Characterization of differential K-theory by hexagon diagram

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Abstract

We prove differential K-theory is determined by the character diagram, thus giving an affirmative answer to a question asked by Simons and Sullivan in [SS10]. Our proof combines the techniques of Bunke and Schick [BS10] with our key observation that there are canonical topologies, induced from the smooth structure of the manifold, on the groups in the character diagram for differential K-theory.

1 Introduction

Differential cohomology was invented by Cheeger and Simons [CS85] as a refinement of singular cohomology theory. It combines singular cochains with differential forms to provide a natural room in the base for geometric invariants of bundles. Two decades later, the idea of enriching topological information by differential forms was adopted by Hopkins and Singer [HS05] to assign a differential refinement to every generalized cohomology theory. In particular a refinement of topological K-theory is given, known as differential K-theory.

A few years after Hopkins and Singer’s work, Simons and Sullivan provided in [SS08] an axiomatic characterization of differential cohomology by showing it is the unique functor $\hat{H}^*$, up to natural equivalence, fitting into the following so-called character diagram with exact diagonals and boundaries:

And later they constructed in [SS10] a geometric model $\hat{K}$ for differential K-theory using vector bundles with connections which is shown to fit into a similar hexagon shape diagram, called character diagram for differential K-theory, with exact diagonals and boundaries:
The sequence along the upper boundary in the above character diagram

\[
\rightarrow H^{\text{od}}(R) \xrightarrow{\text{mod } Z} K_{R/Z}^{-1} \xrightarrow{\beta} K \xrightarrow{c} H^{cv}(R) \rightarrow
\]

is identified, via Chern character (tensored with \(R\)), with the Bockstein exact sequence for complex K-theory associated to the coefficient exact sequence \(0 \rightarrow Z \rightarrow R \rightarrow R/Z \rightarrow 0\), and the sequence along the lower boundary

\[
\rightarrow H^{\text{od}}(R) \xrightarrow{\text{deR}} \Omega^{\text{od}}/\Omega_{U} \xrightarrow{d} \Omega_{BU} \xrightarrow{\text{deR}} H^{cv}(R) \rightarrow
\]

comes from de Rham theory of representing real cohomology classes by differential forms. The groups and morphisms in this diagram will be explained in more details in Section 3.

Naturally Simons and Sullivan asked whether or not the character diagram for differential K-theory axiomatically characterizes the functor \(\hat{K}\) up to natural equivalence, as is the case for differential cohomology.

The question of axiomatically characterizing differential K-theory and more generally differential refinements of generalized cohomology theories is thoroughly analyzed by Bunke and Schick [BS10]. They found a slightly different set of axioms than the character diagram, and deduced uniqueness from their axioms for a large class of differential refinements of generalized cohomology theories including differential K-theory. The major difference between their axioms and the character diagram is the absence of the group \(K_{R/Z}^{-1}\) in their axioms. Recently, an attempt to directly resolve Simons and Sullivan’s question is made in [Mat21] by Ishan Mata who obtained a partial affirmative answer by employing homological algebra.

Our purpose here is to give a complete affirmative answer to Simons and Sullivan’s original question. To state our main result more precisely, we need to make a couple of definitions.

**Definition 1.1.** A differential K-functor is a 5-tuple \((\hat{K}, i, j, \delta, \text{ch})\), consisting of a contravariant functor \(\hat{K}\) from the category of compact manifolds with corners (with smooth mappings) to the category of abelian groups, and natural transformations \((i, j, \delta, \text{ch})\) fitting into the character diagram with exact diagonals.

**Definition 1.2.** A natural transformation from a differential K-functor \((\hat{K}', i', j', \delta', \text{ch}')\) to another \((\hat{K}, i, j, \delta, \text{ch})\) is a natural transformation \(\Phi : \hat{K}' \rightarrow \hat{K}\) such that \(\Phi \circ i = i', \Phi \circ j = j, \delta \circ \Phi = \delta'\) and \(\text{ch} \circ \Phi = \text{ch}'\).

Our main result is:

**Theorem 1.3.** Any two differential K-functors \((\hat{K}', i', j', \delta', \text{ch}')\) and \((\hat{K}, i, j, \delta, \text{ch})\) are naturally equivalent via a natural transformation \(\Phi : \hat{K}' \rightarrow \hat{K}\) satisfying three

\[1\text{ This is the category both [HS05] and [SS10] worked with.}\]
of the commutativity conditions (thus implying $\Phi$ is an isomorphism by five lemma), leaving one last commutativity condition $\Phi \circ j' = j$ to be verified. Note this is exactly the commutativity at the group $K^{-1}_{R/Z}$ missing from Bunke and Schick’s axioms. Our key observation kicks in at this point: all the groups in the character diagram for differential K-theory carry canonical topologies induced from the smooth structure of the manifold. These topologies are either inherited from the Frechét space topology on smooth differential forms or forced upon by requiring that the morphisms in the character diagram to be continuous and that the diagonals and boundaries to be exact in the category of topological groups. Furthermore, $K^{-1}_{R/Z}$ and $\Omega^{odd}/\Omega_U$ can be respectively identified with closure of torsion and identity component of $K$.

With the topology on the character diagram, we can show $\Phi$ is continuous and thus reduce verifying the commutativity at $K^{-1}_{R/Z}$ to its dense subgroup $K^{-1}_{Q/Z}$ and further to $K^{-1}_{Z/n}$ since $K^{-1}_{Q/Z} = \lim_{\rightarrow} K^{-1}_{Z/n}$. In fact, $K^{-1}_{Q/Z}$ can be identified with the torsion subgroup of $K$ and (the image of) $K^{-1}_{Z/n}$ the subgroup of $n$-torsions. After this reduction, the techniques of Bunke and Schick apply again to finish the proof.

We should point out that our argument cannot prove odd differential K-theory is unique under the character diagram axiom. The counter example given in [BS10] continues to hold in our case. So we shall stick to even differential K-theory throughout.

This paper is organized as follows. Section 2 contains a general discussion of abelian topological groups. We shall functorially build hexagon diagrams for “good” topological abelian groups. In Section 3 we topologize the character diagram and prove it is naturally isomorphic to the hexagon diagram for the central group $\hat{K}$. Section 4 is devoted to adapting the tools of [BS10] to our situation. We complete the proof of uniqueness of differential K-theory in Section 5.

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2 Hexagon diagram for abelian topological group

Throughout, subgroups, quotient groups and product groups have subspace, quotient, and product topologies respectively. The field of real numbers $R$ is equipped with its standard topology.

2.1 Strict homomorphism

Recall every group homomorphism $\phi : G \to H$ yields a group isomorphism $\bar{\phi} : G/\ker \phi \cong \text{im} \phi$. In contrast, if $G, H$ are topological groups and $\phi$ is a continuous homomorphism, then we cannot conclude $\bar{\phi}$ is an isomorphism of topological groups: a continuous bijection need not be a homeomorphism. This leads to the following notion.

Definition 2.1. A continuous homomorphism between topological groups is said to be strict if it is open onto its image, i.e. it takes open sets to open subsets of its image.

Lemma 2.2. (i) Let $N$ be a normal subgroup of a topological group $G$. Then the quotient map $\pi : G \to G/N$ is open. In particular $\pi$ is strict.

(ii) Let $\pi : G \to G/N$ be a quotient of topological groups. Then a continuous homomorphism $\phi : G/N \to H$ is strict if and only if $\phi \circ \pi : G \to H$ is strict.

(iii) A continuous homomorphism $\phi : G \to H$ is strict if and only if it induces an isomorphism of topological groups $\bar{\phi} : G/\ker \phi \cong \text{im} \phi$.

Proof. (i) Let $U$ be an open subset of $G$. Since $\pi^{-1}(\pi(U)) = \bigcup_{g \in N} gU$ is open in $G$, by definition of quotient topology $\pi(U)$ is open in $G/N$. 

(ii) The ‘only if’ part follows from the openness of $\pi$ guaranteed by (i). Now assume $\phi \circ \pi$ is strict and $U$ is an open subset of $G/N$. Then by continuity of $\pi$, we see $\phi(U) = \phi \circ \pi(\pi^{-1}(U))$ is open in $\text{im}(\phi \circ \pi) = \text{im} \phi$. Therefore $\phi$ is strict. This proves the ‘if’ part.

(iii) Consider the canonical factorization $\phi : G \xrightarrow{\pi} G/\ker \phi \xrightarrow{\bar{\phi}} \text{im} \phi \xrightarrow{\iota} H$. Then

$\phi$ is strict $\iff$ $\bar{\phi} \circ \pi$ is strict (by definition)
$\iff$ $\bar{\phi}$ is strict (by (ii))
$\iff$ $\bar{\phi}$ is an isomorphism of topological groups.

