CARTAN TORI AND ADE CLASSIFICATION
OF TWO-DIMENSIONAL TOPOLOGICAL
PHASE TRANSITIONS

S.A. Bulgadaev

L.D. Landau Institute for Theoretical Physics,
Moscow, RUSSIA
Max Plank Institute for Physics of Complex Systems
Dresden, GERMANY

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I. INTRODUCTION

The topological phase transition (TPT) or the Berezinskii-Kosterlitz-Thouless (BKT) phase transition (PT) takes place in two-dimensional systems with order parameter \( \psi = e^{2\pi i \phi} \in \mathcal{M} = S^1 \). Among them are \( XY \)-model, superconductors, bose-liquids and many other systems \([2, 18, 14, 12, 21]\). \( XY \)-model on a lattice is defined as follows

\[
\mathcal{Z}_{XY} = \sum \exp(-\beta \mathcal{H}),
\]

\[
\mathcal{H} = -\frac{1}{2} \sum_{<i,j>} J(\psi_i \psi_j^* + \text{c.c.}) = -J \sum_{<i,j>} \cos(\varphi_i - \varphi_j). \tag{1}
\]

Its continuous variant is a nonlinear \( \sigma \)-model (NSM) on \( S^1 \)

\[
\mathcal{Z}_{NS} = \int D\varphi e^{-S[\varphi]}, \quad S_{NS} = \frac{1}{2\alpha} \int |\partial \psi|^2 d^2 x = \frac{1}{2\alpha} \int (\partial \varphi)^2 d^2 x, \tag{2}
\]

\[\alpha \simeq T/2J.\]

A circle \( S^1 \) has a nontrivial homotopic group \( \pi_1 \)

\[\pi_1(S^1) = \mathbb{Z}. \tag{3}\]

Due to this fact the topologically stable excitations, vortices, are possible in these systems. One vortex solution has a form (at large distances)

\[\varphi(x) = \frac{1}{\pi} \arctan \frac{y}{x}, \quad x = (x, y) \in \mathbb{R}^2, \tag{4}\]

An account of vortices means that theory must be considered on the covering space \( \mathbb{R} \) of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \). The energy of one ”vortex” is logarithmically divergent, but the energy \( E_N \) of \( N \) vortices with the full topological charge \( e = \sum_{i=1}^N e_i = 0 \) is finite and equals

\[E_N = \frac{2\pi}{2\alpha} \sum_{i \neq k} e_i e_k \ln \frac{|x_i - x_k|}{a} + C(a) \sum_{i} e_i^2, \tag{5}\]

here \( C(a) \) is some nonuniversal constant, determining ”self-energy” (or core energy) of vortices and depending on type of core regularization.

The partition function \( \mathcal{Z}_{XY} \) can be approximated by product of two partition functions \([14, 12]\)

\[\mathcal{Z}_{XY} \simeq \mathcal{Z}_{sw}\mathcal{Z}_{CG}, \tag{6}\]

where \( \mathcal{Z}_{CG} \) is the grand partition function of dilute Coulomb gase (CG) of topological excitations with minimal charges \( e = \pm 1 \) and \( \mathcal{Z}_{sw} \) is the partition function of free ”spin-waves”. \( \mathcal{Z}_{CG} \) can be represented, in its turn, in the form of effective field theory with sine-Gordon (SG) action \( S_{SG} \) \([12, 21]\)

\[\mathcal{Z}_{CG} = \mathcal{Z}_{SG} = \int D\phi e^{-S_{SG}[\phi]}, \tag{7}\]

\[S_{SG} = \int \left[ \frac{1}{2\alpha} (\partial \phi)^2 - 2\mu^2 \cos \phi \right] d^2 x\]
The TPT takes place in the system of vortices and can be described by the effective SG theory. This system has two different phases:

1) **high-T phase**: plasma-like, massive, with finite correlation length with essential singularity at $T_c$ \[\xi \sim a \exp(A \tau^{-\nu}), \quad \nu = \frac{1}{2}, \quad (8)\]

$$\tau = \frac{T - T_c}{T_c} \to 0, \quad T_c \approx \pi J;$$

2) **low-T phase**: dielectric, massless, with infinite correlation length and algebraically falling correlations \[20, 2, 18\].

**Question:**
Are there any possible generalizations on systems with more complicate group $\pi_1$ and other types of critical behaviour?

