Vertex-Coloring with Star-Defects

Patrizio Angelini¹, Michael A. Bekos¹, Michael Kaufmann¹, and Vincenzo Roselli²

¹Wilhelm-Schickhard-Institut für Informatik, Universität Tübingen, Germany, {angelini,bekos,mk}@informatik.uni-tuebingen.de
²Dipartimento di Ingegneria, Università Roma Tre, Italy, roselli@dia.uniroma3.it

Abstract

Defective coloring is a variant of traditional vertex-coloring, according to which adjacent vertices are allowed to have the same color, as long as the monochromatic components induced by the corresponding edges have a certain structure. Due to its important applications, as for example in the bipartisation of graphs, this type of coloring has been extensively studied, mainly with respect to the size, degree, and acyclicity of the monochromatic components.

In this paper we focus on defective colorings in which the monochromatic components are acyclic and have small diameter, namely, they form stars. For outerplanar graphs, we give a linear-time algorithm to decide if such a defective coloring exists with two colors and, in the positive case, to construct one. Also, we prove that an outerpath (i.e., an outerplanar graph whose weak-dual is a path) always admits such a two-coloring. Finally, we present NP-completeness results for non-planar and planar graphs of bounded degree for the cases of two and three colors.

1 Introduction

Graph coloring is a fundamental problem in graph theory, which has been extensively studied over the years (see, e.g., [3] for an overview). Most of the research in this area has been devoted to the vertex-coloring problem (or coloring problem, for short), which dates back to 1852 [18]. In its general form, the problem asks to label the vertices of a graph with a given number of colors, so that no two adjacent vertices share the same color. In other words, a coloring of a graph partitions its vertices into a particular number of independent sets (each of these sets is usually referred to as a color class, as all its vertices have the same color). A central result in this area is the so-called four color theorem, according to which every planar graph admits a coloring with at most four colors; see e.g. [11]. Note that the problem of deciding whether a planar graph is 3-colorable is NP-complete [10], even for graphs of maximum degree 4 [6].

Several variants of the coloring problem have been proposed over the years. One of the most studied is the so-called defective coloring, which was independently introduced by Andrews and Jacobson [1], Harary and Jones [13], and Cowen et al. [5]. In the defective coloring problem edges between vertices of the same color class are allowed, as long as the monochromatic components induced by vertices of the same color maintain some special structure. In this respect, one can regard the classical vertex-coloring as a defective one in which every monochromatic component is an isolated vertex, given that every color class defines an independent set. In this work we focus on defective colorings in which each component is acyclic and has small diameters. In particular, we call a graph G
tree-diameter-$\lambda$ $\kappa$-colorable if the vertices of $G$ can be colored with $\kappa$ colors, so that all monochromatic components are acyclic and of diameter at most $\lambda$, where $\kappa \geq 1$, $\lambda \geq 0$. Clearly, a classical $\kappa$-coloring corresponds to a tree-diameter-0 $\kappa$-coloring. The diameter of a coloring is defined as the maximum diameter among the monochromatic components.

We present algorithmic and complexity results for tree-diameter-$\lambda$ $\kappa$-colorings for small values of $\kappa$ and $\lambda = 2$. For simplicity, we refer to this problem as $(\text{star}$, $\kappa)$-coloring, as each monochromatic component is a star (i.e., a tree with diameter two; see Figure 1d). Similarly, we refer to the tree-diameter-$\lambda$ $\kappa$-coloring problem when $\lambda = 1$ as $(\text{edge}$, $\kappa)$-coloring problem. By definition, a $(\text{edge}$, $\kappa)$-coloring is also a $(\text{star}$, $\kappa)$-coloring. Figs. 1a-1c show a trade-off between number of colors and structure of the monochromatic components.

Our work can be seen as a variant of the bipartisation of graphs, namely the problem of making a graph bipartite by removing a small number of elements (e.g., vertices or edges), which is a central graph problem with many applications [12, 14]. The bipartisation by removal (a not-necessarily minimal number of) non-adjacent edges corresponds to the $(\text{edge}$, 2)-coloring problem. In the $(\text{star}$, 2)-coloring problem, we also solve some kind of bipartisation by removing independent stars. Note that we do not ask for the minimum number of removed stars but for the existence of a solution.

To the best of our knowledge, this is the first time that the defective coloring problem is studied under the requirement of having color classes of small diameter. Previous research was focused either on their size or their degree [1, 5, 13, 16, 17]. As byproducts of these works, one can obtain several results for the $(\text{edge}$, $\kappa)$-coloring problem. More precisely, from a result of Lovász [16], it follows that all graphs of maximum degree 4 or 5 are $(\text{edge}$, 3)-colorable. However, determining whether a graph of maximum degree 7 is $(\text{edge}$, 3)-colorable is NP-complete [5]. In the same work, Cowen et al. [5] prove that not all planar graphs are $(\text{edge}$, 3)-colorable and that the corresponding decision problem is NP-complete, even in the case of planar graphs of maximum degree 10. Results for graphs embedded on general surfaces are also known [2, 4, 5]. Closely related is also the so-called tree-partition-width problem, which is a variant of the defective coloring problem in which the graphs induced by each color class must be acyclic [7, 9, 21], i.e., there is no restriction on their diameter. Our contributions are:

- In Section 2 we present a linear-time algorithm to determine whether an outerplanar graph is $(\text{star}$, 2)-colorable. Note that outerplanar graphs are 3-colorable [20], and hence $(\text{star}$, 3)-colorable, but not necessarily $(\text{star}$, 2)-colorable. On the other hand, we can always construct $(\text{star}$, 2)-colorings for outerpaths (which form a special subclass of outerplanar graphs whose weak-dual is a path).

