On Fourier multipliers with rapidly oscillating symbols

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April 19, 2022

Abstract

We provide asymptotically sharp bounds for the $L_p$ norms of the Fourier multipliers with symbols $e^{i\lambda \varphi(\cdot/|\xi|)}$, where $\lambda \in \mathbb{R}$ is a large parameter.

1 Setting

The motivation for writing this note comes from Maz’ja’s Problem 4 in [6]. We provide an imprecise citation.

'Consider the singular integral operator $A_\lambda$ with the symbol $\partial B_1 \ni \omega \mapsto \exp(i\lambda \varphi(\omega))$, where $\varphi$ is a smooth real-valued function on $\partial B_1$, and $\lambda$ is a large real parameter. Find the sharp value of the parameter $\kappa$ such that the estimate

$$
\|A_\lambda\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq c|\lambda|^{|d|/2 - 1/p},
$$

(1)

holds true; here $1 < p < \infty$, $c$ depends on $d, p$, and the function $\varphi$.'

This estimate is clearly true when $d = 1$ and $\kappa = 0$. We will solve the problem for larger $d$ by proving the following results.

Proposition 1. Let $d \geq 2$. There exists a smooth $\varphi$ defined on the unit sphere in $\mathbb{R}^d$ and a tiny constant $C$ such that

$$
\|A_\lambda\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \geq C|\lambda|^{|d|/2 - 1/p},
$$

(2)

provided $|\lambda|$ is sufficiently large.

The case $d = 2$ of Proposition 1 had been implicitly considered in [3] (see Theorem 6 of that paper). We also prove that the inequality reverse to (2) always holds true.

Theorem 1. Let $d, p$, and $\varphi$ be fixed. There exists a constant $c$ such that

$$
\|A_\lambda\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq c|\lambda|^{|d|/2 - 1/p}.
$$

(3)

A similar bound

$$
\|A_\lambda\|_{L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)} \leq c|\lambda|^{|d|/2 - 1/p}(\log |\lambda|)^{|d|/2 - 1/p},
$$

(4)

was proved in [5], where such type bounds were utilized to obtain sufficient conditions for the $L^p$-continuity of the operators $f \mapsto \int a(\cdot, \xi)\hat{f}(\xi)\,d\xi$. We will prove Proposition 1 in Sections 2 and 3 by an accurate...
computation that resembles the stationary phase method; the first of the two sections deals with the simpler case \( d = 2 \) and the other covers the more involved case of larger \( d \) (the presentation in Section 3 is slightly less detailed). Though the computations in the case \( d \geq 3 \) are longer, they follow the same route as in the \( d = 2 \) case. The proof of Theorem 1 is a combination of the sharp multiplier theorem in the spirit of Mikhlin and Hormander from [7] and soft interpolation techniques, see the details in Section 4.

Some results of this paper were independently obtained in [2]; namely, Theorem 1 and the cases of even \( d \) in Proposition III are considered there. We note that [2] suggests a more fundamental treatment of the whole circle of problems similar to Maz’ja’s problem discussed here. I wish to thank Vladimir Maz’ja for attracting my attention to the problem and fruitful discussions, and to Vjekoslav Kovač for providing me with the references [2] and [3].

2 Example for \( d = 2 \)

Let \( \Phi \) be a non-negative smooth function of a single variable supported in \([1/2, 3/2]\) and non-zero on \([2/3, 4/3]\). Let \( \chi \) be another smooth function supported in \([-1/2, 1/2]\), non-zero on \([-1/3, 1/3]\), and not exceeding one everywhere. Finally, assume that \( \varphi : \mathbb{R} \to \mathbb{R} \) is a \( 2\pi \)-periodic smooth function whose value coincides with \( 2\pi \theta \) when \( \theta \in [-1, 1] \). Consider the function \( M_\lambda : \mathbb{R}^2 \to \mathbb{R} \),

\[
M_\lambda(\xi, \eta) = e^{i\lambda \varphi(\chi)} \chi(\theta) \Phi(\rho) / \rho, \quad \text{where } \xi = \rho \cos \theta, \eta = \rho \sin \theta, \quad \theta \in (-\pi, \pi), \quad \text{and } \rho > 0,
\]

(we have the polar change of variables in mind). We reserve the names \( x, y \) for the variables on the ‘real’ Fourier side. We also set

\[
x = r \sin \theta_{x,y} \quad \text{and} \quad y = r \cos \theta_{x,y}, \quad \text{where } \theta_{x,y} \in (-\pi, \pi) \text{ and } r > 0,
\]

(6)

for the dual polar coordinates (we interchange the cosine and sine functions for convenience of further computations).

