THE MFF SINGULAR VECTORS IN TOPOLOGICAL CONFORMAL THEORIES

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ABSTRACT
It is argued that singular vectors of the topological conformal (twisted $N=2$) algebra are identical with singular vectors of the $sl(2)$ Kač–Moody algebra. An arbitrary matter theory can be dressed by additional fields to make up a representation of either the $sl(2)$ current algebra or the topological conformal algebra. The relation between the two constructions is equivalent to the Kazama–Suzuki realisation of a topological conformal theory as $sl(2) \oplus u(1)/u(1)$. The Malikov–Feigin–Fuchs (MFF) formula for the $sl(2)$ singular vectors translates into a general expression for the topological singular vectors. The MFF/topological singular states are observed to vanish in Witten’s free-field construction of the (twisted) $N=2$ algebra, derived from the Landau–Ginzburg formalism.
1 Introduction

Associated with singular (‘null’) states in the representations of symmetry algebras encountered in conformal theories, are the irreducible models whose correlators satisfy decoupling equations – i.e. the equations ensuring the decoupling of a given null vector. The decoupling equations do in fact define a class of conformal models \([1, 2, 3]\). Besides, singular vectors and their decoupling equations have been used to relate conformal field-theoretic constructions to other structures in physics and mathematics \([4, 5, 6, 7, 8]\). Singular vectors of the Virasoro algebra (corresponding to the minimal models) and its supersymmetric extensions have been studied in some detail \([9, 10, 11, 12, 13, 14, 16, 17, 18]\), as were singular vectors of the \(sl(2)\) Kač–Moody algebra \([19, 20]\) and their relation with the Virasoro ones \([11]\–[15]\).

In this paper we consider singular vectors of the \(N = 2\) supersymmetry algebra in its topological (twisted) version \([21, 22]\). The ‘topological’ singular vectors are interesting for a number of (not independent) reasons. First, as we will see below, they can be identified with singular vectors of \(sl(2)\) Kač–Moody algebra (the latter being given explicitly by the Malikov–Feigin–Fuchs (MFF) construction \([20]\)). At the same time they have been related \([7, 8]\) to Virasoro constraints of the type of those encountered in matrix models. Both these features have to do with the fact that topological conformal theories can be reduced to ordinary minimal matter in such a way that the topological singular vectors map into the ‘minimal’ ones. Conversely, it is possible to represent a topological conformal model as a result of dressing the minimal matter with (free) gravity multiplet (‘Liouville’ and ghosts). A very interesting generalisation of this and similar constructions consists in the idea \([27]\) of universal string theory \([27, 28, 29]\). Moreover, a similar correspondence might exist between ordinary matter theories and \(sl(2)\) WZW models (based on the fact that the \(sl(2)\) algebra can be built by dressing an ordinary matter theory). The hidden connection, so far observed at the level of singular vectors, between the topological and the \(sl(2)\) algebras, may be interesting, in particular, in the context of universal string theory.

The topological conformal algebra with topological central charge \(c\),

\[
\begin{align*}
[\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n}, \\
[\mathcal{H}_m, \mathcal{H}_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m - n)\mathcal{G}_{m+n}, \\
[\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, \\
[\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2 + m)\delta_{m+n,0},
\end{align*}
\]

admits, for \(c \neq 3\), a construction \([23, 8, 24, 25, 26]\) in terms of ordinary (non-supersymmetric) matter with central charge

\[
d = \frac{(c + 1)(c + 6)}{c - 3},
\]

an auxiliary scalar field, and a couple of \(bc\) ghosts. To be more precise, two such constructions exist \([23, 8]\), one of which reproduces in the (matter) + (scalar \(\phi\)) sector the DDK recipe \([30, 31]\) for dressing matter with the Liouville, which will allow us to call the \(\phi\) scalar the Liouville. The second way to build up the topological algebra out of central charge-\(d\) matter, ghosts, and the Liouville involves ghosts of dimension (‘spin’) 1 (rather than 2 as in the DDK-related version).
This will be referred to as the ‘mirror’ gravity coupled to matter, and it this version that is interesting in application to singular states. The two versions follow as different twistings \[32, 8\] of a free-field realisation of the proper \(N=2\) algebra.

The topological singular states, which we are going to study, are built upon chiral primary states \[33\] \(|\Psi\rangle \equiv |h\rangle\) of the algebra \([1.1]:\)

\[
\begin{align*}
Q_0|h\rangle &= 0, & G_0|h\rangle &= 0, & \mathcal{H}_0|h\rangle &= h|h\rangle, \\
\mathcal{L}_{\geq 0}|h\rangle &= \mathcal{H}_{\geq 1}|h\rangle = G_{\geq 1}|h\rangle = Q_{\geq 1}|h\rangle = 0,
\end{align*}
\]

(1.3)

(the topological \(U(1)\) charge \(h\) being the only parameter that distinguishes between chiral primary states). Singular vectors are those descendants of \(|h\rangle\) that satisfy the highest-weight conditions (1.3) except for the chirality \((G_0)\) and zero-dimension \((\mathcal{L}_0)\). In the following, we will show that they coincide with the \(sl(2)\) Kač–Moody singular states. This points to a closer correspondence between the topological and WZW theories, as is also suggested by a number of recent results \[34, 35, 36\], and might also be interpreted in terms of the appearance of the \(sl(2)\) symmetry in quantum gravity \[37\].

In the next section we recall a free-field construction of the algebra \([1.1]\) in terms of matter, ‘Liouville’, and ghosts. Then, in section 3, we explicitly construct several lowest topological singular vectors. Section 4 contains the necessary preparations for the subsequent evaluation of these singular vectors in terms of the \(sl(2)\) algebra. In section 5 we point out a construction, analogous to the one producing topological algebra via dressing ordinary matter, that allows us to build up the \(sl(2)\) currents out of ordinary matter and a couple of scalars. This construction is basically a variation on the theme of hamiltonian reduction and Wakimoto bosonisation, but we would like to stress that the matter enters only through its Virasoro generators and need not be specified any further. In section 6 we recall briefly what the singular vectors of \(sl(2)\) look like (the Malikov–Feigin–Fuchs formulae \[20\]). Finally, in section 7 we state the main result, namely that the topological vectors from section 3, when evaluated according to the recipe given in section 4, become exactly the MFF states of section 4. In fact, the topological singular states can also be used to ‘resolve’ the MFF ones in the classical limit \(|k| = \infty\), as discussed in section 8, where we also make comparison with a different kind of a free-field construction for the topological algebra, proposed recently by Witten \[38\] on the basis of the Landau–Ginzburg formalism. In section 9 we use this free-field construction to evaluate the MFF/topological singular states in terms of the Landau–Ginzburg theory. Section 10 contains several concluding remarks.

2 Topological algebra: free-field constructions

Both the ‘mirror’ and the ‘ordinary’ versions of the construction presenting the topological algebra as matter dressed with the gravity multiplet, can be obtained by performing the two possible twistings of a similar realisation for the proper (untwisted) \(N=2\) algebra

\[
\begin{align*}
[\mathcal{L}_m, \mathcal{L}_n] &= (m - n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,n,0}, \\
[\mathcal{H}_m, \mathcal{H}_n] &= \frac{c}{3}m\delta_{m,n,0}, \\
[\mathcal{L}_m, G^+_r] &= \left(\frac{m}{2} - r\right)G^+_m, \\
[\mathcal{H}_m, G^+_r] &= \pm G^+_m, \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n}, \\
\{G^+_r, G^+_s\} &= 2\mathcal{L}_{r+s} - (r - s)\mathcal{H}_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0},
\end{align*}
\]

(2.1)
For this algebra, a representation in terms of a central-charge-$d$ matter, spin-$\frac{3}{3}$ bc ghosts, and an extra scalar current $I = \partial \phi$, reads

$$
T(z) = T(z) - \frac{1}{2}I(z)^2 - \frac{1}{4}(Q_L + Q)\partial I(z) + t(z),
$$

$$
\mathcal{H}(z) = i(z) - \frac{1}{2}(Q_L - Q)I(z),
\mathcal{G}^-(z) = b(z),
$$

$$
\mathcal{G}^+(z) = 2c(z)T(z) - 2b(z)\partial c(z) + (Q_L - Q)\partial c(z)\cdot I(z) - Qc(z)\partial I(z) + \frac{c}{3}\partial^2 c(z)
$$

where $T$ and $t$ are the matter and the ghost energy-momentum tensors respectively, while $i$ is the ghost current $i = -bc$:

$$
i(z)i(w) = \frac{1}{(z-w)^2}, \quad I(z)I(w) = \frac{-1}{(z-w)^2}
$$

and (for $c < 3$ for definiteness)

$$
Q = \frac{c+3}{\sqrt{3(3-c)}}, \quad Q_L = \frac{9-c}{\sqrt{3(3-c)}},
$$

The crucial operator product

$$
\mathcal{G}^-(z)\mathcal{G}^+(w) = \frac{2c}{3} \frac{1}{(z-w)^3} + \frac{-\mathcal{H}(w) - \mathcal{H}(z)}{(z-w)^2} + \frac{2T}{z-w},
$$

as well as the other relations of the algebra (2.1) can now be verified provided $d$ is related to the $N=2$ central charge $c$ via eq. (1.2).

There are just two possibilities to twist the algebra (2.1), with either $\mathcal{G}^+$ or $\mathcal{G}^-$ acquiring spin 2 after twisting. The first twisting is accomplished by setting

$$
\mathcal{L}^{(1)}_m = \mathcal{L}_m + \frac{1}{2}(m+1)\mathcal{H}_m,
\mathcal{L}^{(1)}_m = \mathcal{L}_m + \frac{1}{2}(m+1)\mathcal{H}_m,
\mathcal{L}^{(1)}_m = \mathcal{L}_m + \frac{1}{2}(m+1)\mathcal{H}_m.
$$

which gives a ‘spin-1’ (for the spin of the ghosts $b^{(1)}_m c^{(1)}_m$) construction of the topological algebra. We thus find

$$
\mathcal{T}^{(1)}(z) = \mathcal{T}(z) - \frac{1}{2}\partial \mathcal{H}(z)
$$

which renders the background charge of the current $I$ the same as the matter background charge $Q$. As the topological $U(1)$ current $\mathcal{H}$ contains the ghost current, the ghosts’ dimensions are also shifted, by the $\partial i$-term, to 1 and 0 for $b$ and $c$ respectively, producing the ghost energy-momentum tensor

$$
t^{(1)} = t - \frac{1}{2}\partial i = -bc.
$$
The resulting bc ghosts of spin 1 are defined in the usual way (omitting the \( ^{(1)} \) superscript): 

\[
\begin{align*}
    b(z) &= \sum_{n \in \mathbb{Z}} b_n z^{-n-1}, \\
    c(z) &= \sum_{n \in \mathbb{Z}} c_n z^{-n}, \\
    \{b_n, c_m\} &= \delta_{m+n,0}, \\
    b_{\geq 0}|0\rangle_{\text{gh}} &= c_{>0}|0\rangle_{\text{gh}} = 0 .
\end{align*}
\]

Therefore, 

\[
Q^{(1)}(z) = b(z) \tag{2.10}
\]

is now of spin 1, and we also get the spin-2 odd generator 

\[
G^{(1)}(z) = 2c(z)T^{(1)}(z) - 2\sqrt{\frac{3-c}{3}} \partial c(z) \cdot I(z) + \frac{c}{3} \partial^2 c(z) . \tag{2.11}
\]

Henceforth, we will omit the \( ^{(1)} \) superscript, as we are going to deal exclusively with this ‘spin-1’ construction for the twisted algebra \( [1,1] \). The other twisting of the \( N=2 \) algebra \( [2,1] \), leading to spin-2 bc ghosts, can be considered similarly \[8\].

