D1/D5 systems in $\mathcal{N} = 4$ string theories

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Abstract

We propose CFT descriptions of the D1/D5 system in a class of freely acting $\mathbb{Z}_2$ orbifolds/orientifolds of type IIB theory, with sixteen unbroken supercharges. The CFTs describing D1/D5 systems involve $\mathcal{N} = (4, 4)$ or $\mathcal{N} = (4, 0)$ sigma models on $(R^3 \times S^1 \times T^4 \times (T^4)^N/S_N)/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ is diagonal and its precise nature depends on the model. We also discuss D1(D5)-brane states carrying non-trivial Kaluza-Klein charges, which correspond to excitations of two-dimensional CFTs of the type $(R^3 \times S^1 \times T^4)^N/S_N \ltimes \mathbb{Z}_2^N$. The resulting multiplicities for two-charge bound states are shown to agree with the predictions of U-duality. We raise a puzzle concerning the multiplicities of three-charge systems, which is generically present in all vacuum configurations with sixteen unbroken supercharges studied so far, including the more familiar type IIB on $K3$ case: the constraints put on BPS counting formulae by U-duality are apparently in contradiction with any CFT interpretation. We argue that the presence of RR backgrounds appearing in the symmetric product CFT may provide a resolution of this puzzle. Finally, we show that the whole tower of D-instanton corrections to certain “BPS saturated couplings” in the low energy effective actions match with the corresponding one-loop threshold corrections on the dual fundamental string side.

PACS: 11.25.-w, 11.25.Hf, 11.25.Sq

Keywords: D1/D5, U-duality, D-instanton corrections.

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1 Introduction

The D1/D5 system has received much attention in the last few years, especially for its relation with the physics of 5-dimensional Black Holes [1, 2, 3, 4], and, more recently, in the context of AdS/CFT correspondence [5, 6]. The cases which have been most considered in the literature are those of D1/D5 systems in type IIB string theory on $\mathcal{M}$ with $\mathcal{M}$ being $T^4$ or $K3$, where $Q_5$ D5-branes are wrapped on $\mathcal{M}$ and $Q_1$ D1-branes are parallel to the D5-branes and localized on $\mathcal{M}$. In both cases the effective field theory describing the system is expected to flow in the infrared limit to a $\mathcal{N} = (4, 4)$ CFT, whose non-trivial part is a sigma model with target the moduli space of $U(Q_5)$ instantons on $\mathcal{M}$ with instanton number $Q_1$. This space is a smooth resolution of the (singular) symmetric product $\mathcal{M}^N/S_N$ with $N = Q_1Q_5$ and $N = Q_5(Q_1 - Q_5) + 1$ for $T^4$ and $K3$ respectively [1, 7]. At the level of CFT, the resolution of singularities is implemented by turning on certain marginal deformations and therefore, if one is interested in topological quantities like the elliptic genus, one may as well work at a symmetric product point in the moduli space.

In particular, the $T^4$ case has been studied in great detail in [10], where a multiplicity formula for three-charge BPS states ($Q_1, Q_5$ and KK momentum) preserving 1/8 of the bulk supersymmetry has been derived. The formula, valid in the primitive three-charge vector case, was shown to agree with U-duality. In the same paper, the multiplicity formula was found to agree, where it is supposed to, with the counting formula obtained from the dual supergravity description in terms of KK harmonics on $AdS_3 \times S^3 \times T^4$.

A similar detailed analysis for the D1/D5 system in backgrounds with 16 supercharges (in $D < 10$) is still missing \(^2\), although for the case of type IIB theory on $K3$ tests of the proposed symmetric product CFT have been performed in the context of AdS/CFT correspondence [11, 12, 13, 14]. Moreover, in this case, the relation between symmetric products of $K3$ and the moduli space of $U(n)$ instantons on $K3$ is mathematically well understood [16, 15].

Purpose of the present work is to discuss systematically various aspects of this issue in the context of a class of theories with 16 supercharges. These are obtained by orbifolding/orientifolding type IIB theory with freely acting $Z_2$ actions, which involve shifting along some compact direction together with the action of $Z_2$ elements such as the world sheet parity $\Omega$, $(-)^{F_L}$, with $F_L$ the spacetime left-moving

\(^1\)The physical reason for the shift in $Q_1$ in the $K3$ case was first explained in [8].

\(^2\)Not to mention the case of 16 supercharges in $D = 10$, i.e. type I theory, which is beyond the scope of the present work.
fermion number or $I_4$, the reflection of the 4 coordinates of a $T^4$. A prototype of these vacua: type IIB/Ωσ$p$, was introduced in [17] and termed as “type I theory without open strings”.

Compared to the type I case, theories obtained this way have the simplifying feature of avoiding the presence of the open string sector (in the case where the $Z_2$ includes the world-sheet parity operator $Ω$) [17]. Moreover, via the adiabatic argument [18, 19], these actions are expected to commute with $S$-duality. Therefore the U-duality group is still at work, relating in a non-perturbative way various backgrounds. In particular we can relate the D1/D5 system in one theory to fundamental string winding plus KK momentum states in another theory. Also, we obtain non-trivial relations for the three-charge systems in different backgrounds.

We will derive CFTs for the D1/D5 systems in the various cases and test them against the U-duality predictions. We will find agreement for the two-charge states but generically disagreement for the three-charge states. For the latter states, we will point out a similar problem in the case of type IIB theory on $K^3$. We will argue that the presence of non-trivial RR backgrounds in the symmetric product CFT may provide a resolution of this problem.

The paper is organized as follows: in section 2 we will describe the various type IIB orbifold/orientifold backgrounds and the U-duality relations among them that will be relevant to our subsequent analysis. The discussion here will be general for the case where the shift is transverse to the D-branes, but we will partially cover the longitudinal shift case too. We also discuss the role of RR background in the U-duality relations.

In section 3 we will derive the effective field theory for systems of D1-branes, D5-branes and D1/D5-branes and show that they involve symmetric products of $T^4$ and $R^3 \times S^1$ factors, with appropriate $Z_2$ orbifold actions, depending on the background in consideration. For theories which involve orientifolding, the resulting D1/D5 CFT is of type (4,0). In section 4 we will derive the relevant elliptic genus formulae for symmetric products involving both even and odd fields, with respect to the above extra $Z_2$ orbifold actions. In section 5 these results will be used to show that the predictions of the proposed CFTs agree with those of the perturbative string partition functions of the U dual theories, for all two-charge cases, including the D1-D5 case. We also point out a problem concerning the three-charge states (D1-D5-KK), which arises in the models we are considering and also in the more familiar $K^3$ case: there is an apparent clash between U-duality and CFT interpretation of multiplicity formulae.

In section 6 we compute the moduli dependence of low energy couplings involving
the gauge fields arising from KK reduction in various backgrounds. We verify the matching of one-loop expressions on the fundamental string side with D-instanton contributions on the dual side.

In section 7, we will make some conclusions and also comment about the puzzle concerning three-charge states discussed in section 5. In appendix A we present a systematic description of the open string theory living on intersecting D1-D5-branes, in the presence of longitudinal shifts. In appendix B we derive symmetric product partition functions for free fields acted upon by a $Z_2$ orbifold. This complements the more general derivation presented in section 4. In appendix C we include some details of the genus-one modular integral relevant to the computation of low energy couplings of section 6.

2 U-duality chain of type IIB orbifold/orientifolds

2.1 Transverse shift

In this section we construct a series of five-dimensional U-dual models with sixteen unbroken supercharges. We adopt Sen’s fiberwise construction procedure [19] to generate lower dimensional dual pairs from the self-dual (under a subgroup of the full five-dimensional U-duality group that we will still call U) type IIB theory on $T^5$, which is taken to be in the 12345 directions. We will further compactify on an additional $S^1$ of radius $R_6$ in the 6-th direction, to accompany various $Z_2$ orbifold/orientifold actions with a shift of order two along $S^1$. We will restrict ourselves to the case of a geometrical shift by half winding, denoted by $\sigma_{p_i}$ with $i = 1, 6$ according to whether the shift is longitudinal or transverse to the brane system. In the transverse case, which we will consider first, this results in a factor $(-)^{p_6}$ in the untwisted sector lattice sum, $p_6$ being the momentum in the $X_6$ direction. By the adiabatic argument [18], the free nature of these $Z_2$’s makes the orbifold action commuting with S-duality.

We start by defining a U-duality chain that maps into each other the various charges in the perturbative and solitonic spectrum of the toroidal type IIB parent theory. Under these duality transformations, perturbative symmetries of the underlying theory such as $\Omega$ (worldsheet parity operator), $(-)^{F_L}$ (left moving spacetime fermion number) and $I_4$ (reflection in the (2345) plane) are mapped into each other. A prototype of such a duality chain is displayed in table 1.1:
\[
\begin{array}{ccc}
A & \xrightarrow{S} & B & \xrightarrow{T_{345}} & C & \xrightarrow{S} & D \\
NS_{12345} & D_{12345} & D_1 & D_{12345} & F_1 \\
F_1 & D_1 & D_{12345} & F_1 \\
p_1 & p_1 & p_1 & p_1 \\
(-)^{F_L}I_4 & \Omega I_4 & \Omega & (-)^{F_L} \\
I_4 & I_4 & I_4 & I_4 \\
(-)^{F_L} & \Omega & \Omega I_4 & (-)^{F_L}I_4 \\
\end{array}
\]

Table 1.1: D1(D5)-p to fundamental strings

For the time being different columns, labeled by A, B,..., represent equivalent descriptions of type IIB theory on \(T^5\) but in the following they will stand for a triplet of models obtained by orbifolding/orientifolding the toroidal theory by one of the three perturbative symmetries displayed in each column (accompanied by a \(Z_2\) shift \(\sigma_{p_i}\)). Different columns are connected by \(S\) or \(T_{ijk...}\) elements (the indices indicating the direction along which T-duality is performed) of the U-duality group. Winding, momentum, NS-fivebrane and D-brane states are denoted by \(F_i, P_i, NS_{ijklm}, D_{ijk...}\) respectively with the indices specifying the directions along which they are oriented. We will focus on two-charge systems which admit always a U-dual perturbative description in terms of winding-momentum charges.

A bound state of \(N\) D1-strings and \(k\) units of KK momentum \(p_1\) at step C, for example, is mapped through \(S\) (step D) to a fundamental string wrapped \(N\) times on the \(1^{st}\) circle and carrying \(k\) units of momentum. Similarly a D5-p bound state at C is mapped at step A to a fundamental string bound state \(F_1 - p_1\). A D1/D5 bound state, on the other hand, can be mapped again to a fundamental string-momentum bound state through the more involved chain of dualities:
We can now consider the modding out of type IIB theory on $T^4 \times S^1$ by one of the three $\mathbb{Z}_2$'s (let say at step C) generated by $\Omega \sigma_{p_6}$, $I_4 \sigma_{p_6}$ and $\Omega I_4 \sigma_{p_6}$. We will refer to these theories as I, II and III respectively. Accordingly, we will denote the dual descriptions at step H as $I_F$, $II_F$ and $III_F$ respectively. In all the cases the shift along $X_6$ is transverse to the D1-, D5-branes which are wrapped along $X_1, ..., X_5$. Notice that alternative descriptions of these triplets of theories appear at different steps in the two tables 1.1, 1.2.

The fiberwise construction procedure [18, 19] states that a dual pair can be defined (under certain adiabatic hypothesis) by modding out the parent theories by two dual actions, i.e. two symmetry elements in the same line in the U-duality chain above. The inclusion of the shift makes the adiabatic argument applicable. Notice also that the shift $\sigma_{p_6}$ is invariant under all the elements of the U-duality group involved in the above chain.

From table 1.2 we see that D1/D5 bound states in column C are mapped to fundamental string states in column H, where theories $I_F$, $II_F$ and $III_F$ appear. However, we see that exciting KK momentum on the D1/D5 system amounts to excite NS5-brane charges in H.

On the other hand the duality chains above provides also stringent constraints on the three-charge systems (D1-D5-KK) of our three theories:

- by comparing column F with column G we see that D1 and D5 charges are exchanged, with theory $II$ left invariant, while theories $I$ and $III$ are exchanged.
- by comparing column C with column F we see that D1 and $p$ charges are exchanged, with theory $I$ left invariant, while theories $II$ and $III$ are exchanged.
- by comparing column B with column G we see that D5 and $p$ charges are...
exchanged, with theory $III$ left invariant, while theories $I$ and $II$ are exchanged. As we will see, these relations will put severe constraints on the multiplicity formulæ for the three-charge systems and hence on the effective field theory governing them.

2.2 Longitudinal shift

One may ask the question of what happens if the shift is longitudinal to the D-branes, i.e. instead of $\sigma_{p_0}$ we have $\sigma_{p_1}$.

One first notice that $\sigma_{p_1}$ is still invariant under the U-duality transformations involved in the chain 1.1, but now in going from step $D$ to step $E$ in chain 1.2, the half winding shift gets transformed into a half-momentum shift $\sigma_{F_1}$. Consequently the twisted sector of the theory at step $E$ contains half-integer momentum modes localized at fixed points. In going from $E$ to $F$ we expect a non-perturbative phase $\sigma_{D_1}$ for the states carrying D1-brane charge. This cannot be the whole story however. This is clear from a comparison of the perturbative spectrum of states at step $E$ and $F$: in the former case integer (untwisted states) and half-integer (twisted states) momentum modes come with different multiplicities due to the winding shift (see formulas (5.3) below with $F_1 \rightarrow p_1$), while in the dual description the distinction between even and odd modes would not exist, since the above non-perturbative phase leaves invariant the whole perturbative spectrum.

An insight about the correct map can be gained from a careful analysis of the fundamental string partition function at step $E$. Model $II$ at this step is type IIB/(-)$F_L I_4 \sigma_{F_1}$. As we mentioned above, a shift in $F_1$ implies that states in the twisted sector carry half-integer momenta and are localized at fixed points. Under $S$ duality to step $F$, these are mapped to open string states living on the D5-branes ("twisted sector" with respect to $\Omega_{I_4}$), sitting at orientifold 5-planes at 16 fixed points. There are, to begin with, 16 pairs of D5-branes, each pair at a fixed point, giving rise to the gauge group $SO(2)^{16}$. However, due to the presence of half-integer momentum modes $p_1$, we conclude that a $Z_2$ Wilson line along the circle on $X_1$ must be turned on at step $F$, thereby breaking completely $SO(2)^{16}$. If we do a further T-duality along $X_1$ we then have type I’ on $T^5/Z_2$ with 32 4-branes distributed on the 32 fixed points and a completely broken gauge group.

After four T-dualities to step $G$ this, together with the fact that D5-branes sit at sixteen different fixed points of $I_4$, translates into a type I theory on $T^5$ with five Wilson lines turned on to break completely the gauge group. Finally under $S$ duality, we get a perturbative description of the D1/D5 system of model $II$ in
terms of the fundamental heterotic string with a gauge group completely broken by Wilson lines at step H. We will refer to this model as model IV.

We will comment later on the difficulties involved in trying to extend this analysis to the case of longitudinal shift for models I and III.

2.3 $\chi = 1/2$ point

In the following we will be considering the symmetric product CFTs for the D1/D5 systems. It is generally believed that this CFT describes the infrared limit of the D1/D5 system for $Q_5 = 1$ at a point in the moduli space where the R-R scalar $\chi$ and the 4-form $C_{(4)}$ are turned on along the directions transverse to the D1 brane and longitudinal to the D5 brane. Let $V_4$ denote the volume of the $T^4$ along these directions, then this point in the moduli space is given by

$$\chi = \frac{C_{(4)}}{Q_1} = \frac{1}{2}, \quad V_4 = Q_1 \quad (2.1)$$

For non-zero $\chi$ (and $C_{(4)}$) the D1/D5 system becomes a true bound state since the D1 (and D5) brane gets an induced fundamental string charge. The net induced charges for the system of 1 D5 brane and $Q_1$ D1 branes remain zero and therefore it would cost some energy for this system to split. This phenomenon can best be understood by mapping D1/D5 system to the perturbative states by following the chain from step C to step H in the table 1.2. Actually for $\chi = 1/2$, the S-duality in going from step C to D should be replaced by the transformation which takes the complex coupling constant $\tau$ to $\frac{\tau}{\sqrt{2} - 1}$, as pointed out in [20]. This keeps $\chi = \text{Re}(\tau) = 1/2$ invariant. Although one can still follow the chain of dualities, the transformations on the charge vectors are more involved than what is presented in the table. To avoid this complication, for this subsection, we will replace the first three steps in the above chain by the transformation $T_{15}ST_{15}R_{15}$ where $R_{15}$ is the rotation by $\pi/2$ in the 1-5 plane. It is easy to see that this transformation takes the step C to H with the charges transforming as indicated in the table. Moreover, the complex moduli $\tau = \chi + ie^{-\phi}$ and $\tau' = C_{(4)} + iV_4e^{-\phi}$ at step C are mapped to the complex moduli $U$ and $T$ corresponding to the complex structure and the complex Kahler class of the $T^2$ along directions 1 and 5 at step H. The perturbative charge lattice $\Gamma_{(2,2)}$ in the latter are given by the complex left and right moving momenta

$$P_L = \frac{1}{\sqrt{2T_2U_2}}[n_5 + m_5TU + n_1U - m_1T]$$

$$P_R = \frac{1}{\sqrt{2T_2U_2}}[n_5 + m_5TU + n_1U - m_1\bar{T}] \quad (2.2)$$
where $n_i$ are the KK momenta and $m_i$ are the windings along directions 1 and 5. By the duality chain one also sees that for $\chi = C_{(4)} = 0$, $n_1$, $m_1$, $n_5$ and $m_5$ at step $\mathbf{H}$ correspond to the numbers of $D_1$, $D_{12345}$, $F_1$ and $NS_{12345}$ branes respectively. Thus the real parts of $P_L$ and $P_R$ contain the information of the $F_1$ and $NS_{12345}$ charges while the imaginary parts that of $D_1$ and $D_{12345}$ charges. Setting $n_5 = m_5 = 0$ (i.e. setting the sources of $F_1$ and $NS_{12345}$ charges to zero), the real parts then would carry the information of the respective induced charges in the presence of $\chi$ and $C_{(4)}$. For $C_{(4)} = \frac{p}{q}\chi$ with $p$ and $q$ some integers, we see that there are no induced charges of $F_1$ and $NS_{12345}$ branes iff $n_1 = \frac{p}{q}m_1$.

