On linear degeneracy of integrable quasilinear systems in higher dimensions

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Abstract

We investigate \((d + 1)\)-dimensional quasilinear systems which are integrable by the method of hydrodynamic reductions. In the case \(d \geq 3\) we formulate a conjecture that any such system with an irreducible dispersion relation must be linearly degenerate. We prove this conjecture in the 2-component case, providing a complete classification of multi-dimensional integrable systems in question. In particular, our results imply the non-existence of 2-component integrable systems of hydrodynamic type for \(d \geq 6\).

In the second half of the paper we discuss a numerical and analytical evidence for the impossibility of the breakdown of smooth initial data for linearly degenerate systems in \(2 + 1\) dimensions.

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1 Introduction

In this paper we discuss \((d + 1)\)-dimensional quasilinear systems of the form

\[ u_t + \sum_{i=1}^{d} A_i(u)u_{x_i} = 0, \]

where \(t, x^i\) are independent variables, \(u\) is an \(n\)-component column vector and \(A_i(u)\) are \(n \times n\) matrices. Systems of this type play an important role in continuum mechanics and physics, and there is a vast literature covering their analytical, geometric and applied aspects, see e.g. [7, 24, 36, 37, 19, 8, 42]. In this paper we assume that the dispersion relation \(\det(\mu I + \alpha^i A_i) = 0\) defines an irreducible algebraic variety (for \(n = 2\), an irreducible quadric) in the variables \(\mu, \alpha^i\). Our prime goal is the classification of integrable systems of the form (1) in any dimension.

For \(d = 1\), any two-component system of the form

\[ u_t + A(u)u_x = 0 \]

is linearizable by a hodograph transformation which interchanges the dependent and independent variables and, therefore, is automatically integrable. In particular, any such system possesses an infinity of conservation laws and commuting flows.

For \(d = 2\),

\[ u_t + A(u)u_x + B(u)u_y = 0, \]  

the situation is more subtle. Even the very concept of integrability for multi-dimensional systems of the form (1) requires an alternative approach. Such approach, based on the method of hydrodynamic reductions, was outlined in [13]. It was proposed to call a \((2 + 1)\)-dimensional system integrable if it possesses infinitely many multi-phase solutions of the form \(u = u(R^1, ..., R^N)\) where the phases \(R^i(x, y, t)\) satisfy a pair of commuting hydrodynamic type systems

\[ R^i_t = \lambda^i(R) \ R^i_x, \quad R^i_y = \mu^i(R) \ R^i_x; \]

we emphasize that the number of phases \(N\) is allowed to be arbitrary. In what follows, we will refer to the equations (3) as \(N\)-component hydrodynamic reductions. Multi-phase solutions of this type, known in gas dynamics as nonlinear interactions of planar simple waves, date back to [41, 6, 34, 20]. Their first appearance in the context of integrable systems and, in particular, the dispersionless Kadomtsev-Petviashvili hierarchy, is due to [17, 18]. The requirement of the existence of multi-phase solutions imposes strong constraints on the matrices \(A(u)\) and \(B(u)\). In the two-component case these integrability conditions were derived in [14], and are included in Sect. 4.1 of this paper. In particular, they imply that any integrable system of this kind is necessarily conservative, and possesses a dispersionless Lax pair. It was demonstrated in [30, 31] that the integrability conditions can be resolved in generalized hypergeometric functions.

Remarkably, for \(d \geq 3\) the situation simplifies dramatically: the requirement of the integrability proves to be too strong, so that very few examples survive in many dimensions. According to the definition proposed in [15], a \((d + 1)\)-dimensional system of the
form (1) is said to be integrable if it possesses infinitely many $N$-component hydrodynamic reductions parametrized by $(d - 1)N$ arbitrary functions of one variable. Here by a hydrodynamic reduction we mean a decomposition of a $(d + 1)$-dimensional system into $d$ commuting systems of the form (3). Although the derivation of the integrability conditions based on the method of hydrodynamic reductions leads to a quite complicated analysis, there exists a simple way to bypass lengthy calculations. It is based on the requirement that all traveling wave reductions of a $(d + 1)$-dimensional integrable system to $(2 + 1)$-dimensional systems should be themselves integrable. Since the integrability conditions in $2 + 1$ dimensions are explicitly known, this provides a set of strong constraints which are therefore necessary for the integrability. In fact, in our case they turn out to be sufficient. Our main result is the following

**Theorem 1.** There exist no two-component integrable systems of the form (1) in dimensions $6 + 1$ and higher. In $5 + 1$ dimensions, there exists a unique integrable system. Up to linear transformations of the independent variables, it can be brought to the form

$$v_t = w_x + wv_r - vw_r, \quad w_z = v_y + vw_s - vw_s.$$  

(4)

In $4 + 1$ and $3 + 1$ dimensions, any two-component integrable system can be obtained as a traveling wave reduction of the system (4).

The proof of this result is given in Sect. 4.2. The system (4) first appeared in [15], see Sect. 2 for more details. An important property of this example is its linear degeneracy. Recall that a matrix $A(u)$ is said to be linearly degenerate if its eigenvalues (assumed real and distinct) are constant in the direction of the corresponding eigenvectors. Explicitly, $L_r \lambda^i = 0$, no summation, where $L_r$ is the Lie derivative in the direction of the eigenvector $r^i$, and $A r^i = \lambda^i r^i$. A system of the form (1) is said to be linearly degenerate (totally linearly degenerate in the terminology of [24]), if any matrix $A(u) = \alpha^i A_i(u)$ is linearly degenerate for any values of the constants $\alpha^i$. Theorem 1 and other existing examples support the following conjecture.

**Conjecture.** For $d \geq 3$, any $n$-component $(d + 1)$-dimensional integrable system of the form (1) with an irreducible dispersion relation must be linearly degenerate.

According to the conjecture of Majda [24], p. 89, linearly degenerate systems are quite exceptional from the point of view of the global solvability of the Cauchy problem: the shock wave formation, which is a standard scenario for genuinely nonlinear systems, never happens for smooth initial data. Although this fact is well-established in $1 + 1$ dimensions [33, 36, 28, 37], the multi-dimensional situation is still out of reach. We refer to the recent work [25, 26, 27] which utilizes a novel version of the inverse scattering transform to investigate the breakdown phenomena for some $(2 + 1)$-dimensional dispersionless integrable models. In Sect. 3 we provide some further analytical and numerical evidence in support of Majda’s conjecture. As an example, we choose a particular $(2+1)$-dimensional traveling wave reduction of the system (4), namely,

$$v_t + w_x = 0, \quad w_t + w_y + vw_x - vw_x = 0,$$

(5)

see Sect. 3.1 for the discussion of this example. In a somewhat different form, it appeared in [39]. In Sections 3.2–3.3, applying the method of hydrodynamic reductions and other
simple techniques, we construct some global exact solutions of the system \([5]\). In Sect. 3.4 we summarize the results of numerical simulations based on the Friedrichs-symmetrized representation of the system \([5]\). These results clearly indicate that hump-like initial data tend to spread out, and do not break down in finite time.

2 Examples of multi-dimensional integrable systems

In this section we list some examples of integrable \((d+1)\)-dimensional quasilinear systems of the form \([1]\) for \(d \geq 3\). All such systems turn out to be linearly degenerate, namely, an arbitrary matrix of the linear pencil

\[
A(u) = \alpha^t A_t(u),
\]

where \(\alpha^t\) are arbitrary constants, must be linearly degenerate (totally linearly degenerate systems in the terminology of \([24]\)). We recall that there exists a useful criterion of linear degeneracy which can be summarized as follows. Given a matrix \(A(u)\), let us introduce its characteristic polynomial,

\[
det(\lambda E - A(u)) = \lambda^n + f_1(u)\lambda^{n-1} + f_2(u)\lambda^{n-2} + \ldots + f_n(u).
\]

The condition of linear degeneracy is given by \([12]\),

\[
\nabla f_1 A^{n-1} + \nabla f_2 A^{n-2} + \ldots + \nabla f_n = 0,
\]

where \(\nabla\) is the operator of the gradient, \(\nabla f = (\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_n})\), and \(A^k\) denotes \(k\)-th power of the matrix \(A\). Our first example plays a key role in the classification of two-component multi-dimensional quasilinear systems.

