Ambiguity resolution for integrable gravitational charges

Antony J. Speranza

Department of Physics, University of Illinois, Urbana-Champaign, Urbana IL 61801, U.S.A.

E-mail: asperanz@gmail.com

ABSTRACT: Recently, Ciambelli, Leigh, and Pai (CLP) [arXiv:2111.13181] have shown that nonzero charges integrating Hamilton’s equation can be defined for all diffeomorphisms acting near the boundary of a subregion in a gravitational theory. This is done by extending the phase space to include a set of embedding fields that parameterize the location of the boundary. Because their construction differs from previous works on extended phase spaces by a covariant phase space ambiguity, the question arises as to whether the resulting charges are unambiguously defined. Here, we demonstrate that ambiguity-free charges can be obtained by appealing to the variational principle for the subregion, following recent developments on dealing with boundaries in the covariant phase space. Resolving the ambiguity produces corrections to the diffeomorphism charges, and also generates additional obstructions to integrability of Hamilton’s equation. We emphasize the fact that the CLP extended phase space produces nonzero diffeomorphism charges distinguishes it from previous constructions in which diffeomorphisms are pure gauge, since the embedding fields can always be eliminated from the latter by a choice of unitary gauge. Finally, we show that Wald-Zoupas charges, with their characteristic obstruction to integrability, are associated with a modified transformation in the extended phase space, clarifying the reason behind integrability of Hamilton’s equation for standard diffeomorphisms.

KEYWORDS: Classical Theories of Gravity, Gauge Symmetry, Space-Time Symmetries

ArXiv ePrint: 2202.00133
1 Introduction

Hamiltonian charges associated to diffeomorphisms constitute an important set of observables in gravitational theories. While bulk diffeomorphisms are well-known to be gauge transformations in such theories, in the presence of boundaries some of these transformations become physical and their associated charges are nonvanishing. Characterizing the complete set of such charges has been the focus of much recent work, due to various classical and quantum gravitational applications, including black hole entropy [1–8], celestial holography [9–11], entanglement entropy [12–14], and quasilocal descriptions of gravitational subregions [15–18].

Due to bulk diffeomorphism invariance, gravitational charges are given by integrals over a codimension-2 surface $\partial \Sigma$, located at the boundary of a spatial slice $\Sigma$ through the spacetime or subregion under consideration. The charges are determined by defining a symplectic form $\Omega$ for the subregion as the integral of a symplectic current $\omega$ over $\Sigma$, and then evaluating $\Omega$ on a diffeomorphism transformation. In standard constructions, the charges obtained by this procedure are of two types. For diffeomorphisms that map the boundary $\partial \Sigma$ into itself, the contraction of this transformation into the symplectic form yields a total variation, and hence the charges satisfy Hamilton’s equation and generate a symmetry of the subregion phase space. On the other hand, diffeomorphisms with a transverse component to $\partial \Sigma$ do not yield Hamiltonian charges, since the contraction of these transformations into the symplectic form is generically not a total variation. In order to determine a diffeomorphism charge in this case, one must split the resulting contraction into a total variation and flux term representing the obstruction to integrability of Hamilton’s equation. This splitting underlies the Wald-Zoupas procedure for determining localized gravitational charges, and suffers from ambiguities in how to determine a preferred form of the flux. Various proposals for fixing this ambiguity have been considered recently,
and have included covariance requirements [19], stationarity conditions [20, 21], and most comprehensively appeals to the variational principle for the subregion and the associated boundary conditions [8, 22, 23]. After fixing this ambiguity, one is further faced with the issue of determining the algebra satisfied by these localized charges. Often this algebra is defined via the Barnich-Troessaert bracket [24], which has been argued to coincide with the Poisson bracket of the charges as functions on the subregion phase space, subject to certain conditions on how one chooses the flux [22].

Recently, a novel proposal for constructing diffeomorphism charges has been put forward by Ciambelli, Leigh, and Pai (CLP) [25], which avoids the complications of the Wald-Zoupas procedure in determining a preferred form of the flux. This is achieved by enlarging the subregion phase space to include a set of embedding fields \( X \) which parameterize the spacetime location of the surface \( \partial \Sigma \). Their procedure closely parallels the extended phase space introduced by Donnelly and Freidel (DF) in [15] and further generalized by the present author in [16], but differs in a crucial way in their choice of subregion symplectic form. In both cases, the introduction of embedding fields eliminates the obstruction to integrability of Hamilton’s equation when evaluating the symplectic form on a diffeomorphism. However, in the DF construction, the charges vanish identically, making the transformation pure gauge, while the charges are nonzero in the CLP extended phase space. This leads to the intriguing conclusion that the CLP construction is able to define Hamiltonian gravitational charges for all diffeomorphisms acting near the boundary, without resorting to a Wald-Zoupas procedure for splitting off a flux term in Hamilton’s equation.

The fact that diffeomorphisms are not pure gauge in the CLP construction points to a fundamental difference between their proposal and the previous DF construction. This implies that the CLP proposal amounts to a genuine extension of the phase space by new degrees of freedom associated to the embedding fields, while this is not the case for the DF proposal. To clarify this point, in section 3 we show that because all diffeomorphisms are gauge in the DF construction, there always exists a choice of unitary gauge in which the embedding fields are eliminated from the phase space description. In this gauge, the DF extended phase space reduces to the standard phase space constructed only from the dynamical fields. On the other hand, the CLP symplectic form differs from the DF choice by a Jacobson-Kang-Myers (JKM) ambiguity term in the Iyer-Wald construction [2, 26], and this ambiguity does not preserve degeneracy with respect to diffeomorphisms acting near the boundary. Because these diffeomorphisms are no longer gauge, one cannot access the unitary gauge condition through a pure gauge transformation. Hence, the embedding fields cannot be eliminated in the CLP extended phase space, implying that they represent new physical degrees of freedom.

This raises an important question about whether the CLP charges are defined unambiguously. Because CLP differs from the DF construction by an ambiguity term, it is important to determine whether the charges can be shifted by further choices of JKM ambiguities. This question is addressed in section 4, where it is demonstrated that there is considerable freedom to shift the charges by ambiguities, necessitating a further principle for fixing the form of the charges. However, such ambiguities also appear in standard co-
variant phase space constructions without embedding fields, and recent developments have shown that these can be resolved by appealing to the variational principle for the subregion [8, 22, 27–30]. We further demonstrate that this resolution carries over to the extended phase space, and we derive corrected expressions for the gravitational charges that match those obtained in recent works on boundaries in the covariant phase space [8, 17, 22, 30]. This resolution requires that the surface $\partial \Sigma$ be realized as a cut of a bounding hypersurface $N$ for the subregion, and we leave open the question as to whether this dependence on a choice of $N$ can further be eliminated. Unlike the original CLP construction, once correction terms resolving the ambiguities are included in the definition of the phase space, new obstructions to the integrability of Hamilton’s equation can arise. This somewhat diminishes the advantage of the CLP construction, but we argue that in some cases the obstruction is expected to vanish, and even when it is nonvanishing, the Barnich-Troessaert bracket of the charges faithfully reproduces the bracket of the diffeomorphism-generating vector fields.

A final question addressed in section 5 relates to the reason behind integrability of Hamilton’s equation in the CLP construction. In phase space constructions without embedding fields, nonintegrability of Hamilton’s equation has a simple interpretation in terms of the loss of symplectic flux during evolution along the subregion boundary. This argument no longer holds in extended phase space constructions since the effective location of the surface $\partial \Sigma$ does not change relative to the dynamical fields, due to the action of diffeomorphisms on the embedding fields and hence the target location of $\partial \Sigma$. To highlight the difference, we further show that a different transformation can be defined on the CLP extended phase space that is the appropriate analog of the diffeomorphisms on the non-extended phase space. It involves a combination of a diffeomorphism and a change in the embedding map that together fix the target surface $\partial \Sigma$. This transformation is shown to satisfy the modification of Hamilton’s equation that appears in the Wald-Zoupas construction, suitably generalized to include contributions from the embedding fields.

1.1 Notation for field space

The construction of phase spaces in this work will utilize concepts related to the differential geometry of a field configuration space, and we briefly review the notation used for calculations performed in this space, which largely follows that of reference [22]. The field configuration space $\mathcal{F}$ is parameterized by all possible configurations of the dynamical fields $\phi$ on the spacetime manifold $\mathcal{M}$. Hence $\mathcal{F}$ can be viewed as an infinite-dimensional manifold, on which we can define tensorial objects such as vector fields and differential forms. The variations of the dynamical fields at each point $x \in \mathcal{M}$ define a basis of one-forms $\delta \phi(x)$ on $\mathcal{F}$, with the operator $\delta$ playing the role of the exterior derivative. Hence, $\delta$ acting on objects involving one variation will always be taken to be the exterior derivative operator, and implicitly involves an antisymmetrized set of independent variations. We will consistently employ the shorthand $\delta \phi$ for $\delta \phi(x)$. A specific linearized variation $\Phi(x)$ of the dynamical field $\phi$ defines a vector field $\hat{\Phi}$ on $\mathcal{F}$, and we will denote the contraction operation of such a vector field into a differential form by $I_{\hat{\Phi}}$, so that in particular, $I_{\hat{\Phi}} \delta \phi = \Phi(x)$. 

\[ -3 - \]
The field space Lie derivative along a vector field $\hat{\Phi}$ will be denoted $L_{\hat{\Phi}}$, and when acting on field space differential forms satisfies Cartan’s magic formula $L_{\hat{\Phi}} = I_{\hat{\Phi}} + \delta I_{\hat{\Phi}}$.

