Non-degenerate coupling forms on twistor bundles

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Abstract

This work presents new constructions of non-degenerate coupling forms on twistor bundles over homogeneous spaces. It also shows that such forms exist on twistor bundles over even dimensional Grassmanians of maximal rank.

1 Introduction

In [4] and [13] it was shown that symplectic geometry can be used to obtain equations of motion of a classical particle in the presence of a Yang-Mills field, for any gauge group. This procedure uses the connection form as defined on the principal bundle to introduce a symplectic structure on certain associated bundles and is automatically gauge invariant. The constructed symplectic structure is used in a way that any “free Hamiltonian” is considered as a function on the associated bundle. The construction of this fiberwise symplectic form on the associated bundle

\[ F \to P \times_G F \to B \]

due to Sternberg, Weinstein and Lerman [13], [14], [8] is described as follows.

Let \( G \to P \to B \) be a principal bundle with a connection. Let \( \theta \) and \( \Theta \) be the connection one-form and the curvature form of the connection, respectively. Both forms have values in the Lie algebra \( \mathfrak{g} \) of the group \( G \).
Denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by $\langle \cdot, \cdot \rangle$. By definition, a vector $u \in \mathfrak{g}^*$ is \textit{fat}, if the two–form

$$(X,Y) \rightarrow \langle \Theta(X,Y), u \rangle$$

is non-degenerate for all \textit{horizontal} vectors $X,Y$. Note that if a connection admits at least one fat vector then it admits the whole coadjoint orbit of fat vectors.

Let $(M, \omega)$ be a closed symplectic manifold with a Hamiltonian action of a Lie group $G$ and the moment map $\Psi : M \rightarrow \mathfrak{g}^*$. Consider the associated Hamiltonian bundle

$$(M, \omega) \rightarrow E := P \times_G M \rightarrow B.$$ 

Sternberg \cite{St} constructed a certain closed two–form $\Omega \in \Omega^2(E)$ associated with the connection $\theta$. It is called the \textit{coupling form} and pulls back to the symplectic form on each fiber and it is degenerate in general. However, if the image of the moment map consists of fat vectors then the coupling form is non-degenerate, hence symplectic. This was observed by Weinstein in \cite{We}(Theorem 3.2) where he used this idea to give a new construction of symplectic manifolds. The coupling form found by Sternberg can be described explicitly as follows. Consider the product $P \times M$ and choose a $\mathfrak{g}^*$-valued function $\Phi : M \rightarrow \mathfrak{g}^*$ (in the sequel it will be the moment map of the symplectic $G$-action). Define a 1-form $\langle \theta, \Phi \rangle$ on $P \times M$ and consider the pullback $p^*\omega$ of $\omega$ on $P \times M$ with respect to the natural projection $p : P \times M \rightarrow M$. Let $\Pi : P \times M \rightarrow P \times_G M = P \times M/G$ be the natural projection. Then the Sternberg coupling form $\Omega$ is determined by the formula

$$d \langle \theta, \Phi \rangle + p^*\omega = \Pi^*\Omega.$$ 

Also, if $(B, \omega_B)$ is symplectic, one can consider the closed 2-form $\Omega + \pi^*\omega_B$ and the whole family

$$\omega_\varepsilon = \varepsilon\Omega + \pi^*\omega_B.$$ 

The latter form is called the \textit{weak coupling form}. Using it, one can describe the effect of the gauge field on the system when the gauge forces are small in comparison with the other forces on the system (see \cite{We}, Section 1.5 for more details). However, in this paper we are interested in the case when the coupling form is itself non-degenerate, and \textit{we don't assume that the base manifold $B$ is symplectic}. In the sequel, the bundles with a non-degenerate coupling form will be called \textit{symplectically fat}. Let us state the result of Sternberg and Weinstein precisely.
Theorem 1 (Sternberg-Weinstein). Let $(M, \omega)$ be a a symplectic manifold with a Hamiltonian action of a Lie group $G$ and the moment map $\mu : M \to g^*$. Let $G \to P \to B$ be a principal bundle. If there exists a connection in the principal bundle $P$ such that all vectors in $\mu(M) \subset g^*$ are fat, then the coupling form on the total space of the associated bundle

$$M \to P \times_G M \to B$$

is symplectic.

The coupling form is functorial. Let $\pi : X \to B$ be a symplectic fiber bundle. Consider the pull-back

$$
\begin{align*}
X_1 \xrightarrow{\sigma} X \\
\pi_1 \downarrow \quad \quad \pi \downarrow \\
B_1 \xrightarrow{f} B
\end{align*}
$$

It is noted in [4], that the coupling form pulls back to the coupling form. If the pullback is non-degenerate, it can be used in physics to adjoin “internal variables” to a classical dynamical system [15], [9], [5].