The realisation \( [2.7]–[2.11] \) of the topological conformal algebra \( [1.1] \) can be applied to its singular states. Demanding that a singular vector \(|\Upsilon\rangle\) decouple amounts to the vanishing of all the correlators \( \langle \Upsilon(z_a) \prod_{b \neq a} \Psi_b(z_b) \rangle \) where \( \Psi_b \) are chiral primary fields (in the twisted version, which will be understood in the following). Now, having built up a singular vector \(|\Upsilon\rangle\) at level \( l \), we can substitute into it the above expressions for the topological generators in terms of the ‘constituent’ matter, Liouville, and ghosts. Note that \( \Psi \) will be a chiral primary if we set \(|\Psi\rangle = |\text{matter}\rangle \otimes |\text{Liouville}\rangle \otimes |0\rangle_{\text{gh}}\), with \(|\text{matter}\rangle \otimes |\text{Liouville}\rangle\) being the matter primary state dressed by the Liouville according to a certain prescription \[6\]. Thus a reduction to the matter\( \otimes \)Liouville theory is well-defined (and a subsequent reduction to the matter theory alone would reproduce the standard minimal singular states). The singular states thus obtained can also be constructed directly, as null states satisfying the ‘Kontsevich–Miwa’ dressing condition \[4,8\]. Decoupling equations associated with these singular vectors are implemented by order-\( l \) differential operators (where \( l \) is the level):

\[
\mathcal{O}^{(l)}_a \langle \Psi(z_a) \prod_{b \neq a} \Psi_b(z_b) \rangle = 0 \tag{2.13}
\]

and such operators \( \mathcal{O}^{(l)}_a \) factorise (completely, or up to a certain obstruction) through the combination \( \sum_{n \geq -1} z^{-n-2}L_n \) of the Virasoro generators 

\[
\begin{align*}
    L_{p>0} &= \frac{1}{2} \sum_{s=1}^{p-1} \frac{\partial^2}{\partial t_{p-s} \partial t_s} + \sum_{s \geq 1} st_s \frac{\partial}{\partial t_{p+s}} + \frac{1}{2}Q(p+1) \frac{\partial}{\partial t_p}, \\
    L_0 &= \sum_{s \geq 1} st_s \frac{\partial}{\partial t_s}, \\
    L_{-1} &= \sum_{s \geq 1} (s+1)t_{s+1} \frac{\partial}{\partial t_s} .
\end{align*}
\]

\[1\]The Liouville charges \( n_b \) must be related to the topological \( U(1) \) charges as 

\[
h_b = \sqrt{\frac{3-c}{3}} n_b \tag{2.12}
\]
which are written down in terms of the time parameters introduced via the Kontsevich–Miwa transform \[7\]

\[
t_r = \frac{1}{r} \sum_b n_b z_b^{-r}, \quad r \geq 1
\]  

(2.15)

where the \(n_b\) are the Liouville charges of the fields \(\Psi_b\). The background charge \(Q\) of these Virasoro generators, \(Q = \frac{2n_a}{l-1} - \frac{l-1}{n_a}\), turns out to coincide with the matter background charge \(Q = \sqrt{(1 - d)/3}\). A full factorisation occurs only for the \((l,1)\)- or \((1,l)\)-type singular vectors (see the next section), while for the type-\((r,s)\) vectors with both numbers different from 1, there is an obstruction to the full factorisation [8]. The ‘invariant’ mechanism (in terms of the structure of topological singular vectors) behind these factorisation properties is unclear, but it is very likely related to the hidden \(sl(2)\) structure that we study below.

3 Topological singular vectors

Two integers \(r \equiv 2j_1 + 1\) and \(s \equiv 2j_2 + 1\) characterise a topological singular vector \(|\Upsilon\rangle\) in the following way: \(|\Upsilon\rangle\) is the singular vector on level \(l = rs\) upon the chiral primary state \(|h\rangle\) whose topological \(U(1)\) charge \(h\) is related to the topological central charge \(c\) by

\[
h = \frac{c - 3}{3} j_1 + 2 j_2
\]  

(3.1)

(in the language of [17], this is the \(A\) series with zero relative charge; see also [18]).

For example, at level 4, we thus get three possibilities according to the values of \(\{j_1, j_2\}\):

\[
\begin{align*}
\{\frac{3}{2}, 0\}, & \quad h = \frac{1}{2}(c - 3), \\
\{\frac{1}{2}, \frac{1}{2}\}, & \quad h = \frac{1}{6}(c + 3), \\
\{0, \frac{1}{2}\}, & \quad h = 3.
\end{align*}
\]  

(3.2)

The corresponding singular vectors follow from a ‘prototype’ state written out in eq. (A.1) upon substituting into it one of these formulas for the topological \(U(1)\) charge \(h\).

Thus in the first case, that of \(\{j_1, j_2\} = \{\frac{3}{2}, 0\}\) and \(h = (c - 3)/2\), eq. (A.1) takes the form

\[
- \frac{1}{12} (c - 1)(c - 3)^4 |\Upsilon_{\{\frac{3}{2}, 0\}}^{(4)}\rangle,
\]  

(3.3)

where the ‘normalised’ singular vector \(|\Upsilon_{\{\frac{3}{2}, 0\}}^{(4)}\rangle\) (whose normalisation, only possible for \(c \neq 3\), has
been chosen in anticipation of the coincidence with the corresponding MFF state) equals

\[ |\Upsilon^{(4)}_{\{\frac{3}{2}, \frac{3}{2}\}}\rangle = \frac{108}{(c-3)^2} \left( \frac{72(5-2c)}{(c-3)^2} H_{-4} + \frac{2(18+12c-11c^2+c^3)}{(c-3)^2} L_{-4} + \frac{3(2c-13)}{c-3} (G_{-3} Q_{-1} - 2L_{-3} H_{-1}) \right) \\
- \frac{2(51-13c+2c^2)}{(c-3)^2} H_{-3} L_{-1} + \frac{6(1-c)}{c-3} H_{-2} L_{-2} + \frac{4(21-16c+2c^2)}{(c-3)^2} L_{-3} L_{-1} + 3L_{-2}^2 \\
+ \frac{(25+13c-2c^2)}{(c-3)^2} Q_{-3} G_{-1} - \frac{3(33-16c+c^2)}{(c-3)^2} Q_{-2} G_{-2} + \frac{36}{3-c} G_{-2} H_{-1} Q_{-1} + \frac{18(c-2)}{(c-3)^2} G_{-2} L_{-1} Q_{-1} \\
+ \frac{6(3c-2)}{(c-3)^2} (H_{-2} Q_{-1} G_{-1} + 2H_{-2} H_{-1} L_{-1}) + \frac{4(3-5c)}{(c-3)^2} H_{-2} L_{-2}^2 + \frac{36}{3-c} L_{-3}^2 H_{-1}^2 + \frac{60}{3-c} L_{-2} L_{-1} L_{-1} \]  

(3.4)

Next, by substituting into \( \{j_1, j_2\} = \{\frac{1}{2}, \frac{1}{2}\} \), \( h = (c + 3)/6 \), we bring (A.1) to the form

\[ \frac{5}{108} (c-3)^6 |\Upsilon^{(4)}_{\{\frac{3}{2}, \frac{3}{2}\}}\rangle \]  

with the ‘normalised’ topological singular state being given by

\[ |\Upsilon^{(4)}_{\{\frac{3}{2}, \frac{3}{2}\}}\rangle = \frac{36}{(c-3)^4} \left( \frac{72(-45+6c-5c^2)}{(c-3)^2} H_{-4} - \frac{12(135-9c+15c^2-c^3)}{(c-3)^2} L_{-4} \right) \\
+ \frac{6(9-30c+c^2)}{c-3} H_{-3} L_{-1} - \frac{(27-6c-c^2)^2}{c} L_{-2}^2 - \frac{6(15+27c-3c^2+c^3)}{(c-3)^2} H_{-2} L_{-2} \\
+ \frac{3(135+45c-3c^2-c^3)}{(c-3)^2} (L_{-1} H_{-1} - G_{-3} Q_{-1}) - \frac{12(81+9c+3c^2-c^3)}{(c-3)^2} L_{-3} L_{-1} \\
- \frac{3(135-315c-3c^2-c^3)}{(c-3)^2} Q_{-3} G_{-1} + \frac{3(135-63c+33c^2-c^3)}{(c-3)^2} Q_{-2} G_{-2} \\
+ \frac{36(3+c)^2}{(c-3)^2} (L_{-2}^2 H_{-1} - G_{-3} H_{-1} Q_{-1}) + \frac{36(18+3c+c^2)}{(c-3)^2} (H_{-2} Q_{-1} G_{-1} + 2H_{-2} H_{-1} L_{-1}) \\
+ \frac{72(9-3c+c^2)}{(c-3)^2} G_{-2} L_{-1} Q_{-1} + \frac{108(3+c)}{c-3} H_{-2} L_{-2}^2 - \frac{12(135+27c-3c^2+c^3)}{(c-3)^2} L_{-2} H_{-1} L_{-1} \]  

(3.5)

Finally, in the case \( \{j_1, j_2\} = \{0, \frac{3}{2}\} \), \( h = 3 \), eq. (A.1) becomes

\[ \frac{(c+15)(3-c)^7}{7776} \cdot |\Upsilon^{(4)}_{\{0, \frac{3}{2}\}}\rangle \]
where the third level-4 singular state equals

\[ |\Upsilon^{(4)}_{(0,2)}\rangle = \frac{1296}{(c - 3)^4} \left( \frac{216(15 + 7c)}{(3 - c)^3} \mathcal{H}_{-1} - \frac{36(c - 15)}{(c - 3)^2} \mathcal{L}_{-1} - \frac{270}{(c - 3)^2} \mathcal{G}_{-3} \mathcal{Q}_{-1} \right. \]
\[ + \frac{12(207 + c^2)}{(3 - c)^3} \mathcal{H}_{-3} \mathcal{L}_{-1} + \frac{216(15 + c)}{(3 - c)^3} \mathcal{H}_{-2} \mathcal{L}_{-1} - \frac{540}{(c - 3)^2} \mathcal{L}_{-3} \mathcal{H}_{-1} + \frac{12(5c - 6)}{(c - 3)^2} \mathcal{L}_{-3} \mathcal{L}_{-1} \]
\[ + \frac{324}{(c - 3)^2} \mathcal{L}_{-2}^2 + \frac{30(9 + c)}{(c - 3)^2} \mathcal{Q}_{-3} \mathcal{G}_{-1} + \frac{270}{(c - 3)^2} \mathcal{Q}_{-2} \mathcal{G}_{-2} + \frac{3888}{(c - 3)^3} \mathcal{G}_{-2} \mathcal{H}_{-1} \mathcal{Q}_{-1} \]
\[ + \frac{378}{(c - 3)^2} \mathcal{G}_{-2} \mathcal{L}_{-1} \mathcal{Q}_{-1} + \frac{18(87 + 7c)}{(c - 3)^3} (\mathcal{H}_{-3} \mathcal{Q}_{-1} \mathcal{G}_{-1} + 2 \mathcal{H}_{-2} \mathcal{H}_{-1} \mathcal{L}_{-1}) - \frac{24(12 + c)}{(c - 3)^2} \mathcal{H}_{-2} \mathcal{L}_{-1}^2 \]
\[ (3.6) \]
\[ + \frac{3888}{(c - 3)^3} \mathcal{L}_{-2} \mathcal{H}_{-1}^2 - \frac{1080}{(c - 3)^2} \mathcal{L}_{-2} \mathcal{H}_{-1} \mathcal{L}_{-1} + \frac{60}{c - 3} \mathcal{L}_{-2} \mathcal{L}_{-1}^2 - \frac{162}{(c - 3)^2} \mathcal{L}_{-2} \mathcal{Q}_{-1} \mathcal{G}_{-1} \]
\[ - \frac{270}{(c - 3)^2} \mathcal{Q}_{-3} \mathcal{H}_{-1} \mathcal{G}_{-1} + \frac{6(4c - 9)}{(c - 3)^2} \mathcal{Q}_{-2} \mathcal{L}_{-1} \mathcal{G}_{-1} + \frac{1296}{(3 - c)^3} \mathcal{H}_{-1} \mathcal{Q}_{-1} \mathcal{L}_{-1} + \frac{1944}{(3 - c)^3} \mathcal{H}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} \]
\[ + \frac{36}{3 - c} \mathcal{H}_{-1} \mathcal{L}_{-1}^2 + \frac{396}{(c - 3)^2} (\mathcal{H}_{-2} \mathcal{L}_{-1}^2 + \mathcal{H}_{-1} \mathcal{L}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1}) + \mathcal{L}_{-1}^3 + \frac{18}{3 - c} \mathcal{L}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} \] \]

Let us note that pairwise intersections between the three classes of singular vectors occur precisely for those values of \( h \) and \( c \) that satisfy a chosen pair of conditions \( \{3,2\} \) simultaneously.

