A gauge/gravity duality model for gauge theory thermodynamics

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We study a gauge/gravity model for the thermodynamics of a gauge theory with one running coupling. The gravity side contains an ansatz for the metric and a scalar field, on the field theory side one starts by giving an ansatz for the beta function describing the scale dependence of the coupling. The model is based on relating the scale to the extra dimensional coordinate and the beta function to the gravity fields, thereby also determining the scalar field potential. We study three different forms of beta functions of increasing complexity and give semianalytic solutions describing first order or continuous transitions.

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1 Introduction

The prototype gauge/gravity duality relates finite temperature conformally invariant $\mathcal{N} = 4$ supersymmetric Yang-Mills theory to 5-dimensional AdS space (times $S^5$) with a black hole $[1, 2]$. To extend this duality to usual $SU(N_c)$ Yang-Mills theory at finite $T$, on which there is ample lattice Monte Carlo $[3, 4, 5, 6]$ and even (with quarks included) experimental data, one of the first tasks is to see how conformal invariance can be broken, how new scales can be introduced. A large number of models have been suggested for this $[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]$. This article is one more, very closely related to $[12]$. The basic building blocks of the gravity dual discussed here are ($z$ is the 5th coordinate, the boundary is at $z = 0$) the conformal factor of the 5d metric $b(z)$ ($= L/z$, $L =$ AdS radius, in the conformal case), the black hole factor $f(z)$, $f(z_h) = 0 (= 1 - z^4/z_h^4$ in the conformal case), the scalar field $\phi(z) = \log \lambda(z)$ ($= 0$ in the conformal case) and the scalar field potential $V(\phi)$ in the gravity action ($= 12/L^2$ in the conformal case). Assume now the boundary theory one is searching gravity dual for has a running coupling $g^2(\mu)$ with a known beta function: $\mu d g^2(\mu)/d\mu = \beta(g^2)$. The key idea now is to firstly identify $\mu$ (in some units) with $b(z)$ so that large energy scales, ultraviolet, UV, correspond to $z \to 0$, $b(z) \sim 1/z \to \infty$ and small scales, infrared, IR, to large $z$. In the IR $b$ may approach a constant or vanish. Secondly, one identifies $\lambda = e^\phi$ with $g^2$. One thus has, in addition to three Einstein equations, one more equation $b d\lambda/db = \beta(\lambda)$ relating the metric and the scalar. In the analysis of $[10]$-$[13]$ the scalar potential $V(\phi)$ in the gravity action is so constructed that a desired beta function is obtained in the UV and confinement in the IR.

In this article, we take an extreme view and use the two Einstein equations not containing $V$ and the equation $b d\lambda/db = \beta(\lambda)$ to completely solve the problem and simply regard the Einstein equation containing $V(\phi)$ as an equation determining $V(\phi)$. We shall mainly concentrate on studying a simple powerlike beta function ansatz, $\beta(\lambda) = -\beta_0 \lambda^q$. Here $q$ is a parameter, $q \geq 1$ corresponds to a theory confining in the IR, $q = 2$ would correspond to one-loop Yang-Mills running and $q = 10/3$ has in $[18]$ been shown to reproduce the main features of the detailed analysis in $[13]$. Further, we shall modify this ansatz so that its large $\lambda$ behavior is constrained to be $-\frac{3}{2} \lambda (1 + \alpha/\log \lambda + ..)$. First order transitions are obtained for $q > 1$ and $\alpha > 0$, $\alpha = 0$ describes a continuous transition.

The theoretical basis of this model has been laid in $[10]$-$[13]$, starting from a potential $V(\phi)$. The chief virtue of the approach in this article, starting from $\beta(\lambda)$, is that semianalytic formulas, requiring only numerical evaluation of integrals, are obtained. Details of $SU(N_c)$ gauge theory thermodynamics can be described with $q, \alpha$ and two more parameters: a dimensionless $L^3/G_5$ fixed by $p(T)/T^4$ at large $T$ and an energy scale $\Lambda$ fixed by $T_c$, the transition temperature between high and low $T$ phases. Note that, in particular, the numerical value of the parameter $\beta_0$ in the beta function ansatz, never enters for physical quantities.
2 The model

One starts from the gravity + scalar action (in the Einstein frame and in standard notation)

\[
S = \frac{1}{16\pi G_5} \left\{ \int d^5x \sqrt{-g} \left[ R - \frac{4}{3} (\partial_\mu \phi)^2 + V(\phi) \right] - 2 \int d^4x \sqrt{-\gamma} K \right\}.
\]

(1)

By writing

\[
g_{\mu\nu} = e^{\frac{4}{3} \phi} g_{\mu\nu}^E,
\]

(2)

it can also be written in the string frame. One now assumes a metric ansatz

\[
ds^2 = b^2(z) \left[ -f(z) dt^2 + d\mathbf{x}^2 + \frac{dz^2}{f(z)} \right].
\]

(3)

The four functions \(b(z), f(z)\) in the metric, the scalar field \(\phi(z)\) and the potential \(V(\phi(z))\) are then determined as the solutions of the three field equations following from (1):

\[
6 \frac{\dot{b}^2}{b^3} + 3 \frac{\ddot{b}}{b} + 3 \frac{\dot{b} \dot{f}}{b f} = \frac{b^2}{f} V(\phi),
\]

(4)

\[
6 \frac{\dot{b}^2}{b^3} - 3 \frac{\ddot{b}}{b} = \frac{4}{3} \dot{\phi}^2,
\]

(5)

\[
\frac{\ddot{f}}{f} + 3 \frac{\dot{b}}{b} = 0,
\]

(6)

(\(\dot{b} \equiv b'(z)\), etc.) and from a fourth equation,

\[
\beta(\lambda) = b \frac{d\lambda}{db}, \quad \lambda(z) = e^{\phi(z)} \sim g^2 N_c,
\]

(7)

where \(\beta(\lambda)\) is the beta function of the field theory one is seeking the gravity dual for. To begin with, we use the very simple beta function

\[
\beta(\lambda) = -\beta_0 \lambda^q
\]

(8)

and solve the equations in the order (7), (5) and (6). After this, Eq. (4) gives the potential \(V(\phi)\). In Sections 5 and 6 slightly modified beta functions are studied.

