AN ENERGY APPROACH TO UNIQUENESS FOR HIGHER-ORDER GEOMETRIC FLOWS

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ABSTRACT. We demonstrate that the uniqueness of solutions to a broad class of parabolic geometric evolution equations can be proven via a direct and essentially classical energy argument which avoids the DeTurck trick entirely. Previously, we have used a variation of this technique to give an alternative proof and slight extension to the basic uniqueness result for complete solutions to the Ricci flow of uniformly bounded curvature. Here we extend this approach to curvature flows of all orders, including the $L^2$-curvature flow and a class of quasilinear higher-order flows related to the obstruction tensor. We also detail its application to the fully nonlinear cross-curvature flow.

1. Introduction

In this paper, we revisit the problem of uniqueness for parabolic curvature flows on a smooth manifold $M = M^n$. By virtue of their invariance under the action of $\text{Diff}(M)$, such equations are never strictly parabolic, and this basic geometric degeneracy must be overcome or circumvented in some fashion in order to use the methods of standard parabolic theory on this problem. The standard means of circumvention in this situation is the DeTurck trick ([D], [H2]) which, as it pertains to uniqueness, is an exchange of one problem for two: the problem of uniqueness for the geometric flow for separate problems of existence and uniqueness for two auxiliary strictly parabolic systems. In the case of the Ricci flow, for example (see [H2], [CZ]), one must verify the short-time existence of a certain harmonic-map-type heat flow to obtain the so-called DeTurck diffeomorphisms, and then verify the uniqueness of the associated solutions to the Ricci-DeTurck flow.

When, as on compact manifolds, the problems for the auxiliary equations belong to standard parabolic theory, this exchange is purely an advantageous one. (See [BH2], e.g., for a careful treatment of this approach to a class of higher-order equations.) In some settings, however, these auxiliary equations introduce nontrivial questions of their own. In [CZ], for example, en route to proving the uniqueness of solutions to the Ricci flow on noncompact manifolds, the authors must first confront a nonstandard and rather thorny problem of existence for the DeTurck diffeomorphisms. Of course, their approach pays dividends beyond the question of uniqueness, in that it verifies, as a byproduct, the existence of a well-controlled solution to the DeTurck flow associated to any complete solution to the Ricci flow of bounded curvature.

The purpose of this paper is to demonstrate that, insofar as it concerns the question of uniqueness, it is possible to overcome the geometric degeneracy of these
equations without resorting to DeTurck’s trick or considering any auxiliary parabolic systems. We show that, after reframing the problem as one for a suitably prolonged system, the uniqueness in fact follows from an elementary and essentially classical energy argument, whose proof moreover yields a quantitative (albeit rather coarse) bound on the $L^2$-norm of the difference of the solutions. In an earlier paper [Ko], we applied a variation of this technique to the particular case of the Ricci flow and a somewhat more general class of second-order systems. (See also the subsequent paper of Bell [Be] for an application to the conformal Ricci flow.) Here, we generalize this technique to higher-order systems.

For concreteness, we first consider a class of quasilinear higher-order parabolic equations considered in the recent papers of Bahuaud and Helliwell [BH1], [BH2] and give an alternative proof of the uniqueness result in the latter reference. This class includes the $L^2$-curvature flow and a family of equations introduced in [Bo], [BH1], [BH2] related to the ambient obstruction tensor. We then formulate a general uniqueness result, and verify its applicability to a further example, the fully-nonlinear cross-curvature flow of Chow and Hamilton [CH].

We will restrict our attention below to compact manifolds $M$. Our primary aim is to demonstrate an alternative method of overcoming the geometric degeneracy of these equations, and the demonstration is the most transparent in this simpler case. With suitable estimates on the solutions and the addition of sufficiently rapidly decaying weights, however, this basic technique can be extended to the non-compact setting. In [Ko], for example, we have explored this extension already in the context of the Ricci flow, where we use an energy approach to give an alternative proof of the uniqueness of complete solutions of bounded curvature ([H1], [CZ]) and prove a slight extension to solutions with some spatial growth of curvature. We intend to pursue this extension to other equations in future work.

### 2. A class of quasilinear curvature flows

In this section, we will use our energy technique to give an alternative proof of uniqueness for a class of curvature flows considered by Bahuaud and Helliwell [BH1], [BH2]. The equations have the form

\[
\frac{\partial}{\partial t} g = \Theta_{2k}(g) + \Lambda_{2k-1}(g)
\]

where the leading order terms satisfy

\[
\Theta_{2k}(g) \doteq (-1)^{k+1} 2(\Delta^{(k)} \Rc + \alpha \Delta^{(k)} Sg + \beta \Delta^{(k-1)} \nabla \nabla S)
\]

and the lower-order terms $\Lambda_{2k-1}(g)$ are some polynomial expression in $g$, $g^{-1}$ and the covariant derivatives of $R$ up to order $2k - 1$. Here, $\alpha$ and $\beta$ are constants, and $\nabla = \nabla_g$, $\Delta = \Delta_g$, $\Rc = \Rm(g)$, $\Rc = \Rc(g)$, and $S = S(g)$ denote, respectively, the Levi-Civita connection, Laplacian, $(4, 0)$-curvature tensor, Ricci curvature tensor, and scalar curvature associated to $g$. We permit $k = 0$ when $\beta = 0$.

Included in this family of equations are the Ricci flow

\[
\frac{\partial}{\partial t} g = -2 \Rc(g),
\]

and the $L^2$-curvature flow of J. Streets [SI],

\[
\frac{\partial}{\partial t} g_{ij} = 2(\Delta R_{ij} - \frac{1}{2} \nabla_i \nabla_j S) + 2R^{pq} R_{ipqj} - R^p_i R_{pj} + R^p_i R_{jpqr} - \frac{1}{4} |\Rm|^2 g_{ij},
\]
which is the negative gradient flow of the squared $L^2$-norm of $R$.

When $n \geq 4$ is even, $k = n/2 - 1$, and $\beta = -(n - 2)/(2(n - 1))$, for an appropriate choice of $\Lambda$, equation (1) is

$$\frac{\partial}{\partial t} g = c_n \left( O_n + (-1)^{\frac{n}{2}} \left( \alpha + \frac{1}{2(n - 1)} \right) \Delta^{\frac{n}{2} - 1} S g \right),$$

and

$$O_n = \frac{1}{(-2)^{\frac{n}{2}}(n/2 - 2)!} \left( \Delta^{\frac{n}{2} - 1} P - \frac{1}{2(n - 1)} \Delta^{\frac{n}{2} - 2} \nabla \nabla S \right)$$

are the ambient obstruction tensors of Fefferman-Graham [FG]. Here, $P = (n - 2)^{-1}(Rc - (1/(2(n - 1)) S g)$ is the Schouten tensor. When $n = 4$, the tensor $O_4$ is the Bach tensor

$$B_{ij} = \Delta P_{ij} - \nabla_i \nabla_j P - 2P^{kl} W_{kij} + |P|^2 g_{ij} - 4P^2_{ij}.$$ 

These flows were introduced by Bour (in the case $k = 1$) and by Bahuaud and Helliwell in the case $k \geq 2$ as a potential dynamic means of obtaining obstruction flat manifolds. Short-time existence is not known for the flow by the pure obstruction tensor.

The following theorem is due to Bahuaud-Helliwell [BH2]. We will give an alternative proof based on Theorem 2 below.

**Theorem 1** ([BH2], Theorem A). Let $M = M^n$ be a compact manifold and $\alpha > -1/(2(n - 1))$, $\beta \in \mathbb{R}$, and $\Omega > 0$ given constants. If and $g$ and $\tilde{g}$ are solutions to (1) on $M \times [0, \Omega)$ which satisfy that $g(0) = \tilde{g}(0)$, then $g \equiv \tilde{g}$ on $M \times [0, \Omega)$.

The idea of our argument is to cast the problem as one of uniqueness for a prolonged system, centered on the difference $\nabla^{(l)} R - \nabla^{(l)} \tilde{R}$ of the covariant curvature tensors of the solutions $g$ and $\tilde{g}$. These tensors will satisfy evolution equations that are parabolic (or nearly so), but with respect to different operators $\Theta_{2k}(g), \Theta_{2k}(\tilde{g})$ which involve the derivatives $\nabla^{(l)} (g - \tilde{g})$ of the difference of the solutions up to order $2k$ in their respective solutions. We will thus we will add to the system sufficiently many derivatives of the difference of the metrics $g - \tilde{g}$ to control the terms coming from the difference of the operators. The differences of the derivatives of $g - \tilde{g}$, i.e., $g - \tilde{g}$ and the derivatives of the difference of connections $\Gamma - \tilde{\Gamma}$ can in turn be controlled in a pointwise, ordinary differential way via their evolution equations by sufficiently many differences of derivatives of curvature. Provided we include enough of the differences of the derivatives of curvature in our system, we can obtain a closed system of integral inequalities, susceptible to an attack by energy methods. This prolongation procedure has its origins in the work of [AI], [WY].

