INTEGRABLE QFT ENCODED
ON PRODUCTS OF DYNKIN DIAGRAMS

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Abstract

A large class of Thermodynamic Bethe Ansatz equations governing the Renormalization Group evolution of the Casimir energy of the vacuum on the cylinder for an integrable two-dimensional field theory, can often be encoded on a tensor product of two graphs. We demonstrate here that in this case the two graphs can only be of $ADE$ type. We also give strong numerical evidence for a new large set of Dilogarithm sum Rules connected to $ADE \times ADE$ and a simple formula for the ultraviolet perturbing operator conformal dimensions only in terms of rank and Coxeter numbers of $ADE \times ADE$. We conclude with some remarks on the curious case $ADE \times D$.

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1 Introduction

The space of two-dimensional Quantum field Theories is in connection with that of Conformal Field Theories (CFT$_2$) in the sense that the ultraviolet (UV) and infrared (IR) limit of the former must lie in the space of theories of the latter. In other words, the space of two-dimensional QFT consists of conformal submanifolds (fixed points) of given central charge and of Renormalization Group (RG) flows connecting them. Many of these flows are not integrable, but there exists a subclass of integrable flows that defines the space of two-dimensional (euclidean) Integrable Quantum Field Theories (IQFT$_2$).

The challenge to classify all IQFT$_2$ is one of the most attractive of modern Quantum Field Theory. However, this task is very far from completion and at the present stage we can only hope to get partial classifications similar to what happens in CFT$_2$. There also, the whole classification is far from completed, but for example, the subclass of rational Conformal field theories (RCFT) is (strongly) conjectured to be organized in series of coset models $G/H$, thus putting their description in correspondence with general properties of the Lie algebras and their affine generalizations.

Analogously in IQFT$_2$ it would be interesting to identify some kind of general patterns organizing the vast zoology of integrable models, in other words some kind of “Mendeleev table”, able to show hierarchical structures to be later interpreted as effect of some symmetry underlying the integrable theories. To recognize this organizing criterion can be useful as the first step to figure out some general property of the yet misterious symmetries governing IQFT$_2$’s.

In the present paper we deal with deformations of some coset CFT $G_k/H_{1k}$ by one of its relevant scalar operators, say $\Phi(x)$ of (left) conformal dimension $\Delta_p < 1$. Here $G$ is a compact Lie Algebra and $H$ a proper subalgebra embedded in $G$ with index $I$. The subscripts on $G$ and $H$ denote the level of the corresponding Kac-
Moody algebra. The theories we are interested in are defined by the action

\[ \mathcal{A} = [G_k/H_{Ik}] + \lambda \int d^2x \Phi(x) \]  

(1.1)

where \( \lambda \) is a bare perturbing parameter of dimension \( y = 2(1 - \Delta_P) \) and the symbol \([G_k/H_{Ik}]\) stands for the action of the \( G_k/H_{Ik} \) coset CFT. Such a theory can develop a mass gap and flow to a massive IR theory as the scale increases, or alternatively keep some of its states massless and flow towards a non-trivial infrared CFT. Zamolodchikov’s c-theorem \([2]\) implies, for unitary theories, that the central charge of the IR CFT must be smaller than that of the UV one.

An IQFT\(_2\) possesses by definition an infinite set of local conserved charges. This implies \([3]\) that the N-particle S-matrix is factorizable in terms of 2-particle ones. There are various ways to conjecture the form of the S-matrix once the conserved local and non-local charges are known. If the theory is massless, one can circumvent the problems with the definition of the S-matrix by considering asymptotic left and right movers, as recently proposed in a series of papers \([4, 5, 6, 7]\).

The so far most effective method to recover the UV behaviour of a theory whose factorizable S-matrix is given is the so called Thermodynamic Bethe Ansatz (TBA). That the thermodynamics of a scattering theory can be reconstructed completely from its S-matrix was known since the end of the sixties \([8]\), and the use of Bethe Ansatz techniques to implement this program for integrable theories was issued in a seminal paper by Yang and Yang \([9]\) for a non-relativistic scattering problem. More recently Al.Zamolodchikov has proposed this method to investigate factorized scattering theories corresponding to IQFT\(_2\) \([10]\).

