \frac{1}{2} \int_a^b u(x,0)^2 \, dx.
\end{equation}
By Cauchy-Schwarz,
\[ \int_0^T \int_a^b f u \, dx \, dt \leq \| f \|_{L^2_{T,x}} \| u \|_{L^2_{x,T}}. \]
Thus
\[ \frac{1}{2} \| u(T) \|_{L^2}^2 + \| u_x \|_{L^2_{x,T}}^2 \leq \| f \|_{L^2_{T,x}} \| u \|_{L^2_{x,T}} + \frac{1}{2} \| u(0) \|_{L^2}^2. \]

Lemma 6.2. Suppose $u : [a, b] \times [0, T] \to \mathbb{R}$ satisfies
\[ \frac{\partial u}{\partial t} - \Delta u = Df(x,t), \]
\[ u(a, t) = u(b, t) = 0, \]
\[ u(x, 0) = 0, \]
where $D$ is the Fourier multiplier associated to differentiation in $x$ and $f(\cdot, t) \in L^2$ for $t \in [0, T]$. Then
\[ \frac{1}{2} \| u(T) \|_{L^2}^2 + \| u_x \|_{L^2_{x,T}}^2 \leq \| f \|_{L^2_{T,x}}^2. \]

Proof. We apply the same argument as in the proof of Lemma 6.1 with $f$ replaced by $Df$. Using self-adjointness of the Fourier multiplier,
\[ \int_0^T \int_a^b Df u \, dx \, dt = \int_0^T \int_a^b f D u \, dx \, dt \leq \| f \|_{L^2_{T,x}} \| u_x \|_{L^2_{x,T}}. \]
This, the initial condition $u(x, 0) = 0$, and (6.4) yield
\[ \frac{1}{2} \| u(T) \|_{L^2}^2 + \| u_x \|_{L^2_{x,T}}^2 \leq \| f \|_{L^2_{T,x}} \| u_x \|_{L^2_{x,T}}. \]
Omitting the first term implies
\[ \| u_x \|_{L^2_{x,T}}^2 \leq \| f \|_{L^2_{T,x}} \| u_x \|_{L^2_{x,T}} \]
from which we deduce
\[ \| u_x \|_{L^2_{x,T}} \leq \| f \|_{L^2_{T,x}} \]
and therefore
\[ \frac{1}{2} \| u(T) \|_{L^2}^2 + \| u_x \|_{L^2_{x,T}}^2 \leq \| f \|_{L^2_{T,x}}^2. \]
Observe that omitting the gradient term also implies
\[ \frac{1}{2} \max_{0 \leq s \leq T} \| u(s) \|_{L^2}^2 \leq \| f \|_{L^2_{T,x}}^2. \]
\[ \square \]
Lemma 6.3. If \( u \) solves \( u_t - \Delta u = f \) with initial conditions \( u(x, 0) = 0 \) and Dirichlet conditions on the boundary, then

\[
\|u(t)\|_{L^1} \leq \int_0^t \|f(s)\|_{L^1} ds
\]

and

\[
\int_0^t \|u(s)\|_{L^1} ds \leq \int_0^t (t - s)\|f(s)\|_{L^1} ds \leq t \int_0^t \|f(s)\|_{L^1} ds.
\]

Proof. Both statements follow from Duhamel’s formula

\[
u(t) = \int_0^t e^{(t-s)\Delta} f(s) ds.
\]

For the first statement, we use the triangle inequality to argue that

\[
\int_a^b |u(t, x)| dx \leq \int_a^b \int_0^t [e^{(t-s)\Delta}|f(s)|](x) ds dx.
\]

At this point, we note that the heat equation preserves non-negativity. Moreover, if \( h(x) \) is a non-negative function on \([a, b]\) we have

\[
\frac{d}{ds} \int_a^b [e^{s\Delta}h](x) dx = \int_a^b \Delta[e^{s\Delta}h](x) dx = \left[ \frac{d}{dx} e^{s\Delta}h \right](b) - \left[ \frac{d}{dx} e^{s\Delta}h \right](a).
\]

However, since non-negativity is preserved and the solution vanishes at the boundary, we have that \( \frac{d}{dx} [e^{s\Delta}h](b) \leq 0 \) and \( \frac{d}{dx} [e^{s\Delta}h](a) \geq 0 \) which shows that the heat equation cannot increase the \( L^1 \)-norm of a positive function and

\[
\|u(t)\|_{L^1} \leq \int_0^t \|e^{(t-s)\Delta} f(s)\|_{L^1} ds \leq \int_0^t \|f(s)\|_{L^1} ds.
\]

