Explicit triangular decoupling of the separated vector wave equation on Schwarzschild into scalar Regge-Wheeler equations

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Abstract. We consider the vector wave equation on the Schwarzschild spacetime, which can be considered as coming from the harmonic (or Lorenz) gauge fixed Maxwell equations. After a separation of variables, the radial mode equations form a complicated system of coupled linear ODEs. We outline a precise abstract strategy to decouple this system into triangular form, where the diagonal blocks consist of spin-$s$ scalar Regge-Wheeler equations, with $s = 0$ or 1. This strategy is then implemented to give an explicit transformation of the radial mode equations (with nonzero frequency and angular momentum) into this triangular form. Our decoupling goes a step further than previous results in the literature by making the triangular form explicit and reducing it as much as possible. Also, with the help of our abstractly formulated decoupling strategy, we have significantly streamlined both the presentation of the final results and the intermediate calculations. Finally, we note that the vector wave equation is a simple model for more complicated equations, like harmonic (or de Donder) gauge fixed linearized gravity, and backgrounds, like Kerr, where we expect the same abstract decoupling strategy to work as well.

1. Introduction

The study of linear wave-like equations on a Schwarzschild black hole background has a long history and many applications [1, 2], both for scalar fields as well as higher rank tensor fields. In this work, we will concentrate on the vector wave equation. This equation, besides its purely mathematical significance as a model for more general tensor wave equations, can be seen as the harmonic (or Lorenz) gauge-fixed version of Maxwell equations [3, Sec.15.1], [4], or also as the residual gauge equation for harmonic (or de Donder) gauge-fixed linearized gravity [5, 6, 7].

While the vector wave equation admits a complete separation of variables on Schwarzschild, the resulting radial mode equations still represent a complicated system of coupled ordinary differential equations (ODEs) in the radial variable. What is worse is that, though this ODE system is formally self-adjoint (it can be put into matrix Sturm-Liouville form), it naturally defines a symmetric unbounded operator only on a Krein space (a topological vector space with an indefinite scalar product). This indefiniteness is directly traced to the indefiniteness of the Lorentzian background metric and does not occur in the analogous Riemannian problem. Thus, the abstract spectral theory of symmetric and self-adjoint operators on a Hilbert space is not applicable in this case, which significantly complicates important and fundamental questions about this ODE system. For example: Can we prove or disproved that its frequency spectrum is
purely real? Can we prove that its generalized eigenfunctions are complete? Can we construct a spectral representation for the radial Green function?

To be able to answer these questions, one must appeal to some very special structural properties of the vector wave equation. Namely, in this work, we will show that the radial mode equations (at least for modes with non-zero frequency and angular momentum) can be decoupled into a triangular system, where the diagonal blocks are the spin-s Regge-Wheeler equations, with \( s = 0 \) or 1. These Regge-Wheeler equations are then of standard scalar Sturm-Liouville form, with very well understood spectral properties. Such a decoupling has been previously discussed in the literature in [7] and [4], though not in as explicit and conceptually clear terms as we present below. In fact, the full explicit details of how the original mode equations transform into the decoupled form and back are not easy to extract from these references. The strategy that we follow is basically the one of [7].\(^1\) Though, our results go a step further, by reducing the resulting triangular form as much as possible.

In principle, starting from the triangular decoupled form, the spectral properties of the Regge-Wheeler equations can completely determine the spectral properties of the original radial mode equations. However, the details of such a spectral analysis are left for future works.

In Section 2 we fix our notation, introduce the formal properties of differential equations and operators, and formulate our abstract decoupling strategy. In Section 3 we present a complete separation of variables for the vector wave equation on Schwarzschild spacetime and apply our strategy to give an explicit triangular decoupling of the resulting radial mode equations into spin-0 and spin-1 Regge-Wheeler scalars. In Section 4, we discuss in more detail the novel aspects of our results, as well as several avenues of further investigation.

2. Formal properties of differential equations and operators

2.1. Morphisms and cochain maps between differential equations

For the purposes of this work, a differential operator, say \( f \), is always linear, with smooth coefficients, which we will write as \( f[u] \), where \( u \) is a possibly vector valued function. Thus \( f \) can always be thought of as a matrix of scalar differential operators acting on the components of \( u \). By a scalar differential operator, we mean an operator that takes possibly vector valued functions into scalar valued functions (corresponding to a matrix with a single row). Scalars are real or complex numbers (for the sake of generality we will generally allow complex scalars). Differential operators could be of any order, including order zero, which just corresponds to multiplication by some matrix valued function. While the operators can be thought of as acting on smooth (vector valued) functions, we will be mostly concerned with composition identities among operators, and so we will not bother specifying precisely the domain or codomain of each operator. This information can always be deduced from the context. Also, for the abstract discussion below, it is also not necessary to fix the number of independent variables, but we will be mostly concerned with applications to ordinary differential operators (i.e., acting on functions of a single independent variable).

Below, we state some basic definitions and results concerning differential equations and morphisms between them, which essentially correspond to differential operators that map solutions to solutions. The presentation is logically self-contained and does not extend far beyond what is needed in the rest of the paper. However, for proper context, these ideas can be seen as simple special cases of the more general frameworks of \( D \)-modules [8, Sec.10.5], the category of differential equations [9, Sec.VII.5] or homological algebra [10]. We will not delve

\(^1\) It is by no means a simple task to extract this strategy from [7], as it only becomes apparent as the common pattern in the detailed and explicit calculations done for a sequence of examples of increasing complexity. Still, that work should be credited as (to our knowledge) the first to carry out this strategy explicitly for the example of the vector wave equation as well as the significantly more complicated Lichnerowicz equation, the latter corresponding to harmonic gauge fixed linearized gravity, on Schwarzschild.
into the precise connection with these larger frameworks, but it is useful to mention that all arrows/morphisms need to be reversed when our statements are interpreted in the language of $D$-modules.

When referring to a morphism between differential equations, we essentially mean a differential operator that maps solutions to solutions. Sometimes given a differential equation $e[u] = 0$ we will also refer to the operator $e$ as the equation. This should not lead to any confusion.

**Definition 2.1.** Consider two differential equations $e[u] = 0$ and $\bar{e}[\bar{u}] = 0$.

(a) A differential operator $\bar{u} = k[u]$ is a morphism from $e[u] = 0$ to $\bar{e}[\bar{u}] = 0$ when for any scalar differential operator of the form $\bar{p} \circ \bar{e}[\bar{u}]$ there exists a scalar differential operator of the form $q \circ e[u]$ such that

$$\bar{p} \circ \bar{e}[k[u]] = q \circ e[u],$$

and hence the map $p \mapsto k_p$ given by the identity

$$p \circ k = k_p - h_p \circ e$$

is well-defined on equivalence classes modulo $(\cdots) \circ \bar{e}$ and $(\cdots) \circ e$ respectively.

(b) A pair of differential operators $\bar{u} = k[u]$ and $\bar{v} = g[v]$ is a cochain map from $e[u] = 0$ to $\bar{e}[\bar{u}] = 0$ when they satisfy the identity $\bar{e} \circ k = g \circ \bar{e}$. It is easy to check that a cochain map is always a morphism and, as we will see shortly, a morphism always gives rise to a cochain map, but not necessarily uniquely. The two definitions are complementary to each other. Since a cochain map comes with more structure, it is often more convenient for algebraic manipulations. On the other hand, some properties of morphisms may be easier to check. We illustrate morphisms and cochain maps respectively as

\[
\begin{array}{cccc}
\bullet & \xrightarrow{k} & \bullet & \quad 0 \\
\downarrow & & \downarrow & \\
e & \quad \bar{e} & \quad e & \\
\bullet & \quad \xrightarrow{\cdots} & \bullet & \\
\end{array}
\quad \text{and} \quad
\begin{array}{cccc}
\bullet & \xrightarrow{k} & \bullet & \quad 0 \\
\downarrow & & \downarrow & \\
e & \quad \bar{e} & \quad e & \\
\bullet & \quad \xrightarrow{\cdots} & \bullet & \\
\downarrow & & \downarrow & \\
g & & & \quad 0 \\
\end{array}
\]  

