GLOBAL WELL-POSEDNESS AND INVISCID LIMITS OF THE GENERALIZED OLDROYD TYPE MODELS

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Abstract. We obtain the global small solutions to the generalized Oldroyd-B model without damping on the stress tensor in $\mathbb{R}^n$. Our result gives positive answers partially to the question proposed by Elgindi and Liu (Remark 2 in Elgindi and Liu [J Differ Equ 259:1958–1966, 2015]). The proof relies heavily on the trick of transferring dissipation from $u$ to $\tau$, and a new commutator estimate which may be of interest for future works. Moreover, we prove a global result of inviscid limit of two dimensional Oldroyd type models in the Sobolev spaces. The convergence rate is also obtained simultaneously.

1. Introduction and the main results

The Oldroyd-B model is a typical prototypical model for viscoelastic fluids, which describes the motion of some viscoelastic flows:

$$
\begin{aligned}
&u_t + u \cdot \nabla u - \nu \Delta u + \nabla \pi = K_1 \text{div} \tau, \\
&\tau_t + u \cdot \nabla \tau + \beta \tau + Q(\tau, \nabla u) = K_2 D(u), \\
&\text{div} u = 0,
\end{aligned}
$$

where $\nu > 0$, $\beta \geq 0$, $K_1 \geq 0$, $K_2 \geq 0$, $u$ stands for the velocity of the fluid and $\pi$ for the pressure and $\tau$ is a symmetric tensor. The parameters $\nu$, $K_2$ correspond respectively to $\frac{\theta}{\text{Re}}$ and $\frac{2(1-\theta)}{\text{Re} \cdot \text{We}}$, where $\text{Re}$ is the Reynolds number, $\theta$ is the ratio between the so-called relaxation and retardation times and $\text{We}$ is the Weissenberg number which measures the elasticity of the fluid. We denote by $\Omega(u)$ the vorticity tensor and $D(u)$ the deformation tensor, namely

$$
\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T), \quad D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).
$$

$Q(\tau, \nabla u)$ is a given bilinear form which can be chosen as

$$
Q(\tau, \nabla u) = \tau \Omega(u) - \Omega(u) \tau + b(D(u)\tau + \tau D(u)),
$$

$b$ is a parameter in $[-1, 1]$.

The Oldroyd-B model is a classical model for dilute solutions of polymers suspended in a viscous incompressible solvent (see [2]). The system (1.1) describes the motion of the incompressible fluid satisfying the Oldroyd constitutive law (see [23]). About the derivation of the system (1.1), the interested readers can refer to [18], here we omit it.

The study of the incompressible Oldroyd-B model started by a pioneering paper by Guillopé and Saut in [12], the authors proved that the strong solutions are local well-posed in the Sobolev space $H^s$. They [13] also showed that these solutions are global if the coupling parameter and the initial data are small enough. The extensions of these results to the $L^p$-setting were given by Fernández-Cara, Guillén and Ortega. In 2001, Chemin and Masmoudi [3] studied the local and global well-posedness of (1.1) in $\mathbb{R}^n$. But for the global result in $L^p$ framework, they needed the small coupling parameter. This gap was filled in a recent work by Zi, Fang and Zhang [28] with the method based on Green’s matrix of the linearized system. Furthermore, some blowup criteria for local solutions were also established in [3]. An improvement of the Chemin and Masmoudi’s blow-up criterion was presented by Lei, Masmoudi and Zhou [16]. In addition, we
can refer to [5] for more global existence results in generalized spaces. Very recently, for the case $\nu = 0$, adding an exact diffusion $-\mu \Delta \tau$ in the second equation of system (1.1), Elgindi and Rousset [9] proved the global existence of smooth solutions in $\mathbb{R}^2$ with the arbitrary initial data for $Q(\tau, \nabla u) = 0$, $K_2 \in \mathbb{R}$, $K_1, \beta \geq 0$ and small initial data for $Q(\tau, \nabla u) \neq 0$, $K_1 > 0, K_2 > 0, \beta > 0$, respectively. For the same model considered in [9], the global small solutions in $\mathbb{R}^3$ was further obtained by Elgindi and Liu [8] in Sobolev spaces $H^s(\mathbb{R}^3), s > \frac{5}{2}$. Most recently, by constructing two special time-weighted energies, Zhu [27] obtained the global smooth solutions to system (1.1) with $\beta = 0, \mu > 0, K_1 > 0, K_2 > 0$ in $\mathbb{R}^3$. For the same model considered in [27], Chen and Hao [4] established the global small solutions in critical Besov spaces for more general dimension $\mathbb{R}^n(n \geq 2)$. This result was further extended by Zhai [25] to a critical $L^p$ framework which implies that a class of highly oscillating initial data are allowed. There are other lots of excellent works have been done to Oldroyd model and related models (see [5], [7], [15], [17]–[21], [26])

It should mention that, in Remark 2 of the paper by Elgindi and Liu [8], the authors expect the global small solutions of the generalized Oldroyd type models with $\Lambda^{\alpha_1}$ dissipation on $u$ and $\Lambda^{\alpha_2}$ dissipation on $\tau$ in the two and three dimensions, which motivates us to consider the following generalized Oldroyd-B model without damping mechanism:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nu \Lambda^\alpha u + \nabla \pi - K_1 \text{div} \tau &= 0, \\
\partial_t \tau + u \cdot \nabla \tau + Q(\tau, \nabla u) - K_2 D(u) &= 0,
\end{align*}
\]

(1.2)

with $\Lambda \overset{\text{def}}{=} \sqrt{-\Delta}$, and initial data satisfy

\[
u(x, 0) = u_0(x), \quad \tau(x, 0) = \tau_0(x).
\]

Now, we can state the main theorem of the paper:

**Theorem 1.1.** Let $n \geq 2, \nu, K_1, K_2 > 0$. For any $1 < \alpha \leq 2$, $\tau_0 \in \dot{B}_{2,1}^{\frac{n}{2} + 1 - \alpha} \cap \dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)$, $u_0 \in \dot{B}_{2,1}^{\frac{n}{2} + 1 - \alpha}(\mathbb{R}^n)$ with $\text{div} u_0 = 0$. If there exists a positive constant $c_0$ such that,

\[
\|u_0\|_{\dot{B}_{2,1}^{\frac{n}{2} + 1 - \alpha}} + \|\tau_0\|_{\dot{B}_{2,1}^{\frac{n}{2} + 1 - \alpha} \cap \dot{B}_{2,1}^{\frac{n}{2}}} \leq c_0,
\]

(1.3)

then the system (1.2) has a unique global solution $(u, \tau)$ so that for any $T > 0$

\[
\begin{align*}
&u \in C_b([0, T]; \dot{B}_{2,1}^{\frac{n}{2} + 1 - \alpha}(\mathbb{R}^n)) \cap L^1([0, T]; \dot{B}_{2,1}^{1 + \frac{n}{2}}(\mathbb{R}^n)), \\
&\tau \in C_b([0, T]; \dot{B}_{2,1}^{\frac{n}{2} + 1 - \alpha} \cap \dot{B}_{2,1}^{\frac{n}{2}}(\mathbb{R}^n)), \\
&\left(\Lambda^{-1} \mathbb{P} \text{div} \tau\right)^t \in L^1([0, T]; \dot{B}_{2,1}^{\frac{n}{2} + 1}(\mathbb{R}^n)), \\
&\left(\Lambda^{-1} \mathbb{P} \text{div} \tau\right)^h \in L^1([0, T]; \dot{B}_{2,1}^{\frac{n}{2} + 2 - \alpha}(\mathbb{R}^n)).
\end{align*}
\]

Here $\mathbb{P} = \mathcal{I} - \mathcal{Q} \overset{\text{def}}{=} \mathcal{I} - \nabla \Delta^{-1} \text{div}$ is the projection operator. One refers to Section 2 for the definitions of the Besov space $\dot{B}_{2,1}^s(\mathbb{R}^n)$ and $f^t, f^h$.

**Remark 1.2.** In Remark 1.3 of the paper by Elgindi and Rousset in [9], they expect the global solutions of the generalized version of (1.2) in $\mathbb{R}^2$ with $\Lambda^{2\alpha_1} u$, $\Lambda^{2\alpha_2} \tau$ and $\alpha_1 + \alpha_2 = 1$, if neglecting the effect of the quadratic form $Q(\tau, \nabla u)$. To our knowledge, this problem is still open. The Theorem 1.1 brings us closer to this more interesting case, hope we can give a positive answer about this problem in the further.

**Remark 1.3.** If $\alpha = 2$, the above Theorem 1.1 coincides with the results by Chen and Hao [4] and Zhai [25]. Compared to [4], [25], we obtain the global small solutions with less dissipation for $u$. 

Remark 1.4. For any $0 < \alpha \leq 1$, we cannot obtain any smoothing effect of $\Lambda^{-1} \Pi \div \tau$, thus, we don't close our energy estimates, a new method need to be introduced to deal with this difficulty.

Remark 1.5. Let us finally say a few words on our functional setting. As we all know that an important quantity for the wellposedness problem of the fluid is $\|\nabla u\|_{L^1_t(L^\infty)}$. By using the embedding relation $\dot{B}^{\frac{n}{2}}_{2,1}(R^n) \hookrightarrow L^{\infty}(R^n)$, we only need to control the norm of $\|\nabla u\|_{L^1_t(\dot{B}^{\frac{n}{2}}_{2,1})}$.