Integrating this inequality immediately implies the second inequality since

\[
\int_0^t \|u(s)\|_{L^1} ds \leq \int_0^t (t - s)\|f(s)\|_{L^1} ds \leq t \int_0^t \|f(s)\|_{L^1} ds.
\]

\[\square\]

6.3. Combining the estimates. Consider the system

\[
\begin{cases}
\mu_t - \Delta \mu = -\lambda D \sqrt{|w|}, \\
w_t - \Delta w = -\lambda \mu_x \sqrt{|w|} + (\mu_x)^2, \\
\mu_x(a) = \mu_x(b) = 0, \\
w(a) = w(b) = 0.
\end{cases}
\]

We work in the space \( X = H^1_{x,T} \times L^1_{x,T} \) with norm

\[
\|(e, f)\|^2_X = \frac{1}{100 \lambda} \cdot \|e\|^2_{H^1_{x,T}} + \|f\|^2_{L^1_{x,T}}.
\]

Suppose we are given two pairs of functions

\((e, f) \in H^1_{x,T} \times L^1_{x,T} \) and \((g, h) \in H^1_{x,T} \times L^1_{x,T}\).
such that \( e(x,0) = g(x,0) \) and \( f(x,0) = h(x,0) \). We use these as two right-hand sides for our system of equations and denote the corresponding solutions by \((\mu_1, w_1)\) for \((e, f)\) as right-hand side and \((\mu_2, w_2)\) for \((g, h)\) as right-hand side. Since both equations are linear, the difference between these two solutions \( u = \mu_1 - \mu_2 \) and \( v = w_1 - w_2 \) satisfy

\[
\begin{align*}
  u_t - \Delta u &= -\lambda D\sqrt{|f|} + \lambda D\sqrt{|h|}, \\
  v_t - \Delta v &= -\lambda e_x \sqrt{|f|} + (e_x)^2 + \lambda g_x \sqrt{|h|} - (g_x)^2, \\
  u(x, 0) &= 0 = v(x, 0), \\
  u_x(a) = u_x(b) &= 0, \\
  v(a) = v(b) &= 0.
\end{align*}
\]

Lemma (6.2) implies

\[
\frac{1}{2} \| u(T) \|^2_{L^2_x} + \| u_x \|^2_{L^2_{x,T}} \leq \lambda^2 \left\| \sqrt{|f|} - \sqrt{|h|} \right\|^2_{L^2_{x,T}}
\]

from which we immediately infer

\[
\| u_x \|_{L^2_{x,T}} \leq \lambda \left\| \sqrt{|f|} - \sqrt{|h|} \right\|_{L^2_{x,T}},
\]

\[
\frac{1}{2} \| u(T) \|^2_{L^2_x} \leq \lambda^2 \left\| \sqrt{|f|} - \sqrt{|h|} \right\|^2_{L^2_{x,T}}.
\]

The last inequality applies with \( T \) replaced by \( t \in [0, T] \) so

\[
\max_{0 \leq t \leq T} \frac{1}{2} \| u(t) \|^2_{L^2_x} \leq \lambda^2 \left\| \sqrt{|f|} - \sqrt{|h|} \right\|^2_{L^2_{x,T}}.
\]

Trivially,

\[
\left( \sqrt{|f|} - \sqrt{|h|} \right)^2 \leq \left| \left( \sqrt{|f|} - \sqrt{|h|} \right) \left( \sqrt{|f|} + \sqrt{|h|} \right) \right| = \| f - h \| \leq | f - h |
\]

and so

\[
\| u_x \|_{L^2_{x,T}} \leq \lambda \cdot \| f - h \|_{L^1_{x,T}},
\]

as well as

\[
\| u \|^2_{L^2_{x,T}} \leq T \max_{0 \leq t \leq T} \| u(t) \|^2_{L^2_x} \leq 2 T \lambda^2 \cdot \| f - h \|^2_{L^1_{x,T}}.
\]

Altogether, we have

\[
\| u \|^2_{L^2_{x,T}} + \| u_x \|^2_{L^2_{x,T}} \leq (2T + 1) \lambda^2 \| f - h \|^2_{L^1_{x,T}}
\]

and therefore

\[
(6.5) \quad \| u \|^2_{L^2_{x,T}} \leq \lambda \sqrt{2T + 1} \cdot \| f - h \|_{L^1_{x,T}}.
\]