- In Section 3 we prove that the $(\text{star}$, 2)-coloring problem is NP-complete, even for graphs of maximum degree 5 (note that the corresponding $(\text{edge}$, 2)-coloring prob-

\[1\] Recall that the weak-dual of a plane graph is the subgraph of its dual induced by neglecting the face-vertex corresponding to its unbounded face.
lem is NP-complete, even for graphs of maximum degree 4 [5]). Since all graphs of maximum degree 3 are (edge, 2)-colorable [16], this result leaves open only the case for graphs of maximum degree 4. We also prove that the (star, 3)-coloring problem is NP-complete, even for graphs of maximum degree 9 (recall that the corresponding (edge, 3)-coloring problem is NP-complete, even for graphs of maximum degree 7 [5]). Since all graphs of maximum degree 4 or 5 are (edge, 3)-colorable [16], our result implies that the computational complexity of the (star, 3)-coloring problem remains unknown only for graphs of maximum degree 6, 7, and 8. For planar graphs, we prove that the (star, 2)-coloring problem remains NP-complete even for triangle-free planar graphs (recall that triangle-free planar graphs are always 3-colorable [15], while the test of 2-colorability can be done in linear time).

2 Coloring Outerplanar Graphs and Subclasses

In this section we consider (star, 2)-colorings of outerplanar graphs. To demonstrate the difficulty of the problem, we first give an example (see Figure 2a) of a small outerplanar graph not admitting any (star, 2)-coloring. Therefore, in Theorem 1 we study the complexity of deciding whether a given outerplanar graph admits such a coloring and present a linear-time algorithm for this problem; note that outerplanar graphs always admit 3-colorings [20]. Finally, we show that a notable subclass of outerplanar graphs, namely outerpaths, always admit (star, 2)-colorings by providing a constructive linear-time algorithm (see Theorem 2).

Lemma 1. There exist outerplanar graphs that are not (star, 2)-colorable.

Proof. We prove that the outerplanar graph of Figure 2a is not (star, 2)-colorable. In particular, we show that in any 2-coloring of this graph there exists a monochromatic path of four vertices. Assume w.l.o.g. that vertex $u$ has color gray. Then, at least two vertices out of $u_1, \ldots, u_8$ are gray, as otherwise there would be a path of four white vertices. Hence, $u$ is the center of a gray star.

Next, we observe that either $u_2$ is white or the path $u_2, \ldots, u_8$ must consist of only white vertices. Similarly, we observe that either $u_3$ is white or the path $u_3, \ldots, u_8$ must consist of only white vertices. If both $u_2$ and $u_3$ are white, then either one of paths $u_2, \ldots, u_8$ and $u_3, \ldots, u_8$ consists only of gray vertices, or there exists a path from one of $u_2, \ldots, u_8$ via $u_2$ and $u_3$ to one of $u_3, \ldots, u_8$, that consists only of white vertices. Clearly, all aforementioned cases lead to a monochromatic path of four vertices.

Figure 2: (a) An outerplanar graph that is not (star, 2)-colorable. (b) An outerpath, whose spine edges are drawn as dashed segments. Dotted arcs highlighted in gray correspond to edges belonging to the fan of each spine vertex. Note that $|f_6| = 0$.

Lemma 1 implies that not all outerplanar graphs are (star, 2)-colorable. In the following we give a linear-time algorithm to decide whether an outerplanar graph is (star, 2)-colorable and in case of an affirmative answer to compute the actual coloring.
Theorem 1. Given an outerplanar graph $G$, there exists a linear-time algorithm to test whether $G$ admits a (star, 2)-coloring and to construct a (star, 2)-coloring, if one exists.

Proof. We assume that $G$ is embedded according to its outerplanar embedding. We can also assume that $G$ is biconnected. This is not a loss of generality, as we can always reduce the number of cut-vertices by connecting two neighbors $a$ and $b$ of a cut-vertex $c$ belonging to two different biconnected components with a path having two internal vertices. Clearly, if the augmented graph is (star, 2)-colorable, then the original one is (star, 2)-colorable.

For the other direction, given a (star, 2)-coloring of the original graph, we can obtain a corresponding coloring of the augmented graph by coloring the neighbors of $a$ and $b$ with different color than the ones of $a$ and $b$, respectively.

Denote by $T$ the weak dual of $G$ and root it at a leaf $\rho$ of $T$. For a node $\mu$ of $T$, we denote by $G(\mu)$ the subgraph of $G$ corresponding to the subtree of $T$ rooted at $\mu$. We also denote by $f(\mu)$ the face of $G$ corresponding to $\mu$ in $T$. If $\mu \neq \rho$, consider the parent $\nu$ of $\mu$ in $T$ and their corresponding faces $f(\nu)$ and $f(\mu)$ of $G$, and let $(u, v)$ be the edge of $G$ shared by $f(\nu)$ and $f(\mu)$. We say that $(u, v)$ is the attachment edge of $G(\mu)$ to $G(\nu)$. The attachment edge of the root $\rho$ is any edge of face $f(\rho)$ that is incident to the outer face (since $G$ is biconnected and $\rho$ is a leaf, this edge always exists). Consider a (star, 2)-coloring of $G(\mu)$. In this coloring, each of the endpoints $u$ and $v$ of the attachment edge of $G(\mu)$ plays exactly one of the following roles: (i) center or (ii) leaf of a colored star; (iii) isolated vertex, that is, it has no neighbor with the same color; or (iv) undefined, that is, the only neighbor of $u$ (resp. $v$) which has its same color is $v$ (resp. $u$). Note that if the only neighbor of $u$ (resp. $v$) which has its same color is different from $v$ (resp. from $u$), we consider $u$ (resp. $v$) as a center. Two (star, 2)-colorings of $G(\mu)$ are equivalent w.r.t. the attachment edge $(u, v)$ of $G(\mu)$ if in the two (star, 2)-colorings each of $u$ and $v$ has the same color and plays the same role. This definition of equivalence determines a partition of the colorings of $G(\mu)$ into a set of equivalence classes. Since both the number of colors and the number of possible roles of each vertex $u$ and $v$ are constant, the number of different equivalence classes is also constant (note that, when the role is undefined, $u$ and $v$ must have the same color).