In the usual type IIB on $T^4$ or $K3$, or the model $II$ considered here, $\chi$ and $C_{(4)}$ are moduli fields and therefore one can continuously go from the point $\chi = C_{(4)} = 0$ to any other point. Furthermore, since the subspace of the moduli space where the system is at threshold is of real co-dimension greater than 1, one expects that the quantities such as elliptic genus would not depend on the value of $\chi$. In the models $I$ and $III$, however, the $\chi$ and $C_{(4)}$ fields are projected out, and therefore, the $\chi = 1/2$ and/or $C_{(4)} = 1/2$ point (which fortunately is invariant under $\Omega$) cannot be connected to the trivial point. In fact, in general the models obtained by $\Omega$ projection at $\chi = 1/2$ may be quite different from the ones at $\chi = 0$. To illustrate this, consider the usual type $\Gamma'$ theory on $S^1 \times T^4$ (i.e. $\text{IIB}/\Omega I_4$ where $I_4$ acts on $T^4$ with no shift) at $\chi = 1/2$ and $C_{2345} = 0$. By following the duality chain of table 1.2, at step $\mathbf{H}$, this becomes IIB on $S^1 \times T^4/I_4$ where the effect of $\chi$ is a $Z_2$ mixing of $\Gamma_{(1,1)}$ and $\Gamma_{(4,4)}$ lattices of momenta and windings on $S^1$ and $T^4$. More explicitly, $\chi = 1/2$ and $C_{(4)} = 0$ implies that $Re(U) = 1/2$ and $Re(T) = 0$ in eq(2.2). The $I_4$ action which reflects the real parts of $P_L$ and $P_R$ is still an automorphism of the resulting $\Gamma_{(5,5)}$ lattice. The set of $I_4$ invariant vectors now form a sublattice $\Gamma$ of the self-dual lattice $\Gamma_{(1,1)}$ with $\Gamma_{(1,1)}/\Gamma = Z_2$. By modular transformation from the sector $\text{tr}I_4$ over the untwisted Hilbert space, one finds that in the twisted sector the charge vectors are contained in the dual lattice $\Gamma^*$ of $\Gamma$ and the multiplicities are now given by number of fixed points divided by the square root of the order of $\Gamma^*/\Gamma$ which in the present case gives $16/2 = 8$. In particular, there will be 8 massless 5-dimensional gauge fields coming from the twisted sector instead of the usual 16 (for $\chi = 0$) when there is no $Z_2$

3For $\chi = 1/2$ the equation $n_1 = \frac{p}{q}m_1$ need be true modulo even numbers since we can turn on $n_5$ to set the real parts of $P_L$ and $P_R$ equal to zero. Turning on $n_5$ would mean turning on a source of $F_1$ in the original system. In particular this means that at the $\chi = 1/2$ point with $p/q = Q_1$, the bound state of 1 D5 and $Q_1$ D1 branes has a mass equal to the sum of the masses of $(1D_5, (Q_1 - 2r)D_1, rF_1)$ bound state and $(2rD_1, -rF_1)$ bound state. Here the integer $r$ represents the source of $F_1$ charge. It is not clear to us why the existence of such channels does not introduce singularity in the CFT.
mixing of $\Gamma_{(1,1)}$ and $\Gamma_{(4,4)}$. At the level of type I’ what this means is that out of the 16 orientifold fix planes, 12 come with positive charge while the remaining 4 come with negative charge. As a result, there are only 8 pairs of space filling D5 branes (instead of 16 in the absence of $\chi$). This is not surprising, since by two T-dualities followed by an S-duality, type I’ model is mapped to IIB on $T^2$ modded by $\Omega(-1)^F I_2$ in the presence of a $Z_2$ discrete NS B field. The latter is known to give rise to 3 orientifold planes with positive charge and 1 with negative charge, resulting in a rank 8 gauge group coming from the open string sector [20]. Note that the single D1 brane now is non-BPS, since it carries induced F1 charge. At step H this is evident, since for $(n_1, m_1) = (1, 0)$ the real parts of $P_L$ and $P_R$, which are reflected by $I_4$ (and hence not associated to a conserved charge), are non-vanishing and therefore the corresponding state must be non-BPS. These states would decay to BPS states by emitting pairs of twisted states.

Let us now return to the D1/D5 system for model III, considered here at $\chi = 1/2$. In the present case since $\Omega I_4$ is accompanied by a transverse shift there are no additional gauge fields coming from the twisted sectors in the U-dual theory at step H. The multiplicities of the D1/D5 system can be read off from the orbifold group invariant untwisted states of the latter. Note that while $\chi = 1/2$, $C_{(4)} = 0$ or 1/2 (modulo integers), depending on whether $Q_1$ is even or odd. At step H, this means that while real part of the modulus $U$ is 1/2, the real part of $T$ is 0 or 1/2 in these two cases. As a result, the corresponding charge vector is always in the invariant sublattice $\Gamma$. Therefore the multiplicities of D1/D5 system can be read off from the untwisted sector of the U-dual theory at step H.

For the 3 charge system, however, the U-duality map between the model III at step C and model II at step F which exchanges D1 and KK charges will not give much information. Indeed, assuming that the symmetric product CFT at step C describes the physics at $\chi = C_{(4)}/Q_1 = 1/2$, the U-duality maps it to a system of 1 D5 brane and $Q_1$ units of KK charge in the presence of $B_{15}^{NS} = C_{1234}/Q_1 = 1/2$. The presence of such background field would break the Lorentz invariance of the world sheet which is along 01 direction. Moreover, as we argued above, a single D5 brane is not BPS (for $C_{(4)} = Q_1/2$ with odd $Q_1$), since it has an induced $Q_1/2$ units of $F_1$ charge at step C. At step F this implies that the single D5 brane carries $Q_1/2$ units of $F_5$ charge. Since this charge can be obtained by turning on an electric field along 05 directions in the D5 brane world volume theory, it would appear as the momentum mode of the Wilson line along the 5th direction. The induced charge therefore should mean a $Z_2$ shift in the momentum lattice of the Wilson line. Since the orbifold group reflects this Wilson line, the shifted lattice for $Q_1$ odd will have no invariant vector, implying that the single D5 brane is not
Moreover, since the D1 branes at step C also carry the same induced charges in such a way that, for $Q_1$ (modulo even numbers) D1 branes it cancels the induced charge of D5 brane, we conclude that there should be a shift in the Wilson line lattice which is proportional to the total momentum along direction 1. In a CFT the latter is given by $L_0 - \bar{L}_0$. Thus, there should be a very non-standard coupling between Wilson line lattice vectors and $L_0 - \bar{L}_0$. These observations might give a hint on how to go about modifying the symmetric product CFT to take into account the $B_{15}^{NS} = C_{1234}/Q_1 = 1/2$ background, although in the following we will not attempt to do this.

### 3 Effective World Volume CFT

In this section we will try to obtain the effective fields theories for pure D5-branes, pure D1-branes and D1/D5 system for each of the models described in the previous section that are related by U-duality.

#### 3.1 Transverse Shift

We first discuss the case when the shift is transverse to the brane system. In all the models we are considering we have $Z_2$ orbifolding of type IIB theory compactified on $T^4 \times S^1 \times S^1$ where the $Z_2$ is generated by an element of the form $g \cdot \sigma$ with $g$ being a combination of $\Omega$ and reflection of $T^4$ and $\sigma$ is a shift on the last $S^1$ factor. We are considering the system of D5-and D1-branes where the D5-brane is wrapped on the $T^4$ and the first $S^1$ factor and the D1 brane is wrapped on the first $S^1$ factor. If we have $Q_5$ D5-branes and $Q_1$ D1-branes in the quotient space, then in the covering space there will be two identical sets of $(Q_5, Q_1)$ systems which are placed at $X^6$ and $X^6 + \pi R_6$ with the world volume theory on the two sets being identified via the $Z_2$ action $g$. Let $\Phi_1$ and $\Phi_2$ be the world volume fields on the two systems at $X^6$ and $X^6 + \pi R_6$ respectively, then the identification is given by $\Phi_2 = \hat{g}\Phi_1$, where $\hat{g}$ is the $Z_2$ action induced by $g$ on the world volume fields. This essentially means that the effective world volume theory is described by just one set of fields (say $\Phi_1$) since the other set is not independent. This would just be the field content of a single set of $(Q_5, Q_1)$ system in type IIB theory compactified on $T^4 \times S^1 \times S^1$ with $R_6$ being the radius of the last $S^1$. Where then does one see the effect of $Z_2$ orbifolding of the underlying IIB theory? To understand this, note that among the fields $\Phi_1$ there is one which corresponds to the common center of mass position of D1/D5-branes along the $X^6$ direction. We shall denote this field by $X^6$. As one changes the value of $X^6$, the entire system of the two sets of
branes moves along this direction. In particular when one moves $X^6$ all the way to $X^6 + \pi R_6$ and the remaining fields $\Phi_1$ to $\hat{g}\Phi_1$, then this system is equivalent to the original system (in a description where one uses $\Phi_2$ as the independent field). Thus there is a $Z_2$ gauging on the effective world volume theory defined by the action $\hat{g}\sigma$, with $\sigma$ being the shift on the center of mass world volume field $X^6$.

Now we will apply these general considerations to the cases of pure D5-branes, D1-branes (carrying KK-momenta) and D5/D1 system for the three models I, II and III respectively:

**D5-brane world volume theory**

The low energy effective world volume theory on D5 branes in type IIB theory is just the 6-dimensional $\mathcal{N} = (1,1)$ supersymmetric $U(Q_5)$ gauge theory. Let $X^0, X^1, ..., X^5$ be the directions along the D5-brane world volume, of which the 4 directions $X^2, ..., X^5$ are compactified on a torus. We can now carry out a dimensional reduction so that the fields depend only on $X^0$ and $X^1$. Let us denote by $\mu, \nu = 0, 1, i, j = 2, ..., 5$ and $a, b = 6, ..., 9$. The world volume fields on the D5-branes are $A_\mu, A_i, X^a$ which are in the adjoint representation of $U(Q_5)$ and their fermionic partners $\psi$. Sometimes for brevity of notation we will use $A_M, M = 0, 1, ..., 9$ to denote all the bosonic fields. Let us denote by $g_1, g_2$ and $g_3$ the $Z_2$ actions $\Omega, I_{2345}$ and $\Omega, I_{2345}$ respectively. We start by determining the $\hat{g}_1, \hat{g}_2, \hat{g}_3$ induced actions on worldvolume fields, once one accompanies the $Z_2$ elements above with a geometric shift $X^6 \rightarrow X^6 + \pi R_6$. For the (4,4) model this follows directly from the interpretation of $A_i$ as describing the Wilson lines along $T^4$ and from supersymmetry:

$$\hat{g}_2 : \quad X^6 \rightarrow X^6 + \pi R_6, \quad A_i \rightarrow -A_i, \quad \psi \rightarrow \Gamma_{2345}\psi$$

To understand the action $\hat{g}_1$, let us reconsider the system of D5-branes. We have a total of $2Q_5$ D5-branes in the covering space, where $Q_5$ of them are sitting at the center of mass position $X^6$ and the remaining $Q_5$ at $X^6 + \pi R_6$. Thus in the resulting system $U(2Q_5)$ is broken to $U(Q_5) \times U(Q_5)$. The gauge fields can then be represented in terms of $Q_5 \times Q_5$ blocks:

$$A_M = \begin{pmatrix} A_M & 0 \\ 0 & A'_M \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi & 0 \\ 0 & \psi' \end{pmatrix}$$

---

4We will throughout this paper assume that the radii of this torus are of the order of string scale so that in the low energy effective action one can ignore the KK modes for D5 branes as well as the winding modes for the D1 branes along these directions.
where $A_M$ and $A'_M$ are the $U(Q_5)$ gauge fields on the two sets of branes. These two gauge fields are of course not independent of each other; they should be related by the $Z_2$ action in the underlying string theory. The shift exchanges the two sets of branes and therefore exchanges $A_M$ with $A'_M$, while the $\Omega$ symplectic projection acts on the Chan-Paton indices with the result

$$\mathcal{A}_M = \mp \Omega_5 A'_M \Omega_5, \quad \Psi = -\Gamma^{(7)} \Omega_5 \Psi' \Omega_5$$

with $\Omega_5$ written in terms of $Q_5 \times Q_5$ blocks as

$$\Omega_5 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $\mp$ sign means $-$ for $M = 0, \ldots, 5$ and $+$ for $M = 6, \ldots, 9$. In the above $\Gamma^{(7)} = \Gamma_{012345}$. More explicitly the above projection implies

$$A'_\mu = -A'_\mu, \quad A'_i = -A'_i, \quad X'_a = X'_a, \quad \Psi' = -\Gamma^{(7)} \Psi$$

We can now take the independent set of fields to be $A_M$ and $\psi$. However the $Z_2$ action which takes the values of these fields to that of $A'_M$ and $\psi'$ will give rise to a configuration which is indistinguishable from the original one. Thus the induced $Z_2$ gauging on the set of fields $A_M$ and $\psi$ is given by

$$\hat{g}_1 : A_\mu \rightarrow -A'_\mu, \quad A_i \rightarrow -A'_i, \quad X_a \rightarrow X'_a, \quad \psi \rightarrow -\Gamma^{(7)} \psi$$

Finally since $g_3 = g_1 \cdot g_2$ it follows that the induced action $\hat{g}_3$ is given by:

$$\hat{g}_3 : A_\mu \rightarrow -A'_\mu, \quad A_i \rightarrow +A'_i, \quad X_a \rightarrow X'_a, \quad \psi \rightarrow -\Gamma_{01} \psi$$

In the Coulomb branch the moduli space is given by taking diagonal matrices for $A_i, X_a$ and $\psi$, up to the Weyl group which is the permutation group $S_{Q_5}$ acting on the $Q_5$ eigenvalues. In particular, the diagonal entries in $X_a$ describe the transverse positions of each of the D5-branes. If one moves any one of these branes along the $X_6$ direction by an amount $\pi R_6$ and simultaneously changes the field on that brane by the action of $\hat{g}$, then this system would be indistinguishable from the original one. Thus the conformal field theory describing the Coulomb branch can be written as

$$\mathcal{M}_{\text{coulomb}} = (\mathbb{R}^3 \times S^1 \times T^4)^N / S_N \times Z_2^N$$

where $N = Q_5$, $S^1$ denotes the circle along $X^6$ direction and $T^4$ (dual of the original $T^4$), appearing from the Wilson lines, will be coordinatized by $A^2, \ldots, A^5$. 


The $Z_2^{Q_5}$ orbifold group is generated in models $I$, $II$ and $III$ by (3.6), (3.1) and (3.7) respectively, acting on each of the $Q_5$ copies of $R^3 \times S^1 \times T^4$. Of course this $Z_2^{Q_5}$ action does not commute with the permutation group $S_{Q_5}$ and indeed in (3.8) we have a semidirect product. This may seem a bit puzzling since $S_{Q_5}$ is the remnant of the $U(Q_5)$ gauge symmetry of the system before going to the Coulomb branch. The point is that at the level of $U(Q_5)$ gauge theory, where one ignores the massive modes coming from the strings stretched between the D5-branes at $X^6$ and their images at $X^6 + \pi R_6$, this $Z_2^{Q_5}$ symmetry is broken. However it is easy to see that this should be the symmetry of the theory when one includes these massive states that transform in the $(Q_5, \bar{Q}_5)$ representations of $U(Q_5) \times U(Q_5)$ gauge symmetry. In the infrared limit in the Coulomb branch, when one ignores all the massive off-diagonal modes, the theory has manifest $Z_2^{Q_5}$ symmetry.

Acting on the transverse fields entering in (3.8), the induced $Z_2$ actions (3.6), (3.1) and (3.7) can be written as

$$\hat{g}_1 = (-)^{F_L} I_{2345} \sigma_{p_6}$$

$$\hat{g}_2 = I_{2345} \sigma_{p_6}$$

$$\hat{g}_3 = (-)^{F_L} \sigma_{p_6}$$

(3.9)

where we recognize in $\Gamma_{01}$ the world sheet chirality operator $(-)^{F_L}$ and in $I_4$ and $\Gamma_{2345}$ the reflection operator acting on $T^4$’s bosons and fermions respectively.

Note that for $Q_5 = 1$ the effective theories we have obtained are just the type IIB $Z_2$ orbifolds generated by (3.9) in the static gauge. Indeed from the U-duality map table 1.1, we recognize the corresponding fundamental sides of the D5-KK systems in the string vacua defined by the $\Omega$, $I_4$ and $\Pi I_4$ projections respectively.

Note that $Z_2$ orbifolding breaks the (8,8) supersymmetry of the parent IIB system down to (4,4) for models $I$ and $II$, while it breaks to (8,0) for model $III$. This is to be expected since by T-dualities along 2,3,4 and 5 directions the D5-brane in the model $III$ is mapped to D1-brane in model $I$, which is just a type I-like theory.

**D1-brane world volume theory**

One can follow the above reasonings also for D1-branes. The only difference is that the $\Omega$ projection now gives an extra negative sign also for 2, 3, 4 and 5 directions since, unlike the D5 brane case, they are now transverse. It is easy to see that this just exchanges $\hat{g}_1$ with $\hat{g}_3$ while leaving the $\hat{g}_2$ unchanged. Once again for $Q_1 = 1$ this reproduces the fundamental side (under column D), for these three
cases. Moreover, this is also consistent with T-duality since by four T-dualities along 2,3,4 and 5 directions, the D5 brane is exchanged with D1 brane and model I is exchanged with model III.

- We summarize the various $Z_2$ actions entering in (3.8) for pure D5- ($N = Q_5$) and pure D1- ($N = Q_1$) brane systems in the following table:

| Model | $D1 - KK$ | $D5 - KK$ |
|-------|-----------|-----------|
| I     | $(-)^{F_L}$ | $(-)^{F_L} \cdot I_4$ |
| II    | $I_4$     | $I_4$     |
| III   | $(-)^{F_L} \cdot I_4$ | $(-)^{F_L}$ |

Table 2.1: $D1$-$KK$, $D5$-$KK$ bound states: Induced $Z_2$ actions entering in (3.8)

In the comparison with the fundamental string multiplicities, we will restrict ourselves to two-charge systems and therefore we should focus on the untwisted sectors of (3.8). $Z_2$-twisted states correspond, in each case, to states carrying additional charges corresponding to D1(D5)-branes oriented along the direction $X_6$. In the dual picture, this corresponds to a fundamental string carrying winding-momentum along $X^1$, together with an additional odd winding charge $F_6$. Indeed degeneracies for fundamental string states carrying odd $F_6$ winding charges can be read off from the twisted sector amplitudes in A or D and with a little effort can be seen to match the ones coming from the three-charge bound states. In the following we will however limit ourselves to the two-charge system, and (3.8) will be always understood (for transverse shifts) restricted to the $Z_2$-untwisted sector.

**D1/D5 system**

Now let us add $Q_1$ D-strings to the system of $Q_5$ D5-branes. D1-branes are along the $X^1$ direction while D5-branes are along the $X^1, ..., X^5$ directions. In the covering space again this system would split into two copies of the D1/D5 system sitting at $X^6$ and $X^6 + \pi R_6$. For each of these two sets, the 1+1 dimensional common world volume theory is just a supersymmetric sigma model on the moduli space $\mathcal{M}$ of $Q_1$ instantons of $\mathcal{N} = 4 \ U(Q_5)$ gauge theory on $T^4$, times the center of mass fields corresponding to the common transverse directions $R^3 \times S^1$. This is exactly the model appearing in the type IIB context. Since each of the $Z_2$ actions on the D5-brane world volume gauge fields described above leaves the self-duality equations invariant, it follows that it induces an action $\hat{g}$ on the instanton moduli space $\mathcal{M}$. Following the logic described above, it then follows that the effective
world volume theory is the $Z_2$ gauging of the theory in the IIB case where the $Z_2$ is generated by $\hat{g} \cdot \sigma$.

In the infrared limit the (4,4) supersymmetric sigma model on $\mathcal{M} \times R^4$ would flow to a (4,4) CFT. It is conjectured that this CFT is a symmetric product space $R^4 \times T^4 \times (T^4)^N / S_N$. There has been much critical discussion of this conjecture in the literature [9, 23, 24]. As mentioned in the previous section, it is generally believed that this is true for the $Q_5 = 1$ case at the point $\chi = C_{2345}/Q_1 = 1/2$ and $V_4 = Q_1$ in the moduli space, where $\chi$ and $C_{2345}$ are the RR 0-form and 4-form fields and $V_4$ is the volume of the $T^4$ along 2345 directions. For other values of $Q_5$ with $Q_1$ relatively coprime, this CFT with $N = Q_1 \cdot Q_5$ perhaps describes the system at some point in the moduli space of the IIB theory on $T^4$, which is related to the point for $Q_5 = 1$ case by a U-duality that maps the $(Q_5, Q_1)$ system to the $(1, Q_1, Q_5)$ system [24]. Note that the point $\chi = C_{2345}/Q_1 = 1/2$, at which the theory is described by the symmetric product CFT, is invariant under the action of $\Omega$ and $I_4$. In model II, $\chi$ and $C_{2345}$ are moduli fields and therefore one can continuously go from this point to the point where these fields are switched off, and one expects that the quantities such as elliptic genus will not change in this process. However, for models I and III, these moduli fields are projected out and as a result these models are frozen at the $\chi = C_{2345}/V_4 = 1/2$ point. Therefore, the elliptic genus computed at this point may not be the same as the one at the trivial point. We will comment on this in the conclusions, but for the rest of the paper we will be working at the point in the moduli space where the symmetric product CFT is the correct description.

