A Comment on the Spectrum of H-Monopoles

S. Sethi*

Lyman Laboratory of Physics  
Harvard University  
Cambridge, MA 02138, USA

and

M. Stern†

Department of Mathematics  
Duke University  
Durham, NC 27706, USA

We consider the question of ground states in the supersymmetric system that arises in the search for the missing H-monopole states. By studying the effective theory near certain singularities in the five-brane moduli space, we find the remaining BPS states required by the conjectured S-duality of the toroidally compactified heterotic string.
1. Introduction

The toroidally compactified heterotic string provides a natural string theory in which to conjecture the existence of an exact strong-weak coupling duality. Some of the testable consequences of such a duality have been described in detail by Sen [1]. Indeed, substantial evidence has accumulated for S-duality in the field theoretic limit; yet, attempts at a truly stringy verification of S-duality have created some puzzles. We shall consider the heterotic theory at a general point in its moduli space, where the gauge group is abelian. The electrically charged states of interest are constructed by tensoring the right moving ground state with an arbitrary left moving state. To saturate the BPS bound, the constraint,

\[ N_L - 1 = \frac{1}{2} (p_R^2 - p_L^2), \]

must be satisfied [2]. Our interest resides in the twenty-four predicted H-monopole states with \( N_L = 1 \). The number twenty-four just corresponding to the choice of left moving oscillator. The first study of the H-monopole spectrum, performed in [3], encountered problems reconciling restrictions on the allowed structure of the H-monopole moduli space with the required number of normalizable modes. Recently, Witten shed a great deal of light on the question of the H-monopole spectrum in his study of small instantons [4]. By a careful study of supersymmetric ground states in five-brane quantization at generic points in the moduli space, Witten finds eight of the desired twenty-four states. The purpose of this letter is to show that the remaining sixteen states arise from the singularities in the five-brane moduli space where the \( SU(2) \) gauge symmetry, arising from small instantons, is still broken to \( U(1) \), but a charged hypermultiplet becomes light, as conjectured in [4]. There are sixteen such singularities, so we desire a single normalizable ground state in the effective theory near the singularity.

The problem involves a potential with flat directions extending to infinity which somewhat complicates the analysis. There is a further issue of gauge invariance since, near the singularity, we must include the charged hypermultiplet in our analysis. The particular system that arises in studying the question of the missing H-monopole states is a special case of a more general class of theories. The Hamiltonian for these systems takes the general form,

\[ H = \frac{1}{2} \text{Tr}(p^i p^j) + V(x) + H_F. \] (1.1)

The bosonic potential \( V(x) \) is polynomial in \( x \), and generally has flat directions. The term \( H_F \) is quadratic in the fermions and linear in \( x \). The coordinates \( x \) are charged under
the gauge group, which is generally non-abelian, and the trace is over the gauge indices. Models of this kind appear, for instance, in the study of D-brane bound states [5]. We shall not undertake an analysis of the most general case. Such an analysis is very interesting, but quite subtle. Rather, we shall use certain features of this particular model that simplify the counting of the number of normalizable ground states for the effective theory near the singularity.

Consequently, we shall restrict to the the case where the gauge group is abelian. Since there are charged fields in the model under consideration, the supersymmetry algebra no longer closes on the Hamiltonian. Rather, the supersymmetry algebra closes if a constraint, $C = C^b + C^f$, following from Gauss’ law, is set to zero. The gauge constraint, $C$, splits into two $U(1)$ generators: one generates rotations of the charged bosons, $C^b$, while the other generates rotations of the charged fermions, $C^f$. As usual, we are interested in counting the number of $L^2$ ground states weighted by $(-1)^F$ where $F$ is the fermion number. Therefore, we wish to compute the index,

$$\text{Ind} = \int dx \lim_{\beta \to \infty} \text{tr} (-1)^F e^{-\beta H}(x, x),$$

$$= n_B - n_F,$$

where the trace is over the gauge invariant spectrum of the Hamiltonian i.e. states $|\psi(x)\rangle$ satisfying $C|\psi(x)\rangle \geq 0$. Since the space of scalars is non-compact and the potential has flat directions, the integral depends on $\beta$, and usually, we cannot just consider the more easily evaluated $\beta \to 0$ limit. Our first task is to implement the projection onto gauge invariant states explicitly, so we can trace over the full, unconstrained spectrum. Let us denote the operator generating a gauge transformation $g(\theta)$ on the fermions by $\Pi(g(\theta))$ where we shall drop the explicit dependence on $\theta$. To project onto gauge invariant states, we insert:

$$I(\beta) = \int_{U(1)} d\theta \int dx \text{tr} e^{i\theta C} (-1)^F e^{-\beta H}(x, x),$$

