An arithmetic theorem and its demonstration

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The Theorem which I have taken here to propose and demonstrate I had previously communicated to colleagues, in which it was seen to be fully worthy of attention, especially since its demonstration is hardly obvious and may perhaps have been investigated by many in vain. I stated the theorem in the following way:

If however many distinct numbers \(a, b, c, d\) etc. are given, and from them fractions are formed whose common numerator is unity, and such that the denominator of each is the product of all the differences of the number and each of the remaining numbers, so that the fractions are

\[
\frac{1}{(a - b)(a - c)(a - d) \text{ etc.}}, \quad \frac{1}{(b - a)(b - c)(b - d) \text{ etc.}}, \quad \frac{1}{(c - a)(c - b)(c - d) \text{ etc.}},
\]

then the sum of all these fractions is always equal to 0.

Thus for example for the given numbers 2, 5, 7, 8, the four fractions

\[
\frac{1}{-3 \cdot -5 \cdot -6}, \quad \frac{1}{3 \cdot -2 \cdot -3}, \quad \frac{1}{5 \cdot -2 \cdot -1}, \quad \frac{1}{6 \cdot 3 \cdot 1},
\]

are to be thence formed, which reduce to these

\[
-\frac{1}{3 \cdot 5 \cdot 6}, \quad +\frac{1}{3 \cdot 2 \cdot 3}, \quad -\frac{1}{5 \cdot 2 \cdot 1}, \quad +\frac{1}{6 \cdot 3 \cdot 1},
\]

and on the strength of the theorem

\[
-\frac{1}{90} + \frac{1}{18} - \frac{1}{10} + \frac{1}{18} = 0.
\]
So as not to make complications with negative signs, these fractions can be formed so that, arranging the given terms according to order of magnitude, either increasing or decreasing, for each the differences with it and all the other terms are taken, and taking these for the denominators in fractions with numerator unity, the signs are taken alternately + and −.

For instance, if the given numbers are

\[3, 8, 12, 15, 17, 18,\]

from each the denominators are thus collected:

\[
\begin{array}{l}
3 \cdot 5 \cdot 9 \cdot 12 \cdot 14 \cdot 15 = 113400 \\
8 \cdot 5 \cdot 4 \cdot 7 \cdot 9 \cdot 10 = 12600 \\
12 \cdot 9 \cdot 4 \cdot 3 \cdot 5 \cdot 6 = 3240 \\
15 \cdot 12 \cdot 7 \cdot 3 \cdot 2 \cdot 1 = 1512 \\
17 \cdot 14 \cdot 9 \cdot 5 \cdot 2 \cdot 1 = 1260 \\
18 \cdot 15 \cdot 10 \cdot 6 \cdot 3 \cdot 1 = 2700 \\
\end{array}
\]

and it will be

\[
\frac{1}{113400} - \frac{1}{12600} + \frac{1}{3240} - \frac{1}{1512} + \frac{1}{1260} - \frac{1}{2700} = 0
\]

or multiplying each by 36

\[
\frac{1}{3150} - \frac{1}{350} + \frac{1}{90} - \frac{1}{42} + \frac{1}{35} - \frac{1}{75} = 0,
\]

and reducing these fractions to the single denominator 3150, it is clear by itself that

\[
\frac{1 - 9 + 35 - 75 + 90 - 42}{3150} = 0.
\]

Indeed, in the case in which only two numbers are given the theorem needs no demonstration, since it is transparent that

\[
\frac{1}{a - b} + \frac{1}{b - a} = 0;
\]

but even in the case of three numbers \(a, b, c\) it is now more subtle, for it is not immediately clear that

\[
\frac{1}{(a - b)(a - c)} + \frac{1}{(b - a)(b - c)} + \frac{1}{(c - a)(c - b)} = 0;
\]

and for larger numbers, and even more so in general for any number whatsoever, finding the truth for simpler cases offers scant help.