In order to test whether $G$ admits a (star, 2)-coloring, we perform a bottom-up traversal of $T$. When visiting a node $\mu$ of $T$ we compute the maximal set $C(\mu)$ of equivalence classes such that, for each class $C \in C(\mu)$, graph $G(\mu)$ admits at least a coloring belonging to $C$. Note that $|C(\mu)| \leq 38$. In order to compute $C(\mu)$, we consider the possible equivalence classes one at a time, and check whether $G(\mu)$ admits a (star, 2)-coloring in this class, based on the sets $C(\mu_1), \ldots, C(\mu_h)$ of the children $\mu_1, \ldots, \mu_h$ of $\mu$ in $T$, which have been previously computed. In particular, for an equivalence class $C$ we test the existence of a (star, 2)-coloring of $G(\mu)$ belonging to $C$ by selecting an equivalence class $C_i \in C(\mu_i)$ for each $i = 1, \ldots, h$ in such a way that:

1. the color and the role of $u$ in $C_1$ are the same as the ones $u$ has in $C$;
2. the color and the role of $v$ in $C_h$ are the same as the ones $v$ has in $C$;
3. for any two consecutive children $\mu_i$ and $\mu_{i+1}$, let $x$ be the vertex shared by $G(\mu_i)$ and $G(\mu_{i+1})$. Then, $x$ has the same color in $C_i$ and $C_{i+1}$ and, if $x$ is a leaf in $C_i$, then $x$ is isolated in $C_{i+1}$ (or vice-versa); and
4. for any three consecutive children $\mu_{i-1}$, $\mu_i$, and $\mu_{i+1}$, let $x$ (resp. $y$) be the vertex shared by $G(\mu_{i-1})$ and $G(\mu_i)$ (resp. by $G(\mu_i)$ and $G(\mu_{i+1})$). Then, $x$ (resp. $y$) has the same color in $C_i$ and $C_{i+1}$ (resp. $C_{i+1}$); also, if $x$ and $y$ are both undefined in $C_i$, then in $C_{i-1}$ and $C_{i+1}$ none of $x$ and $y$ is a leaf, and at least one of them is isolated.
Note that the first two conditions ensure that the coloring belongs to \( C \), while the other two ensure that it is a \((\text{star, 2})\)-coloring. Since the cardinality of \( C(\mu_i) \) is bounded by a constant, the test can be done in linear time. If the test succeeds, add \( C \) to \( C(\mu) \).

Once all 38 equivalence classes are tested, if \( C(\mu) \) is empty, then we conclude that \( G \) is not \((\text{star, 2})\)-colorable. Otherwise we proceed with the traversal of \( T \). At the end of the traversal, if \( C(\rho) \) is not empty, we conclude that \( G \) is \((\text{star, 2})\)-colorable. A \((\text{star, 2})\)-coloring of \( G \) can be easily constructed by traversing \( T \) top-down, by following the choices performed during the bottom-up visit. \( \square \)

In the following, we consider a subclass of outerplanar graphs, namely outerpaths, and we prove that they always admit \((\text{star, 2})\)-colorings. Note that the example that we presented in Lemma 1 is “almost” an outerpath, meaning that the weak-dual of this graph contains only degree-1 and degree-2 vertices, except for one specific vertex that has degree 3 (see the face of Figure 2a highlighted in gray). Recall that the weak-dual of an outerpath is a path (hence, it consists of only degree-1 and degree-2 vertices).

Let \( G \) be an outerpath (see Figure 2b). We assume that \( G \) is inner-triangulated. This is not a loss of generality, as any \((\text{star, 2})\)-coloring of a triangulated outerpath induces a \((\text{star, 2})\)-coloring of any of its subgraphs. We first give some definitions. We call spine vertices the vertices \( v_1, v_2, \ldots, v_m \) that have degree at least four in \( G \). We consider an additional spine vertex \( v_{m+1} \), which is the (unique) neighbor of \( v_m \) along the cycle delimiting the outer face that is not adjacent to \( v_{m-1} \). Note that the spine vertices of \( G \) induce a path, that we call spine of \( G \). The fan \( f_i \) of a spine vertex \( v_i \) consists of the set of neighbors of \( v_i \) in \( G \), except for \( v_{i-1} \) and for those following and preceding \( v_i \) along the cycle delimiting the outer face.\(^2\) Note that \( |f_i| \geq 1 \) for each \( i = 1, \ldots, m \), while \( |f_{m+1}| = 0 \). For each \( i = 1, \ldots, m+1 \), we denote by \( G_i \) the subgraph of \( G \) induced by the spine vertices \( v_1, \ldots, v_i \) and by the fans \( f_1, \ldots, f_{i-1} \). Note that \( G_{m+1} = G \). We denote by \( c_i \) the color assigned to spine vertex \( v_i \), and by \( c(G_i) \) a coloring of graph \( G_i \). Finally, we say that an edge of \( G \) is colored if its two endpoints have the same color.

