Fast convergence techniques in the study of Lie groupoid representations

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Abstract

In this short communication of results whose detailed proofs will appear elsewhere, we outline an inductive method for constructing effective representations of proper Lie groupoids. Our method relies on an analytic recursive averaging technique for groupoid connections.

Introduction

The theory of Lie groupoid representations is still a poorly understood subject. On the one hand, it is known that some of the foundational results in the classical theory of Lie group representations, for example the Peter–Weyl theorem, cease to be valid in the more general setting of groupoids [LO01, JM13]. On the other hand, new notions emerge in this setting which do not have non-trivial counterparts in the classical theory, which are just beginning to be explored. The notion of effective representation, discovered by the author, is one such. It plays a major role in the Tannakian duality theory of proper Lie groupoids [Tre10]. Its relevance, however, is not limited to that context.

Summarily speaking, an effective representation is one whose kernel solely consists of ineffective arrows, i.e., isotropic arrows whose infinitesimal effects [Tre15, §1] are trivial. On the basis of the author’s work [Tre10 §4], it is possible to show that whenever a proper Lie groupoid admits effective representations, it can be presented as an extension, by a locally trivial bundle of compact Lie groups, of a Lie groupoid which is Morita equivalent to the translation groupoid associated to some compact Lie group action. This result, which should be regarded as a generalization of the classical presentation theorem for effective orbifolds [MM03, §2.4], features a short exact sequence of Lie groupoids which is strongly reminiscent of that described by Moerdijk [Moe03] in the regular case;

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it is tempting to speculate that such result may lead to a classification theorem analogous to that of loc.cit. (We refer the reader to the introduction of [Tre15] for a discussion on related issues.) Information about the existence of effective representations bears information about the existence of faithful representations as well. In fact, every effectively representable, proper Lie groupoid which is essentially connected—i.e. gives rise to a connected orbit space—and which, over at least one base point, has no non-identical ineffective arrows must be faithfully representable. Existence of faithful representations is a basic question in representable equivariant $K$-theory [EM09]; very few explicit results about the existence of such representations are known to date, most of which pertain to the étale case [LO01, HM04].

Our approach to the construction of effective representations stems from the following remarks. If $G$ is any Lie group acting smoothly on a manifold $M$, the correspondence $(g, m) \mapsto T_m(g \cdot)$ defines an effective representation of the translation groupoid $G \ltimes M \Rightarrow M$ on the tangent bundle $TM$. This representation turns out to be an instance of what we call the effect of a multiplicative connection (see definitions below). The effect of a multiplicative connection is always an effective representation. This fact provides the basic link between Lie groupoid representation theory and Lie groupoid connection theory. The question of when a Lie groupoid admits multiplicative connections presents itself as a natural attack point on understanding the representation theory of Lie groupoids. This question is interesting in its own right, as multiplicative connections enter as a key ingredient in several constructions [Beh05, Tan06, CSS], as in the case of representations, very few explicit existence criteria are known.

The present note is partly intended to serve as an introduction to our recent preprint [Tre]. Its main purpose is to provide a rigorous description of some new applications of the results of loc.cit., which we simply announce below without proof. These new applications include an inductive method for the construction of multiplicative connections on source-proper Lie groupoids; it will be pointed out that the general case reduces to the solution of a series of extension problems which only involve the regular case.

**Overall conventions.** In what follows, all manifolds will be $(C^\infty)$-differentiable, of constant dimension, separated, and will possess a countable basis of open sets. All maps between manifolds, as well as all vector bundles, will be differentiable. For any vector bundle $E$ over a manifold $X$, we shall let $\Gamma^\infty(X; E)$ denote the vector space formed by all differentiable cross-sections of $E$. We shall regard $\Gamma^\infty(X; E)$ as a Fréchet space, its topology being that of uniform convergence on compact sets up to all orders of derivation (this being usually known as the $C^\infty$-topology). We shall assume familiarity with the theory of Lie groupoids at the level say of Chapters 5–6 of the standard textbook [MM03].

