A PARTIALLY OBSERVED NON-ZERO SUM DIFFERENTIAL GAME OF FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND ITS APPLICATION IN FINANCE

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Abstract. In this article, we study a class of partially observed non-zero sum stochastic differential game based on forward and backward stochastic differential equations (FBSDEs). It is required that each player has his own observation equation, and the corresponding Nash equilibrium control is required to be adapted to the filtration generated by the observation process. To find the Nash equilibrium point, we establish the maximum principle as a necessary condition and derive the verification theorem as a sufficient condition. Applying the theoretical results and stochastic filtering theory, we obtain the explicit investment strategy of a partial information financial problem.

1. Introduction.

1.1. Historical contribution. The general theory of backward stochastic differential equation (BSDE) was first introduced by Pardoux and Peng [17]. For the BSDE coupled with a forward stochastic differential equation (SDE), it is the so-called forward and backward stochastic differential equation (FBSDE), which has important applications in many areas in our society. In stochastic control area, the Hamiltonian system is one of the form of FBSDEs. More essentially in financial market, the famous Black-Scholes option pricing formula can be deduced by a certain FBSDE. Some research based on FBSDE is surveyed by Ma and Yong [12].

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In stochastic control theory, one can use control to reach a maximum or minimum objection based on stochastic differential system. Peng [18] firstly considered the maximum principle of convex domain forward-backward stochastic control system. Xu [33] dealt with a case that the control domain does not need to be convex and there is no control variable in diffusion coefficients in the forward equation. For recent progress for the maximum principle of forward-backward stochastic control systems, refer to Yong [34], Wu [30], Hu [8] and Nie et al. [15]. All these previous work were based on the “complete information” case, meaning that the control variable is adapted to the original full filtration. In reality, there are many cases the controller can only obtains “partial information”, reflecting in mathematics that the control variable is adapted to a filtration smaller than the complete one. It can be generated by a Brownian motion or an observation process available to the controller. Based on this phenomenon, Xiong and Zhou [32] dealt with a mean-variance problem in financial market that the investor’s optimal portfolio is only based on the stock and bond process he observed. This assumption of partial information is indeed natural in financial market.

Wang and Wu [23] considered the Kalman-Bucy filtering equation of FBSDE system. Huang, Wang and Xiong [9] dealt with the backward stochastic control system under partial information. Wang and Wu [24], Wu [29], Shi and Wu [19], Wang, Wu and Xiong [25] solved the partially observed case of forward and backward stochastic control system.

The game theory was firstly constructed by Von Neumann and Morgenstern [22]. Nash [14] made the fundamental contribution in non-cooperate games. The notion of the equilibrium point was defined in these work. According to that, many articles on forward stochastic differential games appeared, such as Hamadène [6], El Karoui and Hamadène [5], Wu [28], An and Øksendal [1]. For the backward system, Yu and Ji [36] studied the linear quadratic (LQ) case. Wang and Yu [26] gave the maximum principle of the general backward case. Øksendal and Sulem [16], Hui and Xiao [10] had a research on the maximum principle of forward-backward system. Recently, Tang and Meng [21] solved the partial information case of zero-sum forward and backward system. Wang and Yu [27] solved the partial information case of non-zero sum backward system.

In our article, we generate the control system in [25] to the non-zero sum game system, and allow control processes to enter into observation equations. More importantly, we suppose every player has his own observation equation. In Section 1, we introduce some historical contributions and give the problem formulation. In Section 2, we establish a necessary condition (maximum principle) and a sufficient condition (verification theorem) for the existence of the Nash equilibrium point. In Section 3, we consider a practical financial investment problem and use theoretical results in Section 2 to obtain the Nash equilibrium point and derive the explicit observable solution for the investment problem.

1.2. Basic Notions. Throughout our article, we denote by \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) the complete probability space, on which \((W(\cdot), Y_1(\cdot), Y_2(\cdot))\) is a standard 3-dimensional \(\mathcal{F}_t\) Brownian motion. Let \(\mathcal{F}_t^W, \mathcal{F}_t^1, \mathcal{F}_t^2\) be the natural filtration generated by \(W(\cdot), Y_1(\cdot), Y_2(\cdot)\), respectively. We set \(\mathcal{F}_t = \mathcal{F}_t^W \otimes \mathcal{F}_t^1 \otimes \mathcal{F}_t^2.\) For fixed terminal time \(T, \mathcal{F} = \mathcal{F}_T.\) We denote the 1-dimensional Euclidean space by \(\mathbb{R}\), the Euclidean norm by \(|\cdot|\), the transpose of matrix \(A\) by \(A^\tau\), and the partial derivative of function \(f(\cdot)\) with respect to \(x\) by \(f_x(\cdot)\). We also use \(L^2_{\mathcal{F}}(0, T; S)\) to denote the set of \(S\)-valued, \(\mathcal{F}_t\)-adapted square integrable process (i.e. \(\mathbb{E}\int_0^T |x(t)|^2 \, dt < \infty\)), and the \(L^2_{\mathcal{F}}(\Omega; S)\) representing the set of \(S\)-valued, \(\mathcal{F}\)-measured square integrable random variable. In
the following discussion, all processes are 1-dimensional unless specifically stated otherwise.

1.3. **Problem formulation.** We consider a partially observed stochastic differential game problem of forward-backward stochastic systems, focusing on the necessary and sufficient conditions for the existence of open-loop Nash equilibrium point.

We formulate the controlled forward and backward stochastic differential equation (FBSDE) as

\[
\begin{aligned}
  dx(t) &= b(t, x(t), v_1(t), v_2(t))dt + \sigma(t, x(t), v_1(t), v_2(t))dW(t) \\
  &\quad + \sigma_1(t, x(t), v_1(t), v_2(t))dW_1^\tau_1,\tau_2(t) \\
  &\quad + \sigma_2(t, x(t), v_1(t), v_2(t))dW_2^\tau_1,\tau_2(t), \\
  -dy(t) &= f(t, x(t), y(t), z(t), z_1(t), z_2(t), v_1(t), v_2(t))dt - z(t)dW(t) \\
  x(0) &= x_0, \quad y(T) = g(x(T)),
\end{aligned}
\]

where \( v_1(\cdot), v_2(\cdot) \) are two control processes taking values in convex sets \( U_1 \subset \mathbb{R}, U_2 \subset \mathbb{R} \) respectively. \( W_1^\tau_1,\tau_2(\cdot) \) and \( W_2^\tau_1,\tau_2(\cdot) \) are controlled stochastic processes taking values in \( \mathbb{R} \). \( b, \sigma, \sigma_1, \sigma_2 : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}, f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}, g : \Omega \times \mathbb{R} \to \mathbb{R} \) are continuous maps. \( x_0 \in \mathbb{R}, g(x(T)) \) is an \( \mathcal{F}_T \) measurable square integrable random variable. (1) can be seen as a generalization of [25] from the control system to the game system. Here for simplicity, we omit the variable \( \omega \) in each process.

We regard \( v_1(\cdot), v_2(\cdot) \) as two strategies of players 1 and 2. Both players are not able to observe the processes \( x(\cdot), y(\cdot), z_1(\cdot), z_2(\cdot) \) directly. However, they can observe their own related processes \( Y_1(\cdot), Y_2(\cdot) \), which satisfy the following equation\(^1\)

\[
\begin{aligned}
  dY_i(t) &= h_i(t, x(t), v_1(t), v_2(t))dt + dW_1^\tau_1,\tau_2(t), \\
  Y_i(0) &= 0 \quad (i = 1, 2),
\end{aligned}
\]

where \( h_i : \Omega \times [0, T] \times \mathbb{R} \times U_1 \times U_2 \to \mathbb{R}, i = 1, 2 \) is a continuous map. We define the filtration \( \mathcal{F}_t^i = \sigma \{ Y_i(s) | 0 \leq s \leq t \} \), \( i = 1, 2 \) as the information that player \( i \) obtains at time \( t \), and the admissible control \( v_i(\cdot) \) as

\[
\begin{aligned}
  v_i(\cdot) &\in \mathcal{U}_i = \{ v_i(\cdot) \in U_i | v_i(t) \in \mathcal{F}_t^i \} \quad \text{and} \\
  \sup_{0 \leq t \leq T} \mathbb{E}[|v_i(t)|^3] &< \infty \quad (i = 1, 2),
\end{aligned}
\]

where \( \mathcal{U}_i, i = 1, 2 \) is called the open-loop admissible control set for player \( i \).