The second diagram is meant to illustrate the terminology cochain map, which comes from homological algebra \[10\]. There, the primary objects are cochain complexes, which are sequences of linear spaces (or more generally modules, or even just abelian groups) connected by linear maps, of which any two successive ones compose to zero. What is of interest about a cochain complex is its cohomology (the quotient of the kernel of one linear map by the image of the preceding one). In our case the property of being a cochain complex is trivially satisfied by the columns of the second diagram and the cohomologies correspond to the spaces $\ker e$ and $\coker e$ of our differential operator $e$. Morphisms between cochain complexes are sequences of maps that commute with the linear maps of each complex. In our case, this property is illustrated by the commutativity (the identity $\bar{e} \circ k = g \circ \bar{e}$) of the middle square of the second diagram. It is straightforward to check that cochain maps induce well-defined maps on cohomology, which in our case corresponds to $k$ mapping solutions of $e[u] = 0$ ($u \in \ker e$) to solutions of $\bar{e}[\bar{u}] = 0$.
Definition 2.3. A cochain map $k$ is called a cochain map if $\bar{k}$ is on-shell injective (surjective) if the map $p \mapsto k_p$ is surjective (injective) from equivalence classes modulo $(\cdots) \circ \bar{e}$ to equivalence classes modulo $(\cdots) \circ e$. We say on-shell bijective to mean both on-shell injective and surjective. We say that $k$ is on-shell vanishing when $p \circ k$ belongs to the same equivalence class as 0 modulo $(\cdots) \circ e$ for any operator $p$.

Lemma 2.1. A morphism $k$ from $e[u] = 0$ to $\bar{e}[\bar{u}] = 0$ always induces a cochain map $k, g$.

Proof. Choosing $\bar{p} = \text{id}$ in the definition of a morphism (Definition 2.1(a)), implies that there exists an operator $g$ such that $\bar{e} \circ k = g \circ e$. The last identity also implies that $k, g$ is a cochain map (Definition 2.1(b)).

Of course in general the operator $g$ completing the morphism $k$ to a cochain map may not be unique. The next two definitions introduce some parallel properties of morphisms and cochain maps.

Definition 2.2. A morphism $k$ from $e[u] = 0$ to $\bar{e}[\bar{u}] = 0$ is called on-shell injective (surjective) if the map $p \mapsto k_p$ is surjective (injective) from equivalence classes modulo $(\cdots) \circ \bar{e}$ to equivalence classes modulo $(\cdots) \circ e$. We say on-shell bijective to mean both on-shell injective and surjective. We say that $k$ is on-shell vanishing when $p \circ k$ belongs to the same equivalence class as 0 modulo $(\cdots) \circ e$ for any operator $p$.

Definition 2.3. A cochain map $k, g$ from $e[u] = 0$ to $\bar{e}[\bar{u}] = 0$ is said to be induced by a homotopy if $k = h \circ e$ and $g = \bar{e} \circ h$ for some operator $h$ referred to as a (cochain) homotopy:

We say that $k, g$ has a left (right) inverse up to homotopy if there exists a cochain map $\bar{k}, \bar{g}$ from $\bar{e}[\bar{u}] = 0$ to $e[u] = 0$ such that

\[
\bar{k} \circ k = \text{id} - h \circ e \quad (k \circ \bar{k} = \text{id} - \bar{h} \circ \bar{e}) \\
\bar{g} \circ g = \text{id} - e \circ h \quad (g \circ \bar{g} = \text{id} - \bar{e} \circ \bar{h})
\]

for some homotopy $h$ of $e[u] = 0$ (h of $\bar{e}[\bar{u}] = 0$) into itself. A cochain map $k, g$ that has a two-sided inverse up to homotopy is called an equivalence up to homotopy.

The last two sets of definitions are related but not exactly equivalent. In general, the existence of a left or right inverse is a stronger property than being on-shell injective or surjective. Though, under certain hypotheses, the link could be made stronger.

Definition 2.4. Consider a differential equation $e[u] = 0$. Any operator $p$ such that $e \circ p = 0$ is called a gauge symmetry (generator) of $e$. Any operator $q$ such that $q \circ e = 0$ is called a Noether identity (or compatibility operator) of $e$. The equation is said to be under-determined if it has a non-zero gauge symmetry and over-determined if it has a non-zero Noether identity. The equation is said to be over-under-determined if it has both non-zero gauge symmetries and Noether identities and determined (or normal) if it has neither.

Lemma 2.2. Consider equations $e[u] = 0$ and $\bar{e}[\bar{u}] = 0$.

(a) When $e$ is not over-determined, a morphism $\bar{u} = k[u]$ induces a unique cochain map $k, g$.

(b) A cochain map $k, g$ has a left (right) inverse up to homotopy $\iff k$ is on-shell injective (surjective). If $k, g$ is induced by a homotopy, then $k$ is on-shell vanishing.
Proof. (a) From Lemma 2.1, we already know that there exists at least one operator $g$ such that $k,g$ is a cochain map. Let $g'$ be another such operator. Then $(g' - g) \circ e = \bar{e} \circ (k - k) = 0$. Hence, if $e$ has no non-vanishing Noether identities (Definition 2.4), we must have $g' = g$.

(b) Suppose that $\tilde{k}, \tilde{g}$ is a left inverse to $k, g$ up to homotopy. Then, for any operator $q[u]$, the operator $\tilde{p} = q \circ \tilde{k}$ satisfies

$$\tilde{p} \circ k = q \circ (\tilde{k} \circ k) = q - (q \circ h) \circ e. \tag{5}$$

Hence $k$ is on-shell injective.

On the other hand, suppose that $\tilde{k}, \tilde{g}$ is a right inverse to $k, g$ up to homotopy. Then any operator $\tilde{p}$ such that $\tilde{p} \circ k = q \circ e$, we have the identity

$$p \circ (id - \bar{h} \circ \bar{e}) = (p \circ k) \circ \bar{k} = q \circ (e \circ \bar{k}) = (q \circ \bar{g}) \circ \bar{e}. \tag{6}$$

Hence we must have $\tilde{p} = (q \circ \bar{g} + p \circ h) \circ \bar{e}$ and therefore $k$ is on-shell surjective.

Finally if $k = h \circ e$ and $g = \bar{e} \circ h$, then for any operator $p$ we have $p \circ k = (p \circ h) \circ e$, meaning that $k$ is on-shell vanishing.

(c) By on-shell injectivity, there must exist operators $\bar{k}$ and $h$ such that $\tilde{k} \circ k = id - h \circ e$. Then

$$(id - k \circ \bar{k}) \circ k = k \circ (id - \tilde{k} \circ k) = (k \circ h) \circ e. \tag{7}$$

Hence, by on-shell surjectivity, there must exist an operator $\bar{h}$ such that $k \circ \bar{k} = id - \bar{h} \circ \bar{e}$. Now, it remains only to check that $\bar{k}$ is actually a morphism. For that, note the identity

$$(e \circ \bar{k}) \circ k = e \circ (\bar{k} \circ k) = e - e \circ h \circ e = (id - e \circ h) \circ e. \tag{8}$$

But by on-shell surjectivity of $k$ this means that there exists an operator $\bar{g}$ such that $e \circ \bar{k} = \bar{g} \circ \bar{e}$, meaning that $\tilde{k}, \tilde{g}$ is a cochain map and hence $\tilde{k}$ itself is a morphism that is a two-sided on-shell inverse to $k$.

