Jacobi fields and odular structure of affine manifolds

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Abstract

The connection between Jacobi fields and odular structures of affine manifold is established. It is shown that the Jacobi fields generate the natural geoodular structure of affinely connected manifolds.

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1 Introduction

The recent development of geometry has shown the importance of non-associative algebraic structures, such as quasigroups, loops and odules. For instance, it is possible to say that the nonassociativity is the algebraic equivalent of the differential geometric concept of the curvature. The corresponding construction may be described as follows. In a neighbourhood of an arbitrary point on a manifold with the affine connection one can introduce the geodesic local loop, which is uniquely defined by means of the parallel translation of geodesics along geodesics (Kikkawa, 64; Sabinin 72, 77). The family of local loops constructed in this way uniquely defines the space with affine connection, but not every family of geodesic loops on a manifold yields an affine connection. It is necessary to add some algebraic identities connecting loops in different points. This additional algebraic structures (so-called geoodular structures) were introduced and the equivalence of the categories of geoodular structures and of affine connections was shown by Sabinin (77, 81). The development of this line gave a start to a new approach to manifold with affine coonection - Loopuscular and Odular Geometry. The main algebraic structures arising in this approach are related to nonassociative algebra and theory of quasigroups and loops.

In our paper we investigate the connection between Jacobi fields and odular structure of affinely connected manifold. In Section 2 we outline the main consructions of the loopuscular and odular geometry. In Section 3 we proof our main result (Theorem 3.3).

2 Odular structure of manifolds with affine connection

Below we outline the main facts from the loopuscular geometry following (Sabinin, 77, 81, 87, 88, 98).

Definition 2.1. Let \( \langle M, \cdot, \varepsilon \rangle \) be a partial magma with a binary operation \( (x, y) \mapsto x \cdot y \) and the neutral element \( \varepsilon, \ x \cdot \varepsilon = \varepsilon \cdot x = x; \) \( M \) be a smooth manifold (at least \( C^1 \)-smooth) and the operation of multiplication (at least \( C^1 \)-smooth) be defined in some neighbourhood \( U_\varepsilon \), then \( \langle M, \cdot, \varepsilon \rangle \) is called a partial loop on \( M \).
Remark 2.1. The operation of multiplication is locally left and right invertible. This means, if \( x \cdot y = L_x y = R_y x \), then there exist \( L_x^{-1} \) and \( R_y^{-1} \) in some neighbourhood of the neutral element \( \varepsilon \):

\[
L_a(L_a^{-1}x) = x, \quad R_a(R_a^{-1}x) = x.
\]

The vector fields \( A_j \) defined in \( U_\varepsilon \) by

\[
A_j(x) = \left( (L_x)_{*x,\varepsilon} \right)^j \frac{\partial}{\partial x^i}
\]

are called the left basic fundamental fields. Similarly, the right basic fundamental fields \( B_j \) are defined by

\[
B_j(x) = \left( (R_x)_{*x,\varepsilon} \right)^j \frac{\partial}{\partial x^i}.
\]

The solution of the equation

\[
\frac{d f^i(t)}{dt} = L^i_j(f(t))x^j, \quad f(0) = \varepsilon
\]

yields the exponential map

\[
\text{Exp} : X \in T_\varepsilon(\mathfrak{m}) \longrightarrow \text{Exp}X \in \mathfrak{m}.
\]

The unary operation

\[
t x = \text{Exp} t \text{Exp}^{-1} x,
\]

based on the exponential map, is called the left canonical unary operation for \( (\mathfrak{m}, \cdot, \varepsilon) \). A smooth loop \( (\mathfrak{m}, \cdot, \varepsilon) \) equipped with its canonical left unary operations is called the left canonical preodule \( (\mathfrak{m}, \cdot, (t)_{t \in \mathbb{R}}, \varepsilon) \). If one more operation is introduced

\[
x + y = \text{Exp}(\text{Exp}^{-1} x + \text{Exp}^{-1} y),
\]

then we obtain the canonical left prediodule of a loop, \( (\mathfrak{m}, \cdot, +, (t)_{t \in \mathbb{R}}, \varepsilon) \). A canonical left preodule (prediodule) is called the left odule (diodule), if the monoassociativity property

\[
t x \cdot u x = (t + u) x
\]

is satisfied. In the smooth case for an odule the left and the right canonical operations as well as the exponential maps coincide.
Definition 2.2. Let $\mathcal{M}$ be a smooth manifold and

$$ L : (x, y, z) \in \mathcal{M} \mapsto L(x, y, z) \in \mathcal{M} $$

a smooth partial ternary operation, such that $x_ay = L(x, a, z)$ defines in the some neighbourhood of the point $a$ the loop with the neutral $a$, then the pair $(\mathcal{M}, L)$ is called a loopuscular structure (manifold).