Now from the maximal smoothing effect of the fractional heat equation in (1.2), the best space of the initial velocity should be $\dot{B}^{\frac{n}{2}+1-\alpha}_{2,1}(R^n)$.

Next, we shall prove a global result of inviscid limit of the following model in the Sobolev spaces:

$$
\begin{cases}
\partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla \pi_\nu - \div \tau_\nu = 0, \\
\partial_t \tau_\nu + u_\nu \cdot \nabla \tau_\nu - \Delta \tau_\nu - D(u_\nu) = 0, \\
\div u_\nu = 0, \\
(u_\nu, \tau_\nu)|_{t=0} = (u_0, \tau_0).
\end{cases}
$$

Let $\nu = 0$ in (1.4), that is, considering the following system:

$$
\begin{cases}
\partial_t u + u \cdot \nabla u + \nabla \pi - \div \tau = 0, \\
\partial_t \tau + u \cdot \nabla \tau - \Delta \tau - D(u) = 0, \\
\div u = 0, \\
(u, \tau)|_{t=0} = (u_0, \tau_0).
\end{cases}
$$

Elgindi and Rousset [9] obtained the following theorem of (1.5):

Theorem 1.6. (see [9]) Assume that $(u_0, \tau_0) \in H^s(R^2)$ with $s \in (2, \infty)$. Then (1.5) has a unique global solution $(u, \tau)$ satisfying, for any $T > 0$,

$$(u, \tau) \in C([0, T]; H^s(R^2)).$$

The aim of this paper is to analyze the inviscid limit problem and we will show strong convergence of the solutions $(u_\nu, \tau_\nu)$ of the system (1.4) to the one of (1.5) in the same space of initial data. More precisely, we obtain the following theorem:

Theorem 1.7. Let $\nu > 0$ and $(u_0, \tau_0) \in H^\sigma(R^2)$ with $\sigma > 4$. For any $T > 0$, there exists $\nu_0 = \nu_0(T) > 0$ such that, for $0 < \nu \leq \nu_0$, (1.4) has a unique global solution satisfying $(u_\nu, \tau_\nu) \in C([0, T]; H^\sigma(R^2))$. Moreover, for any $0 \leq s \leq \sigma - 2$ and $0 < \nu \leq \nu_0$, we have

$$
\lim_{\nu \to 0} \|(u_\nu, \tau_\nu) - (u, \tau)\|_{L^s_{t}(H^s)} = 0.
$$

Remark 1.8. From the proof of the above theorem, we can get the precise convergence rate in the $H^s$ space. More precisely, we have

$$
\|(u_\nu, \tau_\nu) - (u, \tau)\|_{H^s} \leq C(T)\nu,
$$

where $C(T)$ is a constant dependent on $T$ and $\|(u, \tau)\|_{L^{\infty}([0, T]; H^s)}$.

Notations : Let $A$, $B$ be two operators, we denote $[A, B] = AB - BA$, the commutator between $A$ and $B$. For $a \lesssim b$, we mean that there is a uniform constant $C$, which may be different on different lines, such that $a \leq C b$. We shall denote by $\langle a, b \rangle$ the $L^2(R^d)$ inner product of $a$ and $b$. For $X$ a Banach space and $I$ an interval of $R$, we denote by $C(I; X)$ the set of continuous functions on $I$ with values in $X$. For $q \in [1, +\infty]$, the notation $L^q(I; X)$ stands for the set of measurable functions on $I$ with values in $X$, such that $t \to \|f(t)\|_X$ belongs to $L^q(I)$. We always let $(d_j)_{j \in \mathbb{Z}}$ be a generic element of $\ell^1(\mathbb{Z})$ so that $\sum_{j \in \mathbb{Z}} d_j = 1$. 

2. Preliminaries

The result of the present paper rely on the use of a dyadic partition of unity with respect to the Fourier variables, the so-called the Littlewood-Paley decomposition. Here, we only give an homogeneous Littlewood-Paley decomposition $(\hat{\Delta}_j)_{j \in \mathbb{Z}}$ that is a dyadic decomposition in the Fourier space for $\mathbb{R}^n$. One may for instance set $\hat{\Delta}_j \overset{\text{def}}{=} \varphi(2^{-j}D)$, $\hat{S}_j \overset{\text{def}}{=} \chi(2^{-j}D)$ with $\varphi(\xi) \overset{\text{def}}{=} \chi(\frac{\xi}{2}) - \chi(\xi)$, and $\chi$ a non-increasing nonnegative smooth function supported in $B(0, \frac{4}{3})$, and with value 1 on $B(0, \frac{2}{3})$ (see [1], Chap. 2 for more details).

The Littlewood-Paley decomposition is foundational to the definition of Besov spaces.

Definition 2.1. For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, the homogeneous Besov space $\dot{B}^s_{p,1}(\mathbb{R}^n)$ is the set of tempered distributions $f$ satisfying

$$\lim_{j \to -\infty} \|\hat{S}_j f\|_{L^\infty} = 0, \quad \text{and} \quad \|f\|_{\dot{B}^s_{p,1}} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \|\hat{\Delta}_j f\|_{L^p} < \infty.$$ 

For any $f \in \mathcal{S}'(\mathbb{R}^n)$, the lower and higher oscillation parts can be expressed as

$$f^\ell \overset{\text{def}}{=} \sum_{j \leq N_0} \hat{\Delta}_j f \quad \text{and} \quad f^h \overset{\text{def}}{=} \sum_{j > N_0} \hat{\Delta}_j f$$

for a large integer $N_0 \geq 0$. The corresponding truncated semi-norms are defined as follows:

$$\|f\|_{\dot{B}^s_{p,1}} \overset{\text{def}}{=} \|f^\ell\|_{\dot{B}^s_{p,1}} \quad \text{and} \quad \|f\|_{\dot{B}^s_{p,1}} \overset{\text{def}}{=} \|f^h\|_{\dot{B}^s_{p,1}}.$$ 

As we shall work with time-dependent functions valued in Besov spaces, we introduce the norms:

$$\|f\|_{L^q_T(\dot{B}^s_{p,1})} \overset{\text{def}}{=} \left\| \|f(t, \cdot)\|_{\dot{B}^s_{p,1}} \right\|_{L^q(0,T)}.$$ 

Moreover, in the present paper, we frequently use the so-called "time-space" Besov spaces or Chemin-Lerner space first introduced by Chemin and Lerner [1].

Definition 2.2. Let $s \in \mathbb{R}$ and $0 < T \leq +\infty$. We define

$$\|f\|_{L^q_T(\dot{B}^s_{p,1})} \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T \|\hat{\Delta}_j f(t)\|_{L^p} dt \right)^\frac{q}{q}$$

for $q, p \in [1, \infty)$ and with the standard modification for $p, q = \infty$.

Remark 2.3. For any $1 \leq q \leq \infty$, from the Minkowski’s inequality, one can deduce that

$$\|f\|_{L^q_T(\dot{B}^s_{p,1})} \leq \|f\|_{L^q_T(\dot{B}^s_{p,1})}.$$ 

The following lemma describes the way derivatives act on spectrally localized functions.

Lemma 2.4. Let $B$ be a ball and $C$ a ring of $\mathbb{R}^n$. A constant $C$ exists so that for any positive real number $\lambda$, any non-negative integer $k$, any smooth homogeneous function $\sigma$ of degree $m$, and any couple of real numbers $(p, q)$ with $1 \leq p \leq q$, there hold

$$\text{Supp } \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1} \lambda^{k+m\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p},$$

$$\text{Supp } \hat{u} \subset \lambda C \Rightarrow C^{-k-1} \lambda^k \|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p},$$

$$\text{Supp } \hat{u} \subset \lambda C \Rightarrow \|\sigma(D) u\|_{L^q} \leq C_{\sigma,m} \lambda^{m+n\left(\frac{1}{p} - \frac{1}{q}\right)} \|u\|_{L^p}.$$
Next we recall a few nonlinear estimates in Besov spaces which may be obtained by means of paradifferential calculus. Here, we recall the decomposition in the homogeneous context:

\[ uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v), \]  

where \( \hat{T}_u v \triangleq \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u, v) \triangleq \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \hat{\Delta}_j v, \) and \( \hat{\Delta}_j v \triangleq \sum_{|j-j'| \leq 1} \hat{\Delta}_j v. \)

The paraproduct \( \hat{T} \) and the remainder \( \hat{R} \) operators satisfy the following continuous properties.

**Lemma 2.5** ([1]). For all \( s \in \mathbb{R}, \sigma \geq 0, \) and \( 1 \leq p, p_1, p_2 \leq \infty, \) the paraproduct \( \hat{T} \) is a bilinear, continuous operator from \( \dot{B}^{-\sigma}_{p,1} \times \dot{B}^{s-\sigma}_{p_1,1} \) to \( \dot{B}^{-\sigma+s}_{p,1} \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \) The remainder \( \hat{R} \) is bilinear continuous from \( \dot{B}^{s_1}_{p_1,1} \times \dot{B}^{s_2}_{p_2,1} \) to \( \dot{B}^{s_1+s_2}_{p_1,1} \) with \( s_1 + s_2 > 0, \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \)

Next, we give the important product acts on homogenous Besov spaces and composition estimate which will be also often used implicitly throughout the paper.