**Hypothesis(H1).** Suppose that the functions \( b, \sigma, \sigma_1, \sigma_2, h_1, h_2, f, g \) are continuously differentiable in \( (x, y, z, z_1, z_2, v_1, v_2) \). The partial derivatives \( b_x, b_y, \sigma_x, \sigma_y, \sigma_{jx}, \sigma_{jy}, h_{1x}, h_{1y}, f_x, f_y, g_x, g_y, i, j = 1, 2 \) are uniformly bounded. Further, we assume that there is a constant \( C \) such that \( |h(t, x, v_1, v_2)| + |\sigma_1(t, x, v_1, v_2)| + |\sigma_2(t, x, v_1, v_2)| \leq C \) for any \( (t, x, v_1, v_2) \in [0, T] \times \mathbb{R} \times U_1 \times U_2 \).

\(^1\)Here we assume that the control variables \( v_1(\cdot), v_2(\cdot) \) are explicitly appeared in the observation function \( h_i(\cdot) \).
From the Hypothesis(H1), we can defined a new probability measure \( \mathbb{P}^{v_1,v_2} \) by

\[
\frac{d\mathbb{P}^{v_1,v_2}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = Z^{v_1,v_2}(t),
\]

(4)

where \( Z^{v_1,v_2}(t) \) is an \( \mathcal{F}_t \)-martingale satisfying

\[
Z^{v_1,v_2}(t) = \exp \left( \sum_{j=1}^{2} \int_0^t h_j(s,x(s),v_1(s),v_2(s))dY_j(s) - \frac{1}{2} \sum_{j=1}^{2} \int_0^t h_j^2(s,x(s),v_1(s),v_2(s))ds \right),
\]

(5)

Equivalently, it can be written in the SDE form

\[
\begin{cases}
    dZ^{v_1,v_2}(t) = \sum_{j=1}^{2} h_j(t,x(t),v_1(t),v_2(t))Z^{v_1,v_2}(t)dY_j(t), \\
    Z^{v_1,v_2}(0) = 1.
\end{cases}
\]

(6)

By using the Girsanov theorem, we transform \( Y_i(\cdot), i=1,2 \) to the classical form of stochastic observation process under new probability \( \mathbb{P}^{v_1,v_2} \). \( (W(\cdot),W_1^{v_1,v_2}(\cdot),W_2^{v_1,v_2}(\cdot)) \) becomes a 3-dimensional standard Brownian motion defined on the new probability space \( (\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t \geq 0},\mathbb{P}^{v_1,v_2}) \), where \( (W_1^{v_1,v_2}(\cdot),W_2^{v_1,v_2}(\cdot)) \) is a 2 dimensional controlled Brownian motion.

Based on the new probability space, we define two cost functionals

\[
J_i(v_1(\cdot),v_2(\cdot))
= \mathbb{E}^{v_1,v_2} \left[ \int_0^T l_i(t,x(t),y(t),z(t),z_1(t),z_2(t),v_1(t),v_2(t))dt + \Phi_i(x(T)) + \gamma_i(y(0)) \right]
\]

\[
= \mathbb{E} \left[ \int_0^T Z^{v_1,v_2}(t)l_i(t,x(t),y(t),z(t),z_1(t),z_2(t),v_1(t),v_2(t))dt + \mathbb{E}[Z^{v_1,v_2}(T)\Phi_i(x(T)) + \gamma_i(y(0))] \right],
\]

(7)

for two players \( i=1,2 \), where \( \mathbb{E}^{v_1,v_2} \) is the corresponding expectation. \( l_i : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U_1 \times U_2 \rightarrow \mathbb{R} \), \( \Phi_i : \Omega \rightarrow \mathbb{R} \), \( \gamma_i : \mathbb{R} \rightarrow \mathbb{R} \) are continuous maps. In this cost functional, it contains the running cost part representing a utility in duration, and the terminal and initial terms representing the restrictions on the endpoints.

**Hypothesis(H2).** Suppose that the functions \( l_i, \Phi_i, \gamma_i, i=1,2 \) are continuously differentiable in \( (x,y,z,z_1,z_2,v_1,v_2) \), \( x,y \) respectively, the partial derivatives \( l_{ix},l_{iy},l_{iz},l_{iz_1},l_{iz_2},l_{iv_1},l_{iv_2},i,j=1,2 \) are bounded by \( C(1 + |y| + |z| + |z_1| + |z_2| + |v_1| + |v_2|) \) where \( C \) is a constant.

For each of the player, his goal is to minimise his own cost functional. Namely, we need to find \( (u_1(\cdot),u_2(\cdot)) \in U_1 \times U_2 \) such that

\[
\begin{cases}
    J_1(u_1(\cdot),u_2(\cdot)) = \min_{v_1(\cdot) \in U_1} J_1(v_1(\cdot),u_2(\cdot)), \\
    J_2(u_1(\cdot),u_2(\cdot)) = \min_{v_2(\cdot) \in U_2} J_2(u_1(\cdot),v_2(\cdot)).
\end{cases}
\]

(8)
In this definition, \((u_1(\cdot), u_2(\cdot))\) is the well-known open-loop Nash equilibrium point of the partially-observed forward-backward non-zero sum system, and \((x, y, z, z_1, z_2, Z)\) is the corresponding equilibrium state process. We denote the whole problem above as Problem\((\text{NEP})\).

In particular, if we set \(J(v_1(\cdot), v_2(\cdot)) = J_1(v_1(\cdot), v_2(\cdot)) = -J_2(v_1(\cdot), v_2(\cdot))\), then (8) is equivalent to
\[
J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)),
\]
for \(\forall (v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\).

In that case, the reward of player 1 is actually the cost of player 2, and the sum is always zero. We can regard it as a special case of the non-zero sum game. We define this problem of our system as Problem\((\text{EP})\).

Remark 1. Note that the formulation of Problem \((\text{NEP})\) is inspired by [3], and it implies that the determination of the control variable \(v_i(\cdot)\) depends on the observations \(Y_i(\cdot)\), while \(Y_i(\cdot)\) is independent of \(v_i(\cdot)\). Therefore a coupled circle will arise and there is an immediate difficulty in handling the problem (see, e.g., [7], [2]). More clearly, if we suppose \(W_j^j(t) = W_j^{v_1,v_2}(t)\) to be an \(\mathcal{F}_t\)-Brownian motion under \(\mathbb{P}\), then the distribution of the observation process \(Y_i(\cdot)\) will depend on the control process. In that way, the admissible control will be adapted to a controlled filtration. Technically, there exists great difficulty in dealing with the model with controlled Brownian motion when applying the variational technique, and in most cases, the filtration should be specified a priori (i.e., independent of the control).

Here, we break through the circulation by Girsanov theorem, making observation process to be an uncontrolled stochastic process and depict the controlled Brownian motion under equivalent probability measure. Along this line, there are many articles discussing about the control systems similar to ours (see, e.g., [24, 29, 25]). Nevertheless, our problem is different, partly due to the fact that we consider the game system and the observation noise \(W_j^{v_1,v_2}(\cdot)\) enters into state equation (1), and the observation coefficient \(h_i(\cdot)\) in (2) contains the control variables. Due to these new features, the derivation of the maximum principle is technically more challenging.

2. Maximum principle. In this section, we will establish the necessary condition (maximum principle) for an open-loop Nash equilibrium point in problem \((\text{NEP})\), and derive a sufficient condition (verification theorem) under certain conditions.

2.1. Variational equation. Let \((v_1(\cdot), v_2(\cdot)) \in \mathbb{L}_{\mathbb{F}^1}^{s}(0, T; \mathbb{R}) \times \mathbb{L}_{\mathbb{F}^2}^{s}(0, T; \mathbb{R})\) be such that \((u_1(\cdot) + v_1(\cdot), u_2(\cdot) + v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\). For any \(\epsilon \in [0, 1]\), we make the variational controls as
\[
\begin{align*}
u_1'(\cdot) &= u_1(\cdot) + \epsilon v_1(\cdot), \\u_2'(\cdot) &= u_2(\cdot) + \epsilon v_2(\cdot).
\end{align*}
\]

Because \(\mathcal{U}_1, \mathcal{U}_2\) are convex sets, we have \((\nu_1'(\cdot), \nu_2'(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2\). We denote \(\phi^u(\cdot), \phi = x, y, z, z_1, z_2, Z(i = 1, 2)\) as the corresponding state trajectories of variation \((u_1', u_2')\) or \((u_1, u_2')\).

