Stability analysis of discrete population balance model for bubble growth and shrinkage

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Abstract
The stability condition for solving the population balance equation (PBE) involving bubble growth and shrinkage within the Eulerian framework is proposed. The particle flux weighted average Courant number, $\overline{C}_{\text{CFL}}$, which reflects the propagation of particle state of the entire size classes on internal coordinates is first derived. Stability conditions of internal convection for PISO and PIMPLE algorithms are obtained by evaluating a series of tests concerning the constant bubble growth. The proposed stability conditions are then applied to simulate two kinds of laboratory experiments, that is, the bubble growth in stagnant liquid and condensing steam-water pipe flows. The results show that the stable PBE solutions hold for the cases using either PISO or PIMPLE algorithms combined with the corresponding stability condition. Meanwhile, the conservation of the zeroth and first moments of the number density function can be guaranteed when the stability condition is satisfied. In addition, the effect of heat transfer coefficient correlations is also discussed.

KEYWORDS
class method, discretization, internal CFL condition, phase change, population balance equation, stability analysis

1 | INTRODUCTION

Gas–liquid two-phase flow is important from the environmental, chemical, and many other industrial processes point of view as it provides large interfacial area for mass and heat transfer. By focusing on the interfacial structure, it is noticed that such multiphase flow in general shows a spectrum of different bubble sizes, which originates from various interfacial interaction, such as nucleation, coalescence and breakup, growth, and shrinkage.\textsuperscript{1,2} Special attention should be paid to the local bubble size distribution (BSD) and its change as well, considering their remarkable effects on the interfacial transfer, thus thermal and hydrodynamic features of both gas and liquid phases. In Eulerian framework of two-phase flow modeling, bubble size changes are normally quantified by coupling the population balance model (PBM) together with the classical two-fluid model (TFM). As of yet, the coupled TMF-PBM has been extensively applied in two-phase flow simulation.\textsuperscript{3–9}

Ramkrishna\textsuperscript{10} derived the population balance equation (PBE) based on the generalized continuum mechanical concept to describe the density of a suitable extensive variable, usually the particle number. Because of the complexity of
aforementioned density-changing processes, that is, coalescence and breakup, growth and shrinkage, the source and sink terms in the PBE often contain integrals, and its analytical solution is possible in limited cases. Numerous work has been performed to improve the PBM on basis of the fundamental work mentioned above. The efforts have been devoted in two main fields. One is developing kernel functions that describe the rate of bubble generation and disappearance, and the other is improving the numerical methods for solving the PBE. In adiabatic bubbly flows without mass and heat transfer, bubble coalescence and breakup play a major role in determining BSDs. A large number of kernels have been proposed, ranging from the classical ones of Coulaloglou and Tavlarides,11 Prince and Blanch,12 Luo and Svendsen13 to recent developments of Liao et al.,14 Shen and Hibiki,15 Shi et al.,16 Karimi and Andersson,17 Zhang et al.,18 and Gong et al.19 Systematic reviews concerning coalescence and breakup were presented by Liao and Lucas.20,21 In case of bubbly flows with phase change, for example, subcooled wall boiling, appropriate closure models for nucleation, growth, and shrinkage should be considered additionally.22,23 Despite huge efforts the performance of kernel functions is far from satisfying, which is beyond the scope of this article.

In view of numerical aspects, a variety of methods have been proposed for the solution of the integrodifferential PBE.24 Among them sectional, stochastic and moment-based methods are representative ones.25-27 The reader is referred to Ramkrishna,1 Rigopoulos,28 Tadmor,29 Bayraktar,30 and She et al.31 for detailed reviews. Note that in this work we restrict our attention to the sectional method (or class method) as it is widely used in the coupled TFM-PBM application for gas–liquid flows. The class method is based on the discretization of internal coordinate space into a number of “classes,” over which the distribution of variables is approximated by their mean values in the classes. On this point the PBE is also referred to as discrete PBE. Here the internal coordinate denotes the particle size, while the external one for the physical space. A wide range of discretization schemes and corresponding consistency and convergence analyses for the source and sink terms resulting from coalescence and breakup are well documented.32-37 The multiple size group (MUSIG) model is one example of the discrete PBM.38,39 It was extended by Lucas et al.40 to include bubble growth and shrinkage due to phase change. Recently, Liao et al.41 and Liao42 updated the MUSIG model by proposing a consistent discrete formulation, which guarantees the preservation of both number and mass of bubbles during the coalescence and breakup, while the original one in ANSYS CFX preserves only the mass.

On the other hand, the stability analysis, which sets limits on the space and time intervals of numerical grids, is rare in the literature. As is well known, the particle state is transported through the space of both internal and external coordinates simultaneously in PBE. Given an evaporation process, the growth of a bubble represents motion along the coordinate of increasing size, while condensation indicates the opposite. In the absence of bubble coalescence and breakup, if we increase the time step while keeping the resolution of internal coordinates fixed (or decrease the resolution while keeping the time step fixed) under a bubble growth/shrinkage rate known a priori, the class method may turn to be unstable and produce incorrect results. This indicates that the choice of resolution of internal grids (or time step) may not be independent of time step (or resolution of internal grids). A necessary condition, similar to the Courant–Friedrichs–Lewy (CFL) condition, which so far is routinely used for external coordinates, is required here to ensure the stability and convergence of the class method.

The objective of this article is to present such a stability condition for solving PBE involving bubble growth and shrinkage. The finite difference scheme of PBE in this article comes from the multiphaseEulerFoam solver in OpenFOAM, which is based on the class method mentioned above.2,41 A series of numerical experiments are carried out to obtain the stability condition, and to throw some light upon the stability mechanism. Note that OpenFOAM uses the finite volume method to discretize a set of fundamental transport equations ascribed on any 3D unstructured polyhedral grids, while the finite difference approximation is applied to deal with the one-dimensional PBE. The remaining article is organized as follows. Section 2 reviews the PBM as well as the finite difference scheme in OpenFOAM. Section 3 contains a set of numerical experiments with constant bubble growth rates. Section 4 provides the simulation of bubble growth in stagnant liquid and condensing steam-water flows based on the presented stability condition. Conclusions are summarized in Section 5.

2 MATHEMATICAL MODEL

The numerical study is performed in the framework of coupled TFM-PBM. Considering that the TFM theory and its fundamental transport equations have been described at length in many references, for example, Drew,42 Ishii and Hibiki,43 Liao et al.,44 the mathematical description is focused on the PBM, that is, the class method, as well as the interphase mass and heat transfer modeling related to bubble growth and shrinkage in phase change.
2.1 The class method

As discussed before, the class method partitions the dispersed phase into a number of size classes. The physical phenomena affecting size change such as evaporation and condensation can be considered for each class. To account for the size-dependence of phase velocity or interphase forces, the size classes can be further grouped into different velocity groups, which are treated as individual phases and governed by own transport equations. According to the number of velocity groups, it is often referred to as homogeneous or inhomogeneous class method, see Figure 1. The symbols $v_i, x_i$ denote the boundary and representative values of the class $i$, respectively.