Indeed, I have extended this theorem more widely, and it can be stated in the following way.
A General Theorem

If however many distinct numbers $a$, $b$, $c$, $d$, $e$, $f$ etc. are given, whose number $= m$, and the following products are formed from the differences of each with the others

$$(a - b)(a - c)(a - d)(a - e)(a - f) \text{ etc.} = A,$$
$$(b - a)(b - c)(b - d)(b - e)(b - f) \text{ etc.} = B,$$
$$(c - a)(c - b)(c - d)(c - e)(c - f) \text{ etc.} = C,$$
$$(d - a)(d - b)(d - c)(d - e)(d - f) \text{ etc.} = D,$$
$$(e - a)(e - b)(e - c)(e - d)(e - f) \text{ etc.} = E$$

each of which is made from $m - 1$ factors, then not only will it be as before

$$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D} + \frac{1}{E} + \text{etc.} = 0$$

but even in this general way

$$\frac{a^n}{A} + \frac{b^n}{B} + \frac{c^n}{C} + \frac{d^n}{D} + \frac{e^n}{E} + \text{etc.} = 0,$$

providing that the exponent $n$ is a positive integral number less than $m - 1$.

Thus in the above example where the given numbers are $3, 8, 12, 15, 17, 18$, not only is it there

$$\frac{1}{113400} - \frac{1}{12600} + \frac{1}{3240} - \frac{1}{1512} + \frac{1}{1260} - \frac{1}{2700} = 0,$$

but one even obtains the truth of the following fractions

$$\frac{3}{113400} - \frac{8}{12600} + \frac{12}{3240} - \frac{15}{1512} + \frac{17}{1260} - \frac{18}{2700} = 0,$$
$$\frac{3^2}{113400} - \frac{8^2}{12600} + \frac{12^2}{3240} - \frac{15^2}{1512} + \frac{17^2}{1260} - \frac{18^2}{2700} = 0,$$
$$\frac{3^3}{113400} - \frac{8^3}{12600} + \frac{12^3}{3240} - \frac{15^3}{1512} + \frac{17^3}{1260} - \frac{18^3}{2700} = 0,$$
$$\frac{3^4}{113400} - \frac{8^4}{12600} + \frac{12^4}{3240} - \frac{15^4}{1512} + \frac{17^4}{1260} - \frac{18^4}{2700} = 0;$$

but one is not permitted to continue this to higher powers, since each denominator is made from five factors.

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3Translator: Euler in fact showed this in his Institutionum calculi integralis volumen secundum, 1769, E366, §1169.
Demonstration of the theorem

I found this theorem from the consideration of the formula

\[
x^n \frac{1}{(x-a)(x-b)(x-c)(x-d) \text{ etc.}},
\]

which, whenever the exponent \( n \) is a positive integral numbers less than the number of factors in the denominator, is such that it can always be resolved into simple fractions as such

\[
\frac{A'}{x-a} + \frac{B'}{x-b} + \frac{C'}{x-c} + \frac{D'}{x-d} + \text{ etc.,}
\]

whose denominators are the very factors of the denominator, and whose numerators are constant quantities, not depending on \( x \), each of which can be defined in the following way. Since the given form is equal to these simple fractions, by multiplying by \( x-a \) we will have

\[
x^n \frac{1}{(x-b)(x-c)(x-d) \text{ etc.}} = A' + \frac{B'(x-a)}{x-b} + \frac{C'(x-a)}{x-c} + \frac{D'(x-a)}{x-d} + \text{ etc.}
\]

This equality holds for any value taken for \( x \), since the letters \( A', B', C', D' \), etc. do not depend on \( x \). Therefore this equation will be true if one takes \( x = a \), whence

\[
\frac{a^n}{(a-b)(a-c)(a-d) \text{ etc.}} = A',
\]

and thus the value of \( A' \) is known. One similarly sees that

\[
B' = \frac{b^n}{(b-a)(b-c)(b-d) \text{ etc.}}, \quad C' = \frac{c^n}{(c-a)(c-b)(c-d) \text{ etc.}},
\]

and so on for the others. Then, moving the simple fractions to the one side,

\[
x^n \frac{1}{(x-a)(x-b)(x-c)(x-d) \text{ etc.}} + \frac{A'}{a-x} + \frac{B'}{b-x} + \frac{C'}{c-x} + \frac{D'}{d-x} + \text{ etc.} = 0,
\]

and in any case we will have, viewing the number \( x \) as the last of the numbers \( a, b, c, d, \ldots, x \),

\[
\frac{a^n}{(a-b)(a-c)(a-d) \cdots (a-x)} + \frac{b^n}{(b-a)(b-c)(b-d) \cdots (b-x)} + \cdots + \frac{x^n}{(x-a)(x-b)(x-c) \cdots (x-v)} = 0,
\]