An important set of linearized variations are those corresponding to infinitesimal diffeomorphisms. These will be denoted $\hat{\xi}$ where $\xi^a$ is a spacetime vector field, and they satisfy $I_{\hat{\xi}} \delta \phi = L_{\xi} \phi$, where $L_{\xi}$ is the Lie derivative. The field space Lie bracket of two such vector fields $\hat{\xi}, \hat{\zeta}$, constructed from generically field-dependent vectors $\xi^a, \zeta^a$, is given by \[ [\hat{\xi}, \hat{\zeta}] = -[\xi, \zeta] + I_{\hat{\xi}} \delta \zeta^a + I_{\hat{\zeta}} \delta \xi^a. \] (1.1)

As we will deal with diffeomorphism-covariant theories, it is useful to define an operator $\Delta_{\hat{\xi}}$ that measures the failure of a field-space differential form to transform covariantly. This is defined to be \[ \Delta_{\hat{\xi}} = L_{\hat{\xi}} - L_{\xi} - I_{\hat{\delta} \xi}. \] (1.2)

2 Embedding fields and the extended phase space

The construction of diffeomorphism charges in this work utilizes an extended phase space in which embedding fields $X$ are included as additional degrees of freedom in the theory. Their inclusion into covariant phase space constructions was initially proposed by Donnelly and Freidel in [15] in the case of vacuum general relativity with zero cosmological constant, and the generalization to arbitrary diffeomorphism-invariant theories was subsequently derived by the present author in [16]. Here, we briefly review the standard covariant phase space construction [1, 2, 33–37], as well as the extended construction involving the inclusion of embedding fields [15, 16].

The main input for the covariant phase space is the Lagrangian $L[\phi]$, taken to be a spacetime differential form of maximal degree constructed from the dynamical fields, collectively denoted $\phi$. These fields will be taken to be tensor fields on spacetime, and can consist of the metric $g_{ab}$ and any additional matter fields. Varying the Lagrangian with respect to the dynamical fields determines the equations of motion $E[\phi]$ and symplectic potential current $\theta[\phi; \delta \phi]$ according to \[ \delta L = E \cdot \delta \phi + d\theta. \] (2.1)

Taking a second variation of $\theta$ yields the symplectic current $\omega = \delta \theta$, and its integral over a Cauchy surface $\Sigma$ defines the symplectic form for the theory, \[ \Omega = \int_{\Sigma} \omega. \] (2.2)

The current $\omega$ is conserved on-shell, $d\omega = -\delta E \cdot \delta \phi$, and the covariant phase space is defined on the subspace of $\mathcal{F}$ of all solutions to the equations of motion, on which this conservation law holds.

Under a diffeomorphism $Y : \mathcal{M} \rightarrow \mathcal{M}$, the dynamical fields transform via pullbacks $\phi \rightarrow Y^* \phi$. Diffeomorphism-invariance of the theory implies then that the Lagrangian
transforms covariantly under this transformation,

\[ L[Y^* \phi] = Y^* L[\phi]. \]  

(2.3)

This is simply the statement that the Lagrangian does not depend on any nondynamical background fields. This covariance property of the Lagrangian allows one to derive a Noether current

\[ J_\xi = I_{\xi} \theta - i_\xi L \]  

(2.4)

that is conserved on-shell, \( dJ_\xi = 0 \). Furthermore, since this equation holds for all vector fields \( \xi^a \), one can show that \( J_\xi \) can be expressed in terms of a potential \( Q_\xi \) by the equation \[2, 39, 40\]

\[ J_\xi = dQ_\xi + C_\xi, \]  

(2.5)

where \( C_\xi = 0 \) are combinations of the equations of motion that define the constraints of the theory. Hence, the Noether current \( J_\xi \) is exact on-shell.

The gauge symmetries of the theory can be discerned by examining the degenerate directions of the symplectic form (2.2). These correspond to diffeomorphisms acting in the interior of \( \Sigma \). This is seen by employing the standard on-shell Iyer-Wald identity

\[ I_{\xi} \omega = -d(\delta Q_\xi - Q_\delta \xi - i_\xi \theta), \]  

which implies

\[ -I_{\xi} \Omega = \int_{\partial \Sigma} (\delta Q_\xi - Q_\delta \xi - i_\xi \theta). \]  

(2.6)

Since this contraction localizes to a boundary integral, it is immediately apparent that all diffeomorphisms except for those with support near \( \partial \Sigma \) define gauge transformations of the symplectic form. On the other hand, diffeomorphisms acting at \( \partial \Sigma \) define physical transformations of the subregion phase space. This suggests that the presence of a boundary for the Cauchy surface has promoted some pure gauge transformations to physical degrees of freedom \[41\].

In order to explicitly parameterize these new boundary degrees of freedom, Donnelly and Freidel proposed an extension of the dynamical fields of vacuum general relativity to include a set of embedding fields \( X \) \[15\]. These fields describe the embedding of a neighborhood of \( \Sigma \) into spacetime, and were included in the theory to enforce that the symplectic form be fully diffeomorphism-invariant. Their construction was reformulated and extended to arbitrary diffeomorphism-invariant theories in \[16\] by employing the Iyer-Wald formalism \[2\]. As explained in \[15, 16, 42\], the embedding fields can be viewed as a diffeomorphism \( X : M_r \rightarrow M \) from a reference spacetime \( M_r \) into \( M \). The embedding fields are coupled to the theory only through the pullbacks of the dynamical fields \( X^* \phi \), in order to preserve covariance of the theory under diffeomorphisms. The variation of any pulled back field satisfies the identity \[15, 16, 42\]

\[ \delta X^* \phi = X^*(\delta \phi + L_\chi \phi) \]  

(2.7)

\[ ^1 \text{With slight modifications, the formalism can also handle theories whose Lagrangian is only covariant up to boundary terms [22, 38].} \]
where \( \chi^a \) is a spacetime vector field constructed from the variation of the embedding map, and hence is a one-form on field space. Its variation satisfies the equation

\[
\delta \chi^a + \frac{1}{2} [\chi, \chi]^a = 0,
\]

giving it the interpretation of a flat connection on field space, viewed as a \( \text{Diff}(\mathcal{M}) \) fiber bundle [43]. Under a diffeomorphism \( Y \), the embedding map transforms to its pullback under \( Y^{-1} \), \( X \to (Y^{-1} \circ X) \). This transformation law ensures that the pulled back fields \( X^*\phi \) are fully invariant under diffeomorphisms, since \( X^*\phi \to (Y^{-1} \circ X)^*Y^*\phi = X^*(Y^{-1})^*Y^*\phi = X^*\phi \). Infinitesimally, this implies from equation (2.7) that

\[
I_\xi \delta \chi^a = -\xi^a, \tag{2.9}
\]

since \( 0 = I_\xi \delta X^*\phi = X^*(\mathcal{L}_\xi \phi + \mathcal{L}(I_\xi \phi)) \).

Diffeomorphism invariance of the pulled back fields then yields a straightforward prescription to couple the embedding fields to theory [16]: simply write the Lagrangian in terms of the pulled back fields \( X^*\phi \), and full invariance under diffeomorphisms is guaranteed. The variation of the extended Lagrangian \( L[X^*\phi] \) can then be expressed as

\[
\delta L[X^*\phi] = E[X^*\phi] \cdot \delta X^*\phi + d\theta[X^*\phi; \delta X^*\phi], \tag{2.10}
\]

which shows that \( \theta[X^*\phi; \delta X^*\phi] \) serves as a symplectic potential in the extended phase space. Using equation (2.7) and the on-shell identity \( J_\xi = dQ_\xi \), one can show that the extended symplectic potential can be expressed as

\[
\theta_X \equiv \theta[X^*\phi; \delta X^*\phi] = X^*(\theta + i\chi L + dQ_\chi), \tag{2.11}
\]

where the quantities on the right hand side are functionals of \( \phi, \delta \phi \). The left hand side of this expression is both manifestly invariant under diffeomorphisms and horizontal on field space, \( I_\xi \theta[X^*\phi; \delta X^*\phi] = 0 \), and hence the symplectic form derived from it will share these properties. The resulting symplectic form for the extended phase space derived in [16] is given by

\[
\Omega_X = \int_{X^*\Sigma} \delta \theta_X = \int_{\Sigma} \omega + \int_{\partial \Sigma} \left( i_\chi \theta + \frac{1}{2} i_\chi i_\chi L + \delta Q_\chi + \mathcal{L}_\chi Q_\chi \right), \tag{2.12}
\]

and can be shown to reduce to the expression derived by Donnelly and Freidel [15] when specializing to vacuum general relativity with \( \Lambda = 0 \). Note that the integrals over \( \Sigma \) and \( \partial \Sigma \) depend on the embedding field to determine their location in spacetime, so that \( \int_{\Sigma} \alpha = \int_{X^*\Sigma} X^*\alpha \). When computing variations of such integrals, we will always hold the source location \( X^*\Sigma \) or \( X^*\partial \Sigma \) of the embedding map fixed, and hence the variation of any such integral will always be understood to including a variation of the embedding map:

\[
\delta \int_{\Sigma} \alpha = \int_{X^*\Sigma} \delta X^*\alpha = \int_{\Sigma} (\delta \alpha + \mathcal{L}_\chi \alpha) \tag{2.13}
\]

applying (2.7).

