Top Quark Pair Production at Threshold – Uncertainties and Relativistic Corrections

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Abstract
In this talk it is shown how nonrelativistic QCD (NRQCD) can be used to determine next-to-next-to-leading order relativistic and short-distance contributions to the total $t\bar{t}$ production cross section in the threshold regime at lepton colliders. A recipe for the calculation of all such contributions for the total photon mediated production cross section is presented and a review of the already known Abelian next-to-next-to-leading results is given.

Introduction
The production of $t\bar{t}$ pairs in the threshold region at future lepton colliders like the NLC (Next Linear Collider) or the FMC (First Muon Collider) offers a unique opportunity to carry out precision tests of QCD in a completely new environment. Due to the large top mass\(^1\) ($M_t \approx 175$ GeV), which allows for the decay channel $t \to Wb$, hadronization effects can be neglected in a first approximation\(^2\). This makes the $t\bar{t}$ production cross section in the threshold regime (including various distributions) calculable from perturbative QCD (and electroweak interactions), which then allows for precise extractions of the top quark mass and the strong coupling once the cross section is measured. In fact, experimental simulations for the NLC and the FMC\(^3\) have shown that experimental errors of around 100 MeV for the top quark mass and of around 0.002 for $\alpha_s(M_Z)$ can be expected for a cross section measurement with a total integrated luminosity of $50 - 100 fb^{-1}$. In particular the prospect

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for the error in the top quark mass measurement beats any hadron collider experiment. However, the errors for $M_t$ and $\alpha_s$ given above do not contain any theoretical uncertainties. At this point it is illustrative to recall that the standard present day formalism used for describing $t\bar{t}$ production at threshold consists of solving the nonrelativistic Schrödinger equation with a QCD potential which for small distances is given by perturbative QCD up to one loop \cite{4,5} and for large and intermediate distances by fits to quarkonia spectra (and leptonic decay widths) \cite{6,7}. The results are then modified by various $O(\alpha_s)$ short-distance corrections which makes the results correct at the next-to-leading order (NLO) level, i.e., they properly include all $O(\alpha_s)$ corrections.\footnote{The solutions of the nonrelativistic Schrödinger equation with the one-loop corrected QCD potential contains, in the language of Feynman diagrams, the resummation of terms $\propto (\alpha_s/v)^n \times [1, \alpha_s]$, $n = 0, 1, \ldots, \infty$, ($v$ being the top quark c.m. velocity) to all orders in $\alpha_s$. Because in the threshold region $|v| \lesssim \alpha_s$ we count all terms $\propto \alpha_s/v$ of order one.} NNLO ($O(\alpha_s^2)$) corrections have never been taken into account so far. Their contributions, however, can be sizable. As an example, consider the $O(\alpha_s^2)$ relativistic corrections to the total cross sections which can lead to a shift in the location of the 1S peak of order $M_t\alpha_s^4 \sim 150$ MeV and a corrections of order $\alpha_s^2 \sim 3\%$ in the size of the cross section (for $\alpha_s \sim \alpha_s(M_t\alpha_s) \sim 0.17$). Further, even the $O(\alpha_s^2)$ short-distance corrections normalizing the total cross sections might be large if the huge size of the $O(\alpha_s)$ corrections of order $-20\%$ is taken into account. From this point of view it is clear that the theoretical uncertainties in the present day analyses are certainly not negligible and that full control over all NNLO effects should be gained.

Unfortunately the formalism described above is constructed in a way that makes a systematic and rigorous implementation of all NNLO effects from first principles QCD conceptually difficult if not impossible – a consequence of the use of phenomenological information in the potential for large and intermediate distances. In principle, this formalism has to be considered as a (sophisticated) potential model approach which cannot be improved in a rigorous way at all. I therefore propose to rely on perturbative QCD only. This means that one applies the perturbative QCD potential for all distances. In fact, such a decision seems to be just natural if one takes into account that the $t\bar{t}$ system is almost insensitive to the large distance (i.e. non-perturbative) contributions in the QCD potential. This makes the framework in which effects beyond the NLO level shall be determined more transparent and still leaves the possibility to incorporate the non-perturbative effects later as a perturbation (see e.g. \cite{8} for an approach of this sort).

