REILLY-TYPE UPPER BOUNDS FOR THE $p$-STEKLOV PROBLEM ON SUBMANIFOLDS

JULIEN ROTH and ABHITOSH UPADHYAY

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Abstract

We prove Reilly-type upper bounds for the first nonzero eigenvalue of the Steklov problem associated with the $p$-Laplace operator on submanifolds of manifolds with sectional curvature bounded from above by a nonnegative constant.

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1. Introduction

Let $(M^n, g)$ be an $n$-dimensional compact, connected, oriented manifold without boundary, isometrically immersed into the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. The spectrum of the Laplacian of $(M, g)$ is an increasing sequence of real numbers

$$0 = \lambda_0(\Delta) < \lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \cdots \leq \lambda_k(\Delta) \leq \cdots \rightarrow +\infty.$$ 

The eigenvalue 0 (corresponding to constant functions) is simple and $\lambda_1(\Delta)$ is the first positive eigenvalue. In [16], Reilly proved the following well-known upper bound for $\lambda_1(\Delta)$:

$$\lambda_1(\Delta) \leq \frac{n}{V(M)} \int_M H^2 \, dv_g, \quad (1.1)$$

where $H$ is the mean curvature of the immersion. He also proved an analogous inequality involving the higher order mean curvatures: for $r \in \{1, \ldots, n\}$,

$$\lambda_1(\Delta) \left( \int_M H_{r-1} \, dv_g \right)^2 \leq V(M) \int_M H_r^2 \, dv_g, \quad (1.2)$$

where $H_r$ is the $r$th mean curvature, defined by the $r$th symmetric polynomial of the principal curvatures.

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Inequalities (1.1) and (1.2) have been generalised in many ways for submanifolds of any codimension of Euclidean spaces and spheres [9, 16], submanifolds of hyperbolic spaces [7, 9, 11], other differential operators of divergence-type [1, 17] and Paneitz-like operators [18], as well as for different types of Steklov problems. In particular, Ilias and Makhoul [12] proved the following upper bound for the first eigenvalue $\sigma_1$ of the Steklov problem:

$$\sigma_1 V(\partial M)^2 \leq n V(M) \int_{\partial M} \|H\|^2 \, dv_g,$$

where $(M^n, g)$ is a compact submanifold of $\mathbb{R}^N$ with boundary $\partial M$ and $H$ denotes the mean curvature of $\partial M$. They also proved analogous inequalities involving higher order mean curvatures as in (1.2). Recently, Manfio and the authors have extended this inequality for submanifolds of any Riemannian manifold of bounded sectional curvature in [15].

Let us consider $(M^n, g)$, a compact Riemannian manifold with a possibly nonempty boundary $\partial M$. For $p \in (1, +\infty)$, we consider the so-called $p$-Laplacian defined by

$$\Delta_p u = -\text{div}(\|\nabla u\|^{p-2}\nabla u)$$

for any $C^2$ function $u$. For $p = 2$, $\Delta_2$ is just the Laplace–Beltrami operator of $(M^n, g)$. This operator $\Delta_p$ and especially its spectrum have been intensively studied, mainly for Euclidean domains with Dirichlet or Neumann boundary conditions (see for instance [13] and references therein) and also on Riemannian manifolds [14]. Later, Du and Mao [6] gave analogues of the Reilly inequalities (1.1) and (1.2) for the $p$-Laplacian on submanifolds of Euclidean spaces and spheres and it was extended by Chen and Wei [5] for submanifolds of hyperbolic space. Very recently, Chen [3] and Chen and Gui [4] have obtained upper bounds for submanifolds of manifolds with curvature bounded from above, generalising to the $p$-Laplacian the result of Heintze for the Laplacian.

In the present paper, we will consider the Steklov problem associated with the $p$-Laplacian on submanifolds with boundary of the Euclidean space. That is, we consider the $p$-Steklov problem which is the following boundary value problem:

$$\begin{align*}
\Delta_p u &= 0 & \text{in } M, \\
\|\nabla u\|^{p-2} \frac{\partial u}{\partial \nu} &= \sigma |u|^{p-2} u & \text{on } \partial M,
\end{align*}$$