On the other hand, the functoriality of the coupling form is important also inside symplectic geometry. More precisely, if $\Omega$ is the coupling form for a Hamiltonian connection in a bundle

$$(M^{2n}, \omega) \to E \xrightarrow{\pi} B$$

then the fibre integrals $\mu_k := \pi_1[\Omega^{n+k}] \in H^{2k}(B \text{Ham}(M, \omega))$ define Hamiltonian characteristic classes. Thus, if the coupling form is nondegenerate then the fibre integral of the top power is a nonzero top cohomology class of the base. If the base is a sphere, then the corresponding fibre integral in the universal bundle is an indecomposable class. This is the first step to understanding the ring structure of $H^*(B \text{Ham}(M, \omega))$. Some results in this direction are contained in [6]. Fiber bundles with non-degenerate coupling forms yield families of symplectic manifolds with various prescribed properties serving as important testing examples [3], [12].

Although the method of fat bundles seems interesting and useful, both, in symplectic geometry and in mathematical physics, it is extremely difficult to even find examples satisfying the fatness condition, provided that one does not assume the symplecticness of the base. Surprisingly, the only known examples of it are the following classes of bundles (see [6]).
1. Bundles of the form

\[ \frac{H}{K} \to \frac{G}{K} = \frac{G \times_H (H/K)}{G} \to \frac{G}{H} \]

where \( K = Z_G(T) \) for some torus in \( G \);

2. twistor bundles over even-dimensional Riemannian manifolds \( M^{2n} \) of pinched curvature with sufficiently small pinching

\[ SO(2n)/U(n) \to T(M) \to M; \]

3. locally homogeneous complex \( \Gamma \setminus \frac{G}{V} \) manifolds fibered over locally symmetric Riemannian manifolds as follows:

\[ \frac{K}{V} \to \frac{\Gamma \setminus G}{V} \to \frac{\Gamma \setminus G}{K}, \]

where \( G \) is a semisimple Lie group of non-compact type, \( \Gamma \) is a uniform lattice in \( G \), \( K \) a maximal compact subgroup in \( G \), and \( V = Z_G(T) \) where \( T \) is a compact torus in \( G \).

Note that if the Riemannian metric on \( M^{2n} \) has pinched positive curvature, than \( M^{2n} \) is diffeomorphic to a sphere.

We see that basically everything in sight is homogeneous. Thus, it is tempting to find some other classes of fat bundles, as well as to understand the reason why they are so rare from mathematical as well as physical point of view. In this paper we approach the problem restricting ourselves to a more general but still tame case of \( G \)-structures over homogeneous spaces \( K/H \) of semisimple Lie groups. In particular:

1. we find sufficient conditions ensuring that a \( G \)-structure

\[ G \to P \to K/H \]

over compact reductive homogeneous space \( K/H \) admits a symplectically fat associated bundle

\[ \frac{G}{G_\xi} \to P \times_{G_\xi} (\frac{G}{G_\xi}) \to K/H \]

with coadjoint orbits \( \frac{G}{G_\xi} \) as fibers (Theorem 6 and Corollary 1) in terms of the isotropy representation;
2. Theorem 6 yields conditions on the isotropy representation ensuring that the twistor bundle over even-dimensional $K/H$ of maximal rank is symplectic (Theorem 7);

3. Theorem 7 yields also new examples of twistor bundles over homogeneous non-symplectic bases of exceptional Lie groups (Example 1 and Example 2):

$$F_4/\text{SO}(9), \ G_2/\text{SU}(3).$$

Using these results we prove the main theorem of this paper.

**Theorem 2.** *The twistor bundles over even dimensional Grassmannians of maximal rank*

$$SO(2n + 2m)/\text{SO}(2n) \times \text{SO}(2m), \ m, n \neq 1$$

$$SO(2(n + m) + 1)/\text{SO}(2n) \times \text{SO}(2m + 1), n \neq 1$$

$$\text{Sp}(n + m)/\text{Sp}(n) \times \text{Sp}(m)$$

$$\text{U}(m + n)/\text{U}(m) \times \text{U}(n)$$

*are symplectically fat.*

Symplecticness of twistor bundles may be of independent interest, because they generalize the approach of Penrose [11] which enables one to construct particular Einstein metrics on the base (see [1], Chapter 13). However, we want to stress that in this paper we don’t propose or investigate any new physical models related to the coupling construction. Rather, we try to understand its natural limits concentrating on the existence problem. The constructed classes of symplectically fat fiber bundles are not covered by the previously known results. In what follows we use the theory of Lie groups and Lie algebras closely following [2] and [10] without further explanations.

## 2 Lerman’s theorem

We need to introduce some notation which will be used throughout this work. We denote by $\mathfrak{g}$ the Lie algebra of a semisimple Lie group $G$. The symbol $\mathfrak{g}^c$ denotes the complexification. Let $\mathfrak{t}$ be a maximal abelian subalgebra in $\mathfrak{h}$. Then $\mathfrak{t}^c$ is a Cartan subalgebra in $\mathfrak{g}^c$. We denote by $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{t}^c)$ the root
system of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$. Under these choices the root system for $\mathfrak{h}^c$ is a subsystem of $\Delta$. Denote this subsystem as $\Delta(\mathfrak{h})$.