In [12, 13] the starting point was the potential

\[
V(\phi) = \frac{12}{\mathcal{L}^2} \left\{ 1 + V_0 \lambda + V_1 \lambda^{4/3} [\log(1 + V_3 \lambda^2)]^{1/2} \right\},
\]

(9)

\[
\approx \frac{12}{\mathcal{L}^2} \left[ 1 + V_0 \lambda + V_1 \sqrt{V_3} \lambda^{7/3} + \mathcal{O}(\lambda^{13/3}) \right],
\]

(10)

where the parameter values used were

\[
V_0 = 0.04128, \quad V_1 = 14.3, \quad V_3 = 170.4, \quad \lambda(z = 1) = 0.0242254.
\]

(11)
V_0 matters only in far UV and can be set to zero for thermodynamics, which is dominated by
the \lambda^{7/3} term \[18\]. The equations of motion (4)-(6) were solved numerically in \[13\] and (8) can
then be used to determine the beta function \[18\]. Here we proceed in the opposite direction,
starting from the beta function. The prototype values of q would be 
q = 2 corresponding to
the \lambda term in (10) and 
q = 10/3 corresponding to the thermodynamically dominant \lambda^{7/3}
term (see Eq. (29) below). In general, q > 1 and the limit q = 1 plays a special role, e.g., the
latent heat vanishes in this limit.

2.1 The fourth equation.
First, from Eq.(7) and for q > 1
\[
\log \frac{b}{b_0} = \frac{1}{(q - 1)\beta_0 \lambda^{q-1}} = Q,
\]
where \( b_0 \) is a constant, the analogue of \( \Lambda_{QCD} \), the scale at which \( \lambda \) diverges. It will prove to
be convenient to use Q as the extra dimensional variable and relate \( b, \lambda, f \) and \( z \) to it. The
numerical value of \( \beta_0 \) and the normalisation of \( \lambda \) do not matter, only the combination in \[12\]
enters. We shall also abbreviate
\[
a \equiv \frac{16}{9(q - 1)^2}, \quad A \equiv \frac{1}{2} \sqrt{a} = \frac{2}{3(q - 1)}.
\]
An important characteristic of \[12\] is that in the IR, \( \lambda \to \infty \), \( Q \to 0 \), \( b \) approaches a non-
zero constant \( b_0 > 0 \). We shall in Section 5 study the case when \( b \) vanishes proportionally to
a power of \( Q \) and in Section 6 the case of \( b \) vanishing \( \sim \) powers of \( Q \) and \( \log Q \). Then the
simple relation \[12\] between \( b \) and \( Q \) is replaced by the more complicated one in \[73\].

2.2 The second equation.
To integrate the second equation one introduces
\[
W = -\dot{b}/b^2,
\]
which simplifies \[5\] to the form
\[
\dot{b}W = \frac{4}{5} \dot{\phi}^2 = \frac{4}{5} \dot{\lambda}^2/\lambda^2.
\]
Writing \( \dot{W} = dW/d\lambda \cdot \dot{\lambda} \) cancels one power of \( \dot{\lambda} \) and replacing the remaining \( \dot{\lambda} \) by the beta
function using the basic equation \[7\] just produces to the right hand side the definition \[14\]
of \( W \). The equation then integrates to give
\[
W(\lambda) = W(0) \exp \left( -\frac{4}{9} \int_0^\lambda d\lambda \frac{\beta(\lambda)}{\lambda^2} \right) = \frac{1}{\mathcal{L}} \exp \frac{4}{9(q - 1)^2 \log(b/b_0)} \equiv \frac{1}{\mathcal{L}} \exp \frac{a}{4Q}.
\]
The normalisation \( W(0) = 1/\mathcal{L} \) follows from the requirement that the boundary be asymptot-
ically AdS, \( V(0) = 12/\mathcal{L}^2 \) (see Eq. \[27\] below). Now that \( W \) is known, the second integration
leading to \( b = b(z) \) is performed by writing \[14\] in the form
\[
dz = \frac{db}{-b^2 W} = \frac{dQ}{-bW} = -\frac{\mathcal{L}}{b_0} dQ \exp \left[ -Q - \frac{a}{4Q} \right].
\]
or, if one wants \( \lambda = \lambda(z) \),

\[
\frac{d \lambda}{dz} = \frac{d b \beta}{dz} = -\beta(\lambda)b(\lambda)W(\lambda).
\]

(17)

Here the important energy scale

\[
\Lambda = \frac{b_0}{\mathcal{L}}
\]

(18)

has appeared and one has

\[
\Lambda z = I(Q, \frac{1}{4} a),
\]

(19)

where

\[
I(Q, a) = \int_Q^\infty dy \exp \left[ -y - \frac{a}{y} \right]
\]

(20)

\[
= \sum_{n=0}^\infty \frac{1}{n!} (-a)^n \Gamma(1-n,Q)
\]

(21)

\[
= \exp \left[ -Q - \frac{a}{Q} \right] \left( 1 + \frac{a}{Q^2} - \frac{2a}{Q^3} + \frac{a^2 + 6a}{Q^4} - \frac{6a^2 + 24a}{Q^5} + .. \right),
\]

(22)

\[
I(0, a) = 2\sqrt{a}K_1(2\sqrt{a}),
\]

(23)

A few terms of the asymptotic expansion at large \( Q \) are exhibited here. The behavior of the integral \( I(Q, a) \) is as shown in Fig. 1. The \( Q = 0 \) limit together with (23) implies that

\[
z \leq \frac{1}{\Lambda} \sqrt{a}K_1(\sqrt{a}) = \frac{4}{\Lambda} \frac{1}{9(q-1)} K_1 \left( \frac{4}{9(q-1)} \right);
\]

(24)

a "wall" has been dynamically generated via the assumption [7].