Therefore, any IQFT\(_2\) must have a (possibly infinite) set of TBA equations. It can be seen as a set of equations governing non-perturbatively and exactly the evolution of the Casimir vacuum energy of the theory put on a cylinder along the Renormalization Group flow. In sect.2 we present various interesting systems of TBA equations for some classes of well known coset CFT perturbed by relevant operators in integrable directions and we shall illustrate how in most cases the re-
sulting TBA equations can be encoded in a sort of tensor product of two graphs. This encoding is even more evident by passing from TBA equations to an equivalent system of functional equations (the so-called Y-system) (sect.3). For the Y-system to be consistent with TBA interpretation, the two graphs cannot be chosen arbitrarily. We shall see in sect.3 how one can prove that they must be both restricted to the set of Dynkin diagrams of ADE type (or their foldings and their extensions). We also mention two properties of the $ADE \times ADE$ Y-system, that we checked numerically, namely a set of Dilogarithm sum rules and a periodicity formula. The amusing fact is that they can be expressed in terms of properties of Dynkin diagrams only.

The next obvious step is to explore (sect. 4) the set of would be flows suggested by the various $ADE$ choices for the two graphs in the product, and recognize them with known flows or interpret them as signal of new, yet unexplored IQFT$_2$’s. We are conscious that the observation of a consistent set of equations does not prove the effective existence of the flow it seems to describe (although many reasonable checks like those of sect.4 can be proposed in this direction). However, it can be used as a leading tool to investigate, conjecture and then, when possible, verify the existence of new IQFT$_2$’s and better understand the relations among them and the pattern organizing IQFT$_2$. In this respect we trust it as a very productive method.

Of course the set of TBA-like equations we explore here does not exhaust even the set of IQFT$_2$ of the form $[1,1]$. Our goal is to understand as far as possible the relation between the $G$ and $H$ algebras in the definition of the coset UV CFT, and the two Dynkin graphs appearing in the TBA equations, and possibly find extensions to this set able to incorporate other models.

## 2 TBA equations on $G \times H$

In this section we want to recall some facts known in the literature about TBA equations and their encoding on products of Dynkin diagrams.
For a clear exposition of the deduction of TBA equations from the diagonal S-matrix of a purely elastic scattering theory we refer the reader to the original paper of Al.Zamolodchikov \[10\] and also to ref. \[12, 13\]. In this case, the equations turn out to have the form

\[
\nu_a(\theta) = \varepsilon_a(\theta) + \frac{1}{2\pi} \sum_b [\phi_{ab} \ast \log(1 + e^{-\varepsilon_b})](\theta)
\]

(2.1)

where the indices \(a, b = 1, \ldots, r\) label the different species of particles of masses \(m_a\) respectively, the symbol * stands for convolution and the kernel \(\phi_{ab}(\theta)\) is determined from the S-matrix via the formula

\[
\phi_{ab}(\theta) = -i \frac{d}{d\theta} \log S_{ab}(\theta)
\]

(2.2)

\(\theta\) is the rapidity parametrizing the energy (momentum) of a particle as \(m \cosh \theta\) (\(m \sinh \theta\)). The massive behaviour of the theory is encoded in the energy terms \(\nu_a(\theta) = m_a R \cosh \theta\) (\(R\) being the radius of the cylinder on which the theory is put, or, by modular invariance, the inverse temperature). The unknowns of this set of integral equations are the pseudoenergies \(\varepsilon_a(\theta)\). These in turn determine the Casimir energy of the vacuum on the cylinder via the equations

\[
E(R) = -\frac{\pi c(r)}{6R} \quad \text{and} \quad c(r) = \frac{3}{\pi^2} \sum_a \int_{-\infty}^{+\infty} d\theta \nu_a(\theta) L_a(\theta)
\]

(2.3)

where \(L_a(\theta)\) is short for \(\log(1 + e^{-\varepsilon_a(\theta)})\), and \(r = m_1 \cosh \theta\) (\(m_1\) is the mass of the lightest particle) is such that the UV limit corresponds to \(r \to 0\) and the IR one to \(r \to \infty\). The function \(c(r)\) is even more interesting than \(E(R)\), as it interpolates directly the central charges of the UV and IR theories: \(c(0) = c_{UV}\), \(\lim_{r \to \infty} c(r) = c_{IR}\) (if the theory is massive \(c_{IR} = 0\)).