If the Killing form $B$ is nondegenerate on $\mathfrak{h}$ then the subspace

$$m := \{ X \in \mathfrak{g} \mid B(X, Y) = 0, \text{ for all } Y \in \mathfrak{h} \}$$

defines a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus m.$$ 

The decomposition is $\text{ad}_H$-invariant and the restriction of the Killing form to $m$ is nondegenerate (see Theorem 3.5 in Section X of [7]). The decomposition complexifies to $\mathfrak{g}^c = \mathfrak{h}^c \oplus \mathfrak{m}^c$. Thus, we have root decompositions:

$$\mathfrak{g}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

$$\mathfrak{h}^c = \mathfrak{t}^c + \sum_{\alpha \in \Delta(\mathfrak{h})} \mathfrak{g}^\alpha,$$

$$\mathfrak{m}^c = \sum_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} \mathfrak{g}^\alpha.$$ 

Since $G$ is semisimple, the Killing form $B$ defines an isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ between the Lie algebra of $G$ and its dual. If the Killing form is nondegenerate on $\mathfrak{h}$, the composition

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}^* \rightarrow \mathfrak{h}^*$$

is an $\text{Ad}_H$-equivariant isomorphism. Let us denote this isomorphism by $u \mapsto X_u$. Let $C \subset \mathfrak{t}$ be the Weyl chamber and let $C_\alpha$ denote its wall determined by the root $\alpha$ (i.e the set of vectors in $C$ annihilated by $\alpha$).

**Theorem 3.** Let $G$ be a semisimple Lie group, and $H \subset G$ a compact subgroup of maximal rank. Suppose that the Killing form $B$ of $G$ is nondegenerate on the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ of the subgroup $H$. The following conditions are equivalent

1. A vector $u \in \mathfrak{h}^*$ is fat with respect to the the canonical invariant connection in the principal bundle

$$H \rightarrow G \rightarrow G/H.$$
2. The vector $X_u$ does not belong to the set

$$Ad_H(\cup_{\alpha \in \Delta \backslash \Delta(b)} C_{\alpha}).$$

3. The isotropy subgroup $V \subset H$ of $u \in \mathfrak{h}^*$ with respect to the coadjoint action is the centralizer of a torus in $G$.

This theorem is a generalization of the theorem of Lerman [8], and is proved in [6]. Note that the cited result yields conditions on the fatness of vector $u$, and therefore, on the whole coadjoint orbit of this vector.

3 Symplecticness of bundles associated with $G$-structures over homogeneous spaces

Here we use the notation, terminology, and results formulated as Theorems 4 and 5 below, from [7].

Let $M$ be a smooth manifold of dimension $n$, and let

$$G \to P \to M$$

be a $G$-structure, that is, a reduction of the frame bundle $L(M) \to M$ to a Lie group $G$. Any diffeomorphism $f \in Diff(M)$ acts on $L(M)$ by the formula

$$f(u) := (df_x X_1, ... , df_x X_n)$$

for any frame $u = (X_1, ..., X_n), X_i \in T_x M$ over a point $x \in M$. By definition, $f$ is called an automorphism of the given $G$-structure, if this action commutes with the action of $G$.

Let $M = K/H$ be a homogeneous space with a connected Lie group $K$ and the exact isotropy representation $\lambda$. Assume that $M$ is equipped with a $K$-invariant $G$-structure. The latter means that any left translation $\tau(k) : K/H \to K/H, \tau(k)(aH) = kaH$ lifts to an automorphism. In what follows we assume that $K/H$ is reductive, that is, on the Lie algebra level there is a decomposition

$$\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}, [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

We say that a connection $\theta$ in (1) is $K$-invariant, if for any $k \in K$ the lift of $\tau(k)$ preserves it.
**Theorem 4.** Let there be given a $K$-invariant $G$-structure over a reductive homogeneous space $M = K/H$. There is a one-to-one correspondence between the $K$-invariant connections in it, and $\text{Ad}(H)$-invariant linear maps

$$\Lambda_m : m \rightarrow g.$$ 

Recall that a connection in the given $K$-invariant $G$-structure is called canonical if it corresponds to the map $\Lambda_m = 0$. Since $G$ acts on frames, one can identify each frame $u$ with a map $u : \mathbb{R}^n \rightarrow T_oM$, where $o \in M$ is projection of $u$ on the base of the given principal bundle. One can define the Lie group homomorphism by the formula

$$\lambda : H \rightarrow G, \lambda(h) = u_o \circ dh \circ u_0^{-1}$$

where $u_o$ defines a fixed point in the fiber over $o$. We denote the differential of this homomorphism (which is the Lie algebra homomorphism) by the same symbol.