where $\partial u/\partial \nu$ is the derivative of the function $u$ with respect to the outward unit normal $\nu$ to the boundary $\partial M$. Note that for $p = 2$, $(S_p)$ is the usual Steklov problem. (See, for example, [8] for an overview of results about the spectral geometry of the Steklov problem.) It has been observed that very little is known about the spectrum of the $p$-Steklov problem. If $M$ is a domain of $\mathbb{R}^N$, there exists a sequence of positive eigenvalues $\sigma_{0,p} = 0 < \sigma_{1,p} \leq \sigma_{2,p} \leq \cdots \leq \sigma_{k,p} \leq \cdots$ in the variational spectrum obtained by the Ljusternik–Schnirelmann theory (see [13, 20] and also [2] for details of the Ljusternik–Schnirelmann principle). Note that, as mentioned in [14, Remark 1.1],
the arguments used in [13] can be extended to domains on Riemannian manifolds and there exists a nondecreasing sequence of variational eigenvalues obtained by the Ljusternik–Schnirelman principle. Moreover, the eigenvalue 0 is simple with constant eigenfunctions and isolated, that is, there is no eigenvalue between 0 and $\lambda_1$. The first positive eigenvalue of the $p$-Steklov problem is $\sigma_{1,p}$ and it has the variational characterisation

$$\sigma_{1,p} = \inf \left\{ \frac{\int_M \|\nabla u\|^p \, dv_g}{\int_{\partial M} |u|^p \, dv_h} \mid u \in W^{1,p}(M) \setminus \{0\}, \int_{\partial M} |u|^{p-2} u \, dv_h = 0 \right\},$$

where $\nabla$ is the gradient on $M$, and $dv_g$ and $dv_h$ are the Riemannian volume forms respectively associated with the metric $g$ on $M$ and the induced metric $h$ on $\partial M$. All the other eigenvalues $\sigma_{k,p}$ of this sequence also have a variational characterisation but we do not know if all the spectrum is contained in this sequence.

Recently, Verma obtained upper bounds for the first eigenvalue $\sigma_{1,p}$ of the $p$-Steklov problem ($S_p$) for Euclidean domains [21]. She proved that for a bounded domain $\Omega$ with smooth boundary, $\sigma_{1,p} \leq 1/R^{p-1}$ if $1 < p < 2$ and $\sigma_{1,p} \leq n^{p-2}/R^{p-1}$ if $p \geq 2$, where $R > 0$ satisfies $V(\Omega) = V(B(R))$ and $B(R)$ is a ball of radius $R$. After that, in [19], the first author proved the following upper bounds of Reilly-type for $\sigma_{1,p}$ for submanifolds with boundary of the Euclidean space:

$$\sigma_{1,p} \leq N^{2-\rho/2} n^{\rho/2} \left( \int_{\partial M} \|H\|^{\rho/(p-1)} \, dv_h \right)^{p-1} \frac{V(M)}{V(\partial M)^{\rho/2}}$$

and, more generally,

$$\sigma_{1,p} \left( \int_{\partial M} \text{tr} (T) \, dv_h \right)^p \leq N^{2-\rho/2} n^{\rho/2} \left( \int_{\partial M} \|H_T\|^{\rho/(p-1)} \, dv_h \right)^{p-1} V(M),$$

where $T$ is a symmetric and divergence-free $(1,1)$-tensor on $\partial M$. The aim of the present paper is to prove an inequality for submanifolds with boundary of Riemannian manifolds of sectional curvature bounded from above by a nonnegative constant. We prove the following result.

**Theorem 1.1.** Let $\delta \geq 0$, $p > 1$ be real numbers and $(\tilde{M}^N, \tilde{g})$ an $N$-dimensional Riemannian manifold of sectional curvature bounded from above by $\delta$. Let $(M^n, g)$ be a compact $n$-dimensional Riemannian manifold with nonempty boundary $\partial M$ isometrically immersed into $(\tilde{M}, \tilde{g})$ and let $S$ be a symmetric, positive definite and divergence-free $(1,1)$-tensor on $\partial M$.

(1) If $\delta = 0$, then

$$\sigma_{1,p} \left( \int_{\partial M} \text{tr} (S) \, dv_h \right)^p \leq N^{2-\rho/2} n^{\rho/2} V(M) \left( \int_{\partial M} \|H_S\|^{\rho/(p-1)} \, dv_h \right)^{p-1}.$$
(2) If $\delta > 0$ and $M$ is contained in a ball of radius $R \leq \pi/4\sqrt{\delta}$, then

a) for $1 < p < 2$, we have

$$\sigma_{1,p} \leq \delta^{(p/2)-1}(N + 1)^{(2-p)/2}n^{p/2}V(M)\left(\delta + \frac{\int_{\partial M} \|H_S\|^2 \, dv_g}{\inf(\tr(S))^2 V(\partial M)}\right);$$

b) for $p \geq 2$, we have

$$\sigma_{1,p} \leq (N + 1)^{(p-2)/2}n^{p/2}V(M)\left(\delta + \frac{\int_{\partial M} \|H_S\|^2 \, dv_g}{\inf(\tr(S))^2 V(\partial M)}\right)^{p/2}.$$