(d) By part (c), we know that an on-shell bijective morphism $k$ has a two sided inverse morphism $\bar{k}$, satisfying

$$\bar{k} \circ k = id - h \circ e \text{ and } k \circ \bar{k} = id - \bar{h} \circ \bar{e}, \tag{9}$$

for some operators $h$ and $\bar{h}$. By part (a), they can both be uniquely completed to cochain maps, $k, g$ and $\bar{k}, \bar{g}$. It remains to check the relation between $g$ and $\bar{g}$. Consider the identities

$$(\bar{g} \circ g) \circ e = \bar{g} \circ (g \circ e) = (\bar{g} \circ \bar{e}) \circ k = e \circ (\bar{k} \circ k) = (id - e \circ h) \circ e, \tag{10}$$

$$(g \circ \bar{g}) \circ \bar{e} = g \circ (\bar{g} \circ \bar{e}) = (g \circ e) \circ \bar{k} = \bar{e} \circ (k \circ \bar{k}) = (id - \bar{e} \circ \bar{h}) \circ \bar{e}. \tag{11}$$

Since neither $e$ nor $\bar{e}$ has non-vanishing Noether identities, we must conclude that

$$\bar{g} \circ g = id - e \circ h \text{ and } g \circ \bar{g} = id - \bar{e} \circ \bar{h}. \tag{12}$$

Hence, the cochain map $\bar{k}, \bar{g}$ is a two-sided to $k,g$ up to homotopy. □
2.2. Block-triangular decoupling

In this section, we show some equations can be transformed (or decoupled) into block triangular form. These results will be the building blocks of our abstract decoupling strategy, which will be discussed in Section 2.5. Note that we do not state these results in their most general form, but only in sufficient generality to be useful later on.

**Lemma 2.3.** Consider the differential equations $e[u] = 0$ and $f[v] = 0$. A morphism $v = k[u]$ induces an on-shell bijective morphism from $e[u] = 0$ to $\bar{e}[\bar{u}] = 0$, with $\bar{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ and $\bar{e}$ in upper block-triangular form, namely

\[
\begin{bmatrix}
  \begin{bmatrix} 0 \\ k \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

where the last two morphisms are mutually on-shell inverse.

We say that the morphism $k$ decouples $f$ from $e$. Once $f$ has been decoupled into a lower diagonal block, any transformations of remaining upper diagonal block transform the block-triangular form as follows.

**Lemma 2.4.** Consider the equations $e[u] = 0$, $\bar{e}[\bar{u}] = 0$, $f[v] = 0$, and

\[
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} u \\ v \end{bmatrix}
\end{bmatrix} = 0,
\]

where the operator $\Delta$ has the property that for any Noether identity $n \circ e = 0$ we have $n \circ \Delta = m \circ f$ for some operator $m$. An on-shell bijective morphism $k$ between $e$ and $\bar{e}$, which more precisely satisfies the identities (which are slightly weaker than equivalence up to homotopy)

\[
\begin{bmatrix}
  \begin{bmatrix} k_0 & 0 \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} \bar{g} & m \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

with a Noether identity $n \circ e = 0$, induces the following mutually on-shell bijective morphisms of equations in triangular form, where $\bar{\Delta} = g \circ \Delta$:

\[
\begin{bmatrix}
  \begin{bmatrix} k & 0 \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} g & 0 \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} \bar{g} & m \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} \bar{g} & 0 \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} k & 0 \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} g & 0 \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  \begin{bmatrix} \bar{g} & m \\ 0 & id \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} e & \Delta \\ 0 & f \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
  \begin{bmatrix} 0 \\ g \end{bmatrix}
\end{bmatrix}
\]
Of course, we should note that the block-triangular equations obtained in the earlier Lemma 2.3 satisfy the hypotheses of Lemma 2.4.

Proof. Consider the following identities:

\[
\begin{bmatrix}
\bar{e} \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
k \\
0
\end{bmatrix}
- \begin{bmatrix}
g \\
0
\end{bmatrix}
\Delta
= \begin{bmatrix}
\bar{e} \circ k - g \circ \Delta \\
0
\end{bmatrix},
\]

\[
\begin{bmatrix}
e \\
0
\end{bmatrix}
\begin{bmatrix}
\bar{k} \\
0
\end{bmatrix}
- \begin{bmatrix}
g \\
0
\end{bmatrix}
\begin{bmatrix}
e \\
0
\end{bmatrix}
\Delta
= \begin{bmatrix}
e \circ \bar{k} - g \circ \Delta \\
0
\end{bmatrix}.
\]

They show that the diagrams in the Lemma show actual cochain maps. It is also straightforward to see that they are mutually inverse up to homotopy.

We also have the analogous Lemma 2.5.

Consider equations \( e[u] = 0, \bar{e}[\bar{u}] = 0, f[v] = 0 \) and

\[
\begin{bmatrix}
e \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
u \\
v
\end{bmatrix} = 0,
\]

An equivalence up to homotopy \( k, g \) between \( e \) and \( \bar{e} \)

\[
\begin{bmatrix}
e \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
u \\
v
\end{bmatrix} \leftrightarrow \begin{bmatrix}
\bar{e} \\
\bar{v}
\end{bmatrix}, \quad \bar{k} \circ k = \text{id} - h \circ e,
\]

\[
\bar{g} \circ g = \text{id} - e \circ h.
\]

induces the following mutually on-shell bijective morphisms of equations in triangular form, where \( \Delta = \bar{\Delta} \circ k \):

\[
\begin{bmatrix}
f \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
u \\
v
\end{bmatrix} \iff \begin{bmatrix}
f \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
\bar{v} \\
\bar{v}
\end{bmatrix}.
\]

Proof. Consider the following identities:

\[
\begin{bmatrix}
f \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
\text{id} \\
0
\end{bmatrix} - \begin{bmatrix}
\text{id} \\
0
\end{bmatrix}
\Delta
= \begin{bmatrix}
f - f \\
0
\end{bmatrix}
\Delta
= \begin{bmatrix}
f - f \\
0
\end{bmatrix}
\Delta
\begin{bmatrix}
\text{id} \\
0
\end{bmatrix}.
\]

They show that the diagrams in the Lemma show actual cochain maps. It is also straightforward to see that they are mutually inverse up to homotopy.
2.3. Reducing the triangular form
Given an equation that is in block-triangular form, when is it equivalent to an equation in block-diagonal form? That is, when can an equation in triangular form be further reduced?

In the sequel, we will only need the following result.

Lemma 2.6. Consider the mutually inverse up to homotopy cochain maps

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \text{id} & \delta & 0 \\
0 & \varepsilon & \text{id} & 0 \\
\varepsilon & 0 & 0 & \text{id}
\end{bmatrix}
\quad \Longleftrightarrow \quad
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \text{id} & -\delta & 0 \\
0 & -\varepsilon & \text{id} & 0 \\
\varepsilon & 0 & 0 & \text{id}
\end{bmatrix}
\]

The two squares in these diagrams actually commute (that is, we can kill the off-diagonal block \( f \)) if and only if the operators \( \delta \) and \( \varepsilon \) satisfy the identity

\[
e \circ \delta = f + \varepsilon \circ \bar{e}.
\]

Essentially, the operators \( \delta \) and \( \varepsilon \) allow us to solve the equation is \( e[u] = f[\bar{u}] \) by \( u = \delta[\bar{u}] \) whenever \( \bar{e}[\bar{u}] = 0 \).

Proof. The desired result follows immediately from the identity

\[
\begin{bmatrix}
e & 0 \\
0 & \bar{e}
\end{bmatrix}
\circ
\begin{bmatrix}
\text{id} & \delta \\
0 & \text{id}
\end{bmatrix}
- \begin{bmatrix}
\text{id} & \varepsilon \\
0 & \text{id}
\end{bmatrix}
\circ
\begin{bmatrix}
e & f \\
0 & \bar{e}
\end{bmatrix}
= \begin{bmatrix}
e - e \circ \delta - f - \varepsilon \circ \bar{e} \\
0 & \bar{e} - \bar{e}
\end{bmatrix}.
\]

The reverse direction is completely analogous. \( \Box \)

2.4. Formal adjoints
Recall that we are only dealing with linear ordinary differential operators, with derivative operator \( \partial_r \), where each operator can be written as a matrix of scalar differential operators with smooth coefficients. Given a matrix linear differential operator \( k \), we define its formal adjoint \( k^* \) using the following rules. If \( k \) is a zero-th order scalar operator acting on a single scalar argument, then \( k^* = \partial_r \) if \( k \) is multiplication by the complex conjugate function \( k(r)^* \). If \( k^* = \partial_r \), acting on a single scalar argument, then \( (k + g)^* = k^* + g^* \) and \( (k \circ g)^* = g^* \circ k^* \). Finally, if \( k = [k_{ij}] \) is a matrix of scalar differential operators, then \( (k^*)_{ij} = k_{ji}^* \).

