Examples of non exact 1-subexponential $C^*$-algebras

by

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Abstract

This is a supplement to our previous paper on the arxiv [13]. We show that there is a non-exact $C^*$-algebra that is 1-subexponential, and we give several other complements to the results of that paper. Our example can be described very simply using random matrices: Let $\{X_j^{(m)} | j = 1, 2, \cdots \}$ be an i.i.d. sequence of random $m \times m$-matrices distributed according to the Gaussian Unitary Ensemble (GUE). For each $j$ let $u_j(\omega)$ be the block direct sum defined by

$$u_j(\omega) = \oplus_{m \geq 1} X_j^{(m)}(\omega) \in \oplus_{m \geq 1} M_m.$$ 

Then for almost every $\omega$ the $C^*$-algebra generated by $\{u_j(\omega) | j = 1, 2, \cdots \}$ is 1-subexponential but is not exact.

The GUE is a matrix model for the semi-circular distribution. We can also use instead the analogous circular model.

Consider the direct sum $B = \oplus_{m \geq 1} M_m$. By definition, for any $x = \oplus_{m \geq 1} x(m) \in B$ we have $\|x\| = \sup_{m \geq 1} \|x(m)\|$. We equip $M_m$ with its normalized trace $\tau_m$.

Let $u_j = \oplus_m u_j(m)$ be elements of $B$. Let $\mathcal{A}$ be the unital $C^*$-algebra generated by $u_1, u_2, \cdots, u_n$. For simplicity we set $u_0 = 1$. Let $\mathcal{C}$ be a unital $C^*$-algebra that we assume generated by $c_1, c_2, \cdots$ and equipped with a faithful tracial state $\tau$. We again set $c_0 = 1$.

We say (following [5]) that $\{u_j(m) | 0 \leq j \leq n\}$ tends strongly to $\{c_j | 0 \leq j \leq n\}$ when $m \to \infty$ if it tends weakly (meaning “in moments” relative to $\tau_m$ and $\tau$) and moreover $\|P(u_i(m))\| \to \|P(c_i)\|$ for any (non-commutative) polynomial $P$. This implies that for any $n+1$-tuple of such polynomials $P_0, P_1, \cdots, P_n$, for any $k$ and any $a_j \in M_k$ we have

$$(0.1) \quad \lim_{m \to \infty} \| \sum_0^n a_j \otimes P_j(u_i(m)) \| = \| \sum_0^n a_j \otimes P_j(c_i) \|.$$

In particular we have

$$(0.2) \quad \lim_{m \to \infty} \| \sum_0^n a_j \otimes u_j(m) \| = \| \sum_0^n a_j \otimes c_j \|.$$

Let $I_0 \subset B$ denote the ideal of sequences $(x_m) \in B$ that tend to zero in norm (usually denoted by $c_0(\{M_N\})$). Let $Q : B \to B/I_0$ be the quotient map. It is easy to check that for any polynomial $P$ we have $\|Q(P(u_j))\| = \|P(c_j)\|$. So that, if we set $I = I_0 \cap \mathcal{A}$, we have a natural identification $\mathcal{A}/I = \mathcal{C}.$
Let $P_d$ denote the linear space of all polynomials of degree $\leq d$ in the non commutative variables $(X_1, \cdots, X_n, X_1^*, \cdots, X_n^*)$. We will need to consider the space $M_k \otimes P_d$. It will be convenient to systematically use the following notational convention:

$$\forall 1 \leq j \leq n \quad X_{n+j} = X_j^*.$$ 

A typical element of $M_k \otimes P_d$ can then be viewed as a polynomial $P = \sum a_J \otimes X^J$ with coefficients in $M_k$. Here the index $J$ runs over the disjoint union of the sets $\{1, \cdots, 2n\}^i$ with $1 \leq i \leq d$. We also add symbolically the value $J = 0$ to the index set and we set $X^0 = 1$ equal to the unit.

We denote by $P(u(m)) \in M_k \otimes M_m$ (resp. $P(c) \in M_k \otimes C$) the result of substituting $\{u_j(m)\}$ (resp. $\{c_j\}$) in place of $\{X_j\}$. It follows from the strong convergence of $\{u_j \mid 0 \leq j \leq n\}$ to $\{c_j \mid 0 \leq j \leq n\}$ that for any $d$ and any $P \in M_k \otimes P_d$ we have

$$\|P(u(m))\| \rightarrow \|P(c)\|.$$ 

With a similar convention we will write e.g. $P(c) = \sum a_J \otimes c^J$.

