The Euclidean numbers

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Abstract
We introduce axiomatically a Nonarchimedean field $\mathbb{E}$, called the field of the Euclidean numbers, where a transfinite sum is defined that is indicized by ordinal numbers less than the first inaccessible $\Omega$. Thanks to this sum, $\mathbb{E}$ becomes a saturated hyperreal field isomorphic to the so-called Kiesler field of cardinality $\Omega$, and there is a natural isomorphic embedding into $\mathbb{E}$ of the semiring $\Omega$ equipped by the natural ordinal sum and product. Moreover a notion of limit is introduced so as to obtain that transfinite sums be limits of suitable $\Omega$-sequences of their finite subsums.

Finally a notion of numerosity satisfying all Euclidean common notions is given, whose values are nonnegative nonstandard integers of $\mathbb{E}$. Then $\mathbb{E}$ can be charachterized as the hyperreal field generated by the real numbers together with the semiring of numerosities (and this explains the name “Euclidean” numbers).

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Introduction
In this paper we introduce a numeric field denoted by $\mathbb{E}$, which we name the field of the Euclidean numbers. The theory of the Euclidean numbers combines the Cantorian theory of ordinal numbers with Non Standard Analysis (NSA).
From the algebraic point of view, the Euclidean numbers are a non-Archimedean field with a supplementary structure (the Euclidean structure), which characterizes it. This Euclidean structure is introduced axiomatically by the operation of transfinite sum: more precisely, in Section 2 we introduce sums of the type $\sum_{k<\alpha} a_k$ where the $a_k$s are real numbers, while $k$ and $\alpha$ are ordinal numbers in $\Omega$, the set of the ordinals smaller than the first inaccessible ordinal. We give in Subsection 2.1 five natural axioms that rule the behaviour of these transfinite sums.

We list the main peculiarities of the Euclidean numbers that we deduce from the axioms on transfinite sums:

- The field $\mathbb{E}$ is saturated with respect to the order relation, actually it is the unique saturated real closed field having the cardinality of $\Omega$ (see Subsection 3.3). This property implies that every ordered field having cardinality less than or equal to $\Omega$ is (isomorphic to) a subfield of $\mathbb{E}$.

- Every Euclidean number can be obtained as a transfinite sum of real numbers; more generally, a transfinite sum of Euclidean numbers is well defined in Subsection 2.1 and it can be obtained as limit of ordinal-indexed partial sums, under an appropriate notion of limit, given in Subsection 2.3.

- Any accessible ordinal $\alpha \in \Omega$ can be identified with the transfinite sum of $\alpha$ ones in $\mathbb{E}$; this identification is consistent with the natural ordinal operations $+$ and $\cdot$ (see Subsection 2.2), so the field of the Euclidean numbers can be considered as a sort of natural extension of the semiring of the (accessible) ordinal numbers.

- The Euclidean numbers are a hyperreal field; more precisely $\mathbb{E}$ is isomorphic to the hyperreal Keisler field introduced in [20]; the Keisler field is the unique saturated (in the sense of NSA) hyperreal field having the cardinality of $\Omega$ (see Subsection 3.4).

- The Euclidean numbers are strictly related to the notion of numerosity, introduced in [2, 4, 6] and developed in [7, 16, 13, 19], so as to save the five Euclidean common notions (see Section 4). In fact, $\mathbb{E}$ can be characterized as the hyperreal field generated by the real numbers and the semiring of numerosities, provided that the numerosity is defined on a coherent family of labelled sets containing the accessible ordinal
numbers. The numerosity theory provided by the Euclidean numbers satisfies the following properties which altogether are not shared by other numerosity theories (see Subsections 4.1-4.2):

- each set (in this theory) is equinumerous to a set of ordinals,
- the set of numerosities $\mathcal{N}$ is a positive subsemiring of nonstandard integers, that generates the whole $\mathbb{Z}^*$.

We have chosen to call $\mathbb{E}$ the field of the Euclidean numbers for two main reasons: firstly, this field arises in a numerosity theory (including the subsets of $\Omega$), whose main aim is to save all the Euclidean common notions, including the fifth

*The whole is greater than the part,*

in contrast to the Cantorian theory of cardinal numbers.

The second reason is that, in our opinion, the field $\mathbb{E}$ describes the Euclidean continuum better than the *real field* $\mathbb{R}$, at least when looking for a set theoretic interpretation of the Euclidean geometry. This last point has been dealt with in [9] and will be shortly outlined in the Appendix.

1 Notation and preliminary notions

Let $\Omega$ be the least (strongly) *inaccessible* cardinal. Or better, taking into account that in what follows the ordinals are viewed “à la Cantor” as *atomic* numbers, which are not identified with, but rather considered as the *order types of* the set of all the smaller ordinals, let $\Omega$ be the *set of all accessible ordinals*, and in general $\Omega_\alpha = \{\beta \in \Omega \mid \beta < \alpha\}$.

1.1 Operations on $\Omega$

Since we use the ordinary symbols $\cdot$ and $+$ for the operations on the Euclidean numbers that we shall define in section 4.2 and among them we shall include the ordinals, the usual ordinal multiplication and addition on $\Omega$ will be denoted by $\circ$ and $\oplus$, respectively, whereas $\cdot$ and $+$ will correspond to the so called natural operations, see below.

Given ordinals $x, j \in \Omega$, there exist uniquely determined ordinals $k \in \Omega$ such that $s < 2^j$, such that

$$x = (2^j \circ k) \oplus s.$$
Recall that each ordinal has a unique base-2 normal form

\[ x = \sum_{n=1}^{N} 2^{j_n} \]

where \( n_1 < n_2 \Rightarrow j_{n_1} > j_{n_2} \).

As we identify the ordinals in \( \Omega \) with numbers of the field \( \mathbb{E} \), we shall simply write the normal form \( x = \sum_{n=1}^{N} 2^{j_n} \), independently of the ordering of the exponents. But one has to be careful: sum and product agree with the ordinary ordinal operations only when the exponents are decreasing and the integer coefficients are put on the right side. On the other hand, the exponentiation between ordinals is intended as the ordinal exponentiation, and so it differs from the nonstandard extension of the real exponentiation.

In particular \( 2^\omega = \omega \), and the power \( 2^\alpha = \omega^\alpha \) whenever \( \alpha = \omega \odot \alpha \). It follows that the fixed points of the function \( \alpha \mapsto 2^\alpha \) are \( \omega \) and and the so called \( \varepsilon \)-numbers \( \varepsilon \) such that \( \omega^\varepsilon = \varepsilon \).

### 1.2 Finite sets of ordinals

The usual antilexicographic wellordering of the finite sets of ordinals is defined by

\[ L_1 < L_2 \text{ if and only if } \max (L_1 \triangle L_2) \in L_2. \]

In this ordering \( 2^\alpha \) is the order type of the set \( \mathcal{P}_{\text{fin}}(\Omega_\alpha) \) of all finite sets of ordinals less than \( \alpha \), hence the family \( \mathcal{L} = \mathcal{P}_{\text{fin}}(\Omega) \) of all finite subsets of \( \Omega \) can be isomorphically indexed by \( \Omega \). Therefore we shall denote by \( L_\alpha \) the \( \alpha \)th set of ordinals, namely

\[ L_\alpha = \{\alpha_1, \ldots, \alpha_n\} \text{ for } \alpha = \sum_{1}^{n} 2^{\alpha_i}. \]

In particular

\[ L_0 = \emptyset, \quad L_2^\alpha = \{\alpha\}, \quad \text{and } \quad L_{2^\alpha+\beta} = \{\alpha\} \cup L_\beta \text{ for all } \beta < 2^\alpha. \]

The order isomorphism \( \alpha \mapsto L_\alpha \) between \( \Omega \) and \( \mathcal{L} \) allows to single out two restrictions of the ordinal ordering on \( \Omega \), that will be useful in the following sections: the formal membership \( \prec \) that corresponds to ordinary membership, and the formal inclusion \( \sqsubseteq \), that corresponds to ordinary inclusion:
Definition 1.1.

**formal membership**  Given $\alpha, \beta \in \Omega$ we say that $\beta$ is a *formal member* of $\alpha$, written $\beta \lessdot \alpha$, if and only if $\beta \in L_{\alpha}$, or equivalently if and only if $2^\beta$ appears in the base-2 normal form of $\alpha$.

Hence $\alpha = \sum_{\beta \lessdot \alpha} 2^\beta$.

**formal inclusion**  Given $\alpha, \beta \in \Omega$ we say that $\alpha$ is *formally included* in $\beta$ (written $\alpha \subseteq \beta$) if and only if $L_{\alpha} \subseteq L_{\beta}$.

Hence $\alpha = \sum_{i \in I} 2^j \subseteq \beta = \sum_{h \in H} 2^j \iff I \subseteq H$.

So the formal members of $\alpha$ are the ordinary members of $L_{\alpha}$, while the formal inclusion $\beta \subseteq \alpha$ reflects the ordinary inclusion between the corresponding finite sets $L_{\beta} \subset L_{\alpha}$. In particular the following useful properties hold:

- $0 \subseteq \alpha$ for all $\alpha \in \Omega$.
- Both $\{|\xi | \xi \subseteq \alpha\} = 2^{|L_{\alpha}|}$ and $\{|\xi | \xi \lessdot \alpha\} = |L_{\alpha}|$ are finite for all $\alpha \in \Omega$;
- The relation $\subseteq$ equips $\Omega$ with a natural *lattice* structure where $L_{\alpha} \cup L_{\beta} = L_{\alpha \lor \beta}$ and $L_{\alpha \land \beta} = L_{\alpha} \cap L_{\beta}$.
- For all $h, k, \alpha \in \Omega$ one has $h, k \lessdot \alpha \iff 2^h \lor 2^k \subseteq \alpha$.

In order to deal with the Euclidean numbers, we single out the following class of ordinals (or better of finite sets of ordinals)

**Definition 1.2.** Let $\eta = 2^\eta$ be a fixed point ordinal, and let $h$ be an ordinal in $\Omega$. An ordinal $\alpha$ (and the corresponding set $L_{\alpha}$) is *$(\eta, h)$-complete* if

$\{ k < \eta \mid k \lessdot \alpha \} \iff \eta \lor h + k \lessdot \alpha$.

The important property of the complete ordinals is the following:

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1 The name formal inclusion should recall that the respective base-2 normal forms are indeed contained one inside of the other one.
Lemma 1.3. Let

\[ D(\eta, h) = \{ \alpha \in \Omega \mid \forall k < \eta. k < \alpha \iff \eta \odot h + k < \alpha \} \]

be the set of all \((\eta, h)\)-complete ordinals, and let

\[ C(\beta) = \{ \alpha \mid \beta \subseteq \alpha \} = \{ \alpha \mid L_\beta \subseteq L_\alpha \} \]

be the cone over \(\beta\) with respect to the formal inclusion.

Then the family

\[ D = \{ D(\eta, h) \mid h, \eta = 2^\eta \in \Omega \} \cup \{ C(\beta) \mid \beta \in \Omega \} \]

enjoys the finite intersection property \(\text{FIP}\).

Hence \(D\) generates a filter \(Q\) over \(\Omega\) (and correspondingly over \(\mathcal{L}\)).

**Proof.** Let \(E = \{ \eta_1 < \eta_2 < \ldots < \eta_n \}\) be a set of fixed point ordinals \(\eta_i = 2^n\), and let \(H = \{ h_1, \ldots, h_m \}\) be a set of ordinals. Define \(D(E, H; \beta) \subseteq \Omega\) as the set of all ordinals \(\alpha \supseteq \beta\) that satisfy the following conditions for all \(i \leq n\) and all \(j \leq m\):

- for all \(k < \eta_i. k < \alpha \iff \eta_i \odot h_j + k < \alpha\).

One has \(C(\beta) \cap C(\beta') = C(\beta \lor \beta')\); hence it suffices to show that each \(D(E, H; \beta)\) is nonempty.

Define inductively the ordinals \(\alpha_i\) as follows:

- In order to obtain \(L_{\alpha_1}\), firstly add to \(L_\beta\) the ordinals \(k < \eta_1\) such that, for some \(j, \eta_1 \odot h_j + k \in L_\beta\), and then, for all \(k < \eta_1\) now in \(L_{\alpha_1}\), add all ordinals \(\eta_i \odot h_j + k\);

- similarly, in order to obtain \(L_{\alpha_{i+1}}\), firstly add to \(L_{\alpha_i}\) the ordinals \(k < \eta_{i+1}\) such that, for some \(j, \eta_{i+1} \odot h_j + k \in L_{\alpha_i}\), and then, for all \(k < \eta_{i+1}\) now in \(L_{\alpha_{i+1}}\), add all ordinals \(\eta_{i+1} \odot h_j + k\).