Translator: cf. Gauss, Carl Friedrich Gauss Werke, Band III, pp. 265–268.

Translator: The denominator here has only \( m-1 \) factors: it is \((x-a)(x-b)(x-c)(x-d) \cdots (x-v)\), where \( v \) is the second last of the numbers. For the moment \( x \) is a variable, but later we will take \( x \) to be the last of the numbers.

Translator: Partial fractions: see Euler’s Introductio in analysin infinitorum, vol. I, §46.
with $\nu$ denoting the second last of the numbers.

This is the demonstration of the proposed theorem, which is thus not quite obvious, so that it seems that this truth should be counted among the common ones, whose rule is easily perceived, unless perhaps another easier demonstration could be found, but the nature of the rule hardly lets one hope for this, because this theorem is not true unless the exponent $n$ is a positive integral number less than the number of factors in each of the denominators.

Then, since taking a greater number for $n$, the sum of these fractions no longer vanishes, from the same source from which we drew this theorem, for each case we will be able to assign the value of the sum, namely by taking the number of factors to be $= m - 1$ and hence the number of all the given numbers $a, b, c, d, \ldots, x$ to be $= m$. If $n = m - 1$, or $n = m$, or $n > m$, the fraction

$$\frac{x^n}{(x-a)(x-b)(x-c)(x-d) \text{ etc.}}$$

used in the demonstration should be seen as improper, and contains as it were an integral part, and the sum of the fractions will be equal to that very part.

Thus in the case in which $n = m - 1$, the integral part is unity, and so too the sum of the fractions $= 1$. Hence in the example treated above, where the signs are changed according to the demonstration, it will be

$$\frac{18^5}{2700} - \frac{17^5}{1260} + \frac{15^5}{1512} - \frac{12^5}{3240} + \frac{8^5}{12600} - \frac{3^5}{113400} = 1.$$ 

But if $n = m$, the integral part arising from the fraction is

$$x + a + b + c + d + \text{etc.},$$

that is, the sum of all the given numbers. Therefore since in the above example the sum of all the given numbers is $= 73$, it will be

$$\frac{18^6}{2700} - \frac{17^6}{1260} + \frac{15^6}{1512} - \frac{12^6}{3240} + \frac{8^6}{12600} - \frac{3^6}{113400} = 73.$$ 

One can easily see from here how the further sums are to be found. Namely, first the sum of all the given numbers $a, b, c, d, \ldots, x$ is taken, which one lets $= P$, then the sum of the products from two, which one lets $= Q$, next the sum of the products from three, which one lets $= R$, likewise from four $= S$, from five $= T$, and so on. Now with this done, one forms the series

$$1 + A + B + C + D + \text{etc.}$$

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7Translator: Then not only would the statement of the proof be easily perceived, but also its proof would be easily perceived?
8Translator: cf. vol. I, §46 of the Introductio. If $\deg M \geq \deg N$ then there is a polynomial part in the partial fraction decomposition of $\frac{M}{N}$. In §38 Euler defines an “improper” fraction of polynomials.
9Translator: Rather one forms the generating function $1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$
such that
\[ \mathcal{A} = P, \quad \mathcal{B} = \mathcal{A}P - Q, \quad \mathcal{C} = \mathcal{B}P - \mathcal{A}Q + R, \]
\[ \mathcal{D} = \mathcal{C}P - \mathcal{B}Q + \mathcal{A}R - S \]
eq etc.
and then

| case | sum of the fractions |
|------|----------------------|
| \( n = m - 1 \) | 1, |
| \( n = m \) | \( \mathcal{A} = P, \) |
| \( n = m + 1 \) | \( \mathcal{B} = P^2 - Q, \) |
| \( n = m + 2 \) | \( \mathcal{C} = P^3 - 2PQ + R, \) |
| \( n = m + 3 \) | \( \mathcal{D} = P^4 - 3P^2Q + 2PR + Q^2 - S, \) |
| \( n = m + 4 \) | \( \mathcal{E} = P^5 - 4P^3Q + 3P^2R + 3PQ^2 - 2PS - 2QR + T \) |
eq etc.

