Strong Law of Large Numbers for Iterates of Some Random-Valued Functions

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Abstract. Assume \((\Omega, \mathcal{A}, P)\) is a probability space, \(X\) is a compact metric space with the \(\sigma\)-algebra \(\mathcal{B}\) of all its Borel subsets and \(f : X \times \Omega \to X\) is \(\mathcal{B} \otimes \mathcal{A}\)-measurable and contractive in mean. We consider the sequence of iterates of \(f\) defined on \(X \times \Omega^\infty\) by \(f^0(x, \omega) = x\) and \(f^n(x, \omega) = f(f^{n-1}(x, \omega), \omega_n)\) for \(n \in \mathbb{N}\), and its weak limit \(\pi\). We show that if \(\psi : X \to \mathbb{R}\) is continuous, then for every \(x \in X\) the sequence \(\left(\frac{1}{n} \sum_{k=1}^{n} \psi(f^k(x, \cdot))\right)_{n \in \mathbb{N}}\) converges almost surely to \(\int_X \psi d\pi\). In fact, we are focusing on the case where the metric space is complete and separable.

Mathematics Subject Classification. 37H12, 39B12, 60B12, 60F15.

Keywords. Random-valued functions, Iterates, Strong law of large numbers, Convergence in law, Almost sure convergence.

1. Introduction

Fix a probability space \((\Omega, \mathcal{A}, P)\) and a metric space \(X\).

Let \(\mathcal{B}\) denote the \(\sigma\)-algebra of all Borel subsets of \(X\). We say that \(f : X \times \Omega \to X\) is a random-valued function (shortly: an rv-function) if it is measurable with respect to the product \(\sigma\)-algebra \(\mathcal{B} \otimes \mathcal{A}\). The iterates of such an rv-function are given by

\[
  f^0(x, \omega_1, \omega_2, \ldots) = x, \quad f^n(x, \omega_1, \omega_2, \ldots) = f(f^{n-1}(x, \omega_1, \omega_2, \ldots), \omega_n)
\]

for \(n \in \mathbb{N}\), \(x \in X\) and \((\omega_1, \omega_2, \ldots)\) from \(\Omega^\infty\) defined as \(\Omega^\mathbb{N}\). Note that \(f^n : X \times \Omega^\infty \to X\) is an rv-function on the product probability space \((\Omega^\infty, \mathcal{A}^\infty, P^\infty)\). More exactly, for \(n \in \mathbb{N}\) the \(n\)-th iterate \(f^n\) is \(\mathcal{B} \otimes \mathcal{A}_n\)-measurable, where \(\mathcal{A}_n\) denotes the \(\sigma\)-algebra of all sets of the form

\[
  \{(\omega_1, \omega_2, \ldots) \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}
\]
with $A$ from the product $\sigma$-algebra $\mathcal{A}^n$. See [10, Sec. 1.4], [8].

A result on a.s. convergence of $(f^n(x, \cdot))_{n\in\mathbb{N}}$ for $X$ being the unit interval can be found in [10, Sec. 1.4B]. The paper [7] brings theorems on the convergence a.s. and in $L^1$ of those sequences of iterates in the case where $X$ is a closed subset of a separable Banach lattice. A simple criterion for the convergence in law of $(f^n(x, \cdot))_{n\in\mathbb{N}}$ to a random variable independent of $x \in X$ was proved in [1], assuming that $X$ is complete and separable. In [2] it has been strengthened and applied to obtain a weak law of large numbers for iterates of random-valued functions. In the present paper we are interested in a strong law of large numbers. We will be based on the following Brunk-Prokhorov-type theorem, see [11, Theorem 3.3.1] and [6, Corollary 3.1].

\[ (C) \text{ Let } (F_n)_{n\in\mathbb{N}} \text{ be an increasing sequence of sub-$\sigma$-algebras of } \mathcal{A} \text{ and } (\xi_n)_{n\in\mathbb{N}} \text{ a sequence of random variables such that } \xi_n \text{ is } F_n\text{-measurable and } E(\xi_{n+1}|F_n) = 0 \text{ for each } n \in \mathbb{N}. \text{ If } (a_n)_{n\in\mathbb{N}} \text{ is an increasing and unbounded sequence of positive reals and} \]

\[ \sum_{n=1}^{\infty} \frac{E(|\xi_n|^2)}{a_n^2} < \infty, \]

\[ \text{then} \]

\[ \lim_{n\to\infty} \frac{1}{a_n} \sum_{k=1}^{n} \xi_k = 0 \text{ a.s.} \]

2. A Scheme

Assume $X$ is a metric space and $f : X \times \Omega \to X$ an rv-function.