**Example 1.** The six-dimensional two-component integrable system \([4]\),

\[
\begin{align*}
v_t &= w_x + vw_t - vw_r, \\
w_z &= v_y + vw_s - wv_s,
\end{align*}
\]

first appeared in \([15]\). It can also be obtained as the condition of commutativity of two parameter-dependent vector fields,

\[
\partial_z - v\partial_x - \lambda\partial_t + \lambda v\partial_r, \\
\partial_y - w\partial_s - \lambda\partial_t + \lambda w\partial_r \}
\]

It was demonstrated in \([15]\) that system \([4]\) possesses infinitely many \(n\)-component hydrodynamic reductions parametrized by \(4n\) arbitrary functions of one variable. This example is a result of the following construction. In \([14]\), we proposed a characterization of two-component \((2+1)\)-dimensional integrable systems of the form

\[
\begin{pmatrix} v \\ w \end{pmatrix}_t + A(v, w) \begin{pmatrix} v \\ w \end{pmatrix}_x + B(v, w) \begin{pmatrix} v \\ w \end{pmatrix}_y = 0.
\]

The integrability conditions constitute a complicated overdetermined system of second order PDEs for the \(2 \times 2\) matrices \(A\) and \(B\) which are presented in Sect. 4.1. In the particular case when \(A\) is taken to be linearly degenerate,

\[
A = \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix},
\]

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where $f$ is a cubic polynomial, $f(v) = \alpha v^3 + \beta v^2 + \gamma v + \delta$, and $\alpha, \beta, \gamma, \delta$ are arbitrary constants. A remarkable property of this example is that any matrix in the linear pencil $B + \mu A$ is also linearly degenerate. Explicitly, one has

$$B = \begin{pmatrix} f(w) - \alpha w^2 & f(v) \\ f(w) & f(v) - \alpha v^2 \end{pmatrix},$$

where $f$ is a cubic polynomial, $f(v) = \alpha v^3 + \beta v^2 + \gamma v + \delta$, and $\alpha, \beta, \gamma, \delta$ are arbitrary constants.

Notice that an arbitrary linear combination of matrices $A$ and $B_1, B_2, B_3, B_4$ is automatically linearly degenerate. This can be verified directly using the criterion (6). In the new variables $V = v + w$, $W = vw$, our system reduces to

$$V_t + W_x + WV_r - VW_r = 0, \quad W_t + VW_x - WV_x + V_y + W_z + VW_s - WV_s = 0,$$

taking a fully symmetric form (4) after an obvious linear transformation of the independent variables. We would like to emphasize that there was no a priori reason for the $(5 + 1)$-dimensional system obtained in this way to be integrable: this is a miracle which is due to the linear degeneracy. Notice that a particular dimensional reduction of the system

$$m_t = n_x, \quad n_t = m_y + mn_x - nm_x,$$

known as the Pavlov equation, has been extensively discussed in the context of hydrodynamic chains and the ‘universal hierarchy’ [33, 39, 40, 27].

**Example 2.** The following four-dimensional three-component integrable system appeared in [11] in the context of hyper-Hermitian conformal structures,

$$AN_x + B_1 N_{y^1} + B_2 N_{y^2} + B_3 N_{y^3} = 0,$$

here $N$ is a three-component column vector, $N = (N_1, N_2, N_3)^t$, and $A, B_1, B_2, B_3$ are $3 \times 3$ matrices,

$$A = \begin{pmatrix} 1 & -N_3 & N_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplying by $A^{-1}$ one can bring this system to an evolutionary form, moreover, any linear combination of matrices $A^{-1}B_i$ is linearly degenerate. Notice, however, that the dispersion relation of this system is not irreducible.

A general scheme for constructing multi-dimensional integrable systems based on the commutativity of vector fields was discussed in [3]. A broad class of new examples of integrable linearly degenerate systems in $2 + 1$ dimensions was constructed in [32].
Global solutions of linearly degenerate systems

In 1 + 1 dimensions, linearly degenerate systems are known to be quite exceptional from the point of view of the solvability of the Cauchy problem: generic smooth initial data do not develop shocks in finite time \[35, 36, 23, 37\]. The conjecture of Majda \[24\], p. 89, suggests that the same statement should be true in higher dimensions, namely, for linearly degenerate systems the shock formation never happens for smooth initial data. For a particular case of the Pavlov equation, this conjecture was confirmed in \[27\] based on the novel version of the inverse scattering transform. In this section we discuss another dimensional reduction of the system (4), namely, the system (5),

\[ v_t + w_x = 0, \quad w_t + wy + vw_x - wv_x = 0. \]

In contrast to the Pavlov equation (9), this system has a well-posed initial value problem. We also refer to \[4, 5\] for a detailed discussion of the \(10 \times 10\) linearly degenerate Born-Infeld system which is a nonlinear version of Maxwell’s equations. We hope that our linearly degenerate example will be more appropriate for establishing the global existence type results.

Some useful properties of system (5) are discussed in Sect. 3.1: these include the conservative representation and the Friedrichs-symmetrized form, which is most suitable for numerical simulations. In Sect. 3.2 and 3.3 we construct globally defined exact solutions of system (5) by reducing it to known integrable equations in \(1 + 1\) dimensions. This includes reductions to the sinh-Gordon and KdV equations (Sect. 3.2), as well as the reduction to a pair of \(N\)-component linearly degenerate systems (Sect. 3.3). Sect. 3.4 contains numerical simulations of the system (5) which support the conjecture that smooth initial data do not break down in finite time. We emphasize that all these properties are not specific to the system (5), and hold for any linearly degenerate integrable system.

### 3.1 Properties of System (5)

One can show that system (5) possesses exactly three zero order conservation laws,

\[ v_t + w_x = 0, \]

\[ (v^2 - w)_t + (vw)_x - w_y = 0, \]

\[ (1/w)_t + (v/w)_x + (1/w)_y = 0. \]

It can be written in a Friedrichs-symmetrized form,

\[
\begin{pmatrix}
-w & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix}_t
+
\begin{pmatrix}
0 & -w \\
-w & v
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix}_x
+
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
v \\
w
\end{pmatrix}_y
= 0,
\]

which shows that one should expect at least finite time existence results in the region \(w < 0\). Introducing the variables \(p = -2v/w^2\), \(q = 1/w^2\), one can rewrite (5) in the Godunov form \[19\],

\[ (f_p)_t + (g_p)_x + (h_p)_y = 0, \quad (f_q)_t + (g_q)_x + (h_q)_y = 0, \]
where
\[ f = \frac{-p^2}{4q} - 2\sqrt{q}, \quad g = \frac{p}{\sqrt{q}}, \quad h = -2\sqrt{q}. \]

**Remark.** The system (5) possesses a Lax pair which can be represented as the condition of commutativity of two vector fields,
\[ [\partial_t + \partial_y + (v + \lambda)\partial_x, w\partial_x - \lambda\partial_t] = 0. \] (11)
Lax pairs of this kind form a basis of the novel generalization of the inverse scattering transform [25, 26, 27].

### 3.2 Reductions to integrable equations

The system (5) possesses a number of remarkable reductions to soliton equations in 1 + 1 dimensions. Below we describe two particular reductions, to the sinh-Gordon and the KdV equations, respectively. For convenience, we make a change of variables \( x, y, t \rightarrow x, y, \tau = t - y \) which brings system (5) into the form
\[ v_\tau + w_x = 0, \quad w_y + vw_x - wv_x = 0. \] (12)

**Reduction to the sinh-Gordon equation.** Applying the procedure of [10] to the Lax pair (11) one can obtain a class of exact solutions of the system (12) in the form
\[ v = u_y \cos x, \quad w = -\sinh u \sin x + \cosh u, \] (13)
where \( u(y, \tau) \) satisfies the sinh-Gordon equation
\[ u_{y\tau} = \sinh u. \]

In the original variables, it takes the form \( u_{yt} + u_{tt} = \sinh u \). Taking a traveling wave solution in the form
\[ u = \arccosh \left( e - (e - 1) \text{sn}^2[\sigma(y - ct) \mid m] \right) \text{sign} \left( \cos \frac{\pi(y - ct)}{T} \right), \]
\[ T = 2\sigma^{-1}K(m), \quad \sigma = \sqrt{\frac{e + 1}{2c(1 - c)}}, \quad m = \frac{e - 1}{e + 1}, \] (14)
where \( \text{sn}[\cdot \mid m] \) is a Jacobi elliptic function and \( K(m) \) is the complete elliptic integral of the first kind (e.g. [11]), \( 0 < c < 1 \) and \( e > 1 \), one obtains bounded doubly periodic solutions \( v(x, y, t) = (u_y + u_t) \cos x \) and \( w(x, y, t) = -\sinh u \sin x + \cosh u \) of the system (5) shown in Fig. 1 for \( c = 1/3 \) and \( e = 1.1 \) (plotted using Mathematica [28]). These solutions remain non-singular for all values of \( t \).

