When non-extensive entropy becomes extensive

Wada Tatsuaki\textsuperscript{a,*} and Saito Takeshi\textsuperscript{b,1}

\textsuperscript{a}Department of Electrical and Electronic Engineering, Ibaraki University, Hitachi, Ibaraki, 316-8511, Japan
\textsuperscript{b}Complex Functional Robot Laboratories, Graduate School of Science and Engineering, Ibaraki University, Hitachi, Ibaraki, 316-8511, Japan

Abstract

Tsallis’ non-extensive entropy $S_q$ enables us to treat both a power and exponential evolutions of underlying microscopic dynamics on equal footing by adjusting the variable entropic index $q$ to proper one $q^\ast$. We propose an alternative constraint of obtaining the proper entropic index $q^\ast$ that the non-additive conditional entropy becomes additive if and only if $q = q^\ast$ in spite of that the associated system cannot be decomposed into statistically independent subsystems. Long-range (time) correlation expressed by $q$-exponential function is discussed based on the nature that $q$-exponential function cannot be factorized into independent factors when $q \neq 1$.

Key words: non-extensivity, Tsallis’ entropy, pseudo-additivity, power law

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1 Introduction

There has been growing interest in the non-extensive statistical mechanics [1,2] based on Tsallis’ generalized entropy (in $k_B = 1$ unit):

\begin{equation}
S_q = 1 - \frac{\sum_i p_i^q}{q - 1}, \quad (\sum_i p_i = 1; \quad q \in \mathbb{R})
\end{equation}

* Corresponding author.

Email addresses: wada@ee.ibaraki.ac.jp (Wada Tatsuaki), saito@kif.co.jp (Saito Takeshi).

1 Present address: KIF & Co., Ltd., Tokyo Dia Bldg. #5, 15 floor, 1-28-23 Shinkawa Chuo-ku, Tokyo, 104-0033, Japan
At least formally, Tsallis’ entropy is an extension of conventional Boltzmann-Shannon (BS) entropy with one real parameter of \( q \). In the limit of \( q \to 1 \), Tsallis’ entropy Eq. (1) reduces to BS entropy, 
\[
S_1 \equiv -\sum_i p_i \ln p_i, \quad \text{since} \quad p_q^{q-1} = e^{(q-1)\ln p_i} \approx 1 + (q - 1) \ln p_i.
\]

The parameter \( q \) may be interpreted as a quantity characterizing the degree of non-extensivity of Tsallis’ entropy through the so-called pseudo-additivity:
\[
S_q(A, B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B), \quad (2)
\]

where \( A \) and \( B \) denote two statistically independent sub-systems. It is worth while to realize that the pseudo-additivity of \( S_q \) is one of the crucial ingredients of Tsallis’ non-extensive statistical mechanics. In fact, the uniqueness of Tsallis’ entropic form is proved [4] for an entropy that fulfills the generalization of the Shannon-Kinchin axioms [5] based on the pseudo-additive conditional entropy obeying the pseudo-additivity instead of additivity. Then what is a role of the pseudo-additivity? By rewriting Eq. (1) as the following form,
\[
S_q = -\sum_i p_i^q \ln_q p_i, \quad (3)
\]

we see that the pseudo-additivity of Tsallis’ entropy comes from the \( q \)-logarithmic function, which is defined by
\[
\ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q}, \quad (4)
\]

since it equips the pseudo-additivity as
\[
\ln_q(xy) = \ln_q x + \ln_q y + (1 - q) \ln_q x \ln_q y. \quad (5)
\]

The inverse function of the \( q \)-logarithmic function is \( q \)-exponential one, which is defined by
\[
\exp_q(x) \equiv [1 + (1 - q)x]^{\frac{1}{1-q}}, \quad (6)
\]

for \( 1 + (1 - q)x > 0 \) and otherwise \( \exp_q(x) = 0 \).

As Tsallis [3] has already pointed out, the parameter \( q \) plays a similar role as the light velocity \( c \) in special relativity or Planck’s constant \( \hbar \) in quantum mechanics in the sense of a one-parameter extension of classical mechanics. Unlike \( c \) or \( \hbar \), however, \( q \) does not seem to be a universal constant. Thus it is a natural question whether \( q \) is merely an adjustable parameter or not in
the non-extensive statistical mechanics. In some cases the parameter $q$ has no physical meaning, but when it is used as an adjustable parameter the resulting distributions give excellent agreement with experimental data. In other but a few cases [6–10], $q$ is uniquely determined by the constraints of the problem and thereby $q$ may have a physical meaning. Recent studies of the characterization of mixing in one-dimensional (1D) dissipative maps [6–8] and of symbolic sequences [9] seems to provide a positive answer to the above question since there exists the special value $q^*$ such that $S_{q^*}$ becomes linear. For example, in the studies [6,7] of mixing in simple logistic map, $q^*$ can be obtained by three different methods based on: i) the upper bound of a power-law sensitivity $\xi$ to initial conditions; ii) the singularity indices in multi-fractal structure; and iii) the rate of information loss in terms of $S_q$. The remarkable fact is that all methods lead to the same value of $q^* \simeq 0.24$, which may shed some light on the physical meaning of $q$. They established some connections among the sensitivity $\xi$ to initial conditions, Tsallis’ entropy $S_q$ and the proper entropic index $q^*$. In particular we focus on the work of Buiatti et al. [9], in which they have shown, for the symbolic sequences with length of $N$, that the generalized block entropy [11] is proportional to $N$ when the proper entropic index $q^*$ is used. In other words, there may exist a proper entropic index $q^*$, which may statistically characterize a non-extensive system.