2.3 The third equation.

The third equation (together with the first one) always has the trivial solution \( \dot{f} = 0, f = 1 \). This will be the low temperature phase with zero free energy. It is confining for \( q > 1 \):
the confinement criterion is that in the string frame the metric factor \( b_s = b \lambda^{2/3} \) have a minimum at some \( z \) (see Fig. 2). Using (8) this converts to the condition that the equation \( \beta(\lambda) + \frac{2}{3} \lambda = 0 \) have a solution. This is so for \( q > 1 \) and one finds [18], e.g., the string tension

\[
\sigma = \frac{\mathcal{L}^2 \Lambda^2}{2\pi \alpha'} \left( \frac{3e}{2\beta_0} \right)^{4/(3q-3)}.
\]

To have a nontrivial solution, one changes \( dz \) to \( dQ \) using (16) and directly integrates the third equation (6) to give

\[
f(z) = C_1 + C_2 \int_0^z \frac{d\bar{z}}{b^3(\bar{z})} = C_1 + C_2 \frac{\mathcal{L}}{b_0^2} I(4Q,a)
\]

\[
= 1 - \int_0^{z_h} \frac{d\bar{z}}{b^3(\bar{z})} / \int_0^{z_h} \frac{d\bar{z}}{b^3(\bar{z})} = 1 - \frac{I(4Q,a)}{I(4Q_h,a)},
\]

where \( C_2 \) is fixed by introducing a scale \( Q_h = Q(z_h) \) at which \( f \) vanishes, \( f(Q_h) = 0 \), and \( C_1 = 1 \) by the fact that \( b(z) = \mathcal{L}/z \), \( z \to 0 \) demands \( f(z = 0) = f(Q = \infty) = 1 \).
\[ V(Q, Q_h) \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{\( V(Q) \) from (28) for \( q = 10/3 \). The dashed line is the full potential from [13], indistinguishable from the \( Q > Q_h \) limit (29).}
\end{figure}

2.4 The first equation, the potential \( V(\phi) \).

Now that we have all the functions solved, we can insert them to the first equation (4) and obtain the potential

\[
V(\lambda) = 12fW^2 \left[ 1 - \left( \frac{\beta}{3\lambda} \right)^2 \right] - 3\frac{f}{b}W
\]

\[ = \frac{12}{\mathcal{L}^2} \exp \left( \frac{a}{4Q} \right) \left[ \left( 1 - \frac{I(4Q,a)}{f(4Q_h,a)} \right) \exp \left( \frac{a}{4Q} \right) \left( 1 - a \frac{1}{16Q^2} \right) + \frac{\exp(-4Q)}{f(4Q_h,a)} \right] \]

\[ = \frac{12}{\mathcal{L}^2} \left( 1 + \frac{a}{8Q} + \frac{2a^2 - a}{16Q^2} + \ldots \right) = \frac{12}{\mathcal{L}^2} \left( 1 + \frac{8\beta_0}{9(q-1)} \lambda^{q-1} + \ldots \right). \tag{29} \]

Eq. (28) is plotted in Fig. 3 for \( a = 16/[9(q-1)^2] = 16/49 \) by choosing some values for \( Q_h \). \( V \) is always positive in the form (28): there is a range beyond the horizon where \( f(1 - (\beta/(3\lambda))^2) < 0 \) but the last term cancels this negative part. One sees how the potential agrees with the potential in [12, 13] for \( Q > Q_h, z < z_h \). Analytically, the \( Q \gg Q_h, \lambda \ll 1, z \ll z_h \) limit (29) coincides with the same limit of the potential in [12, 13]. Beyond the horizon \( V \) starts growing rapidly, in fact, for \( Q \ll Q_h, V(Q) \sim Q^{-2} \exp[a/(2Q)] \sim \lambda^{q-2} \exp[8\beta_0/(9(q-1)) \lambda^{q-1}]. \)

The solution \( b(z), \phi(z), f(z) \) for the thermal case is given concretely by the equations above. To get the vacuum solutions \( b_0(z), \phi_0(z), f = 1 \) one should now solve Eqs. (4) and (5) with \( f = 1 \) and using the potential (28). We do not need these solutions, only that they exist.

3 Thermodynamics

Now that \( f(z) \) is explicitly known in (28), the standard formula \( 4\pi T = -f'(z_h) \) with \( dz/dQ \) from (16) gives the horizon temperature (when \( Q_h \) is a dummy variable we leave out the
index \( h \). Thus

\[
\frac{1}{4\pi T} = b^3 \int_Q^{\infty} dQ \frac{1}{b^4 W}
\]

(30)

or

\[
\frac{\pi T(Q)}{\Lambda} = \frac{\exp(-3Q)}{I(4Q, a)}
\]

(31)

\[
= \exp \left( Q + \frac{a}{4Q} \right) \left[ 1 - \frac{a}{16Q^2} + \frac{a}{32Q^3} - \frac{3a}{128Q^4} + \frac{12a + a^2}{512Q^5} + .. \right].
\]

(32)

Conversely, Eq. (31) gives \( Q_h = Q_h(\pi T/\Lambda) \). The large-\( T \) asymptotic expansion of this relation is, inverting (32) and denoting \( L \equiv \log(\pi T/\Lambda) \),

\[
Q_h = L - \frac{a}{4L} + \frac{a}{16L^2} - \frac{a + 2a^2}{32L^3} + \frac{12a + 25a^2}{512L^4} + ..
\]

(33)

A particular role is also played by \( dT/dQ \). In fact, below in (43) we shall prove that \( dT/dQ = 3c_2^2(T)/T \).