It is straight forward to check that this gives rise to a consistent and unique definition, which satisfies \( (k \circ g)^* = g^* \circ k^* \) and \( k^{**} = k \), now with any composable operators \( k \) and \( g \), and agrees with the common notion of formal adjoint. When dealing with a family of operators \( k_\omega \) parametrized by a possibly complex parameter \( \omega \), we adopt the convention that the family of adjoint operators is parametrized as \( (k^*)_\omega = (k_\omega^*)^* \). With this convention, \( (k^*)_\omega = (k_\omega^*)^* \) for real values \( \omega = \omega^* \) of the parameter. An operator satisfying \( k^* = k \) is called (formally) self-adjoint.

Sometimes, it will be convenient for us, when introducing a new differential operator, to write it directly as \( k^* \), even if we have not explicitly introduced the operator \( k \). This is possible, since any differential operator is the formal adjoint of some differential operator. This convention allows us to highlight some special features of the equations and differential operators that we will be working with, and also reigns in a bit the proliferation of new notation.
2.5. Gauge fixing, gauge-invariant fields and triangular decoupling

In this section, we give an abstract overview of the triangular decoupling procedure that will be illustrated on a specific example in Section 3. The abstract approach allows us to separate the general strategy from the calculational details of a specific example, so that it could be applied in other examples as well. One benefit of having a general strategy is that it eliminates much trial-and-error and unnecessary calculations in more complex examples.

We start with a system of differential equations $E[u] = 0$ and aim to find explicit equivalence of this system to one that is in block upper-triangular form, with as many off-diagonal terms set to zero (or reduced) as possible. Such a triangular decoupling will be possible because of the large amount of structure that we can assume about our equation. Essentially, we expect $E$ to be the gauge-fixed form of a more fundamental self-adjoint equation $E_0[u] = 0$ that has both gauge symmetries and (hence necessarily) Noether identities. The gauge fixing and the modifications of $E$ from $E_0$ are chosen such that $E$ remains self-adjoint and it induces a nice residual gauge equation. We will refer to such a gauge fixing condition as harmonic, because harmonic or generalized harmonic gauges tend to satisfy all of these requirements.

While we do try to keep track of self-adjointness of our equations, which is helpful at least as an error-check on explicit calculations, adjoint operators do not play a specific role in the decoupling strategy outlined below. Hence, it would work just as well if the operators that appear as adjoints are replaced by independent operators, which satisfy all the same composition identities.

The general strategy will be to show that the original dependent variables separate into gauge modes, gauge invariant modes and constraint violating modes, which are coupled to each other in a hierarchical (hence triangular) and minimal way.

Let us introduce the following differential operators and identities among them:

- $E_0$—self-adjoint equation with gauge symmetry,
- $D$—gauge symmetry generator, $E_0 \circ D = 0$,
- $D^*$—Noether identity, $D^* \circ E_0 = 0$,
- $T$—harmonic gauge fixing condition, $T \circ D = R \circ D_D$,
- $D_D$—self-adjoint residual gauge equation,
- $R$—self-adjoint gauge fixing correction, $R^* = R$,
- $E = E_0 + T^* \circ R \circ T$—self-adjoint gauge-fixed equation, $R = R^{-1}$,
- $\Phi_0$—gauge invariant field, $\Phi_0 \circ D = 0$,
- $D_\Phi$—self-adjoint dynamical equation for gauge-invariant fields, $D_\Phi \circ \Phi_0 = \tilde{\Phi}_0^* \circ E_0$, with $E \circ D = T^* \circ D_D$ and $D_\Phi^* \circ T = D^* \circ E$.

The operators

\begin{align}
D[\phi_0], \quad \Phi_0, \quad T
\end{align}

respectively separate out, in a sense that will become clear by the end of this section, the gauge modes, the gauge invariant modes and the constraint violating modes. The general strategy will be to show that these modes couple to each other only in a hierarchical (hence triangular) way, and to find slight modifications to them that reduce this coupling as much as possible. The subscript $s$ on $\phi_s$ and $\psi_s$ indicates the spin of the corresponding Regge-Wheeler operators $D_s$ that appear on the diagonal of the decoupled form.

The above operators fit into the following commutative diagrams:

\begin{align}
D_D \quad E \quad D^* \quad D_\Phi \quad \Phi_0
\end{align}
Even though we have $D^*_D = D_D$, we use this notation to formally distinguish between the dynamical equation satisfied by the longitudinal/constraint-violating modes $T$ and the residual gauge modes $D$. Note that the operator $\Phi^*_0$ is not uniquely fixed by completing the commutative diagram between $E_0$, $\Phi_0$ and $D_\Phi$. If $\Phi^*_1 = \Phi^*_0 + N$ is any other such operator, then $N \circ E_0 = 0$ and hence $N$ is a Noether identity for $E_0$.

As a first step, we use the operators $T$ and $D^*$ and Lemma 2.3 to decouple $D^*_D$ from $E$. We have the following morphism from $E$ to a triangular form:

\[
\begin{array}{ccc}
\bullet & \overset{[\text{id}]_{T}}{\longrightarrow} & \bullet \\
E & \downarrow & \downarrow \\
\bullet & \overset{[T \quad -\text{id}]_{D^*_D}}{\longrightarrow} & \bullet
\end{array}
\iff
\begin{array}{ccc}
\bullet & \overset{[\text{id} \quad 0]}{\longrightarrow} & \bullet \\
E & \downarrow & \downarrow \\
\bullet & \overset{[T \quad -\text{id}]_{D^*_D}}{\longrightarrow} & \bullet
\end{array}
\]

(29)

where the last two morphisms are on-shell mutually inverse.

Next, we would like to use the operators $\Phi_0$ and $\Phi^*_0$ to decouple $D_\Phi$ from $[E \ T]$. Let us introduce the operator

\[
\Delta_\Phi = \Phi^*_0 \circ T^* \circ \bar{R},
\]

which is necessary to decouple $D_\Phi$, since it completes the commutative diagram

\[
\begin{array}{ccc}
\bullet & \overset{\Phi_0}{\longrightarrow} & \bullet \\
[E \ T] & \downarrow & \downarrow \\
\bullet & \overset{D_\Phi \quad -\Delta_\Phi}{\longrightarrow} & \bullet
\end{array}
\]

(31)

Once again, applying Lemma 2.3 we get the mutually on-shell inverse morphisms

\[
\begin{array}{ccc}
\bullet & \overset{[\text{id}]_{\Phi_0}}{\longrightarrow} & \bullet \\
[E \ T] & \downarrow & \downarrow \\
\bullet & \overset{[T \quad 0 \quad \Phi_0 \quad -\text{id}]_{D_\Phi}}{\longrightarrow} & \bullet
\end{array}
\iff
\begin{array}{ccc}
\bullet & \overset{[\text{id} \quad 0]}{\longrightarrow} & \bullet \\
[E \ T] & \downarrow & \downarrow \\
\bullet & \overset{[T \quad 0 \quad \Phi_0 \quad -\text{id}]_{D_\Phi}}{\longrightarrow} & \bullet
\end{array}
\]

(32)

The operator $\Delta_\Phi$ is an obstacle to decoupling $D_\Phi$ directly from $E$. This obstacle becomes trivial when $\Delta_\Phi = D_\Phi \circ q$ for some operator $q$, since then replacing $\Phi_0$ with $\Phi_0 + q^* \circ T$ and setting $\Delta_\Phi = 0$ does the required job. If we had used $\Phi_0 + q \circ T$ from the start, then the final triangular form that we will arrive at will not have any coupling between $T$ and $\Phi_0 + q \circ T$. We will actually be able to achieve such a reduced triangular form, by using the strategy from Section 2.3 at the very end.
Note that due to the identities involving $D$, $T$, $\Phi_0$ and $E$, the operator $D$ remains a morphism from $D_D$ to each of the $E$, $E-T$ and $E-T-\Phi_0$ equations. We will proceed under the hypothesis that in that last case the morphism $D$ is actually on-shell bijective. That is, by an application of Lemmas 2.2 and 2.1 we have the following mutually on-shell inverse morphisms