Klassen and Melzer \[12\] explored with this method the whole set of \(G = A, D, E\) purely elastic minimal S-matrices, confirming that they describe the set of theories with action

\[
\mathcal{A} = \frac{[G_1 \times G_1]}{G_2^{\text{diag}}} + \lambda \int d^2 x \phi_{\text{id, id}}^\ast (x)
\]

(2.4)
where the perturbing operator is here labelled by representations of \( \mathcal{G} \times \mathcal{G} \) (upper indices) and \( \mathcal{G}^{\text{diag}} \) (lower index) as usual in coset CFT. The link between S-matrix and ADE Lie algebras can be made explicit in the TBA equations thanks to a transformation proposed by Al. Zamolodchikov in the very interesting paper ref. \([11]\). For all the ADE theories explored in \([12]\) the TBA can be recasted in the following form

\[
\nu_a(\theta) = \varepsilon_a(\theta) + \frac{1}{2\pi} \{ \phi_g \ast \sum_b \mathcal{G}_{ab} \ast [\nu_b - \Lambda_b] \}(\theta)
\]  

(2.5)

where \( \Lambda_b = \log(1 + e^{\varepsilon_b}) \), \( \phi_g(\theta) = g/2 \cosh(g\theta/2) \) is a universal kernel depending on the Coxeter number \( g \) of \( \mathcal{G} \) only, and all the dependence of the kernel from the indices \( a, b \) has been confined in the incidence matrix of the Dynkin diagram of the Lie algebra \( \mathcal{G} \), that we shall denote in this paper as \( \mathcal{G}_{ab} \). Hence we can think of (2.5) as a set of equations where the unknowns couple to each other obeying the connectivity of the \( \mathcal{G} \) Dynkin diagram, and in this sense they can be thought as uniquely defined once the diagram is given. The masses of the particles are known to be encoded on the Perron-Frobenius eigenvector \( \psi_G \) of \( \mathcal{G} \):

\[
m_a = m_1 \psi_G^a.
\]

A large set of IQFT_2 is known where the S-matrix is not diagonal. The deduction of TBA equations in this case is much more cumbersome, and one has to resort to Higher Level Bethe Ansatz techniques to diagonalize the ”color” transfer matrix appearing in the Bethe equations. In spite of these difficulties, Al. Zamolodchikov was able to work out some simple case of TBA and then to conjecture a generalization valid for the whole set of minimal models (the \((A_1)_k \times (A_1)_1/(A_1)^{\text{diag}}_{k+1}\) coset CFT’s) perturbed by their least relevant \( \phi_{13} \) operator \([13, 4]\). These equations are of the form

\[
\nu_i(\theta) = \varepsilon_i(\theta) + \frac{1}{2\pi} \sum_j (A_k)_{ij} (\phi_{2} \ast L_j)(\theta)
\]

(2.6)

where the indices \( i, j = 1, \ldots, k \) run on the incidence matrix of the \( A_k \) Dynkin diagram. The most curious feature of this system is the form of the energy terms. For negative \( \lambda \) the behaviour is massive and can be reproduced by choosing \( \nu_i = \delta^1_{1} mR \cosh \theta \). For positive \( \lambda \) these theories describe massless interpolating flows between the \( k \)-th
and $k-1$-th minimal models. In spite of the conceptual difficulties to define an S-matrix for scattering of massless objects, one can still present consistent sets of TBA equations with the concept of particle substituted by that of left and right movers, of energy $mRe^\theta/2$ and $mRe^{-\theta}/2$ respectively. The sets of TBA equations are basically the same as (2.6), but now $\nu_i = \frac{mR}{2}(\delta_i^1e^\theta + \delta_i^ke^{-\theta})$. In both cases, almost all nodes on the diagram are attached vanishing energy terms. This is a quite general feature of many IQFT$_2$. From the few known examples, one can argue that the Bethe Ansatz procedure usually performs a sort of ”factorization” of the scattering process in two subprocesses: first a pure elastic scattering of the particles as they had no internal indices (colors), then a pure exchange of color between particles, that one is free to think as mediated by virtual objects with no energy nor momentum called magnons, somehow in connection with the nodes of the diagram having zero energy term, that are therefore called magnonic nodes.