For bounding the second function, we recall Lemma 6.3 and argue

\[
\int_0^T \| v(s) \|_{L^1} ds \leq T \int_0^T \int_a^b \left| -\lambda e_x \sqrt{|f|} + (e_x)^2 + \lambda g_x \sqrt{|h|} - (g_x)^2 \right| dx ds
\]
\[
\begin{align*}
    \leq T & \int_0^T \int_a^b |e_x|^2 - |g_x|^2 \, dx \, ds + \lambda T \int_0^T \int_a^b |e_x \sqrt{|f|} - g_x \sqrt{|h|}| \, dx \, ds \\
    \leq T & \|e_x - g_x\|_{L^2_{z,T}} \|e_x + g_x\|_{L^2_{z,T}} + \lambda T \int_0^T \int_a^b |e_x \sqrt{|f|} - g_x \sqrt{|h|}| \, dx \, ds.
\end{align*}
\]

We continue by estimating the remaining integral as
\[
\begin{align*}
    \int_0^T \int_a^b |e_x \sqrt{|f|} - g_x \sqrt{|h|}| \, dx \, ds &= \int_0^T \int_a^b |e_x \sqrt{|f|} - g_x \sqrt{|f|} + g_x \sqrt{|f|} - g_x \sqrt{|h|}| \, dx \, ds \\
    &\leq \int_0^T \int_a^b |e_x \sqrt{|f|} - g_x \sqrt{|f|}| \, dx \, ds \\
    &\quad + \int_0^T \int_a^b |g_x \sqrt{|f|} - g_x \sqrt{|h|}| \, dx \, ds \\
    &\leq \|e_x - g_x\|_{L^2_{z,T}} \|f\|_{L^1_{z,T}} + \|g_x\|_{L^2_{z,T}} \|\sqrt{|f|} - \sqrt{|h|}\|_{L^1_{z,T}} \\
    &\leq \|e_x - g_x\|_{L^2_{z,T}} \|f\|_{L^1_{z,T}} + \|g_x\|_{L^2_{z,T}} \|f - h\|_{L^1_{z,T}}.
\end{align*}
\]

Altogether, we obtain
\[
\begin{align*}
    \|v\|_{L^1_{z,T}} &\leq T(1 + \lambda) \|e_x - g_x\|_{L^2_{z,T}} \left( \|e_x\|_{L^2_{z,T}} + \|g_x\|_{L^2_{z,T}} + \|f\|_{L^1_{z,T}} \right) \\
    &\quad + T\lambda \|g_x\|_{L^2_{z,T}} \|f - h\|_{L^1_{z,T}}.
\end{align*}
\]

6.4. An Iteration scheme. We define the iteration scheme
\[
\begin{align*}
    \frac{\partial}{\partial t} \mu_{n+1} - \Delta \mu_{n+1} &= -\lambda D \sqrt{|w_n|}, \\
    \frac{\partial}{\partial t} w_{n+1} - \Delta w_{n+1} &= -\lambda (\mu_{n+1})_x \sqrt{|w_n|} + (\mu_n)^2, \\
    (\mu_{n+1})_x(a, t) &= (\mu_{n+1})_x(b, t) = 0, \\
    w_{n+1}(a, t) &= w_{n+1}(b, t) = 0,
\end{align*}
\]
with the initial conditions \(\mu_0(x, 0) = \mu_0(x, 0)\) and \(\sigma_n(x, 0) = \sigma_0(x, 0)\).

Our goal is to show that \((\mu_n)^\infty_{n=1}\) is a convergent sequence in \(H^1_{z,T}\) and that \((w_n)^\infty_{n=1}\) is a convergent sequence in \(L^1_{z,T}\) provided \(T\) is sufficiently small (but strictly positive). This is done by showing that they are Cauchy sequences. We recall the estimates (6.5)-(6.6) which imply, in this setting,
\[
\|\mu_{n+1} - \mu_n\|_{H^1_{z,T}} \leq \lambda \sqrt{2T + 1} \|w_n - w_{n-1}\|_{L^1_{z,T}}
\]
as well as
\[
\begin{align*}
    \|w_{n+1} - w_n\|_{L^1_{z,T}} &\leq T(1 + \lambda) \|\mu_n - \mu_{n-1}\|_{H^1_{z,T}} \cdot \left( \|(\mu_n)_x\|_{L^2_{z,T}} + \|(\mu_{n-1})_x\|_{L^2_{z,T}} \right) \\
    &\quad + T(1 + \lambda) \|\mu_n - \mu_{n-1}\|_{H^1_{z,T}} \cdot \|w_n\|_{L^1_{z,T}} \\
    &\quad + T\lambda \|(\mu_{n-1})_x\|_{H^1_{z,T}} \cdot \|w_n - w_{n-1}\|_{L^1_{z,T}}.
\end{align*}
\]