For $Q_5 = 1$ the various factors appearing in the CFT have a clear interpretation: the D-flatness condition sets all the bifundamental fields coming from 1-5 open string states to zero, leaving behind only the Cartan directions of the 1-1 $U(Q_1)$ adjoint states. The latter have the interpretation of the positions of the D-strings inside $T^4$. Thus the factor $(T^4)^Q_1 / S_{Q_1}$ represents the positions of the $Q_1$ instantons, while the center of mass factors $R^4$ and $T^4$ represent the transverse position of the D5 brane and its $U(1)$ Wilson lines on the $T^4$ (so more precisely this should be the dual torus). In our case, of course since the transverse direction along $X^6$ is compactified on a circle, $R^4$ should be replaced by $R^3 \times S^1$.

With the physical interpretation of the various fields appearing in the CFT being clear, we are now in a position to deduce the induced $Z_2$ action on the instanton

\footnotetext{5}{$V_4$ is set at $Q_1/Q_5$ in order to minimize the mass of the bound state. This amounts to setting the asymptotic value of the $V_4$ equal to its fixed value at the horizon of the D1/D5 system. Note that precisely for this choice the supergravity solution admits a constant $\chi$ and $C_{2345}$ [25].}
moduli space for each of the three models. Let us denote by \( A_i \) for \( i = 2, \ldots, 5 \) the four \( U(1) \) Wilson lines of the D5-brane gauge field and by \( X_i^{(\ell)} \) for \( \ell = 1, \ldots, Q_1 \) the positions of the \( Q_1 \) instantons on \( T^4 \). Finally we denote by \( X_a \) for \( a = 6, \ldots, 9 \) the coordinates of the center of mass transverse position \( S^1 \times R^3 \). The little group \( SO(4) \equiv SU(2)_A \times SU(2)_Y \) acts on the tangent space of \( S^1 \times R^3 \). In the type IIB theory the resulting CFT has \((4,4)\) supersymmetry. The left and right moving supercharges come with definite chiralities with respect to the little group \( SO(4) \). Specifically the left moving supercharges are two \( SU(2)_A \) doublets while the right moving ones are two \( SU(2)_Y \) doublets. The supermultiplets then are

\[
\begin{array}{|c|c|c|}
\hline
\text{Bosons} & \text{Left-moving Fermions} & \text{Right-moving Fermions} \\
\hline
X_a & \psi_A & \bar{\psi}_A \\
A_i & \psi_Y & \bar{\psi}_A \\
X_i^{(\ell)} & \psi_Y^{(\ell)} & \bar{\psi}_A^{(\ell)} \\
\hline
\end{array}
\]

\( \hat{g}_1 \) leaves \( X_a \) and \( X_i^{(\ell)} \) invariant, since these are respectively the center of mass position and the positions of D1-branes in \( T^4 \). It however takes \( A_i \) to \(-A_i\), since \( \Omega \) projects out the \( U(1) \) gauge field. The easiest way to understand its action on the fermions is to use the fact that in this theory the D1/D5 system should preserve \((4,0)\) supersymmetry. As a result, \( \psi_A \) and \( \psi_Y^{(\ell)} \) should remain unchanged while \( \psi_Y \) must pick up a minus sign. On the right-moving fermions the action is exactly the reverse of this, i.e. \( \bar{\psi}_Y \) and \( \bar{\psi}_A^{(\ell)} \) should pick up a minus sign while \( \bar{\psi}_A \) should remain unchanged. This is because \( X_a \) and \( A_i \) are D5-brane fields and as one can see from table 2.1 or eqs. (3.9) \( \hat{g}_1 \) acts on the fermions by \( \Gamma^{(7)} = (-)^{F_L} I_4 \). Thus \( SU(2)_A \) and \( SU(2)_Y \) doublets must appear with opposite signs. On the other hand, \( X_i^{(\ell)} \) are the D1-brane fields and on the fermions \( \hat{g}_1 \) acts as \( \Gamma_{01} \). Thus the left and right moving fermions appear with opposite signs. To summarize \( \hat{g}_1 \) maps

\[
\begin{align*}
(X_a, X_i^{(\ell)}, \psi_A, \psi_Y^{(\ell)}, \bar{\psi}_A) & \rightarrow (X_a + \delta_{a6} \pi R_6, X_i^{(\ell)}, \psi_A, \psi_Y^{(\ell)}, \bar{\psi}_A) \\
(A_i, \psi_Y, \bar{\psi}_A^{(\ell)}, \bar{\psi}_Y) & \rightarrow -(A_i, \psi_Y, \bar{\psi}_A^{(\ell)}, \bar{\psi}_Y)
\end{align*}
\] (3.10)

In model II the induced action is more straightforward to see. In this case the D1/D5 system preserves the full \((4,4)\) supersymmetry of the parent type IIB system. Thus it is sufficient to specify the \( \hat{g}_2 \) action on the bosonic fields. Since \( g_2 \) is the inversion \( I_{2345} \), it follows that it gives negative sign to \( A_i \) and \( X_i^{(\ell)} \) and all their fermionic partners. Finally, \( \hat{g}_3 \) is just obtained as product of \( \hat{g}_1 \cdot \hat{g}_2 \). We conclude then that D1/D5 gauge theories flow in the infrared to orbifolds CFT of the type

\[
\mathcal{M}_{\text{higgs}} = (R^3 \times S^1 \times T^4 \times (T^4)^N / S_N) / Z_2
\] (3.11)
with \( N = Q_1 Q_5 = Q_1 \) and \( Z_2 \) acting diagonally in the way specified in the following table:

| Model | D1-D5 |
|-------|-------|
| I     | \((-)^{FL} I_4^{c.m.}\) |
| II    | \(I_4^{c.m.} I_4^{sp}\) |
| III   | \((-)^{FL} I_4^{sp}\) |

*Table 2.2: D1/D5 systems: \( Z_2 \) actions entering in (3.11)*

We denote by \((-)^{FL}\) the total left moving fermionic number, \(I_4^{c.m.}\) the reflection of the fields corresponding to the first \( T^4 \) factor in (3.11) and by \(I_4^{sp}\) the diagonal \( Z_2 \) reflecting bosonic and fermionic fields in the symmetric product.

Let us finally observe that, both in models I and III the \( T^4 \) components of the NS-NS B-field, as well as the \( T^4 \) components of the R-R 4-form and the axion, are projected out from the massless spectrum, due to the \( \Omega \)-projection. But in the \( T^4 \) case the self-dual part of the above B-field and a combination of four-form and axion are the moduli that, when switched on, render the D1/D5 system a bound state below threshold. They in turn correspond, in the effective gauge theory to Fayet-Iliopoulos D-terms and theta-term for the \( U(1) \) gauge field. The effective gauge theories derived in sub-section 3.1 agree with the spectrum above, in the sense that in models I and III, due to the \( \Omega \) action on Chan-Paton factors, there is no room for Fayet-Iliopoulos D-terms in the potential for hypermultiplets and/or theta-term for the \( U(1) \) gauge field. (the \( U(1) \) generators are traceless).

On the other hand, in model II the D1/D5 gauge theory agrees with the fact that the above moduli are actually there.

### 3.2 Longitudinal Shift

We now consider the shift along the common world volume direction \( X_1 \). The general discussion in this case is very similar to the transverse shift case. If \( \Phi \) denotes the set of world volume fields in the type IIB theory, and \( \hat{g} \) the induced \( Z_2 \) actions, then the worldvolume fields satisfy the condition:

\[
\Phi(\pi R_1, X_0) = \hat{g} \Phi(X_1, X_0)
\]

Taking the interval of \( X_1 \) to be \( \pi R_1 \), what this condition says is that the fields on which \( \hat{g} \) acts as \(-1\) are anti-periodic along the \( X_1 \) direction and the ones on which the \( \hat{g} \) action is \(+1\) are periodic. This means that the CFTs are again given by (3.8), (3.11) with \( Z_2 \) actions specified by tables 2.1 and 2.2, but with the twists
now oriented in the $\sigma$-direction, i.e. the $Z_2$ twisted sectors of (3.8), (3.11). Notice that, in contrast with the transverse case, this represents still a two-charge system.

Another way to see that the above proposal must be correct, is to start from the effective field theories in the transverse case and consider the threshold corrections due to the single (i.e. minimal unit) D5- or D1-instanton, obtained by wrapping the time directions of these systems on the $X_6$ circle by a length $\pi R_6$ (i.e. half winding). The resulting amplitude is just the one loop amplitude in the orbifold sector given by the insertion of the operator $\hat{g}$. This is because in the path-integral formulation, it is in this sector that there is a half winding, corresponding to the shift along the time direction. On the other hand, by a modular transformation, we can exchange $t$ and $\sigma$, and the resulting path integral should be interpreted as that of the field theory living in the single D5-aligned along directions 23456 or a single D1-brane along direction 6, with longitudinal shift along $X_6$. This field theory is just the twisted sector ($\hat{g}$ twist along $\sigma$ direction) of the $Z_2$ orbifold of $R^4 \times T^4$. Putting $N$ copies of these together should, in the Coulomb branch, reproduce the conjecture of the previous paragraph. In the case of the D1/D5 system the same argument applies in a more straightforward way since there is only a single copy of the center of mass $R^4$. We conclude then that in the longitudinal shift case the CFT descriptions are given again in terms of (3.8), (3.11) but now are the $Z_2$-twisted sectors which are relevant to our discussion.

There is however an apparent puzzle we would like to discuss here. Consider D1-brane in model $I$. $\Omega\sigma_p$ projects the $U(N)$ group to $SO(N)$ so that $SO(N)$ gauge fields are periodic while the remaining ones, that are in the symmetric tensor representation of $SO(N)$, are anti-periodic. Now let us put D5-branes. The Gimon-Polchinski consistency condition [26] would at first sight imply that the $\Omega$-projection on D5-brane Chan-Paton factors should be chosen to be symplectic. This would mean a doubling phenomenon, i.e. one would need an even number of type IIB D5-branes. If this would be the case, then the whole towers of states with odd windings at step $H$ would be missing in the dual description.

There are various ways of seeing the Gimon-Polchinski consistency condition. One of them involves a consideration of the Dirac charge quantization condition. This was one way to see that, in the usual type I theory, if there is a single D-string, then D5-branes should be paired (with respect to type IIB counting) because of the fact that in the 1-5 sector there is a factor of $1/2$ due to the $\Omega$ projection. Indeed, this additional factor rescales the D1, D5 charges by $1/\sqrt{2}$ with respect to their type IIB cousins, and the Dirac quantization condition $Q_e Q_m = 2\pi$ requires therefore the claimed pairing of D5-branes. Now let us discuss our case. Consider
the D5-brane to be longitudinal to the direction $X_1$ along which the shift acts. Then the Poincare' dual B-field, which enters in Dirac quantization condition, would refer to the D-string which is transverse to $X_1$. But in this case we have actually 2 D-strings (one sitting at say $X_1$ and the other at $X_1 + \pi R_1$). So quantization condition is satisfied just with one D5-brane wrapped on the circle with circumference $2\pi R_1$. Similarly, if one takes the D-string longitudinal to $X_1$, then its Poincare' dual involves a D5-brane that is transverse to $X_1$, in which case again we have 2 D5-branes, showing that it suffices to have just one D-string.

The other way to see the appearance of this condition is to consider the action of $\Omega^2$ on the open string states. In the usual Type I theory, Gimon-Polchinski showed [26] that on the 1-5 open strings $\Omega^2$ picks up an extra minus sign, due to the fact that these states involve a twist field along the four directions longitudinal to D5-brane and transverse to the D1-brane. Including the action of $\Omega$ on the Chan-Paton indices we have

$$\Omega^2 : |\alpha, \mu \rangle \rightarrow - (\gamma^t \gamma^{-1})_{\alpha \beta} |\beta, \nu \rangle (\gamma'^t \gamma'^{-1})_{\nu \mu}$$

(3.13)

where $\alpha, \beta$ and $\mu, \nu$ are the Chan-Paton indices on D1- and D5-branes respectively and $\gamma$ and $\gamma'$ are $\Omega$ actions on the D1- and D5-brane Chan-Paton indices. Due to the extra minus sign above, one concludes that if $\gamma$ is symmetric then $\gamma'$ must be anti-symmetric and vice versa. So, if one system is projected onto the Orthogonal Group then the other must be projected onto the Symplectic Group.

In our case however $\Omega$ is accompanied by the shift $\sigma_1$. Thus $g_1^2 = \Omega^2 \sigma_1^2$ and we can take both systems to have Orthogonal projections provided $\sigma_1^2 = -1$ on the 1-5 string states. This means that 1-5 string states will carry half-integer momenta along the $X_1$ circle. Since 1-5 states are bi-fundamentals this can be thought of as turning on of a $Z_2$ Wilson line in one of the systems along the circle.

In Appendix A we construct the open string theory living on D1/D5 system in the presence of a longitudinal shift. We follow the open string descendant techniques developed in [27]. The consistency condition translates in this formalism in the requirement that Klein bottle, Annulus and Moebius amplitudes in the transverse channel admit a sensible interpretation as closed string exchanges between boundaries (D-branes) and crosscaps (orientifold planes). Once again, one finds that the Orthogonal assignements for both D1 and D5 Chan-Paton indices are allowed provided that 1-5 string states carry half-integer momenta along the $X_1$ circle. This is the same condition that we found in the previous paragraph.

This result might seem a little surprising, since the D1/D5 system can be thought of as Y-M instantons in the D5-brane world volume. It is known that in the context
of the ADHM construction, $SO(N)$ instantons have $Sp(k)$ symmetry where $k$ is the instanton number, and vice versa. This is indeed the result for the standard $\Omega$ projection. In our case however $\Omega$ is accompanied by a shift. What this means is that we are looking for $U(N)$ instantons in the 4 directions spanned by $X_2, \ldots, X_5$, and the moduli of the instantons are slowly varying functions of $X_1$ in such a way that

$$A_\mu(X_1 + \pi R_1) = -g^{-1}A_\mu^i(X_1)g \quad \mu = 2, 3, 4, 5$$

(3.14)

where $g \in U(N)$ is a slowly varying function of $X_1$. The periodicity condition (up to a possible Wilson line $h \in U(N)$) as $X_1 \rightarrow X_1 + 2\pi R_1$ implies

$$g^* \cdot g = h$$

(3.15)

For orthogonal and symplectic projections $h = +1$ and $h = -1$ respectively and in these two cases we can take $g$ to be $+1$ and the symplectic matrix $J$ respectively. In the latter case of course $N$ must be even.

Let us now see how this condition is translated on the ADHM data (here we will take the 4-dimensional space where the instanton is sitting to be $R^4$ since the discussion of the doubling phenomenon should not depend on whether the space is $T^4$ or $R^4$). The ADHM data consists of a $(N + 2k) \times 2k$ matrix $\Delta$ defined as

$$\Delta_{\lambda,i\hat{\alpha}} = a_{\lambda,i\hat{\alpha}} + b_{\lambda,i}x_{a\hat{\alpha}}$$

(3.16)

where $x_{a\hat{\alpha}} = x_\mu \sigma_{a\hat{\alpha}}^\mu$ and the indices $\lambda = u + j\beta$ with $u$ running over $N$ indices and $i, j$ run over $k$ indices. $\Delta$ satisfies the quadratic constraint

$$\bar{\Delta}_{\lambda,i\hat{\alpha}} \Delta_{\lambda,j}\hat{\beta} = \delta_{\lambda,j} f_{i\hat{\beta}}^{-1}$$

(3.17)

where $f$ is a $k \times k$ hermitian matrix.

The self-dual gauge fields are then given by

$$A_\mu = \bar{U} \partial_\mu U$$

(3.18)

where $U$ is an $(N + 2k) \times N$ matrix which satisfies the equations

$$\bar{U}U = 1, \quad \bar{\Delta}U = \bar{U} \Delta = 0$$

(3.19)

From the above it follows that the projection operator $U\bar{U} = 1 - \Delta f \bar{\Delta}$.

The symmetries of these equations are

$$U \rightarrow B \cdot U \cdot g, \quad \Delta \rightarrow B \cdot \Delta \cdot (C \times 1_{2 \times 2})$$

(3.20)
where $g$ is a local $U(N)$ transformation while $B$ and $C$ are independent of $X_\mu$ and are in $U(N+2k)$ and $GL(k)$ respectively. Using this freedom in defining the data, we can set $b_{\alpha,i}^\mu = 0$ and $b_{\beta,i}^\mu = \delta_{ij}\delta_{\beta}^\alpha$. Then the instanton moduli are contained in the matrix $a$ which, as follows from eq.(3.17), satisfies the constraint that $a_{\mu}^\alpha$ is in the adjoint representation of $U(k)$, where $a^\mu$ is defined via $a_{\mu}^\alpha = a_{\mu}^\alpha a^\mu a^\alpha$. There are also the $3k^2$ D-term constraints that are quadratic in $a$, that follows from (3.17), but they will not concern us here. With this canonical choice for $\Delta$, the global symmetry group $U(N+2k) \times GL(k)$ reduces to $U(N) \times U(k)$. Explicitly, this corresponds to taking $C$ in eq. (3.20) to be in $U(k)$ and

$$B = \begin{pmatrix} D \\ C^{-1} \times 1_{2 \times 2} \end{pmatrix}$$

(3.21)

with $D \in U(N)$. Note that the moduli $a_{ij}^\mu$ transforming in the adjoint representation of $U(k)$ are part of the 1-1 string states that define the position of the D1-brane inside the D5-brane, while $a_{u,i} = w_{u,i}$ are the 1-5 string states that are bi-fundamentals of $U(N)$ and $U(k)$. The spinorial index $\dot{\alpha}$ just refers to the fact that the bosonic 1-5 string states are spinors of the $SO(4)$ acting on $X^2, ...X^5$.

However at this point we still have two $U(N)$ actions: the local $U(N)$ action on $U$ on the right and the global $U(N)$ action on the left. The instanton gauge field $A_\mu$, which lives on the D5-brane sees only the local action, while the ADHM data $w$ see only the global action. In order to relate this system to the D1/D5 system, we must identify these two $U(N)$ actions. The basic point is to choose a particular gauge for the instanton solution that fixes the local $U(N)$ symmetry. We will choose the singular gauge [28] which is described as follows. Writing $U$ as an $N \times N$ block $V$ and $2k \times N$ block $U'$, the condition $U \bar{U} = 1 - \Delta f \Delta$ implies $V \bar{V} = 1 - w \bar{w}$. Given a solution for $V, V \cdot g$ will also solve this equation, for $g$ being a local $U(N)$ transformation. Choosing the singular gauge amounts to taking $V$ to be one of the $2^N$ matrix square roots of the right-hand side $(1 - w \bar{w})^{1/2}$. With this choice, it is clear that a transformation $w \rightarrow Dw$ implies $V \rightarrow DVD^{-1}$ and the two $U(N)$’s are identified.