Other progresses have been made on the same path. Putting masses on the $k$-th node or left/right movers on $k$-th/l-th nodes of the $A_{k+l-1}$ diagram, one obtains a TBA set of equations describing the massive (negative $\lambda$) and massless (positive $\lambda$) behaviour of all the coset theories $(A_1)_k \times (A_1)_l/(A_1^{diag})_{k+l}$ perturbed by their $\phi_{11}^{11}$ relevant operator of dimension $\Delta_P = 1 - \frac{2}{k+l+1}$ [16].

One can also ask if it is possible to use diagrams different from $A_k$ for this "magnonic" TBA’s. In ref. [17] the $Z_k$ parafermions (in coset language the $(A_1)_k/U(1)$ RCFT’s) deformed by their $\psi_1\bar{\psi}_1 + c.c.$ operator were studied. The TBA system proposed there is similar to (2.6) but encoded on $D_k$. See ref. [17] for details. There is an elegant way to write in a unified form all the TBA systems cited above and also other generalizations. The idea is to introduce two graphs, one encoding the particle (or kink) species ($G$), the other the magnonic structure of the colors ($H$). The TBA is then fixed by giving the graph tensor product $G \times H$ and specifying the form of the energy term $\nu_a^i(\theta)$ attached to each node $(a, i)$ (the labels $a, b = 1, \ldots r = \text{rank}(G)$ run here on the $G$ graph and $i, j = 1, \ldots k = \text{rank}(H)$ run
Here $g$ is simply some real number whose connection with the graph $G$ shall specified better below. The rationale under this generalization comes from the study of the $\phi_{\text{adj}}$ perturbations of the large class of coset models $G_k \times G_l / G_k + l$ ($G = A_n, D_n, E_{6,7,8}$), which are the most straightforward generalizations of the minimal models perturbed by $\phi_{13}$. They are the first evident example where one has to introduce both particle and magnonic indices. The TBA equations can be encoded on the product of diagrams $G \times A_{k+l-1}$, the first describing the connectivity of the particle indices, the second that of the magnonic ones. The cases previously discussed can all be encoded in this general class (2.7), with suitable choices of $G$ and $H$. One is naturally lead to ask how large is the scope of integrable flows described by system (2.7) and if there are restrictions on the choice of $G$ and $H$. We deal with this subject in the next section.

### 3 The Y-system

Here we wish to recall a very important result of paper [11] (or better its generalization [18] to the more general system (2.7)), that perhaps has not sufficiently been emphasised and appreciated by part of the S-matrix community, but that will be the main instrument of our investigation, i.e. the fact that all solutions of the system (2.7) are also solutions of the system of functional equations

$$Y^i_a(\theta - \frac{i\pi}{g}) Y^i_a(\theta + \frac{i\pi}{g}) = \prod_b (1 + Y^i_b(\theta))^{G_{ab}} \prod_j (1 + Y^j_a(\theta) - H_{ij})^{-H_{ij}}$$

often referred as the Y-system. The relation with the TBA equations is given by $Y^i_a(\theta) = e^{\varepsilon_a(\theta)}$. Here the encoding on the $G$ and $H$ graphs is even more evident. This system has been encountered in many applications of TBA, but such kind of objects appear in other areas of physics and mathematics too, for example in lattice
integrable models, $\tau$-functions for Toda lattices, etc..., so the importance to study such mathematical object is larger than the TBA problem we are considering here.

The derivation from TBA (2.7) given in [18] is valid for $\mathcal{G}$ any $ADE$ Dynkin diagram, $g = \text{cox}(\mathcal{G})$ and $\mathcal{H}$ any connected unoriented graph. One can however consider the Y-system for any $\mathcal{G} \times \mathcal{H}$ graph and any $g \in R$ and then ask when it is consistent with a TBA interpretation, in other words when it can be used to represent an integrable Renormalization Group flow (this does not mean that the flow exists, simply that it is mathematically possible). In this connection, we shall prove the following two basic statements:

1. The Y-system (3.1) can be consistent with a TBA interpretation only if the graph $\mathcal{G}$ is a Dynkin diagram of a simply-laced simple Lie algebra or a folding of it ($ADET$). This algebra encodes the mass ratios of the particles (or the relative crossover scales of the left and right movers) of the integrable QFT underlying it.