The extended symplectic form \( \Omega_X \) annihilates all diffeomorphisms, \( I_\xi \Omega_X = 0 \), and hence on this phase space all charges associated with diffeomorphisms are trivial. The
nontrivial charges that arise in this construction are associated with diffeomorphisms of the reference space $Z : \mathcal{M}_r \rightarrow \mathcal{M}_r$, which are defined to leave the dynamical fields $\phi$ invariant. These transformations act on the embedding fields via $X \rightarrow X + Z$, and one can show that the infinitesimal transformation corresponding to a vector field $w^a$ on $\mathcal{M}_r$ acts via

$$I_\hat{w} \delta \phi = 0, \quad I_\hat{w} \delta \phi = (X, w)^a \equiv W^a, \quad I_\hat{w} \delta X^a \phi = \mathcal{L}_w X^a \phi. \quad (2.14)$$

Such transformations induce a change in the target location of the embedding map while holding fixed the dynamical fields, and hence can be viewed as physical evolution within a fixed spacetime. Contracting this transformation into the extended symplectic form yields

$$- I_\hat{w} \Omega_X = \delta \int_{\partial \Sigma} Q_W - \int_{\partial \Sigma} \left( Q_{\delta W + [\chi, W]} - i W (\theta + i \chi L + dQ \chi) \right) \quad (2.15)$$

Here, for transformations $W^a$ that are purely tangential at $\partial \Sigma$ and which are field-independent in the sense $\delta W^a + [\chi, W]^a = (X, \delta w) = 0$ will satisfy Hamilton’s equation with Hamiltonian $H_\hat{w} = \int_{\partial \Sigma} Q_W$. For more general transformations, such as those which deform the surface $\partial \Sigma$ in a transverse direction, the remaining terms in (2.15) prevent the Hamilton’s equations from being satisfied for the transformation $I_\hat{w}$. These obstruction terms are similar to those appearing in equation (2.6) for diffeomorphism charges in the non-extended phase space.

3 Unitary gauge for embedding fields

The similarity between the charges (2.15) in the extended phase space and the diffeomorphism charges (2.6) is not a coincidence; rather, it arises due to an equivalence between the two descriptions of the subregion phase space. This equivalence is due to the full diffeomorphism-invariance of the extended symplectic form $\Omega_X$, and it can be shown that the non-extended phase space is simply a gauge fixing of the extended one. Hence, the extended phase space construction of Donnelly and Freidel is fully equivalent to one in which no additional embedding fields are introduced.

The reason for this equivalence lies in the fact that the introduction of embedding fields is essentially the Stueckelberg trick for diffeomorphisms (see, e.g., [44]). Equation (2.6) indicates that the presence of a boundary has caused some diffeomorphisms to become physical. The embedding fields are introduced to restore diffeomorphism invariance, just as Stueckelberg fields can be used to restore gauge invariance in theories with massive vector bosons. However, there always exists a choice of unitary gauge, in which the embedding field is set to a fixed value $X = X_0$. In this gauge, variations of $X$ are also set to zero, which further implies $\chi^a = 0$. Since all the boundary terms in the extended symplectic form (2.12) depend on $\chi^a$, we see that in this gauge these boundary contributions drop out, and the symplectic form reduces to the standard expression in the absence of embedding fields (2.2).

Note that the $\hat{w}$ transformation defined by (2.14) is not consistent with the unitary gauge condition $\chi^a = 0$. However, we are free to redefine this transformation to include an
arbitrary diffeomorphism, since these are pure gauge in the extended phase space. Choosing this diffeomorphism to be given by $\xi_w = X_\ast w$, we see that the new transformation
\[ \hat{w}_0 = \hat{w} + \hat{\xi}_w \] (3.1) satisfies
\[ I_{\hat{w}_0} \chi^a = (I_{\hat{w}} + I_{\hat{\xi}_w}) \chi^a = (X_\ast w)^a - (X_\ast w)^a = 0, \] (3.2) consistent with the unitary gauge condition. Hence, $\hat{w}_0$ is the appropriate transformation from which to obtain the physical charges in unitary gauge, and it is straightforward to verify the contraction of this transformation into the symplectic form simply reproduces (2.6). This demonstrates that upon restricting to unitary gauge, the physical charges constructed in the extended phase space reduce to ordinary diffeomorphism charges in the non-extended phase space.

This equivalence between the extended phase space in unitary gauge and the standard, non-extended phase space raises the question as to whether the introduction of embedding fields is necessary. In a recent work [25], Ciambelli, Leigh, and Pai exhibited a modified construction of an extended phase space in which diffeomorphisms acting near the boundary are not pure gauge. This then implies that the unitary gauge condition in which the embedding map is fixed to a constant is not accessible via pure gauge transformations, invalidating the argument for the equivalence between the extended and non-extended phase spaces. The construction of CLP utilizes an ambiguity in the Iyer-Wald formalism [2, 26] to define a modified symplectic potential whose corresponding symplectic form no longer treats boundary diffeomorphisms as gauge transformations. This ambiguity comes from the freedom to shift $\theta$ by exact terms $\theta \rightarrow \theta + d\nu$, where $\nu$ is a phase space one-form. Their proposal for the extended symplectic form is
\[ \theta_{CLP} = X^\ast (\theta + i\chi L), \] (3.3) which differs from the previous symplectic potential (2.11) by the term $dX^\ast Q_\chi$,
\[ \theta_{CLP} = \theta_X - dX^\ast Q_\chi. \] (3.4)

The ambiguity term $dQ_\chi$ depends explicitly on $\chi^a$ without a corresponding variation of a dynamical field, unlike the variations appearing in $\theta_X$ in which all instances of $\chi^a$ arise from variations of pulled back fields, $\delta X^\ast \phi = X^\ast (\delta \phi + L_\chi \phi)$. This explicit dependence on $\chi^a$ is responsible for breaking the full diffeomorphism-invariance of the symplectic form, ultimately leading to the conclusion that the embedding fields cannot be eliminated by gauge-fixing this phase space. Hence, unlike the phase space constructed from $\theta_X$, the embedding fields represent genuinely new degrees of freedom in the extended phase space of CLP.

The main result of [25] is that the phase space constructed from this modified symplectic potential yields nonzero charges satisfying Hamilton’s equation for all diffeomorphisms.
acting near the boundary, including those which do not preserve the location of the surface \(\partial \Sigma\). This is in contrast to standard phase space constructions, in which there is an obstruction to integrating Hamilton’s equation for such surface-deforming diffeomorphisms, as is evident in equations (2.15) and (2.6). To see how this arises, we first note that (3.4) implies that the symplectic form constructed from \(\theta_{\text{CLP}}\) can be written as

\[
\Omega_{\text{CLP}} = \Omega_X - \delta \int_{\partial \Sigma} Q_X
\]

with \(\Omega_X\) given in equation (2.12). It was previously demonstrated that diffeomorphisms are degenerate directions of \(\Omega_X\), which immediately implies

\[
-I_\xi \Omega_{\text{CLP}} = I_\xi \delta \int_{\partial \Sigma} Q_X.
\]

To proceed, we note the following general identity satisfied by a pulled-back phase space form \(X^* \alpha\),

\[
I_\xi \delta X^* \alpha = X^* \left( I_\xi \delta \alpha - \mathcal{L}_\xi \alpha - \mathcal{L}_\chi I_\xi \alpha \right)
\]

\[
= X^* \left( I_\xi \delta \alpha - I_\xi^* \delta \alpha - \mathcal{L}_\chi I_\xi \alpha \right)
\]

\[
= -\delta X^* I_\xi \alpha + X^* \left( \Delta_\xi \alpha + I_\xi^* \delta \alpha \right)
\]

where the anomaly operator \(\Delta_\xi\) was defined in (1.2). Taking \(\alpha = Q_X\), we note that because the Noether potential \(Q_\xi\) is covariantly constructed from the dynamical fields and the vector \(\xi^a\), and because \(\chi^a\) is a covariant one-form on phase space,\(^3\) we have that \(\Delta_\xi Q_X = 0\), and so for field-independent diffeomorphisms (\(\delta \xi^a = 0\)), equation (3.6) reduces to

\[
-I_\xi \Omega_{\text{CLP}} = \delta \int_{\partial \Sigma} Q_\xi.
\]

showing that this transformation satisfies Hamilton’s equation with a Hamiltonian charge

\[
H_\xi = \int_{\partial \Sigma} Q_\xi.
\]