**Lemma 2.6.** Let \( s_1 \leq \frac{n}{2}, \) \( s_2 \leq \frac{n}{2} \) and \( s_1 + s_2 > 0. \) For any \( u \in \dot{B}^{s_1}_{2,1}(\mathbb{R}^n), \) \( v \in \dot{B}^{s_2}_{2,1}(\mathbb{R}^n), \) we have

\[ \|uv\|_{\dot{B}^{s_1+s_2}_{2,1}} \lesssim \|u\|_{\dot{B}^{s_1}_{2,1}} \|v\|_{\dot{B}^{s_2}_{2,1}}. \]

**Lemma 2.7.** (Lemma 2.100 in [1]) Let \( -1 - \frac{n}{2} < s \leq 1 + \frac{n}{2}, v \in \dot{B}^{s}_{2,1}(\mathbb{R}^n) \) and \( u \in \dot{B}^{0}_{2,1}(\mathbb{R}^n) \) with \( \text{div} \, u = 0. \) Then there holds

\[ \| [\hat{\Delta}_j, u \cdot \nabla] v \|_{L^2} \lesssim d_j 2^{-js} \|\nabla u\|_{\dot{B}^{s}_{2,1}} \|v\|_{\dot{B}^{s}_{2,1}}. \]

**Remark 2.8.** Let \( A(D) \) be a zero-order Fourier multiplier. For any \( 0 \leq \alpha < n + 2, \) from the above lemma, we can easily get

\[ \| [\hat{\Delta}_j A(D), u \cdot \nabla] v \|_{L^2} \lesssim d_j 2^{-(\frac{n}{2}+1-\alpha)j} \|\nabla u\|_{\dot{B}^{s}_{2,1}} \|v\|_{\dot{B}^{s+1-\alpha}_{2,1}}. \]

Finally, we present a new commutator estimate in the Besov spaces which can be regarded as a partial generalization of the commutator estimate of Kato and Ponce [14]. In fact, it was proved in Kato and Ponce [14], for \( s \geq 0 \) and \( 1 < p < \infty, \) that,

\[ \left\| J^s (u \cdot \nabla v) - u \cdot \nabla J^s v \right\|_{L^p} \leq C(\|\nabla u\|_{L^\infty} \|J^{s-1} \nabla v\|_{L^p} + \|\nabla v\|_{L^\infty} \|J^s u\|_{L^p}), \quad (2.3) \]

with \( J^s f \triangleq \mathcal{F}^{-1} [(1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)]. \)

Especially, let \( p = 2 \) and \( s > \frac{n}{2} \) in (2.3), one can get

\[ \| J^s (u \cdot \nabla v) - u \cdot \nabla J^s v \|_{L^2} \leq C(\|\nabla u\|_{H^s} \|v\|_{H^s} + \|u\|_{H^s} \|\nabla v\|_{H^s}). \quad (2.4) \]

The estimate (2.4) was further improved by Fefferman et al. in [10] to

\[ \| \Lambda^s (u \cdot \nabla v) - u \cdot \nabla \Lambda^s v \|_{L^2} \leq C \|\nabla u\|_{H^s} \|v\|_{H^s}, \quad s > \frac{n}{2}. \quad (2.5) \]

Moreover, they exhibited a counterexample in Appendix A in [10] to show that the commutator estimate (2.5) does not hold in the case \( s = \frac{n}{2}, \) at least for \( n = 2, \) even if \( u \) and \( v \) are required to be divergence-free. In the Sobolev spaces, the estimate (2.5) seems optimal as the relation \( H^s \rightarrow L^\infty \) is not valid. However, if we consider the commutator estimate in the Besov spaces, we can verify that the estimate is still valid for any dimension \( n \geq 2 \) if \( s = \frac{n}{2} \) and \( \text{div} \, u = 0. \) More precisely, we shall prove the following commutator estimate:
Lemma 2.9. Let \( n \geq 2 \), \( \Lambda \overset{\text{def}}{=} \sqrt{-\Delta} \) and \(-1 \leq s < n+1\). For any \( v \in \dot{B}^s_{2,1}(\mathbb{R}^n) \), \( \nabla u \in \dot{B}^s_{2,1}(\mathbb{R}^n) \) with \( \text{div} \, u = 0 \), there exists a constant \( C \) such that
\[
\|\Lambda^s(u \cdot \nabla v) - u \cdot \nabla \Lambda^s v\|_{\dot{B}^s_{2,1}} \leq C\|\nabla u\|_{\dot{B}^s_{2,1}} \|v\|_{\dot{B}^s_{2,1}}.
\] (2.6)

Proof. First, according to the definition of the commutator, one can write
\[
\hat{\Delta}_j(\Lambda^s, u \cdot \nabla) v = [\hat{\Delta}_j \Lambda^s, u \cdot \nabla] v - [\hat{\Delta}_j, u \cdot \nabla] \Lambda^s v.
\] (2.7)

Applying Lemma 2.7, we can get the estimate of the second term of the right-hand side of the above equation:
\[
\|[\hat{\Delta}_j, u \cdot \nabla] \Lambda^s v\|_{L^2} \lesssim d_j 2^{-(\frac{n}{2} - s)} \|\nabla u\|_{\dot{B}^s_{2,1}} \|v\|_{\dot{B}^s_{2,1}}, \quad -1 \leq s < n+1.
\] (2.8)

In the following, we mainly concern with the term about \([\hat{\Delta}_j \Lambda^s, u \cdot \nabla] v\). With the aid of the notion of para-products (2.2), we can easily write
\[
[\hat{\Delta}_j \Lambda^s, u \cdot \nabla] v = I_j^1 + I_j^2 + I_j^3,
\]
where
\[
I_j^1 = \sum_{|k-j| \leq 4} [\hat{\Delta}_j \Lambda^s, \hat{S}_{k-1} u \cdot \nabla] \hat{\Delta}_k v,
\]
\[
I_j^2 = \sum_{|k-j| \leq 4} \hat{\Delta}_j \Lambda^s(\hat{\Delta}_k u \cdot \nabla \hat{S}_{k-1} v) - \sum_{k \geq j+1} \hat{\Delta}_k u \cdot \nabla \hat{S}_{k-1} \hat{\Delta}_j \Lambda^s v,
\]
\[
I_j^3 = \sum_{k \geq j-3} \hat{\Delta}_j \Lambda^s(\hat{\Delta}_k u \cdot \nabla \hat{\Delta}_k v) - \sum_{|k-j| \leq 2} \hat{\Delta}_k u \cdot \nabla \hat{\Delta}_k \hat{\Delta}_j \Lambda^s v.
\]

with \( \hat{\Delta}_k = \hat{\Delta}_{k-1} + \hat{\Delta}_k + \hat{\Delta}_{k+1} \).

Since
\[
\hat{\Delta}_j \Lambda^s f = \varphi(2^{-j} \xi) |\xi|^s \hat{f},
\]
one has
\[
\hat{\Delta}_j \Lambda^s f = 2^{j(n+s)} h(2^j \cdot) \ast f, \quad \text{for some } h \in \mathcal{S}.
\]

From the definition of Bony’s decomposition and the mean value theorem, one can write \( I_j^1 \) into
\[
I_j^1 = \sum_{|k-j| \leq 4} [\hat{\Delta}_j \Lambda^s, \hat{S}_{k-1} u_m] \partial_m \hat{\Delta}_k v
\]
\[
= 2^{j(n+s)} \sum_{|k-j| \leq 4} \int_{\mathbb{R}^n} h(2^j y) (\hat{S}_{k-1} u_m(x-y) - \hat{S}_{k-1} u_m(x)) \partial_m \hat{\Delta}_k v(x-y) dy
\]
\[
= -2^{j(n+s)} \sum_{|k-j| \leq 4} \int_{\mathbb{R}^n} h(2^j y) \left( \int_0^1 y \cdot \nabla \hat{S}_{k-1} u_m(x - \tau y) d\tau \right) \partial_m \hat{\Delta}_k v(x-y) dy,
\]
from which we can deduce that
\[
\sum_{j \in \mathbb{Z}} 2^{j(n+s)} \|I_j^1\|_{L^2} \leq C \sum_{j \in \mathbb{Z}} 2^{j(n-1)} \sum_{|k-j| \leq 4} \|\nabla \hat{S}_{k-1} u\|_{L^\infty} \|\nabla \hat{\Delta}_k v\|_{L^2}
\]
\[
\leq C \|\nabla u\|_{L^\infty} \sum_{j \in \mathbb{Z}} 2^{\frac{n}{2} j} \sum_{|k-j| \leq 4} \|\hat{\Delta}_k v\|_{L^2} \leq C \|\nabla u\|_{L^\infty} \|v\|_{\dot{B}^s_{2,1}}.
\] (2.9)
For the last two terms, we get by Hölder’s inequality and Young’s inequality that
\[
\sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} \| I_j^3 \|_{L^2} \leq C \sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} \sum_{|k-j| \leq 4} 2^{j(s+1)} \| \hat{S}_{k-1} v \|_{L^\infty} \| \hat{A}_k u \|_{L^2} \\
+ C \sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} \sum_{k \geq j+1} 2^{j(s+1)} \| \hat{A}_k u \|_2 \| \hat{A}_j v \|_{L^\infty} \\
\leq C \| v \|_{B_{x,t}^{\frac{3}{2} + 1}} \sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} + 1\right)} \left( \sum_{|k-j| \leq 4} \| \hat{A}_k u \|_2 + \sum_{k \geq j+1} \| \hat{A}_k u \|_{L^2} \right) \\
\leq C \| u \|_{B_{x,t}^{\frac{3}{2} + 1}} \| v \|_{B_{x,t}^{\frac{3}{2} + 1}}.
\]
\[ (2.10) \]
Similarly,
\[
\sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} \| I_j^3 \|_{L^2} \leq C \sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} \sum_{k \geq j-3} 2^{j(s+1)} \| \hat{A}_k u \|_2 \| \hat{A}_j v \|_{L^\infty} \\
+ C \sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} \sum_{|k-j| \leq 2} 2^{j(s+1)} \| \hat{A}_k u \|_2 \| \hat{A}_j v \|_{L^\infty} \\
\leq C \| v \|_{B_{x,t}^{\frac{3}{2} + 1}} \sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} + 1\right)} \left( \sum_{k \geq j-3} \| \hat{A}_k u \|_2 + \sum_{|k-j| \leq 2} \| \hat{A}_k u \|_{L^2} \right) \\
\leq C \| u \|_{B_{x,t}^{\frac{3}{2} + 1}} \| v \|_{B_{x,t}^{\frac{3}{2} + 1}}.
\]
\[ (2.11) \]
Adding up (2.9), (2.10), and (2.11), we arrive at
\[
\sum_{j \in \mathbb{Z}} 2^{j \left(\frac{3}{2} - s\right)} [\hat{A}_j \Lambda^s, u \cdot \nabla] v \lesssim \| \nabla u \|_{B_{x,t}^{\frac{3}{2}} \cap B_{x,t}^{\frac{3}{2} + 1}} \| v \|_{B_{x,t}^{\frac{3}{2} + 1}}.
\]
\[ (2.12) \]
Combining (2.8) and (2.12), we can complete the proof of this lemma. \[ \square \]