Then clearly \(\alpha_n\) belongs to \(D(E, H; \beta)\). \(\square\)

**Remark 1.1.** If \(\beta < \eta = 2^n\) and \(E, H \subseteq \Omega_\eta\), then the ordinal \(\alpha_n\) of the above proof is smaller than \(\eta\). Hence the family \(D_\eta = \{ D(E, H; \beta) \cap \Omega_\eta \mid E, H \subseteq \Omega_\eta, \beta < \eta \}\) enjoys the \(\text{FIP}\), and generates a filter \(Q_\eta\) on \(\Omega_\eta\).
2 The field of the Euclidean numbers

We introduce the field of the Euclidean numbers as constituted of all transfinite sums of real numbers, of length equal to some accessible ordinal in $\Omega$. In order to avoid the antinomies and paradoxes that might affect the summing up of infinitely many numbers, it seems essential to consider only sums of indexed elements. In particular, the choice of ordinal numbers as indices seems particularly appropriate, given their natural ordered structure. We ground on the properties of the formal inclusion $\sqsubseteq$ and of the formal membership $\prec$, introduced in Subsection 1.2.

2.1 Axiomatic introduction of the Euclidean numbers as infinite sums

Let $E$ be an ordered superfield of the reals $\mathbb{R}$ and assume that a transfinite sum

$$\sum_{k<\beta} \xi_k$$

is defined for all $\beta \in \Omega$ and all $\xi = (\xi_k \mid k \in \Omega) \in E^\Omega$.

We make the natural assumption that a transfinite sum coincides with the ordinary sum of the field $E$ when the number of non-zero summands is finite.

We call $E$ the field of the Euclidean numbers if the following axioms are satisfied.

**LA Linearity Axiom:**

The transfinite sum is $\mathbb{R}$-linear, i.e.

$$s \sum_{h<\beta} \xi_h + t \sum_{h<\beta} \zeta_h = \sum_{h<\beta} (s\xi_h + t\zeta_h) \text{ for all } s, t \in \mathbb{R} \text{ and } \xi_h, \zeta_h \in E.$$

**RA Real numbers Axiom:**

For all $\xi \in E$ there exist $\beta \in \Omega$ and $x \in \mathbb{R}^\Omega$ such that

$$\xi = \sum_{h<\beta} x_h.$$
Grounding on the axiom RA, in the following axioms we restrict ourselves to considering transfinite sums of real numbers. Firstly, an axiom for comparing transfinite sums:

**CA Comparison Axiom**

For all \( x, y \in \mathbb{R}^\Omega \),

\[
\text{if } \sum_{k<\alpha} x_k \leq \sum_{k<\alpha} y_k \text{ for all } \alpha \geq \beta, \text{ then } \sum_{k<\beta} x_k \leq \sum_{k<\beta} y_k
\]

(Remark that the sums \( \sum_{k<\alpha} \) are ordinary finite sums of real numbers.)

We define also a double sum:

\[
\sum_{h,k<\beta} x_{hk} = \sum_{j<\beta} y_j, \text{ where } y_j = \sum_{h,k<^* j} x_{hk}.
\]

Here \( h,k <^* j \) means that at most one between \( h \) and \( k \) may belong to some \( i \sqsubseteq j \), so as to have that \( \sum_{h,k<\alpha} x_{hk} = \sum_{j<\alpha} y_j \).

The double sum allows to compute the products according to the following axiom:

**PA Product axiom:**

\[
(\sum_{h<\beta} x_h)(\sum_{k<\beta} y_k) = \sum_{h,k<\beta} x_h y_k.
\]

In general, the double sum is different from the corresponding sum of sums, as we shall see below. So we give an axiom in order to simplify a sum of sums:

**SA Sum axiom:** If \( \beta < \eta = 2^\eta \), then

\[
\sum_{h<\beta} \sum_{k<\beta} x_{hk} = \sum_{i<\eta \circ \beta} y_i, \text{ where } y_i = \begin{cases} x_{hk} & \text{if } i = \eta \circ h + k \\ 0 & \text{otherwise} \end{cases}.
\]

**CAVEAT:** It is not true, in general, that the double sum equals the corresponding sum of sums! *E.g.* let \( \langle x_{hk} \in \{0,1\} \mid h,k < \omega \rangle \) be chosen so as to have:
\[ \xi_k = \sum_{h<\omega} x_{hk} = 1 \text{ for all } k, \quad \eta_h = \sum_{k<\omega} x_{hk} = 2 \text{ for all } h, \]

and

\[ \sum_{h,k<\eta^*} x_{hk} = \begin{cases} 1 & \text{if } j \equiv 0 \mod 2, \\ 0 & \text{otherwise}. \end{cases} \]

Then

\[ \sum_{h<\omega} \sum_{k<\omega} x_{hk} = 2 \cdot \sum_{k<\omega} \sum_{h<\omega} x_{hk} = 4 \cdot \sum_{h,k<\omega} x_{hk}. \]

Surprisingly enough, these simple and natural axioms are all that is needed in order to endow \( E \) with a very rich structure, as we shall show in the sequel.

We begin with a few simple consequences.

- **Double sum comparison:**
  If \( \sum_{h,k<\alpha} a_{hk} \leq \sum_{h,k<\beta} b_{hk} \) for all \( \alpha \equiv \beta \), then \( \sum_{h,k<\beta} a_{hk} \leq \sum_{h,k<\beta} b_{hk} \).

- **Translation invariance:**
  If \( \eta = 2^n > \beta \), then, for all \( h \in \Omega \),

\[
\sum_{k<\beta} x_k = \sum_{i<\eta \odot h + \beta} y_i, \quad \text{where } y_i = \begin{cases} x_k & \text{if } i = \eta \odot h + k \\ 0 & \text{otherwise} \end{cases}.
\]

In fact, put \( x_{jk} = \begin{cases} x_k & \text{if } j = 0, h \\ 0 & \text{otherwise} \end{cases} \).

Then \( x_{hk} = y_{\eta \odot h + k} \) and so, by the axioms LA and SA,

\[ \sum_{j} \sum_{k} x_{jk} = 2 \sum_{k} x_k = \sum_{k} x_k + \sum_{i} y_i. \]
• **Associativity:**

For $\alpha < \beta \in \Omega$ denote

$$\sum_{\alpha \leq k < \beta} \xi_k = \sum_{k \in [\alpha, \beta)} \xi_k = \sum_{k < \beta} \xi_k - \sum_{k < \alpha} \xi_k$$

Then the axiom SA yields, for $\beta < \eta = 2^n$,

$$\sum_{k < \eta \cap \beta} x_k = \sum_{h < \beta} \sum_{\eta \cap h \leq k < \eta \cap (h+1)} x_k.$$

**Remark 2.1.** The initial assumption that *sums of finitely many non-zero elements receive their natural values in $E$* could be deduced from the axioms LA, RA, PA, and CA: in fact such sums are equal to the constant sequence of their sum $s$ in $E$, by comparison, hence equal to a multiple $s \cdot \xi$, by linearity. Then $\xi^2 = \xi$ follows by the product formula, and so $\xi = 1$.

### 2.2 Ordinal numbers as Euclidean numbers

An important consequence of the axioms is the existence of a natural isomorphic embedding of $\Omega$ (as ordered semiring with natural sum and product) into $E$:

**Theorem 2.2.** Define $\Psi : \Omega \rightarrow E$ by $\Psi(\alpha) = \sum_{k < \alpha} 1_k$, where $1_k$ denotes the constant 1. then:

$$\Psi(\alpha + \beta) = \Psi(\alpha) + \Psi(\beta) \quad \text{and} \quad \Psi(\alpha \cdot \beta) = \Psi(\alpha) \cdot \Psi(\beta).$$

**Proof.** In order to prove that $\Psi(\alpha + \beta) = \Psi(\alpha) + \Psi(\beta)$, it suffices to show that $\Psi$ preserves the (decreasing) base-2 normal form, i.e.

$$\alpha = \sum_{i=1}^{n} 2^{j_i} (j_1 > j_2 > \ldots > j_n) \implies \Psi(\alpha) = \sum_{i=1}^{n} \Psi(2^{j_i}),$$

and this follows by repeated application of translation invariance

$$\Psi(\sum_{i=1}^{h} 2^{j_i}) + \Psi(2^{j_h+1}) = \sum_{k < \sum_{i=1}^{h} 2^{j_i}} 1_k + \sum_{k < 2^{j_h+1}} 1_k = \sum_{k < \sum_{i=1}^{h+1} 2^{j_i}} 1_k = \Psi(\sum_{i=1}^{h+1} 2^{j_i}).$$
Remark that the equality holds also when \( j = h \), thus giving

\[
2\Psi(2^j) = \Psi(2^j) + \Psi(2^j) = \sum_{i < (2^j + 2^j)} 1_i = \Psi(2^j + 2^j) = \Psi(2^j \cdot 2).
\]

Moreover, the multiplicative property \( \Psi(\alpha \cdot \beta) = \Psi(\alpha) \cdot \Psi(\beta) \) needs to be proved only for ordinals of the form \( 2^\alpha = 2^\sum_{n=1}^{N} 2^{2n} = \prod_{n=1}^{N} 2^{2n} \), by distributivity.

Now, for pure powers \( 2^{2j}, 2^{2h} \) the equality \( \Psi(2^{2j} \cdot 2^{2h}) = \Psi(2^{2j})\Psi(2^{2h}) \) follows by applying associativity and product formula:

\[
\Psi(2^{2j+2h}) = \sum_{k \in [0, 2^{2j} \cdot 2^{2h})} 1_k = \sum_{i \in [0, 2^{2h})} \sum_{k \in [2^{2j} \odot i, 2^{2j} \odot (i+1))} 1_k =
\]

\[
= \sum_i \sum_k \chi_{2^{2j}}(k) \chi_{2^{2h}}(i) = \Psi(2^{2j})\Psi(2^{2h})
\]

By virtue of this theorem, we shall identify each ordinal \( \alpha \in \Omega \) with the corresponding Euclidean number \( \Psi(\alpha) \), so as to obtain that \( \Omega \subseteq \mathbb{E} \), exactly as we have assumed \( \mathbb{R} \subseteq \mathbb{E} \). Since we prefer to have the field \( \mathbb{E} \) as a set of atoms, we have viewed from the beginning each ordinal number \( \alpha \in \Omega \) “à la Cantor” as the order-type of, and not “à la Von Neumann” as identified with the corresponding initial segment \( \Omega_\alpha = \{ \beta \in \Omega \subseteq \mathbb{E} \mid \beta < \alpha \} \).

**Remark 2.3.** The meaning of the natural product between ordinal numbers, defined through order types, is quite involved and not intuitive at all. On the contrary, thinking of an ordinal number as a Euclidean number, namely as a transfinite sum of ones, makes appear quite natural the meaning of the product, as given by the product formula. Also the natural ordering of the ordinals coincides with that induced by \( \mathbb{E} \), by the Comparison axiom, because they are transfinite sums of ones without zeroes in between.

### 2.3 The counting functions

Let

\[
\mathcal{F}(\Omega, \mathbb{E}) = \{ \xi \in \mathbb{E}^\Omega \mid \exists \beta \in \Omega, \forall j \geq \beta, \xi_j = 0 \}
\]
be the set of all eventually zero $\Omega$-sequences of elements of $E$, and define the sum map $\Sigma : \mathcal{S}(\Omega, E) \to E$ by $\Sigma(\xi) = \sum_{k<\beta} \xi_k$, with any $\beta$ such that $\xi_j = 0$ for $j \geq \beta$.

In the sequel, we shall simply write $\Sigma(\xi) = \sum_k \xi_k$ whenever the $\Omega$-sequence $\xi$ is eventually zero.

We now associate to each $x \in \mathcal{S}(\Omega, \mathbb{R}) = \mathcal{S}(\Omega, E) \cap \mathbb{R}^\Omega$ an $\Omega$-sequence of real numbers, its counting function.

**Definition 2.1.** The counting function of the $\Omega$-sequence $x \in \mathcal{S}(\Omega, \mathbb{R})$ is the function $\varphi_x : \Omega \to \mathbb{R}$ such that $\varphi_x(\alpha) = \sum_{k<\alpha} x_k$ for all $\alpha \in \Omega$.

Given $j \in \Omega$ and any set $X$, call a function $\psi : \Omega \to X$ $j$-periodic if

$\psi(0) = 0$ and $\psi(2^j \circ h + \alpha) = \psi(\alpha)$ for all $\alpha < 2^j$ and all $h \in \Omega$.