The introduction of embedding fields therefore has been shown to yield genuine Hamiltonian charges associated with all diffeomorphisms. This is to be contrasted with constructions that involve no embedding fields, in which case diffeomorphisms which move the surface are associated with Wald-Zoupas charges, which satisfy a modification of Hamilton’s

\(^3\)Covariance of \(\chi^a\) follows from the definition of \(\Delta_\xi\) acting on it, which gives

\[
\Delta_\xi \chi^a = I_\xi \delta \chi^a + \delta I_\xi \chi^a - \mathcal{L}_\xi \chi^a - I_\xi^* \delta \chi^a = -\frac{1}{2} I_\xi [\chi^a, \chi^a] - \delta \xi^a - [\xi, \chi^a] + \delta \xi^a = 0,
\]

applying equation (2.8) for the variation \(\delta \chi^a\). Although not necessary for this work, the fact that \(\chi^a\) can be interpreted as a flat \text{Diff}(\mathcal{M})\ connection on field space [43] suggests that one could consider a generalization in which the connection has curvature, in which case (2.8) would be modified to \(\delta \chi^a + \frac{1}{2} [\chi, \chi]^a = \rho^a\), where \(\rho^a\) is a field-space two form defining the curvature of the connection. The proof of covariance continues to go through [45], since (3.10) would be corrected by a term \(I_\xi^* \rho^a\), which vanishes since the curvature of a connection is always a horizontal form [46].
equations that allows for additional flux contributions [8, 20, 22]. Note that it was already apparent in the extended phase space of Donnelly and Freidel that embedding fields yield Hamiltonian diffeomorphism charges; however, in their construction, all of these charges vanish identically. The important novelty of the CLP construction is their modification of the phase space symplectic form to arrive at nonzero charges while simultaneously preserving the Hamiltonian property.

Since these charges satisfy Hamilton’s equation, the Poisson bracket between two of them can be computed by contracting the corresponding vector fields into the symplectic form. One verifies straightforwardly that

$$\{H_\xi, H_\zeta\} = -I_\xi \delta H_\zeta = -\int_{\partial \Sigma} (I_\xi \delta Q_\zeta - E_\xi Q_\zeta) = -\int_{\partial \Sigma} \Delta_\xi Q_\zeta = \int_{\partial \Sigma} Q_{[\xi, \zeta]} \quad (3.13)$$

where the last equality uses that the only noncovariance in $Q_\zeta$ comes from its dependence on the fixed vector $\zeta^a$, so $\Delta_\xi Q_\zeta = Q_\Delta_\xi = -Q_{[\xi, \zeta]} - I_\xi \delta_\zeta$, and then noting that $\delta \zeta^a = 0$. This reproduces the result of [25, 47] that Poisson brackets of the charges yield a representation of the vector field algebra under Lie brackets.

The nonzero charges correspond to diffeomorphisms tangential to the surface $\partial \Sigma$, point-wise $SL(2, \mathbb{R})$ transformations in the normal plane, and two independent surface deformations that move the surface in the transverse directions. This algebra has the structure of the Lie algebra of the group $\text{Diff}(\partial \Sigma) \times (SL(2, \mathbb{R}) \ltimes \mathbb{R}^2)^{\partial \Sigma}$, and was first identified in [16], and subsequently explored and expanded upon in [38, 47]. Furthermore, this algebra appears universally in any diffeomorphism-invariant theory [16], provided certain choices are made to resolve the ambiguities in the Iyer-Wald formalism. Resolving these ambiguities leads to important corrections to the charges, and hence we next turn to understanding how this resolution can be applied in the new proposal of CLP.

### 4 Ambiguities and their resolution

Ambiguities arise in the Iyer-Wald formulation of the covariant phase space due to certain quantities being defined only up to the addition of spacetime exact forms. Specifically, the Lagrangian $L$ and symplectic potential $\theta$ can both be shifted according to

$$L \rightarrow L + da \quad (4.1)$$
$$\theta \rightarrow \theta + \delta a + dv \quad (4.2)$$

without affecting the relation (2.1). Resolving these ambiguities is of crucial importance when constructing charges, since, for example, the difference between the DF and CLP extended phase spaces is given by such an ambiguity. To further emphasize the issue presented by these ambiguities, note that the existence of a covariant spacetime vector field $\chi^a$ constructed from variations of the embedding fields allows for the construction of a wide variety of covariant two-forms $A_\chi$ that can be used as ambiguities to change the extended phase space symplectic form. The new symplectic potential would then be $\theta_A = \theta_{\text{CLP}} + dX^* A_\chi$, resulting in a new extended symplectic form

$$\Omega_A = \Omega_{\text{CLP}} + \delta \int_{\partial \Sigma} A_\chi. \quad (4.3)$$
Covariance implies that $\Delta_\xi A_\chi = 0$, and hence the derivation of section 3 can be repeated to derive a shifted diffeomorphism charge

$$
-I_\xi \Omega_A = \delta H_\xi^A \quad (4.4)
$$

$$
H_\xi^A = \int_{\partial \Sigma} (Q_\xi - A_\xi) \quad (4.5)
$$

Additionally, any ambiguity terms $\nu$ constructed solely from the dynamical fields will also shift the charges. We further allow for the ambiguity terms to be generically noncovariant, but as demonstrated below, this tends to spoil the integrability of Hamilton's equation.

Recently, it has been understood that these ambiguities can be resolved by appealing to the variational principle for a subregion in spacetime [8, 22, 27–30]. This resolution requires $\partial \Sigma$ to arise as a cut of a hypersurface $\mathcal{N}$ bounding the spacetime subregion $\mathcal{U}$ under consideration. Given the additional structure provided by the hypersurface $\mathcal{N}$, one looks for a decomposition of the pullback of $\theta$ to $\mathcal{N}$, denoted $\tilde{\theta}$, of the form

$$
\tilde{\theta} = -\delta \ell + d\beta + \mathcal{E}. \quad (4.6)
$$

The flux term $\mathcal{E}$ is the quantity that would be set to zero by boundary conditions in the variational principle for the subregion. Note that the identification of the boundary condition is used to single out a preferred form of $\mathcal{E}$, but we do not assume such boundary conditions have been imposed, since they restrict the dynamics in finite subregions. To obtain an unambiguous decomposition, criteria must be given for fixing the form of the flux. For example, in general relativity, given a subregion bounded by a timelike surface, we can require the flux be in Dirichlet form $\mathcal{E} = \pi^{ij} \delta h_{ij}$. The term $\ell$ appearing in (4.6) is then used as the boundary term when constructing the subregion action,

$$
S = \int_{\mathcal{U}} L + \int_{\mathcal{N}} \ell + \ldots \quad (4.7)
$$

where the dots denote additional terms at past, future, or higher codimension boundary components. The symplectic form for the subregion also receives a correction from the quantity $\beta$ in (4.6),

$$
\Omega = \int_{\Sigma} \omega - \delta \int_{\partial \Sigma} \beta. \quad (4.8)
$$

One can demonstrate that this symplectic form is invariant under the ambiguities described in equations (4.1) and (4.2), and leads to ambiguity-free expressions for the charges as well [8, 22, 30].

This resolution of the ambiguities continues to apply in the context of the extended phase space, and we will show that it justifies the procedure employed by CLP in their modification of the subregion symplectic form. Furthermore, we will find that specific corrections to the charges appear, which match the corrections to charges explored in several recent works [8, 17, 22, 30, 38]. Beginning with the extended Lagrangian $L[X^* \phi] = X^* L[\phi]$, we can determine the extended symplectic potential from the variation

$$
\delta X^* L = X^*(\delta L + d_{\chi} L) = X^*(E \cdot \delta \phi) + dX^*(\theta + i_{\chi} L). \quad (4.9)
$$
Here, $\theta_{\text{CLP}} = X^*(\theta + i\chi L)$ appears naturally when parametrizing the variations in terms of $\delta\phi$ as opposed to $\delta X^*\phi$ appearing in equation (2.10). We then proceed to carry out the decomposition (4.6) for $\theta_{\text{CLP}}$ pulled back to the boundary hypersurface $\mathcal{N}$. Taking $\ell$, $\beta$, and $E$ to be defined by the decomposition for $\theta$ alone, we have that

$$
\theta_{\text{CLP}} = X^* (\delta\ell + d\beta + E + i\chi L) \quad (4.10)
$$

giving the expressions for the boundary, corner, and flux terms in the extended phase space,

$$
\ell_{\text{ex}} = X^* \ell \quad (4.13)
$$

$$
\beta_{\text{ex}} = X^* (\beta + i\chi \ell) \quad (4.14)
$$

$$
E_{\text{ex}} = X^* (E + i\chi (L + d\ell)) \quad (4.15)
$$

To argue for the uniqueness of this decomposition, we must examine the boundary condition implied by this choice of the flux $E_{\text{ex}}$. The first term $X^*E$ is simply the pullback of the flux term that appears before adding the embedding fields, and hence will vanish for the same set of boundary conditions that would make $E$ itself vanish. The second term $X^*\chi (L + d\ell)$ involves the flux due to variations of the embedding field $X$. Because it depends only on $\chi$ and not its derivatives, this term takes a Dirichlet form for the embedding field $X$, since it vanishes if the Dirichlet condition $\delta X = 0$ is imposed at the boundary.\footnote{To see the how the vanishing of $\chi$ is related to the Dirichlet condition for the embedding fields, we can take $X^\mu$ to be a set of scalar functions defining the embedding map. Then the coordinate expression for the pullback $X^*\chi^a$ is given by $[42]$}

$$
X^*\chi^a = \delta X^\mu (X^* \partial^\mu_a) \quad (4.16)
$$

We will see presently that taking the Dirichlet form for the embedding field flux leads to the expected expression for the improved charges.

