THIN KNOTS AND THE CABLING CONJECTURE

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Abstract. The Cabling Conjecture of González-Acuña and Short holds that only cable knots admit Dehn surgery to a manifold containing an essential sphere. We approach this conjecture for thin knots using Heegaard Floer homology, primarily via immersed curves techniques inspired by Hanselman’s work on the Cosmetic Surgery Conjecture. We show that almost all thin knots satisfy the Cabling Conjecture, with possible exception coming from a (conjecturally non-existent) collection of thin, hyperbolic, L-space knots. This result serves as a reproof that the Cabling Conjecture is satisfied by alternating knots, and also a new proof that thin, slice knots satisfy the Cabling Conjecture.

1. Introduction

For a knot $K$ in $S^3$, let $S^3_r(K)$ denote $r$-sloped Dehn surgery along $K$. If $S^3_r(K)$ is a reducible manifold, meaning it contains an essential 2-sphere, we will call $r$ a reducing slope. The primary example of a reducible surgery to keep in mind is when $K$ is the $(p,q)$-cable of some knot $K'$ and $r$ is given by the cabling annulus. In this case, we have $S^3_{pq}(K) \cong L(p,q)\#S^3_K(K')$. The Cabling Conjecture asserts that this is the only example of a reducible surgery.

Conjecture 1.1 (Cabling Conjecture, Gonzalez-Acuña – Short [GAnS86]). If $K$ is a knot in $S^3$ which has a reducible surgery, then $K$ is a cabled knot and the reducing slope is given by the cabling annulus.

The Cabling Conjecture is satisfied by many classes of knots. Torus knots, as cables of the unknot, were shown to satisfy the conjecture in [Mos71]. Additionally, satellite [Sch90] and alternating knots [MT92] satisfy the conjecture as well. Then to establish this conjecture in full, it remains to show that it is satisfied by hyperbolic knots. Our aim is shorter, as we consider thin, hyperbolic knots.

We will present knot Floer homology in more detail in Section 2, but for now recall that $\widehat{\text{HFK}}(K)$ with coefficients in $\mathbb{F}_2$ is bigraded with Alexander and Maslov gradings, respectively $A$ and $M$. A knot $K$ is Floer homologically thin if the generators of $\widehat{\text{HFK}}(K)$ all have the same $\delta = A - M$ grading. This family contains alternating knots [OS03b], and the more generalized quasi-alternating knots [OS05b]. We say $K$ is an L-space knot if it admits a surgery to a (Heegaard Floer) L-space, which is a manifold with the simplest Heegaard Floer homology. Using Heegaard Floer homology via immersed curves techniques, we show that

Theorem 1.2. If a thin, hyperbolic knot $K$ in $S^3$ admits a reducible surgery, then $K$ is an L-space knot and the reducing slope must be $r = 2g(K) - 1$ after mirroring $K$ if necessary.

It is conjectured that the only thin, L-space knots are the torus knots $T(2,n)$. Provided this is true, there would not exist thin, hyperbolic, L-space knots and so Theorem 1.2 would show that all thin knots satisfy the Cabling Conjecture. While stated for thin, hyperbolic knots, this theorem holds more generally for non-cabled knots. This is because we use the Matignon-Sayari genus bound, stated blow, to only need to consider $r \leq 2g(K) - 1$. The case where $r > 2g(K) - 1$ can be handled using the techniques in this paper to conclude that $K = T(2, n)$, but perhaps more immediate is the result of Dey that cables of non-trivial knots are not thin [Dey19]. Since the only alternating, L-space knots are the $T(2,n)$'s [OS05a], Theorem 1.2 provides an immersed curves reproof that alternating knots satisfy the Cabling Conjecture.

Corollary 1.3. Alternating knots satisfy the Cabling Conjecture.

The next corollary follows because the only thin, slice, L-space knot is the unknot.

Corollary 1.4. Thin, slice knots satisfy the Cabling Conjecture.
Corollary 1.5. Thin knots cannot admit two reducible surgeries.

Part of the proof strategy for Theorem 1.2 involves obstructing an $\mathbb{RP}^3$ connected summand, and so we get the following corollary with identical proof to that of [HLZ15, Corollary 1.5].

Corollary 1.6. If $K$ is a thin, hyperbolic knot, then $S^3 \setminus \nu K$ does not contain properly embedded punctured projective planes.

When $K$ is a non-trivial knot in $S^3$ with reducible surgery $S^3_r(K)$, the surgery decomposes as a connected sum and the reducing slope satisfies $r \neq 0$ [Gab87]. We saw from the cabled knot example that the reducing slope is an integer and one of the connected summands is a lens space. The former and latter conditions occur for all reducible surgeries due to [GL87] and [GL89], respectively. A reducible surgery can admit at most three connected summands due to the combined efforts of [Say98][VS99][How02], in which case two summands are lens spaces and the remaining summand is an integer homology sphere. Since $S^3_r(K)$ must have a non-trivial lens space summand, the integral reducing slope $r$ satisfies $r \neq -1, 0, 1$. In [MS03], Matignon and Sayari provide the following genus bound if $K$ is non-cabled:

$$1 < |r| \leq 2g(K) - 1.$$ 

Heegaard Floer homology satisfies a Künneth formula for connected sums, and has proved very useful in general for studying Dehn surgery. If surgery along $K$ produces precisely a connected sum of two lens spaces, then $K$ must be a cabled knot due to [Gre15]. Further, [Gre15] together with [BZ98] shows that a hyperbolic knot in $S^3$ cannot admit both a lens space surgery and a reducible surgery. Hom, Lidman, and Zufelt showed that a hyperbolic, $L$-space knot can admit at most one reducing slope, and the slope must be $2g(K) - 1$ after mirroring the knot to make the slope positive [HLZ15]. They also established a periodicity structure to the Heegaard Floer homology of a reducible surgery, which is invaluable to the proof strategy of Theorem 1.2. We will involve these constraints via bordered invariants in the form of immersed curves.

Lipshitz, Ozsváth, and Thurston introduced bordered Heegaard Floer invariants for manifolds with torus boundary in [LOT18]. With $M_0 \cup_h M_1$ denoting a gluing of two manifolds along their torus boundaries, they prove a pairing theorem involving the two bordered invariants that recovers the Heegaard Floer homology of such a gluing. Hanselman, Rasmussen, and Watson reinterpreted these bordered invariants as collections of immersed curves in the punctured torus, and proved an analogous pairing theorem. In [Han19], Hanselman used this package to obtain obstructions for cosmetic surgeries along knots in $S^3$, and our approach in this paper is largely inspired by this work.

Organization

We only consider surgeries with positive slopes, and mirror knots to achieve this whenever necessary. All manifolds are assumed to be compact, connected, oriented 3-manifolds, unless stated otherwise, and the coefficients in Floer homology are taken to belong to $\mathbb{F} = \mathbb{F}_2$. We will denote closed manifolds by $X$ or $Y$, and manifolds with (typically torus) boundary by $M$. Figures containing immersed curves invariants will have the curves for $S^3 \setminus \nu K$ in blue and the curves for the filling solid torus in red/purple.

Section 2 summarizes the relevant background from knot Floer homology and Heegaard Floer homology. It also contains an overview of immersed curves invariants, their general properties and form for thin knots, as well as their associated pairing theorem and how to compute Maslov grading differences.

Section 3 expands on the relative Maslov grading for immersed curves invariants of complements of thin knots. Along the way we set up formulas for components of the grading difference formula in terms of $\tau(K)$.

Section 4 uses these relations to generate obstructions to periodicity for various cases of $r$ in relation to $\tau(K)$ and $g(K)$. It hosts a sizable collection of lemmas for the cases with $|\tau(K)| < g(K)$. Section 5 resolves the remaining cases where $|\tau(K)| = g(K)$, including some that use absolute grading information. Afterward, all lemmas are collected to handle the proof of the theorem.
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2. Background Material

We will assume the reader is familiar with the \( \widehat{HF} \) and \( HF^+ \) constructions of Heegaard Floer homology for 3-manifolds [OS04b], and knot Floer homology \( \widehat{HF}k \) for knots in \( S^3 \) (with associated full knot Floer complex \( CFK^\infty \)) [OS04b] [Ras03].

2.1. \( \widehat{HF} \) for reducible surgeries. Let us identify \( \text{Spin}^c(S^3(K)) \) with \( \mathbb{Z}/r\mathbb{Z} \) as in [OS08] Subsection 2.4, and denote the correspondence using \( \{ s \} \in \text{Spin}^c(S^3(K)) \) for \( \{ s \} \in \mathbb{Z}/r\mathbb{Z} \). We will also choose equivalence classes for elements of \( \mathbb{Z}/r\mathbb{Z} \) as centered about 0, so that for example \( \mathbb{Z}/r\mathbb{Z} = \{-\frac{r-1}{2}, \ldots, 0, \ldots, \frac{r-1}{2}\} \) if \( r \) is odd. As an abuse of notation, we will commonly use \( s \) for the representative of \( \{ s \} \) that falls within this range.

The following lemma is a simplified version of a more general Floer homology periodicity result for \( HF^+ \) of a general reducible 3-manifold from [HLZ15]. Basically, we should expect to see repeated behavior among the \( \text{spin}^c \) summands of \( \widehat{HF}(S^3(K)) \) if the surgery is reducible.

Lemma 2.2. Suppose \( S^3(K) \equiv X \# Y \), where \( X \) is an L-space and \( |H^2(X)| = k < \infty \). Then for any \( \{ s \} \in \text{Spin}^c(S^3(K)) \) and \( \alpha \in H^2(S^3(K)) \cong \mathbb{Z}/r\mathbb{Z} \), we have \( \widehat{HF}(S^3(K), \{ s + k\alpha \}) \cong \widehat{HF}(S^3(K), \{ s \}) \) as relatively-graded \( \mathbb{F} \) vector spaces.

Proof. Let \( \{ s \} \in \text{Spin}^c(S^3(K)) \) restrict to \( \{ s \} \in \text{Spin}^c(X) \) and \( \{ s \} \in \text{Spin}^c(Y) \). We see that \( HF(X, \{ s \}) \cong F \) since \( X \) is an L-space, and so the Künneth formula for \( \widehat{HF} \) [OS04c] Theorem 1.5 implies

\[
\widehat{HF}(S^3(K), \{ s \}) \cong H_*(\widehat{CF}(X, \{ s \}) \otimes \widehat{CF}(Y, \{ s \}))
\]

\[
\cong \widehat{HF}(Y, \{ s \}).
\]

For any \( \alpha \in \mathbb{Z}/r\mathbb{Z} \), we have that \( \{ s + k\alpha \} \) restricts to \( \{ s \} \) in \( \text{Spin}^c(Y) \). Then because \( \widehat{HF}(S^3(K), \{ s \}) \) is independent of \( \{ s \} \), we obtain

\[
\widehat{HF}(S^3(K), \{ s + k\alpha \}) \cong \widehat{HF}(Y, \{ s \}) \cong \widehat{HF}(S^3(K), \{ s \})
\]

as relatively-graded \( \mathbb{F} \) vector spaces. \( \square \)

We also need to gather some integral invariants of \( K \) involved with the Mapping Cone Formula that relates \( CFK^\infty(K) \) to \( HF^+(S^3(K)) \) [OS03]. For \( s \in \mathbb{Z} \), recall the subcomplexes and quotient complexes of the \( \mathbb{Z} \oplus \mathbb{Z} \)-filtered full knot Floer complex \( CFK^\infty(K) \)

\[
A^+_s = C \{ \text{max} \{ i, j - s \} \geq 0 \},
\]

\[
B^+_s = C \{ i \geq 0 \}.
\]

Notice \( B^+_s \cong CF^+(S^3) \) by definition. There are also chain maps \( v^+_s : A^+_s \to B^+_s \) and \( h^+_s : A^+_s \to B^+_{s+1} \) between these subcomplexes. Take homology to obtain \( A^+_s = H_*(A^+_s) \) and \( B^+_s = H_*(B^+_s) \cong HF^+(S^3) \), and induced maps \( v^+_s \) and \( h^+_s \). Let \( T^+ \) denote \( HF^+(S^3) \), and notice that \( U^N(A^+_s) \cong T^+ \) for arbitrarily large \( N \). By restricting both \( v^+_s \) and \( h^+_s \) to this submodule, we obtain \( v^+_s \) and \( h^+_s \). The integral invariants of \( K \) that we desire are due to [NW15], and are defined by

\[
V_s = \text{rank}(\text{ker} v^+_s),
\]

\[
H_s = \text{rank}(\text{ker} h^+_s).
\]

These terms have simple behaviour when \( K \) is alternating because of the “staircase” part of \( CFK^\infty(K) \) due to [OS03]. This holds more generally for thin knots due to [Pet13], but we will have an alternative geometric way of computing these terms later in Subsection 2.2. By [HLZ15] Lemma 2.3, the maps \( v^+_s \) and \( h^+_{s-1} \) agree on homology after identifying \( A^+_s \cong A^-_{s-1} \) (essentially...
reversing the roles of $i$ and $j$ above) so that $V_s = H_{-s}$. They are by definition non-negative, and also satisfy the following lemma.