The relation \( T = T(Q) \) is plotted in Fig. 4 in the small \( Q \) region. At large \( Q \) the asymptotic expansion (32) is very accurate and \( T \sim \exp(Q) \). At \( Q = 0 \)

\[
\frac{\pi T(0)}{\Lambda} = \frac{1}{2\sqrt{a}K_1(2\sqrt{a})}, \quad \frac{\pi}{\Lambda} \frac{dT(0)}{dQ} = -3 \frac{\pi}{\Lambda} T(0).
\]

(34)

Thus \( dT/dQ \) has to change sign at some \( Q = Q_{\text{min}} \). At this point there is a minimum of \( T(Q) \) and \( c_2^2 = 0 \). In the more general analysis of [12] the branch with \( dT/dQ < 0 \) is called the small black hole branch, the one with \( dT/dQ > 0 \) is the big black hole one. In Section 6 we shall discuss the case when \( T \) can grow to infinity in the small BH branch, now it is bounded by (34).

From the form (3) of the metric it follows that the black hole entropy density is

\[
s = \frac{S}{V_3} = \frac{1}{4G_5} b^3(Q_h) = \frac{b_0^3}{4G_5} \exp(3Q_h) = \frac{L^3}{4G_5} \Lambda^3 \exp(3Q_h(T)).
\]

(35)

Scaling with \( T \), from (31), (32) and (33),

\[
\frac{s}{T^3} = \left( \frac{\pi L}{4G_5} \right)^3 \left[ \exp(4Q_h)I(4Q_h, a) \right]^3
\]

\[
= \left( \frac{\pi L}{4G_5} \right)^3 \exp \left( -3a \frac{4Q_h}{4Q_h} \right) \left( 1 + \frac{3a}{16Q_h^2} - \frac{3a}{32Q_h^3} + \frac{3a^2 + 9a}{256Q_h^4} \right) \left( \frac{9a^2 + 36a}{512Q_h^5} + .. \right)
\]

(36)

\[
= \left( \frac{\pi L}{4G_5} \right)^3 \left( 1 - \frac{3a}{4L} + \frac{9a^2 + 6a}{32L^2} - \frac{9a^3 + 24a^2 + 12a}{128L^3} + .. \right), \quad L \equiv \log \frac{\pi T}{\Lambda}.
\]

(37)

The full bulk thermodynamics is known when the (minus) free energy density or pressure \( p(T) \) is known. The rest then follows from relations like

\[
s(T) = p'(T), \quad \epsilon(T) = Ts - p, \quad \frac{\epsilon - 3p}{T^4} = T \frac{\partial}{\partial T} \frac{p}{T^4}, \quad c_2^2 = \frac{dp}{d\epsilon} = \frac{s}{Ts'(T)}.
\]

(38)
Figure 4: $T(Q)$ from (31) (left panel) and $p(T)/T^4$ from (39) (right panel) with numbers corresponding to $q = 10/3$, $a = 16/49$. The pressure vanishes at $p(Q_c = 0.363) = 0$, the corresponding value of $T$ is $\pi T_c / \Lambda = 1.67$. At this point $c_s^2 = 0.707/3$. The dashed line is the leading large $Q$ approximation $\exp[Q + a/(4Q)]$.

Now that $s(T)$ is known, $p(T)$ can be obtained by integrating $s(T) = p'(T)$. The key observation in [12, 13] now is that the correct constant of integration is obtained by integrating over the entire curve $T = T(Q)$ starting the integration at the value $T(0)$ corresponding to $Q = 0$ in Fig. 4. Thus one has

$$p(T) = \int^T dT s(T) = \frac{b_0^3}{4G_5} \int_0^{Q(T)} dQ \frac{dT}{dQ} \exp(3Q), \quad b_0 = \mathcal{L}\Lambda. \quad (39)$$

Since $dT/dQ$ starts negative, $p(T)$ starts decreasing (see Fig. 4), but soon starts growing and crosses the value zero at some $T$ which defines the transition temperature between the phase we are discussing and the phase corresponding to $f = 1$ in the metric ansatz (3). Since this low $T$ phase has $p = 0$, the transition temperature is determined by

$$p(Q_c) = p(Q(T_c)) = 0, \quad (40)$$

where the value of $T_c = \text{number} \times \Lambda$ is obtained from (31).

From (39) one further computes

$$\epsilon(T) = \frac{b_0^3}{4G_5} [T \exp(3Q) - p] = \frac{b_0^3}{4G_5} \left[ 3 \int_0^{Q(T)} dQ T(Q) \exp(3Q) + T(0) \right], \quad (41)$$

$$\epsilon(T) - 3p(T) = \frac{b_0^3}{4G_5} \left[ 3 \int_0^{Q(T)} dQ \left( T - \frac{dT}{dQ} \right) \exp(3Q) + T(0) \right] \quad (42)$$

and for the sound velocity

$$c_s^2 = \frac{1}{3T} \frac{dT}{dQ} = \frac{1}{3} \left[ \frac{4 \pi T}{\Lambda} \exp \left( -Q - \frac{a}{4Q} \right) - 3 \right]. \quad (43)$$
Figure 5: Left panel: $\epsilon/T^4$, $3p/T^4$ and the scaled interaction measure up to $30T_c$ for $q = 10/3$. Here and in other bulk thermo figures quantity/$N_c^2$ is plotted. The dashed line shows the limit at $T \to \infty$.

