Gauge-invariant Lagrangian of the soft-collinear effective theory
and its application to soft-collinear currents

Junegone Chay and Chul Kim

Department of Physics, Korea University, Seoul 136-701, Korea

Abstract

We construct the soft-collinear effective Lagrangian which is manifestly gauge invariant order by order. Field redefinitions of collinear gauge fields and a proper decomposition of quark fields are necessary to make the Lagrangian gauge invariant. We can obtain the effective Lagrangian order by order in SCET$_I$ and SCET$_{II}$ by adopting appropriate power counting methods. Various types of current operators can be investigated using this formalism, and we choose the soft-collinear current as a specific example to present explicit radiative corrections. The hard-collinear, collinear, and ultrasoft modes in SCET$_I$ and the collinear and soft modes in SCET$_{II}$ reproduce the infrared divergences of the full theory and the matching can be performed. We show our results explicitly using two types of regularization schemes. The first scheme is the dimensional regularization both for the ultraviolet and the infrared divergences with on-shell particles, and the second scheme is regulating the infrared divergence with the off-shellness of external particles. We discuss other types of operators in terms of the two-step matching.

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*Electronic address: chay@korea.ac.kr
†Electronic address: chulkim@korea.ac.kr
I. INTRODUCTION

Soft-collinear effective theory (SCET) is a useful tool to extract important physics from a system with energetic light quarks \[1, 2, 3, 4, 5, 6, 7, 8\]. In describing energetic light particles, we decompose the momentum of a collinear particle as

\[ p^\mu = \frac{\mathbf{n} \cdot p}{2} n^\mu + p^\mu_\perp + \frac{n \cdot p}{2} \mathbf{m}^\mu, \]  

where \(n^\mu\) is the light-like direction in which the collinear particle moves, and \(\mathbf{m}^\mu\) is a light-like vector in the opposite direction, satisfying \(n^2 = m^2 = 0, n \cdot m = 2\). And \(p^\mu_\perp\) is the momentum perpendicular to \(n^\mu\) and \(\mathbf{m}^\mu\). Here the component \(n \cdot p\) is the largest scale of order \(Q\), \(p^\mu_\perp\) is the intermediate scale, and \(n \cdot \mathbf{m}\) is the smallest component.

For a system with energetic light quarks, the relevant scales are a large energy \(Q\) and the typical hadronic scale \(\Lambda \sim \Lambda_{QCD}\), and we describe the power counting in terms of the ratio of these two scales. We will eventually describe the light quarks with \(p^2 \sim \Lambda^2\), which is appropriate for the quarks to form light hadrons. In this case, the collinear momentum scales as \((\mathbf{n} \cdot p, p_\perp, n \cdot p) \sim (Q, \Lambda, \Lambda^2/Q)\). However, we consider the intermediate effective theory, called SCET1, in which the momentum of the collinear particles scales as \((\mathbf{n} \cdot p, p_\perp, n \cdot p) \sim (Q, \sqrt{Q\Lambda}, \Lambda)\) such that \(p^2 \sim Q\Lambda\). In the literature \[9, 10\], these modes are sometimes labelled as “hard-collinear” modes. There are soft particles with momentum \(p^\mu_s \sim (\sqrt{Q\Lambda}, \sqrt{Q\Lambda}, \sqrt{Q\Lambda})\) in SCET1, but these particles do not interact with collinear particles, and we do not consider them here. There are also ultrasoft (usoft) particles with momentum \(p^\mu_{us} \sim (\Lambda, \Lambda, \Lambda)\) with \(p^2_{us} \sim \Lambda^2\). The intermediate effective theory SCET1 can be obtained by integrating out the hard degrees of freedom of order \(Q\) from the full theory. Then we obtain the final effective theory SCETII by integrating out the degrees of freedom of order \(p^2 \sim Q\Lambda\). In SCETII, the collinear particles scale as \(p_c \sim (Q, \Lambda, \Lambda^2/Q)\), and the soft momentum, which was labelled as the usoft momentum in SCET1, now scales as \(p_s \sim (\Lambda, \Lambda, \Lambda)\). The small expansion parameter in SCET is chosen as the ratio \(p_\perp/\mathbf{n} \cdot p \sim p_\perp/Q\). In SCET1, the expansion parameter is \(\sqrt{\Lambda/Q}\), and in SCETII, it is of order \(\Lambda/Q\).

This systematic method to obtain the effective theories is a two-step matching \[11\]. It is convenient and transparent to use this two-step matching where there are some contributions of order \(p^2 = Q\Lambda\), as in the spectator contributions for \(B\) decays \[12, 13, 14, 15, 16, 17\]. The construction of the operators, and the computation of the hard-collinear contributions in SCET1 can be systematically performed, and we can easily go down to SCETII. On the
other hand, we can directly match the full theory to the final effective theory SCET_{II}, but proper care should be taken. This is discussed in detail in Sec. IV.

In constructing the effective Lagrangian in SCET, we require that the gauge invariance under collinear and (u)soft gauge transformations be preserved order by order. This has been partially accomplished in Ref. [11] by treating the $n \cdot A_{us}$ component of the usoft gauge fields as a background field for collinear gauge transformations, and by redefining the collinear gauge field to make the collinear Lagrangian gauge invariant order by order. As a result, the collinear Lagrangian in SCET$_I$ is gauge invariant at each order, and the usoft-collinear Lagrangian is also gauge invariant as a byproduct, but it is not clear whether the usoft Lagrangian at higher orders is gauge invariant.

In order to make the Lagrangian gauge invariant at each order, we treat all the components of the usoft gauge field as background fields for collinear gauge transformations, and we also redefine the collinear gauge field accordingly. The fact that all the components of the usoft gauge field should act as background fields is also important in factorizing the usoft contributions. And when we decompose the quark field in the full theory into the collinear quark fields and the usoft quark field, we put an additional phase factor to satisfy the transformation properties under the collinear and usoft gauge transformations. Then the collinear gauge fields are redefined by factorizing the usoft interactions in SCET$_I$ to obtain the gauge-invariant Lagrangian order by order in SCET$_I$ and we go down to SCET$_{II}$. We employ the appropriate power counting methods in each effective theory and list the first few terms of the Lagrangian in each sector.

The main ingredient in constructing effective theories is that we modify the ultraviolet behavior of the theory, but the infrared behavior should be intact. This is physically obvious since both theories share the same infrared region. It means that if there is infrared divergence in the full theory, it should be reproduced in the effective theory. Only after we assure that the infrared behavior is the same in both theories, we can match the two theories at the boundary, and compute the Wilson coefficients and the anomalous dimensions of various operators because the infrared divergences cancel in matching.

Recently it has been argued that the collinear and the soft degrees of freedom in the final effective theory SCET$_{II}$ may not be enough to reproduce the infrared divergence in the full theory at one loop [9, 18]. In this paper, we consider radiative corrections to soft-collinear current operators and show that the infrared divergences in the full theory are fully
reproduced both in SCET I and SCET II using the conventional hard-collinear, collinear, and usoft modes in SCET I and the collinear, soft modes in SCET II.

In classifying the collinear and the soft degrees of freedom in SCET II, note that the collinear momentum scales as \( p_c \sim (Q, \Lambda, \Lambda^2/Q) \). For the momentum to be collinear, the component \( \mathbf{n} \cdot p_c \) must be of order \( Q \), not smaller than that, and the remaining components are determined by requiring that \( p_c^2 \sim \Lambda^2 \). However the soft momentum is different. We generically say that the soft momentum scales as \( p_s \sim (\Lambda, \Lambda, \Lambda) \), but it means that \( \Lambda \) is the maximum fluctuation. When we refer to soft dynamics, we consider the dynamics with scales less than \( \Lambda \). Therefore further classification of the soft momentum, in which momenta with \( p^2 < \Lambda^2 \) are labelled differently from the soft momentum, may be conceptually helpful, but it is not necessary in the construction of the effective theory and radiative corrections.

In this paper, we consider a specific example of soft-collinear current operators and compute their radiative corrections in the full QCD, SCET I and SCET II. We can employ dimensional regularization both for the ultraviolet and the infrared divergences with external particles on their mass shell. In this case it is difficult to extract the infrared divergent part alone. In order to separate the infrared divergence from the ultraviolet divergence, we use the off-shell regularization scheme for the infrared divergence and the dimensional regularization for the ultraviolet divergence. In this case, care must be taken in using the factorized form for the usoft interactions. After we separate the infrared and the ultraviolet divergences, we use the on-shell regularization scheme with the dimensional regularization both for the ultraviolet and the infrared divergences in order to calculate the anomalous dimensions of the soft-collinear current operators in each effective theory.

The organization of the paper is as follows: In Section III, we construct the effective Lagrangian in SCET I, which is gauge invariant under both collinear and usoft gauge transformations at each order in \( \sqrt{\Lambda/Q} \). And we evolve down to SCET II and construct the effective Lagrangian in SCET II employing an appropriate power counting. In Section III, we construct the effective soft-collinear current operators in SCET and compute the radiative corrections at one loop. We show that the infrared divergences in the full QCD are the same both in SCET I and in SCET II, and compute the Wilson coefficients and the anomalous dimension of the soft-collinear current operator. In Sec. IV, we compare the scaling behavior of the soft-collinear current with that of the back-to-back collinear current and the heavy-to-light current. Finally in Sec. V, we present a conclusion.
II. EFFECTIVE LAGRANGIAN IN SCET

A. Effective Lagrangian in SCET

The effective Lagrangian in SCET is derived by integrating out the hard degrees of freedom of order \(Q\) in the full theory. In SCET there are the collinear modes with \(p^2 \sim QA\) and the usoft modes with \(p^2 \sim \Lambda^2\). We decompose the gauge field \(A^\mu\) in the full QCD into the collinear gauge field \(A^\mu_n\), the soft field \(A^\mu_s\), and the usoft field \(A^\mu_us\). The collinear gauge field scales as \(A^\mu_n = (\pi \cdot A_n, A^\mu_{n\perp}, n \cdot A_n) \sim (Q, \sqrt{Q\Lambda}, \Lambda)\), the soft gauge field scales as \(A^\mu_s \sim (\sqrt{Q\Lambda}, \sqrt{Q\Lambda}, \sqrt{Q\Lambda})\) and the usoft field scales as \(A^\mu_us \sim (\Lambda, \Lambda, \Lambda)\). Since the soft fields do not couple to the collinear sector, we will not consider them from now on.