**Groupoid representations and multiplicative connections**

Let $\Gamma \Rightarrow M$ be a Lie groupoid. Let $s, t : \Gamma \to M$ denote the groupoid source, resp., target map. Let $E$ be a vector bundle over $M$. A pseudo-representation
of $\Gamma \rightrightarrows M$ on $E$ is a vector bundle morphism from $s^*E$ to $t^*E$, in other words, a global cross-section of the vector bundle $L(s^*E, t^*E)$. To each arrow $g \in \Gamma$, a pseudo-representation $\lambda : s^*E \to t^*E$ assigns a linear map $\lambda_g : E_{sg} \to E_{tg}$ between the fibers of $E$ corresponding to the source and to the target of $g$. We call $\lambda$ invertible if $\lambda_g$ is for each $g$ a linear isomorphism of $E_{sg}$ onto $E_{tg}$. We call $\lambda$ unital if $\lambda_{1_x} = \text{id}_{E_x}$ for all $x$ in $M$. We call $\lambda$ a representation if $\lambda$ is unital and the identity $\lambda_{g'g} = \lambda_{g'} \lambda_g$ holds for every pair of arrows $g', g$ which are composable (i.e. satisfy the condition $sg' = tg$).

A connection on $\Gamma \rightrightarrows M$ is a right splitting $\eta$ for the following short exact sequence of morphisms of vector bundles over the manifold $\Gamma$:

$$0 \longrightarrow \ker ds \longrightarrow T\Gamma \xrightarrow{ds \eta} s^*TM \longrightarrow 0.$$ 

We identify $\eta$ with the subbundle $H = \text{im} \eta$ of $T\Gamma$, and refer to $\eta = (ds|H)^{-1}$ as the horizontal lift associated to $H$, also written $\eta^H$. Letting $1 : M \to \Gamma$ denote the groupoid unit map $x \mapsto 1_x$, we call $\eta$ unital if the condition $\eta_{1_x} = T_x1$ is satisfied for all $x$ in $M$. Lie groupoids always admit unital connections. One can compose the horizontal lift $\eta^H : s^*TM \to T\Gamma$ associated to $H$ with the vector bundle morphism $dt : T\Gamma \to t^*TM$ and thus obtain a pseudo-representation $\lambda^H$ of $\Gamma \rightrightarrows M$ on $TM$. We call $\lambda^H$ the effect of $H$. By an effective connection we mean one whose effect is a representation.

We let $\text{Conn}(\Gamma)$ denote the space of all connections on $\Gamma \rightrightarrows M$. This is the affine subspace of $\Gamma^{\infty}(\Gamma; L(s^*TM, T\Gamma))$ formed by all those differentiable cross-sections $\eta$ of the vector bundle $L(s^*TM, T\Gamma)$ which are solutions for the equation $ds \circ \eta = \text{id}_{s^*TM}$. We view $\text{Conn}(\Gamma)$ as a Fréchet manifold. We also let $\text{Conu}(\Gamma)$ denote the subset of $\text{Conn}(\Gamma)$ formed by all unital connections.

The tangent groupoid of a Lie groupoid $\Gamma \rightrightarrows M$ is the Lie groupoid $T\Gamma \rightrightarrows TM$ whose structure maps are obtained by differentiating those of $\Gamma \rightrightarrows M$. A connection $H$ on $\Gamma \rightrightarrows M$ is said to be multiplicative if $H \subset T\Gamma$ constitutes a subgroupoid (by necessity, over the whole of $TM$) of the tangent groupoid $T\Gamma \rightrightarrows TM$. Trivially, multiplicative connections are unital. An arbitrary unital connection $H$ on $\Gamma \rightrightarrows M$ is multiplicative if and only if the identity below holds for every composable pair of arrows $g', g$ for every tangent vector $v \in T_{sg}M$.

$$\eta^H_{g'g}v = (\eta^H_g \lambda^H_g v) \eta^H_g v$$

Multiplicative connections are effective, as can be seen by applying the linear map $T_g t$ to both sides of this identity.