2. Preliminaries

Let $(\tilde{M}^N, \tilde{g})$ be an $N$-dimensional Riemannian manifold with sectional curvature $K_{\tilde{M}} \leq \delta$. For $q$ a fixed point in $\tilde{M}$, we denote by $r(x)$ the geodesic distance between $x$ and $q$, and we define the vector field $Z$ by $Z(x) := s_{\delta}(r(x))(\nabla r)(x)$, where $s_{\delta}$ is the function defined by

$$s_{\delta}(r) = \begin{cases} \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}r) & \text{if } \delta > 0, \\ r & \text{if } \delta = 0, \\ \frac{1}{\sqrt{|\delta|}} \sinh(\sqrt{|\delta|}r) & \text{if } \delta < 0. \end{cases}$$

We also define

$$c_{\delta}(r) = \begin{cases} \cos(\sqrt{\delta}r) & \text{if } \delta > 0, \\ 1 & \text{if } \delta = 0, \\ \cosh(\sqrt{|\delta|}r) & \text{if } \delta < 0. \end{cases}$$

Hence, $c_{\delta}^2 + \delta s_{\delta}^2 = 1$, $s_{\delta}' = c_{\delta}$ and $c_{\delta}' = -\delta s_{\delta}$.

To prove Theorem 1.1, we recall some key lemmas. The first, in some sense, extends the Hsiung–Minkowki formulas to spaces of nonconstant curvature.

**Lemma 2.1** (Grosjean, [10]). Let $(\Sigma, g)$ be a compact submanifold of $(\tilde{M}, \tilde{g})$ and $S$ be a symmetric, positive definite and divergence-free $(1, 1)$-tensor on $\Sigma$. Then

1. $\sum_{i=1}^{N} \langle S \nabla Z_i, Z_i \rangle \leq \tr(S) - \delta \langle S\perp Z, Z\perp \rangle$;
2. $\div(SZ^\top) \geq (c_{\delta}(r)\tr(S) + \langle Z, H_S \rangle)$.

If in addition, $\Sigma$ has no boundary,

3. $\int_{\Sigma} c_{\delta}(r)\tr(S) \, dv_g \leq \int_{\Sigma} \|H_S\|s_{\delta}(r) \, dv_g$;
4. $\delta \int_{\Sigma} \langle SZ^\top, Z^\top \rangle \, dv_g \geq \int_{\Sigma} (c_{\delta}^2(r)\tr(S) - \|H_S\|s_{\delta}(r)c_{\delta}(r)) \, dv_g$. 
Here, $H_S$ denotes $\text{tr}(B \circ S)$ and so is a normal vector field and $Z^\top$ is the part of $Z$ tangent to $\Sigma$. Note that if $S = \text{Id}$, we recover the classical inequalities proved by Heintze [11].

To prove the desired upper bounds, we will use the variational characterisation (1.3) of $\sigma_{1,p}$. For this, we need to use appropriate test functions. As usual, for eigenvalue upper bounds for submanifolds, the candidates for test functions are the coordinate functions and their analogues in nonconstant curvature $Z_i = (s_\delta(r)/r)x_i, \ 1 \leq i \leq N$, which are the coordinates of $Z$ in a normal frame $\{e_1, \ldots, e_N\}$. To be eligible to be test functions, we need to ‘centre’ these functions using the following lemma given by Chen in [3].

**Lemma 2.2 [3].** Let $p \in (1, +\infty)$ and assume that $\Sigma$ is a submanifold of $\bar{M}$ contained in a convex ball $B \subset \bar{M}$. Then, there exists $q_0 \in B$ such that for any $i \in \{1, \ldots, N\}$,

$$\int_{\Sigma} \left| \frac{s_\delta(r)}{r}x_i \right|^p \frac{s_\delta(r)}{r}x_i \, dv_g = 0,$$

where $r$ is the distance function to $q_0$ in $\bar{M}$.

For the case $\delta > 0$, we need another test function $c_\delta$. To use it as a test function, we need to translate it appropriately. For this, we recall the following elementary lemma, also given by Chen in [3].

**Lemma 2.3.** Let $\delta > 0$, $p \in (1, +\infty)$ and assume that $\Sigma$ is a submanifold of $\bar{M}$ contained in a ball of centre $q_0$ and radius $\rho \leq \pi/2\sqrt{\delta}$. Then, there exists a constant $c \in [0, 1]$ so that

$$\int_{\Sigma} \left| \frac{c_\delta(r) - c}{\sqrt{\delta}} \right|^p \frac{c_\delta(r) - c}{\sqrt{\delta}} \, dv_g = 0,$$

where $r$ is the distance function to $q_0$ in $\bar{M}$.