In particular this implies (actually this already follows from weak convergence)

$$\forall k \forall d \forall P \in M_k \otimes P_d \quad \|P(c)\| \leq \liminf_{m \rightarrow \infty} \|P(u(m))\|. \tag{0.3}$$

**Remark 0.1.** Let us write $P$ as a sum of monomials $P = \sum a_J \otimes X^J$ as above. We will assume that the operators $\{c^J\}$ are linearly independent. From this assumption follows that there is a constant $c_2(n,d)$ such that

$$\sum_j \|a_J\| \leq c_2(n,d)\|P(c)\|.$$ 

Indeed, since the span of the $c^J$'s is finite dimensional, the linear form that takes $P$ to its $c^J$-coefficient is continuous, and its norm (that depends obviously only on $(n,d)$) is the same as its c.b. norm. Of course this depends also on the distribution of the family $\{c_j\}$ but we view this as fixed from now on.

We will consider the following assumption:

$$\sum_1^n \tau(|c_j|^2) > \sum_1^n \|u_j \otimes \bar{c}_j\|_{\mathcal{A} \otimes_{\min} C}. \tag{0.4}$$

**Notation.** Let $\alpha \subset \mathbb{N}$ be a subset (usually infinite in the sequel). We denote

$$B(\alpha) = \oplus_{m \in \alpha} M_m.$$ 

$$u_j(\alpha) = \oplus_{m \in \alpha} u_j(m) \in B(\alpha).$$ 

We will denote by $A(\alpha) \subset B(\alpha)$ the unital $C^*$-algebra generated by $\{u_j(\alpha) \mid 1 \leq j \leq n\}$. With this notation $\mathcal{A} = A(\mathbb{N})$.

We also set $E_d(\alpha) = \{P(u(\alpha)) \mid P \in P_d\}$. It will be convenient to set also $u_j^d(m) = 0$ whenever $m \not\in \alpha$.

Fix a degree $d \geq 1$. Then for any real numbers $m \geq 1$ and $t \geq 1$ we define

$$C_d(m,t) = \sup_{m' \geq m} \sup_{k \leq t} \{\|P(u(m'))\| \mid P \in M_k \otimes P_d, \|P(c)\| \leq 1\}.$$
Theorem 0.2. Assume that for any $d \geq 1$ there are $a > 0$ and $D > 0$ such that $C_d(aN^D, N) \to 1$ when $N \to \infty$. Assume moreover that (0.3) holds. Then for any subset $\alpha \subset \mathbb{N}$ the unital $C^*$-algebra $A(\alpha)$ generated by $\{u_j(\alpha) \mid 1 \leq j \leq n\}$ is 1-subexponential. Moreover, if we assume (0.4) then it is not exact.

Proof. For subexponentiality, we need to show that for any fixed $\varepsilon > 0$ and any finite dimensional subspace $E \subset A(\alpha)$ the growth of $N \mapsto K_E(N, 1+\varepsilon)$ is subexponential. Since the polynomials in $\{u_j(\alpha)\}$ are dense in $A(\alpha)$, by perturbation it suffices to check this for $E \subset E_d(\alpha)$. Thus we may as well assume $E = E_d(\alpha)$.

Then we may choose $N_0$ large enough so that $C_d(aN^D, N) < 1 + \varepsilon$ for all $N \geq N_0$. We claim that for all $N \geq N_0$ we have $K_E(N, (1+\varepsilon)^2) \in \mathcal{O}(N^{2D})$ when $N \to \infty$. To verify this, let $P \in M_N \otimes P_d$.

Then, recalling (0.3), we have

$$\|P(e)\| \leq \sup_{m \geq aN^D} \|P(u(m))\| \leq C_d(aN^D, N)\|P(e)\|.$$  

Let $\alpha' = \alpha \cap [1, aN^D)$. Let $T : E \to B(\alpha') \oplus \mathcal{C}$ be the linear mapping defined for all $P$ in $P_d$ by

$$T(P(u(\alpha))) = P(u(\alpha')) \oplus P(e).$$

We may assume $\alpha$ infinite (otherwise the subexponentiality is trivial). Then (0.3) shows that $\|T\|_{cb} \leq 1$. Conversely, by (0.5) we have

$$\|(T^{-1})_N\| \leq C_d(aN^D, N) < 1 + \varepsilon.$$ 

Let $\hat{E}$ be the range of $T$. This shows that $d_N(E, \hat{E}) < 1 + \varepsilon$. We have $\hat{E} \subset \bigoplus_{k \leq aN^D} M_k \oplus \hat{E}'$ where $\hat{E}'$ is a finite dimensional subspace of $\mathcal{C}$ (included in the span of polynomials of degree $d$). Since $\mathcal{C}$ is exact, there is an integer $K$ such that $\hat{E}'$ is completely $(1 + \varepsilon)$-isomorphic to a subspace of $M_K$, so that $\hat{E}$ is completely $(1 + \varepsilon)$-isomorphic to a subspace of $\bigoplus_{k \leq aN^D} M_k \oplus M_K$. Therefore we have for any $N \geq N_0$

$$K_E(N, (1+\varepsilon)^2) \leq 1 + 2 + \cdots + [aN^D] + K$$

and hence our claim follows, proving the 1-subexponentiality.