Let us consider the gauge symmetries in SCET starting from the full QCD. The full QCD possesses an \(SU(3)\) color gauge symmetry, and the gauge field \(A^\mu\) transforms as

\[
A^\mu \rightarrow UA^\mu U^\dagger + \frac{i}{g} U[\partial^\mu U^\dagger],
\]

where \(U\) is an \(SU(3)\) gauge transformation and the bracket means that the differential operator \(\partial^\mu\) acts only inside the bracket. In SCET, there are collinear, soft and usoft gauge transformations, which are subsets of the gauge transformations of the full theory. The collinear (usoft) gauge transformation \(U_c\) (\(U_{us}\)) is the gauge transformation which satisfies \(\partial^\mu U_c \sim (Q, \sqrt{Q\Lambda}, \Lambda)\) \([\partial^\mu U_{us} \sim (\Lambda, \Lambda, \Lambda)]\). We decompose the gauge field \(A^\mu\) as

\[
A^\mu = A^\mu_n + A^\mu_us,
\]

and we extract the label momentum from the collinear gauge field \(A^\mu_n\) as

\[
A^\mu_n(x) = \sum_q e^{-iq \cdot x} A^\mu_{n,q}(x),
\]

where \(q^\mu = \pi \cdot qn^\mu/2 + q^\mu_\perp = \mathcal{O}(Q) + \mathcal{O}(\sqrt{Q\Lambda})\) is the label momentum, and the derivative acting on \(A^\mu_{n,q}(x)\) gives the momentum of order \(\Lambda\). We also decompose the collinear gauge transformation as

\[
U_c = \sum_p e^{-ip \cdot x} U_{c,p}.
\]

Since the large label momentum is extracted, the derivative acting on \(U_{c,p}\) gives \(\partial^\mu U_{c,p} \sim \Lambda\). From now on, we will drop the label index and the summation over the label momentum unless they are necessary.
Since the fluctuation of the usoft gauge field is of order $1/\Lambda$, the variation is smooth compared to that of the collinear fields. Therefore the usoft gauge field $A^\mu_{us}$ acts as a classical background field under collinear gauge transformations, and $A^\mu_{n}$ transforms homogeneously under usoft gauge transformations. That is, the gauge fields transform as

$$A^\mu_{n} \rightarrow U_c(A^\mu_{n} + A^\mu_{us})U^\dagger_c + \frac{1}{g}U_c[(P^\mu + i\partial^\mu)U^\dagger_c] - A^\mu_{us}, \quad A^\mu_{us} \rightarrow A^\mu_{us},$$

under collinear gauge transformations, where $P^\mu$ is the label momentum operator with $n \cdot P = 0$. Under usoft gauge transformations, the gauge fields transform as

$$A^\mu_{n} \rightarrow U_{us}A^\mu_{n}U^\dagger_{us}, \quad A^\mu_{us} \rightarrow U_{us}A^\mu_{us}U^\dagger_{us} + \frac{i}{g}U_{us}[\partial^\mu U^\dagger_{us}].$$

As a first step toward the gauge invariant formulation of the Lagrangian order by order, we redefine the collinear gauge fields as

$$gA^\mu_{n} = g\hat{A}^\mu_{n} + \hat{W}[iD^\mu_{us}, \hat{W}^\dagger],$$

where $iD^\mu_{us} = i\partial^\mu_{us} + gA^\mu_{us}$. The derivative $\partial^\mu$ is written as $\partial^\mu = \partial^\mu_c + \partial^\mu_{us}$, where $\partial^\mu_c$ ($\partial^\mu_{us}$) picks out collinear (usoft) momentum of order $\Lambda$ from collinear (usoft) fields, and gives zero when applied to usoft (collinear) fields. In SCET$_I$, $i\partial^\mu_c$ is negligible compared to $P^\mu$, but it is kept to make the transition to SCET$_{II}$ transparent.

The hatted fields $\hat{A}^\mu_{n}$ are newly defined fields and $\hat{W}$ is given by

$$\hat{W} = WZ^\dagger = \left[ \sum_{\text{perm}} \exp\left( -\frac{g}{P} \mathbf{\Pi} \cdot \hat{A}_{n} \right) \right],$$

where $P = \mathbf{n} \cdot P$, and the Wilson lines $W$ and $Z$ are defined as

$$W = P \exp\left( ig \int_{-\infty}^{0} ds \mathbf{n} \cdot (A_{n} + A_{us})(\mathbf{n}s + x) \right),$$

$$Z = P \exp\left( ig \int_{-\infty}^{0} ds \mathbf{n} \cdot A_{us}(\mathbf{n}s + x) \right).$$

The hatted gauge field $\hat{A}^\mu_{n}$ transforms under collinear gauge transformations as

$$g\hat{A}^\mu_{n} \rightarrow U_c g\hat{A}^\mu_{n}U^\dagger_c + U_c[(P^\mu + i\partial^\mu_c)U^\dagger_c],$$

which shows that $\hat{A}^\mu_{n}$ transforms like a collinear gauge field under collinear gauge transformations, and the transformation does not involve any usoft gauge fields.
The definition of $\hat{A}_n^\mu$ is similar to the definition in Ref. [11], but in their definition, $n \cdot \hat{A}_n (= n \cdot A_n)$ does not change, and $\hat{A}_n^\mu$ transforms as

$$g\hat{A}_n^\mu \rightarrow U_c g\hat{A}_n^\mu U_c + U_c \left[ P^\mu \pi^\mu \pi + \partial^\mu + in \cdot D_{us} \pi^\mu \pi U_c \right]$$

which still involves usoft gauge fields. In some sense, this definition looks reasonable because only the $n \cdot A_n$ component is of order $\Lambda$, while other components such as $\pi \cdot A_n$ and $A_{n,\perp}^\mu$ are larger than $\Lambda$. Therefore we may decompose the fields of order $\Lambda$ into the quantum field $n \cdot A_n$ and the background field $n \cdot A_{us}$ for collinear gauge transformations. However, small fluctuations of order $\Lambda$ are allowed for the components $n \cdot A_n$ and $A_{us,\perp}^\mu$ though the characteristic scales are of order $Q$, and $\sqrt{Q\Lambda}$ respectively. Therefore we can put $\pi \cdot A_{us}$ and $A_{us,\perp}^\mu$ as background fields for $\pi \cdot A_n$ and $A_{n,\perp}^\mu$ for those small fluctuations. From our definition, the usoft gauge field truly becomes a background field for collinear gauge transformations, and it does not spoil the power counting. Furthermore, it is critical to keep all the components of the usoft gauge field as background fields for collinear gauge transformations in showing the decoupling of the usoft interactions with the collinear particles and in writing the gauge fixing term and the ghost Lagrangian, in which the collinear and the usoft modes are decoupled.

In deriving the effective Lagrangian for the quark sector, we also decompose the quark field $\psi$ in the full QCD into the usoft field $q_{us}$ and the collinear fields $\xi_{n,p}$ and $\xi_{\pi,p}$. We express the collinear fields extracting the label momentum as $\sum_p e^{-ip^x} q_{n,p}(x)$, and $\xi_{n,p} = (\pi\pi/4) q_{n,p}$.

| Fields | Collinear $U_c$ | Usoft $U_{us}$ |
|--------|----------------|----------------|
| $\xi_n$ | $U_c \xi_n$ | $U_{us} \xi_n$ |
| $\xi_\pi$ | $U_c \xi_\pi$ | $U_{us} \xi_\pi$ |
| $gA_n^\mu$ | $U_c g(A_n^\mu + A_{us}^\mu) U_c^\dagger - gA_{us}^\mu + U_c [(P^\mu + i\partial^\mu) U_c^\dagger]$ | $U_{us} A_n^\mu U_{us}^\dagger$ |
| $g\hat{A}_n^\mu$ | $U_c g\hat{A}_n^\mu U_c^\dagger + U_c [(P^\mu + i\partial^\mu) U_c^\dagger]$ | $U_{us} \hat{A}_n^\mu U_{us}^\dagger$ |
| $q_{us}$ | $q_{us}$ | $U_{us} q_{us}$ |
| $gA_{us}^\mu$ | $gA_{us}^\mu$ | $U_{us} gA_{us} U_{us}^\dagger + U_{us} [i\partial_{us} U_{us}^\dagger]$ |
| $\hat{W}$ | $U_c \hat{W}$ | $U_{us} \hat{W} U_{us}^\dagger$ |
| $Y$ | $Y$ | $U_{us} Y$ |

**TABLE I**: Gauge transformation properties of the quantities in SCET$_1$ under collinear and usoft gauge transformations.
\[ \xi_{\pi, p} = (m_4/4)q_{n, p} \]. We also drop the label index \( p \) and the summation from now on. The collinear quark fields \( \xi_n \), \( \xi_\pi \) transform under collinear and usoft gauge transformations as

\[
\begin{align*}
\text{collinear:} \quad \xi_n &\rightarrow U_c \xi_n, \quad \xi_\pi \rightarrow U_c \xi_\pi, \quad q_{us} \rightarrow q_{us} \\
\text{usoft:} \quad \xi_n &\rightarrow U_{us} \xi_n, \quad \xi_\pi \rightarrow U_{us} \xi_\pi, \quad q_{us} \rightarrow U_{us} q_{us}.
\end{align*}
\]

(13)

The gauge transformation properties of the gauge fields and the quark fields in SCET1 are summarized in Table I.

At first sight, we may decompose the quark field as

\[ \psi = \xi_n + \xi_\pi + q_{us}, \quad (14) \]

but this decomposition is too naive. In general, the usoft field can have an additional phase factor. In order to determine the relative phase of \( q_{us} \) with respect to the collinear fields, note that the quark field \( \psi \) and the covariant derivative \( iD^\mu = i\partial^\mu + gA^\mu \) in the full QCD transform as

\[ \psi \rightarrow U \psi, \quad iD^\mu \rightarrow UiD^\mu U^\dagger, \quad iD^\mu \psi \rightarrow UiD^\mu \psi. \]

(15)

Since the collinear and usoft gauge transformations are subsets of the full gauge transformations, \((iD^\mu_c + iD^\mu_{us})\psi \) should transform as in Eq. (15) with \( U \) replaced by \( U_c \) or \( U_{us} \). (Here we can neglect the soft modes since it does not change the result.) We find that the combination

\[ \psi = q_c + \hat{W} q_{us} = \xi_n + \xi_\pi + \hat{W} q_{us} \]

(16)

satisfies this property. With the transformation properties in Table I the sum of the covariant derivatives \( iD^\mu_c + iD^\mu_{us} \) transforms homogeneously under collinear and usoft gauge transformations as

\[
\begin{align*}
(iD^\mu_c + iD^\mu_{us}) &\rightarrow \\
&\begin{cases} 
U_c(iD^\mu_c + iD^\mu_{us})U^\dagger_c, & \text{collinear}, \\
U_{us}(iD^\mu_c + iD^\mu_{us})U^\dagger_{us}, & \text{usoft}.
\end{cases}
\end{align*}
\]

(17)

Then it follows that

\[ (iD^\mu_c + iD^\mu_{us})(q_c + \hat{W} q_{us}) \rightarrow \\
\begin{cases} 
U_c(iD^\mu_c + iD^\mu_{us})(q_c + \hat{W} q_{us}), & \text{collinear}, \\
U_{us}(iD^\mu_c + iD^\mu_{us})(q_c + \hat{W} q_{us}), & \text{usoft},
\end{cases} \]

(18)

as desired.
Now the starting point to derive the effective Lagrangian from the full QCD Lagrangian

\[ \mathcal{L} = \bar{\psi} i \gamma \cdot D \psi \]

reads as

\[
\mathcal{L} = \xi_n \frac{\nabla}{n} \cdot D \xi_n + \xi_n i \gamma_\perp \xi \pi + \xi_n i \hat{W} q_{us} + \bar{\xi}_{us} \hat{W}^\dagger i \gamma \cdot D \xi_n + \bar{\xi}_{us} \hat{W}^\dagger i \gamma \cdot D \xi_{\perp},
\]

and the equation of motion for \( \xi_n \) now becomes

\[
\xi_n = -\frac{1}{\nabla \cdot D^2} (i \gamma_\perp \xi_n + i \hat{W} q_{us}),
\]

which satisfies the collinear gauge transformation property in Table I. If there is no \( \hat{W} \) in front of \( q_{us} \), \( \xi_n \) does not transform as \( \xi_n \to U \xi_n \).

Before we proceed further, it would be useful to compare our approach with the formulations by Bauer et al. \([11]\) and by Beneke et al. \([7,8]\). In Ref. \([11]\), they use the collinear gauge transformation, Eq. \([12]\), in which only the \( n \cdot A_{us} \) acts as a classical background field. They redefine \( \hat{A}_n^\mu \) as shown in Eq. \([8]\) except that \( n \cdot \hat{A}_n = n \cdot A_n \). It makes the collinear Lagrangian gauge invariant order by order. But they do not introduce the relative phase \( \hat{W} \) in front of \( q_{us} \) when the quark field in the full theory is decomposed in SCET_1, which does not make the usoft Lagrangian gauge invariant at higher orders. Beneke et al. \([7,8]\) start from the same collinear gauge transformation properties as ours, in which all the the usoft field is a background field. And they also introduce the relative phase \( \hat{W} \) in front of \( q_{us} \). Then the collinear quark and gluon fields are redefined to satisfy the homogenized gauge transformation given by Eq. \([12]\). The redefined collinear fields involve Wilson lines made of usoft gluons after choosing a light-cone gauge. The \( \pi \cdot A_n \) component of the collinear gauge field remains unchanged, while the \( n \cdot A_n \) component does not change in Ref. \([11]\).