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D \\
D_D
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
E \\
T \\
\Phi_0
\end{array}
\end{array}
\end{array}
\end{array}
\Leftrightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D_0 \\
D_D
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
E \\
T \\
\Phi_0
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
-T_0^* -\Delta_T -\Delta_\Phi
\end{array}
\end{array}
\end{array}
\end{array}
\]

for some operators $T_0^*$, $\Delta_T$ and $\Delta_\Phi$. It would be convenient if the above diagrams actually described an equivalence up to homotopy. In general that will not be true, but only due to the restricted notion of up to homotopy that we have adopted in this work. On the other hand, we make another hypothesis that the inverse relationship between the above cochain maps takes on the following slightly more general form, which we have actually already seen in the hypotheses of Lemma 2.4:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bar{D}_0 \circ D = \text{id} - h_D \circ D_D , \\
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bar{T}_0^* -\Delta_T -\Delta_\Phi \circ \begin{bmatrix} T^* \\ R \\ 0 \end{bmatrix} = \text{id} - D_D \circ h_D , \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

\[
D \circ \bar{D}_0 = \text{id} - \begin{bmatrix} h_E & \bar{T}_2 & \bar{\Phi}_2 \end{bmatrix} \circ \begin{bmatrix} E \\ T \\ \Phi_0 \end{bmatrix} ,
\]

\[
\begin{bmatrix} T^* \\ R \\ 0 \end{bmatrix} \circ \begin{bmatrix} \bar{T}_0^* -\Delta_T -\Delta_\Phi \end{bmatrix} = \begin{bmatrix} \text{id} & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \text{id} \end{bmatrix} - \begin{bmatrix} E \\ T \\ \Phi_0 \end{bmatrix} \circ \begin{bmatrix} h_E & \bar{T}_2 & \bar{\Phi}_2 \end{bmatrix} - \bar{n} ,
\]

where $\bar{n} \circ \begin{bmatrix} 0 & 0 \\ 0 & -\text{id} \\ -\text{id} & 0 \end{bmatrix} = \begin{bmatrix} \Phi_2^* & \bar{m}_{T,\Phi} & \bar{m}_{T,T} \\ \bar{m}_{T,\Phi} & \bar{m}_{\Phi,\Phi} & \bar{m}_{T,T} \\ \bar{m}_{\Phi,\Phi} & \bar{m}_{T,T} & \text{id} \end{bmatrix} \begin{bmatrix} D_D \\ \Delta_\Phi \end{bmatrix} ,
\]

for some choice of the newly introduced operators that satisfy the above identities. Note that, as in the hypotheses of Lemma 2.4, we require the operator $\bar{n}$ to be a Noether identity for the $E-T-\Phi_0$ system that also satisfies the factorization identity given by the last equation.

Next applying Lemma 2.4 we lift the above equivalence first from the $E-T-\Phi_0$ system to the block-triangular system (29),

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{bmatrix} D & \Phi_2 \\ 0 & \text{id} \end{bmatrix} \\
\begin{bmatrix} D_D \\ \Delta_\Phi \end{bmatrix}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{bmatrix} E \\ T \\ \Phi_0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -\text{id} \end{bmatrix}
\end{array}
\end{array}
\end{array}
\end{array}
\Leftrightarrow
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{bmatrix} D_0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} D_D \\ \Delta_\Phi \end{bmatrix}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{bmatrix} E \\ T \\ \Phi_0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ -\text{id} \end{bmatrix}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{bmatrix} T_0^* \\ \bar{m}_{T,\Phi} \\ \bar{m}_{\Phi,\Phi} \end{bmatrix} \\
\begin{bmatrix} 0 \\ \text{id} \end{bmatrix}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{bmatrix} 0 \\ -\Delta_T \\ -\Delta_\Phi \end{bmatrix}
\end{array}
\end{array}
\end{array}
\end{array}
\]
and then to the block-triangular system (32),

\[
\begin{bmatrix}
D \Phi_2 & T^*_2 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0
\end{bmatrix}
\quad \implies \quad \begin{bmatrix}
E & 0 \\
T & 0
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0
\end{bmatrix}
\quad \implies \quad \begin{bmatrix}
D_D & \Phi_2 & T^*_2 \\
0 & D_D & \Phi_2 \\
0 & 0 & D_D
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0 & \Phi_0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Next, to arrive at an equivalence up to homotopy with the original \(E\) equation, we repeatedly apply Lemma 2.4 and compose the appropriate morphisms from (32) and (29) to first arrive at

\[
\begin{bmatrix}
D \Phi_2 & T^*_2 \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0
\end{bmatrix}
\quad \implies \quad \begin{bmatrix}
E & 0 \\
T & 0
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0
\end{bmatrix}
\quad \implies \quad \begin{bmatrix}
D_D & \Phi_2 & T^*_2 \\
0 & D_D & \Phi_2 \\
0 & 0 & D_D
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0 & \Phi_0 \\
0 & 0 & 0
\end{bmatrix}.
\]

and then finally arrive at

\[
\begin{bmatrix}
D_D & \Phi_2 & T^*_2 \\
0 & D_D & \Phi_2 \\
0 & 0 & D_D
\end{bmatrix} = \begin{bmatrix}
\Phi_0 & \delta \\
0 & 0 & \Phi_0 \\
0 & 0 & 0
\end{bmatrix}.
\]

By construction the above morphisms are mutually on-shell inverse and in fact are an equivalence up to homotopy. The last claim would follow from Lemma 2.2(d) when neither \(E\) nor the triangular decoupled system is over-determined (Definition 2.4). In fact, in cases that are of interest to us, both systems will actually be determined.

Next, using the idea from Section 2.3, we can reduce the triangular form (42) further provided we can find \(\delta\) and \(\varepsilon\) operators satisfying the identities

\[
D_D \circ \Phi = \Delta_\Phi + \varepsilon_\Phi \circ D_\Phi, \quad \delta \Phi = \delta_\Phi + \varepsilon_\Phi \circ \delta_\Phi^*, \quad D_D \circ \delta_T = (\Delta_T - \Delta) + \varepsilon_T \circ D_\Phi^*, \quad \text{where} \quad \Delta_T = \Delta_T + \varepsilon_\Phi \circ \Delta_\Phi,
\]

\[
\delta \Phi_2 = \Delta_\Phi + \varepsilon_\Phi \circ \delta_\Phi_2, \quad \delta_T = \delta_T + \varepsilon_T \circ \delta_\Phi_2^*.
\]
for some operator $\Delta$. The result is the following simplified triangular form:

\[
\begin{bmatrix}
\text{id} & \delta_T & 0 \\
0 & \text{id} & \delta_T \\
0 & 0 & \text{id}
\end{bmatrix} \circ \begin{bmatrix}
\mathcal{D} & \Delta \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\text{id} & -\delta_T - \delta_T \circ \delta_T \\
0 & \text{id} \\
0 & 0 & \text{id}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\text{id} & \varepsilon_T & 0 \\
0 & \text{id} & \varepsilon_T \\
0 & 0 & \text{id}
\end{bmatrix} \circ \begin{bmatrix}
\mathcal{D} & \Delta \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\text{id} & -\varepsilon_T - \varepsilon_T \circ \varepsilon_T \\
0 & \text{id} \\
0 & 0 & \text{id}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} \circ \begin{bmatrix}
\text{id} & \delta_T & 0 \\
0 & \text{id} & \delta_T \\
0 & 0 & \text{id}
\end{bmatrix} \circ \begin{bmatrix}
\mathcal{D} & \Delta \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\text{id} & -\delta_T - \delta_T \circ \delta_T \\
0 & \text{id} \\
0 & 0 & \text{id}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} \circ \begin{bmatrix}
\text{id} & \varepsilon_T & 0 \\
0 & \text{id} & \varepsilon_T \\
0 & 0 & \text{id}
\end{bmatrix} \circ \begin{bmatrix}
\mathcal{D} & \Delta \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\text{id} & -\varepsilon_T - \varepsilon_T \circ \varepsilon_T \\
0 & \text{id} \\
0 & 0 & \text{id}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]

Thus, we arrive at the final form of the equivalence up to homotopy of our original $E$ equation with the simplified triangular form:

\[
\begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix} = \begin{bmatrix}
\mathcal{D} & \Phi \\
0 & \mathcal{D} \\
0 & 0 & \mathcal{D}^*_1
\end{bmatrix}
\]
\[
[T^* \Phi_1^* \hat{D}_1^*] = [T^* \Phi_2^* \hat{D}_2^*] \circ \begin{bmatrix}
\text{id} & -\varepsilon_\Phi & -\varepsilon_T + \varepsilon_\Phi \circ \varepsilon_\Phi \\
0 & \text{id} & -\varepsilon_\Phi \\
0 & 0 & \text{id}
\end{bmatrix},
\] (51)

In the next section, we apply the above strategy to an explicit example.