In this talk I demonstrate how NRQCD \cite{9} can be used to determine NNLO relativistic and short-distance corrections to the total $t\bar{t}$ production cross section. For simplicity only the production through a virtual photon is considered. The generalization for production through different currents is straightforward. I want to stress that I do not talk about the peculiar NNLO finite width effects coming from the off-shellness of the top quark, time dilatation effects and the interaction among the decay products. These effects have been addressed previously in a number of publications \cite{10,11,12}, but no rigorous and consistent description of them for the $t\bar{t}$ cross section has been found yet. To achieve full NNLO accuracy these finite width effects should eventually be included. For now they remain as an unsolved problem. As far as the NNLO relativistic corrections discussed in this talk are concerned I will use the naive replacement

\begin{equation}
E \equiv \sqrt{s} - 2M_t \rightarrow \tilde{E} = E + i\Gamma_t
\end{equation}

in the spirit of \cite{1} in order to examine their size and properties, where $\Gamma_t$ represents a constant which is not necessarily the decay width of a free top quark.
The \( t\bar{t} \) Cross Section in NRQCD

NRQCD is an effective field theory of QCD designed to handle nonrelativistic systems of heavy quark–antiquark pairs to in principle arbitrary precision. It is based on the separation of long- and short-distance effects\(^3\) by reformulating QCD in terms of a non-renormalizable Lagrangian containing all possible operators in accordance to the symmetries in the nonrelativistic limit. The NRQCD Lagrangian reads

\[
\mathcal{L}_{\text{NRQCD}} = -\frac{1}{2} \mathrm{Tr} G^\mu\nu G_{\mu\nu} + \sum_{q=u,d,s,c,b} \bar{q} i\slashed{D} q + \psi^\dagger \left[ iD_t + a_1 \frac{D^2}{2M_t} + a_2 \frac{D^4}{8M_t^2} + \frac{a_3 g}{2M_t} \sigma \cdot B \right. \\
\left. + \frac{a_4 g}{8M_t^2} (D \cdot E - E \cdot D) + \frac{a_5 g}{8M_t^2} i\sigma (D \times E - E \times D) \right] \psi + \ldots.
\]  

The gluonic and light quark degrees of freedom are described by the conventional relativistic Lagrangian, whereas the top and antitop quarks are described by the Pauli spinors \( \psi \) and \( \chi \), respectively. For convenience all color indices are suppressed. The straightforward antitop bilinears are omitted and only those terms relevant for the NNLO cross section are displayed. \( D_t \) and \( D \) are the time and space components of the gauge covariant derivative \( D_\mu \) and \( E^i = G^{0i} \) and \( B^i = \frac{1}{2} \epsilon^{ijk} G_{jk} \) the electric and magnetic components of the gluon field strength tensor (in Coulomb gauge). The short-distance coefficients \( a_1, \ldots, a_5 \) are normalized to one at the Born level.

To formulate the normalized total \( t\bar{t} \) production cross section (via a virtual photon) \( R = \sigma(\mu^+\mu^- \to \gamma^* \to t\bar{t})/\sigma_{pt} \) \((\sigma_{pt} = 4\pi\alpha^2/3s)\) in the nonrelativistic region at NNLO in NRQCD we start from the fully covariant expression for the cross section

\[
R(q^2) = \frac{4\pi Q_t^2}{s} \text{Im} \langle 0 \mid T \tilde{j}_\mu(q) \tilde{j}^\mu(-q) \mid 0 \rangle,
\]

where \( Q_t = 2/3 \) is the electric charge of the top quark. We then expand the electromagnetic current (in momentum space) \( \tilde{j}_\mu(\pm q) = (\tilde{\gamma}^\mu \tilde{t})(\pm q) \) which produces/annihilates a \( t\bar{t} \) pair with c.m. energy \( \sqrt{q^2} \) in terms of \( ^3S_1 \) NRQCD currents up to dimension eight \( (i = 1, 2, 3)\)\(^4\)

\[
\tilde{j}_i(q) = b_1 \left( \tilde{\psi}^\dagger \sigma_i \tilde{\chi} \right)(q) - \frac{b_2}{6M_t^2} \left( \tilde{\psi}^\dagger \sigma_i (\tilde{\Delta} D)^2 \tilde{\chi} \right)(q) + \ldots,
\]

where the constants \( b_1 \) and \( b_2 \) are short-distance coefficients normalized to one at the Born level. Inserting expansion \((4)\) back into Eq. \((3)\) leads to the NRQCD expression of the nonrelativistic cross section at the NNLO level