**Reduction to the KdV equation.** Let \( v(x, y, \tau) \) be a solution to a pair of commuting flows
\[ v_\tau = (v^2)_x, \quad v_y = \frac{1}{4}v_{xxx} - 3v_x, \]
where $\varphi$ satisfies the system

$$\varphi_{xx} = 2v\varphi, \quad \varphi_y = \frac{1}{2} v_x \varphi - v \varphi_x.$$ 

Notice that the last two equations constitute a Lax pair for the KdV equation (at zero value of the spectral parameter), and the consistency condition $v_{\tau y} = v_{y\tau}$ holds identically modulo the equations for $\varphi$. Setting

$$w(x, y, \tau) = -\varphi^2, \quad (15)$$

one can see that $v(x, y, \tau)$ and $w(x, y, \tau)$ solve the system $\{12\}$. Exact non-singular multi-soliton solutions of these equations can be constructed in the form

$$v = -\frac{d^2}{dx^2} \ln W(\psi_1, ..., \psi_n), \quad \varphi = \frac{1}{\prod k_i} \frac{W(\psi_1, ..., \psi_n, 1)}{W(\psi_1, ..., \psi_n)}; \quad (16)$$

here $W$ denotes the Wronskian of $n$ functions $\psi_i$ where $\psi_{2j-1} = \cosh(k_{2j-1}x + k_{2j-1}^3y + \tau/k_{2j-1})$ and $\psi_{2j} = \sinh(k_{2j}x + k_{2j}^3y + \tau/k_{2j})$, $k_i = \text{const}$ satisfying $k_1 < k_2 < ... < k_N$ (e.g., $38$). Taking $n = 2$, $k_1 = 1.05$, $k_2 = 1.2$, and setting $\tau = t - y$, we obtain a two-soliton solution of the system $\{5\}$ shown in Fig. 2.

Further differential reductions of the system $\{5\}$ can be obtained from the general approach of $39$. 

Figure 1: Solution $\{13\}, \{14\}$ to the system $\{5\}$ for $v$ and $w$ at $t = 0$. 

Figure 2: Solution $\{15\}, \{16\}$ to the system $\{5\}$ for $v$ and $w$ at $t = 0$. 

Further differential reductions of the system $\{5\}$ can be obtained from the general approach of $39$. 

8
3.3 Reduction to a pair of (1+1)-dimensional linearly degenerate systems

The method of hydrodynamic reductions, originally proposed in [17, 18] in the context of the dispersionless KP hierarchy, was subsequently generalized to a whole class of multidimensional quasilinear systems in [13, 14, 15]. We will use the example of system (5) to illustrate the method, see also [33, 40]. Let us seek N-phase solutions of the system (5) in the form
\[
v = v(R^1, ..., R^N), \quad w = w(R^1, ..., R^N)
\]
where the phases (Riemann invariants) \( R^i(x, y, t) \) satisfy a pair of compatible (1 + 1)-dimensional systems
\[
R^i_t = \lambda^i(R)R^i_x, \quad R^i_y = \mu^i(R)R^i_x.
\]
(18)

Recall that the commutativity conditions of the flows (18) take the form [42],
\[
\frac{\partial_j \lambda^i}{\lambda^i - \lambda^j} = \frac{\partial_j \mu^i}{\mu^j - \mu^i},
\]
(19)
\[i \neq j, \quad \partial_j = \partial/\partial R^j, \text{ no summation}.
\]

The substitution of the ansatz (17) into (5) implies
\[
\partial_i w = -\lambda^i \partial_i v, \quad \mu^i = -\lambda^i - v - \frac{w}{\lambda^i},
\]
which, taking into account the commutativity condition (19), leads to \( \partial_j \log \lambda^i = \partial_j \log w, \partial_i \partial_j \log w = 0, \partial_i \partial_j v = 0, \ i \neq j. \) These equations are straightforward to integrate:
\[
v = \sum_k \eta^k(R^k), \quad w = \prod_k f^k(R^k), \quad \lambda^i = \varphi^i(R^i) \prod_k f^k(R^k),
\]
here \( f^i(R^i), \eta^i(R^i) \) and \( \varphi^i(R^i) \) are arbitrary functions of one variable such that \( (\eta^i)' = -(f^i)'/(f^i \varphi^i) \). After a reparametrization \( R^i \to f^i(R^i) \) this simplifies to
\[
v = \sum_k \eta^k(R^k), \quad w = \prod_k R^k, \quad \lambda^i = \varphi^i(R^i) \prod_k R^k, \quad (\eta^i)' = -1/(R^i \varphi^i).
\]
From now on we fix a particular choice \( \varphi^i(R^i) = 1/R^i \) under which both systems in (18) become linearly degenerate:
\[
R^i_t = \left( \prod_{k \neq i} R^k \right) R^i_x, \quad R^i_y = \left( \sum_{k \neq i} R^k - \prod_{k \neq i} R^k \right) R^i_x.
\]
(20)

Once a solution to equations (20) is known, the variables \( v \) and \( w \) can be reconstructed by the formulae
\[
v = -\sum_k R^k, \quad w = \prod_k R^k.
\]
Thus, the \((2 + 1)\)-dimensional system (5) is decoupled into a pair of \((1 + 1)\)-dimensional linearly degenerate systems (20) with an arbitrarily large number of dependent variables \( N \). Since the gradient catastrophe for one-dimensional linearly degenerate hyperbolic
systems does not occur \[35, 36, 23, 37\], generic solutions arising from this construction will not break down in finite time. The general solution of Eqs. (20) can be represented in the following parametric form,

\[
\int R_1^{\xi_{N-1}}\frac{d\xi}{f_1(\xi)} + \cdots + \int R_N^{\xi_{N-1}}\frac{d\xi}{f_N(\xi)} = x, \\
\int R_1^{\xi_{N-2}}\frac{d\xi}{f_1(\xi)} + \cdots + \int R_N^{\xi_{N-2}}\frac{d\xi}{f_N(\xi)} = -y, \\
\int \int R_1^{\xi_k}\frac{d\xi}{f_1(\xi)} + \cdots + \int R_N^{\xi_k}\frac{d\xi}{f_N(\xi)} = 0, \quad k = 1, 2, \ldots, N - 3, \\
\int R_1^{\xi_k}\frac{d\xi}{f_1(\xi)} + \cdots + \int R_N^{\xi_k}\frac{d\xi}{f_N(\xi)} = (-1)^N(y - t),
\]

(21)

see [12], where \(f_i(\xi)\) are \(N\) arbitrary functions of one variable. In the next subsections we utilize this representation to construct particular solutions of system (5) representing various multiple-wave interactions.

3.3.1 Reduction to the Jacobi inversion problem

With the choice \(f_i(\xi) = \sqrt{P_{2N+1}(\xi)}\), \(i = 1, \ldots, N\), where \(P_{2N+1}(\xi)\) is a polynomial of degree \(2N + 1\) in \(\xi\), Eqs. (21) reduce to the Jacobi inversion problem which can be solved using multi-dimensional theta functions.