3. Vector wave equation on Schwarzschild

Consider the exterior Schwarzschild spacetime \((\mathcal{M}, g)\) of mass \(M > 0\), where \(\mathcal{M} \cong \mathbb{R}^2 \times S^2\). If \((S^2, \Omega)\) is the unit round sphere and \((t, r)\), with \(-\infty < t < \infty\) and \(2M < r < \infty\), are coordinates on the \(\mathbb{R}^2\) factor, then the Schwarzschild metric is

\[
^4g := g + r^2 \Omega, \quad g := -f(r) \, dt^2 + \frac{dr^2}{f(r)}, \quad f := 1 - \frac{2M}{r}.
\] (52)

For convenience we also define

\[
f_1 := rf' = \frac{2M}{r}.
\] (53)

Following the 2 + 2 formalism of [11], spacetime tensors decompose into spherical and radiotemporal sectors according to

\[
v_\mu \rightarrow \left(\frac{v_a}{ru_A}\right), \quad v^\nu \rightarrow \left(\frac{v^b}{r}u^B\right),
\] (54)

which is consistent both with raising-lowering \(v_\mu \leftrightarrow v^\mu\) with \(^4g_{\mu\nu}\), as well as raising-lowering \(v_a \leftrightarrow v^a\), \(u_A \leftrightarrow u^A\) with \(g_{ab}, \Omega_{AB}\), respectively. If \(D_A, \epsilon_{AB}\) and \(Y^{lm}\) are respectively the Levi-Civita connection, the volume form and the unit-normalized spherical harmonics on \((S^2, \Omega)\), the following tensors define a basis for the even \((Y)\) and odd \((X)\) vector harmonics:

\[
Y_A = D_A Y, \quad X_A = \epsilon_{BA} D_B Y.
\] (55)

The tensor spherical harmonic decomposition of a covariant vector field is then

\[
v_\mu = v_\mu^{\text{even}} + v_\mu^{\text{odd}},
\] (56)

\[
v_\mu^{\text{even}} \rightarrow \sum_{lm} \left(\frac{v_a}{ru_A} Y^{lm}\right),
\] (57)

\[
v_\mu^{\text{odd}} \rightarrow \sum_{lm} \left(\frac{0}{ru} X^{lm}\right).
\] (58)

For conciseness, we can omit the \(lm\) indices from now on.

If \(^4\nabla_\mu\) is the spacetime Levi-Civita connection on \((\mathcal{M}, g)\), the vector wave equation is

\[
^4\Box v_\mu := ^4\nabla_\nu ^4\nabla^\nu v_\mu = 0.
\] (59)

A separated mode

\[
v_\mu \rightarrow \left(\frac{v_t}{ru} Y_A + \frac{v_r}{ru} Y_A\right) e^{-i\omega t} + \left(\frac{0}{ru} X_A\right) e^{-i\omega t},
\] (60)
where $v_l = v_l(r)$, $v_r = v_r(r)$, $u = u(r)$ and $w = w(r)$, satisfies the vector wave equation when

$$\Box_v \begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix} = \begin{bmatrix} -\frac{1}{f} r^2 f \partial_r v_l \\ \partial_r v_r \\ \partial_r u \end{bmatrix} + \left( \frac{\omega^2}{f} - \frac{E_l}{r^2} \right) \begin{bmatrix} -\frac{1}{f} r^2 v_l \\ f r^2 v_r \\ f r^2 u \end{bmatrix} + i \omega f \frac{1}{f} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 f^2 & 2 B_l f \\ 2 B_l f & B_l f_1 \end{bmatrix} \begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix} = 0, \quad (61)$$

$$\Box_o w := \partial_r B_l r^2 f \partial_r w + \left( \frac{\omega^2}{f} - \frac{E_l}{r^2} \right) B_l r^2 w + B_l f_1 w = 0, \quad (62)$$

where for convenience we have defined

$$B_l = l(l + 1). \quad (63)$$

These are our radial mode equations. Note that we have written these equations in manifestly self-adjoint form, $\Box_v = \Box_o$ and $\Box_o^* = \Box_v^*$ (cf. Section 2.4).

We may think of the vector wave equation as the harmonic gauge-fixed Maxwell equations

$$4 \Box v_\mu = 4 \nabla^\nu (4 \nabla_\nu v_\mu - 4 \nabla_\mu v_\nu) + 4 \nabla_\mu 4 \nabla^\nu v_\nu. \quad (64)$$

We will take the harmonic gauge condition and the gauge modes to respectively be

$$\psi_0 = r^4 \nabla^\mu v_\mu = 0, \quad v_\mu = 4 \nabla_\mu \frac{1}{r} \phi_0. \quad (65)$$

After the triangular decoupling, we will show that the vector wave equation is equivalent to a system with spin-$s$ Regge-Wheeler (RW) operators $D_s$ [4, Eq.(23)] on the diagonal, where

$$D_s \phi := \partial_r f \partial_r \phi - \frac{1}{r^2} [B_l (1 - s^2) f_1] \phi + \frac{\omega^2}{f} \phi = 0. \quad (66)$$

In the following calculations, we will freely permit division by $\omega$ and $B_l = l(l + 1)$. Hence our results will hold for the $\omega \neq 0$, $l > 0$ modes. A separate treatment would be required when either of those conditions does not hold.

3.1. Odd sector

This sector is particularly simple, since it does not contain any gauge modes and is not constrained by the harmonic gauge condition. The transformation

$$\phi_1 = -i \omega r w, \quad w = -\frac{1}{i \omega r} \phi_1, \quad (67)$$

puts $\Box_o w = 0$ in direct equivalence with a $s = 1$ RW equation:

$$\Box_o w = B_l i \omega r \frac{1}{\omega^2} D_1 \phi_1, \quad \frac{1}{\omega^2} D_1 \phi_1 = \frac{1}{B_l i \omega r} \Box_o w. \quad (68)$$

In diagrammatic form, we have

$$\begin{align*}
\Box_o &\quad \begin{array}{c}
\bullet \\
\bullet
\end{array} \\
\Box_o &\quad \begin{array}{c}
\bullet
\end{array}
\end{align*} \quad \begin{align*}
\frac{1}{\omega} D_1, \quad &\left( \frac{1}{\omega} i \omega r \right) \left( -i \omega r \right) = \text{id}, \\
\frac{1}{B_l i \omega r} D_1, \quad &\left( \frac{1}{B_l i \omega r} \right) \left( B_l i \omega r \right) = \text{id}. \quad (69)
\end{align*}$$
3.2. Even sector

This sector is more complicated and is the first prototype for the abstract approach to triangular decoupling that we outlined earlier in Section 2.5. Let us now explicitly introduce some of the auxiliary operators that we will need, following the notation established earlier:

\[ E := \Box_e, \]
\[ D := \frac{1}{\omega^2 r^2} \left[ \begin{array}{c} -i \omega r \\ r^2 \partial_r \frac{1}{r} \end{array} \right], \quad T^* := \left[ \begin{array}{c} i \omega r \\ f r^2 \partial_r \frac{1}{r} \end{array} \right], \]
\[ T := \left[ \begin{array}{cc} -i \omega r f & \frac{1}{r} \end{array} \right], \quad D^* := \frac{1}{\omega^2 r^2} \left[ \begin{array}{cc} i \omega r & -r \partial_r \end{array} \right], \]
\[ D_D := \frac{1}{\omega^2} D_0, \quad D_D^* := \frac{1}{\omega^2} D_0, \]
\[ \Phi_0 := \left[ \begin{array}{c} 0 \\ -f \\ f \partial_r \end{array} \right], \quad \Phi_0^* := \frac{1}{r^2} \left[ \begin{array}{cc} 0 & -1 \\ \frac{1}{B_l^2} r^2 \partial_r \frac{1}{r} \end{array} \right], \]
\[ D_\Phi := \frac{1}{\omega^2} D_1, \quad R = \bar{R} := -i d. \]