\[
P_{t\bar{t}}^{\text{NNLO}}(\tilde{E}) = \frac{\pi Q_t^2}{M_t^2} C_1(\mu_{\text{hard}}, \mu_{\text{fac}}) \text{Im} \left[ A_1(\tilde{E}, \mu_{\text{soft}}, \mu_{\text{fac}}) \right]
\]

\[
- \frac{4\pi Q_t^2}{3M_t^4} C_2(\mu_{\text{hard}}, \mu_{\text{fac}}) \text{Im} \left[ A_2(\tilde{E}, \mu_{\text{soft}}, \mu_{\text{fac}}) \right] + \ldots,
\]

\(^3\) In this context “long-distance” is not equivalent to “non-perturbative”.

\(^4\) Only the spatial components of the current contribute. The expansion of \( j_\mu(-q) \) is obtained from Eq. \((4)\) via charge conjugation symmetry.
where

\[ A_1 = \langle 0 | (\bar{\psi}^\dagger \vec{\sigma} \chi ) (\bar{\chi}^\dagger \vec{\sigma} \bar{\psi}) | 0 \rangle, \quad (6) \]

\[ A_2 = \frac{1}{2} \langle 0 | (\bar{\psi}^\dagger \vec{\sigma} \chi ) (\bar{\chi}^\dagger \vec{\sigma} (-\frac{i}{2} \vec{D}^2 \bar{\psi}) + \text{h.c.} | 0 \rangle. \quad (7) \]

The cross section is expanded in terms of a sum of absorptive parts of nonrelativistic current-current correlators (containing long-distance physics) multiplied by short-distance coefficients \( C_i (i = 1, 2, \ldots) \).

In Eq. (5) I have also shown the dependences on the various renormalization scales: the soft scale \( \mu_{\text{soft}} \) and the hard scale \( \mu_{\text{hard}} \) are governing the perturbative expansions of the correlators and the short-distance coefficients \( C_i \) respectively, whereas the factorization scale \( \mu_{\text{fac}} \) essentially represents the boundary between hard (i.e. of order \( M_t \)) and soft momenta. Because this boundary can in principle be chosen freely, both, correlators and the short-distance coefficients, in general depend on it (leading to new anomalous dimensions). Because the term in the second line in Eq. (5) is already of NNLO (i.e. suppressed by \( v^2 \)) we can set \( C_2 = 1 \) and ignore the factorization scale dependence of the correlator \( A_2 \).

The calculation of all terms in expression (5) proceeds in two basic steps.

1. **Calculation of the nonrelativistic correlators.** – Determination of the correlators in Eq. (5) by taking into account the interactions up to NNLO displayed in \( \mathcal{L}_{\text{NRQCD}} \).

2. **Matching calculation.** – Calculation of the constant \( C_1 \) up to \( \mathcal{O}(\alpha_s^2) \) by matching expression (5) to the fully covariant cross section at the two-loop level in the (formal) limit \( \alpha_s \ll v \ll 1 \) where an expansion in (first) \( \alpha_s \) and (then) \( v \) is feasible.

*Calculation of the nonrelativistic correlators:* In Coulomb gauge the gluon propagation is separated into a longitudinal, instantaneous (i.e. energy-independent) and a transverse, non-instantaneous (i.e. energy-dependent) propagation. The longitudinal exchange between the \( t\bar{t} \) pair is described by an instantaneous potential. (The Coulomb potential is just the LO interaction caused by the longitudinal exchange.) The transverse exchange, however, leads to a temporally retarded interaction, closely related to Lamb-shift type effects known in QED. Fortunately, because the \( t\bar{t} \) pair is produced/annihilated in a color singlet (S-wave) configuration, the energy dependence of the transverse gluon exchange leads only to NNNLO (i.e. \( \mathcal{O}(\alpha_s^3) \) relative to the effects of the LO Coulomb exchange) contributions.