Lemma 2. The inequality

\[
|r M_\lambda(x, y) - e^{-2\pi i \lambda \theta_{x,y}} \Phi(\lambda/r) \chi(-\theta_{x,y})| \leq 0.1
\]

(7)

holds true provided \( r \in [3\lambda/4, 3\lambda/2], \theta_{x,y} \in [-1/10, 1/10], \text{ and } \lambda \text{ is sufficiently large.} \)

Proof. We write down the definition of the Fourier transform and perform the polar change of variables:

\[
\hat{M}_\lambda(x, y) = \int_{\mathbb{R}^2} M_\lambda(\xi, \eta) e^{-2\pi i (x\xi + y\eta)} \, d\xi \, d\eta = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda \varphi(\chi)} \chi(\theta) \Phi(\rho) e^{-2\pi i (x\rho \cos \theta + y\rho \sin \theta)} \, d\rho \, d\theta =
\]

\[
\int_{\mathbb{R}} \hat{\Phi}(x \cos \theta + y \sin \theta) e^{i\lambda \varphi(\chi)} \chi(\theta) \, d\theta = \int_{\mathbb{R}} \hat{\Phi}(r \sin(\theta + \theta_{x,y})) e^{i\lambda \varphi(\chi)} \chi(\theta) \, d\theta =
\]

\[
\int_{\mathbb{R}} \hat{\Phi}(r \sin \theta) e^{i\lambda \varphi(\theta - \theta_{x,y})} \chi(\theta - \theta_{x,y}) \, d\theta = \int_{\mathbb{R}} \hat{\Phi}(r \sin \theta) e^{2\pi i \lambda (\theta - \theta_{x,y})} \chi(\theta - \theta_{x,y}) \, d\theta,
\]

(8)

by our requirements on the function \( \varphi \). We multiply this identity by \( r \) and make yet another change of variable:

\[
r M_\lambda(x, y) = e^{-2\pi i \lambda \theta_{x,y}} \int_{\mathbb{R}} \hat{\Phi}(r \sin \theta) e^{2\pi i \lambda s} \chi\left(\frac{s}{r} - \theta_{x,y}\right) \, ds.
\]

(9)
We note that since $\chi$ is supported in $[-1/2, 1/2]$ and $|\theta_{x,y}| \leq 1/10$, the integrand is non-zero only when $|s| \leq r$. In particular, $|r \sin s/r| \geq s/10$. Let $R$ be a fixed large number such that
\[
|\hat{\Phi}(s)| \leq |s|^{-2}, \quad \text{when } |x| > R, \quad \text{and} \quad \int_{|x| > R} \frac{dx}{x^2} < 10^{-5}. \tag{10}
\]

Then, since we assume $\|\chi\|_{L_\infty} \leq 1$,
\[
\left| \int_{|s| > R} \hat{\Phi} r \sin \frac{S}{r} e^{2\pi i \lambda x} \chi \left( \frac{S}{r} - \theta_{x,y} \right) ds \right| \leq \int_{|s| > R} \frac{100}{s^2} ds < \frac{1}{100}. \tag{11}
\]
Thus, we need to show that
\[
\int_{|s| < R} \hat{\Phi} r \sin \frac{S}{r} e^{2\pi i \lambda x} \chi \left( \frac{S}{r} - \theta_{x,y} \right) ds \tag{12}
\]
is close to $\Phi(\lambda/r)\chi(-\theta_{x,y})$. For that, we wish to replace $r \sin \frac{\theta}{r}$ by $s$. Note that $|s| < R$ and $r$ is large. Therefore,
\[
\left| \hat{\Phi} r \sin \frac{S}{r} - \hat{\Phi}(s) \right| \leq \|\Phi\|_{\text{Lip}} \frac{|r|}{r^3} = \|\hat{\Phi}\|_{\text{Lip}} \frac{|s|^3}{s^2}. \tag{13}
\]
What is more,
\[
r^{-2} \int_{|s| < R} |s|^3 \chi \left( \frac{S}{r} - \theta_{x,y} \right) ds = O(|\lambda|^{-2}) \to 0, \tag{14}
\]
when $\lambda \to \infty$ and $r \approx \lambda$. Thus, it suffices to prove
\[
\int_{|s| < R} \hat{\Phi}(s) e^{2\pi i \lambda x} \chi \left( \frac{S}{r} - \theta_{x,y} \right) ds \tag{15}
\]
is close to $\Phi(\lambda/r)\chi(-\theta_{x,y})$. Using the same estimate as in (11), we see that this quantity is $1/100$-close to
\[
\int_{R} \hat{\Phi}(s) e^{2\pi i \lambda x} \chi \left( \frac{S}{r} - \theta_{x,y} \right) ds, \tag{16}
\]
which converges to the desired value. \qedhere

**Remark 3.** The convergence is uniform with respect to $\theta_{x,y} \in [-1/10, 1/10]$ and $r/\lambda \in [3/4, 3/2]$.

