WITTEN DEFORMED EXTERIOR DERIVATIVE AND BESSEL FUNCTIONS

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Abstract

In a recent paper we investigated the internal space of Bessel functions associated with their orders. We found a formula (new) unifying Bessel functions of integer and of real orders. In this paper we study the deformed exterior derivative system $H = d_{\lambda}$ on the punctured plane as a tentative to understand the origin of the formula and find that indeed similar formula occurs. This is no coincidence as we will demonstrate that generating functions of integer order reduced Bessel functions and of real orders are respectively eigenstates of the usual exterior derivative and its deformation. As a direct consequence we rediscover the unifying formula and learn that the system linear in $d_{\lambda}$ is related to Bessel theory much as the system quadratic in $(d_{\lambda} + d_{\lambda}^*)$ is related to Morse theory.

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1 Introduction

Early studies [e.g.,[1],[2]] proposed a unifying scheme for special functions showing that some of these functions may originate from the same structure. For Bessel functions of concern here, generating functions of integer orders are representation "states" of derivative and integral operators of arbitrary orders. More precisely we have the "inner" structure

\[ \partial_{\mid m\mid} = \frac{\partial}{z\partial z} \cdots \frac{\partial}{z\partial z} \]

\[ \partial_{-\mid m\mid} = \int zdz \int zdz \cdots \int zdz \]

\[ \partial_m \Phi(z,t) = (-t)^{-m} \Phi(z,t), m \in \mathbb{Z} \]

\[ \Phi(z,t) = \sum_{n=-\infty}^{n=\infty} \phi_n(z) t^n \]

where we extend the index \( m \) to negative values by introducing the symbol \( \int dz \) to denote a "truncated" primitive i.e. in defining the integral we omit the constant of integration \( \int \frac{df}{dz} dz = f \) and where \( \phi_n(z) \) stands for the reduced Bessel function \( \phi_n(z) = \frac{J_n(z)}{z^n} \) of integer order. For the polynomials such as Hermite and Laguerre for instance, the generating functions only involve the realization of the set \( \mathbb{N} \) of positive integers, with slight modifications of the derivative operators to account for the conventions used in defining these polynomials. It is important to note that although this common structure only set up the \( z \)-dependence of the generating functions, it is the "dynamical" part of the scheme so to say. The \( t \) dependence is simply set by imposing some given desired properties. For Bessel functions for example we require a "symmetry" between positive and negative integer indices that is \( J_{-n} = (-1)^n J_n \), while for the polynomials it is the natural property of orthonormality that is invoked.

In a recent paper[3] we intuitively applied a mechanism to generate real numbers out of integers in order to unify (reduced) Bessel functions and showed by direct analytic computation that indeed, Bessel functions fit into the scheme and therefore integer orders are mapped to real orders (\( \lambda \) is real
through the formula

$$J_{n+\lambda}(z) = \exp \left[ -\frac{\lambda}{z} \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{\partial_m}{m} \right] J_n(z)$$

(2)

Let us summarize the mechanism to see how it works to convert an integer into a real. Suppose we are given an abstract state $|n\rangle$ an integer $n \in \mathbb{Z}$ and a set of raising $\Pi_m$, $m > 0$ and lowering $m < 0$ operators. Then it is easy to show, given that data, that the state $|n + \lambda\rangle$ is related to the state $|n\rangle$ through the following formula

$$|n + \lambda\rangle = \exp \left[ -\lambda \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^m \Pi_m}{m} \right] |n\rangle$$

(3)

Fourier transforming the $|n\rangle$ state as

$$|n\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{in\theta} |\theta\rangle$$

(4)

where the $\Pi_m$ operators act on the $|\theta\rangle$ state by simple multiplication by the factor $e^{im\theta}$ we get

$$|n + \lambda\rangle = \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} e^{in\theta} \exp \left[ -\lambda \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^m \Pi_m}{m} \right] |\theta\rangle$$

(5)

Compare states in (4) to states in (3). In deriving the last line use have been made of the known formula

$$\sum_{m=1}^{\infty} (-1)^m \frac{\sin m\theta}{m} = -\frac{\theta}{2}, \quad -\pi < \theta < \pi$$

(6)

Formula (2) is a new formula (to be added to the huge literature on Bessel functions) which is shown to apply to Neumann and Hankel functions as well [4]. It is to be noted that although formula (2) is shown to be true, we
didn’t know why the above mechanism should apply to Bessel functions. Our guess of the above relation was based on the following correspondence \[ \phi_n(z) \iff |n > \\
\phi_{n+\lambda}(z) \iff |n + \lambda > \]