Here we treat the usoft field as a background field for collinear gauge transformations, and put an additional phase \( \hat{W} \) as in Ref. \([7,8]\). Then we redefine the collinear gauge field as in Eq. \([8]\) as a first step to make the Lagrangian invariant order by order. And we express the Lagrangian in terms of the hatted collinear gauge field, \( \hat{W} \), and the unchanged collinear and usoft quark fields.

The Lagrangian can be decomposed into the collinear, the usoft-collinear, and the usoft Lagrangian \( \mathcal{L} = \mathcal{L}_c + \mathcal{L}_\xi q + \mathcal{L}_{us} \), depending on the quark fields, and can be written as

\[
\mathcal{L}_c = \frac{\nabla}{2} \left( in \cdot D + i \gamma_\perp \frac{1}{in \cdot D} i \gamma_\perp \right) \xi,
\]
\[ \mathcal{L}_{\xi q} = \xi \bar{q} \gamma^\mu W_{\mu} q + \frac{\xi \bar{q} i \gamma^\mu}{\mathcal{D}} i \partial^\mu \bar{q} + \frac{1}{\mathcal{N}} \frac{1}{\frac{1}{\mathcal{D}} \hat{W} q_{us} + \text{h.c.}}, \]
\[ \mathcal{L}_{us} = \bar{q}_{us} \hat{W} \gamma^\mu \bar{q}, \]
\[ \hat{W} = \frac{1}{\mathcal{D}} \hat{W} q_{us} - \frac{1}{\mathcal{D}} \hat{W} q_{us}, \quad (21) \]
where we also drop the index \( n \) for the collinear quark field to simplify the notation. In terms of the hatted field, the covariant derivative is written as
\[ i \hat{D}^\mu = i D_\mu + i \hat{D}_\mu = i \hat{D}_\mu + \hat{W} i D_\mu \hat{W} \]
\[ \hat{W} = \frac{1}{\mathcal{D}} \hat{W} q_{us} - \frac{1}{\mathcal{D}} \hat{W} q_{us}, \quad (22) \]
where \( i \hat{D}_\mu = \mathcal{P}^\mu + i \partial^\mu \mathcal{A}_\mu + g \mathcal{A}_\mu \)

\[ \text{Before we go down to SCET}_\text{II}, \text{we decouple the usoft interactions with the collinear particles by factorizing the usoft interactions. This is achieved by the field redefinitions} \]
\[ \xi = Y \xi^{(0)}, \quad \mathcal{A}_n = Y \mathcal{A}_n^{(0)} Y^\dagger, \quad (23) \]
where the Wilson line \( Y \) is given by
\[ Y(x) = P \exp \left(i g \int_{-\infty}^{0} ds n \cdot A_{us}(ns + x) \right). \quad (24) \]
This is the second and final redefinition of the collinear fields, and these fields will be matched to the corresponding fields in SCET\_II. In SCET\_I, the usoft interactions with collinear particles do not put the intermediate states off the mass shell, hence this factorization does not correspond to integrating out off-shell modes. However in SCET\_II, the corresponding soft interactions put the intermediate collinear particles off the mass shell by \( p^2 \sim Q \Lambda \), therefore these should be integrated out in SCET\_II. This is easily achieved by replacing \( Y \) by \( S \), where the Wilson line \( S \) is obtained by replacing \( A_{us}^\mu \) of \( Y \) in SCET\_I by \( A_s^\mu \) in SCET\_II.

In Ref. 3, the usoft factorization for the collinear fields is illustrated nicely by attaching usoft gluons to collinear quarks and gluons. Here the attached usoft gluons are regarded as background fields and when we sum over the radiation of usoft gluons to all orders, it gives a path-ordered exponential \( Y \), which depends only on \( n \cdot A_{us} \). This eikonal summation is possible only when all the components of the usoft gluons are treated as background fields. Therefore our treatment of the usoft field as a background field for collinear gauge transformations is also justified in factorizing the usoft interactions.

The fields \( \xi^{(0)} \) and \( \mathcal{A}_n^{(0)} \) are the fundamental objects and they transform in a peculiar way under gauge transformations. Under collinear gauge transformations, \( \xi^{(0)} \) and \( g \mathcal{A}_n^{(0)} \)
\[ \xi^{(0)} \rightarrow Y^\dagger U_c Y \xi^{(0)}, \]
\[ g\hat{A}_n^{\mu,(0)} \rightarrow Y^\dagger U_c Y \hat{A}_n^{\mu,(0)} Y^\dagger U_c Y, \]
\[
= (Y^\dagger U_c Y)g\hat{A}_n^{\mu,(0)}(Y^\dagger U_c Y)^\dagger + (Y^\dagger U_c Y)(\mathcal{P}^{\mu} + i\partial^{\mu}_c)(Y^\dagger U_c Y)^\dagger, \quad (25)
\]
where the last equality is obtained, using the fact that \( Y \) commutes with the collinear momentum operator \( \mathcal{P}^{\mu} \) and \( i\partial^{\mu}_c \). Under usoft gauge transformations, they transform as
\[ \xi^{(0)} \rightarrow Y^\dagger U^\dagger us U us \xi^{(0)}, \]
\[ g\hat{A}_n^{\mu,(0)} \rightarrow g\hat{A}_n^{\mu,(0)}. \quad (26) \]
Therefore \( \xi^{(0)} \) and \( \hat{A}_n^{\mu,(0)} \) transform under the modified collinear gauge transformations \( Y^\dagger U_c Y \) as \( \xi \) and \( \hat{A}_n^{\mu} \) transform under the collinear gauge transformations \( U_c \), while they do not transform at all under usoft gauge transformations. On the other hand, the usoft fields \( q_{us} \) and \( A_{us}^{\mu} \) do not transform under the modified collinear transformations. Therefore the collinear fields transform only under collinear transformation, while the usoft fields transform only under usoft gauge transformations. This makes the construction of the gauge fixing terms simple.

The Lagrangian for the gauge fields is given by
\[
\mathcal{L}_g = \frac{1}{2g^2} \text{Tr} \left( [iD^{\mu}, iD^{\nu}][iD^{\mu}, iD^{\nu}] \right)
= \frac{1}{2g^2} \text{Tr} \left( \left[ i\hat{D}^{\mu}_c + WY^\dagger iD^{\mu}_{us} Y, i\hat{D}^{\nu}_c + WY^\dagger iD^{\nu}_{us} Y \right]^2 \right)
= \frac{1}{2g^2} \text{Tr} \left( \left[ i\hat{D}^{\mu}_c, i\hat{D}^{\nu}_c \right]^2 \right) + \frac{1}{2g^2} \text{Tr} \left( \left[ iD^{\mu}_{us}, iD^{\nu}_{us} \right]^2 \right) + \cdots, \quad (27)
\]
where the dots represent the interaction between collinear and usoft gluons and we can systematically expand the Lagrangian in powers of \( \sqrt{\Lambda} \). Since the collinear fields and the usoft fields transform separately under collinear and usoft gauge transformations, it is simple to fix the gauge and to construct the ghost Lagrangian. We can choose the gauge fixing term as
\[
\mathcal{L}_{\text{fix}} = -\frac{1}{2\alpha_i} \left( F^a_i \right)^2, \quad (i = c, us), \quad (28)
\]
where
\[
F^a_c = -i(\mathcal{P}^{\mu} + i\partial^{\mu}_c)\hat{A}_{\mu,0}^{(0),a}, \quad F^a_{us} = \partial^{\mu}_{us} A^{a}_{us,\mu}. \quad (29)
\]
And the ghost Lagrangian becomes
\[
\mathcal{L}_{\text{ghost}} = \eta^a \left( (\mathcal{P}^\mu + i\partial^\mu_c)^2 \delta^{ac} + ig f^{abc}(\mathcal{P}^\mu + i\partial^\mu_c) \hat{\Lambda}_{mu}^{(0)a} \right) \eta^c \\
+ \overline{\epsilon^a} (-\partial^2_{us} \delta^{ac} - g f^{abc} \partial^\mu_{us} \hat{A}^b_{mu}) c^c,
\] (30)
where \( \eta^a \) (\( \epsilon^a \)) is the collinear (usoft) ghost field.

According to the transformation Eq. (23), we have \( \hat{W} = Y \hat{W}^{(0)} Y^\dagger \), and from now on, we will drop the superscript “(0)”. The effective Lagrangian can be expanded Lagrangian in powers of \( \sqrt{\Lambda/Q} \). The first three terms of the collinear Lagrangian are given by
\[
\mathcal{L}_c^{(0)} = \xi \left( \frac{1}{2}(in \cdot \hat{D}_c + \hat{\phi}_{c\perp}) \right) \xi, \\
\mathcal{L}_c^{(1)} = \xi \left( \frac{1}{2} \hat{\phi}_{c\perp} \hat{W} Y^{\dagger} \hat{\phi}_{us} Y \hat{W}^\dagger + \hat{W} Y^{\dagger} \hat{\phi}_{us} Y \hat{W}^\dagger \frac{1}{im \cdot \hat{D}_c} \hat{\phi}_{c\perp} \right) \xi, \\
\mathcal{L}_c^{(2)} = \xi \left( \hat{W} Y^{\dagger} \hat{\phi}_{us} Y \hat{W}^\dagger \frac{1}{im \cdot \hat{D}_c} \hat{W} Y^{\dagger} \hat{\phi}_{us} Y \hat{W}^\dagger \hat{\phi}_{c\perp} \right) \xi.
\] (31)
In deriving \( \mathcal{L}_c^{(0)} \), we use the following relations
\[
\frac{\xi}{2} \frac{1}{2} \left( \frac{1}{2}(in \cdot \hat{D}_c + \hat{\phi}_{c\perp}) \right) \xi = \frac{\xi}{2} \left( \frac{1}{2}(in \cdot \hat{D}_c + \hat{\phi}_{c\perp}) \right) \xi \\
= \frac{\xi}{2} \left( \frac{1}{2}(in \cdot \hat{D}_c + \hat{\phi}_{c\perp}) \right) \xi = \frac{\xi}{2} \left( \frac{1}{2}(in \cdot \hat{D}_c + \hat{\phi}_{c\perp}) \right) \xi,
\] (32)
where we use the fact that \( in \cdot \hat{D}_c Y = Y in \cdot \partial_{us} \) and the usoft derivative operator applied to the collinear fields in \( \hat{W} \) and \( \xi \) yields zero.

The usoft-collinear Lagrangian is given by
\[
\mathcal{L}_{\xi q}^{(1)} = \xi \hat{\phi}_{c\perp} Y \hat{\phi}_{us} q_{us} + \text{h.c.}, \\
\mathcal{L}_{\xi q}^{(2)} = \xi \hat{\phi}_{c\perp} Y \hat{\phi}_{us} q_{us} + \text{h.c.}.
\] (33)
and the usoft quark Lagrangian is written as
\[
\mathcal{L}_{us}^{(0)} = \bar{q}_{us} \hat{\phi}_{us} q_{us}, \\
\mathcal{L}_{us}^{(3)} = \bar{q}_{us} Y \hat{\phi}_{us} Y \hat{\phi}_{us} q_{us}, \\
\mathcal{L}_{us}^{(4)} = \bar{q}_{us} \frac{1}{2} \hat{W} Y^{\dagger} \left( in \cdot \hat{D}_c + \hat{\phi}_{c\perp} \frac{1}{im \cdot \hat{D}_c} \right) \hat{W}^{\dagger} q_{us}.
\] (34)
In the present form, the effective Lagrangian at each order is manifestly gauge invariant under collinear and usoft gauge transformations. The importance of the relative phase $\hat{W}$ of $q_{us}$ in decomposing the quark fields is clearly seen in $L_{us}^{(3),(4)}$. If there were no relative phase $\hat{W}$, these Lagrangians would not be gauge invariant, and the gauge invariance can be sustained only after we include $\hat{W}$ in front of $q_{us}$.