For example, in the case of the $L^2$-curvature flow [3], our prolonged system will consist of the quantities $g - \tilde{g}, \nabla (\Gamma - \tilde{\Gamma}), \nabla \tilde{R} - R, \nabla^{(2)} R - \nabla^{(2)} \tilde{R}$. The rest of the argument is just a direct computation to show that

$$\mathcal{E} \geq \int_M \left( |g - \tilde{g}|^2_{g(t)} + |\nabla (\Gamma - \tilde{\Gamma})|_{g(t)}^2 + |R - \tilde{R}|_{g(t)}^2 + |\nabla^{(2)} R - \nabla^{(2)} \tilde{R}_{g(t)}|_{g(t)}^2 \right) d\mu_{g(t)}$$

satisfies the differential inequality $\dot{\mathcal{E}} \leq C \mathcal{E}$ for some $C$, from which the asserted uniqueness follows.
2.1. Setup and statement of results. In this section, we assume that $M$ is compact, and that $g(t)$ and $\hat{g}(t)$ are two smooth solutions to (1) for some fixed $k \geq 1$ on $M \times [0, \Omega)$, with Levi-Civita connections $\nabla$ and $\tilde{\nabla}$, and curvature tensors $R_m$ and $\tilde{R}_m$, respectively. For any $t$, we define

$$h \doteq g - \hat{g}, \quad A \doteq \nabla - \tilde{\nabla}, \quad X^{(t)} \doteq \nabla^{(t)} R_m - \tilde{\nabla}^{(t)} \tilde{R}_m, \quad Z^{(t)} \doteq \nabla^{(t)} S - \tilde{\nabla}^{(t)} \tilde{S}.$$ 

Here $A$ is the family of $(2, 1)$-tensors given in local coordinates by $A_{ij}^k = \Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k$. We will use one of the solutions, $g(t)$, as a reference metric and, for each $t \in [0, \Omega)$, consider the $L^2(d\mu_{g(t)})$-inner product induced on various tensor bundles over $M$. Given a bundle $Z$ over $M$ and smooth sections $U, V \in Z$, we will write

$$(U, V) \doteq (U, V)_{L^2(d\mu_{g(t)})} \doteq \int_M \langle U, V \rangle_{g(t)} \, d\mu_{g(t)}, \quad ||U||^2 \doteq (U, U).$$

For the most part, we will suppress the dependency of these norms on $g(t)$ and use the unadorned notation $| \cdot | \doteq | \cdot |_{g(t)}$ to represent the norms induced by $g(t)$ on the tensor bundles we will encounter. We will also use the convention that $U * V$ denotes a linear combination of contractions of $U \otimes V$ by the metric $g$. (Should we wish to emphasize the role of the metric in the contraction or denote contractions by the metric $\hat{g}$, we will use instead the notation $U *_g V$ and $U *_{\hat{g}} V$.)

Next, we define the quantities

$$G(t) \doteq ||h||^2 + \|\nabla^{(k)} A\|^2, \quad H(t) \doteq ||X^{(0)}||^2 + \|X^{(2k)}\|^2, \quad K(t) \doteq \|Z^{(2k)}\|^2.$$ 

We will prove Theorem [1] by verifying the following inequalities.

**Theorem 2.** For any $\epsilon > 0$, there exists a constant $C$ such that the quantities $G$, $H$, and $K$ satisfy the differential inequalities

(4) $\dot{G} \leq C(G + H) + 2\epsilon \|\nabla^{(k+1)} X^{(2k)}\|^2$

(5) $\dot{H} \leq C(G + H) - 2(1 - \epsilon)\|\nabla^{(k+1)} X^{(2k)}\|^2 - 2\alpha \|\nabla^{(k+1)} Z^{(2k)}\|^2$

(6) $\dot{K} \leq C(G + H) + 2\epsilon \|\nabla^{(k+1)} X^{(2k)}\|^2 - 2(1 + 2\alpha(n - 1)) \|\nabla^{(k+1)} Z^{(2k)}\|^2$

on the interval $[0, \Omega]$.

**Corollary 3.** For any $r \in \mathbb{R}$ and any $\epsilon > 0$, there is a constant $C$ depending on $\epsilon$ and the solutions $g$ and $\hat{g}$ such that $E \doteq G + H + rK$ satisfies

(7) $E(t) \leq C E + a(\epsilon, r) \|\nabla^{(k+1)} X^{(2k)}\|^2 + b(n, r) \|\nabla^{(k+1)} Z^{(2k)}\|^2$

on $[0, \Omega]$ where

$$a(\epsilon, r) = -2(1 - \epsilon(r + 2)), \quad b(n, r) = -2(\alpha + r(1 + 2\alpha(n - 1))).$$

In particular, when $\alpha + 1/(2(n - 1)) > 0$, choosing $r > -\alpha/(1 + 2\alpha(n - 1))$ and $\epsilon < 1/(r + 2)$, we have $a, b < 0$, so $E(t) \leq C E(t)$ on $[0, \Omega]$ and

$$E(t_2) \leq e^{C(t_2 - t_1)} E(t_1)$$

for any $0 \leq t_1 \leq t_2 \leq \Omega$.

When $\alpha \geq 0$, as, e.g., in the $L^2$-curvature flow, the term $K$ can be dropped from the quantity $E$ altogether. It is included only to balance the potentially positive contribution proportional to $\|\nabla^{(k+1)} Z^{(2k)}\|^2$ in the evolution equation (5) for $H$. A similar device was used by Bour in [10] to obtain $L^2$-Bernstein-Bando-Shi-type estimates for solutions to equation (1) with $k = 1$. 


2.2. **Evolution equations and commutation relations.** In order to prove Theorem 2, we will need to compute the evolution equations for the connection and for the covariant derivatives of the curvature tensor of a solution of (1). We will use the notation \( P^l_g(R) \) to denote a polynomial expression formed from sums of contractions of the tensor products of various factors of \( R, \nabla R, \ldots, \nabla^{(l)} R \) taken with respect to the metric \( g \). We begin with two standard commutation identities.

**Lemma 4.** Suppose \( g = g(t) \) is a smooth family of metrics on \( M \) evolving according to (1) and \( W = W(t) \) is a smooth family of tensor fields. Then, for any multi-indices \( \alpha \) and \( \beta \) of lengths \( l \) and \( m \), respectively, we have the commutation relations

\[
[(\nabla)_\alpha, (\nabla)_\beta] W = \sum_{p=0}^{m+l-2} g^{-1} \nabla^{(m+l-2-p)} R \nabla^{(p)} W,
\]

where \((\nabla)_\alpha\) is shorthand for \( \nabla_{\alpha_1} \nabla_{\alpha_2} \cdots \nabla_{\alpha_l} \),

\[
\left[ \frac{\partial}{\partial t}, (\nabla)^r \right] W = \sum_{p=1}^r P_{g}^{2k+p+1}(R) \nabla^{(r-p)} W
\]

for all \( r \geq 0 \).

**Proof.** For the first identity, note that \( [\nabla_{\alpha_1}, \nabla_{\beta_1}] W = g^{-1} R W \). The special case of (8) where \( \beta \) has length 1 then follows by induction from the identity

\[
[\nabla_{\alpha_1}, \nabla_{\alpha_2} \cdots \nabla_{\alpha_{l+1}}, \nabla_{\beta_1}] W = \nabla_{\alpha_1} [\nabla_{\alpha'}, \nabla_{\beta_1}] W + \nabla_{\beta_1} [\nabla_{\alpha_1}, \nabla_{\beta}] W
\]

where \( \alpha' \) represents a multi-index of length \( l \). The general case of (8) follows from this case from a separate induction argument and the identity

\[
[(\nabla)_\alpha, (\nabla_{\beta_1} \nabla_{\beta_2} \cdots \nabla_{\beta_m})] W = [(\nabla)_\alpha, (\nabla_{\beta_1})] (\nabla_{\beta'}) W + \nabla_{\beta_1} [(\nabla)_\alpha, (\nabla_{\beta'}) W]
\]

where \( \beta' \) represents a multi-index of length \( m \).

For (9), we combine the general formula for the evolution of the Christoffel symbols with the equation (1), to obtain

\[
\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{nk} \left\{ \nabla_i \frac{\partial}{\partial t} \beta_{jm} + \nabla_j \frac{\partial}{\partial t} \beta_{im} - \nabla_m \frac{\partial}{\partial t} \beta_{ij} \right\} = P_{g}^{2k+1}(R).
\]

Then we have

\[
\left[ \frac{\partial}{\partial t}, (\nabla)^r \right] W = \frac{\partial}{\partial t} \Gamma \ast W = P_{g}^{2k+1}(R) \ast W,
\]

from which (9) follows, with the identity

\[
\left[ \frac{\partial}{\partial t}, (\nabla)^{r+1} \right] W = \left[ \frac{\partial}{\partial t}, (\nabla)^{r} \right] (\nabla)^{r} W + \nabla \left[ \frac{\partial}{\partial t}, (\nabla)^{r} \right],
\]

and an induction argument. \( \square \)

Now we determine the structure of the evolution equations satisfied by the covariant derivatives of the curvature tensor associated to a solution of (1).