We now show that $A(\alpha)$ is not exact. Recall the notation $B(\alpha) = \bigoplus_{m \in \alpha} M_m$. By Kirchberg’s results (see e.g. [11] p. 286), if $A(\alpha)$ is exact then the inclusion map $V : A(\alpha) \to B(\alpha)$ satisfies the following: for any $C^*$-algebra $\mathcal{C}$ the mapping $V \otimes Id \mathcal{C} : A(\alpha) \otimes_{\min} C \to B(\alpha) \otimes_{\max} \mathcal{C}$ is bounded (and is actually contractive). Let $\mathcal{U}$ be any free ultrafilter on $\alpha$. Let $\mathcal{M}^\mathcal{U}$ denote the von Neumann algebra ultraproduct of $\{M_m \mid m \in \alpha\}$, with each $M_m$ equipped with $\tau_m$. Recall that $\mathcal{M}^\mathcal{U}$ is finite (cf. e.g. [11] p. 211). We may view $\mathcal{C}$ as embedded in $\mathcal{M}^\mathcal{U}$. Let $\mathcal{M}$ be the von Neumann algebra generated by $\mathcal{C}$. Note that $\mathcal{M}$ can also be identified (as von Neumann algebra) with the von Neumann algebra generated by $\mathcal{C}$ when we view it as embedded in $B(L_2(\tau))$.

We have a quotient map $Q_1 : B(\alpha) \to \mathcal{M}^\mathcal{U}$ and a (completely contractive) conditional expectation $Q_2$ from $\mathcal{M}^\mathcal{U}$ to $\mathcal{M}$. Let $q : A(\alpha) \to \mathcal{M}$ be the composition $q = Q_2Q_1V$. By the above, $q \otimes Id \mathcal{C} : A(\alpha) \otimes_{\min} \mathcal{C} \to \mathcal{M} \otimes_{\max} \mathcal{C}$ must be bounded (and actually contractive). However, if we take $C = \hat{\mathcal{C}}$, this implies since $c_j = q(u_j)$

$$\|\sum_{1}^{n} c_j \otimes \hat{c}_j\|_{\mathcal{M} \otimes_{\max} \hat{\mathcal{C}}} \leq \|\sum_{1}^{n} u_j \otimes \hat{c}_j\|_{A(\alpha) \otimes_{\min} \hat{\mathcal{C}}} \leq \|\sum_{1}^{n} u_j \otimes \hat{c}_j\|_{A \otimes_{\min} \hat{\mathcal{C}}}.$$ 

But now using the fact that left and right multiplication acting on $L_2(\tau)$ are commuting representations on $\mathcal{M}$, we immediately find

$$\sum_{1}^{n} \tau(|c_j|^2) \leq \|\sum_{1}^{n} c_j \otimes \hat{c}_j\|_{\mathcal{M} \otimes_{\max} \hat{\mathcal{C}}}.$$ 

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and this contradicts (0.4). This contradiction shows that $A(\alpha)$ is not exact. □

Remark 0.3. Let $Y^{(m)}$ denote a random $m \times m$-matrix with i.i.d. complex Gaussian entries with mean zero and second moment equal to $m^{-1/2}$, and let $(Y^{(m)}_j)$ be a sequence of i.i.d. copies of $Y^{(m)}$. We will use the matrix model formed by these matrices (sometimes called the “Ginibre ensemble”), for which it is known ([15]) that we have weak convergence to a free circular family $\{c_j\}$. Moreover, by [5] we have also almost surely strong convergence of the random matrices to the free circular system. Actually, the inequalities from [5] that we will crucially use are stated there mostly for the GUE ensemble, i.e. for self adjoint Gaussian matrices with a semi-circular weak limit. These can be defined simply by setting

$$X^{(m)}_j = \sqrt{2} \Re(Y^{(m)}_j).$$

Note we also have an identity in distribution $s_j = \sqrt{2} \Re(c_j)$. We call this the self-adjoint model. However, as explained in [5], it is easy to pass from one setting to the other by a simple “2×2-matrix trick”. Since we prefer to work in the circular setting, we will now indicate this trick.