Lemma 1. If $\varphi : X \to \mathbb{R}$ is Borel and $\varphi \circ f^n(x, \cdot)$ is integrable for $P^\infty$ for each $x \in X$ and $n \in \mathbb{N}$, then the function $\alpha : X \to \mathbb{R}$ defined by

\[ \alpha(x) = \int_{\Omega} \varphi(f(x, \omega))P(d\omega) \]  

is Borel and

\[ E(\varphi \circ f^{n+1}(x, \cdot)|\mathcal{A}_n) = \alpha \circ f^n(x, \cdot) \text{ for } x \in X \text{ and } n \in \mathbb{N}. \]

Proof. Since $\varphi \circ f$ is $\mathcal{B} \otimes \mathcal{A}$-measurable, by Fubini’s theorem $\alpha$ is Borel. Consequently, for every $x \in X$ and $n \in \mathbb{N}$ the function $\alpha \circ f^n(x, \cdot)$ is $\mathcal{A}_n$-measurable and for each $A \in \mathcal{A}^n$ we have

\[ \int_{\{\omega \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}} \varphi(f^{n+1}(x, \omega))P^\infty(d\omega) = \int_{\{\omega \in \Omega^\infty : (\omega_1, \ldots, \omega_n) \in A\}} \varphi(f^n(x, \omega), \omega_{n+1})P^\infty(d\omega) \]
\[
\int_{\{\omega \in \Omega : (\omega_1, \ldots, \omega_n) \in A\}} \left( \int_{\Omega} \varphi \left( f^n(x, \omega), \omega_{n+1} \right) P(d\omega_{n+1}) \right) P^\infty(d\omega)
= \int_{\{\omega \in \Omega : (\omega_1, \ldots, \omega_n) \in A\}} \alpha(f^n(x, \omega)) P^\infty(d\omega).
\]

The following theorem is in fact a scheme of proving a strong law of large numbers for iterates of random-valued functions.

**Proposition 1.** Let \( \psi : X \to \mathbb{R} \) and assume that there exists a Borel and bounded \( \varphi : X \to \mathbb{R} \) such that

\[
\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega) + \psi(x) \quad \text{for } x \in X. \tag{2}
\]

If \((a_n)_{n \in \mathbb{N}}\) is an increasing and unbounded sequence of positive reals such that

\[
\sum_{n=1}^{\infty} \frac{1}{a_n^2} < \infty,
\]

then, for every \( x \in X \),

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=1}^{n} \psi \circ f^k(x, \cdot) = 0 \quad \text{a.e. for } P^\infty. \tag{3}
\]

**Proof.** Define \( \alpha : X \to \mathbb{R} \) by (1). Since \( \varphi \) is bounded, \(|\varphi(x)| \leq M \) for every \( x \in X \) with an \( M \in (0, \infty) \). Obviously also \(|\alpha(x)| \leq M \) for every \( x \in X \). Fix \( x \in X \) and put

\[
\xi_n = \varphi \circ f^n(x, \cdot) - \alpha \circ f^{n-1}(x, \cdot) \quad \text{for } n \in \mathbb{N}. \tag{4}
\]

Then \(|\xi_n| \leq 2M\) and by Lemma 1, \( \mathbb{E}(\xi_{n+1} | \mathcal{A}_n) = 0 \) for each \( n \in \mathbb{N} \). It now follows from Brunk-Prokhorov-type theorem (C) that

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=1}^{n} \left( \varphi \circ f^k(x, \cdot) - \alpha \circ f^{k-1}(x, \cdot) \right) = 0 \quad \text{a.e. for } P^\infty. \tag{5}
\]

Since \( \psi = \varphi - \alpha \), for every \( n \in \mathbb{N} \) we have

\[
\sum_{k=1}^{n} \psi \circ f^k(x, \cdot) = \sum_{k=1}^{n} \left( \varphi \circ f^k(x, \cdot) - \alpha \circ f^{k-1}(x, \cdot) \right)
+ \sum_{k=1}^{n} \left( \alpha \circ f^{k-1}(x, \cdot) - \alpha \circ f^k(x, \cdot) \right),
\]

i.e.,

\[
\sum_{k=1}^{n} \psi \circ f^k(x, \cdot) = \sum_{k=1}^{n} \left( \varphi \circ f^k(x, \cdot) - \alpha \circ f^{k-1}(x, \cdot) \right) + \alpha(x) - \alpha \circ f^n(x, \cdot)
\]

for every \( n \in \mathbb{N} \). Moreover, \(|\alpha \circ f^n(x, \cdot)| \leq M \). Consequently (3) holds. \( \square \)
3. The Weak Limit

Assume now the following hypothesis (H).