The model has been widely used for poly-dispersed gas–liquid flows, for example, by Cheung et al., Deju, Guillen et al., Liao et al., for isothermal vertical pipe flows; by Yeoh and Tu, Krepper and Rzehak, Peltola et al., for subcooled wall boiling; by Lifante et al., Liao et al., Liao and Lucas for steam condensation in subcooled water; by Liao and Lucas for evaporating steam-water flow under depressurization conditions. Nevertheless, no discussions on the stability regarding time step and class width are available. We envisage a process of bubble growth and shrinkage arising only from phase change, the governing equation for size fraction in PBM can be written as:

$$\frac{\partial}{\partial t}(\alpha_g \phi_i) + \nabla_X \cdot (\alpha_g \phi_i u_g) + \int_{v_i}^{v_{i+1}} \nabla_v (x_i n(v) G(v)) dv = 0,$$

where $\nabla_X$ and $\nabla_v$ denote the partial derivatives with respect to the external and internal coordinates $X$ and $v$, respectively. $\phi_i$ represents the size fraction of class $i$, which belongs to the velocity group $j$. The last term on the left-hand side of Equation (1) represents the direct mass transfer rate between size classes due to bubble growth or shrinkage. $n(v)$ and $G(v)$ represent the volume fraction and the growth function of a bubble, respectively.
G(ν) symbolize the number density and volume change rate, respectively. They are defined as:

\[ n = \frac{dN}{dv}, \quad (2) \]

where \( N \) represents the number concentration.

\[ G = \frac{dv}{dt}. \quad (3) \]

It is related to the interfacial mass transfer flux, which will be explained in detail later.

### 2.2 Discretization of PBE

In order to solve the Equation (1) numerically, the last term involving bubble growth and shrinkage mechanisms has to be discretized regarding to the internal coordinate \( v \), which is not always straightforward.\(^{56,57}\) Most of existing discretization methods either suffer from numerical diffusion or preserve only one moment. In the PBM in OpenFOAM, the method proposed by Kumar\(^{56}\) is applied.

\[
\int_{\nu_i}^{\nu_{i+1}} \nabla_v (x_i n(\nu) G(\nu)) d\nu = x_i \left( w_i \frac{G_{i-1} N_{i-1}}{x_i - x_{i-1}} - p_i G_i N_i + e_i \frac{G_{i+1} N_{i+1}}{x_i - x_{i+1}} \right),
\]

where \( G_{i-1}, G_i, G_{i+1} \) denote the mean growth rate of the size class \( i-1, i \) and \( i+1 \), respectively. The coefficients \( w_i \) and \( e_i \) are defined as:

\[
w_i = \begin{cases} 0 & \text{if } i = 0 \text{ or } G_i \leq 0, \\ 1 & \text{else}, \end{cases} \quad (5)
\]

\[
e_i = \begin{cases} 0 & \text{if } i = n \text{ or } G_i \geq 0, \\ 1 & \text{else}, \end{cases} \quad (6)
\]

and

\[
p_i = \begin{cases} (x_{i+1} - x_i)^{-1} & \text{if } i < I \text{ and } G_i > 0, \\ (x_{i-1} - x_i)^{-1} & \text{if } i > 0 \text{ and } G_i < 0, \\ (-x_i)^{-1} & \text{else}. \end{cases} \quad (7)
\]

The method is similar to the first-order upwind differencing scheme, but it is capable of preserving the first two moments of the number density function provided that there are no bubbles present in the largest size class (growth) or in the smallest one in case of shrinkage. Furthermore, the consistent feature is independent of computational grids used in the computation with respect to the internal coordinate. On the other hand, the upwind discretization loses the preservation of number concentration while applied on nonhomogeneous grids or with a nonconstant growth rate. Nevertheless, both discretizations suffer from numerical diffusion. A fine grid is often necessary for reliable solutions as discussed in Kumar and Ramkrishna,\(^{35}\) Qamar and Warnecke,\(^{57}\) and Bertin et al.\(^{37}\) In this work, we will unveil another important point regarding numerical instability. That is, the refinement of grids has to be done together with time step restriction. Otherwise, the simulation may become unstable and the preservation of moments will be violated. A criterion similar to the CFL condition needs to be introduced. The connection between intervals of size class and time step will be discussed in detail below.

### 2.3 Stability condition of internal convection

As we can see from Equation (1), the two convection terms (external and internal) share a same time step, but have a different advection velocity and space coordinate. In OpenFOAM, the middle term on the right-hand side of Equation (4)
are treated implicitly while the other terms are solved explicitly. Lapidus and Pinder\textsuperscript{58} pointed out that most explicit finite difference approximates are conditionally stable and have restrictive bounds on the mesh widths, above which the numerical solution is unstable. For generally convective-dominated partial differential equations, numerical instability of external convective term arising from explicit treatment can be obviated by making use of the CFL condition, which has the following form:\textsuperscript{59}

\[ C_{\text{CFL},X} = \frac{1}{2} \frac{\Delta t}{\Delta V} \sum_{f} |\phi_f| \leq C_{\text{CFL},\text{max}}, \]  

(8)

where $\Delta t$ is the time step, $\Delta V$ is the cell volume, $\phi_f = \mathbf{u}_f \cdot \mathbf{S}_f$ is the volumetric face flux, with $\mathbf{u}_f$ being the face-center velocity and $\mathbf{S}_f$ being the surface-normal area vector, and $\sum_f$ denotes the summation over all cell faces. The CFL condition commonly provides an upper limit on the time step of a given resolution. The external Courant number $C_{\text{CFL},X}$ is typically less than or equal to unity, which implies that the information cannot propagate through more than one physical grid cell at each time step. On the other hand, the condition also ensures that the numerical domain of dependence of solution is larger than the physical one.

As it was mentioned before, information of particle states is transported by two “passages,” that is, the external physical space and the internal property space. The CFL condition in Equation (8) which limits the external propagation cannot be directly applied to the internal one even within the same time domain. Therefore, we seek to remedy this situation by proposing a similar stability condition for the internal convection of explicit treatment. The local Courant number of a size class can be written as:

\[ C_{\text{CFL},i} = \frac{\Delta t}{\Delta V} \frac{G_i}{\Delta V_i}, \]  

(9)

where $G_i$ and $\Delta V_i$ represent the volume change rate and the width of class $i$.

We first notice that each size class has its own Courant number. It is commensurate with the practical situation owing to the fact that a size class may differ from one to another in growth/shrinkage rate or width of size classes. However, the CFL condition is nonlocal.\textsuperscript{60} It means that the local Courant number cannot guarantee the satisfying of CFL condition within the whole particle property space. The most straightforward way to ensure the stability condition is to enable it globally using the largest growth/shrinkage rate and the smallest interval in the entire internal coordinates. While this approach can be extremely computationally intensive as it forces those size classes, which are stable even with a relatively large time step, to advance now with a much smaller one.

We introduce a computationally efficient alternative which is based on a weighted average of particle flux as follows:

\[ \bar{C}_{\text{CFL}} = \sum_{i=1}^{I} \varphi_i C_{\text{CFL},i} \]  

(10)

with

\[ \varphi_i = \frac{G_i N_i}{\sum_{i=1}^{M} G_i N_i} \]  

(11)

The averaged Courant number $\bar{C}_{\text{CFL}}$ weighted by particle flux fraction $\varphi_i$ involves the contribution of information propagation of all size classes. Under the condition of constant growth/shrinkage rate and uniform width of size classes, the averaged Courant number is identical to the general one in Equation (9).