In this work we study a reason why the generalized block entropy in the work of Buiatti et al. [9] is proportional to the length $N$ of symbolic sequences when we use the proper entropic index $q^*$. In particular we focus our attention on a role of the pseudo-additivity of the conditional entropy in characterizing a non-extensive system with the proper entropic index $q^*$. We reformulate the constraint of obtaining the proper entropic index $q^*$ as follows: the pseudo-additive conditional entropy becomes additive with respect to $N$ when the proper entropic index $q^*$ is used. In other words, for the special value $q^*$ of entropic index, the additivity of the conditional entropy is held in spite of that the involved subsystems are not statistically independent of each other.

The rest of the paper is organized as follows: in the next section we explain the constraint of obtaining the proper entropic index $q^*$ in the work [9] of Buiatti et al., and propose our constraint of obtaining $q^*$. We then show the equivalence of the two constraints and discuss the underlying simple mechanism of why the conditional entropy becomes additive for the proper entropic index $q^*$ under the assumption of equi-probability. In Section 3, we discuss long-range correlation expressed by $q$-exponential function. Section 4 is devoted to our conclusions.
2 How to determine a proper entropic index

Buiatti et al. [9] showed, by studying a symbolic binary sequence \( \{\sigma_1, \sigma_2, \cdots \} \) with a long-range correlation, that for the probability \( p(\sigma_1, \cdots, \sigma_N) \) of each path with length of \( N \), the generalized block entropy [11],

\[
S_q(N) \equiv \frac{1 - \sum_{\sigma_1, \cdots, \sigma_N} p(\sigma_1, \cdots, \sigma_N)^q}{q - 1},
\]

is proportional to \( N \) if and only if the variable index \( q \) equals the proper entropic index \( q^* \).

We reformulate this constraint as the following. The pseudo-additive conditional entropy [12], which is defined by

\[
S_q(N|1) \equiv \frac{S_q(N + 1) - S_q(1)}{1 + (1 - q)S_q(1)},
\]

should satisfy the additivity,

\[
S_q(N|1) = S_q(N - 1|1) + S_q(1|1),
\]

when \( q \) is equal to the proper entropic index \( q^* \). At first sight our constraint seems to be paradoxical, since \( S_q \) is pseudo-additive in general. We explain in the followings that the equivalence of the two constraints which determine \( q^* \), and discuss an underlying simple mechanism connecting the variable entropic index \( q \) with the proper index \( q^* \).

2.1 Proof of the equivalence of the two constraints

The method of Buiatti et al. is rephrased as follows: \( S_q(N) \) is a linear function of \( N \) when \( q = q^* \), i.e.

\[
S_{q^*}(N) = (N - 1)L_{q^*} + S_{q^*}(1),
\]

where \( L_{q^*} \) is a proportional constant, or equivalently

\[
S_{q^*}(N + 1) - S_{q^*}(N) = S_{q^*}(2) - S_{q^*}(1) = L_{q^*}, \quad \text{for } N > 1.
\]
Subtracting $S_{q^*}(1)$ from the both sides and after a little bit algebra, Eq. (11) is rewritten as

$$S_{q^*}(N+1) - S_{q^*}(1) = S_{q^*}(N) - S_{q^*}(1) + S_{q^*}(2) - S_{q^*}(1).$$ \hfill (12)

Dividing the both sides of Eq. (12) by $1 + (1 - q^*)S_{q^*}(1)$ and using the definition of the conditional entropy of Eq. (8), it is obvious that Eq. (11) is equivalent to Eq. (9). Hence we have reformulated the method of Buiatti et al. as follows: the conditional entropy becomes additive when we use the proper entropic index $q^*$.

### 2.2 A reason why $S_{q^*}(N|1)$ is proportional to $N$

Having explained the equivalence of Buiatti et al. and our methods of obtaining a proper entropic index $q^*$, we now consider why the conditional entropy $S_{q}(N|1)$ is proportional to the length $N$ of symbolic sequences when $q = q^*$.