2. If the $\mathcal{G}$ graph is $ADET$ then the $\mathcal{H}$ graph (that is known to encode the “color” structure of the theory) can only be $ADET$ or at least an extended Dynkin diagram $\hat{A}\hat{D}\hat{E}$ or a folding of it.

### 3.1 Proof of statement 1

The Y-system (3.1) can be depicted on the tensor product diagram $\mathcal{G} \times \mathcal{H}$, organized in $\mathcal{G}$-rows reproducing the $\mathcal{G}$ diagram, and $\mathcal{H}$-columns reproducing the $\mathcal{H}$ diagram. We wish to prove that a physically relevant system of this kind can only exist for $\mathcal{G}$ in the set of simply-laced Dynkin diagrams or foldings of them $ADET$. Physically relevant means that we are interested in systems of the kind (3.1) that allow an interpretation in terms of a scattering theory; i.e. they must come from some TBA, able to reproduce sensible physical behaviours at UV and IR. As $Y^i_a(\theta) = e^{\nu^i_a(\theta)}$ and the asymptotic behaviours for large $R$ and large $\theta$ of the pseudoenergies are driven by the corresponding energy terms $\nu^i_a(\theta)$, the possible behaviours of $Y^i_a(\theta)$ are of
the following two types:

\[
Y^i_a(\theta)_{R, \theta \to +\infty} \sim \begin{cases} 
    e^{m_i^a R e^{\theta}/2} & \text{if the node } (a, i) \text{ is massive or left mover} \\
    e^{\text{const.}} & \text{if the node } (a, i) \text{ is magnonic or right mover}
\end{cases}
\]  

(3.2)

In what follows we denote the nodes having the massive or left mover behaviour as \textit{black}; those with magnonic or right mover behaviour as \textit{white}.

If a system of type (3.1) has white nodes only, i.e. if the corresponding TBA has all the \( \nu^i_a(\theta) \equiv 0 \), it degenerates completely to have the totally trivial solution \( c(r) \equiv 0 \). Therefore a physically sensible Y-system must have at least one black node. We now prove that if a \( G \)-row contains a black node, then all the nodes in that row must be black. Indeed, consider a would-be white node \( a \) on the same \( G \)-row, connected to the black node \( b \). Connected means that \( G_{ab} \neq 0 \). On the left hand side of (3.1) we then have a constant asymptotic behaviour. On the right hand side the product over \( H \) also behaves as a constant irrespective of the color of the nodes adjacent to \( a \) in the \( H \)-column direction. However, the product over \( G \) contains at least one term (the one connecting \( a \) with \( b \)) that has the exponential behaviour typical of black nodes. The constant behaviour is exponentially depressed compared to the “black” one, therefore, comparing terms of order \( e^{\theta} \) in the asymptotic of (3.1) we would get the condition

\[ 0 = m_i^a R e^{\theta}/2 \]  

(3.3)

which would imply that the \( b \) node has zero mass, in contradiction with the hypothesis that it is black. Hence we conclude that for a given \( G \)-row the nodes are all black or all white.

We said that the Y-system must have at least one black node: this implies that indeed there must be at least one black \( G \)-row. With this result at hand, we proceed further by picking up a black \( G \)-row for a fixed \( H \) index \( i \), and considering the asymptotics of system (3.1) for this row. As before, the \( H \) product on the r.h.s. drops having constant behaviour, while the \( e^{\theta}/2 \) terms give the asymptotic
consistency condition

$$2m^i_a \cos \frac{\pi}{g} = \sum_{b=1}^{r} G_{ab} m^i_b$$  \hspace{1cm} (3.4)

From this equation some very important facts follow:

1. if $G = A_1$, i.e. the $1 \times 1$ matrix $0$, then the r.h.s. of (3.4) vanishes. For the l.h.s to vanish too, we must have $g = 2$.

2. if $G$ is the incidence matrix of a non-trivial connected graph, then the r.h.s. of (3.4) is positive, therefore, to have a positive l.h.s. we must require $g > 2$.

