Polynomial 6j –symbols and states sums

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For a given 2r–th root of unity ξ, we give explicit formulas of a family of 3–variable Laurent polynomials Ji,j,k with coefficients in Z[ξ] that encode the 6j–symbols associated with nilpotent representations of Uξ(sl(2)). For a given abelian group G, we use them to produce a state sum invariant τr(M, L, h1, h2) of a quadruplet (compact 3–manifold M, link L inside M, homology class h1 ∈ H1(M, Z), homology class h2 ∈ H2(M, G)) with values in a ring R related to G. The formulas are established by a “skein” calculus as an application of the theory of modified dimensions introduced by the authors and Turaev in [4]. For an oriented 3–manifold M, the invariants are related to τ(M, L, φ ∈ H1(M, C*)) defined by the authors and Turaev in [3] from the category of nilpotent representations of Uξ(sl(2)). They refine them as τ(M, L, φ) = ∑h1 τr(M, L, h1, φ̄) where φ̄ correspond to φ with the isomorphism H2(M, C*) ≅ H1(M, C*).

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Introduction

The 6j–symbols are tensors describing the associativity of the tensor product in a tensor category. Formulas exist for the classical and quantum 6j–symbols associated to the defining representations of sl(2) and its powers (see Kirillov and Reshetikhin [6] and Masbaum and Vogel [7]). At a root of unity ξ, new representations appear for the quantum group Uξ(sl(2)). There are essentially two new families: the nilpotent and the cyclic representations. Unlike the cyclic family, the nilpotent representations can be enriched to form a ribbon category. For ξ a fourth root of unity, this was already observed by O Viro in [11] who used these representations to construct a ribbon graph invariant related to the multivariable Alexander polynomial.

The theory of modified dimensions developed with V Turaev by the authors in [4; 3] produces a family of modified 6j–symbols that share properties similar with usual 6j–symbol. Nevertheless this family has a very different nature than previously defined 6j–symbols. Indeed, the whole family of nilpotent representation can be thought as a unique module with parameters. This module is then a nontrivial one parameter
deformation of the so called Kashaev module. For this reason, the $6j$–symbols can be described by a finite set of parameterized functions and more precisely by a family of 3–variable Laurent polynomials $J_{i,j,k}$ with coefficients in $\mathbb{Z}[\xi]$.

These Laurent polynomials have wonderful properties. They have some symmetries (see (3)) and satisfy a Biedenharn–Elliott type identity (see (5)) and an orthonormality relation (see (6)). These three identities, imply that from a triangulation of a 3–manifold, one can compute a state sum, that is a weighted sum of product of these $J$ polynomials associated with the tetrahedra of the triangulation, which is a topological invariant of $M$. Furthermore, F Costantino and J Murakami [2] show that the asymptotical behavior of these polynomial $6j$–symbols is related to the volume of truncated tetrahedra.

The main substance of this paper is the careful computation of the $6j$–symbols associated with nilpotent representations of $U_{\xi}(\mathfrak{sl}_2)$. This is done in Section 3. But once the $6j$–symbols are identified with certain values of the $J$ polynomials, all the machinery of tensor category can be forgotten. This is what we want to highlight by the structure of this document. Hence the first part only defines the $J$ polynomials and announces their properties. Here the tensor categories does not appear except in the fact that we do not have, without them, a direct proof of the identities. The second part is a short exposition of how the polynomials can be used to construct a Turaev–Viro [10] type invariant, following and refining their ideas and those of the authors with Turaev [3].

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1 Polynomial $6j$–symbols

Fix a nonzero positive integer $r'$. In this section, we define a set of formal $6j$–symbols $J_{i_1,i_2,i_3}^{r'}$ for $i_1,i_2,i_3 \in \mathbb{Z}$ and give some of their properties. Since $r'$ is fixed, we write $J_{i_1,i_2,i_3}$ for $J_{i_1,i_2,i_3}^{r'}$.

Let $\mathbb{N}$ be the set of positive integers including zero. Let $r = 2r' + 1$ and $\xi = e^{i m \pi/r}$ for $m$ coprime with $2r$. Let $\mathfrak{L} = \mathbb{Z}[\xi][q_1^{\pm 1}, q_2^{\pm 1}, q_3^{\pm 1}]$ be the ring of Laurent polynomials in three variables, with coefficients in $\mathbb{Z}[\xi]$. We denote with a bar the involutive ring automorphism of $\mathfrak{L}$ defined by $\bar{\xi} = \xi^{-1}, \bar{q}_1 = q_1^{-1}, \bar{q}_2 = q_2^{-1}$ and $\bar{q}_3 = q_3^{-1}$. For
any invertible element $X$ of a ring and $N \in \mathbb{N}$ let $\langle X \rangle$ and $F_N(X)$ be analogues of quantum integer and quantum factorial, given by 

$$
\langle X \rangle = X - X^{-1}, \quad F_N(X) = \prod_{i=0}^{N-1} \langle \xi^i X \rangle = \langle X \rangle \langle \xi X \rangle \cdots \langle \xi^{N-1} X \rangle.
$$

Also, for $n, N \in \mathbb{N}$ and $i_1, i_2, i_3 \in \{-r', -r' + 1, \ldots, r'\}$ such that $n \leq N$ we set 

$$
\{n\} = \langle \xi^n \rangle = \xi^n - \xi^{-n}, \quad \{N\}! = \{1\} \{2\} \cdots \{N\}, 
$$

$$
\begin{bmatrix}
N \\
\{n\}
\end{bmatrix} = \frac{\{N\}!}{\{n\}! \{N-n\}!}, \quad \{i_1, i_2, i_3\} = \frac{\{2r'\}!}{\{r'-i_1\}! \{r'-i_2\}! \{r'-i_3\}!}.
$$

Remark that $F_k(X^{-1}) = (-1)^k F_k(X)$ and $\{2r'\}! = (-1)^r r'$. For $N = r$ notice that 

$$
F_r(X) = \prod_{i=0}^{r-1} \langle \xi^i X - \xi^{-i} X^{-1} \rangle = \prod_{i=0}^{r-1} \xi^i X^{-1}(X^2 - \xi^{-2i}).
$$

After multiplying the right hand side of this equation by $X^r$ we see that this polynomial has roots $\pm \xi^{-i}$ for $i = 0, \ldots, r - 1$, and so up to the sign $(-1)^{r'} = \xi^{r(r-1)/2}$ is equal to $X^{2r} - 1$. Thus, we have shown that $F_r(X) = \prod_{i=0}^{r-1} \xi^i X^{-1}(X^2 - \xi^{-2i}) = (-1)^{r'} \langle X^r \rangle$.

Let us consider the finite set 

$$
\mathcal{H}_{r'} = \{(i_1, i_2, i_3) \in \mathbb{N} : -r' \leq i_1, i_2, i_3, i_1 + i_2 + i_3 \leq r'\}.
$$

One can easily show that $\text{card}(\mathcal{H}_{r'}) = \frac{1}{3} r (2 r^2 + 1)$. It can be useful to have in mind the action of the tetrahedral group $\mathcal{S}_4$ on $\mathcal{H}_{r'}$ by permuting $i_1, i_2, i_3$ and $i_4 = -(i_1 + i_2 + i_3)$.

For all $(i_1, i_2, i_3) \in \mathcal{H}_{r'}$, we define a Laurent polynomial 

$$
J_{i_1,i_2,i_3}(q_1, q_2, q_3) \in \mathcal{Q}
$$

as follows.

- If $i_1, i_3 \leq i_1 + i_2 + i_3$ then let $N = r' - i_1 - i_2 - i_3$ and define 

$$(1) \quad J_{i_1,i_2,i_3}(q_1, q_2, q_3) = \{i_1, i_2, i_3\} F_{i_2+i_3}(q_1 \xi^{-i_3-r'}) F_{i_1+i_2}(q_3 \xi^{-i_2-r'}) \times \left( \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} F_{N-n}(q_2 \xi^{i_3+r'+1}) F_{N-n}(q_2 \xi^{i_3+i_2-i_1-r'}) \times F_n(q_1 \xi^{-2i_3-N}) F_n(q_3 \xi^{i_1+r'+1}) F_{r'-i_2}(q_2 \xi^{-i_1-r'-n}) \right).$$
If \( i_2, i_3 \geq i_1 + i_2 + i_3 \) then let \( N = r' + i_1 + i_2 + i_3 \) and define

\[
(2) \quad J_{i_1,i_2,i_3}(q_1,q_2,q_3) = \frac{F_{r'+i_2-N}(q_3 \bar{q}_1 \xi^{N-2i_2+1})}{\{N\}!} \left( \sum_{n=0}^{N} \binom{N}{n} F_n(q_2 \xi^{-i_1+r'+1}) \right) \\
\times F_{N-n}(q_2 \xi^{i_1-i_2+n+1})F_{r'+i_3}(q_1 \bar{q}_2 \xi^{N-n-2i_3+1})F_{r'+i_1}(q_2 \bar{q}_3 \xi^{n-2i_1+1})
\]

For other \((i_1, i_2, i_3) \in \mathcal{H}_{r'}\), the polynomial \( J_{i_1,i_2,i_3} \) is obtained from Equation (3), below.

The definition of these symbols come from the \( 6j \)-symbols associated to nilpotent representations of \( U_\xi(\mathfrak{sl}(2)) \) (see Definition 20 and Theorem 29). Theorem 29 shows that Equations (1) and (2) agree when both conditions are satisfied. We also extend the definition for \((i_1, i_2, i_3) \in \mathbb{Z}^3 \setminus \mathcal{H}_{r'}\) by \( J_{i_1,i_2,i_3} = 0 \). Theorem 19 implies that \( J_{i_1,i_2,i_3}(q_1,q_2,q_3) \) is an element of \( \Sigma \).

It is well known that the \( 6j \)-symbols satisfy certain relations. We use Equation (27) to show the family of polynomials defined above satisfy equivalent relations. Indeed the theory of modified \( 6j \)-symbols developed in [3] shows that these identities are satisfied as functions over some open dense subset of \( \mathbb{C}^n \). Therefore, since the elements \( J_{*,*,*} \) are Laurent polynomials they satisfy the identities formally.

Let us now discuss these relations. Since the \( 6j \)-symbols have tetrahedral symmetry we have

\[
(3) \quad J_{i_1,i_2,i_3}(q_1,q_2,q_3) = J_{i_2,i_1,i_3}(\bar{q}_2, \bar{q}_1, \bar{q}_3) = J_{i_2,i_3,i_4}(q_1 \bar{q}_2 \xi^{-2i_3}, q_1 \bar{q}_3 \xi^{2i_2}, q_1)
\]

where \( i_4 = -i_1 - i_2 - i_3 \). These two equalities generate the 24 symmetries of the tetrahedral group. In particular, if \( \sigma \) is permutation of the set \( \{1,2,3\} \) then \( J_{i_1,i_2,i_3}(q_1,q_2,q_3) = J_{i_1,i_3,i_2}(q_1^{e_1(q_2)\cdot q_3^{e_2(q_1)\cdot q_3^{e_3(q_2)}}) \) where \( e = e(\sigma) \) is the signature of \( \sigma \).

The other relations involve a function called a modified dimension (see [3]). We introduce the following polynomial in \( q_1 \) which is a formal analog of the inverse of this function:

\[
(4) \quad D(q_1) = F_{2r'}(q_1 \xi) = (-1)^{r'} \frac{q_1^{-r'} - q_1^{-r}}{q_1 - q_1^{-1}}.
\]
The $J$ polynomials satisfy the Biedenharn–Elliott identity: For $x \in \mathbb{Z}$ let $\bar{x}$ be the element of $\{-r',-r'+1,\ldots,r'\}$ congruent to $x$ modulo $r$. For any \( i_1, i_2, i_3, i_4, i_5, i_6 \in \mathbb{Z}, \)
\[
(5) \quad J_{i_1, i_2, i_3}(q_1, q_2, q_3) J_{i'_1, i'_3, i'_5}(q_0 \xi^{2i_4}/q_1, q_0 \xi^{2i_5}/q_2, q_0 \xi^{2i_6}/q_3) = \sum_{n=-r'}^{r'} J_{i_1, i_2, i_3}(q_0 \xi^{2n}, q_2, q_3) J_{i_4, i_5, i_6}(q_0 \xi^{2n}, q_2, q_1) J_{i_5, i_6, i'_4}(q_0 \xi^{2n}, q_1, q_2) D(q_0 \xi^{2n})
\]
where
\[
\begin{align*}
i'_1 &= -i_1 + i_5 - i_6, \\
i'_2 &= -i_2 + i_6 - i_4, \\
i'_3 &= -i_3 + i_4 - i_5, \\
i'_4 &= i_4 - n, \\
i'_5 &= i_5 - n, \\
i'_6 &= i_6 - n,
\end{align*}
\]
and $q_0$ is an independent variable.

For any \((i_1, i_2, i_3) \in H_{r'}\) and any \(i'_1 \in \mathbb{Z}\) the orthonormality relation is expressed as
\[
(6) \quad \sum_{n=-r'}^{r'} \frac{J_{i_1, i_2, i_3}(q_1 \xi^{2n}, q_2, q_3) J_{-i'_1, n-i_2, i'_3-n}(\bar{q}_1 \xi^{-2n}, \bar{q}_2, \bar{q}_3)}{D(q_2 \bar{q}_3 \xi^{-2i_1}) D(q_1 \xi^{2n})} = \delta_{i_1, i'_1}
\]
where $\delta_{i_1, i'_1}$ is the Kronecker symbol.

## 2 3–Manifold invariant

In this section we derive a set of topological invariant of links in a closed 3–manifold $M$ from the family $J_{\ast \ast \ast}(q_1, q_2, q_3)$. These invariants are indexed by element of $H_1(M, \mathbb{Z})$ and they refine the invariant constructed in [3, Section 10.4].

Let $M$ be a closed 3–manifold and $L$ a link in $M$. Here we follow the exposition of [3] inspired by Baseilhac and Benedetti [1]. A *quasiregular triangulation of $M$* is a decomposition of $M$ as a union of embedded tetrahedra such that the intersection of any two tetrahedra is a union (possibly, empty) of several of their vertices, edges, and (2–dimensional) faces. Quasiregular triangulations differ from usual triangulations in that they may have tetrahedra meeting along several vertices, edges, and faces. Nevertheless, the edges of a quasiregular triangulation have distinct ends. A *Hamiltonian link* in a quasiregular triangulation $\mathcal{T}$ is a set $\mathcal{L}$ of unoriented edges of $\mathcal{T}$ such that every vertex of $\mathcal{T}$ belongs to exactly two edges of $\mathcal{L}$. Then the union of the edges of $\mathcal{T}$ belonging to $\mathcal{L}$ is a link $L$ in $M$. We call the pair $(\mathcal{T}, \mathcal{L})$ an $H$–triangulation of $(M, L)$.