**Lemma 2.2** ([NW15 Lemma 2.4]). The $V_s$ form a non-increasing sequence and the $H_s$ form a non-decreasing sequence, so that

$$V_s \geq V_{s+1} \text{ and } H_s \leq H_{s+1} \text{ for all } s \in \mathbb{Z}.$$

For a rational homology sphere $Y$, we can write $HF^+(Y,s) \cong \mathcal{T}^+ \oplus HF_{red}(Y,s)$, where $\mathcal{T}^+ \cong \mathbb{F}[U,U^{-1}]/\mathbb{F}[U]$ denotes the “tower” submodule. The $d$-invariants $d(Y,s)$, sometimes called the Heegaard Floer correction terms, record the smallest absolutely graded element of $\mathcal{T}^+ \subseteq HF(Y,s)$ [OS03a]. These invariants satisfy a few symmetries, such as spin$^c$ conjugation symmetry $d(Y,s) = d(Y,\overline{s})$ and orientation-reversal $d(-Y,s) = -d(Y,s)$, as well as additivity for connected sums. It is normalized so that $d(S^3, s_0) = 0$, and is recursively determined for lens spaces in [OS03a Proposition 4.8]. In [NW15], the $d$-invariants of rational surgeries are related to those of $d(L(p,q), [s])$ and the $H$’s and $V$’s. We state a special case of the more general result for our purposes.

**Proposition 2.3** ([NW15 Proposition 1.6]). Suppose $r$ is integral and positive, and fix $0 \leq s < r - 1$. Then

$$d(S^3(K), [s]) = d(L(r,1), [s]) - 2 \max \{V_s, V_{r-s}\}.$$

Among many of its applications, this result enables the following lemma.

**Lemma 2.4** ([HLZ15 Lemma 2.5]). For all $s \in \mathbb{Z}$, the integers $V_s$ and $H_s$ are related by

$$H_s - V_s = s.$$

We will involve the $d$-invariants later in Section 5.1 when necessary.

2.2. $\widehat{HF}$ via immersed curves. Bordered Heegaard Floer homology, introduced by Lipshitz, Ozsváth, and Thurston, provides a cut-and-paste style of computing $\widehat{HF}$ for a 3-manifold. This is done by decomposing along a surface, and then recovering Floer homology by a suitable means of pairing the relative Floer invariants for the decomposed pieces [LOT18]. Dubbed the pairing theorem, we will invoke it in immersed curves form due to Hanselman, Rasmussen, and Watson [HRW16].

Let $M$ be an orientable 3-manifold with torus boundary. Choose $\alpha, \beta$ in $\partial M$ so that $(\alpha, \beta)$ forms a parameterization of $\partial M$, and fix a basepoint $z \in \partial M$. The invariant $\widehat{HF}(M)$ is a collection of immersed curves in $T_M = \partial M \setminus z$, possibly decorated with local systems, and up to regular homotopy of the curves. When $M = S^3 \setminus \nu K$, we will use a preferred meridian-longitude basis $\{\mu, \lambda\}$.

The manifolds in this paper all happen to be loop type (see [HW15]), which means that their associated immersed curves invariant has trivial local systems. If the invariant has multiple curve components, then they are connected by pairs of edges which we denote with a grading arrow. These are presented in Figure 1 and they carry an integral weight $w$ useful for determining Maslov grading differences. Let $M = S^3 \setminus \nu K$. We can lift $\widehat{HF}(M)$ to the infinite cylindrical cover $\overline{T}_M$, where each lifted marked point resides within a neighborhood of the lift of the meridian $\overline{\mu}$. One of the curves wraps around the cylinder, and we will use $\overline{\gamma}$ to denote this component. Figure 2 shows a centered lift of the invariant for the complement of a hypothetical example of a thin knot $K$ with $g(K) = 2$ and $\tau(K) = 1$. 

![Figure 1. Edges of a grading arrow either follow or oppose the orientations of the attached curve components.](image-url)
The lifts of the marked points will be taken to lie at purely half-integral heights, so that curve components cross at integral heights. With this at hand, the lifted curve invariant also encodes a few numerical and concordance invariants of $K$. For example, the Seifert genus is given by the height of the tallest curve component. Additionally, the height around which $\tau$ wraps is precisely the Ozsváth-Szabó invariant $\tau(K)$. Hom’s $\epsilon$ invariant may also be determined by observing what $\tau$ does next. It curve turns downwards, upwards, or continues straight corresponding to $\epsilon(K)$ being 1, -1, and 0, respectively. These two invariants determine the slope $\tau$ outside of a thin vertical strip surrounding the lifts of the marked point, given by $2\tau(K) - \epsilon(K)$.

Recall that $\widehat{HF}(K)$ detects $g(K)$ due to [OS04a]. Looking in $T_M$, genus detection manifests itself in $\widehat{HF}(M)$ by ensuring that some curve component crosses at height $g(K)$. The immersed curves also satisfy a very powerful constraint related to a conjugation symmetry. For invariants of knot complements of $S^3$, this means that the curves are invariant under rotation by $\pi$.

**Theorem 2.5 ([HRW18 Theorem 7]).** The invariant $\widehat{HF}(M)$ is symmetric under the elliptic involution of $\partial M$. Here, the involution is chosen so that $z$ is a fixed point.

If $K$ admits either a horizontally or vertically simplified basis for $\text{CFK}^{-}(K)$, then the procedure of [HRW18 Proposition 47] allows one to construct $\widehat{HF}(M)$ from $\text{CFK}^{-}(K)$. The special case when $\text{CFK}^{-}(K)$ is both horizontally and vertically simplified enables us to quickly generate these curves. This condition holds if $K$ is a thin knot, and in particular we have that every arrow encoding the differential in $\text{CFK}^{-}(K)$ has length one. These properties imply two features of $\widehat{HF}(M)$, which are effectively the immersed curves analog of [Pet13 Lemma 7]:

- The essential component $\gamma$ winds between adjacent basepoints determined by $\tau(K)$, before ultimately wrapping around the cylinder.
- Every other component is a simple figure-eight, enclosing vertically adjacent basepoints.

**Definition 2.6.** Let $e_n$ denote the number of simple figure-eight components at height $n$ of $\widehat{HF}(M)$, viewed in $T_M$.

We have $e_{-n} = e_n$ due to Theorem 2.5 and Figure 2 provides an example with $e_0 = 0$ and $e_{-1} = e_1 = 1$. With individual properties of the curve invariants handled, we turn to the main reason for their involvement.

**Theorem 2.7 ([HRW18 Theorem 2]).** Consider the gluing $M_1 \cup_h M_2$, where the $M_i$ are compact, oriented 3-manifolds with torus boundary and $h : \partial M_2 \to \partial M_1$ is an orientation reversing homeomorphism for which $h(z_2) = z_1$. Then

$$\widehat{HF}(M_1 \cup_h M_2) \cong HF(\widehat{HF}(M_1), h(\widehat{HF}(M_2))),$$

where intersection Floer homology is computed in $T_M$, and the isomorphism is one of relatively graded vector spaces that respects the Spin$^c$ decomposition.

More precisely, $HF(\widehat{HF}(M_1), h(\widehat{HF}(M_2)))$ decomposes over Spin$^c$ structures and carries a relative Maslov grading on each Spin$^c$ summand. Theorem 2.7 places these in correspondence with the Spin$^c$ decomposition on $\widehat{HF}(M_1 \cup_h M_2)$, and also ensures the relative Maslov gradings agree. This is best seen when viewing Dehn surgery as such a gluing, continuing to use $M$ for $S^3 \setminus \nu K$. We have $S^3_r(K) = M \cup_{h_r}(D^2 \times S^1)$ with $h_r$ the slope-$r$ gluing map. Then Theorem 2.7 provides

$$\widehat{HF}(S^3_r(K)) \cong HF(\widehat{HF}(M), h_r(\widehat{HF}(D^2 \times S^1))).$$

The Spin$^c$ decomposition is recovered by using $r$ vertically-adjacent lifts of $h_r(\widehat{HF}(D^2 \times S^1))$, which is the precise number required to lift every intersection from $T_M$ to $T_M$ without duplicates. This is motivated by the example in Figure 3 showing the pairing of curves that recovers $\widehat{HF}(S^3_r(T(2,5)))$. The invariant for the solid torus simply consists of a horizontal essential
curve, and so \( h_4(\hat{HF}(D^2 \times S^1)) \) is a slope 4 curve in the punctured torus. We have four lifts of \( h_4(\hat{HF}(D^2 \times S^1)) \), each generating intersections in correspondence with the four spin\(^c\) summands of \( \hat{HF}(S^3(K)) \). These lifts are selected at heights in correspondence with the selected representatives of \( \mathbb{Z}/r\mathbb{Z} \) from Section 2. These are \(-1, 0, 1, \) and 2 for the example in Figure 3, and motivate the following definition when lifting further to the tiled-plane cover \( \tilde{T} \).

**Definition 2.8.** Let \( l^s_r = h_r(\hat{HF}(D^2 \times S^1)) \) denote the slope- \( r \) line in \( \tilde{T} \) that crosses between lifts of basepoints at heights congruent to \( s \) (mod \( r \)). These are selected so that each \( l^s_r \) crosses at height \( s \) in the same column of \( \tilde{T} \), with \( s \) taken to be the representative of \( [s] \) that falls within the \( \mathbb{Z}/r\mathbb{Z} \) range.

In this way, Theorem 2.7 implies

\[
\hat{HF}(S^3(K), [s]) \cong HF(\hat{HF}(S^3 \setminus \nu K), l^s_r).
\]

After using regular homotopy of curves to remove any possible intersections that do not contribute to homology, we end up with \( \dim (\hat{HF}(S^3(K), [s])) \) amounting to counting intersections between \( \hat{HF}(M) \) and \( l^s_r \). However, we have yet to incorporate the relative Maslov grading. We can compute grading differences between generators belonging to the same spin\(^c\) structure using a formula from [Han19].

Suppose \( x \) and \( y \) are two intersections belonging to the same \( [s] \in \text{Spin}^c(S^3(K)) \), arising from intersections between \( \hat{HF}(M) \) and \( l^s_r \). Further, let \( P \) be the bigon from \( y \) to \( x \) whose boundary consists of a (not necessarily smooth) path from \( y \) to \( x \) in \( \hat{HF}(M) \), concatenated with a path from \( x \) to \( y \) in \( l^s_r \). Defined this way, the boundary of \( P \) is a closed path that is smooth apart from right corners at \( x \) and \( y \), and possibly one or more cusps. The following formula follows from the conversion of bordered invariants into immersed curves, keeping track of grading contributions from relevant Reeb chords [HRW18, Section 2.2].