Let $\Gamma_{\times}$ denote the submanifold of $\Gamma \times \Gamma$ formed by all pairs of arrows $g, h$ such that $sg = sh$; we call any such pair divisible. Let $q_{\times} : \Gamma_{\times} \to \Gamma$ denote the map $(g, h) \mapsto gh^{-1}$. For any divisible pair of arrows $w_1, w_2$ in the tangent groupoid $T\Gamma \rightrightarrows TM$, let $w_1 \div w_2$ denote the “ratio” $w_1w_2^{-1} = Tq_{\times}(w_1, w_2)$. A unital connection $H$ on $\Gamma \rightrightarrows M$ is multiplicative if and only if for every divisible pair of arrows $g, h$ in $\Gamma \rightrightarrows M$ and for every tangent vector $v \in T_{sg=sh}M$

$$\eta^H_{gh^{-1}}(\lambda^H_h v) = \eta^H_g v \div \eta^H_h v.$$
The averaging operator

Let \( \Gamma \rightrightarrows M \) be a Lie groupoid. We say that a connection \( H \) on \( \Gamma \rightrightarrows M \) is non-degenerate if its effect \( \lambda^H \) is an invertible pseudo-representation. We let \( \text{Coni}(\Gamma) \) denote the subset of \( \text{Conn}(\Gamma) \) formed by all non-degenerate connections.

**Definition 1.** For any non-degenerate connection \( H \) on \( \Gamma \rightrightarrows M \) we set

\[
\delta^H(g, h) := (\eta^H_g \div \eta^H_h) \circ (\lambda^H_g)^{-1} \in L(T_{th}M, T_{gh^{-1}}\Gamma)
\]

for every divisible pair \((g, h) \in \Gamma \times \Gamma\). Letting \( s_\gamma \) denote the map of \( \Gamma \) into \( M \) given by \((g, h) \mapsto th\), we refer to the global cross-section

\[
\delta^H \in \Gamma^\infty(\Gamma_x; L(s_\gamma^*TM, q_\gamma^*T\Gamma))
\]

as the division cocycle associated to \( H \).

From now on, we assume that \( \Gamma \rightrightarrows M \) is proper and thus can be endowed with a normalized Haar system \( \nu = \{\nu_x\} \). Recall that any such system assigns to each base point \( x \) in \( M \) a positive Radon measure \( \nu_x \) on the target fiber \( \Gamma_x = t^{-1}(x) \) in such a way that the following three conditions are satisfied:

(A) For some “differentiably varying” family \( \tau = \{\tau_x\} \) of volume densities on the target fibers, and for some differentiable non-negative function \( c \) on \( M \) with the property that for each compact subset \( K \) of \( M \) the intersection \( \text{supp} \ c \cap \Gamma K \) is compact, one has \( d\nu_x = (c \circ s_x) d\mu_x \) for all \( x \), where \( \mu_x \) denotes the positive Radon measure on \( \Gamma_x \) associated to the volume density \( \tau_x \) and where \( s_x \) denotes the restriction of the map \( s \) to \( \Gamma_x \).

(B) For every arrow \( g \), and for all Borel subsets \( A \) of the target fiber \( \Gamma^s_g \),

\[
\nu_{tg}(gA) = \nu_{sg}(A).
\]

(C) \( \nu_x(\Gamma_x) = 1 \) for all \( x \).

The first condition implies that \( C^0(\Gamma_x) \subset L^1(\nu_x) \). The second condition, left invariance, can be rephrased by saying that for every continuous function \( \varphi \) on \( \Gamma^s_g \)

\[
\int \varphi(g^{-1}h) \ d\nu_{tg}(h) = \int \varphi(k) \ d\nu_{sg}(k).
\]

Let \( f : P \to M \) be a map from some manifold of “parameters” \( P \) into the base of the groupoid. If \( E \) is any vector bundle over \( P \) then, letting \( pr_P \) denote the projection from the fiber product \( P \times_\Gamma \Gamma = \{(y, h) \in P \times \Gamma \mid f(y) = th\} \) onto \( P \), every cross-section \( \vartheta \) of the vector bundle \( pr_P^*E \) can be turned into a cross-section \( \int \vartheta \ d\nu \) of \( E \) by ‘integration along the target fibers’:

\[
P \ni y \mapsto (\int \vartheta \ d\nu)(y) := \int \vartheta(y, h) \ d\nu_f(y)(h) \in E_y
\]