**Lemma 5.** Suppose \( g(t) \) is a solution to (1). For any \( m = 0, 1, 2, \ldots \), the \( m \)-fold covariant derivative \( (\nabla)^{(m)} R \) of \( g \) evolves by the equation

\[
\frac{\partial}{\partial t} (\nabla)^{(m)} R = (-1)^k (\Delta^{(k+1)} (\nabla)^{(m)} R + \alpha (\nabla \nabla \Delta^{(k)} (\nabla)^{(m)} S) \circ g) + P_{g}^{2k+m+1}(R)
\]
and the \(m\)-fold covariant derivative of the scalar curvature satisfies an equation of the form

\[
\frac{\partial}{\partial t} \nabla^{(m)} S = (1 + 2\alpha(n - 1))\Delta^{(k+1)} \nabla^{(m)} S + P_g^{2k+m+1}(R)
\]
on \(M \times [0, \Omega]\).

Here, for \(m = 0\),

\[
(\nabla \nabla \Delta^{(k)} S \circ g)_{abcd} = \nabla_a \nabla_d \Delta^{(k)} S_{gbc} + \nabla_b \nabla_c \Delta^{(k)} S_{gad} - \nabla_a \nabla_c \Delta^{(k)} S_{gbd}
\]

is the Kulkarni-Nomizu product of \(\nabla \nabla \Delta^{(k)} S\) and \(g\), and, for \(m \geq 1\), we define

\[
(\nabla \nabla \Delta^{(k)} \nabla^{(m)} S \circ g)_{abcd} = \nabla_a \nabla_d \Delta^{(k)} (\nabla)_{gbc} + \nabla_b \nabla_c \Delta^{(k)} (\nabla)_{gad} - \nabla_a \nabla_c \Delta^{(k)} (\nabla)_{gbd} - \nabla_b \nabla_d \Delta^{(k)} (\nabla)_{gac}
\]

for any multi-index \(\gamma\) of length \(m\).

**Proof.** The proof is a standard computation, some details of which we include for completeness. We start with the case \(m = 0\) and the formula

\[
DR_g[h]_{ijkl} = -\frac{1}{2} (\nabla_i \nabla_l h_{jk} + \nabla_j \nabla_k h_{il} - \nabla_i \nabla_k h_{jl} - \nabla_j \nabla_l h_{ik}) + (R \ast g h)_{ijkl}.
\]

for the linearization \(DR_g\) of \(R = \text{Rm}\) at the metric \(g\) in the direction of a given symmetric two tensor \(h\). By the Bianchi identities,

\[
DR_g(2 \nabla \nabla \Delta^{(k)} Rc) = -\Delta R + P_g^{(2k)}(R),
\]

so, using (8), we have

\[
DR_g(2 \Delta^{(k)} Rc) = -\Delta^{(k+1)} R + P_g^{2k}(R) + [DR_g, \Delta^{(k)}] R
\]

\[
= -\Delta^{(k+1)} R + P_g^{2k}(R).
\]

Similarly, using (12), we have

\[
DR_g(2 \Delta^{(k)} S) = -\nabla \nabla \Delta^{(k)} S \circ g + P_g^{2k}(R).
\]

The diffeomorphism invariance of \(R\) implies that \(DR_g[\mathcal{L}_X g] = \mathcal{L}_X R_g\) for any vector field \(X\), and, since

\[
\Delta^{(k-1)} \nabla \nabla S = \nabla \nabla \Delta^{(k-1)} S + P_g^{2k-2}(R)
\]

by (8), it follows that

\[
DR_g[\Delta^{(k-1)} \nabla \nabla S] = P_g^{2k}(R),
\]

so that the third term in \(\Theta_{2k}(g)\) contributes only lower order terms to the evolution of \(R\). Combining these computations with the observation that \(DR_g[\Lambda_{2k-1}(g)] = P_g^{2k+1}(R)\), yields (10) in the case \(m = 0\). Together with the commutation relations (8) and (9), this case implies (10) for \(m > 0\).

We will also need to express the the difference of \(m\)-fold covariant derivatives relative to \(g\) and \(\tilde{g}\) in terms of the tensor \(A\) and its derivatives.
Lemma 6. Suppose \( \nabla \) and \( \tilde{\nabla} \) are two connections on \( TM \) and \( W \) is a smooth tensor field, and \( A = \nabla - \tilde{\nabla} \). Then, for any \( m \geq 0 \),

\[
(13) \quad \nabla^{(m)} W - \tilde{\nabla}^{(m)} W = \sum_{p=1}^{m} Q^p(A) \ast \tilde{\nabla}^{(m-p)} W
\]

where \( Q^p(A) \) represents some linear combination of simple (non-metric) contractions of tensors of the form \( \nabla^{(m_1)} A \otimes \nabla^{(m_2)} A \otimes \cdots \otimes \nabla^{(m_k)} A \) where \( m_1 + m_2 + \cdots + m_k + k = p \).

Proof. The proof is by induction. For \( m = 1 \), we have \( \nabla W - \tilde{\nabla} W = A \ast W \).

Supposing the claim to be true for some \( m \geq 1 \), we can verify it for \( m + 1 \) by observing that

\[
\nabla^{(m+1)} W - \tilde{\nabla}^{(m+1)} W = \nabla (\nabla^{(m)} W - \tilde{\nabla}^{(m)} W) + (\nabla - \tilde{\nabla}) \tilde{\nabla}^{(m)} W
\]

\[
= \nabla \left( \sum_{p=1}^{m} Q^p(A) \ast \tilde{\nabla}^{(m-p)} W \right) + A \ast \tilde{\nabla}^{(m)} W
\]

\[
= \sum_{p=1}^{m} \left( \nabla (Q^p(A)) \ast \tilde{\nabla}^{(m-p)} W + Q^p(A) \ast A \ast \tilde{\nabla}^{(m-p)} W + Q^p(A) \ast \tilde{\nabla}^{(m-p+1)} W \right)
\]

\[
+ Q^1(A) \ast \tilde{\nabla}^{(m)} W
\]

\[
= \sum_{p=1}^{m} \left( Q^{p+1}(A) \ast \tilde{\nabla}^{(m-p)} W + Q^p(A) \ast \tilde{\nabla}^{(m-p+1)} W \right) + Q^1(A) \ast \tilde{\nabla}^{(m)} W
\]

\[
= \sum_{p=1}^{m+1} Q^p(A) \ast \tilde{\nabla}^{(m-p)} W,
\]

which is of the desired form. \( \square \)

2.3. Evolution equations for \( h, \nabla^{(l)} A, \) and \( X^{(l)} \). We will only need the precise form of the terms of highest order in the evolution equations below; it will be enough to observe that the lower-order terms are polynomial expressions involving at least one factor of the elements of our system. Since \( M \) is compact, and we are working in the interval of smooth existence of the solutions, the remaining factors will be uniformly bounded.

To simplify our presentation, it will be convenient to use the notation \( C \) to denote any smooth, uniformly bounded tensor whose internal structure is immaterial to our argument. This tensor will vary in rank and composition from line-to-line (and even term-to-term) in our expressions.

Lemma 7. Suppose that \( g \) and \( \tilde{g} \) are solutions to \( \Pi \) on \( M \times [0, \Omega) \). Then, for any \( l \geq 0 \), \( h \) and \( \nabla^{(l)} A \) satisfy the evolution equations

\[
(14) \quad \frac{\partial}{\partial t} h = C \ast_g h + \sum_{p=0}^{2k} C \ast_g X^{(p)},
\]

\[
(15) \quad \frac{\partial}{\partial t} \nabla^{(l)} A = C \ast_g h + \sum_{p=0}^{l} C \ast_g \nabla^{(p)} A + \sum_{p=0}^{l+1} \sum_{q=0}^{2k} C \ast_g \nabla^{(p)} X^{(q)},
\]
and $X^{(l)}$ and $Z^{(l)}$ satisfy
\begin{equation}
\frac{\partial}{\partial t} X^{(l)} = (-1)^k \left( \Delta^{(k+1)} X^{(l)} + \alpha(\nabla \Delta^{(k)} Z^{(l)}) \right) + C \ast g \ h + \sum_{p=0}^{m(k,l)} C \ast g \ \nabla^{(p)} A + \sum_{p=0}^{l+1} \sum_{q=0}^{2k} C \ast g \ \nabla^{(p)} X^{(q)},
\end{equation}
(16)
\begin{equation}
\frac{\partial}{\partial t} Z^{(l)} = (-1)^k (1 + 2\alpha(n-1)) \Delta^{(k+1)} Z^{(l)} + C \ast g \ h
\end{equation}
where $m(k,l) \equiv \max\{2k+1, l\}$.