**Theorem 5.** The curvature form of the canonical connection in $P$ is given by the formula

$$\Omega(X, Y) = -\lambda([X,Y]_h), X, Y \in m.$$ 

Consider the following notation. Let $\lambda : \mathfrak{h} \rightarrow \mathfrak{g}$ be the monomorphism of Lie algebras. Denote by $\lambda^* : \mathfrak{g}^* \rightarrow \mathfrak{h}^*$ the dual map defined by $\lambda^*(f)(H) = f(\lambda(H))$. Let $B_g$ and $B_h$ be some non-degenerate bilinear invariant forms on $\mathfrak{g}$ and $\mathfrak{h}$ (these may be the Killing forms, for example, if the corresponding Lie algebras are semisimple). We use them to get the natural pairings between $\mathfrak{g}^*$ and $\mathfrak{g}$; $\mathfrak{h}^*$ and $\mathfrak{h}$. Thus

$$B_g(f, X) = \langle f, X \rangle, \quad B_h(g, Y) = \langle g, Y \rangle$$

for $f \in \mathfrak{g}^*, X \in \mathfrak{g}, g \in \mathfrak{h}^*, Y \in \mathfrak{h}$. If $\lambda^*(f) \in \mathfrak{h}^*$, then the $B_h$-dual of $\lambda^*(f)$ will be denoted by $X_f^\lambda$, that is

$$B_h(X_f^\lambda, Y) := \langle \lambda^*(f), Y \rangle.$$ 

Denote by $B_t$ the Killing form of $t$, and by $B_h$ the restriction of $B_t$ on $\mathfrak{h}$. Consider now the additional assumption that we are given a homogeneous space $K/H$ which is naturally reductive and $K$ is a semisimple Lie group. The
latter means that \( \mathfrak{k} \) admits a decomposition \( \mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m} \) such that \([\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}, \mathfrak{h} \) and \( \mathfrak{m} \) are orthogonal with respect to \( B_\mathfrak{h} \), and that \( B_\mathfrak{h} \) is non-degenerate (see Theorem 3.5, Chapter X of [7]). Note that for compact Lie groups, we may (and will) always assume the natural reductivity of \( K/H \) without further notice.

**Theorem 6.** Let \( K \) be a semisimple Lie group, and \( H \subset K \) a compact subgroup of maximal rank. Suppose that the Killing form \( K \) is non-degenerate on the Lie algebra \( \mathfrak{h} \subset \mathfrak{g} \) of the subgroup \( H \). Let there be given a \( G \)-structure over \( K/H \). Assume that the isotropy representation \( \lambda \) is exact. Then \( v \in \mathfrak{g}^* \) is fat with respect to the canonical connection, if the 2-form

\[
B_\mathfrak{t}(X^\lambda_v, [X, Y]), \; X, Y \in \mathfrak{m}
\]

is non-degenerate.

**Proof.** By definition \( v \in \mathfrak{g}^* \) is fat with respect to the canonical connection, if and only if the 2-form

\[
\langle v, \Omega(X, Y) \rangle = \langle v, \lambda([X, Y]) \rangle
\]

is non-degenerate. Therefore

\[
\langle v, \lambda([X, Y]) \rangle = \langle \lambda^* v, [X, Y] \rangle = \\
B_\mathfrak{h}(X^\lambda_v, [X, Y]) = B_\mathfrak{t}(X^\lambda_v, [X, Y]) = B_\mathfrak{t}(X^\lambda_v, [X, Y]).
\]

Note that the last equality is proved by repeating the corresponding proof of Theorem \( \text{3} \) in the previous section (as in [6]), applied to \( \mathfrak{t} \).

\( \square \)

**Corollary 1.** Under the assumptions of the previous theorem, \( v \in \mathfrak{g}^* \) is fat, if

\[
X^\lambda_v \not\in \text{Ad}(H)(\cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha).
\]

**Proof.** Again, the argument follows by repeating verbatim the proof of Theorem \( \text{3} \) from the previous section (as in [6]), since one has to prove the fatness of the vector \( X^\lambda_v \in \mathfrak{h} \subset \mathfrak{t} \).

\( \square \)
4 Symplectic fatness of twistor bundles over homogeneous spaces

A twistor bundle over an even dimensional Riemannian manifold \((M, g)\) is the bundle of complex structures in the tangent spaces \(T_p M\). More precisely, it is a bundle associated with the orthonormal frame bundle to \(M\) with fibre \(SO(2n)/U(n)\). It generalizes a construction of Penrose in dimension four [1], [11]. We see that if the base \(M = K/H\) is homogeneous, and \(g\) is \(K\)-invariant, the corresponding twistor bundle is the \(SO(2n)\)-structure over \(K/H\), and one can apply the results of the previous section.