The corner terms $\beta_{\text{ex}}$ in (4.14) defines a correction to the extended symplectic form, which is now given by

$$
\Omega_{\text{ex}} = \Omega_{\text{CLP}} - \delta \int_{\partial \Sigma} (\beta + i\chi \ell). \quad (4.17)
$$

To determine the diffeomorphism charges, we combine the expression (3.11) for the CLP symplectic form with the general relation (3.9) to derive

$$
-I_{\xi} \Omega_{\text{ex}} = \delta \int_{\partial \Sigma} h_{\xi} + \int_{\partial \Sigma} \left( \Delta_{\xi} (\beta + i\chi \ell) - h_{d\xi} \right) \quad (4.18)
$$

$$
h_{\xi} = Q_{\xi} + i\xi \ell - I_{\xi} \beta \quad (4.19)
$$

where we have retained the terms involving $\delta\xi$. In order for Hamilton’s equation to be satisfied, not only does the diffeomorphism need to be field-independent $\delta\xi = 0$, the quantity $\Delta_{\xi} (\beta + i\chi \ell)$ must also vanish. The charge is then given by the integral of the charge density $h_{\xi}$ (4.19), which includes corrections coming from $\ell$ and $\beta$. This corrected charge coincides
with the expression derived by Harlow and Wu in their work on covariant phase space with boundaries [30] and has appeared subsequently in a variety of other contexts [8, 17, 22, 38].

Since the improved charges have nontrivial integrability conditions in order to satisfy Hamilton’s equation, it is interesting to investigate when this condition is satisfied. While less is know for generic diffeomorphism-invariant theories, in the case of general relativity it has been shown that when choosing $E$ to be of Dirichlet form, $\Delta^\xi_\beta = 0$ for timelike and null boundaries if the transformation generated by $\xi^a$ preserves the bounding hypersurface $\mathcal{N}$. Such transformations also satisfy $\Delta^\xi_\ell = 0$ in the timelike case, but this term can be nonzero when working with null surfaces [8]. In fact, a nonzero value of $\Delta^\xi_\ell$ has been shown to be a necessary ingredient for extensions to appear in the Poisson bracket algebra of the charges, so it is interesting here to see it appearing as an obstruction to integrability of Hamilton’s equation within the extended phase space. For transformations with a transverse component to $\mathcal{N}$, in general we would expect both $\Delta^\xi_\beta$ and $\Delta^\xi_\ell$ to be nonzero. Another context in which there may be a nonzero contribution from $\Delta^\xi_\beta$ is in higher curvature theories, where in order to arrive at the universal embedding subalgebra $\text{Diff}(\partial \Sigma) \ltimes (\text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2)^{\partial \Sigma}$ as the only nonzero charges, a specific contribution to $\beta$ must be added that is not spacetime covariant [16]. It would be interesting to investigate the extent to which this obstructs the construction of unambiguous Hamiltonian charges in the higher curvature context.

In the case that the obstruction
\[
\mathcal{F}_\xi = -\int_{\partial \Sigma} \left( \Delta^\xi_\beta + i \chi^\xi \right) - h^\xi
\]  
(4.20)
is nonzero, we can still consider the quantity
\[
H_\xi = \int_{\partial \Sigma} h^\xi
\]  
(4.21)
as a function on the subregion phase space. It will not satisfy Hamilton’s equation for the diffeomorphism transformation due to the obstruction term (4.20). Instead, equation (4.18) represents a modification of Hamilton’s equation, where the obstruction appears as a flux of the local charge. In this case, one can still seek to compute the Poisson brackets of the charges $H_\xi$, in a similar manner to the procedure explained in section 3 of [22]. Letting $\mathcal{A}, \mathcal{B}, \ldots$ denote abstract indices on phase space and choosing an inverse $\Omega^{\mathcal{A} \mathcal{B}}_{\text{ex}}$ for the symplectic form, this Poisson bracket is computed to be
\[
\{H_\xi, H_\zeta\} = \Omega^{\mathcal{A} \mathcal{B}}_{\text{ex}} (\delta H_\xi)_{\mathcal{A}} (\delta H_\zeta)_{\mathcal{B}}
\]  
(4.22)
\[= \Omega^{\mathcal{A} \mathcal{B}}_{\text{ex}} \left( \Omega^{\mathcal{C} \mathcal{D}}_{\mathcal{A}} \hat{\xi}^\mathcal{C} + (\mathcal{F}_\xi)_{\mathcal{A}} \right) \left( \Omega^{\mathcal{E} \mathcal{F}}_{\mathcal{B}} \hat{\zeta}^\mathcal{E} + (\mathcal{F}_\zeta)_{\mathcal{B}} \right)
\]  
(4.23)
\[= \{H_\xi, H_\zeta\}_{\text{BT}} + \Omega^{\mathcal{A} \mathcal{B}}_{\text{ex}} (\mathcal{F}_\xi)_{\mathcal{A}} (\mathcal{F}_\zeta)_{\mathcal{B}}
\]  
(4.24)
where we have defined the Barnich-Troessaert (BT) bracket [24]
\[
\{H_\xi, H_\zeta\}_{\text{BT}} = -I_\xi \delta H_\zeta + I_\zeta F_\xi.
\]  
(4.25)

\(^5\)Due to degeneracies, $\Omega^{\mathcal{A} \mathcal{B}}_{\text{ex}}$ is not invertible, but we can construct a partial inverse that satisfies $\Omega^{\mathcal{C} \mathcal{D}}_{\text{ex}} \Omega^{\mathcal{E} \mathcal{F}}_{\text{ex}} \hat{\xi}^\mathcal{E}$. 

\(-13\)
The BT bracket will coincide with the Poisson bracket, provided that the flux terms in (4.24) can be shown to drop out. In [22], it was argued that this term will vanish if the flux term in the decomposition of the symplectic potential is in Dirichlet form. In the present context, if all terms appearing in the integrand (4.20) for $F_{\xi}$ only involve undifferentiated variations of the dynamical fields and $\chi^a$, we would similarly expect the term quadratic in the flux in (4.24) to drop out. For example, if we specialize to general relativity, consider a field-independent transformation $\delta \xi^a = 0$, and take the vector field $\xi^a$ to be tangent to the hypersurface $N$, the only remaining term in the flux will be the integral of $\chi^a \Delta(\hat{\xi} \ell)$. The contribution involving $F_{\xi}$ in (4.24) will vanish then as long as
\[ \Omega_{ab}^c \chi^a \chi^b = 0, \]
which holds assuming that the embedding fields commute among themselves.\footnote{Intriguingly, there remains a possibility that this commutator not vanish if one considered a noncommutative geometry setup, as occurs in some approaches to quantum gravity and string theory (see, e.g. [48–50]), in which case the Poisson bracket of localized charges would be corrected from the BT bracket.}

\[ \{ X^\mu, X^\nu \} = 0. \]

The BT bracket of the charges can be evaluated directly from equation (4.25) to obtain a charge representation theorem. Using that $h_{\xi} = -I_{\xi}(Q_{\chi} + i\chi_{\ell} + \beta)$ and $[\Delta_{\xi}, I_{\xi}] = -I_{[\xi, \xi]} + I_{I_{\xi_{\xi}}}$ [22], and recalling that $\Delta_{\xi}Q_{\chi} = 0$, we find
\[ \{ H_{\xi}, H_{\zeta} \}_{BT} = -\int_{\partial \Sigma} \left( I_{\xi} \delta h_{\zeta} - \mathcal{L}_{\xi} h_{\zeta} + I_{\xi} \Delta_{\zeta}(\beta + i\chi_{\ell}) - I_{\zeta} h_{\delta \xi} \right) \]
\[ = -\int_{\partial \Sigma} \left( -\Delta_{\xi} I_{\xi}(Q_{\chi} + i\chi_{\ell} + \beta) + I_{\xi} \Delta_{\zeta}(\beta + i\chi_{\ell}) - h_{I_{\xi_{\xi}}} \right) \]
\[ = \int_{\partial \Sigma} h_{[\xi, \zeta]} = H_{[\xi, \zeta]} \]
where the modified Lie bracket $[\cdot, \cdot]$ of field-dependent vector fields was defined in (1.1). Hence, the BT bracket of the localized charges reproduces the algebra satisfied by the vector fields under the bracket $[\cdot, \cdot]$. Crucially, no extension terms appear in the bracket, unlike the examples of localized charges constructed without introducing embedding fields. For the non-extended phase spaces, the BT bracket of the charges instead satisfies the relation $\{ H_{\xi}, H_{\zeta} \}_{BT} = H_{[\xi, \zeta]} + K_{\xi, \zeta}$, where the extension $K_{\xi, \zeta}$ is generically nonzero. This therefore generalizes the results of [25, 47] that the diffeomorphism charges represent the vector field algebra without extensions to the case of ambiguity-free charges for subregions with boundary.