It is noteworthy that when using the variational technique, we require the Brownian motion is not affected by the control process. Then our state equation can be
Lemma 2.2. \[ dx(t) = [b(t, \Theta(t)) - \sum_{j=1}^{2} \sigma_j(t, \Theta(t))h_j(t, \Theta(t))]dt + \sigma(t, \Theta(t))dW(t) \]
\[ + \sum_{j=1}^{2} \sigma_j(t, \Theta(t))dY_j(t), \]
\[ -dy(t) = f(t, x(t), y(t), z(t), z_1(t), z_2(t), v_1(t), v_2(t))dt - z(t)dW(t) \]
\[ x(0) = x_0, \quad y(T) = g(x(T)), \]
where \( \Theta(\cdot) = (x(\cdot), v_1(\cdot), v_2(\cdot)), (W(t), Y_1(t), Y_2(t)) \) is \( \mathcal{F}_t \)-Brownian motion under \( \mathbb{P} \).

Under Hypothesis (H1), we can derive the following estimates whose technique are classical (see e.g. [18, 29]). Thus we omit the details and only state the main result for simplicity.

**Lemma 2.1.**
\[
\sup_{0 \leq t \leq T} \mathbb{E}|x(t)|^8 \leq C(1 + \sup_{0 \leq t \leq T} \mathbb{E}|v(t)|^8),
\]
\[
\sup_{0 \leq t \leq T} \mathbb{E}|y(t)|^2 \leq C(1 + \sup_{0 \leq t \leq T} \mathbb{E}|v(t)|^2),
\]
\[
\mathbb{E}\left( \int_0^T |z(t)|^2 dt \right) + \int_0^T |z_1(t)|^2 dt + \int_0^T |z_2(t)|^2 dt \leq C(1 + \sup_{0 \leq t \leq T} \mathbb{E}|v(t)|^2),
\]
\[
\sup_{0 \leq t \leq T} \mathbb{E}|Z^{v_1,v_2}(t)| \leq K,
\]
where \( C, K \) are constants independent of \( \epsilon \).

**Lemma 2.2.**
\[
\sup_{0 \leq t \leq T} \mathbb{E}|x^{u_1}(t) - x(t)|^8 \leq C\epsilon^8, \quad \sup_{0 \leq t \leq T} \mathbb{E}|y^{u_1}(t) - y(t)|^2 \leq C\epsilon^2,
\]
\[
\mathbb{E}\int_0^T |z^{u_1}(t) - z(t)|^2 dt \leq C\epsilon^2, \quad \mathbb{E}\int_0^T |z_1^{u_1}(t) - z_1(t)|^2 dt \leq C\epsilon^2,
\]
\[
\mathbb{E}\int_0^T |z_2^{u_1}(t) - z_2(t)|^2 dt \leq C\epsilon^2, \quad \mathbb{E}\sup_{0 \leq t \leq T} |Z^{u_1}(t) - Z(t)|^2 \leq C\epsilon^2,
\]
for \( i = 1, 2, \) where \( C \) is constant independent of \( \epsilon \).

For notation simplicity, we set
\[ \zeta(t) = \zeta(t, x(t), u_1(t), u_2(t)), \quad \zeta = b, \sigma, \sigma_i, h_i \quad (i = 1, 2), \]
\[ \psi(t) = \psi(t, x(t), y(t), z(t), z_1(t), z_2(t), u_1(t), u_2(t)), \quad \psi = f, l_i \quad (i = 1, 2). \]
We introduce the following variational equations

\[
\begin{align*}
    dx^i_1(t) &= \left\{ [b_x(t) - \sum_{j=1}^{2} (\sigma_{jx}(t)h_j(t) + \sigma_j(t)h_{jx}(t))]x^i_1(t) + b_{vi}(t) \right. \\
    &\quad - \sum_{j=1}^{2} (\sigma_{jv}(t)h_j(t) + \sigma_j(t)h_{jv}(t))]v_i(t) \big\} dt + [\sigma_x(t)x^i_1(t) \\
    &\quad + \sigma_{vi}(t)v_i(t)]dW(t) + \sum_{j=1}^{2} [\sigma_{jx}(t)x^i_1(t) + \sigma_{jv}(t)v_i(t)]dY_j(t), \\
\end{align*}
\]

\(i,j \in \mathbb{N}, \quad i,j = 1,2\)

\[
-dy^i_1(t) = \left\{ f_x(t)x^i_1(t) + f_y(t)y^i_1(t) + f_z(t)z^i_1(t) + \sum_{j=1}^{2} f_{zj}(t)z^j_1(t) \\
    + f_{vi}(t)v_i(t)]dt - z^i_1(t)dW(t) - \sum_{j=1}^{2} z^j_1(t)dY_j(t),
\]

\(x^1_1(0) = 0, \quad y^1_1(T) = g_x(x(T))x^1_1(T) \quad (i = 1, 2),\)

and

\[
\begin{align*}
    dZ^i_1(t) &= \sum_{j=1}^{2} [Z^j_1(t)h_j(t) + Z(t)(h_{jx}(t)x^i_1(t) + h_{jv}(t)v_i(t))]dY_j(t), \\
    Z^i_1(0) &= 0 \quad (i = 1, 2).
\end{align*}
\]

From Hypothesis (H1), we know that \(12\) and \(13\) admit a unique solution.

Next, we introduce the notations

\[
\hat{\phi}^i(t) = \frac{\phi^u(t) - \phi(t)}{\epsilon} - \phi^i_1(t), \quad \text{for} \quad \phi = x, y, z, z_1, z_2, Z \quad (i = 1, 2),
\]

and

\[
\hat{\phi}(t) = \phi^u(t) - \phi(t), \quad \text{for} \quad \phi = x, y, z, z_1, z_2, Z, b, \sigma, \sigma_1, \sigma_2, h_1, h_2, Z \quad (i = 1, 2).
\]

Then, similar to \([25]\), we can also obtain the following estimation.

\textbf{Lemma 2.3.} For \(i,j = 1,2,\)

\[
\begin{align*}
    \lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|x^i_1(t)|^4 &= 0, \quad \lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|Z^i_1(t)|^2 = 0, \\
    \lim_{\epsilon \to 0} \sup_{0 \leq t \leq T} \mathbb{E}|y^i_1(t)|^2 &= 0, \quad \lim_{\epsilon \to 0} \mathbb{E} \int_{0}^{T} |z^i_1(t)|^2 dt = 0, \\
    \lim_{\epsilon \to 0} \mathbb{E} \int_{0}^{T} |z^j_1(t)|^2 dt &= 0.
\end{align*}
\]

\textbf{2.2. Variational inequality.} From the definition of the Nash equilibrium point \((u_1(\cdot), u_2(\cdot))\) in Problem (NEP), it is clear that

\[
\begin{align*}
    \epsilon^{-1} [J_1(u^*_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot))] &\geq 0, \\
    \epsilon^{-1} [J_2(u_1(\cdot), u^*_2(\cdot)) - J_2(u_1(\cdot), u_2(\cdot))] &\geq 0.
\end{align*}
\]
Let \( \Gamma_i(\cdot) = Z_i^1(\cdot)Z_i^{-1}(\cdot), i = 1, 2 \). By Itô’s formula, we obtain

\[
\begin{aligned}
d\Gamma_i(t) &= \sum_{j=1}^{2} [h_{jx}(t)x_i^1(t) + h_{jv_i}(t)v_i(t)]dW_j^{u_{i1},u_{i2}}(t), \\
\Gamma_i(0) &= 0 \quad (i = 1, 2).
\end{aligned}
\] (15)

From (14), we can derive the variational inequality.