The case when topological central charge \( c = 3 \) is discussed in section 8.

Continuing with the examples, let us go down to level 3. Here, for \( j_1 = 1, j_2 = 0, h = (c - 3)/3 \), the ‘normalised’ topological singular vector reads

\[ |\Upsilon^{(3)}_{(1,0)}\rangle = \frac{12}{(c - 3)^2} \left( \frac{2c \mathcal{L}_{-1}}{(c - 3)} \mathcal{H}_{-1} \mathcal{L}_{-1} - \frac{12 \mathcal{H}_{-1} \mathcal{L}_{-2} + 12 \mathcal{L}_{-2} \mathcal{L}_{-1}}{(3 - c)} \right. \]
\[ + \frac{3(15 - c)}{3 - c} \mathcal{Q}_{-2} \mathcal{G}_{-1} - 6 \mathcal{Q}_{-1} \mathcal{G}_{-2} + \frac{36}{c - 3} \mathcal{H}_{-1} \mathcal{L}_{-1}^2 + \frac{54}{3 - c} \mathcal{H}_{-1} \mathcal{L}_{-1}^2 \]
\[ + \frac{36}{3 - c} \mathcal{H}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} + \frac{18}{c - 3} \mathcal{L}_{-1}^3 + \frac{27}{3 - c} \mathcal{L}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} \] \]

(3.7)

Similarly, at \( j_1 = 0, j_2 = 1, h = 2 \) we find another singular vector

\[ |\Upsilon^{(3)}_{(0,1)}\rangle = \frac{216}{(c - 3)^3} \left( \frac{12(9 + c)}{(c - 3)^2} \mathcal{L}_{-3} - \frac{6(9 + c)}{(c - 3)^2} \mathcal{H}_{-2} \mathcal{L}_{-1} - \frac{144}{(c - 3)^2} \mathcal{H}_{-1} \mathcal{L}_{-2} \right. \]
\[ + \frac{24}{c - 3} \mathcal{L}_{-2} \mathcal{L}_{-1} + \frac{6}{c - 3} \mathcal{Q}_{-2} \mathcal{G}_{-1} - \frac{72}{(c - 3)^2} \mathcal{Q}_{-1} \mathcal{G}_{-2} + \frac{72}{(c - 3)^2} \mathcal{H}_{-1} \mathcal{L}_{-1} \]
\[ + \frac{18}{3 - c} \mathcal{H}_{-1} \mathcal{L}_{-1}^2 + \frac{72}{(c - 3)^2} \mathcal{H}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} + \mathcal{L}_{-1}^3 + \frac{9}{3 - c} \mathcal{L}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} \] \]

(3.8)

The situation is yet simpler at level 2. There exist two topological singular states, one at \( h = (c - 3)/6 \), which reads

\[ |\Upsilon^{(2)}_{(1,0)}\rangle = \frac{6}{c - 3} (\mathcal{L}_{-2} + \frac{6}{c - 3} \mathcal{H}_{-1} \mathcal{L}_{-1} + \frac{6}{c - 3} \mathcal{L}_{-1}^2 + \frac{3}{3 - c} \mathcal{Q}_{-1} \mathcal{G}_{-1}) \] \]

(3.9)

and the other at \( h = 1 \),

\[ |\Upsilon^{(2)}_{(0,1)}\rangle = \frac{36}{(c - 3)^2} (\mathcal{L}_{-2} - \frac{6}{c - 3} \mathcal{H}_{-1} \mathcal{L}_{-1} + \mathcal{L}_{-1}^2 - \frac{3}{c - 3} \mathcal{Q}_{-1} \mathcal{G}_{-1}) \] \]

(3.10)

(these two coincide at \( c = 9, h = 1 \)). This completes our construction of lower-level topological singular states.
4 \( sl(2)_k \) and matter coupled to gravity

The above singular vectors can be ‘split’ according to the decomposition \( t = m \oplus l \oplus [bc] \) of the topological algebra, where \( m, l \) and \([bc]\) are the matter, (‘mirror’) Liouville, and ghost theories respectively (see section 2). By further suppressing ghosts, we get those singular states in the matter + Liouville theory which satisfy the ‘Kontsevich–Miwa’ conditions, and which can be arrived at independently of the topological considerations [8]. Dropping down the ghosts is a meaningful procedure, since ghost-independent field operators represent chiral primary fields, and thus only the ghost-independent part of a given singular vector contributes to the decoupling equations for correlators comprising chiral primary fields. It is to these decoupling equations that the Kontsevich–Miwa transform can be applied, showing complete factorisation for the \((l, 1)\) or \((1, l)\) states.

Another, and not unrelated, approach allowing us to investigate the structure of topological singular vectors is provided by revealing a hidden \( sl(2) \) Kač–Moody algebra. This can be done using the (twisted) Kazama–Suzuki model [39] \( sl(2)_k \oplus u(1)/u(1) \), where \( sl(2)_k \) denotes the level-\( k \) algebra

\[
J^0(z)J^\pm(w) = \pm \frac{J^\pm}{z-w}, \quad J^0(z)J^0(w) = \frac{k/2}{(z-w)^2},
\]

with the twisted Sugawara energy-momentum tensor

\[
\tilde{T}^s = \frac{1}{k+2} \left( J^0J^0 - \frac{1}{2} (J^+J^- + J^-J^+) \right) + \partial J^0.
\]

The Kazama–Suzuki ‘numerator’ \( u(1) \) algebra fermionises into a couple of spin-1 ghosts, denoted in the following as \( BC \) (cf. [25]), which allows us to build up the topological algebra generators in the standard way [21, 41, 40, 42]: The odd generators \( Q \) and \( G \) are given by

\[
Q = \sqrt{\frac{2}{k+2}} BJ^+, \quad G = -\sqrt{\frac{2}{k+2}} CJ^-,
\]

while the topological \( U(1) \) current and the energy-momentum tensor take the form

\[
\mathcal{H} = -\frac{k}{k+2} BC - \frac{2}{k+2} J^0,
\]

\[
\mathcal{T} = -\frac{1}{k+2}(J^+J^-) + \frac{k}{k+2} \partial B \cdot C + \frac{2}{k+2} BCJ^0.
\]

Generators (4.3), (4.4), and (4.5) close to the algebra (1.1) with topological central charge

\[
\mathcal{c} = \frac{3k}{k+2}.
\]

We thus have a mapping

\[
t \rightarrow \mathcal{U}(sl(2)_k \oplus u(1)),
\]
where $t$ is the topological algebra and $\mathcal{U}$ denotes the universal enveloping. On the other hand, as we have discussed above, it is possible to view the topological algebra $t$ as $t = m \oplus l \oplus [bc]$. We have short exact sequences (omitting $\mathcal{U}$)

\[
0 \rightarrow m \oplus l \oplus [bc] \rightarrow 0
\]

\[
0 \rightarrow \text{sl}(2) \rightarrow A \rightarrow u(1)_{v} \rightarrow 0
\]

(4.8)

where $u(1)_{v}$ is the Kazama–Suzuki denominator $u(1)$ algebra, which is generated by the current

\[
\partial v = \sqrt{\frac{2}{k+2}} (J^0 - BC)
\]

(4.9)

that decouples from the generators (4.3), (4.4), (4.5). The vertical sequence splits as well as the horizontal one does, which gives us two ways to describe the algebra $A$. First of all, this algebra is just $\text{sl}(2) \oplus u(1)$, but at the same time it is $m \oplus l \oplus [bc] \oplus u(1)_{v}$, and in these latter terms the splitting of the horizontal sequence is accomplished by representing the ‘Kazama–Suzuki’ ghosts as

\[
B = bc \sqrt{\frac{2}{k+2}} (v-\phi), \quad C = c \sqrt{\frac{2}{k+2}} (v-\phi)
\]

(4.10)

Also, the embedding $\text{sl}(2) \hookrightarrow m \oplus l \oplus [bc] \oplus u(1)_{v}$ is given by the following explicit formulae:

\[
J^+ = c \sqrt{\frac{2}{k+2}} (v-\phi), \quad J^0 = -i + \sqrt{\frac{2}{k+2}} \frac{k}{\sqrt{2(k+2)}} \partial v,
\]

\[
J^- = \{- (k+2)(T + T_L) + i^2 - (k+1)d \partial i - \sqrt{2(k+2)}(\phi-v)\}
\]

(4.11)

(recall that $i = -bc$ is the ghost current, $\partial \phi = I$ is the Liouville current, and $T$ and $T_L$ are the matter and Liouville energy-momentum tensors respectively; it should also be kept in mind that we are dealing with the ‘mirror’ version in which the Liouville energy-momentum tensor has the form

\[
T_L = -\frac{1}{2} \partial \phi \partial \phi - \frac{1}{2} Q \partial^2 \phi
\]

(4.12)

with the background charge $Q = \sqrt{(1-d)/3}$. A remarkable fact (which was to be expected, though) is that upon dropping the ghosts, the current $J^-$ reduces to the matter and Liouville energy-momentum tensors dressed with an appropriate vertex operator of dimension zero.