Right panel: $p(T)$ at very large $T$ scaled by the $T \to \infty$ limit. The two data points at $T = 300T_c$ and $T = 3 \cdot 10^7 T_c$ are from [20], the third is the highest from [3]. The dashed lines are the asymptotic expansions (44) including only the $-3a/4 \log(T)$ term.

The large $Q$ or large $L = \log(\pi T/\Lambda)$ expansions are

$$\frac{p}{T^4} = \frac{(\pi L)^3}{4G_5} \left( \frac{1}{4} - \frac{3a}{16Q} + \frac{9a^2}{128Q^2} - \frac{9a^3 + 12a}{512Q^3} + \frac{27a^4 + 96a^2}{8192Q^4} + .. \right)$$

$$\frac{\epsilon}{T^4} = 3 \left( \frac{(\pi L)^3}{4G_5} \left( \frac{1}{4} - \frac{3a}{16L} + \frac{9a^2}{128L^2} - \frac{9a^3 + 24a^2 + 12a}{512L^3} + .. \right) \right)$$

$$\frac{\epsilon - 3p}{T^4} = 3 \left( \frac{(\pi L)^3}{4G_5} \left( \frac{a}{16Q^2} - \frac{3a^2}{64Q^4} + \frac{9a^3 + 8a^2 + 12a}{512Q^3} + .. \right) \right)$$

$$\frac{c_s^2}{T s'(T)} = \frac{1}{3} \left( 1 - \frac{a}{4Q^2} + \frac{a}{8Q^3} + ... \right)$$

Note that to plot the asymptotic expansions vs $T/T_c$ one must write

$$\log \frac{\pi T}{\Lambda} = \log \left( \frac{T}{T_c} \frac{\pi T_c}{\Lambda} \right)$$

and determine $\pi T_c/\Lambda$ from (40).
Figure 6: $\epsilon/T^4$, $3p/T^4$ and the interaction measure up to $5T_c$ for $q = 10/3$ (left) and $q = 2$ (right). For $q = 10/3$ the interaction measure decreases $\sim 1/T^{2.8}$ above $T_c$ (dashed line), for $q = 2$ it decreases $\sim 1/T^2$ for $T_c < T < 2T_c$ (dashed line), in agreement with data, but more slowly for $T > 2T_c$ (see also Fig. 8).

Figure 7: Left panel: The latent heat plotted as a function of $q$. $L$ vanishes at $q = 1$. The lattice value at $N_c \to \infty$ is $0.34(6)$ [4] and is obtained for $q = 2$. Right panel: Lattice data for $N_c \gg 1$ (dashed lines) [6] compared with model prediction for $q = 2$.

4 Comparison with lattice data for hot SU($N_c$) gauge theory

Our model is so far based on a very simple powerlike in $\lambda$ (exponential in $\phi$) ansatz for the beta function, from which the scalar potential is derived, and one cannot expect detailed agreement with data. Nevertheless, we find that on a qualitative level the main features of the data are reproduced to surprising extent.

To analyse the bulk quantities one has to fix the value of the parameter $L^3/G_5$. We fix it
to the pressure of ideal gluon gas,

$$p_{SB} = 2(N_c^2 - 1)\frac{\pi^2}{90}T^4.$$  (49)

Comparing with (44) one has

$$\frac{(\pi L)^3}{4G_5} = \frac{4\pi^2}{45}N_c^2.$$  (50)

This is less by a factor 8/45 relative to the value for $\mathcal{N} = 4$ SYM, 2/15 comes from the reduction of the number of degrees of freedom and 4/3 from going from strongly interacting to ideal systems. Bulk thermodynamic quantities are very accurately proportional to $N_c^2$ \cite{4, 5, 6} so that we shall simply divide by this factor.

Using this normalisation Figs. 5 and 6 show $\epsilon/T^4$, $p/T^4$, $(\epsilon - 3p)/T^4$ as functions of $T/T_c$ evaluated numerically from the above equations for $q = 10/3$ and $q = 2$. Fig. 5 also shows $p(T)$ at very large $T$ scaled by the $T \rightarrow \infty$ limit (49).

At $T_c$, $p = p(T_c) = 0$ but $\epsilon(T_c)$ has the nonzero value given by (41). Since $p = \epsilon = 0$ for the low $T$ phase, $p$ is continuous, by construction, but $\epsilon$ jumps. This gives the latent heat

$$\frac{L}{T_c^4} = \frac{\epsilon(T_c)}{T_c^4} = \frac{s(T_c)}{T_c^3} = \frac{4\pi^2}{45}N_c^2 [\exp(4Q_c)I(4Q_c, a)]^3.$$  (51)

This is plotted in Fig. 7 as a function of $q$. The latent heat vanishes for $q = 1$, the transition goes over to a second order one. The lattice value \cite{4} $L/T_c^4 = 0.34(6)N_c^2$ would be obtained for $q \approx 2.2$. Choosing this value of $q$, Fig. 7 also compares the $T$ dependence of the bulk quantities with lattice data extrapolated to $N_c \rightarrow \infty$ \cite{6}.

One well known characteristic of $(\epsilon - 3p)/T^4$ is that it decreases $\sim 1/T^2$ for $T \gtrsim T_c$. This is checked in Fig. 8 for the data and for the model with $q = 2.2$. As seen in Fig. 6 for $q = 10/3$
the model result decreases faster, $\sim 1/T^{2.8}$. Continuing to still larger $T$, according to (46) the leading large $T$ behaviour of the interaction measure is

$$\frac{\epsilon - 3p}{T^4} = \frac{4\pi^2 N_c^2}{135(q - 1)^2 \log^2(\pi T/\Lambda)},$$

(52)
a slight generalisation of Eq. (I.3) in [12].