**B. Effective Lagrangian in SCET$_{II}$**

In SCET$_{II}$, we rename the usoft fields as the soft fields and we will replace $Y$ by $S$. We can write down the effective Lagrangian in SCET$_{II}$ in powers of $\Lambda$. However, there is change in organizing the Lagrangian because the power counting of the fields changes. In SCET$_{II}$, the collinear field $\xi$ scales as $\sim \Lambda$, the soft quark field $q_s$ (also the heavy quark field $h$ if any) scales as $\sim \Lambda^{3/2}$. For the collinear gauge field, it scales as $\hat{A}^u = (Q, \Lambda, \Lambda^2/Q)$, and for the soft gauge field $A_s^u \sim (\Lambda, \Lambda, \Lambda)$. In the action, the power counting for the integration measure $d^4x$ also changes. For the Lagrangian which involves collinear fields only or soft fields only, the integration measure becomes $d^4x \sim \Lambda^{-4}$, but if collinear fields and soft fields are mixed, the integration measure scales as $d^4x \sim \Lambda^{-3} [20, 21]$. With this in mind, the effective Lagrangian in SCET$_{II}$, which is gauge invariant under collinear and soft gauge transformations, can be organized in powers of $\Lambda$.

The collinear quark Lagrangian is given as

$$L_{c,II}^{(0)} = \xi (in \cdot \hat{D}_c + i\hat{\Psi}^\dagger_{c\perp} \hat{W} 1\!\!1 \hat{W}^\dagger i\hat{\Psi}_{c\perp}) \frac{\mu}{2} \xi,$$

$$L_{c,II}^{(1)} = (\xi \hat{W}) (S^\dagger i\hat{\Psi}^\dagger_s S) \frac{1}{P} \hat{W}^\dagger i\hat{\Psi}_{c\perp} \frac{\mu}{2} \xi +(\xi \hat{W}) (S^\dagger i\hat{\Psi}^\dagger_s S) \frac{1}{P} \hat{W} (S^\dagger i\hat{\Psi}^\dagger_s S) \frac{\mu}{2} (\hat{W} \xi) + (\xi \hat{W}) S^\dagger i\hat{\Psi}^\dagger_s 1\!\!1 i\hat{\Psi}_{c\perp} S \frac{\mu}{2} (\hat{W} \xi). \quad \text{(35)}$$

Note that the third term in $L_{c,II}^{(1)}$ previously belonged to $L_{c}^{(2)}$ in SCET$_{I}$, but now it is included in $L_{c,II}^{(1)}$ due to the different power counting. The soft-collinear quark Lagrangian is given by

$$L_{sc,II}^{(1/2)} = (\xi \hat{W}) (\hat{W}^\dagger i\hat{\Psi}^\dagger_{c\perp} \hat{W}) S^\dagger q_{s} + \text{h.c.},$$

$$L_{sc,II}^{(3/2)} = (\xi \hat{W}) \frac{\mu}{2} (\hat{W}^\dagger i\hat{D}_c \hat{W} + \hat{W}^\dagger i\hat{\Psi}_{c\perp} \hat{W} 1\!\!1 \hat{W}^\dagger i\hat{\Psi}_{c\perp} \hat{W}^\dagger $$

$$+ \hat{W}^\dagger i\hat{\Psi}^\dagger_{c\perp} \hat{W} 1\!\!1 S^\dagger i\hat{\Psi}^\dagger_s S + S^\dagger i\hat{\Psi}^\dagger_s S \frac{1}{P} \hat{W}^\dagger i\hat{\Psi}^\dagger_c \hat{W} \xi) S^\dagger q_{s}. \quad \text{(36)}$$

Note that the superscripts have changed reflecting that $L_{sc,II}^{(1/2)}$ is suppressed by $\Lambda^{1/2}$ compared to $L_{c}^{(0)}$, and the orders are rearranged with the power counting $iD_{c}^\perp \sim iD_{s}^\perp \sim \Lambda$. The soft
The higher-order terms in the Lagrangian in SCET$_1$ are redistributed according to the power counting rules in SCET$_{II}$. And there can be many-quark operators by integrating out the off-shell fields, which we do not present here. Formally in order to obtain the effective Lagrangian in SCET$_{II}$, we introduce hard-collinear auxiliary fields, which scale as $(Q, \sqrt{Q\Lambda}, \Lambda)$, and they are integrated out. The effective action in SCET$_{II}$ is given by

$$e^{i\Gamma_{\text{eff}}^{\text{II}}} = \int D\xi X D\xi X DA_{n}X e^{iS_{I}(\xi+\xi+\xi+\xi, A_{n}+A_{n}^{\mu}, \text{softfields})},$$

where $S_{I}$ is the action in SCET$_{I}$ and all the fields with the subscript $X$ are the auxiliary fields of order $\sim (Q, \sqrt{Q\Lambda}, \Lambda)$, which are to be integrated out.

The effective Lagrangians in Eqs. (35)–(37) are the effective Lagrangian in SCET$_{II}$. However, care must be taken in deriving Feynman rules if we expand each Lagrangian in powers of $g$. Especially when the Wilson lines $\hat{W}$ and $S$ are expanded, we have to take the momentum conservation into account. Note that in the soft-collinear interactions, there should be more than two collinear particles and two soft particles to conserve momentum. For example, consider $L_{sc, II}^{(1/2)}$. At leading order in $g$, it contains a collinear quark $\xi$, a collinear gluon $A_{n}^{\mu}$ and a soft quark $q_{us}$. But this is not allowed since the momentum is not conserved. Though we add any number of collinear momenta in the $n^{\mu}$ direction, we cannot make a soft momentum. Therefore the nonzero leading term in $L_{sc, II}^{(1/2)}$ starts from order $g^2$, in which a soft gluon is added by expanding $S$, conserving momentum. In $L_{s, II}^{(1)}$, it starts with a single collinear gluon, which is not allowed either, and there should be at least two collinear gluons.

The selecting procedure described above is somewhat cumbersome, and it would be desirable to express the Lagrangian in which the gauge invariance and the momentum conservation are shown manifestly. One way to accomplish this is to choose a specific gauge, in which the Lagrangian takes a simpler form. For example, if we choose the gauge $\hat{W} = S = 1$, all the multi-fields from the Wilson lines vanish and we can easily impose the momentum conservation. In $L_{c, II}^{(1)}$, only the last term survives. No terms survive in $L_{sc, II}^{(1/2)}$, and only the last two terms survive in $L_{sc, II}^{(3/2)}$. In the soft Lagrangian, $L_{s, II}^{(1)}$ vanishes. The construction of the gauge-invariant and momentum-conserving Lagrangian in SCET$_{II}$ without choosing a specific gauge, and the comparison with the Lagrangian with a specific gauge will be discussed elsewhere.
III. SOFT-COLLINEAR CURRENT

As a specific example, we consider the radiative corrections for the soft-collinear current of the form $\xi \Gamma q$ from the current $\bar{\psi} \Gamma \psi$ in the full theory, where $\Gamma$ is a Dirac matrix. The usoft-collinear current operators in SCET$_I$ can be obtained by replacing the quark field in the full theory by Eq. [106], and it is given by

$$\bar{q}\Gamma q \rightarrow (\xi_n + \xi_{\perp} + \bar{q}_{us} \hat{W}^\dagger)\Gamma (\xi_n + \xi_{\perp} + \hat{W}_{us}).$$

(39)

Using the equation of motion for $\xi_{\perp}$, and expanding the currents in powers of $\sqrt{\Lambda}$, we can obtain the current operators in SCET$_I$. We list the first few current operators at tree level after we factorize the usoft interactions by redefining $\xi \rightarrow Y \xi$, $\hat{A}_n \rightarrow Y \hat{A}_n^\mu Y^\dagger$. The collinear-collinear current operators are given by

$$J_c^{(0)} = \bar{\xi} \Gamma \xi,$$
$$J_c^{(1)} = \bar{\xi} \left( \frac{1}{2} i \bar{\psi}_c \Gamma + \Gamma \frac{1}{2} \frac{1}{i \sigma \cdot D_c} \psi_c \right) \xi,$$
$$J_c^{(2)} = \bar{\xi} \left( i \bar{\psi}_c \Gamma + \Gamma \frac{1}{2} \frac{1}{i \sigma \cdot D_c} \psi_c \right) \xi + \frac{1}{2} \bar{W} Y^\dagger i \bar{\psi}_{us} Y \hat{W}^\dagger \left( \frac{1}{i \sigma \cdot D_c} \Gamma + \Gamma \frac{1}{2} \frac{1}{i \sigma \cdot D_c} \hat{W} Y^\dagger i \bar{\psi}_{us} Y \hat{W}^\dagger \frac{1}{2} \right) \xi,$$

(40)

and the usoft-collinear current operators are given by

$$J_{\xi q}^{(2)} = \bar{\xi} \hat{W} Y^\dagger q_{us} + \text{h.c.},$$
$$J_{\xi q}^{(3)} = -\bar{\xi} \left( i \bar{\psi}_c \Gamma + \Gamma \frac{1}{2} \frac{1}{i \sigma \cdot D_c} \psi_c \right) \hat{W} Y^\dagger q_{us} + \text{h.c.},$$

(41)

and the usoft current operators are given by

$$J_{us}^{(4)} = \bar{q}_{us} \Gamma q_{us},$$
$$J_{us}^{(5)} = -\bar{q}_{us} \hat{W}^\dagger \left( \Gamma \frac{1}{i \sigma \cdot D_c} \frac{1}{2} \frac{1}{i \sigma \cdot D_c} \psi_c + i \psi_c \Gamma \frac{1}{2} \frac{1}{i \sigma \cdot D_c} \Gamma \right) \hat{W} q_{us}.$$  

(42)

Note that the power counting of these operators is performed in SCET$_I$, that is, in powers of $\sqrt{\Lambda/Q}$, and the superscripts in the current operators show the power dependence with respect to $J_c^{(0)}$. When we go down to SCET$_II$, $Y$ should be replaced by $S$, and the power counting in SCET$_II$ should be applied. In SCET$_I$, $iD_c^\mu \sim (Q, \sqrt{Q}, \Lambda)$, $iD_{us} \sim (\Lambda, \Lambda, \Lambda)$, but in SCET$_II$, $iD_c^\mu \sim (Q, \Lambda, \Lambda^2/Q)$, $iD_{us} \sim (\Lambda, \Lambda, \Lambda)$. Therefore $iD_c^\perp \sim iD_{us}^\perp$ in SCET$_II$,
while they have different power behavior in SCET$_I$. As an example, the first term in $J_c^{(2)}$ is suppressed compared to the other terms in SCET$_{II}$.

We compute the vertex corrections at one loop. We consider a special case with $\Gamma = \gamma^\mu$ at the end of the computation, but we keep $\Gamma$ when the results hold for an arbitrary $\Gamma$. In our computation, two regularization schemes are employed. The first one is the on-shell regularization in which the external particles are on their mass shell, and we use the dimensional regularization both for the infrared and the ultraviolet divergences. However, there can be cancellation between the infrared and the ultraviolet divergences in this scheme. This happens especially in SCET$_{II}$ and all the radiative corrections are zero due to this cancellation. In order to separate the infrared and the ultraviolet divergences, we use the off-shell regularization scheme in which the infrared divergence is regulated by the off-shellness of the external particles and we use the dimensional regularization for the ultraviolet divergence. The computation in this scheme is more complicated, but we can explicitly check if the infrared divergences are reproduced in the effective theories. After we show that the infrared divergences of the full theory are reproduced in SCET$_I$ and SCET$_{II}$, we can match the full theory and SCET since the infrared divergence is canceled in matching. We match the full theory and SCET$_I$ at $\mu \sim Q$, and match SCET$_I$ and SCET$_{II}$ at $\mu \sim \sqrt{Q\Lambda}$, using the on-shell regularization scheme to compute the Wilson coefficients and the anomalous dimensions of the $(u)$soft-collinear current operator.