3. The Proof of the Theorem 1.1

Now, we begin to prove the global small solutions of (1.2). We first denote the initial energy
\[
E_0(0) \overset{\text{def}}{=} \| (u_0, \tau_0) \|_{B_{x,t}^{s+1-\alpha}} + \| \tau_0 \|_{B_{x,t}^{s+1-\alpha} \cap B_{x,t}^{\frac{s+1}{2}-1-\alpha}},
\]
and then define the total energy
\[
E(t) \overset{\text{def}}{=} E_1(t) + E_2(t),
\]
with
\[
E_1(t) \overset{\text{def}}{=} \| (u, \tau) \|_{L_t^\infty(B_{x,t}^{s+1-\alpha})} + \| \tau \|_{L_t^\infty(B_{x,t}^{s+1})},
\]
\[
E_2(t) \overset{\text{def}}{=} \| u \|_{L_t^1(B_{x,t}^{s+1})} + \| (\Lambda^{-1} \nabla \tau) \|_{L_t^1(B_{x,t}^{s+1})} + \| (\Lambda^{-1} \nabla \tau)^h \|_{L_t^1(B_{x,t}^{s+1-\alpha})}.
\]

We shall derive the a priori estimates of $E_1(t)$ and $E_2(t)$ separately.

3.1. The estimates of $E_1(t)$. Applying $\hat{A}_j \nabla$ to the first equation and $\hat{A}_j$ to the second equation in (1.2), respectively, we discover that
\[
\begin{aligned}
\partial_t \hat{A}_j u + u \cdot \nabla \hat{A}_j u + \hat{A}_j \Lambda^a u - \hat{A}_j \nabla \tau = [u \cdot \nabla, \hat{A}_j \nabla] u, \\
\partial_t \hat{A}_j \tau + u \cdot \nabla \hat{A}_j \tau + \hat{A}_j Q(\nabla u) - \hat{A}_j D(u) = [u \cdot \nabla, \hat{A}_j] \tau.
\end{aligned}
\]
\[ (3.1) \]
Applying Lemma 2.6 gives
\( (3.5) \), and summing over
\( \dot{\Delta}_j \tau \), respectively, yields
\[
\frac{1}{2} \frac{d}{dt} \| \dot{\Delta}_j u \|_{L^2}^2 + \| \dot{\Delta}_j \Lambda \dot{\Phi} u \|_{L^2}^2 = \langle \dot{\Delta}_j \mathbb{P} \text{div} \tau, \dot{\Delta}_j u \rangle + \langle [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u, \dot{\Delta}_j u \rangle, \tag{3.2}
\]
and
\[
\frac{1}{2} \frac{d}{dt} \| \dot{\Delta}_j \tau \|_{L^2}^2 = \langle \dot{\Delta}_j D(u), \dot{\Delta}_j \tau \rangle + \langle [u \cdot \nabla, \dot{\Delta}_j \tau], \dot{\Delta}_j \tau \rangle - \langle \dot{\Delta}_j Q(\tau, \nabla u), \dot{\Delta}_j \tau \rangle, \tag{3.3}
\]
in which we have used the following fact:
\[
\langle u \cdot \nabla \dot{\Delta}_j u, \dot{\Delta}_j u \rangle = 0, \quad \langle u \cdot \nabla \dot{\Delta}_j \tau, \dot{\Delta}_j \tau \rangle = 0.
\]
Summing up (3.2) and (3.3) and using the cancellation relation
\[
\langle \dot{\Delta}_j \mathbb{P} \text{div} \tau, \dot{\Delta}_j u \rangle + \langle \dot{\Delta}_j D(u), \dot{\Delta}_j \tau \rangle = 0,
\]
we can get
\[
\frac{1}{2} \frac{d}{dt} \| (\dot{\Delta}_j u, \dot{\Delta}_j \tau) \|_{L^2}^2 + \| \dot{\Delta}_j \Lambda \dot{\Phi} u \|_{L^2}^2
= \langle [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u, \dot{\Delta}_j u \rangle + \langle [u \cdot \nabla, \dot{\Delta}_j \tau], \dot{\Delta}_j \tau \rangle + \langle \dot{\Delta}_j Q(\tau, \nabla u), \dot{\Delta}_j \tau \rangle, \tag{3.4}
\]
which implies
\[
\frac{1}{2} \frac{d}{dt} \| (\dot{\Delta}_j u, \dot{\Delta}_j \tau) \|_{L^2}^2 \lesssim \| [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u, \dot{\Delta}_j u \| + \| [u \cdot \nabla, \dot{\Delta}_j \tau], \dot{\Delta}_j \tau \| + \| \dot{\Delta}_j Q(\tau, \nabla u), \dot{\Delta}_j \tau \|. \tag{3.5}
\]
Hence dividing (formally) (3.5) by \( \| (\dot{\Delta}_j u, \dot{\Delta}_j \tau) \|_{L^2} \), multiplying by \( 2^{(\frac{n}{2}+1-\alpha)j} \), integrating (3.5), and summing over \( j \), we obtain
\[
\| (u, \tau) \|_{L^\infty_t (B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha})} \lesssim \| (u_0, \tau_0) \|_{B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha}} + \sum_{j \in \mathbb{Z}} 2^{(\frac{n}{2}+1-\alpha)j} \| \dot{\Delta}_j Q(\tau, u) \|_{L^1_t(L^2)}
+ \| [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u \|_{L^1_t(L^2)} + \| [u \cdot \nabla, \dot{\Delta}_j \tau] \|_{L^1_t(L^2)}. \tag{3.6}
\]
Applying Lemma 2.6 gives
\[
\sum_{j \in \mathbb{Z}} 2^{(\frac{n}{2}+1-\alpha)j} \| \dot{\Delta}_j Q(\tau, u) \|_{L^1_t(L^2)} \lesssim \int_0^t \| \nabla u \|_{B_{2 \alpha, 1}^{\frac{n}{2}}} \| \tau \|_{B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha}} ds. \tag{3.7}
\]
Next we see, thanks to Lemma 2.7 that
\[
\sum_{j \in \mathbb{Z}} 2^{(\frac{n}{2}+1-\alpha)j} (\| [u \cdot \nabla, \dot{\Delta}_j \mathbb{P}] u \|_{L^1_t(L^2)} + \| [u \cdot \nabla, \dot{\Delta}_j \tau] \|_{L^1_t(L^2)}) \lesssim \int_0^t \| \nabla u \|_{B_{2 \alpha, 1}^{\frac{n}{2}}} \| \tau \|_{B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha}} ds. \tag{3.8}
\]
Taking estimates (3.7) and (3.8) into (3.6) gives
\[
\| (u, \tau) \|_{L^\infty_t (B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha})} \lesssim \| (u_0, \tau_0) \|_{B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha}} + \int_0^t \| \nabla u \|_{B_{2 \alpha, 1}^{\frac{n}{2}}} \| (u, \tau) \|_{B_{2 \alpha, 1}^{\frac{n}{2}+1-\alpha}} ds. \tag{3.9}
\]
From the first equation in (1.2), we can get similarly to estimate (3.6) that
\[
\| \tau \|_{L^\infty_t (B_{2 \alpha, 1}^{\frac{n}{2}})} \lesssim \| \tau_0 \|_{B_{2 \alpha, 1}^{\frac{n}{2}}} + \sum_{j \in \mathbb{Z}} 2^{\frac{n}{2}} \| \dot{\Delta}_j D(u) \|_{L^1_t(L^2)}
+ \sum_{j \in \mathbb{Z}} 2^{\frac{n}{2}} \| [u \cdot \nabla, \dot{\Delta}_j] \tau \|_{L^1_t(L^2)} + \sum_{j \in \mathbb{Z}} 2^{\frac{n}{2}} \| \dot{\Delta}_j Q(\tau, \nabla u) \|_{L^1_t(L^2)}. \tag{3.10}
\]
Thanks to Lemmas 2.6, 2.7, we have
\[
\sum_{j \in \mathbb{Z}} 2^{\frac{3\alpha}{2}} \| \hat{\Delta}_j Q(\tau, \nabla u) \|_{L^1_t(L^2)} + \sum_{j \in \mathbb{Z}} 2^{\frac{3\alpha}{2}} \| \hat{\Delta}_j ([u \cdot \nabla, \hat{\Delta}_j] \tau) \|_{L^1_t(L^2)} \lesssim \int_0^t \| \nabla u \|_{B_{2,1}^{\frac{3\alpha}{2}}} \| \tau \|_{B_{2,1}^{\frac{3\alpha}{2}}} ds. \tag{3.11}
\]