Another remarkable feature is that the matter only enters in eqs. (4.11) (as well as in the construction of the topological algebra from section 2) through its Virasoro generators and therefore can be arbitrary. There is in (4.11) one field more than necessary for representing three $\text{sl}(2)$ currents, and the construction can be ‘straightened up’ as shown in the next section.  

\[\text{No hamiltonian reduction is being performed!}\]
Corresponding to the diagram (4.8), we have the following decomposition of the energy-momentum tensors:

\[
\tilde{T} - B \partial C = T + \frac{1}{2} \partial v \partial v + \frac{k + 1}{\sqrt{2(k + 2)}} \partial^2 v ,
\]

(4.13)

where \(T\) is the topological energy-momentum tensor. The realisation of the topological theory as matter dressed with ‘mirror’ gravity allows us to read the formula (4.13) as a representation for the twisted \(sl(2)_k\) theory together with the \(BC\) ghosts as a sum of two minimal models \(M_{k+2,1}\), a system of weight-(1,0) ghosts, and a Liouville scalar whose background charge coincides with that associated to the \(M_{k+2,1}\) minimal theory:

\[
\tilde{sl}(2)_k \oplus [BC] = \underbrace{M_{k+2,1}^{\text{matter}} \oplus \text{Liouville} \oplus [bc]}_{\text{Matter model}} \oplus \underbrace{M_{k+2,1}^v}_{\text{Liouville}}
\]

(4.14)

The \(M_{k+2,1}^v\) theory ‘absorbs the central charge’ (leaving 0 to the topological model) according to

\[
\left( \frac{3k}{k + 2} - 6k \right) - 2 = 0 + \left( 1 - 3 \frac{2(k + 1)^2}{k + 2} \right).
\]

(4.15)

Next, we would like to represent states in two ways, as

\[
| \rangle_{sl(2)} \otimes | \rangle_{BC} = |h \rangle \otimes |v \rangle
\]

(4.16)

with \(|h\rangle\) being a topological state. This is indeed possible for primary states, as

\[
|\{j_1, j_2\} \rangle \otimes |0\rangle_{BC} = |h_{j_1} \rangle \otimes |V_{j_2} \rangle
\]

(4.17)

where \(|0\rangle_{BC}\) is the ghost vacuum:

\[
B_{\geq 1} |0\rangle = C_{\geq 0} |0\rangle = 0
\]

(4.18)

and \(h_j\) is to be expressed through \(\{j_1, j_2\}\) via eq. (3.1). From (4.4) and the highest-weight conditions we find

\[
h = h_j = - \frac{2j}{k + 2}.
\]

(4.19)

Combined with (4.6), eq. (4.19) allows us to read the equality (3.1) as the parametrisation (6.3) for the highest weight of the \(sl(2)\) affine algebra. Note also that the background charge \(Q = \sqrt{(1 - d)/3}\) evaluates from (4.6) and (1.2) as \(Q = \sqrt{2(k + 1)}/\sqrt{k + 2}\).

Further, to find out what the \(V_j\) operator is, consider the balance of dimensions. Since the twisted \(N = 2\) dimension of \(|h\rangle\) is zero, the role of the dressing by \(V_j = \exp \rho_j v\) is to provide the state \(|j\rangle_{sl(2)}\) with the correct (twisted Sugawara) dimension

\[
\tilde{\delta}[j] = j(j + 1) - j.
\]

(4.20)

Together with \(|\rangle_{BC} = |0\rangle\) this determines \(\rho_j\) as

\[
\rho_j = j \sqrt{\frac{2}{k + 2}} \quad \text{or} \quad (k + 1 - j) \sqrt{\frac{2}{k + 2}}.
\]

(4.21)

The first of these values (to be used in eq. (10.1) below) coincides with minus the Liouville \(U(1)\) charge, which is determined from (3.1) and (2.12) as \(n_a = -j \sqrt{\frac{2}{k + 2}}\).
5 An ‘invariant’ Wakimoto construction and hamiltonian reduction

The construction (4.11) expresses the $sl(2)$ currents in terms of three independent scalar fields and a central-charge-$d$ matter. As in the construction (2.7)–(2.11) for the topological conformal algebra, matter need not be bosonised, since it enters only through its Virasoro generators. Anyway, the total of ‘almost’ four bosonic currents is redundant for representing three $sl(2)$ currents. To remove this redundancy from eqs. (4.11), we have to find a combination of the $bc$ ghost, Liouville, and $\partial v$-currents that decouples from the $sl(2)$ algebra. According to (4.8) and (4.10), this is the $BC$ ghost current

$$\partial F = i + \sqrt{\frac{2}{k+2}}(\partial v - \partial \phi).$$

(5.1)

Two other independent fields are introduced along with $\partial F$ via the appropriate mixing, as

$$\partial \chi = \sqrt{\frac{k+2}{k}}\partial \phi - \sqrt{\frac{2}{k}}i,$n

$$\partial \psi = \sqrt{\frac{k}{k+2}}\partial v + \frac{2}{\sqrt{k(k+2)}}\partial \phi - \sqrt{\frac{2}{k}}i$$

(5.2)

(with signatures $-$ and $+$ respectively, cf. eq. (2.3)). In terms of these, the $sl(2)$ currents (4.11) take the form

$$J^+ = e^{\sqrt{\frac{2}{k}}(\psi - \chi)}, \quad J^0 = \sqrt{\frac{k}{2}}\partial \psi,$n

$$J^- = \left[-(k+2)T + k\left(\frac{1}{2}\partial \chi \partial \chi + \frac{k+1}{\sqrt{2k}}\partial^2 \chi\right)\right]e^{-\sqrt{\frac{2}{k}}(\psi - \chi)}.$$n

(5.3)

where, as before, $T$ is the matter energy-momentum tensor with central charge evaluated from (1.2) and (4.6) as

$$d = 13 - 6(k+2) - \frac{6}{k+2}.$$n

(5.4)

Thus an arbitrary matter theory with central charge $d \leq 1$ or $d \geq 25$ can be dressed so as to make up the $sl(2)$ Kač–Moody algebra. The matter need not be specified any further beyond its Virasoro algebra.

However, if one bosonises the matter through a scalar with the appropriate background charge, then, by a simple exercise in linear algebra, one can explicitly map the formulae (5.3) onto the more standard Wakimoto bosonisation [45, 46, 47]: a (hyperbolic) rotation in the space of the currents allows one to identify a weight-1 $\beta \gamma$ system, bosonised according to the recipe [48] $\beta = be^{\varphi}$, $\gamma = c\partial e^{\varphi}$, and an independent scalar $\phi$. In this way, the currents take the standard Wakimoto form

$$J^+ = \beta, \quad J^0 = \beta \gamma + \sqrt{\frac{k+2}{2}}\partial \phi,$n

$$J^- = \beta \gamma^2 + \sqrt{2(k+2)}\partial \phi \cdot \gamma + k\partial \gamma.$$n

(5.5)

We do not present the details, partly because they have already appeared in [49], and partly because we prefer the form (5.3) as being more ‘intrinsic’ (i.e. not requiring a bosonisation of the
matter theory). In fact, once the bosonisation is allowed, the entire set of four fields from eq. (1.11) can be organised to make up a Wakimoto-bosonised \(osp(2,1)\) Kač–Moody algebra (cf. [50]). Also, the matter + \(\partial \chi\) theory from (5.3) is made equivalent to \(\mathbb{Z}_k\)-parafermions, as it should be after ‘suppressing’ the \(J^0\) current in the \(sl(2)_k\) WZW model.

As an aside, note that equations (5.3) suggest themselves for the hamiltonian reduction (cf. [13, 14]). This amounts to projecting out the \(\psi\) and \(\chi\) scalars, and we thus get from the theory with the energy-momentum tensor (4.13) precisely the matter \(M_{k+2,1}\) theory (see (4.14)), plus the \(BC\) ghosts. The ghosts are not interesting in this context, since they remain a direct summand and are not involved in the reduction. Thus, the matter theory is recovered by performing the hamiltonian reduction of the \(sl(2)_k\) theory, which gives the matter an ‘invariant’ meaning in the context of decomposing a given topological conformal theory as \(t = m \oplus l \oplus [bc]\).

6 The MFF vectors

We will use the mapping (4.7) for an explicit evaluation of the topological singular states. Namely, we are going to show that singular vectors of the topological conformal algebra coincide with those of the \(sl(2)\) Kač–Moody algebra.

These latter are well known: singular vectors of \(sl(2)_k\) are labelled by two integers \(r\) and \(s\), and can be written in the MFF form [20] as

\[
|MFF_{rs}\rangle = (J_0^-)^{r+(s-1)(k+2)}(J_+^0)^{r+(s-2)(k+2)}(J_0^-)^{r+(s-3)(k+2)} \ldots \
\times (J_+^0)^{r-(s-2)(k+2)}(J_0^-)^{r-(s-1)(k+2)}|\{\frac{r-s}{2}, \frac{s+1}{2}\}\rangle
\]

where \(|\{j_1, j_2\}\rangle\) is a highest-weight state:

\[
J_+^{n}|\{j_1, j_2\}\rangle = 0, \quad n \geq 0, \\
J_-^{n}|\{j_1, j_2\}\rangle = 0, \quad n > 0, \\
J_0^{n}|\{j_1, j_2\}\rangle = j|\{j_1, j_2\}\rangle,
\]

with

\[
j = j_1 - j_2(k + 2).
\]

The formula (6.2) is not so innocuous as it might appear: it is simple only when \(s = 1\) (being reduced then to \((J_0^-)^r\)), while, for instance, writing out the state \(|MFF14\rangle\) as a polynomial in the currents one finds

\[
(k + 3)(6(k + 1)(k + 2)^2(3 + 2k)(7 + 3k)J_{-3} + 12k(k + 2)(5 + 2k)(7 + 3k)J_{-2}J_0^0 \\
- 4(7 + 3k)(4 - 9k - 9k^2 - 2k^3)J_0^0J_0^- + 6(k + 2)(7 + 3k)(2 + 7k + 2k^2)J_0^0J_{-1}^- \\
- 2(7 + 3k)(4 - 9k - 4k^2)J_{-2}J_0^-J_0^0 + 3(k + 2)(k + 4)(7 + 3k)J_{-1}J_0^-J_0^0 \\
+ 12(k + 2)(5 + 2k)(7 + 3k)J_{-1}J_0^0J_0^- + 4(7 + 3k)(2 + 17k + 6k^2)J_0^0J_{-1}^-J_0^- \\
- (8 - 3k - 2k^2)[2(3k + 3)J_{-3}J_0^-J_0^0 + 2(7 + 3k)J_{-2}J_{-1}J_0^-] + 4(7 + 3k)(14 + 5k)J_{-1}J_{-1}J_0^-J_0^- \\
- 4(1 - 13k - 5k^2)J_{-2}J_{-1}J_0^-J_0^- + 5(5 + 2k)(7 + 3k)J_0^0J_{-1}^-J_0^- \\
- 8(9 - 2k - 2k^2)J_{-2}J_{-1}J_0^-J_0^- + 2(14 + 5k)J_{-1}J_{-1}J_0^-J_0^- + 4(29 + 11k)J_0^0J_{-1}^-J_0^-J_0^- \\
+ 4(k - 2)J_{-2}J_{-1}J_0^-J_0^-J_0^0 + 12J_0^0J_{-1}J_{-1}J_0^-J_0^- + J_{-2}J_{-1}J_0^-J_0^-J_0^-J_0^-J_0^0)
\]
(which is understood to act on the corresponding highest-weight state), and a similar expression for $|MFF15\rangle$ contains already as many as 42 different terms. All of them appear as a result of the repeated use of the following identities:

\begin{align}
(J_0^-)^q J_m^0 &= q J_m^-(J_0^-)^{q-1} + J_m^0 (J_0^-)^q, \\
(J_0^-)^q J_m^+ &= J_m^+(J_0^-)^q + 2q J_m^0 (J_0^-)^{q-1} + q(q-1) J_m^-(J_0^-)^{q-2}, \\
J_m^0 (J_{-1}^+)^q &= q(J_{-1}^+)^{q-1} J_{m-1}^+ + (J_{-1}^+)^q J_m^0, \\
J_m^- (J_{-1}^+)^q &= (J_{-1}^+)^q J_m^- + 2q(J_{-1}^+)^{q-1} J_{m-1}^0 + q(q-1)(J_{+1}^-)^{q-2} J_{m-2}^+ - kq\delta_{m-1}(J_{+1}^-)^{q-1}.
\end{align}

The MFF states satisfy the highest-weight conditions

\begin{align}
J_n^+ |S\rangle &= 0, \quad n \geq 0, \\
J_n^0 |S\rangle &= 0, \quad n \geq 1, \\
J_n^- |S\rangle &= 0, \quad n \geq 2.
\end{align}

Let us also note that in order to end up with singular states which are $J^0$-neutral and belong to level $l = rs$, one has to act on an MFF state $|MFFrs\rangle$ with $(J_{-1}^+)^r$.