**Proposition 2.9.** Suppose \( x, y, \) and \( P \) are defined as above. Let Rot\((P)\) denote \( \frac{1}{2\pi} \) times the total counterclockwise rotation along the smooth sections of \( P \), let Wind\((P)\) denote the net winding number of \( P \) around enclosed basepoints, and finally let Wght\((P)\) be the sum of weights (counted with sign) of all grading arrows traversed by \( P \). Then

\[
M(x) - M(y) = 2\text{Wind}(P) + 2\text{Wght}(P) - 2\text{Rot}(P).
\]

If \( l^s_r \) intersects a simple figure-eight component at height \( n \), it generates a **right intersection** \( y^n \) and a **left intersection** \( x^n \). Figure 4 shows off the three types of bigons that will typically appear. The first type has \( P \) connecting a right and left intersection of the same simple figure-eight.
The bigon encloses a single basepoint with positive winding number, total counterclockwise rotation along smooth sections as \( \pi \), and no contribution from traversed grading arrows. These traits imply \( M(x^n) - M(y^n) = 1 \). The second and third types are the more interesting ones, and have the same winding number of enclosed basepoints, but the rotation and grading arrow contributions to \( M(y^n) - M(a^s) \) initially appear to be different. We will see later that for these bigons, the \( 2\text{Wght}(P) - 2\text{Rot}(P) \) component of the grading difference is the same.

\[\text{Figure 4.} \text{ Bigons used to determine the relative Maslov grading. Example (a) does not involve a grading arrow, while (b) and (c) (with a cusp) do.}\]

3. Thin knots and Maslov grading differences

Throughout this section, let \( K \) be a thin, hyperbolic knot and let \( M \) denote \( S^3 \setminus \nu K \). To enable swift grading comparisons later on, let us designate a reference intersection associated to \( [s] \in \text{Spin}^c(S^3_r(\nu K)) \). We will define a \textit{vertical intersection} to be an intersection between \( l^s_\nu \) and a vertical segment of \( \gamma \), within the neighborhood of \( \vec{\nu} \), provided they exist. If \( s \) satisfies \( 0 \leq |s| < |\tau(K)| \), then such an intersection occurs and we will denote it using \( a^s \). Alternatively, if \( |s| \geq \tau(K) \geq 0 \) then any intersection between \( l^s_\nu \) and \( \gamma \) is outside any neighborhood of the lifts of the marked point in \( T_M \). In this case \( l^s_\nu \) intersects \( \gamma \) once if \( \tau(K) \geq 0 \), and so \( a^s \) will denote this lone intersection. When \( \tau(K) < 0 \) and \( s \geq 0 \), we let \( a^s \) denote the intersection between \( l^s_\nu \) and \( \gamma \) to the left of \( \vec{\nu} \). Analogously when \( \tau(K) < 0 \) and \( s < 0 \), we will have \( a^s \) be the intersection between \( l^s_\nu \) and \( \gamma \) to the right of \( \vec{\nu} \). It is likely helpful to reference Figure 5 for these different possibilities. While cumbersome, this scheme allows us to label the intersection that often corresponds via the Pairing theorem to a generator with the the least Maslov grading.

\[\text{Figure 5.} \text{ The possibilities for the reference intersection } a^s. \text{ (a) has } \tau(K) = 0, \text{ (b) has } \tau(K) > 0 \text{ and } |s| < \tau(K), \text{ and (c) has } \tau(K) < 0 \text{ with two curves representing } s \geq 0 \text{ in red and } s < 0 \text{ in purple. The case when } \tau(K) > 0 \text{ and } |s| \geq \tau(K) \text{ is similar to (a).}\]

It will also be particularly useful to know the winding number of enclosed lifts of the marked point of specific regions. Consider the neighborhood of \( \vec{\nu} \) in \( \overline{T}_M \) that contains the lifts of the
marked points, which is also wide enough to enclose the vertical segments of \( \gamma \). Intersect \( \gamma \) with a horizontal line \( l^s \) slightly longer than this neighborhood at height \( s \), so that these segments together bound regions enclosing basepoints. We will define \( \tilde{H}_s \) to be the number of enclosed lifts of the marked point in the region bounded above by \( l^s \), on the side(s) by the neighborhood of \( \mu \), and elsewhere by \( \gamma \). If the region is empty, then \( \tilde{H}_s = 0 \). This number coincides with the invariant \( H_s \) from the mapping cone formula for all examples encountered by the author. However, we will use this makeshift definition in lieu of a HF\(^*\) immersed curves theory.

Analogously, there is often a region where \( l^s \) bounds from below and the number of enclosed lifts of the marked point of such a region will be denoted by \( \tilde{V}_s \). These are depicted in Figure 6.

Due to Theorem 2.5, we recover both \( \tilde{H}_s - \tilde{V}_s \) and \( \tilde{H}_s - \tilde{H}_{-s} = \frac{1}{2}(s - (-s)) = s \).

From the discussion in the previous section, we know that the form of \( \hat{HF}(M) \) is very restricted. Our goal is to leverage this to constrain gradings on \( \hat{HF}(S^3_r(K), [s]) \cong HF(\hat{HF}(M), l^s) \) to obstruct reducible surgeries. We use multisets, which are sets with repetition allowed, to collect these relative Maslov gradings.

**Definition 3.1.** Let \([s] \in \text{Spin}^c(S^3_r(K))\) be arbitrary with reference intersection \( a^s \). For any intersection \( y \) of \( HF(\hat{HF}(M), l^s) \), let \( M_{rel}(y) \) denote the grading difference \( M(y) - M(a^s) \). We define the desired multiset by

\[
MR^{[s]} := \left\{ M_{rel}(y) \mid y \in \hat{HF}(M) \cap l^s \right\}.
\]

Further, let \( \text{Width}(MR^{[s]}) \) denote the difference between the largest and smallest elements of this multiset.

As defined, \( MR^{[s]} \) is only an invariant up to translation, which means its width is an invariant. Next, we establish lemmas that enable us to swiftly compute grading differences. For a bigon \( P \) used to calculate a grading difference, we will first determine the contribution due to \( 2\text{Wght}(P) - 2\text{Rot}(P) \) using the knot Floer homology of \( K \). This is performed using an analogous bigon \( P_K \) for \( \hat{HF}_K \), and then we show that the same properties hold for the bigon \( P \) for \( \hat{HF} \).

**Lemma 3.2.** Let \( y^n \) be a right intersection belonging to a simple figure-eight at height \( n \) of \( \hat{HF}(M) \), let \( a \) be an intersection from a different component of \( \hat{HF}(M) \) and \( l^s \), and suppose \( P \) is a bigon between them. If \( K \) is thin, then \( 2\text{Wght}(P) - 2\text{Rot}(P) = -1 - \tau(K) - |n| \).
Proof. In the infinite cylinder $\overline{T}_M$, we can represent $\overline{\mu}$, the lift of the meridian of $T_M$, as the vertical line that pierces each lift of the marked point in $\overline{T}_M$. Let $a^{-\tau(K)}$ be the last intersection that $\gamma$ makes with $\overline{\mu}$ before wrapping around $\overline{T}_M$. Because $\widehat{HF}(M)$ is invariant under the action by the hyperelliptic involution, the weights of the grading arrows connecting $\gamma$ to the simple figure-eights at heights $n$ and $-n$ are equivalent. From this we can assume that $n$ is non-negative, and use $|n|$ in future formulas otherwise.

Lift $\widehat{HF}(M)$ to $\overline{T}$ for convenience, and intersect it with $\overline{\mu}$. If we place $z$ and $w$ basepoints to the left and right, respectively, of every lift of the marked point, then $\widehat{HF}(M) \cong HF(\widehat{HF}(M), \overline{\mu})$ due to [HRW18, Theorem 51]. This pairing is depicted in Figure 7. The formula in Definition 2.29 still holds with the adjustment that Wind is modified to count the net winding number of enclosed $w$ basepoints, denoted Wind$_w$.

![Figure 7](image)

Since $\widehat{HF}(M)$ has a simple figure-eight component at height $n$, there must be a generator $\eta$ of $\widehat{HF}(K)$ with $A(\eta) = n + 1$. Let $P_K$ be the bigon from $a^{-\tau(K)}$ to $\eta$ that traverses the grading arrow connecting the relevant components of $\widehat{HF}(M)$, visible in Figure 7 with $\tau(K) \geq 0$ and $\tau(K) < 0$, respectively. To determine Wght($P_K$) directly would require care for the orientations of the grading arrow. However since we are after a different term, we can abuse notation by having every grading arrow connect to the right side of a simple figure-eight, regardless of its orientation. Essentially, any change that Wght($P_K$) experiences between the two ways of attaching the grading arrow is inverted and absorbed by Rot($P$), so that $2\text{Wght}(P_K) - 2\text{Rot}(P_K)$ remains unchanged.

If $\tau(K) \geq 0$ so that $A(a^{-\tau(K)}) < n$, we have

$$M(\eta) - M(a^{-\tau(K)}) = 2\text{Wind}_w(P_K) + 2\text{Wght}(P_K) - 2\text{Rot}(P_K).$$

However since $K$ is thin, it follows that

$$M(\eta) - M(a^{-\tau(K)}) = A(\eta) - A(a^{-\tau(K)}) = A(\eta) + \tau(K).$$

Then $2\text{Wght}(P_K) - 2\text{Rot}(P_K) = A(\eta) - 2\text{Wind}(P_K) + \tau(K)$. Since Wind($P_K$) = $A(\eta) + \tau(K)$, we have

$$2\text{Wght}(P_K) - 2\text{Rot}(P_K) = A(\eta) - \tau(K) = -1 - \tau(K) - n.$$
traverses the grading arrow in reverse, visible in Figure 7. Traveling the grading arrow in reverse means that we have \( M(a^{-\tau(K)}) - M(\eta) = 2\text{Wind}_w(P_K) - 2\text{Wght}(P_K) - 2\text{Rot}(P_K) \), and so
\[
-2\text{Wght}(P_K) - 2\text{Rot}(P_K) = M(a^{-\tau(K)}) - M(a^n) - 2\text{Wind}_w(P) \\
= -\tau(K) - (n + 1) - 2(-\tau(K) - (n + 1)) \\
= 1 + \tau(K) + n.
\]

Due to the shape of \( P_K \), the bigon has a cusp near the grading arrow regardless of how it connects these components, and so \( \text{Rot}(P_K) = 0 \). Then we have \( 2\text{Wght}(P_K) - 2\text{Rot}(P_K) = 2\text{Wght}(P_K) + 2\text{Rot}(P_K) = -1 - \tau(K) - n \), as claimed.

With the formula established for \( P_K \), we will now show that it is satisfied for a bigon between generators of \( \widehat{HF}(\widehat{HF}(M), l^*_r) \) with similar attributes. Let \( y^n \) be a right intersection from the simple figure-eight at height \( n \), and let \( a \) be an intersection from a vertical segment of \( \overline{\tau} \) and \( l^*_r \). With \( P \) denoting the bigon from \( a \) to \( y^n \), we see that \( P \) must traverse the same grading arrow that \( P_K \) traversed, and so \( \text{Wght}(P) = \text{Wght}(P_K) \). Additionally, it is straightforward to see that \( \text{Rot}(P) = \text{Rot}(P_K) \) after tilting the bigons as well, with visual given in Figure 8. This completes the proof. □

![Figure 8](image.png)

**Figure 8.** Tilting bigons to show they have equivalent net clockwise rotation along their boundaries. (a) The bigon \( P_K \) from \( a^{-\tau(K)} \) to \( \eta \). (b) The bigon \( P \) from \( a \) to \( y^n \).

The following proposition considers left and right intersections of a simple figure-eight whose height \( n \) is less than \(|\tau(K)|\). There is then a nearby vertical intersection \( a^n \), and we will see that these three intersections have little difference in grading.

**Proposition 3.3.** Let \( K \) be thin and have \( M \) denote \( S^3 \setminus \nu K \). Further, let \( x^n \) and \( y^n \) be left and right intersections belonging to a simple figure-eight of \( \widehat{HF}(M) \) with height \( 0 \leq n < |\tau(K)| \), and let \( a^n \) be the nearby vertical generator. Then \( -1 \leq M(y^n) - M(a^n) \leq 0 \) and \( 0 \leq M(x^n) - M(a^n) \leq 1 \).