**Theorem 2.** Every outerpath admits a \((\text{star, 2})\)-coloring, which can be computed in linear time.

**Proof.** Let \( G \) be an outerpath with spine \( v_1, \ldots, v_k \). We describe an algorithm to compute a \((\text{star, 2})\)-coloring of \( G \). At each step \( i = 1, \ldots, k \) of the algorithm we consider the spine edge \((v_{i-1}, v_i)\), assuming that a \((\text{star, 2})\)-coloring of \( G_i \) has already been computed satisfying one of the following conditions (see Figure 3):

\( Q_0 \): The only colored vertex is \( v_1 \);

\( Q_1 \): \( c_i \neq c_{i-1} \), vertex \( v_{i-1} \) is the center of a star with color \( c_{i-1} \), and no colored edge is incident to \( v_i \);

\( Q_2 \): \( c_i = c_{i-1} \), and no colored edge other than \((v_{i-1}, v_i)\) is incident to \( v_{i-1} \) or \( v_i \);

\( Q_3 \): \( c_i \neq c_{i-1} \), vertex \( v_{i-1} \) is a leaf of a star with color \( c_{i-1} \), and no colored edge is incident to \( v_i \);

\( Q_4 \): \( c_i \neq c_{i-1} \), vertex \( v_{i-1} \) is the center of a star with color \( c_{i-1} \), and vertex \( v_i \) is the center of a star with color \( c_i \); further, \( i < k \) and \(|f_i| > 1|\);

\(^2\)Note that the spine of \( G \) coincides with the spine of the caterpillar obtained from the outerpath \( G \) by removing all the edges incident to its outer face, neglecting the additional spine vertex \( v_{m+1} \).

\(^3\)Fan \( f_i \) contains all the leaves of the caterpillar incident to \( v_i \), plus the following spine vertex \( v_{i+1} \).
Figure 3: Schematization of the algorithm. Each node represents the (unique) condition satisfied by $G_i$ at some step $0 \leq i \leq k$. An edge label $0, 1, e, o$ represents the fact that the cardinality of a fan $f_i$ is 0, 1, even $\neq 0$, or odd $\neq 1$. If the label contains two characters, the second one describes the cardinality of $f_{i+1}$. An edge between $Q_j$ and $Q_h$ with label $x \in \{1, e, o\}$ (with label $xy$, where $y \in \{0, 1, e, o\}$) represents the fact that, if $G_i$ satisfies condition $Q_j$ and $|f_i| = x$ (resp. $|f_i| = x$ and $|f_{i+1}| = y$), then $f_i$ is colored so that $G_{i+1}$ satisfies $Q_h$.

$Q_5$: $c_i = c_{i-1}$, vertex $v_{i-1}$ is the center of a star with color $c_{i-1}$, and no colored edge other than $(v_{i-1}, v_i)$ is incident to $v_i$; further, $i < k$ and $|f_i| = 1$.

Next, we color the vertices in $f_i$ in such a way that $c(G_{i+1})$ is a (star, 2)-coloring satisfying one of the conditions; refer to Figure 3 for a schematization of the case analysis. In the first step of the algorithm, we assign an arbitrary color to $v_1$, and hence $c(G_1)$ satisfies $Q_0$. For $i = 1, \ldots, k$ we color $f_i$ depending on the condition satisfied by $c(G_i)$.

**Coloring $c(G_1)$ satisfies $Q_0$:** Independently of the cardinality of $f_i$, we color its vertices with alternating colors so that $c_{i+1} \neq c_i$. In this way, the only possible colored edges are incident to $v_i$ and not to $v_{i+1}$. So, $c(G_{i+1})$ satisfies condition $Q_1$.

**Coloring $c(G_1)$ satisfies $Q_1$:** In this case we distinguish the following subcases, based on the cardinality of $f_i$.

- If $|f_i| = 0$, we have that $i = k$ and hence $G_k = G$. It follows that $c(G_k)$ is a (star, 2)-coloring of $G$.
- If $|f_i| = 1$ (that is, $f_i$ contains only $v_{i+1}$; see Figure 4a), we set $c_{i+1} = c_i$. Since the only neighbor of $v_{i+1}$ in $G_{i+1}$ different from $v_i$ is $v_{i-1}$, whose color is $c_{i-1} \neq c_i$, and since $v_i$ has no neighbor with color $c_i$ other than $v_{i+1}$, by condition $Q_1$, coloring $c(G_{i+1})$ is a (star, 2)-coloring satisfying condition $Q_2$.
- If $|f_i| > 1$ (see Figure 4b), we color the vertices in $f_i$ with alternating colors so that $c_{i+1} \neq c_i$. This implies that every colored edge of $G_{i+1}$ not belonging to $G_i$ is incident either to $v_i$, if its color is $c_i$, or to $v_{i-1}$, if its color is $c_{i-1}$; the latter case only happens if $|f_i|$ is odd. Thus, $v_i$ (resp. $v_{i-1}$) is the center of a star of color $c_i$ (resp. $c_{i-1}$) in $G_{i+1}$. Since $v_i$ has no neighbor with color $c_i$ in $G_i$, while $v_{i-1}$ is a center also in $G_i$, coloring $c(G_{i+1})$ is a (star, 2)-coloring. Finally, since $v_{i+1}$ has no neighbors with color $c_{i+1} \neq c_i$, by construction, $c(G_{i+1})$ satisfies condition $Q_1$.