The asymptotic expansion (44) looks much like a perturbative expansion in $g^2$ and it is of interest to compare the leading corrections quantitatively - to higher orders the finite $T$ perturbative expansion proceeds in powers of $g$. Consider first the interaction measure. In perturbation theory, to leading order, by taking $T\partial/\partial T$ of Eq. (6.1) of [19],

$$\epsilon - 3p = \frac{(N_c^2 - 1) N_c}{144} T \frac{dg^2(T)}{dT} = \frac{\pi^2 (N_c^2 - 1)}{66} \frac{1}{\log^2(\pi T/\Lambda_{\text{MS}})}.$$

(53)

This is very much similar to (52), in fact, coincides for $q = \sqrt{88/45} + 1 \approx 2.4$. Consider then the pressure, for which the two expansions are

$$p_{\text{SB}} = 1 - \frac{5g^2 N_c}{16\pi^2} + \frac{80}{\sqrt{3}} \left(\frac{g^2 N_c}{16\pi^2}\right)^{3/2} + .. = 1 - \frac{4}{3(q - 1)^2 \log(\pi T/\Lambda_{\text{MS}}) + ..}.$$

(54)

where also the $g^3$ plasmon term is included in the perturbative expansion. Comparing the leading terms with 1-loop running for $g^2(T)$, the two expansions again coincide for $q = \sqrt{88/45} + 1 \approx 2.4$. However, the plasmon term is quantitatively very important, smaller than the $g^2$ term only for $T > 10^5 T_c$, which explains why the $q = 10/3$ term agrees with the perturbative result, see Fig. 5. This figure also shows to what extent the leading order term of the expansion (44) agrees with the numerical computation. How still further terms improve the agreement is seen from Fig. 8.

From the above one sees that $q = 10/3$ gives a good fit to SU($N_c$) thermodynamics for $T \gtrsim 10 T_c$; for smaller $T$ the curves are too high and the latent heat too big. This indicates that one has to modify the ansatz in the IR large $\lambda$ region, the excellent fit in [13] contained several parameters. In the next two sections we present systematic generalisations of (8), which will be seen to lead first to vanishing latent heat, a continuous transition and then again to a first order transition.

### 5 Continuous or second order transition

As discussed above, the transition approaches a continuous one when $q \to 1$. To study this case it is more realistic not to take $q \to 1$ in (8) but to take a beta function which only approaches the marginally confining $q = 1$ form $-\frac{1}{2} \lambda$ at large $\lambda$:

$$\beta(\lambda) = \frac{-\beta_0 \lambda^q}{1 + \frac{2}{3} \beta_0 \lambda^{q-1}}.$$

(55)

Even for this case the solution of the gravity equations (4)-(7) is surprisingly simple. Denoting again (but now this is not $\log(b/b_0)$)

$$Q = \frac{1}{(q - 1)\beta_0 \lambda^{q-1}}$$

(56)
and recalling the abbreviations

\[ A = \frac{1}{2} \sqrt{a} = \frac{2}{3(q-1)} \]  

(57)

we have

\[ b = b_0 e^Q \left( \frac{Q}{Q_0} \right)^A, \]  

(58)

\[ W = \frac{1}{\mathcal{L}} \left( 1 + \frac{A}{Q} \right)^A, \]  

(59)

where \( Q_0 \) is the scale at which \( b \) has the value \( b_0 e^{Q_0} \). In Section 6 we shall choose \( Q_0 = A \). Note that now \( b \) vanishes in the IR proportionally to a power of \( Q \) or of \( 1/\lambda \).

Using \( db/dz = -b^2W \) from (16) and \( 4\pi T = -\dot{f}(z) \) from (26) one finds the temperature

\[ \frac{\Lambda}{\pi T} \frac{1}{4Q_0^3} = e^{3Q} Q^3 A \int_Q^\infty dy \frac{y^{-3A-1}}{(y + A)^{3A-1}} e^{-4y} = \frac{1}{T(Q, A)}, \]  

(60)

the pressure (\( Q_0 \) cancels from here)

\[ \frac{p}{T^4} = \frac{(4\pi \mathcal{L})^3}{4G_5} \frac{1}{T^4} \int_0^Q dQ d\dot{T} e^{3Q} Q^3 A, \]  

(61)

and sound velocity

\[ c_s^2 = \frac{1}{3} \frac{Q}{A + Q} \frac{dT}{dQ}. \]  

(62)

A numerical example of the scaled bulk quantities for \( q = 10/3 \) is shown in Fig. 9.

Now that the transition is of second order the minimum of \( T(Q) \) is at \( Q = 0 \), beyond that \( dT/dQ > 0 \). \( T(Q = 0) \) is calculable and defines the critical temperature

\[ \frac{\pi T_c}{\Lambda} = \frac{3}{4} \left( \frac{A}{Q_0} \right)^A. \]  

(63)

To find the behavior of \( T(Q) \) near \( Q = 0 \) or near \( T_c \) one must do partial integrations with \( dy^{-3A} \) to increase the power of \( y \) in (60). For \( 0 < 3A < 1, q > 3 \) one partial integration is enough to make the integral converge for \( Q \to 0 \). Expanding the remainder in \( Q \) one obtains

\[ \frac{T(Q, A)}{T(0, A)} - 1 = T/T_c - 1 \equiv t \]

\[ = A^{A-1} \int_0^\infty dy y^{-3A} e^{-4y} \frac{5A - 1 + 4y}{(A+y)^A} Q^{3A} - \frac{6A}{1 - 3A} Q + \mathcal{O}(Q^2), \]  

(64)

where the integral is positive for \( 0 < 3A < 1 \). For this range of \( 3A \) the first term is dominant in the critical region \( t \to 0 \) so that

\[ t \sim Q^{3A}, \quad Q \sim t^{1/3A} \quad 0 < 3A < 1. \]  