**Proof.** Schematically, both $\Delta^{(l)} R$ and $\nabla \nabla \Delta^{(l-1)} S$ are of the form $(g^{-1})^{*l} \ast \nabla^{(2l)} R$ and $\Delta^{(l)} S g$ is of the form $(g^{-1})^{*(l+2)} \ast \nabla^{(2l)} R \ast g$. So, using that $g^{ij} \sim \tilde{g}^{ij} = -g^{ik} \tilde{g}^{kj} h_{kl}$, we have
\[
\Theta_{2k}(g) - \Theta_{2k}(\tilde{g}) = (g^{-1})^{*(k+1)} \ast \nabla^{(2k)} R - (\tilde{g}^{-1})^{*(k-1)} \ast \nabla^{(2k)} \tilde{R}
\]
\[
+ (g^{-1})^{*(k+2)} \ast \nabla^{(2k)} R \ast g - (\tilde{g}^{-1})^{*(k+2)} \ast \nabla^{(2k)} \tilde{R} \ast \tilde{g}
\]
\[
= C \ast h + C \ast X^{(2k)}.
\]
Since $\Lambda_{2k-1}(g)$ has polynomial structure in $g, g^{-1}, R, \ldots, \nabla^{(2k-1)} R$, we have
\[
\Lambda_{2k-1}(g) - \Lambda_{2k-1}(\tilde{g}) = C \ast h + C \ast X^{(1)} + C \ast X^{(2)} + \ldots + C \ast X^{(2k-1)}
\]
and (14) follows.

Next, since $\frac{\partial}{\partial t} \Gamma = g^{-1} \ast \nabla \frac{\partial}{\partial t} g$, we compute that
\[
\frac{\partial}{\partial t} A = g^{-1} \ast \nabla \frac{\partial}{\partial t} g - \tilde{g}^{-1} \ast \nabla \frac{\partial}{\partial t} \tilde{g} = C \ast h + C \ast A + C \ast \nabla \frac{\partial}{\partial t} h
\]
\[
= C \ast h + C \ast A + \sum_{p=0}^{2k} \nabla (C \ast X^{(p)}),
\]
where we have used (14), together with the identity $\nabla_k h_{ij} = -A^p_{ki} \tilde{g}_{pj} - A^p_{kj} \tilde{g}_{pj}$.

Now we use the commutator identity (9) to deduce that
\[
\frac{\partial}{\partial t} \nabla^{(l)} A = \left[ \frac{\partial}{\partial t} \nabla^{(l)} \right] A + \nabla^{(l)} \frac{\partial}{\partial t} A
\]
\[
= \sum_{p=0}^{l-1} C \ast \nabla^{(p)} A + \nabla^{(l)} \left( C \ast h + C \ast A + \sum_{q=0}^{2k} \nabla (C \ast X^{(q)}) \right)
\]
\[
= C \ast h + \sum_{p=0}^{l-1} C \ast \nabla^{(p)} A + \sum_{p=0}^{l+1} \sum_{q=0}^{2k} C \ast \nabla^{(p)} X^{(q)},
\]
which is (13).

For (16), we consider the difference of the highest-order terms in (10) first. By (13), $\nabla^{(l)} W - \tilde{\nabla}^{(l)} W = \sum_{p=0}^{l-1} C \ast \nabla^{(p)} A$ for any tensor $W$, so
\[
\Delta^{(k+1)} \nabla^{(l)} R - \tilde{\Delta}^{(k+1)} \tilde{\nabla}^{(l)} \tilde{R} = C \ast h + \sum_{p=0}^{2k+1} C \ast \nabla^{(p)} A + \Delta^{(k+1)} X^{(l)}
\]
and similarly,
\[
\nabla \nabla \Delta^{(k)} \nabla^{(l)} S \otimes g - \nabla \nabla \Delta^{(k)} \nabla^{(l)} \tilde{S} \otimes \tilde{g} = \nabla \nabla \Delta^{(k)} \nabla^{(l)} Z + C \ast h + \sum_{p=0}^{2k+1} C \ast \nabla^{(p)} A.
\]

For the difference of the lower-order terms in (10), we note that
\[
P^{2k+1}_g(R) - P^{2k+1}_g(\tilde{R}) = C \ast h + \sum_{q=0}^{2k+1} C \ast X^{(q)}
\]
where we have used (13) to obtain the second line. Combined with our earlier expressions for the higher order terms, we obtain (16). Equation (17) follows similarly, using (11). □

2.4. \(L^2\)-inequalities. In the proof of Theorem 2, we will make use of a version of Gårding’s inequality and a version of the Gagliardo-Nirenberg inequality.

Lemma 8. For any tensor bundle \(T^s_b(M)\), there exists a constant \(C\) depending on \(a, b, l, n\) and the covariant derivatives of \(R\) up to order \(l - 1\), such that
\[
\left|(-1)^l \left(\Delta^{(l)} W, W\right) - \|\nabla^{(l)} W\|^2\right| \leq C \sum_{p=0}^{l-1} \|\nabla^{(p)} W\|^2.
\]

for any section \(W \in C^\infty(T^s_b(M))\). Moreover, for any \(0 \leq l < k\), and any \(\epsilon > 0\), there exists a constant \(C = C(\epsilon, a, b, k, l, n)\) such that
\[
\|\nabla^{(l)} W\|^2 \leq C(\epsilon)\|W\|^2 + \epsilon\|\nabla^{(k)} W\|^2.
\]

Proof. The proof of (18) is essentially standard. For \(l = 1\), one has \(-\Delta W, W) = \|W\|^2. Proceeding by induction, and assuming the inequality to hold for \(l - 1\) for some \(l > 1\), one integrates by parts and uses (8) to obtain
\[
(-1)^l \left(\Delta^{(l)} W, W\right) = (-1)^{l-1} \left(\Delta^{(l-1)} \nabla W, \nabla W\right) + (-1)^{l-1} \left(\nabla, \Delta^{(l-1)} W\right) = (-1)^{l-1} \left(\Delta^{(l-1)} W, \nabla W\right) + \sum_{p+q=2l-3} (C \ast \nabla^{(p)} R \ast \nabla^{(q)} W, \nabla W)
\]

Since the sum of the orders of the derivatives in each term in the sum on the right is \(2l - 2\), one can integrate by parts to achieve that no derivative of order greater than \(l - 1\) appears on any one factor. Using the induction hypothesis on the first term establishes the inequality for \(l\).

Inequality (19) is also standard. Let \(\epsilon > 0\) be given. The case \(k = 1, l = 0\) is trivial. Proceeding by induction on \(k\), and assuming the inequality to hold for all \(0 \leq l < k - 1\) where \(k > 1\), we let \(0 \leq l < k\). Integrating by parts, we obtain
\[
\|\nabla^{(k-1)} W\|^2 \leq \|\nabla^{(k-2)} W\|\|\nabla^{(k)} W\|
\]
so, using Cauchy-Schwarz and the induction hypothesis,
\[
\|\nabla^{(k-1)} W\|^2 \leq C\|W\|^2 + \frac{1}{2}\|\nabla^{(k-1)} W\|^2 + \frac{\epsilon}{2}\|\nabla^{(k)} W\|^2
\]
for some $C = C(\epsilon)$. Hence

$$\|\nabla^{(k-1)} W\|^2 \leq C\|W\|^2 + \epsilon\|\nabla^{(k)} W\|^2. \tag{20}$$

This handles the case $l = k-1$. If $0 \leq l < k-1$, we can use the induction hypothesis again and (20) to obtain

$$\|\nabla^{(l)} W\|^2 \leq C\|W\|^2 + \|\nabla^{(k-1)} W\|^2 \leq C\|W\|^2 + \epsilon\|\nabla^{(k)} W\|^2. \tag{21}$$

\[\Box\]

In the next two lemmas, we use the inequality (19) to obtain interpolation inequalities for the elements of our system.