In addition to the stability condition for PBE, the numerical algorithm also requires careful attention as it affects the numerical stability strongly.\textsuperscript{61} Two of the most well-known segregated algorithms are pressure-implicit with splitting of operators (PISO)\textsuperscript{62} and PIMPLE,\textsuperscript{63} the latter is a combination of PISO and SIMPLE, where SIMPLE stands for semi-implicit method for pressure linked equations.\textsuperscript{64} Both PISO and PIMPLE algorithms follow the same procedure for discretization but differ in the outer correctors, which enables looping over the entire system of equations within one time step. In OpenFOAM, there is a parameter called number of outer correctors (NOC) that controls the switch between PISO (NOC = 1) and PIMPLE (NOC > 1). In Section 3, the acceptable range of the $\bar{C}_{\text{CFL}}$ combining with different algorithms is evaluated by carrying out a series of numerical tests.
2.4 Interphase mass and heat transfer

The volume change rate of a bubble in the class $i$, $G_i$, is related to the volumetric mass flow rate $I_i$ in the way:

$$G_i = \frac{I_i}{\rho_g N_i}$$

(12)

with $I_i$ further connected to the interphase mass transfer rate:

$$I_i = f_i d_{32g} \Gamma_{ig,j},$$

(13)

where $\Gamma_{ig,j}$, $d_{32g}$, respectively denote the mass transfer rate from liquid to the liquid–vapor interface and the Sauter mean diameter of the velocity group $j$, to which the class $i$ belongs. For the description of phase change processes, the thermal phase change model is adopted, which envisions the phase change (condensation or evaporation) induced by interphase heat transfer. Furthermore, the vapor phase and the interface are assumed to be always saturated. In other words, only the liquid-side heat transfer contributes to the phase change process. Based on total heat balance the mass flow rate is derived as:

$$\Gamma_{lg,j} = \frac{h_{lg}(T_i - T_{sat})A_{lg,j}}{L_{g,sat} - L_{l,inter}},$$

(14)

where $T_{sat}$, $L_{g,sat}$ represent the saturation temperature and vapor saturation enthalpy, $L_{l,inter}$ denotes the interfacial value of liquid enthalpy. It is set to the corresponding saturation value in the case of condensation while to local liquid enthalpy in evaporation. By assuming a spherical bubble shape, the interfacial area density $A_{lg,j}$ is given by:

$$A_{lg,j} = \frac{6a_{lg,j}}{d_{32g,j}}.$$

(15)

The Sauter mean bubble diameter for each velocity group can be calculated by means of its definition:

$$d_{32g,j} = \frac{1}{\sum_i I_i d_{lg,i}^2},$$

(16)

where $I$ denotes the total number of size classes in the velocity group $j$. In the application of thermal phase change model for describing phase change processes, an appropriate correlation for the heat transfer coefficient $h_{lg,j}$ is of crucial importance, which is often characterized using the Nusselt number $N_{u_{lg,j}}$:

$$N_{u_{lg,j}} = \frac{h_{lg} d_{32g,j}}{\lambda_l}.$$

(17)

There have been a large body of work on interfacial heat transfer analyses. Numerous analytical and empirical correlations for the calculation of interphase heat transfer coefficient are available. Some of them are listed in Table 1. They correlated the Nusselt number further with other dimensionless numbers characterizing the hydrodynamic and thermodynamic conditions like the Jakob number $Ja$, Reynolds number $Re_{lg,j}$, and Péclet number $Pe_{lg,j}$. For conduction-dominated heat transfer, the effect of slip velocity between spherical vapor bubbles and surrounding liquid can be neglected, and the analytical correlations are able to predict the bubble growth in short-time intervals. While for those cases where the momentum effect plays a key role on interphase heat transfer, one may expect a better prediction using the empirical models presented in Table 1. For example, the Ranz–Marshall correlation, which derived from the experiment on spherical water droplets evaporating in warm air, has been used in subcooled nucleate boiling, flashing cases, and many others, although the valid range of Reynolds number is limited. Tomiyama proposed a similar formulation which enables simulations of high $Re_{lg,j}$ cases. Recently, a mechanistic model considering heat conduction, convection, and turbulence intensification was developed by Liao et al. The total Nusselt number is formulated by cumulating the effect of the three
TABLE 1 Correlations for evaluating interphase heat transfer coefficient

| Reference          | Correlation                           | Mechanism           | Note       |
|--------------------|---------------------------------------|---------------------|------------|
| Fritz and Ende    | \[ \text{Nu}_{ij} = \frac{4}{\pi} J_a \]  | Conduction          | Analytical |
| Forster and Zuber | \[ \text{Nu}_{ij} = \pi J_a \]         | Conduction          | Analytical |
| Plesset and Zwick | \[ \text{Nu}_{ij} = \frac{12}{\pi} J_a \] | Conduction          | Analytical |
| Olek et al.       | \[ \text{Nu}_{ij} = \frac{J_a}{\frac{\pi}{2} \left[ 1 + \left( 1 + \frac{2 \pi}{3} \right)^{1/2} \right]^2} \] | Conduction          | Analytical |
| Ranz and Marshall | \[ \text{Nu}_{ij} = 2 + 0.6 \text{Re}^{1/3} \text{Pr}^{1/3} \text{Peg}^{1/3} \text{Pet}^{1/3} \] | Conduction, convection | Empirical |
| Tomiyama          | \[ \text{Nu}_{ij} = 2 + 0.15 \text{Re}^{1/3} \text{Pr}^{1/3} \text{Peg}^{1/3} \text{Pet}^{1/3} \] | Conduction, convection | Empirical |
| Liao et al.       | \[ \text{Nu}_{ij} = \frac{12}{\pi} J_a + \frac{2}{\sqrt{\pi}} \text{Pe}_{ij}^{1/2} + \frac{2}{\sqrt{\pi}} \text{Pe}_{t}^{1/2} \] | Conduction, convection, turbulence | Heuristic |

mechanisms. The associated dimensionless numbers in the correlations are defined by:

\[
\begin{align*}
\text{Re}_{gd} & = \frac{\rho_l |\textbf{u}_{gd} - \textbf{u}| d_{32g}}{\mu_l} \\
\text{Pr}_{l} & = \frac{c_p l \mu_l}{\lambda_l} \\
\text{Ja} & = \frac{\rho_l c_p l |T_{sat} - T_l|}{\rho_g L} \\
\text{Pe}_{t} & = \frac{l_t u_t}{D_t} \\
\text{Pe}_{g} & = \frac{d_{32g} |\textbf{u}_{gd} - \textbf{u}|}{D_t}
\end{align*}
\]

where \( \text{Pe}_{t} \) represents the turbulent Péclet number; the length and velocity scale are determined in terms of turbulent kinetic energy \( k_t \) and dissipation rate \( \epsilon_t \):

\[
\begin{align*}
l_t & = C_{\mu}^{3/4} k_t^{3/2} \\
u_t & = C_{\mu}^{1/4} k_t^{3/2}
\end{align*}
\]

with the value of 0.09 for the constant \( C_{\mu} \). The correlations in Table 1 are adopted in this work for different test simulations shown below.

3 | STABILITY ANALYSIS

Stability conditions combined with PISO and PIMPLE algorithms are determined by performing a series of numerical tests based on the multiphaseEulerFoam solver in OpenFOAM. A cubic domain with single-cell resolution is utilized as the physical space to eliminate the effect of discretization of external coordinates. The buoyancy is deactivated by setting the gravity acceleration to zero. There is no relative motion between gas and liquid phases, thus the BSDs can be specified through several size classes but one velocity group, that is, the homogeneous class method applied here. Another noteworthy aspect of the numerical tests is that the underrelaxation strategy is not involved as it is beyond the scope of the present work.