(the integrand being a vector-valued differentiable function on \( \Gamma^\infty_f(y) \) with values in the finite-dimensional vector space \( E_y \)). The “integration functional”

\[
\Gamma^\infty(\Gamma; pr_P^*E) \to \Gamma^\infty(P; E), \vartheta \mapsto \int \vartheta \ d\nu
\]

is continuous (as a linear map between Fréchet spaces).
**Definition 2.** Let \( H \) be a non-degenerate connection on \( \Gamma \rightrightarrows M \). For every arrow \( g \) we let \( \hat{\eta}^H_g \) denote the linear map \[ T_{sg}M \ni v \mapsto \hat{\eta}^H_g v := \int_{tk = sg} \delta^H(gk, k)v \, dk \in T_g\Gamma. \] (3)

[This expression makes sense because \( \delta^H(gk, k) \) is a linear map of \( T_{tk = sg}M \) into \( T_{gkk^{-1}}=g\Gamma \) for all \( g \leftarrow k \leftarrow \).] We refer to the global cross-section \[ \hat{\eta}^H \in \Gamma^\infty(\Gamma; L(s^*TM, T\Gamma)) \] (4) as the multiplicative average of \( H \) (relative to our choice of normalized Haar systems on \( \Gamma \rightrightarrows M \)).

**Lemma 3.** The multiplicative average \( \hat{\eta}^H \) of a non-degenerate connection \( H \) is itself the horizontal lift for a unique connection \( \hat{H} \) on \( \Gamma \rightrightarrows M \). The connection \( \hat{H} \) is always unital. It too will be called the multiplicative average of \( H \).

We have the following integral formula for \( \hat{\lambda}^H_g := T_g t \circ \hat{\eta}^H_g \) (the effect of the multiplicative average of \( H \)) in terms of the effect of \( H \).

\[ \hat{\lambda}^H_g = \int_{tk = sg} \lambda^H_g (\lambda^H_k)^{-1} \, dk \] (5)

**Proposition 4.** Let \( \Phi \) be an effective (a fortiori, non-degenerate) connection on a proper Lie groupoid \( \Gamma \rightrightarrows M \). Then its multiplicative average \( \hat{\Phi} \) (relative to any choice of normalized Haar systems) is a multiplicative connection.

A connection \( \Phi \) on a proper Lie group bundle is always effective. (Indeed, by definition, for any such groupoid the target map equals the source map so \( \lambda^\Phi_g = id \) for all \( g \).) More in general, the same is true of any connection on a proper Lie groupoid whose associated Lie algebroid has zero anchor map.

**Corollary 5.** Any proper Lie group bundle—more in general, any proper Lie groupoid whose associated Lie algebroid has zero anchor map—admits multiplicative connections.

The next result, which is a generalization of the preceding one, is a slightly less trivial application of our averaging method. Recall that a regular Lie groupoid is one whose associated anchor map is a vector-bundle morphism of constant rank. The longitudinal tangent bundle of a regular Lie groupoid \( \Gamma \rightrightarrows M \) is the vector subbundle of \( TM \) that coincides with the image of the anchor map.

**Corollary 6.** Every proper, regular, Lie groupoid whose longitudinal tangent bundle is trivializable—in particular, every proper, transitive, Lie groupoid over a parallelizable base manifold—admits multiplicative connections.
The process that to each non-degenerate connection $\Phi$ on an arbitrary proper Lie groupoid $\Gamma \rightrightarrows M$ endowed with a normalized Haar system assigns the corresponding multiplicative average $\hat{\Phi}$ gives rise to a continuous operator

$$\text{Coni}(\Gamma) \to \text{Conu}(\Gamma), \Phi \mapsto \hat{\Phi},$$

which carries effective connections into multiplicative connections. Multiplicative connections lie within the fixed-point set of this operator. Every connection on the line segment $\{\Phi + t(\hat{\Phi} - \Phi) \mid 0 \leq t \leq 1\}$ is effective whenever so is $\Phi$.

**Corollary 7.** For any proper Lie groupoid the space of all multiplicative connections is a strong deformation retract of the space of all effective connections.

Next, for a generic non-degenerate connection $\Phi$, we want to consider the sequence of connections $\hat{\Phi}, \hat{\hat{\Phi}}, \ldots$ which one obtains by repeatedly averaging $\Phi$ (provided this sequence is at all defined), and understand its limiting behavior.

