First order signatures in 4D pure compact U(1) gauge theory with toroidal and spherical topologies.

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Abstract

We study the pure compact U(1) gauge theory with the extended Wilson action ($\beta, \gamma$ couplings) by finite size scaling techniques, in lattices ranging from $L=6$ to $L=24$ in the region of $\gamma \leq 0$ with toroidal and spherical topologies. The phase transition presents a double peak structure which survives in the thermodynamical limit in the torus. In the sphere the evidence supports the idea of a weaker, but still first order, phase transition. For $\gamma < 0$ the transition becomes weaker and larger lattices are needed to find its asymptotic behaviour. Along the transient region the behaviour is the typical one of a weak first order transition for both topologies, with a region where $1/d < \nu < 0.5$, which becomes $\nu \approx 1/4$ when larger lattices are used.
1 Introduction

The four dimensional pure compact U(1) gauge theory is the simplest gauge abelian interaction we can describe in the lattice. This model is known to possess a phase transition (PT) line separating a Confined phase from a Coulomb one. Big efforts have been devoted to the study of the order of this PT, which turned out to be a controversial issue. The implementation of pure compact U(1) in the lattice with the Wilson action:

\[ S_W = -\sum_P \beta \cos \theta_P \] (1)

has been studied for a long time. At first, in the eighties, the transition was believed to be continuous [1, 2], but, as simulations in larger lattices became accessible to computer resources, the onset of metastabilities revealed the first order nature of this transition [3, 4, 5].

It was suggested some time ago [6] that the order of the PT could be altered when using an extended Wilson action, including a term proportional to the plaquette squared.

\[ S_{\text{ext}} = -\sum_P [\beta \cos \theta_P + \gamma \cos 2\theta_P] \] (2)

It was then found [7] that for positive values of \(\gamma\) the first order signatures appear for lattices smaller than the ones needed to state the first order nature of the PT with the Wilson action (\(\gamma = 0\)). In this way, \(\gamma\) could be used as a parameter to reinforce (positive \(\gamma\)) or weaken (negative \(\gamma\)) the transition, and some authors [6, 8] conjecture about the existence of a tricritical point at some negative \(\gamma\) value where the order of the transition changes, becoming a continuous one.

However, numerical simulations showed that metastabilities appear for negative values of \(\gamma\) as well. At this stage, it was pointed out that one usually works on the lattice with a toroidal topology, and it was questioned whether or not these metastabilities survive in the thermodynamic limit, or they are rather lattice artifacts due to the toroidal topology [9]. The closed monopole loops appearing on the torus were initially supposed to be responsible for such double peak structures. This hypothesis led authors to work on lattices homotopic to the sphere, since in those lattices all monopole loops can be contracted to a point [8, 9, 10]. In fact, the authors of [8], working on the
hyper-surface of a 5D cube, which is homotopic to the sphere, do not observe any signal of metastability.

This situation is quite disturbing for the lattice community. One would expect that the topology of the lattice does not affect the physics of the model, since its contribution behaves as a surface term, which vanishes in the thermodynamical limit.

To shed some light on this problem, we have studied numerically the extended Wilson action for several values of the parameter $\gamma$, both on the torus and, following [8], on the sphere. On the torus we run simulations up to lattice sizes $L=24$, finding a clear first order phase transition. On the sphere we work at $\gamma = 0$ and at $\gamma = -0.2$, and we, as Jersak et al. [8], do not see any two peak signals when we measure in the lattice sizes investigated by them, but when we go to larger sizes, we find that the behaviour of the transition is that expected for a first order one [16], where for small lattices $\nu$, being lower than $1/2$, is larger than the first order value $1/4$, but approaches that value monotonically as the lattice size is increased.

We have worked with the plaquette energy defined as

$$E = \frac{1}{N_P} \langle \sum_P \cos \theta_P \rangle$$

(3)

and the specific heat

$$C_v = \frac{\partial}{\partial \beta} E$$

(4)

where $N_P$ stands for the number of plaquettes of the system.