We therefore ignore the energy dependence of the transverse gluons which allows us to formulate all interactions contained in the NRQCD Lagrangian in terms of instantaneous potentials. In other words, as far as the nonrelativistic correlators at NNLO in Eq. (5) are concerned, NRQCD reduces to a two-body (top-antitop) Schrödinger theory. The potential in the resulting Schrödinger equation is determined by considering \( t\bar{t} \rightarrow t\bar{t} \) one gluon exchange t-channel scattering amplitudes

5 The scales \( \mu_{\text{soft}} \) and \( \mu_{\text{hard}} \) arise from the light degrees of freedom in \( \mathcal{L}_{\text{NRQCD}} \) and are already present in full QCD, whereas \( \mu_{\text{fac}} \) is generated by “new” UV divergences in NRQCD diagrams. It is crucial to consider \( \mu_{\text{soft}} \) and \( \mu_{\text{hard}} \) as independent scales. Because both, nonrelativistic correlators and short-distance coefficients, are calculated perturbatively a residual dependence on \( \mu_{\text{soft}} \) and \( \mu_{\text{hard}} \) remains.

6 This can be seen by either using formal counting rules (see e.g. [13, 14]) or from positronium results where this phenomenon is well known. From the physical point of view the suppression comes from the fact that the transverse gluon is radiated after the \( t\bar{t} \) pair is produced and absorbed before the \( t\bar{t} \) pair is annihilated. This process is already suppressed by \( v^2 \) due to the dipole matrix element for transverse gluon radiation/absorption. If the gluon also carries energy, another (phase space) factor \( v \) arises because the gluon essentially becomes real. For the same reason no top quark self energy or crossed ladder diagrams have to be considered.

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in NRQCD. To NNLO (i.e. including potentials suppressed by at most \( \alpha_s^2 \), \( \alpha_s/M_t \) or \( 1/M_t^2 \) relative to the Coulomb potential) all potentials are already known and read (\( a \equiv C_F \alpha_s(\mu_{\text{soft}}) \), \( C_F = 4/3 \), \( C_A = N_c = 3 \), \( r \equiv |\vec{r}| \))

\[
V_{\text{BF}}(\vec{r}) = \frac{a \pi}{M_t^2} \left[ 1 + \frac{8}{3} \vec{S}_t \vec{S}_t \right] \delta(3)(\vec{r}) + \frac{a}{2 M_t^2 r} \left[ \vec{\nabla}^2 + \frac{1}{r^2} \vec{r}(\vec{r} \nabla)\nabla \right] - \frac{3a}{M_t^2 r^3} \left[ \frac{1}{3} \vec{S}_t \vec{S}_t - \frac{1}{r^2} \left( \vec{S}_t \vec{r} \right) \left( \vec{S}_t \vec{r} \right) \right] + \frac{3a}{2 M_t^2 r^3} \vec{L} \left( \vec{S}_t + \vec{S}_t \right),
\]

\[
V_{N_A}(\vec{r}) = -\frac{C_A}{2 M_t r^2} a^2,
\]

where \( \vec{S}_t \) and \( \vec{S}_t \) are the top and antitop spin operators and \( \vec{L} \) is the angular momentum operator. \( V_{\text{BF}} \) is the Breit-Fermi potential known from positronium and \( V_{N_A} \) a purely non-Abelian potential generated through non-analytic terms in one-loop NRQCD (or QCD) diagrams containing the triple gluon vertex (see e.g. [13] for an older reference). The Coulomb potential \( V_c(\vec{r}) = a/r \left[1 + \text{corrections up to } O(\alpha_s^2) \right] \) is not displayed due to lack of space. Its \( O(\alpha_s) \) and \( O(\alpha_s^2) \) corrections have been determined in [3, 6] and [14], respectively.

The nonrelativistic correlators are directly related to the Green function of the Schrödinger equation

\[
\left( -\frac{\vec{\nabla}^2}{M_t} - \frac{V_c(\vec{r})}{4 M_t^2} + V_{\text{BF}}(\vec{r}) + V_{N_A}(\vec{r}) - \tilde{E} \right) G(\vec{r}, \vec{r}', \tilde{E}) = \delta^{(3)}(\vec{r} - \vec{r}'),
\]

where \( V_{\text{BF}} \) is evaluated for the \( ^3S_1 \) configuration only. The correlators read

\[
\mathcal{A}_1 = 6 N_c \left[ \lim_{|\vec{r}|,|\vec{r}'| \to 0} G(\vec{r}, \vec{r}', \tilde{E}) \right],
\]

\[
\mathcal{A}_2 = M_t \tilde{E} \mathcal{A}_1.
\]