Compositions of strict homomorphisms need not be strict. For example, let $\iota_A : A \to G$ be an embedding of topological groups and $\pi_B : G \to G/B$ be a quotient by a normal subgroup $B$. Both $\iota_A$ and $\pi_B$ are clearly strict, but their composition $\pi_B \circ \iota_A$ is not strict in general, for example take $G = \mathbb{R}^2$, $B = \mathbb{Z}^2$ and $A$ to be a line passing through the origin with irrational slope.

Here we state an easy but useful sufficient condition that ensures $\pi_B \circ \iota_A$ is strict.

**Lemma 2.3.** If either $A$ or $B$ is an open subgroup of $G$, then the composition $\pi_B \circ \iota_A : A \to G/B$ is open. In particular, $\pi_B \circ \iota_A$ is strict and induces an isomorphism of topological groups $A/A \cap B \cong AB/B$.

**Proof.** If $A$ is open, then both $\iota_A$ and $\pi_B$ are open, hence $\pi_B \circ \iota_A$ is open. If $B$ is open, then since open subgroups are also closed, $B$ is clopen (i.e. closed and open) in $G$. Recall the quotient of a topological group by a clopen normal subgroup is discrete, so $G/B$ is discrete and therefore all maps into $G/B$ are open.

**Remark 2.4.** The above discussion shows extra conditions are required for the first and second isomorphism theorems to hold in the category of topological groups. Strange enough, the third isomorphism theorem always holds.

**Definition 2.5.** A sequence of continuous homomorphisms among topological groups is strictly exact if it is an exact sequence of groups in which all the homomorphisms are strict.

**Example 2.6.** Let $A, B$ be normal subgroups of a topological group $G$. Assume either $A$ or $B$ is open, then it follows from Lemma 2.3 that the following commutative diagram has strictly exact rows and columns.

![Hexagon Diagram](image)

**2.2 Hexagon diagram**

Let us specialize Example 2.6 to the case where $G$ is a “nice” abelian topological group and to its two special closed subgroups: the identity (connected) component $G_\text{e}$ and the closure of torsion $T = \text{Tor}(G)$. We now wish to build a hexagon-shape diagram for $G$, but technical assumptions are needed.

\[ \begin{array}{c}
1 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\end{array} \quad \begin{array}{c}
A \cap B \\
\downarrow \\
B \\
\downarrow \\
B/A \cap B \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\end{array} \quad \begin{array}{c}
A \\
\downarrow \\
G \\
\downarrow \\
G/A \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\end{array} \quad \begin{array}{c}
A/A \cap B \\
\downarrow \\
G/B \\
\downarrow \\
G/AB \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\end{array} \quad \begin{array}{c}
1 \\
\downarrow \\
1 \\
\downarrow \\
1 \\
\end{array} \]

\[ \text{2.2 Hexagon diagram} \]

This example is inspired by the work of Ishan Mata in [Mata21].
Assumption 1. Assume $G$ is a locally connected, $G_e \cap T$ is connected and $G/(G_e + T)$ is torsion-free.

Recall the quotient of a topological group by a closed subgroup is Hausdorff. Thus $\pi_0 G = G/G_e$ and $G/T$ are both Hausdorff. Further, since $G$ is assumed to be locally connected, $G_e$ is not only closed but also open in $G$, thus the condition of Example 2.6 is satisfied. Moreover since $G_e$ is clopen, we see $\pi_0 G$ is discrete.

**Lemma 2.7.** Under assumption 1 we have canonical isomorphisms and equalities of abelian topological groups:

(i) $G_e \cap T = \text{Tor}(G_e) = T_e$

(ii) $G_e/(G_e \cap T) \cong (G_e + T)/T = (G/T)_e$

(iii) $T/(G_e \cap T) \cong (G_e + T)/G_e = \text{Tor}(\pi_0 G)$

**Proof.** (i) $G_e$ being clopen in $G$ implies $G_e \cap T$ is clopen in $T$. This means $G_e \cap T$ is a union of connected components of $T$. But $G_e \cap T$ is assumed to be connected, therefore $G_e \cap T = T_e$.

The second equality follows from $\text{Tor}(G_e) = G_e \cap \text{Tor}(G)$ by taking closure and noticing $G_e$ is clopen.

(ii) The first isomorphism follows from Lemma 2.3. Then the connectedness of $G_e$ implies both $G_e/(G_e \cap T)$ and $(G_e + T)/T$ are connected. We claim $(G_e + T)/T$ is clopen in $G/T$, then $(G_e + T)/T = (G/T)_e$ follows immediately from the connectedness of $(G_e + T)/T$. To prove the claim, notice since $\pi_0 G$ is discrete, its quotient $G/(G_e + T)$ must also be discrete. Therefore $(G_e + T)/T$, being the kernel of $G/T \to G/(G_e + T)$, is clopen in $G/T$.

(iii) Again the first isomorphism follows from Lemma 2.3. Now observe that $(T + G_e)/G_e$ is the image of $T$ under the projection $G \to \pi_0 G$. Since $\text{Tor}(G)$ is dense in $T$, and group homomorphisms take torsion elements to torsion elements, we have $(T + G_e)/G_e \subseteq \text{Tor}(\pi_0 G) = \text{Tor}(\pi_0 G)$ (recall $\pi_0 G$ is discrete). On the other hand, since $G/(G_e + T)$ is assumed to be torsion-free, we see $(G_e + T)/G_e$, being the kernel of $\pi_0 G \to G/(G_e + T)$, must be contained in $\text{Tor}(\pi_0 G)$. This proves $(G_e + T)/G_e = \text{Tor}(\pi_0 G)$.

**Corollary 2.8.** Under assumption 1 we have the following commutative diagram with strictly exact rows and columns.

\[
\begin{array}{ccccccc}
0 & \to & T_e & \to & T & \to & \pi_0 T & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & G_e & \to & G & \to & \pi_0 G & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & G_e/T_e & \to & G/T & \to & \pi_0 (G/T) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 & \\
\end{array}
\]

Now if we rotate the above diagram counter-clockwise by 45 degrees, thus placing $T_e$ and $\pi_0(G/T)$ on the far left and far right respectively, and also draw an arrow connecting $T$ to $\pi_0 G$ standing for the composition, and similarly connect $G_e$ to $G/T$, then a hexagon is formed, centered around the group $G$. This diagram now looks similar to the character diagram. However note that in the character diagram, the groups on the far left and right are real vector spaces. For this reason, we consider two further constructions: the universal cover of $T_e$ and $\pi_0 G \otimes Z R$.

Of course, extra assumptions have to be made.

**Assumption 2.** Assume $T_e$ is connected, locally path-connected and semilocally simply connected so that it has a universal cover $T_e$. Assume $\pi_0(G/T)$ is finitely generated.
The universal covering map \( \tilde{T}_e \to T_e \), precomposed with the inclusions \( T_e \to T \) and \( T_e \to G_e \), gives rise to two strict homomorphisms \( \tilde{T}_e \to T \), and \( \tilde{T}_e \to G_e \).

Next we consider \( \pi_0 G \otimes \mathbb{Z} \mathbb{R} \). Since \( \pi_0 T \) is torsion, the exact sequence of discrete groups \( 0 \to \pi_0 T \to \pi_0 G \to \pi_0 (G/T) \to 0 \) gives an isomorphism \( \pi_0 G \otimes \mathbb{Z} \mathbb{R} \cong \pi_0 (G/T) \otimes \mathbb{R} \). Moreover, since \( \pi_0 (G/T) \cong \pi_0 G/\pi_0 T \cong \pi_0 G/\text{Tor}(\pi_0 G) \) is torsion free, we see \( \pi_0 (G/T) \otimes \mathbb{R} \cong \pi_0 G \otimes \mathbb{R} \). Note that \( \pi_0 (G/T) \) is assumed to be finitely generated, the group \( \pi_0 G \otimes \mathbb{R} \) is a finite dimensional \( \mathbb{R} \)-vector space that inherits a topology from \( \mathbb{R} \), in which \( \pi_0 (G/T) \) is identified as a discrete lattice. So the inclusion \( \pi_0 (G/T) \hookrightarrow \pi_0 G \otimes \mathbb{R} \), composed with the quotient maps \( \pi_0 G \to \pi_0 (G/T) \) and \( G/T \to \pi_0 (G/T) \), gives rise to two strict homomorphisms \( \pi_0 G \to \pi_0 G \otimes \mathbb{R} \), and \( G/T \to \pi_0 G \otimes \mathbb{R} \).