In what follows we will always assume that \(K\) is a semisimple Lie group, \(H \subset K\) is a compact subgroup of maximal rank, and the Killing form of \(K\) is non-degenerate on the Lie algebra \(\mathfrak{h} \subset \mathfrak{g}\). Denote by \(J\) the matrix in \(\mathfrak{so}(2n)\) consisting of \(n\) blocks of the form

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

**Theorem 7.** Let there be given the twistor bundle

\[
SO(2n)/U(n) \rightarrow \mathcal{T}(K/H) \rightarrow K/H
\]

over the reductive homogeneous space \(K/H\). Let \(\lambda : \mathfrak{h} \rightarrow \mathfrak{g} = \mathfrak{so}(2n)\) be the isotropy representation. Let \(J^* \in \mathfrak{so}(2n)^*\) denote the dual to \(J\) with respect to the Killing form \(B_{\mathfrak{g}}\) of \(\mathfrak{g}\). Assume \(X_{J^*}^\lambda \in \mathfrak{h}\) has the property

\[X_{J^*}^\lambda \not\in \text{Ad}(H)(\bigcup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})}(\mathfrak{c}_\alpha)).\]

Then the total space \(\mathcal{T}(K/H)\) admits a non-degenerate coupling form.

**Proof.** The proof follows from the equality

\[(X_{J^*}^\lambda)^* = \lambda^*(J^*).\]

The latter means that \(J^*\) is fat and, as a result, the coadjoint orbit of \(J^*\) (or an adjoint orbit of \(J\)) is also fat. Thus the fiber \(SO(2n)/U(n)\) has a fat image under the moment map.

\[\square\]

**Theorem 8.** Consider the twistor bundle over the reductive homogeneous space \(K/H\). Assume that the following assumptions hold
1. $K$ is semisimple, $H$ is compact simple, and the Killing form of $k$ restricted to $h$ is non-degenerate;

2. $K/H$ is a reductive homogeneous space of maximal rank;

3. there exists $T \in t \subset h$ in the Cartan subalgebra $t$ of $h$ and $k$ such that
   \[(ad T|_m)^2 = -id, T \notin \cup_{\alpha \in \Delta \setminus \Delta(h)} C_\alpha.\]

Then $\mathcal{T}(K/H)$ admits a non-degenerate coupling form.

**Proof.** Begin with a straightforward remark on duality: if $\lambda : V \to W$ is a monomorphism of vector spaces endowed with non-degenerate bilinear forms $B_V$ and $B_W$ such that $B_W(\lambda(u), \lambda(v)) = B_V(u, v)$ then, for the duality determined by $B_V$ and $B_W$, the following holds
   \[\text{if } \lambda(v) = w, \text{ then } \lambda^*(w^*) = v^*.\]  \(1\)

Let $B_h$ be the restriction of $B_k$ to $h$. It follows from the assumption that $B_h$ is an invariant non-degenerate form. Since $G = SO(2n)$ is a compact Lie group, the restriction $\tilde{B}_h$ of the Killing form $B_g$ of $g$ to $h := \lambda(h)$ is also non-degenerate. Because $H$ is simple, every bilinear invariant form on $h$ is equal to the Killing form (modulo constant). Define
   \[\tilde{B}(X,Y) := B_h(\lambda(X), \lambda(Y)), \ X, Y \in h.\]

Since $\lambda$ is a Lie algebra monomorphism, $\tilde{B}$ is a non-degenerate invariant bilinear form on $h$. We have
   \[\tilde{B} = s \cdot B_h, \text{ where } s \neq 0.\]

Put $B_g := \frac{1}{s} \cdot B_g$. We obtain
   \[B_g(\lambda(X), \lambda(Y)) = \frac{1}{s} B_h(\lambda(X), \lambda(Y)) = \frac{1}{s} sB_h(X, Y) = B_h(X,Y).\]

Thus we can apply observation \((1)\) to $B_h$ and $B_g$.

Let $\lambda(T) = ad T|_m = J$, where $J : m \to m$. Note that the latter follows from the fact, that for the canonical connection on $K/H$ the isotropy representation coincides with the adjoint representation restricted on $m$ (see [7],}
Chapter 10). It follows from observation (1) and our choice of $B_g$ that the dual $X^*_{J^*}$ of $\lambda^*(J^*)$ must be $T$:

$$T = X^*_{J^*}.$$ 

Therefore, $J^* \in \mathfrak{so}^*(2n)$ is fat and by the assumption 3, $J^2 = -id$. Notice that $J$ is skew-symmetric with respect to the Killing form $B_k$. Thus $J \in \mathfrak{so}(m)$ and represents some complex structure on the vector space $m$. Since the coadjoint orbit dual to the adjoint orbit of $J$ consists of fat vectors, the proof follows.