Finally, we recall the following technical lemma proved by Manfio and the authors in [15] which will be useful at the end of the proof of Theorem 1.1.

**Lemma 2.4 [15].** Let $(\bar{M}^N, \bar{g})$ be a Riemannian manifold with sectional curvature bounded from above by $\delta$, $\delta > 0$. Let $(\Sigma, g)$ be a closed Riemannian manifold isometrically immersed into $(\bar{M}^N, \bar{g})$ and assume that $\Sigma$ is contained in a geodesic ball of radius $R \leq \pi/2\sqrt{\delta}$. Let $S$ be a symmetric, divergence-free and positive definite $(1, 1)$-tensor on $\Sigma$. Then, we have

$$1 - \left( \frac{\int_{\Sigma} c_\delta(r) \, dv_g}{V(\Sigma)} \right)^2 \geq \frac{1}{1 + \frac{\int_{\Sigma} \|H_S\|^2 \, dv_g}{\delta \inf(\text{tr}(S))^2 V(\Sigma)}}.$$
3. Proof of Theorem 1.1

3.1. The case \( \delta = 0 \). To use the coordinate functions as test functions in the variational characterisation of \( \sigma_{1,p} \), we need to place the coordinate centre at a good point. Therefore, we apply Lemma 2.2 to \( \Sigma = \partial M \) and we consider \( r \) as the distance to the point \( q_0 \) given in Lemma 2.2. Thus, we are able to prove the following lemma.

**Lemma 3.1.** For any \( p \in (1, +\infty) \),

\[
\sigma_{1,p} \int_{\partial M} r^p \, dv_h \leq N^{\frac{p-2}{2}} n^{p/2} V(M).
\]

**Proof.** From Lemma 2.2, we can consider the functions

\[
Z_i = \left( s_\delta (r) / r \right) x_i, \quad 1 \leq i \leq N,
\]

as test functions in the variational characterisation (1.3) of \( \sigma_{1,p} \). For \( \delta = 0 \), we have \( Z_i = x_i \). Taking the summation for \( i \) from 1 to \( N \),

\[
\sigma_{1,p} \int_{\partial M} \sum_{i=1}^{N} |Z_i|^p \, dv_h \leq \int_{M} \sum_{i=1}^{N} \|\nabla Z_i\|^p \, dv_g. \tag{3.1}
\]

We will discuss the cases \( p \geq 2 \) and \( 1 < p < 2 \), separately.

**Case 1:** \( 1 < p < 2 \). Since \( p < 2 \),

\[
r^p = (r^2)^{p/2} = \left( \sum_{i=1}^{N} Z_i^2 \right)^{p/2} \leq \sum_{i=1}^{N} |Z_i|^p. \tag{3.2}
\]

However, by the Hölder inequality (for vectors),

\[
\sum_{i=1}^{N} \|\nabla Z_i\|^p \leq N^{(2-p)/p} \left( \sum_{i=1}^{N} \|\nabla Z_i\|^2 \right)^{p/2}, \tag{3.3}
\]

which gives with Lemma 2.1(1) and \( \delta = 0 \),

\[
\sum_{i=1}^{N} \|\nabla Z_i\|^p \leq N^{(2-p)/p} n^{p/2}. \tag{3.4}
\]

From (3.1), (3.2) and (3.4),

\[
\sigma_{1,p} \int_{\partial M} r^p \, dv_h \leq \sigma_{1,p} \int_{\partial M} \sum_{i=1}^{N} |Z_i|^p \, dv_h \leq \int_{M} \sum_{i=1}^{N} \|\nabla Z_i\|^p \, dv_g \leq N^{(2-p)/2} n^{p/2} V(M). \tag{3.5}
\]

**Case 2:** \( p \geq 2 \). By the Hölder inequality,

\[
r^2 = \sum_{i=1}^{N} |Z_i|^2 \leq N^{(p-2)/p} \left( \sum_{i=1}^{N} |Z_i|^p \right)^{2/p},
\]

\[
\sum_{i=1}^{N} \|\nabla Z_i\|^p \leq N^{(2-p)/p} n^{p/2}. \tag{3.6}
\]
which gives

\[ r^p = N^{(p-2)/2} \left( \sum_{i=1}^{N} |Z_i|^p \right). \]  

(3.6)

Moreover, since \( p \geq 2 \),

\[ \sum_{i=1}^{N} \| \nabla Z_i \|^p \leq \left( \sum_{i=1}^{N} \| \nabla Z_i \|^2 \right)^{p/2}. \]  

(3.7)