A smooth manifold $\mathcal{M}$ with a smooth partial ternary operation $L$ and smooth binary operations $\omega_t : (a, b) \in \mathcal{M} \times \mathcal{M} \mapsto \omega_t(a, b) = t_a b \in \mathcal{M}$, $(t \in \mathbb{R})$, such that $x_ay = L(x, a, y)$ and $t_az = \omega_t(a, z)$ determine in some neighborhood of an arbitrary point $a$ the odule with the neutral element $a$, is called a left odular structure (manifold) $(\mathcal{M}, L, (\omega_t)_{t \in \mathbb{R}})$. Let $(\mathcal{M}, L, (\omega_t)_{t \in \mathbb{R}})$ and $(\mathcal{M}, \Lambda, (\omega_t)_{t \in \mathbb{R}})$ be odular structures then $(\mathcal{M}, L, \Lambda, (\omega_t)_{t \in \mathbb{R}})$ is called a diodular structure (manifold). If $x_ay = \Lambda(x, a, y)$ and $t_ax = \omega_t(a, x)$ define a vector space, then such a diodular structure is called a linear diodular structure.

Definition 2.3. (Sabinin, 77, 81) Let $\mathcal{M}$ be a $C^k$-smooth ($k \geq 3$) affinely connected manifold and the following operations are defined on $\mathcal{M}$:

$$ L(x, a, y) = x_ay = \exp_x \tau^a_x \exp_y^{-1}y, $$

$$ \omega_t(a, z) = t_az = \exp_a \tau^a_t \exp_z^{-1}z, $$

$$ \Lambda(x, a, y) = x_\Lambda(y) = \exp_a (\exp_y^{-1}x + \exp_y^{-1}y), $$

Exp$_x$ being the exponential map at the point $x$ and $\tau^a_x$ the parallel translation along the geodesic going from $a$ to $x$. The construction above is called a natural linear geodiodular structure of an affinely connected manifold $(\mathcal{M}, \nabla)$.

Remark 2.2. Any $C^k$-smooth ($k \geq 3$) affinely connected manifold can be considered as a geoodular structure.

Definition 2.4. (Sabinin, 77, 81, 86) Let $(\mathcal{M}, L)$ be a loopuscular structure of a smooth manifold $\mathcal{M}$. Then the formula

$$ \nabla_{X_a} Y = \left\{ \frac{d}{dt} \left[ \left[ (L^g(t))_{*a} \right]^{-1} Y_{g(t)} \right] \right\}_{t=0}, $$

$g(0) = a, \quad \dot{g}(0) = X_a,$

$Y$ being a vector field in the neighbourhood of a point $a$, defines the tangent affine connection.
In coordinates the components of affine connection are written as

$$\Gamma^i_{jk}(a) = -\left[\frac{\partial^2 (x_ay)^i}{\partial x^j \partial x^k}\right]_{x=y=a}$$

The equivalence of the categories of geoodular (geodiodular) structures and of affine connections is formulated as follows (Sabinin, 77, 81).

**Proposition 2.1.** The tangent affine connection $\nabla$ to the natural geoodular (geodiodular) structure of an affinely connected manifold $(\mathfrak{M}, \nabla)$ coincides with $\nabla$.

**Proposition 2.2.** The natural geoodular (geodiodular) structure $\mathfrak{M}$ of the tangent affine connection to a natural geoodular (geodiodular) structure $\mathfrak{M}$ coincides with $\mathfrak{M}$.

## 3 Jacobi fields and odular structure

A vector field $X$ along a geodesic $\gamma$ is called a *Jacobi field* if it satisfies the Jacobi differential equation (the geodesic deviation equation)

$$\frac{D^2 X}{dt^2} + R(X, Y)Y = 0,$$

(10)

$D/dt$ being the operator of the parallel translation along $\gamma$, $R(X, Y)$ the curvature operator and $Y = d\gamma/dt$ the tangent vector to the geodesic $\gamma(t)$.

As is known the Jacobi equation has $2n$ linearly independent solutions, which are completely determined by the initial conditions: $X(0)$ and $DX(0)/dt \in T_\gamma(\mathfrak{M})$. Besides, every geodesic admits two natural Jacobi fields. The first one is defined as follows: $X_1 = Y$, and the second one: $X_2 = tY$; $t$ being canonical parameter along the geodesic (Kobayashi and Nomizu, 69; Milnor, 73).

**Definition 3.1.** (Kobayashi and Nomizu, 69) A one-parametric family of geodesics $\alpha(s, t)$, $-\varepsilon < s < \varepsilon$, such that $\alpha(0, t) = \gamma(t)$, $0 \leq t \leq 1$, is called a variation of the geodesic $\gamma(t)$.

This means, that there is a $C^\infty$-differentiable map from $[0, 1] \times (-\varepsilon, \varepsilon)$ to $\mathfrak{M}$ such that:

(i) for each given $s \in (-\varepsilon, \varepsilon)$ $\alpha(s, t)$ is the geodesic;
\( \alpha(0, t) = \gamma(t) \) for \( 0 \leq t \leq 1 \).

An infinitesimal variation \( X \) of the geodesic \( \gamma(t) \) is defined as follows:

\[
X|_x = \frac{\partial \alpha(s, t)}{\partial s} \quad \text{for} \quad 0 \leq t \leq 1, \tag{11}
\]

where \( \alpha(s, t) \) is a variation. Further the following theorem is important.