When working in the self-adjoint model, of course we consider only polynomials of degree $d$ in $(X_1, \cdots, X_n)$. Fix $k$. Then the set of polynomials of degree $\leq d$ with coefficients in $M_k$ of the form $P(X^{(m)}_j)$ is included in the corresponding set of polynomials of degree $\leq d$ of the form $P(Y^{(m)}_j)$. Conversely, any $P(Y^{(m)}_j)$ can be viewed as a polynomial of degree $\leq d$ in $(X_1^{(m)}, \cdots, X_2^{(m)}_n)$. Indeed, the real and imaginary parts of $Y^{(m)}_j$ are independent copies of $X^{(m)}_j$. This is clear when the coefficients are arbitrary in $M_k$. However, the results of [5] are stated for self-adjoint coefficients $a_J$ in $M_k$. Then the trick consists in replacing the general coefficients $a_J$ by self-adjoint ones defined by

$$\hat{a}_J = \begin{pmatrix} 0 & a_J \\ a_J^* & 0 \end{pmatrix} \in M_{2k}.$$

Let $\hat{P} = \sum \hat{a}_J \otimes X^J$. One then notes that $\|\hat{P}(s)\| = \|P(s)\|$ and similarly $\|\hat{P}(X^{(m)}_j)\| = \|P(X^{(m)}_j)\|$. Thus by simply passing from $k$ to $2k$ we can deduce the strong convergence for general coefficients, as expressed in (0.1) and (0.2) from the case of self-adjoint coefficients. The following Lemma is well known.

Lemma 0.4. Let $F$ be any scalar valued random variable that is in $L_p$ for all $p < \infty$. Fix $\alpha > 0$. Assume that

$$\sup_{p \geq 1} p^{-\alpha} \|F\|_p \leq \sigma.$$

Then

$$\forall t > 0 \quad \mathbb{P}\{|F| > t\} \leq e \exp - (e\sigma)^{-1/\alpha} t^{1/\alpha}.$$

Proof. By Tchebyshev’s inequality, for any $t > 0$ we have $t^p \mathbb{P}\{|F| > t\} \leq (\sigma p^\alpha)^p$, and hence

$$\mathbb{P}\{|F| > t\} \leq (t^{-1/\alpha} \sigma p^\alpha)^p \leq \exp - p \log (t/\sigma p^\alpha).$$

Assuming $t/(e\sigma) \geq 1$, we can choose $p = (t/(e\sigma))^{1/\alpha}$ and then we find

$$\mathbb{P}\{|F| > t\} \leq \exp -(e\sigma)^{-1/\alpha} t^{1/\alpha}$$

and, a fortiori, the inequality holds. Now if $t/(e\sigma) < 1$, we have $\exp -(e\sigma)^{-1/\alpha} t^{1/\alpha} > e^{-1}$ and hence $e \exp -(e\sigma)^{-1/\alpha} t^{1/\alpha} > 1$ so that the inequality trivially holds. □

We will use concentration of measure in the following form:

Lemma 0.5. There is a constant $c_1(n, d) > 0$ such that for any $k$ and any $P \in M_k \otimes P_d$ with $\|P(c)\| \leq 1$, we have

$$\forall t > 0 \quad \mathbb{P}\{|P(Y^{(m)})| - \mathbb{E}\|P(Y^{(m)})\| > t\} \leq e \exp -(t^{2/d} m^{1/d} / c_1(n, d)).$$
Proof. This follows from a very general concentration inequality for Gaussian random vectors, that can be derived in various ways. We choose the following for which we refer to [7]. Consider any sufficiently smooth function (meaning a.e. differentiable) \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and let \( \mathbb{P} \) denote the canonical Gaussian measure on \( \mathbb{R}^n \). Assuming \( f \in L_p(\mathbb{P}) \) we have
\[
\| f - \mathbb{E} f \|_p \leq (\pi/2) \| D f(x) \|_{L_p(\mathbb{P}(dx) \mathbb{P}(dy))}.
\]
Let \( \gamma(p) \) denote the \( L_p \)-norm of a standard normal Gaussian variable (in particular \( \gamma(p) = \| f \|_p \) for \( f(x) = x_1 \)). Recall that \( \gamma(p) \in O(\sqrt{p}) \) when \( p \rightarrow \infty \). Thus the last inequality implies that there is a constant \( \beta \) such that
\[
\| f - \mathbb{E} f \|_p \leq \beta \sqrt{p} \| D f(x) \|_{L_p(\mathbb{P}(dx))},
\]
where \( \| D f(x) \|_2 \) denotes the Euclidean norm of the gradient of \( f \) at \( x \). Clearly this remains true for any \( f \) on \( \mathbb{C}^n \) (with the gradient computed on \( \mathbb{R}^{2n} \)).