**Proof.** If \( P \) is the bigon between \( a^n \) and \( y^n \), we have \( 2\text{Wght}(P) - 2\text{Rot}(P) = -1 - \tau(K) - |n| \) due to Lemma 3.2. Due to the hyperelliptic involution invariance of \( \widehat{HF}(M) \), we can take \( 0 \leq n < |\tau(K)| \). We have \( \text{Wind}(P) \) is \( \tilde{H}_n \) if \( \tau(K) \geq 0 \) or \( \tilde{V}_n \) if \( \tau(K) < 0 \), the values of which depend on the parity of \( n \) and \( \tau(K) \) when \( K \) is thin. The simple structure of \( \overline{\tau} \) for a thin knot together with a counting argument for \( \tau(K) > 0 \) yields

\[
\tilde{H}_n = \begin{cases} 
\frac{n + \tau(K)}{2} & \text{parity}(n) = \text{parity}(\tau(K)) \\
\frac{n + \tau(K) + 1}{2} & \text{parity}(n) \neq \text{parity}(\tau(K)).
\end{cases}
\]
Then for \( \tau(K) > 0 \) we have \( M(y^n) - M(a^n) = 2\widetilde{H}_n - 1 - \tau(K) - n \) implies \( M(y^n) - M(a^n) \) is either -1 or 0. Since \( M(x^n) - M(y^n) = 1 \), we see that \( M(x^n) - M(a^n) \) is either 0 or 1, handling the \( \tau(K) > 0 \) case.

When \( \tau(K) < 0 \), the bigon \( P \) runs from \( y^n \) to \( a^n \), encloses \( \tilde{V}_n \) lifts of the marked points, traverses the grading arrow in reverse, and has \( \text{Rot}(P) = 0 \). Figure \ref{fig:grading-difference} shows that \( \tilde{V}_n \) with \( \tau(K) < 0 \) is the same as \( \tilde{V}_n = \tilde{H}_{-n} \) with \( \tau(K) \geq 0 \), except using \( -\tau(K) \) or \( -\tau(K) - 1 \) in the formula above. Using Lemma \ref{lem:grading-difference} and the \( -\tau(K) \) modified formula for \( H_{-n} \), we have \( M(a^n) - M(y^n) = 2\tilde{H}_{-n} + 1 + \tau(K) + n \). This is either 1 or 0, and so \( M(y^n) - M(a^n) \) is either \(-1 \) or 0 and analogously \( M(x^n) - M(a^n) \) is either 0 or 1. \( \square \)

Because \( M(x^n) - M(y^n) = 1 \), these possibilities happen in pairs. A simple figure-eight at height \( n < |\tau(K)| \) contributes either \( \{ M_{rel}(a^n), M_{rel}(a^n) - 1, M_{rel}(a^n) \} \subseteq MR_{[1]} \) or \( \{ M_{rel}(a^n), M_{rel}(a^n), M_{rel}(a^n) + 1 \} \subseteq MR_{[1]} \). An example of this to keep in mind is when looking at large surgery on the figure-eight knot \( 4_1 \).

This situation we have \( \{ 0, -1, 0 \} = MR_{[0]} \), and the right intersection contributing \(-1 \) to \( MR_{[0]} \) actually has the smallest relative Maslov grading. This proposition then allows us to determine which intersection associated to \( [s] \in \text{Spin}^c(S^3_1(K)) \) has the smallest relative Maslov grading depending on parity(\( \tau(K) \)):

- If \( \tau(K) \geq 0 \) and parity(\( s \)) = parity(\( \tau(K) \)), and there is a right intersection \( y^n \), then \( M_{rel}(y^n) = -1 \) is the smallest relative grading of \( MR_{[1]} \).

- If \( \tau(K) \geq 0 \) and parity(\( s \)) = parity(\( \tau(K) \)), and there is no simple figure-eight at height \( s \), then \( M_{rel}(a^n) = 0 \) is the smallest relative grading of \( MR_{[1]} \).

- If \( \tau(K) \geq 0 \) and parity(\( s \)) \neq parity(\( \tau(K) \)), then \( M_{rel}(a^n) = 0 \) is the smallest relative grading of \( MR_{[1]} \).

- If \( \tau(K) < 0 \), then \( M_{rel}(a^n) = 0 \) is the smallest relative grading of \( MR_{[1]} \).

The last component of the grading difference formula to handle is \( \text{Wind}(P) \). Lift both \( \widehat{HF}(M) \) and each \( l^r_\ast \) to the tiled plane \( \widehat{T} \), and let the 0th column be the neighborhood of the lift \( \widehat{\mu} \) for which each \( l^r_\ast \) intersects \( \widehat{\mu} \) at height \( [s] \). For \( [s] \in \mathbb{Z}/r\mathbb{Z} \) define \( w_s = \frac{n - [s]}{r} \), with \( n \) the largest natural number satisfying \( 0 \leq n \leq g(K) - 1 \) and \( n \equiv [s] \pmod{r} \). This number represents the number of columns of marked points in \( \widehat{T} \) between \( a^n \) and a potential furthest right intersection \( y^n \). Further, because the slopes we consider satisfy \( r \leq 2g(K) - 1 \), we have \( w_s \geq 0 \).

It is certainly possible that a simple figure-eight component may not exist at this height, it is still sufficient for the following strategy to suppose otherwise.

**Proposition 3.4.** For a given \( [s] \in \mathbb{Z}/r\mathbb{Z} \), let \( a^n \) be the chosen reference intersection and \( y^n \) be a right intersection of a furthest possible figure-eight component. If \( \tau(K) \geq 0 \), then

\[
\text{Wind}(P) = \widehat{H}_s + \sum_{i=1}^{w_s} (s + ir).
\]

If \( \tau(K) < 0 \), then

\[
\text{Wind}(P) = \begin{cases} 
\sum_{i=0}^{w_s} (s + ir) & [s] \geq 0 \\
\sum_{i=1}^{w_s} (s + ir) & [s] < 0,
\end{cases}
\]

where all sums are taken to be zero if empty.

When \( \tau(K) \geq 0 \), the contribution to \( \text{Wind}(P) \) from the 0th column of \( \widehat{T} \) is \( \widehat{H}_s \). The contribution from the \( i \)-th column is \( \widehat{H}_{s+ir} - \widehat{V}_{s+ir} = s + ir \), and is shown in Figure \ref{fig:grading-difference}. When \( \tau(K) < 0 \), we have the different choices for \( a^n \) depending on \( s \) influencing whether the contribution from
the 0th column is non-trivial. However in every column, the contribution to $\text{Wind}(P)$ is $H_{s+ir} - V_{s+ir} = s + ir$. Since these terms are always non-negative, it follows that the smallest relative grading belongs to an intersection in the 0th column.

$$\tilde{H}_s + \text{id} - \tilde{V}_s + \text{id} = s + \text{id}. $$ Since these terms are always non-negative, it follows that the smallest relative grading belongs to an intersection in the 0th column.

![Figure 9. Example bigons $P$ between $a^s$ and $y^n$, showing the contributions from each column to $\text{Wind}(P)$ for (a) $\tau(K) \geq 0$ and (b) $\tau(K) < 0$ with $s \geq 0$.](image)

4. Cases with $|\tau(K)| < g(K)$

Our objective is to build a collection of lemmas required to prove the main theorem. These vary depending on $r$ in relation to $g(K)$, and on $\tau(K)$ and its parity. The primary technique involves comparing the various $\text{Width}(MR^{[s]})$ to obstruct periodicity, typically done by showing that $\text{Width}(MR^{[s]})$ is maximal if $[s']$ is the spin structure associated to the line that crosses height $g(K) - 1$. At other times the widths will agree up to translation, but the multiplicity of specific elements of the grading multisets will not.

Use Theorem 2.7 to identify $\hat{HF}(S^3_r(K), [s]) \equiv HF(\hat{HF}(M), l^r_s)$. In order to halve the amount of comparisons to make, we leverage the fact that $\hat{HF}(S^3_r(K), [s]) \equiv HF(\hat{HF}(S^3_r(K), [-s])$ [OS04c].

In immersed curves form, Theorem 2.5 implies that intersections between $\hat{HF}(M)$ and $l^r_s$ in negative columns of $\tilde{T}$ are in correspondence with intersections of $\hat{HF}(M)$ and $l^-_s$ that belong to positive columns of $\tilde{T}$ (see Figure 10). Also, the self-conjugate spin structure(s) [0] (and possibly $[r/2]$) are symmetric in this way by default.

Recall that the smallest element of $MR^{[s]}$ is the relative grading of an intersection belonging to the 0th column of $\tilde{T}$, which is either the reference intersection $a^s$ or a nearby right/left intersection. This means that we can capture $\text{Width}(MR^{[s]})$ by considering non-negative intersections associated to both $[s]$ and $[-s]$. Note that since parity($[s]$) = parity($[-s]$), the need to translate a multiset by 1 is consistent if it arises.

**Definition 4.1.** The multiset $MR^{[s]}_+$ consists of the relative gradings of intersections between $\hat{HF}(M)$ and $l^r_s$ that belong to non-negative columns of $\tilde{T}$. We define $MR^{[s]}_-$ analogously, and notice that $\text{Width}(MR^{[s]}) = \max \left\{ \text{Width}(MR^{[s]}_+), \text{Width}(MR^{[s]}_-) \right\}$.

Due to how genus detection is expressed by $\hat{HF}(M)$, either $\gamma$ achieves height $g(K)$ (equivalent to $|\tau(K)| = g(K)$), or only a simple figure-eight at height $g(K) - 1$ achieves this desired height.
We see that \( \tau \) we will handle the which is the easier starting point. \( \tau \)

Subcase A1a: compare widths, we compute either equality depending on whether \( a \) depends on whether

\( \hat{\tau} \) is nearly determined by \( a \) is the smallest relatively graded intersection. To

This reason, we will suppose that \( 2 \)Wind(\( P \)) term satisfies 2Wind(\( P \)) \( \geq 2n \) while the other term is \( -1 - \tau(K) - n \). For this reason, we will suppose that \( \hat{H}(M) \) has a simple figure-eight at height \( n \), taken to be the largest integer satisfying both \( n \leq g(K) - 1 \) and \( n \equiv s \) (mod \( r \)). Let \( P' \) be the bigon between \( a^s \) and \( y^{s-1} \), and \( P \) the bigon between \( a^s \) and \( y^n \). Because the choice of \( a^s \) depends on \( \tau(K) \), we will handle the \( \tau(K) \geq 0 \) subcase first before handling the \( \tau(K) < 0 \) subcase.

Subcase A1a: \( \tau(K) \geq 0 \). Due to Lemma 3.2 Proposition 3.3 and Proposition 3.4 Width(\( MR^{[s]}_+ \)) is nearly determined by \( M_{rel}(y^n) \). We have \( M_{rel}(y^n) \leq \text{Width}(MR^{[s]}_+) \leq M_{rel}(y^n) + 1 \), with either equality depending on whether \( a^s \) is the smallest relatively graded intersection. To compare widths, we compute

\[
M_{rel}(y^{n-1}) = 2 \left( \hat{H}_{s'} + \sum_{i=1}^{w_{s'}} (s' + ir) \right) - 1 - (s' + w_{s'}r),
\]

and likewise

\[
M_{rel}(y^n) = 2 \left( \hat{H}_{s} + \sum_{i=1}^{w_{s}} (s + ir) \right) - 1 - (s + w_{s}r).
\]
Their difference is then
\[
M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2 \left( \tilde{H}_{s'} - \frac{w_{s'}}{w_s} \sum_{i=1}^{w_{s'}} (s' + ir) - \frac{w_s}{w_s} \sum_{i=1}^{w_s} (s + ir) \right) \\
- (s' + w_{s'}r - (s + w_s) - (s' - s) - r (w_{s'} - w_s).
\]

If \( s < s' \) so that \( w_s = w_{s'} \), then
\[
M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2((\tilde{H}_{s'} - \tilde{H}_s) + (2w_{s'} - 1)(s' - s) - (s' - s) - r(w_{s'} - w_s) \\
\geq 1,
\]
since \( w_{s'} > 0 \) and \( s' > s \) implies that \( \tilde{H}_{s'} \geq \tilde{H}_s \).