**Proposition 1** [1, Proposition 4.20] Any pair (a closed connected 3–manifold $M$, a nonempty link $L \subset M$) admits an $H$–triangulation.
The language of both triangulation and skeleton are useful here. In particular, it is convenient to use triangulation to give the notion of a Hamiltonian link and skeleton to define the state sum. A skeleton of $M$ is a 2–dimensional polyhedron $\mathcal{P}$ in $M$ such that $M \setminus \mathcal{P}$ is a disjoint union of open 3–balls and locally $\mathcal{P}$ looks like a plane, or a union of 3 half-planes with common boundary line in $\mathbb{R}^3$, or a cone over the 1–skeleton of a tetrahedron (see, for instance [1; 9]). A typical skeleton of $M$ is constructed from a triangulation $\mathcal{T}$ of $M$ by taking the union $P_T$ of the 2–cells dual to its edges. This construction establishes a bijective correspondence $T \leftrightarrow P_T$ between the quasiregular triangulations $T$ of $M$ and the skeletons $P$ of $M$ such that every 2–face of $P$ is a disk adjacent to two distinct components of $M \setminus \mathcal{P}$ and no connected component of the 1–dimensional strata of $\mathcal{P}$ is a circle. To specify a Hamiltonian link $L$ in a triangulation $T$, we provide some faces of $P_T$ with dots such that each component of $M \setminus P_T$ is adjacent to precisely two (distinct) dotted faces. These dots correspond to the intersections of $L$ with the 2–faces.

Let $R$ be a commutative ring with a morphism $\mathbb{Z}[\xi] \to R$. We still denote by $\xi$ the image of $\xi$ in $R$ and we assume that the group of $2r$–th root of 1 in $R$ is of order $2r$ generated by $\xi$. Let $R^\times$ be the group of units of $R$ and consider any subgroup $G$ of $\{x^r : x \in R^\times\}$, for example $(R, G) = (\mathbb{C}, \mathbb{C}^*)$. Clearly any element $x \in G$ has exactly $r$ $r$–th roots in $R$. They form a set $\text{Root}_r(x) = \{y\xi^{-2r}, \ldots, y, y\xi^2, \ldots, y\xi^{2r}\}$ for some $y \in R$ such that $y^r = x$. We call $(R, G)$ a coloring pair.

Let $(\mathcal{T}, \mathcal{L})$ be a $H$–triangulation of $(M, L)$. Let $\mathcal{P}_T$ be a skeleton dual to $T$. The skeleton $\mathcal{P}_T$ gives $M$ a cell decomposition $M_P$. So a $n$–chain of cellular homology with coefficients in $G$ can be represented by a map from the oriented $n$–cells of $M_P$ to $G$.

By a $G$–coloring of $T$ (or of $\mathcal{P}_T$), we mean a $G$–valued 2–cycle $\Phi$ on $\mathcal{P}_T$, that is a map from the set of oriented faces of $\mathcal{P}_T$ to $G$ such that

1. the product of the values of $\Phi$ on the three oriented faces adjacent to any oriented edge of $\mathcal{P}_T$ is 1,

2. $\Phi(-f) = \Phi(f)^{-1}$ for any oriented face $f$ of $\mathcal{P}_T$, where $-f$ is $f$ with opposite orientation.

Each $G$–coloring $\Phi$ of $T$ represents a homology class $[\Phi] \in H_2(M, G)$. When $M$ is oriented, a $G$–coloring of $T$ can be seen as a 1–cocycle (a map on the set of oriented edges of $T$; see [3]). In general, it can also be interpreted as a map on the set of co-oriented edges of $T$ but we prefer to adopt the point of view of $\mathcal{P}_T$.

A state $\varphi$ of a $G$–coloring $\Phi$ is a map assigning to every oriented face $f$ of $\mathcal{P}_T$ an element $\varphi(f)$ of $\text{Root}_r(\Phi(f))$ such that $\varphi(-f) = \varphi(f)^{-1}$ for all $f$. The set of all
states of \( \Phi \) is denoted \( \text{St}(\Phi) \). A state \( \varphi \) can also be seen as a 2–chain on \( P_T \) with values in \( R^\times \) but its boundary, the 1–chain \( \delta \varphi \) might not be trivial. Nevertheless, as \( \varphi^r = \Phi \), we have \((\delta \varphi)^r = 1\). We call the height of \( \varphi \) the unique map \( h_\varphi \) assigning to every oriented edge \( e \) of \( P_T \) an element of \( \{-r', -r', r', r'\} \) such that \((\delta \varphi)(e) = \xi 2h_\varphi(e)\). It follows that modulo \( r \), \( h_\varphi \) is a 1–cycle on \( P_T \). In the case when \( h_\varphi \) is also a 1–cycle with integer coefficients, let us denote its homology class by \([h_\varphi] \in H_1(M, \mathbb{Z})\). For \( h \in H_1(M, \mathbb{Z}) \), set \( \text{St}_h(\Phi) = \{ \varphi \in \text{St}(\Phi) : \delta h_\varphi = 0 \text{ and } [h_\varphi] = h \}\).

Given a \( G \)–coloring \( \Phi \) of \((\mathcal{T}, \mathcal{L})\), we define a certain partition function (state sum) as follows: For each vertex \( x \) of \( P_T \), fix a little 3–ball \( \mathcal{B} \) centered at \( x \) whose intersection with \( P_T \) is homeomorphic to the cone on the 1–skeleton of a tetrahedron. The trace of \( P_T \) on \( \partial \mathcal{B} \) gives a triangulation of this sphere whose one skeleton is a tetrahedron with four vertices \( v_1, v_2, v_3, v_4 \). Let \( f_1, f_2, f_3 \) be the regions of \( P_T \) contained in the triangles \( xv_2v_3, xv_3v_1, xv_1v_2 \), respectively (see Figure 1). Also, let \( e_1, e_2, e_3, e_4 \) be the segments of \( P_T \) contained in \( xv_1, xv_2, xv_3, xv_4 \), respectively. The segment \( xv_i \) is oriented from \( x \) to \( v_i \) and induces an orientation on \( e_i \). Similarly, the triangles above induce orientations on \( f_1, f_2, f_3 \). For each \( \varphi \in \text{St}(\Phi) \), if \( h_\varphi \) does not satisfy the cycle condition at \( x \) (ie if \( \sum_i h_\varphi(e_i) \neq 0 \)) we set \( J(\varphi, x) = 0 \), otherwise define

\[
J(\varphi, x) = J_{h_\varphi(e_1), h_\varphi(e_2), h_\varphi(e_3)}(\varphi(f_1), \varphi(f_2), \varphi(f_3)) \in R.
\]

Equation (3) implies that \( J(\varphi, x) \) does not depend of the choice of ordering of the vertices \( v_1, v_2, v_3, v_4 \). For example, if one chooses the ordering \( v_2, v_1, v_3, v_4 \) then \( e_1 \) and \( e_2 \) are exchanged, \( xv_2v_3 \) becomes \( xv_1v_3 \) and so \( f_1 \) becomes \( -f_2 \), etc… and the first equality of (3) implies that the two expressions for \( J(\varphi, x) \) are equal.

We say that \( g \in G \) is admissible if \( \langle g \rangle = g - g^{-1} \) is invertible in \( R \). We call a \( G \)–coloring \( \Phi \) admissible if it takes admissible values. If \( \varphi \) is a state of an admissible coloring \( \Phi \), and \( f \) is an unoriented face of \( P_T \), then we define

\[
d(\varphi, f) = D(g)^{-1} = \frac{(-1)^{r'} \langle g \rangle}{\langle g^r \rangle} \in R
\]
where \( g \) is \( \varphi(f) \) for any orientation \( f \) of \( f \). Note that \( d(\varphi, f) \) does not depend on the orientation of \( f \), as \( D(g) = D(g^{-1}) \).

**Lemma 2** Let \((R, G)\) be a coloring pair with the following property:

\[(\ast) \text{ For all } g_1, \ldots, g_n \in G \text{ there exists } x \in G \text{ such that } xg_1, \ldots, xg_n \text{ are all admissible.}\]

Then for any \( H \)-triangulation \((T, \mathcal{L})\) of \((M, L)\) and for any homology class \( h_2 \in H_2(M, G) \), there exists an admissible \( G \)-coloring \( \Phi \) on \( T \) representing \( h_2 \).

**Proof** Take any \( G \)-coloring \( \Phi \) of \( T \) representing \( h_2 \). For \( f \) an oriented face of \( \mathcal{P}_T \) and \(-f\) the same face with opposite orientation, we have that \( \langle \Phi \rangle(-f) = -\langle \Phi \rangle(f) \) and thus \( \Phi(-f) \) is admissible if and only if \( \Phi(f) \) is admissible. As mentioned above \( M \setminus \mathcal{P}_T \) is the disjoint union of open 3–balls. We say that a such a 3–ball \( b \) is bad for \( \Phi \) if there is a oriented face \( f \) in \( T \) incident to \( b \) such that \( \Phi(f) \) is not admissible. It is clear that \( \Phi \) is admissible if and only if \( \Phi \) has no bad 3–balls. We show how to modify \( \Phi \) in its homology class to reduce the number of bad 3–balls. Let \( b \) be a bad 3–ball for \( \Phi \) and let \( E_b \) be the set of all oriented faces of \( T \) which are oriented away from \( b \). From Property \((\ast)\) of the lemma, there exists \( x \in G \) such that \( x\Phi(f) \) is admissible for all \( f \in E_b \). Let \( c \) be the \( G \)-valued 3–chain on \( M_\mathcal{P} \) assigning \( x \) to \( b \) and \( 1 \) to all other 3–balls (recall \( M_\mathcal{P} \) is the cell decomposition of \( M \) coming from \( \mathcal{P}_T \)). Taking the boundary of this 3–chain we obtain a \( G \)-valued 2–chain \( \delta c \Phi \) on \( M_\mathcal{P} \). The 2–cycle \((\delta c)\Phi \) on \( \mathcal{P}_T \) takes values in \( \{x\Phi(f); (x\Phi(f))^{-1} : f \in E_b\} \) which are admissible on all faces of \( \mathcal{P}_T \) incident to \( b \) and takes the same values as \( \Phi \) on all other faces of \( T \). Here we use the fact every 2–face of \( \mathcal{P}_T \) is a disk adjacent to two distinct components of \( M \setminus \mathcal{P}_T \). The transformation \( \Phi \mapsto (\delta c)\Phi \) decreases the number of bad 3–balls. Repeating this argument, we find a 2–cycle without bad 3–balls. \( \square \)

Let \( \Phi \) be an admissible \( G \)-coloring of \( T \) and \( h_1 \in H_1(M, \mathbb{Z}) \). Then we define

\[
\tau(T, \mathcal{L}, h_1, \Phi) = r^{-2N} \sum_{\varphi \in St_{h_1}(\Phi)} \prod_{f \in \mathcal{P}_2 \setminus \mathcal{L}} d(\varphi, f) \prod_{x \in \mathcal{P}_0} J(\varphi, x) \in R
\]

where \( \mathcal{P}_2 \setminus \mathcal{L} \) is the set of unoriented faces of \( \mathcal{P}_T \) without dots, \( \mathcal{P}_0 \) is the set of vertices of \( \mathcal{P}_T \) and \( N \) is the number of connected component of \( M \setminus \mathcal{P}_T \) (that is the number of vertices in \( T \)).

When the coloring pair does not satisfy Property \((\ast)\) of Lemma 2, we explain how to perturb a nonadmissible \( G \)-coloring \( \Phi \): Consider the set \( S \) of element of \( R[X^{\pm 1}] \) that
are monic polynomials in \(X\) (ie Laurent polynomials whose leading coefficient is 1). Then \(S\) is a multiplicative set that does not contain zero divisor, hence we can form \(R' = S^{-1}R[X^{\pm 1}] \supset R\). Let \(G'\) be the multiplicative group generated by \(G\) and \(X^r\). Then \((R', G')\) is a coloring pair with property (\(\ast\)) as any \((X^r h)\) with \(h \in G\) and \(k \in \mathbb{Z}^*\) is invertible in \(R'\). Using the inclusion above we can view \(\Phi\) and \([\Phi]\) as taking values in \(G'\). Then Lemma 2 implies there exists a \(2\)--boundary \(x\) with values in \(G'\) such that \(\Phi' = x\Phi\) is an admissible \(G'\)--coloring. We say that \(\Phi'\) is a perturbation of \(\Phi\).

**Theorem 3** Let \(L\) be a link in a \(3\)--manifold \(M\), \((R, G)\) be a coloring pair and \((h_1, h_2) \in H_1(M, \mathbb{Z}) \times H_2(M, G)\). Choose any \(H\)--triangulation \((T, \mathcal{L})\) of \((M, L)\) and let \(\Phi\) be any admissible (or perturbation of a) \(G\)--coloring representing \(h_2\). Then \(\tau_R(M, L, h_1, h_2) = \tau(T, \mathcal{L}, h_1, \Phi)\) belongs to \(R\) and it is an invariant of the diffeomorphism class of the four-tuple

\((M, L, h_1 \in H_1(M, \mathbb{Z}), h_2 \in H_2(M, G))\).

**Proof** First, let us assume that \((R, G)\) satisfy (\(\ast\)) of Lemma 2, so there exists an admissible \(\Phi\) representing \(h_2\). In this case the proof is essentially the same as the proof of Theorem 22 in [3]. Here we sketch the main steps:

(I) In [1] it is shown that any two \(H\)--triangulation are related by a finite sequence of so called elementary \(H\)--moves. One can then colors this sequence and makes it a sequence of “colored \(H\)--moves.”

(II) One shows that the state sum \(\tau(T, \mathcal{L}, h_1, \Phi)\) is invariant under an elementary “admissible colored \(H\)--move,” ie an elementary \(H\)--move where the colors of the \(H\)--triangulation on both sides of the move are admissible. The main point here is that Equation (5) implies that if one performs a so called Pachner \(2 \rightarrow 3\) move (which consists in replacing in \(T\) two tetrahedra glued along a face with 3 tetrahedra having a common edge) the state sum is unchanged. Similarly, Equation (6) imply the invariance of the state sum under the lune move which consists in removing two tetrahedra which have 2 common faces and then gluing by pairs the orphan faces.

Here the following observation makes the refinement with \(h_1\) possible: if two states \(\varphi, \varphi'\) of \(\Phi\) differ only on a set of faces then \(h_{\varphi}, h_{\varphi'}\) differ only on the set \(E\) of edges adjacent to these faces. Assume that the set \(E\) is included in a simply connected part of \(\mathcal{P}_T\). Then if \(\varphi\) and \(\varphi'\) have nontrivial contributions in the state sum (which implies \(\delta h_{\varphi} = \delta h_{\varphi'} = 0\)), we have \([h_{\varphi}] = [h_{\varphi'}]\) since \(h_{\varphi}\) and \(h_{\varphi'}\) are equal outside the simply connected space containing \(E\). Hence the colored \(H\)--moves do not modify the partial state sum associated to any homology class \(h_1\).
We now consider a universal example: Let $H$ be the first fundamental example is obtained when $h_2$ is the only component of $\Phi'$ which depends on $X \in R'$ is the boundary $\delta$ and as the state sum depend of the coloring only up to a boundary, it does not depend on $X$. To be more precise, let $\rho: R' \to R'$ be the ring morphisms which is the identity on $R$ and sends $X$ to $X^2$. Then $\Phi'' = \rho(\Phi')$ is also an admissible $R'$–coloring of $T$. Moreover, $\rho(x)$ is a boundary and $\rho(\Phi) = \Phi$, hence $\Phi'/\Phi''$ is a boundary. But from above we know that two admissible colorings representing the same homology class give equal state sums. Thus, $\tau(T, L, h_1, \Phi') = \tau(T, L, h_1, \Phi'') = \rho(\tau(T, L, h_1, \Phi'))$ which implies that $\tau(T, L, h_1, \Phi) \in R$. 