**Remark 4.** One may prove the asymptotic formula
\[
r \hat{M}_\lambda(x, y) \to e^{-2\pi i \lambda \theta_{x,y}} \Phi(\lambda/r)\chi(-\theta_{x,y}) \tag{17}
\]
by choosing $R$ slowly tending to infinity as $\lambda \to \infty$.

**Corollary 5.** Let us denote the Fourier multiplier with the symbol $M_\lambda$ by the same symbol. Then, there exists a tiny positive constant $c$ such that
\[
\|M_\lambda\|_{L_p \to L_p} \geq c\lambda^{\frac{p}{p-1}}, \quad p \in [1, \infty], \tag{18}
\]
provided $\lambda$ is sufficiently large.
Since the norms are the \( \tilde{\Phi} \):
The function use (7):
\[
\|K\|_{L^p} \geq c\lambda, \quad \text{where} \quad \lambda \text{ is so large that (7) holds true.}
\]

Then, by our assumptions about the functions \( \Phi \) and \( \chi \),
\[
\tilde{M}(x, y) \gtrsim \lambda^{-1}, \quad (x, y) \in \mathcal{R}, \quad \text{where}
\]
\[
\mathcal{R} = \left\{ (x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} \in [4\lambda/5, 5\lambda/4] \text{ and } |x|/|y| < \frac{1}{100} \right\}.
\]

The notation \( A \gtrsim B \) means \( A \geq CB \) for a uniform constant \( C \) (with respect to \( \lambda \)). The area of \( \mathcal{R} \) is bounded away from zero by \( c\lambda^2 \), which leads to the desired estimate.

The estimate for the \( L_p \rightarrow L_p \) norm is a little bit trickier. Let \( f \) be the characteristic function of a small ball centered at the origin. We will show that
\[
|Kf(x, y)| \gtrsim \lambda^{-1}, \quad \text{when } (x, y) \in \mathcal{R},
\]
and the parameter \( \lambda \) is sufficiently large, and the radius of the ball is sufficiently small (1/100 suffices). We use (7):
\[
|Kf(x, y)| = \left| \int_{(u, z) \in B_{0,01}(x, y)} K\chi(u, z) \, du \, dz \right| \geq \int_{(u, z) \in B_{0,01}(x, y)} e^{-2\pi\lambda\theta r^{-1}\Phi(\lambda/r)\chi(-\theta)} \, du \, dz - \frac{1}{1000\lambda}.
\]

here \( r = \sqrt{u^2 + z^2} \) and \( \theta = \theta_{u,z} \). The absolute value of the latter integral may be rewritten as
\[
\left| \int_{(u, z) \in B_{0,01}(x, y)} e^{-2\pi\lambda(\theta - \theta_{x,y}) r^{-1}\Phi(\lambda/r)\chi(-\theta)} \, du \, dz \right|.
\]

Since \( |\lambda(\theta - \theta_{x,y})| < 1/10 \) on the domain of integration (since the function \( (u, z) \mapsto \lambda\theta_{u,z} \) is 10-Lipschitz there), the absolute value of the integral is comparable to \( (x^2 + y^2)^{-1/2}\Phi(\lambda/\sqrt{x^2 + y^2})\chi(-\theta_{x,y}) \) and (21) is proved.

By (21), we have
\[
\|Kf\|_{L_p} \gtrsim \lambda^{-1}|\mathcal{R}|^{1/p} = \lambda^{2/p - 1},
\]
which finishes the proof.

Proof of Proposition 7. The function \( (\xi, \eta) \mapsto \chi(\theta)\Phi(\rho)/\rho \) is smooth and compactly supported, denote the Fourier multiplier with this symbol by \( T \). Then, \( M = TA \) and \( \|M\| \leq \|T||A\| \lesssim \|A\| \), where the norms are the \( L_p \rightarrow L_p \) norms of the operators.

3 Example for \( d \geq 3 \)

Consider the function
\[
M\lambda(\xi) = e^{2\pi i \lambda \varphi(\xi/|\xi|)} \hat{\Phi}(\xi/|\xi|) \hat{\chi}(\xi/|\xi|), \quad \xi \in \mathbb{R}^d.
\]

The function \( \hat{\Phi} : \mathbb{R} \rightarrow \mathbb{R} \) is a smooth non-negative function supported inside \([1/2, 3/2]\). The functions \( \varphi \) and \( \hat{\chi} \) are defined on the unit sphere. We suppose \( \hat{\chi} \) is smooth, non-negative, and supported in a tiny neighborhood \( U_\varepsilon \) of \((1, 0, \ldots, 0) \). As for the function \( \varphi \), we provide an explicit formula:
\[
\varphi(\zeta) = |\zeta_1 - 1|^2 + \sum_{j=2}^d \zeta_j^2 = 2 - 2\zeta_1, \quad \zeta \in S^{d-1}.
\]
The main feature we will use is that this linear function possesses certain 'curvature' when restricted to the unit sphere. In particular, if we restrict the function \( \varphi \) to the intersection of \( S^{d-1} \) with a linear hyperplane passing through \( U_c \), \( \varphi \) attains the minimal value only at the point that has maximal first coordinate on this intersection.