We will apply this to a function \( f \) defined on \( (\mathbb{C}^{m^2})^n \). We need to first clarify the notation. We identify \( \mathbb{C}^{m^2} \) with \( M_m \). Then we define \( f \) on \( (\mathbb{C}^{m^2})^n \) by
\[
f(w_1, \ldots, w_n) = \|g(w_1, \ldots, w_n)\|
\]
with
\[
g(w_1, \ldots, w_n) = P(m^{-1/2}w_1, \ldots, m^{-1/2}w_n, m^{-1/2}w_1^*, \ldots, m^{-1/2}w_n^*).
\]
Note that for this choice of \( f \) the derivative \( D_z \) in any direction \( z \) satisfies \( D_z f \leq \| D_z g \| \) and hence taking the sup over \( z \) in the Euclidean unit sphere, we have pointwise
\[
\| D f \|_2 \leq \sup_z \| D_z g \|.
\]

We now invoke Remark 0.1. Using the bound in that remark, we are reduced to the case when \( P = Y^J \). Then \( D_z g \) is the sum of at most \( d \) terms of the form \( m^{-1/2}a z_i b \) so that \( \| m^{-1/2}a z_i b \| \leq m^{-1/2}\|a\|\|z_i\|\|b\| \) and hence since \( \|z_i\| \leq \|z_i\|_2 \), \( \| m^{-1/2}a z_i b \| \leq m^{-1/2}\|a\|\|b\| \). Recollecting all the terms, this yields a pointwise estimate at the point \( w \in M^*_m \)
\[
\sup_z \| D_z g \| \leq c_3(n, d)m^{-1/2}\sup\{m^{-1/2}w_j \mid 1 \leq j \leq n\}^{d-1}.
\]
Thus we obtain
\[
\| f - \mathbb{E} f \|_p \leq \beta \sqrt{pc_3(n, d)m^{-1/2}} \sup_{1 \leq j \leq n} \| Y_j^{(m)} \|^{d-1} \| p,\]
and a fortiori
\[
\| f - \mathbb{E} f \|_p \leq \beta \sqrt{pc_3(n, d)m^{-1/2}} \sum_{1 \leq j \leq n} \| Y_j^{(m)} \|^{d-1} \| p \leq \beta \sqrt{pc_3(n, d)m^{-1/2}n} \| Y_1^{(m)} \|^{d-1} \| p,\]
Now by general results on integrability of Gaussian vectors (see [7, p. 134]), we know that there is an absolute constant \( c_5 \) such that
\[
\| \| Y_1^{(m)} \|^{d-1} \| p = \| Y_1^{(m)} \|^{d-1} \|_{L_p(d-1)(M_m)} \leq (c_5 \sqrt{p(d-1)} \| E \| Y_1^{(m)} \|)^{d-1}
\]
and since we know that \( E \| Y_1^{(m)} \| \rightarrow 2 \) when \( m \rightarrow \infty \) it follows that \( \| \| Y_1^{(m)} \|^{d-1} \| p \leq (c_{10} \sqrt{p(d-1)})^{d-1} \). Thus we obtain
\[
\| f - \mathbb{E} f \|_p \leq c_4(n, d)m^{-1/2}p^{d/2},\]
and the conclusion follows from the preceding Lemma. \( \square \)
Thus we obtain

\[ E = \|P(Y^{(m)})\| \] and let \( t_m = E \|P(Y^{(m)})\| \), so that we know \( \forall t > 0 \quad P\{F > t + t_m\} \leq \psi_m(t) \) with

\[ \psi_m(t) = e \exp - (t^2/m^{1/d} c_1(n, d)). \]

We have

\[ E \left( \frac{F}{2} - t_m \right) \mathbb{1}_{\{F/2 > t_m\}} = \int_{t_m}^{\infty} P\{F/2 > t\} dt \leq \int_{t_m}^{\infty} P\{F > t + t_m\} dt \leq \int_{t_m}^{\infty} \psi_m(t) dt \]

and hence

\[ E F \mathbb{1}_{\{F/2 > t_m\}} \leq 2 t_m P\{F/2 > t_m\} + 2 \int_{t_m}^{\infty} \psi_m(t) dt. \]  \hspace{1cm} (0.6)

The next result is a consequence of the results of Haagerup and Thorbjørnsen [5] and of them with Schultz [6]. Let us first recall the result from [5] that we crucially need.