Relation (13) can be easily inferred by taking into account that the Green function \( G(\vec{r}, \vec{r}', \tilde{E}) \) describes the propagation of a top-antitop pair which is produced and annihilated at distances \( |\vec{r}| \) and \( |\vec{r}'| \), respectively. (A more formal derivation can be found e.g. in [3].) Relation (12), on the other hand, is obtained through the equation of motion (10). Because the exact solution of Eq. (10) seems to be an impossible task and we start with the well-known Coulomb Green function \( G_c(0) \) \( \left( V_c(0)(\vec{r}) \equiv -a/r \right) \),

\[
\left( -\frac{\vec{\nabla}^2}{M_t} - V_c(0)(\vec{r}) - \tilde{E} \right) G_c(\vec{r}, \vec{r}', \tilde{E}) = \delta^{(3)}(\vec{r} - \vec{r}'),
\]

and incorporate the higher order terms using Rayleigh-Schrödinger time-independent perturbation theory (TIPT). It should be noted that the limit \( |\vec{r}|, |\vec{r}'| \to 0 \) causes UV divergences which have to be regularized using either an explicit short-distance cutoff or dimensional regularization. The freedom in the choice of the regularization parameter is the origin of the factorization scale dependence mentioned before.

**Matching calculation:** The determination of the short-distance coefficient \( C_1 \) up to \( O(\alpha_s^2) \) can be carried out in two ways: One either calculates the constants \( b_1 \) in Eq. (1) through two-loop matching at the amplitude level for the electromagnetic vertex in full QCD and NRQCD or one matches expression (3)
directly to the two-loop cross section calculated in full QCD. Both ways of matching must be carried out for stable top quarks \((\Gamma_t = 0)\) and are performed in the (formal) limit \(\alpha_s \ll v \ll 1\) where NRQCD and full QCD are applicable in the conventional multiloop approximation.\(^7\) In our case one has to match at the two-loop level including terms up to NNLO in an expansion in \(v\). Technically the second way of matching, called “direct matching” \(^8\), is simpler because it allows for a sloppier treatment of the regularization procedure. (In fact, using the first way one has to be very careful to use exactly the same regularization for the matching calculation as in the calculation of the correlators. This is a quite tricky task if one wants to avoid solving the Schrödinger equation in \(D\) dimensions.) The disadvantage of direct matching, however, is that matching is carried out at the level of the final result which practically eliminates the possibility to use the calculated short-distance coefficient for any other process. Further, it requires that a multiloop expression for the cross section is at hand. For convenience, I use the “direct matching” method.

Some Explicit Results

This talk would not be complete without making the somewhat general discussion before more explicit. In the following I will carry out the program described above for all Abelian contributions, i.e., for those effects which also exist in QED. I will not present any technical details and refer the interested reader to \(^{20}\), where the calculations have actually been carried out.\(^8\) I start with the well-known expression for the Coulomb Green function \(G_c^{(0)}\) \(^{17}\)

\[
G_c^{(0)}(0, \vec{r}, \vec{E}) = -i \frac{M_t^2}{2\pi} \bar{v} e^{iM_t\bar{v}r} \int_0^\infty dt e^{2iM_t\bar{v}rt} \left( \frac{1 + \frac{t}{M_t}}{\bar{v}} \right)^\mu, \quad \bar{v} \equiv \left( \frac{\vec{E}}{M_t} \right)^\frac{1}{2},
\]

(14)

The NNLO corrections coming from the kinetic term \(-\nabla^4/4M_t^3\) and the Breit-Fermi potential \(V_{BF}\) are calculated via TIPT and read

\[
\left[ \delta G(0, 0, \vec{E}) \right]_{\text{NNLO, Abel}} = \int d\vec{x} G_c^{(0)}(0, \vec{x}, \vec{E}) \left[ \frac{\nabla^4}{4M_t^3} - V_{BF}(\vec{x}) \right] G_c^{(0)}(\vec{x}, 0, \vec{E}).
\]

(15)

Abelian corrections coming from the one- and two-loop contributions in the Coulomb potential do not exist because we define \(\alpha_s \equiv \alpha_{s, MS}^{(n_l=5)}\). As mentioned before, taking the limit \(|\vec{r}|, |\vec{r}'| \to 0\) in Eq. \((13)\) and the integral \((14)\) leads to UV divergences which I regularize with the short-distance cutoff \(\mu_{\text{fac}}\). This leads to the following result for the correlator \(A_1\) at NNLO

\[
\left[ A_1 \right]_{\text{Abel}} = 3 a M_t^2 \left\{ i \bar{v} - a \left[ \ln\left( -i \frac{M_t \bar{v}}{\mu_{\text{fac}}} \right) + \gamma + \Psi \left( 1 - i \frac{a}{2\bar{v}} \right) \right] \right\}^2 \]