Lemma 9. Let $h$ and $A$ be as defined above. Then, for $0 \leq l \leq k$, there exists a constant $C$ such that

$$\|\nabla^{(l)} A\|^2 \leq C\|h\|^2 + C\|\nabla^{(k)} A\|^2. \tag{21}$$

Proof. Inequality (21) will follow immediately from (19) once we prove an inequality of the form $\|A\|^2 \leq C\|h\|^2 + C\|\nabla^{(k)} A\|^2$. For this, we use the identities

$$\nabla h = -\nabla \tilde{g} = A * \tilde{g}, \quad A = \tilde{g}^{-1} * \nabla h \tag{22}$$

Differentiating the first of these identities $k$ times and using (19) yields a constant $C_1$ such that

$$\|\nabla^{(k+1)} h\|^2 \leq C_1 (\|A\|^2 + \|\nabla^{(k)} A\|^2).$$

Using this together with (19) again, we obtain a constant $C_2$ such that

$$\|\nabla^{(2)} h\|^2 \leq C_2 \left(\|h\|^2 + \|A\|^2 + \|\nabla^{(k)} A\|^2\right).$$

Hence, using the identities in (22) and integrating by parts, we obtain

$$\|A\|^2 \leq C_3 (\|h\|\|\nabla^{(2)} h\| + \|h\|\|A\|) \leq C_3^2 \left(\frac{C_2}{2} + 1\right) \|h\|^2 + \frac{1}{2C_2} \|\nabla^{(2)} h\|^2 + \frac{1}{4} \|A\|^2 \leq C\|h\|^2 + \frac{3}{4} \|A\|^2 + \frac{1}{2} \|\nabla^{(k)} A\|^2$$

so $\|A\|^2 \leq C\|h\|^2 + C\|\nabla^{(k)} A\|^2$ as desired. \[\Box\]

Lemma 10. Let $X^{(q)}$ and $Z^{(q)}$ be as defined above. For any nonnegative integers $p$, $q$ with $0 \leq p \leq k$ and $0 \leq q \leq 2k$, and any $\epsilon > 0$, there exists a constant $C$ depending on those parameters such that

$$\|\nabla^{(p)} X^{(q)}\|^2 \leq C \left(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(0)}\|^2 + \|\nabla^{(k)} X^{(2k)}\|^2\right) \tag{23}$$

$$\|\nabla^{(p)} Z^{(q)}\|^2 \leq C \left(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|Z^{(0)}\|^2 + \|\nabla^{(k)} Z^{(2k)}\|^2\right) \tag{24}$$

$$\leq C \left(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(0)}\|^2 + \|\nabla^{(k)} X^{(2k)}\|^2\right) \tag{25}$$
Proof. For (23), we first integrate by parts and use (13) to obtain
\[\|\nabla^{(k)} X^{(0)}\|^2 = (C \ast \nabla^{(2k)} X^{(0)}, X^{(0)}) = (C \ast X^{(2k)} + C \ast (\nabla^{(2k)} - \nabla^{(2k)}) \tilde{R}, X^{(0)})\]
\[= \left( C \ast X^{(2k)}, X^{(0)} \right) + \left( \sum_{r=0}^{2k-1} C \ast \nabla^{(r)} A, X^{(0)} \right)\]
\[= \left( C \ast X^{(2k)}, X^{(0)} \right) + \left( \sum_{r=0}^{k} C \ast \nabla^{(r)} A, X^{(0)} \right) + \left( \nabla^{(k)} A, \sum_{r=1}^{k-1} C \ast \nabla^{(r)} X^{(0)} \right)\]
which, using (19) and Cauchy Schwarz, implies that for all \( \epsilon > 0 \), there is a \( C = C(\epsilon) \) such that
\[\|\nabla^{(k)} X^{(0)}\|^2 \leq C(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(0)}\|^2) + \epsilon \|X^{(2k)}\|^2.\]
Using \( X^{(k)} = \nabla^{(k)} X^{(0)} + (\nabla^{(k)} - \nabla^{(k)}) \tilde{R} \) with (19), (21), and (20), we obtain
\[\|X^{(k)}\|^2 \leq C(\|\nabla^{(k)} X^{(0)}\|^2 + \|h\|^2 + \|\nabla^{(k)} A\|^2)\]
\[\leq C(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(0)}\|^2) + \epsilon \|X^{(2k)}\|^2,\]
and, in the same way, using \( X^{(2k)} = \nabla^{(k)} X^{(k)} + (\nabla^{(k)} - \nabla^{(k)}) \nabla^{(k)} \tilde{R} \), we obtain
\[\|X^{(2k)}\|^2 \leq \|X^{(k)}\|\|\nabla^{(k)} X^{(2k)}\| + C \sum_{p=0}^{k} \|\nabla^{(p)} A\|\|X^{(2k)}\|\]
\[\leq C(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(k)}\|^2 + \|\nabla^{(k)} X^{(2k)}\|^2).\]
Thus, choosing \( \epsilon \) sufficiently small, we obtain
\[\|X^{(2k)}\|^2 \leq C(\|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(0)}\|^2 + \|\nabla^{(k)} X^{(2k)}\|^2).\]
In combination with (19), inequalities (20), (21), and (23) establish the estimate (23) in the case \( p = 0, q = k \), the case \( p = 0, q = 2k \), and the case \( 0 \leq p \leq k, q = 0 \).
We consider the remaining cases in turn. First, if \( p = 0 \), and \( 0 \leq q \leq k \), we have
\[X^{(q)} = \nabla^{(q)} X^{(0)} + \sum_{r=0}^{q-1} C \ast \nabla^{(r)} A,\]
from which the desired inequality is obtained as a consequence of (19), (21), and the cases considered earlier. Second, if \( p = 0 \), and \( k < q \leq 2k \), then
\[X^{(q)} = \nabla^{(q-k)} X^{(k)} + \sum_{r=0}^{q-k-1} C \ast \nabla^{(r)} A.\]
The second term can be controlled by (21), and, working as above, we obtain
\[\|\nabla^{(q-k)} X^{(k)}\|^2 \leq C \left( \|X^{(k)}\|^2 + \|\nabla^{(k)} X^{(k)}\|^2 \right)\]
\[\leq C \left( \|X^{(k)}\|^2 + \|X^{(2k)}\| + (\nabla^{(k)} - \nabla^{(k)}) \nabla^{(k)} \tilde{R}\|^2 \right)\]
\[\leq C \left( \|h\|^2 + \|\nabla^{(k)} A\|^2 + \|X^{(k)}\|^2 + \|X^{(2k)}\|^2 \right)\]
from which we see that the first term on the right of (29) can also be estimated by cases already considered, thus the desired inequality follows for this range of \( p \) and \( q \) as well.
The remaining case is $0 < p \leq k$, $0 < q \leq 2k$. When $q = 2k$ and $p = k$, the desired inequality is trivial, and the case when $p < k$ and $q = 2k$ follows from the case $p = 0$, $q = 2k$ and (19). The case $q < 2k$ and $p + q > 2k$ may be reduced to the case $p < k$ and $q = 2k$ by means of the identity

$$\nabla^p X^q = \nabla^{p+q-2k} X^{2k} + \sum_{r=0}^{2k-q-1} C \nabla^r A,$$

and the remaining case, $q < 2k$ and $p + q \leq 2k$ may be reduced to the case $p = 0$ and $q < 2k$ by means of the identity

$$\nabla^p X^q = X^{p+q} + \sum_{r=0}^{p+q-1} C \nabla^r A.$$

The inequality (24) may be obtained from the above by the same argument, substituting $Z^q(t)$ for $X^q(t)$. Inequality (25) follows from (24) in view of the estimate

$$\|\nabla^{(k+1)} Z^l\| \leq C(\|h\| + \|\nabla^k A\| + \|\nabla^{(k+1)} X^l\|),$$

which is a simple consequence of the identity $Z^l = C * h + C * X^l$.

□

2.5. Proof of Theorem 2. We first compute the derivative of $\mathcal{G}$. Since $M$ is compact, in the computations below, the effect of differentiating the norms $| \cdot | = | \cdot |_{g(t)}$ and measure $d\mu_{g(t)}$ will only generate contributions that are uniformly proportional to the original quantity. Thus, using (14), (15), and Lemmas 9 and 10 in conjunction with (19), we have

$$\dot{\mathcal{G}} \leq C \mathcal{G} + \frac{\partial}{\partial t} h, h + \frac{\partial}{\partial t} \nabla^{(k)} A, \nabla^{(k)} A$$

$$\leq C \mathcal{G} + \left( C \ast h + \sum_{p=0}^{2k} C \ast X^{(p)}, h \right)$$

$$+ \left( C \ast g h + \sum_{p=0}^{k} C \ast g \nabla^{(p)} A + \sum_{p=0}^{k+1} \sum_{q=0}^{2k} C \ast g \nabla^{(p)} X^{(q)}, \nabla^{(k)} A \right)$$

$$\leq C(\varepsilon)(\mathcal{G} + \mathcal{H}) + 2\varepsilon \|\nabla^{(k+1)} X^{(2k)}\|^2$$

for any $\varepsilon > 0$. This is (4).
2.5.1. The case \( \alpha = 0 \). We first compute the derivative of \( H \) in the simpler case that \( \alpha = 0 \). Using (15) and (18), we have that, for any \( 0 \leq l \leq 2k \),

\[
\frac{d}{dt} \|X^{(l)}\|^2 \leq C \|X^{(l)}\|^2 + 2(-1)^k ( \Delta^{(k+1)} X^{(l)}, X^{(l)} ) \\
+ \left( C \ast h + \sum_{p=0}^{2k+1} C \ast \nabla^{(p)} A + \sum_{p=0}^{l+1} \sum_{q=0}^{2k} C \ast \nabla^{(p)} X^{(q)}, X^{(l)} \right) \\
\leq -2 \|\nabla^{(k+1)} X^{(l)}\|^2 + C \sum_{p=0}^{k} \|\nabla^{(p)} X^{(l)}\|^2 + C \left( \|h\| + \sum_{p=0}^{k} \|\nabla^{(p)} A\| \right) \|X^{(l)}\| \\
+ C \sum_{p=0}^{k+1} \|\nabla^{(k)} A\| \|\nabla^{(p)} X^{(l)}\| + C \sum_{p=0}^{k+1} \sum_{q=0}^{2k} \|\nabla^{(p)} X^{(q)}\| \|X^{(l)}\| \\
+ C \sum_{r=1}^{l-k} \sum_{q=0}^{2k} \|\nabla^{(k+1)} X^{(q)}\| \|\nabla^{r} X^{(l)}\|.
\]