We wish to compute the value \( \tilde{M}_A(x) \). More specifically, we will prove the following analog of Lemma 2.

It yields the case \( d \geq 3 \) in Proposition 1 in the same way as Lemma 2 yields Corollary 5 and the case \( d = 2 \) of Proposition 1.

**Proposition 6.** There exists a parallelepiped \( R_\lambda \) with dimensions \( \gtrsim \lambda \) and two functions \( L_\lambda : R_\lambda \to \mathbb{R} \) and \( A_\lambda : R_\lambda \to \mathbb{R}_+ \) such that

\[
\left| \tilde{M}_A(x) - e^{2\pi i L_\lambda(x)} A_\lambda(x) \right| \lesssim \lambda^{-\frac{d}{2}-1},
\]

the function \( L_\lambda \) is \( O(1) \)-Lipschitz, and \( A_\lambda(x) \gtrsim \lambda^{-d/2} \) when \( x \in R_\lambda \), provided \( \lambda \) is sufficiently large.

**Proof.** Let \( r = |x| \) and assume \( x/|x| \) is close to the vector \( (0, 1, 0, 0, \ldots, 0) \) (the specific vector is chosen for notational convenience, the only important thing is that it is orthogonal to \((1, 0, 0, \ldots, 0)\)). Later we will see that \( x/|x| \) should be close, but not equal to \((0, 1, 0, \ldots, 0)\). We will need to use another spherical coordinate system, we will call it 'new' (its choice depends on \( r \), and the one we have worked in before is called 'stationary' (the stationary system is simply the usual Euclidean coordinate system, not a spherical one). The coordinates in the new system will be \((\rho, \tilde{\theta})\), where \( \rho \in \mathbb{R}_+ \) and \( \tilde{\theta} \in S^{d-1} \). In the new coordinate system, \( x \) has coordinates \((r, 0, 1, 0, \ldots, 0)\). Let \( z_x \) be the point on the intersection of the linear hyperplane \( x^+ \) with the unit sphere at which \( \varphi \) attains its minimum (on this intersection).

One may see \( z_x \) is close to \((1, 0, 0, \ldots, 0) \) (in the stationary system). We choose the new system in such a way that \( z_x \) has coordinates \((1, 1, 0, 0, \ldots, 0)\). This information completely defines the new coordinate system in the case \( d = 3 \); in higher dimensions we make some choice of possible new coordinate systems.

We go further and parametrize \( \tilde{\theta} \) with the points on the tangent plane at the point \( z_x \):

\[
\tilde{\theta} = \left( \cos \theta_2 \cos \theta_3 \ldots \cos \theta_d, \sin \theta_2, \cos \theta_2 \sin \theta_3, \ldots, \cos \theta_2 \cos \theta_3 \ldots \sin \theta_d \right).
\]

Here \( \tilde{\theta} = (\theta_2, \theta_3, \ldots, \theta_d) \in \mathbb{R}^{d-1} \) lies inside a small ball \( B_r(0) \). We have

\[
\tilde{M}_A(x) = \int_{B_r(0)} \int_{\frac{2\pi}{d}} e^{2\pi i (-r \rho \sin \theta_2 + \lambda \varphi_x(\theta))} \tilde{\Phi}(\rho) \tilde{\chi}(\tilde{\theta}) J(\rho, \theta) d\rho d\tilde{\theta}.
\]

The function \( J(\rho, \theta) = \rho^{d-1} \tilde{J}(\tilde{\theta}) \) comes from the spherical change of variables. Note that \( \tilde{J}(\tilde{\theta}) \neq 0 \) on \( B_r(0) \). Let us specify the function \( \varphi_x : B_r \to \mathbb{R} \). Let \((1, 0, 0, \ldots, 0)\) in the stationary coordinates correspond to \((1, \alpha)\), \( \alpha \in S^{d-1} \), in the new coordinates. Then, with the notation \( \langle \cdot, \cdot \rangle \) for the Euclidean scalar product,

\[
\varphi_x(\theta) = 2 - 2 \langle \theta, \alpha \rangle = 2 - 2\alpha_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_d - 2\alpha_2 \sin \theta_2 - 2\alpha_3 \cos \theta_2 \sin \theta_3 - \ldots - 2\alpha_d \cos \theta_2 \cos \theta_3 \ldots \sin \theta_d.
\]