**Lemma 2.4.**

\[
\mathbb{E}^{u_{i1},u_{i2}} \left\{ \int_{0}^{T} [l_{ix}(t)x_i^1(t) + l_{iy}(t)y_i^1(t) + l_{iz}(t)z_i^1(t) + \sum_{j=1}^{2} l_{iz_j}(t)z_{j1}(t) + l_i(t)\Gamma_i(t) \right. \\
+ l_{iv_i}(t)v_i(t)]dt + \Phi_{ix}(x(T))x_i^1(T) + \gamma_{iy}(y(0))y_i^1(0) + \Phi_1(x(T))\Gamma_i(T) \left. \right\} \geq 0.
\]

**Proof.** We only consider the case \( i = 1 \). From (7), we have

\[
\begin{aligned}
\epsilon^{-1}[J_1(u_1^{\epsilon}(\cdot),u_2^{\epsilon}(\cdot)) - J_1(u_1(\cdot),u_2(\cdot))] &= \epsilon^{-1}\mathbb{E}\left[ \int_{0}^{T} (Z^{u_1^{\epsilon}}(t)l_{i1}^{u_1^{\epsilon}}(t) - Z(t)l_1(t))dt \\
+ Z^{u_1^{\epsilon}}(T)\Phi_{11}(x(T)) - Z(T)\Phi_1(x(T)) + \gamma_{11}(y(0)) - \gamma_1(y(0)) \right] \geq 0.
\end{aligned}
\]

According to Lemma 2.3 and Hypothesis (H2),

\[
\epsilon^{-1}[\gamma_{11}(y(0)) - \gamma_1(y(0))]
= \int_{0}^{1} \gamma_{1y}(y(0)) + \lambda(y^{u_1^{\epsilon}}(0) - y(0)))d\lambda \left( \frac{y^{u_1^{\epsilon}}(0) - y(0)}{\epsilon} \right) \rightarrow \gamma_{1y}(y(0))y_1^1(0),
\] (16)

and

\[
\begin{aligned}
\epsilon^{-1}\mathbb{E}[Z^{u_1^{\epsilon}}(T)\Phi_{11}(x(T)) - Z(T)\Phi_1(x(T))]
&= \epsilon^{-1}\mathbb{E}[Z^{u_1^{\epsilon}}(T)\Phi_{11}(x(T)) - \Phi_1(x(T))) + \Phi_1(x(T))(Z^{u_1^{\epsilon}}(T) - Z(T))] \\
&= \mathbb{E}[Z^{u_1^{\epsilon}}(T)(\int_{0}^{1} \Phi_{ix}(x(T) + \lambda(x^{u_1^{\epsilon}}(T) - x(T)))d\lambda \left( \frac{x^{u_1^{\epsilon}}(T) - x(T)}{\epsilon} \right) \\
+ \mathbb{E}[\Phi_1(x(T))(Z^{u_1^{\epsilon}}(T) - Z(T))]
\rightarrow \mathbb{E}[Z(T)\Phi_{ix}(x(T))x_i^1(T) + \Phi_1(x(T))Z_i^1(T)].
\end{aligned}
\] (17)

Similarly, we have

\[
\begin{aligned}
\epsilon^{-1}\mathbb{E}\left[ \int_{0}^{T} (Z^{u_1^{\epsilon}}(t)l_{i1}^{u_1^{\epsilon}}(t) - Z(t)l_1(t))dt \\
+ \mathbb{E}\left[ \int_{0}^{T} Z(t)(l_{ix}(t)x_i^1(t) + l_{iy}(t)y_i^1(t) + l_{iz}(t)z_i^1(t) + \sum_{j=1}^{2} l_{iz_j}(t)z_{j1}(t) + l_{iv_i}(t)v_i(t))dt \\
+ l_{iv_i}(t)v_i(t))]dt + \int_{0}^{T} l_i(t)Z_i^1(t)dt.\right.
\end{aligned}
\] (18)

From the definition of \( \Gamma(\cdot) \) and (16)-(18), we derive the variational inequality. \(\square\)
2.3. A necessary condition (maximum principle). In the following, we ignore the superscript of \(W_j^{\alpha_1,\alpha_2}(\cdot), j = 1, 2\) for notational simplification.

We formulate adjoint equations under probability measure \(\mathbb{P}^{\alpha_1,\alpha_2}\) for \(i = 1, 2\):

\[
\begin{align*}
-dP_i(t) &= l_i(t)dt - Q_i(t)dW(t) - \sum_{j=1}^{2} Q_{ji}(t)dW_j(t), \\
\end{align*}
\]

\(P_i(T) = \Phi_i(x(T)),\)

and

\[
\begin{align*}
-dq_i(t) &= \left\{ b_x(t) - \sum_{j=1}^{2} \sigma_j(t)h_{xj}(t)q_i(t) + \sigma_x(t)k_i(t) + \sum_{j=1}^{2} \sigma_{xj}(t)k_{ji}(t) \\
&+ h_{xj}(t)Q_{ji}(t) - f_x(t)p_i(t) - l_{iz}(t)\right\}dt - k_i(t)dW(t) \\
&- \sum_{j=1}^{2} k_{ji}(t)dW_j(t), \\
p_i(0) = -\gamma_i(y(0)), \quad q_i(T) = -g_x(x(T))p_i(T) + \Phi_{ix}(x(T)).
\end{align*}
\]

\[
\text{(20)}
\]

It is noteworthy here that \((W_1(\cdot), W_2(\cdot))\) is the Brownian motion under probability measure \(\mathbb{P}^{\alpha_1,\alpha_2}\). Due to the observation equation (2) and the appearance of \(W_j^{\alpha_1,\alpha_2}(\cdot)\) and \(Y_j(\cdot), j = 1, 2\) in the forward-backward state equation (1), equation (20) is not the classical form any more. Also, we introduce equation (19) to deal with the controlled probability measure \(\mathbb{P}^{\alpha_1,\alpha_2}\) or expectation \(E^{\alpha_1,\alpha_2}\) related to the observation process when using the variational technique.

Now we give the necessary condition.

**Theorem 2.5.** Suppose that (H1) and (H2) hold, \((u_1(\cdot), u_2(\cdot))\) is a Nash equilibrium point of problem (NEP), and \((x, y, z, z_1, z_2)\) is the corresponding state process. Then

\[
\begin{align*}
E^{\alpha_1,\alpha_2}[H_{1,v_1}(t)(v_1 - u_1(t))] &\geq 0, \\
E^{\alpha_1,\alpha_2}[H_{2,v_2}(t)(v_2 - u_2(t))] &\geq 0,
\end{align*}
\]

\[
\text{(21)}
\]

for any \((v_1, v_2) \in U_1 \times U_2\), a.e.. Here we set

\[
\begin{align*}
\tilde{H}_{iv}(t) &= \tilde{H}_{iv}(t, x, y, z, z_1, z_2, u_1, u_2; q_i, k_i, k_{1i}, k_{2i}, p_i, Q_{1i}, Q_{2i}) \\
&= H_{iv}(t, x, y, z, z_1, z_2, u_1, u_2; q_i, k_i, k_{1i}, k_{2i}, p_i, Q_{1i}, Q_{2i}) \\
&- \sum_{j=1}^{2} q_i(t)\sigma_j(t, x(t), u_1(t), u_2(t))h_{xj}(t, x(t), u_1(t), u_2(t)),
\end{align*}
\]

\[
\text{(22)}
\]

where

\[
\begin{align*}
H_i(t, x, y, z, z_1, z_2, v_1, v_2; q_i, k_i, k_{1i}, k_{2i}, p_i, Q_{1i}, Q_{2i}) \\
= b(t, x, v_1, v_2)q_i(t) + \sigma(t, x, v_1, v_2)k_i(t) + \sum_{j=1}^{2} [\sigma_j(t, x, v_1, v_2)k_{ji}(t)
\end{align*}
\]
\begin{equation}
\begin{aligned}
&+ h_j(t, x, v_1, v_2)Q_{j_1}(t) - [f(t, x, y, z, z_1, z_2, v_1, v_2)
\quad - \sum_{j=1}^{2} h_j(t, x, v_1, v_2)z_j(t)]p_i(t) + l_i(t, x, y, z, z_1, z_2, v_1, v_2).
\end{aligned}
\end{equation}
\tag{23}

**Proof.** We only consider the \( i = 1 \) case. Applying Itô's formula to \( q_1(\cdot)x_1^1(\cdot) \), \( p_1(\cdot)y_1^1(\cdot) \), and \( \Gamma_1(\cdot) \) respectively, we obtain
\begin{equation}
\begin{aligned}
&\mathbb{E}^{u_1, u_2}[\Phi_1(x(T)) - g_x(x(T))p_1(T)x_1^1(T)] \\
&= \mathbb{E}^{u_1, u_2} \int_0^T [q_1(t)([b_x(t) - \sum_{j=1}^{2} \sigma_j(t)h_{jx}(t)]x_1^1(t) + 2\sum_{j=1}^{2} \sigma_j(t)h_{jx}(t)v_1(t))]
\quad + \sum_{j=1}^{2} \mathbb{E}^{u_1, u_2} \int_0^T k_j(t)(\sigma_{jx}(t)x_1^1(t) + \sigma_{jv_1}(t)v_1(t))dt]
\quad + \mathbb{E}^{u_1, u_2}[p_1(T)y_x(x(T))x_1^1(T) + \gamma_0(y(0))y_1^1(0)]
\quad + \mathbb{E}^{u_1, u_2} \int_0^T [-p_1(t)(f_x(t)x_1^1(t) + f_y(t)y_1^1(t) + f_z(t)z_1^1(t) + f_{v_1}(t)v_1(t))]
\quad + \sum_{j=1}^{2} (f_{j}(t) - h_j(t))z_j^1(t)) + y_1^1(t)(f_y(t)p_1(t) - l_{iy}(t))dt]
\quad + \mathbb{E}^{u_1, u_2} \int_0^T z_1^1(t)(f_{z}(t) - h_j(t))p_1(t) - l_{i1z}(t)dt]
\quad + \sum_{j=1}^{2} \mathbb{E}^{u_1, u_2} \int_0^T z_1^1(t)((f_{j}(t) - h_j(t))p_1(t) - l_{i1z}(t))dt],
\end{aligned}
\end{equation}
\tag{25}

and
\begin{equation}
\begin{aligned}
&\mathbb{E}^{u_1, u_2}[\Phi_1(x(T))\Gamma_1(T)] \\
&= \mathbb{E}^{u_1, u_2} \int_0^T -\Gamma_1(t)l_1(t)dt] + \sum_{j=1}^{2} \mathbb{E}^{u_1, u_2}[\int_0^T Q_{j_1}(t)(h_{jx}(t)x_1^1(t) + h_{jv_1}(t)v_1(t))]dt.
\end{aligned}
\end{equation}
\tag{26}