(H) $(X, \rho)$ is a complete and separable metric space and $f : X \times \Omega \to X$ is an rv-function such that

$$\int_{\Omega} \rho(f(x, \omega), f(z, \omega)) P(d\omega) \leq \lambda \rho(x, z) \quad \text{for } x, z \in X$$

with a $\lambda \in (0, 1)$, and

$$\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) < \infty \quad \text{for } x \in X.$$  

Then (see [1, Theorem 3.1]) there exists a probability Borel measure $\pi^f$ on $X$ such that for every $x \in X$ the sequence of distributions of $f^n(x, \cdot)$, $n \in \mathbb{N}$, converges weakly to $\pi^f$. See also [3, Lemma 2.2] and [9, Corollary 5.6 and Lemma 3.1].

This limit distribution $\pi^f$ plays an important role in solving functional equations, in particular in the class of Hölder continuous functions. We call a function $\psi : X \to \mathbb{R}$ Hölder continuous with exponent $\delta \in (0, 1]$ if there is a constant $L \in [0, \infty)$ such that

$$|\psi(x) - \psi(z)| \leq L \rho(x, z)^\delta \quad \text{for } x, z \in X.$$  

Moreover we call a function Hölder continuous if it is Hölder continuous with an exponent $\delta \in (0, 1]$. The following theorem (see [3, Theorem 2.1] and [4, Corollary 2.6]) will be useful to us.

(B) Assume (H). If $\psi : X \to \mathbb{R}$ is Hölder continuous with exponent $\delta \in (0, 1]$, then it is integrable for $\pi^f$ and if additionally

$$\int_X \psi(x) \pi^f(dx) = 0,$$  

then there exists a Hölder continuous with exponent $\delta$ function $\varphi : X \to \mathbb{R}$ such that (2) holds.

4. Main Results

In what follows $(X, \rho)$ is a metric space and $f : X \times \Omega \to X$ is an rv-function.

We start with a simple consequence of Proposition 1 and (B). It is a special case of Theorem 2 given below, but shows our approach without technical details.

**Theorem 1.** If $(X, \rho)$ is complete and separable with finite diameter and (7) holds with a $\lambda \in (0, 1)$, then for every Hölder continuous $\psi : X \to \mathbb{R}$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ f^k(x, \cdot) = \int_X \psi d\pi^f \quad \text{a.e. for } P^\infty.$$  

(10)
Proof. Fix a Hölder continuous \( \psi : X \to \mathbb{R} \). Replacing \( \psi \) by \( \psi - \int_X \psi d\pi_f \) we may assume that (9) holds. By (B) there is a Hölder continuous \( \varphi : X \to \mathbb{R} \) satisfying (2). Since \( X \) is bounded, so is \( \varphi \). Applying now Proposition 1 with \( a_n = n \) for \( n \in \mathbb{N} \) we obtain (3) which ends the proof. \( \square \)

Since continuous real functions defined on a compact metric space can be uniformly approximated by Lipschitz functions (see [5, 11.2.4]), Theorem 1 implies the following corollary.

**Corollary 1.** If \((X, \rho)\) is compact and (7) holds with a \( \lambda \in (0,1) \), then we have (10) for every continuous \( \psi : X \to \mathbb{R} \) and for each \( x \in X \).

**Theorem 2.** Assume (H). Let \( x \in X \) and

\[
\sum_{n=1}^{\infty} \int_{\Omega^\infty} \rho(f^n(x, \omega), x)^{2\delta} P^\infty(d\omega) < \infty
\]

with a \( \delta \in (0,1] \) and an increasing and unbounded sequence \((a_n)_{n \in \mathbb{N}}\) of positive reals. If \( \psi : X \to \mathbb{R} \) is Hölder continuous with exponent \( \delta \), then

\[
\lim_{n \to \infty} \frac{1}{a_n} \sum_{k=1}^{n} \left( \psi \circ f^k(x, \cdot) - \int_X \psi d\pi_f \right) = 0 \quad \text{a.e. for } P^\infty. \tag{11}
\]

The proof will be based on three lemmas.

Assume that \((X, \rho)\) is separable, (7) holds with a \( \lambda \in (0,1) \), (8) is satisfied and \( \varphi : X \to \mathbb{R} \) is Hölder continuous with exponent \( \delta \in (0,1] \), i.e.,

\[
|\varphi(x) - \varphi(z)| \leq L \rho(x, z)^{\delta} \quad \text{for } x, z \in X \tag{12}
\]

with an \( L \in [0, \infty) \).