3.1 | Stability condition for PISO algorithm

The test cases with constant bubble growth rates are listed in Table 2, in which the analytical solutions are available from Ramabhadran et al.. Note that the local and averaged Courant number can be varied by changing either the time step \( \Delta t \) or the resolution of internal grids \( I \) according to Equations (9) and (10). For a specific case, we perform a set of tests covering a wide range of time steps (Cases 1–6) or resolution (Cases 7 and 8) to identify the Courant number \( C_{\text{CFL}} \) or \( \overline{C}_{\text{CFL}} \). In addition, Cases 1–3 and 7 possess equally distributed internal coordinates, and the width of their classes \( \Delta \nu \) is therefore constant. In other words, their local Courant number \( C_{\text{CFL}} \) is equal to their averaged one. For convenience, we use the averaged Courant number \( \overline{C}_{\text{CFL}} \) to represent the internal convection for these cases. Exponential distribution of internal coordinates are specified for Cases 4–6 and 8. All the results are obtained at the run time \( t \), which is determined
| Case | $x_i$ (m$^3$) | $[x_{\text{min}}, x_{\text{max}}]$ | $I$ or $\Delta t$ or $\xi$ | $n_i$ (m$^{-3}$m$^{-3}$) | $G$ (m$^3$s$^{-1}$) | $[\Delta t_{\text{min}}, \Delta t_{\text{max}}]$ | $t$ (s) |
|------|--------------|----------------|-----------------|----------------|----------------|-----------------|-----|
| 1    | $x_{\text{min}} + \frac{x_{\text{max}} - x_{\text{min}}}{n_i}$ | [0.001, 0.016] | 31  | \[1 \quad 0.002 \leq x_i \leq 0.004, \] \[0 \quad \text{else} \] | 0.001 | [2.5 \times 10^{-5}, \ 3] | 6   |
| 2    | $x_{\text{min}} + \frac{x_{\text{max}} - x_{\text{min}}}{n_i}$ | [1 \times 10^{-5}, 0.5] | 31  | $\exp \left( -\frac{x_i}{0.05} \right)$ | 0.005 | [2 \times 10^{-4}, \ 20] | 20  |
| 3    | $x_{\text{min}} + \frac{x_{\text{max}} - x_{\text{min}}}{n_i}$ | [1 \times 10^{-5}, 0.5] | 31  | $\frac{x_i}{0.05} \exp \left( -\frac{x_i}{0.05} \right)$ | 0.005 | [2 \times 10^{-4}, \ 20] | 20  |
| 4    | $6.81 \times 10^{-6}$ exp[0.38(i + 1)] | [1 \times 10^{-5}, 10] | 37  | $\exp \left( -\frac{x_i}{0.05} \right)$ | 0.005 | [1 \times 10^{-5}, \ 10] | 20  |
| 5    | $6.81 \times 10^{-6}$ exp[0.38(i + 1)] | [1 \times 10^{-5}, 10] | 37  | $\frac{x_i}{0.05} \exp \left( -\frac{x_i}{0.05} \right)$ | 0.005 | [1 \times 10^{-5}, \ 10] | 20  |
| 6    | $6.31 \times 10^{-6}$ exp[0.46(i + 1)] | [1 \times 10^{-5}, 10] | 31  | \[1 \times 10^5 \quad 3 \times 10^{-5} \leq x_i \leq 8.2 \times 10^{-4}, \] \[0 \quad \text{else} \] | 0.001 | [1 \times 10^{-4}, \ 5] | 10  |
| 7    | $x_{\text{min}} + \frac{x_{\text{max}} - x_{\text{min}}}{n_i}$ | [0.001, 0.016] | 16,9376 | 1 \quad 0.002 \leq x_i \leq 0.004, \] \[0 \quad \text{else} \] | 0.001 | 0.01 | 6   |
| 8    | $6.31 \times 10^{-6}$ exp[0.46(i + 1)] | [1 \times 10^{-5}, 10] | 16,3841 | \[1 \times 10^5 \quad 3 \times 10^{-5} \leq x_i \leq 8.2 \times 10^{-4}, \] \[0 \quad \text{else} \] | 0.001 | 2 \times 10^{-4} | 10  |
according to the size of the internal coordinates and the growth rate of each case. The parameter NOC is set to 1 for all cases, implying that the PISO algorithm is used to solve the discretized PBE.

Figure 2 shows the simulation results for the cases which have uniform distribution of internal coordinates. The vertical axis represents the Sauter mean bubble diameter $d_{32}$, normalized by that of the smallest $C_{\text{CFL}}$ considered in that case. Note that the averaged $C_{\text{CFL}}$ indicated in horizontal axis is identical to the local $C_{\text{CFL}}$ due to the constant growth rate and uniform internal coordinates. It is observed that the calculated Sauter mean bubble diameter maintains stable until $C_{\text{CFL}}$ approaches 0.01. With further increase in $C_{\text{CFL}}$ (by increasing time step $\Delta t$ or decreasing width of size classes), the predicted bubble diameter deviates largely from the stable value, leading to the so-called numerical instability.

Results for the cases having exponential distribution of internal coordinates are plotted in Figure 3. The prediction of the normalized bubble diameter for these cases gets worse as the averaged $C_{\text{CFL}}$ exceeds a critical value, that is, $C_{\text{CFL}_{\text{cr}}} = 0.01$. If we take a close look at the local $C_{\text{CFL}}$ of each size class, we find that for those stable cases which have $C_{\text{CFL}}$ smaller than 0.01, their local $C_{\text{CFL}}$s are allowed partially to be higher than the $C_{\text{CFL}_{\text{cr}}}$. For example, in Case 5 where $C_{\text{CFL}} = 2.54 \times 10^{-3}$, the local $C_{\text{CFL}}$ of the smallest size class is of the order $O(10^1)$. It thus turns out that the stability condition evaluated by $C_{\text{CFL}}$ is more computationally efficient than that by forcing the local $C_{\text{CFL}}$ of all size classes to be lower than a certain value.

Furthermore, we check how the internal Courant number affects the time development of the moments during bubble growth. Similar to Kumar and Ramkrishna, the first two moments of number density function, that is, the total bubble number and total bubble volume, are defined as:

\begin{align}
M_0(t) &= \sum_{i=1}^{I} N_i, \\
M_1(t) &= \sum_{i=1}^{I} x_i N_i,
\end{align}

where $I$ represents the total number of size classes. Figure 4 depicts the time evolution of total bubble number for Cases 1 and 7, respectively. The ratio $M_0(t)/M_0(0)$, which reflects the extent of bubble aggregation, should keep unchanged due to the absence of bubble coalescence and breakup. For the cases with $C_{\text{CFL}}$ smaller than $C_{\text{CFL}_{\text{cr}}}$, the ratios of zeroth moment $M_0(t)/M_0(0)$ maintain stable, even though slight deviations from the analytical solution are observed. While as the $C_{\text{CFL}}$ rises beyond the $C_{\text{CFL}_{\text{cr}}}$, the solutions tend to lose the stability as time progresses. Similar situation can also be found in Figure 5, which shows the evolution of total bubble volume. Clearly, the discretization method has failed in preserving both the total bubble number and total bubble volume once the stability condition $C_{\text{CFL}} \leq C_{\text{CFL}_{\text{cr}}} = 0.01$ is
Figure 3 shows the local Courant number \( C_{\text{CFL}} \) versus representative volume \( x \) and Sauter mean bubble diameter \( d_{32} \) normalized by \( d_{32}(C_{\text{CFLmin}}) \) versus averaged Courant number \( \overline{C}_{\text{CFL}} \) for cases with exponential distribution of internal coordinates. The blue solid lines with point markers correspond to the secondary axes which represent the normalized Sauter mean bubble diameter as a function of \( \overline{C}_{\text{CFL}} \). For a specific \( \overline{C}_{\text{CFL}} \), the distributions of \( C_{\text{CFL}} \) within internal coordinates are plotted on the primary axes [Colour figure can be viewed at wileyonlinelibrary.com]

violated. Bearing in mind that we do not attempt to compare the simulation result with the analytical one at this stage, but it will be discussed in the following.