Inserting the above estimate into (3.10), we can get
\[
\| \tau \|_{L^\infty_t(B_{2,1}^{\frac{3\alpha}{2}})} \lesssim \| \tau_0 \|_{B_{2,1}^{\frac{3\alpha}{2}}} + \int_0^t \| u \|_{B_{2,1}^{1+\frac{\alpha}{2}}} ds + \int_0^t \| \nabla u \|_{B_{2,1}^{\frac{3\alpha}{2}}} \| \tau \|_{B_{2,1}^{\frac{3\alpha}{2}}} ds. \tag{3.12}
\]

Together with (3.9) and (3.12), one has
\[
\|(u, \tau)\|_{L^\infty_t(B_{2,1}^{\frac{3\alpha}{2}+1-\alpha})} + \| \tau \|_{L^\infty_t(B_{2,1}^{\frac{3\alpha}{2}})} \lesssim \|(u_0, \tau_0)\|_{B_{2,1}^{\frac{3\alpha}{2}+1-\alpha}} + \| \tau_0 \|_{B_{2,1}^{\frac{3\alpha}{2}}}
+ \int_0^t \| u \|_{B_{2,1}^{1+\frac{\alpha}{2}}} ds + \int_0^t \| \nabla u \|_{B_{2,1}^{\frac{3\alpha}{2}}} (\|(u, \tau)\|_{B_{2,1}^{\frac{3\alpha}{2}+1-\alpha}} + \| \tau \|_{B_{2,1}^{\frac{3\alpha}{2}}}) ds \tag{3.13}
\]
which implies
\[
\mathcal{E}_1(t) \leq C\mathcal{E}_0(0) + \mathcal{E}_2(t) + C\mathcal{E}_1(t)\mathcal{E}_2(t). \tag{3.14}
\]

Due to lack of dissipation of $\tau$ in (1.2), in the above basic energy-type argument, we have to abandon the smooth effect of $u$ tentatively (i.e. a good term $\| \hat{\Delta}_j \Lambda^{\frac{3\alpha}{2}} u \|_{L^2}$ from (3.4) to (3.6)). We will get back the smooth effect of $u$ and the hidden dissipation of $\mathbb{P}\div \tau$ in the next subsection.

3.2. The estimates of $\mathcal{E}_2(t)$. As discussed in the above subsection, the aim of this subsection is to find the dissipation of $u$ and $\tau$. In fact, we can only get the dissipation of $u$ and $\mathbb{P}\div \tau$. A key observation is that the equations about $u$ and $\Lambda^{-1}\mathbb{P}\div \tau$ have a good structure which can make the dissipation transfer from $u$ to $\Lambda^{-1}\mathbb{P}\div \tau$. In order to express our idea more precisely, we first apply project operator $\mathbb{P}$ on both sides of the first two equations in (1.2) to get
\[
\begin{cases}
\partial_t u + \mathbb{P}(u \cdot \nabla u) + \Lambda^{\alpha} u - \mathbb{P}\div \tau = 0, \\
\partial_t \mathbb{P}\div \tau + \mathbb{P}\div (u \cdot \nabla \tau) - \Delta u + \mathbb{P}\div (Q(\tau, \nabla u)) = 0.
\end{cases} \tag{3.15}
\]

In the following, we introduce two magic new quantities:
\[
\phi \overset{\text{def}}{=} \Lambda^{-1}\mathbb{P}\div \tau, \quad w \overset{\text{def}}{=} \Lambda^{\alpha-1} \phi - u.
\]
Let us note that the velocity $u$ has good smoothing effect, we can get information on $\phi$ from information on $w$.

By using the new definitions of $\phi$ and $w$, we deduce from (3.15) that
\[
\begin{cases}
\partial_t \phi + u \cdot \nabla \phi + \Lambda u = f, \\
\partial_t u + u \cdot \nabla u + \Lambda^{\alpha} u - \Lambda \phi = g, \\
\partial_t w + u \cdot \nabla w + \Lambda \phi = F,
\end{cases} \tag{3.16}
\]
in which
\[
f \overset{\text{def}}{=} -[\Lambda^{-1}\mathbb{P}\div, u \cdot \nabla] \tau - \Lambda^{-1}\mathbb{P}\div (Q(\tau, \nabla u)), \quad g \overset{\text{def}}{=} -[\mathbb{P}, u \cdot \nabla] u, \quad F \overset{\text{def}}{=} -[\Lambda^{\alpha-1}, u \cdot \nabla] \phi + \Lambda^{\alpha-1} f - g.
\]
Considering the linear system of (3.16), we find $\phi, u$ satisfy some kind of damped wave equations and have enough decay. The result in [27] is just based on this fact by the time-weighted energy method. However, the method used in [27] seems not to be work in the
lower regularity spaces here. We shall use the localization technique and different commutator argument to get desired a priori estimates.

We begin to apply $\hat{\Delta}_j$ to the first equation in (3.16) that
\begin{equation}
\partial_t \hat{\Delta}_j \phi + u \cdot \nabla \hat{\Delta}_j \phi + \hat{\Delta}_j Au = -[\hat{\Delta}_j, u \cdot \nabla] \phi + \hat{\Delta}_j f.
\end{equation}

Taking $L^2$ inner product of $\hat{\Delta}_j \phi$ with (3.17) and using integrating by parts, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Delta_j \phi\|_{L^2}^2 + \int_{\mathbb{R}^n} \hat{\Delta}_j \Lambda \phi \cdot \hat{\Delta}_j u \, dx = -\int_{\mathbb{R}^n} [\hat{\Delta}_j, u \cdot \nabla] \phi \cdot \hat{\Delta}_j \phi \, dx + \int_{\mathbb{R}^n} \hat{\Delta}_j f \cdot \hat{\Delta}_j \phi \, dx.
\end{equation}

Similarly, from the second and third equations in (3.16)
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Delta_j \Lambda^2 \phi\|_{L^2}^2 - \int_{\mathbb{R}^n} \hat{\Delta}_j \Lambda \phi \cdot \hat{\Delta}_j u \, dx
= -\int_{\mathbb{R}^n} [\hat{\Delta}_j, u \cdot \nabla] u \cdot \hat{\Delta}_j u \, dx + \int_{\mathbb{R}^n} \hat{\Delta}_j g \cdot \hat{\Delta}_j u \, dx,
\end{equation}
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Delta_j \Lambda^2 \phi\|_{L^2}^2 - \int_{\mathbb{R}^n} \hat{\Delta}_j \Lambda \phi \cdot \hat{\Delta}_j u \, dx
= -\int_{\mathbb{R}^n} [\hat{\Delta}_j, u \cdot \nabla] w \cdot \hat{\Delta}_j w \, dx + \int_{\mathbb{R}^n} \hat{\Delta}_j F \cdot \hat{\Delta}_j w \, dx,
\end{equation}
in which we have used the fact:
\begin{equation}
\int_{\mathbb{R}^n} \hat{\Delta}_j \Lambda u \cdot \hat{\Delta}_j u \, dx = \|\Delta_j \Lambda^2 \phi\|_{L^2}^2 - \int_{\mathbb{R}^n} \hat{\Delta}_j \Lambda \phi \cdot \hat{\Delta}_j u \, dx.
\end{equation}

Let $0 < \eta < 1$ be a small constant which will be determined later on. Summing up (3.18)–(3.20) and using the Hölder inequality and Berntson’s lemma, we have
\begin{equation}
\frac{1}{2} \frac{d}{dt} (\|\Delta_j \phi\|_{L^2}^2 + (1 - \eta)\|\Delta_j u\|_{L^2}^2 + \eta\|\Delta_j w\|_{L^2}^2) + (1 - \eta)2^{\alpha j}\|\Delta_j u\|_{L^2}^2 + \eta2^{\alpha j}\|\Delta_j \phi\|_{L^2}^2
\lesssim \|\Delta_j \phi\|_{L^2} (\|\Delta_j, u \cdot \nabla\|_{L^2} + \|\Delta_j f\|_{L^2})
+ \|\Delta_j u\|_{L^2} (\|\Delta_j, u \cdot \nabla\|_{L^2} + \|\Delta_j g\|_{L^2}) + \|\Delta_j w\|_{L^2} (\|\Delta_j, u \cdot \nabla\|_{L^2} + \|\Delta_j F\|_{L^2}).
\end{equation}

By Lemma 2.4 and $w \overset{\text{def}}{=} \Lambda^{\alpha - 1}\phi - u$, we can get
\begin{equation}
\|\Delta_j w\|_{L^2} \leq \|\Delta_j (\Lambda^{\alpha - 1}\phi)\|_{L^2} + \|\Delta_j u\|_{L^2} \leq C2^{(\alpha - 1)j}\|\Delta_j \phi\|_{L^2} + \|\Delta_j u\|_{L^2}.
\end{equation}