**Remark 2.9.** It is necessary to assume \( \pi_0 (G/T) \) is finitely generated, for in general the inclusion \( \pi_0 (G/T) \hookrightarrow \pi_0 G \otimes \mathbb{R} \) is not an embedding of topological groups. For example, take \( \pi_0 G = \pi_0 (G/T) = \mathbb{Q} \) with discrete topology, then the above inclusion \( \mathbb{Q} \hookrightarrow \mathbb{Q} \otimes \mathbb{R} \) is the usual inclusion of \( \mathbb{Q} \) into \( \mathbb{R} \). But the subspace topology on \( \mathbb{Q} \) is not discrete.

**Proposition 2.10.** Under assumption 1 and assumption 2, the boundary and diagonal exact sequences in the following commutative diagram are strictly exact.

\[
\begin{array}{ccc}
0 & \to & T \\
& \searrow & \downarrow \pi_0 G \\
& & \pi_0 G \\
\tilde{T}_e & \to & G \\
G_e & \to & G/T \\
& \nwarrow & \downarrow \pi_0 G \otimes \mathbb{Z} \mathbb{R} \\
0 & \to & 0
\end{array}
\]

**Definition 2.11.** The diagram in Proposition 2.10 is called the hexagon diagram for \( G \), provided \( G \) satisfies assumption 1 and assumption 2. The hexagon diagram is clearly functorial in \( G \).

### 3 Topology on character diagram

The goal of this section is to show the smooth structure on a manifold induces natural topologies on the groups in the character diagram so that all the homomorphisms in the character diagram are continuous and strict. Moreover, with such topology the character diagram is canonically isomorphic to the hexagon diagram for \( \hat{K} \).

Let \( M \) be a compact smooth manifold with corners. We now proceed to topologize each of the groups in the character diagram for \( M \). Along the way, we will prove all the homomorphisms in the character diagram are continuous and strict. Furthermore, we will identify \( \Omega_{BU} \) and \( K_{\mathbb{R}/\mathbb{Z}}^{-1} \) with the component of identity of \( \hat{K} \) and the closure of torsion of \( \hat{K} \) respectively.

All functors in this section are applied to \( M \) unless otherwise stated. For simplicity, we shall sometimes drop \( M \) from our notation.

#### 3.1 Differential forms

The space of (real-valued smooth) differential \( k \)-forms \( \Omega^k(M) \) has a Fréchet space topology as follows. Choose a Riemannian metric and a connection on \( M \). If \( \omega \) is a \( k \)-form, denote its \( j \)-th
covariant derivative by $D^j\omega$. Then
\[
\|\omega\|_n = \sum_{j=0}^n \sup |D^j\omega|
\]
(where $|\cdot|$ is induced by the Riemannian metric) is a family of seminorms making $\Omega^k(M)$ into a Fréchet space. Since $M$ is compact, different choices of metrics and connections yield a bounded change of each seminorm $\|\cdot\|_n$, so the topology on $\Omega^k(M)$ is independent of such choices. If $f : N \to M$ is a smooth map from a compact manifold $N$ into $M$, then by compactness of $M, N$ and standard estimates, the pull-back homomorphism
\[
f^* : \Omega^k(M) \to \Omega^k(N)
\]
is a bounded linear operator. Similarly by standard estimates the exterior differential
\[
d : \Omega^k(M) \to \Omega^{k+1}(M)
\]
is a bounded linear operator.

**Definition 3.1.** An oriented smooth cycle in $M$ is a pair $(V, f)$ consisting of a closed (i.e. compact without boundary) oriented smooth manifold $V$ and a smooth mapping $f : V \to M$.

**Proposition 3.2.** A closed form $\omega$ on $M$ is exact if and only if $\int_V f^* \omega = 0$ for all oriented smooth cycles $(V, f)$.

**Proof.** The ‘only if’ part follows from Stokes theorem. The ‘if’ part follows from de Rham’s theorem that integration induces an isomorphism $H^*_dR(M) \cong \text{Hom}(H_k(M; \mathbb{Z}), \mathbb{R})$ and from Tho54 that $\mathbb{Z}$-homology classes, after multiplied by some positive integer (depending only on $k$), can be represented by oriented smooth cycles.

**Corollary 3.3.** $\Omega^k_{cl} = \{\text{closed $k$-forms}\}$ and $d\Omega^{k-1} = \{\text{exact $k$-forms}\}$ are closed vector subspaces of $\Omega^k(M)$. In particular, $\Omega^0_{cl}$ and $d\Omega^{k-1}$ are Fréchet spaces.

**Proof.** Since $d$ is bounded, $\Omega^k_{cl} = \ker d$ is closed. For each oriented smooth $k$-dimensional cycle $(V, f)$, $\omega \mapsto \int_V f^* \omega$ is a bounded linear functional on $\Omega^k_{cl}$. By Proposition 3.2 $d\Omega^{k-1}$ is the intersection of the kernels of these functionals, hence $d\Omega^{k-1}$ is closed. The second assertion follows from that closed vector subspaces of Fréchet spaces are Fréchet.

### 3.2 De Rham and singular cohomology

The de Rham cohomology groups $H^*_dR(M) = \Omega^*_dR/M^*_dR$ have induced topologies as subquotients of differential forms. Since quotients of Fréchet spaces by closed vector subspaces are Fréchet, by Corollary 3.3 $H^*_dR(M)$ is a Fréchet space.

The singular cohomology with $\mathbb{R}$-coefficients $H^*(M; \mathbb{R})$ can be topologized as follows. The $k$-th singular homology $H_k(M; \mathbb{Z})$ is a finitely generated abelian group, on which equip the discrete topology. Next we equip $C(H_k(M; \mathbb{Z}), \mathbb{R}) = \mathbb{R}^{H_k(M; \mathbb{Z})}$, the set of continuous functions on $H_k(M; \mathbb{Z})$, with the compact-open topology, or equivalently the product topology on $\mathbb{R}^{H_k(M; \mathbb{Z})}$. This way $C(H_k(M; \mathbb{Z}), \mathbb{R})$ is a (Hausdorff) topological vector space. Then through the isomorphism
\[
H^k(M; \mathbb{R}) \cong \text{Hom}(H_k(M; \mathbb{Z}), \mathbb{R}) \subseteq C(H_k(M; \mathbb{Z}), \mathbb{R}),
\]
we can now endow $H^k(M; \mathbb{R})$ with the subspace topology, making it into a finite dimensional topological vector space. We can similarly topologize $H^k(M; \mathbb{Q})$ and it is not hard to see $H^k(M; \mathbb{R})$ is dense in $H^k(M; \mathbb{Q})$. Indeed, choose a basis $e_i$ for the free part of $H_k(M; \mathbb{Z})$, then a real cohomology class $l \in H^k(M; \mathbb{R})$ is determined by the real numbers $l(e_i)$. The class $l$ belongs to $H^k(M; \mathbb{Q})$ if and only if $l(e_i) \in \mathbb{Q}$ for all $i$.

Now that both $H^*_dR$ and $H^*(\mathbb{R})$ are finite dimensional topological vector spaces, the de Rham isomorphism $H^*_dR \cong H^*(\mathbb{R})$ must also be a homeomorphism. We shall therefore not distinguish $H^*_dR(M)$ from $H^*(M; \mathbb{R})$ from now on.

However, it is probably worth pointing out that the topology we put on $H^k(\mathbb{R})$ is clearly functorial with respect to continuous maps, while the topology on $H^*_dR(M)$ is functorial with respect to smooth maps, for the de Rham groups and the de Rham isomorphisms depend a priori on the smooth structure.
3.3 Unitary forms

**Definition 3.4.** A smooth SAC cycle $(V, f)$ in $M$ is a closed stably almost complex (SAC) manifold $V$ together with a smooth mapping $f : V \rightarrow M$. The period of a closed form $\omega$ over $(V, f)$ is

$$\int_V f^* \omega \cdot Td(V)$$

where $Td(V)$ is the Todd class of $V$, which may be represented by a total even closed form by enriching the stable tangent bundle of $V$ with a unitary connection.

Since (according to Chern-Weil theory) different choices of connections result in cohomologous differential form representatives of $Td(V)$, and since $\omega$ is closed, Stokes theorem ensures the period of $\omega$ over $(V, f)$ is independent of choices of connections on $V$.