We have chosen our coordinate system in such a fashion that \( \varphi_x \), when restricted to the set

\[
x^+ \cap S^{d-1} = \{ \tilde{\theta} \mid \tilde{\theta}_2 = 0 \}
\]

(which is the same as \( \theta_2 = 0 \)), attains its minimum at the origin. Thus, \( \alpha_3 = \alpha_4 = \ldots = \alpha_d = 0 \). We may compute \( \alpha_2 = \langle x/|x|, (1, 0, 0, \ldots, 0) \rangle = x_1/|x| \). Recall we also have \( \alpha_1^2 + \alpha_2^2 = 1 \), so this equation completely defines \( \alpha \). Thus,

\[
\varphi_x(\theta) = 2 - 2\alpha_1 \cos \theta_2 \cos \theta_3 \ldots \cos \theta_d - 2\alpha_2 \sin \theta_2, \quad \alpha_2 = \frac{x_1}{|x|}, \quad \alpha_1 = \frac{\sqrt{|x|^2 - x_1^2}}{|x|}.
\]
Later we will require \( \alpha_2 \) to be non-zero.

Let us write \( \psi = \varphi_x, \Phi(\rho) = \rho^{2\pi} \hat{\Phi}(\rho), \) and \( \chi(\theta) = \chi(\hat{\theta}) \hat{J}(\theta) \) for brevity. Note that the function \( \chi \) depends on \( x/|x| \), we will only use that since \( \chi \) is non-zero on a small neighborhood of \( (1,0,0,\ldots,0) \) in the stationary system, \( \chi(0) \neq 0 \), provided \( x/|x| \) is sufficiently close to \( (0,1,0,\ldots,0) \). We wish to compute

\[
\int_{B_r(0)} \int r^2 e^{2\pi i (-r \sin \theta_2 + \lambda \psi(\theta))} \Phi(\rho) \chi(\theta) \, d\rho \, d\theta = \int_{B_r(0)} e^{2\pi i \lambda \psi(\theta)} \hat{\Phi}(r \sin \theta_2) \chi(\theta) \, d\theta. \tag{33}
\]

Let us call the latter integral simply \( I \). We will be using the notation \( \theta = (\theta_2, \theta_{[2]}), \) where \( \theta_{[2]} = (\theta_3, \theta_4, \ldots, \theta_d) \in \mathbb{R}^{d-2} \). We pick a large natural number \( N \) and write Taylor’s expansion

\[
\psi(\theta) = \psi_0(\theta_{[2]}) + \theta_2 \psi_1(\theta_{[2]}) + \sum_{j=2}^{N} \theta_2^{j} \psi_j(\theta_{[2]}) + R_N^\psi(\theta), \tag{34}
\]

where \( R_N^\psi(\theta) = O(|\theta_{[2]}|^{N+1}) \) as \( \theta_2 \to 0 \), the functions \( \psi_j \) are smooth, and \( \theta \in B_r(0) \). In particular,

\[
\psi_0(\theta_{[2]}) = \psi(0, \theta_{[2]}) \quad \text{and} \quad \psi_1(\theta_{[2]}) = \frac{\partial \psi}{\partial \theta_2}(0, \theta_{[2]}), \tag{35}
\]

which, in view of (33), turns into

\[
\psi_0(\theta_{[2]}) = -2\alpha_1 \cos \theta_1 \cos \theta_4 \ldots \cos \theta_d; \tag{36}
\]

\[
\psi_1(\theta_{[2]}) = -2\alpha_2, \tag{37}
\]

here \( \theta_{[2]} \in B_\delta(0) \), where \( \delta \) is a small number. We perform dilation with respect to \( \theta_2 \) (similar to (9)):

\[
r I = \int \exp \left( 2\pi i \lambda \left( \sum_{j=0}^{N} \left( \theta_2/r \right)^j \psi_j(\theta_{[2]}) + R_N^\psi(\theta_2/r, \theta_{[2]}) \right) \right) \hat{\Phi}(r \sin(\theta_2/r)) \chi(\theta_2/r, \theta_{[2]}) \, d\theta. \tag{38}
\]