**Lemma 2.** For every \( x \in X \) and \( n \in \mathbb{N} \) we have

\[
\int_{\Omega^\infty} \rho(f^n(x, \omega), x) P^\infty(d\omega) \leq \frac{1}{1 - \lambda} \int_{\Omega} \varrho(f(x, \omega), x) P(d\omega),
\]

\[
\int_{\Omega^\infty} |\varphi(f^n(x, \omega))| P^\infty(d\omega) \leq L \left( \int_{\Omega^\infty} \rho(f^n(x, \omega), x) P^\infty(d\omega) \right)^{\delta} + |\varphi(x)|.
\]

**Proof.** Fix \( x \in X \), \( n \in \mathbb{N} \) and assume for the inductive proof that

\[
\int_{\Omega^\infty} \rho(f^n(x, \omega), x) P^\infty(d\omega) \leq \sum_{k=0}^{n-1} \lambda^k \int_{\Omega} \varrho(f(x, \omega), x) P(d\omega).
\]
Then, applying Fubini’s theorem, (7) and the above inequality, we obtain
\[
\begin{align*}
\int_{\Omega} \rho(f^{n+1}(x, \omega), x) P^\infty(d\omega) \\
\leq \int_{\Omega} \rho(f^n(x, \omega, \omega_1, \ldots, \omega_{n+1}), f(x, \omega_{n+1})) P^\infty(d(\omega_1, \omega_2, \ldots)) \\
+ \int_{\Omega} \rho(f(x, \omega_{n+1}), x) P(d\omega_{n+1}) \\
\leq \lambda \int_{\Omega} \rho(f^n(x, \omega), x) P^\infty(d\omega) + \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \\
\leq \sum_{k=0}^{n} \lambda^k \int_{\Omega} \rho(f(x, \omega), x) P(d\omega)
\end{align*}
\]
which ends the proof of the first part. To get the second one observe that by (12) and Jensen’s inequality for every \(x \in X\) and \(n \in \mathbb{N}\) we have
\[
\begin{align*}
\int_{\Omega} |\varphi(f^n(x, \omega))| P^\infty(d\omega) &\leq L \int_{\Omega} \rho(f^n(x, \omega), x)^\delta P^\infty(d\omega) + |\varphi(x)| \\
&\leq L \left( \int_{\Omega} \rho(f^n(x, \omega), x) P^\infty(d\omega) \right)^\delta + |\varphi(x)|.
\end{align*}
\]

Lemma 2 makes sense to define a Borel function \(\alpha : X \to \mathbb{R}\) by (1).

Lemma 3. For every \(x \in X\) and \(n \in \mathbb{N}\) we have
\[
\begin{align*}
\int_{\Omega} |\varphi(f^n(x, \omega)) - \alpha(f^{n-1}(x, \omega))|^2 P^\infty(d\omega) \\
\leq 8L^2 \int_{\Omega} \rho(f^n(x, \omega), x)^{2\delta} P^\infty(d\omega).
\end{align*}
\]

Proof. Since, for every \(\omega \in \Omega^\infty\) and \(\omega' \in \Omega\),
\[
|\varphi(f^n(x, \omega)) - \varphi(f(f^{n-1}(x, \omega), \omega'))| \leq L \rho(f^n(x, \omega), f(f^{n-1}(x, \omega), \omega'))^\delta \\
\leq L \left( \rho(f^n(x, \omega), x)^\delta + \rho(f(f^{n-1}(x, \omega), \omega'), x)^\delta \right),
\]
for every \(\omega \in \Omega\) we have
\[
\begin{align*}
|\varphi(f^n(x, \omega)) - \alpha(f^{n-1}(x, \omega))|^2 \\
= \left| \int_{\Omega} (\varphi(f^n(x, \omega)) - \varphi(f(f^{n-1}(x, \omega), \omega'))) P(d\omega') \right|^2 \\
\leq L^2 \left( \rho(f^n(x, \omega), x)^\delta + \int_{\Omega} \rho(f(f^{n-1}(x, \omega), \omega'), x)^\delta P(d\omega') \right)^2 \\
\leq 4L^2 \left( \rho(f^n(x, \omega), x)^{2\delta} + \left( \int_{\Omega} \rho(f(f^{n-1}(x, \omega), \omega'), x)^\delta P(d\omega') \right) \right)^2.
\end{align*}
\]
Hence, applying Jensen’s inequality and Fubini’s theorem,
\[
\int_{\Omega_\infty} |\varphi(f^n(x,\omega)) - \alpha(f^{n-1}(x,\omega))|^2 \, P^\infty(d\omega)
\leq 4L^2 \left( \int_{\Omega_\infty} \rho(f^n(x,\omega), x)^{2\delta} \, P^\infty(d\omega) \right)
+ \int_{\Omega_\infty} \left( \int_{\Omega} \rho(f^{n-1}(x,\omega'), x)^{2\delta} \, P(d\omega') \right) P^\infty(d\omega)
= 8L^2 \int_{\Omega_\infty} \rho(f^n(x,\omega), x)^{2\delta} \, P^\infty(d\omega).
\]
\[\square\]

**Lemma 4.** Let \((b_n)_{n\in\mathbb{N}}\) be a converging to zero sequence of positive reals. If \(x \in X\) and there is a \(p \in (0, \infty)\) such that
\[
\sum_{n=1}^{\infty} b_n \int_{\Omega_\infty} \rho(f^n(x,\omega), x)^p \, P^\infty(d\omega) < \infty,
\]
then
\[
\lim_{n \to \infty} b_n \alpha \circ f^n(x, \cdot) = 0 \quad \text{a.e. for } P^\infty.
\]