Recall by definition $\Omega_{BU}(M)$ (resp. $\Omega_{U}(M)$) is the group of closed even (resp. odd) forms on $M$ having periods in $\mathbb{Z}$ over all even (resp. odd) dimensional smooth SAC cycles. It is proved in [SS10] that $\Omega_{BU}$ contains exactly those closed even forms cohomologous to Chern characters of unitary vector bundles on $M$. Similarly, $\Omega_U$ consists of those closed odd forms cohomologous to pull-backs by maps of $M$ into the unitary group $U$ (i.e. union of $U(n)$) of the transgressed Chern character. We therefore refer to them as unitary forms. The spaces of unitary forms $\Omega_U(M)$ and $\Omega_{BU}(M)$, as subspaces of differential forms, are equipped with subspace topologies.

By Stokes theorem, exact forms have vanishing periods over smooth SAC cycles, so in particular exact forms are unitary forms. Also notice that given any enriched SAC cycle $(V, f)$, the map

$$\Omega_{cl} \rightarrow \mathbb{R}, \quad \omega \mapsto \int_V f^* \omega \cdot Td(V)$$

is a bounded linear functional. Therefore since $\mathbb{Z}$ is closed in $\mathbb{R}$, $\Omega_{BU}$ (resp. $\Omega_U$) is closed in $\Omega_{cl}^e$ (resp. $\Omega_{cl}^o$). In particular, $\Omega_{od}/\Omega_U$ is Hausdorff.

**Remark 3.5.** By replacing Thom’s theorem with the Conner-Floyd theorem [CF66] in the proof of Proposition 3.2, one can prove a closed form is exact if and only if it has vanishing periods over all smooth SAC cycles.

**Proposition 3.6.** The sequence

$$H^{od}(\mathbb{R}) \xrightarrow{\text{def}R} \Omega^{od}/\Omega_U \xrightarrow{d} \Omega_{BU} \xrightarrow{\text{def}R} H^{ev}(\mathbb{R})$$

is strictly exact.

We will see shortly (at the beginning of the next subsection) that the kernel of $H^{od}(\mathbb{R}) \xrightarrow{\text{def}R} \Omega^{od}/\Omega_U$ is a lattice $L_U$ of full rank (i.e. the rank of $L_U$ is the dimension of $H^{od}(\mathbb{R})$). Similarly the image of $\Omega_{BU} \xrightarrow{\text{def}R} H^{ev}(\mathbb{R})$ is a lattice $L_{BU}$ of full rank. Granted these, we can proceed to prove our proposition.

**Proof.** We know already from [SS10] this sequence is exact, it remains to show each map is continuous and strict.

(1) $H^{od}(\mathbb{R}) \xrightarrow{\text{def}R} \Omega^{od}/\Omega_U$ is continuous since it is the composition of the continuous maps $H^{od}(\mathbb{R}) = \Omega^{od}_{cl}/d\Omega^{ev} \hookrightarrow \Omega^{od}/d\Omega^{ev}$ and $\Omega^{od}/d\Omega^{ev} \rightarrow \Omega^{od}/\Omega_U$. Therefore, it induces a continuous bijection $H^{od}(\mathbb{R})/L_U \rightarrow \text{im}(\text{def}R)$. Notice that $H^{od}(\mathbb{R})/L_U$ is a finite dimensional torus which is in particular compact, and $\text{im}(\text{def}R)$ as a subspace of the Hausdorff space $\Omega^{od}/\Omega_U$ is Hausdorff. Recall a continuous bijection from a compact space to a Hausdorff space must be a homeomorphism, we conclude $H^{od}(\mathbb{R})/L_U \rightarrow \text{im}(\text{def}R)$ is an isomorphism of topological groups. So by Lemma 2.2 $H^{od}(\mathbb{R}) \xrightarrow{\text{def}R} \Omega^{od}/\Omega_U$ is strict.

(2) $\Omega_{BU} \xrightarrow{\text{def}R} H^{ev}(\mathbb{R})$ is continuous since it is the composition of the continuous maps $\Omega_{BU} \hookrightarrow \Omega^{ev}_{cl}$ and $\Omega^{ev}_{cl} \rightarrow \Omega^{ev}/d\Omega^{od} = H^{ev}(\mathbb{R})$. Meanwhile its image $L_{BU}$ is discrete, therefore $\Omega_{BU} \xrightarrow{\text{def}R} H^{ev}(\mathbb{R})$ is trivially strict.

(3) Finally we show $\Omega^{od}/\Omega_U \xrightarrow{d} \Omega_{BU}$ is continuous and strict. By Lemma 2.2 and definition of quotient topology, it suffices to show $d : \Omega^{od} \rightarrow \Omega_{BU}$ is continuous and strict, which follows from the continuity of $d$ and the openness of $d : \Omega^{od} \rightarrow d\Omega^{od}$ guaranteed by the open mapping theorem for Fréchet spaces. □
3.4 Complex K-group

The (even) complex K-group $K(M) = K^0(M)$ is a finitely generated abelian group, we equip it with the discrete topology so that all the maps out of $K(M)$ are automatically continuous. Meanwhile a map into $K(M)$ is continuous if and only if for all $x \in K(M)$ the preimage of $x$ is clopen.

Recall the Chern character map $K \to H^{ev}(Q)$ is an isomorphism when tensored with $Q$. Hence the rational $\mathbb{K}$-theory $K_\mathbb{Q}$ is isomorphic to $H^{ev}(Q)$. Also the image of the Chern character map is a discrete integral lattice $L_{BU}$ in $H^{ev}(Q)$ whose rank is the same as the dimension of $H^{ev}(Q)$. From the character diagram, this lattice $L_{BU}$ is also the image of $\Omega_{BU} \xrightarrow{\text{def}} H^{ev}(R)$. Similar analysis applies word-by-word to the Chern character map for the odd $K$-group $K^{-1} \to H^{od}(Q)$. We denote the corresponding lattice in $H^{od}(Q) \subset H^{od}(R)$ by $L_U$.

Consider the coefficient long exact sequences for $K$-theory induced by $0 \to Z \to Q \to Q/Z \to 0$ and $0 \to Z \to R \to R/Z \to 0$, and identify rational (resp. real) $K$-groups with the rational (resp. real) cohomology by means of Chern character, we then obtain the following commutative diagram with exact rows:

$$
\begin{array}{cccc}
K^{-1} & \to & H^{od}(Q) & \to & K^{-1}_{Q/Z} & \to & K & \to & H^{ev}(Q) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K^{-1} & \to & H^{od}(R) & \xrightarrow{\text{mod } Z} & K^{-1}_{R/Z} & \xrightarrow{\beta} & K & \xrightarrow{c} & H^{ev}(R)
\end{array}
$$

From this commutative diagram, we can deduce several useful facts.

**Lemma 3.7.** (1) $K^{-1}_{Q/Z} \to K^{-1}_{R/Z}$ is an injection, and it identifies $K^{-1}_{Q/Z}$ with $\text{Tor}(K^{-1}_{R/Z})$.

(2) The image of the Bockstein $\beta : K^{-1}_{R/Z} \to K$ is $\text{Tor}(K)$.

(3) $\text{mod } Z : H^{od}(R) \to K^{-1}_{R/Z}$ factors as $H^{od}(R) \xrightarrow{\beta} H^{od}(R)/L_U \xrightarrow{\epsilon} K^{-1}_{R/Z}$, where the first map is a universal covering map.

**Proof.** (1) The injectivity follows from the five-lemma, we hence view $K^{-1}_{Q/Z}$ as a subgroup of $K^{-1}_{R/Z}$. Now tensoring the first row with $Q$ yields an exact sequence

$$
K^{-1} \otimes Q \to H^{od}(Q) \to K^{-1}_{Q/Z} \otimes Q \to K \otimes Q \to H^{ev}(Q),
$$

where the first and last maps are isomorphisms. Hence $K^{-1}_{Q/Z} \otimes Q = 0$, that is to say $K^{-1}_{Q/Z}$ is a torsion group. Meanwhile, if $x \in K^{-1}_{R/Z}$ is a torsion, then $\beta(x) \in K$ is also a torsion, hence it is mapped to zero by $K \to H^{ev}(Q)$. By the exactness of the first row, we see $x$ in fact belongs to $K^{-1}_{Q/Z}$. This proves $K^{-1}_{Q/Z} = \text{Tor}(K^{-1}_{R/Z})$.

(2) Since $\text{im } \beta = \ker c$ and $H^{ev}(R)$ is torsion-free, we see $\text{Tor}(K) \subseteq \text{im } \beta$. On the other hand, $\text{im } \beta$ coincides with the image of $K^{-1}_{Q/Z} \to K$ by an easy diagram tracing. But we know from (1) that $K^{-1}_{Q/Z}$ is a torsion group, so $\text{im } \beta \subseteq \text{Tor}(K)$.