Note that since \( \hat{\Phi} \) is a Schwartz function, we may ’cut the Schwartz tail’ similar to (10):

\[
r I = \int_{|\theta_2| \leq r^\kappa} \exp \left( 2\pi i \lambda \left( \sum_{j=0}^{N-1} \left( \theta_2/r \right)^j \psi_j(\theta_{[2]}) + R_N^\psi(\theta_2/r, \theta_{[2]}) \right) \right) \hat{\Phi}(r \sin(\theta_2/r)) \chi(\theta_2/r, \theta_{[2]}) \, d\theta + O(r^{d-1}), \tag{39}
\]

here \( \kappa \) is a fixed small positive number to be specified later. Let us assume \( N \) is odd. We write the expansion

\[
\hat{\Phi}(r \sin \frac{\theta_2}{r}) = \hat{\Phi}(\theta_2 + \sum_{j=2}^{N-1} \frac{\theta_2^{j+1}}{(j+1)!r^j} + rR(\theta_2/r)), \tag{40}
\]

where \( R(t) = O(|t|^{N+1}) \) when \( t \to 0 \) and \( R \) is a smooth function. We require \( \kappa(N+1) \leq 1/10 \) (we choose \( \kappa \) sufficiently small), and, thus, may assume \( rR(\theta_2/r) = O(r^{-N+\kappa}) \) uniformly on the domain \( |\theta_2| \leq r^\kappa \). Expanding further (with Taylor’s formula for the function \( \Phi \)), we arrive at

\[
\hat{\Phi}(r \sin \frac{\theta_2}{r}) = \hat{\Phi}(\theta_2) + \sum_{j=1}^{N-1} r^{-j} \Psi_j(\theta_2) + R_N^\psi(\theta_2, r). \tag{41}
\]
Here \( R_N^q(\theta_2, r) = O(r^{-N + \frac{1}{2}}) \) uniformly when \(|\theta_2| \leq r^\kappa\), and, moreover, the \( \Psi_j \) are Schwartz functions. What is more, \( \chi(\theta_2/r, \theta_2[\cdot]) = \chi(0, \theta_2[\cdot]) + \sum_{j=1}^{N-1} \theta_2^j r^{-j} \chi_j(\theta_2[\cdot]) + R_N^q(\theta) \),

\[
\chi(\theta_2/r, \theta_2[\cdot]) = \chi(0, \theta_2[\cdot]) + \sum_{j=1}^{N-1} \theta_2^j r^{-j} \chi_j(\theta_2[\cdot]) + R_N^q(\theta),
\]

where \( R_N^q(\theta) = O((\theta_2/r)^N) \); in particular

\[
R_N^q(\theta) = O(r^{-N + \frac{1}{2}}) \quad \text{when} \quad |\theta_2| \leq r^\kappa.
\]

We may write yet another asymptotic formula

\[
\exp \left( 2\pi i \lambda \left( \sum_{j=2}^{N} \theta_2^j r^{-j} \psi_j(\theta_2[\cdot]) + R_N^q(\theta_2/r, \theta_2[\cdot]) \right) \right) - 1 = \sum_{k=1}^{N-1} \frac{(2\pi i \lambda)^k}{k!} \left( \sum_{j=2}^{N} \theta_2^j r^{-j} \psi_j(\theta_2[\cdot]) + R_N^q(\theta_2/r, \theta_2[\cdot]) \right)^k + O\left( \frac{(\lambda |\theta_2|^2)^N}{r^{2N}} \right) = \sum_{k=1}^{N-1} \sum_{j=2k}^{N} \lambda^k r^{-j} \psi_{k,j}(\theta) + O(r^{-N + \frac{1}{2}}),
\]

since we assume \( \lambda \asymp r \), \( |\theta_2| \leq r^\kappa \) and \( \kappa(N + 1) \leq 1/10 \). The functions \( \psi_{k,j}(\theta) \) are of the form \( p(\theta_2)q(\theta_2[\cdot]) \), where both factors are smooth. Plugging (41), (42), and (44) into (39), we obtain

\[
\Theta_k,j = \sum_{\alpha} a\alpha(\theta_2) b\alpha(\theta_2[\cdot]),
\]

where the \( a\alpha \) are Schwartz functions of a single variable, the \( b\alpha \) are smooth and compactly supported, and the index \( \alpha \) runs through a finite set. Recall \( \lambda/r \asymp 1 \) and let \( \lambda, r \to \infty \). We claim that the terms with the indices \( k,j \) in the expansion (46) behave as \( \lambda^{-d/2} \lambda^{-j+k} \). From now on we set \( N = 2d, \kappa = 1/(30d) \) and forget about the remainder term. Let us compute the asymptotically sharp value for the leading term of (45) and then say that the computation for lower order terms is similar. Here the computation is:

\[
\int_{|\theta_2| \leq r^\kappa} e^{2\pi i \lambda (\psi(\theta_2[\cdot]) + \theta_2 \psi_1(\theta_2[\cdot]))} \Phi(\theta_2) \chi(0, \theta_2[\cdot]) d\theta =
\]