They can be obtained from a mode decomposition of the spacetime operators we discussed earlier, up to simple multiplicative normalizations. We have chosen the normalizations so that all operators have dimensionless components, under the convention that each of \( r^{-1}, \omega, \partial_r \) has the dimension of inverse length, while \( f, f_1 \) and \( B_l \) are dimensionless. The operators

\[ \left[ \begin{array}{c} v_t \\ v_r \\ u \end{array} \right] = D[\phi_0], \quad \phi_1 = \Phi_0 \left[ \begin{array}{c} v_t \\ v_r \\ u \end{array} \right], \quad \psi_0 = T \left[ \begin{array}{c} v_t \\ v_r \\ u \end{array} \right] \]

respectively separate out the gauge modes (\( \phi_0 \)), the gauge invariant modes (\( \phi_1 \)) and the constraint violating modes (\( \psi_0 \)).

Computing the composition

\[ \bar{\Delta}_\Phi := \Phi_0^* \circ T^* \circ \bar{R} = -\frac{f_1}{\omega^2 r^2} \]

is enough to get us to the \( E\cdot T\cdot \Phi_0 \) triangular form (32).

Next, we must invert the morphism \( D \) of gauge modes satisfying \( D_D[\phi_0] = 0 \) into the \( E\cdot T\cdot \Phi_0 \) system. At this step it is helpful to use a computer algebra system. Any equivalence class of scalar operators acting on the \( v_t, v_r, u \) variables modulo the \( E\cdot T\cdot \Phi_0 \) equations can be uniquely represented by a linear combination of \( v_t \) and \( \partial_r v_t \) (all second and higher derivatives are eliminated by \( E, \partial_r v_r \) is eliminated by \( T \) and \( \partial_r u \) is eliminated by \( \Phi_0 \)). Precomposing each of them with \( D \), we get representatives of equivalence classes of scalar operators acting on the \( \phi_0 \) variable modulo the \( D_D \) equation. These, in turn, are uniquely represented by \( \phi_0 \) and \( \partial_r \phi_0 \). Algebraically inverting the relationship between \( v_t, \partial_r v_t \) and \( \phi_0, \partial_r \phi_0 \) gives us the operator \( \bar{D}_0 \) from (32). Keeping track of how higher derivatives are eliminated by the equations also allows us to complete \( D_0 \) to a cochain map. The result is

\[ \bar{D}_0 = \left[ \begin{array}{cc} i \omega r & 0 \\ 0 & 0 \end{array} \right], \]
\[ \left[ \bar{T}_0 \quad -\Delta_T \quad -\Delta_\Phi \right] = \frac{1}{\omega^2 r^2} \left[ \begin{array}{ccc} -i \omega r f & -f_1 & 0 \\ -f_1 f r^2 \partial_r \frac{1}{r} & B_l f_1 \end{array} \right]. \]
Further explicit calculations give us the homotopy corrections needed to proceed with the decoupling:

\[ h_D = 0, \]  
\[ \tilde{h}_E \tilde{T}_2 \Phi_2 = \frac{1}{\omega^2 r^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & f r^2 \partial_r \frac{f}{r} & -B_l r \partial_r f \\ 0 & 0 & f & -f r \partial_r \\ f & 0 & 0 & 0 \end{bmatrix}, \]  
\[ \tilde{n} = \begin{bmatrix} 0 & 0 & -f r^2 \partial_r \frac{f}{r} & B_l f \\ B_l f & B_l r \partial_r f & f & 0 \\ 0 & f & 0 & 0 \end{bmatrix}, \]  
\[ \Phi_2 \tilde{D}_2 \tilde{T}_2 \tilde{D}_2 \tilde{T}_2 = \begin{bmatrix} 0 & 0 & -B_l f & f r^2 \partial_r \frac{f}{r} \\ B_l f & B_l r \partial_r f & 0 & 0 \\ 0 & -f & 0 & 0 \end{bmatrix}. \]  

From the above operators, we can construct the desired triangular decoupling (42) of \( E \):

\[ \begin{bmatrix} \tilde{D}_0 \\ \Phi_0 \\ T \end{bmatrix} = \begin{bmatrix} i \omega r & 0 & 0 \\ 0 & -f & f \partial_r r \\ -\frac{i \omega r}{f} & \frac{r^2 \partial_r f}{f} & B_l \end{bmatrix}, \]  
\[ \begin{bmatrix} \tilde{T}_0 \\ \tilde{\Phi}_0 \\ \tilde{D}^* \end{bmatrix} = \frac{1}{\omega^2 r^2} \begin{bmatrix} -i \omega r f & -f_1 & 0 \\ 0 & -1 & \frac{1}{B_l} r^2 \partial_r \frac{f}{r} \\ i \omega r & -r \partial_r & 1 \end{bmatrix}, \]  
\[ \begin{bmatrix} D \tilde{\Phi}_2 \tilde{T}_2 \end{bmatrix} = \frac{1}{\omega^2 r^2} \begin{bmatrix} i \omega r & 0 & 0 \\ r^2 \partial_r \frac{f}{r} & -B_l & 0 \\ 1 & -f r \partial_r & f \end{bmatrix}, \]  
\[ \begin{bmatrix} T^* \Phi_2^* \tilde{D}_2^* \end{bmatrix} = \begin{bmatrix} i \omega r & 0 & 0 \\ f r^2 \partial_r \frac{f}{r} & -B_l f & f r^2 \partial_r \frac{f}{r} \\ B_l f & B_l r \partial_r f & B_l \end{bmatrix}. \]  

An explicit calculation also shows that the cochain maps that we have obtained are mutually inverse up to homotopy:

\[ \begin{bmatrix} \tilde{D}_0 \\ \Phi_0 \\ T \end{bmatrix} \circ \begin{bmatrix} D \tilde{\Phi}_2 \tilde{T}_2 \end{bmatrix} = \text{id} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & f & 0 \\ 1 & 0 & f \end{bmatrix} \circ \begin{bmatrix} D_D & \Delta_\Phi & \Delta_T \\ 0 & D_\Phi & \Delta_\Phi \\ 0 & 0 & D_D \end{bmatrix}, \]  
\[ \begin{bmatrix} \tilde{D}_0 \\ \Phi_0 \\ T \end{bmatrix} \circ \begin{bmatrix} D \tilde{\Phi}_2 \tilde{T}_2 \end{bmatrix} = \text{id} - \frac{1}{\omega^2 r^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & f \end{bmatrix} \circ \Omega_\epsilon. \]
Noting the helpful identities
\[
\frac{1}{\omega^2} D_{s_1} \frac{1}{s_1^2 - s_2^2} = \frac{f_1}{r^2} + \frac{1}{(s_1^2 - s_2^2)} \frac{1}{\omega^2} D_{s_2},
\]
\[
\frac{1}{\omega^2} D_0 \left( \frac{f}{2} \right) = \left( \frac{f_1}{\omega^2 r^2} r^2 \partial_r \frac{1}{r} + \frac{f_2}{2 \omega^2 r^2} \right) + \left( \frac{f}{2} \right) \frac{1}{\omega^2} D_0,
\]
we can use the idea of Section 2.3 to further reduce the triangular form with the following operators:
\[
\bar{\delta}_\Phi = -i \text{id}, \quad \delta_\Phi = B_l, \quad \delta_T = \frac{f}{2},
\]
\[
\bar{\varepsilon}_\Phi = -i \text{id}, \quad \varepsilon_\Phi = B_l, \quad \varepsilon_T = \frac{f}{2},
\]
which are needed to kill or simplify the off-diagonal terms
\[
\bar{\Delta}_\Phi = -\frac{f_1}{\omega^2 r^2},
\]
\[
\Delta_\Phi = -\frac{f_1}{\omega^2 r^2} B_l,
\]
\[
\Delta_T = \frac{f_1}{\omega^2 r^2} r^2 \partial_r \frac{1}{r},
\]
and \( \bar{\Delta}_T = \Delta_T + \varepsilon_\Phi \circ \bar{\delta}_\Phi = \frac{f_1}{\omega^2 r^2} \left( f r^2 \partial_r \frac{1}{r} - B_l \right). \)