Figure 6 shows the time evolution of BSDs of Cases 1 and 7 under different \( \overline{C}_{\text{CFL}} \). We follow Kumar and Ramkrishna\(^{33} \) and call the sharp variation of number density as a front, and as can be seen from these figures the front moves to large bubble size at a slower rate with the increase in the \( \overline{C}_{\text{CFL}} \). In other words, increasing the \( \overline{C}_{\text{CFL}} \) gives rise to an under-prediction of bubble size, which is consistent with the observations in Figure 2. The same conclusion is drawn for the cases with nonuniform internal coordinates (Cases 4–6 and 8). It is also found that the numerical diffusion in Case 7 (Figure 6B) is reduced from small \( \overline{C}_{\text{CFL}} \) to large one due to the refinement of the internal grids. In addition, one may expect an over-estimation of bubble size for a shrinkage situation when \( \overline{C}_{\text{CFL}} \geq \overline{C}_{\text{CFL,cr}} = 0.01 \), due to the slow shift of the front from large size to small size.
3.2 Stability condition for PIMPLE algorithm

It can be seen from last section that the stability condition \( \overline{C}_{\text{CFL}} \leq \overline{C}_{\text{CFL,cr}} = 0.01 \) should be strictly satisfied when using PISO algorithm, otherwise, poor solutions together with the violation of conservation can be occurred. The PIMPLE algorithm offers more robustness and efficiency as it provides the outer iteration, which is controlled by specifying the NOC, to improve the stability.\(^76\) Cases 1 and 6 in Table 2 are selected to further investigate the effect of NOC on numerical stability. The configuration of these cases regarding internal grids, BSDs, and growth rates keeps unchanged, while the rest of the setup is updated in Table 3.

Figure 7 shows the normalized Sauter mean bubble diameter as a function of NOC and \( \overline{C}_{\text{CFL}} \), respectively. We first notice that the NOC has limited effect on the cases which satisfy the stability condition \( \overline{C}_{\text{CFL}} \leq \overline{C}_{\text{CFL,cr}} = 0.01 \). For the case with \( \overline{C}_{\text{CFL}} < 0.1 \), NOC \( \subseteq [2, 5] \) is suggested to achieve the numerical stability. When the \( \overline{C}_{\text{CFL}} \) rises up to the order of 0.1 or higher, more than five outer iterations are needed to obtain a stable solution. This pattern can also be reflected in Figure 8, in which the distribution of bubble number density converges to a stable state using NOC = 2 for \( \overline{C}_{\text{CFL}} = 0.05 \), while it requires at least NOC = 5 for \( \overline{C}_{\text{CFL}} = 0.5 \). This is a clear indication that the stability condition is also indispensable for PIMPLE algorithm, even NOC can be taken as a tool to alleviate numerical instability.
Let us now briefly describe the main stability criterion for the solution of PBE: (1) The stability condition $C_{\text{CFL}} \leq C_{\text{CFL,cr}} = 0.01$ is valid for both PISO and PIMPLE algorithms; (2) When $C_{\text{CFL}}$ lies within the range $[0.01, 0.1]$, the PIMPLE algorithm with $NOC \geq 2$ needs to be introduced, and $2 \leq NOC \leq 5$ is suggested to balance the numerical stability and computational efficiency; (3) For cases of $0.1 < C_{\text{CFL}} < 1$, at least five outer iterations ($NOC \geq 5$) are required to obtain a stable solution; (4) The $C_{\text{CFL}}$ is not recommended to exceed 1 because the numerical instability can occur even with large enough NOC. These stability conditions can also be found in Figure 9 which exhibits a stability regime map for both PISO and PIMPLE algorithms.

Furthermore, Case 8 in Table 2 is selected to investigate the difference between the stable solution of PISO algorithm and that of PIMPLE algorithm. The updated configuration of Case 8 is listed in Table 4, and the rest keeps consistent with Table 2. Note that the first three tests of Case 8 employ the PISO algorithm and satisfy the stability condition $C_{\text{CFL}} \leq$...
FIGURE 7  Sauter mean bubble diameter \( \frac{d_{32}}{d_{32}} \) normalized by \( d_{32} \text{(NOC}_{\text{max}}) \) versus NOC for Cases 1 and 6

TABLE 5  Configuration of cases for bubble growth in stagnant liquid

| Case | \( \Delta T \) (K) | \( Ja \) | \( Nu \) | \( [d_{\text{min}}, d_{\text{max}}] \) | \( \Delta t \) (m) | \( \Delta t \) (s) | \( \overline{C}_{\text{CFL}} \) | NOC | Stability condition |
|------|-----------------|------|------|-----------------|--------------------|--------------------|-----------------|------|---------------------|
| 1    | 3.1             | 9.29 | 11.83 | \([1 \times 10^{-5}, 0.002]\) | \(1.99 \times 10^{-5}\) | \(5 \times 10^{-7}\) | 0.009           | 1    | \( \overline{C}_{\text{CFL}} \leq \overline{C}_{\text{CFL,cr}} = 0.01 \) |
| 2    | 3.1             | 9.29 | 11.83 | \([1 \times 10^{-5}, 0.002]\) | \(1.99 \times 10^{-5}\) | \(5 \times 10^{-6}\) | 0.09            | 3    | \( 2 \leq \text{NOC} \leq 5 \) for \( 0.01 < \overline{C}_{\text{CFL}} < 0.1 \) |
| 3    | 3.1             | 9.29 | 11.83 | \([1 \times 10^{-5}, 0.002]\) | \(1.99 \times 10^{-5}\) | \(5 \times 10^{-5}\) | 0.9             | 1    | Unsatisfied         |
| 4    | 4.5             | 13.48 | 17.17 | \([1 \times 10^{-4}, 0.005]\) | \(4.9 \times 10^{-5}\) | \(5 \times 10^{-7}\) | 0.003           | 1    | \( \overline{C}_{\text{CFL}} \leq \overline{C}_{\text{CFL,cr}} = 0.01 \) |
| 5    | 4.5             | 13.48 | 17.17 | \([1 \times 10^{-4}, 0.005]\) | \(4.9 \times 10^{-5}\) | \(5 \times 10^{-6}\) | 0.03            | 3    | \( 2 \leq \text{NOC} \leq 5 \) for \( 0.01 < \overline{C}_{\text{CFL}} < 0.1 \) |
| 6    | 4.5             | 13.48 | 17.17 | \([1 \times 10^{-4}, 0.005]\) | \(4.9 \times 10^{-5}\) | \(5 \times 10^{-5}\) | 0.3             | 1    | Unsatisfied         |

\( \overline{C}_{\text{CFL,cr}} = 0.01 \), and the following tests make use of the PIMPLE algorithm and also meet the stability condition \( 2 \leq \text{NOC} \leq 5 \) for \( 0.01 < \overline{C}_{\text{CFL}} < 0.1 \). Figure 10 shows the distribution of bubble number density as well as the normalized Sauter mean bubble diameter for different resolution of internal grids. It is observed that the numerical diffusion is obviously improved due to the increase in the number of size classes. Moreover, the results obtained by PISO algorithm coincide with that by PIMPLE algorithm, indicating that stable solutions can be achieved by either PISO or PIMPLE algorithm on the premise of satisfying the corresponding stability conditions.