The equation \(\mu^2 = \xi \prod_{i=1}^{2N}(\xi - \lambda_i)\) with \(\lambda_i \in \mathbb{R}, i = 1, \ldots, 2N, \lambda_i \neq \lambda_j\) for \(i \neq j\) and \(\lambda_i \neq 0\) defines a hyperelliptic Riemann surface \(\mathcal{L}\) of genus \(N\) with simple branch points at the \(\lambda_i, 0\) and at infinity. In addition we choose \(\sum_{i=1}^{N} \lambda_i = 0\). We order the branch points in a way that \(\lambda_1 < \lambda_2 < \ldots < 0 < \ldots < \lambda_{2N}\). The differentials

\[
\eta_k = \frac{\xi^{k-1}d\xi}{\mu}, \quad k = 1, \ldots, N,
\]

form a basis for the holomorphic one-forms on \(\mathcal{L}\). On this surface we introduce a basis of the homology, where the \(a\)-cycles are in the upper sheet each of them encircling exactly one of the cuts between \([\lambda_1, \lambda_2], [\lambda_3, \lambda_4], \ldots, [\lambda_{2N-2}, \lambda_{2N-1}]\); all \(b\)-cycles start at the cut \([\lambda_{2N}, \infty]\), and each of them intersects exactly one \(a\)-cycle such that \(a_i \cdot b_j = \delta_{ij}\). A normalized basis of holomorphic differentials \(\omega_i, i = 1, \ldots, N - 1\) is given by

\[
\omega_i = \sum_{j=1}^{N} c_{ij}\eta_j, \quad i = 1, \ldots, N - 1
\]

where the \(c_{ij}\) are fixed by the normalization condition

\[
\int_{a_i} \omega_j = 2\pi i \delta_{ij}, \quad i, j = 1, \ldots, N.
\]

1More general hyperelliptic Riemann surfaces can be used in this context, but since we only want to illustrate this approach, we took this choice leading to simple formulae. With this choice, \(v\) will be a solution to the KdV equation
With this normalization the matrix

$$B_{ij} = \int_{b_i} \omega_j, \quad i, j = 1, \ldots, N,$$

is a Riemann matrix, i.e., it is symmetric and has a negative definite real part (see for instance [2]). Therefore the theta function with half integer characteristics $[\alpha] = [\alpha'; \alpha''] \in \mathbb{R}^{2N}$ defined by

$$\Theta[\alpha](x|B) = \sum_{n \in \mathbb{Z}^N} \exp \left( \frac{1}{2} \langle n + \alpha', B(n + \alpha') \rangle + \langle n + \alpha', x + 2\pi i\alpha'' \rangle \right),$$

where the angular brackets denote the Euclidean scalar product, is an entire function for all $z \in \mathbb{C}^N$.

With this setting the Jacobi inversion problem following from (21) can be written in the form

$$\sum_{j=1}^N \int_{R_j} \omega_j = c_{iN}x - c_{i(N-1)}y + c_{i1}(-1)^N(y - t) =: z_i.$$  

Notice that an arbitrary constant can be added to each $z_i$ reflecting the freedom to choose a lower limit of the integrals on the left-hand side. Formulae for the symmetric functions $\sum_{j=1}^N R_j$ and $\prod_{j=1}^N R_j$ in terms of theta functions can be found for instance in [2]: the former appears in the hyperelliptic solutions to the KdV equation, the latter (as a logarithm) in the hyperelliptic solutions of the sine-Gordon equation, which is related to the sinh-Gordon equation by the transformation $u \rightarrow iu$. Thus not surprisingly both equations which appear as reductions of the studied systems also play a role in the context of hyperelliptic solutions. We have

$$v = -\sum_{i=1}^N R_i = -(c + \partial_{xx} \ln \Theta[\alpha](z)), \quad w = \prod_{i=1}^N R_i = d + \partial_{xt} \ln \Theta[\alpha^*](\int_0^\infty \omega),$$

where the arbitrary non-singular constant characteristic reflects the possibility to change $z$ by an arbitrary constant, and where $c$ and $d$ are constants with respect to the physical coordinates,

$$c = \frac{1}{2\pi i} \sum_{i=1}^N c_{iN} \int_{a_i} \frac{\xi d\xi}{\mu}, \quad d = \partial_{xt} \ln \Theta[\alpha^*] \left( \int_0^\infty \omega \right),$$

where $[\alpha^*]$ is an arbitrary odd half integer characteristic. It can be seen that $v_t + w_x = 0$ by construction. The validity of the second equation in (22) is checked by applying derivatives to the theta functions and computing these numerically. Since this relation is not implemented in the code, its validity provides a strong test on the quality of the numerics. The code typically satisfies this test to the order of machine precision. We use the code [16] developed for hyperelliptic solutions to KdV, Kadomtsev-Petviashvili and other equations. In Fig. 3 one can see a typical genus 3 solution. As already mentioned, $v$ solves the KdV equation in $y$ and $x$, the time $t$ playing just the role of a parameter here. In Fig. 4 the same situation as in Fig. 3 can be seen for $t = 0.1$ thus illustrating the time
Figure 3: Solutions $v$ and $w$ \cite{22} for $N = 3$ on the surface with branch points $[-3 - \epsilon, -3, -\epsilon, 0, 1, 1 + 2\epsilon, 4]$ for $\epsilon = 1$ at $t = 0$. The characteristics of the theta function are chosen to be $\alpha'_i = 1/2$ and $\alpha''_i = 0$ for $i = 1, 2, 3$. 
Figure 4: Solutions $v$ and $w$ for $N = 3$ on the surface with branch points $[-3 - \epsilon, -3, -\epsilon, 0, 1, 1 + 2\epsilon, 4]$ for $\epsilon = 1$ at $t = 0.1$. The characteristics of the theta function are chosen to be $\alpha_i' = 1/2$ and $\alpha_i'' = 0$ for $i = 1, 2, 3$. 
evolution of the pattern. If the $\epsilon$ in Fig. 3 tends to 0, the surface degenerates which leads to the solitonic limit of the KdV solution $v$. This can be seen in Fig. 5 where a 3-soliton solution to KdV appears. The characteristic phase shift at the point of collision between the solitons can be clearly recognized. In this sense the solutions on a non-degenerate surface can be loosely interpreted as an infinite train of 3-solitons. Going to higher genus, one obtains $N$-phase solutions which show with increasing $N$ more and more structure, as can be seen for instance in Fig. 6.

3.3.2 Parametric plots of 3-wave interactions

In the case $N = 3$ Eqs. (20) reduce to

$$ R_1^1 = R_2^2 R_3^3 R_x^1, \quad R_1^2 = R_3^1 R_3^2 R_x^2, \quad R_1^3 = R_3^1 R_2^2 R_x^3 $$

and

$$ R_1^1 = (R_2^2 + R_3^3 - R_2^2 R_3^3) R_x^1, \quad R_1^2 = (R_1^1 + R_3^3 - R_1^1 R_3^3) R_x^2, \quad R_1^3 = (R_1^1 + R_2^2 - R_1^1 R_2^2) R_x^3, $$
Figure 6: Solutions $v$ and $w$ \cite{22} for $N = 4$ on the surface with branch points $[-4, -3, -2, -1, 0, 1, 2, 3, 4]$ at $t = 0$. The characteristics of the theta function are chosen to be $\alpha'_i = 1/2$ and $\alpha''_i = 0$ for $i = 1, 2, 3, 4$. 
respectively. The formulae (21) simplify to
\[ R^1 \int \xi^2 f(\xi) + R^2 \int \xi^2 g(\xi) + R^3 \int \xi^2 h(\xi) = x, \]
\[ R^1 \int \xi f(\xi) + R^2 \int \xi g(\xi) + R^3 \int \xi h(\xi) = -y, \]
\[ R^1 \int \frac{d\xi}{f(\xi)} + R^2 \int \frac{d\xi}{g(\xi)} + R^3 \int \frac{d\xi}{h(\xi)} = t - y, \]
where \( f, g, h \) are arbitrary functions. For our purposes, it will be more convenient to work with an equivalent parametric representation of the general solution of (23)–(24), namely,
\[ R^1 = F(a), \quad R^2 = G(b), \quad R^3 = H(c), \]
where \( a, b, c \) are functions of \( x, y, t \) defined implicitly via
\[ \int \frac{F^2(a)}{F(a) - 1} da + \int \frac{G^2(b)}{G(b) - 1} db + \int \frac{H^2(c)}{H(c) - 1} dc = x, \]
\[ \int \frac{F(a)}{F(a) - 1} da + \int \frac{G(b)}{G(b) - 1} db + \int \frac{H(c)}{H(c) - 1} dc = -y, \]
\[ a + b + c = -t; \]
here \( F, G, H \) are three arbitrary functions, and all integrals are understood as indefinite.