Call $\psi$ periodic if it is $j$-periodic for some $j < \Omega$.

The counting functions are strictly connected to the real valued periodic functions, namely

**Theorem 2.4.**

1. Let $x \in \mathcal{S}(\Omega, \mathbb{R})$ be such that $x_k = 0$ for all $k \geq j$. Then, if $\alpha < 2^j$,

$$
\sum_{k<\alpha} x_k = \sum_{k<(2^j \circ h + \alpha)} x_k + \sum_{k<\alpha} x_k = \sum_{k<(2^j \circ h)} x_k \text{ for all } h \in \Omega.
$$

Hence

$$
\varphi_x(2^j \circ h + \alpha) = \varphi_x(2^j \circ h) + \varphi_x(\alpha) = \varphi_x(\alpha),
$$

for all $\alpha < 2^j$ and all $h \in \Omega$, so $\varphi_x$ is $j$-periodic.

2. Conversely, for all $j$-periodic $\psi \in \mathbb{R}^\Omega$, there exists a unique $x \in \mathcal{S}(\Omega, \mathbb{R})$ such that $\psi(2^\alpha) = \varphi_x(2^\alpha)$ for all $\alpha \in \Omega$.

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3 The relevance of the counting functions will result below, when it will become apparent that the counting function plays (for transfinite sums) the role played by the sequence of the partial sums for the usual series.

On the other hand, the qualification “counting” is due to their meaning in the theory of numerosities developed in Section.
Proof.
The first assertion follows from the definition of formal membership \( \prec \), because, for \( \alpha < 2^j \), one has \( L_{2j} \cont \alpha = L_{2j} \cup L_\alpha \) and all members of \( L_{2j} \cont \alpha \) are greater than or equal to \( j \).

In order to prove the converse, recall that \( L_{2\alpha} = \{ \alpha \} \), so \( \varphi_x(\beta) = \sum_{k < \beta} x_k \) is uniquely defined for \( \beta = 2^\alpha + \gamma, \gamma < 2^\alpha \), once \( \varphi_x(2^\delta) = \psi(2^\delta) \) is fixed for all \( \delta \leq \alpha \). Finally, \( \psi(2^j \cont h) = \psi(0) = 0 \) for all \( h = \sum_n 2^{h_n} \), because \( \psi \) is \( j \)-periodic, hence \( x_\beta = 0 \) for \( \beta = 2^j + h \). So \( \varphi_x(2^j \cont h + \alpha) = \varphi_x(\alpha) \) for \( \alpha < 2^j \), hence \( x_\beta = 0 \) for \( \beta \geq 2^j \).

\[ \square \]

The above theorem has an important consequence:

**Theorem 2.5.** Let \( \mathcal{B}(\Omega, \mathbb{R}) \) be the set of all real valued periodic functions. Define the map \( J : \mathcal{B}(\Omega, \mathbb{R}) \to \mathbb{E} \) by \( J(\psi) = J(\varphi_x) = \Sigma(x) \), with \( \varphi_x(2^\alpha) = \psi(2^\alpha) \). Then \( J \) is an \( \mathbb{R} \)-algebra homomorphism onto the ordered field \( \mathbb{E} \).

**Proof.** First of all, by Theorem 2.4, for any \( \psi \in \mathcal{B}(\Omega, \mathbb{R}) \) there is a unique \( x \in \mathcal{S}(\Omega, \mathbb{R}) \) such that \( \varphi_x(2^\alpha) = \psi(2^\alpha) \). Hence the map \( J \) is well defined.

The map \( \Sigma \) being \( \mathbb{R} \)-linear, \( J \) preserves linear combinations over \( \mathbb{R} \). Moreover, by the axiom RA, the range of \( J \) is the whole field \( \mathbb{E} \).

Finally, given \( x, y \in \mathcal{S}(\Omega, \mathbb{R}) \) such that \( x_h = y_h = 0 \) for \( h \geq i \), put \( z_j = \sum_{h, k < j} x_h y_k \). Then, by the Product axiom PA, we have

\[
\Sigma(x) \cdot \Sigma(y) = \sum_{h, k} x_h y_k = \sum_j z_j = \Sigma(z).
\]

Hence one obtains, for all \( \alpha \in \Omega \),

\[
(\varphi_x \cdot \varphi_y)(\alpha) = \varphi_x(\alpha) \cdot \varphi_y(\alpha) = \sum_{h < \alpha} x_h \cdot \sum_{k < \alpha} y_k = \sum_{h, k < \alpha} x_h y_k = \sum_{j < \alpha} z_j = \varphi_z(\alpha)
\]

Therefore also the products are preserved. \( \square \)

**Remark 2.6.** The kernel of \( J \) is a maximal ideal determined by its idempotents, so there is an ultrafilter \( \mathcal{U}(\Omega) \) on \( \Omega \) such that

\[
\varphi \in \ker J \iff \{ \beta \in \Omega \mid \varphi(\beta) = 0 \} \in \mathcal{U}(\Omega), \quad (U.1)
\]
or equivalently
\[ \sum_{k} x_k = 0 \iff \{ \alpha \in \Omega \mid \sum_{k \in \alpha} x_k = 0 \} \in \mathcal{U}(\Omega). \quad (U.2) \]

This fact will be basic in the construction of the Euclidean field given in Subsection 5.2.

We could extend the definition of the counting function \( \varphi_\xi : \Omega \to \mathbb{E}^\Omega \) to the set \( \mathcal{S}(\Omega, \mathbb{E}) \) of all eventually zero \( \Omega \)-sequences of Euclidean numbers in the natural way:

\[ \varphi_\xi(\alpha) = \sum_{k \in \alpha} \xi_k \text{ for all } \alpha \in \Omega. \]

The defining sums are ordinary sums in the field \( \mathbb{E} \), so we could naturally extend the homomorphism \( J \) to the whole algebra \( A(\Omega, \mathbb{E}) = \{ \varphi_\xi \mid \xi \in \mathcal{S}(\Omega, \mathbb{E}) \} \), and obtain a \( \mathbb{R} \)-linear application \( J_\mathbb{E} \) onto the ordered field \( \mathbb{E} \), such that

\[ J_\mathbb{E}(\varphi_\xi) = \sum_{k} \xi_k \text{ for all } \xi \in \mathcal{S}(\Omega, \mathbb{E}). \]

**Caveat.** The Comparison Axiom CA does not hold for transfinite sums of Euclidean numbers, so the map \( J_\mathbb{E} \) is not an algebra homomorphism. The kernel of \( J_\mathbb{E} \) is a subspace of \( A(\Omega, \mathbb{E}) \) such that \( (\ker J_\mathbb{E}) \cap \mathcal{B}(\Omega, \mathbb{R}) = \ker J \). But it is not an ideal, *a fortiori* it is not definable through an ultrafilter on \( \Omega \) by extending the conditions \((U.1), (U.2)\) above to transfinite sums of general Euclidean numbers.

In fact there exist sequences \( \xi_n \) in \( \mathbb{E} \) such that \( \sum_{n<\omega} \xi_n = 0 \), while all partial sums \( \sum_{k<n} \xi_k \) are greater than zero: e.g. take \( \xi_0 = \omega - 1, \xi_k = -1 \) for \( 0 < k < \omega \). Then \( \sum_{n<\omega} \xi_n = \omega - \omega = 0 \), but \( \sum_{k<n} \xi_k = \omega - |L_n| > 0 \) for all all \( n \in \mathbb{N} \).

Given the eventually zero \( \Omega \)-sequence \( x = (x_k \mid k \in \Omega) \in \mathcal{S}(\Omega, \mathbb{R}) \), the counting function \( \varphi_x \) satisfies

\[ \varphi_x(\alpha) = \sum_{k \in \alpha} x_k. \quad (2.1) \]

Thus it could be viewed as the \( \Omega \)-sequence of the partial sums of the transfinite sum \( \sum_k x_k \), and we would like to write

\[ \sum_k x_k = \lim_{\alpha \uparrow \Lambda} \varphi_x(\alpha) \quad (2.2) \]

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where the limit should be taken towards an appropriate “point at infinity” \(\Lambda\).

Call \(\Lambda\)-limit the limit so defined. Then the following properties hold by definition:

(A.1) **Existence and uniqueness:**
Every \(\varphi \in \mathcal{B}(\Omega, \mathbb{R})\) has a unique \(\Lambda\)-limit \(\lim_{\eta \uparrow \Lambda} \varphi(\eta) = \xi \in E\),
and every \(\xi \in E\) is the \(\Lambda\)-limit of some net \(\varphi \in \mathcal{B}(\Omega, \mathbb{R})\).

(A.2) **Real numbers preservation:**
\[ (\exists \eta_0 \in \Omega \ \forall \eta \sqsupseteq \eta_0 \cdot \varphi(\eta) = r) \implies \lim_{\eta \uparrow \Lambda} \varphi(\eta) = r \]

(A.3) **Sum and product preservation:**
For all \(\varphi, \psi \in \mathcal{B}(\Omega, \mathbb{R})\)
\[ \lim_{\eta \uparrow \Lambda} \varphi(\eta) + \lim_{\eta \uparrow \Lambda} \psi(\eta) = \lim_{\eta \uparrow \Lambda} (\varphi(\eta) + \psi(\eta)) \]
\[ \lim_{\eta \uparrow \Lambda} \varphi(\eta) \cdot \lim_{\eta \uparrow \Lambda} \psi(\eta) = \lim_{\eta \uparrow \Lambda} (\varphi(\eta) \cdot \psi(\eta)) \]

The properties (A.1-3) are assumed as axioms in \(\mathbb{R}\) (with an appropriate directed set \(\Lambda\) replacing \(\Omega\)), thus providing a different approach to Nonstandard Analysis, called \(\Lambda\)-theory, useful for the applications. The theory of the Euclidean numbers, having more structure, is a fortiori suitable to this aim: the next section is devoted to this development. However the product preservation in (A.3) fails if extended to arbitrary \(\varphi, \psi \in \mathcal{A}(\Omega, E)\), as shown by the example given in the Caveat above.

### 3 Euclidean numbers and Nonstandard Analysis

In this section we show that the Euclidean numbers are hyperreal numbers, actually they are the **unique saturated field of hyperreal numbers with the cardinality of \(\Omega\).**

#### 3.1 Hyperreal fields

Many different approaches to Nonstandtard Analysis can be found in the literature, see in particular [23, 20, 8] and the bibliography therein. For
completeness, we briefly recall here the basic definitions of the superstructure approach.

**Definition 3.1.** For any set $X$ of atoms, the *superstructure over* $X$ is the set

$$V_\omega(X) = \bigcup_{n \in \mathbb{N}} V_n(X)$$

where

$$V_0(X) = X \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X))$$

**Definition 3.2.** Given a field $F \supset \mathbb{R}$, a *nonstandard embedding* is a mapping

$$*: V_\omega(\mathbb{R}) \to V_\omega(F); \quad (3.1)$$

that satisfies the *Leibniz transfer principle*, i.e.

$$\rho(a_1, \ldots, a_n) \iff \rho(a_1^*, \ldots, a_n^*)$$

for all *bounded quantifier formula* $\rho(x_1, \ldots, x_n)$ and all $a_1, \ldots, a_n \in V_\omega(\mathbb{R})$.

Moreover, it is assumed that $r^* = r$ for every $r \in \mathbb{R}$, that $\mathbb{R}^* = F$, and that $F$ is a set of *atoms*.

Given a nonstandard embedding $*: V_\omega(\mathbb{R}) \to V_\omega(F)$, the triple $(*, \mathbb{R}, F)$ is called *hyperreal number system*, and the field $F$ is called *hyperreal field*.

### 3.2 The Euclidean numbers as hyperreal numbers

In this section we show that $E$ is a *hyperreal field* by giving an explicit definition of the map $*$ in (3.1). This is one of the reasons why we have assumed that the Euclidean numbers are *atoms*.

**Definition 3.3.** Given any set $S$, a real algebra of functions $\mathfrak{F}(S, \mathbb{R}) \subset \mathbb{R}^S$ is called *composable* if

$$\forall f \in \mathbb{R}^\mathbb{R} \quad \forall \varphi \in \mathfrak{F}(S, \mathbb{R}). f \circ \varphi \in \mathfrak{F}(S, \mathbb{R}).$$

Recall the following theorem of Benci and Di Nasso:

---

4 By *bounded quantifier formula* we mean a first-order formula in the language $L = \{ \in \}$ of set theory, where all quantifiers occur in the bounded forms $\forall x \in y$ or $\exists x \in y$. 