### A. Full QCD calculation

In full QCD, we consider the vertex correction to the soft-collinear current, which is shown in Fig. 1. The Feynman graph yields

$$M_F = -ig^2C_F \int \frac{d^Dl}{(2\pi)^D} \frac{\gamma(\not{l} + \not{p})\gamma_\mu(\not{l} + \not{k})\gamma^\alpha}{l^2(l + p)^2(l + k)^2},$$

where $D = 4 - 2\epsilon$, $C_F = (N^2 - 1)/(2N)$ is the $SU(3)$ color factor with $N = 3$.

If we put the external particles on the mass shell ($k^2 = p^2 = 0$), and use the dimensional regularization both for the infrared and the ultraviolet divergences, the one-loop correction in this scheme is given by

$$M_F^D = \frac{\alpha_sC_F}{4\pi} \gamma_\mu I_F(q^2),$$

16
where \( q^2 = (p - k)^2 = -2p \cdot k \), and \( I_F(q^2) \) is given as

\[
I_F(q^2) = \left( \frac{-q^2}{\mu^2} \right)^{-\epsilon} \left( \frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon^2} - \frac{4}{\epsilon} - 8 + \frac{\pi^2}{6} \right)
= \frac{1}{\epsilon_{UV}} - \frac{2}{\epsilon^2} + \left(-4 + 2 \ln \frac{-q^2}{\mu^2}\right) \frac{1}{\epsilon} + 3 \ln \frac{-q^2}{\mu^2} - \ln^2 \frac{-q^2}{\mu^2} - 8 + \frac{\pi^2}{6}.
\] (45)

The pole \( 1/\epsilon_{UV} \) comes from the ultraviolet divergence, which will be canceled by the wave function renormalization due to the current conservation, and other \( 1/\epsilon \) poles are of the infrared origin.

If we put the external particles off the mass shell \( p^2, k^2 \neq 0 \), the calculation is more complicated, but the graph in Fig. 1 gives

\[
M_C^{\mu} = \frac{\alpha_s C_F}{4\pi} \gamma_\mu \left( \frac{1}{\epsilon_{UV}} - \ln \frac{-q^2}{\mu^2} - 2 \ln \frac{p^2}{q^2} \ln \frac{k^2}{q^2} - 2 \ln \frac{p^2}{q^2} - 2 \ln \frac{k^2}{q^2} - 2 \ln \frac{\mu^2}{q^2} - \frac{2\pi^2}{3} \right),
\] (46)

where the pole \( 1/\epsilon_{UV} \) is the ultraviolet divergence, and the infrared divergences are regulated by the off-shellness. Here \( q^2 = (p - k)^2 = -2p \cdot k + p^2 + k^2 \), but \( p^2 \) and \( k^2 \) can be neglected compared to \( p \cdot k \).

**B. Calculation in SCET**

In SCET1, the calculation of the vertex correction is more complicated, but it can be performed systematically. In constructing the effective Lagrangian in SCET1, we have not distinguished the collinear modes with \( p^\mu_c \sim (Q, \Lambda, \Lambda^2/Q) \) and the hard-collinear modes with \( p^\mu_{hc} \sim (Q, \sqrt{Q\Lambda}, \Lambda) \). However, it is conceptually convenient to distinguish these two contributions because the hard-collinear contributions will be integrated out in order to obtain SCETII, but the collinear contributions still remain in SCETII. This point becomes transparent when we use the off-shellness of the external particles as an infrared cutoff in the computation. If we have a loop momentum \( l^\mu \) and an external collinear momentum \( p^\mu \sim (Q, \Lambda, \Lambda^2/Q) \), there is a propagator whose denominator is given by \((l + p)^2\). When the
loop momentum is hard-collinear with \( l^\mu \sim (Q, \sqrt{Q\Lambda}, \Lambda) \), it is given by

\[
(l + p)^2 = l^2 + \pi \cdot p n \cdot l + (p^2 + n \cdot \pi n \cdot l + 2l_\perp \cdot p_\perp) = \mathcal{O}(Q\Lambda) + \mathcal{O}(Q\Lambda) + \mathcal{O}(Q\Lambda \sqrt{\frac{\Lambda}{Q}}). \tag{47}
\]

On the other hand, if the loop momentum is collinear with \( l^\mu \sim (Q, \Lambda, \Lambda^2/Q) \), it becomes

\[
(l + p)^2 = l^2 + \pi \cdot p n \cdot l + (p^2 + n \cdot \pi n \cdot l + 2l_\perp \cdot p_\perp) = \mathcal{O}(\Lambda^2) + \mathcal{O}(\Lambda^2) + \mathcal{O}(\Lambda^2). \tag{48}
\]

Therefore when the loop momentum \( l^\mu \) is hard-collinear, we do not have to introduce the infrared cutoff since the denominator can never reach the infrared region near \( \Lambda^2 \). But when \( l^\mu \) is collinear, the integral can be infrared divergent from this propagator, and we need an infrared cutoff to regulate the infrared divergence.

In SCET\(_1\), we consider the contributions from the collinear modes with \( p^\mu_c \sim (Q, \Lambda, \Lambda^2/Q) \), the usoft modes with \( p^\mu_{us} \sim (\Lambda, \Lambda, \Lambda) \), and the hard-collinear modes with \( p^\mu_{hc} \sim (Q, \sqrt{Q\Lambda}, \Lambda) \). The Feynman diagrams for the vertex correction are shown in Fig. 2. Fig. 2 (a) shows the collinear contribution, in which the loop momentum is collinear, and Fig. 2 (b) is the usoft contribution. The remaining four diagrams Fig. 2 (c) – (f) are the hard-collinear

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Vertex correction to the soft-collinear current in SCET\(_1\). The square box denotes the inserted current operators. The curly line with a solid line in (a) is a collinear gluon, and the curly line in (b) is a soft gluon. The curly line with a solid line in (c)–(f) is a hard-collinear gluon. The solid line is a collinear quark and the dashed line is an usoft quark.}
\end{figure}
contributions. The insertions of the current operators and the Lagrangian are explicitly shown in the figure. We express the results using the off-shellness \((p^2, k^2 \neq 0)\) in order to see if the infrared divergence in the full theory in Eq. (46) is reproduced in SCET\(_1\).

Before we compute the radiative corrections in Fig. 2, let us consider the propagators which appear in the calculation. The external collinear particle has the collinear momentum \(p^\mu\) which scales as \(p^\mu \sim (Q, \Lambda, \Lambda^2/Q)\) with \(p^2 \sim \Lambda^2\). The usoft momentum \(k^\mu\) scales as \(k^\mu \sim (\Lambda, \Lambda, \Lambda)\) with \(k^2 \sim \Lambda^2\). The propagator of the collinear quark is given from \(L_c(\mathcal{O})\) as

\[
\frac{i}{2} \frac{1}{n \cdot p + p_\perp^2 / \pi \cdot p} = \frac{i}{2} \frac{\pi \cdot p}{p^2}.
\]

(49)

And the internal quark propagators in Fig. 2 have \((l + p)^2\) or \((l + k)^2\) in the denominator. In the collinear contribution as in Fig. 2(a), \(l^\mu \sim (Q, \Lambda, \Lambda^2/Q)\), and all the terms in \((l + p)^2\) are of the same order and it should be written as \((l + p)^2\) according to Eq. (48). Here the off-shellness \(p^2\) acts as an infrared cutoff. In the usoft contribution of Fig. 2(b), \(l^\mu \sim (\Lambda, \Lambda, \Lambda)\), and all the terms in \((l + k)^2\) are of the same order and it should be written as \((l + k)^2\) and \(k^2\) acts as an infrared cutoff. In the hard-collinear contributions as in Fig. 2(c) – (f), the loop momentum \(l^\mu\) is hard-collinear, scaling as \(l^\mu \sim (Q, \sqrt{Q}\Lambda, \Lambda)\) with \(l^2 \sim Q\Lambda\). In this case, \((l + p)^2\) can be written as

\[
(l + p)^2 = l^2 + n \cdot l \pi \cdot p + \mathcal{O}(Q\Lambda \sqrt{\frac{\Lambda}{Q}}),
\]

(50)

where the first two terms are at leading order \(\mathcal{O}(Q\Lambda)\), and there is no need for the infrared cutoff \(p^2\) since the leading term can never reach the region \(\sim \Lambda^2\). Similarly, \((l + k)^2\) can be written as

\[
(l + k)^2 \approx l^2 + \pi \cdot l n \cdot k + \mathcal{O}(Q\Lambda \sqrt{\frac{\Lambda}{Q}}),
\]

(51)

where \(l^2 \approx \pi \cdot l n \cdot k \sim Q\Lambda\). Therefore the infrared cutoff \(k^2\) does not have to be kept in the denominator either for the hard-collinear contributions.

Note that the off-shellness \(p^2\) is of order \(\Lambda^2\) for external collinear particles. However, since the off-shellness \(p^2\) acts only as an infrared cutoff, we can relax the off-shellness as \(p^2 < \Lambda^2\) as long as \(\pi \cdot p \sim Q\) is fixed. As long as \(\pi \cdot p \sim Q\), we can use the power counting as in Eq. (50). Since \(p^2\) is an infrared cutoff, we can make it as small as we want and we can show that the infrared divergences are fully reproduced using any small \(p^2\), and there is no need to include any other degrees of freedom corresponding to smaller \(p^2\).
FIG. 3: Feynman rules for the subleading Lagrangian necessary for the one-loop corrections to the soft collinear current. The momentum $q^\mu$ is incoming.

The Feynman rules for the subleading Lagrangian, which appears in Fig. 2, are listed in Fig. 3. The Feynman rules for the currents can be derived from Eqs. (40), (41) and (42), and we list those necessary for the one-loop corrections of the soft-collinear current in Fig. 4. Note that the Feynman rule in Fig. 4 (a) for the usoft-collinear current with an usoft gluon is derived from the current $\tilde{\xi} \tilde{W} Y^\dagger q_{us}$, in which the usoft Wilson line $Y$ is obtained by putting the external particles on the mass shell. Therefore we have to modify the Feynman rules for the usoft-collinear current when we put the particles off the mass shell to extract the infrared divergence separately. The vertex for the usoft-collinear current with a single usoft gluon in SCET$_I$ with the off-shell external particles can be computed from the Feynman

FIG. 4: Feynman rules for the current operators necessary for the one-loop corrections to the soft collinear current. The momentum $q^\mu$ is incoming.
FIG. 5: Single-gluon vertex for the usoft-collinear current operator in SCET\textsubscript{I} when the external particle is off the mass shell. The usoft momentum $q$ is incoming.

diagram in Fig. 5 and the amplitude is given as

$$\bar{\xi}(ign \cdot A_{us} \frac{\pi}{2}) \frac{q \cdot (p - q)}{2(p - q)^2} \Gamma q_{us} \rightarrow -g\bar{\xi} \frac{n \cdot A_{us} \pi \cdot p}{-\pi \cdot pn \cdot q + p^2} \Gamma q_{us}, \quad (52)$$

where $q^\mu$ is the incoming usoft momentum, and the leading terms are kept after the arrow.