The answer is nontrivial. The simplest generalization of circle $S^1$ is a torus $T^n$ with the homotopy group

$$\pi_1(T^n) = \bigoplus_{i=1}^{n} \mathbb{Z}_i = \mathbb{Z}^n, \quad (9)$$

where $i$-th component describes maps of the boundary $S^1$ into $i$-th circle of $T^n$.

The maps into different components cannot be transformed into or annihilate each other. Consequently, one can introduce in $\pi_1(T^n)$ and in space of corresponding topological charges a vector structure: a vector basis and a metric. In case of $T^n$ it is an usual euclidean structure with canonical basis $\{e_i\}$ and metrics $g_{ik} = \sum_{a=1}^{n} e_i^a e_k^a = \delta_{ik}$. Then the topological charges, corresponding to different $S^1$, do not interact! Thus, the theories with $\mathcal{M} = T^n$ simply replicate the case $\mathcal{M} = S^1$ and reduce to it.

Moreover, all tori $T_L = \mathbb{R}^n / \mathbb{L}$, \[\mathbb{L} = \sum_{i=1}^{n} n_i e_i, \quad n_i \in \mathbb{Z}_i, \quad e_i \in \{e_i\}_L, \quad g_{ik} = \sum_{a=1}^{n} e_i^a e_k^a, \quad (10)\]

where $\mathbb{L}$ is $n$-dimensional lattice in $\mathbb{R}^n$ ($\{e_i\}_L, i = 1, ..., n$ forms a basis of lattice $\mathbb{L}$), and $g_{ik}$ is an effective metric determined by the theory action, reduce in the same sense to the case $S^1$ also \[4\]. It means that

the torus properties have some hardness (or stability):

the smooth deformations in general position do not change them.

This fact is connected with corresponding deformation of effective metric in space of topological charges. In terms of critical properties of TPT it means that for all systems, having such tori as a vacuum space, the TPT will have the same critical properties \[4\].

**Proposal:**
II. CARTAN SUBGROUP AND DEGENERATED TORI

The Cartan torus $T_G$, the maximal abelian subgroup of group $G$, consists of elements

$$g = e^{2\pi i(H \phi)} \quad H = \{H_1, ..., H_n\} \in \mathcal{C}, \quad [H_i, H_j] = 0,$$

where $n$ is a rank of $G$, $\mathcal{C}$ is a maximal commutative Cartan subalgebra of the Lie algebra $\mathcal{G}$ of the group $G$. It is assumed here that $\langle H \phi \rangle$ is a usual euclidean scalar product. All $H_i$ can be diagonalized simultaneously. Their eigenvalues, the weights $w$ (or quantum numbers), depend on the concrete representation of $G$ and $\mathcal{C}$.

All possible weights $w$ of the simply connected group $G$ (or of the universal covering group $\tilde{G}$ of the non-simply connected group $G$) form a lattice of weights $L_w$. In this basis all $H_i$ (and any element $g \in T_G$) get a diagonal form

$$g_\tau = \text{diag}(e^{2\pi i(w_1 \phi)}, ..., e^{2\pi i(w_p \phi)}) \quad (11)$$

The main differences of this form from the usual representation of $T_L$ type tori are:

1) a dimension of diagonal matrices coincides with dimension $p$ of $\tau$-representation, which is usually larger, than rank of $G$;

2) the set of weights $\{w\}_\tau$ has a discrete Weyl (or crystallographic) symmetry, which results in the next two properties

$$\sum_{a=1}^{p} w_a = 0, \quad g_{ik} = \sum_{a=1}^{p} w_i^a w_k^a = B_\tau \delta_{ik}, \quad (12)$$

where constant $B_\tau$ depends on representation. Now the effective metrics $g_{ik}$ is proportional to the euclidean one.

Tori $T_G$ can be considered as appropriately constrained (reduced) torus $T^N = T_{U(N)}$ with large enough $N$. They give the examples of degenerated tori $T_L$, which we define, in general case, by diagonal matrices of form (11), containing all minimal vectors of lattice $\mathbb{L}$. A dimension of these matrices $p$, equals to the number of all minimal vectors of lattice $\mathbb{L}$, which is usually larger than dimension of lattice.