**Theorem 0.7** ([5] [6]). Let \( \chi_d(k, m) \) denote the best constant such that for any \( P \in M_k \otimes P_d \) we have

\[ \mathbb{E} \|P(Y_j^{(m)})\| \leq \chi_d(k, m) \|P(c)\|. \]

Then for any \( 0 < \delta < 1/4 \)

\[ \lim_{m \to \infty} \chi_d([m^\delta], m) = 1. \]

**Proof.** By homogeneity we may assume \( \|P(c)\| \leq 1 \). Then by Remark 0.1 we also have

\[ \sum J \|a_J\| \leq c_2(n, d). \] \hspace{2cm} (0.7)

Fix \( \varepsilon > 0 \) and \( t > 1 + \varepsilon \). Consider a function \( \varphi \in C^\infty_c(\mathbb{R}, \mathbb{R}) \) with values in \([0, 1]\) such that \( \varphi = 0 \) on \([-1, 1]\) and \( \varphi(x) = 1 \) for all \( x \) such that \( 1 + \varepsilon < |x| < t \) and \( \varphi(x) = 0 \) for \( |x| > 2t \). Let \( P^{(m)} = P(X_j^{(m)}) \) and \( P^{(\infty)} = P(s_j) \). By Remark 0.3 we can reduce our estimate to the case of a polynomial in \( (X_j^{(m)}) \) with self-adjoint coefficients and with \( (s_j) \) in place of \( (c_j) \). Thus we now assume \( \|P(s)\| \leq 1 \). Clearly we still have a bound of the form (0.7). Then by [6] (and by very carefully tracking the dependence of the various constants in [6]) we have for \( m \geq c_{13}(n, d) \)

\[ \mathbb{E} \left\{ (\tau_k \otimes \tau_m) \varphi(P^{(m)}) \right\} = (\tau_k \otimes \tau) \varphi(P^{(\infty)}) + R_m(\varphi) \] \hspace{2cm} (0.8)

where

\[ |R_m(\varphi)| \leq k^3 m^{-2} c_9(n, d) c_\varepsilon t^3 \] \hspace{2cm} (0.9)

where \( c_\varepsilon \) depends only on \( \varepsilon \). Note \( \varphi(P^{(\infty)}) = 0 \). Therefore

\[ \mathbb{E} \left\{ (\tau_k \otimes \tau_m) \varphi(P^{(m)}) \right\} \leq k^3 m^{-2} c_9(n, d) c_\varepsilon t^3. \] \hspace{2cm} (0.10)

Since \( \|P^{(m)}\| \in (1 + \varepsilon, t) \Rightarrow (\tau_k \otimes \tau_m) \varphi(P^{(m)}) \geq 1/(km) \) by Tchebyshev’s inequality we find

\[ P\{\|P^{(m)}\| \in (1 + \varepsilon, t)\} \leq (km) k^3 m^{-2} c_9(n, d) c_\varepsilon t^3 = k^4 m^{-1} c_9(n, d) c_\varepsilon t^3. \]

Thus we obtain

\[ \mathbb{E} \|P^{(m)}\| \leq 1 + \varepsilon + k^4 m^{-1} c_9(n, d) c_\varepsilon t^4 + \mathbb{E}(\|P^{(m)}\| 1_{\{\|P^{(m)}\| > t\}}). \]
We will now invoke (0.6): choosing \( t = 2t_m = 2\mathbb{E}\|P^{(m)}\| \) we find
\[
\mathbb{E}\|P^{(m)}\| \leq 1 + \varepsilon + k^4m^{-1}c_9(n, d)c_\varepsilon t_m^4 + 2t_m\psi_m(t_m) + 2\int_{t_m}^{\infty} \psi_m(t) dt.
\]
Now by (0.7) and by H"older we have
\[
t_m \leq c_2(n, d) \sup_j \mathbb{E}\|X^{(m)}_j\| \leq c_2(n, d) \sup_j \mathbb{E}\|X_1^{(m)}\|^{\|J\|}
\]
but by a well known result essentially due to Geman [3] (cf. e.g. [14] Lemma 6.4), for any \( d \) we have
\[
c_9(d) = \sup_m \mathbb{E}(\|X_1^{(m)}\|^d) < \infty.
\]
Therefore we have \( t_m \leq c'_2(n, d) \). We may assume \( t_m > 1 \) (otherwise there is nothing to prove) and hence we have proved
\[
\mathbb{E}\|P^{(m)}\| \leq 1 + \varepsilon + k^4m^{-1}c'_9(n, d)c_\varepsilon + 2c'_2(n, d)\psi_m(1) + 2\int_{1}^{\infty} \psi_m(t) dt.
\]
Thus for any \( \varepsilon > 0 \) we conclude
\[
\chi_d(k, m) \leq 1 + \varepsilon + k^4m^{-1}c'_9(n, d)c_\varepsilon + 2c'_2(n, d)\psi_m(1) + 2\int_{1}^{\infty} \psi_m(t) dt.
\]
From this estimate it follows clearly that for any \( 0 < \delta < 1/4 \)
\[
\limsup_{m \to \infty} \chi_d([m^d], m) \leq 1 + \varepsilon.
\]
\[\square\]