For any $2^j \leq N_0$, we can find an $\eta > 0$ small enough such that
\begin{equation}
\|\Delta_j \phi\|_{L^2}^2 + (1 - \eta)\|\Delta_j u\|_{L^2}^2 + \eta\|\Delta_j w\|_{L^2}^2 \geq \frac{1}{C}(\|\Delta_j \phi\|_{L^2}^2 + \|\Delta_j u\|_{L^2}^2).
\end{equation}

From (3.21) and (3.23), one can deduce that
\begin{equation}
\|(\Delta_j u, \Delta_j \phi)\|_{L^2}^2 + 2^{\alpha j} \int_{0}^{t} \|(\Delta_j u, \Delta_j \phi)\|_{L^2} ds
\lesssim \|(\Delta_j u_{0}, \Delta_j \phi_{0})\|_{L^2} + \int_{0}^{t} \|(\Delta_j f, \Delta_j g, \Delta_j F)\|_{L^2} ds
\end{equation}
\begin{equation}
+ \int_{0}^{t} (\|\Delta_j, u \cdot \nabla\|_{L^2} + \|\Delta_j, u \cdot \nabla\|_{L^2} + \|\Delta_j, u \cdot \nabla\|_{L^2} ds).
\end{equation}
Multiplying by $2^{(\frac{5}{2}+1-\alpha)j}$ and summing over $2^j \leq N_0$, we can further get

$$
\|(u^t, \phi^t)\|_{L^\infty_t(B_{2^j}^{s+\frac{\alpha}{2}-\alpha})} + \int_0^t \|(u^t, \phi^t)\|_{B_{2^j}^{s+\frac{\alpha}{2}-\alpha}} ds
\lesssim \|(u_0^t, \phi_0^t)\|_{B_{2^j}^{s+\frac{\alpha}{2}-\alpha}} + \int_0^t \|(f, g)^t\|_{B_{2^j}^{s+\frac{\alpha}{2}-\alpha}} ds
+ \int_0^t \sum_{2^j \leq N_0} 2^{(1+\frac{\alpha}{2}-\alpha)j} \|(\hat{\Delta}^j, u \cdot \nabla)\phi\|_{L^2} + \|[\hat{\Delta}^j, u \cdot \nabla]u\|_{L^2} + \|[\hat{\Delta}^j, u \cdot \nabla]\Lambda^{\alpha-1}\phi\|_{L^2}) ds.
$$

(3.25)

In the following, we mainly concern the estimates for the high frequency part of the solution. We get similarly to (3.19), (3.20) that

$$
\frac{1}{2} \frac{d}{dt} \|\hat{\Delta}^j u\|^2_{L^2} + \|\hat{\Delta}^j \Lambda^{\frac{\alpha}{2}} u\|^2_{L^2} - \|\hat{\Delta}^j \Lambda^{\frac{\alpha}{2}} u\|^2_{L^2}
= -\int_{\mathbb{R}^n} [\hat{\Delta}^j, u \cdot \nabla] u \cdot \hat{\Delta}^j u \, dx + \int_{\mathbb{R}^n} \hat{\Delta}^j \Lambda^{2-\alpha} w \cdot \hat{\Delta}^j u \, dx + \int_{\mathbb{R}^n} \hat{\Delta}^j g \cdot \hat{\Delta}^j u \, dx,
$$

(3.26)

$$
\frac{1}{2} \frac{d}{dt} \|\hat{\Delta}^j w\|^2_{L^2} + \|\hat{\Delta}^j \Lambda^{1-\frac{\alpha}{2}} w\|^2_{L^2}
= -\int_{\mathbb{R}^n} [\hat{\Delta}^j, u \cdot \nabla] w \cdot \hat{\Delta}^j w \, dx - \int_{\mathbb{R}^n} \hat{\Delta}^j \Lambda^{2-\alpha} u \cdot \hat{\Delta}^j w \, dx + \int_{\mathbb{R}^n} \hat{\Delta}^j F \cdot \hat{\Delta}^j w \, dx,
$$

(3.27)

in which we have used:

$$
\int_{\mathbb{R}^n} \hat{\Delta}^j \Lambda^\alpha \cdot \hat{\Delta}^j u \, dx = \|\hat{\Delta}^j \Lambda^{1-\frac{\alpha}{2}} u\|^2_{L^2} + \int_{\mathbb{R}^n} \Lambda^{2-\alpha} \hat{\Delta}^j w \cdot \hat{\Delta}^j u \, dx
$$

$$
\int_{\mathbb{R}^n} \hat{\Delta}^j \Lambda^\alpha \cdot \hat{\Delta}^j w \, dx = \|\hat{\Delta}^j \Lambda^{1-\frac{\alpha}{2}} w\|^2_{L^2} + \int_{\mathbb{R}^n} \Lambda^{2-\alpha} \hat{\Delta}^j u \cdot \hat{\Delta}^j w \, dx.
$$

Summing up (3.26), (3.27) and choosing a suitable $N_0$ (for example $N_0 = 2^{(\frac{\alpha}{2}-1)}$) such that for any $2^j \geq N_0$ there holds $C(2^{\alpha j} - 2^{(2-\alpha)j}) \geq \frac{C}{2}(2^{(2-\alpha)j})$, then we have

$$
\frac{1}{2} \frac{d}{dt} (\|\hat{\Delta}^j u\|^2_{L^2} + \|\hat{\Delta}^j w\|^2_{L^2}) + 2^{(2-\alpha)j} (\|\hat{\Delta}^j u\|^2_{L^2} + \|\hat{\Delta}^j w\|^2_{L^2})
\lesssim \|\hat{\Delta}^j u\|_{L^2} (\|\hat{\Delta}^j g\|_{L^2} + \|[\hat{\Delta}^j, u \cdot \nabla]u\|_{L^2} + (\|\hat{\Delta}^j F\|_{L^2} + \|[\hat{\Delta}^j, u \cdot \nabla]w\|_{L^2})) \|\hat{\Delta}^j w\|_{L^2}.
$$

(3.28)

Multiplying by $2^{(\frac{5}{2}+1-\alpha)j}$ and summing over $2^j \geq N_0$ imply that

$$
\|(u^h, w^h)\|_{L^\infty_t(B_{2^j}^{s+\frac{\alpha}{2}-\alpha})} + \int_0^t \|(u^h, w^h)\|_{B_{2^j}^{s+\frac{\alpha}{2}-\alpha}} ds
\lesssim \|(u_0^h, w_0^h)\|_{B_{2^j}^{s+1-\alpha}} + \int_0^t \|(f, g)^h\|_{B_{2^j}^{s+1-\alpha}} ds
+ \int_0^t \sum_{2^j \geq N_0} 2^{(\frac{5}{2}+1-\alpha)j} (\|[\hat{\Delta}^j, u \cdot \nabla]u\|_{L^2} + \|[\hat{\Delta}^j, u \cdot \nabla]w\|_{L^2}) ds.
$$

(3.29)

Due to $\phi = \Lambda^{1-\alpha} w + \Lambda^{1-\alpha} u$ and $1 < \alpha \leq 2$, we can get

$$
\|\phi^h\|_{L^\infty_t(B^{s+\frac{\alpha}{2}-\alpha}_{2^j})} = \|(\Lambda^{1-\alpha} w + \Lambda^{1-\alpha} u)^h\|_{L^\infty_t(B^{s+\frac{\alpha}{2}-\alpha}_{2^j})} \lesssim \|(u^h, w^h)\|_{L^\infty_t(B^{s+\frac{\alpha}{2}-\alpha}_{2^j})},
$$

$$
\|\phi^h\|_{L^1(B^{s+2-\alpha}_{2^j})} = \|(\Lambda^{1-\alpha} w + \Lambda^{1-\alpha} u)^h\|_{L^1(B^{s+2-\alpha}_{2^j})} \lesssim \|(u^h, w^h)\|_{L^1(B^{s+3-2\alpha}_{2^j})},
$$
from which and (3.29), we yield
\[ ||u^h||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + ||\phi^h||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \int_0^t (||u^h||_{B^{\frac{3}{2}}_{2,1}} + ||\phi^h||_{B^{\frac{3}{2}}_{2,1}}) ds \]
\[ \lesssim ||u_0^h||_{B^{\frac{3}{2}}_{2,1}} + ||\phi_0^h||_{B^{\frac{3}{2}}_{2,1}} + \int_0^t ||(F, g)^h||_{B^{\frac{3}{2}}_{2,1}} ds + \int_0^t \sum_{2^j \geq N_0} 2(\frac{3}{2} + 1 - \alpha)j ||[\hat{\Delta}_j, u \cdot \nabla]u||_{L^2} ds. \] (3.30)

From the equation \( \partial_t u + u \cdot \nabla u + \Lambda^\alpha u - \Lambda \phi = g \) and the smoothing effect of the parabolic equation with fractional derivative, one can find that we have lost "2\(\alpha - 2"\) order regularity for \(u^h\) in (3.30). The reason why this case happened is that \(u\) and \(\phi\) have different order smoothing effect in the high frequencies part. Although we have lost "2\(\alpha - 2"\) order regularity for \(u^h\) in (3.30), we have obtained the whole dissipation of \(\phi^h\) luckily. From the second equation in (3.16), we can get similarly to (3.29) that
\[ ||u^h||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \int_0^t ||u^h||_{B^{\frac{3}{2}}_{2,1}} ds \]
\[ \lesssim ||u_0^h||_{B^{\frac{3}{2}}_{2,1}} + \int_0^t ||\phi^h||_{B^{\frac{3}{2}}_{2,1}} ds + \int_0^t ||g^h||_{B^{\frac{3}{2}}_{2,1}} ds + \int_0^t \sum_{2^j \geq N_0} 2(\frac{3}{2} + 1 - \alpha)j ||[\hat{\Delta}_j, u \cdot \nabla]u||_{L^2} ds. \] (3.31)