7 The evaluation

Lay down the book, and I will allow you half a day to give a probable guess at the grounds of this procedure.

LAWRENCE STERNE, Tristram Shandy (1760)

Now we are ready to formulate the main observation: evaluating the topological singular states $|\Upsilon\rangle$ in the Kač–Moody terms via the ‘Kazama–Suzuki mapping’ (4.7), we find the following identifications:

\begin{align}
|\Upsilon\rangle \otimes |V\rangle_v &= |S\rangle_{sl(2)} \otimes |0\rangle_{BC}
\end{align}

where the $BC$ oscillators drop from the RHS and $|S\rangle$ is an $sl(2)$ singular vector. In this formula, $V$ is a primary chosen as explained in the end of section 4; in particular, there are no $\partial v$-oscillators, and therefore the topological singular vector can be identified with the $sl(2)$ one.

The identity between topological singular states and the MFF vectors (6.1) has been checked explicitly for levels 2 through 4. As a by-product of the computation, we obtain a useful form of the MFF vectors that involves the Sugawara Virasoro generators; these are not present explicitly in the original formulation (4.1), but are quite helpful when writing down the decoupling equations corresponding to singular states: using the standard Ward identities for the Virasoro generators, the $sl(2)$ decoupling equations are then written as differential.

We find the equalities, presented below, between singular vectors of the algebras (1.1) and (4.1) very convincing, considering the relative complexity of some of the expressions that follow. A general proof might be easier in the direction from $sl(2)$ to the topological algebra. It is in fact straightforward to see that any $sl(2)$ primary state, which satisfies eqs. (6.6), would correspond to a topological primary via eq. (4.17). Indeed, using the mode expansions $Q(z) = \sum Q_n z^{-n-1}$ and
\[ G(z) = \sum \mathcal{G}_n z^{-n-2} \], we find from (1.3) and (4.18) that, on our highest-weight state, \( Q_{\geq 0} \sim 0 \) and \( \mathcal{G}_{\geq 1} \sim 0 \). Similarly, eqs. (4.18) and (4.4) imply \( \mathcal{H}_{\geq 1} \sim 0 \), whence all the topological highest-weight conditions follow. Further, the relations (4.6) and (4.15) allow us to rewrite the formula (6.3) as (3.4), thus recovering the \( A \) series with zero ‘relative charge’ from ref. [17]. However, this does not immediately imply that the \( sl(2) \) singular vectors would be those with respect to the topological conformal algebra, since it is not clear (to the author) how one can independently characterise those \( sl(2) \)-descendants that translate into the topological descendants.

Anyway, along with the results on explicit evaluation of topological singular vectors in the \( sl(2) \) terms, this partial argument in favour of the one-to-one correspondence between the singular states strongly suggests that the \( sl(2) \) Kač–Moody algebra and the topological conformal algebra possess identical singular vectors (when \( c \neq 3 \), see the next section). Recall also that ‘the same’ ordinary matter theory enters our constructions for the topological algebra (section 2) and the \( sl(2) \) algebra (section 5), which are the two algebras whose singular states are identical. We thus arrive at the diagram

\[
\begin{array}{ccc}
\text{topological} & \overset{\sim}{\rightarrow} & \text{\( sl(2) \) singular states} \\
\text{singular states} & & \text{singular states} \\
\text{\( \rightarrow \)} & & \text{\( \leftarrow \)} \\
\text{matter} & & \\
\text{singular states} & & \\
\end{array}
\]

As noted above, it would be interesting to understand these relations not just between the singular states, but rather between the corresponding algebras, in the context of universal string theory [27, 28, 29].

In the remaining part of this section, we present the ‘experimental evidence’ on the identity between the topological and \( sl(2) \) singular vectors. By a direct (although quite lengthy) calculation we can evaluate that, for instance, level-4 state (3.4) gives rise, according to (7.1), to the following \( sl(2) \)-state:

\[
\begin{align*}
|S^{(4)}_{(4,0)}\rangle &= (k+2) \left[ 3(-112 - 102k - 11k^2 + 3k^3) J^0_{-4} + 3(8 + 20k + 3k^2 - k^3) \bar{L}^S_{-4} \\
&\quad - 3(6 - 4k + k^2) J^+_{-3} J^-_{-1} + 3(4 - k)(k+2) J^-_{-2} J^+_{-2} - 3(96 + 59k + 4k^2) J^0_{-1} \bar{J}^0_{-1} \\
&\quad + 3(k - 4)(k + 2)(7 + 2k) J^0_{-3} \bar{L}^S_{-1} + 3(-6 - 7k + k^2) \bar{J}^0_{-2} J^0_{-2} + (4 - k)(11 + 3k) J^+_{-4} \bar{J}^0_{-1} \\
&\quad + 3(22 + 6k - k^2) J^+_{-3} J^-_{-1} + 9(k+2)(k+4) \bar{L}^S_{-3} J^-_{-1} + (k+2)(28 - 4k - 3k^2) \bar{L}^S_{-3} \bar{L}^S_{-1} \\
&\quad + 9(k+2)(\bar{L}^S_{-2})^2 + 9(k+4) J^-_{-2} J^0_{-2} J^-_{-1} + 3(4 - k)(k+2) J^-_{-2} \bar{L}^S_{-1} - 18(k+4) J^0_{-2} J^0_{-2} J^-_{-1} \\
&\quad + 6(k-4)(k+2) J^0_{-2} J^0_{-1} \bar{L}^S_{-1} + (28 + 26k + 3k^2) J^0_{-2} \bar{J}^0_{-2} + (32 + 13k) J^+_{-3} J^0_{-2} \bar{J}^0_{-1} \\
&\quad + (k+2)(8 + 5k) J^0_{-3} \bar{L}^S_{-1} \bar{J}^0_{-1} - 3(k+2)(k+6) J^-_{-2} \bar{L}^S_{-2} \bar{J}^0_{-2} + 3(k+4) J^+_{-2} J^0_{-2} J^0_{-2} J^-_{-1} \\
&\quad - 10(k+2)^2 \bar{L}^S_{-2} (\bar{L}^S_{-1})^2 + 8(k+2) J^-_{-2} J^0_{-1} \bar{L}^S_{-1} \bar{J}^0_{-2} + 6(k+2)^2 J^+_{-2} (\bar{L}^S_{-1})^2 \bar{J}^0_{-1} + (k+2)^3 (\bar{L}^S_{-1})^2 \{ \frac{3}{2}, 0 \} \right].
\end{align*}
\]

---

\(^3\)Even the converse (that a topological descendant is an \( sl(2) \) descendant) may not be true for any vector other than a singular one, because, when rewriting a topological state by using eqs. (1.3), (1.4), and (1.5), it is not at all obvious that the \( BC \) modes would drop out. For instance, even for the expression (A.1), which is ‘very close’ to being a singular state, the \( BC \) ghosts do not decouple unless we substitute one of the relations (B.2), thus reducing (A.1) to a singular vector.
where $\tilde{L}_m^S$ are modes of the twisted Sugawara energy-momentum tensor $[14, 22]$. Evaluating these in terms of the currents, we bring eq. (7.3) to the MFF form

$$|S^{(4)}_{(2, 0)}\rangle = J_{-1}^+ J_{-1}^- J_{-1}^+ J_{0}^- J_{0}^- J_{0}^- |\{\frac{3}{2}, 0\}\rangle = J_{-1}^+ J_{-1}^- J_{-1}^+ J_{-1}^+ |\text{MFF41}\rangle$$  (7.4)

For the other level-4 topological singular vectors we observe equally dramatic cancellations. The singular vector (7.5) is mapped in the same way into

$$|S^{(4)}_{(\frac{1}{2}, \frac{1}{2})}\rangle = (k + 2)^2 \left(2(k + 2)(13 + 14k + 9k^2 + 2k^3)J_{0}^6 - (k + 2)(5 + 2k)(2 + 2k + k^2)\tilde{L}_4^S \right.

+ (1 + 6k + k^2 - 2k^3)J_{-3} J_{-1}^- J_{-2}^+ J_{-2}^+ + \left. \right.

+ (3 + 6k + 47k^2 + 14k^3 + k^4)J_{0}^6 J_{-1}^0 J_{-1}^0 + (k + 2)(3 + 2)(4 + 2k + k^2)J_{0}^6 \tilde{L}_1^S \right.

+ (7 + 8k + 6k^2 + 2k^3)J_{0}^2 J_{-2}^0 J_{-2}^0 - (20 + 18k + 7k^2 + k^3)J_{-4}^0 J_{-6}^- - (5 + 4k + 3k^2 + k^3)J_{-3}^0 J_{-3}^- \right.

- (k + 1)(k + 1)^2(k + 2)(k + 4)\tilde{L}_3^S \tilde{L}_1^S \left. - (k + 2)(6 + 10k + 6k^2 + k^3)\tilde{L}_3^S \tilde{L}_1^S \right. \right.

+ (k + 1)^2(k + 2)^2(\tilde{L}_2^S)^2 - (k + 1)(k + 1)(k + 2)(k + 4)J_{-2}^0 J_{-1}^0 J_{-1}^0 + (k + 2)(4 + 2k + k^2)J_{-2}^0 J_{-1}^0 \tilde{L}_1^S \right.

+ 2(k + 1)(k + 2)(k + 4)J_{-2}^0 J_{-1}^0 J_{-1}^0 + 2(k + 2)(4 + 2k + k^2)J_{-2}^0 J_{-1}^0 \tilde{L}_1^S \right.

+ 2(3 + 3k + k^2)J_{-2}^0 J_{-2}^0 J_{0}^- + (2 - 6k - 5k^2 - k^3)J_{3}^0 J_{-1}^0 J_{-1}^0 - (k + 2)(10 + 18k + 8k^2)J_{3}^0 \tilde{L}_1^S \tilde{L}_1^S \right.

+ (2 + 10k + 6k^2 + k^3)J_{-2}^0 J_{-2}^0 J_{-2}^0 - (2k + 3)(4 + 2k + k^2)\tilde{L}_3^S \tilde{L}_1^S \tilde{L}_1^S + (k + 2)(4 + 4k)J_{-2}^0 J_{-1}^0 J_{-1}^0 \right.

- 4(k + 2)J_{-2}^0 J_{-1}^0 J_{-1}^0 - (k + 2)(4 + 2k + k^2)J_{-4}^0 J_{-6}^0 \tilde{L}_1^S \tilde{L}_1^S J_{0}^- + (k + 2)^2(\tilde{L}_1^S)^4 \right) |\{\frac{1}{2}, \frac{1}{2}\}\rangle. \quad (7.5)

The reader can check that by inserting now the Sugawara Virasoro generators, this rewrites as

$$|S^{(4)}_{(\frac{1}{2}, \frac{1}{2})}\rangle = J_{-1}^+ J_{-1}^+ |\text{MFF22}\rangle.$$

The third level-4 singular vector (3.0) (for which $\{j_1, j_2\} = \{0, \frac{3}{2}\}$, $j = (-6 - 3k)/2$) evaluates similarly as

$$|S^{(4)}_{(0, \frac{3}{2})}\rangle = (k + 2)^4 \left(\tilde{L}_1^S \right)^4 - 6(k + 2)(11 + 6k)J_{0}^4 - 6(k + 2)(5 + 2k)\tilde{L}_4^S + 9(k + 2)(3 + 4k)J_{-3} J_{-1} \right.