(3) The factorization follows from the exactness of the bottom row. Also the map $H^{od}(R) \to H^{od}(R)/L_U$, being the quotient map of a finite dimensional real vector space by a lattice of maximal rank, is clearly a universal covering map. More precisely, it is the universal covering map of a finite dimensional torus.

**Remark 3.8.** Using the coefficient exact sequence associated to $0 \to Z \xrightarrow{x_n} Z \to Z/n \to 0$, the same argument shows the image of $K^{-1}_{Z/n} \to K^{-1}_{Z/Z}$ (induced by $Z/n \hookrightarrow R/Z$) is the subgroup consisting of $n$-torsions of $K^{-1}_{R/Z}$.
Corollary 3.9. $d\Omega^{od}$ is the identity component of $\Omega_{BU}$, and $\pi_0(\Omega_{BU}) \cong L_{BU}$.

Proof. By Proposition 3.6, $\Omega_{BU}/d\Omega^{od}$ is homeomorphic to the image of $\Omega_{BU} \xrightarrow{\text{def}} H^{od}(\mathbb{R})$, which we have seen is $L_{BU}$. Since $L_{BU}$ is discrete, we conclude $d\Omega^{od}$ is clopen in $\Omega_{BU}$. Now $d\Omega^{od}$, being a Fréchet space, is (arcwise) connected, so $d\Omega^{od}$ must be the identity component of $\Omega_{BU}$. It follows $\pi_0(\Omega_{BU}) = \Omega_{BU}/d\Omega^{od} \cong L_{BU}$. ■

3.5 Odd K-group with $\mathbb{R}/\mathbb{Z}$-coefficients

From Lemma 3.7 we have the following induced commutative diagram in which each row is exact.

\[
\begin{array}{ccccccc}
0 & \to & H^{od}(\mathbb{Q})/L_U & \to & K^{-1}_{Q/Z} & \to & \text{Tor}(K) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^{od}(\mathbb{R})/L_U & \xrightarrow{\text{mod } \mathbb{Z}} & K^{-1}_{R/Z} & \xrightarrow{\beta} & \text{Tor}(K) & \to & 0 \\
\end{array}
\]

Proposition 3.10. There is a unique topology on $K^{-1}_{R/Z}$ making it into an abelian topological group such that

\[0 \to H^{od}(\mathbb{R})/L_U \xrightarrow{\text{mod } \mathbb{Z}} K^{-1}_{R/Z} \xrightarrow{\beta} \text{Tor}(K) \to 0\]

is strictly exact. Moreover, under such topology $K^{-1}_{Q/Z}$ is dense in $K^{-1}_{R/Z}$.

Proof. Indeed, in order so, mod $\mathbb{Z}$ must induce a homeomorphism between $H^{od}(\mathbb{R})/L_U$ and $\beta^{-1}(0)$. In particular $\beta^{-1}(0)$ must be connected. Also $\beta$ must be a quotient map, so for each $x \in \text{Tor}(K)$, $\beta^{-1}(x)$ should be clopen in $K^{-1}_{R/Z}$. Moreover since the topology is required to be compatible with the group structure, $\beta^{-1}(x)$ should be homeomorphic to $\beta^{-1}(0)$ by translation. Therefore $K^{-1}_{R/Z} = \prod_{x \in K} \beta^{-1}(x)$ must be the partition of $K^{-1}_{R/Z}$ into its connected components, and each component is by translation homeomorphic to the identity component $\beta^{-1}(0)$. This in turn completely determines the topology on $K^{-1}_{R/Z}$ and this topology clearly makes the above sequence strictly exact.

In order to check $K^{-1}_{Q/Z}$ is dense in $K^{-1}_{R/Z}$, it suffices to check the intersection of $K^{-1}_{Q/Z}$ with each connected component of $K^{-1}_{R/Z}$ is dense in that component. This, by the above diagram and by translation homeomorphism, is reduced to verifying the density of $H^{od}(\mathbb{Q})/L_U$ in $H^{od}(\mathbb{R})/L_U$ which is now obvious. ■

Therefore we equip $K^{-1}_{R/Z}$ with the topology discussed in the proof of the above proposition.

3.6 Differential K-group

By the same argument used in the proof of Proposition 3.10 we have

Proposition 3.11. There is a unique topology on the differential K-group $\hat{K}(M)$ compatible with its group structure such that the sequence

\[0 \to \Omega^{od}/\Omega_U \xrightarrow{j} \hat{K} \xrightarrow{\delta} K \to 0\]

is strictly exact. Moreover, with such topology $\Omega^{od}/\Omega_U$ is the identity component of $\hat{K}$. ■

Therefore we equip $\hat{K}$ with such unique topology as in Proposition 3.11. Note that $\Omega^{od}$ is locally connected, hence so are $\Omega^{od}/\Omega_U$ and $\hat{K}$.

Proposition 3.12. The sequence

\[0 \to K^{-1}_{R/Z} \xrightarrow{j} \hat{K} \xrightarrow{\text{ch}} \Omega_{BU} \to 0\]

is strictly exact. Moreover, $j$ maps $K^{-1}_{R/Z}$ isomorphically onto $\text{Tor}(\hat{K})$.
Proof. First we show $j$ is continuous and strict. Since $j$ is a group homomorphism and connected components are open in $K^{-1}_{R/Z}$, it suffices to prove $j$ restricted to the identity component of $K^{-1}_{R/Z}$ is continuous and strict. From the proof of Proposition 3.10 and the character diagram, such restriction coincides with the composition $H^{od}(R)/L_U \xrightarrow{\text{delR}} \Omega^{od}/\Omega_U \xrightarrow{\delta} \hat{K}$, whose continuity and strictness follow from the continuity and strictness of $\text{deR}$ and $i$.

Similarly, to prove $ch$ is continuous and strict, we only need to check $ch$ restricted to the identity component of $\hat{K}$ is continuous and strict. Such restriction, by Proposition 3.11 and the character diagram, coincides with $\Omega^{od}/\Omega_U \xrightarrow{\delta} \hat{K} \xrightarrow{\hat{c}} L_{BU}$, which we have seen is continuous and strict.

Lastly we show $\text{im} j = \text{Tor}(\hat{K})$. Since $\Omega_{BU}$, being a subgroup of the torsion-free group $\Omega^{ev}$, is torsion-free, we see $\text{Tor}(\hat{K}) \subseteq \ker ch = \text{im} j$. Now that $ch$ is continuous and $\Omega_{BU}$ is Hausdorff, we know $\text{im} j = \ker j$ is closed, hence $\text{Tor}(\hat{K}) \subseteq \text{im} j$. Meanwhile we have $j(\text{Tor}(K^{-1}_{R/Z})) \subseteq \text{Tor}(K)$, so by density of $\text{Tor}(K^{-1}_{R/Z}) = K^{-1}_{Q/Z}$ in $K^{-1}_{R/Z}$ we get $\text{im} j \subseteq \text{Tor}(\hat{K})$. 

\section{3.7 Character diagram as hexagon diagram}

In summary, we have the following theorem, from which it should be clear that the character diagram coincides with the hexagon diagram for $\hat{K}$.

**Theorem 3.13.** There is a unique topology on $\hat{K}$, functorial with respect to smooth mappings, so that all the exact sequences in the character diagram are strictly exact. With such topology, we have the following commutative diagram with strictly exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & \to & H^{od}(R)/L_U & \xrightarrow{\text{modZ}} & K^{-1}_{R/Z} & \xrightarrow{\beta} & \text{Tor} K & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \Omega^{od}/\Omega_U & \xrightarrow{i} & \hat{K} & \xrightarrow{\delta} & K & \to & 0 \\
\downarrow d & & \downarrow & & \downarrow c & & \downarrow & & \downarrow \\
0 & \to & d\Omega^{od}/\Omega_U & \xrightarrow{\text{deR}} & \Omega_{BU} & \to & 0 \\
\end{array}
\]

Moreover each row is isomorphic to the strictly exact sequence associated to the identity component, and each column is isomorphic to the strictly exact sequence associated to closure of torsion. 