Substituting (24)-(26) into the variational inequality, we get
\begin{equation}
\begin{aligned}
&\mathbb{E}^{u_1, u_2} \int_0^T (b_v(t) - \sum_{j=1}^{2} \sigma_j(t)h_{jv_1}(t))q_1(t) - f_{v_1}(t)p_1(t) + \sigma_{v_1}(t)k_1(t)
\quad + \sum_{j=1}^{2} (\sigma_{jv_1}(t)k_j(t) + h_{jv_1}(t)Q_{j}(t)) + l_{v_1}(t)v_1(t)]dt \geq 0,
\end{aligned}
\end{equation}
for any \(v_1(\cdot)\) such that \(u_1(\cdot) + v_1(\cdot) \in U_1\).

Let \(w_1(\cdot) = u_1(\cdot) + v_1(\cdot)\), then above equation implies that

\[ E^{u_1,u_2}[\tilde{H}_{1v_1}(t)(w_1(t) - u_1(t))] \geq 0, \quad \text{a.e.} \]

We set \(w_1(t) = v_11_A + u_1(t)1_{A^c}, \ \forall v_1 \in U_1, \ \forall A \in \mathcal{F}_t^1\). Then \(v_1(t) = (v_1 - u_1(t))1_A, \ \forall v_1 \in U_1, \ \forall A \in \mathcal{F}_t^1\), and hence, \(E^{u_1,u_2}[1_A\tilde{H}_{1v_1}(t)(v_1 - u_1(t))] \geq 0, \ \forall v_1 \in U_1, \ \forall A \in \mathcal{F}_t^1\). This implies

\[ E^{u_1,u_2}[\tilde{H}_{1v_1}(t)(v_1 - u_1(t))|\mathcal{F}_t^1] \geq 0, \ \forall v_1 \in U_1. \]

The proof of case \(i = 2\) is similar. \(\square\)

**Remark 2.** The appearance of the second part on the right-side of equation (22) is caused by the control variable \((v_1(\cdot), v_2(\cdot))\) in observation equation (2).

**Corollary 1.** Suppose \((H1)\) and \((H2)\) hold, \((u_1(\cdot), u_2(\cdot))\) is a saddle point of problem \((EP)\). Then,

\[
E^{u_1,u_2}[\tilde{H}_{1v_1}(t)(v_1 - u_1(t))|\mathcal{F}_t^1] \geq 0, \\
E^{u_1,u_2}[\tilde{H}_{1v_2}(t)(v_2 - u_2(t))|\mathcal{F}_t^2] \leq 0,
\]

for any \((v_1, v_2) \in U_1 \times U_2\).

**Remark 3.** If \((u_1(\cdot), u_2(\cdot))\) is a Nash equilibrium point of Problem \((NEP)\) and \((u_1(\cdot), u_2(\cdot))\) is an interior point of \(U_1 \times U_2\) for all \(t \in [0, T]\), then the inequality in Theorem 2.5 is equivalent to

\[
E^{u_1,u_2}[\tilde{H}_{1v_1}(t)|\mathcal{F}_t^1] = 0, \\
E^{u_1,u_2}[\tilde{H}_{1v_2}(t)|\mathcal{F}_t^2] = 0,
\]

for any \((v_1, v_2) \in U_1 \times U_2\).

In the following remarks, we discuss some particular cases in the system of our problem \((NEP)\).

**Remark 4.** If the form of the forward equation \(x(\cdot)\) in (1) satisfies \(\sigma_j(\cdot) \equiv 0\) for \(j = 1, 2\). According to Theorem 2.5, the necessary condition becomes:

\[
E^{u_1,u_2}[\tilde{H}_{1v_i}(t, x, y, z, z_1, z_2, u_1, u_2; q_i, k_i, p_i, Q_{1i}, Q_{2i})|\mathcal{F}_t^j] = 0 \quad (i = 1, 2),
\]

where

\[
\tilde{H}_i(t, x, y, z, z_1, z_2, v_1, v_2; q_i, k_i, p_i, Q_{1i}, Q_{2i}) \\
= b(t, x, v_1, v_2)q_i(t) + \sigma(t, x, v_1, v_2)k_i(t) + \sum_{j=1}^{2} h_j(t, x, v_1, v_2)Q_{ji}(t) \\
- [f(t, x, y, z, z_1, z_2, v_1, v_2) - \sum_{j=1}^{2} h_j(t, x, v_1, v_2)z_j(t)]p_i(t) \\
+ l_i(t, x, y, z, z_1, z_2, v_1, v_2).
\]
The adjoint process becomes

$$\begin{aligned}
dp_i(t) &= \left[f_p(t)p_i(t) - l_{i_y}(t)\right]dt + \left[f_z(t)p_i(t) - l_{i_z}(t)\right]dW(t) \\
&\quad + \sum_{j=1}^{2}\left[(f_z(t) - h_j(t))p_i(t) - l_{i_z}(t)\right]dW_j(t), \\
dq_i(t) &= \left[b_z(t)q_i(t) + \sigma_x(t)k_i(t) + \sum_{j=1}^{2}h_{jx}(t)Q_{ji}(t) - f_x(t)p_i(t) + l_{ix}(t)\right]dt \\
&\quad - k_i(t)dW(t), \\
p_i(0) &= -\gamma_i(y(0)), \\
p_i(T) &= -g_x(x(T))p_i(T) + \Phi_{ix}(x(T)), \quad (i = 1, 2),
\end{aligned}$$

and

$$\begin{aligned}
-dP_i(t) &= l_i(t)dt - Q_i(t)dW(t) - \sum_{j=1}^{2}Q_{ji}(t)dW_j(t), \\
P_i(T) &= \Phi_i(x(T)), \quad (i = 1, 2).
\end{aligned}$$

**Remark 5.** If $h_j(t, x, v_1, v_2) = h_j(t, x)$ for $j = 1, 2$ and $t \in [0, T]$, then the Hamiltonian in Theorem 2.5 satisfies:

$$\begin{aligned}
H_{iv}(t) &= H_{iv}(t, x, y, z, z_1, z_2, u_1, u_2; q_i, k_i, k_{i1}, k_{i2}, p_i, Q_{i1}, Q_{i2}), \\
H_i(t, x, y, z, z_1, z_2, v_1, v_2; q_i, k_i, k_{i1}, k_{i2}, p_i, Q_{i1}, Q_{i2}) \\
&= b(t, x, v_1, v_2)q_i(t) + \sigma(t, x, v_1, v_2)k_i(t) + \sum_{j=1}^{2}\sigma_j(t, x, v_1, v_2)k_{ji}(t) \\
&\quad + h_j(t, x)Q_{ji}(t) - [f(t, x, y, z, z_1, z_2, v_1, v_2) - \sum_{j=1}^{2}h_{jx}(t)z_{j}(t)]p_i(t) \\
&\quad + l_i(t, x, y, z, z_1, z_2, v_1, v_2),
\end{aligned}$$

and the corresponding adjoint equations (19)-(20) are unchanged.

**Remark 6.** As mentioned in Remark 1, the main obstacle in the construction of the partially observed system is that if observation variable $Y_j(\cdot)(j = 1, 2)$ relies on control variables, the control will be adapted to a controlled filtration, and the $dY_j(\cdot)$ part will be affected by convex variational approach. Here, if we consider the special case that $h_j(\cdot, x(\cdot), v_1(\cdot), v_2(\cdot)) = h_j(\cdot)$, then $Y_j(\cdot)$ is an uncontrolled process. So the adjoint process $(P_i(\cdot), Q_i(\cdot), Q_{i1}(\cdot), Q_{i2}(\cdot))$ is needless because the control does not affect the observation any more. We can find it is equivalent to the condition that we set $W_j^{v_1, v_2}(\cdot) = W_j(\cdot)$ to be the B.M. under probability measure $P^{v_1, v_2} = P$ directly.