A hint to this correspondence came from the fact that we indeed have a set of raising and lowering operators \( \Pi_m = (-1)^m \partial_m \) \( m \in \mathbb{N} \) \[ \text{and for negative } m \text{'s we just have to replace derivative operators by integral operators defined in } \frac{1}{\pi} \text{. That this correspondence works is quite intriguing and therefore further investigations of it are needed. In this paper we answer the point. In section 2 we will study the deformed exterior derivative on the punctured plane and will realize that the converting (or deforming) mechanism is closely tied to the deformation of the exterior derivative. In section 3 we will demonstrate directly that real order reduced Bessel functions are the deformed versions of integer order reduced Bessel functions in the same fashion as } d_\lambda \text{ is the deformed version of the exterior derivative } d. \text{ As a consequence formula } 3 \text{ will show up more elegantly.}

## 2 Deformed exterior derivative on \( \mathbb{R}^2/(0) \)

Let \( M \) be a Riemannian manifold of dimension \( n \). Let \( V_p, p = 0, 1, \ldots, n \) be the space of \( p \)-forms. Let \( d \) and \( d^* \) be the usual exterior derivative which define the De Rham cohomology of \( M \) and its adjoint. Let \( V \) be a smooth function \( V : M \to \mathbb{R} \) or \( \mathbb{C} \) (called prepotential in the language of topological quantum field theories) and \( \lambda \) a real number. Define

\[
d_\lambda[V] = e^{-\lambda V} d e^{\lambda V}
\]

Evidently we have \( d_\lambda^2 = d^*_\lambda = 0 \). E.Witten\(^7\) had shown that \( V \) plays the role of a Morse function and his consideration of the system

\[
H_\lambda = d_\lambda d^*_\lambda + d^*_\lambda d_\lambda
\]

4
had led to a new proof of Morse inequalities. Let us note at this point that there exists another version of $d$-deformation which is related to the fixed point theorems for Killing vector fields

$$d_s = d + s iK$$

where $s$ is an arbitrary number and where $K$ is a killing vector field—the infinitesimal generator of an isometry of $M$. In this context $K$ is regarded as an operator $iK$ on differential forms acting by interior multiplication and hence maps a $p$-form into a $(p - 1)$-form. Since we are interested in functions (0-form) such a deformation is not relevant as $d_s$ coincide with $d$ on the space of functions. In this section and the subsequent section we investigate the simpler system

$$H = d_\lambda$$

and show that it gives informations on the index structure of Bessel functions. Let us note at once that the above system is topological in the sense that $d_\lambda$, like $d$ or $d_s$, can be defined purely in terms of differential topology without choosing a metric in $M$. Now to proceed we need to know the appropriate form of $V$. The system in 9 has also been investigated, in another context, by Baulieu et al. to get informations on topological invariants. Their analysis of topological quantum mechanics on the punctured plane $R^2/(0)$ had selected the prepotential $V = k\theta$ which we later generalized. We have shown that the most general prepotential compatible with the topology of the punctured plane (first homotopy group $\sim Z$) has necessary the form.

$$V(\theta) = k\theta + \phi(\theta) \quad (10)$$

where $\theta$ is the polar angle on the plane, $k$ a constant and $\phi(\theta)$ any function but periodic, (recall that the polar angle $\theta$ is not a periodic function). It is that form $10$ that we plug into $d_\lambda$. On the restricted space of functions which depend only on the angle, the exterior derivative simplifies to $d = d\theta \partial_\theta$ (there is no $r$ dependance on which $d$ acts). Inserting the specific form of the prepotential $V$ into $9$ and rewriting the twisted operator as $d_\lambda = d\theta \partial_\theta^\lambda$ we find

$$\partial_\theta^\lambda = e^{-\lambda \phi} \partial_\theta e^{\lambda \phi} \partial_\theta + \lambda k = \partial_\theta + \lambda k + \lambda \partial_\theta \phi$$
Fourier transforming the periodic function $\phi(\theta)$

$$\partial_\theta \phi(\theta) = i \sum_{m \in \mathbb{Z}/0} \rho_m e^{-im\theta}$$

we get

$$\partial_\theta^\lambda = \partial_\theta + i\lambda \sum_{m \in \mathbb{Z}} \rho_m e^{-im\theta} \quad (11)$$

where we inserted the constant $k = i\rho_0$ into the sum. In the punctured plane the operator $\partial_\theta$ and $\partial_\theta^\lambda$ have the natural interpretation respectively of the winding number operator and the effective or perturbed winding number operator. We thus write them as $W = -i\partial_\theta$ and $W_\lambda = -i\partial_\theta^\lambda$. We also introduce the operator $\Pi_m = e^{im\theta}$ with evident action on the basis $| n >$ defined in 4. For the operator $W$ and $\Pi_m$ we have $W | n > = n | n >$ and $\Pi_m | n > = | n + m >$. In this new basis (11) takes the form

$$W_\lambda = W + \lambda \sum_{m \in \mathbb{Z}} \rho_m \Pi_m$$

This is an example of a very simple topological quantum mechanical system where $W_\lambda$ is the perturbed Hamiltonian, $\Pi_m$ a set of operators responsible for the interactions, $\lambda \rho_m$ a set of coupling constants and $W$ is the unperturbed Hamiltonian. The eigenstates of the effective winding number which we denote $| n, \lambda, \rho >$ are shown to be related to the unperturbed one, through the formula (11)

$$| n, \lambda, \rho >= \exp \left[ -\lambda \sum_{m \in \mathbb{Z}/(0)} \frac{\rho_m \Pi_m}{m} \right] | n > \quad (12)$$