**Theorem 3.1.** (Kobayashi and Nomizu 69) A vector field \( X \) along the geodesic \( \gamma \) is a Jacobi field if and only if it is an infinitesimal variation for \( \gamma \).

Let us consider a geodesic variation \( \alpha(s, t) \) of a geodesic \( \gamma(t) \) starting at a point \( a \) such that \( \Gamma(s) = \alpha(s, 0) \) be a geodesic passing through the point \( a \), as well, and the infinitesimal variation \( X = \partial \alpha(s, t)/\partial s \) satisfies

\[
X|_\Gamma \equiv X(s, 0) \neq 0, \quad \frac{DX(s, 0)}{dt} \equiv \frac{DX}{dt} \bigg|_\Gamma = 0. \tag{12}
\]

Let \( \zeta = \partial \alpha(s, t)/\partial s|_{s=t=0} \) and \( \xi = \partial \alpha(s, t)/\partial t|_{s=t=0} \) be tangent vectors at the point \( a \) to geodesics \( \Gamma(s) \) and \( \gamma(t) \) respectively. The following lemma is valid.

**Lemma 3.2.** Let \( \tau^a_x \) be the parallel translation along \( \Gamma \), then it holds

\[
\alpha(s, t) = \text{Exp}_{x(s)} \tau^a_{x(s)} \text{Exp}^{-1}_a y(t),
\]

where \( x(s) = \text{Exp}_a(s\zeta) \) and \( y(t) = \text{Exp}_a(t\xi) \).

**Proof.** Any geodesic \( \tilde{\alpha}(x, t) \) passing through a point \( x \in \Gamma \) can be presented as \( \tilde{\alpha}(x, t) = \text{Exp}_x(t\eta) \), \( \eta \) being the tangent vector to \( \tilde{\alpha} \) at the point \( x \), and \( t \) the canonical parameter. Representing \( \eta \) as \( \eta = \tau^a_x \xi \), where \( \tau^a_x \) is the parallel translation along \( \Gamma \), we find

\[
\tilde{\alpha}(x, t) = \text{Exp}_x(t\tau^a_x \xi) = \text{Exp}_x \tau^a_x(t\xi).
\]

Introducing \( y(t) = \text{Exp}_a(t\xi) \) and \( x = x(s) = \alpha(s, 0) \), we obtain

\[
\tilde{\alpha}(x(s), t) = \text{Exp}_{x(s)}(t\tau^a_{x(s)} \xi) = \text{Exp}_{x(s)} \tau^a_{x(s)} \text{Exp}^{-1}_a y(t).
\]

On the other hand, \( \tilde{\alpha}(x(s), t) \) can be obtained as the result of the variation of the geodesic \( \gamma(t) \). This yields \( \tilde{\alpha}(x(s), t) = \alpha(s, t) \) and, obviously,

\[
\alpha(s, t) = \text{Exp}_{x(s)} \tau^a_{x(s)} \text{Exp}^{-1}_a y(t).
\]

The proof is completed. \( \square \)
Theorem 3.3. (Nesterov, 1989) Jacobi fields generate the natural linear geodiodular structure of an affinely connected space \((\mathfrak{m}, \nabla)\).

**Proof.** Taking into account Lemma 3.2 we define the operation \(L(x(s), a, y(t))\) in the following way:

\[
L(x(s), a, y(t)) = \alpha(s, t) = \text{Exp}_{x(s)} \tau^{a}_{x(s)} \text{Exp}_{a}^{-1} y(t).
\]

Actually, this operation is defined for arbitrary points \(x, y\) in the some neighbourhood \(U_{a}\) of the point \(a\). Thus, it can be represented as

\[
L(x, a, y) = \text{Exp}_{x} \tau^{a}_{x} \text{Exp}_{a}^{-1} y,
\]

(see **Def. 2.3**, (7)).

Now let us consider the following variation of geodesic \(\gamma(t)\):

\[
\beta(s, t) = \text{Exp}_{a}(t\xi + st\eta).
\]

The infinitesimal variation \(X = \partial \beta(s, t) / \partial s\) satisfies

\[
X(s, 0) = 0, \quad \frac{DX(s, 0)}{dt} \neq 0.
\]

Defining \(x(t) = \text{Exp}_{a}(t\xi)\) and \(y(s, t) = \text{Exp}_{a}(st\eta)\), we obtain

\[
\beta(s, t) = \text{Exp}_{a}(\text{Exp}_{a}^{-1} x(t) + \text{Exp}_{a}^{-1} y(s, t)).
\]

In fact, this is valid for arbitrary points \(x, y \in U_{a}\). Thus, one can identify the operation (9) as the geodesic variation \(\beta\):

\[
\Lambda(x, a, y) = \beta(s, t) = \text{Exp}_{a}(\text{Exp}_{a}^{-1} x + \text{Exp}_{a}^{-1} y),
\]

The operation (8) is related to the existence of the affine parameter along a geodesic. The proof is completed.

\[\square\]

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