Let us now consider the case where the coloring pair does not satisfy property $(\ast)$. We will prove that the perturbed state sum belongs to $R$. Let $\Phi' = x\Phi$ be a perturbation of any $G$–coloring $\Phi$ representing $h_2$. The idea is that the only component of $\Phi'$ which depends on $X \in R'$ is the boundary $\delta$ and as the state sum depend of the coloring only up to a boundary, it does not depend on $X$. To be more precise, let $\rho: R' \to R'$ be the ring morphisms which is the identity on $R$ and sends $X$ to $X^2$. Then $\Phi'' = \rho(\Phi')$ is also an admissible $R'$–coloring of $T$. Moreover, $\rho(x)$ is a boundary and $\rho(\Phi) = \Phi$, hence $\Phi'/\Phi''$ is a boundary. But from above we know that two admissible colorings representing the same homology class give equal state sums. Thus, $\tau(T, L, h_1, \Phi') = \tau(T, L, h_1, \Phi'') = \rho(\tau(T, L, h_1, \Phi'))$ which implies that $\tau(T, L, h_1, \Phi) \in R$. 

We also define

$$\tau_R(M, L, h_2) = \sum_{h_1 \in H_1(M, \mathbb{Z})} \tau_R(M, L, h_1, h_2) \in R$$

This sum is finite because $\tau_R(M, L, h_1, h_2) = 0$ for all but finitely many $h_1$. Indeed, if an admissible coloring $\Phi$ represent $h_2$ then we have $St(\Phi) = \bigcup_{h_1} St_{h_1}(\Phi)$ is finite and thus $St_{h_1}(\Phi) = \emptyset$ for all but finitely many $h_1 \in H_1(M, \mathbb{Z})$.

The first fundamental example is obtained when $(R, G) = (\mathbb{C}, \mathbb{C}^*)$. It is easy to see that this coloring pair satisfies Property $(\ast)$ of Lemma 2 since $1, -1 \in \mathbb{C}^*$ are the only nonadmissible elements. In this case, if $M$ is oriented we denote the Poincaré dual of $h_2 \in H_2(M, \mathbb{C}^*)$ by $h_2^* \in H^1(M, \mathbb{C}^*)$. Then

$$\tau_\mathbb{C}(M, L, h_2) = \tau(M, L, h_2^*)$$

where $\tau(M, L, h_2^*)$ is the invariant defined in [3, Section 10.4].

We now consider a universal example: Let $H = H_1(M, \mathbb{Z})$, and assume that $M$ is oriented so that for any abelian group $G$ we have $H_2(M, G) \simeq H^1(M, G) \simeq \text{Hom}(H, G)$. Then $H_2(M, H)$ has a particular universal element $\eta$ whose image in $\text{Hom}(H, H)$ is the identity. We will assume that the order of the torsion of $H$ is
coprime with \( r \). Then multiplication by \( r \) is an injective morphism \( m_r \colon H \to H \). Denote the image of \( m_r \) by \( rH \), then \( (\mathbb{Z}[\xi][H], rH) \) is a coloring pair and we consider \( \tau(M, L, r\eta) \in \mathbb{Z}[\xi][H] \).

**Proposition 4**  The invariant \( \tau(M, L, r\eta) \) takes values in \( \mathbb{Z}[\xi][rH] \) making it possible to define

\[
\tau(M, L) = m_r^* (\tau(M, L, r\eta)) \in \mathbb{Z}[\xi][H].
\]

Then for any pair \( (R, G) \) as above and any \( \psi \in \text{Hom}(H, G) \) we have

\[
\tau(M, L, \overline{\psi}) = \psi_*(\tau(M, L))
\]

where \( \overline{\psi} \) is the image of \( \psi \) in \( H_2(M, G) \).

**Proof**  First, let us show that for each \( h_1 \in H_1(M, \mathbb{Z}) \) we have \( \tau(M, L, h_1, r\eta) \in \mathbb{Z}[\xi][rH] \). We choose a base of the free part of \( H \); that is we write \( H = \text{Tor}(H) \oplus \mathbb{Z} x_1 \oplus \cdots \oplus \mathbb{Z} x_k \). Then define the ring morphism \( \rho_i : \mathbb{Z}[\xi][H] \to \mathbb{Z}[\xi][H] \) as the identity on this basis except that \( \rho_i(e^{xi}) = \xi e^{xi} \). Clearly, the set of states and thus the state sum is invariant by \( \rho_i \) for any \( i \) and thus belongs to \( \mathbb{Z}[\xi][rH] \). The last point follows from the fact that \( \psi_*(\eta) = \psi \).

**Remark 5**  Suppose that \( T \) is not a quasiregular triangulation but a generalized triangulation where some edges might be loops. Then not all homology classes of \( H_2(M, G) \) can be represented by admissible colorings on \( T \).

Nevertheless, suppose that an admissible coloring \( \Phi \) is given on \( T \). Then one can prove that the state sum \( \tau(T, L, h_1, \Phi) \) as above is still equal to the invariant \( \tau(M, L, h_1, [\Phi]) \). This might be useful for effective computations. This can be proven using the fact that up to perturbing the coloring, the triangulation \( T \) can be transformed into a quasiregular one by a sequence of elementary moves such that at each step, the locally modified coloring is admissible.

### 3  Skein calculus

#### 3.1  The category \( C^H \) of \( U_{\xi}^H(\mathfrak{sl}(2)) \) weight modules

For \( x \in \mathbb{C} \) we extend the notation \( \xi^x \) by setting \( \xi^x = e^{\frac{ix\pi}{r}} \). Also, if \( (\alpha, k) \in \mathbb{C} \times \mathbb{N} \),

\[
\{\alpha\} = \xi^\alpha - \xi^{-\alpha} \quad \text{and} \quad \{\alpha; k\}! = F_k (\xi^\alpha) = \{\alpha\} \{\alpha + 1\} \cdots \{\alpha + k - 1\}.
\]

Many computations in this section use the identity

\[
\{x + z\} \{y + z\} - \{x\} \{y\} = \{x + y + z\} \{z\}.
\]
Let $U_{\xi}^H(\mathfrak{sl}(2))$ be the “unrolled” quantization of $\mathfrak{sl}(2)$, i.e., the $\mathbb{C}$–algebra with generators $E, F, K, K^{-1}, H$ and the following defining relations:

$$KK^{-1} = K^{-1} K = 1, \quad KEK^{-1} = \xi^2 E, \quad KFK^{-1} = \xi^{-2} F,$$

$$HK = KH, \quad [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = \frac{K - K^{-1}}{\xi - \xi^{-1}}.$$

This algebra is a Hopf algebra with coproduct $\Delta$, counit $\varepsilon$, and antipode $S$ defined by the formulas

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \varepsilon(E) = 0, \quad S(E) = -EK^{-1},$$

$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \varepsilon(F) = 0, \quad S(F) = -KF,$$

$$\Delta(K) = K \otimes K, \quad \varepsilon(K) = 1, \quad S(K) = K^{-1},$$

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \varepsilon(H) = 0, \quad S(H) = -H.$$

Following [4], we define $\tilde{U}_{\xi}^H(\mathfrak{sl}(2))$ to be the quotient of $U_{\xi}^H(\mathfrak{sl}(2))$ by the relations $E^r = F^r = 0$. It is easy to check that the operations above turn $\tilde{U}_{\xi}^H(\mathfrak{sl}(2))$ into a Hopf algebra.

Let $V$ be a $\tilde{U}_{\xi}^H(\mathfrak{sl}(2))$–module. An eigenvalue $\lambda \in \mathbb{C}$ of the operator $H$: $V \to V$ is called a weight of $V$ and the associated eigenspace $E_\lambda(V)$ is called a weight space. We call $V$ a weight module if $V$ is finite-dimensional, splits as a direct sum of weight spaces, and $\xi^H = K$ as operators on $V$.

Let $\mathcal{C}^H$ be the tensor category of weight $\tilde{U}_{\xi}^H(\mathfrak{sl}(2))$–modules. By Section 6.2 of [4], $\mathcal{C}^H$ is a ribbon Ab-category with ground ring $\mathbb{C}$. The braiding $c_{V,W}: V \otimes W \to W \otimes V$ on $\mathcal{C}^H$ is defined by $v \otimes w \mapsto \tau(R(v \otimes w))$ where $\tau$ is the permutation $x \otimes y \mapsto y \otimes x$ and $R$ is the operator of $V \otimes W$ defined by

$$R = \xi^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{\{1\}^{2n}}{\{n\}!} \xi^n \xi^{-n-1/2} E^n \otimes F^n.$$

The inverse of the twist on a weight module $V$ is given by the operator

$$\theta_V^{-1} = K^{r-1} \xi^{-H^2/2} \sum_{n=0}^{r-1} (-1)^n \frac{\{1\}^{2n}}{\{n\}!} \xi^{3n(n-1)/2} F^n K^{-n} E^n$$

(also see Ohtsuki [8, Chapter 4.5] where this formula is given with $\xi = \xi^2$ instead of $\xi$).

For an isomorphism classification of simple weight modules over the usual quantum $\mathfrak{sl}(2)$, see for example Kassel [5, Chapter VI]. This classification implies that simple
weight $\bar{U}^H_\xi(\mathfrak{sl}(2))$–modules are classified up to isomorphism by highest weights. For $\alpha \in \mathbb{C}$, we denote by $V_\alpha$ the simple weight $\bar{U}^H_\xi(\mathfrak{sl}(2))$–module of highest weight $\alpha + r - 1$. This notation differs from the standard labeling of highest weight modules. Note that $V_{-r+1} = \mathbb{C}$ is the trivial module and $V_0$ is the so called Kashaev module.

The well-known Reshetikhin–Turaev construction defines a $\mathbb{C}$–linear functor $F$ from the category of $\mathcal{C}^H$–colored ribbon graphs with coupons to $\mathcal{C}^H$. Let $B = (\mathbb{C} \setminus \mathbb{Z}) \cup r \mathbb{Z}$. The modules $\{V_\alpha\}_{\alpha \in B}$ are called typical and all have dimension $r = 2r' + 1$. Note that $F$ is trivial on all closed $\mathcal{C}^H$–colored ribbon graph that have at least one color in $B$. In [4], the definition of $F$ is extended to a nontrivial map $F'$ defined on closed $\mathcal{C}^H$–colored ribbon graphs with at least one edge colored by a typical module. Let us recall how one can compute $F'$. If $T \subset \mathbb{R} \times [0, 1]$ is a $\mathcal{C}^H$–colored (1-1)-tangle with the two ends colored by the same typical module $V_\alpha$, we can form its “braid closure” $\hat{T}$. Then we say that $T$ is a cutting presentation of the closed $\mathcal{C}^H$–colored ribbon graph $\hat{T}$. In this situation, $F(T)$ is an endomorphism of $V_\alpha$ that is a scalar. Then $F'(\hat{T})$ is this scalar multiplied by the modified dimension of $V_\alpha$ which is given by

$$d(V_\alpha) = (-1)^{r'} \frac{\{\alpha\}}{\{r\alpha\}} = \prod_{k=1}^{2r'} \frac{1}{\{\alpha + k\}}.$$

It can be shown that $F'(\hat{T})$ does not depend on the cutting presentation $T$ of $\hat{T}$ (see [4]).

For $\alpha \in B$ let us consider the basis of $V_\alpha$ given by $(v_i = F^i v_0)_{i=0..2r'}$ where $v_0$ is a highest weight vector of $V_\alpha$. Then the $U^H_\xi(\mathfrak{sl}(2))$–module structure of $V_\alpha$ is given by

$$H . v_i = (\alpha + 2(r' - i)) v_i, \quad E . v_i = \frac{\{i\} \{i - \alpha\}}{\{1\}^2} v_{i-1}, \quad F . v_i = v_{i+1}.$$

**Remark 6** The family of module indexed by $B$ can be seen as a vector bundle $\mathcal{E} \to B$ on which elements of $U^H_\xi(\mathfrak{sl}(2))$ act by continuous linear transformations. Then the $v_i$ are sections of this vector bundle that form a trivialization $\mathcal{E} \simeq B \times \mathbb{C}^r$. In fact one can extend $\mathcal{E}$ to a unique vector bundle $\mathcal{E}'$ over $\mathbb{C} \supset B$ with an action of $U^H_\xi(\mathfrak{sl}(2))$ but the fiber over $k \in \mathbb{Z} \setminus r \mathbb{Z}$ is not an irreducible module.

Let $\nu$ be the 2–dimensional simple weight $U^H_\xi(\mathfrak{sl}(2))$–module of highest weight 1 and basis $(v_0, v_1)$ with $E . v_1 = v_0$ and $v_1 = \bar{E} . v_0$. The categorical dimension of $V_\alpha$ is zero, while that of $\nu$ is equal to $q\dim(\nu) = \{-2\}/\{1\} = -\xi - \xi^{-1}$.
3.2 Duality in $C^H$

As in [3], the ribbon structure of $C^H$ induce the existence of functorial left and right duality given by $V^* = \text{Hom}_C(V, \mathbb{C})$ and the morphisms
\[
\begin{align*}
    b_V: \mathbb{C} &\to V \otimes V^* \quad \text{is given by} \quad 1 \mapsto \sum j v_j \otimes v_j^*, \\
d_V: V^* \otimes V &\to \mathbb{C} \quad \text{is given by} \quad f \otimes w \mapsto f(w), \\
d'_V: V \otimes V^* &\to \mathbb{C} \quad \text{is given by} \quad v \otimes f \mapsto f(K^{1-r} v), \\
b'_V: \mathbb{C} &\to V^* \otimes V \quad \text{is given by} \quad 1 \mapsto \sum j v_j^* \otimes K^{r-1} v_j.
\end{align*}
\]

For $\alpha \in B$, the classification of simple modules implies that $V^*_\alpha$ is isomorphic to $V_\alpha$. We consider the isomorphism $w_\alpha: V_\alpha \to V^*_\alpha$ given by
\[
    v_i \mapsto -\xi^{i^2-i+1-\alpha} v_{2r^-i}^*.
\]

The isomorphism $w_\alpha$ is the unique map up to a scalar that sends $v_0$ to $-\xi v_{2r}$ and $v_i = F^i v_0$ to
\[
-\frac{1}{\xi} v_{2r}^- \circ (-KF)^i = -\frac{1}{\xi} v_{2r}^- \circ ((-K)^i \xi i(i-1) F^i) = \frac{-1}{\xi} v_{2r}^- = \xi^{i^2-i+1-\alpha} v_{2r^-i}.
\]

Let $w_v = w_{1-2r}: v \mapsto \tilde{v} \mapsto v^*$ be the isomorphism given by $v_0 \mapsto -\xi v_1^*$ and $v_1 \mapsto v_0^*$. 