**Proof.** If \(n \in \mathbb{N}\) and \(\omega \in \Omega\), then by (1), (12), Jensen’s inequality and (7) we have
\[
|\alpha(f^n(x,\omega))| \leq \int_{\Omega} |\varphi(f^n(x,\omega),\omega')| \, P(d\omega')
\leq L \int_{\Omega} \rho(f^n(x,\omega),\omega') \, P(d\omega')
+ L \int_{\Omega} \rho(f(x,\omega'), x)^\delta \, P(d\omega') + |\varphi(x)|
\leq L\lambda^\delta \rho(f^n(x,\omega), x)^\delta + L \left( \int_{\Omega} \rho(f(x,\omega), x) \, P(d\omega) \right)^\delta + |\varphi(x)|.
\]
Now to finish the proof it is enough to show that \(\lim_{n \to \infty} b_n \xi_n = 0\) a.e. for \(P^\infty\), where \(\xi_n = \rho(f^n(x,\cdot), x)^\delta\) for \(n \in \mathbb{N}\). To this end observe that by Markov’s inequality for every \(n \in \mathbb{N}\) and \(\varepsilon > 0\) we have
\[
P^\infty(b_n \xi_n \geq \varepsilon) \leq \frac{\mathbb{E}(\xi_n^p)}{(\varepsilon/n)^p} = \frac{1}{\varepsilon^p} b_n^p \mathbb{E}(\xi_n^p).
\]
Hence it follows from the assumption of the lemma that for every \(\varepsilon > 0\) the series \(\sum_{n=1}^{\infty} P^\infty(b_n \xi_n \geq \varepsilon)\) converges. Consequently, \(\lim_{n \to \infty} b_n \xi_n = 0\) a.e. for \(P^\infty\).
\[\square\]

**Proof of Theorem 2.** Fix a Hölder continuous with exponent \(\delta\) function \(\psi : X \to \mathbb{R}\). Replacing \(\psi\) by \(\psi - \int_X \psi d\pi f\) we may assume that (9) holds. By (B) there is a Hölder continuous with exponent \(\delta\) function \(\varphi : X \to \mathbb{R}\) satisfying...
(2). Now using Lemma 2 define a Borel function \( \alpha : X \to \mathbb{R} \) by (1). Since \( \psi = \varphi - \alpha \), (6) follows. Applying Lemmas 1 and 3, and the Brunk-Prokhorov-type theorem (C) to the sequence of random variables \((\xi_n)_{n \in \mathbb{N}}\) defined by (4), we have (5). Finally, by Lemma 4 with \( b_n = \frac{1}{a_n} \), \( n \in \mathbb{N} \), and \( p = 2 \),

\[
\lim_{n \to \infty} \frac{1}{a_n} \alpha \circ f^n(x, \cdot) = 0 \quad \text{a.e. for } P^\infty.
\]

This, (5), (6) and (9) give (11). \( \square \)

**Corollary 2.** Assume (H). If \( \psi : X \to \mathbb{R} \) is Hölder continuous with an exponent \( \delta \leq \frac{1}{2} \), then we have (10) for each \( x \in X \).

**Proof.** It is enough to observe that by Jensen’s inequality and Lemma 2 for every \( x \in X \) we have

\[
\int_{\Omega} \rho(f^n(x, \omega), x)^{2\delta} P^\infty(d\omega) \leq \left( \int_{\Omega} \rho(f^n(x, \omega), x) P^\infty(d\omega) \right)^{2\delta} \leq \left( \frac{1}{1 - \lambda} \int_{\Omega} \rho(f(x, \omega), x) P(d\omega) \right)^{2\delta},
\]

and then to apply Theorem 2 with \( a_n = n, \ n \in \mathbb{N} \). \( \square \)

To get a result for exponents \( \delta > \frac{1}{2} \) we accept the following hypothesis \((H_\delta)\) with parameter \( \delta \in (0, \infty) \).

\((H_\delta)\) \((X, \rho)\) is a complete and separable metric space, \( f : X \times \Omega \to X \) is an rv-function such that

\[
\rho(f(x, \omega), f(z, \omega)) \leq \xi(\omega) \rho(x, z) \quad \text{for } \omega \in \Omega \text{ and } x, z \in X,
\]

where \( \xi : \Omega \to [0, \infty) \) is a random variable for which \( \mathbb{E}(\xi^{2\delta}) < 1 \), and

\[
\int_{\Omega} \rho(f(x_0, \omega), x_0)^{2\delta} P(d\omega) < \infty
\]

with an \( x_0 \in X \).