If \( s > s' \) so that \( w_s = w_{s'} - 1 \), then shifting \( P \) one column to the right in \( \bar{T} \) (see Figure 11) provides
\[
M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2 \left( (\tilde{H}_{s'} - \tilde{H}_s) + \frac{w_{s'}}{w_s} \sum_{i=1}^{w_{s'}} (s' + ir) - \frac{w_s}{w_s} \sum_{i=1}^{w_s} (s + (i-1)r) \right) \\
- (s' + r - s) \\
= 2 \left( (\tilde{H}_{s'} + s - \tilde{H}_s) + (s' + r - s) + (w_{s'} - 1)(s' + r - s) \right) \\
- (s' + r - s) \\
= 2 \left( (\tilde{H}_{s'} + s - \tilde{H}_s) + (s' + r - s) + (w_{s'} - 1)(s' + r - s) \right) \\
- (s' + r - s)
\]
\[= 2(\tilde{H}_{s'} + (s - \tilde{H}_s)) + (2w_{s'} - 1)(s' + r - s)\]
\[= 2(\tilde{H}_{s'} - \tilde{V}_s) + (2w_{s'} - 1)(s' + r - s)\]
\[= 2(\tilde{H}_{s'} - \tilde{H}_{s}) + (2w_{s'} - 1)(s' + r - s).\]

Notice that \(s' + s - 1 \leq 2(\tilde{H}_{s'} - \tilde{H}_{s}) \leq s' + s\) depending on the parities of \(s\) and \(s'\) together with \(s > s'\). Then we have
\[
M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2(\tilde{H}_{s'} - \tilde{H}_{s}) + (2w_{s'} - 1)(s' + r - s)
\geq s' + s - 1 + (2w_{s'} - 1)(s' + r - s)
\geq s' + s - 1 + s' + r - s
= 2s' - 1 + r
> 1,
\]
since \(w_{s'} > 0\) and \(s' < \frac{r - 1}{2}\) if there exists an \(s > s'\).

![Figure 11. Example bigons \(P'\) (split-shaded green and pink) and \(P\) (shaded pink) when \(w_{s'} = 1\). (a) has \(s < s'\), while (b) has \(s > s'\) together with the single column shift to the right.](image)

In both situations, we see that \(M_{rel}(y^{g-1}) - M_{rel}(y^n) \geq 1\). If this difference is greater than one, then
\[
\text{Width}(MR_{\pm}^{[s']}) \geq \text{Width}(MR_{\pm}^{[s]}) \geq M_{rel}(y^{g-1}) > M_{rel}(y^n) + 1 \geq \text{Width}(MR_{\pm}^{[s]}).
\]
This already handles the possibility where we need to translate \(MR_{\pm}^{[s]}\) by 1, so suppose \(M_{rel}(y^{g-1}) - M_{rel}(y^n) = 1\). This is possible only if \(\tilde{H}_s = \tilde{H}_{s'}, w_{s'} = 1\), and \(s = s' - 1\), which altogether imply that \(s = \tau(K)\). However, the widths only match if \(\text{Width}(MR_{\pm}^{[s]}) = M_{rel}(y^n) + 1\). This condition is equivalent to having \(\text{parity}(s) \neq \text{parity}(\tau(K))\), which is a contradiction. Therefore \(\text{Width}(MR_{\pm}^{[s']}) > \text{Width}(MR_{\pm}^{[s]})\), which completes the \(\tau(K) \geq 0\) subcase.

**Subcase A1b: \(\tau(K) < 0\).** Recall that the reference intersection \(a^s\) has no nearby left/right intersections belonging to a simple figure-eight. This means that \(a^s\) has the smallest relative
grading of $MR^{[s]}$, and so $\text{Width}(MR^{[s]}) = M_{rel}(y^n) + 1$. From Proposition 3.4 we see

$$\text{Wind}(P) = \begin{cases} 
s + \sum_{i=1}^{w_s}(s + ir) & s \geq 0, \\
-\sum_{i=1}^{w_s}(s + ir) & s < 0. 
\end{cases}$$

If $0 \leq s < s'$, then proceeding as before we have

$$M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2 \left( s' + \sum_{i=1}^{w_{s'}}(s' + ir) - \left( s + \sum_{i=1}^{w_s}(s + ir) \right) \right) - (s' - s) - r(w_{s'} - w_s)$$

$$= 2(s' + w_{s'}(s' - s) - (s' - s)$$

$$= (2w_{s'} + 1)(s' - s) \geq 3.$$ 

If $s < s' \leq 0$, then

$$M_{rel}(y^{g-1}) - M_{rel}(y^n) = (2w_{s'} - 1)(s' - s) \geq 1.$$ 

If $s > s'$, then as before we have $w_s = w_{s'} - 1$. If $s > s' \geq 0$, then

$$M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2 \left( s' - s + \sum_{i=1}^{w_{s'}}(s' + ir) - \sum_{i=1}^{w_s}(s + ir) \right) - (s' - s) - r(w_{s'} - w_s)$$

(column shift) $$= 2 \left( s' - s + \sum_{i=1}^{w_{s'}}(s' + ir) - \sum_{i=2}^{w_{s'}}(s + (i-1)r) \right) - (s' + r - s)$$

$$= 2(s' + (s' + r - s) + (w_{s'} - 1)(s' + r - s)) - (s' + r - s)$$

$$= 2s' + (2w_{s'} - 1)(s' + r - s) \geq 1.$$ 

In the event that $0 \geq s > s'$, we get

$$M_{rel}(y^{g-1}) - M_{rel}(y^n) = 2 \left( \sum_{i=1}^{w_{s'}}(s' + ir) - \sum_{i=1}^{w_s}(s + ir) \right) - (s' - s) - r(w_{s'} - w_s)$$

(column shift) $$= 2 \left( \sum_{i=1}^{w_{s'}}(s' + ir) - \sum_{i=2}^{w_{s'}}(s + (i-1)r) \right) - (s' + r - s)$$

$$= 2(s' + r + (w_{s'} - 1)(s' + r - s)) - (s' + r - s)$$

$$= 2(s + w_{s'}(s' + r - s) - (s' + r - s)$$

$$= 2s + (2w_{s'} - 1)(s' + r - s) \geq 2s + s' + r - s$$

$$\geq (s' + s) + r$$

$$\geq 1.$$ 

In every inequality we have $M_{rel}(y^{g-1}) > M_{rel}(y^n)$. Then

$$\text{Width}(MR^{[s]}) \geq \text{Width}(MR^{[s']}) = M_{rel}(y^{g-1}) + 1 > M_{rel}(y^n) + 1 = \text{Width}(MR^{[s]}),$$

for each $[s] \in \text{Spin}^e(S^3(K))$. This completes the $\tau(K) < 0$ subcase, and the proof. \hfill $\square$

**Case A2: $w_{s'} = 0$.** Recall that in this case we have $r \geq 2(g(K) - 1)$, so let us consider $r = 2g(K) - 1$ first. When $\tau(K) \geq 0$, the surgery slope is large enough so that every intersection lies in the 0th column of $\bar{T}$. Width alone as an invariant won’t be enough, so we will also
need to appeal to the multiplicities of the elements of the relative grading multisets. They will be used to show that only spin' structures with the same parity are obstructed. When we assume that \( S_2^2(K) \) is reducible later on, the fact that \( r \) is odd will provide a contradiction with periodicity. When \( \tau(K) < 0 \), we need far less sublety.

**Lemma 4.3.** Suppose \( K \) is thin, \( 0 \leq \tau(K) < g(K) \), and \( r = 2g(K) - 1 \). Then there exists an \([s'] \in \text{Spin}'(S_2^2(K))\) for which \([s] \neq [\pm s']\) satisfies \( MR[s] \cong MR[s'] \) up to translation only if parity\((s) = \text{parity}(s')\).

**Proof.** The spin' structure \([s']\) we want to consider has \([s] = g(K) - 1\). Suppose for the sake of contradiction that some \([s] \neq [\pm s']\) satisfies \( MR[s] \cong MR[s'] \) up to translation and parity\((s) \neq \text{parity}(s').\) We know that \( r > 1 \) forces \( g(K) > 1 \), and also that each \(-l_s^a\) intersects \( \tilde{HF}(M) \) exactly once due to this large surgery slope. Because the choice of reference generator \( a' \) depends on \( \tau(K) \), let us split into two cases: \( \tau(K) \geq 0 \) and \( \tau(K) < 0 \).

Assume \( \tau(K) \geq 0 \). Because all intersections lie within the 0th column of \( \tilde{T} \), we will instead use the hyperelliptic involution invariance of \( \tilde{HF}(M) \) to only consider \( s \geq 0 \). If \( \tilde{HF}(M) \) has no simple figure-eight at height \( s \), then \( \text{Width}(MR[s]) = 0 \) immediately does not match \( \text{Width}(MF[s']) \geq 1 \), so we may as well assume that there is a simple figure-eight at height \( s \). We have \( M_{rel}(y^s') = 2\tilde{H}_{s'} - 1 - \tau(K) - s' = s' - 1 - \tau(K) \) by Lemma 3.2 and Proposition 3.4 since \( \tilde{H}_{s'} = s' \) when \( \tau(K) \leq g(K) - 1 = s' \). Further,

\[
M_{rel}(y^s') - M_{rel}(y^s) = 2\tilde{H}_{s'} - 1 - \tau(K) - s' - (2\tilde{H}_s - 1 - \tau(K) - s) = 2(\tilde{H}_{s'} - \tilde{H}_s) - (s' - s).
\]

If \( s > \tau(K) \), then \( \tilde{H}_s = s \) implies that \( M_{rel}(y^s') - M_{rel}(y^s) = s' - s \geq 1 \). But then

\[
\text{Width}(MR[s]) = M_{rel}(y^s') + 1 > M_{rel}(y^s) + 1 \geq \text{Width}(MR[s]),
\]

so we must have \( s \leq \tau(K) \) together with \( \text{Width}(MR[s']) = 1 \). Notice that \( \text{Width}(MR[s']) = \text{Width}(y^s') + 1 = s' - \tau(K) > 1 \) if \( \tau(K) < s' - 1 \), and so we are also forced to have either \( \tau(K) = s' - 1 \) or \( \tau(K) = s' \). In both cases we have \( \text{Width}(MR[s']) = 1 \). Since using width as an invariant has been exhausted, let us count multiplicities of elements of the \( MR[s] \)'s next.

Recall that \( e_n \) denotes the number of simple figure-eights at height \( n \) of \( \tilde{HF}(M) \). Further, we need \( e_{s'} = e_{s'} \) in order to have \( |MR[s]| = |MR[s']| \). We have assumed that \( \text{parity}(s) \neq \text{parity}(s') \), so one of these two multisets contains \(-1\) and must be translated by \( 1 \) to make \( 0 \) the smallest element. This translated multiset will then contain \( 0 \) with multiplicity \( e_{s'} \), while the other multiset will contain \( 0 \) with multiplicity \( e_{s'} + 1 \). This is the desired contradiction. \( \square \)

When \( r = 2(g(K) - 1) \), we will end up having \( \text{Width}(MR[s]) = 1 \) for every \([s]\) if \( \tau(K) \) is large enough. This means relative grading information alone will not be enough, and so we will return to such cases in Section 5.

**Lemma 4.4.** Suppose \( K \) is thin, \( 0 \leq \tau(K) < g(K) - 2 \), and \( r = 2(g(K) - 1) \). Then there exists an \([s'] \in \text{Spin}'(S_2^2(K))\) for which every \([s] \neq [\pm s']\) satisfies \( MR[s] \neq MR[s'] \) up to translation.

**Proof.** We again use \( s' = g(K) - 1 \), and notice that when \( \tau(K) < g(K) - 2 \), we have

\[
M_{rel}(y^s') = 2(g(K) - 1) - 1 - \tau(K) - (g(K) - 1) = g(K) - 2 - \tau(K) > 0.
\]

This shows that \( \text{Width}(MR[s]) = M_{rel}(y^s') + 1 > 1 \) when \([s] \neq [\pm s']\) with \( |s| \leq \tau(K) \) has \( \text{Width}(MR[s]) = 1 \) due to Proposition 3.3 so suppose \( \tau(K) < |s| < s' \). In this case, \( \text{Width}(MR[s]) \leq M_{rel}(y^s) + 1 \), but we also have \( M_{rel}(y^s') - M_{rel}(y^s) = s' - |s| > 0 \). Then \( \text{Width}(MR[s]) < \text{Width}(MR[s']) \), which completes the proof. \( \square \)

When \( \tau(K) < 0 \), the fact that the reference intersection \( a' \) lies outside of the neighborhood of \( \bar{\mu}_0 \) is very convenient. This is an example of a non-vertical intersection, which is an intersection between \(-l^a_s\) and \( \tau \) that lies outside of a neighborhood of a lift \( \bar{\mu} \).