The corresponding solution of (5) is given by the formulae
\[ v = -F(a) - G(b) - H(c), \quad w = F(a)G(b)H(c). \]

Relations (26) can be derived from (25) by setting
\[ R^1 = F(a), \quad R^2 = G(b), \quad R^3 = H(c) \]
and
\[ \int \frac{(\xi - 1)d\xi}{f(\xi)} = a, \quad \int \frac{(\xi - 1)d\xi}{g(\xi)} = b, \quad \int \frac{(\xi - 1)d\xi}{h(\xi)} = c, \]
respectively. Let us choose, for instance, three soliton-like profiles
\[ F(a) = \alpha - \frac{\epsilon}{\cosh^2 \mu a}, \quad G(b) = \beta - \frac{\epsilon}{\cosh^2 \mu b}, \quad H(c) = \gamma - \frac{\epsilon}{\cosh^2 \mu c}, \]
where \( \alpha, \beta, \gamma \) are distinct constants and \( \epsilon \) is a small parameter. In this case equations (26) take the form
\[ \frac{\alpha^2 a}{\alpha - 1} + \frac{\beta^2 b}{\beta - 1} + \frac{\gamma^2 c}{\gamma - 1} + \epsilon(\ldots) = x, \quad \frac{\alpha a}{\alpha - 1} + \frac{\beta b}{\beta - 1} + \frac{\gamma c}{\gamma - 1} + \epsilon(\ldots) = -y, \quad a + b + c = -t, \]
where dots denote certain explicit functions of \( a, b, c \) which are bounded along with their first order derivatives. Thus, for sufficiently small \( \epsilon \) the change of variables \( a, b, c \to x, y, t \)
is nonsingular and well-defined globally, so that the corresponding \( v \) and \( w \) will also be nonsingular. Expressing \( c \) in the form \( c = -t - a - b \) and substituting into the formulas \([27]\) for \( v \), \( w \), and the first two equations \([26]\), one obtains \( v, w, x, y \) as functions of \( a \) and \( b \). Thus, one can construct parametric plots of \( v(x, y, t) \) and \( w(x, y, t) \). Figs. 7 and 8 show snapshots of \( v \) and \( w \) at \( t = 0 \) and \( t = 6 \) for the parameter values \( \alpha = 1 - \sqrt{\frac{1 + \tan(\phi + \pi/3)}{1 + \tan(\phi - \pi/3)}} \), \( \beta = 1 - \sqrt{\frac{1 + \tan(\phi)(1 + \tan(\phi - \pi/3))}{1 + \tan(\phi + \pi/3)}} \), \( \gamma = 1 - \sqrt{\frac{1 + \tan(\phi)(1 + \tan(\phi + \pi/3))}{1 + \tan(\phi - \pi/3)}} \), where \( \phi = \pi/9 \), and \( \mu = 1, \epsilon = 0.05 \). At \( t = 0 \) initial profiles represent three solitons having a triple collision at the origin (parameters are chosen in such a way that solitons meet at equal angles \( \pi/3 \)). The triple collision then disintegrates into pairwise interactions, shown at \( t = 6 \).

Figure 7: Solution \([27], [28]\) to the system \([5]\) for \( v \) at \( t = 0 \) and \( t = 6 \).

Figure 8: Solution \([27], [28]\) to the system \([5]\) for \( w \) at \( t = 0 \) and \( t = 6 \).

Unfortunately, for \( N > 3 \) one cannot reduce Eqs. \([21]\) to parametric formulas, and has to solve them numerically.

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3.3.3 Plots of $N$-wave interactions

To obtain a more convenient form for explicit integration, we rewrite Eqs. (21) as

$$\int F_N^{-1}(p_1) dp_1 + \cdots + \int F_N^{-1}(p_N) dp_N = x,$$

$$\int F_N^{-2}(p_1) dp_1 + \cdots + \int F_N^{-2}(p_N) dp_N = -y,$$

$$\int F_k(p_1) dp_1 + \cdots + \int F_k(p_N) dp_N = 0, \quad k = 1, 2, \ldots, N - 3,$$

$$p_1 + \cdots + p_N = (-1)^N(y - t).$$

(29)

Here $F_i(p_i)$ are arbitrary functions, and all integrals are understood as indefinite. The corresponding solution of (5) is given by the formulae

$$v = -\sum F_i(p_i), \quad w = \prod F_i(p_i).$$

Formule (21) and (29) are identified upon setting $R_i = F_i(p_i)$, $\int_{R_i}^R d\xi f_i(\xi) = p_i$. Equations (29) can be used to construct nonlinear interactions of $N$ soliton-like profiles by taking, e.g., $F_i(p_i) = \alpha_i + \epsilon \cosh^2 \mu_i p_i$, where $\epsilon$ is sufficiently small and the constants $\alpha_i$ are chosen in such a way that equations (29) are uniquely solvable for $p_i(x, y, t)$.

This approach can in principle be used to build hump-like initial data by letting $N$ tend to infinity, thus providing in a loose sense a non-linear generalization of Fourier analysis for the studied PDE. We illustrate this concept below by studying examples for finite $N$ numerically. The idea is to construct for $t = 0$ a central hump from $N$ elementary solutions of the form of the KdV soliton.

The angles between these elementary solutions are chosen to be equal. From the second relation in (20) it can be seen that a condition on these angles is equivalent to a condition on the $\alpha_i$,

$$\sum_{j=1}^N \alpha_j - \alpha_i - \frac{1}{\alpha_i} \prod_{j=1}^N \alpha_j = \tan \psi_i, \quad i = 1, \ldots, N.$$

(30)

To ensure that $w(x, y, 0) < 0$ we choose $F_i = \alpha_i + \frac{\epsilon}{\cosh^2 \mu_i p_i}$ for $i = 1, \ldots, N - 2$ and $F_i = \alpha_i$ for $i = N - 1, N$. In addition we fix $\sum_{j=1}^N \alpha_j = 0$ and $\prod_{j=1}^N \alpha_j = -1$, thus assuring that we are studying solutions on the zero background for $v$ and on the $-1$ background for $w$. The first $N - 2 \alpha_i$ then follow from (30) via the quadratic relations,

$$1/\alpha_i - \alpha_i = \tan \psi_i, \quad i = 1, \ldots, N - 2.$$

The remaining two $\alpha_i$ are a consequence of the condition of vanishing sum of the $\alpha_i$ and product equal to $-1$,

$$\alpha_{N-1,N} = \frac{1}{2} \left( \sum_{j=1}^{N-2} \alpha_j \pm \sqrt{ \left( \sum_{j=1}^{N-2} \alpha_j \right)^2 - \frac{4}{\prod_{j=1}^{N-2} \alpha_j} } \right).$$
The $\psi_i$ are chosen in a way to generate equal angles between the elementary solutions in the $(x, y)$–plane for $t = 0$. For numerical reasons one has to avoid $\psi = \pi/2$ since the tangent diverges there. To obtain finite $\psi_i$ also for even $N$, we shift them by a common factor, thus

$$
\psi_n = \begin{cases} 
\frac{n\pi}{N-2} & N \text{ odd}, \\
\frac{n\pi}{N-2} + \frac{\pi}{2(N-2)} & N \text{ even}.
\end{cases}
$$

With this choice we assure that the values for $\tan \psi$ are as small as possible. This is numerically important since the $\alpha_i$ appear in an $N \times N$ Vandermonde determinant, i.e., there will be $N$th powers of each $\alpha_i$. For the same reason we also have to choose the signs carefully to avoid too large or too small values of $\alpha_{N-1,N}$. With the above choice of the $\psi_i$, $i = 1, \ldots, N-2$, we have assured that with some $\alpha_j$ a solution to one of the equations (30), also $-\alpha_j$ and $\pm \alpha_j$ are solutions to these equations (either for $\psi_j$ or for $\pi - \psi_j$). We fix the signs in a way to obtain a minimal $\sum_{i=1}^{N-2} \alpha_i$ and a negative $\prod_{i=1}^{N-2} \alpha_i$. This is done by alternating the signs in the $\alpha_i$, $i = 1, \ldots, N_{\text{max}}$ ($N_{\text{max}} = N/2 - 1$ for $N$ even and $N_{\text{max}} = (N - 1)/2$ for $N$ odd). The remaining signs are chosen in a way that the $\alpha_j$ corresponding to $\pi - \psi_i$ is equal to $1/\alpha_i$. Thus we assure that $\alpha_{N-1}$ and $\alpha_N$ are real and that $1/|\alpha_{N_{\text{max}}}| \leq |\alpha_i| \leq |\alpha_{N_{\text{max}}}|$. We give the values of the $\alpha_i$ for two examples below. Since $\alpha_{N_{\text{max}}} \sim 2(N - 2)/\pi$, we can in double precision reach values of $N = 14$ in which case the conditioning of the Vandermonde matrix of the $\alpha_i$ is of the order of $10^{-14}$ (the absolute value of the ratio of the smallest to the largest eigenvalue of the matrix). To treat higher values of $N$ more than double precision is needed which is, however, beyond the scope of this paper.