### 4 | APPLICATION OF THE STABILITY CONDITION

The stability condition derived in last section will be applied to two kinds of laboratory experiments. One is bubble growth in stagnant liquid and the other is condensing steam-water pipe flow, which have been investigated in many references.

#### 4.1 | Bubble growth in stagnant liquid

Two experimental cases involving stationary bubble growth in water–steam system are simulated to further validate the stability condition. The experimental data are taken from the work of Plesset and Zwick.\(^73\) As illustrated in Table 5, bubble growth is controlled by evaporation under superheat degrees of 3.1 and 4.5 K, respectively. The saturation conditions and corresponding material properties are calculated based on the IAPWS-IF97 database, which represents the accurate equation of state for water and steam properties. The analytical correlation from Fritz and Ende\(^71\) in Table 1 is applied to
obtain the $Nu$, leading to different bubble growth rates according to Equations (12), (13), and (17). The internal coordinates are uniformly distributed based on the bubble diameter. Apart from Cases 3 and 6, the rest of the cases satisfy the specific stability condition for PISO (NOC = 1) and PIMPLE (NOC > 1) algorithm.

Figure 11 shows the time evolution of the simulated and experimental bubble radii. Stable numerical solutions are obtained when stability conditions are satisfied. Both Cases 3 and 6 underpredict the bubble radius due to the violation of stability condition, and it is also related to the lag of front shifting along the direction of increasing bubble size, which has been explained in Section 3.1. It can be seen from Figures 12 and 13 that the total bubble number and total bubble volume preserve well for all cases except Cases 3 and 6. Recall that for the constant bubble growth in Section 3, the bubble size has no effect on bubble growing. While when it turns to the growth rate controlled by phase change, the underprediction of bubble diameter (Figure 11) will inevitably increase the interphase mass transfer rate by increasing the interfacial area density according to Equation (15), the gas volume fraction is therefore over-estimated, resulting in the unconservation of the total bubble volume for Cases 3 and 6.

We now return to bubble growth curve in Figure 11, at the early stage of bubble growth (0–2 ms), the measured bubble size is well captured by the current heat transfer correlation, while it underpredicts the bubble growth rate substantially with the increase in bubble size whether the $\overline{C}_{CFL}$ satisfies the stability condition. It is conceivable that the accuracy of the
Figure 10 Bubble number density \( n \) versus representative bubble volume \( x \) and Sauter mean bubble diameter \( d_{32}(NOC = 3) \) normalized by \( d_{32}(NOC = 1) \) versus number of size classes \( I \). The blue solid lines (empty squares) correspond to the secondary axes which represent the normalized Sauter mean bubble diameter as a function of \( I \). For a specific \( I \), the distributions of \( n \) within internal coordinates are plotted on the primary axes using black solid lines (solid symbols) [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 11 Bubble radius \( r \) versus time \( t \) for bubble growth in stagnant liquid

predictions depends not only on the resolution of the internal coordinates, but also on the applicability of the interphase heat transfer models.

4.2 Condensing steam-water pipe flow

Under practical conditions, bubbly flow is often turbulent, and interphase heat and mass transfer as well as bubble growth and shrinkage are commonly believed to be a result of joint effects of heat conduction, convection, and turbulence. The stability condition with homogeneous class method involving complete interactions between phases are extended to simulate the condensing steam-water flow.
4.2.1 | TOPFLOW facility

The condensation experiments were performed at the TOPFLOW facility of the Helmholtz-Zentrum Dresden-Rossendorf. The test section of the facility consists of a vertical pipe with an inner diameter of 195.3 mm and a height about 8 m. On one hand, the saturated steam is injected into the pipe via orifices which distribute equally over the circumference of the pipe wall. On the other hand, the subcooled water is supplied from the bottom of the pipe by means of an isolating valve and a 90° bend. A cocurrent upward condensation flow occurs as the steam contacts the subcooled water. The evolution of gas volume fraction and BSDs are measured by the wire-mesh sensors (WMS). The distance between gas injection and WMS can be varied by adjusting the location of the former, that is, the so-called variable gas injection method. In this way, the flow developing along the pipe can be reproduced and the results are comparable to those obtained by moving the WMS while fixing the injection. The height of each level relative to the steam injection is given in Table 6. There are in total around 100 test cases covering a broad range of operating parameters, such as injection pressure, superficial velocity, degree of initial subcooling, and diameter of the gas injection orifices. The reader is referred to Lucas and Prasser and Lucas et al. for more information. One of those cases with the combination of $p = 10$ bar, $U_t = 1.017$ m s$^{-1}$, $U_g = 0.219$ m s$^{-1}$, and $\Delta T_{in} = 5.0$ K is investigated numerically in the present work.
TABLE 6  Heights relative to location of gas injection

| Level | A  | C  | F  | I  | L  | O  |
|-------|----|----|----|----|----|----|
| Height (m) | 0.221 | 0.335 | 0.608 | 1.552 | 2.595 | 4.531 |

(A) Schematic diameter of the axially symmetrical wedge  (B) A quasi 2D mesh

FIGURE 14  Computational domain

4.2.2  Computational domain and boundary conditions

To avoid unnecessary computational costs, a wedge covering 5° of the circumferences is selected as the physical domain (Figure 14A), and a quasi-2D mesh with one layer of cells in the circumferential direction is applied (Figure 14B). This simplification is reasonable since both the geometry and the flow field inside the pipe are axially symmetrical. The boundary conditions are the same as that in the work of Liao and Lucas. Considering the vortex perturbations below level A, measurements at level A are used as the inlet condition. Meanwhile, level O is regarded as the outlet as beyond which the gaseous phase is already exhausted. The inlet boundary conditions including volume fraction, gas velocity, BSDs as well as liquid temperature are defined according to experimental data. While due to the lack of data for liquid phase, the inlet velocity, which satisfies the liquid superficial velocity, is assumed to have an identical profile as the measured gas velocity with accompanying velocity differences. In addition, the turbulence parameters are postulated to be that of fully developed single phase flow.

4.2.3  Closure models and discretization

The TFM framework coupling with the homogeneous class method are used to describe the kinetic and thermodynamic behaviors of the gas and liquid flow. The main closures are listed in Table 7. The Ranz–Marshall correlation, Tomiyama correlation, and Liao correlation listed in Table 1 are applied to calculate the heat transfer coefficient. According to Liao and Lucas, the variation of bubble sizes in the selected case is dominated by condensation; therefore, bubble coalescence and breakup are not included in the closure models.