---
Theorem 3.1 ([5], Thm. 3.3). A field $\mathbb{F}$ is a hyperreal field if and only if there exist a set $S$, a composable algebra of functions $\mathcal{F}(S, \mathbb{R}) \subset \mathbb{R}^S$, and a surjective homorphism $J : \mathcal{F}(S, \mathbb{R}) \rightarrow \mathbb{F}$.

Applying this theorem together with Theorem 2.5, we immediately get:

Theorem 3.2. The algebra $\mathcal{B}(\Omega, \mathbb{R})$ is composable, hence $\mathbb{E}$ is a hyperreal field. \hfill $\square$

Now we define explicitly the map $\ast$ and some other notions of Nonstandard Analysis by applying the $\Lambda$-limit introduced in Subsection 2.3.

Definition 3.4. Given a periodic $\Omega$-sequence $\varphi \in \mathcal{B}(\Omega, V_n(\mathbb{R}))$, define by induction its $\Lambda$-limit $\lim_{\eta \uparrow \Lambda} \varphi(\eta)$ as follows:

- for $n = 0$, the limit $\lim_{\eta \uparrow \Lambda} \varphi(\eta) = J(\varphi)$ has been defined by the condition (2.2) in Subsection 2.3.
- so, assuming the limit defined for $n$, put, for $\varphi \in \mathcal{B}(\Omega, V_{n+1}(\mathbb{R}))$:
  - $\lim_{\eta \uparrow \Lambda} \varphi(\eta) = \left\{ \lim_{\eta \uparrow \Lambda} \psi(\eta) \mid \psi \in \mathcal{B}(\Omega, V_n(\mathbb{R})) \text{ and } \forall \eta \in \Omega, \psi(\eta) \in \varphi(\eta) \right\}$.
- A set in $V_\omega(\mathbb{E})$ which is the $\Lambda$-limit of an $\Omega$-sequence is called internal.
- A mathematical entity (number, set, function or relation), when identified with a set in $V_\omega(\mathbb{E})$, is called internal if the corresponding set is internal.

Now we can define the $\ast$-map.

Definition 3.5. If $r \in \mathbb{R}$, then $r^* = r$. If $E \in V_\omega(\mathbb{R})$ is a set, then the star extension $E^*$ of $E$ is

$$E^* := \lim_{\eta \uparrow \Lambda} c_E(\eta) = \left\{ \lim_{\eta \uparrow \Lambda} \psi(\eta) \mid \psi(\eta) \in E \right\} = \{ J(\varphi) \mid \varphi \in \mathcal{B}(\Omega, E) \}$$

where $c_E(\xi)$ is the sequence identically equal to $E$.

This approach to Nonstandard Analysis being based on the notion of limit, it is natural to formulate the Leibniz principle in the following apparently stronger form:
Proposition 3.6. Let $\rho(x_1,\ldots,x_n)$ be a bounded quantifier formula and let $\varphi_1,\ldots,\varphi_n$ be $\Omega$-sequences in $\mathcal{R}(\Omega, V_N(\mathbb{R})), with N \in \mathbb{N};$ then

$$\left( \exists Q \in \mathcal{U}(\Omega) \forall \xi \in Q. \rho(\varphi_1(\xi),\ldots,\varphi_n(\xi)) \right) \iff \rho\left(\lim_{\xi \uparrow \Lambda} \varphi_1(\xi),\ldots,\lim_{\xi \uparrow \Lambda} \varphi_n(\xi)\right)$$

Proof. The proof is a simple adaptation of the usual proof (see e.g. [20]). □

Clearly the Leibniz principle as formulated in Def. 3.2 follows by taking constant sequences in the above proposition. Let us deduce two important corollaries:

Corollary 3.7. The Euclidean number field $\mathbb{E}$ is a real closed field.

Proof. Since $\mathbb{R}$ is real closed, then also $\mathbb{E} = \mathbb{R}^*$ is real closed. □

Another immediate consequence of Leibniz principle is that $\mathbb{E}$, as ordered field, is unique in the following sense:

Corollary 3.8. If $\mathbb{F}$ is a real closed field having cardinality $\Omega$, then $\mathbb{F}$ is isomorphic to $\mathbb{E}$.

Proof. Apply Coroll. 3.7 and the fact that two real closed fields are isomorphic if and only if they share the same absolute trascendency degree. (See e.g. [14], p. 348.) □

Remark 3.3. If, instead of the first inaccessible number, we take $\Omega$ to be the class of all ordinals, then the field of Euclidean numbers $\mathbb{E}$ is isomorphic to the field of surreal numbers $\mathbb{No}$ of Conway [15]; so, following Ehrlich [17] they form an absolute arithmetic continuum.

3.3 Saturation

Recall the usual definiton of saturated field in Nonstandard Analysis:

Definition 3.9. A hyperreal number system $(\star, \mathbb{R}, \mathbb{F})$ is saturated if any family of internal sets $\mathcal{E} = \{E_k \mid k \in K\}$ of size $|K| < |\mathbb{F}|$, with the finite intersection property $\mathcal{FIP}$, has a nonempty intersection.

5 The $\mathcal{FIP}$ says that every finite subfamily of $\mathcal{E}$ has nonempty intersection.
Theorem 3.4. The Euclidean number system \((*, \mathbb{R}, \mathbb{E})\) is saturated.

Proof. Let \(\{E_k\}_{k \in K}, |K| < |\Omega|,\) be a family of internal sets with the FIP. Recall that each internal set is the \(\Lambda\)-limit of a periodic function \(\varphi \in \mathcal{B}(\Omega, \mathcal{P}(\mathbb{R}))\). The ordinal \(\Omega\) being regular and greater than \(2^{\aleph_0}\), the periods are bounded by \(2^i\), say. So we assume w.l.o.g. that \(E_k = \lim_{\alpha \uparrow \Lambda} E_{k,\alpha}\), with \(\mathbb{R} \supset E_{k,\alpha} = E_{k,2^ih+\alpha}\) for all \(k \in K, h \in \Omega,\) and all \(\alpha < 2^i\). For sake of simplicity we assume that also the index set is a segment of ordinals, say \(K = \{k \in \Omega \mid k < 2^i\}\), and we put

\[
E_k = E_{2^ih+k}\text{ for all } h \in \Omega \text{ and all } k \in K
\]

so as to have \(E_k\) periodically defined for all \(k \in \Omega\).

By FIP, the internal sets \(B_\beta := \bigcap_{k < \beta} E_k\) are nonempty for all \(\beta \in \Omega\). Remark that we have extended the definition of the sets \(E_k\) to all \(k \in \Omega\) in such a way that \(B_{2^ih+\beta}\) coincides with \(B_\beta\) for all \(\beta < 2^i\) and all \(h \in \Omega\). The sets \(B_\beta\) being internal and nonempty, there exists for each \(\beta\) a family of nonempty sets \(\langle B_\beta, \xi \subseteq \mathbb{R} \mid \xi \in \Omega \rangle\) such that

\[
B_\beta = \lim_{\xi \uparrow \Lambda} B_{\beta, \xi}.
\]

So for each \(\beta\) there exists a sequence

\[
\varphi_\beta \in \mathcal{B}(\Omega, \mathbb{R}) \text{ such that } \forall \xi. \varphi_\beta(\xi) \in B_{\beta, \xi}
\]

Moreover, by our assumptions, we may choose the functions \(\varphi_\beta\) so that

\[
\varphi_{2^ih+\beta}(2^ik + \xi) = \varphi_\beta(\xi) \quad \forall \beta, \xi < 2^i \forall h, k \in \Omega.
\]

Now define \(\psi \in \mathcal{B}(\Omega, \mathbb{R})\) by putting, for \(s < 2^i\),

\[
\psi(2^i \odot \beta + s) := \varphi_\beta(s), \text{ so that } \forall \beta \in \Omega \forall s < 2^i . \psi(2^i \odot \beta + s) \in B_{\beta, s}.
\]

Then \(\psi(\xi) \in B_{\beta, \xi}\) for all \(\xi \in \Omega\), hence

\[
b = \lim_{\xi \uparrow \Lambda} \psi(\xi) \in \lim_{\xi \uparrow \Lambda} B_{\beta, \xi} = B_\beta \text{ for all } \beta \in \Omega,
\]

and so \(b \in \bigcap_{\beta \in \Omega} B_\beta = \bigcap_{k \in K} E_k\), which is therefore nonempty. \(\square\)
3.4 The Keisler hyperreal field

Following Keisler we give the following definition of isomorphism between hyperreal fields:

**Definition 3.10.** Let \((\ast, \mathbb{R}, \mathbb{R}^\ast)\) and \((\ast\ast, \mathbb{R}, \mathbb{R}^{\ast\ast})\) be hyperreal number systems with the same real part \(\mathbb{R}\). A map \(h : \mathbb{R}^\ast \to \mathbb{R}^{\ast\ast}\) is an isomorphism if the following conditions are fulfilled:

- (i) \(h(r) = r\) for each \(r \in \mathbb{R}\),
- (ii) \(h\) is an ordered field isomorphism from \(\mathbb{R}^\ast\) onto \(\mathbb{R}^{\ast\ast}\),
- (iii) For each real function \(f\) of \(n\) variables and all \(x_1, ..., x_n \in \mathbb{R}^\ast\),

\[
f^{\ast\ast}(h(x_1), ..., h(x_n)) = h(f^\ast(x_1, ..., x_n))
\]

Two hyperreal number systems are isomorphic if there is an isomorphism between them.

In the set theory ZFC plus the Axiom of Inaccessibility, one can prove the following theorem:

**Theorem 3.5** ([20], p.196). There is a definable\(^6\) hyperreal number system \((\ast; \mathbb{R}; \mathbb{R}^\ast)\) which is saturated and such that the cardinality of \(\mathbb{R}^\ast\) is the first uncountable inaccessible cardinal.

We shall refer to the field \(\mathbb{R}^\ast\) as the hyperreal Keisler field.

According to Thm. 3.4, we have the following interesting result:

**Corollary 3.11.** The Euclidean number system \((\ast, \mathbb{R}, \mathbb{E})\) is isomorphic to the hyperreal Keisler field \((\ast; \mathbb{R}; \mathbb{R}^\ast)\).

3.5 \(\Omega\) versus \(\mathbb{N}^*\)

As an ordered field, the Euclidean field \(\mathbb{E}\) is unique, and it defines a hyperreal number system \((\ast, \mathbb{R}, \mathbb{E})\) which is unique up to isomorphism, by Corollary 3.11. In addition, the Euclidean field \(\mathbb{E}\) has two main extra properties which are not shared by other hyperreal fields:

\(^6\) Recall that a set \(X\) is (first order) definable if there is a first order formula \(\rho(x)\) such that \(X\) is the unique set such that \(\rho(X)\) holds.
• the sum of infinitely many hyperreal numbers is well defined;
• the semiring of the accessible ordinals \( \Omega \), with the natural sum and product, is isomorphically embedded in \( E \) in a natural way.

The combination of these features creates new phenomena which we now investigate.

**Remark 3.6.** According to Theorem 2.2 we have identified each ordinal \( \alpha \in \Omega \) with the Euclidean number given by the natural embedding \( \Psi : \Omega \to E \) defined by

\[
\Psi(\alpha) = \sum_k \chi_\alpha(k) = \lim_{\xi \uparrow \Lambda} \varphi_\alpha(\xi)
\]

where

\[
\varphi_\alpha(\xi) = \sum_{k < \xi} \chi_\alpha(k) = |\{ k < \alpha \mid k < \xi \}|.
\]  

(3.2)

Actually, \( \Psi \) is an isomorphic embedding of \( \Omega \) into \( \mathbb{N}^* \) (as ordered semirings): in fact, by definition,

\[
\mathbb{N}^* = \left\{ \lim_{\xi \uparrow \Lambda} \varphi(\xi) \mid \varphi \in \mathscr{B}(\Omega, \mathbb{N}) \right\} \subseteq E
\]

so \( \Psi[\Omega] \subseteq \mathbb{N}^* \), because the counting function \( \varphi_\alpha \) of \( \Psi(\alpha) = \sum_k \chi_\alpha(k) \) takes its values in \( \mathbb{N} \).

Thus the ordinal numbers in \( \Omega \subseteq E \) can be viewed as “special” hypernatural numbers. In order to investigate the relation of general hypernatural numbers with ordinal numbers, put, for \( \xi \in \Omega \),

\[
K(\xi) = \{ \alpha \in \Omega \mid \alpha < \xi \}, \quad m(\xi) = |K(\xi)|, \quad \text{and} \quad \mathbb{N}_m = \{ n \in \mathbb{N} \mid n < m \}.
\]

Let \( j_\xi : \mathbb{N}_{m(\xi)} \to K(\xi) \) be the order-preserving bijection; in particular \( j_\xi(0) = 0 \) and \( j_\xi(m(\xi) - 1) = \xi \) for all \( \xi \in \Omega \).