The goal of this subsection is to construct for $t = 0$ a single hump for $v$ with value $c = \text{const}$ at $x = y = 0$. To this end we choose $\epsilon = c/(N - 2)$ since $N - 2$ elementary solutions will overlap at the origin of the $(x, y)$ plane with the above choices. Thus we take $\epsilon = c/(N - 2)$ and take $c = 0.1$. The constant $c$ has to be chosen in a way that $w$ remains negative which seems to be the case in the numerical experiments discussed below if $c < 1$. A smaller value of $c$ leads to a more rapid convergence of the iteration to be described below.

For the given form of the $F_i$, $i = 1, \ldots, N$, we have $(n \geq 1)$

$$
\int F_i^n(p_i)dp_i = \alpha_i^n p_i + \sum_{j=1}^{n} \binom{n}{j} e^{j} \alpha_i^{n-j} I_j(p_i),
$$
where \( I_n(p) = \int dp' / \cosh^2 n p' \). Equations (29) thus take the form \((k = 1, 2, \ldots, N - 3)\)

\[
\sum_{i=1}^{N-2} \left\{ \alpha_i^{N-1} p_i \right\} + \sum_{j=1}^{N-1} \left( \begin{array}{c} N - 1 \\ j \end{array} \right) e^j \alpha_i^{N-1-j} I_j(p_i) \right\} + \alpha_{N-1}^{N-1} p_{N-1} + \alpha_N^{N-1} p_N = x,
\]

\[
\sum_{i=1}^{N-2} \left\{ \alpha_i^{N-2} p_i \right\} + \sum_{j=1}^{N-2} \left( \begin{array}{c} N - 2 \\ j \end{array} \right) e^j \alpha_i^{N-2-j} I_j(p_i) \right\} + \alpha_{N-1}^{N-2} p_{N-1} + \alpha_N^{N-2} p_N = -y
\]

\[
\sum_{i=1}^{N-2} \left\{ \alpha_i^k p_i \right\} + \sum_{j=1}^{k} \left( \begin{array}{c} k \\ j \end{array} \right) e^j \alpha_i^{k-j} I_j(p_i) \right\} + \alpha_{N-1}^k p_{N-1} + \alpha_N^k p_N = 0,
\]

\[
p_1 + \ldots + p_N = (-1)^N (y - t),
\]

This system can be solved iteratively. Let \( V \) be the Vandermonde determinant of the \( \alpha_i \).
Then we solve iteratively for the vector \( \vec{p} \), where \( \vec{p}_{n+1} \) follows from the \( n \)th iterate via

\[
V \vec{p}_{n+1} = \vec{J}(\vec{p}_n),
\]

with

\[
\vec{J}(\vec{p}_n) = -\sum_{i=1}^{N-2} \sum_{j=1}^{N-1} \left( \begin{array}{c} N - 1 \\ j \end{array} \right) e^j \alpha_i^{N-1-j} I_j(p_i) + x,
\]

etc, and

\[
\vec{J}(\vec{p}_0) = (x, -y, 0, \ldots, 0, (-1)^N (y - t)).
\]

The iteration is stopped once \( |\vec{p}_{n+1} - \vec{p}_n| < \delta \) (we use \( \delta = 10^{-6} \)). For a value of \( c \ll 1 \), this iteration converges rapidly (typically less than 4 iterations). If \( c \sim 1 \) the iteration converges only slowly or not at all.

The results of the above procedure for \( N = 6 \) (four elementary solutions) for \( t = 0 \) can be seen in Fig. 9. Since this approach is concerned with hump-like structures in \( v \), the outcome for \( w \) is not obvious, but can be seen in the lower part. The central hump for \( v \) at the origin of the \((x, y)\)-plane disintegrates with time as can be seen in Fig. 10.

Since we fixed the maximum of the absolute value of \( v \) at the center to be of the order of \( c \), the relative maximum of the modulus of the elementary waves tends with large \( N \) to 0. This is illustrated in Fig. 11 for \( N = 13 \). The central hump for \( v \) at the origin of the \((x, y)\)-plane disintegrates with time as can be seen in Fig. 12. Compare this figure with the time evolution of hump-like initial data for \( v \) studied numerically in the next subsection.

### 3.4 Cauchy problems and numerical evaluations

We will use the Friedrichs-symmetrized form (10), which is well-posed for \( w < 0 \), for numerical simulations of system (5). The task is to investigate the conjecture that solutions to the system (5) for generic initial data do not develop a point of gradient catastrophe in finite time.

To this end we consider initial data in the Schwartzian class of rapidly decreasing functions. Such functions can be treated for numerical purposes as effectively periodic if
Figure 9: Solutions \( v \) and \( w \) to the system (29) for \( N = 6 \) at \( t = 0 \) and \( \vec{\mu} = (1, 100, 1, 1) \). The values of \( \vec{\mu} \) have been chosen in a way that the lateral extension of the elementary waves is of a similar order. This implies \( \vec{\alpha} \approx (0.81, -2.77, -0.36, 1.23, -0.59, 1.69) \).
Figure 10: Solutions $v$ and $w$ to the system \cite{20} for $N = 6$ at $t = 6$ and $\vec{\mu} = (1, 100, 1, 1)$. 

The period is chosen large enough that the studied functions decrease below the machine precision (which is of the order $10^{-16}$ for double precision computations in MATLAB \cite{20}) at the ends of the intervals. The Gibbs phenomenon in such a case is of the order of the rounding error and thus negligible. This setting allows the use of Fourier spectral methods which are known for their excellent approximation properties for smooth functions. Thus we will approximate the spatial dependence of the functions $v$ and $w$ via discrete Fourier series. We find that the solutions to the system \cite{5} stay in the Schwartzian class in contrast, e.g., to solutions to the dispersionless Kadomtsev-Petviashvili (dKP) equation, see \cite{21} for a numerical study.

The time dependence is treated via a standard fourth order Runge-Kutta scheme. The quality of the numerics is controlled via the conserved quantity

$$I := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (v^2 - (w - w_\infty)) \, dx \, dy$$

of system \cite{5}, where $w_\infty = \lim_{x,y \to \infty} w(x, y, t)$. Since the conservation of this quantity is not implemented in the code, it will not be exactly observed in actual numerical experiments due to unavoidable numerical errors. Thus the numerical conservation of $I$ can serve as an indicator for the achieved accuracy. In our experiments the error in the relative conservation of this quantity was below $10^{-6}$. It is known (see for instance \cite{22}) that integral quantities overestimate the accuracy in a $L_\infty$ sense of numerical solutions to a partial differential equations by one to two orders of magnitude. Thus the results
Figure 11: Solutions $v$ and $w$ to the system (29) for $N = 13$ at $t = 0$ and $\vec{\mu} = (100, 100, 10, 10^4, 10, 10^5, 10, 10^3, 100)$. The values of $\vec{\mu}$ have been chosen in a way that the lateral extension of the elementary waves is of a similar order. This implies $\vec{\alpha} \approx (1.00, -1.16, 0.73, -1.73, 0.39, -7.10, -0.14, 2.58, -0.58, 1.37, -0.86, -0.18, 5.68)$. 
Figure 12: Solutions $v$ and $w$ to the system (29) for $N = 13$ at $t = 2$ and $\vec{\mu} = (100, 100, 10, 10^4, 10, 10^{10}, 1, 10^5, 10, 10^3, 100)$. 
presented below are accurate to at least $10^{-4}$ which is well beyond plotting accuracy. Comparing the behavior of the numerical solutions to solutions to the dKP equation in [21] which show blow up, we see in the experiments performed below no indication of an increasing gradient of the solutions. Thus it seems that at least for the class of initial data considered, the numerical solutions will have global existence in time.

As a concrete example we study the initial data $v(x,y,0) = \exp(-(x^2 + y^2))$ and $w(x,y,0) = -1$. It can be seen in Fig. 13 that the solution for $v$ stays smooth and localized in both spatial directions.