In this case, the Hamiltonian in Theorem 2.5 becomes the classical form:

$$E[H_{iv}(t, x, y, z, z_1, z_2, u_1, u_2; q_i, k_i, k_{i1}, k_{i2}, p_i)|\mathcal{F}_t^i] = 0, \quad (i = 1, 2),$$

where

$$\begin{aligned}
H_i(t, x, y, z, z_1, z_2, v_1, v_2; q_i, k_i, k_{i1}, k_{i2}, p_i) \\
&= b(t, x, v_1, v_2)q_i(t) + \sigma(t, x, v_1, v_2)k_i(t) + \sum_{j=1}^{2}\sigma_j(t, x, v_1, v_2)k_{ji}(t)
\end{aligned}$$
where

\[ - \left[ f(t, x, y, z, z_1, z_2, v_1, v_2) - \sum_{j=1}^{2} h_j(t)z_j(t) \right] p_i(t) \]

\[ + l_i(t, x, y, z, z_1, z_2, v_1, v_2), \]

and the adjoint process satisfies

\[
\begin{align*}
    dp_i(t) &= -H_{iy}(t)dt - H_{iz}(t)dW(t) - \sum_{j=1}^{2} H_{izj}(t)dW_j(t), \\
    -dq_i(t) &= H_{ix}(t)dt - k_i(t)dW(t) - \sum_{j=1}^{2} k_{ij}(t)dW_j(t), \\
    p_i(0) &= -\gamma_iy(y(0)), \\
    q_i(T) &= -g_i(x(T))p_i(T) + \Phi_{ix}(x(T)) \quad (i = 1, 2).
\end{align*}
\]

Here, we set \( H_{ix}(t) = H_{ix}(t, x, y, z, z_1, z_2, u_1, u_2; q_1, k_i, k_{11}, k_{21}, p_i) \), etc.

### 2.4. A sufficient condition (verification theorem)

In the following, we establish the sufficient condition under the case where the observation process is not affected by the control process. As Remark 6, we suppose \( h(t, x(t), v_1(t), v_2(t)) = h(t) \) and \((W(\cdot), W_1(\cdot), W_2(\cdot))\) is a standard B.M. on probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})\).

**Hypothesis (H3).** We suppose that the derivative \( \gamma_{iy}(\cdot) \geq 0 \), \( i = 1, 2 \), and \( l_i(t, x, y, z, z_1, z_2, v_1, v_2) = l_i(t, x, v_1, v_2), \quad i = 1, 2 \) for all \( t \in [0, T] \).

**Theorem 2.6.** Suppose that Hypothesis (H1)-(H3) hold, and the adjoint equation (27) admits a solution \((p_1(\cdot), q_1(\cdot), k_1(\cdot), k_{11}(\cdot), k_{21}(\cdot))\) \( \in L^2_2(0, T; \mathbb{R}^5) \) for \( i = 1, 2 \).

Suppose

\[
\begin{align*}
    &\mathbb{E}[H_1(t)|\mathcal{F}_1] = \min_{v_1 \in U_1} \mathbb{E}[H^{v_1}_1(t)|\mathcal{F}_1], \\
    &\mathbb{E}[H_2(t)|\mathcal{F}_2] = \min_{v_2 \in U_2} \mathbb{E}[H^{v_2}_2(t)|\mathcal{F}_2],
\end{align*}
\]

where

\[
\begin{align*}
    H_i(t) &= H_i(t, x, y, z, z_1, z_2, u_1, u_2; q_1, k_i, k_{11}, k_{21}, p_i), \quad (i = 1, 2), \\
    H^{v_1}_i(t) &= H_i(t, x, y, z, z_1, z_2, v_1, u_2; q_1, k_i, k_{11}, k_{21}, p_i), \quad (i = 1, 2), \\
    H^{v_2}_i(t) &= H_i(t, x, y, z, z_1, z_2, u_1, v_2; q_2, k_i, k_{11}, k_{21}, p_i).
\end{align*}
\]

Suppose that \( \mathbb{E}[H^{v_i}_i(t)|\mathcal{F}_1] \) is continuous at \( v_i = u_i(t) \) \( (i = 1, 2) \). Further, functions

\[
\begin{align*}
    (x, y, z, z_1, z_2, v_1) &\mapsto H^{v_i}_i(t) \quad (i = 1, 2), \\
    x &\mapsto g(x) \quad (i = 1, 2), \\
    x &\mapsto \Phi_i(x) \quad (i = 1, 2), \\
    y &\mapsto \gamma_i(y) \quad (i = 1, 2),
\end{align*}
\]

are convex. Then, \((u_1(\cdot), u_2(\cdot))\) is a Nash equilibrium point.

**Proof.** We only prove the case of \( i = 1 \). For \( \forall v_1(\cdot) \in U_1 \), we have

\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) = A + B + C,
\]

with

\[
\begin{align*}
    A &= \mathbb{E} \int_0^T [l^{v_1}_1(t) - l_1(t)]dt, \\
    B &= \gamma_1(y^{v_1}(0)) - \gamma_1(y(0)), \\
    C &= \mathbb{E} [\Phi_1(x^{v_1}(T)) - \Phi_1(x(T))].
\end{align*}
\]
From the convexity of $\gamma_1(\cdot)$ on $y$,
\[ B \geq \gamma_{1y}(y(0))(y^{\nu_1}(0) - y(0)). \]

Apply Itō’s formula to $p_1(\cdot)(y^{\nu_1}(\cdot) - y(\cdot))$, we obtain
\[
B \geq \mathbb{E} \int_0^T \left[ -p_1(t)(f^{\nu_1}(t) - f(t) - \sum_{j=1}^{2}(z_j^{\nu_1}(t) - z_j(t))h_j(t)) - (y^{\nu_1}(t) - y(t))H_{1y}(t) - (z^{\nu_1}(t) - z(t))H_{1z}(t) \right. \\
- \left. \sum_{j=1}^{2}(z_j^{\nu_1}(t) - z_j(t))H_{1z_j}(t) \right] dt - \mathbb{E}p_1(T)(g(x^{\nu_1}(T)) - g(x(T))).
\] (29)

From assumption (H3) and the adjoint equation (27), we have $p_1(T) \leq 0$. Together with the convexity of $g(\cdot)$ on $x$, we get
\[
B \geq \mathbb{E} \int_0^T \left[ -p_1(t)(f^{\nu_1}(t) - f(t) - \sum_{j=1}^{2}(z_j^{\nu_1}(t) - z_j(t))h_j(t)) - (y^{\nu_1}(t) - y(t))H_{1y}(t) - (z^{\nu_1}(t) - z(t))H_{1z}(t) \right. \\
- \left. \sum_{j=1}^{2}(z_j^{\nu_1}(t) - z_j(t))H_{1z_j}(t) \right] dt - \mathbb{E}p_1(T)g_x(x(T))(x^{\nu_1}(T) - x(T)).
\] (30)

From the convexity of $\Phi_1(\cdot)$ on $x$, we have
\[ C \geq \mathbb{E}\Phi_{1x}(x(T))(x^{\nu_1}(T) - x(T)). \]

Apply Itō’s formula to $q_1(\cdot)(x^{\nu_1}(\cdot) - x(\cdot))$, we obtain
\[
C \geq \mathbb{E} \int_0^T \left[ q_1(t)(b^{\nu_1}(t) - b(t)) - (x^{\nu_1}(t) - x(t))H_{1x}(t) + k_1(t)(\sigma^{\nu_1}(t) - \sigma(t)) \right. \\
+ \left. \sum_{j=1}^{2}k_{j1}(t)(\sigma_j^{\nu_1}(t) - \sigma_j(t)) \right] dt + \mathbb{E}p_1(T)g_x(x(T))(x^{\nu_1}(T) - x(T)).
\] (31)

Moreover,
\[
A = \mathbb{E} \int_0^T [H_1^{\nu_1}(t) - H_1(t)] dt - \mathbb{E} \int_0^T [(b^{\nu_1}(t) - b(t))q_1(t) + (\sigma^{\nu_1}(t) - \sigma(t))k_1(t) \\
+ \sum_{j=1}^{2}(\sigma_j^{\nu_1}(t) - \sigma_j(t))k_{j1}(t) - [f^{\nu_1}(t) - f(t) - \sum_{j=1}^{2}(z_j^{\nu_1}(t) - z_j(t))h_j(t)]p_1(t)] dt.
\]