In the form (11) all $g \in T_G$ are periodic with a lattice of periods $L_\tau^L$, inverse to the lattice $L_\tau$, generated by weights $w_a(a = 1, ..., p)$ of $\tau$-representation. $L_\tau^L$ forms a set of all topological charges of $\tau$-representation of $T_G$.

The lattice $L_\tau^L$ satisfies the next restriction

$$L_w^* \supseteq L_\tau^L \supseteq L_v.$$

where $L_w^*$ is a weight lattice of dual group $G^*$, $L_v$ is a lattice of dual roots. For $\tau = \text{min}$ a lattice $L_\tau^L = L_v$, for $\tau = \text{ad}$ a lattice $L_\tau^L = L_w^*$. The lattices $L_v$ and $L_w^*$ differ by a factor, which is isomorphic to the centre $Z_G$ of group $G$

$$L_w^*/L_v = Z_G.$$
Thus the set of minimal topological charges \( \{ q \}_\tau \) can vary from the set of minimal vectors of the weight lattice till that of the dual root lattice. All possible cases are determined by subgroups of the centre \( Z_G \). For groups \( G \) with \( Z_G = 1 \) the lattices \( L_v \) and \( L_{w^*} \) coincide, i.e. they are self-dual \(( G = E_8 )\).

When \( L'_L = L_w \) (or \( L'_{\tau} = L_{w^*} \)) all weights (i.e. quantum numbers) of group \( G \) (or \( G^* \)) can be reproduced as the vector topological charges of vortices!

Analogously, a lattice of topological charges \( L'_L \) of degenerated torus \( T_L \) belongs to a reciprocal lattice \( L^{-1} \). Since the integer-valued lattices \( L \) also belong to their own inverse lattices \( L^{-1} \), their lattices of topological charges, in general, are even larger than \( L \). For this reason they can contain fractional (in this basis) topological charges. Only for degenerated tori \( T_L \), connected with self-dual lattices, \( L'_L \) exactly coincide with \( L \). Consequently, for tori associated with the integer-valued lattices all their "quantum numbers" always have a topological interpretation.

### III. NS-MODELS ON \( T_G \), DUALITY AND EFFECTIVE THEORIES

The euclidean two-dimensional NSM on \( T_G \), have the following form

\[
S = \frac{1}{2\alpha} \int d^2x Tr_\tau(t_{\nu}^{-1}t_\nu) = \left( \frac{2\pi}{2\alpha} \right)^2 \int d^2x Tr_\tau(H\varphi_\nu)^2 = \frac{(2\pi)^2}{2\alpha} B_\tau \int d^2x (\varphi_\nu)^2,
\]

where \( \varphi_\nu = \partial_\nu \varphi, \nu = 1,2 \). An including of a factor \( B_\tau \) into trace \( Tr_\tau \) gives a canonical euclidean metric in space of topological charges.

These theories are invariant under direct product of right \((R)\) and left \((L)\) groups \( N_G = T_G \times W_G \), which are a semi-direct product of \( T_G \) and \( W_G \)

\[
N_G = T_G \times W_G.
\]

The group \( N_G \in G \) is called a normalizator of \( T_G \) and is a symmetry group of torus \( T_G \), thus the theories (13) can be considered also as chiral NSM on group \( G \) with symmetry breaking \( G \rightarrow N_G \). The NSM on \( T_G \) have properties analogous to those of \( XY \)-model:

1) a zero beta-functions \( \beta(\alpha) \) due to flatness of \( T_G \);
2) non-trivial homotopy group \( \pi_1 \) and corresponding vortex solutions with topological charges \( q \in L'_L \).

In quasi-classical approximation (or in low \( T \) expansion) a partition function of the \( \sigma \)-model on \( T_G \) can be represented as a grand partition function of classical neutral Coulomb gas of vortices with topological charges \( q_i \in \{ q \}_\tau \)

\[
Z = Z_0 Z_{CG}, \quad Z_{CG} = \sum_{N=0}^{\infty} \frac{\mu^{2N}}{N!} \sum_{\{ q \}'} Z_N(\{ q \}_\beta).
\]

Here \( Z_0 \) is a partition function of free massless isovectorial boson field which corresponds to "spin waves" of \( XY \)-model.
This gives an embedding of the compact $\sigma$-models on $T_G$ into noncompact generalized SG theories

$$\mathcal{Z}_{CG} = \int D\phi e^{-S_{\text{eff}}}, \quad S_{\text{eff}} = \int \frac{1}{2\beta}(\partial \phi)^2 - \mu^2 V(\phi),$$

(16)

$$V(\phi) = \sum_{\{q\}} \exp i(q\phi).$$

where $\sum_{\{q\}}$ goes over the set of minimal topological charges, and $\phi \in \mathbb{R}^n$. The initial NSM correspond to some relation between parameters $\mu$ and $\beta$.