**Lemma 4.5.** Suppose \( K \) is thin, \(-g(K) < \tau(K) < 0 \), and \( r \geq 2(g(K) - 1) \). Then there exists an \([s'] \in \text{Spin}'(S_2^2(K))\) for which every \([s] \neq [\pm s']\) satisfies \( MR[s] \neq MR[s'] \) up to translation.
Proof. Since \( w_{s'} = 0 \), we again have \( [s'] = g(K) - 1 \). Notice that each \( l''_s \) gives rise to only two non-vertical intersections around the 0th column and intersections at height \( s \) when \( [s] \neq [\pm s'] \). We have \( s' \) maximal when \( w_{s'} = 0 \), so use hyperelliptic involution invariance to assume \( 0 \leq s < s' \).

Recall that \( \text{Width}(MR^{[s]}) = M_{rel}(y^s) + 1 \) under the assumptions that \( \tau(K) < 0 \). The formula for \( \text{Wind}(P) \) does not depend on \( \tau(K) \), which means

\[
M_{rel}(y^{s'}) - M_{rel}(y^s) = 2s' - 1 - \tau(K) - s' - (2s - 1 - \tau(K) - s)
= s' - s.
\]

Then \( \text{Width}(MR^{[s']}) = M_{rel}(y') + 1 > M_{rel}(y^s) + 1 = \text{Width}(MR^{[s]}) \), which implies \( MR^{[s]} \neq MR^{[s']} \).

In the following section we address the remaining cases involving \( |\tau(K)| = g(K) \), as well as the few unresolved cases of this section. In particular, the cases with \( g(K) - 2 \leq \tau(K) < g(K) \) and \( r = 2(g(K) - 1) \) are handled in Lemma 5.5.

5. Remaining Cases and Absolute Gradings

With the case analysis for \( |\tau(K)| < g(K) \) out of the way, we turn to the more difficult part.

Case B: \( |\tau(K)| = g(K) \). When \( |\tau(K)| \) is at its largest, the essential curve \( \gamma \) suffices to indicate \( g(K) \) and we are not guaranteed a simple figure-eight at height \( g(K) - 1 \). For these cases we still choose \([s']\) so that \( g - 1 \equiv s' \pmod{r} \) and continue to use \( w_s \), except now modifying it to just be the largest multiple of \( r \) so that \( s + w_s r < g(K) \). The \( \tau(K) = -g(K) \) case is easier, so we start there.

Case B1: \( \tau(K) = -g(K) \).

Lemma 5.1. Suppose \( K \) is thin with \( \tau(K) = -g(K) \), and let \( 1 < r \leq 2g(K) - 1 \). Then there exists an \([s'] \in \text{Spin}^c(S^3_r(K))\) for which every \([s] \neq [\pm s']\) satisfies \( MR^{[s]} \neq MR^{[s']} \) up to translation.

Proof. Recall the labeling scheme from Figure 5. The reference intersection \( a^s \) is immediately to the left of the 0th column if \( s \geq 0 \), and is similarly immediately to the right of the 0th column if \( s < 0 \). In general we will label these generators \( x^s \) and \( x^s ' \), respectively. Let us dispense with the \( w_{s'} = 0 \) case first.

Notice that each \( MR^{[s]} \) contains two elements whose difference is precisely \( 2|s| \). These arise from \( M_{rel}(x^s) - M_{rel}(x^s') = 2H_s - 1 - (2V_s - 1) = 2(H_s - V_s) = 2s \). We also see that \( 2H_s - 1 \leq \text{Width}(MR^{[s]}) \leq 2H_s \) if \( s \leq 0 \) and \( 2V_s - 1 \leq \text{Width}(MR^{[s]}) \leq 2V_s \) if \( s > 0 \), with the even equalities achieved if an appropriate generator from a simple figure-eight exists at height \( s \).

So if some \([s] \neq [\pm s']\) is to achieve \( MR^{[s]} \equiv MR^{[s']} \) up to translation, we should see that the widths of these multisets agree and that there exist pairs with grading differences \( 2|s| \) and \( 2|s'| \). These are only possibly simultaneously true if \( \bar{V}_s = \bar{H}_{s'} \) and \( \bar{H}_s = \bar{V}_{s'} \), which forces \( s = -s' \) with \( K \) thin. Thus, \( w_{s'} > 0 \).

If \( w_{s'} > 0 \), we can appeal to \( MR^{[s']} \) achieving maximal width once again. Due to the formula for \( \text{Wind}(P) \) when \( \tau(K) < 0 \), the grading difference between consecutive vertical intersections between \( \gamma \) and \( l''_s \) around the \( i \)th column is \( 2(s + ir) \). As before this happens as \( 2\bar{H}_{s+ir} - 1 - (2\bar{V}_{s+ir} - 1) = 2(\bar{H}_{s+ir} - \bar{V}_{s+ir}) = 2(s + ir) \), and is positive. For this reason it is often the case that the vertical intersection on the left side of the \( w_s \)th column, which we now denote by \( b^s \), has the largest relative grading in \( MR^{[s]} \). As before, by appealing to the hyperelliptic involution invariance of \( \bar{HF}(M) \) we see that \( \text{Width}(MR^{[s]}) \) is either \( M_{rel}(b^s) \) or \( M_{rel}(b^{-s}) \) when \( w_s > 0 \). We will obtain our desired contradiction by comparing \( M_{rel}(b^s) \) to every \( M_{rel}(b^s) \) with \( s \neq \pm s' \), just as in the lemmas of the previous section.
Chaining the grading differences of vertical intersection pairs from $b^s$ back to $a^s$, we see that

$$
M_{rel}(b^s) = \begin{cases} 
2 \left( \sum_{i=0}^{w_s-1} (s + ir) + \overline{H}_{s+w_s} \right) - 1 & \text{if } s \geq 0 \\
2 \left( \sum_{i=1}^{w_s-1} (s + ir) + \overline{H}_{s+w_s} \right) - 1 & \text{if } s \leq 0,
\end{cases}
$$

with empty sums taken to be zero as before. Since it can be hectic determining when such a sum is empty, we break into more cases.

When $s < s'$ we have $w_s = w_{s'}$, and it is straightforward to check that

$$M_{rel}(b^{s'}) - M_{rel}(b^s) \geq 2(\bar{H}_{s'+w_{s'}} - \bar{H}_{s+w_s}) > 0.$$ 

This follows because the various multiples of $(s' - s)$ are positive if they appear, and because $\bar{H}_{s'+w_{s'}} > \bar{H}_{s+w_s}$ when $s < s'$.

Let us begin the $s' < s$ cases with $w_{s'} = 1$. For $0 \leq s' < s$ we can once again use a column shift to see

$$M_{rel}(b^{s'}) - M_{rel}(b^s) = 2(s' + H_{s'+r} - H_s) > 0,$$

since $s' \geq 0$ and $H_{s'+r} > H_s$. The same inequality holds if $s' \leq 0 < s$, together with dropping the $s'$ term. For $s' < s \leq 0$ with $w_s = 0$, we are forced to have Width$(MR^{s'}) = 2\bar{H}_s - 1$ if $s \geq 0$ and Width$(MR^{s'}) = 2\bar{V}_s - 1$ if $s \leq 0$, since Width$(MR^{s'})$ is guaranteed to be odd. For the former we get

$$M_{rel}(b^{s'}) - M_{rel}(b^s) = 2(\bar{H}_{s'+r} - \bar{H}_s) > 0,$$

since $s < s' + r$. The latter yields

$$M_{rel}(b^{s'}) - M_{rel}(b^s) = 2(\bar{H}_{s'+r} - \bar{V}_s) > 0,$$

since $s > s'$.

Finally we are left with $w_{s'} > 1$ with $s' < s$. If we have $0 \leq s' < s$, then the fact that $\bar{H}_{s'+w_{s'}}$ is maximal ensures

$$M_{rel}(b^{s'}) - M_{rel}(b^s) = 2 \left( \sum_{i=0}^{w_{s}-1} (s' + ir) - \sum_{i=0}^{w_{s'}-1} (s + ir) \right) + 2(\bar{H}_{s+w_s} - \bar{H}_{s+(w_s-1)r})$$

(Column shift) $$= 2 \left( s' + \sum_{i=1}^{w_{s'}-1} (s' + ir) - \sum_{i=1}^{w_{s}-1} (s + (i-1)r) \right)$$

$$+ 2(\bar{H}_{s+w_s} - \bar{H}_{s+(w_s-1)r})$$

$$= 2s' + 2(w_{s'} - 1)(s' + r - s) + 2(\bar{H}_{s+w_s} - \bar{H}_{s+(w_s-1)r})$$

$$> 0,$$

Analogously, the same inequality holds true if $s' \leq 0 < s$ by dropping the $2s'$ term. For $s' < s \leq 0$ a single $(s' + r - s)$ term disappears, but the inequality holds since $s' + r - s > 0$ and $\bar{H}_{s+w_{s'}} - \bar{H}_{s+(w_s-1)r} > 0$.

Then since $M_{rel}(b^{s'}) > M_{rel}(b^s)$ for every configuration of $s$ relative to $s'$ for $w_{s'} > 0$, we have Width$(MR^{s'}) > Width(MR^{s'})$. Together with the argument for $w_{s'} = 0$, this completes the proof.

**Case B2:** $\tau(K) = g(K)$. Let us consider $1 < r < 2(g(K) - 1)$ first, delaying the penultimate slope to Lemma 5.5 and the maximal slope to Lemma 5.3. If $r < 2(g(K) - 1)$, then $l^s_r$ intersects $\tau$ more than once for $s' \equiv g(K) - 1 \pmod{r}$. Our approach involves different arguments depending on whether $l^s_r$ makes vertical intersections on both sides of the 0th column.
**Lemma 5.2.** Suppose $K$ is thin with $\tau(K) = g(K)$, the surgery slope satisfies $1 < r < 2(g(K) - 1)$, and that there exists a $k$ properly dividing $r$ so that every $[s] \in \text{Spin}^c(S^3(K))$ satisfies $MR^{[s]} \cong MR^{[s+k]}$ up to translation.

- If $r < g(K) - 1$, then $MR^{[s]} \cong MR^{[s]}$ up to translation only if $[s] = [s']$.
- If $r \geq g(K) - 1$, then $\tau(K) = g(K) = r = 3$.

**Proof.** If $r < g(K) - 1$, then the slope of $l^{s'}_r$ is small enough so that intersecting it with $\gamma$ produces vertical intersections in at least 3 columns of $\tilde{T}$. We know $w_{s'} > 0$ since $r < 2(g(K) - 1)$, so suppose $w_{s'} = 1$. We have vertical intersections with $\gamma$ to the left and right of this column, which we can label $c^{s'}$ and $b^{s'}$, respectively. Then $M_{rel}(c^{s'}) = 2V_{s'} - 1$ and $M_{rel}(b^{s'}) = 2H_{s'} - 1$, and so $2H_{s'} - 1 \leq \text{Width}(MR^{[s']}) \leq 2H_{s'}$ since $s' > 0$ yields $H_{s'} > V_{s'}$. If some $[s] \neq [\pm s']$ satisfies $MR^{[s]} \cong MR^{[s]}$ up to translation then the parities of their widths must agree. Under the same labeling convention for intersections associated to $[s]$, we see that $2V_{s} - 1 \leq \text{Width}(MR^{[s]}) \leq 2V_{s}$ if $s \leq 0$ and $2H_{s} - 1 \leq \text{Width}(MR^{[s]}) \leq 2H_{s}$ if $s \geq 0$.

For $0 < s' < s$, we compute for either parity of width that

$$\text{Width}(MR^{[s']}) - \text{Width}(MR^{[s]}) = 2(H_{s'} - V_{s}).$$

This implies that $\tilde{V}_s = \tilde{H}_{s'}$, which is impossible when $s' > 0$. Similarly, if $s < -s'$ then the analogous statement holds true using $\tilde{H}_s$.