Combining (29)-(31) with (28), we get
\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq \mathbb{E} \int_0^T \left[ (H_1^{\nu_1}(t) - H_1(t)) - (x^{\nu_1}(t) - x(t))H_{1x}(t) \\
- (y^{\nu_1}(t) - y(t))H_{1y}(t) - (z^{\nu_1}(t) - z(t))H_{1z}(t) - \sum_{j=1}^{2}(z_j^{\nu_1}(t) - z_j(t))H_{1z_j}(t) \right] dt.
\]
Due to \((x, y, z, z_1, z_2, v_1) \mapsto H_1^{v_1}(t)\) being convex, we can derive
\[
J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \geq \mathbb{E} \int_0^T H_{1v_1}(t)(v_1(t) - u_1(t))dt
= \mathbb{E} \int_0^T \mathbb{E}[H_{1v_1}(t)(v_1(t) - u_1(t))|\mathcal{F}_t^1]dt.
\]

From the assumption that \(v_1 \mapsto \mathbb{E}[H_1^{v_1}(t)|\mathcal{F}_t^1]\) is minimal at \(v_1 = u_1(t)\) for \(\forall t \in [0, T]\) and \(H_{1v_1}^{u_1}(t)\) is continuous on \(v_1\), we arrive at
\[
\mathbb{E}[H_{1v_1}(t)(v_1(t) - u_1(t))|\mathcal{F}_t^1] = \left(\frac{\partial}{\partial v_1}\mathbb{E}[H_1(t)|\mathcal{F}_t^1]\right)(v_1(t) - u_1(t)) \geq 0.
\]

Thus, it implies that
\[
J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot) \in U_1} J_1(v_1(\cdot), u_2(\cdot)).
\]

Similarly, we can prove the case when \(i = 2\),
\[
J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot) \in U_2} J_2(u_1(\cdot), v_2(\cdot)).
\]

\(\square\)

**Remark 7.** From the proof procedure in equations (29)-(31), we can find that if \(g(x(T)) = Mx(T), M \in \mathbb{R}\), we do not need Hypothesis (H3) and can draw the same conclusion. This assumption is coincide with Theorem 2.2 in [20].

3. **An example in finance.** In this section, we consider a practical investment problem in financial market. We can solve it by using the necessary and sufficient condition we derived in Section 2, and obtain the Nash equilibrium point explicitly.

We assume that there are \(n + 1\) assets can be continuously traded in financial market: One bond whose price is governed by
\[
\begin{aligned}
dB(t) &= r(t)B(t)dt, \\
B(0) &= 1,
\end{aligned}
\]
and \(n\) stocks whose prices satisfy the following SDEs:
\[
\begin{aligned}
dS_i(t) &= \mu_i(t)S_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t)S_i(t)dW_j(t), \\
S_i(0) &= 1 \quad (i = 1, 2, \ldots, n),
\end{aligned}
\]
where \(W(\cdot) = (W_1(\cdot), \ldots, W_n(\cdot))^T\) is an \(n\)-dimensional standard B.M. defined on probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). \(\mu(\cdot) = (\mu_1(\cdot), \ldots, \mu_n(\cdot))^T\) is the appreciation rate of the stock process. The \(n \times n\) matrix valued process \(\Sigma(t) = (\sigma_{ij}(t))^T\) is the volatility coefficients of the stock process. We set \(S(\cdot) = (S_1(\cdot), \ldots, S_n(\cdot))^T\).

We make the following assumptions.

**Hypothesis (H4).** \(\mu(\cdot)\) is an \(\mathcal{F}_t\)-adapted bounded process. \(r(\cdot)\) and \(\sigma_{ij}(\cdot)\) are deterministic bounded coefficients. \(\Sigma(\cdot)\) has full rank for \(\forall t \in [0, T]\), and the inverse matrix \(\Sigma(\cdot)^{-1}\) is also bounded. They are all continuous with respect to \(t\).

We suppose that a company hires two managers. Each of them can observe a few stocks from \(S(\cdot)\) as follows:
Manager 1: 

\[ dS^1_t(t) = \mu^1_t(t)S^1_t(t)dt + \sum_{j=1}^{n_1} \sigma^1_{1j}(t)S^1_t(t)dW^1_j(t) \quad (i = 1, \ldots, n_1), \]

Manager 2: 

\[ dS^2_t(t) = \mu^2_t(t)S^2_t(t)dt + \sum_{j=1}^{n_2} \sigma^2_{1j}(t)S^2_t(t)dW^2_j(t) \quad (i = 1, \ldots, n_2), \]

where \( n_k \) is the dimension of the observed stocks \( S^k(\cdot) = (S^k_1(\cdot), \ldots, S^{kn_k}(\cdot))' \) for \( k = 1, 2 \), which are parts of real stock process \( S(\cdot) \) corresponding to the two managers respectively. We denote the rest of both unobservable part of stock process as \( S^0(\cdot) = (S^{01}(\cdot), \ldots, S^{0n_0}(\cdot))' \), which can also be invested by the company. Here \( W^k(\cdot) = (W^1_k(\cdot), \ldots, W^{kn_k}_k(\cdot))' \), \( k = 0, 1, 2 \) are the corresponding mutually independent \( n_k \)-dimensional Brownian motions. For calculation simplicity, we might as well suppose there is no common B.M. among the vectors \( W^k(\cdot), k = 0, 1, 2 \) and no common observed stock between two managers. In this case, we have \( n_0 + n_1 + n_2 = n \). We set \( \mu^k(\cdot) = (\mu^1_k(\cdot), \ldots, \mu^{kn_k}_k(\cdot))' \), \( k = 0, 1, 2 \) and \( \Sigma^k(\cdot) = (\sigma^1_{ij}(\cdot))_{i,j = 1, \ldots, n_k} \), \( k = 0, 1, 2 \) to be the corresponding appreciation rate and volatility of stock process \( S^k(\cdot) \). Merton [13] has shown that the expected returns are so hard to estimate in the financial market. Thus, we consider the stochastic appreciation rate\(^2\) and make the assumption that \( \mu^k(t) \) is unobservable for decision makers at time \( t \) (see e.g. [32]). We set \( a^k_{ij}(\cdot) = \sum_{l=1}^{n_k} \sigma^{1l}(\cdot)\sigma^{2l}_{ij}(\cdot), i, j = 1, \ldots, n_k, k = 0, 1, 2 \), 

\[ A^k(\cdot) = (a^k_{11}(\cdot), \ldots, a^k_{nn_k}(\cdot))^\tau. \]

Moreover, we set

\[ dY^k(\cdot) = \Sigma^k(\cdot)^{-1}d\log S^k(\cdot) \quad (k = 1, 2). \]

By using Itô's formula, our observation equation turns to

\[ dY^1(t) = \eta^1(t)dt + dW^1(t), \quad (32) \]

\[ dY^2(t) = \eta^2(t)dt + dW^2(t), \quad (33) \]

where \( Y^k(\cdot) = (Y^1_k(\cdot), \ldots, Y^{kn_k}_k(\cdot))' \) \( (k = 1, 2) \), and

\[ \eta^k(\cdot) = \Sigma^k(\cdot)^{-1}(\mu^k(t) - \frac{1}{2}A^k(\cdot)) \quad (k = 1, 2). \quad (34) \]

Let

\[ \mathcal{F}^k_t = \sigma\{B(s), Y^k(s); 0 \leq s \leq t\} \quad (k = 1, 2) \quad (35) \]

be the available filtration to each manager. In that case, \( r(\cdot), \Sigma(\cdot) \) are all completely observable, while \( \mu(\cdot) \) is unobservable.