**Lemma 0.8.** Fix integers \( d, k, m \). Let \( \chi_d(k, m) \) denote the best constant appearing in Theorem (0.7). Then for any \( \varepsilon > 0 \) there are positive constants \( c_7(n, d, \varepsilon) \) and \( c_8(n, d, \varepsilon) \) such that if \( k \) is the largest integer such that \( m \geq c_7(n, d, \varepsilon)k^2d \) the set
\[
\Omega_{d, \varepsilon}(m) = \{ \forall P \in M_k \otimes P_d \ | \ \|P(Y^{(m)}(\omega))\| \leq (1 + \varepsilon)(\chi_d(k, m) + \varepsilon)\|P(c)\| \}
\]
satisfies
\[
\mathbb{P}(\Omega_{d, \varepsilon}(m)^c) \leq e \exp \left(-m^{1/d}/c_8(n, d, \varepsilon)\right).
\]
\[\text{Proof.}\] For any \( P \in M_k \otimes P_d \) with \( \|P(c)\| \leq 1 \), we have by Lemma (0.5) for any \( t > 0 \)
\[
\mathbb{P}\{\|P(Y^{(m)})\| > t + \chi_d(k, m)\} \leq e \exp(-(t^{2/d}m^{1/d}/c_1(n, d))).
\]
Let \( \mathcal{N} \) be a \( \delta \)-net in the unit ball of the space \( P_d \) equipped with the norm \( P \mapsto \|P(c)\| \). Since \( \dim(M_k \otimes P_d) = c_6(n, d)k^2 \) for some \( c_6(n, d) \), it is known that we can find such a net with
\[
|\mathcal{N}| \leq (1 + 2/\delta)^{c_6(n, d)k^2}.
\]
Let \( \Omega_1 = \{ \forall a \in \mathcal{N}, \|P(Y^{(m)})\| > t + \chi_d(k, m)\} \). Clearly
\[
\mathbb{P}(\Omega_1) \leq |\mathcal{N}|e \exp(-(t^{2/d}m^{1/d}/c_1(n, d))) \leq e \exp\left(2c_6(n, d)\delta^{-1}k^2 - t^{2/d}m^{1/d}/c_1(n, d)\right).
\]
Thus if we choose \( m \) so that (roughly) \( t^{2/d}m^{1/d}/c_1(n, d) = 4c_6(n, d)\delta^{-1}k^2 \) we find an estimate of the form
\[
\mathbb{P}(\Omega_1) \leq e^{\exp\left(-t^{2/d}m^{1/d}/2c_1(n, d)\right)}.
\]
Note that on the complement of \( \Omega_1 \) we have
\[
\forall P \in \mathcal{N} \quad \|P(Y^{(m)})\| \leq t + \chi_d(k, m).
\]
By a well known result (see e.g. [10, p. 49-50]) we can pass from the set \( \mathcal{N} \) to the whole unit ball at the cost of a factor close to 1, namely we have on the complement of \( \Omega_1 \)
\[
\forall P \in M_k \otimes P_d \quad \|P(Y^{(m)})\| \leq (1 - \delta)^{-1}(t + \chi_d(k, m))\|P(c)\|.
\]
Thus if we set \( t = \varepsilon \) and \( \delta \approx \varepsilon/2 \), we obtain that if \( m = c_7(n, d, \varepsilon)k^{2d} \) we have a set \( \Omega_1' = \Omega_1^\varepsilon \) with
\[
\mathbb{P}(\Omega_1') > 1 - e^{\exp\left(-\varepsilon^{2/d}m^{1/d}/2c_1(n, d)\right)},
\]
such that for any \( \omega \in \Omega_1' \) we have
\[
\forall P \in M_k \otimes P_d \quad \|P(Y^{(m)}(\omega))\| \leq (1 + \varepsilon)(\chi_d(k, m) + \varepsilon)\|P(c)\|.
\]