![Figure 13](image)

Figure 13: Solution $v$ to the system (5) for the initial data $v(x,y,0) = \exp(-(x^2 + y^2))$ and $w(x,y,0) = -1$ for several values of $t$.

A similar behavior can be observed for the solution $w$ in Fig. 14. This solution starts with $w = -1$ and develops peaks above and below this initial surface. It can be seen that both functions $v$ and $w$ approach an elliptic shape which could indicate the existence of an elliptic localized solution which acts as an attractor for a wide class of initial data.

The elliptic shape seems to be also present if one starts with several humps. In Fig. 15 we consider initial data of the form $v(x,y,0) = \exp(-(x-3)^2 + y^2)) + \exp(-(x+3)^2 + y^2))$ and $w(x,y,0) = -1$. As can be seen both humps develop into an ellipse. The behavior for $w$ is as follows: since it starts as identically constant, two humps of opposite sign are forming. Both develop then an elliptic shape as seen in Fig. 15.
Figure 14: Solution $w$ to the system (5) for the initial data $v(x, y, 0) = \exp(-x^2 + y^2)$ and $w(x, y, 0) = -1$ for several values of $t$. 
Figure 15: Solutions $v$ and $w$ to the system (5) for the initial data $v(x, y, 0) = \exp(-((x - 3)^2 + y^2)) + \exp(-((x + 3)^2 + y^2))$ and $w(x, y, 0) = -1$ for $t = 4$. 
4 Proof of Theorem 1

In Sect. 4.1 we summarize the integrability conditions for two-component (2+1)-dimensional quasilinear systems. Their differential consequences are used in Sect. 4.2 to establish Theorem 1.

4.1 Integrability conditions for 2-component systems in 2 + 1 dimensions

Here we present the integrability conditions for 2-component quasilinear systems of the form (2). Setting $u = (v, w)^t$ and diagonalizing the matrix $A$, which is always possible in the $2 \times 2$ case, one obtains the equations

$$
\begin{pmatrix}
v \\
w
\end{pmatrix}_t + \begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix} \begin{pmatrix}
v \\
w
\end{pmatrix}_x + \begin{pmatrix}
p & q \\
r & s
\end{pmatrix} \begin{pmatrix}
v \\
w
\end{pmatrix}_y = 0.
$$

The integrability conditions constitute a complicated set of second order differential constraints for the coefficients $a, b, p, q, r, s$ which were derived in [14] based on the method of hydrodynamic reductions.

Equations for $a$:

$$
a_{vv} = \frac{qa_v b_v + 2qa_v^2 + (s-p)a_v a_w - ra_w^2}{(a-b)q} + \frac{a_v r_v}{r} + \frac{2a_v p_w - a_w p_v}{q},
$$

$$
a_{vw} = \frac{a_v a_w + b_w}{a-b} + \frac{a_v(q_w + r_w)}{q},
$$

$$
a_{ww} = \frac{qa_v b_v + (s-p)a_v b_w + ra_w^2}{(a-b)r} + \frac{a_v s_w}{r} + \frac{a_w q_w}{q}.
$$

Equations for $b$:

$$
b_{vv} = \frac{ra_w b_w + (p-s)a_v b_w + q b_w^2}{(b-a)q} + \frac{b_w p_v}{q} + \frac{b_v r_v}{r},
$$

$$
b_{vw} = \frac{b_w a_v + b_v}{b-a} + \frac{b_w(q_v + r_v)}{q},
$$

$$
b_{ww} = \frac{ra_w b_w + 2rb_w^2 + (p-s)b_v b_w - q b_w^2}{(b-a)r} + \frac{b_w q_w}{q} + \frac{2b_w s_v - b_v s_w}{r}.
$$
Equations for $p$:

\[ p_{v} = 2r(a_v b_w - a_w b_v) + (s - p)a_v b_v + \frac{r_v p_v}{r} + \frac{p_v p_w}{q} + \frac{2q_v a_w - 2a_v q_w + a_w p_w}{b - a} - b_v p_v + 2r_v a_w - 2a_v(s_v + p_v + r_w) + \frac{p_v - s}{q}(2p_v a_w - a_v p_w) \]

\[ p_{w} = \frac{2(s - p)a_v b_w}{(a - b)^2} - \frac{b_w p_v}{b - a} + p_v \left(\frac{q_w}{q} + \frac{r_w}{r}\right) \]  \hspace{1cm} (34)

\[ p_{v} = 2(s - p)a_v b_w + (s - p)a_v b_v + \frac{(p - s)b_w p_v - q b_w p_v - 2r s_w a_w - r a_w p_w}{(b - a)r} + \frac{p_v s_w}{q} + \frac{q a p_w}{r} \]

Equations for $s$:

\[ s_{v} = \frac{2r(a_v b_w - a_w b_v) + (p - s)a_v b_v}{(a - b)^2} + \frac{(s - p)a_v s_w - r a_w s_w - 2q p_v b_v - q b_v s_v}{(a - b)q} + p_v s_w + \frac{a v s_v}{q} + \frac{r_v s_v}{r} \]

\[ s_{w} = \frac{2(p - s)a_v b_w}{(a - b)^2} - \frac{a_v s_w + (2p_w + s_v)b_w}{a - b} + s_w \left(\frac{q_w}{q} + \frac{r_w}{r}\right) \]  \hspace{1cm} (35)

\[ s_{w} = \frac{2q(a_v b_w - a_w b_v)}{(a - b)^2} + \frac{q w s_w}{q} + \frac{s_v s_w}{r} + \frac{2q_v b_w - 2b_w (p_w + s_w + q_v) + \frac{s_v r}{q}(2s_w b_v - b_w s_v)}{a - b} \]
Equations for \( q \) and \( r \):

\[
q_{rw} + rq_{ww} = 2(p-s) \frac{(p-s)a_w b_w + q(a_w b_w - a_w b_v)}{(a-b)^2} + \frac{r_w q b_v + (s-p)b_w}{a-b} + \frac{(s-p)2a_w s_w + 2b_w p_w + b_w q_v}{a-b} + \frac{r(a_w - 2b_w)q_w}{a-b} + \frac{q_w r w + b_v(2p_v + 2s_w + q_v) - 2b_w(r_w + p_v + s_v)}{a-b} + \frac{r}{q} \frac{q^2 + q r q_v + q w r^2 + s_w (2p_w + q_v)}{r(a-b)},
\]

\[
r_{qw} = (p-s) \frac{r a_w b_w + (p-s) a_v b_w + q a_v b_v}{q(a-b)^2} + \frac{r v r_w + p_v s_w + b_w(r_q + q r_v) + (p-s)(a_v s_w + b_w p_v) + r a_w s_w + q p_v b_v}{q(b-a)},
\]

\[
q_{rv} + rq_{uv} = 2(s-p) \frac{(s-p)a_v b_v + r(a_v b_w - a_w b_v)}{(a-b)^2} + \frac{r q w r a_w + (p-s)a_v}{b-a} + \frac{(p-s)2b_v p_v + 2a_v s_v + a_v r_w}{b-a} + \frac{q(b_v - 2a_v)r_v}{b-a} + \frac{r b_v q_v + a_w(2s_v + 2p_v + r_w) - 2a_v(q_v + s_w + p_w)}{b-a} + \frac{q}{r} \frac{r^2 + r q p_v q_w - r v q_v + p_v(2s_v + r_w)}{q(b-a)};
\]

These formulas are completely symmetric under the identification \( v \leftrightarrow w \), \( a \leftrightarrow b \), \( p \leftrightarrow s \), \( q \leftrightarrow r \). Our next goal is to rewrite these equations in a somewhat different form which will be used in the proof of Theorem 1. First of all, Eqs. (32) and (33) can be cast into the form

\[
d \ln q r = \Omega, \quad \Omega = \left( \frac{b_{vw}}{b_w} + \frac{a_v + b_v}{a - b} \right) \, dv + \left( \frac{a_{vw}}{a_v} + \frac{a_w + b_w}{b - a} \right) \, dw.
\]