The arrangement of the external mesh (or coordinates) is taken from the work of Liao and Lucas, in which 25 and 625 nodes are employed in the X and Y directions, respectively. The reader is referred to Liao and Lucas for a detailed description. To obtain a stable external solution, a series of simulations of single phase flow under different Δt are performed. The stability of the external coordinates can be achieved by setting Δt ≤ 0.001 s.
### TABLE 7  Closures implemented in TFM framework

| Closure         | Subterm                  | Model                                                   |
|-----------------|--------------------------|---------------------------------------------------------|
| Mass            | Mass flux                | Thermal phase change model (Equation 13)                |
| Momentum        | Drag                     | Ishii and Zuber model<sup>79</sup>                     |
|                 | Lift                     | Tomiyama model<sup>80</sup>                            |
|                 | Turbulent dispersion     | Burns model<sup>81</sup>                               |
|                 | Virtual mass             | Constant virtual mass coefficient set to 0.5           |
|                 | Wall lubrication         | Hosokawa model<sup>82</sup>                            |
| Heat            | Heat transfer coefficient| Ranz–Marshall correlation<sup>66</sup>                 |
|                 |                          | Tomiyama correlation<sup>70</sup>                      |
|                 |                          | Liao correlation<sup>46</sup>                          |
| Turbulence      | Liquid                   | k – ω SST model<sup>83</sup>                           |
|                 | Bubble induced turbulence| Ma model<sup>84</sup>                                  |
|                 | Gas                      | Laminar                                                 |
| Poly-dispersity | Growth and collapse      | Homogeneous class method (Equation 1)                   |
|                 | Coalescence and breakup  | --                                                      |

### TABLE 8  Configuration of cases for condensing steam-water pipe flows

| Case | Heat transfer correlation | [d<sub>min</sub>, d<sub>max</sub>] | Δd (m) | Δt (s) | C<sub>CFL</sub> | NOC | Stability condition                                                                 |
|------|---------------------------|-------------------------------------|--------|--------|-----------------|-----|-------------------------------------------------------------------------------------|
| 1    | Ranz–Marshall correlation | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 1 × 10<sup>-4</sup> | 0.002           | 1   | C<sub>CFL</sub> ≤ C<sub>CFL,cr</sub> = 0.01                                               |
| 2    | Ranz–Marshall correlation | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 5 × 10<sup>-4</sup> | 0.008           | 3   | C<sub>CFL</sub> ≤ C<sub>CFL,cr</sub> = 0.01                                               |
| 3    | Ranz–Marshall correlation | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 0.001  | 0.017           | 4   | 2 ≤ NOC ≤ 5 for 0.01 < C<sub>CFL</sub> < 0.1                                         |
| 4    | Tomiyama correlation      | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 2 × 10<sup>-4</sup> | 0.021           | 4   | 2 ≤ NOC ≤ 5 for 0.01 < C<sub>CFL</sub> < 0.1                                         |
| 5    | Liao correlation          | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 0.001  | 0.034           | 4   | 2 ≤ NOC ≤ 5 for 0.01 < C<sub>CFL</sub> < 0.1                                         |
| 6    | Ranz–Marshall correlation | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 0.005  | 0.083           | 1   | Unsatisfied                                                                         |
| 7    | Ranz–Marshall correlation | [5 × 10<sup>-4</sup>, 0.019]       | 5 × 10<sup>-4</sup> | 0.010  | 0.165           | 5   | Unsatisfied                                                                         |
The discretization of the internal coordinates are presented in Table 8. One velocity group, that is, the homogeneous class method is considered here. The size classes are divided equally based on bubble diameter, resulting in nonuniform distribution of bubble volume. The averaged $C_{\text{CFL}}$ is estimated through the initial subcooling degree and the initial Sauter mean bubble diameters, both of which are expected to decrease with the development of the condensing flow, resulting in the decline in $C_{\text{CFL}}$. Therefore, once the initial $C_{\text{CFL}}$ satisfies the stability condition, the solution can maintain stable in the following calculation. Cases 1 to 3 utilize the Ranz–Marshall correlation, and PISO algorithm is applied to Case 1 and PIMPLE algorithm for Cases 2 and 3. Cases 4 and 5 employ different heat transfer correlations but same solution algorithms. The stability conditions are satisfied for Cases 1 to 5 while violated for Cases 6 and 7. Note that, the last two cases which possess high $C_{\text{CFL}}$ directly crashed at the beginning of calculation, so only the results of stable cases are provided in the following discussion. On the other hand, we keep them to remind us that the importance of the stability conditions for complicated simulations. According to the experimental results, the condensing steam-water flow will ultimately turn into a steady state. In the simulation a maximum equation residual of $10^{-5}$ and a maximum global mass imbalance of $10^{-3}$ are defined as the convergence criteria.

4.2.4 Results

Figure 15 shows the development of the cross-sectional averaged Sauter mean bubble diameter and cross-sectional averaged gas volume fraction $\alpha_g$ versus pipe height $H$ for condensing steam-water flows. The Sauter mean bubble diameter calculated in Case 1 is apparently lower than that in Cases 2 and 3. It must be kept in mind that the aforementioned stability condition is established for the solution of PBE. For pure bubble growth cases (see in Sections 3 and 4.1) where only PBE is involved, stable solutions can be guaranteed as long as the stability condition is satisfied. While for practical cases such as the condensing flow, not only the PBE but also other governing equations with explicit difference approximation are coupled in the whole system of equations, and the stability of those equations should also be considered. The discrepancy of solutions for Case 1 arises due to the focus only on the PBE but neglecting other equations. This situation is improved in Cases 2 and 3 by employing the PIMPLE algorithm, in which the NOC allows the outer iterations to promote the stability of the entire system. A similar evolution trend of the cross-sectional averaged gas volume fraction is observed for all cases due to the same heat transfer correlation is applied. Considering the fact that the gas volume fraction is relatively small at the middle to upper part of the pipe, the effect of Sauter mean bubble diameter on the condensation rate at these regions is limited according to Equations (14) and (15). But one may expect an profound effect for evaporation cases as illustrated in Section 4.1. Up to this point, the PIMPLE algorithm combined with the proposed stability condition is more appropriate for the simulation of practical situations, which involve multiple interactions among phases.
FIGURE 16  Cross-sectional averaged Sauter mean bubble diameter $d_{32}$ versus pipe height $H$ and cross-sectional averaged gas volume fraction $\alpha_g$ versus pipe height $H$ for different heat transfer correlations.

FIGURE 17  Radial gas volume fraction versus pipe radius.
Another noticeable point is that the gas volume fraction are underpredicted by Cases 1 to 3 compared with the experimental values as a consequence of small heat transfer coefficients predicted by the Ranz–Marshall correlation.66 Cases 4 and 5 provide the results using the other two correlations, that is, the Liao46 and Tomiyama70 correlations listed in Table 1. As shown in Figure 16, the gas volume fraction obtained by Liao46 and Tomiyama70 correlations conform well with the measurements due to the higher condensation rates, and consequently the Sauter mean bubble diameter becomes smaller than the experimental values as well as the Ranz–Marshall predictions. Note that the experimental bubble diameter increases from level C to level F and then decreases continuously at the upper part. The increase of bubble size mainly arise from the effect of bubble coalescence, which is not included in the modeling work. Therefore, this trend at the beginning of the pipe cannot be captured by the simulation. The decrease of bubble size and gas volume fraction is condensation dominated, which is consistent with the simulation results.