Take the \( \Lambda \)-limits

\[
K := \lim_{\xi \uparrow \Lambda} K(\xi) \subseteq \Omega^*, \quad \mu = \lim_{\xi \uparrow \Lambda} m(\xi), \quad \mathbb{N}^*_\mu := \lim_{\xi \uparrow \Lambda} \mathbb{N}_{m(\xi)}, \quad \text{and} \quad j := \lim_{\xi \uparrow \Lambda} j_\xi.
\]

(3.3)

Then, by Leibniz principle,

\[
\Omega \subset K \subseteq E, \quad \mathbb{N}^*_\mu = \{ k \in \mathbb{N}^* \mid k < \mu \}, \quad \text{and} \quad j : \mathbb{N}^*_\mu \to K
\]

is an order-preserving surjection that is the identity when restricted to \( \Omega \).
3.6 Hyperfinite sums v/s transfinite sums

In Nonstandard Analysis one deals with particular infinite sums, usually indexed by closed initial segments of the set $\mathbb{N}^*$ of the nonstandard natural numbers. We consider in this subsection the relations between these hyperfinite sums of the NSA and the transfinite sums of the Euclidean field $\mathbb{E}$ introduced in Section 2.

**Definition 3.12.** An internal set $F \in V_\omega(\mathbb{E})$ is called hyperfinite if it is the $\Lambda$-limit of finite sets, namely

$$ F := \lim_{\xi \uparrow \Lambda} F_\xi = \left\{ \lim_{\xi \uparrow \Lambda} x_\xi \mid x_\xi \in F_\xi \right\} $$

where the $F_\xi$s are finite sets in $V_\omega(\mathbb{R})$.

E.g., for $\mu = \lim_{\xi \uparrow \Lambda} m_\xi \in \mathbb{N}^*$, the set $\mathbb{N}_\mu^* = \{ \nu \in \mathbb{N}^* \mid \nu < \mu \}$ is hyperfinite, since $\mathbb{N}_\mu^* = \lim_{\xi \uparrow \Lambda} \{ n \in \mathbb{N} \mid n < m_\xi \}$.

The notion of hyperfinite set is basic in defining the notion of hyperfinite sum, which is the usual notion of infinite sum of hyperreal numbers:

**Definition 3.13.** Given a hyperfinite set of hyperreal numbers $F \subset \mathbb{E}$, the hyperfinite sum of the elements of $F$ is defined as follows:

$$ \sum^* x = \lim_{\xi \uparrow \Lambda} \left( \sum_{x \in F_\xi} x_\xi \right) $$

Transfinite sums and hyperfinite sums are strictly related, as the next theorem shows.

**Theorem 3.7.** If $a \in \mathcal{S}(\Omega, \mathbb{R})$, then

$$ \sum_k a_k = \sum_{x \in F^a} x $$

where the hyperfinite set $F^a$ is defined as

$$ F^a = \lim_{\xi \uparrow \Lambda} F_\xi^a \quad \text{where} \quad F_\xi^a = \{ a_k \mid k \leq \xi \} \quad (3.4) $$

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Proof. We have

\[
\sum_{k} a_k = \lim_{\xi \uparrow \Lambda} \sum_{k < \xi} a_k = \lim_{\xi \uparrow \Lambda} \sum_{x \in F^a_{\xi}} x_\xi = \sum_{x \in F^a} x.
\]

By the Leibniz principle, we have that a set \( F \subset \mathbb{E} \) is hyperfinite if and only if there exist \( a \in \mathbb{R}^{\mathbb{N}^*} \) and \( \mu \in \mathbb{N}^* \) such that

\[
F = F^a_{\mu} := \{ a_\nu \mid \nu \in \mathbb{N}^*, \nu < \mu \}.
\]

This fact suggests the following notation:

\[
\sum_{\nu \in \mathbb{N}^*_\mu} a_\nu = \sum_{x \in F^a_{\mu}} x.
\]

Given a sequence \( a \in \mathbb{R}^\mathbb{N} \), put \( S(n) = \sum_{k < n} a_k \). Denote by \( a^*, S^* \) the *-extensions of \( a, S \) respectively: then, for any hypernatural number \( \mu \in \mathbb{N}^* \), the corresponding hyperfinite sum is

\[
\sum_{\nu \in \mathbb{N}^*_\mu} a^*_\nu = S^*(\mu),
\]

by the Leibniz principle. In particular we have

**Theorem 3.8.**

\[
\sum_{k < \omega} a_k = \sum_{\nu \in \mathbb{N}^*_\omega} a^*_\nu \quad \text{for all} \quad a \in \mathbb{R}^\mathbb{N}.
\]

**Proof.** Let \( \varphi \) be the counting function of \( \sum_{k < \omega} a_k \), so

\[
\varphi(\omega \odot h + n) = \sum_{k < \omega \odot h + n} a_k \chi_\omega(k) = \sum_{k < n} a_k = S(n) \quad \text{whence} \quad \sum_{k < \omega} a_k = \lim_{\omega \odot h + n \uparrow \Lambda} S(n).
\]

We have taken \( \omega = \sum_k \chi_\omega(k) = \lim_{\omega \odot h + n \uparrow \Lambda} n : \) then we have that

\[
\sum_{k < \omega} a_k = S^* \left( \lim_{\omega \odot h + n \uparrow \Lambda} n \right) = S^*(\omega) = \sum_{\nu \in \mathbb{N}^*_\omega} a^*_\nu.
\]

\( \square \)
E.g., compute the sum $\sum_{k \in \mathbb{N}} \frac{1}{2^k}$:

$$S(n) = \sum_{k<n} \frac{1}{2^k} = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}, \quad \text{hence} \quad \sum_{k<\omega} \frac{1}{2^k} = 2 - \frac{1}{2^{\omega-1}} \quad (3.5)$$

This computation is “more accurate” than the usual value given to the series $\sum_{k=0}^{+\infty} 2^{-k} = 2$, which neglects the infinitesimal $2^{1-\omega}$ in (3.5).

4 Numerosities

In the history of Mathematics the problem of comparing the size of objects has been extensively studied. In particular different methods of measuring sets, by associating to them suitable kinds of numbers, have been exploited.

A satisfactory notion of measure for sets should be submitted to the famous five common notions of Euclid’s Elements, which traditionally embody the properties of magnitudes (see [18]):

1. Things equal to the same thing are also equal to one another.
2. And if equals be added to equals, the wholes are equal.
3. And if equals be subtracted from equals, the remainders are equal.
4. Things [exactly] applying onto one another are equal to one another$^7$
5. The whole is greater than the part.

To be sure, the Euclidean common notions seem prima facie unsuitable for measuring the size of arbitrary sets: the third and fifth notions are known to be incompatible with the very ground of the Cantorian theory of cardinality, namely the so called Hume’s Principle

\[ \text{(HP) Two sets have the same size if and only if there exists a biunique correspondence between them.} \]

$^7$ Here we translate εφαρμοζωντα by “[exactly] applying onto”, instead of the usual “coinciding with”. As pointed out by T.L. Heath in his commentary [18], this translation seems to give a more appropriate rendering of the mathematical usage of the verb εφαρμοζων.
This principle amounts to encompass the largest possible class of size preserving applications, namely all bijections. This fact might seem natural, and even implicit in the notion of counting; but it strongly violates the equally natural Euclid’s principle

(EP) A set is greater than its proper subsets,

which in turn seems implicit in the notion of size. Be it as it may, the spectacular development of set theory in the entire twentieth century has put Euclid’s principle in oblivion. Only the new millennium has seen a limited resurgence of proposals including (EP) at the cost of some limitations of (HP) (see e.g. [2, 4, 6, 7], or the excellent survey [22] and the references therein).

4.1 Saving the five Euclidean common notions

It is worth noticing that traditional geometry satisfies the Euclidean common notions because there is a restricted class of “exact applications”: e.g., the rigid equidecompositions of polygons, or, more generally, when considering metric spaces, the isometries, i.e. distance preserving bijections. But, when dealing with general set theory, an appeal to the notion of distance seems inappropriate. So the question arises as to which correspondences can be taken as “exact applications” in order to fulfill the five Euclidean common notions.

Cantor himself, besides his cardinality theory based on general bijections, introduced another way of assigning numbers to sets, namely refined cardinal numbers to ordinal numbers. In this case one considers sets endowed with a wellordering, and restricts the “exact applications” to the order preserving bijections. However, while the ordinal arithmetic may respect the third common notion, nevertheless the Euclid’s principle (EP) still badly fails.

The Euclidean numbers have been introduced above as an extension of the ordinals, suitable to provide a notion of size satisfying all the Euclidean common notions for an appropriate class of “labelled sets”. A labelled set $E$ comes together with a suitable labelling map $\ell$ such that $\ell^{-1}(x)$ is a finite set for all $x \in E$.

The original idea is that by putting an appropriate labelling on arbitrary sets, the label preserving bijections (intended as “exact applications”) might be used in defining an appropriate Euclidean notion of size, that produces exactly the “nonnegative integers” of the Euclidean numbers (whence their
Remark that a labelled set can be viewed as a generalization of a wellordered set, because the latter can be naturally labelled by the unique order isomorphism with an (initial segment of an) ordinal. In fact we shall see below that any labelled set is “equinumerous” to a set of ordinals.

Let us state the basic definitions.

**Definition 4.1.** A labelled set is a pair $(E, \ell)$, where

- $E$ is a set of cardinality less than $\Omega$, the set of the ordinals smaller than the first inaccessible cardinal,

- $\ell : E \to \Omega$ is a function (the labelling function) such that
  1. the set $\ell^{-1}(x)$ is finite for all $x \in \Omega$,
  2. $\ell(x) = x$ for all $x \in E \cap \Omega$.

- Two labelled sets $(E_1, \ell_1)$ and $(E_2, \ell_2)$ are coherent if
  $$\ell_1(x) = \ell_2(x) \text{ for all } x \in E_1 \cap E_2.$$

- Two labelled sets $(E_1, \ell_1)$ and $(E_2, \ell_2)$ are isomorphic if there is a biunique map $\phi : E_1 \to E_2$ such that
  $$\ell_2(\phi(x)) = \ell_1(x) \text{ for all } x \in E_1.$$

**Remark 4.1.** By Point 2, each initial segment $\Omega_\alpha$ receives the identity as labelling function. Thus a labelled set can be viewed as a generalization of a well ordered set, because the latter can be naturally labelled by the unique order isomorphism with an (initial segment of an) ordinal. More generally, we shall see below that any labelled set is “equinumerous” to a set of ordinals.

Now we are ready to introduce our main notion:

---

8 A notion of numerosity for countable “labelled sets”, whose elements come with suitable labels (given by natural numbers) was first presented in [2], and later developed in the paper [3]. It provides a notion of “number of elements” that fulfills the fifth Euclidean common notion, and produces particular nonstandard integers.

9 We consider the ordinals in $\Omega$ as Euclidean numbers in $E$, hence atoms, according to our stipulation following Thm.2.2; however, as far as the numerosity theories are concerned, the Von Neumann ordinals would be equivalent.
Definition 4.2. The Euclidean numerosity of a labelled set \((E, \ell)\) is the number

\[ n(E, \ell) = \sum_k |\ell^{-1}(k)| \in \mathbb{E}. \]

The Euclidean numerosity satisfies the five Euclidean common notions (whence the name), when they are interpreted in the natural way:

1. Two labelled sets are considered equal (in size) if they have the same numerosity;
2. The addition of two labelled sets \((E_1, \ell_1)\) and \((E_2, \ell_2)\) with \(E_1 \cap E_2 = \emptyset\) is given by

\[(E_1 \cup E_2, (\ell_1 \cup \ell_2)) \text{ where } (\ell_1 \cup \ell_2)(x) = \begin{cases} \ell_1(x) & \text{if } x \in E_1 \\ \ell_2(x) & \text{if } x \in E_2 \end{cases} \]

3. The subtraction of two coherent labelled sets \((E_1, \ell_1)\) and \((E_2, \ell)\) with \(E_1 \subset E_2\) is given by

\[(E_2 \setminus E_1, \ell_{|E_2\setminus E_1}) \]

4. Two labelled sets [exactly] apply onto one another is equivalent to say that they are isomorphic;
5. A part of a labelled set is just a (coherent) subset.