**Fast convergence theorems**

Let $E$ be an arbitrary vector bundle over the base $M$ of a proper Lie groupoid $\Gamma \rightrightarrows M$. Let $\text{Psr}(\Gamma; E)$ denote the space of all pseudo-representations of $\Gamma \rightrightarrows M$ on $E$, i.e., $\Gamma^\infty(\Gamma; L(s^*E, t^*E))$. Also let $\text{Psu}(\Gamma; E)$ denote the subset of $\text{Psr}(\Gamma; E)$ formed by all unital pseudo-representations, and let $\text{Psi}(\Gamma; E) \subset \text{Psr}(\Gamma; E)$ denote the set of all invertible pseudo-representations. Motivated by the formula (5), for every $\lambda \in \text{Psi}(\Gamma; E)$ and for every arrow $g$ we set

$$\hat{\lambda}(g) := \int_{tk=sg} \lambda(gk)\lambda(k)^{-1} \, dk.$$  

It follows from the fundamental properties of Haar integrals depending on parameters recalled previously that $\hat{\lambda}$ belongs to $\text{Psu}(\Gamma; E)$ i.e. is a unital pseudo-representation of $\Gamma \rightrightarrows M$ on $E$.

Let us endow the vector bundle $E$ with some metric $\phi$ of class $C^\infty$ (Riemannian or Hermitian, depending on whether $E$ is real or complex). For each pair of points $x, y$ in $M$, we have a norm $\| \|_{x,y}$ on $L(E_x, E_y)$ given by

$$\| \lambda \|_{x,y} := \sup_{|v|_x=1} |\lambda v|_y$$

for all linear maps $\lambda : E_x \to E_y$. [Here of course $| \cdot |_x$ denotes the norm on $E_x$ given by $|v|_x = \sqrt{\phi_x(v, v)}$.] For every $\lambda \in \text{Psr}(\Gamma; E)$ we set

$$b(\lambda) := \sup_{g \in \Gamma} \| \lambda(g) \|_{sg,tg} \quad \text{and} \quad c(\lambda) := \sup_{(g',g) \in \Gamma \times \Gamma} \| \lambda(g'g) - \lambda(g')\lambda(g) \|_{sg,tg'}.$$

For every $\lambda \in \text{Psu}(\Gamma; E)$ which satisfies the condition $c(\lambda) < 1$ it is possible to show that $\lambda$ is invertible and that the following estimates hold.

$$\| \hat{\lambda}(g) \|_{sg,tg} \leq \frac{b(\lambda)}{1 - c(\lambda)}$$

$$\| \hat{\lambda}(g'g) - \hat{\lambda}(g')\hat{\lambda}(g) \|_{sg,tg'} \leq 2c(\lambda)^2 \frac{b(\lambda)^2}{(1 - c(\lambda))^2}.$$
Lemma 8. Let \( \{b_0, b_1, \ldots, b_l\} \) and \( \{c_0, c_1, \ldots, c_l\} \) be two finite sequences of non-negative real numbers, of length, say, \( l + 1 \geq 2 \). Suppose that for every index \( i \) between 0 and \( l - 1 \) the following implication is true.

\[
c_i < 1 \Rightarrow \begin{cases} 
    b_{i+1} \leq \frac{b_i}{1 - c_i} \text{ and } \\
    c_{i+1} \leq 2c_i^2 \frac{b_i}{1 - c_i} 
\end{cases}
\]

Also suppose that \( b_0 \geq 1 \) and that \( \varepsilon := 6b_0^2c_0 \leq \frac{2}{3} \). Then, the following inequalities hold for every index \( i = 0, 1, \ldots, l \):

\[
c_i \leq \frac{\varepsilon^2}{6b_0^2} \quad \text{and} \quad \frac{b_i}{1 - c_i} \leq \sqrt{3}b_0.
\]

For an arbitrary open subset \( U \) of \( M \) we let \( \lambda \mid U \) denote the pseudo-representation of the open subgroupoid \( \Gamma \mid U \Rightarrow U \) of \( \Gamma \Rightarrow M \) on \( E \mid U \) induced by \( \lambda \). The preceding lemma motivates our next definition.

Definition 9. A unital pseudo-representation \( \lambda \in \text{Psu}(\Gamma; E) \) is nearly multiplicative or a near representation if for each point in \( M \) one can find an invariant open neighborhood \( U = \Gamma U \) with the property that the inequality below holds for some choice of \( C^\infty \) metrics on \( E \mid U \).

\[
c(\lambda \mid U) \leq \frac{1}{9}b(\lambda \mid U)^{-2}
\]

A unital connection \( \Phi \in \text{Conu}(\Gamma) \) is nearly effective if the associated pseudo-representation \( \lambda^\Phi \in \text{Psu}(\Gamma; TM) \), i.e. the effect of \( \Phi \), is a near representation.