We introduce two sequences of real numbers via
\[
a_0 = \|\mu_0\|_{H^1_{z,T}}, \quad b_0 = \|w_0\|_{L^1_{z,T}},
\]
and then, for \( n \geq 0 \),

\[
a_{n+1} = \|\mu_{n+1} - \mu_n\|_{H^1_{x,T}} \quad \text{and} \quad b_{n+1} = \|w_{n+1} - w_n\|_{L^1_{x,T}}.
\]

Note that, with this notation, we have

\[
\|\mu_n\|_{H^1_{x,T}} \leq \|\mu_0\|_{H^1_{x,T}} + \sum_{k=1}^{n} \|\mu_k - \mu_{k-1}\|_{H^1_{x,T}} = \sum_{k=0}^{n} a_k
\]

as well as

\[
\|w_n\|_{L^1_{x,T}} \leq \|w_0\|_{L^1_{x,T}} + \sum_{k=1}^{n} \|w_k - w_{k-1}\|_{L^1_{x,T}} = \sum_{k=0}^{n} b_k.
\]

Therefore the inequalities (6.7)-(6.8) imply that

\[
(6.9) \quad a_{n+1} \leq \lambda \sqrt{2T + 1} b_n
\]

and

\[
(6.10) \quad b_{n+1} \leq T(1 + \lambda)a_n \left(2 \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k\right) + T\lambda b_n \sum_{k=0}^{n} a_k.
\]

Lemma 6.4. For any \( a_0, b_0, \lambda \geq 0 \) there exists \( T > 0 \) such that any sequences \((a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}\) of non-negative real numbers satisfying (6.9)-(6.10) have the property that

\[
\sum_{k=0}^{\infty} a_k \quad \text{and} \quad \sum_{k=0}^{\infty} b_k \quad \text{converge.}
\]

Proof. We first note that (6.9) implies that when \( \sum b_k \) converges, then \( \sum a_k \) converges as well. Substituting (6.9) into (6.10), we deduce that

\[
b_{n+1} \leq T(1 + \lambda)a_n \left(2 \sum_{k=0}^{n} a_k + \sum_{k=0}^{n} b_k\right) + T\lambda b_n \sum_{k=0}^{n} a_k
\]

\[
\leq T(1 + \lambda)\lambda \sqrt{2T + 1} b_{n-1} \left(2a_0 + (2\lambda \sqrt{2T + 1} + 1)\sum_{k=0}^{n} b_k\right)
\]

\[
+ T\lambda b_n \left(a_0 + \lambda \sqrt{2T + 1}\sum_{k=0}^{n} b_k\right).
\]

Using \( \varepsilon_T \) to denote a quantity depending on \( a_0, b_0, \lambda \) and \( T \) that tends to 0 as \( T \to 0 \), we can simplify this as

\[
(6.11) \quad b_{n+1} \leq \varepsilon_T (b_{n-1} + b_n) \left(1 + \sum_{k=0}^{n} b_k\right).
\]

Consider a geometric sequence \( d_n = d_0 2^{-n} \) and choose \( d_0 \) so that \( b_0 < d_0 \) and \( b_1 = \|w_1 - w_0\|_{L^1_{x,T}} < d_1 \) for \( T \leq 1 \). We have

\[
d_{n+1} \geq \varepsilon_T (d_{n-1} + d_n) \left(1 + \sum_{k=0}^{n} d_k\right)
\]
if and only if
\begin{equation}
\frac{1}{2} \geq \varepsilon_T 3(1 + 2d_0(1 - 2^{-n-1})).
\end{equation}

Now we choose $T \leq 1$ so small that $\varepsilon_T$ is so small that (6.13) holds for all $n \geq 1$, and, therefore, (6.12) holds for all $n \geq 1$. This and (6.11) imply by induction that $b_n \leq d_n$ for $n \geq 0$, so $\sum_n b_n < \infty$. $\square$

This completes the proof that $(\mu_n)_{n=1}^\infty$ is a convergent sequence in $H^1_{x,T}$ and $(w_n)_{n=1}^\infty$ is a convergent sequence in $L^1_{x,T}$ and, therefore, completes the proof of Theorem 4.1.

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