**Coloring $c(G_1)$ satisfies $Q_2$:** We again distinguish subcases based on $|f_i|$.
Figure 4: Graph $G_{i+1}$ after coloring $f_i$ when $c(G_i)$ satisfies: $Q_1$ and (a) $|f_i| = 1$ or (b) $|f_i| > 1$; $Q_2$ and (c) $|f_i| = 0$, or (d) $|f_i| = e$ and (e) $|f_{i+1}| = 0$, (f) $|f_{i+1}| = 1$, or (g) $f_{i+1} > 1$. Shaded regions represent $G_i$. Bold edges connect vertices with the same color, while spine edges are dashed.

Figure 5: Graph $G_{i+1}$ after coloring $f_i$ when $c(G_i)$ satisfies: $Q_3$ and (a) $|f_i| = 1$, or (b) $|f_i| = e$: $Q_4$ (c) or $Q_5$ (d). Shaded regions represent $G_i$. Bold edges connect vertices with the same color, while spine edges are dashed.

- If $|f_i| = 0$, we have that $i = k$ and hence $c(G_k)$ is a (star, 2)-coloring of $G = G_k$.
- If $|f_i| = e$ and $|f_{i+1}| = 0$, we have that $i = k$ and hence $c(G_k)$ is a (star, 2)-coloring of $G = G_k$.
- If $|f_i| = 1$, or $|f_i| = e$, we color the vertices of $f_i$ with alternating colors in such a way that $c_{i+1} \neq c_i$. By construction, $c(G_{i+1})$ is a (star, 2)-coloring satisfying condition $Q_1$.
- If $|f_{i+1}| = 0$ : Note that in this case $i = k$ holds (see Figure 4d). We color the vertices of $f_i$ with alternating colors so that $c_{i+1} = c_i$. Note that the unique neighbor of $v_{i-1}$ in $f_i$ has color different from $c_{i-1}$, since $|f_i|$ is even. Hence, all the new colored edges are incident to $v_i$, which implies that $c(G_{i+1})$ is a (star, 2)-coloring satisfying condition $Q_2$.
- If $|f_{i+1}| = 1$ : Note that $i < k$ and $f_{i+1}$ only contains $v_{i+2}$ (see Figure 4e). We color the vertices of $f_i$ with alternating colors so that $c_{i+1} = c_i$. Since (i) all the new colored edges are incident to $v_i$, (ii) $v_i$ and $v_{i-1}$ have no neighbor with their same color in $G_i$ (apart from each other), (iii) $c_{i+1} = c_i$, and (iv) $i < k$, we have that $c(G_{i+1})$ is a (star, 2)-coloring satisfying condition $Q_3$.
- If $|f_{i+1}| > 1$ : Note that $i < k$ (see Figure 4f). Independently of whether $|f_{i+1}|$ is even or odd, we color the vertices of $f_i$ so that $c_{i+1} \neq c_i$, the unique neighbor of $v_{i+1}$ different from $v_i$ has also color $c_{i+1}$, and all the other vertices have alternating colors. Since each new colored edge is incident to either $v_i$ or $v_{i+1}$, since $c_{i+1} \neq c_i$, and since $i < k$, coloring $c(G_{i+1})$ is a (star, 2)-coloring satisfying condition $Q_4$.

Coloring $c(G_i)$ satisfies $Q_3$:

- If $|f_i| = 0$, we have that $i = k$ and hence $c(G_k)$ is a (star, 2)-coloring of $G = G_k$.
- If $|f_i| = 1$ (that is, $f_i$ contains only $v_{i+1}$; see Figure 5a), we set $c_{i+1} = c_i$. As in the analogous case in which $c(G_i)$ satisfies condition $Q_1$, we can prove that $c(G_{i+1})$ is a (star, 2)-coloring which satisfies condition $Q_2$. 

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• If \( |f_i| \) is even and different from 0 (see Figure 5b), we color the vertices of \( f_i \) with alternating colors in such a way that \( c_{i+1} \neq c_i \). By construction, \( c(G_{i+1}) \) is a (star, 2)-coloring which satisfies condition \( Q_1 \).

• If \( |f_i| \) is odd and different from 1, we again consider the cardinality of \( f_{i+1} \) in order to decide the coloring of \( f_i \). For the four possible classes of values of \( |f_{i+1}| \), the coloring strategy and the condition satisfied by the resulting coloring \( c(G_{i+1}) \) are the same as for the analogous case in which \( c(G_i) \) satisfies \( Q_2 \) and \( |f_i| \) is even.

**Coloring \( c(G_1) \) satisfies \( Q_4 \):** Note that \( |f_i| > 0 \), given that \( i < k \), and \( |f_i| \neq 1 \), by condition \( Q_4 \). Independently of whether \( |f_i| \) is even or odd (see Figure 5c), we color the vertices in \( f_i \) with alternating colors so that \( c_{i+1} \neq c_i \). In this way, the only possible colored edges are incident to \( v_{i-1} \) and to \( v_i \), which are both centers of a star already in \( G_i \), and not to \( v_{i+1} \). Hence, \( c(G_{i+1}) \) is a (star, 2)-coloring satisfying condition \( Q_1 \).