**Theorem 0.9.** For each \( j \) let \( u_j(\alpha)(\omega) \) be the block direct sum defined by
\[
u_j(\alpha)(\omega) = \oplus_{m \in \alpha} Y_j^{(m)}(\omega) \in \oplus_{m \in \alpha} M_m.
\]
Let \( \alpha \subset \mathbb{N} \) be any infinite subset. Then for almost every \( \omega \) the \( C^* \)-algebra \( A(\alpha)(\omega) \) generated by \( \{u_j(\alpha)(\omega) \mid j = 1, 2, \ldots \} \) is 1-subexponential but is not exact.
Moreover, these results remain valid in the self-adjoint setting, if we replace \( u_j(\alpha)(\omega) \) by
\[
\hat{u}_j(\alpha)(\omega) = \oplus_{m \in \alpha} X_j^{(m)}(\omega) \in \oplus_{m \in \alpha} M_m.
\]

**Proof.** We will give the proof for \( \alpha = \mathbb{N} \). The proof for a general subset is identical. By Lemma [18] for any degree \( d \) and \( \varepsilon > 0 \) we have
\[
\sum_m \mathbb{P}(\Omega_{d, \varepsilon}^{(m)}) < \infty.
\]
Therefore the set \( V_{d, \varepsilon} = \liminf_{m \to \infty} \Omega_{d, \varepsilon}^{(m)} \) has probability 1. Furthermore (since we may use a sequence of \( \varepsilon \)'s tending to zero) we have
\[
\mathbb{P}(\cap_{d, \varepsilon} V_{d, \varepsilon}) = 1.
\]
Now if we choose \( \omega \) in \( \cap_{d \geq 1, \varepsilon > 0} V_{d, \varepsilon} \) the operators \( u_j(\alpha)(\omega) \) satisfy the assumptions of Theorem [12] and hence \( A(\alpha)(\omega) \) is 1-subexponential.

Note that
\[
\sum_{1}^{n} \tau(|c_j|^2) = n.
\]
Since \( \|u_j\| = \sup_m \|u_j(m)\| \) and \( \lim_{m \to \infty} \|u_j(m)(\omega)\| = 2 \) a.s. we know that \( \sup_m \|u_j(m)\| < \infty \) a.s. Therefore, by concentration (or by the integrability of the norm of Gaussian random vectors, see [7])
\[
\mathbb{E}(\|u_j\|^2) = \mathbb{E}(\sup_m \|u_j(m)\|^2) < \infty,
\]
and since the \(u_j\)'s have the same distribution \(\mathbb{E}(\|u_j\|^2) = \mathbb{E}(\|u_1\|^2)\). By Fatou’s lemma

\[
\mathbb{E}\liminf_{n \to \infty} n^{-1} \sum_{i=1}^{n} \|u_j\|^2 \leq \liminf_{n \to \infty} n^{-1} \sum_{i=1}^{n} \|u_j\|^2 = \mathbb{E}(\|u_1\|^2) < \infty
\]

and hence there is a measurable set \(\Omega_0 \subset \Omega\) with \(\mathbb{P}(\Omega_0) = 1\) such that

\[
\forall \omega \in \Omega_0 \quad \liminf_{n \to \infty} n^{-1} \sum_{i=1}^{n} \|u_j(\omega)\|^2 < \infty.
\]

Therefore if we choose \(\omega\) in the intersection of \(\cap_{d,e} V_{d,e} \cap \Omega_0\) (which has probability 1) we find almost surely

\[
\left\| \sum_{i=1}^{n} u_j(\omega) \otimes \tilde{c}_j \right\|_{A \otimes \min C} \leq 2 \max\left\{ \left\| \sum_{i=1}^{n} u_j u_j^* \right\|^{1/2}, \left\| \sum_{i=1}^{n} u_j^* u_j \right\|^{1/2} \right\} \leq 2 \left( \sum_{i=1}^{n} \|u_j(\omega)\|^2 \right)^{1/2} \in O(\sqrt{n})
\]

so that (0.4) is satisfied and \(A(\alpha)(\omega)\) is not exact.

Lastly, since \(\{\hat{u}_j(\alpha)(\omega) \mid j \in \alpha\}\) has the same distribution as \(\{\sqrt{2} \Re u_j(\alpha)(\omega) \mid j \in \alpha\}\) the random \(C^*\)-algebra they generate has “the same distribution” as \(A(\alpha)(\omega)\), whence the last assertion. \(\square\)

**Remark 0.10.** It seems clear that our results remain valid if we replace \((Y_j^{(m)})\) by an i.i.d. sequence of uniformly distributed \(m \times m\) unitary matrices, but, at the time of this writing, we have not yet been able to extract the suitable estimates (as in Theorem 0.7) from the proof of Collins and Male [2] that strong convergence holds in this case. However, while this would simplify our example, by eliminating the need for estimates of \(\|u_j\|\), it apparently would not significantly change the picture.

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