When $-s' < s < s'$, something interesting occurs. In addition to $l^{s'}_r$, we see that $l^{s}_r$ successfully makes two non-vertical intersections on both sides of the 0th column. Also since $K$ is thin, it follows that $\tilde{V}_s \leq \tilde{H}_{s'}$ and $\tilde{H}_s \leq \tilde{V}_{s'}$, with equality only possible when $s = -s' + 1$ or $s = s' - 1$, respectively. However, this results in the configuration shown for the latter situation in Figure 12. We see that when the widths of $MR^{[s]}$ and $MR^{[s']}$ can agree if $\tilde{H}_s = \tilde{H}_{s'}$, this necessarily results in $\tilde{V}_s \neq \tilde{V}_{s'}$ since $K$ is thin. This is true vice versa as well, and so the multisets cannot both contain the same relative gradings for their respective vertical intersection pairs. Thus we must consider $w_{s'} > 1$, and we will do so following similar computations to those in Lemma 5.1.

![Figure 12](image.png)

Figure 12. If $\tilde{H}_s = \tilde{H}_{s'}$ and $s = s' - 1$, then $\tilde{V}_s - \tilde{V}_{s'} = 1$ when $K$ is thin.

When $w_{s'} > 1$ the intersection with largest relative grading in $MR^{[s]}$ comes from the furthest non-vertical intersection, which is $b^{s}$ when $s \geq 0$ or $c^{s}$ if $s \leq 0$. Since we can use the hyperelliptic involution invariance of $\text{HF}(M)$ to treat such a $c^{s}$ as $b^{-s}$, let us only compare $M_{rel}(b^{s'})$ to the various $M_{rel}(b^{s})$. Chaining grading differences between adjacent non-vertical intersections from $b^{s}$ back to $a^{s}$, we have

$$M_{rel}(b^{s}) = 2 \left( \tilde{H}_s + \sum_{i=1}^{w_{s}-1} (s + ir) \right) - 1.$$
If $s < s'$ then $w_s = w_{s'}$, and we compute

$$M_{rel}(b') - M_{rel}(b^*) = 2 \left( \bar{H}_{s'} - \bar{H}_s + \sum_{i=1}^{w_{s'}-1} (s + i\tau) - \sum_{i=1}^{w_{s}-1} (s + i\tau) \right)$$

$$= 2(\bar{H}_{s'} - \bar{H}_s + (w_{s'} - 1)(s' - s)) \geq 1.$$  

We have equality only if $\bar{H}_{s'} = \bar{H}_s$, $s = s' - 1$, and $w_{s'} = 2$. In this case, we have $\text{parity}(s' + 2r) = \text{parity}(s')$ regardless of $r$. However parity$(s' + 2r) \neq \text{parity}(\tau(K))$, and so $\text{parity}(s) = \text{parity}(\tau(K))$. This implies $\text{Width}(MR^{[s]}) \leq M_{rel}(b^*)$, which means we cannot have $s < s'$.

If $s > s'$, then with $w_s = w_{s'} - 1$ we obtain

$$M_{rel}(b') - M_{rel}(b^*) = 2 \left( \bar{H}_{s'} - \bar{H}_s + \sum_{i=1}^{w_{s'}-1} (s + i\tau) - \sum_{i=1}^{w_{s}-1} (s + i\tau) \right)$$

$$(\text{column shift}) \geq 2 \left( s' + r + \bar{H}_{s'} - \bar{H}_s + \sum_{i=2}^{w_{s'}-1} (s + i\tau) - \sum_{i=2}^{w_{s}-1} (s + i\tau) \right)$$

$$= 2(s' + r) - 2(\bar{H}_s - \bar{H}_{s'}) + 2(w_{s'} - 2)(s' + r - s).$$  

Now $s - s' \leq 2(\bar{H}_s - \bar{H}_{s'}) \leq s - s' + 1$ when $K$ is thin by careful inspection of these regions. This implies

$$M_{rel}(b') - M_{rel}(b^*) \geq 2(s' + r) - (s - s' + 1) + 2(w_{s'} - 2)(s' + r - s)$$

$$= 2s' + r - 1 + (2w_{s'} - 3)(s' + r - s) > 1.$$  

Altogether, these grading comparisons are enough to see that $MR^{[s]} \neq MR^{[s']}$ up to translation when $r < g(K) - 1$. Next we look at the cases with larger surgery slopes.

If $r = g(K) - 1$, then $w_{s'} = 1$ and $s' = 0$. Due to hyperelliptic involution invariance, we can assume $s \neq s'$ satisfies $s < 0$. Notice that $MR^{[s]}$ contains $2\bar{H}_{s'} - 1$ with multiplicity at least two since $l^s_{s'}$ generates non-vertical intersections on both sides of the 0th column and $\bar{V}_{s'} = \bar{H}_{s'}$. The only way that $MR^{[s]}$ could contain this grading with multiplicity greater than one is if a simple figure-eight component in the 1st column has an intersection with $l^s_{s'}$.

The nearby vertical intersection $a^{s+r}$ has $M_{rel}(a^{s+r}) = 2\bar{H}_{s'} - 2$, and so we would require $\text{parity}(s + r) \neq \text{parity}(\tau(K))$ in order for an intersection with a simple figure-eight to have the desired grading. However $s + r = \tau(K) - 2$, and so $MR^{[s]}$ cannot contain $2\bar{H}_{s'} - 1$ more than once. Thus, no $[s] \neq [s']$ satisfies $MR^{[s]} \cong MR^{[s']}$ up to translation when $r = g(K) - 1$.

We still have $w_{s'} = 1$ if $r > g(K) - 1$, but now $s' < 0$. The crux of the argument in the previous case relied on $\bar{H}_{s'} > 1$. This holds more generally when $r < 2g(K) - 3$, except now $MR^{[s']}$ need only contain $2\bar{H}_{s'} - 1$ once. Since $\text{Width}(MR^{[s']}) \geq 3$, the above argument still applies to show that $MR^{[s]} \cong MR^{[s']}$ up to translation only if $|s| = |s' + 1|$. This forces $k = 1$, which in turn forces $r \leq 3$ so that there cannot exist $|s' + \alpha k|$ with $\text{Width}(MR^{[s' + \alpha k]}) = 1$. The possibility $r = 2$ is handled exactly as in the $r = g(K) - 1$ argument, and so we must have $r = 3$. This means $s' = -1$, and so $\tau(K) = g(K) = r = 3$.

If $r = 2g(K) - 3$, then only $l^s_{s'}$ generates non-vertical intersections between columns of $\bar{T}$. All other $l^s_{s'}$ intersect $\bar{H}(M)$ only in the 0th column, which means that every $\text{Width}(MR^{[s']}) = 1$. Since $\text{parity}(s') = \text{parity}(\tau(K))$ when $r = 2g(K) - 3$, we must have $\epsilon_{s'} = 0$. This is because a simple figure-eight component at this height would contribute an intersection with relative grading $-1$ to $MR^{[s]}$, which would yield $\text{Width}(MR^{[s']}) = 2$ and prevent periodicity. We have $\dim \bar{H}(S^L_{s'}(K), [s']) = 3 + 2\epsilon_{s' + r}$, and so some $[s' + k]$ satisfying $MR^{[s' + k]} \cong MR^{[s']}$ up to
translation forces $1 + 2e_{s'+k} = 3 + 2e_{s'+r}$, or $e_{s'+k} = e_{s'+r} + 1$. If necessary, translate $M^t[3+k]$ so that $0$ is the smallest element. The only way that the multiplicities of 0 and 1 agree is if $MR^s$ contains more 0’s than 1’s, which happens only when $\text{parity}(s' + k) = \text{parity}(\tau(K))$. But $\text{parity}(s') = \text{parity}(\tau(K))$ as well, which is a contradiction since $k$ is odd when $r$ odd. □

We return to the two unhandled cases of $\tau(K) = g(K) = r = 3$ and $r = 2(g(K) - 1)$ shortly in Subsection 5.1 and for now are left with the case where $r = 2g(K) - 1$. Because $\tau(K) = g(K)$, there is no guaranteed simple figure-eight at height $g(K) - 1$. This small difference is enough of an issue if $K$ is an $L$-space knot, since each $MR^s = \{0\}$ means $HF$ cannot provide an obstruction. With existing techniques, we can only show the following:

**Lemma 5.3.** Suppose $K$ is thin with $\tau(K) = g(K)$, and let $r = 2g(K) - 1$. If there exists a $k$ properly dividing $r$ such that every $[s] \in \text{Spin}^c(S^3(K))$ satisfies $HF(S^3(K), [s]) \cong HF(S^3(K), [s + k])$, then $K$ is an $L$-space knot.

**Proof.** Suppose for the sake of contradiction that $K$ is not an $L$-space knot, meaning that $\dim HF(S^3(K), [s]) > 1$ for some $[s] \in \text{Spin}^c(S^3(K))$. Each $l^r_+$ intersects $\tau$ precisely once since $r \geq 2g(K) - 1$. In order to have $\dim HF(S^3_p(K), [s]) > 1$ for some $[s]$, we need for $HF(M)$ to have a simple figure-eight component at height $s$. Let $t$ be the height of the lowest simple figure-eight component. We have $e_t$ many simple figure-eights at height $t$, and so we must also have $e_{t+k} = e_t$ many simple figure-eight components at height $t + k$ to satisfy

$$\dim HF(S^3(K), [t]) = \dim HF(S^3(K), [t + k]).$$

If $\text{parity}(|t|) \neq \text{parity}(|t + k|)$, then one of $MR^t$ or $MR^{t+k}$ contains $-1$ and would need to be translated by 1 to make 0 the smallest element by Proposition 3.3. However, this results in both multisets having unequal multiplicities of 0’s and 1’s. This would lead to $MR^t \neq MR^{t+k}$ up to translation, and so we must have $\text{parity}(|t + k|) = \text{parity}(|t|)$. However this condition implies that $k$ is even, which contradicts $r = 2g(K) - 1$ being odd. Therefore, $K$ must be an $L$-space knot. □

### 5.1. Obstructions from Absolute Gradings.

Until now, we have primarily appealed to information carried by $HF(S^3(K))$ as this is currently the only flavor of Heegaard Floer homology computable by immersed curves techniques. When considering $S^3(K) = Y \# Z$ with $|H^2(Y)| = k < \infty$, we will use properties of the $d$-invariants mentioned in Section 2 in order to obtain a relationship between $r$, $k$, and the $V$’s associated to $K$. We initially settle the curious $\tau(K) = g(K) = r = 3$ case, and afterwards assemble the proof of Theorem 1.2

**Lemma 5.4.** Let $K$ be a thin knot with $\tau(K) = g(K) = 3$. Then $S^3_3(K)$ is irreducible.

**Proof.** If $S^3_3(K)$ is reducible, it must admit an integer homology sphere connected summand $Y$ since $r = 3$ is prime. Using the additivity of the $d$-invariants, we have

$$d(S^3_3(K), [s]) = d(L(3, \pm 1), [s]) + d(Y).$$

Proposition 2.3 then implies $d(Y) = -2V_0(K) = -2V_1(K)$, which in turn forces $V_0(K) = V_1(K)$. However this is true only for thin knots with even $\tau(K)$, which can be seen using the formula in Proposition 3.3 together with $V_0 = H_{-}$. This forms the desired contradiction. □

**Lemma 5.5.** Let $K$ be a thin knot with $\tau(K) \geq g(K) - 2$. Then $S^3_3(K)$ is irreducible when $r = 2g(K) - 1$.

**Proof.** Let $\tau(K) \geq g(K) - 2$, and suppose for the sake of contradiction that $S^3_3(K)$ is reducible for $r = 2g(K) - 1$. Then $S^3_3(K)$ admits connected summands $Y$ a lens space and $Z$ with $|H^2(Z)| = k < \infty$. Since $H_1(S^3_3(K))$ is cyclic and $r$ is even, one of $|H_1(Y)| = |Z|/k$ or $k$ is even. We will show the latter must be true.