Similar quantities, defined with respect to the term $\cos 2\theta_P$, have been measured, but they being highly correlated with the previous ones and their behaviour being qualitatively identical, their results have not been reported.

On the Torus, $N_P = 6L^4$. On the sphere, the number of plaquettes has a less simple expression, and can be computed as a function of $N$, the number of points in one of its 5 dimensions, as $N_P = 60(N-1)^4 + 20(N-1)^2$. In this case, the system is not homogeneous and $N_P$ is not proportional to the number of points on the four dimensional surface, which is $N^5 - (N-2)^5$, some points having a number of surrounding plaquettes less than the possible maximum 12, as opposed to what happens on the torus. In order to allow comparison, we define $L_{\text{eff}} = (N_P/6)^{1/4}$.

If the transition is first order, $C_v$ must scale at the transition point as $N_P$ in both topologies.
We simulate the subgroup \( Z(1024) \subset U(1) \), and we also ran simulations at some points with the whole \( U(1) \) group, the results with both groups being fully compatible for both the spherical and the toroidal topologies.

In order to check the goodness of the simulation, we have used the Schwinger-Dyson equations on the lattice [1], which allows us to extract \( \beta, \gamma \) from the Monte Carlo data, the value of the input couplings having been recovered from the simulations on both the toroidal and spherical topologies.

We intend to give in this letter a schematic presentation of our results. A complete account of our data will be given in a future paper [2].

2 Results for the toroidal topology

We have studied the model on the torus in order to check whether or not the double peak structures observed in the small volumes survive in the thermodynamical limit. The smallest lattice we use is \( L=6 \) and the largest one is \( L=24 \). We update by means of a standard Metropolis algorithm. The statistics range from \( \approx 8 \times 10^5 \) MC iterations for the smallest lattices \( (L=6,8,12) \) to \( \approx 1.4 \times 10^6 \) for the largest ones \( (L=16,20,24) \). The autocorrelation time for the energy ranges from \( O(10^2) \) to \( O(10^3) \).

For every value of \( \gamma = -0.1, -0.2, -0.3, -0.4 \) we consider, we have used the Spectral Density Method [3] to locate the critical coupling \( \beta_c(L) \) at the maximum of the specific heat peak.

We carried out the simulations on the RTNN machine consisting of 32 Pentium Pro processors, the total CPU time employed being the equivalent of 4 Pentium Pro years.

We find that the two-state signal persists for all volumes we consider at all \( \gamma \) values. We plot in figure 1 the histograms for the plaquette energy at \( \gamma = -0.4 \). The transition shows an increasing weakness as we go to more negative \( \gamma \) values. The double peak structure is clearly observed in \( L=6 \) at \( \gamma = -0.1 \), while at \( \gamma = -0.4 \) one has to go to \( L=12 \) to observe an equivalent signal.

From the energy distributions, we measure the latent heat through a cubic spline fit of the peaks. The result is shown in figure 2 (upper figure). The latent heat can be safely extrapolated to a value different from zero in the thermodynamical limit.

In table 1 we quote, for the different \( \gamma \) values on the torus, the value of \( \beta \) at which we have simulated, and the \( \beta_c(L) \) obtained from the maximum of
the specific heat using the Spectral Density Method.

For every lattice size, we have measured the position of the nearby zero of the partition function closest to the real axis \([4]\). The imaginary part of that value is known to scale as \(L^{-1/\nu}\). Following that, we calculated an effective \(\nu\) exponent between consecutive lattice sizes. We see that \(\nu_{\text{eff}}\) goes asymptotically to \(1/4\), but for large negative \(\gamma\) this value is attained for increasingly large \(L\), so evidencing a weaker transition, but still first order for all \(\gamma\) values considered.

From the energy distributions, we can measure an useful quantity in order to determine the order of the phase transition, i.e., the free energy gap \(\Delta F\), which is the difference between the minima and the local maximum of the free energy \([15]\). We use the spectral density method to get, from the measured histograms, a new histogram where both peaks have equal height. We take the logarithm of those histograms and measure the energy gap. If the transition is first order, that gap has to be more pronounced as we go to larger \(L\), while it has to stay constant if the transition is second order.