Substituting \( l = 0 \) and \( l = 2k \) in the above inequality and using Cauchy-Schwarz together with (19) and the interpolation inequalities in Lemmas (10) and (11) we obtain that, for any \( \epsilon > 0 \), there is a constant \( C = C(\epsilon) \) such that

\[
\dot{H} \leq C(\mathcal{G} + H) - 2(1 - \epsilon) \|\nabla^{(k+1)} X^{(2k)}\|^2.
\]

This is (3) in the case \( \alpha = 0 \). The computation for the formula (3) is entirely analogous, using (25) to bound \( \mathcal{K} \) and all positive terms involving \( \|\nabla^{(p)} Z^{(q)}\| \) in terms of \( \mathcal{G}, \mathcal{H} \), and \( \|\nabla^{(k+1)} X^{(2k)}\| \). In fact, the computation for \( \mathcal{K} \) is the same regardless of the value of \( \alpha \), since the leading term in the evolution of \( Z^{(2k)} \) is always only a multiple of \( \Delta^{(k+1)} \).

2.5.2. The case \( \alpha \neq 0 \). The main ingredient is the following lemma concerning the second leading order term in (16).

**Lemma 11.** There is a constant \( C \) such that

\[
\left| \left( \nabla \nabla \Delta^{(k)} Z^{(l)} \odot g, X^{(l)} \right) - \left( \Delta^{(k+1)} Z^{(l)}, Z^{(l)} \right) \right| \\
\leq C \|\nabla^{(k+1)} Z^{(l)}\| \left( \|h\| + \sum_{p=0}^{k} \|\nabla^{(p)} A\| + \sum_{p=0}^{k} \sum_{q=1}^{l} \|\nabla^{(p)} X^{(q)}\| \right)
\]

for any \( t \in [0, \Omega] \).

Using (18) and the interpolation inequalities (19), (21), (23), and (25), we obtain the following result.

**Corollary 12.** For any \( \alpha \) and any \( \epsilon > 0 \), there is a constant \( C \) depending on \( \alpha, \epsilon \), and the solutions \( g \) and \( \tilde{g} \) such that

\[
2\alpha(-1)^k \left( \nabla \nabla \Delta^{(k)} Z^{(l)} \odot g, X^{(l)} \right) \\
\leq -2\alpha \|\nabla^{(k+1)} Z^{(l)}\|^2 + \epsilon \|\nabla^{(k+1)} X^{(l)}\|^2 + C(\mathcal{G} + H)
\]

for any \( t \in [0, \Omega] \).
Using this corollary and carrying over the computations from the case $\alpha = 0$ for all of the other terms in $\mathcal{H}$, we obtain (34) and (35) in the general case, completing the proof of Theorem 2.

**Proof of Lemma 11.** First we note that

\begin{equation}
\bar{\gamma}_\delta \Delta^k Z^{(l)}(\nabla \delta R_{ru} - \nabla \delta \bar{R}_{ru})
\end{equation}

where $\gamma$ and $\delta$ represent multi-indices of length $l$ and $g^{\gamma \delta}(\nabla)_{\gamma} V(\nabla)_{\delta} W$ is shorthand for $g^{\gamma_{i_1} \ldots \gamma_{i_l}} \nabla_{\gamma_{i_1}} \ldots \nabla_{\gamma_{i_l}} V \otimes \nabla_{\delta_{i_1}} \ldots \nabla_{\delta_{i_l}} W$. Hence,

\begin{equation}
\int_M \left( \nabla \nabla \Delta^k Z^{(l)} \right) \otimes g, X^{(l)} \right) d\mu
= \int_M \left( C * h, X^{(l)} \right) + 4g^{ar} g^{du} g^{\gamma \delta} \nabla_a \nabla_d \Delta^k Z^{(l)}(\nabla \delta R_{ru} - \nabla \delta \bar{R}_{ru}) \right) d\mu
\end{equation}

Now, integrating by parts and using the contracted second Bianchi identity, we simplify the second term on the right-hand side as follows:

\begin{equation}
\int_M g^{ar} g^{du} g^{\gamma \delta} \nabla_a \nabla_d \Delta^k Z^{(l)}(\nabla \delta R_{ru} - \nabla \delta \bar{R}_{ru}) d\mu
= \int_M g^{\gamma \delta} \left( \left( \Delta^k Z^{(l)}(g^{ar} g^{du} \nabla_a \nabla_d R_{ru} - \nabla \delta g^{du} \nabla_a \nabla \delta \bar{R}_{ru}) \right)
+ \left( \Delta^k Z^{(l)}, C * h + C * A + C * \nabla A \right) \right) d\mu
= \int_M \left( \frac{1}{2} \left( \Delta^k Z^{(l)}, \Delta Z^{(l)} \right)
+ g^{\gamma \delta} \left( \left( \Delta^k Z^{(l)}, (\nabla \delta, \Delta) S - (\nabla \delta, \Delta \bar{S}) \right)
+ \left( \Delta^k Z^{(l)}, C * h + C * A + C * \nabla A \right) \right) d\mu.
\end{equation}

Using (32), we see that $[\nabla(2), \nabla(l)] = P^l_g(R)$, so the commutator terms in the second line of (34) can be rewritten as

\begin{equation}
g^{ar} g^{du} [\nabla_a \nabla_d, \nabla \delta] R_{ru} - \nabla \delta g^{du} [\nabla_a \nabla \delta, \nabla \delta] R_{ru}
= P^l_g(R) - P^l_g(\bar{R}) = C * h + \sum_{p=0}^{l} C * X^{(p)}.
\end{equation}

The commutator terms in the third line are of the same schematic form and can be simplified in the same way.

Thus, after using integration by parts in the first line of (31) to move the Laplacian from the right side of the inner product to the left, and further integrations-by-parts on the second, third, and fourth lines to move $(k-1)$-covariant derivatives
from the factor of $\Delta^{(k)} Z$ to the opposite factor in the inner product, we obtain
\[
\left| \int_M g^{\alpha \beta} g^{\gamma \delta} \nabla_\alpha \nabla_\beta \Delta^{(k)} Z^l_\delta \left( \nabla_\gamma R_{ru} - \nabla_\delta \tilde{R}_{ru} \right) d\mu - \frac{1}{2} \left( \Delta^{(k+1)} Z^l, Z^l \right) \right| 
\leq C \| \nabla^{(k+1)} Z^l \| \left( \| h \| + \sum_{p=0}^k \| \nabla^{(p)} A \| + \sum_{p=0}^k \sum_{q=0}^l \| \nabla^{(p)} X^{(q)} \| \right).
\]
Substituting this expression into (33), we obtain (30). □

3. A general uniqueness theorem

As we remarked in the introduction, the method of the previous section does not depend on the specific structure of (1), but rather of the structure of the inequalities satisfied by the prolonged system derived from it. The uniqueness assertion for the prolonged system is essentially then a consequence of the standard energy argument for strictly parabolic equations. We formulate a somewhat more general version of the argument below which may be useful for other applications.

3.1. Setup. Let $M = M^n$ be a closed manifold equipped with a family of smooth metrics $g(t)$ for $t \in [0, \Omega]$ and $\mathcal{X}$ and $\mathcal{Y}$ tensor bundles over $M$. For simplicity of notation, we will regard $\mathcal{X}$ and $\mathcal{Y}$ as orthogonal subbundles of $\mathcal{W} \equiv \mathcal{X} \oplus \mathcal{Y}$. Denote by $(U, V)$ the family of $L^2(d\mu_{g(t)})$-inner products
\[
(U, V) \equiv \int_M \langle U, V \rangle_{g(t)} d\mu_{g(t)},
\]
induced by $g(t)$ on $\mathcal{W}$ for $t \in [0, \Omega]$, with $\| U \|^2 \equiv (U, U)$, and and below, let $P = P(T_1, T_2, \ldots, T_{2k})$ and $Q = Q(T_1, T_2, \ldots, T_{k+1})$ denote polynomial expressions in their arguments, that is, finite linear combinations of contractions of tensor products of various nontrivial subsets of their arguments with respect to the metric $g(t)$.