\[
\int_{\mathbb{R}^{d-1}} e^{2\pi i \lambda (\psi(\theta_2[\cdot]) + \theta_2 \psi_1(\theta_2[\cdot]))} \Phi(\theta_2) \chi(0, \theta_2[\cdot]) d\theta + O(r^{-d}) =
\]

\[
\int_{\mathbb{R}^{d-2}} e^{2\pi i \lambda (\psi(\theta_2[\cdot]) + \theta_2 \psi_1(\theta_2[\cdot]))} \Phi(\theta_2) \chi(0, \theta_2[\cdot]) d\theta + O(r^{-d}).
\]

\[
\]
Recall that we have adjusted our new coordinates in the way that $\psi_0$ attains its minimal value at the origin only. One may see $\psi_0$ is smooth and, thus, falls under the scope of the standard stationary phase method (see, e.g., p. 344 in [5]):

$$
\int_{\mathbb{R}^{d-2}} e^{2\pi i \lambda \psi_0(\theta|\phi)} \Phi \left( \frac{\lambda}{r} \psi_1(\theta|\phi) \right) \chi(0, \theta|\phi) \, d\theta|\phi = \frac{c_0}{\sqrt{H}} \lambda^{-\frac{d+2}{2}} e^{2\pi i \lambda \psi_0(0)} \Phi \left( \frac{\lambda}{r} \psi_1(0) \right) \chi(0) + O(\lambda^{-\frac{d}{2}}), \quad \lambda \to \infty, \quad (48)
$$

where $c_0$ is an absolute constant and $H$ is the determinant of the Hesse matrix $\frac{\partial^2 \psi_0}{\partial t^2}(0)$ (one may observe from [53] that this value does not vanish).

Pick a tiny number $\nu > 0$ and consider a small spherical cap close to the vector $(0,1,0,\ldots,0)$ (in the stationary system) on which $x_1 \in (-1.1\nu|x|, -0.9\nu|x|)$. In particular, this cap does not contain the vector $(0,1,0,\ldots,0)$ itself. The number $\nu$ is chosen in such a way that $\tilde{\chi}(z_x) > 0$ when $x$ belongs to the said spherical cap. Then, we have

$$
\hat{M}_\lambda(x) = r^{-1} \lambda^{-\frac{d}{2}+1} e^{2\pi i \lambda \psi_0(0)} \Phi \left( -\frac{2\nu}{r|x|} \right) c_0(x) + O(\lambda^{-\frac{d+2}{2}}), \quad (49)
$$

where $c_0(x)$ is a smooth non-zero function (it equals a constant times a positive function). Indeed, the leading term in (45) is evaluated with the help of (48), whereas the lower order terms do not contribute stronger than $O(\lambda^{-\frac{d+2}{2}})$ by a similar formula.

We adjust $r$ in such a way that $\Phi \left( -\frac{2\nu}{r|x|} \right)$ is non-zero (in other words, $r \in [c_1 \lambda, c_2 \lambda]$, where $c_1$ and $c_2$ are absolute constants). Recall that $x/|x|$ belongs to a fixed spherical cap. This means there exists a parallelepiped $R_\lambda$ of dimensions $\gtrsim \lambda$ such that

$$
\hat{M}_\lambda(x) = e^{2\pi i \lambda \chi}, \quad x \in R_\lambda, \quad (50)
$$

where $L_\lambda$ is a 100-Lipschitz function and $A_\lambda \gtrsim \lambda^{-d/2}$.

4 Estimate

The proof of Theorem 1 relies upon the following sharp version of the Mikhlin–Hormander multiplier theorem. In this theorem, $\Phi$ is an auxiliary smooth radial non-zero function that is compactly supported in $\mathbb{R}^d \setminus \{0\}$.

**Theorem 2** (Theorem 1 in [7]). Let $m$ be a function in $\mathbb{R}^d$, let $M$ be the Fourier multiplier with the symbol $m$. Then,

$$
\|Mf\|_{L_{1,r}} \lesssim \left( \sup_{t>0} \|\Phi(\cdot)m(t\cdot)\|_{B^d_{2/2,1}} \right) \|f\|_{H_1}, \quad (51)
$$

provided $r > 2$.

The spaces $H_1$, $L_{1,r}$, and $B^d_{2,1}$ are the real Hardy class, the Lorentz, and the Besov spaces. The interpolation formulas (see Theorem 5.3.1 in [7] and [1], correspondingly)

$$
L_p = (L_{1,r}, L_2)_{2-2/p, p} \quad \text{and} \quad L_p = (H_1, L_2)_{2-2/p, p}, \quad p \in (1, 2), \quad (52)
$$

lead to the estimate

$$
\|M\|_{L_p \to L_p} \lesssim \|M\|_{H_1 \to L_1}^{2/p-1} \|M\|_{L_2 \to L_2}^{2-2/p} \lesssim \left( \sup_{t>0} \|\Phi(\cdot)m(t\cdot)\|_{B^d_{2,1}} \right)^{2/p-1} \quad (53)
$$
for any \( p \in (1, 2) \) (the constant depends on \( p \)), provided \( m \) is uniformly bounded. Since the symbol of \( A_\lambda \) is homogeneous of order zero, Theorem I follows from the multiplier theorem above and the lemma below.