Let us now return to the $Z_2$ projection condition (3.14). With the two $U(N)$ actions identified, this condition on the ADHM data becomes:

$$w(X_1 + \pi R_1) = gw^*(X_1)(C \times \sigma_2)$$

$$a_\mu(X_1 + \pi R_1) = C^{-1}a_\mu^*(X_1)C$$

(3.22)

where $C \in U(k)$. Here we have used the fact that $x = \sigma_2 x^* \sigma_2$. Repeating this equation twice we find:

$$w(X_1 + 2\pi R_1) = -(g \cdot g^*)w(X_1)(C^*C), \quad a_\mu(X_1 + 2\pi R_1) = (C^*C)^{-1}a_\mu(X_1)(C^*C).$$
As stated earlier, orthogonal and symplectic projections of $U(N)$ correspond to $g = 1$ and $g = J$ respectively. Similarly, the orthogonal and symplectic projections of $U(k)$ are given by choosing $C = 1$ and $C = J$ respectively. The above equations show that if both the groups are projected to Orthogonal or Symplectic Groups, then $w$, which represents 1-5 string states, are anti-periodic as $X_1 \to X_1 + 2\pi R_1$, while if they are projected in the opposite ways the $w$ are periodic. This is exactly the condition we found using Gimon-Polchinski consistency condition.

It is instructive to consider $k = 1$, since in this case we can explicitly solve the ADHM constraints. The result is\[28]:

$$w_{\alpha\dot{\alpha}} = \rho G\begin{pmatrix} 0_{[N-2] \times [2]} \\ 1_{[2] \times [2]} \end{pmatrix}, \quad G \in \frac{SU(N)}{SU(N-2)} \tag{3.23}$$

where $\rho$ is the scale of the instanton. One can then solve for $U$ in the singular gauge satisfying equation (3.19) and obtain the gauge field as:

$$A_\mu = G\begin{pmatrix} 0 & 0 \\ 0 & A^{SU(2)}_\mu \end{pmatrix} G^{-1} \tag{3.24}$$

where $A^{SU(2)}_\mu$ is the standard $SU(2)$ single instanton gauge field in the singular gauge with scale $\rho$ and position $a_\mu$:

$$A^{SU(2)}_\mu = \frac{\rho^2 \eta_{\mu\nu}(X-a)^\nu \sigma^\nu}{(X-a)^2((X-a)^2 + \rho^2)} \tag{3.25}$$

The moduli of the instantons are the position $a_\mu$, the scale $\rho$ and the gauge orientations contained in $G$. These moduli are now slowly varying functions of $X_1$ in such a way that the $Z_2$ projection condition (3.14) is satisfied. It is easy to see that this condition implies that

$$a_\mu(X_1 + \pi R_1) = a_\mu(X_1), \quad \rho(X_1 + \pi R_1) = \rho(X_1), \quad G(X_1 + \pi R_1) = gG^*(X_1)\sigma_2 \tag{3.26}$$

Repeating this twice and using eq. (3.23), we find the condition $w(X_1 + 2\pi R_1) = -(gg^*)w(X_1)$. Taking the orthogonal projection for $U(N)$, namely $h = 1$, we recognize from above that the ADHM data $w$, which represents the 1-5 string states are anti-periodic as $X_1 \to X_1 + 2\pi R_1$. Note that the above equations also show that the $Z_2$ projection acts trivially on the instanton position $a_\mu$ and scale $\rho$ as expected.

The above result has also been obtained in [29], where the possibility to have orthogonal projections on $p$ and $p + 4$ branes simultaneously has been observed.
Indeed by making T-duality along the common world volume direction, we end up with IIA on $S^1$ modded by $\Omega$ times the reflection on $S^1$. The orientifold planes at the two fixed points come with opposite sign $O^+_S$ and $O^-_S$. The D4-branes and D0-branes come with symplectic and orthogonal projections respectively at $O^-_S$ and vice versa at $O^+_S$. Thus taking D4-branes at one fixed point and D0-branes at other fixed point we can obtain orthogonal groups in both the systems. The fact that in order for this to happen, the D4-brane and D0-brane systems must sit at the two different fixed points implies that in the T-dual description there is a relative $Z_2$ Wilson line between the D5-branes and D1-branes which makes the bifundamental fields anti-periodic.

4 Partition functions and symmetric product spaces

In this section we derive a general formula for the character valued string partition function of a symmetric product CFT, involving fields carrying non-trivial boundary conditions. More precisely, we will consider the orbifold CFT defined as the symmetric product $S_N \mathcal{H} \equiv \mathcal{H}^N / S_N$, with $\mathcal{H}$ describing the Hilbert space of closed string excitations, with periodic or antiperiodic boundary conditions along $\sigma$ and $\tau$ directions of the worldsheet torus, which we will denote by $(\sigma_1, \sigma_2)$. We label the boundary conditions by the characteristics $[g_\phi, h_\phi]$, with $g_\phi, h_\phi$ taking values \{0, $\frac{1}{2}$\}, describing the holonomies of a given field $\Phi$ around the two cycles:

$$\Phi(\sigma_1 + 1, \sigma_2) = e^{2\pi i g_\phi} \Phi(\sigma_1, \sigma_2) \quad \Phi(\sigma_1, \sigma_2 + 1) = e^{2\pi i h_\phi} \Phi(\sigma_1, \sigma_2) \quad (4.1)$$

Our results generalize the familiar symmetric product formula [31] to the case where some of the fields carry boundary conditions different from the periodic ones ($g_\phi = h_\phi = 0$ in our notation) and can be associated to sectors of a diagonal $Z_2$ orbifold action on the more familiar symmetric product spaces. As we have seen in the previous section, such CFTs naturally arise in the study of D-brane bound state physics for type IIB orbifolds/orientifolds involving $Z_2$-shifts in the winding-momentum modes.

The derivation of the partition function follows, with slight modifications, the procedure of [31] (see also [22]). We start by specifying the character valued string partition function for single copy of the Hilbert space $\mathcal{H}$

$$Z\left[\frac{g}{h}\right](\mathcal{H}|q, \bar{q}, y) = \text{Tr}_\mathcal{H} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} y_0^\Delta \bar{y}_0^\Delta$$

$$= \sum C\left[\frac{g}{h}\right](\Delta, \bar{\Delta}, \ell, \bar{\ell}) q^\ell \bar{q}^\bar{\ell} y^\ell \bar{y}^\bar{\ell} \quad (4.2)$$
with the sum running over $\Delta, \bar{\Delta}, \ell, \bar{\ell}$. The supertrace runs over string states in the Hilbert space $\mathcal{H}$ and boundary conditions for the various sigma model fields are compactly denoted by $[j]$, $q = e^{2\pi i r}$ describes the genus-one worldsheet modulus, $L_0, \bar{L}_0$ are the Virasoro generators. $J_0, \bar{J}_0$ are the Cartan generators of a $SU(2)_L \times SU(2)_R$ current algebra, present in all our models, to which $y$ and $\bar{y}$ couple respectively.

Our next task is to evaluate the supertrace (4.2) for a Hilbert space constructed by considering $N$ copies of the Hilbert space $\mathcal{H}$ modded out by the permutation group $S_N$. This can be done following the standard non-abelian orbifold techniques developed in [30]. The string partition function is written as the double sum

$$Z_{gh} = \sum_{gh=hg} \frac{1}{C_{g,h}} [h] \Box [g]$$

(4.3)

over orbifold twisted sectors labeled by the conjugacy classes $[g]$ of the permutation group $S_N$

$$[g] = \prod_{L=1}^{n} (L)^{N_L} \quad \text{with} \quad \sum_{L=1}^{n} L N_L = N,$$

(4.4)

and over the conjugacy classes $[h]$ of the centralizer

$$C_g = \prod_{L=1}^{n} S_{N_L} \ltimes Z_L^{N_L}$$

(4.5)

The integer $C_{g,h}$ is the order of the centralizer of $h$ in $C_j$. In writing (4.3) we have used the fact that traces of elements in the same conjugacy class $[h]$ lead to the same result. We will write elements in $[h]$ as

$$[h] = \prod_{L=1}^{n} \prod_{M=1}^{n_L} (M)^{r_M^L} t \in \prod_{L=1}^{n} S_{N_L} \ltimes Z_L^{N_L} \quad \sum_{M=1}^{n_L} M r_M^L = N_L.$$ 

(4.6)

with $t$ an element in $Z_L^{N_L}$ for a given choice of $N_L$’s.

Orbifold group sectors are then parametrized by the integers $\{N_L\}$ (partitions of $N$), $\{r_M^L\}$ (partitions of $N_L$) and $\{t\}$ (elements of $Z_L^{N_L}$). In particular the number of such $[h]$’s is given by

$$C_{g,h} = \prod_{L,M} L r_M^L M r_M^L r_M^L!$$

(4.7)

We can label the $N$ copies of the field in $S_N \mathcal{H}$ by the quintuple of integers $(L,l,M,m,i)$ running over the domains $L = 1, \ldots, n$, $l = 0, 1, \ldots, L-1$ $M = 1, \ldots, n_L$, $m = 0, 1, \ldots, M - 1$ and $i = 1, \ldots, r_M^L$ respectively. Writing $Z_L^N$ elements as $t = (L)^{s_{M,i}}$ the boundary conditions for a field $\Phi_L^{m,i}$ (dependence in $L, M$...
are implicitly understood) along the worldsheet torus cycles can be written as

\[ \Phi_{m;i}^{(1)}(\sigma_1 + 1, \sigma_2) = e^{2\pi i g_\phi} \Phi_{m;i}^{(1)}(\sigma_1, \sigma_2) \]

\[ \Phi_{m;i}^{(2)}(\sigma_1, \sigma_2 + 1) = e^{2\pi i h_\phi} \Phi_{m;i}^{(2)}(\sigma_1, \sigma_2) \]  

(4.8)

After iterating (4.8) one is left with the doubly periodic functions

\[ \Phi_{m;i}^{(1)}(\sigma_1 + L, \sigma_2) = e^{2\pi i L g_\phi} \Phi_{m;i}^{(1)}(\sigma_1, \sigma_2) \]

\[ \Phi_{m;i}^{(2)}(\sigma_1 + s_i, \sigma_2 + M) = e^{2\pi i (s g_\phi + M h_\phi)} \Phi_{m;i}^{(2)}(\sigma_1, \sigma_2) \]  

(4.9)

with \( s_i = \sum_{m=0}^{M-1} s_{m;i} \) (mod \( L \)). The contribution to the orbifold string partition function of a given sector specified by \( \{N_L, r_M, s_{m;i}\} \) can therefore be written in terms of the result for a single copy in a torus with the induced complex structure \( \tilde{\tau} = M_L \tau + \frac{L}{L} \) and spin characteristics (4.9), i.e.

\[ (M)^{r_M}_L \Big( (L)^{N_L} \prod_{i=1}^{r_M} Z \left[ g^{L} \right] g_{S,i} + M h \right)(\tilde{q}_i, \tilde{\bar{q}}_i, y^M, \tilde{y}^M) \]  

(4.10)

with \( \tilde{q}_i = e^{2\pi i \tilde{\tau}_i} = q^{\frac{M}{L}} e^{2\pi i \frac{L}{L}} \)

Plugging this basic trace result into the sum over orbifold sectors specified by (4.4), (4.6), we are left with

\[ Z \left[ g \right] (S_{NM}|q, \tilde{q}, y, \tilde{y}) = \]

\[ \sum_{\{N_L, \{r_M\}, \{s^L\}\}} \prod_{L,M,i} \frac{1}{M^{r_M}_L r^L_M} \prod_{L} Z \left[ g^{L} \right] g_{S,i} + M h \right)(\tilde{q}_i, \tilde{\bar{q}}_i, y^M, \tilde{y}^M) \]

\[ = \sum_{\{N_L, \{r_M\}, \{s^L\}\}} \prod_{L,M} \frac{1}{M^{r_M}_L r^L_M} \]

\[ \times \left( \frac{1}{L} \sum_{s=0}^{L-1} C \left[ g^{L} \right] g_{S} + M h \right)(\Delta, \tilde{\Delta}, \ell, \tilde{\ell}) q^{\frac{M}{L}} \tilde{q}^{\frac{M}{L}} e^{2\pi i \frac{L}{L-\Delta}} y^M \tilde{y}^M \right)^{r^L_M} \]  

(4.11)

with \( C_{[M]}^{[N]}(\Delta, \tilde{\Delta}, \ell, \tilde{\ell}) \) the expansion coefficients (4.2) defined for a single copy of \( \mathcal{H} \).

Before proceeding further, it is worth to make a comment about the BPS content of this formula. We have seen in the previous section that the CFTs describing excitations of the D-brane bound state involve fermionic zero modes. The trace over these modes leads to the vanishing of the quantity inside the bracket in (4.11) for \( y = \tilde{y} = 1 \), corresponding to the fact that bound state excitations organize themselves into supermultiplets of the unbroken supersymmetry. Sectors with
\( r_M^L = 1 \) correspond then to states in ultra-short BPS supermultiplets and the counting formula for these states simplifies to [22]:

\[
Z_{BPS} \left[ \frac{g}{h} \right] (S_N M | q, \bar{q}, y, \bar{y}) = \\
\frac{1}{N} \sum_{s, L, M} C \left[ \frac{gL}{gs + Mh} \right] (\Delta, \bar{\Delta}, \ell, \bar{\ell}) q^{\frac{M\Delta}{L}} \bar{q}^{\frac{M\bar{\Delta}}{L}} e^{2\pi i \frac{1}{L} (\Delta - \bar{\Delta})} y^M \bar{y}^{\bar{M}} \tag{4.12}
\]

with \( N = LM, s = 0, 1, \ldots, L - 1 \). In some of our considerations in the following section, the restriction of the general formula presented below to this sector will be relevant.

Coming back to the general expression (4.11) one can now perform the sum over \( s \). It is easy to see (see the appendix A for similar projection-sum manipulations) that this leads effectively to a projection onto states satisfying the “level matching condition”

\[
\frac{(\Delta - \bar{\Delta})}{L} \in \mathbb{Z} + \delta. \tag{4.13}
\]

with \( \delta = 0, 1/2 \) depending on the different orbifold group sectors and boundary conditions. Introducing, as in [31], a generating function for the symmetric product formulae (4.11)

\[
Z \left[ \frac{g}{h} \right] (p, q, \bar{q}, y, \bar{y}) = \sum_{N \geq 0} p^N Z \left[ \frac{g}{h} \right] (S_N \mathcal{H} | q, \bar{q}, y, \bar{y})\tag{4.14}
\]

with \( p^N = p^{LMr_M^L} \), one can write the final result in the compact form:

\[
Z \left[ \frac{g}{h} \right] (p, q, \bar{q}, y, \bar{y}) = \prod_{\delta = 0} (1 - p^\ell q^{\Delta} \bar{q}^{\bar{\Delta}} y^{\ell} \bar{y}^{\bar{\ell}})^{-C_{\pm}(gL)(\Delta, \bar{\Delta}, \ell, \bar{\ell})} \times \prod_{\delta = g} (1 - (-)^{2h} p^\ell q^{\Delta} \bar{q}^{\bar{\Delta}} y^{\ell} \bar{y}^{\bar{\ell}})^{-C_{\pm}(gL)(\Delta, \bar{\Delta}, \ell, \bar{\ell})}. \tag{4.15}
\]

The products run over all possible \( L, \Delta, \bar{\Delta}, \ell, \bar{\ell} \) satisfying the level matching condition (4.13) with \( \delta \) explicitly indicated in (4.15). The coefficients \( C_{\pm}\{g\} \) are defined by

\[
C_{\pm}\{g\} = \frac{1}{2} \left( C\left[ \frac{g}{0} \right] \pm C\left[ \frac{g}{\frac{1}{2}} \right] \right) \tag{4.16}
\]

and count the number of states in the \( g \)-twisted sector of the starting (the single copy) CFT with \( \pm \) eigenvalues under the \( Z_2 \) orbifold group action. The choice \( g = h = 0 \) corresponds to the case studied in [31] where all fields are periodic in both \( \sigma \) and \( \tau \) directions and the sum over \( s \) results in a projector onto (4.13) with \( \delta = 0 \).
Specifying to ground states in the right moving sector of the symmetric product CFT, the formula (4.15) reduces to

\[
\begin{align*}
Z^{[0]}_{0}(p, q, y, \tilde{y}) &= \prod_{L,k} (1 - p^L q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^0_{[0]}(Lk, \ell, \tilde{\ell})} \\
Z^{[0]}_{\frac{1}{2}}(p, q, y, \tilde{y}) &= \prod_{L,k} (1 - p^L q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^0_{[0]}} (1 + p^L q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^-_{[0]}} \\
Z^{[\frac{1}{2}]}_{0}(p, q, y, \tilde{y}) &= \prod_{L,k} (1 - p^{2L} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^0_{[0]}} (1 - p^{2L-1} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^+_{[0]}} \\
&\quad \times (1 - p^{2L} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C^-_{[0]}} (1 - p^{2L-1} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C^-_{[0]}} \\
Z^{[\frac{1}{2}]}_{\frac{1}{2}}(p, q, y, \tilde{y}) &= \prod_{L,k} (1 - p^{2L} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^0_{[0]}} (1 - p^{2L-1} q^k y^\ell \tilde{y}^{\tilde{\ell}})^{-C^+_{[0]}} \\
&\quad \times (1 + p^{2L} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C^-_{[0]}} (1 + p^{2L-1} q^{k-\frac{1}{2}} y^\ell \tilde{y}^{\tilde{\ell}})^{-C^-_{[0]}}
\end{align*}
\]

where the arguments of the expansion coefficients $C^\pm_{[0]}(nm, \ell, \tilde{\ell})$, with $n$ and $m$ being the powers of $p$ and $q$ respectively have been omitted. The net effect of a non-trivial holonomy $(g, h) \neq (0, 0)$ is then to correlate the parity of excitations in the CFT under the $Z_2$ orbifold group action to the parity of the permutation group orbifold sector and the level of the SCFT specified by $L$ and $k \equiv \Delta L$ respectively\(^6\).

Note that the marginal deformations of the CFTs are given by the ground states in the above partition functions which appear as $p^n(y\tilde{y})^{1-n}$ as one can see by spectral flow from the Ramond to the Neveu-Schwarz sector. One can then verify that for model $I$ and $III$, the partition functions above include the marginal deformations corresponding to the $T^4$ (they appear in the untwisted sector of $S_N$) but they do not include the blowing up modes which arise in the twisted sector of $S_N$. This is in agreement with the discussion at the end of section 3.1.

\section{D1/D5 bound states versus fundamental strings}

In this section we evaluate the partition functions (or elliptic genera) encoding multiplicities and charges of D1/D5 two-charge bound state systems and compare them with the expected results from a U-dual description, in terms of winding-momentum modes of fundamental strings. We are assuming that the effective gauge theories, describing the low energy D1/D5 dynamics, flow to one of the orbifold symmetric product CFTs (3.8), (3.11) with the $Z_2$ actions specified in the tables 2.1 and 2.2.

---

\(^6\)Partition functions somewhat similar to (4.17) appear in [32], where however the symmetric product orbifold includes discrete torsion.
There is an important difference between the symmetric product CFTs (3.8) and (3.11). In the former case, associated to stacks of pure $D1(D5)$-branes bound to KK momentum modes, the position in $\mathbb{R}^4$ is described by the center of mass of $N$ copies of $\mathbb{R}^4$ in the symmetric product CFT. This leads to a subtlety in the counting of BPS excitations, since not all states contributing to the elliptic genus correspond to normalizable ground states of the gauge theory [21]. A careful analysis [21] reveals that among all the states with the right supersymmetry structure to reconstruct a short supermultiplet ($r^L_M = 1$) in (4.12) only those coming from the long string sector $[g] = (N)$ represent truly one-particle states. BPS charges and multiplicities can therefore be read off from the formula (4.12) specifying $M = 1, L = N$, the so called “long string” sector. This is not the case for the symmetric products associated to $D1/D5$ bound states (3.11), where all intermediate strings are needed in (4.17) in order to reproduce the fundamental string degeneracies. The position of the bound state is, in this case, specified by a single coordinate in the non compact transverse $\mathbb{R}^4$ (strictly speaking in our case $\mathbb{R}^4$ is replaced by $\mathbb{R}^3 \times S^1$).