Multiplying by a suitable large constant on both sides of (3.30) and then plusing (3.31), we can get for the high frequency of the solutions
\[ ||u^h||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + ||\phi^h||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \int_0^t (||u^h||_{B^{\frac{3}{2}}_{2,1}} + ||\phi^h||_{B^{\frac{3}{2}}_{2,1}}) ds \]
\[ \lesssim ||u_0^h||_{B^{\frac{3}{2}}_{2,1}} + ||\phi_0^h||_{B^{\frac{3}{2}}_{2,1}} + \int_0^t ||(F, g)^h||_{B^{\frac{3}{2}}_{2,1}} ds + \int_0^t \sum_{2^j \geq N_0} 2(\frac{3}{2} + 1 - \alpha)j ||[\hat{\Delta}_j, u \cdot \nabla]u||_{L^2} ds. \] (3.32)

Combining with (3.25) and (3.32), using the definition of \(w \equiv \Lambda^{-1} \phi - u\), we get
\[ ||(u, \phi^f)||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + ||\phi^f||_{L^\infty_t(B^{\frac{3}{2}}_{2,1})} + \int_0^t ||\phi^h||_{B^{\frac{3}{2}}_{2,1}} ds + \int_0^t ||(u, \phi^f)||_{B^{\frac{3}{2}}_{2,1}} ds \]
\[ \lesssim ||(u_0, \phi_0^f)||_{B^{\frac{3}{2}}_{2,1}} + ||\phi_0^f||_{B^{\frac{3}{2}}_{2,1}} + \int_0^t \sum_{2^j \geq N_0} 2(\frac{3}{2} + 1 - \alpha)j ||[\hat{\Delta}_j, u \cdot \nabla]\phi||_{L^2} ds + \int_0^t \sum_{2^j \geq N_0} \sum_{j \in \mathbb{Z}} 2(\frac{3}{2} + 1 - \alpha)j ||[\hat{\Delta}_j, u \cdot \nabla]u||_{L^2} ds. \] (3.33)

Next, we give the estimates to the terms of the right-hand side of the above inequality. By a simple computation, one has
\[ \hat{\Delta}_j([\mathbb{P}, u \cdot \nabla]u) = [\hat{\Delta}_j \mathbb{P}, u \cdot \nabla]u - [\hat{\Delta}_j, u \cdot \nabla]u. \] (3.34)
As the operator $\mathbb{P}$ is a zero-order Fourier multiplier, we can deduce from the Lemma 2.7 and the Remark 2.8 that

$$\|g\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} = \sum_{j \in \mathbb{Z}} 2^{(\frac{n}{2}+1-\alpha)j} \|\hat{\Delta}_j([\mathbb{P}, u \cdot \nabla]u)\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} 2^{(\frac{n}{2}+1-\alpha)j}(\|\hat{\Delta}_j[\mathbb{P}, u \cdot \nabla]u\|_{L^2} + \|\hat{\Delta}_j(u \cdot \nabla)u\|_{L^2}) \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|u\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}}. \quad (3.35)$$

Due to the operator $\Lambda^{-1}\mathbb{P}\text{div}$ is also a zero-order Fourier multiplier, we can get by a similar derivation of (3.35) that

$$\|[\Lambda^{-1}\mathbb{P}\text{div}, u \cdot \nabla]\tau\|_{B_{2,1}^{\frac{n}{2}}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}}}, \quad (3.36)$$

$$\|[\Lambda^{-1}\mathbb{P}\text{div}, u \cdot \nabla]\tau\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}}. \quad (3.37)$$

From Lemma 2.6, we have

$$\|\Lambda^{-1}\mathbb{P}\text{div}(Q(\tau, \nabla u))\|_{B_{2,1}^{\frac{n}{2}}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}}}, \quad (3.38)$$

$$\|\Lambda^{-1}\mathbb{P}\text{div}(Q(\tau, \nabla u))\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}}. \quad (3.39)$$

The combination of (3.36)–(3.38) gives

$$\|f\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}}, \quad \|\Lambda^{-1}f\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|f\|_{B_{2,1}^{\frac{n}{2}}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}}}. \quad (3.40)$$

By Lemma 2.9, we have

$$\|[\Lambda^{-1}, u \cdot \nabla]\phi\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\phi\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} \|\tau\|_{B_{2,1}^{\frac{n}{2}}},$$

from which and (3.35), (3.40), we can get

$$\|F\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} ((u\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} + \|\tau\|_{B_{2,1}^{\frac{n}{2}}}). \quad (3.41)$$

By using Lemma 2.7, we can get

$$\sum_{j \in \mathbb{Z}} 2^{(\frac{n}{2}+1-\alpha)j} \|[\hat{\Delta}_j, u \cdot \nabla]\phi\|_{L^2} + \|[\hat{\Delta}_j, u \cdot \nabla]u\|_{L^2} + \|[\hat{\Delta}_j, u \cdot \nabla]\Lambda^{-1}\phi\|_{L^2} \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} (\|\phi\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} + \|u\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} + \|\Lambda^{-1}\phi\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}}) \lesssim \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} ((u, \tau)\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} + \|\tau\|_{B_{2,1}^{\frac{n}{2}}}). \quad (3.42)$$

Inserting (3.35), (3.40), (3.41) and (3.42) into (3.33) gives

$$\|(u, \phi^t)\|_{L^\infty_t(B_{2,1}^{\frac{n}{2}+1-\alpha})} + \|\phi^t\|_{L^\infty_t(B_{2,1}^{\frac{n}{2}})} + \int_0^t \|\phi^h\|_{B_{2,1}^{\frac{n}{2}+2-\alpha}}ds + \int_0^t \|(u, \phi^t)\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}}ds \lesssim \|(u_0, \tau_0^t)\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} + \|\tau_0^h\|_{B_{2,1}^{\frac{n}{2}}} + \int_0^t \|\nabla u\|_{B_{2,1}^{\frac{n}{2}}} ((u, \tau)\|_{B_{2,1}^{\frac{n}{2}+1-\alpha}} + \|\tau\|_{B_{2,1}^{\frac{n}{2}}})ds \quad (3.43)$$

which implies that

$$\mathcal{E}_2(t) \leq C\mathcal{E}_0(0) + C\mathcal{E}_1(t)\mathcal{E}_2(t). \quad (3.44)$$
3.3. Proof of the Theorem 1.1. Now, let us give the proof of Theorem 1.1. Multiplying by a suitable large constant on both sides of (3.44) and then plusing (3.14), we can get
\[ \mathcal{E}(t) \leq C_1 \mathcal{E}(0) + C_1 \mathcal{E}(t) \mathcal{E}_2(t) \leq C_2 \mathcal{E}(0) + C_2 \mathcal{E}_2(t) \] (3.45)
for some positive constant $C_2$.

Under the setting of initial data in Theorem 1.1, there exists a positive constant $C_3$ such that $\mathcal{E}(0) + C_2 \mathcal{E}_0(0) \leq C_3 \varepsilon$. Due to the local existence result which can be achieved similarly to [4], there exists a positive time $T$ such that
\[ \mathcal{E}(t) \leq 2C_3 \varepsilon, \quad \forall \ t \in [0, T]. \] (3.46)

Let $T^*$ be the largest possible time of $T$ for what (3.46) holds. Now, we only need to show $T^* = \infty$. By the estimate of total energy (3.45), we can use a standard continuation argument to show that $T^* = \infty$ provided that $\varepsilon$ is small enough. We omit the details here. Hence, we finish the proof of Theorem 1.1.

4. The proof of the Theorem 1.7

In this section, we prove the second theorem of the present paper. The local wellposedness of (1.4) with $(u_0, \tau_0) \in H^s(\mathbb{R}^2)$, $(\sigma > 4)$ can be obtained by using a standard energy argument. Here, we omit it. To proved the global wellposedness, it suffices to obtain a global a priori bound for $\|(u_\nu, \tau_\nu)\|_{H^s}$. In fact, we only need to bound
\[ \int_0^T \| (\nabla u_\nu, \nabla \tau_\nu) \|_{L^\infty} dt < \infty. \]

To do this, we will prove $\|(u_\nu, \tau_\nu)\|_{H^s} < \infty$ for any $0 \leq s \leq \sigma - 2$. By Theorem 1.6, one can deduce that $(u, \tau) \in C([0, T]; H^s(\mathbb{R}^2))$. Thus, in the following, we main concern with the bound of $\|(u_\nu - u, \tau_\nu - \tau)\|_{H^s}$. The proof in the following borrows some idea from [24].