Near representations are always invertible, so for a near representation \( \lambda \) it makes sense to consider the pseudo-representation \( \hat{\lambda} \) defined by the averaging formula (7). One can show that \( \hat{\lambda} \) is itself a near representation. One therefore obtains a whole sequence of averaging iterates \( \hat{\lambda}^{(i)} \) of \( \lambda \), which one constructs recursively by setting \( \hat{\lambda}^{(0)} := \lambda \) and \( \hat{\lambda}^{(i+1)} := (\hat{\lambda}^{(i)})^\wedge \).

Theorem 10 (Fast convergence theorem I). Let \( \Gamma \Rightarrow M \) be a proper Lie groupoid. Let \( \lambda \in \text{Psu}(\Gamma; E) \) be a unital pseudo-representation of \( \Gamma \Rightarrow M \) on some vector bundle \( E \) over \( M \). Suppose that \( \lambda \) is nearly multiplicative. Then, for any choice of normalized Haar systems, the sequence of successive averaging iterates of \( \lambda \) obtained by recursive application of the formula (7)

\[
\hat{\lambda}^{(0)} := \lambda, \quad \hat{\lambda}^{(1)} := \hat{\lambda}, \ldots, \quad \hat{\lambda}^{(i+1)} := (\hat{\lambda}^{(i)})^\wedge, \ldots \in \text{Psu}(\Gamma; E)
\]

converges within the Fréchet space \( \text{Psr}(\Gamma; E) \) (endowed with the \( C^\infty \)-topology) to a unique representation \( \hat{\lambda}^{(\infty)} \) of \( \Gamma \Rightarrow M \) on \( E \).

The formula (5) says that \( \lambda^\Phi \) equals \( (\lambda^\Phi)^\wedge \) for any non-degenerate connection \( \Phi \). We thus have that any nearly effective connection \( \Phi \) gives rise by recursive averaging to a sequence of nearly effective connections \( \hat{\Phi}^{(i)} \).
Theorem 11 (Fast convergence theorem II). Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid. Let $\Psi \in \text{Conu}(\Gamma)$ be a unital connection on $\Gamma \rightrightarrows M$. Suppose that $\Psi$ is nearly effective. Then the sequence of successive averaging iterates of $\Psi$, constructed by recursive application of the averaging operator (arising from an arbitrary choice of normalized Haar systems),

$$\hat{\Psi}(0) := \Psi, \hat{\Psi}(1) := \hat{\Psi}, \ldots, \hat{\Psi}(i+1) := \hat{\Psi}(\hat{\Psi}(i)), \ldots \in \text{Conu}(\Gamma)$$

converges within the affine Fréchet manifold $\text{Conn}(\Gamma)$ to a unique multiplicative connection $\hat{\Psi}(\infty)$ on $\Gamma \rightrightarrows M$.

Applications of the fast convergence theorem

Let $\Gamma \rightrightarrows M$ be a proper Lie groupoid. The orbit $\Gamma x$ corresponding to any base point $x$ is a smooth submanifold of $M$. Let $q$ be an arbitrary integer between zero and $\dim M$. The set of all base points that lie on $q$-dimensional orbits

$$M_q := \{x \in M \mid \dim \Gamma x = q\}$$

is an (invariant, smooth) submanifold of $M$. The restriction $\Gamma \mid M_q \rightrightarrows M_q$ is a regular (proper) Lie groupoid. For each invariant submanifold $S$ of $M$, any multiplicative connection $\Phi$ on $\Gamma \rightrightarrows M$ induces by restriction a multiplicative connection on $\Gamma \mid S \rightrightarrows S$, which we shall designate $\Phi \mid S$ and call the induced connection. Recall that $\Gamma \rightrightarrows M$ is said to be source proper if for each compact set $K \subset M$ the inverse image $s^{-1}(K)$ is compact.