The account of vortices reduces the initial symmetry group $N_G$ into discrete dual group $W_G^* \times \mathbb{Z}_{q}^{-1}$ ($W_G^*$ is a dual Weyl group, $\mathbb{Z}_{q}^{-1}$ is a periodicity lattice of potential $V$). This dual group generalizes the dual group $\mathbb{Z}_2 \times \mathbb{Z}$ of $XY$-model.

Thus, in this semiclassical and long wavelength approximation



| Compact theory on a torus $T_G$ with continuous symmetry $N_G$ appears equivalent (modulo $\mathbb{Z}_0$) to noncompact theory with periodic potential and an infinite discrete symmetry. |

In case $\tau = ad$ the generalized SG theories can describe other systems with symmetry $G$ broken to $N_G$.

**IV. ADE LATTICES, TPT AND SYMMETRIES**

The critical properties of the BKT type PT can be determined by renorm-group method [14, 21]. The new critical properties appear only in case, when each vector $q \in \{q\}$ can be represented as a sum of two other vectors. This condition concides with a definition of the root systems $\{r\}$ of simple groups from series $A, D, E$ or of the root sets of the even integer-valued lattices of $A, D, E$ types. Moreover, the sets of minimal roots (and minimal weights) of all simple groups belong to four series of the integer-valued (in appropriate scale) lattices $A, D, E, Z$. $Z_n$ is an example of the odd self-dual (or unimodular) lattices and contains a minimal vectors with norm equal 1, while the series $A_n, D_n, E_n$ belong to the even lattices with minimal norm equal 2.

Each root set is characterised by the Coxeter number $h_G$

$$h_G = \frac{\text{(number of all roots)}}{\text{(rank of group)}},$$

$$h_{A_n} = n + 1, \quad h_{D_n} = 2(n - 1), \quad h_{E_6,7,8} = 12, 18, 30.$$  

All coefficients of RG equations are expressed only through the Coxeter numbers $h_G$

The phase diagram have two separatrices with next declinations ($u_1$ corresponds to the phase separation line): $u_{1,2} = (1/\pi h_G, -1/2\pi)$. 

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The dashed line of the initial values corresponds to the initial $\sigma$-model. The critical exponent $\nu_G$ is inverse to the Lyapunov index of the separatrix 1 and equals

$$\nu_G = 2/(2 + h_G).$$

It can take the next values \[3, 4\].

| $G$  | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----|------|------|------|------|------|------|------|------|------|
| $\nu_G$ | $\frac{2}{n+3}$ | $\frac{1}{n}$ | $\frac{1}{2}$ | $\frac{1}{n}$ | $\frac{2}{5}$ | $\frac{1}{4}$ | $\frac{1}{7}$ | $\frac{1}{10}$ | $\frac{1}{16}$ |

The critical properties of some systems with different symmetries can coincide due to coincidence of their $h_G$.

**Low-T phase properties**

The low-temperature phase is described by effective free field theory with a renomalized "temperature" $\tilde{\beta}$, depending on initial values $\beta_0$. At the PT point (where $\tilde{\beta} = \beta^\star = 8\pi/r^2 = 4\pi$) an additional logarithmic factors, related with the "null charge" behaviour of the renormalized parameters on the critical separatrix (the phase separation line), can appear.

Free-like behaviour of the low-temperature phase (except logarithmic corrections at criticality) admits for its description the conformal field theories with integer central charges $C = n$, instead of PT points of two-dimensional systems with discrete symmetries, described by conformal theories with rational central charges \[1, 8\]. The BKT PT can be considered as the limiting case $k \rightarrow \infty$ (where $k$ is a level) of the sequence of minimal conformal theories with $C = 1 - 6/(k + 1)(k + 2)$ \[19\]. Analogously, the TPT in $\sigma$-models on $T_G$ are the limiting cases of unitary minimal conformal theories, connected with conformal $W$-algebras \[4\]. There exists a puzzling coincidence of $\nu_G$ with "screening" factor in formulae for central charges of the affine Lie algebras $\hat{G}$ \[13\] at level $k = 2$ (though $T_G$ corresponds to $k = 1$)

$$C_k = \frac{k}{k + h_G} \text{dim}G$$
and of the coset realization of the minimal unitary conformal models \([10]\) at level \(k = 1\)

\[
C_k = r \left(1 - \frac{h_G(h_G + 1)}{(k + h_G)(k + h_G + 1)}\right).
\]