5.1 Fundamental string partition functions

Before proceeding with the study of the spectrum of D-brane bound states, let us evaluate the partition function for the fundamental sides of the duality chain. Two-charge D-brane bound state will be systematically mapped to a fundamental strings carrying both momentum ($p_1$) and winding charges ($F_1$), with no extra charges turned on in one of the three type IIB orbifolds corresponding to $(-)^{F_L} I_4 I_4 \sigma_{pa}$, $(-)^{F_L} I_4 \sigma_{pa}$ and $I_4 I_4 \sigma_{pa}$. We will refer to these theories as $I_F$, $II_F$ and $III_F$, respectively. $\sigma_{pa}$ represents a $Z_2$-shift in the momentum mode along direction $a$, with $a = 1, 6$ in the case of a longitudinal and transverse shift respectively.

The fundamental string partition functions for the BPS states is defined by the supertrace (4.2), restricted to the right-moving ground states. After performing the spin structure sums, these can be written as

$$Z_{I_F}(q, y, \tilde{y}) = \frac{1}{2} \frac{y^2 \vartheta_4^2(y)}{\vartheta_1(y \tilde{y}) \vartheta_1(y \tilde{y}^{-1})} \left( \Gamma_{4,4} \tilde{y}^2 \frac{\vartheta_4^2(\tilde{y})}{\eta^6} \Gamma_{1,1} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] + \tilde{y}^2 \frac{\vartheta_2^2(y)}{\vartheta_2^2(0)} \Gamma_{1,1} \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] + 16 \frac{\vartheta_2^2(y)}{\vartheta_2^2(0)} \Gamma_{1,1} \left[ \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right] \right)$$

$$Z_{II_F}(q, y, \tilde{y}) = \frac{1}{2} \frac{y^2 \tilde{y}^2}{\vartheta_1(y \tilde{y}) \vartheta_1(y \tilde{y}^{-1}) \eta^6} \left( \frac{\vartheta_4^2(y)}{\vartheta_4^2(\tilde{y})} \Gamma_{1,1} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] + \tilde{y}^2 \frac{\vartheta_2^2(y)}{\vartheta_2^2(0)} \Gamma_{1,1} \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] + 16 \frac{\vartheta_2^2(y)}{\vartheta_2^2(0)} \Gamma_{1,1} \left[ \begin{array}{c} \frac{1}{2} \\ 0 \end{array} \right] \right)$$

(5.1)
\[ + \partial_2^2(y) \partial_2^2(\tilde{y}) \Gamma_{1,1} \left[ \begin{array} {c} 0 \\ \frac{1}{2} \end{array} \right] + \partial_4^2(y) \partial_4^2(\tilde{y}) \Gamma_{1,1} \left[ \begin{array} {c} \frac{1}{2} \\ 0 \end{array} \right] + \partial_3^2(y) \partial_3^2(\tilde{y}) \Gamma_{1,1} \left[ \begin{array} {c} \frac{1}{2} \\ 0 \end{array} \right] \] (5.2)

\[ Z_{III}(q, y, \tilde{y}) = \frac{1}{2} \frac{y^2 \partial_1^2(\tilde{y}) \partial_1^2(y^{-1})}{\tilde{y}^2} \Gamma_{1,1} \left[ \begin{array} {c} 0 \\ 0 \end{array} \right] \Gamma_{1,1} \left[ \begin{array} {c} \frac{1}{2} \\ 0 \end{array} \right] + 16 \frac{\partial_2^2(y)}{\partial_2^2(0)} \Gamma_{1,1} \left[ \begin{array} {c} \frac{1}{2} \\ 0 \end{array} \right] + 16 \frac{\partial_3^2(y)}{\partial_3^2(0)} \Gamma_{1,1} \left[ \begin{array} {c} \frac{1}{2} \\ 0 \end{array} \right] \] (5.3)

with \( y_\pm \equiv y^{\frac{1}{2}} \pm y^{-\frac{1}{2}} \) and similarly for \( \tilde{y}_\pm \). \( \Gamma_{4,4} \) is the \( T^4 \) winding-momentum lattice sum and

\[ \Gamma_{1,1} \left[ \frac{g}{h} \right] \equiv \sum_{(p_a, w_a) \in (\mathbb{Z}, \mathbb{Z} + g)} (-1)^{2p_a h} q^{(p_a/R + w_a R)^2} \bar{q}^{(p_a/R - w_a R)^2}. \] (5.4)

is the shifted lattice parallel to \( \sigma_{p_a} \). The hat in the \( \partial \)-functions in the denominators denotes the omission of their zero mode parts, i.e. \( \hat{\partial}_1 \equiv \partial_1^3 \), \( \hat{\partial}_2 \equiv \frac{1}{2} \partial_2 \) and \( \hat{\partial}_{3,4} \equiv \partial_{3,4} \). The completely untwisted sector, common to all three models, corresponds to the partition function of type IIB on \( T^5 \). In the case of a transverse shift, an extra \( \Gamma_{1,1} \left[ \begin{array} {c} 0 \\ 0 \end{array} \right] \) lattice sum, common to all orbifold group sectors, should be included.

Multiplicities for fundamental string states, carrying \( k \) units of momenta and \( N \) units of winding, can be read off from (5.1)-(5.3), once the level matching condition \( (N_R = c_R) \),

\[ kN = N_L - c_L, \] (5.5)

is imposed. Here \( N_L, N_R \) are the left- and right-moving oscillator levels and \( c_L, c_R \) the zero point energies.

The fourth fundamental theory that will be relevant to our next discussion is the toroidal heterotic string with gauge group \( SO(32) \) completely broken by Wilson lines. A possible choice of Wilson lines (in a fermionic representation) can be written as

\[ A_1 : \left[ (+)^{16}, (-)^{16} \right] \]
\[ A_2 : \left[ (+)^8, (-)^8, (+)^8, (-)^8 \right] \]
\[ A_5 : \left[ (+), (-), (+), (-), \ldots (+), (-) \right] \] (5.6)

Alternatively, one can represent this model as a \( \mathbb{Z}^5 \) orbifold of heterotic string on \( T^5 \), where the \( \mathbb{Z}_2 \) generators act simultaneously as a shift in one of the five circles and on the \( SO(32) \) lattice, in the way specified by (5.6). The fundamental string
partition function can then be written as
\[
Z_{IV}(q, y, \tilde{y}) = \frac{1}{2^5} \frac{y^2 \tilde{y}^2}{\vartheta_1(y \tilde{y}) \vartheta_1(y \tilde{y}^{-1}) \eta^{18}} \left( \frac{1}{2} (\vartheta_2^{16} + \vartheta_3^{16} + \vartheta_4^{16}) \Gamma_{5,5} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] + \vartheta_3^5 \vartheta_5^5 \Gamma_{5,5} \left[ \begin{array}{c} 0 \\ \epsilon_i \end{array} \right] + \vartheta_3^5 \vartheta_5^8 \Gamma_{5,5} \left[ \begin{array}{c} 0 \\ \epsilon_i \end{array} \right] \right) \] (5.7)
\]
where the sum over \(\epsilon_i = 0, \frac{1}{2}, \ldots, 5\) is always implicitly understood. We denote by \(\Gamma_{5,5}_{[g, h]}^{[N]}\) the lattice built from five copies of (5.4), with twists specified by \(g_i, h_i\). Notice that among all the \(Z_2^5\) orbifold group elements only the \(\epsilon_i\)-projection (for a fixed \(\epsilon_i\)) leads to a non-trivial result in the \(\epsilon_i\)-twisted sector.

5.2 D1(D5)-KK momenta bound states

In this subsection we compare the CFT results (in the long string sector) for multiplicities of BPS states, i.e. \(\Delta = 0\), in the pure D1(D5)-KK bound state systems, to the ones coming from the fundamental string partition functions (5.1)-(5.3) in the dual theories. The partition function will be evaluated using the CFT proposals (3.8) with \(Z_2^5\) given in table 2.1. As explained before, only the long string sector, \(M = 1, L = N\) in (4.12) is relevant to the counting of one-particle states. The results generalize a similar analysis in [22].

In the transverse shift case, the CFT description of a pure D1-KK or D5-KK systems is associated to the untwisted sector of \(Z_2^5\) orbifolds of \((R^3 \times S^1 \times T^4)^N / S_N\). Specializing to the long string sector in (4.12), with \(g = 0, h = \frac{1}{2}\), one is left with
\[
\frac{1}{2} \left( Z_{long} \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] + Z_{long} \left[ \begin{array}{c} 0 \\ \frac{1}{2} \end{array} \right] \right) = \sum C_{+} \{0\} (kN, \ell, \hat{\ell}) q^k y^\ell \tilde{y}^{\hat{\ell}} \] (5.8)
\]
where \(C_{\pm} \{g\} (kN, \ell, \hat{\ell})\) are the expansion coefficients of the partition function evaluated for a single copy \(N = 1\) in (3.8). We have performed the sum over \(s = 0, 1, \ldots, N - 1\), that projects the sum onto states satisfying the level matching condition \(k = \frac{A}{N} \in \mathbb{Z}\). Charges and multiplicities for a bound state of \(N\) D1- or D5-branes carrying \(k\) units of KK momentum \(p_1\) are then described in each theory by the corresponding expansion coefficients \(C_{+} \{0\} (kN, \ell, \hat{\ell})\). Recalling from our discussion in section 2, that the \(N = 1\) CFTs, and therefore their \(C_{+} \{0\}\) coefficients in table 2.1, coincide in each of the cases with their fundamental descriptions, we conclude that the D1(D5)-KK multiplicities we obtain from the above CFTs agree with those of untwisted states (even windings) with even momenta, in the corresponding fundamental theory (5.1)-(5.3), once the level matching condition (5.5) is imposed. This is precisely what one would expect from the duality map,
since the image of the D1(D5)-KK bound state \((p_6 = F_6 = 0)\) carries no windings and no momenta along the shift.

A similar result can be found in the longitudinal shift case \(\sigma_{p_1}\). Now the long string sector in (4.12), with \(g = \frac{1}{2}, h = 0\) leads to

\[
Z_{\text{long}} \left[ \frac{7}{2} \right] = \sum_{k \in \mathbb{Z}} C_+ \left\{ \frac{N}{2} \right\} (kN, \ell, \tilde{\ell}) q^k y^\ell \tilde{y}^\tilde{\ell} + \sum_{k \in \mathbb{Z} + \frac{1}{2}} C_- \left\{ \frac{N}{2} \right\} (kN, \ell, \tilde{\ell}) q^k y^\ell \tilde{y}^\tilde{\ell} \tag{5.9}
\]

This is again in complete agreement with (5.1)-(5.3). Even(odd) fundamental winding states are mapped to bound states involving an even(odd) number \(N\) of D-branes and multiplicities are described by \(C_\pm\) according to whether the level \(k\) (momentum on the fundamental side) is integer or half-integer.

### 5.3 D1/D5 bound states

In this subsection we consider D1/D5 bound state systems. In order to compare multiplicities of the bound states with those of fundamental strings (pure \(F_1 - p_1\)), one should restrict the attention to ground states \(\Delta = \bar{\Delta} = 0\) in both left- and right-moving sides of the CFT. Once again, the transverse shift results can be expressed as \(Z_2\) orbifolds:

\[
Z(p, y, \tilde{y}) = \sum_{h=0,\frac{1}{2}} Z_{\text{cm}} \left[ \frac{0}{h} \right] (p, y, \tilde{y}) Z_{\text{sym}} \left[ \frac{0}{h} \right] (p, y, \tilde{y}) \tag{5.10}
\]

of the type IIB result

\[
Z_{\text{cm}} \left[ \frac{0}{0} \right] (p, y, \tilde{y}) Z_{\text{sym}} \left[ \frac{0}{0} \right] (p, 0, y, \tilde{y}) = y^- y^2 \frac{\vartheta_1^2 (y | p) \vartheta_1^2 (\tilde{y} | p)}{\vartheta_1 (y \tilde{y} | p) \vartheta_1 (y \tilde{y}^{-1} | p) \eta^8 (p)}, \tag{5.11}
\]

associated to the symmetric product space \(R^3 \times S^1 \times T^4 \times (T^4)^N / S_N\). We denote by \(Z_{\text{cm}}(q, y, \tilde{y})\) the contribution coming from the center of mass, while \(Z_{\text{sym}}(p, q, y, \tilde{y})\) will be associated to the symmetric product of the \(T^4\) tori. Using the data for the single copy the CFT \(T^4 / \hat{g}\) with \(\hat{g}\) specified in table 2.2:

\[
(-)^{F_L} : \sum_{\ell, \tilde{\ell}} C^I_\pm \{0\} (0, 0, \ell, \tilde{\ell}) y^\ell \tilde{y}^\tilde{\ell} = \frac{1}{2} (\tilde{y}_-^2 + y_+^2) \]

\[
I^\text{sp}_4 : \quad \sum_{\ell, \tilde{\ell}} C^{II}_\pm \{0\} (0, 0, \ell, \tilde{\ell}) y^\ell \tilde{y}^\tilde{\ell} = \frac{1}{2} (\tilde{y}_-^2 + y_+^2) \]

\[
(-)^{F_L} I^\text{sp}_4 : \quad \sum_{\ell, \tilde{\ell}} C^{III}_\pm \{0\} (0, 0, \ell, \tilde{\ell}) y^\ell \tilde{y}^\tilde{\ell} = \frac{1}{2} (\tilde{y}_-^2 + y_+^2) \tag{5.12}
\]
into the symmetric product formulae (4.17), we are left with the partition functions

\[
\begin{align*}
Z_I^{0 \left[ \frac{1}{2} \right]}(p, y, \tilde{y}) &= \frac{1}{2} y^2 \tilde{y} \frac{\vartheta_2^2(\tilde{y}|p)}{\vartheta_1(\tilde{y}|p)\vartheta_1(\tilde{y}-1|p)} \frac{\vartheta_2^2(y|p)}{\vartheta_2(0|p)} \\
Z_{II}^{0 \left[ \frac{1}{2} \right]}(p, y, \tilde{y}) &= \frac{1}{2} y^2 \tilde{y} \frac{\vartheta_2^2(\tilde{y}|p)}{\vartheta_1(\tilde{y}|p)\vartheta_1(\tilde{y}-1|p)} \frac{\vartheta_2(y|p)}{\eta^6(p)} \\
Z_{III}^{0 \left[ \frac{1}{2} \right]}(p, y, \tilde{y}) &= \frac{1}{2} y^2 \tilde{y} \frac{\vartheta_2^2(\tilde{y}|p)}{\vartheta_1(\tilde{y}|p)\vartheta_1(\tilde{y}-1|p)} \frac{\vartheta_2(y|p)}{\vartheta_2(0|p)}
\end{align*}
\tag{5.13}
\]

where the contributions of the center of mass \(Z_{cm}^I(y, \tilde{y}) = y^4 \tilde{y}^4\), \(Z_{cm}^{II}(y, \tilde{y}) = Z_{cm}^{III}(y, \tilde{y}) = y^2 \tilde{y} \tilde{y}^2 \tilde{y}^2 \) in the three cases, has been included.

Together with (5.11) the results (5.13) for D1/D5 bound state degeneracies reproduce the multiplicities (5.1)-(5.3) for untwisted fundamental strings with \(p_6 = 0\), as required by the U-duality chain of table 1.2.

Finally, let us compute the D1/D5 bound state spectrum on \(T^4 \times S^1 / I_4 \sigma_{p_1}\) with \(\sigma_{p_1}\) a longitudinal shift. The relevant CFT data are now given in terms of the expansion coefficients for \(T^4 / I_4^{sp}\):

\[
\begin{align*}
\sum_{\ell, \tilde{\ell}} C_{\pm}^{IV}(0, 0, \ell, \tilde{\ell}) y^\ell \tilde{y}^{\tilde{\ell}} &= \frac{1}{2} (y^2 \tilde{y}^2 \pm \tilde{y}^2 y^2) \\
\sum_{\ell, \tilde{\ell}} C_{+}^{IV}(0, 0, \ell, \tilde{\ell}) y^\ell \tilde{y}^{\tilde{\ell}} &= 16 \\
\sum_{\ell, \tilde{\ell}} C_{-}^{IV}(0, 0, \ell, \tilde{\ell}) y^\ell \tilde{y}^{\tilde{\ell}} &= 0
\end{align*}
\tag{5.14}
\]

and \(Z_{cm} = 16 y^2 \tilde{y}^2\). Plugging in (4.17) we are left with

\[
\begin{align*}
Z_{II}^{0 \left[ \frac{3}{2} \right]}(p, y, \tilde{y}) &= 16 y^2 \tilde{y} \frac{1}{\vartheta_1(\tilde{y})\vartheta_1(\tilde{y}-1)\eta^2} \frac{\eta^8}{\vartheta_4(0)} \\
&= \frac{1}{24} y^2 \tilde{y}^2 \frac{1}{\vartheta_1(\tilde{y})\vartheta_1(\tilde{y}-1)\eta^{18}} \frac{\vartheta_8(0)\vartheta_8(0)}{\vartheta_2(0)\vartheta_3(0)} \tag{5.15}
\end{align*}
\]

This is in complete agreement with the fundamental heterotic string degeneracies (5.7), coming from the twisted sector, once the level matching condition (5.5) is imposed. Notice that the expansion of (5.15) reproduces both signs in (5.7), \(p_1\) even or odd, according to whether we expand to integer or half-integer powers in \(p\). That only states in the twisted sector (odd windings) are relevant to the comparison is due to the fact that the CFT proposals are valid only for a single fivebrane, which is mapped in the fundamental side to a single unit of winding.
mode. One can however test multiplicities in the untwisted sector (with $p_1 = 1$) by going after four T-dualities to step B, where the role of the D1 and D5 are exchanged inside model II. The CFT description of those D1/D5 states are, of course, the same as before and multiplicities are again given by (5.15). The fundamental string multiplicities on the other hand lead to apparently two very different results, depending on whether we consider states with $F_1$ odd (twisted sector) or even (untwisted sector). In the former case one finds again (5.15) in agreement with the duality predictions. The multiplicities for even $F_1$ are, on the other hand, given by

$$Z_{F_1 = 1}^{p_1 = 1}(q, y, \tilde{y}) = \frac{1}{2\pi} \frac{y^2 \tilde{y}^2}{\vartheta_1(\vartheta y) \vartheta_1(\vartheta \tilde{y}^{-1})} \eta^{18} \left[ \frac{1}{2} (\vartheta_2^{16} + \vartheta_3^{16} + \vartheta_4^{16}) - \vartheta_3 \vartheta_4 \right]$$

The discrepancy is only apparent, since expression (5.16) coincides with (5.15) after simple manipulations of $\vartheta$-identities. The fact that the heterotic dual model treats on the same footing winding and momentum modes, as required by the U-duality chain, can be taken as a further support for the consistency of the whole picture.

One can try to apply a similar analysis to the D1/D5 systems in models I and III with a longitudinal shift, but one immediately runs into problems. The ground states of the natural CFT proposals in table 2.2 are in these cases either tachyonic or massive and a naive application of the elliptic genus formula leads to meaningless results. A proper description of these $(4,0)$ D1/D5 systems remains an interesting open problem.

### 5.4 Three-charge systems

We will restrict the discussion of 3-charge systems to the transverse shift case, since, as we mentioned before, the D1/D5 CFT description of models I and III in the longitudinal shift case is problematic due to the presence of either tachyonic or massive ground states.

