Structural stability in the model of spatial economy

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Abstract. Urban problems have become very complicated as a result of technological progress and change in the behavior of human beings. The urban systems of our epoch are characterized by increasing spatial and temporal variation. Many urban models have been suggested to explain and forecast urban pattern formation in urban economics, regional science and geography. Within the framework of the spatial economy model, equilibrium states of commodity flows and labor, which are described by two-dimensional vector fields, are considered. These configurations are obtained on the assumption of the validity of divergent and gradient laws, which characterize the influence on the commodity flows and labor by external factors, such as the distance from the places of habitation of labor to industrial objects, the local costs of transportation and so on. The structural stability of such a system for various types of configuration perturbations is investigated. It is very difficult to obtain general characteristics of structurally stable systems, but in the two-dimensional case it turns out to be possible to draw precise picture of structurally stable flows and spatial organization of the economy corresponding to such flows. In the paper possible flow structures for a hyperbolic umbilical for a cubic potential function that depends on three parameters are constructed numerically in accordance with the concept of catastrophe theory. Bifurcation varieties are found in the parameter space, flux fields for various parameter combinations are constructed. A qualitative change in the character of the flow structure in the neighbourhood of a bifurcation manifold is demonstrated.

1. Introduction

Modern science, first of all, is based on a systematic approach to the study of phenomena and processes occurring in the development of various systems. At the present time, such a concept as the self-organization of an open system is increasingly encountered. That is, the system develops according to some of its internal laws under the influence of external influences or influences, adapting at the expense of certain internal resources to existence in the external environment.

When considering economic activity in a particular spatial region, the importance is not only of the geographic location of various objects, but also the structure of settlement of inhabitants in this territory, the location of recreational zones and industrial infrastructure. In addition, transport networks and their capacity in this territory are very important. In fact, all these factors form a single sphere of life of human society, which develops, taking into account each of the factors, within the framework of a single social and production system.

When considering and analyzing each separately chosen territory, special attention should be given to the degree of development of the given area, to take into account the level of development of the transport system. Due to the heterogeneity of the socio-economic development of the territories, it is
necessary to look for ways of optimal and most profitable development of the production, transport and social infrastructure.

When analyzing and modeling the development of individual territories, it is extremely interesting to preserve the characteristics and properties of the territory under consideration as a social and production system with changes in various factors that affect it. One of these properties is the structural stability of the dynamical system, the concept of which was first proposed by A.A. Andronov and L.S. Pontryagin. Dynamic systems are said to be structurally stable if a small perturbation does not qualitatively change the evolution equations. Mathematical aspects of structural stability are discussed in the book by V.I. Arnold [1].

We consider the evolution of the urban system as an analogue of the problem of structural stability of vector fields. Economic activity is characterized by a certain time and place, so it is reasonable to take into account not only temporal, but also spatial dependence in calculations in the modeling of nonlinear development of economic systems. Under the influence of external influences or due to structural instability, as a result of self-consistent evolution, changes in the topology of the urban system can occur. Structural stability of urban formations was studied by T. Puu [2].

The urban system is regarded as a structurally unstable azimuthally symmetric vector field with one critical point of high order. Using computer modeling, we investigated the dynamics of the development of urban centers and demonstrated various urban systems that can exist: single-core systems, dual-core systems and multi-core systems. There are various options for the location of large cities in a single-core system:

- The central location of the capital, as in Spain;
- Displaced location of the capital, as in Austria;
- Peripheral location of large cities, as in Southeast Asia.

A similar situation exists for modern megapolis, where business centers play the role of a core.

2. Basic urban model

When constructing the urban model of urban development, the model proposed by Beckmann and described in the book of Puu [3] was used. The Beckmann model is based on two partial differential equations, expressed in the so-called gradient and divergent laws [4]. The gradient law is expressed by the relation:

\[ k \frac{\phi}{|\phi|} = \nabla \lambda, \]  

(1)

where \( k \) is the local cost of transporting the goods at a given point, \( \phi \) is the vector of the local goods flow taken in the direction \((\sin \alpha, \cos \alpha) = \hat{e}_\phi, |\phi| = \text{quantity of the goods to be moved}, \) \( \lambda \) denotes the price of the goods. All these quantities are functions of the spatial variables \( x, y \) and time \( t \). This equation shows that any flow is directed towards maximum growth in the price of the goods, and that the price of the goods depends on the cost of transportation. From equation (1) it follows that the relationship between the local cost of transporting the goods and the price of the goods is expressed by the equality:

\[ k = |\nabla \lambda|, \]  

(2)

then from equations (1) and (2) we have \( \phi = \chi \nabla \lambda \), where \( \chi(\lambda) \) is some scalar function of \( \lambda \), analogous to the thermal conductivity coefficient.