$$= \int_{U(1)} d\theta \int dx \text{tr} \Pi(g) (-1)^F e^{-\beta H}(gx, x),$$

where the measure for the $U(1)$ integration is chosen so that $\int_{U(1)} d\theta = 1$. The trace is now over the full Hilbert space, including gauge-variant states. Now we can choose a
supersymmetry generator, \(Q\), obeying \(H = Q^2\) under the assumption \(C = 0\). We can then write our index as,

\[
\text{Ind} = \lim_{\beta \to 0} \int dx \int_{U(1)} d\theta \text{tr} \left( (-1)^F e^{-\beta H} \Pi(g)(gx, x) + \int dx \int_{U(1)} d\theta \int_0^\infty d\beta \text{tr} \left( (-1)^F Q^2 e^{-\beta H} \Pi(g)(gx, x) \right) \right. (1.4)
\]

As we shall argue, the second term does not appear to contribute to the index for this particular system. The first term in (1.4) can easily be reduced to quadrature using perturbation theory, as we shall describe in the following section.

2. Counting H-monopole States

The model of interest is the dimensional reduction of abelian supersymmetric Yang-Mills from 5 + 1 dimensions to 0 + 1 with a single charged hypermultiplet. Rather than reduce from six dimensions, we shall, for convenience, reduce N=2 Yang-Mills from four dimensions. This choice hides the symmetry between the scalars coming from the reduction of the six dimensional connection, but it doesn’t affect our analysis in any significant way. From reducing the (four-dimensional) vector multiplet and Higgs field, we obtain three real scalars \(x^a\), and a complex scalar \(y\). The hypermultiplet provides two more complex coordinates \(Q_+\) and \(Q_-\), where the subscripts denote the \(U(1)\) charge. Introducing canonical momenta obeying,

\[
[x, p_x] = i,
\]

we find that the Hamiltonian for this system takes the form:

\[
H = \frac{1}{2} p^a p_a^\dagger + p_y p_y^\dagger + p_+ p_+^\dagger + p_- p_-^\dagger + \frac{1}{2} (Q_+ Q_-^\dagger + Q_- Q_+^\dagger)^2 + (x^a x^a + 2yy^\dagger)(Q_+ Q_-^\dagger + Q_- Q_+^\dagger) + H_F. (2.1)
\]

The term \(H_F\), which we shall describe below, contains operators quadratic in the fermions and linear in the coordinates. The crucial point about the bosonic potential is the existence of flat directions, occurring when \(Q_+ = Q_- = 0\). Such flat directions complicate the counting of ground states since a normalizable ground state decays with a power law along the flat directions, rather than with the usual exponential fall-off.

The situation is actually somewhat more subtle than the preceding comment might imply. If one were to consider just the bosonic theory, then the spectrum for this model
is actually discrete – regardless of the flat directions in the potential. To construct a scattering state along the flat direction, we would want to put the oscillators, which are transverse to the flat directions, into their ground states; however, the zero point energy for these oscillators increases as we travel down the flat direction, prohibiting finite energy scattering states. Unfortunately, this argument is no longer true for the supersymmetric theory, since the additional fermionic oscillators, required by supersymmetry, cancel the bosonic zero point energy.

Let us introduce a set of two component fermions, \( L, N, M_-, M_+ \), each of which obey an anti-commutation relation of the form,

\[ \{ L_\alpha, L_\beta^\dagger \} = \delta_\alpha\beta. \]

The fermionic term, \( H_F \), is given by,

\[
H_F = x^a (M_+^\dagger \sigma^a M_+ - M_-^\dagger \sigma^a M_-) + \sqrt{2}(y M_- t M_+ - y^\dagger M_+^\dagger t M_-^\dagger) \\
+ \sqrt{2}(Q_+ M_+^\dagger t L^\dagger - Q_+^\dagger M_+ t L - Q_- M_-^\dagger t L^\dagger + Q_-^\dagger M_- t L) \\
+ \sqrt{2}(Q_- N t M_+ + Q_+ N t M_- - Q_-^\dagger N^\dagger t M_+^\dagger - Q_+^\dagger N^\dagger t M_-^\dagger),
\]

where \( \sigma^a \) are the Pauli matrices. The matrix \( t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and summation is implied on all indices.

The component of the constraint which generates gauge transformations on the charged bosons, \( C^b \), is proportional to the operator,

\[ Q_-^\dagger \pi_-^\dagger - Q_+^\dagger \pi_+^\dagger + Q_+ \pi_+ - Q_- \pi_- \],

while the component generating transformations on the fermions, \( C^f \), is proportional to,

\[ M_+^\dagger M_+ - M_-^\dagger M_- \].

Our task is to show the existence of a normalizable state satisfying \( H\lvert \psi \rangle = 0 \), and \( C\lvert \psi \rangle = 0 \).