**Coloring \( c(G_1) \) satisfies \( Q_5 \):** Note that \( |f_i| = 1 \), by condition \( Q_5 \) (that is, \( f_i \) only contains \( v_{i+1} \); see Figure 5d). We set \( c_{i+1} \neq c_i \); clearly, \( c(G_{i+1}) \) is a (star, 2)-coloring satisfying condition \( Q_3 \).

Observe that the running time of the algorithm is linear in the number of vertices of \( G \). In fact, at each step \( i = 1, \ldots, k \), the condition \( Q_j \) satisfied by \( c(G_i) \) and the cardinalities of \( f_i \) and \( f_{i+1} \) are known (the cardinality of all the fans can be precomputed in advance), and the coloring strategy to obtain \( c(G_{i+1}) \) and the condition satisfied by this coloring are uniquely determined by these information in constant time.

### 3 NP-completeness for (Planar) Graphs of Bounded Degree

In this section, we study the computational complexity of the (star, 2)-coloring and (star, 3)-coloring problems for (planar) graphs of bounded degree.

**Theorem 3.** It is NP-complete to determine whether a graph admits a (star, 2)-coloring, even in the case where its maximum degree is no more than 5.

**Proof.** The problem clearly belongs to NP; a non-deterministic algorithm only needs to guess a color for each vertex of the graph and then in linear time can trivially check whether the graphs induced by each color-set are forests of stars. To prove that the problem is NP-hard, we employ a reduction from the well-known Not-All-Equal 3-SAT problem or NAESAT for short [19, p.187]. An instance of NAESAT consists of a 3-CNF formula \( \phi \) with variables \( x_1, \ldots, x_n \) and clauses \( C_1, \ldots, C_m \). The task is to find a truth assignment of \( \phi \) so that no clause has all three literals equal in truth value (that is, not all are true). We show how to construct a graph \( G_\phi \) of maximum vertex-degree 5 admitting a (star, 2)-coloring if and only if \( \phi \) is satisfiable. Intuitively, graph \( G_\phi \) reflecting formula \( \phi \) consists of a set of subgraphs serving as variable gadgets that are connected to simple 3-cycles that serve as clause gadgets in an appropriate way; see Figure 6c for an example.

Consider the graph of Figure 6a, which contains two adjacent vertices, denoted by \( u_1 \) and \( u_2 \), and four vertices, denoted by \( v_1, v_2, v_3 \) and \( v_4 \), that form a path, so that each of \( u_1 \) and \( u_2 \) is connected to each of \( v_1, v_2, v_3 \) and \( v_4 \). We claim that in any (star, 2)-coloring of this graph \( u_1 \) and \( u_2 \) have different colors. Assume to the contrary that \( u_1 \) and \( u_2 \) have the same color, say white. Since \( u_1 \) and \( u_2 \) are adjacent, none of \( v_1, v_2, v_3 \) and \( v_4 \) is white. So, \( v_1, \ldots, v_4 \) form a monochromatic component in gray which is of diameter 3; a contradiction. Hence, \( u_1 \) and \( u_2 \) have different colors, say gray and white, respectively. In addition, the colors of \( v_1, v_2, v_3 \) and \( v_4 \) alternate along the path \( v_1 \to v_2 \to v_3 \to v_4 \), as otherwise there
would exist two consecutive vertices \( v_i \) and \( v_{i+1} \), with \( i = 1, 2, 3 \), of the same color, which would create a monochromatic triangle with either \( u_1 \) or \( u_2 \).

For \( k \geq 1 \), we form a chain of length \( k \) that contains \( k \) copies \( G_1, G_2, \ldots, G_k \) of the graph of Figure 6a, connected to each other as follows (see Figure 6b). For \( i = 1, 2, \ldots, k \), let \( u_1^i, u_2^i, v_1^i, v_2^i, v_3^i \) and \( v_4^i \) be the vertices of \( G_i \). Then, for \( i = 1, 2, \ldots, k-1 \) we introduce between \( G_i \) and \( G_{i+1} \) an edge connecting vertices \( v_4^i \) and \( v_4^{i+1} \) (dotted in Figure 6b). This edge ensures that \( v_4^i \) and \( v_4^{i+1} \) are not of the same color, since otherwise we would have a monochromatic path of length four. Hence, the colors of the vertices of the so-called spine-path \( v_1^i \rightarrow v_2^i \rightarrow v_3^i \rightarrow v_4^i \rightarrow \ldots \rightarrow v_1^k \rightarrow v_2^k \rightarrow v_3^k \rightarrow v_4^k \) alternate along this path. In other words, if the odd-positioned vertices of the spine-path are white, then the even-positioned ones will be gray, and vice versa. In addition, all vertices of the spine-path have degree 4 (except for \( v_1^i \) and \( v_4^i \), which have degree 3).