The divergent law in the Beckmann model has the form:

\[ \text{div} \phi + z(\lambda) = 0, \]  

(3)

where \( z(\lambda) \) is the excess of offer. In the nonstationary case, from the gradient and divergent law given in equations (1) and (3), we obtain a parabolic equation for \( \lambda \):

\[ \frac{\partial \lambda}{\partial t} = \text{div}(\chi \nabla \lambda) + z(\lambda). \]  

(4)
3. Results of computer modeling

We present the results of a series of calculations of the self-consistent development of a vector urban configuration. The calculated region was a hexagon. The choice of such a calculated area is based on the classical theory of the central places of the Crystal, it is this territorial arrangement that allows us to achieve the maximum area of paving.

The basic postulate of the theory of central places is that the placement of economic activity is mainly determined by the conditions of supply and demand. Transport accessibility is expected in all directions. In such a territory, the cost of supplying the settlement will depend only on the distance between the place of production of the goods and this settlement. With the increase in costs, the demand for most goods decreases, and therefore it is obvious that as the distance increases, the demand for any product in any area will decrease. And since the population, in turn, is placed evenly and transport costs are proportional to the distance, the sales area of any product will have a circle shape and the place of production of this product will be located in the center of the sales area, that is, it will become a "central" place, and all settlements that are supplied from this center, will be "dependent" places. The concept of the demand cone appears, shown in Figure 1 - the radius of the distribution zone of central goods, the lower limit of which is determined by the threshold size of the market, and the upper limit by the distance beyond which the central place is no longer able to sell its goods.

![Figure 1. Model cone of demand.](image)

The entire study area could be divided into a number of round zones defined by the cone of demand, but here there is a certain difficulty: if the circles touch each other, then there are unattended areas, if the circles on the contrary fill the entire territory, then they must intersect, zone overlap, as shown in Figure 2.

![Figure 2. Examples of the paving of the territory.](image)

Therefore, the most effective form of sales areas is the shape of a regular hexagon. Hexagonal (hexagonal) structure arises from the desire to place on the plane the maximum possible number of cones of demand. Areas in the form of a hexagon fill the entire territory evenly.
Crystaller formulated the revealed regularities as follows: a group of identical central places has hexagonal complementary regions, and the central places themselves form a regular triangular lattice. Placement of cities in the Crystal model ensures optimal movement of consumers of goods and services – to the most central places close to their place of residence. Thus, the market, transport infrastructure and administrative structure are optimized.

On the basis of the previous section, we obtained equation (4), which characterizes the dynamic development of the urban system.

We considered systems with zero local excess of supply and boundary conditions corresponding to optimal trading conditions between the interacting regions of the city, that is, \( \lambda = \lambda_0 + \frac{1}{2}(x^2 + y^2)^{-1/2} \), where \( \lambda_0 = \lambda(x, y, t = 0) \) – is the initial state of the system. In the calculations, we used an explicit difference scheme with a 1500 x 1500 grid.

As an initial state of the system, we considered an urban vector configuration with axial symmetry of the third order. Such a configuration is specified as follows: \( \lambda(x, y, t = 0) = x^3 - 3xy^2 \).

With a similar simulation of the urban system, road analogues are the "lines of force" in the resulting vector model.

\[ \begin{align*}
\text{Figure 3.} & \quad \text{Initial state of the simulated urban system at time } t = 0. \text{ Lines - transport highways.} \\
\text{Figure 4.} & \quad \text{The state of the simulated urban system at time } t = 4.
\end{align*} \]

Figures 3-4 schematically show the evolution of the system in time. It is clearly seen that over time, the "collapse" of one center into three additional ones, the topology of the roads changes.

In the second series of calculations, the same model was used, with the same boundary conditions, and the following was chosen as the initial urban vector configuration: \( \lambda(x, y, t = 0) = x^3 - 3xy^2 + 0.5x \).

Figures 5-6 schematically show what will happen to a similar system in the process of self-consistent development. In this case, a system with two connected centers, over time, is transformed into a system with two independently existing centers.
4. Conclusion
This paper presents the results of mathematical modeling of the development of various urban systems. It is shown that the urban system tends to structurally stable configurations, regardless of the initial and boundary conditions that affect the time necessary for rebuilding. Carrying out such studies will help in researching the development and forecasting of changes in urban structures, which is extremely important in the modern world.

References
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