**Theorem 13.** Suppose that $X = X(t)$ and $Y = Y(t)$ are smooth families of sections of $\mathcal{X}$ and $\mathcal{Y}$ defined for $t \in [0, \Omega]$ which satisfy a system of the form
\[
\frac{\partial X}{\partial t} - \mathcal{L} X = P(X, \nabla X, \ldots, \nabla^{(2k-1)} X, Y)
\]
\[
\frac{\partial Y}{\partial t} = Q(X, \nabla X, \ldots, \nabla^{(k)} X, Y),
\]
where $\mathcal{L} : C^\infty(\mathcal{X}) \to C^\infty(\mathcal{X})$ is a smoothly varying strongly elliptic linear operator of order $2k$ for some $k \geq 1$. Then, $X(0) = 0$ and $Y(0) = 0$ imply $X(t) = 0$ and $Y(t) = 0$ for all $t \in [0, \Omega]$.

We will prove the following slightly more general statement, which in some applications permits simpler choices of $X$ and $Y$ (see e.g., [Ko] and the application to the cross-curvature flow in following section).
Theorem 14. Suppose that $X = X(t)$ and $Y = Y(t)$ are smooth families of sections of $\mathcal{X}$ and $\mathcal{Y}$ defined for $t \in [0, \Omega]$ which satisfy a system of the form

$$\left\| \frac{\partial X}{\partial t} - \mathcal{L}X - F \right\| \leq C \sum_{p=0}^{k-1} \|\nabla^{(p)}X\| + C\|Y\|,$$

(35)

$$\left\| \frac{\partial Y}{\partial t} \right\| \leq C \sum_{p=0}^{k} \|\nabla^{(p)}X\| + C\|Y\|$$

on $M \times [0, \Omega]$ for some constant $C$, where $\mathcal{L}$ is a strongly elliptic linear operator of order $2k$ for some $k \geq 1$, and $F$ is a family of sections of $\mathcal{X}$ satisfying

$$\langle F, X \rangle \leq C\|\nabla^{(k)}X\| \left( \sum_{p=0}^{k-1} \|\nabla^{(p)}X\| + \|Y\| \right) + C \left( \sum_{p=0}^{k-1} \|\nabla^{(p)}X\|^2 + \|Y\|^2 \right).$$

(36)

Then, $X(0) = 0$ and $Y(0) = 0$ implies $X(t) = 0$ and $Y(t) = 0$ for all $t \in [0, \Omega]$.

Theorem 13 indeed follows from this restatement, taking $F = P(X, \nabla X, \ldots, Y)$, since (using the compactness of $M$), we can write

$$(P, X) = \sum_{p=0}^{2k-1} (C * \nabla^{(p)}X + C * Y, X)$$

for some bounded (possibly zero) tensors $C$. Integrating by parts $l$ times on each term with $p > k$ verifies (36). In the next section, in which $\mathcal{L}$ has order 2, we will take $F$ in the form $F = \text{div} U$ where $U$ satisfies $U \leq C(\|X\| + \|Y\|)$.

Proof of Theorem 14. The proof is a trivial modification of the standard version for $Y \equiv 0$. Form the quantity $\mathcal{E}(t) \triangleq \|X\|^2 + \|Y\|^2$. Since $M$ is closed, the contributions of the time-derivatives of $g$, and $d\mu_g$ to the following computation are all bounded, and therefore, integrating by parts and using the Cauchy-Schwarz inequality, we have

$$\dot{\mathcal{E}} \leq C\mathcal{E} + 2 \left( \frac{\partial X}{\partial t} X \right) + 2 \left( \frac{\partial Y}{\partial t} Y \right)$$

$$\leq C\mathcal{E} + 2 \left( \frac{\partial X}{\partial t} - \mathcal{L}X - F, X \right) + 2(\mathcal{L}X, X) + 2 \langle F, X \rangle + 2 \left( \frac{\partial Y}{\partial t}, Y \right)$$

$$\leq C\mathcal{E} + 2(\mathcal{L}X, X) + C \left( \sum_{p=0}^{k} \|\nabla^{(p)}X\| + \|Y\| \right) \left( \sum_{p=0}^{k-1} \|\nabla^{(p)}X\| + \|Y\| \right)$$

for some constant $C$. Since $M$ is compact and $\mathcal{L}$ is strongly elliptic with coefficients varying smoothly in $t$, Gårding’s inequality implies that there is some $\epsilon > 0$ such that $(\mathcal{L}X, X) \leq -\epsilon\|\nabla^{(k)}X\|^2 + C\|X\|^2$ for all $t \in [0, \Omega]$. Using Cauchy-Schwarz, we thus obtain

$$\dot{\mathcal{E}} \leq C\mathcal{E} - \epsilon\|\nabla^{(k)}X\|^2 + C \sum_{p=1}^{k-1} \|\nabla^{(p)}X\|^2$$

for some constant $C_1 = C_1(\epsilon)$. On the other hand, inequality (19) implies that there is a constant $C = C(\epsilon, k, C_1)$ such that, for all $0 < p < k$,

$$\|\nabla^{(p)}X\|^2 \leq \frac{\epsilon}{C_1(k-1)}\|\nabla^{(k)}X\|^2 + C\|X\|^2.$$
Therefore, we obtain $\dot{E} \leq C E$ on $[0, \Omega]$, and the claim follows. □

4. The cross-curvature flow

As an application of Theorem 14, we give a proof of the uniqueness of solutions to the cross-curvature flow of strictly positively or negatively curved metrics on a closed three-manifold $M = M^3$, and give a detailed description of the prolongation procedure. We first need to introduce some notation.

4.1. The equation. Suppose that $g$ has either strictly positive or strictly negative sectional curvature and define $\sigma$ to be 1 if the curvature is positive and $-1$ otherwise. The Einstein tensor $E(g) = Rc(g) - S/2g$ of $g$ will be, respectively, either strictly negative or strictly positive definite. We will use the notation $E$ for the endomorphism $E : T \mapsto T$ given by $E^j_\ i = g^j_k E^k_\ i$.

Let $V \in C^\infty(T^0_2(M))$ be the inverse of $E_\ ij = g^i_k g^j_l E_\ ij$, i.e., the tensor satisfying $V^i_\ j E_j^\ k = \delta^i_\ k$, and let $P = \det(E) = \det(E^i_\ j \det(g^\ i_\ j))$. The cross-curvature tensor of $g$ is then defined to be

$$X_\ ij \equiv PV_\ ij = -\frac{1}{2} E^{pq} R^i_\ pqj.$$  

The cross-curvature flow of a family of metrics $g(t)$ is the equation

$$\frac{\partial}{\partial t} g = -\sigma 2X(g).$$

This equation was introduced by Hamilton and Chow in [CH] as a tool to study three-manifolds of negative curvature. The short-time existence of solutions to the equation beginning at metrics of positive or negative curvature was verified by Buckland [Bu].

Here we give a proof of the following uniqueness assertion.

Theorem 15. Let $M = M^3$ be a closed manifold. Suppose $g(t)$, $\tilde{g}(t)$ are two solutions to (38) on $M \times [0, \Omega]$ with strictly positive or negative sectional curvature and $g(0) = \tilde{g}(0) = \bar{g}$. Then $g(t) = \tilde{g}(t)$ for all $t \in [0, \Omega]$.

The proof in the positively and negatively curved cases are virtually the same, so we will prove only the case that $\bar{g}$ has strictly negative curvature. As we have observed, in this case, the Einstein tensors $E$, $\tilde{E}$ of $g$ and $\tilde{g}$ are positive definite. Let $\lambda > 0$ be a constant such that

$$\lambda g \leq E \leq \lambda^{-1} g,$$

on $M \times [0, T]$. (Since the manifold $M$ is compact, the solutions $g$ and $\tilde{g}$ are uniformly equivalent on $M$ for $t \in [0, \Omega]$.)

4.2. The prolonged system. In view of Theorem 14 it suffices to prove that we can encode the problem of uniqueness into one for a prolonged system of the form (35). This can be done in several different ways. We find it convenient we base the parabolic part (the $X$ component) of the system on the difference of the inverses $V, V_\tilde{}$ of the Einstein tensors of $g$ and $\tilde{g}$. Thus, we define

$$h = g - \tilde{g}, \quad A = \Gamma - \tilde{\Gamma}, \quad W = V - \tilde{V},$$

and introduce the operator $\Box \equiv E^{ab} \nabla_a \nabla_b$. (In general, we take $\Box = -\sigma E^{ab} \nabla_a \nabla_b$.) These quantities together satisfy a closed system of differential inequalities relative
to the norms and connection induced by \( g(t) \), which we take, as before, to be our reference metric.