(65)
For $1 < 3A < 2$, $2 < q < 3$ two partial integrations are needed and
\[
    t = \frac{6A}{3A-1} Q - \text{Integral} Q^{3A} + \mathcal{O}(Q^2),
\]
where the Integral, similar to the one in (64), is positive. Thus now and for all larger values of $A$ and smaller values of $q$:
\[
    t \sim Q, \quad 1 < 3A < 2, \quad 2 < q < 3.
\]
When converted to the critical behavior of the free energy ($= -p$) one has
\[
    f \sim t^2, \quad 3A < 1 \quad f \sim t^{3A+1}, \quad 3A > 1
\]
so that the critical exponent in $C_V \sim f''(t) \sim t^{-\alpha}$ is $\alpha = 0$ ($= 1 - 3A$) for $3A < 1$, ($3A > 1$). Since $f''(t)$ is not divergent, the transition is a continuous one. For the SU(2) finite $T$ transition, of 3d Ising universality class, one has $\alpha = 0.12$ [21], a mild divergence. A continuous transition appears also if one takes SU(3) gauge theory with effectively 3 flavours of infinite mass fermions and reduces the mass of one fermion, i.e., studies one-flavor QCD. For sufficiently small mass the transition becomes a continuous one [22], the end point again being of Ising universality class.

With the modified beta function (55) with $q = 10/3$ we have now reduced the latent heat to zero. The work in [12, 13] indicates what the modification should be if one wants a first order transition but still maintains the leading asymptotic behavior $-\frac{3}{2} \lambda$: the asymptotic behavior at large $\lambda$ should be
\[
    \beta \rightarrow -\frac{3}{2} \lambda \left(1 + \frac{\alpha}{\log \lambda}\right)
\]
with $\alpha > 0$. We shall in the next section derive analytic formulas even for this case. In principle, one could then redo the critical analysis above so that the parameter $\alpha$ would effectively be an external field which could drive the transition to a first order one and more critical indices would appear.

### 6 First order transition again

Consider now the following ansatz:
\[
    \beta(\lambda) = -\frac{\beta_0 \lambda^q}{1 + \frac{2}{3} \beta_0 \lambda^{q-1}} \left[1 + \alpha(q - 1) \frac{\log(1 + \frac{2}{3} \beta_0 \lambda^{q-1})}{\log^2(1 + \frac{2}{3} \beta_0 \lambda^{q-1}) + 1}\right]
\]
\[
    = -\frac{\lambda}{q-1} \frac{1}{Q + A} \left[1 + \alpha(q - 1) \frac{\log(1 + A/Q)}{\log^2(1 + A/Q) + 1}\right],
\]
where $Q$, $A$ are defined as in the previous section. The modification factor relative to (55) is so constructed that in the IR, $\lambda \rightarrow \infty$, it is $1 + \alpha(\log \lambda + ...)$. In the UV it is $1 + \mathcal{O}(\lambda^{q-1})$ and also so that $W$ is integrable in closed form from (15):
\[
    W = \frac{1}{L} \left(1 + \frac{A}{Q}\right)^A \left[\log^2 \left(1 + \frac{A}{Q}\right) + 1\right]^{\frac{1}{3} \alpha},
\]
\[\text{In the notation of [12], this parameter was denoted by } \frac{3}{4} \alpha/(\alpha - 1), \alpha \geq 1.\]
The expression for $b$ from (7),

$$\log \frac{b}{b_0} = \int_{\lambda_0}^{\lambda} d\lambda \frac{d}{\beta} = \int_{A}^{Q} dQ \frac{Q}{(q-1)\beta(\lambda)} = \int_{A}^{Q} dy \left( 1 + \frac{A}{y} \right) \frac{1}{1 + \frac{\alpha(q-1)\log(1+A/y)}{\log^2(1+A/y)+1}},$$

is also surprisingly simple, though not integrable in closed form. The lower limit of the $Q$ integral has here been fixed to be $A$, i.e., $b(Q = A) = b_0$. In the IR, $Q \to 0$, this implies

$$\frac{b}{b_0} = \left( \frac{Q}{A} \right)^A \left( \frac{1}{\alpha(q-1)} \log \frac{A}{Q} + 1 \right)^{\frac{2}{3} \alpha}.$$  \hspace{1cm} (74)$$

This is the key modification relative to (58): $b$ vanishes in the IR but more slowly by a power of $\log Q$ than an ansatz with $\beta \to -\frac{3}{2} \lambda$.

The modified IR behavior has several consequences. Consider first the $z$ dependence of the field configurations. For this one has to integrate $z = z(Q)$ from $dz = -db/(b^2W)$ so that, as a generalisation of (19),

$$\Lambda z = \int_{Q(z)}^{\infty} dQ \frac{d\log b}{dQ} \frac{1}{bW}.$$  \hspace{1cm} (75)$$

Studying the limit $Q(z) \to 0$ one sees that this integral diverges if $\frac{4}{3} \alpha < 1$:

$$\Lambda z \sim \log(A/Q)^{1-\frac{4}{3} \alpha},$$  \hspace{1cm} (76)$$

otherwise it is finite. Thus, if $\alpha < \frac{3}{4}$ the range of $z$ is unbounded.
Similarly, for the temperature we have

\[ \frac{1}{4\pi T} = b^3 \int_0^{z_h} \frac{dz}{b^3} = b^3 \int_{Q(T)}^{\infty} dQ \frac{d\log b}{dQ} \frac{1}{b^4W} \]  