To count the number of ground states with sign, we begin by computing the principal term in (1.4),

\[
I(0) = \lim_{\beta \to 0} \int dx \, dy \, dQ \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \text{tr}(-1)^F e^{-\beta H} \Pi(g) (gQ, Q).
\]
The projection operator $\Pi(g)$ is $e^{i\theta C^f}$. To compute this term, we shall approximate $e^{-\beta H}$ by the operator,

$$\frac{1}{(2\pi\beta)^{9/2}}e^{-(|x-x'|^2 + 2|y-y'|^2 + 2|e^{i\theta}Q_+ - Q_+|^2 + 2|e^{-i\theta}Q_+ - Q_-|^2)/2\beta} \times e^{-\beta V} e^{-\beta H_F},$$

where $V$ is the bosonic potential given in (2.1). Corrections to this approximation are suppressed by powers of $\beta$, and give a vanishing contribution to the principal term. As usual, the inclusion of $(-1)^F$ in the trace creates fermion zero modes which must be absorbed to obtain a non-vanishing contribution. Fermions appear from two sources: $H_F$, and the constraint $C^f$. To obtain the required number of $L$ and $N$ zero modes, all fermion zero modes must be supplied by $H_F$, rather than $C^f$. A non-vanishing contribution then arises when $\Pi(g)$ is set to one. The integral now becomes,

$$\lim_{\beta \to 0} \int \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \beta^{15/4} \frac{1}{(2\pi\beta)^{9/2}} e^{-(|e^{i\theta}Q_+ - Q_+|^2 + |e^{-i\theta}Q_+ - Q_-|^2)/\beta^{3/2}} \times e^{-V} \frac{1}{8!} \text{tr} (-1)^F (H_F)^8,$$

where the explicit trace is now only over the fermion modes. By rescaling all of the scalar coordinates, we can replace the integral by the expression,

$$\lim_{\beta \to 0} \int \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \beta^{15/4} \frac{1}{(2\pi\beta)^{9/2}} e^{-(|e^{i\theta}Q_+ - Q_+|^2 + |e^{-i\theta}Q_+ - Q_-|^2)/\beta^{3/2}} \times e^{-V} \frac{1}{8!} \text{tr} (-1)^F (H_F)^8.$$

We can rewrite $|e^{i\theta}Q_+ - Q_+|^2$ as $2|Q_+|^2(1 - \cos \theta)$, which can further be replaced by $|Q_+|^2\theta^2$ as $\beta$ becomes small. On rescaling $\theta$, we are left with the integral,

$$\int \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{1}{(2\pi)^{9/2}} e^{-\theta^2(|Q_+|^2 + |Q_-|^2)} e^{-V} \frac{1}{8!} \text{tr} (-1)^F (H_F)^8,$$

which fortunately is $\beta$-independent. Note that without the projection onto gauge-invariant states, we would not have arrived at an expression independent of $\beta$. A preliminary investigation of $H_F$, shown in (2.2), shows that the terms proportional to $x$ or $y$ cannot contribute the required number of fermions for a non-vanishing trace. The integrations over $x, y, \theta$ are then Gaussian giving a total factor of,

$$\frac{1}{2(2\pi)^{9/2}} \frac{\pi^2}{(|Q_+|^2 + |Q_-|^2)^{3/2}} \frac{1}{8!}.$$
and leaving only the integration over $Q_+$, and $Q_-$.

We now require the term in $(H_F)^8$ proportional to the ‘volume form’ for the fermion zero modes. A non-vanishing contribution comes from the following terms in $(H_F)^8$,

$$16 \left( \frac{8}{4} \right)^2 \left( \frac{4}{2} \right)^2 (Q_{-\dagger} t M_{- \dagger} L - Q_{+\dagger} M_{+ \dagger} t L)^2 (Q_{+\dagger} M_{+ \dagger} t L\dagger - Q_{-\dagger} M_{- \dagger} t L\dagger)^2 \times$$

$$(Q_{-\dagger} N t M_{+ \dagger} + Q_{+\dagger} N t M_{- \dagger})^2 (Q_{-\dagger\dagger} N t M_{- \dagger\dagger} + Q_{+\dagger\dagger} N t M_{+ \dagger\dagger})^2.$$

Ignoring the overall sign, we obtain a term $16 \cdot 8! (|Q_+|^2 + |Q_-|^2)^4$ multiplied by the ‘volume form’ for the complex fermions. After integrating out the fermions, the final integral reduces to,

$$\int \frac{8\pi^2}{(2\pi)^9/2} (|Q_+|^2 + |Q_-|^2) e^{-\frac{1}{2}(|Q_+|^2 + |Q_-|^2)^2}.$$

After expressing the complex coordinate $Q$ in terms of real coordinates using $Q = \frac{1}{\sqrt{2}} (Q^r + iQ^i)$, we can easily evaluate the integral which gives one for the contribution of the principal term to the index, again neglecting the overall sign.