- 21(k + 2)J_{-2}^+ J_{-2}^+ - 9(k + 2)(6 + 7k)J_{0}^6 J_{-1} - 21(k + 2)^2 J_{0}^6 J_{-1} - 36(k + 1)(k + 2)J_{0}^6 J_{-2} \right.

+ (3k - 5)J_{-2}^0 J_{-2}^0 - 15(k + 2)J_{-3} J_{-1}^+ + 9(k + 2)^2 \tilde{L}_3^S J_{-2}^0 - (k + 2)(4 + 3k)\tilde{L}_3^S \tilde{L}_1^S + 9(k + 2)^2(\tilde{L}_3^S)^2 \right.

+ 9(k + 2)J_{-2}^0 J_{-1}^0 J_{-1}^0 - 21(k + 2)J_{-2}^0 J_{-1}^0 J_{-1}^0 \tilde{L}_1^S - 18(k + 2)J_{0}^6 J_{-1}^0 J_{-1}^0 + 42(k + 2)J_{0}^6 J_{-1}^0 \tilde{L}_1^S \right.

- (31 + 18k)J_{0}^6 J_{-2} J_{0}^- - (2 + 3k)J_{-3} J_{-1}^0 J_{-1}^0 - \left. (k + 10)J_{-3} \tilde{L}_1^S J_{0}^- + 9(k + 2)J_{-2}^0 \tilde{L}_2^S J_{0}^- \right.

- 10(k + 2)\tilde{L}_2^S (\tilde{L}_1^S)^2 - 3J_{-2} J_{-1}^0 J_{-1}^0 J_{0}^- + 8J_{-2} J_{-1}^0 \tilde{L}_1^S J_{0}^- - 6J_{-2}^0 (\tilde{L}_1^S)^2 J_{0}^- \right) |\{0, \frac{3}{2}\}\rangle \quad (7.7)

which is nothing other but

$$|S_{(0, \frac{3}{2})}\rangle = J_{-1}^+ |\text{MFF14}\rangle \quad (7.8)

(the polynomial expression for |MFF14\rangle was written out in (6.4)).

The same situation repeats, in simpler terms, at level 3: the topological singular vector (3.7) evaluates in the Kac–Moody language as

$$|S^{(3)}_{(1, 0)}\rangle = (k + 2)\left(2k \tilde{L}_3^S - 2k J_{-3}^+ J_{0}^- + 4J_{-2}^+ J_{-1}^- + 4(k + 2) \tilde{L}_2^S \tilde{L}_1^S \right.

- 2J_{-2}^0 J_{0}^- J_{-1}^- - 3(k + 2)J_{-2}^0 \tilde{L}_1^S J_{0}^- - (k + 2)^2(\tilde{L}_1^S)^3 \right) |\{1, 0\}\rangle \quad (7.9)

= J_{-1}^+ J_{-1}^+ J_{-1}^+ |\text{MFF31}\rangle$$
The second level-3 singular state \( \langle 3.8 \rangle \) becomes, similarly,
\[
|S^{(3)}_{(0,1)}\rangle = (k+2)^3(-2(k+2)(3+2k)L^S_{-3} - 2(3+2k)J^+_3J^-_0 - 4(k+2)J^+_2J^-_1 \\
+ 4(k+2)L^S_{-2}L^S_{-1} - 2J^+_2J^-_0 + 3J^+_1J^-_1J^-_0 - (L^S_{-1})^3)|\{0,1\}\rangle
\]
\[
= J^+_1|\text{MFF13}\rangle
\]

Finally, at level 2, the first of the two topological null vector rewrites simply as
\[
|S^{(2)}_{\{\frac{1}{2},0\}}\rangle = (k+2)(-\tilde{L}^S_{-2} + J^+_2J^-_0 + (k+2)(\tilde{L}^S_{-1})^2)|\{\frac{1}{2},0\}\rangle
\]
\[
= J^+_1J^-_1|\text{MFF21}\rangle
\]
while the other one, as
\[
|S^{(2)}_{\{0,\frac{1}{2}\}}\rangle = (k+2)^2(-(k+2)\tilde{L}^S_{-2} - J^+_2J^-_0 + (\tilde{L}^S_{-1})^2)|\{0,\frac{1}{2}\}\rangle
\]
\[
= J^+_1|\text{MFF12}\rangle
\]

All these demonstrate a remarkable correspondence with the MFF states.

8 \( k \to \infty \) and alternative free-field constructions

Up to now, the normalisation of the topological singular states has been chosen in anticipation of their coincidence with the MFF states. This obviously failed at \( c = 3 \), which corresponds to the classical limit \( |k| \to \infty \). It is worth mentioning that the construction of topological theories in terms of matter dressed with gravity fails at the same value \( c = 3 \) (see eq. (1.2)). However, from the point of view of the topological algebra as such, there is nothing singular about this value, and the null states continue smoothly to this point: it suffices to multiply the above null states by the minimal power of \((c-3)\) that is necessary, and then set \( c = 3 \). Or, at level 4, for instance, we can simply put \( c = 3 \) in the ‘prototype’ state (A.1) and insert the appropriate value of \( h \) from (3.2). This produces three states given in the Appendix.

It is possible to trace the ‘classical’ degeneration of the Kazama–Suzuki mapping (4.3), (4.4) and (4.5) at \( |k| \to \infty \). While in the original representation (5.1) the limit of very large \( k \) is unclear, it can be evaluated using the ‘Sugawara’ form of the singular states, e.g. eq. (7.3). This requires rescaling the currents as \( J^{0,\pm} \to \sqrt{k}J^{0,\pm} \), after which we are left, at \( |k| = \infty \), with a ‘complex’ scalar current \( J^\pm = (\partial \varphi, \partial \varphi) \) and an independent current \( J^0 \) that decouples from the \( c = 3 \) topological algebra, while the algebra itself is now constructed from \((\partial \varphi, \partial \varphi)\) and the BC ghosts (which are not rescaled). The Sugawara energy-momentum tensor then reduces to the energy-momentum tensor of two bosons, whose central charge \( c = 2 \) is compensated by that of the BC ghosts. The topological energy-momentum tensor, determined from either (4.5) or (4.13), becomes
\[
\mathcal{T} = -\partial \varphi \partial \varphi + \partial B \cdot C \quad (c = 3),
\]
while the other topological generators reduce to
\[
\mathcal{H} = -BC, \quad \mathcal{Q} = \sqrt{2}B \partial \varphi, \quad \mathcal{G} = -\sqrt{2}C \partial \varphi \quad (c = 3).
\]
Thus the topological singular states at \( c = 3 \) can be viewed as a ‘resolution’ of the MFF construction at \( k = \infty \).

Interestingly, the formulae (8.1) and (8.2) represent also the classical limit of Witten’s free-field realisation [38] of the topological conformal algebra, derived from the Landau–Ginzburg theory (to be precise, the formulae given below are obtained from those of ref. [38] by twisting, cf. section 2). Recall that the idea behind the construction of a topological conformal algebra out of matter, Liouville, and spin-1 ghosts [23, 8] was that matter and the (‘mirror’!) Liouville give the total central charge of

\[ 1 - 3Q^2 + 1 + 3Q^2 = 2, \]

which is cancelled by the \( bc \) contribution of \(-2\). A different possibility to have central charge +2 is provided simply by a complex scalar field. We will realise this as a couple of scalars with opposite signatures, combined into \( (\varphi, \overline{\varphi}) \) with the operator product

\[ \partial \varphi(z) \partial \overline{\varphi}(w) = -1/(z - w)^2. \]

This leads to the centreless energy-momentum tensor

\[ \mathcal{T} = -\partial \varphi \partial \overline{\varphi} + \partial \mathcal{B} \cdot \mathcal{C}. \tag{8.3} \]

Here \( \mathcal{B} \) and \( \mathcal{C} \) are ghosts of weight \((0, 1)\). The other topological generators are derived from the Landau–Ginzburg considerations [38] and read

\[ \mathcal{H} = -\frac{c + 3}{6} \mathcal{B} \mathcal{C} - \frac{c - 3}{6} \varphi \partial \overline{\varphi}, \tag{8.4} \]

and

\[ Q = \sqrt{\frac{2}{6}} [(c + 3) \partial \varphi \cdot \mathcal{B} + (c - 3) \varphi \partial \mathcal{B}], \quad G = -\sqrt{2} \mathcal{C} \partial \overline{\varphi} \tag{8.5} \]

This representation of the topological conformal algebra continues smoothly to \( c = 3 \), where it, obviously, coincides with what we had in (8.1)–(8.2) as a result of taking the classical limit of the ‘Kazama–Suzuki mapping’ (4.3)–(4.3). We will return to the construction (8.3)–(8.5) in section 9.

Curiously, the formulae (8.3)–(8.5) are not the only realisation of the topological conformal algebra with the energy-momentum tensor (8.3), in terms of the specified fields \((\varphi; \overline{\varphi}; \mathcal{B}, \mathcal{C})\). Another one can be obtained from the above construction (2.7)–(2.11) if we bosonise the matter through a current \( \partial u \) and then change the basis of fields to \((\varphi; \overline{\varphi}; \mathcal{B}, \mathcal{C})\), introduced via

\[ b = e^{Q \varphi} \mathcal{C}, \quad c = e^{-Q \varphi} \mathcal{B}, \quad \partial u = -\partial \overline{\varphi} + Q \mathcal{B} \mathcal{C} + \frac{1-Q^2}{2} \partial \varphi, \quad I = -\partial \overline{\varphi} + Q \mathcal{B} \mathcal{C} - \frac{1+Q^2}{2} \partial \varphi \tag{8.6} \]

where, as before, \( Q = \sqrt{(1-d)/3} \) and \( \partial \varphi(z) \partial \overline{\varphi}(w) = -1/(z - w)^2, \mathcal{B}(z) \mathcal{C}(w) = 1/(z - w) \). In terms of the new fields, the energy-momentum tensor (2.7) takes precisely the form (8.3), but the other topological generators are in no simple way related to those from eqs. (8.4) and (8.5). It is doubtful that the construction based on (8.6) would have any meaning beyond an exercise in bosonisation.

### 9 (No) MFF states in Landau–Ginzburg theories

In this section we use the construction (8.3)–(8.5) [38] for the topological conformal algebra in order to evaluate the topological/MFF singular states in the \( N=2 \) Landau–Ginzburg theory (see [57]–[61] and references therein; our analysis pertains, obviously, to undeformed Landau–Ginzburg models describing conformal topological theories).
The ‘bosonisation’ (8.3)–(8.5) does not refer explicitly to either a ‘constituent’ matter theory with central charge (1.2) or to an \( sl(2) \) algebra, but instead it bears a Landau–Ginzburg interpretation, as explained in [38]. However, the formulae for \( \mathcal{H} \) and \( Q \) should be handled with caution, since they involve zero mode of the \( \varphi \) field. In particular, the states that the generators are acting on should be defined carefully. Since \( \varphi \) is OPE-isotropic, an obvious possibility is to start with states represented as functions of \( \varphi \) only (and possibly the ghosts). Then, a state satisfying eqs. (1.3) (i.e. a chiral primary state) can be represented by the operator

\[
\Psi_j = \varphi^{6\hbar/(c-3)} = \varphi^{2j} \tag{9.1}
\]

(where \( j \) is the \( sl(2) \) spin, see eq. (4.19)).