3. for $2 < g < \infty$, as $\cos \frac{\pi}{g} < 1$ we have the condition

$$\sum_{b \in G} G_{ab} m^i_b < 2m^i_a \hspace{1cm} (3.5)$$

where the vector $m^i = (m^i_1, m^i_2, ..., m^i_r)$ has all positive components. This is exactly the condition that selects, among all possible connected graphs the list of simply-laced Dynkin diagrams and their possible foldings,

$$G = A_n, D_n, E_{6,7,8}, T_n = A_{2n}/Z_2$$  \hspace{1cm} (3.6)

4. As the vector $m^i$ is an eigenvector of $G$ with all non negative components, it must be proportional to the (unique) Perron-Frobenius eigenvector $\psi^{(G)}$ of $G$

$$m^i_a = m^i \psi^{(G)}_a$$  \hspace{1cm} (3.7)

for each black $G$-row. Moreover, $g$ plays the role of the (dual) Coxeter number of $G$, hence it is a positive integer.

### 3.2 Proof of statement 2

Now we prove another statement, i.e. if $G$ is a simply-laced ($ADET$) Dynkin diagram then $\mathcal{H}$ is or in the set $ADET$ too, or in the set of extended simply-laced Dynkin diagrams or their foldings. This can be done by generalizing an argument given in [14]. Consider stationary solutions (i.e. independent on $\theta$) of the system

10
We know that these solutions enter the calculation of the UV central charge and in particular that at least one solution with real \( \varepsilon^i_a \) must occur (for the simplest cases one can prove that this real solution exists and is unique, we shall assume this as true for all cases in what follows). Reality and finiteness of the \( \varepsilon^i_a \) implies that

\[
y^i_a > 0.
\]

Then

\[
2 \log y^i_a = \sum_b G_{ab} \log(1 + y^j_b) - \sum_j H_{ij} \log(1 + 1/y^j_a)
\]

The quantity \( z^i_a = \log(1 + y^i_a) \) is real and strictly positive, which implies that it satisfies \( \sum_b G_{ab} z^i_b < 2 z^i_a \), as \( G \) is in the list of \( ADET \) Dynkin diagrams. This allows, after few manipulations, to obtain the inequality

\[
\sum_j H_{ij} w^j_a < 2 w^i_a
\]

where \( w^i_a = \log(1 + 1/y^i_a) > 0 \). This proves that \( H \) must also be in the list of \( ADET \) Dynkin diagrams. It can happen that the system degenerates to have some \( y^a_i = 0 \) i.e. \( \varepsilon^i_a = -\infty \). In this case one can prove, using tecniques similar to those of the proof of statement 1, that if one node has \( y^a_i = 0 \), then all nodes have \( y^a_i = 0 \). In this case instead of (3.9) we get the equality \( \sum_j H_{ij} w^j_a = 2 w^i_a \) selecting the extended Dynkin diagram \( \tilde{A}_n, \tilde{D}_n, \tilde{E}_{6,7,8} \) or the vaste set of their foldings. We have to mention here also this somehow degenerate case, as it has physical applications, e.g. in [19].

### 3.3 Some useful formulas on the Y-system

Many authors converge on the opinion that the Y-system seems to encode a great amount of information about the IQFT\(_2\) it is attached to. In particular the connection between the Y-system and the Rogers Dilogarithm function seems to be very productive in reproducing the properties of the UV and IR limits of the integrable model under consideration [20]. One can extremize this point and think to a reconstruction program, i.e. to an answer to the question: given a Y-system and the asymptotic behaviour of its solutions for \( \theta, R \to +\infty \), can we reconstruct
completely an IQFT\textsubscript{2} underlying it? Is this theory unique? This program is up to now very far from completion. However many facts indicate that it is not hopeless. We shall not deal with this very interesting aspect in this paper; we shall be content to comment here about the two simplest quantities that can be extracted from a Y-system and their relation to the graphs $\mathcal{G} \times \mathcal{H}$.

1. The stationary solutions $Y^i_a$ of the Y-system allow to compute the central charge via Dilogarithm sum rules. The main ingredient is the sum

$$s(\mathcal{G} \times \mathcal{H}) = \sum_{a \in \mathcal{G}} \sum_{i \in \mathcal{H}} \mathcal{L}\left(\frac{1}{1 + Y^i_a}\right) = \frac{\pi^2}{6} \frac{rk}{g + h}$$

(3.10)

Here $\mathcal{L}(z)$ denotes the Rogers Dilogarithm function, $g = cox(\mathcal{G})$ and $h = cox(\mathcal{H})$. For $\mathcal{G} = A_r$ and $\mathcal{H} = A_k$ this sum rule is proven ([21]), we checked it numerically to very high precision up to $r = rank(\mathcal{G})$ and $k = rank(\mathcal{H})$ both equal to 50 for the remaining cases, for which this sum rule is new and is proposed to the mathematicians for a rigorous proof.