Denote $\bar{u} = u_\nu - u$, $\bar{\pi} = \pi_\nu - \pi$, and $\bar{\tau} = \tau_\nu - \tau$. From (1.4) and (1.5), we can deduce that $(\bar{u}, \bar{\tau})$ satisfies the following equations:

\[
\begin{aligned}
&\begin{cases}
\partial_t \bar{u} + u + \nabla \bar{u} + \bar{u} \cdot \nabla (u + \bar{u}) - \nu \Delta (u + \bar{u}) + \nabla \bar{\pi} - \text{div} \bar{\tau} = 0, \\
\partial_t \bar{\tau} + u \cdot \nabla \bar{\tau} + \bar{u} \cdot \nabla (\tau + \bar{\tau}) - \Delta \bar{\tau} - D(\bar{u}) = 0, \\
\text{div} \bar{u} = 0, \\
(\bar{u}, \bar{\tau})|_{t=0} = (0, 0).
\end{cases}
\end{aligned}
\] (4.1)

Denote
\[ J^s f \overset{\text{def}}{=} \mathcal{F}^{-1}[(1 + |\xi|^2)^s \hat{f}(\xi)]. \]

Taking the $L^2$ inner product with $J^s \bar{u}, J^s \bar{\tau}$ to the first and second equation of (4.1), respectively, we have
\[
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \| \bar{u} \|_{H^s}^2 + \nu \| \nabla \bar{u} \|_{H^s}^2 \\
&= \int_{\mathbb{R}^2} J^s(\text{div} \bar{\tau}) \cdot J^s \bar{u} \, dx - \nu \int_{\mathbb{R}^2} J^s(\Delta \bar{u}) \cdot J^s \bar{u} \, dx \\
&= - \int_{\mathbb{R}^2} J^s(u \cdot \nabla \bar{u}) \cdot J^s \bar{u} \, dx - \int_{\mathbb{R}^2} J^s(\bar{u} \cdot \nabla u) \cdot J^s \bar{u} \, dx - \int_{\mathbb{R}^2} J^s(\bar{u} \cdot \nabla \bar{u}) \cdot J^s \bar{u} \, dx,
\end{aligned}
\] (4.2)
and
\[
\frac{1}{2} \frac{d}{dt} \|\tau\|^2_{H^s} + \|\nabla \tau\|^2_{H^s} = \int_{\mathbb{R}^2} J^s(D(\tilde{u})) \cdot J^s \tilde{\tau} \, dx - \int_{\mathbb{R}^2} J^s(u \cdot \nabla \tau) \cdot J^s \tilde{\tau} \, dx \\
- \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tau) \cdot J^s \tilde{\tau} \, dx - \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tau) \cdot J^s \tilde{\tau} \, dx. \tag{4.3}
\]

Summing up (4.2) and (4.3) and using the following cancelation,
\[
\int_{\mathbb{R}^2} J^s(\text{div} \, \tilde{\tau}) \cdot J^s \tilde{u} \, dx + \int_{\mathbb{R}^2} J^s(D(\tilde{u})) \cdot J^s \tilde{\tau} \, dx = 0,
\]
we get
\[
\frac{1}{2} \frac{d}{dt} \|\tilde{u}, \tilde{\tau}\|_{H^s}^2 \leq \nu \int_{\mathbb{R}^2} J^s(\Delta u) \cdot J^s \tilde{u} \, dx \\
+ \int_{\mathbb{R}^2} J^s(u \cdot \nabla \tilde{\tau}) \cdot J^s \tilde{u} \, dx - \int_{\mathbb{R}^2} J^s(u \cdot \nabla \tilde{\tau}) \cdot J^s \tilde{\tau} \, dx \\
+ \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tilde{u}) \cdot J^s \tilde{u} \, dx - \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tilde{u}) \cdot J^s \tilde{\tau} \, dx \\
+ \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla u) \cdot J^s \tilde{u} \, dx - \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tilde{u}) \cdot J^s \tilde{\tau} \, dx. \tag{4.4}
\]
Applying the Hölder inequality, we have
\[
\left| \nu \int_{\mathbb{R}^2} J^s(\Delta u) \cdot J^s \tilde{u} \, dx \right| \leq C \nu \|u\|_{H^{s+2}} \|\tilde{u}\|_{H^s}. \tag{4.5}
\]
Due to \(\text{div} \, u = 0\), one can use integrating by part to get
\[
\int_{\mathbb{R}^2} u \cdot \nabla J^s \tilde{u} \cdot J^s \tilde{\tau} \, dx = \int_{\mathbb{R}^2} u \cdot \nabla J^s \tilde{\tau} \cdot J^s \tilde{\tau} \, dx,
\]
from which we can deduce that
\[
\left| \int_{\mathbb{R}^2} J^s(u \cdot \nabla \tilde{\tau}) \cdot J^s \tilde{u} \, dx + \int_{\mathbb{R}^2} J^s(u \cdot \nabla \tilde{\tau}) \cdot J^s \tilde{\tau} \, dx \right|
= \left| \int_{\mathbb{R}^2} (J^s(u \cdot \nabla \tilde{\tau}) - u \cdot \nabla J^s \tilde{u}) \cdot J^s \tilde{u} \, dx + \int_{\mathbb{R}^2} (J^s(u \cdot \nabla \tilde{\tau}) - u \cdot \nabla J^s \tilde{\tau}) \cdot J^s \tilde{\tau} \, dx \right|
\leq C \left( \|J^s(u \cdot \nabla \tilde{\tau}) - u \cdot \nabla J^s \tilde{u}\|_{L^2} \|J^s \tilde{u}\|_{L^2} + \|J^s(u \cdot \nabla \tilde{\tau}) - u \cdot \nabla J^s \tilde{\tau}\|_{L^2} \|J^s \tilde{\tau}\|_{L^2} \right)
\leq C \left( \|\nabla u\|_{L^\infty} \|J^s \tilde{u}\|_{L^2} + \|\nabla \tilde{\tau}\|_{L^\infty} \|J^s u\|_{L^2} \right) \|J^s \tilde{\tau}\|_{L^2}
\leq C \|u\|_{H^{s+2}} (\|\tilde{u}\|^2_{H^s} + \|\tilde{\tau}\|^2_{H^s}) \tag{4.6}
\]
where we have used the following commutator estimate of Kato and Ponce [14]:
\[
\|J^s(u \cdot \nabla v) - u \cdot \nabla J^s v\|_{L^2} \leq C \left( \|\nabla u\|_{L^\infty} \|J^{s-1} \nabla v\|_{L^2} + \|\nabla v\|_{L^\infty} \|J^s u\|_{L^2} \right), \forall s \geq 0.
\]
Exact the same line as (4.6), one has
\[
\left| \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tilde{u}) \cdot J^s \tilde{u} \, dx + \int_{\mathbb{R}^2} J^s(\tilde{\tau} \cdot \nabla \tilde{u}) \cdot J^s \tilde{\tau} \, dx \right| \leq C \|\tilde{u}\|_{H^s} (\|\tilde{u}\|^2_{H^s} + \|\tilde{\tau}\|^2_{H^s}). \tag{4.7}
\]
Using the fact that $H^s(\mathbb{R}^2)$ is an algebra for any $s > 1$, thus, we can get
\[
\left| \int_{\mathbb{R}^2} J^s(\tilde{u} \cdot \nabla u) \cdot J^s\tilde{u} \, dx + \int_{\mathbb{R}^2} J^s(\tilde{u} \cdot \nabla \tau) \cdot J^s\tilde{\tau} \, dx \right|
\leq C(\|J^s(\tilde{u} \cdot \nabla u)\|_{L^2} \|J^s\tilde{u}\|_{L^2} + \|J^s(\tilde{u} \cdot \nabla \tau)\|_{L^2} \|J^s\tilde{\tau}\|_{L^2})
\leq C(\|\nabla u\|_{H^s} \|\tilde{u}\|_{H^s}^2 + \|\nabla \tau\|_{H^s} \|\tilde{u}\|_{H^s} \|\tilde{\tau}\|_{H^s})
\leq C(\|u\|_{H^{s+1}} \|\tilde{u}\|_{H^s}^2 + \|\tau\|_{H^{s+1}} \|\tilde{u}\|_{H^s} \|\tilde{\tau}\|_{H^s}).
\] (4.8)

Inserting (4.5)–(4.8) into (4.4) and letting
\[
G(t) \overset{\text{def}}{=} \|\tilde{u}\|_{H^s} + \|\tilde{\tau}\|_{H^s}, \quad M(t) \overset{\text{def}}{=} \|u\|_{H^{s+1}} + \|\tau\|_{H^{s+1}},
\]
we can get
\[
\frac{d}{dt} G(t) \leq \nu \|u(t)\|_{H^{s+2}} + C_1 M(t) G(t) + C_2 G^2(t).
\]

Multiplying by $e^{-C_1 \int_0^t M(\xi') \, d\xi'}$ on both hand side of the above inequality, we have
\[
\frac{d}{dt} (G e^{-C_1 \int_0^t M(\xi') \, d\xi'}) \leq \nu \|u\|_{H^{s+2}} e^{-C_1 \int_0^t M(\xi') \, d\xi'} + C_2 G^2 e^{-C_1 \int_0^t M(\xi') \, d\xi'}.
\] (4.9)

By Lemma 1.3 of [6], if we set
\[
\nu_0 = \left(8C_2 \int_0^T \|u(t')\|_{H^s} e^{C_1 \int_0^T M(\xi) \, d\xi} \, dt' \right)^{-1},
\]
then, for $0 < \nu \leq \nu_0$ and $0 \leq t \leq T$, we infer from (4.9) that
\[
G(t) \leq 12\nu \int_0^T \|u(t')\|_{H^s} e^{C_1 \int_0^T M(\xi) \, d\xi} \, dt'.
\] (4.10)

From (4.10), one can deduce that
\[
\|(u_{\nu'}, \tau_{\nu'}) - (u, \tau)\|_{H^s} \leq C(T)\nu,
\]
where $C(T)$ is a constant dependent on $T$ and $\|(u, \tau)\|_{L^\infty([0,T];H^s)}$.

Consequently, we have completed the proof of Theorem 1.7.

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