There are two issues that remain to be addressed. The first is whether the second term in (1.4) gives a non-vanishing contribution to the index. One may integrate by parts to show that this second integral is proportional to

$$\lim_{R \to \infty} \int_{|x|=R} \ dx \int_{U(1)} \ d\theta \int_0^\infty \ d\beta \frac{x^i}{R} \psi^i (-1)^F Q e^{-\beta H} \Pi(g) (gx, x),$$

where $\psi^i$ is a fermionic operator, and $x$ denotes all bosonic variables. For a more detailed discussion, see for example, [7]. Therefore, we only need to consider the kernel, $e^{-\beta H}$, at large $R$ where $R = |x|$. Away from the flat points as $R \to \infty$, the potential term, $e^{-\beta V}$, strongly suppresses any boundary contribution. Let us consider the theory in the neighborhood of a flat point. Without the mass perturbation, the bosonic potential takes the form $V \sim -\frac{1}{2} r^2 |Q|^2 + O(|Q|^4)$, where $Q$ parametrizes the transverse directions, and $r$ is a radial coordinate for the flat directions. The Hamiltonian is then essentially a set of bosonic and fermionic harmonic oscillators for the transverse directions, and a free Laplacian along the flat directions. The two systems are coupled through the frequency of the harmonic oscillators which depends on the radial coordinate. For very large $r$, to obtain a finite energy solution, the wavefunction in the transverse directions is approximately the harmonic oscillator ground state, and decays very rapidly. Clearly, the only possible place
for a boundary term to arise is from a small neighborhood of the flat points. There are a number of arguments that suggest that there is no contribution from around these points. For instance, after performing the $\beta$ integration, we obtain

$$\lim_{R \to \infty} \int_{|x|=R} dx \int_{U(1)} d\theta \, \frac{x^i}{R} \text{tr} \psi^j (-1)^F \Pi(g) (-1)^F \Pi(g) (gx, x).$$

Here $H^{-1}$ is the unbounded operator defined to be zero on the kernel of $H$ and to have image orthogonal to the kernel of $H$. In order to show that this boundary term is zero, we construct an approximation $G$ to $H^{-1}$ with $E := I - HG$ decaying polynomially at infinity and show that

$$\lim_{R \to \infty} \int_{|x|=R} dx \int_{U(1)} d\theta \, \frac{x^i}{R} \text{tr} \psi^j (-1)^F QG \Pi(g) (gx, x) = 0.$$  

The error in this approximation is then

$$\lim_{R \to \infty} \int_{|x|=R} dx \int_{U(1)} d\theta \, \frac{x^i}{R} \text{tr} \psi^j (-1)^F QH^{-1} E \Pi(g) (gx, x),$$

which one expects to vanish (but we shall not establish this rigorously at this time). The operator $G$ is obtained in a standard iterative construction, except that rather than Fourier expanding in all variables we Fourier expand in the $x, y$ directions and use the natural harmonic oscillator expansion in the $Q$ directions. Following the approach outlined above, we have found no correction to the index for this model.

So far, we have counted ground states in an index sense. We would like to argue that the index is actually counting the total number of ground states. However, there does not seem to be a simple vanishing theorem to guarantee that there are no ground states which are either fermionic or bosonic in this particular model. Agreement with duality suggests that the ground state that we have found is unique. Nevertheless, we can conclude that there is at least one normalizable ground state from each of the sixteen singularities. These modes provide the missing H-monopole states as required by the conjectured S-duality of the toroidally compactified heterotic string.

**Note added:** While we were completing this work, a paper [8] appeared which discusses this problem from a somewhat different viewpoint.

**Acknowledgements**

We would like to thank A. Lesniewski, S. Mathur and E. Witten for helpful discussions and comments. The work of S. S. was supported in part by a Hertz Fellowship and NSF grant PHY-92-18167; that of M. S. by NSF Grant DMS 9505040.
References

[1] A. Sen, Int. J. Mod. Phys. A9 (1994) 3707.
[2] A. Dabholkar and J. Harvey, Phys. Rev. Lett. 63 (1989) 719.
[3] J. Gauntlett and J. Harvey, “S-Duality and the Spectrum of Magnetic Monopoles in Heterotic String Theory”, hep-th/9407111.
[4] E. Witten, Nucl. Phys. B460 (1996) 541.
[5] E. Witten, Nucl. Phys. B460 (1996) 335.
[6] B. Simon, Ann. Phys. 146 (1983), 209.
[7] C. Callias, Commun. Math. Phys. 62 (1978), 213.
[8] M. Porrati, “How to Find H-monopoles in Brane Dynamics”, hep-th/9607082.