A Two-Component Generalization of the Integrable rdDym Equation

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Abstract. We find a two-component generalization of the integrable case of rdDym equation. The reductions of this system include the general rdDym equation, the Boyer–Finley equation, and the deformed Boyer–Finley equation. Also we find a Bäcklund transformation between our generalization and Bodganov’s two-component generalization of the universal hierarchy equation.

Key words: coverings of differential equations; Bäcklund transformations

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1 Introduction

Recent papers [3, 8, 16] provide two-component generalizations for the hyper-CR Einstein–Weil structure equation [6, 22]

\[ s_{yy} = s_{tx} + s_{y}s_{xx} - s_{x}s_{xy}, \] (1.1)
Plebański’s second heavenly equation [25]

\[ s_{xz} = s_{ty} + s_{xx}s_{yy} - s_{xy}^{2} \] (1.2)

and the universal hierarchy equation [18, 19, 22]

\[ s_{xx} = s_{x}s_{ty} - s_{t}s_{xy}. \] (1.3)

Namely, equations (1.1)–(1.3) appear from systems

\[ s_{yy} = s_{tx} + (s_{y} + r)s_{xx} - s_{x}s_{xy}, \] (1.4)

\[ r_{yy} = r_{tx} + (s_{y} + r)r_{xx} - s_{x}r_{xy} + r_{x}^{2}; \]

\[ s_{xz} = s_{ty} + s_{xx}s_{yy} - s_{xy}^{2} + r, \]

\[ r_{xz} = r_{ty} + s_{yy}r_{xx} + s_{xx}r_{yy} - 2s_{xy}r_{xy}, \] (1.5)

and

\[ s_{xx} = e^{r}(s_{x}s_{ty} - s_{t}s_{xy}), \]

\[ (e^{-r})_{xx} = s_{x}r_{ty} - s_{t}r_{xy}, \] (1.6)

respectively, by substituting for \( r = 0 \). Other reductions for (1.4) are found in [7, 16]: when \( u = 0 \), system (1.4) gives the Khokhlov–Zabolotskaya (or dispersionless Kadomtsev–Petviashvili) equation

\[ v_{yy} = v_{tx} + vv_{xx} + v_{x}^{2}, \]
while substituting for \( v = u_x \) in (1.4) produces the normal form
\[
u_{yy} = u_{tx} + (u_x + u_y)u_{xx} - u_x u_{xy},
\]
for the family of equations studied in [7]. Also, we note the reduction \( v = u_y \) for system (1.4). This reduction yields equation
\[
u_{yy} = u_{tx} - u_x u_{xy}
\]
studied in [9, 14, 17, 21].
As it was shown in [3], the reduction \( s = x \) for system (1.6) gives the Boyer–Finley equation
\[
rt_y = (e^{-r})_{xx}.
\]
(1.7)
The purpose of the present paper is to introduce the two-component generalization for equation
\[
rt_y = u_x u_{xy} - u_y u_{xx},
\]
(1.8)
which is integrable in the following sense: it has the differential covering [2, 11, 12, 13]
\[
p_t = (u_x - \lambda)p_x, \quad p_y = \lambda^{-1}u_y p_x
\]
(1.9)
containing the non-removable parameter \( \lambda \neq 0 \) [20]. We show that reductions of the generalization include the general \( r \)-th dispersionless Dym equation [1]
\[
u_{ty} = u_x u_{xy} + \kappa u_y u_{xx},
\]
(1.10)
the Boyer–Finley equation (1.7), and the deformed Boyer–Finley equation. Also we find a Bäcklund transformation between our generalization and Bodganov’s two-component generalization (1.6) of the universal hierarchy equation (1.3).

2 The two-component generalization
Along with the covering (1.9) equation (1.8) has the covering
\[
qt = (u_x - q)q_x, \quad q_y = u_y q^{-1} q_x,
\]
(2.1)
which can be obtained by the method of [20]. While the coverings (1.9) and (2.1) are not equivalent w.r.t. the pseudo-group of contact transformations, (2.1) can be derived from (1.9) by the following procedure, see, e.g., [24]. We consider the function \( p = p(t, x, y) \) from (1.9) to be defined implicitly by the equation \( q(t, x, y, p(t, x, y)) = \lambda \) with \( q_p \neq 0 \). Then for \((x^1, x^2, x^3) = (t, x, y)\) we have \( q_{x^i} + q_p p_{x^i} = 0 \), so \( p_{x^i} = -q_{x^i} / q_p \). Substituting these into (1.9) yields (2.1).