□

**Corollary 2.** Under the assumptions of Theorem [8] assume that there exists an inner automorphism of $K$ of the form $\text{Ad} t$, $t \in H$, $t = \exp T$ such that

$$(\text{ad} T|_m)^2 = -id, T \not\in \cup_{\alpha \in \Delta \setminus \Delta(h)} C_\alpha.$$ 

Then the twistor space over $K/H$ is symplectic.

The latter corollary yields examples of homogeneous spaces with symplectic twistor bundles over them. To describe these examples, recall that the compact real form of any semisimple complex Lie algebra $\mathfrak{g}^c$ can be written using the following formula

$$\mathfrak{g} = \sum_{\alpha \in \Delta} \mathbb{R}(i\alpha) + \sum_{\alpha \in \Delta} \mathbb{R}((X_\alpha - X_{-\alpha})) + \sum_{\alpha \in \Delta} \mathbb{R}(i(X_\alpha + X_{-\alpha}).$$

Here $\Delta$ denotes the root system for $\mathfrak{g}^c$.

This observation enables us to compute examples of bundles described in Theorem [8]. Assume that we are given a compact homogeneous space $K/H$. Denote by $\mathfrak{k}, \mathfrak{h}$ Lie algebras of $K,H$, and by $\mathfrak{k}^c, \mathfrak{h}^c$ - complexifications of these algebras. Since $\mathfrak{k}$ is of compact type, therefore a restriction of its Killing form to $\mathfrak{h}$ is non-degenerate and $K/H$ is naturally reductive. We also obtain the following decompositions

$$\mathfrak{k}^c = \mathfrak{k}^c + \sum_{\alpha \in \Delta \setminus \Delta(h)} \mathfrak{k}_\alpha + \sum_{\beta \in \Delta \setminus \Delta(h)} \mathfrak{k}_\beta$$

$$\mathfrak{h}^c = \mathfrak{k}^c + \sum_{\alpha \in \Delta(h)} \mathfrak{k}_\alpha.$$
\[ m^c = \sum_{\beta \in \Delta \setminus \Delta(h)} t_\beta. \]

Here \( \Delta \) again denotes the root system for \( t \). Assume that we can choose \( T \in t^c \) satisfying the equations
\[ \alpha(T) = iR, \; \alpha \in \Delta(h) \quad \text{and} \quad \alpha(T) = \pm i, \; \alpha \in \Delta \setminus \Delta(h), \]
where \( i \) or \(-i\) are chosen in a way to ensure that the above system of linear equations has a solution. Note that \( t_c = t + i t \), where \( t \) denotes the real form of \( t^c \) consisting of vectors \( H \in t^c \) such that \( \alpha(H) \in \mathbb{R} \) for all \( \alpha \in \Delta \). It follows that \( T \in i t = \sum_{\alpha \in \Delta} \mathbb{R}(iH_\alpha) \). Therefore, \( T \in t \). But \( (ad T|_m)^2 = -id \), because by construction it satisfies this equality on \( m^c \). Also, it is easy to see that \( T \) does not belong to any wall \( C_\alpha, \alpha \in \Delta \setminus \Delta(h) \). Finally, we see that under the adopted assumptions the twistor bundle over \( K/H \) must be symplectically fat.

**Example 1.** The twistor bundle
\[ SO(16)/U(8) \to T(F_4/SO(9)) \to F_4/SO(9), \]
over the Riemannian space \( K/H \) is symplectically fat, and therefore, admits a non-degenerate coupling form.

**Proof.** It suffice to find an adequate \( T \in t^c \). We have
\[ \Delta = \{ \pm e_i \pm e_j, \; \pm e_i, \; \frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2} \mid 1 \leq i, j \leq 4 \} \]
\[ \Delta(h) = \{ \pm e_i \pm e_j, \; \pm e_i \mid 1 \leq i, j \leq 4 \}, \]
therefore
\[ \Delta \setminus \Delta(h) = \{ \frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2} \}. \]
Take \( T \in t^c \) so that \( e_1(T) = 2i \) and \( e_2(T) = e_3(T) = e_4(T) = 0 \). Then one can easily verify that \( \alpha(T) = \pm i, \; \alpha \in \Delta \setminus \Delta(h) \) and
\[ \alpha(T) = iR, \; \alpha \in \Delta(h). \]
\[ \square \]

In the same fashion one can show that

**Example 2.** The twistor bundle
\[ SO(6)/U(3) \to T(G_2/SU(3)) \to G_2/SU(3), \]
is symplectically fat, and therefore, admits a non-degenerate coupling form.
5 Twistor bundles over Grassmannians

Now we will extend the results from previous sections to a more general setting, and prove Theorem 2.