**Remark 1.** If \( \delta \geq \frac{1}{2} \), then \((H_\delta)\) implies (H).

**Proof.** Assume \((H_\delta)\) with a \( \delta \geq \frac{1}{2} \). By Jensen’s inequality

\[
\mathbb{E}\xi = \mathbb{E}(\xi^{2\delta})^{\frac{1}{2\delta}} \leq (\mathbb{E}(\xi^{2\delta}))^{\frac{1}{2\delta}} < 1
\]

and

\[
\int_{\Omega} \rho(f(x_0, \omega), x_0) P(d\omega) \leq \left( \int_{\Omega} \rho(f(x_0, \omega), x_0)^{2\delta} P(d\omega) \right)^{\frac{1}{2\delta}}.
\]
Moreover, for every $x \in X$,
\begin{align*}
\int_{\Omega} \rho(f(x, \omega), x) P(d\omega) &\leq \int_{\Omega} \rho(f(x, \omega), f(x_0, \omega)) P(d\omega) + \int_{\Omega} \rho(f(x_0, \omega), x_0) P(d\omega) + \rho(x_0, x) \\
&\leq (E\xi + 1) \rho(x, x_0) + \int_{\Omega} \rho(f(x_0, \omega), x_0) P(d\omega) + \rho(x_0, x).
\end{align*}

\textbf{Theorem 3.} Assume (H$_{\delta}$) with $\delta \in [\frac{1}{2}, 1]$. If $\psi : X \to \mathbb{R}$ is Hölder continuous with exponent $\delta$, then we have (10) for each $x \in X$.

\textit{Proof.} By Remark 1 we have (H), and it follows from Theorem 2 that to finish the proof it is enough to show that for every $x \in X$ the sequence
\[
\left( \int_{\Omega} \rho(f^n(x, \omega), x)^{2\delta} P^\infty(d\omega) \right)_{n \in \mathbb{N}}
\]
is bounded. This follows from the lemma that is stated below. $\Box$

Let
\[
\beta_p(x) = \int_{\Omega} \rho(f(x, \omega), x)^p P(d\omega) \quad \text{for } p \in (0, \infty) \text{ and } x \in X.
\]

\textbf{Lemma 5.} Assume (13) holds with a random variable $\xi : \Omega \to [0, \infty)$ and let $p$ be a positive real. If $E(\xi^p) < 1$ and $\beta_p(x_0) < \infty$ for an $x_0 \in X$, then $\beta_p(x) < \infty$ for every $x \in X$ and there exists a constant $c_p \in (0, \infty)$ such that
\[
\int_{\Omega} \rho(f^n(x, \omega), x)^p P^\infty(d\omega) \leq c_p \beta_p(x) \quad \text{for } x \in X \text{ and } n \in \mathbb{N}.
\]

\textit{Proof.} Fix $x \in X$. By (13) for every $\omega \in \Omega$ we have
\[
\rho(f(x, \omega), x)^p \leq 3^p \left( \xi(\omega)^p \rho(x, x_0)^p + \rho(f(x_0, \omega), x_0)^p + \rho(x_0, x)^p \right),
\]
whence
\begin{align*}
\int_{\Omega} \rho(f(x, \omega), x)^p P(d\omega) &\leq 3^p \left( (E\xi^p) + 1 \right) \rho(x, x_0)^p + \int_{\Omega} \rho(f(x_0, \omega)x_0)^p P(d\omega) \\
&< \infty.
\end{align*}

Put now
\[
\eta(\omega) = \rho(f(x, \omega), x) \quad \text{for } \omega \in \Omega,
\]
and
\[
\xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n), \quad \eta_n(\omega_1, \omega_2, \ldots) = \eta(\omega_n)
\]
for \( n \in \mathbb{N} \) and \((\omega_1, \omega_2, \ldots) \in \Omega^\infty\). Then, by induction and (13),
\[
\rho(f^n(x, \omega), x) \leq \sum_{k=1}^{n} \eta_k(\omega)\xi_{k+1}(\omega) \cdot \ldots \cdot \xi_n(\omega) \quad \text{for } \omega \in \Omega^\infty \text{ and } n \in \mathbb{N},
\]
where \( \prod_{j=n+1}^{n} \xi_j(\omega) := 1 \). Consequently,
\[
\int_{\Omega^\infty} \rho(f^n(x, \omega), x)^p P^\infty(d\omega) \leq \mathbb{E}\left(\left(\sum_{k=1}^{n} \eta_k \prod_{j=k+1}^{n} \xi_j\right)^p\right) \quad \text{for } n \in \mathbb{N}.
\]
Moreover, for every integer \( n \geq 2 \) and \( k \in \{1, \ldots, n - 1\} \) the random variables \( \eta_k, \xi_{k+1}, \ldots, \xi_n \) are independent. Hence, if \( p \in (0, 1) \), then for every \( n \in \mathbb{N} \) we have
\[
\int_{\Omega^\infty} \rho(f^n(x, \omega), x)^p P^\infty(d\omega) \leq \mathbb{E}\left(\sum_{k=1}^{n} \eta_k \prod_{j=k+1}^{n} \xi_j^p\right) = \sum_{k=1}^{n} \mathbb{E}(\eta_k^p) \prod_{j=k+1}^{n} \mathbb{E}(\xi_j^p) = \sum_{k=1}^{n} \mathbb{E}(\eta_k^p) \frac{1 - \mathbb{E}(\xi_j^p)^n}{1 - \mathbb{E}(\xi_j^p)} \leq \mathbb{E}(\eta_k^p) \frac{1}{1 - \mathbb{E}(\xi_j^p)} \beta_p(x).
\]
If \( p \in [1, \infty) \), then by Minkowski’s inequality for every \( n \in \mathbb{N} \) we have
\[
\left(\int_{\Omega^\infty} \rho(f^n(x, \omega), x)^p P^\infty(d\omega)\right)^{1/p} \leq \sum_{k=1}^{n} \left(\mathbb{E}(\eta_k) \prod_{j=k+1}^{n} \mathbb{E}(\xi_j^p)\right)^{1/p} \leq \frac{1}{1 - \mathbb{E}(\xi_j^p)} \beta_p(x)^{1/p}.
\]