Our main observation in this paper is that the covering (2.1) allows the generalization
\[
qt = (u_x - q + v)q_x + v_x q, \quad q_y = u_y q^{-1} q_x + v_y.
\]
(2.2)
This system is compatible whenever the two-component system
\[
u_{ty} = (u_x + v)u_{xy} - u_y u_{xx},
\]
\[
v_{ty} = (u_x + v)u_{xy} - u_y u_{xx} + v_x v_y
\]
(2.3)
(2.4)
holds. In other words, (2.2) is a covering for system (2.3), (2.4).
3 Reductions

By the construction, we have the following reduction for system (2.2):

Reduction A. Substituting for \( v = 0 \) in equations (2.3), (2.2) gives equations (1.8) and (2.1), while (2.4) becomes an identity.

Also, we have three other reductions.

Reduction B. If we put \( v = - (\kappa^{-1} + 1)u_x \), then (2.3) gets the form

\[
 u_{ty} = -\kappa^{-1} u_x u_{xy} - u_y u_{xx}, \tag{3.1}
\]

while (2.4) is its differential consequence. The transformation \( u \mapsto -\kappa u \) maps (3.1) to (1.10). The corresponding reduction of (2.2) produces the covering of (1.10) studied in [20, 23].

Reduction C. Taking \( v = -u_x \) in (2.3), (2.4), we obtain

\[
 u_{ty} = -u_y u_{xx}
\]

and its differential consequence. Then we divide this equation by \( u_y \), differentiate w.r.t. \( y \) and put \( u_y = -e^w \). This gives the Boyer–Finley equation [4]

\[
 w_{ty} = (e^w)_{xx} \tag{3.2}
\]

This equation is equation (1.7) in a different notation. Substituting for \( q = e^p \) in the corresponding reduction of (2.2), we have the covering [10, 15, 26] for equation (3.2):

\[
 p_t = w_t - e^p p_x, \quad p_y = e^{w-p}(w_x - p_x).
\]

Reduction D. Finally, when we put \( v = u_y - u_x \) into (2.3) and (2.4), we get the equation

\[
 u_{ty} = u_y (u_{xy} - u_{xx})
\]

and its differential consequence. Then for \( u_y = e^w \) we have the deformed Boyer–Finley equation [5]

\[
 w_{ty} = (e^w)_{xy} - (e^w)_{xx}, \tag{3.3}
\]

and the corresponding reduction of equations (2.2) with \( q = e^s \) gives the covering

\[
 s_t = (e^s - e^w)s_x - w_t, \quad s_y = e^w(s_x - w_x + w_y).
\]

for (3.3). This covering in other notations was found in [5, 20].

4 Bäcklund transformations

The substitution

\[
 u_x = -v + \frac{s_t}{s_x}, \quad u_y = -\frac{e^{-r}}{s_x}, \quad v_x = \frac{r_x s_t}{s_x} - r_t, \quad v_y = -\frac{e^{-r} r_x}{s_x} \tag{4.1}
\]

maps system (2.2) to system

\[
 q_t = \left( \frac{s_t}{s_x} - q \right) q_x + \left( \frac{s_t r_x}{s_x} - r_t \right) q, \quad q_y = -\frac{e^{-r}}{q s_x} (q_x + r_x q) \tag{4.2}
\]

found in [3]. This system is the two-component generalization of the covering

\[
 q_t = \left( \frac{s_t}{s_x} - q \right) q_x, \quad q_y = -\frac{q_x}{q s_x}.
\]
of equation (1.3). The compatibility conditions for (4.2) coincide with (1.6). Solving (4.1) for $s_t$, $s_x$, $r_t$, $r_x$ yields

$$s_t = -(u_x + v)\frac{e^{-r}}{u_y}, \quad s_x = -\frac{e^{-r}}{u_y}, \quad r_t = \frac{v_y}{u_y}, \quad r_x = \frac{(u_x + v)v_y}{u_y} - v_x. \quad (4.3)$$

This system is compatible whenever equations (2.3), (2.4) are satisfied. Thus equations (4.1) define a Bäcklund transformation from (2.3), (2.4) to (1.6) with the inverse transformation (4.3). In particular, when $v = 0$ and $r = 0$, we have a Bäcklund transformation

$$u_x = \frac{s_t}{s_x}, \quad u_y = -\frac{1}{s_x},$$

between (1.1) and (1.3) with the inverse transformation

$$s_t = -\frac{u_x}{u_y}, \quad s_x = -\frac{1}{u_y}.$$  

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