**Theorem 9.** Consider the twistor bundle over the reductive homogeneous space $K/H$. Assume that the following assumptions hold

1. $K$ is semisimple, $H$ is compact non-abelian, and the Killing form of $\mathfrak{k}$ restricted to $\mathfrak{h}$ is non-degenerate;
2. $K/H$ is a reductive homogeneous space of maximal rank;
3. there exists $T \in \mathfrak{t} \subset \mathfrak{h}$ in some simple part of $\mathfrak{h}$ such that

$$(\text{ad} T|_\mathfrak{m})^2 = -id, \quad T \notin \cup_{\alpha \in \Delta \setminus \Delta(\mathfrak{h})} C_\alpha$$

Then $\mathcal{T}(K/H)$ admits a non-degenerate coupling form.

**Proof.** Retain the previous notation. The proof goes essentially unchanged as in Theorem 8 if we show the following implication

$$\text{if } \lambda(T) = J \text{ then } \lambda^*(J^*) = T^*.$$ 

Without loss of generality assume that $\mathfrak{h} = \mathfrak{h}_1 + \mathfrak{h}_2$ where $\mathfrak{h}_1$, $\mathfrak{h}_2$ are ideals in $\mathfrak{h}$ with trivial intersection and $\mathfrak{h}_1$ is a simple Lie algebra. Take $T \in \mathfrak{h}_1$ and assume that $\lambda(T) = J \in \mathfrak{g}$. Denote by $B_\mathfrak{k}$ and $B_\mathfrak{g}$ Killing forms of $\mathfrak{k}$ and $\mathfrak{g}$, respectively. Take $B_\mathfrak{h} := B_\mathfrak{k}|_\mathfrak{h}$. First notice that $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are $B_\mathfrak{h}$-orthogonal. Indeed take $H_1 \in \mathfrak{h}_1$, $H_2 \in \mathfrak{h}_2$. Since both spaces are ideals with trivial intersection we have $[H_1, H_2] = 0$. Moreover since $\mathfrak{h}_1$ is simple there exist $H_{11}, H_{12} \in \mathfrak{h}_1$ such that $[H_{11}, H_{12}] = H_1$. Thus

$$B_\mathfrak{h}(H_1, H_2) = B_\mathfrak{h}([H_{11}, H_{12}], H_2) = B_\mathfrak{h}(H_{11}, [H_{12}, H_2]) = B_\mathfrak{h}(H_{11}, 0) = 0$$

Denote by $\tilde{\mathfrak{h}} := \lambda(\mathfrak{h})$ and set $B_{\tilde{\mathfrak{h}}} := B_\mathfrak{g}|_{\tilde{\mathfrak{h}}}$. Since $\mathfrak{g}$ is a simple Lie algebra of compact type, $B_{\tilde{\mathfrak{h}}}$ is non-degenerate. Put $\tilde{\mathfrak{h}}_1 := \lambda(\mathfrak{h}_1)$ and $\tilde{\mathfrak{h}}_2 := \lambda(\mathfrak{h}_2)$. Because $\lambda$ is the Lie algebra monomorphism we obtain a decomposition

$$\tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_1 + \tilde{\mathfrak{h}}_2,$$
where \( \tilde{h}_1 \) are ideals in \( \tilde{h} \) with trivial intersection and \( \tilde{h}_1 \) is a simple Lie algebra.

By a similar argument \( \tilde{h}_1 \) is \( B_{\tilde{h}} \)-orthogonal to \( \tilde{h}_2 \). Let \( H = H_a + H_b \in h = h_1 + h_2 \). We have

\[
\lambda^*(J^*)(H) = \lambda^*(J^*)(H_a + H_b) = J^*(\lambda(H_a + H_b)) = B_{\tilde{h}}(J, \lambda(H_a + H_b)) =
\]

\[
B_{\tilde{h}}(\lambda(T), \lambda(H_a + H_b)) = B_{\tilde{h}}(\lambda(T), \lambda(H_a + H_b)) = B_{\tilde{h}}(\lambda(T), \lambda(H_a)).
\]

Since \( h_1 \) is a simple Lie algebra, we may - as in the proof of Theorem 1 - assume that \( B_{h}(X, Y) = B_{\tilde{h}}(\lambda(X), \lambda(Y)) \) for any \( X, Y \in h_1 \). Therefore one may continue as follows.

\[
B_{h}(T, H_a) + 0 = B_{h}(T, H_a) + B_{h}(T, H_b) = B_{h}(T, H) = T^*(H).
\]

Now we will examine twistor bundles over oriented Grassmannian homogeneous spaces

\[
SO(n)/SO(n-k) \times SO(k),
\]

\[
SU(n)/SU(n-k) \times SU(k),
\]

\[
Sp(n)/Sp(n-k) \times Sp(k).
\]

Since in this article we treat homogeneous spaces of maximal rank, and of even dimension, our attention is limited to following cases:

\[
SO(2n+1)/SO(2n+1-k) \times SO(k), \quad k \neq 2,
\]

\[
SO(2(n+m))/SO(2n) \times SO(2m), \quad n, m \neq 1,
\]

\[
Sp(n)/Sp(n-k) \times Sp(k).
\]