Lemma 7 For $\alpha \in B$, one has
\[
(9) \quad d_V(w_{-\alpha} \otimes \text{Id}_V) = d'_{V_{-\alpha}}(\text{Id}_{V_{-\alpha}} \otimes w_\alpha)
\]
and similarly $d_V(w_{1-2r} \otimes \text{Id}_V) = d'_{V_{1-2r}}(\text{Id}_{V} \otimes w_{1-2r})$.

Proof Let us denote by $f$ the left hand side of (9) and by $g$ the right hand side of (9).

By a direct computation on $v_i \otimes v_{2r^-i} \in V_{-\alpha} \otimes V_\alpha$,
\[
f(v_i \otimes v_{2r^-i}) = d_V(\xi^{i^2-i+1+\alpha} v_{2r^-i} \otimes v_{2r^-i}) = \xi^{i^2+i\alpha}
\]
and
\[
g(v_i \otimes v_{2r^-i}) = d'_{V_{-\alpha}}(\xi^{(2r^-i)^2-1-2(2r^-i)\alpha} v_i \otimes v_i^*) = \xi^{4r^2-4r^2+1-2(2r^-i)\alpha} v_i^* (K^{2r} v_i) = \xi^{4r^2-4r^2+1-2(2r^-i)\alpha} \xi^{(-2r^-)(-\alpha+2(2r^-i))} = \xi^{i^2-i+1+\alpha}.
\]

The analogous equation for $v$ follows similarly
\[
g(v_0 \otimes v_1) = g v_{1}^*(v_1) = -\xi = \xi^{-2r} = v_0^* (K^{-2r} v_0) = f(v_0 \otimes v_1). \qed
\]
For \( \alpha \in B \), we denote \( d^\alpha \) and \( b^\alpha \) as the following morphisms
\[ d^\alpha = d_{V_{\alpha}} \circ (w_{-\alpha} \otimes \text{Id}_{V_{\alpha}}) : V_{-\alpha} \otimes V_{\alpha} \to \mathbb{C} \]
\[ b^\alpha = (\text{Id}_{V_{\alpha}} \otimes (w_{-\alpha})^{-1}) \circ b_{V_{\alpha}} : \mathbb{C} \to V_{\alpha} \otimes V_{-\alpha} \]

Similarly,
\[ d^v = d_{V} \circ (w_{V} \otimes \text{Id}) : V \otimes V \to \mathbb{C} \quad b^v = (\text{Id} \otimes w_{V}) \circ b_{V} : \mathbb{C} \to V \otimes V. \]

We use the isomorphism \( w_{-\alpha} \) to identify \( V_{\alpha}^* \) with \( V_{-\alpha} \). Under this identification, we get \( d_{V_{\alpha}} \cong d'_{V_{-\alpha}} \cong d^\alpha \) and \( b_{V_{\alpha}} \cong b'_{V_{-\alpha}} \cong b^\alpha \). Similarly, \( d_{V} \cong d'_{V} \cong d^v \) and \( b_{V} \cong b'_{V} \cong b^v \). Graphically, for a \( CH \)–colored ribbon graph \( \Gamma \), this means that one can reverse the orientation of an edge colored by \( V_{\alpha} \) and simultaneously replace its coloring by \( V_{-\alpha} \). Also, if \( \Gamma \) has an oriented edge colored by \( V \), one can forget its orientation. We will represent edges colored by \( V \) with dashed unoriented edges (see for example Figure 2).

![Figure 2: The family of maps X](image)

### 3.3 Multiplicity modules in \( V_{\alpha} \otimes V_{-\alpha} \otimes 1 \otimes \mathbb{C} \)

We consider the following spaces of morphisms of \( CH \) using the notation
\[ H_{U,V,W}^W = \text{Hom}_{CH} (U \otimes V, W), \quad H_{U,V}^U = \text{Hom}_{CH} (W, U \otimes V), \]
\[ H_{U,V,W} = \text{Hom}_{CH} (U \otimes V \otimes W, \mathbb{1}), \quad H_{U, V, W} = \text{Hom}_{CH} (\mathbb{1}, U \otimes V \otimes W), \]

where \( U, V, W \) are weight modules. If there is no ambiguity, for \( \alpha \in B \) we replace \( V_{\alpha} \) with \( \alpha \) in this notation, eg \( H_{V_{\beta}, V_{\gamma}}^{V_{\alpha}} = H_{\beta, \gamma}^{\alpha} \). Also, since \( V_{\alpha}^* \) and \( V_{-\alpha} \) are identified we can replace \( V_{\alpha}^* \) with \( -\alpha \), eg \( H_{V_{\beta}, V_{\gamma}}^{\alpha} = H_{\beta, \gamma}^{-\alpha} \).

We define the symmetric multiplicity module of \( U, V, W \) to be the space \( H(U, V, W) \) obtained by identifying the 12 following isomorphic spaces
\[ H_{U,V,W}^U \cong H_{W,U,V}^W \cong H_{V,W,U}^V \cong H_{W,V,U}^W \cong H_{V,W,U}^{V*} \cong H_{W,V,U}^{W*} \cong H_{V,W,U}^{V*,U} \cong H_{W,V,U}^{W*,U} \cong H_{V,W,U}^{V*,W} \cong H_{W,V,U}^{W*,V} \]

where each of these isomorphisms come from certain duality morphisms (see [9]).
For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ the character formula implies

\begin{equation}
 v \otimes V_\alpha \simeq V_{\alpha - 1} \oplus V_{\alpha + 1}.
\end{equation}

Therefore, for $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, the space $H^v_\beta$ is the zero space if $\beta \neq \alpha \pm 1$ and $H^v_\beta$ has dimension 1 if $\beta = \alpha \pm 1$.

Consider the morphism

\begin{equation}
 v \otimes V_\alpha \simeq V_{\alpha + 1} \rightarrow v \otimes V_\alpha
\end{equation}

given by $v_0 \mapsto v_0 \otimes v_0$ and $v_i \mapsto \xi^{-i} v_0 \otimes v_i + \{i\} v_1 \otimes v_{i-1}$.

This morphism forms a basis of $H^v_{\alpha + 1}$. Thus, this morphism and the cyclic isomorphisms

\begin{equation}
 H^v_{\alpha + 1} \simeq H^v_{\alpha - 1}, \quad H^v_{\alpha + 1, v} \simeq H^v_{\alpha - 1}, \quad H^v_{v, \alpha + 1} \simeq H^v_{\alpha + 1}
\end{equation}

induce a basis on $H^v_{\alpha + 1}, H^v_{\alpha + 1, v},$ and $H^v_{v, \alpha + 1}$. Each of these bases consists of the single morphism which we denote by

\begin{equation}
 v \otimes V_\alpha \simeq V_{\alpha + 1}, \quad v \otimes V_{\alpha + 1, v}, \quad v \otimes V_{v, \alpha + 1}
\end{equation}

respectively. Moreover, the morphism

\begin{equation}
 v \otimes V_\alpha \simeq V_{\alpha + 1}
\end{equation}

and isomorphisms represented in Equation (12) define a basis vector $\omega^-(\alpha)$ for the symmetric module $H(v, \alpha, -\alpha - 1)$.

Similarly, consider the basis of $H^v_{\alpha + 1}$ given by the morphism

\begin{equation}
 v \otimes V_\alpha \simeq V_{\alpha + 1} \rightarrow v \otimes V_\alpha
\end{equation}

given by $v_0 \mapsto v_0 \otimes v_0$ and $v_i \mapsto -\xi^{\alpha - i - 1} \{i\} v_0 \otimes v_i + \xi^{-1} \{i\} v_1 \otimes v_i$.

As above this morphism and the isomorphisms in (12) induce basis of $H^v_{\alpha + 1}, H^v_{\alpha + 1, v}, H^v_{v, \alpha + 1}$ and $H(v, \alpha + 1, -\alpha)$ which each consist of one morphism which we denote by

\begin{equation}
 v \otimes V_\alpha \simeq V_{\alpha + 1, v}, \quad v \otimes V_{\alpha + 1, v}, \quad v \otimes V_{v, \alpha + 1}, \quad \omega^+(\alpha),
\end{equation}

respectively.

The next proposition is illustrated by Figure 3. It computes the pairing of some of the families of morphisms defined above.
Proposition 8 We have the relation
\[
\begin{array}{c}
\begin{array}{c}
\alpha \\
\downarrow \quad \downarrow \\
V_{\alpha + 1} & V_{\alpha + 1}
\end{array}
\end{array}
\quad = \quad \{\alpha + 1\} \cdot \text{Id}_{V_{\alpha}}.
\]

The evaluation of $F'$ on the colored $\Theta$–graph

induce the pairing

\[
H(v, \alpha, -\alpha - 1) \otimes H(v, \alpha + 1, -\alpha) \rightarrow \mathbb{C}
\]
determined by

\[
\langle \omega^-(\alpha), \omega^+(\alpha) \rangle = (-1)^{r'} \frac{\{\alpha\} \{\alpha + 1\}}{\{l\alpha\}} = \prod_{k=2}^{2r'} \frac{1}{\{\alpha + k\}}.
\]

Proof The duality follows from the value of $d(V_{\alpha})$ and from the first statement which is the result of the following computation:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\alpha \\
\downarrow \quad \downarrow \\
V_{\alpha + 1} & V_{\alpha + 1}
\end{array}
\end{array}
\end{align*}
\quad = \quad (d_v \otimes \text{Id}_{V_{\alpha}}) \circ (w_v \otimes \text{Id}_v \otimes \text{Id}_{V_{\alpha}}) \circ \left(\text{Id}_v \otimes \begin{array}{c}
\begin{array}{c}
\alpha \\
\downarrow
\end{array}
\end{array}
\right)^{\alpha+1} \circ \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow \\
V_{\alpha + 1} & V_{\alpha + 1}
\end{array}
\end{array}
\quad = \quad (d_v \otimes \text{Id}) \circ (w_v \otimes \text{Id} \otimes \text{Id})(-\xi^{\alpha - 1} \{1\} v_0 \otimes v_1 \otimes v_0 + \xi^{-1} \{\alpha\} v_1 \otimes v_0 \otimes v_0)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\alpha \\
\downarrow \quad \downarrow \\
V_{\alpha + 1} & V_{\alpha + 1}
\end{array}
\end{array}
\end{align*}
\quad = \quad (d_v \otimes \text{Id})(\xi^\alpha \{1\} v_1^* \otimes v_1 \otimes v_0 + \xi^{-1} \{\alpha\} v_0^* \otimes v_0 \otimes v_0)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\alpha \\
\downarrow \quad \downarrow \\
V_{\alpha + 1} & V_{\alpha + 1}
\end{array}
\end{align*}
\quad = \quad \xi^\alpha \{1\} v_0 + \xi^{-1} \{\alpha\} v_0 = \{\alpha + 1\} v_0.
\end{align*}
\]
Here the first equality corresponds to the following isotopy:

\[
\begin{array}{c}
\begin{array}{c}
\text{\star} \\
\text{-} \\
\text{\star}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{\star} \\
\text{\star}
\end{array}
\end{array}
\]

This completes the proof. \(\square\)

**Remark 9** If \(\alpha \neq \beta\) are in \(\mathbb{C} \setminus \mathbb{Z}\) then \(V_\alpha\) and \(V_\beta\) are nonisomorphic simple modules and we have \(\text{Hom}(V_\alpha, V_\beta) = 0\). Thus,

\[
\begin{array}{c}
\begin{array}{c}
\text{\star}, \text{\star} \\
\text{\star}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{\star} \\
\text{-} \\
\text{\star}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{\star} \\
\text{\star}
\end{array}
\end{array}
\quad = 
\begin{array}{c}
\begin{array}{c}
\text{\star} \\
\text{-} \\
\text{\star}
\end{array}
\end{array}
\quad = 0.
\]

Here and after, the stars \(\star\) shall be replaced by any element in \(\mathbb{C} \setminus \mathbb{Z}\) such that the morphisms are defined.

**Corollary 10** (Fusion rule) For any \(\alpha \in \mathbb{C} \setminus \mathbb{Z}\),

\[
\{\alpha\} \text{Id}_v \otimes V_\alpha = \frac{v, \alpha}{\alpha-1} \alpha+1 \frac{v, \alpha}{\alpha+1} v.
\]

**Proof** This is a direct consequence of Proposition 8 and the fact that \(v \otimes V_\alpha\) split into a direct sum of simple modules as in (13). (Also see Remark 9.) \(\square\)

**Lemma 11** For all \(\alpha \in \mathbb{C} \setminus \mathbb{Z}\), one has

\[
\left(\text{Id} \otimes \frac{\alpha, v}{\alpha+1}\right) \circ \frac{v, \alpha+1}{\alpha+2} = \left(\frac{v, \alpha}{\alpha+1} \otimes \text{Id}\right) \circ \frac{\alpha+1, v}{\alpha+2}
\]

and similarly

\[
\left(\text{Id} \otimes \frac{\alpha, v}{\alpha-1}\right) \circ \frac{v, \alpha-1}{\alpha-2} = \left(\frac{v, \alpha}{\alpha-1} \otimes \text{Id}\right) \circ \frac{\alpha-1, v}{\alpha-2}.
\]

**Proof** The first equality is true because both side are maps \(V_{\alpha+2} \to v \otimes V_\alpha \otimes v\) determined by \(v_0 \mapsto v_0 \otimes v_0 \mapsto v_0 \otimes v_0 \otimes v_0\). Similarly, an easy computation gives that the other two maps \(V_{\alpha-2} \to v \otimes V_\alpha \otimes v\) are determined by \(v_{2r'} \mapsto \xi^{-2} \{\alpha\} \{\alpha - 1\} v_1 \otimes v_{2r'} \otimes v_1\). \(\square\)

If \(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}\), we will use the following family of operators

\[
X: V_\alpha \otimes V_\beta \to V_{\alpha+1} \otimes V_{\beta+1}
\]

given by

\[
X = \frac{1}{\{1\}} (\text{Id} \otimes d^v \otimes \text{Id}) \circ \left(\frac{\alpha+1, v}{\alpha} \otimes \frac{v, \beta+1}{\beta}\right).
\]

The following lemma shows that the denominator of \(X\) disappears.
Lemma 12 For \( \alpha, \beta \in \mathbb{C} \setminus \mathbb{Z} \) the map \( X: V_\alpha \otimes V_\beta \to V_{\alpha+1} \otimes V_{\beta+1} \) is given by

\[
X: v_i \otimes v_j \mapsto \xi^{\beta+i-j-1}\{\alpha - i\}v_i \otimes v_{j+1} + \xi^{-1}\{\beta - j\}v_{i+1} \otimes v_j
\]

where \( v_{2r'+1} \) should be understood as 0.