It is worth mentioning a related recent construction in the context of a non-extended phase space in which the bracket of the charges also represents the vector field bracket without extension [38]. This work defines a bracket of the charges that subtracts the extension term $K_{\xi, \zeta}$ from the BT bracket, so that, by definition, the bracket yields a representation of the modified vector field bracket $[\cdot, \cdot]$ without extension. While the original construction of this bracket did not arise from a Poisson bracket on a phase space,
it has been pointed out in [45] that it does arise in the CLP extended phase space applied to covariant charges, and simply coincides with the Poisson bracket of the diffeomorphism charges. The representation theorem (4.30) extends this result to the improved, ambiguity-free charges, with the caveat that the BT bracket computed in the relation may not coincide with the Poisson bracket on the subregion phase space unless the flux terms in (4.24) commute.

As a final aside, we mention that an additional ambiguity can arise due to the fact that fixing $E$ does not uniquely specify $\ell$ and $\beta$ in the decomposition (4.6), since shifts of the form $\ell \rightarrow \ell + de, \beta \rightarrow \beta + \delta e$ do not affect this equation. A proposal for resolving this final ambiguity was made in [8, 22], in which one must further decompose $\beta$ as

$$\beta = -\delta c + \varepsilon$$  \hspace{1cm} (4.31)

and give a criterion for determining the corner flux $\varepsilon$. This then leads to a correction to the charges, and this contribution carries over to the charges in the extended phase space. Noting that $[\delta, \Delta \hat{\xi}] = \Delta \hat{\delta}_\xi$ [22], we find that

$$\int_{\partial \Sigma} \Delta \hat{\xi} \beta = \int_{\partial \Sigma} \left( -\delta \Delta \hat{\xi} c + \Delta \hat{\delta}_\xi c + \Delta \hat{\xi} \varepsilon \right)$$  \hspace{1cm} (4.32)

$$\int_{\partial \Sigma} \Delta \hat{\xi} c + \int_{\partial \Sigma} \left( \Delta \hat{\xi} \varepsilon + i \chi dc + \Delta \hat{\delta}_\xi \varepsilon \right)$$  \hspace{1cm} (4.33)

Then defining the improved charge density

$$\tilde{h}_\xi = h_\xi - \Delta \hat{\delta}_\xi,$$  \hspace{1cm} (4.34)

we see that equation (4.18) can be re-expressed as

$$- I_\xi \Omega_{\text{ex}} = \delta \tilde{H}_\xi - \tilde{F}_\xi$$  \hspace{1cm} (4.35)

with

$$\tilde{H}_\xi = \int_{\partial \Sigma} \tilde{h}_\xi$$  \hspace{1cm} (4.36)

$$\tilde{F}_\xi = - \int_{\partial \Sigma} \left( \Delta \hat{\xi} \left( \varepsilon + i \chi (\ell + dc) \right) - \tilde{h}_\delta \xi \right)$$  \hspace{1cm} (4.37)

It would be interesting to study these corner improvements in more detail, since, for example, they may provide a way of eliminating the dependence of the ambiguity-free charges on the choice of the hypersurface in which $\partial \Sigma$ is embedded.

5 Wald-Zoupas charges

One aspect of the Hamiltonian charges obtained in the CLP extended phase space that is initially surprising is that the charges satisfy Hamilton’s equation for surface-deforming diffeomorphisms that move the bounding surface $\partial \Sigma$. In standard constructions that do not employ embedding fields, such surface deformations fail to satisfy Hamilton’s equation, and there is a simple physical explanation why this occurs. Surface deformations describe
transformations of the dynamical fields corresponding to evolution along the boundary of the subregion, and during this evolution one generically expects flux of gravitational and matter degrees of freedom through the boundary. This flux appears as an obstruction to integrability of Hamilton’s equation, and hence one does not expect to be able to obtain Hamiltonian charges in this case. Instead, one can employ a Wald-Zoupas procedure to isolate a term in this equation that can be identified as the localized charge, and the obstruction term is used to construct the flux which parameterizes the failure of the local charge to be conserved as one evolves along the boundary [8, 20–22]. This procedure yields well-defined charges as long as a criterion for specifying the flux is given, and making such a choice is equivalent to determining the decomposition (4.6) of the symplectic potential. For example, one can appeal to the action principle for the subregion to fix the form of the flux [8, 22].

In the extended phase space, this physical argument for nonintegrability of Hamilton’s equation no longer applies. The reason is that the location of the surface $\partial \Sigma$ is now determined as the image of the embedding map $X$, which implies that a diffeomorphism now changes the location of the surface in addition to evolving the dynamical fields. Hence, when viewed relationally to the dynamical fields, the effective location of the surface is fixed, leaving no room for loss of symplectic flux during the diffeomorphism transformation. This holds in both the DF extended phase space in which diffeomorphisms are pure gauge, as well as in the CLP extended phase space containing nonzero diffeomorphism charges.

To better understand how fixing the effective relational location of the surface produces integrable Hamiltonian charges, it is helpful to see how one can obtain Wald-Zoupas charges in the extended phase space, which instead satisfy the modified Hamilton equation involving a flux term. These charges would arise from a transformation on phase space that changes the relational location of the surface relative to the dynamical fields. This can be achieved by changing the embedding map $X$ while holding the dynamical fields fixed. In the standard extended phase space, this corresponds to the $\hat{w}$ transformation described in equation (2.14), resulting in the desired modified Hamilton’s equation (2.15). This equation can further be modified to include corrections for resolving the ambiguities in phase space.

On the other hand, the $\hat{w}$ transformation alone does not yield the appropriate relation for the CLP extended phase space, as might be expected due to the difference in symplectic forms. Instead, the appropriate transformation is a combination of a spacetime diffeomorphism and a change in the embedding map which together fix the spacetime location of the target surface $\partial \Sigma$. This turns out to be none other than the $\hat{w}_0$ transformation defined in equation (3.1) in the unitary gauge description of the extended phase space. The condition that it fix the target location is equivalent to equation (3.2).

To confirm this is the desired transformation, we evaluate the contraction

$$- I_{\hat{w}_0} \Omega_{\text{ex}} = - I_{\xi_w} \Omega_{\text{ex}} - I_{\hat{w}} \Omega_{\text{ex}}.$$  \hspace{1cm} (5.1)

The first term is given by equation (4.18). The second term can be evaluated by noting that the bulk contribution to $\Omega_{\text{ex}}$ depends only on variations of the dynamical fields, which have zero contraction with $\hat{w}$. The remaining boundary contributions can be evaluated
using that the boundary term in $\Omega_{\text{CLP}}$ is given by the integral of $i\chi \theta + \frac{1}{2} i\chi i\chi L$ [25], which combines with the remaining terms in the expression (4.17) for $\Omega_{\text{ex}}$ to give

$$-I_{\hat{w}} \Omega_{\text{ex}} = -I_{\hat{w}} \int_{\partial \Sigma} \left( i\chi \theta + \frac{1}{2} i\chi i\chi L + \delta \beta + L \chi \beta - \frac{1}{2} i[x, \chi] \ell - i\chi \delta \ell + L \chi i\chi \ell \right)$$

(5.2)

$$= - \int_{\partial \Sigma} \left( i\xi \theta + i\xi i\chi L + L \xi \beta - i[x, \chi] \ell - i\xi \delta \ell + L \xi i\chi \ell - L \xi i\xi \ell \right)$$

(5.3)

$$= - \int_{\partial \Sigma} \left( i\xi \mathcal{E} + i\xi i\chi (L + d\ell) \right),$$

(5.4)

where we employed the shorthand $\xi$ for $\xi_w$, and the last line applied the decomposition (4.6) of $\theta$. Combining with (4.18), we arrive at the expression for the contraction of $\hat{w}_0$ into the extended symplectic form,

$$-I_{\hat{w}_0} \Omega_{\text{ex}} = \delta \int_{\partial \Sigma} h_\xi - \int_{\partial \Sigma} \left( i\xi \mathcal{E} + i\xi i\chi (L + d\ell) - \Delta_\xi (\beta + i\chi \ell) + h_\xi \right).$$

(5.5)

Note that the combinations $\mathcal{E} + i\chi (L + d\ell)$ and $\beta + i\chi \ell$ are the same as those appearing in the flux and corner terms $\mathcal{E}_{\text{ex}}$ (4.15) and $\beta_{\text{ex}}$ (4.14) in the decomposition of the extended symplectic form $\theta_{\text{CLP}}$.