**Proposition 1.** The twistor bundle

\[
SO(4nm)/U(2nm) \rightarrow T(SO(2n+2m)/SO(2n) \times SO(2m))
\]

\[
\rightarrow SO(2n+2m)/SO(2n) \times SO(2m), \quad n, m \neq 1
\]

over the Riemannian space \( K/H \) is symplectically fat, and therefore, admits a non-degenerate coupling form.
Proof. We will choose an adequate $T \in \mathfrak{t}^e \cap \mathfrak{so}(2m)$. We have

$$\Delta = \{ \pm e_i \pm e_j, \ 1 \leq i,j \leq n+m \}$$

$$\Delta(\mathfrak{h}) = \{ \pm e_i \pm e_j \mid 1 \leq i,j \leq n \} \cup \{ \pm e_i \pm e_j \mid n+1 \leq i,j \leq n+m \}.$$ 

Take $T \in \mathfrak{so}(2m)$ so that

$$e_i(T) = \begin{cases} 0 & \text{for } 1 \leq i \leq n, \\ i & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

Since any root $\alpha \in \Delta \setminus \Delta(\mathfrak{h})$ is of the form $\pm e_i \pm e_j$ or $\pm e_i$, $i \leq n; j > n$, thus $\alpha(T) = \pm e_j(T) = \pm i$ and

$$\alpha(T) = i\mathbb{R}, \alpha \in \Delta(\mathfrak{h}).$$

Proposition 2. The twistor bundle

$$SO(4nm+2n)/U(2nm+n) \rightarrow \mathcal{T}(SO(2(n+m)+1)/SO(2n) \times SO(2m+1))$$

$$\rightarrow SO(2(n+m)+1)/SO(2n) \times SO(2m+1), n \neq 1,$$

over the Riemannian space $K/H$ is symplectically fat, and therefore, admits a non-degenerate coupling form.

Proof. We will choose an adequate $T \in \mathfrak{t}^e \cap \mathfrak{so}(2n)$. We have

$$\Delta = \{ \pm e_i \pm e_j, \pm e_i \mid 1 \leq i,j \leq n+m \}$$

$$\Delta(\mathfrak{h}) = \{ \pm e_i \pm e_j \mid 1 \leq i,j \leq n \} \cup \{ \pm e_i \pm e_j, \pm e_i \mid n+1 \leq i,j \leq n+m \}.$$ 

Take $T \in \mathfrak{so}(2n)$ so that

$$e_i(T) = \begin{cases} i & \text{for } 1 \leq i \leq n, \\ 0 & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

Since any root $\alpha \in \Delta \setminus \Delta(\mathfrak{h})$ is of the form $\pm e_i \pm e_j$ or $\pm e_i$, $i \leq n; j > n$, thus $\alpha(T) = \pm e_i(T) = \pm i$ and

$$\alpha(T) = i\mathbb{R}, \alpha \in \Delta(\mathfrak{h}).$$

If $m = 0$ then it is sufficient to take $e_i(T) = i$ for $1 \leq i \leq n$. \qed
Proposition 3. The twistor bundle

\[ SO(4nm)/U(2nm) \to \mathcal{T}(Sp(n+m)/Sp(n) \times Sp(m)) \]
\[ \to Sp(n+m)/Sp(n) \times Sp(m), \]

over the Riemannian space \( K/H \) is symplectically fat, and therefore, admits a non-degenerate coupling form.

Proof. We will choose an adequate \( T \in \mathfrak{t} \cap \mathfrak{sp}(m) \). We have

\[ \Delta = \{ \pm e_i \pm e_j \pm 2e_i \mid 1 \leq i, j \leq n+m \} \]
\[ \Delta(\mathfrak{h}) = \{ \pm e_i \pm e_j, \pm 2e_i \mid 1 \leq i, j \leq n \} \cup \{ \pm e_i \pm e_j, \pm 2e_i \mid n+1 \leq i, j \leq n+m \}. \]

Take \( T \in \mathfrak{sp}(m) \) so that

\[ e_i(T) = \begin{cases} 0 & \text{for } 1 \leq i \leq n, \\ i & \text{for } n+1 \leq i \leq m. \end{cases} \]

Since any root \( \alpha \in \Delta \setminus \Delta(\mathfrak{h}) \) is of the form \( \pm e_i \pm e_j \leq n; j > n \), thus

\[ \alpha(T) = \pm e_j(T) = \pm i \] and

\[ \alpha(T) = i\mathbb{R}, \alpha \in \Delta(\mathfrak{h}). \]

Now we can complete the proof of Theorem 2 applying Propositions 1, 2, 3.

The case of complex Grassmannian does not require a separate proof, because \( U(n+m)/U(m) \times U(n) \) is Kaehler, hence, symplectic, and the existence of the non-degenerate coupling form follows, for example, from [4].

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