4.2 The Euclidean numerosity theories

There are three main operations which produce (possibly new) labelled sets:

Definition 4.3. The basic operations on labelled sets are the following:

1. **Subset** - A subset of a labelled set \((E, \ell)\) is a labelled set \((F, \ell_{|F})\) where \(F \subset E\);
2. **Union** - The union of two coherent labelled sets \((E_1, \ell_1)\), \((E_2, \ell_2)\) is the labelled set

\[(E_1 \cup E_2, \ell) \text{ where } \ell(x) = \begin{cases} \ell_1(x) & \text{if } x \in E_1 \\ \ell_2(x) & \text{if } x \in E_2 \end{cases} \]

3. **Cartesian product** - The Cartesian product of two labelled sets \((E_1, \ell_1)\), \((E_2, \ell_2)\) is the labelled set

\[(E_1 \times E_2, \ell) \text{ where } \ell(x_1, x_2) = \ell_1(x_1) \lor \ell_2(x_2) \]
Definition 4.4.

- A family \((\mathcal{A}, \ell)\) of (accessible) pairwise coherent labelled sets is \textit{closed} if it is closed under the three basic operations of Def. 4.3.
- A \textit{Euclidean numerosity theory} is a pair \((U, n)\), where \(U\) is a closed family of labelled set and \(n: U \rightarrow \mathbb{E}\) is the Euclidean numerosity.

Remark 4.2. Given a coherent family \((\mathcal{A}, \ell)\) of labelled sets, there exists a \textit{least closed family} including \(\mathcal{A}\), denoted by \(\mathcal{U}(\mathcal{A}, \ell)\) and called the \textit{closure} of \((\mathcal{A}, \ell)\) (or the closed family \textit{generated by} \((\mathcal{A}, \ell)\)). We omit the labelling function \(\ell\) if it is clear from the context.

In particular, if \(\mathcal{A} = \{X\}\), then we write \(\mathcal{U}[X]\) for \(\mathcal{U}(\mathcal{A})\). We also write \(\mathcal{U}(\Omega)\) for \(\mathcal{U}((\{\Omega_\alpha \mid \alpha \in \Omega}\))\).

Let us see some examples. Recall that we identify the natural numbers with the finite ordinal numbers, and the accessible ordinals with the corresponding Euclidean numbers.

- Let \(F \subset \Omega\) be a finite set, then \(\mathcal{U}[F]\) contains only finite sets and \(n(E)\) is just the cardinality of the finite set \(E\); in this case
  \[n(\mathcal{U}[F]) = \mathbb{N} \subset \mathbb{E};\]

- The “simplest” numerosity theory containing \textit{infinite} sets is given by \((\mathcal{U}[\mathbb{N}], n)\); in this case we have that
  \[n(\mathcal{U}[\mathbb{N}]) \subseteq \{\phi(\omega) \in \mathbb{E} \mid \phi \in \mathcal{B}(\Omega, \mathbb{N}), \phi(\omega + n) = \phi(n)\};\]

- The \textit{canonical numerosity theory} is given by \((\mathcal{U}(\Omega), n) = (\mathcal{U}(\{\Omega_\alpha \mid \alpha \in \Omega}\), n)\) : this is the “simplest” theory which contains all the (accessible) ordinal numbers.

Let us state the main properties of a \textit{Euclidean numerosity theory}.

Theorem 4.3. Let \((U, n)\) be a Euclidean numerosity theory. Then

- each set in \(U\) is equinumerous to a set of ordinals, and one has, for all \(A, B \in U:\)
  \[\text{Sum-Difference} \quad n(A \cup B) = n(A) + n(B) - n(A \cap B);\]
Part-Whole  
- \( A \subset B \implies n(A) < n(B) \);

Cartesian Product  
- \( n(A \times B) = n(A) \cdot n(B) \);

Comparison  
- if the 1-to-1 map \( T : A \to B \) preserves labels, then \( n(A) \leq n(B) \).

Moreover

- If \( \Omega_\alpha \in A \) for all \( \alpha \in \Omega \), then the set of numerosities \( \mathcal{N} = n(\bigcup A) \) is a positive subsemiring of nonstandard integers, that generates the whole \( \mathbb{Z}^* \), and one has

Identity  
- \( \forall \alpha \in \Omega, \ n(\Omega_\alpha) = \alpha \);

Cartesian product  
- \( \forall \alpha, \beta \in \Omega, \ n(\Omega_\alpha \times \Omega_\beta) = \alpha \beta \);

Translation invariance  
- \( \forall E \subseteq \Omega_\theta, \ n \left( \left\{ 2^\theta + \xi \mid \xi \in E \right\} \right) = n(E) \);

Homothety invariance  
- \( \forall E \subseteq \Omega_\theta, \ n \left( \left\{ 2^\theta \xi \mid \xi \in E \right\} \right) = n(E) \).

Proof.

All the assertions are straightforward consequences of the definitions and of the axioms of \( \mathbb{E} \), except the first one. In order to prove that each set \( E \in \bigcup A \) is equinumerous to some set of ordinals, let \( \theta \) be an ordinal greater than all labels of the set \( E \in \bigcup A \); we define a suitable subset of \( \Omega_{2^\theta} \) equinumerous to \( E \).

Let \( n_k \) be the number of elements of \( E \) with label \( k \), so \( n(E) = \sum_{k<\theta} n_k \).

Let \( a_{kh} = c_{2^\theta \odot k + h} \) be defined as

\[
    a_{kh} = \begin{cases} 
        1 & \text{if } 0 \leq h < n_k \text{ and } k < \theta \\
        0 & \text{otherwise}
    \end{cases}
\]

Then \( \sum_h a_{kh} = n_k \) for all \( k \), and \( \sum c_l = \sum_k \sum_h a_{kh} = \sum_k n_k \). Clearly the latter is the numerosity of the set

\[
    L = \{ l < 2^\theta \theta \mid l = 2^\theta k + h, \ h < n_k, \ n_k > 0 \},
\]

and so \( n(E) = n(L) \).

In particular, every transfinite sum \( \sum_k n_k \) of nonnegative integers is the numerosity of a subset of some \( \Omega_\alpha \). Then Theorem 2.4 yields that any periodic \( \Omega \)-sequence in \( \mathcal{B}(\Omega, \mathbb{Z}) \) is the difference between the counting functions of two sets of ordinals. Hence \( n(\bigcup \Omega) \) generates the whole \( \mathbb{Z}^* \).

\[\square\]

Remark 4.4. Another interesting operation on labelled sets is
**Finite parts** - The set of the finite parts of the labelled set \((E, \ell)\) is the labelled set \((P_{\omega}(E), \lor \ell)\) where \(\lor \ell\) is defined as follows:

\[
\lor \ell \left( \{a_1, \ldots, a_n\} \right) = \bigvee_{k=1}^{n} \ell(a_k)
\]

If, as usual in axiomatic set theory, we identify the ordered pair \((a, b)\) with the “Kuratowski doubleton” \(\{\{a\}, \{a, b\}\}\), then, by the above definition,

\[
\lor \ell (a, b) = \lor \ell \left( \{\{a\}, \{a, b\}\} \right) = \ell(a) \lor \ell(b).
\]

Hence, if \((E_1, \ell_1)\) and \((E_2, \ell_2)\) are labelled sets, their Cartesian product is precisely the labelled set \((E_1 \times E_2, \ell_1 \lor \ell_2)\).

It is easily seen that, if the family \(\mathbb{U}\) of the Euclidean numerosity theory \((\mathbb{U}, n)\) is closed also under the operation of finite parts, then one has

\[
n(P_{\omega}(A)) = 2^{n(A)}.
\]

### 4.3 Euclidean numerosity v/s Aristotelian size

The Euclidean numerosity theory \((\mathbb{U}(\Omega), n)\) of the preceding section might be compared with the “Aristotelian” numerosity theory introduced in [6], where in particular every set of ordinals \(A\) receives a numerosity \(s(A)\) belonging to the non-negative part of an ordered ring \(\mathfrak{A}\) so that the following conditions are fulfilled:

1. **(sp)** If \(A \cap B = \emptyset\), then \(s(A \cup B) = s(A) + s(B)\);
2. **(up)** \(s(\{\xi\}) = 1\) for all \(\xi\);
3. **(fpp)** If \(A \subseteq \Omega_{\theta^+}\) and \(B \subseteq \Omega_{\theta^+}\) with \(\theta = 2^\theta\) and \(h < \omega\), then

\[
s(A) \cdot s(B) = s(\{\theta^\gamma \odot \beta + \alpha \mid \alpha \in A, \beta \in B\}).
\]

(For convenience we use here the notation of this article. Moreover we restrict ourselves to sets of accessible ordinals less than \(\Omega\), so as to avoid the use of proper class-functions.)

---

10 Here \(P_{\omega}(E)\) denotes the family of all finite subsets of \(E\), also denoted by \([E]^{<\omega}\).
Theorem 4.3 immediately implies that these properties are fulfilled by the Euclidean numerosity. Conversely, one can define a transfinite sum of non-negative integers in the ring $\mathfrak{A}$, by proceeding as in the proof of Theorem 4.3:

$$\sum_{k<\theta} n_k = s(L), \text{ where } L = \{2^\theta \circ k + m \mid 0 \leq m < n_k \neq 0, k < \theta\}$$

This transfinite sum may be uniquely extended to arbitrary integers by considering separately positive and negative summands. We may assume w.l.o.g. that the ring $\mathfrak{A}$ is generated by the set of the numerosities, i.e. that any $a \in \mathfrak{A}$ is the difference $s(A) - s(B)$ of the numerosities of two sets of ordinals: then $\mathfrak{A}$ becomes the set of all transfinite sums of integers.

It turns out that the axiom CA holds in $\mathfrak{A}$ for the transfinite sums of integers. In fact, put $A_\alpha = \{x \in A \mid x \prec \alpha\}$: then $|A_\alpha| = |B_\alpha|$ for all $\alpha \subseteq \beta$ implies $A \setminus A_\beta = B \setminus B_\beta$ and $|A_\beta| = |B_\beta|$, hence $s(A) = s(B)$.

Moreover the property (fpp) can be used in defining the numerosity of Cartesian products. Then a natural strengthening of the property (fpp) could be the assumption that $s$ is definable on arbitrary sets of pairs of ordinals by putting

$$s(E) = s(\{2^\theta \cdot \beta + \alpha \mid (\alpha, \beta) \in E\}) \text{ for all } E \subseteq \Omega_\theta \times \Omega_\theta$$

On the other hand, the natural labelling of pairs given in the preceding subsection would give to $E$ the numerosity

$$s(E) = \sum_k |E(k)| \text{ where } E(k) = \{(\alpha, \beta) \in E \mid \alpha \lor \beta = k\},$$

i.e., according to the above definition,

$$s(E) = s(\{2^\theta \cdot k + m \mid 0 \leq m < |E(k)|, k < \theta\}).$$

So we are led to the following

**Definition 4.5.** Call *multiplicative* an Aristotelian numerosity $s$ such that following equality holds for all $E \subseteq \Omega_\theta \times \Omega_\theta$

\[\text{Notice that we use here the natural product of ordinals, which agrees with the product of the field } \mathbb{E}, \text{ in order to avoid absorption phaenomena.}\]
(dsp) \( s(\{2^\theta \cdot \beta + \alpha \mid (\alpha, \beta) \in E\}) = s(\{2^\theta \cdot (\alpha \lor \beta) + m \mid 0 \leq m < |E(\alpha \lor \beta)|\}) \).

Clearly, the equalities (dsp) allow for consistently extending \( s \) to all sets of pairs of ordinals, obtaining in particular that \( s(E \times F) = s(E) \cdot s(F) \).

Then we have

**Theorem 4.5.** Let \((U, s)\) be a multiplicative Aristotelian numerosity.

Let \( \psi \) map each set \( E \in U \) to the counting function \( \psi_E \in \mathcal{B}(\Omega, \mathbb{N}) \) such that \( \psi_E(\alpha) = |E_\alpha| \), where \( E_\alpha = E \cap \{ \beta \in \Omega \mid \beta \prec \alpha \} \).

Let \( \mathcal{I} \) be the kernel of the homomorphism \( J : \mathcal{B}(\Omega, \mathbb{R}) \to \mathbb{E} \) of Thm. 2.3, and let \( i = \mathcal{I} \cap \mathcal{B}(\Omega, \mathbb{Z}) \) be the restriction of \( \mathcal{I} \) to \( \mathcal{B}(\Omega, \mathbb{Z}) \).