Finally, from (3.1), using (3.6), (3.7) and Lemma 2.1(1),

\[
\sigma_{1,p} \int_{\partial M} r^p \, dv_h \leq \sigma_{1,p} N^{(p-2)/2} \int_{\partial M} \sum_{i=1}^{N} |Z_i|^p \, dv_h \\
\leq N^{(p-2)/2} \int_{M} \sum_{i=1}^{N} \| \nabla Z_i \|^p \, dv_g \\
\leq N^{(p-2)/2} \int_{M} \left( \sum_{i=1}^{N} \| \nabla Z_i \|^2 \right)^{p/2} \, dv_g \\
\leq N^{(p-2)/2} n^{p/2} V(M). 
\]

Since \( \delta = 0 \), we have \( c_\delta \equiv 1 \) and Lemma 2.1(3) reduces to

\[
\int_{\partial M} \text{tr}(S) \, dv_h \leq \int_{\partial M} s_\delta(r) \| H_S \| \, dv_h.
\]

Thus,

\[
\sigma_{1,p} \left( \int_{\partial M} \text{tr}(S) \, dv_h \right)^p \leq \sigma_{1,p} \left( \int_{\partial M} s_\delta(r) \| H_S \| \, dv_h \right)^p \\
\leq \sigma_{1,p} \left( \int_{\partial M} s_\delta^p(r) \, dv_h \right) \left( \int_{\partial M} \| H_S \|^{p/(p-1)} \, dv_h \right)^{p-1} \\
\leq N^{(p-2)/2} n^{p/2} V(M) \left( \int_{\partial M} \| H_S \|^{p/(p-1)} \, dv_h \right)^{p-1},
\]

where we have used first the H"older inequality and then Lemma 3.1 since \( s_\delta(r) = r \) when \( \delta = 0 \).

### 3.2. The case \( \delta > 0 \)

In the case \( \delta > 0 \), in addition to the \( Z_i \) terms, we need another test function. For this, from the assumption that \( M \) is contained in a ball of radius \( R \leq \pi/4 \sqrt{\delta} \) and since the point \( q_0 \) in Lemma 2.2 belongs to this ball, we can conclude that \( M \) is contained in a ball of centre \( q_0 \) and radius smaller than or equal to \( \pi/2 \sqrt{\delta} \). Therefore, we can apply Lemma 2.3 to get a constant \( c \in [0, 1] \) so that

\[
\int_{\Sigma} \left( \frac{c_\delta(r) - c}{\sqrt{\delta}} \right)^{p-2} \frac{c_\delta(r) - c}{\sqrt{\delta}} \, dv_g = 0.
\]
The function \( C = (c_\delta(r) - c)/\sqrt{\delta} \) can be used as a test function. From the variational characterisation (1.3) of \( \sigma_{1,p} \) using \( C \) and the \( Z_i \) terms as test functions, we get

\[
\sigma_{1,p} \int_{\partial M} \left( |C|^p + \sum_{i=1}^N |Z_i|^p \right) d\nu_h \leq \int_M \left( \|\nabla C\|^p + \sum_{i=1}^N \|\nabla Z_i\|^p \right) d\nu_r. \tag{3.8}
\]

Moreover,

\[
C^2 + \sum_{i=1}^N Z_i^2 = \left( \frac{c_\delta(r) - c}{\sqrt{\delta}} \right)^2 + \sum_{i=1}^N \left( \frac{s_\delta(r)}{r^i} x_i \right)^2
\]

\[
= s_\delta(r) + \frac{c_\delta^2(r) + c^2 - 2cc_\delta(r)}{\delta}
\]

\[
= 1 + c^2 - 2cc_\delta(r), \tag{3.9}
\]

where we have used \( c_\delta^2 + \delta s_\delta^2 = 1 \). However, we also have

\[
\nabla C = \nabla \left( \frac{c_\delta(r) - c}{\sqrt{\delta}} \right) = -\sqrt{\delta} s_\delta(r) \nabla r = \sqrt{\delta} Z^\perp,
\]

so that

\[
\|\nabla C\|^2 + \sum_{i=1}^N \|\nabla Z_i\|^2 = \delta \|Z^\perp\|^2 + \sum_{i=1}^N \|\nabla Z_i\|^2 \leq \delta \|Z^\perp\|^2 + (n - \delta \|Z^\perp\|^2) = n, \tag{3.10}
\]

where we have used Lemma 2.1(1). Note here that \( Z^\perp \) is the part of \( Z \) tangent to \( M \). We now consider the cases \( 1 < p < 2 \) and \( p \geq 2 \) separately.