Now it becomes possible to evaluate in terms of the construction (8.3)–(8.5) the singular vectors constructed above. They turn out to \textit{vanish}. The derivation is straightforward, starting from the definition of the mode action

\[
(L_n\Psi)(w) = \oint (z-w)^{n+1} T(z) \cdot \Psi(w), \quad (H_n\Psi)(w) = \oint (z-w)^n \mathcal{H}(z) \cdot \Psi(w),
\]

\[
(Q_n\Psi)(w) = \oint (z-w)^n Q(z) \cdot \Psi(w), \quad (G_n\Psi)(w) = \oint (z-w)^{n+1} G(z) \cdot \Psi(w),
\]

and evaluating operator products in the integrands. As a sample calculation, consider the term \((Q_{-1}G_{-1}\Psi)(w)\) (which is a part of the level-2 vectors (3.3) and (3.10)): we find

\[
(Q_{-1}\Psi)(w) = \oint G(z)\Psi(w) = 2j\sqrt{2} C(w) \varphi(w)^{2j-1} \tag{9.3}
\]

and then,

\[
Q(z)(Q_{-1}\Psi)(w) = \frac{4j}{k+2} \left( \frac{\varphi(z)\varphi(w)^{2j-1}}{(z-w)^2} + (k+1) \frac{\varphi(z)\varphi(w)^{2j-1}}{z-w} \right) + (k+1)\partial\varphi(z) B(z) C(w) \varphi(w)^{2j-1} - \varphi(z) \partial B(z) C(w) \varphi(w)^{2j-1} \tag{9.4}
\]

so that

\[
(Q_{-1}G_{-1}\Psi)(w) = \oint (z-w)^{-1} Q(z)(Q_{-1}\Psi)(w) = \frac{2j}{k+2} \left[ (2k+3)\partial^2 \varphi(w) \varphi(w)^{2j-1} - 2\partial B(w) C(w) \varphi(w)^{2j-1} \right]
\tag{9.5}
\]

By doing similarly the other terms, we check that they add up to zero for the corresponding values of the \( sl(2) \) spin \( j = (r, s) \equiv (r-1)/2 - (k+2)(s-1)/2 \).

For the \( |MFFr1\rangle \) vectors, the details are as follows: We start with singular vectors at levels \( r = 2, 3, \) and 4 in the topological guise, as constructed in section 3. When evaluated on a \( \varphi^{2j} \) vacuum, they become, respectively,

\[
|Y^{(2)}_{\{1,0\}}\rangle = (2j-1)(k+2) \left( -\partial_\varphi \partial_\varphi \varphi^{2j} + \partial B C \varphi^{2j} + 2j(k+3)\partial_\varphi \partial_\varphi \varphi^{2j-2} \right), \tag{9.6}
\]

\[
|Y^{(3)}_{\{1,0\}}\rangle = 2(2j-1)(k+2) \left( -2\partial_\varphi \partial_\varphi \varphi^{2j+1} + 2j(10+3k)\partial_\varphi \partial_\varphi \varphi^{2j-1} - (k+2)\partial_\varphi \partial^2 \varphi \varphi^{2j} + 4\partial B C \partial_\varphi \varphi^{2j+1} - 2j(10+3k)\partial B C \partial_\varphi \varphi^{2j-1} + (k+2)\partial B \partial C \varphi^{2j} + (k-2)\partial^2 \varphi \partial_\varphi \varphi^{2j} + 2(1-2j)j(k+3)(k+4)\partial_\varphi \partial_\varphi \partial_\varphi \varphi^{2j-3} + (2-k)\partial^2 C \varphi^{2j} \right), \tag{9.7}
\]
and

\[ |\Phi_{(2,0)}^{(1)}| = \frac{1}{2}(2j-3)(k+2)(6(k+1)(k+2)BC \partial \phi^2 \phi^{2j} - 6(k+1)(k+2)B \partial C \partial \phi \phi^{2j} - 12 \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+2} + 2(-2 + 80j - k + 22j k) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-2} - 14(k+2) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 36\delta BC \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+2} + 8(1-2j)j(43 + 2k + 3k^2) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-2} + 2j(k+2)(31 + 8k) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-1} - (k+2)(k+4) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 8j(2j-1)(43 + 23k + 3k^2) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-2} + 2j(k+2)(17 + 6k) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-1} - 4j(k+2)(24 + 7k) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-1} + (k+2)(k+4) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 2(-22 + 7k) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 8j(24 - k - 2k^2) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-1} + 2(k+2)(3k-10) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 8j(24 + 7k) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j-1} + 2(10 - 3k)(k+1) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 2(1-k)(k-4) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j+1} + 2(k-4)(k-1) \partial \phi \partial \phi \partial \phi \partial \phi \phi^{2j} + (9.10) \]

These formulae abuse the notation, since, strictly speaking, the singular states \( \Phi_{(r,0)}^{(1)} \) have been defined only upon the state that now takes the form \( \phi^{-1} \), while in \( \Phi_{(j,1)}^{(1,0)} \) we have \( \phi^{2j} \) with an arbitrary \( j \) instead. Anyway, we observe the vanishing of the above states for the ‘true’ value \( j = j(r,1) \equiv (r-1)/2 \). The same holds true for the Landau–Ginzburg mapping of the other topological/MFF singular states considered in this paper. For instance, the second level-2 vector maps, in the same way, into

\[ |\Phi_{(2,0)}^{(2)}| = (k+2)^2(2 + 2j + k) \left( \partial \phi \partial \phi \phi^{2j} - \partial \phi \partial \phi \phi^{2j} \right), \]  

which vanishes for the corresponding value of the \( sl(2) \) spin \( j = j(1,2) = -\frac{1}{2}(k+2) \).

It may be interesting to look more carefully at the states obtained by dropping the factor \( (j - j(r,s)) \) in the \( |MFF|s \) states evaluated on the \( \phi^{2j} \) vacuum (with \( j \) set to \( j(r,s) \) afterwards). At level 2, for instance, these read

\[ (k+2)(-\partial \phi \partial \phi \phi + \partial BC \phi + (k+3)\partial \phi \partial \phi \phi^{-1}) \]  

and

\[ (k+2)^2(\partial \phi \partial \phi \phi^{-k-2} - \partial BC \phi^{-k-2}). \]

These, as well as the other states (for \( rs \leq 4 \) thus obtained are BRST-closed, while being at the same time non-polynomial in \( \phi \).

10 Concluding remarks

The fact that the topological conformal theories share their singular states with the \( sl(2) \) theories points to a close relation between these two classes of theories. This relation generalises the results of \( \mathbb{L}, \mathbb{R} \) on the reduction from WZW to minimal models, by ‘lifting it up’ to the \( N = 2 \) case. The null states are just equal, while the reduction of the decoupling equations performed in
gets replaced with the decomposition $t \rightarrow m \oplus I \oplus [bc]$ of the topological theory into matter, Liouville, and ghosts, considered in section 2. It can thus be applied to the whole algebra, not only to its singular states. Even from the point of view of the correspondence between $sl(2)$ and minimal singular vectors, incorporating the ghosts and the Liouville simplifies it greatly, allowing us to work at the level of singular states as such, rather than the decoupling equations.

Note that when taking singular states in the MFF form, the associated decoupling equations do not have the usual form of differential equations; this, however, can be restored by using the ‘Sugawara’ form (see, e.g., eq. (7.3)) of the same singular vectors (which must be tantamount, eventually, to repeatedly using the Knizhnik–Zamolodchikov equations for the correlators and other related identities elaborated in [14]).

Exploiting the usual Ward identities for the Sugawara Virasoro generators $\tilde{L}^S_m$, we get, for the topological/MFF states, the decoupling equations implemented by differential operators with coefficients from the Lie algebra $sl(2)$. Alternatively, we could start with the singular states written in the topological guise and use the construction for the topological algebra generators in terms of matter, Liouville, and ghosts. Here, as has been noted above, it would make sense to discard all the $bc$ ghost contributions, since, at the level of decoupling equations, this would correspond to taking all the fields in a given correlator as chiral primary ones. It is these ‘ghost-free’ reduced decoupling operators that are related to the Virasoro constraints introduced in terms of the times $(2.13)$ [1, 3]. An interesting question is which form the ‘hidden’ $sl(2)$ structure (due to the MFF description) takes in these reduced decoupling operators. The answer can be expected in terms of the formalism developed recently in refs. [51]–[56]. It can be seen in simple examples (although the argument is by far not complete) that the currents $J^{\pm,0}$ which enter in these operators on top of the twisted Sugawara Virasoro generators $\tilde{L}^S_m$, arrange into the combinations

$$(J^{-n}, J_0)_{sl(2)} \sim -\sum_{b \neq a} P_{ab} \left( \frac{z_b - z_a}{n} \right),$$

with $P_{bc} = (t_b, t_c)_{sl(2)} \equiv t_0^b t_0^c - \frac{1}{2} (t_b^+ t_c^- + t_b^- t_c^+)$, where $(, )_{sl(2)}$ is the Killing form and $t_{\pm,0}$ are the Lie algebra generators, with the standard notation $t_0^b$ used for $t^a$ in the $b$-th position in $1 \otimes \ldots \otimes 1 \otimes t^a \otimes 1 \otimes \ldots \otimes 1$ (the RHS is understood to act on a correlator $\langle \Psi(z_a) \prod_{b \neq a} \Psi_b(z_b) \rangle$).

It turns out further that, by imposing a ‘generalised symmetry’ condition $P_{ab} \sim j_a j_b$, where the $j_b$ are $sl(2)$ spins of insertions in the correlator, the decoupling operators become exactly those of the matter + Liouville theory (i.e., the reduced ‘ghost-free’ ones). This points to the relation with certain other extensions of differential operators, considered recently in a number of papers [51]–[56]. There, our ‘reduced’ decoupling operators have been encountered in the analysis of the Calogero model and they, too, appeared as a reduction of more general operators that extend the differential ones by the group that permutes insertions in the correlator (see, in particular, ref. [56]). Relation of these operators with the Knizhnik–Zamolodchikov equations has also been pointed out in [57].

Since the MFF formula (6.1) gives the general (modulo the remarks following (6.3)) form of $sl(2)$ singular vectors, by feeding into it the construction (4.11) for the currents (or, its ‘irreducible’ version (5.3)), we get, formally, a general form of the topological singular states, in the case when the topological theory is interpreted as the appropriately dressed matter theory. As to the

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4The algebraic nature of the reduction from $sl(2)$- to minimal singular states has been revealed in [15] (I thank V. Petkova for pointing this paper out to me). As we see now, the BRST formalism developed there extends naturally to the $N=2$ framework.
practical use of these formulae (which is not at all straightforward), note that in the \( m \oplus l \oplus [bc] \) interpretation of the topological theory, the chiral primary states (I.17) can be represented by the following ghost-independent operators:

\[
\Psi_j = e^{j \sqrt{\frac{2}{k+2}(v-\phi)}} \psi_j
\]  

(10.1)

where \( \psi_j \) is a matter primary state. Now, when evaluating the operator products such as \( J^\pm(z) \cdot \Psi_j(w) \), the fusion of the two exponentials does not produce a pole, while with respect to the energy-momentum tensors entering \( J^- \), the field \( \Psi_j \) has dimension zero. It is very interesting if other, more effective, free-field constructions for the \( sl(2) \) current algebra exist that would make the MFF formulae (6.1) directly applicable (i.e. would allow a direct evaluation of, say, \( (J^-)^{r+p(k+2)} \)).