Proof First, a direct computations shows that

\[
d^\alpha(v_i \otimes v_{2r'-j}) = -\delta_j^i \xi^{i\alpha+i^2-1},
\]

\[
b^\alpha(1) = \sum_{i=0}^{2r'} -\xi^{-i\alpha+i^2}v_{2r'-i} \otimes v_i.
\]

Then using

\[
\alpha+1, \nu \frac{\alpha}{\nu} = (\text{Id} \otimes d^\alpha) \circ (\text{Id} \otimes \frac{\nu, -\alpha}{-\alpha-1} \otimes \text{Id}) \circ (b^{\alpha+1} \otimes \text{Id}),
\]

we have

\[
\nu_{\alpha+1, \nu} \left( v_i \right) = -\xi^i \{\alpha - i\}v_i \otimes v_1 - \xi^{-1}\{1\}v_{i+1} \otimes v_0.
\]

On the other hand, by definition we have

\[
\nu, \beta+1 \frac{\nu, \beta}{\beta} \left( v_j \right) = -\xi^{\beta-j-1}\{\beta - j\}v_0 \otimes v_{j+1} + \xi^{-1}\{\beta - j\}v_1 \otimes v_j.
\]

Combining these equalities with \( d^\nu(v_0 \otimes v_1) = -\xi \) and \( d^\nu(v_1 \otimes v_0) = 1 \) the result follows. \( \square \)

3.4 Multiplicity modules in \( V_\alpha \otimes V_\beta \otimes V_\gamma \)

It is well-known that, in the quantum plane \( \mathbb{Z}(x, y)_{/ (x, y) = \xi^2} \), one has

\[
(x + y)^i = \sum_{k=0}^{i} \xi^{k(i-k)} \binom{i}{k} x^k y^{i-k}
\]

for all \( i \in \mathbb{N} \). Applying this to \( y = K^{-1} \otimes F \) and \( x = F \otimes 1 \), we get

\[
(\Delta F)^i = (x + y)^i = \sum_{k=0}^{i} \xi^{k(i-k)} \binom{i}{k} F^k K^{k-i} \otimes F^{i-k}.
\]
The character formula for typical modules (see [4]) also implies that for all \( \alpha, \beta \in B \) with \( \alpha + \beta \not\in \mathbb{Z} \),

(19) \[ V_\alpha \otimes V_\beta = \sum_{k=-r'}^{r'} V_{\alpha+\beta+2k}. \]

Hence, for \( \alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z} \),

\[
\dim(H_\alpha^{\beta, \gamma}) = \begin{cases} 
1 & \text{if } \beta + \gamma - \alpha \in \{-2r', -2r' + 2, \ldots, 2r'\}, \\
0 & \text{else.}
\end{cases}
\]

Now for \( \alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z} \) with \( \beta + \gamma - \alpha = 2k \in \{-2r', -2r' + 2, \ldots, 2r'\} \) we define a map

\[
\begin{aligned}
\beta, \gamma \\
\alpha \\
2k
\end{aligned}
\]

which will form a basis for the 1–dimensional space \( H_\alpha^{\beta, \gamma} \). First, suppose \( \beta + \gamma - \alpha = -2r' \) then

\[
\begin{aligned}
\beta, \gamma \\
\alpha \\
-2r'
\end{aligned}
: V_\alpha \to V_\beta \otimes V_\gamma
\]

\[
v_0 \mapsto v_0 \otimes v_0
\]

\[
v_n \mapsto (\Delta F)^n v_0 \otimes v_0 = \sum_{k=0}^{n} \xi(n-k)(k-\beta-2r') \left[ \begin{array}{c} n \\ k \end{array} \right] v_k \otimes v_{n-k}
\]

where the last equality follows from Equation (18). Now, let \( n = r' + k \) and define

\[
\begin{aligned}
\beta, \gamma \\
\alpha \\
2k
\end{aligned}
= X^{\circ n} \circ \begin{aligned}
\beta-n, \gamma-n \\
\alpha \\
-2r'
\end{aligned}
: V_\alpha \to V_\beta \otimes V_\gamma.
\]

We now show that these bases are compatible with the cyclic isomorphisms defining the symmetric multiplicity modules. Let \( \mathcal{R} \) be the cyclic isomorphism

(20) \[ \mathcal{R}: H_\alpha^{\beta, \gamma} \to H_\beta^{\gamma, -\alpha}, \]

\[ f \mapsto (\delta^B \otimes \text{Id} \otimes \text{Id}) \circ f \circ (\text{Id} \otimes \text{Id} \otimes b^\alpha). \]

**Remark 13** The family of maps

\[
\begin{aligned}
\beta, \gamma \\
\alpha \\
-2r'
\end{aligned}
\]

can be seen as a section of the vector bundle \( \mathcal{E}_{-2r'} \) which is a restriction of \( \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E}^* \) to the subset of \( B^3 \) defined by the equation \( \beta + \gamma - \alpha = -2r' \). The cyclic isomorphism \( \mathcal{R} \)
is a lift to this vector bundle of the permutation on the basis: $(\alpha, \beta, \gamma) \mapsto (-\beta, \gamma, -\alpha)$.

The following proposition means that the section

\[
\begin{array}{c}
\beta, \gamma \\
\alpha \\
-2r'
\end{array}
\]

is a fixed point of the cyclic isomorphism $\mathcal{R}: \mathcal{E}_{-2r'} \to \mathcal{E}_{-2r'}$.

**Proposition 14** For all $(\alpha, \beta, \gamma) \in B^3$ with $\beta + \gamma - \alpha = -2r'$, we have

\[
\mathcal{R}\left(\begin{array}{c}
\beta, \gamma \\
\alpha \\
-2r'
\end{array}\right) = \begin{array}{c}
\gamma, -\alpha \\
-\beta \\
-2r'
\end{array}.
\]

**Proof** Let

\[
f_1 = \begin{array}{c}
\gamma, -\alpha \\
-\beta \\
-2r'
\end{array}
\quad \text{and} \quad
f_2 = \mathcal{R}\left(\begin{array}{c}
\beta, \gamma \\
\alpha \\
-2r'
\end{array}\right).
\]

Since $f_1(v_0) = v_0 \otimes v_0$, then $f_2$ is determined by its value on $v_0 \in V_{-\beta}$ which must be a multiple of the unique weight vector $v_0 \otimes v_0 \in V_{\gamma} \otimes V_{-\alpha}$. Because of this, we don’t need to compute all the terms to see that $f_2(v_0) = v_0 \otimes v_0$. In particular, from the facts

- $b_{V_{\alpha}}: 1 \mapsto v_{2r'} \otimes v_{2r'}^* + \cdots$
- $w_{-\alpha}^{-1}(v_{2r'}^*) = -\xi v_0$
- $w_{-\beta}(v_0) = -\xi^{-1} v_{2r'}$
- $\begin{array}{c}
\beta, \gamma \\
\alpha \\
-2r'(v_{2r'}) = (\Delta F)^{2r'}(v_0 \otimes v_0) = v_{2r'} \otimes v_0 + \cdots
\end{array}$
- (because $(\Delta F)^{2r'} = F^{2r'} \otimes 1 + \cdots$)
- $d_{V_{\beta}}(v_{2r'}^* \otimes v_{2r'}) = 1$

one can see that

\[
f_2(v_0) = (d_{V_{\beta}} \otimes \text{Id} \otimes \text{Id}) \circ \left( w_{-\beta} \otimes \begin{array}{c}
\beta, \gamma \\
\alpha \\
-2r' \otimes w_{-\alpha}^{-1}
\end{array}\right) \circ (\text{Id} \otimes b_{V_{\alpha}})(v_0)
\]

is equal to $v_0 \otimes v_0$. Thus, $f_1 = f_2$. \qed

To establish the same statement for the maps

\[
\begin{array}{c}
\beta, \gamma \\
\alpha \\
k
\end{array}
\]

we will need the two following lemmas.
Lemma 15  Let $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$ such that $\alpha + \beta + \gamma = 2 - 2r'$ then
\[
(X \otimes \text{Id}) \circ \left( \text{Id} \otimes \delta^{\beta - 1, \gamma}_{1 - r'} \right) \circ b^{\alpha - 1} = (\text{Id} \otimes X) \circ \left( \text{Id} \otimes \delta^{\beta - 1, \gamma - 1}_{1 - \alpha} \right) \circ b^\alpha .
\]

Proof  Both side of this equality are invariant maps $\mathbb{C} \rightarrow V_\alpha \otimes V_\beta \otimes V_\gamma$. Let $Z_l$ and $Z_r$ be the maps on the right and left hand sides, respectively. The space $H_{\alpha, \beta, \gamma}$ has dimension 1 so the maps $Z_l$ and $Z_r$ are proportional. Thus, to show they are equal it is enough to show the functions $(d^\alpha \otimes \text{Id} \otimes d^\gamma)(v_0 \otimes Z_i(1) \otimes v_{2r'})$, for $i = l, r$, are equal.

First, let us work on the left hand side. By considering the formulas for $d^\alpha$ and $X$ the only terms of $b^{\alpha - 1}(1)$ that contribute nontrivially to the function are $-\xi v_{2r'} \otimes v_0$ and $-\xi^{1 - \alpha} v_{2r' - 1} \otimes v_1$. Therefore, we only need to consider
\[
\begin{align*}
\delta^{* \ast}_{2r'}(v_0) &= v_0 \otimes v_0 \quad \text{and} \quad \delta^{* \ast}_{2r'}(v_1) = v_1 \otimes v_0 + \cdots
\end{align*}
\]
where the other term(s) contained in the $\cdots$ can be disregarded since $d^\gamma(v_i \otimes v_{2r'})$ is nonzero if and only if $i = 0$. So $Z_l(1)$ is equal to
\[
-\xi (X \otimes \text{Id})(v_{2r'} \otimes v_0 \otimes v_0) - \xi^{1 - \alpha}(X \otimes \text{Id})(v_{2r' - 1} \otimes v_1 \otimes v_0) + \cdots
\]
\[
= -\xi^{\beta - 2}\{\alpha\}(v_{2r'} \otimes v_1 \otimes v_0) - \xi^{-\alpha}\{\beta - 2\}(v_{2r'} \otimes v_1 \otimes v_0) + \cdots
\]
where as above the term(s) contained in the $\cdots$ can be disregarded since they do not contribute nontrivially to the function. So, we have
\[
(d^\alpha \otimes \text{Id} \otimes d^\gamma)(v_0 \otimes Z_l(1) \otimes v_{2r'}) = -(\xi^{\beta - 4}\{\alpha\} + \xi^{-\alpha - 2}\{\beta - 2\})v_1
\]
\[
= -\xi^{-2}\{\alpha + \beta - 2\}v_1
\]
since $d^X(v_0 \otimes v_{2r'}) = -\xi^{-1}$. Similarly,
\[
(d^\alpha \otimes \text{Id} \otimes d^\gamma)(v_0 \otimes Z_r(1) \otimes v_{2r'}) = -\xi^{-2}\{\gamma - 1\}v_1 = -\xi^{-2}\{\alpha + \beta - 2\}v_1 .
\]

Lemma 16  Let $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$, then $X \otimes \text{Id}$ and $\text{Id} \otimes X$ commute:
\[
(X \otimes \text{Id}) \circ (\text{Id} \otimes X) = (\text{Id} \otimes X) \circ (X \otimes \text{Id}): V_{\beta - 1} \otimes V_{\alpha - 2} \otimes V_{\gamma - 1} \rightarrow V_\beta \otimes V_\alpha \otimes V_\gamma .
\]

Graphically, this is illustrated by the following:

\[
\begin{array}{c}
\begin{array}{c}
\text{\Large \uparrow} \\
\text{\Large \uparrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\Large \downarrow} \\
\text{\Large \downarrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\Large \uparrow} \\
\text{\Large \uparrow}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{\Large \downarrow} \\
\text{\Large \downarrow}
\end{array}
\end{array}
\]

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Proof Composing both side of the second equality of Lemma 11 with
\[
\beta \\
\beta - 1, v 
\otimes \text{Id}_{\nu} \otimes 
\gamma \\
\alpha, v, \gamma - 1
\]
gives the result.

Proposition 17 For all $(\alpha, \beta, \gamma) \in (\mathbb{C} \setminus \mathbb{Z})^3$ with $\beta + \gamma - \alpha = k \in \{-2r', -2r' + 2, \ldots, 2r'\}$ we have
\[
\mathcal{R} \left( \begin{array}{c} \beta, \gamma \\ \alpha \end{array} \right) = \begin{array}{c} \gamma, -\alpha \\ -\beta \end{array}
\]
where $\mathcal{R} : H_{\alpha}^{\beta, \gamma} \rightarrow H_{-\beta}^{\gamma, -\alpha} \text{ is given in Equation (20)}$.

Proof We first give a reformulation of Proposition 14: tensoring the equality with $\text{Id}_{\nu}$ on the left and composing on the right with $b^\beta$, we have
\[
\left( \text{Id} \otimes \begin{array}{c} \gamma, -\alpha \\ -2r' \end{array} \right) \circ b^\beta = \left( \begin{array}{c} \beta, \gamma \\ -\beta \end{array} \otimes \text{Id} \right) \circ b^\alpha
\]
for all $\alpha, \beta, \gamma \in (\mathbb{C} \setminus \mathbb{Z})^3$ with $\beta + \gamma - \alpha = -2r'$. Let $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$ such that $\alpha + \beta + \gamma = 2 - 2r'$ then from Lemmas 15 and 16 we have that for $p, q \in \mathbb{N}$,
\[
(X^{\circ p + 1} \otimes \text{Id}) \circ (\text{Id} \otimes X^{\circ q}) \circ \left( \text{Id} \otimes \begin{array}{c} \beta - 1, \gamma \\ 1 - \alpha \end{array} \right) \circ b^{\alpha - 1} \\
= (X^{\circ p} \otimes \text{Id}) \circ (\text{Id} \otimes X^{\circ q + 1}) \circ \left( \text{Id} \otimes \begin{array}{c} \beta - 1, \gamma - 1 \\ -\alpha \end{array} \right) \circ b^\alpha.
\]
Then by induction, for any $n \in \mathbb{N}$ and for any $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$ such that $\alpha + \beta + \gamma = 2n - 2r'$, one has
\[
(X^{\circ n} \otimes \text{Id}) \circ \left( \text{Id} \otimes \begin{array}{c} \beta - n, \gamma \\ n - \alpha \end{array} \right) \circ b^{\alpha - n} = (\text{Id} \otimes X^{\circ n}) \circ \left( \text{Id} \otimes \begin{array}{c} \beta - n, \gamma - n \\ -\alpha \end{array} \right) \circ b^\alpha.
\]
Therefore, for $n = k/2 + r' \in \mathbb{N}$ we have
\[
\mathcal{R} \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) = (d^\beta \otimes \text{Id} \otimes \text{Id}) \circ (\text{Id} \otimes \left( X^{\circ n} \circ \left( \begin{array}{c} \beta - n, \gamma - n \\ \alpha - n \end{array} \right) \otimes \text{Id} \right) \circ (\text{Id} \otimes b^\alpha) \\
= (d^\beta \otimes \text{Id} \otimes \text{Id}) \circ (\text{Id} \otimes \text{Id} \otimes X^{\circ n}) \circ \left( \text{Id} \otimes \begin{array}{c} \beta, \gamma - n \\ -2r' \otimes \text{Id} \end{array} \right) \circ (\text{Id} \otimes b^{\alpha - n})
\]
\[ X^n \circ (d^B \otimes \text{Id} \otimes \text{Id}) \circ \left( \text{Id} \otimes \left( \begin{array}{c} -2r' \\ \beta, \gamma - n \\ \alpha - n \end{array} \right) \otimes \text{Id} \right) \circ (\text{Id} \otimes b^{a-n}) \]

\[ = X^n \circ \left( \begin{array}{c} -2r' \\ \beta \\ \gamma - n \\ n - \alpha \end{array} \right) = \left( \begin{array}{c} -\beta \\ \gamma \end{array} \right) \]

where the second to last equality is given by Proposition 14.