**Case 1:** \( 1 < p < 2 \). Since \( p < 2 \),

\[
|C|^p + \sum_{i=1}^N |Z_i|^p = \frac{1}{\delta^{p/2}} \left( |\cos(\sqrt{\delta} r) - c|^p + \sum_{i=1}^N \left| \sin(\sqrt{\delta} x_i/r) \right|^p \right). \tag{3.11}
\]

Since \( |\sin(\sqrt{\delta}) x_i/r| \leq 1 \), \( |\cos(\sqrt{\delta} r) - c| < 1 \) and \( 1 < p < 2 \),

\[
\left| \sin(\sqrt{\delta}) \frac{x_i}{r} \right|^p \geq \left| \sin(\sqrt{\delta}) \frac{x_i}{r} \right|^2 \quad \text{and} \quad |\cos(\sqrt{\delta} r) - c|^p \geq |\cos(\sqrt{\delta} r) - c|^2,
\]

which after substituting into (3.11) gives

\[
|C|^p + \sum_{i=1}^N |Z_i|^p \geq \frac{1}{\delta^{p/2}} \left( |\cos(\sqrt{\delta} r) - c|^2 + \sum_{i=1}^N \left| \sin(\sqrt{\delta} x_i/r) \right|^2 \right)
\]

\[
= \frac{1}{\delta^{(p/2)-1}} \left( C^2 + \sum_{i=1}^N Z_i^2 \right)
\]

\[
= \frac{1}{\delta^{p/2}} (1 + c^2 - 2cc_\delta(r)), \tag{3.12}
\]
where we have used (3.9) for the last line. However, by the Hölder inequality,

$$
\|\nabla C\|_p^p + \sum_{i=1}^N \|\nabla Z_i\|_p^p \leq (N + 1)^{(2-p)2} \left( \|\nabla C\|^2 + \sum_{i=1}^N \|Z_i\|^2 \right)^{p/2}
\leq (N + 1)^{(2-p)2} \, n^{p/2}
$$

(3.13)

by using (3.10). From (3.8) together with (3.12) and (3.13),

$$
\sigma_{1,p} \int_{\partial M} (1 + c^2 - 2cc_\delta(r)) \, dv_h \leq \delta^{p/2} (N + 1)^{(2-p)/2} n^{p/2} V(M).
$$

(3.14)

Moreover,

$$
\int_{\partial M} (1 + c^2 - 2cc_\delta(r)) \, dv_h = V(\partial M) \left( 1 + c^2 - 2c \frac{\int_{\partial M} c_\delta(r) \, dv_h}{V(\partial M)} \right)
= V(\partial M) \left( 1 + \left( c - \frac{\int_{\partial M} c_\delta(r) \, dv_h}{V(\partial M)} \right)^2 - \left( \frac{\int_{\partial M} c_\delta(r) \, dv_h}{V(\partial M)} \right)^2 \right)
\geq V(\partial M) \left( 1 - \left( \frac{\int_{\partial M} c_\delta(r) \, dv_h}{V(\partial M)} \right)^2 \right).
$$

(3.15)

Substituting this into (3.14) yields

$$
\sigma_{1,p} \int_{\partial M} (1 - \left( \frac{\int_{\partial M} c_\delta(r) \, dv_h}{V(\partial M)} \right)^2) \, dv_h \leq \delta^{p/2} (N + 1)^{(2-p)/2} n^{p/2} V(M).
$$

(3.16)

Finally, by Lemma 2.4,

$$
\sigma_{1,p} \leq \delta^{p/2-1} (N + 1)^{(2-p)/2} n^{p/2} V(M) \left( \frac{V(M)}{\inf(\text{tr}(S))^2 V(\partial M)} \right).
$$

Case 2: $p \geq 2$. Since $p \geq 2$,

$$
\|\nabla C\|^p + \sum_{i=1}^N \|Z_i\|^p \leq \left( \|\nabla C\|^2 + \sum_{i=1}^N \|Z_i\|^2 \right)^{p/2} \leq n^{p/2},
$$

(3.17)

where we have used (3.10). However, by the Hölder inequality,

$$
\left( C^2 + \sum_{i=1}^N Z_i^2 \right)^{p/2} \leq (N + 1)^{(p-2)/2} \left( |C|^p + \sum_{i=1}^N |Z_i|^p \right).
$$

(3.18)
Thus, using successively (3.9), (3.18), (3.8) and (3.17),
\[
\sigma_{1,p/2} \int_{\partial M} \left( \frac{1 + c^2 - 2cc_\delta(r)}{\delta} \right)^{p/2} d\nu_h = \sigma_{1,p} \int_{\partial M} \left( C^2 + \sum_{i=1}^{N} Z_i^2 \right)^{p/2} d\nu_h \\
\leq \sigma_{1,p} (N + 1)^{(p-2)/2} \int_{\partial M} \left( |C|^p + \sum_{i=1}^{N} |Z_i|^p \right) d\nu_h \\
\leq (N + 1)^{(p-2)/2} \int_{M} \left( |\nabla C|^p + \sum_{i=1}^{N} |\nabla Z_i|^p \right) d\nu_h \\
\leq (N + 1)^{(p-2)/2} n^{p/2} V(M). \quad (3.19)
\]