**Properties of massive phase**

In this phase all additional vector charges will be shielded like in plasma. All excitations are massive. There is another enlargement of the isotopic symmetry of the initial NSM on separatrix 2. \(\sigma\)-model on \(T_G\) has at classical level two continuous symmetries: 1) scale (or conformal) symmetry, 2) isotopic global symmetry \(N_G = T_G \times W_G\). Both symmetries are spontaneously broken in IR region by vortices. For this reason \(\sigma\)-model has in massive phase a finite correlation length \(\xi \sim m^{-1}\), where \(m\) is some characteristic mass scale of the theory. This mass can depend on the coupling constant \(\beta\).

On separatrix 2 one obtains

\[
m \sim \Lambda \exp (-1/2\pi gh_G), \quad \Lambda \sim a^{-1}.
\]

This mass scale is defined only by \(K_G \sim h_G\) (note that here \(G = ADE\) and coincides in main approximation with those for chiral models on groups \(G\) \([17, 16, 7]\) and for fermionic models with symmetry group \(G\) \([7]\).)

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**Mass scale of NSM on \(T_G\)** coincides on separatrix 2 with that for all \(G\)-invariant theories, the chiral as well as fermionic (at least for \(G = A, D, E\)).

It means a possible restoration of the full isotopic symmetry group \(G\) in \(\sigma\)-models with symmetry group \(N_G\) in massive phase.

**ADE classification**

There are a number of other integer-valued lattices, which can serve as a lattice of topological charges. Their classification is not completed at present (except some low-dimensional cases) \([3]\). But, if one confines himself with NSM on tori \(T_L\) with integer-valued lattices of topological charges (their importance was noted above), then all possible types of critical behaviour will belong only to \(ADEZ\) series. This conclusion follows from the Witt theorem, proving that the minimal vector sets of any integer-valued lattice must be a direct sum of the root systems \([3]\). But the last can be only of \(A, D, E, Z\) types. Consequently, all NS-models on tori \(T_L\) with integer-valued lattices of topological charges \(L^1_L\) with minimal norm equal 1 can have critical properties only of \(XY\)-model (or of \(Z^n\) lattice) type, while all NS-models on tori \(T_L\) with integer-valued lattices \(L^1_L\) with minimal norm equal 2 can have critical properties only of \(A, D, E\) lattice types. In this relation it is worth to note that analogous ADE classifications take place in the theory of singularities \([\ ]\) and in the string theory \([\ ]\). In general case of integer-valued lattices the different components of the minimal vector sets can have different \(h_G\). Then one can have in NSM on tori \(T_L\) with general integer-valued lattice \(L^1_L\) a series of PTP, taking place separately in each component (with critical properties, depending on \(h_G\)). For even
self-dual lattices with minimal norm 2 all components must have the same Coxeter number [6], and, consequently, the PTP in all components take place simultaneously [5].

The TPT in NSM on degenerated tori can have an application to the description of partial space decompactification in string theories [5].

V. CONCLUSIONS

1. It is shown that one must consider deformed tori for obtaining the interacting vector topological charges.
2. Vector topological charges form a lattice and in some cases can reproduce all quantum numbers of the corresponding groups.
3. A sequence of approximately equivalent transformations of 2d models is constructed

\[
\text{General GL Theory} \rightarrow \text{NS Model} \rightarrow \text{TopExcGas} \rightarrow \text{General SG Theory}
\]

It simplifies a problem and extracts all necessary long-wave properties of these theories! Here last theory has a pure group-theoretical structure and is universal for whole class of theories with the same symmetries.

4. All critical properties are classified by integer-valued lattices from series A, D, E, Z and are characterised by the corresponding Coxeter numbers.
5. The possible scenarios of the dynamical enlargement of the initial internal symmetry groups is discussed.
6. Some applications of these TPT for cosmological and string theories is proposed.

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