In what follows we assume that all quantities \( a_v, a_w, b_v, b_w \) are nonzero, and parametrize the first order derivatives of \( q \) and \( r \) as

\[
\frac{q_w}{q} = \frac{a_w}{a_w} + \frac{a_w}{b-a} - n, \quad \frac{r_w}{r} = \frac{b_{vw}}{b_v} + \frac{b_v}{a-b} - m,
\]

\[
\frac{q_v}{q} = \frac{b_{vw}}{b_w} - \frac{b_v}{a-b} + m, \quad \frac{r_v}{r} = \frac{a_{vw}}{a_v} - \frac{a_w}{a_w} + \frac{b_w}{b-a} + n,
\]

\[30\]
where \( m \) and \( n \) are two auxiliary functions. Solving the remaining equations (32), (33) for the first order derivatives of \( p \) and \( s \) one gets

\[
\begin{align*}
p_v &= \frac{b_v}{b_w} q_m + \frac{a_w}{a - b} r + \frac{a_v}{a - b} (p - s), \\
s_w &= \frac{a_w}{a_v} r_n + \frac{b_v}{b - a} q + \frac{b_w}{b - a} (s - p), \\
p_w &= \frac{a_w}{b - a} (s - p) + \frac{a_v^2}{a_v(a - b)} r + \frac{1}{2} q \left( \left( \ln \frac{a_v}{b_v} \right)_v + \left( 1 + \frac{a_w b_v}{a_v b_w} \right) m + 2 \frac{a_v + b_v}{b - a} \right), \\
s_v &= \frac{b_v}{a - b} (p - s) + \frac{b_v^2}{b_w(b - a)} q + \frac{1}{2} r \left( \left( \ln \frac{b_w}{a_w} \right)_w + \left( 1 + \frac{a_w b_v}{a_v b_w} \right) n + 2 \frac{a_w + b_v}{a - b} \right).
\end{align*}
\]

Substituting (39) into the equations for \( q_{vw} \) and \( r_{vw} \) one gets

\[
\begin{align*}
\ln q_{vw} &= \frac{a_v}{a - b} - \frac{a_v(a_w + b_w)}{(a - b)^2} + \frac{a_w b_v}{a_v b_w} mn + 2 \frac{a_v^2 r_n}{a_v(a - b) q}, \\
\ln r_{vw} &= \frac{b_v}{b - a} - \frac{b_v(a_v + b_v)}{(a - b)^2} + \frac{a_w b_v}{a_v b_w} mn + 2 \frac{b_v^2 q_m}{b_w(b - a) r},
\end{align*}
\]

respectively. Finally, differentiating the equation (39)\(_1\) with respect to \( w \) and comparing the result with (34)\(_2\), as well as differentiating (39)\(_2\) with respect to \( v \) and comparing with (35)\(_2\), we obtain

\[
\begin{align*}
\left( \frac{b_v}{b_w} q_m \right)_w - \frac{b_v}{b_w} q_m \left( \frac{a_v}{a_w} - \frac{a_v}{a - b} \right) + 2 n r \frac{a_v}{b - a} + 3 q \frac{a_w b_v}{(a - b)^2} &= 0, \\
\left( \frac{a_w}{a_v} r_n \right)_v - \frac{a_w}{a_v} r_n \left( \frac{b_v}{b_w} - \frac{b_v}{b - a} \right) + 2 m q \frac{b_v}{a - b} + 3 r \frac{a_w b_v}{(a - b)^2} &= 0.
\end{align*}
\]

Equations (37), (40) and (41) will be crucial for the proof of Theorem 1.

### 4.2 Proof of Theorem 1

For definiteness, let us consider a two-component \((3 + 1)\)-dimensional system of the form (1). We assume that the generic matrix of the family \( \alpha^t A_i \) has a simple spectrum, and that the dispersion relation \( \det(\mu I + \alpha^t A_i) = 0 \) defines an irreducible quadric. Diagonalizing the first matrix, we obtain the equations

\[
\mathbf{u}_t + A(\mathbf{u}) \mathbf{u}_x + B(\mathbf{u}) \mathbf{u}_y + \tilde{B}(\mathbf{u}) \mathbf{u}_z = 0
\]

where \( \mathbf{u} = (v, w)^t \), and the matrices \( A, B, \tilde{B} \) have the form

\[
A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{p} & \tilde{q} \\ \tilde{r} & \tilde{s} \end{pmatrix}.
\]
Due to the irreducibility of the dispersion relation, one cannot have $q = \tilde{q} = 0$ or $r = \tilde{r} = 0$. Thus, by taking appropriate linear combinations of $B$ and $\tilde{B}$, one can assume $q, r, \tilde{q}, \tilde{r}$ to be nonzero. Since all traveling wave reductions of our system must be integrable, the matrices $B, \tilde{B},$ and $B + \tilde{B}$ must satisfy the $(2 + 1)$-dimensional integrability conditions from Sect. 4.1 (with the same matrix $A$). There are three cases to consider:

**Case 1.** A generic matrix of the family $\alpha A + \beta B + \beta \tilde{B}$ has constant eigenvalues. In particular, all coefficients of the dispersion relation must be constant. This implies that $a, b, p, s, \tilde{p}, \tilde{s}, qr, \tilde{q}r, q\tilde{r} + r\tilde{q}$ must be constant, and the integrability conditions of Sect. 4.1, applied to pairs $(A, B)$, $(A, \tilde{B})$ and $(A, B + \tilde{B})$, show that any such system must be linearizable by a transformation of the form $v \to f(v), \ w \to g(w)$.

**Case 2.** A generic matrix of the family $\alpha A + \beta B + \beta \tilde{B}$ is linearly degenerate. Then, due to the irreducibility of the dispersion relation, one cannot have a situation when a generic matrix of this family has one constant and one non-constant eigenvalue. Thus, we can assume that the matrix $A$ is linearly degenerate, and has non-constant eigenvalues. In this case $A$ can be brought to the form (7), and both matrices $B$ and $\tilde{B}$ must be of the form (8): see Example 1 of Sect. 2. Any such system is therefore a traveling wave reduction of the system (5).

**Case 3.** A generic matrix of the family $\alpha A + \beta B + \beta \tilde{B}$ is genuinely nonlinear (again, due to the irreducibility of the dispersion relation, one cannot have a ‘mixed’ situation when a generic matrix of the family has one linearly degenerate, and one genuinely nonlinear eigenvalue). In particular, $a_v \neq 0$ and $b_w \neq 0$. In this case we will prove that matrices $A, B$ and $\tilde{B}$ must be linearly dependent, so that such system cannot be essentially $(3 + 1)$-dimensional. Applying the condition (37) to the matrices $B, \tilde{B}$ and $B + \tilde{B}$ one obtains

$$d \ln qr = d \ln \tilde{q}\tilde{r} = d \ln (q + \tilde{q})(r + \tilde{r}),$$

so that $\tilde{q} = \lambda q, \tilde{r} = \mu r$ where $\lambda$ and $\mu$ are constants. In what follows we assume that $a_w \neq 0$ and $b_v \neq 0$ (if both of these derivatives vanish, one can take $a = v, b = w$, however, with such $A$ the integrability conditions from Sect. 4.1 are not compatible; similarly, due to the irreducibility of the dispersion relation one cannot have a situation when only one of these derivatives vanishes). If $\lambda = \mu$, the equations (39) imply $\tilde{p} = \lambda p + \tau a + \eta, \tilde{s} = \lambda s + \tau b + \eta$, so that $\tilde{B} = \lambda B + \tau A + \eta I$, and our system becomes essentially $(2 + 1)$-dimensional. Thus, we assume $\lambda \neq \mu$. This means that, whenever one obtains a corollary of the integrability conditions which does not depend explicitly on $p$ and $s$, and involves $a, b, q, r$ only, it should still hold after the rescaling $q \to \lambda q, r \to \mu r$. In particular, rescaling in this way the relations (40) one obtains $m = n = 0$. This, however, is inconsistent with relations (41): the last term does not contain $m, n$, and cannot vanish. This finishes the proof.

### 5 Concluding remarks

Our results suggest that there must be a restriction on the dimensions in which integrable $n$-component systems can live. Namely, for any number of components $n$, there must be an integer $d(n)$ such that there exist no $n$-component integrable systems of hydrodynamic type in the dimensions higher than $d(n) + 1$. We proved that $d(2) = 5$, moreover, there exists a unique two-component $(5 + 1)$-dimensional integrable system. It would be of...
interest to find a general formula for \( d(n) \), and classify the systems which attain an upper bound of this inequality.

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