We end this section with two short exact sequences derived from the character diagram. They will be used in Section 3, in constructing universal classes and in proving the uniqueness. Consider the following push-out and the pull-back diagrams:

\[
\begin{array}{ccccccccc}
H^{od}(R)/L_U & \xrightarrow{\text{modZ}} & K^{-1}_{R/Z} & \xrightarrow{i-j} & \hat{K} & \xrightarrow{\hat{c} \delta} & L_{BU} & \to & 0, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^{od}/\Omega_U & \xrightarrow{\text{modZ}/\Omega_U \times K^{-1}_{R/Z}} & \hat{K} & \xrightarrow{\hat{c} \delta} & L_{BU} & \to & 0, \\
\end{array}
\]

**Corollary 3.14.** We have short exact sequences

\[
0 \to \Omega^{od}/\Omega_U \times K^{-1}_{R/Z} \xrightarrow{i-j} \hat{K} \xrightarrow{\hat{c} \delta} L_{BU} \to 0, 
\]
and

\[
0 \to H^{od}(R)/L_U \xrightarrow{\text{delR}} \hat{K} \xrightarrow{\delta} \Omega_{BU} \times H^{ev}(R) K \to 0. 
\]

**Proof.** Exactness follows from a straightforward diagram tracing. Continuity is obvious and strictness follows from Lemma 2.2. \hfill \blacksquare

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4 Technical preparation

The results in this section are slightly modified from those in [BS10] to serve our purposes.

4.1 Approximation of spaces by manifolds

The following proposition generalizes [BS10 Proposition 2.3] by allowing the fundamental group to be finite. This generalization should allow us to handle KO-theory as well (recall $\pi_1 BO = \mathbb{Z}/2$), without leaving the category of compact manifolds.

**Proposition 4.1.** Let $E$ be a pointed connected topological space. Assume $\pi_1 E$ is finite and $\pi_k E$ is finitely generated for $k \geq 2$. Then there exists a sequence of pointed compact manifolds with boundary $\{E_k\}_{k \in \mathbb{N}}$ together with embedding of manifolds $\mu_k : E_k \to E_{k+1}$ and continuous maps $\lambda_k : E_k \to E$ compatible with the embeddings, that is $\lambda_k = \lambda_{k+1} \circ \mu_k$, such that

1. $E_k$ is homotopy equivalent to a $k$-dimensional CW-complex;
2. $\lambda_k$ is $k$-connected, i.e. $(\lambda_k)_* : \pi_* E_k \to \pi_* E$ is an isomorphism for $* < k$ and onto for $* = k$.

**Proof.** We inductively construct $E_k$ by carefully attaching “handles”. Let $\lambda_0 : E_0 \hookrightarrow E$ be the inclusion of the base point of $E$. Suppose $\{\gamma_i : (S^1, *) \to (E, *)\}_{i \in I}$ is a finite set of representatives of elements in $\pi_1 E$, define $F_1 = \bigvee_{i \in I} S^1$ and let $\lambda_1 = \bigvee_{i \in I} \gamma_i$. We may embed $F_1$ in $\mathbb{R}^2$ and let $E_1$ be a compact tubular neighborhood of $F_1$. Since $E_1$ is homotopy equivalent to $F_1$, $\lambda_1$ can be extended to a continuous map $\lambda_1 : E_1 \to E$. Let $\mu_0 : E_0 \to E_1$ be the inclusion of base point. Note $\pi_1(E_1)$ is a finitely generated free group.

Suppose we have constructed $\langle E_{k-1}, \mu_{k-2}, \lambda_{k-1} \rangle$. Embed $(E_{k-1}, \partial E_{k-1})$ into $(\mathbb{R}^n, \mathbb{R}^n_+)$ as a submanifold for some big $n$. Let $U$ be a compact tubular neighborhood of $E_{k-1}$. Since $E_{k-1}$ is homotopy equivalent to $U$, the map $\lambda_{k-1} : E_{k-1} \to E$ can be extended to a map $\tilde{\lambda}_{k-1} : U \to E$. By assumption, $\lambda_{k-1} : \pi_{k-1}(U) \to \pi_{k-1}(E)$ is surjective. Now we claim the kernel of $\lambda_{k-1}$ is finitely generated. Indeed, if $k = 2$ then the kernel is a finite index subgroup of a finitely generated group, hence is finitely generated. If $k \geq 3$, since $U$ is homotopy equivalent to a finite CW complex whose fundamental group is finite, by Serre’s theorem $\pi_{k-1}(U)$ is a finitely generated abelian group, and so its subgroup is also finitely generated.

Now we can choose a finite set of maps $\{\gamma_j : S^{k-1} \to U\}_{j \in J}$ representing the generators of $\ker(\lambda_{k-1})$. Moreover we may choose $\gamma_j$ so that each $\gamma_j$ is an embedding $S^{k-1} \hookrightarrow \partial U$ and the images of $\gamma_j$ are disjoint. Then we let $F_k = U \cup \left( \coprod_{j \in J} D^k \times D^{n-k} \right)$ and then round about the corners. The map $\tilde{\lambda}_{k-1}$ now extends to a map $\lambda_k : F_k \to E$. Next choose a finite set of maps $S^k \times D^{n-k} \to E$ representing the generators of $\pi_k E$. Let $E_k$ be the connected sum of $F_k$ with those $S^k \times D^{n-k}$ and the map $\lambda_k : E_{k-1} \to E$ be the connected sum $\tilde{\lambda}_{k-1}$ with maps $S^k \times D^{n-k} \to E$. There is a natural embedding $\mu_{k-1} : E_{k-1} \to E_k$. It is now a routine to verify that $\{E_k, \mu_k, \lambda_k\}$ as constructed satisfy the requirements. □

For the proof of the following Mittag-Leffler property, we refer the reader to [BS10, Lemma 2.4].

**Corollary 4.2.** Let $(E_k, \mu_k, \lambda_k)$ be as in Proposition 4.1 then as subgroups of $H^*(E_k; A)$ we have

$$
\mu_k^{k+l} : H^*(E_{k+l}; A) = \mu_k^{k+l} : H^*(E_{k+l}; A)
$$

for all $k, l \geq 1, s \geq 0$ and abelian group $A$, where $\mu_k^{k+l} = \mu_{k+l-1} \circ \cdots \circ \mu_k$. □

4.2 Universal classes

Henceforth we fix an approximation $\{E_k, \mu_k, \lambda_k\}$ of $BU$. Let $u \in K(BU) = [BU, \mathbb{Z} \times BU]$ be the class that corresponds to the inclusion of identity component of $\mathbb{Z} \times BU$. Note $c(u) \in H^0(\mu_k ; R)$ is known as the universal Chern character whose degree 0 part is 0 as $u$ is of virtual rank 0.

The proof of the next two propositions are similar to [BS10]. The construction of the universal forms and universal differential $K$-theory classes are obstruction theoretical in nature: one inductively constructs the desired objects and, if necessary, goes one step back to modify the constructed ones for further extensions.
Proposition 4.3. There exists a sequence of differential forms $\omega_k \in \Omega_{BU}(E_k)$ so that $\text{deR}(\omega_k) = \lambda_k^*\iota(u)$ and $\mu_k^*\omega_{k+1} = \omega_k$.

Proof. We inductively construct $\omega_k$. Set $\omega_0 = 0$. Suppose $\omega_{k-1}$ has been constructed. Since $E_k$ is homotopy equivalent to a finite CW-complex, the map $\lambda_k : E_k \to BU$ factors through $BU(n)$ for some big $n$, and thus pulls back a principal $U(n)$-bundle onto $E_k$. Enrich this bundle with a connection and let $\tilde{\omega}_k$ be the Chern-Weil character form of the connection minus $n$. It is clear from construction that $\text{deR}(\tilde{\omega}_k) = \lambda_k^*\iota(u)$. Next, we compare $\mu_k^*\tilde{\omega}_k$ to $\omega_{k-1}$. Now since

$$
\text{deR}(\mu_k^*\tilde{\omega}_k - \omega_{k-1}) = \mu_k^*\text{deR}(\tilde{\omega}_k) - \text{deR}(\omega_{k-1}) = \mu_k^*\lambda_k^*\iota(u) - \lambda_k^*\iota(u) = 0,
$$

there exists $\eta \in \Omega^{ad}(E_{k-1})$ such that $\mu_k^*\tilde{\omega}_k + d\eta = \omega_{k-1}$. Notice $\mu_k : E_{k-1} \to E_k$ is an embedding of manifolds, so we can extend $\eta$ to a form $\tilde{\eta}$ on $E_k$ and define $\omega_k = \tilde{\omega}_k + d\tilde{\eta}$. It is straightforward to check $\omega_k$ satisfies the requirements.