2. The solutions of the Y-system possess a periodicity $Y^i_a(\theta + i\pi P) = Y^i_a(\theta)$ ($P \in \mathbb{Z}/g$) In [11, 4] arguments are given to relate $P$ to the conformal dimension $\Delta_P = 1 - 1/P$ of the UV perturbing operator and that of the IR attracting operator if it is the case. We checked numerically the following formula for the periodicity to be valid for all the $\mathcal{G} \times \mathcal{H}$ Y-systems

$$P = \frac{g + h}{g}$$

(3.11)

4 \textbf{Zoology of }\mathcal{G} \times \mathcal{H} \textbf{ IQFT}\textsubscript{2}

Armed with the properties and formulas of the preceding section, one can explore the whole set of $ADET \times ADET$ cases. Here we only briefly report on a very preliminary exploration where we only computed UV and IR central charges, thanks to (3.10) and conformal dimensions of UV perturbing and IR attracting operators, thanks to (3.11). The standard way to extract these data from the $\mathcal{G} \times \mathcal{H}$ Y-system
(or TBA) is widely explained in the literature, see e.g. [12, 4]. In the following we just comment on some interesting cases. Here \( \mathcal{G} = ADE \).

- \( \mathcal{G} \times A_{k+1} \) is the case discussed in [13]. It corresponds to the best studied class of coset models, namely \( \mathcal{G}_k \times \mathcal{G}_l / \mathcal{G}_{k+l} \), perturbed by the operator \( \phi^{id,id}_{adj} \).
  
  We refer to [13] for the details.

- A complete list of all the \( A_1 \times ADET \) cases is given in [14].

- A set of interesting cases where one is able to reconstruct the S-matrix, that turns out to be diagonal, is \( \mathcal{G} \times T_1 \) [22]. All the cases involving the \( T_n \) diagrams correspond to non-unitary theories. There are curious developments in this direction [23], in particular for \( T_1 \times A_k \).

- A partially new and interesting case is \( \mathcal{G} \times D_k \) with a massive energy term on one node of the fork, or left and right movers on the two nodes of the fork. This case generalizes to higher \( \mathcal{G} \) what has been discussed in [17] for \( A_1 \times D_k \). In that paper this TBA has been used to describe \( Z_k \) parafermionic theories perturbed by their \( \psi \bar{\psi} + \psi^\dagger \bar{\psi}^\dagger \) operators (\( \psi \) being the generating parafermion). In the massless direction these theories flow to an IR limit given by the minimal models \( (A_1)_k \times (A_1)_1 / (A_1)_{k+1} \). Taking the limit for large \( k \) one defines a flow from \( c = 2 \) to \( c = 1 \) which is interpreted as the \( O(3) \) massless sigma model with theta term \( \theta = \pi \), which renormalizes at IR to the \( SU(2)_{k=1} \) WZW model. One would search for a generalization of this class of flows to higher \( \mathcal{G} \). In a certain sense the most obvious candidate is exactly this set \( \mathcal{G} \times D_k \). With the choice of left and right movers on the two nodes of the fork, this TBA does indeed reproduce an IR limit towards \( \mathcal{G}_k \times \mathcal{G}_1 / \mathcal{G}_{k+1} \). The UV CFTs, however, are not the \( \mathcal{G}_k / U(1)^r \) as one would naively expect by generalizing the \( A_1 \) case, but better the list of so called Normal Forms over \( \mathcal{G} \) (for a definition and a list, see the appendix in [24]). This suggests that the corresponding sigma models defined as the \( k \to \infty \) limit of these series
are *quantum* integrable and that it is possible to add some kind of topological term transforming the model in a massless one with IR limit the $G_{k=1}$ WZW model. However, it seems that these sigma models (apart the case $A_1/U(1)$) do not allow topological terms and, to our knowledge, even the quantum integrability is an unclear issue here. Moreover, another fact seems to invite to deepen the study of these models. In the $A_1$ case, it has recently proposed a so-called *sausage model*, i.e. a deformation of the target space of the $O(3)$ sigma model defining a one parameter family of integrable sigma models. A TBA has been proposed for a discrete set of values of this parameter, encoded on $A_1 \times \hat{D}_k$. The main feature is that all the UV central charges of these TBA’s are equal to 1, irrespective of the rank of the $D$ diagram, thus allowing the sausage interpretation. Now, if this interpretation can be extended to higher $G$, one should expect constant central charges, usually equal to some integer. This is not the case for all $G \times \hat{D}_k$ with $G$ other than $A_1$. The generalized sausage models, if they exist, have nothing to do with the sigma models on the normal forms of $G$. We intend to return on this puzzling problem in the future.