Then \( i \) is generated by the differences \( \psi_E - \psi_F \) with \( s(E) = s(F) \), and there exists a unique ordered ring isomorphism \( \sigma : \mathcal{A} \to \mathbb{Z}^* \subset \mathbb{E} \) that makes the following diagram commute\(^{12}\):

\[
\begin{array}{ccc}
U & \xrightarrow{\psi} & \mathcal{B}(\Omega, \mathbb{Z}) \\
\downarrow{s} & & \downarrow{i} \\
\mathcal{A} & \xrightarrow{\sigma} & \mathcal{B}(\Omega, \mathbb{Z})/i \cong \mathbb{Z}^* \xrightarrow{i} \mathbb{E} \cong \mathcal{B}(\Omega, \mathbb{R})/\mathcal{I}
\end{array}
\]

In particular \( n = \sigma \circ s \) is the Euclidean numerosity function, and the Euclidean field \( \mathbb{E} \) is uniquely determined by \( s \).

**Proof.**

Let \( E \) be a set of ordinals: then \( \sum_k \chi_E(k) \) is the Euclidean numerosity of \( E \), by definition, and the corresponding counting function is precisely \( \psi_E \). On the other hand, the transfinite sum of nonnegative integers has been defined above in a consistent way also inside the ring \( \mathcal{A} \), so \( s(E) = \sum_k \chi_E(k) \).

We have already remarked that the axiom CA holds in \( \mathcal{A} \) for transfinite sums of integers. Moreover the equality (dsp) yields that also both axioms SA and PA hold. It follows that \( s(E) = s(F) \) if and only if \( \psi_E \equiv \psi_F \mod i \), and \( \sigma \) is uniquely and consistently defined by \( \sigma(s(E) - s(F)) = [\psi_E - \psi_F] \mod i \).

\( \square \)

\(^{12}\) here \( i \) is the inclusion map, and \( \pi_i \) and \( \pi_3 \) are the projections onto the quotients modulo the ideals \( i \) and \( \mathcal{I} \), respectively.
Remark 4.6. The last point of the above theorem shows that the Euclidean field $\mathbb{E}$ arises quite naturally from a numerosity theory that satisfies very reasonable assumptions which extend and interpret the Euclidean common notions.

5 The existence of the field of the Euclidean numbers

In this section we ground on the particular kind of ordinals, that we called complete ordinals in subsection 1.2: they will be used as convenient “check points” for the values of the transfinite sums, and of the corresponding counting functions, in order to produce a model of the field of the Euclidean numbers.

5.1 The Euclidean ultrafilter $\mathcal{U}$

Let $\eta = 2^n$ be a fixed point ordinal in $\Omega$, and let $\mathcal{F}$ be a filter on the set $\Omega_\eta$ of the ordinals less than $\eta$. Given a family $F = \langle F_h \mid h \in \Omega \rangle$ of sets $F_h \in \mathcal{F}$, put

$$\overline{F} = \{ \alpha = \eta^h \circ \gamma \mid h \in \Omega, \gamma \in F_h \}$$

so

$$i < \alpha = \eta^h \circ \gamma \in \overline{F} \iff \exists k < \gamma. i = \eta \circ h + k.$$ 

Clearly $\overline{F} \cap \overline{F'} = \langle F_h \cap F'_h \rangle$, hence the family $\{ \overline{F} \mid F \subseteq \mathcal{F} \}$ is closed under finite intersection, so it generates a filter $\overline{\mathcal{F}}$ on $\Omega$.

Similarly, the intersections $\overline{F} \cap \Omega_\varepsilon$ generate a filter $\overline{\mathcal{F}}|_\varepsilon$ on $\Omega_\varepsilon$ for all fixed point ordinals $\varepsilon = 2^\varepsilon > \eta$.

Recall that $\mathcal{Q}$ is the filter generated by the family of the $\sqsubseteq$-cones $C(\beta) = \{ \alpha \mid \beta \sqsubseteq \alpha \}$ together with all $(\eta, h)$-complete ordinals (see Lemma 1.3), and that $\mathcal{Q}_\eta$ is the corresponding filter restricted to $\Omega_\eta$.

We may now give an inductive construction of a “Euclidean ultrafilter” on $\Omega$. Since each ordinal $\alpha \in \Omega$ may be considered also as the finite subset $L_\alpha \subseteq \Omega$, we call fine a filter on $\Omega_\eta$ if it is fine as a set of finite subsets of $\Omega_\eta$.

It is worth noticing that this identification works well only for fixed points $\eta = 2^n$, because then $\alpha < \eta$ if and only if $L_\alpha \subseteq \Omega_\eta$.  

Theorem 5.1. There exist a fine ultrafilter $\mathcal{U}$ on $\Omega$, and fine ultrafilters $\mathcal{U}(\eta)$ on $\Omega_\eta$, for all $\eta = 2^n \in \Omega$, such that

- $\mathcal{Q} \subseteq \mathcal{U}$ and $\mathcal{Q}_\eta \subseteq \mathcal{U}(\eta)$;
- $\mathcal{U}(\eta) \subseteq \mathcal{U}$ and $\mathcal{U}(\varepsilon)_{\eta} \subseteq \mathcal{U}(\eta)$ for all $\varepsilon < \eta$.

Proof. Let $\langle \eta_j \rangle_{j \in \Omega}$ be an enumeration of the fixed point ordinals in $\Omega$. We proceed by induction on $j$.

For $j = 0$ let $\mathcal{U}(\omega)$ be a fine ultrafilter on $\Omega_{\eta_0} = \mathbb{N}$.

Now let the ultrafilters $\mathcal{U}(\eta_s)$ be conveniently defined for all $s \leq j$, and consider the filters $\mathcal{Q}_{\eta_{j+1}}$ and $\mathcal{U}(\eta_s)|_{\eta_{j+1}}$, $s \leq j$: they are separately closed under intersection, so we need only to prove that $\mathcal{U} \cap D \neq \emptyset$ for all $U = \{U_h \mid h < \eta_{j+1}\} \subseteq \mathcal{U}(\eta_j)$ and all $D = D(E, H; \beta) \in \mathcal{Q}_{\eta_{j+1}}$, i.e. that there exists $\alpha = \eta_j^h \odot \gamma$ with $h < \eta_{j+1}$ and $\gamma \in U_h$ that belongs to $D$, i.e. such that, for all $h \in H$ and all $\eta_s \in E$,

$$\forall k < \eta_s \cdot \eta_s \odot h + k < \alpha \iff k < \alpha.$$ 

But

$$\alpha = \eta_j^h \odot \gamma \implies \eta_j \odot h + k < \alpha \iff k < \gamma \in U_h$$

So we pick any $\gamma \in D(E\setminus\{\eta_j\}, H; \beta) \cap U_h$, which is nonempty by induction hypothesis. Then we can take $\mathcal{U}(\eta_{j+1})$ to be any ultrafilter containing both families of sets.

For limit $j$, assume that the ultrafilters $\mathcal{U}(\eta_s)$ have been conveniently defined for all $s < j$. Then, for all $r < j$, the family of filters

$$\mathcal{U}(\eta_s) \text{ and } \mathcal{Q}_{\eta_s}, \ s < r$$

is included in $\mathcal{U}(\eta_r)$, by induction hypothesis, hence it has the finite intersection property. Then also the union of all these families has the FIP, because $\eta < \eta_j$ implies $\eta < \eta_r$ for some $r$ strictly in between $s$ and $j$. So any ultrafilter containing this union can be taken to be $\mathcal{U}(\eta_j)$.

Finally, after having defined the ultrafilter $\mathcal{U}(\eta_j)$ for all $j \in \Omega$, define $\mathcal{U}$ as an ultrafilter on $\Omega$ containing the filters $\mathcal{U}(\eta_j)$ and $\mathcal{Q}_{\eta_j}$, for all $j \in \Omega$. The finite intersection property follows by the same argument of the case of limit $j$, and all conditions are fulfilled. \[\square\]
We name Euclidean an ultrafilter satisfying the conditions of $U(\Omega)$, since any such ultrafilter provides a model of the field of the Euclidean numbers, as we show in the next subsection.

### 5.2 A construction of the field of the Euclidean numbers

We may now present a construction of a field enjoying the properties of the Euclidean numbers axiomatized in Section 2.

Let $\mathbb{R}^\Omega$ be the algebra of all real valued functions on $\Omega$, and let

$$
\mathcal{A}(\Omega, \mathbb{R}) = \{ \varphi_x \in \mathbb{R}^\Omega \mid x \in \mathcal{I}(\Omega, \mathbb{R}) \}
$$

be the subalgebra of the counting functions. Then, by Theorem 2.4 every element of $\mathcal{A}(\Omega, \mathbb{R})$ is $j$-periodic for some $j \in \Omega$, and conversely any $j$-periodic $\psi \in \mathcal{B}(\Omega, \mathbb{R})$ agrees with one and only one $\varphi_x \in \mathcal{A}(\Omega, \mathbb{R})$ on all ordinal powers $\beta = 2^\alpha$.

- Let $U$ be a Euclidean ultrafilter on $\Omega$, and let $I$ be the corresponding maximal ideal of $\mathcal{A}(\Omega, \mathbb{R})$

$$
I := \{ \varphi \in \mathcal{A}(\Omega, \mathbb{R}) \mid \exists Q \in U \forall \beta \in Q \cdot \varphi(\beta) = 0 \}
$$

- let

$$
J : \mathcal{A}(\Omega, \mathbb{R}) \rightarrow \mathcal{A}(\Omega, \mathbb{R})/I = E
$$

be the canonical homomorphism onto the corresponding quotient field, which becomes an ordered field with the ordering induced by the natural partial ordering of $\mathcal{A}(\Omega, \mathbb{R})$.

- Denote by $[\varphi]$ the coset $J(\varphi) = \varphi + I \in E$ of the function $\varphi \in \mathcal{A}(\Omega, \mathbb{R})$.

- Given any eventually zero $\Omega$-sequence $x \in \mathcal{I}(\Omega, \mathbb{R})$ define the sum $\sum_k x_k = \Sigma(x)$ as the coset $[\varphi_x] = J(\varphi_x)$ of the corresponding counting function $\varphi_x \in \mathcal{A}(\Omega, \mathbb{R})$.

Then we have

**Theorem 5.2.** The ordered field $E$ satisfies the axiom RA, and the transfinite sum $\Sigma : \mathcal{I}(\Omega, \mathbb{R}) \rightarrow E$ satisfies the axiom CA; moreover $\Sigma$ can be uniquely extended to a $\mathbb{R}$-linear map $\Sigma : \mathcal{I}(\Omega, E) \rightarrow E$ so as to satisfy both axioms PA and SA. Hence $E$ becomes a Euclidean field.
Proof.
• The axiom RA holds by definition.
• The map \( x \mapsto \varphi_x \) is \( \mathbb{R} \)-linear, so \( \Sigma \) is \( \mathbb{R} \)-linear.
• The axiom CA holds because the ultrafilter \( \mathcal{U} \) contains all cones \( C(\beta) \).
• The map \( J \) is an \( \mathbb{R} \)-algebra homomorphism, so
  \[
  (\varphi_x \varphi_y)(\alpha) = \varphi_x(\alpha) \varphi_y(\alpha) = \sum_{h<\alpha} x_h \sum_{k<\alpha} y_k = \sum_{h,k<\alpha} x_h y_k.
  \]
  Hence, putting \( w_i = \sum_{h,k<\star} x_h y_k \), one has \( \sum_{i<\alpha} w_i = \sum_{h,k<\alpha} x_h y_k \), hence
  \[
  (\sum_{h} x_h)(\sum_{k} y_k) = [\varphi_x][\varphi_y] = [\varphi_w] = \sum_{i} w_i = \sum_{h,k} x_h y_k,
  \]
  and the axiom PA holds.
• Now we extend the map \( \Sigma \) to \( \mathcal{S}(\Omega,\mathbb{E}) \) so as to satisfy the axiom SA.
  Let \( \xi_h = \sum_k x_{hk} \) be given, and assume that \( x_{hk} = 0 \) for \( h, k \geq \eta = 2^\eta \).
  Put
  \[
  \Sigma((\xi_h)) = \sum_h \xi_h = \sum_h \sum_{k} x_{hk} = \sum_{i} y_i \quad \text{where} \quad y_{\eta \odot h + k} = x_{hk}.
  \]
  The definition is well posed. Namely, let \( \xi_h = \sum_k x'_{hk} \) be other expressions of the same elements \( \xi_h \in \mathbb{E} \); assume w.l.o.g. that also \( x'_{hk} = 0 \) for \( h, k \geq \eta \), and define \( y'_i \) accordingly. Then, for each \( h < \eta \) there exists a qualified set \( U_h \in \mathcal{U}(\eta) \) such that \( \sum_{k<\gamma} x_{hk} = \sum_{k<\gamma} x'_{hk} \) for all \( \gamma \in U_h \).
  The Euclidean ultrafilter \( \mathcal{U} \) contains the filter \( \overline{\mathcal{U}(\eta)} \), hence all sets
  \[
  \overline{\mathcal{U}} = \{ \alpha = \eta^h \odot \gamma \mid \gamma \in U_h \} \quad \text{for} \quad U = \{ U_h \mid h \in \Omega \} \subseteq \mathcal{U}(\eta).
  \]
  So, for \( \alpha \in \overline{\mathcal{U}} \), there is \( h \in \Omega \) and \( \gamma \in U_h \) such that \( \alpha = \eta^h \odot \gamma \); it follows that \( i = \eta \odot h + k \leftrightarrow k \leq \gamma \), whence
  \[
  \sum_{i<\alpha} y_i = \sum_{k<\gamma} x_{hk} = \sum_{k<\gamma} x'_{hk} = \sum_{i<\alpha} y'_i,
  \]
  and so \( \sum_i y_i = \sum_i y'_i \).
So the existence of the Euclidean field $\mathbb{E}$ is implied by that of Euclidean ultrafilters.