Figure 1: Plaquette energy distribution measured at \(\mathcal{C}^{\text{max}}_\nu(L)\) for \(L=12,16,20,24\) at \(\gamma = -0.4\)
In figure 2 (lower plot) we show the behaviour of the energy gap for the different $\gamma$ values. The gap $\Delta F$ grows up drastically for all $\gamma$ values, supporting the first order nature of the phase transition. The value of $L$ at which $\Delta F$ starts growing is certainly larger as the value of $\gamma$ is more negative, revealing the increasing weakness of the transition as $\gamma$ gets more negative, but there is no suggestion of the existence of a tricritical point at finite $\gamma$. This is in agreement with what one would expect from the behaviour of $\nu_{\text{eff}}$. Also in this case, a pseudo plateau is present for $\Delta F$, larger for larger negative $\gamma$, and if small lattices are used, that could be interpreted as a second order behaviour. It should be also noticed that $\Delta F$ scales for the largest lattices as $L^{d-1}$, as expected in a first order phase transition [15].

In figure 3 we plot $C_v$ for the torus. In the $x$ axis we plot the plaquette number $N_P$. With this scale, a straight line dependence means that $C_v$ scales as the volume, and then $\nu = 1/4$. We superimpose a linear fit to the three last points, which is very good, but would not work at smaller sizes, a behaviour which used to appear in the transient region of weak first order phase transitions, somehow preceding the onset of the true transition [16, 17, 18, 19].
The first order nature of the deconfinement transition having been stated for the torus, we follow Jersak et al.\cite{8} and work on the 4D surface of a 5D cube to check whether the two-state signal does disappear for that topology.

Based on the torus experience, where we have learned that the behaviour of the system is similar for combinations of decreasing $\gamma$ and increasing volume, we have chosen to work at $\gamma = 0$ and at $\gamma = -0.2$ in lattices ranging from $N=6$ to $N=14$. In order to be sure that no topologically induced large metastabilities are present, we have run for larger lattices two independent simulations, starting from cold and hot configurations, the results being unaffected by the initial configuration. We discarded around 20% of the statistics for thermalization.

We have also measured the position of the first Fisher zero, and computed a $\nu_{\text{eff}}$ in the same way as we did for the toroidal topology. The results, together with the autocorrelation time ($\tau$) for the energy, and the statistics in number of $\tau$, $N_{\tau}$, are reported in table 2.

At $\gamma = 0$, (Figure 4, lower part), we do not find evidence for double peak structures up to $N=8$. However, for $N=10$ the Energy distribution presents deviations from a simple gaussian behaviour. The onset of a double-peaked distribution occurs in $N=12$. The values for $\nu_{\text{eff}}$ in table 2 show a trend towards $1/d$, as expected in a first order phase transition. The behaviour of the specific heat, proportional to $N_{\beta}$ for larger lattices (see figure 3) supports the first order too.