The ‘irreducible’ form (5.3) of the construction (4.11) for the \( sl(2) \) currents provides the ‘invariant’ version of the Wakimoto representation that incorporates a central-charge-\( d \) matter theory. Thus any matter with central charge \( d \leq 1 \) or \( d \geq 25 \) can be dressed up into an \( sl(2) \) WZW theory (and, conversely, recovered from the latter via the Hamiltonian reduction). Another dressing, that of ref. [23], makes up a topological conformal theory, while singular vectors of the two resulting theories are identical (see the diagram (7.2)).

An interesting question is whether the relation between the topological and the \( sl(2) \) algebras can be understood in the context of the hidden \( sl(2) \) symmetry of quantum gravity of ref. [37], as, for instance, a ‘covariant’ (involving ghosts, in particular) counterpart of the ‘light-cone’ \( sl(2) \).

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Appendix

Level 4 provides the most representative of the examples of singular states considered in the text, since there exists the ‘difficult’ \( \{ \frac{1}{2}, \frac{1}{2} \} \)-state along with the \( \{ \frac{1}{2}, 0 \} \) and \( \{ 0, \frac{1}{2} \} \) ones. All the three topological singular vectors at level 4 can be derived from the following ‘prototype’ state

\[
\begin{align*}
&36(135 - 54c + 7c^2 + 30h + 58ch)H_{-4} + 3h(351 - 18c - 5c^2 + 246h - 14ch + 96h^2)(G_{-3}Q_{-1} - 2L_{-3}H_{-1}) \\
&+ 2(1215 - 486c + 63c^2 + 108h + 657ch - 60c^2h - c^3h + 252h^2 \\
&\quad - 18ch^2 + 14c^2h^2 + 216h^3 - 24ch^3 - 144h^4)L_{-4} \\
&+ 2(486 - 459c + 156c^2 + c^3 - 585h + 168h - 47c^2h - 378h^2 + 18c^2h + 288h^3)L_{-3}L_{-1} \\
&+ 12h(189 - 54c + c^2 + 84h + 28ch - 24h^2)H_{-2}L_{-2} \\
&+ 2(810 - 81c - 12c^2 - 5c^3 + 72h + 414c - 14c^2h - 144h^2 - 96c^2h)L_{-3}L_{-1} \\
&- 6h(270 - 30c^2 + 279h + 6ch + 11c^2h + 6h^2 + 2ch^2 - 48h^3)L_{-2}^2 \\
&+ (1458 - 1593c - 108c^2 - 5c^3 - 1485h + 588ch + c^2h - 306h^2 - 54c^2h + 288h^3)Q_{-3}G_{-1} \\
&+ 3(-405 + 162c - 21c^2 + 288h - 282ch + 2c^2h + 168h^2 + 56c^2h - 48h^3)Q_{-2}G_{-2} \\
&+ 216h(3 + c + 4h)(G_{-2}H_{-1}Q_{-1} - L_{-2}^2) + 9(-135 + 54c - 7c^2 + 6h - 46ch + 48h^2)G_{-2}L_{-1}Q_{-1} \\
&+ 3(-243 + 18c - 7c^2 - 210h - 70h - 48h^2)(H_{-2}Q_{-1}G_{-1} + 2H_{-2}Q_{-1}L_{-1}) \\
&+ 4(27c + 12c^2 + c^3 - 720h + 276ch + 4c^2h - 252h^2 - 84ch^2 + 144h^3)H_{-2}L_{-1}^2 \\
&+ 12(162 - 18c^2 + 135h + 54c + 11c^2h + 6h^2 + 2c^2h - 48h^3)L_{-2}H_{-1}L_{-1} \\
&+ 2(324 - 972c + 204c^2 + 28c^3 + 1287h - 297ch - 275c^2h - 11c^3h \\
&\quad + 1512h^2 + 72ch^2 + 64c^2h^2 + 180h^3 + 60c^3h + 288h^4)L_{-2}L_{-2}^2 \\
&+ 3(-81 + 162c - 57c^2 + 288h - 30ch + 22c^2h + 156h^2 + 4c^2h - 96h^3)L_{-2}Q_{-1}G_{-1} \\
&- 9(27 - 126c - 5c^2 - 78h + 22ch + 48h^2)Q_{-2}H_{-1}G_{-1} + 108(3 + c + 4h)(2H_{-1}Q_{-1} + 3H_{-1}^2G_{-1}) \\
&+ (1377 - 486c - 207c^2 - 4c^3 - 1458h + 978ch + 56c^2h - 576h^2 - 240c^2h + 288h^3)Q_{-2}L_{-1}G_{-1} \\
&+ 6(81 - 126c + 11c^2 - 6h - 2c^2h)(H_{-2}^2L_{-1}G_{-1} + H_{-1}L_{-1}Q_{-1}G_{-1}) \\
&+ 6(-189 + 27c + 33c^2 + c^3 - 72ch - 8c^2h + 36h^2 + 12c^2h)H_{-1}L_{-1}^3 \\
&+ \frac{1}{6}(-405 + 3024c - 738c^2 - 120c^3 - c^4 - 3294h + 1026ch + 822c^2h \\
&\quad + 38c^3h - 4644h^2 - 72ch^2 - 180c^2h^2 - 648h^3 - 216ch^3 + 864h^4)L_{-1}^4 \\
&+ 3(-189 + 27c + 33c^2 + c^3 - 72ch - 8c^2h + 36h^2 + 12c^2h)L_{-1}^2Q_{-1}G_{-1}
\end{align*}
\]

(A.1)

by simply substituting into it the corresponding relation for the topological \( U(1) \) charge from eq. (3.2). While for the singular vectors thus obtained we have established their identity with the MFF states, the state (A.1), on the contrary, does not demonstrate any good behaviour under the mapping (4.1)–(4.2): the \( BC \) modes would not decouple from the \( sl(2) \) currents unless (A.1) is actually reduced to one of the singular states (3.4), (3.5) or (3.6).
In the ‘classical’ case of topological central charge \( c = 3 \) the mapping to the \( sl(2) \) states degenerates. Nevertheless, we find from (A.1) the corresponding singular vectors of the \( c = 3 \) topological conformal algebra:

\[
\begin{align*}
\mathcal{H}_{-4} + \frac{1}{2} \mathcal{L}_{-4} + \frac{5}{6} \mathcal{H}_{-3} \mathcal{L}_{-1} + \frac{1}{2} \mathcal{L}_{-3} \mathcal{L}_{-1} - \frac{4}{3} \mathcal{Q}_{-3} \mathcal{G}_{-1} - \frac{1}{4} \mathcal{Q}_{-2} \mathcal{G}_{-2} - \frac{1}{4} \mathcal{G}_{-2} \mathcal{L}_{-1} \mathcal{Q}_{-1} - \frac{7}{6} \mathcal{H}_{-2} \mathcal{H}_{-1} \mathcal{L}_{-1} \\
+ \frac{2}{3} \mathcal{H}_{-2} \mathcal{L}_{-1}^2 - \frac{7}{6} \mathcal{H}_{-2} \mathcal{Q}_{-1} \mathcal{G}_{-1} - \frac{4}{3} \mathcal{L}_{-2} \mathcal{Q}_{-1} \mathcal{G}_{-1} - \frac{1}{4} \mathcal{Q}_{-2} \mathcal{H}_{-1} \mathcal{G}_{-1} - \frac{1}{4} \mathcal{H}_{-2} \mathcal{L}_{-1} \mathcal{G}_{-1} + \mathcal{H}_{-1}^3 \mathcal{L}_{-1} \\
- \frac{3}{12} \mathcal{H}_{-1}^2 \mathcal{L}_{-1}^2 + \frac{2}{12} \mathcal{H}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} + \mathcal{H}_{-1} \mathcal{L}_{-1}^3 - \frac{1}{4} \mathcal{H}_{-1} \mathcal{L}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} - \frac{1}{4} \mathcal{L}_{-1}^4 + \frac{1}{2} \mathcal{L}_{-1}^2 \mathcal{Q}_{-1} \mathcal{G}_{-1} \bigg| 0 \bigg>,
\end{align*}
\]

\[
\begin{align*}
\mathcal{H}_{-4} + \frac{1}{2} \mathcal{L}_{-4} + \frac{1}{6} \mathcal{L}_{-3} \mathcal{Q}_{-1} + \frac{1}{3} \mathcal{Q}_{-2} \mathcal{L}_{-2} - \frac{1}{3} \mathcal{L}_{-3} \mathcal{H}_{-1} + \frac{1}{3} \mathcal{L}_{-3} \mathcal{L}_{-1} - \frac{1}{3} \mathcal{L}_{-2}^2 - \frac{1}{3} \mathcal{Q}_{-3} \mathcal{G}_{-1} \\
- \frac{1}{2} \mathcal{Q}_{-2} \mathcal{G}_{-2} + \frac{1}{3} \mathcal{Q}_{-2} \mathcal{H}_{-1} \mathcal{Q}_{-1} - \frac{1}{6} \mathcal{Q}_{-2} \mathcal{L}_{-1} \mathcal{Q}_{-1} - \frac{1}{2} \mathcal{H}_{-2} \mathcal{H}_{-1} \mathcal{L}_{-1} - \frac{1}{2} \mathcal{H}_{-2} \mathcal{Q}_{-1} \mathcal{G}_{-1} \\
- \frac{1}{3} \mathcal{L}_{-2} \mathcal{H}_{-1}^2 + \frac{1}{3} \mathcal{L}_{-2} \mathcal{Q}_{-1} \mathcal{L}_{-1} + \frac{1}{3} \mathcal{L}_{-2} \mathcal{Q}_{-1} \mathcal{G}_{-1} + \frac{1}{3} \mathcal{Q}_{-2} \mathcal{H}_{-1} \mathcal{G}_{-1} - \frac{1}{3} \mathcal{Q}_{-2} \mathcal{L}_{-1} \mathcal{G}_{-1} \\
+ \frac{1}{3} \mathcal{H}_{-1}^3 \mathcal{L}_{-1} - \frac{1}{3} \mathcal{H}_{-1} \mathcal{L}_{-1}^2 + \frac{2}{3} \mathcal{H}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} - \frac{1}{2} \mathcal{H}_{-1} \mathcal{L}_{-1} \mathcal{Q}_{-1} \mathcal{G}_{-1} \bigg| 1 \bigg),
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{H}_{-4} + \frac{1}{2} \mathcal{L}_{-4} + \frac{1}{2} \mathcal{Q}_{-3} \mathcal{G}_{-1} + \frac{1}{2} \mathcal{Q}_{-2} \mathcal{H}_{-1} \mathcal{Q}_{-1} - \frac{1}{3} \mathcal{H}_{-2} \mathcal{H}_{-1} \mathcal{L}_{-1} \\
- \frac{1}{2} \mathcal{H}_{-2} \mathcal{Q}_{-1} \mathcal{G}_{-1} - \frac{1}{2} \mathcal{L}_{-2} \mathcal{L}_{-1}^2 + \frac{1}{6} \mathcal{H}_{-1}^3 \mathcal{C}_{-1} + \frac{1}{4} \mathcal{H}_{-1} \mathcal{L}_{-1} \mathcal{G}_{-1} \bigg| 3 \bigg),
\end{align*}
\]

which can then be viewed as a ‘resolution’ of the corresponding MFF states at \( k = \infty \).

Unlike the case with mapping the ‘prototype’ level-4 state (A.1) to the \( sl(2) \) module, the transformation of (A.2) to the Landau–Ginzburg variables of section 9 can be performed and is found to be proportional to \((2j - 3)(1 + 2j + k)(6 + 2j + 3k)\), thus vanishing for each of the three level-4 values of the \( sl(2) \) spin \( j \), read off from (3.2).
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