Hermiticity of $W_\lambda$ restricts the real “spectral” function $\rho$ to be symmetric $\rho_m = \rho_{-m}$. Comparing with the previous result 3 we see that our application of the formalism to Bessel functions requires the simple choice of the function $\rho_m = (-1)^m$.

3 Relation of $\phi_{n+\lambda}$ to $d_\lambda$

To show the relation, first write the generating functions of integer and of real orders
\[
\Phi(z, t) = \sum_{n=-\infty}^{\infty} \phi_n(z) t^n
\]

\[
t^{-\lambda} \Phi(z, t) = \sum_{n=-\infty}^{\infty} \phi_{n+\lambda}(z) t^n
\]  

We have learned in the particular case of section 2 that eigenstates of \( d_{\lambda} \) (\( W_{\lambda} \)) are deformed versions of the eigenstates of \( d \) (\( W \)) and that the deformation consists in converting the index \( n \) into \( n + \lambda \). We therefore have to look for eigenstates of \( d \) and of \( d_{\lambda} \). The generating function \( \Phi(z, t) \) is an eigenstate of the exterior derivative by use of the recursion formula (fixed \( t \))

\[
d \Phi(z, t) = \left(-\frac{t}{z}\right)^{-1} \Phi(z, t) \, dz
\]

For the eigenstate of \( d_{\lambda} \), the function of interest to look at is \( e^{-\lambda V} \Phi(z, t) \). We will show that, with an appropriate \( V \), it is indeed an eigenstate of the deformed exterior derivative and in the same time generating function of real order Bessel functions. The judicious choice of the operator \( V \) so as to identify this function with the generating function for real orders (remember that the same crucial point of which \( V \) to choose arose in the last section) can be shown to be ¹

\[
V = \sum_{m \in \mathbb{Z}/0} \frac{\partial_m}{m}
\]

In fact we have

\[
e^{-\lambda V} \Phi = \exp \left[ -\lambda \sum_{m \in \mathbb{Z}/0} \frac{\partial_m}{m} \right] \Phi
\]

¹In choosing a (differential) operator for \( V \) instead of a simple function as in B we have implicitly generalized the deformed exterior derivative on flat space. This is enough for our purpose. We do not however, know, the expression of the generalized \( d_{\lambda} \) in the case of a general manifold \( M \). Such expression should be defined so as to be covariant and not to depend on the metric on \( M \) like \( d = dz D_z = dz \partial_z \).
This is the generating function of real order Bessel functions. To come to the second line we applied the recursion formula \( \Phi \) and from the second line to the last we put \( t = e^{i\theta} \) and made use of \( \Phi \). Then acting by \( d_{\lambda} \) on \( e^{-\lambda V} \Phi(z, t) \) we find

\[
d_{\lambda}(e^{-\lambda V} \Phi) = e^{-\lambda V} d\Phi = (-t)^{-1} dz \ e^{-\lambda V}(z \ \Phi(z, t)) = \Gamma(z, t, \lambda) \ dz \ e^{-\lambda V} \Phi
\]  

(15)

The function \( \Gamma(z, t, \lambda) \) is a more involved expression which we will not work out as this is not necessary. Inspection of the third term in (15) shows that \( e^{-\lambda V}(z \ \Phi) \sim \Phi \sim e^{-\lambda V} \Phi \) where the last proportionality comes from the result in (14).

Thus using the fact that the generating function of integer orders \( \Phi(z, t) \) is an eigenstate of \( d \), we show that the function \( e^{-\lambda V} \Phi \) is indeed an eigenstate of \( d_{\lambda} \) and generating function of real orders.

Expanding both sides of (14) as in (13) the above relationship extends to Bessel functions themselves as the operator \( V \) acts only on the \( z \) variable. Hence we recover the unifying formula

\[
\phi_{n+\lambda}(z) = \exp \left[ -\lambda \sum_{m \in \mathbb{Z}/0} \frac{(-1)^m}{m} \partial_m \right] \phi_{n}(z)
\]

The method outlined in this section has the advantage of giving a new check to the unifying formula, in addition it sheds light on the inner structure of Bessel functions showing that the modern concept of deformed or twisted exterior derivative, first introduced by Witten (which has been at the origin of the launching of topological field theories) is already encoded in the index structure of Bessel functions.

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