**Lemma 7.** Let \( \Phi \) be a smooth function compactly supported in \( \mathbb{R}^d \setminus \{0\} \). Then,

\[
\left\| \Phi(\xi)e^{i\lambda \varphi(\xi/|\xi|)} \right\|_{B^{d/2,1}_2} \lesssim |\lambda|^{d/2}, \quad |\lambda| \geq 1.
\]

**Proof.** Let

\[
M_\lambda(\xi) = \Phi(\xi)e^{i\lambda \varphi(\xi/|\xi|)}, \quad \xi \in \mathbb{R}^d.
\]

The estimate

\[
\| \partial^\alpha M_\lambda \|_{L_\infty} \lesssim |\lambda|^{|\alpha|},
\]

where \( \alpha \) is an arbitrary multiindex, follows from direct differentiation. Since \( \Phi \) is compactly supported, (56) yields

\[
\| M_\lambda \|_{W^{d/2}_2} \lesssim |\lambda|^d, \quad |\lambda| \geq 1.
\]

Therefore, it suffices to prove the multiplicative bound

\[
\| G \|_{B^{d/2,1}_2} \lesssim \| G \|_{L_2}^{\frac{1}{2}} \| G \|_{W^d_2}^{\frac{1}{2}}.
\]

A slightly simpler estimate

\[
\| G \|_{W^{d/2}_2} \lesssim \| G \|_{L_2}^{\frac{1}{2}} \| G \|_{W^d_2}^{\frac{1}{2}}
\]

follows from the Cauchy–Schwarz inequality. The inequality (58) is a little bit more tricky. It follows from the interpolation formula

\[(L_2, W^{d/2}_2)_{\frac{1}{2}, 1} = B^{d/2,1}_2
\]

(see [1], Theorem 6.2.4). We provide an elementary proof of (58) in the appendix.

5 Appendix

Let \( B_r(x) \) be the Euclidean ball centered at \( x \in \mathbb{R}^d \) and of radius \( r > 0 \). Let

\[
a_k = \| \hat{G} \|_{L_2(B_{2k}(0)) \setminus B_{2k-1}(0)), \quad k \geq 1,
\]

and \( a_0 = \| \hat{G} \|_{L_2(B_1(0))} \). Then,

\[
\| G \|_{L_2} = \left( \sum_{k \geq 0} a_k^2 \right)^{\frac{1}{2}}, \quad \| G \|_{W^d_2} \asymp \left( \sum_{k \geq 0} 2^{dk} a_k^2 \right)^{\frac{1}{2}}, \quad \text{and} \quad \| G \|_{B^{d/2,1}_2} = \sum_{k \geq 0} 2^{dk/2} a_k.
\]

With the notation \( A = 2^{d/2} \), (58) is rewritten as

\[
\sum_{k \geq 0} A^k a_k \lesssim \left( \sum_{k \geq 0} a_k^2 \right)^{\frac{1}{4}} \left( \sum_{k \geq 0} A^{4k} a_k^2 \right)^{\frac{1}{4}}.
\]
We raise the expression on the left hand side to the fourth power, use the AM-GM inequality together with summation of geometric series:

\[
\sum_{p,q,r,s \geq 0} A^{p+q+r+s} a_p a_q a_r a_s = 24 \sum_{p \geq q \geq r \geq s} A^{p+q+r+s} a_p a_q a_r a_s \leq 12 \sum_{p \geq q \geq r \geq s} \left( A^{-p+3q+r+s} a_q^2 a_s^2 + A^{3p-q+r+s} a_p^2 a_r^2 \right) \leq 12 \sum_{q,s} a_q^2 a_s^2 A^{3q+s} \sum_{p \geq q \geq r \geq s} A^{-p+q+r+s} \sum_{s' \leq s} A^{-q+s'} \leq A \left( \sum_{q,s} a_q^2 a_s^2 A^{3q+s} + \sum_{p,r} a_p^2 a_r^2 A^{3p+r} \right) \leq \sum_{k,l} a_k^2 a_l^2 A^{4l} = \left( \sum_{k \geq 0} a_k^2 \right) \left( \sum_{k \geq 0} A^{4k} a_k^2 \right), \tag{64}
\]

and arrive at the fourth power of the right hand side of (63).

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