To extract the multiplicities for 3-charge systems, i.e. D1/D5 system carrying $k$ units of KK momentum, from our partition functions, we restrict the right-moving part to the ground state ($\Delta = 0$) and consider excitations of the left-moving part (which is non-supersymmetric in models I and III) to level $k$. The resulting partition functions are of the form:

$$Z(p, q, y, \tilde{y}) = \sum_{h = 0, \frac{1}{2}} Z_{cm}^{0} \left[ h \right] (q, y, \tilde{y}) Z_{sym}^{0} \left[ h \right] (p, q, y, \tilde{y})$$

34
Notice that the \( q^0 \) term in \( Z \) corresponds to the partition function of the fundamental sides for the D1/D5 systems and is given in (5.13) for the three theories. Denoting this latter by \( Z_E(p) \), it is convenient to rewrite \( Z \) in (5.17) as:

\[
Z(p, q, y, \tilde{y}) = \sum_{h=0,\frac{1}{2}} Z_{cm}^{[0]}(q, y, \tilde{y}) \hat{Z}_E^{[0]}(p, y, \tilde{y}) \hat{Z}_{sym}^{[0]}(p, q, y, \tilde{y})
\]  

(5.18)

where, as before, the “hat” denotes omission of zero modes.

We have already seen in section 2 that U-duality gives certain relations for the multiplicities of the 3 charge systems. From table 1.2, by comparing columns \( \mathbf{F} \) and \( \mathbf{G} \), we see that models I and III get exchanged together with D1 and D5 charges. The CFTs we have proposed trivially satisfy this symmetry for \( Q_5 = 1 \), since in this case \( N = 1 \) in (3.11) and therefore both CFTs are given by \((R^3 \times S^1 \times T^4 \times T^4/\mathbb{I}_4)/(-)^{F_L}\).

A less trivial relation comes from the comparison between columns \( \mathbf{C} \) and \( \mathbf{F} \): in this case the 3-charge system (D1, D5, KK) of model II is mapped to (KK, D5, D1) in model III. This means that the full elliptic genera corresponding to models II and III must get exchanged if we exchange \( q \) with \( p \). Since:

\[
\hat{Z}_{cm}^{[0]}(q, y, \tilde{y}) = \hat{Z}_{cm}^{[0]}(q, y, \tilde{y}) = \frac{1}{2} \frac{\hat{\vartheta}_2(y|q)}{\hat{\vartheta}_1(y|q)} \frac{\hat{\vartheta}_2(q|y)}{\hat{\vartheta}_1(q|y)}
\]

(5.19)

the symmetry under the \( (p, q) \) exchange implies then that \( \hat{Z}_{sym}^{[0]}(p, q, y, \tilde{y}) \) of model II and III should get exchanged. Notice that we are comparing a (4,4) theory (model II) with a (4,0) theory (model III). Although in the previous discussions we have set \( \Delta = 0 \) while keeping \( y, \tilde{y} \) arbitrary, actually the quantity that is expected to be invariant under deformations of the CFT (the elliptic genus) is obtained by setting \( \tilde{y} = 1 \). However, in order to soak the fermionic zero modes we will take two derivatives in \( \tilde{y} \) and then set \( \tilde{y} = 1 \) (the two derivatives necessarily act on the center of mass CFT). This results in putting the right-moving sector (which is supersymmetric in both cases) on the ground state. Once we set \( \tilde{y} = 1 \), the relevant expressions become (omitting indices II and III):

\[
\hat{Z}_{sym}^{[0]}(p, q, y, \tilde{y} = 1) = \prod_{n,m \geq 1} \left( \frac{1 + p^n q^m y^l}{1 - p^n q^m y^l} \right)^{\frac{1}{2} C^{[0]}_m^{[0]}(nm, l)}.
\]

(5.20)

(5.20) follows from (4.17), using the identities:

\[
C^{[0]}_m^{[0]}(m, l) \equiv \sum_i C_+\{0\}(m, l, i) = - \sum_i C_-\{0\}(m, l, \bar{i}).
\]
The U-duality requirement that, under \((p,q)\) exchange \(\hat{Z}_{\text{sym}}\) for models II and III are exchanged, implies

\[
C_{II} \left[ \frac{0}{2} \right] (m,l) = C_{III} \left[ \frac{0}{2} \right] (m,l), \quad \text{for } m \geq 1.
\]  

(5.21)

This requirement is not satisfied by the proposed CFTs for theories II and III. Indeed in terms of theta functions:

\[
\sum_{m,l} C_{II} \left[ \frac{0}{2} \right] (m,l) q^{m} y^{l} = 4 \frac{\vartheta_{2}^{2}(y|q)}{\vartheta_{2}^{2}(0|q)},
\]

\[
\sum_{m,l} C_{III} \left[ \frac{0}{2} \right] (m,l) q^{m} y^{l} = 4 \frac{\vartheta_{1}^{2}(y|q)}{\vartheta_{2}^{2}(0|q)}.
\]  

(5.22)

from which it follows that \(C_{II} \left[ \frac{0}{2} \right] (m,l) \neq C_{III} \left[ \frac{0}{2} \right] (m,l)\). Remarkably however the equality holds for \(m\) odd!

This problem affects also the seemingly well understood case of the \((4,4)\) CFT \(\mathbb{R}^{4} \times (K3)^{N}/S_{N}\), describing the D1/D5 system in type IIB on \(K3 \times S^{1}\). Using type II/heterotic duality in 6 dimensions, one can relate type IIB on \(K3 \times S^{1}\) to type IIB on \(S^{1} \times T^{4}/\Omega_{I4}\), while exchanging D1 and KK charges. Thus for D5 charge 1, this amounts to exchange \((p,q)\) (at order \(\bar{q}^{0}\)) in the corresponding elliptic genus. Notice that in this case \(Z_{F}(p)\) is just the bosonic oscillator part of the heterotic string and clearly \(\hat{Z}_{\text{sym}}(p,q)\) is symmetric under \((p,q)\) exchange \([31, 33, 10]\). However it is also easy to see that, for instance, the coefficient at order \(q^{1}\) of the elliptic genus does not have a well defined modular property as a function of \(p\). Finally, the same problem is present in the longitudinal shift case for the model II that we have studied before, although \(\hat{Z}(p,q)\) is not \((p,q)\) symmetric in this case.

In the above discussion we have ignored the non-trivial background of RR 0- and 4-form fields in the symmetric product CFT. As pointed out in section 2, in the presence of this background the issue of \(p, q\) exchange symmetry is more subtle. We will make some more comments on this problem in the conclusion.

## 6 One-loop effective gauge couplings

In this section we study the \((T, U)\) moduli dependence of one-loop threshold corrections to \(F^{2k+4}\) gauge couplings in the low energy effective action, associated to the four dimensional string compactifications under consideration. \((T, U)\) are the Kahler and complex structure moduli of a \(T^{2}\) along directions 1,6. The aim
of this section is to extract this information for four-dimensional gauge couplings $\mathcal{F}^{2k+4}$ involving some definite combinations of the eight field strengths,

$$\begin{align*}
\mathcal{F}^\pm_{L_i} &\equiv \partial_{\mu}(G^\pm_{\nu i} + B^\pm_{\nu i}) \\
\mathcal{F}^\pm_{R_i} &\equiv \partial_{\mu}(G^\pm_{\nu i} - B^\pm_{\nu i})
\end{align*}$$

(6.1)

which arise from KK-reductions of the six-dimensional metric and antisymmetric tensor to $D = 4$, with $i = 1, 6$. $\pm$ refers to (anti-)self-dual four-dimensional two-forms. The one-loop threshold corrections for these couplings will be then mapped to two-charge D-instanton contributions in the non-perturbative U-dual descriptions of table 1.1. Related computations in various contexts can be found in [35].

We will introduce a complex (euclidean) notation for spacetime coordinates

$$\begin{align*}
Z^1 &= \frac{1}{\sqrt{2}}(X^0 + iX^3) \\
Z^2 &= \frac{1}{\sqrt{2}}(X^1 + iX^2) \\
\chi^1 &= \frac{1}{\sqrt{2}}(\psi^0 + i\psi^3) \\
\chi^2 &= \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)
\end{align*}$$

(6.2)

with barred quantities corresponding to complex conjugates.

The moduli dependence will be extracted from the string amplitudes:

$$\mathcal{A}_{2k+4} = \langle \prod_{i=1}^{k+2} V(p_1, \xi_i)V(\bar{p}_2, \bar{\xi}_i) \rangle$$

(6.3)

where for simplicity we choose a kynematical configuration where half of the vertices carry momentum $p_1$ and the other half $\bar{p}_2$. In addition all the vertices will be chosen with definite (anti-) self-duality properties. The computation and notations follow closely [36].

The vertex operators for the gauge field strengths (6.1) are given by

$$\begin{align*}
V_L(p, \xi) &= \int d^2z \xi_{\mu i} (\partial X^\mu - ip\chi^\mu)(\bar{\partial} X^i - ip\bar{\chi}^i) e^{ipX} \\
V_R(p, \xi) &= \int d^2z \xi_{\mu i} (\partial X^i - ip\chi^i)(\bar{\partial} X^\mu - ip\bar{\chi}^\mu) e^{ipX}
\end{align*}$$

(6.4)

Notice that each vertex carries at least one power of space-time momentum $p_\mu$, and therefore to the order we are interested in we can keep only linear terms in $p_\mu$.

A representative of such couplings in each of the three models is indicated in the table below, where we also indicate how the various charges are mapped to each others:
\[ IIF \xrightarrow{T_{15ST2345}S} IF \xrightarrow{T_{15ST2345}S} IIIF \]

\[
\begin{array}{cccc}
I_1 & F_1 & I_4 & F_1 \\
(-)^{F_L} \sigma_6 & (\cdot)^{F_L} I_4 \sigma_6 & I_4 \sigma_6 & I_4 \sigma_6 \\
NS_{12345} & P_1 & NS_{12345} & F_1 \\
P_1 & NS_{12345} & F_1 & NS_{12345} \\
(\mathcal{F}_R^+)^2 (\mathcal{F}_R^-)^2 (\mathcal{F}_L^+)^{2k} & (\mathcal{F}_R^+)^2 (\mathcal{F}_L^+)^{2k} & (\mathcal{F}_L^-)^2 (\mathcal{F}_R^-)^2 (\mathcal{F}_L^+)^{2k} & (\mathcal{F}_L^-)^2 (\mathcal{F}_R^-)^2 (\mathcal{F}_L^+)^{2k}
\end{array}
\]

The field strengths \( \mathcal{F}_{L,R} \) are defined in (6.1) and \( \sigma_6 \) is a shift of order 2 along the 6th direction.

The couplings in the table are special in the sense that they receive in each case contributions only from right moving ground states (BPS saturated states). This can be seen by noticing that the insertions exactly soak the number of fermionic zero modes in the right-moving part of the string amplitudes. This will be implicit in most of our discussion.

In the model \( II_F \) this corresponds to the case where the eight right moving fermionic zero modes (once the sums over spin structure have been performed) are soaked up by exactly four insertions \( (\mathcal{F}_R^+)^2 (\mathcal{F}_R^-)^2 (\mathcal{F}_L^+)^{2k} \) of right moving gauge fields.

Similarly, in the models \( I_F \) and \( III_F \) two right-moving insertions of \( (\mathcal{F}_R^+)^2 \) and \( (\mathcal{F}_R^-)^2 \) respectively are needed in order to get a non-trivial result. Vertex operators for self-dual components can therefore be effectively replaced by:

\[
\begin{align*}
\bar{V}_L^+(p_1) &= ip_1 \bar{P}_L \tau_1 \int d^2 \sigma (Z^1 \partial Z^2 - \bar{Z}^1 \bar{\partial} \bar{Z}^2) + ... \\
\bar{V}_L^+(\bar{p}_2) &= i\bar{p}_2 \bar{P}_L \tau_2 \int d^2 \sigma (Z^2 \partial Z^1 - \bar{Z}^2 \bar{\partial} \bar{Z}^1) + ... \\
\bar{V}_R^+(p_1) &= ip_1 \bar{P}_R \tau_2 \int d^2 \sigma (Z^1 \partial Z^2 - \bar{Z}^1 \bar{\partial} \bar{Z}^2) + ... \\
\bar{V}_R^+(\bar{p}_2) &= i\bar{p}_2 \bar{P}_R \tau_2 \int d^2 \sigma (Z^2 \partial Z^1 - \bar{Z}^2 \bar{\partial} \bar{Z}^1) + ...
\end{align*}
\]

(6.5)

where we have grouped the components \( P_L^i = \partial X^i, P_R^i = \bar{\partial} X^i \) with \( i = 1, 2 \) into a two-dimensional vector and \( \partial = \tau_2 (\partial_{\sigma_2} - \bar{\tau} \partial_{\sigma_1}) \). Similar expressions are given for anti-self-dual components, replacing \( Z^2, \bar{Z}^2, \bar{Z}^1, \bar{\partial} \bar{Z}^1 \) by their complex conjugates.

From the expressions for the effective vertex operators in (6.5), we see that their insertion in the correlator (6.3) amounts to inserting factors of \( \bar{P}_{L,R} \).

Since the vertices are quadratic in the quantum fluctuations one can exponentiate...
them into a generating function

\[
G_{\vec{a},\vec{b}}(v, w) = \langle e^{-S_0 - \vec{v} \cdot \vec{V}_L - \vec{w} \cdot \vec{V}_R} \rangle = \int \frac{d^2 \tau}{\tau_2} \sum_{q,h} q^n C \left[ \frac{g}{h} \right] (n, l) \Gamma_{d,d} \left[ \frac{g}{h} \right] (\ell \cdot v, \ell \cdot w) \tag{6.6}
\]

with \( S_0 \) the free string action, \( q = e^{2\pi i \tau} \) (\( \tau \) the genus-one worldsheet modulus) and \( \vec{v}_\pm, \vec{w}_\pm \) are sources for the eight \( U(1) \) gauge fields. The scalar product is defined as usual by \( \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 \), while the dotted product reads \( a \cdot b = a_+ b_+ + a_- b_- \).

In the right-hand side we have introduced the notation \( v_\pm \equiv \vec{v}_\pm \vec{P}_L \) with a similar definition for \( w_\pm \) with \( \vec{P}_L \) replaced by \( \vec{P}_R \) and \( \vec{v} \) by \( \vec{w} \). Vertex insertions are defined by \( \vec{v}_\pm, \vec{w}_\pm \)-derivatives of (6.6).

Finally, we denote as before by \( C_{\left[ \frac{g}{h} \right]}(n, l) \) with \( l \equiv (\ell, \tilde{\ell}, s, \tilde{s}) = (\ell^+, \ell^-, \ell^+_s, \ell^-_s) \) the coefficients in the expansion of the partition function which includes Wilson lines:

\[
G_{\left[ \frac{g}{h} \right]}(q, y) = \text{Tr}_{g-tw} \left( \Theta^h q^{L_0-c/24} q^{\tilde{L}_0-c/24} y^{L_0} \tilde{y}^{\tilde{L}_0} y^{2J_3} \tilde{y}^{\tilde{J}_3} y^{2\tilde{J}_3} \tilde{y}^{2\tilde{J}_3} \right) = C_{\left[ \frac{g}{h} \right]}(n, l) q^n y^\ell \tilde{y}^{\tilde{\ell}} y^s \tilde{y}^{\tilde{s}} \tag{6.7}
\]

\( \Theta \) is the \( \mathbb{Z}_2 \) orbifold generator, \( y = e^{\pi i v} \), \( \tilde{y} = e^{\pi i w} \), and similarly \( y_s, \tilde{y}_s \) are obtained by replacing \( \vec{v}_\pm \) by \( \vec{w}_\pm \). \( L_0, \tilde{L}_0 \) are the Virasoro generators and \( J_3 \)'s are four \( U(1) \) generators to which the corresponding gauge fields couple. Therefore we see that (6.7) has the structure of a helicity supertrace. The four possible twists along the \( \sigma \) and \( \tau \) directions will be denoted by \( \left[ \frac{g}{h} \right] \) with \( g, h = 0, \frac{1}{2} \). The primes denote the omission of the bosonic zero mode contributions which have been displayed explicitly in (6.6). \( \Gamma_{d,d} \) is a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) lattice sum for the completely untwisted sectors in the models \( I_F, III_F \) and all sectors in the model \( II_F \), while reduced to a \( \mathbb{Z}_2 \) lattice for all non-trivial twists in models \( I_F, III_F \). Since we are interested only in the \((T, U)\) moduli dependence of the first torus we will always work in the orbits where neither momentum nor winding modes are excited in the \( \mathbb{Z}_2 \)-lattice. The effects of introducing half-shifts in \( \Gamma_{d,d} \) lattices have been extensively studied in [37]. The perturbed lattice sum can be written as the sum

\[
\Gamma_{2,2} \left[ \frac{g}{h} \right] (\ell \cdot \vec{v}, \ell \cdot \vec{w}) = \frac{T_2}{\tau_2} \sum_M \eta \left[ \frac{g}{h} \right] e^{-\frac{\pi 2g}{\tau_2^2 \tau}} |(1 U) M( \frac{\tau}{2})|^2 \times e^{2\pi i [T \det M + (\ell \cdot v^1, \ell \cdot s^2) M( \frac{\tau}{2}) - (\ell \cdot w^1, \ell \cdot s^2) M( \frac{\tau}{2})]}
\tag{6.8}
\]

over worldsheet instantons

\[
\begin{pmatrix}
X^1 \\
X^6
\end{pmatrix} = M \begin{pmatrix}
\sigma^1 \\
\sigma^2
\end{pmatrix} \equiv \begin{pmatrix}
m_1 & n_1 \\
m_2 & n_2
\end{pmatrix} \begin{pmatrix}
\sigma^1 \\
\sigma^2
\end{pmatrix} \quad m \in \mathbb{Z} + \vec{b}g, \ n \in \mathbb{Z} + \vec{b}h, \tag{6.9}
\]

39
where now the entries in $M$ are integer or half-integer depending on the winding (momentum) shift vector $\vec{a} \ (\vec{b})$. Finally the lattice sum is weighted by the $\eta_{[g]}$ phase

$$\eta_{[g]} = e^{-4\pi i b \cdot g - 2\pi i \bar{a} \cdot (h \bar{m} - g \bar{n})} \quad (6.10)$$

We will consider only shifts involving either a pure momentum or pure winding $\vec{a} \vec{b} = 0$.

Evaluating the partition functions (6.7) in the three orbifold models described above one is left with (after performing the spin structure sums):

$$G_{II_F}[g,h](q,y) = \vartheta_{[g]}^2(q, y) \vartheta_{[h]}^2(\bar{y}) (y^\frac{1}{2} - y^\frac{1}{2})^2$$

$$G_{I_F}[g,h](q,y) = \frac{\vartheta_{[g]}^2(q, y) \vartheta_{[h]}^2(\bar{y})}{\vartheta_{1}(y\bar{y})} (y^\frac{1}{2} - y^\frac{1}{2})^2$$

$$G_{III_F}[g,h](q,y) = \frac{\vartheta_{[g]}^2(q, y) \vartheta_{[h]}^2(\bar{y})}{\vartheta_{1}(y\bar{y})} (y^\frac{1}{2} - y^\frac{1}{2})^2 \quad (6.11)$$

The hat on the $\vartheta$-functions in the denominators denotes as before the omission of their zero mode parts, i.e. $\vartheta_{1}(v) \equiv \frac{1}{v} \vartheta_{1}(v)$, $\vartheta_{2}(v) \equiv \frac{1}{v^2} \vartheta_{2}(v)$ and $\vartheta_{3,4}(v) \equiv \vartheta_{3,4}(v)$.