The cyclic isomorphisms allow us to define the basis

\[ \left( \begin{array}{c} \alpha \\ \beta, \gamma \\ k \end{array} \right) \]

of \( H^\alpha_{\beta, \gamma} \) in two equivalent way: if \( \alpha - \beta - \gamma = k \), let

\[ \left( \begin{array}{c} \alpha \\ \beta, \gamma \\ k \end{array} \right) = (\text{Id} \otimes d^\gamma) \circ \left( \left( \begin{array}{c} \alpha, -\gamma \\ k \end{array} \right) \otimes \text{Id} \right) = (d^{\beta} \otimes \text{Id}) \circ \left( \text{Id} \otimes \left( \begin{array}{c} -\beta, \alpha \\ \gamma \end{array} \right) \right) \]

Similarly, if \( \alpha + \beta + \gamma = k \), we get a vector \( \omega^k(\alpha, \beta, \gamma) \), which forms a canonical basis of the symmetric multiplicity module \( H(\alpha, \beta, \gamma) \).

In what follows we consider ribbons graphs with coupons colored by the elements \( \omega^k(\alpha, \beta, \gamma) \). For such a coupon \( c \), Proposition 17 implies that we do not need to know what edges are attached to the bottom of \( c \) and what edges are attached to its top. Only the information of the cyclic ordering of these edges is needed to compute \( F \) or \( F' \).

The choice of an half twist \( \theta' \) (a family of endomorphisms whose square are given by the twist) produces isomorphisms

\[ H^\alpha_\beta \rightarrow H^\beta_\alpha \]

given by \( f \mapsto \theta_\alpha' \theta_\beta' \theta_\gamma'^{-1} \text{CV}_\alpha \text{CV}_\beta \circ f \)

(for details see [3]). These isomorphism produces isomorphisms \( H(\alpha, \beta, \gamma) \rightarrow H(\beta, \alpha, \gamma) \). The following lemma shows that the bases we have defined above are compatible with these isomorphisms.

**Lemma 18** We can define an half twist on the set of typical modules \( \{ V_\alpha \}_{\alpha \in B} \) by the formula

\[ \theta_\alpha' = \xi(\alpha/2)^2 - r'^2 \text{Id}_{V_\alpha} \]
Let $\alpha, \beta, \alpha + \beta \in \mathbb{C} \setminus \mathbb{Z}$. Then

\[(21)\quad C_{V_{\alpha}, V_{\beta}} \circ \begin{array}{c}
\times \\
\alpha, \beta
\end{array} = \xi^{(1/2)(\alpha+2r')(\beta+2r')} \begin{array}{c}
\times \\
\beta, \alpha
\end{array},
\]

\[(22)\quad C_{V_{\alpha}, v} \circ \begin{array}{c}
\times \\
\alpha, v
\end{array} \circ \begin{array}{c}
\times \\
\alpha-1
\end{array} = \xi^{-(1/2)\alpha} \begin{array}{c}
\times \\
\alpha, \alpha
\end{array} \circ \begin{array}{c}
\times \\
\alpha-1
\end{array},
\]

\[(23)\quad C_{v, V_{\alpha}} \circ \begin{array}{c}
\times \\
\alpha, \alpha
\end{array} \circ \begin{array}{c}
\times \\
\alpha-1
\end{array} = \xi^{-(1/2)\alpha} \begin{array}{c}
\times \\
\alpha, \alpha
\end{array} \circ \begin{array}{c}
\times \\
\alpha-1
\end{array},
\]

\[(24)\quad C_{V_{\alpha}, V_{\beta}} \circ X = \xi^{(-\alpha-\beta+1)/2} X \circ C_{V_{\alpha-1}, V_{\beta-1}},
\]

and for $n = r' + k$,

\[(25)\quad C_{V_{\alpha}, V_{\beta}} \circ \begin{array}{c}
\times \\
\alpha, \beta
\end{array} = \frac{\theta'_{\alpha+\beta-2k}}{\theta'_{\alpha} \theta'_{\beta}} \begin{array}{c}
\times \\
\beta, \alpha
\end{array}.
\]

**Proof** From Formula (8), we have that $\theta_{V_{\alpha}}$ acts on the highest weight vector $v_0 \in V_\alpha$ as

$$K^{-2r'} \xi H^2/2 v_0 = \xi^{-2r'(\alpha+2r')+(\alpha+2r')^2/2} v_0 = \xi^{\alpha^2/2-2r'^2} v_0.$$ 

Hence $\theta'$ is an half twist.

Only the “Cartan” part $\xi^{H \otimes H}$ of the R-matrix (7) acts nontrivially on the tensor product of two highest weight vectors. Hence

$$C_{V_{\alpha}, V_{\beta}} (v_0 \otimes v_0) = \xi^{(1/2)(\alpha+2r')(\beta+2r')} v_0 \otimes v_0 \in V_{\beta} \otimes V_{\alpha}.$$ 

But

$$v_0 \otimes v_0 = \begin{array}{c}
\times \\
\alpha, \beta
\end{array} \otimes \begin{array}{c}
\times \\
\alpha, \beta
\end{array},$$

and this gives Equation (21).

Similarly,

$$C_{V_{\alpha}, v} \circ \begin{array}{c}
\times \\
\alpha, v
\end{array} \circ \begin{array}{c}
\times \\
\alpha+1
\end{array} = \xi^{(1/2)\alpha} \xi^{r'} \begin{array}{c}
\times \\
\alpha, v
\end{array} \circ \begin{array}{c}
\times \\
\alpha+1
\end{array},$$

and Equation (22) follows from the duality of (14). Equation (22) is proved with analogous techniques.

To prove Equation (24), consider the isotopy

$$\begin{array}{c}
\begin{array}{c}
\times \\
\alpha, \beta
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\times \\
\alpha, \beta
\end{array}
\end{array}$$

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which illustrates the fact that \( \{1\} C_{V, \beta} \circ X \) is equal to

\[
(26) \quad \left( \text{Id} \otimes \left( \bigotimes_{\alpha, \beta} \alpha^{-1, \nu} C_{V_{\alpha-1, \nu}} \right) \right) \circ \left( \bigotimes_{\alpha, \beta} \left( C_{V, \beta} \otimes \nu \beta_{-1} \right) \otimes \text{Id} \right) \circ C_{V_{\alpha-1, \beta-1}}.
\]

Here Equation (23) can be used to remove the braiding \( C_{V, \beta} \) in (26). Now Equation (22) implies that

\[
\bigotimes_{\alpha, \beta} \alpha^{-1, \nu} C_{V_{\alpha-1, \nu}} = \xi^{-\alpha/2} \xi^{r'} \bigotimes_{\nu, \alpha-1}
\]

and Equation (24) follows.

Finally, Equation (25) follows from

\[
C_{V, \beta} \circ X^n = \xi^{-(1/2)(\alpha+\beta-1)+\ldots+(\alpha+\beta-2n+1)} X^n \circ C_{V_{\alpha-n, \beta-n}}
\]

\[
= \xi^{-(1/2)n(\alpha+\beta-n)} X^n \circ C_{V_{\alpha-n, \beta-n}}.
\]

Composing this equation with

\[
\bigotimes_{\alpha-n, \beta-n}^{2r', \alpha+\beta-2k} \nu
\]

and applying Equation (21), the result follows. \( \square \)

### 3.5 A Laurent polynomial invariant of planar trivalent graphs

In this section we discuss how to defined maps lead to invariant of planar graphs that are in some sense Laurent polynomial.

Let \( \Gamma \subset \mathbb{R} \times [0, 1] \) be a planar unitrivalent framed graph with trivalent vertices marked by heights, that are integers in \(-2r', -2r' + 2, \ldots, 2r' - 2, 2r'\) and whose set \( \Gamma_u \) of univalent vertices is included in \( \mathbb{R} \times \{0, 1\} \). The heights can be seen as a \( 0- \) chain \( h \) on the CW–complex \( \Gamma \) relative to \( \Gamma_u \). A coloring of \( \Gamma \) is a complex \( 1- \) chain \( c \in C_1(\Gamma, \Gamma_u; \mathbb{C}) \) such that its boundary is \( \delta c = h \). Let \( \text{Col}(\Gamma) \) be the affine space of coloring of \( \Gamma \) and \( \text{Col}_0(\Gamma) \) be the subset of coloring that have no values (no coefficients) in \( \mathbb{Z} \).

Since a coloring is a realization of \( h \) as a boundary we have the set \( \text{Col}(\Gamma) \) is nonempty if and only if \([h] = 0 \in H_0(\Gamma, \Gamma_u; \mathbb{Z})\). This means that the sum of the heights of any connected component of \( \Gamma \) that does not meet \( \Gamma_u \) is zero. Let us assume that this is true and let \( n = \dim H_1(\Gamma, \Gamma_u; \mathbb{C}) \). Then \( \text{Col}(\Gamma) \) is an affine space over \( H_1(\Gamma, \Gamma_u; \mathbb{C}) \).

We then choose a family of \( n \) edges \( e_1, \ldots, e_n \) of \( \Gamma \). We assume that the union of the interior of these edges has a complement in \( \Gamma/\Gamma_u \) which is simply connected. Then the map

\[
\text{Col}(\Gamma) \to \mathbb{C}^n \quad \text{given by} \quad c \mapsto (c(e_1), \ldots, c(e_n))
\]

is bijective.
We will also suppose that every edge of $\Gamma$ is in the support of a relative cycle. Hence, any coloring that takes an integer value on an edge can be infinitesimally modified to a coloring of $\text{Col}_0(\Gamma)$. Then $\text{Col}_0(\Gamma)$ is an open dense subset of $\text{Col}(\Gamma)$.

If $c \in \text{Col}_0(\Gamma)$, we can form a $C$–colored ribbon graph $c(\Gamma)$ as follows. First, we choose an orientation of the edges of $\Gamma$. Color each oriented edge $e$ of $\Gamma$ with $V_c(e)$. Any trivalent vertex of $\Gamma$ with height $k$ is replaced with a trivalent coupon containing the morphism $\omega_k$ previously defined. Positioning the edges around the coupon involves some choice but the value under $F$ (or $F'$ if $\Gamma$ is closed i.e. has no univalent vertices) of the resulting ribbon graph does not depend of these choices.

**Theorem 19** Let $\Gamma$ be a planar unitrivalent framed graph with height $h$ as above. Also, as above choose $n$ edges $e_1, \ldots, e_n$ of $\Gamma$. Supposing that $\Gamma$ is not a circle, for any coloring $c \in \text{Col}_0(\Gamma)$ define $x(c)$ as follows:

1. If $\Gamma$ has univalent vertices, then let $x(c)$ be any fixed coefficient of the matrix in the canonical bases of $F(c(\Gamma))$,

2. else, $\Gamma$ is closed and let $x(c)$ be $F'(c(\Gamma))$.

Then there exists a unique Laurent polynomial

$$P(q_1, \ldots, q_n) \in \mathbb{Z}[\xi][q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$$

such that for any coloring $c \in \text{Col}_0(\Gamma)$, $x(c) = P(\xi^{c(e_1)}, \ldots, \xi^{c(e_n)})$.

**Proof** First consider the case $\Gamma_u \neq \emptyset$. For the existence of the Laurent polynomials, it is sufficient to remark that it is true for the elementary morphisms

$$\begin{align*}
\begin{array}{c}
\includegraphics[height=0.5cm]{elementary_morphisms1}
\end{array},
\begin{array}{c}
\includegraphics[height=0.5cm]{elementary_morphisms2}
\end{array}
\end{align*}$$

and $b^*, d^*$ from (10), (11). Now the uniqueness follows from the general fact that a Laurent polynomials in $n$ variables with complex coefficients which vanishes on an open dense subset of $(\mathbb{C}^*)^n$ must be 0.

In the other case, $\Gamma$ is a closed graph and $x(c) = F'(c(\Gamma))$. To compute $F'(c(\Gamma))$ we open $c(\Gamma)$ on an edge $e$ to get a cutting presentation of $c(\Gamma)$. The invariant of this cutting presentation is then a scalar times the identity of $V_c(e)$. By the previous argument, this scalar is given by a Laurent polynomial $P_e$. $F'(c(\Gamma))$ is by definition this scalar times $d(V_c(e)) = D(\xi^{c(e)})^{-1} = \langle \xi^{c(e)} \rangle / (\langle \xi^{c(e)} \rangle)^r$. This denominator seems to be a problem but in fact it must cancel. Indeed, as $\Gamma$ is not a circle, we have $n \geq 2$. But $F'(c(\Gamma))$ does not depend on where we cut and open $c(\Gamma)$ (see [4, Theorem 3] in Algebraic & Geometric Topology, Volume 11 (2011)).
and Section 6.2]). Hence cutting alternatively on the edges $e_1$, and then $e_2$, we get that there exits polynomials $P_1, P_2 \in \mathbb{Z}[\xi][q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$ such that

$$F'(c(\Gamma)) = \frac{P_1(\xi^{c(e_1)}, \ldots, \xi^{c(e_n)})}{(\xi^{c(e_i)})^{r} - (\xi^{c(e_i)})^{-r}}$$

with $i \in \{1, 2\}$ and thus

$$\frac{P_1}{q_1^r - q_1^{-r}} = \frac{P_2}{q_2^r - q_2^{-r}}.$$ 

Even if $\mathbb{Z}[\xi]$ is not a unique factorization domain, one easily see that this last equality implies that $(1/(q_1^r - q_1^{-r}))P_1 \in \mathbb{Z}[\xi][q_1^{\pm 1}, \ldots, q_n^{\pm 1}]$. \hfill \Box

We now use this theorem applied to the tetrahedron graph to give an alternative definition of the polynomials $J$. This will prove in particular that their coefficients are in $\mathbb{Z}[\xi]$. As we will see Theorem 29 implies that this definition coincides with the formulas given in Section 1.

For $(i, j, k) \in \mathcal{H}_r'$ we consider the planar 1–skeleton $\Gamma$ of the tetrahedron with heights as follows. Let $v_1, v_2, v_3, v_4$ be the vertices of $\Gamma$:

$$\Gamma = \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\downarrow
\end{array}
\end{array}
\end{array}$$

Assign $v_1, v_2, v_3, v_4$ the heights $2i, 2j, 2k, -2i - 2j - 2k$, respectively.