Under the assumption of equi-probability, Tsallis’ entropy can be written in terms of the number of states $W(N)$ for the symbolic sequences with the length of $N$ as

$$S_{q}(N) = \ln_q W(N).$$ \hfill (13)

Then the conditional entropy of Eq. (8) is expressed in terms of $W(N)$ as

$$S_{q}(N|1) = \ln_q \frac{W(N + 1)}{W(1)}.$$ \hfill (14)

Suppose that the number of states $W(N)$ obeys a power-law evolution, which can be well described by

$$W(N + 1) = W(1) \exp_{q^*}(L'_{q^*} N),$$ \hfill (15)

with the proper $q^*$ of a system of interest, where $L'_{q^*}$ is another constant. The relation between $L'_{q^*}$ and $L_{q^*}$ of Eq. (11) is discussed in the next section. Substituting Eq. (15) into Eq. (14), the corresponding conditional entropy can be written as

$$S_{q}(N|1) = \ln_q \left[ \frac{W(N + 1)}{W(1)} \right] = \ln_q [\exp_{q^*}(L'_{q^*} N)].$$ \hfill (16)
Now we readily see that $S_q(N|1)$ is proportional to $N$, if and only if we set $q$ to $q^*$. In other words, if $W$ obeys the $q^*$-exponential evolution of Eq. (15), then it is reasonable to use its inverse function in order to define the conditional entropy.

3 Long-range correlation expressed by $q$-exponential function

Tsallis’ entropic description may be well suited for a long-range correlated system which obeys a power-law evolution described by $q$-exponential function. Then how can $q$-exponential function express long-range correlation? We here explain that long-range correlation may be expressed by the non-factorizability of $q$-exponential function into independent terms. The long-range correlation in this case means the initial condition dependency of long duration. It is known that $q$-exponential function cannot be resolved into a product of independent terms unless $q = 1$. For example $\exp_q(t+s)$ is not resolved into the independent factors as $\exp_q(x) \cdot \exp_q(s)$. Instead it can be expressed as the product of the dependent factors as $\exp_q(x) \cdot \exp_q(s/(1+(1-q)x))$.

Let us focus on the long-range correlation associated with the power-law evolution of $W(N)$ described by Eq. (15). Using Eqs. (10) and (13), the $q$-exponential dependency of $W(N+1)$ can be expressed as

$$W(N+1) = \exp_q[S_q(N+1)] = \exp_q[S_q(1) + L_q N]$$
$$= \exp_q[S_q(1)] \cdot \exp_q[L_q N \left\{1 + (1-q)S_q(1)\right\}]$$
$$= \exp_q[S_q(1)] \cdot \exp_q[L_q \left\{1 + (1-q)S_q(1)\right\}]$$
$$\times \cdots \times \exp_q[L_q \left\{1 + (1-q)(S_q(1) + N-1)\right\}].$$

(17)

Note that $S_q(1)$ appears in all terms and this reflects the initial condition dependency of long duration. This feature is consistent with the single-trajectory approach by Montangero et al. [15] in which they fix a given initial condition in order to obtain the $q^*$ of the non-extensive version of Kolmogorov-Sinai entropy for the dynamics of the logistic map at the chaotic threshold. Because of the initial condition dependency of long duration, an averaging over many different initial conditions is not appropriate.

Now let us focus on the relation between the proportional constants $L_q'$ and
\(L_{q^*}\) in the previous section. From the second line of Eq. (17), we see that

\[
W(N + 1) = W(1) \cdot \exp_q\left[\frac{L_q N}{1 + (1 - q)S_q(1)}\right].
\] (18)

Comparing this with Eq. (15), \(L'_{q^*}\) and \(L_{q^*}\) are related by

\[
L'_{q^*} = \frac{L_{q^*}}{1 + (1 - q)S_{q^*}(1)} = \frac{L_{q^*}}{W(1)^{1-q}}.
\] (19)

which is the same relation [14] of \(\lambda' = \lambda/\tilde{Z}_q^{1-q}\) between the Lagrange multiplier \(\lambda'\) of optimal Lagrange multipliers (OLM) method and that \(\lambda\) of Tsallis-Mendes-Plastino one in canonical ensemble formalism, where \(\tilde{Z}_q\) denotes partition function.

4 Conclusions

We have proposed a constraint of obtaining the proper Tsallis’ entropic index \(q^*\) in describing the evolutions of correlated symbolic sequences with length \(N\). The proper entropic index \(q^*\) can be determined by requiring that the conditional entropy \(S_q(N|1)\) should be proportional to \(N\) if and only if \(q\) equals the proper entropic index \(q^*\). In other words \(S_q\) becomes additive for the proper \(q^*\). It is the non-factorizability of \(q\)-exponential function into independent terms that can express a long-range correlation.

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