Notice the modular invariance of $G(v, w)$ under the SL(2, Z) transformations

$$\tau \rightarrow \frac{p\tau + q}{r\tau + s} \quad v \rightarrow \frac{v}{r \bar{v} + s} \quad w \rightarrow \frac{w}{r \bar{w} + s} \quad M \rightarrow M \left( \begin{array}{cc} s & q \\ r & p \end{array} \right) \quad (6.12)$$

The modular integral (6.6) can then be computed following the standard trick [38], that consists in trading the sum over the $M_\epsilon$ matrices in (6.8) by sums over $SL(2, Z)$ representatives integrated in unfolded domains. We will concentrate here on the contributions of non-degenerated orbits ($\det M \neq 0$) for which representatives can be chosen to be:

$$M = \left( \begin{array}{cc} m_1 & n_1 \\ 0 & n_2 \end{array} \right) \quad (6.13)$$

where $m_1 \in \mathbb{Z} + b_1 g$ and $n_1 \in \mathbb{Z} + b_1 h$. The integral (6.6) is then unfolded to the whole upper half plane.

The modular integral (6.6) is evaluated in the Appendix A. We keep only its leading order in the expansion around $T_2 \rightarrow \infty$. In the dual picture, this corresponds to the classical contribution of the D-instanton background. Higher orders can be interpreted as quantum fluctuations around the instanton background as in [39], but the analysis of these terms is beyond the scope of this work.
The final result, formula (C.4) of the Appendix C, after performing the $m_1$ and $n_2$ sums can be written as:

$$I(a, b) = \ln \mathcal{Z}(a, b) \mathcal{Z}(a, b)$$

(6.14)

with

$$\mathcal{Z}(1, 0) = \prod_{\delta=0}^{\infty} \left(1 - p^{m_1} q^{k_0} \hat{y}^{\ell} \hat{y}_*^{\ell} \right) C_+ \left(\frac{\delta}{2}\right) (km_1, l)$$

$$\times \prod_{\delta=1}^{\infty} \left(1 - p^{m_1} q^{k_0} \hat{y}^{\ell} \hat{y}_*^{\ell} \right) C_- \left(\frac{\delta}{2}\right) (km_1, l)$$

$$\mathcal{Z}(0, 1) = \prod_{\delta=0}^{\infty} \left(1 - p^{2m_1} q^{k_0} \hat{y}^{\ell} \hat{y}_*^{\ell} \hat{\bar{y}}^{\ell} \hat{\bar{y}}_*^{\ell} \right) C_+ \left(\frac{m}{2}\right) (km_1, l)$$

$$\times \prod_{\delta=1}^{\infty} \left(1 - p^{2m_1} q^{k_0} \hat{y}^{\ell} \hat{y}_*^{\ell} \hat{\bar{y}}^{\ell} \hat{\bar{y}}_*^{\ell} \right) C_- \left(\frac{m}{2}\right) (km_1, l)$$

(6.15)

The various quantities entering in (6.15) have the following meaning: $p = e^{2\pi i T}$ and $q = e^{2\pi i U}$, $\hat{y} = e^{\pi i \bar{v}}$ and $\hat{y}_* = e^{\pi i \bar{w}}$ are the induced sources, with $\bar{v} = v_1 U - v_2$ and $\bar{w} = \frac{1}{U}(w_1 \bar{U} - w_2)$ and similar definitions for $\tilde{y}$ and $\tilde{y}_*$. Finally $\tilde{Z}$ is simply the complex conjugate of $Z$ and represents the anti-instanton contributions.

Notice that $\mathcal{Z}(1, 0)$ and $\mathcal{Z}(0, 1)$ in (6.15) are mapped into each other under the simultaneous exchange of the momentum($b_1$)-winding($a_1$) shifts and $k$-$m_1$ modes, as required by T-duality.

This formula is rather more general than what we really need. Still, depending on the model, a certain number of $y$-derivatives should be taken and then the right moving source $y_*$ should be set to zero ($y_0 = \bar{y}_* = 1$). Notice however that already at this stage one can recognize in $\mathcal{Z}(0, 1)$ the symmetric product formula (4.17) for the longitudinal shift elliptic genus. This is enough to ensure, following [34, 22, 39], the agreement between the D-instanton corrections associated to the states counted by (4.17), provided that the orbifold CFT describing the D1(D5)-KK system is constructed out of the symmetric product of $N$ copies of the fundamental theory in the twisted sector (basic unit of winding).

Coming back to our formula, after acting in (6.15) with the appropriate number$^7$ of $w$-derivatives (see [10] for similar manipulations) one is left with:

$$\hat{\mathcal{Z}}(1, 0) = \frac{1}{2} \sum (-)^{m_1 h} \hat{C} \left[ \frac{g}{h} \right] (km_1, \ell, \bar{\ell}) p^{m_1 s} q^{s l} y^{s \ell} \bar{y}^{s \bar{\ell}}$$

$$\hat{\mathcal{Z}}(0, 1) = \frac{1}{2} \sum (-)^{k h} \hat{C} \left[ \frac{g}{h} \right] (km_1, \ell, \bar{\ell}) p^{m_1 s} q^{s l} y^{s \ell} \bar{y}^{s \bar{\ell}}$$

(6.16)

$^7$The minimal one in order to get a non-trivial result.
where
\[ \hat{Z}(a, b) \equiv \frac{1}{m!n!} \frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial \tilde{y}^n} Z(a, b)|_{y^*_s = \tilde{y}^*_s = 1} \] (6.17)
with \( m = 2, n = 0 \) for the model \( I_F \), \( m = n = 2 \) for the model \( II_F \) and \( m = 0, n = 2 \) for the model \( III_F \). The sum run over \( \ell, \tilde{\ell} \) integers, \( g, h = 0, 1, 2 \), \( m_1 \in \mathbb{Z} + b_1 g \) and \( k \in \mathbb{Z} + a_1 g \). Finally, the coefficients \( \hat{C}[\tilde{g}](\Delta, \ell, \tilde{\ell}) \) are similarly defined in terms of the expansion coefficients of (6.11) by
\[ \hat{C}[\tilde{g}](\Delta, \ell, \tilde{\ell}) \equiv \sum_{\ell_*, \tilde{\ell}_*} (-)^{kh} \ell^m \tilde{\ell}^n C[\tilde{g}](\Delta, \ell, \tilde{\ell}, \ell_*, \tilde{\ell}_*) \] (6.18)
and correspond to the expansion coefficients of the chiral supertraces appearing in (6.11).

7 Conclusions and open problems

The main goal of this paper has been the formulation of CFT descriptions of the moduli space of D1/D5-brane systems in a class of models with 16 supercharges. These models were obtained by orbifolding/orientifolding type IIB theory accompanied by a \( \mathbb{Z}_2 \) shift. The presence of the \( \mathbb{Z}_2 \) shift allowed us to apply the adiabatic principle in order to obtain the corresponding CFTs. Our proposed CFTs involved symmetric products of \( R^4 \) and \( T^4 \) factors with additional \( \mathbb{Z}_2 \) actions, whose precise form depends on the background in consideration. For backgrounds involving \( \Omega \) projection, the CFTs turn out to be \((4,0)\).

We have worked out elliptic genus formulae for these modified symmetric products and shown that the resulting multiplicities for D1/D5 bound states were in agreement with those of winding/momentum states in U-dually related theories. There remain however several open problems, which we have already anticipated in the introduction and discussed in Section 5.

Probably the most challenging one concerns the issue of 3-charge systems. As stressed above, our CFTs predict multiplicities which agree with U-duality in the 2-charge cases, i.e. in the CFTs of D1(D5)-KK systems or pure D1/D5 bound states, i.e. \( \Delta = \bar{\Delta} = 0 \). In general, problems arise when we excite momentum in the D1/D5-brane system, that is we let \( \Delta \neq 0 \). This corresponds to exciting states which preserve 1/4 of the 16 bulk supercharges (in the \((4,0)\) case the momentum is excited in the right-moving, non-supersymmetric sector). As we have noted in section 5, U-duality in this case puts constraints which generically are not satisfied by the proposed CFTs. To put it even more dramatically, the predictions of U-duality apparently do not admit any CFT interpretation, as they clash with
modular invariance. Since this problem is generic to all three charge systems in theories with sixteen supercharges studied so far, including the D1/D5 brane system in type IIB theory on $K3$, let us summarize the assumptions involved.

The usual type IIB theory on $K3$ as well as the model II studied in this paper are (4,4) superconformal field theories. In the $K3$ case there are good reasons to believe that the CFT is a deformation of the (4,4) symmetric product CFT times the center of mass CFT. Since model II is closely related to $K3$, it is also reasonable to assume a symmetric product (4,4) CFT. The elliptic genus (after taking 2-derivatives with respect to $\tilde{y}$ and setting it to 1) is then of the form

$$Z(p,q,y) = Z_{cm}(q,y)Z_{sym}(p,q,y) \quad Z_{sym}(p,q,y) = Z_F(p,y)Z_{sym}(p,q,y)$$

(7.1)

where the subscripts $cm$ and $sym$ denote center of mass and the internal theory (which in this case we are assuming is a symmetric product CFT) respectively and in the second equation we have separated out the $q^0$ term in $Z_F$ and the remaining terms in $Z_{sym}$. An inspection of the second equation in (4.17) relevant for the transverse shift case, shows that $Z_{sym}$ is symmetric under the exchange of $p$ and $q$. U-duality which exchanges D1 and KK modes, will exchange $p$ and $q$, with the result that $Z_F$ now becomes the center of mass contribution $Z_{cm}$ and the internal contribution $Z_{in}$ (which we don’t assume to be necessarily a symmetric product CFT) is the product $Z_{cm}(p,y)Z_{sym}(q,p,y)$. Expanding $Z_{in}$ in powers of $p$, at each order $p^n$ the coefficient $Z_{in}'(q,y)$ must describe the internal $(4,0)$ CFT of the system of 1 D5 and $n$ D1 branes in the U-dual theory. In particular for $n = 1$ we find

$$Z_1'(q,y) = Z_1(q,y) - Z_1(0,y) + f_1(y)$$

(7.2)

where $f_1$ is coefficient of $q^1$ in the $q$-expansion of $Z_{cm}(q,y)$ and $Z_1$ is the elliptic genus for the single copy of the (4,4) internal theory. Now $Z_1$ is a modular form in $q$ and $y$ under a suitable subgroup of $SL(2,Z)$ acting in the usual way. Thus, unless $f_1(y) = Z_1(0,y)$, the left hand side $Z_1'(q,y)$ will not be a modular form. On the other hand, $f_1(y)$ cannot be equal to $Z_1(0,y)$, since at $y = 1$ the former vanishes as it describes the bound states of D1/D5 system which forms 1/4 BPS state, while the latter is non-zero since it describes 1/2 BPS bound state. One might wonder that since in the U-dual theory we have only $(4,0)$ supersymmetry, the $SU(2)$ to which $y$ couples is broken. However, this is part of the $SO(4)_E$ which is the little group of the system under consideration and therefore it should have a well defined action on the states. If we assume that at the conformal invariant fixed point this global symmetry should be promoted to Kac-Moody algebra in the internal theory (actually we need only the $U(1)_y$ current algebra to which $y$ couples), then $Z_1'(q,y)$ must be a modular form. Thus we conclude that (4,4)
symmetric product theory, U-duality and the existence of $U(1)_y$ current algebra in the internal part of the U-dual (4,0) CFT cannot be satisfied simultaneously. One may try to argue that the problem is related to the fact that the 1/4-BPS states in theories with 16 supercharges generically are not stable throughout the whole moduli space. As a result the elliptic genus might jump. For example, it is believed [40] that in $\mathcal{N} = 4$ Yang-Mills theory in 4 dimensions 1/4 BPS dyonic states do indeed decay after crossing codimension 1 regions of marginal stability in the moduli space. In other words, single particle BPS states become multiparticle states. One can see from the mass formula for 1/4-BPS states that the region of marginal stability in 4-dimensional string theories is of real codimension 1 while in 5-dimensional theory it is the entire moduli space. This is indeed the case for the 5-dimensional type I’ theory (which is U-dual to IIB on $K^3$) as well as model III where the D1/D5 system is 1/4-BPS. However the D1/D5 system in the model II (or IIB on $K^3$) is 1/2 BPS. This system is at threshold in a codimension 4 subspace and as a result the corresponding (4,4) CFT is non-singular. This would suggest that the elliptic genus for this model should be constant throughout the moduli-space, which in turn would indicate that the 3-charge system is stable everywhere.

The symmetric product CFTs which arise by the adiabatic argument presented in section 3, is presumably valid at $\chi = C_{2345}/Q_1 = 1/2$. However, unlike in model II, the $\chi$ and $C_{(4)}$ fields are projected out in model III, and therefore, the $\chi = 1/2$ point cannot be connected to the point $\chi = 0$. In fact, as discussed at the end of section 2, the models obtained by $\Omega$ projection at $\chi = 1/2$ may be quite different from the one at $\chi = 0$. This might be a possible explanation for why the elliptic genus of the symmetric product theory for model II after $p, q$ exchange, which by U-duality should describe the multiplicities of the 3 charge system of model III at $\chi = C_{2345} = 0$, differs from that of the symmetric product theory for model III which is valid at nonzero $\chi$ and RR 4-form field. Conversely, $\chi$ and RR 4-form $C_{2345}$ of model III, under U-duality are mapped to NS 2-form $B_{15}$ and the RR 4-form $C_{1234}$ in model II. Note that these fields are projected out in model II but their discrete $Z_2$ fluxes are allowed. Therefore under the U-duality the symmetric product CFT of model III should be compared with the 3 charge system in model II in the background of the latter fields, which however breaks

\footnote{Actually model III with transverse shift is a 4-dimensional model and the subspace at which the system becomes threshold is real codimension one in the moduli space of this 4-dimensional theory. However the modulus which takes one away from the singular subspace is a certain combination of the metric $G_{16}$ and the RR 2-form $C_{16}$ which would break the Lorentz invariance in the world sheet along 01 directions.}
the Lorentz invariance of the world sheet along 01 directions.

The near horizon geometries of the D1/D5 systems describe $AdS_3 \times S^3$ times a 4-dimensional internal space. One can use AdS/CFT correspondence to compare the results from the bulk with that of the CFTs studied here. By introducing the concept of degree, [13], this comparison can be extended also to non-chiral primaries. In [44], we show that for model $II$, the CFT results are in complete agreement with the AdS/CFT predictions. On the other hand for model $III$, supergravity result is in contradiction with the $(4,0)$ CFT for the excited states. In fact, it turns out that supergravity elliptic genus is consistent with the U-dual model, namely it reproduces the elliptic genus of model $II$ with $p$ and $q$ exchanged (of course in the regime of validity). This result seems to support the remarks in the previous paragraph, according to which the elliptic genus for model $III$ at $\chi = 0$ should be given by that of model $II$ (after $p$, $q$ exchange). It is however hard to see how the gravity analysis in model $III$ would be affected by the non-trivial backgrounds of $\chi$ and the RR 4-form field. Therefore, a proper understanding of the $(4,0)$ models remains still an important open question.

Acknowledgements

We acknowledge discussions with J. de Boer, E. Kiritsis, J. Maldacena, B. Pioline, E Verlinde and especially with M. Bianchi and G. Thompson at various stages of this work. This project is supported in part by EEC under TMR contracts ERBFMRX-CT96-0090, HPRN-CT-2000-00148, HPRN-CT-2000-00122 and the INTAS project 991590. The work of J.F.M. is supported by the INFN section of University of Rome “Tor Vergata”.

8 Appendix A: $SO(n_1) \times SO(n_5)$ D1/D5 gauge theory

In this appendix we determine the spectrum of open string states living on intersections of D1/D5-branes, in “type I” like backgrounds, where the orientifold group action is accompanied by a shift longitudinal to the world-volume system. Our aim is to show that, unlike for the more familiar type I cousins, where consistency of the underlying open string theory requires that $\Omega$ projection acts with a relative sign between the D1 and D5 gauge groups [26], in the presence of a longitudinal shift $SO(n_1) \times SO(n_5)$ Chan-Paton assignements are allowed. We
adopt the open string descendant techniques systematized in [27]. Being interested in open string theories describing excitations of D-brane bound states rather than vacuum configurations we relax (and generically violate) tadpole cancellation conditions. We will discuss the case of model I, with orientifold group generated by $\Omega p_1$, but minor modifications are required to describe the $T$-dual model $III$ associated to $\Omega I_4 p_1$.

These vacuum configurations are often termed as “type I theory without open strings”, since the Klein bottle tadpole is removed by the presence of the shift and therefore the inclusion of D9(D5)-branes with their corresponding open string excitations are no longer needed [17]. Although we are mainly interested in the study of pure D1/D5 systems, with a little more effort, we can (and we will) include also $n_9$ D9-branes in our analysis. Besides aesthetical reasons, the inclusion of D9-branes will help the comparison with the more familiar type I results.

We start by describing the D1-D5-D9 system in the presence of the standard type I orientifold (O9)-plane. We orient $n_1$ D1- and $n_5$ D5-branes along (01) and (012345) planes respectively. For an homogeneous notation it will be convenient to start by wrapping the whole system on a $S^1 \times T^4 \times \tilde{T}^4$ torus, with directions $(1) \times (2345) \times (6789)$, and only at the end take the volume of $\tilde{T}^4$ to infinity. The annulus, Moebius strip and Klein bottle amplitudes associated to such brane configuration can be written as

$$\mathcal{K} = \frac{1}{2} \rho_{00} (2it) P_1(t) P_4(t) \tilde{P}_4(t)$$

$$\mathcal{A} = \frac{1}{2} \left[ \rho_{00} \left( \frac{it}{2} \right) \left( n_5^2 P_4(t) \tilde{P}_4(t) + n_5^3 P_4(t) \tilde{W}_4(t) + n_5^2 W_4(t) \tilde{W}_4(t) \right) \right.$$
performing the spin structure sums these traces can be written as

\[
\rho_{gh} = \frac{\eta^2 \eta^2}{\eta^6 \eta^2} \left( \frac{\frac{3}{2} + g}{\frac{3}{2} + h} \right) ^2 \quad g, h = \frac{1}{2} \quad \text{for} \quad g, h = A, B
\]

\[
\rho_{gh} = \frac{\eta^4 \eta^4}{\eta^4 \eta^4} \left( \frac{\frac{3}{2} + g}{\frac{3}{2} + h} \right) ^4 \quad g, h = \frac{1}{2} \quad \text{for} \quad g, h = C
\]

(A.2)

Finally momentum and winding lattice sums are given by

\[
P_d(t) = \sum_{m_i \in \mathbb{Z}^d} e^{-\pi t a^2 m_i^2} \quad W_d(t) = \sum_{n_i \in \mathbb{Z}^d} e^{-\pi t a^2 n_i^2}
\]

(A.3)

The basic requirement that these string amplitudes (A.1) should satisfy, is that, after the exchange of \(\sigma\) and \(\tau\) directions, they must admit a sensible interpretation in terms of closed string exchanges between boundaries (D-branes) and crosscaps (O9-planes). More precisely, the sum of closed string amplitudes should reconstruct the whole square

\[
\langle B | e^{-iH} | B \rangle
\]

(A.4)

with

\[
| B \rangle = |O9\rangle + n_9 |D9\rangle + n_5 |D5\rangle + n_1 |D1\rangle.
\]

(A.5)

and \(|O9\rangle, |D9\rangle, |D5\rangle\) and \(|D1\rangle\) representing the boundary state for corresponding brane objects.