In the denominator of Eq. (52), the maximum fluctuation of $\pi \cdot pn \cdot q$ is of order $Q\Lambda$, but it can approach $\Lambda^2$ or smaller since $q$ is usoft. Therefore we need an infrared cutoff $p^2$ and both terms should be kept. We can derive the Feynman rule for the vertex of the usoft-collinear current with an usoft gluon from Eq. (52). When we put $p^2 = 0$, we obtain the Feynman rule in Fig. 3 (a). At higher orders in $\alpha_s$, the modified Feynman rules may be complicated, but it can be derived at each order. Bauer et al. have considered another form of the regulator in SCET\textsubscript{II} to all orders in $\alpha_s$. However, the Feynman rules for the off-shell particles are useful in distinguishing and separating the infrared divergence from a given Feynman diagram. Once we know that the infrared divergence of the full theory is reproduced in SCET, we go back to the dimensional regularization with on-shell particles. So we do not dwell further on the possible construction of the modified Feynman rules for the off-shell particles.

We do not have to modify the Feynman rule in Fig. 4 (b) for the collinear gluon from expanding $\hat{W}$ when the external particles are off the mass shell. In this case, we consider the interaction of an usoft quark and a collinear quark emitting a collinear gluon. In this case the momentum squared of the intermediate collinear quark becomes $(q + k)^2 = \pi \cdot qn \cdot k + \mathcal{O}(\Lambda^2)$, where $q^\mu$ is collinear. Therefore we can safely disregard $k^2$ in the denominator, and the vertex is unaffected.
The collinear contribution comes from Fig. 2(a), and the amplitude is given by

\[ M_{CO} = -ig^2C_F \int \frac{d^Dl}{(2\pi)^D} \frac{2\pi \cdot (l + p)\Gamma}{l^2(l + p)^2\bar{m} \cdot l} = \frac{\alpha_s C_F}{4\pi} \Gamma I_a(p^2), \] (53)

where only the leading terms are collected by considering that the loop momentum is collinear. And \( I_a(p^2) \) is given by

\[ I_a(p^2) = \left( \frac{-p^2}{\mu^2} \right)^{-\epsilon} \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon} + 4 - \frac{\pi^2}{6} \right) \]

\[ = \frac{2}{\epsilon^2} + \frac{2}{\epsilon} \left( 1 - \ln \frac{-p^2}{\mu^2} \right) - 2 \ln \frac{-p^2}{\mu^2} + \ln^2 \frac{-p^2}{\mu^2} + 4 - \frac{\pi^2}{6}. \] (54)

Here we neglected the terms which approach zero as \( p^2 \to 0 \).

As a concrete example, let us show how the calculation of \( M_{CO} \) can be performed using the modified minimal subtraction (\( \overline{\text{MS}} \)) scheme with the spacetime dimension \( D = 4 - 2\epsilon \). We first combine the denominator using the Feynman parametrization technique as

\[ \frac{1}{l^2(l + p)^2\bar{m} \cdot l} = 4 \int_0^1 dx \int_0^\infty du \frac{1}{(x + p - x(p - (1 - x)p^2))^2 - \left(2u\bar{m} \cdot p - x(1 - x)p^2\right)}. \] (55)

Shifting the loop momentum as \( l^\mu \to l^\mu - xp - u\bar{m}^\mu \), \( M_{CO} \) becomes

\[ M_{CO} = -2ig^2C_F 4\Gamma \left( \frac{\mu^2 \epsilon^\gamma}{4\pi} \right) \int_0^1 dx \int_0^\infty du \int \frac{d^Dl}{(2\pi)^D} \frac{(1 - x)\bar{m} \cdot p}{\left(l^2 - \left(2u\bar{m} \cdot p - x(1 - x)p^2\right)^2\right)^{1-\epsilon}} \]

\[ = -\frac{\alpha_s C_F}{4\pi} 4\Gamma \left( \frac{\mu^2 \epsilon^\gamma}{4\pi} \right) \bar{m} \cdot p (1 + \epsilon) \int_0^1 dx (1 - x) \int_0^\infty du \left(2u\bar{m} \cdot p - x(1 - x)p^2\right)^{-1-\epsilon} \]

\[ = -\frac{\alpha_s C_F}{4\pi} 2\Gamma \left( \frac{\mu^2 \epsilon^\gamma}{4\pi} \right) \left( -p^2 \right)^{-\epsilon} (1 + \epsilon) \int_0^1 dx (1 - x)^{1-\epsilon} x^{-1-\epsilon} \]

\[ = \frac{\alpha_s C_F}{4\pi} \Gamma \left( \frac{\mu^2 \epsilon^\gamma}{4\pi} \right) \left( -p^2 \right)^{-\epsilon} \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon} + 4 - \frac{\pi^2}{6} \right) \] \[ = \frac{\alpha_s C_F}{4\pi} \Gamma \left( \frac{\mu^2 \epsilon^\gamma}{4\pi} \right) \left( -p^2 \right)^{-\epsilon} \left( \frac{2}{\epsilon^2} + \frac{2}{\epsilon} \left( 1 - \ln \frac{-p^2}{\mu^2} \right) - 2 \ln \frac{-p^2}{\mu^2} + \ln^2 \frac{-p^2}{\mu^2} + 4 - \frac{\pi^2}{6} \right). \] (56)

The \( 1/\epsilon \) poles are ultraviolet divergences and the infrared divergences appear as logarithms of \( p^2 \). If we put \( p^2 = 0 \), there is no definite scale and the diagram becomes zero. The ultraviolet and the infrared divergences cancel in this case, as is clear from Eq. (54).

The usoft contribution is given by Fig. 2(b), and it is written as

\[ M_{US} = -ig^2C_F \int \frac{d^Dl}{(2\pi)^D} \frac{\pi \cdot p\gamma^\mu(l + k)\eta}{l^2(l + p)^2(\bar{m} \cdot l + p^2)^2} \]

\[ = \frac{\alpha_s C_F}{4\pi} \gamma^\mu \left[ \frac{2}{\epsilon} \left( 1 - \ln \frac{-k^2}{p^2} \right) - 2 \ln \frac{-k^2}{\mu^2} - 2 \ln \frac{p^2}{q^2} \ln \frac{-k^2}{\mu^2} - \ln^2 \frac{p^2}{q^2} + 4 - \frac{2\pi^2}{3} \right]. \] (57)
Here the $1/\epsilon$ poles are ultraviolet divergences and the infrared divergences appear as logarithms of $p^2$.

The remaining contributions from Fig. 2 (c) – (f) are hard-collinear contributions. The contribution from Fig. 2 (c) is given by

$$M_c = -ig^2 C_F \Gamma \int \frac{d^D l}{(2\pi)^D l^2 (l^2 + \pi \cdot p \cdot l) \pi \cdot l} = 0,$$

where the off-shellness $p^2$ is suppressed by $\Lambda$ and it is discarded according to the power counting method described earlier. If we disregard the power counting, we can imagine performing the calculation with an arbitrary infrared regulator in the denominator. Then there appears the dependence on the infrared cutoff. But when we add all the hard-collinear contributions Fig. 2 (c) – (f), we have verified that this dependence on the fictitious infrared cutoff is canceled, which means that the treatment of the off-shellness is consistent.

In Fig. 2 (d), there are two types of contributions in which the currents $J_{\xi q}^{(1)}$, $J_{c(1)}^{(1)}$ contribute, and the currents $J_{\xi q}^{(2)}$, $J_{c(0)}^{(0)}$ contribute. The Feynman diagram in Fig. 2 (d) for the first case gives

$$M_d^{(1,1)} = -ig^2 C_F \int \frac{d^D l}{(2\pi)^D l^2 (l^2 + \pi \cdot p \cdot l) (l^2 + \pi \cdot l \cdot n \cdot k)} \left[ \frac{\gamma_\perp \alpha_\perp l^2}{\pi \cdot (l + p) \pi \cdot l} \left( \frac{\gamma_\perp \alpha_\perp l^2}{\pi \cdot (l + p)} + \frac{\gamma_\perp \alpha_\perp l^2}{\pi \cdot l} \right) \gamma_\perp - 2 \left( \frac{l^2 \gamma_\perp}{\pi \cdot (l + p)} + \frac{\gamma_\perp l^2}{\pi \cdot l} \right) \frac{l^2}{\pi \cdot l} \right]$$

$$= -ig^2 C_F \int \frac{d^D l}{(2\pi)^D l^2 (l^2 + \pi \cdot p \cdot l) (l^2 + \pi \cdot l \cdot n \cdot k)} \left[ (4 - D) \frac{\pi \cdot l}{\pi \cdot (l + p)} l^2 \gamma_\perp - 2 \frac{\pi \cdot (l + p) l^2 \gamma_\perp}{\pi \cdot l} \right],$$

where the Dirac algebra for the perpendicular components in $D$ dimensions is given by

$$\gamma_\perp \gamma_\perp = D - 2, \quad \gamma_\perp \phi_\perp \gamma_\perp = (4 - D) \phi_\perp,$$

$$\gamma_\perp \phi_\perp b_\perp \gamma_\perp = 4 a_\perp \cdot b_\perp + (D - 6) \phi_\perp b_\perp,$$

$$\gamma_\perp \phi_\perp b_\perp c_\perp \gamma_\perp = -2 \phi_\perp b_\perp c_\perp + (6 - D) \phi_\perp b_\perp c_\perp,$$

for arbitrary four-vectors $a^\mu$, $b^\mu$, and $c^\mu$. The first two terms are finite and each term gives $-\alpha_s C_F \gamma_\perp / (4\pi)$, and only the last term involves divergences. In the last term, $l^2_\perp$ can be written as

$$l^2_\perp = l^2_\perp + \pi \cdot ln \cdot (l + k) - \pi \cdot ln \cdot (l + k) = l^2 + \pi \cdot ln \cdot k - \pi \cdot l \cdot n \cdot (l + k).$$
With this decomposition, the last term in Eq. (59) can be written as

\[
2ig^2C_F \int \frac{d^Dl}{(2\pi)^D} \frac{1}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)} \frac{\pi \cdot (l + p) l_{\perp \mu}}{\pi \cdot l}
\]

\[
= 2ig^2C_F \gamma_{\perp \mu} \int \frac{d^Dl}{(2\pi)^D} \frac{\pi \cdot (l + p)}{l^2(l^2 + \pi \cdot p(n \cdot l) + \pi \cdot l \cdot k)} - \int \frac{d^Dl}{(2\pi)^D} \frac{\pi \cdot (l + p)n \cdot (l + k)}{(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)} = \frac{\alpha_s C_F}{4\pi} \gamma_{\perp \mu} I_b(q^2),
\]

(62)

where the first term is zero and \( I_b(q^2) \) is the second term which is given by

\[
I_b(q^2) = \left( -\frac{q^2}{\mu^2} \right)^{-\epsilon} \left( -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 3 + 2 \ln \frac{q^2}{\mu^2} + 3 \ln \frac{-q^2}{\mu^2} - \ln^2 \frac{-q^2}{\mu^2} - 6 + \frac{\pi^2}{6} \right).
\]

(63)

Here \( q^2 = (p - k)^2 \approx -\pi \cdot p(n \cdot k) \sim QA \), neglecting terms of order \( \Lambda^2 \). As a result, \( M_d^{(1,1)} \) is given by

\[
M_d^{(1,1)} = \frac{\alpha_s C_F}{4\pi} (I_b(q^2) - 2).
\]

(64)

The second case of Fig. 2 (d) is given as

\[
M_d^{(2,0)} = -ig^2C_F \frac{\pi}{2} l_{\perp \mu} \int \frac{d^Dl}{(2\pi)^D} \frac{1}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)} \times \left( -\frac{2\pi \cdot (l + p) l^2}{\pi \cdot l} + (D - 2) l_{\perp}^2 \right).
\]

(65)