In addition, from the Hölder inequality (for integrals),
\[
\int_{\partial M} \left( \frac{1 + c^2 - 2cc_\delta(r)}{\delta} \right)^{p/2} d\nu_h \geq V(\partial M)^{(2-p)/2} \left( \int_{\partial M} \frac{1 + c^2 - 2cc_\delta(r)}{\delta} d\nu_h \right)^{p/2}. \quad (3.20)
\]
Hence, we deduce from (3.19) with (3.15) and (3.20),
\[
\sigma_{1,p} \frac{V(\partial M)}{\delta^{p/2}} \left[ 1 - \left( \frac{\int_{\partial M} c_\delta(r) d\nu_h}{V(\partial M)} \right)^2 \right]^{p/2} \leq \sigma_{1,p} V(\partial M)^{(2-p)/2} \left( \int_{\partial M} \frac{1 + c^2 - 2cc_\delta(r)}{\delta} d\nu_h \right)^{p/2} \leq (N + 1)^{(p-2)/2} n^{p/2} V(M). \quad (3.21)
\]
Finally, we use Lemma 2.4 to conclude that
\[
\sigma_{1,p} \leq (N + 1)^{(p-2)/2} n^{p/2} \frac{V(M)}{V(\partial M)} \left( \delta + \frac{\int_{\partial M} ||H||^2 d\nu_g}{\inf_{\partial M} ||\text{tr}(S)||^2 V(\partial M)} \right)^{p/2}.
\]
This completes the proof of Theorem 1.1. \(\square\)

4. New results for the \(p\)-Laplacian when \(\delta > 0\)

We finish by giving similar results for the first eigenvalue of the \(p\)-Laplacian for closed submanifolds when \(\delta > 0\). We will not give all the details of the proof since it is similar to the proof of Theorem 1.1. The difference is that the variational characterisation of \(\lambda_{1,p}\) is
\[
\lambda_1 = \inf \left\{ \frac{\int_{M} ||\nabla u||^p d\nu_g}{\int_{M} |u|^p d\nu_g} \mid u \in W^{1,p}(M) \setminus \{0\}, \int_{M} |u|^{p-2} u d\nu_g = 0 \right\}.
\]
In this case, \(M\) has no boundary and so we can apply Lemmas 2.2 and 2.3 with \(\Sigma = M\) to use the functions \(C\) and \(Z_i, 1 \leq i \leq N\), as test functions. By completely similar computations, we obtain the analogue of (3.16) if \(1 < p < 2\), that is,
\[
\lambda_{1,p} V(M) \left( 1 - \left( \frac{\int_{\partial M} c_\delta(r) d\nu_h}{V(M)} \right)^2 \right) d\nu_h \leq \delta^{p/2} (N + 1)^{(2-p)/2} n^{p/2} V(M),
\]
and of (3.21) if $p \geq 2$, that is,

$$
\lambda_{1,p} \leq \frac{V(M)}{\delta^{p/2}} \left[ 1 - \left( \frac{\int_M c_\delta(r)}{V(M)} \right)^2 \right]^{p/2} \leq (N + 1)^{(p-2)/2} n^{p/2} V(M).
$$

Finally, applying Lemma 2.4 to $M$, we deduce the following result.

**Theorem 4.1.** Let $\delta > 0$, $p \in (1, +\infty)$ and $(\bar{M}^N, \bar{g})$ be a Riemannian manifold of sectional curvature bounded from above by $\delta$. Let $(M^n, g)$ be a closed Riemannian manifold isometrically immersed into $(\bar{M}, \bar{g})$ and $S$ a symmetric, positive definite and divergence-free $(1, 1)$-tensor on $\partial M$. We denote by $\lambda_{1,p}$ the first eigenvalue of the $p$-Laplacian on $M$. Suppose $M$ is contained in a ball of radius $R \leq \pi/4 \sqrt{\delta}$.

1. If $1 < p < 2$,

$$
\lambda_{1,p} \leq \delta^{(p/2)-1} (N + 1)^{(2-p)/2} n^{p/2} \left( \delta + \frac{\int_M \|H_S\|_g^2}{\inf_M (\text{tr} (S))^2 V(M)} \right).
$$

2. If $p \geq 2$,

$$
\lambda_{1,p} \leq (N + 1)^{(p-2)/2} n^{p/2} \left( \delta + \frac{\int_M \|H_S\|_g^2}{\inf_M (\text{tr} (S))^2 V(M)} \right)^{p/2}.
$$

If $S = \text{Id}$, we recover the result of Chen [3].