Proposition 4.4. Let $(K, i, j, \delta, ch)$ be a differential K-functor. Then there exists a sequence $\tilde{u}_k \in \tilde{K}(E_k)$ so that $u_k := \delta(\tilde{u}_k) = \lambda_k^*(u)$, $\text{deR}(\tilde{u}_k) = \omega_k$ and $\mu_k^*\tilde{u}_k = \tilde{u}_k$.

Proof. Since $c(\lambda_k^*u) = \text{deR}(\omega_k)$, by Corollary 3.14 we can independently for each $k$ find $\tilde{u}_k \in \tilde{K}(E_k)$ so that $\delta(\tilde{u}_k) = \lambda_k^*(u)$ and $\text{deR}(\tilde{u}_k) = \omega_k$. We now inductively modify $\tilde{u}_k$ to $\tilde{u}_k = \tilde{u}_k + i \circ \text{deR}(\rho_k)$ by appropriately choosing $\rho_k \in H^{ad}(M; R)$ so that $\mu_k^*\tilde{u}_k + i \circ \text{deR}(\rho_k) = \tilde{u}_k$. Set $u_0 = 0$ and $\tilde{u}_1 = \tilde{u}_1$. Assume by induction that we have found $\tilde{u}_0, \ldots, \tilde{u}_k$ and $\tilde{u}_k$ as desired. Then by Corollary 3.14 $\mu_k^*\tilde{u}_k + \tilde{u}_k = i \circ \text{deR}(\rho)$ for some $\rho \in H^{ad}(E_k; R)$. Thus $\mu_k^*\tilde{u}_k = \tilde{u}_k - i \circ \text{deR}(\rho_k)$. By Corollary 4.2 we can find $\tilde{\rho} \in H^{ad}(E_{k+1}; R)$ such that $\mu_{k+1}^*\tilde{\rho} = \mu_{k+1}^*\rho$. Now set $\tilde{u}_k = \tilde{u}_k + i \circ \text{deR}(\rho_k)$ and $\tilde{u}_k = \tilde{u}_k - i \circ \text{deR}(\rho_k)$. Then we have $\mu_k^*\tilde{u}_k = \tilde{u}_k$ and $\mu_k^*\tilde{u}_k = \tilde{u}_k$. This completes the inductive step.

4.3 Chern-Simons integration formula

Proposition 4.5. Let $(\tilde{K}, i, j, \delta, ch)$ be a differential K-functor. Let $M$ be a compact manifold with corners and $\iota_0, \iota_1$ denote the inclusions of endpoints $M \hookrightarrow [0, 1] \times M$. Then for any $\dot{x} \in \tilde{K}([0, 1] \times M)$, we have

$$
\iota_1^*\dot{x} - \iota_0^*\dot{x} = \iota \left[ \int_{\partial \iota_0, \iota_1} ch(\dot{x}) \right]
$$

where $\int_{\partial \iota_0, \iota_1}$ means integration over the fiber of the projection $p : [0, 1] \times M \to M$.

Proof. Since $\iota_0 \circ p$ is homotopic to the identity map, we have $\delta(\dot{x} - p^*\iota_0^*\dot{x}) = \delta \dot{x} - p^*\iota_0^*(\delta \dot{x}) = 0$ in $\tilde{K}([0, 1] \times M)$. It follows

$$
\dot{x} = p^*\iota_0^*\dot{x} + i([\omega])
$$

for some $\omega \in \Omega^{ad}([0, 1] \times M)$. Therefore $ch(\dot{x}) = ch(p^*\iota_0^*\dot{x}) + ch \circ i([\omega]) = p^*\iota_0^*(ch \dot{x}) + d\omega$. Now write $\omega = \omega_0 + dt \wedge \omega_1$ with $\omega_0(t), \omega_1(t) \in \Omega^*(M)$. Then

$$
\frac{\partial}{\partial t} ch \dot{x} = \frac{\partial}{\partial t} d\omega
$$

(since $p^*\iota_0^*(ch \dot{x})$ is independent of $t$)

$$
= \frac{\partial}{\partial t}(d\omega_0 - dt \wedge d\omega_1)
$$

$$
= \partial_t \omega_0 - d\omega_1,
$$

and

$$
\int_{[0, 1] \times M/M} ch \dot{x} = \omega|_{t=1} - \omega|_{t=0} - d(\int_0^1 \omega_1 dt).
$$

So we have

$$
i \left( \int_{[0, 1] \times M/M} ch \dot{x} \right) = i \left( [\iota_1^*\omega_0 - \iota_0^*\omega_1] \right).
$$
Meanwhile, again using $\tilde{x} = p^* t_0^* \tilde{x} + i(\omega)$ we have

$$i_1^* \tilde{x} - i_0^* \tilde{x} = i_1^* p^* t_0^* \tilde{x} + i_1^* i(\omega) - i_0^* p^* t_0^* \tilde{x} - i_0^* i(\omega)$$

$$= i([t_1^* \omega - t_0^* \omega]) \quad \text{(since } p \circ t_0 = id_M = p \circ i_1)$$

$$= i\left( \left[ \int_{[0,1] \times M/M} ch \tilde{x} \right] \right).$$

\[ \blacksquare \]

5 Uniqueness of differential $K$-functor

We now prove Theorem 1.3.

5.1 Reduction to connected manifolds, virtual rank

Let $(\hat{K}, i, j, \delta, ch)$ be a differential $K$-functor. By compactness, $M$ has only finitely many connected components. The differential $K$-group of $M$ is related to those of its connected components as follows:

**Lemma 5.1.** Let $M_1, \ldots, M_r$ be all connected components of $M$. Then the product of maps $\hat{K}(M) \to \hat{K}(M_i)$ induced by the inclusions is an isomorphism $\hat{K}(M) \cong \hat{K}(M_1) \times \cdots \times \hat{K}(M_r)$ of abelian topological groups.

**Proof.** This follows from $\Omega^*(M) \cong \left( \bigtimes \Omega^*(M_1) \right) \times \cdots \times \left( \bigtimes \Omega^*(M_r) \right)$ and $K(M) \cong \left( K(M_1) \right) \times \cdots \times \left( K(M_r) \right)$ as abelian topological groups. \[ \blacksquare \]

So, in order to prove the uniqueness of differential $K$-functor, we are reduced to considering connected manifolds only. From now on, all manifolds are assumed to be connected.

**Definition 5.2** (virtual rank). For $\hat{x} \in \hat{K}(M)$ we define its virtual rank to be the virtual rank of $\delta(\hat{x})$ in $K(M)$.

Since $\Omega^d(pt) = 0$, we can deduce from the character diagram that $\delta : \hat{K}(pt) \to K(pt)$ is an isomorphism. The mapping $M \to pt$ induces $K(pt) \to K(M)$. Denote the image of $n \in \mathbb{Z} \cong K(pt)$ by $n$, and similarly define $\hat{n} \in \hat{K}(M)$. The virtual ranks of $n$ and $\hat{n}$ are both $n$.

5.2 Natural transformation $\Phi$

Let $(\hat{K}', i', j', \delta', ch')$ be another differential $K$-functor, and choose universal classes $\hat{u}_k \in \hat{K}(E_k)$, $\hat{u}'_k \in \hat{K}'(E_k)$ as in Proposition 4.4.

For a manifold $M$ and $\hat{x}' \in \hat{K}'(M)$ of virtual rank $n$, we consider $\delta'(\hat{x}' - \hat{n}') \in K(M)$.

Since $\delta'(\hat{x}' - \hat{n}')$ has virtual rank 0, then by compactness of $M$ we can find a smooth mapping $f_k : M \to E_k$ so that $\delta'(\hat{x}') - n = f_k^* u_k = \delta'(f_k^* \hat{u}_k)$. Therefore there is a unique $[\eta] \in \Omega^d/L_U(M)$ such that

$$\hat{x}' = \hat{n}' + f_k^* \hat{u}_k + i'(\eta).$$

Define $\Phi : \hat{K}'(M) \to \hat{K}(M)$ by

$$\hat{x} = \Phi(\hat{x}') := \hat{n} + f_k^* \hat{u}_k + i(\eta).$$

**Lemma 5.3.** $\Phi$ is well-defined.

**Proof.** The only choice made is the mapping $f_k$. Two such choices are smooth homotopic if we pass to a bigger $k$. Suppose we have two such mappings $f_{k,i} : M \to E_k$ for $i = 0, 1$ and between which a smooth homotopy $F_{k,i} : [0,1] \times M \to E_k$. Then by Proposition 4.5 for $K'$, we have

$$f_{k,1}^* \hat{u}_k - f_{k,0}^* \hat{u}_k = i'(\int_{[0,1] \times M/M} F_{k,1-i}^* \omega_k).$$

Then by construction we find $\eta_0, \eta_1 \in \Omega^d(M)$ so that

$$\hat{x}' = \hat{n}' + f_{k,0}^* \hat{u}_k + i'(\eta_0)$$

$$= \hat{n}' + f_{k,1}^* \hat{u}_k + i'(\eta_1).$$
It follows that
\[ i'(\eta_1 - \eta_0) = i'(\int_{[0,1] \times M/M} F^*_k \omega_k) \].