Equation (5.5) is the expected form of the modified Hamilton equation satisfied by Wald-Zoupas charges. The first term is the total variation of the localized charge, which is precisely the same form as the ambiguity-free diffeomorphism charge (4.21). The remaining terms represent the flux and parameterize the failure of the $\hat{w}_0$ transformation to preserve the subregion symplectic form. Comparing to the analogous equation for Wald-Zoupas constructions in the non-extended phase space [8, 22], we see that the same terms appear in the flux and corner terms $\mathcal{E}_{\text{ex}}$ (4.15) and $\beta_{\text{ex}}$ (4.14) in the decomposition of the extended symplectic form $\theta_{\text{CLP}}$.

Together, equations (4.18) and (5.5) demonstrate that the CLP extended phase space can describe both diffeomorphism charges and Wald-Zoupas localized charges by changing the details of the phase space transformation. Both charges are given by the integral of the charge density $h_\xi$ defined in (4.19), and in general neither type of charge satisfies Hamilton’s equation due to specific obstruction terms to integrability.

6 Discussion

This work has shown that the gravitational charges constructed in the CLP extended phase space can be improved to ambiguity-free expressions by applying recent results for handling boundaries in the covariant phase space. Although the ambiguity resolution generically produces a new obstruction to integrability of Hamilton’s equation, charges can nevertheless be identified and given an algebra structure through BT bracket. This algebra gives a
representation of the vector field field-dependent Lie bracket without extension. Furthermore, the CLP extended phase space is flexible enough to describe standard diffeomorphism charges, as well as Wald-Zoupas-like charges whose obstruction to integrability is precisely analogous to that encountered in non-extended phase spaces.

An immediate question to be addressed in the present construction is whether the extension of $\partial\Sigma$ to a hypersurface $N$ is necessary in constructing ambiguity-free charges. On the one hand, this choice is natural when constructing an action principle for an open subregion in spacetime $U$ and taking $\partial\Sigma$ to simply be a cut of a component $N$ of the boundary. On the other hand, it seems more natural for the charges to be completely covariant with respect to the codimension-2 surface $\partial\Sigma$, since such a surface determines a subregion of spacetime via the causal development of an infilling hypersurface $\Sigma$. It is also notable that the symmetry group $\text{Diff}(\partial\Sigma) \ltimes (\text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2)\partial\Sigma$ is naturally associated with covariance with respect to the codimension-2 surface [47], while introducing the bounding hypersurface $N$ would be expected to break this down to a subgroup. A possibility for maintaining corner covariance is to use an appropriate corner improvement to the charges as described in equations (4.34) and (4.36), and references [8, 22]. It would be useful to take up this question in more detail.

Another generalization would be to investigate higher curvature theories. The Iyer-Wald formalism employed in this work immediately applies to any diffeomorphism-invariant theory. However, in order to maintain the same universal symmetry group $\text{Diff}(\partial\Sigma) \ltimes (\text{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2)\partial\Sigma$, specific choices must be made for resolving the ambiguities [16]. These resolution terms can introduce noncovariance into the charges and obstruct integrability of Hamilton’s equation. Nevertheless, since the charges and brackets still appear to be well-defined in the presence of these obstructions, one still has a consistent construction of improved gravitational charges. It would be interesting to compute the explicit form of the obstruction to integrability in order to better understand how the CLP construction extends to other diffeomorphism-invariant theories.

At a practical level, the CLP charges provide a possible way to better understand charges constructed at asymptotic boundaries on which leaky boundary conditions are imposed, such as at $\mathcal{I}^+$ in asymptotically flat spacetimes. Since the CLP charges agree with the standard expression (4.21) for charges constructed by a Wald-Zoupas procedure, when applied in the context of, for example, 4D asymptotically flat space, they should reproduce the standard expressions for BMS charges and their generalizations [19, 20, 51–55]. These charges could still fail to satisfy Hamilton’s equation due to the obstruction terms appearing in (4.18); however, this obstruction is fundamentally different from the standard obstruction appearing in the Wald-Zoupas procedure, which involves the news tensor. The news tensor should appear in the quantity $E_{\text{ex}}$, whereas the obstruction in (4.18) involves the noncovariance of the boundary Lagrangian $\ell$ and the corner term $\beta$. In particular, $\Delta_{\xi}\ell$ is the quantity that shows up in the expression for the extension in the bracket of Wald-Zoupas charges [8, 22], and hence for asymptotic symmetries that produce no such extension, it is likely that the corresponding CLP charges integrate Hamilton’s equation for the diffeomorphism. An interesting future direction would be to explore the applications of the CLP extended phase space to asymptotic symmetries an to examine the integrability properties in more detail.
We found that the charges precisely reproduce the algebra satisfied by the vector fields without extension, according to (4.30). However, in some contexts, the extension term in the symmetry generators yields important information about the theory. For example, the Brown-Henneaux central extension in the asymptotic symmetries of AdS \(_3\) \([56]\) determines the central charge of the dual CFT, and extensions of symmetries on black hole horizons can in some cases provide a derivation of the black hole entropy \([3–8]\). The question then arises as to whether any such extension terms can appear in the CLP extended phase space. Perhaps they can arise due to a difference between the BT bracket (4.25) and the Poisson bracket of the charges, due to some noncommutativity of the flux terms in (4.24).

One interpretation of the results on black hole entropy is that the extra degrees of freedom associated with the extended phase space yield a contribution to the entropy of the region outside the horizon. One of the primary motivations for introducing the extended phase space is to attempt to give a definition of entanglement entropy in gravity, which is complicated by the lack of factorization of the classical phase space due to gauge constraints. In this picture, one seeks to construct the global phase space via a gluing procedure of two extended phase spaces associated with a subregion and its complement \([15]\). This gluing procedure involves a symplectic reduction of the product of the two extended phase spaces in which the charges are matched at the boundary to produce zero total charge, which ensures the gauge constraints hold on the global phase space. This same procedure should continue to hold in the CLP extended phase space. In fact, the CLP construction offers several advantages since the charges represent the diffeomorphism algebra of the vector fields without extension, and hence lead to a more direct application of the symplectic reduction procedure. On the other hand, it has been speculated that the existence of central extensions in the charge algebra could be indicative of a reduction procedure that does not fully eliminate all the extended degrees of freedom, and the entropy of the subregion could related to the leftover degrees of freedom that are not eliminated during the reduction \([22]\). It would be interesting how this conjecture plays out for the CLP extended phase space, whose charge algebra never exhibits extension terms. Instead, the object that would serve as an extension appears as an obstruction to integrability of the ambiguity-free charges, and it would be interesting to examine the effects of this obstruction to the symplectic reduction procedure.

Acknowledgments

I thank Rob Leigh for helpful discussions. This work is supported by the Air Force Office of Scientific Research under award number FA9550-19-1-036.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP\(^3\) supports the goals of the International Year of Basic Sciences for Sustainable Development.
References

[1] R.M. Wald, *Black hole entropy is the Noether charge*, Phys. Rev. D **48** (1993) R3427 [gr-qc/9307038] [inSPIRE].

[2] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, Phys. Rev. D **50** (1994) 846 [gr-qc/9403028] [inSPIRE].

[3] A. Strominger, *Black hole entropy from near horizon microstates*, JHEP **02** (1998) 009 [hep-th/9712251] [inSPIRE].

[4] S. Carlip, *Black hole entropy from conformal field theory in any dimension*, Phys. Rev. Lett. **82** (1999) 2828 [hep-th/9812013] [inSPIRE].

[5] S.W. Hawking, M.J. Perry and A. Strominger, *Soft Hair on Black Holes*, Phys. Rev. Lett. **116** (2016) 231301 [arXiv:1601.00921] [inSPIRE].

[6] S. Haco, S.W. Hawking, M.J. Perry and A. Strominger, *Black Hole Entropy and Soft Hair*, JHEP **12** (2018) 098 [arXiv:1810.01847] [inSPIRE].

[7] L.-Q. Chen, W.Z. Chua, S. Liu, A.J. Speranza and B.d.S.L. Torres, *Virasoro hair and entropy for axisymmetric Killing horizons*, Phys. Rev. Lett. **125** (2020) 241302 [arXiv:2006.02430] [inSPIRE].

[8] V. Chandrasekaran and A.J. Speranza, *Anomalies in gravitational charge algebras of null boundaries and black hole entropy*, JHEP **01** (2021) 137 [arXiv:2009.10739] [inSPIRE].

[9] A.B. Prema, G. Compère, L.P. de Gioia, I. Mol and B. Swidler, *Celestial holography: Lectures on asymptotic symmetries*, SciPost Phys. Lect. Notes **47** (2022) 1 [arXiv:2109.00997] [inSPIRE].