**Corollary 3.** Assume that either

(i) \((H_\delta)\) holds with a \( \delta \in [\frac{1}{2}, 1] \) and \( \psi : X \to \mathbb{R} \) is Hölder continuous with exponent \( \delta \),

or

(ii) \((H_{\frac{1}{2}})\) is satisfied and \( \psi : X \to \mathbb{R} \) is Hölder continuous with an exponent

\( \delta \leq \frac{1}{2} \).

Then for every bounded and nonempty \( A \subset X \) and for almost all \( \omega \in \Omega^\infty \) with respect to \( P^\infty \),
\[
\lim_{n \to \infty} \sup \left\{ \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(x, \omega)) - \int_{X} \psi d\pi^f \right\} : x \in A \right\} = 0.
\]

**Proof.** It concerns both, (i) and (ii).
By induction,
\[ \rho(f^n(x, \omega), f^n(z, \omega)) \leq \left( \prod_{k=1}^{n} \xi_k(\omega) \right) \rho(x, z) \]
for \( x, z \in X, \, \omega \in \Omega^\infty \) and \( n \in \mathbb{N} \), with
\[ \xi_n(\omega_1, \omega_2, \ldots) = \xi(\omega_n) \quad \text{for} \quad (\omega_1, \omega_2, \ldots) \in \Omega^\infty \quad \text{and} \quad n \in \mathbb{N}. \]

Hence
\[ |\psi(f^n(x, \omega)) - \psi(f^n(z, \omega))| \leq L \left( \prod_{k=1}^{n} \xi_k(\omega)^\delta \right) \rho(x, z)^\delta \]
for \( x, z \in X, \, \omega \in \Omega^\infty \) and \( n \in \mathbb{N} \), with an \( L \in (0, \infty) \).

Fix \( z \in X \). Since, for every \( x \in X, \, \omega \in \Omega^\infty \) and \( n \in \mathbb{N} \),
\[
\left| \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(x, \omega)) - \int_X \psi d\pi^f \right| \leq \frac{1}{n} \sum_{k=1}^{n} |\psi(f^k(x, \omega)) - \psi(f^k(z, \omega))| + \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(z, \omega)) - \int_X \psi d\pi^f \]
\[
\leq L \frac{1}{n} \sum_{k=1}^{n} \left( \prod_{j=1}^{k} \xi_j(\omega)^\delta \right) \rho(x, z)^\delta + \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(z, \omega)) - \int_X \psi d\pi^f ,
\]
for every \( r \in (0, \infty) \) and for every nonempty subset \( A \) of the ball with center at \( z \) and radius \( r \), for every \( \omega \in \Omega^\infty \) and \( n \in \mathbb{N} \) we have
\[
\sup \left\{ \left| \frac{1}{n} \sum_{k=1}^{n} \psi \circ f^k(x, \omega) - \int_X \psi d\pi^f \right| : x \in A \right\} \leq Lr^\delta \frac{1}{n} \sum_{k=1}^{n} \prod_{j=1}^{k} \xi_j(\omega)^\delta + \frac{1}{n} \sum_{k=1}^{n} \psi(f^k(z, \omega)) - \int_X \psi d\pi^f .
\]
In view of Theorem 3 and Corollary 2, to finish the proof it is enough to show that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \prod_{j=1}^{k} \xi_j^\delta = 0 \quad \text{a.e. for} \quad P^\infty . \quad (14)
\]
To this end observe that, by Jensen’s inequality, in the first case (i) we have
\[ \mathbb{E}(\xi^\delta) = \mathbb{E}\left((\xi^2)^{\frac{\delta}{2}}\right) \leq \left(\mathbb{E}(\xi^2)\right)^{\frac{\delta}{2}} < 1 \]
and in the second one
\[ \mathbb{E}(\xi^\delta) \leq \left(\mathbb{E}(\xi)\right)^{\delta} < 1 .\]
Therefore, applying the monotone convergence theorem and independence of \( \xi_n, n \in \mathbb{N} \), we get
\[
E \left( \sum_{n=1}^{\infty} \prod_{k=1}^{n} \xi_k^\delta \right) = \sum_{n=1}^{\infty} E \left( \prod_{k=1}^{n} \xi_k^\delta \right) = \sum_{n=1}^{\infty} \prod_{k=1}^{n} E (\xi_k^\delta) = \sum_{n=1}^{\infty} \left( E(\xi^\delta) \right)^n < \infty.
\]
Consequently, the series \( \sum_{n=1}^{\infty} \prod_{k=1}^{n} \xi_k^\delta \) converges a.e. for \( P^\infty \) and (14) follows.