For each variable \( x_i \) of \( \phi \), graph \( G_\phi \) contains a so-called variable-chain \( C_{x_i} \) of length \([\frac{n_i-2}{2}]\), where \( n_i \) is the number of occurrences of \( x_i \) in \( \phi \), \( 1 \leq i \leq n \); see Figure 6c. Let \( O[C_{x_i}] \) and \( E[C_{x_i}] \) be the sets of odd- and even-positioned vertices along the spine-path of \( C_{x_i} \), respectively. For each clause \( C_i = (\lambda_j \lor \lambda_k \lor \lambda_l) \) of \( \phi \), \( 1 \leq i \leq m \), where \( \lambda_j \in \{x_k, \neg x_k\} \), \( \lambda_k \in \{x_l, \neg x_l\}, \lambda_l \in \{x_{l'}, \neg x_{l'}\} \) \( i, k, l, l' \in \{1, \ldots, n\} \), graph \( G_\phi \) contains a 3-cycle of corresponding clause-vertices which, of course, cannot have the same color (clause-gadget; highlighted in gray in Figure 6c). If \( \lambda_j \) is positive (negative), then we connect the clause-vertex corresponding to \( \lambda_j \) in \( G_\phi \) to a vertex of degree less than 5 that belongs to set \( E[C_{x_{j_i}}] \) \( O[C_{x_{j_i}}] \) of chain \( C_{x_i} \). Similarly, we create connections for literals \( \lambda_k \) and \( \lambda_l \); see the edges leaving the triplets for clauses \( C_1 \) and \( C_2 \) in Figure 6c. The length of \( C_{x_i} \), \( 1 \leq i \leq n \), guarantees that all connections are accomplished so that no vertex of \( C_{x_i} \) has degree larger than 5. Thus, \( G_\phi \) is of maximum degree 5. Since \( G_\phi \) is linear in the size of \( \phi \), the construction can be done in \( O(n + m) \) time.

We show that \( G_\phi \) is (star, 2)-colorable if and only if \( \phi \) is satisfiable. Assume first that \( \phi \) is satisfiable. If \( x_i \) is true (false), then we color \( E[C_{x_i}] \) white (gray) and \( O[C_{x_i}] \) gray (white). Hence, \( E[C_{x_i}] \) and \( O[C_{x_i}] \) admit different colors, as desired. Further, if \( x_i \) is true (false), then we color gray (white) all the clause-vertices of \( G_\phi \) that correspond to positive literals of \( x_i \) in \( \phi \) and we color white (gray) those corresponding to negative literals. Thus, a clause-vertex of \( G_\phi \) cannot have the same color as its neighbor at the variable-gadget. Since in the truth assignment of \( \phi \) no clause has all three literals true, no three clause-vertices
belonging to the same clause have the same color.

Suppose that $G_\phi$ is (star, 2)-colorable. By construction, each of $E[C_{x_i}]$ and $O[C_{x_i}]$ is either white or gray, $i = 1, \ldots, n$. If $P[C_{x_i}]$ is white, then we set $x_i = true$; otherwise, we set $x_i = false$. Assume, to the contrary, that there is a clause of $\phi$ whose literals are all true or all false. By construction, the corresponding clause-vertices of $G_\phi$, which form a 3-cycle in $G_\phi$, have the same color, which is a contradiction.

We now turn our attention to planar graphs. Our proof follows the same construction as the one of Theorem 3 but to ensure planarity we replace the crossings with appropriate crossing-gadgets. Also, recall that the construction in Theorem 3 highly depends on the presence of triangles (refer, e.g., to the clause gadgets). In the following theorem, we prove that the (star, 2)-coloring problem remains NP-complete, even in the case of triangle-free planar graphs. Note that in order to avoid triangular faces, we use slightly more complicated variable- and clause-gadgets, which have higher degree but still bounded by a constant.

**Theorem 4.** It is NP-complete to determine whether a triangle-free planar graph admits a (star, 2)-coloring.

**Proof.** Membership in NP can be shown as in the proof of Theorem 3. To prove that the problem in NP-hard, we again employ a reduction from NAESAT. To avoid crossings we will construct a triangle-free planar graph $G_\phi$ (with different variable- and clause-gadgets) similar to the previous construction, so that $G_\phi$ admits a (star, 2)-coloring if and only if $\phi$ is satisfiable.

The clause-gadget is illustrated in Fig. 7a. It consists of a $2 \times 3$ grid (highlighted in gray) and one vertex of degree 2 (denoted by $u$ in Fig. 7a) connected to the top-left and bottom-right vertices of the grid. We claim that the clause-vertices of this gadget (denoted by $u$, $u_{11}$ and $u_{23}$ in Fig. 7a) cannot all have the same color. For a proof by contradiction assume that $u$, $u_{11}$ and $u_{23}$ are all black. Since $u_{12}$, $u_{13}$, $u_{21}$ and $u_{22}$ are adjacent either to $u_{11}$ or to $u_{23}$, none of them is black. Hence, $u_{21} \rightarrow u_{22} \rightarrow u_{12} \rightarrow u_{13}$ is a monochromatic path of length three; a contradiction to the diameter of the coloring.

Fig. 7b illustrates the so-called transmitter-gadget, which consists of three copies of the $2 \times 3$ grid (highlighted in gray), each of which gives rise to a clause-gadget with the degree-2 vertices $u_1$, $u_2$ and $u_3$. It also has two additional vertices (denoted by $s$ and $t$ in Fig. 7b), each of which forms a clause-gadget with each of the three copies of the rectangular grid. We claim that in any (star, 2)-coloring of the transmitter-gadget $s$ and $t$ are of the same colors. Otherwise, a simple observation shows that there is a monochromatic path of length three; a contradiction to the diameter of the coloring. A schematization of the transmitter-gadget is given in Fig. 7b.
The variable-gadget is illustrated in Fig. 7c. We claim that in any (star, 2)-coloring of these gadget vertices $x$ and $y$ must be of different colors. Assume to the contrary that $x$ and $y$ are both white. Then, vertices $x_1$, $x_2$ and $x_3$ must also be white, due to the transmitter-gadgets involved. Hence, $x \rightarrow x_1 \rightarrow y \rightarrow x_3 \rightarrow x_1$ is a white-colored cycle; a contradiction to the diameter of the coloring. A schematization of the variable-gadget is given in Fig. 7c. The corresponding one for the chain is given in Fig. 7d.