**Appendix: The Euclidean continuum**

In ancient geometry, lines and segments are not considered as sets of points; on the contrary, in the last two centuries, the reductionistic attitude of modern mathematics has tried and described the Euclidean geometry through a set theoretic interpretation. So the Euclidean continuum has been identified with the Dedekind continuum and the Euclidean line has been identified with (or at least considered isomorphic to) the set of real numbers (once an origin $O$ and a unit segment $OA$ have been fixed). Although this identification be almost universally accepted today, nevertheless it contradicts various theorems of the Euclidean geometry.

As an important example, we cite the Euclidean statement that a segment $AB$ can be divided in two congruent segments $AM$ and $MB$. If $AB$ is identified with the Dedekind continuum then, either $AM$ has a maximum or $MB$ has a minimum. Thus $AM$ and $MB$ are not congruent, hence *stricto sensu* the Dedekind continuum is not a correct model for the Euclidean continuum. In order to construct a consistent model, we are forced to assume that the points $A, B$ and $M$ do not belong to the segment $AB$. So the picture which comes out of the Euclidean straight line is a linearly ordered set $\mathcal{E}$, where the Euclidean segment $AB$ is a subset of $\mathcal{E}$ which cannot be identified with the set theoretical segment

$$ S(A, B) := \{X \in \mathcal{E} \mid A < X < B\}, \text{ because } M \in S(A, B) \setminus AB. $$

So we might better think of the segment $AB$ as a set of atoms with lots of empty spaces between them.

In the Euclidean theory of proportions, a set of magnitudes can be put in biunique correspondence with the lengths of the segments, and the lengths of segments satisfy the axiom of Archimedes. Hence, assuming also this axiom in the set theoretic interpretation of (oriented) segments, they build a field isomorphic to the real field, so that, after a suitable identification,

$$ \mathbb{R} \subset \mathcal{E} \quad (5.1) $$

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and, since $AB \subseteq S(A, B) \setminus \mathbb{R}$, we can assume that $AB = S(A, B) \setminus \mathbb{R}$.

Now take an atom $s$ in $AB$: the distance between $A$ and $s$ cannot be measured by a length, because $S(A, s)$ is not a segment and no segment is congruent to $S(A, s)$. So the lengths cannot be as many as the distances; if the lengths are in birecursive correspondence with $\mathbb{R}$, the distances can be put in biunique correspondence with the full $\mathcal{E}$. Assuming that also the distances form a field, the inclusion (5.1) implies that $\mathcal{E}$ is a non-Archimedean field. So the Euclidean continuum leads to the non-Archimedean geometry as described by Veronese at the end of the XIX century (see [24, 25]). However, there are many non-Archimedean fields which contain the real numbers. Moreover, every non-Archimedean field has gaps. Thus the question arises as to whether a non-Archimedean field provides a more satisfactory model of the Euclidean continuum than the Dedekind continuum.

In a naive way, a continuum is a linearly ordered set without holes. In contrast with our intuition, a set $X$ which satisfies the following property

$$\forall a, b \in X, \quad a < b, \quad \exists c \in X, \quad a < c < b$$

(5.2)

is not a continuum since there are holes (think e.g. to a segment of rational numbers: here the irrational numbers can be considered holes). However, also the Dedekind continuum is not satisfactory: the arguments outlined above yield that the lengths form a Dedekind continuum, but there are distances which are not lengths. Thus in a sense also the Dedekind continuum contains holes, represented by the distances which are not lengths. So we are tempted to give the following definition which generalizes (5.2):

**A linearly ordered “set” $X$ is an continuum if given two subsets $A$ and $B$ such that $\forall a \in A, \forall b \in B, \ a < b$, then**

$$\exists c \in X, \ a < c < b$$

(5.3)

Assuming this definition, a continuum is a proper class in the sense of von Neumann-Bernays-Gödel (NBG) class theory. This is the point of view of Ehrlich [17], where the class of surreal numbers is viewed as the absolute arithmetic continuum. In fact, as far as the order structure is concerned, the surreal numbers have the order type described by (5.3).

We prefer to have the continuum to be a set; so we assume the existence of an inaccessible ordinal and we give the following definition:

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13 Actually Veronese considered also infinitesimal segments, but this is matter of definitions that will be not discussed here.
Definition 5.1. A linearly ordered set $X$ is a Euclidean continuum if it is $\Omega$-saturated, i.e. given any two subsets $A$ and $B$ such that $|A|, |B| < |X| = |\Omega|$ and $\forall a \in A, \forall b \in B, a < b$, we have that

$$\exists c \in X, a < c < b$$

(5.4)

Grounding on this discussion, we are led to assume that the Euclidean line $E$ is an Euclidean continuum equipped with the structure of real closed field, so it is isomorphic to the field of Euclidean numbers $\mathbb{E}$.

The fact that $E$ has to be real closed is a reasonable request: it has to contain not only all Euclidean numbers stricto sensu but also all zeroes of sign-changing polynomials.

At this point it seems appropriate to explain why, in our opinion, the order type of $E$ must be $\eta_\Omega$ rather that $\eta_\alpha$ for some $\alpha < \Omega$.

We sketch two arguments. The first is set theoretical. The condition (5.3) seems implicit in the notion of (absolute) continuum, but working with proper classes meets several technical limitations, which do not arise in working with inaccessible ordinals; so it seems appropriate to “truncate” the universe at the first inaccessible level. In doing this, the Euclidean continuum becomes indiscernible from the absolute continuum of Ehlrich, because then the class of surreal numbers would be a field isomorphic to $\mathbb{E}$.

Moreover, taking into account the role of the Euclidean numbers as numerosities, the use of a saturated real closed field $F$ of accessible cardinality seems inappropriate: it contradicts the natural assumption that a powerset, or a set of functions, has a numerosity in $F$ whenever the original set (the domain) has a numerosity in $F$.

The second argument is “geometric”. The assumption $|E| = \Omega$ yields $E \cong \mathbb{E}$, and hence it inherits a very rich structure. In particular, every Euclidean number being a transfinite sum of reals, one has that any distance is a transfinite (algebraic) sum of lengths. For instance

$$\sum_{k \in \langle 0, \omega \rangle} \frac{1}{2^k} = 2 - \frac{1}{2^{\omega-1}}$$

(5.5)

(see equation (3.3) in subsection 3.6) so adding infinitely many segments of length $2^{-k}$ cannot provide a segment of length 2, but only a quantity (a

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14 I.e. those constructible by ruler and compass.

15 $\eta_\alpha$ is the order type of an $\alpha$-saturated ordered set of size $\alpha$; assuming the generalized continuum hypothesis, such sets exist for every regular $\alpha$. 

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distance) which is (infinitesimally) smaller than 2. In the Euclidean world, Achilles will never properly reach the turtle, remaining forever at infinitesimal distance. Assuming that in the physical world infinitesimal quantities cannot be measured and so do not count, the turtle is reached. In our opinion, this joke might emphasize the fact that non-Archimedean fields (and, in the future, hopefully also the Euclidean field) might be very useful in building models of natural phenomena (see some examples in [1, 3, 10, 12, 11]).

References

[1] Albeverio S., Fenstad J.E., Hoegh-Krohn R., Lindstrom T. - Nonstandard Methods in Stochastic Analysis and Mathematical Physics, Dover Books on Mathematics, 2006.

[2] V. Benci - I numeri e gli insiemi etichettati, in Conferenze del seminario di matematica dell’ Università di Bari, vol. 261, Laterza, Bari 1995, p. 29.

[3] V. Benci - Ultrafunctions and generalized solutions, Adv. Nonlinear Stud. 13 (2013), 461–486, arXiv:1206.2257

[4] V. Benci, M. Di Nasso - Numerosities of labelled sets: a new way of counting, Adv. Math. 21 (2003), 505–67.

[5] V. Benci, M. Di Nasso - A purely algebraic characterization of the hyperreal numbers, Proc. Amer. Math. Soc.

[6] V. Benci, M. Di Nasso, M. Forti - An Aristotelian notion of size, Ann. Pure Appl. Logic 143 (2006), 43–53.

[7] V. Benci, M. Di Nasso, M. Forti - An Euclidean notion of size for mathematical universes, Logique et Analyse 50 (2007), 43–62.

[8] V. Benci, M. Di Nasso, M. Forti - The eightfold path to nonstandard analysis, in Nonstandard Methods and Applications in Mathematics (N. J. Cutland, M. Di Nasso, and D. A. Ross, eds.), Lecture Notes in Logic 25, Association for Symbolic Logic, AK Peters, Wellesley, MA, 2006, 3–44.
[9] V. Benci, P. Freguglia - Alcune osservazioni sulla matematica non archimedea, Matem. Cultura e Soc., RUMI, 1 (2016), 105–122.

[10] V. Benci, L. Horsten, S. Wenmackers - Non-Archimedean probability, Milan j. Math. 81 (2013), 121–151. arXiv:1106.1524.

[11] V. Benci, L. Horsten, S. Wenmackers - Infinitesimal Probabilities, Brit. J. Phil. Sci. (2016), to appear.

[12] V. Benci, L. Luperi Baglini - A model problem for ultrafunctions, in: Variational and Topological Methods: Theory, Applications, Numerical Simulations, and Open Problems, Electron. J. Diff. Eqns., Conference 21 (2014), 11–21.

[13] A. Blass, M. Di Nasso, M. Forti - Quasi-selective ultrafilters and asymptotic numerosities, Adv. Math. 231 (2012), 1462–1486.

[14] C.C. Chang, H.J. Keisler - Model Theory, Dover Books on Mathematics.

[15] J.H. Conway - On numbers and games, 2nd ed., A. K. Peters, Natick 2001.

[16] M. Di Nasso, M. Forti - Numerosities of point sets over the real line, Trans. Amer. Math. Soc. 362 (2010), 5355–5371.

[17] Ph. Ehrlich - The absolute arithmetic continuum and the unification of all numbers great and small, Bull. Symb. Logic 18 (2012), .

[18] Euclid - The Elements (T.L. Heath, translator), 2nd edition (reprint), Dover, New York 1956.

[19] M. Forti, G. Morana Roccasalvo - Natural numerosities of sets of tuples, Trans. Amer. Math. Soc. 367 (2015), 275–292

[20] H.J. Keisler - Foundations of Infinitesimal Calculus, Prindle, Weber & Schmidt, Boston 1976.

[21] Levi-Civita T. - Sugli infiniti ed infinitesimi attuali quali elementi analitici, Atti R. Istit. Veneto Sci. Lett. Arti, Venezia (Serie 7) (1892–93), 1765–1815.
[22] P. Mancosu - Measuring the size of infinite collections of natural numbers: was Cantor’s theory of infinite number inevitable?, *Rev. Symb. Logic* 4 (2009), 612–646.

[23] A. Robinson - Non-standard Analysis, *Proc. Royal Acad. Sci., Amsterdam* (Series A) 64 (1961), 432–440.

[24] G. Veronese - Il continuo rettilineo e l’assioma V di Archimede, *Mem. Reale Accad. Lincei, Classe Sci. Nat. Fis. Mat.* 4 (1889), 603–624.

[25] G. Veronese - Intorno ad alcune osservazioni sui segmenti infiniti o infinitesimi attuali, *Math. Annalen* 47 (1896), 423–432.