(16)

\[
+ \frac{9 M_t^2}{2\pi} \left\{ i \left( \bar{v} + \frac{5}{8} \bar{v}^3 \right) - a \left( 1 + 2 \bar{v}^2 \right) \ln\left( -i \frac{M_t \bar{v}}{\mu_{\text{fac}}} \right) + \gamma + \Psi \left( 1 - i \frac{a (1 + \frac{11}{2} \bar{v}^2)}{2\bar{v}} \right) \right\},
\]

where \(\gamma\) is the Euler constant and \(\Psi\) the digamma function. All power divergences \(\propto \mu_{\text{fac}}/M_t\) are freely dropped and \(\mu_{\text{fac}}\) is defined in a way that expression \((14)\) takes the simple form shown above. For the correlator \(A_2\) only the LO contribution in \((14)\) is relevant and we arrive at

\[
A_2 = \bar{v}^2 \left( 3 a M_t^2 \left\{ i \bar{v} - a \left[ \ln\left( -i \frac{M_t \bar{v}}{\mu_{\text{fac}}} \right) + \gamma + \Psi \left( 1 - i \frac{a}{2\bar{v}} \right) \right] \right\} \right).
\]

(17)

\(^7\) Other matching points like \(v = 0\) are also possible, see e.g. \(^{18}\).

\(^8\) In Ref. \(^{20}\) the calculations have not been formulated in the framework of NRQCD. The results, of course, are not affected by this.
There are no non-Abelian contributions to $A_2$. The Abelian contributions to $C_1$ are calculated by expanding expression (3), expanding (first) for small $\alpha_s$ and (then) $v$, keeping terms up to order $\alpha_s^2$ and NNLO in the $v$ expansion, and demanding equality to the corresponding two-loop cross section calculated in full QCD [20, 21] for $\Gamma_t = 0$ ($v = (E/M_t)^{1/2}$),

$$R_{\text{2-loop QCD}}^{\text{NNLO}}_{\text{Abel}} = N_c Q_t^2 \left\{ \left[ \frac{3}{2} v - \frac{17}{16} v^3 \right] + C_F \frac{\alpha_s}{\pi} \left[ \frac{3 \pi^2}{4} - 6 v + \frac{\pi^2}{2} v^2 \right] \right. + \left. C_F \frac{2^{11/6} \zeta_3}{12} \right\} \left( 49 C_F^2 \frac{\pi^2}{192} + \frac{3}{2} C_{\text{Abel}} - \frac{2}{3} C_F^2 \ln v \right) \right\}, \quad (18)$$

where $C_{\text{Abel}} = C_F^2 \left[ \frac{1}{3} \pi^2 - \frac{3}{2} \ln 2 - \frac{35}{18} \right] + C_F T \left[ \frac{1}{9} \left( \frac{11}{\pi^2} - 1 \right) \right]$. The result for $C_1$ then reads

$$\left[ C_1 \right]_{\text{Abel}} = 1 - 4 C_F \frac{\alpha_s(\mu_{\text{hard}})}{\pi} + \alpha_s^2(\mu_{\text{hard}}) \left[ C_{\text{Abel}} + \frac{2}{3} C_F^2 \ln \left( \frac{M_t}{\mu_{\text{fac}}} \right) \right]. \quad (19)$$

Expression (19) does not contain any energy dependent (or even IR divergent) contributions because NRQCD and QCD have the same low energy behavior, i.e., all the energy dependence is contained in the correlators. Apart from the dependence on the hard scale $\mu_{\text{hard}}$, which remains because $[C_1]_{\text{Abel}}$ represents a truncated perturbative series [5] there is a dependence on the factorization scale $\mu_{\text{fac}}$. As can be seen in Eq. (19), for $\Gamma_t = 0$ this dependence is cancelled by a corresponding $\mu_{\text{fac}}$-dependent term in $[A_1]_{\text{Abel}}$ [4]. For $\Gamma_t \neq 0$ there is a small contribution $\propto a_t \Gamma_t \ln M_t^{\mu_{\text{fac}}}$ which is not cancelled.