### Table 1: Results obtained for the toroidal topology.

| $L$ | $\beta_{\text{sim}}$ | $\beta_{c}(L)$ | $\nu_{\text{eff}}$ | $\beta_{\text{sim}}$ | $\beta_{c}(L)$ | $\nu_{\text{eff}}$ |
|-----|------------------|----------------|------------------|------------------|----------------|------------------|
| 6   | 1.0720           | 1.0716(2)      | -                | 1.1460           | 1.1462(2)      | -                |
| 8   | 1.0783           | 1.0780(2)      | 0.324(13)        | 1.1533           | 1.1539(2)      | 0.342(16)        |
| 12  | 1.0834           | 1.0831(2)      | 0.323(12)        | 1.1582           | 1.1582(2)      | 0.352(21)        |
| 16  | 1.0838           | 1.0827(1)      | 0.288(17)        | 1.15935          | 1.1593(1)      | 0.302(14)        |
| 20  | 1.0833           | 1.0833(1)      | 0.270(16)        | 1.1599           | 1.1599(1)      | 0.292(18)        |
|     |                  |                |                  |                  |                |                  |
| $\gamma = -0.3$ | $\beta_{\text{sim}}$ | $\beta_{c}(L)$ | $\nu_{\text{eff}}$ | $\beta_{\text{sim}}$ | $\beta_{c}(L)$ | $\nu_{\text{eff}}$ |
|-----|------------------|----------------|------------------|------------------|----------------|------------------|
| 6   | 1.2255           | 1.2283(4)      | -                | 1.3090           | 1.3092(4)      | -                |
| 8   | 1.2344           | 1.2349(1)      | 0.359(11)        | 1.3109           | 1.3139(3)      | 0.374(18)        |
| 12  | 1.2395           | 1.2395(2)      | 0.344(12)        | 1.3258           | 1.3259(1)      | 0.372(15)        |
| 16  | 1.2410           | 1.2410(1)      | 0.335(14)        | 1.3275           | 1.3278(1)      | 0.360(12)        |
| 20  | 1.2416           | 1.2415(5)      | 0.321(17)        | 1.3285           | 1.3284(1)      | 0.313(21)        |
| 24  | 1.2417           | 1.24162(5)     | 0.282(13)        | 1.3286           | 1.3287(1)      | 0.270(15)        |

### 3 Results for the spherical topology

The first order nature of the deconfinement transition having been stated for the torus, we follow Jersak et al.\cite{8} and work on the 4D surface of a 5D cube to check whether the two-state signal does disappear for that topology.

Based on the torus experience, where we have learned that the behaviour of the system is similar for combinations of decreasing $\gamma$ and increasing volume, we have chosen to work at $\gamma = 0$ and at $\gamma = -0.2$ in lattices ranging from $N=6$ to $N=14$. In order to be sure that no topologically induced large metastabilities are present, we have run for larger lattices two independent simulations, starting from cold and hot configurations, the results being unaffected by the initial configuration. We discarded around 20% of the statistics for thermalization.

We have also measured the position of the first Fisher zero, and computed a $\nu_{\text{eff}}$ in the same way as we did for the toroidal topology. The results, together with the autocorrelation time ($\tau$) for the energy, and the statistics in number of $\tau$, $N_{\tau}$, are reported in table 2.

At $\gamma = 0$, (Figure 4, lower part), we do not find evidence for double peak structures up to $N=8$. However, for $N=10$ the Energy distribution presents deviations from a simple gaussian behaviour. The onset of a double-peaked distribution occurs in $N=12$. The values for $\nu_{\text{eff}}$ in table 2 show a trend towards $1/d$, as expected in a first order phase transition. The behaviour of the specific heat, proportional to $N_{\beta}$ for larger lattices (see figure 3) supports the first order too.
Figure 3: $C_v^{\text{max}}(L)$ as a function of the plaquette number for the sphere at $\gamma = 0, -0.2$ and for the torus at $\gamma = -0.3, -0.4$. For the sphere at $\gamma = 0$ we plot $C_v^{\text{max}}(L)/2$ for the clarity of the graphic’s sake.

At $\gamma = -0.2$, (Figure 4, upper part), the transition turns out to be much weaker than could be expected from the results obtained with the toroidal topology. In general, a stable sharp double peak structure can only be observed when the lattice size is much larger than the correlation length at the critical point. We do not observe such signals up to $N=14$, yet at $N=14$ the distribution is distinctly non-gaussian, and moreover, its width is practically identical to the one at $N=12$, which means that $C_v$ scales between both lattices as the volume, or equivalently, that $\nu \approx 1/4$.

Also the results for $\nu_{\text{eff}}$ in table 2 show a clear trend towards the first order value, similar to that shown on the torus in table 1. The behaviour of the specific heat, (figure 3) is almost compatible with $C_v \approx N_P$, as expected for a first order phase transition near the asymptotic region.