The radial distributions of gas volume fraction at different height positions are shown in Figure 17. The experimental distribution at level A is used as inlet conditions for simulation. The gas volume fraction at level A exhibits a wall-peak profile since the steam is injected from the pipe wall \((R = 1 \text{ m})\). Furthermore, the peak of the profile migrates slightly toward the pipe center \((R = 0 \text{ m})\) from level A to level F under the joint effect of lift and wall lubrication forces. Satisfying agreements between the simulation and measurements are obtained for Liao46 and Tomiyama70 correlations.

5 CONCLUSIONS

Modeling of gas–liquid two-phase flow in the Eulerian framework has received considerable attention. The poly-dispersity of the phases can be simulated by coupling the TFM together with the PBM. The concept of class method provides an effective way to solve the PBE. For processes involving growth and shrinkage mechanisms, the discretization of PBE encounters numerical difficulties in approximation of partial derivatives. An upwind difference formulation is published in the multiphaseEulerFoam solver in OpenFOAM. In the absence of coalescence and breakup mechanisms, the stability condition for the discretization method is discussed. Similar to the general CFL condition, a local \(C_{\text{CFL}}\) of each size class and a global particle flux weighted \(C_{\text{CFL}}\) is proposed to evaluate the propagation of particle state on the internal coordinates.

The tests focusing on constant bubble growth with uniform and nonuniform internal coordinates are carried out first to determine the stability condition. The effect of solution algorithms including PISO and PIMPLE is also discussed. For PISO algorithm, the simulation results become unstable when \(C_{\text{CFL}}\) approaches the critical value \(C_{\text{CFL,cr}} = 0.01\). The discretization formulation is found to fail in preserving both the total bubble number and the total bubble volume once the stability condition is violated. Meanwhile, the front of bubble number density moves slower from large size to small size with the increase in \(C_{\text{CFL}}\), resulting in an underprediction of the bubble size. Similar conclusions can be drawn for bubble shrinkage by condensation. Using the global \(C_{\text{CFL}}\) instead of the local \(C_{\text{CFL}}\) as the stability criteria improves the computational efficiency because some of the size classes are permitted to have larger \(C_{\text{CFLs}}\). For PIMPLE algorithm, the parameter NOC which determines the number of outer loops over the entire system of equations is introduced. With the increase in \(C_{\text{CFL}}\), more outer loops are required to ensure the numerical stability, and \(2 \leq \text{NOC} \leq 5\) for \(0.01 < C_{\text{CFL}} < 0.1\) is recommended considering the stability and computational efficiency. A stability regime map for the solution of PBE is also obtained as a guide toward the determination of \(C_{\text{CFL}}\) and NOC.

The proposed stability condition combined with the TFM-PBM coupled model are applied in simulating two kinds of laboratory experiments. The first one involves the bubble growth in stagnant liquid, where interphase heat transfer is governed by heat conduction. The underestimation of the bubble size is observed when \(C_{\text{CFL}}\) lies outside the stability condition. Meanwhile, the discretization formulation fails in preserving not only the total bubble number, but also the total bubble volume. For bubble growth controlled by phase change, the mass transfer rate is affected by the variation of the interfacial area density, which is different from the case with a constant bubble growth rate. With the increase in \(C_{\text{CFL}}\), the calculated bubble size decreases, giving rise to a large interfacial area density and a high mass transfer rate. The gas volume fraction or the first moment of the number density function therefore deviates largely from its stable solutions.

The second application is confined to the simulation of turbulent steam-water flows. The condensation experiments were performed at the TOPFLOW facility, and one of those cases is investigated numerically. The results show that the stability condition of the PIMPLE algorithm performs better than that of the PISO algorithm when simulating practical flows. This is essentially because of the increase in the NOC that guarantees the stability of the entire system of equations. The effect of heat transfer correlations on accuracy of condensing simulation is also discussed. Three heat transfer correlations are implemented into OpenFOAM. Although the Liao correlation46 and Tomiyama correlation70 give better
prediction than the classical Ranz–Marshall correlation, development of a general model accounting for all heat transfer mechanisms appropriately demands further efforts. In addition, the discrepancy between simulations and experiments induced by neglect of bubble coalescence and breakup will be the subject for future work.

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DATA AVAILABILITY STATEMENT
The data that support the findings of this study are available from the corresponding author upon reasonable request.

NOMENCLATURE

- $A$: interfacial area density (m$^{-1}$)
- $c_p$: isobaric specific heat capacity (J kg$^{-1}$ K$^{-1}$)
- $C_{\mu}$: standard eddy viscosity model constant (–)
- $d$: bubble diameter (m)
- $d_{32}$: Sauter mean bubble diameter (m)
- $D$: thermal diffusivity (m$^2$ s$^{-1}$)
- $e$: coefficient in discretization formulation of PBE (–)
- $f$: size fraction (–)
- $G$: growth/collapse rate (m$^3$ s$^{-1}$)
- $h$: heat transfer coefficient (W m$^{-2}$ K$^{-1}$)
- $H$: pipe height (m)
- $k$: turbulent kinetic energy (m$^2$ s$^{-2}$)
- $l$: length scale (m)
- $L$: specific enthalpy (J kg$^{-1}$)
- $M_0$: total particle number of number density function (m$^{-3}$)
- $M_1$: total particle volume of number density function (–)
- $n$: number density (m$^{-3}$ m$^{-3}$)
- $N$: number concentration (m$^{-3}$)
- $r$: bubble radius (m)
- $R$: pipe radius (m)
- $S$: surface-normal area vector (m$^2$)
- $t$: time (s)
- $\Delta t$: time step (s)
- $T$: temperature (K)
- $\Delta T$: superheat degree (K)
- $u$: velocity vector (m s$^{-1}$)
- $U$: superficial velocity (m s$^{-1}$)
- $v$: internal space/boundary of representative volume (m$^3$)
- $\Delta v$: width of size classes (m$^3$)
- $\Delta V$: cell volume (m$^3$)
- $w$: coefficient in discretization formulation of PBE (–)
- $x$: representative volume (m$^3$)
- $X$: external space (m)

Greek letters

- $\alpha$: void fraction (–)
- $\Gamma$: mass transfer flux (kg m$^{-3}$ s$^{-1}$)
- $\varepsilon$: turbulent dissipation rate (–)
- $\lambda$: thermal conductivity (W m$^{-1}$ K$^{-1}$)
- $\mu$: dynamic viscosity (Pa s)
\( \rho \) density (kg m\(^{-3}\))
\( \varphi \) particle flux fraction (–)
\( \Phi \) volumetric face flux (m\(^{-3}\)s\(^{-1}\))

**Dimensionless numbers**

- \( C_{\text{CFL}} \) Courant number (–)
- \( \overline{C_{\text{CFL}}} \) averaged Courant number (–)
- \( Ja \) Jakob number (–)
- \( Nu \) Nusselt number (–)
- \( Pe \) Péclet number (–)
- \( Pr \) Prandtl number (–)
- \( Re \) Reynolds number (–)

**Subscripts**

- \( cr \) critical value
- \( f \) index of cell surface
- \( g \) gaseous phase
- \( i \) index of size classes
- \( in \) initial stage
- \( \text{inter} \) interface
- \( I \) number of size classes
- \( j \) index of velocity groups
- \( J \) number of velocity groups
- \( l \) liquid phase
- \( \text{max} \) maximum value
- \( \text{min} \) minimum value
- \( \text{sat} \) saturation
- \( t \) turbulence

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