We assume that the company plans to obtain a terminal wealth \( \xi \), which is an \( \mathcal{F}_T \)-measurable non-negative random variable satisfying \( \mathbb{E}[\xi^2] < \infty \). Now the whole wealth of the company is denoted by \( y(\cdot) \). The first manager invests \( \pi^1_1(t) \) wealth in stock \( S^1_1(t)(i = 1, \ldots, n_1) \) he observed, and the second manager invests \( \pi^2_2(t) \) wealth in stock \( S^2_2(t)(i = 1, \ldots, n_2) \) he focused on. We suppose that there are \( \pi^0_0(t) \) wealth invested by company in unobservable stocks \( S^0_i(t)(i = 1, \ldots, n_0) \) of both managers. So the rest \( y(t) - \sum_{k=0}^{2} \sum_{i=1}^{n_k} \pi^k_i(t) \) wealth is invested in bond. Then,

\(^2\)In financial market, stochastic appreciation rates can be assumed to satisfy the Ornstein-Uhlenbeck process with mean reverting drift, see e.g. [4, 11].
we can establish the wealth equation as

\[
\begin{aligned}
\frac{dy(t)}{dt} &= [r(t)y(t) + \sum_{k=0}^{2n} \sum_{i=1}^{n_k} (\mu_i^k(t) - r(t))\pi_i^k(t) + I_1(t) + I_2(t)]dt \\
&\quad + \sum_{k=0}^{2n} \sum_{i=1}^{n_k} \sum_{j=1}^{n_k} \pi_i^k(t)\sigma_{ij}^k(t)dW_j^k(t), \quad t \in [0,T],
\end{aligned}
\]

where \( I_1(\cdot) \) and \( I_2(\cdot) \) represent the instantaneous capital injection of each manager to guarantee the terminal wealth of the company.

For each of them, the target is to use the minimal capital injection and the minimum start-up capital to make sure the company reach the ultimate wealth. Meanwhile, each one of them has his own utility on their injection process. The more capital he uses, the more risk he undertakes. So we can define the related utility function for each manager as follows:

\[
J_i(I_1(\cdot), I_2(\cdot)) = \mathbb{E} \int_0^T [L_i e^{-\beta t} I_i^2(t) dt + M_i y(0)] \quad (i = 1, 2),
\]

where \( L_i, M_i \) are two positive constants, \( \beta \) is the discount rate. We define the running cost as a criteria for injection utility. To attain the terminal wealth, they want to minimize both of the injection utility and the start-up wealth value. That is

\[
\begin{aligned}
J_1(\tilde{I}_1(\cdot), \tilde{I}_2(\cdot)) &= \min_{I_1(\cdot) \in \mathcal{I}_1} J_1(I_1(\cdot), I_2(\cdot)), \\
J_2(\tilde{I}_1(\cdot), \tilde{I}_2(\cdot)) &= \min_{I_2(\cdot) \in \mathcal{I}_2} J_2(I_1(\cdot), I_2(\cdot)),
\end{aligned}
\]

where we define

\[
\mathcal{I}_i = \{ I_i(\cdot) \in L^2_{T^F}; (0,T; \mathbb{R}); I_i(t) \geq 0, t \in [0,T] \} \quad (i = 1, 2).
\]

We regard \((\tilde{I}_1(\cdot), \tilde{I}_2(\cdot))\) as the Nash equilibrium point of this investment problem.

The wealth equation (36) is a backward case. We denote \( \pi^k(t) = (\pi_1^k(t), \ldots, \pi_{n_k}^k(t))^\top, z^k(t) = (z_1^k(t), \ldots, z_{n_k}^k(t)) = \pi^k(t)^\top \Sigma^k(t) \quad (k = 0,1,2). \) Then, our wealth equation turns into

\[
\begin{aligned}
\frac{dy(t)}{dt} &= [r(t)y(t) + \sum_{k=0}^{2} b^k(t)^\top z^k(t)^\top + I_1(t) + I_2(t)]dt + \sum_{k=0}^{2} z^k(t)dW^k(t), \\
y(T) &= \xi,
\end{aligned}
\]

where

\[
b^k(t) = \Sigma^k(t)^{-1}(\mu^k(t) - r(t)) \quad (k = 0,1,2).
\]

For (39) and (37), we can apply the maximum principle derived in Section 2. The Hamiltonian functions are

\[
H_i(t, y, z^0, z^1, z^2, I_1, I_2; p_i) = (r(t)y(t) + \sum_{k=0}^{2} b^k(t)^\top z^k(t)^\top + I_1(t) + I_2(t))p_i(t)
+ L_i e^{-\beta t} I_i^2(t),
\]
for $i = 1, 2$. The adjoint process $p_i(\cdot)$ satisfies

$$
\begin{cases}
    dp_i(t) = -r(t)p_i(t)dt - \sum_{k=0}^{2} b^k(t)^\tau p_i(t)dW^k(t), \\
    dp_i(0) = -M_i \quad (i = 1, 2).
\end{cases}
$$  \tag{41}

From the necessary condition, we can derive a candidate Nash equilibrium point

$$
\begin{cases}
    \hat{I}_1(t) = -\frac{1}{2} e^{\beta t} L_1^{-1}\hat{p}_1(t), \\
    \hat{I}_2(t) = -\frac{1}{2} e^{\beta t} L_2^{-1}\tilde{p}_2(t),
\end{cases}
$$

where we set $\hat{\phi}(t) = \mathbb{E}[\phi(t)|\mathcal{F}_t^1]$, $\tilde{\psi}(t) = \mathbb{E}[\psi(t)|\mathcal{F}_t^2]$ for $\forall \phi(\cdot), \psi(\cdot) \in \mathcal{F}$.

Now focusing on the $i = 1$ case, from the observation equation (32) and the Kushner-FKK equation in [31], we derive

$$
\begin{cases}
    d\hat{p}_1(t) = -r(t)\hat{p}_1(t)dt + [-b^1(t)^\tau \hat{p}_1(t) + \eta^1(t)^\tau \hat{p}_1(t) - \hat{\eta}^1(t)^\tau \hat{p}_1(t)]d\hat{W}^1(t), \\
    d\hat{p}_1(0) = -M_1,
\end{cases}
$$  \tag{42}

where the innovation process $\hat{W}^1(\cdot)$ satisfies

$$\hat{W}^1(t) = Y^1(t) - \int_0^t \hat{\eta}^1(s)ds.$$

$\hat{W}^1(t)$ is an $\mathcal{F}_t^1$-Brownian motion under probability measure $\mathbb{P}$.

From $\eta^1(\cdot)$ in (34), and $b^1(t)$ in (40), we find that

$$\eta^1(t)\hat{p}_1(t) - \hat{\eta}^1(t)\hat{p}_1(t) = \Sigma^1(t)^{-1}(\mu^1(t)\hat{p}_1(t) - \mu^1(t)\hat{p}_1(t))$$

$$= b^1(t)\hat{p}_1(t) - \hat{b}^1(t)\hat{p}_1(t).$$  \tag{43}

Substituting (43) into (42), we get

$$
\begin{cases}
    d\hat{p}_1(t) = -r(t)\hat{p}_1(t)dt - \hat{b}^1(t)^\tau \hat{p}_1(t)d\hat{W}^1(t), \\
    d\hat{p}_1(0) = -M_1.
\end{cases}
$$

Thus

$$\hat{p}_1(t) = -M_1 \exp\{\int_0^t [-r(s) - \frac{1}{2} \hat{b}^1(s)^2]ds - \int_0^t \hat{b}^1(s)d\hat{W}^1(s)\}.  \tag{44}$$

Similarly, we can prove that $\tilde{p}_2(\cdot)$ satisfies

$$
\begin{cases}
    d\tilde{p}_2(t) = -r(t)\tilde{p}_2(t)dt - \tilde{b}^2(t)^\tau \tilde{p}_2(t)d\tilde{W}^2(t), \\
    d\tilde{p}_2(0) = -M_1,
\end{cases}
$$

where

$$\tilde{W}^2(t) = Y^2(t) - \int_0^t \tilde{\eta}^2(s)ds.$$

Thus

$$\tilde{p}_2(t) = -M_1 \exp\{\int_0^t [-r(s) - \frac{1}{2} \hat{b}^2(s)^2]ds - \int_0^t \hat{b}^2(s)d\tilde{W}^2(s)\}.  \tag{45}$$
Finally, from the linearity of the state processes and the convexity of the cost functions as well as the sufficient condition we discussed above, we know that $(\bar{I}_1(\cdot), \bar{I}_2(\cdot))$ is the Nash equilibrium point satisfying
\[
\begin{aligned}
\bar{I}_1 &= \frac{1}{2} e^{\beta t} L_1^{-1} M_1 \exp\left\{ \int_0^t [-r(s) - \frac{1}{2} \hat{b}_1(s)^2] \, ds - \int_0^t \hat{b}_1(s) d\hat{W}_1(s) \right\}, \\
\bar{I}_2 &= \frac{1}{2} e^{\beta t} L_2^{-1} M_1 \exp\left\{ \int_0^t [-r(s) - \frac{1}{2} \tilde{b}_2(s)^2] \, ds - \int_0^t \tilde{b}_2(s) d\tilde{W}_2(s) \right\}.
\end{aligned}
\]

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