**Proposition 16.** Under the assumptions of Theorem 15, the tensors \( W, h, A \) satisfy the inequalities
\[
\frac{\partial}{\partial t} W - \Box W - \text{div} U \leq C (|h| + |A| + |W| + |\nabla W|)
\]
\[
\frac{\partial}{\partial t} h \leq C (|h| + |W|)
\]
\[
\frac{\partial}{\partial t} A \leq C (|h| + |A| + |W| + |\nabla W|)
\]
for some constant \( C \), where \( U \) is the \((2,1)\)-tensor given by
\[
U_{ij} = (E_{al} - \tilde{E}_{al}) \nabla_a \tilde{V}_{ij} - E_{ab} A_{pi} \tilde{V}_{pj} - E_{ab} A_{aj} \tilde{V}_{ip}.
\]
and satisfies \( |U| \leq C (|A| + |W|) \).

We prove Proposition 16 in the next section.

**Proof of Theorem 15.** Proposition 16 implies that \( \hat{X} = W \) and \( \hat{Y} = h \oplus A \) satisfy
\[
\frac{\partial}{\partial t} \hat{X} - \Box \hat{X} - \text{div} U \leq C (|\hat{X}| + |\nabla \hat{X}| + |\hat{Y}|)
\]
\[
\frac{\partial}{\partial t} \hat{Y} \leq C (|\hat{X}| + |\nabla \hat{X}| + |\hat{Y}|)
\]
where \( |U| \leq C (|\hat{X}| + |\hat{Y}|) \) for some constant \( C \). The operator \( \Box \) is strongly elliptic under our assumptions, and \( F = \text{div}(U) \) satisfies
\[
(F, \hat{X}) = -(U, \nabla \hat{X}) \leq C \| \nabla \hat{X} \| (\| \hat{X} \| + \| \hat{Y} \|).
\]
Hence Theorem 15 follows from Theorem 14.

In fact, when formulated in terms of \( \hat{X}, \hat{Y} \) and \( U \), the uniqueness assertion of Theorem 15 essentially follows from our earlier general theorem in [Kog], modulo the verification of the inequalities in Proposition 16; see Remark 15 in that reference.

### 4.3. Evolution equations and proof of Proposition 16

**Lemma 17.** Let \( g(t) \) be a solution to (38). The Levi-Civita connection and the tensor \( V \) associated to \( g \) evolve according to the equations
\[
\frac{\partial}{\partial t} \Gamma^k_{ij} = g^{mk} \{ \nabla_i (PV_{jm}) + \nabla_j (PV_{im}) - \nabla_m (PV_{ij}) \}
\]
\[
\frac{\partial}{\partial t} V_{ij} = \Box V_{ij} + (E_{al} E_{kb} - 2E_{ab} E_{kl}) \nabla_k V_{ai} \nabla_l V_{bj} - Pg^{kl} (V_{ik} V_{jl} + V_{ij} V_{kl}).
\]

**Proof.** Equation (41) follows from the standard formula for the evolution of the connection and (37). Equation (42) follows from some routine calculations from the formula
\[
\frac{\partial}{\partial t} E^{ij} = \Box E^{ij} - \nabla_k E^{ij} \nabla_l E^{kl} - Pg^{ij} - \text{tr}_g(X) E^{ij}
\]
from [CH], using \( V_{ik} E^{kj} = \delta^i_j \).
Now, the Einstein tensor is divergence-free, i.e., $\nabla_i E^{jk} = 0$, and so we compute that (cf. the computations on pp. 6-7 of [Ko])

$$\Box V_{ij} - \Box \tilde{V}_{ij} = \nabla_a (E^{ab} \nabla_b V_{ij}) - \nabla_a (E^{ab} \tilde{\nabla}_b V_{ij})$$

$$= \Box (V_{ij} - \tilde{V}_{ij}) + \nabla_a U^a_{ij} + A^a \tilde{E}^{ab} \tilde{\nabla}_b V_{ij} + A^a E^{ab} \tilde{\nabla}_b \tilde{V}_{ij}$$

$$- A^a \tilde{E}^{ab} \tilde{\nabla}_b \tilde{V}_{ip} - A^a E^{ab} \tilde{\nabla}_b V_{jp} - A^a \tilde{E}^{ab} \tilde{\nabla}_b \tilde{V}_{ip},$$

where $\tilde{E}^{ab} = \tilde{g}^{ac} \tilde{g}^{bd} \tilde{E}_{cd}$, and

$$U^a_{ij} = (E^{ab} - \tilde{E}^{ab}) \tilde{\nabla}_a \tilde{V}_{ij} - E^{ab} A^a_{ij} \tilde{V}_{ip} - E^{ab} A^a_{ij} \tilde{V}_{ip}.$$

Since $E^{ab} - \tilde{E}^{ab} = -E^{ab} \tilde{E}^{bd} W_{kl}$, we have $|U| \leq C(|A| + |W|)$ for some constant $C$. Returning to \textbf{[12]} and putting things together, we obtain

$$\left| \frac{\partial}{\partial t} W - \Box W - \text{div} U \right| \leq C (|h| + |A| + |W| + |\nabla W|)$$

on $M \times [0, \Omega]$ as desired.

The tensor $h$, meanwhile, evolves according to

$$\frac{\partial}{\partial t} h = 2(X - \tilde{X}) = 2P(V - \tilde{V}) + 2(\tilde{P} - \tilde{P}) \tilde{V}.$$

The first term is just $2PW$, and for the second term, we have $|P - \tilde{P}| \leq C(|h| + |W|)$ for some $C$ since the Einstein tensors of $g$ and $\tilde{g}$ are uniformly strictly positive on $M$. Thus

$$\left| \frac{\partial}{\partial t} h \right| \leq C (|h| + |W|).$$

Finally, from \textbf{[11]}, we have the schematic equation

$$\frac{\partial}{\partial t} \Gamma = Pg^{-1} \ast \nabla V + g^{-1} \ast \nabla P \ast V.$$

Now,

$$\nabla V - \tilde{\nabla} \tilde{V} = \nabla W + A \ast \tilde{V}, \quad \nabla P - \tilde{\nabla} \tilde{P} = P \ast E \ast \nabla V - \tilde{P} \ast \tilde{E} \ast \tilde{\nabla} V,$$

so expanding the last difference on the right into three terms and estimating them as above, we obtain

$$\left| \frac{\partial}{\partial t} A \right| \leq C (|h| + |A| + |W| + |\nabla W|)$$

for some $C$. The proof of Proposition \textbf{[16]} is completed.

**REFERENCES**

[A] S. Alexakis, *Unique continuation for the vacuum Einstein equations*, Preprint (2009), \texttt{arXiv:0907.1131}

[Be] T. Bell, *Uniqueness of Conformal Ricci Flow using Energy Methods*, Preprint (2013), \texttt{arXiv:1301.5052} [math.DG].

[BH1] E. Bahuaud and D. Helliwell, *Short-time existence for some higher-order geometric flows*, Comm. Partial Differential Equations 36 (2011), no. 12, 2189–2207.

[BH2] E. Bahuaud and D. Helliwell, *Uniqueness for some higher-order geometric flows*, Preprint (2014), \texttt{arXiv:1407.4406} [math.DG].

[Bo] V. Bour, *Fourth order curvature flows and geometric applications*, Preprint (2010), \texttt{arXiv:1012.0342} [math.DG].

[Bu] J. Buckland, *Short-time existence of solutions to the cross curvature flow on 3-manifolds*, Proc. Amer. Math. Soc. 134, no. 6 (2006), 1803–1807.
[CH] B. Chow and R. S. Hamilton, *The cross curvature flow of 3-manifolds with negative sectional curvature*, Turkish J. Math. 28 (2004), no. 1, 1–10.

[CZ] B. -L. Chen and X. -P. Zhu, *Uniqueness of the Ricci flow on complete noncompact manifolds*, J. Diff. Geom. 74 (2006), no. 1, 119–154.

[D] D. DeTurck, *Deforming metrics in the direction of their Ricci tensors, improved version*, Collected Papers on Ricci Flow, ed. H.-D. Cao, B. Chow, S.-C. Chu, and S.-T. Yau, Internat. Press, Somerville, MA (2003).

[FG] C. Fefferman and C. R. Graham, *Conformal invariants*, Astérisque (1985), Numero Hors Série, 95–116, The mathematical heritage of Élie Cartan (Lyon, 1984).

[H1] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom. 17 (1982), no. 2, 255–306.

[H2] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in differential geometry, Vol. II, (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.

[Ko] B. Kotschwar, *An energy approach to the problem of uniqueness for the Ricci flow*, Comm. Anal. Geom. 22 (2014), no. 1, 149–176.

[S1] J. D. Streets, *The gradient flow of $\int_M |Rm|^2$*. J. Geom. Anal. 18 (2008), no. 1, 249–271.

[WY] W. W.-Y. Wong and P. Yu, *On strong unique continuation of coupled Einstein metrics*, Int. Math. Res. Not. (2012), no. 3, 544–560.

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