The integrand of the first term in Eq. (63) can be written as

\[
\frac{2\pi \cdot (l + p)\left[ (l^2 + \pi \cdot p(n \cdot k) - \pi \cdot l \cdot (l + k) + \pi \cdot l \cdot l \right]}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)\pi \cdot l}
\]

\[
= -\frac{2\pi \cdot (l + p)}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)\pi \cdot l} + \frac{2\pi \cdot (l + p)n \cdot (l + k)}{2\pi \cdot (l + p)n \cdot l - l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)}.
\]

(66)

Therefore by combining all these contributions, we have

\[
M_d^{(2,0)} = -ig^2C_F \frac{\pi}{2} l_{\perp \mu} \int \frac{d^Dl}{(2\pi)^D} \left[ -\frac{2\pi \cdot (l + p)}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)\pi \cdot l} + \frac{2\pi \cdot (l + p)n \cdot (l + k)}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)} + \frac{(D - 2) l_{\perp}^2 - 2\pi \cdot (l + p)n \cdot l}{l^2(l^2 + \pi \cdot p(n \cdot l))(l^2 + \pi \cdot l \cdot k)} \right] = \frac{\alpha_s C_F}{4\pi} \frac{l_{\perp}^2}{2} \left[ I_b(q^2) + I_c(q^2) \right],
\]

(67)

where the first term here also vanishes. And \( I_c(q^2) \) comes from the last term in Eq. (67), which is given as

\[
I_c(q^2) = 2 + \left( -\frac{q^2}{\mu^2} \right)^{-\epsilon} \left( \frac{2}{\epsilon} + 5 \right) = \frac{2}{\epsilon} - 2 \ln \frac{-q^2}{\mu^2} + 3.
\]

(68)
The Feynman diagram in Fig. 2(e) is an additional diagram with the insertion of $L^{(1)}$, which also contributes at leading order. It is given as

$$M_e = 2ig^2C_F n_\mu \int \frac{d^D l}{(2\pi)^D l^2} \frac{l_\perp \cdot k_\perp (l+p)\pi \cdot l}{l^2 + \pi \cdot p n \cdot l} \times \left[(4-D)\pi \cdot l - 2\pi \cdot (l+p)\right] l_\perp. \quad (69)$$

In the calculation of $M_e$, the result is proportional to $\frac{\not{\pi}}{k_\perp}$, which can be written as

$$\frac{\not{\pi}}{k_\perp} = \frac{\not{\pi}}{2\pi} \cdot k - \frac{\not{n}}{2} n \cdot k. \quad (70)$$

Since this is sandwiched between $\bar{\xi}$ and $q_{us}$, $\frac{\not{n}}{n}$ vanishes due to the equation of motion, and the part with $\not{\pi}$ also vanishes due to $\xi$. And the final result can be written as

$$M_e = \frac{\alpha_s C_F \not{\pi}}{4\pi} \frac{-2 + 2 \ln -q^2/\mu^2 - 5}{2 n_\mu \gamma_\mu} \left(-2 - I_c(q^2)\right). \quad (71)$$

The Feynman diagram in Fig. 2(f) is introduced because it is of the same order as the other hard-collinear contributions, but it is zero.

When we sum over all the hard-collinear contribution from Fig. 2(c)–(f), the hard-collinear contribution is given as

$$M_{HC} = M_c + M_d^{(1,1)} + M_d^{(2,0)} + M_e$$

$$= \frac{\alpha_s C_F \not{\pi}}{4\pi} \gamma_\mu \left[-\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(-3 + \ln -q^2/\mu^2\right) + 3 \ln \frac{-q^2}{\mu^2} - \ln \frac{-q^2}{\mu^2} - \ln \frac{-q^2}{\mu^2} - 8 + \frac{\pi^2}{6}\right]. \quad (72)$$

Here we use the fact that $\gamma_\perp \cdot \pi n_\mu / 2 = \gamma_\mu - \not{\pi} \gamma_\mu / 2$, and the component with $\not{\pi}$ vanishes when attached to the collinear quark field. The hard-collinear contribution $M_{HC}$ looks fortuitously the same as the result in full QCD with the on-shell renormalization scheme if we disregard the origin of the $1/\epsilon$ poles. However, in the result of the full theory in Eq. (44), only the first $1/\epsilon$ pole is the ultraviolet divergence and the remaining divergences are infrared. On the other hand, the divergences of the hard-collinear contribution in Eq. (72) are ultraviolet divergences.

The infrared divergences in Eq. (72) is handled by the off-shellness, but there is no need for the infrared cutoff for the hard-collinear contributions from the power counting analysis. The first terms in Eqs. (62), (67) are those proportional to $M_c$, which is zero. Even though it becomes nonzero, say, by putting a fictitious infrared cutoff, these all cancel when we sum all the hard-collinear contributions. As a result, the hard-collinear contribution contains only the ultraviolet divergences.
If we add the collinear and the soft contributions in Eqs. (56), (57), we obtain

\[
M_{CO} + M_{US} = \frac{\alpha_s C_F}{4\pi} \gamma_{\mu} \left[ \left( \frac{2}{\epsilon^2} + \frac{4}{\epsilon} \right) \left( \frac{-q^2}{\mu^2} \right)^{-\epsilon} - 2 \ln \frac{p^2}{q^2} \ln \frac{k^2}{q^2} - 2 \ln \frac{p^2}{q^2} - 2 \ln \frac{k^2}{q^2} + 8 - \frac{5\pi^2}{6} \right],
\]

from which we can see explicitly that the infrared divergences \([\ln(p^2/q^2), \ln(k^2/q^2)]\) terms are exactly the same as those from the full theory given by Eq. (46). Another interesting feature of the calculation appears when we add all the hard-collinear, collinear and soft contributions, which is given by

\[
M_{HC} + M_{CO} + M_{US} = \frac{\alpha_s C_F}{4\pi} \gamma_{\mu} \left( \frac{1}{\epsilon} - \ln \frac{-q^2}{\mu^2} - 2 \ln \frac{p^2}{q^2} \ln \frac{k^2}{q^2} - 2 \ln \frac{p^2}{q^2} - 2 \ln \frac{k^2}{q^2} - \frac{2\pi^2}{3} \right),
\]

which is exactly the same as the result from the full theory in Eq. (46). It means that, though the calculation in SCET\(_I\) is complicated due to the classification of the effective Lagrangian and the current operators by power counting, the calculation in SCET\(_I\) corresponds to the separation of the full-theory results into different kinematic regions. In fact, there is no hard contribution of order \(Q\) in the full theory because the radiative corrections involve the relativistic invariants, which are \(p \cdot k \sim O(Q\Lambda), p^2, k^2 \sim \Lambda^2\). Therefore there is no change in the behavior of the soft-collinear currents including their renormalization effects near the boundary \(\mu = Q\). Of course, if we consider, for example, spectator effects including the soft-collinear currents as in \(B\) decays, the situation will be completely different.

The behavior of the soft-collinear current changes drastically in SCET\(_{II}\) because there are no hard-collinear contributions, which are already integrated out in SCET\(_I\). There are only collinear and soft contributions. In order to calculate these contributions, we evolve the soft-collinear current down to SCET\(_{II}\) and compute the radiative corrections at one loop. There is no effect from the effective Lagrangian in SCET\(_{II}\) since the inserted Lagrangian starts from \(\alpha_s\), and it does not contribute to the correction at order \(\alpha_s\). Therefore the necessary change is to replace \(Y\) by \(S\), and the Feynman graphs contributing to the radiative correction of the soft-collinear current in SCET\(_{II}\) are shown in Fig. 6.

FIG. 6: Vertex correction to the soft-collinear current in SCET\(_{II}\). (a) collinear contribution, (b) soft contribution.
The results of the collinear and soft contributions are the same as those obtained in SCET$_1$, which are given by

$$M_{CO} + M_S = \frac{\alpha_s C_F}{4\pi} \gamma^\mu \left[ \frac{2}{\epsilon_{UV}} + \frac{4}{\epsilon_{UV}} \left( \frac{-q^2}{\mu^2} \right)^{-\epsilon} - 2 \ln \frac{p^2}{q^2} \ln \frac{k^2}{q^2} - 2 \ln \frac{p^2}{q^2} - 2 \ln \frac{k^2}{q^2} + 8 - \frac{5\pi^2}{6} \right]$$

$$= \frac{\alpha_s C_F}{4\pi} \gamma^\mu \left[ \frac{2}{\epsilon_{UV}} + \frac{2}{\epsilon_{UV}} \left( 2 - \ln \frac{-q^2}{\mu^2} \right) - 4 \ln \frac{-q^2}{\mu^2} + \ln^2 \frac{-q^2}{\mu^2} - 2 \ln \frac{k^2}{q^2} + 8 - \frac{5\pi^2}{6} \right].$$

(75)

Here again, the infrared divergence in SCET$_{II}$ is the same as that in the full theory, and in SCET$_1$, which makes the matching between the full theory and the effective theories possible. However the ultraviolet behavior of the soft-collinear current is different from the current in the full theory or SCET$_1$.

If we calculate the collinear and the soft contributions to the soft-collinear current in SCET$_{II}$ with external particles on the mass shell ($p^2 = k^2 = 0$) and use the dimensional regularization for both the ultraviolet and the infrared divergences, all the Feynman diagrams in Fig. 6 vanish and the infrared divergences are canceled by the ultraviolet divergences since there are no scales involved. However, since we know the ultraviolet divergences explicitly in Eq. (75), we can read off the infrared and the ultraviolet divergences in SCET$_{II}$ as

$$M_{CO} + M_S = \frac{\alpha_s C_F}{4\pi} \gamma^\mu \left[ \frac{2}{\epsilon_{UV}} + \frac{2}{\epsilon_{UV}} \left( 2 - \ln \frac{-q^2}{\mu^2} \right) - \frac{2}{\epsilon_{IR}} - \frac{2}{\epsilon_{IR}} \left( 2 - \ln \frac{-q^2}{\mu^2} \right) \right].$$

(76)

With all this information, we can calculate the Wilson coefficients and the anomalous dimension of the (u)soft-collinear current at leading order in SCET, and to leading-log order in $\alpha_s$. Since the soft-collinear current is a conserved current in the full theory, it is not renormalized. It can be explicitly seen by noting that the $1/\epsilon_{UV}$ pole in the vertex correction is canceled by the wave function renormalization. The radiative corrections in the full theory and in SCET$_1$ are the same when we put the external particles on the mass shell [See Eqs. (44), (72) and (76)], or off the mass shell [See Eqs. (46), (74)], therefore the Wilson coefficient is 1 to order $\alpha_s$ and the soft-collinear current operator does not scale in the region $\sqrt{Q\Lambda} < \mu < Q$ either. However, in SCET$_{II}$, the ultraviolet behavior becomes different and the Wilson coefficient for the soft-collinear operator is given by

$$C_{II}(\mu) = 1 + \frac{\alpha_s(\mu)C_F}{4\pi} \left( 3 \ln \frac{-q^2}{\mu^2} - \ln^2 \frac{-q^2}{\mu^2} - 8 + \frac{\pi^2}{6} \right),$$

(77)

including the contribution from the wavefunction renormalization. Note that the contribution of the wavefunction renormalization in the full theory is the same for the collinear quark.
and the (u)soft quark. Therefore they cancel in matching, but it is needed in calculating the Wilson coefficient. The Wilson coefficient will be matched at $\mu = \sqrt{-q^2} \sim \sqrt{Q\Lambda}$

$$C_{II}(\sqrt{-q^2}) = 1 + \frac{\alpha_s(\sqrt{-q^2})C_F}{4\pi}(-8 + \frac{\pi^2}{6}),$$

and it satisfies the renormalization group equation

$$\mu \frac{d}{d\mu} C_{II}(\mu) = \gamma_{II}(\mu)C_{II}(\mu).$$

The counterterm $Z$ for the current operator is given by

$$Z = 1 + \frac{\alpha_s C_F}{4\pi} \left[ \frac{2}{e^2} + \frac{1}{e} \left( 3 - 2 \ln \frac{-q^2}{\mu^2} \right) \right],$$

including the counterterm from the wave function renormalization. The anomalous dimension is given by

$$\gamma_{II} = \frac{1}{Z} \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} \right) Z = -\frac{\alpha_s C_F}{4\pi} \left( 6 - 4 \ln \frac{-q^2}{\mu^2} \right).$$

IV. COMPARISON WITH OTHER CURRENT OPERATORS

With our understanding of the behavior of the (u)soft-collinear current operators in SCET, we can analyze the behavior of other current operators. Here we discuss the cases of the back-to-back collinear current and the heavy-to-light current. The back-to-back collinear current has been considered in the four-quark operators for nonleptonic $B$ decays [14, 15], and in deep inelastic scattering in the endpoint region where SCET can be applied [23]. The scaling of the momentum components in deep inelastic scattering is a little different from the power counting method described here. We describe the behavior of the back-to-back collinear current using two-step matching with the large scale $Q$, which separates the full theory and SCET$_I$, and the intermediate scale $\sqrt{Q\Lambda}$, which separates SCET$_I$ and SCET$_{II}$. Then we go back to discuss the back-to-back current in deep inelastic scattering.