**Definition 20** Let $J_{i,j,k} \in \mathbb{Z}[\xi][q_1^{\pm 1}, q_2^{\pm 1}, q_3^{\pm 1}] = \mathcal{L}$ be the Laurent polynomial of Theorem 19 associated to $\Gamma$ and the edges $(e_1, e_2, e_3) = (v_2v_3, v_3v_1, v_1v_2)$. Thus, $J_{i,j,k}$ is the unique Laurent polynomial such that for all $\alpha, \beta, \gamma \in \mathbb{C}$ with $\alpha, \beta, \gamma, \alpha - \beta, \beta - \gamma, \gamma - \alpha \notin \mathbb{Z},$

$$\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix} = J_{i,j,k}(\xi^\alpha, \xi^\beta, \xi^\gamma)$$

(27)

$$\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix} = F'(c(\Gamma))$$

where

is the invariant of the graph

$$\Gamma = \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\begin{array}{c}
\downarrow
\end{array}
\end{array}$$

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colored with
\[ c(v_2v_3) = \alpha \quad c(v_3v_1) = \beta \quad c(v_1v_2) = \gamma \]
\[ c(v_1v_4) = \beta - \gamma - 2i \quad c(v_2v_4) = \gamma - \alpha - 2j \quad c(v_3v_4) = \alpha - \beta - 2k. \]

If \(|i|, |j|, |k|\) or \(|i + j + k|\) is \(> r'\), then by convention, set
\[
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix} = 0.
\]

Here we change from the usual notation
\[
\begin{vmatrix}
  j_1 & j_2 & j_3 \\
  j_4 & j_5 & j_6
\end{vmatrix}
\]
of [3] to the notation
\[
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix}.
\]

We use the new notation because it is closely related to the polynomials \(J\) and easily adapts to the computations below. The correspondence between the two notation is given in Figure 4.

![Figure 4: The two notation for the 6j–symbols](image)

with \(j_1 = \alpha\) \(j_2 = -\beta\) \(j_6 = -\gamma\)
\(j_4 = \beta - \gamma - 2i\) \(j_5 = \alpha - \gamma + 2j\) \(j_3 = \alpha - \beta - 2k\)

**3.6 Computations of the 6j–symbols**

The next proposition establishes the unexpected fact that the family of bases of the multiplicity modules constructed in Section 3.4 is self dual. Proposition 21 is illustrated
by Figure 5 where the left hand side may be seen as a cutting presentation of the \(\Theta\)-graph.

\[
\begin{array}{c}
\alpha \\
\beta \\
\gamma \\
-\kappa
\end{array}
\begin{array}{c}
-2r' \\
\beta, \gamma
\end{array}
\begin{array}{c}
\beta-2r', \gamma-2r' \\
\alpha
\end{array}
= d(\alpha)^{-1} \alpha
\]

Figure 5: The duality for \(H(V_\alpha, V_\beta, V_\gamma)\)

**Proposition 21** Let \(\alpha = \beta + \gamma - 2r'\) then

\[
\begin{array}{c}
\alpha \\
\beta, \gamma
\end{array}
\begin{array}{c}
-2r' \\
\beta, \gamma
\end{array}
\begin{array}{c}
\beta-2r', \gamma-2r' \\
\alpha
\end{array}
= d(\alpha)^{-1} \text{Id}_{V_\alpha}
\]

and consequently, if \(\alpha + \beta + \gamma = k\) then

\[
\langle \omega^k(\alpha, \beta, \gamma), \omega^{-k}(-\gamma, -\beta, -\alpha) \rangle = 1
\]

where the duality \(H(\alpha, \beta, \gamma) \otimes H(-\gamma, -\beta, -\alpha) \to \mathbb{C}\) is obtained by the evaluation of \(F'\) on the colored \(\Theta\)-graph

\[
\begin{array}{c}
\gamma \\
\beta \\
\alpha
\end{array}
\begin{array}{c}
\beta \\
\alpha
\end{array}
\]

**Proof** Let us denote by

\[
\Theta = \begin{array}{c}
\gamma \\
\beta \\
\alpha
\end{array}
\begin{array}{c}
\beta \\
\alpha
\end{array}
\]

the \(\Theta\)-graph where the coupons are filled with the morphisms \(\omega^k(\alpha, \beta, \gamma)\) and \(\omega^{-k}(-\gamma, -\beta, -\alpha)\). We use properties of \(F'\) to compute \(F'(\Theta)\) as follows. We have

\[
F'(\Theta) = d(V_\alpha) \left\langle \begin{array}{c}
\alpha \\
-2r' \\
\beta, \gamma
\end{array}
\begin{array}{c}
-2r' \\
\alpha
\end{array}
\begin{array}{c}
\beta-2r', \gamma-2r' \\
2r'-\beta, -\alpha
\end{array}
\right\rangle
= d(V_{2r'}) \left\langle \begin{array}{c}
\gamma-2r' \\
2r'-\beta, -\alpha
\end{array}
\begin{array}{c}
\alpha \\
2r'-\beta, -\alpha
\end{array}
\left(\text{Id} \otimes \begin{array}{c}
\alpha \\
2r'-\beta, -\alpha
\end{array}
\right)
\right\rangle
\circ \left(\text{Id} \otimes X^\otimes 2r'\right) \circ \left(b^{2r'}(1) \otimes \text{Id}_{V_{2r'}}\right)
\]

We compute the bracket of the right hand side of the last equality by evaluating the morphisms on the lowest weight vector \(v_{2r'} \in V_{2r'}\).
First remark that according to Lemma 12, $X$ sends
\[ V_{\delta+i} \otimes V_{\varepsilon+i} \ni v_i \otimes v_{2r'} \mapsto \xi^{-1}\{\varepsilon + i - 2r'\}v_{i+1} \otimes v_{2r'} \in V_{\delta+i+1} \otimes V_{\varepsilon+i+1}. \]

Therefore, $X^{\circ 2r'}(v_i \otimes v_{2r'}) = 0$ if $i \geq 1$ and
\[ X^{\circ 2r'}(v_0 \otimes v_{2r'}) = -\xi \left( \prod_{i=1}^{2r'} \{\varepsilon + i\} \right) v_{2r'} \otimes v_{2r'} \]
where here we use the equalities $\xi^{-2r'} = -\xi$ and $\{\varepsilon + i - 2r'\} = -\{\varepsilon + i + 1\}$. Applying this to $h^{2r'-\beta}(1) \otimes v_{2r'} \in V_{2r'-\beta} \otimes V_{\beta-2r'} \otimes V_{\gamma-2r'}$ we get
\[ \text{Id} \otimes X^{\circ 2r'}(h^{2r'-\beta}(1) \otimes v_{2r'}) = \text{Id} \otimes X^{\circ 2r'} (-\xi v_{2r'} \otimes v_0 \otimes v_{2r'}) \]
\[ = \xi^2 \left( \prod_{i=1}^{2r'} \{\gamma - 2r' + i\} \right) v_{2r'} \otimes v_{2r'} \otimes v_{2r'} \in V_{2r'-\beta} \otimes V_{\beta} \otimes V_{\gamma}. \]

Now using the fact that
\[ *_{-2r'}(v_{2r'}) = v_{2r'} \otimes v_0 + \cdots \]
and $d^*(v_0 \otimes v_{2r'}) = -\xi^{-1}$ we have
\[ *_{-2r'}(v_{2r'} \otimes v_{2r'}) = (\text{Id} \otimes d^*) \circ \left( \begin{array}{c} *_{-2r'} \otimes \text{Id} \end{array} \right) (v_{2r'} \otimes v_{2r'}) = -\xi^{-1} v_{2r'} \]
and we see that the above bracket is equal to $d(V_{\gamma-2r'})^{-1}$. \hfill \Box

Remark that with Theorem 19, the previous result can be restated as saying: the Laurent polynomial associated to the $\Theta$–graph with heights $k, -k$ is constant equal to 1.

**Proposition 22**

(28) \[ \left( \text{Id} \otimes *_{-2r'} \right) \circ *_{-2r'} = \left( \begin{array}{c} *_{-2r'} \otimes \text{Id} \end{array} \right) \circ *_{-2r'} \]
(29) \[ \left( \text{Id} \otimes v_{-2r'} \right) \circ *_{-2r'} = \left( \begin{array}{c} v_{-2r'} \otimes \text{Id} \end{array} \right) \circ *_{-2r'} \]

Here the stars can be replaced by any colors in $\mathbb{C} \setminus \mathbb{Z}$ such that the compositions are matching, the source and target of the maps are the same in both side of the equalities and the colors meeting at a trivalent vertex satisfy the conditions given in Sections 3.3 and 3.4.

**Proof** All these maps send the highest weight vector of the bottom irreducible module $v_0$ to $v_0 \otimes v_0 \otimes v_0$. \hfill \Box
Define the following operators:

- $\overline{-X}: V_\alpha \otimes V_\beta \rightarrow V_{\alpha-1} \otimes V_{\beta+1}$ by
  \[
  \overline{-X} = (\text{Id} \otimes d^\gamma \otimes \text{Id}) \circ \left( \begin{array}{c c}
  \alpha-1, \gamma & \text{Id} \\
  \alpha & \beta
  \end{array} \right),
  \]

- $X^ -: V_\alpha \otimes V_\beta \rightarrow V_{\alpha+1} \otimes V_{\beta-1}$ by
  \[
  X^- = (\text{Id} \otimes d^\gamma \otimes \text{Id}) \circ \left( \begin{array}{c c}
  \alpha+1, \gamma & \text{Id} \\
  \alpha & \beta
  \end{array} \right),
  \]

- $\overline{X}: V_\alpha \otimes V_\beta \rightarrow V_{\alpha-1} \otimes V_{\beta-1}$ by
  \[
  \overline{X} = (\text{Id} \otimes d^\gamma \otimes \text{Id}) \circ \left( \begin{array}{c c}
  \alpha-1, \gamma & \text{Id} \\
  \alpha & \beta
  \end{array} \right).
  \]

From Corollary 10 we have a commutation rule for these operators:

**Lemma 23** We have the following equalities of maps from $V_\alpha \otimes V_\beta$:

- $\overline{-X} \circ X = X \circ -X$ and $X^- \circ X = X \circ X^-$
- $\overline{-X} \circ \overline{X} = \overline{X} \circ -X$ and $X^- \circ \overline{X} = \overline{X} \circ X^-$
- $\overline{-X} \circ X^- - \{1\} \overline{X} \circ X = \{\alpha + 1\} \{\beta\} \text{Id}_{V_\alpha \otimes V_\beta}$

**Proof** Consider the map $\text{End}(v \otimes V_\gamma) \rightarrow \text{Hom}(V_* \otimes V_\gamma, V_* \otimes V_\gamma)$ given by,

\[
y \mapsto \left( \begin{array}{c c}
  *, \gamma & \text{Id} \\
  *, & \text{Id}
  \end{array} \right) \circ (\text{Id} \otimes y) \circ \left( \begin{array}{c c}
  *, \gamma & \text{Id} \\
  *, & \text{Id}
  \end{array} \right).
\]

The identities of the lemma are obtained by composing this map with both sides of Equation (15). \qed

The following proposition describes how these operators act on multiplicity modules.

**Proposition 24** For any $\alpha, \beta, \gamma \in \mathbb{C} \setminus \mathbb{Z}$ and any $k \in \{-r', \ldots, r'\}$, we have the following:

- $\overline{-X} \circ \left( \begin{array}{c c}
  *, \beta-1 \gamma & 2k \\
  * & *
  \end{array} \right) = \{\beta + r' + k\} \left( \begin{array}{c c}
  *, \alpha-1 \gamma & 2k \\
  * & *
  \end{array} \right)$

- $X^- \circ \left( \begin{array}{c c}
  \alpha-1, *, \gamma & 2k \\
  * & *
  \end{array} \right) = \{\alpha + r' + k\} \left( \begin{array}{c c}
  \alpha, *, \gamma & 2k \\
  * & *
  \end{array} \right)$

- $\overline{X} \circ \left( \begin{array}{c c}
  *, *, \gamma & 2k+2 \\
  * & *
  \end{array} \right) = \left( \begin{array}{c c}
  r' + k \gamma & 2k \\
  {1} & *
  \end{array} \right) \left( \begin{array}{c c}
  \gamma + k + r' + 1 \gamma & 2k \\
  * & *
  \end{array} \right)$
**Proof** Let us start with the first equality. If $k = -r'$ it is obtained by composing (29) with

$$\text{Id} \otimes \begin{array}{c} \ast \\ v, \ast \end{array}$$

where the factor $\{1 - \beta\} = \{\beta + 2r'\}$ arises from the duality of (14). Now, for any $k = n - r' \in \{-r', \ldots, r'\}$, we have

$$\begin{align*}
\mathcal{X} \circ \begin{array}{c} \ast, \beta - 1 \\ 2k \end{array} &= \mathcal{X} \circ \mathcal{X}^n \circ \begin{array}{c} \ast - n, \beta - 1 - n \\ -2r' \end{array} = \mathcal{X}^n \circ \mathcal{X} \circ \begin{array}{c} \ast - n, \beta - 1 - n \\ -2r' \end{array} \\
&= \{\beta + 2r' - n\} \mathcal{X}^n \circ \begin{array}{c} \ast - n - 1, \beta - n \\ -2r' \end{array} = \{\beta + r' - k\} \begin{array}{c} \ast - 1, \beta \\ 2k \end{array}
\end{align*}$$

which proves the first equality. The proof of the second identity is similar.

For the third, Lemma 23 implies

$$\{1\} \mathcal{X} \circ \mathcal{X} = \mathcal{X} \circ \mathcal{X} - \{\alpha\} \{\beta - 1\} \text{Id}_{\mathcal{V}_{\alpha - 1} \otimes \mathcal{V}_{\beta - 1}}.$$

Then since

$$\begin{array}{c} \alpha, \beta \\ 2k + 2 \end{array} = \mathcal{X} \circ \begin{array}{c} \alpha - 1, \beta - 1 \\ 2k \end{array},$$

the identity comes from the equality

$$\{\alpha + r' - k\} \{\beta + r' - k - 1\} - \{\alpha\} \{\beta - 1\} = \{\alpha + \beta - k + r' - 1\} \{r' - k\}$$

with $\gamma = \alpha + \beta - 2k - 2$. \qed

**Proposition 25**

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \begin{bmatrix} 2j & \beta \\ 2k & \gamma \end{bmatrix} = \begin{bmatrix} 2k & \gamma \\ 2i & \alpha \end{bmatrix} = \begin{bmatrix} 2k & -\gamma \\ 2i & -\alpha \end{bmatrix}$$

$$= \begin{bmatrix} -2(i + j + k) & -\gamma \\ 2j & \beta - \gamma - 2i \\ 2i & \alpha - \gamma + 2j \end{bmatrix}$$

$$= \begin{bmatrix} -2(i + j + k) & -\beta \\ 2i & \alpha - \beta - 2k \\ 2k & \gamma - \beta + 2i \end{bmatrix} = \begin{bmatrix} -2(i + j + k) & -\alpha \\ 2k & \gamma - \alpha - 2j \\ 2j & \beta - \alpha + 2k \end{bmatrix}$$

**Proof** These identities are exactly the usual symmetries of $6j$–symbols. \qed
Lemma 26  If \( i \leq r' \) and \( j \geq -r' \) then

\[
\begin{pmatrix}
2i - 2 & \alpha \\
2j + 2 & \beta \\
2k & \gamma
\end{pmatrix}
\]

\[
= \{\gamma + i + r' - 1\}\{\alpha + j + r' + 1\}
\begin{pmatrix}
2i & \alpha + 1 \\
2j & \beta + 1 \\
2k & \gamma - 1
\end{pmatrix}
\]

Proof  We give a graphical proof:

\[
\begin{pmatrix}
2j & \alpha \\
2k & \beta \\
2i & \gamma
\end{pmatrix}
\]

then substitute \( \gamma \) with \( \gamma - 1 \).