Or, if one puts the sum of the numbers = \( \varPsi, \) the sum of their squares = \( \Omega, \) the sum of their cubes = \( \mathcal{R}, \) the sum of their fourth powers = \( \mathcal{S}, \) of their fifth powers = \( \mathcal{T}, \) etc., it will be such that
\[ \mathcal{A} = \varPsi, \quad \mathcal{B} = \frac{1}{2}\varPsi^2 + \frac{1}{2}\Omega, \quad \mathcal{C} = \frac{1}{6}\varPsi^3 + \frac{1}{2}\varPsi\Omega + \frac{1}{3}\mathcal{R}, \]
\[ \mathcal{D} = \frac{1}{24}\varPsi^4 + \frac{1}{4}\varPsi^2\Omega + \frac{1}{8}\Omega^2 + \frac{1}{3}\varPsi R + \frac{1}{4}\mathcal{S} \]
eq etc.,
which values proceed according to the law
\[ \mathcal{A} = \varPsi, \]
\[ \mathcal{B} = \frac{1}{2}(\varPsi\mathcal{A} + \Omega), \]
\[ \mathcal{C} = \frac{1}{3}(\varPsi\mathcal{B} + \Omega\mathcal{A} + \mathcal{R}), \]
\[ \mathcal{D} = \frac{1}{4}(\varPsi\mathcal{C} + \Omega\mathcal{B} + \Omega\mathcal{A} + \mathcal{S}) \]
eq etc.

With the truth of our theorem established, I judge that it will not be beside the point if I should carefully investigate the nature of the formulas on which the theorem turns. Therefore, if numbers \( a, b, c, d, \) etc. are given, for each of them one searches for what the character will be of the formula \( (a - b)(a - c)(a - d) \) etc., which is produced from the product of the differences of it with all the others. Then let the number of the given numbers be = \( n \)\(^{11}\) and assuming \( z \) to be a variable quantity, I form this product from it
\[ (z - a)(z - b)(z - c)(z - d)(z - e) \] etc.,

\(^{10}\)Translator: The Opera omnia refers to Euler’s E153, Demonstratio gemina theorematis Neutoniani... in which Euler proves Newton’s identities. Newton’s identities relate the coefficients of a polynomial with the sums of the powers of the roots of the polynomial.  

\(^{11}\)Translator: This was denoted earlier by \( m. \)
which by multiplication yields the expansion

\[ z^n - Pz^{n-1} + Qz^{n-2} - Rz^{n-3} + Sz^{n-4} - \text{etc.} \]

Therefore by dividing by \( z - a \) we will have

\[ (z - b)(z - c)(z - d) \text{ etc.} = \frac{z^n - Pz^{n-1} + Qz^{n-2} - Rz^{n-3} + \text{etc.}}{z - a}. \]

If we now put here \( z = a \), the very form \((a - b)(a - c)(a - d) \text{ etc.} \) will arise which I indicated above with the letter \( A \). Then indeed, for the other side, both the numerator and the denominator go to zero, and therefore its value will be

\[ na^{n-1} - (n - 1)Pa^{n-2} + (n - 2)Qa^{n-3} - (n - 3)Ra^{n-4} + \text{etc.}, \]

which, since it is the case that

\[ a^n - Pa^{n-1} + Qa^{n-2} - Ra^{n-3} + Sa^{n-4} - \text{etc.} = 0, \]

\[ \ldots \]

\[ {^{\text{12}}} \text{Translator: The paper stops here.} \]