5. An Application to Random Affine Maps

Corollary 4. Assume \( X \) is a closed subset of a separable Banach space containing the origin, \( \xi : \Omega \to \mathbb{R} \) and \( \eta : \Omega \to X \) are random variables such that \( \xi(\omega)X + \eta(\omega) \subset X \) for \( \omega \in \Omega \), and
\[
\zeta_n(\omega_1, \omega_2, \ldots) = \sum_{k=1}^{n} \left( \prod_{j=k+1}^{n} \xi(\omega_j) \right) \eta(\omega_k) \quad \text{for } (\omega_1, \omega_2, \ldots) \in \Omega^\infty, \ n \in \mathbb{N}.
\]
If either \( \delta \in (0, \frac{1}{2}] \) and
\[
E|\xi| < 1, \quad E\|\eta\| < \infty,
\]
or \( \delta \in [\frac{1}{2}, 1] \) and
\[
E(|\xi|^{2\delta}) < 1, \quad E(\|\eta\|^{2\delta}) < \infty,
\]
then there exists a probability Borel measure \( \mu \) on \( X \) such that
\[
\int_X \|x\| \mu(dx) < \infty
\]
and for every Hölder continuous with exponent \( \delta \) function \( \psi : X \to \mathbb{R} \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ \zeta_k = \int_X \psi d\mu \quad \text{a.e. for } P^\infty.
\]
Proof. The function \( f : X \times \Omega \to X \) defined by
\[
f(x, \omega) = \xi(\omega)x + \eta(\omega)
\]
is an rv-function. It satisfies (H) in the first case, and (H_\delta) in the second one. By induction,
\[
f^n(x, \omega_1, \omega_2, \ldots) = \left( \prod_{k=1}^{n} \xi(\omega_k) \right) x + \sum_{k=1}^{n} \left( \prod_{j=k+1}^{n} \xi(\omega_j) \right) \eta(\omega_k)
\]
for \( x \in X, (\omega_1, \omega_2, \ldots) \in \Omega^\infty \) and \( n \in \mathbb{N} \). Hence, \( \zeta_n = f^n(0, \cdot) \) for \( n \in \mathbb{N} \), so an application of Corollary 2 and Theorem 3 finishes the proof. \( \square \)
Remark 2. Let $\lambda \in (0,1)$ and let $\eta : \Omega \to [0,1 - \lambda]$ be a random variable. Put
\[ \zeta_n(\omega_1, \omega_2, \ldots) = \sum_{k=1}^{n} \lambda^{n-k} \eta(\omega_k) \]
for $(\omega_1, \omega_2, \ldots) \in \Omega^\infty$ and $n \in \mathbb{N}$. By Corollary 4 there exists a probability Borel measure $\mu$ on $[0,1]$ such that for every Hölder continuous $\psi : [0,1] \to \mathbb{R}$,
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \psi \circ \zeta_k = \int_{[0,1]} \psi d\mu \quad \text{a.e. for } P^\infty. \]
But, as observed in [2, Remark 4.3], if $(\psi \circ \zeta_n)_{n \in \mathbb{N}}$ converges in probability for a Borel $\psi : [0,1] \to \mathbb{R}$ such that
\[ c|x - z| \leq |\psi(x) - \psi(z)| \quad \text{for } x, z \in [0,1] \]
with a constant $c \in (0,\infty)$, then $\eta$ is a.s. for $P$ constant.

Acknowledgements

The research of Karol Baron was supported by the Institute of Mathematics of the University of Silesia in Katowice (Iterative Functional Equations and Real Analysis program). The research of Rafał Kapica was supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of Ministry of Science and Higher Education.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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Received: May 27, 2021.
Accepted: December 15, 2021.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.