It is useful to consider the characteristics of effective theories in order to answer the question about whether a single-step matching and a two-step matching will produce equivalent results. In constructing an effective theory, we integrate out the degrees of freedom above some large energy scale $Q$. Therefore the ultraviolet behavior is altered in the effective theory compared to the full theory, while the infrared behavior remains the same because two theories share the same infrared region. Therefore, if there are some hard contributions
of order $Q$ for some operators in the full theory, they are not reproduced in the effective theory. As a result, nontrivial Wilson coefficients and anomalous dimensions are developed for these operators in the effective theory in matching.

In the context of SCET, if there are hard contributions of order $Q$ for an operator in the full theory, nontrivial Wilson coefficients and anomalous dimensions for the operator appear in SCET. It happens for the back-to-back collinear current, and the heavy-to-light current. For the back-to-back collinear current, the momenta of the external particles are given by $p^\mu_n \sim Q n^\mu$ and $p^\mu_\pi \sim Q \pi^\mu$. The radiative corrections contain the relativistic invariant $p_n \cdot p_\pi \sim Q^2$, which are hard contributions. For the heavy-to-light current, the full theory corresponds to the heavy quark effective theory for the heavy quark with momentum $p^\mu_h \sim Q v^\mu$ and the full QCD for the light quark with momentum $p^\mu_c \sim Q n^\mu$. The radiative corrections contain the relativistic invariant of order $p_h \cdot p_c \sim Q^2$, which are hard contributions. Therefore there is a nontrivial matching for the back-to-back collinear current and the heavy-to-light current between the full theory and SCET.

It is possible to have no hard contributions of order $Q$ in the full theory for some reason, but if there are hard-collinear contributions of order $\sqrt{Q \Lambda}$, the ultraviolet behavior of the full theory is the same as that of SCET, which is given by the hard-collinear contributions in SCET. This happens for the usoft-collinear current. In this case the collinear particle has momentum $p_c \sim Q n^\mu$ and the usoft particle has $p^\mu_\text{us} \sim (\Lambda, \Lambda, \Lambda)$, where the largest relativistic invariant is $p_c \cdot p_\text{us} \sim Q \Lambda$. The radiative corrections in the full theory are hard-collinear contributions and there are no hard contributions of order $Q$. There should be nontrivial hard-collinear contributions for the usoft-collinear current in SCET, and they should give the same contributions as in the full theory. Therefore the ultraviolet behavior is the same in the full theory and in SCET, and the matching between these theories becomes trivial. The Wilson coefficients and the anomalous dimensions are the same as those in the full theory. In other words, there is no change of the behavior of the operators across the boundary between the full theory and SCET.

The criterion for nontrivial matching also applies to the matching between SCET and SCET. If there are hard-collinear contributions of order $\sqrt{Q \Lambda}$ in SCET, the ultraviolet behavior of some operators in SCET and SCET is different because the hard-collinear modes are not present in SCET. And there is a nontrivial matching between SCET and SCET. The soft-collinear current belongs to this class, as explicitly shown in this paper.
FIG. 7: Radiative corrections to the heavy-to-light current in SCET. The double line is a heavy quark. (a) hard-collinear contribution, (b) usoft contribution, (c) collinear contribution.

On the other hand, if there are no hard-collinear contributions of order $\sqrt{Q}$, the behavior of the operators across the boundary between SCET$_1$ and SCET$_{II}$ is the same. This is true for the back-to-back collinear current and the heavy-to-light currents. For the back-to-back collinear current, there is no hard-collinear interaction between the quarks moving in the opposite directions, and the hard-collinear interaction of the quarks with the hard-collinear gluons is given by the same form as Eq. (58), which is zero. For the heavy-to-light current, the radiative corrections to the heavy-to-light current in SCET$_1$ is shown in Fig. 7. Here the hard-collinear contribution in Fig. 7 (a) is zero. However, if there are additional hard-collinear contributions, such as the spectator quark interactions for $B$ decays, a nontrivial matching occurs as we go down to SCET$_1$ and SCET$_{II}$.

We summarize the types of radiative corrections for the three current operators in Table II. From Table II, nontrivial matching between the full theory and SCET$_1$ occurs for the back-to-back collinear current and the heavy-to-light current, and the (u)soft-collinear current needs nontrivial matching between SCET$_1$ and SCET$_{II}$. Therefore even though we apply the two-step matching for these operators, one of the matching processes is irrelevant because there is no change across the boundary, and only one nontrivial matching appears. However, if there is an operator which has hard and hard-collinear contributions in the full theory, and hard-collinear contributions in SCET$_1$, the two-step matching is necessary and there is a nontrivial matching procedure at each boundary. An example of the operator which needs the two-step matching appears in the four-quark operators for nonleptonic $B$ decays.

| Currents       | (u)soft-collinear back-to-back heavy-to-light |
|---------------|-----------------------------------------------|
| hard ($Q$)    | X                                             |
| hard-collinear ($\sqrt{Q}\Lambda$) | O X                                           |

TABLE II: Hard ($\sim Q$), and hard-collinear ($\sim \sqrt{Q}\Lambda$) contributions for various current operators.
If a four-quark operator consists of a heavy quark, a soft quark, and two collinear quarks in the opposite directions, the interactions between the two collinear quarks and between a collinear quark and the heavy quark produce hard contributions of order $Q$. The interaction between a collinear quark and a soft quark produces hard-collinear contributions of order $\sqrt{Q}$. Therefore we need two-step matching in this case.

There will be neither hard, nor hard-collinear contributions in the radiative corrections of the (u)soft-(u)soft current operator in the full theory and in SCET$_I$ though we have not verified explicitly. This is based on the fact that all the relativistic invariants in this case are of order $\Lambda^2$ in all the theories. Then the matching will be trivial without any change of the behavior of the operator from the full theory down to SCET$_II$, but it should be checked explicitly.

The situation in deep inelastic scattering is subtler than the discussion presented above. By choosing different reference frames, different types of hadronic current operators are considered [23]. In the target rest frame, the relevant current is the (u)soft-collinear current operator, and in the Breit frame where the photon has no energy component, the corresponding current is the back-to-back collinear current operator.

In the target rest frame, the incoming quark has momentum $p^\mu = (\pi \cdot p, p_\perp, n \cdot p) \sim (\Lambda, \Lambda, \Lambda)$, which is (u)soft. And the final-state quark has momentum $p_X^\mu \sim (Q^2/\Lambda, \Lambda, (1-x)\Lambda)$ with $p_X^2 \sim Q^2(1-x)$. In this case, the boundary which separates the full theory and SCET$_I$ is $\pi \cdot p_X \sim Q^2/\Lambda$, and the boundary between SCET$_I$ and SCET$_II$ is $\mu^2 \sim \pi \cdot p_X n \cdot p \sim Q^2$. Therefore the matching near $\mu = Q$ corresponds to the matching between SCET$_I$ and SCET$_II$. As explained before, from very high energy to $Q^2/\Lambda$ (from full theory to SCET$_I$), there is no scaling of the soft-collinear current. In the Breit frame, $p^\mu \sim (\Lambda^2/Q, \Lambda, Q)$, and $p_X^\mu \sim (Q, \Lambda, Q(1-x))$ with $p_X^2 \sim Q^2(1-x)$. Here the boundary between the full theory and SCET$_I$ is $\pi \cdot p_X \sim Q$ and the boundary between SCET$_I$ and SCET$_II$ is $\mu^2 \sim Q^2(1-x)$. Therefore the matching near $\mu = Q$ corresponds to the matching between the full theory and SCET$_I$. Therefore the matching procedures in different reference frames correspond to the matching between different effective theories. But the starting scale where nontrivial evolution occurs is $\mu = Q$ in both matching. Therefore the Wilson coefficients and the anomalous dimension are the same in both frames just below $Q$. This interesting result arises from the combination of the behavior of the soft-collinear current and the back-to-back collinear current in SCET$_I$ and SCET$_II$, and the different scales which separate these
effective theories.

V. CONCLUSION

We have constructed the effective Lagrangian for SCET\textsubscript{I} and SCET\textsubscript{II}, which are gauge invariant under collinear and (u)soft gauge transformations at each order. It has been achieved in SCET\textsubscript{I} by treating the usoft field as a background field for collinear gauge transformations, and by redefining the collinear gauge field. Also the usoft interactions with the collinear particles are factorized. The effective Lagrangian in SCET\textsubscript{II} is obtained by integrating out the hard-collinear degrees of freedom in SCET\textsubscript{I}. This is the starting point to investigate various effects of the interactions in SCET\textsubscript{II}, which is the final effective theory for systems with $p^2 \sim \Lambda^2$. For example, at higher orders in $\Lambda$ and in $\alpha_s$, there are interactions which violate factorization properties in $B$ decays. For many processes, these factorization-violating interactions are subleading, but there may be interesting cases in which these interactions can enter as the leading contribution. This will be investigated in detail in a future publication.

There are many points yet to be clarified in the effective Lagrangian in SCET\textsubscript{II}. We have constructed the effective Lagrangian explicitly gauge invariant, but the momentum conservation is not explicitly realized. Therefore, in the current form, we have to consider if a certain vertex conserves collinear and soft momenta. If not, it is not allowed. It would be desirable to have the effective Lagrangian which is explicitly gauge invariant and momentum conserving at each order in $\Lambda$ and $g$. If we choose a specific gauge ($\hat{W} = S = 1$), the momentum conservation can be transparent, but we also have to consider the corresponding change of the gauge fixing Lagrangian. This will be discussed in more detail elsewhere.

We have considered the radiative corrections of the (u)soft-collinear current operators both in SCET\textsubscript{I} and in SCET\textsubscript{II} at one loop and found that we can reproduce the infrared divergence of the full theory in SCET\textsubscript{I} and SCET\textsubscript{II} only in terms of the collinear, usoft modes, and the collinear, soft modes respectively and we do not need additional degrees of freedom. In regulating the infrared divergence with the off-shellness, some modifications are needed for the Feynman rules. But these are relatively easy and we have not invented more complicated regularization schemes \cite{10, 22} except the power counting method in expressing the propagators for various cases.
As we have seen in the behavior of the soft-collinear, back-to-back collinear and heavy-to-light current operators, the presence of the hard contributions of order $Q$ in the full theory, and the hard-collinear contributions of order $\sqrt{Q}\Lambda$ in SCET$_1$ is important in determining which matching gives a nontrivial result. It gives a criterion for using a single-step matching or a two-step matching for the radiative corrections. However, this is determined only at the end of the calculation, and there is no a priori criterion.

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