Now that \( i' \) is injective, we see
\[ \eta_1 - \eta_0 = \int_{[0,1] \times M/M} F^*_k \omega_k \].

Then applying Proposition 4.5 to \( \hat{K} \) we have
\[ f^*_k,1 \hat{u}_k - f^*_k,0 \hat{u}_k = i(\int_{[0,1] \times M/M} F^*_k \omega_k) = i(\eta_1 - \eta_0) \].

This proves \( \Phi \) is well-defined.

Lemma 5.4. \( \Phi \) is natural and \( \Phi \circ i = i' \), \( \delta \circ \Phi = \delta' \), \( ch \circ \Phi = ch' \).

Proof. Straightforward verifications.

Proposition 5.5. \( \Phi \) is a group homomorphism.

Proof. Consider the deviation of \( \Phi \) from being a group homomorphism, that is, a natural transformation
\[
\tilde{B} : \hat{K}' \times \hat{K}' \to \hat{K}
\]
such that
\[
\Phi(\hat{x}' + \hat{y}') = \Phi(\hat{x}') + \Phi(\hat{y}') + \tilde{B}(\hat{x}', \hat{y}').
\]
Clearly \( \tilde{B}(\hat{x}', \hat{y}') = \tilde{B}(\hat{y}', \hat{x}') \). Also from our definition and Lemma 5.4 we have
\[
0 = \Phi(\hat{x}' + i'(\eta)) - \Phi(\hat{x}') = \tilde{B}(\hat{x}', i'(\eta))
\]
\[
0 = \Phi(\hat{x}' + \hat{y}') - \Phi(\hat{x}') = \Phi(\hat{y}') = \tilde{B}(\hat{x}', \hat{y}')
\]
\[
0 = ch(\Phi(\hat{x}' + \hat{y}') - \Phi(\hat{x}') - \Phi(\hat{y}')) = ch(\tilde{B}(\hat{x}', \hat{y}')).
\]

It follows \( \tilde{B} \) factors over a natural transformation
\[
B : K(M) \times K(M) \to H^{od}(M; R).
\]

Then by that every finite CW-complex is homotopy equivalent to a compact manifold with boundary, every continuous map between manifolds is homotopic to a smooth map, Adams’ variant of Brown representation theorem [Ada71] and the Yoneda lemma, we see \( B \) is represented by the homotopy class of some map
\[
(\mathbb{Z} \times BU) \times (\mathbb{Z} \times BU) \to \prod_{i=1}^{\infty} K(R, 2i - 1)
\]
which must be homotopy equivalent to the trivial map since \( H^{od}((\mathbb{Z} \times BU) \times (\mathbb{Z} \times BU); R) = 0 \). This proves the natural transformation \( B = 0 \) and therefore \( \tilde{B} = 0 \).

Corollary 5.6. \( \Phi \) is continuous and strict.

Proof. This follows from \( \Phi \) is a group homomorphism and \( \Phi \) restricted to identity component is continuous and strict.

Corollary 5.7. \( \Phi \) is an isomorphism of abelian topological groups.

Proof. Apply the above corollary and five lemma to the following commutative diagram:

\[
\begin{array}{ccccc}
0 & \longrightarrow & \Omega^{od}/\Omega_U & \longrightarrow & \hat{K}' \longrightarrow & \Phi \\
& & \downarrow & & \Phi \\
0 & \longrightarrow & \Omega^{od}/\Omega_U & \longrightarrow & \hat{K} \longrightarrow & \hat{K}
\end{array}
\]

\(\square\)
5.3 The proof of $\Phi \circ j' = j$

It remains to prove $\Phi \circ j' = j$. By continuity and density of $K_{Q/Z}^{-1}$ in $K_{R/Z}^{-1}$, it suffices to show $\Phi \circ j'|_{K_{Q/Z}^{-1}} = j|_{K_{Q/Z}^{-1}}$.

Since $K_{Q/Z} = \lim_{n} K_{Z/n}$, we consider the composition

$$j_{Z/n} : K_{Z/n}^{-1} \rightarrow K_{Q/Z}^{-1} \xrightarrow{j} K$$

and similarly define $j'_{Z/n} : K_{Z/n}^{-1} \rightarrow K'$. It suffices to show $\Phi \circ j'_{Z/n} = j_{Z/n}$ for all $n$.

Let $GL(Z/n)$ represent the functor $K_{Z/n}^{-1}$ (i.e. the degree $-1$ space in the Omega-spectrum representing the cohomology theory $K_{Z/n}^{*}$). Then we have

$$\pi_{*} GL(Z/n) = \begin{cases} Z/n & \text{is odd,} \\ 0 & \text{is even.} \end{cases}$$

We caution the reader that $GL(Z/n)$ is not the general linear group over the ring $Z/n$.

By Proposition 5.1 we can find an approximation $\{F_{k}, \xi_{k}, \theta_{k}\}$ of $GL(Z/n)$ by compact manifolds-with-boundary.

\[
\begin{array}{ccc}
F_{k} & \xrightarrow{\theta_{k}} & F_{k+1} \\
\downarrow \xi_{k} & & \downarrow \xi_{k+1} \\
GL(Z/n) & & GL(Z/n)
\end{array}
\]

Let $v \in K_{Z/n}^{-1}(GL(Z/n))$ be the class corresponding to the identity map of $GL(Z/n)$, and denote $v_{k} = \xi_{k}^{*}v \in K_{Z/n}^{-1}(F_{k})$.

For each $\rho \in K_{Z/n}^{-1}(M)$, we can find a smooth mapping $g_{k} : M \rightarrow F_{k}$ so that $\rho = g_{k}^{*}v_{k}$. Then by functoriality, the desired identity $\Phi \circ j'_{Z/n}(\rho) = j_{Z/n}(\rho)$ will follow from:

**Proposition 5.8.** $\Phi \circ j'_{Z/n}(v_{k}) = j_{Z/n}(v_{k})$.

**Proof.** Notice $\delta \circ \Phi \circ j'_{Z/n} = \delta' \circ j'_{Z/n}$ coincides with $\delta \circ j_{Z/n}$; both being the composition $K_{Z/n}^{-1} \rightarrow K_{R/Z}^{-1} \xrightarrow{\beta} K$. Hence $\Phi \circ j'_{Z/n}(v_{k}) - j_{Z/n}(v_{k}) \in \ker \delta$. Meanwhile, since $ch \circ \Phi \circ j'_{Z/n} = ch' \circ j'_{Z/n} = 0 = ch \circ j_{Z/n}$, we see $\Phi \circ j'_{Z/n}(v_{k}) - j_{Z/n}(v_{k}) \in \ker ch$. Then from Corollary 3.14 we can find $\sigma_{k} \in H^{od}(F_{k}; R)$ so that

$$i \circ \text{deR}(\sigma_{k}) = \Phi \circ j'_{Z/n}(v_{k}) - j_{Z/n}(v_{k}).$$

Similarly, for all $l \geq 1$, we can find $\sigma_{k+l} \in H^{od}(F_{k+l}; R)$ such that

$$i \circ \text{deR}(\sigma_{k+l}) = \Phi \circ j'_{Z/n}(v_{k+l}) - j_{Z/n}(v_{k+l}).$$

Therefore, we have

$$i \circ \text{deR}(\theta_{k}^{*}\sigma_{k+l}) = i \circ \text{deR}(\sigma_{k}).$$

By injectivity of $i$, we see $\text{deR}(\theta_{k}^{*}\sigma_{k+l}) = \text{deR}(\sigma_{k})$. Now by construction $\xi_{k+l} : F_{k+l} \rightarrow GL(Z/n)$ is $(k+l)$-connected, it follows for $s \leq k$ we have

$$H^{s}(F_{k+l}; R) \cong H^{s}(GL(Z/n); R) = 0.$$ 

Combined with $H^{s}(F_{k}; R) = 0$ for $s \geq k + 1$ (since $F_{k}$ is homotopy equivalent to a $k$-dimensional CW complex), we have $\text{deR}(\sigma_{k}) = 0$. This completes the proof for $\Phi \circ j'_{Z/n}(v_{k}) = j_{Z/n}(v_{k})$. ■
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