Since we proved that the clause-vertices of the clause-gadgets cannot all have the same color and that the variable gadget has two specific vertices of different colors, the rest of the construction is identical to the one of the previous theorem. Note, however, that $G_\phi$ is unlikely to be planar, as required by this theorem. However we can arrange the variable-gadgets and the clause-gadgets so that the only edges that cross are the ones joining the variable-gadgets with the clause-gadgets. Then, we replace every crossing by the crossing-gadget illustrated in Fig. 7e. This particular gadget has the following two properties: (i) its topmost and bottommost vertices must be of the same color (due to the vertical arrangement of the variable-gadgets), which implies that (ii) the leftmost and rightmost vertices must be of the same color as well. Hence, we can replace all potential crossings with the crossing-gadget. Since the number of crossings is quadratic to the number of edges, the size of the construction is still polynomial. Everything else in the construction and in the argument remains the same.

Note that Theorems 3 and 4 have been independently proven by Dorbec et al. [8]. In the following theorem we prove that the (star, 2)-coloring problem remains NP-complete even if one allows one more color and the input graph is either of maximum degree 9 or planar of maximum degree 16. Recall that all planar graphs are 4-colorable.

**Theorem 5.** It is NP-complete to determine whether a graph $G$ admits a (star, 3)-coloring, even in the case where the maximum degree of $G$ is no more than 9 or in the case where $G$ is a planar graph of maximum degree 16.

**Proof.** Membership in NP can be proved similarly to the corresponding one of Theorem 3. To prove that the problem is NP-hard, we employ a reduction from the well-known 3-COLORING problem, which is NP-complete even for planar graphs of maximum vertex-degree 4 [6]. So, let $G$ be an instance of the 3-COLORING problem. To prove the first part of the theorem, we will construct a graph $H$ of maximum vertex-degree 9 admitting a (star, 3)-coloring if and only if $G$ is 3-colorable.

Central in our construction is the complete graph on six vertices $K_6$, which is (star, 3)-colorable; see Figure 8a. We claim that in any (star, 3)-coloring of $K_6$ each vertex is adjacent to exactly one vertex of the same color. For a proof by contradiction, assume that there is a (star, 3)-coloring of $K_6$ in which three vertices, say $u$, $v$ and $w$, have the same color. From the completeness of $K_6$, it follows that $u$, $v$ and $w$ form a monochromatic components of diameter 3, which is a contradiction.

Graph $H$ is obtained from $G$ by attaching a copy of $K_6$ at each vertex $u$ of $G$, and by identifying $u$ with a vertex of $K_6$, which we call attachment-vertex. Hence, $H$ has maximum degree 9. As $H$ is linear in the size of $G$, it can be constructed in linear time.

If $G$ admits a 3-coloring, then $H$ admits a (star, 3)-coloring in which each attachment-vertex in $H$ has the same color as the corresponding vertex of $G$, and the colors of the other vertices are determined based on the color of the attachment-vertices. To prove that a (star, 3)-coloring of $H$ determines a 3-COLORING of $G$, it is enough to prove that any two adjacent attachment-vertices $v$ and $w$ in $H$ have different colors, which clearly holds since both $v$ and $w$ are incident to a vertex of the same color in the corresponding copies of $K_6$ associated with them.
For the second part of the theorem, we attach at each vertex of \( G \) the planar graph of Figure 8b using as attachment its topmost vertex, which is of degree 12 (instead of \( K_6 \) which is not planar). Hence, the constructed graph \( H \) is planar and has degree 16 as desired. Furthermore, it is not difficult to be proved that in any (star, 3)-coloring of the graph of Figure 8b, its attachment-vertex is always incident to (at least one) another vertex of the same color, that is, it has exactly the same property with any vertex of \( K_6 \). Hence, the rest of the proof is analogous to the one of the first part of the theorem.

\[ \square \]

4 Conclusions

In this work, we presented algorithmic and complexity results for the (star, 2)-coloring and the (star, 3)-coloring problems. We proved that all outerpaths are (star, 2)-colorable and we gave a polynomial-time algorithm to determine whether an outerplanar graph is (star, 2)-colorable. For the classes of graphs of bounded degree and planar triangle-free graphs we presented several NP-completeness results. However, there exist several open questions raised by our work.

- In Theorem 3 we proved that it is NP-complete to determine whether a graph of maximum degree 5 is (star, 2)-colorable. So, a reasonable question to ask is whether one can determine in polynomial time whether a graph of maximum degree 4 is (star, 2)-colorable. The question is of relevance even for planar graphs of maximum degree 4. Note that not all planar graphs of maximum degree 4 are (star, 2)-colorable (Figure 8c shows such a counterexample found by extensive case analysis), while all graphs of maximum degree 3 are (edge, 2)-colorable [16].

- Other classes of graphs, besides the outerpaths, that are always (star, 2)-colorable are of interest.

- In Theorem 4 we proved that it is NP-complete to determine whether a graph of maximum degree 9 is (star, 3)-colorable. The corresponding question on the complexity remains open for the classes of graphs of maximum degree 6, 7 and 8. Recall that graphs of maximum degree 4 or 5 are always (star, 3)-colorable.

- One possible way to expand the class of graphs that admit defective colorings, is to allow larger values on the diameter of the graphs induced by the same color class.

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