Proposition 27  •  If \( i + j + k = r' \) then

\[
\begin{pmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{pmatrix}
\]

\[
= \{i, j, k\}\{\alpha - r' - k; r' - i\}!\{\beta - r' - i; r' - j\}!\{\gamma - r' - j; r' - k\}!.
\]
If $i + j + k = -r'$ then
\[
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix} = \{\delta + 1; r' + i\}!\{\epsilon + 1; r' + j\}!\{\varphi + 1; r' + k\}!
\]
where we use the notation
\[
\begin{align*}
\delta &= \beta - \gamma - 2i, \\
\epsilon &= \gamma - \alpha - 2j, \\
\varphi &= \alpha - \beta - 2k.
\end{align*}
\]

**Proof** Remark first that these formulas are symmetric for the action of $\mathfrak{S}_3$ which permute simultaneously \{i, j, k\} and \{\alpha, \beta, \gamma\} and multiplies the last three variables by the signature of the permutation (see Proposition 25).

We will prove these identities by a recurrence on the natural number $n = 2r' - \max(i, j, k) + \min(i, j, k)$. First, we prove Equation (31) when $n = 0$. In this case up to a permutation we have $(i, j, k) = (-r', r', r')$. Hence, we must prove that
\[
\begin{bmatrix}
-2r' & \alpha \\
2r' & \beta \\
2r' & \gamma
\end{bmatrix} = \{\alpha - 2r'; 2r'\}! = d(\alpha)^{-1}.
\]
To do this, recall Equation (28):
\[
\begin{bmatrix}
* & \alpha - \beta - 2r' \\
\circ & -2r' \\
* & \gamma
\end{bmatrix} \circ \begin{bmatrix}
* & \beta \\
\circ & -2r' \\
* & \gamma
\end{bmatrix} = \begin{bmatrix}
\circ & \alpha \\
\circ & -2r' \\
\circ & \gamma
\end{bmatrix}.
\]
Composing this identity with
\[
\begin{bmatrix}
\circ & 2r' \\
\circ & \alpha \\
\circ & \gamma
\end{bmatrix} \circ \begin{bmatrix}
\circ & \alpha \\
\circ & 2r' \\
\circ & \gamma
\end{bmatrix},
\]
we get that
\[
\begin{bmatrix}
-2r' & \alpha \\
2r' & \beta \\
2r' & \gamma
\end{bmatrix} = \begin{bmatrix}
\circ & \alpha \\
\circ & 2r' \\
\circ & \gamma
\end{bmatrix} \circ \begin{bmatrix}
\circ & \alpha \\
\circ & 2r' \\
\circ & \gamma
\end{bmatrix} \circ \begin{bmatrix}
\circ & \alpha \\
\circ & -2r' \\
\circ & \gamma
\end{bmatrix}.
\]

Proposition 21 states that
\[
\begin{bmatrix}
\circ & x \\
\circ & \alpha \\
\circ & -2r'
\end{bmatrix} \circ \begin{bmatrix}
\circ & * \\
\circ & \alpha \\
\circ & -2r'
\end{bmatrix} = d(x)^{-1} \text{Id}.
\]
Applying this identity twice in Equation (34) we arrive at (33). Remark that using the symmetries of the $6j$--symbols, Equation (33) also implies that

$$\begin{bmatrix} 2r' & \alpha \\ -2r' & \beta \\ -2r' & \gamma \end{bmatrix} = \{\beta - \gamma - 2r'; 2r'\}!$$

which is the case $n = 0$ for Equation (32).

Now let

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}$$

be the right hand sides of Equation (31) (respectively Equation (32)). By induction, it is enough to show that

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}$$

satisfy the relation of Lemma 26. Indeed, this relation applied to both sides of Equation (31) (respectively Equation (32)) after a well chosen permutation of $(i, j, k)$ allows us to reduce $n$ by 1 or 2.

- Let us start with Equation (31). Here $i + j + k = r'$ and direct computation shows

$$\{\gamma + i + r' - 1\}\{\alpha + j + r' + 1\} \begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma - 2 \end{bmatrix} + \{\gamma - 1\}\{\alpha - k - r'\} \begin{bmatrix} 2i & \alpha + 1 \\ 2j & \beta + 1 \\ 2k & \gamma - 1 \end{bmatrix}$$

$$= \{i, j, k\}\{\alpha - r' - k, r' - i + 1\}!\{\beta - i - r' + 1, r' - j - 1\}!\{\gamma - j - r' - 1, r' - k\}! \times (-\{\beta - i - r'\}\{\gamma - j + r' - 1\} + \{\beta + k - r'\}\{\gamma - 1\})$$

where we use that $\{x \pm (2r' + 1)\} = -\{x\}$. Now this is equal to

$$\{i + r'\}\{\beta - \gamma - i + 2 + r'\} \begin{bmatrix} 2i - 2 & \alpha \\ 2j + 2 & \beta \\ 2k & \gamma \end{bmatrix}$$

since

$$\{\beta + k - r'\}\{\gamma - 1\} - \{\beta - i - r'\}\{\gamma - j + r' - 1\} = \{r' - j\}\{\beta - \gamma - i + r' + 2\}.$$

Thus,

$$\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}$$

satisfy the relation of Lemma 26 when $i + j + k = r'$. 

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We now deal similarly with the case \( i + j + k = -r' \).

\[
\{ \gamma + i + r' - 1 \} \{ \alpha + j + r' + 1 \} \begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma - 2 \end{bmatrix}' + \{ \gamma - 1 \} \{ \alpha - k - r' \} \begin{bmatrix} 2i & \alpha + 1 \\ 2j & \beta + 1 \\ 2k & \gamma - 1 \end{bmatrix}'
\]

\[
= \{ \beta - \gamma - 2i + 3, i + r' \} \{ \gamma - \alpha - 2j - 1, j + r' \} \{ \alpha - \beta - 2k + 1, k + r' \} \times \left( \{ \gamma + i + r' - 1 \} \{ \alpha + j + r' + 1 \} + \{ \gamma - 1 \} \{ \alpha + i + j \} \right)
\]

But now this is equal to

\[
\{ i + r' \} \{ \beta - \gamma - i + 2 + r' \} \begin{bmatrix} 2i - 2 & \alpha \\ 2j + 2 & \beta \\ 2k & \gamma \end{bmatrix}'
\]

because

\[
\left( \{ \gamma + i + r' - 1 \} \{ \alpha + j + r' + 1 \} + \{ \gamma - 1 \} \{ \alpha + i + j \} \right) = \{ i + r' \} \{ \beta - \gamma + j + 1 - r' \}.
\]

Thus,

\[
\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix}'
\]

satisfy the relation of Lemma 26 when \( i + j + k = -r' \) and this completes the proof. \( \square \)

Let us now rewrite and generalize the relation of Lemma 26:

**Lemma 28** Let \((i, j, k) \in \mathcal{H}_r\), and let \( l = -i - j - k \). Here, we again use the “colors”:

\[
\begin{cases}
\delta = \beta - \gamma - 2i, \\
\varepsilon = \gamma - \alpha - 2j, \\
\varphi = \alpha - \beta - 2k.
\end{cases}
\]

- If \( i + j + k = -l < r' \) and \( k < r' \) then

\[
(36) \quad \begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \frac{1}{\{ k - r' \} \{ -\alpha + k - r' \} \{ \varphi - k - r' \} \{ \delta - l - r' \} \begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k + 2 & \gamma \end{bmatrix} + \{ \varphi - 1 \} \{ -\delta + i - r' \} \begin{bmatrix} 2i & \alpha \\ 2j & \beta - 1 \\ 2k + 2 & \gamma \end{bmatrix}}.
\]
If $N > 0$, $i + j + k = -l \leq r' - N$ and $k \leq r' - N$ then

$$
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix}
= \frac{1}{\{k - r'; N\}!\{-\alpha + k - r'; N\}!} \left( \sum_{n=0}^{N} \binom{N}{n} \{-\varphi - k - r'; N - n\} \right.
\times \{\delta - l - r'; N - n\}!\{\varphi - N; n\}!\{-\delta - i - r'; n\}! \left[
\begin{bmatrix}
2i & \alpha \\
2j & \beta - n \\
2k + 2N & \gamma
\end{bmatrix}
\right]
\end{equation}

$$

If $N > 0$, $i + j + k = -l \geq N - r'$ and $k \geq N - r'$ then

$$
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix}
= \frac{1}{\{l - r'; N\}!\{-\varepsilon + l - r'; N\}!}
\times \left( \sum_{n=0}^{N} \binom{N}{n} \{\varphi - l - r'; N - n\}!\{-\beta - k - r'; N - n\}!
\{\varphi - N; n\}!\{\beta - i - r'; n\}! \left[
\begin{bmatrix}
2i & \alpha \\
2j & \beta + n \\
2k - 2N & \gamma
\end{bmatrix}
\right]
\right).
$$

**Proof** The first relation (36) is obtained from Equation (30) by using the symmetry

$$
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix}
= \begin{bmatrix}
2l & \beta \\
2k & \beta - \gamma - 2i \\
2i & \beta - \alpha + 2k
\end{bmatrix}
$$

and renaming the variables.

The second relation is shown by recurrence on $N$. For $N = 1$ it is just Equation (36). The induction step also follows from Equation (36). The meticulous reader who wants to check this relation carefully will have to use the identity:

$$
\binom{N}{n} \{\varphi + n - N - 1\} + \binom{N}{n - 1} \{\varphi + n - 2N - 2\} = \binom{N + 1}{n} \{\varphi - N - 1\}.
$$

The third identity can be obtained from the second by using the symmetry

$$
\begin{bmatrix}
2i & \alpha \\
2j & \beta \\
2k & \gamma
\end{bmatrix}
= \begin{bmatrix}
2i & \varepsilon \\
2j & -\delta \\
2l & \gamma
\end{bmatrix}
$$

and renaming the variables.

\[\square\]
Theorem 29  •  For any \((i, j, k) \in \mathcal{H}_{r'}\) with \(i, k \leq i + j + k\), let \(N = r' - i - j - k\). Then

\[
\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \{i, j, k\}\{\alpha - r' - k; j + k\}!\{\gamma - r' - j; i + j\}!
\]

\[
\times \left( \sum_{n=0}^{N} \binom{N}{n} \{\beta + k - \alpha + r' + 1; N-n\}!\{\beta + k - \gamma - i + j - r'; N-n\}!
\]

\[
\{\alpha - \beta - 2k - N; n\}!\{\gamma - \beta + i + r' + 1; n\}!\{\beta - r' - i - n; r' - j\}! \right) .
\]

- For any \((i, j, k) \in \mathcal{H}_{r'}\) with \(j, k \geq i + j + k\), let \(N = r' + i + j + k\). Then

\[
\begin{bmatrix} 2i & \alpha \\ 2j & \beta \\ 2k & \gamma \end{bmatrix} = \frac{\{\varepsilon + N + 1; r' + j - N\}!}{\{N\}!} \left( \sum_{n=0}^{N} \binom{N}{n} \{\beta - i - j + n + 1; N-n\}!
\]

\[
\times \{\beta - i + r' + 1; n\}!\{\varphi + N - n + 1; r' + k\}!\{\delta + n + 1; r' + i\}! \right)
\]

where we use the notation

\[
\begin{align*}
\delta &= \beta - \gamma - 2i, \\
\varepsilon &= \gamma - \alpha - 2j, \\
\varphi &= \alpha - \beta - 2k.
\end{align*}
\]

Proof  The first formula is obtained by replacing

\[
\begin{bmatrix} 2i & \alpha \\ 2j & \beta - n \\ 2k + 2N & \gamma \end{bmatrix}
\]

with

\[
\{i, j, k\}\{\alpha - r' - k + N; r' - i\}!\{\beta - r' - i - n; r' - j\}!\{\gamma - r' - j; r' - k - N\}!
\]

in Equation (37) where \(l = -i - j - k = N - r'\). The second formula is obtained from Equation (38) by replacing

\[
\begin{bmatrix} 2i & \alpha \\ 2j & \beta + n \\ 2k - 2N & \gamma \end{bmatrix}
\]

with

\[
\{\delta + n + 1; r' + i\}!\{\varepsilon + 1; r' + j\}!\{\varphi + 2N - n + 1; r' + k - N\}!
\]

where \(l = -i - j - k = r' - N\). □
References

[1] S Baseilhac, R Benedetti, *Quantum hyperbolic invariants of 3–manifolds with PSL(2, C)–characters*, Topology 43 (2004) 1373–1423 MR2081430

[2] F Costantino, J Murakami, *On SL(2, C) quantum 6j–symbols and its relation to the hyperbolic volume*. arXiv:1005.4277

[3] N Geer, B Patureau-Mirand, V Turaev, *Modified 6j–symbols and 3–manifold invariants* arXiv:0910.1624

[4] N Geer, B Patureau-Mirand, V Turaev, *Modified quantum dimensions and renormalized link invariants*, Compos. Math. 145 (2009) 196–212 MR2480500

[5] C Kassel, *Quantum groups*, Graduate Texts in Math. 155, Springer, New York (1995) MR1321145

[6] A N Kirillov, NY Reshetikhin, *Representations of the algebra U_q(sl(2)). q–orthogonal polynomials and invariants of links*, from: “Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988)”, (V G Kac, editor), Adv. Ser. Math. Phys. 7, World Sci. Publ., Teaneck, NJ (1989) 285–339 MR1026957

[7] G Masbaum, P Vogel, *3–valent graphs and the Kauffman bracket*, Pacific J. Math. 164 (1994) 361–381 MR1272656

[8] T Ohtsuki, *Quantum invariants. A study of knots, 3–manifolds, and their sets*, Series on Knots and Everything 29, World Sci. Pub., River Edge, NJ (2002) MR1881401

[9] V G Turaev, *Quantum invariants of knots and 3–manifolds*, de Gruyter Studies in Math. 18, de Gruyter, Berlin (1994) MR1292673

[10] V G Turaev, O Y Viro, *State sum invariants of 3–manifolds and quantum 6j–symbols*, Topology 31 (1992) 865–902 MR1191386

[11] O Y Viro, *Quantum relatives of the Alexander polynomial*, Algebra i Analiz 18 (2006) 63–157 MR2255851

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