Rewriting (A.1) in terms of the closed string variables \(\ell_K = \frac{1}{2\ell}, \ell_A = \frac{2}{\ell}, \ell_M = \frac{1}{2\ell}\) one is left (at the origin of the \(T^4 \times \tilde{T}^4\) lattice sum) with \(\int d\ell\times\)

\[
\tilde{\kappa}_0 = \frac{2^5}{2} \left[ \chi_0 + \chi_V + \chi_S + \chi_C \right] (i\ell) v_1 v_4 \tilde{v}_4 W_1 \left( \frac{\ell}{2} \right)
\]

\[
\tilde{\alpha}_0 = \frac{2^5}{2} \left[ \chi_0 I_0^2 + \chi_V I_V^2 + \chi_S I_S^2 + \chi_C I_C^2 \right] (i\ell) W_1 \left( \frac{\ell}{2} \right)
\]

\[
\tilde{\mathcal{M}}_0 = -\frac{2}{2} \left[ \chi_0 I_0 + \chi_V I_V + \chi_S I_S + \chi_C I_C \right] (i\ell) \sqrt{v_1 v_4 \tilde{v}_4} W_1 \left( \frac{\ell}{2} \right)
\]

(A.6)

\(W_1^{\text{even}}\) is defined like (A.3), with \(n\) restricted to be even. We have introduce the linear combinations of \(\rho_{gh}\) traces

\[
\begin{pmatrix}
\chi_0 \\
\chi_V \\
\chi_S \\
\chi_C
\end{pmatrix}
= \frac{1}{4}
\begin{pmatrix}
+ & + & + & + \\
+ & + & - & - \\
+ & - & - & + \\
+ & - & + & -
\end{pmatrix}
\begin{pmatrix}
\rho_{00} \\
\rho_{0A} \\
\rho_{0B} \\
\rho_{0C}
\end{pmatrix}
\]

(A.7)
and the Chan-Paton dependent combinations

\[
\begin{pmatrix}
I_0 \\
I_V \\
I_S \\
I_C
\end{pmatrix}
= 
\begin{pmatrix}
+ & + & + \\
+ & - & - \\
+ & - & + \\
+ & + & - \\
\end{pmatrix}
\begin{pmatrix}
n_9 \sqrt{v_1 v_4 \tilde{v}_4} \\
n_5 \sqrt{\frac{v_1}{v_4}} \\
n_1 \sqrt{\frac{v_1}{v_4 v_4}}
\end{pmatrix}.
\tag{A.8}
\]

with \( v_1, v_4 \) and \( \tilde{v}_4 \) the volumes of \( S^1, T^4 \) and \( \tilde{T}^4 \) respectively. The relevant modular transformations connecting the two expressions (A.1) and (A.6) can be easily read from (A.2) to be:

\[
\begin{align*}
\rho_{00}(-\frac{1}{i \alpha \ell}) &= (\alpha \ell)^{-4} \rho_{00}(i \alpha \ell) \quad \rho_{00}(\frac{i}{2t} + \frac{1}{2}) = t^{-4} \rho_{00}(\frac{it}{2} + \frac{1}{2}) \\
\rho_{0h}(-\frac{1}{i \alpha \ell}) &= 4(\alpha \ell)^{-2} \rho_{0h}(i \alpha \ell) \quad \rho_{0h}(\frac{i}{2t} + \frac{1}{2}) = -t^{-2} \rho_{0h}(\frac{it}{2} + \frac{1}{2}) \\
\rho_{0c}(-\frac{1}{i \alpha \ell}) &= 16 \rho_{0c}(i \alpha \ell) \quad \rho_{0c}(\frac{i}{2t} + \frac{1}{2}) = \rho_{0c}(\frac{it}{2} + \frac{1}{2}) \\
P_d(\frac{1}{\alpha \ell}) &= v_d(\alpha \ell)^\frac{3}{2} W_d(\alpha \ell)
\end{align*}
\tag{A.9}
\]

with \( h = A, B \) and \( \alpha \) a factors of 2 depending on the kind of one-loop diagram.

Rewritten in the basis (A.7), one can easily recognize in \( \tilde{K}_0 + \tilde{A}_0 + \tilde{M}_0 \) given by (A.6) as the different terms in the square (A.4). Notice that, unlike tadpole cancellation conditions, the requirement that the whole amplitude reconstructs a square, is a restriction on the structure of the entire tower of massive closed string amplitudes. Indeed, this requirement together with the choice of SO gauge groups for D9-branes is sufficient to fix completely the Moebius strip string amplitudes, once the Klein bottle and annulus amplitudes are given [27]. In particular, the relative signs between the \( \Omega \) projections on D5 and D1, D9 Chan-Paton factors are crucial in order to reproduce the square.

Let us consider the same system in the “type I” theory with the shift. First, let us notice that closed string states in (A.6) with odd windings enter only in the annulus amplitudes. This can be attributed to the fact that only these states can be reflected by the standard O9-plane. The situation gets reversed if we now accompany the worldsheet parity operator with a \( \sigma_{p_i} \) momentum shift along the circle. This is done by replacing the lattice sum \( P_1(t) \) in the Klein bottle and Moebius strip amplitudes (A.1) by

\[
P_1(t) \rightarrow P_1\left[ \frac{0}{\frac{1}{2}} \right](t) = \sum_{m_1 \in \mathbb{Z}} (-)^{m_1} e^{-\pi t a' \frac{m_1^2}{\ell^2}}
\tag{A.10}
\]

In the closed string channel this translates into the replacement

\[
W_1^{\text{even}}(\frac{\ell}{2}) \rightarrow W_1^{\text{odd}}(\frac{\ell}{2})
\tag{A.11}
\]

48
and therefore now only odd windings modes are reflected by the orientifold plane. We can see that combining this with a non-trivial Wilson line turned on in the D5 gauge group, one gets the desired result. The Wilson line can be included by replacing the lattice sum $P_1(t)$ accompanying annulus terms linear in $k$

\[
P_1(t) \rightarrow P_1\left[\frac{1}{2}\right] (t) = \sum_{n_1 \in \mathbb{Z}} e^{-\pi \alpha' \left(\frac{n_1 - \frac{1}{2}}{\eta}\right)^2} (A.12)
\]

After these replacements, the Klein bottle and annulus amplitudes can be written in the closed string channel as

\[
\tilde{K}_0 = \frac{2^5}{2} \left[ \chi_O + \chi_V + \chi_S + \chi_C \right] (i\ell) v_1 v_4 \tilde{v}_4 W_{1^{\text{odd}}}(\frac{\ell}{2})
\]

\[
\tilde{A}_0 = \frac{2^{-5}}{2} \left[ \chi_O I_O^2 + \chi_V I_V^2 + \chi_S I_S^2 + \chi_C I_C^2 \right] (i\ell) W_{1^{\text{even}}}^{\text{even}}(\frac{\ell}{2})
\]

\[+ \frac{2^{-5}}{2} \left[ \chi_O I_S^2 + \chi_V I_C^2 + \chi_S I_O^2 + \chi_C I_V^2 \right] (i\ell) W_{1^{\text{odd}}}^{\text{even}}(\frac{\ell}{2}) (A.13)
\]

The complete square is now reconstructed by

\[
\tilde{M}_0 = -\frac{2^5}{2} \left[ \chi_O I_S + \chi_V I_C + \chi_S I_O + \chi_C I_V \right] (i\ell) \sqrt{v_1 v_4} W_{1^{\text{odd}}}(\frac{\ell}{2}) (A.14)
\]

which differ from the ones in (A.6) by the parity of the closed string winding sum $W_{1^{\text{odd}}}$ and in an overall flip of the sign of $k$. This leads in the open string channel to the lattice sum (A.10) and the gauge group $SO(M) \times SO(k) \times SO(N)$ gauge theory. In deriving this result we have used the massless content $\rho_{00} = V + H$, $\rho_{0A} = V - H$. $\rho_{A0} = \frac{1}{2} H$ with $V$, $H$ denoting massless $N = 2$ vector- and hyper-multiplets. For the case we are interested in, we set $n_9 = 0$, which leads, after the above replacements, to the direct amplitudes

\[
K = \frac{1}{2} \rho_{00} (2i\ell) P_1\left[\frac{1}{2}\right] (t) P_4(t) \tilde{P}_4(t)
\]

\[
A = \frac{1}{2} \left[ \rho_{00} \left(\frac{i\ell}{2}\right) (n_5^2 P_4(t) \tilde{W}_4(t) + n_1^2 W_4(t) \tilde{W}_4(t)) P_1(t)
\right.
\]

\[+ 2 n_5 n_1 \rho_{A0} \left(\frac{i\ell}{2}\right) \tilde{W}_4(t) P_1\left[\frac{1}{2}\right] (t) \right] (A.15)
\]

\[
M = -\frac{1}{2} \left[ n_5 \rho_{0B} \left(\frac{i\ell}{2} + \frac{1}{2}\right) P_4(t) + n_1 \rho_{0C} \left(\frac{i\ell}{2} + \frac{1}{2}\right) \right] P_1\left[\frac{0}{\frac{1}{2}}\right] (t) (A.16)
\]

9 Appendix B: Symmetric product orbifold CFT: free field theory realization

In this appendix we give an alternative derivation of the partition function formulae (4.17) for the case where the Hilbert space $\mathcal{H}$ involved in the symmetric
product admits a free field theory (orbifold) description. We follow closely [22].

The oscillator contribution of a given worldsheet field $\Phi$ with boundary conditions (4.1), to the string partition function is given by

$$Z_{\text{osc}} \left[ g_{\phi} \right] (q, y) = \prod_{n=1}^{\infty} \left( 1 - e^{2\pi i n \phi} y^{n} q^{n-g_{\phi}} \right)^{\epsilon_{\phi}} \quad (B.1)$$

with $\epsilon_{\phi} = -1$ for bosons and $\epsilon_{\phi} = 1$ for fermions. The partition function (4.2), can then be written as a product over $\Phi$ of such contributions in the left and right moving-part of the CFT, times a (in general non-holomorphic) zero mode contribution

$$Z \left[ g_{\phi} \right] (\mathcal{H}|q, \bar{q}, y, \tilde{y}) = \tau_{2}^{-\frac{\phi}{2}} \prod_{\phi, \phi} Z_{0} Z_{\text{osc}} \left[ g_{\phi} \right] (q, y) \tilde{Z}_{0} \tilde{Z}_{\text{osc}} \left[ g_{\bar{\phi}} \right] (\bar{q}, \tilde{y}) \quad (B.2)$$

For complex bosonic and fermionic degrees of freedom this zero mode contribution can be written as

$$Z_{0} \left[ 0 \right]_{\text{boson}} (q, y) = q^{\chi_{\phi} - \frac{1}{2} P_{L}^{2}} q_{\phi}$$

$$Z_{0} \left[ g_{\phi} \right]_{\text{boson}} (q, y) = q^{\chi_{\phi}}$$

$$Z_{0} \left[ 0 \right]_{\text{fermi}} (q, y) = q^{-\chi_{\phi}} (y^{\phi} + y^{-\omega_{\phi}} - 2)$$

$$Z_{0} \left[ g_{\phi} \right]_{\text{fermi}} (q, y) = q^{-\chi_{\phi}} \quad (B.3)$$

with similar expressions for the right-moving components in terms of anti holomorphic quantities and $P_{L}$ replaced by $P_{R}$. In the following we will display only holomorphic formulas since the analysis of the antiholomorphic part follows trivially. The boson in (B.3) is understood to be compact. For non compact bosons we should of course simply omit the lattice sum in (B.3), $\omega_{\phi}$ stands for the charge of the field $\phi$ under $J_{0}^{3}$ and $\chi_{\phi}$ represents the contribution of a complex boson with spin characteristics $g_{\phi}, h_{\phi}$ to the zero point energy

$$\chi_{\phi} = -\frac{1}{12} + \frac{1}{2} g_{\phi} (1 - g_{\phi}) \quad (B.4)$$

Orbifold group sectors and the $N$ copies of the field $\Phi$ are labeled following the notation of section 4. Since the $g$ and $h$ twists commute we can diagonalize them simultaneously. In this basis one can write

$$g = e^{2\pi i \frac{l}{N}}$$

$$h = e^{2\pi i \left( \frac{L_{0}}{N L} + \frac{m}{N} \right)} \quad (B.5)$$
Let us now evaluate the basic trace $(4.10)$. For the time being we will concentrate on the left-moving contribution $Z_{osc}\left[\frac{g_0}{h_0}\right](q, y)$. After simple manipulations of the product formulae one is left with

\begin{equation}
(M)^{L_{\text{tr}}} t \times (L)^{N_L} : \prod_{i,l,m} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i (-\frac{\ell m}{L} + \frac{m}{L} + h_0)} y^{\omega_\phi} q^{n-g_0-l/L} \epsilon_\phi \right)
\end{equation}

\begin{equation}
= \prod_{i} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i (s_i g_0 + M h_0)} y^M \omega_\phi (q^\frac{M}{L} e^{2\pi i \frac{s_i}{L}})^{n-g_0 L} \epsilon_\phi \right)
\end{equation}

\begin{equation}
= \prod_{i} Z_{osc} \left[\frac{g_0 L}{g_0 s_i + M h_0}\right] (q^\frac{M}{L} e^{2\pi i \frac{s_i}{L}}, y^M) \quad (B.6)
\end{equation}

The result is in agreement with $(4.10)$. One can follow similar manipulations to show that the zero mode contribution to $S_N \mathcal{H}$ can be again reexpressed as $Z_0[\frac{g_0 L}{g_0 s + M h_0}](q^\frac{M}{L} e^{2\pi i \frac{s_i}{L}}, y^M)$. This is clear for the lattice sum and the fermionic zero mode trace following similar manipulations as before, while for the zero point energy this can be read off from $(B.4)$ (let say in the $(L)^M$-twisted sector) leading to a contribution $q^{\chi_{L,M}}$ with

\begin{equation}
\chi_{L,M} = M \epsilon_\phi \sum_{l=0}^{L-1} \left[\frac{1}{12} - \frac{1}{2} (g_0 + \frac{l}{L})(1 - g_0 - \frac{l}{L}) \right]
\end{equation}

\begin{equation}
= \frac{M}{L} \epsilon_\phi \left[\frac{1}{12} - \frac{1}{2} g_0 L (1 - g_0 L) \right] \quad (B.7)
\end{equation}

as expected.

This concludes our derivation of $(4.10)$ in the free field theory context.

Following the same steps as in section 4, one readily arrives to the symmetric product formulae $(4.17)$.

10 Appendix C: Modular integral with shifts

In this appendix we evaluate the modular integral (6.6).

\begin{equation}
G_{\delta, \delta}(v, w) = \frac{1}{2} \int \frac{d^2 \tau}{\tau_2^{3+2}} \sum_{g,h,n} C \left[\frac{g}{h}\right] (n) \Gamma_{2,2} \left[\frac{g}{h}\right] (v, w) e^{2\pi i n \tau} \quad (C.1)
\end{equation}

with $d = 2(6)$ in the case of models $I_F$, $III_F$ ($II_F$) and $\Gamma_{2,2}[g][h] (\ell \cdot \bar{v}, \ell_s \cdot \bar{w})$ defined by $(6.8)$.

We start by reabsorbing $(d + 2)/2$ powers (the number of $\bar{w}$-insertions) of $\tau_2$ by the rescaling of the left moving source $\bar{w} \rightarrow \frac{\bar{w}}{\tau_2}$. Equalities in the following are
understood to hold once \((d + 2)/2\) \(\vec{w}\)-derivatives are applied to the final result, with the sources \(\vec{w}\) then set to zero.

After performing the \(\tau_1\) gaussian integral we are left with

\[
G_{\vec{a},\vec{b}} = C \frac{(U_2 T_2)^{\frac{1}{2}}}{|m_1|} \int_0^\infty \frac{d\tau_2}{\tau_2^{3/2}} e^{-b_0 - \gamma \tau_2 - \frac{\vec{a}}{\tau_2}} = C \frac{(U_2 T_2)^{\frac{1}{2}}}{|m_1|} \sqrt{\frac{\pi}{\beta}} e^{-b_0 - 2\sqrt{\beta}} \tag{C.2}
\]

with

\[
\begin{align*}
C & = \frac{1}{2} \sum_M \eta \left[ g_{[h]} \right] C \left[ g_{[h]} \right] (n,1) \\
b_0 & = -2\pi i \frac{T_1 m_1 n_2 - 2\pi i \frac{U_1 n_1}{m_1}}{m_2} (n_2 U_1 + n_1) \\
& \quad + 2\pi i \ell \cdot \left[ \vec{n} \vec{v} - v_1 (n_2 U_1 + n_1) \right] - 2\pi i \ell_* \cdot w_1 \left[ m_1 - \frac{U_2}{m_1 T_2} (n + m_1 \ell \cdot v_1) \right] \\
\beta & = \pi n^2 T_2 U_2 + 2\pi i n_2 \ell_* \cdot (w_1 U_1 - w_2) + \frac{\pi U_2}{T_2} (\ell_* \cdot w_1)^2 \\
\gamma & = \frac{m^2}{U_2} \frac{\pi T_2}{U_2} \left[ 1 + \frac{U_2}{m^2 T_2} (n + m_1 \ell \cdot v) \right]^2 \tag{C.3}
\end{align*}
\]

We are interested in the leading order in a \(1/T_2\) expansion of (C.2), which is associated in the dual theory to the semiclassical approximation around the D-instanton background. In this limit all subleading \(\vec{w}\)-dependent terms in the exponential can be discarded since they lead to subleading contributions once they are hit by \(\vec{w}\)-derivatives. At the leading order the result can be written as

\[
G_{\vec{a},\vec{b}} = \sum_{m_1, n_2, g, h} \frac{1}{|n_2|} e^{2\pi i \left[ T m_1 n_2 + \frac{U_1}{m_1} n_2 \ell \cdot \vec{v} - m_1 \ell \cdot \vec{w} \right]} J_{\vec{a},\vec{b}} + h.c. \tag{C.4}
\]

where \(\vec{v} = v_1 U - v_2\) and \(\vec{w} = \frac{1}{U_2} (w_1 \vec{U} - w_2)\) are induced sources and \(h.c.\) denotes the hermitian conjugate contributions coming from anti D-instantons. \(J_{\vec{a},\vec{b}}\) represents the shift dependent sum

\[
J_{\vec{a},\vec{b}} = \frac{1}{2|m_1|} \sum_{n_1, g, h} e^{2\pi i \left( \frac{a_1}{m_1} n_1 - 2a_1 b_1 g h - a_1 h m_1 + a_1 g n_1 \right)} C \left[ g_{[h]} \right] (n,1). \tag{C.5}
\]

Our next task is to evaluate (C.5) in the cases of a winding \(a_1 = 1\) or momentum shift \(b_1 = 1\). The domains of \(\vec{m}, \vec{n}\) are specified by (6.9), while \(n\) is integer in the untwisted sector and both integer and half integer in the twisted sector. Using the identity

\[
C \left[ \frac{1}{2} \right] \left( n,1 \right) = (-)^{2n} C \left[ \frac{1}{2} \right] \left( n,1 \right) \tag{C.6}
\]

52
and performing the geometric sums one can write the final result as
\[
\mathcal{J}_{1,0} = \frac{1}{2} (C \left[ \frac{1}{2} \right] + (-)^{m_1} C \left[ \frac{1}{2} \right] ) (n, l) \mathcal{P}^{Z}_{\frac{m_1}{m_1}} + \frac{1}{2} C \left[ \frac{0}{1/2} \right] (n, l) \mathcal{P}^{Z+\frac{1}{2}}_{\frac{m_1}{m_1}}
\]
\[
\mathcal{J}_{0,1} = \frac{1}{2} (C \left[ \frac{1}{2} \right] + (-)^{n} C \left[ \frac{1}{2} \right] + 2C \left[ 0 \right] ) (n, l) \mathcal{P}^{Z}_{\frac{m_1}{m_1}}
\]
(C.7)

with \( \mathcal{P}^{Z+\delta}_{\frac{m_1}{m_1}} \) a projector onto states with \( \frac{n}{m_1} \in Z + \delta \).

Plugging in (C.4) and introducing the quantum number \( k \equiv \frac{n}{m_1} \in Z + a_1 g \) one obtains, after performing the remaining \( m_1, n_2 \) sums, with the final result (6.15).

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54
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