Proposition 12. Let $\Gamma \rightrightarrows M$ be a source-proper Lie groupoid. Let $U$ be an invariant open neighborhood of the closed set

$$M_{\leq q - 1} := \{x \in M \mid \dim \Gamma x \leq q - 1\}.$$

Suppose given a multiplicative connection $\Phi$ on $\Gamma \mid U \rightrightarrows U$ and a multiplicative connection $\Psi$ on $\Gamma \mid M_q \rightrightarrows M_q$ with the property that on $\Gamma \mid M_q \cap U \rightrightarrows M_q \cap U$ the connection induced by $\Phi$ coincides with the restriction of $\Psi$. Let $V$ be an invariant open neighborhood of $M_{\leq q - 1}$ such that $V \subset U$. Then, for some invariant open neighborhood $U'$ of $M_{\leq q}$ containing $V$, there exists on $\Gamma \mid U' \rightrightarrows U'$ a multiplicative connection $\Phi'$ such that $\Phi' \mid V = \Phi \mid V$ and such that $\Phi' \mid M_q = \Psi$.

We shall in particular refer to $M_0$ as the semi-fixed locus of $\Gamma \rightrightarrows M$ (each of its points lies on a discrete orbit and is thus locally invariant). Taking $q = 0$ and $U = V = \emptyset$ in the proposition, from Corollary 10 we deduce:

Corollary 13. Let $\Gamma \rightrightarrows M$ be a source-proper Lie groupoid. The semi-fixed locus of $\Gamma \rightrightarrows M$ admits some invariant open neighborhood $U$ for which there exist multiplicative connections on $\Gamma \mid U \rightrightarrows U$. 

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Proposition 12, which is a consequence of Theorem 11, constitutes the analytic basis of an inductive technique for the construction of multiplicative connections. Corollary 13 is the start of the induction. The inductive step requires solving the following extension problem: Given a regular, source-proper Lie groupoid $\Delta \Rightarrow N$, an invariant open subset $A$ of $N$, a multiplicative connection $\Psi$ on $\Delta | A \Rightarrow A$ and an invariant open subset $B$ of $N$ with $\overline{B} \subset A$, under what conditions is it possible to extend $\Psi | B$ to a multiplicative connection defined on all of $\Delta \Rightarrow N$? It is not hard to see that the extension problem can always be solved when the groupoid orbits’ dimension is one, provided the groupoid is source connected (i.e. has connected source fibers). As a result:

**Corollary 14.** Let $\Gamma \Rightarrow M$ be a source-proper and source-connected Lie groupoid. It is possible to choose an invariant open neighborhood $U$ of the set

$$M_{\leq 1} = \{ x \in M \mid \dim \Gamma x \leq 1 \}$$

so that the Lie groupoid $\Gamma | U \Rightarrow U$ admits multiplicative connections. In particular, any source-connected, source-proper, Lie groupoid all of whose orbits have dimension not greater than one admits multiplicative connections.

Since the effect of a multiplicative connection is always an effective representation, any result concerning the existence of multiplicative connections will automatically imply a corresponding result concerning the existence of effective representations. The results thus obtained will in general allow of further improvement, both from the point of view of their assumptions and from the point of view of their conclusions; this will fall out by application of standard representation-theoretic techniques, such as Morita equivalence, or other ad hoc arguments. By way of example, we point out that Corollary 13 lends itself to the following generalization; by the minimal dimension locus of a Lie groupoid, we mean the set of all base points that lie on orbits of minimal dimension.

**Corollary 15.** The minimal dimension locus of any proper Lie groupoid $\Gamma \Rightarrow M$ admits an invariant open neighborhood $U$ with the property that the restricted groupoid $\Gamma | U \Rightarrow U$ is effectively representable.

There is a different order of applications of Theorem 11 which is also worth mentioning. Let $\Phi$ be a multiplicative connection on a, say, compact Lie groupoid. If $\Psi$ is another multiplicative connection which is sufficiently close to $\Phi$, all the connections on the line segment $\{ \Psi + t(\Phi - \Psi) \mid 0 \leq t \leq 1 \}$ will be nearly effective. Therefore, by the fast convergence theorem, there will be a smooth path of multiplicative connections deforming $\Psi$ into $\Phi$. The space of multiplicative connections on any compact Lie groupoid is therefore semi-locally contractible. This remark illustrates how Theorem 11 can be used to study the topological properties of the space of multiplicative connections.

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