[10] A.-M. Raclariu, *Lectures on Celestial Holography*, arXiv:2107.02075 [inSPIRE].

[11] S. Pasterski, *Lectures on celestial amplitudes*, Eur. Phys. J. C **81** (2021) 1062 [arXiv:2108.04801] [inSPIRE].

[12] W. Donnelly, *Entanglement entropy and nonabelian gauge symmetry*, Class. Quant. Grav. **31** (2014) 214003 [arXiv:1406.7304] [inSPIRE].

[13] V. Benedetti and H. Casini, *Entanglement entropy of linearized gravitons in a sphere*, Phys. Rev. D **101** (2020) 045004 [arXiv:1908.01800] [inSPIRE].

[14] J.R. David and J. Mukherjee, *Entanglement entropy of gravitational edge modes*, arXiv:2201.06043 [inSPIRE].

[15] W. Donnelly and L. Freidel, *Local subsystems in gauge theory and gravity*, JHEP **09** (2016) 102 [arXiv:1601.04744] [inSPIRE].

[16] A.J. Speranza, *Local phase space and edge modes for diffeomorphism-invariant theories*, JHEP **02** (2018) 021 [arXiv:1706.05061] [inSPIRE].

[17] L. Freidel, M. Geiller and D. Pranzetti, *Edge modes of gravity. Part I. Corner potentials and charges*, JHEP **11** (2020) 026 [arXiv:2006.12527] [inSPIRE].

[18] W. Donnelly, L. Freidel, S.F. Moosavian and A.J. Speranza, *Gravitational edge modes, coadjoint orbits, and hydrodynamics*, JHEP **09** (2021) 008 [arXiv:2012.10367] [inSPIRE].

[19] E.E. Flanagan, K. Prabhu and I. Shehzad, *Extensions of the asymptotic symmetry algebra of general relativity*, JHEP **01** (2020) 002 [arXiv:1910.04557] [inSPIRE].
[20] R.M. Wald and A. Zoupas, A General definition of ‘conserved quantities’ in general relativity and other theories of gravity, *Phys. Rev. D* 61 (2000) 084027 [gr-qc/9911095] [inSPIRE].

[21] V. Chandrasekaran, E.E. Flanagan and K. Prabhu, Symmetries and charges of general relativity at null boundaries, *JHEP* 11 (2018) 125 [arXiv:1807.11499] [inSPIRE].

[22] V. Chandrasekaran, E.E. Flanagan, I. Shehzad and A.J. Speranza, A general framework for gravitational charges and holographic renormalization, arXiv:2111.11974 [inSPIRE].

[23] H. Bart, Quasi-local conserved charges in General Relativity, Ph.D. thesis, Munich University, Germany (2019), arXiv:1908.07504. DOI: 10.5282/edoc.25814 [inSPIRE].

[24] G. Barnich and C. Troessaert, BMS charge algebra, *JHEP* 12 (2011) 105 [arXiv:1106.0213] [inSPIRE].

[25] L. Ciambelli, R.G. Leigh and P.-C. Pai, Embeddings and Integrable Charges for Extended Corner Symmetry, *Phys. Rev. Lett.* 128 (2022) [arXiv:2111.13181] [inSPIRE].

[26] T. Jacobson, G. Kang and R.C. Myers, On black hole entropy, *Phys. Rev. D* 49 (1994) 6587 [gr-qc/9312023] [inSPIRE].

[27] G. Compere and D. Marolf, Setting the boundary free in AdS/CFT, *Class. Quant. Grav.* 25 (2008) 195014 [arXiv:0805.1902] [inSPIRE].

[28] T. Andrade, W.R. Kelly, D. Marolf and J.E. Santos, On the stability of gravity with Dirichlet walls, *Class. Quant. Grav.* 32 (2015) 235006 [arXiv:1507.06081] [inSPIRE].

[29] T. Andrade and D. Marolf, Asymptotic Symmetries from finite boxes, *Class. Quant. Grav.* 33 (2016) 015013 [arXiv:1508.02515] [inSPIRE].

[30] D. Harlow and J.-Q. Wu, Covariant phase space with boundaries, *JHEP* 10 (2020) 146 [arXiv:1906.08616] [inSPIRE].

[31] H. Gomes, F. Hopfmüller and A. Riello, A unified geometric framework for boundary charges and dressings: non-Abelian theory and matter, *Nucl. Phys. B* 941 (2019) 249 [arXiv:1808.02074] [inSPIRE].

[32] F. Hopfmüller and L. Freidel, Null Conservation Laws for Gravity, *Phys. Rev. D* 97 (2018) 124029 [arXiv:1802.06135] [inSPIRE].

[33] E. Witten, Interacting Field Theory of Open Superstrings, *Nucl. Phys. B* 276 (1986) 291 [inSPIRE].

[34] C. Crnkovic and E. Witten, Covariant description of canonical formalism in geometrical theories, in S.W. Hawking and W. Israel eds., *Three Hundred Years of Gravitation*, ch. 16, Cambridge University Press, Cambridge, U.K. (1987) pp. 676–684, http://adsabs.harvard.edu/abs/1987thyg.book..676C.

[35] C. Crnkovic, Symplectic Geometry of the Covariant Phase Space, Superstrings and Superspace, *Class. Quant. Grav.* 5 (1988) 1557 [inSPIRE].

[36] A. Ashtekar, L. Bombelli and O. Reula, The covariant phase space of asymptotically flat gravitational fields, in M. Francaviglia ed., *Mechanics, Analysis and Geometry: 200 Years After Lagrange*, Elsevier Science Publishers B.V., Netherlands (1991).

[37] J. Lee and R.M. Wald, Local symmetries and constraints, *J. Math. Phys.* 31 (1990) 725 [inSPIRE].

[38] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, Extended corner symmetry, charge bracket and Einstein’s equations, *JHEP* 09 (2021) 083 [arXiv:2104.12881] [inSPIRE].
[39] R.M. Wald, *On identically closed forms locally constructed from a field*, *J. Math. Phys.* **31** (1990) 2378.

[40] V. Iyer and R.M. Wald, *A Comparison of Noether charge and Euclidean methods for computing the entropy of stationary black holes*, *Phys. Rev. D* **52** (1995) 4430 [gr-qc/9503052] [SPIRE].

[41] S. Carlip, *Statistical mechanics and black hole entropy*, gr-qc/9509024 [SPIRE].

[42] A.J. Speranza, *Geometrical tools for embedding fields, submanifolds, and foliations*, arXiv:1904.08012 [SPIRE].

[43] H. Gomes and A. Riello, *The observer’s ghost: notes on a field space connection*, *JHEP* **05** (2017) 017 [arXiv:1608.08226] [SPIRE].

[44] N. Arkani-Hamed, H. Georgi and M.D. Schwartz, *Effective field theory for massive gravitons and gravity in theory space*, *Annals Phys.* **305** (2003) 96 [hep-th/0210184] [SPIRE].

[45] L. Freidel, *A canonical bracket for open gravitational system*, arXiv:2111.14747 [SPIRE].

[46] Y. Choquet-Bruhat and C. DeWitt-Morette, *Analysis, Manifolds and Physics, Revised Edition*, Elsevier, Netherlands (2004), https://www.elsevier.com/books/analysis-manifolds-and-physics-revised-edition/choquet-bruhat/978-0-444-86017-0.

[47] L. Ciambelli and R.G. Leigh, *Isolated surfaces and symmetries of gravity*, *Phys. Rev. D* **104** (2021) 046005 [arXiv:2104.07643] [SPIRE].

[48] A. Connes, *Noncommutative Geometry*, Academic Press, U.S.A. (1994).

[49] A.H. Chamseddine and A. Connes, *Universal formula for noncommutative geometry actions: Unification of gravity and the standard model*, *Phys. Rev. Lett.* **77** (1996) 4868 [hep-th/9606056] [SPIRE].

[50] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [SPIRE].

[51] M. Campiglia and A. Laddha, *New symmetries for the Gravitational S-matrix*, *JHEP* **04** (2015) 076 [arXiv:1502.02318] [SPIRE].

[52] M. Campiglia and A. Laddha, *Asymptotic symmetries and subleading soft graviton theorem*, *Phys. Rev. D* **90** (2014) 124028 [arXiv:1408.2228] [SPIRE].

[53] E.E. Flanagan and D.A. Nichols, *Conserved charges of the extended Bondi-Metzner-Sachs algebra*, *Phys. Rev. D* **95** (2017) 044002 [arXiv:1510.03386] [SPIRE].

[54] G. Compère, A. Fiorucci and R. Ruzziconi, *Superboost transitions, refraction memory and super-Lorentz charge algebra*, *JHEP* **11** (2018) 200 [Erratum ibid. **04** (2020) 172] [arXiv:1810.00377] [SPIRE].

[55] L. Freidel, R. Oliveri, D. Pranzetti and S. Speziale, *The Weyl BMS group and Einstein’s equations*, *JHEP* **07** (2021) 170 [arXiv:2104.05793] [SPIRE].

[56] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, *Commun. Math. Phys.* **104** (1986) 207.