**References**

[1] L. J. Alias and J. M. Malacarné, ‘On the first eigenvalue of the linearised operator of the higher order mean curvature for closed hypersurfaces in space forms’, *Illinois J. Math.* 48(1) (2004), 219–240.

[2] F. Browder, ‘Existence theorems for nonlinear partial differential equations’, in: *Global Analysis*, Proceedings of the Symposia in Pure Mathematics, XVI (eds. S.-S. Chern and S. Smale) (American Mathematical Society, Providence, RI, 1970), 1–60.

[3] H. Chen, ‘Extrinsic upper bound of the eigenvalue for $p$-Laplacian’, *Nonlinear Anal.* 196 (2020), Article no. 111833.

[4] H. Chen and X. Gui, ‘Reilly-type inequalities for submanifolds in Cartan–Hadamard manifolds’, Preprint, 2022, arXiv:2206.11164.

[5] H. Chen and G. Wei, ‘Reilly-type inequalities for $p$-Laplacian on submanifolds in space forms’, *Nonlinear Anal.* 184 (2019), 210–217.

[6] F. Du and J. Mao, ‘Reilly-type inequalities for $p$-Laplacian on compact Riemannian manifolds’, *Front. Math. China* 10(3) (2015), 583–594.

[7] A. El Soufi and S. Ilias, ‘Une inégalité du type “Reilly” pour les sous-variétés de l’espace hyperbolique’, *Comment. Math. Helv.* 67(2) (1992), 167–181.

[8] A. Girouard and I. Polterovich, ‘Spectral geometry of the Steklov problem’, *J. Spectr. Theory* 7(2) (2017), 321–359.

[9] J. F. Grosjean, ‘Upper bounds for the first eigenvalue of the Laplacian on compact manifolds’, *Pacific J. Math.* 206(1) (2002), 93–111.

[10] J. F. Grosjean, ‘Extrinsic upper bounds for the first eigenvalue of elliptic operators’, *Hokkaido Math. J.* 33(2) (2004), 319–339.

[11] E. Heintze, ‘Extrinsic upper bounds for $\lambda_1$’, *Math. Ann.* 280(3) (1988), 389–402.
[12] S. Ilias and O. Makhoul, ‘A Reilly inequality for the first Steklov eigenvalue’, *Differential Geom. Appl.* **29**(5) (2011), 699–708.

[13] A. Lê, ‘Eigenvalue problems for the $p$-Laplacian’, *Nonlinear Anal.* **64**(5) (2006), 1057–1099.

[14] B. P. Lima, J. F. B. Montenegro and N. L. Santos, ‘Eigenvalue estimates for the $p$-Laplace operator on manifolds’, *Nonlinear Anal.* **72** (2010), 771–781.

[15] F. Manfio, J. Roth and A. Upadhyay, ‘Extrinsic eigenvalues upper bounds for submanifolds in weighted manifolds’, *Ann. Global Anal. Geom.* **62** (2022), 489–505.

[16] R. C. Reilly, ‘On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space’, *Comment. Math. Helv.* **52** (1977), 525–533.

[17] J. Roth, ‘General Reilly-type inequalities for submanifolds of weighted Euclidean spaces’, *Colloq. Math.* **144**(1) (2016), 127–136.

[18] J. Roth, ‘Reilly-type inequalities for Paneitz and Steklov eigenvalues’, *Potential Anal.* **53**(3) (2020), 773–798.

[19] J. Roth, ‘Extrinsic upper bounds for the first eigenvalue of the $p$-Steklov problem on submanifolds’, *Commun. Math.* **30**(1) (2022), Article no. 5.

[20] O. Torné, ‘Steklov problem with an indefinite weight for the $p$-Laplacian’, *Electron. J. Differential Equations* **87** (2005), 1–8.

[21] S. Verma, ‘Upper bounds for the first nonzero eigenvalue related to the $p$-Laplacian’, *Proc. Indian Acad. Sci.* **130** (2020), Article no. 21.

JULIEN ROTH, Université Gustave Eiffel, CNRS, LAMA UMR 8050, F-77447 Marne-la-Vallée, France

e-mail: julien.roth@univ-eiffel.fr

ABHITOSH UPADHYAY, School of Mathematics and Computer Science, Indian Institute of Technology, Goa 403401, India

e-mail: abhitosh@iitgoa.ac.in