The remaining off-diagonal term in the triangular form (46) is then
\[
\Delta = \Delta_T - \left( \frac{f_1}{\omega^2 r^2} f r^2 \partial_r \frac{1}{r} + \frac{f_2}{2 \omega^2 r^2} \right) = -\frac{f_1}{\omega^2 r^2} \left( B_l + \frac{f_1}{2} \right),
\]
so that our final reduced triangular form is
\[
\Pi_e \begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix} = 0 \iff \frac{1}{\omega^2} \begin{bmatrix} D_0 & 0 & -\frac{f_1}{r^2} \left( B_l + \frac{f_1}{2} \right) \\ 0 & D_1 & 0 \\ 0 & 0 & D_0 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} = 0. \tag{88}
\]

The following operators will provide us with the final pieces necessary to construct a homotopy equivalence between \( \Pi_e \) and the reduced triangular form (88):
\[
\begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} \bar{D} \\ \bar{T} \end{bmatrix} \begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix} = \begin{bmatrix} 1 & B_l & \frac{f}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} i \omega r & 0 & 0 \\ 0 & -f & f \partial_r \tau \\ -\frac{i \omega r}{f} & -\frac{1}{f} \partial_r r^2 B_l \end{bmatrix} \begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix}, \tag{89}
\]
\[
\begin{bmatrix} \bar{T}_1 \\ \bar{\Phi}_1^* \\ D^* \end{bmatrix} = \begin{bmatrix} 1 & B_l & \frac{f}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} -i \omega r f & -f_1 & 0 \\ 0 & -1 & \frac{1}{B_l r^2 \partial_r \tau} \\ i \omega r & -r \partial_r & 1 \end{bmatrix}, \tag{90}
\]
\[
\begin{bmatrix} v_l \\ v_r \\ u \end{bmatrix} = \begin{bmatrix} D \bar{T} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} \frac{-i \omega r}{r^2 \partial_r \frac{1}{r}} & 0 & 0 \\ -B_l & f r^2 \partial_r \frac{1}{r} & 0 \\ -fr \partial_r & f & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & -B_l & -\frac{f}{2} - B_l \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix}, \tag{91}
\]
\[
\begin{bmatrix} T^* \Phi_1^* \bar{D}_1^* \end{bmatrix} = \begin{bmatrix} \frac{i \omega r}{f r^2 \partial_r \frac{1}{r}} & 0 & 0 \\ -B_l f & f r^2 \partial_r \frac{1}{r} & 0 \\ -B_l r \partial_r f & B_l f & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & -B_l & -\frac{f}{2} - B_l \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{92}
\]
4. Discussion
We have shown how an abstract strategy, based on formal properties of linear differential operators, can be used to decouple the radial mode equations (with nonzero frequency \(\omega\) and angular momentum \(l\)) of the vector wave equation on Schwarzschild spacetime into a system of hierarchically (that is triangularly) coupled scalar equations. Essentially, we have explicitly separated out the degrees of freedom of the vector field into pure gauge, gauge invariant and constraint violating modes (with the interpretation that the vector wave equation arose as the harmonic (or Lorenz) gauge fixed Maxwell equations). This strategy was first successfully applied in [7]. However, it should be noted that this decoupling strategy is never made explicit in [7], where it becomes apparent only as a common pattern in the voluminous explicit calculations in a sequence of examples of increasing complexity. One of the main contributions of our work is to make this decoupling strategy explicit and mathematically founded on basic ideas drawn from the theory of \(D\)-modules [8, Sec.10.5], the category theoretic approach to differential equations [9, Sec.VII.5] and homological algebra [10].

It is worth pointing out that, even though our approach to triangular decoupling and reduction of the triangular form owes much to [7], the triangular form (88) appears in [7] only implicitly. Moreover, the reduction of the triangular form in [7] only kills the \(\Delta \Phi\) (84) off-diagonal term. Since we have also killed the \(\bar{\Delta} \Phi\) (83) and simplified the \(\Delta T\) (85) terms, our decoupling goes a step further.

The next step is to apply the same decoupling strategy to linearized gravity in harmonic (or de Donder) gauge on the Schwarzschild spacetime. In fact, this strategy was already successfully applied to that case in [7], which was actually one of the main goals of that work. Unfortunately, this decoupling for linearized gravity was achieved only after some rather voluminous explicit calculations and the final results were not presented in a very economic way. We believe that following the abstract strategy in Section 2.5, can significantly streamline both the presentation of the final results and the intermediate calculations. The details will be discussed elsewhere.

We believe that the same strategy, of triangular decoupling by separating out pure gauge, gauge invariant and constraint violating modes should also work for harmonic gauge field equations (either Maxwell or linearized gravity) on other spacetime geometries where the physical degrees of freedom satisfy decoupled equations. The Kerr spacetime is an example, where at the mode level the gauge invariant degrees of freedom decouple and satisfy Teukolsky or Fackerell-Ipser equations [1, 2]. The main uncertainty there is the degree to which the triangular form can be explicitly reduced using the method of Section 2.3.

As was explained in the Introduction, the complicated form of the radial mode equations (61) and (62) of the vector wave equation is a significant obstacle to the study of their analytical properties, and in particular their spectral theory. Once these equations have been put into the decoupled triangular form (88) and (68), their analytical properties can be more easily deduced from their diagonal parts. These diagonal parts consist of spin-s Regge-Wheeler equations, whose analytical properties are well understood [1]. Such information can then be used to construct both classical and quantum propagators (or Green functions) for the vector wave equation on Schwarzschild spacetime. The same will be true for linearized gravity, once a similar decoupling has been achieved. For that purpose, the \(\omega = 0\) and \(l = 0\) modes would need to be investigated as well. These applications will also be discussed elsewhere.

Finally, it would also be interesting to relate our decoupling strategy to the adjoint operator method of Wald [12, 13]. Recall that we have liberally used formal adjoints (Section 2.4) as part of our notation. This was convenient because some of the operators that we used in the case of the vector wave equation, in Section 3, really were formal adjoints of each other. However, we did not really use this property as part of our decoupling strategy. On the other hand, supposing that we can find a decoupling of a formally self-adjoint differential equation \(e = e^*\) into a triangular form \(\bar{e}\), taking adjoints produces for free another decoupling of \(e\) but this time
into $\bar{e}^*$. This is illustrated in the following commutative diagrams:

\[
\begin{array}{ccc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bar{e} & \bar{e}^* \\
\downarrow & \downarrow \\
\bullet & \bullet
\end{array}
\quad \Longleftrightarrow \quad
\begin{array}{ccc}
\bullet & \bullet \\
\downarrow & \downarrow \\
\bar{e} & \bar{e}^* \\
\downarrow & \downarrow \\
\bullet & \bullet
\end{array}
\]

(93)

If $\bar{e}$ is in upper triangular form, then $\bar{e}^*$ is in lower triangular form. In the vector wave equation example (88), the adjoint equation $\bar{e}^*$ is actually equivalent to the original equation $\bar{e}$ by the simple interchange $\phi_0 \leftrightarrow \psi_0$. In such a case, composing the equivalences of $e$ with $\bar{e}$, of $\bar{e}$ with $\bar{e}^*$, and of $\bar{e}^*$ with $e^*$, we obtain a morphism from $e$ to itself, or more precisely a pair of operators $\hat{k}, \hat{g}$ satisfying $e \circ \hat{k} = \hat{g} \circ e$. In the terminology of [13], $\hat{k}$ is a symmetry operator and it maps solutions of $e$ to solutions of $\bar{e}$ (in this case, isomorphically). Thus, if $\hat{k}$ is not simply proportional to a constant, it gives an interesting way to generate new solutions from known ones. The significance of the existence of such a symmetry operator should be investigated further.

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