Using the immersed curves techniques of the previous section, we see that $\text{Width}(MR^s)$ = 1 for all $[s]$ when $K$ is thin and $\tau(K) \geq g(K) - 2$. Using $s' \equiv g(K) - 1 \pmod r$, we are guaranteed to have $HF(S^3_3(K), [s']) > 1$ since $l^r_+$ either intersects a simple figure-eight at height $g(K) - 1$ when $\tau(K) < g(K)$ or intersects $\tau$ multiple times when $\tau(K) = g(K)$. In order for some $[s' - k]$ to satisfy $MR^{s' - k} \cong MR^{s'}$ up to translation, we also require $\text{parity}(s' - k) = \text{parity}(s')$ so that the multiplicities of 0 and 1 agree. This implies $k$ is even.
Let \( \pi_Y([s]) \) and \( \pi_Z([s]) \) denote the restrictions of \([s]\) to \(\text{Spin}^c(Y)\) and \(\text{Spin}^c(Z)\), respectively. Since \(\text{Spin}^c(S^3_r(K)) \cong \mathbb{Z}/r\mathbb{Z}\) is \(\mathbb{Z}/r\mathbb{Z}\)-equivariant \cite{OS08}, we have both
\[
\pi_Y([s + \frac{r}{2}]) = \pi_Y([s]) \quad \text{and} \quad \pi_Z([s + k]) = \pi_Z([s]).
\]
The two self-conjugate spin\(^c\) structures of \(S^3_r(K)\) must project onto the lone self-conjugate structure of \(L(\frac{r}{2}, q)\), and so
\[
\pi_Y([0]) = \pi_Y([\frac{r}{2}] \in \text{Spin}^c(L(\frac{r}{2}, q)).
\]
Their respective restrictions on \(Z\) are distinct, and so let \(\pi_Z([0]) = u_c\) and \(\pi_Z([\frac{r}{2}]) = u_o\). Due to the additivity of \(d\)-invariants, we have
\[
d(S^3_r(K), [s]) = d(L(\frac{r}{2}, q), \pi_Y([s])) + d(Z, \pi_Z([s])).
\]
Since \(k\) is even, we may apply this to the self-conjugate structures to see
\[
d(S^3_r(K), [0]) - d(S^3_r(K), [\frac{r}{2}]) = (d(L(\frac{r}{2}, q), [0]) + d(Z, u_c)) - (d(L(\frac{r}{2}, q), [0]) + d(Z, u_o))
= d(Z, u_c) - d(Z, u_o).
\]
Observe that \(\pi_Z([\frac{1}{2}]) = u_0\), and so \(\pi_Z([\frac{r+k}{2}]) = u_c\). We likewise have \(\pi_Y([\frac{1}{2}]) = \pi_Y([\frac{r+k}{2}])\), and so
\[
d(S^3_r(K), [\frac{1}{2}]) - d(S^3_r(K), [\frac{r+k}{2}]) = d(Z, u_o) - d(Z, u_c).
\]
Using the inductive formula for \(d(L(p, q))\) \cite[Proposition 4.8]{OS03a}, it follows that \(d(L(r, 1), [s]) = \frac{s^2}{r} - s + \frac{r-1}{4}\). Summing the prior two equations and using Proposition 2.3 (with \(V_{\frac{r+k}{2}} = \max\{V_{\frac{r+k}{2}}, V_{\frac{r}{2}}\} \)) yields
\[
2 \left(V_0 - V_{\frac{r}{2}} + V_{\frac{r+k}{2}} - V_{\frac{r-k}{2}}\right) = d(L(r, 1), [0]) - d(L(r, 1), [\frac{r}{2}]) + d(L(r, 1), [\frac{r+k}{2}]) - d(L(r, 1), [\frac{r+k}{2}])
= - \left(\frac{r^2}{4r} - \frac{r}{2}\right) + \left(\frac{k^2}{4r} - \frac{k}{2}\right) - \left(\frac{(r+k)^2}{4r} - \frac{r+k}{2}\right)
= \frac{r-k}{2},
\]
Therefore, we have the following relationship between \(r, k\), and the \(V\)'s associated to \(K\):
\[
\frac{r-k}{4} = (V_0 - V_{\frac{r}{2}}) + (V_{\frac{r}{2}} - V_{\frac{r-k}{2}}).
\]
Notice that when \(K\) is thin and \(\tau(K) \geq 0\), we have that
\[
V_0 = \begin{cases} 
\frac{\tau(K)}{2} & \text{if parity}(\tau(K)) = 1 \\
\frac{\tau(K)^2}{2} & \text{if parity}(\tau(K)) = 0
\end{cases}
\]
We will use this to generate contradictions, and break into cases since the values of \(V_{\frac{r}{2}}\) and \(V_{\frac{r-k}{2}}\) depend on \(\tau(K)\). It will also be useful to use the fact that \(k \leq \frac{r}{2}\).

**Case C1**: \(\tau(K) = g(K)\). Here we have \(V_{\frac{r}{2}} = 1\) and \(V_{\frac{r-k}{2}} > 0\). We see that \(V_{\frac{r}{2}} - V_{\frac{r-k}{2}}\) is given by half the distance between \(\frac{r-k}{2} - \frac{k}{2}\) since \(K\) is thin. Thus,
\[
V_{\frac{r}{2}} - V_{\frac{r-k}{2}} = \frac{r}{4} - \frac{k}{2}.
\]
This together with Equation 1 above then yields
\[
V_0 = \frac{k}{4} + 1.
\]
If parity(\(\tau(K)\)) = 1, then we have

\[
\frac{\tau(K) + 1}{2} = \frac{k}{4} + 1
\]

\[
\Leftrightarrow \frac{\tau(K) - 1}{2} = \frac{k}{4}
\]

\[
\Leftrightarrow \frac{\tau(K) - 1}{2} \leq \frac{r}{12}
\]

\[
\Leftrightarrow 6(\tau(K) - 1) \leq 2(g(K) - 1)
\]

\[
\Leftrightarrow 6(\tau(K) - 1) \leq 2(\tau(K) - 1)
\]

\[
\Rightarrow 4\tau(K) \leq 4.
\]

However this would imply \(g(K) = \tau(K) = 1 \Rightarrow r = 0\), a clear contradiction. If parity(\(\tau(K)\)) = 0, then similar reasoning yields \(\tau(K) \leq 2\), which forces \(r = \tau(K) = g(K) = 2\). We return to immersed curves techniques to rule out this case by comparing the multiplicity of elements of MR\([0\]) and MR\([1\]). Since parity(\(\tau(K)\)) = 0, we must translate MR\([0\]) by one so that 0 is its smallest element. The multiplicity of 0 in the translated MR\([0\]) is \(e_0\), and the multiplicity of 0 in MR\([1\]) is \(2e_1 + 2\). However if \(S^3_2(K)\) is reducible then Lemma 2.1 forces \(e_0 = 2e_1 + 1\) in order for \(\dim \tilde{HF}(S^3_2(K), [1]) = \dim \tilde{HF}(S^3_2(K), [0])\), generating the desired contradiction.

**Case C2:** \(\tau(K) = g(K) - 1\). Once again \(V_{2g-2} > 0\), and in this case we obtain \(V_0 = \frac{k}{2}\) since \(V_2 = 0\). Together with \(r = 2\tau(K)\), the argument of the previous case yields the contradiction \(4\tau(K) \leq -4\) when \(\text{parity}(\tau(K)) = 1\) or \(4\tau(K) \leq 2\) when \(\text{parity}(\tau(K)) = 0\).

**Case C3:** \(\tau(K) = g(K) - 2\). We still have \(V_2 = 0\), and things are more interesting here since it is possible for \(V_{2g-3} = 0\). This happens only if \(k = 2\), in which case Equation \[\Box\] becomes

\[
\frac{r - 2}{4} = V_0 + V_1.
\]

Curiously enough \(\tau(K) = V_0 + V_1\) for a thin knot, and so this would force \(\tau(K) = \frac{r - 2}{4} = \frac{2(\tau(K) + 1) - 2}{4} \Leftrightarrow 4\tau(K) = 2\tau(K) \Rightarrow \tau(K) = 0\). However this forces \(r = k\), a contradiction. Then we cannot have \(k = 2\), and so \(V_{2g-3} > 0\) and we once again have \(V_0 = \frac{k}{2}\). As with the previous cases, having \(r = 2\tau(K) + 1\) would yield the contradictions \(6\tau(K) + 1 \leq 2\tau(K) + 1\) if \(\text{parity}(\tau(K)) = 1\) or \(4\tau(K) \leq 2\) if \(\text{parity}(\tau(K)) = 0\). This completes the proof. \(\Box\)

### 5.2. Proof of Theorem 1.2

**Proof.** Suppose \(S^3_2(K)\) is reducible for \(K\) thin and hyperbolic. The Matignon-Sayari bound implies that \(1 < r < 2g(K) - 1\), after mirroring the knot if necessary to make the surgery slope positive. Reducibility also gives \(S^3_2(K) \cong Y \# Z\) for some lens space \(Y\) and some \(Z\) with \(|H^2(Z)| = k < \infty\). By Lemma 2.1, we have \(\tilde{HF}(S^3_2(K), [s + \alpha k]) \cong \tilde{HF}(S^3_2(K), [s])\) for arbitrary \([s], \alpha \in \mathbb{Z}/r\mathbb{Z}\). When \(r < 2(g(K) - 1)\), Lemmas 4.2, 5.1 and 5.2 apply to show that there exists an \([s'] \in \text{Spin}^c(S^3_2(K))\) such that either \(\tilde{HF}(S^3_2(K), [s'])\) is relatively-graded isomorphic only to \(\tilde{HF}(S^3_2(K), [-s'])\), or \(\tau(K) = g(K) = r = 3\). The latter is prevented by Lemma 5.4 so we proceed with the former. Since \(Y \neq S^3\), we see that \([s']\) cannot be self-conjugate and also that \(|H_1(Y)| = 2\). This implies \(Y = \mathbb{R}P^3\), as well as \(k = |[s'] - [-s']}| = 2||s'||\). However, together this means 4 divides \(r\), which is impossible when \(S^3_2(K)\) admits an \(\mathbb{R}P^3\) summand with \(H_1(S^3_2(K))\) cyclic.

Therefore, we must have \(r \geq 2(g(K) - 1)\). Lemmas 4.3 and 4.5 cover \(-g(K) < \tau(K) < g(K) - 2\) and Lemma 5.5 covers \(\tau(K) \geq g(K) - 2\) for the possibility that \(r = 2(g(K) - 1)\), and so we must have \(r = 2g(K) - 1\). In this situation \(k\) is odd since \(r\) is odd, which means periodicity will cycle through spin\(^c\) structures with different parities. Then Lemmas 4.3, 4.5 and 5.1 apply to fully obstruct reducibility via the above argument if \(\tau(K) \neq g(K)\). If \(\tau(K) = g(K)\), our techniques have been exhausted and leave just the conclusion of Lemma 5.3 showing that \(K\) must be an \(L\)-space knot. \(\Box\)
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References

[BZ98] S. Boyer and X. Zhang. On Culler-Shalen seminorms and Dehn filling. *Ann. of Math. (2)*, 148(3):737–801, 1998.

[DeY21] Robert DeYeso III. Integral Klein bottle surgeries and Heegaard Floer homology. *https://arxiv.org/abs/2009.10197*, 2021.

[Gab87] David Gabai. Foliations and the topology of 3-manifolds. III. J. Differential Geom., 26(3):479–536, 1987.

[GAoS86] Francisco González-Acuña and Hamish Short. Knot surgery and primeness. *Math. Proc. Cambridge Philos. Soc.*, 99(1):89–102, 1986.

[GL87] C. McA. Gordon and J. Luecke. Only integral Dehn surgeries can yield reducible manifolds. *Math. Proc. Cambridge Philos. Soc.*, 102(1):97–101, 1987.

[GL89] C. McA. Gordon and J. Luecke. Knots are determined by their complements. *J. Amer. Math. Soc.*, 2(2):371–415, 1989.

[Gre15] Joshua Evan Greene. L-space surgeries, genus bounds, and the cabling conjecture. *J. Differential Geom.*, 100(3):491–506, 2015.

[Han19] Jonathan Hanselman. Heegaard Floer homology and cosmetic surgeries in $S^3$. *https://arxiv.org/abs/1906.06773*, 2019.

[HLZ15] Jennifer Hom, Tye Lidman, and Nicholas Zufelt. Reducible surgeries and Heegaard Floer homology. *Math. Res. Lett.*, 22(3):763–788, 2015.

[How02] James Howie. A proof of the Scott-Wiegold conjecture on free products of cyclic groups. *J. Pure Appl. Algebra*, 173(2):167–176, 2002.

[NW15] Yi Ni and Zhongtao Wu. Cosmetic surgeries on knots in $S^3$. *J. Reine Angew. Math.*, 706:1–17, 2015.