This ambiguity arises from our ignorance of a consistent treatment of the NNLO finite width effect mentioned at the beginning. In Fig. 1 the relative size (in %) of the NNLO Abelian contributions $(R_{\text{NNLO}}^{\text{thr}} - R_{\text{NLO}}^{\text{thr}})/R_{\text{NLO}}^{\text{thr}}$ (in $\left( R_{\text{NNLO}}^{\text{thr}} \right)$ contains only the contribution from the Coulomb Green function, Eq. (14), in $A_1$ and the terms in $C_1$ up to $O(\alpha_s)$) is plotted in the energy range $-10 \text{ GeV} < E < 10 \text{ GeV}$ for $M_t = 175 \text{ GeV}, \alpha_s(M_Z) = 0.118$ and $\Gamma_t = 1.56 \text{ GeV}$ (solid line)/0.80 GeV (dashed line). For the scales the choices $\mu_{\text{soft}} = 30 \text{ GeV}$ and $\mu_{\text{hard}} = \mu_{\text{fac}} = M_t$ have been made and two-loop running of the strong coupling has been used. It is evident that the corrections are indeed at the level of several percent and that they are fairly insensitive to the value of $\Gamma_t$, indicating that the size of the Abelian NNLO contributions is not affected by the ignorance of a consistent treatment of the finite width effects. I also would like to note that the largest source of theoretical uncertainty in the cross section $R_{\text{NNLO}}^{\text{thr}}$ comes from the dependence on the soft scale $\mu_{\text{soft}}$ – clearly because it governs the perturbative series describing the long-distance (i.e. low energy) effects for which the convergence can be expected to be worse than for the short-distance coefficients. A thorough examination of this behavior, however, has to wait until all the NNLO relativistic corrections are calculated.

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9 At this point we set $\mu_{\text{soft}} = \mu_{\text{hard}}$ because for $\alpha_s < v < 1$ a distinction between the soft and the hard scale is irrelevant.

10 The dependence on $\mu_{\text{hard}}$ of the $O(\alpha_s)$ term in Eq. (19) is not cancelled by terms in the $O(\alpha_s^2)$ contributions because non-Abelian and massless quark corrections are not considered.

11 The invariance under changing the factorization scale $\mu_{\text{fac}}$ can be used to resum renormalization group logarithms. However, I would like to warn the reader to blindly apply renormalization group methods in the belief this would represent a resummation of “leading logarithms”. Although it is true that a naive resummation of logarithms is possible in this way, the resummed logarithms would certainly not be “leading”. This is a consequence of the fact that $Q\bar{Q}$ systems in the threshold regime are multi-scale problems. At the NNLO level, where all interactions can be treated as instantaneous, only the relative momentum of the top quarks $\sim M_t \alpha_s$ and the top quark mass $M_t$ are relevant scales. At NNNLO, however, also the energy of the top quarks $\sim M_t \alpha_s^2$ arises as a relevant scale and leads to a much more complicated structure of the anomalous dimensions. It is in fact not clear whether not even lower scales $M_t \alpha_s^n$ ($n > 2$) become relevant if effects beyond the NNLO level are considered.
Figure 1: The NNLO Abelian corrections to the cross section in percent for $\Gamma_t = 1.56$ GeV (solid line) and 0.80 GeV (dashed line). See the text for more details.

Conclusion

Due to the large top quark mass the $t\bar{t}$ system in the threshold regime offers a unique opportunity to study strong interactions in heavy-quark–antiquark pairs in the threshold regime using perturbative QCD. In this talk I have shown that NRQCD provides an ideal framework to determine the $t\bar{t}$ cross section at threshold at future lepton colliders. NRQCD, an effective field theory of QCD, allows for the calculation of the cross section (including various distributions) to in principle arbitrary precision by offering a systematic formalism which parameterizes all higher order effects from first principles QCD. In this respect NRQCD is superior to the present day potentialmodel-like approach used for analyses of $t\bar{t}$ production at threshold because NRQCD does not necessarily rely on any phenomenological input. I therefore propose that the potentialmodel-like approach should eventually be abandoned. In this talk I have given a detailed recipe how NNLO relativistic corrections to the total vector current induced cross section can be calculated using NRQCD, and I have presented explicit results for all Abelian NNLO contributions.

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