To conclude, we have explored here a wide class of sets of TBA equations, that incorporate almost all examples shown so far. We have determined definitively the scope of this class, and, exploring the zoology of the models encoded in it, we have found repeated regularities but also some unclear problems to be investigated further. Of course, we are conscious that this class of IQFT’s does not exhaust all the possible deformations of RCFT’s. A well known counterexample is the set of $G_k \times G_l/G_{k+l}$ CFT deformed by their $\phi_{id, id}$ operator, where $G$ is a non-simply-laced algebra. This case needs a TBA, and then a Y-system, which is not included in the set described above. The most reasonable generalizations one can think are

- to allow *different* diagrams $\mathcal{H}_a$ on different nodes of $G$
- to allow for shift terms on the right-hand-side of the Y-system too.
This is indeed the case for the non-simply-laced $\mathcal{G} \times \mathcal{G}/\mathcal{G}$. We shall report on this in a forthcoming publication.

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References

[1] P.Goddard, A.Kent and D.Olive, Comm. Math. Phys. 103 (1986) 105

[2] A.B.Zamolodchikov, JETP Lett. 43 (1986) 730

[3] A.B.Zamolodchikov and Al.B.Zamolodchikov, Ann. Phys. 120 (1979) 253

[4] Al.B.Zamolodchikov, Nucl. Phys. B358 (1991) 524

[5] A.B.Zamolodchikov and Al.B.Zamolodchikov, Nucl. Phys. B379 (1992) 602

[6] P.Fendley, H.Saleur and Al.B.Zamolodchikov, Massless Flows I and II, hep-th 9304050-51, to appear in Int. J. Mod. Phys. A

[7] P.Fendley and H.Saleur, Massless integrable quantum field theories and massless scattering in 1 + 1 dimensions, Los Angeles preprint USC-93-022

[8] R.Dashen, S.-K.Ma and H.J.Bernstein, Phys.Rev. 187 (1969) 345

[9] C.N.Yang and C.P.Yang, J. Math. Phys. 10 (1969) 1115

[10] Al.B.Zamolodchikov, Nucl. Phys. B342 (1990) 695

[11] Al.B.Zamolodchikov, Phys. Lett. B253 (1991) 391

[12] T.Klassen and E.Melzer, Nucl. Phys. B338 (1990) 485

[13] T.Klassen and E.Melzer, Nucl. Phys. B350 (1991) 635
[14] F.Ravanini, R.Tateo and A.Valleriani, Int. J. Mod. Phys. A8 (1993) 1707

[15] Al.B.Zamolodchikov, Nucl. Phys. B358 (1991) 497

[16] Al.B.Zamolodchikov, Nucl. Phys. B366 (1991) 122

[17] V.A.Fateev and Al.B.Zamolodchikov, Phys. Lett. B271 (1991) 91

[18] F.Ravanini, Phys. Lett. B282 (1992) 73

[19] V.Fateev, E.Onofri and Al.B.Zamolodchikov, LPTHE preprint 92-46

[20] W.Nahm, A.Recknagel and M.Therhoeven, Bonn preprint, hep-th/9211034

[21] A.Kirillov, Newton inst. preprints, hepth 9211137 and 9212150

[22] F.Ravanini, R.Tateo and A.Valleriani, Phys. Lett. B293 (1992) 361

[23] F.Ravanini, M.Stanishkov and R.Tateo, in preparation

[24] M.A.Olshanetskii and A.M.Perelomov, Phys. Rep. 94 (1983) 313

[25] A.M.Perelomov, Physica 4D (1981) 1