In view of all this, our hypothesis is that the two peaks of the energy distribution are too close to be discerned up to $N = 14$, but the results for
the scaling of the specific heat start to be significant from $N = 12$ on. From the behaviour observed at $\gamma = 0$ we hope that the splitting of the peaks in the energy distribution shall be visible for $N = 16$. We are running simulations in that lattice to support this conjecture [12].

Uncontrolled finite size effects seem to be much larger for the spherical topology than for the toroidal one. As an example, in $\gamma = 0$, one has to go to the surface of a 5D, $N=12$ cube (which has roughly the same number of points as a $L=20$ toroidal lattice) to observe signals comparable to those in a $L=8$ lattice with toroidal topology. This could be due to the fact that in the cube surface translational invariance is lost, and then the thermodynamical limit is reached only for larger lattices than in the torus topology. This loss of translational invariance, which is more important for smaller lattices, can be somewhat alleviated introducing appropriate weight factors in the edges.
\[ \gamma = 0 \]

\[ N \quad L_{\text{eff}} \quad \beta_{\text{sim}} \quad \tau \quad N_{\tau} \quad \beta_{c}(L) \quad \nu_{\text{eff}} \]

| \( N \) | \( L_{\text{eff}} \) | \( \beta_{\text{sim}} \) | \( \tau \) | \( N_{\tau} \) | \( \beta_{c}(L) \) | \( \nu_{\text{eff}} \) |
|-----|------|------|------|------|------|------|
| 6   | 8.921| 1.0128| 240  | 8300 | 1.01340(1) | - |
| 8   | 12.469| 1.0120| 400  | 2300 | 1.0128(2)  | 0.296(42) |
| 10  | 16.921| 1.0120| 780  | 1200 | 1.01212(3) | 0.264(23) |
| 12  | 19.574| 1.0119| 850  | 1150 | 1.01194(2) | 0.240(9) |

\[ \gamma = -0.2 \]

| \( N \) | \( L_{\text{eff}} \) | \( \beta_{\text{sim}} \) | \( \tau \) | \( N_{\tau} \) | \( \beta_{c}(L) \) | \( \nu_{\text{eff}} \) |
|-----|------|------|------|------|------|------|
| 6   | 8.921| 1.1587| 160  | 2000 | 1.1587(4)  | - |
| 8   | 12.469| 1.1597| 510  | 1000 | 1.1603(2)  | - |
| 10  | 16.921| 1.1602| 680  | 1500 | 1.1604(2)  | 0.376(25) |
| 12  | 19.574| 1.1604| 820  | 1200 | 1.1602(1)  | 0.289(23) |
| 14  | 23.123| 1.1605| 900  | 1100 | 1.16048(5) | 0.264(19) |

Table 2: Results obtained for the spherical topology.

\[ \] , which we did not consider, since such details are not expected to affect the order of the phase transition.

4 Conclusions and outlook

We have studied the 4D pure compact U(1) gauge theory with the extended Wilson action on the torus and on the sphere. On the torus the evidence supports the idea of a first order phase transition between the Confined phase and the Coulomb one, as stated some years ago using RG techniques [20]. On the sphere, our preliminary results point to a first order transition at \( \gamma = 0 \). At \( \gamma = -0.2 \) the transition seems to be weaker, but still first order. This statement is supported by the behaviour of \( \nu_{\text{eff}} \), which goes to 1/4 in both topologies, with numerical coincidence at corresponding sizes (see tables 1 and 2), and by an incipient double peak structure at \( N=14 \). The behaviour of the specific heat, and the scaling of the Fisher zeros, are hardly compatible with a second order phase transition. This result has to be confirmed in larger lattices. The evidence produced in favour of the first order character supports the folklore that both, the torus and the sphere, should exhibit the same behaviour in the thermodynamical limit.

Also, one should be aware of the fact that in weak first order phase transitions an exponent \( \nu_{\text{eff}} \) between 1/d and 1/2 appears during its transitory region [3, 16, 17, 8, 19] and hence it should not be surprising that this be the behaviour exhibited by this model.
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