(77)

and counting IR logs one finds that the power of \( \log(1/Q) \) is \( 2\alpha \) from the outside \( b^3 \), 
\(-8\alpha/3 - 2\alpha/3 \) from the inside \( 1/(b^4W) \) so that in the IR \( T \sim \log^{2\alpha/3}(1/Q) \). Thus \( T \) diverges logarithmically when \( Q \to 0 \); with the beta function \( b^3 \) \( T(Q = 0) \) was finite but there was a minimum, Fig. 3. Thus there again is a minimum of \( T(Q) \) and a branch with \( dT/dQ < 0 \), so that \( p(T) \) behaves as in Fig. 4 and the transition becomes of first order. Full thermodynamics in terms of two parameters, \( q > 1 \) and \( \alpha > 0 \), is now given by the formulas, generalisations of (39) and (41):

\[ s(T) = \frac{1}{4G_5} b^3(Q(T)) \]  

(78)

\[ p(T) = \int dTs(T) - \frac{1}{4G_5} \int_0^{Q(T)} dQ \frac{dT}{dQ} b^3(Q) \]  

(79)

\[ \epsilon(T) = \frac{3}{4G_5} \int_0^{Q(T)} dQ T(Q) b^3(Q) \frac{d\log b}{dQ} \]  

(80)

\[ \epsilon(T) - 3p(T) = \frac{3}{4G_5} \int_0^{Q(T)} dQ \left( T \frac{d\log b}{dQ} - \frac{dT}{dQ} \right) b^3(Q) \]  

(81)

\[ c_s^2 = \frac{1}{3} \frac{dT}{dQ} \frac{dQ}{d\log b} = \frac{1}{3} \left( 4\pi T/bW - 3 \right) \]  

(82)

Note that \( T(0) \) in (41) is actually \( T(0)b^3(0) \) which now vanishes. The procedure again is that one first finds at what value of \( Q = Q_c \) the pressure vanishes; \( p(Q_c) = 0 \). From (77) one next finds the corresponding value of \( T = T_c = T(Q_c) \). Finally one plots parametrically with \( Q > Q_c \) as the parameter the required thermodynamic quantity as the \( y \) axis and \( T(Q)/T_c \) as \( x \) axis.

We refer a detailed comparison with lattice data in the few \( T_c \) range and with QCD perturbation theory at large \( T \) to later work. As a preliminary step, Fig. 10 compares the entropy density scaled by \( N_c^2T^3 \) obtained for \( q = 10/3 \) with the \( N_c \to \infty \) limit of lattice data (known \( [6] \) for \( T \leq 3.4T_c \)) and the QCD perturbative result \( [7] \) at large \( T \). One sees that the too large latent heat observed in Fig. 6 for this value of \( q \) (which leads to good large \( T \) behavior) is cured by the modification of the IR region incorporated in the beta function \( [71] \), if one chooses \( \alpha \) to be in the range 0.1...0.2. There is still some deviation very close to \( T_c \), but overall the agreement is excellent, as already noted in \( [13] \).

Finally, one may note that the latent heat vanishes when \( \alpha \to 0 \) as a power of \( \alpha \):

\[ L \frac{N_c^2T_c^3}{N_c^2T^3} = 1.55 \alpha^{130}, \quad q = \frac{10}{3} \]  

(83)

This is accurate up to \( \alpha = 0.1 \).

\(^2\)The perturbative result is obtained by computing entropy density from the pressure in Eq. (6.4) of \( [19] \), choosing the renormalisation scale \( \bar{\mu} = 2\pi T \), the unknown 4-loop Linde coefficient \( q_c = -3800 \) and a 2-loop running coupling with \( N_c = 3 \).
Figure 10: Entropy density scaled by $T^3 N_c^2$ (the asymptotic $T \to \infty$ value is $4\pi^2/45$) for the beta function (71) with $q = 10/3$ and $\alpha = 0.25, 0.125$ and for $T < 5T_c$ (left panel) or $T < 10^5 T_c$ (right panel). The dashed curve gives the $N_c \to \infty$ limit of lattice data [6] and the dot-dashed one a QCD perturbative result (see text).

7 Conclusions

In this paper, we have studied a simple gauge/gravity duality model for gauge theory thermodynamics. Basically, the model is a modification of the thorough work carried out in [10, 11, 12, 13]: instead of starting from a scalar field potential $V(\phi)$ of the gravity side we start from the beta function on the gauge theory side. Its chief virtue is simplicity, semi-analytic formulas are obtained, no numerical solutions of Einstein’s equations are needed and the only essential parameters are the power $q > 1$ of $\lambda = e^\phi$ in the beta function and the parameter $\alpha > 0$ associated with the approach to the infrared limit. Notable is also that the normalisation of $\lambda$ never enters, everything is expressed using the combination $Q = 1/((q - 1)/\beta_0 \lambda^{q-1})$.

The key element in this analysis was that the three equations determining the three functions $b(z), \phi(z), \lambda(z)$ were taken to be the two Einstein equations not containing the potential $V(\phi)$ and the equation relating the beta function to $b(z), \lambda(z)$. The Einstein equation containing the potential then serves simply to determine the potential.

It is perhaps surprising that such a simple beta function as $\beta = -\beta_0 \lambda^q$ reproduces main features of a first order transition so well. Detailed agreement with lattice data and QCD perturbation theory requires beta functions constrained to approach $-\frac{3}{2} \lambda(1 + \alpha/\log \lambda)$ at large $\lambda$. Here $\alpha > 0$ leads to a first order transition. Even for this case analytic expressions could be given. They become particularly simple for the case $\alpha = 0$, corresponding to a continuous transition.

There are many directions into which one could develop this work. In particular, a study of beta functions of the type $-\beta_0 \lambda^q(1 - \lambda/\lambda_s)$ containing an infrared fixed point $\lambda_s$ suggests itself. Further, the model could be applied to walking technicolor beta functions which only approach a fixed point $\lambda_s$. Perhaps then one looses the virtue of simplicity.
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